Sur les monoïdes des classes de groupes de tresses

Pablo Gonzalez Pagotto

Communauté
UNIVERSITÉ Grenoble Alpes

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Pablo GONZALEZ PAGOTTO

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on the coset monoids of the braid groups

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Monsieur LOUIS FUNAR
DIRECTEUR DE RECHERCHE, CNRS DELEGATION ALPES, Directeur de thèse
Monsieur PAOLO BELLINGERI
MAITRE DE CONFERENCES HDR, UNIVERSITE DE CAEN NORMANDIE, Rapporteur
Monsieur EMMANUEL WAGNER
PROFESSEUR, UNIVERSITE PARIS 7, Rapporteur
Madame CHRISTINE LESCOP
DIRECTRICE DE RECHERCHE, CNRS DELEGATION ALPES, Examinateur
Monsieur JEAN-BAPTISTE MEILHAN
MAITRE DE CONFERENCES, UNIVERSITE GRENOBLE ALPES, Examinateur
Monsieur JOHN GUASCHI
PROFESSEUR, UNIVERSITE DE CAEN NORMANDIE, Président

Abstract

Hurwitz showed that a branched cover $f\colon M \to N$ with branch locus $P \subset N$ determines and is determined, up to inner automorphism of the symmetric group S_m, by a homomorphism $\pi_1(N \setminus P, *) \to S_m$. This result reduces the questions of existence and uniqueness of branched covers to combinatorial problems. For a suitable set of generators for $\pi_1(N \setminus P, *)$, a representation $\pi_1(N \setminus P, *) \to S_m$ determines and is determined by a sequence $(a_1, b_1, \ldots, a_g, b_g, z_1, \ldots, z_k)$ of elements of S_m satisfying $[a_1, b_1] \cdots [a_g, b_g] z_1 \cdots z_k = 1$. The sequence $(a_1, b_1, \ldots, a_g, b_g, z_1, \ldots, z_k)$ of permutations is called a Hurwitz system for f.

Therefore, to understand the classes of branched covers one need to study the orbits of Hurwitz systems by suitable actions on $S_m^n, n = 2g+k$. One of such actions is the simultaneous conjugation that leads to the study of the set of double cosets of symmetric groups.

In Chapter 1 we bring an exposition of the recent work of Neretin on the multiplicative structure on the set $S_\infty \backslash S_\infty^n / S_\infty$.

In Chapter 2 we aim at extending Neretin's results to the group B_∞ of finitely supported braids on infinitely many strands. We prove that $B_\infty \backslash B_\infty^n / B_\infty$ admits such a multiplicative structure and explain how this structure is related to similar constructions in $\mathrm{Aut}(F_\infty)$ and $GL(\infty)$. We also define a one-parameter generalization of the usual monoid structure on the set of double cosets of $GL(\infty)$ and show that the Burau representation provides a functor between the categories of double cosets of B_∞ and $GL(\infty)$.

The last chapter is dedicated to the study of homomorphisms $\pi_1(N \setminus P, *) \to G$, G a discrete group. We give an exposition of the stable classification of such homomorphisms following the work of Samperton and some new results concerning the number of stabilizations necessary to make them equivalent with respect to Hurwitz moves. We also explore a generalization of the classification of finite branched covers by introducing the braid monodromy for surfaces embedded in codimension 2. Following ideas of Kamada we define the braid monodromy associated to braided surfaces, which correspond to $G = B_\infty$ and study the spherical functions associated to braid group representations.

Keywords: branched cover, braid monodromy, double cosets, representation theory, stable classification.

Resumé

Hurwitz a montre qu'un revêtement ramifié $f\colon M \to N$ avec lieu de ramification $P \subset N$ détermine et est déterminé, à un automorphisme intérieur près du groupe symétrique S_m, par un homomorphisme $\pi_1(N \setminus P, *) \to S_m$. Ce résultat réduit les questions d'existence et d'unicité à un problème combinatoire. Pour un ensemble de générateurs convenable pour le groupe $\pi_1(N \setminus P, *)$, une représentation $\pi_1(N \setminus P, *) \to S_m$ détermine et est determinée par une suite $(a_1, b_1, \ldots, a_g, b_g, z_1, \ldots, z_k)$ d'éléments de S_m satisfaisant $[a_1, b_1] \cdots [a_g, b_g] z_1 \cdots z_k = 1$. La suite $(a_1, b_1, \ldots, a_g, b_g, z_1, \ldots, z_k)$ de permutations est appelé un système de Hurwitz pour f.

Par conséquent, pour comprendre les classes de revêtements ramifiés, on doit étudier les orbites des systèmes de Hurwitz par des actions convenables sur $S_m^n, n = 2g + k$. Une de ces actions est la conjugaison simultanée qui conduit à l'étude de l'ensemble des classes doubles des groupes symétriques.

Dans le premier chapitre, nous présentons les travaux récents de Neretin sur la structure multiplicative sur l'ensemble $S_\infty \backslash S_\infty^n / S_\infty$.

Dans le deuxième chapitre, nous visons étendre les résultats de Neretin au groupe B_∞ des tresses à support fini avec un nombre infini de brins. Nous montrons que $B_\infty \backslash B_\infty^n / B_\infty$ admet une telle structure multiplicative et expliquons comment cette structure est liée à des constructions similaires dans $\mathrm{Aut}(F_\infty)$ et $GL(\infty)$. Nous définissons également une généralisation à un paramètre de la structure habituelle de monoïde sur l'ensemble des classes doubles de $GL(\infty)$ et montrons que la représentation de Burau fournit un foncteur entre les catégories des classes doubles de B_∞ et de $GL(\infty)$.

Le dernier chapitre est consacré à l'étude des homomorphismes $\pi_1(N \setminus P, *) \to G$, G un groupe discret. Nous exposons la classification stable de tels homomorphismes selon Samperton et des nouveaux résultats concernant le nombre de stabilisations nécessaires pour les rendre équivalents par rapport aux mouvements de Hurwitz. Nous explorons ensuite une généralisation de la classification des revêtements ramifiés finis en introduisant la monodromie des tresses associée à des surfaces plongées en codimension 2. Suivant des idées de Kamada, nous définissons la monodromie des tresses associée à des surfaces tressées correspondant à $G = B_\infty$ et étudions les fonctions sphériques associées aux représentations des groupes des tresses.

Mots-clés: revêtement ramifié, monodromie des tresses, classes doubles, théorie de représentation, classification stable.

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Contents

Introduction

Consider two connected orientable surfaces M and N. A branched cover of degree m with n branching points is a continuous map $\phi\colon M \to N$ that fails to be a covering map of degree m only on a set $B_\phi \subset N$ of n points and we say that two branched covers $\phi, \psi\colon M \to N$ are equivalent if there exists two homeomorphisms $f\colon M \to M$ and $g\colon N \to N$ such that $f\psi = \phi g$.

Given a branched cover $\phi\colon M \to N$ of degree d and a base point $* \in N \setminus B_\phi$ number the elements of the set $\phi^{-1}(*)$ from 1 to d. To each loop α in $N \setminus B_\phi$ based at $*$ we can associate the permutation $\rho_\phi(\alpha)$ induced by transporting $\phi^{-1}(*)$ along α using the path lifting property. The homomorphism $\rho_\phi\colon\ \pi_1(N \setminus B_\phi) \to S_d$, where S_d denotes the d-symmetric group, is called the monodromy of the branched cover ϕ.

A. Hurwitz [19] showed how to classify branched covers by means of the classification of ordinary coverings via the fundamental group. We have the following results:

Theorem (2.1 of [3])**.** *Two branched covers of degree d $\phi_1\colon M_1 \to N$ and $\phi_2\colon M_2 \to N$ are equivalent if and only if there exists a homeomorphism*

$$h\colon (N, B_{\phi_1}, *) \to (N, B_{\phi_2}, *)$$

and an inner automorphism $\mu\colon S_d \to S_d$ such that

$$\mu\rho_{\phi_1} = \rho_{\phi_2} h_*.$$

Theorem (2.2 of [3])**.** *Let N be a compact, connected, oriented surface. If B is a finite subset of* $\operatorname{int} N$ *and $\rho\colon\ \pi_1(N \setminus B, *) \to S_d$ is a representation such that it is nontrivial on each class represented by a small loop around any single point of B, then there exists a branched cover $\phi\colon M \to N$ with $B_\phi = B$ and $\rho_\phi = \rho$.*

These results reduce questions of existence and uniqueness to virtually combinatorial problems. In fact, given a suitable set of generators for $\pi_1(N \setminus B_\phi, *)$, called a Hurwitz arc system, the representation $\rho_\phi\colon\ \pi_1(N \setminus B_\phi, *) \to S_d$ determines and is determined by a sequence

$$H(\phi) = (\sigma_1, \ldots, \sigma_n, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_k)$$

of elements of S_d subject to the requirement that

$$\sigma_1 \cdots \sigma_n [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_k = 1.$$

This sequence of permutations corresponding to a Hurwitz arc system for N is called a Hurwitz system for ϕ.

Corolary. *Two branched covers of degree d over a given surface N are equivalent if and only if they have Hurwitz systems that are conjugate by an element of S_d.*

Therefore, to understand the classes of branched covers one may study the orbits of the Hurwitz systems by actions on S_d. One of such actions is the simultaneous conjugation that leads to the study of the set of double cosets of the symmetric groups. In fact, since we have a bijection between the set $\mathrm{diag}(S_d^n)\backslash S_d^n/\mathrm{diag}(S_d^n)$ and the set of conjugacy classes $S_d^{n-1}/\!\!/\mathrm{diag}(S_d^{n-1})$ (see Remark 1.4.3) we obtain a one-to-one correspondence

$$\begin{array}{c} \{\text{Hurwitz systems up to simultaneous conjugation}\} \\ \updownarrow \\ \mathrm{diag}(S_d^n)\backslash S_d^n/\mathrm{diag}(S_d^n) \end{array}$$

where n is the length of the Hurwitz system. For any finite integer d the set of double cosets $\mathrm{diag}(S_d^n)\backslash S_d^n/\mathrm{diag}(S_d^n)$ does not have an interesting structure. Therefore we include the groups S_d into their direct limit S_∞ which have additional structure.

In fact, for some infinite-dimensional groups G and suitable subgroups K there exists a monoid structure on the set $K\backslash G/K$ of double cosets of G with respect to K. This can be seen, for example, for the group S_∞ of the finitely supported permutations of $\mathbb{N}_+ = \{1, 2, \ldots\}$, for infinite-dimensional classical Lie groups, for groups of automorphisms of measure spaces and for $\mathrm{Aut}(F_\infty)$, a direct limit of the groups of automorphisms of the free groups F_n.

The study of these structures was pioneered by R. S. Ismagilov, followed by G. I. Olshanski, who used them in the representation theory of infinite-dimensional classical Lie groups ([35, 36, 38–41]). In Chapter 1 we bring an exposition of the recent work of Y. A. Neretin exploring these structures for the infinite tri-symmetric group and $\mathrm{Aut}(F_\infty)$ ([27, 29, 33, 34]).

In Chapter 2 we show that the group B_∞, of the finite braids on infinitely many strands, admits such a structure (Definition 2.3.2). Furthermore, we show how this multiplicative structure is related to similar constructions in $\mathrm{Aut}(F_\infty)$ and $GL(\infty)$. We also define a one-parameter generalization (Definition 2.3.7) of the usual monoid structure on the set of double cosets of $GL(\infty)$ (see [30, 31]) and show that the Burau representation provides a functor between the categories of double cosets of B_∞ and $GL(\infty)$.

The last chapter is divided into three parts. In section 3.1.4 we bring a brief exposition of the theory of stable classification of branched covers following Samperton [43] and some new results concerning the number of stabilizations necessary in order to reach stability (Theorems 3.1.14 and 3.1.16). In section 3.2 we give a explicit relation between the double cosets of S_∞ and branched covers by means of checker surfaces (Section 1.5.1). In section 3.3 we introduce a generalization (Definition 3.3.2) of the concept of braided surface given by Kamada (Definition 16.1 of [20]) and define a braid system (Definition 3.3.9) in analogy to Hurwitz systems for branched covers. We hope to pursuit this analogy in a future work and see if it is possible to take advantage of the multiplicative structure on B_∞ for this purpose. Lastly we present some considerations for future works employing spherical matrix coefficients to construct invariants for double cosets.

Chapter 1

The Infinite Symmetric Group

This chapter should be taken as a survey of several papers by Neretin [14, 27, 28, 30–34]. Here we lay the ideas that motivated the other chapters.

For some infinite-dimensional groups G and suitable subgroups K there exists a monoid structure on the set $K\backslash G/K$ of double cosets of G with respect to K. This can be seen, for example, for the group S_∞ of the finitely supported permutations of $\mathbb{N}_+$, for infinite-dimensional classical Lie groups, for groups of automorphisms of measure spaces and for $\mathrm{Aut}(F_\infty)$, a direct limit of the groups of automorphisms of the free groups F_n.

The study of these structures was pioneered by R. S. Ismagilov, followed by G. I. Olshanski, who used them in the representation theory of infinite-dimensional classical Lie groups ([38, 41]). More recently there is the work of Y. A. Neretin for the infinite tri-symmetric group and $\mathrm{Aut}(F_\infty)$ ([27, 29, 33]).

Representation theory of infinite symmetric groups S_∞ was initiated by two works. The first one was the paper of Elmar Thoma [45], in 1964, where he introduced analogs of characters for the group S_∞ of finitely supported infinite permutations. The second was the paper of Arthur Lieberman [23], in 1972, where he classified all unitary representations of the complete infinite symmetric group.

1.1 The group S_∞

For each $d \in \mathbb{N}_+$, denote by S_d the symmetric group on d symbols, the group consisting of all bijections of the set $\{1, 2, \ldots, d\}$ onto itself with group operation given by function composition. For a fixed $d \in \mathbb{N}_+$ there is a natural injective homomorphism $i_d\colon S_d \to S_{d+1}$. We define the infinite symmetric group S_∞ to be the direct limit of the groups S_d with respect to the homomorphisms i_d,

$$S_\infty = \varinjlim S_n.$$

It is easy to see that S_∞ is the group of all finitely supported bijections of $\mathbb{N}_+$ (all bijections of $\mathbb{N}_+$ onto itself that fails to be the identity only on a finite set) with operation given by function composition. The group S_∞ is a countable discrete group and can be realized as the group of infinite invertible 0-1 matrices (on a 0-1 matrix all entries are either 0 or 1 and each column and each row contains at most one unit and a matrix is invertible if and only if each column and each row contains exactly one unit) such that the sequence of elements of its diagonal is eventually constant equal to 1.

We define a family of subgroups of S_∞ that will be used later. For each $\alpha \in \mathbb{N}$, denote by $S_\infty[\alpha]$ the subgroup consisting of all permutations that fix the set $\{1, 2, \ldots, \alpha\}$ pointwise. In terms of 0-1 matrices this groups consists of the matrices of the form

$$\begin{pmatrix} 1_p & 0 \\ 0 & r \end{pmatrix},$$

for $r \in S_\infty$. We get a descending sequence of subgroups

$$S_\infty = S_\infty[0] \supset S_\infty[1] \supset S_\infty[2] \supset \cdots$$

all of which are isomorphic among themselves.

1.2 Spherical functions and Thoma characters

Let G be a group. A unitary representation of G in a complex Hilbert space H is a homomorphism of G into the group $U(H)$ of unitary operators in H. If T is a representation we often write $H(T)$ for the Hilbert space of T.

We say that two unitary representations T and T' of the same group are equivalent if there exists a surjective map $f\colon H(T) \to H(T')$ such that $fT(g) = T'(g)$ for all $g \in G$. The commutant of T is the set of all bounded operators on $H(T)$ commuting with all the operators $T(g), g \in G$. The commutant is an algebra and it is closed under the action of taking the adjoint operator.

An invariant subspace of a unitary operator T is a closed subspace $H' \subset H(T)$ which is invariant under the action of all operators $T(g)$. If H' is an invariant subspace of T then the orthogonal complement to H' is also an invariant subspace and the restriction of T to an invariant subspace H' gives rise to a subrepresentation of T. Moreover, if T does not admit any proper invariant subspace then T is said to be irreducible and we denote the set of equivalence classes of irreducible representations of G by $\widehat{G}$.

Given a vector $\xi \in H(T)$, there exists a smallest invariant subspace in $H(T)$ containing ξ; This is the closure of the linear span of the orbit $\{T(g)\xi; g \in G\}$ and it is called the cyclic span of ξ. If the cyclic span of ξ coincides with the whole space $H(T)$ then ξ is called a cyclic vector.

For a countable group G, if T is a unitary representation admitting a cyclic vector then $H(T)$ is separable. If T is irreducible then any nonzero vector is cyclic.

Theorem 1.2.1 (Schur lemma, Proposition 8.1 of [6]). *A unitary representation T is irreducible if and only if its commutant reduces to scalar operators.*

Theorem 1.2.2 (Burnside theorem, Proposition 8.2 of [6]). *Let T be a unitary representation of G, $\operatorname{End} H$ be the algebra of all bounded operators on $H = H(T)$, and $A \subset \operatorname{End} H$ be the subalgebra generated by the operators $T(g), g \in G$.*

If T is irreducible then A is dense in $\operatorname{End} H$ in the strong operator topology.

Proposition 1.2.3 (Proposition 8.3 of [6]). *Let T be a unitary representation of a group G and $\xi \in H(T)$ be a nonzero vector. The matrix coefficient $\phi(g) = \langle T(g)\xi, \xi\rangle$ defines a positive definite function $\phi\colon G \to \mathbb{C}$.*

Conversely, if ϕ is a nonzero positive definite function on G then there exists a unitary representation T with a cyclic vector ξ such that the corresponding matrix coefficient coincides with ϕ. Moreover, such a couple (T, ξ) is unique up to the natural equivalence given by: if (T', ξ') is another couple with the same property then there exists a surjective isometry $f\colon H(T) \to H(T')$ such that $f(\xi) = \xi'$ and $fT(g) = T'(g), g \in G$.

Denote by $L(G)$ the set of all positive definite functions on G. This is convex cone in the linear space of functions on G. The base of this cone is the convex set $L_1(G) \subset L(G)$ of those functions that take the unit of G to 1. It is compact with respect to the topology of pointwise convergence and by the Krein-Milman Theorem, $L_1(G)$ is the convex hull of its extreme points.

Theorem 1.2.4 (Corollary 8.5 of [6]). *Let T be a unitary representation of G with a cyclic vector ξ, and $\phi \in L_1(G)$ the corresponding matrix coefficient. Then T is irreducible if and only if ϕ is an extreme point of the convex set $L_1(G)$.*

Definition 1.2.5. *Let G be a group and H, K subgroups of G. A double coset is a subset of G of the form HgK. We denote by $H\backslash G/K$ the set of all double cosets of G with respect to the subgroups H and K.*

Let K be a subgroup of G. If T is a unitary representation of G, we will denote by $H(T)^K$ the subspace of K-invariant vectors in $H(T)$. It is easy to see that if $\xi \in H(T)^K$ then the corresponding matrix coefficient is bi-K-invariant, that is, it is invariant with respect to the two-sided action of the subgroup K on the group G and therefore a well-defined function on $K\backslash G/K$. Conversely, we have,

Proposition 1.2.6 (Proposition 8.6 of [6]). *Let ϕ be a nonzero bi-K-invariant function from $L(G)$ and (T, ξ) be the corresponding unitary representation with cyclic vector. Then this vector is K-invariant.*

We will denote the subset of $L(G)$ formed by all bi-K-invariant functions by $L(K\backslash G/K)$ and identify it with the set of all complex functions on the set of double cosets $K\backslash G/K$.

One way to study these functions is to use spherical functions, or more generally, spherical matrix coefficients (this functions were already introduced on Proposition 1.2.3 but we give a formal definition below).

For a unitary irreducible representation ρ of a group G and K a subgroup of G, we say that ρ is K-spherical if $\dim H(\rho)^K = 1$, that is, $H(\rho)^K = \langle \xi \rangle$. The matrix coefficient associated to ξ, $\phi_\xi(g) = \langle \rho(g)\xi, \xi \rangle$ is called a spherical function. Furthermore, if for every irreducible unitary representation ρ of G, ρ is K-spherical or $H(\rho)^K = \{0\}$ we say that the pair (G, K) is spherical.

On the other hand, if $\dim H(\rho)^K \geq 1$ we define the spherical matrix coefficients associated to ρ by $\phi^\rho_{i,j}(g) = \langle \rho(g)v_i, v_j \rangle$, where $\{v_1, v_2, \ldots\}$ is an orthonormal basis for $H(\rho)^K$. We emphasize the following well known property of the spherical matrix coefficients (see, for example, Remark 1.2.16 of [9] for the finite case) and give a short proof for it:

Proposition 1.2.7. *Let G be a finite or compact group and K one of its subgroups, the set of all spherical matrix coefficients when ρ ranges over all unitary representations of G form a generating set for the bi-K-invariant functions $L(K\backslash G/K)$.*

Proof. Let $L(X)$ be the associated permutation representation of the homogeneous space $X = G/K$. Then for each class (ρ, V_ρ) of unitary irreducible representations of G there is a nonnegative integer m_ρ, the multiplicity of the representation ρ, such that

$$L(X) = \sum_{\rho \in \hat{G}} m_\rho V_\rho.$$

By Wielandt's Lemma (Theorem 3.13.3 of [9]) the number of K-orbits in X is equal to $\sum m_\rho^2$.

On the other hand, there is a one-to-one correspondence between the K-orbits in X and the set of double cosets $K\backslash G/K$. Hence,

$$\sum_{\rho\in\hat{G}} m_\rho^2 = |\{K\text{-orbits in X}\}| = |K\backslash G/K| = \dim L(K\backslash G/K).$$

By the Frobenius reciprocity

$$m_\rho = \dim V_\rho^K$$

and therefore

$$\sum_{\rho\in\hat{G}} (\dim V_\rho^K)^2 = \dim L(K\backslash G/K).$$

By Lemma 3.6.3 of [9], $\mathcal{F}$ is a subset of an orthonormal basis of G and therefore is linearly independent. Since $\mathcal{F}$ is a subset of $L(K\backslash G/K)$ of cardinality $\sum_{\rho\in\hat{G}}(\dim V_\rho^K)^2$ we conclude that they generate $L(K\backslash G/K)$.

The compact case follows as in the finite case from the Peter-Weyl theorem: the matrix coefficients are dense in the space L^2, the spherical functions are spanning the space of L^2-class functions while $L^2(G) = \oplus V_\rho^{\dim V_\rho}$ is the direct sum of all irreducible representations V_ρ of G with multiplicity equal to their dimension. □

A particular class of spherical pairs are Gelfand pairs

Definition 1.2.8. *Let G be a group and K one of its subgroups. Given a unitary representation ρ let P_ρ be the orthogonal projection on $H(\rho)^K$. We say that (G,K) is Gelfand pair if for every unitary representation ρ the operators $P_\rho\rho(g)P_\rho$ commute with each other for all $g\in G$. That is, for every unitary representation ρ*

$$P_\rho\rho(g_1)P_\rho\rho(g_2)P_\rho = P_\rho\rho(g_2)P_\rho\rho(g_1)P_\rho, \quad \forall g_1,g_2\in G.$$

Proposition 1.2.9 (Proposition 8.14 of [6])**.** *Let (G,K) be a Gelfand pair. Then for any irreducible unitary representation ρ of G, the dimension of $H(\rho)^K$ is at most 1.*

Proposition 1.2.10 (Proposition 8.15 of [6])**.** *Let G be a group and K one of its subgroups. Assume that G is the union of an ascending chain of subgroups G_n and K is the union of an ascending sequence of subgroups $K_n\subset G_n$. If (G_n,K_n) is a Gelfand pair for every $n\in\mathbb{N}_+$ then (G,K) is a Gelfand pair.*

Corollary 1.2.11. *If a group K is an ascending chain of finite subgroups K_n then $(K\times K, \operatorname{diag} K)$ is a Gelfand pair.*

We introduce characters for infinite groups following [44] and [6]

Definition 1.2.12 (Definition 1.7 of [6])**.** *A character of the group G is an extreme point of the set $L_1(G)$.*

For finite or compact groups, characters defined in this way have the form $\frac{\chi(h)}{\dim\chi}$, where $\chi(h)$ is the usual irreducible character of the group G.

Theorem 1.2.13 (Thoma character, [44]). *Consider a collection of non-negative real numbers $\alpha_j, \beta_j, \gamma$ satisfying the conditions*

$$\ldots \leq \alpha_2 \leq \alpha_1, \quad \ldots \leq \beta_2 \leq \beta_1, \quad \sum \alpha_j + \sum \beta_j + \gamma = 1.$$

The characters of the group S_∞ are given by

$$\chi_{\alpha,\beta,\gamma}(h) = \prod_{k=2}^{\infty} \left(\sum_j \alpha_j^k + (-1)^{j-1} \sum_j \beta_j^k \right)^{r_k(h)}$$

where $r_k(h)$ is the number of cycles of length k of the permutation h.

Proposition 1.2.14 (Theorem 1.4 of [32]). *The pair $(S_\infty \times S_\infty, \operatorname{diag} S_\infty)$ is spherical (Gelfand). Furthermore, the spherical functions are the characters $\chi_{\alpha,\beta,\gamma}(g_1 g_2^{-1})$*

1.3 The complete symmetric group $\overline{S}_\infty$

Denote by $\overline{S}_\infty$ the group of all permutations of the set $\mathbb{N}_+$ of natural numbers and for each $\alpha \in \mathbb{N}$ let $\overline{S}_\infty[\alpha]$ be the subgroup of $\overline{S}_\infty$ consisting of all permutations that fix the set $\{1, \ldots, \alpha\}$ pointwise. We also convention that $\overline{S}_\infty[0] = \overline{S}_\infty$.

In terms of 0-1 matrices this groups consists of the matrices of the form

$$\begin{pmatrix} 1_\alpha & 0 \\ 0 & r \end{pmatrix},$$

for $r \in \overline{S}_\infty$. We define a totally disconnected topology on $\overline{S}_\infty$ by requiring that the subgroups $\overline{S}_\infty[p]$ are open. In this topology if $(g_j)_j$ is a sequence, we have

(i) $g_j \to g$ in $\overline{S}_\infty$ if for any $k \in \mathbb{N}_+$ $g_j(k) = g(k)$ for sufficiently large j, or;

(ii) $g_j \to g$ if the corresponding sequence of 0-1 matrices converges weakly.

1.3.1 Multiplication of double cosets on $\overline{S}_\infty$

We intent to define a product

$$\overline{S}_\infty[\alpha] \backslash \overline{S}_\infty / \overline{S}_\infty[\beta] \times \overline{S}_\infty[\beta] \backslash \overline{S}_\infty / \overline{S}_\infty[\gamma] \to \overline{S}_\infty[\alpha] \backslash \overline{S}_\infty / \overline{S}_\infty[\gamma].$$

Following [32], consider the following sequence $\theta_j[\beta]$ in $\overline{S}_\infty[\beta]$

$$\theta_j[\beta] = \begin{pmatrix} 1_\beta & 0 & 0 & 0 \\ 0 & 0 & 1_j & 0 \\ 0 & 1_j & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}.$$

Consider two double cosets

$$\mathfrak{p} \in \overline{S}_\infty[\alpha] \backslash \overline{S}_\infty / \overline{S}_\infty[\beta], \quad \mathfrak{q} \in \overline{S}_\infty[\beta] \backslash \overline{S}_\infty / \overline{S}_\infty[\gamma],$$

and let p and q be their respective representatives. Consider the sequence of double cosets in $\overline{S}_\infty[\alpha] \backslash \overline{S}_\infty / \overline{S}_\infty[\gamma]$ given by

$$\mathfrak{r}_j = \overline{S}_\infty[\alpha]\, p \theta_j[\beta] q\, \overline{S}_\infty[\gamma].$$

Proposition 1.3.1. *In the conditions above we have:*

(a) The sequence $(\mathfrak{r}_j)$ defined above is eventually constant.

(b) Let p' and q' be other two representatives of $\mathfrak{p}$ and $\mathfrak{q}$ respectively. Consider the sequence

$$\mathfrak{r}'_j = \overline{S}_\infty[\alpha]\, p'\theta_j[\beta]q'\, \overline{S}_\infty[\gamma].$$

Then there exists an integer $N > 0$ such that

$$\mathfrak{r}'_j = \mathfrak{r}_j, \quad \forall j \geq N.$$

Before we prove Proposition 1.3.1 we need the following Lemma:

Lemma 1.3.2. *Let $p, q \in \overline{S}_\infty$ and $\alpha, \beta \in \mathbb{N}$. Then $\overline{S}_\infty[\beta]p\overline{S}_\infty[\alpha] = \overline{S}_\infty[\beta]q\overline{S}_\infty[\alpha]$ if and only if the sets $S(\alpha,\beta,p) = I_\alpha \cap p^{-1}(I_\beta)$ and $S(\alpha,\beta,q) = I_\alpha \cap q^{-1}(I_\beta)$ are equal and the permutations p and q coincide on these sets, where $I_x = \{1, 2, \ldots, x\}$.*

Proof. If $\overline{S}_\infty[\beta]p\overline{S}_\infty[\alpha] = \overline{S}_\infty[\beta]q\overline{S}_\infty[\alpha]$ then there exist elements $h \in \overline{S}_\infty[\beta]$ and $l \in \overline{S}_\infty[\alpha]$ such that $p = hql$. Hence,

$$\begin{aligned} S(\alpha,\beta,p) = I_\alpha \cap p^{-1}(I_\beta) = I_\alpha \cap (hql)^{-1}(I_\beta) = I_\alpha \cap l^{-1}(q^{-1}h^{-1}(I_\beta)) = l^{-1}(I_\alpha \cap q^{-1}l^{-1}(I_\beta)) \\ = I_\alpha \cap q^{-1}h^{-1}(I_\beta) = I_\alpha \cap q^{-1}(I_\beta) = S(\alpha,\beta,q). \end{aligned}$$

And for $i \in S(\alpha,\beta,p)$, we have $p(i) = hql(i) = hq(i)$, since $i \in p^{-1}(I_\beta)$ we conclude that $p(i) = q(i)$.

Now, suppose that $S(\alpha,\beta,p) = S(\alpha,\beta,q)$ and that p and q agree in these sets. We construct permutations $h \in \overline{S}_\infty[\beta]$ and $l \in \overline{S}_\infty[\alpha]$ such that $p = hql$. Define the injection $\bar{l}\colon I_\alpha \cup p^{-1}(I_\beta) \to \mathbb{N}_+$ by

$$\bar{l}(i) = \begin{cases} i, & i \in I_\alpha, \\ q^{-1}p(i), & i \in p^{-1}(I_\beta). \end{cases}$$

Extend the injection $q\bar{l}$ to a permutation $\tau\colon \mathbb{N}_+ \to \mathbb{N}_+$, and define permutations $h = \tau p^{-1}$ and $l = q^{-1}\tau$. It is easy to see that $h \in \overline{S}_\infty[\beta]$, $l \in \overline{S}_\infty[\alpha]$ and $hp = ql$. Therefore $\overline{S}_\infty[\beta]p\overline{S}_\infty[\alpha] = \overline{S}_\infty[\beta]q\overline{S}_\infty[\alpha]$.

□

Proof. (of Proposition 1.3.1)

(a) For simplicity, set $n_j = p\theta_j[\beta]q$. By the previous Lemma, it suffices to show that for some $j_0 \in \mathbb{N}_+$ we have, $S(\gamma,\alpha,n_j) = S(\gamma,\alpha,n_{j_0}) = S$ and $n_{j|S} = n_{j_0|S}$ for all $j \geq j_0$.

First, notice that

$$S(\gamma,\alpha,n_j) = I_\gamma \cap n_j^{-1}(I_\alpha) = I_\gamma \cap (q^{-1}\theta_j[\beta]^{-1}p^{-1})(I_\alpha) = q^{-1}(q(I_\gamma) \cap \theta_j[\beta]p^{-1}(I_\alpha)).$$

Second, we have

$$\theta_j[\beta]p^{-1}(I_\alpha) = \theta_j[\beta](p^{-1}(I_\alpha) \cap I_\beta) \cup \theta_j[\beta](p^{-1}(I_\alpha) \cap I_\beta^C) = (p^{-1}(I_\alpha) \cap I_\beta) \cup \theta_j[\beta](p^{-1}(I_\alpha) \cap I_\beta^C).$$

Therefore, it is enough to prove that

$$q(I_\gamma) \cap \theta_j[\beta](p^{-1}(I_\alpha) \cap I_\beta^C) = q(I_\gamma) \cap \theta_{j_0}[\beta](p^{-1}(I_\alpha) \cap I_\beta^C)$$

for all $j \geq j_0$. In fact, we are going to prove that $q(I_\gamma) \cap \theta_j[\beta](p^{-1}(I_\alpha) \cap I_\beta^C) = \emptyset$ for all $j \geq j_0$. Consider $j_0 = \max\{m_\gamma, m_\alpha\}$, where $m_\gamma = \max q(I_\gamma)$ and $m_\alpha = \max p^{-1}(I_\alpha)$. For $i \in p^{-1}(I_\alpha) \cap I_\beta^C$ we have $\beta < i \leq m_\alpha \leq j_0 \leq j+\beta$ and therefore $\theta_j[\beta](i) = i+j \geq m_\gamma$. Hence, $\theta_j[\beta](i) \notin q(I_\gamma)$ and we conclude that $q(I_\gamma) \cap \theta_j[\beta](p^{-1}(I_\alpha) \cap I_\beta^C) = \emptyset$ for all $j \geq j_0$. This also shows that, for sufficiently large j,

$$S(\gamma, \alpha, n_j) = I_\gamma \cap q^{-1}(I_\beta \cap p^{-1}(I_\alpha)).$$

We still need to show that $n_{j|S} = n_{j_0|S}$, where $S = S(\gamma, \alpha, n_{j_0})$. Indeed, if $i \in S$ then $q(i) \in I_\beta \cap p^{-1}(I_\alpha) \subset I_\beta$ and therefore

$$n_j(i) = p\theta_j[\beta]q(i) = pq(i) = p\theta_{j_0}[\beta]q(i) = n_{j_0}(i).$$

(b) By the previous Lemma we have that

(i) $S(\beta, \alpha, p) = S(\beta, \alpha, p')$ and p and p' coincide in this set;

(ii) $S(\gamma, \beta, q) = S(\gamma, \beta, q')$ and q and q' coincide in this set.

We need to show that for sufficiently large j, the sets $S(\gamma, \alpha, p\theta_j[\beta]q)$ and $S(\gamma, \alpha, p'\theta_j[\beta]q')$ are equal and that $p\theta_j[\beta]q$ and $p'\theta_j[\beta]q'$ coincide in this set.

By the first part of the proof, there exist $j_0 \in \mathbb{N}_+$ such that, for all $j \geq j_0$,

$$S(\gamma, \alpha, p\theta_j[\beta]q) = I_\gamma \cap q^{-1}(I_\beta \cap p^{-1}(I_\alpha)) \quad \text{and} \quad S(\gamma, \alpha, p'\theta_j[\beta]q') = I_\gamma \cap q'^{-1}(I_\beta \cap p'^{-1}(I_\alpha)).$$

Let us show that these two sets are the same. Let $i \in I_\gamma \cap q^{-1}(I_\beta \cap p^{-1}(I_\alpha))$. Since $i \in I_\gamma \cap q^{-1}(I_\beta) = S(\beta, \gamma, q)$ it follows from the hypothesis that $q(i) = q'(i)$. On the other hand, $i \in q^{-1}(I_\beta \cap p^{-1}(I_\alpha))$ which implies that $q'(i) = q(i) \in I_\beta \cap p^{-1}(I_\alpha) = S(\beta, \alpha, p) = S(\beta, \alpha, p')$ and thus $i \in q'^{-1}(I_\beta \cap p'^{-1}(I_\alpha))$ (notice also that $p'q'(i) = pq(i)$). From these considerations we conclude that $i \in I_\gamma \cap q'^{-1}(I_\beta \cap p'^{-1}(I_\alpha)) = S(\gamma, \alpha, p'\theta_j[\beta]q')$ and hence $S(\gamma, \alpha, p\theta_j[\beta]q) \subset S(\gamma, \alpha, p'\theta_j[\beta]q')$. The other inclusion follows in an analogous way.

It remains to prove that $p\theta_j[\beta]q = p'\theta_j[\beta]q'$ in $S(\gamma, \alpha, p\theta_j[\beta]q)$. Again by the first part we have that, for $i \in S(\gamma, \alpha, p\theta_j[\beta]q)$, $p\theta_j[\beta](i) = pq(i)$ and $p'\theta_j[\beta]q'(i) = p'q'(i)$, but $pq(i) = p'q'(i)$ and the lemma follows.

□

Now we can define the product $\mathfrak{p} \circ \mathfrak{q}$ as

$$\mathfrak{p} \circ \mathfrak{q} = \mathfrak{r}_j \quad \text{for sufficiently large } j.$$

Furthermore, this product is associative.

Proposition 1.3.3. *The product of double cosets defined above is associative.*

Proof. Consider double cosets $\mathfrak{p} = \overline{S}_\infty[\alpha]p\overline{S}_\infty[\beta], \mathfrak{q} = \overline{S}_\infty[\beta]q\overline{S}_\infty[\gamma]$ and $\mathfrak{r} = \overline{S}_\infty[\gamma]r\overline{S}_\infty[\delta]$. Then

$$\mathfrak{p} \circ (\mathfrak{q} \circ \mathfrak{r}) = \overline{S}_\infty[\alpha]p\theta_i[\beta](q\theta_j[\gamma]r)\overline{S}_\infty[\gamma] \quad \text{and} \quad (\mathfrak{p} \circ \mathfrak{q}) \circ \mathfrak{r} = \overline{S}_\infty[\alpha](p\theta_k[\beta]q)\theta_l[\gamma]r\overline{S}_\infty[\gamma],$$

for sufficiently large $i, j, k, l \in \mathbb{N}_+$. As we have seen in the proof of the previous Lemma, we can choose i, j, k and l such that

$$S(\gamma,\delta,p\theta_i[\beta](q\theta_j[\gamma]r)) = I_\delta \cap (q\theta_j[\gamma]r)^{-1}(I_\beta \cap p^{-1}(I_\alpha)) \qquad S(\delta,\beta,q\theta_j[\gamma]r) = I_\delta \cap r^{-1}(I_\gamma \cap q^{-1}(I_\beta))$$
$$S(\gamma,\delta,(p\theta_k[\beta]q)\theta_l[\gamma]r) = I_\delta \cap r^{-1}(I_\gamma \cap (p\theta_k[\beta]q)^{-1}(I_\alpha)) \quad S(\alpha,\gamma,p\theta_k[\beta]q) = I_\gamma \cap q^{-1}(I_\beta \cap p^{-1}(I_\alpha)).$$

From the equations above it is easy to see that $S(\gamma,\delta,(p\theta_k[\beta]q)\theta_l[\gamma]r) = I_\delta \cap r^{-1}(I_\gamma \cap q^{-1}(I_\beta \cap p^{-1}(I_\alpha)))$. Now,

$$\begin{aligned} S(\gamma,\delta,p\theta_i[\beta](q\theta_j[\gamma]r)) &= I_\delta \cap (q\theta_j[\gamma]r)^{-1}(I_\beta \cap p^{-1}(I_\alpha)) = I_\delta \cap (q\theta_j[\gamma]r)^{-1}(I_\beta) \cap (q\theta_j[\gamma]r)^{-1}(p^{-1}(I_\alpha)) \\ &= I_\delta \cap r^{-1}(I_\gamma \cap q^{-1}(I_\beta)) \cap (q\theta_j[\gamma]r)^{-1}(p^{-1}(I_\alpha)) = I_\delta \cap r^{-1}(I_\gamma \cap q^{-1}(I_\beta) \cap \theta_j[\gamma]q^{-1}(p^{-1}(I_\alpha))) \\ &= I_\delta \cap r^{-1}(I_\gamma \cap q^{-1}(I_\beta) \cap q^{-1}(p^{-1}(I_\alpha))) = I_\delta \cap r^{-1}(I_\gamma \cap q^{-1}(I_\beta \cap p^{-1}(I_\alpha))) \end{aligned}$$

Hence, $S(\gamma,\delta,p\theta_i[\beta](q\theta_j[\gamma]r)) = S(\gamma,\delta,(p\theta_k[\beta]q)\theta_l[\gamma]r)$. To see that the elements $(p\theta_k[\beta]q)\theta_l[\gamma]$ and $p\theta_i[\beta](q\theta_j[\gamma]r)$ coincide in this set, it suffices to notice that, for all $t \in S(\gamma,\delta,(p\theta_k[\beta]q)\theta_l[\gamma]r)$ we have $(p\theta_k[\beta]q)\theta_l[\gamma](t) = pqr(t) = p\theta_i[\beta](q\theta_j[\gamma]r)(t)$. ☐

Remark 1.3.4. Notice that all the constructions above for the complete infinite symmetric group are still valid for the infinite symmetric group. In fact, the definition of the product and Propositions 1.3.1 and 1.3.3 are generalizations of Lemmas 3.1 and 3.2 of [32]

1.3.2 Representations of $\overline{S}_\infty$

In 1972, A. Lieberman [23] obtained a complete classification of unitary representations of the groups $\overline{S}_\infty$. This classification is uncomplicated: all irreducible representations are induced from representations $\rho \otimes I$ of subgroups $S_\alpha \times \overline{S}_\infty[\alpha]$, where ρ is irreducible and I is the trivial one-dimensional representation of $\overline{S}_\infty[\alpha]$.

Recall that a unitary representation of a topological group in a Hilbert space is a continuous homomorphism from the group to the group of all unitary operators equipped with the weak operator topology. Consider a unitary representation ρ of the group S_∞ in a Hilbert space H and let $H[\alpha] = H(\rho)^{S_\infty[\alpha]}$.

Proposition 1.3.5 (Proposition 2.1 of [32])**.** *The following conditions are equivalent:*

(i) The representation ρ admits a extension to a continuous representation of the group $\overline{S}_\infty$;

(ii) The set $\cup_\alpha H[\alpha]$ is dense in H.

Moreover, if ρ is irreducible, then these conditions are equivalent to:

(iii) The set $\cup_\alpha H[\alpha]$ is nonempty.

Denote by W_n the semigroup of 0-1 matrices of order n and by W_∞ the semigroup of infinite 0-1 matrices equipped with the weak operator topology. We have

Lemma 1.3.6 (Lemma VIII.1.1 of [29])**.** *The group S_∞ is dense in W_∞.*

For a matrix $p \in W_\infty$, denote by

$$\{p\}_n \in W_n$$

the left upper corner of size $n \times n$.

Lemma 1.3.7 (Lemma 2.3 of [32]). *The map $p \mapsto \{p\}_\beta$ induces a bijection*

$$S_\infty[\beta]\backslash S_\infty/S_\infty[\beta] \longrightarrow W_\beta.$$

Consider the following element of W_∞

$$\theta[\beta] := \begin{pmatrix} 1_\beta & 0 \\ 0 & 0_\infty \end{pmatrix},$$

and notice that

$$\theta[\beta] = \lim_{j\to\infty} \theta_j[\beta].$$

Theorem 1.3.8 (Theorem 2.4 of [32]). *Let ρ be a unitary representation of the group $\overline{S}_\infty$. Then ρ admits a unique extension to a continuous representation $\overline{\rho}$ of the semigroup W_∞. Moreover, for all $p \in W_\infty$ and $\alpha \in \mathbb{N}$, we have*

$$\overline{\rho}(p^*) = \overline{\rho}(p)^* \quad \text{and} \quad \overline{\rho}(\theta[\alpha]) = P[\alpha],$$

where $P[\alpha]$ is the operator of orthogonal projection to $H[\alpha]$.

For $r \in W_n$ consider the element

$$\pi(r) = \begin{pmatrix} r & 0 \\ 0 & 0_\infty \end{pmatrix}$$

of W_∞. Denote by W_n° the subsemigroup of W_∞ consisting of the matrices $\pi(r)$. Clearly, $\pi(r)\theta[n] = \theta[n]\pi(r) = \pi(r)$ for all $r \in W_n$. Applying $\overline{\rho}$ to this identity we get that the operator $\overline{\rho}(\pi(r))$ has the block structure

$$\begin{pmatrix} \xi(r) & 0 \\ 0 & 0 \end{pmatrix} : H[n] \oplus H[n]^\perp \to H[n] \oplus H[n]^\perp,$$

where $\xi(r)$ is an operator in $H[n]$.

Corollary 1.3.9 (Corollary 2.5 of [32]). *For any unitary representation ρ of $\overline{S}_\infty$ and each $n \geq 0$ we have a natural representation ξ_n of the semigroup W_n in the subspace $H[n]$. Furthermore, in $H[n]^\perp$ the semigroup W_n° acts by zero operators.*

Lemma 1.3.10 (Lemma 2.6 of [32]). *If the representation ρ is irreducible, then ξ_n also is irreducible.*

Given $n \in \mathbb{N}_+$ and a irreducible representation ρ of S_n in the space $L = H(\rho)$. Consider the set Ω of all n-element subsets of $\mathbb{N}_+$ and the space V of all L-valued functions on Ω. For each $s \in \overline{S}_\infty$ and $I \in \Omega$ consider the permutation $\sigma(s, I)$ defined as follows: Denote the elements of I by $i_1 < i_2 < \cdots < i_n$ and the elements of sI by $j_1 < j_2 < \cdots < j_n$. If $s(i_m) = j_l$ we set $\sigma(s, I)(m) = l$. Define an action of S_∞ in the space V by

$$T_{n,\rho}(g)f(I) = \rho(\sigma(g, I))f(gI).$$

Equip V with the ℓ_2-inner product,

$$\langle f, g\rangle = \sum_\Omega \langle f(I), g(I)\rangle_L.$$

Theorem 1.3.11 (Lieberman theorem, Theorem 2.8 of [32]). *Each unitary irreducible representation of the group $\overline{S}_\infty$ has the form $T_{n,\rho}$. Furthermore, any of its unitary representations decomposes into a direct sum of irreducible representation and this decomposition is unique.*

1.4 The group $G_3 = S_\infty \times S_\infty \times S_\infty$

1.4.1 Multiplication of double cosets

Consider $G_3 = S_\infty \times S_\infty \times S_\infty$ and denote by K the diagonal subgroup of G_3 and by $K[\alpha]$ the image of $S_\infty[\alpha]$ under the canonical isomorphism $S_\infty \to K$. We will color each component of an element $s \in G_3$ as $s = (s_r, s_g, s_b)$ (here r,g, and b stand for red, green, and blue, respectively). The symbol s_ν will denote one of these three colored permutations. Denote by the same symbol $\theta_j[\beta]$ the image of $\theta_j[\beta] \in S_\infty[\beta]$ under the isomorphism $S_\infty \to K$.

For each $\alpha, \beta, \gamma \in \mathbb{N}$ we define the multiplication of double cosets

$$K[\alpha]\backslash G_3/K[\beta] \times K[\beta]\backslash G_3/K[\gamma] \to K[\alpha]\backslash G_3/K[\gamma]$$

as above. Precisely, for double cosets $\mathfrak{p} \in K[\alpha]\backslash G_3/K[\beta]$ and $\mathfrak{q} \in K[\beta]\backslash G_3/K[\gamma]$, choose representatives $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$ and consider the following sequence of double cosets

$$\mathfrak{r}_j = K[\alpha] p \theta_j[\beta] q K[\gamma].$$

Lemma 1.4.1 (Lemma 3.1 of [32]). *The sequence $\mathfrak{r}_j$ is eventually constant. Its limit does not depend on the choice of representatives of $\mathfrak{p}$ and $\mathfrak{q}$.*

Therefore, for all pairs $(\mathfrak{p}, \mathfrak{q}) \in K[\alpha]\backslash G_3/K[\beta] \times K[\beta]\backslash G_3/K[\gamma]$ we have a well defined product $\mathfrak{p} \circ \mathfrak{q} \in K[\alpha]\backslash G_3/K[\gamma]$ given by

$$\mathfrak{p} \circ \mathfrak{q} = K[\alpha]\, p\theta_j[\beta] q\, K[\gamma],$$

$p \in \mathfrak{p}, q \in \mathfrak{q}$ and j sufficiently large.

Lemma 1.4.2 (Lemma 3.2 of [32]). *The product of double cosets is associative.*

The proofs of Lemmas 1.4.1 and 1.4.2 follow easily from Propositions 1.3.1 and 1.3.3. Upon a closer look one notices that the elements responsible by the equality $\mathfrak{r}_j = \mathfrak{r}_{j+1}$, for sufficiently large j, does not depends on p and q.

The map $g \mapsto g^{-1}$ induces an involution

$$\begin{array}{rcl} K[\alpha]\backslash G_3/K[\beta] & \to & K[\beta]\backslash G_3/K[\alpha] \\ \mathfrak{g} & \mapsto & \mathfrak{g}^*, \end{array}$$

and

$$(\mathfrak{g} \circ \mathfrak{h})^* = \mathfrak{h}^* \circ \mathfrak{g}^*.$$

Remark 1.4.3. For a group G and a subgroup $H \subset G$, we say that $g, g' \in G$ are conjugate with respect to H if there exists $h \in H$ such that $g' = hgh^{-1}$. Denote by $G/\!\!/H$ the set of conjugacy classes with respect to the subgroup H. There is a natural one-to-one correspondence between the sets $H\backslash(G^n \times H)/H$ and $G^n/\!\!/H$ (here H is the image of the subgroup $H \subset G$ by the appropriate diagonal map). In fact, it is easy to see that the function from $G^n \times H$ to G^n given by

$$(g_1, g_2, \ldots, g_n, h) \mapsto (g_1 h^{-1}, g_2 h^{-1}, \ldots, g_n h^{-1}),$$

induces a bijection between the sets $H\backslash(G^n \times H)/H$ and $G^n/\!\!/H$.

In particular, there is in a one-to-one correspondence between $\operatorname{diag} G\backslash G^n/\operatorname{diag} G$ and the set of conjugacy classes of G^{n-1} with respect to $\operatorname{diag} G$. When $G = S_\infty$ or $G = \overline{S}_\infty$ it is possible to use this correspondence to define a monoid structure on the set $G \times G/\!\!/\operatorname{diag} G$ of conjugacy classes.

1.4.2 Representations of G_3

Let ρ be a unitary representation of the group G_3 in a Hilbert space H. Let $H[\alpha] = H^{K[\alpha]}$, $P[\alpha]$ be the orthogonal projection over $H[\alpha]$ and, for each double coset $\mathfrak{p} \in K[\alpha]\backslash G_3/K[\beta]$, define $\tilde{\rho}(\mathfrak{p}) : H[\beta] \to H[\alpha]$ by

$$\tilde{\rho}(\mathfrak{p}) = P[\alpha]\rho(p)_{|H[\beta]},$$

where $p \in \mathfrak{p}$.

Notice that the operator $\tilde{\rho}(\mathfrak{p})$ does not depend on the choice of the representative p. Indeed, if $h \in K[\alpha]$ and $k \in K[\beta]$ then, for all $v \in H[\alpha]$ and $w \in H[\beta]$, we have

$$\langle v, \rho(hpk)w\rangle = \langle v, \rho(h)\rho(p)\rho(k)w\rangle = \langle \rho(h^{-1})v, \rho(p)\rho(k)w\rangle = \langle v, \rho(p)w\rangle.$$

The next theorem, the multiplicativity theorem, is a central result over the articles studied. According to Neretin [32] "In [29] there was observed that a multiplication of double cosets and a multiplicativity theorem are highly general phenomena for infinite-dimensional groups." Although a multiplicativity theorem for the infinite braid group B_∞, which will be studied on the next chapter, is not known we state the multiplicativity theorem for G_3 below and include the proof given in [32] for completion.

Theorem 1.4.4 (Multiplicativity Theorem, Theorem 3.7 of [32])**.** *For each $\alpha, \beta, \gamma \in \mathbb{N}$ and for each $\mathfrak{p} \in K[\alpha]\backslash G_3/K[\beta]$ and $\mathfrak{q} \in K[\beta]\backslash G_3/K[\gamma]$ we have*

$$\tilde{\rho}(\mathfrak{p})\tilde{\rho}(\mathfrak{q}) = \tilde{\rho}(\mathfrak{p} \circ \mathfrak{q}).$$

Moreover,

$$\tilde{\rho}(\mathfrak{p}^*) = (\tilde{\rho}(\mathfrak{p}))^*, \quad \|\tilde{\rho}(\mathfrak{p})\| \leq 1.$$

Proof. First notice that the closure V of the subspace $\cup_\beta H[\beta]$ is G_3-invariant, and therefore its orthogonal complement is also G_3-invariant. In fact, let S_α the subgroup $\tilde{S}_\alpha \times \tilde{S}_\alpha \times \tilde{S}_\alpha$ of G_3, where $\tilde{S}_\alpha$ is the subgroup of S_∞ consisting of all permutations fixing $\alpha+1, \alpha+2, \ldots$. Notice that for all $s \in S_\alpha$ and all $t \in K[\beta]$, with $\beta > \alpha$, we have $st = ts$. Let $v \in H[\alpha]$. Then, $\rho(g)v \in H[\beta]$ for all $g \in S_\alpha$. Indeed, for $s \in K[\beta]$ we have

$$\rho(s)(\rho(g)v) = \rho(sg)v = \rho(g)(\rho(s)v) = \rho(g)v.$$

Therefore, $H[\beta]$ is S_α-invariant for all $\alpha < \beta$. We claim that $\cup_\beta H[\beta]$ is S_α-invariant for all α. In fact, we have that

$$H[0] \subset H[1] \subset \cdots \subset H[\beta] \subset H[\beta+1] \subset \cdots$$

Let $g \in S_\alpha$ for some α and $v \in \cup_\beta H[\beta]$. Then, for sufficiently large β, we have $v \in H[\beta]$ with $\beta > \alpha$. Hence $\rho(g)v \in H[\beta] \subset \cup H[\beta]$. Since $G_3 = \cup_\alpha S_\alpha$ we conclude that $\cup H[\beta]$ is G_3-invariant. By continuity we conclude that V is G-invariant. Now, let $w \in V^\perp$. Then, for all $g \in G_3$ and $v \in V$ we have

$$\langle \rho(g)w, v\rangle = \langle w, \rho(g)^*v\rangle = \langle w, \rho(g^{-1})v\rangle = 0.$$

Therefore $\rho(g)w \in V^\perp$ for all $g \in V^\perp$.

Since V and $V^\perp$ are G_3-invariant, the operators $\tilde{\rho}(\mathfrak{p})$ depend only on the restriction of ρ to the subspace V. Therefore, without loss of generality, we can assume that $V = H$.

By Proposition 1.3.5 the representation of the subgroup $K \simeq S_\infty$ in H extends continuously to the complete symmetric group $\overline{S}_\infty$. Now, applying Theorem 1.3.8 we see that $\rho(\theta_j[\beta])$ converges weakly to $P[\beta]$. Choose representatives p and q of the double cosets $\mathfrak{p}$ and $\mathfrak{q}$ respectively. For sufficiently large j we have that $P[\alpha]\rho(p\theta_j[\beta]q)P[\gamma]$ is constant and therefore

$$\begin{aligned}\tilde{\rho}(\mathfrak{p}\circ\mathfrak{q})P[\gamma] &= P[\alpha]\rho(p\theta_j[\beta]q)P[\gamma] = \lim_{k\to\infty} P[\alpha]\rho(p\theta_k[\beta]q)P[\gamma] \\ &= P[\alpha]\rho(p)\lim_{k\to\infty}\rho(\theta_j[\beta])\rho(q)P[\gamma] = P[\alpha]\rho(p)P[\beta]\rho(q)P[\gamma] = \tilde{\rho}(\mathfrak{p})\tilde{\rho}(\mathfrak{q})P[\gamma].\end{aligned}$$

The limits means the weak operator limit. The other statements follow easily from the definition of $\tilde{\rho}$. □

The next theorem shows how we can use the representations $\tilde{\rho}$ to prove that the pair (G_3, K) is spherical. We also provide the proof found in [32] in more detail.

Theorem 1.4.5 (Sphericity, Theorem 3.8 of [32])**.** *The pair (G_3, K) is spherical.*

Proof. We have seen that every unitary representation ρ of G in a Hilbert space H induces an unitary representation $\tilde{\rho}$ of $K\backslash G/K$ in $H[0]$. To prove the theorem it suffices to show that, if ρ is irreducible, then $H[0]$ has dimension at most one. Suppose that ρ is irreducible. Then, by the proof of Lemma 1.3.10, $\tilde{\rho}$ is also irreducible. Now, since $K\backslash G/K$ is commutative (see Remark 1.5.4) and $\tilde{\rho}$ is compatible with the involution, there exists an abelian group J and an irreducible unitary representation $\bar{\rho}$ of J in $H[0]$ such that the following diagram commutes

$$\begin{tikzcd} K\backslash G/K \arrow[r, "\tilde{\rho}"] \arrow[d] & GL(H[0]) \\ J \arrow[ur, "\bar{\rho}"'] & \end{tikzcd}.$$

Since every irreducible unitary representation of an abelian group is one-dimensional, it follows that $\tilde{\rho}$ is one-dimensional, which proves the theorem. □

1.4.3 Several copies of the infinite symmetric group and the n-symmetric group

Consider the product $G_n = S_\infty \times \cdots \times S_\infty$ of n copies of the infinite symmetric group. denote by K the diagonal subgroup $\operatorname{diag} G_n$ and by $K[\alpha]$ the image of the subgroup $S_\infty[\alpha]$ by the canonical isomorphism $S_\infty \to K$. In this setting we can repeat the construction of the multiplication on G_3 to obtain a well-defined associative multiplication

$$K[\alpha]\backslash G_n/K[\beta] \times K[\beta]\backslash G_n/K[\gamma] \longrightarrow K[\alpha]\backslash G_n/K[\gamma]$$

on G_n.

We define the n-symmetric group $\mathbb{G}_n$ as the subgroup of $\overline{S}_\infty^n$ consisting of all the elements $(g_1, g_2, \ldots, g_n)$ such that

$$g_i g_j^{-1} \in S_\infty \text{ for all } i \neq j.$$

In other words, $\mathbb{G}_n$ is the subgroup of $\overline{S}_\infty^n$ generated by G_n and the diagonal $\mathbb{K}$ of $\overline{S}_\infty^n$. We define a topology in $\mathbb{G}_n$ by requiring that the subgroups $K[\alpha]$ to be open. The quotient space $\mathbb{G}_n/\mathbb{K}$ is countable and equipped with the discrete topology.

As we did before, define the family of subgroups $\mathbb{K}[\alpha]$ of $\mathbb{K}$. The multiplication of double cosets is well defined and the multiplicativity theorem still holds for these groups (see Section 4 of [28]). Moreover, the natural map

$$K[\alpha]\backslash G_n/K[\beta] \longrightarrow \mathbb{K}[\alpha]\backslash \mathbb{G}_n/\mathbb{K}[\beta]$$

is a bijection.

For the trisymmetric group $\mathbb{G}_3$ we have the following result

Lemma 1.4.6 (Lemma 3.10 of [32]). *Let ρ be a unitary representation of the group G_3 in a Hilbert space H.*

(a) The representation ρ admits a continuous extension to the group $\mathbb{G}$ if and only if $\rho_{|K}$ admits a continuous extension to $\mathbb{K}$.

(b) The representation ρ admits a continuous extension to $\mathbb{G}$ if and only if $\cup H[\alpha]$ is dense in H.

Remark 1.4.7. The bisymmetric group. Throughout the articles studied the central object is the trisymmetric group. A representation for the bisymmetric group also exists. The reason for the interest in the trisymmetric group is given by Neretin in [27] as follows:

"The existing representation theory of infinite symmetric groups is mainly the representation theory of the bisymmetric group $\mathbb{G}_2$, see [44], Vershik, Kerov [46], Olshanski [41], Okounkov [35], Kerov, Olshanski, Vershik [21]. The situation was explained by Olshanski in [41]. . . . The group $\mathbb{G}_n$ is outside Olshanski's approach to infinite-dimensional groups, based on imitation of symmetric pairs, see Olshanski [37,41], and also [29]. However, $\mathbb{G}_n$ is a (G, K)-pair in the sense of [29], VIII.5."

1.5 Graphical calculus for the group G_3

1.5.1 Checker surfaces and the product of double cosets

A checker-board is a countable disjoint union of triangulated closed surfaces equipped with the following data:

1. Triangles are colored black and white, neighboring triangles have different colors.
2. Edges are colored red, green and blue. The boundary of any black triangle is colored red-green-blue anti-clockwise; the boundary of a white triangle is colored red-green-blue clockwise.
3. Black (respectively, white) triangles are enumerated by $\mathbb{N}_+$.
4. All but finitely many components are unions of only two triangles, called double-triangles.

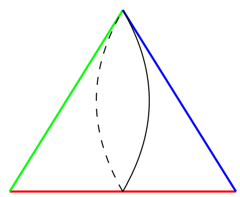


Figure 1.1: A double-triangle

Dually, a checker-board is a triple (P, γ, i) where P is an oriented compact surface, γ is a finite graph in P separating it into triangles and i is a labeling of the set of triangles by $\mathbb{Z}^*$. All but finitely many components are unions of only two triangles, called double-triangles. The edges of the graph are colored red, green and blue in such a way that every triangle has edges of the three colors. We say that a triangle is white if its edges are colored red-green-blue clockwise and black otherwise. Furthermore, labels of black triangles are negative, and of white triangles are positive. We denote the set of all checker-boards by Ξ.

Two checker-boards (P, γ) and (P, γ') are *equivalent* if there is an orientation preserving homeomorphism $P \to P'$ identifying the colored graphs γ and γ'.

Checker boards are useful for the study of the set of double cosets of G_n as we defined earlier. As we will see in Remark 1.5.4 they can be used to prove that the monoide $K[0]\backslash G_3/K[0]$ is commutative and in Theorem 1.6.3 we give an expression for a class of spherical functions of G_3 using these objects. Furthermore, in Chapter 3 we show how checker boards are closely related to branched covers over the sphere (Propositions 3.2.1 and 3.2.2).

1.5.1.1 From permutations to checker-boards

We can use checker-boards to give a graphical interpretation of the group G_3 and its double cosets. We start by establishing a correspondence between the elements of G_3 and the checker-boards. Let

$$p = (p_r, p_g, p_b)$$

denote an element of G. We construct a checker-surface T_p associated to it. Consider a collection of *black* triangles $\{B_j\}_{j\in\mathbb{N}_+}$ and color its sides red, green and blue anti-clockwise. In the same fashion consider a collection of *white* triangles $\{W_j\}_{j\in\mathbb{N}_+}$ and color its sides red, green and blue clockwise. We glue a simplicial complex from these triangles: if $p_r(i) = j$ we glue the red side of the B_i to the red side of W_j; repeat this operation for all $i \in \mathbb{N}_+$ and for the other two permutations p_g and p_b. In this way, we get a disjoint union of 2-dimensional compact closed triangulated surfaces. Notice that only a finite number of components are not the gluing of only two triangles (that is, a double-triangle or sphere).

The inverse construction is also possible. For a checker-board T, we define the permutation $t = (t_r, t_g, t_b)$ as follows: for $i \in \mathbb{N}_+$ let W_i be the white triangle labeled i. It has a common red edge with a black triangle labeled j. We set $t_r(i) = j$. To obtain t we repeat this procedure for every color and every triangle of T.

Proposition 1.5.1 (Theorem 3.3 de [32])**.** *There is a canonical one-to-one correspondence between G_3 and the set of all checker-boards.*

1.5.1.2 Multiplication of checker-boards

Consider two elements,

$$a, b \in \Xi.$$

For each $j \in \mathbb{N}_+$ we cut off the interior of the black triangle of a numbered j and the interior of the white triangle of b numbered j and identify their edges respecting colors and orientation. We obtain a checker-board, but some parts of the surface are glued only by the vertices. To obtain a proper checker surface we equip this object with a natural metric by assuming that all the triangles are equilateral. We proceed to remove all vertices and complete our space with respect to this metric. Denoted this product checker surface by $b \circ a$.

Its easy to see that if $a = T_x$ and $b = T_y$ then $T_y \circ T_x$ is equivalent to T_{yx} and thus the correspondence defined above is in fact an isomorphism.

Remark 1.5.2. (1) If $g \in K$ then Ξ consists of double triangles.

(2) Replacing the infinite families of triangles by finite ones with n triangles we get a correspondence between the group $S_n \times S_n \times S_n$ and the set of equipped surfaces with $2n$-triangles.

(3) Every vertex is incident to edges of two colors only and this edges are 'interlaced'. Colour each vertex with the complementary color to that of the edges. Then we get an $1-1$ correspondence between the red vertices and the cycles of $s_g^{-1}s_b$.

(4) Let Γ_s be the subgroup of S_∞ generated by $s_g^{-1}s_b$ and $s_b^{-1}s_r$. Then the components of the surface associated to s are in one-to-one correspondence with the orbits of Γ_s.

(5) Let $s \in G$ and $h \in K$. The operation of right multiplication $s \mapsto sh$ is reduced to a permutation of $+$ labels on the equipped surface. The same can be said for left multiplication and the $-$ labels.

1.5.1.3 Double cosets and checker-boards

Since we understand completely the behavior of left and right multiplication by elements of K, we can use the construction above to give an interpretation for the double cosets $K[\alpha]\backslash G_3/K[\beta]$. An (α, β)-board is a compact oriented triangulated surface equipped with the following data:

1. Triangles are colored black and white, neighboring triangles have different colors.
2. Edges are colored red, green and blue. The boundary of any black triangle is colored red-green-blue anti-clockwise; the boundary of a white triangle is colored red-green-blue clockwise.
3. There are β distinguished black triangles, enumerated $1, 2, \ldots, \beta$, and α distinguished white triangles, enumerated $1, 2, \ldots, \alpha$.
4. There are no double-triangles without distinguished triangles.

We can obtain an (α, β)-board from a checker-board by erasing labels greater than β from black triangles and labels greater than α from white triangles and deleting double triangles without labels. We denote the set of all (α, β)-boards by $\mathcal{K}(\alpha, \beta)$.

1.5.1.4 From double cosets to (α, β)-boards

Take $\mathfrak{p} \in K(\alpha)\backslash G_3/K(\beta)$ a double coset and let $p \in G_3$ be one of its representatives. Following the construction above we get the checker-board corresponding to p. The (α, β)-board corresponding to $\mathfrak{p}$ is obtained by (i) erasing the numbers greater than β from black triangles and those greater than α from white triangles. (ii) removing the double-triangles without any numbers.

The inverse construction is also possible in this context, by adding an infinite family of double-triangles and completing the labeling to obtain a representative of the double coset.

Proposition 1.5.3 (Theorem 3.4 of [32]). *There is a canonical one-to-one correspondence between the set $K(\alpha)\backslash G_3/K(\beta)$ and $\mathcal{K}(\alpha,\beta)$.*

1.5.1.5 Multiplication of (α,β)-boards

Consider two elements,

$$\mathfrak{a} \in \mathcal{K}(\alpha,\beta), \quad \mathfrak{b} \in \mathcal{K}(\beta,\gamma).$$

For each $j \in \{1,2,\ldots,\beta\}$ we cut off the interior of the black triangle of $\mathfrak{a}$ numbered j and the interior of the white triangle of $\mathfrak{b}$ numbered j and identify their edges respecting colors and orientation (if some parts of the resulting object are glued only by the vertices we cut them following the procedure described for the product of checker-surfaces). We obtain an (α,γ)-board denoted $\mathfrak{a}\circ\mathfrak{b}$.

This operation corresponds to the multiplication of double cosets. In fact, consider two elements $p,q \in G_3$ such that their respective associated surfaces Ω_p and Ω_q coincide with $\mathfrak{a}$ and $\mathfrak{b}$ by taking (α,β)-boards, respectively. Consider $\mathfrak{p} = K[\alpha]pK[\beta]$ and $\mathfrak{q} = K[\beta]qK[\gamma]$ the double cosets associated to p and q respectively. Then,

$$\mathfrak{p}\circ\mathfrak{q} = K[\alpha]\, p\theta_j[\beta]q\, K[\gamma]$$

for sufficiently large j. Since $\theta_j[\beta] \in K[\beta]$ we have that the surface $\Omega_{\theta_j[\beta]q}$ associated to $\theta_j[\beta]q$ is obtained from Ω_q by permuting the minus-labels on triangles. Furthermore, only labels greater than β are permuted. Now, gluing the surfaces $\Omega_{\theta_j[\beta]q}$ and Ω_p in the same fashion described above we obtain a surface Ω_r corresponding to the product $p\theta_j[\beta]q$. Passing to $\mathcal{K}(\alpha,\gamma)$ we have a surface $\widehat{\Omega}_r$. We claim that this surface is the same as $\mathfrak{a}\circ\mathfrak{b}$. In fact, for j sufficiently large, all minus-triangles of $\Omega_{\theta_j[\beta]q}$ with labels greater than β are glued to double triangles of Ω_p and all plus-triangles of Ω_p with labels greater than β are glued to double triangles of $\Omega_{\theta_j[\beta]q}$. Since gluing double triangles does not affect the surface, we have that Ω_r is the same surface as the one obtained by gluing only the triangles with labels not greater than β and permuting the labels of the other triangles. The labels on reglued triangles will be forgotten after passing to $\mathcal{K}(\alpha,\gamma)$.

Thus we get a category $\mathcal{M}$, whose objects are $0,1,2\ldots$ and morphisms $\beta\to\alpha$ are (α,β)-boards.

Remark 1.5.4. The multiplication in the semigroup $K[0]\backslash G/K[0]$ is the operation of disjoint union of surfaces. In particular, this semigroup is commutative.

Proposition 1.5.5 (Proposition 3.6 of [32]). *The involution $\mathfrak{p}\mapsto\mathfrak{p}^*$ corresponds to the change of orientation of the surface and inverting the sign on all labels.*

1.5.2 Extension to G_n

1.5.2.1 n-sided polygons

As we saw previously, we have a well-defined operation on the set of double cosets of G_n with respect to the subgroups $K[\alpha]$. All the constructions and results above may be extended to G^n in the natural way, considering n-gons instead of triangles. Notice that we must fix a cyclical order on the set of copies of S_∞. A permutation of the order of the copies leads to an equivalent theory, but the surface corresponding to certain element may change. We can apply the general construction to the case $n=2$ to get a surface glued from digons. But in this case there is a simpler language (the language of chips, see [32]). Lastly, these constructions are also valid, without any changes, for the n-symmetric group $\mathbb{G}_n$.

1.5.2.2 Pseudomanifolds

We can also extend these constructions to G_n using higher-dimensional objects by the introduction of pseudomanifolds (see [14]).

Consider a disjoint union $\sqcup J_k$ of a finite collection of simplices J_k. We consider a topological quotient space Σ of $\sqcup J_k$ satisfying the following properties

(a) For any simplex J_i the quotient map $\xi\colon J_i \to \Sigma$ is an embedding.

(b) For any pair of distinct simplices J_i, J_k the intersection $\xi_i^{-1}(\xi_i(J_i) \cap \xi_k(J_k))$ is a union of faces of J_i, and the partially defined map

$$J_i \xrightarrow{\xi_i} \Sigma \xrightarrow{\xi_k^{-1}} J_k$$

is affine on each face.

We shall call such quotients simplicial cell complexes.

A pseudomanifold of dimension n is a simplicial cell complex such that

(a) Each face is contained in an n-dimensional face. We call n-dimensional faces chambers.

(b) Each $(n-1)$-dimensional face is contained in precisely two chambers.

Let Σ be a pseudomanifold and Γ its k-dimensional face. Consider all $(k+1)$-dimensional faces Φ_j of Σ containing Γ and choose a point ϕ_j in the relative interior of each face Φ_j. For each face Ψ_m (of any dimension) containing Γ consider the convex hull of all points ϕ_j that are contained in Ψ_m. We define the link of Γ to be the simplicial cell complex whose faces are such convex hulls.

Definition 1.5.6. *A pseudomanifold is normal if the link of any face of codimension greater than one is connected.*

Proposition 1.5.7 (see Section 4.1 of [18])**.** *For any pseudomanifold Σ there is a unique normalization, i.e. a normal pseudomanifold $\tilde{\Sigma}$ and a map $\pi\colon \tilde{\Sigma} \to \Sigma$ such that*

- *The restriction of π to any face of $\tilde{\Sigma}$ is an affine bijective map of faces.*
- *The map π send different n-dimensional and $(n-1)$-dimensional faces to different faces.*

We can give a coloring to a n-dimensional normal pseudomanifold as follows:

1. To any chamber we assign a sign $(+)$ or $(-)$. Chambers adjacent to plus-chambers are minus-chambers and vice-versa.
2. Choose $n+1$ colors. Each vertex of the complex is colored in such a way that the colors of the vertices of each chamber are pairwise different.
3. All $(n-1)$-dimensional faces are colored and in such a way that the colors of faces of a chamber are pairwise different, and the color of a face coincides with the color of the opposite vertex of any chamber containing this face.

Fix $n \geq 1$. We define the category PsBor_n of colored pseudobordisms. Its objects are the non-negative integers. A morphism $\beta \to \alpha$ is a colored n-dimensional pseudomanifold together with the following data: An injective map of the set $\{1, 2, \dots, \alpha\}$ to the set of plus-chambers and an injective map of the set $\{1, 2, \dots, \beta\}$ to the set of minus-chambers. In other words, we assign labels $1, 2, \dots, \alpha$ to some plus-chambers and labels $1, 2, \dots, \beta$ to some minus-chambers.

As was the case for triangles, there is a one-to-one correspondence between the category PsBor_{n-1} and the category of double cosets of G_n with respect to the subgroups $K[\alpha]$. For an element $s = (s_1, \dots, s_n) \in G_n$ the associated pseudomanifold is constructed as follows: Consider the families $\{P_j\}_{j\in\mathbb{N}_+}$ of colored positive chambers and $\{N_j\}_{j\in\mathbb{N}_+}$ of colored negative chambers. If $s_1(\alpha) = \beta$, then glue the chamber P_α to the chamber N_β along the color of this permutation, preserving the coloring of the vertices. The same is done for all remaining colors.

For two pseudobordisms $\Sigma \in \mathrm{Hom}(\beta, \alpha)$ and $\Gamma \in \mathrm{Hom}(\gamma, \beta)$ we define their composition $\Sigma\Gamma$ as follows: Remove the interiors of the labeled minus-chambers of Σ and the interiors of the labeled plus-chambers of Γ. For each $s \leq \beta$, we glue the boundaries of the minus-chambers of Σ with label s with the boundary of the plus-chambers of Γ with label s according the simplicial structure of the boundaries and colorings of $(n-1)$-simplices. Finally, we normalize the resulting pseudomanifold and remove label-less double chambers.

1.6 Representations of the trisymmetric group

Recall that the trisymmetric group is the subgroup $\mathbb{G}_3$ of $\overline{S}_\infty \times \overline{S}_\infty \times \overline{S}_\infty$ consisting of all the triples (s_r, s_g, s_b) such that $s_r s_g^{-1}, s_g s_b^{-1} \in S_\infty$. In other words, $\mathbb{G}_3$ is the subgroup of $\overline{S}_\infty^3$ generated by G_3 and the diagonal $\mathbb{K}$ of $\overline{S}_\infty^3$. In this chapter we show how to construct some families of representations for the trisymmetric, and n-symmetric, groups. In particular, we show that this family induces spherical functions (as defined in Section 1.2) on these groups.

1.6.1 Countable product of Hilbert spaces

For the next sections we are going to make use of countable products of Hilbert spaces.

Let $\{H_j\}_{j=0}^\infty$ be a countable family of Hilbert spaces. For each j fix a unit vector $\xi \in H_j$ and choose an orthonormal basis $\{e_k^{(j)}\}$ of H_j such that $e_1^{(j)} = \xi_j$.

Consider the Hilbert space

$$H := \bigotimes_{j=1}^{\infty} (H_j, \xi_j),$$

whose basis is formed by all formal vectors

$$e_{k_1}^{(1)} \otimes e_{k_2}^{(2)} \otimes e_{k_3}^{(3)} \otimes \cdots ,$$

where the sequences (k_i) are eventually constant equal to 1.

For a sequence $v^{(j)} \in H_j$, we define the vector

$$v^{(1)} \otimes v^{(2)} \otimes v^{(3)} \otimes \cdots$$

by means of expanding it on the basis of H. It is an element of H if and only if the products

$$\prod_{j=1}^{\infty} \|v^{(j)}\|_{H_j} \quad \text{and} \quad \prod_{j=1}^{\infty} \langle v^{(j)}, \xi_j \rangle_{H_j}$$

converge.

Under these conditions, we have

$$\langle \otimes_{j=1}^{\infty} v^{(j)}, \otimes_{j=1}^{\infty} w^{(j)} \rangle = \prod_{j=1}^{\infty} \langle v^{(j)}, w^{(j)} \rangle_{H_j}.$$

The construction does not depend on the choice of basis of H_j, but depends on the distinguished vector ξ_j.

Consider three Hilbert spaces V_r, V_y and V_b. Let $X := V_r \otimes V_y \otimes V_b$ be their tensor product and $\xi \in X$ a unity vector. The tensor product

$$H := \bigotimes_{j=1}^{\infty} (X, \xi).$$

Also denote

$$\mathcal{E} := \xi^{\otimes \infty} \in H.$$

Define an action of $\mathbb{G}_3$ in H as follows: The first copy of S_∞ interchanges red factors V_r, the second copy interchanges yellow factors V_y, the third copy interchanges blue factors V_b and the diagonal subgroup $\mathbb{K} = \overline{S}_\infty$ acts by permutations of the factors (X, ξ).

Remark 1.6.1. (1) In general, we cannot extend the action of the red copy to an action of $\overline{S}_\infty$.

(2) The vector $\mathcal{E}$ is a unique $\mathbb{K}$-fixed vector. Indeed, complete the vector ξ to an orthonormal basis $\{\xi, r_2, r_3 \dots\}$ of X. Let h be a $\mathbb{K}$-fixed vector and $\beta_{(i_j)}$ be the coefficients of its expansion in the basis $\{\otimes_{j=1}^{\infty} r_{i_j}\}$. Thus $\beta_{(i_j)}$ does not change under permutation of subscripts. Now, for each $\beta_{(i_j)}$, if not all i_j are equal to one, then we get a countable number of equal coefficients and therefore they must be zero.

(3) In a similar way $H[\alpha]$ is given by

$$\underbrace{(X, \xi) \otimes \cdots \otimes (X, \xi)}_{\alpha\text{-times}} \otimes \xi \otimes \xi \otimes \cdots$$

(4) Denote by $U(V)$ the group of all unitary operators in a Hilbert space V. Consider in $X := U(V_r) \times U(V_y) \times U(V_b)$ the subgroup Q fixing ξ. Then Q acts on each factor of the product $X \otimes X \otimes \cdots$ and therefore on the whole tensor product. This actions commutes with the representation of $\mathbb{G}_3$.

Proposition 1.6.2 (Proposition 3.14 of [32])**.** *The cyclic $\mathbb{G}_3$-span of the vector $\mathcal{E}$ is an irreducible representation of $\mathbb{G}_3$.*

The action defined above induces, for each choice of $\xi \in X$, a representation ρ_ξ of $\mathbb{G}_3$ into H. Consider the family of spherical functions

$$\mathcal{N}_\xi : \mathbb{G}_3 \to \mathbb{C},$$

given by

$$\mathcal{N}_\xi(\mathfrak{g}) = \langle \rho(g)\mathcal{E}, \mathcal{E} \rangle,$$

where g is a representative of $\mathfrak{g} \in \mathbb{K}\backslash\mathbb{G}_3/\mathbb{K}$.

These spherical functions have an explicit expression making use of the checker boards defined on Section 1.5 as we will see in Theorem 1.6.3. We provide the proof for this theorem given in [32] for completion.

Let $\{e_i^r\}, \{e_j^y\}, \{e_k^b\}$ be orthonormal basis of V_r, V_y and V_b respectively and

$$\xi = \sum \alpha_{i,j,k}(e_i^r \otimes e_j^y \otimes e_k^b).$$

For $\mathfrak{g} \in \mathbb{K}\backslash\mathbb{G}_3/\mathbb{K}$ consider the corresponding $(0,0)$-board $\mathfrak{G} \in \mathcal{K}[0,0]$. For each edge with color ν we assign a basis vector e_i^ν. We call such data an assignment. Fix an assignment ε. For each triangle Δ we denote by $i_r(\Delta), i_y(\Delta)$ and $i_b(\Delta)$ the index of the basis vectors on its sides.

Theorem 1.6.3 (Theorem 3.15 of [32]). *Let $\mathfrak{g} \in \mathbb{K}\backslash\mathbb{G}_3/\mathbb{K}$. If δ^+ and δ^- denotes the set of plus-triangles and the set of minus-triangles of $\mathfrak{g}$, respectively. Then*

$$\mathcal{N}_\xi(\mathfrak{g}) = \sum_\omega \left(\left[\prod_{\Delta\in\delta^+} \alpha_{i_r(\Delta),i_y(\Delta),i_b(\Delta)} \right] \left[\prod_{\Delta\in\delta^-} \overline{\alpha_{i_r(\Delta),i_y(\Delta),i_b(\Delta)}} \right] \right),$$

where ω ranges over all possible assignments

Proof. We choose a representative $g \in \mathbb{G}_3$ of the coset $\mathfrak{g}$ which is finitely supported, let us say $g \in S_N \times S_N \times S_N$. Then it suffices to consider the representation of $S_N \times S_N \times S_N$ in $X^{\otimes N}$ and evaluate $\langle \rho(g)\xi^{\otimes N}, \xi^{\otimes N}\rangle$. For brevity, denote by p, q and r the permutations of g and $(x_i), (y_j)$ and (z_k) the basis of V_r, V_y and V_b respectively. In this notation,

$$\mathcal{E} = \sum_{(i_n),(j_n),(k_n)} \left(\prod_{m\le N} \alpha_{i_m,j_m,k_m} \right) (x_{i_1} \otimes y_{j_1} \otimes z_{k_1}) \otimes (x_{i_2} \otimes y_{j_2} \otimes z_{k_2}) \otimes \cdots \tag{1.1}$$

and

$$\rho(g)\mathcal{E} = \sum_{(i_n),(j_n),(k_n)} \left(\prod_{m\le N} \alpha_{i_m,j_m,k_m} \right) \left(x_{i_{p(1)}} \otimes y_{j_{q(1)}} \otimes z_{k_{r(1)}} \right) \otimes \left(x_{i_{p(2)}} \otimes y_{j_{q(2)}} \otimes z_{k_{r(2)}} \right) \otimes \cdots. \tag{1.2}$$

Now, consider the basis vectors $(x_{i_1} \otimes y_{j_1} \otimes z_{k_1}) \otimes (x_{i_2} \otimes y_{j_2} \otimes z_{k_2}) \otimes \cdots$. Then the coefficients of $\mathcal{E}$ and $\rho(g)\mathcal{E}$ are, respectively,

$$\prod_m \alpha_{i_m,j_m,k_m} \quad \text{and} \quad \prod_m \alpha_{i_{p^{-1}(m)},j_{q^{-1}(m)},k_{r^{-1}(m)}},$$

where $(i_m), (j_m)$ and (k_m) are sequences in $\{1, \ldots, N\}$.

Thus, summands are enumerated by collections of numbers written on edges. Evaluating the inner product of (1.1) and (1.2) we obtain the desired result. $\square$

1.6.1.1 Super-tensor products

The previous constructions admit an extension to the super setting.

Let $V = V^{\bar{0}} \oplus V^{\bar{1}}$ be a linear space decomposed as a sum of an even and an odd parts, we call such an object a super-space. Define the tensor square of V in the usual way. But we change the operator of transposition of summands to

$$(v^{\bar{0}} + v^{\bar{1}}) \otimes (w^{\bar{0}} + w^{\bar{1}}) = (v^{\bar{0}} \otimes w^{\bar{0}}) + (v^{\bar{0}} \otimes w^{\bar{1}}) + (v^{\bar{1}} \otimes w^{\bar{0}}) + (v^{\bar{1}} \otimes w^{\bar{1}}) \mapsto$$
$$\mapsto (w^{\bar{0}} \otimes v^{\bar{0}}) + (w^{\bar{0}} \otimes v^{\bar{1}}) + (w^{\bar{1}} \otimes v^{\bar{0}}) - (w^{\bar{1}} \otimes v^{\bar{1}})$$

Having actions of transpositions $(1\,2), (2\,3), \dots$ we can define an action of the symmetric group S_n in a tensor power $V^{\otimes n}$. A tensor product equipped with such an action is called a super-tensor product.

Now, let V_r, V_y and V_b be super-spaces. We define a structure of super-space on X in an obvious way

$$X^{(0)} := (V_r^{(0)} \otimes V_y^{(0)} \otimes V_b^{(0)}) \oplus (V_r^{(1)} \otimes V_y^{(1)} \otimes V_b^{(0)}) \oplus (V_r^{(1)} \otimes V_y^{(0)} \otimes V_b^{(1)}) \oplus (V_r^{(0)} \otimes V_y^{(1)} \otimes V_b^{(1)})$$

and $X^{(1)}$ is the orthogonal complement of $X^{(0)}$.

Choose a unit vector $\xi \in X^{(0)}$ and consider the super-tensor product $H = \bigotimes_{i=0}^{\infty}(X, \xi)$. The complete symmetric group $\mathbb{K} = \overline{S}_\infty$ acts in H by permutations of factors. Indeed, vectors of type $\otimes_{j=1}^{\infty} v_j$, where v_j are purely even or odd and $v_j = \xi$ for all but finitely many indices j, form a total system in H.

On the other hand an action of an infinite permutation in such a vector is well defined. The action of the three copies of the symmetric group S_∞ is defined as above. As above, $\xi^{\otimes\infty}$ is the unique $\mathbb{K}$-fixed vector, and its cyclic span is an irreducible representation.

1.6.1.2 The Fock representation of the group of isometries of a Hilbert space

By $\mathrm{Isom}(V)$ we denote the group of isometries of a real Hilbert space V. Isometries have the form $U + t_h$ where U is an orthogonal operator (we denote the group of all orthogonal operators by $\mathrm{O}(V)$) and t_h is the translation by the element $h \in V$.

Lemma 1.6.4 (Lemma 3.16 of [32])**.** *Let V be a real Hilbert space. Then there exists a Hilbert space $F(V)$ (the Fock space) and a total system of vectors $\psi_v \in F(V)$, where v ranges in V, such that*

$$\langle \psi_v, \psi_w \rangle_{F(V)} = e^{-\frac{\|v-w\|^2}{2}}.$$

Let R be an isometry of the space V. Consider the map of the system ψ_v to itself determined by

$$\psi_v \mapsto \psi_{Rv}.$$

Then $\langle \psi_{Rv}, \psi_{Rw} \rangle = \langle \psi_v, \psi_w \rangle$, therefore the map can be extended to a certain unitary operator $\sigma(R)$ in the space $F(V)$. As a result we get a unitary representation of the group $\mathrm{Isom}(V)$.

Remark 1.6.5. The space $F(V)$ can be realized as the space of holomorphic functions on the complexification of the space V.

It is easy to see that this representation is $\mathrm{O}(V)$-spherical, the $\mathrm{O}(V)$-fixed vector is ψ_0 and the spherical function is equals to $e^{-\frac{\|h\|^2}{2}}$.

We want to embed $\mathbb{G}_3$ in $\mathrm{Isom}(V)$ for some real Hilbert space V and obtain a representation of $\mathbb{G}_3$ in the Fock space.

As V we take the tensor product $\ell_2 \otimes \ell_2 \otimes \ell_2$. In this space we have a tautological representation $\pi_1 \otimes \pi_2 \otimes \pi_3$ of the direct product $\overline{S}_\infty \otimes \overline{S}_\infty \otimes \overline{S}_\infty$. Consider the formal expression

$$\mathbf{u} := \sum_j e_j \otimes e_j \otimes e_j,$$

which is not an element of $\ell_2 \otimes \ell_2 \otimes \ell_2$, and the following transformation of the space $\ell_2 \otimes \ell_2 \otimes \ell_2$:

$$R_t(g_1, g_2, g_3)v = \pi_1(g_1) \otimes \pi_2(g_2) \otimes \pi_3(g_3)v + t[\pi_1(g_1) \otimes \pi_2(g_2) \otimes \pi_3(g_3)\mathbf{u} - \mathbf{u}],$$

where t is a real parameter. First notice that the expression in the square brackets vanishes if $g_1 = g_2 = g_3$, and it is a element of $\ell_2 \otimes \ell_2 \otimes \ell_2$ if $(g_1, g_2, g_3) \in G_3$. Therefore, for $(g_1, g_2, g_3) \in \mathbb{G}_3$, this expression is contained in $\ell_2 \otimes \ell_2 \otimes \ell_2$. Hence the affine isometric transformations R_t are well defined in $\ell_2 \otimes \ell_2 \otimes \ell_2$. Second, we get an action of the group $\mathbb{G}_3$.

Restricting the Fock representation of the group $\mathrm{Isom}(\ell_2 \otimes \ell_2 \otimes \ell_2)$ to the subgroup $\mathbb{G}_3$, we get a series of representations of the group $\mathbb{G}_3$, denote these representations by $\upsilon_t^{1,1,1}$. Next we have a representation of $\mathbb{G}_3$ in $\ell_2(\mathbb{G}_3/\mathbb{K}) = \ell_2(G_3/K)$. It is spherical and it is reasonable to understand it as $\upsilon_\infty^{1,1,1}$.

The construction can be varied. Consider the action of $\overline{S}_\infty \otimes \overline{S}_\infty \otimes \overline{S}_\infty$ in $\ell_2 \otimes \ell_2$, where the first and the second factors act in the tautological way, and the third factor in the trivial way. We consider the expression $\sum_j e_j \otimes e_j$ and repeat the same construction to obtain a series of representations $\upsilon_s^{1,1,0}$. We can repeat this procedure omitting the first factor or the second factor to obtain representations $\upsilon_t^{0,1,1}$ and $\upsilon_t^{1,0,1}$.

1.6.2 The *train* category and its representation

Consider the category $\mathcal{K}(G_3, K)$, the train of the pair (G_3, K), whose objects are nonnegative integers and morphisms are double cosets, $\mathrm{Hom}(\beta, \alpha) = K[\alpha]\backslash G_3/K[\beta]$. Consider its $*$-representation R. This means that for each positive integer α we have a Hilbert space $V[\alpha]$, and for each morphism $\mathfrak{p} : \beta \to \alpha$ we have a bounded linear operator $R(\mathfrak{p}) : V[\beta] \to V[\alpha]$ such that

$$R(\mathfrak{p} \circ \mathfrak{q}) = R(\mathfrak{p})R(\mathfrak{q}), \quad R(\mathfrak{p}^*) = R(\mathfrak{p})^*, \quad R(u_\alpha) = u_{V[\alpha]}$$

where u_α is the unit automorphism of an object α, and $1_{V[\alpha]}$ is the unit operator in $V[\alpha]$.

Theorem 1.6.6 (Theorem 3.11 of [32])**.** *Any $*$-representation of the category $\mathcal{K}(G, K)$ is equivalent to a representation $\tilde{\rho}$ obtained from some unitary ρ of the group $\mathbb{G}_3$. Moreover, the representation ρ is unique.*

1.6.2.1 The inverse construction

We obtain ρ as a limit of representations of semigroups $\mathrm{Hom}(\alpha, \alpha)$.

Let $\alpha \leq \beta$. Define the following morphisms of our category:

1. $\theta_\alpha^\beta \in \mathrm{Hom}(\beta, \beta)$ is the collection of double triangles with the following labels:

$$(1, 1), (2, 2), \ldots, (\alpha, \alpha), (\alpha + 1, \emptyset), \ldots, (\beta, \emptyset), (\emptyset, \alpha + 1), \ldots, (\emptyset, \beta).$$

2. $\mu_\alpha^\beta \in \mathrm{Hom}(\alpha, \beta)$ is the collection of double triangles with the following labels:

$$(1,1), (2,2), \ldots, (\alpha, \alpha), (\emptyset, \alpha+1), \ldots, (\emptyset, \beta).$$

3. $\nu_\alpha^\beta \in \mathrm{Hom}(\beta, \alpha)$ is defined by $\nu_\alpha^\beta = (\mu_\alpha^\beta)^*$.

We have that $(\theta_\alpha^\beta)^* = \theta_\alpha^\beta = (\theta_\alpha^\beta)^2$. Furthermore,

$$\begin{aligned} \mu_\alpha^\beta \circ \nu_\alpha^\beta &= \theta_\alpha^\beta & \qquad \nu_\alpha^\beta \circ \mu_\alpha^\beta &= 1_\alpha \\ \theta_\alpha^\beta \circ \mu_\alpha^\beta &= \mu_\alpha^\beta & \qquad \nu_\alpha^\beta \circ \theta_\alpha^\beta &= \nu_\alpha^\beta \end{aligned}$$

Applying R to these identities we get that $R(\theta_\alpha^\beta)$ is the operator of orthogonal projection in $V[\beta]$, and $R(\mu_\alpha^\beta)$ is an operator of isometric embedding $V[\alpha] \to V[\beta]$, identifying $V[\alpha]$ with the image of the operator θ_α^β.

Next, we construct an embedding of monoids $\zeta : \mathrm{Hom}(\alpha, \alpha) \to \mathrm{Hom}(\beta, \beta)$. Given a surface in $\mathrm{Hom}(\alpha, \alpha)$ we add a collection of double triangles with labels:

$$(\alpha+1, \emptyset), \ldots, (\beta, \emptyset), (\emptyset, \alpha+1), \ldots, (\emptyset, \beta).$$

We have

$$\zeta(\mathfrak{p}) \circ \theta_\alpha^\beta = \theta_\alpha^\beta \circ \zeta(\mathfrak{p}), \quad \zeta(\mathfrak{p}) \circ \mu_\alpha^\beta = \mu_\alpha^\beta \circ \mathfrak{p}.$$

Decompose $V[\beta]$ as a direct sum $\mathrm{Im} R(\mu_\alpha^\beta) \oplus \mathrm{Im} R(\mu_\alpha^\beta)^\perp$. By the identities above, the operator $R(\zeta(\mathfrak{p}))$ has the following block structure:

$$\begin{pmatrix} R(\mathfrak{p}) & 0 \\ 0 & 0 \end{pmatrix}.$$

Next, notice that for $\alpha \leq \beta \leq \gamma$, we have

$$\mu_\beta^\gamma \circ \mu_\alpha^\beta = \mu_\alpha^\gamma.$$

Consider the following chain of embeddings of Hilbert spaces:

$$\cdots \to V[\alpha] \to V[\alpha+1] \to \cdots.$$

Call V the completion of the limit of this sequence. In each $V[\alpha]$ we have the action of $\mathrm{Hom}(\alpha, \alpha)$. The action is compatible with the embeddings and therefore we get a representation $R^\diamond$ in the limit space of the semigroup Γ, which is obtained as the limit of the chain

$$\cdots \to \mathrm{Hom}(\alpha, \alpha) \to \mathrm{Hom}(\alpha+1, \alpha+1) \to \cdots.$$

By $P[\alpha]$ we denote the projection operator to $V[\alpha]$. Then $P[\alpha]_{|V[\beta]} = R^\diamond(\theta_\alpha^\beta)_{|V[\beta]}$.

Lemma 1.6.7 (Lemma 3.13 of [32]). *Let $p \in \mathbb{G}_3$. Let $\mathfrak{p}_\alpha \in \mathrm{Hom}(\alpha, \alpha)$ be the double coset containing p, we also consider it as an element of Γ. Then the sequence of operators $R^\diamond(\mathfrak{p}_\alpha)$ has a weak limit and its norm in less or equal one.*

For $p \in G$, denote $\rho(p) := \lim_{\alpha\to\infty} R^\diamond(\mathfrak{p}_\alpha)$. Let $p, q \in G$ and $r = pq$. For N sufficiently large we have

$$\mathfrak{p}_\alpha \circ \mathfrak{q}_\beta = \mathfrak{r}_{\min\{\alpha,\beta\}}$$

for all $\alpha, \beta > N$. Applying the functor $R^\diamond$ on both sides and taking the limits

$$\lim_{\alpha\to\infty} \lim_{\beta\to\infty} R^\diamond(\mathfrak{p}_\alpha \circ \mathfrak{q}_\beta) = \lim_{\alpha\to\infty} \lim_{\beta\to\infty} R^\diamond(\mathfrak{r}_{\min\{\alpha,\beta\}})$$

we get $\rho(p)\rho(q) = \rho(r) = \rho(pq)$.

A representation of the group G_3 is constructed. Its restriction to the diagonal is continuous in the topology of the complete symmetric group and therefore it admits a continuous extension to the whole group $\mathbb{G}_3$.

1.6.2.2 Spherical characters and self-similarity

Consider the following elements of a semigroup $\mathrm{Hom}(\alpha, \alpha)$: We take the union of α double triangles with labels $(1, 1), (2, 2), \ldots, (\alpha, \alpha)$ and an arbitrary surface without labels. The semigroup Z_α of all such morphisms is the center of the semigroup $\mathrm{Hom}(\alpha, \alpha)$. The semigroups Z_α are isomorphic for any α, the isomorphism $\pi_\alpha^{\alpha+1} : Z_{\alpha+1} \to Z_\alpha$ is the forgetting of two labels $\alpha + 1$. In other words,

$$\pi_\alpha^{\alpha+1}(\mathfrak{p}) = \nu_\alpha^{\alpha+1} \circ \mathfrak{p} \circ \mu_\alpha^{\alpha+1}.$$

Notice that $Z_0 = K\backslash G_3/K = \mathrm{Hom}(0, 0)$. Let ρ be an irreducible representation of the group $\mathbb{G}_3$ in a Hilbert space H. Then the semigroup Z_α acts on $H[\alpha]$ by scalar operators, that is, we get a homomorphism χ_α from Z_α to the multiplicative semigroup of complex numbers whose norm is less than or equal to one.

For any irreducible unitary representation ρ of $\mathbb{G}_3$ we have

$$\tilde{\rho}(\pi_\alpha^{\alpha+1}(\mathfrak{p})) = \tilde{\rho}(\nu_\alpha^{\alpha+1})\tilde{\rho}(\mathfrak{p})\tilde{\rho}(\mu_\alpha^{\alpha+1}) = \chi_{\alpha+1}(\mathfrak{p}) \cdot 1,$$

and we get the following statement:

Proposition 1.6.8 (Proposition 3.17 of [32])**.** *The character χ does not depend on α.*

We call $\chi : K\backslash G_3/K$ the spherical character of an irreducible representation. In the group $\mathbb{G}_3 \subset \overline{S}_\infty^3$ we define the subgroups

$$\mathbb{G}[\alpha] := \mathbb{G}_3 \cap (\overline{S}_\infty[\alpha] \times \overline{S}_\infty[\alpha] \times \overline{S}_\infty[\alpha]).$$

Clearly, the subgroups $\mathbb{G}_3[\alpha]$ are canonically isomorphic to the group $\mathbb{G}_3$. Consider an irreducible representation ρ of the group $\mathbb{G}_3$ in a Hilbert space H. Restrict it to a subgroup $\mathbb{G}[\alpha]$. If α is sufficiently large, then $H[\alpha] \neq 0$. In $H[\alpha]$ we have the action of the semigroup Z_α of the group $\mathbb{G}_3$, it coincides with the semigroup Z_0 of the group $\mathbb{G}[\alpha]$. It is easy to verify that the $\mathbb{G}[\alpha]$-cyclic span of any non-zero element of $H[\alpha]$ is a spherical representation of $\mathbb{G}[\alpha]$. Moreover, all spherical subrepresentations of the restriction $\rho_{|\mathbb{G}_3}$ are equivalent and their spherical functions coincide with the spherical character of the representation ρ (for all α).

Chapter 2

The Infinite Braid Group

In this chapter we extend the construction of a product on double cosets given in the previous chapter to the infinite braids group B_∞. The content of this chapter has become an article [42].

2.1 The braid group

Braids describe the intuitive concept of intertwining strands whose endpoints are fixed. The first natural approach was formalized as isotopy classes of mutually non-intersecting curves in 3-space. Other formalizations were developed as the fundamental group of configuration spaces, mapping class groups of the punctured disk and as a subgroup of the group of automorphisms of the free group.

Braids were rigorously defined by Artin in 1925 where he introduced his famous presentation for the braid group B_n. In the 30's Burau introduced non-trivial linear representations of the braid group, now named after him, which in the 90's was proven to be unfaithful for $n > 4$, it is faithful for $n = 3$ but the case $n = 4$ remains unanswered.

Magnus proved the relation between braids and mapping class groups followed by the proof by Alexander and Markov that there is a bijection between links and equivalence classes of braids.

In the 2000's it was proved that the braid groups are linear.

2.1.1 Equivalent definitions of braids

2.1.1.1 Isotopy classes

The first definition of the braid groups is in terms of geometric braid diagrams. Let $\{p_1, \dots, p_n\}$ be n distinguished points in $\mathbb{C}$. Let $(f_1, \dots, f_n)$ be a n-tuple of functions

$$f_i \colon [0,1] \to \mathbb{C}$$

satisfying

$$f_i(0) = p_i \quad \text{and} \quad f_i(1) \in \{p_1, \dots, p_n\} \qquad (1 \le i \le n)$$

and such that the paths

$$\begin{aligned} \gamma_i \colon [0,1] &\to \mathbb{C} \times [0,1] \\ t &\mapsto (f_i(t), t), \end{aligned}$$

called strands, have disjoint images. The n-tuple $(\gamma_1, \ldots, \gamma_n)$ is called a braid. The braid group B_n on n-strands is the group of isotopy classes of braids.

The product of two braids is given by the concatenation of paths, that is, given braids $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ we defined their product $\sigma\gamma = (\delta_1, \ldots, \delta_n)$ by

$$\delta_i(t) = \begin{cases} \sigma_i(2t), & 0 \leq t \leq \frac{1}{2} \\ \gamma_j(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

where $\sigma(1) = \gamma(0)$.

2.1.1.2 Generators and relations

The *Artin braid group* on n strings B_n ([2, 10, 15]) has the presentation with $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and the *braid relations*:

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| \geq 2, \quad i, j \in \{1, \ldots, n-1\},$$

and

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \leq i \leq n-2.$$

The generators σ_i are called *elementary braids*. It is convenient to represent the elements of this groups as braid diagrams. In this sense, the elementary braid σ_i correspond to the crossing of the i-th strand over the $(i+1)$-th strand and its inverse σ_i^{-1} correspond to the crossing of the i-th strand under the $(i+1)$-th strand.

Figure 2.1: The elementary braids: (a) σ_i, (b) σ_i^{-1}.

The product of braids correspond to the concatenation of diagrams, from top to bottom.

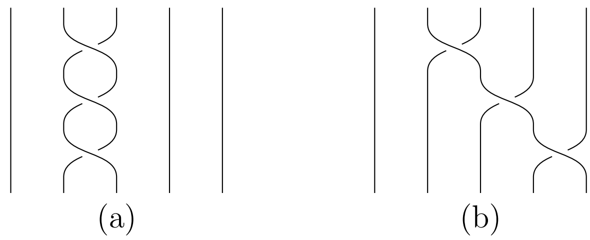


Figure 2.2: Concatenation of braid diagrams: (a) σ_2^3, (b) $\sigma_2\sigma_3\sigma_4$.

2.1.1.3 Configuration spaces

The configuration space of n points on the complex plane $\mathbb{C}$ is

$$\mathcal{C}_{\hat{n}} = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n; z_i \neq z_j, i \neq j\}.$$

The symmetric group acts freely on $\mathcal{C}_{\hat{n}}$, by permuting the coordinates of its points. The orbit space of this action is $\mathcal{C}_n = \mathcal{C}_{\hat{n}}/S_n$ and the orbit space projection is $\tau\colon \mathcal{C}_{\hat{n}} \to \mathcal{C}_n$. Choosing a fixed base point $p \in \mathcal{C}_{\hat{n}}$ we define the pure braid group P_n on n strands and the braid group B_n on n strands to be the fundamental groups

$$P_n = \pi_1(\mathcal{C}_{\hat{n}}, p) \quad \text{and} \quad B_n = \pi_1(\mathcal{C}_n, \tau(p)).$$

While the manifold $\mathcal{C}_n$ has dimension $2n$, the fact that the points $z_1, \ldots, z_n$ are pairwise distinct allows us to think of a point $z \in \mathcal{C}_n$ as a set of n distinct points on $\mathbb{C}$. An element of $\pi_1(\mathcal{C}, \tau(p))$ is then represented by a loop which lifts uniquely to a path $g\colon [0,1] \to \mathcal{C}_{\hat{n}}$, where each coordinate function $g_i\colon [0,1] \to \mathbb{C}$ of g satisfies the condition $g_i(t) \neq g_j(t), i \neq j, t \in [0,1]$ and $g(0) = g(1) = p$.

These two braid groups are related in a simple way. Let $\tau_*\colon P_n \to B_n$ be the homomorphism induced by the projection τ. The orbit space projections is a $n!$-sheeted covering space projection, with S_n as the group of covering translations. From this follows that P_n is a index $n!$ subgroup of B_n, and we get a short exact sequence

$$1 \longrightarrow P_n \xrightarrow{\tau_*} B_n \longrightarrow S_n \longrightarrow 1.$$

2.1.1.4 Mapping class groups

Let $\Sigma = \Sigma_g^{n,b}$ denote a 2-manifold of genus g with b boundary components that we assume to be simple closed curves and n punctures $\{p_1 \ldots, p_n\} \subset \operatorname{int}\Sigma$. Let $\operatorname{Homeo}(\Sigma)$ denote the group of all orientation preserving homeomorphisms $h\colon \Sigma \to \Sigma$ such that h is the identity on each boundary component and $h(\{p_1 \ldots, p_n\}) = \{p_1 \ldots, p_n\}$.

The mapping class group $\operatorname{MCG}(\Sigma) = \pi_0(\operatorname{Homeo}(\Sigma))$ is the set of isotopy classes relative to $\partial\Sigma$ and to $\{p_1 \ldots, p_n\}$, of mappings in $\operatorname{Homeo}(\Sigma)$, with the composition operation.

Let $D_i = \Sigma_0^{i,1}$ be the unit disk with i punctures. Then

$$B_n = \operatorname{MCG}(D_n).$$

We give an intuitive description on how to pass from homeomorphisms to geometric braids following [5]. Choose $h \in \operatorname{Homeo}(D_n)$. Its image by the inclusion $\operatorname{Homeo}(D_n) \to \operatorname{Homeo}(D_0)$ is isotopic to the identity, since $\operatorname{MCG}(D_0)$ is trivial. Let h_t denote this isotopy in the instant t, then the n paths $(h_t(p_1), \ldots, h_t(p_n)) = g(t)$ defined by the traces of the points $\{p_1, \ldots, p_n\}$ under the isotopy sweep out a braid in $D_0 \times [0,1]$.

2.1.1.5 Automorphisms of the free group

Consider B_n as the abstract group of 2.1.1.2.

For each $n \in \mathbb{N}_+$, let $i_n : B_n \to \operatorname{Aut}(F_n)$ be the *Artin representation* of B_n on the free group F_n, given by

$$i_n(\sigma_j)(x_k) = \begin{cases} x_j, & k = j+1 \\ x_j x_{j+1} x_j^{-1}, & k = j \\ x_k, & \text{otherwise.} \end{cases}$$

This representation is faithful and therefore we can identify B_n with the image of i_n in $\operatorname{Aut}(F_n)$.

2.2 The group B_∞

Consider B_n as the abstract group of 2.1.1.2. For each n, consider the monomorphism $i_n : B_n \to B_{n+1}$ sending the k-th elementary braid of B_n to the k-th elementary braid of B_{n+1}. Geometrically this operation corresponds to adding a new string to the right of the braid, without creating any new crossings, as in the picture below:

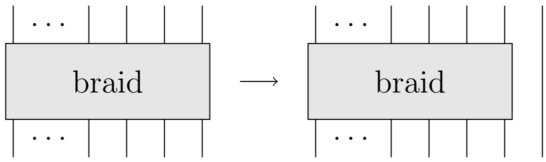


Figure 2.3: The monomorphism i_n.

The direct limit of this sequence of groups, with respect to the homomorphisms i_n, is the *infinite braid group*

$$B_\infty = \varinjlim B_n,$$

consisting of braids with countably many strings and finitely many crossings. This group has the presentation

$$B_\infty = \left\langle \ \sigma_i, i \in \mathbb{N}_+ \ \middle| \ \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{array} \right\rangle.$$

For each non-negative integer α, let $B_\infty[\alpha]$ be the subgroup of B_∞ given by

$$B_\infty[\alpha] = \langle \sigma_j | j > \alpha \rangle.$$

2.3 A product on double cosets of B_∞

2.3.1 Motivation

In the following sections we show that the group B_∞, of the finite braids on infinitely many strands, admits a product on the set of double cosets similar to the product defined on the first chapter. Furthermore, we show how this multiplicative structure is related to similar constructions in $\mathrm{Aut}(F_\infty)$ and $GL(\infty)$. We also define a one-parameter generalization of the usual monoid structure on the set of double cosets of $GL(\infty)$ (see [30,31]) and show that the Burau representation provides a functor between the categories of double cosets of B_∞ and $GL(\infty)$ (the category where the objects are nonnegative integers, the morphisms are the double cosets and the composition of morphisms is given by the product of double cosets).

2.3.1.1 Main results

Consider the double cosets on B_∞ with respect to the subgroups $B_\infty[\alpha]$. Given double cosets $\mathfrak{p} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\beta]$ and $\mathfrak{q} \in B_\infty[\beta]\backslash B_\infty/B_\infty[\gamma]$, we are going to define an element $\mathfrak{p} \circ \mathfrak{q} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\gamma]$. To this purpose, we first introduce the following:

Definition 2.3.1. *For integers $\beta \geq 0$ and $n > 0$, denote by $\tau_i^{(n)}$ the braid*

$$\tau_i^{(n)} = \sigma_{n+\beta+i}\sigma_{n+\beta+i-1}\cdots\sigma_{\beta+i+1}.$$

Further we define the element $\theta_n[\beta] \in B_\infty[\beta]$ as

$$\theta_n[\beta] = \tau_0^{(n)} \tau_1^{(n)} \cdots \tau_{n-1}^{(n)}.$$

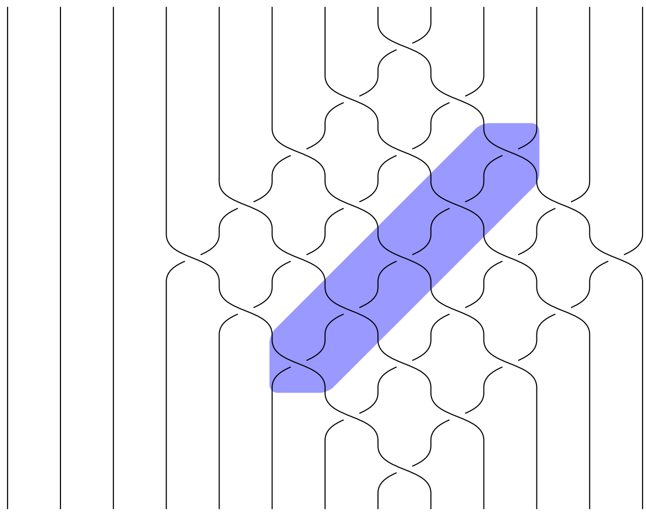


Figure 2.4: The element $\theta_5[3]$. The highlighted region corresponds to $\tau_2^{(5)}$

This braid can be seen as a half twist on the strands $\beta+1$ to $\beta+2n$. Finally, the definition of the product of the double cosets is as follows:

Definition 2.3.2. *Let $\mathfrak{p} \in B_\infty[\alpha] \backslash B_\infty / B_\infty[\beta]$ and $\mathfrak{q} \in B_\infty[\beta] \backslash B_\infty / B_\infty[\gamma]$ be double cosets. Consider $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$ representatives of these double cosets. Then we define their product as*

$$\mathfrak{p} \circ \mathfrak{q} = B_\infty[\alpha]\, p\theta_n[\beta] q \, B_\infty[\gamma],$$

for sufficiently large n.

Remark 2.3.3. The introduction of the element $\theta_k[\beta]$ is essential for our construction of a product on the set of double cosets of B_∞. In fact, for $\mathfrak{p} \in B_\infty[\alpha] \backslash B_\infty / B_\infty[\beta]$ and $\mathfrak{q} \in B_\infty[\beta] \backslash B_\infty / B_\infty[\gamma]$, let $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$ be representatives of these double cosets. The "naive" product $B_\infty[\alpha]\, pq \, B_\infty[\gamma]$ does not always coincide for all choices of p and q. For instance σ_2 and $\sigma_3\sigma_2$ are representatives of the same double coset in $B_\infty[2] \backslash B_\infty / B_\infty[2]$. But σ_2^2 and $\sigma_3\sigma_2\sigma_3\sigma_2$ represent distinct cosets. In order to see this, we consider the permutation associated to each braid. For the braid σ_2^2 it is the identity and for the braid $\sigma_3\sigma_2\sigma_3\sigma_2$ it is (432). Since no braid in $B_\infty[2]$ permutes the point 2, we see that these are in fact distinct double cosets.

However, if we introduce an intermediary braid $\theta_k[\beta]$ that "forces apart" the braids p and q, the double coset $B_\infty[\alpha]\, p\theta_k[\beta] q \, B_\infty[\gamma]$ becomes independent of k for k large enough and its limit does not depend on the choice of the representatives for $\mathfrak{p}$ and $\mathfrak{q}$.

Theorem 2.3.4. *The operation defined above does not depend on the choice of the representatives of the double cosets for n large enough. Moreover, it is associative.*

As a consequence we have that $(B_\infty[\alpha] \backslash B_\infty / B_\infty[\alpha], \circ)$ is a monoid, for each non-negative integer α.

Remark 2.3.5. We will show that there exists some $n_0(\alpha, \gamma, p, q)$ such that, for all $n \geq n_0$, $B_\infty[\alpha]\, p\theta_n[\beta] q \, B_\infty[\gamma] = B_\infty[\alpha]\, p\theta_{n_0}[\beta] q \, B_\infty[\gamma]$. More precisely, $n_0 = \max\{\operatorname{supp} p, \operatorname{supp} q, \alpha, \gamma\} + 1$, where supp is the support of a braid, defined in Definition 2.3.12.

Using the correspondence between double cosets and conjugacy classes given in the Remark 1.4.3 we can define a monoid structure on the set $B_\infty /\!\!/ B_\infty[\alpha]$. In fact, we have an one-to-one correspondence between the sets $B_\infty /\!\!/ B_\infty[\alpha]$ and $B_\infty[\alpha] \backslash (B_\infty \times B_\infty[\alpha])/B_\infty[\alpha]$, the later being a submonoid of $B_\infty[\alpha] \backslash (B_\infty \times B_\infty)/B_\infty[\alpha]$.

Furthermore, as a consequence of the existence of a solution for the conjugacy problem for the braid groups and the fact that the injections i_n do not merge conjugacy classes (see Theorem 1.1 of [17]), we have

Proposition 2.3.6. *The conjugacy problem for B_∞ has a solution.*

Notice that combining the observations above with Proposition 2.3.6, it is possible to devise an algorithm to determine when two elements of $B_\infty \times B_\infty$ belong to the same class in $B_\infty[0] \backslash (B_\infty \times B_\infty)/B_\infty[0]$.

2.3.1.2 The Burau representation of B_∞

The Burau representation ([4, 5]) is the homomorphism $\eta_n : B_n \to GL(n, \mathbb{Z}[t, t^{-1}])$ given by

$$\eta_n(\sigma_i) = \begin{pmatrix} 1_{i-1} & & \\ & \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} & \\ & & 1_{n-i-1} \end{pmatrix}.$$

Denote $GL(n, \mathbb{Z}[t, t^{-1}])$ by $GL(n)$ and consider the homomorphisms $j_n : GL(n) \to GL(n+1)$ given by

$$j_n(T) = \begin{pmatrix} T & 0 \\ 0 & 1 \end{pmatrix}.$$

The group $GL(\infty)$ is the direct limit of $GL(n)$ with respect to the homomorphisms j_n and consists of infinite matrices that differ from the identity matrix only in finitely many entries. Due to the commutativity of the diagram

$$\begin{tikzcd} B_n \arrow[r, "\eta_n"] \arrow[d, "i_n"'] & GL(n) \arrow[d, "j_n"] \\ B_{n+1} \arrow[r, "\eta_{n+1}"'] & GL(n+1) \end{tikzcd}$$

we can construct a representation $\eta : B_\infty \to GL(\infty)$ of B_∞ by taking the limit of the representations η_n. More precisely, η is given by the following formulas:

$$\eta(\sigma_i) = \begin{pmatrix} 1_{i-1} & & \\ & \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} & \\ & & 1_\infty \end{pmatrix}.$$

With this representation in mind, we will define an operation on double cosets of $GL(\infty)$ such that the Burau representation will be functorial between the categories of double cosets.

Now, let $v = (1, t, t^2, \ldots)$ and $u = (1, 1, 1, 1, \ldots)$ and denote by x^T the transpose of the vector x. Consider the subgroup of $GL(\infty)$ given by

$$G[n] = \left\{ \begin{pmatrix} 1_n & \\ & X \end{pmatrix} ; X \in GL(\infty), v^T X = v^T, Xu = u \right\}.$$

It is easy to see that the image of $B_\infty[n]$ by the Burau representation is contained in $G[n]$.

Definition 2.3.7. *Consider the matrix*

$$\Theta_j[k] = \begin{pmatrix} \mathbf{1}_k & 0 & 0 & 0 \\ 0 & V_j & t^j 1_j & 0 \\ 0 & 1_j & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_\infty \end{pmatrix},$$

where

$$V_j = (1-t) \begin{pmatrix} 1 & t & \cdots & t^{j-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t & \cdots & t^{j-1} \end{pmatrix}.$$

Let $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$ be representatives of the double cosets $\mathfrak{p} \in G[n]\backslash GL(\infty)/G[k]$ and $\mathfrak{q} \in G[k]\backslash GL(\infty)/G[m]$. Then we define their product as

$$\mathfrak{p} \star_t \mathfrak{q} = G[n]\, p\Theta_j[k]q\, G[m],$$

for sufficiently large j.

Theorem 2.3.8. *The operation defined above does not depend on the choice of representatives of double cosets for j large enough. Moreover, it is associative.*

Remark 2.3.9. In particular, there exists an integer $j_0(n, m, p, q)$ such that $G[n]\, p\Theta_j[k]q\, G[m] = G[n]\, p\Theta_{j_0}[k]q\, G[m]$ for all $j \geq j_0$. We can make j_0 more precise. In fact let $N \in \mathbb{N}$ be such that p and q can be written as diagonal block matrices $\begin{pmatrix} A & 0 \\ 0 & 1_\infty \end{pmatrix}$, where A is a square matrix of dimension $k+N$. Then $j_0 = \max\{m, n, k+N\}$.

Remark 2.3.10. The operation $\star_t$ generalizes the usual multiplication defined on the double cosets of $GL(\infty)$ in the sense that setting the parameter $t = 1$ we recover the usual multiplication.

Let G be a group and $K[*] = \{K[s]; s \in \mathbb{N}\}$ a family of subgroups of G. We say that there is a well-defined operation on the double cosets of G with relation to the family $K[*]$ when there exists a family of morphisms

$$\mu = \{\mu_{rst} : K[r]\backslash G/K[s] \times K[s]\backslash G/K[t] \to K[r]\backslash G/K[t]; r, s, t \in \mathbb{N}\}$$

satisfying

$$\mu_{rtu}(\mu_{rst} \times 1_{K[t]\backslash G/K[u]}) = \mu_{rsu}(1_{K[r]\backslash G/K[s]} \times \mu_{stu})$$

for all $r, s, t, u \in \mathbb{N}$ and, if $\mathbf{e} \in K[r]\backslash G/K[r]$ denotes the class of the unit element of G, then for all $\alpha \in K[t]\backslash G/K[r]$ and all $\beta \in K[r]\backslash G/K[t]$

$$\mu_{trr}(\alpha, \mathbf{e}) = \alpha \quad \text{and} \quad \mu_{rrt}(\mathbf{e}, \beta) = \beta.$$

In this case, consider the category $\mathcal{K}(G, K)$ of double cosets, where the objects are nonnegative integers and the morphisms are given by $\mathrm{Hom}(r, s) = K[s]\backslash G/K[r]$. Then,

Proposition 2.3.11. *The Burau representation $\eta : B_\infty \to GL(\infty)$ induces a functor between the categories $\mathcal{K}(B_\infty, B_\infty[*])$ and $\mathcal{K}(GL(\infty), G[*])$.*

When G is the bisymmetric (or trisymmetric) group and K is its diagonal subgroup, we get a category called the train category of the pair (G, K). This category encodes information about the representations of the bisymmetric (respectively, trisymmetric) group (see [32, 37] and Section 1.6.2).

2.3.2 Proofs of main results

2.3.2.1 Proof of Theorem 2.3.4

Before proceeding, we introduce the notion of *support*, which will be needed later.

Definition 2.3.12. *Let p be a braid in B_∞. The support of p is*

$$\operatorname{supp} p = \min\{j \in \mathbb{N}_+; p \in \langle \sigma_1, \ldots, \sigma_j \rangle\}.$$

Notice that the decomposition of p into the product of elementary braids does not contain any element of $B_\infty[\operatorname{supp} p]$, hence p commutes with every element of $B_\infty[1 + \operatorname{supp} p]$. Also, we can identify p with an element of $B_{1+\operatorname{supp} p}$. We define $\operatorname{supp} 1 = 0$.

Consider double cosets

$$\mathfrak{p} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\beta] \quad \text{and} \quad \mathfrak{q} \in B_\infty[\beta]\backslash B_\infty/B_\infty[\gamma],$$

and let $p \in \mathfrak{p}$ and $q \in \mathfrak{q}$ be their respective representatives. Setting $\mathfrak{r}_j = B_\infty[\alpha]\, p\theta_j[\beta]q\, B_\infty[\gamma]$, we have a sequence of double cosets in $B_\infty[\alpha]\backslash B_\infty/B_\infty[\gamma]$.

Proposition 2.3.13. *The sequence $(\mathfrak{r}_j)_{j\geq 1}$ defined above is eventually constant.*

Proof. We are going to give a proof in several steps:

Step 1. Given $m > 0$ we have $\tau_i^{(m+1)} = \sigma_{m+\beta+1+i}\tau_i^{(m)}$ for all $0 \leq i \leq m-1$.
In fact, we have the equality

$$\tau_i^{(m+1)} = \sigma_{m+1+\beta+i}\sigma_{m+\beta+i}\sigma_{m+\beta+i-1}\cdots\sigma_{\beta+i+1} = \sigma_{m+1+\beta+i}\tau_i^{(m)}.$$

Step 2. For all $j \leq i$ we have $\sigma_{m+\beta+i+2}\tau_j^{(m)} = \tau_j^{(m)}\sigma_{m+\beta+i+2}$.
Indeed, since $\operatorname{supp}\tau_j^{(m)} = m+\beta+j$ and $\sigma_{m+\beta+i+2} \in B_\infty[1+m+\beta+j]$, we find that $\sigma_{m+\beta+i+2}$ commutes with $\tau_j^{(m)}$.

Step 3. Define $u = (\sigma_{m+\beta+1}\sigma_{m+\beta+2}\cdots\sigma_{2m+\beta})^{-1}$ and $\ell^{-1} = \tau_m^{(m+1)}$. Then $\theta_m[\beta] = u\theta_{m+1}[\beta]\ell$.
In fact, we have

$$\begin{aligned}
u\theta_{m+1}[\beta]\ell =& u(\tau_0^{(m+1)}\cdots\tau_m^{(m+1)})\ell = u\tau_0^{(m+1)}\cdots\tau_{m-1}^{(m+1)} = \\
=& u(\sigma_{m+\beta+1}\tau_0^{(m)})(\sigma_{m+\beta+2}\tau_1^{(m)})\cdots(\sigma_{2m+\beta}\tau_{m-1}^{(m)}) = \\
=& \sigma_{2m+\beta}^{-1}\sigma_{2m+\beta-1}^{-1}\cdots\sigma_{m+\beta+2}^{-1}\tau_0^{(m)}(\sigma_{m+\beta+2}\tau_1^{(m)})\cdots(\sigma_{2m+\beta}\tau_{m-1}^{(m)}) = \\
=& \sigma_{2m+\beta}^{-1}\sigma_{2m+\beta-1}^{-1}\cdots\sigma_{m+\beta+2}^{-1}\sigma_{m+\beta+2}\tau_0^{(m)}\tau_1^{(m)}\cdots(\sigma_{2m+\beta}\tau_{m-1}^{(m)}) = \\
=& \sigma_{2m+\beta}^{-1}\sigma_{2m+\beta-1}^{-1}\cdots\sigma_{m+\beta+3}^{-1}\tau_0^{(m)}\tau_1^{(m)}(\sigma_{m+\beta+3}\tau_2^{(m)})\cdots(\sigma_{2m+\beta}\tau_{m-1}^{(m)}) = \\
& \vdots \\
=& \tau_0^{(m)}\tau_1^{(m)}\tau_2^{(m)}\cdots\tau_{m-1}^{(m)} = \theta_m[\beta].
\end{aligned}$$

Step 4. Let $M = \max\{\operatorname{supp} p, \operatorname{supp} q, \alpha, \gamma\} + 1$. We show that for all $m \geq M$, we have $\mathfrak{r}_m = \mathfrak{r}_{m+1}$ and hence that $\mathfrak{r}_m = \mathfrak{r}_M$ for all $m \geq M$. Let u and ℓ be like in step 3. Since $u, \ell \in B_\infty[m+\beta]$, it follows that $u \in B_\infty[\alpha]$, $\ell \in B_\infty[\gamma]$ and they commute with p and q. Therefore

$$u(p\theta_{m+1}[\beta]q)\ell = p(u\theta_{m+1}[\beta]\ell)q = p\theta_m[\beta]q.$$

Thus $\mathfrak{r}_m = \mathfrak{r}_{m+1}$.

$\square$

The following technical lemma will be used in the proof of Lemma 2.3.16, which in turn is used in Proposition 2.3.18 and more extensively in Theorem 2.3.19.

Lemma 2.3.14. *Let $\{(\upsilon_j^i)_{j=1}^{\ell}\}_{i=1}^{g}$ be a family of sequences of positive integers such that $\upsilon_{j+k}^i < \upsilon_j^{i+n}$ whenever $k+n>0$ with $k,n\in\mathbb{N}_+$; in other words, the sequences $(\upsilon_j^i)_{j\geq 1}$ are decreasing and the sequences $(\upsilon_j^i)_{i\geq 1}$ are increasing. If $\mu_j = \prod_{k=1}^{\ell}\sigma_{\upsilon_k^j}$ and $\lambda_i = \prod_{k=1}^{g}\sigma_{\upsilon_i^k}$, then $\mu_1\cdots\mu_g = P = \lambda_1\cdots\lambda_\ell$.*

Proof. We prove the lemma by induction on the pair (g,ℓ). The statement is trivial for $g=\ell=1$. Assume it is true for (g,ℓ), we prove it is true for $(g+1,\ell)$ and $(g,\ell+1)$.

For $(g+1,\ell)$, notice that

$$\prod_{s=1}^{g+1}\prod_{r=1}^{\ell}\sigma_{\upsilon_r^s} = \left(\prod_{s=1}^{g}\prod_{r=1}^{\ell}\sigma_{\upsilon_r^s}\right)\left(\prod_{r=1}^{\ell}\sigma_{\upsilon_r^{g+1}}\right) = \left(\prod_{r=1}^{\ell}\prod_{s=1}^{g}\sigma_{\upsilon_r^s}\right)\left(\prod_{r=1}^{\ell}\sigma_{\upsilon_r^{g+1}}\right).$$

If $x_r = \prod_{s=1}^{g}\sigma_{\upsilon_r^s}$ we have that $x_r\sigma_{\upsilon_t^{g+1}} = \sigma_{\upsilon_t^{g+1}}x_r$ for $r>t$, this follows from the inequalities $\upsilon_r^s < \upsilon_r^{g+1} < \upsilon_t^{g+1}$ for $s<g+1$. Therefore

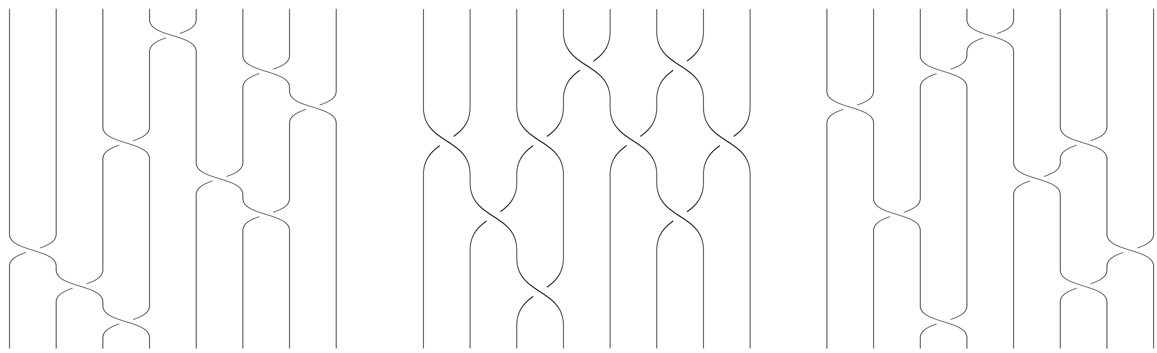
$$\left(\prod_{r=1}^{\ell}x_r\right)\left(\prod_{r=1}^{\ell}\sigma_{\upsilon_r^{g+1}}\right) = x_1\cdots x_\ell\sigma_{\upsilon_1^{g+1}}\cdots\sigma_{\upsilon_\ell^{g+1}} = x_1\sigma_{\upsilon_1^{g+1}}x_2\sigma_{\upsilon_2^{g+1}}\cdots x_\ell\sigma_{\upsilon_\ell^{g+1}} =$$

$$= \prod_{r=1}^{\ell}x_r\sigma_{\upsilon_r^{g+1}} = \prod_{r=1}^{\ell}\prod_{s=1}^{g+1}\sigma_{\upsilon_r^s}.$$

The proof for the case $(g,\ell+1)$ is analogous.

$\square$

Example 2.3.15. Consider the sequences given by $\upsilon_j^i = 3+i-j, 1\leq i,j\leq 3$. Let $\mu_i,\lambda_i, i=1,2,3$ be as in Lemma 2.3.14. By the same Lemma we have that $\mu_1\mu_2\mu_3 = \lambda_1\lambda_2\lambda_3$. These products are depicted in Figure 2.5 (a) and (c). Drawing these braids in a more compact form (Figure 2.5(b)) the equivalence between the products becomes evident.



(a) $\mu_1\mu_2\mu_3$ (b) P (c) $\lambda_1\lambda_2\lambda_3$

Figure 2.5: The equality $\mu_1\mu_2\mu_3 = P = \lambda_1\lambda_2\lambda_3$.

It will be useful to write the product P from Lemma 2.3.14 as a matrix, where the indices increase from right to left and from top to bottom.

$$P = \begin{bmatrix} \upsilon_1^1 & \rightarrow & \upsilon_\ell^1 \\ \downarrow & & \downarrow \\ \upsilon_1^g & \rightarrow & \upsilon_\ell^g \end{bmatrix}.$$

In this way, $\lambda_1 \cdots \lambda_\ell$ is the column-wise product and $\mu_1 \cdots \mu_g$ is the row-wise product.

Consider, for each positive integer m, the homomorphism $C_m : B_\infty \to B_\infty$ given by $C_m(\sigma_j) = \sigma_{m+j}$. Then we have the following lemma.

Lemma 2.3.16. *Let β and j be nonnegative integers with $j > 1$. If $d \in \langle \sigma_{\beta+1}, \ldots, \sigma_{\beta+j-1} \rangle$, then:*

(i) $d\theta_j[\beta] = \theta_j[\beta]C_j(d)$.

(ii) $\theta_j[\beta]d = C_j(d)\theta_j[\beta]$.

Proof. Since C_j is a homomorphism, it is enough to prove both statements of the proposition for the case where $d = \sigma_k$, for some $\beta + 1 \leq k \leq \beta + j - 1$.

(i) Recall that $\theta_j[\beta] = \tau_0^{(j)} \cdots \tau_{j-1}^{(j)}$. We claim that the following holds:

$$\sigma_{k+i}\tau_i^{(j)} = \tau_i^{(j)}\sigma_{k+i+1}, \quad 0 \leq i \leq j-1.$$

Indeed, since σ_{k+i} is a letter of $\tau_i^{(j)}$, but it is different from $\sigma_{j+\beta+i}$, we have

$$\begin{aligned} \sigma_{k+i}\tau_i^{(j)} &= \sigma_{k+i}(\sigma_{j+\beta+i} \cdots \sigma_{\beta+1+i}) = \\ &= \sigma_{j+\beta+i} \cdots \sigma_{k+i+2}\sigma_{k+i}\sigma_{k+i+1}\sigma_{k+i}\sigma_{k+i-1} \cdots \sigma_{\beta+1+i} = \\ &= \sigma_{j+\beta+i} \cdots \sigma_{k+i+2}\sigma_{k+i+1}\sigma_{k+i}\sigma_{k+i+1}\sigma_{k+i-1} \cdots \sigma_{\beta+1+i} = \\ &= \sigma_{j+\beta+i} \cdots \sigma_{\beta+1+i}\sigma_{k+i+1} = \tau_i^{(j)}\sigma_{k+i+1}. \end{aligned}$$

Therefore

$$\begin{aligned} \sigma_k\theta_j[\beta] = \sigma_k\tau_0^{(j)} \cdots \tau_{j-1}^{(j)} = \tau_0^{(j)}\sigma_{k+1}\tau_1^{(j)} \cdots \tau_{j-1}^{(j)} = \cdots = \tau_0^{(j)} \cdots \sigma_{k+j-1}\tau_{j-1}^{(j)} = \\ = \tau_0^{(j)} \cdots \tau_{j-1}^{(j)}\sigma_{k+j} = \theta_j[\beta]\sigma_{k+j}. \end{aligned}$$

(ii) Let $\upsilon_r^s = j + \beta + s - r$ for r and s positive integers. The family $\{(\upsilon_r^s)\}_{r,s=1}^j$ satisfies the hypothesis of Lemma 2.3.14 and therefore $\mu_1 \cdots \mu_j = \lambda_1 \cdots \lambda_j$, where

$$\mu_i = \sigma_{j+\beta+i-1} \cdots \sigma_{\beta+i} \quad \text{and} \quad \lambda_i = \sigma_{j+\beta-i+1} \cdots \sigma_{2j+\beta-i}.$$

Since $\mu_i = \tau_{i-1}^{(j)}$, we see that $\theta_j[\beta] = \lambda_1 \cdots \lambda_j$. As we saw in item (i), we have that

$$\lambda_{j-i}\sigma_{k+i} = \sigma_{k+i+1}\lambda_{j-i}, \quad 0 \leq i \leq j-1.$$

□

Remark 2.3.17. The intuition behind Lemma 2.3.16 is that the element $\theta_j[\beta]$ exchanges braids in the interval between strands $\beta+1$ and $\beta+j$ with braids in the interval between strands $\beta+j+1$ and $\beta+2j$.

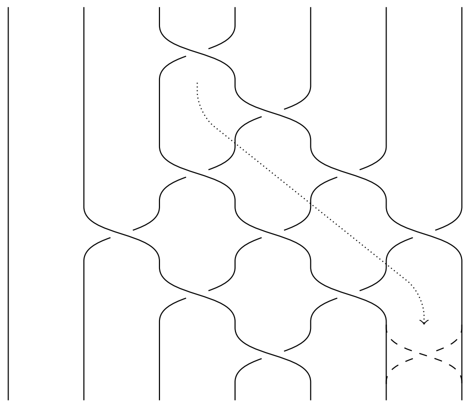


Figure 2.6: The element $\theta_3[1]$ exchanges the elementary braids σ_3 and σ_6.

Our next step is to prove that the product does not depend on the chosen representatives.

Proposition 2.3.18. *Let p' and q' be other two representatives of $\mathfrak{p}$ and $\mathfrak{q}$ respectively. Consider the sequence*

$$\mathfrak{r}'_j = B_\infty[\alpha]\, p'\theta_j[\beta]q'\, B_\infty[\gamma].$$

Then there exists an integer $N > 0$ such that

$$\mathfrak{r}'_j = \mathfrak{r}_j, \quad \textit{for all } j \geq N.$$

Proof. Since p and p' are representatives of the same double coset, there exist $r \in B_\infty[\alpha]$ and $h \in B_\infty[\beta]$ such that $p' = rph$. In a similar way, there exist $k \in B_\infty[\beta]$ and $s \in B_\infty[\gamma]$ such that $q' = kqs$. Therefore,

$$\mathfrak{r}'_j = B_\infty[\alpha]\, p'\theta_j[\beta]q'\, B_\infty[\gamma] = B_\infty[\alpha]\, rph\theta_j[\beta]kqs\, B_\infty[\gamma] = B_\infty[\alpha]\, ph\theta_j[\beta]kq\, B_\infty[\gamma].$$

Consider $N = \max\{\operatorname{supp} p, \operatorname{supp} q, \operatorname{supp} h, \operatorname{supp} k, \alpha, \gamma\} + 1$. Given $j \geq N$, let $\bar{h} = C_j(h^{-1})$ and $\bar{k} = C_j(k^{-1})$. Then $\bar{h}, \bar{k} \in B_\infty[j+\beta]$ and hence $\bar{h} \in B_\infty[\gamma]$ and $\bar{k} \in B_\infty[\alpha]$. Furthermore, $\bar{h}$ commutes with q and k, and $\bar{k}$ commutes with p and h. Now,

$$\begin{aligned} \bar{k}ph\theta_j[\beta]kq\bar{h} = ph\bar{k}\theta_j[\beta]k\bar{h}q = ph\bar{k}C_j(k)\theta_j[\beta]\bar{h}q = \\ phC_j(k^{-1})C_j(k)\theta_j[\beta]\bar{h}q = ph\theta_j[\beta]\bar{h}q = p\theta_j[\beta]C_j(h)\bar{h}q = p\theta_j[\beta]q. \end{aligned}$$

□

Therefore, for all pairs $(\mathfrak{p}, \mathfrak{q}) \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\beta] \times B_\infty[\beta]\backslash B_\infty/B_\infty[\gamma]$ we have a well-defined product $\mathfrak{p} \circ \mathfrak{q} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\gamma]$ given by

$$\mathfrak{p} \circ \mathfrak{q} = B_\infty[\alpha]\, p\theta_j[\beta]q\, B_\infty[\gamma],$$

$p \in \mathfrak{p}, q \in \mathfrak{q}$ and j sufficiently large.

Finally, we are going to prove the associativity of the operation $\circ$.

Proposition 2.3.19. *The product of double cosets is associative.*

Proof. Let $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ and consider $\mathfrak{a} \in B_\infty[\alpha]\backslash B_\infty/B_\infty[\beta]$, $\mathfrak{b} \in B_\infty[\beta]\backslash B_\infty/B_\infty[\gamma]$ and $\mathfrak{c} \in B_\infty[\gamma]\backslash B_\infty/B_\infty[\delta]$. Choose representatives $a \in \mathfrak{a}$, $b \in \mathfrak{b}$ and $c \in \mathfrak{c}$ for the double cosets and consider $k = \max\{\alpha, \beta, \gamma, \delta, \operatorname{supp} a, \operatorname{supp} b, \operatorname{supp} c\} + 1$. Then

$$\mathfrak{a} \circ \mathfrak{b} = B_\infty[\alpha]\, a\theta_k[\beta]b\, B_\infty[\gamma] \quad \text{and} \quad \mathfrak{b} \circ \mathfrak{c} = B_\infty[\beta]\, b\theta_k[\gamma]c\, B_\infty[\delta].$$

If $l = \operatorname{supp}\{a\theta_k[\beta]b\} + 1 = 2k + \beta$ and $l' = \operatorname{supp}\{b\theta_k[\gamma]c\} + 1 = 2k + \gamma$, we have

$$(\mathfrak{a} \circ \mathfrak{b}) \circ \mathfrak{c} = B_\infty[\alpha]\, a\theta_k[\beta]b\theta_l[\gamma]c\, B_\infty[\delta] \quad \text{and} \quad \mathfrak{a} \circ (\mathfrak{b} \circ \mathfrak{c}) = B_\infty[\alpha]\, a\theta_{l'}[\beta]b\theta_k[\gamma]c\, B_\infty[\delta].$$

To prove our claim we are going to show that the double cosets above are the same, by exhibiting two representatives that are equal (figures 2.7 and 2.8 give an example of the process involved). Here we are assuming $\beta \leq \gamma$, the case $\gamma < \beta$ is analogous.

Throughout the rest of the proof we will use the symbol $a \equiv b$ to signify that a and b are representatives of the same double coset of $B_\infty[\alpha]\backslash B_\infty/B_\infty[\gamma]$, that is, we can find elements $h \in B_\infty[\alpha]$ and $k \in B_\infty[\gamma]$ such that $hak = b$.

Using the notation of Lemma 2.3.14 we can write

$$a\theta_k[\beta]b\theta_l[\gamma]c = a \begin{bmatrix} k+\beta & \to & \beta+1 \\ \downarrow & & \downarrow \\ 2k+\beta-1 & \to & k+\beta \end{bmatrix} b \begin{bmatrix} 2k+\beta+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 4k+2\beta+\gamma-1 & \to & 2k+\gamma+\beta \end{bmatrix} c$$

$$a\theta_{l'}[\beta]b\theta_k[\gamma]c, = a \begin{bmatrix} 2k+\gamma+\beta & \to & \beta+1 \\ \downarrow & & \downarrow \\ 4k+2\gamma+\beta-1 & \to & 2k+\gamma+\beta \end{bmatrix} b \begin{bmatrix} k+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 2k+\gamma-1 & \to & k+\gamma \end{bmatrix} c.$$

Using the same lemma, we can see that

$$\theta_k[\beta] = R_1P; \quad P = \begin{bmatrix} k+1 & \to & \beta+1 \\ \downarrow & & \downarrow \\ 2k & \to & k+\beta \end{bmatrix}, \quad R_1 = \begin{bmatrix} k+\beta & \to & k+2 \\ \downarrow & & \downarrow \\ 2k+\beta-1 & \to & 2k+1 \end{bmatrix}.$$

$$\theta_{l'}[\beta] = R_2P_2; \quad P_2 = \begin{bmatrix} k+1 & \to & \beta+1 \\ \downarrow & & \downarrow \\ 3k+\gamma & \to & 2k+\beta+\gamma \end{bmatrix}, \quad R_2 = \begin{bmatrix} 2k+\beta+\gamma & \to & k+2 \\ \downarrow & & \downarrow \\ 4k+2\gamma+\beta-1 & \to & 3k+\gamma+1 \end{bmatrix}.$$

$$\theta_k[\gamma] = P_3R_3; \quad P_3 = \begin{bmatrix} k+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 2k & \to & k+1 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 2k+1 & \to & k+2 \\ \downarrow & & \downarrow \\ 2k+\gamma-1 & \to & k+\gamma \end{bmatrix}.$$

$$\theta_l[\gamma] = P_4R_4; \quad P_4 = \begin{bmatrix} 2k+\beta+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 3k+\beta & \to & k+1 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 3k+\beta & \to & k+2 \\ \downarrow & & \downarrow \\ 4k+2\beta+\gamma-1 & \to & 2k+\beta+\gamma \end{bmatrix}.$$

Since $R_i \in B_\infty[k+1], 1 \leq i \leq 4$, we have

$$aR_1PbP_4R_4c = R_1aPbP_4cR_4 \equiv aPbP_4c \quad aR_2P_2bP_3R_3c = R_2aP_2bP_3cR_3 \equiv aP_2bP_3c.$$

Notice also that $P_4 = R_5W$, where

$$R_5 = \begin{bmatrix} 2k+\beta+\gamma & \to & 2k+2 \\ \downarrow & & \downarrow \\ 3k+\beta & \to & 3k-\gamma+2 \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} 2k+1 & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 3k-\gamma+1 & \to & k+1 \end{bmatrix}.$$

Since $\operatorname{supp} P = 2k$ and $R_5 \in B_\infty[2k+1]$, $R_5P = PR_5$ and we have $aPbR_5Wc = aPR_5bWc = R_5aPbWc \equiv aPbWc$.

Our next objective is to find elements $E, A \in B_\infty$ such that $aP_2bP_3c \equiv aPbEAWc$.

Step 1. $aP_2bP_3c = aPbELP_3c$. Consider the element

$$F = \begin{bmatrix} 2k+1 & \to & k+\beta+1 \\ \downarrow & & \downarrow \\ 3k+\gamma & \to & 2k+\beta+\gamma \end{bmatrix},$$

and notice that $P_2 = PF$. Since $F \in B_\infty[k]$, we see that $bF = Fb$.

Moreover, $F = EL$ where

$$E = \begin{bmatrix} 2k+1 & \to & k+\beta+1 \\ \downarrow & & \downarrow \\ 2k-\beta+\gamma & \to & k+\gamma \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 2k-\beta+\gamma+1 & \to & k+\gamma+1 \\ \downarrow & & \downarrow \\ 3k+\gamma & \to & 2k+\beta+\gamma \end{bmatrix}$$

Step 2. $LP_3c \equiv CP_3c$ for some C. In fact, consider

$$C = \begin{bmatrix} 2k+\gamma-\beta+1 & \to & k+\gamma+1 \\ \downarrow & & \downarrow \\ 3k-\beta+1 & \to & 2k+1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3k-\beta+2 & \to & 2k+2 \\ \downarrow & & \downarrow \\ 3k+\gamma & \to & 2k+\beta+\gamma \end{bmatrix}.$$

Then $L = CD$ and, since $D \in B_\infty[2k+1]$ and $\operatorname{supp} P_3 = 2k$, we have $DP_3c = P_3cD \equiv P_3c$. Hence $LP_3c \equiv CP_3c$.

Step 3. $CP_3 = AW$ for some A. In fact, consider $A = \begin{bmatrix} 2k+\gamma-\beta+1 & \to & 2k+2 \\ \downarrow & & \downarrow \\ 3k-\beta+1 & \to & 3k-\gamma+2 \end{bmatrix}$. Then,

$$\begin{aligned} CP_3 &= \begin{bmatrix} 2k+\gamma-\beta+1 & \to & k+\gamma+1 \\ \downarrow & & \downarrow \\ 3k-\beta+1 & \to & 2k+1 \end{bmatrix} \begin{bmatrix} k+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 2k & \to & k+1 \end{bmatrix} = \\ &= \begin{bmatrix} 2k+\gamma-\beta+1 & \to & 2k+2 \\ \downarrow & & \downarrow \\ 3k-\beta+1 & \to & 3k-\gamma+2 \end{bmatrix} \begin{bmatrix} 2k+1 & \to & k+\gamma+1 \\ \downarrow & & \downarrow \\ 3k-\gamma+1 & \to & 2k+1 \end{bmatrix} \begin{bmatrix} k+\gamma & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 2k & \to & k+1 \end{bmatrix} \\ &= \begin{bmatrix} 2k+\gamma-\beta+1 & \to & 2k+2 \\ \downarrow & & \downarrow \\ 3k-\beta+1 & \to & 3k-\gamma+2 \end{bmatrix} \begin{bmatrix} 2k+1 & \to & \gamma+1 \\ \downarrow & & \downarrow \\ 3k-\gamma+1 & \to & k+1 \end{bmatrix} = AW \end{aligned}$$

Therefore, $aP_2bP_3c = aPbELP_3c \equiv aPbECP_3c = aPbEAWc$. At last, consider

$$\tilde{W} = \begin{bmatrix} 3k-\beta+2 & \to & k+2 \\ \downarrow & & \downarrow \\ 4k-2\beta+\gamma+1 & \to & 2k-\beta+\gamma+1 \end{bmatrix}.$$

Then $AW\tilde{W} = \theta_r[\gamma]$ with $r = 2k-\beta+1$. Hence $aPbEAWc \equiv aPbE\theta_r[\gamma]c$ and, by Lemma 2.3.16, $E\theta_r[\gamma] = \theta_r[\gamma]C_r(E)$. Therefore,

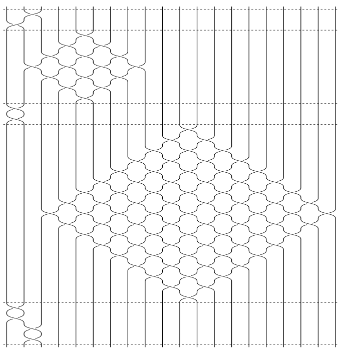
$$aPbE\theta_r[\gamma]c = aPb\theta_r[\gamma]C_r(E)c = aPb\theta_r[\gamma]cC_r(E) \equiv aPb\theta_r[\gamma]c \equiv aPbAWc.$$

Furthermore, since $A \in B_\infty[2k+1]$ and $\operatorname{supp} P = 2k$,

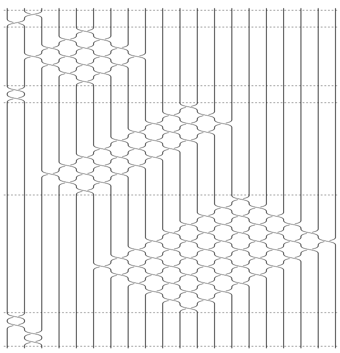
$$aPbAWc = aPAbWc = AaPbWc \equiv aPbWc.$$

$\square$

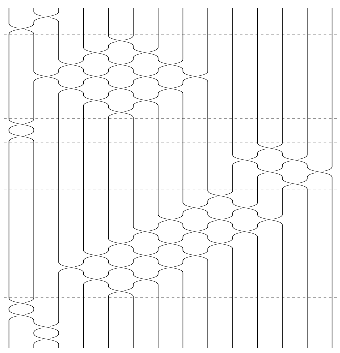
Example 2.3.20. In this example we illustrate the method described in the proof of the theorem above. Here we used $a = \sigma_2^{-1}\sigma_1^{-1}, b = \sigma_1^2, c = \sigma_1^2\sigma_2^2, \alpha = \delta = 3, \beta = 1$ and $\gamma = 2$. In each of the figures below, the diagrams are different representatives of the same double coset, obtained following the steps of the proof of Proposition 2.3.4. The dashed horizontal lines highlight the different braids mentioned in the captions.



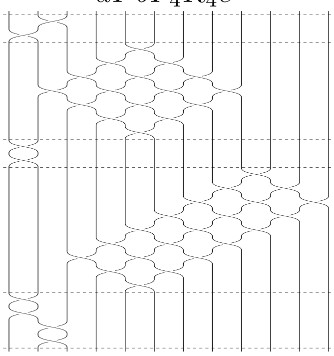
$a\theta_k[\beta]b\theta_l[\gamma]c$



$aPbP_4R_4c$

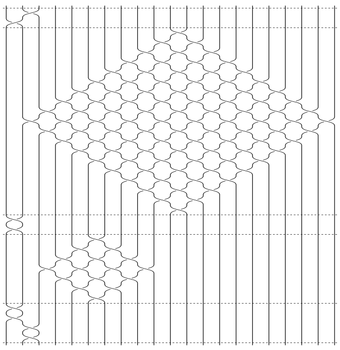


$aPbR_5Wc$

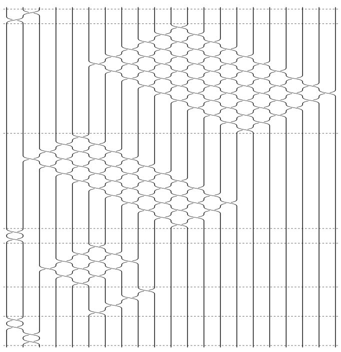


$aPbWc$

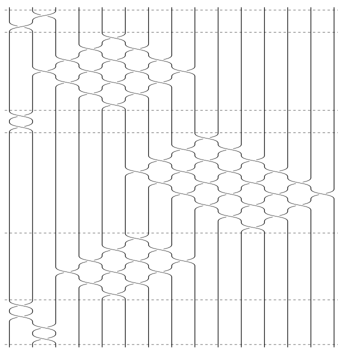
Figure 2.7: The equality $a\theta_k[\beta]b\theta_l[\gamma]c = aPbWc$.



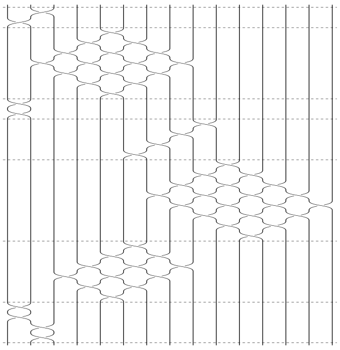
$a\theta_{l'}[\beta]b\theta_k[\gamma]c$



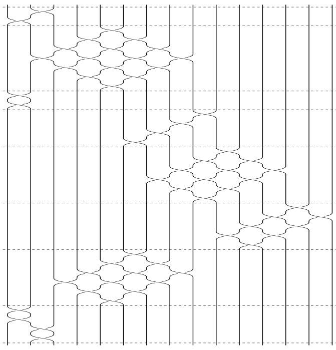
$aR_2P_2bP_3R_3c$



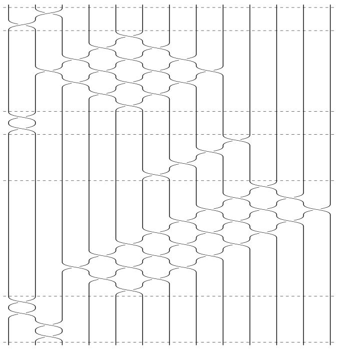
$aPbFP_3c$



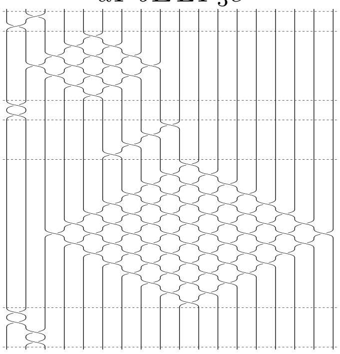
$aPbELP_3c$



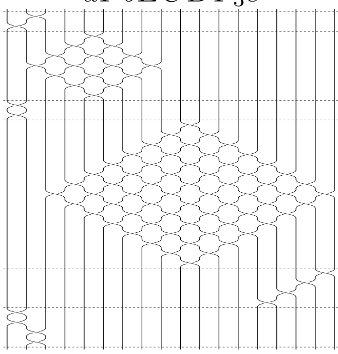
$aPbECDP_3c$



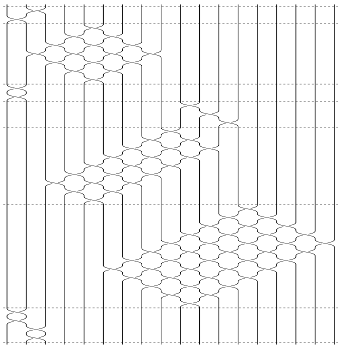
$aPbEAWc$



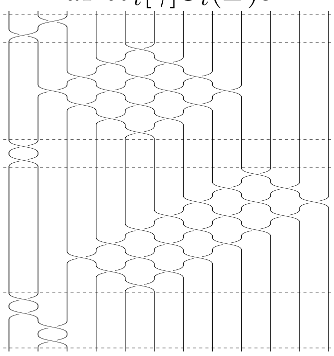
$aPbE\theta_t[\gamma]c$



$aPb\theta_t[\gamma]C_t(E)c$



$aPbAW\tilde{W}c$



$aPbWc$

Figure 2.8: The equality $a\theta_{l'}[\beta]b\theta_k[\gamma]c = aPbWc$.

2.3.2.2 Proof of Proposition 2.3.6

We show that the conjugacy problem for B_∞ can be reduced to a conjugacy problem in B_n, for some n. Given two braids $x, y \in B_\infty$, since these braids are finitely supported, there exists $n \in \mathbb{N}_+$ such that we can consider these braids as elements of B_n. Since the conjugacy problem has a solution in B_n, to prove the proposition it suffices to show that x is conjugate to y in B_n if and only if they are conjugate in B_∞. But this follows from the properties of the direct limit and the fact that the inclusions $i_n : B_n \to B_{n+1}$ do not merge conjugacy classes (see [17]).

2.3.2.3 Proof of Theorem 2.3.8

Let p and q be representatives of the double cosets $\mathfrak{p} \in G[n]\backslash GL(\infty)/G[k]$ and $\mathfrak{q} \in G[k]\backslash GL(\infty)/G[m]$, respectively. Define the sequence of double cosets

$$\mathfrak{r}_j = G[n]\, p\Theta_j[k]q\, G[m],$$

in $G[n]\backslash GL(\infty)/G[m]$.

We remark the following identity:

Lemma 2.3.21. *If $\eta : B_\infty \to GL(\infty)$ is the Burau representation, as defined in subsection 2.3.1.2, the following identity holds*

$$\Theta_j[k] = \eta(\theta_j[k]), \text{ for all } j, k \in \mathbb{N}.$$

Proposition 2.3.22. *The sequence $\mathfrak{r}_j$ above is eventually constant and its limit does not depend on the choice of representatives.*

Proof. Let $N \in \mathbb{N}_+$ be such that $N > \max\{m, k, n\}$ and p and q can be written as square $(k + N + \infty)$-matrices with the following block configuration:

$$p = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}, \qquad q = \begin{pmatrix} x & y & 0 \\ z & w & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}.$$

Suppose that for some $i \geq N$ we have $\mathfrak{r}_i = \mathfrak{r}_N$. We show that $\mathfrak{r}_i = \mathfrak{r}_{i+1}$. As we saw in Proposition 2.3.13, there are elements $u, l \in B_\infty$ such that $\theta_i[k] = u\theta_{i+1}[k]l$. Hence, if $U = \eta(u)$ and $L = \eta(l)$, we have

$$\Theta_i[k] = U\Theta_{i+1}[k]L.$$

Furthermore, U and L have the following block configuration

$$U = \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & 1_i & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}, \quad L = \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & 1_i & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}.$$

Thus,

$$Up = pU \quad \text{and} \quad Lq = qL.$$

Consequently,

$$p\Theta_i[k]q = pU\Theta_{i+1}[k]Lq = Up\Theta_{i+1}[k]qL.$$

Since U and L are elements of the image of the Burau representation η, we have that $U, L \in G[k]$ and therefore

$$\mathfrak{r}_{i+1} = G[n]\, p\Theta_{i+1}[k]q\, G[m] = G[n]\, Up\Theta_{i+1}[k]qL\, G[m] = G[n]\, p\Theta_i[k]q\, G[m] = \mathfrak{r}_i.$$

To show that the limit of the sequence $\mathfrak{r}_\mathfrak{i}$ does not depend on the choice of representatives it suffices to show that, for any H and J in $G[k]$, we have

$$\lim G[n]\, p\Theta_i[k]q\, G[m] = \lim G[n]\, pJ\Theta_i[k]Hq\, G[m].$$

Let $N > 0$ be as before. Consider $M > N$ such that H and J are square $(k + M + \infty)$-matrices with the block configuration:

$$H = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}, \qquad J = \begin{pmatrix} 1_k & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1_\infty \end{pmatrix}.$$

Since H preserves the vector v, we have that $V_M h = V_M$. Similarly, $jV_M = V_M$. Therefore,

$$\begin{aligned} J\Theta_M[k]H &= \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & 1_M & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \begin{pmatrix} 1_k & 0 & 0 & \\ 0 & V_M & t^M 1_M & 0 \\ 0 & 1_M & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & 1_M & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \\ &= \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & jV_M h & t^M h & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} = \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & V_M & t^M h & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \\ &= \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & 1_M & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & V_M & t^M 1_M & 0 \\ 0 & 1_M & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix} \begin{pmatrix} 1_k & 0 & 0 & 0 \\ 0 & 1_M & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}. \end{aligned}$$

Call J' the new matrix containing the block j and H' the new matrix containing the block h. Then we have

$$pJ\Theta_M[k]Hq = pH'\Theta_M[k]J'q = H'p\Theta_M[k]qJ'.$$

Therefore, $p\Theta_M[k]q$ and $pJ\Theta_M[k]Hq$ belong to the same double coset for M sufficiently large. $\square$

Hence we have a well-defined product of the double cosets $\mathfrak{p}$ and $\mathfrak{q}$ given by

$$\mathfrak{p} \star_t \mathfrak{q} = \lim G[n]\, p\Theta_j[k]q\, G[m].$$

Proposition 2.3.23. *The operation defined above is associative. Furthermore, the Burau representation is a functor between the categories of double cosets of $GL(\infty)$ and of B_∞.*

Proof. The proof of the associative property is analogous to the proof of Theorem 2.3.19, using Lemma 2.3.21. The functoriality follows from Lemma 2.3.21. $\square$

2.3.3 Several copies of B_∞ and relations with other groups

We can extend the above constructions to the product $G^{[n]} = B_\infty \times \cdots \times B_\infty$ of n copies of the infinite braid group. Let K be the diagonal subgroup of $G^{[n]}$. Clearly, K is naturally isomorphic to B_∞. Let $K[\alpha]$ be the image of $B_\infty[\alpha]$ under this isomorphism. We define the product of double cosets componentwise.

Corollary 2.3.24. *Consider two double cosets*

$$\mathfrak{p} \in K[\alpha]\backslash G^{[n]}/K[\beta], \quad \mathfrak{q} \in K[\beta]\backslash G^{[n]}/K[\gamma],$$

and let p and q be their respective representatives. Then the operation given by

$$\mathfrak{p} \circ \mathfrak{q} = K[\alpha]\, p\theta_j[\beta]q\, K[\beta],$$

for j sufficiently large, is well-defined and associative.

Proof. It follows from Propositions 2.3.13, 2.3.18 and 2.3.19. $\square$

When there is a surjective homomorphism from B_∞ onto a group A, we have an induced operation on the double cosets of A. More precisely, let $\psi : B_\infty \to A$ be a surjective homomorphism and consider, for each $\alpha \in \mathbb{N}$, the image $A[\alpha]$ of the subgroup $B_\infty[\alpha]$ by ψ. Then the induced product on the double cosets of A with relation to the subgroups $A[\alpha]$ is well-defined. Indeed, this follows from the fact that the sequence used to define the product of double cosets in B_∞ not only converges, it becomes constant.

For integers $\beta \geq 0$ and $n > 0$, denote by $\theta_n^s[\beta]$ the permutation of S_∞ given by

$$\theta_n^s[\beta](i) = \begin{cases} i+n, & \beta < i \leq \beta + n \\ i-n, & \beta + n < i \leq \beta + 2n \\ i, & \text{otherwise.} \end{cases}$$

It is easy to see that this permutation is the distinguished element used to define the product of double cosets of $\overline{S}_\infty$ in Section 1.3.1.

Consider the canonical homomorphism $j : B_\infty \to S_\infty$ that associates to each braid the corresponding permutation of its endpoints. It is clear that this is a surjective homomorphism (it is, up to conjugacy, the only surjective homomorphism from B_∞ to S_∞, see [1]) and hence induces an operation on the double cosets of S_∞ with relation to the subgroups $j(B_\infty[\alpha]), \alpha \in \mathbb{N}$. Furthermore, it is easy to check that $j(B_\infty[\alpha]) = S_\infty[\alpha]$ for each $\alpha \in \mathbb{N}$.

Proposition 2.3.25. *The operation on double cosets of S_∞, with relation to the subgroups $S_\infty[\alpha], \alpha \in \mathbb{N}$, coincides with the operation induced by the group B_∞.*

Proof. In fact, it suffices to check that the elements $\theta_n^s[\beta]$ and $j(\theta_n[\beta])$ coincide for all integers $n > 0$ and $\beta \geq 0$. But this identity follows directly from the definition of $\theta_n[\beta]$ and j. $\square$

As a last remark, we point out some similarities between the multiplicative structure defined in B_∞ and that of $\mathrm{Aut}(F_\infty)$. The group $\mathrm{Aut}(F_\infty)$ is defined as follows: Let F_n be the free group with n generators $x_1, \ldots, x_n$ and denote by $\mathrm{Aut}(F_n)$ the group of automorphisms of F_n. Then

$$\mathrm{Aut}(F_\infty) = \lim \mathrm{Aut}(F_n).$$

The limit is taken with relation to the obvious inclusion $\mathrm{Aut}(F_n) \to \mathrm{Aut}(F_{n+1})$.

For each $\alpha \in \mathbb{N}$ consider the subgroup $H(\alpha)$ of $\mathrm{Aut}(F_\infty)$ of automorphisms h such that $h(x_i) = x_i$ for $i \leq \alpha$. In [33], it is defined a product on the double cosets of $\mathrm{Aut}(F_\infty)$ in the following way: Consider the automorphism $\vartheta_j[\beta] \in \mathrm{Aut}(F_\infty)$ given by

$$\vartheta_j[\beta](x_i) = \begin{cases} x_i, & i \leq \beta, i > 2j + \beta \\ x_{i+j}, & \beta < i \leq \beta + j \\ x_{i-j}, & \beta + j < i \leq \beta + 2j. \end{cases}$$

Then, for p and q in $\operatorname{Aut}(F_\infty)$, the product of the double cosets $H(\alpha)\,p\,H(\beta)$ and $H(\beta)\,q\,H(\gamma)$ is the double coset limit of the sequence $p\vartheta_j[m]q$ in $H(\alpha)\backslash \operatorname{Aut}(F_\infty)/H(\gamma)$. Consider the limit homomorphism $i_\infty : B_\infty \to \operatorname{Aut}(F_\infty)$. The element $\vartheta_j[m]$ is related to the image of the element $\theta_j[m]$ as we see in the following proposition

Proposition 2.3.26. *Let β be a fixed positive integer. For each $k \in \mathbb{N}_+$, consider the element $y_k = x_{\beta+k}x_{\beta+k-1}\cdots x_{\beta+1} \in F_\infty$. Then*

$$i_\infty(\theta_k[\beta])(x_i) = \begin{cases} x_i, & i \leq \beta, i > 2k+\beta \\ y_k^{-1}x_{i+k}y_k, & \beta+1 \leq i \leq k+\beta \\ x_{i-k}, & k+\beta < i \leq 2k+\beta. \end{cases}$$

In other words,

$$i_\infty(\theta_k[\beta])(x_i) = \begin{cases} y_k^{-1}\vartheta_k[\beta](x_i)y_k, & \beta+1 \leq i \leq k+\beta \\ \vartheta_k[\beta](x_i), & \textit{otherwise.} \end{cases}$$

Proof. For $k = 1$ we have that $\theta_1[\beta] = \sigma_{\beta+1}$ and therefore

$$i_\infty(\theta_1[\beta])(x_i) = i_\infty(\sigma_{\beta+1})(x_i) = \begin{cases} x_i, & i \leq \beta, i > \beta+2 \\ x_i^{-1}x_{i+1}x_i, & i = 1+\beta \\ x_{i-1}, & i = 2+\beta. \end{cases}$$

We are going to show the truth of the identity by induction on k. Suppose the identity holds for k. We can write $\theta_{k+1}[\beta]$ as

$$\theta_{k+1}[\beta] = \sigma_{k+\beta+1}\cdots\sigma_{2k+\beta+1}\theta_k[\beta]\sigma_{2k+\beta}\cdots\sigma_{k+\beta+1}.$$

If we put $w = \sigma_{k+\beta+1}\cdots\sigma_{2k+\beta+1}$ and $s = \sigma_{2k+\beta}\cdots\sigma_{k+\beta+1}$ we can re-write the equation above as

$$\theta_{k+1}[\beta] = w\theta_k[\beta]s.$$

We have five cases to analyze:

Case 1 When $\beta+1 \leq i \leq k+\beta$, notice that $i_\infty(s)(x_i) = x_i$ and $i_\infty(\theta_k[\beta])(x_i) = y_k^{-1}x_{i+k}y_k$, therefore $i_\infty(\theta_{k+1}[\beta])(x_i) = i_\infty(w)(y_k^{-1}x_{i+k}y_k)$. Now,

$$\begin{aligned} i_\infty(w)(x_{i+k}) &= i_\infty(\sigma_{k+\beta+1}\cdots\sigma_{i+k-1})i_\infty(\sigma_{i+k})(x_{i+k}) = \\ & i_\infty(\sigma_{k+\beta+1}\cdots\sigma_{i+k-2})i_\infty(\sigma_{i+k-1})(x_{i+k}^{-1}x_{i+k+1}x_{i+k}) = \\ & i_\infty(\sigma_{k+\beta+1}\cdots\sigma_{i+k-3})i_\infty(\sigma_{i+k-2})(x_{i+k-1}^{-1}x_{i+k+1}x_{i+k-1}) = \cdots = \\ & i_\infty(\sigma_{k+\beta+1})(x_{k+\beta+2}^{-1}x_{k+i+1}x_{k+\beta+2}) = x_{k+\beta+1}^{-1}x_{k+i+1}x_{k+\beta+1}. \end{aligned}$$

Hence,

$$i_\infty(w)(y_k^{-1}x_{i+k}y_k) = y_k^{-1}i_\infty(w)(x_{i+k})y_k = y_k^{-1}x_{k+\beta+1}^{-1}x_{k+i+1}x_{k+\beta+1}y_k = y_{k+1}^{-1}x_{k+i+1}y_{k+1}.$$

Case 2 When $i = k+\beta+1$, we have that

$$i_\infty(s)(x_{k+i+\beta}) = x_{k+\beta+1}^{-1}\cdots x_{2k+\beta}^{-1}x_{2k+\beta+1}x_{2k+\beta}\cdots x_{k+\beta+1}.$$

Hence,

$$i_\infty(\theta_k[\beta]s)(x_{k+\beta+1}) = i_\infty(\theta_k[\beta])(x_{k+\beta+1}^{-1}\cdots x_{2k+\beta}^{-1}x_{2k+\beta+1}x_{2k+\beta}\cdots x_{k+\beta+1}) = \\ x_{\beta+1}^{-1}\cdots x_{k+\beta}^{-1}x_{2k+\beta+1}x_{k+\beta}\cdots x_{\beta+1} = y_k^{-1}x_{2k+\beta+1}y_k.$$

Furthermore, $i_\infty(w)(x_{2k+\beta+1}) = x_{k+\beta+1}^{-1}x_{2k+\beta+2}x_{k+\beta+1}$ and hence

$$i_\infty(\theta_{k+1}[\beta])(x_{k+\beta+1}) = i_\infty(w)(y_k^{-1}x_{2k+\beta+1}y_k) = \\ = y_k^{-1}x_{k+\beta+1}^{-1}x_{2k+\beta+2}x_{k+\beta+1}y_k = y_{k+1}^{-1}x_{2k+\beta+2}y_{k+1}.$$

Case 3 When $k+\beta+1 < i \leq 2k+\beta+1$, it is sufficient to notice that $i_\infty(s)(x_i) = x_{i-1}$, $i_\infty(\theta_k[\beta])(x_{i-1}) = x_{i-k-1}$ and $i_\infty(w)(x_{i-k-1}) = x_{i-k-1}$.

Case 4 For the case $i = 2k+\beta+2$ we have $i_\infty(\theta_k[\beta]s)(x_{2k+\beta+2}) = x_{2k+\beta+2}$. Furthermore, $i_\infty(w)(x_{2k+\beta+2}) = x_{k+\beta+1}$.

Case 5 For $i \leq \beta$ or $i > 2k+\beta+2$, we have that $i_\infty(w)(x_i) = i_\infty(\theta_k[\beta](x_i) = i_\infty(s)(x_i) = x_i$ and the result follows.

□

Thus the elements $\vartheta_j[\beta]$ and $i_\infty(\theta_j[\beta])$ are always conjugate in $\mathrm{Aut}(F_\infty)$ (in particular, by an element of $H(\beta)$). Nevertheless i_∞ does not induce a homomorphism between the monoids of double cosets. In fact, consider the braid $\omega = \sigma_2^{-1}\sigma_3\sigma_1\sigma_3\sigma_2$ in B_∞ and its projection $[\omega]$ in $B[2]\backslash B_\infty/B[2]$. Then $i_\infty(\omega\theta_N[2]\omega)$ and $i_\infty(\omega)\vartheta_N[2]i_\infty(\omega)$ do not belong to the same double coset of $H(2)\backslash\mathrm{Aut}(F_\infty)/H(2)$.

As a final remark we show how to define representations similar to the ones in section 1.6.1 for any group G^n. Fix $k \in \mathbb{N}_+ \cup \{\infty\}$ and consider n representations of G, namely $p_i : G \to GL_k$ for $1 \leq i \leq n$.

For each i and $g \in G$ consider a matrix representation of $p_i(g) = (a^i_{m,j})$. Let $a^i_{m,j} : V \to V$ act as scalar multiplication by $a^i_{m,j}$.

Set $H = V^n$ and $X = H^k$. Let $P(g) = P(p_1, p_2, \ldots)(g)$ be the matrix of operators given by

$$(P(g))_{m,j} = \begin{cases} a^i_{x+1,y+1} & m = nx+i \ \&\ j = ny+i \\ 0 & \textit{otherwise} \end{cases}$$

For each $g \in G^n$ the matrix $P(g)$ defines an action on X by: if $v = (v_i)_i$ a element of X let $r(v) = (w_i)_i$ where $w_i = \sum_j P_{i,j}(v_j)$. Denoting by $\mathbf{H} = \otimes^n V$ and $\mathbf{X} = \otimes^k \mathbf{H}$ we define a representation $\rho : \mathbb{G} \to \mathrm{Aut}(\mathbf{X})$ as the representation induces by r on $\mathbf{X}$

Chapter 3

Branched Covers

3.1 Classification of branched covers

3.1.1 Introduction

Let M and N denote closed, connected surfaces. A smooth map $\phi\colon M \to N$ is a degree d branched (or ramified) cover if

1. There exists a finite set $B_\phi \subset N$ such that the restriction map $\phi\colon M \smallsetminus \phi^{-1}(B_\phi) \to N \smallsetminus B_\phi$ is a degree d cover map;

2. For each $x \in B_\phi$, there is a neighborhood V of x, and a neighborhood U_y of each preimage y of x by ϕ, such that the restriction map $\phi_{|U_y}\colon U_y \to V$ is equivalent, up to a topological change of coordinates, to the map $z \to z^{k_y}$ on the unity disk, for some integer $k_y > 0$.

For each $y \in \phi^{-1}(x)$ the integer k_y, called the *branching number* of y, is uniquely determined. In particular, this integer is defined for every point of M and is equal to 1 if and only if ϕ is locally a homeomorphism.

The branching number of a point y can be visualized as the number of preimages, close to y, of a point close to $\phi(y)$. When $k_y > 1$ this point is called a *critical point* and its image a *critical value.* Moreover, for each $x \in N$, we have the relation

$$d = \sum_{y \in \phi^{-1}(x)} k_y$$

In other words, the branching numbers of the preimages of any point in N form an integer partition of d and this partition is non-trivial if and only if this point is a critical value of ϕ. Let $\{x_i; 1 \leq i \leq q\}$ be an enumeration of B_ϕ, then the *branch data* of ϕ is the list $\mathcal{D} = [\Pi_1, \ldots, \Pi_q]$ where Π_i is the non-trivial integer partition of d induced by x_i.

We say that two branched covers $\phi, \psi\colon M \to N$ are *equivalent* if there are homeomorphisms $f\colon M \to M$ and $g\colon N \to N$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\phi"'] & M \arrow[d, "\psi"] \\ N \arrow[r, "g"'] & N \end{tikzcd}$$

We say that the equivalent branched covers ϕ and ψ are *strongly equivalent* if the homeomorphism g is homotopic to the identity.

3.1.2 The realization problem

Consider an integer $d > 1$. Given a closed connected surface N and a list $\mathcal{D}$ of non-trivial integer partitions of d, is there a branched cover $\phi\colon M \to N$ with $\mathcal{D}$ as branch data?

The history of the problem goes back to Hurwitz [19], who essentially showed how to reduce the general question to a problem about realizing partitions by suitable permutations in the symmetric group S_d. It turns out to be very delicate problem, in general, to decide whether or not to exist such an element in S_d for a given $\mathcal{D}$. There are some well-known necessary conditions, which are referred as the *Hurwitz conditions*, and which are consequences of the *Riemann-Hurwitz formula*:

Let $\Pi_i = [k_1, k_2, \ldots, k_n]$ be an integer partition of d. We define the *weight* of Π_i by

$$\nu(\Pi_i) = \sum_{j=1}^{n}(k_j - 1).$$

For a list $\mathcal{D} = [\Pi_1, \ldots, \Pi_q]$ its *total weight* is given by

$$\nu(\mathcal{D}) = \sum_{i=1}^{q} \nu(\Pi_i).$$

Proposition 3.1.1 (Proposition 2.6 of [13])**.** *The total weight has the property that, if $\mathcal{D}$ is the branch data of a branched cover, the $\nu(\mathcal{D})$ is even.*

Theorem 3.1.2 (Riemann-Hurwitz formula, Proposition 2.4 of [13])**.** *Let $\mathcal{D}$ be the branch data of a branched cover $\phi\colon M \to N$ of degree d. Then,*

$$\chi(M) = d\chi(N) - \nu(\mathcal{D}).$$

When the Euler characteristic of N is non-positive and the total weight of $\mathcal{D}$ is even, the answer to the realization problem is positive ([13]):

Theorem 3.1.3 (Theorem 3.3 of [13])**.** *Let N be a closed connected surface with Euler characteristic $\chi(N) \leq 0$, and let $\mathcal{D}$ be a list of partitions of d with $\nu(\mathcal{D})$ even. Then there is a degree d branched cover $\phi\colon M \to N$ with branch data $\mathcal{D}$ and M connected. Furthermore, M is orientable if and only if N is orientable.*

Theorem 3.1.4 (Theorem 5.1 of [13])**.** *If N is the projective plane and $\mathcal{D}$ is a list of partitions of d. Then there exists a branched cover $\phi\colon M \to N$ with M closed and connected with branch data $\mathcal{D}$ if and only if $\nu(\mathcal{D})$ is even and $\nu(\mathcal{D}) > d - 1$. Moreover M can be chosen to be nonorientable.*

The case when $N = \mathbb{S}^2$ remains open, but several partial results exist. We point out some results below (Theorem 3.1.5 corresponds to Propositions 5.2, 5.3 and Theorem 5.4 of [13]):

Theorem 3.1.5 ([13])**.** *Let $\mathcal{D} = \{A_1, \ldots, A_k\}$ be a partition of $d \geq 2$.*

1. *If $A_i = [d]$ for some $1 \leq i \leq k$, $\mathcal{D}$ is realizable if and only if $\nu(\mathcal{D}) \geq 2d - 2$ and even.*

2. *If $A_i = [d-1, 1]$ for some $1 \leq i \leq k$, $\mathcal{D}$ is realizable if and only if $\nu(\mathcal{D}) \geq 2d-2$ and even and $\mathcal{D}$ is not*
   - $\{[2,2], \ldots, [2,2], [3,1]\}(d = 4, k \geq 3)$ *or*
   - $\{[2, \ldots, 2], \ldots, [2, \ldots, 2], [d-1, 1]\}(d = 2r, k = 3)$.
3. *If $d \neq 4$ and $\nu(\mathcal{D}) \geq 3(d-1)$ and even, then $\mathcal{D}$ is realizable.*

Theorem 3.1.6 (Theorem C of [45])**.** *Let $\ell = (a_1, \ldots, a_n)$ be a list of integers with $2 \leq a_i \leq d, \forall i$. Then ℓ is realizable as the list of critical branching numbers of a branched cover $\phi\colon \mathbb{S}^2 \to \mathbb{S}^2$ of the degree d if and only if $\sum_i (a_i - 1) = 2d - 2$.*

3.1.3 Classification

For a branched cover $\phi\colon M \to N$ of degree d, set $N_0 = N \smallsetminus B_\phi$ and $M_0 = M \smallsetminus \phi^{-1}(B_\phi)$. Denote by $\phi_0\colon M_0 \to N_0$ the restriction of ϕ.

The cover ϕ_0 is determined by a homomorphism $\rho_\phi\colon \pi(N_0, *) \to S_d$, where $*$ is a base point. The representation ρ is determined, up to an inner automorphism of S_d, by choosing a one-to-one correspondence between $\phi^{-1}(*)$ and $\{1, 2, \ldots, d\}$ and assigning to each loop α in N_0, based at $*$, the permutation of $\{1, 2, \ldots, d\}$ induced by transporting $\phi^{-1}(*)$ along α using the path lifting property.

Theorem 3.1.7 (Classification, Theorem 2.1 of [3])**.** *Two branched covers $\phi_1\colon M_1 \to N$ and $\phi_2\colon M_2 \to N$ of degree d are equivalent if and only if there exists a homeomorphism*

$$h\colon (N, B_{\phi_1}, *) \to (N, B_{\phi_2}, *)$$

and an inner automorphism $\mu\colon S_d \to S_d$ such that

$$\mu \rho_{\phi_1} = \rho_{\phi_2} h_*.$$

Theorem 3.1.8 (Existence, Theorem 2.2 of [3])**.** *Let N be a compact, connect and oriented surface. If B is a finite subset of $\operatorname{int} N$ and $\rho\colon \pi(N \smallsetminus B, *) \to S_d$ is a representation such that it is nontrivial on each class represented by a small loop around any single point of B, then there exists a branched cover $\phi\colon M \to N$ with $B_\phi = B$ and $\rho_\phi = \rho$.*

Fix an orientation for $\Sigma = \Sigma_g^{n,k}$ and let $* \in \operatorname{int}\Sigma$ be a base point, B the set of marked points and $D \subset \operatorname{int}\Sigma$ a small disk centered at $*$. Let $c_1, \ldots, c_k$ denote the oriented boundary components.

Let $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ be a maximal family of simple closed curves in $\operatorname{int}\Sigma$ such that $a_i \cap a_j = b_i \cap b_j = a_i \cap b_j = \emptyset$ if $i \neq j$, and $a_i \cap b_i$ is a single point of transverse intersection for each i. Orient each of these curves so that the orientation of a_i followed by that of b_i corresponds to the orientation of Σ at $a_i \cap b_i$, and so that the induced orientation on a curve representing the commutator $a_i b_i a_i^{-1} b_i^{-1}$ is the same as that induced by $\Sigma \smallsetminus a_i \cup b_i$ using the convention that the orientation of a boundary component, followed by an inward normal, should coincide with the orientation of Σ. Around each point of B choose a small disk D_i whose boundary is oriented by $\Sigma \smallsetminus D_i$. Choose $n + g + k$ simple arcs r_i with disjoint interiors running from $*$ to the points $a_i \cap b_i, 1 \leq i \leq g$ and to the boundary components $c_1 \ldots, c_k, \partial D_1, \ldots, \partial D_n$ in order. We choose and index the set of arcs r_j so that in the disk D they correspond to distinct radii, indexed cyclically, consistent with the orientation of ∂D. We choose each arc r_j to $a_i \cap b_i$ so that at this point the orientation of a_i followed by the reverse orientation of r_j toward $*$ is positive, while the orientation of b_i followed by the reverse orientation of r_j toward $*$ is negative.

These choices identify the group $\pi_1(\Sigma \setminus B, *)$ as a free group with $n + 2g + k$ generators, say $w_1, \dots, w_n, x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_k$ and subject to the single relation

$$w_1 \cdots w_n [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_k = 1,$$

where $[x_j, y_j] = x_j y_j x_j^{-1} y_j^{-1}$. Such a system of arcs and simple curves is called a Hurwitz arc system for Σ. Now a representation $\pi_1(\Sigma \setminus B, *) \to G$, where G is a group, determines and is determined by a sequence

$$(\sigma_1, \dots, \sigma_n, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_k)$$

of elements of G subject to the requirement that

$$\sigma_1 \cdots \sigma_n [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_k = 1.$$

If $\phi\colon M \to \Sigma$ is a branched cover of degree d between compact, connected oriented surfaces corresponding to a representation $\pi_1(\Sigma \setminus B, *) \to S_d$, then the sequence

$$H(\phi) = (\sigma_1, \dots, \sigma_n, \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \gamma_1, \dots, \gamma_k)$$

of permutations corresponding to a Hurwitz arc system for Σ will be called a Hurwitz system for ϕ.

Corollary 3.1.9 (Theorem 2.3 of [3])**.** *Two branched covers of degree d over a given Σ are equivalent if and only if they have Hurwitz systems that are conjugate by an element of S_d.*

3.1.4 Extension to higher dimensional branching sets

Most of the contents of this section are based on Samperton's paper [43]. Let G be a group and X a space. Suppose that G has a continuous action on X, free and properly discontinuous (and such that $y \mapsto y \cdot g$ is an homeomorphism for all g). If $p\colon X \to X/G$ is a cover, then p is called a G-cover. Furthermore, if the action of the Deck transformations is transitive on each fiber, then p is called a regular G-cover.

Let N be a smooth, connected manifold, possibly with boundary. Let G be a discrete group and $C \subset G$ a conjugacy invariant subset. A *C-branched G-cover of N* consists of the following data:

- A smooth map $\phi\colon M \to N$, where M is a smooth manifold.
- A codimension two properly embedded submanifold $K \subset N$, possibly empty, called *branch locus.*
- A trivialization of the unit disk bundle

  $$N(K) \simeq K \times D^2$$

  where D^2 has the standard orientation inherited from $\mathbb{R}^2$. We identify $N(K)$ with a closed regular neighborhood of K.

This data must satisfy the following conditions:

- The restriction $\phi_{|\phi^{-1}(N \smallsetminus K)}$ is a regular G-cover.
- (C-branched condition) The monodromy homomorphism $\pi_1(N \smallsetminus K) \to G$ associated to the regular G-cover $\phi_{|\phi^{-1}(N \smallsetminus K)}$ sends a counterclockwise loop around the boundary of each fiber D^2 of $N(K)$ into C.
- The restriction $\phi_{|\phi^{-1}(K)}$ is a cover over each component of K.

If N is not connected the a C-branched G-cover of N is simply a C-branched G-cover of each component of K. If we pick a component of K, then for any two fibers of $N(K)$ over that component, their counterclockwise boundary loops map to conjugate elements of G. Thus, to every component of K we associate a conjugacy class in C, called the *branch type* of the component.

When N is 2-dimensional, K is simply a collection of points and the disk bundle $N(K)$ is the union of the unit tangent disks at K. Up to oriented equivalences, a trivialization of a 2-disk bundle over a point is just an orientation of the disk. We conclude that if N is oriented, we can specify a trivialization of $N(K)$ by labeling each branch point in K with a sign $+$ or $-$ to indicate whether the orientation of the normal disk agrees with the orientation of N. Similar remarks apply when N is oriented and 3-dimensional. In this case, K is a link in N, and we can specify an trivialization of $N(K)$ by orienting K.

There are three equivalence relations on C-branched G-covers: equivalence, concordance, and cobordism. Each of these is coarser than the preceding one.

- Two C-branched G-covers M_1 and M_2 of N_1 and N_2, respectively, are *equivalent* if there is a diffeomorphism from M_1 to M_2 that takes $\phi_1^{-1}(K_1)$ diffeomorphically to $\phi_2^{-1}(K_2)$ so that the trivializations are identified, and there is an equivalence of G-covers on the complement of the branch loci. If N_1 and N_2 are oriented, we require equivalences to preserve orientations.
- Two C-branched G-covers M_1 and M_2 are *cobordant* if there exists a manifold W such that $\partial W = N_1 \sqcup N_2$, and a C-branched G-cover $\tilde{W}$ that is equivalent to M_1 when restricted to N_1 and equivalent to M_2 when restricted to N_2. In particular, the trivialization of the branch locus of W has to extend the trivializations of K_1 and K_2. We can also talk about oriented cobordism. In the sequel, when we say cobordism we will always mean oriented cobordism.
- In the previous definition, if $W = M \times I$, then we say M_1 and M_2 are *concordant*. Every equivalence yields a concordance by taking the mapping cylinder, but not all concordances are cylinders because there can be births and deaths of components of the branch loci.

A C-branched G-cover of M is uniquely specified by K, a framing of K and a homomorphism $\pi_1(M \smallsetminus K) \to G$ satisfying the C-branched condition as we can see in the following Lemma

Lemma 3.1.10 (Lemma 2.1 of [43]). *Let K be a codimension 2 properly embedded submanifold of M such that $N(K)$ is trivializable. Then for every choice of framing of K and for every regular G-cover of $M \smallsetminus K$ that satisfies the C-branched condition with respect to the chosen framing, there is a unique (up to equivalence) C-branched G-cover of M with branch locus K with the given framing and G-cover of $M \smallsetminus K$.*

Therefore we can specify a well-defined C-branched G-cover of M by specifying K, a trivialization of $N(K)$, and a G-cover of $M \smallsetminus K$ satisfying the C-branched condition. We describe a G-cover of $M \smallsetminus K$ by picking a base-point and fixing a homomorphism $\pi_1(M \smallsetminus K) \to G$.

3.1.5 Counting stabilizations

3.1.5.1 Stable classification of branched covers

When the total space of the branched covers is a genus g oriented surface with n distinct marked points, we have a *stable* classification of branched covers.

Denote the oriented genus g surface, with n marked points, by Σ_g^n. Let $\phi\colon \pi_1(\Sigma_g^n) \to G$ be a surjective homomorphism and let $K = \{p_1, \dots, p_n\} \subset \Sigma_g$ be the set of punctures. As discussed in the previous section we can specify a framing of K simply by decorating each point p_i with a sign $o_i \in \{+1, -1\}$.

Let $T\colon K \to \{+1, -1\}$ be such that $T(p_i) = o_i$ denote a framing of K and suppose that the homomorphism ϕ satisfies the C-branched condition with respect to T. Lemma 3.1.10 shows that the pair (T, ϕ) determines a C-branched G-cover of Σ_g with branch locus K, which we will refer to simply by (T, ϕ). For each puncture p_i, pick a simple closed loop $\gamma_i \in \pi(\Sigma_g^n)$ winding once counterclockwise around the puncture p_i and not enclosing any other puncture. We define the branching data of the cover (T, ϕ) to be the vector

$$\nu_{(T,\phi)} \in \mathbb{N}^{C/\!\!/G \times \{+1,-1\}}$$

given by

$$\nu_{(T,\phi)}(\bar{c}, o) = \#\{1 \le i \le n; o_i = o, \phi(\gamma_i)^o \in \bar{c}\},$$

where $\bar{c}$ is a conjugacy class in $C/\!\!/G$ and $o \in \{+1, -1\}$. Thus, $\nu_{(T,\phi)}(\bar{c}, o)$ counts the number of branch points such that winding around them in the direction of o yields monodromy in $\bar{c}$.

For a vector ν in $\mathbb{N}^{C/\!\!/G \times \{+1,-1\}}$ we say that its cardinality is the sum of its entries. Given ν of cardinality n, we define

$$R_{g,\nu} = \{(T, \phi); \phi \text{ is onto and } C\text{-branched with respect to } T, \nu_{\phi,T} = \nu\}$$

We define the classifying space BG_C: Let BG be the classifying space of G and denote by LBG the loop space of BG, with the compact-open topology. The set of components of LBG is the space of free homotopy classes of maps $\mathbb{S}^1 \to BG$, which after orienting $\mathbb{S}^1$ can be naturally identified with the set of conjugacy classes $G/\!\!/G$. Let $L^C BG$ be the union of components of LBG corresponding to $C/\!\!/G$. Then

$$BG_C = BG \bigsqcup_{ev} L^C BG \times D^2$$

where $ev\colon L^C BG \times \mathbb{S}^1 \to BG$ is the evaluation map.

Theorem 3.1.11 (Theorem 2.3 of [43])**.** *Fix a discrete group and a conjugacy invariant subset $C \subset G$. Let M be a smooth manifold. Then the homotopy classes of maps from M to BG_C are in natural bijection with concordance classes of C-branched G-covers of M.*

By Theorem 3.1.11, the cover $(T, \phi) \in R_{g,\nu}$ induces a homotopy class of maps

$$(T, \phi)_\#\colon \Sigma_g \to BG_C$$

which in turns induces a homomorphism

$$(T, \phi)_*\colon H_2(\Sigma_g) \to H_2(BG_C).$$

The C-branched Schur invariant of (T, ϕ) is the homology class

$$\mathrm{sch}_C(T, \phi) = (T, \phi)_*([\Sigma_g]) \in H_2(BG_C),$$

where $[\Sigma_g]$ is the orientation of Σ_g.

We define three kinds of stabilization of branched covers. For any $\bar{c} \in C /\!\!/ G$, let $\mathbb{S}^2_{\bar{c}}$ denote any C-branched G-cover of the oriented sphere $\mathbb{S}^2$ such that the branch locus consists of one point with branch data $(\bar{c}, +1)$ and one point with branch data $(\bar{c}, -1)$ and let (T, ϕ) be a C-branched G-cover of Σ_g, then

- A *handle stabilization* of (T, ϕ) is any C-branched G-cover of a surface that is equivalent to the connected sum of (T, ϕ) with the trivial C-branched G-cover over the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

- A *$\bar{c}$-stabilization* of (T, ϕ) is any C-branched G-cover of a surface that is equivalent to the connected sum of (T, ϕ) with a cover of the form $\mathbb{S}^2_{\bar{c}}$ (note that if (T, ϕ) is connected, *i. e.* ϕ is surjective, it does not matter with of the possible choices for $\mathbb{S}^2_{\bar{c}}$ we use, since stabilizing by any of them yields equivalent covers).

- A *puncture stabilization* of (T, ϕ) is some sequence of $\bar{c}$-stabilizations of (T, ϕ) with various $\bar{c} \in C /\!\!/ G$. We call a connected sum of copies of $\mathbb{S}^2_{\bar{c}}$, for possibly varying c, a puncture stabilizing sphere.

Note that if $\phi \in R_{g,\nu}$, then a $\bar{c}$-stabilization of (T, ϕ) has branch data

$$\nu + \delta_{(\bar{c},+1)} + \delta_{(\bar{c},-1)} \in \mathbb{N}^{C /\!\!/ G \times \{+1,-1\}}$$

where δ_x is the delta function on x. In particular, a puncture stabilization never has positive trivialization.

Finally, we say that two C-branched G-covers are *stably equivalent* if they are equivalent after applying some sequence of handle and puncture stabilization to each of them.

Theorem 3.1.12 (Proposition 3.1 of [43])**.** *Suppose that $(T, \phi) \in R_{g,\nu}$ and $(S, \psi) \in R_{h,\omega}$ are connected C-branched G-covers of oriented surfaces. Then $\mathrm{sch}_C(T, \phi) = \mathrm{sch}_C(S, \psi)$ if and only if (T, ϕ) and (S, ψ) are stably equivalent. Moreover, if $g = h$ and C generates G, then handle stabilization is unnecessary, that is, $\mathrm{sch}_C(T, \phi) = \mathrm{sch}_C(S, \psi)$ if and only if (T, ϕ) and (S, ψ) are puncture-stabilization equivalents.*

The theorem above was originally proved by Livingston [25] in the case for which C is empty. It was further improved by Dunfield-Thurston[11], Lönne-Catanese-Perroni[7,8] and Samperton[43] whose version we presented above.

Consider the space $\mathrm{MCG}_*(\Sigma^n_g)$, the pointed mapping class group of Σ^n_g, consisting of isotopy classes of orientation preserving diffeomorphisms of Σ^n_g that fix the base point. This space acts on $R_{g,\nu}$, if $(T, \phi) \in R_{g,\nu}$ then the action on ϕ is the usual action of $\mathrm{MCG}_*(\Sigma^n_g)$ on a homomorphism, and the action on T is induced by the permutation action on K.

Any two $\bar{c}$-stabilizations of a given connected cover are equivalent. Thus, $\bar{c}$-stabilization yields a well-defined map

$$p_{\bar{c}} \colon \frac{R_{g,\nu}}{\mathrm{MCG}_*(\Sigma^n_g)} \to \frac{R_{g,\nu+\delta_{(\bar{c},+1)}+\delta_{(\bar{c},-1)}}}{\mathrm{MCG}_*(\Sigma^{n+2}_g)}.$$

Similarly, any two handle stabilizations are equivalent, thus we have a map

$$h\colon \frac{R_{g,\nu}}{\mathrm{MCG}_*(\Sigma_g^n)} \to \frac{R_{g+1,\nu}}{\mathrm{MCG}_*(\Sigma_{g+1}^n)}.$$

Since two elements of $R_{g,\nu}$ in the same $\mathrm{MCG}_*(\Sigma_g^n)$-orbit are equivalent as C-branched G-covers, they must have the same Schur invariant. Thus the Schur invariant induces a map

$$\mathrm{sch}_C\colon \frac{R_{g,\nu}}{\mathrm{MCG}_*(\Sigma_g^n)} \to H_2(BG_C).$$

All of these maps commute. More precisely we have the commutative diagram

$$\begin{tikzcd}
 & \frac{R_{g,\nu+\delta_{(\bar{c},+1)}+\delta_{(\bar{c},-1)}}}{\mathrm{MCG}_*(\Sigma_g^{n+2})} \arrow[d, "\mathrm{sch}_C"] \arrow[dr, "h"] & \\
\frac{R_{g,\nu}}{\mathrm{MCG}_*(\Sigma_g^n)} \arrow[ur, "p_{\bar{c}}"] \arrow[r, "\mathrm{sch}_C"] \arrow[dr, "h"'] & H_2(BG_C) & \frac{R_{g+1,\nu+\delta_{(\bar{c},+1)}+\delta_{(\bar{c},-1)}}}{\mathrm{MCG}_*(\Sigma_{g+1}^{n+2})} \arrow[l, "\mathrm{sch}_C"'] \\
 & \frac{R_{g+1,\nu}}{\mathrm{MCG}_*(\Sigma_{g+1}^n)} \arrow[u, "\mathrm{sch}_C"] \arrow[ur, "p_{\bar{c}}"'] &
\end{tikzcd}$$

When the group G is finite we have the following theorem about the classification of the orbits of the action of $\mathrm{MCG}_*(\Sigma_g^n)$.

Theorem 3.1.13 (Theorem 1.1 of [43]). *Fix a finite group G and a conjugacy invariant subset $C \subset G$. Let $\nu \in \mathbb{N}^{C/\!\!/G \times \{+1\}}$ be a branching data of cardinality n.*

1. *If ν and g are large enough, then the C-branched Schur invariant is a complete invariant for the orbits of the action of $\mathrm{MCG}_*(\Sigma_g^n)$ on $R_{g,\nu}$.*

2. *If ν is large enough and C generates G, then the C-branched Schur invariant is a complete invariant for the orbits of the action of $\mathrm{MCG}_*(\Sigma_g^n)$ on $R_{g,\nu}$ for all $g \geq 0$.*

3.1.5.2 Stabilizations without punctures

Consider two homomorphisms $\rho_j : \pi_1(\Sigma_{g_j}) \to G, j = 1, 2$. By Theorem 3.1.12 we have that ρ_j are stably (weakly) equivalent if and only the associated classes called Schur invariants agree, namely

$$\mathrm{sch}(\rho_1)[\Sigma_1] = \mathrm{sch}(\rho_2)[\Sigma_2] \in H_2(G)$$

A natural question is how many stabilizations are needed. To do that we first introduce the following invariant.

Consider several tuples $\{(\alpha_i, \beta_i)\}_{i \in I_s}$ of elements of G, where $I_1, I_2, \ldots, I_m \subset \{1, 2, \ldots, g\}$ are disjoint subsets, $|I_s| = g_s$. We suppose that:

$$\prod_{i \in I_s} [\alpha_i, \beta_i] = 1 \in G \tag{3.1}$$

Such data is equivalent to the data of m homomorphisms $\rho_s : \pi_1(\Sigma_{g_s}) \to G$, or alternatively $\rho : \pi_1(\sqcup\Sigma_{g_s}) \to G$.

Assume from now on that ρ_s are bordant, namely that

$$\sum_{s=1}^{m} \mathrm{sch}(\rho_s)[\Sigma_{g_s}] = 0 \in H_2(G) \tag{3.2}$$

If $G = F/R$ is a presentation of the group G, where F is a free group, let consider lifts $\{(\tilde{\alpha}_i, \tilde{\beta}_i)\}_{i\in I_s}$ of these elements to F, namely a lift of ρ to $\tilde{\rho}\colon\ \pi_1(\sqcup\Sigma_{g_s}) \to F$.

We define

$$ocl(\tilde{\rho}) = \min\left\{n; \prod_{i\in I_s}[\tilde{\alpha}_i, \tilde{\beta}_i] = \prod_{j=1}^{n}[f_j, r_j]^{\pm}, f_j \in F, r_j \in R\right\}$$

And set:

$$ocl(\rho) = \min\{ocl(\tilde{\rho}); \tilde{\rho} \text{ lifts } \rho\}$$

Theorem 3.1.14. *The minimal number of stabilizations needed for making $\rho_1 * \rho_2$ weakly equivalent to the trivial representation is $g - (g_1 + g_2)$, where $g = ocl(\rho_1 * \rho_2)$.*

Proof. We first claim that the minimal g which appears above is, the minimal Heegaard genus of a manifold M^3 to which $\rho_1 * \rho_2$ and the trivial one extend to $\rho\colon\ \pi_1(M) \to G$. The fact that it is smaller than this Heegaard genus follows directly from the proof of Dunfield and Thurston.

Conversely, assume that $\rho_1'\colon\ \pi_1(\Sigma_g) \to G$ is a stabilization of $\rho_1 * \rho_2$ and the trivial one ρ_2' are in the same $\mathrm{Aut}(\pi_1(\Sigma_g))$ orbit, namely such that $\rho_2' = \rho_1' \circ \phi$, with $\phi \in \mathrm{Aut}(\pi_1(\Sigma_g))$. We assume that $\pi_1(\Sigma_g) \to \pi_1(\Sigma_{g_1}) * \pi_1(\Sigma_{g_2})$ is the pinch map which kills the generators α_i, β_i, $i \in I_j$. Let then H be the result of adding 2-handles to $\Sigma_g \times [0,1]$ along the curves $\alpha_i, i \in I_1$ and H' the result of adding 2-handles to $\Sigma_g \times [0,1]$ along the curves $\beta_i, i \in I$. Then both H and H' are compression bodies. Moreover, the 3-manifold $M = H \cup_\phi H'$ has $\Sigma_g = H \cap H'$ as a Heegaard surface (nonseparating the boundary components). Observe that there are natural surjective homomorphisms $p_1\colon\ \pi_1(H) \to \pi_1(\Sigma_{g_1}) * \pi_1(\Sigma_{g_2})$ Then the map $\pi_1(\Sigma_g) \to G$ factors as $\pi_1(\Sigma_g) \to \pi_1(H) \to G$ and $\pi_1(\Sigma_g) \to \pi_1(H') \to G$ and hence through $\pi_1(H) *_{\pi_1(\Sigma_g)} \pi_1(H') = \pi_1(M^3)$. This shows that $\rho_1 * \rho_2$ extends to a 3-manifold and the stabilizations are realized on the Heegaard surface.

We now have to prove that the smallest Heegaard genus coincides with the *ocl*. The proof goes similarly with that given by Liechti-Marché [24] for the case of a bordant torus. Consider more generally several bordant ρ_s, as above.

As 2-homology is the bordism in dimension 2 there exists M^3 to which ρ_s extend. Let Σ_g a Heegaard surface in M^3. We denote by H_g the genus g handlebody. Take a basis in $\pi_1(\Sigma_g)$ of the form $\{\alpha_j, \beta_j\}$ such that β_j bound disks in H_g. We adjoin 2-handles to $\Sigma_g \times [0,1]$ over β_j, for all $j \notin \sqcup I_s$. We obtain a compression body diffeomorphic to $H_g \setminus \sqcup H_{g_s}$, where H_{g_s} are embedded in H_g in the standard way.

We can therefore write $M = (H_g \setminus \sqcup H_{g_s}) \cup_\phi \overline{H_g}$. We have then surjective homomorphisms $\pi_1(\Sigma_g) \to \pi_1(H_g)$ and $\pi_1(\Sigma_g) \to \pi_1(H_g \setminus \sqcup H_{g_s})$, while $\pi_1(M) = \pi_1(H_g \setminus \sqcup H_{g_s}) *_{\pi_1(\Sigma_g)} \pi_1(H_g)$. Denote by $\theta\colon\ \pi_1(\Sigma_g) \to \pi_1(M)$ the corresponding surjective map.

Note that $\theta \circ \rho\colon\ \pi_1(\Sigma_g) \to G$ is the stabilization of all ρ_s. Its key property is

$$\theta \circ \rho(\beta_i) = 1, \ \text{ if } i \notin \sqcup I_s$$

Further as $\pi_1(H_g)$ is free, the composition $\pi_1(H_g) \to \pi_1(M) \to G$ lifts to $\tilde{\rho}\colon \pi_1(H_g) \to F$. On the other hand $p\colon \pi_1(\Sigma_g) \to \pi_1(H_g)$ is surjective. Consider the images of $\tilde{\alpha}_j = p\tilde{\rho}(\alpha_j), \tilde{\beta}_j = p\tilde{\rho}(\beta_j)$ of the generators above into F. Then

$$\prod_{i=1}^{g} [\tilde{\alpha}_i, \tilde{\beta}_i] = 1$$

This means that

$$\prod_s \prod_{i \in I_s} [\tilde{\alpha}_i, \tilde{\beta}_i] = \prod_{i \notin \sqcup I_s} [\tilde{\alpha}_i, \tilde{\beta}_i] \tag{3.3}$$

Now $\tilde{\beta}_i \in R$, by definition. This shows that

$$ocl(\rho) + \sum_s g_s \leq g \tag{3.4}$$

Conversely, if we have equation (3.3) then we have defined a well-defined homomorphism $\tilde{\rho}\colon \pi_1(\Sigma_g) \to F$, by $\tilde{\alpha}_j = p\tilde{\rho}(\alpha_j), \tilde{\beta}_j = p\tilde{\rho}(\beta_j)$.

By ([24], Lemma 3.5), such a homomorphism can be described as a composition

$$\tilde{\rho} = f \circ i_* \circ \phi$$

where $i_*\colon \pi_1(\Sigma_g) \to \pi_1(H_g)$ is induced by the inclusion, ϕ is an automorphism of $\pi_1(\Sigma_g)$ and $f\colon \pi_1(H_g) \to F$ is some homomorphism. Let then $M^3 = (H_g \setminus \sqcup H_{g_s}) \cup_\phi \overline{H_g}$. It follows that the map $\tilde{\rho}$ induces $\rho\colon \pi_1(H_g \setminus \sqcup H_{g_s}) \to G$. Further ρ extends to $\rho\colon \pi_1(M) \to G$ if and only if $\rho(\phi^{-1}(\beta_i)) = 1$, which follows from the fact that $i_*(b_i) = 1$. This proves the reverse inequality

$$ocl(\rho) + \sum_s g_s \geq g \tag{3.5}$$

and hence we derive the equality. □

Remark 3.1.15. If M^3 has (at least) two boundary components, then there are several Heegaard decompositions, depending on the number of boundary components required to be on each side. If $\partial M = \Sigma_{g_1} \sqcup \Sigma_{g_2}$ we have a first decomposition $H \setminus (H_1 \cup H_2) \cup \overline{H}$ and another one of the form $(H' \setminus H_1) \cup (\overline{H' \setminus H_2})$. Above we considered the first type of Heegard decomposition (nonseparating boundaries).

3.1.5.3 Stabilization with punctures

We now consider C-branched G-covers. Note that there is a difference between the oriented cobordism of a unbranched G-cover as in [11] and the oriented cobordism of C-branched G-covers, as we can have unbranched G-covers which are oriented cobordant as C-branched G-covers, but not as unbranched G-covers.

Recall that if C generates G, then it follows that two C-branched G-covers of surfaces of the same genus whose reduced Schur invariants agree are equivalent by means of puncture stabilizations alone (see Theorem 3.1.12).

Consider two sets $\{\alpha_i, \beta_i, c_j; 1 \leq i \leq g,\ 1 \leq j \leq k\}$ and $\{\alpha'_i, \beta'_i, c'_j; 1 \leq i \leq g,\ 1 \leq j \leq k'\}$, of elements of G.

We suppose that:

$$\left(\prod_{i=1,g}[\alpha_i,\beta_i]\right)c_1\cdots c_k=\left(\prod_{i=1,g}[\alpha'_i,\beta'_i]\right)c'_1\cdots c'_{k'}=1\in G \tag{3.6}$$

This is equivalent to give two homomorphisms $\rho\colon\ \pi_1(\Sigma_g^k)\to G$, $\rho'\colon\ \pi_1(\Sigma_g^{k'})\to G$ or alternatively $\rho\colon\ \pi_1(\Sigma_g^k\sqcup\Sigma_g^{k'})\to G$.

Assume from now on that $\rho*\rho'$ is C-branched nullbordant, namely that (see [43]):

$$\mathrm{sch}_C(\rho)[\Sigma_g]+\mathrm{sch}_C(\rho')[\Sigma_g]=0\in H_2(BG_C) \tag{3.7}$$

If $G=F/R$ is a presentation of the group G, where F is a free group, let consider lifts $\{\tilde{\alpha}_i,\tilde{\beta}_i,\tilde{c}_j;1\le i\le g,\ 1\le j\le k\}$ and $\{\tilde{\alpha}'_i,\tilde{\beta}'_i,\tilde{c}'_j;1\le i\le g,\ 1\le j\le k'\}$ of these elements to F, namely a lift of $\rho*\rho'$ to $\widetilde{\rho*\rho'}\colon\ \pi_1(\Sigma_g^k\sqcup\Sigma_g^{k'})\to F$.

We define $pcl(\widetilde{\rho*\rho'})$ to be the minimal integer n such that there exists a sequence $\{\tilde{d}_j,\tilde{\ell}_j\}_{j=1}^n\in F$, $1\le j\le n$, with $[\tilde{d}_j,\tilde{\ell}_j]=1\in G$ and $\ell_j\in C$ satisfying the equation

$$\prod_{i=1}^{g}[\tilde{\alpha}_i,\tilde{\beta}_i]\prod_{j=1}^{k}\tilde{c}_j\prod_{r=1}^{g}[\tilde{\alpha}'_r,\tilde{\beta}'_r]\prod_{s=1}^{k'}\tilde{c}'_s=\prod_{t=1}^{n}[\tilde{d}_t,\tilde{\ell}_t]$$

And we set:

$$pcl(\rho*\rho')=\min\{pcl(\widetilde{\rho*\rho'});\tilde{\rho}\text{ lifts }\rho\}$$

Theorem 3.1.16. *Assume that $\rho*\rho'$ has trivial reduced Schur invariant and C generates G. Then the minimal number of punctures stabilizations needed to make $\rho*\rho'$ equivalent to a puncture stabilization of a trivial representation (i.e. the holonomy of a trivial C-branched G-cover) is $pcl(\rho*\rho')$.*

Proof. The C-branched G-cover of $\Sigma_g^k\sqcup\Sigma_g^{k'}$ is nullcobordant. This means that there is a tangle $K\subset\Sigma_g\times[0,1]$ such that there is an extension of the associated branched cover to $\Sigma_g\times[0,1]\setminus K$, and K intersects $\Sigma_g^k\sqcup\Sigma_g^{k'}$ along the punctures.

We can assume that the homological branching type of the two branched covers agree. Therefore the set of punctures $\{c_i,c'_j\}$ admits an involution without fixed points which preserves the label and reverses the framing. We call them dual punctures.

We have an arc of K joining each pair of dual punctures. The remaining components of K are knots in the interior of $\Sigma_g\times[0,1]$, whose union is denoted K°.

We put the tangle K in bridge position with respect to a surface S in $\Sigma_g\times[0,1]$ which does not separate the boundaries. This means that S bounds on one side a handlebody H and $K\cap H$ is a union of properly embedded arcs and the complement $\Sigma_g\times[0,1]\setminus H$ is a compression body which is intersected by K in a set of properly embedded arcs. It follows that $S=\Sigma_{2g}$. Further arcs in H join either points corresponding to dual punctures on $\Sigma_g^k\sqcup\Sigma_g^{k'}$ or else some pairs of new punctures which are also connected by arcs from K° in the complement $\Sigma_g\times[0,1]\setminus H$. Note that the arcs from $K^\circ\cap(\Sigma_g\times[0,1]\setminus H)$ are unknotted and unlinked in the sense that there exist disjoint embedded disks in $\Sigma_g\times[0,1]\setminus H$ whose boundaries consist of the arcs in H union arcs in S. Similarly the arcs $K^\circ\cap H$ are unknotted and unlinked.

Consider $N(K)$ a regular neighborhood of K within $\Sigma_g \times [0,1]$ and $N(K^\circ) \subset N(K)$ be the regular neighborhood of K° obtained by keeping only those components of $N(K)$ containing points of K°. Consider the surface S' obtained by surgery on S as follows:

$$S' = (S \setminus S \cap N(K))_{\partial(S \cap N(K^\circ))} \cup (\partial N(K^\circ) \cap H)$$

Therefore the number of puncture stabilizations arising in this construction is the number n of knot components of the link K°, namely half the number of new punctures on S.

Since we have a C-branched G-cover of $\Sigma_g \times [0,1] \setminus K$, there exists a holonomy map $\psi\colon\ \pi_1(\Sigma_g \times [0,1] \setminus N(K)) \to G$ which extends $\rho * \rho'$.

Remark that $H \setminus N(K)$ is actually homeomorphic to a handlebody, in particular its fundamental group is free. There is a homomorphism $\pi_1(H \setminus N(K^\circ)) \to \pi_1(\Sigma_g \times [0,1] \setminus N(K))$ induced by inclusion. Its composition with ψ admits therefore a lift $\phi\colon\ \pi_1(H \setminus N(K)) \to F$. Recall also that we have a map $\pi_1(S') \to \pi_1(H \setminus N(K))$, also induced by inclusion, whose composition by ϕ will be called $\Phi\colon\ \pi_1(S') \to F$.

Observe that $\pi_1(S^\circ)$ is generated by elements which correspond to $\{\alpha_i, \beta_i, c_j; 1 \leq i \leq g,\ 1 \leq j \leq k\}$, $\{\alpha'_i, \beta'_i, c'_j; 1 \leq i \leq g,\ 1 \leq j \leq k'\}$ and the elements d_j, ℓ_j associated to the surgered 1-handles $N(K_i) \cap H$. The longitude of $N(K_i)$ is specified by the framing and the choice of arcs joining the endpoints of $K_i \cap S$ along S.

Notice that $\phi([d_i, \ell_i]) = 1 \in G$, since the commutator is trivial in $\pi_1(\Sigma_g \times [0,1] \setminus N(K))$. Moreover $\phi(\ell_i) \in C$.

Denote the images of the generators by Φ by the same letters with a tilde. It follows from the relation satisfied by these generators in $\pi_1(S^\circ)$ that

$$\prod_{i=1}^{g} [\tilde{\alpha}_i, \tilde{\beta}_i] \prod_{j=1}^{k} \tilde{c}_j \prod_{r=1}^{g} [\tilde{\alpha}'_r, \tilde{\beta}'_r] \prod_{s=1}^{k'} \tilde{c}'_s \prod_{t=1}^{n} [\tilde{d}_t, \tilde{\ell}_t] = 1 \in F$$

Observe that $[\tilde{d}_j, \tilde{\ell}_j]$ is an element of $[F,F] \cap R$ and $\ell_j \in C$. This proves that *pcl* is at most the number of puncture stabilizations.

For the reverse inequality observe that each element $[\tilde{d}_j, \tilde{\ell}_j]$ represents the class of a C-torus in $H_2(G)$ according to the Hopf formula. Therefore we consider the elements ℓ_j inside $\pi_1(\Sigma_g \times [0,1])$ and represent them by based embedded loops K_j. The formula above shows that there is a map $\pi_1(\Sigma_g \times [0,1] \setminus K) \to G$ with the property that its restriction to $\partial N(K_i)$ is given by two commuting elements $d_j, \ell_j \in G$. On the other hand every C-torus is nullcobordant. Therefore $\rho * \rho'$ is equivalent to a stabilization of the trivial representation by no more than n puncture stabilizations. □

3.2 Branched covers and the n-symmetric group

3.2.1 Branched covers corresponding to double cosets

The correspondence between branched covers and checker surfaces was already known by Neretin (see Section 3.19 remark (b) of [32] and [28]). Here we explore this correspondence and show its relation with Hurwitz systems. Furthermore, we show that this correspondence can be used to define an product on the set of branched covers (Proposition 3.2.3).

Let $\phi\colon M \to \Sigma_g$ be a branched cover of degree d with n branching points $\{p_1, \dots, p_n\}$ and denote by $N_0 = \Sigma_g \backslash \{p_1, \dots, p_n\} \simeq \Sigma_g^n$. We know from Theorem 3.1.8 that the set of strong equivalence classes of such branched covers is in a 1-1 correspondence with the set $\mathrm{Hom}(\pi_1(N_0), S_d)/\,\mathrm{Inn}(S_d)$.

To obtain a correspondence with the (weak) equivalence classes of branched covers we take the quotient of $S_\infty \backslash S_\infty^{2g+n}/S_\infty$ by the action induced by the action of $\mathrm{MCG}(\Sigma_g^n)$ on $\pi(\Sigma_g^n)$ given as follows: Let $f \in \mathrm{MCG}(\Sigma_g^n)$ and $x \in S_\infty \backslash S_\infty^{2g+n}/S_\infty$. If $g : \Sigma_g^n \to \Sigma_g^n$ is a representative of f and $y \in S_\infty^{2g+n}$ is a representative of x then, by the correspondence above, y determines a homomorphism $\rho : \pi(\Sigma_g^n) \to S_\infty$. Let $f \cdot x$ be the double coset that represents the homomorphism $\rho \circ g_*$. This is a well-defined action and its orbits correspond to the weak equivalence classes of n-branched covers of Σ_g.

3.2.1.1 A monoid structure for branched covers

Let $\phi\colon M \to \mathbb{S}^2$ be a branched cover of degree d with n branching points and consider the sphere as the quotient space obtained by gluing two n-sided polygons P_+ and P_-. With this decomposition is possible to easily compute a Hurwitz system for ϕ and equip M with a checker surface structure. Suppose that the branching points of ϕ lie on the vertices of the polygons $P_\pm$. Then $\phi^{-1}(\partial P_+)$ divides M into $2d$ copies of the n-sided polygons. Let $\{A_i\}_{i=1}^d \cup \{B_i\}_{i=1}^d$ denote these polygons, with $\phi(A_i) = P_+$ and $\phi(B_i) = P_-$ for all $i = 1, \dots, d$. Choose a cyclical coloring of the edges of P_+ and color the edges of $\{A_i\} \cup \{B_i\}$ accordingly. These choices determine a checker structure in M. Furthermore, by the theory of classification of branched covers we have

Proposition 3.2.1. *If the checker surface M is represented by the double coset $(s_1, s_2, s_3, s_4, \dots, s_n)$ then the branched cover $f : M \to \Sigma_0$ has a Hurwitz system given by:*

$$(s_2^{-1}s_3, s_3^{-1}s_4, \dots, s_n^{-1}s_1).$$

The inverse construction is also possible, that is, given a double coset $(s_1, s_2, s_3, s_4, \dots, s_n)$ and its associated checker board M, we show how to construct a representative of the strong equivalence class of branched covers with total space M and Hurwitz system given by $(s_2^{-1}s_3, s_3^{-1}s_4, \dots, s_n^{-1}s_1)$.

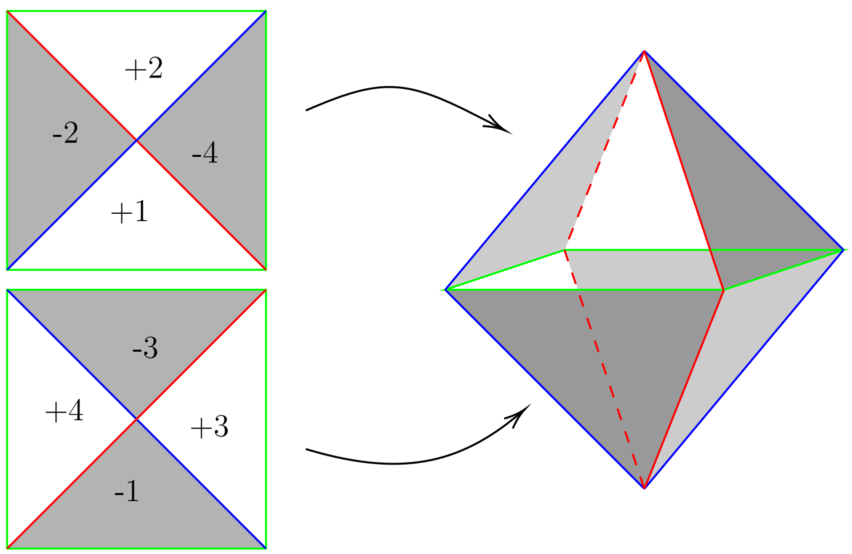


Figure 3.1: The checker board corresponding to the element $((1\ 4)(1\ 3\ 4\ 2)(1\ 2\ 4\ 3)) \in G_3$

In fact, let $\hat{\mathbb{C}}$ the Riemann sphere and consider the set $\{B_j\}_{j\in\mathbb{N}_+} \sqcup \{W_j\}_{j\in\mathbb{N}_+}$ of black and white triangles that constitute M, we identify each black polygon B_j conformaly with the north hemisphere of $\hat{\mathbb{C}}$. In the same way we identify the white triangles with the south hemisphere of the Riemann sphere. If d denotes this identification we have the following diagram

$$\begin{tikzcd} \{B_j\} \sqcup \{W_j\} \arrow[r] \arrow[d, "d"'] & M \arrow[dl, dashed, "\phi_\gamma"] \\ \mathbb{S}^2 & \end{tikzcd} \tag{3.8}$$

The unique map ϕ_γ is a branched cover. Furthermore this correspondence is a monoid homomorphism. In fact, let Bcov_n^+ denote the set of strong equivalence classes of branched covers over the sphere with at most n branching points, since the product in $S_\infty \backslash S_\infty^n / S_\infty$ is given by the disjoint union of checker surfaces, we can define the product in Bcov_n as the disjoint union of branched covers. More specifically, let $f\colon M_f \to \mathbb{S}^2$ and $g\colon M_g \to \mathbb{S}^2$ be elements of Bcov_n. Assume that the singular points of f and g are contained in the set $P = \{p_1, \ldots, p_n\} \subset \mathbb{S}^2$. We define the product $f \sqcup g\colon M_f \sqcup M_g \to \mathbb{S}^2$ as

$$f \sqcup g(x) = \begin{cases} f(x), & x \in M_f \\ g(x), & x \in M_g \end{cases}$$

We can summarize as:

Proposition 3.2.2. *The Hurwitz system yields an monoid-homomorphism*

$$(S_\infty \backslash S_\infty^n / S_\infty, \circ) \longleftrightarrow (\mathrm{Bcov}_n^+, \sqcup)$$

Let Bcov_n denote the set of weak equivalence classes of branched covers over the sphere with at most n branching points. Then we have:

Proposition 3.2.3. *There is a* $1-1$ *correspondence between the sets*

$$\frac{S_\infty \backslash S_\infty^n / S_\infty}{\mathrm{MCG}(\Sigma_0^n)} \longleftrightarrow \mathrm{Bcov}_n$$

3.2.1.2 General pseudo-triangulations of the sphere

Let ϕ be as above and consider a pseudo-triangulation P of $\mathbb{S}^2$ such that its vertices correspond to the branching points of ϕ. Then $P' = \phi^{-1}(P)$ is a pseudo-triangulation of M. Consider D and D' the dual fat graphs of these pseudo-triangulations respectively. Color the vertices of D with colors $c_1, \ldots, c_k$. This coloring induces a coloring of the vertices of D'. Enumerate the vertices of each color in D' to obtain n subsets of vertices $\{c_1^1, c_1^2, \ldots c_1^d\}, \ldots, \{c_k^1, c_k^2, \ldots c_k^d\}$. For each edge e of D, it corresponds d edges of D' and therefore we have a bijection s_e between the sets of the color joined by e. The set $P_\phi = \{s_{e_1}, \ldots, s_{e_m}\}$, up to simultaneous conjugacy, encodes information about the equivalence class of ϕ. In fact, the boundary of every connected component of the complement of D, with the orientation induced by $\mathbb{S}^2$, gives us the permutation corresponding to this branching point in the Hurwitz system. The ribbon graph D carries the fundamental group of Σ_g^d and therefore the elements of any Hurwitz systems for ϕ can be written as words in the alphabet P_ϕ.

On the other hand, given a enumeration of the edges of P and an element $t = (s_{e_1}, \ldots, s_{e_m})$ of G_m. We can construct a pseudomanifold M and a branched cover ϕ_t associated to t by gluing d^k colored triangles accordingly.

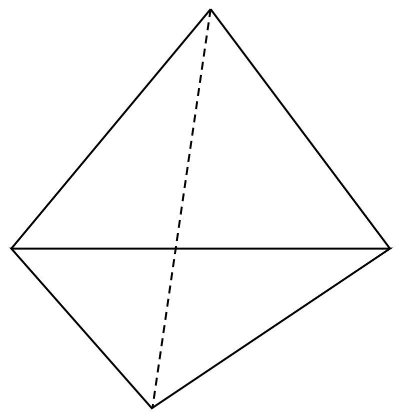


Figure 3.2: A tetrahedron is a pseudotriangulation of the sphere with four vertices

3.2.2 Surfaces with higher genus

Let $\phi\colon M \to \Sigma_g$ be a branched cover of degree d with n branching points. Consider a pseudo-triangulation P of Σ_g such that its vertices correspond to the branching points of ϕ. Then $P' = \phi^{-1}(P)$ is a pseudo-triangulation of M. Consider D and D' the dual fat graphs of these pseudo-triangulations respectively. Color the vertices of D with colors $c_1, \ldots, c_k$. This coloring induces a coloring of the vertices of D'. Enumerate the vertices of each color in D' to obtain n subsets of vertices $\{c_1^1, c_1^2, \ldots c_1^d\}, \ldots, \{c_k^1, c_k^2, \ldots c_k^d\}$. For each edge e of D, it correspond d edges of D' and therefore we have an bijection between the sets of the color joined by e. The set $P_f = \{s_{e_1}, \ldots, s_{e_m}\}$, up to simultaneous conjugacy, encodes information about the equivalence class of f. In fact, the boundary of every connected component of the complement of D, with the orientation induced by Σ_g, gives the permutation corresponding to this branching point in the Hurwitz system. The ribbon graph D carries the fundamental group of Σ_g^d and therefore the elements of any Hurwitz systems for ϕ can be written as words in the alphabet P_ϕ.

On the other hand, given a enumeration of the edges of P and an element $t = (s_{e_1}, \ldots, s_{e_m})$ of G_m. We can construct a pseudomanifold M and a branched cover ϕ_t associated to t by gluing the of d^k colored triangles accordingly.

3.2.3 The case of G-covers

We can perform the same operation for G-covers. In this case the permutation s associated to the lift of the path around the branching point x can be seen as a bijection of the fiber G into itself.

Let G be a group and fix an embedding $i : G \to S_\infty$. We say that a branched cover ϕ is a G-cover if every Hurwitz system for ϕ is in the image of i.

By means of the embedding i we can construct checker surfaces associated to elements of $G\backslash G^n/G$. Furthermore, all the considerations above regarding S_n-covers remain valid for general G-covers.

In particular, the strong equivalence classes of G-covers are classified by the classes of homomorphisms of $\operatorname{Hom}(\pi_1(N_0), G)/\operatorname{Inn}(G)$.

3.3 Braid monodromy

The monodromy action is a important invariant, as we saw for the case of branched cover maps of finite degree this invariant classify them up to a certain equivalence relation. The monodromy action is well-defined for fiber spaces and can be used to classify complex curves up to equisingular equivalence. During the 1980's, Libgober, Moishezon and others observed that the monodromy action of the projection of a complex curve can be identified with a braid, the braid monodromy of the curve. We begin by recalling the definition of monodromy action.

3.3.1 Monodromy action

A fiber space (also called a fiber bundle) with fiber F is a map $f\colon E \to B$ where E is called the total space and B the base space, such that for every $b \in B$ there exists a neighborhood U of b such that $f^{-1}(U)$ is homeomorphic to $U \times F$ and this homeomorphism commutes with the projection over B. Namely, if $h\colon f^{-1}(U) \to U \times F$ is the homeomorphism, then the diagram

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "h"] \arrow[d, "f"'] & U \times F \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

is commutative. A homeomorphism h is called a local trivialization for f.

Let $p\colon E \to B$ be a locally trivial fiber space and let $\gamma\colon [0,1] \to B$ be a path in B with initial point a and end-point b. A trivialization of the fibration defines a homeomorphism T_γ of the fiber $p^{-1}(a)$ onto the fiber $p^{-1}(b), T_\gamma\colon p^{-1}(a) \to p^{-1}(b)$. If the trivialization is modified, then T_γ changes into a homotopically-equivalent homeomorphism; this also happens if γ is changed to a homotopic path. The homotopy type of T_γ is called the monodromy transformation corresponding to a path γ. When $a = b$, that is, when γ is a loop, the monodromy transformation is a homeomorphism of $F = p^{-1}(a)$ into itself (defined up to a homotopy). This mapping, and also the homomorphisms induced by it on the homology and cohomology spaces of F, is also called a monodromy transformation. The correspondence of γ with T_γ gives a representation of the fundamental group $\pi_1(B, a)$ on $H^*(F)$.

If p is branched, we defined the monodromy action to be the one of the unbranched locally trivial fibration $\hat{p}\colon E\backslash p^{-1}(C) \to B\backslash C$, where C is the set of branching points of p

3.3.2 Definition of the braid monodromy

3.3.2.1 For algebraic curves

As a motivation for the definition of braid monodromy of the next section, we present the definition of braid monodromy for algebraic curves following [22].

Let $\mathscr{C}$ be a curve in the complex projective plane $\mathbb{C}P^2$ transversal to the line at infinity. Let $p : \mathbb{C}^2 \to \mathbb{C}$ be a linear projection of the affine portion of $\mathbb{C}P^2$ from a point at infinity. We say that p is a general projection if

1. The fibers of p are transversal to $\mathscr{C}$ except for a finite set $Cr(\mathscr{C}) = \{p_1, \ldots, p_N\}$;

2. The fibers over $Cr(\mathscr{C})$ have simply tangency with $\mathscr{C}$ or pass through the singularities of $\mathscr{C}$ so that these fibers are transversal to the tangent cones of singularities of $\mathscr{C}$;

3. The center of projection p (on the line in infinity) does not belong to $\mathscr{C}$.

Fixed a general projection, a choice of base point $P_0 \in \mathbb{C}\backslash Cr(\mathscr{C})$ and trivializations of the restrictions of $p\colon p^{-1}(\mathbb{C}\backslash Cr(\mathscr{C})) \to \mathbb{C}\backslash Cr(\mathscr{C})$ over loops in $\mathbb{C}\backslash Cr(\mathscr{C})$ based in P_0, defines the map θ from $\pi_1(\mathbb{C}\backslash Cr(\mathscr{C}), P_0)$ into the group of isotopy classes of homeomorphisms of $p^{-1}(P_0)$ preserving the set of $d = \deg(\mathscr{C})$ points $p^{-1}(P_0) \in \mathscr{C}$. These homomorphisms can be chosen to preserve a circle of a sufficiently large radius. Therefore one has a homomorphism θ from $\pi_1(\mathbb{C}\backslash Cr(\mathscr{C}))$ into the braid group on d strings B_d. This homomorphism is the braid monodromy of $\mathscr{C}$.

3.3.2.2 For branched covers

In order to define the braid monodromy of a branched covering we introduce the concepts of nondegenerate map and fold-free maps.

We say that a map $f\colon S \to \Sigma$ is nondegenerate if there exists triangulations of S and Σ such that $f(s)$ is a k-simplex for every k-simplex $s \subset S$.

Furthermore we say that f is fold-free is there exist triangulations of S and Σ such that no two adjacent 2-simplices of S are taken to the same 2-simplex of Σ.

Let S be a surface and consider an embedding $f\colon S \to \Sigma \times D^2$. If $p\colon \Sigma \times D^2 \to \Sigma$ denotes the projection, then the map $\phi = p \circ f$ is a branched cover (see [12]) if and only if there exists triangulations of S and Σ such that f is simplicial, nondegenerate and fold-free.

Remark 3.3.1. We can replace the fold-free condition by a stronger one by requesting f to be a positive embedding. Consider triangulations of S and Σ and for each vertex $v \in S$ consider its oriented link $lk(v)$. We say that the embedding f is positive if ϕ preserves orientations, that is, the links $lk(v)$ and $lk(\phi(v))$ have the same orientation for every vertex $v \in S$.

Definition 3.3.2. *Let $\phi = p \circ f\colon S \to \Sigma$ be a branched cover of degree m obtained as above and assume that ∂S is a closed braid on m strands in $\partial\Sigma \times D^2$. In this case we say that the pair (S, f) (or simply S) is a braided surface of degree m over Σ. Moreover, two braided surfaces S and S' of degree m over a surface Σ are said equivalent if there exists an ambient isotopy $\{h_u\}_{u\in[0,1]}$ of $\Sigma \times D^2$ such that $h_1(S) = S'$, $h_{u|\partial\Sigma\times D^2}$ is the identity map for all $u \in [0,1]$ and for each $u \in [0,1]$ the map h_u is fiber-preserving, that is, there is a homeomorphism $\bar{h}_u\colon \Sigma \to \Sigma$ such that $p \circ h_u = \bar{h}_u \circ p$.*

Notice that if (S_1, f_1) and (S_2, f_2) are two equivalent braided surfaces, then their associated branched covers $\phi_1\colon S_1 \to \Sigma$ and $\phi_2\colon S_2 \to \Sigma$ are equivalent. In fact, recall that $\phi_i = p \circ f_i$ and that following diagram commutes

$$\begin{array}{ccc}
S_1 & \xrightarrow{\tilde{h}_1} & S_2 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
\Sigma \times D^2 & \xrightarrow{h_1} & \Sigma \times D^2 \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\
\Sigma & \xrightarrow{\bar{h}_1} & \Sigma
\end{array}$$

where $\tilde{h}_1\colon S_1 \to S_2$ is the homeomorphism induced by h_1.

Remark 3.3.3. If we restrict ourselves to braided surfaces over D^2 such that ∂S is the trivial closed braid on m strands, such braided surfaces are called surface braids, we have a structure of commutative monoid on the set of equivalence classes of braided surfaces (see Chapter 15 of [20]).

In fact, given two braided surfaces S_1 and S_2 as above, we construct their product as follows: Divide D^2 into two disks E_1 and E_2 by a proper arc in D^2 and fix identification maps

$$D^2 \simeq E_1 \quad \text{and} \quad D^2 \simeq E_2.$$

In other words, we assume that D^2 is obtained from two copies of the disk by identifying a arc on their boundaries. Using this identification we assume that S_i is a surface in $E_i \times D^2$ which is contained in $D^2 \times D^2$. Then the union $S_1 \cup S_2$ in $(E_1 \cup E_2) \times D^2 = D^2 \times D^2$ is a surface braid of degree m over the disk, denoted S_1S_2.

The product-surface S_1S_2 depends on the division $D^2 = E_1 \cup E_2$ and on the identification maps $D^2 \simeq E_i$. However the equivalence class $[S_1S_2]$ depends only on the classes $[S_1]$ and $[S_2]$.

For S a braided surface over Σ with S a closed surface, denote by $P \subset \Sigma$ the set of its critical values. Then the restriction of the map p to $p' \colon \Sigma \times D^2 \backslash S \to \Sigma$ is a branched locally trivial fibration

Theorem 3.3.4. *Let (S, f) be a braided surface of degree n over Σ, $P \subset \Sigma$ the set of its critical points and denote $\phi = p \circ f$. If $Y = \Sigma \times \mathbb{C} \smallsetminus (f(S) \cup p^{-1}(P)$ then the restriction of p to Y, $\rho : Y \to \Sigma \smallsetminus P$, is a locally trivial fibration with fiber $\mathbb{C}_n = \mathbb{C}\backslash\{p_1, \dots, p_n\}$.*

Proof. Let $x \in \Sigma \smallsetminus P$ and consider a small neighborhood U_x of x. A simple computation shows that $\rho^{-1}(U_x) = p^{-1}(U_x)\backslash f(f^{-1}(U_x)) = U_x \times \mathbb{C}\backslash f(\phi^{-1}(U_x))$.

We show that for a certain neighborhood K of x there is an isomorphism of $K \times \mathbb{C}$ such that $f(\phi^{-1}(K)) \simeq K \times \{t_1, \dots, t_n\}$ and therefore $\rho^{-1}(K) \simeq K \times \mathbb{C}_n$.

Since ϕ is a covering map there exists an open neighborhood W of x such that $\phi^{-1}(W)$ is the disjoint union of open sets $W_i \subset S$ such that the restriction $\phi_i \colon W_i \to W$ of ϕ is an homeomorphism for every i. On the other hand, $\Sigma \smallsetminus P$ is a surface and therefore there exists an open neighborhood V of x homeomorphic to the disk $D^2 \subset \mathbb{C}$. Let $g \colon V \to D^2$ denote this homeomorphism and let B be a closed disk in $g(V \cap W)$ centered in $g(x)$. Set $K = g^{-1}(B)$ and $K_i = \phi_i^{-1}(K)$ for every i. Since f is an embedding, $f(\sqcup_j K_j) = \sqcup_j f(K_j)$ and taking its image by the homeomorphism $g \times 1$ we see that, if $B_j = (g \times 1)(f(K_j))$ then $K \times \mathbb{C}\backslash f(\phi^{-1}(K)) \simeq B \times \mathbb{C}\backslash \sqcup_j B_j$. Furthermore, since the projection in the first component takes each B_j homeomorphically onto B and they are compact, there must exist a homeomorphism $\theta \colon K \times \mathbb{C} \to K \times \mathbb{C}$ such that each B_j is taken homeomorphically to a set $B \times \{t_j\}$. Replacing B by its interior the arguments above continue valid and prove the proposition. $\square$

When S is a closed surface the monodromy action of ρ is trivial outside a closed disk containing the holes corresponding to the inverse image of ϕ. Furthermore, since the fiber of ρ is a 2-manifold we have that homotopic homeomorphisms are in fact isotopic. Hence the monodromy action of ρ induces an action $\beta_\phi : \pi_1(\Sigma, *) \to \mathrm{MCG}(D_n, \partial D) = B_n$.

Definition 3.3.5. *Let (S, f) be a braided surface over Σ with S a closed surface. Then the braid monodromy β_S^f of the braided surface S is the monodromy action of the associated locally trivial fibration $\rho : Y \to \Sigma \smallsetminus P$ defined in Theorem 3.3.4.*

Proposition 3.3.6. *Let (S, f) be a braided surface over Σ with S compact. If $\phi = p \circ f$ and ρ_ϕ is its monodromy, then the following diagram commutes*

$$\begin{tikzcd} \pi_1(\Sigma \smallsetminus P, *) \arrow[r, "\beta_S"] \arrow[dr, "\rho_\phi"'] & B_n \arrow[d] \\ & S_n \end{tikzcd}$$

where $B_n \to S_n$ is the canonical projection.

Definition 3.3.7. *Let S and S' be two braided surfaces over the same closed surface Σ and let β_S and $\beta_{S'}$ be their respective braid monodromies. We say that β_S and $\beta_{S'}$ are equivalent if there exists an homeomorphism $g\colon \Sigma \to \Sigma$ fixing the base point $*$ taking the critical points of S to the critical points of S' and such that $\beta_{S'} \circ g_* = \beta_S$.*

We have the following result concerning the invariance of the braid monodromy

Lemma 3.3.8. *Let S and S' be two braided surfaces over the same closed surface Σ. If S and S' are equivalent, then their braid monodromies β_S and $\beta_{S'}$ are equivalent.*

Proof. Since S and S' are equivalent there exist a fiber-preserving ambient isotopy $\{h_u\}_{u\in[0,1]}$ of $\Sigma \times D^2$ which carries S to S'. Therefore there is an ambient isotopy $\{\bar{h}_u\}_{u\in[0,1]}$ of Σ such that $p \circ h_u = \bar{h}_u \circ p$. Let $h = \bar{h}_1$, then h takes the critical points of S to the critical points of S'. The map h yields the desired equivalence. □

Let S be a braided surface of degree m over a closed surface $\Sigma = \Sigma_g$. Let P be the set of critical values of the branched cover associated to S and consider a Hurwitz arc system $w_1, \dots, w_n, x_1, y_1, \dots, x_g, y_g$ of Σ_g^n, assuming that the set of punctures is P. Let β be the braid monodromy of S.

Definition 3.3.9. *The braid system $B(S) \in B_m^{n+2g}$ of S associated to the given Hurwitz arc system is*

$$B(S) = (\beta(w_1), \dots, \beta(w_n), \beta(x_1), \beta(y_1), \dots, \beta(x_g), \beta(y_g)).$$

Notice that the braid system determines, and is determined, by the braid monodromy, as is the case for Hurwitz systems.

One may ask if every homomorphism from $\pi_1(\Sigma\backslash P)$ to B_n arises as the braid monodromy of some braided surface? The answer is negative, since the closure of the braids in the image of the braid monodromy must be completely splittable.

Definition 3.3.10. *A closed braid K in a solid torus $\mathbb{S}^2 \times \mathbb{S}^1$ is completely split if there exists m mutually disjoint convex disks $N_1, \dots, N_m$ in D^2, such that every open solid torus $\operatorname{int} N_i \times \mathbb{S}^1$ contains a component of K, where m is the number of components of K. We say that a closed braid is completely splittable if it is equivalent to a completely split closed braid.*

To see this, suppose that (S, f) is a braided surface over Σ. Let $P \subset \Sigma$ denote the set of its singular points and consider $p_0 \in P$.

Let D be an arbitrarily small disk centered at p_0. The loop γ going around ∂D once in the clockwise sense represents a homotopy class of $\pi_1(\Sigma\backslash P)$. Denote by b the image of γ by the braid monodromy. By construction the set $B = p^{-1}(\gamma) \cap f(S)$ is a geometric braid (a link) corresponding to the closure of b. Let $K = f(S) \cap p^{-1}(D)$ and denote by $p'\colon K \to D$ the restriction of p^{-1}, them p' is a branched covering with one singular point at p_0. Notice that K is a braided surface over D^2. By Lemma 16.12 of [20], K is completely splittable.

Remark 3.3.11. It is not clear if every completely splittable closed braid is the monodromy of a surface braid over a closed surface. However it is the case for braided surfaces over the disk D^2 as was proved by Kamada in his book [20]. Let A_m denote the subset of B_m consisting of the non-trivial braids b such that their closure $\hat{b}$ is represented by a completely split closed braid in the solid torus $D^2 \times \partial D^2$ that is the trivial link in $\partial(D^2 \times D^2)$. For a braid $b \in B_m$, consider the set

$$P_b^n = \{(l_1, \ldots, l_n) \in A_m^n ; l_1 \cdots l_n = b\}.$$

Kamada also proved that P_b^n characterizes the braided surfaces over the disk. We intent to extend this relations for more general braided surfaces in a future work.

3.3.3 Invariants

A key algebraic object in our study is the representation variety in the unbranched case $R = \mathrm{Hom}(\pi, G)/G$, where π is a surface group $\pi_1(\Sigma_g)$ or a free group $\pi_1(\Sigma_g^r)$ and G some given group. In the branched case we have to fix the holonomy around the punctures $R_\nu = \mathrm{Hom}_\nu(\pi, G)/G$, where $\pi = \pi_1(\Sigma_g^r)$ and the holonomy around the puncture p_i is in the conjugacy class of $\nu_i \in G$. There are analogous moduli spaces $\mathcal{M} = \mathrm{MCG}(\Sigma_g)\backslash R$ and $\mathcal{M}_\mu = \mathrm{MCG}(\Sigma_g^r)\backslash R_\mu$. Our aim is to give a description of these sets and to construct invariants for them, in order to apply it in the case where $G = B_n$.

For the remainder of this section suppose that G is a compact connected Lie group. Recall that by Proposition 1.2.7 the spherical matrix coefficients on G^k separate the points of $H\backslash G^k/H$. In fact, since the delta function δ_g at $g \in H\backslash G^k/H$ is an element of $L(H\backslash G^k/H)$ and the set of spherical matrix coefficients $\{\phi_{i,j}^V : V \in \widehat{G}\}$ is a generating set for it we see that δ_g is a combination of the functions $\phi_{i,j}^V$.

This provides an infinite family of functions for the case where $R = G\backslash G^k/G$ corresponds to a punctured surface representation variety. Specifically we consider the set $\{V_i\}_{i\in\widehat{G}}$ of all isomorphisms types of irreducible representations of G. Then for each set of indexes $I = \{i_1, \ldots, i_k\}$ the product $V_{i_1} \otimes \cdots V_{i_k}$ form a representation of G^k. We should restrict to those unitary representations for which $V_{i_1} \otimes \cdots \otimes V_{i_k}$ has a fixed G-vector. For each $u, v \in B_I$ in some basis B_I of the space of G-invariants $H^0(G, V_{i_1} \otimes \cdots \otimes V_{i_k})$ we have the spherical matrix coefficients

$$\psi_{u,v,I}(x) = \langle V_{i_1} \otimes \cdots \otimes V_{i_k}(x)u, v\rangle$$

where $I = (i_1, \ldots, i_k)$. The (infinite) set of all such functions will separate points of R. It is now easy to construct a single function taking values in the series in several variables with matrix coefficients:

$$\Psi = \sum_{I,(u,v)\in B_I} \frac{1}{I!}(\psi_{I,u,v})X^I$$

It is clear that ψ is a complete invariant for R, namely it separates its points. In many cases we can reduce the matrices again to a finite polynomial in more variables.

In particular, for any G as above and R homeomorphic to a finite CW complex, it admits an embedding $\xi\colon R \to \mathbb{R}^n$. The components of ξ form therefore a complete invariant for R and so there is a much simpler invariant than Ψ. Nevertheless we lack an exact form of ξ, in general.

In many interesting cases $G\backslash G^k/G$ has the structure of an (affine) algebraic variety over $\mathbb{C}$. Thus we can expect to have a suitable algebraic embedding ξ. Such an embedding can be obtained from a basis of the algebra of regular functions on R. This is the case of the $G = SU(n)$, for instance.

The computation of Neretin ([26]) for $G = SU(2)$ provides a version of a single algebraic function Ψ which can be expressed as a determinant. Here we know that B_I is indexed by the set of partitions $\alpha = (\alpha_{st})_{s,t=1,\dots,k}$ with

$$\sum_t \alpha_{st} = i_s$$

Then we consider

$$\Psi = \sum_{I,(\alpha_{st})} \frac{1}{\alpha!\beta!} \prod_{s,t} \mathbf{x}^\alpha \mathbf{y}^\beta (\psi_{I,\alpha,\beta})$$

Here we set $\mathbf{x}^\alpha = \prod_{s,t} x_{st}^{\alpha_{st}}$, $\alpha! = \prod_{s,t} \alpha_{st}!$. By [26] we have

$$\Psi = \det(1 - AXA^\perp Y)^{-\frac{1}{2}}$$

where $A \in SU(2)^k$ and X is a matrix of blocks $X_{ij} = \begin{pmatrix} 0 & x_{ij} \\ -x_{ij} & 0 \end{pmatrix}$ and x_{ij} are variables and similarly for Y.

In particular we obtain

Proposition 3.3.12. *For any representation $G \to SU(2)$ we have an invariant of the strong equivalence classes in R by means of Ψ. In particular this is the case when we take $G = B_3$ and the Burau representation restricted on the unit circle, giving invariants of lifted positive surfaces of degree 3 in products.*

To step from strong equivalence to the weak equivalence amounts of studying the action of $\mathrm{MCG}(\Sigma_g^r)$. However the previous approach using pull-backs of spherical functions from compact Lie groups lead to a dead end. In fact we have the following result due to Goldman, Xia and Pickrell [16]:

Theorem 3.3.13. *If G is a compact connected Lie group and Σ_g^r is hyperbolic then the action of $\mathrm{MCG}(\Sigma_g^r)$ on R_ν is ergodic with respect to the quasi-invariant measure.*

In particular there are no continuous functions on R_ν invariant under the $\mathrm{MCG}(\Sigma_g^r)$ action other than the constant.

Thus in order to get further insight by this method we have to step to non-compact Lie groups.

3.3.4 Neretin-type invariants for branched covers

From the discussion above we see that if we consider $G = S_\infty$, then the bi-S_∞-invariant functions on S_∞^n classify the weak equivalence classes of branched covers with at most n branching points. Furthermore, if we consider the action of the mapping class group $\mathrm{MCG}(N_0)$ into $S_\infty \backslash S_\infty^n / S_\infty$ it is clear that the bi-S_∞-invariant functions that are invariant under this action classify the weak equivalence classes of branched covers with at most n branching points.

In Section 1.6 we constructed a class of spherical functions $\mathcal{N}_\nu$, associated to the spherical representations ρ_ν. Unfortunately, as remarked in [32], not all spherical representations are of this form and therefore the set of spherical functions $\mathcal{N}_\nu$ is not guaranteed to separate points of $S_\infty \backslash S_\infty^n / S_\infty$ (and therefore distinguish strong equivalence classes of branched covers). Nevertheless, some (brute force) computations show that for $n = 3$ and for $d < 5$ there exists a choice of distinguished vector ν_0 such that $\mathcal{N}_{\nu_0}$ distinguishes all double cosets of $S_d \backslash S_d^3 / S_d$, but it is not clear if this is a general phenomenon.

An important property of the representation $\rho_\nu : \mathbb{G} \to \operatorname{Aut} H$ is that it is a well-defined representation when restricted to the groups S_n. More precisely,

Proposition 3.3.14. *1. For each positive integer d we have a restriction representation of $S_d \times S_d \times S_d$ given by the restriction $\rho_d : S_d \times S_d \times S_d \to \operatorname{Aut} H_d$ where $H_d = \otimes_{i=1}^d X$;*

2. *The representation ρ_d can be decomposed as the outer tensor product of three copies of the representation $\sigma_d : S_d \to \operatorname{Aut} \otimes_{i=1}^d V$;*

3. *For a permutation s of cycle type $(a_1, a_2, \ldots, a_k)$, set $l(s) = \sum_{i=1}^k a_i - 1$. Then the character function of the representation σ_d at s is given by*

$$\chi(\sigma_d)(s) = \dim(V)^{d-l(s)}.$$

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