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Thèse de Doctorat

présentée par

Xumin WANG

*Pour obtenir le grade de
Docteur en Mathématiques et Applications
de l'Université Bourgogne Franche-Comté*

Functional and harmonic analysis of noncommutative L_p spaces associated to compact quantum groups

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Résumé

Cette thèse a pour but d'étudier l'analyse sur les groupes quantiques compacts. Elle se compose de deux parties. La première partie porte sur la classification des semi-groupes de Markov invariants sur ces espaces homogènes quantiques. Les générateurs de ces semi-groupes sont considérés comme des opérateurs de Laplace sur ces espaces. La sphère classique \mathcal{S}^{N-1} , la sphère libre \mathcal{S}_+^{N-1} et la sphère semi-libérée \mathcal{S}_*^{N-1} sont considérées comme des exemples et les générateurs de semi-groupes de Markov sur ces sphères sont classifiés. Nous calculons aussi les dimensions spectrales des trois familles de sphères en fonction du comportement asymptotique des valeurs propres de leurs opérateurs de Laplace. Dans la deuxième partie, nous étudions la convergence des séries de Fourier pour les groupes non abéliens et les groupes quantiques. Il est bien connu qu'un certain nombre de propriétés d'approximation de groupes peuvent être interprétées comme des méthodes de sommation et de convergence en moyenne de séries de Fourier non commutatives associées. Nous établissons un critère général d'inégalités maximales pour les unités approchées de multiplicateurs non commutatifs de Fourier. En conséquence, nous prouvons que pour tout groupe dénombrable discret moyennable, il existe une suite de fonctions de type positif à support fini, telle que les multiplicateurs de Fourier associés sur les espaces L_p non commutatifs satisfassent à la convergence ponctuelle (presque uniforme). Nos résultats s'appliquent également à la convergence presque partout des séries de Fourier de fonctions L_p sur des groupes compacts non-abéliens. D'autre part, nous obtenons des bornes indépendantes de la dimension pour les inégalités maximales de Hardy-Littlewood non commutatives dans l'espace à valeurs opérateurs $L_p(\mathbb{R}^d, \mathcal{N})$ associées à des corps convexes.

Mots-clefs

Espaces L_p non commutatifs, groupes quantiques compacts, actions quantiques, sphères non commutatives, semi-groupes de Markov, multiplicateur de Fourier, convergence ponctuelle

Abstract

This thesis is devoted to study of analysis on compact quantum groups. It consists of two parts. The first part presents the classification of invariant quantum Markov semigroups on these quantum homogeneous spaces. The generators of these semigroups are viewed as Laplace operators on these spaces. The classical sphere \mathcal{S}^{N-1} , the free sphere \mathcal{S}_+^{N-1} , and the half-liberated sphere \mathcal{S}_*^{N-1} are considered as examples and the generators of Markov semigroups on these spheres are classified. We compute spectral dimensions for the three families of spheres based on the asymptotic behavior of the eigenvalues of their Laplace operator. In the second part, we study the convergence of Fourier series for non-abelian groups and quantum groups. It is well-known that a number of approximation properties of groups can be interpreted as some summation methods and mean convergence of associated noncommutative Fourier series. Based on this framework, this work studies the refined counterpart of pointwise convergence of these Fourier series. We establish a general criterion of maximal inequalities for approximative identities of noncommutative Fourier multipliers. As a result, we prove that for any countable discrete amenable group, there exists a sequence of finitely supported positive definite functions, so that the associated Fourier multipliers on noncommutative L_p -spaces satisfy the pointwise convergence. Our results also apply to the almost everywhere convergence of Fourier series of L_p -functions on non-abelian compact groups. On the other hand, we obtain the dimension free bounds of noncommutative Hardy-Littlewood maximal inequalities associated with convex bodies.

Keywords

Noncommutative L_p -spaces, compact quantum groups, quantum actions, noncommutative spheres, Markov semigroups, Fourier multipliers, maximal inequalities, pointwise convergence.

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Introduction

This thesis investigates analysis on non-abelian groups and quantum groups. After Woronowicz's pioneering work [104], the theory of topological quantum groups has been fruitfully developed in recent decades. It provides a prominent framework for the study of quantum symmetries, noncommutative geometry and abstract harmonic analysis. Functional analytic and probabilistic methods play essential roles in these researches. The main interests in this thesis will be exactly at the intersection of these areas, and in particular we will focus on the study of Markov semigroups and Fourier multipliers, as well as related objects that appear in the geometry and analysis on quantum groups.

The main objective of this work can be explained in the following two aspects. On the one hand, in the classical setting, the study of analytic or differential structures on a manifold often requires interactions with its isometric group. As a fundamental model, viewing classical spheres as homogeneous spaces of special orthogonal groups, Fourier analysis of spherical functions relies heavily on the invariant Laplacians and the representation theory of special orthogonal groups. From a probabilistic viewpoint, this also leads to the study of invariant Markov semigroups such as the heat semigroups on spheres. For a quantized theory in a similar fashion, the quantum symmetry groups and their quantum homogeneous spaces naturally enter the picture. Based on a probabilistic approach, we aim to study the classification of invariant Markov semigroups on quantum homogeneous spaces; as an example, we will give a precise computation of the natural Laplace operators and spectral dimensions of noncommutative spheres.

On the other hand, it is worth mentioning that the theory of topological quantum groups was also motivated by noncommutative generalizations of Fourier analysis. More precisely, the functional analytic attempts have been made to generalize the Pontryagin duality of locally compact abelian groups. This topic of noncommutative Fourier analysis finds one of its origins in the investigation of approximation properties of operator algebras inaugurated by Haagerup's pioneering work [54]. From a viewpoint of harmonic analysis, one can say that these approximation properties are interpretations of Fourier summation methods and mean convergences of Fourier series. Based on this framework, we aim to develop a refined study on the corresponding pointwise convergences of Fourier series. Again our methods will rely on the study of Markov semigroups, as well as tools from noncommutative L_p -spaces and quantum probability.

In the following part of this introduction, we will give a detailed presentation on the background, motivations and main results related to the two aforementioned subjects. After the introduction, we will present our work in two parts. The first part deals with the invariant Markov semigroups on quantum homogeneous spaces, and this is principally based on the author's joint work with Biswarup Das and Uwe Franz [37]. The second part is devoted to the study of convergences of Fourier series, and this is based on the

work joint with Guixiang Hong and Simeng Wang [58].

I Invariant Markov semigroups on quantum homogeneous spaces

Symmetry plays an essential role in many places in mathematics and sciences. Many systems are naturally invariant under the action of some groups, like time or space translations, rotations, or reflections. It is therefore of great interest to characterize and classify all invariant equations for a given group action. See for example the recent books by Ming Liao [75] and Vladimir Dobrev [44], that study invariant Markov processes and invariant differential operators, respectively. Liao's book is motivated by probability theory, whereas Dobrev's book deals with applications to physics.

Quantum groups [102, 103] provide a generalisation of groups and can be considered as a mathematical model for quantum symmetries. Dobrev [45] has also studied invariant differential operators for quantum groups. The quantum groups considered in [45] are q -deformations of semi-simple Lie groups.

But there exist also interesting quantum groups that are not deformations, but rather “liberations” of classical groups, see, e.g., [97, 98, 8]. These “liberated” quantum groups furthermore have actions on interesting “liberated” noncommutative spaces, see, e.g., [6]. This provides an interesting class of examples for noncommutative geometry.

Banica and Goswami investigated how to define Dirac operators on two of these noncommutative spaces: the free sphere \mathcal{S}_+^{N-1} and the half-liberated sphere \mathcal{S}_\times^{N-1} , cf. [6, Theorem 6.4]. The action of the free or the half-liberated orthogonal group yields a natural choice for the eigenspaces, but it does not suggest how to choose the eigenvalues.

In this part we introduce an approach for classifying invariant Markov semigroups on noncommutative space equipped with an action of a compact quantum group. The generators of these semigroups can be considered as natural candidates for Laplace operators. Dirac operators could be obtained via Cipriani and Sauvageot's construction [30] of a derivation from a Dirichlet form, see also [29]. Our method generalizes the case of an action of a classical compact group on a homogeneous space presented in [73, Chapter 3], [74], [75, Chapter 1]. Since here we are dealing only with compact quantum groups and actions on compact quantum spaces, everything can be done on the $*$ -algebraic level. As concrete examples we study the classical sphere \mathcal{S}^{N-1} , the half-liberated sphere \mathcal{S}_*^{N-1} , and the free sphere \mathcal{S}_+^{N-1} . Our approach adds a positivity condition to the invariance condition in [6], and leads to the formula for the eigenvalues of the Laplace operator on the three spheres \mathcal{S}^{N-1} , \mathcal{S}_*^{N-1} , \mathcal{S}_+^{N-1} , see Theorem 0.3 below. A more detailed study of the zeta function could probably be used to introduce further interesting “invariants” for these noncommutative manifolds.

To state our main results, we consider a compact quantum group \mathbb{G} equipped with the Haar state h . Denoted by $\text{Irr}(\mathbb{G})$ the equivalence classes of irreducible representations of \mathbb{G} . For $\pi \in \mathbb{G}$, we fix a representation $u^{(\pi)} \in B(H_\pi) \otimes C(\mathbb{G})$ of the class π where H_π is the Hilbert space with finite dimension n_π . We use $\text{Pol}(\mathbb{G})$ to denote the dense Hopf $*$ -algebra of polynomials in $C(\mathbb{G})$. Let ϵ be the counit and S be the antipode on $\text{Pol}(\mathbb{G})$. Let \mathcal{C} be a $*$ -subalgebra of $\text{Pol}(\mathbb{G})$. We call it an expected left coidealgebra, if the comultiplication Δ sending \mathcal{C} to $\mathcal{C} \otimes \text{Pol}(\mathbb{G})$ and there exists a contractive projection $\mathbb{E}_\ell : \text{Pol}(\mathbb{G}) \rightarrow \mathcal{C}$ with

$h \circ \mathbb{E}_\ell = h$. In [48], the authors showed that the coidealgebra admits an idempotent state Φ , where idempotent state means $(\Phi \otimes \Phi)\Delta = \Phi$.

Let us first give some basic definitions about generator function and Markov semigroup.

Definition 0.1. (Definition 2.12) Let \mathcal{A} be a unital $*$ -algebra and $\phi : \mathcal{A} \rightarrow \mathbb{C}$ be a state. A linear functional $\psi : \mathcal{A} \rightarrow \mathbb{C}$ is called a ϕ -generating functional, if

- (1) ψ is normalised, i.e., $\psi(1) = 0$;
- (2) ψ is hermitian, i.e., $\psi(a^*) = \overline{\psi(a)}$, for all $a \in \mathcal{A}$;
- (3) ψ is ϕ -conditionally positive, i.e., $\psi(a^*a) \geq 0$ for all $a \in \mathcal{A}$ with $\phi(a^*a) = 0$.

Definition 0.2. (Definition 2.8) Let $(T_t)_{t \geq 0}$ be a semigroup of operators on \mathcal{C} .

(1) We call $T : \mathcal{C} \rightarrow \mathcal{C}$ a *Markov operator*, if $T(1) = 1$ and T is completely positive in the sense that $(T_t \otimes \text{id}_n)(x) \geq 0$ for all $x \in M_n(\mathcal{C})_+$ where id_n is the identity map on $M_n(\mathbb{C})$. We call $(T_t)_{t \geq 0}$ a *Markov semigroup* if T_t is a Markov operator for any $t \geq 0$.

(2) We call $(T_t)_{t \geq 0}$ a \mathbb{G} -invariant semigroup, if $(T_t \otimes \text{id}) \circ \Delta|_{\mathcal{C}} = \Delta|_{\mathcal{C}} \circ T_t$ for any $t \geq 0$.

(3) We call $(T_t)_{t \geq 0}$ *strongly continuous* if for any fixed $x \in \mathcal{C}$,

$$T_t(x) \xrightarrow{\text{norm}} T_0(x) \quad \text{as } t \rightarrow 0.$$

We get the following Schoenberg correspondence for invariant Markov semigroups on an expected coidealgebra \mathcal{C} .

Theorem 0.3. (Theorem 2.13) Consider an expected left coidealgebra \mathcal{C} of a compact quantum group \mathbb{G} . Let Φ be the idempotent state corresponding to \mathcal{C} .

(1) Let $(T_t)_{t \geq 0}$ be a strongly continuous \mathbb{G} -invariant Markov semigroups on \mathcal{C} . Then the generator $L := \left. \frac{d}{dt} T_t \right|_{t=0}$ exists and there exists a $(\epsilon \circ T_0)$ -generating functional ψ , which is Φ invariant, such that

$$L = (\psi \otimes \mathbb{E}_\ell) \circ \alpha.$$

(2) Let ψ be a $(\epsilon|_{\mathcal{C}})$ -generating functional which is Φ -invariant. Define $L = (\psi \otimes \mathbb{E}_\ell) \circ \alpha$. Then we can construct a \mathbb{G} -invariant Markov semigroups on \mathcal{C} by

$$T_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} L^{\circ k}$$

where $L^{\circ k} = L \circ L \cdots \circ L(x)$ is the k folds composition of L , and the sum is taking in the sense of pointwise norm convergence.

As examples, we consider some spheres: classical spheres \mathcal{S}^{N-1} , half-liberated spheres \mathcal{S}_*^{N-1} and free spheres \mathcal{S}_+^{N-1} . Let O_N, O_N^*, O_N^+ denote classical, half-liberated and free orthogonal (quantum) groups respectively. We use the notation \mathcal{S}_\times^{N-1} and O_N^\times to denote those three spheres and associated three orthogonal groups respectively with $\times \in \{\emptyset, *, +\}$ (where \emptyset stands for no symbol, i.e. classical case).

Let α be the right action of O_N^\times on \mathcal{S}_\times^{N-1} given by $\alpha(x_i) = \sum_k x_k \otimes u_{ki}$. Let us consider the Podleś $*$ -algebra $\mathcal{O}_{O_N^\times}(\mathcal{S}_\times^{N-1})$ associated to this action α . In [4, Section 5], Banica showed that $\mathcal{O}_{O_N^\times}(\mathcal{S}_\times^{N-1})$ is isomorphism to the $*$ -subalgebra generated by $\{u_{1i} : 1 \leq i \leq N\}$ in $\text{Pol}(O_N^\times)$. Therefore, the Podleś $*$ -algebra $\mathcal{O}_{O_N^\times}(\mathcal{S}_\times^{N-1})$ is a left expected coidealgebra of O_N^\times and let Φ be the corresponding idempotent state on $C(O_N^\times)$. We call the state $h := h_{O_N^\times}|_{\mathcal{O}_{O_N^\times}(\mathcal{S}_\times^{N-1})}$ the Haar state on the sphere.

Let $(u^{(s)})_{s \in \mathbb{N}}$ be a special sequence of representations (may not be irreducible) of O_N^\times labeled by \mathbb{N} (See Proposition 2.15 for the constuction of this sequence $(u^{(s)})_{s \in \mathbb{N}}$). The following lemma is the key to classify invariant Markov semigroups on spheres.

Lemma 0.4. (Lemma 2.16) *For any $s \in \mathbb{N}$, there exists a basis for the Hilbert space K_s where the representation $u^{(s)}$ acts on, such that*

$$\Phi(s) := \left(\Phi(u_{jk}^{(s)}) \right)_{1 \leq j, k \leq d_s} = \delta_{j1} \delta_{k1},$$

if we write $u^{(s)} = (u_{jk}^{(s)})_{1 \leq j, k \leq d_s}$ w.r.t. to this basis.

This lemma shows that $u_{11}^{(s)} \in \text{Pol}[u_{11}]$, i.e. there exists a polynomial $q_s \in \text{Pol}[u_{11}]$ with degree s such that $u_{11}^{(s)} = q_s(u_{11})$. Decompose the Podleś *-algebra $\mathcal{O}_{O_N^\times}(\mathcal{S}_\times^{N-1})$ as following

$$\begin{aligned} H_k &= \text{span}\{x_{i_1} \cdots x_{i_r} : r \leq k\}; \\ D_k &= H_k \cap H_{k-1}^\perp; \quad d_k = \dim D_k, \end{aligned}$$

where the orthogonal \perp is defined by the Haar state h . Then we have the next theorem which classifies \mathbb{G} -invariant Markov semigroups.

Theorem 0.5. (Theorem 2.18) *For any O_N^\times -invariant strongly continuous Markov semigroup $(T_t)_{t \geq 0}$ on $\mathcal{O}_{O_N^\times}(\mathcal{S}_\times^{N-1})$, there exists a pair (b, ν) where b is a positive number and ν is a finite measure on $[-1, 1]$ such that the generator L of $(T_t)_{t \geq 0}$ satisfying*

$$L(x_s) = \lambda_s x_s, \quad x_s \in D_s,$$

with

$$\lambda_s = -b(q_s)'(1) + \int_{-1}^1 \frac{q_s^\times(x) - 1}{x - 1} d\nu(x) \quad (1)$$

where $q_s \in \text{Pol}[u_{11}]$ such that $h(q_n(u_{11})q_m(u_{11})) = C_n \delta_{nm}$ and $q_s^\times(1) = 1$.

The generators of these O_N^\times -invariant semigroups can be considered as natural candidates for Laplace operators on \mathcal{S}_\times^{N-1} . In fact, the arguments above also work for more general sphere $\mathcal{S}_\mathcal{R}^{N-1}$ if there is a quantum subgroup $O_N^\mathcal{R}$ of O_N^+ acting on $\mathcal{S}_\mathcal{R}^{N-1}$ such that the Podleś algebra $\mathcal{O}_{O_N^\mathcal{R}}(\mathcal{S}_\mathcal{R}^{N-1})$ and the *-algebra generated by $\{u_{i1} : 1 \leq i \leq N\}$ in $\text{Pol}(O_N^\mathcal{R})$ are isomorphic.

To make this theorem more complete, we need to show what those polynomials q_s look like. We give the answer case by case (Section 3.1).

- **Classical sphere:** $q_s = J^{\alpha, \beta} / J^{\alpha, \beta}(1)$ where $J^{\alpha, \beta}$ is the Jacobi polynomial with $\alpha = \beta = (N - 3)/2$.
- **Half-liberated sphere:** q_s is defined by

$$\begin{aligned} q_{2k}(x) &= (-1)^k \binom{N+k-2}{k}^{-1} {}_2F_1 \left(\begin{matrix} -k, N+k-1 \\ 1 \end{matrix}; x^2 \right) \\ q_{2k+1}(x) &= x \cdot (-1)^k (k+1) \binom{N+k-2}{k}^{-1} {}_2F_1 \left(\begin{matrix} -k, N+k \\ 2 \end{matrix}; x^2 \right). \end{aligned}$$

- **Free case:** q_s satisfy the following three-term recurrence relation:

$$a_{s+1}q_{s+2}(x) = U_{s+1}(N)q_{s+1}(x)x - a_s q_s(x) \quad \forall s \geq 0$$

where $q_0(x) = 1$, $q_1(x) = x$,

$$a_s = \sum_{k=0}^s (-1)^{s+k} U_k(N) = \begin{cases} U_m(N)(U_m(N) - U_{m-1}(N)) & \text{if } s = 2m, \\ U_m(N)(U_{m+1}(N) - U_m(N)) & \text{if } s = 2m + 1, \end{cases}$$

and where $U_s(N)$ denotes the value of the s^{th} Chebyshev polynomial of the second kind at the point N .

By Theorem 0.5, the generator L of an O_N^\times -invariant Markov semigroup can be determined by a pair (b, ν) . We find that if we set $(b = 1, \nu = 0)$ the associated generator L is indeed $(N - 1)^{-1}\Delta$ where Δ is the Laplace operator on the classical sphere \mathcal{S}^{N-1} . $(N - 1)^{-1}$ is a constant and multiplication by a constant will not make difference for differential structure. So the generator L associated to the special pair $(b = 1, \nu = 0)$ may be a good choice for the Laplace operator on the sphere. Then we can define the Dirac operator D from the Laplace operator such that $D^2 = L$. In other words, the eigenvalues of Dirac operator D are $(\pm\sqrt{\lambda_s})_{s \geq 0}$ where $(\lambda_s)_{s \geq 0}$ are eigenvalues of L given by equation (1). Therefore according to those polynomials q_s described above, we can compute the spectral dimension d_L of these spheres (Section 3.2):

- **Classical sphere:** $d_L = N - 1$.
- **Half-liberated sphere:** $d_L = 2(N - 1)$.
- **Free sphere:**

$$d_L = \begin{cases} 2 & \text{if } N = 2, \\ +\infty & \text{if } N \geq 3. \end{cases}$$

It should be noted that, for $N = 2$, the half-liberated sphere S_*^1 and the free sphere S_+^1 are isomorphic.

II Pointwise convergence of noncommutative Fourier series

The study of convergence of Fourier series goes back to the very beginning of the theory of harmonic analysis. Recall that for a p -integral function f on the unit circle \mathbb{T} with $1 \leq p \leq \infty$, we may define the Fejér means

$$(F_N f)(z) = \sum_{k=-N}^N \left(1 - \frac{|k|}{N}\right) \hat{f}(k) z^k, \quad z \in \mathbb{T}, \quad N \in \mathbb{N},$$

where \hat{f} denotes the Fourier transform. It is well-known that F_N defines a positive and contractive operator on $L_p(\mathbb{T})$, and $F_N f$ converges almost everywhere to f for all $1 \leq p \leq \infty$. Many other summation methods such as the Poisson means and Bochner-Riesz

means have also been widely studied. These operators notably play prominent roles in classical harmonic analysis (see e.g. [52]).

In recent decades, similar topics have been fruitfully discussed in the setting of operator algebras and geometric group theory. The study was initiated in the ground-breaking work of Haagerup [54], motivated by the approximation property of group von Neumann algebras. Indeed, let Γ be a countable discrete group with left regular representation $\lambda : \Gamma \rightarrow B(\ell_2(\Gamma))$ given by $\lambda(g)\delta_h = \delta_{gh}$, where the δ_g 's form the unit vector basis of $\ell_2(\Gamma)$. The corresponding group von Neumann algebra $VN(\Gamma)$ is defined to be the weak operator closure of the linear span of $\lambda(\Gamma)$. For $f \in VN(\Gamma)$ we set $\tau(f) = \langle \delta_e, f\delta_e \rangle$ where e denotes the identity of Γ . Any such f admits a Fourier series

$$\sum_{g \in \Gamma} \hat{f}(g)\lambda(g) \quad \text{with} \quad \hat{f}(g) = \tau(f\lambda(g^{-1})).$$

The convergence and summation methods of these Fourier series are deeply linked with the geometric and analytic properties of Γ , and in the noncommutative setting they are usually interpreted as various *approximation properties* for groups (see e.g. [22, 26]). More precisely, for a function $m : \Gamma \rightarrow \mathbb{C}$ we may formally define the corresponding Fourier multiplier by

$$T_m : \sum_{g \in \Gamma} \hat{f}(g)\lambda(g) \mapsto \sum_{g \in \Gamma} m(g)\hat{f}(g)\lambda(g).$$

Then the group Γ is *amenable* (resp., has the *Haagerup property*) if there exists a family of finitely supported functions (resp. c_0 -functions) m_N on Γ so that T_{m_N} defines a unital completely positive map on $VN(\Gamma)$ and $T_{m_N}f$ converges to f in the w^* -topology for all $f \in VN(\Gamma)$ (equivalently, m_N converges pointwise to 1); it has the *Haagerup property* if there exists a family of c_0 -functions m_N on Γ so that m_N converges pointwise to 1 and T_{m_N} defines a unital completely positive map on $VN(\Gamma)$; it is *weakly amenable* if there exists a family of finitely supported functions m_N on Γ so that T_{m_N} defines a completely bounded map on $VN(\Gamma)$ with $\sup_N \|T_{m_N}\|_{cb} < \infty$ and $T_{m_N}f$ converges to f in the w^* -topology for all $f \in VN(\Gamma)$. If we take $\Gamma = \mathbb{Z}$ and $m_N(k) = (1 - |k|/N)_+$, it is easy to check that $VN(\mathbb{Z})$ is isomorphic to $L_\infty(\mathbb{T})$ and T_{m_N} recovers the Fejér means F_N , which means that the integer group \mathbb{Z} enjoys the above properties. These approximation properties play essential roles in the modern theory of von Neumann algebras, as well as in geometric group theory. For example, the work of Cowling-Haagerup [35] on the weak amenability solves the isomorphism problems of various group von Neumann algebras; the Haagerup property and its opposite Kazhdan property (T) are amongst the central tools in Popa's deformation/rigidity theory [87]; also, the weak amenability is a key ingredient in the modern approach to the strong solidity and uniqueness of Cartan subalgebras [82, 27, 88].

However, despite the remarkable progress in this field, it is worthy mentioning that only the convergence of $T_{m_N}f$ in the w^* -topology was studied in the aforementioned works. A standard argument also yields the convergence in norm in the corresponding noncommutative L_p -spaces for $1 \leq p < \infty$. Note that in classical harmonic analysis, $T_{m_N}f$ converges almost everywhere to f for a number of regular multipliers m_N . Also the analogue of almost everywhere convergence for the noncommutative setting was introduced by Lance in his study of noncommutative ergodic theory [72]; this type of convergence is usually called the *almost uniform convergence* (abbreviated as *a.u. convergence*; see Section 4.2). Keeping in mind the classical theory and impressive results already obtained

from the mean convergence, it is natural to develop a refined theory of pointwise convergence of noncommutative Fourier series, and to seek applications in geometric group theory, operator algebras and harmonic analysis. However, compared to the classical setting, the pointwise convergence remains very much unexplored, up to isolated contributions [69, 25]. The reason for this lack of development might be explained by numerous difficulties one may encounter when dealing with maximal inequalities for noncommutative Fourier multipliers T_{m_N} ; in particular, the kernels of noncommutative Fourier multipliers are no longer classical functions, which cannot be pointwise comparable, and as a result the standard argument of classical Fejér and Bochner-Riesz means does not apply to the noncommutative setting any more.

In this thesis we would like to provide a new approach to the pointwise convergence theorems for noncommutative Fourier series. Our key technical result gives a criterion of maximal inequalities for noncommutative Fourier multipliers. The condition of this criterion only focus on the comparison of symbols of multipliers and those of Markov semigroups. Hence it is relatively easy to verify in the noncommutative setting. This approach is partially inspired by the works of Bourgain and Carbery [14, 15, 16, 23, 42] on the dimension free bounds of Hardy-Littlewood maximal operators. We refer to Section 4 for the notions of noncommutative L_p -spaces $L_p(VN(\Gamma))$ and noncommutative maximal norms $\|\sup_n^+ x_n\|_p$.

Theorem 0.6. *Let Γ be a discrete group as above. Assume that $\ell : \Gamma \rightarrow \mathbb{R}$ is a conditionally negative length function on Γ .*

(1) *Consider a family of self-adjoint Fourier multipliers $(T_{m_N})_{N \in \mathbb{N}}$ on $VN(\Gamma)$ with $\sup_N \|T_{m_N}\| < \infty$ (by self-adjointness we mean that $(T_{m_N}x)^* = T_{m_N}(x^*)$ for all $x \in VN(\Gamma)$). If there exist $\alpha, \beta > 0$ such that for all $g \in \Gamma$ we have*

$$|1 - m_N(g)| \leq \beta \frac{\ell(g)^\alpha}{2^N}, \quad |m_N(g)| \leq \beta \frac{2^N}{\ell(g)^\alpha},$$

then for all $2 \leq p < \infty$ there exists a constant $c > 0$ such that all $f \in L_p(VN(\Gamma))$,

$$\|\sup_{N \in \mathbb{N}}^+ T_{m_N} f\|_p \leq c \|f\|_p \quad \text{and} \quad T_{m_N} f \rightarrow f \text{ a.u. as } N \rightarrow \infty.$$

If moreover $(T_{m_N})_{N \in \mathbb{N}}$ are positive, the property holds for all $1 < p < \infty$ as well.

(2) *Consider a family of self-adjoint Fourier multipliers $(T_{m_N})_{N \in \mathbb{N}}$ on $VN(\Gamma)$ with $\sup_N \|T_{m_N}\| < \infty$. If there exist $\alpha, \beta > 0$ such that for all $g \in \Gamma$ we have*

$$|1 - m_N(g)| \leq \beta \frac{\ell(g)^\alpha}{N}, \quad |m_N(g)| \leq \beta \frac{N}{\ell(g)^\alpha}, \quad |m_{N+1}(g) - m_N(g)| \leq \beta \frac{1}{N},$$

then for all $2 \leq p < \infty$ there exists a constant c such that all $f \in L_p(VN(\Gamma))$,

$$\|\sup_{N \in \mathbb{N}}^+ T_{m_N} f\|_p \leq c \|f\|_p \quad \text{and} \quad T_{m_N} f \rightarrow f \text{ a.u. as } N \rightarrow \infty.$$

If moreover $(T_{m_N})_{N \in \mathbb{N}}$ are positive, the property holds for all $3/2 < p < \infty$ as well.

(3) *Consider a family of unital completely positive Fourier multipliers $(T_{m_t})_{t \in \mathbb{R}_+}$ on $VN(\Gamma)$. If there exists $\alpha > 0$ and $\eta \in \mathbb{N}_+$ such that for all $g \in \Gamma$ and $1 \leq k \leq \eta$ we have*

$$|1 - m_t(g)| \leq \beta \frac{\ell(g)^\alpha}{t}, \quad |m_t(g)| \leq \beta \frac{t}{\ell(g)^\alpha}, \quad \left| \frac{d^k m_t(g)}{dt^k} \right| \leq \beta \frac{1}{t^k},$$

then for all $1 + \frac{1}{2\eta} < p < \infty$ there exists a constant c such that all $f \in L_p(VN(\Gamma))$,

$$\left\| \sup_{t \in \mathbb{R}_+} {}^+T_{m_t} f \right\|_p \leq c \|f\|_p \quad \text{and} \quad T_{m_t} f \rightarrow f \text{ a.u. as } t \rightarrow \infty.$$

The theorem can be extended to many other quantum structures with Fourier expansions, such as twisted crossed products, compact quantum groups and Voiculescu's free semicircular systems. Hence, we will prove the result in a quite general setting in Section 6.

Based on the above theorem, we may study the pointwise convergence problems for a large class of groups with suitable approximation properties. We refer to Section 7 for the notion of groups with the *almost completely positive approximation property* (AC-PAP), which form a large subclass of groups with the Haagerup property and the weak amenability.

Theorem 0.7. (1) *Any countable discrete amenable group Γ admits a sequence of unital completely positive Fourier multipliers $(T_{m_N})_{N \in \mathbb{N}}$ on $VN(\Gamma)$ so that m_N are finitely supported and $T_{m_N} f$ converges to f a.u. for all $f \in L_p(VN(\Gamma))$ with $1 < p \leq \infty$.*

(2) *Any countable discrete group Γ with the ACPAP admits a sequence of completely contractive Fourier multipliers $(T_{m_N})_{N \in \mathbb{N}}$ on $VN(\Gamma)$ so that m_N are finitely supported and $T_{m_N} f$ converges to f a.u. for all $f \in L_p(VN(\Gamma))$ with $2 \leq p \leq \infty$.*

This result holds in the setting of locally compact amenable groups and quantum groups up to standard adaptation as well. Also, we may give more concrete examples as follows:

- i) In the classical Euclidean setting, our approach provides new types of approximate identities. Given any symmetric convex body B in \mathbb{R}^d with volume 1. Let Φ be the inverse Fourier transform of the convolution $\mathbb{1}_B * \mathbb{1}_B$, where $\mathbb{1}_B$ denotes the characteristic function of B , and let $\Phi_t = t^{-d} \Phi(t^{-1} \cdot)$ for $t > 0$. Then Φ is a positive integrable function with integral one. We have

$$\lim_{t \rightarrow 0} \Phi_t * f = f \text{ a.e., } f \in L_p(\mathbb{R}^d)$$

with $3/2 < p < \infty$, and

$$\lim_{j \rightarrow \infty} \Phi_{2^{-j}} * f = f \text{ a.e., } f \in L_p(\mathbb{R}^d)$$

with $1 < p < \infty$. Moreover we can get the almost uniform convergence for a function $f \in L_p(\mathbb{R}^d; L_p(\mathcal{M}))$ with value in a noncommutative L_p -space $L_p(\mathcal{M})$. If B is the unit cube, the convergence of $(\Phi_t * f)_{t>0}$ also holds for $1 < p < \infty$ and we recover the classical Fejér means.

- ii) (Noncommutative Fejér means) Inspired by i), we may introduce the following analogue of Fejér means on non-abelian discrete amenable groups. Let $(K_N)_{N \in \mathbb{N}}$ be a Følner sequence, that is, $K_N \subset \Gamma$ are subsets so that

$$m_N(g) := \frac{\mathbb{1}_{K_N} * \mathbb{1}_{K_N}(g)}{|K_N|} = \frac{|K_N \cap gK_N|}{|K_N|} \rightarrow 1, \quad \text{as } N \rightarrow \infty,$$

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where $|K_N|$ denotes the cardinality of K_N . Then the associated multipliers T_{m_N} are unital completely positive with m_N finitely supported. And there is a subsequence $(N_j)_{j \in \mathbb{N}}$ such that

$$T_{m_{N_j}} f \rightarrow f \text{ a.u.} \quad f \in L_p(VN(\Gamma))$$

for all $1 < p < \infty$. For instance if Γ is a group of polynomial growth with finite generating set S , we may take $K_N = S^N$. If moreover Γ is a 2-step nilpotent group and $p > 3/2$, we may take $N_j = j$.

- iii) (Noncommutative Bochner-Riesz means) Let Γ be a hyperbolic group so that the word length function $|\cdot|$ is conditionally negative. For example we may take Γ to be a non-abelian free group or a hyperbolic Coxeter group. The following Bochner-Riesz means are studied in [78]: for a fixed $\delta > 1$ we take

$$B_N^\delta f = \sum_{g \in \Gamma: |g| \leq N} \left(1 - \frac{|g|^2}{N^2}\right)^\delta \hat{f}(g) \lambda(g), \quad f \in L_p(VN(\Gamma)).$$

We have $B_N^\delta f \rightarrow f$ almost uniformly for all $2 \leq p \leq \infty$.

We remark that we will indeed establish Corollary 8.6 in the general setting of Woronowicz's *compact quantum groups*. As a particular case our result applies to Fourier series of non-abelian compact groups. Recall that for a compact group G , any function $f \in L_p(G)$ admits a Fourier series

$$f(x) \sim \sum_{\pi \in \text{Irr}(G)} \dim(\pi) \text{Tr}(\hat{f}(\pi) \pi(x)), \quad x \in G \quad \text{with} \quad \hat{f}(\pi) = \int_G f(x) \pi(x^{-1}) dx$$

where $\text{Irr}(G)$ denotes the collection of equivalence classes of irreducible unitary representations of G . The study of pointwise summability of above Fourier series is much more intricate than the abelian case. To our best knowledge, the pointwise convergence theorems in this setting were studied for differentiable or continuous functions in [55, 94, 60], and for some p -integrable functions on compact Lie groups for example in [32, 91]. However, from the viewpoint of amenable quantum groups, our approach easily establish the following pointwise convergence theorem for general p -integral functions. The summation method does not relies on the Lie algebraic structure, and hence it is novel compared to previous works; moreover it can be extended to the general setting of compact quantum groups.

Theorem 0.8. *Let G be a compact second countable group. There exists a sequence of finitely supported functions $m_N : \text{Irr}(G) \rightarrow \mathbb{R}$ so that*

$$(F_N f)(x) := \sum_{\pi \in \text{Irr}(G)} m_N(\pi) \dim(\pi) \text{Tr}(\hat{f}(\pi) \pi(x)), \quad x \in G, f \in L_p(G)$$

defines unital positive operators on $L_p(G)$ and

$$\lim_{N \rightarrow \infty} F_N f = f \text{ a.e.}, \quad f \in L_p(G)$$

for all $1 < p < \infty$. The functions m_N can be explicitly determined by the representation theory of G .

Moreover, there exists an increasing sequence of finite subsets $K_N \subset \text{Irr}(G)$ such that for all $f \in L_2(G)$ we have

$$f(x) = \lim_{N \rightarrow \infty} \sum_{\pi \in K_N} \dim(\pi) \text{Tr}(\hat{f}(\pi)\pi(x)), \quad a.e. x \in G.$$

Before ending the introduction to main results, we remark that the main Theorem 0.6 also implies as a byproduct the dimension free bounds of noncommutative Hardy-Littlewood maximal operators. The noncommutative version of Hardy-Littlewood maximal inequalities was studied in [77] for balls respecting to Euclidean metrics and in [57] for general doubling metric spaces. The dimension free bounds in this noncommutative setting were only studied by the first author in [56] for Euclidean balls; because of various difficulties in noncommutative analysis as mentioned before, the general cases for convex bodies remained unexplored before our work. Our following result establishes the desired maximal inequalities for general convex bodies in \mathbb{R}^d with dimension free estimates.

Corollary 0.9. *Let B be a symmetric convex body in \mathbb{R}^d and \mathcal{N} a semifinite von Neumann algebra. Define $\Phi_r : L_p(\mathbb{R}^d; L_p(\mathcal{N})) \rightarrow L_p(\mathbb{R}^d; L_p(\mathcal{N}))$ by*

$$\Phi_r(f)(\xi) = \frac{1}{\mu(B)} \int_B f(\xi - r\eta) d\eta.$$

Then there exists constant $C_p > 0$ independent of d such that the following holds:

(1) For any $1 < p \leq \infty$,

$$\|\sup_{j \in \mathbb{Z}}^+ \Phi_{2^j}(f)\|_p \leq C_p \|f\|_p, \quad f \in L_p(\mathbb{R}^d; L_p(\mathcal{N})).$$

(2) For any $\frac{3}{2} < p \leq \infty$,

$$\|\sup_{r \geq 0}^+ \Phi_r(f)\|_p \leq C_p \|f\|_p, \quad f \in L_p(\mathbb{R}^d; L_p(\mathcal{N})).$$

(3) If B is the ℓ_q -ball $\{(x_i)_{i=1}^d : \sum_{i=1}^d |x_i|^q \leq 1\}$ with $q \in 2\mathbb{N}$, then for any $1 < p < \infty$,

$$\|\sup_{r \geq 0}^+ \Phi_r(f)\|_p \leq C_p \|f\|_p, \quad f \in L_p(\mathbb{R}^d; L_p(\mathcal{N})).$$

In the end we make some comments on the strategy of the proof. Our argument heavily relies on the theory of noncommutative square functions, noncommutative martingales and ergodic inequalities. Also, Theorem 0.7, Theorem 0.8 and the related examples are crucially based on abstract constructions of conditionally negative functions on (quantum) groups; in hindsight, these methods essentially originated in the study of probabilistic and analytic properties on (quantum) groups [62, 24, 38]. As to the key criterion in Theorem 0.6, the principle method is to estimate the noncommutative square functions via Fourier series. The part of $p \geq 2$ is relatively accessible thanks to the Placherel formula. But the case of $p < 2$ is highly nontrivial, and our strategy is inspired by the interpolation methods developed by Bourgain in [15], as is mentioned previously. However, substantial difficulties arise if we want to transfer Bourgain's method to the noncommutative setting, since the operator space structure of Hilbert spaces is much more intricate.

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We refer to Part II for the detailed proofs and exposition of the above work. In particular, in Chapter 4, we will recall the background and prove some preliminary results on noncommutative vector valued L_p -spaces and pointwise convergence. Chapter 5 is devoted to the calculation of the order of square function of subordinate Poisson semi-groups. In Chapter 6, we will establish the key criterion for maximal inequalities for Fourier multipliers, i.e. Theorem 6.2 and Theorem 6.3. Last in Chapter 8, we will establish various maximal inequalities and pointwise convergence theorem of noncommutative Fourier multipliers.

Part I

Invariant Markov semigroups on quantum homogeneous spaces

Chapter 1

Preliminaries

1.1 Compact quantum groups

Let us first recall the definition and some properties of compact quantum groups. We refer to [104] for more details.

Definition 1.1. Let A be a unital C^* -algebra. If there exists a unital $*$ -homomorphism $\Delta : A \rightarrow A \otimes A$ such that $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ and

$$\{\Delta(a)(1 \otimes b) : a, b \in A\} \quad \text{and} \quad \{\Delta(a)(b \otimes 1) : a, b \in A\}$$

are linearly dense in $A \otimes A$, then (A, Δ) is called a *compact quantum group* and Δ is called the *comultiplication* on A . We denote $\mathbb{G} = (A, \Delta)$ and $A = C(\mathbb{G})$.

Any compact quantum group \mathbb{G} admits a unique *Haar state* h on $C(\mathbb{G})$ such that

$$(h \otimes \text{id}) \circ \Delta(x) = h(x)1 = (\text{id} \otimes h) \circ \Delta(x) \quad \forall x \in C(\mathbb{G}).$$

Let $u \in C(\mathbb{G}) \otimes B(H)$, where H is a Hilbert space with a finite dimension n . If the basis of H was chosen, then we can write $u = (u_{ij})_{i,j=1}^n$. u is called an *n -dimensional representation* of \mathbb{G} , if for any $1 \leq i, j \leq n$ we have

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

Denote by $\text{Irr}(\mathbb{G})$ the set of unitary equivalence classes of irreducible representations of \mathbb{G} , which is called the dual of \mathbb{G} . For each $\pi \in \text{Irr}(\mathbb{G})$, we fix a representation $u^{(\pi)} \in B(H_\pi) \otimes C(\mathbb{G})$ of the class π where H_π is a Hilbert space with finite dimension n_π . Denote

$$\text{Pol}(\mathbb{G}) = \text{span}\{u_{ij}^{(\pi)} : u^{(\pi)} = (u_{ij}^{(\pi)})_{i,j=1}^{n_\pi}, \pi \in \text{Irr}(\mathbb{G})\},$$

which is a dense $*$ -subalgebra of $C(\mathbb{G})$.

For each $\pi \in \text{Irr}(\mathbb{G})$, there exists a unique positive invertible operator $Q_\pi \in B(H_\pi)$ with $\text{Tr}(Q_\pi) = \text{Tr}(Q_\pi^{-1}) := d_\pi$ such that the Haar state can be calculated as following:

$$h(u_{ij}^{(\pi)}(u_{lm}^{(\pi')})^*) = \delta_{\pi\pi'} \delta_{il} \frac{(Q_\pi)_{mj}}{d_\pi}, \quad h((u_{ij}^{(\pi)})^* u_{lm}^{(\pi')}) = \delta_{\pi\pi'} \delta_{jm} \frac{(Q_\pi^{-1})_{mj}}{d_\pi} \quad (1.1)$$

for any $\pi, \pi' \in \text{Irr}(\mathbb{G})$ and $1 \leq i, j \leq n_\alpha$, $1 \leq l, m \leq n_\beta$. The quantum group \mathbb{G} is said to be of *Kac type* if $Q_\pi = \text{id}_\pi$ for all $\pi \in \text{Irr}(\mathbb{G})$. The number d_π is called the *quantum dimension* of π . We have $d_\pi = n_\pi$ if \mathbb{G} is of Kac type.

Let $C_u(\mathbb{G})$ be the universal C^* -algebra generated by $\text{Pol}(\mathbb{G})$. Let $L_2(\mathbb{G}, h)$ ($L_2(\mathbb{G})$ for short) be the Hilbert space defined from the GNS representation of h . Let $L_\infty(\mathbb{G})$ to be the von Neumann algebra generated by $\text{Pol}(\mathbb{G})$ in $B(L_2(\mathbb{G}))$ and define $C_r(\mathbb{G})$ to be the reduced C^* -algebra generated by $\text{Pol}(\mathbb{G})$ in $B(L_2(\mathbb{G}))$. In this thesis, we will be interested in the case that \mathbb{G} is of *Kac type*, in other words, the Haar state h is tracial.

Define the tensor product of representations

$$u^{(\alpha)} \boxtimes u^{(\beta)} = \sum_{i,j,k,l} u_{ij}^{(\alpha)} u_{kl}^{(\beta)} \otimes e_{ij}^{(\alpha)} \otimes e_{kl}^{(\beta)}$$

where $\alpha, \beta \in \text{Irr}(\mathbb{G})$ and $e_{ij}^{(\alpha)}, e_{kl}^{(\beta)}$ denote the matrix units of $B(H_\alpha)$ and $B(H_\beta)$ respectively.

Define linear maps $\epsilon : \text{Pol}(\mathbb{G}) \rightarrow \mathbb{C}$ and $S : \text{Pol}(\mathbb{G}) \rightarrow \text{Pol}(\mathbb{G})$ by

$$\epsilon(u_{ij}^{(\pi)}) = \delta_{ij} \quad \text{and} \quad S(u_{ij}^{(\pi)}) = (u_{ji}^{(\pi)})^*.$$

Then, ϵ is called the *counit* of $\text{Pol}(\mathbb{G})$ and S is called the *antipode* of $\text{Pol}(\mathbb{G})$. The counit ϵ is a $*$ -homomorphism such that

$$(\epsilon \otimes \text{id})\Delta = (\text{id} \otimes \epsilon)\Delta = \text{id}.$$

The antipode is a anti-homomorphism such that $S(S(x^*)^*) = x$ for all $x \in \text{Pol}(\mathbb{G})$.

1.2 Actions of compact quantum groups

In this subsection, we introduce the actions of compact quantum groups on compact quantum spaces, where compact quantum spaces mean unital C^* -algebras. We refer to [39] for a recent survey on actions of compact quantum groups.

Definition 1.2. Let $\mathcal{O}(\mathbb{X})$ be a \mathcal{C} and (\mathcal{A}, Δ) be a Hopf $*$ -algebra. A *Hopf $*$ -algebraic right action* α is a $*$ -homomorphism $\alpha : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{A}$ such that

- $(\text{id}_{\mathcal{C}} \otimes \Delta) \circ \alpha = (\alpha \otimes \text{id}_{\mathcal{A}}) \circ \alpha$,
- $(\text{id}_{\mathcal{C}} \otimes \epsilon) \circ \alpha = \text{id}_{\mathcal{C}}$.

The following definition was first introduced in [86].

Definition 1.3. A *right action* $\mathbb{X} \curvearrowright^\alpha \mathbb{G}$ of a compact quantum group \mathbb{G} on a compact quantum space \mathbb{X} is a unital $*$ -homomorphism

$$\alpha : C(\mathbb{X}) \rightarrow C(\mathbb{X}) \otimes C(\mathbb{G})$$

such that

- the *action property* holds:

$$(\text{id}_{\mathbb{X}} \otimes \Delta) \circ \alpha = (\alpha \otimes \text{id}_{\mathbb{G}}) \circ \alpha,$$

- the *density condition* (also called *Podleś condition*) holds:

$$\overline{\alpha(C(\mathbb{X}))(\mathbf{1}_{C(\mathbb{X})} \otimes C(\mathbb{G}))} = C(\mathbb{X}) \otimes C(\mathbb{G}).$$

Let $\pi \in \text{Irr}(\mathbb{G})$ and $u^{(\pi)} \in C(\mathbb{G}) \otimes B(H_\pi)$ be the irreducible representation associated to π . The *intertwiner space* between π and an action α is defined by

$$\text{Mor}(\pi, \alpha) = \left\{ T : H_\pi \rightarrow C(\mathbb{X}) : \alpha(T\xi) = (T \otimes \text{id}_{\mathbb{G}}) \left(u^{(\pi)}(\xi \otimes 1) \right) \right\}.$$

We call the following subspace π -*spectral subspace*

$$C(\mathbb{X})_\pi = \text{span}\{T\xi : \xi \in H_\pi, T \in \text{Mor}(\pi, \alpha)\} \subset C(\mathbb{X}).$$

The *Podleś subalgebra* of $C(\mathbb{X})$ is defined by

$$\mathcal{O}_{\mathbb{G}}(\mathbb{X}) = \text{span}\{C(\mathbb{X})_\pi : \pi \in \text{Irr}(\mathbb{G})\}. \quad (1.2)$$

If we consider the action $\alpha = \Delta$ of \mathbb{G} on \mathbb{G} itself, then the Podleś subalgebra $\mathcal{O}_{\mathbb{G}}(\mathbb{G}) = \text{Pol}(\mathbb{G})$.

If the restriction of α on the Podleś subalgebra satisfying $\alpha^H = \alpha|_{\mathcal{O}_{\mathbb{G}}(\mathbb{X})} : \mathcal{O}_{\mathbb{G}}(\mathbb{X}) \rightarrow \mathcal{O}(\mathbb{X}) \otimes \text{Pol}(\mathbb{G})$, then α^H is a Hopf*-algebraic action.

Assume $\mathcal{O}_{\mathbb{G}}(\mathbb{X})$ admits a universal completion, denoted by $C_u(\mathbb{X})$, then the Hopf*-algebraic action α^H can extend to a universal action $\alpha^u : C_u(\mathbb{X}) \rightarrow C_u(\mathbb{X}) \otimes C_u(\mathbb{G})$.

An action is called *embeddable*, if $\mathcal{O}_{\mathbb{G}}(\mathbb{X})$ is isomorphic to a *-subalgebra of $\text{Pol}(\mathbb{G})$, such that the action corresponds to the restriction of the comultiplication, i.e., if there exists an injective unital *-homomorphism $\vartheta : \mathcal{O}_{\mathbb{G}}(\mathbb{X}) \rightarrow \text{Pol}(\mathbb{G})$ such that $(\vartheta \otimes \text{id}) \circ \alpha|_{\mathcal{O}_{\mathbb{G}}(\mathbb{X})} = \Delta \circ \vartheta$.

1.3 Quantum homogeneous spaces

Embeddable actions can give a unital *-subalgebras which are also coideals.

Definition 1.4. A *left (right, resp.) coidealgebra* of $\text{Pol}(\mathbb{G})$ is unital *-subalgebra \mathcal{C} of $\text{Pol}(\mathbb{G})$ such that

$$\Delta(\mathcal{C}) \subseteq \mathcal{C} \otimes \text{Pol}(\mathbb{G}), \quad (\Delta(\mathcal{C}) \subseteq \text{Pol}(\mathbb{G}) \otimes \mathcal{C} \text{ resp.}).$$

We call \mathcal{C} an *expected left (right, resp.) coidealgebra*, if there exists a surjective projection $\mathbb{E} : \text{Pol}(\mathbb{G}) \rightarrow \mathcal{C}$ such that $\|\mathbb{E}\| = 1$, $h \circ \mathbb{E} = h$ and $\mathbb{E}(xyz) = x\mathbb{E}(y)z$ for any $x, z \in \mathcal{C}$, $y \in \text{Pol}(\mathbb{G})$. We call this projection \mathbb{E} the *conditional expectation* from $\text{Pol}(\mathbb{G})$ onto \mathcal{C} .

Definition 1.5. Let f, g are two functionals on $C(\mathbb{G})$. The convolution is defined by

$$f * g = (f \otimes g) \circ \Delta.$$

A state Φ on $C(\mathbb{G})$ is called *idempotent* if $\Phi * \Phi = \Phi$.

Remark 1.6. In [49, Section 3], authors showed that for any idempotent state ϕ ,

$$\phi \circ S = \phi \quad (1.3)$$

where S is the antipode.

Franz and Skalski showed that there is a one to one correspondence between expected coidealgebras and idempotent states.

Theorem 1.7. ([48]) *Let \mathbb{G} be a compact quantum group. There is a one-to-one correspondence between the following objects:*

- *idempotent states Φ on $\text{Pol}(\mathbb{G})$;*
- *expected left (equivalently, right) coidealgebras \mathcal{C} in $\text{Pol}(\mathbb{G})$ (denote by $\mathbb{E} : \text{Pol}(\mathbb{G}) \rightarrow \mathcal{C}$ the conditional expectation).*

Given an idempotent state Φ , we get the expected left coidealgebra by $\mathcal{C} = (\Phi \otimes \text{id}) \circ \Delta(\text{Pol}(\mathbb{G}))$. Given an expected coidealgebra $\text{Pol}(\mathbb{G})$, we can recover the idempotent state as $\Phi = \varepsilon \circ \mathbb{E}$.

Definition 1.8. Let \mathbb{K} be a compact quantum group. We call \mathbb{K} is a compact quantum subgroup of \mathbb{G} , if there exists a surjective, unital $*$ -homomorphism $\theta : C_u(\mathbb{G}) \rightarrow C_u(\mathbb{K})$ such that

$$(\theta \otimes \theta) \circ \Delta_{u,\mathbb{G}} = \Delta_{u,\mathbb{K}} \circ \theta$$

where $\Delta_{u,\mathbb{G}}$ and $\Delta_{u,\mathbb{K}}$ are the comultiplications of $C_u(\mathbb{G})$ and $C_u(\mathbb{K})$ respectively.

Definition 1.9. The $*$ -algebra of the left quantum quotient of \mathbb{G} by \mathbb{K} , denoted $\text{Pol}(\mathbb{K} \setminus \mathbb{G})$ is defined by

$$\text{Pol}(\mathbb{K} \setminus \mathbb{G}) := \{x \in \text{Pol}(\mathbb{G}) : (\theta \otimes \text{id})(\Delta_{\mathbb{G}}(x)) = 1_{\mathbb{K}} \otimes x\}.$$

Let $C(\mathbb{K} \setminus \mathbb{G})$ be the norm closure of $\text{Pol}(\mathbb{K} \setminus \mathbb{G})$ in $C(\mathbb{G})$.

Note that $\Delta(\text{Pol}(\mathbb{K} \setminus \mathbb{G})) \subset \text{Pol}(\mathbb{K} \setminus \mathbb{G}) \otimes \text{Pol}(\mathbb{G})$. We define

$$\alpha := \Delta|_{\text{Pol}(\mathbb{K} \setminus \mathbb{G})} : \text{Pol}(\mathbb{K} \setminus \mathbb{G}) \rightarrow \text{Pol}(\mathbb{K} \setminus \mathbb{G}) \otimes \text{Pol}(\mathbb{G}).$$

We can extend this action to $C(\mathbb{G})$ and denote it α again, which is a right action of \mathbb{G} on $\mathbb{K} \setminus \mathbb{G}$.

One can check that $\mathbb{E} := (h_{\mathbb{K}} \otimes \theta) \circ \Delta_{\mathbb{G}}$ is the $h_{\mathbb{G}}$ -preserving conditional expectation from $\text{Pol}(\mathbb{G})$ to $\text{Pol}(\mathbb{K} \setminus \mathbb{G})$. Therefore, $\text{Pol}(\mathbb{K} \setminus \mathbb{G})$ is an expected left coidealgebra.

It is worth to be mentioned here that if G is a classical group, an expected coidealgebra is always an algebra of the continuous functions of a quotient space. But in noncommutative $C(\mathbb{G})$, there are a lot of examples (for instance, see Proposition 3.1) showing that coidealgebra may be NOT a quantum quotient space.

Chapter 2

Invariant Markov semigroups on quantum homogeneous spaces

2.1 Generator of invariant Markov semigroup

Let \mathbb{G} be a compact quantum group and $1 \in \mathcal{C} \subset \text{Pol}(\mathbb{G})$ be an expected left coidealgebra of $\text{Pol}(\mathbb{G})$. Let $\alpha = \Delta|_{\mathcal{C}}$ be the comultiplication restricted on \mathcal{C} , which is a right action of \mathbb{G} on \mathcal{C} . Let \mathbb{E}_ℓ be the conditional expectation from $\text{Pol}(\mathbb{G})$ onto \mathcal{C} and Φ be the idempotent state associated to \mathcal{C} by Theorem 1.7. Then

$$\Phi = \epsilon \circ \mathbb{E}_\ell \quad \text{and} \quad \mathbb{E}_\ell = (\Phi \otimes \text{id}) \circ \Delta$$

where ϵ is the counit and Δ is the comultiplication of \mathbb{G} . Define a right conditional expectation associated to Φ by

$$\mathbb{E}_r := (\text{id} \otimes \Phi) \circ \Delta.$$

We start with a simple lemma which will be used in the following.

Lemma 2.1. *On $\text{Pol}(\mathbb{G})$ the following holds:*

- (1) $(\mathbb{E}_\ell \otimes \text{id}) \circ \Delta = \Delta \circ \mathbb{E}_\ell$; $(\text{id} \otimes \mathbb{E}_r) \circ \Delta = \Delta \circ \mathbb{E}_r$.
- (2) $(\mathbb{E}_r \otimes \text{id}) \circ \Delta = (\text{id} \otimes \mathbb{E}_\ell) \circ \Delta$.

Proof. The identity in (1) is observed in [46]. Indeed,

$$(\mathbb{E}_\ell \otimes \text{id}) \circ \Delta = (\Phi \otimes \text{id} \otimes \text{id}) \circ (\Delta \otimes \text{id}) \circ \Delta = (\Phi \otimes \Delta) \circ \Delta = \Delta \circ \mathbb{E}_\ell.$$

Similarly, we get the second equation for \mathbb{E}_r .

We prove (2):

$$\begin{aligned} (\mathbb{E}_r \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \Phi \otimes \text{id}) \circ (\Delta \otimes \text{id}) \circ \Delta \\ &= (\text{id} \otimes (\Phi \otimes \text{id}) \circ \Delta) \circ \Delta \\ &= (\text{id} \otimes \mathbb{E}_\ell) \circ \Delta. \end{aligned}$$

□

Definition 2.2. (1) We say that a functional f on \mathcal{C} is Φ -invariant if $(f \otimes \Phi) \circ \alpha = f$.

(2) We say that a functional f on $\text{Pol}(\mathbb{G})$ is *bi- Φ -invariant* if $(f \otimes \Phi) \circ \Delta = f = (\Phi \otimes f) \circ \Delta$

Definition 2.3. Let f and g be two functionals on an expected left coidealgebra \mathcal{C} . We define the convolution of f and g , denoted by $f \star_\ell g$, by the following functional on \mathcal{C} :

$$f \star_\ell g := (f \otimes (g \circ \mathbb{E}_\ell)) \circ \alpha.$$

Remark 2.4. We should distinguish this convolution with the convolution for functionals on $\text{Pol}(\mathbb{G})$. Recall that (see Definition 1.5) for two functionals μ and ν on $\text{Pol}(\mathbb{G})$, the convolution is defined by $\mu * \nu := (\mu \otimes \nu) \circ \Delta$.

Proposition 2.5. Let f, g be Φ -invariant functionals on \mathcal{C} . Then $f \star_\ell g$ is Φ -invariant.

Proof. By the action property of α , Φ -invariance of g and Lemma 2.1, we have

$$\begin{aligned} ((f \star_\ell g) \otimes \Phi) \circ \alpha &= (f \otimes g \otimes \Phi) \circ (\text{id} \otimes \mathbb{E}_\ell \otimes \text{id}) \circ (\text{id} \otimes \Delta) \circ \alpha \\ &= (f \otimes g) \circ (\text{id} \otimes \mathbb{E}_\ell) \circ \alpha \\ &= f \star_\ell g. \end{aligned}$$

□

We call a family of linear functionals $\{f_t : \mathcal{C} \rightarrow \mathbb{C}\}_{t \geq 0}$ a *convolution semigroup on \mathcal{C}* if

$$f_s \star_\ell f_t = f_{s+t} \quad \forall s, t \geq 0.$$

Definition 2.6. Let $(f_t)_{t \geq 0}$ be a semigroup of functionals on \mathcal{C} (on $\text{Pol}(\mathbb{G})$, resp.).

(1) We call $(f_t)_{t \geq 0}$ a *convolution semigroup of states*, if for any $t \geq 0$, the functionals f_t are states, i.e. $f_t(1) = 1$ and $f_t(x^*x) \geq 0$ for any $x \in \mathcal{C}$ ($x \in \text{Pol}(\mathbb{G})$, resp.).

(2) We call $(f_t)_{t \geq 0}$ a Φ -invariant (bi- Φ -invariant, resp.) semigroup, if for any $t \geq 0$, f_t is Φ -invariant (bi- Φ -invariant, resp.).

(3) We call that $(f_t)_{t \geq 0}$ is a *continuous* semigroup on \mathcal{C} (on $\text{Pol}(\mathbb{G})$ resp.) if for any fixed $x \in \mathcal{C}$ ($x \in \text{Pol}(\mathbb{G})$, resp.),

$$\lim_{t \rightarrow 0} f_t(x) = f_0(x).$$

Definition 2.7. We say that a linear map $T : \mathcal{C} \rightarrow \mathcal{C}$ is \mathbb{G} -invariant, if

$$\alpha \circ T = (T \otimes \text{id}) \circ \alpha.$$

Definition 2.8. Let $(T_t)_{t \geq 0}$ be a semigroup of operators on \mathcal{C} .

(1) We call $T : \mathcal{C} \rightarrow \mathcal{C}$ a *Markov operator*, if $T(1) = 1$ and T is completely positive in the sense that $(T_t \otimes \text{id}_n)(x) \geq 0$ for all $x \in M_n(\mathcal{C})_+$ where id_n is the identity map on $M_n(\mathbb{C})$. We call $(T_t)_{t \geq 0}$ a *Markov semigroup* if T_t is a Markov operator for any $t \geq 0$.

(2) We call $(T_t)_{t \geq 0}$ a \mathbb{G} -invariant semigroup, if $(T_t \otimes \text{id}) \circ \Delta|_{\mathcal{C}} = \Delta|_{\mathcal{C}} \circ T_t$ for any $t \geq 0$.

(3) We call $(T_t)_{t \geq 0}$ *strongly continuous* if for any fixed $x \in \mathcal{C}$,

$$T_t(x) \xrightarrow{\text{norm}} T_0(x) \quad \text{as } t \rightarrow 0.$$

Proposition 2.9. *The following holds*

(1) *Let $(f_t)_{t \geq 0}$ be a Φ -invariant semigroup of states on \mathcal{C} . Define $\mu_t := f_t \circ \mathbb{E}_\ell$, for all $t \geq 0$. Then $(\mu_t)_{t \geq 0}$ is the unique bi- Φ -invariant semigroup of states on $\text{Pol}(\mathbb{G})$, such that $\mu_t|_{\mathcal{C}} = f_t$ for any $t \geq 0$.*

(2) *Let $(\mu_t)_{t \geq 0}$ be a bi- Φ -invariant semigroup of functionals on $\text{Pol}(\mathbb{G})$. Then $f_t := \mu_t|_{\mathcal{C}}$ is the unique Φ -invariant semigroup of states on \mathcal{C} , such that $f_t \circ \mathbb{E}_\ell = \mu_t$.*

Proof. We prove (1) first. By Lemma 2.1,

$$\mu_t * \mu_s = (f_t \otimes f_s \circ \mathbb{E}_\ell)(\mathbb{E}_\ell \otimes \text{id})\Delta = f_{s+t} \circ \mathbb{E}_\ell = \mu_{t+s}.$$

Since conditional expectation \mathbb{E}_ℓ is positive, and $\mathbb{E}_\ell(1) = 1$, $(\mu_t)_{t \geq 0}$ is a semigroup of states on $\text{Pol}(\mathbb{G})$.

We then prove the bi- Φ -invariance of μ_t , by applying Lemma 2.1.

Left Φ -invariance:

$$\begin{aligned} (\Phi \otimes \mu_t) \circ \Delta &= (\Phi \otimes f_t \circ \mathbb{E}_\ell) \circ \Delta \\ &= (\Phi \otimes f_t)((\mathbb{E}_r \otimes \text{id}) \circ \Delta) \\ &= (\Phi * \Phi \otimes f_t) \circ \Delta \\ &= (\Phi \otimes f_t) \circ \Delta \\ &= f_t \circ \mathbb{E}_\ell \\ &= \mu_t. \end{aligned}$$

Right Φ -invariance:

$$\begin{aligned} (\mu_t \otimes \Phi) \circ \Delta &= (f_t \otimes \Phi)(\mathbb{E}_\ell \otimes \text{id}) \circ \Delta \\ &= (f_t \otimes \Phi)\Delta \circ \mathbb{E}_\ell \\ &= f_t \circ \mathbb{E}_\ell \\ &= \mu_t. \end{aligned}$$

Let ν_t be any bi- Φ -invariant functional on $\text{Pol}(\mathbb{G})$ such that $\nu_t|_{\mathcal{C}} = f_t$. Then using the right Φ -invariance of ν_t we have

$$\begin{aligned} \nu_t(x) &= (\Phi \otimes \nu_t)(\Delta(x)) \\ &= \nu_t(\mathbb{E}_\ell(x)) \\ &= f_t(\mathbb{E}_\ell(x)) = \mu_t(x), \end{aligned}$$

which proves the uniqueness.

Now we prove (2): Since $\alpha = \Delta|_{\mathcal{C}}$, the Φ -invariance of f_t follows from the bi- Φ -invariance of μ_t . The uniqueness can be seen easily. \square

In fact, the following proposition shows that the assumption of Φ -invariance for f_t in Proposition 2.9 is not necessary.

Proposition 2.10. *Let $\{f_t : \mathcal{C} \rightarrow \mathbb{C}\}_{t \geq 0}$ be a convolution semigroup of states on \mathcal{C} . Then f_t is Φ -invariant.*

Proof. Let $\mu_t := f_t \circ \mathbb{E}_\ell$. Proposition 2.9 implies that $(\mu_t)_{t \geq 0}$ is a convolution semigroup of functionals on $\text{Pol}(\mathbb{G})$. Let us first show that μ_t is bi- Φ -invariant.

By definition, for any $t \geq 0$, $(\Phi \otimes \mu_t) \circ \Delta = \mu_t \circ \mathbb{E}_\ell = \mu_t$. Note that μ_0 is an idempotent state on $\text{Pol}(\mathbb{G})$. By equation (1.3), we have

$$(\mu_0 \otimes \Phi) \circ \Delta = ((\Phi \circ S) \otimes (\mu_0 \circ S)) \circ \Delta^{op} = (\Phi \otimes \mu_0) \circ \Delta \circ S = \mu_0 \circ S = \mu_0.$$

This implies that $\mu_t * \Phi = \mu_t * \mu_0 * \Phi = \mu_t$. Hence $(\mu_t)_{t \geq 0}$ is a convolution semigroup of bi- Φ -invariant functionals on $\text{Pol}(\mathbb{G})$. Proposition 2.9 now yields that $f_t = \mu_t|_{\mathcal{C}}$ must be Φ -invariant for each $t \geq 0$. \square

Proposition 2.11. (1) *Let $(f_t)_{t \geq 0}$ be a continuous semigroup of states on \mathcal{C} . Then $T_t := (f_t \otimes \mathbb{E}_\ell) \circ \alpha$ is the continuous \mathbb{G} -invariant Markov semigroup on \mathcal{C} such that $f_t = \epsilon \circ T_t$, for any $t \geq 0$.*

(2) *Let $(T_t)_{t \geq 0}$ be a continuous \mathbb{G} -invariant Markov semigroup on \mathcal{C} . Then $f_t := \epsilon \circ T_t$ is the continuous semigroup of states on \mathcal{C} such that $T_t = (f_t \otimes \mathbb{E}_\ell) \circ \alpha$, for any $t \geq 0$.*

Proof. We prove (1). By action property (see in Definition 1.3) and Lemma 2.1, we have

$$\begin{aligned} (T_t \otimes \text{id}) \circ \alpha &= (f_t \otimes \mathbb{E}_\ell \otimes \text{id})(\alpha \otimes \text{id}) \circ \alpha \\ &= (f_t \otimes \mathbb{E}_\ell \otimes \text{id})(\text{id} \otimes \Delta) \circ \alpha \\ &= (f_t \otimes \alpha \circ \mathbb{E}_\ell) \circ \alpha \\ &= \alpha \circ T_t. \end{aligned}$$

This implies that T_t is a \mathbb{G} -invariant operator for all t . Moreover,

$$\begin{aligned} T_t \circ T_s &= (f_t \otimes f_s \otimes \mathbb{E}_\ell)(\text{id} \otimes \alpha \circ \mathbb{E}_\ell) \alpha \\ &= (f_t \otimes f_s \otimes \text{id})(\text{id} \otimes \mathbb{E}_\ell \otimes \mathbb{E}_\ell)(\text{id} \otimes \Delta) \alpha \\ &= (f_t \star_\ell f_s \otimes \mathbb{E}_\ell) \alpha \\ &= T_{t+s}. \end{aligned}$$

Since f_t is a state, we get T_t is completely positive and $T_t(1) = 1$. Therefore $(T_t)_{t \geq 0}$ is a Markov semigroup. By Proposition 2.10, we get f_t is Φ -invariant on \mathcal{C} for any t . Then $f_t = \epsilon \circ T_t$ follows from the Φ -invariance of f_t immediately. By the definition of T_t , $T_t(x)$ norm-converging to $T_0(x)$ as $t \rightarrow 0$ is obvious, because of the continuity of $(f_t)_t$.

Now, Let us proof (2). For any $t \geq 0$, since T_t is completely positive, $T_t(1) = 1$ and $\epsilon|_{\mathcal{C}}$ is a $*$ -homomorphism on \mathcal{C} , we have that f_t is a state. Moreover,

$$\begin{aligned} (f_t \star_\ell f_s)(x) &= (f_t \otimes f_s \circ \mathbb{E}_\ell) \circ \alpha \\ &= ((\epsilon \circ T_t) \otimes (\epsilon \circ T_s \circ \mathbb{E}_\ell)) \circ \alpha \\ &= (\epsilon \otimes (\epsilon \circ T_s \circ \mathbb{E}_\ell)) \circ (T_t \otimes \text{id}) \alpha \\ &= [\text{id} \otimes (\epsilon \circ T_s \circ \mathbb{E}_\ell)](\epsilon \otimes \text{id}) \circ \alpha \circ T_t \\ &= \epsilon \circ T_s \circ T_t \\ &= f_{t+s}. \end{aligned}$$

By \mathbb{G} -invariance of T_t , we have

$$(f_t \otimes \mathbb{E}_\ell) \circ \alpha = (\epsilon \otimes \mathbb{E}_\ell) \circ \alpha \circ T_t = T_t;$$

$$f_t(x) = \epsilon \circ T_t(x) \xrightarrow{t \rightarrow 0} \epsilon \circ T_0(x) = f_0(x),$$

for any $x \in \mathcal{C}$. □

Definition 2.12. Let \mathcal{A} be a unital $*$ -algebra and $\phi : \mathcal{A} \rightarrow \mathbb{C}$ be a state. A linear functional $\psi : \mathcal{A} \rightarrow \mathbb{C}$ is called a ϕ -generating functional, if

- (1) ψ is normalised, i.e., $\psi(1) = 0$;
- (2) ψ is hermitian, i.e., $\psi(a^*) = \overline{\psi(a)}$, for all $a \in \mathcal{A}$;
- (3) ψ is ϕ -conditionally positive, i.e., $\psi(a^*a) \geq 0$ for all $a \in \mathcal{A}$ with $\phi(a^*a) = 0$.

We have the following Schoenberg correspondence for invariant Markov semigroups on coidealgebras.

Theorem 2.13. Consider an expected left coidealgebra \mathcal{C} of a compact quantum group \mathbb{G} . Let Φ be the idempotent state corresponding to \mathcal{C} .

(1) Let $(T_t)_{t \geq 0}$ be a strongly continuous \mathbb{G} -invariant Markov semigroups on \mathcal{C} . Then the generator $L := \left. \frac{d}{dt} T_t \right|_{t=0}$ exists and there exists a $(\epsilon \circ T_0)$ -generating functional ψ , which is Φ invariant, such that

$$L = (\psi \otimes \mathbb{E}_\ell) \circ \alpha.$$

(2) Let ψ be a $(\epsilon|_{\mathcal{C}})$ -generating functional which is Φ -invariant. Define $L = (\psi \otimes \mathbb{E}_\ell) \circ \alpha$. Then we can construct a \mathbb{G} -invariant Markov semigroups on \mathcal{C} by

$$T_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} L^{\circ k}$$

where $L^{\circ k} = L \circ L \cdots \circ L(x)$ is the k folds composition of L , and the sum is taking in the sense of pointwise norm convergence.

Proof. Let us prove (1):

Set $\mu_t = \epsilon \circ T_t \circ \mathbb{E}_\ell$. By Proposition 2.11 and Proposition 2.9, $(\mu_t)_{t \geq 0}$ is a continuous semigroups of states on $\text{Pol}(\mathbb{G})$. By [47, Theorem 3.3], μ_t is differentiable at the point $t = 0$ and $\tilde{\psi} = \left. \frac{d}{dt} \mu_t \right|_{t=0}$ is a μ_0 -generating functional on $\text{Pol}(\mathbb{G})$. Set $\psi = \tilde{\psi}|_{\mathcal{C}}$, then ψ is a $\epsilon \circ T_0$ -generating functional on \mathcal{C} . Denote $f_t = \mu_t|_{\mathcal{C}} = \epsilon \circ T_t$. Then $T_t = (f_t \otimes \mathbb{E}_\ell) \circ \alpha$. Hence, we have

$$\left. \frac{d}{dt} T_t(x) \right|_{t=0} = \left(\left. \frac{d}{dt} f_t \right|_{t=0} \otimes \mathbb{E}_\ell \right) \circ \alpha(x) = (\psi \otimes \mathbb{E}_\ell) \circ \alpha(x).$$

Since f_t is a semigroup of states, f_t is Φ -invariant by Proposition 2.10, so is ψ .

Now, let us prove the second assertion (2):

Let us prove the \mathbb{G} -invariance of L first.

$$\begin{aligned} (L \otimes \text{id}) \circ \alpha &= (\psi \otimes \mathbb{E}_\ell \otimes \text{id})(\alpha \otimes \text{id}) \circ \alpha \\ &= (\psi \otimes \mathbb{E}_\ell \otimes \text{id})(\text{id} \otimes \Delta) \circ \alpha \\ &= (\psi \otimes \Delta \circ \mathbb{E}_\ell) \circ \alpha \\ &= \alpha \circ L. \end{aligned}$$

Since ψ is Φ -invariant, by the proof of Proposition 2.11, we have $\psi \star_\ell \psi = \epsilon \circ L \circ L$. This implies that

$$(\psi^{\star_\ell k} \otimes \mathbb{E}_\ell) \circ \alpha = ((\epsilon \circ L^{\circ k}) \otimes \mathbb{E}_\ell) \circ \alpha = (\epsilon \otimes \mathbb{E}_\ell) \circ \alpha \circ L^{\circ k} = L^{\circ k}.$$

Let f_t be the convolution exponential of ψ , i.e.

$$f_t(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \psi^{\star k}(x).$$

By [47, Theorem 3.3] and Proposition 2.9, $(f_t)_t$ is a semigroup of states. Thus we can set

$$T_t(x) = (f_t \otimes \mathbb{E}_\ell) \circ \alpha = \sum_{k=0}^{\infty} \frac{t^k}{k!} L^{\circ k}(x).$$

By Proposition 2.11, $(T_t)_{t \geq 0}$ is a \mathbb{G} -invariant Markov semigroup. Let $x = u_{ij}^{(\pi)} \in \mathcal{C} \subset \text{Pol}(\mathbb{G})$. Then

$$T_t(u_{ij}^{(\pi)}) = \sum_{m=1}^{n_\pi} f_t(u_{im}^{(\pi)}) \mathbb{E}_\ell(u_{mj}^{(\pi)}).$$

Note that $f_t(u_{im}^{(\pi)}) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \psi^{\star k}(u_{im}^{(\pi)})$ where the sum is taking in the sense of convergence in \mathbb{C} . On the other hand, for any $x \in \mathcal{C} \subset \text{Pol}(\mathbb{G})$, x can be written as the linear combination of some $u_{ij}^{(\pi)}$. Therefore for any fixed $x \in \mathcal{C}$, the sum in the definition of $T_t(x)$ is taking in the sense of pointwise norm convergence. \square

2.2 Classification of Markov semigroups on real easy quantum spheres

We know that the orthogonal group O_N is the isometry group of the sphere \mathcal{S}^{N-1} . There exist quantum versions, or “liberated” versions, of the orthogonal groups and the spheres. The free orthogonal quantum group O_N^+ was first introduced by Wang in [98], and the free sphere \mathcal{S}_+^{N-1} was studied by Banica in [4]. There are given by their universal C^* -algebras which are defined as follow:

$$C_u(\mathcal{S}_+^{N-1}) = C^* \left(x_1, \dots, x_N \mid x_i = x_i^*, \sum_i x_i^2 = 1 \right);$$

$$C_u(O_N^+) = C^* \left((u_{ij})_{i,j=1,\dots,N} \mid u = \bar{u}, u^t = u^{-1} \right).$$

Banica and Speicher studied easy orthogonal groups in [8]. Weber classified them in [100]. According to these works, we can define more spheres as follow.

$$C_u(\mathcal{S}_{\mathcal{R}}^{N-1}) = C^* \left(x_1, \dots, x_N \mid x_i = x_i^*, \sum_i x_i^2 = 1, \mathcal{R} \right)$$

where \mathcal{R} is a family of some relations between the generators $(x_i)_{1 \leq i \leq N}$.

We call that $\mathcal{S}_{\mathcal{R}}^{N-1}$ is a *real easy quantum sphere*, if there is an easy compact quantum group $O_N^{\mathcal{R}}$, which is a quantum subgroup of O_N^+ , such that ϑ defined below is a unital $*$ -isomorphism and α defined below is an Hopf $*$ -algebraic action:

$$\begin{aligned} \vartheta : \text{Pol}(\mathcal{S}_{\mathcal{R}}^{N-1}) &\rightarrow \mathcal{C}_{\mathcal{R}} & \alpha : \text{Pol}(\mathcal{S}_{\mathcal{R}}^{N-1}) &\rightarrow \text{Pol}(\mathcal{S}_{\mathcal{R}}^{N-1}) \otimes \text{Pol}(O_N^{\mathcal{R}}) \\ x_i &\mapsto u_{1i} & x_i &\mapsto \sum_{k=1}^N x_k \otimes u_{ki}. \end{aligned}$$

where $\text{Pol}(\mathcal{S}_{\mathcal{R}}^{N-1})$ is the $*$ -algebra generated by $\{x_i : 1 \leq i \leq N\}$ and $\mathcal{C}_{\mathcal{R}}$ is the $*$ -algebra generated by $\{u_{1i} : 1 \leq i \leq N\}$ in $\text{Pol}(O_N^{\mathcal{R}})$.

In other words, the action α above is embeddable.

Example 2.14. (1) Let $\mathcal{R} = \emptyset$. The sphere is just the free sphere and the associated easy quantum group is O_N^+ .

(2) Let $\mathcal{R} = \{x_i x_j = x_j x_i : 1 \leq i, j \leq N\}$, we get the classical sphere \mathcal{S}^{N-1} and classical orthogonal group O_N .

(3) (see [6, 4]) Let $\mathcal{R} = \{x_i x_k x_j = x_j x_k x_i : 1 \leq i, k, j \leq N\}$. We get the half-liberated sphere \mathcal{S}_*^{N-1} and the half-liberated quantum orthogonal group O_N^* :

$$C_u(\mathcal{S}_*^{N-1}) = C_u(\mathcal{S}_+^{N-1}) / \langle abc = cba, \forall a, b, c \in x_i \rangle;$$

$$C_u(O_N^*) = C_u(O_N^+) / \langle abc = cba, \forall a, b, c \in u_{ij} \rangle.$$

(4) Let $\mathcal{R} = \{R_p^{S_p} : p \in \Pi\}$ where Π is a set of partitions and $R_p^{S_p}$ is a relation defined by the partition p . The precise definition can be found in [64, Definition 3.8, Definition 3.9, Definition 4.3, Definition 4.4]. These spheres are denoted by $\mathcal{S}^{N-1}(\Pi)$, while the associated quantum groups are denoted by $\mathbb{G}_N(\Pi)$. By [64, Theorem 4.7 and Theorem 5.1], we get that ϑ defined above is a $*$ -isomorphism and α defined above is an action.

We define $\tilde{h} = h \circ \vartheta$. One can easily check that

$$(\tilde{h} \otimes \text{id}) \circ \alpha(x) = \tilde{h}(x)1 \quad \forall x \in \text{Pol}(\mathcal{S}_{\mathcal{R}}^{N-1}).$$

Let $C_r(\mathcal{S}_{\mathcal{R}}^{N-1})$ be the reduced C^* -algebra associated to the \tilde{h} -GNS representation and let $C_r(O_N^{\mathcal{R}})$ be the reduced C^* -algebra associated to the h -GNS representation.

Similar as the proof of [4, Proposition 5.8], $C_r(\mathcal{S}_{\mathcal{R}}^{N-1})$ is isometric isomorphism to the reduced C^* -subalgebra $\overline{\mathcal{C}_{\mathcal{R}}}$. From now on, we regard $\text{Pol}(\mathcal{S}_{\mathcal{R}}^{N-1})$ as a subalgebra of $\text{Pol}(O_N^{\mathcal{R}})$. Therefore, we will not distinguish \tilde{h} and h .

Moreover, since $(\vartheta \otimes \text{id}) \circ \alpha = \Delta \circ \vartheta$, we can view the action α of $\text{Pol}(O_N^{\mathcal{R}})$ on $\text{Pol}(\mathcal{S}_{\mathcal{R}}^{N-1})$ as $\Delta|_{\mathcal{C}_{\mathcal{R}}}$.

Since O_N^+ is of Kac type, so is $O_N^{\mathcal{R}}$. We have a Haar state preserving conditional expectation $\mathbb{E}_{\ell} : \text{Pol}(O_N^{\mathcal{R}}) \rightarrow \mathcal{C}_{\mathcal{R}} \simeq \mathcal{O}(\mathcal{S}_{\mathcal{R}}^{N-1})$. So $\mathcal{O}(\mathcal{S}_{\mathcal{R}}^{N-1})$ is an expected coidealgebra of $\text{Pol}(O_N^{\mathcal{R}})$. Let Φ be the corresponding idempotent state.

Since $\mathcal{O}_{O_N^{\mathcal{R}}}(\mathcal{S}_{\mathcal{R}}^{N-1}) = \text{Pol}(O_N^{\mathcal{R}})$, the Podleś subalgebra

$$\mathcal{O}_{O_N^{\mathcal{R}}}(\mathcal{S}_{\mathcal{R}}^{N-1}) \simeq \mathbb{E}_{\ell}(\text{Pol}(O_N^{\mathcal{R}})) = \mathcal{C}_{\mathcal{R}} \simeq \text{Pol}(\mathcal{S}_{\mathcal{R}}^{N-1}).$$

Now, we can define the associated right conditional expectations

$$\mathbb{E}_r = (\text{id} \otimes \Phi) \circ \Delta$$

and two-sided conditional expectations

$$\mathbb{E}_{bi} = (\Phi \otimes \text{id} \otimes \Phi)(\Delta \otimes \text{id})\Delta.$$

Then

$$\begin{aligned}\mathcal{O}(S_{\mathcal{R}}^{N-1}) &= \mathbb{E}_{\ell}(\text{Pol}(O_N^{\mathcal{R}})) = \ast\text{-alg}\{u_{11}, \dots, u_{1N}\}, \\ S(\mathcal{O}(S_{\mathcal{R}}^{N-1})) &= \mathbb{E}_r(\text{Pol}(O_N^{\mathcal{R}})) = \ast\text{-alg}\{u_{11}, \dots, u_{N1}\}, \\ \mathcal{O}(S_{\mathcal{R}}^{N-1}) \cap S(\mathcal{O}(S_{\mathcal{R}}^{N-1})) &= \mathbb{E}_{bi}(\text{Pol}(O_N^{\mathcal{R}})) = \ast\text{-alg}\{u_{11}\},\end{aligned}$$

where S denotes the antipode of $\text{Pol}(O_N^{\mathcal{R}})$.

We want to compute the eigenvalues and eigenspaces of $O_N^{\mathcal{R}}$ -invariant Markov semigroups on $\mathcal{O}(S_{\mathcal{R}}^{N-1})$. First, we will give decompositions of the Hilbert spaces $L^2(S_{\mathcal{R}}^{N-1}, h)$ and $L^2(O_{N-1}^{\mathcal{R}}, h)$, which are constructed by h -GNS representation. Set

$$\begin{aligned}E_k &= \text{span}\{u_{i_1 j_1} \cdots u_{i_r j_r} : r \leq k\} \subset L^2(S_{\mathcal{R}}^{N-1}, h); \\ H_k &= \text{span}\{x_{i_1} \cdots x_{i_r} : r \leq k\} \subset L^2(O_{N-1}^{\mathcal{R}}, h); \\ V_k &= E_k \cap E_{k-1}^{\perp}; \quad D_k = H_k \cap H_{k-1}^{\perp}; \quad d_k = \dim D_k.\end{aligned}$$

Then $L_2(S_{\mathcal{R}}^{N-1}, h) = \bigoplus_{k \in \mathbb{N}} D_k$ and $L_2(O_N^{\mathcal{R}}, h) = \bigoplus_{k \in \mathbb{N}} V_k$. Furthermore $H_k = \mathbb{E}_{\ell}(E_k)$, and thus $D_k = \mathbb{E}_{\ell}(V_k)$.

Take a complete set $\{u^{\pi} : \pi \in \text{Irr}(O_N^{\mathcal{R}})\}$ of mutually inequivalent, irreducible unitary representations. We know that the generating matrix $u = (u_{ij})$ is an irreducible unitary representation of $O_N^{\mathcal{R}}$. We can decompose its tensor powers $u^{\boxtimes s} = \bigoplus_{\pi \in I_s} n_{\pi}^s u^{\pi}$, where n_{π}^s denotes the multiplicity of u^{π} , and we used the notation $I_s := \{\pi \in \text{Irr}(O_N^{\mathcal{R}}) : n_{\pi}^s \geq 1\}$.

Then, for any $s \geq 2$, we define

$$u^{(s)} := \bigoplus_{\pi \in J_s} u^{\pi}, \quad \text{where } J_s = I_s \setminus \bigcup_{0 \leq i \leq s-1} I_i.$$

In other words, $u^{(s)}$ is the direct sum of the ‘‘new’’ irreducible representations in the decomposition of $u^{\boxtimes s}$, those which did not appear in the decompositions of $u^{\boxtimes i}, \forall i < s$.

Since the linear space spanned by coefficients of $\{u^{\boxtimes i}\}_{0 \leq i \leq s}$ is E_s , by the choice of the set I_s , we have $E_s = \text{span}\{u_{pq}^{(\pi)} : \pi \in I_i, 0 \leq i \leq s\}$. Thus by definition, the linear space spanned by coefficients of $u^{(s)}$ is V_s .

For the free case, by the fusion rule of O_N^+ , we know that $I_s = \{s, s-2, s-4, \dots\}$. Therefore $J_s = \{s\}$. So $u^{(s)}$ is exactly the s^{th} irreducible unitary representation of O_N^+ . For other cases, $u^{(s)}$ defined here may not be irreducible. But it is the direct sum of some mutually inequivalent irreducible unitary representations.

We state the argument above as a proposition:

Proposition 2.15. *There exists a sequence of unitary representations $(u^{(s)})_{s \in \mathbb{N}}$ of $O_N^{\mathcal{R}}$, such that all of non-zero coefficients of $u^{(s)}$ are linearly independent and span V_s . In the free case, $u^{(s)}$ is irreducible.*

Define the degree of u_{11} in $O_N^{\mathcal{R}}$ by

$$\text{deg}(u_{11}) = \max\{s : 1, u_{11}, u_{11}^2, \dots, u_{11}^s \text{ are linearly independent}\}. \quad (2.1)$$

Denote $\text{deg}(u_{11}) = \infty$, if for any $s \in \mathbb{N}$, $1, u_{11}, u_{11}^2, \dots, u_{11}^s$ are linearly independent. One should distinguish the notation u_{11}^s and $u_{11}^{(s)}$, where $u_{11}^s = \underbrace{u_{11}u_{11} \cdots u_{11}}_s$ and $u_{11}^{(s)}$ means

the element lies in the first row and first column of representation $u^{(s)}$.

The following lemma is the main step for characterising the idempotent state Φ .

Lemma 2.16. *Let $(u^{(s)})_{s \in \mathbb{N}}$ be the sequence of representations defined in Proposition 2.15. For any $s \in \mathbb{N}$, there exists a basis for the Hilbert space K_s where the representation $u^{(s)}$ acts on, such that*

$$\Phi(s) := \left(\Phi(u_{jk}^{(s)}) \right)_{1 \leq j, k \leq d_s} = \delta_{j1} \delta_{k1} \cdot \mathbf{1}_{[0, \deg(u_{11})]}(s),$$

if we write $u^{(s)} = (u_{jk}^{(s)})_{1 \leq j, k \leq d_s}$ w.r.t. to this basis. In other words, when $s \leq \deg(u_{11})$, the representation $u^{(s)}$ is unitarily equivalent to one for which applying Φ to it coefficient-wise produces a matrix with entry 1 in the upper left corner and 0 everywhere else.

Proof. Since Φ is idempotent state, we can easily check that

$$\|\Phi_s\| \leq 1 \quad \text{and} \quad \Phi_s^2 = \Phi_s,$$

which means that Φ_s is a projection in $B(K_s)$. We know that every projection matrix can be written as a diagonal matrix with coefficients 1 and 0 by choosing some suitable basis. So

$$\Phi_s = \begin{pmatrix} \left. \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \right\} r_s & & \cdots & 0 \\ & & 0 & \\ & \vdots & & \ddots \\ & 0 & & & 0 \end{pmatrix}.$$

Denote the rank of this matrix by r_s . For all k , we take the basis of K_k as above, so that for $0 \leq i \leq r_k$, $\Phi(u_{ii}^{(s)}) = 1$; otherwise $\Phi(u_{ij}^{(s)}) = 0$. Then for any $s \in \mathbb{N}$,

$$\mathbb{E}_{bi}(u_{ij}^{(s)}) = \sum_{p,q} \Phi(u_{ip}^{(s)}) u_{pq}^{(s)} \Phi(u_{qj}^{(s)}) = \begin{cases} u_{ij}^{(s)} & \text{if } 1 \leq i, j \leq r_s \\ 0 & \text{otherwise.} \end{cases}$$

For $i \leq r_s$, $u_{ii}^{(s)} \neq 0$ since $\Phi(u_{ii}^{(s)}) \neq 0$, while $u_{ij}^{(s)}$ may be 0, $i \neq j$. This implies that

$$r_s \leq \dim(\mathbb{E}_{bi}(V_s)) \leq r_s^2. \quad (2.2)$$

Moreover, the conditional expectation \mathbb{E}_{bi} sends E_s onto

$$\text{Pol}_s(u_{11}) := \{1, u_{11}, u_{11}^2, \dots, u_{11}^s\}.$$

Thus, when $s \leq \deg(u_{11})$,

$$\dim(\mathbb{E}_{bi} \left(\bigoplus_{k=0}^s (V_k) \right)) = \dim(\mathbb{E}_{bi}(E_s)) = \dim(\text{Pol}_s(u_{11})) = s + 1.$$

This implies

$$\dim(\mathbb{E}_{bi}(V_s)) = \dim(\text{Pol}_s(u_{11})) - \dim(\text{Pol}_{s-1}(u_{11})) = 1.$$

By Inequality (2.2), we get $r_s = 1$ for any $s \leq \deg(u_{11})$.

Similarily, when $s \geq \deg(u_{11}) + 1$, $\dim(\mathbb{E}_{bi}(V_s)) = \dim(\text{Pol}_s(u_{11})) - \dim(\text{Pol}_{s-1}(u_{11})) = 0$, therefore $r_s = 0$. \square

This theorem tells us that $u_{11}^{(s)} = \mathbb{E}_{bi}(u_{11}^{(s)}) \in \text{Pol}_s(u_{11})$.

If $\deg(u_{11}) < \infty$, $C_r(\mathcal{S}_{\mathcal{R}}^{N-1})$ is a finite C^* -algebra, since $\mathbb{E}_\ell = (\Phi \otimes \text{id})\Delta$. Let us consider the more interesting case: $\deg(u_{11}) = \infty$, i.e. $(u_{11}^{(s)})_{s \in \mathbb{N}}$ is linearly independent. The algebra $^*\text{-alg}\{u_{11}\}$ as a subalgebra of $\text{Pol}(O_N^{\mathcal{R}})$ can be identified with the algebra of polynomials on the interval $[-1, 1]$. Therefore, there exists $q_k^{\mathcal{R}} \in \text{Pol}[-1, 1]$ such that $q_k^{\mathcal{R}}(u_{11}) = u_{11}^{(k)}$. Since $h_{O_N^{\mathcal{R}}}(q_n^{\mathcal{R}}(u_{11})q_m^{\mathcal{R}}(u_{11})) = h_{O_N^{\mathcal{R}}}(u_{11}^{(n)}u_{11}^{(m)}) = C_n\delta_{nm}$, $(q_s^{\mathcal{R}})_{s \in \mathbb{N}}$ is a family of orthogonal polynomials. The measure for the orthogonality of these polynomials is the probability measure μ where μ is obtained by evaluating the spectral measure of u_{11} in the Haar state. Since u_{11} is hermitian and $\|u_{11}\| \leq 1$, we get that the measure μ is supported on $[-1, 1]$ (which explains why we consider only the values of our polynomials on this interval).

The restriction of the counit to $^*\text{-alg}\{u_{11}\}$ corresponds to evaluation of a polynomial in the boundary point 1, i.e. $\epsilon(p(u_{11})) = p(1)$, for any polynomial $p \in \text{Pol}[-1, 1]$. Therefore we obtain the following result, in the same manner as in [29, Proposition 10.1].

Proposition 2.17. [29, Proposition 10.1]. *Let ψ be a conditionally positive functional on $\text{Pol}[u_{11}]$. Then there exist a unique pair (b, ν) consisting of a real number $b \geq 0$ and a finite measure ν on $[-1, 1]$ such that*

$$\psi(p) = -bp'(1) + \int_{-1}^1 \frac{p(x) - p(1)}{x - 1} d\nu(x)$$

for any polynomial p . Conversely, every ψ of this form is conditionally positive.

Applying the above proposition, we can compute the eigenvalues of Markov semigroups.

Theorem 2.18. *Assume $\deg(u_{11}) = \infty$. For any $O_N^{\mathcal{R}}$ -invariant strongly continuous Markov semigroup $(T_t)_{t \geq 0}$ on $\text{Pol}(S_{\mathcal{R}}^{N-1})$, there exists a pair (b, ν) , with b a positive number and ν a finite measure on $[-1, 1]$, such that the generator L of $(T_t)_{t \geq 0}$ is determined by,*

$$L(x_s) = \lambda_s x_s, \quad x_s \in D_s.$$

Here

$$\lambda_s = -b(q_s^{\mathcal{R}})'(1) + \int_{-1}^1 \frac{q_s^{\mathcal{R}}(x) - 1}{x - 1} d\nu(x);$$

where $q_s^{\mathcal{R}} \in \text{Pol}[u_{11}]$ such that $h(q_n^{\mathcal{R}}(u_{11})q_m^{\mathcal{R}}(u_{11})) = C_n\delta_{nm}$ and $q_s(1) = 1$.

Proof. Theorem 2.13 guarantees that the the generator operator L exists and

$$L = (\psi \otimes \mathbb{E}_\ell) \circ \alpha$$

where ψ is a $(\epsilon \circ T_0)$ -generating functional on $\text{Pol}(S_{\mathcal{R}}^{N-1})$.

By Lemma 2.16, we can compute $\mathbb{E}_\ell(u_{ij}^{(k)}) = \delta_{1i}u_{1j}^{(k)}$, which implies $D_s = \mathbb{E}_\ell(V_k) = \text{span}\{u_{1j}^{(s)} : 1 \leq j \leq d_k\}$. Then for any $u_{1j}^{(s)} \in D_s$,

$$L(u_{1j}^{(s)}) = \sum_k \psi(u_{1k}^{(s)})\mathbb{E}_\ell(u_{kj}^{(s)}) = \psi(u_{11}^{(s)})u_{1j}^{(s)}.$$

Now, we just need to consider $\psi|_{*\text{-alg}(u_{11})}$ which induces the pair (b, ν) by Proposition 2.17. By linearity of L , we can get the eigenvalues for D_s ,

$$\lambda_s = \psi(u_{11}^{(s)}) = -b(q_s^{\mathcal{R}})'(1) + \int_{-1}^1 \frac{q_s^{\mathcal{R}}(x) - 1}{x - 1} d\nu(x),$$

where $q_s^{\mathcal{R}} \in \text{Pol}[u_{11}]$, such that $u_{11}^{(s)} = q_s^{\mathcal{R}}(u_{11})$, and $q_s^{\mathcal{R}}(1) = \varepsilon(u_{11}^{(s)}) = 1$. □

Remark 2.19. We point out here that, in Theorem 2.18, we have different families of orthogonal polynomials $\{q_s^{\mathcal{R}}(x)\}$ associated to $S_{\mathcal{R}}^{N-1}$, since the Haar state $h_{O_{\mathcal{R}}^N}$ depends on \mathcal{R} . Unfortunately, we did not find a general way to calculate these orthogonal polynomials. However, for some special case, we can compute them. We will show these polynomials for classical, half-liberated and free spheres in next chapter.

Chapter 3

Examples: classical, half-liberated and free spheres

In this Chapter, we consider the classical sphere \mathcal{S}^{N-1} , Half-liberated sphere \mathcal{S}_*^{N-1} and Free sphere \mathcal{S}_+^{N-1} (see Example 2.14). We will use the notation \mathcal{S}_\times^{N-1} to denote these three spheres associated to the three quantum orthogonal groups O_N^\times , where $\times \in \{\emptyset, *, +\}$ (here \emptyset stands for no symbol).

We know that in the classical case $\mathcal{S}^{N-1} \cong O_N/O_{N-1}$. Banica, Skalski, and Sołtan [7] have shown that \mathcal{S}_+^{N-1} is not equal to the quotient O_N^+/O_{N-1}^+ . We will now show that the half-liberated and the free spheres can not be obtained as quotient spaces.

Proposition 3.1. *There exists no quantum subgroup \mathbb{K} of O_N^+ (or O_N^* , resp.) such that $\mathcal{S}_+^{N-1} \cong O_N^+/\mathbb{K}$ (or $\mathcal{S}_*^{N-1} \cong O_N^*/\mathbb{K}$, resp.) as left coidealgebras.*

Before to prove this proposition, we introduce the Weingarten formula (or Weingarten calculus) to calculate the Haar state h on O_N^\times (see [4, Theorem 5.4]).

Let $P_2(k)$, $P_2^+(k) = NC_2(k)$ denote pairing and noncrossing pairings among k points respectively. A pairing is called “balanced”, if each pair connects a black leg to a white leg, when we label its legs alternately black and white: $\bullet \circ \bullet \circ \dots$. Denote the set of balanced pairings of k elements by $P_2^*(k)$.

Then the Weingarten fomula is

$$h_{O_N^\times}(u_{i_1 j_1} \cdots u_{i_k j_k}) = \sum_{\pi, \sigma \in P_2^\times(k)} \delta_\pi(i) \delta_\sigma(j) W_{kN}(\pi, \sigma)$$

where $W_{kN} = G_{kN}^{-1}$ with $G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$, δ are Koronecker type symbols.

Now let us prove Proposition 3.1 by Weingarten calculus.

Proof. We start with the free sphere.

If such a quantum subgroup existed, then it would be of Kac type, and therefore its Haar idempotent $\Phi_{\mathbb{K}} = h_{\mathbb{K}} \circ \theta$ would be tracial. We will now show that the idempotent state associated to $\mathcal{O}(\mathcal{S}_+^{N-1})$ by Theorem 1.7 is not a trace.

Let \mathbb{E}_{bi} denote the conditional expectation in $\mathcal{O}(O_N^+)$ onto the *-subalgebra of $\mathcal{O}(O_N^+)$ generated by u_{11} , then we have $\Phi = \varepsilon \circ \mathbb{E}_l = \varepsilon \circ \mathbb{E}_{bi}$.

\mathbb{E}_{bi} is the orthogonal projection onto *-subalgebra generated by u_{11} for the inner product $\langle a, b \rangle = h(a^*b)$, and since we can compute the values of the Haar state on products

of the algebraic generators u_{11}, \dots, u_{NN} using the Weingarten calculus, we can compute \mathbb{E}_{bi} and then Φ . We find

$$\mathbb{E}_{bi}(u_{22}u_{11}u_{22}) = 0$$

since $h_{O_N^+}(u_{11}^k u_{22} u_{11} u_{22}) = 0$ for all $k \in \mathbb{N}$ (there are no matching non-crossing pairings) and

$$\mathbb{E}_{bi}(u_{11}u_{22}^2) = u_{11}\mathbb{E}_{bi}(u_{22}^2) = \frac{(N-2)u_{11} + u_{11}^3}{(N-1)^2}$$

since

$$h_{O_N^+}(u_{22}^2) = \frac{1}{N} \quad \text{and} \quad h_{O_N^+}(u_{11}^2 u_{22}^2) = \frac{1}{N^2 - 1}.$$

It follows that

$$\Phi(u_{11}u_{22}^2) = \varepsilon \left(\frac{(N-2)u_{11} + u_{11}^3}{(N-1)^2} \right) = \frac{1}{N-1} \neq 0 = \Phi(u_{22}u_{11}u_{22}).$$

The case of the half-liberated sphere \mathcal{S}_*^{N-1} is similar. The Weingarten formula for O_N^* uses balanced pairings. The balanced pairings $P_2^*(4)$ and the non-crossing pairings $NC_2(4)$ of four elements are the same. Thus, we get again the same values for Haar state in the half-liberated case,

$$\begin{aligned} h_{O_N^*}(u_{11}^k u_{22} u_{11} u_{22}) &= 0, \quad k = 0, 1, \dots, \\ h_{O_N^*}(u_{22}^2) &= \frac{1}{N} \quad \text{and} \quad h_{O_N^*}(u_{11}^2 u_{22}^2) = \frac{1}{N^2 - 1}, \end{aligned}$$

and we get the same conclusion. □

3.1 Orthogonal polynomials

We will describe these orthogonal polynomials for classical, half-liberated, free cases.

3.1.1 The classical sphere

Here, $(q_s(x))_{s \in \mathbb{N}}$ means the family of the orthogonal polynomials associated to classical sphere. It is well known that the distribution of u_{11}^2 for the classical sphere is the beta distribution with parameters $(1/2, (N-1)/2)$. In other words,

$$h_{\mathcal{S}^{N-1}}(\phi(u_{11}^2)) = C \int_0^1 \phi(t) \frac{1}{\sqrt{t}} (1-t)^{\frac{N-3}{2}} dt,$$

where $C = \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{N-1}{2})}$. The integral vanishes on the odd polynomials, i.e. $h_{\mathcal{S}^{N-1}}(u_{11}^{2k+1}) = 0$. Therefore $h_{\mathcal{S}^{N-1}}\left(\frac{f(u_{11})+f(-u_{11})}{2}\right) = h_{\mathcal{S}^{N-1}}(f(u_{11}))$.

$$\begin{aligned} h_{\mathcal{S}^{N-1}}(f(u_{11})) &= h_{\mathcal{S}^{N-1}}\left(\frac{f(u_{11}) + f(-u_{11})}{2}\right) \\ &= C \int_{-1}^1 f(t) (1-t)^{\frac{N-3}{2}} (1+t)^{\frac{N-3}{2}} dt \end{aligned}$$

The spectral measure of u_{11} is the probability measure on the interval $[-1, 1]$:

$$\mu(dt) = C(1-t)^{\frac{N-3}{2}}(1+t)^{\frac{N-3}{2}} dt,$$

whose family of orthogonal polynomials is well known. Namely, we get the Jacobi polynomials (or ultraspherical polynomials) with parameters $\alpha = \beta = (N-3)/2$, which we will denote by $(J_s)_{s \in \mathbb{N}}$.

Recall that Jacobi polynomials are given by:

$$J_s(x) = \sum_{r=0}^s \binom{s+(N-3)/2}{r} \binom{s+(N-3)/2}{(N-3)/2-r} \left(\frac{x-1}{2}\right)^{s-r} \left(\frac{x+1}{2}\right)^r.$$

Their orthogonality relation is given by

$$\int_{-1}^1 J_k(x) J_m(x) \mu(dt) = \delta_{km} \cdot C \frac{2^{N-2}}{2k+N-2} \frac{\Gamma(k+(N-1)/2)^2}{\Gamma(k+N-2)n!}.$$

Moreover, they satisfy the differential equation

$$(1-x^2)J_s''(x) - x(N-1)J_s'(x) = -s(s+N-2)J_s(x).$$

We need these polynomials in the form $q_s(x) = J_s(x)/J_s(1)$.

Therefore,

$$q_s'(1) = \frac{s(s+N-2)}{N-1}.$$

We can relate our result to the *Morkov sequence problem*. For a given orthonormal basis $\{f_0 = \mathbf{1}, f_1, f_2, \dots\}$ of the L^2 -space of some probability space, this problem of ask for the classification of all sequences $(\lambda_n)_{n \geq 0}$ such that $K(f_n) = \lambda_n f_n$ defines Markov operator cf. [2]. In [13, Theorem 2], Bochner answered this problem for the Jacobi polynomials. Since we found that the Jacobi polynomials are the eigenvectors for any O_N -invariant Markov semigroup on \mathcal{S}^{N-1} , our Theorem 2.18 recovers [13, Theorem 3].

3.1.2 The half-liberated sphere

Next we consider the half-liberated sphere \mathcal{S}_*^{N-1} .

Banica [4, Proposition 6.6] determined the law of $x_{i_1} \cdots x_{i_k}$ with respect to the Haar state $h_{\mathcal{S}_*^{N-1}} = h_{O_N^*}|_{C(\mathcal{S}_*^{N-1})}$ (there is a small misprint in [4, Proposition 6.6], which we correct below).

Proposition 3.2. *The half-liberated integral of $x_{i_1} \cdots x_{i_k}$ vanishes, unless each index a appears the same number of times at odd and even positions in i_1, \dots, i_k . We have*

$$h_{\mathcal{S}_*^{N-1}}(x_{i_1} \cdots x_{i_k}) = \frac{(N-1)! \ell_1! \cdots \ell_n!}{(N + \sum \ell_i - 1)!}$$

where ℓ_i denotes this number of common occurrences of i in the k -tuple (i_1, \dots, i_k) .

This proposition allows to describe the spectral distribution of $u_{11} = x_1$ w.r.t. the Haar state.

Corollary 3.3. *The distribution of u_{11} in the half-liberated case is given by:*

$$h_{\mathcal{S}_*^{N-1}}(f(u_{11})) = \int_{-1}^1 f(t)\mu(dt), \quad \forall f \in C([-1, 1])$$

where $\mu(dt) = (N-1)(1-t^2)^{N-2}|t|dt$.

Proof. This proof repeats the arguments of [4, Proposition 6.5 and Proposition 6.6].

Let $C = \frac{2^N}{2^N \pi^N} \cdot \Gamma(N+1) = \left(\frac{2}{\pi}\right)^N (N-1)!$, then

$$\begin{aligned} h_{\mathcal{S}_*^{N-1}}(x^{2k}) &= \int_{\mathcal{S}_{\mathbb{C}}^{N-1}} |z_1|^{2k} dz \\ &= C \int_{\mathcal{S}_{\mathbb{R}}^{2N-1}} (x_1^2 + y_1^2)^k d(x, y) \\ &= C \int_0^{\pi/2} \int_0^{\pi/2} (\cos^2 \theta_1 + \sin^2 \theta_1 \cos^2 \theta_2)^k \sin^{2N-2} \theta_1 \sin^{2N-3} \theta_2 d\theta_1 d\theta_2 \\ &\quad \cdot \int_0^{\pi/2} \sin^{2N-4} \theta_3 d\theta_3 \cdots \int_0^{\pi/2} \sin \theta_{2N-2} d\theta_{2N-2} \cdot \int_0^{\pi/2} d\theta_{2N-1}. \end{aligned}$$

First we can calculate that

$$\begin{aligned} C' &= C \int_0^{\pi/2} \sin^{2N-4} \theta_3 d\theta_3 \cdots \int_0^{\pi/2} \sin \theta_{2N-2} d\theta_{2N-2} \cdot \int_0^{\pi/2} d\theta_{2N-1} \\ &= \left(\frac{2}{\pi}\right)^N (N-1)! \cdot \left(\frac{\pi}{2}\right)^{N-1} \frac{(2N-4)!! (2N-5)!! \cdots 1!!}{(2N-3)!! (2N-4)!! \cdots 2!!} \\ &= \frac{4}{\pi} (N-1) \end{aligned}$$

where $m!! = (m-1)(m-3)\cdots 1$. Let $t = \sqrt{\cos^2 \theta_1 + \sin^2 \theta_1 \cos^2 \theta_2}$, $u = \cos \theta_1$, then

$$\begin{aligned} h_{\mathcal{S}_*^{N-1}}(x^{2k}) &= C' \int_0^1 \int_0^t t^{2k} (1-t^2)^{N-2} \frac{t}{\sqrt{t^2-u^2}} du dt \\ &= 2(N-1) \int_0^1 t^{2k} \cdot (1-t^2)^{N-2} t dt. \end{aligned}$$

Since the odd moments of u_{11} vanish, we have $h_{\mathcal{S}_*^{N-1}}(f(u_{11})) = h_{\mathcal{S}_*^{N-1}}\left(\frac{f(u_{11})+f(-u_{11})}{2}\right)$. and

$$\begin{aligned} h_{\mathcal{S}_*^{N-1}}(f(u_{11})) &= 2(N-1) \int_0^1 \left(\frac{f(u_{11})+f(-u_{11})}{2}\right) \cdot (1-t^2)^{N-2} t dt \\ &= (N-1) \int_{-1}^1 f(u_{11})(1-t^2)^{N-2}|t|dt. \end{aligned}$$

□

Now we determine the family of orthogonal polynomials associated to the probability measure μ defined in Corollary 3.3. For a general measure, it is difficult to find the orthogonal polynomials. However, in the case which we are considering now, we find that those orthogonal polynomials are very regular, so we can guess the formula and then to prove the orthogonality. That is what we will do in the following.

The standard notation for hypergeometric functions is

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; x \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_r)_n x^n}{(b_1)_n \cdots (b_s)_n n!}$$

where the shifted factorial $(a)_n$ is defined by

$$(a)_n = \begin{cases} a(a+1) \cdots (a+n-1), & n = 1, 2, \dots \\ 1, & n = 0. \end{cases}$$

They satisfy

$$\frac{d}{dx} {}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; x \right) = \frac{\prod_{i=1}^r a_i}{\prod_{j=1}^s b_j} {}_rF_s \left(\begin{matrix} a_1 + 1, \dots, a_r + 1 \\ b_1 + 1, \dots, b_s + 1 \end{matrix}; x \right). \quad (3.1)$$

The following formula is *the Chu–Vandermonde identity*, which is obtained by Newton’s generalized binomial theorem,

$$\sum_{r=0}^s \binom{\alpha}{r} \binom{\beta}{s-r} = \binom{\alpha + \beta}{s} \quad \alpha, \beta \in \mathbb{C}, s \in \mathbb{N}.$$

Using the Chu–Vandermonde identity, we can compute the following value

$${}_2F_1 \left(\begin{matrix} -k, \alpha \\ m \end{matrix}; 1 \right) = \frac{(m - \alpha)_k}{(m)_k} \quad k, m \in \mathbb{N}, \alpha \in \mathbb{C}. \quad (3.2)$$

Definition 3.4. We define the family *half-liberated spherical polynomials* (or “*-polynomials”) by

$$\begin{aligned} P_{2k}(x) &= (-1)^k \binom{N + 2k - 2}{k}^{-1} {}_2F_1 \left(\begin{matrix} -k, N + k - 1 \\ 1 \end{matrix}; x^2 \right) \\ &= \sum_{r=0}^k (-1)^{k+r} \binom{k}{r}^2 \binom{N + 2k - 2}{k-r}^{-1} x^{2r}, \\ P_{2k+1}(x) &= x \cdot (-1)^k (k+1) \binom{N + 2k - 1}{k}^{-1} {}_2F_1 \left(\begin{matrix} -k, N + k \\ 2 \end{matrix}; x^2 \right) \\ &= \sum_{r=0}^k (-1)^{k+r} \binom{k}{r} \binom{k+1}{r+1} \binom{N + 2k - 1}{k-r}^{-1} x^{2r+1}. \end{aligned}$$

Proposition 3.5. *The family of “*-polynomials” satisfies the following three-term recurrence relation:*

$$P_s(x) = xP_{s-1}(x) - \omega_{s-2}P_{s-2}(x) \quad \forall s \geq 2,$$

where $\omega_\ell = \frac{[(\ell+2)/2](N-1+[\ell/2])}{(N+\ell)(N+\ell-1)}$. Moreover, the “*-polynomials” are the orthogonal polynomials for the probability measure $\mu(dt) = (N-1)(1-t^2)^{N-2}|t|dt$.

Proof. We can easily check that for any $k \geq 1$,

$$xP_{2k}(x) - \frac{k(N+k-2)}{(N+2k-2)(N+2k-1)}P_{2k-1}(x) = P_{2k+1}(x),$$

$$xP_{2k-1}(x) - \frac{k(N+k-2)}{(N+2k-2)(N+2k-3)}P_{2k-2}(x) = P_{2k}(x).$$

Therefore the three-term recurrence relation holds.

By the Proposition 3.2 and equation (3.2), we can calculate

$$\begin{aligned} \int_{-1}^1 P_{2k}(x)\mu(dx) &= \sum_{r=0}^k (-1)^{k+r} \binom{k}{r}^2 \binom{N+2k-2}{k-r}^{-1} \frac{(N-1)!r!}{(N+r-1)!} \\ &= (-1)^k \binom{N+2k-2}{k}^{-1} \sum_{r=0}^k \frac{(-1)^r k!(N+k+r-2)!(N-1)!}{(k-r)!(N+k-2)!(N+r-1)!r!}, \\ &= (-1)^k \binom{N+2k-2}{k}^{-1} {}_2F_1 \left(\begin{matrix} -k, N+k-1 \\ N \end{matrix}; 1 \right) \\ &= (-1)^k \binom{N+2k-2}{k}^{-1} \frac{(-k+1)_k}{(N)_k} \\ &= \delta_{0k}; \end{aligned}$$

and all of the odd moments vanish, i.e., $\int_{-1}^1 P_{2k+1}(x)\mu(dx) = 0$.

Inspired by [80, Lemma 5.3], we are now going to prove the orthogonality by induction.

Clearly, $\forall n > 0$, $\int_{-1}^1 P_n(x)P_0(x)\mu(dx) = 0$.

Assume that for any $0 \leq k \leq s$, $\int_{-1}^1 P_n(x)P_k(x)\mu(dx) = 0$ holds for all $n > k$. Then consider $s+1$, and $n > s+1$. Using the three-term recurrence relation, we get

$$\begin{aligned} \int_{-1}^1 P_n(x)P_{s+1}(x)\mu(dx) &= \int_{-1}^1 P_n(x)(xP_s(x) - \omega_{s-1}P_{s-1}(x))\mu(dx) \\ &= \int_{-1}^1 xP_n(x)P_s(x)\mu(dx) + 0 \\ &= \int_{-1}^1 (P_{n+1}(x) + \omega_{n-1}P_{n-1}(x))P_s(x)\mu(dx) \\ &= 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_{-1}^1 P_s^2(x)\mu(dx) &= \int_{-1}^1 P_s(x)(xP_{s-1} - \omega_{s-2}P_{s-2})\mu(dx) \\ &= \int_{-1}^1 (P_{s+1}(x) + \omega_{s-1}P_{s-1}(x))P_{s-1}(x)\mu(dx) \\ &= \omega_{s-1} \int_{-1}^1 P_{s-1}^2(x)\mu(dx) = \omega_0\omega_1 \cdots \omega_{s-1}, \end{aligned}$$

so that

$$\int_{-1}^1 P_m(x)P_n(x)\mu(dx) = \omega_0\omega_1 \cdots \omega_{n-1} \cdot \delta_{mn}.$$

□

Remark 3.6. We change the normalisation of these polynomial to get the sequence $q_s^*(x) = \frac{P_s(x)}{P_s(1)}$ which satisfies the conditions of Theorem 2.18.

We have

$$\begin{aligned}
 P_{2k}(1) &= (-1)^k \binom{N+2k-2}{k}^{-1} {}_2F_1 \left(\begin{matrix} -k, N+k-1 \\ 1 \end{matrix}; 1 \right) \\
 &= (-1)^k \frac{k!(N+k-2)! (2-N-k)_k}{(N+2k-2)! (1)_k} \\
 &= \frac{(N+k-2)!(N+k-2)!}{(N+2k-2)!(N-2)!}; \\
 P_{2k+1}(1) &= (-1)^k (k+1) \binom{N+2k-1}{k}^{-1} {}_2F_1 \left(\begin{matrix} -k, N+k \\ 2 \end{matrix}; 1 \right) \\
 &= (-1)^k (k+1) \frac{k!(N+k-1)! (2-N-k)_k}{(N+2k-1)! (2)_k} \\
 &= \frac{(N+k-1)!(N+k-2)!}{(N+2k-1)!(N-2)!}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 q_{2k}^*(x) &= (-1)^k \binom{N+k-2}{k}^{-1} {}_2F_1 \left(\begin{matrix} -k, N+k-1 \\ 1 \end{matrix}; x^2 \right); \\
 q_{2k+1}^*(x) &= x \cdot (-1)^k (k+1) \binom{N+k-2}{k}^{-1} {}_2F_1 \left(\begin{matrix} -k, N+k \\ 2 \end{matrix}; x^2 \right).
 \end{aligned}$$

The following formula gives the eigenvalues of the generator of the O_N^* -invariant semi-group on the half-liberated sphere S_*^{N-1} associated to the pair $b = 1$ and $\nu = 0$. By analogy with the classical sphere, these values can be considered as the eigenvalues of the Laplace operator of the half-liberated sphere (up to a rescaling by $N - 1$, see Remark 3.11).

Corollary 3.7. *For any $k \geq 0$,*

$$\begin{aligned}
 (q_{2k}^*)'(1) &= \frac{2k(N+k-1)}{N-1}; \\
 (q_{2k+1}^*)'(1) &= \frac{(2k+1)N+2k^2-1}{N-1}.
 \end{aligned}$$

Proof. $q_0'(1) = 0$ is obvious.

For $k \geq 1$, by the equation (3.1) and (3.2), we have

$$\begin{aligned}
 (q_{2k}^*)'(1) &= \frac{2x \frac{d}{dx^2} {}_2F_1 \left(\begin{matrix} -k, N+k-1 \\ 1 \end{matrix}; x^2 \right)}{{}_2F_1 \left(\begin{matrix} -k, N+k-1 \\ 1 \end{matrix}; 1 \right)} \Big|_{x=1} \\
 &= \frac{-2k(N+k-1) {}_2F_1 \left(\begin{matrix} -k+1, N+k \\ 2 \end{matrix}; 1 \right)}{{}_2F_1 \left(\begin{matrix} -k, N+k-1 \\ 1 \end{matrix}; 1 \right)} \\
 &= -2k(N+k-1) \frac{(2-N-k)_{k-1}}{(2)_{k-1}} \frac{(1)_k}{(2-N-k)_k} \\
 &= \frac{2k(N+k-1)}{N-1}; \\
 (q_{2k+1}^*)'(1) &= \frac{\frac{d}{dx} \left(x \cdot {}_2F_1 \left(\begin{matrix} -k, N+k \\ 2 \end{matrix}; x^2 \right) \right)}{{}_2F_1 \left(\begin{matrix} -k, N+k \\ 2 \end{matrix}; 1 \right)} \Big|_{x=1} \\
 &= \frac{{}_2F_1 \left(\begin{matrix} -k, N+k \\ 2 \end{matrix}; x^2 \right) + x \cdot \left(2x \frac{d}{dx^2} {}_2F_1 \left(\begin{matrix} -k, N+k \\ 2 \end{matrix}; x^2 \right) \right)}{{}_2F_1 \left(\begin{matrix} -k, N+k \\ 2 \end{matrix}; 1 \right)} \Big|_{x=1} \\
 &= \frac{{}_2F_1 \left(\begin{matrix} -k, N+k \\ 2 \end{matrix}; 1 \right) + 2 \cdot \frac{-k(N+k)}{2} {}_2F_1 \left(\begin{matrix} -k+1, N+k+1 \\ 3 \end{matrix}; 1 \right)}{{}_2F_1 \left(\begin{matrix} -k, N+k \\ 2 \end{matrix}; 1 \right)} \\
 &= \frac{(2k+1)N + 2k^2 - 1}{N-1}.
 \end{aligned}$$

□

3.1.3 The free sphere

Finally, we consider about the free case.

The family of *Chebyshev polynomials of the second kind* $U_s(x)$ is defined by

$$U_0(x) = 1, \quad U_1(x) = x, \quad U_{s+2}(x) = xU_{s+1}(x) - U_s(x) \quad \forall s \geq 0.$$

In fact, due to the asymptotic semicircle law of $\sqrt{N+2}u_{11}$ when $N \rightarrow \infty$ [5], we expect that $q_s^+(x) \rightarrow U_s(\sqrt{N}x)/\sqrt{N^s}$. Therefore, $\lim_{N \rightarrow \infty} q_s^+(x) = x^s$. So for the special case where the generating functional ψ is associated to the pair $b = 1, \nu = 0$, the eigenvalues for the subspace D_s converge as $N \rightarrow \infty$, $\lim_{N \rightarrow \infty} \lambda_s = -(x^s)'(1) = -s$. We now derive relations between polynomials $(q_s^+)_{s \geq 0}$ for general finite N .

Proposition 3.8. *Let $N \geq 2$. the orthogonal polynomials defined as above satisfy the following three-term recurrence relation:*

$$a_{s+1}q_{s+2}^+(x) = U_{s+1}(N)q_{s+1}^+(x)x - a_s q_s^+(x) \quad \forall s \geq 0$$

where $q_0^+(x) = 1, q_1^+(x) = x$,

$$a_s = \sum_{k=0}^s (-1)^{s+k} U_k(N) = \begin{cases} U_m(N)(U_m(N) - U_{m-1}(N)) & \text{if } s = 2m, \\ U_m(N)(U_{m+1}(N) - U_m(N)) & \text{if } s = 2m + 1, \end{cases}$$

and where $U_s(N)$ denotes the value of the s^{th} Chebyshev polynomial of the second kind at the point N .

Proof. For free orthogonal quantum group, the irreducible corepresentations have the following fusion rule [3]:

$$u^{(s+1)} \otimes u = u^{(s+2)} \oplus u^{(s)}.$$

This implies that $u_{11}^{(s+1)} u_{11} \in V_{s+2} \oplus V_s$. Applying the two-sided conditional expectation \mathbb{E}_{bi} to both sides, we see that $u_{11}^{(s+1)} \cdot u_{11}$ can be written as the linear combination of $u_{11}^{(s+2)}$ and $u_{11}^{(s)}$.

Let λ_s be a number such that the coefficient of the highest degree of the polynomial $\lambda_s q_s^+(x)$ is 1. Since $q_s^+(1) = 1$, we have

$$\lambda_{s+2} q_{s+2}^+(x) = \lambda_{s+1} q_{s+1}^+(x)x - (\lambda_{s+1} - \lambda_{s+2}) q_s^+(x). \quad (3.3)$$

Let d_s be the dimension of $U^{(s)}$. Note that $d_0 = 1$ and $d_1 = N$. By the Fusion rule, we get $n_{s+2} = Nn_{s+1} - n_s = U_s(N)$. By the orthogonality of $(q_s^+(u_{11}))_{s \geq 0}$ and $h_{S_+^{N-1}}((q_s^+(u_{11}))^2) = h_{S_+^{N-1}}\left(\left(u_{11}^{(s)}\right)^2\right) = 1/U_s(N)$, we have

$$\begin{aligned} 0 &= h_{S_+^{N-1}}(\lambda_{s+2} q_{s+2}^+(u_{11}) q_s^+(u_{11})) \\ &= h_{S_+^{N-1}}(\lambda_{s+1} q_{s+1}^+(u_{11}) q_s^+(u_{11}) u_{11}) - (\lambda_{s+1} - \lambda_{s+2}) h_{S_+^{N-1}}((q_s^+(u_{11}))^2) \\ &= \frac{\lambda_{s+1}^2}{\lambda_s} h_{S_+^{N-1}}((q_{s+1}^+(u_{11}))^2) + 0 - (\lambda_{s+1} - \lambda_{s+2}) h_{S_+^{N-1}}((q_s^+(u_{11}))^2) \\ &= \frac{\lambda_{s+1}^2}{\lambda_s U_{s+1}(N)} - \frac{\lambda_{s+1} - \lambda_{s+2}}{U_s(N)}. \end{aligned}$$

Therefore

$$\frac{\lambda_{s+2}}{\lambda_{s+1}} = 1 - \frac{\lambda_{s+1}}{\lambda_s} \cdot \frac{U_s(N)}{U_{s+1}(N)}.$$

Set $a_s = \frac{\lambda_{s+1}}{\lambda_s} \cdot U_s(N)$. By multiplying $\frac{U_{s+1}(N)}{\lambda_{s+1}}$ to Equation (3.3), we get

$$a_{s+1} q_{s+2}^+(x) = U_{s+1}(N) q_{s+1}^+(x)x - a_s q_s^+(x),$$

and

$$a_{s+1} = U_{s+1}(N) - a_s.$$

From the latter equation we can get

$$a_s = \sum_{k=0}^s (-1)^{s+k} U_k(N).$$

□

The following formula gives the eigenvalues of the generator of the O_N^+ -invariant semi-group on the free sphere S_+^{N-1} associated to the pair $b = 1$ and $\nu = 0$. By analogy with the classical sphere, these values can be considered as the eigenvalues of the Laplace operator of the free sphere (up to a rescaling by $N - 1$, see Remark 3.11).

Corollary 3.9.

$$(q_s^+)'(1) = \sum_{r=0}^{s-1} \frac{\sum_{k=0}^r U_k(N)}{\sum_{k=0}^r (-1)^{r+k} U_k(N)} \quad \forall s \geq 1.$$

Proof. Appling Proposition 3.8 and taking derivatives on both sides, we get

$$a_{s+1}(q_{s+2}^+)'(x) = U_{s+1}(N) \left((q_{s+1}^+)'(x)x + q_{s+1}^+(x) \right) - a_s(q_s^+)'(x).$$

Since $q_s^+(1) = 1$, we have

$$a_{s+1}(q_{s+2}^+)'(1) = U_{s+1}(N)(q_{s+1}^+)'(1) + U_{s+1}(N) - a_s(q_s^+)'(1).$$

Rewrite this equation using $U_{s+1}(N) = a_{s+1} + a_s$,

$$a_{s+1} \left((q_{s+2}^+)'(1) - (q_{s+1}^+)'(1) \right) = U_{s+1}(N) + a_s \left((q_{s+1}^+)'(1) - (q_s^+)'(1) \right).$$

Therefore,

$$a_s \left((q_{s+1}^+)'(1) - (q_s^+)'(1) \right) = \sum_{k=0}^s U_k(N).$$

This implies

$$(q_{s+1}^+)'(1) = \sum_{r=0}^s \frac{\sum_{k=0}^r U_k(N)}{a_r}.$$

□

We can get an estimate of these eigenvalues that grows linearly in s .

Corollary 3.10. *For any $N \geq 2$,*

$$s \leq (q_s^+)'(1) \leq \frac{N+2}{N-2}s, \quad \forall s \geq 0,$$

(where the upper becomes $+\infty$ for $N = 2$).

Proof. Using the relation $U_s(N)N = U_{s+1}(N) + U_{s-1}(N)$, we have

$$\sum_{k=0}^m U_{2k}(N) = \frac{1}{2} (U_{2m}(N) + U_0(N)) + \frac{N}{2} \left(\sum_{k=1}^m U_{2k-1}(N) \right),$$

and

$$\sum_{k=0}^m U_{2k+1}(N) = \frac{1}{2} (U_{2m+1}(N) + U_1(N)) + U_0(N) + \frac{N}{2} \left(\sum_{k=1}^m U_{2k}(N) \right).$$

Therefore

$$\frac{\sum_{k=0}^r U_k(N)}{\sum_{k=0}^r (-1)^{r+k} U_k(N)} \leq \frac{N/2 + 1}{N/2 - 1}.$$

□

Remark 3.11. For the classical sphere, we know that the Laplace operator is the operator whose eigenvector are the Jacobi polynomials J_s and whose eigenvalues are $\lambda_s = s(s + N - 2) = -(N - 1)q_s'(1)$. So the generator for classical spheres in Theorem 2.18, is induced from the generating functional ψ associated to the pair $(b, \nu) = (N - 1, 0)$ is the Laplace operator. In the same manner, we may define the Laplace operator Δ_* on the half-liberated sphere and the Laplace operator Δ_+ on free sphere.

Remark 3.12. Recall that we showed in Proposition 2.11 that central convolution semigroups of states on $C_u(\mathbb{G})$ also induce \mathbb{G} -invariant Markov semigroups on any quantum space \mathbb{X} equipped with a right \mathbb{G} -action. The generating functionals of central convolution semigroups of states on $C_u(O_N^+)$ were classified in [29, Corollary 10.3]. This gives the formula

$$\lambda_s = -b \frac{U'_s(N)}{U_s(N)} + \int_{-N}^N \frac{U_s(x) - U_s(N)}{U_s(N)(N-x)} \nu(dx), \quad s = 0, 1, \dots \quad (3.4)$$

with b a positive real number, ν a finite positive measure on the interval $[-N, N]$ and $(U_s)_{s=0}^\infty$ the Chebyshev polynomials of the second kind defined by $U_0(x) = 1$, $U_1(x) = x$, $U_{s+1}(x) = xU_s(x) - U_{s-1}(x)$ for $s \geq 1$.

Recall again that by [5, Theorem 5.3] the distribution of $\sqrt{N+2}u_{11}$ converges uniformly to the semicircle distribution, which is the measure of orthogonality of the Chebyshev polynomials. This suggests that the eigenvalues given by Theorem 2.18 and in Equation (3.4) for the free sphere S_+^{N-1} should be close for large N .

3.2 Spectral dimensions

The Weyl formula for the eigenvalues of the Laplace-Beltrami operator $\Delta_{\mathcal{M}}$ on a compact Riemannian C^∞ -manifold (\mathcal{M}, g) of dimension N states that

$$N(\lambda) \sim_{\lambda \rightarrow +\infty} \frac{\lambda^{N/2} |\mathcal{M}|}{(4\pi)^{N/2} \Gamma\left(\frac{N}{2} + 1\right)}$$

cf. [79], where $|\mathcal{M}|$ denotes the volume of (\mathcal{M}, g) , $N(\lambda)$ denotes the number eigenvalues of the Laplace-Beltrami operator that are less than or equal to λ , and $f \sim_{\lambda \rightarrow +\infty} g$ stands for ‘‘asymptotically equivalent,’’ i.e., for $\lim_{\lambda \rightarrow \infty} \frac{f(\lambda)}{g(\lambda)} = 1$. This implies that the zeta-function $\zeta_{\mathcal{M}}(z) = \sum_{\lambda \in \sigma(\Delta_{\mathcal{M}})} m_\lambda \lambda^z$, where m_λ denotes the multiplicity of the eigenvalue λ , has a simple pole in $\frac{N}{2}$, and that this value is also the abscissa of convergence of the series. For this reason, we define the ‘‘spectral dimension’’ d_L of the spheres \mathcal{S}_*^{N-1} (w.r.t. a generator L) as the abscissa of convergence of the series $\sum_{s=0}^\infty m_s \lambda_s^{-z/2}$, where $(\lambda_s)_{s \geq 0}$ are the eigenvalues of L which we classified in Theorem 2.18. Note that this definition is equivalent to Connes’ definition in [33, 34], if we construct a Dirac operator D_L from L as in [29], since the eigenvalues of D_L will be $(\pm \sqrt{\lambda_s})_{s \geq 0}$.

The spectral dimension d_L is equal to the infimum of all $d > 0$ such that the sum $\sum_s m_s (-\lambda_s)^{-d/2}$ is finite.

For simplicity, we will only consider the special case $b = 1$ and $\nu = 0$ of the eigenvalues given in Theorem 2.18.

3.2.1 The classical sphere

By definition of D_s ,

$$\dim D_s = \dim H_s - \dim H_{s-1},$$

where

$$H_s = \text{span}\{x_1^{k_1} \cdots x_N^{k_N} : k_1 + \cdots + k_N \leq s\}.$$

Since $x_1^2 = 1 - \sum_{i=2}^N x_i^2$, we only need consider $k_1 = 0$ or $k_1 = 1$ in above formula.

Recall that the cardinality measure $|\{(k_1, k_2, \dots, k_n) \in \mathbb{N}^n : k_1 + k_2 + \dots + k_n = M\}| = \binom{M+n-1}{n-1}$.

For $k_1 = 0$,

$$\dim \text{span}\{x_2^{k_2} \cdots x_N^{k_N} : k_2 + \dots + k_N = s\} = \binom{s+N-2}{N-2};$$

and for $k_1 = 1$,

$$\dim \text{span}\{x_1 x_2^{k_2} \cdots x_N^{k_N} : k_2 + \dots + k_N = s-1\} = \binom{s+N-3}{N-2}.$$

Therefore,

$$m_s = \dim D_s = \binom{s+N-2}{N-2} + \binom{s+N-3}{N-2} \asymp s^{N-2}.$$

where the notation $a_s \asymp b_s$ for two sequences of strictly positive numbers means that they are of the same order of magnitude. More precisely, $a_s \asymp b_s$ means that there exist constants $c, C > 0$ such that for all $s \in \mathbb{N}$, $ca_s \leq b_s \leq Ca_s$.

For the eigenvalues we have $-\lambda_s = \frac{s(s+N-2)}{N-1} \asymp s^2$, and therefore we find $d_L = N-1$, as expected.

3.2.2 The half-liberated sphere

Again, $\dim D_s = \dim H_s - \dim H_{s-1}$. Consider first the even case, i.e. $s = 2m$.

Let $X = x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{2m-1}} x_{\ell_{2m}} \in D_{2m}$. Use black dots “●” for odd positions and white dots “○” for even positions, i.e., associate the diagram

$$\bullet \circ \bullet \circ \cdots \bullet \circ.$$

to the monomial X . Since we have the relation $x_{\ell_1} x_{\ell_2} x_{\ell_3} = x_{\ell_3} x_{\ell_2} x_{\ell_1}$ for the generators, we can freely permute the generators $x_{\ell_{2k-1}}$ that are placed on black dots “●” (i.e., in odd positions) among each other. Similarly, generators $x_{\ell_{2k}}$ sitting on white dots “○” (i.e., in even positions) can be permuted among each other..

Write $x_{\ell_{2k-1}} = a_{i_k}$ and $x_{\ell_{2k}} = b_{j_k}$, respectively, for the generators on black and white dots, then $X = a_{i_1} b_{j_1} \cdots a_{i_m} b_{j_m}$.

Since a_{i_k} is commute among each other, we set $a_{i_1} a_{i_2} \cdots a_{i_m} = x_1^{p_1} x_2^{p_2} \cdots x_N^{p_N}$ with $p_1 + p_2 + \cdots + p_N = m$. Similarly, set $b_{i_1} b_{i_2} \cdots b_{i_m} = x_1^{q_1} x_2^{q_2} \cdots x_N^{q_N}$ with $q_1 + q_2 + \cdots + q_N = m$.

Since $x_1^2 = 1 - \sum_{i=2}^N x_i^2$, we can assume $p_1 = 0$ or $q_1 = 0$. Indeed, if both monomials $a_{i_1} a_{i_2} \cdots a_{i_m}$ and $b_{i_1} b_{i_2} \cdots b_{i_m}$ contain the generator x_1 , then we can we could move x_1 to the first position in both the subwords $a_{i_1} a_{i_2} \cdots a_{i_m}$ and $b_{i_1} b_{i_2} \cdots b_{i_m}$, and replace the resulting x_1^2 by $1 - \sum_{i=2}^N x_i^2$. In this way get one monomial that is in H_{s-2} , and in the remaining terms the powers of x_1 in both subwords are reduced by 1. Iterating this procedure we can express X as a linear combination of monomials which have $p_1 = 0$ or $q_1 = 0$.

Therefore,

$$\begin{aligned}
 \dim D_s &= \dim \operatorname{span} \left\{ x_2^{p_2} \cdots x_N^{p_N} : \sum_{k=2}^N p_k = m \right\} \cdot \dim \operatorname{span} \left\{ x_2^{q_2} \cdots x_N^{q_N} : \sum_{k=2}^N q_k = m \right\} \\
 &+ \dim \operatorname{span} \left\{ x_2^{p_2} \cdots x_N^{p_N} : \sum_{k=2}^N p_k = m \right\} \cdot \dim \operatorname{span} \left\{ x_1^{q_1} x_2^{q_2} \cdots x_N^{q_N} : q_1 > 0, \sum_{k=1}^N q_k = m \right\} \\
 &+ \dim \operatorname{span} \left\{ x_1^{p_1} x_2^{p_2} \cdots x_N^{p_N} : p_1 > 0, \sum_{k=1}^N p_k = m \right\} \cdot \dim \operatorname{span} \left\{ x_2^{q_2} \cdots x_N^{q_N} : \sum_{k=2}^N q_k = m \right\} \\
 &= \binom{m+N-2}{N-2}^2 + 2 \binom{m+N-2}{N-2} \binom{m+N-2}{N-1} \asymp m^{2N-3} \asymp s^{2N-3}.
 \end{aligned}$$

Similarly, when $s = 2m + 1$,

$$\begin{aligned}
 \dim D_s &= \binom{m+N-2}{N-2} \binom{m+N-1}{N-2} + \binom{m+N-1}{N-2} \binom{m+N-1}{N-1} \\
 &\quad + \binom{m+N}{N-1} \binom{m+N-2}{N-2} \\
 &\asymp s^{2N-3}.
 \end{aligned}$$

On the other hand, by Corollary 3.7, $-\lambda_s \asymp s^2$. Hence,

$$d_L = 2(N-1).$$

Banica showed in [4, Theorem 1.14]. that $C(\mathcal{S}_*^{N-1})$ can be embedded into the \mathbb{C}^* -algebra $M_2(C(\mathcal{S}_{\mathbb{C}}^{N-1}))$ of continuous functions with values in 2×2 -matrices on the complex sphere $\mathcal{S}_{\mathbb{C}}^{N-1} = \{z = (z_1, \dots, z_N) \in \mathbb{C}^N : \sum_{i=1}^N |z_i|^2 = 1\}$. This embedding sends the generators x_i , $i = 1, \dots, N$, to the functions $\pi(x_i) : \mathcal{S}_{\mathbb{C}}^{N-1} \ni z = (z_1, \dots, z_N) \mapsto \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$. Evaluating these functions in a point $z \in \mathcal{S}_{\mathbb{C}}^{N-1}$ defines a unique 2-dimensional representation $\pi_z : C(\mathcal{S}_{\mathbb{C}}^{N-1}) \rightarrow M_2(\mathbb{C})$. Two of these 2-dimensional representations π_z and π_w , $z, w \in \mathcal{S}_{\mathbb{C}}^{N-1}$, are unitarily equivalent if and only if there exists a complex number λ with $|\lambda| = 1$ such that $z = \lambda w$. This means that the embedding passes to the projective complex sphere $P_{\mathbb{C}}^{N-1} = \mathcal{S}_{\mathbb{C}}^{N-1} / \sim$ (see also [12, Theorem 3.5]), where \sim is the equivalence relation on $\mathcal{S}_{\mathbb{C}}^{N-1}$ defined by

$$z_1 \sim z_2 \Leftrightarrow \exists \lambda \in \mathbb{C}, z_1 = \lambda z_2.$$

Since the dimension of $P_{\mathbb{C}}^{N-1}$ as a real manifold is $2(N-1)$, this provides a heuristic explanation for the value of the spectral dimension d_L for the half-liberated sphere \mathcal{S}_*^{N-1} .

3.2.3 The free sphere

For the free case, $D_s = \operatorname{span}\{u_{1i}^{(s)} : 1 \leq i \leq d_s\}$ where $u_{ij}^{(s)}$ are the coefficients of the s^{th} irreducible corepresentation $u^{(s)}$, which has dimension $d_s = U_s(N)$.

Let us first consider the case $N = 2$. Since $U_s(2) = s + 1$, we get $a_s = [s/2] + 1$ and $\sum_{k=0}^r = \frac{(r+1)(r+2)}{2}$. By Corollary 7.14, we have

$$\begin{aligned} -\lambda_{2k+1} &= (q_{2k+1}^+) \iota(1) = 2k^2 + 4k + 1, \\ -\lambda_{2k} &= (q_{2k}^+) \iota(1) = 2k^2 + 2k. \end{aligned}$$

Therefore,

$$-\lambda_s \asymp s^2, \quad m_s \asymp s.$$

This implies $d_L = 2$ for $N = 2$. For $N = 2$, the defining relation of the free sphere S_+^2 can be written as $x_2^2 = 1 - x_1^2$, which implies $x_1 x_2^2 = x_2^2 x_1$, as well as the other half-commutation relations $x_i x_j x_k = x_k x_j x_i$, $i, j, k \in \{1, 2\}$. So we have $C_u(S_+^2) \cong C_u(S_*^2)$, i.e., the free and the half-liberated two-dimensional spheres coincide.

By Corollary 3.10, $\lambda_s \asymp s$ for $N \geq 3$. Furthermore, in this case $m_s = U_s(N) \asymp N^s$. Hence,

$$d_L = \begin{cases} 2 & \text{if } N = 2, \\ +\infty & \text{if } N \geq 3. \end{cases}$$

This resembles the computation in [29, Remark 10.4], where we found

$$d_D = \begin{cases} 3 & \text{if } N = 2, \\ +\infty & \text{if } N \geq 3. \end{cases}$$

for the spectral dimension of a spectral triple constructed from a central generating functional on the free orthogonal quantum group O_N^+ .

Part II

Pointwise convergence of noncommutative Fourier series

Chapter 4

Preliminaries

Notation: In all what follows, we write $X \lesssim Y$ if $X \leq CY$ for an *absolute* constant $C > 0$, and $X \lesssim_{\alpha, \beta, \dots} Y$ if $X \leq CY$ for a constant $C > 0$ depending only on the parameters indicated. Also, we write $X \asymp Y$ if $C^{-1}Y \leq X \leq CY$ for an *absolute* constant $C > 0$.

Throughout this part of thesis, \mathcal{M} will always denote a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ . Let $\mathcal{S}_{\mathcal{M}+}$ denote the set of all $x \in \mathcal{M}_+$ such that $\tau(\text{supp } x) < \infty$, where $\text{supp } x$ denotes the support projection of x . Let $\mathcal{S}_{\mathcal{M}}$ be the linear span of $\mathcal{S}_{\mathcal{M}+}$. Then S is a w^* -dense $*$ -subalgebra of \mathcal{M} . Given $1 \leq p < \infty$, we define

$$\|x\|_p = [\tau(|x|^p)]^{1/p}, \quad x \in \mathcal{S}_{\mathcal{M}},$$

where $|x| = (x^*x)^{1/2}$ is the modulus of x . Then $(\mathcal{S}_{\mathcal{M}}, \|\cdot\|_p)$ is a normed space, whose completion is the noncommutative L_p -space associated with (\mathcal{M}, τ) , denoted by $L_p(\mathcal{M}, \tau)$ or simply by $L_p(\mathcal{M})$. As usual, we set $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the operator norm. Let $L_0(\mathcal{M})$ denote the space of all closed densely defined operators on H measurable with respect to (\mathcal{M}, τ) , where H is the Hilbert space on which \mathcal{M} acts. Then $L_p(\mathcal{M})$ can be view as closed densely defined operators on H . We refer to [85] for more information on noncommutative L_p -spaces.

4.1 Noncommutative ℓ_∞ -valued L_p -spaces

In the classical analysis, the pointwise properties of measurable functions are often studied by estimating the norms of maximal functions of the form $\|\sup_n |f_n|\|_p$. However, these maximal norms in the noncommutative setting require a specific definition, since $\sup_n |x_n|$ does not make sense for a sequence $(x_n)_n$ of operators. This difficulty is overcome by considering the spaces $L_p(\mathcal{M}; \ell_\infty)$, which are the noncommutative analogs of the usual Bochner spaces $L_p(X; \ell_\infty)$. These spaces were first introduced by Pisier [83] for injective von Neumann algebras and then extended to general von Neumann algebras by Junge [65]. See also [69, Section 2] for more details.

Given $1 \leq p \leq \infty$, we define $L_p(\mathcal{M}; \ell_\infty)$ to be the space of all sequences $x = (x_n)_{n \geq 0}$ in $L_p(\mathcal{M})$ which admit a factorization of the following form: there exist $a, b \in L_{2p}(\mathcal{M})$ and a bounded sequence $y = (y_n) \subset \mathcal{M}$ such that

$$x_n = ay_nb, \quad n \geq 0.$$

The norm of x in $L_p(\mathcal{M}; \ell_\infty)$ is given by

$$\|x\|_{L_p(\mathcal{M}; \ell_\infty)} = \inf \left\{ \|a\|_{2p} \sup_{n \geq 0} \|y\|_\infty \|b\|_{2p} \right\}$$

where the infimum runs over all factorizations of x as above. We will adopt the convention that the norm $\|x\|_{L_p(\mathcal{M}; \ell_\infty)}$ is denoted by $\|\sup_{n \geq 0}^+ x_n\|_p$. As an intuitive description, it is worth remarking that a self-adjoint sequence $(x_n)_{n \geq 0}$ of $L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M}; \ell_\infty)$ if and only if there exists a positive element $a \in L_p(\mathcal{M})$ such that $-a \leq x_n \leq a$ for any $n \geq 0$. In this case, we have

$$\|\sup_{n \geq 0}^+ x_n\|_p = \inf \{ \|a\|_p : a \in L_p(\mathcal{M})_+, -a \leq x_n \leq a, \forall n \geq 0 \}.$$

The subspace $L_p(\mathcal{M}, c_0)$ of $L_p(\mathcal{M}; \ell_\infty)$ is defined as the space of all family $(x_n)_{n \geq 0} \subset L_p(\mathcal{M})$ such that there are $a, b \in L_{2p}(\mathcal{M})$ and $(y_n) \subset \mathcal{M}$ verifying

$$x_n = ay_n b \quad \text{and} \quad \lim_{n \rightarrow \infty} \|y_n\|_\infty = 0.$$

It is easy to check that $L_p(\mathcal{M}, c_0)$ is a closed subspace of $L_p(\mathcal{M}; \ell_\infty)$. It is indeed the closure of the subspace of all finitely supported sequences.

On the other hand, we may also define the space $L_p(\mathcal{M}; \ell_\infty^c)$ for $2 \leq p \leq \infty$. The space $L_p(\mathcal{M}; \ell_\infty^c)$ is defined to be the family of all sequences $(x_n)_{n \geq 0} \subset L_p(\mathcal{M})$ which admits $a \in L_p(\mathcal{M})$ and $(y_n) \subset L_\infty(\mathcal{M})$ such that

$$x_n = y_n a \quad \text{and} \quad \sup_{n \geq 0} \|y_n\|_\infty < \infty.$$

$\|(x_n)\|_{L_p(\mathcal{M}; \ell_\infty^c)}$ is then defined to be the infimum $\{\sup_{n \geq 0} \|y_n\|_\infty \|a\|_p\}$ over all factorization of (x_n) as above. It is easy to check that $\|\cdot\|_{L_p(\mathcal{M}; \ell_\infty^c)}$ is a norm, which makes $L_p(\mathcal{M}; \ell_\infty^c)$ a Banach space. We define similarly the subspace $L_p(\mathcal{M}; c_0^c)$ of $L_p(\mathcal{M}; \ell_\infty^c)$.

Another Banach space $L_p(\mathcal{M}; \ell_1)$ is also defined in [65]. Given $1 \leq p \leq \infty$, a sequence $x = (x_n)$ belongs to $L_p(\mathcal{M}; \ell_1)$ if there are $u_{kn}, v_{kn} \in L_{2p}(\mathcal{M})$ such that

$$x_n = \sum_{k \geq 0} u_{kn}^* v_{kn}, \quad n \geq 0$$

and

$$\|(x_n)\|_{L_p(\mathcal{M}; \ell_1)} := \inf \left\{ \left\| \sum_{k, n \geq 0} u_{kn}^* u_{kn} \right\|_p^{1/2} \left\| \sum_{k, n \geq 0} v_{kn}^* v_{kn} \right\|_p^{1/2} \right\} < \infty.$$

Specially, for a positive sequence $x = (x_n)$, we have

$$\|x\|_{L_p(\mathcal{M}; \ell_1)} = \left\| \sum_{n \geq 0} x_n \right\|_p.$$

The following Proposition will be used in the sequel.

Proposition 4.1 ([69]). *Let $1 \leq p, p' \leq \infty$ and $1/p + 1/p' = 1$.*

(1) $L_p(\mathcal{M}; \ell_\infty)$ is the dual space of $L_{p'}(\mathcal{M}; \ell_1)$. The duality bracket is given by

$$\langle x, y \rangle = \sum_{n \geq 0} \tau(x_n y_n) \quad x \in L_p(\mathcal{M}; \ell_\infty), \quad y \in L_{p'}(\mathcal{M}; \ell_1).$$

In particular for any positive sequence $(x_n)_n$ in $L_p(\mathcal{M}; \ell_\infty)$, we have

$$\left\| \sup_n^+ x_n \right\|_p = \sup \left\{ \sum_n \tau(x_n y_n) : y_n \in L_{p'}^+(\mathcal{M}) \text{ and } \left\| \sum_n y_n \right\|_{p'} \leq 1 \right\}.$$

(2) Each element in the unit ball of $L_p(\mathcal{M}; \ell_\infty)$ (resp. $L_p(\mathcal{M}; \ell_1)$) is a sum of sixteen (resp. eight) positive elements in the same ball.

(3) Let $1 \leq p_0 < p < p_1 \leq \infty$ and $0 < \theta < 1$, such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then the following complex interpolation holds: we have isometrically

$$L_p(\mathcal{M}; \ell_\infty) = (L_{p_0}(\mathcal{M}; \ell_\infty), L_{p_1}(\mathcal{M}; \ell_\infty))_\theta.$$

Similar complex interpolations also hold for $L_p(\mathcal{M}; \ell_\infty^c)$.

We remark that we may define the spaces $L_p(\mathcal{M}; \ell_\infty(I))$, $L_p(\mathcal{M}; c_0(I))$ and $L_p(\mathcal{M}; \ell_\infty^c(I))$, $L_p(\mathcal{M}; c_0^c(I))$ for an uncountable index set I in the same way. The above properties still hold for these spaces. We will simply denote the spaces by the same notation $L_p(\mathcal{M}; \ell_\infty)$, $L_p(\mathcal{M}; c_0)$ and so on if no confusion can occur.

Remark 4.2. It is known that a family $(x_i)_{i \in I} \subset L_p(\mathcal{M})$ belongs to $L_p(\mathcal{M}; \ell_\infty)$ if and only if

$$\sup_{J \subset I \text{ finite}} \left\| \sup_{i \in J}^+ x_i \right\|_p < \infty,$$

and in this case

$$\left\| \sup_{i \in I}^+ x_i \right\|_p = \sup_{J \text{ finite}} \left\| \sup_{i \in J}^+ x_i \right\|_p.$$

Similar observations hold for $L_p(\mathcal{M}; \ell_\infty^c)$. As a result, for any $1 \leq p < \infty$ and any $(x_t)_{t \in \mathbb{R}_+} \in L_p(\mathcal{M}; \ell_\infty)$ such that the map $t \mapsto x_t$ from \mathbb{R}_+ to $L_p(\mathcal{M})$ is continuous,

$$\|(x_t)_{t \in \mathbb{R}_+}\|_{L_p(\mathcal{M}; \ell_\infty)} = \lim_{a \rightarrow 1^+} \|(x_{a^j})_{j \in \mathbb{Z}}\|_{L_p(\mathcal{M}; \ell_\infty)}.$$

Indeed, as mentioned above, it suffices to show that $\|(x_{t_k})_{1 \leq k \leq n}\|_{L_p(\mathcal{M}; \ell_\infty)}$ is dominated by $\lim_{a \rightarrow 1^+} \|(x_{a^j})_{j \in \mathbb{Z}}\|_{L_p(\mathcal{M}; \ell_\infty)}$ for any (finitely many) elements t_1, \dots, t_n . This is obvious since for any $\varepsilon > 0$, by continuity we may find an a sufficiently close to 1 such that $\|x_{t_k} - x_{a^{j_k}}\|_p < \varepsilon/n$ for some $j_k \in \mathbb{Z}$ and for all $1 \leq k \leq n$, which implies

$$\begin{aligned} \|(x_{t_k})_{1 \leq k \leq n}\|_{L_p(\mathcal{M}; \ell_\infty)} &\leq \|(x_{a^{j_k}})_{1 \leq k \leq n}\|_{L_p(\mathcal{M}; \ell_\infty)} + \sum_{1 \leq k \leq n} \|x_{t_k} - x_{a^{j_k}}\|_p \\ &\leq \|(x_{a^j})_{j \in \mathbb{Z}}\|_{L_p(\mathcal{M}; \ell_\infty)} + \varepsilon. \end{aligned}$$

Similarly, for $2 \leq p < \infty$, we have

$$\|(x_t)_{t \in \mathbb{R}_+}\|_{L_p(\mathcal{M}; \ell_\infty^c)} = \lim_{a \rightarrow 1^+} \|(x_{a^j})_{j \in \mathbb{Z}}\|_{L_p(\mathcal{M}; \ell_\infty^c)}.$$

4.2 Maximal inequalities and pointwise convergence

The standard tool in the study of pointwise convergences is the following type of maximal inequalities.

Definition 4.3. Let $1 \leq p \leq \infty$. Consider a family of maps $\Phi_n : L_p(\mathcal{M}) \rightarrow L_0(\mathcal{M})$ for $n \in \mathbb{N}$.

(1) We say $(\Phi_n)_{n \geq 0}$ is of *strong type* (p, p) with constant C if

$$\left\| \sup_{n \geq 0}^+ \Phi_n(x) \right\|_p \leq C \|x\|_p, \quad x \in L_p(\mathcal{M}).$$

(2) We say $(\Phi_n)_{n \geq 0}$ is of *weak type* (p, p) ($p < \infty$) with constant C if for any $x \in L_p(\mathcal{M})$ and any $\alpha > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\|e\Phi_n(x)e\|_\infty \leq \alpha, \quad n \geq 0 \quad \text{and} \quad \tau(e^\perp) \leq \left[C \frac{\|x\|_p}{\alpha} \right]^p.$$

(3) We say $(\Phi_n)_{n \geq 0}$ is of *restricted weak type* (p, p) ($p < \infty$) with constant C if for any projection $f \in \mathcal{M}$ and any $\alpha > 0$, there is a projection $e \in \mathcal{M}$ such that

$$\|e\Phi_n(f)e\|_\infty \leq \alpha \quad n \geq 0 \quad \text{and} \quad \tau(e^\perp) \leq \left(\frac{C}{\alpha} \right)^p \tau(f).$$

We say $(\Phi_n)_{n \geq 0}$ is of *restricted weak type* (∞, ∞) with constant C if $\sup_n \|\Phi_n(f)\|_\infty \leq C$ for any projection $f \in \mathcal{M}$.

It is easy to know that for any $1 < p < \infty$,

$$\text{strong type } (p, p) \Rightarrow \text{weak type } (p, p) \Rightarrow \text{restricted weak type } (p, p).$$

The Marcinkiewicz interpolation theorem plays an important role in studying the maximal inequalities. The noncommutative version was first established by Junge and Xu in [69], and then was generalized in [11] and [43]. We present Dirksen's version here.

Theorem 4.4 ([43, Corollary 5.3]). *Let $1 \leq p < r < q \leq \infty$. Let $(\Phi_n)_{n \geq 0}$ be a family of positive linear maps from $L_p(\mathcal{M}) + L_q(\mathcal{M})$ into $L_0(\mathcal{M})$. If $(\Phi_n)_{n \geq 0}$ is of restricted weak type (p, p) and (q, q) with constants C_p and C_q , then it is of strong type (r, r) with constant*

$$C_r \lesssim \max\{C_p, C_q\} \left(\frac{rp}{r-p} + \frac{rq}{q-r} \right)^2.$$

We need an appropriate analogue for the noncommutative setting of the usual almost everywhere convergence. This is the notion of almost uniform convergence introduced by Lance [72].

Definition 4.5. Let $x_n, x \in L_0(\mathcal{M})$. $(x_n)_{n \geq 0}$ is said to converge *almost uniformly* (a.u. in short) to x if for any $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) < \varepsilon \quad \text{and} \quad \lim_{n \rightarrow \infty} \|(x_n - x)e\|_\infty = 0.$$

$(x_n)_{n \geq 0}$ is said to converge *bilaterally almost uniformly* (b.a.u. in short) to x if for any $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\tau(e^\perp) < \varepsilon \quad \text{and} \quad \lim_{n \rightarrow \infty} \|e(x_n - x)e\|_\infty = 0.$$

It is obvious that the a.u. convergence implies the b.a.u. convergence, so we will be mainly interested in the former. Note that in the commutative case, both notions are equivalent to the usual almost everywhere convergence in terms of Egorov's Theorem.

It is nowadays a standard method to deduce the pointwise convergence from maximal inequalities. We will use the following facts.

Lemma 4.6 ([41]). (1) *If a family $(x_n)_{n \in \mathbb{N}}$ belongs to $L_p(\mathcal{M}, c_0)$ with $1 \leq p < \infty$, then x_n converges b.a.u. to 0.*

(2) *If a family $(x_n)_{n \in \mathbb{N}}$ belongs to $L_p(\mathcal{M}, c_0^c)$ with $2 \leq p < \infty$, then x_n converges a.u. to 0.*

Proposition 4.7. (1) *Let $1 \leq p < \infty$ and let $(\Phi_n)_{n \geq 0}$ be a sequence of positive linear maps on $L_p(\mathcal{M})$. Assume that $(\Phi_n)_{n \geq 0}$ is of weak type (p, p) . If $(\Phi_n x)_{n \geq 0}$ converges a.u. for all elements x in a dense subspace of $L_p(\mathcal{M})$, then $(\Phi_n x)_{n \geq 0}$ converges a.u. for all $x \in L_p(\mathcal{M})$.*

(2) *Fix $2 \leq p < \infty$. Let Φ and $(\Phi_n)_{n \geq 0}$ be linear maps on $L_p(\mathcal{M})$ and C_p be positive constants such that*

$$\|(\Phi_n x)_n\|_{L_p(\mathcal{M}; \ell_\infty^c)} \leq C_p \|x\|_p, \quad x \in L_p(\mathcal{M}).$$

Assume that there exists a dense subspace $B \subset L_p(\mathcal{M})$ such that

$$\lim_{m \rightarrow \infty} \sum_{n \geq m} \|\Phi_n x - \Phi x\|_p^p = 0, \quad x \in B.$$

Then $\Phi_n x$ converges to Φx a.u. for all $x \in L_p(\mathcal{M})$.

Proof. The assertion (1) is given by [28, Theorem 3.1]. The assertion (2) is implicitly proved in [57], for which we provide a brief argument for the convenience of the reader. We first show that $(\Phi_n(y) - \Phi(y))_n \in L_p(\mathcal{M}; \ell_\infty^c)$ for any $y \in B$. Indeed, as m tends to ∞ ,

$$\begin{aligned} \|(\Phi_n(y) - \Phi(y))_{n \geq m}\|_{L_p(\mathcal{M}; \ell_\infty^c)} &\leq \|(|\Phi_n(y) - \Phi(y)|^2)_{n \geq m}\|_{L_{p/2}(\mathcal{M}; \ell_\infty)}^{1/2} \\ &\leq \left\| \left(\sum_{n \geq m} |\Phi_n(y) - \Phi(y)|^p \right)^{\frac{2}{p}} \right\|_{p/2}^{1/2} \\ &\leq \left(\sum_{n \geq m} \|\Phi_n(y) - \Phi(y)\|_p^p \right)^{\frac{1}{p}} \rightarrow 0. \end{aligned}$$

As a result $(\Phi_n(y) - \Phi(y))_{t \geq 0} \in L_p(\mathcal{M}; c_0^c)$, as desired. For any $x \in L_p(\mathcal{M})$ and $\varepsilon > 0$, take $y \in B$ such that $\|x - y\|_p < \varepsilon$, then by the maximal inequality in our assumption, we have $\|((\Phi_n - \Phi)(x - y))_n\|_{L_p(\mathcal{M}; \ell_\infty^c)} \leq C_p \varepsilon$. Thus $((\Phi_n - \Phi)(x))_n \in L_p(\mathcal{M}; c_0^c)$. By Lemma 4.6 we obtain that $\Phi_n x$ converges to Φx almost uniformly. \square

Remark 4.8. If we replace the index set $n \in \mathbb{N}$ by $t \in \mathbb{R}_+$, Proposition 4.7 still holds up to standard adaptations. Indeed, for the assertion (1), it suffices to note that $(\Phi_t(x))_{t \in \mathbb{R}_+}$ converges a.u. if and only if $(\Phi_{t_k}(x))_{k \in \mathbb{N}}$ converges a.u. for an arbitrary subsequence $(t_k)_k$. This can be easily seen from an argument by contradiction. For the assertion (2), it suffices to consider the integral $\int_{t \geq t_0} \|\Phi_t x - \Phi x\|_p^p dt$ instead of the discrete sum over $n \geq m$.

4.3 Noncommutative Hilbert valued L_p -spaces

The noncommutative Hilbert valued L_p -spaces provide a suitable framework for studying square functions in the noncommutative setting. In this thesis we will only use the following concrete representations of these spaces; for a more general description we refer to the seminal paper [66].

First, for a sequence $(x_n)_n \subset L_p(\mathcal{M})$, we define

$$\|(x_n)\|_{L_p(\mathcal{M}; \ell_2^c)} = \left\| \left(\sum_n x_n^* x_n \right)^{1/2} \right\|_p, \quad \|(x_n)\|_{L_p(\mathcal{M}; \ell_2^r)} = \left\| \left(\sum_n x_n x_n^* \right)^{1/2} \right\|_p.$$

We alert the reader that the two norms above are not comparable at all if $p \neq 2$. Let $L_p(\mathcal{M}; \ell_2^c)$ (resp. $L_p(\mathcal{M}; \ell_2^r)$) be the completion of the family of all finite sequence in $L_p(\mathcal{M})$ with respect to $\|\cdot\|_{L_p(\mathcal{M}; \ell_2^c)}$ (resp. $\|\cdot\|_{L_p(\mathcal{M}; \ell_2^r)}$). The space $L_p(\mathcal{M}; \ell_2^{cr})$ is defined in the following way. If $2 \leq p \leq \infty$, we set

$$L_p(\mathcal{M}; \ell_2^{cr}) = L_p(\mathcal{M}; \ell_2^c) \cap L_p(\mathcal{M}; \ell_2^r)$$

equipped with the norm

$$\|(x_n)\|_{L_p(\mathcal{M}; \ell_2^{cr})} = \max\{\|(x_n)\|_{L_p(\mathcal{M}; \ell_2^c)}, \|(x_n)\|_{L_p(\mathcal{M}; \ell_2^r)}\}.$$

If $1 \leq p \leq 2$, we set

$$L_p(\mathcal{M}; \ell_2^{cr}) = L_p(\mathcal{M}; \ell_2^c) + L_p(\mathcal{M}; \ell_2^r)$$

equipped with the norm

$$\|(x_n)\|_{L_p(\mathcal{M}; \ell_2^{cr})} = \inf\{\|(y_n)\|_{L_p(\mathcal{M}; \ell_2^c)} + \|(z_n)\|_{L_p(\mathcal{M}; \ell_2^r)}\}$$

where the infimum runs over all decompositions $x_n = y_n + z_n$ in $L_p(\mathcal{M})$.

Second, for the Borel measure space $\Omega_0 = (\mathbb{R}_+ \setminus \{0\}, \frac{dt}{t})$, we may consider the norms

$$\|(x_t)_t\|_{L_p(\mathcal{M}; L_2^c(\Omega_0))} = \left\| \left(\int_0^\infty x_t^* x_t \frac{dt}{t} \right)^{1/2} \right\|_p, \quad \|(x_t)_t\|_{L_p(\mathcal{M}; L_2^r(\Omega_0))} = \left\| \left(\int_0^\infty x_t x_t^* \frac{dt}{t} \right)^{1/2} \right\|_p.$$

We refer to [66, Section 6.A] for the rigorous meaning of the integral in the above norm. Then we may define the spaces $L_p(\mathcal{M}; L_2^c(\Omega_0))$, $L_p(\mathcal{M}; L_2^r(\Omega_0))$ and $L_p(\mathcal{M}; L_2^{cr}(\Omega_0))$ in a similar way.

We recall the following basic properties.

Proposition 4.9. (1) (Duality) *Let $1 \leq p \leq \infty$ and let $1/p' + 1/p = 1$. Then*

$$(L_p(\mathcal{M}; \ell_2^{cr}))^* = L_{p'}(\mathcal{M}; \ell_2^{cr}).$$

The duality bracket is given by

$$\langle (x_n)_n, (y_n)_n \rangle = \sum_n \tau(x_n y_n), \quad (x_n)_n \in L_p(\mathcal{M}; \ell_2^{cr}), \quad (y_n)_n \in L_{p'}(\mathcal{M}; \ell_2^{cr}).$$

(2) (Hölder inequality) Let $1 \leq p, r, q \leq \infty$ with $1/r = 1/p + 1/q$. For all $(x_n)_n \in L_p(\mathcal{M}; \ell_2^{cr})$ and $(y_n)_n \in L_q(\mathcal{M}; \ell_2^{cr})$ we have

$$\|(x_n y_n)_n\|_{L_r(\mathcal{M}; \ell_2^{cr})} \leq \|(x_n)_n\|_{L_p(\mathcal{M}; \ell_2^{cr})} \|(y_n)_n\|_{L_q(\mathcal{M}; \ell_2^{cr})}.$$

(3) (Complex interpolation) Let $1 \leq p, q \leq \infty$ and $0 < \theta < 1$. Let $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$. Then we have isometrically

$$(L_p(\mathcal{M}; \ell_2^{cr}), L_q(\mathcal{M}; \ell_2^{cr}))_\theta = L_r(\mathcal{M}; \ell_2^{cr}).$$

Similar complex interpolations also hold for $L_p(\mathcal{M}; \ell_2^c)$ and $L_p(\mathcal{M}; \ell_2^r)$.

The following noncommutative Khintchine inequalities are well-known.

Proposition 4.10 ([76]). Let (ε_n) be a sequence of independent Rademacher variables on a probability space (Ω, P) . Let $1 \leq p < \infty$ and (x_n) be a sequence in $L_p(\mathcal{M}; \ell_2^{cr})$.

(1) If $1 \leq p \leq 2$, then there exists an absolute constant $c > 0$ such that

$$c \|(x_n)_n\|_{L_p(\mathcal{M}; \ell_2^{cr})} \leq \left\| \sum_n \varepsilon_n x_n \right\|_{L_p(X; L_p(\mathcal{M}))} \leq \|(x_n)_n\|_{L_p(\mathcal{M}; \ell_2^{cr})}.$$

(2) If $2 \leq p < \infty$, then there exists an absolute constant $c > 0$ such that

$$\frac{1}{\sqrt{2}} \|(x_n)_n\|_{L_p(\mathcal{M}; \ell_2^{cr})} \leq \left\| \sum_n \varepsilon_n x_n \right\|_{L_p(X; L_p(\mathcal{M}))} \leq c \sqrt{p} \|(x_n)_n\|_{L_p(\mathcal{M}; \ell_2^{cr})}.$$

The following proposition will be useful for our further studies.

Proposition 4.11. Let $(x_n)_{n \in \mathbb{N}} \in L_p(\mathcal{M}; \ell_\infty)$. Then there exists an absolute constant $c > 0$ such that for any $1 \leq p \leq \infty$,

$$\|(x_n)_n\|_{L_p(\mathcal{M}; \ell_\infty)} \leq c \|(x_n)_n\|_{L_p(\mathcal{M}; \ell_2^{cr});}$$

and for any $2 \leq p \leq \infty$,

$$\|(x_n)_n\|_{L_p(\mathcal{M}; \ell_\infty^c)} \leq c \|(x_n)_n\|_{L_p(\mathcal{M}; \ell_2^{cr})}.$$

Proof. We first assume that $x_n \geq 0$ for each n and set a sequence of independent Rademacher variables (ε_n) on the probability space $\prod_{n \geq 0} (\Omega_n; P_n)$. Here $\Omega_n = \{\pm 1\}$ and $P_n(\{1\}) = P_n(\{-1\}) = 1/2$. Let $(x_n)_n \in L_p(\mathcal{M}; \ell_1)_+$. Then we have

$$\begin{aligned} \left\| \sum_n x_n \varepsilon_n \right\|_{L_p(\Omega; L_p(\mathcal{M}))}^p &= \int_\Omega \left\| \sum_n \varepsilon_n(\omega) x_n \right\|_p^p dP(\omega) \\ &\leq \int_\Omega \left(\left\| \sum_{\{n: \varepsilon_n(\omega)=1\}} x_n \right\|_p + \left\| \sum_{\{n: \varepsilon_n(\omega)=-1\}} x_n \right\|_p \right)^p dP(\omega) \\ &\leq 2^p \left\| \sum_{n=0}^\infty x_n \right\|_p^p \cdot \int_\Omega dP(\omega) = 2^p \|(x_n)_n\|_{L_p(\mathcal{M}; \ell_1)}^p. \end{aligned}$$

By the noncommutative Khintchine inequalities in the previous proposition, we have for $1 \leq p < \infty$,

$$\|(x_n)_n\|_{L_p(\mathcal{M}; \ell_2^{cr})} \leq c \|(x_n)_n\|_{L_p(\mathcal{M}; \ell_1)}, \quad (4.1)$$

where c is an absolute constant independent of p . Note that it is folklore that $\|(x_n)\|_{L_\infty(\mathcal{M}; \ell_2^{cr})}$ is dominated by $\|\sum_n x_n \varepsilon_n\|_{L_\infty(\Omega; L_p(\mathcal{M}))}$, so the above inequality still holds for $p = \infty$. If x_n is not positive, by Proposition 4.1, we can decompose $(x_n)_n \in L_p(\mathcal{M}; \ell_1)$ into 8 elements $(x_n^{(i)})_n \in L_p(\mathcal{M}; \ell_1)$, such that for each $1 \leq i \leq 8$, $\|(x_n^{(i)})_n\|_{L_p(\mathcal{M}; \ell_1)} \leq \|(x_n)_n\|_{L_p(\mathcal{M}; \ell_1)}$. Therefore, the inequality (4.1) holds for all $(x_n)_n \in L_p(\mathcal{M}; \ell_1)$.

Now let $(y_n)_n \in L_p(\mathcal{M}; \ell_\infty)$. By duality and the above inequality we have

$$\begin{aligned} \|(y_n)_n\|_{L_p(\mathcal{M}; \ell_\infty)} &= \sup \left\{ \frac{\sum_n \tau(x_n y_n)}{\|(x_n)\|_{L_{p'}(\mathcal{M}; \ell_1)}} : (x_n) \in L_{p'}(\mathcal{M}, \ell_1) \right\} \\ &\leq \sup \left\{ 8c \frac{\sum_n \tau(x_n y_n)}{\|(x_n)\|_{L_{p'}(\mathcal{M}; \ell_2^{cr})}} : (x_n) \in L_{p'}(\mathcal{M}; \ell_2^{cr}) \right\} \\ &\leq 8c \|(y_n)\|_{L_p(\mathcal{M}; \ell_2^{cr})}. \end{aligned}$$

This proves the first inequality in the lemma. For the second inequality, it suffices to note that together with the Hölder inequality,

$$\begin{aligned} \|(x_n)\|_{L_p(\mathcal{M}; \ell_\infty^c)} &= \|x_n^* x_n\|_{L_{p/2}(\mathcal{M}; \ell_\infty)}^{1/2} \leq \|x_n^* x_n\|_{L_{p/2}(\mathcal{M}; \ell_2^{cr})}^{1/2} \\ &\leq \|x_n^*\|_{L_p(\mathcal{M}; \ell_2^{cr})}^{1/2} \|x_n\|_{L_p(\mathcal{M}; \ell_2^{cr})}^{1/2} = \|x_n\|_{L_p(\mathcal{M}; \ell_2^{cr})}. \end{aligned}$$

□

Chapter 5

Square functions of subordinate Poisson semigroups

The square function estimates of noncommutative diffusion semigroups has been established in [66]. In this chapter we will slightly adapt the arguments of [66] so as to obtain a refined version of their result for our further purpose. Throughout this chapter, $(S_t)_{t>0}$ always denotes a semigroup of unital completely positive trace preserving and symmetric maps on \mathcal{M} with the infinitesimal generator A , where a map T is said *symmetric* if $\tau(T(x)y) = \tau(yT(x))$ for any $x, y \in \mathcal{S}_{\mathcal{M}}$. Let (P_t) denote the subordinate Poisson semigroup of (S_t) , i.e. the generator of P_t is $-(-A)^{1/2}$.

For notational simplicity, the vector-valued spaces $L_p(\mathcal{M}; L_2^c(\mathbb{R}; \frac{dt}{t}))$, $L_p(\mathcal{M}; L_2^r(\mathbb{R}; \frac{dt}{t}))$ and $L_p(\mathcal{M}; L_2^{cr}(\mathbb{R}; \frac{dt}{t}))$ are denoted respectively by $L_p(L_2^c(\frac{dt}{t}))$, $L_p(L_2^r(\frac{dt}{t}))$ and $L_p(L_2^{cr}(\frac{dt}{t}))$ for short in this section.

To state our theorem, we recall the dilation property. Let $(\mathcal{M}, \tau), (\mathcal{N}, \tau')$ be two von Neumann algebra where τ and τ' are normal faithful semifinite traces. If $\pi: (\mathcal{M}, \tau) \rightarrow (\mathcal{N}, \tau')$ is a normal unital faithful trace preserving $*$ -representation, then it (uniquely) extends to an isometry from $L_p(\mathcal{M})$ into $L_p(\mathcal{N})$ for any $1 \leq p < \infty$. We call the adjoint $\mathbb{E}: \mathcal{N} \rightarrow \mathcal{M}$ of the embedding $L_1(\mathcal{M}) \hookrightarrow L_1(\mathcal{N})$ induced by π *the conditional expectation associated with π* . Moreover $\mathbb{E}: \mathcal{N} \rightarrow \mathcal{M}$ is unital and completely positive.

Definition 5.1. Let $T: \mathcal{M} \rightarrow \mathcal{M}$ be a bounded operator. We say that T satisfies *Rota's dilation property* if there exist a von Neumann algebra \mathcal{N} equipped with a normal semifinite faithful trace, a normal unital faithful $*$ -representation $\pi: \mathcal{M} \rightarrow \mathcal{N}$ which preserves traces, and a decreasing sequence $(\mathcal{N}_m)_{m \geq 1}$ of von Neumann subalgebras of \mathcal{N} such that

$$T^m = \hat{\mathbb{E}} \circ \mathbb{E}_m \circ \pi, \quad m \geq 1, \quad (5.1)$$

where $\mathbb{E}_m: \mathcal{N} \rightarrow \mathcal{N}_m \subset \mathcal{N}$ is the canonical conditional expectation onto \mathcal{N}_m , and where $\hat{\mathbb{E}}: \mathcal{N} \rightarrow \mathcal{M}$ is the conditional expectation associated with π .

We recall the following typical examples of operators with Rota's dilation property.

Lemma 5.2. (1) ([67, 36]) *If \mathcal{M} is a finite von Neumann algebra and τ is a normal faithful state on \mathcal{M} , then for all $t > 0$, the operator S_t satisfies Rota's dilation property.*

(2) ([92]) *If \mathcal{M} is a commutative von Neumann algebra and \mathcal{L} is another semifinite von Neumann algebra, then for all $t > 0$, the operator $S_t \otimes \text{Id}_{\mathcal{L}}$ on $\mathcal{M} \bar{\otimes} \mathcal{L}$ satisfies Rota's dilation property.*

We aim to prove the following square function estimates, which is enough for our further purpose according to Lemma 5.2. The following inequality is essentially established in [66, 61], without specifying the order $(p-1)^{-6}$. However, we will see that the method in [66] is enough to obtain this order. The outline of our proof is slightly different from that of [66], but all the ingredients are already available in the latter.

Proposition 5.3. *Assume that for all $t > 0$, the operator S_t satisfies Rota's dilation property. For all $1 < p < 2$ and $x \in L_p(\mathcal{M})$ we have*

$$\inf \left\{ \left\| \left(\int_0^\infty |t\partial P_t(x^c)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_p + \left\| \left(\int_0^\infty |(t\partial P_t(x^r))^*|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_p \right\} \leq c(p-1)^{-6} \|x\|_p \quad (5.2)$$

where the infimum runs over all $x_c, x_r \in L_p(\mathcal{M})$ such that $x = x_c + x_r$, and $c > 0$ is an absolute constant.

Remark 5.4. When the underlying von Neumann algebra is commutative, it is known from Stein [92] that the optimal order here is $(p-1)^{-1}$. In the noncommutative case, we believe that the order $(p-1)^{-6}$ is not optimal. However, this is sufficient for our purpose of proving the main results in the sequel.

Our study on the semigroup $(P_t)_t$ is based on the analysis of the ergodic averages. Recall that the ergodic averages associated to the semigroup $(S_t)_t$ is defined as follows:

$$M_t = \frac{1}{t} \int_0^t S_u du.$$

We will need the following claim.

Lemma 5.5. *For any $y \in L_p$,*

$$\|(t\partial P_t y)_t\|_{L_p(L_2^c(dt/t))} \leq c \|(t\partial M_t y)_t\|_{L_p(L_2^c(dt/t))},$$

where $c > 0$ is an absolute constant. The inequality remains true if we replace the norm of $L_p(L_2^c(dt/t))$ by $L_p(L_2^r(dt/t))$ or $L_p(L_2^{cr}(dt/t))$.

Proof. This claim is well-known to experts. We only give a sketch of its arguments. Set $\varphi(s) = \frac{1}{2\sqrt{\pi}} \frac{e^{-1/4s}}{s^{3/2}}$. Using integration by parts we have

$$P_t = \frac{1}{t^2} \int_0^\infty \varphi\left(\frac{s}{t^2}\right) (\partial(sM_s)) ds = - \int_0^\infty s\varphi'(s) M_{t^2s} ds.$$

Therefore

$$t\partial P_t = -2 \int_0^\infty t^2 s^2 \varphi'(s) \partial M_{t^2s} ds = -2 \int_0^\infty s\varphi'(s) (t^2 s \partial M_{t^2s}) ds \quad (5.3)$$

which yields the claim by noting that $\|\cdot\|_{L_p(L_2^c(dt/t))}$ is a norm and $s\varphi'(s)$ is absolutely integrable. \square

Before starting the proof we recall a few facts about noncommutative martingales. Let $(\mathbb{E}_n)_{n \geq 0}$ be an increasing sequence of canonical conditional expectations defined on (\mathcal{M}, τ) . A sequence $x = x_n \in L_1(\mathcal{M})$ is called a *martingale* if $\mathbb{E}_n(x_{n+1}) = x_n$ for all $n \geq 0$; it is called a L_p -martingale for some $p \geq 1$ if $(x_n)_n \in L_p(\mathcal{M})$ and we set

$$\|x\|_p := \sup_n \|x_n\|_p < \infty.$$

We denote $dx_n = x_n - x_{n-1}$ for $n \geq 0$ (with the convention that $x_{-1} = 0$) and call them the *martingale differences* of x . We refer to [84, 85] and the references therein for more information on noncommutative martingales and related square functions. We will need the optimal orders for the best constants in several noncommutative martingale inequalities when $p \rightarrow 1$.

Lemma 5.6. *Let $(\mathbb{E}_n)_n$ be an arbitrary monotone sequence of canonical conditional expectations defined on (\mathcal{M}, τ) . Let $1 < p < 2$.*

(1) *(Noncommutative Stein inequality [68, Theorem 8]) For any finite sequence $(x_n)_{n \geq 0} \subset L_p(\mathcal{M})$, we have*

$$\|(\mathbb{E}_n x_n)_{n \geq 0}\|_{L_p(\ell_2^{cr})} \leq c(p-1)^{-1} \|(x_n)_{n \geq 0}\|_{L_p(\ell_2^{cr})},$$

where $c > 0$ is an absolute constant. The inequality remains true if we replace the norm of $L_p(\ell_2^{cr})$ by $L_p(\ell_2^c)$ or $L_p(\ell_2^r)$.

(2) *(Noncommutative martingale transforms [90, Theorem 4.3]) For any L_p -martingale $x = (x_n)_{n \geq 0}$ and for any signs $(\varepsilon_n)_{n \geq 0} \subset \{-1, 1\}$, we have*

$$\left\| \sum_{n \geq 0} \varepsilon_n dx_n \right\|_p \leq c(p-1)^{-1} \|x\|_p,$$

where $c > 0$ is an absolute constant.

We need the following auxiliary results.

Proposition 5.7. *Assume that for all $t > 0$, the operator S_t satisfies Rota's dialtion property. Then for $1 < p < 2$ and $x \in L_p(\mathcal{M})$, we have*

$$\|(t\partial P_t(x))_{t>0}\|_{L_p(L_2^{cr}(dt/t))} \leq c(p-1)^{-2} \|x\|_p, \quad (5.4)$$

where $c > 0$ is an absolute constant.

Proof. This result has been essentially obtained in [66], together with the optimal estimates in Lemma 5.6. Indeed, let $(\mathbb{E}_n)_{n \geq 0}$ be a monotone sequence of conditional expectations on \mathcal{M} and $x_n = \mathbb{E}_{n+1}(x) - \mathbb{E}_n(x)$ be a sequence of martingale differences with $x \in L_p(\mathcal{M})$. By Lemma 5.6(2) and the Khintchine inequality in Lemma 4.10 we have

$$\|(x_n)_{n \geq 0}\|_{L_p(\ell_2^{cr})} \leq \frac{c}{p-1} \|x\|_p,$$

and by Lemma 5.6(1) we have for any sequence $(y_n)_{n \geq 0} \subset L_p(\mathcal{M})$, we have

$$\|(\mathbb{E}_n y_n)_{n \geq 0}\|_{L_p(\ell_2^{cr})} \leq \frac{c}{p-1} \|(y_n)_{n \geq 0}\|_{L_p(\ell_2^{cr})} \quad 1 < p < 2,$$

where $c > 0$ is an absolute constant. Then tracing the order in the proof of [66, Corollary 10.9], we obtain that for all $\varepsilon > 0$,

$$\|(\sqrt{m}D_m^\varepsilon(x))_{m \geq 1}\|_{L_p(\ell_2^{c\varepsilon})} \leq \frac{c'}{(p-1)^2} \|x\|_p,$$

where $c' > 0$ is an absolute constant.

$$D_m^\varepsilon = \rho_m^\varepsilon - \rho_{m-1}^\varepsilon \quad \text{and} \quad \rho_m^\varepsilon = \frac{1}{m+1} \sum_{k=0}^m S_{k\varepsilon}.$$

By a standard discretization argument, see for instance [66, Lemma 10.11], we get the following inequality

$$\|(t\partial M_t y)_t\|_{L_p(L_2^c(dt/t))} \leq c'(p-1)^{-2} \|x\|_p.$$

Then the desired result follows from Lemma 5.5. \square

Estimate (5.4) is weaker than (5.2). The remaining task for proving Proposition 5.3 is to show

$$\inf \left\{ \left\| \left(\int_0^\infty |t\partial P_t(x^c)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_p + \left\| \left(\int_0^\infty |(t\partial P_t(x^r))^*|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_p \right\} \lesssim (p-1)^{-4} \|(t\partial P_t(x))_{t>0}\|_{L_p(L_2^{c\varepsilon}(dt/t))}. \quad (5.5)$$

This inequality is essentially proved in [66, Theorem 7.8]; the order $(p-1)^{-4}$ is not stated there, but it follows immediately after a careful computation on all related constants appeared in the argument therein. For the convenience of the reader, we will recall some part of the proof and clarify all the constants in the proof which are concerned with the precise order.

For any $\theta \in (0, \pi)$, we let

$$\Sigma_\theta = \{z \in \mathbb{C}^* : |\text{Arg}(z)| < \theta\}.$$

We say that a set $\mathcal{F} \subset B(L_p(\mathcal{M}))$ is *Col-bounded* (resp. *Row-bounded*) if there is a constant C such that for any sequence $(T_k)_k \subset \mathcal{F}$ and $(x_k)_k$ in $L_p(\mathcal{M})$, we have

$$\|(T_k x_k)_k\|_{L_p(\mathcal{M}; \ell_2^c)} \leq C \|(x_k)_k\|_{L_p(\mathcal{M}; \ell_2^c)} \quad (5.6)$$

$$\left(\text{resp. } \|(T_k x_k)_k\|_{L_p(\mathcal{M}; \ell_2^r)} \leq C \|(x_k)_k\|_{L_p(\mathcal{M}; \ell_2^r)} \right). \quad (5.7)$$

The least constant C satisfying inequality (5.6) (resp. inequality (5.7)) will be denoted by $\text{Col}(\mathcal{F})$ (resp. $\text{Row}(\mathcal{F})$).

We quote a useful results from [31, Lemma 3.2], see also [66, Lemma 4.2].

Lemma 5.8. *Let $\mathcal{F} \subset B(L_p(\mathcal{M}))$ be a Col-bounded (resp. Row-bounded) collection with $\text{Col}(\mathcal{F}) = M$ (resp. $\text{Row}(\mathcal{F}) = M$). Then the closure of the absolute convex hull of \mathcal{F} in the strong operator topology is also Col-bounded (resp. Row-bounded) with the constant $\text{Col}(\mathcal{F}) \leq 2M$ (resp. $\text{Row}(\mathcal{F}) \leq 2M$).*

The following result was proved in [66, Theorem 5.6] without the estimation of the constant.

Lemma 5.9. *Let $1 < p < 2$. Let $(S_t)_t$ be a semigroup of unital completely positive trace preserving and symmetric maps. Let A be the infinitesimal generator of $(S_t)_t$. Then the set $\mathcal{F}_p = \{A(z - A)^{-1} : z \in \mathbb{C} \setminus \Sigma_\nu\} \subset B(L_p(\mathcal{M}))$ with $\nu = \frac{(p+1)\pi}{4p}$ is Col-bounded and Row-bounded with constants*

$$\text{Col}(\mathcal{F}_p) \leq c(p-1)^{-2} \quad \text{and} \quad \text{Row}(\mathcal{F}_p) \leq c(p-1)^{-2},$$

where c is an absolute constant.

Proof. Let t_1, \dots, t_n be any nonnegative real numbers, x_1, \dots, x_n in $L_q(\mathcal{M})$. For any $2 < q < \infty$, by the Kadison-Schwarz inequality, we have

$$\left\| \left(\sum_{k=1}^n S_{t_k}(x_k)^* S_{t_k}(x_k) \right)^{\frac{1}{2}} \right\|_q = \left\| \sum_{k=1}^n S_{t_k}(x_k)^* S_{t_k}(x_k) \right\|_{\frac{q}{2}} \leq \left\| \sum_{k=1}^n S_{t_k}(x_k^* x_k) \right\|_{\frac{q}{2}}.$$

Let $r = (q/2)' = \frac{q}{q-2}$. By duality and the maximal inequalities for unital completely positive semigroups (see [69, Corollary 5.11 and Theorem 4.1]), we have

$$\begin{aligned} \left\| \left(\sum_k S_{t_k}(x_k^* x_k) \right)^{1/2} \right\|_q^2 &= \|(S_{t_k}(x_k^* x_k))_k\|_{L_{q/2}(\mathcal{M}; \ell_1)} \\ &= \sup_{(y_k) \in L_r(\mathcal{M}; \ell_\infty)} \frac{\sum_{k=1}^n \tau(S_{t_k}(x_k^* x_k) y_k)}{\|\text{sup}_k^+ y_k\|_r} \\ &= \sup_{(y_k) \in L_r(\mathcal{M}; \ell_\infty)} \frac{\sum_{k=1}^n \tau((x_k^* x_k) S_{t_k}(y_k))}{\|\text{sup}_k^+ y_k\|_r} \\ &\leq \|(x_k^* x_k)\|_{L_{q/2}(\mathcal{M}; \ell_1)} \frac{\|\text{sup}_k^+ S_{t_k}(y_k)\|_r}{\|\text{sup}_k^+ y_k\|_r} \\ &\leq cr^2(r-1)^{-2} \left\| \left(\sum_{k=1}^n x_k^* x_k \right)^{1/2} \right\|_q^2. \end{aligned}$$

Therefore, $\mathcal{I}_q = \{S_t : t \geq 0\} \subset B(L_q(\mathcal{M}))$ is Col-bounded with the constant $\text{Col}(\mathcal{I}_q) \leq cq$.

Let $p' = \frac{p}{p-1}$ be the conjugate number of p . We fix some $0 < \beta < \frac{\pi}{p'}$. Let $q = p'(\frac{\pi-2\beta}{\pi-p'\beta})$ and $\alpha = \frac{2\beta}{\pi}$. These numbers satisfy $\frac{1-\alpha}{q} + \frac{\alpha}{2} = \frac{1}{p'}$. By complex interpolation, we know that

$$L_{p'}(\mathcal{M}; \ell_2^c) = [L_q(\mathcal{M}; \ell_2^c), L_2(\mathcal{M}; \ell_2^c)]_\alpha.$$

Let s_1, \dots, s_n be some nonnegative real numbers. For any $z \in \{z \in \mathbb{C}^* : 0 \leq \text{Arg}(z) \leq \frac{\pi}{2}\}$, we define a map $U(z)$ with

$$\begin{aligned} U(z) : L_2(\mathcal{M}; \ell_2^c(n)) \cap L_q(\mathcal{M}; \ell_2^c(n)) &\rightarrow L_2(\mathcal{M}; \ell_2^c(n)) + L_q(\mathcal{M}; \ell_2^c(n)) \\ (x_k)_k &\mapsto (S_{zs_k}(x_k))_k. \end{aligned}$$

This $U(z)$ is well defined, since $S_z(x)$ is well-defined for all z in the right-half complex plain and $x \in L_2(\mathcal{M})$. On the one hand, we have

$$\|U(te^{i\frac{\pi}{2}}) : L_2(\mathcal{M}; \ell_2^c) \rightarrow L_2(\mathcal{M}; \ell_2^c)\| \leq 1, \quad t > 0.$$

On the other hand, by the arguments above, we have

$$\|U(t) : L_q(\mathcal{M}; \ell_2^c(n)) \rightarrow L_q(\mathcal{M}; \ell_2^c(n))\| \leq \text{Col}(\mathcal{I}_q) \leq cq, \quad t > 0.$$

By the ‘sectorial’ form of Stein’s interpolation principle (see for instance [66, Lemma 5.3]), we have

$$\|U(e^{i\beta}) : L_{p'}(\mathcal{M}; \ell_2^c) \rightarrow L_{p'}(\mathcal{M}; \ell_2^c)\| \leq (cq)^{1-\alpha} \leq cq.$$

Thus,

$$\|(S_{s_k e^{i\beta}}(x_k))_k\|_{L_{p'}(\mathcal{M}; \ell_2^c)} \leq cq \|(x_k)\|_{L_{p'}(\mathcal{M}; \ell_2^c)}.$$

Similarly, we have

$$\|(S_{s_k e^{-i\beta}}(x_k))_k\|_{L_{p'}(\mathcal{M}; \ell_2^c)} \leq cq \|(x_k)\|_{L_{p'}(\mathcal{M}; \ell_2^c)}.$$

Namely, $(S_z)_{z \in \partial\Sigma_\beta}$ is Col-bounded. By [66, Lemma 4.8], we get that the set

$$\mathcal{G}_{p'}(\beta) = \{S_z : L_{p'}(\mathcal{M}) \rightarrow L_{p'}(\mathcal{M}) : z \in \Sigma_\beta\}$$

is Col-bounded. Indeed, by some standard arguments (see for instance [101, Proposition 2.8]), we can see that any S_z with $z \in \Sigma_\beta$ can be written as the convex combination of $\{S_z : z \in \partial\Sigma_\beta\}$. Therefore, by Lemma 5.8 we get that $\text{Col}(\mathcal{G}_{p'}(\beta)) \leq 2cq = 2cp'(\frac{\pi-2\beta}{\pi-p'\beta})$.

Set $\theta = \frac{\pi}{2p}$ and $\eta = \frac{\pi}{2} - \theta$. Note that $\eta < \pi/p'$. Thus

$$\text{Col}(\mathcal{G}_{p'}(\eta)) \leq 2cp' \left(\frac{\pi - 2\eta}{\pi - p'\eta} \right) \leq 2c(p-1)^{-1}.$$

By duality, we have

$$\text{Row}(\mathcal{G}_p(\eta)) \leq 2c(p-1)^{-1}. \quad (5.8)$$

Set $w_p = \frac{\pi}{p} - \frac{\pi}{2}$ and $\nu = \frac{(p+1)\pi}{4p}$. Then $0 < \frac{\pi}{2} - \nu < \eta < \frac{\pi}{p'} = \frac{\pi}{2} - w_p$. By the Laplace formula, we have that for any $z \in \mathbb{C} \setminus \overline{\Sigma_{\pi/2}}$,

$$(z - A)^{-1} = - \int_0^\infty e^{tz} S_t dt.$$

Denote $\Gamma_\eta^+ = \{u = te^{i\eta} : t \in \mathbb{R}_+\}$. By [66, Proposition 5.4 and Lemma 3.1], $u \mapsto ze^{uz} S_u$ is analytic on the area $\overline{\Sigma_{\omega_p}}$ for any $z \in \mathbb{C} \setminus \overline{\Sigma_{\pi/2}}$. Then by Cauchy’s theorem, we have that for any $z \in \mathbb{C} \setminus \overline{\Sigma_{\pi/2}}$,

$$z(z - A)^{-1} = - \int_0^\infty ze^{tz} S_t dt = - \int_{\Gamma_\eta^+} ze^{uz} S_u du. \quad (5.9)$$

Note that for any $z \in \{z = re^{i\text{Arg}(z)} : r \geq 0, \nu \leq \text{Arg}(z) \leq \pi/2\}$, we have

$$\text{Re}(uz) = rt \cos(\text{Arg}(z) + \eta) = -rt \sin(\text{Arg}(z) - \theta).$$

Therefore, for any $z \in \{z : \nu \leq \text{Arg}(z) \leq \pi/2\}$,

$$\int_{\Gamma_\eta^+} \|ze^{uz} S_u du\| \leq \sup_{u \in \Gamma_\eta^+} \|S_u\| \int_0^\infty re^{-rt \sin(\text{Arg}(z) - \theta)} dt \leq \frac{\sup_{u \in \Gamma_\eta^+} \|S_u\|}{\sin(\nu - \theta)} < \infty$$

since $0 < \frac{(p-1)\pi}{4p} < \text{Arg}(z) - \theta < \frac{\pi}{2} - \frac{\pi}{2p} < \frac{\pi}{2}$. Hence, the equation (5.9) holds for any $z \in \mathbb{C} \setminus \Sigma_\nu$. By lemma 5.8 and (5.8), we get

$$\text{Row}(\mathcal{F}_p) \leq 2 \left| \int_{\Gamma_\eta^+} z e^{uz} du \right| \cdot \text{Row}(\mathcal{G}_p(\eta)) \lesssim \frac{\text{Row}(\mathcal{G}_p(\eta))}{\nu - \theta} \lesssim p(p-1)^{-2}.$$

A similar proof shows that

$$\text{Col}(\mathcal{F}_p) \lesssim p(p-1)^{-2}.$$

□

Now let us prove the desired proposition.

Proof of Proposition 5.3. Set $F(z) = ze^{-z}$, $G(z) = 4F(z)$ and $\tilde{G}(z) = \overline{G(\bar{z})}$. Let $B = -(-A)^{1/2}$ be the infinitesimal generator of $(P_t)_t$. We have

$$F(tB)x = tBe^{-tB}x = -t \frac{\partial}{\partial t}(P_t(x)).$$

Let $\omega_p = \frac{\pi}{p} - \frac{\pi}{2}$, $\nu = \frac{(p+1)\pi}{4p}$ and $\xi = \frac{(3p+1)\pi}{8p}$. These numbers satisfy $\omega_p < \nu < \xi < \frac{\pi}{2}$. Note that $(P_t)_{t>0}$ is again a semigroup of unital completely positive trace preserving and symmetric maps. By Lemma 5.2, P_t satisfies Rota's dilation property. By [66, Corollary 11.2], $B : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M})$ admits a bounded $H^\infty(\Sigma_\xi)$ functional calculus. Note that $F, G \in H_0^\infty(\Sigma_\xi)$ and $\int_0^\infty G(t)F(t) \frac{dt}{t} = 1$. Then by [66, Theorem 7.6 (1)], B satisfies the dual square function estimate (\mathcal{S}_G^*) . By the proof of [66, Theorem 7.8], we get that

$$\inf \left\{ \left\| \left(\int_0^\infty |t\partial P_t(x^c)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_p + \left\| \left(\int_0^\infty |(t\partial P_t(x^r))^*|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_p \right\} \leq 2C \|(t\partial P_t(x))_{t>0}\|_{L_p(L_2^c(dt/t))} \quad (5.10)$$

where $C = \max\{\|T_c\|, \|T_r\|\}$. Here T_c and T_r are defined by

$$\begin{aligned} T_c : L_p(L_2^c(dt/t)) &\rightarrow L_p(L_2^c(dt/t)) & T_r : L_p(L_2^r(dt/t)) &\rightarrow L_p(L_2^r(dt/t)) \\ (x_t)_t &\mapsto \left(\int_0^\infty F(sB)G(tB)x_t \frac{dt}{t} \right)_s & (x_t)_t &\mapsto \left(\int_0^\infty F(sB)G(tB)x_t \frac{dt}{t} \right)_s. \end{aligned}$$

Let $\gamma \in (\nu, \xi)$. Define

$$\Gamma_\gamma(t) = \begin{cases} -te^{i\gamma}, & t \in \mathbb{R}_-; \\ te^{-i\gamma}, & t \in \mathbb{R}_+. \end{cases}$$

Define

$$\begin{aligned} T_\Phi : L_p(L_2^c(\mathbb{R}, |dt/t|)) &\rightarrow L_p(L_2^c(\mathbb{R}, |dt/t|)) \\ (x_t)_{t \in \mathbb{R}} &\mapsto \left(\frac{\Gamma_\gamma(t)(\Gamma_\gamma(t) - B)^{-1}x_t}{2\pi i} \right)_{t \in \mathbb{R}}. \end{aligned}$$

Set $K_1 = \int_{\Gamma_\gamma} |F(z)| \left| \frac{dz}{z} \right|$ and $K_2 = \int_{\Gamma_\gamma} |G(z)| \left| \frac{dz}{z} \right|$. By the proof of [66, Theorem 4.14], we have

$$\|T_c\| \leq K_1 K_2 \|T_\Phi\|.$$

By the proof of [66, Proposition 4.4], we get that

$$\|T_\Phi\| \leq \text{Col}(\mathcal{O})$$

where $\mathcal{O} = \left\{ \frac{1}{\mu(I)} \int_I \Gamma_\gamma(t) (\Gamma_\gamma(t) - B)^{-1} d\mu(t) : I \subset \mathbb{R}, 0 < \mu(I) < \infty \right\}$. Moreover, by Lemma 5.8,

$$\text{Col}(\mathcal{O}) \leq 2 \text{Col}(\{z(z - B)^{-1} : z \in \Gamma_\gamma\}).$$

Since $\gamma > \nu$, by Lemma 5.9,

$$\text{Col}(\{z(z - B)^{-1} : z \in \Gamma_\gamma\}) \lesssim c(p - 1)^{-2}.$$

On the other hand,

$$K_1 = \int_{\Gamma_\gamma} |F(z)| \left| \frac{dz}{z} \right| = 2 \int_0^\infty |te^{-i\gamma} e^{-t(\cos(\gamma) - i \sin(\gamma))}| \left| \frac{dt}{t} \right| = 2 \int_0^\infty e^{-t \cos(\gamma)} dt \lesssim \frac{1}{\cos(\gamma)},$$

and $K_2 = 4K_1$. Note that $\gamma < \xi = \frac{(3p+1)\pi}{8p}$ and $0 < \frac{\pi}{2} - \xi < \frac{\pi}{16}$. We have

$$\frac{1}{\cos(\gamma)} \leq \frac{1}{\sin(\frac{\pi}{2} - \xi)} \lesssim p(p - 1)^{-1}.$$

Therefore,

$$\|T_c\| \lesssim (p - 1)^{-4}.$$

Similarly,

$$\|T_r\| \lesssim (p - 1)^{-4}.$$

Thus, by (5.10),

$$\inf \left\{ \left\| \left(\int_0^\infty |t\partial P_t(x^c)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_p + \left\| \left(\int_0^\infty |(t\partial P_t(x^r))^*|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_p \right\} \leq c(p-1)^{-4} \|(t\partial P_t(x))_{t>0}\|_{L_p(L_2^{cr}(dt/t))}$$

where c is an absolute constant not depending on p . Then Proposition 5.3 follows from Proposition 5.7. \square

Chapter 6

Maximal inequalities of multipliers on $L_p(\mathcal{M})$

This chapter is devoted to the study of general criteria for maximal inequalities and pointwise convergences given by Theorem 0.6. Our argument does not essentially rely on the group theoretic structure. Note that there are a number of typical structures with Fourier-like expansions in noncommutative analysis, which are not given by group algebras. Hence instead of the framework in Theorem 0.6, we would like to state and prove the results in a quite general setting.

For proceeding the study, we will only require the following simple framework. Throughout this chapter, we fix a von Neumann algebra \mathcal{M} equipped with a normal semifinite faithful trace τ , and an isometric isomorphism of Hilbert spaces $U : L_2(\mathcal{M}) \rightarrow L_2(\Omega, \mu; H)$ for some distinguished regular Borel measure space (Ω, μ) and Hilbert space H . Assume additionally that $U^{-1}(C_c(\Omega; H))$ is a dense subspace in $L_p(\mathcal{M})$ for all $1 \leq p \leq \infty$ (for $p = \infty$ we refer to the w^* -density), where $C_c(\Omega; H)$ denotes the space of H -valued continuous functions with compact supports. Given a measurable function $m \in L_\infty(\Omega; \mathbb{C})$, we denote by T_m the linear operator on $L_2(\mathcal{M})$ determined by

$$T_m : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{M}), \quad U(T_m x) = mU(x), \quad x \in L_2(\mathcal{M}). \quad (6.1)$$

We call m the symbol of the operator T_m . The operator T_m is obviously a generalization of the classical Fourier multiplier. Moreover, for a discrete group Γ , taking

$$\mathcal{M} = VN(\Gamma), \quad (\Omega, \mu) = (\Gamma, \text{counting measure}), \quad H = \mathbb{C}, \quad U : \lambda(g) \mapsto \delta_g,$$

the above framework coincides with that considered in Theorem 0.6.

Example 6.1. Apart from group von Neumann algebras, this framework applies to various typical models in the study of noncommutative analysis. As an illustration we recall briefly some of them.

(1) (Twisted crossed product [9, 10]) Let G be a discrete group with a twisted dynamical system Σ on a von Neumann algebra $\mathcal{N} \subset B(L_2(\mathcal{N}))$. Then we may consider the von Neumann algebra \mathcal{M} generated by the associated regular covariant representations of G and the natural representation of \mathcal{N} on $\ell_2(G; L_2(\mathcal{N}))$. Each $x \in \mathcal{M}$ admits a Fourier series $\sum_{g \in G} \hat{x}(g)\lambda_\Sigma(g)$ with $\hat{x}(g) \in \mathcal{N}$. Take $\Omega = G$, $H = L_2(\mathcal{N})$ and $U : x \mapsto \hat{x}$. It is

easy to see that for any $m \in \ell_\infty(G)$, the multiplier in (6.1) is given by

$$T_m : \sum_{g \in G} \hat{x}(g) \lambda_\Sigma(g) \mapsto \sum_{g \in G} m(g) \hat{x}(g) \lambda_\Sigma(g),$$

which is the usual Fourier multiplier considered in [9]. As a particular case, this also coincides with the Fourier multipliers on quantum tori studied by [25].

(2) (Rigid C*-tensor category [89, 1]) Let \mathcal{C} be a rigid C*-tensor category, $A(\mathcal{C})$ be its Fourier algebra and \mathcal{M} be the von Neumann algebra generated by the image of the left regular representation of $\mathbb{C}[\mathcal{C}]$. Set $(\Omega, \mu) = (\text{Irr}(\mathcal{C}), d)$ where d denotes the intrinsic dimension, and set $U : L_2(\mathcal{M}) \rightarrow \ell_2(\Omega)$ by $U(\alpha) = \delta_\alpha$ for $\alpha \in \text{Irr}(\mathcal{C})$. Then for any $m \in \ell_\infty(\Omega)$, it is easy to check that T_m is the dual map of the multiplication operator $\theta \mapsto m\theta$ for $\theta \in A(\mathcal{C})$, which gives the Fourier multiplier studied in [89, 1].

(3) (Clifford algebras, free semicircular systems and q -deformations) Let $\mathcal{M} = \Gamma_q(H)$ be q -deformed von Neumann algebra in the sense of Bożejko and Speicher [17, 18]. The case $q = 0$ corresponds to Voiculescu's free semicircular systems and the case $q = -1$ corresponds to the usual Clifford algebras. We choose the canonical orthonormal basis of $L_2(\mathcal{M})$ with index set Ω according to the Fock representation $\bigoplus_n H^{\otimes n}$, and denote by U the corresponding isomorphism. We view $m : \mathbb{N} \rightarrow \mathbb{C}$ naturally as a function on Ω by setting the value $m(n)$ on indexes of basis in $H^{\otimes n}$. Then for any such m , the operator T_m is given by $T_m(x_n) = m(n)x_n$ for $x_n \in H^{\otimes n}$, which corresponds to the radial Fourier multiplier studied in [66, Section 9].

(3) (Quantum Euclidean spaces [51]) Let $\mathcal{M} = \mathcal{R}_\Theta$ be the quantum Euclidean space associated with an antisymmetric $n \times n$ -matrix Θ . We take $\Omega = \mathbb{R}^n$ and let $U : L_2(\mathcal{R}_\Theta) \rightarrow L_2(\mathbb{R}^n)$ be the canonical isomorphism. Then the operator T_m coincides with the usual quantum Fourier multipliers on \mathcal{R}_Θ .

(4) The framework also applies to non-abelian compact groups, compact quantum groups and von Neumann algebras of locally compact groups. We will discuss them in more details in the next chapters.

Our criterion is based on comparisons with symbols of a symmetric Markov group. To state our results, we fix a semigroup $(S_t)_{t>0}$ of unital completely positive trace preserving and symmetric maps on \mathcal{M} of the form

$$S_t = T_{e^{-t\ell}} : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{M}), \quad U(S_t x) = e^{-t\ell(\cdot)}(Ux), \quad x \in L_2(\mathcal{M}),$$

for a distinguished continuous function $\ell : \Omega \rightarrow \mathbb{R}_+$. We will also consider the subordinate Poisson semigroup $(P_t)_t$ of $(S_t)_t$, that is,

$$P_t = T_{e^{-t\sqrt{\ell}}} : L_2(\mathcal{M}) \rightarrow L_2(\mathcal{M}), \quad U(P_t x) = e^{-t\sqrt{\ell(\cdot)}}(Ux), \quad x \in L_2(\mathcal{M}).$$

Recall that we are interested in the following types of conditions for a family $(m_N)_{N \in \mathbb{N}}$ (resp. $(m_t)_{t \in \mathbb{R}_+}$) of measurable functions on Ω in Theorem 0.6:

(A1) There exist $\alpha > 0$ and $\beta > 0$ such that for almost all $\omega \in \Omega$ we have

$$|1 - m_N(\omega)| \leq \beta \frac{\ell(\omega)^\alpha}{2^N}, \quad |m_N(\omega)| \leq \beta \frac{2^N}{\ell(\omega)^\alpha}. \quad (6.2)$$

(A2) There exist $\alpha > 0$ and $\beta > 0$ such that for almost all $\omega \in \Omega$ we have

$$|1 - m_N(\omega)| \leq \beta \frac{\ell(\omega)^\alpha}{N}, \quad |m_N(\omega)| \leq \beta \frac{N}{\ell(\omega)^\alpha}, \quad |m_{N+1}(\omega) - m_N(\omega)| \leq \beta \frac{1}{N}. \quad (6.3)$$

(A3) There exist $\alpha > 0$, $\beta > 0$ and $\eta \in \mathbb{N}_+$ such that $t \mapsto m_t(\omega)$ is piecewise η -differentiable for almost all $\omega \in \Omega$, and for all $1 \leq k \leq \eta$ and almost all $\omega \in \Omega$ we have

$$|1 - m_t(\omega)| \leq \beta \frac{\ell(\omega)^\alpha}{t}, \quad |m_t(\omega)| \leq \beta \frac{t}{\ell(\omega)^\alpha}, \quad \left| \frac{d^k m_t(\omega)}{dt^k} \right| \leq \beta \frac{1}{t^k}. \quad (6.4)$$

We split our main theorem into the following two parts for $p \geq 2$ and $p < 2$ respectively.

Theorem 6.2. *Let $2 \leq p < \infty$.*

(1) *If $(T_{m_N})_{N \in \mathbb{N}}$ extends to bounded self-adjoint maps on \mathcal{M} with $\gamma := \sup_N \|T_{m_N}\| < \infty$ and if $(m_N)_N$ satisfies one of the assumptions **(A1)** or **(A2)**, then there exists a constant $c > 0$ depending only on p, α, β and γ , such that for all $x \in L_p(\mathcal{M})$, we have*

$$\|(T_{m_N}x)_N\|_{L_p(\mathcal{M}; \ell_\infty)} \leq c \|x\|_p, \quad \text{and} \quad T_{m_N}x \rightarrow x \text{ a.u. as } N \rightarrow \infty.$$

(2) *If $(T_{m_t})_{t \in \mathbb{R}_+}$ extends to bounded self-adjoint maps on \mathcal{M} with $\gamma := \sup_t \|T_{m_t}\| < \infty$ and if $(m_t)_t$ satisfies **(A3)**, then there exists a constant $c > 0$ depending only on p, α, β and γ , such that for all $x \in L_p(\mathcal{M})$, we have*

$$\|(T_{m_t}x)_t\|_{L_p(\mathcal{M}; \ell_\infty)} \leq c \|x\|_p, \quad \text{and} \quad T_{m_t}x \rightarrow x \text{ a.u. as } t \rightarrow \infty.$$

Theorem 6.3. *Assume that $(T_{m_N})_{N \in \mathbb{N}}$ (resp. $(T_{m_t})_{t \geq 0}$) extends to positive contractions on $L_p(\mathcal{M})$ for all $1 \leq p \leq \infty$. Assume additionally that for all $t > 0$, the operator S_t satisfies Rota's dilation property. There are constants $c_1, c_2 > 0$ depending only on p, α, β , and $c_3 > 0$ on p, α, β, η satisfying the following assertions:*

(1) *If $(m_N)_N$ satisfies the assumptions **(A1)**, then for all $1 < p < \infty$ and all $x \in L_p(\mathcal{M})$,*

$$\|(T_{m_N}x)_N\|_{L_p(\mathcal{M}; \ell_\infty)} \leq c_1 \|x\|_p \quad \text{and} \quad T_{m_N}x \rightarrow x \text{ a.u. as } N \rightarrow \infty.$$

(2) *If $(m_N)_N$ satisfies the assumption **(A2)**, then for all $3/2 < p < \infty$ and all $x \in L_p(\mathcal{M})$,*

$$\|(T_{m_N}x)_N\|_{L_p(\mathcal{M}; \ell_\infty)} \leq c_2 \|x\|_p \quad \text{and} \quad T_{m_N}x \rightarrow x \text{ a.u. as } N \rightarrow \infty.$$

(3) *If $(m_t)_t$ satisfies the assumption **(A3)**, then for all $1 + \frac{1}{2\eta} < p < \infty$ and all $x \in L_p(\mathcal{M})$,*

$$\|(T_{m_t}x)_t\|_{L_p(\mathcal{M}; \ell_\infty)} \leq c_3 \|x\|_p \quad \text{and} \quad T_{m_t}x \rightarrow x \text{ a.u. as } t \rightarrow \infty.$$

The above theorems recover Theorem 0.6. Indeed, note that if \mathcal{M} is a finite von Neumann algebra, the additional assumption on Rota's dilation property is fulfilled by Lemma 5.2(1). Also, for any conditionally negative function $\ell : \Gamma \rightarrow \mathbb{C}$, the associated maps $\lambda(g) \mapsto e^{t\ell(g)}$ forms a semigroup of unital completely positive trace preserving and symmetric maps. On the other hand, for a function $m : \Gamma \rightarrow \mathbb{C}$, the map T_m is τ -preserving; if T_m is unital positive on $VN(\Gamma)$, then it extends to positive contractions to $L_p(VN(\Gamma))$ for all $1 \leq p \leq \infty$ (see e.g. [69, Lemma 1.1]). So the assumption of Theorem 0.6 coincides with that of the above theorems.

Remark 6.4. The completely bounded version of the above two theorems holds true as well. In other words, if \mathcal{N} is another semifinite von Neumann algebra and if we replace T_{m_N} by $T_{m_N} \otimes \text{Id}_{\mathcal{N}}$, T_{m_t} by $T_{m_t} \otimes \text{Id}_{\mathcal{N}}$ and \mathcal{M} by $\tilde{\mathcal{M}} = \mathcal{M} \overline{\otimes} \mathcal{N}$, then the above two theorems still hold true. Indeed, it suffices to consider a larger Hilbert space $\tilde{H} = H \otimes L_2(\mathcal{N})$ and apply the above theorems to \tilde{M} and \tilde{H} .

Before the proof we recall the following well-known fact, which is of essential use for our arguments. The following maximal inequalities and the a.u. convergence on dense subspaces are given in [69], and the a.u. convergence on L_p -spaces then follows from Proposition 4.7(1).

Proposition 6.5. *We have*

$$\|(S_t(x))_t\|_{L_p(\mathcal{M}; \ell_\infty)} \leq c_p \|x\|_p, \quad x \in L_p(\mathcal{M}), \quad 1 < p < \infty,$$

where $c_p \leq C \cdot p(p-1)^{-2}$ with C an absolute constant. Moreover, $(S_t(x))_t$ converges a.u. to x as $t \rightarrow 0$ for all $x \in L_p(\mathcal{M})$ with $1 < p < \infty$.

The following result on mean convergences is an easy consequence of our assumptions.

Proposition 6.6. *Let $(T_{m_N})_N$, $(T_{m_t})_t$ and $1 < p < \infty$ be given as in Theorem 6.2 or Theorem 6.3 which satisfy the assumptions (A1), (A2) or (A3) correspondingly.*

(1) *The family $(T_{m_t})_t$ is strongly continuous on $L_p(\mathcal{M})$, i.e., for any $x \in L_p(\mathcal{M})$ the function $t \mapsto T_{m_t}x$ is continuous from \mathbb{R}_+ to $L_p(\mathcal{M})$.*

(2) *We have*

$$\lim_{N \rightarrow \infty} \|T_{m_N}x - x\|_p = 0, \quad \lim_{t \rightarrow \infty} \|T_{m_t}x - x\|_p = 0, \quad x \in L_p(\mathcal{M}).$$

Proof. Let $x \in U^{-1}(C_c(\Omega; H))$ and $E = \text{supp}(U(x)) \subset \Omega$. By the Hölder inequality, for any $t_0 \geq 0$ and $2 \leq p < \infty$,

$$\begin{aligned} \|T_{m_t}x - T_{m_{t_0}}x\|_p &\leq \|T_{m_t}x - T_{m_{t_0}}x\|_2^{2/p} \|T_{m_t}x - T_{m_{t_0}}x\|_\infty^{1-2/p} \\ &\lesssim \|(m_t - m_{t_0})\mathbf{1}_E\|_\infty^{2/p} \|x\|_2^{2/p} \|x\|_\infty^{1-2/p}. \end{aligned}$$

By the continuity of m_t and the compactness of E , the above quantity tends to 0 as $t \rightarrow t_0$. Similar arguments work for $p < 2$ by using the Hölder inequality with endpoints $p = 1, 2$. Similarly, by the continuity of ℓ and the compactness of E , we have

$$\lim_{t \rightarrow 0} \|T_{m_t}x - x\|_p \rightarrow 0, \quad \lim_{N \rightarrow \infty} \|T_{m_N}x - x\|_p \rightarrow 0.$$

For general elements $x \in L_p(\mathcal{M})$, it suffices to note that $(T_{m_N})_N$ and $(T_{m_t})_t$ extend to bounded operators on $L_p(\mathcal{M})$. Thus the desired results follow from a standard density argument. \square

Now we are ready to proceed the proof of the previous theorems.

6.1 L_2 -estimates under lacunary conditions

Proposition 6.7. *Let $(m_N)_{N \in \mathbb{Z}} \subset \ell_\infty(\Omega)$. Assume that there exist a function $f : \Omega \rightarrow \mathbb{R}_+$ and a positive number $a > 1$ such that*

$$|m_N(\omega)| \leq \beta \frac{a^N f(\omega)}{(a^N + f(\omega))^2} \quad (6.5)$$

Then,

$$\|(T_{m_N} x)_{N \in \mathbb{Z}}\|_{L_2(\mathcal{M}; \ell_2^{\sigma_r})} \leq \beta \sqrt{\frac{a^2}{a^2 - 1}} \|x\|_2.$$

Proof. Note that we have $L_2(\mathcal{M}; \ell_2^{\sigma_r}) = L_2(\mathcal{M} \bar{\otimes} \ell_\infty)$ by the traciality of τ . Then we see that

$$\begin{aligned} \|(T_{m_N} x)_N\|_{L_2(\mathcal{M}; \ell_2^{\sigma_r})}^2 &= \tau \left(\sum_{N \in \mathbb{Z}} |T_{m_N} x|^2 \right) \\ &= \sum_{N \in \mathbb{Z}} \|T_{m_N} x\|_{L_2(\mathcal{M})}^2 = \sum_{N \in \mathbb{Z}} \|m_N U(x)\|_{L_2(\Omega; H)}^2 \\ &= \int_\Omega \sum_{N \in \mathbb{Z}} |m_N(\omega)(Ux)(\omega)|^2 d\mu(\omega) \\ &\leq \left\| \sum_{N \in \mathbb{Z}} |m_N|^2 \right\|_{L_\infty(\Omega)} \|Ux\|_{L_2(\Omega; H)}^2 = \left\| \sum_{N \in \mathbb{Z}} |m_N|^2 \right\|_{L_\infty(\Omega)} \|x\|_{L_2(\mathcal{M})}^2. \end{aligned}$$

However, by (6.5) we see that for all $\omega \in \Omega$,

$$\sum_{N \in \mathbb{Z}} |m_N(\omega)|^2 \leq \sum_{N < \log_a f(\omega)} \beta^2 \frac{a^{2N}}{f(\omega)^2} + \sum_{N \geq \log_a f(\omega)} \beta \frac{f(\omega)^2}{a^{2N}} \lesssim \beta^2 \frac{a^2}{a^2 - 1}.$$

Thus we obtain the desired inequality. \square

6.2 Reduction to lacunary sequences

Lemma 6.8. *Assume that $(m_t)_{t \geq 0}$ satisfies the assumption **(A3)** for $\eta = 1$. Choose an arbitrary function $f : \Omega \rightarrow \mathbb{R}_+$. Define for $j \in \mathbb{Z}$,*

$$a_j = \sup_t \left\{ \sup_{2^{j-2} < \frac{f(\omega)}{t} \leq 2^j} |m_t(\omega)| \right\}, \quad b_j = \sup_t \left\{ \sup_{2^{j-2} < \frac{f(\omega)}{t} \leq 2^j} t \cdot \left| \frac{\partial m_t}{\partial t}(\omega) \right| \right\}.$$

Assume

$$K = \sum_{j \in \mathbb{N}} a_j^{1/2} (a_j^{1/2} + b_j^{1/2}) < \infty.$$

Then for $x \in L_2(\mathcal{M})$, we have the following maximal inequality

$$\|(T_{m_t} x)_t\|_{L_2(\mathcal{M}; \ell_\infty)} \lesssim K \|x\|_2 \quad \text{and} \quad \|(T_{m_t} x)_t\|_{L_2(\mathcal{M}; \ell_{\infty c})} \lesssim K \|x\|_2.$$

Proof. Let $\{\eta_j\}_{j \in \mathbb{Z}}$ be a partition of unity of \mathbb{R}_+ satisfying

$$\sum_j \eta_j = 1, \quad \text{supp } \eta_j \subset [2^{j-2}, 2^j], \quad 0 \leq \eta_j \leq 1 \quad \text{and} \quad |\eta'_j| < C2^{-j}.$$

Define $m_{t,j}(\omega) = m_t(\omega)\eta_j(\frac{f(\omega)}{t}) \in \ell_\infty(\Omega)$. For notational simplicity, denote by $T_{t,j}$ the operators with symbols $m_{t,j}$; that is,

$$U(T_{t,j}x) = m_{t,j}U(x), \quad x \in L_2(\mathcal{M}).$$

Then we have

$$\|\sup_t^+ T_{m_t} x\|_2 = \|\sup_t^+ \left(\sum_{j \in \mathbb{Z}} T_{t,j} x \right)\|_2 \leq \sum_{j \in \mathbb{Z}} \|\sup_t^+ T_{t,j} x\|_2. \quad (6.6)$$

From now on we fix an arbitrary $j \in \mathbb{Z}$ and $A_j > 0$. In the sequel of this proof we denote

$$G_k(\omega) = U(x)(\omega) \cdot \mathbf{1}_{[2^{k-2}, 2^{k+2}]}(f(\omega)), \quad \omega \in \Omega \quad \text{and} \quad x_k = U^*(G_k), \quad x \in L_2(\mathcal{M}).$$

Since $\text{supp } \eta_j \subset [2^{j-2}, 2^j]$, for $v \in \mathbb{Z}$ and $t \in [2^v, 2^{v+1})$ we have

$$m_{t,j}(\omega)U(x)(\omega) = m_t(\omega)\eta_j\left(\frac{f(\omega)}{t}\right)U(x)(\omega) = m_t(\omega)\eta_j\left(\frac{f(\omega)}{t}\right)G_{v+j}(\omega).$$

We may rewrite the above equality as

$$T_{t,j}x = T_{t,j}x_{v+j}, \quad t \in [2^v, 2^{v+1}).$$

We divide the interval $[2^v, 2^{v+1})$ into A_j parts:

$$2^v = \gamma_0 < \gamma_1 < \gamma_2 \cdots < \gamma_{A_j} = 2^{v+1} \quad \text{with} \quad \gamma_{k+1} - \gamma_k = 2^v \cdot A_j^{-1}.$$

For any $t \in [2^v, 2^{v+1})$, there exists $0 \leq k(t) \leq A_j$ such that $t \in [\gamma_{k(t)}, \gamma_{k(t)+1})$. By Kadison's Cauchy-Schwarz inequality, we have

$$\begin{aligned} |T_{t,j}x_{v+j}|^2 &= |U^*(m_{t,j}G_{v+j})|^2 \\ &= \left| U^* \left(\int_{\gamma_{k(t)}}^t \left(\frac{\partial m_{s,j}}{\partial s} \right) G_{v+j} ds + m_{\gamma_{k(t)},j} \cdot G_{v+j} \right) \right|^2 \\ &\leq 2(t - \gamma_{k(t)}) \int_{\gamma_{k(t)}}^t \left| \frac{\partial T_{s,j}}{\partial s}(x_{v+j}) \right|^2 ds + 2|T_{\gamma_{k(t)},j}(x_{v+j})|^2 \\ &\leq 2 \left(\frac{2^v}{A_j} \int_{\gamma_{k(t)}}^{\gamma_{k(t)+1}} \left| \frac{\partial T_{s,j}}{\partial s}(x_{v+j}) \right|^2 ds + |T_{\gamma_{k(t)},j}(x_{v+j})|^2 \right) \\ &\leq 2 \sum_{k=0}^{A_j-1} \left(\frac{2^v}{A_j} \int_{\gamma_k}^{\gamma_{k+1}} \left| \frac{\partial T_{s,j}}{\partial s}(x_{v+j}) \right|^2 ds + |T_{\gamma_k,j}(x_{v+j})|^2 \right). \end{aligned}$$

We denote $y_v = 2 \sum_{k=0}^{A_j-1} \left(\frac{2^v}{A_j} \int_{\gamma_k}^{\gamma_{k+1}} \left| \frac{\partial T_{s,j}}{\partial s}(x_{w+j}) \right|^2 ds + |T_{\gamma_k,j}(x_{w+j})|^2 \right)$. Then

$$|T_{t,j}x_{v+j}|^2 \leq y_v. \quad (6.7)$$

Let us estimate the quantity $\|y_v\|_1$. We have

$$\|y_v\|_1 = \tau(y_v) = 2 \sum_{k=0}^{A_j-1} \left(\frac{2^v}{A_j} \int_{\gamma_k}^{\gamma_{k+1}} \left\| \frac{\partial T_{s,j}}{\partial s}(x_{v+j}) \right\|_2^2 ds + \|T_{\gamma_k,j}(x_{v+j})\|_2^2 \right).$$

Note that

$$\begin{aligned} & \int_{\gamma_k}^{\gamma_{k+1}} \left| \partial_s \left(m_s(\omega) \eta_j \left(\frac{f(\omega)}{s} \right) \right) \right|^2 ds \\ &= \int_{\gamma_k}^{\gamma_{k+1}} \left| \frac{\partial m_s(\omega)}{\partial s} \cdot \eta_j \left(\frac{f(\omega)}{s} \right) - \eta_j' \left(\frac{f(\omega)}{s} \right) \frac{f(\omega)}{s^2} \cdot m_s(\omega) \right|^2 ds \\ &\leq \int_{\gamma_k}^{\gamma_{k+1}} \left(\left| \frac{b_j}{s} \right| + \left| \frac{a_j}{s} \right| \right)^2 ds \quad (\text{since } \text{supp } \eta_j \subset [2^{j-2}, 2^j]) \\ &\leq \frac{1}{2^v A_j} (b_j + a_j)^2. \end{aligned}$$

By Fubini's theorem, we have

$$\begin{aligned} \sum_{k=0}^{A_j-1} \frac{2^v}{A_j} \int_{\gamma_k}^{\gamma_{k+1}} \left\| \frac{\partial T_{s,j}}{\partial s}(x_{v+j}) \right\|_2^2 ds &= \sum_{k=0}^{A_j-1} \frac{2^v}{A_j} \int_{\gamma_k}^{\gamma_{k+1}} \left\| \frac{\partial m_{s,j}}{\partial s} G_{v+j} \right\|_{L_2(\Omega; H)}^2 ds \\ &= \sum_{k=0}^{A_j-1} \frac{2^v}{A_j} \int_{\Omega} \left(\int_{\gamma_k}^{\gamma_{k+1}} \left| \left(\frac{\partial m_{s,j}}{\partial s}(\omega) \right) \right|^2 ds \right) |G_{v+j}(\omega)|^2 d\mu(\omega) \\ &\leq \frac{(b_j + a_j)^2}{A_j} \|x_{v+j}\|_2^2, \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^{A_j-1} \|T_{\gamma_k,j}(x_{v+j})\|_2^2 &= \sum_{k=0}^{A_j-1} \int_{\omega \in \Omega} |m_{\gamma_k}(\omega)|^2 \left| \eta_j \left(\frac{f(\omega)}{\gamma_k} \right) \right|^2 |G_{v+j}(\omega)|^2 d\mu(\omega) \\ &\leq A_j a_j^2 \|x_{v+j}\|_2^2. \end{aligned}$$

Therefore,

$$\|y_v\|_1 \leq 2 \left(\frac{(b_j + a_j)^2}{A_j} + A_j a_j^2 \right) \|x_{v+j}\|_2^2.$$

Recall that the inequality (6.7) asserts that $|T_{t,j}x_{v+j}|^2 \leq y_v \leq \sum_{u \in \mathbb{Z}} y_u$. Since $T_{t,j}x$ is self-adjoint and $T_{t,j}x = T_{t,j}x_{v+j}$, the above estimate gives

$$\begin{aligned} \|\sup_t^+ T_{t,j}x\|_2 &\leq \|\sup_t^+ |T_{t,j}x_{v+j}|\|_2 \leq \left\| \left(\sum_{u \in \mathbb{Z}} y_u \right)^{\frac{1}{2}} \right\|_2 \leq \left(\sum_{u \in \mathbb{Z}} \|y_u\|_1 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{u \in \mathbb{Z}} 2 \left(\frac{(b_j + a_j)^2}{A_j} + A_j a_j^2 \right) \|x_{u+j}\|_2^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{6.8}$$

Note that

$$[2^{j+u-2}, 2^{j+u+2}] = \cup_{l=0}^3 [2^{j+u+l-2}, 2^{j+u+l-1}].$$

We have

$$\begin{aligned} \sum_{u \in \mathbb{Z}} \|x_{u+j}\|_2^2 &= \sum_{u \in \mathbb{Z}} \int_{\Omega} \left| U(x)(\omega) \cdot \mathbb{1}_{[2^j+u-2, 2^j+u-1]}(f(\omega)) \right|^2 d\mu(\omega) \\ &\leq \sum_{l=0}^3 \int_{\Omega} \left| U(x)(\omega) \cdot \sum_{u \in \mathbb{Z}} \mathbb{1}_{[2^j+u+l-2, 2^j+u+l-1]}(f(\omega)) \right|^2 d\mu(\omega) \\ &= \sum_{l=0}^3 \|U(x)\|_2^2 \end{aligned}$$

where the last equality holds since

$$\sum_{u \in \mathbb{Z}} \mathbb{1}_{[2^j+u+l-2, 2^j+u+l-1]}(f(\omega)) = 1.$$

Thus,

$$\sum_{u \in \mathbb{Z}} \|x_{u+j}\|_2^2 \leq 4\|x\|_2^2.$$

Choose $A_j = \frac{a_j+b_j}{a_j}$. Together with the inequalities (6.8), we have

$$\|\sup_n^+ T_{t,j} x\|_2 \lesssim a_j^{1/2} (a_j + b_j)^{1/2} \left(\sum_{u \in \mathbb{Z}} \|x_{u+j}\|_2^2 \right)^{1/2} \lesssim a_j^{1/2} (a_j + b_j)^{1/2} \|x\|_2.$$

By (6.6), the proof is complete for the first maximal inequality.

Note that we have the analogue for the space $L_2(\mathcal{M}; \ell_{\infty}^c)$ of (6.8). In fact,

$$\begin{aligned} \|(T_{t,j} x)_t\|_{L_2(\mathcal{M}; \ell_{\infty}^c)} &= \|(|T_{t,j} x_{v+j}|^2)_t\|_{L_1(\mathcal{M}; \ell_{\infty})}^{1/2} \leq \left\| \left(\sum_{u \in \mathbb{Z}} y_u \right) \right\|_1^{1/2} \leq \left(\sum_{u \in \mathbb{Z}} \|y_u\|_1 \right)^{1/2} \\ &\leq \left(\sum_{u \in \mathbb{Z}} 2 \left(\frac{(b_j + a_j)^2}{A_j} + A_j a_j^2 \right) \|x_{u+j}\|_2^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Repeating the arguments as the proof of the first inequality, we get the desired maximal inequality for $L_2(\mathcal{M}; \ell_{\infty}^c)$. \square

Proposition 6.9. *Assume that $(m_t)_{t \geq 0}$ satisfies the assumption (A3). Then for any $1 \leq q < 2$, $q + \frac{q(2-q)}{q-1+2\eta} \leq p < 2$, we have*

$$\|(T_{m_t})_{t \in \mathbb{R}_+} : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M}; \ell_{\infty})\| \lesssim_{\beta, \eta, p, q} \sup_{1 \leq \delta \leq 2^n} \|(T_{m_{2^j \delta}})_{j \in \mathbb{Z}} : L_q(\mathcal{M}) \rightarrow L_q(\mathcal{M}; \ell_{\infty})\|$$

provided the right hand side is finite.

Proof. By Proposition 6.6, $(T_{m_t}(x))_t$ is strongly continuous on $L_p(\mathcal{M})$ for all $1 < p < \infty$. By Remark 4.2, we have for $x \in L_p(\mathcal{M})$,

$$\begin{aligned} \|\sup_{t \in \mathbb{R}_+}^+ T_{m_t} x\|_p &= \lim_{r \rightarrow \infty} \|\sup_{j \in \mathbb{Z}}^+ T_{m_{2^j/r}} x\|_p \\ &\leq \|\sup_{j \in \mathbb{Z}}^+ T_{m_{2^j}} x\|_p + \sum_{s=0}^{\infty} \left(\|\sup_{j \in \mathbb{Z}}^+ T_{m_{2^j/2^{s+1}}} x\|_p - \|\sup_{j \in \mathbb{Z}}^+ T_{m_{2^j/2^s}} x\|_p \right). \end{aligned}$$

For $s \in \mathbb{N}_0$, we denote $I_s = \{2^{\frac{j}{2^s}} : j \in \mathbb{Z}\}$. Set $\psi_t^{[s]}(\omega) = m_{\frac{1}{2^{2^s+1}t}}(\omega) - m_t(\omega)$ and denote by $\Psi_t^{[s]} = T_{\psi_t^{[s]}}$ the associated multipliers with symbols $\psi_t^{[s]}$. Note that we have the bijection $J : I_s \rightarrow I_{s+1} \setminus I_s$, $2^{j/2^s} \mapsto 2^{(2j+1)/2^{s+1}} = 2^{\frac{1}{2^{s+1}}} 2^{j/2^s}$ and $I_{s+1} = I_s \cup J(I_s)$. Hence for

$$y_t = \begin{cases} T_{m_t} & t \in I_s \\ T_{m_{J^{-1}(t)}} & t \in J(I_s) \end{cases},$$

we have $\|\sup_{t \in I_{s+1}} y_t\| = \|\sup_{t \in I_s} T_{m_t}\|$. Then applying the triangle inequality, we get

$$\|\sup_{t \in I_{s+1}}^+ T_{m_t} x\|_p - \|\sup_{t \in I_s}^+ T_{m_t} x\|_p \leq \|\sup_{t \in I_s}^+ \Psi_t^{[s]}(x)\|_p.$$

Moreover, for any $s_1 \geq 0$ we write

$$\|\sup_{t \in I_{s_1}}^+ \Psi_t^{[s_1]}(x)\|_p = \sum_{s_2=0}^{s_1-1} \left(\|\sup_{t \in I_{s_2+1}}^+ \Psi_t^{[s_1]}(x)\|_p - \|\sup_{t \in I_{s_2}}^+ \Psi_t^{[s_1]}(x)\|_p \right) + \|\sup_{t \in I_0}^+ \Psi_t^{[s_1]}(x)\|_p.$$

Applying the triangle inequality again, we have

$$\|\sup_{t \in I_{s_2+1}}^+ \Psi_t^{[s_1]}(x)\|_p - \|\sup_{t \in I_{s_2}}^+ \Psi_t^{[s_1]}(x)\|_p \leq \|\sup_{t \in I_{s_2}}^+ \Psi_t^{[s_1, s_2]}(x)\|_p$$

where the symbol associated to $\Psi_t^{[s_1, s_2]}$ is defined by

$$\psi_t^{[s_1, s_2]}(\omega) = \psi_{2^{2^{-s_2-1}t}}^{[s_1]}(\omega) - \psi_t^{[s_1]}(\omega).$$

Hence,

$$\|\sup_{t \in \mathbb{R}_+}^+ T_{m_t} x\|_p \leq \|\sup_{j \in \mathbb{Z}}^+ T_{m_{2^j}} x\|_p + \sum_{s_1=0}^{\infty} \|\sup_{t \in I_0}^+ \Psi_t^{[s_1]}(x)\|_p + \sum_{s_1=0}^{\infty} \sum_{s_2=0}^{s_1-1} \|\sup_{t \in I_{s_2}}^+ \Psi_t^{[s_1, s_2]}(x)\|_p.$$

Repeating this process for η times, we get

$$\begin{aligned} \|\sup_{t \in \mathbb{R}_+}^+ T_{m_t} x\|_p &\leq \|\sup_{j \in \mathbb{Z}}^+ T_{m_{2^j}} x\|_p + \sum_{s_1=0}^{\infty} \|\sup_{t \in I_0}^+ \Psi_t^{[s_1]}(x)\|_p + \sum_{s_1=0}^{\infty} \sum_{s_2=0}^{s_1-1} \|\sup_{t \in I_0}^+ \Psi_t^{[s_1, s_2]}(x)\|_p + \\ &\cdots + \sum_{s_1 > s_2 > \cdots > s_{\eta-1}} \|\sup_{t \in I_0}^+ \Psi_t^{[s_1, s_2, \dots, s_{\eta-1}]}(x)\|_p + \sum_{s_1 > s_2 > \cdots > s_{\eta}} \|\sup_{t \in I_{s_{\eta}}}^+ \Psi_t^{[s_1, s_2, \dots, s_{\eta}]}(x)\|_p, \end{aligned} \quad (6.9)$$

where for any $1 \leq v \leq \eta$ we define the multipliers $\Psi_t^{[s_1, s_2, \dots, s_v]} = T_{\psi_t^{[s_1, s_2, \dots, s_v]}}$ with symbols

$$\psi_t^{[s_1, s_2, \dots, s_v]} = \psi_{\delta_v t}^{[s_1, s_2, \dots, s_{v-1}]} - \psi_t^{[s_1, s_2, \dots, s_{v-1}]}$$

where we set $\delta_v = 2^{2^{-s_v-1}}$. Note that $1 < \delta_v < 2$.

We will show that for any $\omega \in \Omega$,

$$\left| \partial_t^{(k)} \left(\psi_t^{[s_1, s_2, \dots, s_v]}(\omega) \right) \right| \lesssim_{\beta} 2^{-(s_1 + s_2 + \cdots + s_v)} \frac{(2k + 2v)^v}{t^k} \quad 0 \leq k \leq \eta - v. \quad (6.10)$$

Let us prove this inequality by induction. Consider first $v = 0$, i.e. $\psi_t = m_t$. The inequality (6.10) holds by assumption. Assume that (6.10) holds for some $0 < v < \eta$ and consider the case of $v + 1$. For any $0 \leq k \leq \eta - (v + 1)$, we have

$$\begin{aligned}
 |\partial_t^{(k)} \psi_t^{[s_1, s_2, \dots, s_{v+1}]}(\omega)| &= |\partial_t^{(k)}(\psi_{\delta_{v+1}t}^{[s_1, s_2, \dots, s_v]}(\omega)) - \partial_t^{(k)}(\psi_t^{[s_1, s_2, \dots, s_v]}(\omega))| \\
 &= |\delta_{v+1}^k \partial_\gamma^{(k)}(\psi_\gamma^{[s_1, s_2, \dots, s_v]}(\omega))|_{\gamma=\delta_{v+1}t} - \partial_t^{(k)}(\psi_t^{[s_1, s_2, \dots, s_v]}(\omega))| \\
 &\lesssim_{\beta, \eta} (\delta_{v+1}^k - 1) 2^{-(s_1+s_2+\dots+s_v)} \frac{(2k+2v)^v}{(\delta_{v+1}t)^k} \\
 &\quad + (\delta_{v+1}t - t) \sup_{t \leq t \leq \delta_{v+1}t} |\partial_t^{(k+1)}(\psi_t^{[s_1, s_2, \dots, s_v]}(\omega))| \\
 &\lesssim_{\beta, \eta} k 2^{-s_v} 2^{-(s_1+s_2+\dots+s_v)} \frac{(2k+2v)^v}{t^k} \\
 &\quad + (\delta_{v+1}t - t) \beta 2^{-(s_1+s_2+\dots+s_v)} \frac{(2k+2+2v)^v}{t^{k+1}} \\
 &\lesssim_{\beta, \eta} 2^{-(s_1+s_2+\dots+s_{v+1})} \frac{(2(k+1+v))^{v+1}}{t^k}.
 \end{aligned}$$

So (6.10) is proved. In particular, setting $k = 0$ and $k = 1$ respectively, we get for any $1 \leq v \leq \eta$

$$|\psi_t^{[s_1, s_2, \dots, s_v]}(\omega)| \lesssim_{\eta, \beta} 2^{-(s_1+s_2+\dots+s_v)},$$

and for any $1 \leq v \leq \eta - 1$,

$$|\partial_t \psi_t^{[s_1, s_2, \dots, s_v]}(\omega)| \lesssim_{\eta, \beta} 2^{-(s_1+s_2+\dots+s_v)} \frac{1}{t}.$$

This also yields

$$\begin{aligned}
 |\partial_t \psi_t^{[s_1, s_2, \dots, s_\eta]}(\omega)| &\leq |\partial_t \psi_{\delta_\eta t}^{[s_1, s_2, \dots, s_{\eta-1}]}(\omega)| + |\partial_t \psi_t^{[s_1, s_2, \dots, s_{\eta-1}]}(\omega)| \\
 &\lesssim_{\eta, \beta} 2^{-(s_1+s_2+\dots+s_{\eta-1})} \frac{1}{t}.
 \end{aligned}$$

On the other hand, by definition

$$\psi_t^{[s_1, s_2, \dots, s_v]}(\omega) = \sum_{\varepsilon \in \{0,1\}^v} (-1)^{(v+\sum_{i=1}^v \varepsilon_i)} m_{\delta^\varepsilon t}(\omega) \quad (6.11)$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_v) \in \{0,1\}^v$ and $\delta^\varepsilon = \delta_1^{\varepsilon_1} \delta_2^{\varepsilon_2} \dots \delta_v^{\varepsilon_v} < 2^v$. By the assumption **(A3)**,

$$|\psi_t^{[s_1, s_2, \dots, s_v]}(\omega)| \leq \sum_{\varepsilon \in \{0,1\}^v} |m_{\delta^\varepsilon t}(\omega)| \leq \sum_{\varepsilon \in \{0,1\}^v} \beta \frac{\delta^\varepsilon t}{\ell(\omega)^\alpha} \leq \beta 2^{2v} \frac{t}{\ell(\omega)^\alpha},$$

$$|\psi_t^{[s_1, s_2, \dots, s_v]}(\omega)| \leq \sum_{\varepsilon \in \{0,1\}^v} |1 - m_{\delta^\varepsilon t}(\omega)| \leq \sum_{\varepsilon \in \{0,1\}^v} \beta \frac{\ell(\omega)^\alpha}{\delta^\varepsilon t} \leq 2^v \beta \frac{\ell(\omega)^\alpha}{t}.$$

Thus, setting

$$a_j^{[s_1, s_2, \dots, s_v]} := \sup_t \left(\sup_{2^{-j-2} < \frac{\ell(\omega)^\alpha}{t} \leq 2^{-j}} |\psi_t^{[s_1, s_2, \dots, s_v]}(\omega)| \right),$$

and

$$b_j^{[s_1, s_2, \dots, s_v]} := \sup_t \left(\sup_{2^{j-2} < \frac{\ell(\omega)^\alpha}{t} \leq 2^j} t \left| \partial_t \psi_t^{[s_1, s_2, \dots, s_v]}(\omega) \right| \right),$$

we have for $1 \leq v \leq \eta - 1$,

$$a_j^{[s_1, s_2, \dots, s_v]} \lesssim_{\eta, \beta} \min\{2^{-(s_1 + s_2 + \dots + s_v)}, 2^{-|j|}\}, \quad b_j^{[s_1, s_2, \dots, s_v]} \lesssim_{\eta, \beta} 2^{-(s_1 + s_2 + \dots + s_v)}.$$

Then by Lemma 6.8, for $1 \leq v \leq \eta - 1$, we have

$$\left\| \sup_{t \in \mathbb{R}_+} \Psi_t^{[s_1, s_2, \dots, s_v]} x \right\|_2 \lesssim K^{[s_1, s_2, \dots, s_v]} \|x\|_2, \quad (6.12)$$

with

$$\begin{aligned} K^{[s_1, s_2, \dots, s_v]} &= \sum_{j \geq \mathbb{Z}} (a_j^{[s_1, s_2, \dots, s_v]})^{1/2} (a_j^{[s_1, s_2, \dots, s_v]} + b_j^{[s_1, s_2, \dots, s_v]})^{1/2} \\ &\lesssim_{\eta, \beta} \left(\sum_{|j| \leq s_1 + s_2 + \dots + s_v} 2^{-(s_1 + s_2 + \dots + s_v)} + \sum_{|j| > s_1 + s_2 + \dots + s_v} 2^{-j} \right) \\ &\lesssim_{\eta, \beta} \frac{s_1 + s_2 + \dots + s_v}{2^{(s_1 + s_2 + \dots + s_v)}}. \end{aligned}$$

Similarly, for $v = \eta$,

$$\left\| \sup_{t \in \mathbb{R}_+} \Psi_t^{[s_1, s_2, \dots, s_\eta]} x \right\|_2 \lesssim K^{[s_1, s_2, \dots, s_\eta]} \|x\|_2, \quad (6.13)$$

with

$$K^{[s_1, s_2, \dots, s_\eta]} \lesssim_{\eta, \beta} \frac{s_1 + s_2 + \dots + s_\eta}{2^{(s_1 + s_2 + \dots + s_{\eta-1}) + \frac{s_\eta}{2}}}.$$

In the following we consider the case $1 \leq q < 2$. Denote

$$A_q = \sup_{1 \leq \delta \leq 2^\eta} \|(T_{m_{\delta 2^j}})_{j \in \mathbb{N}} : L_q(\mathcal{M}) \rightarrow L_q(\mathcal{M}; \ell_\infty)\|.$$

For $1 \leq v < \eta$, by (6.11) we have

$$\left\| \sup_{t \in I_0} \Psi_t^{[s_1, s_2, \dots, s_v]} x \right\|_q \leq \sum_{\varepsilon \in \{0, 1\}^v} \left\| \sup_{t \in I_0} T_{m_{\delta \varepsilon t}} x \right\|_q \leq 2^v A_q \|x\|_q. \quad (6.14)$$

For $v = \eta$, we decompose

$$I_{s_\eta} = \{2^{j/2^{s_\eta}} : j \in \mathbb{Z}\} = \cup_{\alpha=1}^{2^{s_\eta}} \{2^{\frac{(2^{s_\eta}j + \alpha)}{2^{s_\eta}}} : j \in \mathbb{Z}\} = \cup_{\alpha=1}^{2^{s_\eta}} 2^{\frac{\alpha}{2^{s_\eta}}} I_0.$$

By (6.11) and the triangle inequality, we have

$$\begin{aligned} \left\| \sup_{t \in I_{s_\eta}} \Psi_t^{[s_1, s_2, \dots, s_\eta]}(x) \right\|_q &\leq \sum_{\varepsilon \in \{0, 1\}^\eta} \left\| \sup_{t \in I_{s_\eta}} T_{m_{\delta \varepsilon t}} x \right\|_q \\ &\leq \sum_{\varepsilon \in \{0, 1\}^\eta} \sum_{\alpha=1}^{2^{s_\eta}} \left\| \sup_{t \in 2^{\frac{\alpha}{2^{s_\eta}}} I_0} T_{m_{\delta \varepsilon t}} x \right\|_q \\ &\leq \sum_{\varepsilon \in \{0, 1\}^\eta} \sum_{\alpha=1}^{2^{s_\eta}} \left\| \sup_{t \in I_0} T_{m_{\frac{\alpha}{2^{s_\eta}} \delta \varepsilon t}} x \right\|_q \\ &\lesssim_\eta A_q 2^{s_\eta} \|x\|_q. \end{aligned} \quad (6.15)$$

Now the conclusion follows easily from the complex interpolation. Let $1 < p < 2$, $0 < \theta < 1$ with $\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{2}$. By (6.12), (6.14) and interpolation, we see that for $v < \eta$,

$$\left\| \sup_{t \in I_0}^+ \Psi_t^{[s_1, s_2, \dots, s_v]} x \right\|_p \lesssim_{\eta, \beta} A_q^{1-\theta} (s_1 + s_2 \cdots + s_v)^\theta 2^{-\theta(s_1 + s_2 \cdots + s_v)} \|x\|_p.$$

By (6.13), (6.15) and interpolation, for $v = \eta$,

$$\left\| \sup_{t \in I_{s_\eta}}^+ \Psi_t^{[s_1, s_2, \dots, s_\eta]}(x) \right\|_p \lesssim_{\eta, \beta} A_q 2^{(1-\theta)s_\eta} (s_1 + s_2 \cdots + s_\eta)^\theta 2^{-\theta(s_1 + s_2 \cdots + \frac{s_\eta}{2})} \|x\|_p.$$

We apply the above estimate to (6.9). Note that when $v < \eta$,

$$\sum_{s_1 > s_2 > \cdots > s_v} (s_1 + s_2 \cdots + s_v)^\theta 2^{-\theta(s_1 + s_2 \cdots + s_v)} \lesssim_\theta 1.$$

When $v = \eta$,

$$\sum_{s_1 > s_2 > \cdots > s_\eta} 2^{(1-\theta)s_\eta} (s_1 + s_2 \cdots + s_\eta)^\theta 2^{-\theta(s_1 + s_2 \cdots + \frac{s_\eta}{2})} \lesssim_\theta \sum_{s_\eta \geq 0} 2^{(1-\theta)s_\eta} s_\eta^\theta 2^{-(\eta - \frac{1}{2})\theta s_\eta}.$$

Thus the above quantity is finite if $(1 - \theta)s_\eta < (\eta - 1/2)\theta s_\eta$, i.e. $\theta > \frac{2}{2\eta+1}$. Therefore by (6.9), if $\theta > \frac{2}{2\eta+1}$,

$$\left\| \sup_{t \in \mathbb{R}_+}^+ T_{m_t} x \right\|_p \lesssim_{\beta, \eta, \theta} A_q^{1-\theta} \|x\|_p \lesssim_{\beta, \eta, \theta} A_q \|x\|_p.$$

One can verify that for any $q + \frac{q(2-q)}{q-1+2\eta} < p \leq 2$, we can find a $\theta \in (\frac{2}{2\eta+1}, 1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{2}.$$

Since the pair θ is determined by p and q , the constant of maximal inequality depends only on β, η, p and q . The proof is complete. \square

6.3 Proof of Theorem 6.2

We first remark that the assumption **(A2)** leads to a special case of assumption **(A3)**.

Lemma 6.10. *Assume that $(m_N)_{N \in \mathbb{N}}$ satisfies the assumption **(A2)**. Then there is a family of symbols $(\tilde{m}_t)_{t \in \mathbb{R}_+}$ such that $(\tilde{m}_t)_{t \in \mathbb{R}_+}$ satisfies the assumption **(A3)** with $\eta = 1$ and $\tilde{m}_N = m_N$.*

Proof. Set $m_t = m_0 = 0$ if $0 \leq t < 1$. For $t \geq 1$, we write $t = N_t + r_t$, with $N_t \in \mathbb{N}$ and $0 \leq r_t < 1$. Define

$$\tilde{m}_t = (1 - r_t)m_{N_t} + r_t m_{N_t+1}.$$

It is obvious that $(\tilde{m}_t)_{t \in \mathbb{R}_+}$ satisfies the assumption **(A3)** with $\eta = 1$. \square

Now we are ready to conclude Theorem 6.2.

Proof of Theorem 6.2. First assume that $(m_N)_{N \in \mathbb{N}}$ satisfies the assumption **(A1)**. Set $T_{\phi_N} = T_{m_N} - S_{2^{-N/\alpha}}$ with the symbol $\phi_N = m_N - e^{-\frac{\ell(\cdot)}{2^{N/\alpha}}}$. By the assumption **(A1)**, we can easily see that

$$|\phi_N(\omega)| \lesssim_{\alpha, \beta} \frac{2^N \ell(\omega)^\alpha}{(2^N + \ell(\omega)^\alpha)^2}.$$

By Proposition 4.11, Proposition 6.7 and the interpolation theorem, we get for any $2 \leq p < \infty$

$$\|(T_{\phi_N}(x))_N\|_{L_p(\mathcal{M}; \ell_\infty)} \lesssim_{\alpha, \beta, \gamma, p} \|x\|_p, \quad \text{and} \quad \|(T_{\phi_N}(x))_N\|_{L_p(\mathcal{M}; \ell_\infty)} \lesssim_{\alpha, \beta, \gamma, p} c' \|x\|_p.$$

Applying Proposition 6.5, we also get the strong type (p, p) estimate of T_{m_N} with a constant c depending only on α, β, γ, p .

If $(m_t)_{t \in \mathbb{R}_+}$ satisfies the assumption **(A3)**. Again set $T_{\phi_t} = T_{m_t} - S_{t^{-1/\alpha}}$. Then we have the following estimates for almost all $\omega \in \Omega$

$$\begin{aligned} |\phi_t(\omega)| &\leq |1 - m_t(\omega)| + |1 - e^{-\frac{\ell(\omega)}{t^{1/\alpha}}}| \lesssim_{\alpha, \beta} \frac{\ell(\omega)^\alpha}{t}, \\ |\phi_t(\omega)| &\leq |m_t(\omega)| + |e^{-\frac{\ell(\omega)}{t^{1/\alpha}}}| \lesssim_{\alpha, \beta} \frac{t}{\ell(\omega)^\alpha}, \\ \left| \frac{\partial \phi_t(\omega)}{\partial t} \right| &\leq \beta \frac{1}{t} + \frac{1}{t} \cdot \left(\frac{\ell(\omega)}{\alpha t^{1/\alpha}} e^{\frac{\ell(\omega)}{t^{1/\alpha}}} \right) \lesssim_{\alpha, \beta} \frac{1}{t}. \end{aligned}$$

Applying Lemma 6.8, we get

$$\|\sup_t^+ T_{\phi_t} x\|_2 \lesssim K \|x\|_2,$$

where

$$K \lesssim_{\alpha, \beta} \sum_{j \in \mathbb{Z}} (2^{-|j|} (2^{-|j|} + 1))^{1/2} \lesssim_{\alpha, \beta} 1.$$

By the interpolation theorem, for any $2 \leq p < \infty$, $(T_{\phi_t})_{t \in \mathbb{R}}$ is of strong type (p, p) with constant c depending only on α, β, γ, p . Similarly, for any $2 \leq p < \infty$, we have

$$\|(T_{\phi_t}(x))_{t \geq 0}\|_{L_p(\mathcal{M}; \ell_\infty^c)} \lesssim_{\alpha, \beta, \gamma, p} \|x\|_p.$$

Therefore, we conclude the strong type (p, p) estimate of $(T_{m_t})_t$ thanks to Proposition 6.5.

Now the desired a.u. convergence follows immediately from the above maximal inequalities. For instance, we consider the symbols $(m_t)_{t \in \mathbb{R}_+}$ satisfying **(A3)**. Let $x \in U^{-1}(C_c(\Omega; H))$ and set $E = \text{supp } U(x)$. Note that E is a compact set. As the proof of Proposition 6.6, by the assumption **(A3)**, we have

$$\begin{aligned} \|T_{\phi_t} x\|_p &\leq 2\gamma^{1-(2/p)} \left(\|(m_t - 1)\mathbf{1}_E\|_{L_\infty(E)} + \|(e^{\frac{\ell(\omega)}{t^{1/\alpha}}} - 1)\mathbf{1}_E\|_{L_\infty(E)} \right)^{2/p} \|x\|_2^{2/p} \|x\|_\infty^{1-2/p} \\ &\lesssim_{\alpha, \gamma} \frac{\|\ell^\alpha \mathbf{1}_E\|_{L_\infty(E)}^{2/p}}{t^{2/p}} \|x\|_2^{2/p} \|x\|_\infty^{1-2/p}. \end{aligned}$$

By the continuity of ℓ and the compactness of E , we have

$$\begin{aligned} \lim_{M \rightarrow \infty} \int_M^\infty \|T_{m_t} x - x\|_p^p dt &\lesssim_{\alpha, \gamma} \lim_{M \rightarrow \infty} \int_M^\infty \frac{\|\ell^\alpha \mathbf{1}_E\|_{L_\infty(E)}^2}{t^2} \|x\|_2^2 \|x\|_\infty^{p-2} dt \\ &\lesssim_{\alpha, \gamma} \lim_{M \rightarrow \infty} \frac{\|\ell^\alpha \mathbf{1}_E\|_{L_\infty(E)}^2 \|x\|_2^2 \|x\|_\infty^{p-2}}{M} = 0. \end{aligned}$$

By Proposition 4.7 (2), we obtain that $T_{m_t}(x) - S_{t^{-1/\alpha}}x$ converges a.u. to 0 as $t \rightarrow \infty$. On the other hand, Proposition 6.5 asserts that $S_{t^{-1/\alpha}}x$ converges a.u. to x as $t \rightarrow \infty$. We see that $T_{m_t}(x)$ converges a.u. to x as $t \rightarrow \infty$. The case of **(A1)** can be dealt with similarly.

The assertions for **(A2)** follow from those for **(A3)** immediately according to Lemma 6.10. \square

6.4 Proof of Theorem 6.3

Lemma 6.11. *Let $(\Phi_j)_{j \in \mathbb{Z}}$ be a sequence of linear maps on \mathcal{M} with $\sup_j \|\Phi_j\| < \gamma$. Let $1 < p < 2$. Assume that for any $s \in \mathbb{Z}$, there is a decomposition $\Phi_j = \Phi_j^{(s,1)} + \Phi_j^{(s,2)}$ and constants $d \in \mathbb{Z}$, $c_1, c_2 \geq 1$, such that*

$$\|\sup_{j \in \mathbb{Z}}^+ \Phi_j^{(s,1)} x\|_p \leq sc_1 \|x\|_p \quad \text{and} \quad \|\sup_{j \in \mathbb{Z}}^+ \Phi_j^{(s,2)} x\|_2 \leq 2^{-s} c_2 \|x\|_2.$$

Then $(\Phi_j)_{j \geq 0}$ is of restricted weak type (q, q) for any $q > p$ with constant

$$C_q \lesssim_{\gamma} c_2^2 c_1 (q - p)^{-1}.$$

Proof. Let $1 < p < 2$ and $x \in \mathcal{S}_{\mathcal{M}^+}$. Consider a positive integer $s \in \mathbb{N}_+$. Since the strong type (p, p) estimate implies the weak one, we can find two projections $e_1, e_2 \in \mathcal{M}$, such that

$$\begin{aligned} \sup_j \|e_1 \Phi_j^{(s,1)}(x) e_1\|_{\infty} &\leq \lambda \quad \text{and} \quad \tau(e_1^{\perp}) \leq \left(c_1 s \frac{\|x\|_p}{\lambda} \right)^p, \\ \sup_j \|e_2 \Phi_j^{(s,2)}(x) e_2\|_{\infty} &\leq \lambda \quad \text{and} \quad \tau(e_2^{\perp}) \leq \left(c_2 2^{-s} \frac{\|x\|_2}{\lambda} \right)^2. \end{aligned}$$

Set $e = e_1 \wedge e_2$. Since $\Phi_j = \Phi_j^{(s,1)} + \Phi_j^{(s,2)}$ we have,

$$\|e \Phi_j(x) e\|_{\infty} \leq 2\lambda, \quad j \geq 0,$$

and

$$\tau(e^{\perp}) \leq \tau(e_1^{\perp} + e_2^{\perp}) \leq \left(c_1 s \frac{\|x\|_p}{\lambda} \right)^p + \left(2^{-s} c_2 \frac{\|x\|_2}{\lambda} \right)^2.$$

We consider $x_{\lambda}^{\perp} = x \mathbf{1}_{(\lambda, \infty)}(x)$ and $x_{\lambda} = x \mathbf{1}_{[0, \lambda]}(x)$. Applying the above arguments to x_{λ}^{\perp} , we can find a projection $e \in \mathcal{M}$ such that

$$\|e \Phi_j(x_{\lambda}^{\perp}) e\|_{\infty} \leq \lambda, \quad j \in \mathbb{Z}.$$

and

$$\tau(e^{\perp}) \lesssim \left(c_1 s \frac{\|x_{\lambda}^{\perp}\|_p}{\lambda} \right)^p + \left(2^{-s} c_2 \frac{\|x_{\lambda}^{\perp}\|_2}{\lambda} \right)^2. \quad (6.16)$$

Since $x = x_{\lambda} + x_{\lambda}^{\perp}$ and $x_{\lambda} \leq \lambda \cdot 1$, we have

$$\|e(\Phi_j x) e\|_{\infty} \leq \|e(\Phi_j(x_{\lambda})) e\|_{\infty} + \|e(\Phi_j(x_{\lambda}^{\perp})) e\|_{\infty} \leq \gamma \lambda + \lambda \leq (\gamma + 1) \lambda, \quad j \geq 0.$$

Note that for $1 < p \leq 2$,

$$|t\mathbf{1}_{(\lambda, \infty)}(t)|^p = \frac{|t\mathbf{1}_{(\lambda, \infty)}(t)|^2}{t^{2-p}} \leq \frac{|t\mathbf{1}_{(\lambda, \infty)}(t)|^2}{\lambda^{2-p}}.$$

Therefore we have $|x_\lambda^\perp|^p \leq \frac{|x_\lambda^\perp|^2}{\lambda^{2-p}}$. and

$$\frac{\|x_\lambda^\perp\|_p^p}{\lambda^p} \leq \frac{\|x_\lambda^\perp\|_2^2}{\lambda^2}.$$

Choosing

$$s = \lceil \log_2 \left(2 \frac{\lambda^p \|x_\lambda^\perp\|_2^2}{\lambda^2 \|x_\lambda^\perp\|_p^p} \right) \rceil \geq 1,$$

by (6.16) we have

$$\begin{aligned} \tau(e^\perp) &\lesssim \left(c_1 \left[\log_2 \left(2 \frac{\lambda^p \|x_\lambda^\perp\|_2^2}{\lambda^2 \|x_\lambda^\perp\|_p^p} \right) \right] \frac{\|x_\lambda^\perp\|_p}{\lambda} \right)^p + c_2^2 \left(\frac{\|x_\lambda^\perp\|_p}{\lambda^p} \right) \\ &\lesssim c_2^2 c_1^p \left[\log_2 \left(2 \frac{\lambda^p \|x_\lambda^\perp\|_2^2}{\lambda^2 \|x_\lambda^\perp\|_p^p} \right) \right]^p \frac{\|x_\lambda^\perp\|_p^p}{\lambda^p}. \end{aligned} \quad (6.17)$$

From the inequality $\log_2 t \lesssim \frac{t^\delta}{\delta}$, we get

$$\tau(e^\perp) \lesssim c_2^2 c_1^p \delta^{-p} \lambda^{-(2p\delta + p - p^2\delta)} \|x_\lambda^\perp\|_2^{2p\delta} \|x_\lambda^\perp\|_p^{p - p^2\delta}.$$

Consider $q > p$ and set $\delta = \frac{q-p}{p(2-p)} > 0$. Since $0 \leq x_\lambda^\perp \leq x$, the above inequality yields

$$\tau(e^\perp) \lesssim c_2^2 c_1^p (q-p)^{-p} \cdot \lambda^{-q} \|x\|_2^{\frac{2(q-p)}{2-p}} \|x\|_p^{p(1 - \frac{q-p}{2-p})}. \quad (6.18)$$

Let $f \in \mathcal{S}_{\mathcal{M}_+}$ be an arbitrary projection. Consider

$$x = \tau(f)^{-1/q} f \quad \text{and} \quad \tilde{\lambda} = \tau(f)^{-1/q} (\gamma + 1)^{-1} \lambda.$$

Then by the above argument, we can find a projection $e \in \mathcal{M}$ such that (6.18) holds for $\tilde{\lambda}$ and

$$\|e(\Phi_j x)e\|_\infty \leq (\gamma + 1)\tilde{\lambda}, \quad j \geq 0,$$

which means

$$\|e(\Phi_j f)e\|_\infty \leq \lambda, \quad j \geq 0.$$

Note that

$$\|x\|_2^{\frac{2(q-p)}{2-p}} \|x\|_p^{p(1 - \frac{q-p}{2-p})} = 1.$$

By (6.18), we have

$$\tau(e^\perp) \lesssim \frac{c_2^2 c_1^p (\gamma + 1)^q (q-p)^{-p}}{\lambda^q} \tau(f).$$

This implies that $(\Phi_j)_{j \geq 0}$ is of restricted weak type (q, q) with constant

$$C_q \lesssim_\gamma c_2^{2/q} c_1^{p/q} (q-p)^{-p/q} \lesssim_\gamma c_2^2 c_1 (q-p)^{-1}.$$

□

Lemma 6.12. *Let $1 < p < 2$ and $(\Phi_j)_j$ be a sequence of unital positive maps on $L_p(\mathcal{M})$. Denote $\Phi : (x_j)_{j \in \mathbb{Z}} \mapsto (\Phi_j x_j)_{j \in \mathbb{Z}}$. Then*

$$\|\Phi : L_p(\mathcal{M}; \ell_2^{cr}) \rightarrow L_p(\mathcal{M}; \ell_2^{cr})\| \leq \|\Phi : L_{\frac{p}{2-p}}(\mathcal{M}; \ell_\infty) \rightarrow L_{\frac{p}{2-p}}(\mathcal{M}; \ell_\infty)\|^{1/2}.$$

Proof. Since Φ_j are unital positive maps, by Kadison's Cauchy-Schwarz inequality [70], for any self-adjoint element $x_j \in L_p(\mathcal{M})$, we have

$$\Phi_j(x_j)^2 \leq \Phi_j(x_j^2).$$

Assume $1 < p \leq 2$ and $(x_j)_j \in L_p(\mathcal{M}; \ell_2^{cr})$ is a sequence of self-adjoint elements. Then the conjugate index $p' \geq 2$ and

$$\begin{aligned} \|(\Phi_j x_j)_j\|_{L_{p'}(\mathcal{M}; \ell_2^{cr})} &= \left\| \left(\sum_{j \geq 0} \Phi_j x_j^2 \right)^{1/2} \right\|_{p'} \leq \left\| \left(\sum_{j \geq 0} \Phi_j(x_j^2) \right)^{1/2} \right\|_{p'} = \left\| (\Phi_j(x_j^2))_j \right\|_{L_{p'/2}(\mathcal{M}; \ell_1)}^{1/2} \\ &\leq \|\Phi : L_{p'/2}(\mathcal{M}; \ell_1) \rightarrow L_{p'/2}(\mathcal{M}; \ell_1)\|^{1/2} \| (x_j)_j \|_{L_{p'/2}(\mathcal{M}; \ell_2^{cr})}. \end{aligned}$$

For general $(x_j)_j \in L_p(\mathcal{M}; \ell_2^{cr})$, we may decompose it into two sequence of self-adjoint elements. Note that $\|(x_j)_j\|_{L_{p'}(\mathcal{M}; \ell_2^{cr})} = \|(x_j^*)_j\|_{L_{p'}(\mathcal{M}; \ell_2^{cr})}$ for $p' \geq 2$. Therefore,

$$\|\Phi : L_{p'}(\mathcal{M}; \ell_2^{cr}) \rightarrow L_{p'}(\mathcal{M}; \ell_2^{cr})\| \leq \|\Phi : L_{p'/2}(\mathcal{M}; \ell_1) \rightarrow L_{p'/2}(\mathcal{M}; \ell_1)\|^{1/2}.$$

By duality we obtain

$$\|\Phi : L_p(\mathcal{M}; \ell_2^{cr}) \rightarrow L_p(\mathcal{M}; \ell_2^{cr})\| \leq \|\Phi : L_{\frac{p}{2-p}}(\mathcal{M}; \ell_\infty) \rightarrow L_{\frac{p}{2-p}}(\mathcal{M}; \ell_\infty)\|^{1/2},$$

as desired. \square

Lemma 6.13. *Assume that for all $t > 0$, the operator S_t satisfies Rota's dilation property. Let $s \in \mathbb{N}$ and define $\Delta_{\alpha, j} = P_{2^{-(j+2s)/\alpha}} - P_{2^{-(j-2s)/\alpha}}$ for any $\alpha > 0, j \in \mathbb{Z}$. Then for any $1 < p < 2$,*

$$\|(\Delta_{\alpha, j} x)_j\|_{L_p(\mathcal{M}; \ell_2^{cr})} \lesssim_\alpha s(p-1)^{-6} \|x\|_p, \quad x \in L_p(\mathcal{M}).$$

Proof. We may write

$$\Delta_{\alpha, j} x = \int_{2^{-(j+2s)/\alpha}}^{2^{-(j-2s)/\alpha}} \left(-\frac{\partial}{\partial t} P_t(x) \right) dt, \quad x \in L_p(\mathcal{M}).$$

Let $x = x_1 + x_2$ for $x_1, x_2 \in L_p(\mathcal{M})$. Applying Kadison's Cauchy-Schwarz inequality,

$$\begin{aligned} |\Delta_{\alpha, j} x_1|^2 &= \left| \int_{2^{-(j+2s)/\alpha}}^{2^{-(j-2s)/\alpha}} \frac{1}{\sqrt{t}} \left(-\sqrt{t} \frac{\partial}{\partial t} P_t(x_1) \right) dt \right|^2 \\ &\leq \left(\int_{2^{-(j+2s)/\alpha}}^{2^{-(j-2s)/\alpha}} t \left| \left(\frac{\partial}{\partial t} P_t(x_1) \right) \right|^2 dt \right) \left(\int_{2^{-(j-2s)/\alpha}}^{2^{-(j+2s)/\alpha}} \frac{dt}{t} \right) \\ &\lesssim_\alpha s \left(\int_{\tilde{\alpha}^{-j-2s}}^{\tilde{\alpha}^{-j+2s}} t \left| \left(\frac{\partial}{\partial t} P_t(x_1) \right) \right|^2 dt \right), \end{aligned}$$

where $\tilde{\alpha} = 2^{1/\alpha}$. Therefore,

$$\begin{aligned} \|(\Delta_{\alpha,j}x_1)_j\|_{L_p(\mathcal{M};\ell_2^c)} &\lesssim_\alpha \sqrt{s} \left\| \left(\sum_{j=-\infty}^{\infty} \int_{\tilde{\alpha}^{-j-2s}}^{\tilde{\alpha}^{-j+2s}} t \left| \left(\frac{\partial}{\partial t} P_t(x_1) \right) \right|^2 dt \right)^{1/2} \right\|_p \\ &\lesssim_\alpha \sqrt{s} \left\| \left(\sum_{j=-\infty}^{\infty} \sum_{k=-2s}^{2s-1} \int_{\tilde{\alpha}^{-j+k}}^{\tilde{\alpha}^{-j+k+1}} t \left| \left(\frac{\partial}{\partial t} P_t(x_1) \right) \right|^2 dt \right)^{1/2} \right\|_p \\ &\lesssim_\alpha s \left\| \left(\int_0^\infty t \left| \left(\frac{\partial}{\partial t} P_t(x_1) \right) \right|^2 dt \right)^{1/2} \right\|_p \end{aligned}$$

Similarly,

$$\|(\Delta_{\alpha,j}x_2)_j\|_{L_p(\mathcal{M};\ell_2^c)} \lesssim_\alpha s \left\| \left(\int_0^\infty t \left| \left(\frac{\partial}{\partial t} P_t(x_2) \right)^* \right|^2 dt \right)^{1/2} \right\|_p.$$

On the other hand,

$$\|(\Delta_{\alpha,j}x)_j\|_{L_p(\mathcal{M};\ell_2^{cr})} \leq \inf \{ \|(\Delta_{\alpha,j}x_1)_j\|_{L_p(\mathcal{M};\ell_2^c)} + \|(\Delta_{\alpha,j}x_2)_j\|_{L_p(\mathcal{M};\ell_2^c)} \}$$

where the infimum runs over all $x_1, x_2 \in L_p(\mathcal{M})$ such that $x = x_1 + x_2$. Then the conclusion follows from Proposition 5.3. \square

Now, let us prove Theorem 6.3.

Proof of Theorem 6.3. By Lemma 6.10, it is enough to prove the assertions (1) and (3) of Theorem 6.3. The case $p \geq 2$ has been already treated by Theorem 6.2. In this proof, we focus on the case $1 < p < 2$.

(1) Fix a finite index set $J \subset \mathbb{Z}$. Denote

$$\begin{aligned} A(p, \infty) &= \|(T_{m_j})_{j \in J} : L_p(\mathcal{M}; \ell_\infty) \rightarrow L_p(\mathcal{M}; \ell_\infty)\|, \\ A(p, 1) &= \|(T_{m_j})_{j \in J} : L_p(\mathcal{M}; \ell_1) \rightarrow L_p(\mathcal{M}; \ell_1)\|, \\ A(p, 2) &= \|(T_{m_j})_{j \in J} : L_p(\mathcal{M}; \ell_2^{cr}) \rightarrow L_p(\mathcal{M}; \ell_2^{cr})\|. \end{aligned}$$

Since $(T_{m_j})_{j \in J}$ are positive maps, one can easily check that

$$\|(T_{m_j})_{j \in J} : L_p(\mathcal{M}) \rightarrow L_p(\mathcal{M}; \ell_\infty(J))\| = A(p, \infty).$$

Let $1 < p \leq 2$. It is sufficient to show that $A(p, \infty)$ is dominated by a positive constant independent of J . Let $s \in \mathbb{N}$ and $\Delta_j = \Delta_{2\alpha,j} = P_{2^{-(j+2s)/2\alpha}} - P_{2^{-(j-2s)/2\alpha}}$. By Proposition 4.11 and Lemma 6.13, we have

$$\begin{aligned} \|\sup_{j \in J}^+ T_{m_j}(\Delta_j x)\|_p &\leq A(p, 2) \|(\Delta_j x)_j\|_{L_p(\mathcal{M}; \ell_2^{cr})} \\ &\lesssim_\alpha A(p, 2) s (p-1)^{-6} \|x\|_p. \end{aligned}$$

By Proposition 6.5 and Lemma 6.13, we have

$$\begin{aligned} \|\sup_{j \in J}^+ S_{2^{-\alpha/j}}(\Delta_j x)\|_p &\lesssim (p-1)^{-2} \|(\Delta_j x)_{j \in \mathbb{Z}}\|_{L_P(\mathcal{M}; \ell_2^r)} \\ &\lesssim_\alpha s(p-1)^{-8} \|x\|_p. \end{aligned}$$

Hence,

$$\|\sup_{j \in J}^+ T_{\phi_j}(\Delta_j x)\|_p \lesssim_\alpha A(p, 2)(p-1)^{-8} s \|x\|_p. \quad (6.19)$$

Let $\omega \in \Omega$ and let $\delta_j(\omega) = \exp\left(-\frac{\sqrt{\ell(\omega)}}{2^{(j+2s)/2\alpha}}\right) - \exp\left(-\frac{\sqrt{\ell(\omega)}}{2^{(j-2s)/2\alpha}}\right)$ be the symbol of Δ_j .

When $2^j \geq \ell(\omega)^\alpha$, we have $|\phi_j(\omega)| \leq \beta \frac{\ell(\omega)^\alpha}{2^j}$ and

$$\begin{aligned} |1 - \delta_j(\omega)| &\leq \left|1 - \exp\left(-\frac{\sqrt{\ell(\omega)}}{2^{(j+2s)/2\alpha}}\right)\right| + \left|\exp\left(-\frac{\sqrt{\ell(\omega)}}{2^{(j-2s)/2\alpha}}\right)\right| \\ &\lesssim_\alpha \left(\frac{\sqrt{\ell(\omega)}}{2^{(j+2s)/2\alpha}}\right)^\alpha + \left(\frac{2^{(j-2s)/2\alpha}}{\sqrt{\ell(\omega)}}\right)^\alpha \\ &\lesssim_\alpha 2^{-s} \left(\frac{2^j}{\ell(\omega)^\alpha}\right)^{1/2}, \end{aligned}$$

in particular,

$$|\phi_j(\omega)(1 - \delta_j(\omega))| \lesssim_{\alpha, \beta} 2^{-s} \left(\frac{\ell(\omega)^\alpha}{2^j}\right)^{1/2} \lesssim_{\alpha, \beta} 2^{-s} \left(\frac{2^j \ell(\omega)^\alpha}{(2^j + \ell(\omega)^\alpha)^2}\right)^{1/2}.$$

Similarly, when $2^j < \ell(\omega)^\alpha$, we have

$$|\phi_j(\omega)(1 - \delta_j(\omega))| \lesssim_{\alpha, \beta} 2^{-s} \left(\frac{2^j}{\ell(\omega)^\alpha}\right)^{1/2} \lesssim_{\alpha, \beta} 2^{-s} \left(\frac{2^j \ell(\omega)^\alpha}{(2^j + \ell(\omega)^\alpha)^2}\right)^{1/2}.$$

By Proposition 6.7, we have

$$\|\sup_{j \in J}^+ T_{\phi_j}(1 - \Delta_j)x\|_2 \lesssim_{\alpha, \beta} 2^{-s} \|x\|_2 \quad (6.20)$$

Thus by (6.19) (6.20) and Lemma 6.11, we have that $(T_{\phi_j})_{j \in J}$ is of restrict weak type (q, q) for any $q > p$ with constant

$$C'_q \lesssim_{\alpha, \beta} A(p, 2)(p-1)^{-8}(q-p)^{-1}.$$

Set $D = \sup_{1 < p \leq 2} (p-1)^{22} A(p, \infty) < \infty$. Choose a $1 < r \leq 2$ such that

$$(r-1)^{22} A(r, \infty) > \frac{D}{2}.$$

Set $p = \frac{1}{2}(r+1)$ and $q = p + (r-p)/2$. Note that $1 < p < q < r \leq 2$. By Proposition 6.5, the semigroup $(S_t)_t$ is of strong type (q, q) with constant $c(q-1)^{-2}$. Recall that $T_{m_j} = T_{\phi_j} + S_{2^{-j/\alpha}}$, thus $(T_{m_j})_{j \in J}$ is of restricted weak type (q, q) with constant

$$\begin{aligned} C_q &\lesssim_{\alpha, \beta} A(p, 2)(p-1)^{-8}(q-p)^{-1} + (q-1)^{-2} \\ &\lesssim_{\alpha, \beta} A\left(\frac{p}{2-p}, \infty\right)^{1/2} (r-1)^{-9}. \end{aligned}$$

The last inequality above follows from Lemma 6.12 and the values of p and q . Because $(T_{m_j})_j$ is of strong type (∞, ∞) and of restricted weak type of (q, q) , applying Theorem 4.4 we have

$$\|\sup_{j \in J}^+ T_{m_j} x\|_r \leq \max\{C_q, 1\} \left(\frac{rq}{r-q} + r\right)^2 \|x\|_r. \quad (6.21)$$

Note that $(T_{m_j})_j$ are positive maps. We have $A(r, \infty) \lesssim C_q(r-1)^{-2}$. Therefore,

$$(r-1)^{-22} \frac{D}{2} < A(r, \infty) \lesssim_{\alpha, \beta} A\left(\frac{p}{2-p}, \infty\right)^{1/2} (r-1)^{-11}. \quad (6.22)$$

Recall that $A\left(\frac{p}{2-p}, \infty\right) \lesssim_{\alpha, \beta} 1$ if $\frac{p}{2-p} \geq 2$ by Theorem 6.2. Without loss of generality we assume that $1 < \frac{p}{2-p} < 2$. Then the inequality (6.22) yields

$$\begin{aligned} (r-1)^{-22} \frac{D}{2} &\lesssim_{\alpha, \beta} \left(\left(\frac{p}{2-p} - 1 \right)^{-22} D \right)^{1/2} (r-1)^{-11} \\ &\lesssim_{\alpha, \beta} D^{1/2} (p-1)^{-11} (r-1)^{-11}. \end{aligned}$$

Recall that $p = \frac{1}{2}(r+1)$. We have

$$D \lesssim_{\alpha, \beta} 1.$$

In other word, $(p-1)^{22} A(p, \infty) \lesssim_{\alpha, \beta} 1$ for any $1 < p \leq 2$. In particular, this estimate is independent of the finite index set J . So we obtain the desired property.

Note that $T_{m_j} x$ converges a.u. to x as $j \rightarrow \infty$ for $x \in L_2(\mathcal{M})$ by Theorem 6.2 and that $L_2(\mathcal{M}) \cap L_p(\mathcal{M})$ is dense in $L_p(\mathcal{M})$. Applying Proposition 4.7 (1), we get the a.u. convergence on $L_p(\mathcal{M})$.

(3) For any $1 \leq \delta \leq 2^\eta$, for the subsequence $t_j = 2^j \delta$, we have

$$|1 - m_{t_j}(\omega)| \leq \beta \frac{\ell(\omega)^\alpha}{2^j \delta} \leq \beta \frac{\ell(\omega)^\alpha}{2^j} \quad \text{and} \quad |m_{t_j}| \leq 2^\eta \beta \frac{2^j}{\ell(\omega)^\alpha}.$$

By (1), for any $1 < q \leq 2$, we have

$$\sup_{1 \leq \delta \leq 2^\eta} \|\sup_{j \in \mathbb{N}}^+ T_{m_{t_j}} x\|_q \lesssim_{\alpha, \beta, q, \eta} \|x\|_q.$$

Note that $q + \frac{q(2-q)}{q-1+2\eta}$ tends to $1 + \frac{1}{2\eta}$ as $q \rightarrow 1$. Hence, by Proposition 6.9, for any $1 + \frac{1}{2\eta} < p \leq 2$, we get

$$\|\sup_{t \in \mathbb{R}_+}^+ T_{m_t} x\|_p \lesssim_{\alpha, \beta, p, \eta} \|x\|_p.$$

The a.u. convergence is proved similarly as in (1). \square

Chapter 7

Application to Fourier multipliers on compact quantum groups

This chapter will be devoted to the proof of Theorem 0.7. In other words, we will construct Fourier multipliers satisfying the pointwise convergence for groups with suitable approximation properties. As in the previous chapter, only the framework of the form (6.1) is involved in the essential part of our arguments. Since the approximation properties of discrete quantum groups have drawn wide interests in recent years, we would like present the work in a more general setting, that is, Woronowicz's *compact quantum groups*.

We refer to [103, 104, 95] for a complete description of compact quantum groups. In this thesis, it suffices to recall that each compact quantum group \mathbb{G} is an object corresponding to a distinguished von Neumann algebra denoted by $L_\infty(\mathbb{G})$, a unital normal $*$ -homomorphism $\Delta : L_\infty(\mathbb{G}) \rightarrow L_\infty(\mathbb{G}) \overline{\otimes} L_\infty(\mathbb{G})$ (usually called the *comultiplication*), and a normal faithful state $h : L_\infty(\mathbb{G}) \rightarrow \mathbb{C}$ (usually called the *Haar state*) with the following properties. First, the Haar state h is invariant in the sense that

$$(h \otimes \text{id}) \circ \Delta(x) = h(x)1 = (\text{id} \otimes h) \circ \Delta(x), \quad x \in L_\infty(\mathbb{G}).$$

Second, an $n \times n$ matrix $u = (u_{ij})_{i,j=1}^n$ with coefficients $u_{ij} \in L_\infty(\mathbb{G})$ is called an n -dimensional representation of \mathbb{G} if for any $1 \leq i, j \leq n$ we have

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

We denote by $\text{Irr}(\mathbb{G})$ the collection of unitary equivalence classes of irreducible representations of \mathbb{G} , and we fix a representative $u^{(\pi)}$ for each class $\pi \in \text{Irr}(\mathbb{G})$ and denote by d_π its dimension. Then the space

$$\text{Pol}(\mathbb{G}) = \text{span}\{u_{ij}^{(\pi)} : u^{(\pi)} = (u_{ij}^{(\pi)})_{i,j=1}^{d_\pi}, \pi \in \text{Irr}(\mathbb{G})\}$$

is a w^* -dense $*$ -subalgebra of $L_\infty(\mathbb{G})$. Last, for each $\pi \in \text{Irr}(\mathbb{G})$, we set

$$V_\pi = \text{span}\{u_{ij}^{(\pi)} : u^{(\pi)} = (u_{ij}^{(\pi)})_{i,j=1}^{d_\pi}\}.$$

Then for distinct π, π' , the spaces V_π and $V_{\pi'}$ are orthogonal with respect to h . In this thesis, we will be interested in the case that \mathbb{G} is of *Kac type*, in other words, the Haar

state h is tracial. We denote by $L_p(\mathbb{G})$ the noncommutative L_p -spaces $L_p(L_\infty(\mathbb{G}), h)$. In particular we have an orthogonal decomposition

$$L_2(\mathbb{G}) = \ell_2 - \bigoplus_{\pi \in \text{Irr}(\mathbb{G})} V_\pi.$$

Denote by p_π the orthogonal projection from $L_2(\mathbb{G})$ onto V_π .

To be consistent with our framework in Section 6, for each $\pi \in \text{Irr}(\mathbb{G})$ we may formally take an isometric isomorphism $U_\pi : V_\pi \rightarrow \ell_2(I_\pi)$ for some index set I_π and we set $\Omega = \bigcup_{\pi \in \text{Irr}(\mathbb{G})} I_\pi$ and $U = \bigoplus_{\pi} U_\pi$. In fact, this unitary isomorphism can be chosen to be the Fourier transform on $L_2(\mathbb{G})$. Note that $U^*(C_c(\Omega))$ is nothing but the w^* -dense subalgebra $\text{Pol}(\mathbb{G})$. Therefore, this unitary map U satisfies the description at the beginning of Section 6. Each $m : \text{Irr}(\mathbb{G}) \rightarrow \mathbb{C}$ can be naturally viewed as a function on Ω by evaluating $m(\pi)$ at each point in I_π . Then for all $x \in L_2(\mathbb{G})$, the operator T_m in (6.1) is given by

$$T_m x = \sum_{\pi \in \text{Irr}(\mathbb{G})} m(\pi) p_\pi(x).$$

Example 7.1. Let Γ be a discrete group. We may define a comultiplication Δ on the group von Neumann $VN(\Gamma)$ by

$$\Delta(\lambda(g)) = \lambda(g) \otimes \lambda(g), \quad g \in \Gamma.$$

The triple $(VN(\Gamma), \Delta, \tau)$ carries a compact quantum group structure \mathbb{G} satisfying the aforementioned properties, where we take $L_\infty(\mathbb{G}) = VN(\Gamma)$ and $h = \tau$. In this case we usually denote $\mathbb{G} = \widehat{\Gamma}$. The set of unitary equivalence classes of irreducible representations $\text{Irr}(\mathbb{G})$ can be indexed by Γ , so that for every $g \in \Gamma$ the associated representation is of dimension 1 and is given by $u^{(g)} = \lambda(g) \in VN(\Gamma)$. Therefore, the set Ω defined above becomes Γ , and U is the Fourier transform $\mathcal{F} : x \mapsto \widehat{x}$ sending $\lambda(g)$ to δ_g . Hence, for any $m \in \ell_\infty(\Gamma)$, the associated multiplier is

$$T_m : x \mapsto \sum_{g \in \Gamma} m(g) \widehat{x}(g) \lambda(g), \quad x \in \text{Pol}(\widehat{\Gamma}).$$

This comes back to usual Fourier multipliers on groups mentioned in the introduction.

In accordance with the classical notation, we adopt the terminology of the dual discrete quantum group $\widehat{\mathbb{G}}$, which is an object determined by the representation theory of \mathbb{G} (see e.g. [96]). This is only for our notational use and we will not involve the detailed structure of $\widehat{\mathbb{G}}$. We remark that if $\mathbb{G} = \widehat{\Gamma}$ is the quantum group given in Example 7.1 for a discrete group Γ , then the dual $\widehat{\mathbb{G}}$ coincides with Γ . In this language we recall the following approximation properties of quantum groups introduced by De Commer, Freslon and Yamashita in [40]. For simplicity of exposition we always assume in this thesis that $\text{Irr}(\mathbb{G})$ is countable. The general cases can be dealt with by considering the collection of all finitely generated quantum subgroups of $\widehat{\mathbb{G}}$.

Definition 7.2. Let \mathbb{G} be a compact quantum group. $\widehat{\mathbb{G}}$ is said to have the *central almost completely positive approximation property* (central ACPAP for short), if there are two sequences of functions $(\varphi_s)_{s \in \mathbb{N}}, (\psi_k)_{k \in \mathbb{N}}$ from $\text{Irr}(\mathbb{G})$ to \mathbb{C} such that

- (1) For any $s \in \mathbb{N}$, T_{φ_s} is a unital completely positive map on $\text{Pol}(\mathbb{G})$.
- (2) For any $\pi \in \text{Irr}(\mathbb{G})$, $\lim_{s \rightarrow \infty} \varphi_s(\pi) = 1$ and $\lim_{k \rightarrow \infty} \psi_k(\pi) = 1$.

(3) For any $k \in \mathbb{N}$, ψ_k is finitely supported and T_{ψ_k} is completely bounded on $L_\infty(\mathbb{G})$. For any $s \in \mathbb{N}$ and $\varepsilon > 0$, there is a $k = k(s, \varepsilon)$ such that $\|T_{\psi_k} - T_{\varphi_s}\|_{cb} \leq \varepsilon$.

Moreover, if for any $k \in \mathbb{N}$ we may choose T_{ψ_k} to be unital completely positive, then $\widehat{\mathbb{G}}$ is said to be *centrally amenable* (or to have the *central completely positive approximation property*).

Remark 7.3. In the above definition we may always choose T_{ψ_k} to be self-adjoint. In fact, take $\tilde{\psi}_k(\pi) = \psi_k(\bar{\pi})$, then $T_{\tilde{\psi}_k}$ has the same cb-norm as T_{ψ_k} since $T_{\tilde{\psi}_k} = T_{\psi_k} \circ \kappa$, where $\kappa : u_{ij}^{(\pi)} \mapsto (u_{ji}^{(\pi)})^*$ is well-known to be a complete isometry on $L_\infty(\mathbb{G})$. Obviously the map $T_{(\psi_k + \tilde{\psi}_k)/2}$ is self-adjoint.

Lemma 7.4. *Let $(\varphi_s)_{s \geq 0}$ be a sequence of functions on a set Λ . Let $(E_s)_{s \geq 1}$ be any sequence of increasing finite subsets of Λ . If $\varphi_n \rightarrow 1$ pointwise, then there is a subsequence $(s_N)_{N \geq 1}$ of \mathbb{N} , such that*

$$|1 - \varphi_{s_{N+1}}(\rho)| \leq 2^{J(\rho)-N}, \quad \rho \in E_{s_N}$$

where $J(\rho) = \min\{j \in \mathbb{N} : \rho \in E_{s_j}\}$.

Proof. We will construct a sequence $(s_N)_{N \in \mathbb{N}}$ by induction so that

$$|1 - \varphi_{s_{N+1}}(\rho)| < 2^{J_N(\rho)-N}, \quad \rho \in E_{s_N}.$$

where $J_N : E_{s_N} \rightarrow \mathbb{N}$ is defined by $J_N(\rho) = \min\{j \leq N : \rho \in E_{s_j}\}$. First we let $s_0 = 1$. Assume that $(s_j)_{j=0}^N$ has already been defined. Since $(E_s)_s$ is increasing, we can define $V_j = E_{s_j} \setminus E_{s_{j-1}}$. Then for any $\rho \in V_j$, we can find an $s_{N+1}(\rho) > s_N$ large enough, such that any $s \geq s_{N+1}(\rho)$,

$$|1 - \varphi_s(\rho)| < 2^{j-N}.$$

Since E_{s_N} is a finite set, we can set $s_{N+1} = \max\{s_{N+1}(\rho) : \rho \in E_{s_N}\}$. Note that, $J_N(\rho) = k$ if and only if $\rho \in V_j \subset E_{s_N}$. Thus inductively we obtain a sequence s_N such that

$$|1 - \varphi_{s_{N+1}}(\rho)| < 2^{J_N(\rho)-N}, \quad \rho \in E_{s_N}.$$

E_{s_N} is increasing with respect to N , so $J_N(\rho) = J(\rho)$. Therefore we complete the proof. \square

From this subsequence, we can construct a semigroup of unital completely positive maps. The construction is based on the following well-known fact.

Lemma 7.5 ([24, Lemma 6.4]). *Let H be a Hilbert space, $\mathcal{C} \subset H$ be a closed convex set and $L \in B(H)$ be self-adjoint contractions such that $L(\mathcal{C}) \subset \mathcal{C}$. Put $A = \text{id}_H - L$ and let $S_t = e^{-tA}$, $t \geq 0$. Then we have $S_t(\mathcal{C}) \subset \mathcal{C}$.*

Then we may construct the semigroups and multipliers with the desired property.

Theorem 7.6. *Assume $\widehat{\mathbb{G}}$ has the central ACPAP and let $(\psi_k)_{k \in \mathbb{N}}$ be the sequence of finitely supported functions on $\text{Irr}(\mathbb{G})$ satisfying the conditions of Definition 7.2. Then there are increasing subsequence $(k_s)_{s \in \mathbb{N}}$, $(s_N)_{N \in \mathbb{N}}$ of \mathbb{N} and a function $\ell : \text{Irr}(\mathbb{G}) \rightarrow \mathbb{R}_+$ satisfying the following conditions:*

- (1) $S_t : u_{ij}^{(\pi)} \mapsto e^{-t\ell(\pi)} u_{ij}^{(\pi)}$ is a semigroup of unital completely positive trace preserving symmetric maps on $L_\infty(\mathbb{G})$;
 (2) Set $E_s = \cup_{i=0}^s \text{supp } \psi_i$, then

$$\ell(\pi) \asymp \sqrt{2}^{J(\pi)}$$

where $J(\pi) = \min\{j \in \mathbb{N} : \pi \in E_{s_j}\}$.

- (3) Denote $m_N = \psi_{k_{s_N}}$. Then $(m_N)_{N \in \mathbb{N}}$ satisfies

$$|1 - m_N(\pi)| \lesssim \frac{\ell(\pi)^2}{2^N}, \quad |m_N(\pi)| \lesssim \frac{2^N}{\ell(\pi)^2}, \quad \pi \in \text{Irr}(\mathbb{G}).$$

In particular, for any $2 \leq p \leq \infty$,

$$\|\sup_{N \in \mathbb{N}}^+ T_{m_N} x\|_p \lesssim_p \|x\|_p, \quad x \in L_p(\mathbb{G}).$$

For all $x \in L_p(\mathbb{G})$ with $2 \leq p < \infty$, $T_{m_N}(x)$ converges a.u. to x as $N \rightarrow \infty$.

Moreover, if $\widehat{\mathbb{G}}$ is centrally amenable and the maps T_{ψ_k} are completely positive, then the above results hold for any $1 < p < \infty$.

Proof. Let $(\varphi_s)_s$ be the sequence of functions on $\text{Irr}(\mathbb{G})$ satisfying Definition 7.2. Take $k_s \in \mathbb{N}$ such that

$$\|T_{\varphi_s} - T_{\psi_{k_s}}\|_{cb} \leq \frac{1}{2^{s+1}}.$$

Set $E_s = \cup_{i=0}^s \text{supp } \psi_i \subset \text{Irr}(\mathbb{G})$. Let $(s_N)_N$ be the subsequence defined in Lemma 7.4 so that

$$|1 - \varphi_{s_{N+1}}(\rho)| \leq 2^{J(\rho)-N}, \quad \rho \in E_{s_N}.$$

Define

$$\ell(\pi) = \sum_{j \geq 0} \sqrt{2}^j |1 - \varphi_{s_j}(\pi)|.$$

By the definition of J , we have $\pi \in E_{s_{J(\pi)}} \subset E_{s_{j-1}}$ if $j \geq J(\pi) + 1$. Note that $|\varphi_s(\pi)| \leq 1$ and $|\varphi_s(\pi) - \psi_{k_s}(\pi)| \leq \frac{1}{2^s}$. Therefore,

$$\begin{aligned} \ell(\pi) &= \sum_{j=0}^{J(\pi)} \sqrt{2}^j |1 - \varphi_{s_j}(\pi)| + \sum_{j \geq J(\pi)+1} \sqrt{2}^j |1 - \varphi_{s_j}(\pi)| \\ &= \sum_{j=0}^{J(\pi)} 2\sqrt{2}^j + \sum_{j \geq J(\pi)+1} \sqrt{2}^j \cdot 2^{J(\pi)-j} \lesssim \sqrt{2}^{J(\pi)}. \end{aligned}$$

On the other hand, $\pi \notin E_{s_j}$ if $j \leq J(\pi) - 1$. So

$$\begin{aligned} \ell(\pi) &= \sum_{j=0}^{J(\pi)-1} \sqrt{2}^j (|1 - \psi_{s_j}(\pi) + \psi_{s_j}(\pi) - \varphi_{s_j}(\pi)|) + \sum_{j \geq J(\pi)} \sqrt{2}^j |1 - \varphi_{s_j}(\pi)| \\ &\geq \sum_{j=0}^{J(\pi)-1} \sqrt{2}^j \left(1 - \frac{1}{2^{s_j+1}}\right) \gtrsim \sqrt{2}^{J(\pi)}. \end{aligned}$$

Hence ℓ is well defined and the condition (2) is verified.

Now let us show the complete positivity of S_t . For any $n \in \mathbb{N}_+$, let $M_n(\mathbb{C})$ be the space of $n \times n$ matrices. Denote by id_n the identity operator on $M_n(\mathbb{C})$. Note that $\mathcal{C} := L_2(L_\infty(\mathbb{G}) \otimes M_n(\mathbb{C}))_+$ is a closed convex set in $L_2(L_\infty(\mathbb{G}) \otimes M_n(\mathbb{C}))$. Since $T_{\varphi_{s_j}}$ is unital completely positive for each $j \geq 0$, we have $(T_{\varphi_{s_j}} \otimes \text{id}_n)(\mathcal{C}) \subset \mathcal{C}$. Set

$$A_j = 1 - T_{\varphi_{s_j}} \quad \text{and} \quad S_{n,t,j} = e^{-t(A_j \otimes \text{id}_n)} = e^{-tA_j} \otimes \text{id}_n, \quad t \geq 0.$$

By Lemma 7.5, $S_{n,t,j}(\mathcal{C}) \subset \mathcal{C}$. On the other hand, $e^{-t\ell(\pi)} = \prod_{j \geq 0} e^{-t\sqrt{2}^j(1-\varphi_{s_j}(\pi))}$, so for each $x \in \mathcal{C}$,

$$(S_t \otimes \text{id}_n)(x) = \lim_{j \rightarrow \infty} S_{n,\sqrt{2}^0 t,0} \circ S_{n,\sqrt{2}^1 t,1} \circ \cdots \circ S_{n,\sqrt{2}^j t,j}(x).$$

Note that \mathcal{C} is closed. Thus $(S_t \otimes \text{id}_n)(x) \in \mathcal{C}$. Therefore S_t is completely positive.

Denote by $\mathbf{1} \in \text{Irr}(\mathbb{G})$ the trivial representation. Recall that for any N , $T_{\varphi_{s_N}}$ is unital and in particular $\varphi_{s_N}(\mathbf{1}) = 1$. As a consequence we get $\ell(\mathbf{1}) = 0$. Therefore $S_t(\mathbf{1}) = e^{-t\ell(\mathbf{1})}\mathbf{1} = 1$ and

$$h(S_t(x)) = \langle S_t(x), \mathbf{1} \rangle_{L_2(\mathbb{G})} = \sum_{\pi \in \text{Irr}(\mathbb{G})} \langle e^{-t\ell(\pi)} p_\pi(x), \mathbf{1} \rangle_{L_2(\mathbb{G})} = \langle p_{\mathbf{1}}(x), \mathbf{1} \rangle_{L_2(\mathbb{G})} = h(x),$$

which means that S_t is h -preserving. Also, S_t is symmetric since for all $x, y \in L_2(\mathbb{G})$,

$$h(S_t(x)y) = \langle e^{-t\ell(\pi)} U(x)(\pi), U(y)(\pi) \rangle_{\ell_2(I_\pi)} = h(yS_t(x)).$$

We take $m_N = \psi_{\kappa_{s_N}}$. Note that $s_N \geq N$, so we can easily check that

$$|1 - m_N(\pi)| \leq (2^{J(\pi)-N} + 2^{-s_N-1}) \leq 2 \cdot 2^{J(\pi)-N} \lesssim \frac{\ell(\pi)^2}{2^N}, \quad \pi \in \text{Irr}(\mathbb{G}),$$

and

$$|m_N(\pi)| \leq \mathbb{1}_{[0,N]}(J(\pi)) \leq 2^{N-J(\pi)} \lesssim \frac{2^N}{\ell(\pi)^2}, \quad \pi \in \text{Irr}(\mathbb{G}).$$

Note also that the maps T_{m_N} are h -preserving and extend to positive contractions on $L_p(\mathbb{G})$ for all $1 \leq p \leq \infty$ if they are unital completely positive on $L_\infty(\mathbb{G})$ (see e.g. [69, Lemma 1.1]). Applying Theorem 6.2 and Theorem 6.3, we obtain the desired maximal inequalities and a.u. convergences. \square

In particular, we obtain Theorem 0.8 in the general setting of quantum groups.

Corollary 7.7. (1) *Assume that $\widehat{\mathbb{G}}$ has the central ACPAP. Then $\widehat{\mathbb{G}}$ admits a sequence of completely contractive Fourier multipliers $(T_{m_N})_{N \in \mathbb{N}}$ on $L_\infty(\mathbb{G})$ so that m_N are finitely supported and for any $2 \leq p \leq \infty$,*

$$\|\sup_{N \in \mathbb{N}}^+ T_{m_N} x\|_p \lesssim_p \|x\|_p, \quad x \in L_p(\mathbb{G}),$$

and $T_{m_N} x$ converges to x a.u. as $N \rightarrow \infty$ for all $x \in L_p(\mathbb{G})$ with $2 \leq p \leq \infty$.

(2) Assume that $\widehat{\mathbb{G}}$ is centrally amenable. Then $\widehat{\mathbb{G}}$ admits a sequence of unital completely positive Fourier multipliers $(T_{m_N})_{N \in \mathbb{N}}$ on $L_\infty(\mathbb{G})$ so that m_N are finitely supported and for any $1 < p \leq \infty$

$$\|\sup_{N \in \mathbb{N}}^+ T_{m_N} x\|_p \lesssim_p \|x\|_p, \quad x \in L_p(\mathbb{G}),$$

and $T_{m_N} x$ converges to x a.u. as $N \rightarrow \infty$ for all $f \in L_p(\mathbb{G})$ with $1 < p \leq \infty$.

Moreover, we have the following a.u. convergence of Fourier series of Dirichlet type on $L_2(\mathbb{G})$.

Proposition 7.8. *Let $\widehat{\mathbb{G}}$ be a discrete quantum group having the central ACPAP. Then there exists a sequence of increasing subsets $(K_N)_N \subset \text{Irr}(\mathbb{G})$ such that the maps $x \mapsto \sum_{\pi \in K_N} p_\pi(x)$ is of strong type $(2, 2)$. Moreover, for all $x \in L_2(\mathbb{G})$,*

$$\sum_{\pi \in K_N} p_\pi(x) \rightarrow x \text{ a.u. as } N \rightarrow \infty.$$

Proof. Let ℓ , $(k_s)_s$, $(s_N)_N$, $(E_s)_s$ be given by Theorem 7.6. We show that $K_N = E_{s_N}$ satisfies this proposition. Let $D_N = T_{\mathbb{1}_{K_N}}$ be the multiplier associated to the characteristic function $\mathbb{1}_{K_N}$. Define

$$\phi_N(\pi) = \mathbb{1}_{K_N}(\pi) - e^{-\frac{\ell(\pi)}{\sqrt{2}^N}}.$$

Let $N \geq 0$. For any $\pi \in K_N$, i.e. $J(\pi) \leq N$, we have

$$|\phi_N(\pi)| \lesssim \frac{\ell(\pi)}{\sqrt{2}^N} \lesssim \frac{\ell(\pi)2^N}{(2^N + \ell(\pi))^2},$$

where the last inequality follows from the fact that $\frac{\ell(\pi)}{\sqrt{2}^N} \lesssim 1$. Also, for any $\pi \notin K_N$, i.e. $J(\pi) > N$, we have

$$|\phi_N(\pi)| \lesssim \frac{\sqrt{2}^N}{\ell(\pi)} \lesssim \frac{\ell(\pi)2^N}{(2^N + \ell(\pi))^2}.$$

Therefore, applying Proposition 6.7, Proposition 4.11 and Proposition 6.5, we get

$$\|(D_N(x))_N\|_{L_2(\mathbb{G}; \ell_\infty)} \lesssim \|x\|_2, \quad \text{and} \quad \|(D_N(x))_N\|_{L_2(\mathbb{G}; \ell_\infty)} \lesssim \|x\|_2, \quad x \in L_2(\mathbb{G}).$$

Recall that by the choice of $(E_s)_s$, for any finite subset $F \subset \text{Irr}(\mathbb{G})$ there exists $M \geq 1$ with $F \subset K_N$ for all $N \geq M$. Hence for any $x \in \text{Pol}(\mathbb{G})$, there is an index M large enough such that for any $N \geq M$, we have $D_N(x) = x$. In particular

$$\sum_{N \geq M} \|D_N(x) - x\|_2^2 = 0.$$

By Proposition 4.7, we see that $D_N x$ converges a.u. to x as $N \rightarrow \infty$. □

Chapter 8

More concrete examples

8.1 Generalized Fejér means on non-abelian groups and quantum groups

8.1.1 Case of nilpotent groups and amenable groups

Let Γ be a discrete amenable group. Let $(K_N)_{N \in \mathbb{N}}$ be a Følner sequence of Γ , that is

$$\lim_{N \rightarrow \infty} \frac{|K_N \cap gK_N|}{|K_N|} = 0, \quad g \in \Gamma.$$

For convenience we set $K_0 = \{e\}$. We define a sequence of multipliers $(m_N)_{N \in \mathbb{N}}$ by

$$m_N(g) = \frac{|K_N \cap gK_N|}{|K_N|}. \quad (8.1)$$

We remark that if $\Gamma = \mathbb{Z}^d$ and $K_N = [-N, N]^d \cap \mathbb{Z}^d$, then

$$m_N(k_1, k_2, \dots, k_d) = \prod_{i=1}^d \left(1 - \frac{k_i}{2N+1}\right) \mathbb{1}_{[-2N, 2N]}(k_i),$$

which gives the usual Fejér means on d -dimensional tori. As a result, we regard the multipliers $(T_{m_N})_N$ as a noncommutative analogue of Fejér means.

It is easy to see that m_N is finitely supported, indeed $\text{supp } m_N = K_N^2$. By the Følner condition, we have $m_N \rightarrow 1$ pointwise. For any $g \in \Gamma$, we have

$$m_N(g) = \left\langle \lambda(g) \frac{\mathbb{1}_{K_N}}{|K_N|^{1/2}}, \frac{\mathbb{1}_{K_N}}{|K_N|^{1/2}} \right\rangle_{\ell_2(\Gamma)}.$$

Then it is a well-known argument to see that m_N is positive definite and therefore T_{m_N} is unital completely positive on $VN(\Gamma)$ for any $N \in \mathbb{N}$ (see e.g. [22, Theorem 2.5.11]). Note that T_{m_N} is also τ -preserving. In particular, by Theorem 7.6, there exists a subsequence $(N_j)_{j \in \mathbb{N}}$, such that for all $1 < p \leq \infty$ and all $x \in L_p(VN(\Gamma))$,

$$\left\| \sup_{j \in \mathbb{N}}^+ T_{m_{N_j}} x \right\|_p \lesssim_p \|x\|_p \quad \text{and} \quad T_{m_{N_j}} x \rightarrow x \text{ a.u. as } j \rightarrow \infty$$

In the following, we would like to give a refined study in the case of nilpotent groups. First we consider a 2-step (or 1-step) nilpotent group Γ generated by a finite set S . We assume that $e \in S$ and $S = S^{-1}$. Due to [93], we have the following estimates:

$$\beta^{-1}N^d \leq |S^N| \leq \beta N^d \quad \text{and} \quad \beta^{-1}N^{d-1} \leq |S^N \setminus S^{N-1}| \leq \beta N^{d-1}, \quad (8.2)$$

where $d \geq 1$ is called the *degree* of Γ , $\beta < \infty$ is a positive constant depending only on Γ and S .

For an element $g \in \Gamma$, denote by $|g|$ the word length of g with respect to the generator set S , i.e. $|g| = \min\{k : g \in S^k\}$. Let $(m_N)_N$ be a sequence of symbols given by (8.1) with $K_N = S^N$. By the inequalities (8.2), we have

$$1 - m_N(g) = \frac{|gS^N \setminus S^N|}{|S^N|} \leq \frac{|S^{N+|g|} \setminus S^N|}{|S^N|} \leq \beta \frac{\sum_{i=N+1}^{N+|g|} i^{d-1}}{N^d} \leq \beta \frac{|g|}{N}, \quad g \in \Gamma. \quad (8.3)$$

In particular, this shows that $(S^N)_N$ is a Følner sequence. On the other hand,

$$|m_N(g)| \leq \mathbf{1}_{[0,2N]}(|g|) \leq 2 \frac{N}{|g|}, \quad g \in \Gamma. \quad (8.4)$$

Set $J(g) = \min\{j \in \mathbb{N} : g \in S^{2^{j+1}}\}$, i.e. $J(g)$ is the unique integer such that $2^{J(g)} < |g| \leq 2^{J(g)+1}$. We have

$$|1 - m_{2^j}(g)| \leq \beta \frac{|g|}{2^j} \leq 2\beta 2^{J(g)-j}.$$

This shows that the subsequence $(2^j)_{j \in \mathbb{N}}$ satisfies the Theorem 7.4. Define

$$\ell(g) = \sum_{j \geq 0} \sqrt{2^j} |1 - m_{2^j}(g)|.$$

By Lemma 7.6,

$$\ell(g) \asymp \sqrt{2^{J(g)}} \asymp \sqrt{|g|},$$

and $(S_t)_{t \geq 0} : \lambda(g) \mapsto e^{-t\ell(g)} \lambda(g)$ is a semigroup of unital completely positive trace preserving and symmetric maps. Moreover, for any $t \geq 0$, S_t satisfies Rota's dialtion property according to Lemma 5.2. The inequalities (8.3) and (8.4) can be written as

$$|1 - m_N(g)| \lesssim \beta \frac{\ell(g)^2}{N} \quad \text{and} \quad |m_N(g)| \lesssim \beta \frac{N}{\ell(g)^2}.$$

Moreover,

$$m_N(g) = \varphi_N * \varphi_N(g).$$

by (8.2), we obtain

$$\begin{aligned} & m_{N+1}(g) - m_N(g) \\ &= \frac{\int_{\Gamma} \mathbf{1}_{S^{N+1}}(\gamma) \mathbf{1}_{S^{N+1}}(g^{-1}\gamma) d\gamma}{|S^{N+1}|} - \frac{\int_{\Gamma} \mathbf{1}_{S^N}(\gamma) \mathbf{1}_{S^N}(g^{-1}\gamma) d\gamma}{|S^N|} \\ &\leq \frac{\int_{\Gamma} \mathbf{1}_{S^N}(\gamma) [\mathbf{1}_{S^{N+1}}(g^{-1}\gamma) - \mathbf{1}_{S^N}(g^{-1}\gamma)] d\gamma}{|S^N|} + \frac{\int_{\Gamma} [\mathbf{1}_{S^{N+1}}(\gamma) - \mathbf{1}_{S^N}(\gamma)] \mathbf{1}_{S^{N+1}}(g^{-1}\gamma) d\gamma}{|S^N|} \\ &\leq 2 \frac{|S^{N+1} \setminus S^N|}{|S^N|} \lesssim \beta^2 \frac{1}{N+1}. \end{aligned}$$

Therefore m_N satisfies the assumption **(A2)**. Applying Theorem 6.3, we have the following corollary.

Corollary 8.1. *Let Γ be a 2-step (or 1-step) nilpotent group generated by a finite symmetric set of elements S . Define $m_N(g) = \frac{|S^N \cap gS^N|}{|S^N|}$. Then*

(1) $(T_{m_{2^j}})_{j \in \mathbb{N}}$ is of strong type (p, p) for all $1 < p \leq \infty$. Moreover, for any $x \in L_p(VN(\Gamma))$ with $1 < p < \infty$, $T_{m_{2^j}}(x)$ converges a.u. to x as $j \rightarrow \infty$.

(2) $(T_{m_N})_{N \in \mathbb{N}}$ is of strong type (p, p) for all $\frac{3}{2} < p \leq \infty$. Moreover, for any $x \in L_p(VN(\Gamma))$ with $3/2 < p < \infty$, $T_{m_N}(x)$ converges a.u. to x as $N \rightarrow \infty$.

Let us give some remarks on the case of general groups with polynomial growth. Indeed, it is conjectured in [20] that (8.2) remains true for general groups with polynomial growth. If the conjecture has a positive answer, then the above corollary still holds in this general setting by the same arguments. Moreover a partial result was given in [21] about general r -step nilpotent group Γ generated by a finite set S . It asserts that

$$\beta^{-1}N^{d-1} \leq |S^N \setminus S^{N-1}| \leq \beta N^{d-\frac{2}{3r}}, \quad (8.5)$$

where β is a constant depending only on Γ and S . Therefore, as the arguments for 2-step case, we have

$$1 - m_N(g) \leq \frac{|S^{N+|g|} \setminus S^N|}{|S^N|} \lesssim \beta \frac{|g|}{N^{\frac{2}{3r}}}.$$

Let $k = k(r)$ be the minimum integer with $k \geq \frac{3r}{2}$. Then $N_j = 2^{k+k^2+\dots+k^j}$ satisfies Lemma 7.4. Indeed, let $J(g) = \min\{j \in \mathbb{N} : g \in E_{N_j}\}$, i.e. $2N_{J(g)-1} < |g| \leq 2N_{J(g)}$. Then for any $g \in E_{N_j}$, i.e. $J(g) \leq j$.

$$|1 - m_{N_{j+1}}(g)| \lesssim \beta \frac{|g|}{N_{j+1}^{\frac{2}{3r}}} \lesssim \beta \frac{2^{k+k^2+\dots+k^{J(g)}}}{2^{\frac{2}{3r}(k+k^2+\dots+k^{j+1})}} \lesssim \beta 2^{-(k^{J(g)+1}+\dots+k^{j+1})} \lesssim \beta 2^{J(g)-j}.$$

Therefore, by Theorem 7.6, we get a function ℓ such that

$$|1 - m_{N_{j+1}}(g)| \lesssim \beta \frac{\ell(g)^2}{2^j} \quad \text{and} \quad |m_{N_{j+1}}(g)| \lesssim \beta \frac{2^j}{\ell(g)^2}.$$

By Theorem 6.3 (1), we have

Corollary 8.2. *Let Γ be a r -step nilpotent group generated by a finite symmetric set of elements S . Let m_N , k and N_j be defined above. Then for all $1 < p \leq \infty$, $(T_{m_{N_j}})_{j \geq 0}$ is of strong type (p, p) and for any $x \in L_p(VN(\Gamma))$ with $1 < p < \infty$, $T_{m_{N_j}}(x)$ converges a.u. to x as $j \rightarrow \infty$.*

8.1.2 Case of Euclidean spaces

Let B be a symmetric convex body in \mathbb{R}^d such that the interior contains 0. We define the functions φ_t associated to B as

$$\varphi_t(\xi) = \frac{\mu(B_t \cap (\xi + B_t))}{\mu(B_t)},$$

where $B_t = \{\xi \in \mathbb{R}^d : \xi/t \in B\}$ and μ is the Lebesgue measure. By similar arguments as the case of nilpotent groups, we have the following corollary.

Corollary 8.3. *Let B and $(\varphi_t)_t$ be as above.*

(1) $(T_{\varphi_{2^j}})_{j \geq 0}$ is of strong type (p, p) , for any $1 < p \leq \infty$ and for any $f \in L_p(\mathbb{R})$ with $1 < p < \infty$, $T_{\varphi_{2^j}}(f)$ converges a.e. to f as $j \rightarrow \infty$.

(2) $(T_{\varphi_t})_{t \geq 1}$ is of strong type (p, p) , for all $\frac{3}{2} < p \leq \infty$, and for any $f \in L_p(\mathbb{R})$ with $3/2 < p < \infty$, $T_{\varphi_t}(f)$ converges a.e. to f as $t \rightarrow \infty$.

Proof. For any non-zero vector $\xi \in \mathbb{R}$, there is a unique positive number $|\xi|_B$ such that $\xi/|\xi|_B \in \partial B$ since B is convex and its interior contains 0. We make the convention that $|0|_B = 0$. Note that $\mu(B_t) = t^d \mu(B)$. We have

$$1 - \varphi_t(\xi) \leq \frac{\mu(B_{t+|\xi|_B} \setminus B_t)}{\mu(B_t)} \leq \frac{(t + |\xi|_B)^d - t^d}{t^d} \lesssim_d \frac{|\xi|_B}{t}.$$

Also, we have

$$\frac{\partial \varphi_t}{\partial t} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\mathbb{1}_{B_{t+h}} * \mathbb{1}_{B_{t+h}}(\xi)}{\mu(B_{t+h})} - \frac{\mathbb{1}_{B_t} * \mathbb{1}_{B_t}(\xi)}{\mu(B_t)} \right) \lesssim \lim_{h \rightarrow 0} \frac{1}{h} \frac{\mu(B_{t+h}) - \mu(B_t)}{\mu(B_t)} \lesssim_d \frac{1}{t}.$$

Then the remaining arguments are similar as the case of nilpotent groups. We skip it here. \square

Remark 8.4. Let \mathcal{N} be a semifinite von Neumann algebra equipped with a normal semifinite trace ν and $L_p(\mathbb{R}^d; L_p(\mathcal{N}))$ be the Bochner $L_p(\mathcal{N})$ -valued L_p -spaces. We may consider the sequence of multipliers $T_{\varphi_t} \otimes \text{id}_{L_p(\mathcal{N})}$ on $L_p(\mathbb{R}^d; L_p(\mathcal{N}))$ with symbols φ_t . Note that we may view $L_p(\mathbb{R}^d; L_p(\mathcal{N}))$ as a noncommutative L_p -space associated to the von Neumann algebra $L_\infty(\mathbb{R}^d) \overline{\otimes} \mathcal{N}$: for any $1 \leq p < \infty$,

$$L_p(L_\infty(\mathbb{R}^d) \overline{\otimes} \mathcal{N}, \int \otimes \nu) \cong L_p(\mathbb{R}^d; L_p(\mathcal{N})).$$

So we may consider the noncommutative maximal inequalities and the a.u. convergence for the maps $T_{\varphi_t} \otimes \text{id}_{L_p(\mathcal{N})}$. These properties indeed follow from Remark 6.4, and the above corollary still holds for $(T_{\varphi_t} \otimes \text{id}_{L_p(\mathcal{N})})_t$ in this noncommutative setting.

8.1.3 Case of centrally amenable discrete quantum groups

Let \mathbb{G} be a compact quantum group of Kac type. As before, we assume that $\text{Irr}(\mathbb{G})$ is countable for convenience. We keep the notation introduced in Chapter 7. We may study the Følner sequences and the corresponding multipliers in this quantum setting as follows.

Recall that for any $\alpha, \beta \in \text{Irr}(\mathbb{G})$ we have the decomposition

$$u^{(\alpha)} \boxtimes u^{(\beta)} = \bigoplus_{\gamma \in \text{Irr}(\mathbb{G})} N_{\alpha\beta}^\gamma u^{(\gamma)},$$

where $u^{(\alpha)} \boxtimes u^{(\beta)}$ refers to the tensor product representation of the form $(u_{ij}^{(\alpha)} u_{kl}^{(\beta)})_{i,j,k,l}$ and $N_{\alpha\beta}^\gamma \in \mathbb{N}$. We have the following Frobenius reciprocity law ([103])

$$N_{\alpha\beta}^\gamma = N_{\gamma\beta}^\alpha = N_{\alpha\gamma}^\beta \quad (8.6)$$

for all $\alpha, \beta, \gamma \in \text{Irr}(\mathbb{G})$. We write $\gamma \subset \alpha\beta$ if $N_{\alpha\beta}^\gamma > 0$. Denote by $\bar{\pi}$ the equivalent class of the representation $\overline{u^{(\pi)}} = ((u_{ij}^{(\pi)})^*)_{ij}$. The *weighted cardinality* of a finite subset $F \subset \text{Irr}(\mathbb{G})$ is defined to be

$$|F|_w = \sum_{\alpha \in F} d_\alpha^2,$$

where we recall that d_α denotes the dimension of the representation α . On the other hand, for a finite subset $F \subset \text{Irr}(\mathbb{G})$ and a representation $\pi \in \text{Irr}(\mathbb{G})$, the *boundary of F related to π* is defined by

$$\partial_S F = \{\alpha \in F : \exists \beta \in F^c \text{ such that } \beta \subset \alpha\pi\} \cup \{\beta \in F^c : \exists \alpha \in F \text{ such that } \alpha \subset \beta\pi\}.$$

Kyed ([71]) proved that there exists a sequence of finite subsets $(K_n)_{n \in \mathbb{N}} \subset \text{Irr}(\mathbb{G})$ such that for any $\pi \in \text{Irr}(\mathbb{G})$,

$$\frac{|\partial_\pi K_n|_w}{|K_n|_w} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (8.7)$$

as soon as \mathbb{G} is coamenable. Note that the coamenability of \mathbb{G} is nothing but a property equivalent to the central amenability of $\hat{\mathbb{G}}$ (see e.g. [19]). The above sequence $(K_n)_{n \in \mathbb{N}}$ is called a *Følner sequence*. We associate a sequence of multipliers

$$\varphi_n(\pi) = \frac{\sum_{\alpha, \beta \in K_n} N_{\alpha\beta}^\pi d_\alpha d_\beta}{d_\pi (\sum_{\xi \in K_n} d_\xi^2)}, \quad \pi \in \text{Irr}(\mathbb{G}). \quad (8.8)$$

It is easy to see that if $\hat{\mathbb{G}} = \Gamma$ for a discrete group Γ , then the above function coincides with the symbol introduced in (8.1).

Lemma 8.5. (1) *The maps T_{φ_n} are unital completely positive on $L_\infty(\mathbb{G})$ for all $n \in \mathbb{N}$.*
(2) *The functions φ_n converges to 1 pointwise.*

Proof. (1) It is obvious that $\varphi_n(\mathbf{1}) = 1$ and therefore T_{φ_n} is unital.

Denote by $\chi(\pi) = \sum_i u_{ii}^\pi \in \text{Pol}(\mathbb{G})$ the *character* of π . We have for any $\alpha, \beta \in \text{Irr}(\mathbb{G})$,

$$h(\chi(\bar{\beta})\chi(\alpha)) = \delta_{\alpha\beta}1, \quad \chi(\alpha)^* = \chi(\bar{\alpha}) \quad \text{and} \quad \chi(\alpha)\chi(\beta) = \sum_{\gamma \in \text{Irr}(\mathbb{G})} N_{\alpha\beta}^\gamma \chi(\gamma).$$

We write

$$\text{Pol}_0(\mathbb{G}) = \text{span}\{\chi(\pi) : \pi \in \text{Irr}(\mathbb{G})\}$$

and let \mathcal{A}_0 be the w^* -closure of $\text{Pol}_0(\mathbb{G})$ in $L_\infty(\mathbb{G})$. Let $\mathbb{E} : L_\infty(\mathbb{G}) \rightarrow \mathcal{A}_0$ be the canonical conditional expectation preserving the Haar state h . Recall that we have assumed that \mathbb{G} is of Kac type. So the conditional expectation can be given by the following explicit formula (see e.g. [99, Lemma 6.3])

$$\mathbb{E}(u_{ij}^{(\pi)}) = \frac{\delta_{ij}}{d_\pi} \chi(\pi), \quad \pi \in \text{Irr}(\mathbb{G}).$$

Set

$$\chi(K_n) = \frac{1}{|K_n|_w^{1/2}} \sum_{\alpha \in K_n} d_\alpha \chi(\alpha) \in \text{Pol}(\mathbb{G})_0.$$

Then, we have

$$T_{\varphi_n}(x) = (h \otimes \text{id}) [(\chi(K_n)^* \otimes 1) \cdot [(\mathbb{E} \otimes \text{id}) \circ \Delta(x)] \cdot ((\chi(K_n) \otimes 1))], \quad x \in L_\infty(\mathbb{G}).$$

Indeed, by linearity and normality, we only need to check the case where $x = u_{ij}^{(\pi)}$. In this case we see that

$$\begin{aligned} & (h \otimes \text{id}) [(\chi(K_n) \otimes 1) \cdot [(\mathbb{E} \otimes \text{id}) \circ \Delta(u_{ij}^{(\pi)})] \cdot ((\chi(K_n)^* \otimes 1))] \\ &= \frac{\sum_{\alpha, \beta \in K_n} d_\alpha d_\beta \cdot h(\chi(\alpha)\chi(\pi)\chi(\bar{\beta}))}{d_\pi |K_n|_w} u_{ij}^{(\pi)} \\ &= \frac{\sum_{\alpha, \beta \in K_n} d_\alpha d_\beta N_{\alpha\pi}^\beta}{d_\pi |K_n|_w} u_{ij}^{(\pi)} \\ &= T_{\varphi_n}(u_{ij}^{(\pi)}). \end{aligned}$$

Since \mathbb{E} and Δ are completely positive, so is T_{φ_n} .

(2) The support of φ_n is given by

$$\Lambda_n = \{\pi \in \text{Irr}(\mathbb{G}) : \exists \alpha, \beta \in K_n, \text{ such that } \pi \subset \bar{\alpha}\beta\}. \quad (8.9)$$

We denote $\Theta_\pi(K_n) = \{\alpha \in K_n : \forall \beta \in K_n^c, N_{\alpha\pi}^\beta = 0\} \subset K_n$. Note that $N_{\alpha\pi}^\beta = N_{\bar{\alpha}\beta}^\pi$ and that $\alpha \in \Theta_\pi(K_n)$ implies $\sum_{\beta \in K_n} N_{\alpha\pi}^\beta d_\beta = d_\alpha d_\pi$ by the choice of $N_{\alpha\pi}^\beta$. Then

$$\varphi_n(\pi) \geq \frac{\sum_{\alpha \in \Theta_\pi(K_n)} d_\alpha \left(\sum_{\beta \in K_n} N_{\alpha\pi}^\beta d_\beta \right)}{d_\pi \left(\sum_{\xi \in K_n} d_\xi^2 \right)} = \frac{\sum_{\alpha \in \Theta_\pi(K_n)} d_\alpha^2}{\sum_{\xi \in K_n} d_\xi^2} = \frac{|\Theta_\pi(K_n)|_w}{|K_n|_w}.$$

Therefore,

$$1 - \varphi_n(\pi) \leq \frac{|\{\alpha \in K_n : \exists \beta \in K_n^c, \text{ such that } \beta \subset \alpha\pi\}|_w}{|K_n|_w} \leq \frac{|\partial_\pi(K_n)|_w}{|K_n|_w}.$$

By the Følner condition (8.7), $\varphi_n \rightarrow 1$ pointwise. \square

Therefore, by Theorem 7.6 we get the following result.

Corollary 8.6. *Assume that $\hat{\mathbb{G}}$ is centrally amenable and let $(K_n)_{n \in \mathbb{N}} \subset \text{Irr}(\mathbb{G})$ be a Følner sequence. Let φ_n be the symbols given by (8.8). Then there is a subsequence $(n_j)_{j \in \mathbb{N}}$ such that $(T_{\varphi_{n_j}})_j$ is of strong type (p, p) for any $1 < p \leq \infty$. Moreover for all $x \in L_p(\mathbb{G})$ with $1 < p < \infty$, $T_{\varphi_{n_j}}(x)$ converges a.u. to x as $j \rightarrow \infty$.*

8.2 Convergence of Fourier series of L_p -functions on compact groups

We would like to emphasize in this section that our work indeed brings new ideas to the analysis on classical compact groups. Recall that for a compact second countable group G , any function $f \in L_p(G)$ admits a Fourier series

$$f(x) \sim \sum_{\pi \in \text{Irr}(G)} d_\pi \text{Tr}(\hat{f}(\pi)\pi(x)), \quad x \in G \quad \text{with } \hat{f}(\pi) = \int_G f(x)\pi(x^{-1})dx$$

where $\text{Irr}(G)$ denotes the collection of equivalence classes of irreducible unitary representations of G , and d_π denotes the dimension of π . The pointwise analysis on these Fourier series is much more subtle than the abelian case. A particular difficulty is the lack of obvious order and summation methods on $\text{Irr}(G)$ which is suitable for the study of Fourier series. However, from the viewpoint of quantum group theory, the set $\text{Irr}(G)$ is nothing but the underlying object of a centrally amenable discrete quantum group. So the difficulty can be overcome by transferring the method on discrete amenable groups and its quantum counterpart. The spirit is also partially inspired by the recent work [59].

More precisely, the compact group G trivially gives rise to a compact quantum group. Indeed, it suffices to take the triple $(L_\infty(G), \Delta_G, f)$ where we define for all $f \in L_\infty(G)$,

$$\Delta_G(f)(g, h) = f(gh), \quad g, h \in G.$$

Then \hat{G} is centrally amenable since $L_\infty(G)$ is a commutative von Neumann algebra (see e.g. [19]). As a result, all arguments in Section 8.1.3 work on G . In particular, there always exists an increasing sequence $(K_n)_n \subset \text{Irr}(G)$ determined by the representation theory of G , such that the following finitely supported symbols

$$\varphi_n(\pi) = \frac{\sum_{\alpha, \beta \in K_n} N_{\bar{\alpha}\beta}^\pi d_\alpha d_\beta}{d_\pi(\sum_{\xi \in K_n} d_\xi^2)}, \quad \pi \in \text{Irr}(G)$$

converge to 1 pointwise, where $N_{\bar{\alpha}\beta}^\pi$ is the unique number in the decomposition of the tensor product representation $\bar{\alpha} \boxtimes \beta = \bigoplus_{\pi \in \text{Irr}(G)} N_{\bar{\alpha}\beta}^\pi \pi$. Moreover

$$T_{\varphi_n} f = \sum_{\pi \in \text{Irr}(G)} d_\pi \varphi_n(\pi) \text{Tr}(\hat{f}(\pi) \pi(x)), \quad x \in G, f \in L_p(G)$$

defines a unital completely positive map on $L_p(G)$. We may choose a subsequence $(n_j)_k$ inductively by the algorithm in Lemma 7.4. Set $m_j = \varphi_{n_j}$. We may rewrite Corollary 7.7 and Proposition 7.8 in this setting as follows.

Corollary 8.7. (1) *Let $1 < p < \infty$. There exists a constant $c > 0$ such that*

$$\| \sup_j |T_{m_j} f| \|_p \leq c \|f\|_p, \quad f \in L_p(G).$$

For all $f \in L_p(G)$,

$$f(g) = \lim_{j \rightarrow \infty} \sum_{\pi \in \text{Irr}(G)} d_\pi m_j(\pi) \text{Tr}(\hat{f}(\pi) \pi(g)), \quad a.e. g \in G.$$

(2) *For all $f \in L_2(G)$,*

$$f(g) = \lim_{j \rightarrow \infty} \sum_{\pi \in K_{n_j}} d_\pi \text{Tr}(\hat{f}(\pi) \pi(g)), \quad a.e. g \in G.$$

As an illustration, we consider a concrete example. Let $N \geq 2$ and denote by $SU(N)$ the $N \times N$ special unitary group. Its irreducible representations of $SU(N)$ can be labeled by $N - 1$ non-negative integers, and we write set-theoretically $\text{Irr}(SU(N)) = \mathbb{N}^{N-1}$. The representation theory of $SU(N)$ can be computed in terms of operations on Young diagrams, which yields the following fact.

Proposition 8.8 (Appendix). *Let $K_n = \{0, 1, 2, \dots, n\}^{N-1} \subset \mathbb{N}^{N-1}$. Set*

$$\varphi_n(\pi) = \frac{\sum_{\alpha, \beta \in K_n} N_{\alpha\beta}^\pi d_\alpha d_\beta}{d_\pi(\sum_{\xi \in K_n} d_\xi^2)}, \quad \pi = (t_1, \dots, t_{N-1}) \in \mathbb{N}^{N-1}.$$

Then

$$1 - \varphi_n(\pi) \lesssim_N \frac{|\pi|}{n+1} \quad \text{and} \quad \varphi_{n+1}(\pi) - \varphi_n(\pi) \lesssim_N \frac{1}{n+1},$$

where $|\pi| = \|\pi\|_{\ell_\infty(N-1)} = \max\{t_i : 1 \leq i \leq N-1\}$.

As the arguments in the section 8.1.1, applying Theorem 7.6 to the subsequence $(\varphi_{2^j})_j$ we obtain a function ℓ on \mathbb{N}^{N-1} such that $(T_{e^{t\ell}})_t$ is a semigroup of unital completely positive trace-preserving and symmetric maps, and

$$\ell(\pi) \asymp \sqrt{2}^{J(\pi)} \asymp \sqrt{|\pi|}$$

where $J(\pi) = \min\{j \in \mathbb{N} : \pi \in \Lambda_{2^j}\}$ (Λ_n is given by (8.9)). Then combined with Theorem 6.3 and the proof of Proposition 7.8, the above proposition yields the following corollary.

Corollary 8.9. *Let $K_n = \{0, 1, 2, \dots, n\}^{N-1} \subset \mathbb{N}^{N-1}$. Define*

$$\varphi_n(\pi) = \frac{\sum_{\alpha, \beta \in K_n} N_{\alpha\beta}^\pi d_\alpha d_\beta}{d_\pi(\sum_{\xi \in K_n} d_\xi^2)}, \quad \pi \in \mathbb{N}^{N-1}.$$

(1) *Let $1 < p < \infty$. There exists a constant $c > 0$ such that*

$$\left\| \sup_j |T_{\varphi_{2^j}} f| \right\|_p \leq c \|f\|_p, \quad f \in L_p(G).$$

For all $f \in L_p(G)$,

$$f(g) = \lim_{j \rightarrow \infty} \sum_{\pi \in \text{Irr}(G)} d_\pi \varphi_{2^j}(\pi) \text{Tr}(\hat{f}(\pi) \pi(g)), \quad \text{a.e. } g \in G.$$

For all $f \in L_2(G)$,

$$f(g) = \lim_{j \rightarrow \infty} \sum_{\pi \in K_{2^j}} d_\pi \text{Tr}(\hat{f}(\pi) \pi(g)), \quad \text{a.e. } g \in G.$$

(2) *Let $3/2 < p < \infty$. There exists a constant $c > 0$ such that*

$$\left\| \sup_n |T_{\varphi_n} f| \right\|_p \leq c \|f\|_p, \quad f \in L_p(G).$$

For all $f \in L_p(G)$,

$$f(g) = \lim_{n \rightarrow \infty} \sum_{\pi \in \text{Irr}(G)} d_\pi \varphi_n(\pi) \text{Tr}(\hat{f}(\pi) \pi(g)), \quad \text{a.e. } g \in G.$$

8.3 Bochner-Riesz means on some hyperbolic groups

In this section we briefly discuss a noncommutative analogue of Bochner-Riesz means in the setting of hyperbolic groups. We refer to [50, 53] for a complete description of hyperbolic groups. We merely remind that all hyperbolic groups are weakly amenable and the completely bounded radial Fourier multipliers have been characterized in [81, 78]. In particular, we denote by $|\cdot|$ the usual word length function on a hyperbolic group Γ , then the Fourier multipliers

$$B_N^\delta x = \sum_{g \in \Gamma: |g| \leq N} \left(1 - \frac{|g|^2}{N^2}\right)^\delta \hat{x}(g) \lambda(g), \quad x \in VN(\Gamma)$$

defines a family of completely bounded maps on $VN(\Gamma)$ with $\sup_N \|B_N^\delta\|_{cb} < \infty$ as soon as $\delta > 1$ (see [78, Example 3.4]). Let

$$b_N^\delta(g) = \left(1 - \frac{|g|^2}{N^2}\right)^\delta \chi_{[0, N]}(|g|)$$

be the corresponding symbols of the maps B_N^δ . It is easy to check that

$$|1 - b_N^\delta(g)| \leq c \frac{|g|}{N}, \quad |b_N^\delta(g)| \leq c \frac{N}{|g|}, \quad |b_{N+1}^\delta(g) - b_N^\delta(g)| \leq c \frac{1}{N}, \quad g \in \Gamma$$

for some constant $c > 0$. We are interested in the case where the word length function $|\cdot|$ is conditionally negative definite. This is the case if Γ is a non-abelian free group or a hyperbolic Coxeter group. Applying Theorem 6.2, we obtain the following corollary.

Corollary 8.10. *Let Γ and B_N^δ be as above. Assume additionally that the word length function $|\cdot|$ is conditionally negative definite. Let $2 \leq p \leq \infty$. Then there exists a constant c such that*

$$\|\sup_{N \in \mathbb{N}}^+ B_N^\delta x\|_p \leq c \|x\|_p, \quad x \in L_p(VN(\Gamma)),$$

and for any $x \in L_p(VN(\Gamma))$, $B_N^\delta(x)$ converges a.u. to x as $N \rightarrow \infty$.

The pointwise convergence of the above multipliers for $1 < p < 2$ seems to be more delicate. However, we can still construct some finitely supported multipliers satisfying the pointwise convergence in this case. For any $n \in \mathbb{N}$, we define a multiplier p_n on $VN(\Gamma)$ by

$$p_n(x) = \sum_{|g|=n} \hat{x}(g) \lambda(g), \quad x \in VN(\Gamma),$$

which is the projection onto the subspace $\text{span}\{\lambda(g) : |g| = n\}$. Ozawa [81] showed that these operators have the following estimate

$$\|p_n : VN(\Gamma) \rightarrow VN(\Gamma)\| \leq \beta(n+1) \tag{8.10}$$

where β is a positive constant. For any $N \in \mathbb{N}$, we set

$$m_N(g) = \mathbf{1}_{[0, N^2]}(|g|) e^{-\frac{|g|}{N}}, \quad g \in \Gamma.$$

The following corollary shows that $(T_{m_N})_N$ is the sequence of multipliers we desired.

Proposition 8.11. *Let Γ be a hyperbolic group whose word length function $|\cdot|$ is conditionally negative definite and $(m_N)_N$ be defined as above. Let $1 < p \leq \infty$. There exists a constant c such that*

$$\|\sup_{N \in \mathbb{N}}^+ T_{m_N} x\|_p \leq c \|x\|_p, \quad x \in L_p(VN(\Gamma)).$$

For any $x \in L_p(\mathcal{M})$ with $2 \leq p < \infty$, $T_{m_N} x$ converges a.u. to x as $N \rightarrow \infty$. For any $x \in L_p(\mathcal{M})$ with $1 < p < 2$, $T_{m_N} x$ converges b.a.u. to x as $N \rightarrow \infty$.

Proof. For any $t \geq 0$, denote $S_t : \lambda(g) \mapsto e^{-t|g|} \lambda(g)$. By assumption $(S_t)_{t \geq 0}$ is a semigroups of unital completely positive trace preserving and symmetric maps. We write

$$e^{-\frac{|g|}{N}} - m_N(g) = \mathbf{1}_{[N^2+1, \infty)}(|g|) e^{-\frac{|g|}{N}} = \sum_{r=N^2+1}^{\infty} e^{-\frac{r}{N}} \mathbf{1}_r(|g|).$$

Hence, by the estimate (8.10), we have

$$\|S_{1/N} - T_{m_N}\| \leq \sum_{r=N^2+1}^{\infty} \|e^{-\frac{r}{N}} p_r\| \lesssim \beta \sum_{r=N^2+1}^{\infty} \frac{N^6}{r^6} (r+1) \lesssim \beta \frac{1}{(N+1)^2}.$$

By duality and interpolation, for any $1 \leq p \leq \infty$, we have

$$\|S_{1/N} - T_{m_N} : L_p(VN(\Gamma)) \rightarrow L_p(VN(\Gamma))\| \lesssim \beta \frac{1}{N^2}.$$

By Proposition 6.5, $S_{1/N}$ is of strong type (p, p) for any $1 < p \leq \infty$. Hence for any self-adjoint element $x \in L_p(\mathcal{M})$,

$$\begin{aligned} \|\sup_{N \in \mathbb{N}}^+ T_{m_N}(x)\|_p &\leq \|\sup_{N \in \mathbb{N}}^+ S_{1/N}(x)\|_p + \|\sup_{N \in \mathbb{N}}^+ (S_{1/N} - T_{m_N})(x)\|_p \\ &\lesssim_{\beta, p} \|x\|_p + \sum_{N \geq 0} \frac{1}{(N+1)^2} \|x\|_p \\ &\lesssim_{\beta, p} \|x\|_p. \end{aligned}$$

Similarly, for any $2 \leq p < \infty$, we have

$$\|(T_{m_N}(x))_N\|_{L_p(\mathcal{M}; \ell_\infty)} \lesssim_{\beta, p} \|x\|_p.$$

Then it is easy to check the a.u. convergence for $p \geq 2$ by Proposition 4.7. The b.a.u. convergence for $1 < p < 2$ follows similarly by a standard adaptation of Proposition 4.7 in the bilateral setting. \square

Remark 8.12. The above arguments in this proposition can be extended in a quite general setting. Indeed, note that only the estimate (8.10) and the conditionally negative definite property of $|\cdot|$ was needed in the proof. Thus it is easy to see that the similar results hold for groups with the rapid decay property with respect to a conditionally negative definite length function.

8.4 Dimension free bounds of noncommutative Hardy-Littlewood maximal operators

Our results in particular applies to the problem of dimension free estimates of Hardy-Littlewood maximal operators. Let B be a symmetric convex body in \mathbb{R}^d . We consider the associated averages

$$\Phi_t(f)(x) = \frac{1}{\mu(B)} \int_B f(x - \frac{y}{t}) dy, \quad f \in L_p(\mathbb{R}^d), t > 0.$$

Bourgain studied the dimension free bounds of the associated maximal operators, see [14, 15, 16]. Based on our previous arguments, we may extend his results for the corresponding maximal operators on noncommutative L_p -spaces $L_p(\mathbb{R}^d; L_p(\mathcal{N}))$. Indeed, let $m_t = t^d \mu(B)^{-1} \widehat{\mathbb{1}_{t^{-1}B}}$ be the Fourier transform of the kernel of the above operator Φ_t . Then we may view Φ_t as the Fourier multiplier so that $\widehat{\Phi_t f} = m_t \hat{f}$. The following estimates of m_t are given in [16]: there exist constants c and c_k independent of d such that for all $k \geq 1$ and all $\xi \in \mathbb{R}^d$,

$$|1 - m_t(\xi)| \leq c \frac{|\xi|}{t}, \quad |m_t(\xi)| \leq c \frac{t}{|\xi|}, \quad \left| \frac{d^k m_t(\xi)}{dt^k} \right| \leq c_k \frac{1}{t^k}.$$

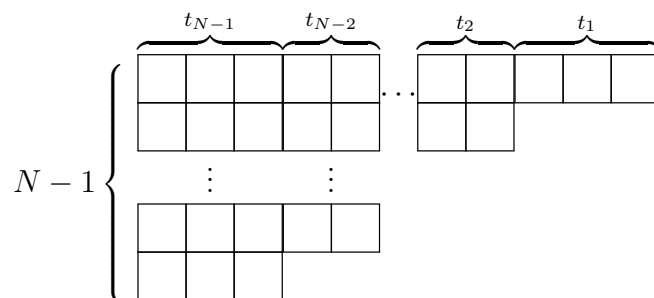
Therefore, applying Theorem 6.3 and Remark 6.4 to the heat semigroup on \mathbb{R}^d , we immediately get the desired result in Corollary 0.9.

Appendix: Følner sequences in the dual of $SU(N)$

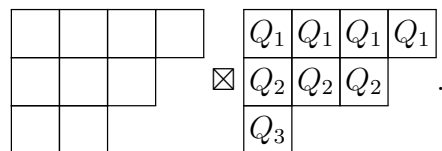
We will prove Proposition 8.8 in this chapter.

Firstly, we recall briefly the representations theory of $SU(N)$. For more details, we refer to [63]. The irreducible representations of $SU(N)$ can be labeled by $N - 1$ non-negative integers, and we write set-theoretically $\text{Irr}(SU(N)) = \mathbb{N}^{N-1}$. And moreover, the decomposition of tensor products into irreducible representations can be described by Young diagrams. Young diagrams consist of boxes: we stick some boxes together so that the number of boxes in each consecutive row (from top to bottom) and each consecutive column (from left to right) does not increase.

Each irreducible representation $u^{(t_1, t_2, \dots, t_{N-1})}$ corresponds to a Young diagrams of the following form:



Let $\alpha, \beta \in \text{Irr}(SU(N))$. The family of irreducible subrepresentations of the tensor product $\alpha \boxtimes \beta$ corresponds to all the Young diagrams which will appear after the following operations. We denote by X_α and X_β the corresponding Young diagrams of α and β . We are going to decompose the tensor product of Young diagrams $X_\alpha \boxtimes X_\beta$:



The prescription goes like this:

(R1) Start by filling the boxes in the top row of X_β with labels ' Q_1 ' and the boxes in the second row with labels ' Q_2 ' and etc (see the above figure).

(R2) Add boxes $\boxed{Q_1} \cdots \boxed{Q_1}, \boxed{Q_2} \cdots \boxed{Q_2}, \boxed{Q_3} \cdots$ etc from X_β to X_α in order according to the following rules:

- (a) We add one box at a time and at each stage the augmented X_α diagram must be a Young diagram which has at most N rows.

- (b) Boxes with the same label, e.g. Q_1 , must not appear in the same column.
- (c) At any given box position, define n_1 to be the number of Q_1 's above and to the right of it. Define n_2 for Q_2 in the similar way, etc. Then we must have $n_1 \geq n_2 \geq n_3$ etc.

(R3) If two diagrams of the same shape are produced by this process they are counted as different only if the labels are differently distributed.

(R4) Cancel columns with N boxes.

For each $\pi \in \text{Irr}(SU(N))$, we denote by X_π its corresponding Young diagram. If in the decomposition of $X_\alpha \boxtimes X_\beta$ there are at most N different operations of the above form yielding X_π , then the multiplicity $N_{\alpha\beta}^\pi$ equals to N .

Moreover, for an irreducible representation $u^{(t_1, t_2, \dots, t_{N-1})}$, the dimension is given by the following formula

$$\begin{aligned}
 d(t_1, t_2, \dots, t_{N-1}) &= (t_1 + 1)(t_2 + 1) \cdots (t_{N-1} + 1) \\
 &\cdot \left(1 + \frac{t_1 + t_2}{2}\right) \left(1 + \frac{t_2 + t_3}{2}\right) \cdots \left(1 + \frac{t_{N-2} + t_{N-1}}{2}\right) \\
 &\cdot \left(1 + \frac{t_1 + t_2 + t_3}{3}\right) \left(1 + \frac{t_2 + t_3 + t_4}{3}\right) \cdots \left(1 + \frac{t_{N-3} + t_{N-2} + t_{N-1}}{3}\right) \\
 &\quad \vdots \\
 &\cdot \left(1 + \frac{t_1 + t_2 + \cdots + t_{N-1}}{N-1}\right).
 \end{aligned} \tag{1}$$

We have the following fact.

Lemma B.1. *Let $N \geq 2$. Set $K_n = \{0, 1, 2, \dots, n\}^{N-1} \subset \text{Irr}(SU(N))$. Then $(K_n)_{n \in \mathbb{N}}$ satisfies the following conditions.*

(1) *We have*

$$(n+1)^{N^2-2} \lesssim_N |E_{n+1}|_w \lesssim_N (n+1)^{N^2-2} \quad \text{where} \quad E_{n+1} = K_{n+1} \setminus K_n.$$

(2) *For any $t, n \in \mathbb{N}$ and any $\alpha \in K_n, \pi \in K_t$, all the irreducible subrepresentations of $\alpha \boxtimes \pi$ are contained in the finite set $K_{n+(N-1)t}$, i.e. $N_{\alpha\pi}^\beta = 0$ if $\beta \notin K_{n+(N-1)t}$.*

(3) *For any $n \in \mathbb{N}, \pi \in \text{Irr}(SU(N))$, we have*

$$\sum_{\alpha \in K_n} \sum_{\beta \in E_{n+1}} N_{\alpha\pi}^\beta d_\alpha d_\beta < \sum_{\alpha \in E_{n+1}} \sum_{\beta \in K_{n+1}^c} N_{\alpha\pi}^\beta d_\alpha d_\beta.$$

Proof. We write $\pi = (t_1, t_2, \dots, t_{N-1}) \in \mathbb{N}^{N-1}$. The conjugate representation is given by $\bar{\pi} = (t_{N-1}, t_{N-2}, \dots, t_1)$

(1) Denote

$$J' := \left\{ \underline{l} = (l_1, l_2, \dots, l_{N-1}) \in \mathbb{N}^{N-1} : \forall 1 \leq j \leq N-1, 1 \leq l_j \leq j(N-j), \sum_{j=0}^{N-1} l_j = \frac{N(N-1)}{2} \right\}.$$

Then we may write

$$d_\pi = \sum_{\underline{l} \in J'} C'_l (t_1 + 1)^{l_1} (t_2 + 1)^{l_2} \cdots (t_{N-1} + 1)^{l_{N-1}},$$

for some constants C'_l depending only on N , and

$$d_\pi^2 = \sum_{\underline{l} \in J} C_l (t_1 + 1)^{l_1} (t_2 + 1)^{l_2} \cdots (t_{N-1} + 1)^{l_{N-1}}$$

with some constants C_l depending only on N and

$$J = \left\{ (l_1, l_2, \dots, l_{N-1}) \in \mathbb{N}^{N-1} : \forall 2 \leq l_j \leq 2j(N-j), \sum_{j=0}^{N-1} l_j = N(N-1) \right\}.$$

Denote

$$E_{n+1}^j = \{(k_1, k_2, \dots, k_{N-1}) \in K_{n+1} : k_j = n+1\}.$$

By definition, $E_{n+1} \subset \cup_{j=1}^{N-1} E_{n+1}^j$. Therefore,

$$\begin{aligned} |E_{n+1}^j|_w &= \sum_{\underline{k} \in E_{n+1}^j} d_{\underline{k}}^2 \\ &= \sum_{\underline{k} \in E_{n+1}^j} \sum_{\underline{l} \in J} C_l (k_1 + 1)^{l_1} (k_2 + 1)^{l_2} \cdots (n+2)^{l_i} \cdots (k_{N-1} + 1)^{l_{N-1}} \\ &\asymp \sum_{\underline{l} \in J} C_l (n+2)^{l_i} (n+2)^{l_1+1} (n+2)^{l_2+1} \cdots (n+2)^{l_{i-1}+1} (n+2)^{l_{i+1}+1} \cdots (n+2)^{l_{N-1}+1} \\ &= \left(\sum_{\underline{l} \in J} C_l \right) (n+2)^{N^2-2} = d_{\underline{0}}^2 (n+2)^{N^2-2}, \end{aligned}$$

where $\underline{0} = (0, 0, \dots, 0)$. Note that $d_{\underline{0}}^2$ is a constant depending only on N . Thus,

$$(n+2)^{N^2-2} \lesssim_N |E_{n+1}^j|_w \leq |E_{n+1}|_w \leq \sum_{j=1}^{N-1} |E_{n+1}^j|_w \lesssim_N (n+2)^{N^2-2}.$$

(2) This follows directly from the rules of decomposition of tensor product of Young diagrams.

(3) Fix $\pi = (t_1, t_2, \dots, t_{N-1}) \in \mathbb{N}^{N-1}$. We will show that for any pair $(\alpha, \beta) \in K_n \times E_{n+1}$ there exists a unique pair $(\xi, \eta) \in E_{n+1} \times K_{n+1}^c$ such that $N_{\alpha\beta}^\pi \leq N_{\xi\eta}^\pi$. This immediately leads to the desired conclusion, since $d_\alpha d_\beta < d_\xi d_\eta$.

We consider

$$\alpha = (a_1, a_2, \dots, a_{N-1}) \in K_n, \quad \beta = (b_1, \dots, b_{k-1}, n+1, b_{k+1}, \dots, b_{N-1}) \in E_{n+1}.$$

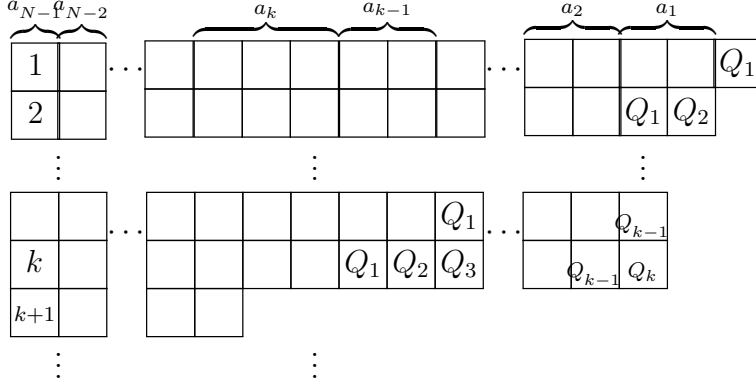
Choose

$$\begin{aligned} \xi &= (a_1, \dots, a_{k-1}, n+1, a_{k+1}, \dots, a_{N-1}) \in E_{n+1}, \\ \eta &= (b_1, \dots, b_{k-1}, 2n+2-a_k, b_{k+1}, \dots, b_{N-1}) \in K_{n+1}^c. \end{aligned}$$

Let \mathcal{O} be an operation satisfying **(R1)**-**(R4)** in the decomposition of $X_\alpha \boxtimes X_\pi$. The operation \mathcal{O} yields a resulting diagram corresponding to β if and only if

$$b_j = a_j + \sum_{i=1}^{N-1} r_j^{Q_i} - \sum_{i=1}^{N-1} r_{j+1}^{Q_i}, \quad (2)$$

where $r_j^{Q_i}$ is the number of Q_i added at the j^{th} row. See the following figure:



In other words, the multiplicity of β is

$$N_{\alpha\pi}^\beta = \# \{ \mathcal{O} : \mathcal{O} \text{ satisfies (2)} \}.$$

Similar observations hold for $N_{\xi\pi}^\eta$. Note that $a_k \leq n + 1$. If an operation \mathcal{O} adding $r_j^{Q_i}$ boxes Q_i at the j^{th} row to the diagram X_α satisfies **(R1)**-**(R4)**, then so does the operation \mathcal{O}' adding $r_j^{Q_i}$ boxes Q_i at the j^{th} row to the diagram X_ξ . This tells us $N_{\alpha\pi}^{\alpha+(\beta-\alpha)} \leq N_{\xi\pi}^{\xi+(\beta-\alpha)}$. By the Frobenius reciprocity law (8.6), we get $N_{\alpha\pi}^\beta \leq N_{\xi\pi}^\eta$. \square

Proposition B.2 (Proposition 8.8). *Let $(K_n)_n$ and $\pi = (t_1, \dots, t_{N-1})$ be as above. Set*

$$\varphi_n(\pi) = \frac{\sum_{\alpha, \beta \in K_n} N_{\alpha\beta}^\pi d_\alpha d_\beta}{d_\pi (\sum_{\xi \in K_n} d_\xi^2)}.$$

Then

$$1 - \varphi_n(\pi) \lesssim_N \frac{|\pi|}{n+1} \quad \text{and} \quad \varphi_{n+1}(\pi) - \varphi_n(\pi) \lesssim_N \frac{1}{n+1},$$

where $|\pi| = \max\{t_i : 1 \leq i \leq N-1\}$.

Proof. Let $E_k = E_k \setminus E_{k-1}$ for all $k \geq 1$. Denote $t = |\pi|$, i.e. $\pi \in E_t$. Let $n \geq 1$. Define

$$\partial_\pi^1 K_n = \{ \alpha \in K_n : \exists \beta \in K_n^c, \text{ such that } \beta \subset \alpha\pi \},$$

$$\partial_\pi^2 K_n = \{ \beta \in K_n^c : \exists \alpha \in K_n, \text{ such that } \alpha \subset \beta\pi \}.$$

Assume first $n - (N-1)t \geq 0$. By Lemma B.1 (2), we know that if $\alpha \in K_{n-(N-1)t}$, all the irreducible subrepresentations of $\alpha \boxtimes \pi$ should belong to the set K_n , which implies $\partial_\pi^1 K_n \cap K_{n-(N-1)t} = \emptyset$. Therefore

$$|\partial_\pi^1 K_n|_w \leq |\cup_{k=n-(N-1)t+1}^n E_k|_w \lesssim_N \sum_{k=n-(N-1)t+1}^n (k+1)^{N^2-1} \lesssim_N t(n+1)^{N^2-2}. \quad (3)$$

On the other hand, if $n - (N - 1)t < 0$, then

$$|\partial_\pi^1 K_n|_w \leq |K_n|_w \lesssim_N (n+1)^{N^2-1} \lesssim_N t(n+1)^{N^2-1}.$$

Also by Lemma B.1 (2), we have $\partial_\pi^2 K_n \subset K_{n+(N-1)t} \setminus K_n$. Therefore,

$$|\partial_\pi^2 K_n|_w \leq |\cup_{k=n+1}^{n+(N-1)t} E_k|_w \lesssim_N \sum_{k=n+1}^{n+(N-1)t} (k+1)^{N^2-1} \lesssim_N t(n+1)^{N^2-2}.$$

Moreover, we have $|K_n|_w = \sum_{i=1}^n |E_i|_w \gtrsim_N (n+1)^{N^2-1}$ and by Lemma B.1 (1) we have $n^{N^2-1} \lesssim_N |K_n|_w \lesssim_N n^{N^2-1}$. Therefore we obtain

$$\frac{|\partial_\pi K_n|_w}{|K_n|_w} \lesssim_N \frac{t(n+1)^{N^2-2}}{(n+1)^{N^2-1}} \lesssim_N \frac{t}{n+1}.$$

For any $\alpha \in K_n \setminus \partial_\pi^1 K_n$, we have $\alpha \boxtimes \pi = \oplus_{\beta \in K_n} N_{\alpha\pi}^\beta \beta$. In particular, we see that $\sum_{\beta \in K_n} N_{\alpha\beta}^\pi d_\beta = d_\pi d_\alpha$ and

$$\begin{aligned} 1 - \varphi_n(\pi) &= \frac{\sum_{\alpha \in K_n} d_\alpha (d_\alpha d_\pi - \sum_{\beta \in K_n} N_{\alpha\beta}^\pi d_\beta)}{d_\pi |K_n|_w} \\ &= \frac{\sum_{\alpha \in \partial_\pi^1 K_n} d_\alpha (d_\alpha d_\pi - \sum_{\beta \in K_n} N_{\alpha\beta}^\pi d_\beta)}{d_\pi |K_n|_w} + 0 \\ &\leq \frac{|\partial_\pi^1 K_n|_w}{|K_n|_w} \lesssim_N \frac{t}{n+1}. \end{aligned}$$

For the second inequality, by Lemma B.1 (3) we have

$$\begin{aligned} \varphi_{n+1}(\pi) - \varphi_n(\pi) &= (1 - \varphi_n(\pi)) - (1 - \varphi_{n+1}(\pi)) \\ &= \frac{\sum_{\alpha \in K_n} d_\alpha \left(\sum_{\beta \in K_{n+1}^c} N_{\alpha\pi}^\beta d_\beta \right)}{d_\pi |K_n|_w} - \frac{\sum_{\alpha \in K_{n+1}} d_\alpha \left(\sum_{\beta \in K_{n+1}^c} N_{\alpha\pi}^\beta d_\beta \right)}{d_\pi |K_{n+1}|_w} \\ &= \frac{|E_{n+1}|_w \left(\sum_{\alpha \in K_n} d_\alpha \left(\sum_{\beta \in K_{n+1}^c} N_{\alpha\pi}^\beta d_\beta \right) \right)}{d_\pi |K_n|_w |K_{n+1}|_w} \\ &\quad + \frac{\sum_{\alpha \in K_n} d_\alpha \left(\sum_{\beta \in K_n^c} N_{\alpha\pi}^\beta d_\beta \right) - \sum_{\alpha \in K_{n+1}} d_\alpha \left(\sum_{\beta \in K_{n+1}^c} N_{\alpha\pi}^\beta d_\beta \right)}{d_\pi |K_{n+1}|_w} \\ &\leq \frac{|E_{n+1}|_w \left(\sum_{\alpha \in K_n} d_\alpha d_\alpha d_\pi \right)}{d_\pi |K_n|_w |K_{n+1}|_w} \\ &\quad + \frac{\sum_{\alpha \in K_n} d_\alpha \left(\sum_{\beta \in E_{n+1}} N_{\alpha\pi}^\beta d_\beta \right) - \sum_{\alpha \in E_{n+1}} d_\alpha \left(\sum_{\beta \in K_{n+1}^c} N_{\alpha\pi}^\beta d_\beta \right)}{d_\pi |K_{n+1}|_w} \\ &\lesssim_N \frac{1}{n+1}. \end{aligned}$$

□

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Résumé :

Cette thèse a pour but d'étudier l'analyse sur les groupes quantiques compacts. Elle se compose de deux parties. La première partie porte sur la classification des semi-groupes de Markov invariants sur ces espaces homogènes quantiques. Les générateurs de ces semi-groupes sont considérés comme des opérateurs de Laplace sur ces espaces. La sphère classique \mathcal{S}^{N-1} , la sphère libre \mathcal{S}_+^{N-1} et la sphère semi-libérée \mathcal{S}_*^{N-1} sont considérées comme des exemples et les générateurs de semi-groupes de Markov sur ces sphères sont classifiés. Nous calculons aussi les dimensions spectrales des trois familles de sphères en fonction du comportement asymptotique des valeurs propres de leurs opérateurs de Laplace. Dans la deuxième partie, nous étudions la convergence des séries de Fourier pour les groupes non abéliens et les groupes quantiques. Il est bien connu qu'un certain nombre de propriétés d'approximation de groupes peuvent être interprétées comme des méthodes de sommation et de convergence en moyenne de séries de Fourier non commutatives associées. Nous établissons un critère général d'inégalités maximales pour les unités approchées de multiplicateurs non commutatifs de Fourier. En conséquence, nous prouvons que pour tout groupe dénombrable discret moyennable, il existe une suite de fonctions de type positif à support fini, telle que les multiplicateurs de Fourier associés sur les espaces L_p non commutatifs satisfassent à la convergence ponctuelle (presque uniforme). Nos résultats s'appliquent également à la convergence presque partout des séries de Fourier de fonctions L_p sur des groupes compacts non-abéliens. D'autre part, nous obtenons des bornes indépendantes de la dimension pour les inégalités maximales de Hardy-Littlewood non commutatives dans l'espace à valeurs opérateurs $L_p(\mathbb{R}^d, \mathcal{N})$ associées à des corps convexes.

Mots-clés : Espaces L_p non commutatifs, groupes quantiques compacts, actions quantiques, sphères non commutatives, semi-groupes de Markov, multiplicateur de Fourier, convergence ponctuelle

Abstract:

This thesis is devoted to study of analysis on compact quantum groups. It consists of two parts. The first part presents the classification of invariant quantum Markov semigroups on these quantum homogeneous spaces. The generators of these semigroups are viewed as Laplace operators on these spaces. The classical sphere \mathcal{S}^{N-1} , the free sphere \mathcal{S}_+^{N-1} , and the half-liberated sphere \mathcal{S}_*^{N-1} are considered as examples and the generators of Markov semigroups on these spheres are classified. We compute spectral dimensions for the three families of spheres based on the asymptotic behavior of the eigenvalues of their Laplace operator. In the second part, we study the convergence of Fourier series for non-abelian groups and quantum groups. It is well-known that a number of approximation properties of groups can be interpreted as some summation methods and mean convergence of associated noncommutative Fourier series. Based on this framework, this work studies the refined counterpart of pointwise convergence of these Fourier series. We establish a general criterion of maximal inequalities for approximative identities of noncommutative Fourier multipliers. As a result, we prove that for any countable discrete amenable group, there exists a sequence of finitely supported positive definite functions, so that the associated Fourier multipliers on noncommutative L_p -spaces satisfy the pointwise convergence. Our results also apply to the almost everywhere convergence of Fourier series of L_p -functions on non-abelian compact groups. On the other hand, we obtain the dimension free bounds of noncommutative Hardy-Littlewood maximal inequalities associated with convex bodies.

Keywords : Noncommutative L_p -spaces, compact quantum groups, quantum actions, non-commutative spheres, Markov semigroups, Fourier multipliers, maximal inequalities, pointwise convergence.

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