



**HAL**  
open science

## Inequalities on Swan conductors

Ammar Yasir Kiliç

► **To cite this version:**

Amman Yasir Kiliç. Inequalities on Swan conductors. Number Theory [math.NT]. Université Paris Saclay (COmUE), 2019. English. NNT : 2019SACLS309 . tel-02612976

**HAL Id: tel-02612976**

**<https://theses.hal.science/tel-02612976>**

Submitted on 19 May 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NNT : 2019SACLS309

*Thèse de doctorat de l'Université Paris-Saclay,  
mention Mathématiques fondamentales*

---

# INEQUALITIES ON SWAN CONDUCTORS

AMMAR YASIR KILIÇ

---

*Thèse préparée à l'Université Paris-Sud,  
dans le Laboratoire de Mathématiques d'Orsay*



*Thèse présentée et soutenue à Orsay,  
le 27 septembre 2019,  
en présence du jury :*

M. BENJAMIN SCHRAEN Université Paris-Sud	Président
M. FARRELL BRUMLEY Université Paris 13	Rapporteur
M. EREZ LAPID Weizmann Institute of Science	Rapporteur
M. COLIN BUSHNELL King's College London	Examineur
M. GUY HENNIART Université Paris-Sud	Directeur de thèse



# ACKNOWLEDGEMENTS

First and foremost, I would like to heartily thank my advisor Guy Henniart. It has been a pleasure to have witnessed your mathematical wisdom and intuition during my PhD journey. Moreover, I will keep your kindness, patience, and generosity as a nice memory of this period in my life.

Secondly, I want to thank Farrell Brumley and Erez Lapid for having read my thesis, as well as for providing useful remarks.

Lastly, I send my gratitude to the remaining members of the jury — Colin Bushnell and Benjamin Schraen — for agreeing to be on my defense committee.



# CONTENTS

<b>Introduction</b>	<b>7</b>
<b>I Background</b>	<b>9</b>
§1 Ramification filtration . . . . .	9
§2 Representations of Weil groups . . . . .	16
§3 Slopes of representations . . . . .	20
<b>II The Adjoint Representation</b>	<b>25</b>
§4 Conjugated characters . . . . .	26
§5 Tame base change . . . . .	29
§6 Lower bound for the conductor . . . . .	31
§7 Carayol Representations . . . . .	34
§8 Application: a result of Lapid . . . . .	45
<b>III Some additional cases</b>	<b>47</b>
§9 Tensor Powers . . . . .	47
§10 Representations of $GL_2(\mathbf{C})$ . . . . .	48
<b>Résumé (français)</b>	<b>53</b>
<b>References</b>	<b>59</b>



# INTRODUCTION

Let  $F$  be a non-Archimedean local field of finite residue characteristic  $\mathfrak{p}$  (no assumptions are made concerning the characteristic of  $F$ ). We fix a separable algebraic closure  $\overline{F}$  of  $F$  and let  $\mathfrak{W}$  denote the Weil group of  $F$  relative to  $\overline{F}/F$ .

We shall be concerned with smooth, complex, semisimple, finite-dimensional representations of  $\mathfrak{W}$ . Via the ramification filtration of  $\mathfrak{W}$ , one can attach to such a representation  $\sigma$  an additive invariant  $\text{sw}(\sigma)$ , known as the *Swan exponent* (3.1). It is a non-negative integer, and depends only on the restriction of  $\sigma$  to the wild ramification subgroup of  $\mathfrak{W}$ .

The central problem in this thesis is the following. For an  $n$ -dimensional representation  $\sigma$  of  $\mathfrak{W}$ , we consider the composition  $\rho \circ \sigma$ , where  $\rho$  denotes an algebraic representation of  $\text{GL}_n(\mathbf{C})$ . We would like to investigate the relations between  $\text{sw}(\rho \circ \sigma)$  and  $\text{sw}(\sigma)$ . After collecting the necessary background, we treat in Chapter II the case where  $\rho$  is given by the adjoint representation of the general linear group. In that case, the composition  $\rho \circ \sigma$  is isomorphic to  $\sigma \otimes \sigma^\vee$ , and we prove the following two theorems.

**6.1 Theorem** — *Let  $\sigma$  be an irreducible representation of  $\mathfrak{W}$  that is minimal and of dimension  $\geq 2$ , and  $\sigma^\vee$  the corresponding contragredient representation. If  $\ell_\sigma$  is the smallest prime number dividing the dimension of  $\sigma$ , then*

$$\zeta(\sigma \otimes \sigma^\vee) \geq (1 - 1/\ell_\sigma)\zeta(\sigma).$$

Here,  $\zeta(\sigma)$  stands for the normalized exponent given by  $\text{sw}(\sigma)/\dim(\sigma)$ . For the definition of minimality, we refer to 4.4.



**7.7 Theorem** — *If  $\sigma$  is an irreducible representation of  $\mathfrak{W}$  that is epipelagic (i.e.,  $\text{sw}(\sigma) = 1$ ), then*

$$\text{sw}(\sigma \otimes \sigma^\vee) = \dim(\sigma) - 1.$$

These results are not new; they were first observed and proved by Bushnell and Henniart in the context of the local Langlands correspondence on the GL-side. (In fact, they have proved a stronger version of the second theorem by showing the equality for any representation  $\sigma$  for which  $\text{sw}(\sigma)$  is not divisible by the residue characteristic.) The proofs given in this thesis, however, do not make use of the Langlands correspondence, but only invoke the Galois theory of  $F$ , and some elementary representation theory.

Motivated by a result of Lapid, we conclude Chapter II with an application of the aforementioned theorems. We prove that  $\dim(\sigma) - 1$  is a lower bound for  $\text{sw}(\sigma \otimes \sigma^\vee)$ , and that the bound is attained if and only if  $\sigma$  is epipelagic.

In Chapter III, we push our techniques further, and consider two other instances of the central problem. In §9, we discuss the case where  $\rho$  is given by a tensor power of the standard representation. More generally, we obtain the following result.

**9.3 Theorem** — *Let  $\sigma: \mathfrak{W} \rightarrow \text{GL}(V)$  be a representation, and  $\Sigma$  its composition with an algebraic representation of  $\text{GL}(V)$ . If  $\sigma$  is  $\varsigma$ -minimal, then*

$$\varsigma(\sigma \otimes \Sigma) \geq \frac{1}{2} \varsigma(\sigma).$$

The definition of  $\varsigma$ -minimality is stated in [9.1](#).

Finally, in §10, we treat the case where the degree of  $\sigma$  is 2, and  $\rho$  is any irreducible algebraic representation of  $\text{GL}_2(\mathbf{C})$ . We obtain lower bounds for the (normalized) Swan exponent; the results are stated in [Theorem 10.3](#). In particular, we observe that in some cases the lower bound is worse than the one obtained for the tensor powers.

# I

## BACKGROUND

In this chapter, we have collected the background that is necessary to understand this thesis. We shall first introduce the ramification filtration on Galois groups (and Weil groups) via Herbrand functions.

In §2, we introduce the notation and terminology related to smooth representations of Weil groups. Here, we also state some elementary results from representation theory that will be used frequently in the subsequent chapters. In the final section, we define the most central notion of this thesis: the Swan exponent. We conclude the chapter with an inequality of Heiermann.

### §1 RAMIFICATION FILTRATION

The theory of Herbrand functions as developed by Serre in [Ser79] can be easily extended to extensions that are not necessarily Galois. Our treatment is mostly based on [Wei73] and [Del84].

We fix a separable closure  $\overline{F}$  of  $F$ . In what follows, all field extensions of  $F$  are supposed to be subextensions of  $\overline{F}$ ; in particular, all extensions are considered to be separable. We let  $\mathfrak{G}$  denote the absolute Galois group of  $F$

(relative to  $\overline{F}$ ), i.e., the group  $\text{Aut}_F(\overline{F})$  of  $F$ -automorphisms of  $\overline{F}$ , equipped with the usual profinite topology.

**1.1 Ramification subsets.** For a finite field extension  $E/F$ , we let  $v_E: E \rightarrow \mathbf{Z} \cup \{+\infty\}$  denote the normalized (additive) valuation. It is convenient to extend  $v_E$  to  $\overline{F}$  as follows. Let  $x \in \overline{F}$ , and let  $E'$  be any finite extension of  $E$  of ramification index  $e'$ , containing  $x$ . We then put  $v_E(x) = v_{E'}(x)/e'$ ; this definition is independent of the choice of  $E'$ , and extends  $v_E$  to a map  $v_E: \overline{F} \rightarrow \mathbf{Q} \cup \{\infty\}$ . Furthermore, the extended valuation map is invariant under the action of  $\mathfrak{G}$ , i.e., conjugate elements in  $\overline{F}$  have the same valuation.

Let  $\mathcal{S}(E/F)$  denote the set of all  $F$ -embeddings  $E \hookrightarrow \overline{F}$ . It is naturally equipped with a transitive action of  $\mathfrak{G}$ ; writing  $\mathfrak{G}_E = \text{Aut}_E(\overline{F})$ , the set  $\mathcal{S}(E/F)$  is isomorphic to  $\mathfrak{G}/\mathfrak{G}_E$  as  $\mathfrak{G}$ -sets.

We want to define a filtration on  $\mathcal{S}(E/F)$ . First, for every element  $\lambda \in \mathcal{S}(E/F)$ , we put

$$\nu_E(\lambda) = \inf_{x \in \mathfrak{o}_E} v_E(\lambda(x) - x).$$

Notice that, choosing a Galois extension  $K/F$  that contains  $E$ , we have  $v_E(\lambda(x) - x) = v_K(\lambda(x) - x)/e' \in (1/e')\mathbf{Z}_+$  for all  $x \in \mathfrak{o}_E$ , where  $e'$  denotes the ramification index of  $K/E$ . So  $\nu_E(\lambda)$  is a rational number  $\geq 0$  or  $+\infty$ ; it is  $+\infty$  if and only if  $\lambda$  is equal to the inclusion  $\varepsilon: E \hookrightarrow \overline{F}$ . Furthermore, if  $x_0$  is an element of  $\mathfrak{o}_E$  that generates  $\mathfrak{o}_E$  as an  $\mathfrak{o}$ -algebra, then we have  $\nu_E(\lambda) = v_E(\lambda(x_0) - x_0)$ .

More generally, we define  $\nu_K(\lambda)$  (with  $\lambda \in \mathcal{S}(E/F)$ ) for any finite extension  $K$  of  $F$  by replacing  $v_E$  with  $v_K$  in the definition given before; then,  $\nu_K = (1/e')\nu_E$  with  $e' = e(K/E)$ .

For  $u \in \mathbf{R}_+$ , we define the ramification subset of (lower) index  $u$  by

$$\mathcal{S}(E/F)_u = \{\lambda \in \mathcal{S}(E/F) \mid \nu_E(\lambda) \geq u + 1\} \subseteq \mathcal{S}(E/F);$$

hence, we obtain a descending filtration of  $\mathcal{S}(E/F)$  that we will refer to as its *ramification filtration*. An index  $u$  is called a *jump* in the filtration if  $\mathcal{S}(E/F)_u \neq \mathcal{S}(E/F)_{u+\epsilon}$  for all  $\epsilon > 0$ . Observe that the number of jumps in the filtration is finite, and  $\mathcal{S}(E/F)_u = \{\varepsilon\}$  if we take  $u$  large enough.

**1.2 Lemma** — *Let  $E/F$  be a finite extension of ramification index  $e = e' \mathfrak{p}^r$  with  $\mathfrak{p} \nmid e'$ , and  $\varepsilon: E \hookrightarrow \overline{F}$  the inclusion map. Then, the following holds.*

i) *If  $E_0/F$  (resp.  $E_1/F$ ) denotes the maximal unramified (resp. maximal tamely ramified) subextension of  $E/F$ , then  $\mathcal{S}(E/F)_0$  (resp.  $\mathcal{S}(E/F)_1$ ) consist of those embeddings that fix the elements of  $E_0$  (resp.  $E_1$ ) pointwise. In particular,  $\#\mathcal{S}(E/F)_0 = e$  and  $\#\mathcal{S}(E/F)_1 = \mathfrak{p}^r$ .*

ii) For all  $g \in \mathfrak{G}$  and  $u \geq 0$ , the set  $g\mathcal{S}(E/F)_u$  is either disjoint from or equal to  $\mathcal{S}(E/F)_u$ . In particular,  $\#\mathcal{S}(E/F)_u$  divides  $\#\mathcal{S}(E/F)$  for all  $u \geq 0$ , and  $\#\mathcal{S}(E/F)_u$  divides  $\#\mathcal{S}(E/F)_v$  for all  $u, v \geq 0$  with  $v \leq u$ .

iii) The wild exponent  $w(E/F)$  of  $E/F$  (i.e.,  $w(E/F) = d(E/F) - e + 1$ ,  $d(E/F)$  denoting the differential exponent of  $E/F$ ), is given by

$$\begin{aligned} w(E/F) &= \sum_{\lambda \neq \varepsilon} \max(\nu_E(\lambda) - 1, 0) \\ &= \int_0^{+\infty} (\#\mathcal{S}(E/F)_u - 1) du. \end{aligned}$$

iv) If  $E/F$  is a Galois extension, then  $\mathcal{S}(E/F)$  can be identified with the Galois group  $\mathcal{G}(E/F)$  of  $E/F$ . Furthermore, all jumps in the ramification filtration of  $\mathcal{G}(E/F)$  are integers, and  $\mathcal{G}(E/F)_u$  is a subgroup of  $\mathcal{G}(E/F)$  for all  $u$ .

*Proof* — We prove the second statement and the second identity of the third statement. The proofs for the other statements can be all found in [Wei73, Ch. VII, §3]. For each  $u \geq 0$ , we define a relation  $\mathcal{R}_u$  on  $\mathcal{S}(E/F)$  as follows:

$$\lambda \equiv \eta \pmod{\mathcal{R}_u} \text{ if and only if } \min_{x \in \mathcal{O}_E} v_E(\lambda(x) - \eta(x)) \geq u.$$

The ultrametric property and the  $\mathfrak{G}$ -invariance of  $v_E$  imply that  $\mathcal{R}_u$  is a  $\mathfrak{G}$ -invariant equivalence relation. Moreover,  $\mathcal{S}(E/F)_u$  is the equivalence class containing the inclusion  $\varepsilon$ , so  $g\mathcal{S}(E/F)_u$  runs through all equivalence classes of  $\mathcal{R}$  when  $g$  runs through  $\mathfrak{G}$ . The first part of the second statement follows. For the second part we note that  $\mathcal{R}_u$  is finer than  $\mathcal{R}_v$  if  $v \leq u$ .

The second identity for  $w(E/F)$  can be obtained as a consequence of the first one:

$$\begin{aligned} \sum_{\lambda \neq \varepsilon} \max(\nu_E(\lambda) - 1, 0) &= \sum_{\lambda \neq \varepsilon} \int_0^{+\infty} \mathbf{1}_{[0, (\nu_E(\lambda)-1)^+]}(u) du \\ &= \int_0^{+\infty} \left( \sum_{\lambda \neq \varepsilon} \mathbf{1}_{[0, (\nu_E(\lambda)-1)^+]} \right)(u) du \\ &= \int_0^{+\infty} (\#\mathcal{S}(E/F)_u - 1) du. \quad \mathbf{q.e.d.} \end{aligned}$$

If  $E/F$  is Galois, we shall write  $\mathcal{I}(E/F)$  for  $\mathcal{G}(E/F)_0$  and  $\mathcal{P}(E/F)$  for  $\mathcal{G}(E/F)_1$ ; these are called the *inertia subgroup* and *wild ramification subgroup* respectively.

The following proposition contains some properties with respect to the ramification filtration of Galois groups and its jumps. They will be used in the subsequent chapters. We refer to [Ser79, Ch. IV, Proposition 10,11] for the proofs.

**1.3 Proposition** — *Let  $E/F$  be a finite Galois extension, and put  $G = \mathcal{G}(E/F)$ . Then the following holds.*

- i) *If  $\lambda \in G_i$ ,  $\kappa \in G_j$ , and  $i, j \geq 1$ , then  $\lambda\kappa\lambda^{-1}\kappa^{-1} \in G_{i+j+1}$ .*
- ii) *All jumps  $\geq 1$  in the filtration of  $\mathcal{G}(E/F)$  are congruent to one another mod  $\mathfrak{p}$ .*

**1.4 Corollary** — *If  $E/F$  is a finite Galois extension that is wildly ramified, then the smallest non-trivial ramification subgroup of  $\mathcal{G}(E/F)$  is a central subgroup of its wild inertia subgroup.*

*Proof* — By the first part of the proposition above, if  $\kappa$  is an element of  $\mathcal{G}(E/F)_1$ , and  $\lambda$  is an element of the smallest non-trivial ramification subgroup, then the commutator  $[\kappa, \lambda]$  is trivial. **q.e.d.**

Next, we state some results that relate the ramification filtration corresponding to an extension  $K/F$  to that of a subextension  $E/F$ .

**1.5 Lemma** — *Let  $F \subseteq K \subseteq E$  be a tower of finite extensions, where  $K \subseteq E$  is of ramification index  $e'$ ; let  $\lambda$  be an embedding  $K \hookrightarrow \overline{F}$ . Then*

$$\nu_K(\lambda) = \sum_{\kappa \mapsto \lambda} \nu_K(\kappa) = (1/e') \sum_{\kappa \mapsto \lambda} \nu_E(\kappa),$$

where both summations run over all embeddings  $\kappa: K \hookrightarrow \overline{F}$  that restrict to  $\lambda$ .

We refer to [Wei73, Ch. VIII, §3] for a proof of this lemma (it is similar to the Galois case).

**1.6 Definition** — *Let  $E/F$  be a finite extension of ramification index  $e$ . The associated Herbrand function  $\phi_{E/F}: \mathbf{R}_+ \mapsto \mathbf{R}_+$  is defined by*

$$\phi_{E/F}(u) = (1/e) \int_0^u \#\mathcal{S}(E/F)_x \, dx.$$

Let us remark that  $\phi_{E/F}$  is continuous, piecewise linear, increasing and concave. In particular, it is bijective, and its inverse function will be denoted by  $\psi_{E/F}$ .

**1.7 Lemma** — *Let  $E/F$  be a finite extension of ramification index  $e$  and wild exponent  $w$  (q.v. part iii of 1.2). Let  $u_0$  be the infimum of all indices  $u \geq 0$  for which  $\mathcal{S}(E/F)_u = \{\varepsilon\}$ . Then, for all  $u \geq u_0$ , we have*

$$\phi_{E/F}(u) = w/e + (1/e)u.$$

*Proof* — Suppose  $u \geq u_0$ , then

$$\begin{aligned} \phi_{E/F}(u) &= (1/e) \int_0^u \#\mathcal{S}(E/F)_x dx \\ &= (1/e) \int_0^u (\#\mathcal{S}(E/F)_x - 1) dx + (1/e) \int_0^u dx \\ &= (1/e) \int_0^\infty (\#\mathcal{S}(E/F)_x - 1) dx + (1/e)u \\ &= w/e + (1/e)u, \end{aligned}$$

where we have used part **iii** of **1.2** to obtain the last equality. **q.e.d.**

With the notation of the lemma above, we shall write  $\phi_{E/F}^\infty$  for the affine function  $\mathbf{R}_+ \rightarrow \mathbf{R}_+ : u \mapsto w/e + (1/e)u$ ; so  $\phi_{E/F}^\infty(u) = \phi_{E/F}(u)$  for  $u$  large enough. In particular, if  $E/F$  is tamely ramified with ramification index  $e$ , then  $\phi_{E/F}(u) = \phi_{E/F}^\infty(u) = (1/e)u$ . So  $\phi_{E/F}$  is linear if  $E/F$  is tame.

**1.8 Proposition (Herbrand's theorem)** — *Let  $F \subseteq K \subseteq E$  be a tower of finite extensions. The surjective map  $\mathcal{S}(E/F) \rightarrow \mathcal{S}(K/F)$  that is obtained via restriction, maps  $\mathcal{S}(E/F)_u$  onto  $\mathcal{S}(K/F)_v$ , with  $v = \phi_{E/K}(u)$ .*

Using **1.5**, the proof in the Galois case of [Ser79, Ch. IV, §3, Lem. 5] can be directly generalized to a proof in the general case.

**1.9 Corollary** — *The Herbrand functions satisfy the transitivity property:*

$$\phi_{K/F} = \phi_{E/F} \circ \phi_{K/E}.$$

The proof is similar to the Galois case [Ser79, Ch. IV, §3, Lem. 5].

**1.10 Upper numbering.** For a finite extension  $E/F$ , we define the *upper numbering* for ramification subsets in the following way. For  $v \in \mathbf{R}_+$ , we put

$$\mathcal{S}(E/F)^v = \mathcal{S}(E/F)_{\psi_{E/F}(v)}.$$

Suppose that  $E$  and  $K$  are finite extensions of  $F$  with  $E \subseteq K$ . Proposition **1.8** then implies that, for all  $v \in \mathbf{R}_+$ , the image of  $\mathcal{S}(K/F)^v$  under the restriction map, is equal to  $\mathcal{S}(E/F)^v$ .

**1.11** Using the upper numbering, we can extend the notion of ramification subsets to all (i.e., not necessarily finite) extensions  $L$  of  $F$ . As before, we let  $\mathcal{S}(L/F)$  denote the set of  $F$ -embeddings of  $L$  into  $\overline{F}$ . For  $v \in \mathbf{R}_+$ , we put  $\mathcal{S}(L/F)^v = \varprojlim \mathcal{S}(E/F)^v$ , where the projective limit is taken with respect to the directed set defined by finite subextension of  $L/F$  and corresponding restriction maps. If  $L/F$  is Galois, we shall identify  $\mathcal{S}(L/F)$  with the corresponding Galois group  $\mathcal{G}(L/F)$ . Then, the ramification subsets  $\mathcal{S}(L/F)^v$

are subgroups of  $\mathcal{G}(L/F)$ . In particular, we obtain a filtration of the absolute Galois group  $\mathfrak{G}$  of  $F$  by ramification subgroups  $\mathfrak{G}^v$ .

The subgroup  $\mathfrak{G}^0$  is the *inertia subgroup* of  $\mathfrak{G}$ , denoted  $\mathfrak{I}$ . It corresponds to the *maximal unramified extension*  $F_{\text{unr}}$  via Galois theory. Note that  $F_{\text{unr}}/F$  is an infinite extension of  $F$ . It is generated over  $F$  by the group  $\mu$  of roots of unity of order prime to  $p$ . Moreover, there exists a unique  $F$ -automorphism of  $F_{\text{unr}}$  that induces the  $q$ -power map  $\zeta \mapsto \zeta^q$  on  $\mu$ , where  $q$  denotes the cardinality of the residue field of  $F$ . We shall denote this automorphism by  $\Phi$ , and refer to it as the Frobenius automorphism of  $F_{\text{unr}}$ .

The closure of the union  $\bigcup_{\epsilon>0} \mathfrak{G}^\epsilon$  is the *wild ramification subgroup* of  $F$ , denoted  $\mathfrak{P}$ . Using the terminology of [Ser02], it is the unique pro- $p$ -Sylow subgroup of  $\mathfrak{I}$ , and corresponds to the *maximal tamely ramified extension*  $F_{\text{tame}}$  via Galois theory.

**1.12 Definition** — *Let  $L/F$  be an extension, and write  $\varepsilon: L \hookrightarrow \overline{F}$  for its inclusion in  $\overline{F}$ . Then, the slope of  $L/F$ , denoted  $\text{sl}(L/F)$ , is defined as the infimum of all numbers  $v \in \mathbf{R}_+$  such that  $\mathcal{S}(L/F)^v = \{\varepsilon\}$ .*

Note that, a priori, the slope is a real number  $\geq 0$  or  $+\infty$ ; however, we shall only use this notion in cases where it defines a finite number.

**1.13 Lemma** — *If  $I$  is a directed partially ordered set, and  $(S_i, \varphi_{ij})$  is a projective system over  $I$  consisting of non-empty finite sets, then*

$$\varprojlim S_i \neq \emptyset.$$

*Proof* — We equip every set  $S_i$  with the discrete topology, so that  $S_i$  is compact and Hausdorff. Then, by Tikhonov's theorem,  $\prod_i S_i$  is compact. For each pair of indices  $i_1, i_2 \in I$  with  $i_1 \geq i_2$  we define the subset  $\mathcal{C}_{i_1 i_2}$  of  $\prod_i S_i$  consisting of elements  $(x_i)_i$  such that  $\varphi_{i_1 i_2}(x_{i_1}) = x_{i_2}$ . Since  $S_{i_2}$  is Hausdorff,  $\mathcal{C}_{i_1 i_2}$  is closed in  $\prod_i S_i$ ; moreover, we have  $\varprojlim S_i = \bigcap_{i \geq j} \mathcal{C}_{ij}$ .

Suppose  $\varprojlim S_i = \emptyset$ , then, by the compactness of  $\prod S_i$ , there exists finitely many subsets  $C_{ij}$  whose intersection is empty. Only a finite number of indices are involved in the latter intersection, so we can find an index  $k$  that is bigger than each of them. Then, choosing an element  $x$  of  $S_k$ , we can use its image under the relevant transition maps to construct an element of the finite intersection. This is a contradiction. **q.e.d.**

**1.14 Lemma** — *Let  $F \subseteq L \subseteq M$  be a tower of (not necessarily finite) extensions. Then, for all  $v \in \mathbf{R}_+$ , we obtain a surjective map*

$$\mathcal{S}(M/F)^v \twoheadrightarrow \mathcal{S}(L/F)^v,$$

*that is defined via restriction.*

This lemma extends Herbrand's theorem (1.8) to infinite extensions.

*Proof* — Choose an element  $\lambda \in \mathcal{S}(L/F)^v$ . For each finite subextension  $E/F$  of  $M/F$ , we consider the subset  $\mathcal{X}_\lambda(E/F)$  of  $\mathcal{S}(E/F)^v$  consisting of embeddings that coincide with  $\lambda$  on  $E \cap L$ . Since we have a surjective map  $\mathcal{S}(E/F)^v \twoheadrightarrow \mathcal{S}((E \cap L)/F)^v$  via restriction (q.v. 1.10), the subset  $\mathcal{X}_\lambda(E/F)$  is nonempty. It is easy to verify that the subsets  $\mathcal{X}_\lambda(E/F)$  give rise to a projective system, and  $\varprojlim \mathcal{X}_\lambda(E/F) \neq \emptyset$  by the previous lemma. Furthermore, each element  $\eta$  of  $\varprojlim \mathcal{X}_\lambda(E/F)$  corresponds to an embedding  $\eta \in (M/F)^v$  that extends  $\lambda$ . The desired statement follows. **q.e.d.**

**1.15 Corollary** — *If  $L/F$  is an extension, then its slope is the infimum of all numbers  $v \in \mathbf{R}_+$  for which  $\mathfrak{G}^v \subseteq \text{Aut}_L(\overline{F})$ . Furthermore, if  $L/F$  is the compositum of a family  $(L_i/F)_i$  of extensions, then*

$$\text{sl}(L/F) = \sup_i \text{sl}(L_i/F).$$

*Proof* — Notice that, by the previous lemma,  $\mathfrak{G}^v \subseteq \text{Aut}_L(\overline{F})$  if and only if  $\mathcal{S}(L/F)^v = 1$ . Both statements of the corollary are direct consequences of this observation. **q.e.d.**

**1.16 Weil groups.** Instead of Galois groups it is convenient to make use of Weil groups. For each Galois extension  $L/F$ , there is an associated Weil group, denoted  $\mathcal{W}(L/F)$ . It is the subgroup of  $\mathcal{G}(L/F)$  containing the elements that act as a power of the Frobenius automorphism  $\Phi$  on the maximal unramified subextension  $L_{\text{unr}}/F$  of  $L/F$ . If  $L/F$  is finite, then  $\mathcal{G}(L_{\text{unr}}/F)$  is the cyclic group generated by the restriction of  $\Phi$  to  $L_{\text{unr}}$ ; hence, for finite Galois extensions  $L/F$ , we have  $\mathcal{W}(L/F) = \mathcal{G}(L/F)$ . In general,  $\mathcal{W}(L/F)$  is a dense subset of  $\mathcal{G}(L/F)$  (where  $\mathcal{G}(L/F)$  is equipped with its natural profinite topology), and it contains the inertia subgroup  $\mathcal{I}(L/F) = \mathcal{G}(L/L_{\text{unr}})$ . We equip the Weil group  $\mathcal{W}(L/F)$  with the topology that induces the usual profinite topology on the inertia subgroup  $\mathcal{I}(L/F)$ , and for which  $\mathcal{I}(L/F)$  is an open subgroup of  $\mathcal{W}(L/F)$ . In particular, taking  $L = \overline{F}$ , we get the absolute Weil group  $\mathfrak{W}$  of  $F$ .

The Weil groups have similar properties as Galois groups. For a finite extension  $E/F$ , the Weil group of  $E$  (relative to  $\overline{F}$ ), denoted by  $\mathfrak{W}_E$ , is the topological subgroup of  $\mathfrak{W}$  consisting of elements that fix  $E$  pointwise. The Weil group  $\mathfrak{W}$  being dense in  $\mathfrak{G}$ , for every element of  $\mathfrak{G}$ , one can find an element of  $\mathfrak{W}$  that restricts to the same embedding of  $E$ . It follows that  $\mathcal{S}(E/F)$  is a transitive  $\mathfrak{W}$ -set, and we obtain an isomorphism  $\mathfrak{W}/\mathfrak{W}_E \rightarrow \mathfrak{G}/\mathfrak{G}_E$  of  $\mathfrak{W}_E$ -sets. In particular, the index of  $\mathfrak{W}_E$  equals the degree of the extension  $E/F$ . Furthermore, by the same argument,  $\mathfrak{W}_E$  is normal in  $\mathfrak{W}$  if and only if  $\mathfrak{G}_E$  is



normal in  $\mathfrak{G}$ ; when this is the case, we shall identify  $\mathfrak{W}/\mathfrak{W}_E$  with the Galois group  $\mathcal{G}(E/F)$ .

Let us note that all ramification groups  $\mathfrak{G}^v$  are contained in  $\mathfrak{J}$ , and therefore in  $\mathfrak{W}$ ; furthermore, the topology on  $\mathfrak{G}^v$  induced from  $\mathfrak{W}$  coincides with the one induced from  $\mathfrak{G}$ . That being so, we shall often use  $\mathfrak{W}^v$  as a substitute notation for  $\mathfrak{G}^v$ . We shall also use the notation  $\mathfrak{W}^{v+}$  to denote the closure in  $\mathfrak{W}$  (or equivalently in  $\mathfrak{G}$ ) of the union  $\bigcup_{\epsilon>0} \mathfrak{W}^{(v+\epsilon)}$ .

**1.17 Local class field theory.** The main theorem of local class field theory establishes the existence of the so-called Artin reciprocity map  $\mathbf{a}: \mathfrak{W} \rightarrow F^\times$ . It can be normalized such that elements restricting to  $\Phi$  are sent to uniformizers of  $F$ . Moreover, it induces a topological isomorphism  $\mathfrak{W}^{\text{ab}} \xrightarrow{\sim} F^\times$ , and allows us to identify a character  $\mathfrak{W} \rightarrow \mathbf{C}^\times$  with the corresponding character  $F^\times \rightarrow \mathbf{C}^\times$ . We shall systematically make use of such identifications in the rest of this thesis, without explicit mention.

Let  $k$  be an integer  $\geq 1$ , and put  $U^k = 1 + \mathfrak{p}^k$ . The *higher ramification theory* of local class field theory states that  $\mathbf{a}$  maps  $\mathfrak{W}^k$  onto  $U^k$ , and  $\mathfrak{W}^{k+}$  onto  $U^{k+1}$ . Hence, under the identification of characters stated above, characters of  $U^k$  correspond to characters of  $\mathfrak{W}^k$  that are trivial on  $\mathfrak{W}^k \cap \mathfrak{W}^{\text{der}}$ , where  $\mathfrak{W}^{\text{der}}$  denotes the (closed) derived group of  $\mathfrak{W}$ .

## §2 REPRESENTATIONS OF WEIL GROUPS

In this section, we introduce the terminology and basic notions related to representations that will appear later in this work. Most of our notions concern representations of (absolute) Weil groups.

**2.1 Conventions.** Unless explicitly stated otherwise, all group representations appearing in this thesis are assumed to be complex, semi-simple, and of finite dimension. For a topological group  $G$ , a representation  $\sigma: G \rightarrow \text{GL}(V)$  is called *smooth* if the corresponding map  $G \times V \rightarrow V$  is continuous (where  $V$  is equipped with the discrete topology). By our assumption concerning dimensions,  $\sigma$  is smooth if and only if its kernel is an open subgroup. Henceforth, when topological groups are involved, representations are understood to be smooth. We shall use the word homomorphism instead of representation if we want to deviate from the smoothness assumption. Also, we shall identify a character  $G \rightarrow \mathbf{C}^\times$  of a group  $G$  with the corresponding homomorphism  $G \rightarrow \text{Aut}(\mathbf{C})$ . Likewise, if  $G$  is equipped with a group topology, such a

character is assumed to be continuous, thereby corresponding to a 1-dimensional representation.

**2.2 Elementary operations.** We shall discuss a number of elementary operations on representations that are used frequently in subsequent sections. Let  $G$  be a group,  $H \subset G$  a subgroup, and let  $\sigma: G \rightarrow \mathrm{GL}(V)$  (resp.  $\tau: H \rightarrow \mathrm{GL}(W)$ ) be a representation of  $G$  (resp.  $H$ ).

The *contragredient representation* associated to  $\sigma$  is the representation  $\sigma^\vee: G \rightarrow \mathrm{GL}(V^\vee)$ , with  $V^\vee$  denoting the vector space of (linear) functionals on  $V$ , defined in the following way:  $\sigma^\vee(g)$  maps an element  $f \in V^\vee$  to the one obtained by precomposing it with  $\sigma(g)^{-1}$ , i.e., to the functional that is given by  $v \mapsto f(\sigma(g)^{-1}v)$ .

The *restricted representation* of  $\sigma$  to  $H$  is the composition of the inclusion map  $H \hookrightarrow G$  with  $\sigma$ , denoted  $\mathrm{Res}_H^G \sigma$  or simply  $\sigma|_H$ .

Suppose that  $H$  is of finite index in  $G$ . Then, the *induced representation* of  $\tau$  to  $G$ , denoted  $\mathrm{Ind}_H^G \tau$  or simply  $\tau^G$ , is defined in the following way. We let  $X$  denote the vector space of functions  $f: G \rightarrow W$  such that

$$f(hg) = \tau(h)f(s), \quad \text{for } h \in H, s \in G.$$

Then  $\mathrm{Ind}_H^G \tau$  is the representation of  $G$  in  $X$  such that  $(\mathrm{Ind}_H^G \tau)(g)$  maps a function  $f \in X$  to the one obtained by precomposing it with right multiplication by  $g$ , i.e., to the function  $s \mapsto f(sg)$ . Note that  $\mathrm{Ind}_H^G \tau$  has dimension  $[G : H] \dim(\tau)$ . A representation is called *primitive* if it is not induced by a proper subgroup and *imprimitive* otherwise.

One verifies easily that these operations indeed define representations in accordance with our conventions in 2.1. The same is true if topological groups are involved: these operations shall only be used if  $H$  is a closed subgroup, and, in that case, the smoothness of  $\sigma$  and  $\tau$  implies the smoothness of representations obtained from them via the described operations.

Finally, we state the following well-known proposition that we will use often in the following sections.

**2.3 Proposition (Mackey's Restriction Formula)** — *Let  $H$  and  $K$  be two closed subgroups of a locally profinite group  $G$ , and let  $\rho: H \rightarrow \mathrm{GL}(W)$  be a representation. Then we have a direct sum decomposition*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G \rho = \bigoplus_{s \in H \backslash G / K} \mathrm{Ind}_{K \cap H^s}^K \mathrm{Res}_{K \cap H^s}^{H^s} \rho^s,$$

where  $s$  runs through a set of representatives for the  $(H, K)$  double cosets of  $G$ ,  $H^s$  denotes the subgroup  $s^{-1}Hs$ , and  $\rho^s$  the representation  $x \mapsto \rho(sxs^{-1})$ .

The proof given in [Ser77, Ch. 7, Prop. 22] can be easily generalized to the context of smooth representations of locally profinite groups. We omit the details.

**2.4** In all that follows, an important role will be played by representation of Weil groups. The following terminology is used throughout. If  $\sigma$  is a representation of  $\mathfrak{W}$ , then it is *unramified* (resp. *tamely ramified*) if  $\mathfrak{I} \subseteq \text{Ker}(\sigma)$  (resp.  $\mathfrak{P} \subseteq \text{Ker}(\sigma)$ ). If the restriction of  $\sigma$  to  $\mathfrak{I}$  (resp.  $\mathfrak{P}$ ) is irreducible, then  $\sigma$  is called *tamely irreducible* (resp. *wildly irreducible*).

**2.5** Note that, in general, the kernel of a  $\mathfrak{W}$ -representation  $\sigma$  is not of finite index (as  $\mathfrak{W}$  is not compact); put equivalently, the image of  $\mathfrak{W}$  under  $\sigma$  is not necessarily finite. However, in this thesis we are often interested in restrictions of  $\mathfrak{W}$ -representations to ramification subgroups. Since the intersection  $\text{Ker}(\sigma) \cap \mathfrak{I}$  is an open subgroup of  $\mathfrak{I}$ , it is equal to  $\mathfrak{I} \cap \mathfrak{W}_E$ , for some finite extension  $E/F$ . Hence, the image  $\sigma(\mathfrak{I})$  can be identified with the inertia subgroup of  $\mathcal{G}(E/F)$ ; in particular,  $\sigma(\mathfrak{I})$  is finite. The same holds for the higher ramification subgroups.

By these considerations, a wildly irreducible  $\mathfrak{W}$ -representation gives rise to an irreducible representation of a finite  $p$ -group on the same vector space. Hence, all such representations are of  $p$ -power dimension.

**2.6 Twisting with unramified characters.** Let  $\sigma$  be an irreducible  $\mathfrak{W}$ -representation. Let  $\Phi \in \mathfrak{W}$  be an element that restricts to the Frobenius automorphism  $\Phi$  of  $F_{\text{unr}}$ . Since  $\sigma(\mathfrak{I})$  is finite (2.5), there exists a positive integer  $k$  for which  $\sigma(\Phi)^k$  commutes  $\sigma(\mathfrak{I})$ , and therefore with the whole of  $\sigma(\mathfrak{W})$ . Hence, by Schur's Lemma,  $\sigma(\Phi)^k$  is equal to a homothety, say of ratio  $c$ . We can choose an unramified  $\mathfrak{W}$ -character  $\chi$  such that  $\chi(\Phi)^k = c$ . Then, the twisted representation  $\sigma \otimes \chi$  has finite image, and extends (uniquely) to a representation of  $\mathfrak{G}$  [BH06, N° 28.6].

**2.7 Restriction to normal subgroups.** We will study  $\mathfrak{W}$ -representations by restricting them to the ramification subgroups. The following elementary lemma will be useful to us.

**2.8 Lemma** — *Let  $H$  be a normal subgroup of a group  $G$ , and  $\sigma$  an irreducible  $G$ -representation. Let  $(\rho, W)$  be an isotypic component of the restriction  $\sigma|_H$ , say a multiple of the isomorphism class defined by an irreducible  $H$ -representation  $\alpha$ , and  $N$  the subgroup of elements  $g \in G$  such that  $g(W) = W$ .*

Then,  $N$  is equal to the stabilizer of  $\alpha$ ; it is of finite index and contains  $H$ . Furthermore, considering  $\rho$  as a representation of  $N$ , we have  $\sigma = \text{Ind}_N^G \rho$ .

Also, if  $\alpha$  extends to an representation  $\hat{\alpha}$  of  $N$ , then there exists a canonical evaluation morphism

$$\begin{aligned} \hat{\alpha} \otimes \text{Hom}_H(\hat{\alpha}, \rho) &\longrightarrow \rho, \\ x \otimes f &\longmapsto f(x) \end{aligned}$$

that is an isomorphism of  $N$ -representations.

We shall apply this lemma in the context of smooth representations of locally profinite groups, where  $H$  is a closed normal subgroup. In that case, the subgroup  $N$  is closed as well.

*Proof* — By transport of structure, one sees that the isotypic components of  $\sigma|_H$  are permuted under the action of  $G$ , and since  $\sigma$  is irreducible, these components are permuted transitively. Moreover, the set of isotypic components is isomorphic, as a  $G$ -set, to the orbit of the class  $\alpha$  under conjugation. This proves that  $N$  is the stabilizer of  $\alpha$ ; moreover, we have  $\sigma = \text{Ind}_N^G \rho$ , where  $\rho$  is now considered to be the natural representation  $N \rightarrow \text{GL}(W)$ .

One proves easily the existence of the evaluation morphism mentioned above. To prove that it is a bijection, it suffices to do so after restriction to  $H$ , and we note that the restricted morphism is the *canonical description* of  $\rho$  as an  $\alpha$ -isotypic representation [Bou12, §4, N<sup>o</sup> 5]. **q.e.d.**

**2.9 Lemma** — *Every irreducible representation of  $\mathfrak{P}$  whose isomorphism class is fixed under conjugation by  $\mathfrak{W}$  is extendable to a representation of  $\mathfrak{W}$ .*

For the proof we refer to [DH81, Lemma 4.11].

**2.10 Lemma** — *Let  $\sigma$  be an irreducible representation of  $\mathfrak{W}$ . If the restriction of  $\sigma$  to  $\mathfrak{I}$  is reducible, then  $\sigma$  is induced from an unramified extension of prime degree. If the restriction of  $\sigma$  to  $\mathfrak{I}$  is irreducible, and its restriction to  $\mathfrak{P}$  is reducible, then  $\sigma$  is induced from a totally tamely ramified extension of prime degree.*

*Proof* — Suppose first that the restriction of  $\sigma$  to  $\mathfrak{I}$  is reducible, and let  $\alpha$  be one of its irreducible components. Lemma 2.8 applied to  $H = \mathfrak{I}$  gives an isomorphism

$$\hat{\alpha} \otimes \text{Hom}_{\mathfrak{I}}(\hat{\alpha}, \sigma_\alpha) \xrightarrow{\sim} \sigma_\alpha,$$

where  $\hat{\alpha}$  denotes an extension of  $\alpha$  to its stabilizer (2.9), and  $\sigma_\alpha$  the  $\alpha$ -isotypic component of  $\sigma|_{\mathfrak{I}}$ . The stabilizer of  $\alpha$  is given by a Weil group  $\mathfrak{W}_K$  for some

unramified extension  $K/F$ , and  $\text{Hom}_{\mathfrak{J}}(\hat{\alpha}, \sigma_\alpha)$  is an irreducible representation factorizing through the cyclic group  $\mathfrak{W}_K/\mathfrak{J}$ . Hence, it is of dimension 1, and in particular the multiplicity of  $\alpha$  in  $\sigma_\alpha$  is 1. This shows that  $K/F$  is a proper unramified extension. We have  $\sigma = \text{Ind}_{K/F} \sigma_\alpha$ , and by considering the structure of unramified extensions and the transitivity of induction, one sees that  $\sigma$  is induced from an unramified extension of prime degree.

Next, we suppose that  $\sigma_{|\mathfrak{J}}$  is irreducible and  $\sigma_{|\mathfrak{P}}$  is reducible, and we let  $\beta$  denote an irreducible component appearing in  $\sigma_{|\mathfrak{P}}$ . Then, applying 2.8 to  $H = \mathfrak{P}$ , we obtain an isomorphism

$$\hat{\beta} \otimes \text{Hom}_{\mathfrak{P}}(\hat{\beta}, \sigma_\beta) \xrightarrow{\sim} \sigma_\beta,$$

where  $\hat{\beta}$  denotes an extension of  $\beta$  to its stabilizer, and  $\sigma_\beta$  the  $\beta$ -isotypic component of  $\sigma_{|\mathfrak{P}}$ . This stabilizer is given by a Weil group  $\mathfrak{W}_T$ , where  $T/F$  is a totally tamely ramified extension. Suppose  $T = F$ . The restriction of  $\text{Hom}_{\mathfrak{P}}(\hat{\beta}, \sigma_\beta)$  to  $\mathfrak{J}$ , being irreducible and factorizing through the pro-cyclic group  $\mathfrak{J}/\mathfrak{P}$ , has dimension 1. This implies  $\sigma_{|\mathfrak{P}} = \sigma_\beta = \beta$ , contradicting the initial reducibility assumption. Hence,  $K/F$  is proper, and the desired statement follows similarly as before. **q.e.d.**

### §3 SLOPES OF REPRESENTATIONS

In this section, we associate to each representation of a Weil group the Swan exponent and other related invariants. We shall use these invariants to define a ramification distance, and derive a triangle inequality on (normalized) Swan exponents.

**3.1 Definition** — *Let  $\sigma: \mathfrak{W} \rightarrow \text{GL}(V)$  be a representation.*

i) *The Swan exponent of  $\sigma$  is the number*

$$\text{sw}(\sigma) = \int_0^{+\infty} \text{codim } V^{\mathfrak{W}^u} \, du,$$

where  $V^{\mathfrak{W}^u}$  denotes the subspace of  $\mathfrak{W}^u$ -invariant vectors.

ii) *The normalized Swan exponent of  $\sigma$  is the number*

$$\zeta(\sigma) = \text{sw}(\sigma) / \dim(\sigma),$$

if  $\sigma$  is a non-zero representation; otherwise we put  $\zeta(\sigma) = 0$ .

iii) *The slope of  $\sigma$ , denoted  $\text{sl}(\sigma)$ , is the slope of the invariant field determined by  $\text{Ker}(\sigma)$ ; in other words, we have*

$$\text{sl}(\sigma) = \inf\{u \in \mathbf{R}_+ : \mathfrak{W}^u \subseteq \text{Ker}(\sigma)\}.$$

**3.2** A priori, the invariants above are elements of  $\mathbf{R}_+ \cup \{+\infty\}$ . However, as explained in 2.5, the restriction to  $\mathfrak{J}$  of a  $\mathfrak{W}$ -representation  $\sigma$  factorizes through the inertia group of some finite extension  $E/F$ . Hence, the slope of  $\sigma$  equals  $\text{sl}(E/F)$ , and, consequently, it is a finite number. This implies that all invariants defined above are actually finite.

**3.3** In the literature, one makes often use of the Artin exponent; the relation with the Swan exponent is given by

$$\text{ar}(\sigma) = \text{codim}(V^{\mathfrak{J}}) + \text{sw}(\sigma).$$

**3.4 Proposition** — *If  $\sigma$  and  $\tau$  are  $\mathfrak{W}$ -representations, then the following statements hold.*

i) *The Swan exponent is additive and invariant under duality:*

$$\begin{aligned} \text{sw}(\sigma \oplus \tau) &= \text{sw}(\sigma) + \text{sw}(\tau); \\ \text{sw}(\sigma) &= \text{sw}(\sigma^\vee). \end{aligned}$$

ii) *We have:*

$$\begin{aligned} \text{sl}(\sigma \oplus \tau) &= \max\{\text{sl}(\sigma), \text{sl}(\tau)\}; \\ \text{sl}(\sigma \otimes \tau) &\leq \max\{\text{sl}(\sigma), \text{sl}(\tau)\}. \end{aligned}$$

Moreover, equality holds in the second formula if  $\text{sl}(\sigma) \neq \text{sl}(\tau)$ .

iii) *If  $\sigma$  is irreducible, then*

$$\begin{aligned} \text{sw}(\sigma) &= \text{dim}(\sigma) \text{sl}(\sigma); \\ \zeta(\sigma) &= \text{sl}(\sigma). \end{aligned}$$

iv) *If  $\sigma = \bigoplus_i \sigma_i$  is the decomposition of  $\sigma$  into irreducible components, then*

$$\zeta(\sigma) = \sum_i c_i \text{sl}(\sigma_i) \quad \text{with} \quad c_i = \frac{\text{dim}(\sigma_i)}{\text{dim}(\sigma)}.$$

In particular, we have  $\zeta(\sigma) \leq \text{sl}(\sigma)$ .

*Proof* — The first two statements are straightforward consequences of the definitions. We shall prove the third statement. Suppose  $\sigma$  is irreducible. Then, since all subgroups  $\mathfrak{W}^u$  are normal, the restriction  $\sigma|_{\mathfrak{W}^u}$  of  $\sigma$  to  $\mathfrak{W}^u$  is a sum of conjugated irreducible  $\mathfrak{W}^u$ -representations (2.8). In particular, such a restriction contains the unit character  $\mathbf{1}_{\mathfrak{W}^u}$  if and only if it is trivial. Hence, if  $u < \text{sl}(\sigma)$ ,  $\sigma|_{\mathfrak{W}^u}$  does not contain any invariant vectors. On the other hand, if  $u > \text{sl}(\sigma)$ , then  $\sigma|_{\mathfrak{W}^u}$  is trivial by definition. Hence, we obtain

$$\text{sw}(\sigma) = \int_0^{\text{sl}(\sigma)} \text{dim}(\sigma) \, du = \text{dim}(\sigma) \text{sl}(\sigma).$$

In particular, the slope and normalized Swan exponent of  $\sigma$  are equal.

The final statement follows from the third one, taking into account the additivity of the Swan exponent. **q.e.d.**

**3.5 Theorem** — *The Swan exponent is a (non-negative) integer for every representation of  $\mathfrak{W}$ .*

This is not at all obvious from the definitions; a proof can be found in [Ser79, Ch. VI, §2] (Serre proves the corresponding statement for the Artin conductor; however, by 3.3, this is equivalent with the statement above).

In the following proposition we state without proof some basic properties of the Swan exponent with respect to restricted and induced representations.

**3.6 Proposition** — *Let  $E/F$  be a finite extension of ramification index  $e$ , inertia degree  $f$ , and wild exponent  $w$  (q.v. 1.2.iii). Let  $\sigma$  (resp.  $\tau$ ) be a representation of  $\mathfrak{W}$  (resp.  $\mathfrak{W}_E$ ). Then, the following statements hold.*

i) *If  $\sigma$  is irreducible and  $\text{sl}(\sigma) \geq \text{sl}(E/F)$ , then*

$$\begin{aligned} \text{sl}(\text{Res}_{E/F} \sigma) &= \psi_{E/F}(\text{sl}(\sigma)); \\ \text{sw}(\text{Res}_{E/F} \sigma) &= \psi_{E/F}(\text{sw}(\sigma)). \end{aligned}$$

ii) *Let  $\phi_{E/F}^\infty: u \mapsto w/e + (1/e)u$  be the affine function that coincides with  $\phi_{E/F}$  for values that are large enough (q.v. 1.7). Then, we have*

$$\begin{aligned} \text{sl}(\text{Ind}_{E/F} \tau) &\geq \text{sl}(E/F); \\ \varsigma(\text{Ind}_{E/F} \tau) &= \phi_{E/F}^\infty(\varsigma(\tau)); \\ \text{sw}(\text{Ind}_{E/F} \tau) &= f(w \dim(\tau) + \text{sw}(\tau)). \end{aligned}$$

**3.7 Ramification distance.** Following [Hei96], we shall introduce an ultrametric on the set of non-zero representations of  $\mathfrak{W}$ . Let  $\sigma$  and  $\tau$  be non-zero representations of  $\mathfrak{W}$ . Then, the *ramification distance* between  $\sigma$  and  $\tau$  is the real number given by

$$\Delta(\sigma, \tau) = \inf\{u \geq 0 : \text{Hom}_{\mathfrak{W}^u}(\sigma, \tau) \neq 0\}.$$

We note that  $\Delta(\sigma, \sigma) = 0$  and  $\Delta(\sigma, \tau) = \Delta(\tau, \sigma) = \Delta(\sigma \otimes \tau^\vee, \mathbf{1}_{\mathfrak{W}})$ , where  $\mathbf{1}_{\mathfrak{W}}$  denotes the unit character of  $\mathfrak{W}$ . Equivalently,  $\Delta(\sigma, \tau)$  is defined as the minimum slope of the irreducible components appearing in the decomposition of  $\sigma \otimes \tau^\vee$ .

**3.8 Lemma** — *Let  $\sigma$  and  $\tau$  be irreducible representations. Then, if  $u > \Delta(\sigma, \tau)$ , the restrictions  $\sigma|_{\mathfrak{W}^u}$  and  $\tau|_{\mathfrak{W}^u}$  are integer multiples of one and the same  $\mathfrak{W}^u$ -representation. If  $0 < u \leq \Delta(\sigma, \tau)$ , then the restrictions  $\sigma|_{\mathfrak{W}^u}$  and  $\tau|_{\mathfrak{W}^u}$  have no irreducible components in common.*

*Proof* — For every  $u \in \mathbf{R}_+$ , the irreducible components that appear in the restriction of  $\sigma$ , resp.  $\tau$ , are all  $\mathfrak{W}$ -conjugate and occur with the same multiplicity (2.8). Hence, for  $u > \Delta(\sigma, \tau)$ , both restrictions give rise to the same  $\mathfrak{W}$ -orbit, and so they are an integer multiple of one and the same  $\mathfrak{W}^u$ -representation. The second statement is a direct consequence of the definitions. **q.e.d.**

**3.9 Proposition** — *If  $\sigma, \tau$  and  $\rho$  are irreducible representations of  $\mathfrak{W}$ , then*

$$\begin{aligned} \Delta(\sigma, \tau) &\leq \max\{\Delta(\sigma, \rho), \Delta(\rho, \tau)\}; \\ \zeta(\sigma \otimes \tau^\vee) &\leq \max\{\zeta(\sigma \otimes \rho^\vee), \zeta(\rho \otimes \tau^\vee)\}. \end{aligned}$$

*Proof* — The first inequality follows directly from the previous lemma. Without loss of generality, we assume  $\Delta = \Delta(\sigma, \tau) \leq \Delta(\sigma, \rho)$ . Next we decompose  $\sigma \otimes \sigma^\vee$  into irreducible components and regroup them according to their slope. So we get  $\sigma \otimes \sigma^\vee = \bigoplus_\lambda \sigma_\lambda$ , where  $\sigma_\lambda$  denotes the sum of components of slope  $\lambda$ . Note that  $\sigma_\lambda$  is zero for almost all  $\lambda$ , and  $\zeta(\sigma_\lambda) = \lambda$  if  $\sigma_\lambda \neq 0$ . Hence, the normalized Swan conductor of  $\sigma \otimes \sigma^\vee$  can be calculated as a weighted average

$$\zeta(\sigma \otimes \sigma^\vee) = \sum_\lambda a_\lambda \lambda, \quad a_\lambda = \dim(\sigma_\lambda) / \dim(\sigma)^2.$$

Similarly, we find an expression  $\zeta(\sigma \otimes \tau^\vee) = \sum_\lambda b_\lambda \lambda$ . By the preceding lemma, we know that  $\sigma$  and  $\tau$  are essentially the same when restricted to  $\mathfrak{W}^u$ , for  $u > \Delta$ . Then, the same holds for the pair of representations consisting of  $\sigma \otimes \sigma^\vee$  and  $\sigma \otimes \tau^\vee$ . In particular, this shows  $b_\lambda = a_\lambda$  for  $\lambda > \Delta$ , and  $b_\Delta = \sum_{\lambda \leq \Delta} a_\lambda$ . Therefore, we obtain

$$\zeta(\sigma \otimes \tau^\vee) = \sum_\lambda \max\{\lambda, \Delta(\sigma, \tau)\}.$$

An analogous formula holds where  $\tau$  is replaced by  $\rho$ . Then, obviously,  $\Delta(\sigma, \tau) \leq \Delta(\sigma, \rho) \leq \Delta(\sigma, \rho)$  implies that  $\zeta(\sigma \otimes \tau^\vee) \leq \zeta(\sigma \otimes \rho^\vee)$ . **q.e.d.**

**3.10 Corollary** — *If  $\sigma$  and  $\tau$  are irreducible representations of  $\mathfrak{W}$ , then*

$$\zeta(\sigma \otimes \sigma^\vee) \leq \zeta(\sigma \otimes \tau^\vee).$$

*Proof* — This follows from 3.9 by replacing  $\tau$  with  $\sigma$ . **q.e.d.**





# II

## THE ADJOINT REPRESENTATION

In this chapter, we start off by studying the slope of conjugated characters. This will naturally lead us to define a notion of minimality for representations of Weil groups, which will play an important role in the rest of the thesis. In §6, we introduce a new invariant for so-called homogeneous representations, which is a refinement of the slope invariant. It will prove its value when dealing with tame base changes.

The first example of our main problem stated in the introduction is presented in §6: we consider the composition of a  $\mathfrak{W}$ -representation  $\sigma$  with the adjoint representation of  $\mathrm{GL}$ , and prove an inequality for its (normalized) Swan exponent in terms of that of  $\sigma$ . In the case where  $\sigma$  is a Carayol representation, one can improve the previous result by providing an exact formula for the Swan exponent of  $\mathrm{Ad}(\sigma) = \sigma \otimes \sigma^\vee$ . We state this as a theorem in §7, but we have only managed to prove it under additional hypotheses. In particular, we shall prove it for representations with Swan exponent equal to 1.

The aforementioned results were first observed by Bushnell and Henniart in the context of the local Langlands correspondences on the  $\mathrm{GL}$ -side. Our arguments, however, only invoke the Galois theory of  $F$ , and some elementary representation theory.

We conclude this chapter by applying our results to obtain a different lower bound for the exponent of  $\mathrm{Ad}(\sigma)$ , thereby sharpening an inequality that was proved by Lapid.

## §4 CONJUGATED CHARACTERS

**4.1** We let  $E/F$  be a cyclic Galois extension of finite degree  $l$ ,  $\lambda$  a generator of  $\mathcal{G}(E/F)$ . The kernel  $V$  of the norm map  $N_{E/F}: E^\times \rightarrow F^\times$  is filtered by the subgroups  $V^i = U_E^i \cap V$  with  $i \geq 0$ . We recall that  $E^\times/U_E^0$  identifies with  $\mathbf{Z}$ ,  $U_E^0/U_E^1$  equals the multiplicative group  $\mathbf{k}_E^\times$  of the residue field  $\mathbf{k}_E$ , and that, for  $i \geq 1$ ,  $U_E^i/U_E^{i+1}$  can be identified with  $\mathfrak{p}_E^i/\mathfrak{p}_E^{i+1}$ , which is a one-dimensional vector space over  $\mathbf{k}_E$  that we will also denote by  $\Omega_E^i$ . Hence  $V^0/V^1$  and  $V^i/V^{i+1}$  are subgroups of  $\mathbf{k}_E^\times$  and  $\Omega_E^i$  ( $i \geq 1$ ) respectively.

By Hilbert's Theorem 90, the map  $x \mapsto \lambda^{-1}x = \lambda x/x$  defines a surjection  $E^\times \twoheadrightarrow V$  whose kernel is given by  $F^\times$ ; the following lemma describes its effect on the filtration subgroups.

**4.2 Lemma** — *With the hypotheses and notation as above, the following statements hold.*

i) *Suppose  $E/F$  is unramified. Then, for any uniformizer  $\varpi'$  of  $E$  we get  $\lambda^{-1}\varpi' = 1$ . Furthermore,  $\lambda - 1$  maps  $U_E^i$  into  $U_E^i$  for all  $i \geq 0$ , and therefore induces, by passage to quotients, homomorphisms  $\mathbf{k}_E^\times \rightarrow \mathbf{k}_E^\times$  and  $\Omega_E^i \rightarrow \Omega_E^i$ . Identifying  $\Omega_E^i$  with  $\mathbf{k}_E \otimes_{\mathbf{k}} \Omega_F^i$  and writing  $\bar{\lambda}$  for the automorphism of  $\mathbf{k}_E$  induced by  $\lambda$ , the latter maps are given by*

$$\begin{aligned} \mathbf{k}_E^\times &\twoheadrightarrow V^0/V^1 \subseteq \mathbf{k}_E^\times \\ \xi &\mapsto \bar{\lambda}^{-1}\xi, \\ \mathbf{k}_E \otimes_{\mathbf{k}} \Omega_F^i &\twoheadrightarrow V^i/V^{i+1} \subseteq \mathbf{k}_E \otimes_{\mathbf{k}} \Omega_F^i \quad (i \geq 1) \\ \xi \otimes z &\mapsto (\bar{\lambda} - 1)\xi \otimes z. \end{aligned}$$

Moreover,  $V^0/V^1$  is the kernel of the norm map of the residue extension, and, for  $i \geq 1$ ,  $V^i/V^{i+1}$  identifies with  $W \otimes \Omega_F^i$  with  $W = \text{Ker}(\text{Tr}_{\mathbf{k}_E/\mathbf{k}})$ .

ii) *Suppose  $E/F$  is totally tamely ramified. Then,  $\lambda - 1$  induces a map*

$$E^\times/U_E^0 \twoheadrightarrow V^0/V^1 \subseteq \mathbf{k}_E^\times,$$

whose image consists of the  $l$ -th roots of unity of  $\mathbf{k}_E^\times$ , and on the residue field it induces the trivial map  $\mathbf{k}_E^\times \xrightarrow{0} \{1\} \subseteq \mathbf{k}_E^\times$ . Furthermore,  $\lambda - 1$  maps  $U_E^i$  into  $U_E^i$  for all  $i \geq 1$ , and we obtain  $\mathbf{k}$ -linear maps

$$\begin{aligned} \Omega_E^i &\xrightarrow{\sim} V^i/V^{i+1} = \Omega_E^i, & \text{if } \mathfrak{p} \nmid i, \\ \Omega_E^i &\twoheadrightarrow V^i/V^{i+1} = \{0\} \subseteq \Omega_E^i, & \text{if } \mathfrak{p} \mid i. \end{aligned}$$

iii) *Suppose  $E/F$  is totally wildly ramified, and  $\delta = \nu_E(\lambda) - 1$  (so  $\delta$  is an integer  $\geq 1$  and  $\lambda \in \mathcal{G}(E/F)_\delta - \mathcal{G}(E/F)_{\delta+1}$ ). Then,  $\lambda - 1$  induces a map*

$$E^\times/U_E^0 \twoheadrightarrow V^\delta/V^{\delta+1} \subseteq \Omega_E^\delta,$$

whose image is a cyclic group of order  $\mathfrak{p}$ . Furthermore, for  $i \geq 1$ ,  $\lambda - 1$  maps  $U_E^i$  into  $U_E^{\delta+i}$ , and we obtain  $\mathbf{k}$ -linear maps

$$\begin{aligned}\Omega_E^i &\xrightarrow{\sim} V^{\delta+i}/V^{\delta+i+1} = \Omega_E^{\delta+i}, & \text{if } \mathfrak{p} \nmid i, \\ \Omega_E^i &\xrightarrow{0} V^{\delta+i}/V^{\delta+i+1} \subseteq \Omega_E^{\delta+i}, & \text{if } \mathfrak{p} \mid i.\end{aligned}$$

(The second arrow denotes the zero-map.) If  $l$  is prime, then  $V^{\delta+i} = V^{\delta+i+1}$  for all  $i$  divisible by  $\mathfrak{p}$ .

*Proof.*

i) The first part of the statement is trivial as any uniformizer  $\varpi'$  of  $E$  is also a uniformizer of  $F$ . Clearly,  $\lambda - 1$  induces the endomorphism  $\bar{\lambda} - 1$  on  $\mathbf{k}_E^\times$ . Also, if  $x = 1 + u\varpi^i$  is an element of  $\Omega_E^i$ ,  $i \geq 1$ , then

$$\begin{aligned}\lambda^{-1}x &= 1 + (\lambda x - x)/x, \\ &\equiv 1 + (\lambda - 1) \cdot u\varpi^i \pmod{\mathfrak{p}_E^{i+1}},\end{aligned}$$

which shows that  $\lambda - 1$  induces the endomorphism  $(\bar{\lambda} - 1) \otimes \text{id}$  on  $\Omega_E^i = \mathbf{k}_E \otimes \Omega_F^i$ .

Any non-trivial class of  $E^\times/F^\times$  can be represented by an element of the form  $u$  or  $(1 + u\varpi^i)$  with  $u \in \mathfrak{o}_E^\times - \mathfrak{o}^\times$ ; from this it easily follows that the endomorphisms described above have image  $V^0/V^1$  and  $V^i/V^{i+1}$  respectively.

Lastly, the final two statements follow from the multiplicative and additive version respectively of Hilbert's 90 theorem applied on the residue field extension.

ii) Let  $\varpi'$  be a uniformizer of  $E$  with  $\varpi'^l \in F$ . Then  $\lambda^{-1}\varpi' = \zeta$  is a primitive  $l$ -th of unity in  $E^\times$ . Since  $E/F$  is assumed to be totally ramified,  $\lambda - 1$  is the trivial map on  $\mathbf{k}_E^\times$ . Also, if  $x = 1 + u\varpi'^i$  is an element of  $\Omega_E^i$ ,  $i \geq 1$ , then

$$\begin{aligned}\lambda^{-1}x - 1 &= (\lambda x - x)/x, \\ &\equiv (u - \lambda u\zeta^i)\varpi'^i \pmod{\mathfrak{p}_E^{i+1}}, \\ &\equiv (1 - \zeta^i)u\varpi'^i \pmod{\mathfrak{p}_E^{i+1}}.\end{aligned}$$

Hence, we obtain a map  $\Omega_E^i \rightarrow \Omega_E^i$  of 1-dimensional  $\mathbf{k}$ -vector spaces that is an isomorphism if  $\mathfrak{p}$  does not divide  $i$ , and the zero-map otherwise.

The above implies that  $\lambda^{-1}\varpi'$  generates  $V^0/V^1$ , and since any non-trivial class of  $E^\times/F^\times$  can be represented by an element  $\varpi'^k$  or  $\varpi'^k(1 + u\varpi'^i)$  with  $k \in \{0, 1, \dots, l-1\}$ ,  $u \in \mathfrak{o}_E^\times$  and  $\mathfrak{p} \nmid i$ , it follows easily that  $\lambda - 1$  maps  $\Omega_E^i$  surjectively onto  $V^i/V^{i+1}$ ; in particular,  $V^i = V^{i+1}$  if  $\mathfrak{p} \mid i$ .

iii) Since a uniformizer  $\varpi'$  of  $E$  generates  $\mathfrak{o}_E$  as an algebra over  $\mathfrak{o}$ , we have

$$\nu_E(\lambda^{-1}\varpi' - 1) = \nu_E(\lambda\varpi' - \varpi') - 1 = \nu_E(\lambda) - 1 = \delta,$$

and, clearly, the image of  ${}^{\lambda-1}\varpi'$  in  $\Omega_E^\delta$  has order  $\mathfrak{p}$ . As in the previous case,  $\lambda - 1$  acts trivially on  $\mathbf{k}_E^\times$ . Also, if  $x = 1 + u\varpi'^i$  is an element of  $U_E^i$ ,  $i \geq 1$ , then, setting  ${}^\lambda\varpi' = \varpi'(1+a)$  with  $a \in \mathfrak{p}_E^\delta$ , we get

$$\begin{aligned} {}^{\lambda-1}x - 1 &= ({}^\lambda x - x)/x, \\ &= \left( {}^\lambda u \varpi'^i (1+a)^i - u \varpi'^i \right) / x \quad \text{mod } \mathfrak{p}_E^{\delta+i+1}, \\ &\equiv \left( {}^\lambda u ((1+a)^i - 1) + ({}^\lambda u - u) \right) \varpi'^i \quad \text{mod } \mathfrak{p}_E^{\delta+i+1}, \\ &\equiv i a {}^\lambda u \varpi'^i \quad \text{mod } \mathfrak{p}_E^{\delta+i+1}. \end{aligned}$$

We get a map  $\Omega_E^i \rightarrow \Omega_E^{\delta+i}$  of 1-dimensional  $\mathbf{k}$ -vector spaces that is an isomorphism if  $\mathfrak{p}$  does not divide  $i$ , and the zero-map otherwise.

The above shows that the image of  ${}^{\lambda-1}\varpi'$  in  $\Omega_E^\delta$  generates  $V^\delta/V^{\delta+1}$ . If  $l$  is prime, then clearly  $l = \mathfrak{p}$ , and every non-trivial class of  $E^\times/F^\times$  can be represented by an element  $\varpi'^k$  or  $\varpi'^k(1+u\varpi'^i)$  with  $k \in \{0, 1, \dots, \mathfrak{p}-1\}$ ,  $u \in \mathfrak{o}_E^\times$  and  $\mathfrak{p} \nmid i$ . Hence,  $\lambda - 1$  induces a surjective map  $\Omega_E^{\delta+i} \twoheadrightarrow V^{\delta+i}/V^{\delta+i+1}$  for all  $i$ , and the final statement follows. **q.e.d.**

**4.3** In the proof above we have determined the greatest index  $i$  for which  ${}^{\lambda-1}x \in U_E^i$ , where  $x$  is an element of  $E^\times$ . This obviously depends on the corresponding greatest index of the element  $x$ , but only up to multiplication with elements of  $F^\times$ . In a similar way, in order to determine the slope of a character  $\chi^{\lambda-1} = \chi^\lambda/\chi$  of  $\mathfrak{W}_E$ , we only need to consider  $\chi$  modulo characters that are restricted from  $\mathfrak{W}$ . These considerations lead to the following definition.

**4.4 Definition** — *Let  $E/F$  be a finite extension. A representation  $\tau$  of  $\mathfrak{W}_E$  is called  $F$ -minimal if  $\text{sl}(\tau \otimes \chi_{|\mathfrak{W}_E}) \geq \text{sl}(\tau)$ , for all characters  $\chi$  of  $\mathfrak{W}$ .*

In case  $E$  equals  $F$ , we shall write minimal instead of  $F$ -minimal.

**4.5 Lemma** — *Let  $E/F$  be a finite Galois extension with Galois group  $\mathcal{G}(E/F)$ , and  $\lambda$  an element of the wild ramification subgroup  $\mathcal{P}(E/F)$  of  $\mathcal{G}(E/F)$ . Let  $\chi$  denote a character of  $\mathfrak{W}_E$  and put  $\delta = \nu_E(\lambda) - 1$ ,  $\varsigma = \varsigma(\chi)$ .*

*If  $\varsigma \leq \delta$ , then  $\varsigma(\chi^{\lambda-1}) = 0$ ; if  $\varsigma > \delta$ , then*

$$\varsigma(\chi^{\lambda-1}) \leq \varsigma - \delta,$$

*with equality if and only if  $\varsigma \not\equiv \delta \pmod{\mathfrak{p}}$ . Furthermore, if equality holds, then  $\chi$  is  $F$ -minimal.*

*Proof* — Let  $F' \subseteq E$  be the invariant subfield corresponding to  $\lambda$ , so  $E/F'$  is a cyclic wildly ramified extension, and we can apply Lemma 4.2.iii.

Hence, if  $\varsigma \leq \delta$ , then  $\lambda - 1$  maps  $\mathbf{U}_E^1$  into  $V^{\delta+1}$  (with  $V = \text{Ker}(N_{E/F'})$ ) and the notation as in 4.2), which is contained in  $\text{Ker}(\chi)$ ; the first statement follows. Next, suppose  $\varsigma > \delta$ . Then, again by 4.2.iii,  $\lambda - 1$  maps  $\mathbf{U}^i$  into  $\mathbf{U}^{\delta+i}$  for all  $i \geq 1$ , and maps  $\mathbf{U}^i$  into  $\mathbf{U}^{\delta+i+1}$  for all  $i \geq 1$  divisible by  $\mathfrak{p}$ . This proves the inequality as well as the necessary and sufficient condition in which equality holds.

We prove the final statement by contradiction: if  $\chi$  is not  $F$ -minimal, then it factorizes through  $N_{E/F'}$ , so  $V^\varsigma \mathbf{U}_E^{\varsigma+1}$  is contained in  $\text{Ker}(\chi)$ . Using Lemma 4.2, our hypothesis on  $\varsigma$  implies  $V^\varsigma \mathbf{U}_E^{\varsigma+1} = \mathbf{U}_E^\varsigma$ , hence  $\mathbf{U}_E^\varsigma \subseteq \text{Ker}(\chi)$ . This is absurd. **q.e.d.**

**4.6 Lemma** — *Let  $E/F$  be a cyclic extension that is of prime degree  $l$  and slope  $\delta = \text{sl}(E/F)$ , and  $\lambda$  a generator of  $\mathcal{G}(E/F)$ .*

*If  $\chi$  is an  $F$ -minimal character of  $\mathfrak{W}_E$  with  $\varsigma = \varsigma(\chi)$ , then*

$$\varsigma(\chi^{\lambda^{-1}}) = \max(\varsigma - \delta, 0).$$

*Proof* — First, suppose that  $E/F$  is unramified. Clearly, it holds  $\varsigma(\chi^{\lambda^{-1}}) \leq \varsigma$ . If  $\varsigma(\chi^{\lambda^{-1}}) < \varsigma$ , then  $\chi$  induces a character on  $\mathbf{\Omega}_E^\varsigma = \mathbf{k}_E \otimes_{\mathbf{k}} \mathbf{\Omega}_F^\varsigma$  whose kernel contains  $W \otimes_{\mathbf{k}} \mathbf{\Omega}_F^\varsigma$ , where  $W = \text{Ker}(\text{Tr}_{\mathbf{k}/k})$  (4.2.i). Hence,  $V^\varsigma$  is contained in the kernel of  $\chi$ , so the restriction of  $\chi$  to  $\mathbf{U}_E^\varsigma$  factorizes through  $N_{E/F}$ . This contradicts the  $F$ -minimality of  $\chi$ .

Next, suppose that  $E/F$  is tamely ramified. As before, we have  $\varsigma(\chi^{\lambda^{-1}}) \leq \varsigma$ . If this inequality is strict, then necessarily  $\varsigma$  is divisible by  $\mathfrak{p}$  (4.2.ii). Then, it holds  $V^\varsigma = V^{\varsigma+1}$ , so the restriction of  $\chi$  to  $\mathbf{U}_E^\varsigma$  factorizes through  $N_{E/F}$ ; we obtain the same contradiction as before.

Finally, we suppose  $E/F$  is wildly ramified. We are then in the situation of lemma 4.5. Hence, it suffices to prove that  $\varsigma > \delta$  implies  $\varsigma \not\equiv \delta \pmod{\mathfrak{p}}$ . If  $\varsigma > \delta$  and  $\varsigma \equiv \delta \pmod{\mathfrak{p}}$ , then  $V^\varsigma = V^{\varsigma+1}$ , and we can finish the proof as in the previous two cases. **q.e.d.**

## §5 TAME BASE CHANGE

**5.1 The  $g$ -invariant.** Let  $\sigma$  be a wild representation of  $\mathfrak{W}$ , say of slope  $s = \text{sl}(\sigma) > 0$ . We suppose that the restriction  $\sigma$  to  $\mathfrak{W}^s$  is a multiple of a single character  $\chi_\sigma$  (i.e.,  $\mathfrak{W}^s$  acts centrally through  $\chi_\sigma$ ). Following [Hen84, N°4], we will call such representations *homogeneous* and we will introduce an invariant for them that is finer than the slope.

By the same considerations as in 2.5, there exists a finite Galois extension  $K/F$ , fixed by  $\mathfrak{W}^s$ , such that the restriction of  $\sigma$  to  $\mathfrak{W}_K$  is a multiple

of a single  $\mathfrak{W}_K$ -character  $\chi$ . Clearly,  $\chi$  extends  $\chi_\sigma$ , and by local class field theory we identify it with a character of  $K^\times$ . On the group  $1 + \mathfrak{p}_K^{\varsigma(\chi)}$ ,  $\chi$  is of the form

$$1 + x \mapsto \Psi \circ \mathrm{Tr}_{K/F}(gx),$$

where  $g$  is a certain element of  $K^\times$ , uniquely determined modulo  $1 + \mathfrak{p}_K$ , and  $\Psi$  is some fixed additive character of  $F$ . Furthermore, since  $\chi$  is stabilized under the conjugation action of  $\mathfrak{W}$ , one has  $g^\sigma \equiv g$  modulo  $1 + \mathfrak{p}_K$  for all elements in  $\mathcal{G}(K/F)$ .

Writing  $C_E = E^\times / (1 + \mathfrak{p}_E)$  for all finite extension  $E/F$ , we have attached above to  $\sigma$  (in a non-canonical way) a  $\mathfrak{W}$ -invariant element  $\bar{g}$  of  $C_K$ . The set of such invariant elements can be identified with the group  $C_F \otimes \mathbf{Z}[1/\mathfrak{p}]$  [Hen84, Lemma 1]. Hence, we associated to  $\sigma$  an element  $g_\sigma$  of  $C_F \otimes \mathbf{Z}[1/\mathfrak{p}]$ , and one proves that  $g_\sigma$  is independent of the choice of  $K$  [Hen84, Lemma 2] (more precisely,  $g_\sigma$  only depends on the restriction of  $\sigma$  to  $\mathfrak{W}^s$ ).

**5.2** A wild representation  $\sigma$  of  $\mathfrak{W}$  that restricts irreducibly to  $\mathfrak{P}$  is homogeneous. This can be seen as follows. After twisting with an unramified character (2.6), we can assume that  $\sigma$  factorizes through a Galois group  $\mathcal{G}(N/F)$  for some finite Galois extension  $N/F$ . Let  $T/F$  denote the maximal tamely ramified subextension of  $N/F$ . Then the last non-trivial ramification subgroup of  $\mathcal{G}(N/F)$  is a central subgroup of its wild inertia subgroup  $\mathcal{G}(N/T)$  (1.4). By assumption,  $\sigma_T$  is irreducible. Hence, by Schur's Lemma, we obtain that  $\sigma$  is homogeneous.

**5.3** For a finite extension  $E/F$ , the valuation map  $v_E: E^\times \rightarrow \mathbf{Z}$  induces a valuation map  $C_E \otimes \mathbf{Z}[1/\mathfrak{p}] \rightarrow \mathbf{Z}[1/\mathfrak{p}]$  in the obvious way; we shall use the same symbol  $v_E$  to denote the latter map. The image of  $C_E$  in  $C_E \otimes \mathbf{Z}[1/\mathfrak{p}]$  is precisely the inverse image of  $\mathbf{Z}$  under  $v_E$ .

**5.4** One can think of the  $g$ -invariant of a homogeneous representation  $\sigma$  as a refinement of the slope, as  $s = \mathrm{sl}(\sigma) (= \varsigma(\sigma))$  can be expressed in terms of  $v_F(g_\sigma)$ . Using the notation of 5.1, we have

$$\begin{aligned} v_K(g_\sigma) &= -(\nu_K + 1) - \varsigma(\chi) \\ &= -e(K/F)(\nu_F + 1) - (w(K/F) + \varsigma(\chi)); \\ v_F(g_\sigma) &= -(\nu_F + 1) - s, \end{aligned} \tag{3.6.ii}$$

where  $\nu_F$  and  $\nu_K$  denote the levels of the additive characters  $\Psi$  and  $\Psi \circ \mathrm{Tr}_{K/F}$ , respectively (here, the *level* of an additive character  $\psi$  of a finite extension  $E/F$  is the largest integer  $i$  such that  $\mathfrak{p}_E^{-i} \subseteq \mathrm{Ker} \psi$ ).

**5.5 Proposition** — *Let  $E/F$  be a tamely ramified finite extension of tame degree  $l$ , and  $\sigma$  a  $\mathfrak{W}$ -representation that is minimal. Then, the restriction of  $\sigma$  to  $\mathfrak{W}_E$  is  $F$ -minimal.*

*Proof* — The proposition is trivial in the case where  $\sigma$  is tame, so we assume that  $\sigma$  is wild. Suppose the restriction  $\sigma' = \sigma|_{\mathfrak{W}_E}$  is non-minimal, then  $\sigma'$  is homogeneous, and, since  $E/F$  is tame, this implies that  $\sigma$  is homogeneous as well. Hence,  $g_\sigma$  and  $g_{\sigma'}$  are defined; moreover, the image of  $g_\sigma$  in  $C_E \otimes \mathbf{Z}[1/\mathfrak{p}]$  is equal to  $g_{\sigma'}$ . Furthermore,  $g_{\sigma'}$  is determined by an element of  $E^\times$ , so it has an integer valuation under  $v_E$ . We have the following commutative diagram

$$\begin{array}{ccc} C_F \otimes \mathbf{Z}[1/\mathfrak{p}] & \xrightarrow{\quad} & C_E \otimes \mathbf{Z}[1/\mathfrak{p}] \\ \downarrow v_F & & \downarrow v_E \\ \mathbf{Z}[1/\mathfrak{p}] & \xrightarrow{\quad \times l \quad} & \mathbf{Z}[1/\mathfrak{p}] \end{array}$$

and since  $\mathfrak{p} \nmid l$ , this shows that  $g_\sigma$  has an integer valuation under  $v_F$ . Hence,  $g_\sigma$  is determined by an element  $c$  of  $F^\times$ . In particular, there exists a character of  $\mathfrak{W}$  that restricts to  $\chi_\sigma$ . This contradicts the minimality of  $\sigma$ . **q.e.d.**

## §6 LOWER BOUND FOR THE CONDUCTOR

The following theorem contains the main result of this chapter.

**6.1 Theorem** — *Let  $\sigma$  be an irreducible representation of  $\mathfrak{W}$  that is minimal and of dimension  $\geq 2$ . If  $\ell_\sigma$  is the smallest prime number dividing the dimension of  $\sigma$ , then*

$$\varsigma(\sigma \otimes \sigma^\vee) \geq (1 - 1/\ell_\sigma) \varsigma(\sigma). \tag{6.1.1}$$

Notice that, in the other direction, the inequality  $\varsigma(\sigma \otimes \sigma^\vee) \leq \varsigma(\sigma)$  holds. Some kind of minimality condition on  $\sigma$  is necessary in the theorem, since twisting by a character can alter its conductor but does not affect the adjoint representation  $\sigma \otimes \sigma^\vee$ . Finally, let us remark that the statement is trivial in case  $\sigma$  is of dimension 1, as both sides of the inequality are then equal to 0.

**6.2 Lemma (tame restriction)** — *We let the assumptions and notation be as in 6.1. We suppose furthermore that  $E/F$  is a tamely ramified extension such that the restriction  $\sigma|_E$  of  $\sigma$  to  $\mathfrak{W}_E$  is irreducible. Then,  $\sigma|_E$  is minimal, and the inequality 6.1.1 holds for  $\sigma$  if and only if it holds for  $\sigma|_E$ .*



*Proof* — The minimality of  $\sigma|_E$  is proved in 5.5. Since  $E/F$  is tame, the ramification subgroups of  $\mathfrak{W}$  and  $\mathfrak{W}_E$  are the same, and their numbering is related via the corresponding Herbrand function, which is the linear function given by  $\psi_{E/F}(u) = ku$ , where  $k$  denotes the tame degree of  $E/F$ . Hence, the statement follows. **q.e.d.**

**6.3 Lemma (tame induction)** — *We let the assumptions and notation be as in 6.1. We suppose furthermore that  $\sigma = \text{Ind}_{E/F} \tau$  for some cyclic tame extension  $E/F$  of prime degree  $l$ , and we write  $\tau = \rho \otimes \chi$  with  $\rho$  a minimal  $\mathfrak{W}_E$ -representation and  $\chi$  a  $\mathfrak{W}_E$ -character. Suppose the inequality 6.1.1 holds for  $\rho$ , then it also holds for  $\sigma$ .*

*Proof* — Using Mackey's Restriction Formula (2.3) we obtain the following decomposition

$$\begin{aligned} \sigma \otimes \sigma^\vee &= \text{Ind}_{E/F}(\tau) \otimes \text{Ind}_{E/F}(\tau^\vee) \\ &= \text{Ind}_{E/F}(\tau \otimes \text{Res}_{E/F} \text{Ind}_{E/F}(\tau^\vee)) \\ &= \text{Ind}_{E/F} \bigoplus_{\lambda \in \mathcal{G}(E/F)} (\tau \otimes (\tau^\vee)^\lambda). \end{aligned}$$

By 3.6, we have

$$\begin{aligned} \varsigma(\sigma) &= \phi_{E/F}^\infty(\varsigma(\tau)); \\ \varsigma(\sigma \otimes \sigma^\vee) &= \phi_{E/F}^\infty\left(1/l \sum_{\lambda} \varsigma(\tau \otimes (\tau^\vee)^\lambda)\right), \end{aligned}$$

where  $\phi_{E/F}^\infty$  is the linear function  $x \mapsto (1/l')x$ , with  $l'$  denoting the tame degree of  $E/F$ .

► Suppose  $\tau$  is minimal, then  $\varsigma(\tau) = \varsigma(\rho)$  and the inequality 6.1.1 is true for  $\tau$ . If  $\dim(\tau) > 1$ , we let  $\ell_\tau$  denote the smallest prime dividing  $\dim(\tau)$ ; otherwise we set  $\ell_\tau = \ell_\sigma$ . Then, for all  $\lambda \in \mathcal{G}(E/F)$ , we have

$$\begin{aligned} \varsigma(\tau \otimes (\tau^\vee)^\lambda) &\geq \varsigma(\tau \otimes \tau^\vee) \\ &\geq (1 - 1/\ell_\tau) \varsigma(\tau) \\ &\geq (1 - 1/\ell_\sigma) \varsigma(\tau), \end{aligned} \tag{3.10}$$

and by taking the average over all  $\lambda$ , we find

$$(1/l) \sum_{\lambda} \varsigma(\tau \otimes (\tau^\vee)^\lambda) \geq (1 - 1/\ell_\sigma) \varsigma(\tau).$$

► Suppose  $\tau$  is not minimal, so we have  $\varsigma(\chi) = \varsigma(\tau) > \varsigma(\rho)$ . Then

$$\tau \otimes (\tau^\vee)^\lambda = \rho \otimes (\rho^\vee)^\lambda \otimes \chi^{1-\lambda}.$$

For all  $\lambda \neq \varepsilon$ , we have  $\varsigma(\chi^{1-\lambda}) = \varsigma(\tau)$  by 4.6, and since  $\rho \otimes (\rho^\vee)^\lambda$  has trivial restriction on  $\mathfrak{W}_E^{\varsigma(\tau)}$ , this implies

$$\varsigma(\tau \otimes (\tau^\vee)^\lambda) = \varsigma(\chi^{1-\lambda}) = \varsigma(\tau),$$

We get the same estimate as before:

$$\begin{aligned} (1/l) \sum_{\lambda} \varsigma(\tau \otimes (\tau^\vee)^\lambda) &\geq (1 - 1/l) \varsigma(\tau) \\ &\geq (1 - 1/\ell_\sigma) \varsigma(\tau). \end{aligned}$$

Hence, in both cases we find

$$\begin{aligned} \varsigma(\sigma \otimes \sigma^\vee) &\geq \phi_{E/F}^\infty((1 - 1/\ell_\sigma) \varsigma(\tau)) \\ &= (1 - 1/\ell_\sigma) \phi_{E/F}^\infty(\varsigma(\tau)) \\ &= (1 - 1/\ell_\sigma) \varsigma(\sigma). \end{aligned} \quad \mathbf{q.e.d.}$$

*Proof (of theorem 6.1)* — If the dimension of  $\sigma$  equals 1, then  $\sigma$  is a tame character, and the statement of the theorem is trivial. Hence, we assume  $\dim(\sigma) > 1$ , and we give a proof by induction on the dimension of  $\sigma$ .

If the restriction of  $\sigma$  to  $\mathfrak{J}$  is reducible, then  $\sigma = \text{Ind}_{E/F} \tau$  for some unramified extension of prime degree (2.10), and the inequality for  $\sigma$  follows via tame cyclic induction (6.3) and the induction hypothesis. So henceforth we assume that  $\sigma_{\mathfrak{J}}$  is irreducible.

Similarly, if  $\sigma_{\mathfrak{B}}$  is reducible, then  $\sigma = \text{Ind}_{E/F} \tau$  for some totally tamely ramified extension  $E/F$  of prime degree (2.10). We put  $l = [E : F]$ , and we let  $E'$  be the normal closure of  $E$  over  $F$ , and  $F'/F$  the maximal unramified subextension of  $E'/F$ . Then  $F'$  is obtained from  $F$  by adjoining a non-trivial  $l$ -th root of unity, and  $E'$  is the compositum of  $E$  and  $F'$ . We let  $\sigma'$  and  $\tau'$  denote the restrictions of  $\sigma$  and  $\tau$  to  $\mathfrak{W}_{F'}$  and  $\mathfrak{W}_{E'}$ , respectively. Since  $\sigma_{\mathfrak{J}}$  is irreducible, also  $\sigma'$  is irreducible; moreover, by 5.5,  $\sigma'$  is minimal. Observe that  $E \cap F' = F$ , hence  $E'/F'$  is cyclic of degree  $l$  and from Mackey's Restriction Formula (2.3) we get  $\sigma' = \text{Ind}_{E'/F'} \tau'$ . By the same reasoning as above, using tame cyclic induction (6.3) and the induction hypothesis, one obtains inequality 6.1.1 for  $\sigma'$ . Then, via tame restriction (6.2), the same inequality also holds for  $\sigma$ .

Hence, we have reduced to the case where  $\sigma_{\mathfrak{B}}$  is irreducible. By twisting  $\sigma$  with an unramified character we can assume that  $\sigma$  factorizes through a finite Galois group, say  $\mathcal{G}(N/F)$  (q.v. 2.6). Tame restriction (6.2) allows us to replace  $\sigma$  by its restriction to the maximal tamely ramified subextension of  $N/F$ . Hence, we can assume  $\sigma(\mathfrak{W}) = \sigma(\mathfrak{B})$ ; in particular,  $\sigma(\mathfrak{W})$  is then a  $\rho$ -group, and  $\sigma$  has a cyclic inducing extension  $E/F$  that is wildly ramified of degree  $\rho$ . We write  $\sigma = \text{Ind}_{E/F} \tau$  for some  $\mathfrak{W}_E$ -representation  $\tau$ , and  $s = \text{sl}(E/F)$ . We shall finish this proof as in 6.3.

The representation  $\sigma \otimes \sigma^\vee$  decomposes in the following way:

$$\sigma \otimes \sigma^\vee = \text{Ind}_{E/F} \bigoplus_{\lambda \in \mathcal{G}(E/F)} (\tau \otimes (\tau^\vee)^\lambda),$$

and, using [3.6.ii](#), we have

$$\begin{aligned} \varsigma(\sigma) &= (1 - 1/\mathfrak{p})s + \varsigma(\tau)/\mathfrak{p}; \\ \varsigma(\sigma \otimes \sigma^\vee) &= (1 - 1/\mathfrak{p})s + \left( \sum_{\lambda} \varsigma(\tau \otimes (\tau^\vee)^\lambda) \right) / \mathfrak{p}^2. \end{aligned}$$

Let us write  $\tau = \rho \otimes \chi$ , where  $\rho$  is some minimal  $\mathfrak{W}_E$ -representation and  $\chi$  a  $\mathfrak{W}_E$ -character. Observe that  $\chi$  is  $F$ -minimal, for if that were not the case, so that  $\chi|_{\mathfrak{W}^s(\chi)}$  were equal to  $\tilde{\chi}|_{\mathfrak{W}^s(\chi)}$  for some  $\mathfrak{W}$ -character  $\tilde{\chi}$ , then  $\sigma \otimes \tilde{\chi} = \text{Ind}_{E/F}(\tau \otimes \tilde{\chi}|_{\mathfrak{W}_E})$  would be a character twist of  $\sigma$  of strictly smaller slope. This contradicts the minimality assumption on  $\sigma$ .

► Suppose  $\varsigma(\rho) \geq \varsigma(\tau) - s$  holds. Then, for all  $g \in \mathcal{G}(E/F)$ , we have

$$\begin{aligned} \varsigma(\tau \otimes (\tau^g)^\vee) &\geq \varsigma(\tau \otimes \tau^\vee) && \text{(3.10)} \\ &\geq (1 - 1/\mathfrak{p})\varsigma(\rho) \\ &\geq (1 - 1/\mathfrak{p})(\varsigma(\tau) - s) \end{aligned}$$

and consequently

$$\begin{aligned} \varsigma(\sigma \otimes \sigma^\vee) &\geq (1 - 1/\mathfrak{p})s + \left( (1 - 1/\mathfrak{p})(\varsigma(\tau) - s) \right) / \mathfrak{p} \\ &= (1 - 1/\mathfrak{p}) \left( (1 - 1/\mathfrak{p})s + \varsigma(\tau)/\mathfrak{p} \right) \\ &= (1 - 1/\ell_\sigma)\varsigma(\sigma) \end{aligned}$$

(note that  $\ell_\sigma = \mathfrak{p}$  in this case).

► Suppose  $\varsigma(\rho) < \varsigma(\tau) - s$  holds, then we have  $\varsigma(\rho \otimes (\rho^\vee)^\lambda) < \varsigma(\tau) - s$ . For all  $\lambda \neq \varepsilon$ ,  $\varsigma(\chi^{1-\lambda}) = \varsigma(\tau) - s$  by [4.6](#), and therefore

$$\varsigma(\tau \otimes (\tau^\vee)^\lambda) = \varsigma(\rho \otimes (\rho^\vee)^\lambda \otimes \chi^{1-\lambda}) = \varsigma(\tau) - s.$$

We obtain the same inequality as we did before:

$$\begin{aligned} \varsigma(\sigma \otimes \sigma^\vee) &\geq (1 - 1/\mathfrak{p})s + (\mathfrak{p} - 1)(\varsigma(\tau) - s) / \mathfrak{p}^2 \\ &= (1 - 1/\ell_\sigma)\varsigma(\sigma). \end{aligned} \quad \text{q.e.d.}$$

## §7 CARAYOL REPRESENTATIONS

**7.1 Definition** — *Let  $\sigma$  be an irreducible representation of  $\mathfrak{W}$ . Then  $\sigma$  is of Carayol type (or simply Carayol) if  $\text{sw}(\sigma)$  is prime to  $\dim(\sigma)$ ; it is epipelagic if  $\text{sw}(\sigma) = 1$ .*

We note that our definition of a Carayol representation is less restrictive than the usual one to be found in the literature (in [BH19] Carayol representations are assumed to be wildly irreducible). Clearly, all epipelagic representations are Carayol.

The proposition below gathers some properties of Carayol and epipelagic representations that will be used in the sequel.

**7.2 Proposition.**

- i) All Carayol representations of  $\mathfrak{W}$  are tamely irreducible (q.v. 2.4).
- ii) Suppose  $\sigma = \text{Ind}_{E/F} \tau$  for some finite extension  $E/F$  and representation  $\tau$  of  $\mathfrak{W}_E$ . If  $\sigma$  is Carayol, then  $\tau$  is also Carayol.
- iii) Let  $n = m\mathfrak{p}^r$  for integers  $m \geq 1$  and  $r \geq 0$  such that  $\mathfrak{p}$  does not divide  $m$ . If  $\sigma$  is an epipelagic  $\mathfrak{W}$ -representation of degree  $n$ , then  $\sigma = \text{Ind}_{K/F} \tau$  for some totally ramified extension  $K/F$  of degree  $m$ , and epipelagic  $\mathfrak{W}_K$ -representation  $\tau$  of degree  $\mathfrak{p}^r$ .
- iv) If an epipelagic representation is of  $\mathfrak{p}$ -power dimension, then it is primitive (q.v. 2.2).

We refer to [BH14] for a proof of these statements.

The following theorem on Carayol representations can be found in the literature [BH19, N<sup>o</sup> 3.5, Lemma 2].

**7.3 Theorem** — *If  $\sigma$  is an irreducible representation of  $\mathfrak{W}$  of Carayol type, then*

$$\text{sw}(\sigma \otimes \sigma^\vee) = (\dim(\sigma) - 1) \text{sw}(\sigma). \tag{7.3.1}$$

Further in this section, we shall prove this theorem under various additional hypotheses. The following two lemmas contain reduction steps that will be used frequently.

**7.4 Lemma (reduction: tame restriction)** — *Let  $\sigma$  be representation of  $\mathfrak{W}$ , and  $K/F$  a finite tame extension. Then formula 7.3.1 holds for  $\sigma$  if and only if it holds for the restriction  $\sigma|_K$  of  $\sigma$  to  $\mathfrak{W}_K$ .*

*Proof* — Since  $K/F$  is tame, the ramification subgroups of  $\mathfrak{W}$  and  $\mathfrak{W}_E$  are the same, and their numbering is related via the Herbrand function  $\psi_{K/F}$  that is given by the linear function  $u \mapsto ku$ , where  $k$  denotes the tame degree of  $K/F$ . The statement easily follows. **q.e.d.**

**7.5 Lemma (reduction: tame induction)** — *Let  $\sigma$  be an irreducible representation of  $\mathfrak{W}$  of Carayol type, and suppose  $\sigma = \text{Ind}_{E/F} \tau$  for some totally*

tamely ramified extension  $E/F$ , and wildly irreducible  $\mathfrak{W}_E$ -representation  $\tau$ . If formula 7.3.1 holds for  $\tau$ , then it holds for  $\sigma$ .

*Proof* — Let  $E'/F$  denote the normal closure of  $E/F$ , and  $F'/F$  its maximal unramified subextension. We put  $\sigma' = \sigma|_{F'}$  and  $\tau' = \tau|_{E'}$ , then  $\sigma'$  and  $\tau'$  are irreducible (7.2), and Mackey's Restriction Formula (2.3) implies  $\sigma' = \text{Ind}_{E'/F'} \tau'$ . The Herbrand function  $\phi_{F'/F}$  is the identity, so  $\sigma'$  is Carayol. Hence, by the previous reduction lemma, we have reduced to the case where  $E/F$  is Galois, and we assume hereafter that we are in this situation.

Put  $m = [E : F]$  and  $\rho^r = \dim(\tau)$ . Since  $E/F$  is totally tame, we have  $\text{sw}(\sigma) = \text{sw}(\tau)$  (3.6.ii). In particular,  $\text{sw}(\tau)$  is coprime to  $\dim(\sigma)$ , and therefore also coprime to  $m$ . Moreover, since  $\tau$  is wildly irreducible, it is homogeneous in the sense of 5.1, and has a well-defined  $g$ -invariant  $g_\tau$ . As explained in 5.4,  $g_\tau$  can be considered as a refinement of the slope  $\zeta(\tau)$ , and we have

$$v_E(g_\tau) = -(\nu_E + 1) - \zeta(\tau) = -m(\nu + 1) - (\text{sw}(\tau)/\rho^r).$$

where  $\nu_E$  and  $\nu$  denote the levels of the additive characters  $\Psi \circ \text{Tr}_{E/F}$  and  $\Psi$ , respectively. In particular, the valuation  $v_E(g_\tau)$  is a rational number coprime to  $m$ . Let us put  $v = v_E(g_\tau)$ .

We choose  $\lambda \in \mathcal{G}(E/F)$ , with  $\lambda \neq \varepsilon$ . Then, the conjugate representation  $\tau^\lambda$  is also homogeneous, and we have  $g_{\tau^\lambda} = \lambda(g_\tau)$ . Furthermore, we have  $\zeta(\tau \otimes \tau^\lambda) < \zeta(\tau)$  if and only if  $\tau$  and  $\tau^\lambda$  restrict to the same character on  $\mathfrak{W}_E^{\zeta(\tau)}$ , or, put differently, if and only if  $g_\tau = \lambda(g_\tau)$ . Now, observe that  $\lambda$  acts on  $g_\tau$  via multiplication with  $\zeta^v$ , where  $\zeta$  denotes a non-trivial  $m$ -th root of unity. We have seen above that  $v$  is coprime with  $m$ , therefore  $\zeta^v$  is non-trivial, and consequently  $g_\tau \neq \lambda(g_\tau)$ . Hence,  $\zeta(\tau \otimes (\tau^\vee)^\lambda) = \zeta(\tau)$ , and we have

$$\text{sw}(\tau \otimes (\tau^\vee)^\lambda) = \rho^r \text{sw}(\tau).$$

We have the following decomposition:

$$\sigma \otimes \sigma^\vee = \text{Ind}_{E/F} \bigoplus_{\lambda \in \mathcal{G}(E/F)} (\tau \otimes (\tau^\vee)^\lambda).$$

Then, assuming formula 7.3.1 holds for  $\tau$ , we find

$$\begin{aligned} \text{sw}(\sigma \otimes \sigma^\vee) &= (\rho^r - 1) \text{sw}(\tau) + (m - 1) \rho^r \text{sw}(\tau) \\ &= (m \rho^r - 1) \text{sw}(\tau) \\ &= (\dim(\sigma) - 1) \text{sw}(\sigma). \end{aligned} \quad \mathbf{q.e.d.}$$

**7.6 Proposition (case: Galois inducing character)** — Let  $\sigma$  be a wildly irreducible Carayol representation of  $\mathfrak{W}$  of dimension  $\rho^r$ , and suppose  $\sigma = \text{Ind}_{E/F} \chi$  for some finite Galois extension  $E/F$  and  $\mathfrak{W}_E$ -character  $\chi$ . Then formula 7.3.1 holds.

*Proof* — Using Mackey’s Restriction Formula (2.3), we obtain the following decomposition:

$$\begin{aligned}\sigma \otimes \sigma^\vee &= \text{Ind}_{E/F} \chi \otimes \text{Ind}_{E/F} \chi^{-1} \\ &= \text{Ind}_{E/F} \bigoplus_{\lambda \in \mathcal{G}(E/F)} \chi^{1-\lambda}.\end{aligned}$$

Note that  $\mathcal{G}(E/F)$  is totally wildly ramified, so all jumps of  $\phi_{E/F}$  are congruent one another modulo  $\mathfrak{p}$  (1.3). Hence, there exists some integer  $a$  such that  $\nu_E(\lambda) - 1 \equiv a \pmod{\mathfrak{p}}$  for all  $\lambda \neq \varepsilon$ . Then, putting  $w = w(E/F)$  and  $\varsigma = \varsigma(\chi)$ , we obtain

$$\text{sw}(\sigma) = w + \varsigma \tag{3.6.ii}$$

$$= \sum_{\lambda \neq \varepsilon} (\nu_E(\lambda) - 1) + \varsigma \tag{1.2.iii}$$

$$\equiv (\mathfrak{p}^r - 1)a + \varsigma \pmod{\mathfrak{p}}$$

$$\equiv \varsigma - a \pmod{\mathfrak{p}},$$

so the Carayol condition of  $\sigma$  implies  $\varsigma \not\equiv a \pmod{\mathfrak{p}}$ .

Now, invoking 4.5, we find  $\varsigma(\chi^{1-\lambda}) = \varsigma - (\nu_E(\lambda) - 1)$ , for all  $\lambda \neq \varepsilon$ . Then, we conclude

$$\text{sw}(\sigma \otimes \sigma^\vee) = \mathfrak{p}^r w + \sum_{\lambda \neq \varepsilon} \varsigma(\chi^{1-\lambda}) \tag{3.6.ii}$$

$$= \mathfrak{p}^r w + \sum_{\lambda \neq \varepsilon} (\varsigma - (\nu_E(\lambda) - 1))$$

$$= (\mathfrak{p}^r - 1)(w + \varsigma) \tag{1.2.iii}$$

$$= (\mathfrak{p}^r - 1) \text{sw}(\sigma) \tag{q.e.d.}$$

**7.7 Theorem (case: epipelagic)** — *If  $\sigma$  is an irreducible representation of  $\mathfrak{W}$  that is epipelagic, then*

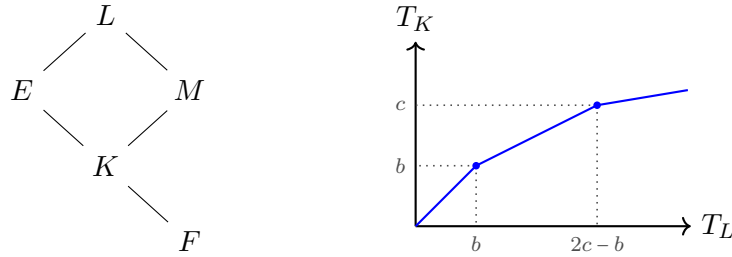
$$\text{sw}(\sigma \otimes \sigma^\vee) = \dim(\sigma) - 1.$$

*Proof* — By 7.2, we can write  $\sigma = \text{Ind}_{K/F} \tau$  for some totally tamely ramified extension  $K/F$  and epipelagic  $\mathfrak{W}_K$ -representation  $\tau$  that is primitive. By reduction via tame induction (7.5), it suffices to prove the formula for  $\tau$ .

The structure of primitive representations is well-known and we use the following fact: there exists a tame extension  $T/K$  such that the restriction  $\tau|_T$  is irreducible of Heisenberg type [Koc77, Theorem 2.2, 4.1]. In particular,  $\tau|_T$  is induced by a character from a totally wild Galois extension. Now, using 7.4 and 7.6, we obtain the desired formula for  $\tau$ . q.e.d.

**7.8 Non-Galois inducing extension.** We shall study formula 7.3.1 for a Carayol representation  $\sigma$  of  $\mathfrak{W}$ , in the easiest case that is not covered by our previous assumptions. More precisely, we suppose  $\sigma = \text{Ind}_{E/F} \chi$ , where  $F$  has residue characteristic 2,  $E/F$  is a non-Galois inducing extension of degree 4, and  $\chi$  is a  $\mathfrak{W}_E$ -character whose Swan exponent is sufficiently large. By our reduction steps (7.4,7.5), we can assume that  $E/F$  has a Galois closure  $L/F$  that is totally wild of degree 8. Then, regarded as a subgroup of the symmetric group on 4 elements, it is clear that  $\mathcal{G}(L/F)$  is isomorphic to the dihedral group of order 8.

The extension  $E/F$  has a unique quadratic subextension that we denote by  $K/F$ , and we put  $a = \text{sl}(K/F)$  and  $\tau = \text{Ind}_{E/K} \chi$ . Furthermore, we let  $\kappa$  denote the non-trivial element of the subgroup  $\mathcal{G}(L/E)$ , and we shall choose an element  $\lambda$  of  $\mathcal{G}(L/F)$  such that  $\lambda|_K$  is a generator of  $\mathcal{G}(K/F)$ , and the commutator  $\gamma = [\lambda, \kappa]$  is the unique non-trivial central element of  $\mathcal{G}(L/F)$ . We write  $M$  for the subfield  $L$  that is invariant under  $\gamma$ . The diagram on the left below depicts the lattice of subextensions of  $L/F$  that are relevant in the sequel.



Since the last non-trivial ramification subgroup corresponds to the center of  $\mathcal{G}(L/F)$  (1.4), it is given by  $\mathcal{G}(L/M)$ , and we have  $\nu_L(\gamma) > \nu_L(\kappa)$ ; in other words,  $L/M$  is strictly more ramified than  $L/E$  (or equivalently:  $M/K$  is strictly less ramified than  $E/K$ ). This implies that the Herbrand function  $\phi_{L/K}$  has two jumps; its graph is displayed in the diagram on the right, and we have set  $b = \text{sl}(M/K)$  and  $c = \text{sl}(E/K)$ . Hence  $b < c$ , and we find  $\text{sl}(L/E) = b$  and  $\text{sl}(L/M) = 2c - b$ .

Now, let us note

$$\sigma \otimes \sigma^\vee = \text{Ind}_{K/F}(\tau \otimes \tau^\vee) \oplus \text{Ind}_{K/F}(\tau \otimes (\tau^\vee)^\lambda),$$

so that, for the desired formula 7.3.1 to hold, it is necessary and sufficient to prove the formula

$$\text{sw}(\tau \otimes (\tau^\vee)^\lambda) = 3(2a + \text{sw}(\tau)) - (4a + \text{sw}(\tau)) = 2(\text{sw}(\tau) - a). \tag{7.8.1}$$

(Here, we have applied twice the formula of 3.6.ii.)

Next, we use elementary formulas of representation theory to derive the following equalities:

$$\begin{aligned}
 \tau \otimes (\tau^\vee)^\lambda &= \text{Ind}_{E/K} \chi \otimes \text{Ind}_{\lambda E/K} \chi^{-\lambda} \\
 &= \text{Ind}_{E/K} (\chi \otimes \text{Res}_{K/E} \text{Ind}_{\lambda E/K} \chi^{-\lambda}) \\
 &= \text{Ind}_{E/K} (\chi \otimes \text{Ind}_{L/E} (\chi^{-\lambda} \circ N_{L/\lambda E})) \\
 &= \text{Ind}_{L/E} ((\chi \circ N_{L/E}) \cdot (\chi^{-\lambda} \circ N_{L/\lambda E})) \\
 &= \text{Ind}_{L/E} ((\chi \circ N_{L/E})^{1-\lambda}).
 \end{aligned}$$

Here, we have used Mackey's Restriction Formula (2.3) to obtain the third equality (together with the fact that  $L/F$  is the compositum of  $E/F$  and  $E^\lambda/F$ ). For the final equality, we have used

$$\lambda \circ N_{L/\lambda E} = N_{L/E} \circ \lambda.$$

*Claim* — Put  $\varsigma = \varsigma(\chi)$ , and  $\varsigma' = \psi_{L/E}(\varsigma) - 2a = 2\varsigma - b - 2a$ . Then, the following holds

$$\varsigma(\chi') = \varsigma'.$$

Let us suppose for the moment that the claim holds. Then, using 3.6.ii, we obtain the following identities

$$\begin{aligned}
 \text{sw}(\tau) &= w(E/K) + \varsigma(\chi) \\
 &= c + \varsigma, \\
 \text{sw}(\tau \otimes (\tau^\vee)^\lambda) &= w(L/K) + \varsigma(\chi') \\
 &= (b + 2c) + (2\varsigma - b - 2a) = 2((c + \varsigma) - a),
 \end{aligned}$$

and this establishes formula 7.8.1

To prove the claim, we consider first the group ring of  $\mathcal{G}(L/F)$  over  $\mathbf{Z}$  and let  $N$  denote its element  $1 + \kappa$ . Clearly, the natural action of  $N$  on  $L$  is given by the norm map  $N_{L/E}$ . Using a commutator trick, we find the following identities in the group ring:

$$\begin{aligned}
 N(1 - \lambda) &= N - N\lambda \\
 &= N - \lambda N + \lambda N - N\lambda \\
 &= (1 - \lambda)N + \lambda\kappa - \kappa\lambda \\
 &= (1 - \lambda)N + \lambda\kappa(1 - \gamma).
 \end{aligned}$$

(We recall that  $\gamma$  denotes the commutator  $[\lambda, \kappa] = [\lambda^{-1}\kappa^{-1}]$ ). The calculation above shows that

$$N_{L/E}(x/\lambda x) = (y/\lambda y) \cdot \lambda\kappa(x/\gamma x), \tag{7.8.2}$$



for all  $x \in L^\times$ , with  $y = N_{L/E}(x)$ . In particular, we see that the character  $\chi'$  can be obtained as the composition  $\chi \circ \Xi$ , where  $\Xi: L^\times \rightarrow E^\times$  is given by

$$\Xi: x \mapsto (y/\wedge y) \cdot \lambda\kappa(x/\wedge x).$$

Hence, it suffices to show that  $\Xi$  maps  $U_L^{\zeta'}$  into  $U_E^\zeta$  and  $U_L^{\zeta'+1}$  into  $U_E^{\zeta+1}$ , and that the map  $\Omega_L^{\zeta'} \rightarrow \Omega_E^\zeta$  obtained by passage to quotients, is an isomorphism. We shall prove this statement in 7.11, but before we do that, we shall verify it in two striking examples in which all computations can be done explicitly.

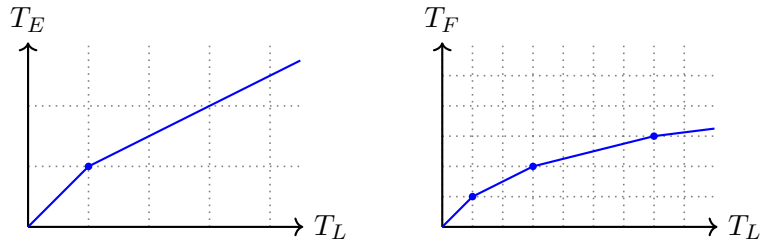
**7.9 First example.** We present our first example by giving the corresponding lattice of field extensions, using the same notation as in 7.8.

$$\begin{array}{ccc} & L = \mathbf{Q}_2(\sqrt[4]{2}, i) & \\ & \swarrow \quad \searrow & \\ E = \mathbf{Q}_2(\sqrt[4]{2}) & & M = \mathbf{Q}_2(\sqrt{2}, i) \\ & \swarrow \quad \searrow & \\ & K = \mathbf{Q}_2(\sqrt{2}) & \\ & & \searrow \\ & & F = \mathbf{Q}_2 \end{array}$$

The non-trivial element of  $\mathcal{G}(K/F)$  maps the uniformizer  $\sqrt{2}$  of  $K$  to  $-\sqrt{2}$ . Since  $v_K(\sqrt{2} - (-\sqrt{2})) = 3$ , the corresponding Herbrand function  $\phi_{E/K}$  has a jump at  $3 - 1 = 2$  (recall the shift with  $-1$  according to Serre's lower numbering convention). Similarly,  $\sqrt[4]{2}$  is a uniformizer of  $E$  that is mapped to  $-\sqrt[4]{2}$  under the action of the Galois group, so  $\phi_{E/K}$  jumps at  $v_E(2\sqrt[4]{2}) - 1 = 4$ . By composing the two Herbrand functions we obtain the Herbrand function corresponding to  $E/F$ ; it has jumps at 2 and 4. Note that  $L/F$  contains  $\mathbf{Q}_2(i)/\mathbf{Q}_2$  as a subextension, and  $\phi_{\mathbf{Q}_2(i)/\mathbf{Q}_2}$  has a jump at 1 [Ser79, Ch. IV, Prop. 17]. Furthermore, by the transitivity of Herbrand functions we have

$$\phi_{\mathbf{Q}_2(i)/\mathbf{Q}_2} \circ \phi_{L/\mathbf{Q}_2(i)} = \phi_{L/F} = \phi_{E/F} \circ \phi_{L/E}.$$

Therefore, both  $\phi_{L/E}$  and  $\phi_{L/F}$  have a jump at 1, and, taking into account the previous results, this fully determines these Herbrand functions. Their graphs are given below (each little square in the grid is of unit measure).



Using the notation of 7.8, we have  $a = \text{sl}(K/F) = 2$ ,  $b = \text{sl}(M/K) = 1$  and  $c = \text{sl}(E/K) = 4$ . Also, we find  $\text{sl}(L/M) = 7$  and  $\text{sl}(L/E) = 1$ . Now, as before, we set  $\varsigma = \varsigma(\chi)$  (recall that  $\varsigma$  is assumed to be sufficiently large), and

$$\varsigma' = \psi_{L/E}(\varsigma) - 2a = (2\varsigma - 1) - 4 = 2\varsigma - 5.$$

Furthermore, since  $\sigma = \text{Ind}_{E/F} \chi$ , the formula of 3.6.ii shows that

$$\text{sw}(\sigma) = w(E/F) + \varsigma(\chi) = 8 + \varsigma.$$

Hence, the Carayol assumption on  $\sigma$  implies  $\varsigma$  is odd.

Next, we shall study the map  $\Xi: L^\times \rightarrow E^\times$  given by

$$\Xi: x \mapsto (y/{}^\lambda y) \cdot \lambda \kappa(x/{}^\gamma x),$$

where  $y = N_{L/E}(x)$ . We shall separately study the factors on the right hand side. Note that the function  $x \mapsto x/{}^\gamma x$  maps  $U_L^{\varsigma'}$  into  $U_L^{\varsigma'+7} = U_L^{2\varsigma+2}$  (4.2.iii). Since  $\mathcal{G}(L/F)$  is wildly ramified, the same then holds for the map  $x \mapsto \lambda \kappa(x/{}^\gamma x)$ .

The map  $x \mapsto y$  induces an isomorphism  $\Omega_L^{\varsigma'} \xrightarrow{\sim} \Omega_E^{\varsigma-2}$  [Ser79, Ch. V, Prop. 5]. For an element  $y$  of  $U_E$ , we can write

$$y = a + b\sqrt[4]{2} + c\sqrt{2} + d(\sqrt[4]{2})^3, \quad \text{with } a, b, c, d \in \mathfrak{o}.$$

Then, we have

$$v_E(y - 1) = \min(v_E(a - 1), v_E(b) + 1, v_E(c) + 2, v_E(d) + 3). \quad 7.9.1$$

Furthermore, the following identities hold:

$$\begin{aligned} {}^\lambda y &= a + ib\sqrt[4]{2} - c\sqrt{2} - id\sqrt[4]{2}^3 \\ y - {}^\lambda y &= (1 - i)b\sqrt[4]{2} + 2c\sqrt{2} + (1 + i)d(\sqrt[4]{2})^3. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} v_E(y - {}^\lambda y) &= \min\left(v_E((1 - i)b) + 1, v_E(2c) + 2, v_E((1 + i)d) + 3\right) \\ &= \min\left(2 + (v_E(b) + 1), 4 + (v_E(c) + 2), 2 + (v_E(d) + 3)\right) \end{aligned} \quad 7.9.2$$

Hence, for all  $y \in U_E$ , we have

$$v_E(y/{}^\lambda y - 1) = v_E(y - {}^\lambda y) \geq 2.$$

Furthermore, if  $y \in U_E^{\varsigma-2} - U_E^{\varsigma-1}$ , then the valuation  $v_E(y - 1) = \varsigma - 2$  is odd, and therefore, by 7.9.1, it is equal to  $v(b) + 1$  or  $v_E(d) + 3$ . Then, using 7.9.2, we obtain

$$v_E(y/{}^\lambda y - 1) = v_E(y - {}^\lambda y) = v_E(y - 1) + 2 = \varsigma,$$

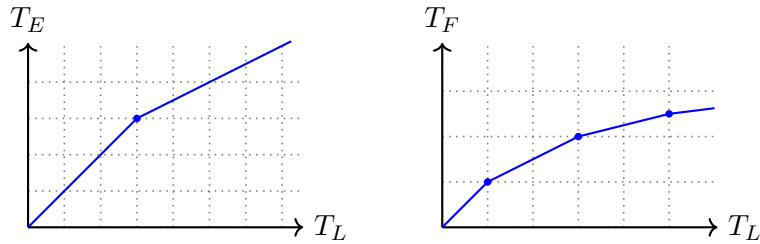
so  $y/\lambda y \in U_L^{2\varsigma} - U_L^{2\varsigma+1}$ . Moreover, if we change  $y$  such that  $y - 1$  is multiplied by a root of unity  $\zeta \in E$  of odd order, then  $y/\lambda y - 1$  is multiplied by the same element  $\zeta$ . Together with the previous results, this shows that  $y \mapsto y/\lambda y$  defines an isomorphism  $\Omega_L^{\varsigma-2} \xrightarrow{\sim} \Omega_L^{2\varsigma}$  of  $\mathbf{k}$ -vector spaces. Thus,  $x \mapsto y/\lambda y$  defines an isomorphism  $\Omega_L^{\varsigma'} \xrightarrow{\sim} \Omega_E^{\varsigma-2} \cong \Omega_L^{2\varsigma-4} \xrightarrow{\sim} \Omega_L^{2\varsigma}$ .

We conclude that  $\Xi$  induces an isomorphism  $\Omega_L^{\varsigma'} \xrightarrow{\sim} \Omega_E^{\varsigma}$ .

**7.10 Second example.** The second example is given by the following lattice of field extensions.

$$\begin{array}{ccc}
 & L = \mathbf{Q}_2(\sqrt{i+1}, \sqrt{2}) & \\
 & \swarrow \quad \searrow & \\
 E = \mathbf{Q}_2(\sqrt{i+1}) & & M = \mathbf{Q}_2(i, \sqrt{2}) \\
 & \swarrow \quad \searrow & \\
 & K = \mathbf{Q}_2(i) & \\
 & \searrow & \\
 & F = \mathbf{Q}_2 &
 \end{array}$$

Observe that the non-trivial element of  $\mathcal{G}(K/F)$  maps the uniformizer  $1+i$  of  $K$  to  $1-i$ ; we have  $v_K((1+i) - (1-i)) = 2$ , so  $\phi_{E/K}$  has a jump at 1. Furthermore,  $\sqrt{1+i}$  is a uniformizer of  $E$  that is mapped to  $-\sqrt{1+i}$  by the non-trivial element of  $\mathcal{G}(E/K)$ , and in a similar way one finds that  $\phi_{E/K}$  jumps at 4. By composing the two Herbrand functions we obtain the Herbrand function of  $E/F$ , which has a jump at position 1 and 4. Observe that  $M = \mathbf{Q}_2(\zeta_8)$  is a cyclotomic field extension, and  $\phi_{M/F}$  has jumps at 1 and 3 [Ser79, Ch. IV, Prop. 17]. Then,  $\phi_{L/E}$  jumps at 3, and with this information we can also determine the Herbrand function of  $L/F$ ; the graphs of  $\phi_{L/K}$  and  $\phi_{L/F}$  are as follows (each little square in the grid is of unit measure):



Hence, using the notation introduced in 7.8, we have  $a = \text{sl}(K/F) = 1$ ,  $b = \text{sl}(M/K) = 3$  and  $c = \text{sl}(E/K) = 4$ . Then,  $\text{sl}(L/M) = 5$  and  $\text{sl}(L/E) = 3$ . We set  $\varsigma = \varsigma(\chi)$ , and

$$\varsigma' = \psi_{L/E}(\varsigma) - 2a = (2\varsigma - 3) - 2 = 2\varsigma - 5.$$

By the Carayol condition on  $\sigma$ , we know that

$$\text{sw}(\sigma) = w(E/F) + \varsigma(\chi) = 6 + \varsigma,$$

is an odd number, hence  $\varsigma$  is also odd.

Next, we study  $\Xi$  in the same way as in the previous example. We note that, by 4.5,  $x \mapsto x/\gamma x$  defines an isomorphism  $\Omega_L^{\varsigma'} \xrightarrow{\sim} \Omega_L^{\varsigma'+5} = \Omega_L^{2\varsigma}$ . Since  $\mathcal{G}(L/F)$  is totally wild, the same also holds for the map  $x \mapsto \lambda\kappa(x/\gamma x)$ .

The map  $x \mapsto y$  maps  $U_L^{\varsigma'}$  into  $U_E^{\varsigma-1}$  [Ser79, Ch. V, Prop. 5]. For an element  $y$  of  $U_E$  we can write

$$y = a + b\sqrt{1+i} + c(1+i) + d(1+i)\sqrt{1+i},$$

with  $a, b, c, d \in \mathfrak{o}$ , and, as in the previous example, we have

$$v_E(y-1) = \min(v_E(a-1), v_E(b)+1, v_E(c)+2, v_E(d)+3).$$

Then, we have the following identities:

$$\begin{aligned} \lambda y &= a + \zeta_8 b\sqrt{1+i} + \zeta_8^2 c(1+i) + \zeta_8^3 d(1+i)\sqrt{1+i} \\ y - \lambda y &= (1 - \zeta_8)b\sqrt{1+i} + (1 - \zeta_8^2)c(1+i) + (1 - \zeta_8^3)d(1+i)\sqrt{1+i}. \end{aligned}$$

Hence, we get

$$\begin{aligned} v_E(y - \lambda y) &= \min\left(v_E((1 - \zeta_8)b) + 1, v_E((1 - \zeta_8^2)c) + 2, v_E((1 - \zeta_8^3)d) + 3\right) \\ &= \min(1 + (v_E(b) + 1), 2 + (v_E(c) + 2), 1 + (v_E(d) + 3)) \quad \mathbf{7.10.1} \end{aligned}$$

This shows that

$$v_E(y/\lambda y) = v_E(y - \lambda y) \geq v_E(y-1) + 1 \geq \varsigma + 1,$$

for all  $y \in U_E$ . Moreover, if  $y \in U_E^{\varsigma-1} - U_E^{\varsigma}$ , then  $v_E(y-1) = \varsigma - 1$  is even. Then, by 7.10.1, it is equal to  $v_E(a-1)$  or  $v_E(c) + 2$ . Hence, 7.10.1 implies that

$$v_E(y/\lambda y - 1) = v_E(y - \lambda y) \geq v_E(y-1) + 2 = \varsigma + 1,$$

so  $y/\lambda y \in U_L^{2\varsigma+2}$ . Thus  $y \mapsto y/\lambda y$  maps  $U_L^{2\varsigma-2}$  into  $U_L^{2\varsigma+2}$ ; consequently,  $x \mapsto y/\lambda y$  maps  $U_L^{\varsigma'}$  into  $U_L^{2\varsigma+2}$ .

We conclude that  $\Xi$  induces an isomorphism  $\Omega_L^{\varsigma'} \xrightarrow{\sim} \Omega_E^{\varsigma}$ .

**7.11 General case: degree 8.** We finish the discussion that was started in 7.8. Let us study the map  $E^\times \rightarrow L^\times$  given by  $y \mapsto y/\lambda y$ . Using Lemma 1.5, we find

$$2\nu_K(\lambda|_K) = \nu_E(\lambda|_E) + \nu_E(\lambda^{-1}|_E).$$

It follows easily from the definitions that the terms on the right are equal, so we obtain  $\nu_E(\lambda|_E) = a + 1$ . Next, we choose a uniformizer  $\varpi$  of  $E$ , and we note that  $\varpi$  generates  $\mathfrak{o}_E$  as an  $\mathfrak{o}$ -algebra. Hence

$$v_E(\lambda \varpi / \varpi - 1) = v_E(\lambda \varpi - \varpi) - 1 = a,$$

so we can write  ${}^\lambda\varpi/\varpi = 1+z$ , for some  $z \in \mathfrak{p}_L^{2a}$ . Now, taking  $y$  to be an element  $1+u\varpi^i$  with  $i \geq 1$ , we find (as in the proof of 4.2.iii)

$$\begin{aligned} y/{}^\lambda y - 1 &= (y - {}^\lambda y)/{}^\lambda y \\ &\equiv -iz^\lambda u \varpi^i \pmod{\mathfrak{p}_L^{2a+2i+1}}. \end{aligned}$$

We get an induced map  $\Omega_L^{2i} \rightarrow \Omega_L^{2a+2i}$  of 1-dimensional  $\mathbf{k}$ -vector spaces that is an isomorphism if  $i$  is odd, and the zero-map otherwise.

Now, let  $\eta$  be a  $\mathfrak{W}_K$ -character whose kernel equals  $\mathfrak{W}_E$ , then  $\eta^{1-\lambda}$  is a  $\mathfrak{W}_K$ -character whose kernel equals  $\mathfrak{W}_M$ . Then, by 4.5, we have

$$b = \varsigma(\eta^{1-\lambda}) \leq \varsigma(\eta) - \nu_K(\lambda|_K) = c - a,$$

with equality if and only if  $c - a$  is odd. Put equivalently, we have

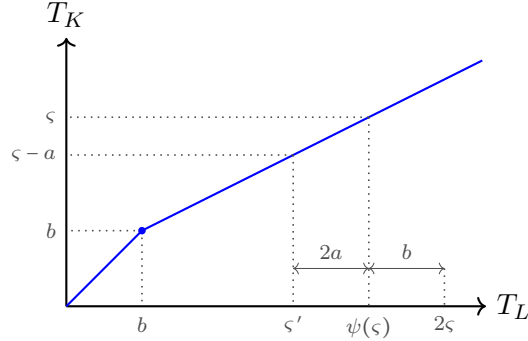
$$c \geq a + b, \text{ with equality if and only if } c - a \text{ is odd.} \quad \mathbf{7.11.1}$$

Furthermore, since  $\tau = \text{Ind}_{E/K} \chi$ , we obtain

$$\varsigma(\tau) = w(E/K) + \varsigma(\chi) = c + \varsigma,$$

using 3.6.ii. Since  $\tau$  is Carayol if  $\sigma$  is Carayol, we find that  $c + \varsigma$  is odd.

We recall that  $\varsigma' = \psi_{L/K}(\varsigma) - 2a = 2\varsigma - b - 2a$ . For convenience, we have given the graph of  $\phi_{L/K}$  below, together with some other variables.



Suppose first that  $\varsigma - a$  is odd. Then, since  $c + \varsigma$  is odd, it follows that  $c - a$  is even. Thus, by 7.11.1,  $c > a + b$ . Hence, the following holds:

- ▶ the map  $x \mapsto x/{}^\lambda x$  induces a map  $\mathbf{U}_L^{\varsigma'} \rightarrow \mathbf{U}_L^{\varsigma'+(2c-b)}$  (4.2.iii), with

$$\varsigma' + (2c - b) = (2\varsigma - b - 2a) + (2c - b) = 2\varsigma + 2(c - a - b) > 2\varsigma;$$

- ▶ the map  $x \mapsto y \mapsto y/{}^\lambda y$  induces an isomorphism

$$\Omega_L^{\varsigma'} \xrightarrow{\sim} \Omega_E^{\varsigma-a} \cong \Omega_L^{2\varsigma-2a} \xrightarrow{\sim} \Omega_L^{2\varsigma}.$$

Conclusion:  $\Xi$  induces an isomorphism  $\Omega_L^{\varsigma'} \xrightarrow{\sim} \Omega_E^\varsigma$ .

Next, we suppose that  $\varsigma - a$  is even. Then,  $c - a$  is odd and  $c = a + b$  (7.11.1). So  $b (= c - a)$  is odd, and therefore  $\varsigma' = 2\varsigma - b$  is odd, and we derive the following statements:

► the map  $x \mapsto x/\gamma x$  induces an isomorphism  $\Omega_L^{\varsigma'} \xrightarrow{\sim} \Omega_L^{\varsigma'+(2c-b)}$  (4.2.iii), with

$$\varsigma' + (2c - b) = (2\varsigma - b - 2a) + (2c - b) = 2\varsigma + 2(c - a - b) = 2\varsigma;$$

► the map  $x \mapsto y \mapsto y/\lambda y$  induces a map

$$U_L^{\varsigma'} \longrightarrow U_E^{\varsigma-a} \longrightarrow U_L^{2\varsigma+1}.$$

Conclusion:  $\Xi$  induces an isomorphism  $\Omega_L^{\varsigma'} \xrightarrow{\sim} \Omega_E^\varsigma$ .

So if the residue characteristic of  $F$  equals 2, the formula 7.3.1 for the Swan exponent of a Carayol representation  $\sigma$  holds in all cases where  $\sigma$  is induced by a character from a quartic extension  $E$  of  $F$ .

## §8 APPLICATION: A RESULT OF LAPID

We shall give an application of the main results of this chapter by proving the following corollary.

**8.1 Corollary** — *If  $\sigma$  is an irreducible representation of  $\mathfrak{W}$ , then*

$$\text{sw}(\sigma \otimes \sigma^\vee) \geq \dim(\sigma) - 1, \quad \mathbf{8.1.1}$$

*with equality if and only if  $\sigma$  is epipelagic.*

*Proof* — The desired inequality is insensitive to twisting  $\sigma$  by a character, so we may assume that  $\sigma$  is minimal. If  $\text{sw}(\sigma) \geq 2$ , then by 6.1, we have

$$\text{sw}(\sigma \otimes \sigma^\vee) \geq \frac{1}{2} \dim(\sigma) \text{sw}(\sigma) \geq \dim(\sigma).$$

If  $\text{sw}(\sigma) = 1$ , then equality holds by 7.7.

**q.e.d.**

The statement in 8.1 was motivated by an inequality of Lapid that was obtained via different methods [Lap19, Prop. 2.3]. His result is stated in terms of the Artin conductor and the so-called twisting number introduced below. We explain hereafter how these two inequalities compare.

**8.2** We define the *twisting number* of a  $\mathfrak{W}$ -representation  $\sigma$  as the order of the group of unramified  $\mathfrak{W}$ -characters  $\chi$  such that  $\sigma \otimes \chi \cong \sigma$ . If  $\sigma$  is irreducible, this number can be characterized as the number of isotypic components of the

restriction  $\sigma|_{\mathfrak{J}}$ . By 2.8 (with  $H = \mathfrak{J}$ ),  $\sigma|_{\mathfrak{J}}$  has  $[\mathfrak{W} : N]$  isotypic components, where  $N$  denotes the stabilizer of one (and therefore each) of the isotypic components. In particular,  $t$  divides  $\dim(\sigma)$ .

Suppose  $\sigma$  is irreducible with twisting number  $t \geq 1$ ; then, by taking 3.3 into account and applying 8.1.1, we find

$$\begin{aligned} \text{ar}(\sigma \otimes \sigma^\vee) &= \text{codim}((\sigma \otimes \sigma^\vee)^{\mathfrak{J}}) + \text{sw}(\sigma \otimes \sigma^\vee) \\ &\geq (\dim(\sigma)^2 - t) + (\dim(\sigma) - 1) \\ &\geq \dim(\sigma)(\dim(\sigma) + 1) - 2t. \end{aligned}$$

In [Lap19, Prop. 2.3], an additional positive term is subtracted from the final expression on the right above.

# III

## SOME ADDITIONAL CASES

In this chapter, we present some additional cases of the main problem stated in the introduction. More specifically, we consider tensor powers of  $\mathfrak{W}$ -representations in §9, and present a lower bound for their Swan exponent.

Finally, in §10, we deal with the algebraic irreducible representations of  $\mathrm{GL}_2(\mathbf{C})$ , and obtain again a lower bound for the Swan exponent.

### §9 TENSOR POWERS

In our discussion regarding Swan exponents of tensor products, we will make use of an inequality that can be found in the literature [BH17, N<sup>o</sup>5, Theorem AS]. We state this result in theorem 9.2 below, and we explain that it can be proved without invoking the local Langlands correspondence. Since it involves a new notion of minimality, we will start off with a definition.

**9.1 Definition** — *A representation  $\sigma$  of  $\mathfrak{W}$  is said to be  $\varsigma$ -minimal if*

$$\varsigma(\chi \otimes \sigma) \geq \varsigma(\sigma),$$

*for all characters  $\chi$  of  $\mathfrak{W}$ .*

Observe that, for irreducible representations, this definition coincides with the notion of minimality introduced in 4.4.



**9.2 Theorem** — *Let  $\sigma$  and  $\tau$  be representations of  $\mathfrak{W}$ . If  $\sigma$  is  $\varsigma$ -minimal, then*

$$\varsigma(\sigma \otimes \tau) \geq \frac{1}{2} \max(\varsigma(\sigma), \varsigma(\tau)).$$

*Proof* — We prove this theorem in the special case where  $\sigma$  and  $\tau$  are both irreducible and  $\varsigma$ -minimal. If that is the case, then, using [3.10](#) and [6.1.1](#), we find

$$\varsigma(\sigma \otimes \tau) \geq \varsigma(\sigma \otimes \sigma^\vee) \geq \frac{1}{2} \varsigma(\sigma).$$

Similarly, we have  $\varsigma(\sigma \otimes \tau) \geq \frac{1}{2} \varsigma(\tau)$ ; the inequality follows.

The result for general  $\sigma$  and  $\tau$  can be obtained by using the inequality in the special case discussed above, and some purely combinatorial arguments. We refer to [\[BH17\]](#) for the details. **q.e.d.**

**9.3 Theorem** — *Let  $\sigma: \mathfrak{W} \rightarrow \mathrm{GL}(V)$  be a representation, and  $\Sigma$  its composition with an algebraic representation of  $\mathrm{GL}(V)$ . If  $\sigma$  is  $\varsigma$ -minimal, then*

$$\varsigma(\sigma \otimes \Sigma) \geq \frac{1}{2} \varsigma(\sigma).$$

*Proof* — We invoke the previous theorem with  $\tau = \Sigma$ . **q.e.d.**

It is straightforward from the definitions that  $\mathrm{sl}(\sigma \otimes \Sigma) \leq \mathrm{sl}(\sigma)$ . In particular, if  $\sigma$  is irreducible,  $\varsigma(\sigma \otimes \Sigma)$  is bounded above by  $\varsigma(\sigma)$ . This upper bound, however, does not hold in general.

A special case of [9.3](#) is stated as a corollary below.

**9.4 Corollary** — *Let  $k$  be an integer  $\geq 1$ . If  $\sigma$  is a representation of  $\mathfrak{W}$  that is  $\varsigma$ -minimal, then*

$$\varsigma(\sigma^{\otimes k}) \geq \frac{1}{2} \varsigma(\sigma),$$

where  $\sigma^{\otimes k}$  denotes the  $k$ -th tensor power of  $\sigma$ .

## §10 REPRESENTATIONS OF $\mathrm{GL}_2(\mathbf{C})$

**10.1** Firstly, let us remark that the irreducible algebraic representations of  $\mathrm{GL}_2(\mathbf{C})$  are given by the symmetric powers of the standard representation, possibly twisted by a power of the determinant character. Hence, if  $\rho$  is any such representation, and  $\sigma$  is a representation  $\mathfrak{W} \rightarrow \mathrm{GL}_2(\mathbf{C})$ , then the composition  $\Sigma = \rho \circ \sigma$  is equal to  $\mathbf{S}^m(\sigma) \otimes \det(\sigma)^l$ , where  $m$  and  $l$  are integers,  $m \geq 1$ , and  $\mathbf{S}^m(\sigma)$  denotes the  $m$ -th symmetric power of  $\sigma$ . In theorem [10.3](#),

we shall present a lower bound for the (normalized) Swan exponent of  $\Sigma$  in terms of that one of  $\sigma$ .

**10.2 Reduction: tame restriction.** As we have seen in the previous chapter, applying a tame base change only affects the numbering of the ramification filtration via a linear Herbrand function (q.v. 6.2, 7.4). Therefore, we shall apply several tame base changes in the proof below, in order to reduce to a situation that we have dealt with beforehand.

**10.3 Theorem** — *Let  $\sigma$  be a representation of  $\mathfrak{W}$  of dimension 2, and let  $\Sigma$  be the  $\mathfrak{W}$ -representation  $\mathbf{S}^m(\sigma) \otimes \det(\sigma)^l$ , for some integers  $m$  and  $l$  such that  $m \geq 1$  (q.v. 10.1). Then, the following statements hold.*

i) *Suppose  $\sigma$  is reducible, i.e.,  $\sigma = \eta \oplus \theta$ , for some characters  $\eta, \theta$  of  $\mathfrak{W}$ . Put  $\varsigma = \varsigma(\theta/\eta)$ , then*

$$\varsigma(\Sigma) \geq \begin{cases} \frac{1}{2} \varsigma & \text{if } m \text{ is odd;} \\ \frac{m}{2(m+1)} \varsigma & \text{if } m \text{ is even.} \end{cases}$$

*Furthermore, if  $\sigma$  is  $\varsigma$ -minimal (9.1), then  $\varsigma(\sigma) = \varsigma/2$ .*

ii) *Suppose  $\sigma$  is irreducible and minimal. If the restriction of  $\sigma$  to  $\mathfrak{P}$  is reducible, then*

$$\varsigma(\Sigma) \geq \begin{cases} \frac{1}{2} \varsigma(\sigma) & \text{if } m \text{ is odd;} \\ \frac{m}{2(m+1)} \varsigma(\sigma) & \text{if } m \text{ is even.} \end{cases}$$

iii) *Suppose  $\sigma$  is wildly irreducible and minimal, then*

$$\varsigma(\Sigma) \begin{cases} = \varsigma(\sigma) & \text{if } m \text{ is odd;} \\ \geq \frac{m}{2(m+1)} \varsigma(\sigma) & \text{if } m \text{ is even.} \end{cases}$$

*Proof* — We shall first prove the theorem assuming  $l = 0$ . In that case  $\Sigma$  is equal to the  $m$ -th symmetric power  $\mathbf{S}^m(\sigma)$  of  $\sigma$ .

i) Note that  $\mathbf{S}^m(\sigma)$  is the sum of  $m + 1$  characters given by

$$\mathbf{S}^m(\sigma) = \eta^m \oplus \eta^{m-1}\theta \oplus \dots \oplus \theta^m.$$

The quotient of two consecutive characters in the decomposition above is given by the character  $\eta/\theta$  of slope  $\varsigma$ . Hence, for each pair of consecutive characters at least one of them has at least slope  $\varsigma$ , and we find

$$\mathrm{sw}(\mathbf{S}^m(\sigma)) \geq \left\lfloor \frac{m+1}{2} \right\rfloor \varsigma.$$

Dividing by  $\dim(\mathbf{S}^m(\sigma))$  gives the desired inequalities.

Similarly, at least one of the characters  $\eta$  and  $\theta$  has at least slope  $\varsigma$ , so  $\varsigma(\sigma) \geq \varsigma/2$ . Note that the twisted representation  $\eta^{-1}\sigma$  is the sum of the unit character and  $\eta/\theta$ ; therefore, it has normalized exponent  $\varsigma/2$ . Hence, if  $\sigma$  is  $\varsigma$ -minimal, then  $\varsigma(\sigma) = \varsigma/2$ .

ii) If  $\sigma|_{\mathfrak{P}}$  is reducible, then  $\sigma = \text{Ind}_{E/F} \chi$  for some tame quadratic extension  $E/F$  and a  $\mathfrak{W}_E$ -character  $\chi$ . Let  $\lambda$  denote the non-trivial element of  $\mathcal{G}(E/F)$ , then

$$\sigma_E = \chi \oplus \chi^\lambda.$$

Since  $\sigma$  is  $\varsigma$ -minimal, it follows that  $\chi$  is  $F$ -minimal. Hence, using 4.5, we find

$$\varsigma(\sigma_E) = \varsigma(\chi) = \varsigma(\chi^\lambda) = \varsigma(\chi/\chi^\lambda).$$

By reduction via tame restriction (10.2), we have reduced to the previous case with  $\varsigma = \varsigma(\sigma)$ . Hence, the desired inequalities follow.

iii) First, observe that  $\mathfrak{W}^{\text{sl}(\sigma)}$  acts via a central character of order 2. Hence, if  $m$  is odd, the same holds for the representation  $\mathfrak{S}^m(\sigma)$  and, in particular,  $\varsigma(\mathfrak{S}^m(\sigma)) = \varsigma(\sigma)$ . Hereafter, we assume  $m$  is even and we write  $m = 2k$ .

Let  $\bar{\sigma}$  be the corresponding projective representation  $\mathfrak{W} \rightarrow \text{PGL}_2(\mathbf{C})$ . Its kernel defines a finite Galois extension  $K/F$ , and we let  $T/F$  denote the maximal tame subextension of  $K/F$ . Then, applying a base change to  $T/F$  (10.2) reduces us to the case where  $\bar{\sigma}(\mathfrak{W}) = \bar{\sigma}(\mathfrak{P})$ . Then,  $\sigma$  is, up to scalars, given by a representation of a pro- $p$ -group. Hence,  $\sigma = \text{Ind}_{E/F} \chi$  for some wild quadratic extension  $E/F$ , and a wildly ramified character  $\chi$  of  $\mathfrak{W}_E$ . We may assume that  $E/F$  is a least ramified inducing extension. Then, by a result of Kutzko [Kut79, Theorem 1.3], we have

$$\varsigma(\chi) \geq 2\text{sl}(E/F). \quad 10.3.1$$

Next, we consider the restriction of  $\mathfrak{S}^m(\sigma)$  to  $\mathfrak{W}_E$ . Letting  $\lambda$  denote the non-trivial element of  $\mathcal{G}(E/F)$ , we have

$$\mathfrak{S}^m(\sigma)|_E = \chi^m \oplus \chi^{m-1}\chi^\lambda \oplus \dots \oplus \chi^k(\chi^\lambda)^k \oplus \dots \oplus (\chi^\lambda)^m.$$

Observe that, except for the character in the middle, they can be paired into  $k$  pairs of characters whose corresponding 1-dimensional subspaces are swapped under the action of  $\lambda$ ; hence, each such pair gives rise to an induced representation from  $E/F$  that is contained in  $\mathfrak{S}^m(\sigma)$ . Moreover, the remaining character of  $\mathfrak{W}_E$  is a restriction of a character  $\eta$  of  $\mathfrak{W}$ . We conclude:

$$\mathfrak{S}^m(\sigma) = \underbrace{\text{Ind}_{E/F} \chi^m \oplus \text{Ind}_{E/F} \chi^{m-1}\chi^\lambda \oplus \dots \oplus \text{Ind}_{E/F} \chi^{k+1}(\chi^\lambda)^{k-1}}_{k \text{ terms}} \oplus \eta.$$

Now, note that for each pair of consecutive characters in the sequence

$$\chi^m, \chi^{m-1}\chi^\lambda, \dots, \chi^{k+1}(\chi^\lambda)^{k-1}, \chi^k(\chi^\lambda)^k, \quad 10.3.2$$

the quotient is given by the character  $\chi^{1-\lambda}$ , and so at least one of them has slope  $\geq \varsigma(\chi^{1-\lambda})$ . Moreover, by the  $\varsigma$ -minimality of  $\sigma$ , we know that  $\chi$  is  $F$ -minimal. Hence, by [4.5](#) and [10.3.1](#), we find

$$\varsigma(\chi^{1-\lambda}) \geq \varsigma(\chi) - \mathrm{sl}(E/F) \geq \mathrm{sl}(E/F). \quad \mathbf{10.3.3}$$

Now assume that  $k$  is even. Then, at least  $k/2$  of the first  $k$  characters in the sequence [10.3.2](#) have slope  $\geq \varsigma(\chi^{1-\lambda})$ . Hence, by applying [3.6](#) to the decomposition of  $\mathbf{S}^k(\sigma)$  given above, we obtain

$$\begin{aligned} \mathrm{sw}(\mathbf{S}^m(\sigma)) &\geq k \mathrm{sl}(E/F) + \frac{k}{2} (\varsigma(\chi) - \mathrm{sl}(E/F)) \\ &= \frac{k}{2} (\mathrm{sl}(E/F) + \varsigma(\chi)) \\ &= \frac{k}{2} \mathrm{sw}(\sigma). \end{aligned}$$

We conclude

$$\varsigma(\mathbf{S}^m(\sigma)) \geq \frac{k}{2k+1} \frac{\mathrm{sw}(\sigma)}{2} = \frac{m}{2(m+1)} \varsigma(\sigma).$$

Next, we assume  $k$  is odd. Our reasoning will focus on the first and last character of the sequence [10.3.2](#). Their quotient is given by the character  $(\chi^{1-\lambda})^k$ , which has slope  $\varsigma(\chi^{1-\lambda})$  as  $k$  is odd. Hence, at least one of these two characters has slope  $\geq \varsigma(\chi^{1-\lambda})$ . We shall consider both cases separately.

► Suppose the first character of [10.3.2](#) has slope  $\geq \varsigma(\chi^{1-\lambda})$ . Then, the same holds for at least  $(k+1)/2$  of the first  $k$  characters of [10.3.2](#), and we find

$$\begin{aligned} \mathrm{sw}(\mathbf{S}^m(\sigma)) &\geq k \mathrm{sl}(E/F) + \frac{k+1}{2} (\varsigma(\chi) - \mathrm{sl}(E/F)) \\ &= \frac{k-1}{2} \mathrm{sl}(E/F) + \frac{k+1}{2} \varsigma(\chi) \\ &\geq \frac{k}{2} (\mathrm{sl}(E/F) + \varsigma(\chi)) \\ &= \frac{k}{2} \mathrm{sw}(\sigma). \end{aligned}$$

Again, this leads to the desired inequality.

► Suppose the last character of [10.3.2](#) has slope  $\geq \varsigma(\chi^{1-\lambda})$ . Note that this character is given by the restriction of  $\eta$  to  $\mathfrak{W}_E$ . Hence, by [10.3.3](#), we obtain

$$\varsigma(\eta|_E) \geq \varsigma(\chi) - \mathrm{sl}(E/F) \geq \mathrm{sl}(E/F),$$

so that  $\varsigma(\eta|_E)$  lies beyond the jump of  $\phi_{E/F}$ . Then, using **3.6.i**, we find

$$\begin{aligned}\varsigma(\eta) &= \phi_{E/F}(\varsigma(\eta|_E)) \\ &\geq \phi_{E/F}^\infty(\varsigma(\chi) - \text{sl}(E/F)) \\ &= \phi_{E/F}^\infty(\varsigma(\chi)) - \frac{1}{2}\text{sl}(E/F) \\ &= \frac{1}{2}\varsigma(\chi),\end{aligned}$$

where  $\phi_{E/F}^\infty$  denotes the final linear piece of  $\phi_{E/F}$  and is given by the function  $u \mapsto \frac{1}{2}\text{sl}(E/F) + \frac{1}{2}u$  (**1.7**). From this, using similar arguments as before, we find

$$\begin{aligned}\text{sw}(\mathbb{S}^m(\sigma)) &\geq k \text{sl}(E/F) + \frac{k-1}{2}(\varsigma(\chi) - \text{sl}(E/F)) + \frac{1}{2}\varsigma(\chi) \\ &= \frac{k+1}{2}\text{sl}(E/F) + \frac{k}{2}\varsigma(\chi) \\ &\geq \frac{k}{2}(\text{sl}(E/F) + \varsigma(\chi)) \\ &= \frac{k}{2}\text{sw}(\sigma).\end{aligned}$$

As before, this implies the desired result. So we have proved the theorem in the case  $\Sigma = \mathbb{S}^m(\sigma)$ .

Finally, we consider the general case, in which  $\Sigma$  can be obtained from  $\mathbb{S}^m(\sigma)$  by twisting with a power of the determinant character.

We make the following crucial observation: given a sequence of characters  $\eta_1, \dots, \eta_n$ , the quotient of two consecutive characters is invariant under twisting with a character. Hence, all arguments given above remain valid for general  $\Sigma$ , and we obtain the same bounds. **q.e.d.**

**10.4 Corollary** — *Let  $\sigma: \mathfrak{W} \rightarrow \text{GL}_2(\mathbf{C})$  be an irreducible representation, and  $\Sigma$  its composition with an irreducible algebraic representation of  $\text{GL}_2(\mathbf{C})$  of dimension  $\geq 2$ . If  $\sigma$  is minimal, then*

$$\varsigma(\Sigma) \geq \frac{1}{3}\varsigma(\sigma).$$

*Proof* — We remark that the function  $m \mapsto m/(2(m+1))$  has minimal value  $1/3$  (which it attains at  $m = 2$ ). **q.e.d.**

# RÉSUMÉ

**Résultats principaux.** Soit  $F$  un corps local non archimédien de caractéristique résiduelle  $p > 0$ . Fixons une clôture algébrique séparable  $\overline{F}$  de  $F$ , et notons  $\mathfrak{W}$  le groupe de Weil correspondant à  $\overline{F}/F$ .

Dans cette thèse, nous nous intéressons aux représentations de  $\mathfrak{W}$ , continues, complexes, semi-simples et de dimension finie. Si  $\sigma$  est une telle représentation, la filtration de ramification de  $\mathfrak{W}$  nous permet de définir l'exposant  $\text{sw}(\sigma)$  de son conducteur de Swan (3.1). Cet invariant est un entier positif, qui ne dépend que de la restriction de  $\sigma$  au sous-groupe d'inertie sauvage de  $\mathfrak{W}$ .

Le problème central de cette thèse est le suivant. Soit  $\sigma$  une représentation de  $\mathfrak{W}$  de dimension  $n$ , et soit  $\rho \circ \sigma$  la composée de  $\sigma$  par une représentation algébrique  $\rho$  de  $\text{GL}_n(\mathbf{C})$ .

*Question* — Quelles sont les relations entre  $\text{sw}(\rho \circ \sigma)$  et  $\text{sw}(\sigma)$  ?

Plus précisément, nous souhaitons minorer  $\text{sw}(\rho \circ \sigma)$  par une quantité qui dépend de  $\text{sw}(\sigma)$ .

Après avoir rappelé des définitions et des lemmes essentiels dans le chapitre I, nous étudions dans le chapitre II le cas où la représentation  $\rho$  est donnée par la représentation adjointe. La représentation  $\rho \circ \sigma$  est alors isomorphe à  $\sigma \otimes \sigma^\vee$ , où  $\sigma^\vee$  dénote la représentation duale. Nous obtenons les deux théorèmes suivants.

**6.1 Théorème** — *Soit  $\sigma$  une représentation irréductible de  $\mathfrak{W}$  qui est minimale et de dimension  $\geq 2$ . Si  $\ell_\sigma$  est le plus petit nombre premier divisant la dimension de  $\sigma$ , alors*

$$\varsigma(\sigma \otimes \sigma^\vee) \geq (1 - 1/\ell_\sigma)\varsigma(\sigma).$$

Ici,  $\varsigma(\sigma)$  désigne l'exposant normalisé défini par  $\text{sw}(\sigma)/\dim(\sigma)$ . Nous renvoyons le lecteur à 4.4 pour la notion de minimalité.

**7.7 Théorème** — *Soit  $\sigma$  une représentation irréductible de  $\mathfrak{W}$  qui est épipélagique (i.e.,  $\text{sw}(\sigma) = 1$ ), alors*

$$\text{sw}(\sigma \otimes \sigma^\vee) = \dim(\sigma) - 1.$$

Ces résultats ont déjà été prouvés par Bushnell et Henniart dans le contexte de la correspondance locale de Langlands du côté GL. (En effet, ils ont prouvé un théorème plus fort en montrant l'égalité ci-dessus pour chaque représentation  $\sigma$  telle que  $\text{sw}(\sigma)$  et  $\dim(\sigma)$  sont premiers entre eux.) Par contre, les démonstrations données dans cette thèse sont uniquement de nature galoisienne, et ne font pas appel à la correspondance de Langlands.

Motivés par un résultat de Lapid, nous concluons le deuxième chapitre avec le corollaire suivant.

**8.1 Corollaire** — *Soit  $\sigma$  une représentation irréductible de  $\mathfrak{W}$ , alors*

$$\text{sw}(\sigma \otimes \sigma^\vee) \geq \dim(\sigma) - 1,$$

*avec égalité si et seulement si  $\sigma$  est épipélagique.*

Ensuite, en utilisant des techniques similaires, nous considérons deux autres cas du problème central dans le chapitre III. Le §9 traite le cas où la représentation  $\rho$  est donnée par une puissance tensorielle de la représentation standard. Plus généralement, on obtient le théorème suivant.

**9.3 Théorème** — *Soient  $\sigma: \mathfrak{W} \rightarrow \text{GL}(V)$  une représentation et  $\Sigma$  sa composée avec une représentation algébrique de  $\text{GL}(V)$ . Si  $\sigma$  est  $\varsigma$ -minimale, alors*

$$\varsigma(\sigma \otimes \Sigma) \geq \frac{1}{2}\varsigma(\sigma).$$

La définition de  $\varsigma$ -minimalité est donnée dans 9.1.

Enfin, le §10 présente des résultats si  $\sigma$  est de degré 2 et  $\rho$  est une représentation algébrique irréductible de  $\mathrm{GL}_2(\mathbf{C})$ .

**10.3 Théorème** — Soit  $\sigma$  une représentation de  $\mathfrak{W}$  de dimension 2, et soit  $\Sigma$  la représentation  $\mathfrak{S}^m(\sigma) \otimes \det(\sigma)^l$  de  $\mathfrak{W}$ , où  $m$  et  $l$  sont des entiers tels que  $m \geq 1$ . Alors, on a les énoncés suivants :

i) Supposons que  $\sigma$  soit réductible, i.e.,  $\sigma = \eta \oplus \theta$ , où  $\eta$  et  $\theta$  sont des caractères de  $\mathfrak{W}$ . Notons  $\varsigma = \varsigma(\theta/\eta)$ , alors

$$\varsigma(\Sigma) \geq \begin{cases} \frac{1}{2}\varsigma & \text{si } m \text{ est impair;} \\ \frac{m}{2(m+1)}\varsigma & \text{si } m \text{ est pair.} \end{cases}$$

De plus, si  $\sigma$  est  $\varsigma$ -minimale (9.1), alors  $\varsigma(\sigma) = \varsigma/2$ .

ii) Supposons que  $\sigma$  soit irréductible et minimale. Si la restriction de  $\sigma$  à  $\mathfrak{P}$  est réductible, alors

$$\varsigma(\Sigma) \geq \begin{cases} \frac{1}{2}\varsigma(\sigma) & \text{si } m \text{ est impair;} \\ \frac{m}{2(m+1)}\varsigma(\sigma) & \text{si } m \text{ est pair.} \end{cases}$$

iii) Supposons que  $\sigma$  soit sauvagement irréductible et minimale, alors

$$\varsigma(\Sigma) \begin{cases} = \varsigma(\sigma) & \text{si } m \text{ est impair;} \\ \geq \frac{m}{2(m+1)}\varsigma(\sigma) & \text{si } m \text{ est pair.} \end{cases}$$

**10.4 Corollaire** — Soit  $\sigma: \mathfrak{W} \rightarrow \mathrm{GL}_2(\mathbf{C})$  une représentation irréductible, et soit  $\Sigma$  sa composée avec une représentation algébrique irréductible de  $\mathrm{GL}_2(\mathbf{C})$  de dimension  $\geq 2$ . Si  $\sigma$  est minimale, alors

$$\varsigma(\Sigma) \geq \frac{1}{3}\varsigma(\sigma).$$

**Contexte et motivation.** Pour un entier strictement positif  $n$ , la correspondance de Langlands locale pour  $\mathrm{GL}_n$  établit une bijection canonique entre l'ensemble des classes d'isomorphisme des représentations cuspidales irréductibles de  $\mathrm{GL}_n(F)$  (le côté GL de la correspondance), et l'ensemble des classes d'isomorphisme des représentations irréductibles de  $\mathfrak{W}$  de dimension  $n$  (le côté galoisien) :

$$\left\{ \begin{array}{l} \text{repr. irréductible cuspidale} \\ \text{du groupe } \mathrm{GL}_n(F) \end{array} \right\} / \sim \xrightarrow{\sim} \left\{ \begin{array}{l} \text{repr. irréductible} \\ \text{de dim } n \text{ du groupe } \mathfrak{W} \end{array} \right\} / \sim$$

Cette bijection peut être caractérisée par le fait qu'elle préserve certains invariants définis des deux côtés. En particulier, ils existent deux notions



indépendantes de conducteur d'Artin (l'une pour les représentations lisses irréductibles de  $GL_n(F)$  et leurs produits tensoriels, due à Jacquet–Piatetski–Shapiro–Shalika, et l'autre, facilement dérivable du conducteur de Swan, pour les représentations lisses du groupe de Weil) qui sont préservées.

Par une application de la correspondance ci-dessus, et en utilisant la classification des représentations (lisses) irréductibles de  $GL_n(F)$  par la théorie des types de Bushnell–Kutzko, l'inégalité du théorème 6.1 a été prouvée par Bushnell et Henniart.

Dans cette thèse, nous fournissons une preuve plus élémentaire et transparente de ce résultat en utilisant uniquement la théorie de Galois, la théorie des corps de classes locaux, et des arguments élémentaires de la théorie des représentations. Il est alors naturel de se demander s'il est possible d'étendre ce résultat pour obtenir des inégalités similaires, ce qui nous a amené à formuler la question centrale de cette thèse comme écrite au-dessus.

**Structure détaillée de la thèse.** La thèse est organisée en trois chapitres. Le premier chapitre contient des définitions préliminaires. Plus précisément, nous rappelons les filtrations de ramification du groupe de Weil, et les fonctions de Herbrand qui relient leurs indices. Ensuite, nous présentons les définitions analogues dans la théorie des représentations, comme le conducteur de Swan et la pente d'une représentation, et nous présentons leurs propriétés de base. Ensuite, nous introduisons quelques arguments élémentaires de la théorie de Mackey qui sont utilisés fréquemment par la suite. Ces résultats sont connus mais ils ne sont pas faciles à trouver dans la littérature. On termine le chapitre avec une inégalité de Heiermann, qui énonce que si  $\sigma$  et  $\tau$  sont des représentations irréductibles, alors l'inégalité  $\text{sw}(\sigma \otimes \tau) \geq \text{sw}(\sigma \otimes \sigma^\vee)$  est satisfaite.

Le chapitre II est consacré aux preuves des théorèmes 6.1 et 7.7. Dans §4 et §5, nous présentons des préliminaires en étudiant l'action par conjugaison d'un groupe de Galois sur ses sous-groupes de ramification, et nous introduisons une notion plus fine que la notion de pente. La preuve du théorème 6.1 est ensuite présentée dans §6, et on procède de la façon suivante. La preuve est par récurrence sur la dimension, ce qui permet de réduire au cas où  $\sigma$  est totalement irréductible. Deux lemmes cruciaux (lemmes 6.2 et 6.3) jouent un rôle important dans l'argument de récurrence. Dans le cadre du §6, ils permettent de réduire la preuve de l'inégalité souhaitée au cas de la restriction de  $\sigma$  à la sous-extension maximale modérée. De là, en utilisant des arguments élémentaires de la théorie de Mackey, la borne de Heiermann et la minimalité de  $\sigma$ , on peut en déduire le résultat.

La preuve du théorème 7.7, donnée dans §7, présente certaines similitudes avec la preuve du théorème 6.1, mais une analyse plus détaillée est

nécessaire. Pour toute représentation irréductible  $\sigma$  qui est de type Carayol, on peut à nouveau utiliser l'induction et la restriction modéré (lemmes [6.2](#) et [6.3](#)) pour réduire au cas où  $\sigma$  est de dimension égale à une puissance de  $p$  et induite par un caractère de  $\mathfrak{W}_E$ , où l'extension  $E$  est totalement ramifiée de degré  $[E : F] = \dim(\sigma)$ . C'est à ce point où nous imposons des hypothèses supplémentaires, notamment que  $E$  soit galois sur  $F$ . En utilisant cette hypothèse, ainsi que certaines propriétés des sauts de la fonction Herbrand, la borne inférieure souhaitée en est déduite. Nous traitons ensuite le cas d'une extension non galoisienne  $E/F$  de degré 4, qui n'est pas couvert par le théorème.

Enfin, le chapitre III est consacré aux deux autres cas du problème central. Le §9 traite le cas où la représentation  $\rho$  est donnée par une puissance tensorielle de la représentation standard. Le §10 est consacré au cas où  $\sigma$  est de degré 2, et  $\rho$  est une représentation algébrique irréductible de  $\mathrm{GL}_2(\mathbf{C})$ . Le résultat principal est énoncé dans le théorème [10.3](#). Nous observons qu'une telle représentation  $\rho$  peut être écrite comme le produit tensoriel d'une puissance symétrique avec une puissance du déterminant. Ensuite, nous présentons un argument facile pour réduire au cas où  $\rho$  est une puissance symétrique. Finalement, un argument de restriction modéré, et une analyse au cas par cas selon la réductibilité de  $\sigma$ , terminent la preuve du théorème.



## REFERENCES

- [BH06] C. J. Bushnell and G. Henniart. *The local Langlands conjecture for  $GL(2)$* , volume 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.
- [BH14] C. J. Bushnell and G. Henniart. Langlands parameters for epipelagic representations of  $GL_n$ . *Math. Ann.*, 358(1-2):433–463, 2014.
- [BH17] C. J. Bushnell and G. Henniart. Strong exponent bounds for the local Rankin-Selberg convolution. *Bull. Iranian Math. Soc.*, 43(4):143–167, 2017.
- [BH19] Colin J. Bushnell and Guy Henniart. Local Langlands correspondence and ramification for Carayol representations. *Compos. Math.*, 155(10):1959–2038, 2019.
- [Bou12] N. Bourbaki. *Éléments de mathématique. Algèbre. Chapitre 8. Modules et anneaux semi-simples*. Springer, Berlin, 2012. Second revised edition of the 1958 edition [MR0098114].
- [Del84] P. Deligne. Les corps locaux de caractéristique  $p$ , limite de corps locaux de caractéristique 0. In *Représentations des groupes réductifs sur un corps local*, Travaux en Cours, pages 150–157. Hermann, Paris, 1984.
- [DH81] P. Deligne and G. Henniart. Sur la variation, par torsion, des constantes locales d'équations fonctionnelles de fonctions  $L$ . *Invent. Math.*, 64(1):89–118, 1981.

- [Hei96] V. Heiermann. Sur l'espace des représentations irréductibles du groupe de Galois d'un corps local. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(6):571–576, 1996.
- [Hen84] G. Henniart. Galois  $\varepsilon$ -factors modulo roots of unity. *Invent. Math.*, 78(1):117–126, 1984.
- [Koc77] H. Koch. Classification of the primitive representations of the Galois group of local fields. *Invent. Math.*, 40(2):195–216, 1977.
- [Kut79] P. C. Kutzko. The irreducible imprimitive local Galois representations of prime dimension. *J. Algebra*, 57(1):101–110, 1979.
- [Lap19] E. Lapid. On the support of matrix coefficients of supercuspidal representations of the general linear group over a local non-archimedean field. In *Representations of reductive groups*, volume 101 of *Proc. Sympos. Pure Math.*, pages 211–218. Amer. Math. Soc., Providence, RI, 2019.
- [Ser77] J.-P. Serre. *Linear representations of finite groups*. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.
- [Ser79] J.-P. Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.
- [Ser02] J.-P. Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.
- [Wei73] A. Weil. *Basic number theory*. Die Grundlehren der mathematischen Wissenschaften, Vol. 144. Springer-Verlag New York, Inc., New York, third edition, 1973.



**Titre :** Inégalités pour les conducteurs de Swan

**Mots Clefs :**

représentations galoisiennes

conducteurs de Swan

programme de Langlands

**Résumé :**

Dans cette thèse, nous nous intéressons aux représentations lisses complexes de groupe de Weil  $\mathfrak{W}$  d'un corps local non archimédien. Si  $\sigma$  est une telle représentation, la filtration de ramification de  $\mathfrak{W}$  nous permet de définir l'exposant  $\text{sw}(\sigma)$  de son *conducteur de Swan*.

Le problème central de cette thèse est le suivant. Soit  $\sigma$  une représentation de  $\mathfrak{W}$  de dimension  $n$ , et soit  $\rho \circ \sigma$  la composée de  $\sigma$  par une représentation algébrique  $\rho$  de  $\text{GL}_n(\mathbf{C})$ . Nous étudions les relations entre  $\text{sw}(\rho \circ \sigma)$  et  $\text{sw}(\sigma)$ . Plus précisément, nous reprovons certains résultats de Bushnell et Henniart dans le cas où la représentation  $\rho$  est donnée par la représentation adjointe, en n'utilisant que la théorie de Galois et des représentations. Ensuite, par des méthodes similaires, nous donnons des résultats quand  $\rho$  est une opération tensorielle. Finalement, nous discutons le cas  $n = 2$ .

**Title:** Inequalities on Swan conductors

**Keywords:**

Galois representations

Swan conductors

Langlands program

**Abstract:**

In this thesis, we are concerned with smooth complex representations of the Weil group  $\mathfrak{W}$  of a non-Archimedean local field. Via the ramification filtration of  $\mathfrak{W}$ , one can attach to such a representation  $\sigma$  an additive invariant  $\text{sw}(\sigma)$ , known as the *Swan exponent*.

The central problem in this thesis is the following. For an  $n$ -dimensional representation  $\sigma$  of  $\mathfrak{W}$ , we consider the composition  $\rho \circ \sigma$ , where  $\rho$  denotes an algebraic representation of  $\text{GL}_n(\mathbf{C})$ . We investigate the relations between  $\text{sw}(\rho \circ \sigma)$  and  $\text{sw}(\sigma)$ . More precisely, we reprove certain results of Bushnell and Henniart in the case where  $\rho$  is the adjoint representation, only invoking Galois theory and elementary representation theory. Using similar methods, we also provide results when  $\rho$  is a tensor operation. Finally, we investigate the case  $n = 2$ .