



An analysis of nonlinear thin structures

Antonio Battista

► To cite this version:

Antonio Battista. An analysis of nonlinear thin structures. Mechanics [physics.med-ph]. Université de La Rochelle; Università degli studi (L'Aquila, Italie), 2019. English. NNT: 2019LAROS017 . tel-02516018

HAL Id: tel-02516018

<https://theses.hal.science/tel-02516018>

Submitted on 23 Mar 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



UNIVERSITÉ DE LA ROCHELLE

en cotutelle avec

**UNIVERSITÀ DEGLI STUDI
DELL'AQUILA**



ÉCOLE DOCTORALE EUCLIDE (URL)
CORSO DI DOTTORATO IN MATEMATICA E MODELLI
LaSIE - M&MoCS

THÈSE présentée par :

Antonio BATTISTA

Soutenance prévue : **29 novembre 2019**

pour obtenir le grade de : **Docteur de l'université de La Rochelle**

Discipline : **Mécanique**

An analysis of nonlinear thin structures

JURY :

Aziz HAMDOUNI
Francesco DELL'ISOLA
Olivier MILLET
Pierre SEPPECHER
Olivier THOMAS
Luca PLACIDI
Emilio TURCO
Giuseppe ROSI

Professeur, Université de La Rochelle, Directeur de thèse
Professeur, Università di Roma La Sapienza, Directeur de thèse
Professeur, Université de La Rochelle, Co-directeur de thèse
Professeur, Université de Toulon
Professeur, ENSAM Lille, Rapporteur
Associate Professor, UNINETTUNO
Associate Professor, UNISS, Rapporteur
MCF, Université Paris-Est Créteil Val de Marne

Contents

1	Résumé	5
1.1	Modèles de poutres non-linéaires	5
1.2	Analyse asymptotique et justification formelle d'un modèle de membrane	11
2	Introduction	15
3	Non linear Euler and Timoshenko beam theories	18
3.1	Introduction	18
3.2	Large deformations of 1D microstructured systems modeled as generalized Timoshenko beams	23
3.3	Large deformations of Timoshenko and Euler beams under dis- tributed load	44
3.4	Extensible beam models in large deformation under distributed loading: a numerical study on multiplicity of solutions	63
4	Nonlinear dynamics of uniformly loaded Elastica	80
4.1	Introduction	80
4.2	Nonlinear dynamics of uniformly loaded Elastica: Experimental and numerical evidence of motion around curled stable equilib- rium configurations	81
5	Asymptotic methods and justification of the Landau-lifshitz membrane model	91
5.1	Introduction	91
5.2	An asymptotic membrane model for wrinkling of very thin films .	94
6	Conclusion	113
	Acknowledgements	115
	Bibliography	116

Chapter 1

Résumé

Le thème principal de cette thèse est l'étude du comportement mécanique de certaines structures minces dans le domaine non-linéaire. Globalement, le travail effectué peut être divisé en deux parties:

- une première partie qui concerne l'analyse de modèles non-linéaires de poutres, généralisant sur différents aspects les modèles de poutre d'Euler et de Timoshenko, en suivant la démarche initiée dans [12];
- une deuxième partie où on justifie formellement par méthode asymptotique un modèle de membrane original, en suivant la démarche présentée dans [29, 30, 31, 24, 25].

Ce travail de thèse est présenté sous forme d'une collection d'articles publiés au cours du doctorat, effectué en co-tutelle entre l'université de La Rochelle (France) et l'Université de L'Aquila (Italie).

1.1 Modèles de poutres non-linéaires

Une des premières tentatives de modélisation mathématique des poutres a été effectuée par Galileo Galilei dans son travail "*Discorsi e Dimostrazioni Matematiche Intorno a Due Nuove Scienze*" en 1638. Cependant, la première formulation mathématique complète du problème est due à Euler [18] dont le travail a été continué ensuite par les frères Bernoulli et par Lagrange [8, 27]. D'autres scientifiques illustres se sont également, par la suite, consacrés à l'étude de ce problème. Il est important de mentionner entre autres Max Born, qui s'est intéressé au problème dans sa thèse de doctorat en 1906 [9]. La nécessité de considérer des poutres dans lesquelles l'épaisseur ne peut être complètement négligée, et qui sont donc sujettes à des déformations en cisaillement suivant l'épaisseur, a conduit, au début du 20ème siècle, à la formulation du modèle de poutre avec cisaillement de Timoshenko ([37, 38]), dont de nombreuses justifications existent à partir de l'élasticité 3D (voir par exemple [32, 33]).

Les deux modèles de poutre mentionnés ci-dessus, ont joué et jouent toujours, un rôle fondamental dans la modélisation mathématique et le comportement mécanique des poutres. Ils suscitent toujours l'intérêt de nombreux chercheurs qui s'intéressent à leur généralisation et leur utilisation dans des problèmes variés structuraux et micro-structuraux (voir par exemple [2, 3]).

Cependant, malgré cela, il n'existe pas à notre connaissance dans la littérature de résultats rigoureux sur l'étude de l'ensemble des formes d'équilibre possibles et de leur stabilité, pour des poutres élastiques en grands déplacements soumises à une charge uniformément répartie dans le cadre de la théorie de poutre d'Euler, et même dans le cas d'un chargement ponctuel dans le cas du modèle de Timoshenko. Il existe néanmoins des résultats numériques intéressants où il est possible de trouver des solutions relativement proches de celle que l'on va étudier mathématiquement dans le cadre de cette thèse (voir par exemple [34, 19]).

D'autre part, il est important de souligner que la formulation du problème de poutre non linéaire étudié dans la première partie de cette thèse n'a jamais fait l'objet d'un traitement mathématique rigoureux. Les résultats obtenus dans ce travail de doctorat représentent une suite du travail de Della Corte et al. [12], où de nouvelles solutions d'équilibre pour une poutre d'Euler encastrée et soumise à une charge répartie ont été étudiés. La caractérisation des extrêmes de l'énergie du système a permis de mettre en évidence l'existence de certaines solutions "bouclées". Enfin, des conditions suffisantes de stabilité et d'instabilité des solutions du problème d'Euler-Lagrange ont été établies.

Dans le premier chapitre de cette thèse, nous étudions une généralisation non-linéaire naturelle du modèle de poutre de Timoshenko avec cisaillement et nous montrons qu'il peut décrire l'énergie de déformation homogénéisée d'un milieu continu 1D composé d'une microstructure simple. Ce travail est synthétisé dans l'article "Large deformations of 1D microstructured systems modeled as generalized Timoshenko beams" publié dans le journal *Zeitschrift für angewandte Mathematik und Physik* [6] et figurant au chapitre 1. Dans celui-ci, nous prouvons le caractère bien posé du problème variationnel correspondant au cas d'une force appliquée générique, et nous discutons les problèmes de la régularité des solutions. L'idée de partir d'une microstructure élémentaire découle de la possibilité, aujourd'hui, d'imprimer des objets avec des imprimantes 3D à des coûts pratiquement insignifiants, afin de pouvoir effectuer une validation expérimentale de la théorie obtenue (travail en cours). Dans [6], nous montrons que que l'homogénéisation formelle de la microstructure représentée à la figure 1.1 permet de retrouver l'énergie d'une poutre de Timoshenko non-linéaire:

$$\int_0^L \left[\frac{K_1}{2} (\phi'(s))^2 + \frac{K_2}{2} (\phi(s) - \theta(s))^2 \right] ds \quad (1.1)$$

où k_1 et k_2 sont respectivement la raideur de cisaillement et la raideur de flexion, ϕ la variable cinématique qui représente le cisaillement et s l'abscisse curviligne le long de la poutre. L'angle entre la tangente à la poutre déformée et l'axe de référence horizontal, noté θ , caractérise la flexion de la poutre. Si on ajoute une force $\vec{F} = F_1 \hat{e}_1 + F_2 \hat{e}_2$, nous obtenons le problème variationnel:

$$\begin{aligned} \min \quad & \left\{ E(\phi, \theta) := \int_0^L \left[\frac{K_1}{2} \phi'^2 + \frac{K_2}{2} (\phi - \theta)^2 + F_1 \cos \theta + F_2 \sin \theta \right] ds \right. \\ & \left. \theta \in L^2[0, L], \quad \phi \in H^1[0, L], \quad \phi(0) = 0 \right\} \end{aligned} \quad (1.2)$$

dont le caractère bien posé est démontré dans [6]. De plus, nous considérons

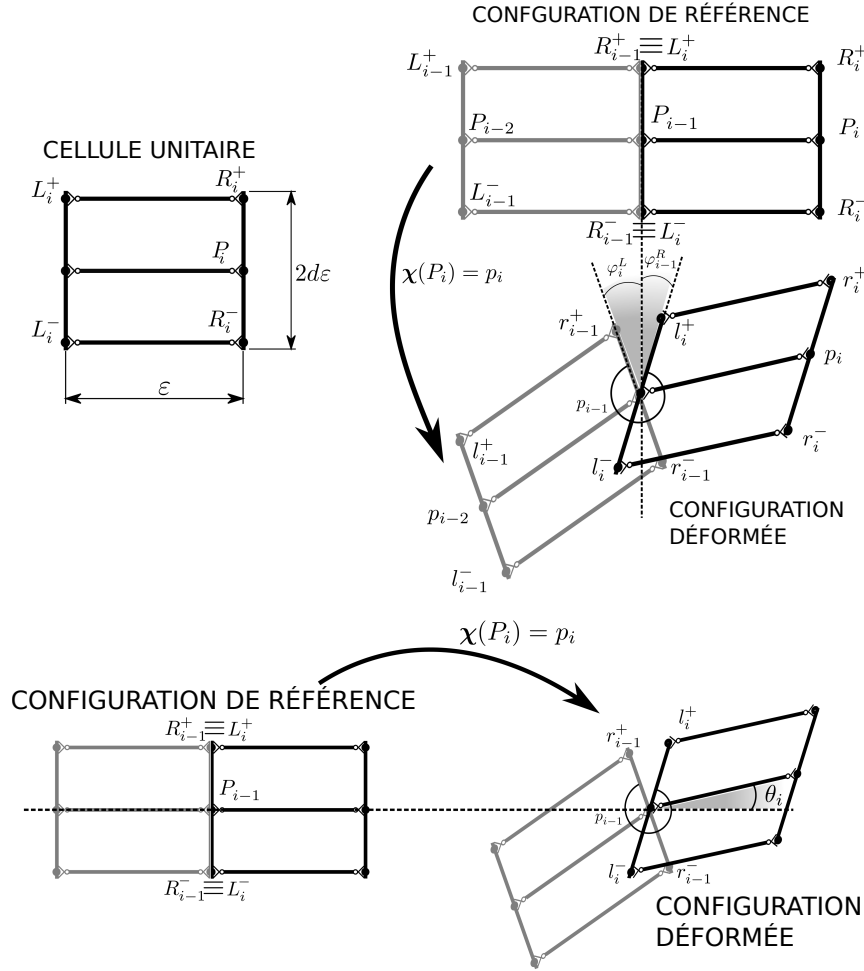


Figure 1.1: Exemple de microstructure d'une poutre déformable par cisaillement reproduisant (après linéarisation) le modèle de poutre de Timoshenko. En haut: la cellule unitaire et une représentation graphique de la variation de l'angle de la section ϕ . En bas: représentation graphique de la variation de l'angle θ entre la tangente à la poutre déformée et l'axe de référence horizontal. Une homogénéisation heuristique de cette microstructure élémentaire conduit à l'énergie (1.1).

un modèle "régularisé" dans lequel la dérivée première de θ apparaît également dans l'énergie et nous analysons son comportement par rapport au modèle décrit ci-dessus au moyen de simulations numériques.

Dans le travail "*Large deformations of Timoshenko and Euler beams under distributed load*" publié dans la revue *Zeitschrift für angewandte Mathematik und Physik* [11], l'attention se porte sur l'aspect de la multiplicité des solutions avec l'augmentation de la force appliquée, d'une part dans le cas de la poutre extensible de Timoshenko, et d'autre part dans le cas limite d'une rigidité infinie dans le cadre du modèle d'Euler conduisant à un modèle inextensible. Les équations d'Euler-Lagrange sont déduites pour les différents modèles et des propriétés de monotonie sont établies. Nous avons choisi de résoudre le problème aux limites pour une poutre encastree-libre avec une technique de tir, au moyen d'un problème de Cauchy paramétrique. Nous rappelons que le problème paramétrique de Cauchy pour une poutre d'Euler peut s'écrire sous la forme simplifiée suivante :

$$\mathcal{P}_k = \begin{cases} \theta'' = -b(1-s)\cos\theta \\ \theta(0) = 0 \\ \theta'(0) = k \end{cases} \quad (1.3)$$

où b est la valeur de la force morte adimensionnée, considérée ici transversale à la poutre dans sa configuration de référence. Les solutions qui satisfont aux conditions aux limites considérées sont obtenues en faisant varier le paramètre k (ici, la longueur de la poutre est adimensionnée et par conséquent nous considérons une poutre de longueur unitaire). Dans le cas d'une poutre soumise à aucun effort ni moment en $s = 1$, nous cherchons des solutions vérifiant $\theta(1) = 0$.

Les variations de $c(k) := \theta'(1)$ en fonction de $k := \theta'(0)$ sont représentées numériquement sur la figure 1.2 pour deux valeurs différentes de b . Le nombre de solutions du problème aux limites (1.3) est égal au nombre d'intersections de la courbe $c(k)$ avec l'axe horizontal. Les 3 solutions possibles existant pour $b = 60$ sont représentées à la figure 1.3 pour une poutre d'Euler inextensible (avec une rigidité en extension infinie).

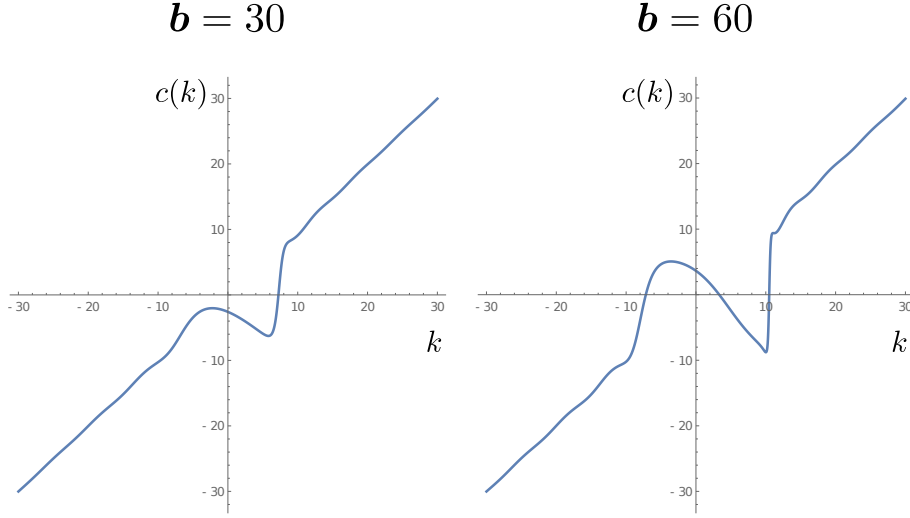


Figure 1.2: Variations de $c(k)$ obtenus numériquement pour deux valeurs différentes de la charge distribuée adimensionnelle dans le cas d'un faisceau d'Euler inextensible. Le problème aux limites (1.3) admet une solution pour $b = 30$ et trois solutions pour $b = 60$.

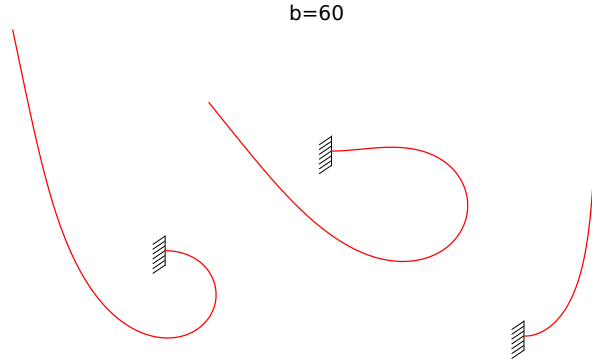


Figure 1.3: Allure générale des 3 positions d'équilibre possibles relatives au cas $b = 60$ pour une poutre d'Euler inextensible, correspondant au deuxième graphique de la figure 1.2 .

Dans l'article [11], nous sommes intéressés à établir certaines propriétés des branches apparaissant lorsque la charge augmente. Des simulations numériques ont également permis de valider les théorèmes établis et de vérifier certaines conjectures. Il a été montré en particulier qu'il ne peut exister de solutions tournant autour de l'origine comme celle représentée sur la figure 1.4, résultat basé sur l'utilisation de la fonction de Liapunov.

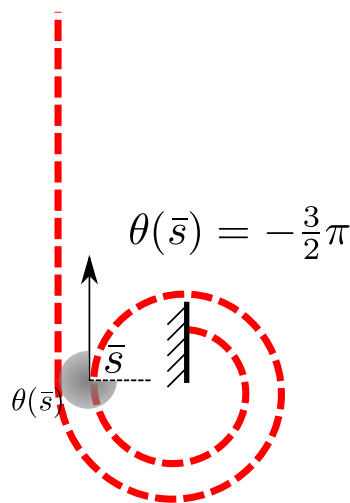


Figure 1.4: Exemple de forme d'équilibre "impossible" pour une poutre d'Euler sous charge répartie. En utilisant un argument basé sur la fonction de Liapunov, on montre qu'une telle configuration, dans laquelle les poutres tournent deux fois autour de la pince, ne peut pas être une configuration d'équilibre.

Dans le troisième article du premier chapitre "*Extensible beam models in large deformation under distributed loading: a numerical study on multiplicity of solutions*" publié dans [13], des simulations numériques ont permis d'effectuer une étude paramétrique des solutions des modèles de poutres extensibles d'Euler et Timoshenko. Dans ce cas, l'application des outils mathématiques utilisés dans les travaux précédent [6, 11] est plus compliquée car le problème variationnel n'est pas autonome. De ce fait, les outils classiques d'analyse fonctionnelle et en particulier les méthodes directes de calcul des variations ne peuvent pas être appliqués. Le problème de l'existence et de la stabilité des configurations d'équilibre pour les modèles d'Euler et Timoshenko *extensibles* en grands déplacements sous charge répartie est toujours ouvert. Certaines considérations heuristiques suggèrent que l'augmentation de l'extensibilité devrait favoriser la stabilité sur l'instabilité.

Dans le deuxième chapitre de la thèse, nous présentons l'article "*Nonlinear dynamics of uniformly loaded Elastica: Experimental and numerical evidence of motion around curled stable equilibrium configurations*" publié dans ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik [5], dans lequel nous avons réalisé une étude dynamique sur une poutre d'Euler inextensible. Les résultats numériques sont comparés aux résultats expérimentaux obtenus sur des configurations simples. Les simulations numériques ont été effectuées à partir d'une discrétisation du faisceau proposée par Henky [26] (Fig. 1.5) dont la Γ -convergence vers une poutre d'Euler inextensible en grands déplacements a été rigoureusement justifiée (voir [1]). Les équations de mouvement ont été obtenues sous forme

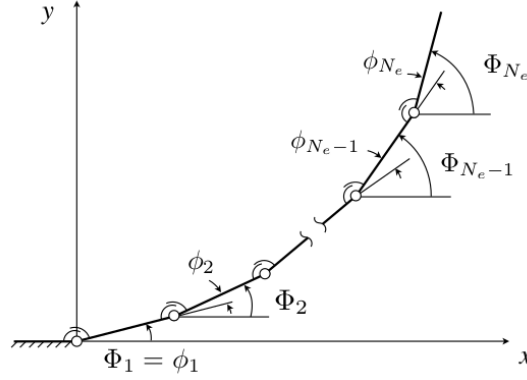


Figure 1.5: Discrétisation de "type Henky" d'une poutre d'Euler.

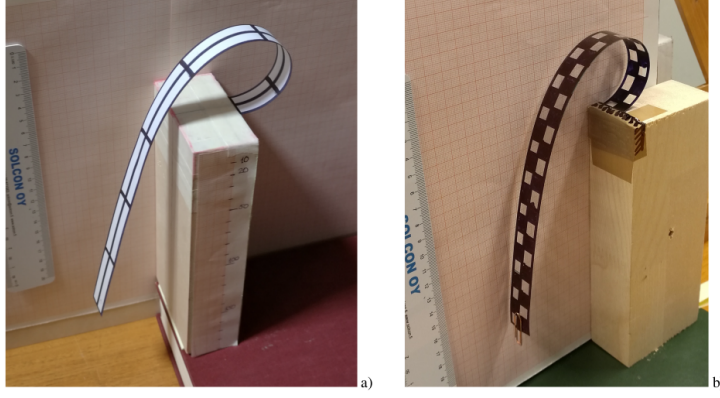


Figure 1.6: Dispositif expérimental: a) poutre en papier. b) poutre en PET avec une force concentrée exercée à l'extrémité libre.

d'équations d'Euler-Lagrange à partir du Lagrangien du système discret équivalent composé de tiges et de ressorts en rotation. Le dispositif expérimental est représenté dans la figure 1.6 où les poutres en papier et en PET (téréphtalate de polyéthylène) sont dans une des configurations (stables) décrites plus en détails dans [12]. De petites oscillations autour de la position d'équilibre ont été observées autour de la configuration stable représentée sur la figure 1.6, et le passage de ce minimum d'énergie local au minimum global a également été obtenu.

1.2 Analyse asymptotique et justification formelle d'un modèle de membrane

Dans le dernier chapitre de cette thèse, nous proposons de justifier rigoureusement par développements asymptotiques, bien que de façon formelle, le modèle de membrane de Landau et Lifschitz obtenu de façon tout à fait heuristique dans [28]. Ce travail a fait l'objet de la publication "*An asymptotic membrane model*

for wrinkling of very thin films” dans *Continuum Mechanics and Thermodynamics* [7].

De façon générale, lorsqu’un paramètre entrant dans la description d’un problème physique est ”petit” par rapport aux autres, la technique des développements asymptotiques permet d’extraire les effets les plus importants en écartant ceux d’ordre inférieur. Elle permet de justifier ainsi rigoureusement les modèles existant en en précisant clairement les domaines de validité, ou encore de construire de nouveaux modèles qui n’existent pas dans la littérature. Le modèle asymptotique obtenu est beaucoup plus simple que le modèle tridimensionnel de départ, bien que capable (d’après les comparaisons expérimentales) de capturer le comportement général du système physique étudié.

Depuis les années 60, de nombreux travaux ont été consacrés à la justification rigoureuse des modèles bidimensionnels de plaques et de coques minces élastiques (voir par exemple [21, 22, 10, 16, 17, 15, 23, 35, 36]), en utilisant principalement des formulations faibles ou variationnelles du problème tridimensionnel élastique de départ. Les travaux plus récents de A. Hamdouni et O. Millet [29, 30, 31, 24, 25], basés sur une formulation locale et une adimensionnalisation du problème tridimensionnel, permettent de définir précisément le domaine de validité des modèles asymptotiques obtenus en fonction du niveau des efforts exercés et de la géométrie de la structure, et également d’en construire de nouveaux.

Dans [7], nous nous sommes intéressés à justifier de façon rigoureuse, par développements asymptotiques des équations non linéaires de l’élasticité tridimensionnelle, un modèle de membrane bidimensionnel capable de décrire les motifs de plissement des membranes précontraintes. L’approche que nous avons utilisée est basée sur une formulation locale et une adimensionnalisation des équations d’équilibre, en suivant l’approche développée dans [29]-[25]. Le modèle de membrane ainsi obtenu est similaire à celui obtenu de façon heuristique par Landau et Lifshitz à partir des équations Föppl-von Kármán [28]. A notre connaissance, il n’a jamais fait l’objet d’une justification rigoureuse dans la littérature et ses propriétés n’ont pas été étudiées de façon précise.

D’après l’équation (14.4-5) p.52 de Landau et Lifshitz [28], la déformation d’une plaque située dans le plan (x_1, x_2) et soumise à de grands déplacements peut être décrite par les équations suivantes :

$$D\Delta^2 u_3 - h \frac{\partial}{\partial x_\beta} \left(\tilde{n}_{\alpha\beta} \frac{\partial u_3}{\partial x_\alpha} \right) = P \quad \alpha, \beta \in \{1, 2\} \quad (1.4)$$

$$\frac{\partial \tilde{n}_{\alpha\beta}}{\partial x_\beta} = 0 \quad (1.5)$$

où $D = \frac{Eh^3}{12(1-\nu^2)}$ représente la rigidité à la flexion (E est le module de Young et ν le coefficient de Poisson), h l’épaisseur de la plaque, u_3 le déplacement transversal, $\tilde{n}_{\alpha\beta}$ le tenseur non linéaire des contraintes membranaires et P la charge appliquée. Le premier terme de (1.4) est négligeable pour des plaques très minces ($h \ll 1$) ou pour des contraintes membranaires très importantes. Dans ce cas, le problème dégénère en un problème de membrane pure (voir

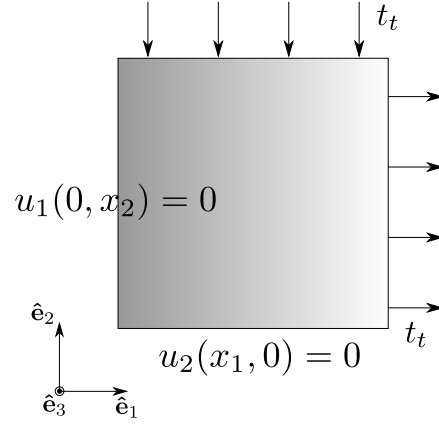


Figure 1.7: Conditions aux limites pour le problème dans le plan donnant lieu à un problème hyperbolique.

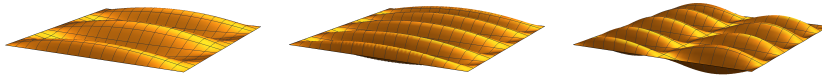


Figure 1.8: Modes possibles pour le déplacement hors plan d'une membrane associée aux valeurs propres déterminés à partir du problème de Sturm-Liouville. De gauche à droite: mode (1, 3), mode (1, 5) et mode (2, 4).

équation (14.8) p.53 dans [28]) :

$$h \frac{\partial}{\partial x_\beta} \left(n_{\alpha\beta} \frac{\partial u_3}{\partial x_\alpha} \right) = -P \quad (1.6)$$

$$\frac{\partial n_{\alpha\beta}}{\partial x_\beta} = 0 \quad (1.7)$$

où $n_{\alpha\beta}$ est la partie linéaire de $\tilde{n}_{\alpha\beta}$.

En plus de fournir une justification formelle mais rigoureuse à ce modèle, nous montrons dans [7], en se basant sur des simulations numériques, qu'il est particulièrement bien adapté pour décrire le froissement de membranes précontraintes très minces. Le point crucial est de constater que le problème (1.6)-(1.7) peut devenir hyperbolique lorsque le tenseur des contraintes membranaires $n_{\alpha\beta}$ est lui-même hyperbolique. De plus, le problème de flexion hors plan (1.6) avec les conditions aux limites (1.7), peut être considéré comme un problème bidimensionnel de Sturm-Liouville pour une membrane simplement supportée, dont certaines solutions sont représentées à la Fig. 1.8. Enfin, on peut constater à la figure 1.9b) que la solution numérique du problème (1.6)-(1.7) contenant un motif de plis typiques d'une membrane étirée et cisailée, concorde très bien avec la photo d'une membrane ridée Fig 1.9a).

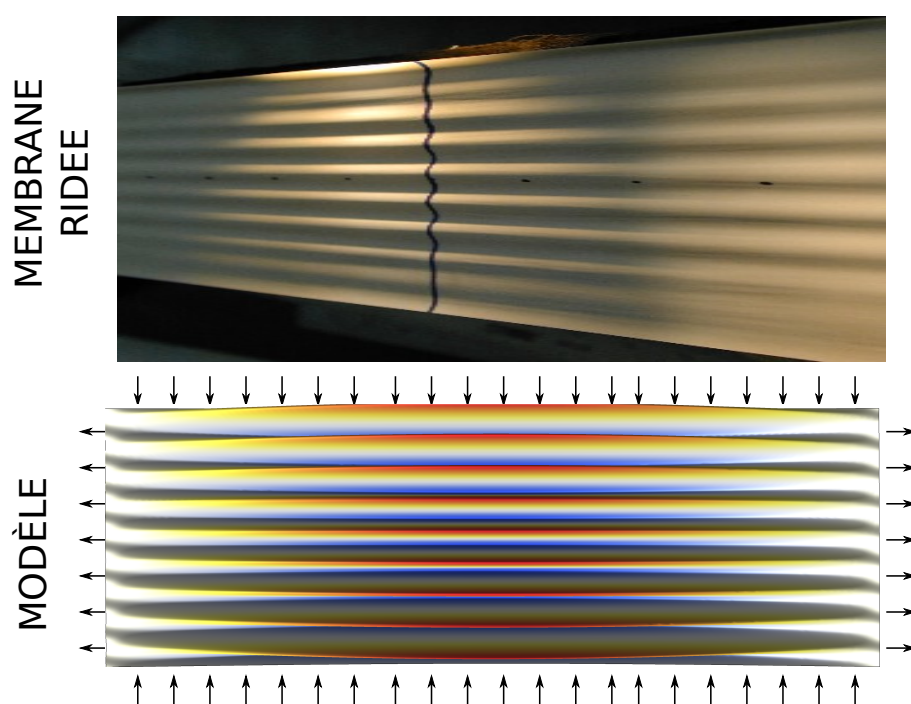


Figure 1.9: Comparaison entre la solution numérique du modèle de membrane dans le cas hyperbolique et une vraie membrane cisallée. a) Solution numérique. b) photo de la membrane étirée et cisallée.

Chapter 2

Introduction

The main topic of this PhD thesis is the study of the mechanical behavior of slender and thin structures in non-linear regime. In particular this work is based on the generalization of Euler and Timoshenko beam theories in large deformations and on the asymptotic expansion applied to nonlinear 3D elasticity equation to formally justify a membrane 2D model.

The interest in generalizing well known beam theories such as Euler and Timoshenko beam theory considering the action of distributed load, lies both in the importance of deepening our knowledge and understanding of such basic structural elements and in the immediate application of our results to the study of more complex systems. One of the first attempt to solve the problem of a cantilever beam with a concentrated load applied at the free end, was formulated by Galiei in his book dated 1638 “Discourses and Mathematical Demonstrations Relating to Two New Sciences” (Discorsi e Dimostrazioni Matematiche Intorno a Due Nuove Scienze). However some recent studies argue that Da Vinci first obtained some crucial results on the problem ¹. A modern formulation of the theory of *Elasticae* was published by Euler in 1744 in [18], with the important contribution of the ideas of Daniel and James Bernoulli [8]. Some year after, Lagrange [27], obtained the stationarity condition for the total energy, by means of variational methods still used nowadays and through this work. Moreover the relative boundary value problems for the resulting ordinary differential equations have been since then extensively used for determining the equilibrium shapes. This preliminary work has started a rich and important research field in Mathematics, Physics and Engennering. Without even trying to provide a complete bibliography, it is worth mentioning that large deformations of the *Elastica* have been studied by many other great scientists (including for instance Max Born in his Doctoral Thesis). Euler’s beam model, in the inextensible case, has been validated by means of rigorous derivation from 3D elasticity [32, 33] and systematically and thoroughly used in structural mechanics.

However, despite its importance as a structural component, we could not find in literature rigorous results on the study of the whole set of equilibrium shapes and of their stability neither for an *Elastica*, neither for a Timoshenko beam in large deformations under a uniformly distributed load. While the numerical methods were more and more developed, the mechanical problems related to

¹Wikipedia

large deformations of beams, with the noticeable exception of problems with concentrated end load, seem to have somewhat escaped rigorous treatment. In the recent past, however, the awareness of the importance of large deformations problems in structural mechanics came back in both theoretical and computational directions, and this importance will probably increase whether further substantial progresses will be achieved, in particular since large deformations play a relevant role in topical research lines as the design of metamaterials. In fact, lot of scientific effort is devoted nowadays to the description of micro structured continua. An important and promising class of this materials is represented by pantographic materials (see for instance [14] and references therein). In this particular class of micro structured continua the micro structure consists of a texture of beams, and the action of the surrounding material on the single fiber, can be seen, in the homogenized limit, as a distributed load. Exactly as in many other conceivable metamaterials, pantographic ones base their exotic behavior on the particular geometrical and mechanical micro-structure of beam lattices and on the deformation energy localization allowed by the onset of large deformations in portion of beams located in some specific areas of the lattice structure. It is therefore clear that, if one is interested in designing and optimizing pantographic metamaterials, a reasonably complete knowledge of the behavior of beams in large deformations is needed. The authors were indeed pushed in the present research direction exactly pursuing the aforementioned aims.

Concerning the state of the art of the problem, a rich literature exists considering both linearized and geometrically nonlinear models, together with generalized 1D elastic models, among which beam models capable to take into account other deformations than deflection, extension and shear. However two aspects in geometrically nonlinear beam theory especially require further investigation: the behavior of the system under a distributed load and the related multiplicity of arising solutions. An analysis in these directions has been started in the work [12], where an inextensible Euler beam has been considered. There were, however, some suggestive numerical results that deserved, in the opinion of the authors, greater attention than they received. Namely, in [19, 34] some numerical results producing equilibrium shapes similar to the ones we will present in this thesis are shown; in these papers the solutions of the boundary value problem are obtained by means of a shooting technique. In the older paper [20], some numerically evaluated stability results on shapes similar to those considered here are presented in case of concentrated end load. The cited papers appeared when the first effective numerical codes capable to deal with nonlinear boundary value problems became available.

The necessity of considering distributed loads in large deformation arises not only in the field of microstructured continua, but is still a very active topic in the field of fluid-structure interaction from the engineering point of view. It was quite surprising, for the authors, that there were not many results (in particular, rigorous ones) for the case of nonlinear deformation of beams under distributed load. For instance, the classical reference work [4] only covers the case of concentrated load.

In this thesis also dynamical studies are performed, with a comparison between the models described in the first part and analyzed only from a static point of view, and some experimental results. The numerical simulations are in agreement with the experimental ones, confirming also some results about the

stability of some solution forecast by the theoretical model. Some important results presented in the first chapter in fact concern the stability of the solutions to the variational problem for a Timoshenko beam under distributed load, representing a continuation of the work started in [14].

The last part of the thesis is devoted to asymptotic methods. This mathematical technique has been applied to 3D nonlinear elasticity, justifying formally a membrane theory envisaged by Landau and Lifshitz in [28]. Some interesting solutions, which seems promising in describing wrinkling patterns in thin films have been found. In the last decades, lot of scientific effort has been devoted to a rigorous justification of two-dimensional plate theories. Historically the set of equations used by engineers for the mathematical description of thin structures, classified in literature as plates, membranes or shells, is deduced by a priori assumptions based on the predominance of two dimensions of the system (the middle surface) over the thickness. This is the usual way of inferring, for instance, the well-known Kirchhoff-Love model, both in linear and nonlinear elasticity, as well as the von Kármán one within geometrically nonlinear theory. In particular we have focused on a two-dimensional membrane model inferred by Landau and Lifshitz in [3], from the Föppl-von Kármán equations, that, according to our knowledge, has rarely attracted the attention of researchers in the existing literature. The thesis is the collection of the publications which represents the results obtained during the 3-year PhD course, and is organized in the following way: the second chapter collects three publications dealing mainly with the static behavior of beam models, well-posedness of the problem and numerical results obtained. The third is represented by a paper dealing with the dynamic of an inextensible Euler beam analyzed both from the numerical and experimental point of view. The fourth chapter consists in a paper on the asymptotic expansion to justify what we have called the Landau-lifshitz membrane model. The paper contains some numerical simulations dealing with the modelling of wrinkling in thin films. In the fifth chapter some general conclusion are given.

Chapter 3

Non linear Euler and Timoshenko beam theories

3.1 Introduction

The first chapter is a collection of papers dealing with the study of beam models in large deformations in the static case. Most of the attention is focused on the Timoshenko beam model, but some results about Euler model are also presented, which constitute mainly a continuation of the work started in [12]. In the first paper “Large deformations of 1D microstructured systems modeled as generalized Timoshenko beams” we analyze the microstructure that once homogenized leads to the Timoshenko beam model (see fig. 3.1). In particular we show how the microstructure found in literature for the linear Timoshenko model produces an homogenized deformation energy which is not coinciding with the usual energy functional found in literature (and reducing to the Euler beam model in the limit of an infinite shear stiffness), i.e.:

$$\int_0^L \left[\frac{K_1}{2} (\phi'(s))^2 + \frac{K_2}{2} (\phi(s) - \theta(s))^2 \right] ds \quad (3.1)$$

where K_1 and K_2 are respectively the shear stiffness and the bending stiffness, ϕ the kinematic variable which represents the shear and s the curvilinear abscissa along the beam. the angle between the tangent to the deformed beam and horizontal reference axis, noted θ , characterizes the bending of the beam. If we add a force $\mathbf{F} = F_1 \hat{\mathbf{e}}_1 + F_2 \hat{\mathbf{e}}_2$, we obtain the variational problem:

$$\min \left\{ E(\phi, \theta) := \int_0^L \left[\frac{K_1}{2} \phi'^2 + \frac{K_2}{2} (\phi - \theta)^2 + F_1 \cos \theta + F_2 \sin \theta \right] ds \right. \\ \left. \theta \in L^2[0, L], \quad \phi \in H^1[0, L], \quad \phi(0) = 0 \right\} \quad (3.2)$$

However, we show a discrete system that when heuristically homogenized leads to the model studied in this paper (3.1).

One of the main reasons that pushed the authors to focus on these discrete models lies in the possibility to 3D print a discrete version of our model in order to provide an experimental comparison and eventually a validation of

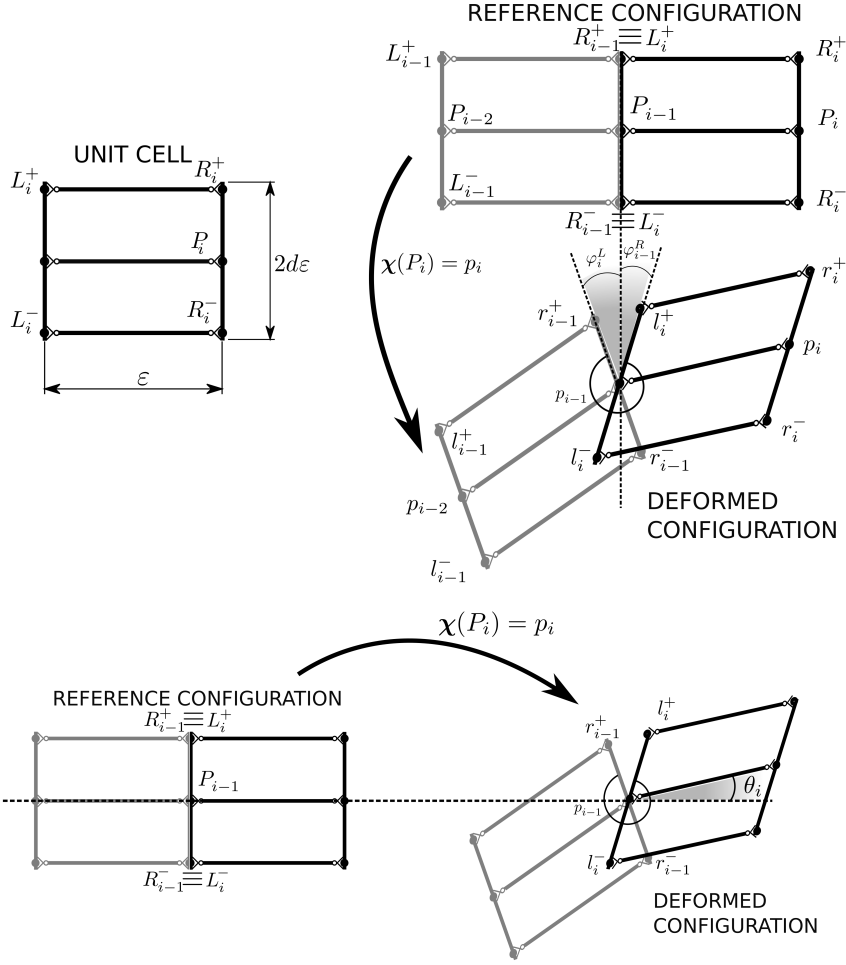


Figure 3.1: Example of microstructure of a shear deformable beam reproducing (after linearization) the Timoshenko beam model. Above: the unit cell and a graphical representation of the variation of the angle of the ϕ section. Bottom: graphical representation of the variation of the angle θ between the tangent to the deformed beam and the horizontal reference axis. A heuristic homogenization of this elementary microstructure leads to the energy (1.1).

the theory. Such an experimental validation will constitute one of the further step of this research work. The non convexity of the energy functional and the absence of the regularizing term (i.e. the derivative of the bending angle) naturally leads to question about the well-posedness of the variational problem given by the equilibria of a clamped beam under end load. This problem is addressed together with some regularity issues, moreover the critical load is evaluated. Finally, some numerical simulation, dealing with the properties of the equilibrium configurations analyzed in the first part of the paper, are presented and discussed.

In the paper “Large deformations of Timoshenko and Euler beams under distributed load” a nonlinear version of the extensible Timoshenko beam model is introduced and the problem of a clamped-free beam is formulated. We derive the Euler-Lagrange equations from the energy functional and presents some numerical simulations. Numerical solutions are obtained with a shooting technique. To be more precise, the variational problems for a clamped-free beam gives a boundary value problem (BVP), we rewrite the BVP as a parametric Cauchy problem:

$$\mathcal{P}_k = \begin{cases} \theta'' = -b(1-s)\cos\theta \\ \theta(0) = 0 \\ \theta'(0) = k \end{cases} \quad (3.3)$$

where b is the value of the adimensionalized dead force, considered here transversal to the beam in its reference configuration. The solutions that satisfy the boundary conditions under consideration are obtained by varying the parameter k (here, the length of the beam is adimensioned and therefore we consider a beam of unit length). In the case of a beam subjected to no effort or moment in $s = 1$, we look for solutions verifying $\theta(1) = 0$, thus we select the solution that satisfy the boundary conditions of the original BVP, which constitute the equilibrium configurations of the energy functional. The variations of $c(k) := \theta(1)$ as a function of $k := \theta'(0)$ are represented numerically in the figure 3.2 for two different values of b . The number of solutions of the BVP (3.3) is equal to the number of intersections of the curve $c(k)$ with the horizontal axis.

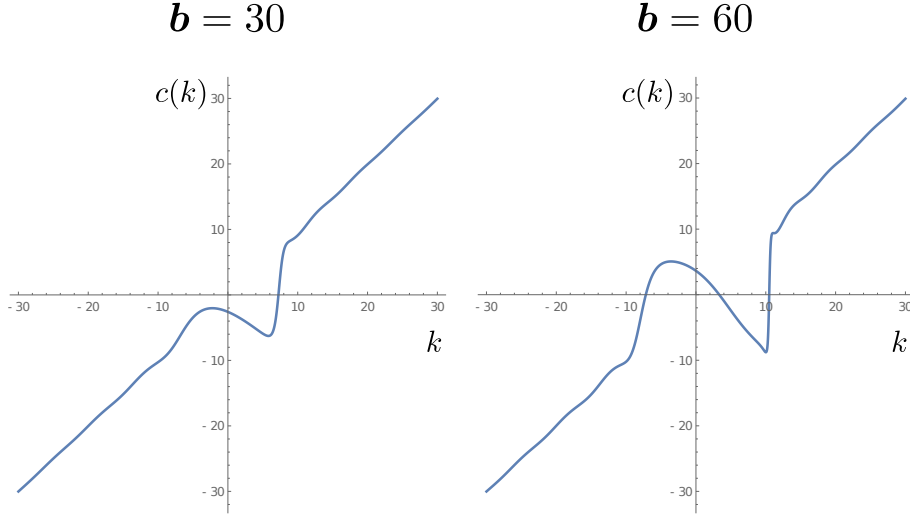


Figure 3.2: Variations of $c(k)$ obtained numerically for two different values of the dimensionless distributed load in the case of an inextensible Euler beam. The BVP (3.3) admits a solution for $b = 30$ and three solutions for $b = 60$.

Moreover we show how this is possible rewriting the variational problem only in terms of the bending angle (we recall that the kinematic variables in Timoshenko model are represented by the bending angle and the shear angle). These results motivate the rigorous study of the properties of the equilibrium solutions done in the paper, in which the Euler-Lagrange equation is studied in the particular case of an infinite shear stiffness, which leads to the nonlinear Euler beam. All the global and local minimizers of this variational problem are characterized and the relative monotonicity and regularity properties are established.

In the paper “Extensible beam models in large deformation under distributed loading: a numerical study on multiplicity of solutions” we focus mostly on the multiplicity of solutions of the problem. While for a clamped-free nonlinear inextensible Euler Elastica the set of stable equilibrium configurations has been completely characterized in [12], the extension of such rigorous results to the extensible Euler beam model, or to the Timoshenko beam model is not straightforward. The problem of existence and stability of equilibrium configurations for extensible Euler and Timoshenko beams in large deformations under distributed load is therefore an open one. Because of this reason, it is interesting to study the behaviour of solutions by means of a systematic collection of parametric studies starting from the inextensible Elastica and approaching more general beam models. This work is aimed at performing such kind of investigation. Our main effort has been to show the variety of different (and at times rather exotic) Equilibrium configurations that can arise when the value of the load is large enough. Moreover, we numerically investigated how fast the number of possible Equilibrium solutions increases with the load and how this particular feature is affected by allowing shear deformation.



Large deformations of 1D microstructured systems modeled as generalized Timoshenko beams

A. Battista , A. Della Corte, F. dell'Isola and P. Seppecher

Abstract. In the present paper we study a natural nonlinear generalization of Timoshenko beam model and show that it can describe the homogenized deformation energy of a 1D continuum with a simple microstructure. We prove the well posedness of the corresponding variational problem in the case of a generic end load, discuss some regularity issues and evaluate the critical load. Moreover, we generalize the model so as to include an additional rotational spring in the microstructure. Finally, some numerical simulations are presented and discussed.

Mathematics Subject Classification. 74B20, 49J45.

Keywords. Nonlinear elasticity, Generalized Timoshenko beam, Microstructured beam, Non-convex variational problems.

1. Introduction

The main aim of this paper is to study the problem of large deformations of a microstructured beam, which can be described at the macroscopic level by means of a generalized Timoshenko model. We focus on the case of a beam clamped at one endpoint and submitted to a concentrated load at the other endpoint.

Timoshenko proposed the first beam model going beyond the classical Euler model. The latter was formulated around mid-18th century [1], and scientists of the caliber of Daniel and Jakob Bernoulli, as well as Lagrange, gave important contributions to its theory [2–4]. The model has been rigorously deduced from 3D elasticity [5, 6] and it is widely applied in many problems of structural mechanics. (A useful reference book is [7].)

Let $\mathcal{E}$ be the affine euclidean plane and $\{\mathbf{e}_1, \mathbf{e}_2\}$ an orthonormal basis. The deformation energy of an inextensible Euler beam can be written as:

$$\int_0^L \frac{k_b}{2} \boldsymbol{\chi}''(s) \cdot \boldsymbol{\chi}''(s) ds = \int_0^L \frac{k_b}{2} \eta^2(s) ds \quad (1)$$

under the inextensibility constraint

$$\|\boldsymbol{\chi}'(s)\| = 1 \quad \forall s \in [0, L] \quad (2)$$

where L is the length of the beam (assumed straight and parallel to $\mathbf{e}_1$ in the reference configuration), $\boldsymbol{\chi}$ is the placement (with components χ_i), i.e., a bijective continuous function mapping material points of the beam (labeled with $s \in [0, L]$) into $\mathcal{E}$, and k_b is a material parameter accounting for the bending stiffness. The dot indicates the usual scalar product of $\mathbb{R}^2$, and the related norm is understood. In the present paper, $(\cdot)'$ denotes differentiation with respect to s . In particular, $\eta(s) = \|\boldsymbol{\chi}''(s)\|$ is the absolute value of the curvature of the current shape of the beam. Note that, although the deformation energy (1) is a quadratic form, the inextensibility constraint (2) is not convex.

Since in most of applications to structural mechanics deflections are small when compared to the length of the beams, it is very usual to consider the approximated linearized model, instead of the previous one. In this case, constraint (2) becomes $\chi_1 = s$ and the deformation energy reads:

$$\int_0^L \frac{k_b}{2} (\chi_2'')^2 ds, \quad (3)$$

Static problems for the commonly employed linearized model lead to fourth-order linear ODEs, while the nonlinear model originally formulated by Euler leads to semilinear fourth-order ODEs.

The nonlinear Euler model can be reformulated in terms of the variable θ satisfying

$$\chi' = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2 \quad (4)$$

The energy becomes:

$$\int_0^L \frac{k_b}{2} \theta'^2(s) ds \quad (5)$$

The beam is said to be clamped at the left extremum if $\chi(0) = \mathbf{0}$ and $\chi'(0) = \mathbf{e}_1$. An external potential $\mathbf{b}(s) \cdot \chi(s)$, where $\mathbf{b}(s)$ is a distributed load, can also be written in terms of θ by performing an integration by parts. The total energy of the system becomes:

$$E(\theta) = \int_0^L \left[\frac{k_b}{2} \theta'^2(s) - B_1 \cos \theta(s) - B_2(s) \sin \theta(s) \right] ds \quad (6)$$

where $\mathbf{B} = B_1 \mathbf{e}_1 + B_2 \mathbf{e}_2$ denotes the primitive of $\mathbf{b}$ verifying $\mathbf{B}(L) = \mathbf{0}$. It is natural to search for minimizers of the energy (6) in the set of functions belonging to $H^1 := W^{1,2}$ verifying $\theta(0) = 0$ in the sense of traces. We remark that the angle $\theta(s)$ is uniquely determined by Eq. (4) if one takes into account the clamp condition $\theta(0) = 0$ and the fact that H^1 functions have continuous representatives. This reformulation automatically takes into account the inextensibility constraint and the non-convexity of the minimization problem appears clearly.

Timoshenko beam model was introduced to describe in a more precise way the shear deformation of the beam. (The problem is addressed in an original way in [8].) It was developed much later, in the early 1920s [9], and motivated by several applications. Specifically, for the static case, it was needed for describing beams that were not so slender, in which case shear deformation is no more negligible. This is also the case with sandwich composite beams ([10]). Timoshenko beam model is still an important tool for current research: the possibility of very precisely manufacturing the inner architecture of beams (e.g., by means of 3D printing) makes it now possible to produce slender objects which display a richer behavior than what can be captured by Euler beam model. (See, e.g., [11–13] for interesting examples, [14–16] for cases in which dynamical/instability problems are addressed and [17–20] for an approach using asymptotic justification; a review of complex structures employing fibers that can be modeled as generalized beams is [21].)

The original model from Timoshenko was established in a linear framework. The deformation energy of an inextensible linear Timoshenko beam reads:

$$\int_0^L \left[\frac{K_1}{2} (\phi'(s))^2 + \frac{K_2}{2} (\phi(s) - u_2'(s))^2 \right] ds \quad (7)$$

where ϕ is an independent kinematic descriptor and K_1 and K_2 are material parameters. In the original interpretation of the model, ϕ was thought to measure the angle between the cross section of the beam and a reference axis. Therefore the model can be seen as a particular case of Cosserat/micropolar continua. The interested reader can see [22] for a historically important reference; a general introduction on the

topic is given, e.g., in [23–25], while some interesting results using Γ -convergence arguments are provided in [26] (concerning 2D plate-like models) and [27] (concerning rod-like models).

Clearly there are many possible nonlinear generalizations of the previous linearized energy model. A natural generalization can be obtained by replacing, similarly to the Euler case, the term u'_2 by the angle θ defined as above:

$$\int_0^L \left[\frac{K_1}{2} (\phi'(s))^2 + \frac{K_2}{2} (\phi(s) - \theta(s))^2 \right] ds \quad (8)$$

We will obtain this energy model with a formal homogenization starting from a microstructure consisting of articulated parallelograms and rotational springs.

The paper is organized as follows: in Sect. 2, various kinds of microstructures are considered and in particular a novel one is introduced which leads to the deformation energy (8); moreover, an additional rotational spring is also considered to introduce a regularizing term which further generalizes the model. In Sect. 3, the well posedness of the variational problem given by the equilibria of a clamped beam under end load is addressed, some regularity issues are discussed, and the critical load is evaluated. In Sect. 4, numerical simulations are presented and discussed. In Sect. 5, some conclusions are provided as well as possible directions for future researches.

2. Microscopic interpretation of a Timoshenko beam

2.1. The linearized case

A traditional way of introducing Timoshenko model in structural mechanics courses is to provide a microscopical interpretation of it by means of a discrete system of rotational and extensional springs. Nowadays, the possibility of accurately 3D printing the microstructures makes these discretizations more than academic examples, but rather possibilities to concretely implement mechanical systems with certain desired properties.

Let us consider the system illustrated in Fig. 1. The length of the deformed extensional springs connecting the points $\mathbf{t}_i^+$ and $\mathbf{t}_{i+1}^+$ can be computed as:

$$\|\mathbf{t}_{i+1}^+ - \mathbf{t}_i^+\|^2 = \varepsilon^2 \{1 + 2d [\sin(\theta_i - \phi_{i+1}) + \sin(\phi_i - \theta_i)] + 2d^2 [1 - \cos(\phi_{i+1} - \phi_i)]\} \quad (9)$$

Now if we assume that the quantities θ and ϕ , which measure the deformation of the system, remain small, using basic trigonometric identities and approximating at the first order the trigonometric functions and the square root, we can write the following approximate identity:

$$\|\mathbf{t}_{i+1}^+ - \mathbf{t}_i^+\| \approx \varepsilon [1 + d(\phi_i - \phi_{i+1})]$$

Let us assume that all the extensional springs are linear (with elastic constant $k_1/(2d^2)$). Performing a formal passage to the limit we have:

$$\frac{1}{2} \frac{k_1}{2d^2} (\|\mathbf{t}_{i+1}^+ - \mathbf{t}_i^+\| - \varepsilon)^2 \approx \frac{1}{2} \frac{k_1}{2} \phi'^2 \varepsilon^4 \quad (10)$$

An analogous contribution in the energy will come from the spring connecting $\mathbf{t}_i^-$ and $\mathbf{t}_{i+1}^-$. The potential energy of the rotational spring connecting $\mathbf{t}_i^- \mathbf{t}_i^+$ and $\mathbf{p}_i \mathbf{p}_{i+1}$ is assumed to be:

$$\frac{1}{2} k_2 (\phi_i - \theta_i)^2 \quad (11)$$

where k_2 is the elastic constant of the spring.

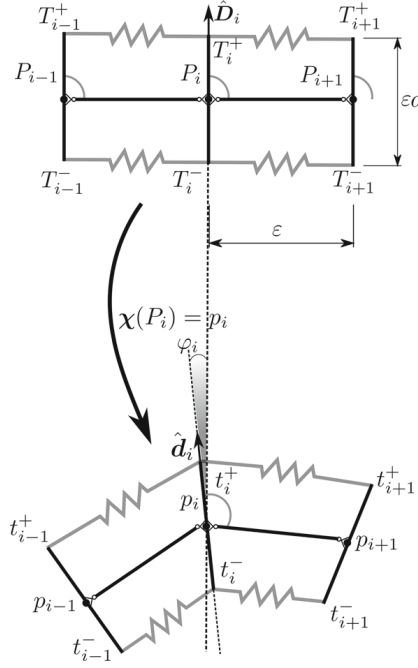


FIG. 1. The bar-spring microstructure usually introduced to get the linear Timoshenko model valid for small deformations (top: reference configuration; bottom: actual configuration). For finite deformations, a heuristic homogenization for this microstructure, leads to the energy (14)

The deformation energy of a microstructured system composed of N elementary cells like the two represented in Fig. 1, in the linearized case, can be therefore written as:

$$\sum_{i=1}^{i=N} \left[\frac{1}{2} k_1 \phi'^2 \varepsilon^4 + \frac{1}{2} k_2 (\phi_i - \theta_i)^2 \right]. \quad (12)$$

In the limit for $\varepsilon \rightarrow 0$, the stiffnesses have to be suitably rescaled: $k_1 = K_1 \varepsilon^{-4}$ and $k_2 = K_2$. Recalling that, in our hypotheses, θ is small, we can approximate it by means of u'_2 , obtaining for the homogenized energy formula (7).

2.2. The nonlinear case

If one does not want to assume that the angles ϕ and θ remain small, the energy of the previous discrete system takes a more complicated form.

Indeed, the continuous form of (9) is:

$$\frac{1}{2} \frac{k_1}{2d^2} (\|t_{i+1}^+ - t_i^+\| - \varepsilon)^2 \approx \frac{1}{2} k_1 \phi'^2 \cos^2(\theta - \phi) \varepsilon^4 \quad (13)$$

and the homogenized energy, with the same scaling as before, becomes

$$\int_0^L \left[\frac{1}{2} K_1 (\phi')^2 \cos^2(\theta - \phi) + \frac{1}{2} K_2 (\phi - \theta)^2 \right] \quad (14)$$

As we can see, the last expression does not coincide with the form of Timoshenko deformation energy given in formula (8) because of the factor $\cos^2(\theta - \phi)$. While the microstructure relative to Fig.1 has an intrinsic physical interest, the homogenized energy (14) has a more complex structure than (8) and we

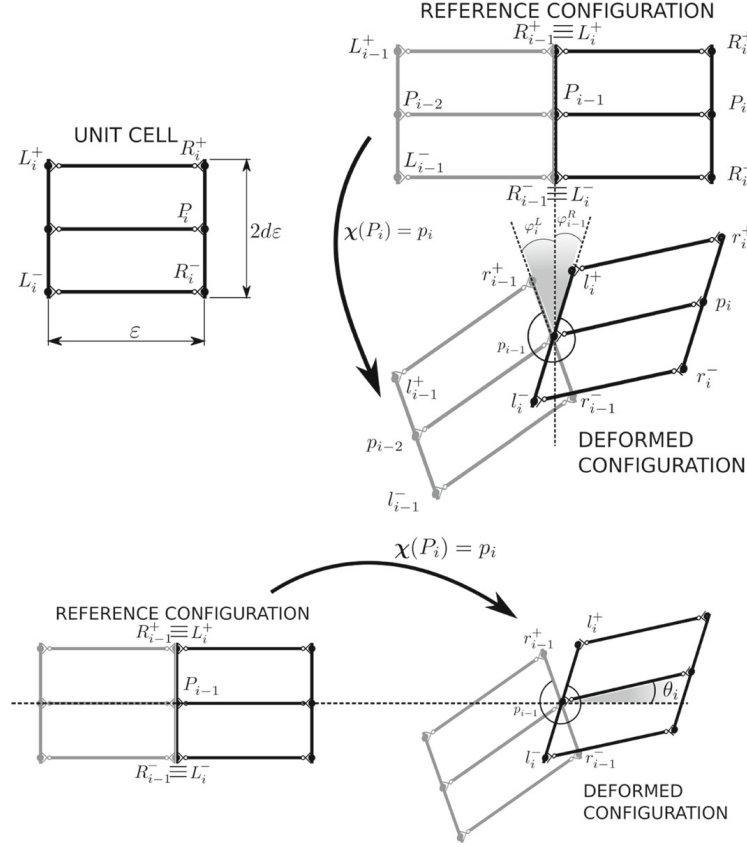


FIG. 2. Another possible microstructure for shear deformable beams producing (after linearization) the Timoshenko beam model. On top: the unit cell and a graphical representation of the variation of the cross-sectional angle ϕ . At the bottom: graphical representation of the variation of angle θ formed by the tangent to the deformed shape). A heuristic homogenization of this microstructure leads to the energy model (8)

consider more suitable to attack the simplest problem first. Indeed, we can recover the energy model (8) by means of a different microstructure (see Fig. 2).

We start by considering an articulated parallelogram in which all the sides are rigid bars, and we add another rigid bar connecting the middle points of two sides. Then we organize them in series (see Fig. 2). The points R_{i-1}^+ and L_i^+ , R_{i-1}^- and L_i^- as well as P_i^r and P_{i-1}^ℓ coincide in the reference configuration, so that the two bars $R_{i-1}^+R_{i-1}^-$ and $L_i^+L_i^-$ are superposed. We describe the mechanical interaction between the bars by means of rotational springs. We introduce two rotational springs:

1. one between the directions of $\overrightarrow{R_iP_i}$ and $\overrightarrow{P_{i+1}P_i}$
2. one between the directions of $\overrightarrow{R_iP_i}$ and $\overrightarrow{L_{i+1}P_{i+1}}$

Assuming that the rotational springs are linear in the angle, the deformation energy takes the form:

$$\sum_{i=1}^N \left[\frac{k_1}{2} (\phi_i - \phi_{i-1})^2 + \frac{k_2}{2} (\phi_i - \theta_i)^2 \right] \quad (15)$$

and setting $K_1 := k_1 \epsilon^4$, $K_2 := k_2$, formula (8) is recovered with a formal homogenization procedure. In the following, we will call the energy model (8) *Nonlinear Timoshenko model*.

Finally, we can introduce an additional rotational spring, i.e.:

3. between the directions of $\overrightarrow{P_iP_{i-1}}$ and $\overrightarrow{P_{i+1}P_i}$.

In this case (assuming that the new rotational spring has stiffness k_3 and setting $K_3 := k_3 \varepsilon^4$), an analogous formal homogenization procedure leads to the following energy functional:

$$\int_0^L \left[\frac{K_1}{2} (\phi')^2 + \frac{K_2}{2} (\phi - \theta)^2 + \frac{K_3}{2} (\theta')^2 \right] ds \quad (16)$$

In the following, we will call the energy model (16) *Regularized Timoshenko model*.

The passage from the discrete system to its homogenized form can be made rigorous in various ways, the most natural being Γ -convergence arguments. This is the subject of a separated investigation, and here we just observe that in our context the passage to the Γ -limit is not trivial, due to the geometrically nonlinear character of the problem which in particular allows finite rotations, possibly around axes that lie in the plane. For instance, the possible overlapping (referring to Fig. 2) of the points R_{I-1}^+ and L_I^- as well as R_{I-1}^- and L_I^+ has to be considered. Similarly, one has to consider the possible overlapping of points indexed by i and points indexed by $i - 2$.

In the following section, we investigate rigorously the proposed heuristic homogenized form to establish the well posedness of the variational problem and regularity properties of the solutions.

3. Properties of the minimization problems

3.1. Nonlinear Timoshenko model

Let us consider the deformation energy (8) in the model case in which a concentrated load $\mathbf{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2$ is applied at the free endpoint. We are led to the following variational problem:

$$\begin{aligned} \min \quad & \left\{ E(\phi, \theta) := \int_0^L \left[\frac{K_1}{2} \phi'^2 + \frac{K_2}{2} (\phi - \theta)^2 + F_1 \cos \theta + F_2 \sin \theta \right] ds \right. \\ & \left. \theta \in L^2[0, L], \quad \phi \in H^1[0, L], \quad \phi(0) = 0 \right\} \end{aligned} \quad (17)$$

3.1.1. Well posedness. We prove in this section that problem (17) is well posed. Let us first rewrite the problem in a non-dimensional form. A change of length and energy units leads to

$$\begin{aligned} \min \quad & \left\{ \tilde{E}(\phi, \theta) := \int_0^1 \left[\frac{1}{2} \phi'^2 + \tilde{K}_2 \left(\frac{1}{2} (\phi - \theta)^2 + \tilde{F} \cos(\theta - \gamma) \right) \right] ds \right. \\ & \left. \theta \in L^2[0, 1], \quad \phi \in H^1[0, 1], \quad \phi(0) = 0 \right\} \end{aligned} \quad (18)$$

where $\tilde{K}_2 := \frac{K_2 L^2}{K_1}$ and $(\tilde{F}, \gamma) \in [0, +\infty) \times [0, 2\pi[$ are defined by:

$$\tilde{F}(\cos(\gamma), \sin(\gamma)) = K_2^{-1}(F_1, F_2).$$

Proposition 1. *Problem (18) admits a solution.*

Proof. Let $G(\theta) := \frac{\theta^2}{2} + \tilde{F} \cos(\theta - \gamma)$ and $G^*(z)$ its convex conjugate, defined by $G^*(z) := \max_{\theta} [z\theta - G(\theta)]$. Since G is continuous and coercive, there exists $\bar{\theta}$ solving the max problem and $\bar{\theta}(z)$ belongs to the subdifferential $\partial G^*(z)$. We note that $G^*(z)$ is not differentiable in correspondence of intervals in which G does not coincide with its lower convex envelop.

Hence:

$$\begin{aligned}
\inf_{\theta, \phi} \tilde{E}(\theta, \phi) &= \inf_{\theta, \phi} \int_0^1 \left[\frac{1}{2} \phi'^2 + \tilde{K}_2 \left(\frac{1}{2} \phi^2 + (-\phi\theta + G(\theta)) \right) \right] ds \\
&= \inf_{\phi} \int_0^1 \left[\frac{1}{2} \phi'^2 + \tilde{K}_2 \left(\frac{1}{2} \phi^2 + \inf_{\theta} (-\phi\theta + G(\theta)) \right) \right] ds \\
&= \inf_{\phi} \int_0^1 \left[\frac{1}{2} \phi'^2 + \tilde{K}_2 \left(\frac{1}{2} \phi^2 - \sup_{\theta} (\phi\theta - G(\theta)) \right) \right] ds = \inf_{\phi} \mathcal{A}(\phi)
\end{aligned}$$

where

$$\mathcal{A}(\phi) := \int_0^1 \left(\frac{1}{2} \phi'^2 + \tilde{K}_2 h(\phi) \right) ds \quad (19)$$

with

$$h(\phi) := \frac{1}{2} \phi^2 - G^*(\phi).$$

Let (θ_n, ϕ_n) be a minimizing sequence. Then $\tilde{E}(\theta_n, \phi_n)$ is a bounded sequence and thus $\int_0^1 \frac{1}{2} (\phi_n')^2$ is also bounded and $\phi_n(0) = 0$. Thus ϕ_n is bounded in H^1 : there exists a subsequence (still denoted ϕ_n) weakly converging in H^1 to some function $\bar{\phi}$. Remind that in the one-dimensional case H^1 -weak convergence imply uniform convergence and note also that G^* is convex, and therefore it is locally Lipschitz (see Theorem 3.7.3 in [28]). Thus $\int_0^1 [\phi_n^2/2 - G^*(\phi_n)] ds$ converges to $\int_0^1 [\bar{\phi}^2/2 - G^*(\bar{\phi})] ds$. On the other hand, as convexity and weak convergence implies $\liminf_n \int_0^1 \phi_n'^2 ds \geq \int_0^1 \bar{\phi}'^2 ds$, the functional $\mathcal{A}$ is H^1 -lower semicontinuous, and $\bar{\phi}$ is a global minimizer for $\mathcal{A}$. Finally let us remark that the constraint $\phi_n(0) = 0$ passes to the uniform limit: $\bar{\phi}(0) = 0$.

The function $\bar{\theta}(s)$, which solves:

$$\max_{\theta} [\bar{\phi}(s)\theta - G(\theta)] \quad (20)$$

is uniquely defined everywhere on $[0, 1]$, by the equation

$$\frac{dG^{**}(\theta)}{d\theta} = \bar{\phi}(s) \quad (21)$$

or equivalently by

$$\theta = \frac{dG^*(\bar{\phi}(s))}{dz} \quad (22)$$

except at the points s such that $G^*(\bar{\phi}(s))$ is not differentiable.

Since $\bar{\phi} \in H^1(0, 1)$ is continuous in $[0, 1]$, it attains a maximum $\phi_{\max}$ and a minimum $\phi_{\min}$, and therefore we are interested in the differentiability of G^* only in the interval $[\phi_{\min}, \phi_{\max}]$.

It is easy to check that the convex conjugate of G is piecewise C^1 and that the set $\{\phi_1, \dots, \phi_k\}$ of the points which belong to $[\phi_{\min}, \phi_{\max}]$ and where G^* is not differentiable is finite. Let us now prove by contradiction that $\bar{\phi}$ takes the values $\phi_1, \dots, \phi_k$ only on a subset of $[0, 1]$ of measure zero. Otherwise there would exist an integer $i \in \{1, \dots, k\}$ such that the measure of $\bar{\phi}^{-1}(\phi_i)$ is positive. Adapting known monotonicity results for autonomous variational problems (see, e.g., Theorem 3.1 in [30]), we know that $\bar{\phi}$ is monotonic¹ and thus that $\bar{\phi}^{-1}(\phi_i)$ is an interval. The following lemma states that for this reason G^* is differentiable at ϕ_i , which contradicts the definition of ϕ_i . In conclusion, $\bar{\theta}$ is uniquely determined

¹The cited theorem is stated for a problem with values prescribed at both ends, but is true also in the present case. One just has to apply it to a problem having $\phi(1) = \bar{\phi}(1)$.

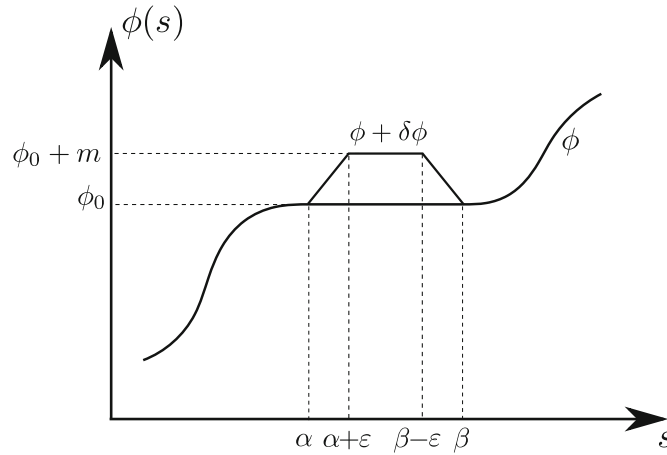


FIG. 3. Representation of ϕ and of the considered variation $\delta\phi$

almost everywhere, it is bounded because G^* is Lipschitz on $[\phi_{\min}, \phi_{\max}]$, and therefore, it belongs to $L^2[0, 1]$. $\square$

Lemma 1. *Let $\bar{\phi}$ be a minimizer of $\mathcal{A}$. Suppose that there exists an interval having positive measure $[\alpha, \beta] \subset [0, 1]$ in which $\bar{\phi}(s) = \phi_0$ is constant. Then G^* is differentiable at ϕ_0 , and $\frac{\partial G^*}{\partial \phi}(\phi_0) = \phi_0$.*

Proof. We first observe that since G^* is convex, it admits left and right derivatives and therefore for every ϕ there exist both the right and left derivatives $\frac{\partial h}{\partial \phi}^+(\phi)$ and $\frac{\partial h}{\partial \phi}^-(\phi)$.

Now let ε be such that $0 < \varepsilon < \frac{\beta - \alpha}{2}$ and let m and $c > 0$ be real numbers such that $\varepsilon = \sqrt{|m|/2c} < (\beta - \alpha)/2$.

Let us consider a variation $\delta\phi$ defined as (see Fig. 3):

$$\delta\phi(s) := \begin{cases} 0 & \text{for } s \in [0, \alpha) \cup (\beta, 1] \\ \frac{m}{\varepsilon}s - \frac{m}{\varepsilon}\alpha & \text{for } s \in [\alpha, \alpha + \varepsilon] \\ -\frac{m}{\varepsilon}s + \frac{m}{\varepsilon}\beta & \text{for } s \in [\beta - \varepsilon, \beta] \\ m & \text{for } s \in (\alpha + \varepsilon, \beta - \varepsilon) \end{cases}$$

Clearly $[\bar{\phi} + \delta\phi] \in H^1$ and $[\bar{\phi} + \delta\phi](0) = 0$. Let us set $\Delta h(\delta\phi) := h(\bar{\phi} + \delta\phi) - h(\bar{\phi})$. We have:

$$\begin{aligned} 0 \leq \Delta \mathcal{A} &:= \mathcal{A}(\bar{\phi} + \delta\phi) - \mathcal{A}(\bar{\phi}) \\ &= \int_{\alpha}^{\beta} \frac{1}{2} \delta\phi'^2 ds + \tilde{K}_2 \left(\int_{\alpha}^{\alpha+\varepsilon} \Delta h(\delta\phi) ds + \int_{\alpha+\varepsilon}^{\beta-\varepsilon} \Delta h(\delta\phi) ds + \int_{\beta-\varepsilon}^{\beta} \Delta h(\delta\phi) ds \right) \end{aligned}$$

Using that h is c -Lipschitz on the compact set $[\phi_{\min} - |m|, \phi_{\max} + |m|]$, one gets:

$$0 \leq \frac{m^2}{\varepsilon} + \tilde{K}_2 ((\beta - \alpha - 2\varepsilon)(h(\phi_0 + m) - h(\phi_0)) + 2c\varepsilon|m|)$$

The last inequality implies:

$$h(\phi_0 + m) - h(\phi_0) \geq -\frac{2\sqrt{2c}}{\tilde{K}_2(\beta - \alpha - 2\varepsilon)}|m|^{3/2} \quad (23)$$

In terms of G^* , the inequality (23) can be written as:

$$G^*(\phi_0 + m) - G^*(\phi_0) \leq \frac{m^2}{2} + \phi_0 m + \frac{2\sqrt{2c}}{\tilde{K}_2(\beta - \alpha - 2\varepsilon)}|m|^{3/2} \quad (24)$$

The left hand side of (24) is a convex function of m , while the right hand side is a C^1 function of m . Both sides coincide when $m = 0$. It is easily seen that a convex function bounded from above by a C^1 function is differentiable at any coinciding point. Therefore G^* is differentiable at $\phi = \phi_0$.

Dividing by m both sides of (24) and letting $m \rightarrow 0^+$ or $m \rightarrow 0^-$, we obtain:

$$\left(\frac{\partial G^*}{\partial \phi}\right)^+(\phi_0) \leq \phi_0 \quad \text{and} \quad \left(\frac{\partial G^*}{\partial \phi}\right)^-(\phi_0) \geq \phi_0. \quad (25)$$

As a consequence

$$\frac{\partial G^*}{\partial \phi}(\phi_0) = \phi_0.$$

□

3.1.2. Regularity. We address now the properties of the function h . In order to establish its regularity properties, we first have to provide some preliminary results.

Let us first remark that h is a 2π -periodic function. Indeed, for any $z \in \mathbb{R}$, let us denote $[z]$ the integer part of $\frac{z-\gamma}{2\pi}$, and let us set $\tilde{z} := z - 2\pi[z]$. We can write

$$\begin{aligned} G^*(z) &= \max_{\theta} \left[z\theta - \frac{\theta^2}{2} - F \cos(\theta - \gamma) \right] \\ &= \max_{\theta} \left[2\pi^2[z]^2 + 2\pi[z]\tilde{z} + \tilde{z}(\theta - 2\pi[z]) - \frac{1}{2}(\theta - 2\pi[z])^2 - F \cos(\theta - \gamma) \right] \\ &= \max_{\alpha} \left[2\pi^2[z]^2 + 2\pi[z]\tilde{z} + \tilde{z}\alpha - \frac{1}{2}\alpha^2 - F \cos(\alpha - \gamma) \right] \\ &= 2\pi^2[z]^2 + 2\pi[z]\tilde{z} + G^*(\tilde{z}). \end{aligned}$$

where we introduced the variable $\alpha := \theta - 2\pi[z]$. Hence:

$$\begin{aligned} h(z) &= \frac{z^2}{2} - G^*(z) \\ &= \frac{\tilde{z}^2}{2} + 2\pi^2[z]^2 + 2\pi[z]\tilde{z} - (2\pi^2[z]^2 + 2\pi[z]\tilde{z} + G^*(\tilde{z})) \\ &= \frac{\tilde{z}^2}{2} - G^*(\tilde{z}) = h(\tilde{z}). \end{aligned}$$

We address now the dependence of h on γ . To this aim let us define

$$G_{\gamma}(\theta) := \frac{\theta^2}{2} + \tilde{F} \cos(\theta - \gamma) \quad (26)$$

and

$$h_{\gamma}(z) = \frac{z^2}{2} - G_{\gamma}^*(z). \quad (27)$$

We have

$$\begin{aligned}
 h_\gamma(z) &= \frac{z^2}{2} - \max_\theta \left[\theta z - \frac{\theta^2}{2} - \tilde{F} \cos(\theta - \gamma) \right] \\
 &= \frac{z^2}{2} - \max_\alpha \left[(\alpha + \gamma)z - \frac{1}{2}(\alpha + \gamma)^2 - \tilde{F} \cos \alpha \right] \\
 &= \frac{z^2}{2} - \gamma z + \frac{\gamma^2}{2} - \max_\alpha \left[(z - \gamma)\alpha - \frac{\alpha^2}{2} - \tilde{F} \cos \alpha \right] \\
 &= \frac{(z - \gamma)^2}{2} - G_0^*(z - \gamma) = h_0(z - \gamma).
 \end{aligned}$$

Therefore the regularity properties of h_γ result from those of h_0 . Remarking that h_0 like G_0^* is clearly even, we are reduced to study h_0 on the interval $[0, \pi]$. We can now address the regularity problem:

Lemma 2. (i) For $\tilde{F} < 1$, h_0 is a C^∞ function,

(ii) For $\tilde{F} \geq 1$, h_0 is a C^∞ function everywhere but at 0 where the jump of derivative is $-2a(\tilde{F})$ with $a(\tilde{F})$ the first positive solution of the equation

$$a(\tilde{F}) = \tilde{F} \sin(a(\tilde{F})) \quad (28)$$

(Note that a is an increasing function from $[1, +\infty)$ onto $[0, \pi]$ and $a(\pi/2) = \pi/2$),

(iii) In any case h_0 is decreasing on the interval $[0, \pi]$. Setting $V(\tilde{F}) := h(0) - h(\pi)$, we have:

$$V(\tilde{F}) = \begin{cases} 2\tilde{F} & \text{if } \tilde{F} < 1 \\ a(\tilde{F})^2/2 + \tilde{F} + \sqrt{\tilde{F}^2 - (a(\tilde{F}))^2} & \text{if } 1 \leq \tilde{F} \leq \pi/2 \\ a(\tilde{F})^2/2 + \tilde{F} - \sqrt{\tilde{F}^2 - (a(\tilde{F}))^2} & \text{if } \pi/2 < \tilde{F} \end{cases}$$

and

$$h'_0(z) = -\sqrt{2}\sqrt{\tilde{F} - h_0(z)} \frac{\sqrt{\tilde{F} + 1 - \sqrt{1 + \tilde{F}^2 - 2h_0(z)}}}{\sqrt{1 - \tilde{F} + \sqrt{1 + \tilde{F}^2 - 2h_0(z)}}}. \quad (29)$$

This lemma is illustrated in Fig. 4.

Proof. Let us start by preliminary remarks. The problem

$$\max_\theta (z\theta - G_0(\theta)) \quad (30)$$

admits at least one solution $\theta(z)$ which has to satisfy $z = G'_0(\theta(z))$ that is

$$z = \theta(z) - \tilde{F} \sin(\theta(z)) \quad (31)$$

When this solution is unique and when $G''_0(\theta(z)) > 0$, local inversion theorem states that $\theta(z)$ is an increasing C^∞ function. Therefore $G_0^*(z) = z\theta(z) - G_0(\theta(z))$ and consequently h_0 are of class C^∞ . In that case, we have

$$\begin{aligned}
 (G_0^*)'(z) &= \theta(z) + (z - G'_0(\theta(z)))\theta'(z) \\
 &= \theta(z) + (z - \theta(z) + \tilde{F} \sin(\theta(z)))\theta'(z) = \theta(z)
 \end{aligned}$$

and so

$$h'_0(z) = z - (G_0^*)'(z) = z - \theta(z) = -\tilde{F} \sin(\theta(z)) \leq 0. \quad (32)$$

We also have

$$h_0(z) = \frac{z^2}{2} - (G_0^*)'(z) = \frac{(z - \theta(z))^2}{2} - \tilde{F} \cos(\theta(z)) \quad (33)$$

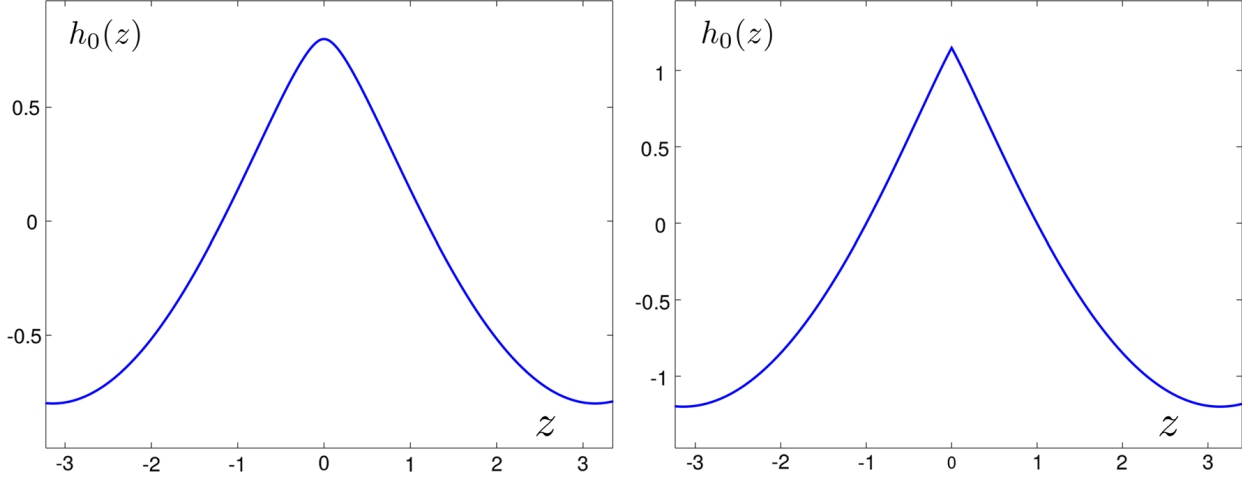


FIG. 4. Plot of the function h_0 in case of $\tilde{F} = 0.8$ (left) and $\tilde{F} = 1.2$ (right). It can be seen that the derivative jumps at $s = 0$ in the second case

and, using (31),

$$h_0(z) = \frac{\tilde{F}^2 (1 - \cos^2(\theta(z)))}{2} + \tilde{F} \cos(\theta(z)) \quad (34)$$

Therefore²

$$\tilde{F} \cos(\theta(z)) = 1 - \sqrt{1 + \tilde{F}^2 - 2h_0(z)} \quad (35)$$

and so³

$$\begin{aligned} h'_0(z) &= -\sqrt{\tilde{F}^2 - (1 - \sqrt{1 + \tilde{F}^2 - 2h_0(z)})^2} \\ &= -\sqrt{2}\sqrt{\tilde{F} - h_0(z)} \frac{\sqrt{\tilde{F} + 1 - \sqrt{1 + \tilde{F}^2 - 2h_0(z)}}}{\sqrt{\tilde{F} - 1 + \sqrt{1 + \tilde{F}^2 - 2h_0(z)}}}. \end{aligned} \quad (36)$$

When $\tilde{F} < 1$, previous remarks apply because the function G_0 is a strictly convex C^∞ function. Point (i) and point (iii) for the case $\tilde{F} < 1$ come directly: we only have to focus on the case $\tilde{F} \geq 1$.

Let us now prove that the restriction of G on the interval $(a(\tilde{F}), \pi)$ is strictly convex. As $G''(\theta) := 1 - \tilde{F} \cos(\theta)$ is strictly increasing, it is enough to check that $G''(a(\tilde{F})) \geq 0$. This is trivial if $\tilde{F} \geq \pi/2$ because

²The justification of the minus sign before the square root involves some cumbersome computations.

³

$$\begin{aligned} h'_0(z) &= -\sqrt{\tilde{F} - 1 + \sqrt{1 + \tilde{F}^2 - 2h_0(z)}} \sqrt{\tilde{F} + 1 - \sqrt{1 + \tilde{F}^2 - 2h_0(z)}} \\ h'_0(z) &= -\frac{\sqrt{(\tilde{F} - 1)^2 - (1 + \tilde{F}^2 - 2h_0(z))}}{\sqrt{\tilde{F} - 1 - \sqrt{1 + \tilde{F}^2 - 2h_0(z)}}} \sqrt{\tilde{F} + 1 - \sqrt{1 + \tilde{F}^2 - 2h_0(z)}} \\ h'_0(z) &= -\sqrt{2}\sqrt{-h_0(z) + \tilde{F}} \frac{\sqrt{\tilde{F} + 1 - \sqrt{1 + \tilde{F}^2 - 2h_0(z)}}}{\sqrt{-F + 1 + \sqrt{1 + \tilde{F}^2 - 2h_0(z)}}}. \end{aligned}$$

$a(\tilde{F}) \in (\pi/2, \pi)$ and so $\cos(a(\tilde{F})) < 0$. This is a bit more complicated if $1 \leq \tilde{F} \leq \pi/2$: $a(\tilde{F}) \in (\pi/2, \pi)$ and we can write

$$G''(a(\tilde{F})) = 1 - \tilde{F} \cos(a(\tilde{F})) = 1 - \sqrt{\tilde{F}^2 - (a(\tilde{F}))^2}$$

The results follow⁴ from the properties of Eq. (28).

We notice now that, by definition of $a(\tilde{F})$,

$$G(\theta) \geq 0 \times \theta + G(a(\tilde{F})),$$

for any θ . On the other hand, $T(\theta) := \pi(\theta - \pi) + G(\pi)$ is tangent to G at $\theta = \pi$ and satisfies, for any $\theta \neq \pi$,

$$G(\theta) - T(\theta) = \frac{1}{2}(\theta - \pi)^2 + \tilde{F}(\cos(\theta) + 1) > 0.$$

As a consequence, owing to strict convexity, the problem (30) admits, for any $z \in (0, \pi]$, a unique solution $\theta(z)$. This solution belongs to $(a(\tilde{F}), \pi]$ and the preliminary remarks apply. The function h_0 is of class C^∞ on $(0, \pi]$ and, as $(G^*)'(0^+) = a(\tilde{F})$, we get the right derivative $h'_0(0^+) = -a(\tilde{F})$. By symmetry, h_0 is also of class C^∞ on the interval $[-\pi, 0)$ and therefore the function h_0 admits a unique singularity at $z = 0$ on the periodicity interval $[-\pi, \pi]$ with a jump of derivative $-2a(\tilde{F})$ at $z = 0$.⁵ Point (ii) is proven.

It remains to compute the variation of h_0 in the case $\tilde{F} \geq 1$. We have $\theta(0) = a(\tilde{F})$ and $\theta(\pi) = \pi$. From (34), we directly deduce

$$V(\tilde{F}) := h(0) - h(\pi) = a(\tilde{F})^2/2 + \tilde{F}(\cos(a(\tilde{F})) + 1)$$

and we get the desired result by taking into account the sign of $\cos(a(\tilde{F}))$. $\square$

⁴In order to prove that $G''(a(\tilde{F})) > 0$, we have to prove that $a(\tilde{F}) > \sqrt{\tilde{F}^2 - 1}$. Owing to the properties of Eq. (28), it is enough to check that $\sqrt{\tilde{F}^2 - 1} < \tilde{F} \sin(\sqrt{\tilde{F}^2 - 1})$ or equivalently that for any $\alpha \in (0, \sqrt{\pi^2/4 - 1})$,

$$\sin(\alpha) - \frac{\alpha}{\sqrt{1 + \alpha^2}} > 0$$

We first notice that, on the interval $(0, \pi)$ owing to Taylor–Lagrange expansion, $\sin(\alpha) \geq \alpha - \frac{\alpha^3}{6}$ and then we are reduced to study the inequality

$$\alpha - \frac{\alpha^3}{6} > \frac{\alpha}{\sqrt{1 + \alpha^2}}$$

since $\alpha < \sqrt{6}$ on the interval $(0, \sqrt{\pi^2/4 - 1})$, the previous relation is equivalent to:

$$1 + \frac{\alpha^4}{36} - \frac{\alpha^2}{3} > \frac{1}{1 + \alpha^2}$$

or

$$0 < 24 - 11\alpha^2 + \alpha^4.$$

The result is ensured by the fact that this inequality is true for any $\alpha^2 \in (0, 3)$.

⁵It may also be useful to remark that, as

$$\theta'(z) = 1/G''(\theta(z)) = 1/(1 - \tilde{F} \cos(\theta(z))) \geq 0,$$

we have

$$h''(z) = 1 - 1/(1 - \tilde{F} \cos(\theta(z))) = -\tilde{F} \cos(\theta(z))/(1 - \tilde{F} \cos(\theta(z))).$$

Hence $h''(z)$ has the opposite sign to $\cos(\theta(z))$. It remains positive on the interval $(a(\tilde{F}), \pi)$ if and only if $a(\tilde{F}) \geq \pi/2$, in otherwords, if $\tilde{F} \geq \pi/2$. In that case h is convex between two successive singularities. If $1 < \tilde{F} < \pi/2$, there exists an inflexion point, and h is concave in a vicinity of the singularities. (We remark that it not very easy to appreciate the inflexion point in Fig. 4, right, because the graph of h_0 is there very similar to a straight line.)

3.1.3. Buckling load. When $\gamma = 0$, the constant function $\phi = 0$ is a critical point of Problem (18). When F is small enough, this trivial solution is the minimizer of the energy but when F increases, other critical solutions may exist. This phenomenon is known as the buckling of the beam under compressive load. We determine in this section the value of the load above which several critical solutions exist.

Lemma 3. *The number of critical solutions of the problem*

$$\min_{\phi} \left\{ \int_0^1 \left(\frac{1}{2} (\phi'(s))^2 + \tilde{K}_2 h_0(\phi(s)) \right) ds; \phi(0) = 0 \right\} \quad (37)$$

with

$$h_0(z) = \frac{(z - \theta(z))^2}{2} + \tilde{F} \cos(\theta(z)) \quad (38)$$

and

$$z = \theta(z) - \tilde{F} \sin(\theta(z)) \quad (39)$$

is larger than one if and only if $\tilde{F} > \frac{1}{1 + \frac{4\tilde{K}_2}{\pi^2}}$.

Proof. It is easy to check that critical solutions of (37) are the solutions, for some constant C , of the differential equation

$$\frac{1}{2} (\phi'(s))^2 - \tilde{K}_2 h_0(\phi(z)) = C \quad (40)$$

with the boundary conditions $\phi(0) = 0$ and $\phi'(1) = 0$.

Let us set $C = \tilde{K}_2 \left(\frac{k^2}{2} - h_0(0) \right)$ or equivalently $|\phi'(0)| = k\sqrt{\tilde{K}_2}$ and let us study for which values of k the solution of (40) with initial conditions $\phi(0) = 0$, $\phi'(0) = k\sqrt{\tilde{K}_2}$ satisfies $\phi'(1) = 0$. If $k^2/2 > V(F) = h_0(0) - h_0(\pi)$, it is clearly impossible.

We let apart the trivial solution $k = 0$, $\phi = 0$, and by symmetry, we focus on the case $\phi'(0) = k\sqrt{\tilde{K}_2} > 0$. We call ϕ_k the solution of (40) and we denote $L(k)$ the smallest positive value of s such that $\phi'_k(s) = 0$. We also denote $z_k = \phi_k(L(k))$ which implies

$$h_0(z_k) = h_0(0) - \frac{k^2}{2}. \quad (41)$$

As ϕ_k is monotonic on $[0, L(k)]$, we can write

$$L(k) = \int_0^{z_k} ((\phi_k)^{(-1)})'(z) dz = \frac{1}{\sqrt{\tilde{K}_2}} \int_0^{z_k} \frac{1}{\sqrt{k^2 + 2(h_0(z) - h_0(0))}} dz$$

Let us first remark that L is a continuous (not necessarily monotonic) function of k which tends to infinity when k tends to $\sqrt{2(h_0(0) - h_0(\pi))}$. Indeed the integral becomes singular when k tends to π . Things are less clear when k tends to zero. Using the change of variable

$$v(z) := \arcsin \left(\frac{\sqrt{2(h_0(0) - h_0(z))}}{k} \right),$$

we get

$$\sqrt{\tilde{K}_2} L(k) = \int_0^{\frac{\pi}{2}} j(k \sin(v)) dv \quad (42)$$

with

$$j(u) := -\frac{u}{h'_0(h_0^{(-1)}(h_0(0) - \frac{u^2}{2}))} = -\frac{u}{h'_0(z_u)}$$

which becomes, using the expressions of h'_0 given by Lemma 2 and Eq. (41),

$$\begin{aligned} j(u) &= \frac{u}{\sqrt{2\tilde{F} - 2h_0(z_u)}} \sqrt{\frac{1 - \tilde{F} + \sqrt{1 + \tilde{F}^2 - 2h_0(z_u)}}{\tilde{F} + 1 - \sqrt{1 + \tilde{F}^2 - 2h_0(z_u)}}} \\ &= \frac{u}{\sqrt{2(\tilde{F} - h_0(0)) + u^2}} \sqrt{\frac{1 - \tilde{F} + \sqrt{1 + \tilde{F}^2 - 2h_0(0) + u^2}}{\tilde{F} + 1 - \sqrt{1 + \tilde{F}^2 - 2h_0(0) + u^2}}} \end{aligned}$$

When $\tilde{F} > 1$, one can easily check that $h_0(0) < \tilde{F}$. The function j has no singularities, $j(0) = 0$ and we can pass to the limit in (42). We obtain $L(0^+) = 0$. The theorem of intermediate values ensures the existence of a k in $(0, +\infty)$ such that $L(k) = 1$;

When $F \leq 1$, we have $h_0(0) = \tilde{F}$ and the expression of j can be simplified

$$j(u) = \sqrt{\frac{1 - \tilde{F} + \sqrt{(1 - \tilde{F})^2 + u^2}}{\tilde{F} + 1 - \sqrt{(1 - \tilde{F})^2 + u^2}}}$$

We see now that j is strictly increasing : L takes values in $(L(0), +\infty)$ and a solution such that $L(k) = 1$ exists if and only if

$$1 > \frac{1}{\sqrt{\tilde{K}_2}} L(0) = \frac{1}{\sqrt{\tilde{K}_2}} \int_0^{\frac{\pi}{2}} j(0) dv = \frac{1}{\sqrt{\tilde{K}_2}} \frac{\pi}{2} \sqrt{\frac{1 - \tilde{F}}{\tilde{F}}}$$

or equivalently

$$\tilde{F} > \tilde{F}_c := \frac{1}{1 + \frac{4\tilde{K}_2}{\pi^2}}.$$

□

It is interesting to write this critical load in terms of the original parameters. It becomes

$$F \geq F_c \frac{1}{1 + \frac{F_c}{K_2}} \quad \text{with} \quad F_c := \frac{\pi^2 K_1}{4L^2}$$

The quantity F_c is known to be the critical load for a clamped-free Euler beam. (The historical reference is [1]; for interesting recent developments see also [36–40].) The previous formula provides the correction to be applied for a Timoshenko beam. As expected the critical value tends to F_c when the Timoshenko coupling parameter K_2 is very large.

Note also that the number of critical solutions increases with $\tilde{F}$. It passes from 1 to 3 when $\tilde{F}$ reaches the critical value $\tilde{F}_c$. It can be shown that more generally, it passes from $2n - 1$ to $2n + 1$ when $\tilde{F}$ reaches the value

$$\frac{1}{1 + \frac{4(2n-1)^2 \tilde{K}_2}{\pi^2}}.$$

To our knowledge, this is the first estimate for the critical load of a beam with energy model (8).

3.1.4. Euler–Lagrange equations. A formal computation starting from (18) leads to the following Euler–Lagrange equations for an inextensible Timoshenko beam under the end load $\mathbf{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2$:

$$\begin{cases} \phi''(s) = \frac{K_2}{K_1}(\phi(s) - \theta(s)) \\ \phi(s) = \theta(s) - \frac{F_1}{K_2} \sin(\theta(s)) + \frac{F_2}{K_2} \cos(\theta(s)) \\ \phi(0) = \phi'(1) = 0 \end{cases} \quad (43)$$

Substituting ϕ in the first equation, we obtain the boundary value problem for $\theta(s)$:

$$\begin{cases} \theta''(s) = \frac{(F_2 \cos \theta - F_1 \sin \theta)(K_2 + K_1 \theta'^2)}{K_1(K_2 - F_1 \cos \theta - F_2 \sin \theta)} \\ \theta(0) = -\frac{F_1}{K_2} \sin(\theta(0)) + \frac{F_2}{K_2} \cos(\theta(0)) \\ \theta'(1) = 0 \end{cases} \quad (44)$$

Of course a minimizer of the problem (18) will not have, in general, enough regularity to solve (43) in a strong sense. A formal computation starting from the functional $\mathcal{A}(\phi)$ defined in (19) leads to the differential equation:

$$\phi'' = \phi - \tilde{K}_2 \frac{dG^*}{d\phi} \quad (45)$$

with $\phi(0) = \phi'(1) = 0$. Clearly, ϕ is at least C^2 on the intervals where G^* is differentiable. (The regularity of G^* is addressed in Lemma 2.) On the other hand, a stationary pair (ϕ, θ) solves the integral equation:

$$\phi(s) = \int_0^s dt \int_t^1 \frac{K_2}{K_1} (\phi - \theta) dx \quad (46)$$

In the intervals on which ϕ is C^2 , the previous integral representation implies C^0 regularity for θ . Hence, the second of Eq. (43) implies $\theta \in C^2$, that together with (46) implies C^∞ regularity for both ϕ and θ almost everywhere, except at the points where G^* is not differentiable. In conclusion, θ solves Eq. (44) in a strong sense on all the intervals in which G^* is differentiable, and we already showed that in case of $\tilde{F} < 1$, this condition holds in all $\mathbb{R}$.

3.2. Regularized Timoshenko model

Since the integrand in (16) is convex and coercive with respect to the highest order derivatives, by a well-known result of calculus of variations (see [31]), the problem:

$$\begin{cases} \min & E(\phi, \theta) := \int_0^1 \left[\frac{K_1}{2} \phi'^2 + \frac{K_2}{2} (\phi - \theta)^2 + \frac{K_3}{2} \theta'^2 + F_1 \cos \theta + F_2 \sin \theta \right] ds \\ \theta \in H^1[0, 1], & \phi \in H^1[0, 1], \quad \theta(0) = \phi(0) = 0 \end{cases} \quad (47)$$

admits at least one solution, say $(\bar{\phi}, \bar{\theta})$. Moreover, the minimizer $(\bar{\phi}, \bar{\theta})$ verifies in a weak form the relative Euler–Lagrange equations:

$$\begin{cases} \phi'' = \frac{K_2}{K_1} (\phi - \theta) \\ \theta'' = \frac{K_2}{K_3} (\theta - \phi) - \frac{F_1}{K_3} \sin \theta + \frac{F_2}{K_3} \cos \theta \end{cases} \quad (48)$$

which means that a minimizing pair $(\bar{\phi}, \bar{\theta})$ is a fixed point for the following nonlinear integral operator:

$$\begin{aligned} (\phi, \theta) &\longmapsto \mathbb{T}(\phi, \theta) := \left(\mathbb{T}_1(\phi, \theta), \mathbb{T}_2(\phi, \theta) \right) \\ \mathbb{T}_1(\phi, \theta) &:= \frac{K_2}{K_1} \int_0^s dx \int_1^x dt [\phi(t) - \theta(t)] \\ \mathbb{T}_2(\phi, \theta) &:= \frac{1}{K_3} \int_0^s dx \int_1^x dt [K_2(\theta(t) - \phi(t)) + F_1 \sin \theta - F_2 \cos \theta] \end{aligned}$$

Clearly $\mathbb{T}(\phi, \theta)$ is in $C^2 \times C^2$ if $(\phi, \theta) \in H^1 \times H^1$. Therefore, $(\bar{\phi}, \bar{\theta})$ verify (48) in a strong (pointwise) sense. Finally, since $(\bar{\phi}, \bar{\theta})$ is a fixed point for $\mathbb{T}$, the previous argument can be applied recursively, which implies C^∞ regularity for the minimizer.

4. Numerical simulations

4.1. Results on the absolute minimizer

In this section, we want to show some numerical results on the absolute minimizers of energy models (18) and (47). The main aim of these numerical investigations is to show the effect of the introduction of the regularizing term in θ'^2 in the deformation energy, and indeed we will “perturb” the problem (18) by means of a term of the type $\varepsilon \theta'^2$ with $\varepsilon \ll 1$.

The numerical procedure to find the minimizer is different in the two cases of models (18) and (47). In the first case, we used a direct numerical minimization (based on finite elements) to find a solution of the problem P (18). We searched for $\tilde{\theta}$ and $\tilde{\phi}$, respectively, in the set of piecewise constant and piecewise linear functions (which are dense, respectively, in L^2 and H^1). As an internal consistency test, we also checked that the solution $\tilde{\theta}$ coincides with the one obtained from $\tilde{\phi}$ by means of (20).

On the other hand, in the case of the problem (47), the regularity proven in Sect. 3.2 allowed us to use the Euler–Lagrange equations. The corresponding boundary value problem was solved by means of a two-dimensional shooting technique. Specifically, an explicit Euler method (with an integration step $\eta = 10^{-3}$) was implemented to solve a set of initial value problems parameterized by means of two parameters:

$$\begin{cases} \phi'' = \frac{K_2}{K_1}(\phi - \theta), & \theta'' = \frac{K_2}{K_3}(\theta - \phi) - \frac{F_1}{K_3} \sin \theta + \frac{F_2}{K_3} \cos \theta, \\ \phi(0) = 0, & \theta(0) = 0, \\ \phi'(0) = A_1, & \theta'(0) = A_2 \end{cases} \quad (49)$$

We studied the behavior of the vector of boundary values $\mathbf{p} := (\phi'(1), \theta'(1))$ as a function of A_1 and A_2 , and employed a direct minimizer to find the points in the space of parameters where $\|\mathbf{p}\|$ is minimal. More precisely, we introduced a square grid (with step length $\ell \approx 10^{-3}$) on the plane (A_1, A_2) and searched for the cells of the grid in which the local minima of $\|\mathbf{p}\|$ are situated.

In the following graphs, we show some relevant results obtained with these two procedures. The parameters employed in the various cases are indicated in the captions (Figs. 5, 6, 7 and 8).

4.2. Results on other stationary points

In this section, we want to show some curled equilibria of the nonlinear Timoshenko beam model (8) and of the regularized model (16), which resemble the ones presented for an inextensible Euler beam (in case

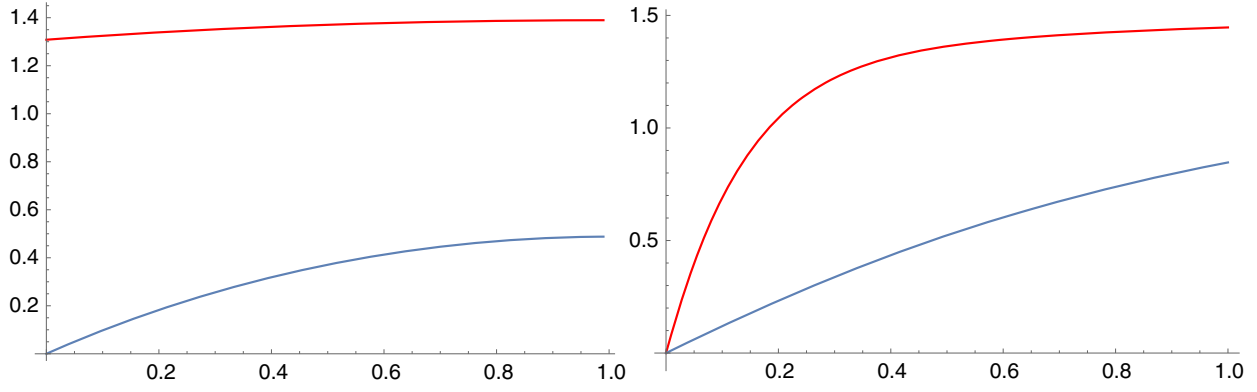


FIG. 5. The minimizing pair $(\tilde{\phi}, \tilde{\theta})$ in case of transverse end load for a nonlinear Timoshenko energy model (left) and regularized model obtained adding to the previous a term in θ'^2 with a small coefficient (right). The parameters are $F_1 = 0$, $F_2 = 10$, $K_1/2 = 1$, $K_2/2 = 1$ (left) and the same with $K_3/2 = 0.1$ (right). Notice that on the left we have $\tilde{\theta}(0) \neq 0$

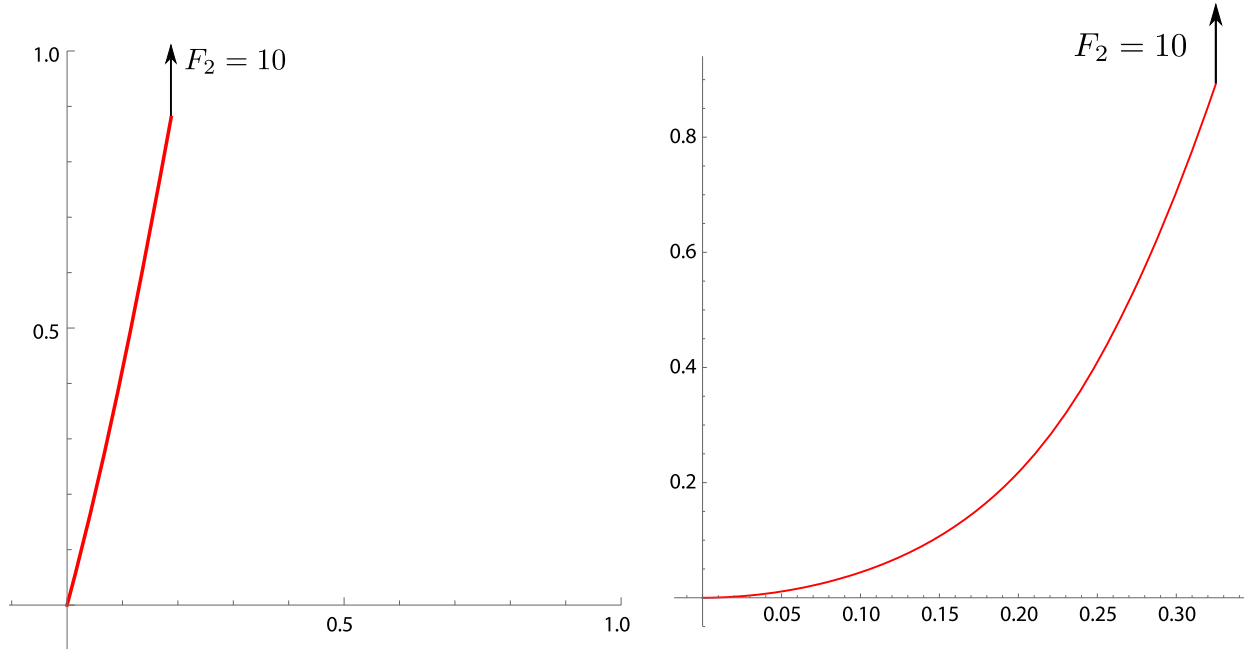


FIG. 6. The deformed shape of the beam corresponding to the case of transverse end load for a nonlinear Timoshenko energy model (left) and regularized model obtained adding to the previous a term in θ'^2 with a small coefficient (right). The parameters are the same as in the previous figure

of distributed load) in [32]. In doing so, we assume that stationary points different from the absolute minimizer solve Euler–Lagrange Eq. (43).

Solving the boundary value problem by means of a shooting technique (similarly to what explained in the previous section), we found the curled solutions shown in Fig. 9. It is interesting to notice that, passing from the simple Euler model to nonlinear Timoshenko and regularized Timoshenko, the property of having such a kind of curled solutions is still kept. The complete study of local minima of the nonlinear Timoshenko beam model is not trivial and would probably require ideas from the theory of phase transitions (see [43, 44]).

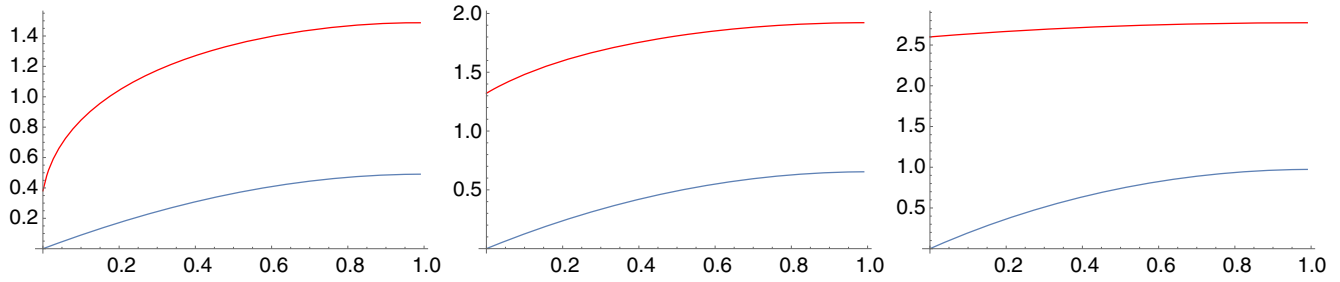


FIG. 7. The minimizing pair $(\tilde{\phi}, \tilde{\theta})$ corresponding to three different compressive end loads for a nonlinear Timoshenko energy model. The parameters are $F_2 = 0$, $K_1/2 = 1$, $K_2/2 = 1$ in the three plots, while $F_1 = 2$ (left), $F_1 = 2.7$ (center) and $F_1 = 10$ (right). Notice that the absolute value of $\tilde{\theta}(0)$ increases with F_1

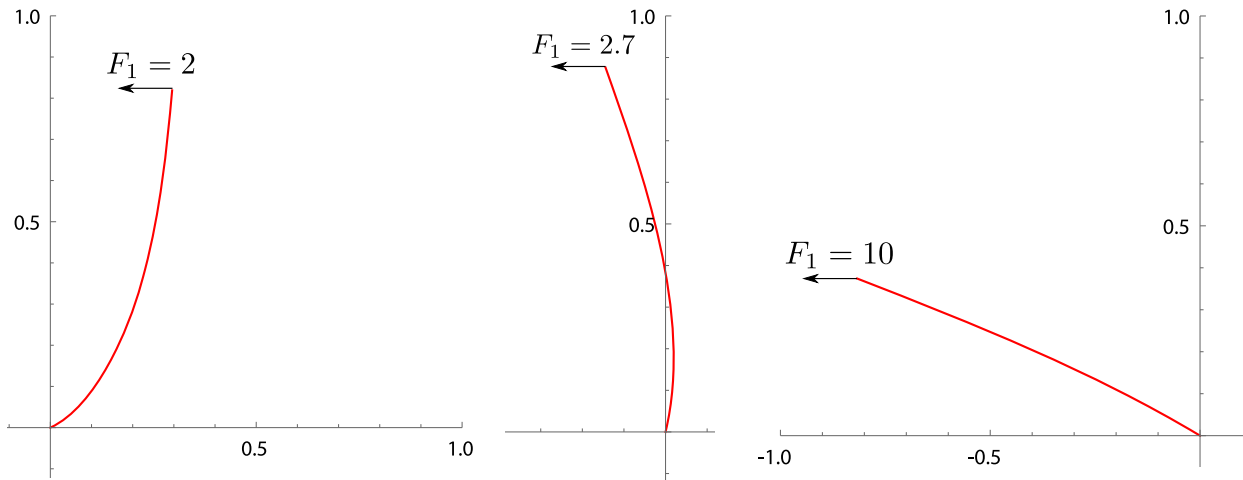


FIG. 8. The deformed shape of the beam corresponding to three different compressive end loads for a nonlinear Timoshenko energy model. The parameters are $F_2 = 0$, $K_1/2 = 1$, $K_2/2 = 1$ in the three plots, while $F_1 = 2$ (left), $F_1 = 2.7$ (center) and $F_1 = 10$ (right)

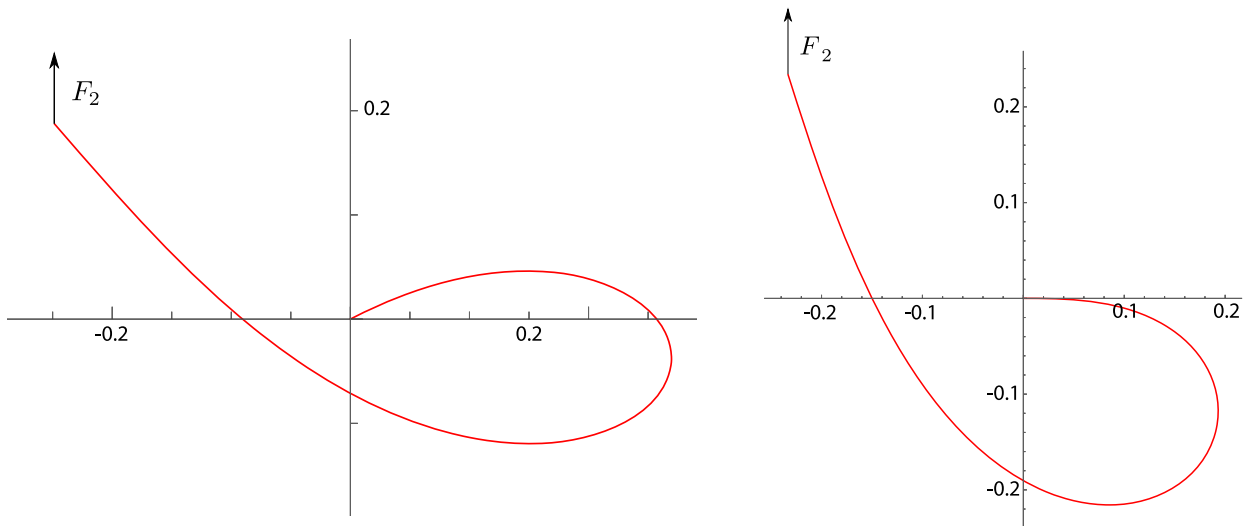


FIG. 9. A curled equilibrium shape for nonlinear Timoshenko model (left); the parameters are: $K_1/2 = 1$, $K_2/2 = 20$, $F_2 = 25$. The same kind of equilibrium shape for a regularized Timoshenko model (right); the parameters are: $K_1/2 = 1$, $K_2/2 = 1$, $K_3/2 = 0.1$, $F_2 = 10$

5. Conclusions

The importance of geometrical nonlinearities is increasingly important in modern structural mechanics (see, e.g., [42]) and in general in elasticity theory (see [45–50]). On the other hand, beam theory, and especially generalized beam theory, is particularly interesting nowadays in view of applications to lattice systems [33, 51, 52], and in this context, pantographic structures are naturally leading to the problem of large deformations of the fibers (see for instance [53–59]).

This paper dealt with the problem of geometrically nonlinear deformations of generalized Timoshenko beam models, obtained by means of a formal homogenization starting from a microstructured 1D system. We considered both a straightforward generalization of the customary linearized model (7) and the model obtained introducing in the microstructure an additional rotational spring entailing a term in θ'^2 in the deformation energy density. We proved well posedness of the variational problem concerning a clamped beam with generic end load, as well as some properties of the minimizers, and presented and discussed some numerical simulations.

The main open problem connected with the content of the work is the generalization of the results to the case of a distributed load, which would allow the use of the result in the modeling of lattice fibrous systems with beams connected each other with finely spaced pivots ([34, 35, 53]). However, this generalization is not trivial as it leads to a non-autonomous variational problem. We remark indeed that the given demonstrations do not generalize trivially to the case of a distributed load, as the monotonicity properties of the minimizer discussed in [30] are not available in general.

Finally, we want to mention that the nonlinear differential equations describing the nonlinear model introduced here are suitable to account for many different phenomena. For instance, they can describe the motion of a pendulum consisting of a weight linked by means of a Hooke's spring to a charge of negligible mass oscillating in an electromagnetic field; in case of distributed load, the electromagnetic field would be time-dependent (see [60]). Also, aspects of the theory of interface instabilities can be interpreted as shear elastic problems and can lead to variational problems presenting similarities with the ones discussed herein (see [61]). Therefore, the interest of studying the nonlinear energy model also in case of distributed load goes probably well beyond 1D elasticity.

References

- [1] Euler, L.: Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti (1744). chapter Additamentum 1, eulerarchive.org E065
- [2] Bernoulli, D.: The 26th letter to Euler. In: Correspondence Mathématique et Physique, vol. 2. P. H. Fuss (1742)
- [3] Bernoulli, J.: Quadratura curvae, e cujus evolutione describitur inflexae laminae curvatura. Die Werke von Jakob Bernoulli, pp. 223–227 (1692)
- [4] Lagrange, J.L.: Mécanique analytique, vol. 1-2. Mallet-Bachelier, Paris (1744)
- [5] Mora, M. G., Müller, S.: A nonlinear model for inextensible rods as a low energy Γ -limit of three-dimensional nonlinear elasticity. In: Annales de l'IHP Analyse non linéaire, vol. 21, pp. 271–293 (2004)
- [6] Pideri, C., Seppecher, P.: Asymptotics of a non-planar rod in non-linear elasticity. Asymptot. Anal. **48**(1, 2), 33–54 (2006)
- [7] Eugster, S.R.: Geometric Continuum Mechanics and Induced Beam Theories, vol. 75. Springer, New York (2015)
- [8] Eugster, S., Glocker, C.: Determination of the transverse shear stress in an Euler–Bernoulli beam using non-admissible virtual displacements. PAMM **14**(1), 187–188 (2014)
- [9] Timoshenko, S.P.: On the correction factor for shear of the differential equation for transverse vibrations of prismatic bar. Philos. Mag. **6**(41), 744 (1921)
- [10] Plantema, F.J.: Sandwich construction; the bending and buckling of sandwich beams, plates, and shells. Wiley, London (1966)

- [11] Turco, E., Barcz, K., Pawlikowski, M., Rizzi, N.L.: Non-standard coupled extensional and bending bias tests for planar pantographic lattices. Part I: numerical simulations. *Zeitschrift für angewandte Mathematik und Physik* **67**(5), 122 (2016)
- [12] Birsan, M., Altenbach, H., Sadowski, T., Eremeyev, V.A., Pietras, D.: Deformation analysis of functionally graded beams by the direct approach. *Compos. Part B Eng.* **43**(3), 1315–1328 (2012)
- [13] Eugster, S.R.: Augmented nonlinear beam theories. In: *Geometric Continuum Mechanics and Induced Beam Theories*, pp. 101–115. Springer, Berlin (2015)
- [14] Piccardo, G., Ferrarotti, A., Luongo, A.: Nonlinear generalized beam theory for open thin-walled members. *Math. Mech. Solids* **22**(10), 1907–1921 (2016). <https://doi.org/10.1177/1081286516649990>. 2017
- [15] Luongo, A., Zulli, D.: *Mathematical Models of Beams and Cables*. Wiley, New York (2013)
- [16] Ruta, G.C., Varano, V., Pignataro, M., Rizzi, N.L.: A beam model for the flexural-torsional buckling of thin-walled members with some applications. *Thin-Walled Struct.* **46**(7), 816–822 (2008)
- [17] Hamdouni, A., Millet, O.: An asymptotic non-linear model for thin-walled rods with strongly curved open cross-section. *Int. J. Nonlin. Mech.* **41**(3), 396–416 (2006)
- [18] Grillet, L., Hamdouni, A., Millet, O.: An asymptotic non-linear model for thin-walled rods. *Comptes Rendus Mécanique* **332**(2), 123–128 (2004)
- [19] Grillet, L., Hamdouni, A., Millet, O.: Justification of the kinematic assumptions for thin-walled rods with shallow profile. *Comptes Rendus Mécanique* **333**(6), 493–498 (2005)
- [20] Hamdouni, A., Millet, O.: An asymptotic linear thin-walled rod model coupling twist and bending. *Int. Appl. Mech.* **46**(9), 1072–1092 (2011)
- [21] dell’Isola, F., Steigmann, D., Della Corte, A.: Synthesis of fibrous complex structures: designing microstructure to deliver targeted macroscale response. *Appl. Mech. Rev.* **67**(6), 060804 (2016)
- [22] Cosserat, E., Cosserat, F.: *Théorie des corps déformables*. Librairie Scientifique A. Hermann et Fils, Paris (1909)
- [23] Forest, S.: *Mechanics of Cosserat Media—An Introduction*, pp. 1–20. Ecole des Mines de Paris, Paris (2005)
- [24] Altenbach, J., Altenbach, H., Eremeyev, V.A.: On generalized Cosserat-type theories of plates and shells: a short review and bibliography. *Arch. Appl. Mech.* **80**(1), 73–92 (2010)
- [25] Eremeyev, V.A., Lebedev, L.P., Altenbach, H.: *Found. Micropolar Mech.* Springer, New York (2012)
- [26] Riey, G., Tomassetti, G.: A variational model for linearly elastic micropolar plate-like bodies. *J. Convex Anal.* **15**(4), 677–691 (2008)
- [27] Riey, G., Tomassetti, G.: Micropolar linearly elastic rods. *Commun. Appl. Anal.* **13**(4), 647–658 (2009)
- [28] Kannan, R., Krueger, C.K.: *Advanced Analysis: On the Real Line*. Springer, New York (2012)
- [29] Rockafellar, R.T.: *Convex Analysis*. Princeton University Press, Princeton (2015)
- [30] Cupini, G., Guidorzi, M., Marcelli, C.: Necessary conditions and non-existence results for autonomous nonconvex variational problems. *J. Differ. Equ.* **243**(2), 329–348 (2007)
- [31] Fonseca, I., Leoni, G.: *Modern Methods in the Calculus of Variations: L^p Spaces*. Springer, New York (2007)
- [32] Della Corte, A., dell’Isola, F., Esposito, R., Pulvirenti, M.: Equilibria of a clamped Euler beam (Elastica) with distributed load: large deformations. *M3AS* (2017), (2016) <https://doi.org/10.1142/S0218202517500221>
- [33] Pipkin, A.C.: Some developments in the theory of inextensible networks. *Q. Appl. Math.* **38**(3), 343–355 (1980)
- [34] Steigmann, D.J., Pipkin, A.C.: Equilibrium of elastic nets. *Philos. Trans. R. Soc. Lond. A Math. Phys. Eng. Sci.* **335**(1639), 419–454 (1991)
- [35] dell’Isola, F., Giorgio, I., Pawlikowski, M., Rizzi, N.L.: Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium. In: *Proceedings of the Royal Society of London A*, vol. 472, no. 2185, p. 20150790. The Royal Society (2016)
- [36] Ferretti, M., D’Annibale, F., Luongo, A.: Flexural-torsional flutter and buckling of braced foil beams under a follower force. *Math. Prob. Eng.* (2017). <https://doi.org/10.1155/2017/2691963>
- [37] Luongo, A., D’Annibale, F.: Double zero bifurcation of non-linear viscoelastic beams under conservative and non-conservative loads. *Int. J. Nonlin. Mech.* **55**, 128–139 (2013)
- [38] Luongo, A., D’Annibale, F.: Bifurcation analysis of damped visco-elastic planar beams under simultaneous gravitational and follower forces. *Int. J. Modern Phys. B* **26**(25), 1246015 (2012)
- [39] Di Egidio, A., Luongo, A., Paolone, A.: Linear and nonlinear interactions between static and dynamic bifurcations of damped planar beams. *Int. J. Nonlin. Mech.* **42**(1), 88–98 (2007)
- [40] Goriely, A., Vandiver, R., Destade, M.: Nonlinear euler buckling. In: *Proceedings of the Royal Society of London A: mathematical, physical and engineering sciences*, vol. 464, no. 2099, pp. 3003–3019. The Royal Society (2008)
- [41] Ball, J.M., Mizel, V.J.: One-dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation. In: *Analysis and Thermomechanics*, pp. 285–348. Springer, Berlin, Heidelberg (1987)
- [42] Fertis, D.G.: *Nonlinear Structural Engineering*. Springer, Berlin, Heidelberg (2006)
- [43] Lawrie, I.D.: Phase transitions. *Contemp. Phys.* **28**(6), 599–601 (1987)
- [44] De Masi, A., Presutti, E., Tsagkarogiannis, D.: Fourier law, phase transitions and the stationary Stefan problem. *Arch. Ration. Mech. Anal.* **201**(2), 681–725 (2011)

- [45] McBride, A.T., Javili, A., Steinmann, P., Bargmann, S.: Geometrically nonlinear continuum thermomechanics with surface energies coupled to diffusion. *J. Mech. Phys. Solids* **59**(10), 2116–2133 (2011)
- [46] Eremeyev, V.A., Pietraszkiewicz, W.: The nonlinear theory of elastic shells with phase transitions. *J. Elast.* **74**(1), 67–86 (2004)
- [47] Steigmann, D.J.: Koiter’s shell theory from the perspective of three-dimensional nonlinear elasticity. *J. Elast.* **111**(1), 91–107 (2013)
- [48] Steigmann, D.J.: A concise derivation of membrane theory from three-dimensional nonlinear elasticity. *J. Elast.* **97**(1), 97–101 (2009)
- [49] Forest, S., Sievert, R.: Nonlinear microstrain theories. *Int. J. Solids Struct.* **43**(24), 7224–7245 (2006)
- [50] Ladevèze, P.: *Nonlinear Computational Structural Mechanics: New Approaches and Non-Incremental Methods of Calculation*. Springer, New York (2012)
- [51] Rivlin, R.S.: Networks of inextensible cords. In: *Collected Papers of RS Rivlin*, pp. 566–579. Springer, New York (1997)
- [52] Pipkin, A.C.: Plane traction problems for inextensible networks. *Q. J. Mech. Appl. Math.* **34**(4), 415–429 (1981)
- [53] Alibert, J.J., Seppecher, P., dell’Isola, F.: Truss modular beams with deformation energy depending on higher displacement gradients. *Math. Mech. Solids* **8**(1), 51–73 (2003)
- [54] Scerrato, D., Giorgio, I., Rizzi, N.L.: Three-dimensional instabilities of pantographic sheets with parabolic lattices: numerical investigations. *Zeitschrift für angewandte Mathematik und Physik* **67**(3), 1–19 (2016)
- [55] Giorgio, I.: Numerical identification procedure between a micro-Cauchy model and a macro-second gradient model for planar pantographic structures. *Zeitschrift für angewandte Mathematik und Physik* **67**(4), 95 (2016)
- [56] Turco, E., Rizzi, N.L.: Pantographic structures presenting statistically distributed defects: numerical investigations of the effects on deformation fields. *Mech. Res. Commun.* **77**, 65–69 (2016)
- [57] Placidi, L., Andreaus, U., Giorgio, I.: Identification of two-dimensional pantographic structure via a linear D4 orthotropic second gradient elastic model. *J. Eng. Math.* **103**(1), 1–21 (2017)
- [58] Barchiesi, E., Placidi, L.: A review on models for the 3D statics and 2D dynamics of pantographic fabrics. In: *Wave dynamics and composite mechanics for microstructured materials and metamaterials*, pp. 239–258. Springer, Singapore (2017)
- [59] Turco, E., Golaszewski, M., Giorgio, I., Placidi, L.: Can a Hencky-type model predict the mechanical behaviour of pantographic lattices? In: *Mathematical Modelling in Solid Mechanics*, pp. 285–311. Springer, Singapore (2017)
- [60] Baker, G.L., Blackburn, J.A.: *The Pendulum: A Case Study in Physics*. Oxford University Press, Oxford (2005)
- [61] De Masi, A., Dirr, N., Presutti, E.: Interface instability under forced displacements. *Ann. Henri Poincaré* **7**(3), 471–511 (2006)

A. Battista
Université de La Rochelle
La Rochelle
France
e-mail: antoniobattista1986@gmail.com

A. Battista, A. Della Corte, F. dell’Isola and P. Seppecher
M&MoCS, Research Center
University of L’Aquila
L’Aquila
Italy
e-mail: alessandro.dellacorte.memocs@gmail.com

F. dell’Isola
e-mail: francesco.dellisola@uniroma1.it

P. Seppecher
e-mail: seppecher@imath.fr

A. Della Corte and F. dell'Isola
University La Sapienza
Rome
Italy

P. Seppecher
IMATH-Université de Toulon
Toulon
France

(Received: December 16, 2017; revised: March 27, 2018)



Large deformations of Timoshenko and Euler beams under distributed load

A. Della Corte , A. Battista, F. dell'Isola and P. Seppecher

Abstract. In this paper, the general equilibrium equations for a geometrically nonlinear version of the Timoshenko beam are derived from the energy functional. The particular case in which the shear and extensional stiffnesses are infinite, which correspond to the inextensible Euler beam model, is studied under a uniformly distributed load. All the global and local minimizers of the variational problem are characterized, and the relative monotonicity and regularity properties are established.

Mathematics Subject Classification. 74B20, 34B15, 49J45.

Keywords. Nonlinear elasticity, Timoshenko beam, Euler beam, Stability of solutions of nonlinear ODEs.

1. Introduction

Since the first mathematically consistent theory of *Elastica*, by Leonhard Euler and Bernoulli brothers (see [1–3]), a great deal of effort has been devoted to the study of beam models, due to their theoretical relevance as a 1D elastic model as well as their importance in structural engineering both as a static and a dynamic element. A rich literature exists considering both linearized and geometrically nonlinear models [4–9] together with generalized 1D elastic models (see [10, 11]), among which beam models capable to take into account other deformations than deflection, extension and shear (see for instance [12–16]). However, two aspects in geometrically nonlinear beam theory especially require further investigation: the behavior of the system under a distributed load and the related multiplicity of arising solutions. An analysis in these directions has been started in the work [17], where an inextensible Euler beam has been considered. In the present work, we extend the results presented in [18], introducing a geometrically nonlinear generalization of an extensible Timoshenko beam under a distributed load and characterizing all the stable equilibrium configurations of the inextensible *Elastica*.

The necessity of considering distributed loads in large deformation arises, for instance, in the field of fluid-structure interaction [19–23] or in the framework of microstructured continua [24, 25], in particular when the microstructure can be modeled as an array of fibers that can be individually modeled as beams undergoing large deformations, while the interaction with the remaining part of the array can be modeled as a distributed load acting on the beam. Possible examples are the structures described in [18, 26–33], which contributed to motivate the present study. The importance of fibrous microstructured systems is increasing in current literature, especially since these objects can be manufactured with great precision and relatively limited costs by means of computer-aided manufacturing (see for instance [34, 35] for useful reference works). Therefore, investigating the behavior of their elementary constituents in geometrically nonlinear deformation is more relevant nowadays. Moreover, equations similar to those describing the equilibrium of beams can also be found in different contexts: That is the case for instance when describing the separation line of self-adhesive polymeric films [36].

Surprisingly, enough there are not many results in the literature, in particular, rigorous ones, concerning nonlinear deformation of beams under distributed load. The classical reference work [4] only covers

the case of concentrated load. Afterward, some numerical results for the inextensible Euler beam under distributed load were published in [37, 38].

The paper is organized as follows: in Sect. 2 a nonlinear version of the extensible Timoshenko beam model is introduced and the problem of a clamped-free beam is formulated. Euler–Lagrange equations are formally derived. In Sect. 3, some numerical results concerning curled equilibrium configurations are shown. These results motivate the analytical study of the properties of the equilibrium solutions done in Sect. 4, in which the Euler–Lagrange equation is studied in the particular case of an infinite shear stiffness, which leads to the nonlinear Euler beam.

2. The model

In this section, we introduce a general Timoshenko model and formally derive the Euler–Lagrange equations associated with the minimization of the total energy.

2.1. Kinematics and deformation energy

We consider a beam lying in a two-dimensional space E in which we fix once and for all an orthonormal reference system $\{\mathbf{e}_1, \mathbf{e}_2\}$. The beam can be understood as the section of a plate which is in planar strain or stress state. The placement of the beam is described by a pair $(\boldsymbol{\chi}, \varphi)$ of functions defined on the interval $[0, L]$ and taking values, respectively, in E and $\mathbb{R}$: $\boldsymbol{\chi}(s)$ represents the position of a point s of the beam while φ is an extra kinematic variable attached to point s . In the classical presentation of Timoshenko linear model in the early 1920s (see [39, 40]), φ is thought to represent the angle of rotation of the sections of the beam (supposed rigid) with respect to the normal to the neutral axis. It is well known that this standard Timoshenko beam model is a particular case of a linear Cosserat 1D continuum in which φ is thought to represent an internal rotational degree of freedom (the original work by Cosserat brothers is [41]; see for instance [42, 42, 43] for interesting applications). The chosen mechanical interpretation of φ is in fact irrelevant. Anyway, we classically call “couple” the dual variable to φ while we call “force” the dual variable to $\boldsymbol{\chi}$.

We assume that, at rest, the beam lies along $\mathbf{e}_1$ so that its placement $(\boldsymbol{\chi}_0, \varphi_0)$ is given $\forall s \in [0, L]$ by $\boldsymbol{\chi}_0(s) = s \mathbf{e}_1$ and $\varphi_0(s) = 0$ so that L stands for the length of the beam at rest. The beam is clamped at the extremity point $s = 0$ (in the sense that $\boldsymbol{\chi}(0) = 0$ and $\varphi(0) = 0$) and this point coincides with the origin of the reference axes.

We introduce the tangent vector to the actual configuration and write it under the form

$$\boldsymbol{\chi}'(s) = \alpha(s) \mathbf{e}(\theta(s)) \quad (1)$$

where

$$\alpha := \|\boldsymbol{\chi}'\| \geq 0, \quad \text{and} \quad \mathbf{e}(\theta) := \cos(\theta) \mathbf{e}_1 + \sin(\theta) \mathbf{e}_2$$

The quantity α accounts for the elongation of the beam, while $\theta(s)$ is the angle that $\boldsymbol{\chi}'(s)$ forms with $\mathbf{e}_1$.

As at rest $\alpha = 1$ and $\varphi = \theta = 0$, we assume that the deformation of the beam can be measured by the quantities $\alpha - 1$, φ' and $\varphi - \theta$ and that the associated energy reads

$$\int_0^L \left\{ k_e f(\alpha(s)) + \frac{k_b}{2} (\varphi'(s))^2 + \frac{k_t}{2} (\varphi(s) - \theta(s))^2 \right\} ds \quad (2)$$

where f is a C^1 function on $]0, +\infty[$ which is: i) positive; ii) convex; iii) tending to $+\infty$ when x tends to 0 or $+\infty$. We extend it on $\mathbb{R}$ by setting $f(x) = +\infty$ when $x \leq 0$ (in the numerical section we will use the function $f(\alpha) = -0.01 \log(\alpha) + (\alpha - 1)^2$ for $x > 0$); as it is easily seen, this function has the desired

properties and is very close to $(\alpha - 1)^2$ except in the vicinity of the origin, where it diverges rapidly). We also introduce the derivative g of f on $]0, +\infty[$.

The three addends in the previous integral are, respectively, called “extensional energy,” “flexural energy” and “shear energy” and the positive material parameters k_e , k_b and k_t are the associated stiffnesses.

It is well known that, in the classical Timoshenko formulations in terms of deflection and rotation, shear energy due to shear strain (i.e., the difference between rotation and gradient of deflection) leads to the so-called numerical shear locking phenomena in standard finite element methods. Traditionally, shear locking has been avoided by introducing reduced numerical integration for the shear term, whereas recently the problem has been overcome by reformulations based on change of variables, for both Timoshenko beams and Reissner–Mindlin plates [44, 45]. The present three-variable formulation (2), with shear strain of the form $\varphi - \theta$, should not be prone to numerical locking either.

The natural functional space for energy (2) is the set of functions $(\alpha, \theta, \varphi) \in \mathcal{V}$ where

$$\mathcal{V} := L^2(0, L) \times L^2(0, L) \times H_0^1(0, L) \quad (3)$$

where H_0^1 stands for set of functions in the Sobolev space H^1 which vanish at $s = 0$. Note that the energy takes values in $[0, +\infty]$ on this functional space as $f \circ \alpha$ may not be integrable. Of course, this will never happen for equilibrium solutions which must have a finite energy.

We suppose that the beam is submitted to a distributed dead load represented by $\mathbf{b} \in L^2((0, L), E)$ and to concentrated load and couple $\mathbf{R}$ and M at the free extremity $s = L$, so that the total energy reads

$$\int_0^L \left\{ k_e f(\alpha(s)) + \frac{k_b}{2} (\varphi'(s))^2 + \frac{k_t}{2} (\varphi(s) - \theta(s))^2 - \mathbf{b}(s) \cdot \boldsymbol{\chi}(s) \right\} ds - \mathbf{R} \cdot \boldsymbol{\chi}(L) - M\varphi(L) \quad (4)$$

Let us introduce the vector $\mathbf{B}(s)$, which may be interpreted as the force exerted by the right side of the beam on the left side at point s , by setting

$$\mathbf{B}(s) := \mathbf{R} + \int_s^L \mathbf{b}(\sigma) d\sigma \quad (5)$$

so that $\mathbf{B}'(s) = -\mathbf{b}(s)$ and $\mathbf{B}(L) = \mathbf{R}$. As we assumed $\boldsymbol{\chi}(0) = 0$ and as, by definition $\mathbf{B}(L) = \mathbf{R}$, integrating by parts gives

$$\int_0^L \mathbf{b}(s) \cdot \boldsymbol{\chi}(s) ds + \mathbf{R} \cdot \boldsymbol{\chi}(L) = \int_0^L \mathbf{B}(s) \cdot \boldsymbol{\chi}'(s) ds.$$

Using (1), this identity reads

$$\int_0^L \mathbf{b}(s) \cdot \boldsymbol{\chi}(s) ds + \mathbf{R} \cdot \boldsymbol{\chi}(L) = \int_0^L \alpha(s) \mathbf{B}(s) \cdot \mathbf{e}(\theta(s)) ds.$$

Therefore, the total energy can be written as a function of α , θ and φ only.

Remark 1. In the following of the paper, we will assume that the distributed load is transverse and uniform ($\mathbf{b}(s) := b\mathbf{e}_2$ for some positive constant b) and that no concentrated load or couple are applied :

$$\mathbf{R} = 0, \quad M = 0 \quad \text{and} \quad \mathbf{B}(s) = B(s)\mathbf{e}_2 \text{ with } B(s) := b(1 - s). \quad (6)$$

Remark 2. The number of parameters in the equilibrium problem can be reduced by adimensionalizing the problem, i.e., by choosing L to be the physical length unit and by choosing the unit of energy such that $k_b = 1$. Hence, in the sequel, we fix $L = 1$ and $k_b = 1$.

2.2. Euler–Lagrange equations

The equilibrium states correspond to stationary points in $\mathcal{V}$ of energy (4). The first variation with respect to φ , α and θ leads to the following boundary value problem¹:

$$\begin{cases} -\varphi'' + k_t(\varphi - \theta) = 0 \\ k_e g(\alpha) - \mathbf{B} \cdot \mathbf{e}(\theta) = 0 \\ k_t(\varphi - \theta) + \alpha \mathbf{B} \cdot \mathbf{e}_\perp(\theta) = 0 \\ \varphi'(1) = 0 \\ \varphi(0) = 0 \end{cases} \quad (7)$$

Remark 3. In the great majority of the textbooks of structural mechanics, the search of stationary points of (4) is performed by directly looking for the solutions of the differential problem (7), assumed as fundamental, without introducing the variational principle (4). The description of contact actions for a Timoshenko beam becomes necessary in this formulation, while starting from a variational principle this delicate issue can be skipped.

Finally, we can add that the variational formulation has several computational advantages (see for instance [46] for a general discussion and [47] for applications to beam problems).

2.3. Particular case of inextensible Euler beam

In the case $k_e = k_t = +\infty$, we have $\alpha = 1$ and $\varphi(s) = \theta(s)$ for all s . The equilibrium problem, in a nondimensional form, then reduces to the minimization of

$$\mathcal{E}(\theta) = \int_0^1 \left\{ \frac{1}{2} (\theta'(s))^2 - b(1-s) \sin(\theta(s)) \right\} ds \quad (8)$$

which corresponds to the case of a clamped inextensible Euler beam in large deformations when submitted to a uniformly distributed transverse force field (see, e.g., [48]). The first variation of (8) with respect to θ gives the boundary value problem:

$$\theta''(s) + b(1-s) \cos(\theta(s)) = 0, \quad \theta(0) = 0, \quad \theta'(1) = 0. \quad (9)$$

This boundary value problem will be studied first numerically in Sect. 3 and then analytically in Sect. 4.

3. Equilibrium configurations of Euler and Timoshenko beams: some numerical results

We focus here on problem (7) and we numerically solve it. Instead of considering problem (7) directly, we remark that, as the function g is the derivative of the convex potential f , it is a bijective function from $]0, +\infty[$ onto $\mathbb{R}$. Hence, from the second equation, we can write α as a function of θ . From the third equation, we can also easily write φ in terms of θ . The first equation becomes a second-order differential equation for θ . As there generally exist more than one value of $\theta(0)$ which ensures $\varphi(0) = 0$ we select the smallest one in absolute value a . It remains to satisfy the condition $\varphi'(1) = 0$. We do this by means of a

¹We introduce here the transverse unit vector $\mathbf{e}_\perp(\theta) := -\sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_2$.

shooting technique : we replace this last condition by $\theta'(0) = k$, numerically solve the resulting Cauchy problem

$$\begin{cases} k_e g(\alpha) - \mathbf{B} \cdot \mathbf{e}(\theta) = 0 \\ k_t (\varphi - \theta) + \alpha \mathbf{B} \cdot \mathbf{e}_\perp(\theta) = 0 \\ -\varphi'' + k_t (\varphi - \theta) = 0 \\ \theta(0) = a \\ \theta'(0) = k \end{cases} \quad (10)$$

by a standard explicit Euler method for different values of the parameter k and finally select the solutions which satisfy (with prescribed accuracy) $\varphi'(1) = 0$.

Note that, in the particular case of an inextensible Euler beam model with uniformly distributed load, this parametric Cauchy problem reduces to:

$$\mathcal{P}_k = \begin{cases} \theta'' = -b(1-s) \cos \theta \\ \theta(0) = 0 \\ \theta'(0) = k. \end{cases} \quad (11)$$

In the sequel, we indicate with $\theta_k(s)$ the solution of (10) or (11). In Fig. 1, we represent $c(k) := (\theta_k)'|_{s=1}$ as a function of the parameter k . The cases $b = 30$ and $b = 60$ are represented. In Fig. 2, we show the same for $b = 200$ and $b = 1000$. It is clear that the values of k such that θ_k is a solution of (9) are, in the case of the inextensible Euler beam, the zeros of the function $c(k)$. We found them numerically with a standard application of the bisection method. These graphs suggest that the number of solutions grows as b increases. Moreover, they suggest that these solutions belong to a neighborhood of the origin depending on b . Indeed, we have:

$$c(k) = (\theta_k)'|_{s=1} = (\theta_k')|_{s=0} + \int_0^1 \theta_k''(t) dt = k - \int_0^1 b(1-t) \cos(\theta_k(t)) dt. \quad (12)$$

Hence $|c(k) - k| \leq b$, which implies that the zeroes of the function c must belong to $[-b, b]$ and that

$$\lim_{k \rightarrow \infty} \frac{c(k)}{k} = 1 \quad (13)$$

This explains why the graph in Figs. 1 and 2 tends to a straight line with slope 1 as k diverges. Remarkably, this happens independently of the value of b .

Let us now consider the deformed shapes of the beam corresponding to equilibrium configurations. We start by showing a solution of the boundary value problem (9) (with $b = 600$) displaying a complex behavior, with three monotonicity intervals for the variable θ . As it is possible to see from Fig. 3, the angle θ never reaches the values $\frac{\pi}{2}$ or $-\frac{3\pi}{2}$ (in the last part it comes quite close to the first value). This is consistent with Proposition 1 of the following section.

In Fig. 4, we show the deformed shapes corresponding to the three solutions relative to the right panel of Fig. 1 for an inextensible Euler beam, while in Fig. 5 we show analogous equilibrium solutions for an inextensible Timoshenko beam. A comparison between the two, under the same adimensional load $b = 60$, shows² that the second model presents an overall decrease in stiffness as expected due to the presence of shear deformation; in the presented simulations the shear stiffness is set at $k_t = 7 \times 10^3$.

²It has to be remarked that another parameter enters the problem for the Timoshenko model, i.e., shear stiffness.

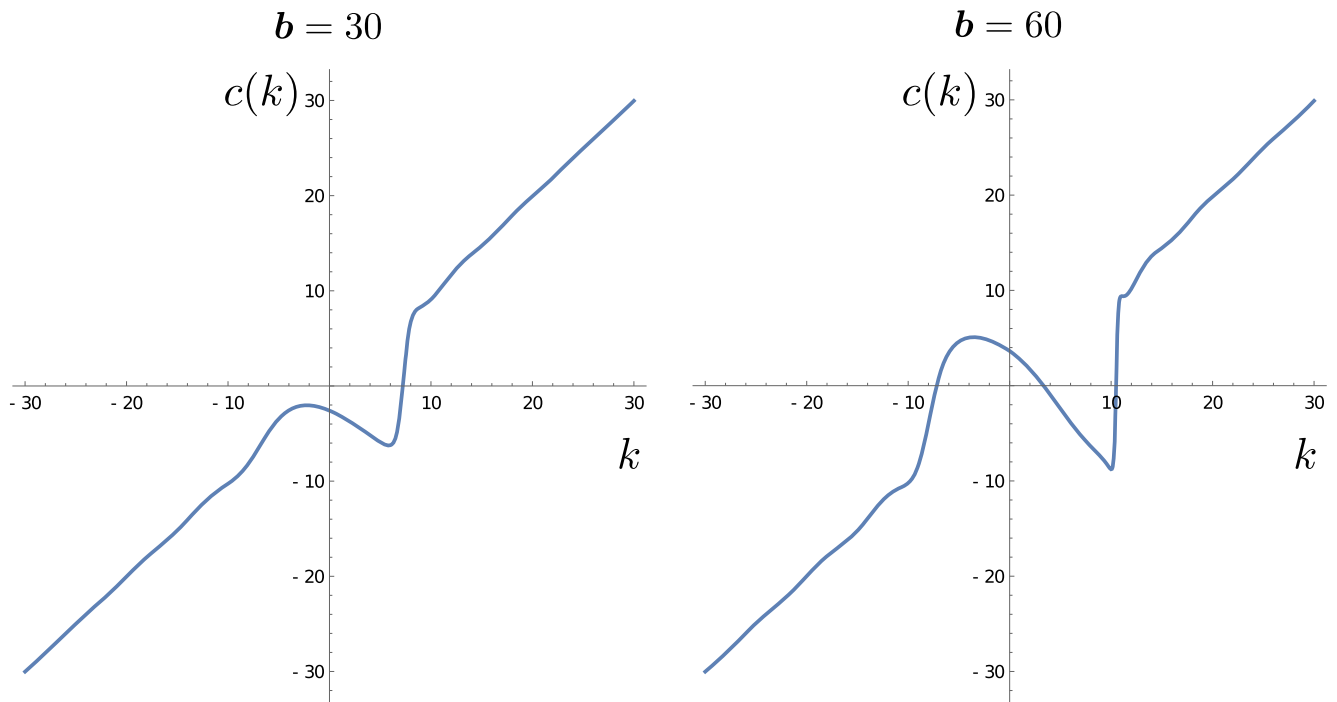


FIG. 1. Parametric shooting: $c(k)$ is plotted for two different values of the adimensional distributed load in the case of an inextensible Euler beam. The plots provide numerical evidence that the boundary value problem (9) admits one solution for $b = 30$ and three solutions for $b = 60$

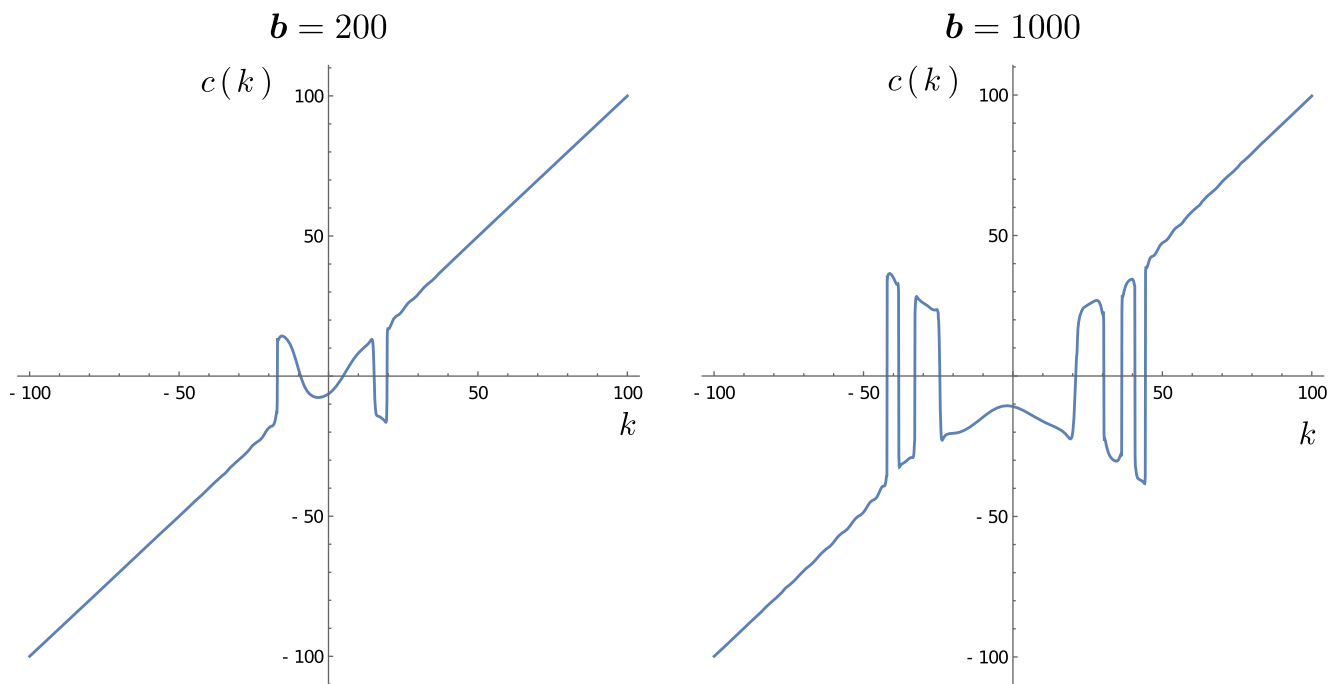


FIG. 2. Parametric shooting: $c(k)$ is plotted for two different values of the adimensional distributed load in the case of an inextensible Euler beam. The plots provide numerical evidence that the boundary value problem (9) admits five solutions for $b = 200$ and nine solutions for $b = 1000$

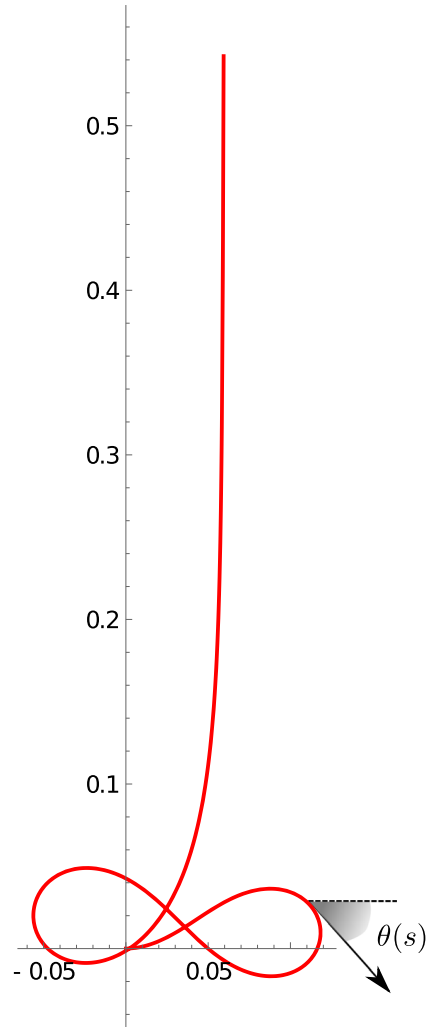


FIG. 3. A solution of the boundary value problem (9) with $b = 600$. The solution is consistent with Proposition 1. Indeed, it is possible to see that the angle θ never reaches the values $\frac{\pi}{2}$ or $-\frac{3\pi}{2}$ (in the last part it comes quite close to the first value)

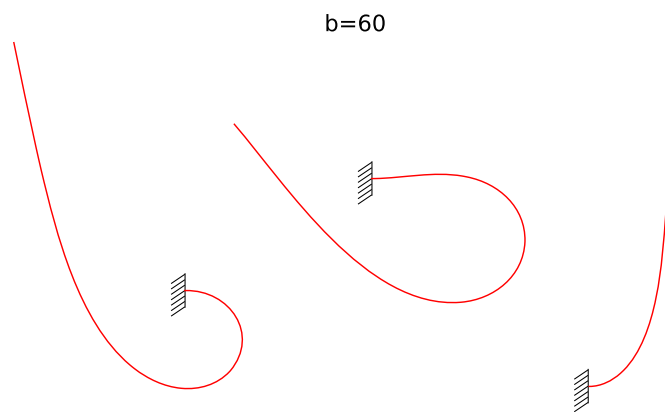


FIG. 4. Deformed shapes corresponding to the three solutions of the boundary value problem (9) with $b = 60$

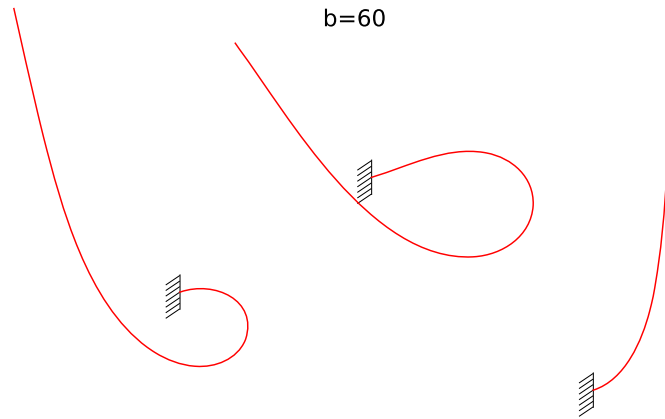


FIG. 5. The three solutions of the boundary value problem (7) with $k_e = +\infty$, $M = 0$ and $k_t = 7 \times 10^3$

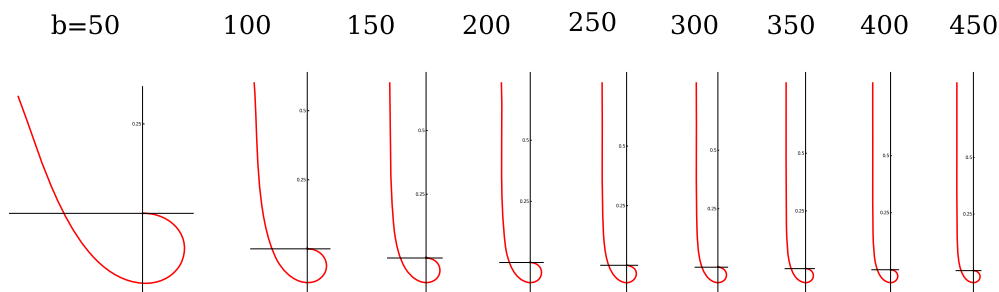


FIG. 6. Parametric study along a branch of solutions for an inextensible Euler beam model varying the transversal load b . The values of the load are indicated in the figure

Now we perform parametric studies which show the influence of the applied load b for the models considered in the paper. Four models are considered, the parameters of which are summarized in the following table. Recalling that $k_b = 1$ we consider:

	Euler inextensible Fig. 6	Euler extensible Fig. 7	Timoshenko inextensible Fig. 9	Timoshenko extensible Fig. 10
k_t	∞	∞	1.8×10^3	1.8×10^3
k_e	∞	5×10^4	∞	2×10^3

In Fig. 6, we show a set of deformed shapes of an inextensible Euler beam. The deformed shapes all belong to a branch of solutions turning around the clamped edge (see the following section for a precise definition of branches). We start from $b = 50$ and gradually increase the load up to $b = 450$.

In Fig. 7, we show a set of deformed shapes of an extensible Euler beam starting from $b = 50$, and gradually increasing the load up to $b = 450$. In this case, another parameter enters the description of the problem, i.e., the extensional stiffness k_e . We recall that the function f describing how the energy density depends on α assumed for all the simulations involving extensibility is $f = -0.01 \log(\alpha) + (\alpha - 1)^2$.

In Fig. 8, we show the plot of the functions θ and α relative to the last simulation presented in Fig. 7. In particular, it is emphasized in the figure that α attains its minimum (corresponding to the maximal local compression) in the point $\bar{s}$ that verify $\theta(\bar{s}) = -\pi/2$, and $\alpha = 1$ in the point $\tilde{s}$ such that $\theta(\tilde{s}) = -\pi$.

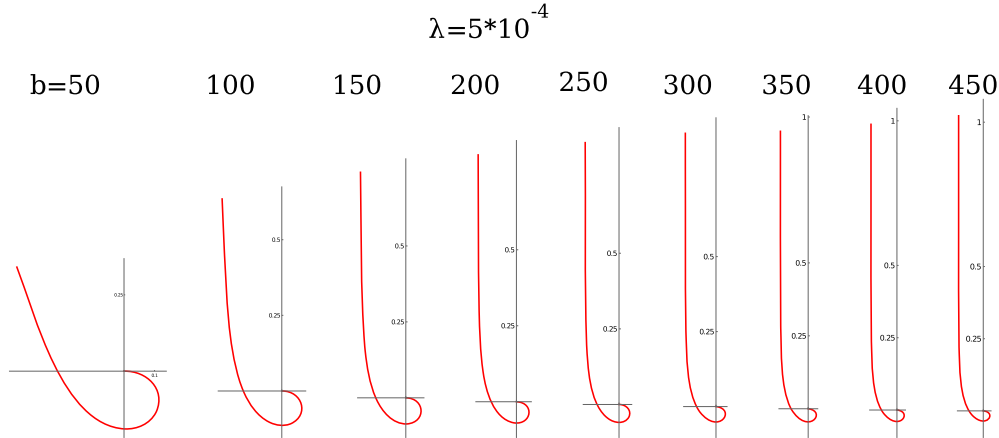


FIG. 7. Parametric study along a branch of solutions for an extensible Euler beam, varying the transversal load b . The values of the load are indicated in the figure. In these simulations, we have set $k_e = 5 \times 10^4$

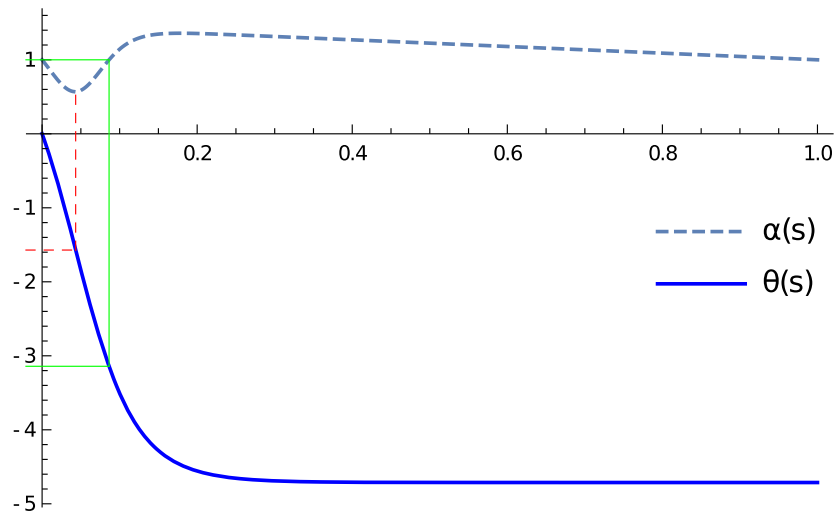


FIG. 8. θ and α for an extensible Euler beam with a transversal load $b = 450$

In Fig. 9, we show a set of deformed shapes of an inextensible Timoshenko beam starting from $b = 50$, and gradually increasing the load up to $b = 450$. Here again, another parameter is needed for the description of the problem, i.e., k_t , the shear stiffness. In these simulations, we have set $k_t = 1.8 \times 10^3$.

In Fig. 10, we show a set of deformed shapes of an extensible Timoshenko beam starting from $b = 50$, and gradually increasing the load up to $b = 450$. In this case, the two parameters introduced above, k_e and k_t are necessary. In these simulations, we have set $k_t = 1.8 \times 10^3$ and $k_e = 2 \times 10^3$.

In Fig. 11, we show the plot of the functions θ , φ and α relative to the last simulation presented in Fig. 10. As for the extensible Euler case, it is emphasized in the figure that α is minimum at the point $\bar{s}$ that verify $\theta(\bar{s}) = -\pi/2$, and $\alpha = 1$ at the point $\tilde{s}$ such that $\theta(\tilde{s}) = -\pi$.

Summarizing our results, we have numerical evidence that there exists a family of “curled” equilibrium configurations, that is configurations in which $|\theta(s)|$ becomes larger than π for some s . It has to be pointed out that the configurations are shown in Figs. 6, 7, 9 and 10 can be also found by a numerical minimization procedure applied to a discretized beam of the type presented in the numerical section of [49]. This suggests the conjecture that the minimizers of the energy (i.e., stable equilibria) are characterized

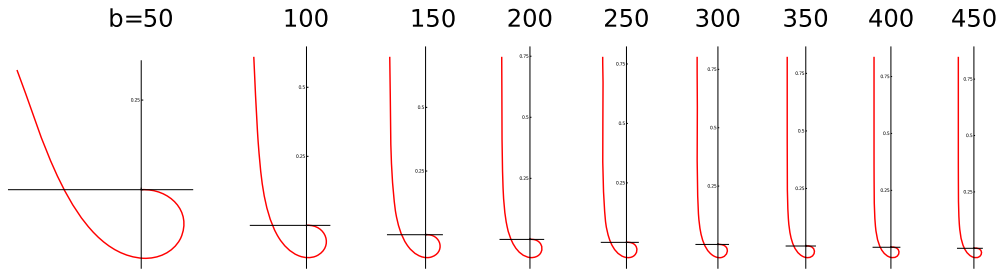


FIG. 9. Parametric study along a branch of solutions for an inextensible Timoshenko beam model varying the transversal load b . The values of the load are indicated in the figure

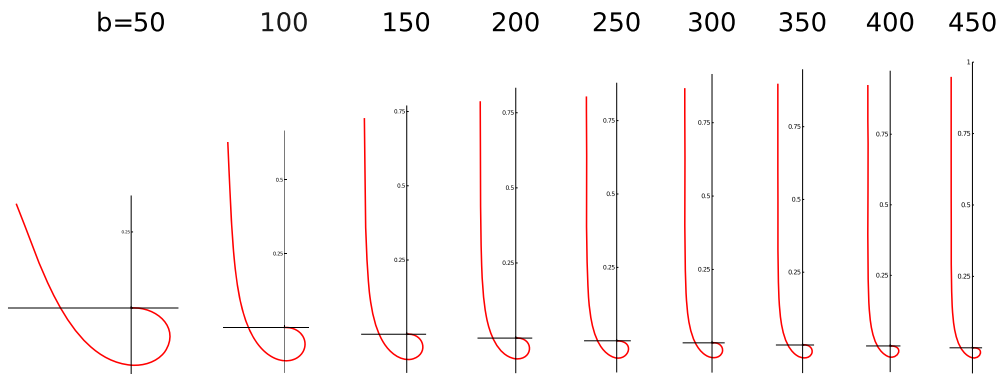


FIG. 10. Parametric study along a branch of solutions for an extensible Timoshenko beam model varying the transversal load b . The values of the load are indicated in the figure

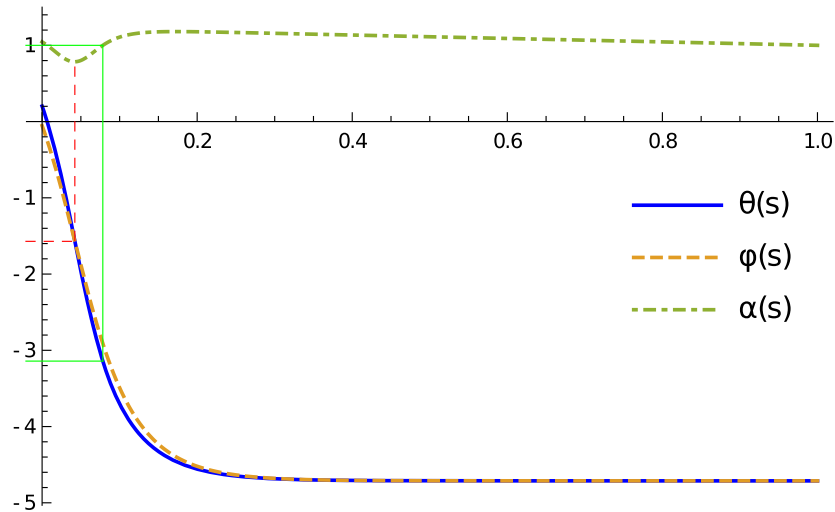


FIG. 11. θ , α and φ for an extensible Timoshenko beam with a transversal load $b = 450$

by the fact that $\theta \leq 0$ everywhere (and never reaches $\theta = -3\pi/2$). In the following section, we formalize and prove these results in the case of an inextensible Euler beam. We extend the results of [17] showing that there can exist only two branches of stable equilibrium configurations.

4. Inextensible Euler beam under distributed load: analytical results

In this section, we prove what we have seen in the numerical simulations shown in Sect. 3, i.e., that there exist a family of curled stable equilibrium configurations for the *Elastica* under distributed load. Our starting point will be the boundary value problem (9).

We classify the different types of solutions by the number of zeros of the function $\sin(\theta)$ on the interval $[0, 1]$. We say that *a solution belongs to branch n of the set of equilibrium solutions if there are exactly n distinct values $0 = s_1 < s_2 < \dots < s_n$ satisfying $\sin(\theta(s_i)) = 0$.*

4.1. A priori bound for the range of stationary points

Here, we prove that, for any stationary point of the functional (8), we have $-\frac{3}{2}\pi < \theta < \frac{\pi}{2}$. This means, informally speaking, that the beam cannot make a complete turn around the clamped point at the equilibrium (see Fig. 12). More precisely, we establish the following:

Proposition 1. *If θ is a solution of (9), then $\sin(\theta(s)) \neq 1$ for every $s \in [0, 1]$.*

Proof. Let us define the function:

$$V(s) := \frac{1}{2}(\theta'(s))^2 + b(1-s)\sin\theta(s) - b(1-s)$$

A direct computation gives

$$V'(s) = \theta'(s)[\theta''(s) + b(1-s)\cos\theta(s)] + b(1-\sin\theta(s))$$

and, using (9),

$$V'(s) = b(1-\sin\theta(s)) \geq 0.$$

Thus V is nondecreasing and, as $V(1) = 0$, V is nonpositive in $[0, 1]$. Assume by contradiction that there exists $\bar{s} \in (0, 1) : \sin(\theta(\bar{s})) = 1$. Then $V(\bar{s}) = \frac{1}{2}(\theta'(\bar{s}))^2 \leq 0$ and thus $\theta'(\bar{s}) = 0$.

The constant function $\theta = \theta_0$ is clearly solution of the Cauchy problem made by Eq. (9) with data $\theta(\bar{s}) = \theta_0$ with $\sin(\theta_0) = 1$ and $\theta'(\bar{s}) = 0$. Uniqueness stated by Cauchy–Lipschitz theorem proves that no such a solution can satisfy $\theta(0) = 0$. $\square$

A graphical representation of a deformed shape that is prohibited as an equilibrium configuration for the clamped *Elastica* under uniform distributed load is shown in Fig. 12.

Remark 4. The function $V(s)$ allows us to give an estimate of the curvature at $s = 0$. Indeed, from $V(0) = \frac{1}{2}(\theta'(0))^2 - b \leq 0$, we get $|\theta'(0)| \leq \sqrt{2b}$.

Remark 5. The previous reasoning holds true as well if we replace $b(1-s)$ with any C^1 , positive and decreasing function, which corresponds to a density of force which has always the same verse (i.e., it is pointing always “upwards” or always “downwards”).

4.2. Study of equilibrium configurations for an inextensible Euler beam

In the following, we will set $B(s) := b(1-s)$. As $\sin(\theta)$ is a continuous function which does not reach 1 when θ takes values in $(-\frac{3\pi}{2}, \frac{\pi}{2})$, $\theta(s_i)$ is either 0 or $-\pi$. On an interval $[s_i, s_{i+1}]$ four situations can arise:

- **S1:** $\theta(s_i) = 0$ and $\theta'(s_i) > 0$: then θ is positive on the considered interval and $\theta(s_{i+1}) = 0$. Owing to (9), we know that θ is strictly concave, it reaches a local maximum at some unique $t_i \in (s_i, s_{i+1})$ and $\theta'(s_{i+1}) < 0$.

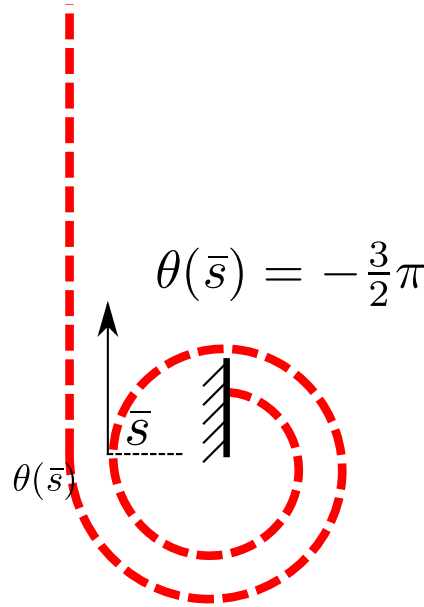


FIG. 12. Example of an “impossible” equilibrium shape for an Euler beam under distributed load: Proposition 1 proves that such a deformed shape, in which the beams turns two times around the clamp, cannot be an equilibrium configuration

- **S2:** $\theta(s_i) = 0$ and $\theta'(s_i) \leq 0$: by integrating (9) we get

$$\frac{(\theta'(s))^2}{2} = \frac{(\theta'(s_i))^2}{2} - B(s) \sin(\theta(s)) - \int_{s_i}^s B'(t) \sin(\theta(t)) dt$$

and thus $(\theta'(s))^2 > (\theta'(s_i))^2$ for $s \in (s_i, s_{i+1}]$. Therefore, θ' cannot vanish, θ is strictly decreasing, $\theta(s_{i+1}) = -\pi$ and $\theta'(s_{i+1}) < 0$.

- **S3:** $\theta(s_i) = -\pi$ and $\theta'(s_i) \leq 0$: then θ belongs to $(-\frac{3\pi}{2}, -\pi)$ on the considered interval and $\theta(s_{i+1}) = -\pi$. Owing to (9), we know that θ is strictly convex, it reaches a local minimum at some unique $t_i \in (s_i, s_{i+1})$ and $\theta'(s_{i+1}) > 0$.
- **S4:** $\theta(s_i) = -\pi$ and $\theta'(s_i) > 0$: For the same reason as in situation (S2), θ' cannot vanish, θ is strictly increasing, $\theta(s_{i+1}) = 0$ and $\theta'(s_{i+1}) > 0$.

On the interval $[s_n, 1]$, only two situations can arise which allow for θ' to vanish.

- **S1':** $\theta(s_n) = 0$ and $\theta'(s_n) > 0$: then θ is positive and strictly concave on the considered interval. It reaches its maximum at $s = 1$ where $\theta'(1) = 0$.
- **S3':** $\theta(s_n) = -\pi$ and $\theta'(s_n) \leq 0$: then θ belongs to $(-\frac{3\pi}{2}, -\pi)$, is strictly concave on the considered interval and reaches its minimum at $s = 1$ where $\theta'(1) = 0$.

Therefore, an equilibrium solution in branch n is made by a sequence of intervals in the order $S1 - S2 - S3 - S4 - S1 \dots$ ending with $S1'$ or $S3'$ and starting, if it is made by more than one interval, with $S1$ or $S2$. For instance, branches 1, 2 and 3 have, respectively, the structure $S1'$, $S2 - S3'$ and $S1 - S2 - S3'$. In general, critical points of a functional do not correspond to local minima: In that case, they correspond to unstable equilibrium solutions. To check if a solution is unstable one usually checks, if it is *linearly unstable* by computing the second Gateaux differential of the functional (8):

$$\int_0^1 [(h'(s))^2 + 2b(1-s) \sin(\theta(s))(h(s))^2] ds \quad (14)$$

The fact that this quadratic form is nonnegative is a necessary condition for stability. Actually, in the particular case of our functional, the fact that this bilinear form is positive is a sufficient condition for stability. Indeed, we have:

Lemma 1. *Let θ be a critical point of (8). If there exists a constant $C > 0$ such that, for every $h \in H^1$ verifying $h(0) = 0$,*

$$\int_0^1 \left[\frac{(h'(s))^2}{2} + b(1-s) \sin(\theta(s)) \frac{(h(s))^2}{2} \right] ds \geq C \|h\|_{H^1}^2 \quad (15)$$

then θ is a local minimizer of $\mathcal{E}$ (defined by (8)) for the H^1 topology and therefore for the uniform norm too.

Proof. A direct computation gives

$$\begin{aligned} \mathcal{E}(\theta + h) &= \mathcal{E}(\theta) + \int_0^1 [\theta' h' - B \cos(\theta) h] ds \\ &\quad + \int_0^1 \left[\frac{(h')^2}{2} + B \sin(\theta) \frac{h^2}{2} - B \left(\sin(\theta) (\cos(h) - 1 + \frac{h^2}{2}) + \cos(\theta) (\sin(h) - h) \right) \right] ds \end{aligned}$$

Integrating by parts, the second addend can be rewritten

$$\int_0^1 [\theta' h' - B \cos(\theta) h] ds = \theta'(1)h(1) - \theta'(0)h(0) - \int_0^1 [\theta''(s) - B \cos(\theta(s))] h(s) ds$$

Let θ be a critical point for $\mathcal{E}$. It satisfies (9) and, as $h(0) = 0$, all terms in the above sum vanish. Using moreover inequality (15), we get the estimation

$$\begin{aligned} \mathcal{E}(\theta + h) - \mathcal{E}(\theta) &\geq C \|h\|_{H^1}^2 \\ &\geq - \int_0^1 B \left(\sin(\theta) (\cos(h) - 1 + \frac{h^2}{2}) + \cos(\theta) (\sin(h) - h) \right) ds \\ &\geq -b \int_0^1 \left[\left| \cos(h) - 1 + \frac{h^2}{2} \right| + |\sin(h) - h| \right] ds \end{aligned}$$

Let η be such that $0 < \eta < \frac{C}{2b}$. There exists $\epsilon > 0$ such that $0 < x < \epsilon$ implies $|\cos(x) - 1 + \frac{x^2}{2}| < \eta x^2$ and $|\sin(x) - x| < \eta x^2$. Hence, for every h such that $\|h\|_{H^1} < \epsilon$, which implies $\|h\|_{L^\infty} < \epsilon$, we have:

$$\int_0^1 \left[\left| \cos(h) - 1 + \frac{h^2}{2} \right| + |\sin(h) - h| \right] ds < 2\eta \int_0^1 h^2 < 2\eta \|h\|_{H^1}^2.$$

Therefore, we obtain that, for any h with $\|h\|_{H^1} < \epsilon$,

$$\mathcal{E}(\theta + h) - \mathcal{E}(\theta) \geq (C - 2b\eta) \|h\|_{H^1}^2$$

□

In the following propositions, we will completely characterize the stable equilibrium configurations of the *Elastica* under distributed load. Indeed, we show that branch 1 corresponds to global minimizers of the functionals and thus to stable equilibrium solutions, that branch 2 corresponds, for b large enough, to other stable equilibrium solutions while all other branches are unstable.

Proposition 2. *For any b , there exists a unique corresponding solution in branch 1. This solution is the global minimizer of the functional (8).*

Proof. The existence of a global minimizer θ of the functional (8) is assured by the coercivity and lower semi-continuity of the functional. The function $\tilde{\theta} := -\frac{\pi}{2} + |\frac{\pi}{2} + \theta|$ has the same energy as θ and takes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$. The truncated function $\bar{\theta} := \max(0, \tilde{\theta})$ (which clearly is in H_0^1) has a lower energy (strictly lower if θ does not belong to branch 1) and takes values in $[0, \frac{\pi}{2})$. Hence, any global minimizer takes values in $(0, \frac{\pi}{2})$: It must belong to branch 1. Let us now prove that, for a given b , there exists at most one solution in branch 1. By contradiction, let us consider two distinct solutions θ_1 and θ_2 . We have $\theta'_1(0) \neq \theta'_2(0)$ owing to Cauchy–Lipschitz uniqueness property. We can therefore suppose that $\theta'_1(0) < \theta'_2(0)$. We then introduce $s_1 > 0$ as the maximum value such that $\theta_1 < \theta_2$ on the interval $(0, s_1)$. On $(0, s_1)$ we have $\cos(\theta_1) > \cos(\theta_2)$, thus $-B \cos(\theta_1) < -B \cos(\theta_2)$, $\theta''_1 < \theta''_2$ and so $\theta'_1 < \theta'_2$. As a consequence $\theta_1(s_1) < \theta_2(s_1)$ which is in contradiction with the definition of s_1 and the continuity of functions θ_1 and θ_2 unless $s_1 = 1$. But in that case $\theta'_1 < \theta'_2$ implies also $\theta'_1(1) < \theta'_2(1)$ which is in contradiction with the fact that $\theta'_1(1) = \theta'_2(1) = 0$. $\square$

To prove that any solution θ in branch n for $n > 2$ is unstable, let us remark that it must contain an interval of type $S1$ or $S3$. Hence, there are at least two distinct points t_i, t_j such that $\theta'(t_i) = \theta'(t_j) = 0$. This prevents linear stability. Indeed, we have:

Proposition 3. *Any solution containing a part defined over $[s_1, s_2] \subset [0, 1]$ such that $\theta'(s_1) = \theta'(s_2) = 0$ is unstable. Thus only branches 1 and 2 can be stable.*

Proof. Let us check the second variation

$$V := \int_0^1 \left[\frac{(h'(s))^2}{2} + b(1-s) \sin(\theta(s)) \frac{(h(s))^2}{2} \right] ds$$

with $h = \theta' 1_{[s_1, s_2]}$ (here by 1_X we denote the characteristic function of the set X). We have:

$$V = \int_{s_1}^{s_2} \left[\frac{(\theta''(s))^2}{2} + \frac{1}{2} (b(1-s) \theta'(s)) (\sin(\theta(s)) \theta'(s)) \right] ds$$

and after integrating by parts the second term:

$$V = \int_{s_1}^{s_2} \left[\frac{(\theta''(s))^2}{2} + \frac{1}{2} (b(1-s) \theta''(s) - b \theta'(s)) \cos(\theta(s)) \right] ds$$

Recalling Eq. (7), we have:

$$\begin{aligned} V &= - \int_{s_1}^{s_2} \left[\frac{1}{2} b \theta'(s) \cos(\theta(s)) \right] ds \\ &= - \int_{s_1}^{s_2} \left[\frac{1}{2} \frac{b(1-s)}{1-s} \theta'(s) \cos(\theta(s)) \right] ds \end{aligned}$$

Recalling again Eq. (7) and integrating by parts once more:

$$\begin{aligned} V &= + \int_{s_1}^{s_2} \left[\frac{1}{2} \frac{\theta''(s)\theta'(s)}{1-s} \right] ds \\ &= - \int_{s_1}^{s_2} \left[\frac{1}{4} \frac{(\theta'(s))^2}{(1-s)^2} \right] ds < 0 \end{aligned}$$

□

It remains to study branch 2. We prove that, for b large enough, the energy possesses some local minimizers which belong to branch 2 and therefore are stable solutions.

Proposition 4. *For b large enough, branch 2 contains stable solutions.*

Proof. We first remark that the energy minimization problem

$$\min_{\theta} \int_0^1 \left(\frac{(\theta'(s))^2}{2} + b(1-s)(1 - \sin(\theta(s))) \right) ds$$

can be rewritten, by setting $u = \frac{\pi}{4} + \frac{\theta}{2}$ and rescaling, as $\min_{u \in \mathcal{S}} F(u)$ where

$$F(u) := \int_0^1 ((u'(s))^2 + 2b(1-s)\cos^2(u(s))) ds$$

and the set $\mathcal{S}$ of admissible u is now the set of functions in $H^1(0,1)$ taking values in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and satisfying $u(0) = \frac{\pi}{4}$.

Let us assume $b^{1/4} > 100$ (which is not an optimal value) and denote $\delta := b^{-1/4}$. Consider the set $\mathcal{O}$ of functions u in $\mathcal{S}$ satisfying $u(\delta) < 0$ and its closure $\overline{\mathcal{O}}$, i.e., the set of functions u in $\mathcal{S}$ satisfying $u(\delta) \leq 0$. We will prove that there exists a local minimizer of the energy in $\mathcal{O}$. Such a local minimizer cannot correspond to a critical solution in branch $n > 2$ as we already know that such a solution is unstable. Neither it can belong to branch 1 as $u(\delta) \leq 0$. Therefore, it must belong to branch 2. To prove that there exists a local minimizer of the energy in $\mathcal{O}$, we consider a minimizer $\bar{u}$ of the energy in $\overline{\mathcal{O}}$ (its existence is clearly ensured as $\overline{\mathcal{O}}$ is closed with respect to the H^1 topology) and check that it actually belongs to $\mathcal{O}$.

We first establish an upperbound for $F(\bar{u})$. Setting $s_0 := \frac{\ln(\sqrt{2}+1)}{\sqrt{b}}$, the function v defined by $v(s) := \frac{\pi}{2} - 2 \arctan(e^{\sqrt{b}(s-s_0)})$ belongs to $\overline{\mathcal{O}}$. Indeed, $v(s) < 0$ as soon as $s > s_0$ and this is the case for $s = \delta$. Moreover, noticing that $v' = -\sqrt{b} \cos(v)$, we can upperbound $F(v)$ by

$$F(v) \leq \int_0^1 ((v'(s))^2 + b \cos^2(v(s))) ds = -2 \int_0^1 \left(\sqrt{b} \cos(v(s)) v'(s) \right) ds \leq 4\sqrt{b}.$$

Therefore $F(\bar{u}) \leq F(v) \leq 4\sqrt{b}$.

Let us now assume, by contradiction, that $\bar{u} \notin \mathcal{O}$. This means $\bar{u}(\delta) = 0$. Using previous estimation, we get

$$\int_0^\delta b(1-s) \cos^2(\bar{u}(s)) ds \leq \int_0^\delta b(1-s) \cos^2(\bar{u}(s)) ds \leq 4\sqrt{b}.$$

Thus, there exists some $t_1 \in (0, \delta)$ such that

$$\cos^2(\bar{u}(t_1)) \leq \frac{4\sqrt{b}}{b(1-\delta)\delta} \leq \frac{4}{99}.$$

and $|\sin(\bar{u}(t_1))| \geq \sqrt{\frac{95}{99}} \geq \frac{95}{99}$. For a similar reason, there exists some $t_2 \in (\delta, 2\delta)$ such that $|\sin(\bar{u}(t_2))| \geq \frac{94}{98}$. We can estimate $F(\bar{u})$ on the different intervals $[0, t_1]$, $[t_1, \delta]$, $[\delta, t_2]$. On each one, we use the estimate:

$$\begin{aligned} \int_x^y ((\bar{u}'(s))^2 + b(1-s)\cos^2(\bar{u}(s))) \, ds &\geq 2 \int_x^y ((\bar{u}'(s))^2 + b(1-y)\cos^2(\bar{u}(s))) \, ds \\ &\geq 2 \int_x^y \sqrt{b(1-y)} |\cos(\bar{u}(s))\bar{u}'(s)| \, ds \\ &\geq 2\sqrt{b(1-y)} |\sin(\bar{u}(y)) - \sin(\bar{u}(x))|. \end{aligned}$$

We obtain the contradiction :

$$\begin{aligned} F(\bar{u}) &\geq 2\sqrt{b(1-2\delta)} \left(|\sin(\bar{u}(t_1)) - \sin(\frac{\pi}{4})| + |\sin(\bar{u}(t_1))| + |\sin(\bar{u}(t_2))| \right) \\ &\geq 2 \times \frac{98}{100} \left(\frac{95}{99} - \frac{\sqrt{2}}{2} + \frac{95}{99} + \frac{94}{98} \right) \sqrt{b} \geq 4\sqrt{b}. \end{aligned}$$

which completes the proof. $\square$

5. Conclusion

We have classified the equilibria of nonlinear Euler and Timoshenko beams subjected to uniformly distributed load. We have identified sequences of equilibria among which two at most are stable. The methods and techniques used here are rather simple and could be efficiently generalized to attack more general theories of beams formulated in order to take into account different kinds of effects. We think of beams with deformation of the section or differentiated deformations of different layers in composite beams [50], strain concentration in thin-walled beams, piezoelectric activated deformations [51].

Another possible extension of the presented result could be the study of the problem introduced in [52] of equilibria of beams constrained to remain on a given smooth surface.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Euler, L., Carathéodory, C.: *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes Sive Solutio Problematis Isoperimetrici Latissimo Sensu Accepti*, vol. 1. Springer, Berlin (1952)
- [2] Bernoulli, D.: The 26th letter to Euler. *Corresp. Math. Phys.* **2**, 1742 (1843)
- [3] Bernoulli, J.: *Quadratura curvae, e cujus evolutione describitur inflexae laminae curvatura*. Die Werke von Jakob Bernoulli **1691**, 223–227 (1692)
- [4] Antman, S.S., Renardy, M.: Nonlinear problems of elasticity. *SIAM Rev.* **37**(4), 637 (1995)
- [5] Steigmann, D.J.: *Finite Elasticity Theory*. Oxford University Press, Oxford (2017)
- [6] Bisshopp, K.E., Drucker, D.C.: Large deflection of cantilever beams. *Q. Appl. Math.* **3**(3), 272–275 (1945)
- [7] Fertis, D.G.: *Nonlinear Structural Engineering*. Springer, Berlin (2006)
- [8] Ladevèze, P.: *Nonlinear Computational Structural Mechanics: New Approaches and Non-incremental Methods of Calculation*. Springer, Berlin (2012)

- [9] Steigmann, D.J.: Invariants of the stretch tensors and their application to finite elasticity theory. *Math. Mech. Solids* **7**(4), 393–404 (2002)
- [10] Nizette, M., Goriely, A.: Towards a classification of Euler–Kirchhoff filaments. *J. Math. Phys.* **40**(6), 2830–2866 (1999)
- [11] Goriely, A., Nizette, M., Tabor, M.: On the dynamics of elastic strips. *J. Nonlinear Sci.* **11**(1), 3–45 (2001)
- [12] Hamdouni, A., Millet, O.: An asymptotic non-linear model for thin-walled rods with strongly curved open cross-section. *Int. J. Non Linear Mech.* **41**(3), 396–416 (2006)
- [13] Luongo, A., Zulli, D.: *Mathematical Models of Beams and Cables*. Wiley, New York (2013)
- [14] Piccardo, G., D’Annibale, F., Luongo, A.: A perturbation approach to the nonlinear generalized beam theory. In: 4th Canadian Conference on Nonlinear Solid Mechanics (CanCNSM 2013) (2013)
- [15] Taig, G., Ranzi, G., D’annibale, F.: An unconstrained dynamic approach for the generalised beam theory. *Contin. Mech. Thermodyn.* **27**(4–5), 879 (2015)
- [16] Piccardo, G., Ranzi, G., Luongo, A.: A complete dynamic approach to the generalized beam theory cross-section analysis including extension and shear modes. *Math. Mech. Solids* **19**(8), 900–924 (2014)
- [17] Della Corte, A., dell’Isola, F., Esposito, R., Pulvirenti, M.: Equilibria of a clamped euler beam (elastica) with distributed load: large deformations. *Mathem. Models Methods Appl. Sci.* **27**, 1–31 (2016)
- [18] dell’Isola, F., Giorgio, I., Pawlikowski, M., Rizzi, N.L.: Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium. *Proc. R. Soc. A* **472**, 20150790 (2016)
- [19] Bungartz, H.-J., Schäfer, M.: *Fluid–Structure Interaction: Modelling, Simulation, Optimisation*, vol. 53. Springer, Berlin (2006)
- [20] Bazilevs, Y., Takizawa, K., Tezduyar, T.E.: *Computational Fluid–Structure Interaction: Methods and Applications*. Wiley, New York (2013)
- [21] Solaria, G., Pagnini, L.C., Piccardo, G.: A numerical algorithm for the aerodynamic identification of structures. *J. Wind Eng. Ind. Aerodyn.* **69**, 719–730 (1997)
- [22] Pagnini, L.C.: A numerical approach for the evaluation of wind-induced effects on inclined, slender structural elements. *Eur. J. Environ. Civ. Eng.* **21**, 1–20 (2016)
- [23] Liberge, E., Pomarede, M., Hamdouni, A.: Reduced-order modelling by pod-multiphase approach for fluid–structure interaction. *Eur. J. Comput. Mech. Revue Eur. Méc. Numér.* **19**(1–3), 41–52 (2010)
- [24] Pideri, C., Seppecher, P.: A second gradient material resulting from the homogenization of an heterogeneous linear elastic medium. *Contin. Mech. Thermodyn.* **9**(5), 241–257 (1997)
- [25] Forest, S., Sievert, R.: Nonlinear microstrain theories. *Int. J. Solids Struct.* **43**(24), 7224–7245 (2006)
- [26] Atai, A.A., Steigmann, D.J.: On the nonlinear mechanics of discrete networks. *Arch. Appl. Mech.* **67**(5), 303–319 (1997)
- [27] Boutin, C., Giorgio, I., Placidi, L., et al.: Linear pantographic sheets: asymptotic micro-macro models identification. *Math. Mech. Complex Syst.* **5**(2), 127–162 (2017)
- [28] Placidi, L., Andreas, U., Giorgio, I.: Identification of two-dimensional pantographic structure via a linear d4 orthotropic second gradient elastic model. *J. Eng. Math.* **103**(1), 1–21 (2017)
- [29] Scerrato, D., Giorgio, I., Rizzi, N.L.: Three-dimensional instabilities of pantographic sheets with parabolic lattices: numerical investigations. *Z. Angew. Math. Phys.* **67**(3), 53 (2016)
- [30] Turco, E., Golaszewski, M., Cazzani, A., Rizzi, N.L.: Large deformations induced in planar pantographic sheets by loads applied on fibers: experimental validation of a discrete Lagrangian model. *Mech. Res. Commun.* **76**, 51–56 (2016)
- [31] Turco, E., Golaszewski, M., Giorgio, I., D’Annibale, F.: Pantographic lattices with non-orthogonal fibres: experiments and their numerical simulations. *Compos. Part B Eng.* **118**, 1–14 (2017)
- [32] Turco, E., Golaszewski, M., Giorgio, I., Placidi, L.: Can a Hencky-type model predict the mechanical behaviour of pantographic lattices? In: Dell’Isola, F. (ed.) *Mathematical Modelling in Solid Mechanics*, pp. 285–311. Springer, Berlin (2017)
- [33] Placidi, L., Barchiesi, E., Turco, E., Rizzi, N.L.: A review on 2d models for the description of pantographic fabrics. *Z. Angew. Math. Phys.* **67**(5), 121 (2016)
- [34] Kalpakjian, S., Vijai Sekar, K.S., Schmid, S.R.: *Manufacturing Engineering and Technology*. Pearson, London (2014)
- [35] Misra, A., Placidi, L., Scerrato, D.: A review of presentations and discussions of the workshop computational mechanics of generalized continua and applications to materials with microstructure that was held in Catania 29–31 October 2015. *Math. Mech. Solids* **9**, 1891–1904 (2016)
- [36] Nase, M., Rennert, M., Naumenko, K., Eremeyev, V.A.: Identifying tractionseparation behavior of self-adhesive polymeric films from in situ digital images under t-peeling. *J. Mech. Phys. Solids* **91**, 40–55 (2016)
- [37] Faulkner, M.G., Lipsett, A.W., Tam, V.: On the use of a segmental shooting technique for multiple solutions of planar elastica problems. *Comput. Methods Appl. Mech. Eng.* **110**(3–4), 221–236 (1993)
- [38] Raboud, D.W., Faulkner, M.G., Lipsett, A.W.: Multiple three-dimensional equilibrium solutions for cantilever beams loaded by dead tip and uniform distributed loads. *Int. J. Non Linear Mech.* **31**(3), 297–311 (1996)
- [39] Timoshenko, S.P.: Lxvi. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Lond. Edinb. Dublin Philos. Mag. J. Sci.* **41**(245), 744–746 (1921)

- [40] Timoshenko, S.P.: X. On the transverse vibrations of bars of uniform cross-section. Lond. Edinb. Dublin Philos. Mag. J. Sci. **43**(253), 125–131 (1922)
- [41] Cosserat, E., Cosserat, F., et al.: *Théorie des corps déformables*. A. Hermann et fils, Paris (1909)
- [42] Altenbach, H., Birsan, M., Eremeyev, V.A.: Cosserat-type rods. In: Altenbach, H. (ed.) *Generalized Continua from the Theory to Engineering Applications*, pp. 179–248. Springer, Berlin (2013)
- [43] Altenbach, J., Altenbach, H., Eremeyev, V.A.: On generalized cosserat-type theories of plates and shells: a short review and bibliography. Arch. Appl. Mech. **80**(1), 73–92 (2010)
- [44] Balobanov, V., Niiranen, J.: Locking-free variational formulations and isogeometric analysis for the Timoshenko beam models of strain gradient and classical elasticity. Comput. Methods Appl. Mech. Eng. **339**, 137–159 (2018)
- [45] Beirao da Veiga, L., Hughes, T.J.R., Kiendl, J., Lovadina, C., Niiranen, J., Reali, A., Speleers, H.: A locking-free model for Reissner–Mindlin plates: analysis and isogeometric implementation via nurbs and triangular nurps. Math. Models Methods Appl. Sci. **25**, 1519–1551 (2015)
- [46] Capobianco, G., Eugster, S.R.: Time finite element based Moreau-type integrators. Int. J. Numer. Methods Eng. **114**(3), 215–231 (2018)
- [47] Eugster, S.R., Hesch, C., Betsch, P., Glocker, C.: Director-based beam finite elements relying on the geometrically exact beam theory formulated in skew coordinates. Int. J. Numer. Methods Eng. **97**(2), 111–129 (2014)
- [48] Eugster, S.R., et al.: *Geometric Continuum Mechanics and Induced Beam Theories*, vol. 75. Springer, Berlin (2015)
- [49] Alibert, J.-J., Della Corte, A., Giorgio, I., Battista, A.: Extensional elastica in large deformation as Γ -limit of a discrete 1D mechanical system. Z. Angew. Math. Phys. **68**(2), 42 (2017)
- [50] Birsan, M., Altenbach, H., Sadowski, T., Eremeyev, V.A., Pietras, D.: Deformation analysis of functionally graded beams by the direct approach. Compos. Part B Eng. **43**, 1315–1328 (2012)
- [51] Chróścielewski, J., Schmidt, R., Eremeyev, V.A.: Nonlinear finite element modeling of vibration control of plane rod-type structural members with integrated piezoelectric patches. Contin. Mech. Thermodyn. **31**, 1–42 (2018)
- [52] Javili, A., McBride, A., Steinmann, P.: Thermomechanics of solids with lower-dimensional energetics: on the importance of surface, interface, and curve structures at the nanoscale. a unifying review. Appl. Mech. Rev. **65**, 010802 (2013)

A. Della Corte, A. Battista, F. dell’Isola and P. Seppecher
M&MoCS, Research Center
University of L’Aquila
L’Aquila
Italy
e-mail: alessandro.dellacorte.memocs@gmail.com

A. Battista
e-mail: antoniobattista1986@gmail.com

F. dell’Isola
e-mail: fdellisola@gmail.com

P. Seppecher
e-mail: seppecher@imath.fr

A. Battista
Université de La Rochelle
La Rochelle
France

F. dell’Isola
DISG
University La Sapienza
Rome
Italy

P. Seppecher
IMATH
Université de Toulon
Toulon
France

(Received: July 26, 2018; revised: February 21, 2019)

Extensible beam models in large deformation under distributed loading: a numerical study on multiplicity of solutions

Journal Title

XX(X):1–17

©The Author(s) 2016

Reprints and permission:

sagepub.co.uk/journalsPermissions.nav

DOI: 10.1177/ToBeAssigned

www.sagepub.com/

SAGE

Francesco dell'Isola^{1,2}, Alessandro Della Corte¹ and Antonio Battista^{1,3}

Abstract

In this paper we present numerical solutions to a geometrically nonlinear version of the extensible Timoschenko beam model under distributed load. The particular cases in which: i) extensional stiffness is infinite (inextensible Timoshenko model), ii) shear stiffness is infinite (extensible Euler model) and iii) extensional and shear stiffnesses are infinite (inextensible Euler model) will be numerically explored. Parametric studies on the axial stiffness in both the Euler and Timoshenko cases will also be shown and discussed.

Keywords

Timoshenko beam. Large deformation of beams. Extensional beam model. Shooting technique.

¹M&MoCS, Research Center, University of L'Aquila, L'Aquila, Italy.

²DISG, University La Sapienza, Rome, Italy.

³LaSIE, Université de La Rochelle, La Rochelle, France.

Corresponding author:

Antonio Battista, LaSIE, Université de La Rochelle, La Rochelle, France.

Email: antonio.battista@univ-lr.fr

Introduction

In the recent literature the behaviour of a clamped-free nonlinear inextensible Euler *Elastica* (introduced in ^{1–3}; see ^{4,5} for recent general reference works) has been mathematically investigated under distributed load ⁶. In particular, the set of stable equilibrium configurations has been completely characterized in [SEPPECHER SOTTOMESSO]. On the other hand, extending such rigorous results to the extensible Euler beam model, or to the Timoshenko beam model is not straightforward. Indeed, the presence of the additional kinematical descriptor accounting for extensibility gives rise to various new mathematical difficulties. The most basic one is that it changes the functional set in which the problem is naturally collocated and in particular prevents it from being a vector space, since a strictly positive local axial deformation has to be prescribed. Of course one can obtain this making suitable assumptions on the energy, but in any case the non-autonomous variational problem that arises in the case of a distributed load will present new difficulties with respect to the inextensible case ⁷.

The problem of existence and stability of equilibrium configurations for extensible Euler and Timoshenko beams in large deformations under distributed load is therefore an open one. Some heuristic considerations would suggest that allowing extensibility should favor stability over instability. Because of this reason, it is interesting to study the behaviour of solutions by means of a systematic collection of parametric studies starting from the inextensible *Elastica* and approaching more general beam models. The present work is aimed at performing such kind of investigation. Our main effort will be to show the variety of different (and at times rather exotic) Equilibrium configurations that can arise when the value of the load is large enough. Moreover, we want to numerically investigate how fast the number of possible Equilibrium solutions increases with the load and how this particular feature is affected by allowing shear deformation.

The study of these exotic configurations is particularly important, nowadays, due to the enhancement of computational methods that make nonlinearity for structural members more practically relevant ^{8–12}. For instance, they are becoming fashionable as an elementary constituent of microstructured objects manufactured with computer-aided techniques ^{13–18}. These objects are potentially advantageous for their mechanical characteristics ^{19–21} and their theoretical study requires tools from homogenization theory ^{22–24} as well as nonlinear elasticity of beams. In this last regard, certainly a full understanding of the onset and the characteristics of multiple solution for the static problem under distributed load would be an important step forward, and possibly more general beam models will also have to be considered ^{25–27} because of the exotic properties of microstructured continua ^{28–30}.

The paper is organized as follows: we introduce the general model for a nonlinear version of the Timoshenko beam; we deduce the equilibrium equations by means of the Lagrange multipliers method. Then, imposing stationarity to the energy functional, we deduce an expression for the total energy in stationary points which depends only on the angles formed by the tangent to the deformed shape. Then we show and discuss numerical results on the multiplicity of solutions with large value of the load and on the effect of

releasing extensional stiffness in both the Euler and the Timoshenko case. Finally we propose some future research directions.

The model

Kinematics and Deformation Energy

Let $\{\mathbf{D}_1, \mathbf{D}_2\}$ be an orthogonal reference system in which the beam lies in the unstressed configuration along $\mathbf{D}_1$. We will denote by s the abscissa along the beam, by the apex $'$ the differentiation with respect to the reference abscissa^{*} and by $\chi(s)$ the placement function. The tangent vector to the current configuration of the beam is then:

$$\chi' = \alpha(s) [\cos(\theta(s))\mathbf{D}_1 + \sin(\theta(s))\mathbf{D}_2] := \alpha(s)\mathbf{e}(\theta(s)) \quad (1)$$

where $\mathbf{e}(\theta(s))$ represents the unit vector parallel to χ' . Therefore $\alpha(s)$ describes the local elongation of the beam:

$$\|\chi'(s)\| = \alpha(s) \quad (2)$$

while $\theta(s)$ is the angle between $\chi'(s)$ and $\mathbf{D}_1$.

We will assume the following energy functional:

$$\mathcal{E}^{\text{def}}(\chi, \varphi) := \int_0^L \left\{ \underbrace{\frac{k_e}{2} (\|\chi'(s)\| - 1)^2}_{\text{extensional energy}} + \underbrace{\frac{k_b}{2} (\varphi'(s))^2}_{\text{flexural energy}} + \underbrace{\frac{k_t}{2} (\varphi(s) - \theta(s))^2}_{\text{shear energy}} \right\} ds \quad (3)$$

Here φ is the angle between sections of the beam (supposed rigid) and the normal to the neutral axes, while k_b , k_e and k_t are respectively the bending, extensional and shear stiffness. The energy can be rewritten as:

$$\mathcal{E}^{\text{def}} = \int_0^L \left\{ \frac{k_e}{2} (\alpha - 1)^2 + \frac{k_b}{2} (\varphi'(s))^2 + \frac{k_t}{2} (\varphi(s) - \theta(s))^2 \right\} ds. \quad (4)$$

We introduce now a uniformly distributed load $\mathbf{b}(s)$ and a concentrated load and couple in the endpoint of the beam $s = L$, denoted respectively by $\mathbf{R}(L)$ and $\mathbf{M}(L)$. The total energy of the system is then:

$$\mathcal{E}^{\text{tot}} = \int_0^L \left\{ \frac{k_e}{2} (\alpha - 1)^2 + \frac{k_b}{2} (\varphi'(s))^2 + \frac{k_t}{2} (\varphi(s) - \theta(s))^2 - \mathbf{b} \cdot \chi \right\} ds - \mathbf{R} \cdot \chi(L) - \mathbf{M} \varphi(L) \quad (5)$$

The beam model described by the functional (5) is a geometrically nonlinear version of the Timoshenko beam model (introduced in^{32,33}), which is a particular case of Cosserat

^{*}Notice that this means that, denoting by θ the angle formed by the tangent to the deformed shape and a reference axis, θ' does not coincide with the geometrical curvature but with the so-called Chebyshev curvature (see³¹).

continuum (introduced in³⁴; for general references and interesting results see e.g.^{35–38}). A periodic mechanical system whose homogenized limit is the model (5) (in the particular case $\alpha \equiv 1$) is shown in⁷. It is in fact a microstructured 1D system whose unit cell is an articulated parallelogram and equipped with suitably placed rotational springs, and it can be easily obtained by means of 3D printing. Of course also other (possibly more complex) microstructured systems can have a similar homogenized version (on microstructured continua see e.g.^{39–41}).

Lagrange multipliers method

Given the total energy of the system, equilibrium configurations are found as stationary points of the energy functional (5). A synthetic formulation of the problem, taking into account together the constrain of equation (1) and the total energy, is obtained with the introduction of a Lagrange multiplier $\Lambda(s)$. We get the following functional formulation:

$$\mathcal{E}^{\text{tot}} = \int_0^L \left\{ \frac{k_e}{2} (\alpha - 1)^2 + \frac{k_b}{2} (\varphi'(s))^2 + \frac{k_t}{2} (\varphi(s) - \theta(s))^2 - \mathbf{b} \cdot \boldsymbol{\chi} + \right. \\ \left. + \Lambda \cdot (\boldsymbol{\chi}' - \alpha [\cos(\theta(s)) \mathbf{D}_1 + \sin(\theta(s)) \mathbf{D}_2]) \right\} ds - \mathbf{R} \cdot \boldsymbol{\chi}(L) - \mathbf{M} \varphi(L) \quad (6)$$

i.e., the energy is a function of the fields $\alpha(s), \varphi(s), \theta(s), \boldsymbol{\chi}(s), \Lambda(s)$. The first variation of the energy with respect to these fields (considered independent) gives the two boundary value problems (BVPs):

$$\begin{cases} -k_b \varphi'' + k_t (\varphi - \theta) = 0 \\ k_b \varphi'(L) \delta \varphi(L) = \mathbf{M} \delta \varphi(L) \\ k_b \varphi'(0) \delta \varphi(0) = 0 \end{cases} \quad (7)$$

and

$$\begin{cases} -\Lambda' - \mathbf{b} = 0 \\ \Lambda(L) \cdot \delta \boldsymbol{\chi}(L) = \mathbf{R} \cdot \delta \boldsymbol{\chi}(L) \\ \Lambda(0) \cdot \delta \boldsymbol{\chi}(0) = 0, \end{cases} \quad (8)$$

as well as the algebraic relations:

$$k_t (\varphi - \theta) + \alpha \Lambda \cdot \mathbf{e}_\perp(\theta) = 0 \quad (9)$$

$$k_e (\alpha - 1) - \Lambda \cdot \mathbf{e}(\theta) = 0 \quad (10)$$

and the kinematic constraint eq. (1).

In the previous relation we introduced:

$$\mathbf{e}_\perp(\theta(s)) = -\sin(\theta(s)) \mathbf{D}_1 + \cos(\theta(s)) \mathbf{D}_2$$

and the two boundary conditions in 0 are imposed considering a cantilever beam (lying along $\mathbf{D}_1$ in the reference configuration and clamped in the extreme $s = 0$).

Expression of the energy of equilibrium configurations as a functional of the field $\theta(s)$

We can exploit the properties of (at least regular) equilibria of the beam to write the energy of the system in terms of the only field $\theta(s)$. For this, we have to use Equations (8) and (9-10), verified in every suitably regular stationary point. From equation (8) one gets, after integration between s and L :

$$\Lambda(L) - \Lambda(s) = - \int_s^L \mathbf{b}(\sigma) d\sigma \quad (11)$$

that together with the boundary condition gives:

$$\Lambda(s) = \int_s^L \mathbf{b}(\sigma) d\sigma + \mathbf{R} =: \mathbf{B}(s). \quad (12)$$

Next, let us consider equation (10) from which we derive, with the help of equation (12):

$$(\alpha - 1) = \frac{\mathbf{B}(s)}{k_e} \cdot \mathbf{e}(\theta(s)) \quad (13)$$

Let us now consider the term

$$\int_0^L \mathbf{b} \cdot \chi d\sigma \quad (14)$$

of equation (5). After equation (12) we have

$$\mathbf{B}'(s) = -\mathbf{b}(s),$$

and recalling the boundary conditions in equations (8) we can write (here and in the following we will consider $\mathbf{R} = 0$):

$$\int_0^L \mathbf{b} \cdot \chi ds = - \int_0^L \mathbf{B}' \cdot \chi ds = - [\mathbf{B} \cdot \chi]_0^L + \int_0^L \mathbf{B} \cdot \chi' ds = -\mathbf{R} \cdot \chi(L) + \int_0^L \mathbf{B} \cdot \chi' ds. \quad (15)$$

Therefore, when equations (12) hold, i.e. in the stationary points of the system, the energy can be written as a functional depending only on the scalar field θ as follows:

$$\begin{aligned} \mathcal{E}_{\text{eq}}^{\text{tot}}(\theta) = \int_0^L \left\{ \frac{k_b}{2} \left(-\frac{\mathbf{B} \cdot \mathbf{e}_{\perp} \left(\frac{-\mathbf{b} \cdot \mathbf{e} + \theta' \mathbf{B} \cdot \mathbf{e}_{\perp}}{k_e} \right)}{k_t} + \frac{\mathbf{b} \cdot \mathbf{e}_{\perp} \left(\frac{\mathbf{B} \cdot \mathbf{e}}{k_e} + 1 \right)}{k_t} + \frac{\theta' \mathbf{B} \cdot \mathbf{e} \left(\frac{\mathbf{B} \cdot \mathbf{e}}{k_e} + 1 \right)}{k_t} + \theta' \right)^2 + \right. \\ \left. + \frac{\left(\frac{\mathbf{B} \cdot \mathbf{e}}{k_e} + 1 \right)^2 (\mathbf{B} \cdot \mathbf{e}_{\perp})^2}{2k_t} + \frac{(\mathbf{B} \cdot \mathbf{e})^2}{2k_e} - \mathbf{B} \cdot \mathbf{e} \left(\frac{\mathbf{B} \cdot \mathbf{e}}{k_e} + 1 \right) \right\} ds \end{aligned} \quad (16)$$

The Euler-Lagrange equation for the functional (16) represents therefore the equilibrium condition for the general case of a clamped-free extensible Timoshenko beam under distributed load. It is quite cumbersome and therefore we show it in the Appendix.

Remark The functional (16) has no clear physical meaning when applied to a generic configuration of the beam defined by a function θ which does not correspond to an equilibrium configuration.

Numerical simulations

Numerical methods

An increasingly popular approach for the numerical study of nonlinear beams is isogeometric analysis (see e.g. ^{42–46}), which is a suitable variant of the finite element method. This method is very powerful and relatively light from a computational point of view, but just like every energy-related method it is not very suitable to study the multiplicity of arising solutions. For this reason, the numerical technique used here is the same as in ⁷. Indeed, the boundary value problem for the clamped-free Euler and Timoshenko beams has been solved by means of a shooting technique. We introduce a family of Cauchy problems [†]:

$$\mathcal{P}_k = \begin{cases} \theta'' = -b(1-s)\cos\theta \\ \theta(0) = 0 \\ \theta'(0) = k \end{cases} \quad (17)$$

depending on the parameter k . Then we selected the solutions of (17) which satisfy (with prescribed accuracy) $\theta'(1) = 0$, so as to obtain a numerical solution of the equilibrium condition $\delta\mathcal{E}^{tot} = 0$. Clearly the solution for the Cauchy problem exists and is unique for every initial datum k .

In Fig. 3, we show the plot of $\theta'(1)$ as a function of $k := \theta'(0)$ for an inextensible Euler beam model for $b = 250$. The graph intersects in five different points the horizontal axis, which means that in this case we have five different solutions of the boundary value problem with $\theta(0) = 0$ and $\theta'(1) = 0$. In the next section, the solutions will be shown in the same order as they appear as intersections between $\theta'(1)$ and the horizontal axis. Therefore the absolute minimum of the total energy will be always the rightmost configuration and, of course, the only solution for b small enough.

The number of equilibrium configurations when the load increases

It is generally very difficult to address theoretically the problem of evaluating how fast the number of solutions of a nonlinear parametric dynamical system increases with the parameter ⁴⁷, in our case, the external load. It is therefore interesting, as a preliminary step, to address the problem numerically. In Fig. 1 the number of solutions for an inextensible (left) and extensible (right) Euler beam model is plotted as a function of the non-dimensional external load. As expected, the behavior is that of a step-function, as it is clear that new branches of solutions arise only when the external load overcomes specific thresholds. The number of equilibrium configurations increases significantly with the load, and again as expected it reaches slightly larger values in case an additional kinematic degree of freedom (i.e. α) is included.

Figure 2 is consistent with this. Indeed, in this case inextensible (left) and extensible (right) Timoshenko beam models are considered, which produces an even more rapid increase of the possible equilibrium configurations. It has to be remarked, however, that a straight comparison with the Euler case is difficult because adimensionalizing the load is a different procedure in the two cases.

[†] We consider here the inextensible Euler beam for simplicity, but everything is analogous for the general case.

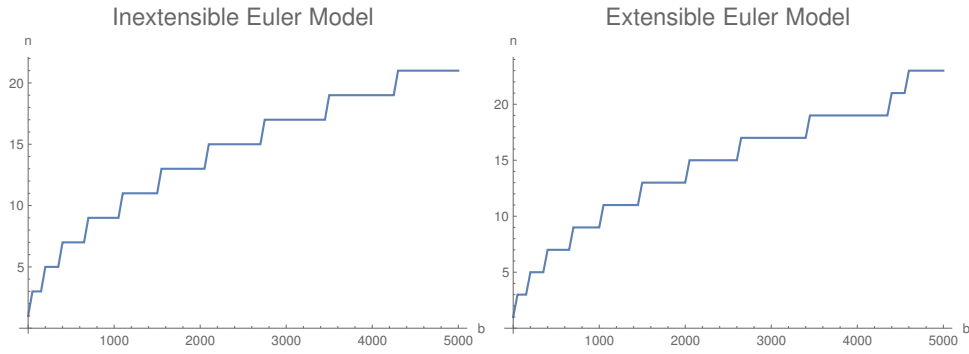


Figure 1. Number of solutions as a function of the transversal applied load. Left panel: inextensible Euler model. Right panel: extensible Euler model ($k_e = 1.3 \times 10^3$).

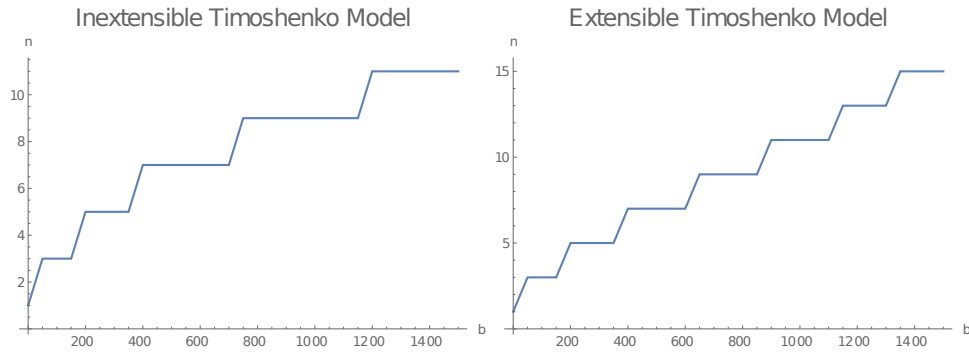


Figure 2. Number of solutions as a function of the transversal applied load. Left panel: inextensible Timoshenko model ($k_t = 1.8 \times 10^4$). Right panel: extensible Timoshenko model ($k_t = 1.8 \times 10^4$, $k_e = 2 \times 10^3$).

Equilibrium configurations

In the first gallery of equilibrium solutions we will show the full set of equilibrium configurations when the nondimensional load is $b = 60$ and $b = 250$. We will consider inextensible Euler (Figs. 4-5), extensible Euler (Figs. 6-7), inextensible Timoshenko (Figs. 8-9) and extensible Timoshenko (Figs. 11-12) beam models. For the chosen value of the others parameters, we have the same number of solutions corresponding to the same value of the nondimensional load for the four beam models. It has to be remarked that no branch of solutions appear to bifurcate. Instead, new branches appear at some thresholds in the external loads. It can be seen that $\theta(s)$ is a positive monotonic function for the rightmost solution in all the cases. For $b = 60$ the central equilibrium shape is non monotonic, while the leftmost is a negative monotonic function. This is a general trend as the load increases. Indeed, branches that arise with increasingly large values of the load will be made of progressively more numerous monotonic pieces. Of course in the Timoshenko case in general it is $\theta(0) \neq 0$. As the boundary datum for θ in 0 can have more than one solution, we have always chosen the smallest one in absolute values (see also the Appendix). The shear deformation relative to Figs. 9 and 12 are shown respectively in Figs. 10 and 13.

It is not clear whether the solutions shown herein for the extensible Euler and Timoshenko models can be stable (while in [SEPPECHER-SOTTOMESSO] it has been proved that for the inextensible Euler case only the left and right configurations of Fig. 4 and Fig. 5 can be stable).

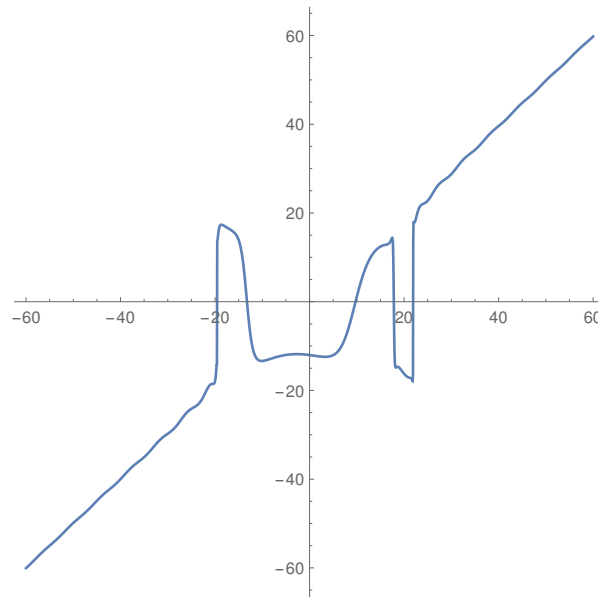


Figure 3. $\theta'(1)$ as a function of $k := \theta'(0)$ for an inextensible Euler beam with a transverse applied load $b = 250$. The solutions to the BVP problem (17) correspond to the intersections of the curve with the horizontal axis.



Figure 4. Clamped inextensible Euler beam with a transversal applied load $b = 60$

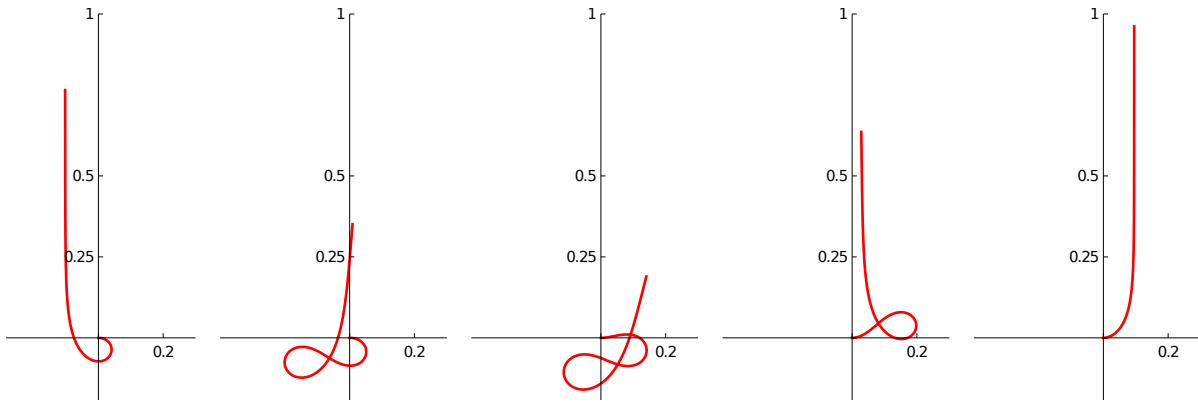


Figure 5. Clamped inextensible Euler beam with a transversal applied load $b = 250$

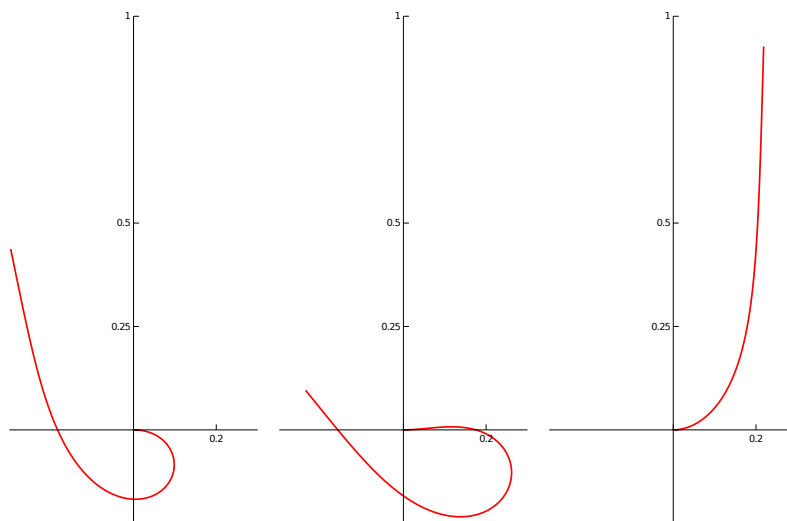


Figure 6. Clamped extensible Euler beam with a transversal applied load $b = 60$ ($k_e = 3000$)

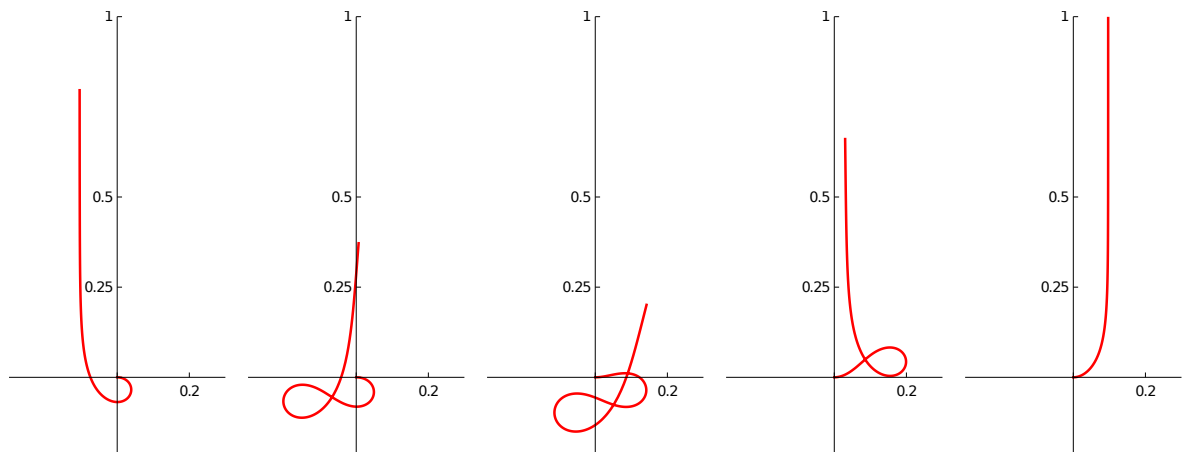


Figure 7. Clamped extensible Euler beam with a transversal applied load $b = 250$ ($k_e = 3000$)

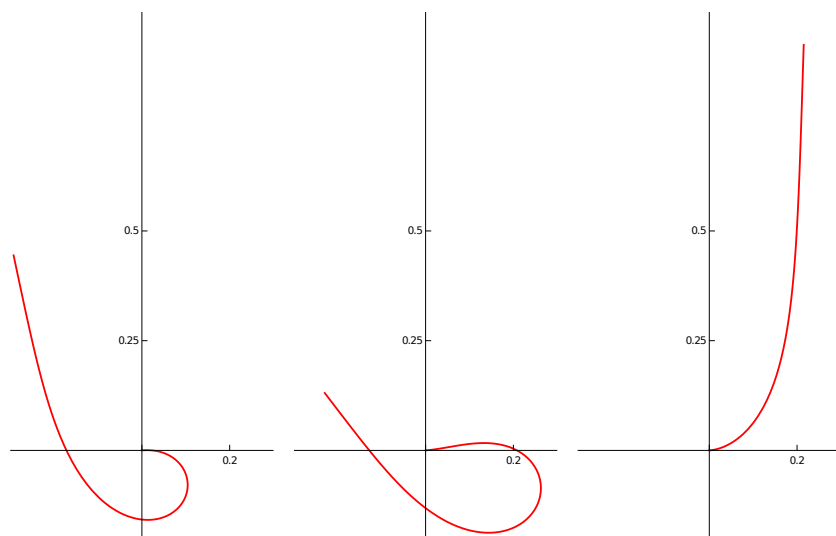


Figure 8. Clamped inextensible Timoshenko beam with a transversal applied load $b = 60$ ($k_t = 7000$)

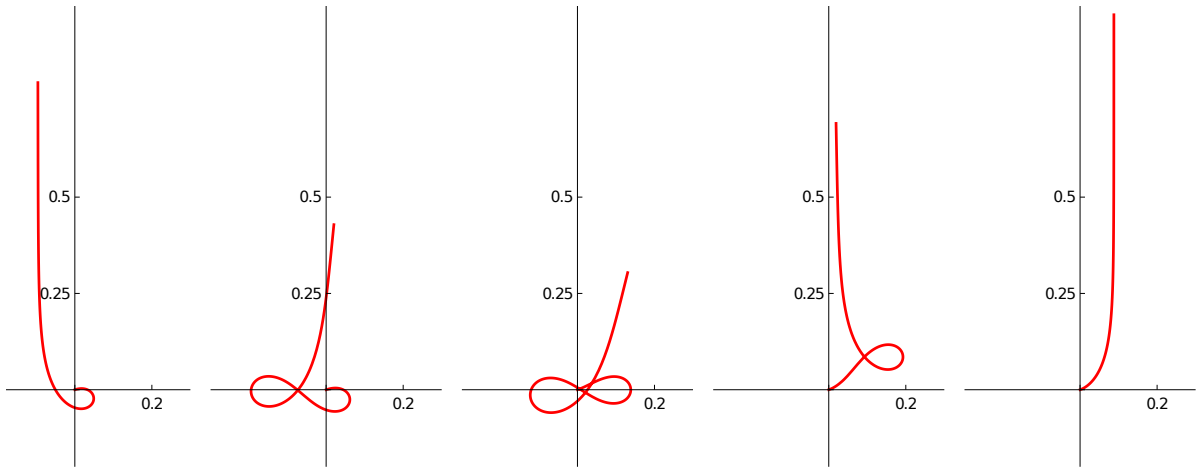


Figure 9. Clamped inextensible Timoshenko beam with a transversal applied load $b = 250$ ($k_t = 7000$)

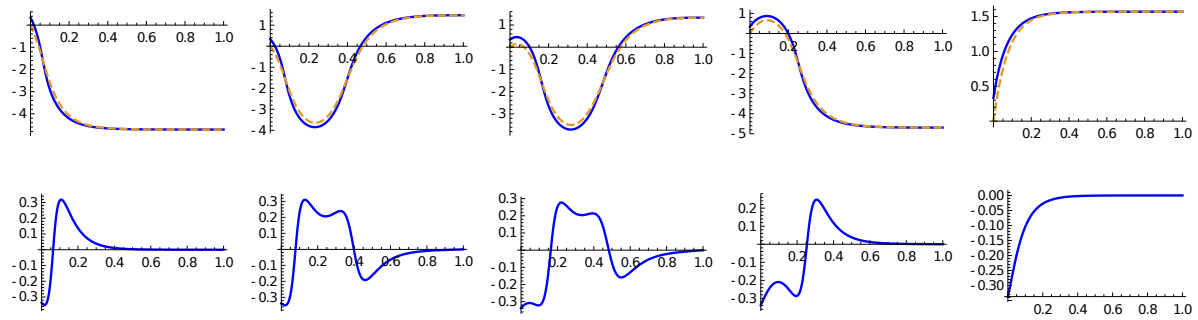


Figure 10. Above: plot of ϕ (dotted) and θ for the five configurations shown in Fig.9. Below: the corresponding plot of $\phi - \theta$.

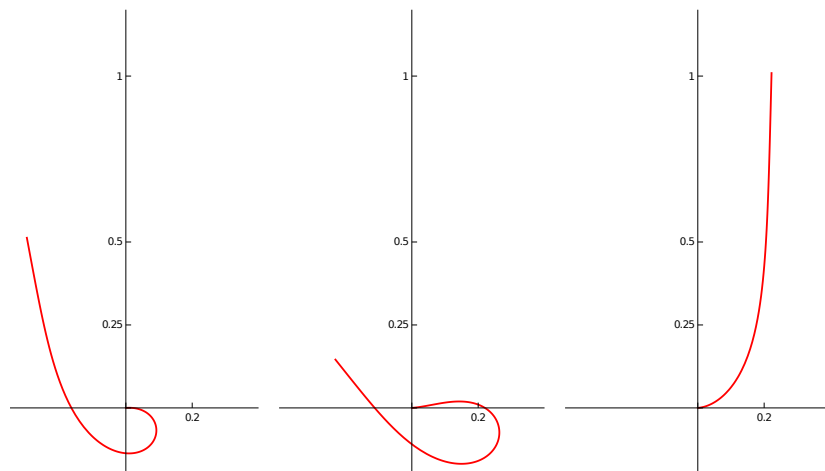


Figure 11. Clamped extensible Timoshenko beam with a transverse applied load $b = 60$ ($k_t = 7000$, $k_e = 3000$)

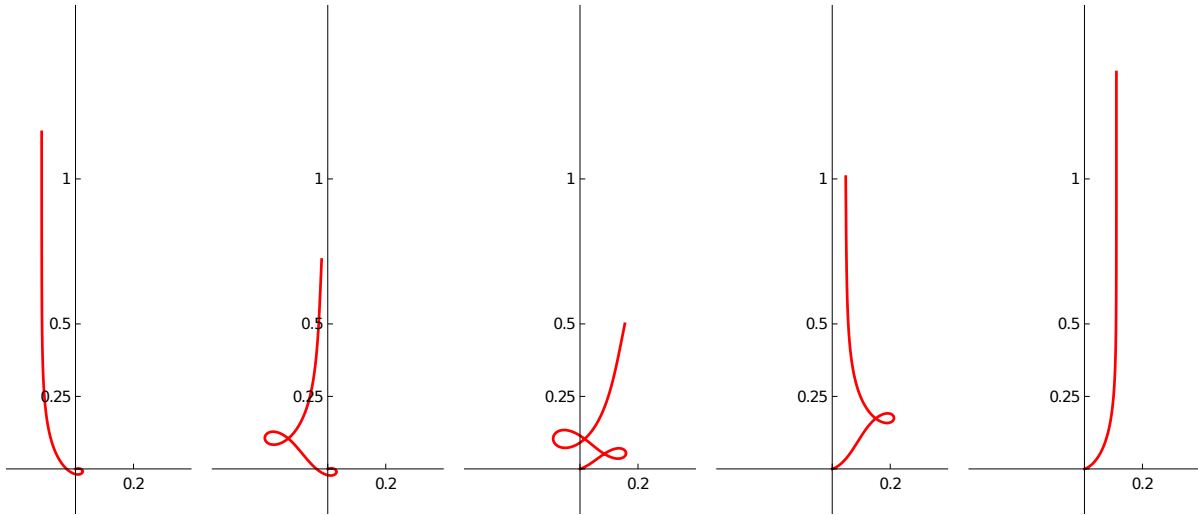


Figure 12. Clamped extensible Timoshenko beam with a transverse applied load $b = 250$ ($k_t = 7000$, $k_e = 3000$)

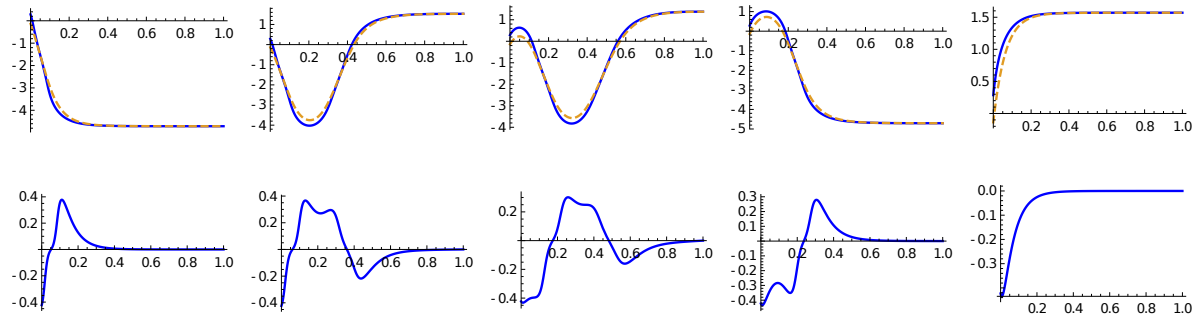


Figure 13. Above: plot of ϕ (dotted) and θ for the five configurations shown in Fig.12. Below: the corresponding plot of $\phi - \theta$.

Parametric study on the extensional stiffness

When the parameter k_e diverges, the beam model tends to inextensibility. The effect of decreasing k_e is evaluated for the Euler and Timoshenko models respectively in Figs14 and 16 for $b = 120$. In Figs.15 and 17 the local elongation $\alpha(s)$ is shown for the two previous cases respectively. The elongation reaches its maximum (minimum) value where the beam lies parallel to the load with the same (opposite) direction. Instead, it is close to 1 where the beam lies orthogonal to the load. While in the Euler case the change in k_e causes a minimal change in the deformed shape, in the Timoshenko case there is a much more relevant influence of k_e on the configuration. In particular, decreasing k_e allows a much larger maximum value of the local geometrical curvature $\gamma := \frac{\theta'}{\alpha}$ of the beam. This maximum is attained when $\theta(s) = -\pi/2$; we will define s_0 the point at which this occurs. It has to be noted, however, that the Chebyshev curvature $\theta'(s_0)$ takes similar values in the two cases. For instance, in the rightmost simulation of Fig.14 we have $\theta'(s_0) \approx -16.4$ and $\gamma(s_0) \approx -17.9$, while in the rightmost simulation of Fig.16 we have $\theta'(s_0) \approx -16.3$ and $\gamma(s_0) \approx -111.9$ (we point out that $s_0 \approx 0.303$ in the right panel of Fig.14 and $s_0 \approx 0.288$ in the right panel of Fig.16).

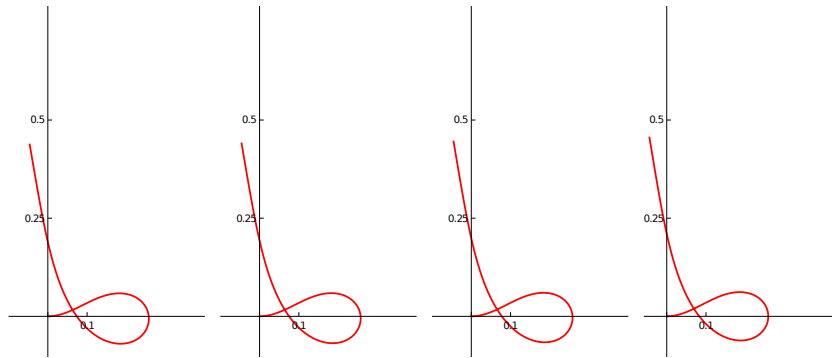


Figure 14. Parametric study on a clamped extensible Euler beam with a transversal applied load $b = 120$ with $k_e = 2500$, $k_e = 2000$, $k_e = 1500$, $k_e = 1000$.

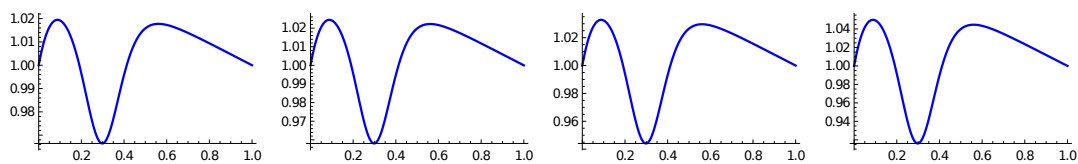


Figure 15. The local elongation $\alpha(s)$ relative to the equilibrium shapes shown in Fig.14.

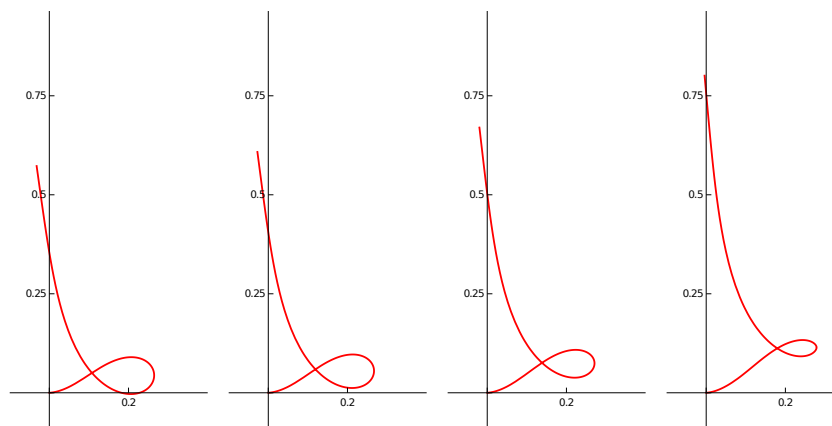


Figure 16. Parametric study on a clamped extensible Timoshenko beam with a transversal applied load $b = 120$ with $k_e = 2500$, $k_e = 2000$, $k_e = 1500$, $k_e = 1000$ ($k_t = 18000$).

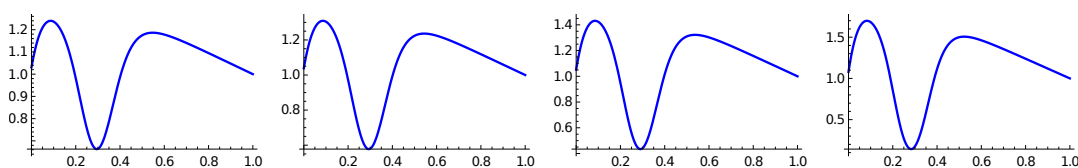


Figure 17. The local elongation $\alpha(s)$ relative to the equilibrium shapes shown in Fig.16.

Conclusions

In this paper we numerically studied clamped-free Euler and Timoshenko beams in large deformation under distributed load. Extensibility has been taken into account and results on the static behavior of the beam under different values of the load and of the axial stiffness has been shown. The main interest of the results consists in the multiplicity of solutions that arise as the load increases, not as a bifurcation of existing branches of solutions but as new branches that arise when the load overcomes a series of progressively larger threshold-values. Future investigations are required to establish whether these multiple solutions can be stable. In this regard, an analysis of the small oscillations of the beam around candidate stable equilibria would be useful to assess numerically the question. Moreover, addressing theoretically the dynamical behavior of the nonlinear version of the Timoshenko beam model proposed here is a challenging task. The results developed in ⁴⁸⁻⁵² may prove useful in this direction.

Appendix

We show here the Euler-Lagrange boundary value problem associated with the functional (16). We recall that the solutions $\theta(s)$ of this BVP are the scalar fields of angles formed by the tangent to the deformed configuration and a reference axis at the equilibrium for a clamped-free, extensible Timoshenko beam, in large deformation regime, under distributed load.

$$\begin{aligned} \theta'' &= [k_b (bk_e(1-s) \sin(\theta) - (b(1-s))^2(\cos(2\theta)) + k_e k_t)]^{-1} \\ &[bk_b(1-s)\theta^2 \cos(\theta)(4b(1-s) \sin(\theta) + k_e) - 2bk_b\theta(2b(1-s) \cos(2\theta) + \\ &-k_e \sin(\theta)) + \cos(\theta) (2b^2 k_b \sin(\theta) - bk_t(1-s)(b(1-s) \sin(\theta) + k_e))] \end{aligned} \quad (18)$$

with boundary conditions:

$$\begin{aligned} \left(\theta - \frac{b^2(1-s)^2 \sin(\theta) \cos(\theta)}{k_e k_e} + \frac{b(1-s) \cos(\theta)}{k_e} \right) \Big|_{s=0} &= 0 \\ \left(\theta - \frac{b^2(1-s)^2 \sin(\theta) \cos(\theta)}{k_e k_e} + \frac{b(1-s) \cos(\theta)}{k_e} \right)' \Big|_{s=1} &= 0 \end{aligned}$$

Remark The boundary conditions in the previous problem have in general more than one solution. In the numerical simulations, we always considered the value $\theta(0)$ which was smaller in absolute value.

Acknowledgements

Grant RUSSO???????

References

1. Euler L and Carathéodory C. *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive solutio problematis isoperimetrici latissimo sensu accepti*, volume 1. Springer Science & Business Media, 1952.

2. Bernoulli D. The 26th letter to euler. *Correspondence Mathématique et Physique* 1843; 2.
3. Bernoulli J. Quadratura curvae, e cujus evolutione describitur inflexae laminae curvatura. *Die Werke von Jakob Bernoulli* ; 223227: 1692.
4. Luongo A and Zulli D. *Mathematical models of beams and cables*. John Wiley & Sons, 2013.
5. Eugster SR et al. *Geometric continuum mechanics and induced beam theories*, volume 75. Springer, 2015.
6. Della Corte A, dell'Isola F, Esposito R et al. Equilibria of a clamped euler beam (*Elastica*) with distributed load: Large deformations. *Mathematical Models and Methods in Applied Sciences* 2016; : 1–31.
7. Battista A, Della Corte A, dell'Isola F et al. Large deformations of 1d microstructured systems modeled as generalized timoshenko beams. *Zeitschrift für angewandte Mathematik und Physik* 2018; 69(3): 52.
8. Fertis DG. *Nonlinear structural engineering*. Springer.
9. Ladevèze P. *Nonlinear computational structural mechanics: new approaches and non-incremental methods of calculation*. Springer Science & Business Media, 2012.
10. Antman SS and Renardy M. Nonlinear problems of elasticity. *SIAM Review* 1995; 37(4): 637.
11. Eugster S, Hesch C, Betsch P et al. Director-based beam finite elements relying on the geometrically exact beam theory formulated in skew coordinates. *International Journal for Numerical Methods in Engineering* 2014; 97(2): 111–129.
12. Steigmann DJ. *Finite Elasticity Theory*. Oxford University Press, 2017.
13. Atai AA and Steigmann DJ. On the nonlinear mechanics of discrete networks. *Archive of Applied mechanics* 1997; 67(5): 303–319.
14. dell'Isola F, Steigmann D and Della Corte A. Synthesis of fibrous complex structures: designing microstructure to deliver targeted macroscale response. *Applied Mechanics Reviews* 2016; 67(6): 21–pages.
15. dell'Isola F, Giorgio I, Pawlikowski M et al. Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium. In *Proc. R. Soc. A*, volume 472. The Royal Society, p. 20150790.
16. Turco E, Golaszewski M, Cazzani A et al. Large deformations induced in planar pantographic sheets by loads applied on fibers: experimental validation of a discrete lagrangian model. *Mechanics Research Communications* 2016; 76: 51–56.
17. Turco E, Golaszewski M, Giorgio I et al. Pantographic lattices with non-orthogonal fibres: Experiments and their numerical simulations. *Composites Part B: Engineering* 2017; 118: 1–14.
18. Milton G, Briane M and Harutyunyan D. On the possible effective elasticity tensors of 2-dimensional and 3-dimensional printed materials. *Mathematics and Mechanics of Complex Systems* 2017; 5(1): 41–94.
19. Boutin C, Giorgio I, Placidi L et al. Linear pantographic sheets: Asymptotic micro-macro models identification. *Mathematics and Mechanics of Complex Systems* 2017; 5(2): 127–162.
20. Giorgio I, Rizzi N and Turco E. Continuum modelling of pantographic sheets for out-of-plane bifurcation and vibrational analysis. *Proc R Soc A* 2017; 473(2207): 20170636.
21. Scerrato D, Giorgio I and Rizzi NL. Three-dimensional instabilities of pantographic sheets with parabolic lattices: numerical investigations. *Zeitschrift für angewandte Mathematik und Physik* 2016; 67(3): 53.

22. Boubaker BB, Haussy B and Ganghoffer J. Discrete models of woven structures. macroscopic approach. *Composites Part B: Engineering* 2007; 38(4): 498–505.
23. Dos Reis F and Ganghoffer J. Construction of micropolar continua from the asymptotic homogenization of beam lattices. *Computers & Structures* 2012; 112: 354–363.
24. Reda H, Rahali Y, Ganghoffer JF et al. Wave propagation in 3d viscoelastic auxetic and textile materials by homogenized continuum micropolar models. *Composite Structures* 2016; 141: 328–345.
25. Diyaroglu C, Oterkus E, Oterkus S et al. Peridynamics for bending of beams and plates with transverse shear deformation. *International Journal of Solids and Structures* 2015; 69: 152–168.
26. Challamel N. Variational formulation of gradient or/and nonlocal higher-order shear elasticity beams. *Composite Structures* 2013; 105: 351–368.
27. Challamel N, Zhang Z and Wang C. Nonlocal equivalent continua for buckling and vibration analyses of microstructured beams. *Journal of Nanomechanics and Micromechanics* 2013; 5(1): A4014004.
28. Misra A, Lekszycki T, Giorgio I et al. Pantographic metamaterials show atypical poynting effect reversal. *Mechanics Research Communications* 2018; 89: 6–10.
29. dell’Isola F, Sepecher P, Alibert JJ et al. Pantographic metamaterials: an example of mathematically driven design and of its technological challenges. *Continuum Mechanics and Thermodynamics* 2018; : 1–34.
30. Diyaroglu C, Oterkus E and Oterkus S. An euler–bernoulli beam formulation in an ordinary state-based peridynamic framework. *Mathematics and Mechanics of Solids* 2017; : 1081286517728424.
31. Chebyshev P. Sur la coupe des vetements. *Complete works by PL Chebyshev* 1878; 5: 165–170.
32. Timoshenko SP. Lxvi. on the correction for shear of the differential equation for transverse vibrations of prismatic bars. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 1921; 41(245): 744–746.
33. Timoshenko SP. X. on the transverse vibrations of bars of uniform cross-section. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 1922; 43(253): 125–131.
34. Cosserat E, Cosserat F et al. Théorie des corps déformables 1909; .
35. Altenbach J, Altenbach H and Eremeyev VA. On generalized cosserat-type theories of plates and shells: a short review and bibliography. *Archive of Applied Mechanics* 2010; 80(1): 73–92.
36. Forest S. Mechanics of cosserat media - an introduction. *Ecole des Mines de Paris, Paris* 2005; : 1–20.
37. Altenbach H, Bîrsan M and Eremeyev VA. Cosserat-type rods. In *Generalized Continua from the Theory to Engineering Applications*. Springer, 2013. pp. 179–248.
38. Eremeyev VA and Pietraszkiewicz W. Material symmetry group and constitutive equations of micropolar anisotropic elastic solids. *Mathematics and Mechanics of Solids* 2016; 21(2): 210–221.
39. Barchiesi E, Spagnuolo M and Placidi L. Mechanical metamaterials: a state of the art. *Mathematics and Mechanics of Solids* 2018; : 1081286517735695.

40. Engelbrecht J, Berezovski A, Pastrone F et al. Waves in microstructured materials and dispersion. *Philosophical Magazine* 2005; 85(33-35): 4127–4141.
41. Barchiesi E, Dell'Isola F, Laudato M et al. A 1d continuum model for beams with pantographic microstructure: Asymptotic micro-macro identification and numerical results. In *Advances in Mechanics of Microstructured Media and Structures*. Springer, 2018. pp. 43–74.
42. Balobanov V and Niiranen J. Locking-free variational formulations and isogeometric analysis for the timoshenko beam models of strain gradient and classical elasticity. *Computer Methods in Applied Mechanics and Engineering* 2018; 339: 137–159.
43. Niiranen J, Balobanov V, Kiendl J et al. Variational formulations, model comparisons and numerical methods for euler–bernoulli micro-and nano-beam models. *Mathematics and Mechanics of Solids* ; : 1081286517739669.
44. Cazzani A, Malagù M and Turco E. Isogeometric analysis of plane-curved beams. *Mathematics and Mechanics of Solids* 2016; 21(5): 562–577.
45. Greco L, Cuomo M, Contrafatto L et al. An efficient blended mixed b-spline formulation for removing membrane locking in plane curved kirchhoff rods. *Computer Methods in Applied Mechanics and Engineering* 2017; 324: 476–511.
46. Dortdivanlioglu B, Javili A and Linder C. Computational aspects of morphological instabilities using isogeometric analysis. *Computer Methods in Applied Mechanics and Engineering* 2017; 316: 261–279.
47. Guckenheimer J and Holmes P. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, volume 42. Springer Science & Business Media, 2013.
48. Berezovski A, Yildizdag M and Scerrato D. On the wave dispersion in microstructured solids. *Continuum Mechanics and Thermodynamics* 2018; : 1–20DOI:10.1007/s00161-018-0683-1.
49. Luongo A and D'Annibale F. Double zero bifurcation of non-linear viscoelastic beams under conservative and non-conservative loads. *International Journal of Non-Linear Mechanics* 2013; 55: 128–139.
50. Piccardo G, Tubino F and Luongo A. A shear–shear torsional beam model for nonlinear aeroelastic analysis of tower buildings. *Zeitschrift für angewandte Mathematik und Physik* 2015; 66(4): 1895–1913.
51. Taig G, Ranzi G and D'annibale F. An unconstrained dynamic approach for the generalised beam theory. *Continuum Mechanics and Thermodynamics* 2015; 27(4-5): 879.
52. Piccardo G, Pagnini LC and Tubino F. Some research perspectives in galloping phenomena: critical conditions and post-critical behavior. *Continuum Mechanics and Thermodynamics* 2015; 27(1-2): 261–285.

Chapter 4

Nonlinear dynamics of uniformly loaded Elastica

4.1 Introduction

The article presented in this chapter, “Nonlinear dynamics of uniformly loaded Elastica: Experimental and numerical evidence of motion around curled stable equilibrium configurations” deals with the dynamics of an uniformly loaded Euler beam from the numerical and experimental point of view. From the numerical point of view, we use a discretization of the beam following a discrete model similar to the one used by Henky in his PhD thesis [26]. The discrete beam is modelled as periodic monodimensional continuum where the cell is represented by a rotational spring and a rigid bar. In the paper is shown how the stiffness of the springs is linked to the bending stiffness of the continuous beam. This discretization allows to write a Lagrangian and to derive the relative equations of motions. We have numerically studies the dynamics of the beam near equilibrium curled solutions whose existence was proved in [12]. A comparison with some experiences is also presented in this work where we use a paper beam and a PET beam with a mass at his free extremity (end load). The experimental apparatus is described in the paper.

Nonlinear dynamics of uniformly loaded *Elastica*: Experimental and numerical evidence of motion around curled stable equilibrium configurations

Djebar Baroudi¹, Ivan Giorgio^{2,3,4*}, Antonio Battista^{5,2,4}, Emilio Turco^{6,2,4}, and Leonid A. Igumnov⁴

¹ Aalto University School of Engineering, Aalto University, FIN-00076 Aalto, Finland.

² International Research Center for the Mathematics and Mechanics of Complex Systems, Università degli studi dell'Aquila, 67100 L'Aquila, Italy.

³ Department of Mechanical and Aerospace Engineering, SAPIENZA Università di Roma, 00184 Rome, Italy.

⁴ Research Institute for Mechanics, Lobachevsky State University of Nizhni Novgorod, Nizhni Novgorod, Russian Federation.

⁵ Université de La Rochelle, La Rochelle, France.

⁶ Department of Architecture, Design and Urban planning, University of Sassari, 07041 Alghero, Italy.

Received XXXX, revised XXXX, accepted XXXX

Published online XXXX

Key words Nonlinear beam, Hencky bar-chain, discrete modelling

MSC (2010) 00-xx

“Frustra fit per plura quod fieri potest per pauciora.”
(It is pointless to do with more what can be done with fewer.)
In Summa Totius Logicae by William of Ockham

In this paper, we show that equilibrium configurations of a clamped beam under distributed load, resembling a curled pending wire—whose existence has been mathematically established—can be obtained experimentally using ‘soft’ beams, *i.e.* beams for which the ratio between amplitude of the load and bending stiffness is large enough. Moreover, we introduce a Hencky-type discrete model, *i.e.* a finite dimensional Lagrangian model, for the ‘soft’ *Elastica* and build a numerical code for determining its motion, in the most general nonlinear regime. This code is able to qualitatively describe observed nonlinear dynamical behavior.

Copyright line will be provided by the publisher

1 Introduction

Since the introduction of the *Elastica* by Bernoulli and Euler [11, 23], beam theory has attracted the attention of various scientists due to its importance both from the mathematical point of view [12] and in its applications to structural mechanics. The literature on the subject is huge and many relevant problems have been studied with sufficient completeness [5, 7, 22, 24, 32, 37, 38]. However, the richness of the model is such that there are still interesting and practically significant problems, that lack a complete study, especially in case of a beam in large deformation under distributed load—which leads to non-autonomous variational problems/Euler-Lagrange equations. In the paper [14] the existence of non-trivial (curled) equilibrium shapes of a clamped Euler beam under uniformly distributed dead load orthogonal to the straight reference configuration has been shown and a study of the properties of the global and local minimizers of the total energy has been performed. The last years have been characterized by a rediscovery of old models conjectured to study, with very simple tools, classical mechanical problems such as, one example for all, the computation of the buckling load of a beam. Hencky, in his work [25] which dates back to 1920, proposed a very simple, but extremely sharp, road to compute the buckling load of a rectilinear planar beam.¹ The keynote was to consider the beam as an assemblage of rigid links and elastic joints. In this way the equilibrium equations, or the stationarity condition of the potential energy, allow to estimate the buckling load quickly. The accuracy of such an estimate improves by increasing the number of elastic joints and rigid links. One of the more attractive point of Hencky’s concept is that of avoiding the necessity of a continuum model since the problem

* Corresponding author, E-mail: ivan.giorgio@uniroma1.it

¹ It has to be remarked that an outline of Hencky’s idea can also be found in the work of Gabrio Piola almost one century before, see [15, 18].

naturally arises in a discrete environment. Moreover, recent Γ -convergence results have shown that Hencky's model is a fully reliable approximation of continuous inextensible [3] and extensible [2] Euler beams. Of note, this discrete model is intrinsically nonlinear and for this reason naturally avoids issues related to the objectivity of the energy when a linearization (around a given deformed configuration) is performed.

Limiting us to mechanical problems, the application of Hencky's idea can be seen in a series of recent works which treat beams, see [20, 26, 34, 35, 39], assemblage of beams, see [21, 40, 45], and specifically designed materials, see [42]. This last research line, fairly trend in last decades, is the additional reason which suggests to take into consideration Hencky's models. Indeed, this kind of problems are characterized by a very large number of structural elements, therefore, the use of models as simple as possible is a forced road to follow. On the other hand, the study of complex metamaterials [9, 19, 28], such as pantographic structures, based on models which can be seen as generalization of that proposed by Hencky have proven to be fairly effective for predicting the mechanical behavior in static problems in large displacements, see [40, 43], and also the onset of failure phenomena [41]. The importance of studying nonlinear beam under distributed load has become clear in the recent past. Let us cite for instance the very active field of microstructured continua and in particular its significant branch in which the fundamental element constituting the microstructure is represented by a beam. In particular, in pantographic continua [4, 16, 17] the single fiber, in a first approximation, can be modeled as a beam interacting with the other fibers, in the homogenized limit, a distributed load.

Some examples of studies which consider the nonlinear dynamics of beams described by simple discrete models are [21, 26, 45]. The present work aims at contributing in this direction. Indeed, we consider some very simple physical experiments reproducible without expensive tools concerning the curled equilibrium shape of a highly flexible beam under a gravity load. We also investigate the planar nonlinear motion in the neighborhood of this configuration.

We assume the same hypotheses characterizing the continuous beam model: i) the elastic stored energy depends quadratically (a more general dependence could easily be considered) upon the curvature; ii) the axis of the beam is inextensible; iii) the shear deformation of the cross-section with respect to the axis is negligible; iv) the cross-section is assumed undeformable. As we will see Hencky's model is consistent with these usual assumptions.

The paper, after this brief introduction, describes the main ingredients necessary to build the discrete model in Sec. 2. Successively, in Sec. 3, are reported firstly a complete description of two physical experiments along with the estimated data and successively the comparison between physical and numerical experiments both for static and dynamic cases. Section 4 closes the paper discussing the main results along with possible extensions of this work.

2 Naturally discrete model of *Elastica*

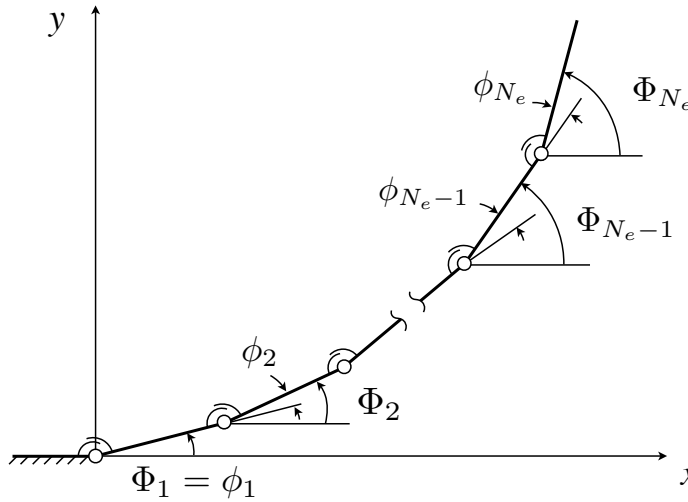


Fig. 1 Hencky-type discrete model for a highly flexible cantilever beam.

In order to describe the behavior of a ‘soft’ cantilever beam under gravity load, we follow the Hencky technique of discretization [33, 44, 46] and consider a discrete system which consists of an articulated chain of N_e rigid rods of length η connected each other by means of zero-torque hinges. Each joint is equipped by a rotational spring in order to model the resistance to be bent of the system (see Fig. 1). The configurations of the introduced system are completely defined by

specifying the evolution of N_e Lagrangian coordinates, $\Phi_i(t)$, which represent the orientation of the rigid rods with respect to the x -axis pointing along the horizontal direction while the y -axis is directed vertically upwards. The system prior to deformation is straight and disposed along the x -axis. Each rigid segment is characterized by a mass, m_i , and a moment of inertia, J_i with respect to an axis orthogonal to the plane of the motion and passing through the mass center. Therefore, the position of the mass center for each segment can be easily written as

$$\begin{cases} x_i(t) = \sum_{k=1}^i \eta \left(1 - \frac{\delta_{ik}}{2}\right) \cos(\Phi_k(t)) \\ y_i(t) = \sum_{k=1}^i \eta \left(1 - \frac{\delta_{ik}}{2}\right) \sin(\Phi_k(t)) \end{cases} \quad (1)$$

where δ_{ik} is the Kronecker delta and, by a differentiation with respect to time, the velocities of the mass centers are evaluated as follows

$$\begin{cases} \dot{x}_i(t) = -\sum_{k=1}^i \eta \left(1 - \frac{\delta_{ik}}{2}\right) \dot{\Phi}_k(t) \sin(\Phi_k(t)) \\ \dot{y}_i(t) = \sum_{k=1}^i \eta \left(1 - \frac{\delta_{ik}}{2}\right) \dot{\Phi}_k(t) \cos(\Phi_k(t)) \end{cases} \quad (2)$$

The adopted kinematic description of the discrete beam leads to a convenient formulation of the motion equations derived from the following Lagrangian

$$\mathcal{L} = \mathfrak{K} - \Psi \quad (3)$$

where $\mathfrak{K}$ and Ψ are the kinetic and potential energies of the system, respectively. Specifically, the kinetic energy, using König's theorem and after some algebraic manipulations, assumes the form

$$\mathfrak{K} = \sum_{i=1}^{N_e} \frac{1}{2} m_i \left\{ \left[\sum_{k=1}^i \eta \left(1 - \frac{\delta_{ik}}{2}\right) \dot{\Phi}_k \sin(\Phi_k) \right]^2 + \left[\sum_{k=1}^i \eta \left(1 - \frac{\delta_{ik}}{2}\right) \dot{\Phi}_k \cos(\Phi_k) \right]^2 \right\} + \frac{1}{2} J_i \dot{\Phi}_i^2 \quad (4)$$

The potential energy Ψ consists of two contributions, namely an elastic term Ψ_{el} which is assumed to be

$$\Psi_{el} = \sum_{i=1}^{N_e} \kappa_{bi} [\cosh(\phi_i) - 1] \quad (5)$$

where we introduce the relative angles between rods, *i.e.* $\phi_1 = \Phi_1$ (due to the clamping constraint) and $\phi_i = \Phi_i - \Phi_{i-1}$ for $i \geq 2$ and a uniform lumped bending stiffness κ_{bi} related to the rotational springs [16, 40], and a gravitational term Ψ_{wg} which is

$$\Psi_{wg} = \sum_{i=1}^{N_e} g m_i \left[\sum_{k=1}^i \eta \left(1 - \frac{\delta_{ik}}{2}\right) \sin(\Phi_k) \right] \quad (6)$$

being g the magnitude of the gravity acceleration. We remark that the potential energy in Eq. (5) should be only positive definite, hence, any convex function can be employed for this purpose, not necessarily a simple quadratic function. It is also worth noting that the first relevant term in a Taylor expansion of each addend of the assumed potential Eq. (5) is the classical quadratic form expressed in terms of the relative angle.

In order to take into account also a possible viscous dissipation (for a more general framework, see *e.g.* [6, 13]) occurring during the motion due to the interaction of the 'soft' beam with the air, we introduce, as a first approximation, the Rayleigh dissipation function as follows

$$\mathfrak{R} = \sum_{i=1}^{N_e} \frac{1}{2} c_{bi} \dot{\phi}_i^2 \quad (7)$$

The Euler-Lagrange equations of motion, thus, can be deduced as

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathfrak{K}}{\partial \dot{\Phi}_i} \right) - \frac{\partial \mathfrak{K}}{\partial \Phi_i} + \frac{\partial \Psi_{el}}{\partial \Phi_i} + \frac{\partial \Psi_{wg}}{\partial \Phi_i} + \frac{\partial \mathfrak{R}}{\partial \dot{\Phi}_i} = 0 \quad \text{for } i = 1 \dots N_e \quad (8)$$

Equations (8) are solved numerically by means of the computing system Wolfram Mathematica with a differential-algebraic system of equations (DAEs) solver after a proper transformation to a standard form.

3 Physical experiments and numerical simulations

The previously introduced Lagrangian model is tested in this section with two relevant experimental cases. Specifically, we consider two samples made up of two different materials and sizes to reproduce the equilibrium configurations related to a local minimum for the energy, and we investigate also the dynamic behavior of such specimens in a regime of large oscillation around the curled stable equilibrium configurations found out. In the first case, we examine a paper strip of size 329×20 mm, whose thickness is about 0.17 mm and its mass is 0.62 ± 0.02 g. The second case involves a similar strip of a thin isotropic sheet of polyethylene terephthalate, namely PET, of size 220×20 mm and thickness about 0.15 mm. The mass of the strip is 0.91 ± 0.02 g; moreover, we add at the free end of the specimen a further mass, *i.e.* a ‘paper clip’ of 0.41 ± 0.02 g (see Fig. 2).

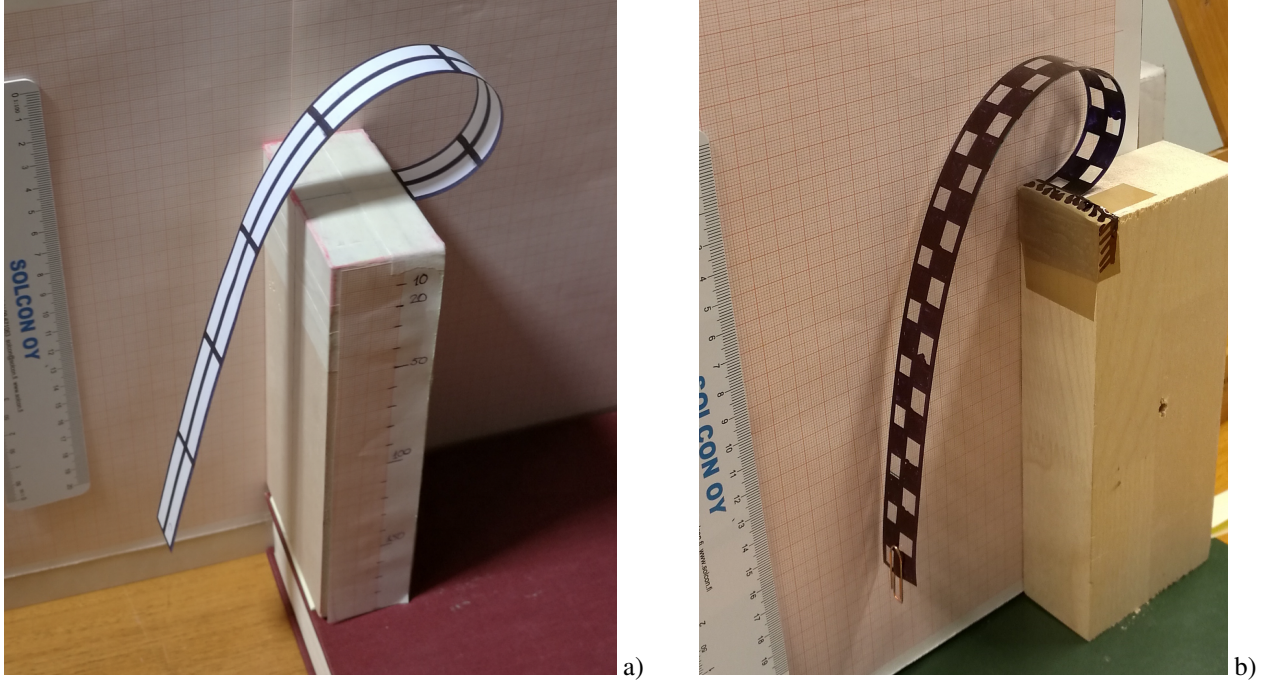


Fig. 2 Curled stable equilibrium configurations: paper beam **a**; PET beam with a mass on the tip **b**.

3.1 Using the curled static configuration to estimate mechanical parameters

First of all, let us consider the local-minimum energy configurations for the two specimens under test (see Fig. 2). Figure 3 shows the pictures of the equilibrium shapes for the beams and also the equilibrium configurations obtained by finding the corresponding local minimum of the potential energy Ψ in the examined cases for $N_e = 30$.

In order to obtain the agreement exhibit in Fig. 3, we use for the lumped bending stiffness the following expression

$$\kappa_{bi} = Y_b j_b / \eta \quad (9)$$

where j_b is the second moment of area of the beam cross section and Y_b , *i.e.* the Young modulus of the material, is used as a material parameter which we identify to fit the ‘measured’ shape. Particularly, we found for the paper beam that $Y_b = 1.25$ GPa and for the PET beam that $Y_b = 2.5$ GPa, starting from an initial guess near to the known values of the elastic moduli of the considered materials.

3.2 Large oscillations around curled stable equilibrium configurations

As illustrative examples of the foregoing, we consider some in-plane oscillating motions around the curled equilibrium configurations which are presented before. From the experiments, we evaluate the Lagrangian coordinates of the effective initial configurations (see Fig. 4 with the overlap between the computed piecewise linear curve and the picture of the beam for both the specimens treated) and then, specifying these values and zero angular velocities as initial conditions, we solve the Eqs. (8) to obtain the motions that originate from those. The evolutions in terms of orientations $\Phi_i(t)$ and angular velocities $\dot{\Phi}_i(t)$ are shown in Fig. 5 and Fig. 6, respectively, for the two cases examined of the paper and the PET beam.

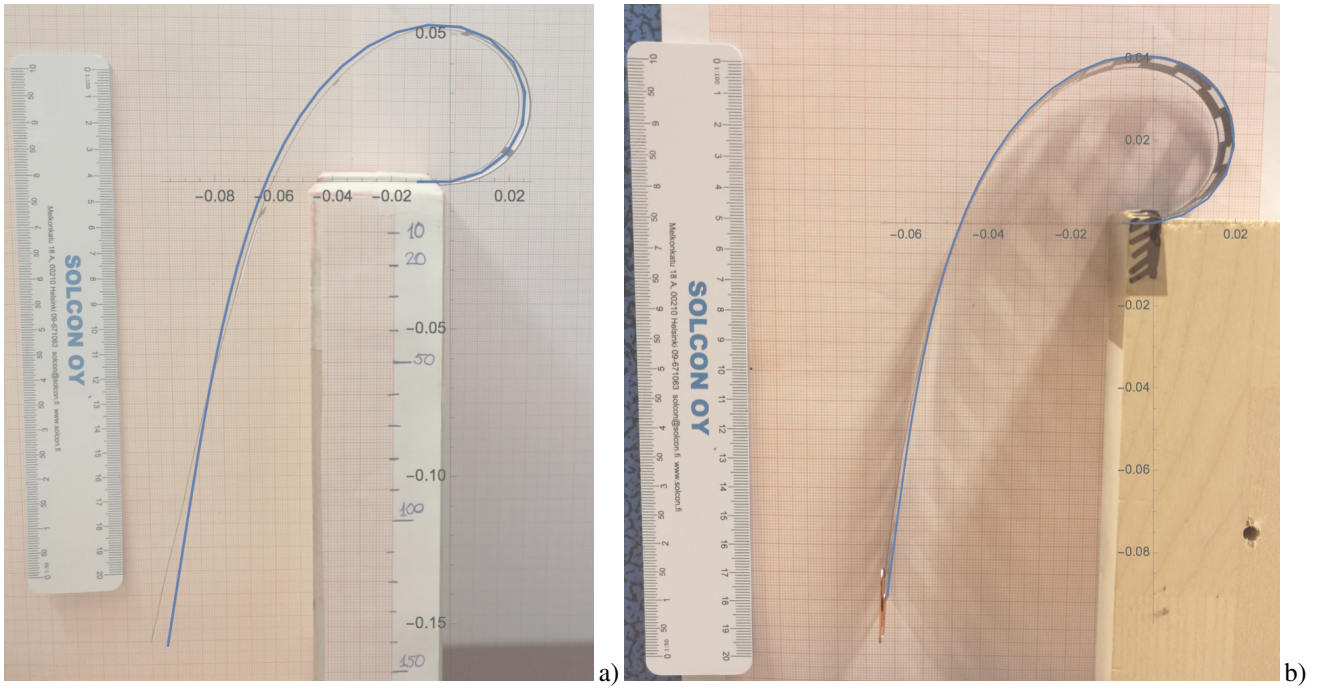


Fig. 3 Comparison between measured and simulated equilibrium configurations: paper beam **a**; PET beam with a mass on the tip **b**.

Figure 6 also displays a zoom of the initial part of the motion. Since the initial shapes and the equilibrium configurations, towards which the evolutions approach, are both curves where the curvature never changes its sign (*i.e.* the curve never crosses its tangent, and therefore the coordinates $\Phi_i(t)$ for a given instant do not decrease with the index i), we note that the plots of Lagrangian coordinates, in Fig. 5, result in an almost increasing order from $\Phi_1(t)$ to $\Phi_{Ne}(t)$ and tend to compact each other for the last part of the beam. Indeed, this terminal part remains almost undeformed. As shown in Fig. 6, the same sequence in the disposition of the plots disappears for the angular velocities. These last exhibit a maximum value around less than a quarter of the beam length at the very beginning of the motion and subsequently the trend of all the histories is governed by the dissipation (the curves are neatly superimposed on top of each other by the lowest index, and hence it is possible to see which segment presents the velocity peak). In the performed simulations, the viscous coefficients are roughly estimated to fit the dissipative behavior of the real ‘soft’ beams. Specifically, we found $c_n = 10^{-5}$ N m s for the beam made up of paper and $c_n = 5 \times 10^{-5}$ N m s in the case of the PET. In Fig. 7 are plotted the evolution, along the motion, of the kinetic and potential energy for the paper and for the PET beam. Their trends are almost counter-phase and consistent with what we expected; indeed, the kinetic energy tends to vanish because of the dissipation while the potential energy approaches the value related to the local minimum around which the beams oscillate. For the sake of brevity, Fig. 8 shows the trajectory in phase space of the Lagrangian coordinate $\Phi_{Ne}(t)$ for both the cases investigated, since the qualitative behavior of the trajectories related to the other coordinates is very similar. In the initial part of the motion the nonlinear behavior is remarkable while towards the end, for the effect of viscous dissipation, the system become almost linear and a point of stable equilibrium is easily detectable.

4 Concluding remarks and future challenges

Herein, the local minimum configurations for highly flexible beams predicted in [14] and characterized by curled shapes are shown experimentally and predicted numerically. Besides, using a Hencky-type discrete model for describing such mechanical systems, we adopt a Lagrangian formulation, which is computationally efficient for determining the motion in the most general nonlinear regime, and compare the solutions of the obtained differential equations with experimental tests with a good qualitative agreement (see also the supplementary data). We expect, therefore, that when a more accurate measurement campaign will be performed and when the lumped parameters in the considered Lagrangian functions will be suitably fitted a perfect quantitative agreement will become possible (see *e.g.* [1, 29, 30]). Considered their great efficiency, we also expect that similar codes will be useful in the study of the nonlinear dynamics of Timoshenko beams [10] and lattice systems including many beams in large deformations as pantographic metamaterials [8, 27, 31, 36, 43].

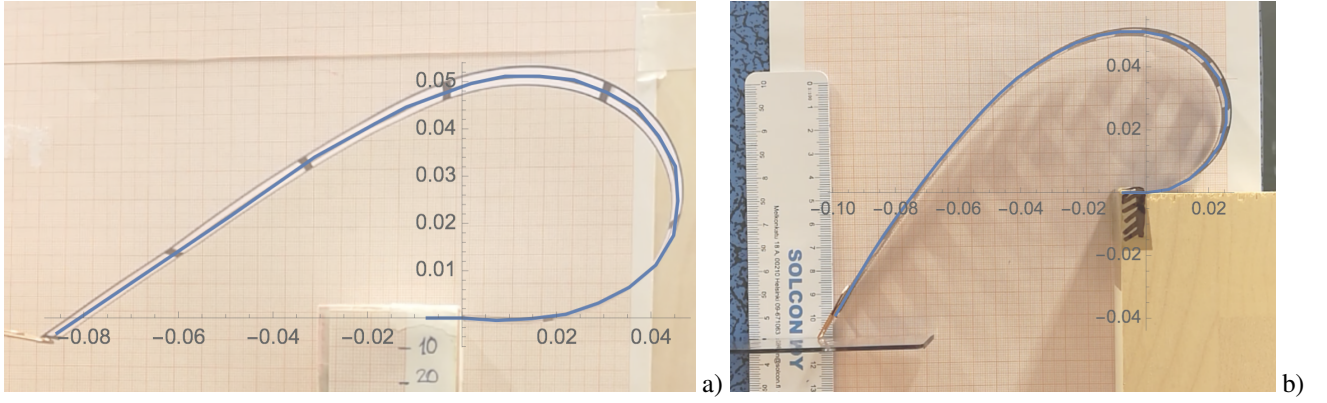


Fig. 4 Initial configurations: paper beam **a**; PET beam with a mass on the tip **b**.

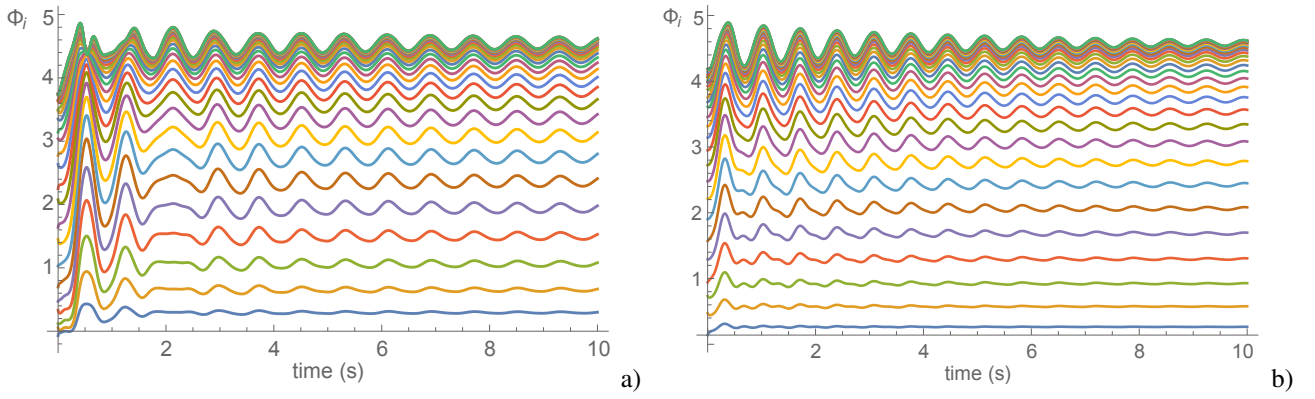


Fig. 5 Large oscillations, $\Phi_i(t)$, around the equilibrium configurations: paper beam **a**; PET beam with a mass on the tip **b**. The plots of Lagrangian coordinates result in an increasing order from the $\Phi_1(t)$ to $\Phi_{N_e}(t)$.

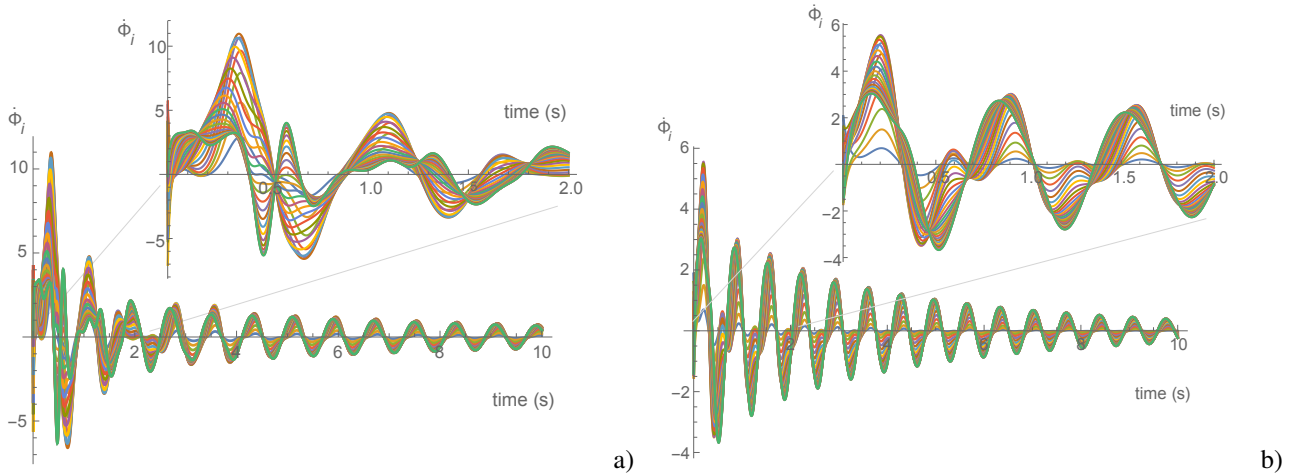


Fig. 6 Histories of the angular velocities, $\dot{\Phi}_i(t)$ with a zoomed initial part: paper beam **a**; PET beam with a mass on the tip **b**.

It is worth noting that, in case of PET beam, the presence of the point mass, *i.e.* the paper clip, at the end of the beam makes stable the ‘curled equilibrium’. Of course, we can add length to the strip to achieve the stable configuration under the distributed own weight but, on the other hand, removing the tip mass and hence, reducing the gravity load, we can show that the only minimum is, in this circumstance, the global one (see for more details [14]). Indeed, it can be proven that below a critical value of the external load the only minimum for the energy is related to the classical equilibrium configuration which resembles the pending wire shape of a cantilever beam. Specifically, Fig. 9 exhibits, for the PET beam without tip

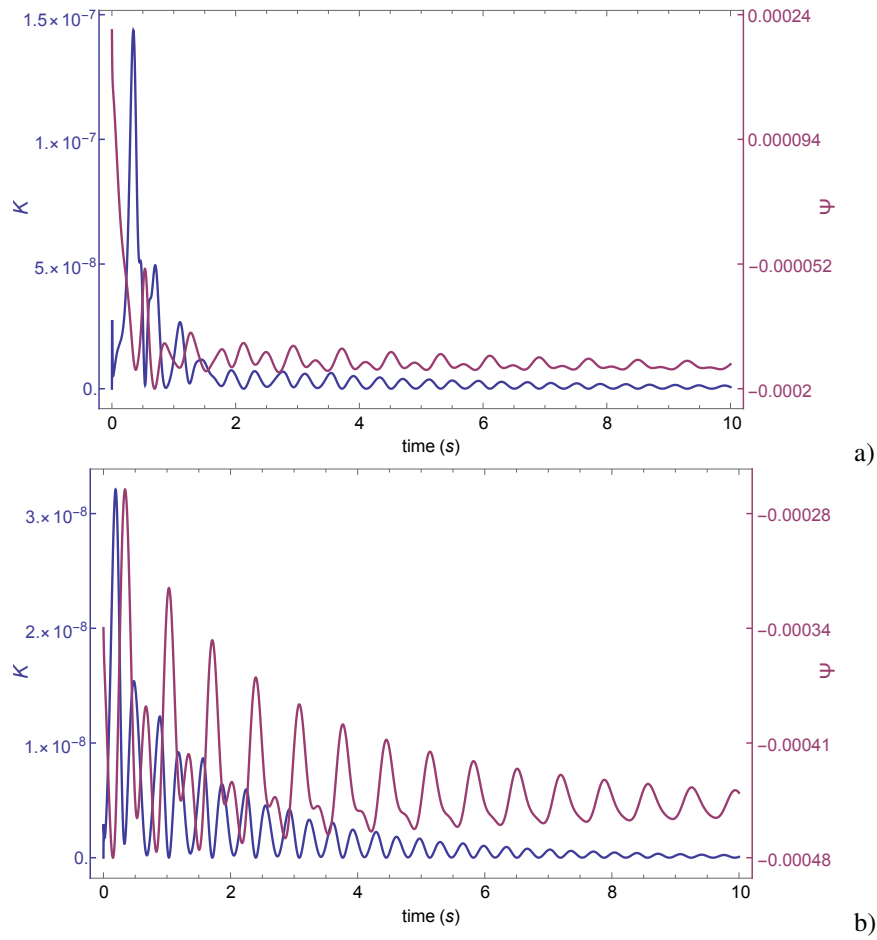


Fig. 7 Histories of the kinetic energy $\mathfrak{K}$ (blue solid line) and potential energy Ψ (purple solid line): paper beam **a**; PET beam with a mass on the tip **b**.

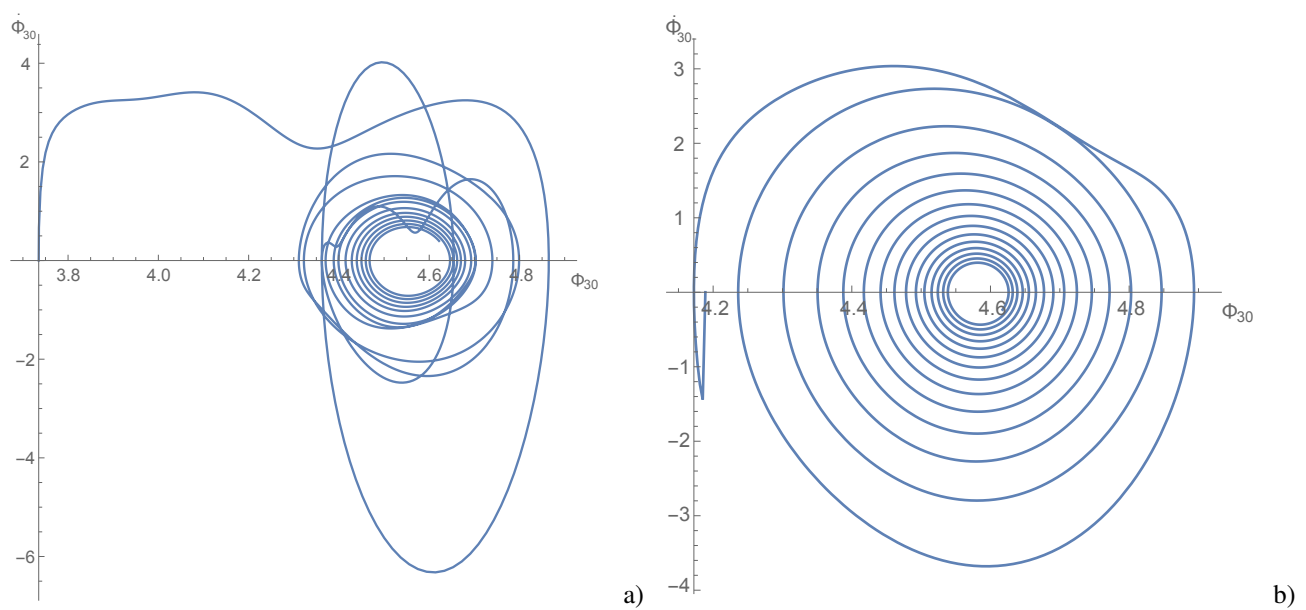


Fig. 8 Phase path of the Lagrangian coordinate related to the last rigid segment: paper beam **a**; PET beam with a mass on the tip **b**.

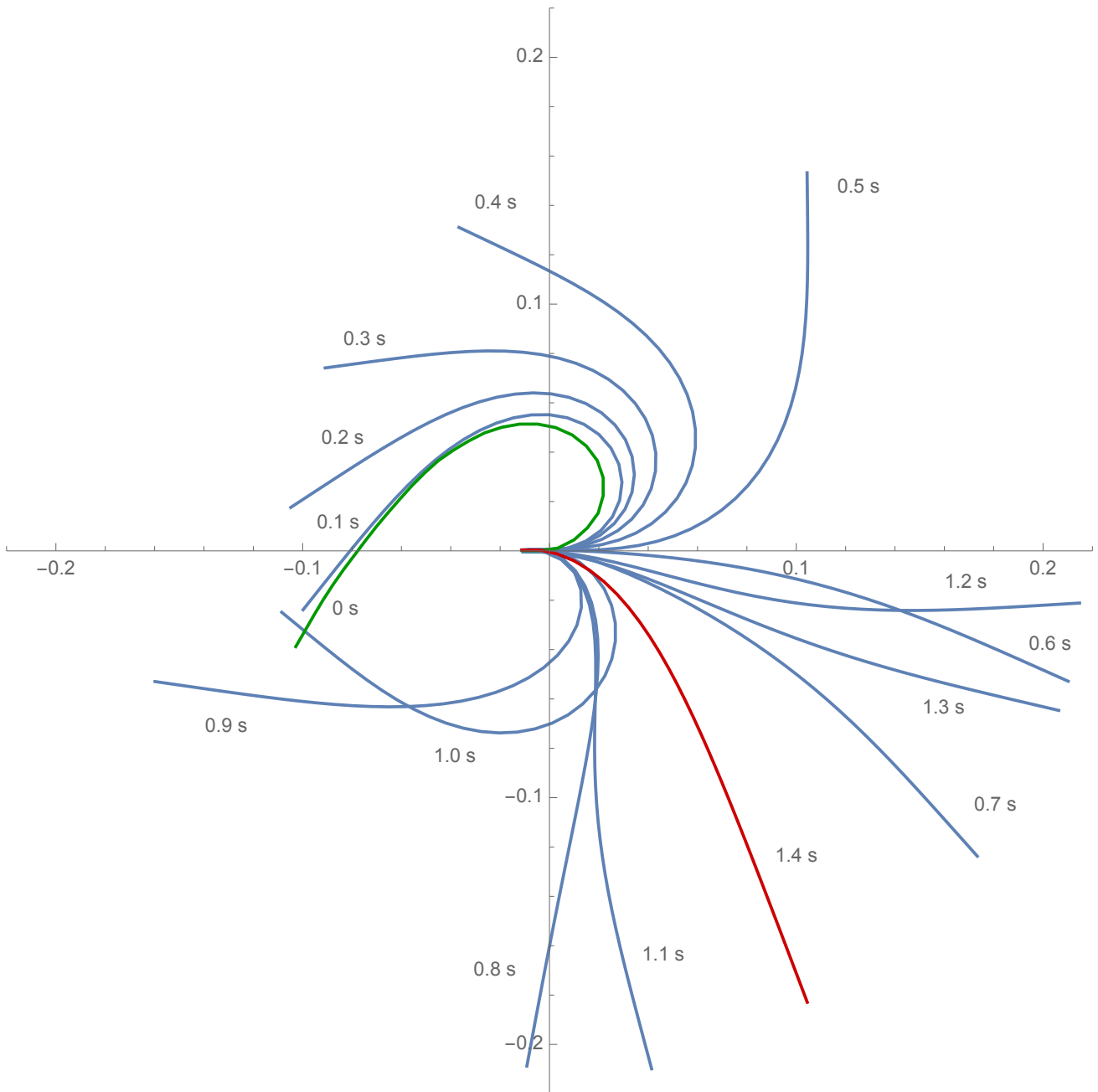


Fig. 9 Stroboscopic motion of the PET beam without mass on the tip. The initial shape is highlighted in green, while the final configuration is red. Near each configuration is specified the corresponding time.

mass, a stroboscopic motion sampled at the rate of 10 images per second from the initial configuration (green solid line), the same displayed also in Fig. 4b, to the image of the current configuration captured at 1.4 s (red solid line).

Acknowledgements IG, AB, ET and LAI are supported by a grant from the Government of the Russian Federation (No. 14.Y26.31.0031).

References

- [1] B.E. Abali, C.C. Wu, and W.H. Müller, An energy-based method to determine material constants in nonlinear rheology with applications, *Continuum Mechanics and Thermodynamics* **28**(5), 1221–1246 (2016).
- [2] J.J. Alibert, A. Della Corte, I. Giorgio, and A. Battista, Extensional elastica in large deformation as Γ -limit of a discrete 1D mechanical system, *Zeitschrift für angewandte Mathematik und Physik* **68**(2), 19 pages (2017).
- [3] J.J. Alibert, A. Della Corte, and P. Seppecher, Convergence of Hencky-type discrete beam model to Euler inextensible elastica in large deformation: rigorous proof, in: *Mathematical Modelling in Solid Mechanics*, (Springer, 2017), pp. 1–12.
- [4] J.J. Alibert, P. Seppecher, and F. dell’Isola, Truss modular beams with deformation energy depending on higher displacement gradients, *Mathematics and Mechanics of Solids* **8**(1), 51–73 (2003).
- [5] H. Altenbach, M. Birsan, and V.A. Eremeyev, Cosserat-type rods, in: *Generalized Continua from the Theory to Engineering Applications*, (Springer, 2013), pp. 179–248.
- [6] H. Altenbach and V.A. Eremeyev, On the constitutive equations of viscoelastic micropolar plates and shells of differential type, *Mathematics and Mechanics of Complex Systems* **3**(3), 273–283 (2015).
- [7] S.S. Antman, Kirchhoff’s problem for nonlinearly elastic rods, *Quarterly of applied mathematics* **32**(3), 221–240 (1974).
- [8] E. Barchiesi, G. Ganzosch, C. Liebold, L. Placidi, R. Grygoruk, and W.H. Müller, Out-of-plane buckling of pantographic fabrics in displacement-controlled shear tests: experimental results and model validation, *Continuum Mechanics and Thermodynamics* pp. 1–13 (2018).
- [9] E. Barchiesi, M. Spagnuolo, and L. Placidi, Mechanical metamaterials: a state of the art, *Mathematics and Mechanics of Solids*(doi: 10.1177/1081286517735695) (2018).
- [10] A. Battista, A. Della Corte, F. dell’Isola, and S. P., Large deformations of 1D microstructured systems modeled as generalized Timoshenko beams, *Zeitschrift für angewandte Mathematik und Physik* **69**, 22 pages (2018).
- [11] D. Bernoulli, Letter from Daniel Bernoulli to Euler, <http://eulerarchive.maa.org/correspondence/letters/OO0147.pdf>, 20 October 1742.
- [12] M. Born, Untersuchungen über die Stabilität der elastischen Linie in Ebene und Raum, unter verschiedenen Grenzbedingungen, PhD thesis, University of University of Göttingen, 1906.
- [13] M. Cuomo, Forms of the dissipation function for a class of viscoplastic models, *Mathematics and Mechanics of Complex Systems* **5**(3), 217–237 (2017).
- [14] A. Della Corte, F. dell’Isola, R. Esposito, and M. Pulvirenti, Equilibria of a clamped Euler beam (Elastica) with distributed load: Large deformations, *Mathematical Models and Methods in Applied Sciences* **27**(08), 1391–1421 (2017).
- [15] F. dell’Isola, U. Andreaus, A. Cazzani, R. Esposito, L. Placidi, U. Perego, G. Maier, and P. Seppecher, The complete works of Gabrio Piola: Volume II Commented English Translation (Springer Publishing Company, Incorporated, 2019).
- [16] F. dell’Isola, I. Giorgio, M. Pawlikowski, and N.L. Rizzi, Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium, *Proceedings of the Royal Society A* **472**(2185), 23 pages (2016).
- [17] F. dell’Isola, T. Lekszycki, M. Pawlikowski, R. Grygoruk, and L. Greco, Designing a light fabric metamaterial being highly macroscopically tough under directional extension: first experimental evidence, *Zeitschrift für angewandte Mathematik und Physik* **66**(6), 3473–3498 (2015).
- [18] F. dell’Isola, G. Maier, U. Perego, U. Andreaus, R. Esposito, and S. Forest, The complete works of Gabrio Piola: Volume I Commented English Translation (Springer Publishing Company, Incorporated, 2014).
- [19] F. dell’Isola, D. Steigmann, and A. Della Corte, Synthesis of fibrous complex structures: designing microstructure to deliver targeted macroscale response, *Applied Mechanics Reviews* **67**(6), 21 pages (2015).
- [20] W.J. Duncan, A critical examination of the representation of massive and elastic bodies by systems of rigid masses elastically connected, *The Quarterly Journal of Mechanics and Applied Mathematics* **5**(1), 97–108 (1952).
- [21] M. Dupac and D.B. Marghitu, Nonlinear dynamics of a flexible mechanism with impact, *Journal of sound and vibration* **289**(4-5), 952–966 (2006).
- [22] S.R. Eugster, C. Hesch, P. Betsch, and C. Glocker, Director-based beam finite elements relying on the geometrically exact beam theory formulated in skew coordinates, *International Journal for Numerical Methods in Engineering* **97**(2), 111–129 (2014).
- [23] L. Euler, De curvis elasticis, Additamentum I to his Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes (Lausanne and Geneva, 1744).
- [24] L. Greco and M. Cuomo, Consistent tangent operator for an exact Kirchhoff rod model, *Continuum Mechanics and Thermodynamics* **27**(4-5), 861–877 (2015).
- [25] H. Hencky, Über die angenäherte lösung von stabilitätsproblemen im raum mittels der elastischen gelenkkette, *Der Eisenbau* **11**, 437–452 (1920).
- [26] F.A. Leckie and G.M. Lindberg, The effect of lumped parameters on beam frequencies, *The Aeronautical Quarterly* **14**(3), 224–240 (1963).
- [27] F. Maurin, F. Greco, and W. Desmet, Isogeometric analysis for nonlinear planar pantographic lattice: discrete and continuum models, *Continuum Mechanics and Thermodynamics* pp. 1–14 (2018).
- [28] G.W. Milton, M. Briane, and D. Harutyunyan, On the possible effective elasticity tensors of 2-dimensional and 3-dimensional printed materials, *Mathematics and Mechanics of Complex Systems* **5**(1), 41–94 (2017).
- [29] A. Misra and P. Poorsolhjoui, Identification of higher-order elastic constants for grain assemblies based upon granular micromechanics, *Mathematics and Mechanics of Complex Systems* **3**(3), 285–308 (2015).

- [30] L. Placidi, U. Andreaus, A. Della Corte, and T. Lekszycki, Gedanken experiments for the determination of two-dimensional linear second gradient elasticity coefficients, *Zeitschrift für angewandte Mathematik und Physik* **66**(6), 3699–3725 (2015).
- [31] L. Placidi, E. Barchiesi, E. Turco, and N. L. Rizzi, A review on 2D models for the description of pantographic fabrics, *Zeitschrift für angewandte Mathematik und Physik* **67**(5), 121 (2016).
- [32] E. Reissner, On one-dimensional finite-strain beam theory: the plane problem, *Zeitschrift für angewandte Mathematik und Physik ZAMP* **23**(5), 795–804 (1972).
- [33] D. Rubinstein, Dynamics of a flexible beam and a system of rigid rods, with fully inverse (one-sided) boundary conditions, *Computer methods in applied mechanics and engineering* **175**(1-2), 87–97 (1999).
- [34] E. Ruocco, H. Zhang, and C. M. Wang, Hencky bar-chain model for buckling analysis of non-uniform columns, *Structures* **6**, 73–84 (2016).
- [35] S. Šalinić, An improved variant of Hencky bar-chain model for buckling and bending vibration of beams with end masses and springs, *Mechanical Systems and Signal Processing* **90**, 30–43 (2017).
- [36] D. Scerrato, I. A. Zhurba Eremeeva, T. Lekszycki, and N. L. Rizzi, On the effect of shear stiffness on the plane deformation of linear second gradient pantographic sheets, *ZAMM - Zeitschrift für Angewandte Mathematik und Mechanik* **96**(11), 1268–1279 (2016).
- [37] M. Spagnuolo and U. Andreaus, A targeted review on large deformations of planar elastic beams: extensibility, distributed loads, buckling and post-buckling, *Mathematics and Mechanics of Solids*(doi: 10.1177/1081286517737000) (2018).
- [38] D. J. Steigmann and M. G. Faulkner, Variational theory for spatial rods, *Journal of Elasticity* **33**(1), 1–26 (1993).
- [39] E. Turco, Discrete is it enough? The revival of Piola–Hencky keynotes to analyze three-dimensional *Elastica*, *Continuum Mechanics and Thermodynamics*(doi: 10.1007/s00161-018-0656-4:1–19) (2018).
- [40] E. Turco, F. dell’Isola, A. Cazzani, and N. L. Rizzi, Hencky-type discrete model for pantographic structures: numerical comparison with second gradient continuum models, *Zeitschrift für angewandte Mathematik und Physik* **67**(4), 28 pages (2016).
- [41] E. Turco, F. dell’Isola, N. L. Rizzi, R. Grygoruk, W. H. Müller, and C. Liebold, Fiber rupture in sheared planar pantographic sheets: Numerical and experimental evidence, *Mechanics Research Communications* **76**, 86–90 (2016).
- [42] E. Turco, I. Giorgio, A. Misra, and F. dell’Isola, King post truss as a motif for internal structure of (meta) material with controlled elastic properties, *Royal Society open science* **4**(10), 20 pages (2017).
- [43] E. Turco, M. Golaszewski, A. Cazzani, and N. L. Rizzi, Large deformations induced in planar pantographic sheets by loads applied on fibers: experimental validation of a discrete lagrangian model, *Mechanics Research Communications* **76**, 51–56 (2016).
- [44] C. M. Wang, H. Zhang, R. P. Gao, W. H. Duan, and N. Challamel, Hencky bar-chain model for buckling and vibration of beams with elastic end restraints, *International Journal of Structural Stability and Dynamics* **15**(07), 16 pages (2015).
- [45] Y. Wang and R. L. Huston, A lumped parameter method in the nonlinear analysis of flexible multibody systems, *Computers & structures* **50**(3), 421–432 (1994).
- [46] H. Zhang, C. M. Wang, and N. Challamel, Buckling and vibration of Hencky bar-chain with internal elastic springs, *International Journal of Mechanical Sciences* **119**, 383–395 (2016).

Chapter 5

Asymptotic methods and justification of the Landau-lifshitz membrane model

5.1 Introduction

The paper “An asymptotic membrane model for wrinkling of very thin films” presented in this section, deals with the formal justification, by means of asymptotic expansion, of a membrane model heuristically deduced by Landau and Lifshitz in [28]. Asymptotic expansion represents an important and very useful tool in mathematical and engineering modelling. Generally speaking, if some parameter entering the description of a physical problem is small compared to the others, the technique of asymptotic expansion allows to extract from the model the most relevant effects, discarding lower order ones related to the smallness of one or more parameters. In this way, one ends up with a simpler model capable, in most cases, to capture the overall behavior of a physical system. In this chapter we show how we have used this technique to derive a membrane model capable to describe wrinkling patterns in pre-stressed membranes. In particular, we have been interested in rigorously justifying, by asymptotic expansions of nonlinear equations of three-dimensional elasticity, a two-dimensional membrane model capable of describing the folding patterns of prestressed membranes. The approach we used is based on local formulation and adimensionalization of the equilibrium equations, following the approach developed in [29, 25]. The membrane model thus obtained is similar to that obtained heuristically by Landau and Lifshitz from the equations Föppl-von Kármán in [28]. To our knowledge, it has never been rigorously justified in the literature and its properties have not been studied in a precise way. According to the equation (14.4-5) p.52 of Landau and Lifshitz [28], the deformation of a plate located in the plane (x_1, x_2) and subjected to great displacements can be

described by the following equations:

$$D\Delta^2 u_3 - h \frac{\partial}{\partial x_\beta} \left(\tilde{n}_{\alpha\beta} \frac{\partial u_3}{\partial x_\alpha} \right) = P \quad \alpha, \beta \in \{1, 2\} \quad (5.1)$$

$$\frac{\partial \tilde{n}_{\alpha\beta}}{\partial x_\beta} = 0 \quad (5.2)$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ represents the bending stiffness (E is the Young's modulus and ν the Poisson's ratio), h the thickness of the plate, u_3 the transverse displacement, $\tilde{n}_{\alpha\beta}$ the nonlinear tensor of the membrane stresses and P the applied load. The first term of (5.1) is negligible for very thin plates ($h \ll 1$) or for very important membrane constraints. In this case, the problem degenerates into a pure membrane problem (see equation (14.8) p.53 in [28]):

$$h \frac{\partial}{\partial x_\beta} \left(n_{\alpha\beta} \frac{\partial u_3}{\partial x_\alpha} \right) = -P \quad (5.3)$$

$$\frac{\partial n_{\alpha\beta}}{\partial x_\beta} = 0 \quad (5.4)$$

where $n_{\alpha\beta}$ is the linear part of $\tilde{n}_{\alpha\beta}$.

We justify the previous model considering a 3D plate, where the small parameter imposed by the geometry is represented by the ratio between the diameter and the thickness. Performing a power series development of all the dependent variables and identifying the coefficients at the order, we show how we have deduced the 2D membrane model. In addition to providing a formal but rigorous justification for this model, we show in [7], based on numerical simulations, that the model is particularly well suited to describe the wrinkling of very thin prestressed membranes (see figures 5.1 and 5.2).



Figure 5.1: Possible modes for the out-of-plane displacement of a membrane associated with eigenvalues determined from the Sturm-Liouville problem (see [7]). From left to right: (1,3) mode, (1,5) mode, and (2,4) mode.

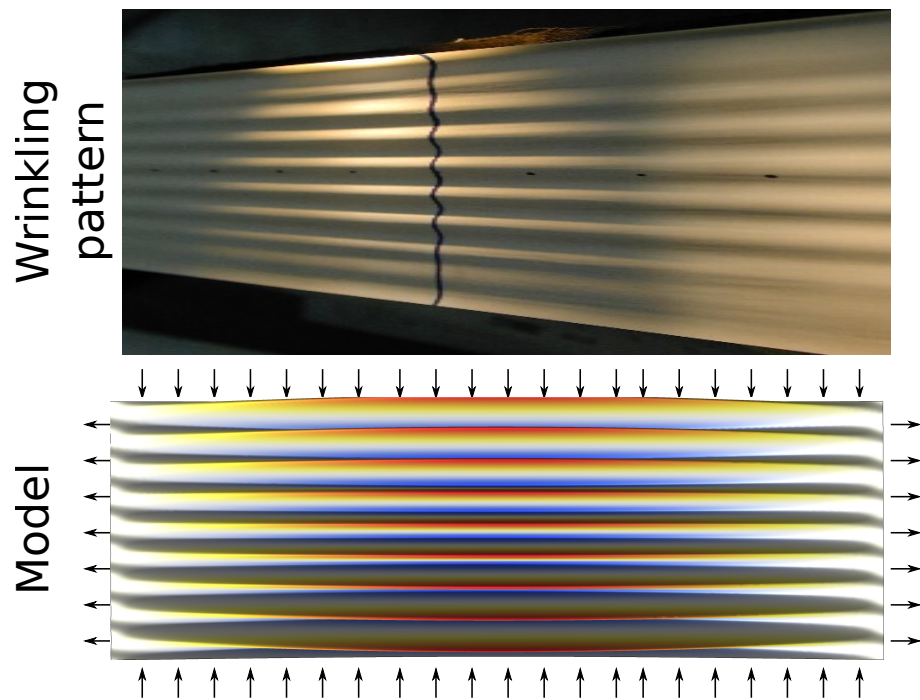


Figure 5.2: Comparison between the numerical solution of the membrane model in the hyperbolic case and a real membrane. a) Numerical solution. b) Photo of the stretched and sheared membrane.

ORIGINAL ARTICLE

Antonio Battista  · Aziz Hamdouni · Olivier Millet

An asymptotic membrane model for wrinkling of very thin films

Received: 22 December 2017 / Accepted: 28 April 2018
© Springer-Verlag GmbH Germany, part of Springer Nature 2018

Abstract In this work, a formal deduction of a two-dimensional membrane theory, similar to Landau–Lifshitz model, is performed via an asymptotic development of the weak formulation of the three-dimensional equations of elasticity. Some interesting aspects of the deduced model are investigated, in particular the property of obtaining a hyperbolic equation for the out-of-plane displacement under a certain class of boundary conditions and loads. Some simple cases are analyzed to show the relevant aspects of the model and the phenomenology that can be addressed. In particular, it is shown how this mathematical formulation is capable to describe instabilities well known as wrinkling, often observed for the buckling of very thin membranes.

Keywords Asymptotic methods · Dimensional analysis · Plate theory · Membrane theory · Hyperbolic problem

1 Introduction

In the last decades, lot of scientific effort has been devoted to a rigorous justification of two-dimensional plate theories (see for instance [1, 2]). Historically the set of equations used by engineers for the mathematical description of thin structures, classified in literature as plates, membranes or shells, is deduced by a priori assumptions based on the predominance of two dimensions of the system (the middle surface) over the thickness. This is the usual way of inferring, for instance, the well-known Kirchhoff–Love model, both in linear and nonlinear elasticity, as well as the von Kármán one within geometrically nonlinear theory.

This work focuses on a two-dimensional membrane model inferred by Landau and Lifshitz in [3], from the Föppl–von Kármán equations, that, according to our knowledge, has rarely attracted the attention of researchers in the existing literature.

Communicated by Francesco dell’Isola.

A. Battista (✉) · A. Hamdouni · O. Millet
Laboratoire des Sciences de l’Ingénieur pour l’Environnement UMR - 7356 CNRS, Université de La Rochelle, La Rochelle, France
E-mail: antonio.battista@univ-lr.fr

A. Hamdouni
E-mail: aziz.hamdouni@univ-lr.fr

O. Millet
E-mail: olivier.millet@univ-lr.fr

A. Battista
M&MoCS, Università degli Studi dell’Aquila, L’Aquila, Italy

Published online: 09 May 2018

Following Landau and Lifshitz, the deformation of a plate undergoing large deflections can be described by the following equations (i.e., (14.4–5), p. 52 in [3]):

$$D\Delta^2 u_3 - h \frac{\partial}{\partial x_\beta} \left(\tilde{n}_{\alpha\beta} \frac{\partial u_3}{\partial x_\alpha} \right) = P \quad \alpha, \beta \in \{1, 2\} \quad (1)$$

$$\frac{\partial \tilde{n}_{\alpha\beta}}{\partial x_\beta} = 0 \quad (2)$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ represents the flexural rigidity (E is the Young modulus and ν the Poisson ratio), h is the thickness of the plate, u_3 the transverse displacement, $n_{\alpha\beta}$ the component of the in-plane stress, and P the applied load, for a plate lying in the $\{x_1, x_2\}$ plane. The first term in (1) is negligible for very thin plates ($h \ll 1$) or equivalently for very important in-plane stresses. In these cases, the problem (1)–(2) degenerates to the membrane problem (see equation (14.8) p. 53 [3]):

$$h \frac{\partial}{\partial x_\beta} \left(n_{\alpha\beta} \frac{\partial u_3}{\partial x_\alpha} \right) = -P \quad (3)$$

$$\frac{\partial n_{\alpha\beta}}{\partial x_\beta} = 0 \quad (4)$$

where $n_{\alpha\beta}$ is the linear part of $\tilde{n}_{\alpha\beta}$. Indeed, the author neglected “the additional longitudinal stress caused by the bending of the plate” [3, p. 53], which corresponds to neglect the nonlinear part of $n_{\alpha\beta}$. We will refer to the degenerate problem (3)–(4) as Landau–Lifshitz model. Classically, equation (3) will be elliptic, hyperbolic or parabolic according to the signs of the eigenvalues of $n_{\alpha\beta}$ that we will indicate with λ_1 and λ_2 . In particular, it will be hyperbolic if $\lambda_1 \lambda_2 < 0$. In this paper, we will focus on the hyperbolic problem that can explain and reproduce, in a simple way, well-known phenomena as wrinkling or out-of-plane buckling.

Recently, the problem of the description of wrinkling formation has attracted the attention of many authors due to the fact that, often, the derive toward miniaturization in technology leads to very thin components (see for instance [4–9]). The formation of wrinkling patterns can be regarded as a consequence of out-of-plane buckling induced via different mechanisms. In [10, 11], such mechanism is related to the interplay between in-plane bending and out-of-plane bending and twist of the fibers constituting the considered body. In other words when the fibers, which can store bending energy, find energetically more convenient to have an out plane deformation, then wrinkling instabilities occur after buckling. In Landau–Lifshitz model, the out-of-plane displacement is governed by an equation where the lower-order plain stress appears as an elasticity tensor. Therefore, the principal directions of in-plane stresses characterize a set of curves playing the same role as reinforcement fibers introduced in pantographic sheets (see [12–14]).

One of the main aim of this work is to give a formal, but rigorous, justification of Landau–Lifshitz model via asymptotic expansion of the weak nonlinear 3D elastic problem. In our derivation, we will follow the asymptotic approach already used for plates and shells in [15–21]. In this approach, all the known quantities of the problem are related to a small parameter $\varepsilon = h/L$ (h and L being, respectively, the thickness and the diameter of the plate) and a bidimensional model is derived by asymptotic expansion of 3D nonlinear equations of elasticity, depending on the level of applied forces. One of the point of strength of this method relies on the possibility to estimate the scope of validity of each of the deduced two-dimensional models on the basis of the geometry and of the external forces applied on the system. Following these works, we present, in the first part of this paper, the formal deduction of Landau–Lifshitz model for applied forces of ε order, starting from the weak formulation of the equilibrium equation of three-dimensional nonlinear elasticity. Note that Landau–Lifshitz model has been already justified in [19] by asymptotic expansion of geometrically nonlinear elasticity, but with a more formal approach based on local formulation and for fully clamped boundary conditions.

In the first part of this work we will perform, in a more rigorous way, but still formal, the asymptotic expansion of the weak formulation of the 3D nonlinear elasticity problem. The order of magnitude of the displacements are chosen according to [19]. Moreover, we will consider more general boundary conditions, so as the examples considered in the second part of this paper to be consistent with the asymptotic membrane problem obtained. However, we postpone to future works the proof of convergence results and the study of boundary layers that may result from the asymptotic expansion of the equations.

In the second part of this work, we will focus on the membrane model investigating in particular the interesting property of the derived equations, to give rise to a hyperbolic problem for the out-of-plane displacement in particular conditions of the in-plane stress state. We will show how the hyperbolic model reproduces

rather accurately wave like patterns originating in very thin films under specific loads action (see for instance [22–27]).

2 The three-dimensional problem

In what follows, star will indicate dimensional quantities, while unstarred quantities are nondimensional. Moreover, we assume Latin indexes to be defined in $\{1, 2, 3\}$ and Greek indexes in $\{1, 2\}$, summation over repeated indexes will be understood.

Let us fix a Cartesian coordinate system in the Euclidean space $\mathbb{R}^3$, we call middle surface of the body, in the reference configuration, the bounded connected region $\omega^* \in \mathbb{R}^2$, of diameter L with suitably regular boundary constituted by the curve γ^* , contained in the plane spanned by the vectors $\hat{e}_\alpha$ and we assume that the body occupies the region $\Omega^* = \omega^* \times]-h/2, h/2[\in \mathbb{R}^3$ (see Fig. 1). In the following, we will consider only thin plates, so we assume $h \ll L$.

We indicate with $\partial\Omega^* = \Gamma^{*+} \cup \Gamma^{*-} \cup \Gamma^*$ the boundary of the region occupied by the body, where $\Gamma^{*\pm}$ denotes the upper and lower boundary surfaces, and Γ^* the lateral surface, i.e.:

$$\Gamma^{*\pm} = \omega^* \times \left\{ \pm \frac{h}{2} \right\} \quad \text{and} \quad \Gamma^* = \gamma^* \times \left[-\frac{h}{2}, \frac{h}{2} \right].$$

Moreover, we decompose the lateral surface Γ^* in:

$$\Gamma^* = \Gamma^{*c} \cup \Gamma^{*b},$$

where

$$\Gamma^{*c} = \gamma^{*c} \times \left[-\frac{h}{2}, \frac{h}{2} \right]$$

is the part of lateral surface where clamping condition is prescribed, and

$$\Gamma^{*b} = \gamma^{*b} \times \left[-\frac{h}{2}, \frac{h}{2} \right]$$

the part where an external force is applied.

With respect to the chosen reference system, the position of a material particle of Ω^* will be written as

$$\mathbf{X}^* = x_\alpha^* \hat{e}_\alpha + x_3^* \hat{e}_3 = \mathbf{x}^* + x_3^* \hat{e}_3 \quad (5)$$

Its associated displacement $\mathbf{U}^* : \Omega^* \rightarrow \mathbb{R}^3$ is then given by:

$$\mathbf{U}^* = u_\alpha^* \hat{e}_\alpha + u_3^* \hat{e}_3 = \mathbf{u}^* + u_3^* \hat{e}_3 \quad (6)$$

Note that every vector through the paper will be written following this decomposition.

Moreover in the sequel, the following notation will be understood: $\partial/\partial \mathbf{X}^*$ or Grad^* , Div^* and $\mathbf{I}_3$ will denote, respectively, the gradient, the divergence and the identity operator of $\mathbb{R}^3$ while $\partial/\partial \mathbf{x}^*$ or grad^* , div^* and $\mathbf{I}_2$ we will indicate, respectively, the gradient, the divergence and the identity operator of $\mathbb{R}^2$. The symbol $(\cdot)^\top$ and $\text{Tr}(\cdot)$ will denote, respectively, transposition and trace of an operator. Finally, we will indicate with

$$\mathbf{f}^* = \mathbf{f}_t^* + f_3^* \hat{e}_3 : \Omega^* \rightarrow \mathbb{R}^3, \quad \mathbf{g}^{*\pm} = \mathbf{g}_t^{*\pm} + g_3^{*\pm} \hat{e}_3 : \Gamma^{*\pm} \rightarrow \mathbb{R}^3, \quad \mathbf{t}^* = \mathbf{t}_t^* + t_3^* \hat{e}_3 : \Gamma^{*b} \rightarrow \mathbb{R}^3$$

the body and surface forces to which the plate is subjected (see Fig. 1).

The explicit form of all the operators is given in “Appendix”.

Within the framework of nonlinear elasticity, the three-dimensional equilibrium equations can be written as:

$$\text{Div}^*(\boldsymbol{\Sigma}^* \mathbf{F}^{*\top}) + \mathbf{f}^* = 0 \quad (7)$$

with boundary conditions (B.C.)

$$\begin{cases} (\mathbf{F}^* \boldsymbol{\Sigma}^*) \cdot \mathbf{v} \big|_{\Gamma^{*\pm}} = \mathbf{g}^{*\pm} \\ (\mathbf{F}^* \boldsymbol{\Sigma}^*) \cdot \mathbf{v} \big|_{\Gamma^{*b}} = \mathbf{t}^* \\ \mathbf{U}^* \big|_{\Gamma^{*c}} = \mathbf{0} \end{cases} \quad (8)$$

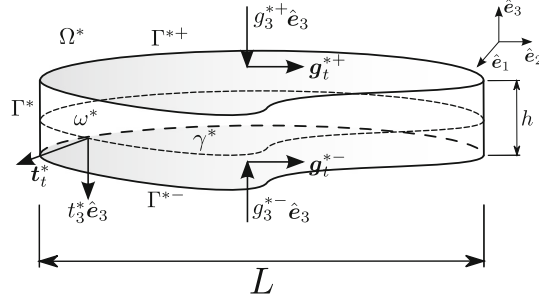


Fig. 1 3D body

where Σ^* denotes the second Piola–Kirchhoff tensor, F^* the gradient of the map between the reference and the actual configuration and $\mathbf{v}$ the outer normal to the boundary.

In the following, we will limit our problem to Hookean materials, defined by the constitutive law

$$\Sigma^* = \lambda \text{Tr}(\mathbf{E}) \mathbf{I}_3 + 2\mu \mathbf{E} \quad (9)$$

where $\mathbf{E}^*$ represents the nonlinear strain tensor, and λ, μ the Lamé coefficients of the material.

The problem defined by (7) and (8) admits the following weak formulation:

$$\begin{aligned} \text{Find } \mathbf{U}^* \in \mathbb{H} = \left\{ \mathbf{U}^* \in \{\mathcal{V}\}^3 \text{ such that } \mathbf{U}^*|_{\Gamma^{*c}} = \mathbf{0} \right\} \\ \int_{\Omega^*} \text{Tr}([\Sigma^* \bar{F}^{*\top}] \mathbf{G}^*(\mathbf{V}^*)) d\mathbf{X}^* - \int_{\Omega^*} \mathbf{f}^* \cdot \mathbf{V}^* d\mathbf{X}^* - \\ - \int_{\omega^*} (\mathbf{g}^{*+} + \mathbf{g}^{*-}) \cdot \mathbf{V}^* d\mathbf{x}^* - \int_{\Gamma^{*b}} \mathbf{t}^* \cdot \mathbf{V}^* dx_3^* ds^* = 0 \\ \forall \mathbf{V}^* \in \mathbb{H} \end{aligned} \quad (10)$$

where $\mathcal{V}$ represents the space of Sobolev functions $W^{1,4}(\Omega^*)$ in order to have a unique solution¹ for problem (2), and ds^* the line element along the curve γ^{*b} . In the next, we use $\mathbf{G}^*(\mathbf{V}^*)$ to write $\text{Grad}^*(\mathbf{V}^*)$ in a more compact form (see “Appendix”).

To Eq. (10), we add the injectivity condition for the map F^* :

$$\exists c_0 \in \mathbb{R} \text{ and } c_0 > 0 \text{ such that } \det F^* \geq c_0. \quad (11)$$

3 Deduction of Landau–Lifshitz model by asymptotic expansion

3.1 Dimensional analysis

Let us define the following nondimensional variables:

$$\begin{aligned} \mathbf{u} = \frac{\mathbf{u}^*}{\|\mathbf{u}_r\|}, \quad u_3 = \frac{u_3^*}{u_{3r}}, \quad \mathbf{x} = \frac{\mathbf{x}^*}{L}, \quad x_3 = \frac{x_3^*}{h}, \quad \Sigma = \frac{\Sigma^*}{\mu} \\ f_3 = \frac{f_3^*}{f_{3r}}, \quad g_3^\pm = \frac{g_{3r}^{*\pm}}{g_{3r}}, \quad t_3^\pm = \frac{t_{3r}^{*\pm}}{t_{3r}}, \quad f_t = \frac{f_t^*}{\|f_{tr}\|}, \quad g_t^\pm = \frac{g_{tr}^{*\pm}}{\|g_{tr}\|}, \quad t_t = \frac{t_t^*}{\|t_{tr}\|}, \end{aligned}$$

¹ Being interested in a formal deduction, we consider sufficiently regular spaces.

where the variables indexed with r are the reference ones². In the framework of nonlinear elasticity, it would be natural to consider $\|\mathbf{u}_r\| = L$ and $u_{3r} = L$. However, for the level forces considered further in our development, this would lead, in the ambit of the asymptotic development, to null first-order term displacements, and to a rescaling of the reference variables as $\|\mathbf{u}_r\| = h$ and $u_{3r} = h$ as showed in [19]. Therefore in our treatment, we will consider directly the correct scaling from the beginning of the calculations.

Let us now write the nondimensional form of the weak formulation of the problem introduced above. On the basis of the quantities introduced in this section (of course, the same arguments for the nondimensional analysis of $\mathbf{U}^*$ hold for $\mathbf{V}^*$), we obtain:

$$\begin{aligned} & \mu L^2 h \int_{\Omega} \text{Tr}([\boldsymbol{\Sigma} \mathbf{F}^T] \mathbf{G}(\mathbf{V})) d\mathbf{X} - L^2 h^2 \left[\|\mathbf{f}_{tr}\| \int_{\Omega} \mathbf{f}_t \cdot \mathbf{v} d\mathbf{X} + f_{3r} \int_{\Omega} f_3 v_3 d\mathbf{X} \right] - \\ & - L^2 h \left[\|\mathbf{g}_{tr}\| \int_{\omega} \mathbf{p}_t \cdot \mathbf{v} d\mathbf{x} + g_{3r} \int_{\omega} p_3 v_3 d\mathbf{x} \right] - \\ & L h^2 \left[\|\mathbf{t}_{tr}\| \int_{\Gamma^b} \mathbf{t}_t \cdot \mathbf{v} dx_3 ds + t_{3r} \int_{\Gamma^b} t_3 v_3 dx_3 ds \right] = 0. \end{aligned} \quad (12)$$

where we have set $\mathbf{p}_t = \mathbf{g}_t^+ + \mathbf{g}_t^-$ and $p_3 = g_3^+ + g_3^-$. Moreover, for the type of problem we are interested in the sequel (in particular in Sect. 4), we consider that $t_3 = 0$. From (12), it comes straightforward:

$$\begin{aligned} & \int_{\Omega} \text{Tr}([\boldsymbol{\Sigma} \mathbf{F}^T] \mathbf{G}(\mathbf{V})) d\mathbf{X} - \frac{h \|\mathbf{f}_{tr}\|}{\mu} \int_{\Omega} \mathbf{f}_t \cdot \mathbf{v} d\mathbf{X} - \frac{h f_{3r}}{\mu} \int_{\Omega} f_3 v_3 d\mathbf{X} - \\ & - \frac{\|\mathbf{g}_{tr}\|}{\mu} \int_{\omega} \mathbf{p}_t \cdot \mathbf{v} d\mathbf{x} + \frac{g_{3r}}{\mu} \int_{\omega} p_3 v_3 d\mathbf{x} - \frac{\varepsilon \|\mathbf{t}_{tr}\|}{\mu} \int_{\Gamma^b} \mathbf{t}_t \cdot \mathbf{v} dx_3 ds = 0 \end{aligned} \quad (13)$$

while condition (11) simply becomes:

$$\det \mathbf{F} \geq c_0. \quad (14)$$

In the formulation of the nondimensional problem, the following nondimensional quantities naturally arise:

$$\mathcal{F}_t = \frac{h \|\mathbf{f}_{tr}\|}{\mu}, \quad \mathcal{F}_3 = \frac{h_0 f_{3r}}{\mu}, \quad \mathcal{G}_t = \frac{\|\mathbf{g}_{tr}\|}{\mu}, \quad \mathcal{G}_3 = \frac{g_{3r}}{\mu}, \quad \mathcal{T}_t = \frac{\|\mathbf{t}_{tr}\|}{\mu}.$$

3.2 Asymptotic expansion of equations

In this section, we will provide a formal deduction of Landau–Lifshitz model taking into account the following forces level:

$$\mathcal{F}_t = 0, \quad \mathcal{F}_3 = \varepsilon^4, \quad \mathcal{G}_t = \varepsilon^2, \quad \mathcal{G}_3 = \varepsilon^3, \quad \mathcal{T}_t = \varepsilon$$

i.e., important tangential surface forces and moderate transversal ones, while the order of magnitude of the body force is chosen in such a way to be comparable with the force of gravity. In fact we have set $\mathcal{F}_t = 0$, while a simple calculation shows that the weight leads to $\mathcal{F}_3 = \varepsilon^4$. The very important order of magnitude of the traction applied on the lateral boundary, i.e., $\mathcal{T}_t$ will make arise the membrane behavior, and is linked to the formal argument used in [3] to deduce the model.

The asymptotic expansion of Eqs. (13)–(14) leads to the following result:

² The nondimensional analysis of the variables leads to the following form for the differential and integral operators:

$$\begin{aligned} \partial_{x_\alpha}^* & \rightarrow L^{-1} \partial_{x_\alpha}, & \partial_{x_3}^* & \rightarrow h^{-1} \partial_{x_3}, \\ \int_{\Omega^*} d\mathbf{X}^* & \rightarrow L^2 h \int_{\Omega} d\mathbf{X}, & \int_{\omega^*} dx^* & \rightarrow L^2 \int_{\omega} dx. \end{aligned}$$

Results 1 For applied forces such as $\mathcal{F}_t = 0$, $\mathcal{F}_3 = \varepsilon^4$, $\mathcal{G}_t = \varepsilon^2$, $\mathcal{G}_3 = \varepsilon^3$, $\mathcal{T}_t = \varepsilon$, the leading term (lowest nonnull) of the asymptotic expansion of the displacement is solution of the membrane model

$$\begin{cases} \operatorname{div}(\mathbf{n}_t) = -\mathbf{p}_t \\ \operatorname{div}(\mathbf{n}_t \operatorname{grad} u_3^0) = -p_3 \end{cases} \quad (15)$$

where we have denoted $\mathbf{n}_t$ the in-plane components of the stress tensor satisfying the following constitutive law:

$$\mathbf{n}_t(\mathbf{u}) = \frac{2\beta}{2+\beta} \operatorname{Tr}(\mathbf{e}_t(\mathbf{u}^0)) + 2\mathbf{e}_t(\mathbf{u}^0)$$

with

$$\mathbf{e}_t(\mathbf{u}^0) = \frac{1}{2}(\operatorname{grad} \mathbf{u}^0 + (\operatorname{grad} \mathbf{u}^0)^\top)$$

The associated boundary conditions are given by:

$$\begin{cases} U_t^0|_{\gamma^c} = 0 \\ \mathbf{n}_t v|_{\gamma^b} = \tilde{\mathbf{t}}_t \end{cases} \quad (16)$$

where $\tilde{\mathbf{t}}_t$ represents the average along x_3 of the traction applied on the lateral surface Γ^b .

Proof We postulate that the displacements of the problem $\mathbf{U}^\varepsilon = (\mathbf{u}^\varepsilon, u_3^\varepsilon)$ admit the power series expansion with respect to ε :

$$(\mathbf{u}^\varepsilon, u_3^\varepsilon) = (\mathbf{u}^0, u_3^0) + (\varepsilon \mathbf{u}^1, \varepsilon u_3^1) + (\varepsilon^2 \mathbf{u}^2, \varepsilon^2 u_3^2) + \dots \quad (17)$$

Substituting this expansion in Eq. (13) and equating to zero each order of the asymptotic expansion, we obtain a self consistent problem for the lower-order terms. In “Appendix” the explicit form for the expansion of all the operators is given. $\square$

Problem at order ε^0 .

At the lowest order, we find for the problem $\mathcal{P}_0$:

$$\int_{\Omega} \operatorname{Tr}([\hat{\Sigma}^0 \hat{\mathbf{F}}^{0\top}] \mathbf{G}^0(V)) d\mathbf{X} = 0 \quad \forall V \in \mathbb{H} \quad (18)$$

That leads to

$$\begin{aligned} \int_{\Omega} \left\{ \left[\left(1 + \frac{\partial u_3^0}{\partial x_3} \right) \frac{\partial \mathbf{u}^0}{\partial x_3} \right] \cdot \frac{\partial \mathbf{v}}{\partial x_3} + \right. \\ \left. + \left[\frac{(2+\beta)}{2} \left(1 + \frac{\partial u_3^0}{\partial x_3} \right) \left(\left\| \frac{\partial \mathbf{u}^0}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3} \right)^2 + 2 \frac{\partial u_3^0}{\partial x_3} \right) \right] \frac{\partial v_3}{\partial x_3} \right\} d\mathbf{X} = 0. \end{aligned} \quad (19)$$

To proceed, we need to establish the following lemma:

Lemma 1 If the generic vector field $\mathbf{R} = \mathbf{r} + r_3 \hat{\mathbf{e}}_3$ satisfies

$$\int_{\Omega} \left(\mathbf{r} \cdot \frac{\partial \mathbf{v}}{\partial x_3} + r_3 \frac{\partial v_3}{\partial x_3} \right) d\mathbf{X} = 0, \quad \forall V \in \mathbb{H} \Rightarrow \mathbf{r} = \mathbf{0}, r_3 = 0. \quad (20)$$

Proof of Lemma 1 We have:

$$\begin{aligned} \int_{\Omega} \left(\mathbf{r} \cdot \frac{\partial \mathbf{v}}{\partial x_3} + r_3 \frac{\partial v_3}{\partial x_3} \right) d\mathbf{X} \\ = \int_{\Omega} \left\{ \frac{\partial}{\partial x_3} (\mathbf{r} \cdot \mathbf{v}) - \frac{\partial \mathbf{r}}{\partial x_3} \cdot \mathbf{v} \right\} d\mathbf{X} + \int_{\Omega} \left\{ \frac{\partial}{\partial x_3} (r_3 v_3) - \frac{\partial r_3}{\partial x_3} v_3 \right\} d\mathbf{X} \end{aligned} \quad (21)$$

Considering any vector $\mathbf{V} = (\mathbf{v}, 0) \in \mathbb{H}$ in (20) and using (21), we obtain:

$$\int_{\Omega} \frac{\partial \mathbf{r}}{\partial x_3} \cdot \mathbf{v} d\mathbf{X} - \int_{\omega} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial}{\partial x_3} (\mathbf{r} \cdot \mathbf{v}) dx_3 d\mathbf{x} = 0 \quad \forall \mathbf{v} \in \mathcal{V}^2$$

Choosing now

$$\mathbf{v} \Big|_{\Gamma^+} = \mathbf{v} \Big|_{\Gamma^-} = 0$$

we get

$$\frac{\partial \mathbf{r}}{\partial x_3} = 0 \tag{22}$$

Next, choosing $\mathbf{v}$ of the form $\mathbf{v} = (x_3 - \frac{1}{2})\mathbf{v}_1 + (x_3 + \frac{1}{2})\mathbf{v}_2$, with $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}^2$, we deduce that $\mathbf{r} = 0$ on $\Gamma^{\pm}$. Since $\mathbf{r}$ is independent with respect to x_3 according to (22), we conclude that $\mathbf{r} = 0$ in Ω . In an analogous way, we can prove $v_3 = 0$. $\square$

From Lemma 1, Eq. (19) implies that:

$$\begin{cases} \left(1 + \frac{\partial u_3^0}{\partial x_3}\right) \frac{\partial \mathbf{u}^0}{\partial x_3} = 0 \\ \frac{(2 + \beta)}{2} \left(1 + \frac{\partial u_3^0}{\partial x_3}\right) \left(\left\| \frac{\partial \mathbf{u}^0}{\partial x_3} \right\|^2 + \left(\frac{\partial u_3^0}{\partial x_3} \right)^2 + 2 \frac{\partial u_3^0}{\partial x_3} \right) = 0. \end{cases} \tag{23}$$

To conclude, let us invoke also the injectivity condition (14) which leads at first order to $\det \hat{\mathbf{F}}^0 > 0$, that is equivalent to (see (69) in appendix):

$$\left(1 + \frac{\partial u_3^0}{\partial x_3}\right) > 0.$$

Therefore, (23) leads trivially to

$$\frac{\partial \mathbf{u}^0}{\partial x_3} = \mathbf{0} \quad \text{and} \quad \frac{\partial u_3}{\partial x_3} = 0.$$

At the first order, we find that the leading terms of the series expansion of the displacements are independent from x_3 :

$$\mathbf{u}^0 = \mathbf{u}^0(x_1, x_2) \quad \text{and} \quad u_3^0 = u_3^0(x_1, x_2). \tag{24}$$

Problem at order ε^1 .

On the basis of the results obtained at order 0, we have

$$\hat{\boldsymbol{\Sigma}}^0 = \mathbf{0} \quad \text{and} \quad \hat{\mathbf{F}}^0 = \mathbf{I}_3$$

and problem $\mathcal{P}_1$ reduces to:

$$\int_{\Omega} \text{Tr} (\hat{\boldsymbol{\Sigma}}^1 \mathbf{G}^0(\mathbf{V})) d\mathbf{X} = 0 \quad \forall \mathbf{V} \in \mathbb{H} \tag{25}$$

Now we have (see ‘‘Appendix’’):

$$\hat{\boldsymbol{\Sigma}}^1 = \hat{\boldsymbol{\Sigma}}_l^1 + \hat{\boldsymbol{\Sigma}}_{nl}^1 = \boldsymbol{\Sigma}_l^1(\mathbf{U}^0) + \boldsymbol{\Sigma}_l^0(\mathbf{U}^1) + 2\boldsymbol{\Sigma}_{nl}^0(\mathbf{U}^0, \mathbf{U}^1) + \boldsymbol{\Sigma}_{nl}^1(\mathbf{U}^0, \mathbf{U}^0) \tag{26}$$

However, the independence of $\mathbf{U}^0$ with respect to x_3 obtained as (24) implies that the last two terms in the sum (26) are null.

Equation (25) then reduces to

$$\int_{\Omega} \left(\frac{\partial \mathbf{u}^1}{\partial x_3} + \text{grad } u_3^0 \right) \cdot \frac{\partial \mathbf{v}}{\partial x_3} + \left[(2 + \beta) \frac{\partial u_3^1}{\partial x_3} + \beta \text{div } \mathbf{u}^0 \right] \frac{\partial v_3}{\partial x_3} d\mathbf{X} = 0 \quad (27)$$

which implies, using Lemma 1, that:

$$\begin{cases} \frac{\partial \mathbf{u}^1}{\partial x_3} + \text{grad } u_3^0 = 0 \\ (2 + \beta) \frac{\partial u_3^1}{\partial x_3} + \beta \text{div } \mathbf{u}^0 = 0 \end{cases} \quad (28)$$

Problem at order ε^2 .

For the problem $\mathcal{P}_2$, we obtain:

$$\begin{aligned} & \int_{\Omega} \left\{ \text{Tr} \left([\hat{\Sigma}^1 \hat{F}^1] + \Sigma^2 \right) \mathbf{G}^0(\mathbf{V}) \right\} + \text{Tr} \left(\hat{\Sigma}^1 \mathbf{G}^1(\mathbf{V}) \right) d\mathbf{X} - \\ & - \int_{\omega} \mathbf{p} \cdot \mathbf{v} d\mathbf{x} - \int_{\Gamma^b} \mathbf{t}_t \cdot \mathbf{v} dx_3 ds = 0 \quad \forall \mathbf{V} \in \mathbb{H} \end{aligned} \quad (29)$$

Using the expansion of the operators detailed in “Appendix”, the first term involving $\mathbf{G}^0(\mathbf{v})$ provides the relation:

$$\int_{\Omega} \left\{ \mathbf{A} \cdot \frac{\partial \mathbf{v}}{\partial x_3} + a \frac{\partial v_3}{\partial x_3} \right\} d\mathbf{X} = 0 \quad (30)$$

with

$$\begin{aligned} \mathbf{A} &= \frac{\partial \mathbf{u}^2}{\partial x_3} + \text{grad } u_3^1 + \left(\frac{\partial \mathbf{U}^0}{\partial x_1} \cdot \frac{\partial \mathbf{U}^1}{\partial x_3}, \frac{\partial \mathbf{U}^0}{\partial x_2} \cdot \frac{\partial \mathbf{U}^1}{\partial x_3} \right)^{\top} + \mathbf{n}_t \text{grad } u_3^0 \\ a &= (2 + \beta) \frac{\partial u_3^2}{\partial x_3} + \beta \text{div } \mathbf{u}^1 + \frac{(\beta + 2)}{2} \left\| \frac{\partial \mathbf{U}^1}{\partial x_3} \right\|^2 + \frac{\beta}{2} \left(\left\| \frac{\partial \mathbf{U}^0}{\partial x_1} \right\|^2 + \left\| \frac{\partial \mathbf{U}^0}{\partial x_2} \right\|^2 \right) \end{aligned} \quad (31)$$

Using Lemma 1, we obtain:

$$\mathbf{A} = 0 \quad \text{and} \quad a = 0 \quad (32)$$

Next, let us consider the remaining part of the functional (29). We have:

$$\int_{\Omega} \text{Tr} \left(\hat{\Sigma}^1 \mathbf{G}^1(\mathbf{V}) \right) d\mathbf{X} - \int_{\omega} \mathbf{p} \cdot \mathbf{v} d\mathbf{x} - \int_{\Gamma^b} \mathbf{t}_t \cdot \mathbf{v} dx_3 ds = 0 \quad \forall \mathbf{V} \in \mathbb{H} \quad (33)$$

Using the expansion of $\hat{\Sigma}^1$ and $\mathbf{G}^1(\mathbf{V})$ given in “Appendix”, it comes:

$$\int_{\Omega} \text{Tr} \left(\mathbf{n}_t \text{grad } \mathbf{v} \right) d\mathbf{X} - \int_{\omega} \mathbf{p} \cdot \mathbf{v} d\mathbf{x} - \int_{\Gamma^b} \mathbf{t}_t \cdot \mathbf{v} dx_3 ds = 0 \quad \forall \mathbf{v} \in \mathcal{V}^2. \quad (34)$$

If we consider $\mathbf{v} \in \mathcal{V}^2$ independent of x_3 , and we apply Stokes formula, (34) is equivalent to:

$$- \int_{\omega} \text{div } \mathbf{n}_t \mathbf{v} d\mathbf{X} + \int_{\gamma^b} \mathbf{n}_t \mathbf{v} \cdot \mathbf{v} ds - \int_{\omega} \mathbf{p} \cdot \mathbf{v} d\mathbf{x} - \int_{\gamma^b} \tilde{\mathbf{t}}_t \cdot \mathbf{v} ds = 0 \quad (35)$$

where

$$\tilde{\mathbf{t}}_t = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{t}_t dx_3 \quad (36)$$

indicates the average of $\mathbf{t}_t$ along the x_3 direction.

Finally, we obtain at this order the in-plane membrane equation of result 1 with the associated BC:

$$\begin{cases} \operatorname{div} \mathbf{n}_t = -\mathbf{p}_t & \text{in } \omega \\ \mathbf{n}_t(\mathbf{u}^0) \cdot \mathbf{v} = \tilde{\mathbf{t}}_t & \text{on } \gamma^b. \end{cases} \quad (37)$$

Problem at order ε^3 .

The computation for $\mathcal{P}_3$ returns:

$$\begin{aligned} & \int_{\Omega} \left\{ \operatorname{Tr} \left([\hat{\Sigma}^1 \hat{\mathbf{F}}^1 + \hat{\Sigma}^2] \mathbf{G}^1(V) \right) + \operatorname{Tr} \left([\hat{\Sigma}^2 \hat{\mathbf{F}}^1 + \hat{\Sigma}^1 \hat{\mathbf{F}}^2 + \hat{\Sigma}^3] \mathbf{G}^0(V) \right) \right\} d\mathbf{X} - \\ & - \int_{\omega} p_3 v_3 d\mathbf{x} = 0 \quad \forall \mathbf{V} \in \mathbb{H} \end{aligned} \quad (38)$$

Let us consider only test displacements independent of x_3 . According to (66), we have $\mathbf{G}^0(V) = 0$ and the remaining terms involved in (38) reduce to:

$$\begin{aligned} & \int_{\Omega} \left\{ \frac{\partial \mathbf{u}^2}{\partial x_3} + \operatorname{grad} u_3^1 + \left(\frac{\partial \mathbf{U}^0}{\partial x_1} \cdot \frac{\partial \mathbf{U}^1}{\partial x_3}, \frac{\partial \mathbf{U}^0}{\partial x_2} \cdot \frac{\partial \mathbf{U}^1}{\partial x_3} \right)^{\top} \right\} \cdot \operatorname{grad} v_3 d\mathbf{X} - \\ & - \int_{\omega} p_3 v_3 d\mathbf{x} = 0 \end{aligned} \quad (39)$$

This equation can be rewritten with the help of equation (32), using Stokes formula and the boundary conditions (8), as:

$$- \int_{\omega} \operatorname{div} (\mathbf{n}_t(\mathbf{u}^0) \operatorname{grad} u_3^0) v_3 d\mathbf{x} - \int_{\omega} p_3 v_3 d\mathbf{x} = 0. \quad (40)$$

which constitutes the bending equation of results 1. Equation (40) can also be written in an equivalent strong form:

$$\operatorname{div} (\mathbf{n}_t^0 \operatorname{grad} u_3^0) = -p_3 \quad \text{in } \omega \quad (41)$$

This completes the proof of Eq. (15).

4 Applications

In this section, we will focus on a particular set of in-plane stress tensors giving rise to analytical solution for the hyperbolic problem associated to the out-of-plane displacement of a membrane, in order to show the phenomenology that can be addressed in the framework of this model. The authors are aware that even simple generalization of the geometry or of the boundary conditions would give rise to the impossibility of finding analytical solutions.

First, before considering the physical examples treated in this section, let us come back to the dimensional variables of Result 1. The dimensional membrane problem reads:

$$\begin{cases} h \operatorname{div}^*(\mathbf{n}_t^*) = -\mathbf{p}_t^* & (42a) \\ h \operatorname{div}^*(\mathbf{n}_t^* \operatorname{grad}^* u_3^*) = -p_3^* & (42b) \end{cases}$$

with constitutive law:

$$\mathbf{n}_t^*(\mathbf{u}^*) = \frac{2\lambda}{2\mu + \lambda} \operatorname{div}^* \mathbf{u}^* \mathbf{I}_2 + \mu (\operatorname{grad}^* \mathbf{u}^* + \operatorname{grad}^* \mathbf{u}^{*\top}) \quad (43)$$

and associated boundary conditions given by:

$$\begin{cases} \mathbf{U}^* \Big|_{\gamma^{c*}} = 0 \\ \mathbf{n}_t^* \cdot \mathbf{v} \Big|_{\gamma^{b*}} = \tilde{\mathbf{t}}_t^* \end{cases} \quad (44)$$

In the following sections, we omit the $*$, but we consider the dimensional problem.

4.1 Constant plane stress

Let us consider the stretched-compressed membrane problem depicted in Fig. 2a, i.e., the boundary value problem for a squared membrane in its reference configuration occupying the region $\omega =]0, 1[\times]0, 1[$:

$$\begin{cases} \operatorname{div} \mathbf{n}_t = 0 & \text{in } \omega \\ u_1|_{x_1=0} = 0 \\ u_2|_{x_2=0} = 0 \\ \mathbf{n}_t \cdot \mathbf{v}|_{x_1=1} = t_t \\ \mathbf{n}_t \cdot \mathbf{v}|_{x_2=1} = -t_t \end{cases} \quad (45)$$

It is easy to verify that the in-plane stress tensor, solution of the membrane problem, is

$$\mathbf{n}_t = \begin{pmatrix} t_t & 0 \\ 0 & -t_t \end{pmatrix} \quad (46)$$

which is obviously hyperbolic. A straightforward integration gives for the displacements:

$$u_1 = \frac{2\mu + \lambda}{2\lambda} t_t x_1 \quad u_2 = -\frac{2\mu + \lambda}{2\lambda} t_t x_2$$

If we fix $t_t = 0.1 \frac{2\lambda}{2\mu + \lambda}$, we obtain the following equation for the out-of-plane displacement (no transverse load applied):

$$\begin{cases} \frac{\partial^2 u_3}{\partial x_1^2} - \frac{\partial^2 u_3}{\partial x_2^2} = 0 \end{cases} \quad (47)$$

To solve this hyperbolic problem, we chose the following boundary conditions to ensure its well-posedness³ (see Fig. 2b):

$$\begin{cases} u_3|_{x_2=0} = 0 \\ \frac{\partial u_3}{\partial x_2}|_{x_2=0} = \sin(20\pi x_1) \\ u_3|_{x_1=0} = 0 \\ u_3|_{x_1=1} = 0 \end{cases} \quad (48)$$

They correspond physically to a membrane simply posed on three sides, with an applied moment on one of these sides, and totally free on the last side.

The solution u_3 of problem (47)–(48), stands:

$$u_3(x_1, x_2) = \frac{\sin(20\pi x_1) \sin(20\pi x_2)}{20\pi}$$

and the actual deformed configuration is shown in Fig. 3. Of course, different type of boundary conditions could have been chosen, for instance $u_3 = 0$ on all sides of the square. This would have lead to an eigenvalue problem solved in Sect. 4.2 (Fig. 9).

In the last part of this section, we present some interesting results relative to a problem similar to the one addressed previously with different boundary conditions. Instead of (48), we use the piecewise defined function (see Fig. 4)

$$\frac{\partial u_3}{\partial x_2}|_{x_2=0} = \begin{cases} x & x \in [0, 0.2[\\ -x + 0.4 & x \in [0.2, 0.4[\\ 0 & x \in [0.4, 0.6[\\ x - 0.6 & x \in [0.6, 0.8[\\ -x + 1 & x \in [0.8, 1] \end{cases} \quad (49)$$

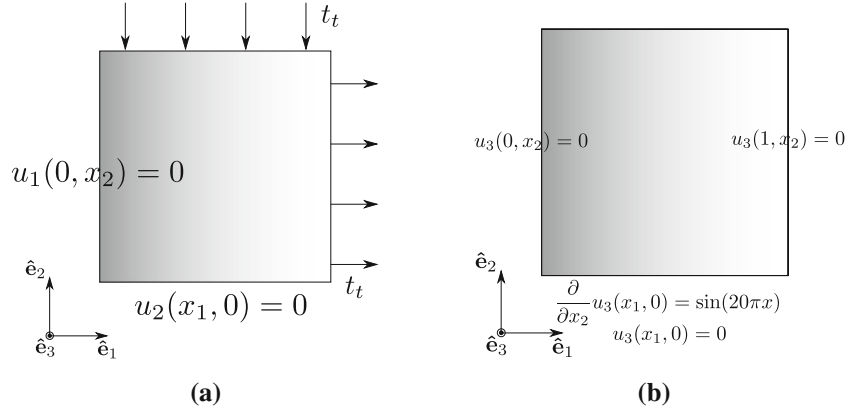


Fig. 2 Boundary conditions for the in-plane problem **a** and for the out-of-plane problem **b** for a squared membrane occupying the region $\omega =]0, 1[\times]0, 1[$

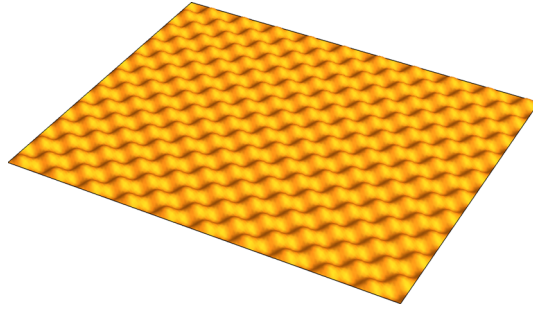


Fig. 3 Actual deformed configuration associated to the membrane problem (45) and (48)

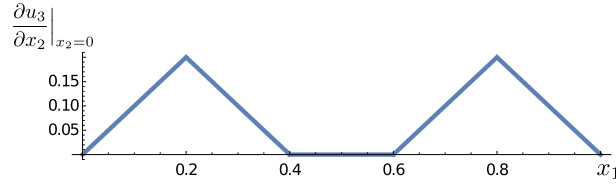


Fig. 4 Representation of the piecewise function relative to Eq. (49)

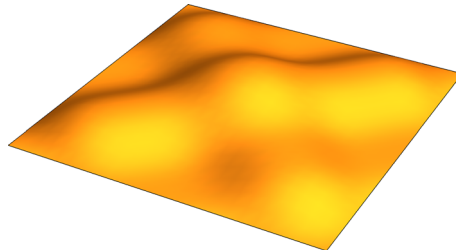


Fig. 5 Actual deformed configuration associated to the membrane problem with (49) as boundary conditions

The deformed actual configuration is shown in Fig. 5.

This example is interesting because it shows clearly the well-known property of propagating (in this case in space) a perturbation applied on the boundary for a hyperbolic PDE.

³ They correspond to classical Cauchy boundary conditions.

4.2 Airy functions and polynomial plane stress

4.2.1 Solution of membrane problem with Airy functions

In what follows, the idea is to develop a class of polynomial solutions to the plane equilibrium problem more general than the particular case of constant plane stress addressed in the previous examples.

In the absence of in-plane surface forces, i.e., $\mathbf{p}_t = 0$, the in-plane elasticity problem (42a) reduces to :

$$\operatorname{div}(\mathbf{n}_t) = 0. \quad (50)$$

It is easily verified that Eq. (50) can be identically satisfied, in a simply connected domain, with the introduction of a scalar function $\phi(x_1, x_2)$ (known as the Airy stress function), defined as follows:

$$n_{t11} = \partial_{22}^2 \phi, \quad n_{t22} = \partial_{11}^2 \phi, \quad n_{t12} = -\partial_{12}^2 \phi. \quad (51)$$

Moreover, the stresses have to satisfy well-known integrability conditions in order to give rise to physically meaningful in-plane displacements. Such relations, known as Beltrami–Mitchell equations, can be written for this problem as:

$$\Delta(n_{t11} + n_{t22}) = 0. \quad (52)$$

The whole in-plane elasticity problem is then reduced to the solution of the compatibility Eq. (52) that, in terms of ϕ , reads:

$$\Delta^2 \phi = 0 \quad (53)$$

In other words, ϕ is a biharmonic function. Next, let us consider the additional conditions on the solutions in order to obtain a hyperbolic problem for the out-of-plane displacement solution of (42b). The in-plane stress tensor can be written in the form

$$\mathbf{n}_t = \begin{pmatrix} \partial_{22}^2 \phi & -\partial_{12}^2 \phi \\ -\partial_{12}^2 \phi & \partial_{11}^2 \phi \end{pmatrix} \quad (54)$$

In order to have hyperbolicity for the out-of-plane displacement, the following condition must hold:

$$\frac{\partial^2 \phi}{\partial x_1^2} \frac{\partial^2 \phi}{\partial x_2^2} - \left(\frac{\partial^2 \phi}{\partial x_1 \partial x_2} \right)^2 < 0. \quad (55)$$

Intuitively this condition is linked with the membrane being subjected to compressive and tensile in-plane stresses along two independent directions. It will be studied in more detail in the next section.

4.2.2 Polynomial solution for ϕ and associated Sturm–Liouville problem for u_3

The goal of this section is to find a possible class of solutions that lead to a hyperbolic problem for the out-of-plane displacement. To do that, we will determine the boundary conditions giving rise to the membrane stresses (54) and satisfying (55), as for an inverse problem.

A class of solutions to the proposed problem can be expressed in form of polynomials. In particular, we will focus on fourth-order polynomials. According to the conditions formulated in the previous section, together with the stress tensor involving second- or higher-order derivatives, the terms lower than second degree will play no role in what follows. Therefore, we can consider a polynomial stress function of the form:

$$\phi = \sum_{n,m} A_{nm} x_1^n x_2^m \quad \text{with } 2 \leq n+m \leq 4 \quad n, m \in \mathbb{N} \quad (56)$$

Moreover, in this first approach, we will restrict this class of functions to the one that gives a diagonal stress tensor. This last requirement is easily verified if we take stress functions of the form:

$$\phi(x_1, x_2) = f(x_1) + g(x_2) \quad (57)$$

In particular, we will consider

$$\begin{aligned} f(x_1) &= \frac{K}{24} x_1^4 + \frac{A_1}{6} x_1^3 + \frac{A_2}{2} x_1^2 \\ g(x_2) &= -\frac{K}{24} x_2^4 + \frac{B_1}{6} x_2^3 + \frac{B_2}{2} x_2^2 \end{aligned} \quad (58)$$

where the coefficients of the fourth-order terms are chosen in order to automatically satisfy equation (53). Condition (55), that ensures hyperbolicity of Eq. (42b), then becomes:

$$f''(x_1)g''(x_2) = \left(\frac{K}{2}x_1^2 + A_1x_1 + A_2\right)\left(-\frac{K}{2}x_2^2 + B_1x_2 + B_2\right) < 0. \quad (59)$$

A sufficient condition to verify (59) is that none of the two parabolas has real zeroes (being the coefficient of the quadratic term of opposite sign)⁴ It is worth noting that in studying pantographic structures with inextensible fibers, hyperbolic equations are derived as a consequence of the inextensibility constraint. That leads to a kinematic constraint exactly equivalent to assumption (57) [28,29], which leads in the homogenized limit to second gradient continua [30–35]. On the basis of what exposed above, Eq. (42b) for the out-of-plane displacement reduces to:

$$g''(x_2)\frac{\partial^2 u_3}{\partial x_1^2} + f''(x_1)\frac{\partial^2 u_3}{\partial x_2^2} = 0 \quad (60)$$

and the boundary conditions satisfied by the stresses are recovered from (54).

For completeness, we present some solution to equation (60) that can be easily integrated via separation of variable technique. In fact assuming for $u_3(x_1, x_2)$ the form

$$u_3(x_1, x_2) = u_{31}(x_1)u_{32}(x_2),$$

and substituting in Eq. (60), we obtain

$$\frac{u_{31}''(x_1)}{f''(x_1)u_{31}(x_1)} = -\frac{u_{32}''(x_2)}{g''(x_2)u_{32}(x_2)} = -C,$$

where C is any constant. If we consider a simply supported square membrane occupying the region $\omega =]-1, 1[\times]-1, 1[$, the two problems can be recast as follows:

$$\begin{cases} u_{31}'' = -C(\frac{K}{2}x_1^2 + A_1x_1 + A_2)u_{31}(x_1) \\ u_{31}(-1) = u_{31}(1) = 0 \end{cases} \quad (61)$$

and

$$\begin{cases} u_{32}'' = -C(\frac{K}{2}x_2^2 - B_1x_2 - B_2)u_{32}(x_2) \\ u_{32}(-1) = u_{32}(1) = 0 \end{cases} \quad (62)$$

Each of the two problems represents a regular Sturm–Liouville (S–L) eigenvalue problem (recalling that (59) ensures that the weight functions do not vanish in the interval of definition of the problem). To solve the problem, let us fix the parameters in order to satisfy (59), i.e., we choose

$$K = 2, \quad A_1 = B_1 = 0, \quad A_2 = -B_2 = 3/4$$

With this choice of the parameters, we obtain two identical one dimensional S–L problems (we do not need here to distinguish between x_1 and x_2):

$$\begin{cases} u_3'' = -C(x^2 + 3/4)u_3(x) \\ u_3(-1) = u_3(1) = 0 \end{cases} \quad (63)$$

On the basis of the theory underlying S–L problems, we have solved the problem numerically, by a shooting technique.⁵ The first nine eigenvectors are shown in Fig. 6. In Figs. 7 and 8, we show some possible shapes for the out of plane displacement.

To finish, in Fig. 9, we show a qualitative comparison between the results of the model and the behavior of a stretched rectangular membrane. In the numerical simulation (Fig. 9), the in-plane stress is constant (as in the case treated of Sect. 4.1), but the boundary conditions are that of a simple posed membrane on all the sides, i.e., $u_3 = 0$. This leads to a S–L problem similar to the one treated in this section. Comparing this numerical result with some experimental result presented in [4] (in particular Fig. 1, page 2), we observe an analogous behavior of wrinkling, well depicted by our simple membrane model.

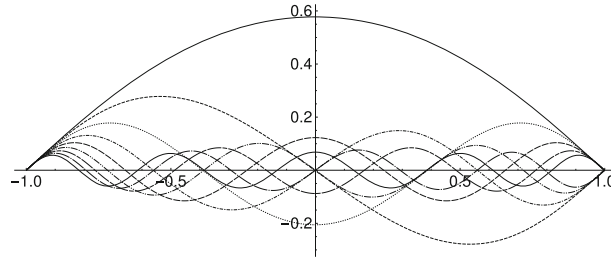


Fig. 6 First nine eigenvalues relative to the Sturm–Liouville problem (63)

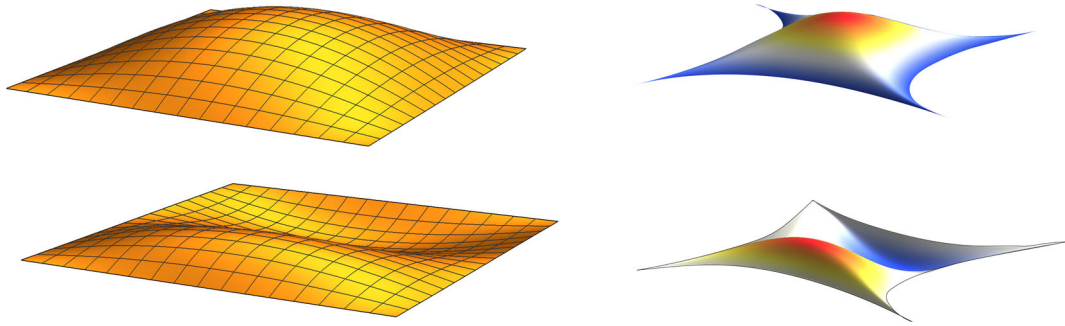


Fig. 7 First two deformation modes, solution of the problems (61) and (62). On the left: the solutions of the out-of-plane problem. On the right: the whole deformation of the membrane

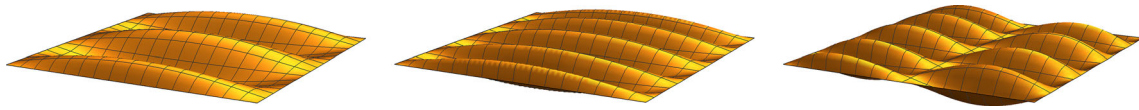


Fig. 8 Possible modes for the out-of-plane displacement for a square membrane associated to the eigenvalues numerically determined (63). From the left to the right: mode (1, 3), mode (1, 5) and mode (2, 4)

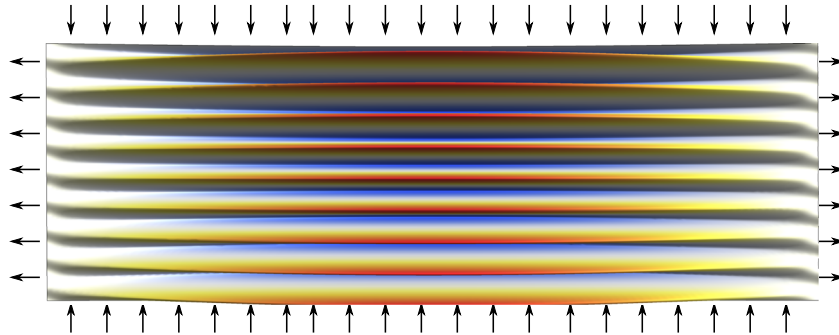


Fig. 9 Qualitative comparison between the numerical results presented above and the typical shape of a stretched membrane (see [4], Fig. 1)

We close this section about the numerical results presenting a qualitative comparison between the membrane 2D model deduced in the paper (Fig. 10-left) and a finite element simulation of the buckling of a very thin simply supported 3D plate subject to a traction compression load on the opposites sides (Fig. 10-right). The 3D plate is modeled as an Hookean material, and the numerical simulation is performed with second-order Lagrangian tetrahedral elements, moreover a linear buckling analysis is performed, and the first normalized bifurcation mode is shown in Fig. 10-right. Of course, as stated in the introduction, a quantitative study implies a systematic treatment of the boundary layers, a study that we deserve for future works.

⁴ Of course, this is not a necessary condition.

⁵ The authors are aware that such a simple problem admits even analytical solutions. But we found the numerical solution more convenient to use in this context.

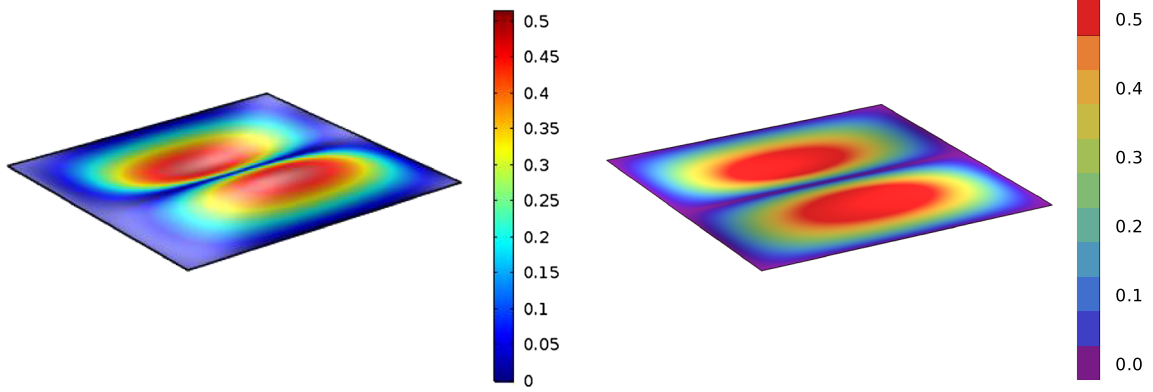


Fig. 10 Comparison between the membrane model (left) and a 3D finite element computation (right), in colors the absolute value of the out-of-plane displacement (in meters, for a square with a side of 2m) (color figure online)

5 Conclusion

We presented in the paper, a formal deduction of a two-dimensional membrane theory performed via an asymptotic development of the weak formulation of the three-dimensional equations of elasticity. This membrane model is similar to that of Landau and Lifshitz established in [3] from a priori simplifications, whose domain of validity is not clearly specified. In the approach presented here, the membrane model is deduced rigorously from asymptotic expansion, with certain order of magnitude of the applied forces, and its domain of validity can be more clearly specified. Moreover, some interesting aspects of the deduced model are investigated, in particular the property of obtaining a hyperbolic equation for the out-of-plane displacement under a certain class of boundary conditions and loads. Some simple cases are analyzed to show the relevant aspects of the model and the phenomenology that can be addressed. In particular it is shown how this mathematical formulation is capable to describe instabilities well known as wrinkling, often observed for the buckling of very thin membranes.

Future investigations will focus, from the theoretical point of view, on the deep connection between the nonuniqueness of the solution for the out-of-plane displacement and the bifurcation related to buckling phenomena. From an applicative point of view, more general situations will be investigated, in order to provide an efficient and ready tool to investigate the class of phenomena linked to instabilities in thin structures.

Appendix: Explicit form of the operators

We fix a coordinate system in $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) . Let us consider the three-dimensional vector $\mathbf{W}$, and let us decompose it in the following way:

$$\mathbf{W}^* = w_\alpha^* \hat{\mathbf{e}}_\alpha + w_3^* \hat{\mathbf{e}}_3 = \mathbf{w}^* + w_3^* \hat{\mathbf{e}}_3 \quad (64)$$

Using the reference variables $w_r = h$ and $w_{3r} = h$ for $\mathbf{w}^*$ and w_3^* , the dimensional analysis of the gradient of $\mathbf{W}^*$ may be written in the explicit form:

$$\text{Grad}^* \mathbf{W}^* = \text{Grad} \mathbf{W} =: \quad (65)$$

$$\mathbf{G}^\varepsilon(\mathbf{W}) = \mathbf{G}^0(\mathbf{W}) + \varepsilon \mathbf{G}^1(\mathbf{W}) = \begin{pmatrix} \varepsilon \text{grad} \mathbf{w} & \frac{\partial \mathbf{w}}{\partial x_3} \\ \varepsilon (\text{grad} w_3)^\top & \frac{\partial w_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \frac{\partial \mathbf{w}}{\partial x_3} \\ \mathbf{0} & \frac{\partial w_3}{\partial x_3} \end{pmatrix} + \varepsilon \begin{pmatrix} \text{grad} \mathbf{w} & \mathbf{0} \\ (\text{grad} w_3)^\top & 0 \end{pmatrix} \quad (66)$$

where we have used the definition of the two-dimensional gradient:

$$\text{grad} \mathbf{w} = \begin{pmatrix} \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial x_2} \\ \frac{\partial w_2}{\partial x_1} & \frac{\partial w_2}{\partial x_2} \end{pmatrix} \quad \text{and} \quad \text{grad} w_3 = \begin{pmatrix} \frac{\partial w_3}{\partial x_1} \\ \frac{\partial w_3}{\partial x_2} \end{pmatrix} \quad (67)$$

In what follows, we give the dimensional form of the quantities that are useful in the calculations.

Gradient of a map

Given a map $\Psi^* : \Omega(\subset \mathbb{R}^3) \rightarrow \mathbb{R}^3$ such that $\Psi^*(X^*) = X^* + U^*(X^*)$, we define its gradient:

$$F^*(W^*) =: F^\varepsilon(W) = \quad (68)$$

$$I_3 + G(W) = F^0(W) + \varepsilon F^1(W) = \begin{pmatrix} I_2 & \frac{\partial \mathbf{w}}{\partial x_3} \\ \mathbf{0} & 1 + \frac{\partial w_3}{\partial x_3} \end{pmatrix} + \varepsilon \begin{pmatrix} \text{grad } \mathbf{w} & \mathbf{0} \\ (\text{grad } w_3)^\top & 0 \end{pmatrix} \quad (69)$$

Nonlinear strain tensor

The nonlinear strain tensor is defined with respect to a vector $\mathbf{W}$ in the following way:

$$E^*(W^*) = E(W) = \frac{1}{2}[G(W) + G^\top(W)] + \frac{1}{2}[G(W)^\top G(W)]$$

To simplify the notation, we introduce the two operators:

$$\mathbf{e}^*(W^*) = \mathbf{e}(W) = \frac{1}{2}[G(W) + G^\top(W)] \quad \text{and} \quad \xi^*(W^*) = \xi(W) = \frac{1}{2}[G(W)^\top G(W)] \quad (70)$$

We have in particular for the linear part of the strain tensor:

$$\mathbf{e}^\varepsilon(W) = \mathbf{e}^0(W) + \varepsilon \mathbf{e}^1(W) = \frac{1}{2} \begin{pmatrix} \mathbf{0} & \frac{\partial \mathbf{w}}{\partial x_3} \\ \frac{\partial \mathbf{w}}{\partial x_3}^\top & 2 \frac{\partial w_3}{\partial x_3} \end{pmatrix} + \frac{\varepsilon}{2} \begin{pmatrix} \text{grad } \mathbf{w} + (\text{grad } \mathbf{w})^\top \text{grad } w_3 \\ (\text{grad } w_3)^\top & 0 \end{pmatrix} \quad (71)$$

while, for the nonlinear part

$$\xi^\varepsilon(W) = \xi^0(W) + \varepsilon \xi^1(W) + \varepsilon^2 \xi^2(W) \quad (72)$$

with:

$$\xi^0(W) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2} \left\| \frac{\partial \mathbf{W}}{\partial x_3} \right\|^2 \end{pmatrix} \quad (73)$$

$$\xi^1(W) = \begin{pmatrix} \mathbf{0} & \frac{1}{2} \begin{pmatrix} \frac{\partial \mathbf{W}}{\partial x_1} \cdot \frac{\partial \mathbf{W}}{\partial x_3} \\ \frac{\partial \mathbf{W}}{\partial x_2} \cdot \frac{\partial \mathbf{W}}{\partial x_3} \end{pmatrix} \\ \frac{1}{2} \begin{pmatrix} \frac{\partial \mathbf{W}}{\partial x_1} \cdot \frac{\partial \mathbf{W}}{\partial x_3} \\ \frac{\partial \mathbf{W}}{\partial x_2} \cdot \frac{\partial \mathbf{W}}{\partial x_3} \end{pmatrix}^\top & 0 \end{pmatrix} \quad (74)$$

$$\xi^2(W) = \frac{1}{2} \begin{pmatrix} \left\| \frac{\partial \mathbf{W}}{\partial x_1} \right\|^2 & \frac{\partial \mathbf{W}}{\partial x_1} \cdot \frac{\partial \mathbf{W}}{\partial x_2} & 0 \\ \frac{\partial \mathbf{W}}{\partial x_1} \cdot \frac{\partial \mathbf{W}}{\partial x_2} & \left\| \frac{\partial \mathbf{W}}{\partial x_2} \right\|^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (75)$$

For further development, we notice that each of the three operators defined above can be regarded as bilinear forms of the vectors $\mathbf{T}$, $\mathbf{Z}$, then we set:

$$\xi^0 = \varphi^0(\mathbf{T}, \mathbf{Z}), \quad \xi^1 = \varphi^1(\mathbf{T}, \mathbf{Z}) \quad \text{and} \quad \xi^2 = \varphi^2(\mathbf{T}, \mathbf{Z}). \quad (76)$$

more over we notice that ξ^0 is symmetric, while ξ^1 and ξ^2 are not.

Stress tensor

We define the stress tensor as:

$$\Sigma^*(E^*) = L_{\lambda,\mu} \mathbf{e}(W) + L_{\lambda,\mu} \xi(W) \quad (77)$$

where the linear operator $L_{\lambda,\mu}$ acts on the generic tensor $\mathbf{P}$ in the following way (here λ and μ represents the Lamé constants):

$$L_{\lambda,\mu} \mathbf{P}^* = \lambda \text{Tr}(\mathbf{P}^*) I_3 + 2\mu \mathbf{P}^* \quad (78)$$

Moreover, we define in the following the linear and nonlinear part of the stress tensor:

$$\Sigma_l^*(W^*) = L_{\lambda,\mu} \mathbf{e}(W) \quad \text{and} \quad \Sigma_{nl}^*(W^*) = L_{\lambda,\mu} \xi(W) \quad (79)$$

If we fix the parameter μ as a reference stress, we can write:

$$\boldsymbol{\Sigma}(\mathbf{W}) = \frac{\boldsymbol{\Sigma}^*(\mathbf{W}^*)}{\mu} \quad (80)$$

Defining now $\beta = \frac{\lambda}{\mu}$, we can rewrite the preceding relations substituting:

$$\boldsymbol{\Sigma}^*(\mathbf{W}^*) \rightarrow \boldsymbol{\Sigma}(\mathbf{W}) \quad L_{\lambda,\mu} \mathbf{P} \rightarrow L_\beta \mathbf{P} = \beta \operatorname{Tr}(\mathbf{P}) \mathbf{I}_3 + 2\mathbf{P} \quad (81)$$

To finish, we give the explicit form for the operator introduced above, i.e.:

$$\boldsymbol{\Sigma}_l^\varepsilon(\mathbf{W}) = \boldsymbol{\Sigma}_l^0(\mathbf{W}) + \varepsilon \boldsymbol{\Sigma}_l^1(\mathbf{W}) = \quad (82)$$

$$\left(\beta \frac{\partial w_3}{\partial x_3} \mathbf{I}_2 \quad \frac{\partial \mathbf{w}}{\partial x_3} \right) + \varepsilon \begin{pmatrix} \beta \operatorname{div} \mathbf{w} \mathbf{I}_2 + \operatorname{grad} \mathbf{w} + (\operatorname{grad} \mathbf{w})^\top & \operatorname{grad} w_3 \\ (\frac{\partial \mathbf{w}}{\partial x_3})^\top & (\beta + 2) \frac{\partial w_3}{\partial x_3} \end{pmatrix} + \varepsilon \begin{pmatrix} \beta \operatorname{div} \mathbf{w} \mathbf{I}_2 + \operatorname{grad} \mathbf{w} + (\operatorname{grad} \mathbf{w})^\top & \operatorname{grad} w_3 \\ (\operatorname{grad} w_3)^\top & \beta \operatorname{div} \mathbf{w} \end{pmatrix} \quad (83)$$

and

$$\boldsymbol{\Sigma}_{nl}^\varepsilon(\mathbf{W}) = \boldsymbol{\Sigma}_{nl}^0(\mathbf{W}) + \varepsilon \boldsymbol{\Sigma}_{nl}^1(\mathbf{W}) + \varepsilon^2 \boldsymbol{\Sigma}_{nl}^2(\mathbf{W}) \quad (84)$$

with:

$$\boldsymbol{\Sigma}_{nl}^0(\mathbf{W}) = \begin{pmatrix} \frac{\beta}{2} \left\| \frac{\partial \mathbf{W}}{\partial x_3} \right\|^2 \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \frac{(\beta+2)}{2} \left\| \frac{\partial \mathbf{W}}{\partial x_3} \right\|^2 \end{pmatrix} \quad (85)$$

$$\boldsymbol{\Sigma}_{nl}^1(\mathbf{W}) = \begin{pmatrix} \mathbf{0} & \begin{pmatrix} \frac{\partial \mathbf{W}}{\partial x_1} \cdot \frac{\partial \mathbf{W}}{\partial x_3} \\ \frac{\partial \mathbf{W}}{\partial x_2} \cdot \frac{\partial \mathbf{W}}{\partial x_3} \end{pmatrix} \\ \left(\frac{\partial \mathbf{W}}{\partial x_1} \cdot \frac{\partial \mathbf{W}}{\partial x_3} \right)^\top & 0 \end{pmatrix} \quad (86)$$

$$\begin{aligned} \boldsymbol{\Sigma}_{nl}^2(\mathbf{W}) &= \begin{pmatrix} \frac{(\beta+2)}{2} \left\| \frac{\partial \mathbf{W}}{\partial x_1} \right\|^2 + \beta \left\| \frac{\partial \mathbf{W}}{\partial x_2} \right\|^2 & \frac{\partial \mathbf{W}}{\partial x_1} \cdot \frac{\partial \mathbf{W}}{\partial x_2} & 0 \\ \frac{\partial \mathbf{W}}{\partial x_1} \cdot \frac{\partial \mathbf{W}}{\partial x_2} & \frac{(\beta+2)}{2} \left\| \frac{\partial \mathbf{W}}{\partial x_2} \right\|^2 + \beta \left\| \frac{\partial \mathbf{W}}{\partial x_1} \right\|^2 & 0 \\ 0 & 0 & \frac{\beta}{2} \left(\left\| \frac{\partial \mathbf{W}}{\partial x_1} \right\|^2 + \left\| \frac{\partial \mathbf{W}}{\partial x_2} \right\|^2 \right) \end{pmatrix} \\ & \quad (87) \end{aligned}$$

For further development, we notice that each of the three operators defined above can be regarded as bilinear forms of the vectors $\mathbf{T}$ and $\mathbf{Z}$, then we set:

$$\boldsymbol{\Sigma}_{nl}^0 = \Phi^0(\mathbf{T}, \mathbf{Z}), \quad \boldsymbol{\Sigma}_{nl}^1 = \Phi^1(\mathbf{T}, \mathbf{Z}) \quad \text{and} \quad \boldsymbol{\Sigma}_{nl}^2 = \Phi^2(\mathbf{T}, \mathbf{Z}). \quad (88)$$

Moreover, we notice that $\boldsymbol{\Sigma}^0$ is symmetric, while $\boldsymbol{\Sigma}^1$ and $\boldsymbol{\Sigma}^2$ are not.

ε -series development

If we suppose that the vector $\mathbf{W}$ can be developed in a ε -series as

$$\mathbf{W}^\varepsilon = \mathbf{W}^0 + \varepsilon \mathbf{W}^1 + \varepsilon^2 \mathbf{W}^2 + \dots$$

then all the operators introduced above admit a similar development. We will adopt the following notations:

$$- \mathbf{F}^\varepsilon(\mathbf{W}^\varepsilon) = \hat{\mathbf{F}}^0 + \varepsilon \hat{\mathbf{F}}^1 + \varepsilon^2 \hat{\mathbf{F}}^2 + \dots$$

with:

$$\hat{\mathbf{F}}^0 = \mathbf{F}^0(\mathbf{W}^0)$$

$$\hat{\mathbf{F}}^1 = \mathbf{F}^0(\mathbf{W}^1) + \mathbf{F}^1(\mathbf{W}^0)$$

$$\hat{\mathbf{F}}^2 = \mathbf{F}^0(\mathbf{W}^2) + \mathbf{F}^1(\mathbf{W}^1)$$

$$- \mathbf{e}^\varepsilon(\mathbf{W}^\varepsilon) = \hat{\mathbf{e}}^0 + \varepsilon \hat{\mathbf{e}}^1 + \varepsilon^2 \hat{\mathbf{e}}^2 + \dots$$

with:

$$\hat{\mathbf{e}}^0 = \mathbf{e}^0(\mathbf{W}^0)$$

$$\hat{\mathbf{e}}^1 = \mathbf{e}^0(\mathbf{W}^1) + \mathbf{e}^1(\mathbf{W}^0)$$

$$\hat{\mathbf{e}}^2 = \mathbf{e}^0(\mathbf{W}^2) + \mathbf{e}^1(\mathbf{W}^1)$$

$$- \boldsymbol{\xi}(\mathbf{W}^\varepsilon) = \hat{\boldsymbol{\xi}}^0 + \varepsilon \hat{\boldsymbol{\xi}}^1 + \varepsilon^2 \hat{\boldsymbol{\xi}}^2 + \dots$$

with:

$$\hat{\boldsymbol{\xi}}^0 = \boldsymbol{\xi}^0(\mathbf{W}^0, \mathbf{W}^0)$$

$$\hat{\boldsymbol{\xi}}^1 = 2\boldsymbol{\xi}^0(\mathbf{W}^0, \mathbf{W}^1) + \boldsymbol{\xi}^1(\mathbf{W}^0, \mathbf{W}^0)$$

$$\hat{\boldsymbol{\xi}}^2 = \boldsymbol{\xi}^0(\mathbf{W}^1, \mathbf{W}^1) + 2\boldsymbol{\xi}^0(\mathbf{W}^0, \mathbf{W}^2) + \boldsymbol{\xi}^1(\mathbf{W}^0, \mathbf{W}^1) + \boldsymbol{\xi}^1(\mathbf{W}^1, \mathbf{W}^0) + \boldsymbol{\xi}^2(\mathbf{W}^0, \mathbf{W}^0)$$

$$- \boldsymbol{\Sigma}_l^\varepsilon(\mathbf{W}^\varepsilon) = \hat{\boldsymbol{\Sigma}}_l^0 + \varepsilon \hat{\boldsymbol{\Sigma}}_l^1 + \varepsilon^2 \hat{\boldsymbol{\Sigma}}_l^2 + \dots$$

with:

$$\hat{\boldsymbol{\Sigma}}_l^0 = \boldsymbol{\Sigma}_l^0(\mathbf{W}^0)$$

$$\hat{\boldsymbol{\Sigma}}_l^1 = \boldsymbol{\Sigma}_l^0(\mathbf{W}^1) + \boldsymbol{\Sigma}_l^1(\mathbf{W}^0)$$

$$\hat{\boldsymbol{\Sigma}}_l^2 = \boldsymbol{\Sigma}_l^0(\mathbf{W}^2) + \boldsymbol{\Sigma}_l^1(\mathbf{W}^1)$$

$$- \boldsymbol{\Sigma}_{nl}^\varepsilon(\mathbf{W}^\varepsilon) = \hat{\boldsymbol{\Sigma}}_{nl}^0 + \varepsilon \hat{\boldsymbol{\Sigma}}_{nl}^1 + \varepsilon^2 \hat{\boldsymbol{\Sigma}}_{nl}^2 + \dots$$

with:

$$\hat{\boldsymbol{\Sigma}}_{nl}^0 = \boldsymbol{\Sigma}_{nl}^0(\mathbf{W}^0, \mathbf{W}^0)$$

$$\hat{\boldsymbol{\Sigma}}_{nl}^1 = 2\boldsymbol{\Sigma}_{nl}^0(\mathbf{W}^0, \mathbf{W}^1) + \boldsymbol{\Sigma}_{nl}^1(\mathbf{W}^0, \mathbf{W}^0)$$

$$\hat{\boldsymbol{\Sigma}}_{nl}^2 = \boldsymbol{\Sigma}_{nl}^0(\mathbf{W}^1, \mathbf{W}^1) + 2\boldsymbol{\Sigma}_{nl}^0(\mathbf{W}^0, \mathbf{W}^2) + \boldsymbol{\Sigma}_{nl}^1(\mathbf{W}^0, \mathbf{W}^1) + \boldsymbol{\Sigma}_{nl}^1(\mathbf{W}^1, \mathbf{W}^0) + \boldsymbol{\Sigma}_{nl}^2(\mathbf{W}^0, \mathbf{W}^0)$$

References

1. Friedrichs, K.O., Dressler, R.F.: A boundary-layer theory for elastic plates. *Commun. Pure Appl. Math.* **14**(1), 1–33 (1961)
2. Ciarlet, P.G., Destuynder, P.: Une justification du modèle bi-harmonique en théorie des plaques. *C. R. Acad. Sci. Paris* **285**, 851–854 (1977)
3. Landau, L.D., Lifshitz, E.M.: *Theory of Elasticity*, 3rd edn. Pergamon Press, Oxford, UK (1986)
4. Vandeparre, H., Piñeirua, M., Brau, F., Roman, B., Bico, J., Gay, C., Bao, W., Lau, C.N., Reis, P.M., Damman, P.: Wrinkling hierarchy in constrained thin sheets from suspended graphene to curtains. *Phys. Rev. Lett.* **106**(22), 224301 (2011)
5. Hure, J., Roman, B., Bico, J.: Stamping and wrinkling of elastic plates. *Phys. Rev. Lett.* **109**(5), 054302 (2012)
6. Takei, A., Brau, F., Roman, B., Bico, J.: Stretch-induced wrinkles in reinforced membranes: From out-of-plane to in-plane structures. *EPL (Europhys. Lett.)* **96**(6), 64001 (2011)
7. Altenbach, H., Eremeyev, V.A.: On the shell theory on the nanoscale with surface stresses. *Int. J. Eng. Sci.* **49**(12), 1294–1301 (2011)
8. Altenbach, H., Eremeyev, V.A., Morozov, N.F.: Linear theory of shells taking into account surface stresses. In *Doklady Physics* **54**(12), 531 (2009). (SP MAIK Nauka/Interperiodica)
9. Autieri, C.: Antiferromagnetic and xy ferro-orbital order in insulating SrRuO₃ thin films with SrO termination. *J. Phys. Condens. Matter* **28**(42), 426004 (2016)
10. Giorgio, I., Corte, A.Della, dell'Isola, F., Steigmann, D.J., Steigmann, D.J.: Buckling modes in pantographic lattices. *C. R. Mecanique* **344**(7), 487–501 (2016)
11. Giorgio, I., Grygoruk, R., dell'Isola, F., Steigmann, D.J.: Pattern formation in the three-dimensional deformations of fibered sheets. *Mech. Res. Commun.* **69**, 164–171 (2015)
12. Alibert, J.J., Seppecher, P., dell'Isola, F.: Truss modular beams with deformation energy depending on higher displacement gradients. *Math. Mech. Solids* **8**(1), 51–73 (2003)
13. Seppecher, P., Alibert, J.J., dell'Isola, F.: Linear elastic trusses leading to continua with exotic mechanical interactions. *J. Phys. Conf. Ser.* **319**, 012018 (2011)
14. Madeo, A., Ferretti, M., dell'Isola, F., Boisse, P.: Thick fibrous composite reinforcements behave as special second-gradient materials: three-point bending of 3D interlocks. *Zeitschrift für angewandte Mathematik und Physik* **66**(4), 2041–2060 (2015)
15. Millet, O., Hamdouni, A., Cimetière, A., Elamri, K.: Analyse dimensionnelle de l'équation de Navier et application à la théorie des plaques minces. *Journal de Physique III* **7**(10), 1909–1925 (1997)
16. Millet, O., Hamdouni, A., Cimetière, A.: Dimensional analysis and asymptotic expansions of equilibrium equations in nonlinear elasticity. Part I: the membrane model. *Arch. Mech.* **50**(6), 953–973 (1998)

17. Millet, O., Hamdouni, A., Cimetière, A.: Dimensional analysis and asymptotic expansions of equilibrium equations in nonlinear elasticity. Part II: the two-dimensional von karman model. *Arch. Mech.* **50**(6), 975–1001 (1998)
18. Millet, O., Hamdouni, A., Cimetière, A.: Construction d'un modèle eulérien de plaques en grands déplacements par méthode asymptotique. *Comptes Rendus de l'Académie des Sciences-Series IIB-Mechanics-Physics-Chemistry-Astronomy* **325**(5), 257–261 (1997)
19. Millet, O., Hamdouni, A., Cimetière, A.: A classification of thin plate models by asymptotic expansion of non-linear three-dimensional equilibrium equations. *Int. J. Non-Linear Mech.* **36**(1), 165–186 (2001)
20. Hamdouni, A., Millet, O.: Classification of thin shell models deduced from the nonlinear three-dimensional elasticity. Part I: the shallow shells. *Arch. Mech.* **55**(2), 135–176 (2003)
21. Hamdouni, A., Millet, O.: Classification of thin shell models deduced from the nonlinear three-dimensional elasticity. Part II: the strongly curved shells. *Arch. Mech.* **55**(2), 177–220 (2003)
22. Audoly, B., Roman, B., Pocheau, A.: Secondary buckling patterns of a thin plate under in-plane compression. *Eur. Phys. J. B Condens. Matter Complex Syst.* **27**(1), 7–10 (2002)
23. Audoly, B.: Stability of straight delamination blisters. *Phys. Rev. Lett.* **83**(20), 4124 (1999)
24. Hutchinson, J.W., He, M.Y., Evans, A.G.: The influence of imperfections on the nucleation and propagation of buckling driven delaminations. *J. Mech. Phys. Solids* **48**(4), 709–734 (2000)
25. Audoly, B., Boudaoud, A.: Buckling of a stiff film bound to a compliant substrate-Part I: formulation, linear stability of cylindrical patterns, secondary bifurcations. *J. Mech. Phys. Solids* **56**(7), 2401–2421 (2008)
26. Audoly, B., Boudaoud, A.: Buckling of a stiff film bound to a compliant substrate-Part II: a global scenario for the formation of herringbone pattern. *J. Mech. Phys. Solids* **56**(7), 2422–2443 (2008)
27. Audoly, B., Boudaoud, A.: Buckling of a stiff film bound to a compliant substrate-Part III: herringbone solutions at large buckling parameter. *J. Mech. Phys. Solids* **56**(7), 2444–2458 (2008)
28. Rivlin, R.S.: Plane strain of a net formed by inextensible cords. In: *Collected Papers of RS Rivlin*, pp. 511–534. Springer, New York (1997)
29. dell'Isola, F., Della Corte, A., Greco, L., Luongo, A.: Plane bias extension test for a continuum with two inextensible families of fibers: a variational treatment with lagrange multipliers and a perturbation solution. *Int. J. Solids Struct.* **81**, 1–12 (2016)
30. Placidi, L., Greco, L., Bucci, S., Turco, E., Rizzi, N.L.: A second gradient formulation for a 2D fabric sheet with inextensible fibres. *Zeitschrift für angewandte Mathematik und Physik* **67**(5), 114 (2016)
31. Placidi, L., Andreaus, U., Della Corte, A., Lekszycki, T.: Gedanken experiments for the determination of two-dimensional linear second gradient elasticity coefficients. *Zeitschrift für angewandte Mathematik und Physik* **66**(6), 3699–3725 (2015)
32. Placidi, L., Andreaus, U., Giorgio, I.: Identification of two-dimensional pantographic structure via a linear D4 orthotropic second gradient elastic model. *J. Eng. Math.* **103**, 1–21 (2016)
33. Alibert, J.J., Della Corte, A.: Second-gradient continua as homogenized limit of pantographic microstructured plates: a rigorous proof. *Zeitschrift für angewandte Mathematik und Physik* **66**(5), 2855–2870 (2015)
34. Turco, E., dell'Isola, F., Cazzani, A., Rizzi, N.L.: Hencky-type discrete model for pantographic structures: numerical comparison with second gradient continuum models. *Zeitschrift für Angewandte Mathematik und Physik* **67**(4), 1–28 (2016)
35. Turco, E.: Discrete is it enough? The revival of Piola–Hencky keynotes to analyze three-dimensional *Elastica*. *Contin. Mech. Thermodyn.* pp. 1–19 (2018). <https://doi.org/10.1007/s00161-018-0656-4>

Chapter 6

Conclusion

In this thesis we have presented a collection of work which represent the main results obtained in this three year PhD. The research activities have mainly focused on the study of beam theories in large deformation under distributed load. The main results concern a generalization of Timoshenko beam theory.

The importance of geometrical nonlinearities is increasingly important in modern structural mechanics and in general in elasticity theory. On the other hand, beam theory, and especially generalized beam theory, is particularly interesting nowadays in view of applications to lattice systems, and in this context pantographic structures are naturally leading to the problem of large deformations of the fibers. Within this framework, we faced the problem of geometrically nonlinear deformation of generalized Timoshenko beam models, obtained by means of a formal homogenization starting from a microstructured 1D system. We considered both a straightforward generalization of the customary linearized model and the model obtained introducing in the microstructure an additional rotational spring. We proved well-posedness of the variational problem concerning a clamped beam with generic end load, as well as some properties of the minimizers, and presented and discussed some numerical simulations. The main open problem connected with what presented here, is the generalization of the results to the case of a distributed load. However, this generalization is not trivial as it leads to a non-autonomous variational problem.

Moreover, we numerically studied clamped-free Euler and Timoshenko beams in large deformation under distributed load. We also took into account extensibility and results on the static behavior of the beam under different values of the load and of the axial stiffness has been shown. The main interest of the results consists in the multiplicity of solutions that arise as the load increases, not as a bifurcation of existing branches of solutions but as new branches that arise when the load overcomes a series of progressively larger threshold-values. Future investigations are required to establish whether these multiple solutions can be stable. In this regard, we also addressed the analysis of the small oscillations of the beam around candidate stable equilibria. In the paper presented in chapter 3, the local minimum configurations for highly flexible beams predicted in [12] and characterized by curled shapes are shown experimentally and predicted numerically. Besides, using a Hencky-type discrete model for describing such mechanical systems, we adopted a Lagrangian formulation, which is computationally efficient for determining the motion in the most general non-

linear regime, and compare the solutions of the obtained differential equations with experimental tests with a good qualitative agreement (see also the supplementary data). We expect, therefore, that when a more accurate measurement campaign will be performed and when the lumped parameters in the considered Lagrangian functions will be suitably fitted a perfect quantitative agreement will become possible. Considered their great efficiency, we also expect that similar codes will be useful in the study of the nonlinear dynamics of Timoshenko beams.

Finally, we presented, a formal deduction of a two-dimensional membrane theory performed via an asymptotic development of the weak formulation of the three-dimensional equations of elasticity. This membrane model is similar to that of Landau and Lifshitz established in [28] from a priori simplifications, whose domain of validity is not clearly specified. In the approach presented here, the membrane model is deduced rigorously from asymptotic expansion, with certain order of magnitude of the applied forces, and its domain of validity can be more clearly specified. Moreover, some interesting aspects of the deduced model are investigated, in particular the property of obtaining a hyperbolic equation for the out-of-plane displacement under a certain class of boundary conditions and loads. Some simple cases are analyzed to show the relevant aspects of the model and the phenomenology that can be addressed. In particular it is shown how this mathematical formulation is capable to describe instabilities well known as wrinkling, often observed for the buckling of very thin membranes. Future investigations will focus, from the theoretical point of view, on the deep connection between the nonuniqueness of the solution for the out-of-plane displacement and the bifurcation related to buckling phenomena. From an applicative point of view, more general situations will be investigated, in order to provide an efficient and ready tool to investigate the class of phenomena linked to instabilities in thin structures.

Acknowledgements

I would like to thank first of all the laboratories where I have been welcome during my PhD. Thanks to the LaSIE and to all the people with which I shared the most of the working and free time! Thanks to the M&mocs and to all the people i meet in this fantastic organization!

Thanks to Prof. Aziz Hamdouni, one of the best person I meet during my career at the University. Thanks for his History, cinema, literature, politics and cultural lectures. Of course I cannot avoid to mention his infinite knowledge in the field of Math, Physics and engineering, I will never forget our dialogs in "Paix-Dieu"! I hope to have been able to keep a small part of his knowledge and passion for knowledge, it would be enough for me!

Thanks to Leila, for her kindness and humanity, I hope to welcome you back in Italy soon...and to visit your beautiful Country too as soon as possible.

Thanks to Prof. Olivier Millet, an excellent professor and scientist and above all a good man! Thanks for your mechanics and life lectures, but I want to tank you also for welcoming me in La Rochelle, and for arranging all the stuff for making my arrival and stay there easier.

Thanks to Prof. Francesco dell'Isola the person that made possible this beautiful route and the realization of my PhD thesis. His scientific value has no equals, but I appreciated also his History, Evolution and Social sciences lectures! I will make a treasure of it!

Thanks to Alessandro Della Corte, a colleague that soon has become a friend! The hidden tutor of my PhD thesis. Thanks for having shown me your competence and passion for science and research, and also for good kitchen! You will be one of the examples I will try to follow in my life. Thanks for everything!

Thanks to Ivan Giorgio, one of the most powerful war machine of Italian research..."the never tired and always available man"! Another friend I have gained during this period.

I want to thank of course also all the colleagues Professor (and friends). The only reason for which I don't spend dedicated words for any of them is the fact that acknowledgments cannot be longer than the thesis. Tanks to Daria Scerrato, Cristian Carrillo, Emilio Barchiesi, Luca Placidi, Emilio Turco, Victor Eremeyev, Pierre Seppecher, Mario Spagnuolo and Leopoldo Greco...and of course the ones I forgot in this list!

Next I pass to all the people that shared the office with me at LaSIE:

Thanks to Jana for the always interesting exchange of cultural and gastronomic information and for all the Lebanese recipes. I will think to you every time I prepare Tabulè...But I want to thank you also for having welcome me at the lab and for having introduced me to all the people and "mechanism" of the

University of La Rochelle. One of the best PR I ever meet! Good luck and see you soon!

Thanks to Rolando, "the Peruvian" to his wife Armandine and to the little Clementine. I spent with you beautiful moments! Thanks for your guitar lectures and for the interesting conversations about religion, "hand of the devil", and about how Vietnamese people can be "mèchant"!

Thanks to Hien, "the Vietnamese"...what to say, after a cold start (of a couple of days) he become one of the points of reference for me in La Rochelle. Thanks for the matches and beers drunk together at the happy hour of the General, and for having never said "no" when I invited you for a coffee. Thanks of course for having introduced me to badminton (now I can confess that I let you win...you are so little). Thanks to his wife Thau for the beautiful moments spent together. Thanks to Ramiz, "the Azerbaijani"...not really an office colleague (maybe he is a Vampire since he preferred to be at the lab in night). Anyway when the sun was not too strong we managed to spend together some time and it was great! Thanks for your "homemade" pizza followed by the tea with your Mather's strawberries marmalade!...I am still waiting for the recipe.

To finish I switch to Italian to thank some special people.

Grazie a tutta la mia famiglia per avermi sostenuto lungo questo percorso iniziato dal liceo e finito con questa tesi di dottorato che dedico a loro: a mamma Luisa, papà Michele, le mie sorelle Ilaria e Martina, Nonna Nocolina e Zia Giovanna. Senza di loro non sarei qui a scrivere questi ringraziamenti. Grazie di cuore!

In ultimo, ma non per importanza, voglio ringraziare la mia fidanzata e compagna di vita Maddalena, il suo immancabile sostegno, le sue spinte ed il suo spronarmi sono stati fondamentali per il raggiungimento di questo (e molti altri traguardi)! Purtroppo non posso ringraziarti abbastanza in queste poche righe, ma ho tutta la vita per farlo...spero di riuscirci! Grazie, grazie, grazie anche a te! E grazie anche a Jack...La sua compagnia è stata di aiuto per superare con leggerezza i momenti più complicati!

Un ringraziamento a tutti i miei amici che sono talmente tanti che non posso nemmeno nominare uno ad uno...Grazie a tutti ed al prossimo traguardo!

Merci beaucoup à tous,

Antonio.

Bibliography

- [1] Alibert, J.-J., Della Corte, A., and Seppecher, P. (2017). Convergence of hencky-type discrete beam model to euler inextensible elastica in large deformation: Rigorous proof. In *Mathematical Modelling in Solid Mechanics*, pages 1–12. Springer.
- [2] Alibert, J.-J., Seppecher, P., and Dell’Isola, F. (2003). Truss modular beams with deformation energy depending on higher displacement gradients. *Mathematics and Mechanics of Solids*, 8(1):51–73.
- [3] Andreaus, U., Spagnuolo, M., Lekszycki, T., and Eugster, S. R. (2018). A ritz approach for the static analysis of planar pantographic structures modeled with nonlinear euler–bernoulli beams. *Continuum Mechanics and Thermodynamics*, 30:1103–1123.
- [4] Antman, S. S. (2005). Problems in nonlinear elasticity. *Nonlinear Problems of Elasticity*, pages 513–584.
- [5] Baroudi, D., Giorgio, I., Battista, A., Turco, E., and Igumnov, L. A. (2019). Nonlinear dynamics of uniformly loaded elastica: Experimental and numerical evidence of motion around curled stable equilibrium configurations. *ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik*, page e201800121.
- [6] Battista, A., Della Corte, A., dell’Isola, F., and Seppecher, P. (2018). Large deformations of 1d microstructured systems modeled as generalized timoshenko beams. *Zeitschrift für angewandte Mathematik und Physik*, 69(3):52.
- [7] Battista, A., Hamdouni, A., and Millet, O. (2019). An asymptotic membrane model for wrinkling of very thin films. *Continuum Mechanics and Thermodynamics*, 31(1):189–207.
- [8] Bernoulli, J. (1999). Quadratura curvae, e cujus evolutione describitur inflexae laminae curvatura. *Die Werke von Jakob Bernoulli*, 223227:1692.
- [9] Born, M. (1906). *Untersuchungen über die stabilität der elastischen Linie in Ebene und Raum: Unter verschiedenen Grenzbedingungen*. Dieterich.
- [10] Ciarlet, P. G. and Miara, B. (1992). Justification of the two-dimensional equations of a linearly elastic shallow shell. *Communications on pure and applied mathematics*, 45(3):327–360.

- [11] Della Corte, A., Battista, A., dell’Isola, F., and Seppecher, P. (2019). Large deformations of timoshenko and euler beams under distributed load. *Zeitschrift für angewandte Mathematik und Physik*, 70(2):52.
- [12] Della Corte, A., dell’Isola, F., Esposito, R., and Pulvirenti, M. (2017). Equilibria of a clamped euler beam (elastica) with distributed load: Large deformations. *Mathematical Models and Methods in Applied Sciences*, 27(08):1391–1421.
- [13] dell’Isola, F., Della Corte, A., Battista, A., and Barchiesi, E. Extensible beam models in large deformation under distributed loading: a numerical study on multiplicity of solutions. In *Higher Gradient Materials and Related Generalized Continua*.
- [14] Dell’Isola, F., Giorgio, I., Pawlikowski, M., and Rizzi, N. (2016). Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium. *Proc. R. Soc. A*, 472(2185):20150790.
- [15] Destuynder, P. (1980). *Sur une justification des modèles de plaques et de coques par les méthodes asymptotiques*.
- [16] Destuynder, P. (1982). On non linear membrane theory. *Computer methods in applied mechanics and engineering*, 32(1-3):377–399.
- [17] Destuynder, P. (1985). A classification of thin shell theories. *Acta Applicandae Mathematica*, 4(1):15–63.
- [18] Euler, L. (1952). *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive solutio problematis isoperimetrici latissimo sensu accepti*, volume 1. Springer Science & Business Media.
- [19] Faulkner, M., Lipsett, A., and Tam, V. (1993). On the use of a segmental shooting technique for multiple solutions of planar elastica problems. *Computer methods in applied mechanics and engineering*, 110(3-4):221–236.
- [20] Fried, I. (1981). Stability and equilibrium of the straight and curved elastica-finite element computation. *Computer Methods in Applied Mechanics and Engineering*, 28:49–61.
- [21] Friedrichs, K. and Dressler, R. (1961). A boundary-layer theory for elastic plates. *Communications on Pure and Applied Mathematics*, 14(1):1–33.
- [22] G. Ciarlet, P. and Destuynder, P. (1977). Une justification du modèle biharmonique en théorie linéaire des plaques. *Comptes Rendus Hebdomadaires des Séances de l’Académie des Sciences, Série A*, 285.
- [23] Gol’denveizer, A. (1963). Derivation of an approximate theory of shells by means of asymptotic integration of the equations of the theory of elasticity. *Journal of Applied Mathematics and Mechanics*, 27(4):903–924.
- [24] Hamdouni, A. and Millet, O. (2003a). Classification of thin shell models deduced from the nonlinear three-dimensional elasticity. part i: the shallow shells. *Archives of Mechanics*, 55(2):135–176.

- [25] Hamdouni, A. and Millet, O. (2003b). Classification of thin shell models deduced from the nonlinear three-dimensional elasticity. part i: the shallow shells. *Archives of Mechanics*, 55(2):135–176.
- [26] Hencky, H. (1921). *Über die angenäherte Lösung von Stabilitätsproblemen im Raum mittels der elastischen Gelenkkette*. PhD thesis, Verlag nicht ermittelbar.
- [27] Lagrange, J. L. (1853). *Mécanique analytique*, volume 1. Mallet-Bachelier.
- [28] Landau, L. D. and Lifshitz, E. (1986). Theory of elasticity, vol. 7. *Course of Theoretical Physics*, 3:109.
- [29] Millet, O., Hamdouni, A., and Cimetière, A. (1998a). Dimensional analysis and asymptotic expansions of equilibrium equations in nonlinear elasticity. part i: The membrane model. *Archives of Mechanics*, 50(6):953–973.
- [30] Millet, O., Hamdouni, A., and Cimetière, A. (1998b). Dimensional analysis and asymptotic expansions of equilibrium equations in nonlinear elasticity. part ii: The two-dimensional von karman model. *Archives of Mechanics*, 50(6):975–1001.
- [31] Millet, O., Hamdouni, A., and Cimetière, A. (2001). A classification of thin plate models by asymptotic expansion of non-linear three-dimensional equilibrium equations. *International journal of non-linear mechanics*, 36(1):165–186.
- [32] Mora, M. G. and Müller, S. (2004). A nonlinear model for inextensible rods as a low energy γ -limit of three-dimensional nonlinear elasticity. In *Annales de l'Institut Henri Poincaré (C) Non Linear Analysis*, volume 21-3, pages 271–293. Elsevier Masson.
- [33] Pideri, C. and Seppecher, P. (2006). Asymptotics of a non-planar rod in non-linear elasticity. *Asymptotic Analysis*, 48(1, 2):33–54.
- [34] Raboud, D., Faulkner, M., and Lipsett, A. (1996). Multiple three-dimensional equilibrium solutions for cantilever beams loaded by dead tip and uniform distributed loads. *International journal of non-linear mechanics*, 31(3):297–311.
- [35] Sanchez-Palencia, E. (1989). Statique et dynamique des coques minces. ii: Cas de flexion pure inhibée. approximation membranaire. *Comptes rendus de l'Académie des sciences. Série 1, Mathématique*, 309(7):531–537.
- [36] Sanchez-Palencia, E. (1990). Passage à la limite de l'élasticité tridimensionnelle à la théorie asymptotique des coques minces. *Comptes rendus de l'Académie des sciences. Série 2, Mécanique, Physique, Chimie, Sciences de l'univers, Sciences de la Terre*, 311(8):909–916.
- [37] Timoshenko, S. P. (1921). On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Phil. Mag.*, 41:744–746.
- [38] Timoshenko, S. P. (1922). X. on the transverse vibrations of bars of uniform cross-section. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 43(253):125–131.

Une analyse de structures minces non linéaires

Le thème principal de cette thèse est l'étude du comportement mécanique de structures minces élancées dans le domaine non-linéaire. Ce travail de thèse est présenté sous la forme d'une collection d'articles publiés au cours du doctorat et est divisé en deux parties. La première partie concerne l'analyse de modèles non-linéaires de poutres inextensibles et extensibles, généralisant sur différents aspects les modèles de poutre d'Euler et de Timoshenko. Une étude théorique de l'existence et de l'unicité de solutions est complétée de simulations numériques mettant en évidence l'existence de solutions multiples avec l'augmentation de la force appliquée. Une étude numérique de la multiplicité de solutions d'un modèle de poutre extensible en grands déplacements est également effectuée. La deuxième partie concerne la justification formelle par méthodes asymptotiques d'un modèle de membrane original présentant une multiplicité de solutions pour des chargements particuliers, pouvant modéliser les plissements de certaines structures très minces sollicitées en cisaillement.

Mots clés: Théories de poutre, élasticité non linéaire, modèle de membrane, méthodes asymptotiques.

An analysis of nonlinear thin structures

The main theme of this thesis is the study of the mechanical behavior of thin slender structures in the nonlinear domain. This thesis work is presented in the form of a collection of articles published during the Ph.D. and is divided into two parts. The first part deals with the analysis of nonlinear models of inextensible and extensible beams, generalizing on different aspects the beam models of Euler and Timoshenko. A theoretical study of the existence and uniqueness of solutions is completed by numerical simulations highlighting the existence of multiple solutions with the increase of the applied force. A numerical study of the multiplicity of solutions of an extensible beam model in large displacements is also carried out. The second part concerns the formal justification by asymptotic methods of an original membrane model presenting a multiplicity of solutions for particular loads, able to model the wrinkling of some very thin structures with a shear stress applied.

Keywords: Beam theory, nonlinear elasticity, membrane model, asymptotic methods..

LaSIE

**Laboratoire des Sciences de l'Ingénieur pour
l'Environnement (LaSIE) - UMR CNRS 7356
Pôle Sciences et Technologie
Avenue Michel Crépeau
17042 LA ROCHELLE Cedex 1 - France**