Patching on Berkovich Spaces and the Local-Global Principle

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Abstract. Field patching, introduced by Harbater and Hartmann in [**33**], and extended by the aforementioned authors and Krashen in [**34**], has recently seen numerous applications. We present an extension of this technique to the setting of Berkovich analytic geometry and applications to the local-global principle.

In particular, we show that this adaptation of patching can be applied to Berkovich analytic curves, and as a consequence obtain local-global principles over function fields of curves defined over complete ultrametric fields. Because of the connection between the points of a Berkovich analytic curve and the valuations that its function field can be endowed with, one of these local-global principles is given with respect to completions, thus evoking some similarity with more classical versions. As an application, we obtain local-global principles for quadratic forms and results on the u-invariant. These findings generalize those of [**34**].

As a starting point for higher-dimensional patching in the Berkovich setting, we show that this technique is applicable around certain fibers of a relative Berkovich analytic curve. As a consequence, we prove a local-global principle over the germs of meromorphic functions on said fibers. By showing that said germs of meromorphic functions are algebraic, we also obtain local-global principles over function fields of algebraic curves defined over a larger class of ultrametric fields.

Résumé. **Recollement sur les espaces de Berkovich et principe local-global.** Le recollement sur les corps, introduit par Harbater et Hartmann dans [**33**], et étendu par ces auteurs et Krashen dans [**34**], a récemment trouvé de nombreuses applications. Nous présentons ici une extension de cette technique au cadre de la géométrie analytique de Berkovich et des applications au principe local-global.

Nous montrons que cette adaptation du recollement peut s'appliquer aux courbes analytiques de Berkovich, et par conséquent obtenons des principes locaux-globaux sur les corps de fonctions de courbes définies sur des corps ultramétriques complets. Grâce à la connexion entre les points d'une courbe analytique de Berkovich et les valuations dont on peut munir son corps de fonctions, nous obtenons un principe local-global par rapport à des complétés du corps de fonctions considéré, ce qui présente une ressemblance avec des versions plus classiques. En application, nous établissons des principes locaux-globaux dans le cas plus précis des formes quadratiques et en déduisons des bornes sur l'u-invariant de certains corps. Nos résultats généralisent ceux de [**34**].

Comme point de départ pour le recollement en dimension supérieure dans un cadre d'espaces de Berkovich, nous montrons que cette technique peut s'appliquer autour de certaines fibres d'une courbe analytique relative. Nous l'utilisons ensuite pour démontrer un principe local-global sur les germes des fonctions méromorphes sur ces fibres. En montrant que ces germes de fonctions méromorphes sont algébriques, nous obtenons aussi des principes locaux-globaux sur les corps de fonctions des courbes algébriques définies sur une famille plus vaste de corps ultramétriques.

Contents

Introduction

In this thesis we use the language of *Berkovich's theory* to prove results on the *local-global principle* as well as applications to quadratic forms and a related invariant. We do this by using the crucial tool of *patching*. This technique has seen many applications, and has recently become the main instrument in an ongoing series of papers. We extend patching from an algebraic setting to one of Berkovich spaces.

With the Berkovich point of view, patching becomes of highly geometric nature: it can be interpreted as the sheaf-theoretical gluing of meromorphic functions, thus providing clarity into the overall strategy of proof. This is one of the reasons why we believe this approach to be a nice framework for further generalizations.

More precisely, we show that patching is applicable to Berkovich analytic curves, and thus obtain a local-global principle over function fields of curves, generalizing the results of the founding paper [**34**]. We recall that a variety X defined over a field F is said to satisfy the local-global principle if there exists a family $(F_i)_i$ of fields containing F (from now on referred to as *overfields*) such that $X(F) \neq \emptyset$ if and only if $X(F_i) \neq \emptyset$ for all i. We provide two possible families of overfields in this setting: one appearing quite naturally in Berkovich's theory (germs of meromorphic functions), and one of more classical nature consisting of completions of the function field. The connection between the two is a consequence of the connection between the points of a Berkovich analytic curve and the valuations that its function field can be endowed with, which we make precise.

Said local-global principle is applicable to quadratic forms. This, combined with the nice algebraic properties of local rings of Berkovich analytic curves, allows us to obtain applications on the u-invariant.

As a first step towards higher dimensional versions of this technique, we show that patching is possible around certain fibers of a relative Berkovich analytic curve. This way, we obtain a local-global principle over the germs of meromorphic functions on said fibers, which is applicable to quadratic forms. As before, there are two possible families of overfields: the germs of meromorphic functions on the points of the fiber, and completions of the field of meromorphic germs. In particular, we show that the latter are algebraic.

By using the theory of projective limits of schemes, we also obtain a local-global principle over function fields of algebraic curves over a larger class of ultrametric fields (which aren't necessarily complete).

Presentation of the major directions

Local-global principle. The local-global principle first appeared in the '20s under the name *Hasse-Minkowski principle*, which states that a rational quadratic form has non-trivial solutions over $\mathbb{Q}$ if and only if it has non-trivial solutions over $\mathbb{R}$ and $\mathbb{Q}_p$ for any prime number p. Modern versions deal with varieties defined over a field K, which have

a K-rational point if and only if they have K_i rational points for all i, where $(K_i)_i$ is a family of overfields of K.

The local-global principle does not always hold. Amongst the first counter-examples was one given by Reichardt and Lind, who showed that the equation $2Y^2 = X^4 - 17Z^4$ has solutions over all the completions of $\mathbb{Q}$, but no rational solutions. There have since been found many other counter-examples. Determining for which fields, overfields, and varieties there is a local-global principle and studying the obstructions to this property has been an active area of research for decades (see for example [**9**] and [**51**]).

The development of arithmetic geometry brought powerful new techniques to the picture, causing the main focus to shift upon questions that have some geometrical meaning. More precisely, using the notation above, K is taken to be the function field of some algebraic variety, and the family of overfields are interpreted in a geometrical setting. So far, known results cover only special cases, with the majority concerning curves (see *e.g.* [**34**] and [**39**]). Moreover, typically, the family of overfields is one of completions of K with respect to discrete valuations (which can be read from a "fine enough" model of the curve).

A particular class of varieties that behaves well with respect to the local-global principle is the class of *homogeneous* varieties over certain linear algebraic groups (*e.g.* see [**16**] for a survey). We recall that given a field F, a variety X/F is said to be *homogeneous* over a linear algebraic group G/F if G acts on X and the group $G(\overline{F})$ acts transitively on the set $X(\overline{F})$, where $\overline{F}$ is an algebraic closure of F. For example, it was shown in [**9**] that, under certain additional conditions, the only obstruction to a local-global principle for homogeneous varieties is the so-called *Brauer-Manin obstruction* introduced by Manin in [**51**].

A new approach to local-global principles for homogeneous varieties over function fields of curves defined over complete discretely valued fields was introduced by Harbater, Hartmann, and Krashen in [**34**] via *patching*.

Patching. Patching techniques were introduced as one of the main approaches to inverse Galois theory. Originally of purely formal and geometric nature, this method provided a way to obtain a global Galois covering from local ones, see for example [**32**]. This is how the inverse Galois problem for $\mathbb{Q}_p(T)$, where p is a prime number, was shown to have an affirmative answer. Formal patching was translated to rigid geometry by Liu in [**50**]. Another example is [**61**], where Poineau used patching on analytic curves in the Berkovich sense and consequently generalized results shown by Harbater in [**30**] and [**31**].

In [**33**], Harbater and Hartmann combined formal patching with algebraic patching in the sense of [**40**], and this way extended the technique to structures over fields, while constructing a setup of heavily algebraic flavor. Since then, patching over fields has seen many applications and is the crucial ingredient in an ongoing series of papers (see *e.g.* [**34**], [**35**], [**39**], [**36**], [**17**]).

One of the main points of focus of these works are local-global principles over function fields of algebraic curves defined over complete discretely valued fields. Namely, it was this form of patching that provided a new approach to the local-global principles of homogeneous varieties over certain linear algebraic groups (for example see [**34**] and [**39**]).

In particular, in [**34**], Harbater, Hartmann, and Krashen (from now on referred to as HHK) obtained local-global principles for quadratic forms and results on the u-invariant,

generalizing those of Parimala and Suresh [**58**], which were proven through different methods. Another source for results on the u-invariant is Leep's article [**47**]. In [**34**], the authors apply the obtained local-global principles also to central simple algebras.

Let us briefly describe the overfields appearing in the local-global principles proven in [**34**]. Let k be a complete discretely valued field, and k° the corresponding valuation ring. Let π denote a uniformizer of k°. Let C/k be an algebraic curve. Let $\mathcal{C}$ be a normal irreducible projective flat model of C over k° with special fiber $\mathcal{C}_s$. Let F denote the function field of $\mathcal{C}$ (and hence of C). For any point $P \in \mathcal{C}_s$, set $R_P = \mathcal{O}_{\mathcal{C},P}$. We denote by $\widehat{R_P}$ the completion of the local ring R_P with respect to its maximal ideal. Set $F_P = \text{Frac } \widehat{R_P}$. Let U be a strict subset of an irreducible component of $\mathcal{C}_s$. Set $R_U = \bigcap_{P \in U} R_P$. We denote by $\widehat{R_U}$ the π-adic completion of R_U. Set $F_U = \text{Frac } \widehat{R_U}$.

Let $\mathcal{P}$ be any finite set of closed points of $\mathcal{C}_s$ containing all points at which the different irreducible components of $\mathcal{C}_s$ intersect. Let $\mathcal{U}$ be the set of connected components of $\mathcal{C}_s \backslash \mathcal{P}$. Then, the overfields in question are $\{F_P, F_U : P \in \mathcal{P}, U \in \mathcal{U}\}$. More precisely, HHK show that for a variety X/F satisfying certain conditions:

$$X(F) \neq \emptyset \iff X(F_P) \neq \emptyset, X(F_U) \neq \emptyset \text{ for all } P \in \mathcal{P}, U \in \mathcal{U}.$$

See subsection 3.3.2 for a somewhat more detailed account of the local-global principle of [**34**].

For a survey on the historic development of different variants of patching, see [**37**]. We have adapted field patching to the setting of *Berkovich spaces*.

Berkovich spaces. Tate's study of elliptic curves with bad reduction over $\mathbb{Q}_p$ in the '60s led to him developing the first approach to non-Archimedean analytic geometry, the so-called *rigid geometry* ([**63**]). Since $\mathbb{Q}_p$ is totally disconnected as a topological space, the naive approach of defining analytic functions to be locally given by convergent power series does not work because we wind up with too many analytic functions. An example of this is the function $f : \mathbb{Q}_p \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 0, \text{ if } |x|_p \leqslant 1 \\ 1, \text{ otherwise} \end{cases}$$

which would be analytic. In order to avoid this issue, Tate allows only certain opens and certain covers. Consequently, rigid spaces don't possess a genuine topology, but only a *Grothendieck* one.

Since then, there have been several other approaches to non-Archimedean analytic geometry: Raynaud's theory of formal models, Berkovich spaces, and Huber's adic geometry.

Developped in the late '80s (see [**6**]), Berkovich's approach was originally motivated by questions in spectral theory. Roughly speaking, Berkovich spaces are obtained by adding points to rigid spaces. This way one obtains topological spaces with nice properties such as local compactness and local arcwise-connectedness. As a consequence, these objects can be thought of geometrically. As opposed to rigid geometry, Berkovich spaces can also be defined over trivially valued fields.

An analogy can be drawn with the classical complex setting: analytic functions over certain analytic domains are convergent power series, there is a maximum modulus principle, a principle of analytic continuation, and GAGA-type theorems. There is also an analogy with algebraic geometry in the sense that Berkovich spaces, just like schemes,

have building blocks. The latter are called *affinoid spaces*. The main difference between these two settings (the algebraic and Berkovich analytic one) is that the building blocks of Berkovich spaces are compact, meaning also Hausdorff, so not always open. This is also a source of many difficulties in Berkovich's theory, seeing as there isn't a basis of open neighborhoods for which the sheaf of analytic functions is easy to describe.

Since its appearance, the theory of Berkovich spaces has been extended in several directions (*e.g.* Berkovich spaces over $\mathbb{Z}$ [**59**]), and many applications have been obtained, most of which, thanks to the GAGA theorems, to arithmetic geometry. These include: dynamical systems, the theory of p-adic *dessins d'enfants*, Bruhat-Tits buildings, inverse Galois theory, *etc.* See [**23**] and [**19**] for more. Recently, connections have been made between Berkovich's theory and other domains such as tropical geometry (*e.g.* [**2**]) and model theory (*e.g.* [**38**]).

Organization of the manuscript

The first chapter is dedicated to an introduction of the theory of Berkovich spaces. In Chapter 2, field patching is extended to a general formal setup that corresponds to Berkovich spaces. Chapter 3 deals with patching over Berkovich analytic curves and the corresponding applications to the local-global principle; its contents gave rise to an article titled "Patching over Berkovich Curves and Quadratic Forms", see [**54**]. Lastly, in Chapter 4, we show patching to be possible around certain fibers of relative analytic curves and obtain local-global principles as a consequence; the contents of this chapter will be the topic of an upcoming paper.

Here is a more detailed description of the organization of this manuscript.

Chapter 1: Introduction to Berkovich Spaces.

This chapter is aimed at giving an introduction to the theory of Berkovich spaces with the purpose of making the manuscript more self-contained. We give an overview of the construction of these objects starting from the basic algebraic setup on which it relies. The latter is a generalization by Berkovich of the algebraic counterpart of Tate's rigid geometry (more precisely, a generalization of the theory of Tate affinoid algebras).

A point of particular focus is the case of analytic curves, which is, arguably, the class of Berkovich spaces that is best understood, and also of most interest to us. We show some properties for them that will be needed for the next chapters. In particular, their graph-like structure gives rise to nice topological properties, which we use throughout the manuscript.

Another point of focus is the sheaf of meromorphic functions, which is crucial for the work presented in this manuscript seeing as patching is interpreted as the gluing of meromorphic functions over certain Berkovich spaces. Its construction is similar to the sheaf of meromorphic functions for schemes and so are the properties it satisfies.

We also provide a detailed description of a typical example of a Berkovich space, the *analytic affine line* $\mathbb{A}^{1,\mathrm{an}}$, and its points (see section 1.2 and subsection 1.8.4). For a complete ultrametric field $(k, |\cdot|)$, $\mathbb{A}^{1,\mathrm{an}}_k$ is the set of all the multiplicative semi-norms on $k[T]$ extending the norm of k. In particular, k is embedded in $\mathbb{A}^{1,\mathrm{an}}_k$ via $a \mapsto |\cdot|_a$, where for any polynomial $P(T) \in k[T]$, $|P(T)|_a := |P(a)|$. The set $\mathbb{A}^{1,\mathrm{an}}_k$ is endowed with a topology of pointwise convergence.

The analytic affine line has a tree-like structure with infinite branching. The following is an illustration of this Berkovich analytic space. By adding an "∞" point to the tree, we obtain the *analytic projective line* $\mathbb{P}^{1,\mathrm{an}}_k$. The k-points are situated in the extremities of the tree, or in other words, they are *leaves* of the tree. Of particular importance to us will be the non-extremal points where there is no branching (an example of such a point is given by x in Figure 1).

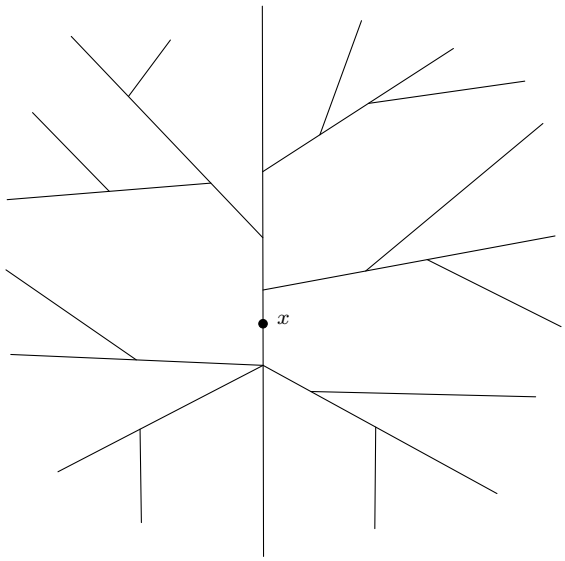


Figure 1: $\mathbb{A}^{1,\mathrm{an}}_k$

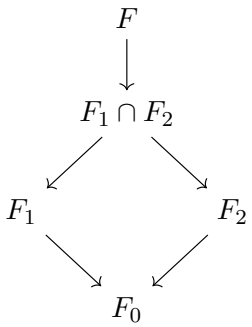
Most of the results of this chapter are well-known in the field and we only provide references for them. Others are more specialized and, to our knowledge, not found in the litterature, so we give proofs.

Chapter 2: Patching.

The general abstract setup for patching is the following.

Let the diagram below be a tower of fields. Suppose we are given *algebraic structures* $\mathcal{A}_1$ and $\mathcal{A}_2$ over F_1 and F_2, respectively. The goal is to find conditions under which they induce an algebraic structure of the same kind over $F_1 \cap F_2$. Typically, these algebraic structures are F_1, resp. F_2-rational points of some variety defined over the smaller field F. Another example would be zero-cycles of degree one.

(1)



We focus on the case where these algebraic structures are rational points of some variety H/F. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible over F_0, then they lift to $F_1 \cap F_2$. However, if this is not the case, then we can't in general lift them to $F_1 \cap F_2$. One way of approaching this problem is to find a way to *render* the rational points compatible on F_0.

Ideally, there exists a linear algebraic group G/F acting on H in such a way that these rational points (or more generally, algebraic structures) can always be altered just enough via the action of G in order to be *made* compatible over F_0. To make this work, we need not only a special action of G on H (which we give in Definition 3.2.1), but also that G itself satisfy certain conditions. The latter is the point of interest of this chapter; let us now make it more precise.

Let G/F be a linear algebraic group. The condition we need for the setup above is the following: for any $g \in G(F_0)$, there exist $g_i \in G(F_i)$, $i = 1, 2$, such that $g = g_1 \cdot g_2$ in $G(F_0)$. To see this, suppose g is such that $g \cdot \mathcal{A}_2 = \mathcal{A}_1$ in $H(F_0)$. Set $\mathcal{A}_1' := g_1^{-1} \cdot \mathcal{A}_1 \in H(F_1)$ and $\mathcal{A}_2' := g_2 \cdot \mathcal{A}_2 \in H(F_2)$. Then, by construction, $\mathcal{A}_1' = \mathcal{A}_2'$ in $H(F_0)$, so they lift to $F_1 \cap F_2$. The existence of an element $g \in G(F_0)$ satisfying $g \cdot \mathcal{A}_2 = \mathcal{A}_1$ is at the source of a hypothesis we will adopt on the action of G over H (see Definition 3.2.1).

From now on, we will refer to the "matrix decomposition" property of the paragraph above as *patching*. The following class of linear algebraic groups will be shown to satisfy patching (with a certain choice of fields appearing in the tower above).

Definition. A linear algebraic group G/F is said to be *rational* (over F) if there exists a Zariski open subset of G isomorphic to a Zariski open subset of $\mathbb{A}^n_F$ for some $n \in \mathbb{N}$.

We fix a general formal setup (Setting 2.1.1) over which we show the main result of this chapter (see Theorem 2.1.10). The latter is fundamental to patching. It is a generalization of [**34**, Theorem 3.2]. The main difference is that the objects considered in *loc.cit.* are defined over a complete *discretely valued* field, whereas we don't require the discretness assumption. A rather direct consequence is that patching is true in a Zariski neighborhood of the identity of G. The proof (and statement) of Theorem 2.1.10 is of very technical nature, and follows the main lines of the proof of [**34**, Theorem 3.2].

The interest of the formal setting over which we work is that it is realised in a natural (and very geometrical) way in Berkovich's theory.

In particular, we show that in a special case of Setting 2.1.1, which is realised by Berkovich analytic curves, Theorem 2.1.10 can be strengthened to show that patching is true in G (Theorem 2.2.3). This is the fundamental tool to showing that patching is possible over Berkovich analytic curves.

Chapter 3: Patching over Berkovich Curves and Quadratic Forms.

In this chapter we show that patching is possible over Berkovich analytic curves and that it can be interpreted as the gluing of meromorphic functions. More concretely, we show that the fields F_i of diagram (1) can be chosen to be fields of meromorphic functions of certain parts (called *analytic domains*) of an analytic curve. We then use this to prove a local-global principle and provide applications to quadratic forms and the u-invariant. The results we obtain generalize those of [**34**].

Before presenting the main results of this chapter, let us introduce some terminology.

Definition (HHK). Let K be a field. Let X be a K-variety, and G a linear algebraic group over K. We say that G acts *strongly transitively* on X if G acts on X, and for any field extension L/K, either $X(L) = \emptyset$ or $G(L)$ acts transitively on $X(L)$.

In general, asking that G act strongly transitively on X is more restrictive than asking that X be homogeneous over G. However, it is shown in [**34**, Remark 3.9] that if G is a reductive linear algebraic group over K and X/K is a projective variety, then the two notions are equivalent.

Our main results, the local-global principles we show, are:

THEOREM. *Let k be a complete non-trivially valued ultrametric field. Let C be a normal irreducible projective k-algebraic curve. Denote by F the function field of C. Let X be an F-variety, and G a connected rational linear algebraic group over F acting strongly transitively on X.*

Let $V(F)$ be the set of all non-trivial rank 1 valuations on F which either extend the valuation of k or are trivial when restricted to k.

Denote by C^{an} the Berkovich analytification of C, so that $F = \mathscr{M}(C^{\mathrm{an}})$, where $\mathscr{M}$ denotes the sheaf of meromorphic functions on C^{an}. Then, the following local-global principles hold:

- *(Theorem 3.2.11) $X(F) \neq \emptyset \iff X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C^{\mathrm{an}}$.*
- *(Corollary 3.2.18) If* $\operatorname{char} k = 0$ *or X is a smooth variety, then:*
$$X(F) \neq \emptyset \iff X(F_v) \neq \emptyset \text{ for all } v \in V(F),$$
where F_v denotes the completion of F with respect to v.

The statement above remains true for affinoid curves if $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$, where $\sqrt{|k^\times|}$ denotes the divisible closure of the value group $|k^\times|$. Being a local-global principle with respect to completions, the second equivalence evokes some resemblance to more classical versions of local-global principles. The statement can be made to include trivially valued base fields, even though in this case we obtain no new information (since at least one of the overfields will be equal to F).

REMARK. In order to prove our main results, we need *less* than strong transitivity. More precisely, it suffices to assume that for any completion $\widehat{F}$ of F with respect to a valuation extending that of k, either $X(\widehat{F}) = \emptyset$ or the group $G(\widehat{F})$ acts transitively on the set $X(\widehat{F})$. We may even restrict to only certain completions, namely those for which $\deg \operatorname{tr}_{\widetilde{k}}\widetilde{\widehat{F}} = 0$ and $\operatorname{rank}_{\mathbb{Q}}|\widehat{F}^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q} = 1$, where $\widetilde{k}$ resp. $\widetilde{\widehat{F}}$, is the residue field of k, resp. $\widehat{F}$. (These are the completions with respect to the valuations induced by the *type 3 points* of the curve, see Definition 1.8.1).

We recall that for any finitely generated field extension F/k of transcendence degree 1, there exists a unique normal projective k-algebraic curve with function field F. Thus, the result of the theorem above is applicable to any such field F.

To show the local-global principles above, we construct certain covers of curves over which patching can be realised (the so-called *nice covers*, see Definition 3.1.6). For this, *type 3 points* are crucial. A type 3 point has nice topological and algebraic properties. More precisely, an analytic curve is a real graph over which a type 3 point has arity 2; also, the stalk of a type 3 point is a field. In Figure 1, x is a type 3 point. The existence of such points is equivalent to the condition $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$, which is why Theorem 3.2.11 is first shown under this hypothesis. The result is then shown in all generality by using arguments from model theory. Here is a quick outline of the proof in the case $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$.

We recall that an *affinoid domain* of a Berkovich analytic space is an analytic domain (meaning the structural sheaf is well-defined over it), isomorphic to an affinoid space.

Moreover, the intersection of affinoid domains of an analytic curve (and, more generally, of any *separated* Berkovich analytic space) is again an affinoid domain. Let us start by introducing the crucial notion of a nice cover.

Definition (Definition 3.1.6). A finite cover $\mathcal{U}$ of a k-analytic curve will be called *nice* if:

(1) the elements of $\mathcal{U}$ are connected affinoid domains with only type 3 points in their topological boundaries;
(2) for any different $U, V \in \mathcal{U}$, $U \cap V = \partial U \cap \partial V$, or equivalently, $U \cap V$ is a finite set of type 3 points;
(3) for any two different elements of $\mathcal{U}$, neither is contained in the other.

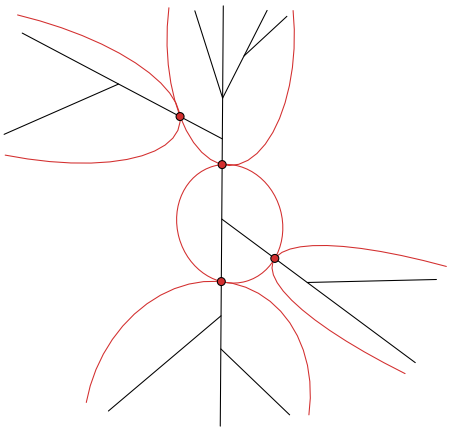


Figure 2: an example of a nice cover for the Berkovich projective line

For a normal irreducible projective analytic curve C, let U, V be connected affinoid domains of C such that $U \cap V$ is a single type 3 point $\{\eta\}$. (This is a special case of a nice cover.) Then, following the notation of diagram (1), set $F = \mathscr{M}(C)$, $F_0 = \mathscr{M}(\{\eta\})$, $F_1 = \mathscr{M}(U), F_2 = \mathscr{M}(V)$. We start by showing that patching is possible with this choice of fields for any rational linear algebraic group G/F. This is then generalized to any nice cover of the analytic projective line. To obtain the same generalization for any normal irreducible projective analytic curve C, we make use of the *Weil restriction of scalars* in order to "descend" to the case of the projective line (as HHK do in [**34**]).

Finally, once patching is shown to be possible, the local-global principle of Theorem 3.2.11 is a rather direct consequence.

To obtain Corollary 3.2.18 from Theorem 3.2.11, we establish a precise connection between the points of a Berkovich analytic curve and the valuations that its function field can be endowed with. This is done in Proposition 3.2.14. The rest is then a consequence of the nice algebraic properties of the fields $\mathscr{M}_x, x \in C$, namely their Henselianity.

While HHK work over models of an algebraic curve, we work directly over analytic curves. Remark that we put no restrictions on the complete valued base field k. Apart from the framework, this is one of the fundamental differences with Theorem 3.7 of [**34**], where the base field needs to be complete with respect to a discrete valuation. Another difference lies in the nature of the overfields, which here are completions or fields of meromorphic

functions. Section 4 shows that the latter contain the ones appearing in HHK's article, and thus that [**34**, Theorem 3.7] is a direct consequence of the local-global principle stated in Theorem 3.2.11. Moreover, we show the converse is true as well provided we choose a "fine" enough model. The proof of the theorem above is based on the patching method, but used in a different setting from the one of [**34**].

As a consequence of our main results, in the context of quadratic forms, we obtain the following theorem, which is a generalization of [**34**, Theorem 4.2]. This is because the projective variety determined by a quadratic form satisfies all of the hypotheses of Theorem 3.2.11 (including the existence of a connected rational linear algebraic group acting strongly transitively on it).

THEOREM. *Let k be a complete non-trivially valued ultrametric field. Let C be a normal irreducible projective k-algebraic curve. Denote by F the function field of C. Suppose $char(F) \neq 2$. Let q be a quadratic form over F of dimension different from 2.*

Let $V(F)$ be the set of all non-trivial rank 1 valuations on F which either extend the valuation of k or are trivial when restricted to k.

Let C^{an} be the Berkovich analytification of C, so that $F = \mathscr{M}(C^{\mathrm{an}})$, where $\mathscr{M}$ is the sheaf of meromorphic functions on C^{an}.

1. *(Theorem 3.4.1) The quadratic form q is isotropic over F if and only if it is isotropic over $\mathscr{M}_x$ for all $x \in C^{\mathrm{an}}$.*
2. *(Corollary 3.4.2) The quadratic form q is isotropic over F if and only if it is isotropic over F_v for all $v \in V(F)$, where F_v is the completion of F with respect to v.*

As mentioned in the introduction of [**34**], it is expected that for a "nice enough" field K the u-invariant remains the same after taking finite field extensions, and that it becomes $2^d u(K)$ after taking a finitely generated field extension of transcendence degree d. Since we work only in dimension one, this explains the motivation behind the following:

DEFINITION. Let K be a field.

1. [Kaplansky] The *u-invariant* of K, denoted by $u(K)$, is the maximal dimension of anisotropic quadratic forms over K. We say that $u(K) = \infty$ if there exist anisotropic quadratic forms over K of arbitrarily large dimension.
2. [HHK] The *strong u-invariant* of K, denoted by $u_s(K)$, is the smallest real number m, such that:
   - $u(E) \leqslant m$ for all finite field extensions E/K;
   - $\frac{1}{2}u(E) \leqslant m$ for all finitely generated field extensions E/K of transcendence degree 1.

   We say that $u_s(K) = \infty$ if there exist such field extensions E of arbitrarily large u-invariant.

The theorem above leads to applications on the u-invariant. Let k be a complete non-Archimedean valued field with residue field $\widetilde{k}$, such that $\mathrm{char}(\widetilde{k}) \neq 2$. Suppose that either $|k^\times|$ is a free $\mathbb{Z}$-module with $\mathrm{rank}_{\mathbb{Z}}|k^\times| =: n$, or, more generally, that $\dim_{\mathbb{Q}} \sqrt{|k^\times|} =: n$, where n is a non-negative integer. This is yet another difference with the corresponding results of HHK in [**34**], where the requirement on the base field is that it be complete discretely valued, *i.e.* that its value group be a free $\mathbb{Z}$-module of rank 1. We obtain an upper bound on the u-invariant of a finitely generated field extension of k with transcendence

degree at most 1, which depends only on $u_s(\widetilde{k})$ and n. More precisely, in terms of the strong u-invariant:

COROLLARY (Corollary 3.4.28). *Let k be a complete ultrametric field. Suppose $char(\widetilde{k}) \neq 2$.*

(1) *If $\dim_{\mathbb{Q}} \sqrt{|k^\times|} =: n, n \in \mathbb{Z}$, then $u_s(k) \leqslant 2^{n+1} u_s(\widetilde{k})$.*
(2) *If $|k^\times|$ is a free $\mathbb{Z}$-module with $\operatorname{rank}_{\mathbb{Z}} |k^\times| =: n, n \in \mathbb{Z}$, then $u_s(k) \leqslant 2^n u_s(\widetilde{k})$.*

Corollary 3.4.28 is a consequence of the local-global principle we obtained for quadratic forms (Theorem 3.4.1), and also of the very nice algebraic properties that the local rings of an analytic curve satisfy, especially Henselianity.

It is unknown to the author whether there is equality in the corollary above. This is true in the particular case of $n = 1$ by using [**34**, Lemma 4.9], whose proof is independent of patching. This way we recover [**34**, Theorem 4.10], which is the main result of [**34**] on quadratic forms. It also provides one more proof that $u(\mathbb{Q}_p(T)) = 8$, where p is a prime number different from 2, originally proven in [**58**].

COROLLARY (Corollary 3.4.30). *Let k be a complete discretely valued field such that $char(\widetilde{k}) \neq 2$. Then, $u_s(k) = 2u_s(\widetilde{k})$.*

Chapter 4: Patching over Analytic Fibers and the Local-Global Principle.

In this chapter we show that patching is possible "around" certain fibers of relative analytic curves. This is then applied to obtain a local-global principle over the field of overconvergent meromorphic functions on said fibers. We also show that the latter can be interpreted as the function field of a particular algebraic curve. As before, the local-global principles obtained are applicable to quadratic forms (provided the setting is one of characteristic different from 2).

The goal of this chapter is twofold:

(1) to establish the very first steps of a strategy for higher dimensional patching and the corresponding applications to the local-global principle;
(2) to generalize the results obtained in Chapter 3; more precisely, to show a local-global principle over algebraic curves (*i.e.* their function fields) defined over a larger class of ultrametric fields (which aren't necessarily complete).

One of the main results we show is the following (see Theorem 4.6.8 for the exact statement):

THEOREM (Theorem 4.6.8). *Let k be a complete non-trivially valued ultrametric field. Let S, C be good k-analytic spaces such that S is normal. Suppose that $\dim S < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$. Suppose there exists a morphism $\pi : C \to S$ that makes C a proper flat relative S-analytic curve. For any affinoid domain Z of S, set $C_Z := \pi^{-1}(Z)$, and $F_Z := \mathscr{M}(C_Z)$, where $\mathscr{M}$ denotes the sheaf of meromorphic functions on C. Let $x \in S$ be such that $\mathcal{O}_{S,x}$ is a field. Let $\mathscr{F}_x$ denote the fiber of x in C.*

Assume there exists a connected affinoid neighboorhood Z_0 of x such that: (1) all the fibers of π on Z_0 are normal irreducible projective analytic curves; (2) C_{Z_0} is normal; (3) $\pi_{|C_{Z_0}} : C_{Z_0} \to Z_0$ is algebraic.

Let G/F_{Z_0} be a connected rational linear algebraic group acting strongly transitively on a variety H/F_{Z_0}. Then, the following local-global principle holds:

$$H(\varinjlim_{x \in Z} F_Z) \neq \emptyset \iff H(\mathscr{M}_{C,u}) \neq \emptyset \text{ for all } u \in \pi^{-1}(x),$$

where the direct limit is taken with respect to connected affinoid neighborhoods $Z \subseteq Z_0$ of x.

Remark that the direct limit appearing on the left side of the local-global principle above is the field of germs of meromorphic functions on the fiber of x in C.

We work only over fibers of points for which the local ring is a field. The set of such points is dense. In fact, in the case of curves, if x is any point that is not *rigid* (see Definition 1.5.10; rigid points are those that we see in rigid spaces), then $\mathcal{O}_x$ is a field. Although this might not appear explicitely during the chapter, the reason behind this hypothesis is that to make the transition from Chapter 2 to patching "around" the fiber, we need the fiber to *not* be a divisor.

To show Theorem 4.6.8, as fibers of an analytic relative curve are endowed with the structure of an analytic curve, we follow a similar line of reasoning as in the one-dimensional case. However, there are many additional technical difficulties that appear in this relative setting. Here is a brief outline of the proof.

We construct particular covers of a neighborhood of the fiber over which patching is possible (the so-called *relative nice covers*); this is a relative analogue of nice covers as introduced in Chapter 3. We first treat the case of $\mathbb{P}^{1,\mathrm{an}}_S$ - the relative projective analytic line over S. To do this, we use the notion of *thickening* of an affinoid domain, the idea for which (in the case of $\mathbb{P}^{1,\mathrm{an}}$) appears in some unpublished notes of Jérôme Poineau. Given an affinoid domain U in the fiber $\mathscr{F}_x$ of x in $\mathbb{P}^{1,\mathrm{an}}_S$, a *$Z$-thickening* of U is an affinoid domain U_Z of $\mathbb{P}^{1,\mathrm{an}}_Z$ such that $U_Z \cap \mathscr{F}_x = U$, where Z is an affinoid neighborhood of x in S. Thickenings of affinoid domains of $\mathscr{F}_x$ exist and have good properties provided we choose Z small enough.

Let $\mathcal{U}$ be any nice cover of the fiber $\mathscr{F}_x$. Then, there exists an affinoid neighborhood Z of x such that for any $U \in \mathcal{U}$, the Z-thickening U_Z of U exists. Let $\mathcal{U}_Z$ denote the set of these Z-thickenings of the elements of $\mathcal{U}$. We show that for a small enough Z, $\mathcal{U}_Z$ satisfies the necessary properties for the results of Chapter 2 to be applicable. In that case, $\mathcal{U}_Z$ is said to be a *Z-relative nice cover* of $\mathbb{P}^{1,\mathrm{an}}_Z$. In particular, we remark that type 3 points play once again an important role. Their existence on the fiber is guaranteed by the hypothesis on the dimension of S. We then show that patching *can* be applied to relative nice covers in the case of $\mathbb{P}^{1,\mathrm{an}}$.

By using pullbacks of finite morphisms towards $\mathbb{P}^{1,\mathrm{an}}$, a notion of relative nice cover can be constructed more generally for the case of normal relative proper curves. By adding to this the Weil restriction of scalars, patching is shown to be possible over relative nice covers in this more general framework as well.

Finally, once patching is shown to be possible around the fiber, the local-global principle of Theorem 4.6.8 can be obtained as a consequence, albeit not as direct as in the one-dimensional case.

There is a connection between the points of the fiber and the valuations that the field of its overconvergent meromorphic functions can be endowed with. We make this precise in Proposition 4.6.6. As in the one-dimensional case, combined with the Henselianity of the fields $\mathscr{M}_{C,z}, \pi(z) = x$, this connection allows us to obtain a local-global principle with respect to completions. Before stating this result precisely, let us recall that the field $\mathcal{O}_{S,x}$ is naturally endowed with a valuation $|\cdot|_x$.

THEOREM (Theorem 4.6.8'). *Using the same notation as in the statement of Theorem 4.6.8 above, set* $F_{\mathcal{O}_x} = \varinjlim_Z \mathscr{M}(C_Z)$. *Let* $V(F_{\mathcal{O}_x})$ *denote the set of non-trivial rank 1 valuations on* $F_{\mathcal{O}_x}$ *which induce either* $|\cdot|_x$ *or the trivial valuation on* $\mathcal{O}_x$. *For* $v \in V(F_{\mathcal{O}_x})$, *let* $F_{\mathcal{O}_x,v}$ *denote the completion of the field* $F_{\mathcal{O}_x}$ *with respect to* v.

If $\operatorname{char} k = 0$ *or* H *is smooth, then the following local-global principle holds:*

$$H(F_{\mathcal{O}_x}) \neq \emptyset \iff H(F_{\mathcal{O}_x,v}) \neq \emptyset \text{ for all } v \in V(F_{\mathcal{O}_x}).$$

Remark that, with the same notation as in the theorem above, $\mathcal{O}_{S,x} = \varinjlim_Z \mathcal{O}_S(Z)$, where the direct limit is taken with respect to affinoid neighborhoods Z of x in S. Using Grothendieck's work on projective limits of schemes to construct a relative algebraic curve over $\mathcal{O}(Z)$ from an algebraic curve over $\mathcal{O}_x$, as a consequence of the theorem above, we obtain the following generalization of Corollary 3.2.18.

THEOREM (Theorem 4.6.9). *Let* S *be a good normal* k*-analytic space such that* $\dim S < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$. *Let* $x \in S$ *be such that* $\mathcal{O}_x$ *is a field. Let* $C_{\mathcal{O}_x}$ *be a smooth geometrically irreducible algebraic curve over the field* $\mathcal{O}_x$. *Let* $F_{\mathcal{O}_x}$ *denote the function field of* $C_{\mathcal{O}_x}$.

Let $G/F_{\mathcal{O}_x}$ *be a connected rational linear algebraic group acting strongly transitively on a variety* $H/F_{\mathcal{O}_x}$. *Then, if* $\operatorname{char} k = 0$ *or* H *is smooth:*

$$H(F_{\mathcal{O}_x}) \neq \emptyset \iff H(F_{\mathcal{O}_x,v}) \neq \emptyset \text{ for all } v \in V(F_{\mathcal{O}_x}),$$

where $V(F_{\mathcal{O}_x})$ *is given as in Theorem 4.6.8' above.*

A crucial element for showing Theorem 4.6.9, and more generally, to highlight the interest of this chapter, is that, in the setting of Theorem 4.6.8, meromorphic functions around the fiber of x are algebraic. More precisely, the field of overconvergent meromorphic functions on the fiber of x is the function field of an algebraic curve over $\mathcal{O}_x$ (which is basically an "algebraization" of a neighborhood of the fiber succeeded by a base change to $\mathcal{O}_x$; see Corollary 4.4.15). To show this non-trivial result, we use GAGA-type theorems for the sheaf of meromorphic functions (see Theorem 1.7.8).

At the end of this chapter we provide some examples of local rings of analytic spaces that are fields and over which the results above can be applied. More precisely, we calculate the stalks of the points of $\mathbb{A}^{1,\mathrm{an}}$ for which the corresponding local ring is a field. In addition to that, we also give a description of the stalk of a certain point of $\mathbb{A}^{n,\mathrm{an}}, n \in \mathbb{N}$. Here is an example of such a field, corresponding to a type 3 point of the analytic affine line.

EXAMPLE. Let $(k, |\cdot|)$ be a complete ultrametric field. Let $r \in \mathbb{R}_{>0} \backslash \sqrt{|k^\times|}$. Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a multiplicative semi-norm on $k[T]$ such that $|T|_x = r$ (in fact, x is the unique such point of $\mathbb{A}_k^{1,\mathrm{an}}$).

For any $r_1, r_2 \in \mathbb{R}_{>0}$ such that $r_1 < r < r_2$, set

$$A_{r_1,r_2} := \left\{ \sum_{n \in \mathbb{Z}} a_n T^n : a_n \in k, \lim_{n \to +\infty} |a_n| r_2^n = 0, \lim_{n \to -\infty} |a_n| r_1^n = 0 \right\}.$$

Then, $\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{an}},x} = \varinjlim_{r_1 < r < r_2} A_{r_1,r_2}$.

As in Chapter 3, seeing as the projective variety determined by a quadratic form satisfies the hypotheses of the results presented, the prime example to which the statements of this chapter can be applied is the case of quadratic forms (under the assumption $\operatorname{char} k \neq 2$).

Introduction (in French)

Dans cette thèse nous utilisons le langage de la *théorie de Berkovich* pour démontrer des résultats sur le *principe local-global* et obtenons des applications aux formes quadratiques et à un invariant qui leur est lié. Pour ce faire, nous utilisons l'outil fondamental du *recollement*. Cette technique admet déjà plusieurs applications, et apparaît récemment comme l'ingrédient principal dans une série d'articles en cours. Nous étendons le recollement d'un cadre algébrique au cadre des espaces de Berkovich.

En adoptant ce point de vue, le recollement acquiert une nature très géométrique : il peut s'interpréter comme le recollement faisceautique de fonctions méromorphes, éclairant ainsi la stratégie globale de preuve. Ceci nous laisse penser que cette approche est propice à des généralisations futures.

Plus précisément, nous montrons que le recollement peut s'appliquer aux courbes analytiques de Berkovich, et obtenons ainsi un principe local-global sur les corps de fonctions de telles courbes, généralisant de cette façon les résultats de l'article fondateur [**34**]. Nous rappelons qu'une variété X définie sur un corps F satisfait le principe local-global s'il existe une famille $(F_i)_i$ de corps contenant F (qu'on appelera des *surcorps*) telle que $X(F) \neq \emptyset$ si et seulement si $X(F_i) \neq \emptyset$ pour tout i. Nous définissons deux familles de surcorps : une qui apparaît très naturellement dans la théorie de Berkovich (des germes de fonctions méromorphes), et une de nature plus classique constituée de complétés du corps de fonctions. Nous établissons une connexion entre ces deux familles en rendant précis le lien qui existe entre les points d'une courbe analytique de Berkovich et les valuations dont on peut munir son corps de fonctions.

Le principe local-global obtenu peut s'appliquer aux formes quadratiques. Ceci, combiné avec les bonnes propriétés algébriques des anneaux locaux d'une courbe analytique de Berkovich, nous permet d'obtenir des application à l'u-invariant.

Comme premier pas vers des versions en dimension supérieure de cette technique, nous montrons que le recollement est possible autour de certaines fibres d'une courbe relative analytique de Berkovich. Ainsi, nous obtenons un principe local-global sur les germes de fonctions méromorphes sur ces fibres, qui peut de nouveau s'appliquer aux formes quadratiques. Comme précédemment, il y a deux familles possibles de surcorps : les germes de fonctions méromorphes en les points de la fibre, et des complétés du corps des germes méromorphes. En particulier, nous montrons que ces derniers sont algébriques.

En utilisant la théorie des limites projectives de schémas, nous obtenons aussi un principe local-global sur le corps de fonctions d'une courbe algébrique définie sur des corps ultramétriques qui ne sont pas nécessairement complets.

Présentation des directions majeures

Principe local-global. Le principe local-global est apparu pour la première fois dans les années '20 sous le nom de *principe de Hasse-Minkowski*, et énonce alors qu'une forme

quadratique rationnelle a une solution non triviale sur $\mathbb{Q}$ si et seulement si elle a des solutions non triviales sur $\mathbb{R}$ et $\mathbb{Q}_p$ pour tout nombre premier p. Les versions modernes traitent plus généralement de variétés définies sur un corps K qui ont un point K-rationnel si et seulement si elles ont des points K_i-rationnels poir tout i, où $(K_i)_i$ est une famille de surcorps de K.

Le principe local-global n'est pas toujours vrai. Parmi les premiers contre-exemples historiques, mentionnons celui obtenu par Reichardt et Lind, qui ont montré que l'équation $2Y^2 = X^4 - 17Z^4$ a des solutions sur tous les complétés de $\mathbb{Q}$, mais pas de solutions rationnelles. Depuis, beaucoup d'autres contre-exemples ont été trouvés. Détérminer pour quels corps, surcorps et variétés il y a un principe local-global et étudier les obstructions à cette propriété a été un domaine actif de recherche depuis des décennies (*cf.* par exemple [**9**] et [**51**]).

Le dévelopement de la géométrie arithmétique a apporté de nouvelles techniques puissantes qui s'appliquent à l'étude du principe local-global, mettant ainsi au premier plan des questions possédant un sens géométrique. Plus précisement, en utilisant la même notation que précédemment, K est le corps de fonction d'une variété algébrique, et la famille de surcorps est interprétée dans un cadre géométrique. Jusqu'à présent, les résultats connus ne couvrent que des cas spécifiques, la majorité concernant les courbes (*cf.* par exemple [**34**] et [**39**]). De plus, en général, la famille de surcorps considérée contient des complétés de K par rapport à des valuations discrètes (qui peuvent se lire sur un modèle "assez fin" de la courbe).

Une classe particulière de variétés qui se comportent bien par rapport au principe local-global est celle des variétés *homogènes* sous certains groupes linéaires algébriques (voir *e.g.* [**16**] pour une brève exposition du sujet). Nous rappelons qu'étant donné un corps F, une variété X/F est dite *homogène* sous un groupe linéaire algébrique G/F si G agit sur X et si le groupe $G(\overline{F})$ agit transitivement sur l'ensemble $X(\overline{F})$, où $\overline{F}$ est une clôture algébrique de F. Il a été démontré dans [**9**] que, sous certaines conditions additionnelles, la seule obstruction au principe local-global pour les variétés homogènes est l'*obstruction de Brauer-Manin* introduite par Manin dans [**51**].

Une nouvelle approche au principe local-global pour les variétés homogènes sur des corps de fonctions de courbes définies sur des corps complets discrètement valués a été introduite par Harbater, Hartmann et Krashen dans [**34**] via *le recollement*.

Le recollement. La méthode de recollement a été introduite comme une des approches principales à la théorie inverse de Galois. Originellement de nature formelle et géométrique, cette technique a fourni une façon d'obtenir un revêtement galoisien global à partir de revêtements locaux (voir par exemple [**32**]). Par cette technique, il a été démontré que le problème inverse de Galois pour $\mathbb{Q}_p(T)$, où p est un nombre premier, admet une réponse affirmative. Le recollement formel a été traduit dans le langage de la géométrie rigide par Liu dans [**50**]. Un autre exemple est [**61**], où Poineau utilise le recollement sur des courbes analytiques au sens de Berkovich et généralise ainsi les résultats montrés par Harbater dans [**30**] et [**31**].

Dans [**33**], Harbater et Hartmann ont combiné le recollement formel avec le recollement algébrique au sens de [**40**], et ont ainsi étendu la technique aux structures définies sur des corps, en construisant un cadre de nature très algébrique. Depuis, le recollement sur les corps a trouvé de nombreuses applications et est l'ingrédient crucial dans une série d'articles en cours (voir par exemple [**34**], [**35**], [**39**], [**36**], [**17**]).

Parmi les principaux problèmes abordés dans ces travaux, on retrouve les principes locaux-globaux sur des corps de fonctions de courbes algébriques définies sur des corps complets discrètement valués. C'est cette forme de recollement qui a fourni une nouvelle approche au principe local-global pour les variétés homogènes sous certains groupes linéaires algébriques (voir par exemple [**34**] et [**39**]).

En particulier, dans [**34**], Harbater, Hartmann et Krashen (abrégé par la suite en HHK) obtiennent un principe local-global pour les formes quadratiques et des résultats sur l'u-invariant, généralisant ainsi ceux de Parimala et Suresh [**58**], qui ont été montrés en utilisant d'autres méthodes. Une autre source pour des résultats sur l'u-invariant est l'article [**47**] de Leep. Dans [**34**], les auteurs appliquent aussi les résultats locaux-globaux obtenus aux algèbres centrales simples.

Décrivons brièvement les surcorps qui apparaissent dans les principes locaux-globaux démontrés dans [**34**]. Soit k un corps complet discrètement valué, et k° l'anneau de valuation correspondant. Soit π une uniformisante de k°. Soit C/k une courbe algébrique. Soit $\mathcal{C}$ un modèle plat normal irréductible projectif de C sur k° avec fibre spéciale $\mathcal{C}_s$. On note avec F le corps de fonctions de $\mathcal{C}$ (et donc de C). Pour tout point $P \in \mathcal{C}_s$, soit $R_P = \mathcal{O}_{\mathcal{C},P}$. On note par $\widehat{R_P}$ le complété de l'anneau local R_P par rapport à son idéal maximal. Soit $F_P = \text{Frac } \widehat{R_P}$. Soit U un sous-ensemble propre d'une composante irréductible de $\mathcal{C}_s$. Soit $R_U = \bigcap_{P\in U} R_P$. On note $\widehat{R_U}$ le complété π-adique de R_U. Soit $F_U = \text{Frac } \widehat{R_U}$.

Soit $\mathcal{P}$ un ensemble fini de points fermés de $\mathcal{C}_s$ contenant tous les points d'intersection des composantes irréductibles de $\mathcal{C}_s$. Soit $\mathcal{U}$ l'ensemble des composantes connexes de $\mathcal{C}_s \backslash \mathcal{P}$. Alors les surcorps en question sont $\{F_P, F_U : P \in \mathcal{P}, U \in \mathcal{U}\}$. Plus précisement, HHK montrent que pour une variété X/F satisfaisant certaines conditions :

$$X(F) \neq \emptyset \iff X(F_P) \neq \emptyset, X(F_U) \neq \emptyset \text{ pour tout } P \in \mathcal{P}, U \in \mathcal{U}.$$

Voir la sous-section 3.3.2 pour plus de détails sur le principe local-global de [**34**].

Pour un survol sur le développement historique des différentes versions du recollement, voir [**37**]. Nous avons adapté le recollement sur les corps au cadre des *espaces de Berkovich*.

Les espaces de Berkovich. L'étude de Tate des courbes elliptiques avec mauvaise réduction sur $\mathbb{Q}_p$ dans les années '60 a mené au développement d'une première approche à la géométrie analytique non-archimédienne, appelée *géométrie rigide* ([**63**]). Comme $\mathbb{Q}_p$ est totalement discontinu en tant qu'espace topologique, l'approche naïve de définir une fonction analytique comme étant localement développable en série entière ne fonctionne pas : on obtiendrait trop de fonctions. Par exemple, la fonction $f : \mathbb{Q}_p \to \mathbb{R}$, définie par

$$f(x) = \begin{cases} 0, \text{ si } |x|_p \leqslant 1 \\ 1, \text{ sinon} \end{cases}$$

serait alors analytique. Pour éviter ce problème, Tate ne permet que certains ouverts et recouvrements. Par conséquent, les espaces rigides ne sont pas dotés d'une vraie topologie, mais seulement d'une *topologie de Grothendieck*.

Depuis, il y a eu plusieurs autres approches à la géométrie analytique non-archimédienne : la théorie de Raynaud des modèles formels, les espaces de Berkovich et la géométrie adique de Huber.

Developpée à la fin des années '80 (voir [**6**]), l'approche de Berkovich a originellement été motivée par des question de théorie spectrale. Grossièrement, les espaces de Berkovich

sont obtenus en ajoutant des points aux espaces rigides. De cette manière on obtient des espaces topologiques avec de bonnes propriétés comme la compacité locale et la connexité par arcs locale. Par conséquent, on peut penser à ces objets géométriquement. Contrairement à la géométrie rigide, les espaces de Berkovich peuvent aussi être définis sur des corps trivialement valués.

Une analogie peut être établie avec le cadre classique complexe : les fonctions analytiques sur certains domaines analytiques sont des séries entières convergentes, il existe un principe de prolongement analytique, ainsi que des théorèmes du type GAGA. On peut aussi observer un parallèle avec la géométrie algébrique dans le sens où les espaces de Berkovich, comme les schémas, sont construits à partir de blocs de base. Ces derniers sont appelés *espaces affinoïdes*. La différence principale entre ces deux cadres (algébrique et Berkovich) est que les blocs de base des espaces de Berkovich sont compacts, donc aussi Hausdorff, et par conséquent pas toujours ouverts. Ceci est une source de nombreuses difficultés dans la théorie de Berkovich, puisqu'il n'y a pas de base d'ouverts pour lesquels on peut décrire facilement le faisceau de fonctions analytiques.

Depuis son apparition, la théorie des espaces de Berkovich a trouvé plusieurs applications, la plupart, grâce aux théorèmes GAGA, à la géométrie arithmétique, et a été étendue dans plusieurs directions (par exemple les espaces de Berkovich sur $\mathbb{Z}$ [**59**]). Celles-ci incluent : les systèmes dynamiques, la théorie des dessins d'enfants p-adiques, les immeubles de Bruhat-Tits, la théorie inverse de Galois, *etc.* Voir [**23**] et [**19**] pour plus d'exemples. Récemment, des connexions ont été établies entre la théorie de Berkovich et d'autres domaines comme la géométrie tropicale (*e.g.* [**2**]) et la théorie de modèles (*e.g.* [**38**]).

Organisation du manuscrit

Le premier chapitre est dédié à l'introduction de la théorie des espaces de Berkovich. Dans le chapitre 2, le recollement sur les corps est étendu à un cadre général formel qui correspond aux espaces de Berkovich. Le chapitre 3 traite du recollement sur les courbes analytiques de Berkovich et de ses applications au principe local-global ; son contenu a donné lieu à un article intitulé "Patching over Berkovich Curves and Quadratic Forms", voir [**54**]. Finalement, dans le chapitre 4, nous montrons que le recollement est possible autour de certaines fibres de courbes analytiques relatives et en déduisons des principes locaux-globaux ; le contenu de ce chapitre sera le sujet d'un futur article.

Voici une description plus détaillée de l'organisation de ce manuscrit.

Chapitre 1 : Introduction aux espaces de Berkovich.

Ce chapitre a comme but de donner une introduction à la théorie des espaces de Berkovich afin de rendre le manuscrit plus auto-suffisant. Nous donnons un rappel rapide de la construction des principaux objets qui interviennent dans cette théorie en commençant par le cadre algébrique sur lequel elle se base. Ce dernier est une généralisation par Berkovich du pendant algébrique de la géométrie rigide de Tate (plus précisement, une généralisation de la théorie des algèbres affinoïdes de Tate).

Nous nous attardons particulièrement sur le cas des courbes analytiques, qui est sans doute la famille d'espaces de Berkovich la mieux comprise, ainsi que celle qui présente le plus d'intérêt pour nous. Nous en montrons quelques propriétés qui nous seront nécessaires dans les chapitres suivants. En particulier, leur structure de graphe est source de nombreuses bonnes propriétés topologiques, que nous utiliserons dans ce manuscrit.

Un autre point important que nous traitons est le faisceau des fonctions méromorphes, qui est crucial pour le travail presenté ici vu que le recollement est interprété comme le recollement de fonctions méromorphes sur certains espaces de Berkovich. Sa construction, ainsi que les propriétés qu'il satisfait, sont similaires à celles du faisceau de fonctions méromorphes sur les schémas.

Nous donnons aussi une description d'un exemple typique d'espace de Berkovich, la *droite analytique affine* $\mathbb{A}^{1,\mathrm{an}}$, et de ses points (voir la section 1.2 et la sous-section 1.8.4). Pour un corps ultramétrique complet $(k, |\cdot|)$, $\mathbb{A}^{1,\mathrm{an}}_k$ est l'ensemble des semi-normes multiplicatives sur $k[T]$ qui étendent la norme de k. En particulier, k se plonge dans $\mathbb{A}^{1,\mathrm{an}}_k$ via $a \mapsto |\cdot|_a$, où pour tout polynôme $P(T) \in k[T]$, $|P(T)|_a := |P(a)|$. L'ensemble $\mathbb{A}^{1,\mathrm{an}}_k$ est muni d'une topologie de convergence simple.

La droite analytique affine a une structure d'arbre infiniment branché. La Figure 3 est une illustration de cet espace de Berkovich. En ajoutant un point "∞" à cet arbre, nous obtenons *la droite analytique projective* $\mathbb{P}^{1,\mathrm{an}}_k$. Les k-points sont situés aux extrémités de l'arbre, ce sont des *feuilles* de l'arbre. Un rôle particulièrement important pour nous est joué par les points non extrémaux où il n'y a pas de branchement (un exemple d'un tel point est donné par x dans la Figure 3).

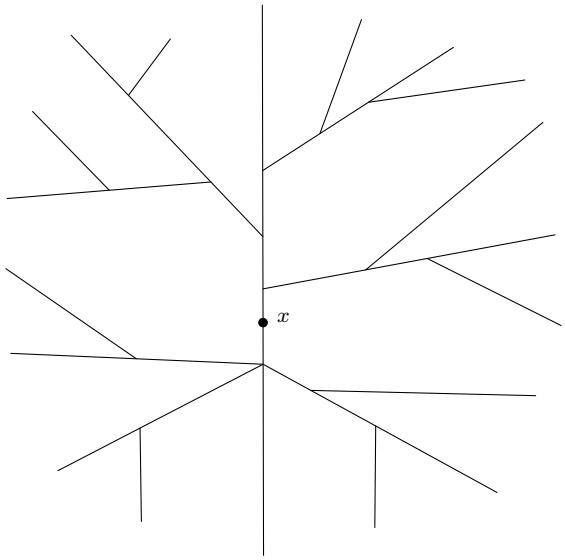


Figure 3 : $\mathbb{A}^{1,\mathrm{an}}_k$

La plupart des résultats de ce chapitre sont bien connus dans le domaine et nous nous contentons de donner des références pour leurs preuves. Certains autres, plus spécialisés, ne sont à notre connaissance pas présents dans la littérature, nous en proposons donc des démonstrations.

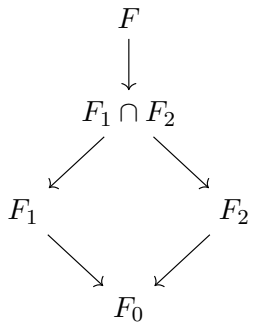
Chapitre 2 : Le recollement.

Le cadre général abstrait pour le recollement est le suivant.

Considérons la tour de corps illustrée dans le diagramme (2) ci-dessous. Supposons qu'on a des *structures algébriques* $\mathcal{A}_1$ et $\mathcal{A}_2$ sur F_1 et F_2, respectivement. Le but est de trouver des conditions sous lesquelles elles induisent une structure algébrique du même

type sur $F_1 \cap F_2$. Typiquement, ces structures algébriques sont des points rationnels sur F_1, respectivement F_2, d'une variété définie sur un corps plus petit F. Un autre exemple est le cas des zéro-cycles de degré un.

(2)



Nous nous concentrons sur le cas où ces structures algébriques sont des points rationnels sur une variété H/F. Si $\mathcal{A}_1$ et $\mathcal{A}_2$ sont compatibles sur F_0, alors elles se relèvent sur $F_1 \cap F_2$. Si ce n'est pas le cas, on ne peut pas généralement les relever à $F_1 \cap F_2$. Une manière d'approcher ce problème est de trouver une façon de *rendre* ces points rationnels compatibles sur F_0.

Idéalement, il existe un groupe linéaire algébrique G/F qui agit sur H de façon à ce que ces points rationnels (ou, plus généralement, ces structures algébriques) puissent toujours être déplacés par l'action de G et rendus compatibles sur F_0. Pour que cette idée fonctionne, il nous faut à la fois une action particulière de G sur H (donnée dans la définition 3.2.1), et des conditions sur le groupe G. Ces conditions font l'objet principal de ce chapitre.

Plus précisément, soit G/F un groupe linéaire algébrique. La condition dont nous avons besoin est la suivante : pour tout $g \in G(F_0)$, il existe $g_i \in G(F_i), i = 1, 2$, tels que $g = g_1 \cdot g_2$ dans $G(F_0)$. En effet, supposons que g est tel que $g \cdot \mathcal{A}_2 = \mathcal{A}_1$ dans $H(F_0)$. Soit $\mathcal{A}_1' := g_1^{-1} \cdot \mathcal{A}_1 \in H(F_1)$ et $\mathcal{A}_2' := g_2 \cdot \mathcal{A}_2 \in H(F_2)$. Alors, par construction, $\mathcal{A}_1' = \mathcal{A}_2'$ dans $H(F_0)$, donc elles peuvent se relever sur $F_1 \cap F_2$. L'existence d'un élément $g \in G(F_0)$ qui satisfait $g \cdot \mathcal{A}_2 = \mathcal{A}_1$ est à l'origine de l'hypothèse que nous adoptons sur l'action de G sur H (voir la définition 3.2.1).

Par la suite, nous appellerons la propriété de "décomposition matricielle" décrite dans le paragraphe précédent *recollement*. Nous montrerons que la famille suivante de groupes linéraires algébriques satisfait le recollement (avec un certain choix de corps dans la tour du diagramme (2)).

Définition. Un groupe linéaire algébrique G/F est dit *rationnel* (sur F) s'il existe un ouvert de Zariski de G isomorphe à un ouvert de Zariski de $\mathbb{A}^n_F$ pour un certain $n \in \mathbb{N}$.

Nous fixons un cadre général formel (Setting 2.1.1) sur lequel nous montrons le résultat principal de ce chapitre (voir le théorème 2.1.10). Ce dernier est fondamental pour le recollement. Il est une généralisation de [**34**, Theorem 3.2]. La différence principale est que les objets considérés dans *loc.cit.* sont définis sur un corps complet *discrètement valué*, alors que nous ne demandons pas aux valuations d'être discrètes. Une conséquence assez directe est que le recollement est vrai sur un voisinage de Zariski de l'identité de G. La preuve (ainsi que l'enoncé) du théorème 2.1.10 est très technique, et suit les lignes principales de la preuve de [**34**, Theorem 3.2].

L'intérêt du cadre formel sur lequel on travaille est qu'il est réalisé de façon naturelle (et très géométrique) dans la théorie de Berkovich.

En particulier, nous montrons que dans un cas particulier du Setting 2.1.1, realisé par les courbes analytiques de Berkovich, le théorème 2.1.10 peut se généraliser pour montrer que le recollement est vrai dans G (théorème 2.2.3). Ceci est l'outil fondamental pour obtenir un recollement sur les courbes analytiques de Berkovich.

Chapitre 3 : Recollement sur les courbes de Berkovich et formes quadratiques.

Dans ce chapitre nous montrons que le recollement est possible sur les courbes analytiques de Berkovich et qu'il peut s'interpréter comme le recollement de fonctions méromorphes. Plus concrètement, nous montrons que les corps F_i du diagramme (2) peuvent être choisis comme les corps de fonctions méromorphes de certaines parties (appelées *domaines analytiques*) d'une courbe analytique. Nous utilisons ensuite ce résultat pour démontrer un principe local-global et donner des applications aux formes quadratiques et à l'u-invariant. Les résultats obtenus généralisent ceux de [**34**].

Avant de présenter les résultats principaux de ce chapitre, nous introduisons un peu de terminologie.

Définition. Soit K un corps. Soit X une K-variété et G un groupe linéaire algébrique sur K. On dit que G agit *fortement transitivement* sur X si G agit sur X et que pour toute extension de corps L/K, soit $X(L) = \emptyset$ soit $G(L)$ agit transitivement sur $X(L)$.

En général, demander que G agisse fortement transitivement sur X est plus restrictif que demander que X soit homogène sous G. En revanche, il est montré dans [**34**, Remark 3.9] que si G est un groupe linéaire algébrique sur K et X/K est une variété projective, alors les deux notions sont équivalentes.

Nos résultats principaux, les principes locaux-globaux que nous montrons, sont :

Théorème. *Soit k un corps ultramétrique complet non trivialement valué. Soit C une courbe k-algébrique normale irréductible projective. On note F le corps de fonctions de C. Soit X une F-variété et G un groupe linéaire algébrique connexe rationnel sur F qui agit fortement transitivement sur X.*

Soit $V(F)$ l'ensemble de toutes les valuations non triviales de rang 1 sur F qui ou bien prolongent la valuation de k ou bien sont triviales sur k.

Soit C^{an} l'analytifié au sens de Berkovich de C ; alors $F = \mathscr{M}(C^{\mathrm{an}})$, où $\mathscr{M}$ désigne le faisceau de fonctions méromorphes sur C^{an}. Alors les principes locaux-globaux suivants sont vrais :

- *(Theorem 3.2.11) $X(F) \neq \emptyset \iff X(\mathscr{M}_x) \neq \emptyset$ pour tout $x \in C^{\mathrm{an}}$.*
- *(Corollary 3.2.18) Si* car $k = 0$ *ou X est une variété lisse, alors :*

$$X(F) \neq \emptyset \iff X(F_v) \neq \emptyset \text{ pour tout } v \in V(F),$$

où F_v désigne le complété de F par rapport à v.

L'énoncé ci-dessus reste vrai pour les courbes affinoïdes si $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$, où $\sqrt{|k^\times|}$ est la clôture divisible du groupe des valeurs $|k^\times|$ de k. Étant un principe local-global par rapport aux complétés, la deuxième équivalence ressemble à des versions plus classiques de principes locaux-globaux. L'énoncé peut se formuler de façon à inclure les corps de base trivialement valués, même si cela ne nous donnerait pas de nouvelles informations puisqu'au moins un des surcorps serait alors égal à F.

Remarque. Pour démontrer nos résultats principaux, nous avons besoin de *moins* que la transitivité forte. Plus précisement, il suffit de supposer que pour tout complété $\widehat{F}$ de F par rapport à une valuation qui prolonge celle de k, soit $X(\widehat{F}) = \emptyset$ soit le groupe $G(\widehat{F})$ agit transitivement sur l'ensemble $X(\widehat{F})$. Nous pouvons même nous restreindre à certains de ces complétés, notamment à ceux pour lesquels $\deg \mathrm{tr}_{\widetilde{k}}\widetilde{\widehat{F}} = 0$ et $\mathrm{rang}_{\mathbb{Q}}|\widehat{F}^{\times}|/|k^{\times}| \otimes_{\mathbb{Z}} \mathbb{Q} = 1$, où $\widetilde{k}$, respectivement $\widetilde{\widehat{F}}$, est le corps résiduel de k, respectivement $\widehat{F}$. (Ce sont les complétés par rapport aux valuations induites par les points *de type 3* de la courbe, voir la définition 1.8.1).

Nous rappelons que pour toute extension de corps F/k finiment engendrée de degré de transcendance 1, il existe une unique courbe normale projective algébrique sur k avec corps de fonctions F. Donc, le résultat du théorème ci-dessus s'applique à tout tel corps F.

Pour montrer les principes locaux-globaux énoncés ci-dessus, nous construisons des recouvrements particuliers des courbes sur lesquels le recollement peut se réaliser (appelés *bons recouvrements*, voir la définition 3.1.6). Lors de cette étape, les *points de type 3* sont cruciaux. Un point de type 3 a de bonnes propriétés topologiques et algébriques. Plus précisément, une courbe analytique est un graphe réel sur laquelle un point de type 3 est d'arité 2 ; de plus, l'anneau local du faisceau structural d'un point de type 3 est un corps. Dans la Figure 3, x est un point de type 3. L'existence de tels points est équivalente à la condition $\sqrt{|k^{\times}|} \neq \mathbb{R}_{>0}$; ceci est la raison pour laquelle le théorème 3.2.11 est d'abord montré sous cette hypothèse. Le résultat est ensuite démontré dans toute sa généralité en utilisant des arguments de théorie des modèles. Voici un bref résumé de la preuve dans le cas où $\sqrt{|k^{\times}|} \neq \mathbb{R}_{>0}$.

Nous rappelons qu'un *domaine affinoïde* d'un espace analytique de Berkovich est un domaine analytique (*i.e.* le faisceau structural y est bien défini), isomorphe à un espace affinoïde. De plus, une intersection finie de domaines affinoïdes d'une courbe analytique (et plus généralement, de tout espace de Berkovich *séparé*) est encore un domaine affinoïde. Nous commençons par introduire la notion cruciale de bon recouvrement.

Définition (Definition 3.1.6). Un recouvrement fini $\mathcal{U}$ d'une courbe k-analytique est appelé *bon* si :

(1) les éléments de $\mathcal{U}$ sont des domaines affinoïdes ne contenant que des points de type 3 dans leur bord topologique ;
(2) pout tous $U, V \in \mathcal{U}, U \neq V$, $U \cap V = \partial U \cap \partial V$, ou, de façon équivalente, $U \cap V$ est un ensemble fini de points de type 3 ;
(3) aucun élément de $\mathcal{U}$ n'est contenu dans un autre élément de $\mathcal{U}$.

Voir la Figure 4 ci-dessous pour un exemple de bon recouvrement de la droite analytique projective.

Pour une courbe normale irréductible projective k-analytique C, soient U, V des domaines affinoïdes connexes de C tels que $U \cap V$ soit un seul point de type 3, noté $\{\eta\}$. (Ceci est un cas particulier d'un bon recouvrement.) Alors, en suivant les notations du diagramme (2), on note $F := \mathscr{M}(C)$, $F_0 := \mathscr{M}(\{\eta\})$, $F_1 := \mathscr{M}(U), F_2 := \mathscr{M}(V)$. Nous commençons par montrer que le recollement est possible avec ce choix de corps pour tout groupe linéaire algébrique rationnel G/F. Ceci est ensuite généralisé à un bon recouvrement quelconque de la droite analytique projective. Pour étendre le résultat à toute courbe analytique normale irréductible projective C, nous utilisons la restriction des scalaires de

Weil pour "descendre" au cas de la droite analytique projective (cette idée est aussi employée par HHK dans [**34**]).

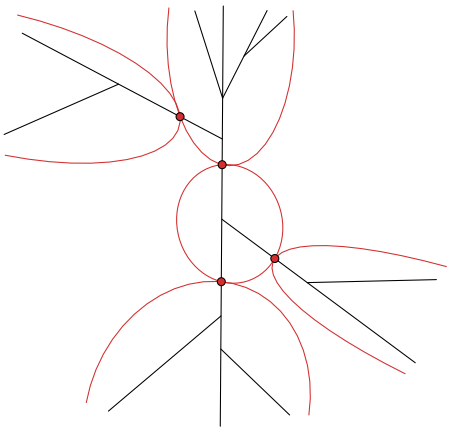


Figure 4 : exemple d'un bon recouvrement pour $\mathbb{P}_k^{1,\mathrm{an}}$

Finalement, après avoir démontré que le recollement est possible, le principe local-global du théorème 3.2.11 est obtenu comme conséquence relativement directe du recollement.

Pour obtenir le corollaire 3.2.18 à partir du théorème 3.2.11, nous établissons une connexion précise entre les points d'une courbe analytique de Berkovich et les valuations dont on peut munir son corps de fonctions. Ceci est l'objet de la proposition 3.2.14. Le reste est alors une conséquence des bonnes propriétés algébriques des corps $\mathscr{M}_x, x \in C$, notamment de leur henselianité.

À la différence de HHK qui travaillent sur des modèles d'une courbe algébrique, nous travaillons directement sur des courbes analytiques. Remarquons que nous n'imposons pas de conditions supplémentaires au corps complet ultramétrique de base k. Ceci est une des différences fondamentales avec le Theorem 3.7 de [**34**], où le corps de base doit être complet par rapport à une valuation discrète. Une autre différence est dans la nature des surcorps, qui ici sont des complétés ou des corps de fonctions méromorphes. La section 3.4 montre que ces derniers contiennent ceux qui apparaissent dans l'article de HHK, et donc que [**34**, Theorem 3.7] est une conséquence du principe local-global énoncé dans le théorème 3.2.11. De plus, nous montrons que l'inverse est aussi vrai si on choisit un modèle "assez fin". La preuve du théorème ci-dessus est basé sur la méthode de recollement, mais utilisé dans un cadre différent de celui de [**34**].

Comme conséquence de nos résultats principaux, dans le contexte des formes quadratiques, nous obtenons le théorème suivant, qui généralise [**34**, Theorem 4.2]. Il est un corollaire direct des principes locaux-globaux mentionnés précédemment car la variété projective définie par une forme quadratique satisfait toutes les hypothèses du théorème 3.2.11 (en particulier l'existence d'un groupe linéaire algébrique connexe rationnel qui agit fortement transitivement sur elle).

THÉORÈME. *Soit k un corps ultramétrique complet non trivialement valué. Soit C une courbe normale irréductible projective algébrique sur k. On note F le corps de fonctions*

de C. Supposons que $\mathrm{car}(F) \neq 2$. *Soit q une forme quadratique sur F de dimension différente de* 2.

Soit $V(F)$ l'ensemble de toutes les valuations non triviales de rang 1 sur F qui soit prolongent la valuation de k soit sont triviales sur k.

Soit C^{an} l'analytifié au sens de Berkovich de C. Alors $F = \mathscr{M}(C^{\mathrm{an}})$, où $\mathscr{M}$ est le faisceau de fonctions méromorphes sur C^{an}.

(1) *(Theorem 3.4.1) La forme quadratique q est isotrope sur F si et seulement si elle est isotrope sur $\mathscr{M}_x$ pout tout $x \in C^{\mathrm{an}}$.*
(2) *(Corollary 3.4.2) La forme quadratique q est isotrope sur F si et seulement si elle est isotrope sur F_v pour tout $v \in V(F)$, où F_v est le complété de F par rapport à v.*

Comme mentionné dans l'introduction de [**34**], on s'attend à ce que pour des corps assez "gentils" K, l'u-invariant reste le même pour les extensions finies, et devienne $2^d u(K)$ pour les extensions finiment engendrées de degré de transcendence d. Puisque nous ne travaillons qu'en dimension 1 ici, ceci explique la motivation derrière la définition suivante :

Définition. Soit K un corps.

(1) (Kaplansky) L'*u-invariant* de K, noté $u(K)$, est la dimension maximale des formes quadratiques anisotropes sur K. On dit que $u(K) = \infty$ s'il existe des formes quadratiques anisotropes sur K de dimension arbitrairement grande.
(2) (HHK) L'*u-invariant fort* de K, noté $u_s(K)$, est le plus petit nombre réel m tel que :
   - $u(E) \leqslant m$ pour toute extension finie de corps E/K ;
   - $\frac{1}{2}u(E) \leqslant m$ pour toute extension de corps finiment engendrée E/K de degré de transcendance 1.

   On dit que $u_s(K) = \infty$ s'il existe de telles extensions E/K d'u-invariant arbitrairement grand.

Le théorème ci-dessus mène à des applications sur l'u-invariant. Soit k un corps ultramétrique complet avec corps résiduel $\widetilde{k}$ tel que $\mathrm{car}(\widetilde{k}) \neq 2$. On suppose que soit $|k^\times|$ est un $\mathbb{Z}$-module libre avec $\mathrm{rang}_{\mathbb{Z}}|k^\times| =: n$, ou, plus généralement, que $\dim_{\mathbb{Q}} \sqrt{|k^\times|} =: n$, où n est un entier naturel. Ceci est encore une différence avec les résultats correspondants de HHK dans [**34**], où l'hypothèse sur le corps de base est qu'il soit complet par rapport à une valuation discrète, c'est-à-dire que son groupe de valeurs soit un $\mathbb{Z}$-module libre de rang 1. Nous obtenons une borne supérieure sur l'u-invariant d'une extension finiment engendrée de k de degré de transcendance au plus 1, qui ne dépend que de $u_s(\widetilde{k})$ et n. Plus précisement, en terme de l'u-invariant fort :

Corollaire (Corollary 3.4.28). *Soit k un corps ultramétrique complet. Supposons que* $\mathrm{car}(\widetilde{k}) \neq 2$.

(1) *Si $\dim_{\mathbb{Q}} \sqrt{|k^\times|} =: n, n \in \mathbb{Z}$, alors $u_s(k) \leqslant 2^{n+1} u_s(\widetilde{k})$.*
(2) *Si $|k^\times|$ est un $\mathbb{Z}$-module libre avec $\mathrm{rang}_{\mathbb{Z}}|k^\times| =: n, n \in \mathbb{Z}$, alors $u_s(k) \leqslant 2^n u_s(\widetilde{k})$.*

Le corollaire 3.4.28 est une conséquence du principe local-global obtenu pour les formes quadratiques (théorème 3.4.1) et des bonnes propriétés algébriques que les anneaux locaux d'une courbe analytique satisfont, notamment l'hensélianité.

L'auteure ignore s'il y a égalité dans le corollaire ci-dessus. Ceci est vrai dans le cas particulier où $n = 1$ en utilisant [**34**, Lemma 4.9], dont la preuve n'utilise pas le

recollement. De cette manière nous obtenons [**34**, Theorem 4.10], qui est un des résultats principaux de [**34**] sur les formes quadratiques. Ceci est une nouvelle preuve du fait que $u(\mathbb{Q}_p(T)) = 8$, où p est un nombre premier différent de 2, originellement démontré dans [**58**].

Corollaire (Corollary 3.4.30). *Soit k un corps complet par rapport à une valuation discrète tel que* $\mathrm{car}(\widetilde{k}) \neq 2$. *Alors* $u_s(k) = 2u_s(\widetilde{k})$.

Chapitre 4 : Recollement sur des fibres analytiques et le principe local-global.

Dans ce chapitre nous montrons que le recollement est possible "autour" de certaines fibres d'une courbe analytique relative. Ceci est ensuite appliqué pour obtenir un principe local-global sur le corps des fonctions méromorphes surconvergentes sur ces fibres. Nous montrons aussi que ce dernier peut s'interpréter comme le corps de fonctions d'une certaine courbe algébrique. Comme précédemment, les principes locaux-globaux obtenus peuvent s'appliquer aux formes quadratiques (si le corps de base est de caractéristique différente de 2).

Ce chapitre a deux objectifs principaux :

(1) établir un premier pas vers une stratégie pour le recollement en dimension supérieure et les applications correspondantes au principe local-global ;
(2) généraliser les résultats obtenus dans le Chapitre 3 ; plus précisément, démontrer un principe local-global sur des courbes algébriques (c'est-à-dire leurs corps de fonctions) définies sur une famille plus grande de corps ultramétriques (qui ne sont pas nécessairement complets).

Les résultats principaux que nous démontrons sont les suivants (voir le théorème 4.6.8 pour l'énoncé précis) :

Théorème (Theorem 4.6.8). *Soit k un corps ultramétrique complet non trivialement valué.*

Soient S, C deux bons espaces k-analytiques tels que S soit normal. On suppose que $\dim S < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$. *On suppose qu'il existe un morphisme $\pi : C \to S$ tel que C est une courbe propre plate relative sur S. Pour tout domaine affinoïde Z de S, on pose $C_Z := \pi^{-1}(Z)$ et $F_Z := \mathscr{M}(C_Z)$, où $\mathscr{M}$ désigne le faisceau des fonctions méromorphes sur C. Soit $x \in S$ tel que $\mathcal{O}_{S,x}$ soit un corps. On désigne par $\mathscr{F}_x$ la fibre de x dans C.*

Supposons qu'il existe un voisinage connexe affinoïde Z_0 de x tel que : (1) toutes les fibres de π sur Z_0 sont des courbes analytiques projectives normales irréductibles ; (2) C_{Z_0} est normale ; (3) $\pi_{|C_{Z_0}} : C_{Z_0} \to Z_0$ est algébrique.

Soit G/F_{Z_0} un groupe linéaire algébrique connexe rationnel qui agit fortement transitivement sur une variété H/F_{Z_0}. Alors le principe local-global suivant est vrai :

$$H(\varinjlim_{x \in Z} F_Z) \neq \emptyset \iff H(\mathscr{M}_{C,u}) \neq \emptyset \text{ pour tout } u \in \pi^{-1}(x),$$

où la limite directe est prise sur tous les voisinages affinoïdes connexes $Z \subseteq Z_0$ de x.

Nous remarquons que la limite directe qui apparaît à gauche du principe local-global est le corps des germes de fonctions méromorphes sur la fibre de x dans C.

Nous ne travaillons que sur les fibres de points en lesquels l'anneau local est un corps. L'ensemble de tels points est toujours dense. En particulier, dans le cas des courbes, si x

est un point qui n'est pas *rigide* (voir la définition 1.5.10 ; les points rigides sont ceux qu'on voit dans les espaces rigides), alors $\mathcal{O}_x$ est un corps. Même si cela n'apparaît pas explicitement pendant le chapitre, la raison derrière cette hypothèse est que pour pouvoir appliquer les résultats du Chapitre 2 et recoller "autour" de la fibre, nous avons besoin que la fibre *ne soit pas* un diviseur.

Pour montrer le théorème 4.6.8, comme les fibres d'une courbe analytique relative sont munies d'une structure de courbe analytique, nous suivons un raisonnement similaire à celui du cas de dimension un. Cependant, beaucoup de difficultés techniques apparaissent dans ce cadre relatif. Voici un bref résumé de la preuve.

Nous construisons des recouvrements particuliers d'un voisinage de la fibre sur lesquels le recollement est possible (qu'on appelle *bons recouvrements relatifs*) ; ils représentent l'analogue relatif des bons recouvrements introduits dans le Chapitre 3. Nous traitons d'abord le cas de $\mathbb{P}^{1,\mathrm{an}}_S$ - la droite analytique projective relative sur S. Pour ce faire, nous utilisons la notion d'*épaississement* d'un domaine affinoïde, dont l'idée apparaît dans un texte non publié de Jérôme Poineau. Si U est un domaine affinoïde de la fibre $\mathscr{F}_x$ de x dans $\mathbb{P}^{1,\mathrm{an}}_S$, un *$Z$-épaississement* de U est un domaine affinoïde U_Z de $\mathbb{P}^{1,\mathrm{an}}_Z$ tel que $U_Z \cap \mathscr{F}_x = U$, où Z est un voisinage affinoïde de x dans S. Les épaississements des domaines affinoïdes de $\mathscr{F}_x$ existent et ont de bonnes propriétés si Z est choisi assez petit.

Soit $\mathcal{U}$ un bon recouvrement de la fibre $\mathscr{F}_x$. Alors, il existe un voisinage affinoïde Z de x tel que pour tout $U \in \mathcal{U}$, le Z-épaississement U_Z de U existe. Soit $\mathcal{U}_Z$ l'ensemble de tous ces Z-épaississements d'éléments de $\mathcal{U}$. Nous montrons que pour Z assez petit, $\mathcal{U}_Z$ satisfait les propriétés pour que les résultats du Chapitre 2 puissent s'appliquer. Dans ce cas, $\mathcal{U}_Z$ est dit être un *bon recouvrement Z-relatif* de $\mathbb{P}^{1,\mathrm{an}}_Z$. En particulier, nous remarquons que les points de type 3 jouent encore un rôle très important. Leur existence dans la fibre est garantie par l'hypothèse sur la dimension de S. Nous montrons alors que le recollement *peut* s'appliquer aux bons recouvrements relatifs dans le cas de $\mathbb{P}^{1,\mathrm{an}}$.

En utilisant des tirés en arrière de morphismes finis vers $\mathbb{P}^{1,\mathrm{an}}$, une notion de bon recouvrement relatif peut se construire plus généralement dans le cas des courbes normales propres relatives. En rajoutant à ceci la restriction des scalaires à la Weil, on montre que le recollement est possible sur des bons recouvrements relatifs dans ce cadre plus général.

Finalement, après avoir démontré que le recollement est possible autour de la fibre, le principe local-global du théorème 4.6.8 peut être démontré. Ceci nécessite toutefois plus de travail que dans le cas de la dimension un.

Il existe une connexion entre les points d'une fibre et les valuations dont son corps de fonctions méromorphes surconvergentes peut être muni. Nous rendons cela précis dans la proposition 4.6.6. Comme dans le cas de la dimension un, combiné avec l'hensélianité des corps $\mathscr{M}_{C,z}, \pi(z) = x$, cette connexion nous permet d'obtenir un principe local-global par rapport aux complétés. Avant d'énoncer ce résultat, nous rappelons que le corps $\mathcal{O}_{S,x}$ est naturellement muni d'une valuation $|\cdot|_x$.

Théorème (Theorem 4.6.8'). *Avec les mêmes notations que dans l'énoncé du théorème 4.6.8, on note $F_{\mathcal{O}_x} = \varinjlim_Z \mathscr{M}(C_Z)$. Soit $V(F_{\mathcal{O}_x})$ l'ensemble des valuations non triviales de rang 1 sur $F_{\mathcal{O}_x}$ qui induisent ou bien $|\cdot|_x$ ou bien la valuation triviale sur $\mathcal{O}_x$. Pour $v \in V(F_{\mathcal{O}_x})$, soit $F_{\mathcal{O}_x,v}$ le complété de $F_{\mathcal{O}_x}$ par rapport à v.*

Si car $k = 0$ *ou H est lisse, alors le principe local–global suivant est vrai :*

$$H(F_{\mathcal{O}_x}) \neq \emptyset \iff H(F_{\mathcal{O}_x,v}) \neq \emptyset \text{ pour tout } v \in V(F_{\mathcal{O}_x}).$$

Nous remarquons que, avec les mêmes notations que dans le théorème ci-dessus, $\mathcal{O}_{S,x} = \varinjlim_Z \mathcal{O}_S(Z)$, où la limite directe est prise sur les voisinages affinoïdes Z de x dans S. En utilisant les travaux de Grothendieck sur les limites projectives de schémas pour construire une courbe relative algébrique sur $\mathcal{O}(Z)$ à partir d'une courbe algébrique sur $\mathcal{O}_x$ et comme conséquence du théorème ci-dessus, nous obtenons la généralisation suivante du corollaire 3.2.18.

THÉORÈME (Theorem 4.6.9). *Soit S un bon espace normal k-analytique tel que* $\dim S < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$. *Soit $x \in S$ tel que $\mathcal{O}_x$ soit un corps. Soit $C_{\mathcal{O}_x}$ une courbe algébrique lisse géométriquement irréductible sur le corps $\mathcal{O}_x$. On note $F_{\mathcal{O}_x}$ le corps de fonctions de $C_{\mathcal{O}_x}$.*

Soit $G/F_{\mathcal{O}_x}$ un groupe linéaire algébrique connexe rationnel qui agit fortement transitivement sur une variété $H/F_{\mathcal{O}_x}$. Si car $k = 0$ *ou H est lisse, alors :*

$$H(F_{\mathcal{O}_x}) \neq \emptyset \iff H(F_{\mathcal{O}_x,v}) \neq \emptyset \text{ pour tout } v \in V(F_{\mathcal{O}_x}),$$

où $V(F_{\mathcal{O}_x})$ est comme dans le théorème 4.6.8' ci-dessus.

Un élément crucial pour démontrer le théorème 4.6.9, et plus généralement, pour mettre en valeur l'interêt de ce chapitre, est que, dans le cadre du théorème 4.6.8, les fonctions méromorphes autour de la fibre de x sont algébriques. Plus précisement, le corps des fonctions méromorphes surconvergentes sur la fibre de x est le corps de fonctions d'une courbe algébrique sur $\mathcal{O}_x$ (qui est l'algébrisation d'un voisinage de la fibre suivi par un changement de base à $\mathcal{O}_x$; voir le corollaire 4.4.15). Pour montrer ce résultat non trivial, nous utilisons un théorème du type GAGA pour le faisceau des fonctions méromorphes (voir le théorème 1.7.8).

À la fin de ce chapitre nous calculons quelques exemples d'anneaux locaux des espaces analytiques de Berkovich qui sont des corps et auxquels les résultats ci-dessus peuvent s'appliquer. Plus précisément, nous calculons les anneaux locaux en les points de $\mathbb{A}^{1,\mathrm{an}}$ pour lesquels l'anneau local correspondant est un corps. De plus, nous donnons une description de l'anneau local en certains points de $\mathbb{A}^{n,\mathrm{an}}$, $n \in \mathbb{N}$. Voici un exemple d'un tel corps, qui correspond à un point de type 3 d'une droite analytique affine.

EXEMPLE. Soit $(k, |\cdot|)$ un corps ultramétrique complet. Soit $r \in \mathbb{R}_{>0} \backslash \sqrt{|k^\times|}$. Soit $x \in \mathbb{A}_k^{1,\mathrm{an}}$ une semi-norme multiplicative sur $k[T]$ telle que $|T|_x = r$ (en fait, x est le seul tel point de $\mathbb{A}_k^{1,\mathrm{an}}$).

Pour tous $r_1, r_2 \in \mathbb{R}_{>0}$ tels que $r_1 < r < r_2$, on pose

$$A_{r_1,r_2} := \left\{ \sum_{n\in\mathbb{Z}} a_n T^n : a_n \in k, \lim_{n\to+\infty} |a_n| r_2^n = 0, \lim_{n\to-\infty} |a_n| r_1^n = 0 \right\}.$$

Alors, $\mathcal{O}_{\mathbb{A}_k^{1,\mathrm{an}},x} = \varinjlim_{r_1<r<r_2} A_{r_1,r_2}$.

Comme dans le Chapitre 3, puisque la variété projective définie par une forme quadratique satisfait les hypothèses des résultats présentés, l'exemple naturel auquel les énoncés de ce chapitre peuvent s'appliquer est celui des formes quadratiques (sous l'hypothèse car $k \neq 2$).

CHAPTER 1

Introduction to Berkovich Spaces

The aim of this chapter is to give an overview of the construction of Berkovich spaces and prove certain properties for them that we need for the next parts of the manuscript. The content of the first six sections is classical, so we only provide references for most of the results. The exposition follows [**6**], [**42**], and [**64**]. The last two sections are more specialized, containing for the most part results (including proofs) that will be applied in the next chapters.

We start by presenting the algebraic setting on which Berkovich's theory is founded. Since the objects in question are analytic, their algebraic counterparts will be rings endowed with some norm structure with respect to which they are complete (*i.e. Banach rings*). We make this more precise in Section 1.1, where the notion of (semi-)norm is defined and some of its properties are presented. Finally, we give the notion of Berkovich analytic spaces over Banach rings. More precisely, the analogue of the affine spectrum, called *Berkovich spectrum*, is discussed.

In Section 1.2, we give a classical example (on a topological level) of a Berkovich space, the *analytic affine line*. This is one of the most understood objects in the theory, and its properties are a good indicator of the kind of properties that analytic curves satisfy.

Like schemes, Berkovich spaces have "building blocks", called *affinoid spaces*. Roughly, affinoid spaces are to Berkovich spaces (or at least a certain subclass thereof known as *good Berkovich spaces*) what finite type affine schemes are to a finite type scheme. One fundamental difference between the two is that affinoid spaces are compact (meaning Hausdorff and hence not necessarily open), which is the cause of many technical difficulties in the Berkovich setting. Another particularity is that there are several notions of boundary for affinoid spaces. These objects are the topic of Section 1.4. We start with their algebraic counterpart, *affinoid algebras* (which are a generalization of Tate's affinoid algebras) in Section 1.3, and continue with the construction of the structural sheaf.

In Section 1.5, good analytic spaces and some of their main properties are presented. Local properties as being reduced, normal, regular, *etc.*, exist for Berkovich analytic spaces and are briefly discussed in this section. Several classes of morphisms in the category of good Berkovich spaces and their main properties are presented. Most of these notions are similar to their algebraic counterparts, but not all; this is due to the (sometimes) bad behaviour of *boundary points*. Berkovich spaces possess very nice topological properties (such as being locally arcwise-connected).

An analytification functor and GAGA-type theorems exist for Berkovich spaces and are the main tool for applying Berkovich's theory to arithmetic geometry. A topology is induced from a scheme to its *Berkovich analytification*, called the *Zariski topology* with respect to which the notion of irreducibility can be obtained. This is coarser than the Berkovich topology, but behaves quite nicely with respect to GAGA-principles. These facts are the topic of Section 1.6.

In Section 1.7, we discuss a somewhat less classical topic, the sheaf of meromorphic functions for Berkovich spaces. The definition and main properties resemble those of the sheaf of meromorphic functions for schemes. Seeing as later on we interpret patching as the gluing of meromorphic functions, this notion is crucial to our work. In particular, we give a detailed proof of a GAGA-type result coming from a MathOverflow thread (see [**57**]).

Finally, in Section 1.8, a more detailed account of analytic curves is given, seeing as it is the case of most interest to us. It includes some general results on curves (most of which shown in [**20**]), and some specific statements that we will need later. This is arguably the class of Berkovich spaces that is best understood and most studied: in [**20**], Ducros shows that Berkovich analytic curves have a(n infinite) graph-like structure; many of the notions defined on Berkovich spaces are much better behaved in the case of curves than in general (*e.g.* boundaries); an algebraic classification of points can be given, and it is usually possible to interpret it topologically. At the end of this section, we give a description of the points of the analytic affine line.

1.1. Banach rings and the Berkovich spectrum

All rings considered here are assumed to be commutative with unity.

1.1.1. Valued Fields. Let k be a field.

Definition 1.1.1. An *absolute value* on k is a function $|\cdot| : k \to \mathbb{R}_{\geqslant 0}$ such that:

(1) $|1| = 1$,
(2) for $x \in k$, $|x| = 0$ if and only if $x = 0$,
(3) $\forall x, y \in k$, $|xy| = |x||y|$,
(4) $\forall x, y \in k, |x - y| \leqslant |x| + |y|$.

We will say that $(k, |\cdot|)$ is a field endowed with an absolute value.

If instead of (4), the following stronger condition is satisfied,

(4') $\forall x, y \in k, |x - y| \leqslant \max(|x|, |y|)$,

then we say that $|\cdot|$ is a *non-Archimedean* or *ultrametric* absolute value on k. In that case, we will say that $(k, |\cdot|)$ is a non-Archimedean valued (or ultrametric) field. When there is no risk of ambiguity, we will simply say that k is an *ultrametric field*.

An absolute value which is not ultrametric will be called *Archimedean*.

Example 1.1.2. (1) For any field k, the function $|\cdot|_0 : k \to \mathbb{R}_{\geqslant 0}$, $x \mapsto 1$ if $x \neq 0$, and $x \mapsto 0$ otherwise, determines an absolute value on k. We call $|\cdot|_0$ the *trivial valuation* on k. Remark that $|\cdot|_0$ is ultrametric. The trivial valuation is the only absolute value with which a finite field can be endowed.

(2) The standard Euclidean norm $|\cdot|_\infty$ is an absolute value in $\mathbb{C}$. It is Archimedean.

(3) Let p be a prime number. For $x \in \mathbb{Q}^\times$, let a_x, b_x be the unique integers such that $b_x > 0$, $(a_x, b_x) = 1$, and $x = \frac{a_x}{b_x}$. Then, the function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geqslant 0}$,

$$x \mapsto \begin{cases} p^{-(v_p(a_x) - v_p(b_x))} & \text{if } x \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where for $c \in \mathbb{Z}\backslash\{0\}$, $c = p^{v_p(c)} \cdot d$ and $p \not| d$, is a non-Archimedean absolute value on $\mathbb{Q}$. It is said to be the *p-adic valuation* on $\mathbb{Q}$.

(4) Let p be a prime number. Then, the function $|\cdot|_T : \mathbb{F}_p(T) \to \mathbb{R}_{\geqslant 0}$, $0 \mapsto 0$, and $x \mapsto \alpha^{-v_T(x)}$ for $x \neq 0$, where $\alpha > 1$ and $v_T(x)$ is the largest integer n for which $x \in (T)^n$, is a non-Archimedean absolute value on $\mathbb{F}_p(T)$.

The following tells us how to distinguish between Archimedean and non-Archimidean absolute values.

PROPOSITION 1.1.3. *An absolute value $|\cdot|$ on k is non-Archimedean if and only if $|n| \leqslant 1$ for all $n \in \mathbb{Z}$.*

Remark that an absolute value on k determines a metric, and thus a topology on k. The following definition gives a criterion for when two absolute values induce the same topology on the underlying field.

DEFINITION 1.1.4. Two absolute values $|\cdot|_1, |\cdot|_2$ are said to be equivalent if there exists $\alpha > 0$ such that $|\cdot|_1 = |\cdot|_2^{\alpha}$.

Ostrowski's Theorem (see *e.g.* [**44**, Theorem 1, Section 1.2]) tells us that, up to equivalence, $\mathbb{Q}$ can only be endowed with the following absolute values: $|\cdot|_0$, $|\cdot|_\infty$, and $|\cdot|_p$, p a prime number.

Let $|\cdot|$ be an absolute value on k. We will say that k is *complete with respect to* $|\cdot|$ (or simply *complete* when there is no risk of ambiguity) if it is complete with respect to the metric $|\cdot|$ induces on k. As usual, one can define the completion $(\widehat{k}, |\cdot|)$ of $(k, |\cdot|)$ by using the Cauchy sequence construction. Then, $(\widehat{k}, |\cdot|)$ is complete.

EXAMPLE 1.1.5. (1) The field k is complete with respect to its trivial absolute valuation $|\cdot|_0$.

(2) The field $\mathbb{C}$ is complete with respect to $|\cdot|_\infty$. By the Gelfand-Mazur theorem (*cf.* [**15**], VI, 6, n°4, Théorème 1), the only complete fields with respect to Archimedean absolute values are $\mathbb{R}$ and $\mathbb{C}$.

(3) Let p be a prime number. We denote the completion of $(\mathbb{Q}, |\cdot|_p)$ by $\mathbb{Q}_p$. It is called *the field of p-adic numbers.*

(4) Let p be a prime number. Then, the completion of $(\mathbb{F}_p(T), |\cdot|_T)$ is the field $\mathbb{F}_p((T))$ of Laurent series over $\mathbb{F}_p$.

THEOREM 1.1.6. *Let $(k, |\cdot|)$ be a complete field. Let us fix an algebraic closure $\overline{k}$ of k.*

1. *The absolute value $|\cdot|$ can uniquely be extended to $\overline{k}$.*
2. *The completion $\widehat{\overline{k}}$ of $(\overline{k}, |\cdot|)$ is a complete algebraically closed field.*

For a prime number p, we will denote by $\mathbb{C}_p$ the field $\widehat{\overline{\mathbb{Q}_p}}$. It is the p-adic analogue of $\mathbb{C}$.

Finally, let us mention a few topological particularities of non-Archimedean valued fields.

PROPOSITION 1.1.7. *Let $(k, |\cdot|)$ be a non-Archimedean valued field.*

1. *For any $x, y \in k$ such that $|x| \neq |y|$, one has $|x + y| = \max(|x|, |y|)$.*
2. *A closed disc of positive radius in k is open.*
3. *For any $a \in k$ and $r \in \mathbb{R}_{\geqslant 0}$, let $B(a, r)$ denote the closed disc in k centered at a and of radius r. Then, for any $b \in B(a, r)$, one has $B(a, r) = B(b, r)$.*
4. *The field k is a totally disconnected topological space (with respect to the topology induced by $|\cdot|$).*

NOTATION 1.1.8. For any field k endowed with an absolute value $|\cdot|$, we denote by $|k|$ the set $\{r \in \mathbb{R}_{\geqslant 0} : \exists a \in k, |a| = r\}$. Set $|k^\times| = \{r \in |k| : r \neq 0\}$. This is a multiplicative subgroup of $\mathbb{R}_{>0}$. We denote by $\sqrt{|k^\times|}$ its divisible closure.

Let k be a complete ultrametric field. We will say that K/k is a *complete ultrametric field extension* if: (1) K is a field extension of k; (2) K is complete with respect to an absolute value that extends that of k.

1.1.2. Semi-normed rings. Let A be a ring.

DEFINITION 1.1.9. A *semi-norm* on A is a function $|\cdot| : A \to \mathbb{R}_{\geqslant 0}$ such that

(1) $|0| = 0, |1| = 1$,
(2) $\forall x, y \in A, |xy| \leqslant |x||y|$,
(3) $\forall x, y \in A, |x - y| \leqslant |x| + |y|$.

If condition (2) is strengthened: $\forall x, y \in A$, $|xy| = |x||y|$, we will say that $|\cdot|$ is a *multiplicative semi-norm* on A.

If $\ker |\cdot| = \{0\}$, we will say that $|\cdot|$ is a *norm* on A. If $|\cdot|$ is a multiplicative semi-norm and a norm, we will say that it is a *multiplicative norm* on A.

If instead of condition (3) we take the following stronger hypothesis:

(3') $\forall x, y \in A$, $|x - y| \leqslant \max(|x|, |y|)$,

then $|\cdot|$ is said to be *non-Archimedean*.

We will sometimes say that A is a *(semi-)normed ring*.

EXAMPLE 1.1.10. (1) The function $|\cdot|_0 : A \to \mathbb{R}_{\geqslant 0}$, $x \mapsto 1$ if $x \neq 0$, and $x \mapsto 0$ otherwise, determines an multiplicative norm on A, called the *trivial norm*.

(2) The Euclidean norm $|\cdot|_\infty$ is a multiplicative norm on $\mathbb{Z}$. For a prime number p, the p-adic absolute value $|\cdot|_p$ determines a non-Archimedean multiplicative norm on $\mathbb{Z}$.

(3) Let $(k, |\cdot|)$ be a field endowed with an absolute value. Let $a \in k$. Then, the function $|\cdot|_a : k[T] \to \mathbb{R}_{\geqslant 0}, P(T) \mapsto |P(a)|$, is a multiplicative semi-norm on $k[T]$. It is non-Archimedean if $|\cdot|$ is so.

(4) Let $M_n(\mathbb{C})$ be the ring of $n \times n$ matrices over $\mathbb{C}$. Let $|||\cdot||| : M_n(\mathbb{C}) \to \mathbb{R}_{\geqslant 0}$ be the function given by $M \mapsto \sup_{v \in \mathbb{C}^n \setminus \{0\}} \frac{\|Mv\|}{\|v\|}$, where $\|\cdot\| : \mathbb{C}^n \to \mathbb{R}_{\geqslant 0}$ is the Euclidean norm, *i.e.* $(t_1, t_2, \ldots, t_n) \mapsto \sqrt{|t_1|_\infty^2 + |t_2|_\infty^2 + \cdots |t_n|_\infty^2}$. Then, $|||\cdot|||$ is a norm on $M_n(\mathbb{C})$.

As before, a semi-norm $|\cdot|$ on A determines a topology on A. Remark that this topology is Hausdorff if and only if $|\cdot|$ is a norm. A sufficient condition for two semi-norms to induce the same topology on A is given by the following:

DEFINITION 1.1.11. Two semi-norms $|\cdot|_1, |\cdot|_2$ on A are said to be equivalent if there exist real numbers $C_1, C_2 > 0$ such that $C_1|\cdot|_1 \leqslant |\cdot|_2 \leqslant C_2|\cdot|_1$.

In particular, remark that $\ker|\cdot|_1 = \ker|\cdot|_2$, so $|\cdot|_1$ is a norm if and only if $|\cdot|_2$ is a norm.

LEMMA 1.1.12. *A multiplicative semi-norm $|\cdot|$ on A is non-Archimedean if and only if $\mathbb{Z}$ is bounded.*

Let I be an ideal of A. Then, any semi-norm $|\cdot|$ on A induces a semi-norm on A/I via $x + I \to \inf\{|y| : x - y \in I\}$, called the *quotient semi-norm* on A/I induced by $|\cdot|$. If $|\cdot|$ is non-Archimedean, then so is the quotient semi-norm.

Let us introduce the class of morphisms between semi-normed rings that will be of interest to us.

DEFINITION 1.1.13. Let $(A, |\cdot|_A), (B, |\cdot|_B)$ be two semi-normed rings (*i.e.* rings endowed with semi-norms). A morphism $f : A \to B$ is said to be *bounded* (with respect

to these semi-norms on A, B, respectively) if there exists a real number $C > 0$, such that for any $a \in A$, $|f(a)|_B \leqslant C \cdot |a|_A$.

The morphism f is said to be *admissible* if the quotient semi-norm on $A/\ker f$ induced by $|\cdot|_A$ is equivalent to the semi-norm induced on $\mathrm{Im}(f)$ by $|\cdot|_B$.

Let $|\cdot|$ be a norm on A. As usual, we will say that $(A, |\cdot|)$ is *complete* if any Cauchy sequence in A has a limit in A (with respect to $|\cdot|$). In that case, A is said to be a *Banach ring*. In Berkovich's theory, Banach rings plays a role analoguous to commutative rings in algebraic geometry.

In Example 1.1.10, $(\mathbb{Z}, |\cdot|_\infty), (M_n(\mathbb{C}), |||\cdot|||)$ are Banach rings.

Definition 1.1.14. Let $(A, |\cdot|_A), (B, |\cdot|_B)$ be Banach rings, such that B is an A-algebra. We say that $(B, |\cdot|_B)$ is a *Banach A-algebra* if the morphism $A \to B$ is bounded.

Lemma 1.1.15. *Let I be a closed ideal of A. If $|\cdot|$ is a norm on A, then the quotient semi-norm $|\cdot|_{A/I}$ on A/I induced by $|\cdot|$ is a norm on A/I. Moreover, if $(A, |\cdot|)$ is a Banach ring, then so is $(A/I, |\cdot|_{A/I})$.*

If $|\cdot|$ is merely a semi-norm on A, it does not make sense to speak of completeness seeing as the Cauchy sequences in A may have more than one accumulation point. There is, however, a notion of completion.

Theorem 1.1.16. *Let $(A, |\cdot|)$ be a semi-normed ring. The set of equivalence classes of Cauchy sequences in A forms a ring $\widehat{A}$ naturally endowed with a norm $|\cdot|'$. Then, $(\widehat{A}, |\cdot|')$ is a Banach ring, and there is a natural admissible morphism $A \to \widehat{A}$ such that the image of A is dense. Moreover, $|\cdot|$ is non-Archimedean if and only if $|\cdot|'$ is non-Archimedean.*

We will say that $(\widehat{A}, |\cdot|')$ is the *completion* (in some texts referred to as *separated completion*) of $(A, |\cdot|)$. The admissible morphism $A \to \widehat{A}$ is an embedding if and only if $|\cdot|$ is a norm. It is an isomorphism if and only if $(A, |\cdot|)$ is a Banach ring. Remark that $(\widehat{A}, |\cdot|')$ is the completion of $A/\ker|\cdot|$ with respect to the quotient semi-norm.

1.1.3. The spectral radius. Let $(A, |\cdot|)$ be a Banach ring. We present here a canonical way to obtain from $|\cdot|$ a semi-norm with particularly nice properties (*e.g.* as close as possible to being multiplicative).

Lemma 1.1.17 (Fekete's Lemma). *Let $x \in A$. Then, $\rho_A(x) := \lim_{n\to+\infty} |x^n|^{1/n}$ exists. Moreover, $\rho_A(x) = \inf_{n\in\mathbb{N}} |x^n|^{1/n}$.*

Definition 1.1.18. For any $x \in A$, $\rho_A(x)$ is called the *spectral radius* of x.

Lemma 1.1.19. *(1)* $\rho_A(1) = 1$;
(2) $\forall x, y \in A, \rho_A(x-y) \leqslant \rho_A(x) + \rho_A(y)$;
(3) $\forall x, y \in A, \rho_A(xy) \leqslant \rho_A(x)\rho_A(y)$;
(4) $\forall n \in \mathbb{N}$, $\forall x \in A, \rho_A(x^n) = \rho_A(x)^n$.
If A is non-Archimedean, then $\forall x, y \in A, \rho_A(x-y) \leqslant \max(\rho_A(x), \rho_A(y))$.

We will call ρ_A the *spectral semi-norm of A*.

Another important property of the spectral semi-norm is that it remains the same even if $|\cdot|$ is replaced by an equivalent norm.

1.1.4. Semi-normed modules. Let $(A, |\cdot|)$ be a ring endowed with a semi-norm. Let M be an A-module.

DEFINITION 1.1.20. A *semi-norm* on the A-module M is a function $\|\cdot\| : M \to \mathbb{R}_{\geqslant 0}$, satisfying:

(1) $\forall x, y \in M, \|x - y\| \leqslant \|x\| + \|y\|$;
(2) there exists a real number $C > 0$ such that $\forall a \in A, \forall m \in M$, $\|am\| \leqslant C|a|\|m\|$.

If $\ker\|\cdot\| = 0$, we will say that $\|\cdot\|$ is a *norm* on M.

If condition (1) is strengthened to: $\forall x, y \in M$, $\|x - y\| \leqslant \max(\|x\|, \|y\|)$, then $\|\cdot\|$ is said to be *non-Archimedean*.

Remark that for any semi-norm $\|\cdot\|$ on M, $\|0\| = 0$ (because of condition (2) above and the fact that $|0| = 0$).

A semi-norm $\|\cdot\|$ defines a topology on M, which is Hausdorff if and only if $\|\cdot\|$ is a norm. There is once again a notion of equivalent semi-norms which gives a sufficient condition for semi-norms to induce the same topology on M: two semi-norms $\|\cdot\|_1, \|\cdot\|_2$ are said to be equivalent if there exist positive real numbers C_1, C_2 such that $C_1\|\cdot\|_1 \leqslant \|\cdot\|_2 \leqslant C_2\|\cdot\|_1$.

Hence, $\|\cdot\|$ can be replaced by an equivalent semi-norm for which in condition (2) of Definition 1.1.20 we can take $C = 1$.

Let M' be an A-submodule of M. Then, a (non-Archimedean) semi-norm $\|\cdot\|$ on M induces a (non-Archimedean) semi-norm on M/M' via $x + M' \mapsto \inf\{\|y\| : x - y \in M'\}$, called the *quotient semi-norm*.

We can define in the same way as in Definition 1.1.13 the notion of bounded and admissible morphisms between semi-normed A-modules.

Let $\|\cdot\|$ be a norm on M. Then, M is said to be *complete* with respect to $\|\cdot\|$ if any Cauchy sequence in M has a limit in M. In that case, we will say that $(M, \|\cdot\|)$ is a *Banach A-module*.

LEMMA 1.1.21. *Let M' be a closed A-submodule of M. If $\|\cdot\|$ is a norm on M, then the quotient semi-norm $\|\cdot\|_{M/M'}$ on M/M' induced by $\|\cdot\|$ is a norm on M/M'. Moreover, if $(M, \|\cdot\|)$ is a Banach A-module, then so is $(M/M', \|\cdot\|_{M/M'})$.*

With the same remarks as in the case of semi-normed rings, there is a notion of completion.

THEOREM 1.1.22. *Let $(M, \|\cdot\|)$ be a semi-normed A-module. The set of equivalence classes of Cauchy sequences in M forms an $\widehat{A}$-module $\widehat{M}$ naturally endowed with a norm $\|\cdot\|'$. Then, $(\widehat{M}, \|\cdot\|')$ is a Banach module over both A and $\widehat{A}$, and there is a natural admissible morphism $M \to \widehat{M}$ such that the image of M is dense. Moreover, $\|\cdot\|$ is non-Archimedean if and only if $\|\cdot\|'$ is non-Archimedean.*

We will say that $(\widehat{M}, \|\cdot\|')$ is the *completion* (in some texts referred to as *separated completion*) of $(M, \|\cdot\|)$. The admissible morphism $M \to \widehat{M}$ is an embedding if and only if $\|\cdot\|$ is a norm. It is an isomorphism if and only if $(M, \|\cdot\|)$ is a Banach A-module. Remark that $(\widehat{M}, \|\cdot\|')$ is the completion of $M/\ker\|\cdot\|$ with respect to the quotient semi-norm.

1.1.5. Complete tensor product of modules in the non-Archimedean case. Let $(A, |\cdot|)$ be a non-Archimedean normed ring. Let $(M, |\cdot|_M), (N, |\cdot|_N)$ be two normed

non-Archimedean A-modules. We can endow the tensor product $M \otimes_A N$ with the semi-norm

$$\|x\| := \inf\left\{ \max_{i=1,\dots,n} |m_i|_M |n_i|_N : x = \sum_{i=1}^{n} m_i \otimes n_i \right\}.$$

DEFINITION 1.1.23. The *complete tensor product* of M and N, denoted $M\widehat{\otimes}_A N$, is the completion of $M \otimes_A N$ with respect to the semi-norm $\|\cdot\|$. It is a Banach module over both A and $\widehat{A}$.

The complete tensor product satisfies a universal property.

DEFINITION 1.1.24. Let $(P_1, |\cdot|_1), (P_2, |\cdot|_2), (P_3, |\cdot|_3)$ be normed A-modules. A morphism $\varphi : P_1 \times P_2 \to P_3$ of A-modules is said to be a *bounded bilinear morphism* if there exists $C > 0$ such that for any $x \in P_1, y \in P_2$, $|\varphi(x,y)|_3 \leqslant C|x|_1|y|_2$.

PROPOSITION 1.1.25 ([**11**, 2.1.7/1]). *Let P be a Banach A-algebra. Any bounded bilinear morphism $M \times N \to P$ is uniquely factorised through $M\widehat{\otimes}_A N$.*

Other useful properties of this construction that we need are the following (non-trivial) results:

THEOREM 1.1.26 ([**29**], Section 3, Thm. 1(4)). *Let k be a complete ultrametric field. Let M, N be non-Archimedean k-Banach vector spaces. Then, the canonical map $M \otimes_k N \to M\widehat{\otimes}_k N$ is an embedding with a dense image.*

THEOREM 1.1.27. [**41**, Appendix D.4.2] *Let A be a non-Archimedean Banach ring. Let M, N, M', N' be Banach A-modules. If there exist A-linear maps $\varphi : M \to M'$ and $\psi : N \to N'$ that are surjective and admissible, then $\varphi\widehat{\otimes}_A\psi : M\widehat{\otimes}_A N \to M'\widehat{\otimes}_A N'$ is surjective and admissible.*

For a detailed treatment of complete tensor products, see [**11**, 2.1.7].

1.1.6. The Berkovich spectrum. All rings considered here are assumed to be commutative with unity. We now define and explore the Berkovich analogue of the affine spectrum, defined in [**6**, Section 1.2].

DEFINITION 1.1.28. Let $(A, \|\cdot\|)$ be a ring endowed with a semi-norm. A semi-norm $|\cdot|$ on A is said to be $\|\cdot\|-$*bounded* if there exists a positive real number C, such that $|\cdot| \leqslant C\|\cdot\|$.

When there is no risk of ambiguity, we will simply say that $|\cdot|$ is a bounded semi-norm on A.

DEFINITION 1.1.29 (The Berkovich Spectrum). Let $(A, \|\cdot\|)$ be a Banach ring. The *Berkovich spectrum* of A, denoted $\mathcal{M}(A)$, is the set of all bounded multiplicative semi-norms on A.

We endow $\mathcal{M}(A)$ with the coarsest topology for which the function $v_y : \mathcal{M}(A) \to \mathbb{R}_{\geqslant 0}$, $|\cdot| \mapsto |f|$, is continuous for all $f \in A$.

REMARK 1.1.30. For any $|\cdot| \in \mathcal{M}(A)$, $|\cdot| \leqslant \|\cdot\|$. To see this, fix $|\cdot| \in \mathcal{M}(A)$, and let $C > 0$ be such that $|\cdot| \leqslant C\|\cdot\|$. Then, for any $a \in A$ and any $n \in \mathbb{N}$, we have $|a|^n = |a^n| \leqslant C\|a^n\| \leqslant C\|a\|^n$, so $|a| \leqslant \sqrt[n]{C}\|a\|$. By taking $n \to +\infty$, we obtain $|a| \leqslant \|a\|$, so $|\cdot| \leqslant \|\cdot\|$.

Remark that because of their multiplicativity, $\mathcal{M}(A)$ does not contain equivalent semi-norms. Also, $\mathcal{M}(A)$ does not change if we replace $\|\cdot\|$ by an equivalent norm.

CONVENTION 1.1.31. For a point x of the space $\mathcal{M}(A)$, we will also use the notation $|\cdot|_x$ when considering it as a semi-norm on A.

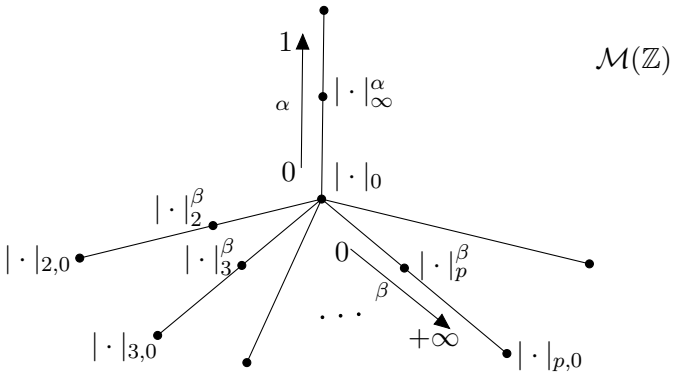
EXAMPLE 1.1.32. (1) Let $(k, |\cdot|)$ be a complete ultrametric field. Then, $\mathcal{M}(k)$ is the single point $\{|\cdot|\}$.

(2) Let us briefly describe $\mathcal{M}(\mathbb{Z})$, where $\mathbb{Z}$ is endowed with $|\cdot|_\infty$. This can be done using Ostrowski's theorem on the classification, up to equivalence, of all absolute values that $\mathbb{Q}$ can be endowed with.

Let $|\cdot|_x$ be a multiplicative norm on $\mathbb{Z}$. Then, this defines an absolute value on $\mathbb{Q}$, so we have the following possibilities:

- $|\cdot|_x = |\cdot|_0$-the trivial norm on $\mathbb{Z}$;
- $|\cdot|_x = |\cdot|_\infty^\alpha$ for some $\alpha > 0$, and in order for this to be an absolute value on $\mathbb{Q}$, $\alpha \in (0, 1]$;
- there exists a prime number p such that $|\cdot|_x = |\cdot|_p^\beta$ for some $\beta > 0$; here $\beta \in (0, +\infty)$.

Suppose that $|\cdot|_x$ is a multiplicative semi-norm on $\mathbb{Z}$ that is not a norm. Then, $\ker|\cdot|_x$ determines a non-zero prime ideal P_x of $\mathbb{Z}$. Consequently, there exists a prime number $p \in \mathbb{Z}$, such that $P_x = p\mathbb{Z}$. The quotient semi-norm on $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$ is a norm, so it is trivial. Consequently, $|\cdot|_x = |\cdot|_{p,0}$, where $|x|_{p,0} := 1$ if $p \not| x$, and $|x|_{p,0} := 0$ otherwise. (Remark that, informally, "$\lim_{\beta\to+\infty} |\cdot|_p^\beta = |\cdot|_{p,0}$".)



Finally, $\mathcal{M}(\mathbb{Z}) = \{|\cdot|_0, |\cdot|_\infty^\alpha, |\cdot|_p^\beta, |\cdot|_{p,0} : 0 < \alpha \leqslant 1, \beta > 0, p - \text{prime}\}$. This gives us the following "tree-like" illustration of $\mathcal{M}(\mathbb{Z})$. For any prime p, there is a "branch" associated to it that is homeomorphic to an interval. There is an additional branch associated to the Euclidean norm $|\cdot|_\infty$, which is also homeomorphic to an interval. All of these branches come together at the "central" point of the tree, corresponding to the trivial valuation. The open neighborhoods of $|\cdot|_0$ have only finitely many boundary points, meaning they contain all but a finite amout of the branches of the tree.

We will see other important examples later. Berkovich spaces will be defined over a complete ultrametric field, for which the space $\mathcal{M}(k)$ will be relevant. There also exist Berkovich spaces defined over $\mathbb{Z}$ (*i.e.* over $\mathcal{M}(\mathbb{Z})$), which include at the same time elements of the Archimedean and non-Archimedean worlds (see [**59**]). This is an area of research that is in the rise.

Let A be a Banach ring, and $x \in \mathcal{M}(A)$. Then, $\ker |\cdot|_x$ is a closed and prime ideal of A, so x induces a semi-norm $|\cdot|'_x$ on the domain $A/\ker|\cdot|_x$. Moreover, for any $f, g \in A$ such that $|f-g|_x = 0$, we have that $|f|_x = |g|_x$. Thus, $|f|_x = |\widetilde{f}|'_x$, where $\widetilde{f}$ is the image of f in $A/\ker|\cdot|_x$. Consequently, the quotient semi-norm in $A/\ker|\cdot|_x$ is a multiplicative norm, and can thus be extended to Frac $(A/\ker|\cdot|_x)$.

DEFINITION 1.1.33. Let A be a Banach ring. For any $x \in \mathcal{M}(A)$, let us denote by $\mathcal{H}(x)$ the completion of $\mathrm{Frac}(A/\ker|\cdot|_x)$ with respect to the quotient norm. We call it the *completed residue field* of x.

Remark that there is a canonical isometric embedding $k \hookrightarrow \mathcal{H}(x)$, implying $\mathcal{H}(x)$ is a complete *ultrametric* field.

The completed residue fields are important objects in Berkovich's theory. These are the fields where analytic functions take their values.

Before stating the main properties of the Berkovich spectrum, let us present another construction of it, which draws an analogy with the affine spectrum.

DEFINITION 1.1.34. Let A be a Banach ring. Let K, L be two complete ultrametric fields, such that there exist bounded maps $\phi_K : A \to K, \phi_L : A \to L$.

The morphisms ϕ_K, ϕ_L are said to be *equivalent* if there exists a complete field M, a bounded morphism $A \to M$, and embeddings $M \hookrightarrow L, M \hookrightarrow K$ corresponding to complete ultrametric field extensions, such that the following diagram commutes.

$$\begin{array}{ccccc} K & \xleftarrow{\phi_K} & A & \xrightarrow{\phi_L} & L \\ & \searrow & \downarrow & \swarrow & \\ & & M & & \end{array}$$

LEMMA 1.1.35. [**6**, Remark 1.2.2(ii)] *Let A be a Banach ring. The points of $\mathcal{M}(A)$ are the equivalence classes (Definition 1.1.34) of bounded morphisms $A \to K$, where K is a complete ultrametric field.*

For a proof of Lemma 1.1.35, see [**41**, pg. 7, Algebraic Characters].

A crucial property of the Berkovich spectrum is the following:

THEOREM 1.1.36. [**6**, Theorem 1.2.1] *Let A be a Banach ring. Then, $\mathcal{M}(A)$ is a non-empty compact space.*

REMARK 1.1.37. In this text, compact will always mean quasi-compact and Hausdorff.

The fact that $\mathcal{M}(A)$ is compact (and hence Hausdorff) is one of the main differences with the algebraic setting. As we will later see, the spectra of certain Banach rings form the building blocks of Berkovich spaces, and the fact that the building blocks are compact is a source of technical difficulties. Namely, the structural sheaf of analytic functions will be defined over these building blocks and will be "nice" there, but this will generally not be the case for opens.

Another very important property is the following:

THEOREM 1.1.38 (The maximum modulus principle, [**6**, Theorem 1.3.1]). *Let A be a Banach ring. Then, for any $a \in A$, $\rho_A(a) = \max_{x \in \mathcal{M}(A)} |a|_x$, where ρ_A is the spectral radius associated to the norm on A.*

Information can be obtained on the Banach ring A by looking at the infinite vector $(|\cdot|_x)_{x\in\mathcal{M}(A)}$. Here is an example.

LEMMA 1.1.39 ([**42**, Corollary 3.15]). *Let A be a Banach ring. An element $a \in A$ is invertible if and only if $|a|_x \neq 0$ for all $x \in \mathcal{M}(A)$.*

Let us finish the construction of a functor from the category of Banach rings to the category of topological spaces ($A \mapsto \mathcal{M}(A)$) by giving the class of morphisms that we consider.

LEMMA 1.1.40. *Let A, B be Banach rings. Any bounded morphism $A \to B$ gives rise to a continuous map $\varphi : \mathcal{M}(B) \to \mathcal{M}(A)$.*

For a proof, see [**41**, pg. 8, Induced maps].

REMARK 1.1.41. Let $f : A \to B$ be a bounded morphism of Banach rings. Let $\varphi : \mathcal{M}(B) \to \mathcal{M}(A)$ be the induced continuous morphism of their spectra. For any $x \in \mathcal{M}(B)$, there exists a natural isometric embedding $\mathcal{H}(y) \hookrightarrow \mathcal{H}(x)$, where $y := \varphi(x)$.

Here is a description of the fibers of these morphisms.

LEMMA 1.1.42. *Let A, B be Banach rings. Let $f : A \to B$ be a bounded morphism, and $\varphi : \mathcal{M}(B) \to \mathcal{M}(A)$ the induced morphism of their spectra. Then, for any $x \in \mathcal{M}(A)$, the fiber $\varphi^{-1}(x)$ is homeomorphic to $\mathcal{M}(B\widehat{\otimes}_A\mathcal{H}(x))$.*

For a proof, see [**41**, pg. 14, Fibers].

Finally, the following important result will be very useful.

PROPOSITION 1.1.43 ([**6**, Corollary 1.3.6]). *Let k be a complete field. Let A be a Banach k-algebra. Set $G = \mathrm{Gal}(k^s/k)$, where k^s is the separable closure of k. Then, G acts on $\widehat{\overline{k}}$ and $\mathcal{M}(A\widehat{\otimes}_k\widehat{\overline{k}})$. Moreover, there is a homeomorphism $\mathcal{M}(A\widehat{\otimes}_k\widehat{\overline{k}})/G \to \mathcal{M}(A)$.*

1.2. The Analytic Affine Line

Before continuing with an overview on the construction of Berkovich spaces, we make a digression in order to describe in detail (only as a topological space) a fundamental example of these objects. As we will later see, said example illustrates well the main geometric properties of Berkovich analytic curves. The objects presented here were originally defined in [**6**, Section 1.5].

1.2.1. The analytic affine space. Let A be a Banach ring. Set

$$\mathbb{A}_A^{n,\mathrm{an}} = \{\text{multiplicative semi-norms on } A[T_1, T_2, \ldots, T_n] \text{ that are bounded on } A\}.$$

We endow the above set with the coarsest topology for which the map $\mathbb{A}_A^{n,\mathrm{an}} \to \mathbb{R}_{\geqslant 0}, x \mapsto |f|_x$, is continuous for all $f \in A[T_1, T_2, \ldots, T_n]$.

The space $\mathbb{A}_A^{n,\mathrm{an}}$ is called the *n-dimensional analytic affine space* over A. If $n = 1$, we say that $\mathbb{A}_A^{1,\mathrm{an}}$ is the *analytic affine line* over A. The analytic affine space has nice topological properties (*cf.* [**59**, Théorème 1.1.13]).

EXAMPLE 1.2.1. If A is $\mathbb{C}$ endowed with $|\cdot|_\infty$, then by the Gelfand-Mazur theorem we obtain the usual complex affine n-dimensional space $\mathbb{A}_{\mathbb{C}}^{n,\mathrm{an}}$ homeomorphic to $\mathbb{C}^n$.

The case of most interest to us is when A is a complete ultrametric field k and $n = 1$. Arguably, these are the (non-trivial) Berkovich spaces that can be described the best. We now focus on that.

DEFINITION 1.2.2. For any $x \in \mathbb{A}_k^{1,\mathrm{an}}$, let $\mathcal{H}(x)$ be the completion of $\mathrm{Frac}(k[T]/\ker|\cdot|_x)$ with respect to the quotient semi-norm. Remark that $\mathcal{H}(x)$ is a complete ultrametric field and there is a canonical isometric embedding $k \hookrightarrow \mathcal{H}(x)$. We denote by $\widetilde{k}$, resp. $\widetilde{\mathcal{H}(x)}$, the residue field of k, resp. $\mathcal{H}(x)$.

1.2.2. $\mathbb{A}_k^{1,\mathrm{an}}$: the trivially valued case. Let k be a trivially valued field. For any $x \in \mathbb{A}_k^{1,\mathrm{an}}$, let $p_x := \ker|\cdot|_x$.

If $p_x \neq 0$, then there exists an irreducible polynomial $P(T) \in k[T]$ such that $p_x = (P)$. Then, $\mathcal{H}(x) = k[T]/(P)$, and since k is trivially valued, we obtain that $\mathcal{H}(x)$ is trivially valued. Consequently, x determines the following semi-norm on $k[T]$: $Q \mapsto 0$ if $P|Q$, and $Q \mapsto 1$ otherwise. We denote $\eta_{P,0} := x$.

Suppose $p_x = 0$. Then, $|\cdot|_x$ is a multiplicative norm on $k[T]$ (and hence on $k(T)$) which when restricted to k is the trivial norm. This implies that $|\cdot|_x$ is non-Archimedean (see Lemma 1.1.12).

- If $|T|_x \leqslant 1$, then for any polynomial P over k, $|P|_x \leqslant 1$. Let $m_x := \{P \in k[T] : |P|_x < 1\}$. If $m_x = 0$, then x is the trivial norm on $k[T]$, which we denote by $\eta_{T,1}$. If $m_x \neq 0$, then there exists an irreducible polynomial Q over k, such that $m_x = (Q)$. Set $r := |Q|_x \in (0,1)$. Then, for any $P \in k[T]\backslash\{0\}$, $|P|_x = r^{v_Q(P)}$, where $v_Q(P)$ is the largest $n \in \mathbb{N}$ such that $Q^n|P$. We will denote this point by $\eta_{Q,r}$. Remark that $\mathcal{H}(\eta_{Q,r})$ is the completion of $k[T]$ with respect to the Q-adic valuation.
- Suppose $|T|_x > 1$, and set $s := |T|_x > 1$. Then, for any $P \in k[T]\backslash\{0\}$, $|P|_x = s^{\deg P}$. We denote this point by $\eta_{T,s}$. The field $\mathcal{H}(\eta_{T,s})$ is then isomorphic to the field of Laurent series $k((T^{-1}))$.

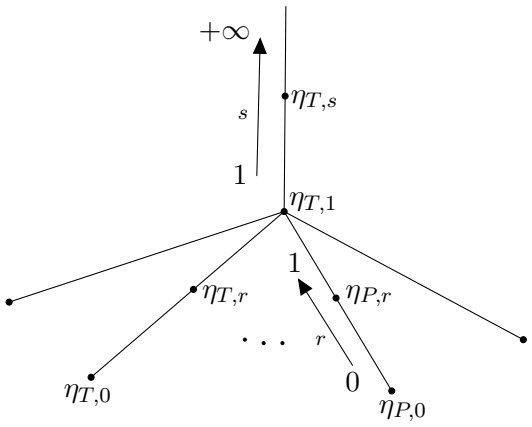


Figure 1: $\mathbb{A}_k^{1,\mathrm{an}}$ for k-trivially valued

We obtain the above "tree-like" illustration of $\mathbb{A}_k^{1,\mathrm{an}}$. To each irreducible polynomial over k, a *branch* homeomorphic to an interval is associated. There is an additional branch corresponding to $\eta_{T,s}, s > 1$, which is also homeomorphic to an interval. They all come together at the "central point" $\eta_{T,1}$ - the trivial valuation. An open neighborhood of $\eta_{T,1}$ contains all but a finite number of these branches (*i.e.* it has finite boundary).

1.2.3. $\mathbb{A}_k^{1,\mathrm{an}}$: the algebraically closed case ([6, 1.4.4]). Let $(k, |\cdot|)$ be a complete non-trivially valued ultrametric field that is algebraically closed.

- For any $a \in k$, let $\eta_{T-a,0} : k[T] \to \mathbb{R}_{\geqslant 0}$, $P(T) \mapsto |P(a)|$. This determines a multiplicative semi-norm on $k[T]$ extending the absolute value on k, so $\eta_{T-a,0}$ is a point of $\mathbb{A}_k^{1,\mathrm{an}}$; it is said to be a *type 1* point. Remark that $\mathcal{H}(x) = k$. We will sometimes denote this point by $\eta_{a,0}$.
- For any $a \in k$ and $r \in \mathbb{R}_{>0}$, let $\eta_{T-a,r} : k[T] \to \mathbb{R}_{\geqslant 0}, P(T) = \sum_n a_n(T-a)^n \mapsto \max_n |a_n| r^n$. This is a multiplicative norm on $k[T]$ extending the absolute value on k, so $\eta_{T-a,r} \in \mathbb{A}_k^{1,\mathrm{an}}$. Remark that $\eta_{T-a,r}$ does not depend on a, but only on the closed disc $B(a,r)$ in k centered at a and of radius r (*i.e.* for any $b \in B(a,r)$, $\eta_{T-b,r} = \eta_{T-a,r}$). We will sometimes simply denote $\eta_{a,r}$.

  These kinds of points behave differently depending on r. If $r \in |k^\times|$, $\eta_{a,r}$ is said to be a *type 2* point. In that case, $\widetilde{\mathcal{H}(x)} \cong \widetilde{k}(T)$ and $|\mathcal{H}(x)| = |k|$.

  If $r \notin |k|$, then $\eta_{a,r}$ is said to be a *type 3* point. If that is the case, then $\widetilde{\mathcal{H}(x)} = \widetilde{k}$, and $|\mathcal{H}(x)^\times|$ is generated by $|k^\times|$ and r.

Let $\mathscr{B} := (B_n)_{n\in\mathbb{N}}$ be a sequence of decreasing closed discs in k, *i.e.* $B_{n+1} \subseteq B_n$ for all $n \in \mathbb{N}$. Let $a_n \in k$ and $r_n \in \mathbb{R}_{>0}$ be such that $B_n = B(a_n, r_n)$. Let us denote by $|\cdot|_{B_n}$ the (unique) point η_{a_n,r_n} of $\mathbb{A}_k^{1,\mathrm{an}}$ determined by B_n as above. Then, $\mathscr{B}$ determines a point of $\mathbb{A}_k^{1,\mathrm{an}}$ as follows: $|\cdot|_{\mathscr{B}} : k[T] \to \mathbb{R}_{\geqslant 0}, P(T) \mapsto \inf_n |P(T)|_{B_n}$.

Berkovich showed that all of the points of $\mathbb{A}_k^{1,\mathrm{an}}$ are of the form described above. In particular, remark that: (1) if $\bigcap_n B_n$ is a single point $a \in k$, then $|\cdot|_{\mathscr{B}}$ is the type 1 point $\eta_{a,0}$; (2) if $\bigcap_n B_n$ is a closed disc centered at $a \in k$ and of radius $r \in \mathbb{R}_{>0}$, then $|\cdot|_{\mathscr{B}}$ is the point $\eta_{a,r}$, which is of type 2 or 3 depending on the nature of r.

However, this does not always cover all of the possibilites. Namely, it could happen that $\bigcap_n B_n = \emptyset$, in which case we say that the field k is *spherically complete*. If this is the case, $\mathscr{B}$ gives rise to a point said of *type 4* (these are in general the most complicated points to work with). Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a type 4 point. By [**6**, 1.4.4], $|\mathcal{H}(x)| = |k|$, and $\widetilde{\mathcal{H}(x)} = \widetilde{k}$.

The above is an illustration of $\mathbb{A}_k^{1,\mathrm{an}}$ in this case. Remark that it is an "infinitely branched tree". The types of points can be read in the drawing. Namely:

- type 2 points are the points of branching in the tree (*e.g.* $\eta_{T,1} = \eta_{T-1,1}$);
- type 3 points are those where there is no branching (*e.g.* $\eta_{T,r}$ with $r \notin |k|$);
- type 1 and 4 points are the "leaves" of the tree.

The topology on $\mathbb{A}_k^{1,\mathrm{an}}$ is quite complicated. All of the injective "paths" are isomorphic to a segment. For points of type 2, neighborhoods resemble somewhat to the neighborhoods of the "central" point in the trivially valued case.

1.2.4. $\mathbb{A}_k^{1,\mathrm{an}}$: the general case. Suppose k is a complete ultrametric field (not necessarily algebraically closed).

There is a canonical continuous open surjective morphism $\varphi : \mathbb{A}_{\widehat{k}}^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$. Moreover, let G denote the absolute Galois group of k, *i.e.* $G = \mathrm{Gal}(k^s/k)$, where k^s is the separable closure of k. Then, G acts on $\mathbb{A}_{\widehat{k}}^{1,\mathrm{an}}$ preserving the types of points, and by Proposition 1.1.43, there is an isomorphism $\mathbb{A}_{\widehat{k}}^{1,\mathrm{an}}/G \cong \mathbb{A}_k^{1,\mathrm{an}}$.

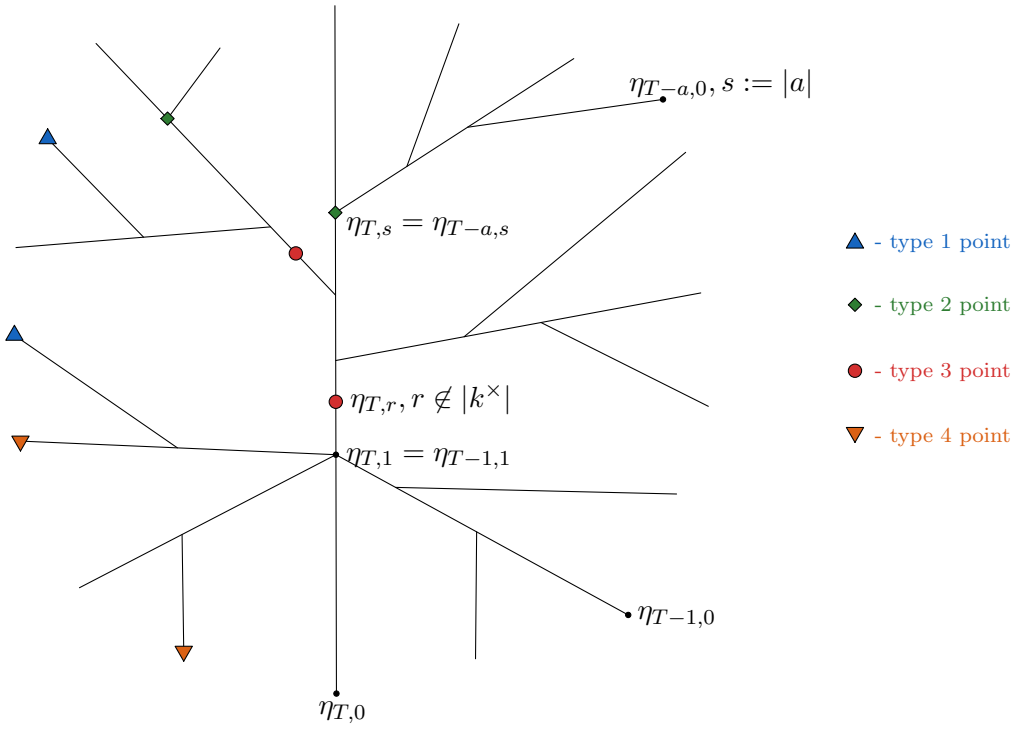


Figure 2: $\mathbb{A}_k^{1,\mathrm{an}}$ for k algebraically closed

A point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ is said to be of *type* i if there exists $y \in \varphi^{-1}(x)$ that is of type i, $i = 1, 2, 3, 4$. (This is well defined seeing as then all of the points of $\varphi^{-1}(x)$ will be of type i.)

For any $a \in k$ and any $r \in \mathbb{R}_{\geqslant 0}$, we denote by $\eta_{a,r}$ the point $\varphi(\eta_{a,r})$ of $\mathbb{A}_k^{1,\mathrm{an}}$.

Definition 1.2.3. A point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ is said to be *rigid* if $\mathcal{H}(x)/k$ is a finite field extension.

Lemma 1.2.4. *There is a bijection between the rigid points of* $\mathbb{A}_k^{1,\mathrm{an}}$ *and the irreducible polynomials over* k *(up to multiplication by an element of* $k^\times$*), given by* $x \mapsto ker|\cdot|_x$.

Proof. By their definition, the rigid points of $\mathbb{A}_k^{1,\mathrm{an}}$ are exactly the ones that determine semi-norms on $k[T]$ of non-zero kernel. Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a rigid point. Then, $\ker|\cdot|_x$ is a non-zero proper prime ideal of $k[T]$, and as such is generated by a (unique up to multiplication by an element of $k^\times$) irreducible polynomial P_x over k.

Let $P \in k[T]$ be an irreducible polynomial. Let $\alpha \in \widehat{\overline{k}}$ be a root of P. Then, $|P|_{\varphi(\eta_{\alpha,0})} = 0$, so $x_P := \varphi(\eta_{\alpha,0}) \in \mathbb{A}_k^{1,\mathrm{an}}$ is a rigid point, and $\ker|\cdot|_{x_P} = (P)$. □

For any irreducible polynomial P over k, let us denote the unique rigid point $x \in \mathbb{A}_k^{1,\mathrm{an}}$ such that $|P|_x = 0$ by $\eta_{P,0}$. By the lemma above, the set of rigid points of $\mathbb{A}_k^{1,\mathrm{an}}$ is given by $\{\eta_{P,0} : P \text{ an irreducible polynomial over } k\}$.

The illustration of $\mathbb{A}_k^{1,\mathrm{an}}$ in the general case is very similar to that in the algebraically closed case. It is an infinitely branched tree, where type 2 points are the branching ones,

type 3 points the non branching ones, and type 1 and 4 the "leaves". In particular, $\eta_{P,0}$ is a leaf of the tree for any irreducible polynomial P over k.

REMARK 1.2.5. With the terminology introduced here, remark that in the trivially valued case, the only type 2 point is $\eta_{0,1}$. The type 1 points are the rigid points, and all the rest are type 3 points of $\mathbb{A}_k^{1,\mathrm{an}}$.

Also, remark that in the algebraically closed case, the type 1 points are the rigid points. This is only true if k is algebraically closed or trivially valued.

We give a more detailed description of the points of $\mathbb{A}_k^{1,\mathrm{an}}$ in part 1.8.4.

1.2.5. The analytic projective line.

DEFINITION 1.2.6. As usual, one can obtain the *n-dimensional projective analytic space* over k, denoted $\mathbb{P}_k^{n,\mathrm{an}}$, by gluing $n+1$ copies of $\mathbb{A}_k^{n,\mathrm{an}}$, or equivalently, compactifying $\mathbb{A}_k^{n,\mathrm{an}}$.

In particular, for $n = 1$, the *projective analytic line* over k is obtained by adding an ∞ point to $\mathbb{A}_k^{1,\mathrm{an}}$.

The ∞ point of $\mathbb{P}_k^{1,\mathrm{an}}$ is a rigid (even rational) point (it is the unique point for which $|1/T| = 0$). We will say that ∞ is a *type 1 point* of $\mathbb{P}_k^{1,\mathrm{an}}$. For any other point $x \in \mathbb{P}_k^{1,\mathrm{an}}$, we will say that x is of *type i* if it is of type i as a point of $\mathbb{A}_k^{1,\mathrm{an}}$, $i = 1, 2, 3, 4$.

We give a more detailed description of the points of $\mathbb{A}_k^{1,\mathrm{an}}$ (and thus of $\mathbb{P}_k^{1,\mathrm{an}}$) in 1.8.4.

1.3. Affinoid Algebras

Affinoid algebras in Berkovich's theory are the analogue of finite type algebras in algebraic geometry. Throughout this section, let $(k, |\cdot|)$ denote a complete ultrametric field.

1.3.1. Definition and some properties.

DEFINITION 1.3.1. Let $n \in \mathbb{N}$ and $r_1, r_2, \ldots, r_n \in \mathbb{R}_{>0}$. We denote

$$k\{r_1^{-1}T_1, r_2^{-1}T_2, \ldots, r_n^{-1}T_n\} := \left\{ \sum_{l \in \mathbb{N}^n} a_l \underline{T}^l : a_l \in k, \lim_{|l| \to +\infty} |a_l| \underline{r}^l = 0 \right\},$$

where for any $l = (l_1, l_2, \ldots, l_n) \in \mathbb{N}^n$, $|l| := l_1 + l_2 + \cdots + l_n$, and for any n-tuple $\underline{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\underline{\alpha}^l := \prod_{i=1}^n \alpha_i^{l_i}$. We sometimes use the notation $k\{\underline{r}^{-1}\underline{T}\}$ for $k\{r_1^{-1}T_1, r_2^{-1}T_2, \ldots, r_n^{-1}T_n\}$. This is a k-algebra.

For $r_1 = r_2 = \cdots = r_n = 1$, we obtain the so-called *Tate affinoid algebra* $k\{\underline{T}\}$.

LEMMA 1.3.2. *For any $n \in \mathbb{N}$ and any $\underline{r} \in \mathbb{R}_{>0}^n$, the map $\|\cdot\| : k\{\underline{r}^{-1}\underline{T}\} \to \mathbb{R}_{\geqslant 0}$, $\sum_{l \in \mathbb{N}^n} a_l \underline{T}^l \mapsto \max_{l \in \mathbb{N}^n} |a_l| \underline{r}^l$, defines a non-Archimedean multiplicative norm on $k\{\underline{r}^{-1}\underline{T}\}$, which satisfies $\|\cdot\|_{|k} = |\cdot|$. Moreover, $(k\{\underline{r}^{-1}\underline{T}\}, \|\cdot\|)$ is a Banach k-algebra.*

For a proof, see [**42**, Lemma 4.8]. In fact, $k\{\underline{r}^{-1}\underline{T}\}$ is the completion of $k[\underline{T}]$ with respect to the norm $\|\cdot\|$ introduced in Lemma 1.3.2.

DEFINITION 1.3.3. A Banach k-algebra A is said to be a *k-affinoid algebra* if there exist $n \in \mathbb{N}, \underline{r} \in \mathbb{R}_{>0}^n$, and a surjective admissible morphism $k\{\underline{r}^{-1}\underline{T}\} \twoheadrightarrow A$.

The Banach k-algebra A is said to be a *strict* k-affinoid algebra if there exist $n \in \mathbb{N}$, and a surjective admissible morphism $k\{\underline{T}\} \twoheadrightarrow A$.

Before anything else, let us note that for any complete ultrametric field extension K of k, the Banach K-algebra $A\widehat{\otimes}_k K$ is a K-affinoid algebra. Also, the completed tensor product of two k-affinoid algebras is also a k-affinoid algebra (for both of these statements, recall Theorem 1.1.27).

Remark that the norm of an affinoid algebra always has a representative in its equivalence class that is non-Archimedean. Before saying anything else about affinoid algebras, let us remark the following.

LEMMA 1.3.4. *Let $(A, \|\cdot\|)$ be a k-affinoid algebra, where we suppose that $\|\cdot\|$ is the norm obtained from some surjective admissible morphism $k\{\underline{r}^{-1}\underline{T}\} \twoheadrightarrow A$. The norm $\|\cdot\|$ is k-linear, meaning: $\forall a \in k, \forall x \in A, \|ax\| = |a|\|x\|$. Moreover, $\|\cdot\|_{|k} = |\cdot|$.*

In practice, when working with k-affinoid algebras, we usually (in this text, always) only encounter norms obtained like the one in Lemma 1.3.2.

Strict affinoid algebras are the central algebraic object in Tate's rigid geometry (where they are simply called "affinoid algebras"). As a consequence, they have been extensively studied (*e.g.* in [**11**]). They share many algebraic properties with finite type algebras, *e.g.* there is a Nullstellensatz and a Noether Normalization Lemma (meaning for any strict k-affinoid algebra A, there exists a finite bounded morphism $k\{T_1, T_2, \ldots, T_n\} \hookrightarrow A$).

LEMMA 1.3.5 ([**11**, 6.1.5/4]). *A k-affinoid algebra A is strict if and only if there exist $n \in \mathbb{N}, r_i \in \sqrt{|k^\times|}$, $i = 1, 2, \ldots, n$, and a surjective admissible morphism $k\{\underline{r}^{-1}\underline{T}\} \twoheadrightarrow A$.*

There is a "trick" to study affinoid algebras by using known information on strict affinoid algebras. The following example gives us the main tool for this trick.

EXAMPLE 1.3.6 ([**6**, pg. 21], [**22**, 1.2]). Let $n \in \mathbb{N}$ and $\underline{r} \in \mathbb{R}^n_{>0}$. Let $K_{\underline{r}}$ denote the k-algebra

$$\left\{ \sum_{l \in \mathbb{Z}^n} a_l \underline{T}^l : a_l \in k, \lim_{|l| \to \infty} |a_l| \underline{r}^l = 0 \right\}.$$

The map $\|\cdot\| : \sum_{l \in \mathbb{Z}^n} a_l \underline{T}^l \mapsto \max_l |a_l| \underline{r}^l$ determines a multiplicative norm on $K_{\underline{r}}$.

Moreover, there is an isometric isomorphism of k-algebras $K_{\underline{r}} \cong k\{\underline{r}^{-1}\underline{X}, \underline{r}\ \underline{Y}\}/(X_i Y_i - 1)_i$, where the norm on the right is the quotient one, so $K_{\underline{r}}$ is a k-affinoid algebra.

If $r_1, r_2, \ldots, r_n$ are linearly independent over $\sqrt{|k^\times|}$, then $K_{\underline{r}}$ is a field.

The construction above can be obtained by starting with $n = 1$ and then iterating. This is because $K_{\underline{r}} \cong K_{r_1} \widehat{\otimes}_k K_{r_2} \widehat{\otimes}_k \cdots \widehat{\otimes}_k K_{r_n}$.

LEMMA 1.3.7. *For any k-affinoid algebra A, there exists a non-trivially field $K_{\underline{r}}$ constructed as in Example 1.3.6, such that $A\widehat{\otimes}_k K_{\underline{r}}$ is a strict $K_{\underline{r}}$-affinoid algebra.*

For a proof of the above, see [**41**, pg. 30].

PROPOSITION 1.3.8 ([**6**, Proposition 2.1.2]). *Let A be a k-affinoid algebra and $r \notin \sqrt{|k^\times|}$.*

1. *The map $A \to A\widehat{\otimes}_k K_r$ is an isometric embedding.*
2. *Let B, C be two k-affinoid algebras. Then, $A \to B \to C$ is exact and admissible if and only if $A\widehat{\otimes}_k K_r \to B\widehat{\otimes}_k K_r \to C\widehat{\otimes}_k K_r$ is exact and admissible.*

For a proof, see [**41**, pg. 29, Proof of Proposition 2.1.2].

A few of the most important algebraic properties of affinoid algebras are the following:

THEOREM 1.3.9. *(1) Any ideal of a k-affinoid algebra is closed.*
(2) A k-affinoid algebra is Noetherian and excellent.

For the Noetherianity and the fact that all ideals are closed, see [**6**, Proposition 2.1.3]. The property of being excellent was shown to be true more recently, in [**21**, Théorème 2.13].

In particular, part (1) of Theorem 1.3.9 tells us that for any $n \in \mathbb{N}$ and $\underline{r} \in \mathbb{R}^n_{>0}$, the k-algebra $k\{\underline{r}^{-1}\underline{T}\}/I$, where I is an ideal of $k\{\underline{r}^{-1}\underline{T}\}$, is a k-affinoid algebra (with respect to the quotient norm). More generally, any quotient of a k-affinoid algebra is a k-affinoid algebra with respect to the quotient norm. (We will see later, in Section 4.2, that a k-affinoid algebra can be endowed with the structure of a Banach k-algebra in a unique way. This is proven for strict affinoid algebras in [**11**, 6.1.3/2]. The general case follows quickly from that.)

The set of k-affinoid algebras endowed with bounded morphisms forms a category.

DEFINITION 1.3.10. For a k-affinoid algebra $(A, |\cdot|)$, $n \in \mathbb{N}$, and $\underline{r} \in \mathbb{R}^n_{>0}$, let $A\{\underline{r}^{-1}\underline{T}\} := \left\{\sum_{l\in\mathbb{N}^n} a_l\underline{T}^l : a_l \in A, \lim_{|l|\to+\infty} |a_l|\underline{r}^l = 0\right\}$.

We endow this k-algebra with the norm:

$$\left\|\sum_{l\in\mathbb{N}^n} a_l\underline{T}^l\right\| = \max_{l\in\mathbb{N}^n} |a_l|\underline{r}^l,$$

which makes it a Banach A-algebra.

DEFINITION 1.3.11. Let A be a k-affinoid algebra. A Banach A-algebra is said to be an *A-affinoid algebra* if there exist $n \in \mathbb{N}$, $\underline{r} \in \mathbb{R}^n_{>0}$, and a surjective admissible morphism $A\{\underline{r}^{-1}\underline{T}\} \twoheadrightarrow B$.

The following is a consequence of Theorem 1.1.27.

LEMMA 1.3.12. *Let A be a k-affinoid algebra. Any A-affinoid algebra is a k-affinoid algebra.*

Another useful result is the following:

LEMMA 1.3.13 ([**6**, Corollary 2.1.5]). *Let K_1, K_2 be two complete ultrametric field extensions of k. Let $f : A \to B$ be a bounded k-linear morphism going from a K_1-affinoid algebra to a K_2-affinoid algebra.*

Let $b_i \in B$ and $r_i \in \mathbb{R}_{>0}$, $i = 1, 2, \ldots, n$, such that $\rho_B(b_i) \leqslant r_i$ for all i, where ρ_B is the spectral semi-norm on B. There exists a unique bounded morphism $g : A\{\underline{r}^{-1}\underline{T}\} \to B$ extending f, such that $T_i \mapsto b_i$ for all i.

1.3.2. Affinoid algebras and the spectral radius. As we saw in Subsection 1.1.3, the spectral radius ρ_A of a Banach ring A determines naturally a non-Archimedean semi-norm on A (which doesn't depend on the representative of the equivalence class of the norm on A).

PROPOSITION 1.3.14 ([**6**, Corollary 2.1.6]). *A k-affinoid algebra A is strict if and only if $\rho_A(a) \in \sqrt{|k^\times|} \cup \{0\}$ for all $a \in A$.*

The following is a good example of why the spectral semi-norm is very useful.

PROPOSITION 1.3.15. *Let A be a k-affinoid algebra, and ρ_A the corresponding spectral semi-norm.*

(1) [**3**, Proposition 2.7.3(2)] *Let $f \in A$. Then, $\rho_A(f) = 0$ if and only if f is nilpotent. The spectral semi-norm ρ_A on A is a norm if and only if A is reduced.*
(2) [**42**, Proposition 9.13] *If A is reduced, then ρ_A is equivalent to the norm on A.*

1.3.3. Finite modules/algebras over a k-affinoid algebra.

DEFINITION 1.3.16. Let A be a k-affinoid algebra. A Banach A-module M is said to be *finite* if there exists a surjective admissible morphism $A^n \twoheadrightarrow M$.

A Banach A-algebra B is said to be *finite* if it is a finite A-module.

THEOREM 1.3.17 ([**6**, Proposition 2.1.9]). *Let A be a k-affinoid algebra. The forgetful functor induces an isomorphism between the categories of finite Banach A-modules (with bounded A-linear maps) and finite A-modules (with A-linear maps).*

The same result remains true when replacing *module* by *algebra* (see [**6**, Proposition 2.1.12]).

The following are properties that we will use in the next chapters.

PROPOSITION 1.3.18 ([**6**, Proposition 2.1.14(i)]). *A k-affinoid algebra that is an integral domain is Japanese.*

PROPOSITION 1.3.19. *Let A be a k-affinoid algebra. Any finite A-algebra is a k-affinoid algebra.*

The above is shown for strict affinoid algebras in [**11**, 6.1.3, Proposition 4]. The general case can be deduced from [**6**]: Proposition 2.1.11 and Corollary 2.1.8.

REMARK 1.3.20. As there is a Banach Open Mapping Theorem for any non-trivially valued complete ultrametric field k, if A, B are k-affinoid algebras, any bounded surjective morphism $A \twoheadrightarrow B$ is admissible.

1.4. Affinoid Spaces

Throughout this section, let $(k, |\cdot|_k)$ be a complete ultrametric field.

1.4.1. A first definition. In order to simplify the terminology we will soon use, let us, for now, fix the following (we hold off on making this a definition until the construction of the sheaf of analytic functions):

CONVENTION 1.4.1. A *k-affinoid space* is the Berkovich spectrum of a k-affinoid algebra.

A k-affinoid space X is said to be *strict* if there exists a strict k-affinoid algebra A such that $X = \mathcal{M}(A)$.

A *morphism* $X \to Y$ of k-affinoid spaces is one induced by a bounded k-linear morphism $A_Y \to A_X$ of the corresponding k-affinoid algebras.

LEMMA 1.4.2. *Let A be a k-affinoid algebra. For any $x \in \mathcal{M}(A)$, the multiplicative semi-norm $|\cdot|_x$ is non-Archimedean. Moreover, $|\cdot|_{x|k} = |\cdot|_k$.*

1.4.2. Affinoid domains. The goal here is to present, for Berkovich spaces, the analogue of an open affine subscheme. This is also a crucial step for the construction of the structural sheaf. Historically, the notion of an affinoid domain appears in Tate's rigid geometry. The main difference is that here these are closed subsets (even compact), whereas in rigid geometry they are open.

CONVENTION 1.4.3. Let $\psi : A \to B$ be a bounded morphism of Banach rings. From now on, we will denote by ψ' the induced continuous morphism $\mathcal{M}(B) \to \mathcal{M}(A)$.

Let $(k, |\cdot|)$ be a complete ultrametric field.

DEFINITION 1.4.4. Let A be a k-affinoid algebra, and X the corresponding k-affinoid space. An *affinoid domain* in X is a pair (V, A_V) such that:

(1) V is a closed subset of X, and A_V is a k-affinoid algebra;
(2) there exists a bounded morphism $\phi : A \to A_V$, such that $\phi'(\mathcal{M}(A_V)) \subseteq V$;
(3) the following universal property is satisfied: for any bounded k-linear morphism $\varphi : A \to B$ such that $\varphi'(\mathcal{M}(B)) \subseteq V$, where B is a K-affinoid algebra for some complete ultrametric field extension K/k, there exists a unique bounded morphism $A_V \to B$ such that the following diagram is commutative.

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A_V \\ & {\scriptstyle\varphi}\searrow \quad \swarrow & \\ & B & \end{array}$$

We will say that (V, A_V) is a *strict affinoid domain* in X if A_V is a strict k-affinoid algebra.

We start by giving important examples of these objects (that come from rigid geometry). Let A be a k-affinoid algebra, and X the corresponding k-affinoid space.

EXAMPLE 1.4.5 (Weierstrass domains). Let $n \in \mathbb{N}$, $f_1, f_2, \ldots, f_n \in A$, and $r_1, r_2, \ldots, r_n \in \mathbb{R}_{>0}$. Then, $V := \{x \in X : |f_i|_x \leqslant r_i, i = 1, 2, \ldots, n\}$ is called a *Weierstrass domain* of X. Set $A_V := A\{\underline{r}^{-1}\underline{T}\}/(T_i - f_i)_i$. This is an A-affinoid algebra with respect to the quotient norm (and thus also a k-affinoid algebra).

We will see in Lemma 1.4.8 that (V, A_V) is an affinoid domain in X.

EXAMPLE 1.4.6 (Laurent domains). Let $m, n \in \mathbb{N}$, $f_i, g_j \in A$, and $r_i, s_j \in \mathbb{R}_{>0}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. The set $V := \{x \in X : |f_i|_x \leqslant r_i, |g_j|_x \geqslant r_j, i, j\}$ is called a *Laurent domain* of X. Set $A_V := A\{\underline{r}^{-1}\underline{T}, \underline{s}S\}/(T_i - f_i, g_j S_j - 1)_{i,j}$. This is an A-affinoid algebra with respect to the quotient norm (and thus also a k-affinoid algebra).

We will see in Lemma 1.4.8 (by applying Lemma 1.4.9) that (V, A_V) is an affinoid domain in X. Remark that a Weierstrass domain is a Laurent domain.

Laurent domains form a basis of neighborhoods of the topology on X. To see this, recall that the topology on X is the coarsest one for which the map $X \to \mathbb{R}_{\geqslant 0}, x \mapsto |a|_x$, is continuous for all $a \in A$.

EXAMPLE 1.4.7 (Rational domains). Let $n \in \mathbb{N}, g, f_i \in A$, $i = 1, 2, \ldots, n$, be such that $(g, f_1, \ldots, f_n) = A$. Let $r_i \in \mathbb{R}_{>0}, i = 1, 2, \ldots, n$. Then, the set $V := \{x \in X : |f_i|_x \leqslant r_i|g|_x, i = 1, 2, \ldots, n\}$ is said to be a *rational domain* in X. Set $A_V := A\{\underline{rT}^{-1}\}/(gT_i - f_i)_i$. This is an A-affinoid algebra with respect to the quotient norm (and thus also a k-affinoid algebra). We will soon see that Laurent (and hence Weierstrass) domains are rational domains.

LEMMA 1.4.8. *The pair (V, A_V) from Example 1.4.7 is an affinoid domain in X.*

PROOF. The subset V is clearly closed (again, recall the topology on X). Let the morphism $\phi : A \to A_V$ be the canonical one. Then, for any $x \in \mathcal{M}(A_V)$, $|g|_x|T_i|_x = |f_i|_x$ for all i. Let $\|\cdot\|$ denote the (quotient) norm on A_V. Then, $\|T_i\| \leqslant r_i$ for all i, so $|T_i|_x \leqslant r_i$

for all $x \in \mathcal{M}(A_V)$, implying $|f_i|_x = |T_i|_x|g|_x \leqslant r_i|g|_x$. Consequently, $|f_i|_{\phi'(x)} \leqslant r_i|g|_{\phi'(x)}$ for all i, so $\phi'(x) \in V$, and $\phi'(\mathcal{M}(A_V)) \subseteq V$.

Let $\varphi : A \to B$ be a bounded k-linear morphism such that $\varphi'(\mathcal{M}(B)) \subseteq V$, where B is a K-affinoid algebra for some complete ultrametric field extension K/k. This means that for any $x \in \mathcal{M}(B)$, $|\varphi(f_i)|_x \leqslant r_i|\varphi(g)|_x$ for all i. In turn, this implies that $\varphi(g)$ is invertible in B: otherwise, by Lemma 1.1.39, there would exist $y \in \mathcal{M}(B)$ such that $|\varphi(g)|_y = 0$, implying $|\varphi(f_i)|_y = 0$ for all i. Thus, $|g|_{\varphi'(y)} = |f_i|_{\varphi'(y)} = 0$ for all i, which contradicts the assumption $(g, f_1, \ldots, f_n) = A$. Hence, $\varphi(g)$ is invertible in B, and $\frac{\varphi(f_i)}{\varphi(g)} \in B$ for all i. Remark that for any $x \in \mathcal{M}(B)$, $\left|\frac{\varphi(f_i)}{\varphi(g)}\right|_x \leqslant r_i$, so $\rho_B(\frac{\varphi(f_i)}{\varphi(g)}) \leqslant r_i$, $i = 1, 2, \ldots, n$. By Lemma 1.3.13, there is a unique bounded morphism $A\{\underline{r}^{-1}\underline{T}\} \to B, T_i \mapsto \frac{\varphi(f_i)}{\varphi(g)}$, $i = 1, 2, \ldots, n$, extending the morphism $A \to B$. Clearly, this factorizes through A_V via bounded morphisms: $A\{\underline{r}^{-1}\underline{T}\} \to A_V \to B$. The uniqueness of the obtained morphism $A_V \to B$ is clear from the construction. □

Lemma 1.4.9. *A Laurent domain is a rational domain.*

The proof comes down to showing that the intersection of rational domains is a rational domain. For this, see [**11**, 7.2.3/7].

Let us describe precisely the relationship between V and A_V.

Proposition 1.4.10 ([**6**, Proposition 2.2.4]). *Let A be a k-affinoid algebra and X the corresponding k-affinoid space. Let (V, A_V) be an affinoid domain in X.*

1. *The morphism $\mathcal{M}(A_V) \to X$ is injective with image V. In other words, $\mathcal{M}(A_V) = V$. In particular, the morphism $A \to A_V$ is uniquely determined by V.*
2. *The morphism $A \to A_V$ is flat.*

Affinoid domains behave well (these can be checked using the definition):

Proposition 1.4.11.
1. *Let $\phi' : X \to Y$ be a morphism of k-affinoid spaces, where $X = \mathcal{M}(B)$ and $Y = \mathcal{M}(A)$. Let (V, A_V) be an affinoid domain in Y. Then, $(\phi'^{-1}(V), A_V\widehat{\otimes}_A B)$ is an affinoid domain in X.*
2. *Let X be a k-affinoid space and A its corresponding k-algebra. Let $(U, A_U), (V, A_V)$ be affinoid domains in X. Then, $(U \cap V, A_U\widehat{\otimes}_A A_V)$ is an affinoid domain in X.*
3. *Let X be a k-affinoid space. Let (U, A_U) be an affinoid domain in X. Let (V, A_V) be an affinoid domain in U. Then, (V, A_V) is an affinoid domain in X.*

The first two parts of the statement above remain true when replacing *affinoid* by *Weierstrass* or *Laurent* or *rational*. The third one remains true for Weierstrass and rational domains, but not always for Laurent ones.

As a direct consequence of the universal property for affinoid domains, we can show that the completed residue field of a point does not depend on the affinoid domain containing it.

Lemma 1.4.12. *Let X be a k-affinoid space. Let (V, A_V) be an affinoid domain in V. For any $x \in V$, let us denote by $\mathcal{H}(x)$ (resp. $\mathcal{H}_V(x)$) the completed residue field of x when considering x as a point in X (resp. V). Then, there is an isometric isomorphism $\mathcal{H}(x) \to \mathcal{H}_V(x)$.*

Remark 1.4.13. From now on, we will denote an affinoid domain simply by V (instead of (V, A_V)). This causes no ambiguity considering Proposition 1.4.10.

The following outstanding result is used to simplify the construction of the structural sheaf (it had not yet been shown when Tate wrote [**63**]). In the rigid case it was proven by Gerritzen and Grauert, and it was generalized to the setting of Berkovich spaces by both Ducros and Temkin (see *e.g.* [**65**]).

THEOREM 1.4.14 (Gerritzen-Grauert). *Let X be a k-affinoid space. Any affinoid domain in X is a finite union of rational domains.*

The following two results will be useful to us in the next chapters.

PROPOSITION 1.4.15 ([**6**, Proposition 2.2.3(iii)]). *Suppose k is non-trivially valued. Let X be a strict k-affinoid space. Then, the strict affinoid domains in X form a basis of neighborhoods of the topology on X.*

LEMMA 1.4.16 ([**6**, Corollary 2.2.10]). *Let $V \to X$ be a morphism of k-affinoid spaces such that V is a rational domain in X. Let us denote by A_V, resp. A_X, the corresponding k-affinoid algebras. Set $S_V := \{a \in A_X : |a|_x \neq 0 \text{ for all } x \in V\}$. Then, $S_V^{-1}A_X$ is dense in A_V.*

1.4.3. The structural sheaf. Recall that $(k, |\cdot|)$ denotes a complete ultrametric field. Let A be a k-affinoid algebra and X the corresponding k-affinoid space.

The Gerritzen-Grauert Theorem is a very useful tool for proving the next results as it allows one to reduce to the case of rational domains.

THEOREM 1.4.17 (Tate's Acyclicity Theorem, [**6**, Proposition 2.2.5]). *Let $(V_i, A_{V_i})_{i=1}^n$ be a cover of X, where (V_i, A_{V_i}) is an affinoid domain in X for all i. The following Čech complex is exact and admissible:*

$$0 \longrightarrow \prod_i A_{V_i} \longrightarrow \prod_{i,j} A_{V_i \cap V_j} \longrightarrow \cdots$$
$$(f_i)_i \mapsto ((f_i - f_j)_{|A_{V_i} \cap A_{V_j}})_{i,j} \mapsto \cdots$$

Let $S(X)$ denote the family of finite unions of affinoid domains in X. An element V of $S(X)$ is said to be a *special* subset of X. Then, the special subsets of X determine a G-topology on X, and Tate's Acyclicity theorem allows us to construct a sheaf for it.

COROLLARY 1.4.18 ([**6**, Corollary 2.2.6]). *For any $V \in S(X)$, set*

$$A_V := \ker\left(\prod_i A_{V_i} \to \prod_{i,j} A_{V_i \cap V_j}\right),$$

where $(V_i)_{i=1}^n$ is a finite cover by affinoid domains of V. Then:

- *A_V is a Banach k-algebra which does not depend on the affinoid cover $(V_i)_{i=1}^n$ of V;*
- *$V \mapsto A_V$ determines a sheaf on the G-topology of X;*
- *V is an affinoid domain if and only if A_V is a k-affinoid algebra and $V \cong \mathcal{M}(A_V)$.*

Let us mention one result that will be important for the next chapters. As we will see later, it is true in much more generality in the case of analytic curves.

COROLLARY 1.4.19 ([**6**, Corollary 2.2.7]). *Let U, V be closed disjoint subsets of X. Then, $W := U \cup V$ is an affinoid domain in X if and only if U and V are affinoid domains. In that case, the corresponding affinoid algebras satisfy: $A_W \cong A_U \times A_V$.*

We are finally in a position to give the following:

DEFINITION 1.4.20. For any open $U \subseteq X$ (with respect to the Berkovich topology), set

$$\mathcal{O}_X(U) := \varprojlim_{\substack{V \subseteq U \\ V-\text{special}}} A_V$$

in the category of k-algebras. This determines a sheaf on X which we call the *sheaf of analytic functions* on X.

Remark that any $f \in \mathcal{O}_X(U)$ can be seen as a function in the sense $f : U \to \prod_{x \in U} \mathcal{H}(x)$, $x \mapsto (|f|_x)_{x \in U}$.

1.4.4. The stalks. As before, X denotes a k-affinoid space.

LEMMA 1.4.21. *(1) For any $x \in X$, $\mathcal{O}_{X,x} = \varinjlim_{x \in V} A_V$, where the limit is taken over neighborhoods V of x that are affinoid domains in X. The ring $\mathcal{O}_{X,x}$ inherits a multiplicative semi-norm induced by the point x on A_V, with V as above. We will continue to denote it by $|\cdot|_x$.*

(2) The ring $\mathcal{O}_{X,x}$ is local with maximal ideal $m_x := \{f \in \mathcal{O}_{X,x} : |f|_x = 0\}$.

PROOF. The first part is a consequence of the fact that Laurent (hence affinoid) domains form a basis of neighborhoods of the Berkovich topology on X.

For the second part, clearly m_x is a proper ideal of $\mathcal{O}_{X,x}$ (for instance, $|1|_x = 1$). It suffices to show that for any $f \in \mathcal{O}_{X,x}$ such that $|f|_x \neq 0$, f is invertible.

Let $g \in \mathcal{O}_{X,x}$ be such that $|g|_x \neq 0$. There exists a neighborhood (W, A_W) of x that is an affinoid domain such that $g \in A_W \backslash \{0\}$. Let $r > 0$ be such that $|g|_x > r$. Then, the Laurent domain $W' := \{y \in W : |g|_y \geqslant r\}$ of W contains x. Remark that $A_{W'} := \mathcal{O}_X(W') = A_W\{rT\}/(gT - 1)$, and $g \in A_{W'}$ is invertible. Consequently, g is invertible in $\mathcal{O}_{X,x}$. □

The field $\kappa(x) := \mathcal{O}_{X,x}/m_x$ is said to be the *residue field* of x.

Another very important property for the next chapters is the relationship between $\mathcal{O}_{X,x}$ and $\mathcal{H}(x)$, which is a direct consequence of Lemmas 1.4.12 and 1.4.21 (recall also Definition 1.1.33).

LEMMA 1.4.22. *For any $x \in X$, there is a canonical embedding $\kappa(x) \hookrightarrow \mathcal{H}(x)$. Moreover, $\kappa(x)$ is dense in $\mathcal{H}(x)$.*

The stalks of $\mathcal{O}_X$ are crucial for our work in the next chapters. In Section 4.7, we will see some examples of them. For now, let us mention that they have very nice algebraic properties.

THEOREM 1.4.23. *For any $x \in X$, $\mathcal{O}_{X,x}$ is a Noetherian, Henselian, and excellent local ring.*

Noetherianity is shown in [**4**, Theorem 2.1.4], and Henselianity in [**4**, Theorem 2.1.5]. The property of being excellent was shown more recently in [**21**, Théorème 2.13].

1.4.5. Back to affinoid spaces. We may now revisit Convention 1.4.1.

DEFINITION 1.4.24. A *k-affinoid space* X is the Berkovich spectrum of a k-affinoid algebra endowed with the sheaf $\mathcal{O}_X$ constructed above. The space X is said to be *strict* if there exists a strict k-affinoid algebra A such that $X = \mathcal{M}(A)$.

These are the building blocks of good Berkovich spaces. Remark that for any k-affinoid space X, there exists a canonical morphism $X \to \mathcal{M}(k)$.

EXAMPLE 1.4.25. Let n be a positive integer, and $\underline{r} \in \mathbb{R}_{>0}$. Then, $\mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$ is said to be the *closed Berkovich polydisc of polyradius* $\underline{r}$.

In particular, if $n = 1$, then we obtain the *closed Berkovich disc of radius* r.

DEFINITION 1.4.26. A *morphism of k-affinoid spaces* $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of k-locally ringed spaces which comes from a bounded morphism of the corresponding k-affinoid algebras.

Remark that the category of k-affinoid spaces (endowed with the morphisms above), denoted *k-aff*, is the opposite of the category of k-affinoid algebras with respect to bounded k-linear morphisms.

The category *k-aff* admits fiber products (the relationship between fiber products and completed tensor products for k-affinoid spaces in Berkovich's theory is the same as that of fiber products and tensor products for affine schemes in algebraic geometry).

DEFINITION 1.4.27. One can also define the category *aff-k*, where the objects are K-affinoid spaces over a complete ultrametric field extension K/k, and the morphisms are those of locally ringed spaces induced by a bounded k-linear morphism of the corresponding affinoid algebras.

For an object of *aff-k*, we will say that it is an *affinoid k-space*. For $X, Y \in$ *aff-k*, and a morphism $\varphi : Y \to X$ of the same category, we will simply say that $\varphi : Y \to X$ is a *morphism of affinoid k-spaces*.

The category *aff-k* admits base change by complete ultrametric field extensions of k. However, it does not in general admit fiber products.

1.4.6. The boundaries of an affinoid space. Since affinoid spaces are compact (so closed), we want to have a notion of boundary for them.

1.4.6.1. *Relative interior and boundary.* These are amongst the rare notions that are exclusive to Berkovich's theory, meaning there is no analogue in rigid/complex/algebraic geometry. See [**6**, 2.5] for more details.

DEFINITION 1.4.28. Let $\varphi : Y \to X$ be a morphism of k-affinoid spaces, where $X = \mathcal{M}(A_1)$ and $Y = \mathcal{M}(A_2)$. The *Berkovich relative interior* of φ, denoted $\mathrm{Int}_B(Y/X)$, is the set of points $y \in Y$ such that there exist $n \in \mathbb{N}, r_i \in \mathbb{R}_{>0}, i = 1, 2, \ldots, n$, and a surjective admissible A_1-linear morphism $\psi : A_1\{\underline{r}^{-1}\underline{T}\} \twoheadrightarrow A_2$ satisfying $|\psi(T_i)|_y < r_i$ for all i.

The set $Y \backslash \mathrm{Int}_B(Y/X)$ is called the *Berkovich relative boundary* of φ, and is denoted by $\partial_B(Y/X)$. We say that φ is *boundaryless* if $\partial_B(Y/X) = \emptyset$ (Berkovich calls this *closed*).

If $Y \to \mathcal{M}(k)$ is the canonical morphism, then we denote by $\mathrm{Int}_B(Y)$ (resp. $\partial_B(Y)$) the set $\mathrm{Int}_B(Y/\mathcal{M}(k))$ (resp. $\partial_B(Y/X)$), and call it the *Berkovich interior* (resp. *Berkovich boundary*) of Y. If $\partial_B(Y) = \emptyset$, then Y is said to be *boundaryless*.

Remark that in the litterature these objects are called *relative interior, relative boundary*, and *absolute interior, absolute boundary*, respectively, and are denoted as above but without the index "B".

As Temkin remarks in [**64**, Remark 3.4.10], the geometric interpretation of the definition above is that the morphism ψ induces a *closed immersion* of Y into the *closed polydisc over A_1 of polyradius* $\underline{r}$ (*i.e.* into $\mathcal{M}(A_1\{\underline{r}^{-1}\underline{T}\})$) such that the image of Y is contained

in the *open polydisc over* A_1 *of polyradius* $\underline{r}$ (*i.e.* in $\{x \in \mathcal{M}(A_1\{\underline{r}^{-1}\underline{T}\}) : |T_i|_x < r_i, i = 1, 2, \ldots, n\}$).

Here are some of the main properties of these notions. The first one is immediate from the definition.

THEOREM 1.4.29. *Let $\varphi : Y \to X$ be a morphism of k-affinoid spaces.*

(1) $\mathrm{Int}_B(Y/X)$ *is open, and* $\partial_B(Y/X)$ *is closed in* Y.
(2) [**6**, Prop. 2.5.8(iii)] *Let $\phi : Z \to Y$ be a morphism of k-affinoid spaces. This induces a morphism $\varphi \circ \phi : Z \to X$. Then,* $\mathrm{Int}_B(Z/X) = \mathrm{Int}_B(Z/Y) \cap \phi^{-1}(\mathrm{Int}_B(Y/X))$.
(3) [**6**, Cor. 2.5.13(i)] $\partial_B(Y/X) = \emptyset$ *if and only if φ is finite, i.e. the corresponding morphism $A_1 \to A_2$ is finite.*
(4) [**6**, Cor. 2.5.13(ii)] *If Y is an affinoid domain of X, then* $\mathrm{Int}_B(Y/X)$ *is the topological interior of Y in X.*

Remark that by part (3) of Theorem 1.4.29, a k-affinoid space is boundaryless if and only if the corresponding k-affinoid algebra is a finite k-algebra.

1.4.6.2. *Shilov boundary.* As usual, let k be a complete ultrametric field, and A a k-affinoid algebra. Set $X = \mathcal{M}(A)$.

A closed subset Γ of X is said to be a *boundary* of X if any element $f \in A$ attains its maximum at a point of Γ. The set of boundaries of X forms a partially ordered set (via inclusion). By Zorn's Lemma, there exist minimal boundaries of X. If there exists a unique minimal boundary, then it is said to be the *Shilov boundary* of X, and is denoted by $\Gamma(X)$.

PROPOSITION 1.4.30 ([**6**, Corollary 2.4.5]). *The Shilov boundary $\Gamma(X)$ of X exists and is finite.*

In particular, this means that for any $f \in A$, $\rho_A(f) = \max_{x \in \Gamma(X)} |f|_x$, where ρ_A denotes the spectral semi-norm of A.

The following is a useful property (see the proof of [**22**, Lemme 2.1]):

LEMMA 1.4.31. *If A is integral, then for any $f \in A \backslash \{0\}$, $|f|_x \neq 0$ for any $x \in \Gamma(X)$.*

The two kinds of boundaries we have just seen are related as follows:

PROPOSITION 1.4.32 ([**6**, Proposition 2.5.20]). *For any affinoid domain (V, A_V) of X,*

$$\Gamma(X) \cap V \subseteq \Gamma(V) \subseteq \partial_B(V/X) \cup (\Gamma(X) \cap V).$$

We will later see (Lemma 1.8.8) that in the case of curves the Berkovich and Shilov boundaries coincide for a k-affinoid space.

1.4.7. The reduction map. The notion of the reduction map will be very useful to us for comparing our results from Chapter 3 to others in the litterature.

Recall that $(k, |\cdot|)$ is a complete ultrametric field. Let A, B be two Banach k-algebras. Let $\varphi : A \to B$ be a bounded morphism. Let ρ_A (resp. ρ_B) denote the spectral radius of A (resp. B). Then, for any $a \in A$, $\rho_B(\varphi(a)) \leqslant \rho_A(a)$.

Set $A^\circ = \{a \in A : \rho_A(a) \leqslant 1\}$ and $A^{\circ\circ} = \{a \in A : \rho_A(a) < 1\}$; A° is a subring of A, and $A^{\circ\circ}$ is an ideal of A°. Set $\widetilde{A} = A^\circ/A^{\circ\circ}$. Let B°, resp. $B^{\circ\circ}$, resp. $\widetilde{B}$, be the analoguous notations for B. By the paragraph above, the morphism $\varphi : A \to B$ induces canonically morphisms $\varphi^\circ : A^\circ \to B^\circ$ and $\widetilde{\varphi} : \widetilde{A} \to \widetilde{B}$. In the case of k-affinoid algebras, a lot of information can be obtained on φ through $\widetilde{\varphi}$ and vice-versa (*cf.* [**6**, 2.4]).

In particular, for any $x \in \mathcal{M}(A)$, the canonical morphism $\chi_x : A \to \mathcal{H}(x)$ induces a morphism $\widetilde{\chi_x} : \widetilde{A} \to \widetilde{\mathcal{H}(x)}$. Clearly, $\ker \widetilde{\chi_x}$ is a prime ideal of $\widetilde{A}$. This gives rise to a map $r : \mathcal{M}(A) \to \operatorname{Spec} \widetilde{A}, x \mapsto \ker \widetilde{\chi_x}$.

DEFINITION 1.4.33. Suppose A is a Banach k-algebra. Then, the map $r : \mathcal{M}(A) \to \operatorname{Spec} \widetilde{A}$ constructed as above is called the *reduction map* of A.

PROPOSITION 1.4.34 ([**6**, Proposition 2.4.4]). *Let A be a strict k-affinoid algebra.*

(1) *The reduction map r of A is surjective and anticontinuous.*
(2) $r^{-1}((\operatorname{Spec} \widetilde{A})_{\mathrm{gen}}) = \Gamma(\mathcal{M}(A))$, *where $(\operatorname{Spec} \widetilde{A})_{\mathrm{gen}}$ is the set of generic points of the irreducible components of $\operatorname{Spec} \widetilde{A}$.*

Temkin generalized this to any affinoid space using what is referred to in the litterature as *Temkin's graded reduction* [**66**]. The (graded) reduction map does not have nice gluing properties, so it is only defined over affinoid spaces.

1.5. Good Berkovich analytic spaces

Throughout this section, let k be a complete ultrametric field.

1.5.1. The category of good analytic spaces.

DEFINITION 1.5.1 (Non-rigorous). A *good k-analytic space* is a locally ringed space $(X, \mathcal{O}_X)$, where each point has a neighborhood isomorphic to a k-affinoid space. The structural sheaf $\mathcal{O}_X$ is said to be the sheaf of *analytic functions* on X.

The analytic space X is said to be *strict* if any point has a neighborhood isomorphic to a strict k-affinoid space.

The morphisms between good k-analytic spaces are the morphisms of locally ringed spaces induced by morphisms of k-affinoid spaces. Remark there is a canonical morphism $X \to \mathcal{M}(k)$.

The good k-analytic spaces with the above morphisms form a category, which we denote by k-An.

The precise definition requires a condition (which we will continue to omit without consequences) on the G-topology we mentioned before.

The fiber product exists in the category of good k-analytic spaces seeing as it exists for k-affinoid spaces.

The fact that affinoid spaces are compact (hence closed) is the reason why the definition above is not enough for all Berkovich spaces (in the sense that there are rigid spaces that don't have a Berkovich analogue). In general, it can happen that a point does not have a neighborhood isomorphic to an affinoid space, but is only contained in the boundary of subsets isomorphic to affinoid spaces. The theory of general Berkovich spaces was developed by Berkovich afterwards, in [**4**]. We will later see that analytic curves are always good analytic spaces.

EXAMPLE 1.5.2. Remark that for any $n \in \mathbb{N}$, and $\underline{r} \in \mathbb{R}^n_{>0}$, we have an embedding $\mathcal{M}(k\{\underline{r}^{-1}\underline{T}\}) \subseteq \mathbb{A}^{n,\mathrm{an}}_k$. To see this, recall that $(k\{\underline{r}^{-1}\underline{T}\}, \|\cdot\|)$ (with $\|\cdot\|$ defined as in Lemma 1.3.2) is the completion of $k[\underline{T}]$. On the other hand, because of the same reason, for any $x \in \mathbb{A}^{n,\mathrm{an}}_k$, if $|T_i|_x = s_i$, $i = 1, 2, \ldots, n$, then $x \in \mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$ for any $\underline{r} \in \mathbb{R}^n_{>0}$ such that $r_i \geqslant s_i$ for all i. Consequently, $\mathbb{A}^{n,\mathrm{an}}_k = \bigcup_{\underline{r} \in \mathbb{R}^n_{>0}} \mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$.

For the compatibility of topologies, it is direct from the definition that the set-wise embedding $\iota_{\underline{r},n} : \mathcal{M}(k\{\underline{r}^{-1}\underline{T}\}) \hookrightarrow \mathbb{A}_k^{n,\mathrm{an}}$ is continuous. It is also a direct consequence of the definition that $\mathbb{A}_k^{n,\mathrm{an}}$ is a Hausdorff space. Thus, $\iota_{\underline{r},n}$ is an injective continuous morphism from a compact space to a Hausdorff one. Consequently, it is a homeomorphism onto its image.

The two paragraphs above can be used to endow $\mathbb{A}_k^{n,\mathrm{an}}$ (and hence $\mathbb{P}_k^{n,\mathrm{an}}$) with the structure of a good k-analytic space. Moreover, if k is non-trivially valued, by Lemma 1.3.5, these spaces are also strict.

A full classification of points of Berkovich closed discs can thus be deduced from that of $\mathbb{A}_k^{1,\mathrm{an}}$ (see part 1.2.4).

Remark 1.5.3. Using the universal property for affinoid domains, one can show that the notation $\mathcal{H}(x)$ as introduced in Definition 1.2.2 for the points of $\mathbb{A}_k^{1,\mathrm{an}}$ is coherent with the one of completed residue fields.

As in the affinoid case, one can also define a larger category:

Definition 1.5.4. Let *An-k* denote the category whose objects are good K-analytic spaces, where K is a complete ultrametric field extension K/k, and the morphisms are morphisms of k-locally ringed spaces induced by those of the category *aff-k*. An object of An-k will be said to be a *good analytic k-space*.

For $X, Y \in$ *An-k*, and a morphism $\varphi : Y \to X$ of the same category, we will simply say that $\varphi : Y \to X$ is a *morphism of good analytic k-spaces* or sometimes just a *morphism of good analytic spaces* when there is no risk of ambiguity.

The category *An-k* admits base change by complete ultrametric field extensions of k, but does not in general admit fiber products (as in the case of *aff-k*).

The following is a class of subsets of X over which the structural sheaf is defined (examples are opens and *affinoid domains*).

Definition 1.5.5. A morphism of good k-analytic spaces $\varphi : Y \to X$ is said to be an *analytic domain* in X if φ induces a homeomorphism of Y with its image in X, and for any morphism of good analytic k-spaces $\psi : Z \to X$ such that $\psi(Z) \subseteq \varphi(Y)$, there exists a unique morphism of good analytic k-spaces $\sigma : Z \to Y$ such that $\psi = \varphi \circ \sigma$.

If Y is isomorphic to a (strict) k-affinoid space, then Y is said to be a *(strict) affinoid domain* of X.

We identify an analytic domain with its image in the corresponding analytic space. It is clear from the definition that the property of being an analytic (resp. affinoid) domain is transitive.

Let X be a good k-analytic space. By definition, for any $x \in X$, there exists a neighborhood of x isomorphic to $(\mathcal{M}(A), \mathcal{O}_{\mathcal{M}(A)})$ for some k-affinoid algebra A. Remark that the affinoid domains of $\mathcal{M}(A)$ are affinoid domains of X. Hence, affinoid domains form a basis of neighborhoods for the topology of any good k-analytic space.

Remark 1.5.6. Let $\varphi : X \to Y$ be a morphism of good k-analytic spaces. Let U be an analytic domain in Y. Then, the topological space $\varphi^{-1}(U)$ is homeomorphic to $U \times_Y X$, and the two can be identified. It is shown as a direct application of Definition 1.5.5 that $\varphi^{-1}(U)$ is an analytic domain of X.

Similarly, if V, W are analytic domains of the good k-analytic space Z, $U \cap V$ is identified with the good k-analytic space $V \times_Z W$ which is an analytic domain of X.

Along the same lines (and very useful for the next chapters):

PROPOSITION 1.5.7. *Let $\varphi : X \to Y$ be a morphism of good k-analytic spaces. Let $y \in Y$. The fiber product $X_y := X \times_Y \mathcal{M}(\mathcal{H}(y))$ exists in the category Aff-k, and is homeomorphic to $\varphi^{-1}(y)$.*

REMARK 1.5.8. It can be shown from the universal property of fiber products that $X_y \cong (X \times_k \mathcal{M}(\mathcal{H}(y))) \times_{Y \times_k \mathcal{M}(\mathcal{H}(y))} \mathcal{M}(\mathcal{H}(y))$, so $X \times_Y \mathcal{M}(\mathcal{H}(x))$ exists as a good $\mathcal{H}(x)$-analytic space. Consequently, the fiber of any point $y \in Y$ can be endowed with the structure of a good $\mathcal{H}(y)$-analytic space. From now on, we will always identify the two.

REMARK 1.5.9. More generally, let $Y \to X$ be a morphism of good k-analytic spaces, and $Z \to X$ a morphism of good analytic spaces in the category An-k. Suppose Z is a good K-analytic space, where K/k is a complete ultrametric field extension. Then, $Y \times_X Z$ exists as a good K-analytic space. To show this, one can check via the universal property of fiber products that $Y \times_X Z \cong (Y \times_k K) \times_{X \times_k K} Z$, and the latter exists.

From now on, we will usually denote $X \times_Y \mathcal{H}(x)$ (resp. $X \times_k Y$) instead of $X \times_Y \mathcal{M}(\mathcal{H}(x))$ (resp. $X \times_{\mathcal{M}(k)} Y$).

DEFINITION 1.5.10. Let X be a good k-analytic space. A point $x \in X$ is said to be *rigid* if the field extension $\mathcal{H}(x)/k$ is finite.

Rigid points are those that we see when considering Tate's rigid spaces.

PROPOSITION 1.5.11 ([**6**, Proposition 2.1.15]). *Suppose k is non-trivially valued. The set of rigid points of a strict k-affinoid space is dense. Consequently, the set of rigid points in a strict good k-analytic space is dense.*

CONVENTION 1.5.12. We will say that a neighborhood of a point which is an affinoid domain is *an affinoid neighborhood* of the point. A cover $\mathcal{U}$ of a good analytic space is said to be *affinoid* if for any $U \in \mathcal{U}$, U is an affinoid domain thereof.

1.5.2. Examples of affinoid domains. Let us give a few examples of affinoid domains in $\mathbb{A}_k^{n,\mathrm{an}}$ and $\mathbb{P}_k^{n,\mathrm{an}}$.

EXAMPLE 1.5.13. By Example 1.5.2, for any $n \in \mathbb{N}$ and any $\underline{r} \in {\mathbb{R}_{>0}}^n$, the closed polydisc $\mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$ of polyradius $\underline{r}$ can be identified with $\{x \in \mathbb{A}_k^{n,\mathrm{an}} : |T_i|_x \leqslant r_i, i = 1, 2, \ldots, n\}$, and its ring of analytic functions with $k\{\underline{r}^{-1}\underline{T}\}$. The closed polydisc of polyradius $\underline{r}$ is an affinoid domain in $\mathbb{A}_k^{n,\mathrm{an}}$.

The open subspace $\{x \in \mathbb{A}_k^{n,\mathrm{an}} : |T_i|_x < r_i, i = 1, 2, \ldots, n\}$ of $\mathbb{A}_k^{n,\mathrm{an}}$ is said to be a *Berkovich polydisc of polyradius $\underline{r}$*. If $n = 1$, it is said to be a *Berkovich disc of radius r*.

Let $\underline{s} \in \mathbb{R}_{>0}^n$ be such that $s_i \leqslant r_i$ for all $i = 1, 2, \ldots, n$. Again, using Example 1.5.2 we obtain that $\{x \in \mathbb{A}_k^{n,\mathrm{an}} : s_i \leqslant |T_i|_x \leqslant r_i\}$ is an affinoid domain in $\mathbb{A}_k^{n,\mathrm{an}}$. It can be identified with $\mathcal{M}(k\{\underline{r}^{-1}\underline{T}, \underline{s}\ \underline{S}\}/(S_iT_i - 1)_i)$, and its ring of analytic functions with the k-affinoid algebra $k\{\underline{r}^{-1}\underline{T}, \underline{s}\ \underline{S}\}/(S_iT_i - 1)_i$. Remark that if $s_i = r_i$ for all i, this coincides with $K_{\underline{r}}$ from Example 1.3.6.

Since these subsets are contained in $\mathbb{P}_k^{n,\mathrm{an}}$, all of the above remains true when replacing $\mathbb{A}_k^{n,\mathrm{an}}$ by $\mathbb{P}_k^{n,\mathrm{an}}$. For $n = 1$, let us fix the following notations: $\mathbb{D}(0, r) := \{x \in \mathbb{A}_k^{1,\mathrm{an}} : |T|_x \leqslant r\} = \{x \in \mathbb{P}_k^{1,\mathrm{an}} : |T|_x \leqslant r\}$, and $C(0; s, r) := \{x \in \mathbb{A}_k^{1,\mathrm{an}} : s \leqslant |T|_x \leqslant r\} = \{x \in \mathbb{P}_k^{1,\mathrm{an}} : s \leqslant |T|_x \leqslant r\}$. The affinoid space $C(0; s, r)$ is said to be the *closed Berkovich annulus of inner radius s and outer radius r*.

The open subspace $\{x \in \mathbb{A}_k^{1,\mathrm{an}} : s < |T|_x < r\}$ is said to be a *Berkovich annulus of inner radius s and outer radius r*.

EXAMPLE 1.5.14. Let $P \in k[T]$. Also, let $r, s \in \mathbb{R}_{>0}$ be such that $s \leqslant r$. Set $D_1 = \{x \in \mathbb{A}_k^{1,\mathrm{an}} : |P|_x \leqslant r\}$ and $D_2 = \{x \in \mathbb{A}_k^{1,\mathrm{an}} : s \leqslant |P|_x \leqslant r\}$.

Seeing as P is bounded in these sets, T has to be so as well. Consequently, there exists $t > 0$ such that $D_1, D_2 \subseteq \mathbb{D}(0,t)$. By Example 1.4.6, D_1, D_2 are affinoid domains in $\mathbb{D}(0,t)$, so they are affinoid domains in $\mathbb{A}_k^{1,\mathrm{an}}$.

As $D_1, D_2 \subseteq \mathbb{P}_k^{1,\mathrm{an}}$, all of the above remains true when replacing $\mathbb{A}_k^{1,\mathrm{an}}$ by $\mathbb{P}_k^{1,\mathrm{an}}$.

In Section 4.2, we will see that $\mathcal{O}(D_1) = k\{r^{-1}T\}[X]/(P(T) - X)$ and $\mathcal{O}(D_2) = k\{r^{-1}T, sS\}[X]/(TS - 1, P(T) - X)$.

EXAMPLE 1.5.15. Let $r \in \mathbb{R}_{>0}$. Let us fix a copy of $\mathbb{A}_k^{1,\mathrm{an}}$ of $\mathbb{P}_k^{1,\mathrm{an}}$ so that we have a coordinate T. The subset $\{x \in \mathbb{P}_k^{1,\mathrm{an}} : |T|_x \geqslant r\}$ is an affinoid domain in $\mathbb{P}_k^{1,\mathrm{an}}$.

To see this, remark that $\{x \in \mathbb{P}_k^{1,\mathrm{an}} : |T|_x \geqslant r\} = \{x \in \mathbb{P}_k^{1,\mathrm{an}} : |1/T|_x \leqslant 1/r\}$, which is an affinoid domain in the copy of $\mathbb{A}_k^{1,\mathrm{an}}$ with coordinate $1/T$. Consequently, it is an affinoid domain in $\mathbb{P}_k^{1,\mathrm{an}}$.

The isomorphism $k[T] \to k[1/T]$ induces an isomorphism between the corresponding analytic affine lines (with coordinate T, resp. $1/T$), which induces an isomorphism between $\{x \in \mathbb{A}_k^{1,\mathrm{an}} : |T|_x \leqslant 1/r\}$ and $\{x \in \mathbb{A}_k^{1,\mathrm{an}} : |1/T|_x \leqslant 1/r\}$. Taking this into account, we obtain that $\mathcal{O}(\{x \in \mathbb{P}_k^{1,\mathrm{an}} : |T|_x \geqslant r\}) = k\{rT^{-1}\}$, where $k\{rT^{-1}\}$ is the k-affinoid algebra $\{\sum_n \frac{a_n}{T^n} : a_n \in k, \lim_{n\to\infty} |a_n| r^{-n} = 0\}$.

EXAMPLE 1.5.16. Let $P(T) \in k[T]$, and $r \in \mathbb{R}_{>0}$. Set $D = \{x \in \mathbb{P}_k^{1,\mathrm{an}} : |P|_x \geqslant r\}$. The finite morphism $k[T] \to k[T], T \mapsto P(T)$, induces a finite morphism $\varphi : \mathbb{P}_k^{1,\mathrm{an}} \to \mathbb{P}_k^{1,\mathrm{an}}$ such that $\varphi^{-1}(\{x \in \mathbb{P}_k^{n,\mathrm{an}} : |T|_x \geqslant r\}) = D$. By Example 1.5.15 and Proposition 1.5.34(1), we obtain that D is an affinoid domain in $\mathbb{P}_k^{1,\mathrm{an}}$. We will talk about its corresponding k-affinoid algebra in detail in Section 4.2.

1.5.3. Local properties and dimension of good analytic spaces. Most of the notions that exist for schemes also exist for Berkovich analytic spaces. We will mostly focus on defining those that we use in the next chapters. Recall that k is a complete ultrametric field.

DEFINITION 1.5.17. Let X be a good k-analytic space. Let $x \in X$. Then, x is said to be a *reduced* (resp. *normal, Cohen-Macaulay, regular*) point if $\mathcal{O}_{X,x}$ is a *reduced* (resp. *normal, Cohen-Macaulay, regular*) ring.

The analytic space X is said to be *reduced* (resp. *normal, Cohen-Macaulay, regular*) if for any $x \in X$, x is reduced (resp. normal, Cohen-Macaulay, regular).

REMARK 1.5.18. In [**21**, Théorème 3.4], it is shown that an analytic domain of a reduced (resp. normal, Cohen-Macaulay, regular) good analytic space is also reduced (resp. normal, Cohen-Macaulay, regular).

Let A be a k-affinoid algebra. Using Noether's Normalization Lemma for strict affinoid algebras, one obtains that for any complete ultrametric field extension K/k such that $A\widehat{\otimes}_k K$ is a strict K-affinoid algebra (by Lemma 1.3.7, such a K exists), the Krull dimension of $A\widehat{\otimes}_k K$ is constant [**22**, 1.5].

DEFINITION 1.5.19. The *dimension* of $X := \mathcal{M}(A)$, denoted $\dim(X)$, is the Krull dimension of the ring $A\widehat{\otimes}_k K$.

The *dimension* of any good k-analytic space Y is denoted by $\dim(Y)$, and defined to be $\sup_V \dim(V)$, where the supremum is taken with respect to all the affinoid domains V in Y.

For any $x \in Y$, the *dimension* of x in Y, denoted $\dim_x(Y)$, is $\inf_{x\in V} \dim(V)$, where the infimum is taken with respect to all affinoid domains V in Y that are neighborhoods of x.

The space Y is said to be *pure-dimensional* if $\dim_x(Y) = \dim(Y)$ for all $x \in Y$.

Dimension is invariant with respect to base change by a complete ultrametric field extension.

EXAMPLE 1.5.20. The following is an example showing the necessity of base change to calculate the dimension of a k-affinoid space: for $r \in \mathbb{R}_{>0}\backslash\sqrt{|k^\times|}$, let $D := \{x \in \mathbb{A}^{1,\mathrm{an}}_k : |T|_x = r\}$. Recall that $\mathcal{O}(D) = K_r$ - the field of Example 1.3.6. Then, the Krull dimension of K_r is 0, whereas the dimension of D is 1.

The notion of dimension brings us to the introduction of very important invariants of points.

DEFINITION 1.5.21. Let X be a good k-analytic space. Recall that for any $x \in X$, there is a canonical isometric embedding $k \hookrightarrow \mathcal{H}(x)$. For any $x \in X$, set $s_x := \deg tr_{\widetilde{k}}\widetilde{\mathcal{H}(x)}$ and $t_x := \mathrm{rank}_{\mathbb{Q}}|\mathcal{H}(x)^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\widetilde{\mathcal{H}(x)}$ (resp. $\widetilde{k}$) denotes the residue field of $\mathcal{H}(x)$ (resp. k).

LEMMA 1.5.22 ([**4**, Lemma 2.5.2], [**22**, 1.14]). *Let X be a good k-analytic space. Then,* $\dim X = \sup_{x\in X}(s_x + t_x)$.

DEFINITION 1.5.23 (Abhyankar points). Let X be a good k-analytic space. Then, $x \in X$ is said to be an *Abhyankar point* if $s_x + t_x = \dim_x(X)$.

In Example 3.2.10 of [**18**], it is shown that if $x \in X$ is an Abhyankar point, then $\mathcal{O}_{X,x}$ is Artinian. Consequently, if $\mathcal{O}_{X,x}$ is reduced (for example if X is reduced), then it is a field.

PROPOSITION 1.5.24 ([**60**, Proposition 4.7]). *Let X be a k-affinoid space. Then, any point of the Shilov boundary of X is an Abhyankar point. In particular, Abhyankar points are dense in X.*

DEFINITION 1.5.25. Let $\varphi : X \to Y$ be a morphism of good analytic spaces. For any $y \in Y$, the *relative dimension* of φ at y is the dimension of $\varphi^{-1}(y)$ as a good $\mathcal{H}(y)$-analytic space.

We will say that X is a *relative curve* over Y if $\varphi^{-1}(y)$ is of pure dimension 1 as an $\mathcal{H}(y)$-analytic space for all $y \in Y$ (we use this in Chapter 4).

The notions of dimension and relative dimension of a morphism are extensively studied by Ducros in [**22**].

1.5.4. Morphisms, relative boundary and interior. We briefly mention some classes of morphisms between good analytic spaces, and a generalization of the notions of relative boundary and interior.

DEFINITION 1.5.26. Let $\varphi : Y \to X$ be a morphism of good k-analytic spaces. The *Berkovich relative interior of* φ, denoted $\mathrm{Int}_B(Y/X)$, is the set of points $y \in Y$, such that there exist affinoid neighborhoods Y' of y and X' of $\varphi(x)$ with $\varphi(Y') \subseteq X'$ and $y \in \mathrm{Int}_B(Y'/X')$. The *Berkovich relative boundary* of φ, denoted $\partial_B(Y/X)$ is the complement of $\mathrm{Int}_B(Y/X)$ in Y. If $\partial_B(Y/X) = \emptyset$, φ is said to be *boundaryless*.

If $X = \mathcal{M}(k)$, then we write $\mathrm{Int}_B(Y)$, resp. $\partial_B(Y)$, and call them the *Berkovich interior*, resp. *Berkovich boundary*, of Y. If $\partial_B(Y) = \emptyset$, Y is said to be *boundaryless*.

THEOREM 1.5.27 ([**4**, Prop. 1.5.5]). *Let $\varphi : Y \to X$ be a morphism of good k-analytic spaces.*

(1) *Let $\phi : Z \to Y$ be a morphism of good k-analytic spaces. This induces a morphism $\varphi \circ \phi : Z \to X$. Then,* $\mathrm{Int}_B(Z/X) = \mathrm{Int}_B(Z/Y) \cap \phi^{-1}(\mathrm{Int}_B(Y/X))$.
(2) *If Y is an analytic domain of X, then* $\mathrm{Int}_B(Y/X)$ *is the topological interior of Y in X.*

EXAMPLE 1.5.28. By Example 1.5.2, for any $n \in \mathbb{N}$, $\mathbb{A}_k^{n,\mathrm{an}}$ (and hence $\mathbb{P}_k^{n,\mathrm{an}}$) is a boundaryless space. For any $r, s \in \mathbb{R}_{>0}$, $r < s$, $\partial_B(\mathbb{D}(0,r)) = \{\eta_{0,r}\}$, and $\partial_B(C(0;r,s)) = \{\eta_{0,r}, \eta_{0,s}\}$ (see [**20**, 3.6.4.1, 3.6.4.3]).

DEFINITION 1.5.29 (Immersions). A morphism of good k-analytic spaces $\varphi : Y \to X$ is an *open immersion* if it induces an isomorphism of Y with an open subset of X (endowed with the induced analytic structure from X).

The morphism φ is said to be a *closed immersion* if Y is homeomorphic to a closed subset of X, $\varphi_*\mathcal{O}_Y$ is a coherent sheaf of $\mathcal{O}_X$-modules, and the morphism $\mathcal{O}_X \to \varphi_*\mathcal{O}_Y$ is surjective.

As usual, there is a bijection between the closed immersions of a good k-analytic space Y and the ideal sheaves $\mathcal{I}$ of $\mathcal{O}_Y$ (*cf.* Proposition 3.1.4(ii) of [**6**]).

DEFINITION 1.5.30 (Proper and finite). A morphism of good k-analytic spaces $\varphi : Y \to X$ is said to be *proper* if it is compact and boundaryless. A good k-analytic space Y is said to be *proper* if the canonical morphism $Y \to \mathcal{M}(k)$ is proper.

A morphism of affinoid spaces $\mathcal{M}(B) \to \mathcal{M}(A)$ is said to be *finite* if the corresponding bounded morphism $A \to B$ is finite.

For any $x \in X$, the morphism $\varphi : Y \to X$ of good k-analytic spaces is said to be *finite at x* if there exists an affinoid neighborhood V of x such that $\varphi^{-1}(V) \to V$ is a finite morphism of affinoid spaces. The morphism φ is said to be *finite* if it is finite at any point of X.

Finite morphisms will be of particular importance in the rest of the manuscript.

PROPOSITION 1.5.31. *Let $\varphi : Y \to X$ be a finite morphism of good k-analytic spaces. Then, $\partial_B(Y/X) = \emptyset$, i.e. φ is boundaryless.*

Proposition 1.5.31 is immediate from Theorem 1.4.29(3), and the fact that being an element of the Berkovich interior $\mathrm{Int}_B(Y/X)$ is a local property.

REMARK 1.5.32. In [**4**, Lemma 1.3.7] it is shown that if $\varphi : Y \to X$ is a finite morphism of k-analytic spaces, then for any affinoid domain V of X, the induced morphism $\varphi^{-1}(V) \to V$ is a finite morphism of k-affinoid spaces.

As usual, for any $y \in Y$, the induced homomorphism of local rings $\mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$ is finite. Finite morphisms have finite fibers.

EXAMPLE 1.5.33. For any $n \in \mathbb{N}$, $\mathbb{P}_k^{n,\mathrm{an}}$ is proper.

Remark that a k-affinoid space is boundaryless, and hence proper, if and only if its corresponding k-affinoid algebra is a finite k-algebra.

PROPOSITION 1.5.34 ([**6**, Prop. 3.1.7]). *(1) The pre-image of an affinoid domain by a finite morphism is an affinoid domain.*

(2) Let X be a good analytic space. There is an equivalence between the category of finite morphisms over X and the category of coherent $\mathcal{O}_X$-algebras.

(3) A closed immersion is a finite morphism.

(4) A finite morphism is proper.

DEFINITION 1.5.35 (Separated). A morphism $\varphi : Y \to X$ of k-analytic spaces is said to be *separated* if the canonical induced morphism $Y \to Y \times_X Y$ is a closed immersion.

A good k-analytic space Y is said to be *separated* if the canonical morphism $Y \to \mathcal{M}(k)$ is separated.

A good k-analytic space is locally Hausdorff by construction, but, in general, not necessarily Hausdorff. Separatedness is equivalent to Hausdorff by [**6**, Proposition 3.1.5]. As a consequence:

COROLLARY 1.5.36 ([**6**, Cor. 3.1.6]). *If a good k-analytic space is separated, then affinoid domains are closed, and the intersection of two affinoid domains is an affinoid domain.*

The following is a central object of this manuscript:

DEFINITION 1.5.37. A (good) k-analytic space is said to be a *k-analytic curve* if it is separated and of pure dimension 1.

The hypothesis of goodness can be omitted from the above definition. In [**20**, Proposition 3.3.7] it is shown that any k-analytic curve is a good k-analytic space.

DEFINITION 1.5.38 (Flatness). A morphism of analytic spaces $\varphi_1 : Y_1 \to X_1$ is said to be *naively flat at $y \in Y_1$* if for $x := \varphi_1(y)$, $\mathcal{O}_{Y_1,y}$ is a flat $\mathcal{O}_{X_1,x}$-module. (Naive flatness is generally not stable with respect to base change.)

A morphism $\varphi : Y \to X$ of good k-analytic spaces is said to be *flat at $y \in Y$* if for any morphism $X' \to X$ of good analytic k-spaces, and any $y' \in Y' := Y \times_X X'$ lying above y, the induced morphism $Y' \to X'$ is naively flat at $y' \in Y'$. The morphism φ is said to be *flat* if it is flat at all $y \in Y$.

In [**18**, Theorem 8.3.4], it is shown that for $y \in \mathrm{Int}_B(Y/X)$, the morphism φ is flat at y if and only if it is naively flat at y. Consequently, if φ is boundaryless, then the notion of flatness above coincides with that of flatness for locally ringed spaces.

Flat morphisms have been extensively studied by Ducros in [**18**]. A special case is the class of (quasi-)smooth morphisms.

DEFINITION 1.5.39 (Quasi-smoothness, [**18**, Definition 5.2.4]). A morphism $\varphi : Y \to X$ of good k-analytic spaces is said to be *quasi-smooth at $y \in Y$* if there exists an affinoid neighborhood V of y such that $V \to X$ factorizes through a closed immersion $V \to W$, where W is an affinoid domain of $\mathbb{A}_X^{n,\mathrm{an}} := \mathbb{A}_k^{n,\mathrm{an}} \times_k X$ for some $n \in \mathbb{N}$ (and satisfies a sort of Jacobian-presentation condition). The morphism φ is said to be *quasi-smooth* if it is quasi-smooth at all $y \in Y$.

A good k-analytic space X is said to be *quasi-smooth* if the canonical morphism $X \to \mathcal{M}(k)$ is quasi-smooth.

DEFINITION 1.5.40 (Smoothness). A morphism $\varphi : Y \to X$ of good k-analytic spaces is said to be *smooth at* $y \in Y$ if it is quasi-smooth at y and $y \notin \partial_B(Y/X)$.

A good k-analytic space X is said to be *smooth* if the canonical morphism $X \to \mathcal{M}(k)$ is smooth.

Berkovich gave a different definition of smoothness in [**4**, 3.5]. Seeing as said definition excludes boundary points from consideration, this led to the introduction of the more general notion of quasi-smoothness by Ducros (see [**18**, Chapter 5] for a detailed treatment). In [**18**, Corollary 5.4.8], it is shown that Berkovich's definition is equivalent to Definition 1.5.40.

All the morphisms that we have defined above are stable under composition, base change, and ground field extension.

1.5.5. Topological properties. By construction, good analytic spaces are locally compact. Let us mention some other very remarkable topological properties of these objects.

DEFINITION 1.5.41. Let X be a topological space. Let $x, y \in X$. A continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(y) = 1$ is said to be a *path in X connecting x and y*. If, moreover, γ induces a homeomorphism with its image in X, then it is said to be an *arc in X connecting x and y*.

We will say that a topological space X is *path-connected* (resp. *arcwise-connected*) if for any different $x, y \in X$, there exists a path (resp. an arc) in X connecting x and y. If, moreover, the arc connecting x and y is unique, then X is said to be *uniquely arcwise-connected*.

THEOREM 1.5.42 ([**6**, Thm. 3.2.1]). *A connected good analytic space is path-connected. Consequently, a good analytic space is locally path-connected.*

By [**70**, Corollary 31.6], a Hausdorff topological space that is path-connected is arcwise-connected.

COROLLARY 1.5.43. *A separated connected good analytic space is arcwise-connected. Consequently, a separated good analytic space is locally arcwise connected.*

The following theorem is shown in [**5**].

THEOREM 1.5.44. *A smooth (good) analytic space is locally contractible.*

The result above was generalized by Hrushovski and Loeser to a larger class of analytic spaces. This was done in their ground-breaking work [**38**], where they use model theory to study Berkovich spaces. The interplay between these two fields is a rising domain of research.

1.6. Analytification functor and GAGA theorems

As in the complex setting, there exists an analytification functor and GAGA-type theorems for Berkovich spaces. For the most part of this manuscript, we will work with good analytic spaces that are obtained from finite type schemes over a complete ultrametric field.

1.6.1. The kernel map. The following gives a strong connection between the Berkovich and affine spectra.

DEFINITION 1.6.1. Let A be a k-affinoid algebra. There is a canonical morphism $\mathcal{M}(A) \to \operatorname{Spec} A$, $x \mapsto \ker|\cdot|_x$, which will be called the *kernel map* and will be denoted by $\ker_A$.

LEMMA 1.6.2 ([**6**, Remark 1.2.5(i)]). *Let A be a k-affinoid algebra. The kernel map $\ker_A$ is continuous.*

Remark that there is a bijective correspondence between the closed immersions of $\operatorname{Spec} A$ and those of $\mathcal{M}(A)$.

The kernel map has very nice properties.

PROPOSITION 1.6.3 ([**4**, Prop. 2.1.1, Thm. 2.1.4]). *Let A be a k-affinoid algebra. Then, $\ker_A$ is faithfully flat.*

THEOREM 1.6.4. [**4**, Thm. 2.2.1] *Let A be a k-affinoid algebra. Set $X = \mathcal{M}(A)$ and $\mathcal{X} = \operatorname{Spec} A$. Let $\mathcal{P}_X$ be the set of points on X which are reduced (resp. normal, Cohen-Macaulay, regular), and similarly for $\mathcal{P}_{\mathcal{X}}$. Then, $\mathcal{P}_X = \ker_A^{-1}(\mathcal{P}_{\mathcal{X}})$.*

In particular, we obtain from the above that a k-affinoid space X is reduced (resp. normal) if and only if $\mathcal{O}(X)$ is reduced (resp. normal).

1.6.2. Analytification over a field. Let k be a complete ultrametric field.

DEFINITION 1.6.5. Let $\mathcal{X}$ be a scheme of locally finite type over k. The *Berkovich analytification of X*, denoted $\mathcal{X}^{\mathrm{an}}$, is a good k-analytic space together with a morphism of k-locally ringed spaces $\mathcal{X}^{\mathrm{an}} \to \mathcal{X}$, which represents the functor An-$k \to \mathrm{Set}, Y \mapsto \mathrm{Hom}_k(Y, \mathcal{X})$, where $\mathrm{Hom}_k(\cdot,\cdot)$ denotes morphisms in the category of k-locally ringed spaces.

THEOREM 1.6.6 ([**6**, Thm. 3.4.1, 3.5.1]). *Let $\mathcal{X}$ be a scheme of locally finite type over k. The Berkovich analytification $X := \mathcal{X}^{\mathrm{an}}$ of $\mathcal{X}$ exists.*

1. *For any complete ultrametric field extension K/k, $X(K) \cong \mathcal{X}(K)$. Moreover, the canonical morphism $\pi : X \to \mathcal{X}$ is surjective, and induces a bijection between the rigid points of X and the closed points of $\mathcal{X}$.*
2. *For any $x \in X$, the canonical morphism $\pi_x : \mathcal{O}_{\mathcal{X},\pi(x)} \to \mathcal{O}_{X,x}$ is faithfully flat. Furthermore, if x is a rigid point, then π_x induces an isomorphism of completions $\widehat{\mathcal{O}_{\mathcal{X},\pi(x)}} \to \widehat{\mathcal{O}_{X,x}}$.*

Let us briefly describe how the space X is constructed.

If $\mathcal{X} = \mathbb{A}^n_k$ for some $n \in \mathbb{N}$, then its Berkovich analytification is $\mathbb{A}^{n,\mathrm{an}}_k$, and the canonical map is a *kernel map* $\pi : \mathbb{A}^{n,\mathrm{an}}_k \to \mathbb{A}^n_k, x \mapsto \ker|\cdot|_x$. This also allows us to construct the analytifications of closed subschemes of $\mathbb{A}^n_k$. Namely, for any finitely generated k-algebra A, the analytification of $\mathcal{X} := \operatorname{Spec} A$ is given by

$$X := \{\text{multiplicative semi-norms on } A \text{ which extend the norm on } k\}.$$

The canonical map $\pi : X \to \mathcal{X}$ is still a kernel map; if $\mathcal{I}$ is the ideal sheaf corresponding to $\mathcal{X}$ as a Zariski closed subset of $\mathbb{A}^{n,\mathrm{an}}_k$, then the analytic structure on X is given by $\mathcal{O}_X := \mathcal{O}_{\mathbb{A}^{n,\mathrm{an}}_k}/\pi^*\mathcal{I}$.

In the general case, *i.e.* when $\mathcal{X}$ is any locally finite type scheme over k, we obtain X and the canonical map by gluing the analytifications and the canonical maps of any open affine cover of $\mathcal{X}$.

Remark that the analytification of a locally finite type scheme over k is boundaryless. This follows from the construction: it is true for $\mathbb{A}^n_k$, and it is true for its Zariski closed subsets by part (1) of Theorem 1.5.27. Finally, the general case is an immediate consequence of the latter, seeing as being boundaryless is a local property. If k is non-trivially valued, the same remains true when replacing *boundaryless* by *strict* (recall Example 1.5.2).

REMARK 1.6.7. Let $\mathcal{X} = \operatorname{Spec} A$ be a finite type affine scheme over k, and denote by X its Berkovich analytification (which can be described via multiplicative semi-norms in a similar way to $\mathbb{A}^{1,\mathrm{an}}_k$, see above). For $x \in A$, let $\mathcal{H}'(x)$ be the completion of $\mathrm{Frac}(A/\ker|\cdot|_x)$ with respect to the quotient norm induced by $|\cdot|_x$. As in Lemma 1.4.12, using the universal property for affinoid domains, we can show that $\mathcal{H}'(x) = \mathcal{H}(x)$ - the completed residue field of x.

Moreover, if $\mathcal{X}$ is a curve, then there is a bijection between the rigid points of X and the maximal ideals of A. If $x \in X$ is a rigid point, then $\mathcal{H}(x)/k$ is a finite field extension, implying $\ker|\cdot|_x$ is a maximal ideal of A. Let P be a maximal ideal of A. Then, by the surjectivity of the kernel map (*i.e.* analytification) $X \to \mathcal{X}$ there exists $x \in X$ such that $\ker|\cdot|_x = P$. Since A/P is a finite field extension of k, we obtain that x is a rigid point of X.

EXAMPLE 1.6.8. The analytification of $\mathbb{P}^n_k$ is $\mathbb{P}^{n,\mathrm{an}}_k$. The canonical map can be described using the canonical maps corresponding to the copies of $\mathbb{A}^n_k$ in $\mathbb{P}^n_k$.

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of locally finite type schemes over k. Set $X = \mathcal{X}^{\mathrm{an}}$ and $Y = \mathcal{Y}^{\mathrm{an}}$. We have a morphism $X \to \mathcal{X} \to \mathcal{Y}$ of k-locally ringed spaces. By the universal property of analytification, this induces a morphism of good k-analytic spaces $X \to Y$. Remark that, by construction, the following induced diagram is commutative.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \uparrow & & \uparrow \\ X & \longrightarrow & Y \end{array}$$

DEFINITION 1.6.9. Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of locally finite type schemes over k. The corresponding morphism $Y \to X$ of their analytifications (constructed in the paragraph above) will be denoted by f^{an}, and called the *analytification* of f.

Let $\mathcal{F}$ be any coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-modules. Let π denote the canonical morphism $X \to \mathcal{X}$. Then, $\mathcal{F}^{an} := \pi^*\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules. It is called *the analytification* of $\mathcal{F}$.

We now mention the main results that compare properties in the algebraic and analytic setting.

THEOREM 1.6.10 ([**6**, Prop 3.4.6, 3.4.7]). *Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism of locally finite type schemes over k. Then, f is: (1) flat, (2) separated, (3) surjective, (4) injective, (5) smooth, (6) an open immersion if and only if f^{an} possesses the same property.*

If f is of finite type, then it is: (1') dominant, (2') a closed immersion, (3') proper, (4') finite if and only f^{an} has the same property.

THEOREM 1.6.11 ([**6**, Prop. 3.4.3]). *A scheme of locally finite type over k is reduced, normal, Cohen-Macaulay, regular, smooth or of dimension n if and only if its analytification satisfies the corresponding property.*

THEOREM 1.6.12 ([**6**, Cor. 3.4.10, Prop. 3.4.11]). *Let $\mathcal{X}$ be a proper scheme over k.*

(1) *Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. Then, for any integer $p \geqslant 0$, the canonical morphism $H^p(\mathcal{X}, \mathcal{F}) \to H^p(\mathcal{X}^{\mathrm{an}}, \mathcal{F}^{\mathrm{an}})$ is an isomorphism.*
(2) *The functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$, induces an equivalence of categories between coherent sheaves of $\mathcal{O}_{\mathcal{X}}$-modules and the coherent sheaves of $\mathcal{O}_{\mathcal{X}^{\mathrm{an}}}$-modules.*

If k is trivially valued, then the properness assumption in Theorem 1.6.12 is not needed (*cf.* [**6**, Theorem 3.5.1(iii)]).

Since we will be working with curves in Chapter 3, the following result is very useful:

PROPOSITION 1.6.13 ([**20**, Théorème 3.7.2]). *For any proper k-analytic curve X, there exists a projective algebraic curve $\mathcal{X}$ over k, such that $\mathcal{X}^{\mathrm{an}} \cong X$.*

Hence, we will often refer to reduced proper k-analytic curves as being reduced and *projective*. The above result was first shown for reduced curves in [**6**, Cor. 3.4.14].

1.6.3. Analytification over an affinoid space. The construction of the analytification functor can be given more generally, over affinoid spaces. We will need this for Chapter 4.

Let k be a complete ultrametric field and A a k-affinoid algebra. Set $\mathcal{X} := \mathrm{Spec}\, A$, and $X := \mathcal{M}(A)$. We will say that a good analytic k-space Y is a *good X-analytic space* if there is a morphism of good analytic k-spaces $\varphi_Y : Y \to X$. A *morphism* $f : Y \to Z$ of X-analytic spaces is a morphism of analytic k-spaces such that $\varphi_Z \circ f = \varphi_Y$. Let us denote the category of good X-analytic spaces by X-An.

Let $\mathcal{Y}$ be a locally finite type scheme over $\mathcal{X}$. Let F denote the functor X-$An \to \mathrm{Set}$, $Z \mapsto \mathrm{Hom}_{\mathcal{X}}(Z, \mathcal{Y})$, where $\mathrm{Hom}_{\mathcal{X}}(\cdot, \cdot)$ denotes the set of morphisms in the category of $\mathcal{X}$-locally ringed spaces.

THEOREM 1.6.14 ([**4**, Prop. 2.6.1]). *The functor F is represented by a good X-analytic space $\mathcal{Y}^{\mathrm{an}}$ that is a good k-analytic space, and a morphism $\mathcal{Y}^{\mathrm{an}} \to \mathcal{Y}$ of $\mathcal{X}$-locally ringed spaces.*

The space $\mathcal{Y}^{\mathrm{an}}$ is called the *Berkovich analytification of $\mathcal{Y}$ over $\mathcal{X}$* (when there is no risk of ambiguity, we will simply say that $\mathcal{Y}^{\mathrm{an}}$ is the analytification of $\mathcal{Y}$). Remark that the analytification of $\mathcal{X}$ over $\mathcal{X}$ is simply X.

Remark that as in the case of fields, if $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ is a morphism of locally finite type $\mathcal{X}$-shemes, it gives rise to a morphism $f^{\mathrm{an}} : \mathcal{Y}_1^{\mathrm{an}} \to \mathcal{Y}_2^{\mathrm{an}}$ of good X-analytic spaces, called *the analytification of f*. This induces the following commutative diagram:

$$\begin{array}{ccc} \mathcal{Y}_1 & \xrightarrow{f} & \mathcal{Y}_2 \\ \uparrow & & \uparrow \\ \mathcal{Y}_1^{\mathrm{an}} & \xrightarrow{f^{\mathrm{an}}} & \mathcal{Y}_2^{\mathrm{an}} \end{array}$$

Analytification of schemes over strict affinoid algebras was introduced by Köpf [**45**] in the setting of rigid geometry, and several GAGA-type properties were shown in the same paper. In [**4**, 2.6], Berkovich defined and studied the analytification of schemes over

general affinoid algebras in the setup of Berkovich spaces. It has since been systematically studied by Ducros in several of his papers (*e.g.* [**21**], [**18**], [**61**, Annexe A]).

All of the properties that we mentioned for the analytification of schemes over fields (with the exception of Proposition 1.6.13) hold in this setting as well. However, seeing as the proofs are dispersed in several papers, we will directly give the references when using certain (most) of these results.

EXAMPLE 1.6.15. $(\mathbb{A}^n_{\mathcal{X}})^{\mathrm{an}} = \mathbb{A}^{n,\mathrm{an}}_k \times_k X =: \mathbb{A}^{n,\mathrm{an}}_X$, and $(\mathbb{P}^n_{\mathcal{X}})^{\mathrm{an}} = \mathbb{P}^{n,\mathrm{an}}_k \times_k X =: \mathbb{P}^{n,\mathrm{an}}_X$ for any $n \in \mathbb{N}$.

The examples of affinoid domains we saw in part 1.5.2 are applicable (using the same arguments) to $\mathbb{A}^{n,\mathrm{an}}_X$ and $\mathbb{P}^{n,\mathrm{an}}_X$ as well.

Let us make a few useful remarks for the next chapters.

LEMMA 1.6.16. *Analytification commutes with respect to affinoid base change. More precisely, let $Z \to X$ be a morphism of affinoid k-spaces. Set $\mathcal{Z} := Spec\ \mathcal{O}(Z)$. Let $\mathcal{Y}$ be any locally finite type scheme over $\mathcal{X}$. Set $Y := \mathcal{Y}^{\mathrm{an}}$. Then, $(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z})^{\mathrm{an}} = Y \times_X Z$ as good Z-analytic spaces.*

PROOF. By Remark 1.5.9, $Y \times_X Z$ exists. It is clearly a good Z-analytic space.

We only needs to check that the universal property is satisfied. Let T be a good Z-analytic space and $T \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ a morphism of $\mathcal{Z}$-locally ringed spaces. This induces a morphism $T \to \mathcal{Y}$ contained in $\mathrm{Hom}_{\mathcal{X}}(T, \mathcal{Y})$, and a morphism of analytic spaces $T \to Z \to X$, meaning $T \to Z$ can be interpreted as a morphism in X-*An*. By the universal property of analytification, we obtain a morphism $T \to Y$ in X-*An*. Consequently, there is a morphism of good Z-analytic spaces $T \to Y \times_X Z$, thus implying the statement. □

Recall that the fiber of a morphism of analytic spaces can be identified with a good analytic space (see Proposition 1.5.7).

COROLLARY 1.6.17. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of $\mathcal{X}$-schemes of locally finite type. For $x \in \mathcal{Y}_2^{\mathrm{an}}$, let $\overline{x}$ denote its image via $\mathcal{Y}_2^{\mathrm{an}} \to \mathcal{Y}_2$. Then, the fiber $(f^{\mathrm{an}})^{-1}(x)$ is isomorphic to $(f^{-1}(\overline{x}) \times_{\kappa(\overline{x})} \mathcal{H}(x))^{\mathrm{an}}$ as an $\mathcal{H}(x)$-analytic space, where $\kappa(\overline{x})$ is the residue field of $\overline{x}$ in $\mathcal{Y}_2$.*

PROOF. By Proposition 1.5.7, $(f^{\mathrm{an}})^{-1}(x)$ is identified to the good $\mathcal{H}(x)$-analytic space $\mathcal{Y}_1^{\mathrm{an}} \times_{\mathcal{Y}_2^{\mathrm{an}}} \mathcal{H}(x)$. By Lemma 1.6.16:

$$\mathcal{Y}_1^{\mathrm{an}} \times_{\mathcal{Y}_2^{\mathrm{an}}} \mathcal{H}(x) = (\mathcal{Y}_1 \times_{\mathcal{Y}_2} \mathcal{H}(x))^{\mathrm{an}} = (\mathcal{Y}_1 \times_{\mathcal{Y}_2} \kappa(\overline{x}) \times_{\kappa(\overline{x})} \mathcal{H}(x))^{\mathrm{an}} = (f^{-1}(\overline{x}) \times_{\kappa(\overline{x})} \mathcal{H}(x))^{\mathrm{an}}.$$

□

1.6.4. The Zariski topology. Let k be a complete ultrametric field.

DEFINITION 1.6.18. Let $X = \mathcal{M}(A)$ be a k-affinoid space. The *Zariski topology on X* is the topology induced by the kernel map $\ker_A : X \to \mathrm{Spec}\ A$ (see Definition 1.4.12).

More generally, let Y be a good k-analytic space. A *Zariski closed* subset of Y is the zero-locus of a coherent ideal of sheaves over Y (remark that these are exactly the closed immersions to Y, and in particular analytic spaces themselves). These are the closed sets of a topology on Y, called the *Zariski topology*.

Remark that the Berkovich topology is finer than the Zariski one.

If $\mathcal{X}$ is a finite type scheme over k, then the Zariski topology on $\mathcal{X}^{\mathrm{an}}$ is the one induced by the canonical analytification map $\mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ (see Theorem 1.6.10 (3')).

DEFINITION 1.6.19. Let X be a good k-analytic space. A Zariski closed subset of X is said to be *irreducible* if it is irreducible for the Zariski topology on X.

REMARK 1.6.20. Let $\mathcal{X}$ be a locally finite type scheme over k. Then, $\mathcal{X}$ is irreducible if and only $\mathcal{X}^{\mathrm{an}}$ is irreducible.

Moreover, in [**18**, Proposition 2.7.16], Ducros showed that the irreducible components of $\mathcal{X}^{\mathrm{an}}$ are the analytifications of the irreducible components of $\mathcal{X}$ in the more general case of analytic spaces over an affinoid space (we won't define what an irreducible component of a good analytic space is; for that, see [**18**, Definition 1.5.2]).

REMARK 1.6.21. In [**21**, Proposition 5.14], it is shown that if X is a normal good k-analytic space, then its irreducible components are its connected components.

Taking Remark 1.5.18 into account, this means that a connected analytic domain of a normal good k-analytic space is always irreducible, and so integral.

DEFINITION 1.6.22. We will say that a good k-analytic space is *integral* if it is reduced and irreducible.

Remark that a k-affinoid space is integral if and only if $\mathcal{O}(X)$ is a domain. (This does not necessarily imply that all the stalks are domains.)

REMARK 1.6.23. One can also define the notion of codimension for the Zariski topology, see for example [**18**, 1.5.15]. In particular, for a good k-analytic space X, *a divisor* on X is a Zariski closed subset of X of codimension 1. If X is the analytification of a locally finite type scheme $\mathcal{X}$, then a divisor on X is simply the pullback of a divisor on $\mathcal{X}$ via the canonical map $X \to \mathcal{X}$ see [**18**, Corollary 2.7.13].

The Zariski topology on Berkovich spaces has been extensively studied by Ducros in several of his papers (see *e.g.* [**18**]).

PROPOSITION 1.6.24 (Analytic continuation). *Let X be an integral good k-analytic space.*

(1) Let Y be any analytic domain of X. The restriction morphism $\mathcal{O}(X) \to \mathcal{O}(Y)$ is injective.

(2) For any $x \in X$, the restriction morphism $\mathcal{O}(X) \to \mathcal{O}_x$ is injective.

PROOF. Let $f \in \mathcal{O}(X)$ be such that its restriction to Y is zero. Let Z denote the Zariski closed subset of X determined by the zero locus of f. Then, $Y \subseteq Z$, so by [**21**, Corollaire 4.14], $Z = X$. This means that for any $x \in X$, $|f|_x = 0$. Let V be an affinoid neighborhood of x in X. Then, $|f|_y = 0$ for all $y \in V$, which by Proposition 1.3.15(1) (see also Theorem 1.1.38) implies that $f_{|V}$ is nilpotent in $\mathcal{O}(V)$. By Remark 1.5.18, the reducedness of X implies that of V, so $f_{|V} = 0$. We have shown that for any $x \in X$, $f = 0$ in $\mathcal{O}_{X,x}$, so $f = 0$ in $\mathcal{O}(X)$, and the restriction morphism to Y is injective.

The second part is a direct consequence of the first one. □

1.7. Complement I: The sheaf of meromorphic functions

As in the complex setting, a sheaf of meromorphic functions can be defined satisfying similar properties. Moreover, its definition resembles heavily that of the sheaf of meromorphic functions for schemes (including the subtleties of the latter, see [**43**]).

Let k denote a complete ultrametric field.

DEFINITION 1.7.1. Let X be a good k-analytic space. Let $\mathcal{S}_X$ be the presheaf of functions on X, which associates to any analytic domain U the set of analytic functions on U whose restriction to any affinoid domain in it is not a zero-divisor. Let $\mathscr{M}_-$ be the presheaf on X that associates to any analytic domain U the ring $\mathcal{S}_X(U)^{-1}\mathcal{O}_X(U)$. The sheafification $\mathscr{M}_X$ of the presheaf $\mathscr{M}_-$ is said to be the *sheaf of meromorphic functions on* X.

It is immediate form the definition that for any analytic domain U of X, $\mathcal{S}_X(U)$ contains no zero-divisors of $\mathcal{O}_X(U)$.

REMARK 1.7.2. The sheaf of meromorphic functions for schemes is given as in Definition 1.7.1 when replacing *affinoid* and *analytic domain* with *open subset*. Recall that if $\mathcal{X}$ is an integral scheme, then the ring of global sections of the sheaf of meromorphic functions on $\mathcal{X}$ coincides with its function field. See [**48**, 7.1.1] for a treatment of meromorphic functions in the algebraic setting.

PROPOSITION 1.7.3. *Let X be a good k-analytic space. Let U be an analytic domain of X. Then,*

(1) $\mathcal{S}_X(U) = \{f \in \mathcal{O}_X(U) : f$ *is a non-zero-divisor in* $\mathcal{O}_{U,x}$ *for all* $x \in U\}$.

(2) $\mathcal{S}_X(U) = \{f \in \mathcal{O}_X(U) : f$ *is a non-zero-divisor in* $\mathcal{O}_U(G)$ *for any open subset* G *of* $U\}$.

PROOF. (1) By a direct application of the definition, the elements of $\mathcal{S}_X(U)$ are non-zero-divisors on $\mathcal{O}_{U,x}$ for all $x \in U$.

Let $f \in \mathcal{O}_X(U)$ be such that f is a non-zero-divisor in $\mathcal{O}_{U,x}$ for all $x \in U$. This means that $\mathcal{O}_{U,x} \to \mathcal{O}_{U,x}, a \mapsto f \cdot a$, is an injective map for $x \in U$.

Let V be any affinoid domain in U. By [**18**, 4.1.11], for any $x \in V$, the morphism $\mathcal{O}_{U,x} \to \mathcal{O}_{V,x}$ is flat. Consequently, the map $\mathcal{O}_{V,x} \to \mathcal{O}_{V,x}, b \mapsto f \cdot b$, is injective, or equivalently, f is a non-zero-divisor in $\mathcal{O}_{V,x}$. Suppose there exists $c \in \mathcal{O}_U(V)$ such that $f \cdot c = 0$. Then, $c = 0$ in $\mathcal{O}_{V,x}$ for all $x \in V$, implying $c = 0$ in $\mathcal{O}_U(V)$. As a consequence, f is a non-zero-divisor in $\mathcal{O}_U(V)$. We have shown that $f \in \mathcal{S}_X(U)$, concluding the proof of the first part of the statement.

Finally, (2) is a direct consequence of (1). □

LEMMA 1.7.4. *Let X be a good k-analytic space. Let U be an affinoid domain in X. Then, $\mathcal{S}_X(U)$ is the set of non-zero divisors of $\mathcal{O}_X(U)$.*

PROOF. By definition, the elements of $\mathcal{S}_X(U)$ are not zero-divisors in $\mathcal{O}_X(U)$.

Let f be an element of $A_U := \mathcal{O}_X(U)$ that is a non-zero-divisor, *i.e.* such that the map $A_U \to A_U$, $a \mapsto f \cdot a$, is injective. Let $V \subseteq U$ be any affinoid domain. Set $A_V := \mathcal{O}_X(V)$. Then, by Proposition 1.4.10(2), the restriction map $A_U \to A_V$ is flat. Consequently, the map $A_V \to A_V, b \mapsto f \cdot b$, remains injective, meaning f is not a zero divisor in A_V. This implies that $f \in \mathcal{S}_X(U)$, proving the statement. □

The proof of the following statement resembles the proof of its algebraic analogue.

COROLLARY 1.7.5. *Let X be a good k-analytic space. Then, for any $x \in X$, $\mathcal{S}_{X,x}$ is the set of elements of $\mathcal{O}_{X,x}$ that are non-zero-divisors.*

PROOF. Let $x \in X$. Clearly, the elements of $\mathcal{S}_{X,x}$ are not zero divisors in $\mathcal{O}_{X,x}$.

Let $f \in \mathcal{O}_{X,x}$ be a non-zero-divisor. By restricting to an affinoid neighborhood of x if necessary, we may assume, without loss of generality, that X is an affinoid space and

$f \in \mathcal{O}_X(X)$. Set $A = \mathcal{O}_X(X)$. Set $I = \{a \in A : f \cdot a = 0\}$. This is an ideal of A, and gives rise to the following short exact sequence

$$0 \to I \to A \to A,$$

where $A \to A$ is given by $a \mapsto f \cdot a$. Seeing as f is a non-zero-divisor in $\mathcal{O}_{X,x}$, we obtain that $I\mathcal{O}_{X,x} = 0$.

The ring A is an affinoid algebra, and hence Noetherian (*cf* Theorem 1.3.9). Consequently, I is finitely generated. Let $a_1, a_2, \ldots, a_n \in A$ be such that $I = (a_1, a_2, \ldots, a_n)$. By the above, the germs $a_{i,x} \in \mathcal{O}_{X,x}$ of a_i at x are zero for all $i \in \{1, 2, \ldots, n\}$. Consequently, there exists an affinoid neighborhood V of x in X such that $a_{i|V} = 0$ for all i, implying $I\mathcal{O}_X(V) = 0$.

Set $A_V := \mathcal{O}_X(V)$. By Proposition 1.4.10(2), the restriction morphism $A \to A_V$ is flat, so the short exact sequence above induces the following short exact sequence:

$$0 \to I \otimes_A A_V \to A_V \to A_V,$$

where $A_V \to A_V$ is given by $b \mapsto f_{|V} \cdot b$. Seeing as A_V is a flat A-module, $I \otimes_A A_V$ is isomorphic to $IA_V = 0$. Consequently, multiplication by $f_{|V}$ is injective in A_V, or equivalently $f_{|V}$ is a non-zero-divisor in A_V. By Lemma 1.7.4, this implies that $f_{|V} \in \mathcal{S}_X(V)$, and finally that $f \in \mathcal{S}_{X,x}$. □

By Corollary 1.7.5, if X is a good k-analytic space, then for any $x \in X$, $\mathscr{M}_{X,x}$ is the total ring of fractions of $\mathcal{O}_{X,x}$. In particular, if $\mathcal{O}_{X,x}$ is a domain, then $\mathscr{M}_{X,x} = \text{Frac } \mathcal{O}_{X,x}$. When there is no risk of confusion, we will simply denote $\mathcal{O}$, resp. $\mathscr{M}$, for the sheaf of analytic, resp. meromorphic functions on X. We make note of the following, well known, fact:

LEMMA 1.7.6. *Let X be an integral k-affinoid space. Then, $\mathscr{M}(X) = \text{Frac } \mathcal{O}(X)$.*

PROOF. Since $\mathcal{O}(X)$ is an integral domain, Frac $\mathcal{O}(X) \subseteq \mathscr{M}(X)$ by the definition of $\mathscr{M}$. Let $f \in \mathscr{M}(X)$. The sheaf $f\mathcal{O} \cap \mathcal{O} \subseteq \mathscr{M}$ is non-zero and coherent, so by Kiehl's Theorem [**6**, Proposition 2.3.1], it has a non-zero global section x. Then, there exists $y \in \mathcal{O}(X)\backslash\{0\}$, for which $f = \frac{x}{y} \in \text{Frac } \mathcal{O}(X)$.

□

A very important non-trivial result for us is the following:

PROPOSITION 1.7.7 (Hurwitz-Weierstrass Theorem, [**6**, Prop. 3.6.2]). *Let $\mathcal{X}$ be a reduced proper scheme over k. Then, the canonical map $\mathscr{M}_{\mathcal{X}}(\mathcal{X}) \to \mathscr{M}_{\mathcal{X}^{\text{an}}}(\mathcal{X}^{\text{an}})$, where $\mathscr{M}_{\mathcal{X}}$ denotes the sheaf of meromorphic functions on $\mathcal{X}$, is an isomorphism.*

This can be generalized to schemes over an affinoid algebra. It is a non-trivial result for which GAGA-type theorems (*cf.* [**45**], [**61**, Annexe A]) are crucial. The arguments to prove the following result were given in a Mathoverflow thread (see [**57**]).

Let us first mention some brief reminders on the notion of depth. Let R be a ring, I an ideal of R, and M a finitely generated R-module. An *M-regular sequence of length d over I* is a sequence $r_1, r_2, \ldots, r_d \in I$ such that r_i is not a zero divisor in $M/(r_1, \ldots, r_{i-1})M$ for $i = 1, 2, \ldots, d$. The *depth* of M over I, denoted $\text{depth}_R(I, M)$ in [**13**, Section 1], is

- ∞ if $IM = M$,
- the supremum of the length of M-regular sequences over I, otherwise.

In what follows, when $M = R$, we will denote $\mathrm{depth}_R(I, R)$ by $\mathrm{depth}_I R$. Remark that $\mathrm{depth}_I R > 0$ if and only if I contains a non-zero divisor of R.

THEOREM 1.7.8. *Let k be a complete ultrametric field. Let A be a k-affinoid algebra. Let X be a proper scheme over Spec A. Let $X^{\mathrm{an}}/\mathcal{M}(A)$ denote the Berkovich analytification of X. Then, $\mathscr{M}_{X^{\mathrm{an}}}(X^{an}) = \mathscr{M}_X(X)$, where $\mathscr{M}_{X^{\mathrm{an}}}$ (resp. $\mathscr{M}_X$) denotes the sheaf of meromorphic functions on X^{an} (resp. X).*

When there is no risk of ambiguity and the ambient space is clear from context, we will simply write $\mathscr{M}$ for the sheaf of meromorphic functions.

PROOF. As in Definition 1.7.1, let $\mathcal{S}_{X^{\mathrm{an}}}$ denote the presheaf of analytic functions on X^{an}, which associates to any analytic domain U the set of analytic functions on U whose restriction to any affinoid domain in it is not a zero divisor. By Corollary 1.7.5, for any $x \in X^{\mathrm{an}}$, $\mathcal{S}_{X^{\mathrm{an}},x}$ is the set of non-zero-divisors of $\mathcal{O}_{X^{\mathrm{an}},x}$.

Let $\mathcal{I}$ be a coherent ideal sheaf on X^{an} that locally on X^{an} contains a section of $\mathcal{S}_{X^{\mathrm{an}}}$. This means that for any $x \in X^{\mathrm{an}}$, $\mathcal{S}_{X^{\mathrm{an}},x} \cap \mathcal{I}_x \neq \emptyset$. Let $s \in \mathcal{S}_{X^{\mathrm{an}},x} \cap \mathcal{I}_x$. Then, s is a non-zero divisor in $\mathcal{O}_{X^{\mathrm{an}},x}$, which implies $\mathrm{depth}_{\mathcal{I}_x} \mathcal{O}_{X^{\mathrm{an}},x} > 0$. Suppose, on the other hand, that $\mathcal{I}$ is a coherent ideal sheaf on X^{an} such that $\mathrm{depth}_{\mathcal{I}_x} \mathcal{O}_{X^{\mathrm{an}},x} > 0$ for all $x \in X^{\mathrm{an}}$. Then, there exists at least one element $s \in \mathcal{I}_x$ which is a non-zero-divisor in $\mathcal{O}_{X^{\mathrm{an}},x}$, implying $s \in \mathcal{S}_{X^{\mathrm{an}},x}$. To summarize, a coherent ideal sheaf $\mathcal{I}$ on X^{an} contains locally on X^{an} a section of $\mathcal{S}_{X^{\mathrm{an}}}$ if and only if $\mathrm{depth}_{\mathcal{I}_x}(\mathcal{O}_{X^{\mathrm{an}},x}) > 0$ for all $x \in X^{\mathrm{an}}$.

Let us show that for any coherent ideal sheaf $\mathcal{I}$ on X^{an} containing locally on X^{an} a section of $\mathcal{S}_{X^{\mathrm{an}}}$, there is an embedding $\mathrm{Hom}_{X^{\mathrm{an}}}(\mathcal{I}, \mathcal{O}_{X^{\mathrm{an}}}) \subseteq \mathscr{M}_{X^{\mathrm{an}}}(X^{\mathrm{an}})$, where $\mathrm{Hom}_{X^{\mathrm{an}}}(\mathcal{I}, \mathcal{O}_{X^{\mathrm{an}}})$ denotes the global sections on X^{an} of the hom sheaf $\mathscr{H}om(\mathcal{I}, \mathcal{O}_{X^{\mathrm{an}}})$. Let $\varphi \in \mathrm{Hom}_{X^{\mathrm{an}}}(\mathcal{I}, \mathcal{O}_{X^{\mathrm{an}}})$. For any $x \in X^{\mathrm{an}}$, φ induces a morphism $\varphi_x : \mathcal{I}_x \to \mathcal{O}_{X^{\mathrm{an}},x}$. Let $s_x \in \mathcal{S}_{X^{\mathrm{an}},x} \cap \mathcal{I}_x$, and set $a_x = \varphi_x(s_x)$. There exists a neighborhood U_x of x, such that $s_x \in \mathcal{I}(U_x) \cap \mathcal{S}_{X^{\mathrm{an}}}(U_x), a_x \in \mathcal{O}_{X^{\mathrm{an}}}(U_x)$, and $\varphi(U_x)(s_x) = a_x$. Set $f_x = \frac{a_x}{s_x} \in \mathcal{S}_{X^{\mathrm{an}}}(U_x)^{-1}\mathcal{O}_{X^{\mathrm{an}}}(U_x) \subseteq \mathscr{M}_{X^{\mathrm{an}}}(U_x)$ (the presheaf $\mathcal{S}^{-1}_{X^{\mathrm{an}}}\mathcal{O}_{X^{\mathrm{an}}}$ is separated, so $\mathcal{S}^{-1}_{X^{\mathrm{an}}}\mathcal{O}_{X^{\mathrm{an}}} \subseteq \mathscr{M}_{X^{\mathrm{an}}}$).

Let U_y, U_z be any non-disjoint elements of the cover $(U_x)_{x \in X^{\mathrm{an}}}$ of X^{an}. Then, considering φ is a morphism of sheaves of $\mathcal{O}_{X^{\mathrm{an}}}$-modules, $\varphi(U_y \cap U_z)(s_y \cdot s_z) = s_y \cdot a_z = a_y \cdot s_z$ in $\mathcal{O}_{X^{\mathrm{an}}}(U_y \cap U_z)$. Consequently, $f_{y|U_y \cap U_z} = f_{z|U_y \cap U_z}$ in $\mathscr{M}_{X^{\mathrm{an}}}(U_y \cap U_z)$, implying there exists $f \in \mathscr{M}_{X^{\mathrm{an}}}(X^{\mathrm{an}})$ such that $f_{|U_x} = f_x$ in $\mathscr{M}_{X^{\mathrm{an}}}(U_x)$ for all $x \in X^{\mathrm{an}}$.

We associate to φ the meromorphic function f. Remark that if $f = 0$, then $a_x = 0$ for all x. This implies that for any $\alpha \in \mathcal{I}_x$, $\varphi_x(s_x \cdot \alpha) = s_x \cdot \varphi_x(\alpha) = a_x \cdot \varphi_x(\alpha) = 0$, which, taking into account $s_x \in \mathcal{S}_{X^{\mathrm{an}},x}$ a non-zero-divisor, means that $\varphi_x(\alpha) = 0$. Consequently, $\varphi_x = 0$ for all $x \in X^{\mathrm{an}}$, so $\varphi = 0$. Thus, the map $\psi_{\mathcal{I}} : \mathrm{Hom}_{X^{\mathrm{an}}}(\mathcal{I}, \mathcal{O}_{X^{\mathrm{an}}}) \to \mathscr{M}_{X^{\mathrm{an}}}(X^{\mathrm{an}})$ we have constructed is an embedding.

Remark that the set of coherent ideal sheaves on X^{an} containing locally on X^{an} a section of $\mathcal{S}_{X^{\mathrm{an}}}$ forms a directed set with respect to reverse inclusion (*i.e.* if $\mathcal{I}, \mathcal{J}$ satisfy these properties, then so does $\mathcal{I} \cdot \mathcal{J} \subseteq \mathcal{I}, \mathcal{J}$). Thus, by the paragraph above, there is an embedding $\varinjlim_{\mathcal{I}} \mathrm{Hom}_{X^{\mathrm{an}}}(\mathcal{I}, \mathcal{O}_{X^{\mathrm{an}}}) \hookrightarrow \mathscr{M}_{X^{\mathrm{an}}}(X^{\mathrm{an}})$, where the direct limit is taken with respect to the same kind of coherent ideal sheaves $\mathcal{I}$ as above. Let us show that this embedding is an isomorphism.

For any $f \in \mathscr{M}_{X^{\mathrm{an}}}(X^{\mathrm{an}})$, define the ideal sheaf D_f as follows: for any analytic domain U of X^{an}, set $D_f(U) = \{s \in \mathcal{O}(U) : s \cdot f \in \mathcal{O}_{X^{\mathrm{an}}}(U) \subseteq \mathscr{M}_{X^{\mathrm{an}}}(U)\}$. This is a coherent ideal sheaf on X^{an}. Since $\mathscr{M}_{X^{\mathrm{an}},x} = \mathcal{S}^{-1}_{X^{\mathrm{an}},x}\mathcal{O}_{X^{\mathrm{an}},x}$ for any $x \in X^{\mathrm{an}}$, there exist $s_x \in \mathcal{S}_{X^{\mathrm{an}},x}$ and $a_x \in \mathcal{O}_{X^{\mathrm{an}},x}$ such that $f_x = \frac{a_x}{s_x}$ in $\mathscr{M}_{X^{\mathrm{an}},x}$. Considering $D_{f,x} = \{s \in \mathcal{O}_{X^{\mathrm{an}},x} : s \cdot f_x \in$

$\mathcal{O}_{X^{\mathrm{an}},x}\}$, we obtain that $s_x \in D_{f,x}$, so D_f contains locally on X^{an} a section of $\mathcal{S}_{X^{\mathrm{an}}}$. To $f \in \mathscr{M}_{X^{\mathrm{an}}}(X^{\mathrm{an}})$ we associate the morphism $\varphi_f : D_f \to \mathcal{O}_{X^{\mathrm{an}}}$ which corresponds to multiplication by f (*i.e.* for any open subset U of X^{an}, $D_f(U) \to \mathcal{O}_{X^{\mathrm{an}}}(U), s \mapsto f \cdot s$). Clearly, $\psi_{D_f}(\varphi_f) = f$, implying the embedding $\varinjlim_{\mathcal{I}} \mathrm{Hom}_{X^{\mathrm{an}}}(\mathcal{I}, \mathcal{O}_{X^{\mathrm{an}}}) \hookrightarrow \mathscr{M}_{X^{\mathrm{an}}}(X^{\mathrm{an}})$ is surjective, so an isomorphism.

Let $\mathcal{S}_X$ denote the presheaf on X through which $\mathscr{M}_X$ is defined (see [**48**, Section 7.1.1]). Remark that since A is Noetherian ([**6**, Proposition 2.1.3]), the scheme X is locally Noetherian. Under this assumption, for any $x \in X$, $\mathcal{S}_{X,x}$ is the set of all non-zero-divisors of $\mathcal{O}_{X,x}$ (see [**48**, 7.1.1, Lemma 1.12(c)]). Taking this into account, all the reasoning above does not make use of the fact that X^{an} is an analytic space, and can be applied *mutatis mutandis* to the scheme X and its sheaf of meromorphic functions $\mathscr{M}_X$. Thus, $\mathscr{M}_X(X) \cong \varinjlim_{\mathcal{J}} \mathrm{Hom}_X(\mathcal{J}, \mathcal{O}_X)$, where the direct limit is taken with respect to coherent ideal sheaves $\mathcal{J}$ on X, for which $\mathrm{depth}_{\mathcal{J}_{X,x}} \mathcal{O}_{X,x} > 0$ for all $x \in X$.

Consequently, to show the statement, we need to show that $\varinjlim_{\mathcal{J}} \mathrm{Hom}_X(\mathcal{J}, \mathcal{O}_X) = \varinjlim_{\mathcal{I}} \mathrm{Hom}_{X^{\mathrm{an}}}(\mathcal{I}, \mathcal{O}_{X^{\mathrm{an}}})$, where the direct limits are taken as above.

By [**61**, Annexe A] (which was proven in [**45**] in the case of rigid geometry), there is an equivalence of categories between the coherent sheaves on X and those on X^{an}. Let us show that this induces an equivalence of categories between the coherent ideal sheaves on X and those on X^{an}. To see this, we only need to show that if $\mathcal{F}$ is a coherent sheaf on X such that $\mathcal{F}^{\mathrm{an}}$ is an ideal sheaf on X^{an}, then $\mathcal{F}$ is an ideal sheaf on X. By [**61**, A.1.3], we have a sheaf isomorphism $\mathscr{H}om(\mathcal{F}, \mathcal{O})^{\mathrm{an}} \cong \mathscr{H}om(\mathcal{F}^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$, so $\mathscr{H}om(\mathcal{F}, \mathcal{O})^{\mathrm{an}}$ has a non-zero global section ι corresponding to the injection $\mathcal{F}^{\mathrm{an}} \subseteq \mathcal{O}_{X^{\mathrm{an}}}$. By [**61**, Théorème A.1(i)], $\mathscr{H}om(\mathcal{F}, \mathcal{O})^{\mathrm{an}}(X^{\mathrm{an}}) \cong \mathscr{H}om(\mathcal{F}, \mathcal{O})(X)$. Let $\iota' \in \mathscr{H}om(\mathcal{F}, \mathcal{O})(X)$ denote the element corresponding to ι. Then, the analytification of $\iota' : \mathcal{F} \to \mathcal{O}_X$ is the morphism $\iota : \mathcal{F}^{\mathrm{an}} \hookrightarrow \mathcal{O}_{X^{\mathrm{an}}}$. By flatness of $X^{\mathrm{an}} \to X$, we obtain that $(\ker \iota')^{\mathrm{an}} = \ker \iota'^{\mathrm{an}} = \ker \iota$, so $(\ker \iota')^{\mathrm{an}} = 0$, implying $\ker \iota' = 0$. Consequently, there exists an embedding $\mathcal{F} \hookrightarrow \mathcal{O}_X$, implying $\mathcal{F}$ is an ideal sheaf on X.

If to a coherent ideal sheaf $\mathcal{J}$ on X we associate the coherent ideal sheaf $\mathcal{J}^{\mathrm{an}}$ on X^{an}, then as seen above $\mathrm{Hom}_X(\mathcal{J}, \mathcal{O}_X) \cong \mathrm{Hom}_{X^{\mathrm{an}}}(\mathcal{J}^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$.

Let us also show that a coherent ideal sheaf $\mathcal{J}$ on X satisfies $\mathrm{depth}_{\mathcal{J}_x} \mathcal{O}_{X,x} > 0$ for all $x \in X$ if and only if $\mathrm{depth}_{\mathcal{J}_y^{\mathrm{an}}} \mathcal{O}_{X^{\mathrm{an}},y} > 0$ for all $y \in X^{\mathrm{an}}$. To see this, recall that by [**4**, Proposition 2.6.2], the morphism $\phi : X^{\mathrm{an}} \to X$ is surjective and for any $y \in X^{\mathrm{an}}$, the induced morphism of local rings $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},y}$ is faithfully flat, where $x := \phi(y)$. By [**13**, 1.3, Proposition 6], $\mathrm{depth}_{\mathcal{J}_x} \mathcal{O}_{X,x} = \mathrm{depth}_{\mathcal{J}_x \mathcal{O}_{X^{\mathrm{an}},y}} \mathcal{O}_{X^{\mathrm{an}},y} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}$. At the same time, seeing as the morphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X^{\mathrm{an}},y}$ is flat, $\mathcal{J}_y^{\mathrm{an}} = \mathcal{J}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X^{\mathrm{an}},y} = \mathcal{J}_x \mathcal{O}_{X^{\mathrm{an}},y}$, so $\mathrm{depth}_{\mathcal{J}_x} \mathcal{O}_{X,x} = \mathrm{depth}_{\mathcal{J}_y^{\mathrm{an}}} \mathcal{O}_{X^{\mathrm{an}},y}$.

From the above, $\varinjlim_{\mathcal{J}} \mathrm{Hom}_X(\mathcal{J}, \mathcal{O}_X) = \varinjlim_{\mathcal{I}} \mathrm{Hom}_{X^{\mathrm{an}}}(\mathcal{I}, \mathcal{O}_{X^{\mathrm{an}}})$, where the direct limits are taken with respect to coherent ideal sheaves $\mathcal{J}$ on X (resp. $\mathcal{I}$ on X^{an}), for which $\mathrm{depth}_{\mathcal{J}_x} \mathcal{O}_{X,x} > 0$ for all $x \in X$ (resp. $\mathrm{depth}_{\mathcal{I}_x} \mathcal{O}_{X^{\mathrm{an}},x} > 0$ for all $x \in X^{\mathrm{an}}$). Finally, this implies that $\mathscr{M}_X(X) = \mathscr{M}_{X^{\mathrm{an}}}(X^{\mathrm{an}})$. □

As an immediate consequence of the theorem above, we obtain that for any integral k-affinoid space Z, $\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z) = \mathscr{M}(Z)(T)$.

1.8. Complement II: Analytic curves

Arguably, analytic curves in the sense of Berkovich form the class of Berkovich spaces that is understood the best and on which the most progress has been made. In [**20**], Ducros proves the semi-stable reduction theorem using only Berkovich's theory, which is why one can find an extensive study of these spaces in *loc.cit.* Among other things, the relationship between formal models of an algebraic curve and its Berkovich analytification is explored. Also, it is shown that Berkovich curves have a graph-like structure preserving their homotopy type, which then led to studies of combinatorial nature of these objects. Namely, connections have been made between tropical and Berkovich's geometry, and this is a rising area of research.

We give here a brief overview on analytic curves, focusing mainly on: major results or ones that are needed for the next chapters and which exist in the litterature (with references), and results that we need (which to our knowledge are not in the litterature) for which we provide proofs. There remain certain results of the latter type which will be proven when needed in the next chapters (we consider this to be a more natural presentation).

Throughout this section, let k denote a complete ultrametric field.

1.8.1. The points of an analytic curve. Recall the definition of a good analytic curve (Definition 1.5.37) and the remark below it. The latter justifies the fact that we will simply say *analytic curves* from now on (instead of *good analytic curves*).

As in the case of $\mathbb{A}_k^{1,\mathrm{an}}$, for any analytic curve there is a full classification of points. Let C be a k-analytic curve. In Definition 1.5.21, for any $x \in C$, we introduced the invariants $s_x := \deg \mathrm{tr}_{\widetilde{k}}\widetilde{H(x)}$ and $t_x := \dim_{\mathbb{Q}} |\mathcal{H}(x)^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathcal{H}(x)$ is the completed residue field of x, and $\widetilde{k}, \widetilde{\mathcal{H}(x)}$ are the residue fields of $k, \mathcal{H}(x)$, respectively. By Lemma 1.5.22, for any $x \in C$, $s_x + t_x \leqslant 1$. Let us fix an algebraic closure $\overline{k}$ of k. The absolute value of k extends uniquely to $\overline{k}$. We denote by $\widehat{\overline{k}}$ the completion of $\overline{k}$ with respect to said absolute value.

Definition 1.8.1. The point x is said to be

(1) of *type 1* if $\mathcal{H}(x) \subseteq \widehat{\overline{k}}$; remark that $s_x = t_x = 0$;
(2) of *type 2* if $s_x = 1$;
(3) of *type 3* if $t_x = 1$;
(4) of *type 4* if $s_x = t_x = 0$ and x is not of type 1.

Remark that type 2 and 3 points are the Abhyankar points of C. By Proposition 1.5.24, for any affinoid domain in C, the points in its Shilov boundary are of type 2 or 3. Also, rigid points are type 1 points (but not necessarily vice-versa unless k is algebraically closed or trivially valued). Definition 1.8.1 is compatible with the terminology we introduced in Subsection 1.2.4.

Here are a few results that we will need for the next chapters.

Lemma 1.8.2. *Let $f : C_1 \to C_2$ be a finite morphism of k-analytic curves. If $x \in C_2$ is a type i point, then for any $y \in f^{-1}(x)$, y is a type i point in $C_1, i = 1, 2, 3, 4$.*

Proof. This is immediate from the fact that for any $y \in C_1$, the morphism $\mathcal{O}_{C_2, f(y)} \to \mathcal{O}_{C_1, y}$ is finite. □

LEMMA 1.8.3 ([**18**, Lm. 4.4.5]). *Let C be a reduced k-analytic curve. Then, for any $x \in C$ that is not rigid, $\mathcal{O}_{C,x}$ is a field.*

LEMMA 1.8.4. *Let C be a normal k-analytic curve. Then, for any $x \in C$, $\mathcal{O}_{C,x}$ is either a field or a discrete valuation ring. Moreover, $\mathcal{O}_{C,x}$ is a discrete valuation ring if and only if x is a rigid point of C.*

PROOF. By Corollary 3.2.9 of [**18**], $\dim \mathcal{O}_{C,x} \leqslant \dim_x(C) = 1$. If $\dim \mathcal{O}_{C,x} = 0$, then $\mathcal{O}_{C,x}$ is a normal local ring of Krull dimension 0, so it is a field. If $\dim \mathcal{O}_{C,x} = 1$, then $\mathcal{O}_{C,x}$ is a Noetherian normal local ring of Krull dimension 1, so it is a discrete valuation ring.

To show the last part of the statement, by Lemma 1.8.3, it suffices to prove that if $\mathcal{O}_{C,x}$ is a field for some $x \in C$, then x is not a rigid point. This is a consequence of [**18**, Corollary 3.2.9]. □

LEMMA 1.8.5. *Let C be an irreducible k-analytic curve. Then, any proper Zariski closed subset of C is discrete and contains only rigid points.*

PROOF. Recall that a Zariski closed subset of C is a k-analytic space, so it makes sense to speak of its dimension.

Let Z be a proper Zariski closed subset of C. By [**21**, Cor. 4.14], $\dim Z = 0$. Hence, for any $x \in Z$, $\dim_x(Z) = 0$, which by [**22**, 1.21] is equivalent to the fact that x is a rigid and isolated point. □

LEMMA 1.8.6. *Let k be a trivially valued field. Let C be an integral projective k-analytic curve. Then, C contains exactly one type 2 point x, and $\mathcal{O}_x = \mathscr{M}(C)$.*

PROOF. By Proposition 1.6.13, the curve C is the analytification of an integral projective algebraic curve $\mathcal{C}$ over k. Let $\pi : C \to \mathcal{C}$ denote the canonical morphism.

Let U be any affine open subset of $\mathcal{C}$. Then, by Lemma 1.8.5, all type 2 points are contained in $\pi^{-1}(U) = U^{\mathrm{an}}$. Seeing as U is affine, U^{an} is the set of multiplicative seminorms on $\mathcal{O}_{\mathcal{C}}(U)$ which extend the absolute value of k. Remark that the trivial norm on $\mathcal{O}_{\mathcal{C}}(U)$ determines a unique point of U^{an}.

Let $x \in U$ be any type 2 point. Then, seeing as $\sqrt{|k^\times|} = \{1\}$, we obtain that $\sqrt{|\mathcal{H}(x)^\times|} = |\mathcal{H}(x)^\times| = \{1\}$, *i.e.* x induces the trivial norm on $\mathcal{O}_{\mathcal{C}}(U)$. Consequently, by the description of the points of U^{an}, there is exactly one type 2 point in U^{an}, and hence in C. By Lemma 1.4.12, $\mathcal{H}(x)$ is the completion of the residue field $\kappa(x)$ with respect to the trivial norm, implying $\kappa(x) = \mathcal{H}(x)$. As x is of type 2, $\kappa(x) = \mathcal{O}_x$.

By Remark 1.6.7, $\mathcal{H}(x)$ is the completion of Frac $\mathcal{O}_{\mathcal{C}}(U)$ with respect to the trivial norm $|\cdot|_x$, so $\mathcal{H}(x) =$ Frac $\mathcal{O}_{\mathcal{C}}(U)$. As C is irreducible, $\mathscr{M}(C) =$ Frac $\mathcal{O}_{\mathcal{C}}(U) = \mathcal{H}(x)$. □

Type 3 points are crucial for the constructions we will make, which is why it is very important to know when they exist.

PROPOSITION 1.8.7. *Let C be a k-analytic curve. Type 3 points exist in C if and only if $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$, in which case they are dense.*

Moreover, the family of connected affinoid domains with only type 3 points in their topological boundaries forms a basis of neighborhoods of the Berkovich topology on C.

PROOF. Since curves are good analytic spaces, we may assume that C is an affinoid space. Let U be an open neighborhood of x in C. There exists an open neighborhood

of x in U given by $\{|f_i| < r_i, |g_j| > s_j : i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$, where f_i, g_j are analytic functions on C and $r_i, s_j \in \mathbb{R}_{>0}$.

Let $r_i', s_j' \in \mathbb{R}_{>0} \backslash \sqrt{|k^\times|}$, such that $r_i' < r_i$ and $s_j' > s_j$, and $|f_i(x)| < r_i', |g_j(x)| > s_j'$, for all i and j. Set $V = \{|f_i| \leqslant r_i', |g_j| \geqslant s_j'\}$. It is an affinoid domain of C and a neighborhood of x contained in U.

As $\{|f_i| < r_i', |g_j'| > s_j\}$ is open, it is contained in $\mathrm{Int}(V)$, so $\partial V \subseteq \bigcup_{i=1}^n \{|f_i| = r_i'\} \cup \bigcup_{j=1}^m \{|g_j| = s_j'\}$. Let $y \in \bigcup_{i=1}^n \{|f_i| = r_i'\} \cup \bigcup_{j=1}^m \{|g_j| = s_j'\}$. Since there exists an analytic function f on C such that $|f(y)| \notin \sqrt{|k^\times|}$, the point y is of type 3, implying that the boundary of V contains only type 3 points. □

1.8.2. Boundaries in dimension 1.

LEMMA 1.8.8. *Let V be a k-affinoid curve. The following sets are equal:*

(1) *the Berkovich boundary $\partial_B(V)$ of V;*
(2) *the Shilov boundary $\Gamma(V)$ of V.*

PROOF. If V is strictly affinoid, this is [**69**, Lemma 2.3]. The proof can be extended to the general case by replacing classical reduction with Temkin's graded reduction (see Propositions 3.3 and 3.4 of [**67**]). □

REMARK 1.8.9. If C is a k-analytic curve and U an analytic domain of C, then by Theorem 1.5.27, $\partial U \subseteq \partial_B(U)$.

PROPOSITION 1.8.10. *Let C be a k-analytic curve such that $\partial_B(C) = \emptyset$. Let V be an affinoid domain of C. The three following sets coincide:*

(1) *the topological boundary ∂V of V in C;*
(2) *the Berkovich relative boundary $\partial_B(V/C)$ of V in C;*
(3) *the Shilov boundary $\Gamma(V)$ of V.*

PROOF. By Theorem 1.5.27(2), $\partial_B(V/C) = \partial V$. By Theorem 1.5.27(1), since C is boundaryless, $\partial_B(V/C) = \partial_B(V)$. Finally, in view of Lemma 1.8.8, $\partial V = \partial_B(V/C) = \Gamma(V)$. □

In particular, the results above tell us that the topological (and Berkovich) boundary of any affinoid domain of an analytic curve is finite. As an immediate consequence:

COROLLARY 1.8.11. *Let C be a k-analytic curve. For any affinoid domain U in C, $\overline{(\mathrm{Int}\ U)} = U$.*

The following is a direct consequence of [**60**, Lemme 4.4] and Proposition 1.3.14.

PROPOSITION 1.8.12. *Let V be a k-affinoid curve. Then, V is strict if and only if $\Gamma(V)$ contains only type 2 points.*

1.8.3. Some general results on curves. Informally, a *real graph* is an infinite graph where there can be "infinite branching" even locally (for example, the tree corresponding to $\mathbb{A}_k^{1,\mathrm{an}}$ is a real graph). The precise notion has many nice properties, *e.g.* it is locally uniquely arcwise-connected. For the precise definition, see [**20**, 1.3.1].

THEOREM 1.8.13 ([**20**, Théorème 3.5.1]). *An analytic curve is a real graph.*

Type 3 points are the points of arity 2 in the real graph associated to the analytic curve. Type 1 and 4 points have arity 1, and type 2 points infinity.

Here is a useful (for the next chapters) application of this.

PROPOSITION 1.8.14. *Let C be a compact k-analytic curve. For any $x, y \in C$, there exist only finitely many arcs in C connecting x and y.*

PROOF. By [**20**, Théorème 3.5.1], C is a real graph. By [**20**, 1.3.13], for any $z \in C$, there exists an open neighborhood U_z of z such that: (1) U_z is uniquely arcwise-connected; (2) the closure $\overline{U_z}$ of U_z in C is uniquely arcwise-connected; (3) the boundary ∂U_z is finite, implying in particular $\partial U_z = \partial \overline{U_z}$. Seeing as C is compact, the finite open cover $\{U_z\}_{z \in C}$ admits a finite subcover $\mathcal{U} := \{U_1, U_2, \ldots, U_n\}$. Set $S := \bigcup_{i=1}^{n} \partial U_i$. This is a finite subset of C.

Let x, y be any two points of C. Let $\gamma : [0,1] \to C$ be any arc in C connecting x and y. Set $S_\gamma := S \cap \gamma([0,1]) \backslash \{x, y\}$. It is a finite (possibly empty) subset of C. For any $\alpha \in S_\gamma$, there exists a unique $a \in [0,1]$ such that $\gamma(a) = \alpha$. This gives rise to an ordering of the points of S_γ. Set $S_\gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ such that the order of the points is the following: $\alpha_1 < \alpha_2 < \cdots < \alpha_m$ (meaning $\gamma^{-1}(\alpha_1) < \gamma^{-1}(\alpha_2) < \cdots < \gamma^{-1}(\alpha_m)$). To the arc γ we associate the finite sequence $\overline{\gamma} := (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of points of S_γ. Set $\alpha_0 = x$, and $\alpha_{m+1} = y$.

For any $i \in \{0, 1, \ldots, m+1\}$, set $\gamma_i := \gamma([\gamma^{-1}(\alpha_i), \gamma^{-1}(\alpha_{i+1})])$. This is an arc in C connecting α_i and α_{i+1}. By construction, for any i, $\gamma_i \cap S \subseteq \{\alpha_i, \alpha_{i+1}\}$. Remark that $\gamma([0,1]) = \bigcup_{i=0}^{m+1} \gamma_i$.

Let us show that for any $i \in \{0, 1, \ldots, m\}$, there exists a unique arc $[\alpha_i, \alpha_{i+1}]_0$ in C connecting α_i and α_{i+1} such that $[\alpha_i, \alpha_{i+1}]_0 \cap S \subseteq \{\alpha_i, \alpha_{i+1}\}$. Let $[\alpha_i, \alpha_{i+1}]$ be any such arc (the existence is guaranteed by the paragraphs above). Let $j \in \{1, 2, \ldots, n\}$ be such that $[\alpha_i, \alpha_{i+1}] \cap U_j \neq \emptyset$. Let $z \in [\alpha_i, \alpha_{i+1}] \cap U_j$; since $[\alpha_i, \alpha_{i+1}] \cap U_j$ is open in $[\alpha_i, \alpha_{i+1}]$, we may choose z such that $z \notin \{\alpha_i, \alpha_{i+1}\}$. Let us denote by $[\alpha_i, z]$, resp. $[z, \alpha_{i+1}]$ the arc in C induced by $[\alpha_i, \alpha_{i+1}]$ connecting α_i and z, resp. z and α_{i+1}. Clearly, $[\alpha_i, \alpha_{i+1}] = [\alpha_i, z] \cup [z, \alpha_i]$.

Suppose there exists $u \in [\alpha_i, \alpha_{i+1}] \backslash \overline{U_j}$. Again, as $[\alpha_i, \alpha_{i+1}] \backslash \overline{U_j}$ is open in $[\alpha_i, \alpha_{i+1}]$, we may assume that $u \notin \{\alpha_i, \alpha_{i+1}\}$. Without loss of generality, let us suppose that $u \in [\alpha_i, z]$. Let $[\alpha_i, u]$, resp. $[u, z]$, be the induced arcs connecting α_i and u, resp. u and z. Seeing as $z \in U_j$ and $u \notin U_j$, $[z, u] \cap \partial U_j \neq \emptyset$. At the same time, $\emptyset \neq [z, u] \cap \partial U_j \subseteq [\alpha_i, \alpha_{i+1}] \cap \partial U_j \subseteq [\alpha_i, \alpha_{i+1}] \cap S \subseteq \{\alpha_i, \alpha_{i+1}\}$, which contradicts the injectivity of $[\alpha_i, \alpha_{i+1}]$.

Consequently, $[\alpha_i, \alpha_{i+1}] \subseteq \overline{U_j}$. Seeing as $\overline{U_j}$ is uniquely arcwise-connected, we obtain that the arc $[\alpha_i, \alpha_{i+1}]$ in C connecting α_i and α_{i+1}, and satisfying the property $[\alpha_i, \alpha_{i+1}] \cap S \subseteq \{\alpha_i, \alpha_{i+1}\}$, is unique. Thus, $\gamma_i = [\alpha_i, \alpha_{i+1}]$, and the arc γ is uniquely determined by its associated ordered sequence $\overline{\gamma}$.

Seeing as S is finite, the set of all finite sequences $(\beta_l)_l$ over S such that $\beta_{l'} \neq \beta_{l''}$ whenever $l' \neq l''$, is also finite. Consequently, the set of arcs in C connecting x and y is finite. □

The following result is crucial for our work in the next chapters. It is among the main reasons why the case of curves is the most pleasant one to treat. The proof is obtained by applying Théorème 6.1.3 of [**20**].

THEOREM 1.8.15. *(1) An irreducible compact k-analytic curve is either projective or an affinoid space.*

(2) The finite union of affinoid domains in an irreducible k-analytic curve is the curve itself or an affinoid domain.

PROOF. (1) Let C be an irreducible compact k-analytic curve. Then, if $\partial_B(C) \neq \emptyset$, C is an affinoid curve by [**20**, Théorème 6.1.3]. If $\partial_B(C) = \emptyset$, then the canonical map $C \to \mathcal{M}(k)$ is proper, so C is a projective curve.

(2) Let C be an irreducible k-analytic curve. It suffices to show the result for two affinoid domains. Let U_1, U_2 be affinoid domains in C. Set $X = U_1 \cup U_2$. This is an analytic domain in the sense of [**4**, 1.3] by definition. As X is an analytic curve, by [**20**, Proposition 3.3.7], it is a good k-analytic space. It can be shown that X is also an analytic domain of C in the sense of Definition 1.5.5 (see [**4**, pg. 23]). If $\partial_B(X) = \emptyset$, then by Remark 1.8.9, $\partial X = \emptyset$. As X is a compact, hence closed, subset of the connected curve C, this is possible if and only if $X = C$. In particular, C is projective (as $\partial_B(C) = \emptyset$).

Suppose $\partial_B(X) \neq \emptyset$. Let X_0 be an irreducible component of X, meaning it is a Zariski closed subset of X, and the inclusion $X_0 \to X$ is a closed immersion, hence finite. By Proposition 1.5.31 and Theorem 1.5.27, $\partial_B(X_0) = X_0 \backslash \mathrm{Int}_B(X)$, meaning $\partial_B(X_0) \subseteq \partial_B(X)$.

If $\partial_B(X_0) = \emptyset$, by [**20**, 3.2.3], $X_0 = C$, so $X = C$, and X is proper, which is in contradiction with the assumption $\partial_B(X) \neq \emptyset$. Thus, for any irreducible component X_0 of X, $\emptyset \neq \partial_B(X_0) \subseteq \partial_B(X)$. We conclude by [**20**, Théorème 6.1.3] that X is an affinoid space which is an analytic domain of C, hence an affinoid domain in C. □

Here is another way to obtain affinoid domains on a curve. This is again an application of [**20**, Théorème 6.1.3].

LEMMA 1.8.16. *Let C be a normal irreducible projective k-analytic curve. Let U be a connected affinoid domain of C such that its boundary contains only type 3 points. Then, for any $S \subseteq \partial U$, $U \backslash S$ is connected.*

PROOF. Suppose that C is generically quasi-smooth. Since ∂S contains only type 3 points, all of the points of S are quasi-smooth in C.

Let $x, y \in \mathrm{Int}\, U$. Since U is connected, there exists an arc $[x, y] \subseteq U$ connecting x and y. Let $z \in S$. We aim to show that $z \notin [x, y]$, implying $[x, y] \subseteq U \backslash S$, and thus the connectedness of $U \backslash S$.

By [**20**, Théorème 4.5.4], there exists an affinoid neighborhood V of z in U such that it is a closed virtual annulus, and its Berkovich boundary is $\partial_B(V) = \{z, u\}$ for some $u \in U$. We may assume that $x, y \notin V$. Since V is an affinoid domain in U, by Theorem 1.5.27, the topological boundary $\partial_U V$ of V in U is a subset of $\partial_B(V) = \{z, u\}$. Since V is a neighborhood of z, $\partial_U V = \{u\}$.

Suppose $z \in [x, y]$. Then, we could decompose $[x, y] = [x, z] \cup [z, y]$. Since $x, y \notin V$, and $z \in V$, the sets $[x, z] \cap \partial_U V$, $[z, y] \cap \partial_U V$ are non-empty, thus implying u is contained in both $[x, z]$ and $[z, y]$, which contradicts the injectivity of $[x, y]$. Consequently, $U \backslash S$ is connected.

Let us get back to the general case. Let C^{alg} denote the algebraization of C (*i.e.* the normal irreducible projective algebraic curve over k whose analytification is C). Since it is normal, there exists a finite surjective morphism $C^{\mathrm{alg}} \to \mathbb{P}^1_k$. This induces a finite field extension $k(T) \hookrightarrow k(C^{\mathrm{an}}) = \mathscr{M}(C)$ of their function fields. Let F denote the separable closure of $k(T)$ in $k(C)$. Then, there exists an irreducible normal algebraic curve X over k such that $k(X) = F$. Seeing as $k(T) \hookrightarrow k(C)$ is separable, the induced morphism $X \to \mathbb{P}^1_k$ is generically étale, so X is generically smooth. On the other hand, the finite field extension $k(C)/F$ is purely inseparable, implying the corresponding finite morphism $C^{\mathrm{alg}} \to X$ is a homeomorphism.

Finally, the analytification X^{an} is a normal irreducible projective k-analytic curve that is generically quasi-smooth, and there is a finite morphism $f : C \to X^{\mathrm{an}}$ that is a homeomorphism. By [**20**, Proposition 4.2.14], $f(U)$ is a connected proper closed analytic domain of X^{an}. By [**20**, Théorème 6.1.3], $f(U)$ is an affinoid domain of X^{an}. Clearly, $\partial f(U) = f(\partial U)$. Let $S \subseteq \partial U$, and set $S' = f(S)$. As shown above, $f(U)\backslash S'$ is connected. Consequently, $U\backslash S$ is connected. □

COROLLARY 1.8.17. *Let C be a normal irreducible k-analytic curve. Let U be an affinoid domain in C containing only type 3 points in its boundary. If $Int(U) \neq \emptyset$, then $(Int\ U)^c$ is an affinoid domain in C containing only type 3 points in its boundary.*

PROOF. Seeing as U is an affinoid domain, it has a finite number of connected components, and by Corollary 1.4.19, they are all affinoid domains in C. Furthermore, each of the connected components of U contains only type 3 points in its boundary. Consequently, by Lemma 1.8.16, $\mathrm{Int}(U)$ has only finitely many connected components. Thus, by [**20**, Proposition 4.2.14], $(\mathrm{Int}\ U)^c$ is a closed proper analytic domain of C. By [**20**, Théorème 6.1.3], it is an affinoid domain in C. □

Until the end of this part, we briefly mention some of the cornerstones of the theory of Berkovich curves.

Triangulations. In [**20**], Ducros introduces the notions of *virtual discs* and *virtual annuli* (resp. *closed virtual discs* and *closed virtual annuli*), which are generalizations of discs and annuli (resp. closed discs and annuli) as we saw them in Examples 1.4.25 and 1.5.13; more precisely, a (closed) virtual disc, resp. annulus, becomes isomorphic to a disjoint union of (closed) discs, resp. annuli, after a base change. The Berkovich boundary of a virtual disc (resp. annulus) is a single point (resp. a set of 2 points), and the same remains true if we replace *disc* (resp. *annulus*) by *closed disc* (resp. *closed annulus*). Using these spaces, Ducros provides bases of neighborhoods for all of the types of points of a quasi-smooth Berkovich curve. In particular, he shows that a basis of neighborhoods of type 3 points of a quasi-smooth curve is given by closed virtual annuli.

In [**20**], the notion of *triangulation* of a quasi-smooth k-analytic curve is introduced; it is a locally finite set of type 2 and 3 points of the curve satisfying certain topological and analytic properties (see [**20**, 5.1.13]). To any triangulation we can associate a locally finite graph, called a *skeleton* of the curve, and there is a strong deformation retraction from the curve to this skeleton. If T is a triangulation of a k-analytic curve C, then the connected components of $C\backslash T$ are virtual discs and virtual annuli.

The specialization map. There is a more thorough treatment of the following content in Section 3.3. We give here a brief overview (without the relevant references, which will be given in Section 3.3).

For a complete ultrametric field k, let k° denote its ring of integers. Given an adequate formal model $\mathscr{C}$ of an algebraic curve over k°, there is a notion of an *analytic generic fiber* C of $\mathscr{C}$, where C is a k-analytic curve. This is the analytification of the algebraic generic fiber of $\mathscr{C}$ if the latter is projective. In general, there exists a *specialization map* (sometimes called *reduction map*, which we will avoid because of Subsection 1.4.7) $C \to \mathscr{C}_s$, where $\mathscr{C}_s$ is the special fiber of $\mathscr{C}$. The specialization map is anti-continuous.

Let C be a normal irreducible projective k-analytic curve. Let C^{alg} be the algebraic curve over k such that $(C^{\mathrm{alg}})^{\mathrm{an}} = C$ (recall Proposition 1.6.13). Ducros showed in [**20**] that under certain conditions, for any finite set of type 2 points S of C, there exists a model $\mathscr{C}$ of C^{alg} over k° such that the corresponding specialization map induces a bijection between

S and the generic points of the irreducible components of the special fiber of $\mathscr{C}$. We will need this connection between fibers when comparing the statements of Chapter 3 to those of [**34**]. This result is shown more generally for certain subsets S of C which are called *vertex sets* (or in French, *ensemble sommitaux*, see [**20**, 6.3.17] for the definition).

1.8.4. Additional properties of $\mathbb{P}^{1,\mathrm{an}}$. Let k be a complete ultrametric field.

Recall that (Definition 1.5.41) that a topological space X is said to be *uniquely arcwise-connected* if for any $x, y \in X$, there exists a unique arc in X connecting x and y.

PROPOSITION 1.8.18. *The analytic affine line $\mathbb{A}^{1,\mathrm{an}}_k$ is locally compact, uniquely arcwise-connected, and contractible.*

PROPOSITION 1.8.19. *The projective analytic line $\mathbb{P}^{1,\mathrm{an}}_k$ is compact, uniquely arcwise-connected, and contractible.*

For a proof, see [**20**, 3.4.20].

For any $x, y \in \mathbb{P}^{1,\mathrm{an}}_k$, let us denote by $[x, y]$ the unique arc connecting them. The next few properties are a direct consequence of Proposition 1.8.19.

LEMMA 1.8.20. *Let $A \subseteq \mathbb{P}^{1,\mathrm{an}}_k$. Then, A is connected if and only if for any $x, y \in A$, $[x, y] \subseteq A$. Furthermore, the intersection of any two connected subsets of $\mathbb{P}^{1,\mathrm{an}}_k$ is connected.*

LEMMA 1.8.21. *Let U, V be two non-disjoint connected affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_k$, such that they have disjoint interiors. Then, $U \cap V$ is a single point.*

PROOF. Since $U \cap V = \partial U \cap \partial V$, it is a finite set of points. At the same time, by Lemma 1.8.20, $U \cap V$ is connected, so it must be a single point. □

We will now give a more precise description of points in $\mathbb{A}^{1,\mathrm{an}}_k$ (or equivalently, $\mathbb{P}^{1,\mathrm{an}}_k$).

Let us fix an algebraic closure $\overline{k}$ of k. There is a canonical surjective open continuous morphism $\varphi : \mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}} \to \mathbb{A}^{1,\mathrm{an}}_k$ induced by the inclusion $k[T] \hookrightarrow \widehat{\overline{k}}[T]$. Let G denote $\mathrm{Gal}(k^s/k)$, where k^s is a separable closure of k. Then, by Proposition 1.1.43, G acts on $\mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$, and φ induces a homeomorphism $\mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}}/G \cong \mathbb{A}^{1,\mathrm{an}}_k$. Remark that for any $a \in \overline{k}$, $r', r \in \mathbb{R}_{\geqslant 0}$, the elements of G act on $\{x \in \mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}} : r' \leqslant |T - a|_x \leqslant r\}$ by sending it homeomorphically to $\{x \in \mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}} : r' \leqslant |T - b|_x \leqslant r\}$, where b is a conjugate of a.

LEMMA 1.8.22. *Let $P(T)$ be a monic irreducible polynomial over k. Let $\alpha \in \overline{k}$ be such that $P(\alpha) = 0$. Then, for any $s, s' \in \mathbb{R}_{\geqslant 0}$ such that $s' \leqslant s$, there exist unique $r, r' \in \mathbb{R}_{\geqslant 0}$ such that $r' \leqslant r$, satisfying: $\varphi(\{x \in \mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}} : r' \leqslant |T - \alpha|_x \leqslant r\}) = \{y \in \mathbb{A}^{1,\mathrm{an}}_k : s' \leqslant |P|_y \leqslant s\}$. Moreover, $s > 0$ (resp. $s' > 0$) if and only if $r > 0$ (resp. $r' > 0$). Consequently,*

$$\varphi^{-1}(\{y \in \mathbb{A}^{1,\mathrm{an}}_k : s' \leqslant |P|_y \leqslant s\}) = \bigcup_{\substack{\alpha \in \overline{k} \\ P(\alpha) = 0}} \{x \in \mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}} : r' \leqslant |T - \alpha|_x \leqslant r\}.$$

PROOF. Let us suppose $s' = 0$. The general statement can be shown using the same type of argument.

For any root β of $P(T)$ over $\overline{k}$, let n_β be its multiplicity, so that $P(T) = \prod_{P(\beta)=0}(T - \beta)^{n_\beta}$. The function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}, t \mapsto t^{n_\alpha} \prod_{P(\beta)=0, \alpha \neq \beta} \max(t, |\alpha - \beta|)^{n_\beta}$, is strictly increasing and continuous, so bijective. Thus, for any $s \in \mathbb{R}_{\geqslant 0}$, there exists a unique positive real

number $r \in \mathbb{R}_{\geqslant 0}$, such that $f(r) = s$. Moreover, $r = 0$ if and only if $s = 0$. Remark that $r^{\deg P} \leqslant s$.

Let $x \in \mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$ be such that $|T-\alpha|_x \leqslant r$. Then, $|P(T)|_{\varphi(x)} = \prod_{P(\beta)=0} |T-\beta|_x^{n_\beta} \leqslant r^{n_\alpha} \prod_{P(\beta)=0,\alpha\neq\beta} |T-\beta|_x^{n_\beta}$. Also, since $|\cdot|_x$ is non-Archimedean, we obtain that $|T-\beta|_x \leqslant \max(|T-\alpha|_x, |\alpha-\beta|_x) \leqslant \max(r, |\alpha-\beta|)$, implying $|P(T)|_{\varphi(x)} \leqslant s$. Hence, $\varphi(x) \in \{y : |P|_y \leqslant s\}$.

Let $y \in \mathbb{A}^{1,\mathrm{an}}_k$ be such that $|P(T)|_y \leqslant s$. Let $x \in \varphi^{-1}(y)$. Suppose that for any root β of $P(T)$ we have $|T-\beta|_x > r$. Then, $s \geqslant |P(T)|_y = \prod_{P(\beta)=0} |T-\beta|_x^{n_\beta} > r^{\deg P}$, which is impossible. Thus, there exists a root β_0 of P in $\overline{k}$ such that $|T-\beta_0|_x \leqslant r$. By the Galois action, there exists $x' \in \varphi^{-1}(y)$ such that $|T-\alpha|_{x'} \leqslant r$.

The last part of the statement is a direct consequence of the Galois action on $\mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$. □

Corollary 1.8.23. *Let $P(T)$ be an irreducible polynomial over k. Let $\alpha \in \overline{k}$ be such that $P(\alpha) = 0$. Let s, s' be positive real numbers such that $s \geqslant s'$, and r, r', with $r \geqslant r'$, the corresponding positive real numbers obtained by applying Lemma 1.8.22.*

Then, the (topological or Shilov or Berkovich) boundary of $\{x : |P|_x \leqslant s\}$ (resp. $\{x : s' \leqslant |P|_x \leqslant s\}$) is $\{\varphi(\eta_{\alpha,r})\}$ (resp. $\{\varphi(\eta_{\alpha,r'}), \varphi(\eta_{\alpha,r})\}$).

Proof. We may, without loss of generality, assume that P is monic (recall Lemma 1.4.2). Seeing as φ is open and continuous, by Lemma 1.8.22, $\varphi^{-1}(\partial\{x : |P|_x \leqslant s\}) = \partial\bigcup_{P(\beta)=0}\{y : |T-\beta|_y \leqslant r\}$. By [**20**, 3.6.4.1], for any $\beta \in \overline{k}$ such that $P(\beta) = 0$, the Shilov boundary of $\{y : |T-\beta|_y \leqslant r\}$ is $\{\eta_{\beta,r}\}$. Thus, $\varphi^{-1}(\partial\{x : |P|_x \leqslant s\}) = \{\eta_{\beta,r} \in \mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}} : P(\beta) = 0\}$.

Seeing as $\eta_{\beta,r}$ is the topological boundary of $\{x : |T-\beta|_x \leqslant r\}$ in $\mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$, we obtain that $\varphi(\eta_{\beta,r}) = \varphi(\eta_{\alpha,r})$ for any $\beta \in \overline{k}$ such that $P(\beta) = 0$. Finally, $\partial\{x : |P|_x \leqslant s\} = \{\varphi(\eta_{\alpha,r})\}$.

We can conclude seeing as $\{x : |P|_x \leqslant s\}$ and $\{y : |T|_y \leqslant r\}$ are affinoid domains in $\mathbb{A}^{1,\mathrm{an}}_{\widehat{\overline{k}}}$, resp. $\mathbb{A}^{1,\mathrm{an}}_k$, and taking into account Proposition 1.8.10 and Example 1.5.2.

The other case is shown similarly. □

Definition 1.8.24. Let $P \in k[T]$ be any irreducible polynomial. Recall that we denote by $\eta_{P,0}$ the only (type 1) point of $\mathbb{A}^{1,\mathrm{an}}_k$ for which $|P| = 0$. For $s \in \mathbb{R}_{>0}$, we will denote by $\eta_{P,s}$ the point of $\mathbb{A}^{1,\mathrm{an}}_k$ that is the Shilov boundary of the affinoid domain $\{|P| \leqslant s\} \subseteq \mathbb{A}^{1,\mathrm{an}}_k$.

Proposition 1.8.25. *For any point $\eta \in \mathbb{A}^{1,\mathrm{an}}_k$ of type 2 or 3, there exist an irreducible polynomial $P \in k[T]$ and $r \in \mathbb{R}_{>0}$, such that $\eta = \eta_{P,r}$. Then, $|P|_\eta = r$ and:*

1. *$r \in \sqrt{|k^\times|}$ if and only if η is a type 2 point;*
2. *$r \notin \sqrt{|k^\times|}$ if and only if η is a type 3 point, in which case η is the only element of $\mathbb{A}^{1,\mathrm{an}}_k$ for which $|P| = r$.*

Proof. This was shown to be true in Subsection 1.2.2 if k is trivially valued, so let us assume that that is not the case.

We recall that the projective line $\mathbb{P}^{1,\mathrm{an}}_k$ can be obtained by adding a rigid point ∞ to $\mathbb{A}^{1,\mathrm{an}}_k$. Let A be a connected component of $\mathbb{P}^{1,\mathrm{an}}_k \backslash \{\eta\}$ that doesn't contain ∞. In particular, $A \subseteq \mathbb{A}^{1,\mathrm{an}}_k$. Seeing as it is open, by Proposition 1.5.11, there exists a rigid point η_0 in A. By Lemma 1.2.4, there exists a unique irreducible polynomial $P \in k[T]$, such that $\eta_0 = \eta_{P,0}$. Then, $\eta \in [\eta_{P,0}, \infty]$.

Let ϕ be the finite morphism $\mathbb{P}^{1,\mathrm{an}}_k \to \mathbb{P}^{1,\mathrm{an}}_k$ determined by the map $k[T] \to k[T]$, $T \mapsto P(T)$. Seeing as $\phi(\eta_{P,0}) = \eta_{T,0}$ and $\phi(\infty) = \infty$, $[\eta_{P,0}, \infty]$ is mapped by ϕ to $[\eta_{T,0}, \infty]$. Set $\eta' = \phi(\eta)$. The arc connecting $\eta_{T,0}$ to ∞ in $\mathbb{P}^{1,\mathrm{an}}_k$ is $\{\eta_{T,s} : s \in \mathbb{R}_{\geqslant 0}\} \cup \{\infty\}$. For any $s \geqslant 0, |T|_{\eta_{T,s}} = s$, and if $\eta_{T,s}$ is a type 3 point, then it is the only one in $\mathbb{P}^{1,\mathrm{an}}_k$ for which $|T| = s$. Furthermore, $\eta_{T,s}$ is a type 2 (resp. type 3) point if and only if $s \in \sqrt{|k^\times|}$ (resp. $s \notin \sqrt{|k^\times|}$).

Thus, there exists $r > 0$, such that $\eta' = \eta_{T,r}$. Since $\phi(\eta) = \eta_{T,r}$, by construction, $\eta = \eta_{P,r}$ and $|P|_{\eta_{P,r}} = r$. Seeing as a finite morphism preserves the type of the point (*i.e.* $\eta_{T,r}$ is a type 2 (resp. 3) point if and only if $\eta_{P,r}$ is so), we obtain (1) and the first part of (2).

To prove the second part of (2), we need to show that if $r \notin \sqrt{|k^\times|}$, $\eta_{P,r}$ is the only point in $\mathbb{A}^{1,\mathrm{an}}_k$ for which $|P| = r$. Since P is irreducible, by [**20**, 3.4.24.3], $|P|$ is strictly increasing in $[\eta_{P,0}, \infty)$, and locally constant elsewhere. Hence, $\eta_{P,r}$ is the only point in $[\eta_{P,0}, \infty)$ for which $|P| = r$, and since it is a type 3 point (*i.e.* $\mathbb{A}^{1,\mathrm{an}}_k$ has exactly two connected components), it is the only such point in $\mathbb{A}^{1,\mathrm{an}}_k$. □

Remark 1.8.26. As we saw in the proof of Proposition 1.8.24, the unique arc connecting $\eta_{P,0}$ to ∞ in $\mathbb{P}^{1,\mathrm{an}}_k$ is $\{\eta_{P,s} : s \in \mathbb{R}_{\geqslant 0}\} \cup \{\infty\}$.

CHAPTER 2

Patching

The purpose of this chapter is to develop the necessary tools for proving a "matrix decomposition" statement generalizing [**34**, Theorem 2.5] and applicable to a Berkovich framework. To do this, we follow along the lines of proof and reasoning of [**34**, Section 2.1] making the necessary adjustements.

We work over a general formal setup (Setting 2.1.5), which is partly why the content of this chapter is of very technical nature. It will be shown in the next parts of this manuscript that the hypotheses we adopt here are satisfied in a very natural way in Berkovich's geometry. After showing the main result (Theorem 2.1.10), we focus on a somewhat more restrictive formal setup (which is realised by curves) over which we prove that patching is possible.

2.1. The general case

SETTING 2.1.1. Let k be a complete non-trivially valued ultrametric field. Let R be an integral domain containing k, endowed with a non-Archimedean (submultiplicative) norm $|\cdot|_R$. Suppose that for any $a \in R$ and $b \in k$, $|ab|_R = |a|_R \cdot |b|$.

Remark that the last assumption implies the norm $|\cdot|_R$ extends $|\cdot|$.

For $p \in \mathbb{N}$, and indeterminates $X_1, \ldots, X_p$, let us use the notation $\underline{X}$ for the p-tuple $(X_1, \ldots, X_p)$. Following [**34**, Section 2], set $A := R[\underline{X}]$ and $\widehat{A} := R[[\underline{X}]]$. For any $M \geq 1$, set

$$\widehat{A_M} := \left\{ \sum_{l \in \mathbb{N}^p} c_l \underline{X}^l \in \widehat{A} : \ \forall l \in \mathbb{N}^p, |c_l|_R \leqslant M^{|l|} \right\},$$

where for $l = (l_1, l_2, \ldots, l_n) \in \mathbb{N}^p$, $\underline{X}^l := \prod_{i=1}^p X_i^{l_i}$ and $|l| := l_1 + l_2 + \cdots + l_p$.

This is a subring of $\widehat{A}$, and for any $M', M'' \geq 1$, if $M' \leqslant M''$ then $\widehat{A_{M'}} \subseteq \widehat{A_{M''}}$. Furthermore, $\widehat{A_M}$ is complete with respect to the $(\underline{X})$-adic topology: if $(f_n)_n$ is a Cauchy sequence in $\widehat{A_M}$, then for any $l \in \mathbb{N}^p$ and large enough n, $f_{n+1} - f_n \in (\underline{X})^{|l|}$, implying that f_n and f_{n+1} have the same "first few" coefficients (the larger $|l|$, the more "first few" coefficients that are the same).

Remark also that for any element $f = \frac{g}{h}$ of the local ring $R[\underline{X}]_{(\underline{X})}$, where $g, h \in R[\underline{X}], h(0) \neq 0$, if $h(0) \in R^\times$, then f can be expanded into a formal power series over R, meaning in this case $f \in \widehat{A}$.

The following two lemmas are generalizations of Lemmas 2.1 and 2.3 of [**34**] (and their proofs follow the line of reasoning of the latter). For any $n \in \mathbb{N}$, we keep the notation $|\cdot|_R$ for the max norm on R^n induced by the norm of R. For $a := (a_1, a_2, \ldots, a_n) \in R^n$ and $l := (l_1, l_2, \ldots, l_n) \in \mathbb{N}^n$, we denote $a^l := a_1^{l_1} \cdots a_n^{l_n}$. Clearly, $a^l \in R$.

LEMMA 2.1.2. (1) *Let $u = \sum_{l\in\mathbb{N}^p} c_l \underline{X}^l \in \widehat{A_M}$. If $a \in R^p$ is such that $|a|_R < M^{-1}$, then the series $\sum_{l\in\mathbb{N}^p} c_l a^l$ is convergent in R. Let us denote its sum by $u(a)$.*

(2) *For $M \geqslant 1$, let $v, w \in \widehat{A_M}$ be such that w and vw are polynomials. If $a \in R^p$ is such that $|a|_R < M^{-1}$, then $vw(a) = v(a)w(a)$.*

(3) *Let $f = \frac{g}{h} \in R[\underline{X}]_{(\underline{X})}, g, h \in R[\underline{X}], h(0) \neq 0$, be such that $g(0) = 0$ and $h(0) \in R^\times$. There exists $M \geq 1$ such that $f \in \widehat{A_M}$ and $h \in \widehat{A_M}^\times$.*

Let $f = \sum_{l\in\mathbb{N}^p} c_l \underline{X}^l$ be the series representation of f. Then, for any $a \in R^p$ with $|a|_R < M^{-1}$, the series $\sum_{l\in\mathbb{N}^p} c_l a^l$ is convergent in R and $f(a) = \frac{g(a)}{h(a)}$.

PROOF. (1) Set $m = |a|_R < M^{-1}$. Then, $|c_l a^l|_R \leqslant (Mm)^{|l|}$. Since $Mm < 1$, $c_l a^l$ tends to zero as $|l|$ tends to $+\infty$, implying $\sum_{l\in\mathbb{N}^p} c_l a^l$ converges in R.

(2) Let $d > \deg vw$, and $C := \max_{l\in\mathbb{N}^p}(|vw_l|_R, |w_l|_R)$, where $vw_l, w_l, l \in \mathbb{N}^p$, are the coefficients of the polynomials vw, w, respectively. Let $v = \sum_{l\in\mathbb{N}^p} b_l \underline{X}^l$ be the series representation of v. For any $s \in \mathbb{N}$, set $v_s = \sum_{|l|<s} b_l \underline{X}^l$. By the first part, the sequence $(v_s(a))_{s\in\mathbb{N}}$ converges in R, and we denote limit by $v(a)$. For $s \geq d$, $r_s := v_s w - vw = (v_s - v)w$ is a polynomial whose monomials are of degree at least s. The coefficient C_j corresponding to any degree $j \geqslant s$ monomial of r_s is a finite sum of products of coefficients of $v_s - v$ and w. Since R is non-Archimedean, $M \geqslant 1$, and $v_s - v \in \widehat{A_M}$, we obtain $|C_j|_R \leqslant M^j C$ (recall the definition of C above).

Set $m = |a|_R$. By the paragraph above, every degree j monomial of r_s evaluated at a has absolute value at most $(mM)^j C$. Since $j \geqslant s$ and $Mm < 1$, using the fact that R is non-Archimedean, we obtain $|r_s(a)|_R \leqslant (Mm)^s C$, implying $r_s(a) \to 0, s \to \infty$. Consequently, $v_s(a)w(a) \to vw(a)$ when $s \to \infty$, *i.e.* $v(a)w(a) = vw(a)$.

(3) Set $b = h(0)$. Then, $b - h \in (\underline{X})$, and thus $1 - b^{-1}h \in (\underline{X})$. Set $e = 1 - b^{-1}h$, so that $b^{-1}h = 1 - e$ with $e \in (\underline{X})$. This implies $(b^{-1}h)^{-1} = bh^{-1} = \frac{1}{1-e} = \sum_{i\in\mathbb{N}} e^i \in \widehat{A}$, and so $h^{-1} = \sum_{i\in\mathbb{N}} b^{-1}e^i \in \widehat{A}$. Consequently, $f = gh^{-1} = \sum_{i\in\mathbb{N}} b^{-1}ge^i \in \widehat{A}$.

Set $M = \max_{l\in\mathbb{N}^p}(1, |b^{-1}|_R, \sqrt[|l|]{|g_l|_R}, \sqrt[|l|]{|e_l|_R}, \sqrt[|l|]{|h_l|_R})$, where g_l (resp. e_l, h_l), $l \in \mathbb{N}^p$, are the coefficients of the polynomial g (resp. e, h). Then, $b^{-1}, g, e \in \widehat{A_M}$, and since $\widehat{A_M}$ is a ring, $b^{-1}e^i$, $b^{-1}ge^i \in \widehat{A_M}$ for any $i \in \mathbb{N}$. Finally, since $\widehat{A_M}$ is complete with respect to the $(\underline{X})$−adic norm, $h^{-1}, f \in \widehat{A_M}$, and so $h \in \widehat{A_M}^\times$.

The rest is a direct consequence of the first two parts of the statement.

□

Let $n \in \mathbb{N}$ and $S_i, T_i, i = 1, 2, \ldots, n$, be indeterminates. As before, we use the notation $\underline{S}$ (resp. $\underline{T}$) for the n-tuple $(S_1, \ldots, S_n)$ (resp. $(T_1, \ldots, T_n)$). For $l, m \in \mathbb{N}^n$, we denote by $|(l, m)|$ the sum $|l| + |m|$, where $|l|$ (resp. $|m|$) is the sum of coordinates of l (resp. m). Also, $\underline{S}^l := \prod_{i=1}^n S_i^{l_i}$ and $\underline{T}^m := \prod_{i=1}^n T_i^{m_i}$. For any vector $a \in R^n$, we denote by a_i the i-th coordinate of a, $i = 1, 2, \ldots, p$, meaning $a = (a_1, a_2, \ldots, a_n), a_i \in R$. As before, $a^l := a_1^{l_1} \cdots a_n^{l_n}$.

LEMMA 2.1.3. *Let $f = \frac{h_1}{h_2} \in R[\underline{S}, \underline{T}]_{(\underline{S},\underline{T})}$, $h_1, h_2 \in R[\underline{S}, \underline{T}], h_2(0) \neq 0$, be such that $h_2(0) \in R^\times$. Suppose there exists $i \in \{1, 2, \ldots, n\}$ such that $f(a, 0) = f(0, a) = a_i$ for any $a \in R^n$ for which $f(a, 0)$ and $f(0, a)$ converge in R.*

Then, there exists $M \geq 1$ such that $f \in \widehat{A_M}$ and its series representation is:

$$f = S_i + T_i + \sum_{|(l,m)|\geq 2} c_{l,m}\underline{S}^l\underline{T}^m.$$

PROOF. Set $g = f - S_i - T_i$. Then, $g(a,b) = 0$ if it is well defined and $a = 0$ or $b = 0$. By Lemma 2.1.2, there exists $M \geq 1$ such that $f \in \widehat{A_M}$, implying $g \in \widehat{A_M}$. Let $g = \sum_{(l,m)\in\mathbb{N}^{2n}} c_{l,m}\underline{S}^l\underline{T}^m$, $|c_{l,m}|_R \leqslant M^{|(l,m)|}$ for all $(l,m) \in \mathbb{N}^{2n}$, be the series representation of g. Since $g(0,0) = 0$, $c_{0,0} = 0$. It remains to show that $c_{l,m} = 0$ for all $(l,m) \in \mathbb{N}^{2n}$ for which $|(l,m)| = 1$.

We proceed by contradiction. Let us assume, without loss of generality, that for $l_0 := (1,0,\ldots,0)$ and $m_0 := (0,\ldots,0)$, $c_{l_0,m_0} \neq 0$. Let $0 < q < 1$ be such that $q \leqslant |c_{l_0,m_0}|_R \leqslant M$. Let $\alpha \in k^\times$ be such that $|\alpha| < \frac{q}{M^2}$ (it exists seeing as k is non-trivially valued). Set $a := (\alpha, 0, \cdots, 0) \in R^n$, $b = (0,\ldots,0) \in R^n$, and $v := (a,b)$. Then, $g(v) = 0$.

Let L be the part of g that has degree 1. Then, $L(v) = c_{l_0,m_0}\alpha \neq 0$. At the same time, $|L(v)|_R = |c_{l_0,m_0}|_R|\alpha| \geqslant q|\alpha|$.

Let $h = c_{l,m}\underline{S}^l\underline{T}^m$ be any non-zero monomial of g of degree $j := |(l,m)| \geqslant 2$. Let us show that $|h(v)|_R < |L(v)|_R$. If $h(v) = 0$, this is clear. If $h(v) \neq 0$ then $h = c_{l,m}S_1^j$ and $|h(v)|_R = |c_{l,m}|_R|\alpha|^j$. Consequently:

$$\frac{|L(v)|_R}{|h(v)|_R} \geqslant \frac{q|\alpha|}{|c_{l,m}|_R|\alpha|^j} \geqslant \frac{q}{M^j|\alpha|^{j-1}} > \frac{q}{M^j}\cdot\left(\frac{M^2}{q}\right)^{j-1} = \frac{M^{j-2}}{q^{j-2}} \geq 1.$$

Consequently, $|h(v)|_R < |L(v)|_R$ implying $|g(v)|_R = |L(v)|_R$, which is impossible seeing as $|g(v)|_R = 0$ and $|L(v)|_R \neq 0$. □

REMARK 2.1.4. Lemma 2.1.3 is the only reason behind the hypothesis that k is non-trivially valued.

We now introduce a general formal setting on which patching results will be proven. As already mentioned, these are hypotheses that are satisfied naturally in the framework of Berkovich's geometry that we will work in.

SETTING 2.1.5. Let $(k, |\cdot|)$ be a complete non-trivially valued ultrametric field. Let $R_i, i = 0,1,2$, be an integral domain containing k, endowed with a non-Archimedean (submultiplicative) norm $|\cdot|_{R_i}$ with respect to which it is complete. Assume that $|\cdot|_{R_i}$ is k-linear, meaning for any $a \in k$ and any $b \in R_i$, $|ab|_{R_i} = |a|\cdot|b|_{R_i}$. In particular, $|\cdot|_{R_i}$ extends $|\cdot|$. Suppose there exist bounded morphisms $R_j \hookrightarrow R_0, j = 1,2$. Set $F_i = \operatorname{Frac} R_i, i = 0,1,2$. Let F be an infinite field embedded in both F_1 and F_2.

Let A_j be a finite R_j-module such that $A_j \subseteq F_j, j = 1,2$. Suppose that there exist embeddings $A_j \hookrightarrow R_0$. Let us endow A_j with the quotient semi-norm induced from a surjective morphism $\varphi_i : R_j^{n_i} \twoheadrightarrow A_j, j = 1,2$; we assume that these semi-norms are norms. Assume that A_j is complete and the morphism $A_j \hookrightarrow R_0$ is bounded for $j = 1,2$. Suppose the induced map $\psi : A_1 \oplus A_2 \to R_0$ is surjective. Finally, suppose the norm of R_0 is equivalent to the quotient norm induced by the surjective morphism $\psi : A_1 \oplus A_2 \twoheadrightarrow A_0$, where $A_1 \oplus A_2$ is endowed with the usual max norm $|\cdot|_{\max}$, i.e. that the morphism ψ is admissible (see Definition 1.1.13).

Before giving an analogue to [**34**, Theorem 2.5] (which is fundamental to patching) in this setting, let us give some motivation behind its interest to us.

DEFINITION 2.1.6. Let K be a field. A *rational* variety over K is a K-variety that has a Zariski open isomorphic to an open of some $\mathbb{A}^n_K$.

REMARK 2.1.7. The definition above does not coincide with the standard notion of rational variety. We adopt it here because we will only use it for linear algebraic groups, in which case a connected rational linear algebraic group is rational in the traditional sense (*i.e.* birationally equivalent to some $\mathbb{P}^n$). We make this distinction because there are certain statements we will show that don't require connectedness and others that do.

Using the same notation as in Setting 2.1.5, let G/F be a rational linear algebraic group (rational here means that G is a rational variety over F). Our main goal will be to show that under certain conditions (which we will interpret geometrically in the next chapters), for any $g \in G(F_0)$, there exist $g_j \in G(F_j), j = 1, 2$, such that $g = g_1 \cdot g_2$ in $G(F_0)$.

REMARK 2.1.8. Let K/F be any field extension. Since G has a non-empty Zariski open subset S' isomorphic to an open subset S of an affine space $\mathbb{A}^n_K$, by translation we may assume that the identity element of G is contained in S', that $0 \in S$, and that the identity is sent to 0. Let us denote the isomorphism $S' \to S$ by φ.

Let m be the multiplication in G, and set $\widetilde{S}' = m^{-1}(S') \cap (S' \times S')$, which is an open of $G \times G$. It is isomorphic to an open $\widetilde{S}$ of $\mathbb{A}^{2n}_K$, and $m_{|\widetilde{S}'}$ gives rise to a map $\widetilde{S} \to S$, i.e. to a rational function $f : \mathbb{A}^{2n}_K \dashrightarrow \mathbb{A}^n_K$ (see the diagram below). Note that for any $(x, 0), (0, x) \in \widetilde{S}$, this function sends them both to x.

$$\begin{array}{ccc} & \overset{m_{|\widetilde{S}'}}{\longrightarrow} & \\ \widetilde{S}' & & S' \\ \Big\downarrow{\scriptstyle (\varphi \times \varphi)_{|\widetilde{S}'}} & & \Big\downarrow{\scriptstyle \varphi} \\ \widetilde{S} & \underset{f}{\longrightarrow} & S \end{array}$$

The result we are interested in can be interpreted in terms of the map f. Theorem 2.1.10 below shows that (under certain conditions) said result is true on some neighborhood of the origin of an affine space.

Let us start with an auxiliary lemma. Referring to Setting 2.1.5, let $|\cdot|_{\inf}$ be the norm on R_0 obtained from the admissible morphism $\psi : A_1 \oplus A_2 \twoheadrightarrow R_0$. Since it is equivalent to $|\cdot|_{R_0}$, there exist positive real numbers C_1, C_2 such that $C_1|\cdot|_{R_0} \leqslant |\cdot|_{\inf} \leqslant C_2|\cdot|_{R_0}$.

Since the morphisms $A_j \hookrightarrow R_0, j = 1, 2$, are bounded, there exists $C > 0$ such that for any $x_j \in A_j$, one has $|x_j|_{R_0} \leqslant C|x_j|_{A_j}$. By changing to an equivalent norm on A_j if necessary, we may assume that $C = 1$.

LEMMA 2.1.9. *There exists $d \in (0, 1)$ such that for all $c \in R_0$, there exist $a \in A_1$, $b \in A_2$, for which $\psi(a + b) = c$ and $d \cdot \max(|a|_{A_1}, |b|_{A_2}) \leqslant |c|_{R_0}$.*

PROOF. Suppose $c \neq 0$. Let D be a real number, such that $D > \max(1, 1/C_2)$. Then, for any $c \in R_0$, there exist $a \in A_1, b \in A_2$, with $\psi(a + b) = c$ and $\max(|a|_{A_1}, |b|_{A_2}) \leqslant D \cdot |c|_{\inf}$. Otherwise, for any $x \in A_1$ and any $y \in A_2$ for which $\psi(x + y) = c$, we would have

$$|x + y|_{\max} := \max(|x|_{A_1}, |y|_{A_2}) > D \cdot |c|_{\inf}.$$

But, $|c|_{\inf} = \inf_{x,y} |x+y|_{\max} \geq D \cdot |c|_{\inf}$, where $D > 1$ and $|c|_{\inf} \neq 0$, so this is impossible.

Set $d' = 1/D < C_2$. Then, $d' \cdot \max(|a|_{A_1}, |b|_{A_2}) \leqslant |c|_{\inf} \leqslant C_2 \cdot |c|_{R_0}$. We obtain the wanted result by setting $d = \frac{d'}{C_2} \in (0,1)$.

If $c = 0$, the statement is true regardless of the choice of d. □

From now on, instead of writing $\psi(x+y) = c$ for $x \in A_1, y \in A_2, c \in R_0$, we will simply put $x + y = c$ when there is no risk of ambiguity.

In what follows, for any positive integer n, let us endow R_0^n with the max norm induced from the norm on R_0, and let us also denote it by $|\cdot|_{R_0}$. For a normed ring A and $\delta > 0$, we denote by $D_A(0,\delta)$ the open disc in A centered at 0 and of radius δ.

THEOREM 2.1.10. *For $n \in \mathbb{N}$, let $f : \mathbb{A}^n_{F_0} \times \mathbb{A}^n_{F_0} \dashrightarrow \mathbb{A}^n_{F_0}$ be a rational map defined on a Zariski open $\widetilde{S}$, such that $(0,0) \in \widetilde{S}$, and $f(x,0) = f(0,x) = x$ whenever $(x,0),(0,x) \in \widetilde{S}$. Write $f = (f_1, f_2, \ldots, f_n)$, where $f_i = \frac{g_i}{h_i}$ for some $g_i, h_i \in R_0[\underline{S},\underline{T}]$, $i = 1,2,\ldots,n$. Suppose $h_i(0) \in R_0^\times$ for all i.*

Let $M \geqslant 1$ be such that $f_i \in \widehat{A_M}$ and $h_i \in \widehat{A_M}^\times$ for all i (applying Lemma 2.1.2 with $R = R_0$). Suppose there exists $\delta > 0$ such $D_{R_0^{2n}}(0,\delta) \subseteq \widetilde{S}(F_0)$. Let d be as in Lemma 2.1.9. Let $\varepsilon > 0$ be such that $\varepsilon < \min(\frac{d}{2M}, \frac{d^3}{M^4}, \frac{d\delta}{2})$. Then, for any $a \in \mathbb{A}^n(F_0)$ with $a \in R_0^n$ and $|a|_{R_0} \leqslant \varepsilon$, there exist $u \in A_1^n$ and $v \in A_2^n$ for which $(u,v) \in \widetilde{S}(F_0)$ and $f(u,v) = a$.

PROOF. Since $f_i(0,0) = 0$ for all i, the functions g_i belong to the maximal ideal $(\underline{S},\underline{T})$ of $R_0[\underline{S},\underline{T}]$. From Lemmas 2.1.2 and 2.1.3:

(1) we can see these rational functions as elements of $R_0[[\underline{S},\underline{T}]]$;
(2) the constant M is such that

$$f_i = S_i + T_i + \sum_{|(l,m)|\geq 2} c^i_{l,m} \underline{T}^l \underline{S}^m \in R_0[[\underline{S},\underline{T}]],$$

with $|c^i_{l,m}|_{R_0} \leqslant M^{|(l,m)|}$, for $i = 1,2,\ldots,n$ and $(l,m) \in \mathbb{N}^{2n}$, where $|(l,m)|$ is the sum of the coordinates of (l,m).

By the choice of δ, for any $(x,y) \in R_0^{2n}$ satisfying $|(x_0,y_0)|_{R_0} < \delta$, $(x,y) \in \widetilde{S}(F_0)$, so the function $f(x,y)$ is well-defined (meaning the functions f_i are well-defined for all i).

Set $\varepsilon' = \frac{\varepsilon}{d}$. Then, $0 < \varepsilon' < \min\{1/2M, d^2/M^4, \delta/2\}$. Since $\varepsilon < \varepsilon' < \min(1/M, \delta/2)$, for any $(x,y) \in \widetilde{S}(F_0)$ satisfying $(x,y) \in R_0^{2n}$ and $|(x,y)|_{R_0} \leqslant \varepsilon'$, $f(x,y)$ is well defined, and by Lemma 2.1.2, the series f_i is convergent in R_0 at (x,y), $i = 1,2,\ldots,n$.

Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{A}^n(F_0)$ be such that $a \in R_0^n$ and $|a|_{R_0} \leqslant \varepsilon$. Let $u_0 = 0 \in A_1^n$, and $v_0 = 0 \in A_2^n$. Using induction, one constructs sequences $(u_s)_s$ in A_1^n, and $(v_s)_s$ in A_2^n, such that the following conditions are satisfied:

(1) $|u_s|_{A_1}, |v_s|_{A_2} \leqslant \varepsilon'$ for all $s \geq 0$;
(2) $|u_s - u_{s-1}|_{A_1}, |v_s - v_{s-1}|_{A_2} \leqslant {\varepsilon'}^{\frac{s+1}{2}}$ for all $s \geq 1$;
(3) $|f(u_s,v_s) - a|_{R_0} \leqslant d{\varepsilon'}^{\frac{s+2}{2}}$ for all $s \geq 0$.

The first terms u_0 and v_0 satisfy conditions 1 and 3. We notice that the first condition implies $|(u_s,v_s)|_{R_0} \leqslant \varepsilon'$, so $f(u_s,v_s)$ is well-defined, and f_i is convergent in R_0 at (u_s,v_s) for $i \in \{1,2,\ldots,n\}$. Suppose that for $j \geq 0$, we have constructed u_j and v_j satisfying all three conditions above. Then, $d_j := a - f(u_j,v_j) \in R_0^n$ is well defined, and $|d_j|_{R_0} \leqslant d{\varepsilon'}^{\frac{j+2}{2}}$.

From Lemma 2.1.9, there exist $u'_j \in A_1^n$ and $v'_j \in A_2^n$, such that $d_j = u'_j + v'_j$, and $d \cdot \max(|u'_j|_{A_1}, |v'_j|_{A_2}) \leqslant |d_j|_{R_0} \leqslant d{\varepsilon'}^{\frac{j+2}{2}}$.

Set $u_{j+1} = u_j + u'_j$ and $v_{j+1} = v_j + v'_j$. Then, $|u_{j+1}|_{A_1} \leqslant \max\left(\varepsilon', {\varepsilon'}^{\frac{j+2}{2}}\right) = \varepsilon'$, and the same is true for v_{j+1}. Also, $|u_{j+1} - u_j|_{A_1} = |u'_j|_{A_1} \leqslant {\varepsilon'}^{\frac{j+2}{2}}$, and similarly, $|v_{j+1} - v_j|_{A_2} \leqslant {\varepsilon'}^{\frac{j+2}{2}}$.

For $r \in \mathbb{N}$, $i \in \{1, 2, \ldots, r\}$, and $\lambda \in F_0^r$, let λ_i be the i-th coordinate of λ. For $p = (p_1, p_2, \ldots, p_r) \in \mathbb{N}^r$, set $\lambda^p := \prod_{i=1}^r \lambda_i^{p_i}$. For any $\nu, \nu' \in \mathbb{N}^r$, $\nu \leqslant \nu'$ will mean $\nu_i \leqslant \nu'_i$ for all $i = 1, 2, \ldots, r$. Then, for the third condition,

$$\begin{aligned}
|f_i(u_{j+1}, v_{j+1}) - a_i|_{R_0} &= \left| u_{j+1,i} + v_{j+1,i} - a_i + \sum_{|(l,m)|\geq 2} c_{l,m}^i u_{j+1}^l v_{j+1}^m \right|_{R_0} \\
&= \left| u_{j,i} + v_{j,i} + u'_{j,i} + v'_{j,i} - a_i + \sum_{|(l,m)|\geq 2} c_{l,m}^i u_{j+1}^l v_{j+1}^m \right|_{R_0} \\
&= \left| f_i(u_j, v_j) - a_i + u'_{j,i} + v'_{j,i} + \sum_{|(l,m)|\geq 2} c_{l,m}^i (u_{j+1}^l v_{j+1}^m - u_j^l v_j^m) \right|_{R_0} \\
&= \left| -d_{j,i} + u'_{j,i} + v'_{j,i} + \sum_{|(l,m)|\geq 2} c_{l,m}^i (u_{j+1}^l v_{j+1}^m - u_j^l v_j^m) \right|_{R_0} \\
&= \left| \sum_{|(l,m)|\geq 2} c_{l,m}^i (u_{j+1}^l v_{j+1}^m - u_j^l v_j^m) \right|_{R_0} \\
&\leqslant \max_{|(l,m)|\geq 2} |c_{l,m}^i|_{R_0} \cdot |u_{j+1}^l v_{j+1}^m - u_j^l v_j^m|_{R_0}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
u_{j+1}^l v_{j+1}^m - u_j^l v_j^m &= (u_j + u'_j)^l (v_j + v'_j)^m - u_j^l v_j^m \\
&= \sum_{\substack{0\leqslant\beta\leqslant l \\ 0\leqslant\gamma\leqslant m}} A_\beta B_\gamma u_j^\beta u_j'^{l-\beta} v_j^\gamma v_j'^{m-\gamma} - u_j^l v_j^m \\
&= \sum_{0\leqslant\alpha\leqslant(l,m)} \sum_{\substack{\beta+\gamma=\alpha \\ 0\leqslant\beta\leqslant l \\ 0\leqslant\gamma\leqslant m}} A_\beta B_\gamma u_j^\beta u_j'^{l-\beta} v_j^\gamma v_j'^{m-\gamma} - u_j^l v_j^m \\
&= \sum_{0\leqslant\alpha<(l,m)} \sum_{\substack{\beta+\gamma=\alpha \\ 0\leqslant\beta\leqslant l \\ 0\leqslant\gamma\leqslant m}} A_\beta B_\gamma u_j^\beta u_j'^{l-\beta} v_j^\gamma v_j'^{m-\gamma}
\end{aligned}$$

where A_β, B_γ are integers (implying they are of norm at most one on R_0). Finally, since the norm $|\cdot|_{R_0}$ is non-Archimedean:

$$\begin{aligned}|u_{j+1}^l v_{j+1}^m - u_j^l v_j^m|_{R_0} &\leqslant \max_{\substack{0\leqslant\beta+\gamma<(l,m)\\ 0\leqslant\beta\leqslant l, 0\leqslant\gamma\leqslant m}} |u_j^\beta|_{R_0}|v_j^\gamma|_{R_0}|u_j'^{l-\beta}|_{R_0}|v_j'^{m-\gamma}|_{R_0}\\ &\leqslant \max_{\substack{0\leqslant\beta+\gamma<(l,m)\\ 0\leqslant\beta\leqslant l, 0\leqslant\gamma\leqslant m}} \varepsilon'^{|(\beta,\gamma)|}(\varepsilon'^{\frac{j+2}{2}})^{|(l,m)|-|(\beta,\gamma)|},\end{aligned}$$

so $|u_{j+1}^l v_{j+1}^m - u_j^l v_j^m|_{R_0} \leqslant \max_{0\leqslant\theta<|(l,m)|} \varepsilon'^\theta \cdot (\varepsilon'^{\frac{j+2}{2}})^{|(l,m)|-\theta}$. This, combined with $|c_{l,m}^i|_{R_0} \leqslant M^{|(l,m)|}$, implies that:

$$\begin{aligned}|f_i(u_{j+1}, v_{j+1}) - a_i|_{R_0} &\leqslant \max_{\substack{|(l,m)|\geq 2\\ 0\leqslant\theta<|(l,m)|}} M^{|(l,m)|}\varepsilon'^\theta \cdot (\varepsilon'^{\frac{j+2}{2}})^{|(l,m)|-\theta}\\ &= \max_{\substack{|(l,m)|\geq 2\\ 0\leqslant\theta<|(l,m)|}} (M\varepsilon')^\theta \cdot (M\varepsilon'^{\frac{j+2}{2}})^{|(l,m)|-\theta}.\end{aligned}$$

Since $\varepsilon' \geq \varepsilon'^{\frac{j+2}{2}}$, we have: $|f_i(u_{j+1}, v_{j+1}) - a_i|_{R_0} \leqslant \max_{|(l,m)|\geq 2}(M\varepsilon')^{|(l,m)|-1} \cdot (M\varepsilon'^{\frac{j+2}{2}})$. Since $M\varepsilon' < 1$, one obtains: $\max_{|(l,m)|\geq 2}(M\varepsilon')^{|(l,m)|-1} \cdot (M\varepsilon'^{\frac{j+2}{2}}) \leqslant M\varepsilon' \cdot M\varepsilon'^{\frac{j+2}{2}}$. We have shown that

$$|f_i(u_{j+1}, v_{j+1}) - a_i|_{R_0} \leqslant M\varepsilon' \cdot M\varepsilon'^{\frac{j+2}{2}} = M^2\varepsilon'^{1+\frac{j+2}{2}}.$$

At the same time, seeing as $\varepsilon' < \frac{d^2}{M^4}$, we obtain $M^2 \cdot \varepsilon'^{1+\frac{j+2}{2}} = (\frac{M^2}{d}\varepsilon'^{1/2})d\varepsilon'^{\frac{j+3}{2}} \leqslant d\varepsilon'^{\frac{j+3}{2}}$, which concludes the induction argument.

The second property of the sequences $(u_s)_s, (v_s)_s$ tells us that they are Cauchy (hence convergent) in the Banach modules A_1^n, A_2^n, respectively. Let $u \in A_1^n$ and $v \in A_2^n$ be the corresponding limits. The first property implies that $|(u,v)|_{R_0} \leqslant \varepsilon' < \delta$, so $(u,v) \in \widetilde{S}(F_0)$, and $f(u,v)$ is well-defined. Lastly, the third property implies that $f(u,v) = a$. □

Using the same notation, we have proven:

PROPOSITION 2.1.11. *Suppose $h_i(0) \in R_0^\times$ and there exists an open disc of R_0^{2n} centered at 0 that is contained in $\widetilde{S}$. Then, there exists $\varepsilon > 0$ such that for any $g \in S'(F_0)$ with $\varphi(g) \in R_0^n$ and $|\varphi(g)|_{R_0} \leqslant \varepsilon$, there exist $g_i \in G(F_i), i = 1, 2$, satisfying $g = g_1 \cdot g_2$ in $G(F_0)$.*

2.2. A special case fundamental for patching over curves

Proposition 2.1.11 can significantly be strengthened under a few additional hypotheses. This setup is of fundamental importance for patching over analytic curves.

The following is a result shown in [**34**] that we will need.

LEMMA 2.2.1 ([**34**, Lemma 3.1]). *Let G be a rational linear algebraic group defined over an infinite field F. Let F_0/F be a field extension and $g \in G(F_0)$. There exists a Zariski open subset U of G isomorphic to a Zariski open subset of some $\mathbb{A}_F^n$ and such that $g \in U(F_0)$.*

Under the hypotheses of Setting 2.1.5, let G/F be a rational linear algebraic group. As in Remark 2.1.8, let S' be a Zariski open subset of G isomorphic to an open subset S of an affine space $\mathbb{A}^n_F$. By translation we may assume that the identity element of G is contained in S', that $0 \in S$, and that the identity is sent to 0. Let us denote the isomorphism $S' \to S$ by φ. Let the diagram below and the corresponding notations be as in Remark 2.1.8. As noted there, the vertical arrows are isomorphisms.

$$\begin{array}{ccc} \widetilde{S}' & \xrightarrow{m_{|\widetilde{S}'}} & S' \\ \downarrow{\scriptstyle (\varphi\times\varphi)_{|\widetilde{S}'}} & & \downarrow{\scriptstyle \varphi} \\ \widetilde{S} & \xrightarrow[f]{} & S \end{array}$$

Convention 2.2.2. Let us fix once and for all an embedding of G into $\mathbb{A}^m_F$ for some $m \in \mathbb{N}$. Let K/F be a field extension, and $M \subseteq K$. Set $G_K = G \times_F K$. Let U be a Zariski open subset of G_K. Seeing as G is affine, there is a notion of "M-points" of U. More precisely, these are the points in $U(K)$ whose coordinates are in M. Let us denote this set by $U(M)$.

An outline of proof (which we follow) of a special case of the following result is given in the proof of [**34**, Theorem 3.2]. Recall that for a normed ring A and $r > 0$, we denote by $D_A(0,r)$ the open disc in A centered at 0 and of radius r.

Theorem 2.2.3. *Under the hypotheses of Setting 2.1.5, suppose $F_0 = R_0$. Suppose that F_1 is dense in F_0. Then, for any $g \in G(F_0)$, there exist $g_1 \in G(F_1)$ and $g_2 \in G(F_2)$ such that $g = g_1 \cdot g_2$ in $G(F_0)$.*

Proof. As already noted in Remark 2.1.8, the function f satisfies the properties of Theorem 2.1.10. Seeing as F_0 is a normed field, if F_0^{2n} is endowed with the max-norm (which we still denote $|\cdot|_{F_0}$), then the induced topology in F_0^{2n} is finer than the Zariski one. Consequently, there exists $\delta > 0$ such that $D_{F_0^{2n}}(0,\delta) \subseteq \widetilde{S}(F_0)$. Hence, all of the hypotheses of Theorem 2.1.10 are satisfied, implying there exists $\varepsilon > 0$ such that for any $a \in \mathbb{A}^n(F_0)$ satisfying $|a|_{F_0} \leqslant \varepsilon$, there exist $u \in A_1^n$ and $v \in A_2^n$ such that $f(u,v) = a$.

(1) Suppose $g \in S'(F_0)$ and $|\varphi(g)|_{F_0} \leqslant \varepsilon$. Then, by the paragraph above, there exist $g_i \in G(A_i) \subseteq G(F_i), i = 1,2$, such that $g = g_1 \cdot g_2$ in $G(F_0)$. Similarly, there exist $g_i' \in G(F_i)$, $i = 1,2$, such that $g = g_2' \cdot g_1'$ in $G(F_0)$.

(2) Suppose $g \in S'(F_0)$ with no further restrictions. Remark that $gS' \cap S'$ is a non-empty (seeing as $g \in gS' \cap S'$) Zariski open subset of G. Let $\psi : gS' \cap S' \to \mathbb{A}^n_{F_0}$ be the morphism given by $h \mapsto \varphi(g^{-1}h)$. Remark $0 \in Im(\psi)$. The preimage $\psi^{-1}(D_{F_0^n}(0,\varepsilon))$ is open in $(gS' \cap S')(F_0)$. As F_1 is dense in F_0, we obtain that $(gS' \cap S')(F_1)$ is dense in $(gS' \cap S')(F_0)$ with respect to the topology induced by $|\cdot|_{F_0}$. Hence, there exists $h \in (gS' \cap S')(F_1) \cap \psi^{-1}(D_{F_0^n}(0,\varepsilon))$. More precisely, $h \in (gS' \cap S')(F_1) \subseteq G(F_1)$, and $|\varphi(g^{-1}h)|_{F_0} \leqslant \varepsilon$. Then, by part (1), there exist $g_1' \in G(F_1)$ and $g_2' \in G(F_2)$ such that $g^{-1}h = g_2' \cdot g_1'$ in $G(F_0)$. Set $g_1 := h \cdot g_1'^{-1} \in G(F_1)$, and $g_2 := g_2'^{-1}$ in $G(F_2)$. Then, $g = g_1 \cdot g_2$ in $G(F_0)$.

(3) Suppose $g \in G(F_0)$ with no further restrictions. By Lemma 2.2.1, there exists a Zariski open subset U of G isomorphic to a Zariski open U' of some $\mathbb{A}^n_F$ such that $g \in U(F_0)$. As F is infinite, there exists $\alpha \in U(F)$. Set $S = \alpha^{-1}U$. It is a Zariski open

subset of G containing the identity, which is isomorphic to an open subset of $\mathbb{A}^n_F$. By translating (over F) if necessary, we may assume that this isomorphism sends the identity to 0 in $\mathbb{A}^n_F$.

By part (2), there exist $g_1' \in G(F_1)$ and $g_2 \in G(F_2)$ such that $\alpha^{-1}g = g_1' \cdot g_2$ in $G(F_0)$. Set $g_1 = \alpha \cdot g_1' \in G(F_1)$. Then, $g = g_1 \cdot g_2$ in $G(F_0)$. □

CHAPTER 3

Patching over Berkovich Curves and Quadratic Forms

In this chapter we show that patching can be applied to analytic curves. As a consequence, we obtain a local-global principle for function fields of curves, which is applicable to quadratic forms. Moreover, we also obtain applications to the u-invariant of function fields of curves. The results of this chapter generalize those of [**34**].

In Section 3.1 we study a special class of covers of an analytic curve, called *nice covers*. The motivation for their study comes from the fact that patching (or, more precisely, a generalized form of patching as seen in Chapter 2) can be applied to these covers. We start by exhibiting a special case to which the results of Chapter 2 are directly applicable, and then use it to obtain said generalization. More precisely, let C be a k-analytic curve. Let U, V be connected affinoid domains in C such that $W = U \cap V$ is a single type 3 point. We show that given two reasonable algebraic structures over $\mathscr{M}(U)$, $\mathscr{M}(V)$, and a suitable group action on them, they can be patched to give the same type of algebraic structure over $\mathscr{M}(U \cup V)$. Roughly, nice covers are a generalization of this situation (*cf.* Definition 3.1.6). In particular, note that type 3 points play a very important role, and so their existence is crucial. We study the properties of these covers, and show that any open cover can be refined by a nice cover.

The second section of this chapter contains its main results. We show a local-global principle (Theorem 3.2.11) for fields of meromorphic functions of normal projective k-analytic curves (or, equivalently, the function fields of such algebraic curves). In the simplest cases, the proofs use patching on nice covers and induction on the number of elements of said covers. We first prove these results over a complete ultrametric base field k such that $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$. This is then generalized for projective curves over any complete ultrametric field using a descent argument that is based on results of model theory. We also prove similar results for affinoid curves. Finally, we show that there is a connection between the points of a Berkovich analytic curve and the valuations on its function field (*i.e.* its field of meromorphic functions). We then use this to prove a local-global principle with respect to completions (Corollary 3.2.18, Corollary 3.4.2).

In Section 3.3, using the theory of generic analytic fibers, we interpret the overfields of HHK's [**34**] in the Berkovich setting. Then, we use a result of Bosch to show that the local-global principle [**34**, Theorem 3.7] is a consequence of the local-global principle obtained here (Theorem 3.2.11). Using the theory of analytic curves developed by Ducros, we prove that the converse is true as well provided one works over a "fine" enough model.

The purpose of Section 3.4 is to give applications to quadratic forms and the u-invariant of a field. We start by applying Theorem 3.2.11 to obtain a local-global principle for quadratic forms (Theorem 3.4.1).

Then, we find conditions under which there is local isotropy of a quadratic form q over analytic curves. The setup will be somewhat more general, which is partly why it is the most technical section of the chapter. The idea is to find a nice enough representative of

the isometry class of q to work with and then use Henselianity conditions. The hypotheses on the base field become stronger here. Namely, we require our complete valued non-Archimedean base field k to be such that the dimension of the $\mathbb{Q}$-vector space $\sqrt{|k^\times|}$ be finite (a special case being when $|k^\times|$ is a free module of finite rank over $\mathbb{Z}$), and the residue characteristic unequal to 2. The restriction on the value group is not very strong: when working over a complete ultrametric field k satisfying this property, for every k-analytic space X and every point $x \in X$, the completed residue field $\mathcal{H}(x)$ of x satisfies it as well.

Finally, we combine the local-global principle for quadratic forms and these local isotropy conditions to give a condition for global isotropy of a quadratic form over an analytic curve. From there we deduce applications to the (strong) u-invariant of a complete ultrametric field k with residue characteristic different from 2, and such that the dimension of the $\mathbb{Q}$-vector space $\sqrt{|k^\times|}$ is finite.

3.1. Nice covers

In this section we construct a class of covers of analytic curves over which we can apply patching, and study some of their properties.

3.1.1. An interpretation of patching over analytic curves. We present here the main example of Setting 2.1.5 which we will be dealing with in this Chapter. Let k be a complete ultrametric field.

Let us start with a couple of auxuliary results.

LEMMA 3.1.1. *Let V be a reduced affinoid space containing a single point x. Then, $\mathcal{O}(V) = \mathscr{M}(V) = \mathcal{H}(x)$ - the completed residue field of x.*

PROOF. Let $(A, \|\cdot\|)$ be the corresponding affinoid algebra of V. Let $f \in A\backslash\{0\}$. If $|f|_x = 0$, then by Proposition 1.3.15, f is nilpotent in A. As A is reduced, $f = 0$, contradiction. Thus, $|f|_x \neq 0$, so by Lemma 1.1.39 f is invertible in A. This means that A is a field, *i.e.* $\mathcal{O}(V) = \mathscr{M}(V)$. Remark that we have also shown that $|\cdot|_x$ is a norm on A.

Since A is reduced, by Proposition 1.3.15, the spectral norm ρ_A of A is equivalent to $\|\cdot\|$. Remark that the Shilov boundary $\Gamma(V)$ of V is $\{x\}$. Hence, by Theorem 1.1.38, $\rho_A = |\cdot|_x$. Consequently, the field A is complete with respect to $|\cdot|_x$, implying $\mathcal{H}(x) = A = \mathcal{O}(V)$. □

LEMMA 3.1.2. *Let C be a normal irreducible k-analytic curve. Let U, V be connected affinoid domains of C, such that $U \cap V = \{\eta\}$, where η is a point of type 3. Then, the images of $\mathscr{M}(U)$ and $\mathscr{M}(V)$ in $\mathscr{M}(\{\eta\})$ are dense.*

PROOF. The subset $\{\eta\}$ is an affinoid domain in both U and V by Corollary 1.5.36. By the Gerritzen-Grauert theorem (Theorem 1.4.14), it is a rational domain. By [**21**, Théorème 3.4], $U, V, \{\eta\}$ are normal. Hence, as they are connected, they are irreducible. In particular, $\mathcal{O}(U), \mathcal{O}(V)$ are integral domains.

Set $S_U := \{f \in \mathcal{O}(U) : |f|_\eta \neq 0\}$. By Lemma 1.4.16, $S_U^{-1}\mathcal{O}(U)$ is dense in $\mathcal{O}(\{\eta\})$.

Suppose $f \in \mathcal{O}(U)$ is such that $|f|_\eta = 0$. As η is a type 3 point, by Lemma 1.8.3, $\mathcal{O}_\eta$ is a field, implying $f = 0$ there. By Proposition 1.6.24, this implies that $f = 0$ in $\mathcal{O}(U)$. Thus, $S_U = \mathcal{O}(U)\backslash\{0\}$, meaning Frac $\mathcal{O}(U)$ is dense in $\mathcal{O}(\{\eta\})$. By Lemmas 3.1.1 and 1.7.6, this is the same as saying that the image of $\mathscr{M}(U)$ is dense in $\mathscr{M}(\{\eta\})$.

The same is true for V. □

The example of Setting 2.1.5 we will be working with is the following:

PROPOSITION 3.1.3. *Let C be a normal irreducible k-analytic curve. Set $F_C = \mathscr{M}(C)$. Let D be an effective divisor of degree n on C. Take two connected affinoid domains U, V in C, such that $W = U \cap V = \{\eta\}$, where η is a type 3 point. Set $R_U = \mathcal{O}(U), F_U = \operatorname{Frac} R_U$, $R_V = \mathcal{O}(V), F_V = \operatorname{Frac} R_V$, and $F_W = R_W = \mathcal{O}(W)$. Set $A_U = \mathcal{O}(D)(U)$, $A_V = \mathcal{O}(D)(V)$.*

For large enough n such that $H^1(C, \mathcal{O}(D)) = 0$, the conditions of Setting 2.1.5 are satisfied for $R_0 := R_W, R_1 := R_U, R_2 := R_V$, $A_1 := A_U, A_2 := A_V$, and $F := F_C$. Moreover, $F_0 = R_0$ and F_U, F_V are dense in F_0.

PROOF. As U, V, W are connected affinoid domains of a normal curve, they are integral, so R_U, R_V, R_W are integral k-affinoid algebras, meaning they are integral domains that are complete with respect to non-Archimedean norms. Moreover, k is contained in R_U, R_V, R_W and the norms of the latter are k-linear. Since $R_W = \mathcal{O}(W) = \mathcal{H}(\eta)$ (Lemma 3.1.1), the normed ring R_W is a field. By Lemma 1.7.6, $\mathscr{M}(U) = F_U$, $\mathscr{M}(V) = F_V$, and $\mathscr{M}(W) = F_W = R_W$. This shows the existence of embeddings of F_C in F_U, F_V, and F_W. The restriction morphisms $R_U, R_V \to R_W$ are bounded by construction. Clearly, F_C is an infinite field.

Notice that for $Z \in \{U, V, W\}$, $\mathcal{O}(Z) \hookrightarrow \mathcal{O}(D)(Z) \hookrightarrow \mathscr{M}(Z)$. In particular, this means that $\mathcal{O}(D)(W) = \mathcal{O}(W) = \mathscr{M}(W)$. Since $\mathcal{O}(D)$ is a coherent sheaf, A_U (resp. A_V) is a finite R_U-module (resp. R_V-module). The completness of A_U (resp. A_V) follows from the fact that ideals of affinoid algebras are closed (see [**6**, Proposition 2.1.3]). The morphism $\mathcal{O}(D)(U) = A_U \hookrightarrow R_W = \mathcal{O}(D)(W)$ is the restriction morphism of the sheaf $\mathcal{O}(D)$, so it is bounded. The same is true for $A_V \hookrightarrow R_W$.

If $U \cup V$ is not the entire C, it is an affinoid domain thereof (see Theorem 1.8.15). By Tate's Acyclicity Theorem (Theorem 1.4.17),

$$0 \to H^0(U \cup V, \mathcal{O}(D)) \to H^0(U, \mathcal{O}(D)) \oplus H^0(V, \mathcal{O}(D)) \to H^0(U \cap V, \mathcal{O}(D)) \to 0$$

is an exact admissible sequence, from which we obtain the surjective admissible morphism $A_U \oplus A_V \twoheadrightarrow \mathcal{O}(D)(W) = F_W$.

Suppose $U \cup V = C$. Since C is then compact and integral, by Theorem 1.8.15, it is either an affinoid space (a case we dealt with in the paragraph above) or a projective curve. If C is projective, by [**49**, Section 7.5, Proposition 5.5] for large enough n, $H^1(U \cup V, \mathcal{O}(D)) = 0$. The Mayer-Vietoris exact sequence now produces a bounded surjective morphism $A_U \oplus A_V \twoheadrightarrow \mathcal{O}(D)(W) = F_W$. Admissibility follows from Banach's Open Mapping Theorem if k is non-trivially valued (for a proof see [**14**]), and by a change of basis followed by the Open Mapping Theorem if it is (see Proposition 1.3.8).

The remaining part of the statement was shown in Lemma 3.1.1 and Lemma 3.1.2. □

REMARK 3.1.4. Other examples of Setting 2.1.5 can be obtained by taking instead of $\mathcal{O}(D)$ any coherent sheaf $\mathcal{F}$ of $\mathcal{O}$-algebras that is a subsheaf of $\mathscr{M}$, for which $H^1(C, \mathcal{F}) = 0$.

We make note of the fact that Proposition 3.1.3 assumes the existence of a point of type 3, which is equivalent to $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$.

REMARK 3.1.5. Remark 2.1.4 tells us that the only reason k was assumed to be non-trivially valued in Chapter 2 is for the proof Lemma 2.1.3 to work. As we saw in Proposition 3.1.3, in the case of curves, the role of the ring R_0 is played by the field of meromorphic functions of a type 3 point. This means that R_0 is a non-trivially valued complete ultrametric field. As a consequence, the proof of Lemma 2.1.3 is valid regardless of the absolute value k is endowed with.

Thus, in the case of curves, patching is true even if the base field is trivially valued. As we will see, the trivially valued case provides no new information on the local-global principle, which is why we continue to disregard it.

Keeping the same notation as in Proposition 3.1.3, let G/F_C be a rational linear algebraic group. By Theorem 2.2.3, for any $g \in G(\mathscr{M}(\{\eta\}))$, there exist $g_U \in G(\mathscr{M}(U))$ and $g_V \in G(\mathscr{M}(V))$ such that $g = g_U \cdot g_V$ in $G(\mathscr{M}(\{\eta\}))$. We will generalize this result to one that applies to certain covers of the analytic curve. The latter generalize the the conditions of Proposition 3.1.3.

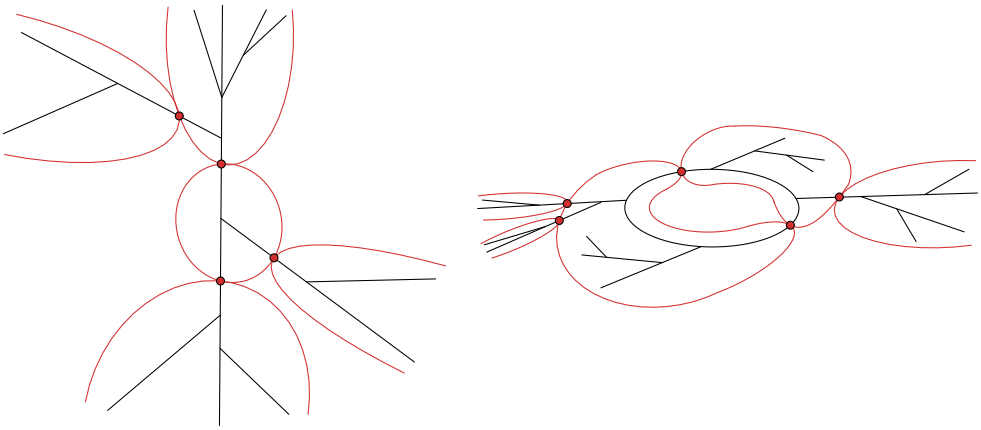
The above should serve as motivation for the following:

DEFINITION 3.1.6. A finite cover $\mathcal{U}$ of a k-analytic curve will be called *nice* if:

(1) the elements of $\mathcal{U}$ are connected affinoid domains with only type 3 points in their topological boundaries;
(2) for any different $U, V \in \mathcal{U}$, $U \cap V = \partial U \cap \partial V$, or equivalently, $U \cap V$ is a finite set of type 3 points;
(3) for any two different elements of $\mathcal{U}$, neither is contained in the other.

Let $\mathcal{V}$ be a cover of a k-analytic curve. We will say that a cover $\mathcal{U}$ of the same curve is a *nice refinement* of $\mathcal{V}$ if it is a refinement of $\mathcal{V}$ that is a nice cover.

We recall that we use the term *boundary* for the topological boundary.



a) *b*)

Figure 3. Examples of nice covers for:
a) a uniquely arcwise-connected curve; *b*) a non-uniquely arcwise-connected curve.

The definition above (as well as Proposition 3.1.3) highlights the importance of type 3 points. To insure their existence, for the rest of this section we assume that $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$ (recall Proposition 1.8.7).

We start our study of these covers by showing that, under certain conditions, for any open cover of a k-analytic curve, there exists a *nice refinement*.

3.1.2. Nice covers of $\mathbb{P}^{1,\mathrm{an}}_k$. Recall that $\mathbb{P}^{1,\mathrm{an}}_k$ is uniquely arcwise-connected. For any $x, y \in \mathbb{P}^{1,\mathrm{an}}_k$, let us denote by $[x, y]$ the unique arc connecting them. The next few properties of the projective line will be essential to the remainder of this section.

We can now show a special case of the result we prove in this section.

LEMMA 3.1.7. *Let C, D be connected affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_k$ with only type 3 points in their boundaries. There exists a nice refinement $\{C_1, \dots, C_n, D\}$ of the cover $\{C, D\}$ of $C \cup D$, such that $C_i \cap C_j = \emptyset$ for any $i \neq j$.*

PROOF. If $C = D$, it is straightforward. Otherwise, suppose $C \not\subseteq D$. By Corollary 1.8.17, $C \backslash \mathrm{Int}\ D$ is an affinoid domain in $\mathbb{P}^{1,\mathrm{an}}_k$ containing only type 3 points in its boundary. Let $C'_1, C'_2, \dots, C'_m$ be its connected components. They are mutually disjoint connected affinoid domains with only type 3 points in their boundaries. Furthermore, for any i, $C'_i \cap D \subseteq (\mathrm{Int}\ D)^c \cap D \subseteq \partial D$ is a finite set of type 3 points. As C'_i and D are connected, by Lemma 1.8.20, so is $C'_i \cap D$, meaning it is either empty or a single type 3 point. By construction, $C'_i \cap C'_j = \emptyset$ for all $i \neq j$, and $\{C'_1, C'_2, \dots, C'_m, D\}$ is a refinement of $\{C, D\}$. For any i, if C'_i is a single point, *i.e.* $C'_i \subseteq D$, we remove it from $\{C'_1, C'_2, \dots, C'_m\}$, and if not, we keep it there. Let $C_1, C_2, \dots, C_n$ be the remaining connected components of $C \backslash \mathrm{Int}\ D$. Then, $\{C_1, C_2, \dots, C_n, D\}$ is a nice refinement of the cover $\{C, D\}$ of $C \cup D$. □

The main result of this section in the case of the projective line is the following generalization:

PROPOSITION 3.1.8. *For any $n \in \mathbb{N}$, let $\{U_i\}_{i=1}^n$ be a set of affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_k$ with only type 3 points in their boundaries. Set $V_n = \bigcup_{i=1}^n U_i$. Then, there exists a nice cover of V_n that refines $\{U_i\}_{i=1}^n$, satisfying the following properties:*

1. *the intersection of any two of its elements is either empty or a single type 3 point;*
2. *if two domains of the refinement intersect, there is no third one that intersects them both.*

PROOF. We will use induction on the number of affinoids domains n. For $n = 1$, the statement is trivial. Suppose the proposition is true for any positive integer smaller or equal to some $n - 1$. Let $\{U_i\}_{i=1}^n$ be affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_k$ with only type 3 points in their boundaries. If they are all of empty interior, *i.e.* unions of points, then the statement is trivially true. Otherwise, let $i_0 \in \{1, 2, \dots, n\}$ be any index for which U_{i_0} has non-empty interior. To simplify the notation, suppose $i_0 = n$. By removing the U_i's contained in U_n if necessary, we may assume that for all i, $U_i \not\subseteq U_n$.

From Lemmas 1.8.17 and 3.1.7, $\mathcal{U} := \{U_n\} \cup \{U_i \cap (\mathrm{Int}\ U_n)^c\}_{i=1}^{n-1}$ is a refinement of $\{U_i\}_{i=1}^n$ containing affinoid domains with only type 3 points in their boundaries. Let $\{W_l\}_{l=1}^s$ be a nice refinement of $\{U_i \cap (\mathrm{Int}\ U_n)^c\}_{i=1}^{n-1}$. Then, for any l, $U_n \cap W_l \subseteq \partial U_n$. By removing those W_l for which $W_l \subseteq U_n$ if necessary, we obtain that $\{U_n\} \cup \{W_l\}_{l=1}^s$ is a nice refinement of $\{U_i\}_{i=1}^n$. The first condition of the statement is a direct consequence of Lemma 1.8.21.

We have proven that for any positive integer n, there exists a nice refinement of $\{U_i\}_{i=1}^n$, which satisfies the first property of the statement. Property 2 is immediate from the following:

LEMMA 3.1.9. *Let W_1, W_2, W_3 be three connected affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_k$ with non-empty interiors and only type 3 points in their boundaries. Suppose their interiors are mutually disjoint. Then, at least one of $W_1 \cap W_2, W_2 \cap W_3, W_3 \cap W_1$ is empty.*

PROOF. Suppose that $W_1 \cap W_2, W_2 \cap W_3$, and $W_3 \cap W_1$ are all non-empty. If $W_1 \cap W_2 \cap W_3 \neq \emptyset$, then by Lemma 1.8.20, it is a single type 3 point $\{z\}$. Since $\mathbb{P}^{1,an}_k \backslash \{z\}$ has exactly

two connected components, and the interiors of W_1, W_2, W_3 are non-empty and mutually disjoint, this is impossible. Hence, $W_1 \cap W_2 \cap W_3 = \emptyset$, and so $W_1 \cap W_2, W_2 \cap W_3$ and $W_3 \cap W_1$ are all non-empty and different. Since $W_1 \cap W_2 \neq \emptyset$, $W_1 \cup W_2$ is a connected affinoid domain with only type 3 points in its boundary. Furthermore, $\mathrm{Int}(W_3) \cap \mathrm{Int}(W_1 \cup W_2) \subseteq (W_3 \cap W_1) \cup (W_3 \cap W_2)$, and since this is a finite set of type 3 points, $\mathrm{Int}(W_3) \cap \mathrm{Int}(W_1 \cup W_2) = \emptyset$.

Thus, the interior of $W_1 \cup W_2$ is disjoint to the interior of W_3. By Lemma 1.8.21, $(W_1 \cup W_2) \cap W_3$ is a single type 3 point. But, $W_1 \cap W_3$ and $W_2 \cap W_3$ were both assumed to be non-empty and shown to be different, implying $(W_1 \cap W_3) \cup (W_2 \cap W_3) = (W_1 \cup W_2) \cap W_3$ contains at least two different points, contradiction.

Thus, at least one of $W_1 \cap W_2, W_2 \cap W_3, W_3 \cap W_1$ must be empty. □

This completes the proof of the proposition. □

In view of Theorem 1.8.7, we obtain:

THEOREM 3.1.10. *Any open cover of a compact subset of $\mathbb{P}^{1,\mathrm{an}}_k$ has a nice refinement.*

The following will be needed later:

LEMMA 3.1.11. *Let A be a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_k$. Let S be a finite subset of $\mathrm{Int}(A)$ containing only type 3 points. There exists a nice cover $\mathcal{A}$ of A, such that the set of points of intersection of different elements of $\mathcal{A}$ is S.*

PROOF. Seeing as S consists of type 3 points, they are all contained in a copy of $\mathbb{A}^{1,\mathrm{an}}_k$ in $\mathbb{P}^{1,\mathrm{an}}_k$. Thus, for any element $\eta \in S$, there exists an irreducible polynomial P over k and a real number $r \notin \sqrt{|k^\times|}$, such that $\eta = \eta_{P,r}$ (*cf.* Proposition 1.8.25).

Let us prove the statement using induction on the cardinality of S. If S is empty, then the statement is trivially true. Suppose we know the statement is true if the cardinality of S is equal to some $n-1$.

Let us assume S contains n points. Fix some element $\eta_{P,r} \in S$. Let $\mathcal{U}$ be a nice cover of A that satisfies the properties of the statement for $S' := S \backslash \{\eta_{P,r}\}$. There exists a unique $U \in \mathcal{U}$, such that $\eta_{P,r} \in U$, in which case $\eta_{P,r} \in \mathrm{Int}(U)$. Then, $\{U \cap \{|P| \leqslant r\}, U \cap \{|P| \geqslant r\}\} \cup \{V \in \mathcal{U} : V \neq U\}$ is a nice cover that fulfills our requirements. □

3.1.3. Nice Covers of a Berkovich Curve.

PROPOSITION 3.1.12. *Let C be an irreducible projective generically smooth k-analytic curve. There exists a type 3 point η in C such that $C \backslash \{\eta\}$ has exactly two connected components E_1, E_2. Furthermore, $E_1 \cup \{\eta\}, E_2 \cup \{\eta\}$ are affinoid domains of C.*

PROOF. By [**20**, Théorème 3.7.2], there exists an algebraic projective curve C^{alg}/k such that $(C^{\mathrm{alg}})^{\mathrm{an}} = C$. By [**6**, Theorem 3.4.1], there is a bijection between the closed points of C^{alg} and the rigid points of C, meaning the latter are Zariski dense in C. As C is generically smooth, by [**21**, Théorème 3.4], the smooth locus of C is a non-empty Zariski open of C. Consequently, there exists η_0 - a rigid smooth point in C.

By [**20**, Théorème 4.5.4], there exists a neighborhood D' of η_0 in C which is a virtual disc. By density of type 3 points in C (Proposition 1.8.7), there exists a type 3 point $\eta \in D'$. By [**20**, 3.6.34], D' is uniquely arcwise-connected with a single boundary point x. By [**20**, 1.4.21], $D := \overline{D'}$ - the closure of D in C, is uniquely arcwise-connected. Remark that $\partial D = \{x\}$, and $D = D' \cup \{x\}$.

As it is of type 3, by [**20**, 4.2.11.2], there exist at most two *branches* coming out of η, and there are exactly two if and only if $\eta \in \mathrm{Int}_B(D)$. As $\eta \in \mathrm{Int}(D) = \mathrm{Int}_B(D)$ (Theorem 1.5.27), there are two branches coming out of η. As D is uniquely arcwise-connected, by [**20**, 1.3.12], this means that $D\backslash\{\eta\}$ has exactly two connected components. Let us denote them by A and B, and assume, without loss of generality, that $x \in B$. Remark that $A \subseteq D'$.

Set $E := (C\backslash D)\cup\{x\} = C\backslash D'$. Let us show that E is connected. Let $a, b \in E$. Since C is connected, by Corollary 1.5.43, there exists an arc $[a, b]$ in C connecting a and b. Suppose $[a, b] \cap D' \neq \emptyset$. Let $d \in [a, b] \cap D'$. Then, $[a, b]$ induces arcs $[a, d]$ and $[d, b]$ in C connecting a and d, resp. d and b. As $a, b \notin D'$ and $d \in D'$, we obtain that $[a, d] \cap \partial D, [d, b] \cap \partial D \neq \emptyset$, so $x \in [a, d]$ and $x \in [d, b]$. This contradicts the injectivity of $[a, b]$ unless $x = d$, which is impossible seeing as $x \notin D'$. Thus, $[a, b] \cap D' = \emptyset$, *i.e.* $[a, b] \subseteq E$, implying E is connected.

As B, E are connected, and $B \cap E = \{x\}$, $G := B \cup E$ is a connected subset of C. Remark that $A \cap G = (A \cap B) \cup (A \cap E) \subseteq D' \cap E = \emptyset$. Also, $A \cup G \cup \{\eta\} = A \cup B \cup E \cup \{\eta\} = D \cup E = C$.

It only remains to show that $A' := A \cup \{\eta\}$ and $G' := G \cup \{\eta\}$ are affinoid domains in C. By [**20**, Proposition 4.2.14], they are both closed analytic domains in C. As C is projective, it is boundaryless, so $\partial_B(A') = \partial A' = \{\eta\}$, and the same is true for G' (Proposition 1.8.10). Let I be an irreducible component of A' (resp. G'). By [**20**, 3.2.3], if $\partial_B(I) = \emptyset$, then $I = C$, implying A' (resp. G') is C, which is false. Hence, $\partial_B(I) \neq \emptyset$.

As I is a Zariski closed subset of A' (resp. G'), there exists a closed immersion (hence, a finite morphism) $I \to A'$ (resp. $I \to G'$). By Proposition 1.5.31 and Theorem 1.5.27, $\partial_B(I)$ is a subset of $I\backslash\mathrm{Int}_B(A')$ (resp. $I\backslash\mathrm{Int}_B(G')$). Hence, $\partial_B(I)$ is a non-empty subset of $\partial_B(A')$ (resp. $\partial_B(G')$). We conclude by [**20**, Théorème 6.1.3]. □

Remark 3.1.13. In general, $C\backslash\{\eta\}$ has at most two connected components "around" η, and it might happen that it has exactly one (for example in a Tate curve), see also [**20**, 4.2.11.2] and the remarks made after Lemma 3.2.7.

Proposition 3.1.14. *Let C be a normal connected projective k-algebraic curve. Then, there exists a nice cover $\{U_1, U_2\}$ of C^{an} - the Berkovich analytification of C, such that $U_1 \cap U_2$ is a single type 3 point.*

Proof. Let $C \to \mathbb{P}^1_k$ be a finite morphism. It induces an embedding of function fields $k(\mathbb{P}^1_k) \hookrightarrow k(C)$. Let K be the separable closure of $k(\mathbb{P}^1_k)$ in $k(C)$. There exists a connected normal projective algebraic curve Y over k, such that $k(Y) = K$. Since the field extension $K/k(\mathbb{P}^1_k)$ is separable, the induced morphism $Y \to \mathbb{P}^1_k$ is generically étale, so Y is a generically smooth curve. In particular, this implies that the k-analytic curve Y^{an} is generically smooth ([**21**, Théorème 3.4]). At the same time, since the finite extension $k(C)/K$ is purely inseparable, the induced finite type morphism $C \to Y$ is a homeomorphism. Consequently, by [**6**, Proposition 3.4.6], its analytification $f : C^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a finite morphism that is a homeomorphism.

By Proposition 3.1.12, there exists a nice cover $\{U'_1, U'_2\}$ of Y^{an}, such that $U'_1 \cap U'_2$ is a single type 3 point. Seeing as f is finite and a homeomorphism, $U_i := f^{-1}(U'_i), i = 1, 2$, is a connected affinoid domain, and $U_1 \cap U_2$ is a single type 3 point. □

Definition 3.1.15. For a nice cover $\mathcal{U}$ of a k-analytic curve, let us denote by $S_{\mathcal{U}}$ the finite set of type 3 points that are in the intersections of different elements of $\mathcal{U}$.

Remark that for a nice cover $\mathcal{U}$ of a k-analytic curve C, if $s \in S_{\mathcal{U}}$, the set $\{s\}$ is an affinoid domain of C. This is because $\{s\}$ is a connected component of the intersection of two affinoid domains.

The following notion will be needed in what follows.

DEFINITION 3.1.16. Let C be a k-analytic curve. Let $\mathcal{U}$ be a nice cover of C. A function $T_{\mathcal{U}} : \mathcal{U} \to \{0,1\}$ will be called *a parity function for* $\mathcal{U}$ if for any different $U', U'' \in \mathcal{U}$ such that $U' \cap U'' \neq \emptyset$, one has $T_{\mathcal{U}}(U') \neq T_{\mathcal{U}}(U'')$.

LEMMA 3.1.17. *For any $n \in \mathbb{N}$, let $U_1, U_2, \ldots, U_n$ be affinoid domains in $\mathbb{P}^{1,an}_k$ such that $\mathcal{U}_n := \{U_i\}_{i=1}^n$ is a nice cover of $K_n := \bigcup_{i=1}^n U_i$. Then, there exists a parity function $T_{\mathcal{U}_n}$ for $\mathcal{U}_n$.*

PROOF. It suffices to prove the result under the assumption that K_n is connected. We will use induction on the cardinality n of $\mathcal{U}_n$. If $n = 1$, the statement is trivially true. Suppose it to be true for some $n-1$.

LEMMA 3.1.18. *Let Z be a topological space. For any positive integer m, let $\{W_i\}_{i=1}^m$ be a set of closed connected subsets of Z. Suppose $\bigcup_{i=1}^m W_i$ is connected. Then, there exists $i_0 \in \{1, 2, \ldots, m\}$, such that $\bigcup_{i \neq i_0} W_i$ is connected.*

PROOF. Let l be the largest integer such that $l < m$ and there exist $W_{i_1}, W_{i_2}, \ldots, W_{i_l}$, with $\bigcup_{j=1}^l W_{i_j}$ connected. As all the W_i are connected, $l > 0$. Set $J = \{1, 2, \ldots, m\} \backslash \{i_1, i_2, \ldots, i_l\}$. If $l < m-1$, then for any $p \in J$, we obtain $W_p \cap \bigcup_{j=1}^l W_{i_j} = \emptyset$. This implies that $\left(\bigcup_{p \in J} W_p\right) \cap \left(\bigcup_{j=1}^l W_{i_j}\right) = \emptyset$, which contradicts the connectedness of $\bigcup_{i=1}^m W_i$. Thus, $l = m-1$. □

Seeing as $\bigcup_{i=1}^n U_i$ is connected, from Lemma 3.1.18, there exist $n-1$ elements of $\mathcal{U}_n$ whose union remains connected. For simplicity of notation, assume them to be the elements of $\mathcal{U}_{n-1} := \{U_1, U_2, \ldots, U_{n-1}\}$. Then, $\mathcal{U}_{n-1}$ is a nice cover of the connected set $K_{n-1} := \bigcup_{i=1}^{n-1} U_i$. Let $T_{\mathcal{U}_{n-1}}$ be a parity function for $\mathcal{U}_{n-1}$. By Lemma 1.8.21, $U_n \cap \bigcup_{i=1}^{n-1} U_i$ is a single type 3 point, so by Lemma 3.1.9, U_n intersects exactly one of the elements of $\mathcal{U}_{n-1}$. Without loss of generality, suppose it to be U_{n-1}. Define $T_{\mathcal{U}_n}$ as follows:

(1) for any $U \in \mathcal{U}_{n-1}$, $T_{\mathcal{U}_n}(U) := T_{\mathcal{U}_{n-1}}(U)$;
(2) $T_{\mathcal{U}_n}(U_n) := 1 - T_{\mathcal{U}_{n-1}}(U_{n-1})$.

The function $T_{\mathcal{U}_n}$ is a parity function for $\mathcal{U}_n$. □

PROPOSITION 3.1.19. *Let Y, Z be k-analytic curves with Y normal and Z compact. Let $f : Z \to Y$ be a finite surjective morphism. Suppose $\mathcal{V}$ is a nice cover of Y. Then, the connected components of $f^{-1}(V), V \in \mathcal{V}$, form a nice cover $\mathcal{U}$ of Z, such that $f^{-1}(S_{\mathcal{V}}) = S_{\mathcal{U}}$.*

Furthermore, if $T_{\mathcal{V}}$ is a parity function for $\mathcal{V}$, then the function $T_{\mathcal{U}}$ that to an element $U \in \mathcal{U}$ associates $T_{\mathcal{V}}(f(U))$, is a parity function for $\mathcal{U}$.

PROOF. Since Z is compact and Y is Hausdorff, f is a closed morphism. By [**20**, 3.5.12], f is open.

If V is any connected affinoid domain of Y, for any connected component V_0' of $f^{-1}(V)$, $f(V_0') = V$. To see this, recall that by [**4**, Lemma 1.3.7], $f_{|f^{-1}(V)} : f^{-1}(V) \to V$ is a finite morphism of affinoid spaces, and by [**21**, Théorème 3.4], as Y is normal, so is V. Thus, $f_{|f^{-1}(V)}$ is open and closed. Seeing as V_0' is a connected component of $f^{-1}(V)$, it is both

open and closed in $f^{-1}(V)$, so its image is both open and closed in V. As V is connected, $f(V_0') = V$.

The connected components of $f^{-1}(V)$ for all $V \in \mathcal{V}$ form a finite cover $\mathcal{U}$ of Z consisting of affinoid domains (see Corollary 1.4.19). As f is open, for any $V \in \mathcal{V}$, $\partial(f^{-1}(V)) = f^{-1}(\partial V)$. Since a finite morphism preserves the type of point (Lemma 1.8.2), $\partial f^{-1}(V)$ is an affinoid domain containing only type 3 points in its boundary. Thus, the elements of $\mathcal{U}$ are connected affinoid domains containing only type 3 points in their boundaries.

Let $U_1, U_2 \in \mathcal{U}$ be such that $U_1 \cap U_2 \neq \emptyset$. Set $V_i = f(U_i), i = 1, 2$. Then, $V_1, V_2 \in \mathcal{V}$, and $V_1 \neq V_2$. To see the second part, if $V_1 = V_2$, then U_1, U_2 would be connected components of $f^{-1}(V_1)$, thus disjoint, which contradicts the assumption $U_1 \cap U_2 \neq \emptyset$. Seeing as $U_1 \cap U_2 \subseteq f^{-1}(V_1 \cap V_2)$, $U_1 \cap U_2$ is a finite set of type 3 points. Hence, $U_1 \cap U_2 = \partial U_1 \cap \partial U_2$. The third condition of a nice cover is trivially satisfied. Since $f^{-1}(\partial V) = \partial f^{-1}(V)$ for all $V \in \mathcal{V}$, it follows that $f^{-1}(S_{\mathcal{V}}) = S_{\mathcal{U}}$. Finally, $T_{\mathcal{U}}(U_1) = T_{\mathcal{V}}(V_1) \neq T_{\mathcal{V}}(V_2) = T_{\mathcal{U}}(U_2)$, so $T_{\mathcal{U}}$ is a parity function for $\mathcal{U}$. □

COROLLARY 3.1.20. *Let C be a normal projective k-analytic curve or a strict k-affinoid curve. Any open cover of C has a nice refinement.*

PROOF. By Theorem 1.8.7, we may assume that the open cover only contains elements with finite boundary consisting of type 3 points. Since C is compact, there is a finite subcover $\mathcal{U}$ of the starting open cover. Set $S := \bigcup_{U \in \mathcal{U}} \partial U$. Suppose C is projective. Then, there exists a finite surjective morphism $C \to \mathbb{P}^{1,\mathrm{an}}_k$. Set $S' = f(S)$. By Lemma 3.1.11, there exists a nice cover $\mathcal{D}$ of $\mathbb{P}^{1,\mathrm{an}}_k$, such that $S_{\mathcal{D}} = S'$. We conclude by applying Proposition 3.1.19.

If C is a strict k-affinoid curve, by Noether's Normalization Lemma there exists a finite surjective morphism $C \to \mathbb{D}$, where $\mathbb{D}$ is the closed unit disc in $\mathbb{P}^{1,\mathrm{an}}_k$. We conclude as above. □

3.2. A Local-Global Principle over Berkovich Curves

Unless mentioned otherwise, throughout this section we assume that k is a complete non-trivially valued ultrametric field such that $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$.

DEFINITION 3.2.1 (HHK). Let F be a field. A linear algebraic group G over F acts *strongly transitively* on an F-variety X if G acts on X and for any field extension E/F, either $X(E) = \emptyset$ or the action of $G(E)$ on $X(E)$ is transitive.

We start by showing some patching results over nice covers. Recall that we denote by $\mathscr{M}$ the sheaf of meromorphic functions.

3.2.1. Patching over nice covers. We show a generalized form of patching (with respect to the one seen in Chapter 2) with is applicable to nice covers.

PROPOSITION 3.2.2. *Let D be $\mathbb{P}^{1,\mathrm{an}}_k$ or a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_k$. Let $\mathcal{D}$ be a nice cover of D, and $T_{\mathcal{D}}$ a parity function for $\mathcal{D}$. Let $G/\mathscr{M}(D)$ be a rational linear algebraic group. Then, for any $(g_s)_{s \in S_{\mathcal{D}}} \in \prod_{s \in S_{\mathcal{D}}} G(\mathscr{M}(\{s\}))$, there exists $(g_U)_{U \in \mathcal{D}} \in \prod_{U \in \mathcal{D}} G(\mathscr{M}(U))$, satisfying: for any $s \in S_{\mathcal{D}}$, if U_0, U_1 are the elements of $\mathcal{D}$ containing s, and $T_{\mathcal{D}}(U_0) = 0$, then $g_s = g_{U_0} \cdot g_{U_1}^{-1}$ in $G(\mathscr{M}(\{s\}))$.*

PROOF. We will use induction on the cardinality n of a nice cover. If $n = 2$, then this is Theorem 2.2.3 (considering Proposition 3.1.3 with $\mathcal{O}(D) = \mathcal{O}$). Suppose the result is true

for some $n-1$. If $\mathcal{D} := \{U_1, U_2, \dots, U_n\}$, since $\bigcup_{i=1}^{n} U_i$ is connected, from Lemma 3.1.18, there exist $n-1$ elements of $\mathcal{U}$ whose union remains connected. For simplicity of notation, suppose them to be the elements of $\mathcal{D}' := \{U_1, U_2, \dots, U_{n-1}\}$. By Lemma 1.8.21, $\bigcup_{i=1}^{n-1} U_i \cap U_n$ is single type 3 point, so by Lemma 3.1.9, U_n intersects exactly one of the elements of $\mathcal{D}'$. To simplify the notation, suppose it to be U_{n-1}. Set $\{\eta\} := U_{n-1} \cap U_n$, so that $S_{\mathcal{D}} = S_{\mathcal{D}'} \cup \{\eta\}$.

Let $(g_s)_{s\in S_{\mathcal{D}}}$ be any element of $\prod_{s\in S_{\mathcal{D}}} G(\mathscr{M}(\{s\}))$. By the induction hypothesis, for $(g_s)_{s\in S_{\mathcal{D}'}} \in \prod_{s\in S_{\mathcal{D}'}} G(\mathscr{M}(\{s\}))$, there exists $(g_U)_{U\in\mathcal{D}'} \in \prod_{U\in\mathcal{D}'} G(\mathscr{M}(U))$, satisfying the conditions of the statement.

- Suppose $T_{\mathcal{D}}(U_n) = 0$. By Theorem 2.2.3, there exist $a \in G(\mathscr{M}(U_n))$ and $b \in G(\mathscr{M}(\bigcup_{i=1}^{n-1} U_i))$ such that $g_\eta \cdot g_{U_{n-1}} = a \cdot b$ in $G(\mathscr{M}(\{\eta\}))$. For any $i \neq n$, set $g'_{U_i} := g_{U_i} \cdot b^{-1}$ in $G(\mathscr{M}(U_i))$. Also, set $g'_{U_n} := a$ in $G(\mathscr{M}(U_n))$.
- Suppose $T_{\mathcal{D}}(U_n) = 1$. By Theorem 2.2.3, there exist $c \in G(\mathscr{M}(\bigcup_{i=1}^{n-1} U_i))$ and $d \in G(\mathscr{M}(U_n))$ such that $g_{U_{n-1}}^{-1} \cdot g_\eta = c \cdot d$ in $G(\mathscr{M}(\{\eta\}))$. For any $i \neq n$, set $g'_{U_i} := g_{U_i} \cdot c$ in $G(\mathscr{M}(U_i))$. Also, set $g'_{U_n} := d^{-1}$ in $G(\mathscr{M}(U_n))$.

The family $(g'_{U_i})_{i=1}^{n} \in \prod_{i=1}^{n} G(\mathscr{M}(U_i))$ satisfies the conditions of the statement for $(g_s)_{s\in S_{\mathcal{D}}}$. □

PROPOSITION 3.2.3. *Let Y be an integral strict k-affinoid curve. Set $K = \mathscr{M}(Y)$. Let G/K be a connected rational linear algebraic group. For any open cover $\mathcal{V}$ of Y, there exists a nice refinement $\mathcal{U}$ of $\mathcal{V}$ with a parity function $T_{\mathcal{U}}$, such that for any given $(g_y)_{y\in S_{\mathcal{U}}} \in \prod_{y\in S_{\mathcal{U}}} G(\mathscr{M}(\{y\}))$, there exists $(g_U)_{U\in\mathcal{U}} \in \prod_{U\in\mathcal{U}} G(\mathscr{M}(U))$, satisfying: for any $y \in S_{\mathcal{U}}$, there are exactly two elements U', U'' of $\mathcal{U}$ containing s, and if $T_{\mathcal{U}}(U') = 0$, then $g_y = g_{U'} \cdot g_{U''}^{-1}$ in $G(\mathscr{M}\{y\})$.*

PROOF. By Proposition 1.8.7, we may assume that the cover $\mathcal{V}$ only contains elements with finite boundary consisting of only type 3 points. Since Y is compact, we may also assume that $\mathcal{V}$ is finite.

Let $f : Y \to \mathbb{D}$ be a finite surjective morphism we obtain from Noether's Normalization Lemma, where $\mathbb{D}$ is the closed unit disc in $\mathbb{P}_k^{1,\mathrm{an}}$. Set $S = f(\bigcup_{V\in\mathcal{V}} \partial V)$. It is a finite set of type 3 points. By Lemma 3.1.11, there exists a nice cover $\mathcal{D}$ of $\mathbb{D}$ such that $S_{\mathcal{D}} = S$. Let $T_{\mathcal{D}}$ be a parity function for $\mathcal{D}$ (it exists by Lemma 3.1.17). From Proposition 3.1.19, the connected components of $f^{-1}(Z')$, $Z' \in \mathcal{D}$, form a nice cover $\mathcal{U}$ of Y such that $f^{-1}(S_{\mathcal{D}}) = S_{\mathcal{U}}$, and $T_{\mathcal{D}}$ induces a parity function $T_{\mathcal{U}}$ for $\mathcal{U}$.

Let us show that $\mathcal{U}$ refines $\mathcal{V}$. Suppose, by contradiction, that $Z \in \mathcal{U}$ is such that there does not exist an element of $\mathcal{V}$ containing it. Then, there must exist $a \in \bigcup_{V\in\mathcal{V}} \partial V \subseteq S_{\mathcal{U}}$ such that $a \in \mathrm{Int}(Z)$. Since $a \in S_{\mathcal{U}}$ there exists $U \in \mathcal{U}$, such that $a \in \partial U$. But then, $Z \cap U \neq \partial Z \cap \partial U$, which contradicts the fact that $\mathcal{U}$ is a nice cover of Y. Consequently, $\mathcal{U}$ must refine $\mathcal{V}$.

Suppose that for $s \in S_{\mathcal{U}}$ there exist different $U_1, U_2, U_3 \in \mathcal{U}$ containing s. Then, $f(s) \in V_1 \cap V_2 \cap V_3$, where $V_i := f(U_i) \in \mathcal{D}$, $i = 1, 2, 3$, (the fact that $V_i \in \mathcal{D}$ was shown in the beginning of the proof of Proposition 3.1.19). By Lemma 3.1.9, this is only possible if at least two of the V_1, V_2, V_3 coincide. Suppose, without loss of generality, that $V_1 = V_2$. Then, U_1, U_2 are connected components of $f^{-1}(V_1)$, so $U_1 \cap U_2 = \emptyset$, contradiction. Hence, for any $s \in \mathcal{S}_{\mathcal{U}}$, there exist at most two elements of $\mathcal{U}$ containing s. Considering the definition of $S_{\mathcal{U}}$, there must exist exactly two.

Set $G' = \mathcal{R}_{K/\mathscr{M}(\mathbb{D})}(G)$ - the Weil restriction of scalars from K to $\mathscr{M}(\mathbb{D})$ of G. It is still a connected rational linear algebraic group (see [**12**, 7.6] or [**55**, Section 1], this is where the connectedness assumption is necessary).

LEMMA 3.2.4. *For any point s of type 3 in $\mathbb{D}$, $\mathscr{M}(\{s\}) \otimes_{\mathscr{M}(\mathbb{D})} \mathscr{M}(Y) = \prod_{x\in f^{-1}(s)} \mathscr{M}(\{x\})$.*

PROOF. Seeing as s is a type 3 point, the set $f^{-1}(s)$ is finite consisting of only type 3 points. Hence, $\mathcal{O}(\{s\}) = \mathscr{M}(\{s\})$, and $\mathcal{O}(\{x\}) = \mathscr{M}(\{x\})$ for all $x \in f^{-1}(s)$ (recall Lemma 3.1.1).

Set $A := \mathcal{O}(\mathbb{D}), B := \mathcal{O}(Y)$, and $C := \mathcal{O}(\{s\})$. Let us denote by T the set of non-zero elements of A. We know that $C \otimes_A B = \prod_{x\in f^{-1}(s)} \mathcal{O}(\{x\}) = \prod_{x\in f^{-1}(s)} \mathscr{M}(\{x\})$. Then, localizing on both sides, we obtain: $T^{-1}(C \otimes_A B) = C \otimes_{S^{-1}A} T^{-1}B$ and $T^{-1}\left(\prod_{x\in f^{-1}(s)} \mathscr{M}(\{x\})\right) = \prod_{x\in f^{-1}(s)} \mathscr{M}(\{x\})$.

Since B is a finite A-module, $T^{-1}B$ is a domain that is a finite dimensional $T^{-1}A$-vector space. Then, for any $b \in B\backslash\{0\}$, the map $T^{-1}B \to T^{-1}B, \alpha \mapsto b\alpha$ is injective, so surjective. Thus, there exists $b' \in T^{-1}B$ such that $bb' = 1$, implying $T^{-1}B = \text{Frac } T$. Consequently, $T^{-1}(C \otimes_A B) = \mathscr{M}(\{s\}) \otimes_{\mathscr{M}(\mathbb{D})} \mathscr{M}(Y)$. □

By the universal property of the Weil restrictio of scalars, for any $s \in S_{\mathcal{D}}$, $G'(\mathscr{M}(\{s\})) = G(\mathscr{M}(\{s\}) \otimes_{\mathscr{M}(\mathbb{D})} \mathscr{M}(Y))$. By the lemma above, $G'(\mathscr{M}(\{s\})) = \prod_{x\in f^{-1}(s)} G(\mathscr{M}(\{x\}))$.

Consequently, $(g_y)_{y\in S_{\mathcal{U}}} \in \prod_{y\in S_{\mathcal{U}}} G(\mathscr{M}(\{y\}))$ determines uniquely an element $(h_s)_{s\in S_{\mathcal{D}}}$ of $\prod_{s\in S_{\mathcal{D}}} G'(\mathscr{M}(\{s\}))$. By Proposition 3.2.2, there exists $(h_Z)_{Z\in\mathcal{D}} \in \prod_{Z\in\mathcal{D}} G'(\mathscr{M}(Z))$, such that if for two different $Z_0, Z_1 \in \mathcal{D}$ with $T_{\mathcal{D}}(Z_0) = 0$, $s \in Z_0 \cap Z_1$, then $h_s = h_{Z_0} \cdot h_{Z_1}^{-1}$ in $G'(\mathscr{M}(\{s\}))$.

For any $Z \in \mathcal{D}$, let $Z_1, Z_2, \ldots, Z_r$ be the connected components of $f^{-1}(Z)$. The application $\mathscr{M}(Z) \otimes_{\mathscr{M}(\mathbb{D})} \mathscr{M}(Y) \to \prod_{i=1}^r \mathscr{M}(Z_i)$ induces a map $G'(\mathscr{M}(Z)) = G(\mathscr{M}(Z) \otimes_{\mathscr{M}(\mathbb{D})} \mathscr{M}(Y)) \to \prod_{i=1}^r G(\mathscr{M}(Z_i))$, which sends h_Z to an element $(g_{Z_1}, g_{Z_2}, \ldots, g_{Z_r})$ of $\prod_{i=1}^r G(\mathscr{M}(Z_i))$. Thus, for any $U \in \mathcal{U}$, we have an element $g_U \in G(\mathscr{M}(U))$. It remains to show that given different $U_0, U_1 \in \mathcal{U}$ with $T_{\mathcal{U}}(U_0) = 0$, such that $y \in U_0 \cap U_1$ for some $y \in S_{\mathcal{U}}$, we have $g_y = g_{U_0} \cdot g_{U_1}^{-1}$ in $G(\mathscr{M}(\{y\}))$. This is a consequence of the analogue result for $(h_s)_{s\in S_{\mathcal{D}}}$ and $(h_Z)_{Z\in\mathcal{D}}$, the relation between $T_{\mathcal{D}}$ and $T_{\mathcal{U}}$, and of the commutativity of the following diagram for any $Z \in \mathcal{D}$ and any $s \in Z$ of type 3:

$$\begin{array}{ccc} \mathscr{M}(Z) & \longrightarrow & \mathscr{M}_{\mathbb{D}}(\{s\}) \\ \downarrow & & \downarrow \\ \prod_{i=1}^r \mathscr{M}(Z_i) & \longrightarrow & \prod_{y\in f^{-1}(s)} \mathscr{M}_Y(\{y\}) \end{array}$$

□

3.2.2. Local-global principles over analytic curves. We now apply patching over nice covers to obtain local-global principles. Throughout this section, unless mentioned otherwise, k denotes a non-trivially valued complete ultrametric field such that $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$.

PROPOSITION 3.2.5. *Let Y be a normal irreducible strict k-affinoid curve. Set $K = \mathscr{M}(Y)$. Let X/K be a variety, and G/K a connected rational linear algebraic group acting strongly transitively on X. The following local-global principles hold:*

- *$X(K) \neq \emptyset \iff X(\mathscr{M}_x) \neq \emptyset$ for all $x \in Y$;*
- *for any open cover $\mathcal{P}$ of Y, $X(K) \neq \emptyset \iff X(\mathscr{M}(U)) \neq \emptyset$ for all $U \in \mathcal{P}$.*

PROOF. Since Y is irreducible and normal, $\mathcal{O}_x$ is a domain for all $x \in Y$, and $\mathscr{M}_x = \operatorname{Frac} \mathcal{O}_x$.

Seeing as $K \hookrightarrow \mathscr{M}_x$ for all $x \in Y$, the implication " $\Rightarrow$ " is true.

Suppose $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in Y$. Then, there exists an open cover $\mathcal{V}$ of Y such that for any $V \in \mathcal{V}$, $X(\mathscr{M}(V)) \neq \emptyset$. Let $\mathcal{U}$ be a nice refinement of $\mathcal{V}$ given by Proposition 3.2.3, and $T_{\mathcal{U}}$ its associated parity function. Remark that for any $U \in \mathcal{U}$, we have $X(\mathscr{M}(U)) \neq \emptyset$.

For $U \in \mathcal{U}$, let $x_U \in X(\mathscr{M}(U))$. For any $y \in S_{\mathcal{U}}$, there exists exactly one element $U_i \in \mathcal{U}$, with $T_{\mathcal{U}}(U_i) = i$, $i = 0, 1$, containing y. From the transitivity of the action of G, there exists $g_y \in G(\mathscr{M}(\{y\}))$ such that $x_{U_0} = g_y \cdot x_{U_1}$ in $G(\mathscr{M}(\{y\}))$. This gives us an element $(g_y)_{y \in S_{\mathcal{U}}} \in \prod_{y \in S_{\mathcal{U}}} G(\mathscr{M}(\{y\}))$. By Proposition 3.2.3, there exists $(g_U)_{U \in \mathcal{U}} \in \prod_{U \in \mathcal{U}} G(\mathscr{M}(U))$, satisfying: for any different $U', U'' \in \mathcal{U}$ containing some point $y \in S_{\mathcal{U}}$ such that $T_{\mathcal{U}}(U') = 0$ (implying $T_{\mathcal{U}}(U'') = 1$), $g_y = g_{U'} \cdot g_{U''}^{-1}$ in $G(\mathscr{M}\{y\})$.

For any $U \in \mathcal{U}$, set $x'_U = g_U^{-1} \cdot x_U \in X(\mathscr{M}(U))$. We have construced a meromorphic function over U for any $U \in \mathcal{U}$. Let us show that these meromorphic functions are compatible, *i.e.* that they coincide on the intersections of the elements of $\mathcal{U}$. Let $D, E \in \mathcal{U}$ be such that $D \cap E \neq \emptyset$. Suppose $T_{\mathcal{U}}(D) = 0$. For any $s \in D \cap E$, $x'_E = g_E^{-1} \cdot x_E = g_D^{-1}(g_D g_E^{-1}) \cdot x_E = g_D^{-1}(g_s \cdot x_E) = g_D^{-1} x_D = x'_D$ in $X(\mathscr{M}(\{s\}))$. Consequently, $x'_E = x'_D$ in $X(\mathscr{M}(E \cap D))$.

Compatibility of these meromorphic functions implies they can be glued to give a meromorphic function on the entire Y. Thus, $X(K) = X(\mathscr{M}(Y)) \neq \emptyset$.

The second version of this local-global principle is a direct consequence of the first one. □

Let us show the same result (Theorem 3.2.9) for any k-affinoid space. Recall that we denote by $\Gamma(\cdot)$ the Shilov boundary of an affinoid space.

LEMMA 3.2.6. *Let k be a complete ultrametric field. Let E be a k-affinoid space. Let e be any point of E. Then, the following statements are equivalent:*

1. *there exists an affinoid neighborhood N_0 of e in E such that $e \in \Gamma(N_0)$;*
2. *for any affinoid neighborhood N of e in E, $e \in \Gamma(N)$;*
3. *$e \in \Gamma(E)$.*

PROOF. Suppose there exists an affinoid neighborhood N_0 of e in E, such that $e \in \Gamma(N_0)$. By [**6**, Proposition 2.5.20], $\Gamma(N_0) \subseteq \partial_B(N_0/E) \cup (\Gamma(E) \cap N_0)$. Since $\partial_B(N/E)$ is the topological boundary of N_0 in E (see [**6**, Corollary 2.5.13 (ii)]), we obtain that $e \notin \partial_B(N_0/E)$, implying $e \in \Gamma(E) \cap N_0 \subseteq \Gamma(E)$.

On the other hand, if $e \in \Gamma(E)$, for any affinoid neighborhood N of e in E, since $\Gamma(E) \cap N \subseteq \Gamma(N)$ (see [**6**, Proposition 2.5.20]), we obtain $e \in \Gamma(N)$. □

LEMMA 3.2.7. *Let Y be an integral k-affinoid curve. Let $y \in Y$ be any point of type 3, and Z a connected affinoid neighborhood of y in Y. Then,*

1. *the subspace $Y \backslash \{y\}$ has at most two connected components at the neighborhood of y; it is connected at the neighborhood of y if and only if $y \in \Gamma(Y)$;*

(2) *if* $y \in \Gamma(Y)$, *then there exist connected affinoid domains* A, B *of* Y, *such that* A *is a neighborhood of* y *in* Z, $\Gamma(Y) \cap A = \{y\}$, $A \cup B = Y$, *and* $A \cap B$ *is a single type 3 point;*
(3) *if* k *is non-trivially valued and* $y \notin \Gamma(Y)$, *there exists a strict affinoid neighborhood of* y *in* Y.

PROOF. Let p denote the characteristic exponent of k. Then, by [**21**, Théorème 6.10], there exists n such that $Y' := (Y \times k^{1/p^n})_{\mathrm{red}}$ is geometrically reduced. Since $k^{1/p^n}/k$ is a purely inseparable field extension, the map $f : Y' \to Y$ is a homeomorphism (see [**21**, Remarque 0.5]). As Y' is geometrically reduced, the set of its smooth points is a non-empty Zariski-open subset (see [**21**, Théorème 3.4]), *i.e.* by Lemma 1.8.5, the complement of a set of rigid points. Consequently, since $y' := f^{-1}(y)$ is non-rigid, it is smooth in Y'. Remark also that by [**20**, Proposition 4.2.14], the image (resp. preimage) of a connected affinoid domain is a connected analytic domain, and thus by [**20**, Théorème 6.1.3], a connected affinoid domain. Finally, for any affinoid domain U of Y', we have that $\Gamma(U) = f^{-1}(\Gamma(f(U)))$: by Proposition 1.5.31 and Theorem 1.5.27 (while taking into account Proposition 1.8.10), this is true for finite morphisms, and taking the reduction of an affinoid space does not change its Shilov boundary. Set $Z' := f^{-1}(Z)$. It suffices to prove the statement for Y', y', Z'.

(1) By [**20**, Théorème 4.5.4], y' has an affinoid neighborhood A' in Y' that is a closed virtual annulus, implying $\partial_B(A')$ contains exactly 2 points. We may assume, seeing as type 3 points are dense in Y' (Proposition 1.8.7), that $\partial A'$ constists of only type 3 point.

Thus, A' has at most two connected components at the neighborhood of y', and it is connected there if and only if $y' \in \Gamma(A')$.

Finally, Y' has at most two connected components at the neighborhood of y', and by Lemma 3.2.6, it is connected there if and only if $y' \in \Gamma(Y')$.

(2) Suppose furthermore that $y' \in \Gamma(Y')$, implying $y' \in \Gamma(A')$. Set $\Gamma(A') = \{y', z'\}$, where z' is a type 3 point. Then, $\partial A' = \{z'\}$ and by Theorem 1.8.15, $B' := (Y' \backslash A') \cup \{z'\}$ is an affinoid domain. We have: $A' \cup B' = Y', A' \cap B' = \{z'\}$ (which implies B' is connected). Finally, by shrinking A' if necessary, we can always assume $z' \notin \Gamma(Y')$, and since $\Gamma(Y') \cap A' \subseteq \Gamma(A')$, this implies $\Gamma(Y') \cap A' = \{y'\}$.

(3) If $y' \notin \Gamma(Y')$, then $y' \notin \Gamma(A')$, and for the non-trivially valued field k^{1/p^n}, the statement follows from the fact that A' is a closed virtual annulus. □

By the terminology introduced in [**20**, Section 1.7] and [**20**, Théorème 3.5.1], the first part of Lemma 3.2.7 shows that points of type 3 of certain k-analytic curves have at most two branches. Furthermore, in view of Lemma 1.8.8 and Theorem 1.5.27(1), it has one branch if and only if it is in the Berkovich boundary of the curve.

The following argument will be used often in what follows.

LEMMA 3.2.8. *Let* k *be a complete ultrametric field. Let* C *be a normal irreducible* k*-analytic curve. Set* $F = \mathscr{M}(C)$. *Let* X/F *be a variety, and* G/F *a connected rational linear algebraic group acting strongly transitively on* X.

(1) *Suppose* $X(\mathscr{M}_x) \neq \emptyset$ *for all* $x \in C$. *Let* Z *be any affinoid domain of* C. *Then,* $G_Z := G \times_F \mathscr{M}(Z)$ *is a connected rational linear algebraic group over* $\mathscr{M}(Z)$ *acting strongly transitively on the* $\mathscr{M}(Z)$*-variety* $X_Z := X \times_F \mathscr{M}(Z)$. *Furthermore,* $X_Z(\mathscr{M}_{Z,x}) \neq \emptyset$ *for all* $x \in Z$, *where* $\mathscr{M}_Z$ *is the sheaf of meromorphic functions over* Z.

(2) *Let U_1, U_2 be connected affinoid domains of C such that $U_1 \cap U_2 = \{s\}$, where s is a type 3 point. If $X(\mathscr{M}(U_i)) \neq \emptyset, i = 1, 2$, then $X(\mathscr{M}(U_1 \cup U_2)) \neq \emptyset$.*

PROOF. (1) That $G_Z = G \times_F \mathscr{M}(Z)$ is still a connected rational linear algebraic group acting strongly transitively on the variety $X_Z = X \times_F \mathscr{M}(Z)$ is immediate. Also, seeing as X, Z are normal, $\mathscr{M}_x$ and $\mathscr{M}_{Z,x}$ are fields, so the restriction morphism $\mathscr{M}_x \hookrightarrow \mathscr{M}_{Z,x}$ is injective for all $x \in Z$. Thus, $X(\mathscr{M}_x) \neq \emptyset$ implies $X(\mathscr{M}_{Z,x}) = X_Z(\mathscr{M}_{Z,x}) \neq \emptyset$ for any $x \in Z$.

(2) Let $x_i \in X(\mathscr{M}(U_i)), i = 1, 2$. By the transitivity of the action of G, there exists $g \in G(\mathscr{M}(\{s\}))$, such that $x_1 = g \cdot x_2$ in $X(\mathscr{M}(\{s\}))$. By Theorem 2.2.3, there exist $g_i \in G(\mathscr{M}(U_i))$ such that $g = g_1 \cdot g_2$ in $G(\mathscr{M}(\{s\}))$. Thus $g_1^{-1} \cdot x_1 = g_2 \cdot x_2$ in $X(\mathscr{M}(\{s\}))$. Set $x_1' = g_1^{-1} \cdot x_1$ and $x_2' = g_2 \cdot x_2$. They represent meromorphic functions over U_1 and U_2, respectively, whose restrictions to $U_1 \cap U_2$ are compatible. Thus, they can be glued to give a meromorphic function x over $\mathscr{M}(U_1 \cup U_2)$, where $x \in X(\mathscr{M}(U_1 \cup U_2))$, implying $X(\mathscr{M}(U_1 \cup U_2)) \neq \emptyset$. □

Recall that unless mentioned otherwise, k is a complete non-trivially valued ultrametric field such that $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$.

THEOREM 3.2.9. *Let Y be a normal irreducible k-affinoid curve. Set $K = \mathscr{M}(Y)$. Let X/K be a variety, and G/K a connected rational linear algebraic group acting strongly transitively on X. The following local-global principles hold:*

- *$X(K) \neq \emptyset \iff X(\mathscr{M}_x) \neq \emptyset$ for all $x \in Y$;*
- *for any open cover $\mathcal{P}$ of Y, $X(K) \neq \emptyset \iff X(\mathscr{M}(U)) \neq \emptyset$ for all $U \in \mathcal{P}$.*

PROOF. Seeing as $K \hookrightarrow \mathscr{M}_x$ for any $x \in Y$, the implication " $\Rightarrow$" is true.

For the other one, let us use induction on the number n of type 3 points in the Shilov boundary of Y. If $n = 0$, then by Proposition 1.8.12, Y is a strict k-affinoid curve, in which case the statement has already been proven in Proposition 3.2.5. Assume we know the statement for any positive integer not larger than $n - 1, n > 0$.

Suppose $\Gamma(Y)$ contains n type 3 points. Let $u \in \Gamma(Y)$. Since $X(\mathscr{M}_u) \neq \emptyset$, there exists a connected affinoid neighborhood U_1' of u in Y, such that $X(\mathscr{M}(U_1')) \neq \emptyset$. By Lemma 3.2.7(2), there exist two connected affinoid domains U_1, U_2 of Y, such that U_1 is a neighborhood of u in U_1', $\Gamma(Y) \cap U_1 = \{u\}$, $U_1 \cup U_2 = Y$, and $U_1 \cap U_2 = \{s\}$, where s is a type 3 point. Since $U_1 \subseteq U_1'$, we obtain $X(\mathscr{M}(U_1')) \subseteq X(\mathscr{M}(U_1))$, so $X(\mathscr{M}(U_1)) \neq \emptyset$. Let U_s be a connected strict affinoid neighborhood of s in Y (see Lemma 3.2.7(3)). Set $Z_i := U_i \cup U_s, i = 1, 2$. It is an integral affinoid domain. Let us show $\Gamma(Z_2)$ contains at most $n - 1$ type 3 points.

For any $y \in U_s$ of type 3, seeing as $\Gamma(U_s)$ doesn't contain any type 3 points, $y \notin \Gamma(U_s)$. Taking into account $\Gamma(Z_i) \cap U_s \subseteq \Gamma(U_s)$, we obtain $y \notin \Gamma(Z_i)$. Similarly, for any $y \in U_i \backslash \Gamma(U_i)$, we have $y \notin \Gamma(Z_i)$. Thus, if z is a type 3 point in the Shilov boundary of Z_i, then $z \notin U_s \cup \bigcup_{i=1}^2 (U_i \backslash \Gamma(U_i))$, implying $z \in \Gamma(U_i)$. For a subset S of Y, let us denote by S_3 the set of type 3 points contained in S. We have just shown that $\Gamma(Z_i)_3 = \Gamma(U_i)_3 \backslash \{s\}, i = 1, 2$. At the same time, $\Gamma(Y)_3$ is a disjoint union of $\Gamma(U_i)_3 \backslash \{s\}, i = 1, 2$. By construction, $u \in \Gamma(U_1)_3 \backslash \{s\}$, so the cardinality of $\Gamma(Z_2)_3$ is at most $n - 1$.

By the first part of Lemma 3.2.8, $X_{Z_2}(\mathscr{M}_{Z_2,x}) \neq \emptyset$ for any $x \in Z_2$. In view of the paragraph above and the induction hypothesis, $X(\mathscr{M}(Z_2)) = X_{Z_2}(\mathscr{M}(Z_2)) \neq \emptyset$. Seeing as $\mathscr{M}(Z_2) \subseteq \mathscr{M}(U_2)$, we obtain $X(\mathscr{M}(U_2)) \neq \emptyset$. Considering we also have $X(\mathscr{M}(U_1)) \neq \emptyset$, we can conclude by applying the second part of Lemma 3.2.8.

The second version of this local-global principle is a direct consequence of the first one. □

We are now able to prove the following:

THEOREM 3.2.10. *Let k be a complete ultrametric field such that $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$. Let C be a normal irreducible projective k-analytic curve. Set $F = \mathscr{M}(C)$. Let X/F be a variety, and G/F a connected rational linear algebraic group acting strongly transitively on X. The following local-global principles hold:*

- *$X(F) \neq \emptyset \iff X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$;*
- *for any open cover $\mathcal{P}$ of C, $X(F) \neq \emptyset \iff X(\mathscr{M}(U)) \neq \emptyset$ for all $U \in \mathcal{P}$.*

PROOF. Since $F \hookrightarrow \mathscr{M}_x$ for any $x \in C$, the direction " $\Rightarrow$ " is true.

Suppose k is non-trivially valued. By Proposition 3.1.14, there exists a nice cover $\{Z_1, Z_2\}$ of C, such that $Z_1 \cap Z_2$ is a single type 3 point η. By the first part of Lemma 3.2.8, G_{Z_i} is a connected rational linear algebraic group acting strongly transitively on the variety X_{Z_i}, and $X_{Z_i}(\mathscr{M}_{Z_i,x}) \neq \emptyset$ for any $x \in Z_i, i = 1, 2$. Thus, by Theorem 3.2.9, $X(\mathscr{M}(Z_i)) = X_{Z_i}(\mathscr{M}(Z_i)) \neq \emptyset$. We now conclude by the second part of Lemma 3.2.8.

Suppose k is trivially valued. Being a projective analytic curve over a trivially valued field, the curve C has exactly one type 2 point x (see Lemma 1.8.6). In that case, $\mathscr{M}_x = F$, so the statement is trivially satisfied.

The second version of this local-global principle is a direct consequence of the first one. □

The condition on the value group of k can be removed using model-theoretic arguments. We are very grateful to Antoine Ducros for bringing this to our attention.

THEOREM 3.2.11. *Let k be a complete ultrametric field. Let C be an irreducible normal projective k-analytic curve. Set $F = \mathscr{M}(C)$. Let X/F be a variety, and G/F a connected rational linear algebraic group acting strongly transitively on X. The following local-global principles hold:*

- *$X(F) \neq \emptyset \iff X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$;*
- *for any open cover $\mathcal{P}$ of C, $X(F) \neq \emptyset \iff X(\mathscr{M}(U)) \neq \emptyset$ for all $U \in \mathcal{P}$.*

PROOF. If $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$, then the statement was already proven in Theorem 3.2.10. Let us show that we can always reduce to this case.

Since $F \hookrightarrow \mathscr{M}_x$ for all $x \in C$, the direction " $\Rightarrow$ " is clear. Assume $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$. Since C is compact, there exists a finite cover $\mathcal{V}$ of C containing only affinoid domains, such that $\{\mathrm{Int}(V) : V \in \mathcal{V}\}$ is also a cover of C, and $X(\mathscr{M}(V)) \neq \emptyset$ for all $V \in \mathcal{V}$. Let $x_V \in X(\mathscr{M}(V))$.

Recall that for any V, $\mathscr{M}(V)$ is the fraction field of an algebra of convergent series over k. Hence, C, X, G, the action of G on X, the isomorphism of a Zariski open of G to an open of some $\mathbb{A}^n_F$, and $x_V, V \in \mathcal{V}$, are all determined by countably many elements of k. Let $S \subseteq k$ denote a countable subset containing all these elements.

Let k_0 be the prime subfield of k. Let k_1 be the field extension of k_0 generated by S. Remark that k_1 is countable. By [**52**, Theorem 2.3.7], there exists a subfield k_2 of k that is a countable extension of k_1, such that $k_2 \subseteq k$ is an elementary embedding in the language of valued fields.

Then, by [**52**, Theorem 2.5.36], there exists a field extension K of k, such that $K = k_2^I/D$, where I is an index set and D is a non-principal ultra-filter on I. Furthermore, by [**52**, Exercise 2.5.22], it is an elementary extension.

Since k_2 is a countable subfield of k, the value group of k_2 with respect to the valuation induced by that of k satisfies $\sqrt{|k_2^\times|} \neq \mathbb{R}_{>0}$. Let k' be the completion of k_2 with respect to this valuation. Then, $\sqrt{|k'^\times|} \neq \mathbb{R}_{>0}$.

Since C is defined over k', there exists a connected compact normal k'-analytic curve C', such that $C' \times_{k'} k = C$. Since C is projective, by Theorem 1.8.15, C' is projective as well.

Set $F' = \mathscr{M}(C')$. By construction, there exists an F'-variety X', and a connected rational linear algebraic group G'/F' acting on X', such that $X = X' \times_{F'} F$, $G = G' \times_{F'} F$, and the action of G induced on X is the one given in the statement. Let us show that G' acts strongly transitively on X'. Let L/F' be any field extension such that $X'(L) \neq \emptyset$. Set $L_1 = L^I/D$. This is a field containing F' and k (since $k \subseteq k'^I/D \subseteq L_1$), so it is a field extension of F. Consequently, $G'(L_1) = G(L_1)$ acts transitively on $X'(L_1) = X(L_1)$, and since by [**52**, Exercise 2.5.22], $L \subseteq L_1$ is an elementary embedding, $G'(L)$ acts transitively on $X'(L)$.

For any $V \in \mathcal{V}$, let V' denote the image of V with respect to the projection morphism $C \to C'$. By construction, $X'(\mathscr{M}(V')) \neq \emptyset$. Hence, $X'(\mathscr{M}_x) \neq \emptyset$ for all $x \in C'$, implying $X'(F') \neq \emptyset$, thus in particular $X'(F') = X(F') \subseteq X(F) \neq \emptyset$.

The second part of the statement is a direct consequence of the first one. □

3.2.3. Valuations, Berkovich Curves, and the local-global principle. Because of the relation of Berkovich points to valuations of the function field of a curve, as a result of Theorem 3.2.11 we will obtain a local-global principle with respect to completions, thus evoking some resemblance to "classical local-global principles". Let us start by making said relation precise.

Definition 3.2.12. Let k be a complete ultrametric field. Let F be a field extension of k. For any valuation v on F, we denote by R_v the valuation ring of F with respect to v, and m_v its maximal ideal. We denote by F_v the completion of F with respect to v. We use the following notations:

- $V_k(F)$ is the set of all rank 1 valuations v on F that extend the valuation of k;
- $V_0(F)$ is the set of all non-trivial rank 1 discrete valuations on F that when restricted to k are trivial;
- for a k-subalgebra R of F, $R \neq k$, $V'_R(F)$ is the set of valuations $v \in V_0(F)$ such that $R \subseteq R_v$;
- $V(F) := V_k(F) \cup V_0(F)$;
- for a k-subalgebra R of F, $R \neq k$, $V_R(F) := V_k(F) \cup V'_R(F)$.

Remark that if k is trivially valued, then $V(F)$ contains the trivial valuation on F.

Remark 3.2.13. Let C be a normal irreducible k-analytic curve. Then, for any point $x \in C$, $\mathcal{O}_x$ is either a field or a discrete valuation ring (see Lemma 1.8.4). If $\mathcal{O}_x$ is a field, then $\mathscr{M}_x = \mathcal{O}_x \hookrightarrow \mathcal{H}(x)$, so we endow $\mathscr{M}_x$ with the valuation induced from $\mathcal{H}(x)$. If $\mathcal{O}_x$ is a discrete valuation ring, then we endow $\mathscr{M}_x$ with the corresponding discrete valuation.

Proposition 3.2.14. *Let k be a non-trivially valued complete ultrametric field. Let C be a normal irreducible k-analytic curve.*

(1) *Suppose there exists an affine curve S over k, such that $S^{\mathrm{an}} = C$. Let F denote the function field of S. Then, there exists a bijective correspondence $C \longleftrightarrow V_{\mathcal{O}(S)}(F)$.*
(2) *If C is projective, set $F = \mathscr{M}(C)$. Then, there exists a bijective correspondence $C \longleftrightarrow V(F)$.*

In either case, if to $x \in C$ is associated the valuation v of F, then $\widehat{\mathscr{M}_x} = F_v$, where the completion of $\mathscr{M}_x$ is taken with respect to the valuation introduced in Remark 3.2.13.

Proof. (1) Let $x \in C$. If x is a non-rigid point, then $\mathcal{O}_x$ is a field (Lemma 1.8.3), so $|\cdot|_x$ is a norm on $A := \mathcal{O}(S)$ extending that of k. Consequently, it extends to $F = \mathrm{Frac}\, A$ and defines a valuation v_x on F extending that of k, *i.e.* $v_x \in V_k(F)$. If x is a rigid point, $\mathcal{O}_{C,x}$ is a dvr (Lemma 1.8.4), and $k^\times \subseteq \mathcal{O}_{C,x}^\times$, so the embedding $A \hookrightarrow \mathcal{O}_{C,x}$ induces a discrete valuation on A whose restriction to k is trivial, *i.e.* a discrete valuation v_x on F whose restriction to k is trivial. Moreover, $A \subseteq R_{v_x}$ by definition, so $v_x \in V_A'(F)$.

Let us look at the function $C \longrightarrow V_A(F)$, $x \mapsto v_x$. It is injective by the paragraph above. It is also surjective: if $v \in V_k(F)$, then it determines a norm on A that extends that of k, so it corresponds to a non-rigid point of C; if $v \in V_A'(F)$, then $A \subseteq R_v$, and $P := A \cap m_v$ is a prime ideal of A, so it corresponds to a rigid point x of C for which $\ker|\cdot|_x = P$ (see Theorem 1.6.6(1)).

If $x \in C$ is non-rigid, then $\widehat{\mathscr{M}_x} = \mathcal{H}(x)$, which is the completion of F with respect to v_x (see Remark 1.6.7). If x is a rigid point of C, and P its corresponding prime ideal in A, then by Theorem 1.6.6(2), $\widehat{\mathcal{O}_{C,x}} = \widehat{A_P} = \widehat{A}$, where $\widehat{A}$ denotes the completion of A with respect to the ideal P. Consequently, $\widehat{\mathscr{M}_x} = \mathrm{Frac}\, \widehat{A} = F_{v_x}$.

(2) Suppose C is projective. Let C^{alg} be the normal irreducible projective k-algebraic curve such that its Berkovich analytification is C, and $\pi : C \to C^{\mathrm{alg}}$ the canonical analytification morphism. Let $x \in C$. Let S' be an affine Zariski open of C^{alg} containing $\pi(x)$. Since C is irreducible, the function field of S' is F. By (1), there exists an injective map: $C \longrightarrow V(F)$, $x \mapsto v_x$.

Let us show it is also surjective. Let $v \in V(F)$ such that $v_{|k}$ is the starting valuation on k. Then, by taking any affine Zariski open subset S' of C^{alg} (as in the paragraph above), seeing as its function field is F, we obtain that v corresponds to some non-rigid point of $S'^{\mathrm{an}} \subseteq C$.

Suppose $v \in V(F)$ is such that $v_{|k}$ is trivial. Let us consider an embedding $C^{\mathrm{alg}} \to \mathbb{P}_k^n = \mathrm{Proj}\ k[x_0, x_1, \ldots, x_n]$. Let $\{U_i := \mathrm{Spec}\ k[x_j/x_i]_{j\neq i}/I_i\}_{i=1}^n$ be a cover of C^{alg} by standard open sets. Let i_0 be such that $|x_{i_0}|_v \geqslant |x_i|_v$ for all i. Since $|x_i/x_{i_0}|_v \leq 1$, $\mathcal{O}(U_{i_0}) \subseteq R_v$, so by (1), v corresponds to a rigid point of $U_{i_0}^{\mathrm{an}} \subseteq C$.

That $\widehat{\mathscr{M}_x} = F_{v_x}$ for all $x \in C$ follows from part (1) by taking an affine Zariski open containing the point x. □

Remark 3.2.15. Proposition 3.2.14 shows that if X/k is a normal irreducible projective *algebraic* curve over k with function field F, then there is a bijective correspondence $X^{\mathrm{an}} \to V(F), x \mapsto v_x$, and $F_{v_x} = \widehat{\mathscr{M}_{C,x}}$.

Let us now show a local-global principle with respect to all such completions of the field F.

We are very greatful to the referee (of the article that arose from the contents of this chapter) for bringing to our attention the following lemma:

LEMMA 3.2.16. *Let K be a complete valued field and K_0 a dense Henselian (called* quasicomplete *in* [**4**, Definition 2.3.1]*) subfield. Let F be a subfield of K_0 and X an F-variety. Then, if F is perfect or X is smooth,*

$$X(K_0) \neq \emptyset \iff X(K) \neq \emptyset.$$

PROOF. Since K_0 is a subfield of K, the implication "$\Rightarrow$" is clear. Suppose $X(K) \neq \emptyset$.

Suppose F is perfect. By taking the reduction of X if necessary, we may assume that X is reduced. Let $a \in X(K)$. Denote by X' the (reduced) Zariski closure of $\{a\}$ in X. Since F is perfect, the smooth locus X'' of X' is a dense Zariski open subset of X' containing a. Thus, X'' is a smooth F-variety such that $X''(K) \neq \emptyset$, implying it suffices to prove the statement in the case X is smooth.

Suppose X is smooth. Let $a \in X(K)$. Since X is smooth, there exists a neighborhood U of a in X, such that there exists an étale morphism $\varphi : U \to \mathbb{A}^d_F$ for some $d \in \mathbb{N}$. Let $\varphi_K : U_K \to \mathbb{A}^d_K$ be the tensorization by K, and let us look at its analytification φ_K^{an}. Since a is a rational point, φ_K^{an} induces an isomorphism between a neighborhood V of x in U_K^{an} and an open V' of $\mathbb{A}_K^{d,\mathrm{an}}$. Since K_0 is dense in K, there exists b in V', such that $b \in \mathbb{A}^d(K) = K^d$ has coordinates over K_0 (recall Theorem 1.6.6(1)). Let c be the only pre-image of b in V. Then, c is a K-rational point over b.

$$\begin{array}{ccc} U_K & \longrightarrow & U_{K_0} \\ {\scriptstyle \varphi_K}\downarrow & & \downarrow{\scriptstyle \varphi_{K_0}} \\ \mathbb{A}^d_K & \xrightarrow[g]{} & \mathbb{A}^d_{K_0} \end{array}$$

Set $b' := g(b) \in \mathbb{A}^d_{K_0}$. By commutativity of the diagram, since b has coordinates over K_0, b' is a closed point of $\mathbb{A}^d_{K_0}$ which is in the image of φ_{K_0}.

Since φ is étale, $\varphi_{K_0}^{-1}(b')$ is a disjoint union $\bigsqcup_i \operatorname{Spec} F_i$, where F_i are separable finite field extensions of $\kappa(b') = K_0$. At the same time, $\varphi_K^{-1}(b) = \bigsqcup_i F_i \otimes_{K_0} K$. Set $\widehat{F}_i := F_i \otimes_{K_0} K$. It is a field by [**4**, Proposition 2.4.1].

We know that $\varphi_K^{-1}(b)(K) \neq \emptyset$. Then, there exists i, such that $(\operatorname{Spec} \widehat{F}_i)(K) \neq \emptyset$, so $\widehat{F}_i = K$. By Proposition 2.4.1 of [**4**], this implies that $F_i = K_0$, and so $\varphi_{K_0}^{-1}(b')(K_0) \neq \emptyset$, implying $X(K_0) \neq \emptyset$. □

COROLLARY 3.2.17. *Let k be a complete ultrametric field. Let C be a normal irreducible k-analytic curve. Set $F = \mathscr{M}(C)$. Let X be an F-variety. Then, if* $\operatorname{char} k = 0$ *or X is smooth:*

$$X(\mathscr{M}_x) \neq \emptyset \iff X(\widehat{\mathscr{M}_x}) \neq \emptyset$$

for all $x \in C$, where the completion $\widehat{\mathscr{M}_x}$ of $\mathscr{M}_x$ is taken with respect to the valuations introduced in Remark 3.2.13.

PROOF. Remark that F is perfect if and only if $\operatorname{char} k = 0$.

If $\mathcal{O}_x$ is a field, then $\mathscr{M}_x$ is Henselian by [**4**, Theorem 2.3.3]. If $\mathcal{O}_x$ is not a field, then it is a discrete valuation ring that is Henselian (see [**4**, Theorem 2.1.5]), so $\mathscr{M}_x$ is Henselian by [**4**, Proposition 2.4.3]. We conclude by Lemma 3.2.16. □

Recall once again that an irreducible compact analytic curve is either projective or affinoid (see Theorem 1.8.15).

COROLLARY 3.2.18. *Let k be a complete ultrametric valued field. Let C be a compact irreducible normal k-analytic curve. Set $F = \mathscr{M}(C)$. Let X/F be a variety, and G/F a connected rational linear algebraic group acting strongly transitively on X. The following local-global principles hold if* $\operatorname{char} k = 0$ *or X is smooth:*

(1) *if C is affinoid and $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$,*

$$X(F) \neq \emptyset \iff X(F_v) \neq \emptyset \text{ for all } v \in V_{\mathcal{O}(C)}(F);$$

(2) *if C is projective,*

$$X(F) \neq \emptyset \iff X(F_v) \neq \emptyset \text{ for all } v \in V(F).$$

PROOF. If k is trivially valued, then the trivial valuation v_0 of F is in $V_{\mathcal{O}(C)}(F)$ (resp. $V(F)$), and since $F_{v_0} = F$, the statement is clear in this case.

Otherwise, it is a consequence of Theorem 3.2.9, and Theorem 3.2.11 in view of Proposition 3.2.14 and Corollary 3.2.17. □

REMARK 3.2.19. Recall that for any finitely generated field extension F/k of transcendence degree 1, there exists a unique normal projective k-algebraic curve C^{alg} with function field F. Let C be the analytification of C^{alg}. Then, $\mathscr{M}(C) = F$ (see [**6**, Proposition 3.6.2]), so the local-global principles above are applicable to any such field F.

In particular, Corollaries 3.2.18 and 3.4.2 can be stated independently from Berkovich's theory.

By Corollary 3.8 of [**34**], if G_1 and G_2 are linear algebraic groups such that $G_1 \times G_2$ is a connected rational linear algebraic group, then all the results proven in this section remain true for G_1 and G_2.

3.3. Comparison of Overfields

The purpose of this section is to draw a comparison between one of the local-global principles we proved (Theorem 3.2.11) and the one proven in ([**34**, Theorem 3.7]). More precisely, we will interpret what the overfields appearing in [**34**] represent in the Berkovich setting, and show that [**34**, Theorem 3.7] can be obtained as a consequence of Theorem 3.2.11. When working over a "fine" enough model, we show that the converse is also true.

Throughout this section, for a non-Archimedean valued field E, we will denote by E° the ring of integers of E, $E^{\circ\circ}$ the maximal ideal of E°, and by $\widetilde{E}$ the residue field of E.

Until the end of this section, we assume k to be a complete discretely valued field.

3.3.1. Analytic generic fiber and the specialization map. We will be using the notion of generic fibre in the sense of Berkovich. To see the construction in more detail and under less constrictive conditions, we refer the reader to [**7**, Section 1] and [**8**, Section 1].

Let $\mathscr{X} = \operatorname{Spec} A$ be a flat finite type scheme over k°. Then, the formal completion $\widehat{\mathscr{X}}$ of $\mathscr{X}$ along its special fiber is $\operatorname{Spf}(\widehat{A})$, where $\widehat{A}$ is a topologically finitely presented ring over k° (*i.e.* isomorphic to some $k^\circ\{T_1, \ldots, T_n\}/I$, where I is a finitely generated ideal). Remark that $\widehat{A} \otimes_{k^\circ} k$ is a strict k-affinoid algebra.

The *analytic generic fiber* of $\widehat{\mathscr{X}}$, denoted by $\widehat{\mathscr{X}_\eta}$, is defined to be $\mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$, where $\mathcal{M}(\cdot)$ denotes the Berkovich spectrum. There exists a *specialization map* $\pi : \widehat{\mathscr{X}_\eta} \to \widehat{\mathscr{X}_s}$

(often called *reduction map* in the litterature, which we avoid because of Subsection 1.4.7), where $\widehat{\mathscr{X}_s}$ is the special fiber of $\widehat{\mathscr{X}}$, which is anti-continuous, meaning the pre-image of a closed subset is open. We remark that $\widehat{\mathscr{X}_s} = \mathscr{X}_s$, where $\mathscr{X}_s$ is the special fiber of $\mathscr{X}$. Let us describe π more explicitly.

There are embeddings $A \hookrightarrow \widehat{A} \hookrightarrow (\widehat{A} \otimes_{k^\circ} k)^\circ$, where $(\widehat{A} \otimes_{k^\circ} k)^\circ$ is the set of all elements f of $\widehat{A} \otimes_{k^\circ} k$ for which $|f|_x \leqslant 1$ for all $x \in \mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$ (if ρ is the spectral seminorm on $\widehat{A} \otimes_{k^\circ} k$, this is equivalent to asking that $\rho(f) \leqslant 1$). Let $x \in \mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$. This point then determines a bounded morphism $A \to \mathcal{H}(x)^\circ$, which induces an application $\varphi_x : A \otimes_{k^\circ} \widetilde{k} \to \widetilde{\mathcal{H}(x)}$. The specialization map π sends x to $\ker \varphi_x$.

The following commutative diagram, where $\phi : \operatorname{Spec}(\widetilde{\widehat{A} \otimes_{k^\circ} k}) \to \operatorname{Spec}(A \otimes_{k^\circ} \widetilde{k})$ is the canonical map, gives the relation between the specialization map and the reduction map from [**6**, Section 2.4]. The morphism ϕ is finite and dominant (see [**11**, 6.1.2 and 6.4.3] and [**68**, pg. 17]).

$$\begin{array}{ccc} \mathcal{M}(\widehat{A} \otimes_{k^\circ} k) & \xrightarrow{r} & \operatorname{Spec}(\widetilde{\widehat{A} \otimes_{k^\circ} k}) \\ & \searrow^{\pi} & \downarrow \phi \\ & & \operatorname{Spec}(A \otimes_{k^\circ} \widetilde{k}) \end{array} \tag{3}$$

The construction above has nice gluing properties. Let $\mathscr{X}$ be a finite type scheme over k°, and $\widehat{\mathscr{X}}$ its formal completion along the special fiber. Then, the *analytic generic fiber* $\widehat{\mathscr{X}_\eta}$ of $\widehat{\mathscr{X}}$ is the k-analytic space we obtain by gluing the analytic generic fibers of an open affine cover of the formal scheme $\widehat{\mathscr{X}}$. In general, $\widehat{\mathscr{X}_\eta}$ is a compact analytic domain of the Berkovich analytification $\mathscr{X}^{\mathrm{an}}$ of $\mathscr{X}$. If $\mathscr{X}$ is proper, then $\mathscr{X}^{\mathrm{an}} = \widehat{\mathscr{X}_\eta}$ (see [**56**, 2.2.2]). Similarly, there exists an anti-continuous *specialization map* $\pi : \widehat{\mathscr{X}_\eta} \to \mathscr{X}_s$, where $\mathscr{X}_s$ is the special fiber of $\mathscr{X}$.

Recall k is assumed to be discretely valued. A property we will need is the following:

PROPOSITION 3.3.1. *With the same notation as above, suppose A is a normal domain. Then, $\widehat{A} = (\widehat{A} \otimes_{k^\circ} k)^\circ$, and the finite morphism ϕ from the diagram above is a bijection.*

PROOF. Let us denote by t a uniformizer of k°, and by I the ideal $t\widehat{A}$. Recall that $\widehat{A}$ is the completion of A with respect to the ideal tA (and is isomorphic to some $k^\circ\{T_1, T_2, \ldots, T_n\}/P$; remark that then $\widehat{A} \otimes_{k^\circ} k$ is isomorphic to the k-affinoid algebra $k\{T_1, T_2, \ldots, T_n\}/P$).

Set $B = (\widehat{A} \otimes_{k^\circ} k)^\circ$ and $J = (\widehat{A} \otimes_{k^\circ} k)^{\circ\circ}$ - the elements f of $\widehat{A} \otimes_{k^\circ} k$ such that $|f|_x < 1$ for all $x \in \mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$ (*i.e.* $\rho(f) < 1$, where the ρ is the spectral norm on $\widehat{A} \otimes_{k^\circ} k$).

Remark that for any maximal ideal m of A, $t \in m$ (*i.e.* the closed points of Spec A are in the special fiber). This means that tA is contained in the Jacobson radical of A. Considering this and the fact that A is excellent and normal, by [**26**, 7.8.3.1], $\widehat{A}$ is also normal. At the same time, by [**11**, 6.1.2, 6.3.4], B is the integral closure of $\widehat{A}$ in $\widehat{A} \otimes_{k^\circ} k$. Since Frac $\widehat{A}$ = Frac B, we obtain $\widehat{A} = B$.

Let us look at the canonical map $A/t = \widehat{A}/I \to B/J$ inducing ϕ. Let $|\cdot|$ be the norm on the affinoid algebra $\widehat{A} \otimes_{k^\circ} k$.

Remark that $\sqrt{I} = J$: let $x \in J$, so that $\rho(x) = \lim_{n\to\infty} |x^n|^{1/n} < 1$, implying $|x^n| \to 0$, $n \to +\infty$. Thus, for large enough n, $x^n \in I$, so $J \subseteq \sqrt{I}$. The other containment is clear seeing as $\rho(\cdot) \leqslant |\cdot|$. This means that any prime ideal of $\widehat{A}$ contains I if and only if it contains J, and thus that ϕ is a bijection. □

3.3.2. The setup of HHK's [34]. Let us start by recalling HHK's framework (see [**34**, Notation 3.3]):

NOTATION 3.3.2. Let $T = k^\circ$ be a complete discrete valuation ring with uniformizer t, fraction field k, and residue field $\widetilde{k}$. Let $\mathscr{C}$ be a flat normal irreducible projective T-curve with function field F. Let us denote by $\mathscr{C}_s$ the special fiber of $\mathscr{C}$.

For any point $P \in \mathscr{C}_s$, set $R_P = \mathcal{O}_{\mathscr{C},P}$. Since T is complete discretely valued, R_P is an excellent ring. Let us denote by $\widehat{R_P}$ the completion of R_P with respect to its maximal ideal. Since R_P is normal and excellent, $\widehat{R_P}$ is also a domain. Set $F_P = \operatorname{Frac} \widehat{R_P}$.

Let U be a proper subset of one of the irreducible components of $\mathscr{C}_s$. Set $R_U = \bigcap_{P\in U} \mathcal{O}_{\mathscr{C},P}$. Let us denote by $\widehat{R_U}$ the t-adic completion of R_U. By [**34**, Notation 3.3], for any $Q \in U$, $\widehat{R_U} \subseteq \widehat{R_Q}$. Thus, $\widehat{R_U}$ is an integral domain. Set $F_U = \operatorname{Frac} \widehat{R_U}$.

Let $\mathscr{P}$ be a finite set of closed points of $\mathscr{C}_s$ containing all points at which distinct irreducible components of $\mathscr{C}_s$ meet. Let $\mathscr{U}$ be the set of all irreducible components of $\mathscr{C}_s \backslash \mathscr{P}$ (which here are also its connected componenets).

The following is the local-global principle proven by HHK in [**34**] and [**35**]:

THEOREM 3.3.3 ([**34**, Theorem 3.7], [**35**, Theorem 9.1]). *Let G be a connected rational linear algebraic group over F that acts strongly transitively on an F-variety X. The following statements are equivalent:*

(1) $X(F) \neq \emptyset$;
(2) $X(F_P) \neq \emptyset$ *for all* $P \in \mathscr{P}$ *and* $X(F_U) \neq \emptyset$ *for all* $U \in \mathscr{U}$;
(3) $X(F_Q) \neq \emptyset$ *for all* $Q \in \mathscr{C}_s$.

The implication (1) ⇒ (2) is immediate seeing as F is embedded into F_P and F_U for all $P \in \mathscr{P}$ and $U \in \mathscr{U}$. Considering for any $U \in \mathscr{U}$ and any $Q \in U$, $F_U \subseteq F_Q$, we obtain that (2) ⇒ (3).

We now proceed to show that the remaining implication (3) ⇒ (1) is a consequence of Theorem 3.2.11. To do this, a comparison will be drawn between the fields $F_Q, Q \in \mathscr{C}_s$, and the ones appearing in Theorem 3.2.11.

3.3.3. The comparison. Let us denote by C the Berkovich analytification of the generic fiber of $\mathscr{C}$. It is a normal irreducible projective k-analytic curve. By [**6**, Proposition 3.6.2], $\mathscr{M}(C) = F$, where $\mathscr{M}$ is the sheaf of meromorphic functions on C. Since $\mathscr{C}$ is projective, $C = \widehat{\mathscr{C}_\eta}$. Let $\pi : C \to \mathscr{C}_s$ be the specialization map.

Let μ be the generic point of one of the irreducible components of $\mathscr{C}_s$. Then, $\mathcal{O}_{\mathscr{C},\mu}$ is a discrete valuation ring with fraction field F, whose valuation extends that of k. Considering the residue field of μ is of transcendence degree one over $\widetilde{k}$, μ determines a unique type 2 point x_μ on the Berkovich curve C (recall the classification of points in a curve, Definition 1.8.1). Moreover:

LEMMA 3.3.4. *Let μ be the generic point of one of the irreducible components of $\mathscr{C}_s$. Then, $\pi^{-1}(\mu) = \{x_\mu\}$.*

PROOF. Let $U = \operatorname{Spec} A$ be an open affine neighborhood of μ in $\mathscr{C}$. Since $\mathscr{C}$ is irreducible, we obtain that $\operatorname{Frac} A = F$. By [**7**, pg. 541], $\pi^{-1}(U_s) = \widehat{U_\eta}$, and the restriction of π on $\widehat{U}_\eta$ is the specialization map $\widehat{U}_\eta \to U_s$. Explicitly, we have $\pi : \mathcal{M}(\widehat{A} \otimes_{k^\circ} k) \to \operatorname{Spec}(A \otimes_{k^\circ} \widetilde{k})$, where $x \in \mathcal{M}(\widehat{A} \otimes_{k^\circ} k)$ is sent to the kernel of the map $A \otimes_{k^\circ} \widetilde{k} = A/k^{\circ\circ}A \to \widetilde{\mathcal{H}(x)}$.

By construction, for any $x \in \pi^{-1}(\mu)$ and any $f \in A$, $f(\mu) = 0$ if and only if $|f|_x < 1$, and $f(\mu) \neq 0$ if and only if $|f|_x = 1$. As a consequence, $|f|_{x_\mu} < 1$ if and only if $|f|_x < 1$, and $|f|_{x_\mu} = 1$ if and only if $|f|_x = 1$. This implies that x and x_μ define the same norm on A (and hence on F), so $x_\mu = x$ in C, and $\pi^{-1}(\mu) = \{x_\mu\}$. □

PROPOSITION 3.3.5. *Let μ be the generic point of one of the irreducible components of $\mathscr{C}_s$. Set $\{x_\mu\} := \pi^{-1}(\mu)$. Then, $F_\mu = \mathcal{H}(x_\mu)$. Let X be an F-variety. If $X(F_\mu) \neq \emptyset$, then $X(\mathscr{M}_{C,x_\mu}) \neq \emptyset$.*

PROOF. Remark that $F_\mu = \operatorname{Frac} \widehat{\mathcal{O}_{\mathscr{C},\mu}}$ is the completion of F with respect to the valuation x_μ. Seeing as x_μ is of type 2, $\mathcal{O}_{C,x_\mu} = \mathscr{M}_{C,x_\mu}$, and by Proposition 3.2.14, $F_\mu = \widehat{\mathscr{M}_{C,x_\mu}} = \mathcal{H}(x_\mu)$.

If X is smooth or $\operatorname{char} k = 0$, we can conclude by Corollary 3.2.17.

Otherwise, the restriction morphism of the sheaf of meromorphic functions gives us $\operatorname{Frac} \mathcal{O}_{\mathscr{C},\mu} = F = \mathscr{M}(C) \hookrightarrow \mathcal{O}_{C,x_\mu}$, so there exist embeddings $\mathcal{O}_{\mathscr{C},\mu} \subseteq \mathcal{O}_{C,x_\mu} \subseteq \mathcal{H}(x_\mu)$. Seeing as all elements of $\mathcal{O}_{\mathscr{C},\mu}$ have norm at most 1, $R_\mu = \mathcal{O}_{\mathscr{C},\mu} \subseteq \mathcal{O}^\circ_{C,x_\mu}$ - the valuation ring of $\mathcal{O}_{C,x_\mu}$.

By the proof of [**35**, Proposition 5.8], $X(F_\mu) \neq \emptyset$ implies $X(\widehat{R_\mu}) \neq \emptyset$. The ring $R_\mu = \mathcal{O}_{\mathscr{C},\mu}$ is excellent, so by Artin's Approximation Theorem ([**1**, Theorem 1.10]), $X(R^h_\mu) \neq \emptyset$, where R^h_μ denotes the henselization of the local ring R_μ. Seeing as $\mathcal{O}^\circ_{C,x_\mu}$ is Henselian ([**4**, Thm. 2.3.3, Prop. 2.4.3]), $R_\mu \subseteq R^h_\mu \subseteq \mathcal{O}^\circ_{C,x_\mu} \subseteq \mathscr{M}_{C,x_\mu}$. Consequently, $X(\mathscr{M}_{C,x_\mu}) \neq \emptyset$. □

We recall that the specialization map is anti-continuous. For any analytic domain U of C, let us denote $|\cdot|_{sup} := \sup_{x \in U} |\cdot|_x$.

PROPOSITION 3.3.6. *Let P be a closed point of $\mathscr{C}_s$. Then, $\widehat{R_P} = \mathcal{O}^\circ_C(\pi^{-1}(P))$, where $\mathcal{O}^\circ$ is the sheaf of analytic functions f such that $|f|_{sup} \leqslant 1$. Consequently, if $X(F_P) \neq \emptyset$, then $X(\mathscr{M}(\pi^{-1}(P))) \neq \emptyset$.*

PROOF. Let $V = \operatorname{Spec} A$ be an open integral affine neighborhood of P in $\mathscr{C}$. As $\mathscr{C}$ is normal, so is A. Note that $P \in V_s$, where V_s is the special fiber of V.

Let π denote the specialization map correspoding to $\mathscr{C}$. By *cf.* [**7**, pg. 541], $\pi^{-1}(V) = \widehat{V_\eta}$ - the analytic generic fiber of V, and the restriction of π to $\widehat{V}_\eta$ is the specialization map $\widehat{V}_\eta \to V_s$ of V. Thus, $\pi^{-1}(P) \subseteq \widehat{V}_\eta$. Let us come back to the commutative diagram 3:

$$\begin{array}{ccc} \widehat{V}_\eta = \mathcal{M}(\widehat{A} \otimes_{k^\circ} k) & \xrightarrow{\ r\ } & \operatorname{Spec}(\widetilde{\widehat{A} \otimes_{k^\circ} k}) \\ & \searrow^{\pi} & \downarrow \phi \\ & & \operatorname{Spec}(A \otimes_{k^\circ} \widetilde{k}) = V_s \end{array}$$

Set $B = (\widehat{A} \otimes_{k^\circ} k)^\circ$. By Proposition 3.3.1, ϕ is a bijection, and $B = \widehat{A}$. Let m_P be the maximal ideal of A corresponding to the point P on the special fiber, and $\widehat{m_P}$ the corresponding ideal in $\widehat{A}$, *i.e.* the completion of m_P along the special fiber. Then, $\phi^{-1}(P)$ is a closed point of $\widetilde{\mathrm{Spec}(\widehat{A} \otimes_{k^\circ} k)}$ corresponding to the maximal ideal $\widehat{m_P}$ of $B = \widehat{A}$.

Since $k^{\circ\circ}A \subseteq m_P$, $\widehat{A}^{m_P} = \widehat{\widehat{A}}^{\widehat{m_P}} = \widehat{B}^{\widehat{m_P}}$, where the notation $\widehat{R}^S$ is used for the completion of a ring R with respect to the topology induced by an ideal S.

As V is reduced, so is its analytification V^{an} ([**21**, Théorème 3.4]). Since $\widehat{V}_\eta$ is an analytic domain of V^{an}, it is reduced (see [**21**, Théorème 3.4]). By a theorem of Bosch (see [**53**, Theorem 3.1], [**10**, Theorem 5.8]),

$$\widehat{B}^{\widehat{m_P}} = \mathcal{O}^\circ_{\widehat{V}_\eta}(r^{-1}(\phi^{-1}(P))) = \mathcal{O}^\circ_{\widehat{V}_\eta}(\pi^{-1}(P)).$$

As P is a closed point of $\mathscr{C}_s$ (resp. V_s), $\pi^{-1}(P)$ is an open subset of C (resp. $\widehat{V_\eta}$), implying $\mathcal{O}^\circ_{\widehat{V}_\eta}(\pi^{-1}(P)) = \mathcal{O}^\circ_C(\pi^{-1}(P))$.

As a consequence,

$$\widehat{R_P} = \widehat{\mathcal{O}_{\mathscr{C},P}} = \widehat{A}^{m_P} = \widehat{B}^{\widehat{m_P}} = \mathcal{O}^\circ_C(\pi^{-1}(P)).$$

This implies that $F_P = \mathrm{Frac}\, \mathcal{O}^\circ(\pi^{-1}(P)) \subseteq \mathscr{M}(\pi^{-1}(P))$. The last part of the statement is now immediate. □

We are now able to state and prove the following argument, thus concluding the proof that HHK's local-global principle (Theorem 3.3.3) can be obtained as a consequence of Theorem 3.2.11.

PROPOSITION 3.3.7. *Using the same notation as in Theorem 3.3.3,* (3) ⇒ (1).

PROOF. Let x be any point of C. Recall π denotes the specialization map $C \to \mathscr{C}_s$.

(1) If $\pi(x) = \mu \in \mathscr{C}_s$ is the generic point of one of the irreducible components of $\mathscr{C}_s$, then by Proposition 3.3.5, $X(F_\mu) \neq \emptyset$ implies $X(\mathscr{M}_{C,x}) \neq \emptyset$.
(2) If $\pi(x) = P \in \mathscr{C}_s$ is a closed point, by Proposition 3.3.6, $F_P \subseteq \mathscr{M}(\pi^{-1}(P))$. Since $x \in \pi^{-1}(P)$ and $\pi^{-1}(P)$ is open, we obtain $\mathscr{M}(\pi^{-1}(P)) \subseteq \mathscr{M}_{\pi^{-1}(P),x} = \mathscr{M}_{C,x}$. Hence, $X(F_P) \neq \emptyset$ implies $X(\mathscr{M}_{C,x}) \neq \emptyset$.

Finally, seeing as $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$, by Theorem 3.2.11, $X(F) \neq \emptyset$. □

Lastly, using Ducros' work on semi-stable reduction in the analytic setting (see [**20**], in particular Chapter 6), we can say something in the other direction as well:

PROPOSITION 3.3.8. *Let F be a finitely generated field extension of k of transcendence degree* 1. *Let C be the normal irreducible projective Berkovich k-analytic curve for which $F = \mathscr{M}(C)$. Let X/F be a variety. Then, there exists a flat normal irreducible projective model $\mathscr{C}'$ over $T = k^\circ$ of F, such that*

$$X(\mathscr{M}_x) \neq \emptyset \text{ for all } x \in C \;\Rightarrow\; X(F_P) \neq \emptyset \text{ for all } P \in \mathscr{C}'_s,$$

where $F_P = \widehat{\mathcal{O}_{\mathscr{C}',P}}$, and $\mathscr{C}'_s$ is the special fiber of $\mathscr{C}'$.

Consequently, a local-global principle with respect to the overfields $F_P, P \in \mathscr{C}'_s$, implies a local-global principle with respect to the $\mathscr{M}_x, x \in C$.

PROOF. Suppose $X(\mathscr{M}_x) \neq \emptyset$ for all $x \in C$. As the analytification of an algebraic curve, C is strict, so the strict affinoid domains form a basis of neighborhoods ([**6**, Proposition 2.2.3(iii)]). Taking into account C is compact, there exists a finite cover $\mathcal{U}$ of C such that:

(1) for any $U \in \mathcal{U}$, U is a connected strict affinoid domain in C;
(2) $\bigcup_{U\in\mathcal{U}} \mathrm{Int}(U) = C$;
(3) for any $U \in \mathcal{U}, X(\mathscr{M}(U)) \neq \emptyset$.

Let S be the set of all boundary points of the elements of $\mathcal{U}$. By construction, S is a finite set of type 2 points.

Let us show that S is a *vertex set* of C using [**20**, Théorème 6.3.15] (see [**20**, 6.3.17] for the definition of a vertex set, which is called *ensemble sommital* there). Since C is projective (implying boundaryless) and irreducible, conditions $\alpha), \beta)$ and $\gamma)$ of [**20**, Théorème 6.3.15 (ii)] are satisfied. Finally, condition $\delta)$ is a consequence of the fact that S contains only type 2 points (see [**20**, Commentaire 6.3.16]).

By [**20**, 6.3.23], this implies the existence of an irreducible projective model $\mathscr{C}'$ of F over T with special fiber $\mathscr{C}'_s$, and specialization map $\pi : C \to \mathscr{C}'_s$, such that π induces a bijection between S and the generic points of the irreducible components of $\mathscr{C}'_s$. Furthermore, by [**20**, 6.3.9.1], since k is discretely valued and C reduced, $\mathscr{C}'$ is locally topologically finitely presented. Finally, by [**20**, 6.3.10], since C is normal, the model $\mathscr{C}'$ is flat and normal.

By Proposition 3.3.6, for any closed point $P \in \mathscr{C}'$, $\widehat{\mathcal{O}_{\mathscr{C}',P}} = \mathcal{O}^o(\pi^{-1}(P))$, where $\mathcal{O}^o$ is the sheaf of holomorphic functions f, such that $|f|_{sup} \leqslant 1$. In particular, remark that if V is an affinoid domain of C, since all holomorphic functions are bounded on V, we have $\mathcal{O}^o(V) \subseteq \mathcal{O}(V)$. This implies $\mathrm{Frac}\ \mathcal{O}^o(V) \subseteq \mathscr{M}(V)$. Let $\frac{f}{g} \in \mathscr{M}(V)$, with $f, g \in \mathcal{O}(V)$. Let $\alpha \in k$ be such that $|\alpha f|_{sup}, |\alpha g|_{sup} \leqslant 1$ (it suffices to choose α so that $|f|_{sup}, |g|_{sup} \leqslant |\alpha^{-1}|$, which is possible seeing as k is non-trivially valued). Then, $\frac{f}{g} = \frac{\alpha f}{\alpha g} \in \mathrm{Frac}\ \mathcal{O}^o(V)$, implying $\mathscr{M}(V) = \mathrm{Frac}\ \mathcal{O}^o(V)$. By construction, there exists $U \in \mathcal{U}$ such that $\pi^{-1}(P) \subseteq U$. In particular, $\mathscr{M}(U) = \mathrm{Frac}\ \mathcal{O}^o(U) \subseteq \mathrm{Frac}(\mathcal{O}^o(\pi^{-1}(P))) = F_P$, so $X(F_P) \neq \emptyset$.

If P is a generic point of $\mathscr{C}'_s$, then $\pi^{-1}(P)$ is a single type 2 point x_P, and by Proposition 3.3.5, $\mathscr{M}_{x_P} \subseteq \mathcal{H}(x_P) = F_P$. Thus $X(F_P) \neq \emptyset$.

Since π is surjective ([**53**, Lemma 4.11]), this implies that $X(F_P) \neq \emptyset$ for all $P \in \mathscr{C}'_s$. □

3.4. Applications to Quadratic Forms and the u-invariant

We give applications to quadratic forms, and in particular, to the u-invariant of function fields. The results presented in this section generalize those of [**34**, Section 4].

3.4.1. Local-global principles for quadratic forms. The main example of a setting satisfying the conditions we have seen so far (*e.g.* see Theorem 3.2.11) are quadratic forms.

We can apply Theorem 3.2.11 to the projective variety X defined by a quadratic form q over F. In [**34**, Theorem 4.2], HHK show that for a regular quadratic form q over F, if $\mathrm{char}(F) \neq 2$, $SO(q)$ - the special orthogonal group of q, acts strongly transitively on X when $\dim q \neq 2$, so in that case we can take $G = SO(q)$. If $\dim q = 2$, then X may not be connected and consequently the group $SO(q)$ doesn't necessarily act strongly transitively on X (see [**34**, Example 4.4] and the proof of [**34**, Theorem 4.2]).

We will say that quadratic form q defined over F is *isotropic over a field extension* K/F if there exists a non-zero v over K such that $q(v) = 0$. In other words, q is isotropic over a field K if and only if the projetive variety defined by q has a K-rational point.

Recall that by Theorem 1.8.15, an irreducible compact k-analytic curve is either projective or an affinoid space.

THEOREM 3.4.1. *Let k be a complete ultrametric field. Let C be a compact irreducible normal k-analytic curve. If $|k^\times| = \{1\}$, assume C is projective. Set $F = \mathscr{M}(C)$. Suppose $char(F) \neq 2$. Let q be a quadratic form over F of dimension different from 2.*

1. *The quadratic form q is isotropic over F if and only if it is isotropic over $\mathscr{M}_x$ for all $x \in C$.*
2. *Let $\mathcal{U}$ be an open cover of C. Then, q is isotropic over F if and only if it is isotropic over $\mathscr{M}(U)$ for all $U \in \mathcal{U}$.*

PROOF. By Witt decomposition ([**46**, I.4.1]), $q = q_t \perp q_r$, where q_r is regular and q_t is totally isotropic. If $q_t \neq 0$, then q is isotropic, so we may assume that q is regular. Consequently, Theorem 3.2.9, and Theorem 3.2.11 are applicable, proving the statement. □

COROLLARY 3.4.2. *Let k be a complete non-Archimedean valued field. Let C be a compact irreducible normal k-analytic curve. Set $F = \mathscr{M}(C)$. Suppose $char(F) \neq 2$. Let q be a quadratic form over F of dimension different from 2. The following local-global principles hold:*

1. *If C is affinoid and $\sqrt{|k^\times|} \neq \mathbb{R}_{>0}$, q is isotropic over F if and only if it is isotropic over all completions $F_v, v \in V_{\mathcal{O}(C)}(F)$, of F.*
2. *If C is projective, q is isotropic over F if and only if it is isotropic over all completions $F_v, v \in V(F)$, of F.*

PROOF. If k is trivially valued, then the trivial valuation v_0 of F is in $V_{\mathcal{O}(C)}(F)$ (resp. $V(F)$), and since $F_{v_0} = F$, the statement is clear in this case.

Otherwise, by Witt decomposition ([**46**, I.4.1]), $q = q_t \perp q_r$, where q_r is regular and q_t is totally isotropic. If $q_t \neq 0$, then q is isotropic. Otherwise, q is regular, so smooth, and we conclude by Corollary 3.2.18. □

3.4.2. Local Calculations. In view of the local-global principle we proved for quadratic forms (Theorem 3.4.1), we now want to find sufficient conditions under which there is local isotropy. To do this, we will need to put further restrictions on the base field. Throughout this section, we will suppose the dimension of $\sqrt{|k^\times|}$ as a $\mathbb{Q}$-vector space (*i.e.* the rational rank of $|k^\times|$) is $n \in \mathbb{Z}$. In the special case that $|k^\times|$ is a free $\mathbb{Z}$-module (e.g. if k is a discretely valued field), the sufficient conditions for local isotropy can be refined. The class of such fields is quite broad, especially when it comes to arithmetic questions: if we work over a complete ultrametric base field k satisfying this condition, then for any k-analytic space and any of its points x, the field $\mathcal{H}(x)$ also satisfies it.

For any valued field E, we denote by E° its ring of integers, by $E^{\circ\circ}$ the corresponding maximal ideal, and by $\widetilde{E}$ its residue field.

For the following two propositions, the case of characteristic 2 can be treated uniformly with the general one. Afterwards, we will restrict to residual characteristic different from 2.

PROPOSITION 3.4.3. *Let l be a valued field. Suppose $|l^\times|$ is a free $\mathbb{Z}$-module of finite rank n. Let L be a valued field extension of l. Let q be a non-zero diagonal quadratic form*

over L. Suppose for any non-zero coefficient a of q, $|a| \in |l^\times|$. There exists a family Q of at most 2^n quadratic forms with coefficients in $(L^\circ)^\times$, such that q is L-isometric to $\perp_{\sigma\in Q} C_\sigma \cdot \sigma$, where $C_\sigma \in L^\times$ for any $\sigma \in Q$.

PROOF. Let us fix $\pi_1, \pi_2, \dots, \pi_n \in l^\times$, such that their norms form a basis of the $\mathbb{Z}$-module $|l^\times|$. Set $\mathcal{A} = \{\prod_{i=1}^n \pi_i^{\delta_i} | \delta_i \in \{0,1\}\}$. For any coefficient a of q, let $p_1, p_2, \dots, p_n \in \mathbb{Z}$ be such that $|a| = \prod_{i=1}^n |\pi_i|^{p_i}$. Then, there exist $v_a \in (L^o)^\times$ and $s_a \in \mathcal{A}$, such that $a \equiv v_a s_a \bmod (L^\times)^2$. Consequently, for any $A \in \mathcal{A}$, there exists a diagonal quadratic form σ_A with coefficients in $(L^\circ)^\times$, such that q is L-isometric to $\perp_{A\in\mathcal{A}} A \cdot \sigma_A$. □

The following is the analogue of Proposition 3.4.3 in a more general case.

PROPOSITION 3.4.4. *Let l be a valued field, such that $\dim_{\mathbb{Q}} \sqrt{|l^\times|}$ equals an integer n. Let L be a valued field extension of l. Let q be a non-zero diagonal quadratic form over L. Suppose for any non-zero coefficient a of q, $|a| \in \sqrt{|l^\times|}$. Then, there exists a family Q of at most 2^{n+1} quadratic forms with coefficients in $(L^\circ)^\times$, such that q is L-isometric to $\perp_{\sigma\in Q} C_\sigma \cdot \sigma$, where $C_\sigma \in L^\times$ for any $\sigma \in Q$.*

PROOF. To ease the notation, let us start by introducing the following:

NOTATION 3.4.5. Let M be a multiplicative $\mathbb{Z}$-module, such that the divisible closure $\sqrt{M}$ of M as a group is a finite dimensional $\mathbb{Q}$-vector space. Set $n = \dim_{\mathbb{Q}} \sqrt{M}$. Set $M^2 = \{m^2 : m \in M\}$.

There exist $t_1, t_2, \dots, t_n \in M$, such that for any $t \in M$, there exist unique $p_1, p_2, \dots, p_n \in \mathbb{Q}$, for which $t = \prod_{i=1}^n t_i^{p_i}$. Let us fix such elements $t_1, t_2, \dots, t_n$.

In the particular situation that is of interest to us, $M = |l^\times|$, and there exist $\pi_1, \pi_2, \dots, \pi_n \in l$, with $|\pi_i| = t_i$, such that for any $\epsilon \in \sqrt{|l^\times|}$, there exist unique $p_1, p_2, \dots, p_n \in \mathbb{Q}$, for which $\epsilon = \prod_{i=1}^n |\pi_i|^{p_i}$. Let us fix such elements $\pi_1, \pi_2, \cdots, \pi_n$.

DEFINITION 3.4.6. Let $\epsilon \in M$. Suppose $\epsilon = \prod_{i=1}^n t_i^{\frac{s_i}{r_i}}$, for $\frac{s_i}{r_i} \in \mathbb{Q}$ with s_i, r_i coprime, $i = 1, 2, \dots, n$.

(1) Let r be the least common multiple of $r_i, i = 1, 2, \dots, n$. We will say r is *the order of* ϵ.
(2) Let $\frac{s_i}{r_i} = \frac{s_i'}{r}$, $i = 1, 2, \dots, n$. If there exists i_0, such that $s_{i_0}' = 1$, then t_{i_0} will be said to be *a base of* ϵ.

Let $\epsilon \in M$, and suppose $\epsilon = \prod_{i=1}^n t_i^{p_i}$, for $p_i \in \mathbb{Q}$, $i = 1, 2, \dots, n$. Let α be the order of ϵ.

LEMMA 3.4.7. *If α is odd, then for any $i = 1, 2, \dots, n$, there exist $\delta_i \in \{0, 1\}$, such that $\epsilon \equiv \prod_{i=1}^n t_i^{\delta_i}$ mod M^2.*

PROOF. Remark that since α is odd, $\epsilon \equiv \epsilon^\alpha \bmod M^2$, and $\epsilon^\alpha = \prod_{i=1}^n t_i^{s_i}$, with $s_i \in \mathbb{Z}$ for all i. Let $s_i = 2s_i' + \delta_i$, where $s_i' \in \mathbb{Z}$ and $\delta_i \in \{0, 1\}$. Then, $\epsilon \equiv \prod_{i=1}^n t_i^{\delta_i} \bmod M^2$. □

LEMMA 3.4.8. *If α is even, then there exist $m \in M$, $x_i, y \in \mathbb{Z}$, $i = 1, 2, \dots, n$, with $y > 0$, satisfying:*

(1) *$\epsilon \equiv m$ mod M^2;*
(2) *$m = \prod_{i=1}^n t_i^{x_i/2^y}$;*
(3) *there exists $i_0 \in \{1, 2, \dots, n\}$, such that $x_{i_0} = 1$.*

Remark that t_{i_0} is a base of m and its order is 2^y.

PROOF. Let $\alpha = 2^y \cdot z$, with z odd and $y > 0$. Then, $\epsilon \equiv \epsilon^z \bmod M^2$, and $(\epsilon^z)^{2^y} = \prod_{i=1}^n t_i^{e_i}$, with $e_i \in \mathbb{Z}, i = 1, 2, \ldots, n$. Furthermore, there exists $i_0 \in \{1, 2, \ldots, n\}$, such that e_{i_0} is odd.

Seeing as $(2^y, e_{i_0}) = 1$, there exist $A, B \in \mathbb{Z}$, with A odd, such that $Ae_{i_0} + 2^y B = 1$. Then, $\epsilon^z \equiv \epsilon^z \cdot (\epsilon^z)^{A-1} \bmod M^2$, and $\epsilon^{zA} = t_{i_0}^{1/2^y - B} \cdot \prod_{i \neq i_0} t_i^{Ae_i/2^y}$. Hence, there exists $m'_B \in M$, such that $\epsilon^{zA} \equiv m'_B \bmod M^2$, and

- $m'_B = t_{i_0}^{1/2^y} \prod_{i \neq i_0} t_i^{Ae_i/2^y}$ if B is even;
- $m'_B = t_{i_0}^{1/2^y + 1} \prod_{i \neq i_0} t_i^{Ae_i/2^y}$ if B is odd.

If B is odd, $m''_B := m'_B \cdot {m'_B}^{2^y} t_{i_0}^{-2-2^y} \equiv m'_B \bmod M^2$, and $m''_B = t_{i_0}^{1/2^y} \prod_{i \neq i_0} t_i^{\frac{Ae_i}{2^y}(2^y+1)}$.

Consequently, in either case, there exist $m \in M$ and $x_i \in \mathbb{Z}$, for $i = 1, 2, \ldots n$, with $x_{i_0} = 1$, such that $\epsilon \equiv m \bmod M^2$, and $m = \prod_{i=1}^n t_i^{x_i/2^y}$. □

For $\varepsilon \in L$, such that $|\varepsilon| \in \sqrt{|l^\times|}$, we will say that the order of $|\varepsilon|$ is *the order of* ε. If $|\pi_{i_0}|$ is a base of $|\varepsilon|$, we will say π_{i_0} is *a base of* ε. By applying the last two lemmas to the valued field L, we obtain:

COROLLARY 3.4.9. *Let $\varepsilon \in L^\times$. Suppose $|\varepsilon| = \prod_{i=1}^n |\pi_i|^{p_i}$ for $p_i \in \mathbb{Q}, i = 1, 2, \ldots, n$.*

1. *If the order of $|\varepsilon|$ is odd, then for any $i = 1, 2, \ldots, n$, there exists $\delta_i \in \{0, 1\}$, such that $\varepsilon \equiv \prod_{i=1}^n \pi_i^{\delta_i} \bmod (L^\times)^2 (L^o)^\times$.*
2. *If the order of $|\varepsilon|$ is even, then there exist $\varepsilon' \in L^\times$, $x_i, y \in \mathbb{Z}$, $i = 1, 2, \ldots, n$, with $y > 0$, satisfying:*
   (a) *$\varepsilon \equiv \varepsilon' \bmod (L^\times)^2 (L^o)^\times$;*
   (b) *$|\varepsilon'| = \prod_{i=1}^n |\pi_i|^{x_i/2^y}$;*
   (c) *there exists $i_0 \in \{1, 2, \ldots, n\}$, such that $x_{i_0} = 1$.*

We immediately obtain as a by-product of the proof:

COROLLARY 3.4.10. *Let $\varepsilon \in L^\times$, such that $|\varepsilon| \in \sqrt{|l^\times|}$. Suppose the order of $|\varepsilon|$ is 2^ν, so that there exist $\nu_i \in \mathbb{Z}, i = 1, 2, \ldots, n$, such that $|\varepsilon| = \prod_{i=1}^n |\pi_i|^{\nu_i/2^\nu}$. If $\nu_{i'}$ is odd for some i', then there exists $\varepsilon' \in L^\times$, such that $\varepsilon \equiv \varepsilon' \bmod (L^\times)^2 (L^\circ)^\times$, and $|\pi_{i'}|$ is a base of $|\varepsilon'|$.*

Let q_1 (resp. q_2) be the part of q whose coefficients have odd (resp. even) order. We remark that q_1, q_2 are diagonal quadratic forms over L, and that $q = q_1 \perp q_2$.

Decomposition of q_1: Set $\mathcal{A} = \left\{ \prod_{i=1}^n \pi_i^{\delta_i} | \delta_i \in \{0, 1\} \right\}$. Let e be any coefficient of q_1. By Corollary 3.4.9 (1), there exist $u_e \in (L^\circ)^\times$ and $A_e \in \mathcal{A}$, such that $e \equiv u_e \cdot A_e \bmod (L^\times)^2$. Consequently, for any $A \in \mathcal{A}$, there exists a diagonal quadratic form σ_A with coefficients in $(L^\circ)^\times$, such that q_1 is L-isometric to $\perp_{A \in \mathcal{A}} A \cdot \sigma_A$.

Decomposition of q_2: We first need an auxiliary result, which requires the following:

DEFINITION 3.4.11. Let $\varepsilon \in L^\times$ be such that there exist $p_i \in \mathbb{Q}$, $i = 1, 2, \ldots, n$, for which $|\varepsilon| = \prod_{i=1}^n |\pi_i|^{p_i}$. Let $I \subseteq \{0, 1 \ldots, n\}$, such that $\{i : p_i \neq 0\} \subseteq I$. We will say that ε is *given in $|I|$ parameters*, where $|I|$ is the cardinality of I, or that ε is *given in parameters over I*.

Notice that $a \in L$ is given in 0 parameters if and only if $a \in (L^\circ)^\times$.

LEMMA 3.4.12. *Let τ be a diagonal quadratic form over L with coefficients of order either 1 or an even number. Let $I \subseteq \{1, 2, \dots, n\}$, with $1 \leqslant |I| = m \leqslant n$, such that the coefficients of τ are given in parameters over I. Then, there exist:*

- *$J \subseteq I$, with $|J| = m - 1$,*
- *$x_1, x_2 \in L^\times$,*
- *diagonal quadratic forms τ_1, τ_2 over L with coefficients of order either 1 or an even number and in parameters over J,*

such that τ is L-isometric to $x_1\tau_1 \perp x_2\tau_2$.

PROOF. Roughly, the idea is to find some i_0 and a partition $A_j, j = 1, 2$, of the set of coefficients, for which there exist $x_j \in L^\times$, satisfying: if $a \in A_j$, there exists $B_a \in L^\times$, such that, modulo squares, $a = x_j \cdot B_a$, and $|B_a| = \prod_{i \neq i_0} |\pi_i|^{p_{i,a}}, p_{i,a} \in \mathbb{Q}$. In what follows, we find suitable representatives of the coefficients modulo squares, from which we can read the factorization $x_j \cdot B_a$.

Without loss of generality, let us assume that $I = \{1, 2, \dots, m\}$. Suppose the coefficients of τ are all of order 1. If they are given in zero parameters, the statement is clear. Otherwise, suppose that there is a coefficient given over a set of parameters containing t_1.

Let d be any coefficient of the quadratic form. There exist $s_i \in \mathbb{Z}, i = 1, 2, \dots, n$, such that $|d| = \prod_{i=1}^n |\pi_i|^{s_i}$. As a consequence, there exist $d' \in L^\times$ and $s_i' \in \mathbb{Z}, i = 2, \dots, n$, for which $d \equiv d' \bmod (L^\times)^2(L^\circ)^\times$, and either $|d'| = \prod_{i=2}^n |\pi_i|^{s_i'}$ or $|d'| = |\pi_1| \cdot \prod_{i=2}^n |\pi_i|^{s_i'}$. Hence, there exist diagonal quadratic forms τ_1, τ_2, whose coefficients are all of order 1, in parameters over $\{2, 3, \dots, m\}$, such that τ is L-isometric to $\pi_1\tau_1 \perp \tau_2$.

Suppose there exists at least one coefficient of τ of even order. Let τ' be the quadratic form obtained from τ by:

(1) leaving the coefficients of order 1 intact;
(2) applying Corollary 3.4.9 (2) to the coefficients of even order to substitute them by elements of $L^\times$ that satisfy properties 2 and 3 of the lemma.

We remark that due to the proof of Corollary 3.4.9 (2) (*i.e.* Lemma 3.4.8), the set of parameters over which the coefficients of τ' are given doesn't change. The quadratic form τ' is L-isometric to τ. Let us fix a', one of the coefficients of τ' with largest order. Suppose the order of a' is $2^{\alpha'}$. Without loss of generality, we may assume that π_1 is a base of a'. For $i = 2, \dots, m$, let $\alpha_i \in \mathbb{Z}$ be such that $|a'| = |\pi_1|^{1/2^{\alpha'}} \cdot \prod_{i=2}^m |\pi_i|^{\alpha_i/2^{\alpha'}}$.

Let c be any other coefficient of τ'. Let π_{i_0} be a base of c, and $2^\gamma, \gamma \geqslant 0$, its order. For $i = 1, 2, \dots, m$, let $\gamma_i \in \mathbb{Z}$ be such that $|c| = \prod_{i=1}^m |\pi_i|^{\gamma_i/2^\gamma}$.

- Suppose $\alpha' > \gamma$. Set $c' = c \cdot a'^{(2^\gamma - \gamma_1) \cdot 2^{\alpha' - \gamma}}$ Then, $c' \equiv c \bmod (L^\times)^2(L^\circ)^\times$, and $|c'| = |\pi_1| \cdot \prod_{i=2}^m |\pi_i|^{\frac{\gamma_i + \alpha_i(2^\gamma - \gamma_1)}{2^\gamma}}$.
- Suppose $\alpha' = \gamma$ and γ_1 is odd. By Corollary 3.4.10, there exist $\alpha_i' \in \mathbb{Z}, i = 2, 3, \dots, n$, and $c'' \in L^\times$ of order $2^{\alpha'}$, having π_1 as a base, such that $c'' \equiv c \bmod (L^\times)^2(L^\circ)^\times$ and $|c''| = |\pi_1|^{1/2^{\alpha'}} \cdot \prod_{i=2}^m |\pi_i|^{\alpha_i'/2^{\alpha'}}$.
- Suppose $\alpha' = \gamma$ and γ_1 is even. Let $\gamma_1'/2^\delta$ be the reduced form of $\gamma_1/2^\gamma$, meaning γ_1' is odd. Set $c''' = c \cdot a'^{(2^\delta - \gamma_1') \cdot 2^{\alpha' - \delta}}$. Then, $c''' \equiv c \bmod (L^\times)^2(L^\circ)^\times$, and $|c'''| = |\pi_1| \cdot \prod_{i=2}^m |\pi_i|^{\frac{\gamma_i + \alpha_i(2^\gamma - \gamma_1)}{2^\gamma}}$.

To summarize, there exist $\bar{c} \in L^\times$ and $\epsilon_2, \cdots, \epsilon_m \in \mathbb{Z}$, such that $c \equiv \bar{c} \bmod (L^\times)^2(L^\circ)^\times$, and either $|\bar{c}| = |\pi_1|^{1/2^\alpha} \cdot \prod_{i=2}^m |\pi_i|^{\epsilon_i/2^{\alpha'}} = |a'| \cdot \prod_{i=2}^m |\pi_i|^{\frac{\epsilon_i - \alpha_i}{2^{\alpha'}}}$ or $|\bar{c}| = |\pi_1| \cdot \prod_{i=2}^m |\pi_i|^{\epsilon_i/2^{\alpha'}}$.

Therefore, there exist diagonal quadratic forms τ_1, τ_2 over L, such that $\tau \cong \pi_1\tau_1 \perp a'\tau_2$, and for any coefficient h of τ_1 or τ_2, the order of h is either 1 or an even integer. Furthermore, h is with parameters over $\{2, 3, \ldots, m\}$. □

Using induction, an immediate consequence of Lemma 3.4.12 is that there exists a family T of 2^n quadratic forms with coefficients in $(L^\circ)^\times$, such that τ is L-isometric to $\perp_{\sigma\in T} B_\sigma \cdot \sigma$, where $B_\sigma \in L^\times$ for any $\sigma \in T$.

Finally, by combining the decomposition results of q_1 and q_2, we obtain the statement of Proposition 3.4.4. □

The following framework corresponds to Berkovich curves:

Setting 3.4.13. Let k be a complete ultrametric field. Let $k \subseteq R$ be a Henselian valuation ring with maximal ideal m_R, and fraction field $F_R = \operatorname{Frac} R$. Set $L' = R/m_R$, and suppose it is endowed with a valuation making it a Henselian (called *quasicomplete* in [**4**]) valued field extension of k. Let L/L' be an immediate Henselian extension. Set $t = \operatorname{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = \operatorname{rank}_{\mathbb{Q}}(|L'^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q})$ and $s = \deg \operatorname{tr}_{\widetilde{k}} \widetilde{L} = \deg \operatorname{tr}_{\widetilde{k}} \widetilde{L'}$. Suppose $s + t \leqslant 1$.

The motivation behind this setup is:

Example 3.4.14. Let C be any k-analytic curve, and $x \in C$ any point. The hypotheses of the setting above are satisfied for $R = \mathcal{O}_x$, $F_R = \mathscr{M}_x$, $L' = \kappa(x)$, and $L = \mathcal{H}(x)$.

For any quadratic form σ with coefficients in R, let us denote by σ_L (resp. $\sigma_{L'}$) its image over L (resp. L').

We recall:

Definition 3.4.15. Let K be a field.

(1) [Kaplansky] The *u-invariant* of K, denoted by $u(K)$, is the maximal dimension of anisotropic quadratic forms over K. We say that $u(K) = \infty$ if there exist anisotropic quadratic forms over K of arbitrarily large dimension.
(2) [HHK] The strong u-invariant of K, denoted by $u_s(K)$, is the smallest real number m, such that:
   - $u(E) \leqslant m$ for all finite field extensions E/K;
   - $\frac{1}{2}u(E) \leqslant m$ for all finitely generated field extensions E/K of transcendence degree 1.

   We say that $u_s(K) = \infty$ if there exist such field extensions E of arbitrarily large u-invariant.

Notation 3.4.16. From now on, let k be a complete ultrametric field, such that $\dim_{\mathbb{Q}} \sqrt{|k^\times|}$ equals an integer n. Also, suppose $\operatorname{char} \widetilde{k} \neq 2$.

Proposition 3.4.17. *Let L/k be a valued field extension, such that* $\operatorname{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$ *and* $\deg \operatorname{tr}_{\widetilde{k}} \widetilde{L} = 0$. *Let τ be a quadratic form over L, with* $\dim \tau > 2^{n+1} u_s(\widetilde{k})$.

(1) *Suppose L is Henselian. Then, τ is isotropic.*
(2) *Under the same hypotheses as in Setting 3.4.13, let q be a diagonal quadratic form over R, such that $q_L = \tau$. Then, q is isotropic over F_R.*

Proof. Since $\operatorname{char}(L) \neq 2$, we may assume that τ is a diagonal quadratic form. Seeing as $\dim_{\mathbb{Q}} \sqrt{|L^\times|} = n$, by Proposition 3.4.4 there exists a set Q of at most 2^{n+1} quadratic

forms with coefficients in $(L^\circ)^\times$, such that τ is L-isometric to $\perp_{\sigma\in Q} C_\sigma \cdot \sigma$, with $C_\sigma \in L^\times$ for every $\sigma \in Q$.

Since $\dim \tau > 2^{n+1} u_s(\widetilde{k})$, there exists $\tau' \in Q$, such that $\dim \tau' > u_s(\widetilde{k})$. Let $\widetilde{\tau'}$ be the image of τ' over $\widetilde{L}$. Seeing as the coefficients of τ' are all in $(L^\circ)^\times$, $\dim \widetilde{\tau'} = \dim \tau' > u_s(\widetilde{k})$. Since $\deg \mathrm{tr}_{\widetilde{k}} \widetilde{L} = 0$, the extension $\widetilde{L}/\widetilde{k}$ is algebraic. Let E be the finite field extension of $\widetilde{k}$ generated by the coefficients of $\widetilde{\tau'}$. Then, $u(E) \leqslant u_s(\widetilde{k}) < \dim \widetilde{\tau'}$, implying $\widetilde{\tau'}$ is isotropic over E, and hence over $\widetilde{L}$. Since L is Henselian, τ' is isotropic over L, and thus so is τ.

For the second part, if $\tau = q_L$ for some diagonal R-quadratic form q, seeing as $\widetilde{\tau'}$ is isotropic over $\widetilde{L} = \widetilde{L'}$, the image of q in $\widetilde{L'}$ is so as well. From Henselianity of L', we obtain that the image of q in L' is isotropic there. Finally, from Henselianity of R, the quadratic form q is isotropic over F_R. □

The bound $2^{n+1} u_s(\widetilde{k})$ in Proposition 3.4.17 will remain the same regardless of whether we demand $|k^\times|$ to be a free $\mathbb{Z}$-module or not. The reason behind this is that in any case, the hypotheses of said proposition tell us only that $\dim_{\mathbb{Q}} \sqrt{|L^\times|} = n$, but not necessarily that $|L^\times|$ is a free $\mathbb{Z}$-module.

PROPOSITION 3.4.18. *Let L/k be a valued field extension, such that* $\mathrm{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$ *and* $\deg \mathrm{tr}_{\widetilde{k}} \widetilde{L} = 1$. *Let τ be a quadratic form over L, with* $\dim \tau > 2^{n+2} u_s(\widetilde{k})$.

1. *Suppose L is Henselian. Then, τ is isotropic.*
2. *Under the same hypotheses as in Setting 3.4.13, let q be a diagonal quadratic form over R, such that $q_L = \tau$. Then, q is isotropic over F_R.*

If $|L^\times|$ is a free $\mathbb{Z}$-modules of dimension n, the statement is true for $\dim \tau > 2^{n+1} u_s(\widetilde{k})$.

PROOF. Since $\mathrm{char}(L) \neq 2$, we may assume that τ is a diagonal quadratic form. Again, let $\perp_{\sigma\in Q} C_\sigma \sigma$ be the L-quadratic form isometric to τ obtained from Proposition 3.4.4 (resp. Proposition 3.4.3), where Q has cardinality at most 2^{n+1} (resp. 2^n). Then, there exists $\tau' \in Q$, such that $\dim \tau' > 2u_s(\widetilde{k})$. Let $\widetilde{\tau'}$ be the image of τ' over $\widetilde{L}$. Since the coefficients of τ' are all in $(L^o)^\times$, $\dim \widetilde{\tau'} = \dim \tau' > 2u_s(\widetilde{k})$.

As the extension $\widetilde{L}/\widetilde{k}$ is finitely generated of transcendence degree 1, one obtains $u(\widetilde{L}) \leqslant 2u_s(\widetilde{k}) < \dim \tau'$. This implies that τ' is isotropic over $\widetilde{L}$. Since L is Henselian, the quadratic form τ' is isotropic over L, and thus so is τ.

For the second part, if $\tau = q_L$ for some diagonal quadratic form q over R, we conclude by using the same argument as in Proposition 3.4.17, seeing as $\widetilde{\tau'}$ is isotropic over $\widetilde{L'}$. □

PROPOSITION 3.4.19. *Let L/k be a valued field extension, such that* $\mathrm{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = 1$ *and* $\deg \mathrm{tr}_{\widetilde{k}} \widetilde{L} = 0$. *Let τ be a quadratic form over L, with* $\dim \tau > 2^{n+2} u_s(\widetilde{k})$.

1. *Suppose L is Henselian. Then, τ is isotropic.*
2. *Under the same hypotheses as in Setting 3.4.13, let q be a diagonal quadratic form over R, such that $q_L = \tau$. Then, q is isotropic over F_R.*

If $|k^\times|$ is a free $\mathbb{Z}$-module, the statement is true for $\dim \tau > 2^{n+1} u_s(\widetilde{k})$.

PROOF. Since $\mathrm{char}(L) \neq 2$, we may assume that τ is a diagonal quadratic form. Since $\mathrm{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = 1$, there exists $\rho \in \mathbb{R}_{>0} \backslash \sqrt{|k^\times|}$, such that the group $|L^\times|$ is generated by $|k^\times|$ and ρ. Let T be an element of L with $|T| = \rho$. Then, for any $a \in L^\times$, there exist $m \in \mathbb{Z}$, $p_i \in \mathbb{Q}$ (resp. $p_i \in \mathbb{Z}$), $i = 1, 2, \ldots, n$, such that $|a| = |T|^m \cdot \prod_{i=1}^n |\pi_i|^{p_i}$.

Consequently, there exist diagonal quadratic forms q_1, q_2 over L, for which τ is isometric to $q_1 \perp T q_2$, where the coefficients of q_1, q_2 have norms in $|k^\times|$.

By applying Proposition 3.4.4 (resp. Proposition 3.4.3) to q_1 and q_2, we obtain a family S of at most 2^{n+2} (resp. 2^{n+1}) diagonal quadratic forms with coefficients in $(L^\circ)^\times$, such that τ is isometric to $\perp_{\sigma\in S} C_\sigma \cdot \sigma$, where $C_\sigma \in L^\times$ for every $\sigma \in S$. Thus, there exists $\tau' \in S$, such that $\dim \tau' > u_s(\widetilde{k})$. Let $\widetilde{\tau'}$ be the image of τ' in $\widetilde{L}$. Seeing as the coefficients of τ' are all in $(L^\circ)^\times$, $\dim \widetilde{\tau'} = \dim \tau' > u_s(\widetilde{k})$.

The extension $\widetilde{L}/\widetilde{k}$ is finite algebraic, so $u(\widetilde{L}) \leqslant u_s(\widetilde{k}) < \dim \widetilde{\tau'}$, implying $\widetilde{\tau'}$ is isotropic over $\widetilde{L}$. Since L is Henselian, τ' is isotropic over L, and thus so is τ.

For the second part, if $\tau = q_L$ for some q, as $\widetilde{\tau'}$ is isotropic over $\widetilde{L'}$, we conclude as in Proposition 3.4.17. □

Keeping the same notation, the three propositions above can be summarized into:

THEOREM 3.4.20. *Let L/k be a valued field extension. Suppose that the inequality* $\operatorname{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) + \deg \operatorname{tr}_{\widetilde{k}} \widetilde{L} \leqslant 1$ *holds. Let τ be a quadratic form over L, with* $\dim \tau > 2^{n+2} u_s(\widetilde{k})$.

(1) *Suppose L is Henselian. Then, τ is isotropic.*
(2) *Under the same hypotheses as in Setting 3.4.13, let q be a diagonal quadratic form over R, such that $q_L = \tau$. Then, q is isotropic over F_R.*

If $|k^\times|$ is a free $\mathbb{Z}$-module, and $|L^\times|$ is a free $\mathbb{Z}$-module with $\operatorname{rank}_{\mathbb{Z}}|L^\times| = n$ *if* $\deg \operatorname{tr}_{\widetilde{k}} \widetilde{L} = 1$ *and* $\operatorname{rank}_{\mathbb{Q}}(|L^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$, *then the statement is true for* $\dim \tau > 2^{n+1} u_s(\widetilde{k})$.

A result we will be using often in what follows:

LEMMA 3.4.21. *Suppose $|k^\times|$ is a free $\mathbb{Z}$-module of dimension n. Let k'/k be a valued field extension, such that $|k'^\times|$ is finitely generated over $|k^\times|$, and $|k'^\times|/|k^\times|$ is a torsion group. Then, $|k'^\times|$ is also a free $\mathbb{Z}$-module of dimension n.*

Suppose k'/k is a finite field extension. Let τ be a diagonal quadratic form over k' with $\dim \tau > 2^n u_s(\widetilde{k})$. *Then, q is k'-isotropic.*

PROOF. Seeing as $|k'^\times|/|k^\times|$ is a torsion group, its rank as a $\mathbb{Z}$ module is 0. Considering $\operatorname{rank}_{\mathbb{Z}}|k'^\times| = \operatorname{rank}_{\mathbb{Z}}|k'^\times|/|k^\times| + \operatorname{rank}_{\mathbb{Z}}|k^\times|$, we obtain $\operatorname{rank}_{\mathbb{Z}}|k'^\times| = n$. Furthermore, being a finitely generated torsion-free module over $\mathbb{Z}$, it is free.

Let $\perp_{\sigma\in Q} C_\sigma \cdot \sigma$ be the quadratic form k'-isometric to τ obtained by applying Proposition 3.4.3. There exists $\sigma_0 \in Q$ with coefficients in $(k'^\circ)^\times$, such that $\dim \widetilde{\sigma_0} = \dim \sigma_0 > u_s(\widetilde{k})$, where $\widetilde{\sigma_0}$ is the image of σ_0 over $\widetilde{k'}$. Suppose k'/k is a finite field extension. Seeing as then $\widetilde{k'}/\widetilde{k}$ is also finite, $\widetilde{\sigma_0}$ is $\widetilde{k'}$-isotropic. From Henselianity of k', we obtain that σ_0 is k'-isotropic, thus so is τ. □

The following shows that if $|k^\times|$ is a free finitely generated $\mathbb{Z}$-module of dimension n, the last conditions of Theorem 3.4.20 are satisfied in the Berkovich setting.

COROLLARY 3.4.22. *Suppose $|k^\times|$ is a free $\mathbb{Z}$-module with* $\operatorname{rank}_{\mathbb{Z}}|k^\times| = n$. *Let C be a k-analytic curve. If $x \in C$ is a type 2 point, then $|\mathcal{H}(x)^\times|$ is a free $\mathbb{Z}$-module and* $\operatorname{rank}_{\mathbb{Z}}(|\mathcal{H}(x)^\times|) = n$.

PROOF. Since x is an Abhyankar point, $|\mathcal{H}(x)^\times|$ is finitely generated over $|k^\times|$, and since it is of type 2, $|\mathcal{H}(x)^\times|/|k^\times|$ is a torsion group, so this follows from Lemma 3.4.21. □

Another result we will be needing in what is to come:

LEMMA 3.4.23. *Under the same hypotheses as in Setting 3.4.13, suppose R is a discrete valuation ring. Let q be a diagonal quadratic form over F_R. Then, there exist diagonal F_R-quadratic forms q_1, q_2 with coefficients in R, and $a \in F_R^\times$, such that:*

- *q is isometric to $q_1 \perp a q_2$;*
- *$q_{i,L}$ has coefficients in $(L^\circ)^\times, i = 1, 2$;*
- *there exists $i_0 \in \{1, 2\}$, such that $\dim q_{i_0,L} \geqslant \frac{1}{2} \dim q$.*

In particular, if either of q_1, q_2 is isotropic over F_R, then so is q.

PROOF. Let π be a uniformizer of R. For any coefficient b of q, either $b \equiv 1 \bmod (F_R^\times)^2 (F_R^\circ)^\times$ or $b \equiv \pi \bmod (F_R^\times)^2 (F_R^\circ)^\times$. Hence, there exist diagonal F_R-quadratic forms q_1, q_2 with coefficients in $(F_R^\circ)^\times = R^\times$, such that q is F_R-isometric to $q' = q_1 \perp \pi q_2$. Then, $\dim q = \dim q'$, and there exists i_0, such that $\dim q_{i_0} \geqslant \frac{1}{2} \dim q$. Since the coefficients of q_1, q_2 are in $R^\times$, their images over L are of same dimension, so $\dim q_{i_0,L} \geqslant \frac{1}{2} \dim q$. Finally, the last sentence of the statement is obvious. □

The following theorem gives the motivation behind the hypotheses we put upon R, L' and L.

THEOREM 3.4.24. *Suppose $char(\widetilde{k}) \neq 2$. Let C be a normal irreducible k-analytic curve. Set $F = \mathscr{M}(C)$. Let q be a quadratic form over F of dimension d, with $d > 2^{n+2} u_s(\widetilde{k})$. Then, for any $x \in C$, the quadratic form q is isotropic over $\mathscr{M}_x$ for all $x \in C$.*

If $|k^\times|$ is a free $\mathbb{Z}$-module, the statement is true for $d > 2^{n+1} u_s(\widetilde{k})$.

PROOF. Seeing as $\mathrm{char}(\widetilde{k}) \neq 2$, neither of the overfields of k has characteristic 2. In particular, $\mathrm{char}(F) \neq 2$, so there exists a diagonal quadratic form q' over F isometric to q. By replacing q with q' if necessary, we may directly assume that q is a diagonal quadratic form.

Recall that $\mathcal{O}_x$ and $\kappa(x)$ are Henselian [**4**, Sections 2.1 and 2.3]. Furthermore, $\mathcal{H}(x)$ is the completion of $\kappa(x)$, so it is a Henselian immediate extension. We know that for any $x \in C$, the field $\mathcal{H}(x)$ is either a finite extension of k or a completion of F with respect to some valuation extending that of k. Abhyankar's inequality tells us that $\mathrm{rank}_{\mathbb{Q}}(|\mathcal{H}(x)^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) + \deg \mathrm{tr}_{\widetilde{k}} \widetilde{\mathcal{H}(x)} \leqslant 1$. We will apply part 2 of Theorem 3.4.20 by taking $R = \mathcal{O}_x$, $F_R = \mathscr{M}_x$, $L' = \kappa(x)$, and $L = \mathcal{H}(x)$.

If $\mathcal{H}(x)/k$ is finite, *i.e.* if x is a rigid point, then $\mathcal{H}(x) = \kappa(x) = \mathcal{O}_x/m_x$. Being a normal Noetherian local ring with Krull dimension one, $\mathcal{O}_x$ is a discrete valuation ring. By Lemma 3.4.23, there exists a diagonal $\mathscr{M}_x$-quadratic form τ with coefficients in $\mathcal{O}_x$, such that $\dim \tau_L \geqslant \frac{1}{2} \dim q > 2^{n+1} u_s(\widetilde{k})$ (resp. $\dim \tau_L \geqslant \frac{1}{2} \dim q > 2^n u_s(\widetilde{k})$) and the isotropy of τ implies that of q. Seeing as $\mathrm{rank}_{\mathbb{Q}}(|\mathcal{H}(x)^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}) = \deg \mathrm{tr}_{\widetilde{k}} \widetilde{\mathcal{H}(x)} = 0$, we can apply Proposition 3.4.17 (resp. Lemma 3.4.21) to τ.

Otherwise, $\mathcal{O}_x = \kappa(x)$ is a field, and $\mathcal{H}(x)$ is its completion. In the general case, we conclude by a direct application of Theorem 3.4.20. In particular, if $|k^\times|$ is a free $\mathbb{Z}$-module, then this is an application of Theorem 3.4.20 in view of Corollary 3.4.22. □

We also obtain:

COROLLARY 3.4.25. *Suppose $char(\widetilde{k}) \neq 2$. Let C be a normal irreducible k-analytic curve. Let x be any point of C. Let q be a quadratic form over $\mathcal{H}(x)$, such that $\dim q > 2^{n+2} u_s(\widetilde{k})$. Then, q is isotropic.*

If $|k^\times|$ is a free $\mathbb{Z}$-module, then the statement is true for $\dim q > 2^{n+1}u_s(\widetilde{k})$.

PROOF. This is a direct consequence of part (1) of Theorem 3.4.20 (in view of Corollary 3.4.22 for the special case). □

3.4.3. The applications. We will now apply the results obtained in the previous section to the (strong) u-invariant.

We recall:

DEFINITION 3.4.26. Let K be a field.

(1) [Kaplansky] The *u-invariant* of K, denoted by $u(K)$, is the maximal dimension of anisotropic quadratic forms over K. We say that $u(K) = \infty$ if there exist anisotropic quadratic forms over K of arbitrarily large dimension.
(2) [HHK] The strong u-invariant of K, denoted by $u_s(K)$, is the smallest real number m, such that:
   - $u(E) \leqslant m$ for all finite field extensions E/K;
   - $\frac{1}{2}u(E) \leqslant m$ for all finitely generated field extensions E/K of transcendence degree 1.

   We say that $u_s(K) = \infty$ if there exist such field extensions E of arbitrarily large u-invariant.

Let k be a complete ultrametric field.

THEOREM 3.4.27. *Suppose $char(\widetilde{k}) \neq 2$. Let F be a finitely generated field extension of k of transcendence degree* 1. *Let q be a quadratic form over F of dimension d.*

(1) *If* $\dim_{\mathbb{Q}} \sqrt{|k^\times|} =: n \in \mathbb{N}$ *and* $d > 2^{n+2}u_s(\widetilde{k})$, *then q is isotropic.*
(2) *If $|k^\times|$ is a free $\mathbb{Z}$-module with* $\operatorname{rank}_{\mathbb{Z}}|k^\times| =: n \in \mathbb{N}$ *and* $d > 2^{n+1}u_s(\widetilde{k})$, *then q is isotropic.*

PROOF. There exists a connected normal projective k-analytic curve C such that $F = \mathscr{M}(C)$. By Theorem 3.4.1, the quadratic form q is isotropic over F if and only if it is isotropic over $\mathscr{M}_x$ for all $x \in C$. The statement now follows in view of Theorem 3.4.24. □

COROLLARY 3.4.28. *Suppose $char(\widetilde{k}) \neq 2$.*

(1) *If* $\dim_{\mathbb{Q}} \sqrt{|k^\times|} =: n \in \mathbb{N}$, *then* $u_s(k) \leqslant 2^{n+1}u_s(\widetilde{k})$.
(2) *If $|k^\times|$ is a free $\mathbb{Z}$-module with* $\operatorname{rank}_{\mathbb{Z}}|k^\times| =: n \in \mathbb{N}$, *then* $u_s(k) \leqslant 2^n u_s(\widetilde{k})$.

PROOF. Let l/k be a finite field extension. Let q be an l-quadratic form of dimension $d > 2^{n+1}u_s(\widetilde{k})$ (resp. $d > 2^n u_s(\widetilde{k})$). Since $\operatorname{char}(\widetilde{k}) \neq 2$, we may assume q to be diagonal. In view of part 1 of Proposition 3.4.17 (resp. Lemma 3.4.21), q is l-isotropic, so $u(l) \leqslant 2^{n+1}u_s(\widetilde{k})$ (resp. $u(l) \leqslant 2^n u_s(\widetilde{k})$). In combination with Theorem 3.4.27, this completes the proof of the statement. □

COROLLARY 3.4.29. *Suppose $char(\widetilde{k}) \neq 2$. Let C be a normal irreducible k-analytic curve. Let x be any point of C.*

(1) *If* $\dim_{\mathbb{Q}} \sqrt{|k^\times|} =: n \in \mathbb{N}$, *then* $u(\mathcal{H}(x)) \leqslant 2^{n+2}u_s(\widetilde{k})$.
(2) *If $|k^\times|$ is a free $\mathbb{Z}$-module with* $\operatorname{rank}_{\mathbb{Z}}|k^\times| =: n \in \mathbb{N}$, *then* $u(\mathcal{H}(x)) \leqslant 2^{n+1}u_s(\widetilde{k})$.

PROOF. See Corollary 3.4.25. □

In particular, when k is discretely valued we obtain the upcoming corollary. It is the most important result on quadratic forms in [**34**], and from it we obtain that $u(\mathbb{Q}_p(T)) = 8$ when $p \neq 2$, originally shown in [**58**].

COROLLARY 3.4.30. *Let k be a complete discretely valued field, such that $char(\widetilde{k}) \neq 2$. Then, $u_s(k) = 2u_s(\widetilde{k})$.*

PROOF. The inequality $u_s(k) \leqslant 2u_s(\widetilde{k})$ is a special case of Corollary 6.2. For the other direction, a proof that is independent of the patching method and relies on the theory of quadratic forms is given in [**34**, Lemma 4.9]. □

CHAPTER 4

Patching over Analytic Fibers and the Local-Global Principle

In this chapter we generalize patching to neighborhoods of certain fibers of a relative proper analytic curve. As an application, we obtain a local-global principle for the germs of meromorphic functions on said fibers.

We treat the case of the relative projective line $\mathbb{P}^{1,\mathrm{an}}$ first. In Section 4.1, we construct the notion of *relative nice covers* around a fiber of $\mathbb{P}^{1,\mathrm{an}}$, analoguous to (and a generalization of) nice covers for curves, and show that it possesses good properties, *i.e.* properties that are necessary for patching. To do this, we start by showing some complementary properties of affinoid domains in the analytic projective line that allow us to deduce a particular *writing* for them. This writing makes it possible to construct affinoid domains in a neighborhood of a fiber (of a relative $\mathbb{P}^{1,\mathrm{an}}$) from an affinoid domain on said fiber. We call this process *thickening*[1] of an affinoid domain. A relative nice cover of the neighborhood of a fiber (of a relative $\mathbb{P}^{1,\mathrm{an}}$) is the thickening of a nice cover of the fiber.

In order to be able to apply the results of Chapter 2 to this setting, it is necessary to constantly "shrink" to smaller neighborhoods of the fiber. Because of this, we need some uniform boundedness results and explicit norm comparisons, which is the topic of Section 4.2. As a consequence, this is the most technical section of Chapter 4. It also contains an explicit description of the Banach algebras of analytic functions on certain affinoid domains of the relative projective line.

In Section 4.3, we show that the results of Chapter 2 are indeed applicable to relative nice covers of fibers of the relative $\mathbb{P}^{1,\mathrm{an}}$, and that patching (in the sense of Chapter 2) can be obtained as a consequence thereof. This is then extended (in the sense of Proposition 3.2.2) to include the level of generality necessary for proving the analoguous result around fibers of relative analytic curves. The arguments used in this section are of very topological nature.

In Section 4.4, we study the properties of the class of relative analytic curves over which we know how to apply patching around certain fibers. The condition that is required is not too restrictive; namely, the relative proper curve is assumed to be normal and algebraic around the fiber, so this is satisfied for the Berkovich analytification of any normal proper relative *algebraic* curve. Using Grothendieck's work on the projective limit of schemes, we show that smooth geometrically irreducible projective algebraic curves defined over certain fields give rise to a proper relative analytic curve satisfying this condition. In particular, this makes it possible to generalize some results from Chapter 3.

In Section 4.5, we construct covers (also called *relative nice covers*) on a neighborhood of fibers of a relative proper analytic curve and show that they satisfy the necessary

[1]The idea for thickenings of affinoid domains of $\mathbb{P}^{1,\mathrm{an}}$ originally appears in some unpublished notes of Jérôme Poineau.

properties for patching to be applicable. For this, we use pullbacks of relative nice covers in the case of $\mathbb{P}^{1,\mathrm{an}}$. Once again, the arguments that are employed are of very topological nature. We then use these covers, as well as the corresponding result in the case of relative $\mathbb{P}^{1,\mathrm{an}}$, to prove that patching is possible in this setting.

Finally, in Section 4.6, we apply patching to prove local-global principles for the germs of meromorphic functions on a fiber of a proper relative curve. As in the case of curves, we first show a local-global principle where the overfields are the stalks of the sheaf of meromorphic functions, and then apply this to obtain a local-global principle with respect to completions. In order to show the latter from the former, we first prove there is a connection between the points of a fiber and the valuations on the field of germs of its meromorphic functions (which we show to have nice algebraic properties; namely, it can be realised as the function field of a certain algebraic curve).

The fibers around which we apply patching are those over points for which their corresponding stalk is a field. In Section 4.7, we calculate some examples of these fields.

4.1. Nice covers for the relative projective line

As in the case of curves, we construct covers around fibers of the relative projective line over which a generalized form of patching as seen in Proposition 3.2.2 will be possible. More precisely, we construct relative analogues of nice covers (Definition 3.1.6).

4.1.1. Some results on the analytic projective line. Let us start with a couple of auxiliary results on the analytic projective line. Recall the nature of the points of $\mathbb{P}^{1,\mathrm{an}}$ presented in Subsection 1.8.4.

Proposition 4.1.1. *Let K be a complete ultrametric field. Let U be a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_K$ with only type 3 points in its boundary. Suppose U is not a point. Let us fix a copy of $\mathbb{A}^{1,\mathrm{an}}_K$ and a coordinate T on it. Let $\partial U = \{\eta_{R_i,r_i} : i = 1,2,\ldots,n\}$, where $R_i \in K[T]$ are irreducible polynomials and $r_i \in \mathbb{R}_{>0}\backslash\sqrt{|K^\times|}$. Then, $U = \bigcap_i \{x : |R_i|_x \bowtie_i r_i\}$, where $\bowtie_i \in \{\leqslant, \geqslant\}, i = 1,2,\ldots,n$.*

Proof. We need the following two auxiliary results:

Lemma 4.1.2. *For any $i \in \{1,2,\ldots,n\}$, either $U \subseteq \{x : |R_i|_x \leqslant r_i\}$ or $U \subseteq \{x : |R_i|_x \geqslant r_i\}$.*

Proof. To see this, assume that the open subsets $V_1 := U \cap \{x : |R_i|_x < r_i\}$ and $V_2 := U \cap \{x : |R_i|_x > r_i\}$ of U are non-empty. As intersections of two connected subets of $\mathbb{P}^{1,\mathrm{an}}_K$, both V_1 and V_2 are connected. Assume $V_j \cap \mathrm{Int}(U) = \emptyset, j = 1,2.$, This implies $V_j \subseteq \partial U$, and since V_j is connected, it is a single type 3 point $\{\eta_j\}$. But then, this would be an isolated point of U, which is in contradiction with the connectedness of U. Consequently, there exist $x_j \in V_j \cap \mathrm{Int}(U), j = 1,2$. By Lemma 1.8.16, $\mathrm{Int}(U)$ is a connected set, so there exists a unique arc $[x_1, x_2]$ connecting x_1, x_2 that is entirely contained in $\mathrm{Int}(U)$. Since $|R_i|_{x_1} < r_i$, $|R_i|_{x_2} > r_i$, there exists $x_0 \in [x_1, x_2]$ such that $|R_i|_{x_0} = r_i$. Since there is a unique point satisfying this condition (Proposition 1.8.25), and it is η_{R_i,r_i}, we obtain that $\eta_{R_i,r_i} \in [x_1, x_2] \subseteq \mathrm{Int}(U)$, which is in contradiction with the fact that $\eta_{R_i,r_i} \in \partial U$. Thus, there exists $j \in \{1,2\}$ such that $V_j = \emptyset$, implying the statement. □

Lemma 4.1.3. *For $n \in \mathbb{N}$, let $W_i := \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |P_i| \bowtie_i r_i\}$, where $P_i \in K[T]$ is irreducible, $r_i \in \mathbb{R}_{>0}\backslash\sqrt{|k^\times|}$, $\bowtie_i \in \{\leqslant, \geqslant\}$, $i \in \{1,2,\ldots,n\}$. Suppose for all $i \neq j$, $W_i \not\subseteq Int(W_j)$. Then, for $V := \bigcap_{i=1}^n W_i$, $\partial V = \bigcup_{i=1}^n \partial W_i$.*

PROOF. Since $\mathrm{Int}(V) = \bigcap_{j=1}^n \mathrm{Int}(W_j)$, we obtain that $\partial V = \left(\bigcap_{j=1}^n W_j\right) \setminus (\bigcap_{i=1}^n \mathrm{Int}(W_i)) = \bigcup_{i=1}^n \bigcap_{j=1}^n (W_i \backslash \mathrm{Int}(W_j))$. Suppose there exist $i, j \in \{1, 2, \ldots, n\}$ such that $W_i \backslash \mathrm{Int}(W_j) = \emptyset$. Then, $W_i \subseteq \mathrm{Int}(W_j)$, contradicting the hypothesis of the statement.

Hence, for any i, j, $W_i \backslash \mathrm{Int}(W_j) \neq \emptyset$. In particular, this means that $W_i \cap \mathrm{Int}(W_j)$ is a strict open subset of W_i, so contained in $\mathrm{Int}(W_i)$. Consequently, $\{\eta_{P_i,r_i}\} = W_i \backslash \mathrm{Int}(W_i) \subseteq W_i \backslash (W_i \cap \mathrm{Int}(W_j)) \subseteq W_i \backslash \mathrm{Int}(W_j)$. This implies that for any i, $\bigcap_{j=1}^n (W_i \backslash \mathrm{Int}(W_j)) = \{\eta_{P_i,r_i}\}$.

Finally, $\partial V = \{\eta_{P_i,r_i} : i = 1, 2, \ldots, n\}$, proving the statement. □

If $U \subseteq \{x : |R_i|_x \leqslant r_i\}$ (resp. $U \subseteq \{x : |R_i|_x \geqslant r_i\}$), set $U_i = \{x : |R_i|_x \leqslant r_i\}$ (resp. $U_i = \{x : |R_i|_x \geqslant r_i\}$). Remark that for all i, U_i is connected and contains U. Set $V = \bigcap_{i=1}^n U_i$. Let us show that $\partial V = \partial U$. Assume there exist i, j such that $U_i \subseteq \mathrm{Int}(U_j)$. Then, $\eta_{R_j,r_j} \not\in U_i$, so $\eta_{R_j,r_j} \notin U$, contradiction. Thus, Lemma 4.1.3 is applicable, and so $\partial V = \{\eta_{R_i,r_i}\} = \partial U$.

Remark that V is a connected affinoid domain (as an intersection of connected affinoid domains) of $\mathbb{P}^{1,\mathrm{an}}_K$. Also, $U \subseteq V$ and $\partial U = \partial V$. Let us show that $U = V$. Suppose there exists some $x \in V \backslash U$. Then, $x \in \mathrm{Int}(V)$. Let $y \in \mathrm{Int}(U) \subseteq \mathrm{Int}(V)$. The unique arc $[x, y]$ in $\mathbb{P}^{1,\mathrm{an}}_K$ connecting x and y is contained in $\mathrm{Int}(V)$ (by connectedness of the latter, see Lemma 1.8.16). At the same time, since $x \notin U$ and $y \in U$, the arc $[x, y]$ intersects $\partial U = \partial V$, contradiction. Thus, $U = V = \bigcap_{i=1}^n U_i$. □

In particular, the result above implies that every connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_K$ with only type 3 points in its boundary is a rational domain.

Recall Proposition 1.8.19. For any $x, y \in \mathbb{P}^{1,\mathrm{an}}_K$, we denote by $[x, y]$ the unique arc in $\mathbb{P}^{1,\mathrm{an}}_K$ connecting x and y.

LEMMA 4.1.4. *Let K be a complete ultrametric field. Let U, V be connected affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_K$ containing only type 3 points in their boundaries, such that $U \cap V = \partial U \cap \partial V$ is a single type 3 point $\{\eta_{R,r}\}$ (i.e. R is an irreducible polynomial over K and $r \in \mathbb{R}_{>0} \backslash \sqrt{|K^\times|}$).*

- *If $U \subseteq \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |R|_x \leqslant r\}$ (resp. $U \subseteq \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |R|_x \geqslant r\}$), then $V \subseteq \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |R|_x \geqslant r\}$ (resp. $V \subseteq \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |R|_x \leqslant r\}$).*
- *Suppose $U \subseteq \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |R|_x \leqslant r\}$. Set $\partial U = \{\eta_{R,r}, \eta_{P_i,r_i}\}_{i=1}^n$ and $\partial V = \{\eta_{R,r}, \eta_{P'_j,r'_j}\}_{j=1}^m$, so that $U = \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |R|_x \leqslant r, |P_i|_x \bowtie_i r_i, i\}$ and $V = \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |R|_x \geqslant r, |P'_j|_x \bowtie'_j r'_j, j\}$, where $\bowtie_i, \bowtie'_j \in \{\leqslant, \geqslant\}$, $P_i, P'_j \in K[T]$ are irreducible, and $r_i, r'_j \in \mathbb{R}_{>0} \backslash \sqrt{|K^\times|}$ for all i, j.*

  Then, $U \cup V = \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |P_i|_x \bowtie_i r_i, |P'_j|_x \bowtie'_j r'_j, i = 1, \ldots, n, j = 1, \ldots, m\}$. If $n = m = 0$, this means that $U \cup V = \mathbb{P}^{1,\mathrm{an}}_K$.

PROOF. (1) Remark that if $U \subseteq V$, then $U = \{\eta_{R,r}\}$, so the statement is trivially satisfied. The same is true if $V \subseteq U$. Let us suppose that neither of U, V is contained in the other.

Suppose $U \subseteq \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |R|_x \leqslant r\}$ and $V \subseteq \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |R|_x \leqslant r\}$. Let $u \in U \backslash V$ and $v \in V \backslash U$. Since $u, v \in \{x : |R|_x < r\}$ - which is a connected set (Lemma 1.8.16), $[u, v] \subseteq \{x : |R|_x < r\}$. At the same time, since $[u, \eta_{R,r}] \subseteq U$ and $[\eta_{R,r}, v] \subseteq V$, $[u, \eta_{R,r}] \cap [\eta_{R,r}, v] = \{\eta_{R,r}\}$, so the arc $[u, v] = [u, \eta_{R,r}] \cup [\eta_{R,r}, v]$ contains the point $\eta_{R,r}$. This is in contradiction with the fact that $[u, v] \subseteq \{x :$

$|R|_x < r\}$. The case $U, V \subseteq \{x \in \mathbb{P}^{1,\mathrm{an}}_K : |R|_x \geqslant r\}$ is shown to be impossible in the same way. (This property is true regardless of whether $\partial U \backslash \{\eta_{R,r}\}$ and $\partial V \backslash \{\eta_{R,r}\}$ contain only type 3 points or not.)

(2) The statement is clearly true if $m = n = 0$, so we may assume that is not the case.

Remark that $\partial(U \cup V) \subseteq \partial U \cup \partial V$. Let $\eta \in \partial U \backslash V$. Let G be any neighborhood of η in $\mathbb{P}^{1,\mathrm{an}}_K$. Since V is closed, there exists a neighborhood $G' \subseteq G$ of η such that $G' \cap V = \emptyset$. Since $\eta \in \partial U$, G' contains points of both U and U^C. Consequently, G', and thus G, contain points of both $U \cup V$ and $U^C \cap V^C = (U \cup V)^C$. Seeing as this is true for any neighborhood G of η, we obtain that $\eta \in \partial(U \cup V)$, implying $\partial U \backslash V \subseteq \partial(U \cup V)$. Similarly, $\partial V \backslash U \subseteq \partial(U \cup V)$. It only remains to check for the point $\eta_{R,r}$.

Let $x \in \mathrm{Int}(U) \subseteq \mathrm{Int}(U \cup V)$ and $y \in \mathrm{Int}(V) \subseteq \mathrm{Int}(U \cup V)$. Remark that $x \notin V$ and $y \notin U$. Furthermore, $|R|_x < r$ and $|R|_y > r$. Consequently, $\eta_{R,r} \in [x, y]$. Since $U \cup V$ is a connected affinoid domain containing only type 3 points in its boundary, its interior is connected (see Lemma 1.8.16). Consequently, $[x, y] \subseteq \mathrm{Int}(U \cup V)$, and hence $\eta_{R,r} \in \mathrm{Int}(U \cup V)$.

We have shown that $\partial(U \cup V) = \{\eta_{P_i,r_i}, \eta_{P'_j,r'_j} : i, j\}$. Since $U \subseteq \{x : |P_i|_x \bowtie_i r_i\}$ and $V \subseteq \{x : |P'_j|_x \bowtie'_j r'_j\}$ for all i, j, we obtain that

$$U \cup V = \{x : |P_i|_x \bowtie_i r_i, |P'_j|_x \bowtie'_j r'_j, i, j\}.$$

□

4.1.2. The general setting.

NOTATION 4.1.5. Let S be a normal *good* k-analytic space (*i.e.* affinoid domains form a basis of the Berkovich topology on S). Suppose that $\dim S < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$. Let us denote by π the structural morphism $\mathbb{P}^{1,\mathrm{an}}_S \to S$. Let $x \in S$ be such that $\mathcal{O}_{S,x}$ is a field. Let F_x be the fiber of x on $\mathbb{P}^{1,\mathrm{an}}_S$, which can be endowed with the analytic structure of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ (see Proposition 1.5.7).

Remark that a connected affinoid domain of S is integral.

Let us explain the hypothesis on the dimension of S in Notation 4.1.5. As in Chapter 3, type 3 points play a very important role for obtaining patching results *around* the fiber F_x. Hence, their existence *on* the fiber is crucial and, as will be seen in the next lemma, this is guaranteed by the condition we imposed on the dimension of S. Recall that for a complete ultrametric field K, a K-analytic curve contains type 3 points if and only if $\sqrt{|K^\times|} \neq \mathbb{R}_{>0}$.

LEMMA 4.1.6. *Let Y be a k-analytic space such that $\dim Y < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$. Then, for any $y \in Y$, $\sqrt{|\mathcal{H}(y)^\times|} \neq \mathbb{R}_{>0}$.*

PROOF. For any $y \in Y$, we have

$$\dim_{\mathbb{Q}} |\mathcal{H}(y)^\times|/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q} \leqslant d(\mathcal{H}(y)/k) \leqslant \dim Y < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Consequently, $\sqrt{|\mathcal{H}(y)^\times|} \neq \mathbb{R}_{>0}$. □

By Lemma 4.1.6, in Notation 4.1.5, $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ contains type 3 points.

LEMMA 4.1.7. *Let U be a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ with only type 3 points in its boundary. Then, all the polynomials R_i from Proposition 4.1.1 can be chosen so that their coefficients are in $\mathcal{O}_x$.*

PROOF. Let $\eta \in \partial U$. It suffices to show that there exist $P \in \mathcal{O}_x[T]$ irreducible over $\mathcal{H}(x)$ and $p > 0$, such that $\eta = \eta_{P,p}$.

The connected components of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} \backslash \{\eta\}$ are virtual discs. Let us fix one that does not contain the point ∞. We need to show it contains a rigid point $\eta_{R,0}$ with $R \in \mathcal{O}_x[T]$ with R irreducible over $\mathcal{H}(x)$. This follows immediately from the density of $\mathcal{O}_x$ in $\mathcal{H}(x)$. □

REMARK 4.1.8. Let U be a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ containing only type 3 points in its boundary. Then, there exist polynomials $R_i \in \mathcal{O}_x[T]$ irreducible over $\mathcal{H}(x)$ and positive real numbers $r_i, i = 1, 2, \ldots, n$, such that $U = \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R_i|_u \bowtie_i r_i, i = 1, 2, \ldots, n\}$, where $\bowtie_i \in \{\leqslant, \geqslant\}$ for all i. Consequently, there exists some connected affinoid neighborhood Z of x in S, such that $R_i \in \mathcal{O}(Z)[T]$ for all i. Hence, the affinoid domain U can be *thickened* to an affinoid domain $\{u \in \mathbb{P}^{1,\mathrm{an}}_{Z} : |R_i|_u \bowtie_i r_i, i = 1, 2, \ldots, n\}$ of $\pi^{-1}(Z) = \mathbb{P}^{1,\mathrm{an}}_{Z}$. The role of nice covers in this relative setting will be played by covers that are constructed by thickening affinoid domains of the fiber $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. We now study some properties of such domains which make patching possible.

4.1.3. A Theorem: Thickenings of Type 3 Points. Following Notation 4.1.5, the goal of this part is to show:

THEOREM 4.1.9. *Let $\eta_{R,r}$ be a type 3 point of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$, where $R \in \mathcal{O}_x[T]$ is irreducible over $\mathcal{H}(x)$ and $r \in \mathbb{R}_{>0} \backslash \sqrt{|\mathcal{H}(x)^\times|}$. There exists a connected affinoid neighborhood Z_0 of x in S, such that*

- *$R \in \mathcal{O}(Z_0)[T]$,*
- *for any connected affinoid neighborhood $Z \subseteq Z_0$ of x, the set $\{u \in \mathbb{P}^{1,\mathrm{an}}_{Z} : |R|_u = r\}$ is a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_{Z}$.*

PROOF. Without loss of generality, since $\mathcal{O}_x$ is a field, we may assume that $R(T)$ is a unitary polynomial.

To prove the statement, we need several auxiliary lemmas.

LEMMA 4.1.10. *Let K be a complete ultrametric field. Let $R(T)$ be a split unitary polynomial over K. Let $r \in \mathbb{R}_{>0}$. Then, for any root α of $R(T)$ there exists a unique positive real number s_α such that $\{y \in \mathbb{P}^{1,\mathrm{an}}_{K} : |R(T)|_y = r\} = \bigcup_{R(\alpha)=0} \{y \in \mathbb{P}^{1,\mathrm{an}}_{K} : |T - \alpha|_y = s_\alpha\}$. The point η_{α,s_α} is the only point y of the arc $[\eta_{\alpha,0}, \infty]$ in $\mathbb{P}^{1,\mathrm{an}}_{K}$ for which $|R(T)|_y = r$. Furthermore, $r = s_\alpha \cdot \prod_{R(\beta)=0, \alpha \neq \beta} \max(s_\alpha, |\alpha - \beta|)$.*

PROOF. Remark that if $y \in \mathbb{P}^{1,\mathrm{an}}_{K}$ is such that $|R(T)|_y = 0$, then $\prod_{R(\alpha)=0} |T - \alpha|_y = 0$, meaning there exists a root α_0 of $R(T)$ such that $|T - \alpha_0|_y = 0$ (notice that we haven't assumed $R(T)$ to be separable, *i.e.* there could be roots with multiplicities). This determines the unique point $\eta_{\alpha_0,0}$ in $\mathbb{P}^{1,\mathrm{an}}_{K}$. Thus, the zeros of $R(T)$ in $\mathbb{P}^{1,\mathrm{an}}_{K}$ are $\eta_{\alpha,0}, R(\alpha) = 0$. Remark also that R has only one pole in $\mathbb{P}^{1,\mathrm{an}}_{K}$ and that is the point ∞.

By [**20**, 3.4.23.1], the analytic function $R(T)$ on $\mathbb{P}^{1,\mathrm{an}}_{K}$ is locally constant everywhere outside of the finite graph $\Gamma := \bigcup_{R(\alpha)=0} [\eta_{\alpha,0}, \infty]$. Furthermore, its variation is compatible

with the canonical retraction $d : \mathbb{P}^{1,\mathrm{an}}_K \to \Gamma$ in the sense that $|R(T)|_y = |R(T)|_{d(y)}$ for any $y \in \mathbb{P}^{1,\mathrm{an}}_K$ (*cf.* [**20**, 3.4.23.8]). By [**20**, 3.4.24.3], $R(T)$ is continuously strictly increasing in all the arcs $[\eta_{\alpha,0}, \infty], R(\alpha) = 0$, where $|R(T)|_{\eta_{\alpha,0}} = 0$ and $|R(T)|_\infty = +\infty$. Consequently, $|R(T)|$ attains the value r exactly one time on each arc $[\eta_{\alpha,0}, \infty]$. Suppose s_α is the unique positive real number for which $|R(T)|_{\eta_{\alpha,s_\alpha}} = r$. Then, $\prod_{R(\beta)=0} |T-\beta|_{\eta_{\alpha,s_\alpha}} = s_\alpha \cdot \prod_{R(\beta)=0,\alpha\neq\beta} \max(s_\alpha, |\alpha-\beta|) = r$.

We have shown that there exist positive real numbers s_α such that $\{y \in \Gamma : |R|_y = r\} = \{\eta_{\alpha,s_\alpha} : R(\alpha) = 0\}$. As mentioned before, the variation of R is compatible with the canonical retraction d of $\mathbb{P}^{1,\mathrm{an}}_K$ to Γ. Since $d^{-1}(\eta_{\alpha,s_\alpha}) = \{y \in \mathbb{P}^{1,\mathrm{an}}_K : |T-\alpha|_y = s_\alpha\}$, we finally obtain that $\{y \in \mathbb{P}^{1,\mathrm{an}}_K : |R|_y = r\} = \bigcup_{R(\alpha)=0}\{y \in \mathbb{P}^{1,\mathrm{an}}_K : |T-\alpha|_y = s_\alpha\}$ with s_α as above. □

Let Z_1 be some connected affinoid neighborhood of x in S such that $R \in \mathcal{O}(Z_1)[T]$. Let E be a finite field extension of $\mathscr{M}(Z_1)$ on which $R(T)$ splits. Since $\mathcal{O}(Z_1)$ is Japanese (see [**6**, Proposition 2.1.14]), its integral closure in E is a finite $\mathcal{O}(Z_1)$-algebra, and in particular, an integral k-affinoid algebra (see Proposition 1.3.19). Let us denote by Z' the corresponding integral k-affinoid space.

By construction, we have a finite morphism $\varphi : Z' \to Z_1$ inducing a finite morphism $\psi : \mathbb{P}^{1,\mathrm{an}}_{Z'} \to \mathbb{P}^{1,\mathrm{an}}_{Z_1}$, and the polynomial $R(T)$ is split over $\mathcal{O}(Z')$. Set $\{x_1, x_2, \ldots, x_t\} := \varphi^{-1}(x)$. Let us study the affinoid domain $|R(T)| = \prod_{R(\alpha)=0} |T-\alpha| = r$ in $\mathbb{P}^{1,\mathrm{an}}_{Z'}$, i.e. the affinoid $\{u \in \mathbb{P}^{1,\mathrm{an}}_{Z'} : \prod_{R(\alpha)=0} |T-\alpha|_u = r\}$.

Since φ is a finite morphism, $\sqrt{|\mathcal{H}(x)^\times|} = \sqrt{|\mathcal{H}(x_i)^\times|}$ for any $i = 1, 2, \ldots, t$, so $r \notin \sqrt{|\mathcal{H}(x_i)^\times|}$. By Lemma 4.1.10, there exist positive real numbers s_{α,x_i}, $R(\alpha) = 0$, such that $\{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x_i)} : |R|_u = r\} = \bigcup_{R(\alpha)=0}\{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x_i)} : |T-\alpha|_u = s_{\alpha,x_i}\}$. Since $r \notin \sqrt{|\mathcal{H}(x_i)^\times|}$, $\{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x_i)} : |R|_u = r\}$ cannot contain any type 2 points, so $s_{\alpha,x_i} \notin \sqrt{|\mathcal{H}(x_i)^\times|} \cup \{0\}$ and $\{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x_i)} : |R|_u = r\} = \{\eta_{\alpha,s_{\alpha,x_i}} : R(\alpha) = 0\}$ (for $a \in k, r \in \mathbb{R}_{>0}$, recall the notation $\eta_{a,r}$ in Subsection 1.2.4).

LEMMA 4.1.11. *For any $i \in \{1, 2, \ldots, t\}$, and any root α of $R(T)$, there exists a connected affinoid neighborhood Z'_i of x_i and a continuous function $s^i_\alpha : Z'_i \to \mathbb{R}_{>0}$ such that for any $y \in Z'_i$,*

$$\{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(y)} : |R|_u = r\} = \bigcup_{R(\alpha)=0} \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(y)} : |T-\alpha|_u = s^i_\alpha(y)\}.$$

Furthermore, we may assume that for any $j \neq i, x_j \notin Z'_i$.

PROOF. Let us fix an $i \in \{1, 2, \ldots, t\}$ and a root α of $R(T)$ of multiplicity m. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the rest of the roots (with multiplicity) of $R(T)$, ordered in such a way that for any $j \leqslant l$, $|\alpha-\alpha_j|_{x_i} \leqslant |\alpha-\alpha_l|_{x_i}$. As remarked above, $s_{\alpha,x_i} \notin \sqrt{|\mathcal{H}(x_i)^\times|} \cup \{0\}$, so $s_{\alpha,x_i} \neq |\alpha-\alpha_j|_{x_i}$ for all $j = 1, 2, \ldots, n$. Set $\alpha_0 := \alpha$. Then, there exists a unique $j_0 \in \{0, 1, \ldots, n\}$, such that $|\alpha-\alpha_j|_{x_i} < s_{\alpha,x_i} < |\alpha-\alpha_l|_{x_i}$ for all j, l for which $j \leqslant j_0 < l$ (in particular, if $j_0 = 0$, this means that $0 < s_{\alpha,x_i} < |\alpha-\alpha_1|_{x_i}$, and if $j_0 = n$, that $|\alpha-\alpha_n|_{x_i} < s_{\alpha,x_i}$). Since in $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x_i)}$:

$$r = |R|_{\eta_{\alpha,s_{\alpha,x_i},x_i}} = |T-\alpha|^m_{\eta_{\alpha,s_{\alpha,x_i},x_i}} \prod_{j=1}^n |T-\alpha_i|_{\eta_{\alpha,s_{\alpha,x_i},x_i}} = s^m_{\alpha,x_i} \cdot \prod_{j=1}^n \max(s_{\alpha,x_i}, |\alpha-\alpha_j|_{x_i}),$$

we obtain that $s_{\alpha,x_i} = \sqrt[j_0+m]{\frac{r}{\prod_{j=j_0+1}^{n}|\alpha-\alpha_j|_{x_i}}}$ (this means that $s_{\alpha,x_i} = \sqrt[n+m]{r}$ if $j_0 = n$.) Note that $|\alpha-\alpha_j|_{x_i} \neq 0$ for all $j > j_0$ seeing as $s_{\alpha,x_i} < |\alpha-\alpha_j|_{x_i}$.

Since the function $Z' \to \mathbb{R}_{>0}$, $y \mapsto |\alpha-\alpha_j|_y$ is continuous for all $j = 1, 2, \ldots, n$, there exists a connected affinoid neighborhood $Z_{i,1}$ of x_i in Z' such that $|\alpha-\alpha_j|_y \neq 0$ for all $j > j_0$ and all $y \in Z_{i,1}$.

Let us define $s_\alpha^i : Z_{i,1} \to \mathbb{R}_{>0}$ by $y \mapsto \sqrt[j_0+m]{\frac{r}{\prod_{j=j_0+1}^{n}|\alpha-\alpha_j|_y}}$. It is a continuous function and $s_{\alpha,x_i} = s_\alpha^i(x_i)$. Also, $|\alpha-\alpha_j|_{x_i} < s_\alpha^i(x_i) < |\alpha-\alpha_l|_{x_i}$ for all j, l for which $j \leqslant j_0 < l$. Since on all sides of these strict inequalities we have continuous functions, there exists a connected affinoid neighborhood Z_i' of x_i in $Z_{i,1}$ such that for all $y \in Z_i'$, $s_\alpha^i(y)$ is positive and $|\alpha-\alpha_j|_y < s_\alpha^i(y) < |\alpha-\alpha_l|_y$ for all j, l for which $j \leqslant j_0 < l$.

Consequently, for $y \in Z_i'$, in $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(y)}$, $|R(T)|_{\eta_{\alpha,s_\alpha^i(y)}} = s_\alpha^i(y)^{j_0+m} \cdot \prod_{j=j_0+1}^{n}|\alpha-\alpha_j|_y = r$. We can now conclude by using Lemma 4.1.10.

Finally, the last part of the statement is a direct consequence of the fact that Z' is Hausdorff. □

Remark 4.1.12. Lemma 4.1.11 is clearly true for any connected affinoid neighborhood of x_i contained in Z_i'.

Let Z_i be any connected affinoid neighborhood of x_i such that $Z_i \subseteq Z_i'$. In view of Lemma 4.1.11, for any $i \in \{1, 2, \ldots, t\}$, $\{u \in \mathbb{P}^{1,\mathrm{an}}_{Z_i} : |R(T)|_u = r\} = \bigcup_{R(\alpha)=0}\{u \in \mathbb{P}^{1,\mathrm{an}}_{Z_i} : |T-\alpha|_u = s_\alpha^i(\pi(u))\}$. For any root α of $R(T)$, set $S_{\alpha,Z_i} := \{u \in \mathbb{P}^{1,\mathrm{an}}_{Z_i} : |T-\alpha|_u = s_\alpha^i(\pi(u))\}$.

Lemma 4.1.13. *For $i \in \{1, 2, \ldots, t\}$, the set S_{α,Z_i} is connected.*

Proof. Seeing as s_α^i is a continuous function, S_{α,Z_i} is a closed and hence compact subset of $\mathbb{P}^{1,\mathrm{an}}_{Z_i}$. Suppose that S_{α,Z_i} is not connected and assume it can be written as a disjoint union of two closed subsets S'_{α,Z_i} and S''_{α,Z_i}. Since S_{α,Z_i} is compact in $\mathbb{P}^{1,\mathrm{an}}_{Z_i}$, so are S'_{α,Z_i} and S''_{α,Z_i}. Since the morphism π is proper, $\pi(S'_{\alpha,Z_i})$ and $\pi(S''_{\alpha,Z_i})$ are both compact subsets of Z_i. Also, $\pi(S_{\alpha,Z_i}) = Z_i$, implying $Z_i = \pi(S'_{\alpha,Z_i}) \cup \pi(S''_{\alpha,Z_i})$. Assume that $\pi(S'_{\alpha,Z_i}) \cap \pi(S''_{\alpha,Z_i}) \neq \emptyset$. This means that there exists a point $y \in Z_i$, such that both $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(y)} \cap S'_{\alpha,Z_i}$ and $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(y)} \cap S''_{\alpha,Z_i}$ are non-empty. But then, the connected domain $\{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(y)} : |T-\alpha|_u = s_\alpha^i(y)\}$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(y)}$ can be written as the union of two disjoint closed subsets, which is impossible. Thus, $\pi(S'_{\alpha,Z_i}) \cap \pi(S''_{\alpha,Z_i}) = \emptyset$, so Z_i can be written as a disjoint union of two closed subsets. This is impossible seeing as Z_i is connected. Consequently, S_{α,Z_i} is connected. □

Recall that the finite morphism $Z' \to Z_1$ was denoted by φ. Let $U_i \subseteq Z_i'$ be open neighborhoods of x_i in Z', $i = 1, 2, \ldots, n$. Then, by [**25**, Lemma I.1.2], there exists a neighborhood U of x in Z, such that $\varphi^{-1}(U) \subseteq \bigcup_{i=1}^t U_i \subseteq \bigcup_{i=1}^t Z_i'$. Let $Z_0 \subseteq U$ be any connected affinoid neighborhood of x. Then, $\varphi^{-1}(Z_0)$ (which is a subset of $\bigcup_{i=1}^t Z_i'$) is an affinoid domain of Z'.

Any connected component C of $\varphi^{-1}(Z_0)$ is mapped surjectively onto Z_0. To see this, remark that φ is at the same time a closed and open morphism (see [**6**, Lemma 3.2.4]). Consequently, $\varphi(C)$ is a closed and open subset of Z_0. Since Z_0 is connected, $\varphi(C) = Z_0$. Thus, for any i, there exists exactly one connected component Z_i of $\varphi^{-1}(Z_0)$ containing x_i and $\varphi^{-1}(Z_0) = \bigcup_{i=1}^t Z_i$. By construction, $Z_i \subseteq Z_i'$.

Let us look at the induced finite morphism $\psi : \mathbb{P}^{1,\mathrm{an}}_{\varphi^{-1}(Z_0)} = \bigsqcup_{i=1}^{t} \mathbb{P}^{1,\mathrm{an}}_{Z_i} \to \mathbb{P}^{1,\mathrm{an}}_{Z_0}$. The preimage of $\{u \in \mathbb{P}^{1,\mathrm{an}}_{Z_0} : |R|_u = r\}$ by ψ is the affinoid $\{u \in \mathbb{P}^{1,\mathrm{an}}_{\varphi^{-1}(Z_0)} : |R|_u = r\}$. Recall that for any i, $\{u \in \mathbb{P}^{1,\mathrm{an}}_{Z_i} : |R|_u = r\} = \bigcup_{R(\alpha)=0} S_{\alpha,Z_i}$, so

$$\{u \in \mathbb{P}^{1,\mathrm{an}}_{\varphi^{-1}(Z_0)} : |R|_u = r\} = \bigcup_{i=1}^{t} \bigcup_{R(\alpha)=0} S_{\alpha,Z_i}.$$

By Lemma 4.1.13, each of the S_{α,Z_i} is connected, and thus so is $\psi(S_{\alpha,Z_i})$. Since $S_{\alpha,Z_i} \cap \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x_i)} : |R|_u = r\} \neq \emptyset$, we also have $\psi(S_{\alpha,Z_i}) \cap \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_u = r\} \neq \emptyset$. Consequently, the type 3 point $\eta_{R,r} \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ is contained in all of the S_{α,Z_i}.

Finally, seeing as $\{u \in \mathbb{P}^{1,\mathrm{an}}_{\varphi^{-1}(Z_0)} : |R|_u = r\}$ can be written as a finite union of connected sets, all of which contain a common point, it is connected.

It is immediate from the constructions we made that the same is true for any other connected affinoid neighborhood of x contained in Z_0. □

4.1.4. Towards Relative Nice Covers. We construct here a relative version of nice covers around the fiber. We keep Notation 4.1.5.

Definition 4.1.14. Let $P_i \in \mathcal{O}_x[T]$ be irreducible over $\mathcal{H}(x)$ and $r_i \in \mathbb{R}_{\geqslant 0}, i = 1, 2, \ldots, n$. The set $A = \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |P_i|_u \bowtie_i r_i, i = 1, 2, \ldots, n\}$, where $\bowtie_i \in \{\leqslant, \geqslant\}$, is an affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. For any affinoid neighborhood Z of x for which $P_i \in \mathcal{O}(Z)[T]$ for all $i = 1, 2, \ldots, n$, we will denote by A_Z the affinoid domain $\{u \in \mathbb{P}^{1,\mathrm{an}}_{Z} : |P_i|_u \bowtie_i r_i, i = 1, 2, \ldots, n\}$ of $\mathbb{P}^{1,\mathrm{an}}_{Z}$ and call it the *Z-thickening* of A.

Remark 4.1.15. The thickening of an affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ depends on the polynomials we choose to represent its boundary points. Hence, from now on, when speaking of the thickening of such an affinoid, we will, unless it plays a specific role (in which case we mention it explicitely), always assume that a writing of the boundary points was fixed *a priori*.

Recall Notation 4.1.5.

Let U and V be connected affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ containing only type 3 points in their boundaries. Suppose that $U \cap V$ is a single type 3 point $\{\eta\}$. This means that $U \cap V = \partial U \cap \partial V = \{\eta\}$. By Lemma 4.1.7, there exist $R(T) \in \mathcal{O}_x[T]$ irreducible over $\mathcal{H}(x)$ and $r \in \mathbb{R}_{>0} \backslash \sqrt{|\mathcal{H}(x)|}$ such that $\eta = \eta_{R,r}$.

By Lemma 4.1.2, either $U \subseteq \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_u \leqslant r\}$ or $U \subseteq \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_u \geqslant r\}$. Without loss of generality, let us assume $U \subseteq \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_u \leqslant r\}$. Then, by Lemma 4.1.4, $V \subseteq \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_u \geqslant r\}$. Set $\partial U = \{\eta_{R,r}, \eta_{P_i,r_i}\}_{i=1}^{n}$ and $\partial V = \{\eta_{R,r}, \eta_{P'_j,r'_j}\}_{j=1}^{m}$, where $P_i, P'_j \in \mathcal{O}_x[T]$ are irreducible over $\mathcal{H}(x)$, and $r_i, r'_j \in \mathbb{R}_{>0} \backslash \sqrt{|\mathcal{H}(x)^\times|}$, for all i and j. By Proposition 4.1.1:

$$U = \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_u \leqslant r, |P_i|_u \bowtie_i r_i, i = 1, 2, \ldots, n\},$$

$$V = \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_u \geqslant r, |P'_j|_u \bowtie'_j r'_j, j = 1, 2, \ldots, m\},$$

where $\bowtie_i, \bowtie'_j \in \{\leqslant, \geqslant\}$ for all i, j. There exists a connected affinoid neighborhood Z of x in S, such that $P_i, P'_j, R \in \mathcal{O}(Z)[T]$ for all i, j. Let us study the relationship between the Z-thickenings U_Z, V_Z of U and V, respectively.

PROPOSITION 4.1.16. *There exists a connected affinoid neighborhood $Z' \subseteq Z$ of x such that:*

(1) $U_{Z'} \cap V_{Z'} = (U \cap V)_{Z'} = \{u \in \mathbb{P}^{1,\mathrm{an}}_{Z'} : |R|_u = r\}$;
(2) $U_{Z'} \cup V_{Z'} = (U \cup V)_{Z'} = \{u \in \mathbb{P}^{1,\mathrm{an}}_{Z'} : |P_i|_u \bowtie_i r_i, |P'_j|_u \bowtie'_j r'_j, i, j\}$ *(see Lemma 4.1.4). If $n = m = 0$, this means that $U_{Z'} \cup V_{Z'} = \mathbb{P}^{1,\mathrm{an}}_{Z'}$.*

The same is true for any connected affinoid neighborhood $Z'' \subseteq Z'$ of x.

PROOF. Recall that we denote by F_x the fiber of x with respect to the morphism π. We will make use of the following:

LEMMA 4.1.17. *Let A, B, C be closed subsets of $\mathbb{P}^{1,\mathrm{an}}_Z$ such that $A \cap B \cap F_x = C \cap F_x$. Suppose there exists an open W of $\mathbb{P}^{1,\mathrm{an}}_Z$ such that $A \cap B \cap W = C \cap W$ and $C \cap F_x \subseteq W$. Then, there exists a connected affinoid neighborhood $Z' \subseteq Z$ of x such that for any connected affinoid neighborhood $Z'' \subseteq Z'$ of x, $A \cap B \cap \pi^{-1}(Z'') = C \cap \pi^{-1}(Z'')$.*

PROOF. Set $F_1 = A \cap B \cap W^c$, and $F_2 = C \cap W^c$, where W^c is the complement of W in $\mathbb{P}^{1,\mathrm{an}}_Z$. Remark that F_i is a closed hence compact set, and that $F_i \cap F_x = \emptyset, i = 1, 2$. Since π is proper, $\pi(F_i)$ is a closed subset of Z, and it does not contain x. Thus, there exists a connected affinoid neighborhood $Z' \subseteq Z$ of x such that $Z' \cap \pi(F_i) = \emptyset, i = 1, 2$. Consequently, $\pi^{-1}(Z') \cap F_i = \emptyset$.

Remark that $\pi^{-1}(Z') \cap F_1 = \pi^{-1}(Z') \cap A \cap B \cap W^c = \emptyset$, so $\pi^{-1}(Z') \cap A \cap B \subseteq W$. Similarly, $\pi^{-1}(Z') \cap C \subseteq W$. Finally, $A \cap B \cap \pi^{-1}(Z') = A \cap B \cap \pi^{-1}(Z') \cap W = C \cap W \cap \pi^{-1}(Z') = C \cap \pi^{-1}(Z')$. Clearly, the same remains true when replacing Z' by any connected affinoid neighborhood $Z'' \subseteq Z'$. □

(1) Set $W = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |P_i|_u \trianglelefteq_i r_i, |P'_j|_u \trianglelefteq'_j r'_j, i, j\}$, where $\trianglelefteq_i$ (resp. $\trianglelefteq'_j$) is the strict version of $\bowtie_i$ (resp. $\bowtie'_j$), meaning for example if $\bowtie_i$ is $\leqslant$ then $\trianglelefteq_i$ is $<$. Set also $A = U_Z$,$B = V_Z$, and $C = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |R|_u = r\}$. Remark that: W is open, A, B, C are closed, $A \cap B \cap W = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |R|_u = r, |P_i|_u \trianglelefteq_i r_i, |P'_j|_u \trianglelefteq'_j r'_j, i, j\} = C \cap W$, and $A \cap B \cap F_x = U \cap V = \{\eta_{R,r}\} = C \cap F_x$. By Lemma 4.1.17, there exists a connected affinoid neighborhood Z' of x such that $U_{Z'} \cap V_{Z'} = \{u \in \mathbb{P}^{1,\mathrm{an}}_{Z'} : |R|_u = r\} = (U \cap V)_{Z'}$, and the same remains true when replacing Z' with any connected affinoid neighborhood $Z'' \subseteq Z'$ of x.
(2) Set $W = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |P'_j|_u \trianglelefteq'_j r'_j, j = 1, \ldots, m\}$, where $\trianglelefteq'_j$ is the strict version of $\bowtie'_j$. Set also $A = C = U_Z$ and $B = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |P_i|_u \bowtie_i r_i, |P'_j|_u \bowtie'_j r'_j, i, j\}$. Clearly, W is open and A, B, C are closed. Also, $A \cap B \cap W = \{u \in \mathbb{P}^{1,\mathrm{an}}_{Z'} : |R|_u \leqslant r, |P_i|_u \bowtie_i r_i, |P'_j|_u \trianglelefteq'_j r'_j, i, j\} = C \cap W$.

Let us look at the affinoid domain $V_1 := \{y \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |P'_j|_y \bowtie'_j r'_j, j = 1, \ldots, m\}$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. As $\partial V = \{\eta_{R,r}, \eta_{P'_j,r'_j}\}_{j=1}^m$, for any $i \neq j$, $\{|P'_i| \bowtie'_i r'_i\} \not\subseteq \{|P'_j| \trianglelefteq'_j r'_j\}$. Otherwise, $V \subseteq \{|P'_i| \bowtie'_i r'_i\} \subseteq \{|P'_j| \trianglelefteq'_j r'_j\}$, implying $\eta_{P'_j,r'_j} \notin V$, contradiction. By Lemma 4.1.3, $\partial V_1 = \{\eta_{P'_j,r'_j}\}_{j=1}^m$, and so $\mathrm{Int}(V_1)$ is $W \cap \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. Remark that $V \subseteq V_1$, so $\eta_{R,r} \in V_1$.

LEMMA 4.1.18. $\{y \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_y \leqslant r\} \subseteq \mathrm{Int}(V_1)$.

PROOF. Suppose there exists $w \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ such that $|R|_w < r$ and $w \notin V_1$. Then, there exists $j_0 \in \{1, 2, \ldots, m\}$, such that $|P'_{j_0}|_w \rtimes'^C_{j_0} r'_{j_0}$, where $\rtimes'^C_{j_0}$ is the inverse sign to $\rtimes'_{j_0}$ (*e.g.* if $\rtimes'_{j_0}$ is $\leqslant$ then $\rtimes'^C_{j_0}$ is $>$). Let $v \in \mathrm{Int}(V) \subseteq \mathrm{Int}(V_1)$, so that $|R|_v > r$ and $|P'_{j_0}|_v \underline{\rtimes}'_{j_0} r'_{j_0}$. Let us look at the unique arc $[v, w]$ in $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. Since $|R|_v > r$ and $|R|_w < r$, $\eta_{R,r} \in [v, w]$. The same is true for $\eta_{P'_{j_0}, r'_{j_0}}$.

We have that $[w, v] = [w, \eta_{P'_{j_0}, r'_{j_0}}] \cup [\eta_{P'_{j_0}, r'_{j_0}}, v]$. Since $|R|_{\eta_{P'_{j_0}, r'_{j_0}}} > r$ (recall $\eta_{P'_{j_0}, r'_{j_0}} \in V$ and the only point of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ satisfying $|R| = r$ is $\eta_{R,r}$) and $|R|_w < r$, we obtain that $\eta_{R,r} \in [w, \eta_{P'_{j_0}, r'_{j_0}}]$. Thus, we can write the following decomposition of the arc connecting v and w: $[w, v] = [w, \eta_{R,r}] \cup [\eta_{R,r}, \eta_{P'_{j_0}, r'_{j_0}}] \cup [\eta_{P'_{j_0}, r'_{j_0}}, v]$. Similarly, $|P'_{j_0}|_{\eta_{R,r}} \underline{\rtimes}'_{j_0} r_j$ and $|P'_{j_0}|_z \rtimes'^C_{j_0} r'_{j_0}$, so $\eta_{P'_{j_0}, r'_{j_0}} \in [w, \eta_{R,r}]$, which is in contradiction with the injectivity of $[w, v]$. Thus, $\{y \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_y \leqslant r\} = \{y \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_y < r\} \cup \{\eta_{R,r}\} \subseteq V_1$.

We showed before that $\partial V_1 = \{\eta_{P'_j, r'_j}\}_{j=1}^n$. Since for any j, $\eta_{P'_j, r'_j} \in V$, $|R|_{\eta_{P'_j, r'_j}} > r$. This implies that $\eta_{P'_j, r'_j} \notin \{y \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_y \leqslant r\}$. Consequently, $\partial V_1 \cap \{y \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_y \leqslant r\} = \emptyset$, implying $\{y \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_y \leqslant r\} \subseteq \mathrm{Int}(V_1)$. □

From the lemma above, $U \subseteq \{y \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_y \leqslant r\} \subseteq \mathrm{Int}(V_1) = W \cap \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. Thus, $A \cap B \cap F_x = \{y \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R|_y \leqslant r, |P_i|_u \rtimes_i r_i, |P'_j|_u \rtimes'_j r'_j, i, j\} = U \cap V_1 = U = C \cap F_x \subseteq V_1 \subseteq W$. This means that Lemma 4.1.17 is applicable, so there exists a connected affinoid neighborhood $Z'_1 \subseteq Z$ of x such that $U_Z \cap B \cap \pi^{-1}(Z'_1) = U_Z \cap \pi^{-1}(Z'_1)$, implying $U_{Z'_1} \subseteq B \cap \pi^{-1}(Z'_1)$, and the same remains true for any connected affinoid neighborhood $Z''_1 \subseteq Z'_1$ of x.

Using similar arguments one shows that there exists a connected affinoid neighborhood $Z'_2 \subseteq Z$ of x such that $V_{Z'_2} \subseteq B \cap \pi^{-1}(Z'_2)$, and the same remains true for any connected affinoid neighborhood $Z''_2 \subseteq Z'_2$ of x.

Thus, there exists a connected affinoid neighborhood $Z' \subseteq Z$ of x such that $U_{Z'} \cup V_{Z'} \subseteq B_{Z'} := \{u \in \mathbb{P}^{1,\mathrm{an}}_{Z'} : |P_i|_u \rtimes_i r_i, |P'_j|_u \rtimes'_j r'_j, i, j\}$, and the same is true for any connected affinoid neighborhood $Z'' \subseteq Z'$ of x. Let $u \in B_{Z''} := B_{Z'} \cap \pi^{-1}(Z'')$. If $|R|_u \leqslant r$, then $u \in U_{Z''}$. If $|R|_u \geqslant r$, then $u \in V_{Z''}$. Consequently, $u \in U_{Z''} \cup V_{Z''}$, and $U_{Z''} \cup V_{Z''} = B_{Z''}$.

□

Let us show that this construction of affinoid domains in $\mathbb{P}^{1,\mathrm{an}}_Z$, where Z is a connected affinoid neighborhood of x, gives us a family of neighborhoods of the points of F_x in $\mathbb{P}^{1,\mathrm{an}}_Z$ (given we choose Z small enough).

LEMMA 4.1.19. *Let A be an open subset of $\mathbb{P}^{1,an}_S$ such that $A \cap F_x \neq \emptyset$. Let $U = \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |P_i|_u \rtimes_i r_i, i = 1, 2, \ldots, n\}$, $\rtimes_i \in \{\leqslant, \geqslant\}$, be any affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ contained in $A \cap F_x$, where $P_i \in \mathcal{O}_x[T]$ is irreducible over $\mathcal{H}(x)$ and $r_i \in \mathbb{R}_{\geqslant 0}, i = 1, 2, \ldots, n$. Then, there exists a connected affinoid neighborhood Z of x, such that $P_i \in \mathcal{O}(Z)[T]$ for all i, and $U_Z \subseteq A$. The same is true for any connected affinoid neighborhood $Z' \subseteq Z$ of x.*

PROOF. Let Z_0 be a connected affinoid neighborhood of x for which the thickening U_{Z_0} exists. Suppose $U_{Z_0} \not\subseteq A$. Then, $U_{Z_0}\backslash A$ is a non-empty compact subset of $\mathbb{P}^{1,\mathrm{an}}_S$. This implies that $\pi(U_{Z_0}\backslash A)$ is a compact subset of S. Furthermore, since $U \subseteq A$, $x \notin \pi(U_{Z_0}\backslash A)$, so there exists a connected affinoid neighborhood $Z \subseteq Z_0$ of x such that $Z \cap \pi(U_{Z_0}\backslash A) = \emptyset$. This implies that for any connected affinoid neighborhood $Z' \subseteq Z$ of x, $U_{Z'}\backslash A = \pi^{-1}(Z') \cap (U_{Z_0}\backslash A) = \emptyset$, and finally that $U_{Z'} \subseteq A$. □

Let $\mathcal{U}_x$ be a nice cover of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. Let $S_{\mathcal{U}_x} = \{\eta_1, \eta_2, \ldots, \eta_t\}$ be the set of intersection points of the elements of $\mathcal{U}_x$. For any $\eta_i \in S_{\mathcal{U}_x}, i = 1, 2, \ldots, t$, there exist $R_i \in \mathcal{O}_x[T]$ irreducible over $\mathcal{H}(x)$ and $r_i \in \mathbb{R}_{>0}\backslash\sqrt{|\mathcal{H}(x)^\times|}$, such that $\eta_i = \eta_{R_i,r_i}$. Since $\bigcup_{U\in\mathcal{U}_x} \partial U = S_{\mathcal{U}_x}$, all pieces of $\mathcal{U}_x$ are a combination of intersections of the affinoid domains $\{|R_i| \bowtie_i r_i\}$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$, where $\bowtie_i \in \{\leqslant, \geqslant\}, i = 1, 2, \ldots, t$.

For any affinoid neighborhood Z_a of x such that $R_i \in \mathcal{O}(Z_a)[T]$ for all i, let us denote by $\mathcal{U}_{Z_a}$ the set of Z_a-thickenings of the elements of $\mathcal{U}_x$. Let Z' be a fixed connected affinoid neighborhood of x such that $R_i \in \mathcal{O}(Z')[T]$ for all $i = 1, 2, \ldots, t$.

THEOREM 4.1.20. *There exists a connected affinoid neighborhood $Z \subseteq Z'$ of x such that the set $\mathcal{U}_Z$ is a cover of $\mathbb{P}^{1,\mathrm{an}}_Z$, and*

1. *for any $U \in \mathcal{U}_x$, the Z-thickening U_Z is a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_Z$;*
2. *for any different $U, V \in \mathcal{U}_x$, either $U_Z \cap V_Z = \emptyset$ or there exists a unique $j \in \{1, \ldots, t\}$ such that $U_Z \cap V_Z = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |R_j|_u = r_j\} = (U \cap V)_Z$ is a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_Z$; in particular, $U_Z \cap V_Z \neq \emptyset$ if and only if $U \cap V \neq \emptyset$;*
3. *for any $U_Z, V_Z \in \mathcal{U}_Z$, $U_Z \cup V_Z$ is either $\mathbb{P}^{1,\mathrm{an}}_Z$ or a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_Z$ that is the Z-thickening of $U \cup V$.*

The statement is true for any connected affinoid neighborhood $Z'' \subseteq Z$ of x.

PROOF. By Theorem 4.1.9, there exists a connected affinoid neighborhood Z of x, such that $R_i \in \mathcal{O}(Z)[T]$ and the affinoid domains $\{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |R_i|_u = r_i\}$ are all connected. We may also assume that for any two non-disjoint elements $U = \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |P_i|_u \bowtie_i r_i, |R|_u \leqslant r : i = 1, \ldots, n\}$ and $V = \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |P'_j|_u \bowtie'_j r'_j, |R|_u \geqslant r : j = 1, \ldots, m\}$ of $\mathcal{U}_x$, Proposition 4.1.16 holds.

Let $\mathcal{U}_x = \{U_1, U_2, \ldots, U_n\}$. By Lemma 3.1.18, there exist $n-1$ elements of $\mathcal{U}_x$ whose union is connected. Without loss of generality, let us assume that $V := \bigcup_{l=1}^{n-1} U_l$ is connected. By Theorem 1.8.15, this is a connected affinoid domain, and $V \cup U_n = \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. Since V, U_n, and $U_n \cup V$ are connected subsets of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$, $U_n \cap V$ is a non-empty connected set, hence a single type 3 point $\{\eta_{R_j,r_j}\}$ for some $j \in \{1, 2, \ldots, t\}$. In particular, this implies that $U_n = \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R_j|_u \bowtie r_j\}$, where $\bowtie \in \{\leqslant, \geqslant\}$. Let us assume without loss of generality that $U_n = \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R_j|_u \geqslant r_j\}$. Then, $V = \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |R_j|_u \leqslant r_j\}$ (see Lemma 4.1.4 to recall what the inequalities for the union of two non-disjoint elements of a nice cover look like). Consequently, $U_{n,Z} = \{u \in \mathbb{P}^{1,an}_Z : |R_j|_u \geqslant r_j\}$ and by Proposition 4.1.16, $V_Z = \left(\bigcup_{l=1}^{n-1} U_i\right)_Z = \bigcup_{i=1}^{n-1} U_{i,Z} = \{u \in \mathbb{P}^{1,an}_Z : |R_j|_u \leqslant r_j\}$, so $U_{n,Z} \cup V_Z = \mathbb{P}^{1,an}_Z$, and $\mathcal{U}_Z$ is a cover of $\mathbb{P}^{1,\mathrm{an}}_Z$.

Let $U \neq V \in \mathcal{U}_x$. Clearly, if $U_Z \cap V_Z = \emptyset$, then $U \cap V = \emptyset$. Assume $U \cap V = \emptyset$. Suppose $A := U_Z \cap V_Z \neq \emptyset$. Remark that $A \cap F_x = \emptyset$. Since A is compact and π proper, $\pi(A)$ is a

compact subset of Z not containing x. Thus, there exists a connected affinoid neighborhood $Z_1 \subseteq Z$, such that $A \cap \pi^{-1}(Z_1) = \emptyset$, and $U_{Z_1} \cap V_{Z_1} = \emptyset$. Thus, we may assume that for any disjoint $U, V \in \mathcal{U}_x$, $U_Z \cap V_Z = \emptyset$, which, taking into account Proposition 4.1.16, shows that property (2) of the statement is true.

Property (3) is a consequence of [**6**, Corollary 2.2.7(i)] if $U_Z \cap V_Z = \emptyset$, and of Proposition 4.1.16 if not. Let Z be such that property (2) is satisfied. Suppose there exists $U \in \mathcal{U}_x$ such that U_Z is not connected. Let C be a connected component of U_Z that doesn't intersect F_x, and B the connected component that does. For any $V \in \mathcal{U}_x$ for which $U \cap V = \emptyset$, $C \cap V_Z \subseteq U_Z \cap V_Z = \emptyset$. For any $V \in \mathcal{U}_x$ for which $U \cap V \neq \emptyset$, there exists a unique j such that $U_Z \cap V_Z = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |R_j|_u = r_j\}$ is a connected affinoid domain, so $U_Z \cap V_Z = B \cap V_Z$. Consequently, $C \cap V_Z = \emptyset$. This means that $C \cap \left((U_Z \backslash C) \cup \bigcup_{V \in \mathcal{U}_x, U \neq V} V_Z\right) = \emptyset$, and $C \cup \left((U_Z \backslash C) \cup \bigcup_{V \in \mathcal{U}_x, U \neq V} V_Z\right) = \mathbb{P}^{1,\mathrm{an}}_Z$, implying $\mathbb{P}^{1,\mathrm{an}}_Z$ is not connected, contradiction. This proves the first part of the statement.

The last part is immediate from the nature of the proof. □

Finally:

Definition 4.1.21. Let $\mathcal{U}_x$ be a nice cover of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$, and Z a connected affinoid neighborhood of x such that the Z-thickening of all of the elements of $\mathcal{U}_x$ exist. Let us denote this set by $\mathcal{U}_Z$. We will say it is a *Z-thickening* of $\mathcal{U}_x$. The set $\mathcal{U}_Z$ will be said to be a *Z-relative nice cover* of $\mathbb{P}^{1,\mathrm{an}}_Z$ if the statement of Theorem 4.1.20 is satisfied.

Remark 4.1.22. Whenever taking the thickening of a nice cover $\mathcal{U}_x$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ to obtain a Z-relative nice cover of $\mathbb{P}^{1,\mathrm{an}}_Z$ for a suitably chosen Z, we will suppose that a writing was fixed simultaneously for all of the points of $\bigcup_{U \in \mathcal{U}_x} \partial U$, and that constructions were made based on this "compatible" writing of the boundary points (as we did *e.g.* in Proposition 4.1.16 and Theorem 4.1.20). The same principle goes for any family of affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ whose Z-thickenings we consider simultaneously.

We have shown:

Theorem 4.1.23. *Let $\mathcal{U}_x$ be a nice cover of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. There exists a connected affinoid neighborhood Z of x such that the Z-thickening of $\mathcal{U}_x$ exists and is a Z-relative nice cover of $\mathbb{P}^{1,\mathrm{an}}_Z$. The same is true for any other connected affinoid neighborhood $Z' \subseteq Z$ of x.*

Corollary 4.1.24. *Let U be a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ containing only type 3 points in its boundary. There exists an affinoid neighborhood Z of x in S such that the Z-thickening U_Z exists and is connected. The same is true for any connected affinoid neighborhood $Z' \subseteq Z$ of x.*

Proof. If U is a type 3 point, then this is Theorem 4.1.9. Suppose this is not the case. By Lemma 3.1.11, there exists a nice cover $\mathcal{U}_x$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ such that $U \in \mathcal{U}_x$. Let Z be a connected affinoid neighborhood of x such that the Z-thickening $\mathcal{U}_Z$ exists and is a Z-relative nice cover. Then, $U_Z \in \mathcal{U}_Z$ is connected. The last part of the statement is clear since the same property is true in Theorem 4.1.20. □

Remark 4.1.25. The notion of a Z-relative nice cover can be extended to connected affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_Z$ that are Z-thickenings of affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$.

4.2. A norm comparison

As seen in the previous section, when constructing relative nice covers we often have to restrict to smaller neighborhoods of the fiber. The same phenomenon appears when trying to apply the patching results of Chapter 2 to this setting. This is why we need some uniform-boundedness-type results.

Recall Notation 4.1.5. Let Z be any connected affinoid neighborhood of x in S. Set $A_Z = \mathcal{O}(Z)$. The k-algebra A_Z is a k-affinoid algebra, and since Z is connected and reduced (recall S is normal), A_Z is an integral domain. By Proposition 1.3.15(2), the spectral norm ρ_Z of A_Z is equivalent to the norm of A_Z, and it satisfies: for all $f \in A_Z$, $|f|_{\rho_Z} = \max_{y\in Z} |f|_y$. In this section, for any connected affinoid neighborhood Z of x in S, we endow the corresponding affinoid algebra A_Z with its spectral norm ρ_Z.

For any positive real number r, we will use the notation $A_Z\{rT^{-1}\}$ (where T is a fixed variable on $\mathbb{P}^{1,\mathrm{an}}_Z$) for the A_Z-affinoid algebra $\left\{\sum_{n\geqslant 0} \frac{a_n}{T^n} : a_n \in A_Z, \lim_{n\to+\infty} |a_n|_{\rho_Z} r^{-n} = 0\right\}$ with corresponding submultiplicative norm $|\sum_{n\geqslant 0} \frac{a_n}{T^n}| := \max_n |a_n|_{\rho_Z} r^{-n}$.

Remark 4.2.1. In what follows we suppose that the coefficient r is not an element of $\sqrt{|k^\times|}$. The only reason behind this assumption is to be able to guarantee the connectedness of the affinoid domains that are considered. If we assume connectedness, then the rest works the same regardless of whether $r \in \sqrt{|k^\times|}$ or not.

4.2.1. The case of degree one polynomials. Let $r \in \mathbb{R}_{>0}\backslash\sqrt{|\mathcal{H}(x)^\times|}$.

(1) Set $X_{|T|\leqslant r,Z} = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |T|_u \leqslant r\}$. It is an affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_Z$, and $\mathcal{O}(X_{|T|\leqslant r,Z}) = A_Z\{r^{-1}T\}$, where

$$A_Z\{r^{-1}T\} = \{\sum_{n\geqslant 0} a_n T^n, a_n \in A_Z, \lim_{n\to+\infty} |a_n|_{\rho_Z} r^n = 0\}$$

and it is endowed with the norm $|\sum_{n\geqslant 0} a_n T^n|_{|T|\leqslant r,Z} := \max_{n\geqslant 0} |a_n|_{\rho_Z} r^n$.

(2) Set $X_{|T|\geqslant r,Z} = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |T|_u \geqslant r\}$. It is an affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_Z$ and $\mathcal{O}(X_{|T|\geqslant r,Z}) = A_Z\{rT^{-1}\}$, where

$$A_Z\{rT^{-1}\} = \{\sum_{n\geqslant 0} \frac{a_n}{T^n} : a_n \in A_Z, \lim_{n\to+\infty} |a_n|_{\rho_Z} r^{-n} = 0\}$$

and it is endowed with the norm $|\sum_{n\geqslant 0} a_n T^n|_{|T|\geqslant r,Z} := \max_{n\geq 0} |a_n|_{\rho_Z} r^{-n}$.

(3) Set $X_{|T|=r,Z} = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |T|_u = r\}$. It is an affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_Z$ and $\mathcal{O}(X_{|T|=r,Z}) = A_Z\{r^{-1}T, rT^{-1}\}$, where

$$A_Z\{r^{-1}T, rT^{-1}\} = \{\sum_{n\in\mathbb{Z}} a_n T^n : a_n \in A_Z, \lim_{n\to\pm\infty} |a_n|_{\rho_Z} r^n = 0\}$$

and it is endowed with the norm $|\sum_{n\in\mathbb{Z}} a_n T^n|_{|T|=r,Z} := \max_{n\in\mathbb{Z}} |a_n|_{\rho_Z} r^n$.

By Corollary 4.1.24, there exists a connected affinoid neighborhood Z_T of x in S such that for any connected affinoid neighborhood $Z \subseteq Z_T$ of x, the affinoids $X_{|T|\leqslant r,Z}, X_{|T|\geqslant r,Z}$ and $X_{|T|=r,Z}$ are connected (and hence integral). For the rest of this subsection, we suppose $Z \subseteq Z_T$.

Lemma 4.2.2. *The norms $|\cdot|_{|T|\leqslant r,Z}, |\cdot|_{|T|\geqslant r,Z}, |\cdot|_{|T|=r,Z}$ defined above are equal to the spectral norms on $A_Z\{r^{-1}T\}, A_Z\{rT^{-1}\}, A_Z\{r^{-1}T, rT^{-1}\}$, respectively.*

PROOF. By Theorem 1.1.38, for any affinoid space X, its associated spectral norm ρ_X has the property that $|f|_{\rho_X} = \max_{y\in X} |f|_y$ for all $f \in \mathcal{O}(X)$.

Let $f = \sum_{n\geqslant 0} a_n T^n$ be any element of $A_Z\{r^{-1}T\}$. Let $\rho_{|T|\leqslant r,Z}$ denote the spectral norm on the integral affinoid space $X_{|T|\leqslant r,Z}$. We will show that $|f|_{|T|\leqslant r,Z} = |f|_{\rho_{|T|\leqslant r,Z}}$. By the remark in the paragraph above, $|f|_{\rho_{|T|\leqslant r,Z}} = \max_{u\in X_{|T|\leqslant r,Z}} |f|_u$. For any $y \in Z$, the fiber of $X_{|T|\leqslant r,Z}$ over y is the disc $\{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(y)} : |T|_y \leqslant r\}$, whose Shilov boundary is the singleton $\{\eta^y_{0,r}\}$ (*i.e.* the point $\eta_{0,r} \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(y)}$). Consequently, in the fiber of $X_{|T|\leqslant r,Z}$ over y, the function f attains its maximum on the point $\eta^y_{0,r}$, implying $|f|_{\rho_{|T|\leqslant r,Z}} = \max_{y\in Z} |f|_{\eta^y_{0,r}}$ (see also Lemma 4.2.24).

Since $|f|_{\eta^y_{0,r}} = |\sum_{n\geqslant 0} a_n T^n|_{\eta^y_{0,r}} = \max_{n\geqslant 0} |a_n|_y r^n$, we obtain that

$$|f|_{\rho_{|T|\leqslant r,Z}} = \max_{y\in Z} \max_{n\geqslant 0} |a_n|_y r^n.$$

At the same time, $|f|_{|T|\leqslant r,Z} = \max_{n\geqslant 0} |a_n|_{\rho_Z} r^n = \max_{n\geqslant 0} \max_{y\in Z} |a_n|_y r^n$, implying the equality of the statement.

The result is proven in the same way for the norms $|\cdot|_{|T|\geqslant r,Z}$ and $|\cdot|_{|T|=r,Z}$. □

COROLLARY 4.2.3. *Let $Z_1 \subseteq Z$ be a connected affinoid neighborhood of x. The restriction morphism $\mathcal{O}(X_{|T|\bowtie r,Z}) \to \mathcal{O}(X_{|T|\bowtie r,Z_1})$ is a contraction with respect to the corresponding norms $|\cdot|_{|T|\bowtie r,Z}$ and $|\cdot|_{|T|\bowtie r,Z_1}$, $\bowtie \in \{\leqslant, =, \geqslant\}$.*

LEMMA 4.2.4. *The restriction maps $A_Z\{r^{-1}T\}, A_Z\{rT^{-1}\} \hookrightarrow A_Z\{r^{-1}T, rT^{-1}\}$ are isometries with respect to the corresponding norms $|\cdot|_{|T|\leqslant r,Z}, |\cdot|_{|T|\geqslant r,Z}$, and $|\cdot|_{|T|=r,Z}$.*

PROOF. Let $f = \sum_{n\geqslant 0} a_n T^n \in A_Z\{r^{-1}T\}$. Then, $|f|_{|T|=r,Z} = \max_n |a_n|_{\rho_Z} r^n = |f|_{|T|\leqslant r,Z}$. The same is true for $A\{rT^{-1}\}$. □

Since $H^1(X_{|T|\leqslant r,Z} \cup X_{|T|\geqslant r,Z}, \mathcal{O}) = H^1(\mathbb{P}^{1,\mathrm{an}}_Z, \mathcal{O}) = 0$, we have the following exact sequence:

$$0 \to A_Z \to A_Z\{r^{-1}T\} \oplus A_Z\{rT^{-1}\} \to A_Z\{r^{-1}T, rT^{-1}\} \to 0,$$

which gives us a surjective morphism $A_Z\{r^{-1}T\} \oplus A_Z\{rT^{-1}\} \twoheadrightarrow A_Z\{r^{-1}T, rT^{-1}\}$. Admissibility follows from Banach's Open Mapping Theorem if k is non-trivially valued (for a proof see [**14**]), and by a change of basis followed by the Open Mapping Theorem if it is (see Proposition 1.3.8).

LEMMA 4.2.5. *For any $c \in A_Z\{r^{-1}T, rT^{-1}\}$, there exist $a \in A_Z\{r^{-1}T\}, b \in A_Z\{rT^{-1}\}$ such that $a + b = c$ and $|c|_{|T|=r,Z} = \max(|a|_{|T|\leqslant r,Z}, |b|_{|T|\geqslant r,Z})$.*

PROOF. Let $c = \sum_{n\in\mathbb{Z}} a_n T^n \in A_Z\{r^{-1}T, rT^{-1}\}$. Set $a = \sum_{n\geqslant 0} a_n T^n$ and $b = \sum_{n<0} a_n T^n$. Clearly, $a \in A_Z\{rT^{-1}\}, b \in A_Z\{r^{-1}T\}$ and $a + b = c$. Furthermore, $|a|_{|T|\leqslant r,Z} = \max_{n\geqslant 0} |a_n|_{\rho_Z} r^n \leqslant \max_{n\in\mathbb{Z}} |a_n|_{\rho_Z} r^n = |c|_{|T|=r,Z}$, and the same is true for b. Consequently, $\max(|a|_{|T|\leqslant r,Z}, |b|_{|T|\geqslant r,T}) \leqslant |c|_{|T|=r,Z}$. At the same time, $|c|_{|T|=r,Z} \leqslant \max(|a|_{|T|=r,Z}, |b|_{|T|=r,Z})$ and by Lemma 4.2.4, this is equal to $\max(|a|_{|T|\leqslant r,Z}, |b|_{|T|\geqslant r,Z})$. □

REMARK 4.2.6. All of the results of this subsection remain true if we replace T by $T - \alpha$ for any $\alpha \in A_Z$.

4.2.2. The general case. Let $P(T)$ be a unitary polynomial over $\mathcal{O}_x$, irreducible over $\mathcal{H}(x)$, and of degree bigger than 1. Then, there exists an affinoid neighborhood Z' of x such that $P(T) \in \mathcal{O}(Z')[T]$. The connected affinoid neighbrohood Z of x in this subsection will always be assumed to satisfy $Z \subseteq Z' \cap Z_T$.

NOTATION 4.2.7. Let $r \in \mathbb{R}_{>0} \backslash \sqrt{|\mathcal{H}(x)^\times|}$. Set $X_{|P|\leqslant r,Z} = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |P|_u \leqslant r\}$, $X_{|P|\geqslant r,Z} = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |P|_u \geqslant r\}$ and $X_{|P|=r,Z} = \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |P|_u = r\}$. These are affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_Z$ (furthermore, $X_{|P|\leqslant r,Z}$ and $X_{|P|=r,Z}$ are affinoid domains of $\mathbb{A}^{1,\mathrm{an}}_Z$). By Corollary 4.1.24, there exists an affinoid neighborhood Z_P of x such that for any connected affinoid neighborhood $Z \subseteq Z_P$, $X_{|P|\leqslant r,Z}, X_{|P|\geqslant r,Z}$ and $X_{|P|=r,Z}$ are connected (hence integral). For the rest of this subsection, we assume that $Z \subseteq Z' \cap Z_T \cap Z_P$.

The rings $\mathcal{O}(X_{|P|\leqslant r,Z})$ and $\mathcal{O}(X_{|P|=r,Z})$ have been studied extensively and under more general conditions by Poineau in [**59**, Chapter 5]. Restricted to our setting, the following is shown:

LEMMA 4.2.8. *Let Z be a connected affinoid neighborhood of x, such that $Z \subseteq Z' \cap Z_T \cap Z_P$. Then, $\mathcal{O}(X_{|P|\leqslant r,Z}) \cong \mathcal{O}(X_{|T|\leqslant r,Z})[X]/(P(X)-T) = A_Z\{r^{-1}T\}[X]/(P(X)-T)$, and $\mathcal{O}(X_{|P|=r,Z}) = \mathcal{O}(X_{|T|=r,Z})[Y]/(P(Y)-T) = A_Z\{r^{-1}T, rT^{-1}\}[Y]/(P(Y)-T)$.*

PROOF. The statement can be seen by considering the finite morphism $\mathbb{P}^{1,\mathrm{an}}_Z \to \mathbb{P}^{1,\mathrm{an}}_Z$ induced by $A_Z[T] \to A_Z[T], T \mapsto P(T)$. □

LEMMA 4.2.9. *Let j_P denote the restriction morphism $\mathcal{O}(X_{|P|\leqslant r,Z}) \hookrightarrow \mathcal{O}(X_{|P|=r,Z})$. Then, the following diagram commutes and $j_P(X) = Y$.*

$$\begin{array}{ccc} A_Z\{r^{-1}T\}[X]/(P(X)-T) & \xrightarrow{j_P} & A_Z\{r^{-1}T, rT^{-1}\}[Y]/(P(Y)-T) \\ \uparrow & & \uparrow \\ A_Z\{r^{-1}T\} & \xrightarrow{\qquad j_T \qquad} & A_Z\{r^{-1}T, rT^{-1}\} \end{array}$$

Taking this into account, we will from now on write $A_Z\{r^{-1}T\}[X]/(P(X)-T)$ and $A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T)$ (i.e. using the same variable X).

PROOF. This follows again from the work of Poineau in [**59**, Chapter 5]. Remark that the finite morphism $A_Z[T] \to A_Z[T], T \to P(T)$, induces a finite morphism $\varphi : X_{|P|\leqslant r,Z} \to X_{|T|\leqslant r,Z}$ and $\varphi^{-1}(X_{|T|=r,Z}) = X_{|P|=r,Z}$. The vertical maps of the diagram above are induced by φ, which implies its commutativity. Remark that $j_T(T) = T$. Also, since $\varphi^{-1}(X_{|T|=r,Z}) = X_{|P|=r,Z}$, we have that $\mathcal{O}(X_{|P|=r,Z}) = \mathcal{O}(X_{|P|\leqslant r,Z}) \otimes_{\mathcal{O}(X_{|T|\leqslant r})} \mathcal{O}(X_{|T|=r,Z})$. The restriction morphism j_P is given by $f \mapsto f \otimes 1$, implying $j_P(X) = Y$. □

Recall that $\mathcal{O}(X_{|P|\leqslant r,Z})$, $\mathcal{O}(X_{|P|\geqslant r,Z})$, and $\mathcal{O}(X_{|P|=r,Z})$ are affinoid algebras, meaning they are naturally endowed with submultiplicative norms $|\cdot|_\leqslant, |\cdot|_\geqslant$ and $|\cdot|_=$, respectively. (These norms are uniquely determined only up to equivalence.) We start by giving an explicit choice for $|\cdot|_\leqslant$ and $|\cdot|_=$.

The morphism $A_Z[T] \to A_Z[T], T \mapsto P(T)$ induces a finite morphism $\varphi_Z : \mathbb{P}^{1,\mathrm{an}}_Z \to \mathbb{P}^{1,\mathrm{an}}_Z$, for which $\varphi_Z^{-1}(X_{|T|\bowtie r,Z}) = X_{|P|\bowtie r,Z}$, where $\bowtie \in \{\leqslant, =, \geqslant\}$. In particular, this gives rise to a finite morphism $X_{|P|\bowtie r,Z} \to X_{|T|\bowtie r,Z}$, hence to a finite morphism $\mathcal{O}(X_{|T|\bowtie r,Z}) \to \mathcal{O}(X_{|P|\bowtie r,Z})$. The latter gives rise to a surjective morphism $\psi_1 : \mathcal{O}(X_{|T|\bowtie r,Z})^n \twoheadrightarrow \mathcal{O}(X_{|P|\bowtie r,Z})$

for some $n \in \mathbb{N}$. Let $|\cdot|'_{\bowtie}$ denote the norm (determined up to equivalence) on $\mathcal{O}(X_{|P|\bowtie r,Z})$ obtained by ψ_1, *i.e.* making ψ_1 admissible.

PROPOSITION 4.2.10. *The norms* $|\cdot|_{\bowtie}$ *and* $|\cdot|'_{\bowtie}$ *are equivalent,* $\bowtie \in \{\leqslant, =, \geqslant\}$.

PROOF. By Lemma 1.3.7, there exists a complete non-trivially valued field extension K of k such that $\mathcal{O}(X_{|T|\bowtie r,Z})\widehat{\otimes}_k K =: \mathcal{O}(X_{|T|\bowtie r,Z_K})$ and $\mathcal{O}(X_{|P|\bowtie r,Z})\widehat{\otimes}_k K =: \mathcal{O}(X_{|T|\bowtie r,Z_K})$ are strict K-affinoid algebras, where $Z_K := Z \times_k K$. Moreover, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}(X_{|T|\bowtie r,Z}) & \xrightarrow{T\mapsto P(T)} & \mathcal{O}(X_{|P|\bowtie r,Z}) \\ \downarrow{\scriptstyle \widehat{\otimes}_k K} & & \downarrow{\scriptstyle \widehat{\otimes}_k K} \\ \mathcal{O}(X_{|T|\bowtie r,Z_K}) & \xrightarrow{T\mapsto P(T)} & \mathcal{O}(X_{|P|\bowtie r,Z_K}) \end{array}$$

which gives rise to the following commutative diagram, where ψ_2 is a surjective admissible morphism induced by ψ_1:

$$\begin{array}{ccccc} \mathcal{O}(X_{|T|\bowtie r,Z}) & \longrightarrow & \mathcal{O}(X_{|T|\bowtie r,Z})^n & \xrightarrow{\psi_1} & \mathcal{O}(X_{|P|\bowtie r,Z}) \\ \downarrow{\scriptstyle \widehat{\otimes}_k K} & & \downarrow{\scriptstyle \widehat{\otimes}_k K} & & \downarrow{\scriptstyle \widehat{\otimes}_k K} \\ \mathcal{O}(X_{|T|\bowtie r,Z_K}) & \longrightarrow & \mathcal{O}(X_{|T|\bowtie r,Z_K})^n & \xrightarrow{\psi_2} & \mathcal{O}(X_{|P|\bowtie r,Z_K}) \end{array}$$

Let $|\cdot|_{\psi_2}$ be the norm (determined up to equivalence) on $\mathcal{O}(X_{|P|\bowtie r,Z_K})$ induced by the morphism ψ_2. Then, $\mathcal{O}(X_{|P|\bowtie r,Z_K})$ is a Banach K-algebra with respect to $|\cdot|_{\psi_2}$.

Since $\mathcal{O}(|X|_{|T|\bowtie r,Z}) \hookrightarrow \mathcal{O}(X_{|T|\bowtie r,Z_K})$ is an isometry (see [**60**, Lemme 3.1]), the diagram above implies that $(\mathcal{O}(|X|_{|P|\bowtie r,Z}), |\cdot|'_{\bowtie}) \hookrightarrow (\mathcal{O}(X_{|P|\bowtie r,Z_K}), |\cdot|_{\psi_2})$ is also an isometry.

Let $|\cdot|_{\bowtie,K}$ denote the norm that the K-affinoid algebra $\mathcal{O}(X_{|P|\bowtie r,Z_K})$ is naturally endowed with. Then, $(\mathcal{O}(|X|_{|P|\bowtie r,Z}), |\cdot|_{\bowtie}) \hookrightarrow (\mathcal{O}(X_{|P|\bowtie r,Z_K}), |\cdot|_{\bowtie,K})$ is an isometry (again, see [**60**, Lemme 3.1]).

Since $\mathcal{O}(X_{|P|\bowtie r,Z_K})$ is a strict K-affinoid algebra, by [**11**, 6.1.3/2], there is a unique way to define the structure of a Banach K-algebra on it. Hence, $|\cdot|_{\psi_2}$ is equivalent to $|\cdot|_{\bowtie,K}$, so the norms $|\cdot|'_{\bowtie}$, resp. $|\cdot|_{\bowtie}$, they induce on $\mathcal{O}(X_{|P|\bowtie r,Z})$, are equivalent. □

NOTATION 4.2.11. Set $d = \deg P$. Since $P(X)$ is unitary, any $f \in A_Z\{r^{-1}T\}[X]/(P(X)-T)$ (resp. $f \in A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T)$) has a unique representation of the form $\sum_{i=0}^{d-1} \alpha_i X^i$, where $\alpha_i \in A_Z\{r^{-1}T\}$ (resp. $\alpha_i \in A_Z\{r^{-1}T, rT^{-1}\}$) for all $i = 0, 1, \ldots, d-1$. Set $|f|_{|P|\leqslant r,Z} := \max_i(|\alpha_i|_{|T|\leqslant r,Z})$ (resp. $|f|_{|P|=r,Z} := \max_i(|\alpha_i|_{|T|=r,Z})$). By Proposition 4.2.10, we can take $|\cdot|_{\leqslant} = |\cdot|_{|P|\leqslant r,Z}$ and $|\cdot|_{=} = |\cdot|_{|P|=r,Z}$. (This kind of norm is called $||\cdot||_{U,\mathrm{div}}$ in [**59**, 5.2]; here U is $X_{|T|\leqslant r,Z}$ or $X_{|T|=r,Z}$.)

Let us now find a good representative for $\mathcal{O}(X_{|P|\geqslant r,Z})$ and its norm. In what follows, we identify the k-affinoid algebras $\mathcal{O}(X_{|P|\leqslant r,Z})$ and $\mathcal{O}(X_{|P|\geqslant r,Z})$ with A_Z-subalgebras of $\mathcal{O}(X_{|P|=r,Z})$ via the respective restriction morphisms. As before, since $H^1(\mathbb{P}^{1,\mathrm{an}}_Z, \mathcal{O}) = 0$, we have the following short exact sequence:

$$0 \to A_Z \to \mathcal{O}(X_{|P|\leqslant r,Z}) \oplus \mathcal{O}(X_{|P|\geqslant r,Z}) \to \mathcal{O}(X_{|P|=r,Z}) \to 0. \tag{4}$$

Let $f \in \mathcal{O}(X_{|P|=r,Z}) = A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T)$. Suppose its unique representative of degree $< d$ in X is $f_0 = \sum_{i=0}^{d-1}\sum_{n\in\mathbb{Z}} a_{n,i}T^n X^i$, where $\sum_{n\in\mathbb{Z}} a_{n,i}T^n \in$

$A_Z\{r^{-1}T, rT^{-1}\}$ for all i. Then, we can write the following decomposition for f_0:

$$f_0 = a_{0,0} + \underbrace{\left(\sum_{n\geqslant 1} a_{n,0}T^n + \sum_{i=1}^{d-1}\sum_{n\geqslant 0} a_{n,i}T^nX^i\right)}_{\alpha_f} + \underbrace{\left(\sum_{i=0}^{d-1}\sum_{n\leqslant -1} a_{n,i}T^nX^i\right)}_{\beta_f}.$$

Remark that $\alpha_f \in A_Z\{r^{-1}T\}[X]/(P(X)-T)$.

PROPOSITION 4.2.12. *The A_Z-subalgebra $\mathcal{O}(X_{|P|\geqslant r,Z})$ of $\mathcal{O}(X_{|P|=r,Z})$ is equal to*

$$B := \left\{ f \in A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T) : f = a_{0,0} + \sum_{i=0}^{d-1}\sum_{n\geqslant 1}\frac{a_{n,i}}{T^n}X^i \right\}.$$

PROOF. Let us first show that B is closed with respect to multiplication. Let $f = a_{0,0} + \sum_{i=0}^{d-1}\sum_{n\geqslant 1}\frac{a_{n,i}}{T^n}X^i, g = b_{0,0} + \sum_{i=0}^{d-1}\sum_{n\geqslant 1}\frac{b_{n,i}}{T^n}X^i \in B$. For any m such that $d \leqslant m < 2d$, the coefficient corresponding to X^m in the product fg is of the form $\sum_{n\geqslant 2}\frac{c_{n,m}}{T^n}$ where $c_{n,m} \in A_Z$ for all n, m. By using Euclidian division, since $P(X)$ is unitary, we obtain $X^m = P(X)Q(X) + R(X)$ where $Q, R \in A_Z[X]$, $\deg R < d$ and $\deg Q = m - d < d$. Hence, $\sum_{n\geqslant 2}\frac{c_{n,m}}{T^n}X^m = \sum_{n\geqslant 2}\frac{c_{n,m}}{T^n}P(X)Q(X) + \sum_{n\geqslant 2}\frac{c_{n,m}}{T^n}R(X) = \sum_{n\geqslant 1}\frac{c_{n,m}}{T^n}Q(X) + \sum_{n\geqslant 2}\frac{c_{n,m}}{T^n}R(X)$ in $A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T)$, which is an element of B seeing as $\deg Q, \deg R < d$. Consequently, $fg \in B$, and B is an A_Z-algebra.

Let us consider the restriction morphism $\psi : A_Z = \mathcal{O}(\mathbb{P}^{1,\mathrm{an}}_Z) \to \mathcal{O}(X_{|P|\geqslant r}, Z)$, a section of which is given as follows: for any $f \in \mathcal{O}(X_{|P|\geqslant r}, Z)$, let f_∞ denote the restriction of f to the Zariski closed subset $\mathcal{Z} := \{x \in X_{|P|\geqslant r,Z} : |T^{-1}|_x = 0\}$. Remark that in the copy of $\mathbb{A}^{1,\mathrm{an}}_Z$ in $\mathbb{P}^{1,\mathrm{an}}_Z$ with coordinate T^{-1}, $\mathcal{Z} = \{u \in \mathbb{A}^{1,\mathrm{an}}_Z : |T^{-1}|_u = 0\}$, so $\mathcal{O}(\mathcal{Z}) = A_Z$.

The morphism $s : \mathcal{O}(X_{|P|\geqslant r}, Z) \to A_Z, f \mapsto f_\infty$, is a section of ψ. Let $\mathcal{O}(X_{|P|\geqslant r,Z})_\infty$ denote the kernel of s. Then, $\mathcal{O}(X_{|P|\geqslant r,Z}) = A_Z \oplus \mathcal{O}(X_{|P|\geqslant r}, Z)_\infty$.

Let us consider the following commutative diagram that is obtained from the short exact sequence 4 above.

$$\begin{array}{ccc} & \mathcal{O}(X_{|P|\leqslant r,Z}) \oplus \mathcal{O}(X_{|P|\geqslant r,Z}) & \\ {\scriptstyle h'}\nearrow & & \searrow{\scriptstyle h''} \\ \mathcal{O}(X_{|P|\leqslant r,Z}) \oplus \mathcal{O}(X_{|P|\geqslant r,Z})_\infty & \xrightarrow{\quad h \quad} & \mathcal{O}(X_{|P|=r,Z}) \end{array}$$

Let $f \in \mathcal{O}(X_{|P|=r,Z})$. By the surjectivity of h'' (from the short exact sequence 4) there exist $f_1 \in \mathcal{O}(X_{|P|\leqslant r,Z})$ and $f_2 \in \mathcal{O}(X_{|P|\geqslant r,Z})$ such that $f_1 + f_2 = f$. Let $f_2' \in A_Z$ and $f_2'' \in \mathcal{O}(X_{|P|\geqslant r,Z})_\infty$ be such that $f_2 = f_2' + f_2''$ (as we saw above, such f_2', f_2'' are unique). Set $f_1' := f_1 + f_2'$ and remark that $f_1' \in \mathcal{O}(X_{|P|\leqslant r,Z})$. By the commutativity of the diagram, $h(f_1', f_2'') = f$, i.e. h is surjective. Let us also show it is injective. Suppose $h(a, b) = 0$ for some $a \in \mathcal{O}(X_{|P|\leqslant r,Z})$ and $b \in \mathcal{O}(X_{|P|\geqslant r,Z})_\infty \subseteq \mathcal{O}(X_{|P|\geqslant r,Z})$. Then, $a + b = h''(a, b) = 0$, and the exact sequence 4 implies that $a = -b \in A_Z$. Since $b \in A_Z$ and $b \in \mathcal{O}(X_{|P|\geqslant r,Z})_\infty$, we obtain that $b = 0$ and $a = 0$, i.e. h is injective.

By Lemma 4.2.9, the map $s' : \mathcal{O}(X_{|P|=r,Z}) \to \mathcal{O}(X_{|P|\leqslant r,Z})$, which to an element $f_0 := \sum_{i=0}^{d-1}\sum_{n\in\mathbb{Z}} d_{n,i}T^nX^i$ associates the element $f_\geqslant := \sum_{i=0}^{d-1}\sum_{n\leqslant -1} d_{n,i}T^nX^i$, is a section of the isomorphism $\mathcal{O}(X_{|P|\leqslant r,Z}) \oplus \mathcal{O}(X_{|P|\geqslant r,Z})_\infty \to \mathcal{O}(X_{|P|=r,Z})$. Consequently, $\mathcal{O}(X_{|P|\geqslant r,Z})_\infty = \left\{ f \in \mathcal{O}(X_{|P|=r,Z}) : f = \sum_{i=0}^{d-1}\sum_{n\leqslant -1} a_{n,i}T^nX^i \right\}$.

Finally, since $\mathcal{O}(X_{|P|\geqslant r,Z}) = A_Z \oplus \mathcal{O}(X_{|P|\geqslant r,Z})_\infty$, we get:

$$\mathcal{O}(X_{|P|\geqslant r,Z}) = \left\{ f \in A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T) : f = a_{0,0} + \sum_{i=0}^{d-1}\sum_{n\geqslant 1} \frac{a_{n,i}}{T^n}X^i \right\}.$$

□

REMARK 4.2.13. Let I be the ideal of $A_Z\{rT^{-1}\}$ generated by T^{-1}. Denote by $I[X]^{d-1}$ the polynomials on X with coefficients in I and degree at most $d-1$. Then, the k-affinoid algebra B can be written as $(A_Z \oplus I[X]^{d-1})/(P(X)T^{-1}-1)$, where multiplication is done using Euclidian division, just like in B.

NOTATION 4.2.14. The morphism $A_Z\{rT^{-1}\} \to B$, $T^{-1} \mapsto \frac{1}{T}$ is finite (it is the one induced by $A_Z[T] \to A_Z[T], T \mapsto P(T)$), and $1, X, \ldots, X^{d-1}$ is a set of generators of B as an A_Z-module. Let $|\cdot|_{|P|\geqslant r,Z}$ be the norm on B induced by the norm $|\cdot|_{|T|\geqslant 1,Z}$ on $A_Z\{rT^{-1}\}$. By Theorem 1.3.9, B is complete with respect to this norm. As before, by Proposition 4.2.10, we can take $|\cdot|_\geqslant := |\cdot|_{|P|\geqslant r,Z}$. Explicitely, for any $f := a_{0,0} + \sum_{i=0}^{d-1}\sum_{n\geqslant 1}\frac{a_{n,i}}{T^n}X^i = \sum_{i=0}^{d-1}\alpha_i X^i \in B$, $|f|_{|P|\geqslant r,Z} = \max_i |\alpha_i|_{|T|\leqslant r,Z}$.

LEMMA 4.2.15. *The restriction maps from $A_Z\{r^{-1}T\}[X]/(P(X)-T)$ and B to $A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T)$ are isometries with respect to the corresponding norms $|\cdot|_{|P|\leqslant r,Z}$, $|\cdot|_{|P|\geqslant r,Z}$ and $|\cdot|_{|P|=r,Z}$.*

PROOF. Let $f = \sum_{i=0}^{d-1}\sum_{n\geqslant 0} a_{n,i}T^nX^i \in A_Z\{r^{-1}T\}[X]/(P(X)-T)$. Then, by Lemma 4.2.4, $|f|_{|P|=r,Z} = \max_i |\sum_{n\geqslant 0} a_{n,i}T^n|_{|T|=r,Z} = \max_i |\sum_{n\geqslant 0} a_{n,i}T^n|_{|T|\leqslant r,Z} = |f|_{|P|\leqslant r,Z}$. The statement for B is proven in the same way. □

The exact sequence 4 above gives rise to a surjection $A_Z\{r^{-1}T\}[X]/(P(X)-T)\oplus B \twoheadrightarrow A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T)$. Admissibility follows from Banach's Open Mapping Theorem if k is non-trivially valued (for a proof see [**14**]), and by a change of basis followed by the Open Mapping Theorem if it is (see Proposition 1.3.8).

LEMMA 4.2.16. *For any $c \in A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T)$, there exist $a \in A_Z\{r^{-1}T\}[X]/(P(X)-T)$ and $b \in B$ such that $a+b=c$ and $\max(|a|_{|P|\leqslant r,Z}, |b|_{|P|\geqslant r,Z}) = |c|_{|P|=r,Z}$.*

PROOF. There exists a unique degree $< d$ polynomial $c_0(X)$ over $A_Z\{r^{-1}T, rT^{-1}\}$ such that $c = c_0$ in $A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T)$. Let $c_0 = \sum_{i=0}^{d-1}\sum_{n\in\mathbb{Z}} a_{n,i}T^nX^i$. Let a and b be given as follows:

$$c_0 = \underbrace{\left(\sum_{i=0}^{d-1}\sum_{n\geqslant 0} a_{n,i}T^nX^i\right)}_{a} + \underbrace{\left(\sum_{i=0}^{d-1}\sum_{n\leqslant -1} a_{n,i}T^nX^i\right)}_{b}.$$

Clearly, $a \in A_Z\{r^{-1}T\}[X]/(P(X)-T)$ and $b \in B$.

Then,

$$\begin{aligned} |a|_{|P|\leqslant r,Z} &= \max_i |\sum_{n\geqslant 0} a_{n,i}T^n|_{|T|\leqslant r,Z} = \max_i \max_{n\in\mathbb{N}} |a_{n,i}|_{\rho_Z} r^n \\ &\leqslant \max_i \max_{n\in\mathbb{Z}} |a_{n,i}|_{\rho_Z} r^n = |a|_{|P|=r,Z}, \end{aligned}$$

and the same is true for $|b|_{|P|\geqslant r,Z}$. Consequently, $\max(|a|_{|P|\leqslant r,Z}, |b|_{|P|\geqslant r,Z}) \leqslant |c|_{|P|=r,Z}$.

On the other hand, $c = a + b$, so $|c|_{|P|=r,Z} \leqslant \max(|a|_{|P|=r,Z}, |b|_{|P|=r,Z})$, which, by Lemma 4.2.15, is the same as $\max(|a|_{|P|\leqslant r,Z}, |b|_{|P|\geqslant r,Z})$.

□

Let $Z_1 \subseteq Z$ be a connected affinoid neighborhood of x.

LEMMA 4.2.17. *The restriction morphism $\mathcal{O}(X_{|P|=r,Z}) \hookrightarrow \mathcal{O}(X_{|P|=r,Z_1})$ is a contraction with respect to the corresponding norms $|\cdot|_{|P|=r,Z}$ and $|\cdot|_{|P|=r,Z_1}$.*

PROOF. Let the restriction morphism $\mathcal{O}(X_{|P|=r,Z}) \hookrightarrow \mathcal{O}(X_{|P|=r,Z_1})$ be denoted by $j_{P,1}$. Similarly to Lemma 4.2.9, the following diagram is commutative and $j_{P,1}(X) = Y$ (remark that $j_{T,1}(T) = T, j_{T,1}(T^{-1}) = T^{-1}$, and the restriction of $j_{T,1}$ to A_Z is the restriction morphism $A_Z \to A_{Z_1}$).

$$\begin{array}{ccc} A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T) & \xrightarrow{j_{P,1}} & A_{Z_1}\{r^{-1}T, rT^{-1}\}[Y]/(P(Y)-T) \\ \uparrow & & \uparrow \\ A_Z\{r^{-1}T, rT^{-1}\} & \xrightarrow{\qquad j_{T,1} \qquad} & A_{Z_1}\{r^{-1}T, rT^{-1}\} \end{array}$$

Let $f = \sum_{i=0}^{d-1} \sum_{n\in\mathbb{Z}} a_{n,i} T^n X^i \in \mathcal{O}(X_{|P|=r,Z}) = A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T)$. Then, $|f|_{|P|=r,Z_1} = \max_i \max_n |a_{n,i}|_{\rho_{Z_1}} r^n$. Since A_Z and A_{Z_1} are equipped with their respective spectral norms, $|a_{n,i}|_{\rho_{Z_1}} \leqslant |a_{n,i}|_{\rho_Z}$, implying $|f|_{|P|=r,Z_1} \leqslant \max_i \max_n |a_{n,i}|_{\rho_Z} r^n = |f|_{|P|=r,Z}$. □

REMARK 4.2.18. By applying the above to the case when S is a point (*i.e.* if everything is defined over a complete ultrametric field), it makes sense to speak of the affinoid domains $X_{|P|\bowtie r,x}$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$, and their norms $|\cdot|_{|P|\bowtie r,x}$, for $\bowtie \in \{\leqslant, =, \geqslant\}$, which satisfy all of the properties we have proven so far.

Furthermore, if P is a unitary polynomial of degree d over A_Z that is irreducible over $\mathcal{H}(x)$, then there exists a "restriction morphism" $(\mathcal{O}(X_{|P|\bowtie r,Z}), |\cdot|_{|P|\bowtie r,Z}) \to (\mathcal{O}(X_{|P|\bowtie r,x}), |\cdot|_{|P|\bowtie r,x})$ on the fiber (corresponding to base change), which is a contraction. To see this, let $f = \sum_{i=0}^{d-1} \sum_{n\in\mathbb{Z}} a_{n,i} T^n X^i \in \mathcal{O}(X_{|P|\bowtie r,Z})$ (with certain $a_{n,i}$ possibly 0 depending on what $\bowtie$ is). Then, $|f|_{|P|\bowtie r,x} = \max_i \max_n |a_{n,i}|_x r^n \leqslant \max_i \max_n |a_{n,i}|_{\rho_Z} r^n = |f|_{|P|\bowtie r,Z}$.

4.2.3. The explicit norm comparison. The following is mainly a special case of [**59**, 5.2] (or a rather direct consequence thereof), which we summarize here with an emphasis on the results that interest us.

Let P be a unitary polynomial of degree $d > 1$ over $\mathcal{O}_x$ that is irreducible over $\mathcal{H}(x)$. Also, let $r \in \mathbb{R}_{>0} \backslash \sqrt{|\mathcal{H}(x)^\times|}$. As before, let Z be any connected affinoid neighborhood of x contained in $Z' \cap Z_T \cap Z_P$.

For $t \in \{x, Z\}$ (we understand here that t can be x or any connected affinoid neighborhood of x with the property we just mentioned), let $(R_t, |\cdot|_{r,t})$ be $(A_Z\{r^{-1}T, rT^{-1}\}, |\cdot|_{|T|=r,Z})$ if $t = Z$ and $(\mathcal{H}(x)\{r^{-1}T, rT^{-1}\}, |\cdot|_{|T|=r,x})$ otherwise. Remark that $(R_t, |\cdot|_{r,t})$ is an affinoid algebra over A_Z if $t = Z$ and over $\mathcal{H}(x)$ if $t = x$. As mentioned in Remark 4.2.18, there is a contraction $R_Z \hookrightarrow R_x$ induced from the restriction $A_Z \hookrightarrow \mathcal{O}_x \hookrightarrow \mathcal{H}(x)$.

For any $s \in \mathbb{R}_{>0}$, let $|\cdot|_{t,s}$ denote the norm on $R_t[X]$ induced from the R_t-affinoid algebra $R_t\{s^{-1}X\}$. Let $|\cdot|_{t,s,\mathrm{res}}$ denote the residue norm on $R_t[X]/(P(X)-T)$ induced by $|\cdot|_{t,s}$.

LEMMA 4.2.19. *For any $t \in \{x, Z\}$, there exists $v'_t > 0$, such that for any $s \geqslant v'_t$, the norm $|\cdot|_{t,s,res}$ is equivalent to $|\cdot|_{|P|=r,t}$. Explicitely, for any $f \in R_t[X]/(P(X)-T)$,*

$$|f|_{t,s,res} \leqslant |f|_{|P|=r,t} \leqslant C_t \max_{1\leqslant i\leqslant d-1}(s^{-i})|f|_{t,s,res},$$

where $C_t = \max(2, 2v_t'^{-d})$.

Fix a connected affinoid neighborhood $Z_0 \subseteq Z' \cap Z_T \cap Z_P$ of x. There exist $v', C' > 0$ such that the statement is true for any $s \geqslant v'$ and any $t \in \{x, Z : Z \subseteq Z_0\}$.

PROOF. For the first part of the statement, see [**59**, Lemme 5.2.3]. The norm $|\cdot|_{t,s,res}$ is the analogue of what in *loc.cit.* is denoted by $|\cdot|_{U,w,res}$ (here U is $X_{|T|=r,t}$ and $s = w$).

To see the last part of the statement, let us describe v'_t explicitely. Let $\alpha_0, \dots, \alpha_{d-1} \in A_Z$ be the coefficients of P, and $\beta_0, \dots, \beta_{d-1} \in A_Z[T] \subseteq R_Z$ the coefficients of $P(X) - T$ (*i.e.* $\beta_0 = \alpha_0 - T, \beta_i = \alpha_i$ for $1 \leqslant i \leqslant d-1$). By the proof of Théorème 5.2.1 of [**59**], we only require that $v'_t > 0$ satisfy $\sum_{i=0}^{d-1} |\beta_i|_{r,t} v'_t \leqslant \frac{1}{2}$. Set $v' := v'_{Z_0}$. Then, $\sum_{i=0}^{d-1} |\beta_i|_{r,Z_0} v' \leqslant \frac{1}{2}$. By Lemma 4.2.17 and Remark 4.2.18, $|\beta_i|_{r,t} \leqslant |\beta_i|_{r,Z_0}$ for any $t \in \{x, Z : Z \subseteq Z_0\}$, so

$$\sum_{i=0}^{d-1} |\beta_i|_{r,t} v' \leqslant \sum_{i=0}^{d-1} |\beta_i|_{r,Z_0} v' \leqslant \frac{1}{2}.$$

Set $C' = \max(2, 2v'^{-d})$. The statement is true with this choice of v' and C'. □

THEOREM 4.2.20. *Let Z_0 be as in the previous lemma. There exist $m, s, C' > 0$ such that for any $t \in \{x, Z : Z \subseteq Z_0\}$ and any $f \in R_t[X]/(P(X)-T)$:*

$$|f|_{\rho_{|P|=r,t}} \leqslant |f|_{|P|=r,t} \leqslant C' \max_{1\leqslant i\leqslant d-1}(s^{-i}) \frac{d^2(2s)^{d^2-d}}{m} |f|_{\rho_{|P|=r,t}},$$

where $\rho_{|P|=r,t}$ is the spectral norm on $R_t[X]/(P(X)-T) = \mathcal{O}(X_{|P|=r,t})$.

PROOF. The first inequality is immediate from the definition of the spectral norm.

By the previous lemma, there exist $v' > 0$ and $C' > 0$ such that for any $s \geqslant v'$ and any $t \in \{x, Z : Z \subseteq Z_0\}$, $|\cdot|_{|P|=r,t} \leqslant C' \max_{1\leqslant i\leqslant d-1}(s^{-i})|\cdot|_{t,s,res}$. Thus, it suffices to compare the norm $|\cdot|_{t,s,res}$ to the spectral one. For a fixed t, this is done in [**59**, Proposition 5.2.7] as follows.

Let $\mathrm{Res}(\cdot,\cdot)$ denote the resultant of two polynomials (we assume the ambient ring is unambiguously determined). Let us show that $\mathrm{Res}(P(X)-T, P'(X)) \neq 0$ in $A_{Z_0}[T]$. Otherwise, the polynomials $P(X)-T$ and $P'(X)$ would have a common divisor of positive degree, *i.e.* there would exist $Q, R, R_1 \in A_{Z_0}[T][X]$, with $deg_X Q > 0$ such that $P(X)-T = Q(X,T)R(X,T)$ and $P'(X) = Q(T,X)R_1(T,X)$. The second expression implies that the degree in T of Q and R_1 is 0, meaning $Q, R_1 \in A_{Z_0}[X]$. Consequently, $P(X) - T = Q(X)R(X,T)$, which is impossible if $\deg_X Q > 0$. Finally, this means that $\mathrm{Res}(P(X) - T, P'(X)) \neq 0$ in $A_{Z_0}[T]$. As the resultant doesn't depend on the ring in which it is computed, $\mathrm{Res}(P(X) - T, P'(X)) \neq 0$ in R_t, so $|\mathrm{Res}(P(X) - T, P'(X))|_{r,t} \neq 0$ for any t.

Let $\alpha_0, \beta_1, \dots, \beta_{d-1} \in A_{Z_0}$ be the coefficients of $P(X)$, and $\beta_0 := \alpha_0 - T, \beta_1, \dots, \beta_{d-1} \in A_{Z_0}[T] \subseteq R_{Z_0}$ the coefficients of $P(X)-T$. Set $v''_t := \max_{1\leqslant i\leqslant d-1}(|\beta_i|_{r,t}^{\frac{1}{d-i}})$. Set $v_t = \max(v', v''_t)$. Let $m_t > 0$ be such that $|\mathrm{Res}(P(X) - T, P'(X))|_{r,t} > m_t$ (such an m_t exists by the paragraph above).

Let $s > v_t$. Then, for any $f \in R_t[X]/(P(X) - T)$ (see [**59**, Proposition 5.2.7]):

$$|f|_{t,s,res} \leqslant \frac{d^2(2s)^{d^2-d}}{m_t}|f|_{\rho_{|P|=r,t}}.$$

By Lemma 4.2.17 and Remark 4.2.18, for any $t \in \{x, Z : Z \subseteq Z_0\}$, $v''_t \leqslant v''_{Z_0}$. Set $v = \max(v', v''_{Z_0})$, so that for any t, $v_t \leqslant v$.

Set $m = m_x$. Note that for any t,

$$0 < m < |\mathrm{Res}(P(X) - T, P'(X))|_{r,x} \leqslant |\mathrm{Res}(P(X) - T, P'(X))|_{r,t}.$$

Consequently, for any $t \in \{x, Z : Z \subseteq Z_0\}$ and any $s \geqslant v$,

$$|f|_{t,s,res} \leqslant \frac{d^2(2s)^{d^2-d}}{m}|f|_{\rho_{|P|=r,t}}.$$

From Lemma 4.2.19, $|f|_{|P|=r,t} \leqslant C' \max_{1\leqslant i\leqslant d-1}(s^{-i})|f|_{t,s,res}$ for all t, so finally

$$|f|_{|P|=r,t} \leqslant C' \max_{1\leqslant i\leqslant d-1}(s^{-i})\frac{d^2(2s)^{d^2-d}}{m}|f|_{\rho_{|P|=r,t}},$$

for all $f \in R_t[X]/(P(X) - T)$ and all $t \in \{x, Z : Z \subseteq Z_0\}$. □

Remark 4.2.21. The previous theorem gives an explicit comparison between the norms $|\cdot|_{|P|=r,t}$ and $\rho_{|P|=r,t}$ with a constant that is valid for all $t \in \{x, Z : Z \subseteq Z_0\}$. By Lemma 4.2.2, in the case of degree one polynomials, this constant is simply 1.

Set $C = \max\left(1, C' \max_{1\leqslant i\leqslant d-1}(s^{-i})\frac{d^2(2s)^{d^2-d}}{m}\right)$. We have shown the following:

Corollary 4.2.22. *Let $P(T)$ be a unitary polynomial in $\mathcal{O}_x[T]$ irreducible over $\mathcal{H}(x)$ and $r \in \mathbb{R}_{>0}\backslash\sqrt{|\mathcal{H}(x)^\times|}$. There exists a connected affinoid neighborhood Z_0 of x in S such that for any $t \in \{x, Z : Z \subseteq Z_0$ is a connected affinoid neighborhood of $x\}$,*

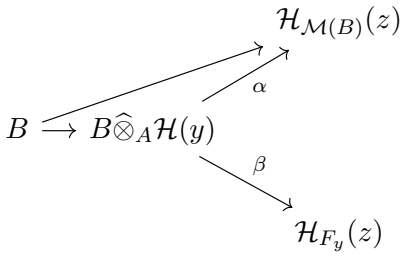
$$|\cdot|_{\rho_{|P|=r,t}} \leqslant |\cdot|_{|P|=r,t} \leqslant C|\cdot|_{\rho_{|P|=r,t}}.$$

Remark 4.2.23. From now on, whenever we consider spaces of the form $X_{|P|\bowtie r,t}$, $t \in \{x, Z\}$, $\bowtie \in \{\leqslant, =, \geqslant\}$, we will always assume its corresponding affinoid algebra to be endowed with the norm $|\cdot|_{|P|\bowtie r,t}$ defined in Notation 4.2.11, resp. Notation 4.2.14.

4.2.4. A useful proposition. Recall the notion of complete residue field of a point (Definition 1.1.33, Lemma 1.4.22). We will need the following:

Lemma 4.2.24. *Let $Y_1 = \mathcal{M}(A)$ be a k-affinoid space. Let $Y_2 = \mathcal{M}(B)$ be a relative affinoid space over Y_1 and $\phi : Y_2 \to Y_1$ the corresponding morphism. Let $y \in Y_1$ and set $F_y := \phi^{-1}(y)$, which we identify with the $\mathcal{H}(y)$-analytic space $\mathcal{M}(B\widehat{\otimes}_A\mathcal{H}(y))$. For any $z \in F_y$, $\mathcal{H}_{\mathcal{M}(B)}(z) = \mathcal{H}_{F_y}(z)$, where $\mathcal{H}_N(z)$ is the completed residue field of z when regarded as a point of N, $N \in \{\mathcal{M}(B), F_y\}$.*

Proof. Considering the bounded embedding $\mathcal{H}(y) \hookrightarrow \mathcal{H}_{\mathcal{M}(B)}(z)$, we have the following commutative diagram where all the maps are bounded:



The proof is based on the identification of F_y to $\mathcal{M}(B\widehat{\otimes}_A\mathcal{H}(x))$. Remark that the map α induces on $B\widehat{\otimes}_A\mathcal{H}(y)$ the semi-norm determined by z, implying there is a bounded embedding $\mathcal{H}_{F_y}(z) \hookrightarrow \mathcal{H}_{\mathcal{M}(B)}(z)$ on the diagram above. Similarly, since the map β induces on B the semi-norm determined by z, we obtain that $\mathcal{H}_{F_y}(z) = \mathcal{H}_{\mathcal{M}(B)}(z)$. □

COROLLARY 4.2.25. *With the same notation as in Lemma 4.2.24 and with Y_2 integral, if $\mathcal{O}_y$, $\mathcal{O}_{F_y,z}$ are fields and z is a smooth point of Y_2, then $\mathcal{O}_{Y_2,z}$ is a field.*

PROOF. Suppose that $\mathcal{O}_{Y_2,z}$ is not a field. Then, its maximal ideal is non-zero, meaning there exists a non-zero $f \in \mathcal{O}_{Y_2,z}$ such that $f(z) = 0$ in $\mathcal{H}(z)$. As we saw in Lemma 4.2.24, this field is the same regardless of which ambient space we consider z in. In particular, this means that the image f_y of f in $\mathcal{O}_{F_y,z}$ satisfies $f_y(z) = 0$ in $\mathcal{H}(z)$. Since $\mathcal{O}_{F_y,z}$ was assumed to be a field, this means that $f_y = 0$ in $\mathcal{O}_{F_y,z}$ so there exists a neighborhood of z in F_y where $f = 0$. By [**18**, Proposition 6.3.1], which is where the smoothness assumption is needed, this means that there exists a neighborhood of z in Y_2 on which $|f| = 0$, implying $f = 0$, which is in contradiction with the assumptions we made. Consequently, $\mathcal{O}_{Y_2,z}$ is a field. □

Applied to our setting, this means that for any type 3 point η of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$, the stalk $\mathcal{O}_{\mathbb{P}^{1,\mathrm{an}}_S,\eta}$ is a field. We aim to show the same for the stalks $\mathcal{O}_{X_{|P|=r,Z},\eta}$. The corollary above does not apply, since the smoothness condition is no longer satisfied.

REMARK 4.2.26. Recall Notation 4.1.5. Let P be a unitaty polynomial in $\mathcal{O}_x[T]$ irreducible over $\mathcal{H}(x)$, and $r \in \mathbb{R}_{>0}\backslash\sqrt{|\mathcal{H}(x)^\times|}$. Let $\eta := \eta_{P,r} \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. As seen in Lemma 4.2.8 (*cf.* also Remark 4.2.18), $\mathcal{H}(x)\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}}(\{\eta\})$. By Lemma 3.1.1, $\mathcal{O}_{\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}}(\{\eta\}) = \mathcal{H}(\eta)$. By Proposition 4.2.10 (see also Remark 4.2.18), $|\cdot|_{|P|=r,x}$ is equivalent to the norm $|\cdot|_\eta$ on $\mathcal{H}(\eta)$.

Following Notation 4.2.7, let $Z_0 \subseteq Z' \cap Z_T \cap Z_P$ be a connected affinoid neighborhood of x.

Let us consider the following commutative diagram for any connected affinoid neighborhood $Z \subseteq Z_0$ of x:

$$\begin{array}{ccc} A_Z\{r^{-1}T, rT^{-1}\} & \xrightarrow{\text{finite}} & A_Z\{r^{-1}T, rT^{-1}\}[X]/(P(X)-T) \\ \downarrow{\scriptstyle \widehat{\otimes}_{A_Z}\mathcal{H}(x)} & & \downarrow{\scriptstyle \widehat{\otimes}_{A_Z}\mathcal{H}(x)} \\ \mathcal{H}(x)\{r^{-1}T, rT^{-1}\} & \xrightarrow{\text{finite}} & \mathcal{H}(x)\{r^{-1}T, rT^{-1}\}[Y]/(P(Y)-T) \end{array} \tag{5}$$

The horizontal arrows are induced by the finite morphism $T \mapsto P(T)$. The vertical arrows correspond to taking the restriction of analytic functions on $X_{|T|=r,Z}$, resp. $X_{|P|=r,Z}$, to the fiber F_x. In particular, remark that $X \mapsto Y$, so we will use the same variable X.

We start by showing an auxiliary result.

LEMMA 4.2.27. *The family $\{X_{|P|=r,Z} : Z \subseteq Z_0\}$ (where Z is always considered to be a connected affinoid neighborhood of x) forms a basis of neighborhoods of η in $X_{|P|=r,Z_0}$.*

PROOF. Let U be an open neighborhood of η in $X_{|P|=r,Z_0}$. There exists a connected affinoid neighborhood $Z \subseteq Z_0$ of x such that $X_{|P|=r,Z} \subseteq U$. To see this, remark that $X_{|P|=r,Z_0} \backslash U$ is a compact subset of $\mathbb{P}^{1,\mathrm{an}}_{Z_0}$, so $\pi(X_{|P|=r,Z_0} \backslash U)$ is a compact subset of Z_0. Furthermore, $x \notin \pi(X_{|P|=r,Z_0} \backslash U)$, so there exists a connected affinoid neighborhood $Z \subseteq Z_0$ of x such that $Z \cap \pi(X_{|P|=r,Z_0} \backslash U) = \emptyset$. Consequently, $X_{|P|=r,Z} \backslash U = \pi^{-1}(Z) \cap (X_{|P|=r,Z_0} \backslash U) = \emptyset$, so $X_{|P|=r,Z} \subseteq U$. □

PROPOSITION 4.2.28. *The local ring $\mathcal{O}_{X_{|P|=r,Z_0},\eta}$ is a field.*

PROOF. Suppose that $\mathcal{O}_{X_{|P|=r,Z_0},\eta}$ is not a field. Then, its maximal ideal is non-zero, so there exists $f \in \mathcal{O}_{X_{|P|=r,Z_0},\eta}$ such that $f \neq 0$ and $f(\eta) = 0$ in $\mathcal{H}(\eta)$ (*i.e.* $|f|_\eta = 0$). By Lemma 4.2.27, there exists a connected affinoid neighborhood $Z \subseteq Z_0$ of x such that $f \in \mathcal{O}(X_{|P|=r,Z})$.

By Lemma 4.2.24, evaluating $f \in \mathcal{O}(X_{|P|=r,Z})$ at the point $\eta \in \mathcal{O}(X_{|P|=r,Z})$ is the same as evaluating the restriction of f to the fiber (see the vertical map on the right of the diagram 5 above) at the point η on the fiber. Consequently, since the norm $|\cdot|_\eta$ is equivalent to $|\cdot|_{|P|=r,x}$ (see Proposition 4.2.10 and Remark 4.2.18), we obtain that $|f|_{|P|=r,x} = 0$.

Let $f = \sum_{i=0}^{d-1} \sum_{n \in \mathbb{Z}} a_{n,i} T^n X^i \in \mathcal{O}(X_{|P|=r,Z})$. Then, $|f|_{|P|=r,x} = \max_i \max_n |a_{n,i}|_x r^n$. If $|f|_{|P|=r,x} = 0$, this implies that for any n and any i, $|a_{n,i}|_x = 0$, and since $\mathcal{O}_x$ is a field, we obtain $a_{n,i} = 0$ in A_Z. Consequently, $f = 0$ over $X_{|P|=r,Z}$.

By Lemma 4.2.27, this means that $f = 0$ in $\mathcal{O}_{X_{|P|=r,Z_0},\eta}$, contradiction. Hence, the local ring $\mathcal{O}_{X_{|P|=r,Z_0},\eta}$ is a field. □

4.3. Patching on the Relative Projective Line

The goal of this section is to prove a relative analogue of Proposition 3.2.3. As before, let k be a complete ultrametric field.

4.3.1. A few preliminary results. Recall Notation 4.1.5.

REMARK 4.3.1. By Theorem 1.7.8, for any integral k-affinoid space Z, $\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z) = \mathscr{M}(Z)(T)$.

LEMMA 4.3.2. *Let X be an integral k-affinoid space with corresponding affinoid algebra R_X. Set $F_X = \mathscr{M}(X)$. Let $z \in X$ be such that $\mathcal{O}_z$ is a field.*

The function $|\cdot|_{F_X} := \max(|\cdot|_y : y \in \Gamma(X) \cup \{z\})$ defines a submultiplicative norm on F_X which when restricted to R_X gives the spectral norm ρ_X.

Let X' be an integral k-affinoid space such that X is a rational domain of X'. Set $F_{X'} = \mathscr{M}(X')$. The field $F_{X'}$ is dense in $(F_X, |\cdot|_{F_X})$.

PROOF. Remark that z (since $\mathcal{O}_z$ is a field) and all $y \in \Gamma(X)$ (because of Lemma 1.4.31) determine multiplicative norms on R_X, and hence also on F_X.

As a consequence, $|\cdot|_{F_X}$ is well-defined. That it is a submultiplicative norm on F_X extending ρ_X follows from the fact that $|\cdot|_{\rho_X} = \max(|\cdot|_y : y \in \Gamma(X))$. Since X is reduced, ρ_X is equivalent to the norm on the affinoid algebra R_X (Proposition 1.3.15).

By Lemma 1.4.16, for $S_X := \{g \in \mathcal{O}(X') : |g|_x \neq 0 \text{ for all } x \in X\}$, the set $S_X^{-1}\mathcal{O}(X')$ is dense in $\mathcal{O}(X) = R_X$. As $S_X \subseteq \mathcal{O}(X')\backslash\{0\}$, by Lemma 1.7.6, $S_X^{-1}\mathcal{O}(X') \subseteq \mathscr{M}(X') = F_{X'}$, so $R_X \cap F_{X'} \subseteq F_X$ is a dense subset of R_X.

Let $f = \frac{u}{v} \in F_X$, where $u, v \in R_X$. Then, by the above, u, v can be approximated by some $u_0, v_0 \in R_X \cap F_{X'}$. We will show that $\frac{u_0}{v_0}$ approximates $\frac{u}{v}$ in F_X, implying (since $\frac{u_0}{v_0} \in F_{X'}$) that $F_{X'}$ is dense in F_X.

Since both $|u - u_0|_{\rho_X}$ and $|v - v_0|_{\rho_X}$ may be assumed to be arbitrarily small, we may suppose that $|u|_y = |u_0|_y$ and $|v_0|_y = |v|_y$ for all $y \in \Gamma(X) \cup \{z\}$. Then, $|\frac{1}{v}|_{F_X} = |\frac{1}{v_0}|_{F_X}$. Finally, $|f - \frac{u_0}{v_0}|_{F_X} \leqslant |uv_0 - u_0v|_{F_X} \cdot |\frac{1}{v}|^2_{F_X} = |uv_0 - u_0v|_{R_X} \cdot |\frac{1}{v}|^2_{F_X} \to 0$ when $u_0 \to u$ and $v_0 \to v$ in R_X. □

The following is an example of Setting 2.1.5 which we will be working with.

PROPOSITION 4.3.3. *Let U, V be connected affinoid domains of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ containing only type 3 points in their boundaries such that $U \cap V$ is a single type 3 point $\{\eta\}$. Let Z be a connected affinoid neighborhood of x in S such that there exist Z-thickenings U_Z, V_Z of U, V, respectively. Assume that Z is such that the statement of Proposition 4.1.16 is satisfied. Then, the conditions of Setting 2.1.5 are satisfied for: $F := \mathscr{M}(Z)(T)$, $R_0 := \mathcal{O}(U_Z \cap V_Z)$, $R_1 = A_1 := \mathcal{O}(U_Z), R_2 = A_2 := \mathcal{O}(V_Z)$, and $F_i := \mathrm{Frac}\ R_i, i = 0, 1, 2$.*

PROOF. The field F is clearly infinite and embeds in both F_1 and F_2. Also, the rings R_i, $i = 0, 1, 2$, are integral domains containing k and endowed with a non-Archimedean submultiplicative norm that extends that of k and is k-linear. The morphisms $R_j \hookrightarrow R_0$, $j = 1, 2$, are bounded seeing as they are restriction morphisms.

Remark that regardless of whether $U_Z \cup V_Z$ is an affinoid domain or all of $\mathbb{P}^{1,\mathrm{an}}_Z$, $H^1(U_Z \cup V_Z, \mathcal{O}) = 0$. Consequently, as usual, there exists a surjective admissible morphism $R_1 \oplus R_2 \twoheadrightarrow R_0$. □

NOTATION 4.3.4. In addition to Notation 4.1.5, let G be a rational linear algebraic group defined over $\mathcal{O}_x(T)$. Let $H/\mathcal{O}_x(T)$ be a variety on which G acts strongly transitively (Definition 3.2.1).

Seeing as $\mathcal{O}_x(T) = \varinjlim_Z \mathscr{M}(Z)(T)$, where the direct limit is taken with respect to connected affinoid neighborhoods of x, there exists such a Z_G for which G is a rational linear algebraic group defined over $\mathscr{M}(Z_G)(T)$. The same remains true for any connected affinoid neighborhood $Z \subseteq Z_G$ of x.

4.3.2. Patching over $\mathbb{P}^{1,\mathrm{an}}$. We now have all the necessary elements to show that patching is possible over $\mathbb{P}^{1,\mathrm{an}}_Z$ for a well-enough chosen affinoid neighborhood Z of x (both in the sense of Chapter 2 and of Proposition 3.2.2).

For the rest of this section, we assume that k is a complete non-trivially valued ultrametric field. Recall Notation 4.1.5.

REMARK 4.3.5. In order for the results of Section 4.2 to be applicable, from now on, whenever taking a thickening of an affinoid domain with respect to a certain writing of its boundary points (see Definition 4.1.14), we will always assume that the corresponding polynomials were chosen to be unitary (since $\mathcal{O}_x$ is a field, this can be done without causing any restrictions to our general setting).

SETTING 4.3.6. Let η be a type 3 point of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. There exists a unitary polynomial $P \in \mathcal{O}_x[T]$ that is irreducible over $\mathcal{H}(x)$ and a real number $r \in \mathbb{R}_{>0} \backslash \sqrt{|\mathcal{H}(x)^\times|}$ such that $\eta = \eta_{P,r}$. Let Z_0 be a connected affinoid neighborhood of x in S such that $P \in \mathcal{O}(Z_0)[T]$ and the Z_0-thickenings of $\{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |T|_u \bowtie r\}$, $\{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |P|_u \bowtie r\}, \bowtie \in \{\leqslant, \geqslant\}$, are connected. Let $Z \subseteq Z_0$ be any connected affinoid neighborhood of x.

As before, set $X_{|T|\bowtie r,Z} := \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |T|_u \bowtie r\}$, and $X_{|P|\bowtie r,Z} := \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |P|_u \bowtie r\}$, where $\bowtie \in \{\leqslant, =, \geqslant\}$. Set $(R_{0,Z}, |\cdot|_{R_{0,Z}}) := (\mathcal{O}(X_{|P|=r,Z}), |\cdot|_{|P|=r,Z})$, $(R_{1,Z}, |\cdot|_{R_{1,Z}}) := (\mathcal{O}(X_{|P|\leqslant r,Z}), |\cdot|_{|P|\leqslant r,Z})$ and $(R_{2,Z}, |\cdot|_{R_{2,Z}}) := (\mathcal{O}(X_{|P|\geqslant r,Z}), |\cdot|_{|P|\geqslant r,Z})$ (see Remark 4.2.23). Also, set $F_{i,Z} := \mathrm{Frac}(R_{0,i}), i = 0, 1, 2$, and $F := \mathscr{M}(Z)(T)$.

Assume that Z_0 is chosen so that all of the results of Section 4.2 are satisfied. Moreover, assume $Z_0 \subseteq Z_G$ (see Notation 4.3.4).

Throughout this subsection, suppose we are in the situation of Setting 4.3.6.

PARAMETER 1. Since $H^1(\mathbb{P}^{1,\mathrm{an}}_Z, \mathcal{O}) = 0$, there is an admissible surjection $R_{1,Z} \oplus R_{2,Z} \twoheadrightarrow R_{0,Z}$. Furthermore, by Lemmas 4.2.5 and 4.2.16, for any $c \in R_{0,Z}$, there exist $a \in R_{1,Z}$ and $b \in R_{2,Z}$ such that $\frac{1}{2}\max(|a|_{R_{1,Z}}, |b|_{R_{2,Z}}) < |c|_{R_{0,Z}}$. Set $d = \frac{1}{2}$.

As mentioned in Remark 2.1.8, since G is a rational linear algebraic group over $F := \mathscr{M}(Z_0)(T)$, by definition there exists a Zariski open S' of G which is isomorphic (via a morphism φ) to an open S'' of some $\mathbb{A}^n_F$. If we denote by m the multiplication on G, this leads to the following commutative diagram (which is defined over F), where $\widetilde{S'} := m^{-1}(S') \cap (S' \times S')$ is an open of $G \times G$, $\widetilde{S''}$ is an open of $\mathbb{A}^{2n}_F$, the vertical maps are isomorphisms, and f is the map induced from m:

$$\begin{array}{ccc} \widetilde{S'} & \xrightarrow{m_{|\widetilde{S'}}} & S' \\ {\scriptstyle (\varphi\times\varphi)_{|\widetilde{S'}}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \widetilde{S''} & \xrightarrow[f]{} & S'' \end{array} \tag{6}$$

Furthermore, by translating if necessary, we may assume that the identity I of G is in S' and that $\varphi(I) = 0$. Then, $0 \in \widetilde{S''}$, and f is a rational morphism $\mathbb{A}^{2n}_F \dashrightarrow \mathbb{A}^{2n}_F$ defined over the open $\widetilde{S''}$. In particular, this means that $f = (f_1, \ldots, f_n)$, where $f_i = \frac{g_i}{h_i}$ for some $g_i, h_i \in F[S_1, \ldots, S_n, T_1, \ldots, T_n]_{(S_1,\ldots,S_n,T_1,\ldots,T_n)} =: F[\underline{S}, \underline{T}]_{(\underline{S},\underline{T})}$, $i = 1, 2, \ldots, n$. Remark also that $f(x, 0) = f(0, x) = x$ whenever $(0, x), (x, 0) \in \widetilde{S''}$.

PARAMETER 2. Let us look at the diagram above over the field F_{0,Z_0}. We may suppose that $g_i, h_i \in R_{0,Z_0}[\underline{S}, \underline{T}]$ for all i. Since $h_i(0) \neq 0$ and $\mathcal{O}_{X_{|P|=r,Z_0},\eta}$ is a field, $|h_i(0)|_\eta \neq 0$. Consequently, by Lemma 4.2.27, there exists a connected affinoid neighborhood $Z_1 \subseteq Z_0$ of x such that $|h_i(0)|_u \neq 0$ for all $u \in X_{|P|=r,Z_1}$, i. By Lemma 1.1.39, $h_i(0) \in R^\times_{0,Z_1}$ for all i. This implies that $h_i(0) \in R^\times_{0,Z}$ for all connected affinoid neighborhoods $Z \subseteq Z_1$ of x.

By Lemmas 2.1.2 and 2.1.3, there exists $M \geqslant 1$ such that

$$f_i = S_i + T_i + \sum_{|(l,m)|\geqslant 2} c^i_{l,m} \underline{S}^l \underline{T}^m \in R_{0,Z_1}[[\underline{S}, \underline{T}]],$$

and $|c^i_{l,m}|_{R_{0,Z_1}} \leqslant M^{|(l,m)|}$ for all i, and all $(l,m) \in \mathbb{N}^{2n}$ such that $|(l,m)| \geqslant 2$, where $|(l,m)|$ is the sum of the coordinates of (l,m).

By Lemma 4.2.17 (see also Corollary 4.2.3), for any connected affinoid neighborhood $Z \subseteq Z_1$ of x, $f_i = S_i + T_i + \sum_{|(l,m)|\geqslant 2} c^i_{l,m} \underline{S}^l \underline{T}^m \in R_{0,Z}[[\underline{S},\underline{T}]]$, and $|c^i_{l,m}|_{R_{0,Z}} \leqslant |c^i_{l,m}|_{R_{0,Z_1}} \leqslant M^{|(l,m)|}$ for all i and all $(l,m) \in \mathbb{N}^{2n}$ such that $|(l,m)| \geqslant 2$,.

Parameter 3. Since $\widetilde{S''}$ is a Zariski open of $\mathbb{A}^{2n}_F$ and $F \hookrightarrow \mathcal{H}(\eta)$, we have that $\widetilde{S''}(\mathcal{H}(\eta))$ is a Zariski open of $\mathcal{H}(\eta)^{2n}$. Since the topology induced by the norm on $\mathcal{H}(\eta)$ is finer than the Zariski one and $0 \in \widetilde{S''}$, there exists $\delta > 0$ such that the open disc $D_{\mathcal{H}(\eta)^{2n}}(0,\delta)$ in $\mathcal{H}(\eta)^{2n}$ (with respect to the max-norm), centered at 0 and of radius δ, is contained in $\widetilde{S''}(\mathcal{H}(\eta)) \subseteq \widetilde{S''}$.

Then, for any connected affinoid neighborhood $Z \subseteq Z_0$ of x, the open disc $D_{R^{2n}_{0,Z}}(0,\delta)$ in $R^{2n}_{0,Z}$ (with respect to the max-norm), centered at 0 and of radius δ, satisfies: $D_{R^{2n}_{0,Z}}(0,\delta) \subseteq D_{\mathcal{H}(\eta)^{2n}}(0,\delta) \subseteq \widetilde{S''}$. This is clear seeing as for any $a \in R_{0,Z}$, $|a|_\eta \leqslant |a|_{\rho_{X_{|P|=r,Z}}} \leqslant |a|_{R_{0,Z}}$, where $\rho_{X_{|P|=r,Z}}$ is the spectral norm on $X_{|P|=r,Z}$.

Remark 4.3.7. Putting Parameters 1, 2, 3 together, let $\varepsilon > 0$ be such that $\varepsilon < \min\left(\frac{d}{2M}, \frac{d^3}{M^4}, \frac{d\delta}{2}\right)$. Then, all of the conditions of Theorem 2.1.10 are satisfied for $R_0 := R_{0,Z}, A_1 := R_{1,Z}, A_2 := R_{2,Z}$, $F_0 = \operatorname{Frac} R_0$. where Z is any connected affinoid neighborhood of x contained in Z_1, with Z_1 as in Parameter 2.

Proposition 4.3.8. *Let $g \in G(F_{0,Z_1})$ (with Z_1 as in Parameter 2). Suppose $g \in S'$ (see diagram 6), and $|\varphi(g)|_\eta < \frac{\varepsilon}{C}$, where C is the constant obtained in Corollary 4.2.22 corresponding to the polynomial P. Then, there exists a connected affinoid neighborhood $Z \subseteq Z_1$ of x, and $g_i \in G(F_{i,Z}), i = 1,2$, such that $g = g_1 \cdot g_2$ in $G(F_{0,Z})$.*

Proof. Since $\varphi(g) \in \mathbb{A}^n_F(F_{0,Z_1}) = F^n_{0,Z_1}$, there exist $\alpha_i, \beta_i \in R_{0,Z_1}$ such that $\varphi(g) = (\alpha_i/\beta_i)_{i=1}^n$. Since $\beta_i \neq 0$, by Proposition 4.2.28, $|\beta_i|_\eta \neq 0$. Thus, by Lemma 4.2.27, there exists a connected affinoid neighborhood $Z' \subseteq Z_1$ of x such that $|\beta_i|_u \neq 0$ for all $u \in X_{|P|=r,Z_1}$ and all i. By Lemma 1.1.39, $\beta_i \in R^\times_{0,Z'}$ for all i. In particular, this means that $\varphi(g) \in R^n_{0,Z'}$. Remark that for any connected affinoid neighborhood $Z \subseteq Z'$ of x, $\varphi(g) \in R^n_{0,Z}$.

Since $|\varphi(g)|_\eta < \varepsilon/C$, there exists a connected affinoid neighborhood $Z \subseteq Z'$ of x such that $|\varphi(g)|_u < \varepsilon/C$ for all $u \in X_{|P|=r,Z}$. Consequently, $|\varphi(g)|_{\rho_{X_{|P|=r,Z}}} < \varepsilon/C$, where $\rho_{X_{|P|=r,Z}}$ is the spectral norm on $X_{|P|=r,Z}$. By Corollary 4.2.22, this means that $|\varphi(g)|_{R_{0,Z}} < \varepsilon$.

By Remark 4.3.7, the conditions of Theorem 2.1.10 are satisfied, meaning there exist $g_i \in G(F_{i,Z}), i = 1,2$, such that $g = g_1 \cdot g_2$ in $G(F_{0,Z})$. □

Remark that in the proposition above, we can in the same way show that there exist $g'_i \in G(F_{i,Z}), i = 1,2$, such that $g = g'_2 \cdot g'_1$ in $G(F_{0,Z})$.

We recall the following from Chapter 2:

Convention 4.3.9. Let us fix once and for all an embedding of G into $\mathbb{A}^m_F$ for some $m \in \mathbb{N}$. Let K/F be a field extension, and $M \subseteq K$. Set $G_K = G \times_F K$. Let U be a Zariski open subset of G_K. Seeing as G is affine, there is a notion of "M-points" of U. More precisely, these are the points in $U(K)$ whose coordinates are in M. Let us denote this set by $U(M)$.

PROPOSITION 4.3.10. *With the same notation as in Proposition 4.3.8, let $g \in G(F_{0,Z_1})$. Suppose $g \in S'$. Then, there exists a connected affinoid neighborhood $Z \subseteq Z_1$ of x, and $g_i \in G(F_{i,Z}), i = 1, 2$, such that $g = g_1 \cdot g_2$ in $G(F_{0,Z})$.*

PROOF. We will reduce to the first case (*i.e.* Proposition 4.3.8). Recall that the fields F_{0,Z_1} can be endowed with a submultiplicative norm $|\cdot|_{F_{0,Z_1}}$ as in Lemma 4.3.2, where the role of the point z is played by η here.

Let $\psi : gS' \cap S' \to \mathbb{A}^n_{F_{0,Z_1}}$ be the morphism given by $h \mapsto \varphi(g^{-1}h)$. Remark $0 \in Im(\psi)$. The preimage $\psi^{-1}(D_{F^n_{0,Z_1}}(0, \varepsilon/C))$ is open in $(gS' \cap S')(F_{0,Z_1})$.

Since $X_{|P|=r,Z_1}$ is a rational domain in $X_{|P|\leqslant r,Z_1}$, by Lemma 4.3.2, F_{1,Z_1} is dense in F_{0,Z_1}, so $(gS' \cap S')(F_{1,Z_1})$ is dense in $(gS' \cap S')(F_{0,Z_1})$ (see Convention 4.3.9). This means there exists $h \in (gS' \cap S')(F_{1,Z_1}) \subseteq G(F_{1,Z_1})$ such that $|\varphi(g^{-1}h)|_{F_{0,Z_1}} < \varepsilon/C$, implying that $|\varphi(g^{-1}h)|_\eta < \varepsilon/C$.

By Proposition 4.3.8, there exists a connected affinoid neighborhood $Z \subseteq Z_1$ of x and $g'_1 \in G(F_{1,Z})$, $g'_2 \in G(F_{2,Z})$, such that $g^{-1}h = g'_2 \cdot g'_1$ in $G(F_{0,Z})$. Hence, there exist $g_1 := hg'^{-1}_1 \in G(F_{1,Z})$ and $g_2 := g'^{-1}_2 \in G(F_{2,Z})$ such that $g = g_1 \cdot g_2$ in $G(F_{0,Z})$. □

THEOREM 4.3.11. *Recall Setting 4.3.6. For any $g \in G(F_{0,Z_0})$, there exists a connected affinoid neighborhood $Z \subseteq Z_0$ of x, and $g_i \in G(F_{i,Z}), i = 1, 2$, such that $g = g_1 \cdot g_2$ in $G(F_{0,Z})$.*

PROOF. Recall the construction of the connected affinoid neighborhood $Z_1 \subseteq Z_0$ of x in Parameter 2. By [**34**, Lemma 3.1], there exists a Zariski open S'_1 of G isomorphic to an open S''_1 of $\mathbb{A}^n_F$ such that $g \in S'_1(F_{0,Z_1})$. Since F is infinite and S'_1 is isomorphic to an open of some $\mathbb{A}^n_F$, there exists $\alpha \in S'_1(F)$. Set $S_1 := \alpha^{-1}S'_1$. Then, $I \in S_1$, and S_1 is isomorphic to an open subset of $\mathbb{A}^n_F$. By translation, we may assume that this isomorphism sends I to $0 \in \mathbb{A}^(F)$. Set $g' := \alpha^{-1}g \in S_1(F_{0,Z_1})$. Then, by Proposition 4.3.10, there exists a connected affinoid neighborhood $Z \subseteq Z_1$ of x, and $g'_1 \in G(F_{1,Z})$, $g_2 \in G(F_{2,Z})$, such that $g' = g'_1 \cdot g_2$ in $G(F_{0,Z})$. Consequently, for $g_1 := \alpha \cdot g'_1 \in G(F_{1,Z})$, we obtain that $g = g_1 \cdot g_2$ in $G(F_{0,Z})$. □

As a consequence, the following, which is the main tool for showing a local-global principle over the relative $\mathbb{P}^{1,\mathrm{an}}$, can be shown.

Recall that in this chapter we are working in Setting 4.1.5.

PROPOSITION 4.3.12. *Let U, V be connected affinoid domains in $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ containing only type 3 points in their boundaries, such that $U \cap V$ is a single type 3 point $\{\eta_{P,r}\}$, with $P \in \mathcal{O}_x[T]$ irreducible over $\mathcal{H}(x)$ and $r \in \mathbb{R}_{>0} \backslash \sqrt{|\mathcal{H}(x)^\times|}$. Set $W := U \cap V$.*

Let G be as in Notation 4.3.4, and Z_0 as in Setting 4.3.6. Let $Z' \subseteq Z_0$ be a connected affinoid neighborhood of x for which the Z'-thickenings $U_{Z'}, V_{Z'}, W_{Z'}$ exist, are connected, and Proposition 4.1.16 is satisfied.

Then, for any $g \in G(\mathscr{M}(W_{Z'}))$ (resp. $g \in G(\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_{Z'},\eta})$), there exists a connected affinoid neighborhood $Z \subseteq Z'$ of x, and $g_U \in G(\mathscr{M}(U_Z)), g_V \in G(\mathscr{M}(V_Z))$, such that $g = g_U \cdot g_V$ in $G(\mathscr{M}(W_Z)) = G(\mathscr{M}(U_Z \cap V_Z))$.

PROOF. Remark that for any $g \in G(\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_{Z'},\eta})$, by Lemma 4.1.19, there exists a connected affinoid neighborhood $Z \subseteq Z'$ of x, such that $g \in G(\mathscr{M}(W_Z))$. Thus, it suffices to show the result for any $g \in G(\mathscr{M}(W_{Z'}))$.

By Theorem 4.3.11, there exists a connected affinoid neighborhood $Z \subseteq Z'$ of x, and $g_i \in G(F_{i,Z}), i = 1, 2$, such that $g = g_1 \cdot g_2$ in $G(\mathscr{M}(W_Z))$ (once again, recall Setting 4.3.6). Set $\partial U = \{\eta_{P,r}, \eta_{P_j,r_j}, j = 1, 2, \ldots, n\}$, where $P_j \in \mathcal{O}_x[T]$ are unitary polynomials that are irreducible over $\mathcal{H}(x)$, and $r_j \in \mathbb{R}_{>0} \backslash \sqrt{|\mathcal{H}(x)^\times|}$, for all j.

Seeing as $U = \{u \in \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} : |P|_u \bowtie r, |P_j|_u \bowtie_j r_j, j\}$, where $\bowtie, \bowtie_j \in \{\leqslant, \geqslant\}$ for all j (Proposition 4.1.1), $U_Z \subseteq \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |P|_u \bowtie r\}$. Without loss of generality, suppose that $\bowtie$ is $\leqslant$. Then, $U_Z \subseteq \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |P|_u \leqslant r\}$ and $V_Z \subseteq \{u \in \mathbb{P}^{1,\mathrm{an}}_Z : |P|_u \geqslant r\}$ (see Lemma 4.1.4).

Consequently, for $g_U := g_{1|U_Z} \in G(\mathscr{M}(U_Z))$ and $g_V := g_{2|Z} \in G(\mathscr{M}(V_Z))$, $g = g_U \cdot g_V$ in $G(\mathscr{M}(W_Z)) = G(\mathscr{M}(U_Z \cap V_Z))$. □

4.3.3. Patching over relative nice covers. Proposition 4.3.12 is enough in itself to directly show a local-global principle over the relative projective line. However, just like in the one-dimensional case, when showing a local-global principle for relative projective curves, we use arguments that make it possible to descend to the line. The goal of this part is to present the necessary arguments to make this descent.

Recall Notation 4.1.5.

THEOREM 4.3.13. *Let $\mathcal{U}_x$ be a nice cover of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$, and $T_{\mathcal{U}_x}$ a parity function corresponding to $\mathcal{U}_x$ (see Definition 3.1.16). Let $S_{\mathcal{U}_x}$ be the set of intersection points of the different elements of $\mathcal{U}_x$. Let Z_0 be a connected affinoid neighborhood of x such that the Z_0-thickening $\mathcal{U}_{Z_0}$ of $\mathcal{U}_x$ exists and is a Z_0-relative nice cover of $\mathbb{P}^{1,\mathrm{an}}_{Z_0}$.*

Let $G/\mathscr{M}(Z_0)(T)$ be a rational linear algebraic group. Then, for any element $(g_s)_{s\in S_{\mathcal{U}_x}}$ of $\prod_{s\in S_{\mathcal{U}_x}} G\left(\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_{Z_0},s}\right)$, there exists a connected affinoid neighborhood $Z \subseteq Z_0$ of x, and $(g_{U_Z})_{U\in\mathcal{U}_x} \in \prod_{U\in\mathcal{U}_x} G(\mathscr{M}(U_Z))$, satisfying: for any $s \in S_{\mathcal{U}_x}$, there exist exactly two $U_s, V_s \in \mathcal{U}_x$ containing s, $g_s \in G(\mathscr{M}(U_{s,Z} \cap V_{s,Z}))$, and if $T_{\mathcal{U}_x}(U_s) = 0$, then $g_s = g_{U_{s,Z}} \cdot g^{-1}_{V_{s,Z}}$ in $G(\mathscr{M}(U_{s,Z} \cap V_{s,Z}))$.

PROOF. Set $\mathcal{U}_x = \{U_1, U_2, \ldots, U_n\}$. If $n = 1$ there is nothing to prove. Otherwise, using induction we will show the following statement for all i such that $2 \leqslant i \leqslant n$:

STATEMENT 1. Let $I \subseteq \{1, 2, \ldots, n\}$ be such that $|I| = i$ and $\bigcup_{h\in I} U_h$ is connected. Let S_I ($\subseteq S_{\mathcal{U}_x}$) denote the set of intersection points of the different elements of $\{U_h\}_{h\in I}$. Let $Z' \subseteq Z_0$ be any connected affinoid neighborhood of x. Then, for any $(g_s)_{s\in S_I} \in \prod_{s\in S_I} G(\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_{Z'},s})$, there exists a connected affinoid neighborhood $Z_I \subseteq Z'$ of x and $(g_{U_h,Z_I})_{h\in I} \in \prod_{h\in I} G(\mathscr{M}(U_{h,Z_I}))$, satisfying: for any $s \in S_I$ there exist exactly two elements $U_s, V_s \in \{U_h\}_{h\in I}$ containing s, $g_s \in G(\mathscr{M}(U_{s,Z_I} \cap V_{s,Z_I}))$, and if $T_{\mathcal{U}_x}(U_s) = 0$, then $g_s = g_{U_s,Z_I} \cdot g^{-1}_{V_s,Z_I}$ in $G(\mathscr{M}(U_{s,Z_I} \cap V_{s,Z_I}))$. The same is true for any connected affinoid neighborhood $Z'' \subseteq Z_I$ of x.

For $i = 2$, this is Proposition 4.3.12. Suppose it is true for some $i - 1, 2 < i < n$, and let us show that it is true for i. Without loss of generality, we may assume that $I = \{1, 2, \ldots, i\}$, i.e. that $\bigcup_{h=1}^{i} U_h$ is connected. By Lemma 3.1.18, there exist $i - 1$ elements of $\{U_h\}_{h=1}^{i}$ whose union is connected. Without loss of generality, let us assume that $\bigcup_{h=1}^{i-1} U_h$ is connected. Set $I' = I\backslash\{i\}$.

Let us start by making a comparison between S_I and $S_{I'}$. Set $V_{i-1} = \bigcup_{h=1}^{i-1} U_h$. This is a connected affinoid domain containing only type 3 points in its boundary. Since V_{i-1}, U_i,

and $V_{i-1} \cup U_i$ are connected subsets of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$, $V_{i-1} \cap U_i$ is non-empty and connected (see Lemma 1.8.20). Furthermore, since $V_{i-1} \cap U_i \subseteq S_{\mathcal{U}_x}$ (*i.e.* it is contained in a finite set of type 3 points), $V_{i-1} \cap U_i$ is a single type 3 point $\{\eta\}$. Hence, there exists $h_0 \in I'$ such that $U_{h_0} \cap U_i \neq \emptyset$. By Lemma 3.1.9, such an h_0 is unique. Consequently, $S_I = S_{I'} \cup \{\eta\}$.

For some $Z' \subseteq Z_0$ as in Statement 1, let $(g_s)_{s\in S_I} \in \prod_{s\in S_I} G(\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_{Z'},s})$. From the induction hypothesis, for $(g_s)_{s\in S_{I'}} \in \prod_{s\in S_{I'}} G(\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_{Z'},s})$, there exist a connected affinoid neighborhood $Z_{I'} \subseteq Z'$ of x and $(g_{U_{h,Z_{I'}}})_{h\in I'} \in \prod_{h\in I'} G(\mathscr{M}(U_{h,Z_{I'}}))$, satisfying: for any $s \in S_{I'}$, there exist exactly two $U_s, V_s \in \{U_h\}_{h\in I'}$ containing s, $g_s \in G(\mathscr{M}(U_{s,Z_{I'}} \cap V_{s,Z_{I'}}))$, and if $T_{\mathcal{U}_x}(U_s) = 0$, $g_s = g_{U_{s,Z_{I'}}} \cdot g^{-1}_{V_{s,Z_{I'}}}$ in $G(\mathscr{M}(U_{s,Z_{I'}} \cap V_{s,Z_{I'}}))$.

Remark that the affinoid domains V_{i-1} and U_i satisfy the properties of Proposition 4.3.12 with $V_{i-1} \cap U_i = \{\eta\}$. As seen above, there exist exactly two elements of $\{U_h\}_{h\in I}$ containing η. Also, since $g_\eta \in G(\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_{Z'},\eta})$, by Lemma 4.1.19, we may assume that $g_\eta \in G(\mathscr{M}(V_{i-1,Z'} \cap U_{i,Z'}))$. Hence, we may also assume that for any connected affinoid domain $Z''' \subseteq Z'$ of x, $g_\eta \in G(\mathscr{M}(V_{i-1,Z'''} \cap U_{i,Z'''}))$.

- Suppose $T_{\mathcal{U}_x}(U_i) = 0$. By Proposition 4.3.12, there exists a connected affinoid neighborhood $Z_I \subseteq Z_{I'} \subseteq Z'$ of x, and $a \in G(\mathscr{M}(U_{i,Z_I})), b \in G(\mathscr{M}(V_{i-1,Z_I}))$, such that $g_\eta \cdot g_{U_{i-1},Z_I} = a \cdot b$ in $G(\mathscr{M}(U_{i,Z_I} \cap V_{i-1,Z_I}))$. For any $h \in I'$, set $g'_{U_h,Z_I} := g_{U_h,Z_I} \cdot b^{-1}$ in $G(\mathscr{M}(U_{h,Z_I}))$. Also, set $g'_{U_i,Z_I} := a$ in $G(\mathscr{M}(U_{i,Z_I}))$.
- Suppose $T_{\mathcal{U}_x}(U_i) = 1$. By Proposition 4.3.12, there exists a connected affinoid neighborhood $Z_I \subseteq Z_{I'} \subseteq Z'$ of x and $c \in G(\mathscr{M}(V_{i-1,Z_I})), d \in G(\mathscr{M}(U_{i,Z_I}))$, such that $g^{-1}_{U_{i-1},Z_I} \cdot g_\eta = c \cdot d$ in $G(\mathscr{M}(V_{i-1,Z_I} \cap U_{i,Z_I}))$. For any $h \in I'$, set $g'_{U_h,Z_I} := g_{U_h,Z_I} \cdot c$ in $G(\mathscr{M}(U_{h,Z_I}))$. Also, set $g'_{U_i,Z_I} := d^{-1}$ in $G(\mathscr{M}(U_{i,Z_I}))$.

The family $(g'_{U_h,Z_I})_{h\in I} \in \prod_{h\in I} G(\mathscr{M}(U_{h,Z_I}))$ satisfies the conditions of Statement 1 for the given $(g_s)_{s\in S_I}$. The last part of Statement 1 is obtained directly by taking restrictions of $g'_{U_{h,Z_I}}$ to $G(\mathscr{M}(U_{h,Z''}))$, $h \in I$.

In particular, for $i = n$, we obtain the result that was announced. □

4.4. Relative proper curves

Throughout this section, let k denote a complete ultrametric field. Let us fix and study the following framework.

Setting 4.4.1. Let S, C be good k-analytic spaces such that S is normal. Suppose there exists a morphism $\pi : C \to S$ that makes C a proper flat relative analytic curve (*i.e.* all the fibers are curves) over S. Assume π is surjective. Let $x \in S$ be such that the stalk $\mathcal{O}_x$ is a field.

Assume there exists a connected affinoid neighborhood Z_0 of x such that:

(1) for any $y \in Z_0$, the fiber $\pi^{-1}(y)$ is a normal irreducible projective $\mathcal{H}(y)$-analytic curve C_y;
(2) there exists a normal proper scheme $C_{\mathcal{O}(Z_0)}$ over Spec $\mathcal{O}(Z_0)$, such that the analytification of the structural morphism $\pi_{\mathcal{O}(Z_0)} : C_{\mathcal{O}(Z_0)} \to$ Spec $\mathcal{O}(Z_0)$ (in the sense of Subsection 1.6.3) is the projection $C_{Z_0} := C \times_S Z_0 \to Z_0$.

Let us mention some immediate consequences of Setting 4.4.1.

For any connected affinoid neighborhood $Z \subseteq Z_0$ of x, set $C_{\mathcal{O}(Z)} = C_{\mathcal{O}(Z_0)} \times_{\mathcal{O}(Z_0)} \mathcal{O}(Z)$. Let us denote by $\pi_{\mathcal{O}(Z)}$ the structural morphism $C_{\mathcal{O}(Z)} \to \text{Spec } \mathcal{O}(Z)$. Seeing as it is a base change of $\pi_{\mathcal{O}(Z_0)}$, $\pi_{\mathcal{O}(Z)}$ is proper.

Let C_Z denote the Berkovich analytification of $C_{\mathcal{O}(Z)}$ (in the sense of Subsection 1.6.3). Remark that by Lemma 1.6.16, $C_Z = (C_{\mathcal{O}(Z)})^{\text{an}} = (C_{\mathcal{O}(Z_0)} \times_{\mathcal{O}(Z_0)} \mathcal{O}(Z))^{\text{an}} = C_{Z_0} \times_{Z_0} Z = C \times_S Z$. Let $\pi_Z : C_Z \to Z$ denote the structural morphism (*i.e.* the analytification of $\pi_{\mathcal{O}(Z)}$). By [**4**, Proposition 2.6.9], π_Z is proper.

Before exploring in more depth the properties of Setting 4.4.1, let us present a particular situation which leads to this setup, and which allows us to generalize the results of Chapter 3.

4.4.1. Example: Realization of an algebraic curve over $\mathcal{O}_x$ as the thickening of an analytic curve over $\mathcal{H}(x)$.

NOTATION 4.4.2. Let S' be a normal good k-analytic space. Let $x \in S'$ be such that $\mathcal{O}_x$ is a field. Let $C_{\mathcal{O}_x}$ be a smooth geometrically irreducible projective algebraic curve over $\mathcal{O}_x$. Let us denote by π_x the structural morphism $C_{\mathcal{O}_x} \to \text{Spec } \mathcal{O}_x$.

Remark that $\mathcal{O}_x = \varinjlim_Z \mathcal{O}(Z)$, where the limit is taken over connected affinoid neighborhoods Z of x in S, implying $\text{Spec } \mathcal{O}_x = \varprojlim_Z \text{Spec } \mathcal{O}(Z)$. By [**27**, Théorème 8.8.2], there exists a connected affinoid neighborhood Z_0 of x, such that for any connected affinoid neighborhood $Z \subseteq Z_0$ of x, there exists a finitely presented scheme $C_{\mathcal{O}(Z)}$ over $\text{Spec } \mathcal{O}(Z)$ satisfying $C_{\mathcal{O}(Z)} \times_{\text{Spec } \mathcal{O}(Z)} \text{Spec } \mathcal{O}_x = C_{\mathcal{O}_x}$. Let us denote by $\pi_{\mathcal{O}(Z)}$ the structural morphism $C_{\mathcal{O}(Z)} \to \text{Spec } \mathcal{O}(Z)$.

Remark that π_x is a proper smooth surjective morphism. The affinoid domain Z_0 can be chosen so that for any connected affinoid neighborhood $Z \subseteq Z_0$ of x, the morphism $\pi_{\mathcal{O}(Z)} : C_{\mathcal{O}(Z)} \to \text{Spec } \mathcal{O}(Z)$ remains proper, surjective (by [**27**, Théorème 8.10.5]), and smooth (by [**62**, Tag 0CNU]). Furthermore, by [**62**, Tag 0EY2], we may assume that $C_{\mathcal{O}(Z)}$ is a relative curve over $\mathcal{O}(Z)$. Let C_Z (defined over Z) denote the Berkovich analytification of the finite type scheme $C_{\mathcal{O}(Z)}$ over $\text{Spec } \mathcal{O}(Z)$ (in the sense of Subsection 1.6.3). We denote by $\pi_Z : C_Z \to Z$ the analytification of $\pi_{\mathcal{O}(Z)}$.

PROPOSITION 4.4.3. *Let $Z \subseteq Z_0$ be a connected affinoid neighborhood of x.*

1. *The morphism $\pi_Z : C_Z \to Z$ is quasi-smooth, proper, and surjective. Furthermore, C_Z is a relative curve over Z.*
2. *The spaces $C_{\mathcal{O}(Z)}, C_Z$ are normal.*

PROOF. Surjectivity of π_Z can be obtained as in the proof of [**6**, Proposition 3.4.6(7)] from the surjectivity of $\pi_{\mathcal{O}(Z)}$. Properness is given by [**4**, Proposition 2.6.9]. Quasi-smoothness is a consequence of the smoothness of $\pi_{\mathcal{O}(Z)}$ via [**18**, 5.2.14]. The dimension property is given by [**18**, Proposition 2.7.7].

Since $\pi_{\mathcal{O}(Z)} : C_{\mathcal{O}(Z)} \to \text{Spec } \mathcal{O}(Z)$ is smooth, for any point $y \in C_{\mathcal{O}(Z)}$, there exists an open neighborhood U of y such that there is a factorization of $U \to \text{Spec } \mathcal{O}(Z)$ as: $U \to \mathbb{A}^d_{\mathcal{O}(Z)} \to \text{Spec } \mathcal{O}(Z)$ for some $d \in \mathbb{N}$, where $U \to \mathbb{A}^d_{\mathcal{O}(Z)}$ is étale. Moreover, by [**28**, II, Remarque 1.5], $d = 1$. By [**28**, I, Théorème 9.5], U is normal at y if and only if $\mathbb{A}^1_{\mathcal{O}(Z)}$ is normal at its image.

Seeing as S is normal, so is Z (by [**21**, Théorème 3.4]). This implies that $\mathcal{O}(Z)$ is an integrally closed domain (recall Z is connected in a normal space, so it is irreducible), hence

so is $\mathcal{O}(Z)[T]$ (where T is an indeterminate), implying $\mathbb{A}^1_{\mathcal{O}(Z)}$ is normal. Consequently, by the above paragraph, $C_{\mathcal{O}(Z)}$ is normal. By [**21**, Théorème 3.4], its analytification C_Z is also normal. □

Seeing as a quasi-smooth morphism is flat (see [**18**, Theorem 5.3.4]), it remains to show that property (1) of Setting 4.4.1 is satisfied.

NOTATION 4.4.4. Let $Z \subseteq Z_0$ be any connected affinoid neighborhood of x.

- For any $y \in Z$, the fiber $\pi_Z^{-1}(y)$ can be endowed with the structure of an $\mathcal{H}(y)$-analytic curve $C_y := C_Z \times_Z \mathcal{H}(y)$ (see Proposition 1.5.7). Remark that C_y does not depend on Z.
- For any $y' \in \operatorname{Spec} \mathcal{O}(Z)$, the fiber $\pi_Z^{-1}(y')$ can be endowed with the structure of a $\kappa(y')$-algebraic curve $C_{\mathcal{O}(Z),\kappa(y')} := C_{\mathcal{O}(Z)} \times_{\mathcal{O}(Z)} \kappa(y')$, where $\kappa(y')$ denotes the residue field of y' in $\operatorname{Spec} \mathcal{O}(Z)$. We will use the notation $C_{\kappa(y')}$ whenever there is no risk of ambiguity.

Since $\operatorname{Spec} \mathcal{O}(Z_0)$ is Noetherian, the proper morphism $\pi_{\mathcal{O}(Z_0)}$ is of finite presentation. Since it is smooth, $\pi_{\mathcal{O}(Z_0)}$ is flat. By [**27**, Théorème 12.2.4], the set

$$A := \{u \in \operatorname{Spec} \mathcal{O}(Z_0) : C_{\mathcal{O}(Z_0),\kappa(u)} \text{ is geometrically integral and smooth}\}$$

is Zariski open in $\operatorname{Spec} \mathcal{O}(Z_0)$.

Let x' denote the image of x via the analytification $Z_0 \to \operatorname{Spec} \mathcal{O}(Z_0)$. Since $\mathcal{O}_x$ is a field, there is a natural embedding $\kappa(x') \hookrightarrow \mathcal{O}_x$, from where we obtain that $C_{\kappa(x')} \times_{\kappa(x')} \mathcal{O}_x = C_{\mathcal{O}_x}$. Since $C_{\mathcal{O}_x}$ is smooth and geometrically irreducible, it is geometrically normal and integral, implying so is $C_{\kappa(x')}$. Consequently, $x' \in A$, so A is a non-empty Zariski open subset of $\operatorname{Spec} \mathcal{O}(Z_0)$.

LEMMA 4.4.5. *Let ψ denote the analytification $Z_0 \to \operatorname{Spec} \mathcal{O}(Z_0)$. For any $y \in Z_0$ such that $\psi(y) \in A$, C_y is a geometrically irreducible smooth projective $\mathcal{H}(y)$-analytic curve. The same is true for any connected affinoid neighborhood $Z \subseteq Z_0$ of x.*

PROOF. Let $y \in Z_0$ be such that $y' := \psi(y) \in A$, *i.e.* that $C_{\kappa(y')}$ is geometrically integral. By Corollary 1.5.7, C_y is isomorphic to the analytification of $C_{\kappa(y')} \times_{\kappa(y')} \mathcal{H}(y)$, so C_y is an $\mathcal{H}(y)$-analytic curve that is geometrically integral, hence geometrically irreducible. Since π_Z is proper, C_y is a proper curve. Since π_Z is quasi-smooth, C_y is quasi-smooth (by [**18**, Theorem 5.3.4]). As it is proper, it is boundaryless, so smooth (see [**18**, Corollary 5.4.8]).

The last part of the statement is a direct consequence of the fact that C_y does not depend on Z_0 (*i.e.* remains the same for any connected affinoid neighborhood $Z \subseteq Z_0$ of x containing y). □

The preimage of A with respect to the analytification morphism $\psi : Z_0 \to \operatorname{Spec} \mathcal{O}(Z_0)$ is a Zariski open in Z_0. Consequently, there exists a connected affinoid neighborhood $Z_1 \subseteq Z_0$ of x, such that $Z_1 \subseteq \psi^{-1}(A)$. This means that for any connected affinoid neighborhood $Z \subseteq Z_1$ of x, the fiber C_y of any $y \in Z$ in C_Z is a geometrically irreducible smooth projective $\mathcal{H}(y)$-analytic curve. Consequently, Setting 4.4.1 is satisfied for $S = Z_1$ and $C = C_{Z_1}$.

4.4.2. Consequences of Setting 4.4.1. Recall that for any affinoid neighborhood Z of x in S, we denote by π_Z the structural morphism $C_Z = C \times_S Z \to Z$.

PROPOSITION 4.4.6. *Let $Z \subseteq Z_0$ be a connected affinoid neighborhood of x.*

(1) *The space C_Z is a normal proper flat relative analytic curve over Z. Furthermore, π_Z is surjective. The same properties are true for $C_{\mathcal{O}(Z)}$ and $\pi_{\mathcal{O}(Z)}$.*
(2) *Any connected affinoid domain of C_Z is normal and irreducible.*

PROOF. Since π_Z is obtained by a base change of $\pi : C \to S$, we immediately obtain that π_Z is proper, surjective, flat, and of relative dimension 1.

Seeing as C_{Z_0} is the analytification of the normal proper $\mathcal{O}(Z_0)$-scheme $C_{\mathcal{O}(Z_0)}$, it is normal by [**21**, Théorème 3.4]. Seeing as $C_Z = \pi_{Z_0}^{-1}(Z)$ is an analytic domain of the normal analytic space C_{Z_0}, by *loc.cit.*, it is normal. By the same result, $C_{\mathcal{O}(Z)}$ is also normal.

The morphism $\pi_{\mathcal{O}(Z)}$ was already remarked to be proper, as a base change of a proper morphism. Surjectivity of $\pi_{\mathcal{O}(Z)}$ can be obtained from the surjectivity of π_Z as in Proposition 3.4.6(7) of [**6**]. The relative dimension of $\pi_{\mathcal{O}(Z)}$ is the same as that of π_Z by [**18**, Proposition 2.7.7]. Its flatness is a consequence of [**18**, Lemma 4.2.1].

Any connected affinoid domain of C_Z is normal by [**21**, Théorème 3.4] and irreducible by [**21**, Théorème 5.17]. □

The object the following lemma deals with will be central for the rest of this chapter.

LEMMA 4.4.7. *Set $C_{\mathcal{O}_x} := C_{\mathcal{O}(Z_0)} \times_{\mathcal{O}(Z_0)} \mathcal{O}_x$. Then, $C_{\mathcal{O}_x}$ is an irreducible normal projective k-algebraic curve.*

PROOF. Let C_x denote the fiber of $\pi_{Z_0} : C_{Z_0} \to Z_0$. It is a normal irreducible projective $\mathcal{H}(x)$-curve by definition. Let $\overline{x}$ denote the image of x via the analytification morphism $\psi : Z_0 \to \operatorname{Spec} \mathcal{O}(Z_0)$. By Corollary 1.6.17, $C_x \cong (C_{\kappa(\overline{x})} \times_{\kappa(\overline{x})} \mathcal{H}(x))^{\mathrm{an}}$, where $\kappa(\overline{x})$ denotes the residue field of x in $\mathcal{O}(Z_0)$, and $C_{\kappa(\overline{x})} := C_{\mathcal{O}(Z_0)} \times_{\mathcal{O}(Z_0)} \kappa(\overline{x})$ - the algebraic fiber of $\overline{x}$ with respect to $C_{\mathcal{O}(Z_0)} \to \operatorname{Spec} \mathcal{O}(Z_0)$.

Set $C_x^{\mathrm{alg}} := C_{\kappa(\overline{x})} \times_{\kappa(\overline{x})} \mathcal{H}(x)$. Seeing as $\psi(x) = \overline{x}$ and $\mathcal{O}_x$ is a field, there is a canonical embedding $\kappa(\overline{x}) \hookrightarrow \mathcal{O}_x$. Consequently, $C_{\mathcal{O}_x} = C_{\kappa(\overline{x})} \times_{\kappa(\overline{x})} \mathcal{O}_x$, and

$$C_x^{\mathrm{alg}} = C_{\kappa(\overline{x})} \times_{\kappa(\overline{x})} \mathcal{H}(x) = C_{\kappa(\overline{x})} \times_{\kappa(\overline{x})} \mathcal{O}_x \times_{\mathcal{O}_x} \mathcal{H}(x) = C_{\mathcal{O}_x} \times_{\mathcal{O}_x} \mathcal{H}(x).$$

As $(C_x^{\mathrm{alg}})^{\mathrm{an}} \cong C_x$, and C_x is a normal irreducible $\mathcal{H}(x)$-analytic curve, C_x^{alg} is a connected ([**6**, Thm. 3.5.8(iii)]) normal algebraic curve ([**6**, Prop. 3.4.3]) over $\mathcal{H}(x)$.

Consequently, $C_{\mathcal{O}_x}$ is connected, and by [**26**, Corollaire 6.5.4], it is normal. Properness is immediate seeing as $C_{\mathcal{O}_x} \to \operatorname{Spec} \mathcal{O}_x$ is a base change of a proper morphism. □

Recall Notation 4.4.4, which is applicable here. A very important property for the constructions we make is the following:

LEMMA 4.4.8. *For any non-rigid point η of C_x, the local ring $\mathcal{O}_{C,\eta}$ is a field. If $\eta \in C_x$ is rigid, then $\mathcal{O}_{C,\eta}$ is a discrete valuation ring.*

In particular, this implies that for any type 3 point $\eta \in C_x$, the local ring $\mathcal{O}_{C,\eta}$ is a field.

PROOF. Seeing as $x \in \operatorname{Int} Z_0$, for any $\eta \in C_x$, $\eta \in \operatorname{Int} C_{Z_0}$, so $\mathcal{O}_{C,\eta} = \mathcal{O}_{C_{Z_0},\eta}$, and we can use the two interchangeably.

The morphism $\pi_{Z_0} : C_{Z_0} \to Z_0$ is proper, so boundaryless. As π_{Z_0} is flat, by the proof of [**18**, Lemma 4.5.11], $\dim \mathcal{O}_{C,\eta} = \dim \mathcal{O}_{C_x,\eta} + \dim \mathcal{O}_x$. Since $\mathcal{O}_x$ is a field, we obtain $\dim \mathcal{O}_{C,\eta} = \dim \mathcal{O}_{C_x,\eta}$.

By [**18**, Lemma 4.4.5], if $\eta \in C_x$ is not rigid, then $\mathcal{O}_{C_x,\eta}$ is a field, implying $\dim \mathcal{O}_{C,\eta} = 0$, so $\mathcal{O}_{C,\eta}$ is a field (recall C_{Z_0} is normal). If $\eta \in C_x$ is rigid, by *loc.cit.* $\mathcal{O}_{C_x,\eta}$ is a discrete valuation ring, implying $\dim \mathcal{O}_{C,\eta} = 1$. Hence, $\mathcal{O}_{C,\eta}$ is a Noetherian normal local ring with Krull dimension 1, meaning a discrete valuation ring. □

We proved a result somewhat similar to Lemma 4.4.8 in Corollary 4.2.25 and applied it to $\mathbb{P}^{1,\mathrm{an}}$. Note that Lemma 4.4.8 is also applicable to the relative projective line.

LEMMA 4.4.9. *Let $Z \subseteq Z_0$ be a connected affinoid neighborhood of x. For any pair of different points $u_1, u_2 \in C_Z$, there exist neighborhoods B_1 of u_1 and B_2 of u_2 in C_Z, such that $B_1 \cap B_2 = \emptyset$.*

PROOF. Seeing as $\pi_{\mathcal{O}(Z)}$ is proper, it is separated, so by [**4**, Corollary 2.6.7], π_Z is separated. Seeing as Z is Hausdorff, by [**6**, Proposition 3.1.5], $Z \to \mathcal{M}(k)$ is separated. Consequently, the canonical morphism $C_Z \to \mathcal{M}(k)$ is separated, and we can conclude by *loc.cit.* □

LEMMA 4.4.10. *Let $Z \subseteq Z_0$ be a connected affinoid neighborhood of x. The spaces $C_Z, C_{\mathcal{O}(Z)}$ are irreducible.*

PROOF. Since all the fibers of $C_Z \to Z$ are connected, C_Z is connected: if, by contradiction, C_Z can be written as the disjoint union of two closed (hence compact) subsets U and V, then $Z = \pi_Z(U) \cup \pi_Z(V)$. Since $\pi_Z(U)$ and $\pi_Z(V)$ are compact, and Z is connected, their intersection is non-empty. Consequently, there exists $y \in Z$, such that $C_y \cap U \neq \emptyset$ and $C_y \cap V \neq \emptyset$. Since C_y is connected and covered by the compacts $C_y \cap U$, $C_y \cap V$, this is a contradiction.

Thus, C_Z is a connected normal analytic space. By [**21**, Proposition 5.14], it is irreducible. Then, by [**18**, Proposition 2.7.16], $C_{\mathcal{O}(Z)}$ is also irreducible. □

PROPOSITION 4.4.11. *There exists a connected affinoid neighborhood $Z_1 \subseteq Z_0$ of x such that for any connected affinoid neighborhood $Z \subseteq Z_1$ of x, there exists a finite surjective morphism $f_Z : C_Z \to \mathbb{P}^{1,\mathrm{an}}_Z$, satisfying:*

1. *f_Z is the analytification of a finite surjective morphism $f_{\mathcal{O}(Z)} : C_{\mathcal{O}(Z)} \to \mathbb{P}^{1,\mathrm{an}}_{\mathcal{O}(Z)}$;*
2. *for any connected affinoid neighborhood $Z' \subseteq Z$ of x, $f_Z \times_Z Z' = f_{Z'}$, i.e. the following diagram (where the horizontal arrows correspond to the base change $Z' \hookrightarrow Z$) is commutative.*

$$\begin{array}{ccc} C_{Z'} & \longrightarrow & C_Z \\ {\scriptstyle f_{Z'}}\downarrow & & \downarrow{\scriptstyle f_Z} \\ \mathbb{P}^{1,\mathrm{an}}_{Z'} & \longrightarrow & \mathbb{P}^{1,\mathrm{an}}_Z \end{array}$$

PROOF. Remark that $\mathcal{O}_x = \varinjlim_Z \mathcal{O}(Z)$, where the limit is taken with respect to connected affinoid neighborhoods $Z \subseteq Z_0$ of x. Consequently, $\operatorname{Spec} \mathcal{O}_x = \varprojlim_Z \operatorname{Spec} \mathcal{O}(Z)$, and $C_{\mathcal{O}_x} = C_{\mathcal{O}(Z_0)} \times_{\mathcal{O}(Z_0)} \mathcal{O}_x = C_{\mathcal{O}(Z_0)} \times_{\mathcal{O}(Z_0)} \varprojlim_Z \mathcal{O}(Z) = \varprojlim_Z C_{\mathcal{O}(Z)}$. Recall that $C_{\mathcal{O}_x}$ is an irreducible normal projective curve (see Lemma 4.4.7).

Let $f_{\mathcal{O}_x} : C_{\mathcal{O}_x} \to \mathbb{P}^1_{\mathcal{O}_x}$ be any finite non-constant (hence surjective) morphism. By [**27**, Théorème 8.8.2], we may assume that Z_0 is such that for any connected affinoid neighborhood $Z \subseteq Z_0$ of x, there exists a morphism $f_{\mathcal{O}(Z)} : C_{\mathcal{O}(Z)} \to \mathbb{P}^1_{\mathcal{O}(Z)}$, such that the following diagram (where the horizontal arrows are the corresponding base changes) is commutative for any connected affinoid neighborhood $Z' \subseteq Z$ of x.

$$\begin{array}{ccccc} C_{\mathcal{O}_x} & \longrightarrow & C_{\mathcal{O}(Z')} & \longrightarrow & C_{\mathcal{O}(Z)} \\ \downarrow{\scriptstyle f_{\mathcal{O}_x}} & & \downarrow{\scriptstyle f_{\mathcal{O}(Z')}} & & \downarrow{\scriptstyle f_{\mathcal{O}(Z)}} \\ \mathbb{P}^1_{\mathcal{O}_x} & \longrightarrow & \mathbb{P}^1_{\mathcal{O}(Z')} & \longrightarrow & \mathbb{P}^1_{\mathcal{O}(Z)} \end{array}$$

Furthermore, by [**27**, Théorème 8.10.5], Z_0 can be chosen so that for any connected affinoid neighborhood $Z \subseteq Z_0$ of x, the morphism $f_{\mathcal{O}(Z)}$ is finite and surjective.

Let $f_Z : C_Z \to \mathbb{P}^{1,\mathrm{an}}_Z$ denote the Berkovich analytification of $f_{\mathcal{O}(Z)}$ in the sense of Subsection 1.6.3. Then, as in [**6**, Proposition 3.4.6(7)], f_Z is surjective; by [**4**, Proposition 2.6.9], it is finite.

Part (2) is a direct consequence of the commutativity of the diagram above. □

Remark that the finite surjective morphism $f_Z : C_Z \to \mathbb{P}^{1,\mathrm{an}}_Z$ induces a finite surjective morphism $f_z : C_z \to \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(z)}$ between the fibers of $z \in Z$ in C_Z and $\mathbb{P}^{1,\mathrm{an}}_Z$, respectively (recall Notation 4.4.4 which is applicable here).

Proposition 4.4.12. *Let $Z \subseteq Z_0$ be a connected affinoid neighborhood of x. Let y be a type 3 point in the fiber $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ of x on $\mathbb{P}^{1,\mathrm{an}}_Z$. Let $\{z_1, z_2, \ldots, z_n\} := f_Z^{-1}(y)$. Then, $\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_Z,y} \otimes_{\mathscr{M}(Z)(T)} \mathscr{M}(C_Z) = \prod_{i=1}^n \mathscr{M}_{C_Z,z_i}$.*

Proof. Let us look at the finite surjective morphism $f_{\mathcal{O}(Z)} : C_{\mathcal{O}(Z)} \to \mathbb{P}^1_{\mathcal{O}(Z)}$ of $\mathcal{O}(Z)$-schemes. Let y' be the image of y via the analytification $\psi : \mathbb{P}^{1,\mathrm{an}}_Z \to \mathbb{P}^1_{\mathcal{O}(Z)}$. Let $\mathcal{A} := \operatorname{Spec} A$ be an open affine neighborhood of y' in $\mathbb{P}^1_{\mathcal{O}(Z)}$. Its preimage by ψ is a Zariski open $\mathcal{A}'$ of $\mathbb{P}^{1,\mathrm{an}}_Z$ containing y.

Let $\mathcal{B} := \operatorname{Spec} B$ be the pre-image of $\mathcal{A}$ by $f_{\mathcal{O}(Z)}$. It is an affine open subset of $C_{\mathcal{O}(Z)}$, and $f_{\mathcal{O}(Z)}$ induces a finite surjective morphism $\mathcal{B} \to \mathcal{A}$. By construction, $\mathcal{B}$ contains $f^{-1}_{\mathcal{O}(Z)}(y')$. By the proof of [**4**, Proposition 2.6.10], there is an isomorphism $\prod_{i=1}^n \mathcal{O}_{C_Z,z_i} = \mathcal{O}_{\mathbb{P}^{1,\mathrm{an}}_Z,y} \otimes_A B$. Since $C_{\mathcal{O}(Z)}$ and $\mathbb{P}^1_{\mathcal{O}(Z)}$ are irreducible, the function field of $C_{\mathcal{O}(Z)}$ is Frac B, and the function field of $\mathbb{P}^1_{\mathcal{O}(Z)}$ is Frac A.

By Theorem 1.7.8, we obtain that $\mathscr{M}(C_Z) = \text{Frac } B$, and $\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z) = \text{Frac } A$. Since B is a finite A-module, by the last paragraph of the proof of Lemma 3.2.4, $\prod_{i=1}^n \mathcal{O}_{C_Z,z_i} = \mathcal{O}_{\mathbb{P}^{1,\mathrm{an}}_Z,y} \otimes_{\text{Frac } A} \text{Frac } B$, so $\prod_{i=1}^n \mathcal{O}_{C_Z,z_i} = \mathcal{O}_{\mathbb{P}^{1,\mathrm{an}}_Z,y} \otimes_{\mathscr{M}(Z)(T)} \mathscr{M}(C_Z)$. Finally, since y and $z_i, i = 1, 2, \ldots, n$, are type 3 points in $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ and C_x, respectively, $\mathcal{O}_{\mathbb{P}^{1,\mathrm{an}}_Z,y} = \mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_Z,y}$, and $\mathcal{O}_{C_Z,z_i} = \mathscr{M}_{C_Z,z_i}$ for all i, concluding the proof of the statement. □

Proposition 4.4.13. *For any connected affinoid neighborhoods $Z, Z' \subseteq Z_0$ of x such that $Z' \subseteq Z$, the base change morphism $\iota_{Z,Z'} : C_{\mathcal{O}(Z')} \to C_{\mathcal{O}(Z)}$ is dominant. Furthermore, if η_Z (resp. $\eta_{Z'}$) is the generic point of $C_{\mathcal{O}(Z)}$ (resp. $C_{\mathcal{O}(Z')}$), then $\iota_{Z,Z'}(\eta_{Z'}) = \eta_Z$.*

Proof. By Lemma 4.4.10, $C_{\mathcal{O}(Z)}, C_{\mathcal{O}(Z')}$ are irreducible, so it makes sense to speak of their generic points $\eta_Z, \eta_{Z'}$, respectively. It suffices to show that η_Z is in the image

of $\iota_{Z,Z'}$. Let α be any point of C_Z. Let α' be its image in $C_{\mathcal{O}(Z)}$ via the analytification $\phi : C_Z \to C_{\mathcal{O}(Z)}$. Let U be an open affine neighborhood of α' in $C_{\mathcal{O}(Z)}$. Then, $\eta_Z \in U$, and the closure of $\{\eta_Z\}$ in U is U.

By [**4**, Proposition 2.6.8], $\phi^{-1}(U) = U^{\mathrm{an}}$-the analytification of U. Remark that U^{an} is an open subspace of C_Z. Let B_α be any open neighborhood of α in C_Z. Then, since $\alpha \in U^{\mathrm{an}}$, $B_\alpha \cap U^{\mathrm{an}}$ is an open neighborhood of α in U^{an}, so by [**4**, Lemma 2.6.5], there exists a point $\beta \in B_\alpha \cap U^{\mathrm{an}} \subseteq B_\alpha$, such that $\phi(\beta) = \eta_Z$. Thus, for any point $\alpha \in C_Z$ and any open neighborhood B_α of α in C_Z, there exists $\beta \in B_\alpha$, such that $\phi(\beta) = \eta_Z$. In other words, $\overline{\phi^{-1}(\{\eta_Z\})} = C_Z$.

$$\begin{array}{ccc} C_{Z'} & \xrightarrow{\phi'} & C_{\mathcal{O}(Z')} \\ {\scriptstyle \theta_{Z,Z'}}\downarrow & & \downarrow{\scriptstyle \iota_{Z,Z'}} \\ C_Z & \xrightarrow{\phi} & C_{\mathcal{O}(Z)} \end{array}$$

Let us now look at the commutative diagram above, where the horizontal lines correspond to analytification, and the vertical ones to base change. In particular, remark that since $C_Z = \pi^{-1}(Z)$ and $C_{Z'} = \pi^{-1}(Z')$, we have $C_{Z'} \subseteq C_Z$, so $\theta_{Z',Z}$ is an inclusion. Let $\gamma \in \pi^{-1}(\mathrm{Int}(Z'))$ (which is non-empty considering $x \in \mathrm{Int}(Z')$). Let B_γ be an open neighborhood of γ in the open $\pi^{-1}(\mathrm{Int}(Z'))$. Then, B_γ is open in both $C_{Z'}$ and C_Z. By the paragraph above, there exists $\gamma' \in B_\gamma$ such that $\phi(\theta_{Z,Z'}(\gamma')) = \phi(\gamma') = \eta_Z$. By the commutativity of the diagram, η_Z is in the image of $\iota_{Z,Z'}$, so $\iota_{Z,Z'}$ is dominant.

Let $\eta_{Z'}$ be the generic point of $C_{\mathcal{O}(Z')}$. Since $C_{\mathcal{O}(Z)}, C_{\mathcal{O}(Z')}$ are integral schemes, this means $\iota_{Z,Z'}(\eta_{Z'}) = \eta_Z$. □

Recall that $C_{\mathcal{O}_x} = C_{\mathcal{O}(Z_0)} \times_{\mathcal{O}(Z_0)} \mathcal{O}_x = \varprojlim_Z C_{\mathcal{O}(Z)}$, where the limit is taken with respect to the connected affinoid neighborhoods $Z \subseteq Z_0$ of x. By the lemma above, the generic points η_Z of $C_{\mathcal{O}(Z)}$ determine a unique point $\eta \in C_{\mathcal{O}_x}$.

PROPOSITION 4.4.14. *The curve $C_{\mathcal{O}_x}$ is integral with generic point η.*

PROOF. Note that $C_{\mathcal{O}_x}$ was already shown to be integral in Lemma 4.4.7.

For any connected affinoid neighborhoods $Z, Z' \subseteq Z_0$ of x such that $Z' \subseteq Z$, the base change $\iota_{Z,Z'} : C_{\mathcal{O}(Z')} = C_{\mathcal{O}(Z)} \times_{\mathcal{O}(Z)} \mathcal{O}(Z') \to C_{\mathcal{O}(Z)}$ is an affine morphism. Furthermore, since $C_{\mathcal{O}(Z)}$ is normal, it is reduced.

By [**62**, Tag 0CUG], $\varprojlim_Z \overline{\{\eta_Z\}}_{\mathrm{red}} = \overline{\{\eta\}}_{\mathrm{red}}$. Seeing as $\overline{\{\eta_Z\}}_{\mathrm{red}} = C_{\mathcal{O}(Z)}$, we obtain that $\overline{\{\eta\}}_{\mathrm{red}} = \varprojlim_Z C_{\mathcal{O}(Z)} = C_{\mathcal{O}_x}$, so $C_{\mathcal{O}_x}$ is reduced and irreducible, *i.e.* integral, with generic point η. □

Let F_N denote the function field of the integral scheme C_N, where $N \in \{\mathcal{O}_x, \mathcal{O}(Z) : Z \subseteq Z_0\}$ (Z is as usual considered to be a connected affinoid neighborhood of x).

COROLLARY 4.4.15. *$F_{\mathcal{O}_x} = \varinjlim_Z F_{\mathcal{O}(Z)}$, where the limit is taken with respect to connected affinoid neighborhoods $Z \subseteq Z_0$ of x.*

PROOF. The projective system of integral schemes $\{C_{\mathcal{O}(Z)}\}_Z$ gives rise to a direct system of fields $\{F_{\mathcal{O}(Z)}\}_Z$. For connected affinoid neighborhoods $Z, Z' \subseteq Z_0$ of x such that $Z' \subseteq Z$, let us denote the corresponding transition morphism $F_{\mathcal{O}(Z)} \to F_{\mathcal{O}(Z')}$ by $\chi_{Z',Z}$. Let us denote by F' the field $\varinjlim_Z F_{\mathcal{O}(Z)}$.

The projections $\iota_Z : C_{\mathcal{O}_x} \to C_{\mathcal{O}(Z)}$ give rise to maps $\chi'_Z : F_{\mathcal{O}(Z)} \to F_{\mathcal{O}_x}$. Since for any $Z' \subseteq Z$, $\iota_Z = \iota_{Z,Z'} \circ \iota_{Z'}$, we have that $\chi'_Z = \chi'_{Z'} \circ \chi_{Z',Z}$. Consequently, there is a map $F' \to F_{\mathcal{O}_x}$. To show that this is an equality it suffices to show that for any field K and morphisms $\lambda_Z : F_{\mathcal{O}(Z)} \to K$ such that for any $Z' \subseteq Z$, $\lambda_Z = \lambda_{Z'} \circ \chi_{Z',Z}$, there is a map $\lambda : F_{\mathcal{O}_x} \to K$, satisfying $\lambda_Z = \lambda \circ \chi'_Z$.

The maps $\lambda_Z : F_{\mathcal{O}(Z)} \to K$ give rise to maps $\lambda'_Z : \operatorname{Spec} K \to \operatorname{Spec} F_{\mathcal{O}(Z)} \to C_{\mathcal{O}(Z)}$, where the image of λ'_Z is the generic point $\{\eta_Z\}$ of $C_{\mathcal{O}(Z)}$. Consequently, by Proposition 4.4.13, for any $Z' \subseteq Z$, we have $\lambda'_Z = \iota_{Z,Z'} \circ \lambda'_{Z'}$, implying there is a morphism $\lambda' : \operatorname{Spec} K \to C_{\mathcal{O}_x}$ that satisfies $\lambda'_Z = \iota_Z \circ \lambda'$ for all Z. In turn, this gives rise to a morphism $\lambda : F_{\mathcal{O}_x} \to K$, which satisfies $\lambda_Z = \lambda \circ \chi'_Z$. □

COROLLARY 4.4.16. *$F_{\mathcal{O}_x} = \varinjlim_Z \mathscr{M}(C_Z)$, where the limit is taken over connected affinoid neighborhoods $Z \subseteq Z_0$ of x.*

PROOF. This is a direct consequence of Corollary 4.4.15 and Theorem 1.7.8. □

4.5. Nice Covers of a Relative Proper Curve and Patching

We work under the hypotheses of Setting 4.4.1 and the notations we have introduced along the way. Here is a summary:

NOTATION 4.5.1. In addition to Setting 4.4.1, for any connected affinoid neighborhood $Z \subseteq Z_0$ of x, let $C_x := C_Z \times_Z \mathcal{H}(x)$, $C_Z := C \times_S Z$, $C_{\mathcal{O}(Z)} := C_{\mathcal{O}(Z_0)} \times_{\mathcal{O}(Z_0)} \mathcal{O}(Z)$, and $C_{\mathcal{O}_x} := C_{\mathcal{O}(Z_0)} \times_{\mathcal{O}(Z_0)} \mathcal{O}_x$. Moreover, we denote by π_Z, resp. $\pi_{\mathcal{O}(Z)}$, the structural morphisms $C_Z \to Z$, resp. $C_{\mathcal{O}(Z)} \to \operatorname{Spec} \mathcal{O}(Z)$.

Finally, let $f_Z : C_Z \to \mathbb{P}^{1,\mathrm{an}}_Z$, $f_{\mathcal{O}_Z} : C_{\mathcal{O}(Z)} \to \mathbb{P}^{1,\mathrm{an}}_{\mathcal{O}(Z)}$ be finite surjective morphisms such that $f^{\mathrm{an}}_{\mathcal{O}(Z)} = f_Z$, and for any connected affinoid neighborhood $Z' \subseteq Z$ of x, $f_Z \times_Z Z' = f_{Z'}$.

4.5.1. Nice covers of a relative proper curve. As in the case of $\mathbb{P}^{1,\mathrm{an}}$, in addition to Setting 4.4.1, we assume that $\dim S < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$. The reason behind this hypothesis is the same as before: it is sufficient for the existence of type 3 points on the fiber C_x (see Lemma 4.1.6).

Goal: Let $\mathcal{V}$ be an open cover of C_x in C. We construct a refinement of $\mathcal{V}$ and show that it satisfies certain properties which are necessary for patching.

(1) The construction. Remark that the finite surjective morphism $f_{Z_0} : C_{Z_0} \to \mathbb{P}^{1,\mathrm{an}}_{Z_0}$ induces a finite surjective morphism $f_x : C_x \to \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ on the corresponding fibers of x.

Without loss of generality, we may assume that $\mathcal{V}$ is an affinoid cover of C_x in C such that $\{\operatorname{Int} V : V \in \mathcal{V}\}$ is an open cover of C_x in C. Since C_x is compact, we may assume $\mathcal{V}$ is finite. Let $\mathcal{V}_x$ denote the finite affinoid cover $\mathcal{V}$ induces on C_x. Remark that $\mathcal{V}'_x := \{\operatorname{Int}_{C_x} V : V \in \mathcal{V}_x\}$ remains an open cover of C_x. Since $\mathcal{V}_x$ is an affinoid cover, for any $V \in \mathcal{V}_x$, the topological boundary $\partial_{C_x} V$ of V in C_x is finite. Consequently, for any $V \in \mathcal{V}'_x$, $\partial_{C_x} V$ is finite. Set $S' = \bigcup_{V \in \mathcal{V}'_x} \partial_{C_x} V$. This is a finite set of points on C_x.

Seeing as C_x is a connected curve, for any two points u, v of S', there exist finitely many arcs $[u,v]_i$, $i = 1, 2, \ldots, l$, in C_x connecting them (Proposition 1.8.14). Let us take a type 3 point on each $[u,v]_i$, for any two points $u, v \in S'$. We denote this set by S_1. By construction of S_1, since type 3 points are dense in C_x (Proposition 1.8.7) and $f_x^{-1}(f_x(S'))$ is a finite set, we may assume that $S_1 \cap f_x^{-1}(f_x(S')) = \emptyset$.

Since S_1 is a finite set of type 3 points in C_x, $f_x(S_1)$ is a finite set of type 3 points in the fiber $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ of x in $\mathbb{P}^{1,\mathrm{an}}_{Z_0}$. By Lemma 3.1.11, there exists a nice cover $\mathcal{D}_x$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ such that $f(S_1) = S_{\mathcal{D}_x}$ (recall this notation in Definition 3.1.15). Let $T_{\mathcal{D}_x}$ be a parity function (Definition 3.1.16) for $\mathcal{D}_x$ (it exists by Lemma 3.1.17).

LEMMA 4.5.2. *The connected components of $f_x^{-1}(D), D \in \mathcal{D}_x$, form a cover $\mathcal{U}_x$ of C_x which is nice and refines $\mathcal{V}_x$. Furthermore, $S_{\mathcal{U}_x} = f_x^{-1}(S_{\mathcal{D}_x})$, and the map $T_{\mathcal{U}_x} : \mathcal{U}_x \to \{0,1\}$, $U \mapsto f_{\mathcal{D}_x}(f_x(U))$, is a parity function for $\mathcal{U}_x$.*

PROOF. That $\mathcal{U}_x$ is a nice cover of C_x, $S_{\mathcal{U}_x} = f_x^{-1}(S_{\mathcal{D}_x})$, and $T_{\mathcal{U}_x}$ is a parity function for $\mathcal{U}_x$ has been shown in Proposition 3.1.19. It remains to show that $\mathcal{U}_x$ refines $\mathcal{V}_x$. For that, it suffices to show that $\mathcal{U}_x$ refines the open cover $\mathcal{V}'_x$ of C_x.

Let us start by proving that $S_{\mathcal{U}_x} \cap S' = \emptyset$. Suppose, by contradiction, that there exists $a \in S_{\mathcal{U}_x} \cap S' = f_x^{-1}(f_x(S_1)) \cap S'$. Then, $f_x(a) \in f_x(S_1) \cap f_x(S')$, so there exists $b \in S_1$ such that $f_x(a) = f_x(b) \in f_x(S_1) \cap f_x(S')$. Consequently, $b \in f_x^{-1}(f_x(S')) \cap S_1 = \emptyset$, which is impossible, so $S_{\mathcal{U}_x} \cap S' = \emptyset$. Considering $S_{\mathcal{U}_x} = \bigcup_{U \in \mathcal{U}_x} \partial U$ and $S' = \bigcup_{V \in \mathcal{V}'_x} \partial V$, for any $U \in \mathcal{U}_x$ and any $V \in \mathcal{V}'_x$, $\partial U \cap \partial V = \emptyset$.

Let us now show that $\mathcal{U}_x$ refines $\mathcal{V}'_x$. Suppose, by contradiction, that there exists $U \in \mathcal{U}_x$, such that for any $V \in \mathcal{V}'_x$, $U \not\subseteq V$. Let $V_j, j = 1, 2, \ldots, m$, be the elements of $\mathcal{V}'_x$ intersecting U ($m \neq 0$ seeing as $\mathcal{V}'_x$ is a cover of C_x). Then, $U \subseteq \bigcup_{j=1}^m V_j$. Considering $U \not\subseteq V_j$ and U is connected, $U \cap \partial V_j \neq \emptyset$ for all j. If $\bigcup_{j=1}^m U \cap \partial V_j$ is a single point $\{w\}$, then $w \in U \setminus \bigcup_{j=1}^m V_j$ (because the V_j are open), which is impossible seeing as $U \subseteq \bigcup_{j=1}^m V_j$. Let x_1, x_2 be two different points of $\bigcup_{j=1}^m U \cap \partial V_j$. Since $\partial U \cap \partial V_j = \emptyset$ for all j (this was shown in the paragraph above), $x_i \in \mathrm{Int}(U), i = 1, 2$.

Since U is connected, by Lemma 1.8.16, Int U is connected, so there exists an arc $[x_1, x_2]$ connecting x_1 and x_2, which is contained entirely in Int U. But then, by the construction of S_1, since $x_1, x_2 \in S'$, there exists $y \in S_1$ such that $y \in [x_1, x_2] \subseteq$ Int U. Considering $y \in S_1 \subseteq f_x^{-1}(S_{\mathcal{D}_x}) = S_{\mathcal{U}_x}$, there exists $U' \in \mathcal{U}_x$, such that $y \in \partial U'$. But then, $\partial U \cap \partial U' \neq U \cap U'$ which is in contradiction with the fact that $\mathcal{U}_x$ is a nice cover of C_x.

Thus, there must exist $V_U \in \mathcal{V}'_x$ such that $U \subseteq V_U$, implying $\mathcal{U}_x$ refines the cover $\mathcal{V}'_x$. □

The following result will be used several times in what is to come.

LEMMA 4.5.3. *Let $Z \subseteq Z_0$ be a connected affinoid neighborhood of x. Let D' be a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_Z$, such that $D' \cap F_x$ is non-empty and connected, where F_x is the fiber of x with respect to the morphism $\mathbb{P}^{1,\mathrm{an}}_Z \to Z$. Then, the connected components of $f_Z^{-1}(D')$ are connected affinoid domains of C_Z that intersect the fiber C_x of x. Moreover, if U is a connected component of $f_Z^{-1}(D')$, then $f_Z(U) = D'$.*

PROOF. Seeing as f_Z is a finite morphism, $f_Z^{-1}(D')$ is an affinoid domain in C_Z, and thus so are its connected components.

Seeing as C_Z and $\mathbb{P}^{1,\mathrm{an}}_Z$ are irreducible, they are pure-dimensional (see [**21**, Corollaire 4.14]). Seeing as f_Z is finite, its relative dimension is pure and equal to 0 (*i.e.* all its fibers are of dimension 0). By [**18**, 1.4.14(3)], the dimension of C_Z is the same as the dimension of $\mathbb{P}^{1,\mathrm{an}}_Z$. Consequently, by [**6**, Lemma 3.2.4], f_Z is open.

Let U be any connected component of $f_Z^{-1}(D')$. It is an open and a closed subset of $f_Z^{-1}(D')$. Seeing as f_Z is open and closed, $f_Z(U')$ is an open and closed subset of D'.

Considering D' is connected, this implies $D' = f_Z(U)$. Since $D' \cap F_x \neq \emptyset$, we obtain $U \cap C_x \neq \emptyset$. □

Let $Z_{\mathcal{D}} \subseteq Z_0$ be a connected affinoid neighborhood of x, such that the $Z_{\mathcal{D}}$-thickening $\mathcal{D}_{Z_{\mathcal{D}}}$ of $\mathcal{D}_x$ exists and is a $Z_{\mathcal{D}}$-relative nice cover for $\mathbb{P}^{1,\mathrm{an}}_{Z_{\mathcal{D}}}$ (see Theorem 4.1.23).

Let $Z \subseteq Z_{\mathcal{D}}$ be any connected affinoid neighborhood of x. We denote by $\mathcal{U}_Z$ the set of connected components of $f_Z^{-1}(D_Z), D \in \mathcal{D}_x$. By Lemma 4.5.3, $\mathcal{U}_Z$ is a finite affinoid cover of C_Z. Furthermore, for any $U \in \mathcal{U}_Z$, $U \cap C_x \neq \emptyset$ and $f_Z(U) \in \mathcal{D}_Z$. Remark that the nice cover $\mathcal{U}_x$ of Lemma 4.5.2 is obtained by taking the connected components of $U \cap C_x, U \in \mathcal{U}_Z$.

(2) The elements of $\mathcal{U}_Z$ intersect the fiber nicely. We show that the connected affinoid neighborhood $Z \subseteq Z_{\mathcal{D}}$ of x can be chosen such that $U \cap C_x$ is connected for any $U \in \mathcal{U}_Z$, and the same remains true when replacing Z with any connected affinoid neighborhood $Z' \subseteq Z$ of x. Let us start with a couple of auxiliary results.

LEMMA 4.5.4. *Let $Z \subseteq Z_0$ be a connected affinoid neighborhood of x. Let A_1, A_2 be two disjoint compact subsets of C_x. Then, there exist two open subsets B_1, B_2 of C_Z such that $A_i \subseteq B_i, i = 1, 2$, and $B_1 \cap B_2 = \emptyset$.*

PROOF. Let $a \in A_1$. By Lemma 4.4.9, for any $b \in A_2$, there exist an open neighborhood $N_{a,b}$ of a in C_Z, and an open neighborhood $B_{a,b}$ of b in C_Z, such that $N_{a,b} \cap B_{a,b} = \emptyset$. The family $\{B_{a,b}\}_{b \in A_2}$ forms an open cover of A_2. Considering A_2 is a compact subset of C_x, it is compact in C_Z, so there exists a finite subcover $\{B_{a,b_i}\}_{i=1}^m$ of $\{B_{a,b}\}_{b \in A_2}$. Set $N_a = \bigcap_{i=1}^m N_{a,b_i}$ and $B_a = \bigcup_{i=1}^m B_{a,b_i}$. Then, N_a, B_a are open subsets of C_Z, $A_2 \subseteq B_a$, and $N_a \cap B_a = \emptyset$.

The family $\{N_a\}_{a \in A_1}$ is an open cover of A_1. Since A_1 is compact, there exists an open subcover $\{N_{a_j}\}_{j=1}^l$. Set $B_1 = \bigcup_{j=1}^l N_{a_j}$ and $B_2 = \bigcap_{j=1}^l B_{a_j}$. Then, B_1 and B_2 satisfy the statement. □

LEMMA 4.5.5. *Let D be a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ containing only type 3 points in its boundary. Let $Z \subseteq Z_0$ be a connected affinoid neighborhood of x such that the Z-thickening D_Z exists, and for any connected affinoid neighborhood $Z' \subseteq Z$ of x, the Z'-thickening $D_{Z'}$ of D is connected. Let $U_{1,Z}, U_{2,Z}, \ldots, U_{n,Z}$ be the connected components of $f_Z^{-1}(D_Z)$.*

Then, the connected components of $f_{Z'}^{-1}(D_{Z'})$ are the connected components of $U_{i,Z} \cap C_{Z'}$, $i = 1, 2, \ldots, n$.

PROOF. By commutativity of the diagram below, $f_Z^{-1}(D_Z) \cap C_{Z'} = f_{Z'}^{-1}(D_Z \cap \mathbb{P}^{1,\mathrm{an}}_{Z'}) = f_{Z'}^{-1}(D_{Z'})$, so $f_{Z'}^{-1}(D_{Z'}) = \bigsqcup_{i=1}^n U_{i,Z} \cap C_{Z'}$ for any i. The statement follows immediately.

$$\begin{array}{ccc} C_{Z'} & \longrightarrow & C_Z \\ {\scriptstyle f_{Z'}}\downarrow & & \downarrow{\scriptstyle f_Z} \\ \mathbb{P}^{1,\mathrm{an}}_{Z'} & \longrightarrow & \mathbb{P}^{1,\mathrm{an}}_{Z} \end{array}$$

□

We can now show property (2):

PROPOSITION 4.5.6. *Let D be a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ containing only type 3 points in its boundary. Let $Z \subseteq Z_0$ be a connected affinoid neighborhood of x such that the Z-thickening D_Z exists, and for any connected affinoid neighborhood $Z' \subseteq Z$ of x, the Z'-thickening $D_{Z'}$ of D is connected.*

Let $U_{1,Z}, U_{2,Z}, \ldots, U_{n,Z}$ be the connected components of $f_Z^{-1}(D_Z)$. The affinoid neighborhood Z of x can be chosen such that:

- *$U_{i,Z} \cap C_x$ is a non-empty connected affinoid domain of C_x for all i;*
- *there is a bijection between the connected components of $f_Z^{-1}(D_Z)$ and the connected components of $f_x^{-1}(D)$ given by $U_{i,Z} \mapsto U_{i,Z} \cap C_x$;*
- *for any connected affinoid neighborhood $Z' \subseteq Z$ of x, the connected components of $f_{Z'}^{-1}(Z')$ are $U_{i,Z'} := U_{i,Z} \cap C_{Z'}, i = 1, 2, \ldots, n$.*

PROOF. Recall that the finite morphism $f_Z : C_Z \to \mathbb{P}^{1,\mathrm{an}}_Z$ induces a finite morphism $f_x : C_x \to \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ on the corresponding fibers of x. Let $L_1, L_2, \ldots, L_s$ be the connected components of $f_x^{-1}(D)$. They are connected affinoid domains of C_x.

Seeing as (follow the diagram below) $\bigsqcup_{t=1}^s L_t = f_x^{-1}(D) = f_Z^{-1}(D_Z) \cap C_x = \bigsqcup_{i=1}^n U_{i,Z} \cap C_x$, for any t, $L_t \subseteq \bigsqcup_{i=1}^n U_{i,Z}$. Since L_t is connected, there exists a unique i_t such that $L_t \subseteq U_{i_t,Z} \cap C_x$.

$$\begin{array}{ccc} C_x & \longrightarrow & C_Z \\ {\scriptstyle f_x}\downarrow & & \downarrow{\scriptstyle f_Z} \\ \mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)} & \longrightarrow & \mathbb{P}^{1,\mathrm{an}}_Z \end{array}$$

Suppose there exists i_0 such that $U_{i_0,Z} \cap C_x$ is not connected. Suppose, without loss of generality, that $L_1, L_2, \ldots, L_r$ are the connected components of $C_x \cap U_{i_0,Z}$. By Lemma 4.5.4, there exist mutually disjoint open subsets B_t of C_Z such that $L_t \subseteq B_t$, $t = 1, 2, \ldots, r$. The set $U_{i_0,Z} \setminus \bigsqcup_{t=1}^r B_t$ is a compact subset of C_Z that doesn't intersect the fiber C_x. It is a non-empty set: otherwise, $U_{i_0,Z} \subseteq \bigsqcup_{t=1}^r B_t$; seeing as $U_{i_0,Z} \cap B_t \supseteq U_{i_0,Z} \cap L_t \neq \emptyset$ for all $t = 1, 2, \ldots, r$, we obtain that $U_{i_0,Z}$ is not connected, contradiction.

Since π_Z is proper, $\pi_Z(U_{i_0,Z} \setminus \bigsqcup_{t=1}^r B_t)$ is a non-empty compact subset of Z that does not contain x. Thus, there exists a connected affinoid neighborhood $Z_1 \subseteq Z$ of x such that $\pi_Z^{-1}(Z_1) \cap (U_{i_0,Z} \setminus \bigsqcup_{t=1}^r B_t) = \emptyset$, implying $U_{i_0,Z} \cap C_{Z_1} \subseteq \bigsqcup_{t=1}^r B_t$.

Let $V_{1,Z_1}, V_{2,Z_1}, \ldots, V_{e,Z_1}$ be the connected components of $U_{i_0,Z} \cap C_{Z_1}$. By Lemma 4.5.5, V_{j,Z_1}, $j = 1, 2, \ldots, e$, are connected components of $f_{Z_1}^{-1}(D_{Z_1})$, so by Lemma 4.5.3, they all intersect the fiber C_x. Moreover, $\bigsqcup_{j=1}^e V_{j,Z_1} \cap C_x = U_{i_0,Z} \cap C_x = \bigsqcup_{t=1}^r L_t$. Hence, for any t, there exists a unique e_t such that $L_t \subseteq V_{e_t,Z_1} \cap C_x$. By the paragraph above, for any j, there exists a unique t_j, such that $V_{j,Z_1} \subseteq B_{t_j}$, hence a unique L_{t_j} contained in V_{j,Z_1}. Consequently, $r = e$ and $\{V_{j,Z_1} \cap C_x : j = 1, 2, \ldots, r\} = \{L_t : t = 1, 2, \ldots, r\}$. We may assume, without loss of generality, that $V_{j,Z_1} \cap C_x = L_j, j = 1, 2, \ldots, r$. Clearly, this induces a bijection between the connected components of $U_{i_0,Z} \cap C_{Z_1}$ and the connected components of $U_{i_0,Z} \cap C_x$, given by $V_{j,Z_1} \mapsto V_{j,Z_1} \cap C_x = L_j, j = 1, 2, \ldots, r$.

Let us show that for any connected affinoid neighborhood $Z_2 \subseteq Z_1$ of x, $V_{j,Z_1} \cap C_{Z_2}$ remains connected for all $j = 1, 2, \ldots, r$. By Lemma 4.5.5, the connected components of $V_{j,Z_1} \cap C_{Z_2}$ are connected components of $f_{Z_2}^{-1}(D_{Z_2})$, so by Lemma 4.5.3, they all intersect the fiber C_x. Seeing as $L_j = V_{j,Z_1} \cap C_x = V_{j,Z_1} \cap C_{Z_2} \cap C_x$ is connected, $V_{j,Z_1} \cap C_{Z_2}$ has to

be connected for all j. In particular, the bijective correspondence obtained above remains true when replacing Z_1 by Z_2.

We have shown that for any $i = 1, 2, \ldots, n$, there exists a connected affinoid neighborhood $Z^i \subseteq Z_0$ of x, such that the connected components $V_{j,i,Z^i}, j = 1, 2, \ldots, r_i$, of $U_{i,Z} \cap C_{Z^i}$ satisfy: (a) $V_{j,i,Z^i} \cap C_x$ is non-empty and connected for all j; (b) there is a bijection between the connected components of $U_{i,Z} \cap C_{Z^i}$ and the connected components of $U_{i,Z} \cap C_x$, given by $V_{j,i,Z^i} \mapsto V_{j,i,Z^i} \cap C_x$; (c) for any connected affinoid neighborhood $Z' \subseteq Z^i$, $V_{j,i,Z^i} \cap C_{Z'}$ remains connected, implying the connected components of $U_{i,Z} \cap C_{Z'}$ are $V_{j,i,Z^i} \cap C_{Z'}, j = 1, 2, \ldots, r_i$.

Let $Z' \subseteq \bigcap_{i=1}^n Z^i$ be a connected affinoid neighborhood of x. Since $Z' \subseteq Z$, by Lemma 4.5.5, the connected components of $f_{Z'}^{-1}(D_{Z'})$ are the connected components of $U_{i,Z} \cap C_{Z'}, i = 1, 2, \ldots, n$. By the paragraph above, these are $V_{j,i,Z^i} \cap C_{Z'}, j = 1, 2, \ldots, r_i$, $i = 1, 2, \ldots, n$, and they satisfy: (a') $V_{j,i,Z^i} \cap C_{Z'} \cap C_x$ is non-empty and connected for all j, i; (b') for any i, there is a bijection between the connected components of $U_{i,Z} \cap C_{Z'}$ and the connected components of $U_{i,Z} \cap C_x$, given by $V_{i,j,Z^i} \cap C_{Z'} \mapsto V_{i,j,Z^i} \cap C_x$, implying there is a bijection between the connected components of $U_{i,Z} \cap C_{Z'}, i = 1, 2, \ldots, n$ (*i.e.* of $f_{Z'}^{-1}(D_{Z'})$) and the connected components of $U_{i,Z} \cap C_x, i = 1, 2, \ldots, n$ (*i.e.* of $f_x^{-1}(D)$), given by $V_{j,i,Z^i} \cap C_{Z'} \mapsto V_{j,i,Z^i} \cap C_x, j, i$; (c') for any connected affinoid neighborhood $Z'' \subseteq Z'$ of x, by the paragraph above, the connected components of $f_{Z''}^{-1}(D_{Z''})$ are $V_{j,i,Z^i} \cap C_{Z'} \cap C_{Z''} = V_{j,i,Z^i} \cap C_{Z''}$, $j = 1, 2, \ldots, r_i$, $i = 1, 2, \ldots, n$. □

We have shown:

COROLLARY 4.5.7. *There exists a connected affinoid neighbohrood $Z_f \subseteq Z_{\mathcal{D}}$ of x, such that for any $U \in \mathcal{U}_{Z_f}$, $U \cap C_x$ is connected, and $\mathcal{U}_x = \{U \cap C_x : U \in \mathcal{U}_{Z_f}\}$, where $\mathcal{U}_x$ is the nice cover of C_x obtained in the statement of Lemma 4.5.2. Moreover, for any connected affinoid neighborhood $Z' \subseteq Z_f$ of x, $\mathcal{U}_{Z'} = \{U \cap C_{Z'} : U \in \mathcal{U}_{Z_f}\}$.*

REMARK 4.5.8. By Corollary 4.5.7, for any connected affinoid neighborhood $Z \subseteq Z_f$ of x, there is a bijective correspondence between $\mathcal{U}_Z$ and $\mathcal{U}_x$ given by $V \mapsto V \cap C_x$.

Consequently, we will from now on sometimes write U_Z for the unique element of $\mathcal{U}_Z$ corresponding to the element U of $\mathcal{U}_x$. In particular, $\mathcal{U}_Z = \{U_Z : U \in \mathcal{U}_x\}$.

(3) $\mathcal{U}_Z$ refines $\mathcal{V}$. Let $Z \subseteq Z_f$ be a connected affinoid neighborhood of x. Let $U_Z \in \mathcal{U}_Z$. Then, $U := U_Z \cap C_x$ is a connected affinoid domain of C_x and an element of $\mathcal{U}_x$ (recall Remark 4.5.8). By Lemma 4.5.2, there exists $V \in \mathcal{V}$, such that $U \subseteq V_x$, where V_x denotes the intersection of V with the fiber C_x. Assume $U_Z \not\subseteq V$. Then, $U_Z \backslash V$ is a non-empty compact subset of C_Z not intersecting the fiber C_x. Seeing as π_Z is proper, $\pi_Z(U_Z \backslash V)$ is a compact subset of Z not containing x. Thus, there exists a connected affinoid neighborhood $Z_1 \subseteq Z$ of x, such that $\pi_Z^{-1}(Z_1) \cap (U_Z \backslash V) = \emptyset$, *i.e.* $C_{Z_1} \cap (U_Z \backslash V) = \emptyset$, implying $C_{Z_1} \cap U_Z \subseteq V$. Clearly, the same remains true when replacing Z_1 by any connected affinoid neighborhood $Z_2 \subseteq Z_1$ of x. Considering $\mathcal{U}_Z$ is a finite cover, by repeating the same argument for all of its elements, we obtain that there exists a connected affinoid neighborhood $Z' \subseteq Z_f$, such that $\{U_Z \cap C_{Z'} : U \in \mathcal{U}_x\}$ refines $\mathcal{V}$, and the same remains true when replacing Z' with any connected affinoid neighborhood $Z'' \subseteq Z'$. By Corollary 4.5.7, $\mathcal{U}_{Z'} = \{U_Z \cap C_{Z'} : U \in \mathcal{U}_x\}$, implying $\mathcal{U}_{Z'}$ is a refinement of $\mathcal{V}$. The same remains true for any $Z'' \subseteq Z'$ as above.

We have shown:

PROPOSITION 4.5.9. *There exists a connected affinoid neighborhood $Z_r \subseteq Z_f$ of x such that for any connected affinoid neighborhood $Z \subseteq Z_r$, the cover $\mathcal{U}_Z$ refines $\mathcal{V}$.*

(4) The intersection of the elements of $\mathcal{U}_Z$ between themselves. Let $Z \subseteq Z_r$ be a connected affinoid neighborhood of x. Let $D_1, D_2 \in \mathcal{D}_x$ such that $D_1 \cap D_2 \neq \emptyset$. Set $D_1 \cap D_2 = \{y\}$. Then, $f_x^{-1}(y) := \{s_1, s_2, \ldots, s_m\}$ is a subset of $S_{\mathcal{U}_x}$. Set $D = D_1 \cap D_2$. As $Z \subseteq Z_{\mathcal{D}}$ (with $Z_{\mathcal{D}}$ as in part (1)), the Z-thickening D_Z of D is a connected affinoid domain of $\mathbb{P}^{1,\mathrm{an}}_Z$ intersecting the fiber $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ at the single type 3 point y.

Let $W_{i,Z}, i = 1, 2, \ldots, n$, be the connected components of $f_Z^{-1}(D_Z)$. By Proposition 4.5.6, we may assume that: (a) $W_{i,Z} \cap C_x$ is connected for all i; (b) there is a bijective correspondence between the connected components of $f_Z^{-1}(D_Z)$ and the points of $f_x^{-1}(y)$, given by $W_{i,Z} \mapsto W_{i,Z} \cap C_x, i = 1, 2, \ldots, n$; (c) for any connected affinoid neighborhood $Z' \subseteq Z$, the connected components of $f_{Z'}^{-1}(D_{Z'})$ are $W_{i,Z'} = W_{i,Z} \cap C_{Z'}$, $i = 1, 2, \ldots, n$.

For any $s \in f_x^{-1}(y)$, let us denote by $W_{s,Z}$ the (unique) connected component of $f_Z^{-1}(D_Z)$ containing s, (*i.e.* $W_{s,Z} \cap C_x = \{s\}$), so the connected components of $f_Z^{-1}(D_Z)$ are $W_{s,Z}, s \in f_x^{-1}(y)$.

Let $U_{j,Z}, j = 1, 2, \ldots, p$ (resp. $V_{l,Z}, l = 1, 2, \ldots, q$), be the connected components of $f_Z^{-1}(D_{1,Z})$ (resp. $f_Z^{-1}(D_{2,Z})$). Then,

$$\bigsqcup_{j=1}^{p} \bigsqcup_{l=1}^{q} U_{j,Z} \cap V_{l,Z} = f_Z^{-1}(D_{1,Z}) \cap f_Z^{-1}(D_{2,Z}) = f_Z^{-1}(D_Z) = \bigsqcup_{s \in f_x^{-1}(y)} W_{s,Z}.$$

For some j, l, let $s_{j,l} \in U_j \cap V_l$. Since $s_{j,l} \in W_{s_{j,l},Z}$, we obtain that $W_{s_{j,l},Z} \subseteq U_{j,Z} \cap V_{l,Z}$. Consequently, for any j, l, $U_{j,Z} \cap V_{l,Z} = \bigsqcup_{s \in U_j \cap V_l} W_{s,Z}$.

Let $Z' \subseteq Z$ be any connected affinoid neighborhood of x. Considering that the connected components of $f_{Z'}^{-1}(D_{1,Z'})$ (resp. $f_{Z'}^{-1}(D_{2,Z'})$) are $U_{j,Z} \cap C_{Z'}$, $j = 1, 2, \ldots, p$ (resp. $V_{l,Z} \cap C_{Z'}, l = 1, 2, \ldots, q$), the same properties remain true when replacing Z by Z'.

The same argument can be repeated for any two non-disjoint elements of the finite cover $\mathcal{D}_x$. We have shown:

PROPOSITION 4.5.10. *There exists a connected affinoid neighborhood $Z_t \subseteq Z_r$ of x such that for any connected affinoid neighborhood $Z \subseteq Z_t$, for any two non-disjoint elements D_1, D_2 of $\mathcal{D}_x$ with $D_1 \cap D_2 =: \{y\}$,*

$$f_Z^{-1}(D_{1,Z} \cap D_{2,Z}) = \bigsqcup_{s \in f_x^{-1}(y)} W_{s,Z},$$

where $W_{s,Z}$ is a connected affinoid neighborhood of C_Z, and for any s, $W_{s,Z} \cap C_x = \{s\}$. Moreover, for any connected affinoid neighborhood $Z' \subseteq Z$, the connected components of $f_{Z'}^{-1}(D_{1,Z'} \cap D_{2,Z'})$ are $W_{s,Z'} := W_{s,Z} \cap C_{Z'}, s \in f_x^{-1}(y)$.

COROLLARY 4.5.11. *Let $Z \subseteq Z_t$ be a connected affinoid neighborhood of x. For any $U, V \in \mathcal{U}_x$, $U \cap V \neq \emptyset$ if and only if $U_Z \cap V_Z \neq \emptyset$.*

PROOF. If $U_Z \cap V_Z \neq \emptyset$, then $f(U) \cap f(V) \neq \emptyset$, so by Proposition 4.5.10, $U_Z \cap V_Z \cap C_x \neq \emptyset$, *i.e.* $U \cap V \neq \emptyset$. The other direction is immediate. □

In order to invoke more easily the properties we have just shown for $\mathcal{U}_Z$, we introduce the following:

DEFINITION 4.5.12. Let $\mathcal{D}_x$ be a nice cover of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$. For a connected affinoid neighborhood Z of x, a cover $\mathcal{U}_Z$ of C_Z constructed as in (1) and satisfying properties (2), (4), will be called a *Z-relative nice cover of C_Z induced by $\mathcal{D}_x$*.

Remark that $\mathcal{U}_x := \{U \cap C_x : U \in \mathcal{U}_Z\}$ is a nice cover of C_x induced by $\mathcal{D}_x$ as in Lemma 4.5.2. Also, for any connected affinoid neighborhood $Z' \subseteq Z$ of x, $\mathcal{U}_{Z'} = \{U \cap C_{Z'} : U \in \mathcal{U}_Z\}$ is a Z'-relative nice cover of C_Z induced by $\mathcal{D}_x$.

REMARK 4.5.13. We have shown that for any open cover $\mathcal{V}$ of C_x in C, there exists a nice cover $\mathcal{D}_x$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ and a connected affinoid neighborhood Z_t of x, such that the Z_t-relative nice cover $\mathcal{U}_{Z_t}$ of C_{Z_t} induced by $\mathcal{D}_x$ refines $\mathcal{V}$. This remains true when replacing Z_t by any connected affinoid neighborhood $Z \subseteq Z_t$ of x.

4.5.2. Patching over Relative Proper Curves. We now generalize the results of Section 4.3, and obtain an application of patching on relative proper curves.

Throughout this part, let k be a non-trivially valued complete ultrametric field. We continue working with Setting 4.4.1 and Notation 4.5.1. Moreover, we assume that $\dim S < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$, so type 3 points exist in C_x.

As in the case of $\mathbb{P}^{1,\mathrm{an}}$:

NOTATION 4.5.14. Let G be a *connected* rational linear algebraic group defined over $F_{\mathcal{O}_x}$. Since $F_{\mathcal{O}_x} = \varinjlim_Z \mathscr{M}(C_Z)$ (Corollary 4.4.16), there exists a connected affinoid neighborhood $Z_G \subseteq Z_0$ of x, such that G is a connected rational linear algebraic group over $\mathscr{M}(C_{Z_G})$.

The following is an analogue of Proposition 3.2.2.

THEOREM 4.5.15. *For any open cover $\mathcal{V}$ of C_x in C, there exists a connected affinoid neighborhood $Z \subseteq Z_G$ of x and a nice cover $\mathcal{D}_x$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ such that:*

- *the Z-relative nice cover $\mathcal{U}_Z$ of C_Z induced by $\mathcal{D}_x$ refines $\mathcal{V}$;*
- *for any $(g_s)_{s\in S_{\mathcal{U}_x}} \in \prod_{s\in S_{\mathcal{U}_x}} G(\mathscr{M}_{C,s})$, there exists $(g_U)_{U\in\mathcal{U}_x} \in \prod_{U\in\mathcal{U}_x} G(\mathscr{M}(U_Z))$, satisfying: for any $s \in S_{\mathcal{U}_x}$, if U_s, V_s are the elements of $\mathcal{U}_x$ containing s, if $W_{s,Z}$ is the connected component of $U_{s,Z} \cap V_{s,Z}$ containing s, and $T_{\mathcal{U}_x}(U_s) = 0$, then $g_s \in G(\mathscr{M}(W_{s,Z}))$, and $g_s = g_U \cdot g_V^{-1}$ in $G(\mathscr{M}(W_{s,Z}))$.*

The same remains true when replacing Z by any connected affinoid neighborhood $Z' \subseteq Z$ of x.

PROOF. Seeing as for any connected affinoid neighborhood Z of x, $x \in \mathrm{Int}(Z)$, for any $u \in C_x$, $u \in \mathrm{Int}(C_Z)$, so $\mathscr{M}_{C_Z,u} = \mathscr{M}_{C,u}$.

By Remark 4.5.13, there exists a connected affinoid neighborhood $Z \subseteq Z_G$ of x and a nice cover $\mathcal{D}_x$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ which induce a refinement $\mathcal{U}_Z$ of $\mathcal{V}$ obtained as in construction (1) and satisfying properties (2) and (4) of Subsection 4.5.1. Let $\mathcal{U}_x$ denote the corresponding nice cover of C_x, $T_{\mathcal{U}_x}$ its associated parity function, and $S_{\mathcal{U}_x}$ the intersection points of the different elements of $\mathcal{U}_x$.

The proof is organized in three parts: in (a) we explore some properties of the neighborhoods of $s \in S_{\mathcal{U}_x}$; in (b) we make the descent to $\mathbb{P}^{1,\mathrm{an}}$ where the statement has already been proven; in (c) we conclude by using pull-backs.

(a) The neighborhoods of $s \in S_{\mathcal{U}_x}$. We will need the following:

LEMMA 4.5.16. *For $s \in S_{\mathcal{U}_x}$, let B_s be a neighborhood of s in C. There exists a connected affinoid neighborhood Z_1 of x such that for any $s \in S_{\mathcal{U}_x}$, if U_s, V_s are the elements of $\mathcal{U}_x$ containing s, and W_{s,Z_1} is the connected component of $U_{s,Z_1} \cap V_{s,Z_1}$ containing s, then $W_{s,Z_1} \subseteq B_s$. The neighborhood Z_1 can be chosen such that the statement remains true when replacing Z_1 by any connected affinoid neighborhood $Z_2 \subseteq Z_1$ of x.*

PROOF. Let $Z \subseteq Z_t$ be a connected affinoid neighborhood of x, where Z_t is as in Proposition 4.5.10. By Lemma 4.4.9, we may suppose that $B_s \cap S_{\mathcal{U}_x} = \{s\}$ for any $s \in S_{\mathcal{U}_x}$.

Let $y \in S_{\mathcal{D}_x}$. By Lemma 4.1.19, there exists an open neighborhood A_y of y in $\mathbb{P}^{1,\mathrm{an}}_Z$, such that $f_Z^{-1}(A_y) \subseteq \bigsqcup_{s \in f_x^{-1}(y)} B_s$. Let D_1, D_2 be the elements of $\mathcal{D}_x$ containing y. By [**25**, Lemma I.1.2], there exists a connected affinoid neighborhood $Z_1 \subseteq Z_t$ of x, such that $D_{1,Z_1} \cap D_{2,Z_1} = (D_1 \cap D_2)_{Z_1} \subseteq A_y$. Then,

$$f_{Z_1}^{-1}(D_{1,Z_1} \cap D_{2,Z_1}) \subseteq f_{Z_1}^{-1}(A_y) = f_Z^{-1}(A_y) \cap C_{Z_1} \subseteq \bigsqcup_{s \in f_x^{-1}(y)} B_s.$$

Let $W_{s,Z_1}, s \in f_x^{-1}(y)$, be the connected components of $f_{Z_1}^{-1}(D_{1,Z_1} \cap D_{2,Z_1})$, where for any $s \in f_x^{-1}(y)$, $s \in W_{s,Z_1}$ (see Proposition 4.5.10). Seeing as $\bigsqcup_{s \in f_x^{-1}(y)} W_{s,Z_1} \subseteq \bigsqcup_{s \in f_x^{-1}(y)} B_s$ and $B_s \cap S_{\mathcal{U}_x} = \{s\}$ for any $s \in f_x^{-1}(y)$, we obtain that $W_{s,Z_1} \subseteq B_s$.

Let $Z_2 \subseteq Z_1$ be any connected affinoid neighborhood of x. Seeing as the connected components of $f_{Z_2}^{-1}(D_{1,Z_2} \cap D_{2,Z_2})$ are $W_{s,Z_2} = W_{s,Z_1} \cap C_{Z_2}, s \in f_x^{-1}(y)$ (Proposition 4.5.10), all of the above remains true when replacing Z_1 by Z_2.

We obtain the statement by applying the above to all points of $S_{\mathcal{D}_x}$. □

SUMMARY 1. Let $(g_s)_{s \in S_{\mathcal{U}_x}} \in \prod_{s \in S_{\mathcal{U}_x}} G(\mathscr{M}_{C,s})$. For any $s \in S_{\mathcal{U}_x}$, there exists a neighborhood B_s of s in C, such that $g_s \in G(\mathscr{M}(B_s))$. By Lemma 4.5.16, there exists an affinoid neighborhood $Z \subseteq Z_t$ (with Z_t as in Proposition 4.5.10) of x such that for any $s \in S_{\mathcal{U}_x}$, if U_s, V_s are the elements of $\mathcal{U}_x$ containing s, then $W_{s,Z} \subseteq B_s$, where $W_{s,Z}$ is the connected component of $U_{s,Z} \cap V_{s,Z}$ containing s. Consequently, $g_s \in G(\mathscr{M}(W_{s,Z}))$. Seeing as for any connected affinoid neighborhood $Z' \subseteq Z$, $W_{s,Z'} = W_{s,Z} \cap C_{Z'}$, the same remains true when replacing Z by Z'.

(b) The descent to $\mathbb{P}^{1,\mathrm{an}}$. Let Z be as in Summary 1. The finite surjective morphism $f_Z : C_Z \to \mathbb{P}^{1,\mathrm{an}}_Z$ induces a finite field extension $\mathscr{M}(C_Z)/\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)$. Set $G' = \mathcal{R}_{\mathscr{M}(C_Z)/\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)}(G)$ - the Weil restriction of scalars from $\mathscr{M}(C_Z)$ to $\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)$ of G. This is still a connected rational linear algebraic group (see [**12**, 7.6] or [**55**, Section 1]). For any $y \in S_{\mathcal{D}_x}$, by the universal property of $\mathcal{R}$, $G'(\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_Z,y}) = G(\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_Z,y} \otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)} \mathscr{M}(C_Z))$. By Proposition 4.4.12, $G'(\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_Z,y}) = \prod_{s \in f_x^{-1}(y)} G(\mathscr{M}_{C_Z,s})$. Let $(g_s)_{s \in S_{\mathcal{U}_x}} \in \prod_{s \in S_{\mathcal{U}_x}} G(\mathscr{M}_{C_Z,s})$. This determines uniquely an element $(h_y)_{y \in S_{\mathcal{D}_x}} \in \prod_{y \in S_{\mathcal{D}_x}} G'(\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_Z,y})$.

By Theorem 4.3.13, there exists a connected affinoid neighborhood $Z' \subseteq Z$ of x, and $(h_D)_{D \in \mathcal{D}_x} \in \prod_{D \in \mathcal{D}_x} G'(\mathscr{M}(D_{Z'}))$, satisfying: for any $y \in S_{\mathcal{D}_x}$, there exist exactly two $D_y, D'_y \in \mathcal{D}_x$ containing y, $h_y \in G'(\mathscr{M}(D_{y,Z'} \cap D'_{y,Z'}))$, and if $T_{\mathcal{D}_x}(D_y) = 0$, then $h_y = h_{D_y} \cdot h_{D'_y}^{-1}$ in $G'(\mathscr{M}(D_{y,Z'} \cap D'_{y,Z'}))$. The same expression remains true for any connected affinoid neighborhood $Z'' \subseteq Z'$ of x.

For any $D \in \mathcal{D}_x$, let $U_{1,Z'}, U_{2,Z'}, \ldots, U_{n,Z'}$, be the connected components of $f_{Z'}^{-1}(D_{Z'})$. The natural map $\mathscr{M}(D_{Z'}) \otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)} \mathscr{M}(C_Z) \to \prod_{i=1}^n \mathscr{M}(U_{i,Z'})$ (obtained by pull-backs and multiplication), induces a map

$$G'(\mathscr{M}(D_{Z'})) = G(\mathscr{M}(D_{Z'}) \otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)} \mathscr{M}(C_Z)) \to \prod_{i=1}^n G(\mathscr{M}(U_{i,Z'})).$$

Let the image of $h_D \in G'(\mathscr{M}(D_{Z'}))$ by this map be the element $(g_{U_1}, g_{U_2}, \ldots, g_{U_n})$ of $\prod_{i=1}^n G(\mathscr{M}(U_{i,Z'}))$. Thus, for any $U_{Z'} \in \mathcal{U}_{Z'}$, we have an element $g_U \in G(\mathscr{M}(U_{Z'}))$.

(c) The decomposition. Finally, it remains to show that for any $U_0, U_1 \in \mathcal{U}_x$ such that $T_{\mathcal{U}_x}(U_0) = 0$, and $s \in U_0 \cap U_1$, if $W_{s,Z'}$ is the connected component of $U_{0,Z'} \cap U_{1,Z'}$ containing s, then $g_s = g_{U_0} \cdot g_{U_1}^{-1}$ in $G(\mathscr{M}(W_{s,Z'}))$, and that the same expression remains true when replacing Z' by any connected affinoid neighborhood $Z'' \subseteq Z'$ of x.

Let $y \in S_{\mathcal{D}_x}$. Let D_1, D_2 be the elements of $\mathcal{D}_x$ containing y. For any $s \in f_x^{-1}(y)$, let $W_{s,Z'}$ denote the connected component of $f_{Z'}^{-1}(D_{1,Z'} \cap D_{2,Z'})$ containing s. There is a natural bilinear map $\mathscr{M}(D_{1,Z'} \cap D_{2,Z'}) \times \mathscr{M}(C_Z) \to \prod_{s \in f_x^{-1}(y)} \mathscr{M}(W_{s,Z'})$, $(a,b) \mapsto ab$, which induces an application $\mathscr{M}(D_{1,Z'} \cap D_{2,Z'}) \otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)} \mathscr{M}(C_Z) \to \prod_{s \in f_x^{-1}(y)} \mathscr{M}(W_{s,Z'})$ (this is "compatible" with the isomorphism $\mathscr{M}_{\mathbb{P}^{1,\mathrm{an}}_Z,y} \otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)} \mathscr{M}(C_Z) \to \prod_{s \in f_x^{-1}(y)} \mathscr{M}_{C_Z,s}$, *i.e.* they are both induced by multiplication). Finally, this gives rise to a morphism $G'(\mathscr{M}(D_{1,Z'} \cap D_{2,Z'})) = G(\mathscr{M}(D_{1,Z'} \cap D_{2,Z'}) \otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)} \mathscr{M}(C_Z)) \to \prod_{s \in f_x^{-1}(y)} G(\mathscr{M}(W_{s,Z'}))$, which sends (the restriction of) h_y to (the restriction of) $(g_s)_{s \in f_x^{-1}(y)}$.

Let $U_i, i = 1, 2, \ldots, n$, (resp. V_j, $j = 1, 2, \ldots, m$) be the connected components of $f_x^{-1}(D_1)$ (resp. $f_x^{-1}(D_2)$). For any i, j, set $U_i \cap V_j = \{s_\alpha^{i,j} : \alpha = 1, 2, \ldots, l_{i,j}\}$ (if $U_i \cap V_j = \emptyset$ for some i, j, then we take $l_{i,j} = 0$). Remark that $f_x^{-1}(y) = \{s_\alpha^{i,j} : \alpha = 1, \ldots, l_{i,j}, i = 1, \ldots, n, j = 1, \ldots, m\}$. For any i, j, α, let $W_{s_\alpha^{i,j},Z'}$ be the connected component of $U_{i,Z'} \cap V_{j,Z'}$ containing $s_\alpha^{i,j}$.

For any i (resp. j), there is a restriction map $\mathscr{M}(U_{i,Z'}) \to \prod_{j=1}^m \prod_{\alpha=1}^{l_{i,j}} \mathscr{M}(W_{s_\alpha^{i,j},Z'})$ (resp. $\mathscr{M}(V_{j,Z'}) \to \prod_{i=1}^n \prod_{\alpha=1}^{l_{i,j}} \mathscr{M}(W_{s_\alpha^{i,j},Z'})$). This induces a restriction map

$$\prod_{i=1}^n \mathscr{M}(U_{i,Z'}) \to \prod_{i,j,\alpha} \mathscr{M}(W_{s_\alpha^{i,j},Z'}) \quad \left(\text{resp. } \prod_{j=1}^m \mathscr{M}(V_{j,Z'}) \to \prod_{i,j,\alpha} \mathscr{M}(W_{s_\alpha^{i,j},Z'}) \right).$$

The following commutative diagram

$$\begin{array}{ccccc}
\mathscr{M}(D_{1,Z'}) \otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)} \mathscr{M}(C_Z) & \longrightarrow & \mathscr{M}(D_{1,Z'} \cap D_{2,Z'}) \otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)} \mathscr{M}(C_Z) & \longleftarrow & \mathscr{M}(D_{2,Z'}) \otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)} \mathscr{M}(C_Z) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{i=1}^n \mathscr{M}(U_{i,Z'}) & \longrightarrow & \prod_{i,j,\alpha} \mathscr{M}(W_{s_\alpha^{ij},Z'}) & \longleftarrow & \prod_{j=1}^m \mathscr{M}(V_{j,Z'})
\end{array}$$

gives rise to the following (where $\lambda_1, \lambda_2, \lambda_3$ are isomorphisms):

$$\begin{array}{ccccc}
G'(\mathscr{M}(D_{1,Z'})) & \longrightarrow & G'(\mathscr{M}(D_{1,Z'}\cap D_{2,Z'})) & \longleftarrow & G'(\mathscr{M}(D_{2,Z'})) \\
\downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_2} & & \downarrow{\scriptstyle \lambda_3} \\
G(\mathscr{M}(D_{1,Z'})\otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)}\mathscr{M}(C_Z)) & \longrightarrow & G(\mathscr{M}(D_{1,Z'}\cap D_{2,Z'})\otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)}\mathscr{M}(C_Z)) & \longleftarrow & G(\mathscr{M}(D_{2,Z'})\otimes_{\mathscr{M}(\mathbb{P}^{1,\mathrm{an}}_Z)}\mathscr{M}(C_Z)) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{i=1}^n G(\mathscr{M}(U_{i,Z'})) & \longrightarrow & \prod_{i,j,\alpha} G(\mathscr{M}(W_{s_\alpha^{ij},Z'})) & \longleftarrow & \prod_{j=1}^m G(\mathscr{M}(V_{j,Z'}))
\end{array}$$

The factorization result is now a consequence of the analoguous result for $(h_y)_{y\in S_{\mathcal{D}_x}}$ and $(h_D)_{D\in\mathcal{U}_x}$, the relationship between $T_{\mathcal{D}_x}$ and $T_{\mathcal{U}_x}$, and the commutativity of the diagram above. More precisely, $h_y = h_{D_1}\cdot h_{D_2}^{-1}$ in $G'(\mathscr{M}(D_{1,Z'}\cap D_{2,Z'}))$, and h_y is sent to $(g_s)_{s\in f_x^{-1}(y)}$, so for any $s_\alpha^{i,j}\in f_x^{-1}(y)$, $g_{s_\alpha^{i,j}} = g_{U_i}\cdot g_{V_j}^{-1}$ in $G(\mathscr{M}(W_{s_\alpha^{i,j},Z'}))$.

Considering for any connected affinoid neighborhood $Z''\subseteq Z'$ of x, $W_{s,Z''} = W_{s,Z'}\cap C_{Z''}$ for any $s\in S_{\mathcal{U}_x}$, and $U_{Z''} = U_{Z'}\cap C_{Z''}$ for all $U\in\mathcal{U}_x$, the same expressions remain true when replacing Z' by Z''. □

4.6. The Local-Global Principles

Let k be a non-trivially valued ultrametric field. Throughout this entire section, we keep working with the hypotheses of Setting 4.4.1, and the related notations we have introduced (see Notation 4.5.1). As before, we also suppose that $\dim S < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times|\otimes_{\mathbb{Z}}\mathbb{Q}$.

Remark 4.6.1. Recall in particular that for $C_{\mathcal{O}_x} = C_{\mathcal{O}(Z_0)}\times_{\mathcal{O}(Z_0)}\mathcal{O}_x$, its function field was denoted by $F_{\mathcal{O}_x}$. It was shown in Corollary 4.4.16 that $F_{\mathcal{O}_x} = \varinjlim_Z \mathscr{M}(C_Z)$, where $\mathscr{M}$ denotes the sheaf of meromorphic functions on C, and the direct limit is taken with respect to connected affinoid neighborhoods of x in S.

4.6.1. With respect to germs of meromorphic functions. We show here the relative analogue of Theorem 3.2.11.

Recall that C_x denotes the fiber at x of the relative proper curve $C\to S$, and it is a normal irreducible projective $\mathcal{H}(x)$-analytic curve.

Theorem 4.6.2. *Let $H/F_{\mathcal{O}_x}$ be a variety and $G/F_{\mathcal{O}_x}$ a connected rational linear algebraic group acting strongly transitively over H. Then,*

$$H(F_{\mathcal{O}_x})\neq\emptyset \iff H(\mathscr{M}_{C,u})\neq\emptyset \text{ for all } u\in C_x.$$

Proof. ($\Rightarrow$): By Corollary 4.4.16, $F_{\mathcal{O}_x} = \varinjlim_Z \mathscr{M}(C_Z)$, where the limit is taken over connected affinoid neighborhoods $Z\subseteq Z_0$ of x. If $H(F_{\mathcal{O}_x})\neq\emptyset$, there exists a connected affinoid neighborhood $Z\subseteq Z_0$ of x, such that $H(\mathscr{M}(C_Z))\neq\emptyset$. Seeing as $x\in\mathrm{Int}(Z)$, we obtain that for any $u\in C_x$, $u\in\mathrm{Int}(C_Z)$, so $\mathscr{M}_{C_Z,u} = \mathscr{M}_{C,u}$. Consequently, there is a restriction morphism $\mathscr{M}(C_Z)\hookrightarrow\mathscr{M}_{C,u}$ for any $u\in C_x$, implying $H(\mathscr{M}_{C,u})\neq\emptyset$.

($\Leftarrow$): Let us now assume $H(\mathscr{M}_{C,u})\neq\emptyset$ for all $u\in C_x$. This implies that for any $u\in C_x$, there exists an open neighborhood N'_u of u in C, such that $H(\mathscr{M}(N'_u))\neq\emptyset$. Let $\mathcal{V}$ denote the open cover $(N'_u)_{u\in C_x}$ of C_x in C.

By Remark 4.5.13, there exists a connected affinoid neighborhood $Z\subseteq Z_G$ of x (Z_G as in Notation 4.5.14), and a nice cover $\mathcal{D}_x$ of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ such that they induce a refinement $\mathcal{U}_Z$ of $\mathcal{V}$ obtained as in construction (1) and satisfying properties (2) and (4) of Subsection 4.5.1. Let $\mathcal{U}_x$ denote the corresponding nice cover of C_x, $T_{\mathcal{U}_x}$ its associated parity function, and

$S_{\mathcal{U}_x}$ the intersection points of the different elements of $\mathcal{U}_x$. As $\mathcal{U}_Z$ refines $\mathcal{V}$, for any $U \in \mathcal{U}_x$ and any connected affinoid neighborhood $Z' \subseteq Z$ of x, $H(\mathscr{M}(U_{Z'})) \neq \emptyset$.

For any $U \in \mathcal{U}_x$, let us fix an element $U' \in \mathcal{V}$ for which $U_Z \subseteq U'$ for any connected affinoid neighborhood $Z \subseteq Z_t \cap Z_G$ of x (it exists seeing as $\mathcal{U}_Z$ refines $\mathcal{V}$, and for any $Z' \subseteq Z'' \subseteq Z_t$ that are connected affinoid neighborhoods of x, $U_{Z'} = U_{Z''} \cap C_{Z'}$).

(a) Finding good neighborhoods of $s \in S_{\mathcal{U}_x}$. Let $s \in S_{\mathcal{U}_x}$. Let U_s, V_s be the elements of $\mathcal{U}_x$ containing s. Then, $s \in U_s \cap V_s \subseteq U'_s \cap V'_s$. Let $N_s \subseteq U'_s \cap V'_s$ be a neighborhood of s in C_{Z_0} such that $N_s \cap S_{\mathcal{U}_x} = \{s\}$ (this is possible considering Lemma 4.4.9).

Let us fix a connected affinoid neighborhood $Z \subseteq Z_t \cap Z_G$ of x. Remark that for any $y \in S_{\mathcal{D}_x}$, $\bigsqcup_{s \in f_x^{-1}(y)} N_s$ is an open neighborhood of $f_x^{-1}(y)$ in C_{Z_0}, hence in C_Z. By [**25**, Lemma I.1.2], there exists a connected neighborhood A_y of y in $\mathbb{P}^{1,\mathrm{an}}_Z$, such that $f_Z^{-1}(A_y) \subseteq \bigsqcup_{s \in f_Z^{-1}(y)} N_s$. By Lemma 4.1.19 (and restricting to a smaller Z if necessary), we may assume that A_y is the Z-thickening A_Z of a connected affinoid domain A of $\mathbb{P}^{1,\mathrm{an}}_{\mathcal{H}(x)}$ containing only type 3 points in its boundary. By Corollary 4.1.24, we may assume that for any connected affinoid neighborhood $Z' \subseteq Z$ of x, the Z'-thickening $A_{Z'}$ of A is connected.

Let $B_{i,Z}, i = 1, 2, \ldots, m$, be the connected components of $f_Z^{-1}(A_Z)$. By Lemma 4.5.3, for any i, $B_{i,Z} \cap C_x \neq \emptyset$ and $f_Z(B_{i,Z}) = A_Z$, implying $B_{i,Z} \cap f_x^{-1}(y) \neq \emptyset$ for all i. By Proposition 4.5.6, we may assume that $B_{i,Z} \cap C_x$ is connected for all i, and for any connected affinoid neighborhood $Z' \subseteq Z$ of x, the connected components of $f_{Z'}^{-1}(A_{Z'})$ are $B_{i,Z'} = B_{i,Z} \cap C_{Z'}, i = 1, 2, \ldots, n$.

Seeing as $\bigsqcup_{i=1}^n B_{i,Z} \subseteq \bigsqcup_{s \in f_x^{-1}(y)} N_s$, for any i, there exists exactly one $s_i \in f_x^{-1}(y)$ such that $B_{i,Z} \subseteq N_{s_i}$, which implies that $B_{i,Z} \cap f_x^{-1}(y) = \{s_i\}$. As $f_x^{-1}(y) \subseteq \bigsqcup_{i=1}^n B_{i,Z}$ and $B_{i,Z} \cap f_x^{-1}(y) \neq \emptyset$, there exists a bijective correspondence between the points of $f_x^{-1}(y)$ and the connected components of $f_Z^{-1}(A_Z)$. For $s \in f_x^{-1}(y)$, let $B_{s,Z}$ be the corresponding connected component of $f_Z^{-1}(A_Z)$ containing s, so that $B_{s,Z} \subseteq N_s$. Since the connected components of $f_{Z'}^{-1}(A_{Z'})$ are $B_{s,Z} \cap C_{Z'}, s \in f_x^{-1}(y)$, the same remains true when replacing Z by Z'.

(b) The transitivity of the action. For $s \in S_{\mathcal{U}_x}$, we denote by U_s, V_s be the elements of $\mathcal{U}_x$ containing s, and suppose $T_{\mathcal{U}_x}(U_s) = 0$. Then, $s \in B_{s,Z} \subseteq U'_s \cap V'_s$, with $B_{s,Z}$ constructed as in part (a). Let $h_{U_s} \in H(\mathscr{M}(U'_s))$ and $h_{V_s} \in H(\mathscr{M}(V'_s))$. The restrictions of h_{U_s}, h_{V_s} (which we keep denoting by h_{U_s}, h_{V_s}) to $\mathscr{M}(B_{s,Z})$ induce elements of $G(\mathscr{M}(B_{s,Z}))$, and the same remains true for any connected affinoid neighborhood $Z' \subseteq Z$.

LEMMA 4.6.3. *There exists a connected affinoid neighborhood $Z_s \subseteq Z$ of x such that there exists $g_s \in G(\mathscr{M}(B_{s,Z_s}))$ satisfying $h_{U_s} = g_s \cdot h_{V_s}$ in $H(\mathscr{M}(B_{s,Z_s}))$. For any connected affinoid neighborhood $Z' \subseteq Z_s$ of x, $h_{U_s} = g_s \cdot h_{V_s}$ in $H(\mathscr{M}(B_{s,Z'}))$.*

PROOF. Set $L = \varinjlim_Z \mathscr{M}(B_{s,Z})$, where the limit is taken with respect to the connected affinoid neighborhoods $Z \subseteq Z_0$ of x. As shown in Proposition 4.5.6, we may assume that $B_{s,Z}$ is connected for all such $Z \subseteq Z_0$, so that $\mathscr{M}(B_{s,Z})$ are fields. Consequently, L is a field. The restriction morphisms $\mathscr{M}(C_Z) \hookrightarrow \mathscr{M}(B_{s,Z})$ induce an embedding $F_{\mathcal{O}_x} = \varinjlim_Z \mathscr{M}(C_Z) \hookrightarrow L$. Hence, $G(L)$ acts transitively on $H(L)$.

As $h_{U_s}, h_{V_s} \in H(L)$, there exists $g \in G(L)$, for which $h_{U_s} = g_s \cdot h_{V_s}$ in $H(L)$. Consequently, there exists a connected affinoid neighborhood Z_s of x, such that $g_s \in$

$G(\mathscr{M}(B_{s,Z_s}))$ and $h_{U_s} = g_s \cdot h_{V_s}$ in $H(\mathscr{M}(B_{s,Z_s}))$. The same remains true for any connected affinoid neighborhood $Z' \subseteq Z_s$ of x seeing as $B_{s,Z'} = B_{s,Z_s} \cap C_{Z'}$. □

By Lemma 4.5.16, there exists a connected affinoid neighborhood $Z_1 \subseteq Z$ of x, such that for any $s \in S_{\mathcal{U}_x}$, if W_{s,Z_1} is the connected component of $U_{s,Z_1} \cap V_{s,Z_1}$ containing s, then $W_{s,Z_1} \subseteq B_{s,Z}$, so $W_{s,Z_1} \subseteq B_{s,Z} \cap C_{Z_1} = B_{s,Z_1}$. Similarly, for any connected affinoid neighborhood $Z' \subseteq Z_1$, $W_{s,Z'} \subseteq B_{s,Z'}$. Consequently, for any $s \in S_{\mathcal{U}_x}$, the equality $h_{U_s} = g_s \cdot h_{V_s}$ of Lemma 4.6.3 is well defined in $H(\mathscr{M}(W_{s,Z'}))$ for any connected affinoid neighborhood $Z' \subseteq \bigcap_{s\in S_{\mathcal{U}_x}} Z_s \cap Z_1$ of x.

(c) The patching. Let us fix a connected affinoid neighborhood $Z \subseteq Z_t \cap Z_G$ of x, where Z_t is as in Remark 4.5.13, and Z_G as in Notation 4.5.14. Then, $\mathcal{U}_Z$ is a cover of C_Z, so $\{U' \in \mathcal{V} : U \in \mathcal{U}_x\}$ is an open cover of C_Z in C. For any $U' \in \mathcal{V}$, let us fix an element $h_U \in H(\mathscr{M}(U'))$. This gives rise to an element of $H(\mathscr{M}(U_{Z'}))$ for any connected affinoid neighborhood $Z' \subseteq Z$ of x, which we will keep denoting by h_U.

By part (b), there exists $(g_s)_{s\in\mathcal{U}_x} \in \prod_{s\in S_{\mathcal{U}_x}} G(\mathscr{M}_{C,s})$ and a connected affinoid neighborhood $Z_2 \subseteq Z$ of x, such that for any $s \in \mathcal{U}_x$, if U_s, V_s are the elements of $\mathcal{U}_x$ containing s, and $T_{\mathcal{U}_x}(U_s) = 0$, then $g_s \in G(\mathscr{M}(W_{s,Z_2}))$, and $h_{U_s} = g_s \cdot h_{V_s}$ in $H(\mathscr{M}(W_{s,Z_2}))$, where W_{s,Z_2} is the connected component of $U_{s,Z_2} \cap V_{s,Z_2}$ containing s. Moreover, the same remains true when replacing Z_2 by any connected affinoid neighborhood $Z' \subseteq Z_2$ of x.

By Theorem 4.5.15, we may assume that Z_2 is such that there exists an element $(g_U)_{U\in\mathcal{U}_x}$ of $\prod_{U\in\mathcal{U}_x} G(\mathscr{M}(U_{Z_2}))$, such that for any non-disjoint $U, V \in \mathcal{U}_x$ with $T_{\mathcal{U}_x}(U) = 0$, and any $s \in U \cap V$, $g_s = g_U \cdot g_V^{-1}$ in $G(\mathscr{M}(W_{s,Z_2}))$, where W_{s,Z_2} is the connected component of $U_{s,Z_2} \cap V_{s,Z_2}$ containing s. Moreover, the same remains true when replacing Z_2 with any connected affinoid neighborhood $Z' \subseteq Z_2$ of x.

For any $U \in \mathcal{U}_x$, set $h'_U = g_U^{-1} \cdot h_U \in H(\mathscr{M}(U_{Z_2}))$. If U, V are two non-disjoint elements of $\mathcal{U}_x$, and $T_{\mathcal{U}_x}(U) = 0$, for any $s \in U \cap V$, one obtains $h'_V = g_V^{-1} h_V = g_U^{-1}(g_U g_V^{-1}) h_V = g_U^{-1} g_s h_V = g_U^{-1} h_U = h'_U$ in $H(\mathscr{M}(W_{s,Z_2}))$, where W_{s,Z_2} is the connected component of $U_{Z_2} \cap V_{Z_2}$ containing s. Thus, $h'_{U|U_{Z_2}\cap V_{Z_2}} = h'_{V|U_{Z_2}\cap V_{Z_2}}$ in $H(\mathscr{M}(U_{Z_2} \cap V_{Z_2}))$.

To summarize, we have an affinoid cover $\mathcal{U}_{Z_2}$ of C_{Z_2}, and for any $U_{Z_2} \in \mathcal{U}_{Z_2}$, an element $h'_U \in H(\mathscr{M}(U_{Z_2}))$. Moreover, for any $U_{Z_2}, V_{Z_2} \in \mathcal{U}_{Z_2}$, $h'_{U|U_{Z_2}\cap V_{Z_2}} = h'_{V|U_{Z_2}\cap V_{Z_2}}$. Consequently, there exists $h \in H(\mathscr{M}(C_{Z_2}))$ such that $h_{|U_{Z_2}} = h'_U$ for any $U_{Z_2} \in \mathcal{U}_{Z_2}$. Seeing as there is an embedding $\mathscr{M}(C_{Z_2}) \hookrightarrow F_{\mathcal{O}_x}$, we obtain that $H(F_{\mathcal{O}_x}) \neq \emptyset$. □

4.6.2. With respect to valuations. Recall the notations mentioned at the beginning of this Section.

Since $\mathcal{O}_x$ is a field, there is an embedding $\mathcal{O}_x \hookrightarrow \mathcal{H}(x)$, and it induces a valuation on $\mathcal{O}_x$. We will say that this is the valuation induced by x on $\mathcal{O}_x$.

Definition 4.6.4. We denote by $V(F_{\mathcal{O}_x})$ the set of non-trivial rank one valuations v on $F_{\mathcal{O}_x}$, such that either $v_{|\mathcal{O}_x}$ is the valuation induced by x on $\mathcal{O}_x$ or $v_{|\mathcal{O}_x}$ is trivial. Set $V'(F_{\mathcal{O}_x}) = \{v \in V(F_{\mathcal{O}_x}) : v_{|\mathcal{O}_x}$ is the norm induced by x on $\mathcal{O}_x\}$. For any $v \in V(F_{\mathcal{O}_x})$, we denote by $F_{\mathcal{O}_x,v}$ the completion of $F_{\mathcal{O}_x}$ with respect to v.

Remark 4.6.5. For any non-rigid point $y \in C_x$, $\mathcal{O}_{C_x,y}$ is a field, so by Lemma 4.4.8, $\mathcal{O}_{C,y}$ is a field, and there is an embedding $\mathcal{O}_{C,y} = \mathscr{M}_{C,y} \hookrightarrow \mathcal{H}(y)$. We endow $\mathscr{M}_{C,y}$ with the valuation induced from $\mathcal{H}(y)$.

For any rigid point $y \in C_x$, $\mathcal{O}_{C_x,y}$ is a dvr, so by Lemma 4.4.8, $\mathcal{O}_{C,y}$ is a dvr. We endow $\mathscr{M}_{C,y}$ with the corresponding discrete valuation.

PROPOSITION 4.6.6. *There exists a surjective map* $\mathrm{val} : C_x \to V(F_{\mathcal{O}_x})$, $y \mapsto v_y$, *such that: if* $y \in C_x$ *is not rigid, then* $v_{y|\mathcal{O}_x}$ *induces the norm determined by* x *on* $\mathcal{O}_x$, *and* $F_{\mathcal{O}_x,v_y} = \widehat{\mathscr{M}_{C,y}}$; *if* $y \in C_x$ *is rigid, then* v_y *is discrete,* $v_{y|\mathcal{O}_x}$ *is trivial, and* $F_{\mathcal{O}_x,v_y} \hookrightarrow \widehat{\mathscr{M}_{C,y}}$.

Let $C_{x,\mathrm{nrig}}$ *denote the set of non-rigid points on* C_x. *The restriction* $\mathrm{val}_{|C_{x,\mathrm{nrig}}} : C_{x,\mathrm{nrig}} \to V'(F_{\mathcal{O}_x})$ *is a bijection.*

PROOF. **The construction of the map** val**:** Let $y \in C_x$ be a non-rigid point. Then, $\mathcal{O}_{C_x,y}$ is a field, and so is $\mathcal{O}_{C,y}$. Consequently, $\widehat{\mathscr{M}_{C,y}} = \mathcal{H}(y)$, so for any connected affinoid neighborhood Z of x, $\widehat{\mathscr{M}(C_Z)} = \widehat{\mathscr{M}_{C,y}}$, where the completion of $\mathscr{M}(C_Z)$ is taken with respect to the norm induced by the embedding $\mathscr{M}(C_Z) \hookrightarrow \mathcal{H}(y)$. Considering $F_{\mathcal{O}_x} = \varinjlim_Z \mathscr{M}(C_Z) \hookrightarrow \mathscr{M}_{C,y}$, and as $\widehat{\mathscr{M}(C_Z)} = \widehat{\mathscr{M}_{C,y}}$ for any connected affinoid neighborhood $Z \subseteq Z_0$ of x, we obtain that $F_{\mathcal{O}_x,v_y} = \widehat{\mathscr{M}_{C,y}}$. The fact that $v_{y|\mathcal{O}_x}$ is the norm determined by x on $\mathcal{O}_x$ is a direct consequence of the fact that $y \in C_x$.

Let $y \in C_x$ be a rigid point. Then, $\mathcal{O}_{C_x,y}$ is a discrete valuation ring, and by Lemma 4.4.8, so is $\mathcal{O}_{C,y}$. As $\pi(y) = x$, this induces a morphism of local rings $\mathcal{O}_x \to \mathcal{O}_{C,y}$. Furthermore, since $\mathcal{O}_x$ is a field, $\mathcal{O}_x \hookrightarrow \mathcal{O}_{C,y}^\times$. As seen above, there is an embedding $F_{\mathcal{O}_x} \hookrightarrow \mathscr{M}_{C,y}$. Let us endow $\mathscr{M}_{C,y}$ with the discrete valuation arising from the dvr $\mathcal{O}_{C,y}$. This induces a discrete valuation v_y in $F_{\mathcal{O}_x}$. That $v_{y|\mathcal{O}_x}$ is trivial is immediate from the embedding $\mathcal{O}_x \hookrightarrow \mathcal{O}_{C,y}^\times$. Clearly, this gives rise to an embedding $F_{\mathcal{O}_x,v_y} \hookrightarrow \widehat{\mathscr{M}_{C,y}}$.

The map $\mathrm{val}_{|C_{x,\mathrm{nrig}}}$**:** It remains to show that the restriction $\mathrm{val}_{|C_{x,\mathrm{nrig}}} : C_{x,\mathrm{nrig}} \to V'(F_{\mathcal{O}_x})$ is bijective. Let $v \in V'(F_{\mathcal{O}_x})$. Then, since $\mathcal{O}_x \hookrightarrow F_{\mathcal{O}_x}$, there is an embedding $\mathcal{H}(x) \hookrightarrow F_{\mathcal{O}_x,v}$. This implies that there is a morphism $F_{\mathcal{O}_x} \otimes_{\mathcal{O}_x} \mathcal{H}(x) \to F_{\mathcal{O}_x,v}$. Let C_x^{alg} denote the normal irreducible projective algebraic curve over $\mathcal{H}(x)$ whose Berkovich analytification is C_x. Its function field is $\mathscr{M}(C_x)$ by [**6**, Proposition 3.6.2].

Let x' denote the image of x via the morphism $Z_0 \to \mathrm{Spec}\ \mathcal{O}(Z_0)$, where Z_0 is as in Setting 4.4.1. Using Notation 4.4.4, by Corollary 1.6.17, $C_x = (C_{\mathcal{O}(Z_0),\kappa(x')} \times_{\kappa(x')} \mathcal{H}(x))^{\mathrm{an}}$, so $C_x^{\mathrm{alg}} = C_{\mathcal{O}(Z_0),\kappa(x')} \times_{\kappa(x')} \mathcal{H}(x)$. Seeing as $\mathcal{O}_x$ is a field, we have an embedding $\kappa(x') \hookrightarrow \mathcal{O}_x$, so $C_x^{\mathrm{alg}} = C_{\mathcal{O}_x} \times_{\mathcal{O}_x} \mathcal{H}(x)$. This means that its function field is $\mathscr{M}(C_x) = F_{\mathcal{O}_x} \otimes_{\mathcal{O}_x} \mathcal{H}(x)$.

Consequently, there are embeddings $F_{\mathcal{O}_x} \hookrightarrow \mathscr{M}(C_x) \hookrightarrow F_{\mathcal{O}_x,v}$, implying $\widehat{\mathscr{M}(C_x)}^v = F_{\mathcal{O}_x,v}$, where $\widehat{\mathscr{M}(C_x)}^v$ is the completion of $\mathscr{M}(C_x)$ with respect to v. By Proposition 3.2.14, there exists a unique (implying both injectivity and surjectivity of $\mathrm{val}_{|C_{x,\mathrm{nrig}}}$) non-rigid point $y \in C_x$ such that $\widehat{\mathscr{M}_{C,y}} = \mathcal{H}(y) = \widehat{\mathscr{M}_{C_x,y}} = F_{\mathcal{O}_x,v}$. Clearly, $v = \mathrm{val}(y)$.

□

COROLLARY 4.6.7. *With the notation of Theorem 4.6.2, if* $\mathrm{char}\ k = 0$ *or* H *is smooth, then:*

$$H(F_{\mathcal{O}_x}) \neq \emptyset \iff H(F_{\mathcal{O}_x,v}) \neq \emptyset \text{ for all } v \in V(F_{\mathcal{O}_x}).$$

PROOF. ($\Rightarrow$): Seeing as $F_{\mathcal{O}_x}$ embeds in $F_{\mathcal{O}_x,v}$ for all $v \in V(F_{\mathcal{O}_x})$, this direction is immediate.

($\Leftarrow$): We remark that $F_{\mathcal{O}_x}$ is perfect if and only if $\mathrm{char}\ k = 0$. Suppose $H(F_{\mathcal{O}_x,v}) \neq \emptyset$ for all $v \in V(F_{\mathcal{O}_x})$. By Proposition 4.6.6, for any $y \in C_x$, there exists $v \in V(F_{\mathcal{O}_x})$, such

that $F_{\mathcal{O}_x,v} \subseteq \widehat{\mathscr{M}_{C,y}}$. Hence, $H(\widehat{\mathscr{M}_{C,y}}) \neq \emptyset$ for all $y \in C_x$. If y is a non-rigid point of C_x, then $\mathcal{O}_{C,y} = \mathscr{M}_{C,y}$ is a Henselian field by [**4**, Theorem 2.3.3]. If y is rigid point, then $\mathcal{O}_{C,y}$ is a dvr that is Henselian, so by [**4**, Proposition 2.4.3], $\mathscr{M}_{C,y} = \operatorname{Frac} \mathcal{O}_{C,y}$ is Henselian. By Lemma 3.2.16, $H(\mathscr{M}_{C,y}) \neq \emptyset$ for all $y \in C_x$. Finally, by Theorem 4.6.2, this implies that $H(F_{\mathcal{O}_x}) \neq \emptyset$. □

4.6.3. Summary of results. Recall that $(k, |\cdot|)$ denotes a complete non-trivially valued ultrametric field. As usual, we denote by $\mathscr{M}$ the sheaf of meromorphic functions.

Let us summarize the main results we have shown:

THEOREM 4.6.8. *Let S, C be good k-analytic spaces such that S is normal. Suppose $\dim S < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$. Suppose there exists a surjective morphism $\pi : C \to S$ that makes C a proper flat relative analytic curve. Let $x \in S$ be such that $\mathcal{O}_x$ is a field. Set $C_x = \pi^{-1}(x)$.*

Assume there exists a connected affinoid neighboorhood Z_0 of x such that all the fibers of π on Z_0 are normal irreducible projective analytic curves. Suppose that $C_{Z_0} := \pi^{-1}(Z_0)$ is normal, and $C_{Z_0} \to Z_0$ is algebraic, i.e. the analytification of an algebraic morphism $C_{\mathcal{O}(Z_0)} \to \operatorname{Spec} \mathcal{O}(Z_0)$. Set $C_{\mathcal{O}_x} = C_{\mathcal{O}(Z_0)} \times_{\mathcal{O}(Z_0)} \mathcal{O}_x$. Let $F_{\mathcal{O}_x}$ be the function field of $C_{\mathcal{O}_x}$.

For any connected affinoid neighborhood $Z \subseteq Z_0$ of x, let us denote by C_Z the analytic space $C \times_S Z$. Then, $F_{\mathcal{O}_x} = \varinjlim_Z \mathscr{M}(C_Z)$.

Let $G/F_{\mathcal{O}_x}$ be a connected rational linear algebraic group acting strongly transitively on a variety $H/F_{\mathcal{O}_x}$. The following local-global principles hold:

- *$H(F_{\mathcal{O}_x}) \neq \emptyset \iff H(\mathscr{M}_{C,u}) \neq \emptyset$ for all $u \in C_x$;*
- *if $\operatorname{char} k = 0$ or H is smooth,*

$$H(F_{\mathcal{O}_x}) \neq \emptyset \iff H(F_{\mathcal{O}_x,v}) \neq \emptyset \text{ for all } v \in V(F_{\mathcal{O}_x}),$$

where $V(F_{\mathcal{O}_x})$ is given as in Definition 4.6.4.

The theorem above tells us that there is a local-global principle in the neighborhood of certain fibers of relative proper analytic curves. More generally, we have shown that patching is possible in the neighborhood of said fibers. Note that the statement of Theorem 4.6.8 is a local-global principle over the germs of meromorphic functions of a fixed fiber.

Considering Subsection 4.4.1 which provides an example of Setting 4.4.1, we also obtain the following theorem, which is a generalization of Corollary 3.2.18.

THEOREM 4.6.9. *Let S be a good normal k-analytic space such that $\dim S < \dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $x \in S$ be such that $\mathcal{O}_x$ is a field. Let $C_{\mathcal{O}_x}$ be a smooth geometrically irreducible projective algebraic curve over $\mathcal{O}_x$. Let $F_{\mathcal{O}_x}$ denote the function field of $C_{\mathcal{O}_x}$.*

Let $G/F_{\mathcal{O}_x}$ be a connected rational linear algebraic group acting strongly transitively on a variety $H/F_{\mathcal{O}_x}$. Then, if $\operatorname{char} k = 0$ or H is smooth:

$$H(F_{\mathcal{O}_x}) \neq \emptyset \iff H(F_{\mathcal{O}_x,v}) \neq \emptyset \text{ for all } v \in V(F_{\mathcal{O}_x}),$$

where $V(F_{\mathcal{O}_x})$ is given as in Definition 4.6.4.

REMARK 4.6.10. Just as in Chapter 3, if $\operatorname{char} k \neq 2$, the two theorems above can be applied to quadratic forms.

4.7. Examples of fields $\mathcal{O}_x$

To illustrate on which types of fields our local-global principles of this chapter can be applied, we calculate a few examples of local rings $\mathcal{O}_x$ that are fields. To do this, the key is to find a "good" basis of neighborhoods of the point x.

We denote by $(k, |\cdot|)$ a complete ultrametric field such that $\dim_{\mathbb{Q}} \mathbb{R}_{>0}/|k^\times| \otimes_{\mathbb{Z}} \mathbb{Q} = \infty$ (this condition is sufficient to guarantee the existence of type 3 points on the fiber of x). In all of the following examples, x is chosen such that $\mathcal{O}_x$ is a field.

Example 1. Suppose $S = \mathcal{M}(k)$, where $\mathcal{M}(\ \cdot\ )$ denotes the Berkovich spectrum. Then, if $S = \{x\}$, we obtain that $\mathcal{O}_x = k$, so a special case of Theorem 4.6.2 is Theorem 3.2.10.

Example 2. Let $\eta_{T,r} \in \mathbb{A}_k^{1,\mathrm{an}}$ be a type 3 point, meaning $r \notin \sqrt{|k^\times|}$. We can deduce from [**20**, 3.4.19.3], that the family of sets $L_{r_1,r_2} := \{y \in \mathbb{A}_k^{1,\mathrm{an}} : r_1 \leqslant |T|_y \leqslant r_2\}, 0 < r_1 < r < r_2$, forms a basis of neighborhoods of $\eta_{T,r}$ in $\mathbb{A}_k^{1,\mathrm{an}}$. Considering $\mathcal{O}(L_{r_1,r_2}) = \{\sum_{n\in\mathbb{Z}} a_n T^n : a_n \in k, \lim_{n\to+\infty} |a_n| r_2^n = 0, \lim_{n\to-\infty} |a_n| r_1^n = 0\}$, we obtain that

$$\mathcal{O}_x = \left\{ \sum_{n\in\mathbb{Z}} a_n T^n : a_n \in k, \exists r_1, r_2 \in \mathbb{R}_{>0},\ \text{s.t. } r_1 < r < r_2,\ \lim_{n\to+\infty} |a_n| r_2^n = 0,\ \lim_{n\to-\infty} |a_n| r_1^n = 0 \right\}$$

The norm that x induces on $\mathcal{O}_x$ is the following: $|\sum_{n\in\mathbb{Z}} a_n T^n|_x = \max_{n\in\mathbb{Z}} |a_n| r^n$.

Notation 4.7.1. For $\alpha \in k$ and $r \in \mathbb{R}_{\geqslant} 0$, let us denote by $B_k(\alpha, r)$ the closed disc in k centered at a and of radius r. Also, for $P \in k[T]$ irreducible, we denote $\mathbb{D}_k(P, r) := \{y \in \mathbb{A}_k^{1,\mathrm{an}} : |P|_y \leqslant r\}$ (resp. $\mathbb{D}_k^\circ(P, r) := \{y \in \mathbb{A}_k^{1,\mathrm{an}} : |P|_y < r\}$) the closed (resp. open) virtual disc centered at $\eta_{P,0}$ and of radius r. In particular, if there exists $\alpha \in k$ such that $P(T) = T - \alpha$, we will simply write $\mathbb{D}_k(\alpha, r)$ (resp. $\mathbb{D}_k^\circ(\alpha, r)$). When there is no risk of ambiguity, we will forget the index k.

Example 3. Suppose k is algebraically closed. Let $x = \eta_{T-\alpha,r} \in \mathbb{A}_k^{1,\mathrm{an}}$ be a type 2 point, meaning $r \in |k^\times|$. By [**20**, 3.4.19.2] that x has a basis of neighborhoods of the form $A_{R,\alpha_i,r_i,I} := \mathbb{D}(\alpha, R) \setminus \bigsqcup_{i\in I} \mathbb{D}^\circ(\alpha_i, r_i)$, where I is a finite set, $0 < r_i < r$ for all $i \in I$, $R > r$, $\alpha_i \in B(\alpha, r)$, and for any $i, j \in I, i \neq j$, we have $|\alpha_i - \alpha_j| = r$. The subset $A_{R,\alpha_i,r_i,I}$ is an affinoid domain in $\mathbb{A}_k^{1,\mathrm{an}}$. By [**24**, Proposition 2.2.6],

$$\begin{aligned}\mathcal{O}(A_{R,\alpha_i,r_i,I}) = \Big\{ &\sum_{n>0} \sum_{i\in I} \frac{a_{n,i}}{(T-\alpha_i)^n} + \sum_{n\geqslant 0} a_n (T-\alpha)^n : \\ &a_{n,i}, a_n \in k, \lim_{n\to+\infty} |a_{n,i}| r_i^{-n} = 0, i \in I, \lim_{n\to+\infty} |a_n| R^n = 0 \Big\}.\end{aligned}$$

Consequently, $f \in \mathcal{O}_x$ if and only if there exist a finite set $I \subseteq \mathbb{N}$, positive real numbers $R, r_i, i \in I$, such that $r_i < r < R$, and elements $\alpha_i \in B(\alpha, r)$, such that $|\alpha_i - \alpha_j| = r$ for any $i, j \in I, i \neq j$, satisfying $f \in \mathcal{O}(A_{R,\alpha_i,r_i,I})$. The norm induced by x is

$$\left| \sum_{n>0} \sum_{i\in I} \frac{a_{n,i}}{(T-\alpha_i)^n} + \sum_{n\geqslant 0} a_n (T-\alpha)^n \right|_x = \max_{n>0, i\in I} (|a_0|, |a_{n,i}| r^{-n}, |a_n| r^n).$$

Example 4. Suppose k is algebraically closed. Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a type 4 point, meaning it is determined by a strictly decreasing family of closed discs $\mathscr{D} := (B(a_i, r_i))_{i\in\mathbb{N}}$ in k such that $\bigcap_{i\in\mathbb{N}} B(a_i, r_i) = \emptyset$. Then, for any $Q(T) \in k[T]$, $|Q|_x = \inf_i |Q|_{\eta_{a_i,r_i}}$. Let us remark that for any $i \in \mathbb{N}$, $x \in \mathbb{D}(a_i, r_i)$. Moreover, $x \in \mathbb{D}^\circ(a_i, r_i)$. To see the last part, assume, by contradiction, that there exists $j \in \mathbb{N}$ such that $|T - a_j|_x = r_j$. Then, for any $i > j$, $\max(|a_i - a_j|, r_i) = |T - a_j|_{\eta_{a_i,r_i}} \geqslant r_j$, which is impossible seeing as $\mathscr{D}$ is strictly decreasing.

By [**20**, 3.4.19.1], the elements of $\mathscr{D}' := (\mathbb{D}(a_i, r)i))_{i\in\mathbb{N}}$ form a basis of neighborhoods of x. Finally, for any $f \in \mathcal{O}_x$, there exists $i' \in \mathbb{N}$ such that $f \in \mathcal{O}(\mathbb{D}(a_{i'}, r_{i'}))$, meaning $f = \sum_{n\in\mathbb{N}} b_n(T - a_{i'})^n$, where $b_n \in k$ for all n, and $\lim_{n\to+\infty} |b_n| r_{i'}^n = 0$. Then, for any $i \geqslant i'$, $f \in \mathcal{O}(\mathbb{D}(a_i, r_i))$. Finally, the norm induced by x is $|f|_x = \inf_{i\geqslant i'} |f|_{\eta_{a_i,r_i}}$.

Example 5. Let us fix an algebraic closure $\overline{k}$ of k. Let $x \in \mathbb{A}_k^{1,\mathrm{an}}$ be a non-rigid type 1 point. This means that there exists an element $\alpha \in \widehat{\overline{k}}\backslash\overline{k}$, such that the image of $\eta_{\alpha,0}$ with respect to the open surjective morphism $\varphi : \mathbb{A}_{\widehat{\overline{k}}}^{1,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ is x. There exists a sequence $(\alpha_i)_{i\in\mathbb{N}}$ in $\overline{k}$ such that $\lim_{i\to+\infty} \alpha_i = \alpha$. Set $r_i = |\alpha - \alpha_i|$. Then, in $\widehat{\overline{k}}$, the point $\eta_{\alpha,0}$ is determined by the strictly decreasing family of closed discs $(B_{\widehat{\overline{k}}}(\alpha_i, r_i))_{i\in\mathbb{N}}$, meaning for any $Q \in \widehat{\overline{k}}[T]$, $|Q|_{\eta_{\alpha,0}} = \inf_i |Q|_{\eta_{\alpha_i,r_i}}$. As in Example 4, by [**20**, 3.4.19.1], the family $(\mathbb{D}_{\widehat{\overline{k}}}(\alpha_i, r_i))_{i\in\mathbb{N}}$ forms a family of neighborhoods of $\eta_{\alpha,0}$ in $\mathbb{A}_{\widehat{\overline{k}}}^{1,\mathrm{an}}$.

Seeing as φ is an open morphism, $(\varphi(\mathbb{D}_{\widehat{\overline{k}}}(\alpha_i, r_i)))_{i\in\mathbb{N}}$ forms a basis of neighborhoods of the point x in $\mathbb{A}_k^{1,\mathrm{an}}$. For any i, let $P_i \in \mathbb{Q}_p[T]$ denote the minimal polynomial of α_i over k. Then, $\varphi(\mathbb{D}_{\widehat{\overline{k}}}(\alpha_i, r_i)) = \mathbb{D}_k(P_i, s_i)$, where $s_i = \prod_{P_i(\beta)=0} \max(|\alpha_i - \beta|, r_i)$ (Lemma 1.8.22).

Finally, for any $f \in \mathcal{O}_x$, there exists $i_f \in \mathbb{N}$, such that $f \in \mathcal{O}(\mathbb{D}_k(P_{i_f}, s_{i_f}))$. As seen in Lemma 4.2.8, $\mathcal{O}(\mathbb{D}_k(P_{i_f}, s_{i_f}))$ is isomorphic to $\mathcal{O}(\mathbb{D}_k(0, s_{i_f}))[S]/(P_{i_f}(S) - T)$, where $\mathcal{O}(\mathbb{D}_k(0, s_{i_f})) = \{\sum_{n\in\mathbb{N}} b_n T^n : b_n \in k, \lim_{n\to+\infty} |b_n| s_{i_f}^n = 0\}$.

Remark that for any $i \geqslant i_f$, $f \in \mathcal{O}(\mathbb{D}_k(P_i, s_i))$. The norm induced by x on $\mathcal{O}_x$ is given as follows: $|f|_x = \inf_{i\geqslant i_f} |f|_{\eta_{P_i,s_i}}$.

Example 6. Let S, T denote the coordinates of $\mathbb{A}_k^{2,\mathrm{an}}$, and $\varphi : \mathbb{A}_k^{2,\mathrm{an}} \to \mathbb{A}_k^{1,\mathrm{an}}$ the projection to $\mathbb{A}_k^{1,\mathrm{an}}$ with coordinate T. Let $s, t \in \mathbb{R}_{>0}$ be such that $t \notin \sqrt{|k^\times|}$ and $s \notin \sqrt{|\mathcal{H}(\eta_{T,t})^\times|}$. Let $x \in \mathbb{A}_k^{2,\mathrm{an}}$ denote a point such that $|T|_x = t, |S|_x = s$. Then, $x \in \varphi^{-1}(\eta_{T,t})$, and considering the condition on s, x is a type 3 point on the fiber of $\eta_{T,t}$. In particular, x is the only point of $\mathbb{A}_k^{2,\mathrm{an}}$ that satisfies $|T|_x = t, |S|_x = s$.

By Lemma 4.1.19 and Example 2, a basis of neighborhoods of x is given by $\{y \in \mathbb{A}_k^{1,\mathrm{an}} : t_1 \leqslant |T|_y \leqslant t_2, s_1 \leqslant |S|_y \leqslant s_2\}$, where $0 < t_1 < t < t_2$, $0 < s_1 < s < s_2$. Consequently,

$$\mathcal{O}_x = \Big\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} T^m S^n : a_{m,n} \in k, \exists t_1, t_2, s_1, s_2 \in \mathbb{R}_{>0}, \text{ s.t. } t_1 < t < t_2, s_1 < s < s_2, \lim_{m+n\to+\infty} |a_{m,n}| t_2^m s_2^n = 0, \lim_{m+n\to-\infty} |a_{m,n}| t_1^m s_1^n = 0 \Big\}.$$

The norm on $\mathcal{O}_x$ is given by: $|\sum_{m,n\in\mathbb{Z}} a_{m,n} T^m S^n|_x = \max_{m,n\in\mathbb{Z}} |a_{m,n}| t^m s^n$.

By iterating the above, we can calculate the local ring of any point $x \in \mathbb{A}_k^{l,\mathrm{an}}, l \in \mathbb{N}$, satisfying similar properties.

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Résumé. Field patching, introduced by Harbater and Hartmann in [**33**], and extended by the aforementioned authors and Krashen in [**34**], has recently seen numerous applications. We present an extension of this technique to the setting of Berkovich analytic geometry and applications to the local-global principle.

In particular, we show that this adaptation of patching can be applied to Berkovich analytic curves, and as a consequence obtain local-global principles over function fields of curves defined over complete ultrametric fields. Because of the connection between the points of a Berkovich analytic curve and the valuations that its function field can be endowed with, one of these local-global principles is given with respect to completions, thus evoking some similarity with more classical versions. As an application, we obtain local-global principles for quadratic forms and results on the u-invariant. These findings generalize those of [**34**].

As a starting point for higher-dimensional patching in the Berkovich setting, we show that this technique is applicable around certain fibers of a relative Berkovich analytic curve. As a consequence, we prove a local-global principle over the germs of meromorphic functions on said fibers. By showing that said germs of meromorphic functions are algebraic, we also obtain local-global principles over function fields of algebraic curves defined over a larger class of ultrametric fields.

Résumé. **Recollement sur les espaces de Berkovich et principe local-global.** Le recollement sur les corps, introduit par Harbater et Hartmann dans [**33**], et étendu par ces auteurs et Krashen dans [**34**], a récemment trouvé de nombreuses applications. Nous présentons ici une extension de cette technique au cadre de la géométrie analytique de Berkovich et des applications au principe local-global.

Nous montrons que cette adaptation du recollement peut s'appliquer aux courbes analytiques de Berkovich, et par conséquent obtenons des principes locaux-globaux sur les corps de fonctions de courbes définies sur des corps ultramétriques complets. Grâce à la connexion entre les points d'une courbe analytique de Berkovich et les valuations dont on peut munir son corps de fonctions, nous obtenons un principe local-global par rapport à des complétés du corps de fonctions considéré, ce qui présente une ressemblance avec des versions plus classiques. En application, nous établissons des principes locaux-globaux dans le cas plus précis des formes quadratiques et en déduisons des bornes sur l'u-invariant de certains corps. Nos résultats généralisent ceux de [**34**].

Comme point de départ pour le recollement en dimension supérieure dans un cadre d'espaces de Berkovich, nous montrons que cette technique peut s'appliquer autour de certaines fibres d'une courbe analytique relative. Nous l'utilisons ensuite pour démontrer un principe local-global sur les germes des fonctions méromorphes sur ces fibres. En montrant que ces germes de fonctions méromorphes sont algébriques, nous obtenons aussi des principes locaux-globaux sur les corps de fonctions des courbes algébriques définies sur une famille plus vaste de corps ultramétriques.