



HAL
open science

p-adic families of special cycles on a tower of unitary Shimura varieties

Yoël Dadoun

► **To cite this version:**

Yoël Dadoun. p-adic families of special cycles on a tower of unitary Shimura varieties. Algebraic Geometry [math.AG]. Universität Zürich, 2019. English. NNT : 2019SACLS528 . tel-02625264

HAL Id: tel-02625264

<https://theses.hal.science/tel-02625264>

Submitted on 26 May 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

THÈSE DE DOCTORAT

de

L'UNIVERSITÉ PARIS-SACLAY

École doctorale de mathématiques Hadamard (EDMH, ED 574)

Établissement d'inscription : Université Paris-Sud

Laboratoire d'accueil : Laboratoire de mathématiques d'Orsay, UMR 8628 CNRS

Spécialité de doctorat : Mathématiques fondamentales

Yoël DADOUN

p-adic families of special cycles
on a tower of unitary Shimura varieties

Date de soutenance : 18 Décembre 2019

Après avis des rapporteurs : MASSIMO BERTOLINI (Universität Duisburg-Essen)
SOPHIE MOREL (Princeton University)

Jury de soutenance :

JOSEPH AYOUB	(Universität Zürich) Co-directeur de thèse
GAËTAN CHENEVIER	(CNRS, Université Paris-Sud) Examineur
CHRISTOPHE CORNUT	(CNRS, Sorbonne Université) Examineur
JEAN-FRANÇOIS DAT	(Sorbonne Université) Président du jury
OLIVIER FOUQUET	(Université Paris-Sud) Co-directeur de thèse
BENJAMIN SCHRAEN	(Université Paris-Sud) Invité

Résumé:

Nous étudions les propriétés p -adiques d'une famille de 1-cycles algébriques *spéciaux* sur une variété de Shimura unitaire de dimension 3 apparaissant dans le cadre des conjectures de Gan-Gross-Prasad. Ces cycles, introduits par D.Jetchev et étudiés également par Boumasmoud-Brooks-Jetchev et R.Boumasmoud, proviennent du plongement diagonal $U(1, 1) \hookrightarrow U(2, 1) \times U(1, 1)$ associé à une extension CM E/F . Ils satisfont des relations de distribution « horizontales » et « verticales » sur leur conducteur, faisant de cette famille un nouvel exemple de *système d'Euler* géométrique généralisant celui des « points CM » sur la courbe modulaire, dont l'exploitation par Kolyvagin permit une avancée conceptuelle majeure dans l'attaque de la conjecture BSD. La preuve de ces relations locales entre action de Galois et celle de l'algèbre de Hecke de $\mathbf{G} = U(2, 1) \times U(1, 1)$ exploite les propriétés de certains opérateurs agissant sur l'immeuble de Bruhat-Tits de \mathbf{G} , en les places finies de F correspondantes. Nous construisons, en une place τ inerte de F divisant p , une filtration de \mathbf{G} par des sous-groupes ouverts compacts de type *Iwahori* définis comme les stabilisateurs d'une famille croissante de segments d'un même appartement. Nous adaptons au cas des segments la notion d'opérateurs « successeurs » étudiés par Boumasmoud-Brooks-Jetchev et montrons que ceux-ci proviennent de l'algèbre de Hecke-Iwahori locale. Nous démontrons que la tour de variétés de Shimura induite par cette filtration rend compatibles les actions de Galois et Hecke sur les cycles avec les morphismes de changement de niveau. Cette relation *verticale sur le niveau* est un ingrédient en faveur de l'existence d'un système d'Euler en familles p -adiques dans la cohomologie étale en degré médian de la variété de Shimura de groupe \mathbf{G} .

Abstract:

We study the p -adic properties of a family of *special* algebraic 1-cycles defined on a 3-dimensional unitary Shimura variety which appears in the setting of the Gan-Gross-Prasad conjectures. These cycles, introduced by Jetchev and also studied by Boumasmoud-Brooks-Jetchev and Boumasmoud, arise from the diagonal embedding $U(1, 1) \hookrightarrow U(2, 1) \times U(1, 1)$ attached to a CM extension E/F . These satisfy « horizontal » and « vertical » distribution relations for their conductors, making this family a new instance of a geometric *Euler system* generalizing the family of « CM-points » on modular curves, whose use by Kolyvagin provided a major conceptual advance towards the BSD conjecture. The proof of these local relations between the Galois action and the action of the Hecke algebra of $\mathbf{G} = U(2, 1) \times U(1, 1)$ make full use of some operators acting on the local Bruhat-Tits building of \mathbf{G} , at the corresponding finite places of F . We construct a τ -local filtration of \mathbf{G} - for some inert place τ of F above p - by *Iwahori*-type compact open subgroups, which are the stabilizers of an increasing family of segments in a same apartment. We adapt to segments the notion of « successor » operators studied by Boumasmod-Brooks-Jetchev and show that these arise from the local Iwahori-Hecke algebra. We show that the tower of varieties induced by this filtration makes the Galois and Hecke actions compatible with the change-of-level maps. This *level-wise vertical relation* is an ingredient towards the existence of a p -adic family of Euler systems in the middle-degree étale cohomology of the Shimura variety attached to \mathbf{G} .

*Ce que l'on conçoit bien s'énonce clairement,
Et les mots pour le dire arrivent aisément.*

[...]

*Hâtez-vous lentement ; et, sans perdre courage,
Vingt fois sur le métier remettez votre ouvrage :
Polissez-le sans cesse et le repolissez ;
Ajoutez quelquefois, et souvent effacez.*

N.Boileau, *L'Art poétique*, Chant I (1674)

Remerciements

Difficile, sans doute, de croire que l'écriture de cette partie de ma thèse - traditionnellement la seule vraiment facile ou du moins agréable à écrire pour un doctorant - ait pu encore causer quelques tracasseries. Sans doute est-ce parce que nombreuses sont les personnes à qui je souhaiterais témoigner ma gratitude, et que je crains d'oublier, à l'heure de m'éloigner pour quelques temps du monde de la recherche en mathématiques. Je remercie Joseph Ayoub pour l'honneur qu'il m'a fait d'accepter de co-diriger ma thèse, malgré un sujet pas tout à fait au centre de son (gigantesque) domaine d'expertise. Merci pour son accueil très chaleureux durant cette belle année passée à Zürich et pour son soutien sans faille dans mes nombreuses démarches administratives. Merci pour son œil acéré, que mes tentatives d'explications un peu trop bancales n'auront jamais risqué de tromper.

Olivier Fouquet m'a accompagné de toute sa bienveillance depuis le milieu de mon M2, pour m'emmener dans un monde peuplé de systèmes d'Euler, de théorie d'Iwasawa, de familles de Hida et de bien d'autres concepts abstraits aux noms japonisants. Merci pour son enthousiasme et ses patientes explications - devant mon air souvent circonspect - et pour les nombreuses discussions, bien rafraîchissantes, hors des mathématiques. Merci enfin pour sa relecture minutieuse de ce manuscrit qui aura sans doute permis de le rendre un peu plus présentable.

Je remercie sincèrement Sophie Morel et Massimo Bertolini d'avoir accepté d'être rapportrice et rapporteur de cette thèse. Leurs nombreuses remarques auront permis, je l'espère, d'en améliorer le fond et la forme. Christophe Cornut, Jean-François Dat et Gaëtan Chezevier m'ont fait l'honneur d'accepter de faire partie de mon jury, et je les en remercie vivement. Je remercie tout particulièrement le dernier cité, dont la rigueur et l'élégance des cours - ainsi que les conseils à divers moments de ma vie mathématique - ne sont certainement pas étrangers à mon envie de poursuivre l'aventure jusqu'en thèse. Puisse sa grande clarté mathématique continuer de profiter à de nombreux étudiants ! Je remercie Valentin Hernandez pour ses explications précises et enthousiastes sur la théorie de Hida des variétés de Shimura unitaires qui, je l'espère, me permettront de faire encore durer un peu le plaisir en exploitant plus tard les quelques résultats de ce manuscrit.

Je remercie chaleureusement la DGA qui m'accueille en son sein depuis quatre ans - et pour de bon depuis quelques semaines - pour avoir grandement facilité les conditions matérielles de cette thèse. Mon ticket CESFO quotidien ne dira, de son côté, certainement pas le contraire. Merci à Gerard Van der Geer pour son accueil à Amsterdam: l'étude des groupes de Bianchi et des surfaces modulaires de Hilbert sous sa supervision m'aura plongé, en douceur et pour de vrai, dans le grand bain de la recherche mathématique.

Le lecteur avisé remarquera l'importance mathématique de Reda Boumasmoud tout au long de ce texte. Outre un mathématicien de grande qualité - la conjecture de Blasius-Rogawski n'a qu'à bien se tenir ! - Reda aura été à mon égard un ami et un soutien inestimable ces deux dernières années, sans qui le présent manuscrit n'aurait probablement jamais abouti. Je dois à nos nombreuses palabres zürichoises d'avoir, un beau jour, pu enfin comprendre comment une bonne utilisation des immeubles de Bruhat-Tits pourraient me permettre d'entrevoir la lumière. Je remercie également Andrew Kresch pour sa grande gentillesse ainsi que pour les discussions - mathématiques ou non - qui auront égayé cette année zürichoise. Qui dit cotutelle dit double (voire triple) dose de formalités administratives. Je remercie Mmes Brass, Myoupo, Rey et Simon, *Frau* Schütze, M. Kleinberger, MM. Nonnenmacher et Paulin, ainsi que MM. Gillyboeuf et Leriche, pour m'avoir accompagné dans ce périple bureaucratique encore partiellement inachevé.

Ces quatre dernières années ont été l'occasion de nombreuses et belles rencontres, en France comme en Suisse. Je remercie pour leur bonne humeur et leur humour - autour d'une tasse de thé ou d'un repas - Armand, Benjamin, Camille, Cyril, Gabriel, Guillaume C., Guillaume L., Hugo, Jean, Jeanne, Louise, Luc, Lucile, Romain, Sasha, Tiago, Thomas, Xiaozong et tous ceux que j'aurais malencontreusement oubliés de ce côté-ci des Alpes. Un remerciement particulier pour mes collègues du bureau 3D20, Antoine, Lucien, Salim et Seginus. *Merci viemal* à Alberto M., Alberto N., Denis, Jakob et Yohan et aux habitué(e)s de la Mensa, *mit oder ohne Gemüse*. Une pensée également à mes camarades de M2 Vlerë, Felix, Guillaume et Lukas, compagnons d'infortune qui avec moi tentèrent de braver le flot d'*abstract nonsense* qui s'est abattu sur nous sans crier gare. Un clin d'œil amical à mes camarades rameurs, au bon souvenir de nos virées hivernales en 8 sur la Seine, ainsi que pour la joyeuse bande de randonneurs-chanteurs, toujours curieux de m'écouter tenter laborieusement d'expliquer le sujet de ma thèse au détour d'un col. Je salue les membres de la coloc' des Gobelins, et en particulier Damien pour les heures passées à dissenter des textes de Gainsbourg et de Brassens, ou des répliques de Georges Abitbol.

L'adolescent que j'étais il y a un moment à peine fut marqué par la rencontre de plusieurs professeurs, chacun dans un style très différent. Merci à Olivier Schettino d'avoir stimulé en moi - avec son charisme unique - la motivation pour les mathématiques et tout particulièrement le goût pour l'arithmétique. Merci à Alexandre Cardinal de m'avoir fait travailler la géométrie du plan (de swing) avec force balles blanches. Une pensée émue pour Charles Einhorn, qui a sans doute déjà trouvé là-haut une mélodie et quelques bons jeux de mots pour compléter ce manuscrit un peu trop fade. Merci à Roland Triay et Claudio Murolo pour leurs conseils et leur soutien à des moments clés de mon orientation.

Mon grand-oncle Roland est sans doute le « matheux » le plus ancien (et le plus sympathique !) de ma famille. Je remercie cette personne exceptionnelle d'avoir contribué, par

son exemple et son extrême bienveillance à mon égard, à entretenir en moi la flamme de la curiosité. Mes parents, mon frangin et Mimine (sans oublier « tata » Nenesse) ont été mes points de repère, mes guides - et parfois mes aiguillons - depuis un beau jour d'octobre. Ma gratitude envers eux va au-delà de ce qui pourrait être exprimé en seulement quelques lignes. Dernière arrivée, ma douce $\alpha_\ell(ix)$ a surgi comme par magie, supportant les nombreux week-ends studieux en y répondant par un soutien logistique et affectif sans faille.

Quelques jours à peine avant la soutenance de cette thèse est survenue la terrible et brutale nouvelle du décès du seul *docteur ès mathématiques* de ma famille, que j'espérais présent pour célébrer mon entrée dans le « club ». A mon cousin Jérôme Monnot, qui fut pour moi une source d'inspiration, un modèle de curiosité mathématique et un exemple de courage, d'optimisme et d'abnégation au quotidien, je voudrais dédier ce manuscrit.

Table of contents

0	Introduction (french version)	11
1	Special diagonal cycles on $\text{Sh}_K(\mathbf{G}, X)$.	25
1.1	The unitary Shimura varieties $\text{Sh}(\mathbf{G}, X)$ and $\text{Sh}(\mathbf{H}, Y)$.	25
1.1.1	Vector spaces V and W .	25
1.1.2	The F -algebraic groups $U(V)$ and $U(W)$.	27
1.1.2.1	The groups $U(V)$ and $U(W)$ split over E .	28
1.1.3	Hermitian symmetric domains X_V and X_W .	31
1.1.4	The Shimura varieties $\text{Sh}_K(\mathbf{G}, X)$ and $\text{Sh}_{K\mathbf{H}}(\mathbf{H}, Y)$.	35
1.1.4.1	The Deligne axioms SV1 - SV6.	35
1.1.4.2	The Shimura varieties.	39
1.1.5	Reflex fields, Galois descent and reflex norm-map.	44
1.1.5.1	The reflex fields $E(\mathbf{G}_\star, X_\star)$, $\star \in \{V, W, \emptyset\}$.	44
1.1.5.2	Interlude on Galois descent.	45
1.1.5.3	Reflex map and canonical models.	47
1.2	The set of special cycles on $\text{Sh}_K(\mathbf{G}, X)$.	49
1.2.1	The normalizer of \mathbf{H} in \mathbf{G} .	49
1.2.2	The set $\mathcal{Z}_K(\mathbf{G}, \mathbf{H})$ of \mathbf{H} -special cycles.	50
1.2.3	Smooth integral models, and a choice for K .	52
1.2.3.1	The level K_0 .	56
1.2.3.2	Allowable inert places τ .	56
1.2.4	A base cycle $\mathcal{Z}_{K_0}(g_0)$.	58
1.2.5	Hecke action on $\mathbb{Z}[\mathcal{Z}_K(\mathbf{G}, \mathbf{H})]$.	59
1.3	Transfer fields, Galois traces and distribution relations.	60
1.3.1	The global Artin map.	61
1.3.2	Restriction to finite idèles.	63
1.3.3	The case of F and E and the transfer map.	65
1.3.4	Galois action on $\pi_0(\text{Sh}_{K\mathbf{H}}(\mathbf{H}, Y))$.	67
1.3.4.1	The determinant map.	67
1.3.4.2	The zero-dimensional Shimura variety $\pi_0(\text{Sh}_{K\mathbf{H}}(\mathbf{H}, Y))$.	70
1.3.4.3	The adelic reflex map $r_{\det(\mu_{W, \mathbb{C}}), \mathbb{A}}$.	70

1.3.5	The field $E(\infty)$ and the map Art_E^1	73
1.3.6	The Galois action on $\mathcal{Z}_K(\mathbf{G}, \mathbf{H})$	74
1.3.7	Orders, ring class fields and transfer fields.	76
1.3.8	The field \mathcal{K}	82
1.3.8.1	Smooth models for \mathbf{Z} , \mathbf{T} and \mathbf{T}^1 over \mathcal{O}_F^Σ	83
1.3.8.2	The \mathcal{K} -transfer fields.	84
2	Bruhat-Tits building and local distribution relations	89
2.1	The Bruhat-Tits building of $\text{U}(3)$	89
2.1.1	The buildings $\tilde{\mathbf{B}}_V$ and \mathbf{B}_V via self-dual ultrametric norms.	90
2.1.2	The graph structure on \mathbf{B}_V / \sim	93
2.1.3	The distance in $ \mathbf{B}_V $	101
2.1.4	Apartments are geodesic lines.	104
2.1.5	Counting the number of neighbours.	108
2.1.6	The sub-building \mathbf{B}_W	110
2.1.6.1	Projection onto \mathbf{B}_W and <i>invariants map</i>	112
2.2	Computing the local conductor of special cycles.	114
2.2.1	The <i>standard situation</i>	114
2.2.2	Special apartments.	116
2.2.3	Interlude: filtration of \mathcal{O}_{E_τ} by local orders.	120
2.2.4	The local conductor formula.	122
2.3	The Iwahori filtration and the <i>level-wise</i> vertical relations.	126
2.3.1	Global and local Hecke algebras.	127
2.3.2	The Hecke polynomial.	129
2.3.3	The <i>partial Hecke operators</i> \mathcal{U} , \mathcal{V} and \mathcal{S}	130
2.3.4	Allowable segments.	134
2.3.5	The <i>Iwahori</i> filtration $(\mathbf{I}_\tau^{(n)})_{n \geq 0}$	137
2.3.5.1	Expression of $\mathcal{U}^{(n)}$ in terms of the n -Iwahori-Hecke algebra.	141
2.3.5.2	The filtration $(H_n)_{n \geq 0}$ of H_τ	143
2.3.5.3	Trace operators in $\mathbb{Z}[\text{Hyp}^{(n)}]$	147
2.3.6	An horizontal distribution relation in $ \mathbf{B}_V $	148
3	Global family of cycles and equivariant cohomology classes	153
3.1	Going back to cycles.	153
3.1.1	Local distribution relations.	153
3.1.1.1	Interlude on local orders, II.	153
3.1.1.2	From $\mathbb{Z}[\text{Hyp}^{(n)}]$ to $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbf{I}_\tau^{(n)}]$	156
3.1.1.3	The map $\det^\#$ and the trace operators $\text{Tr}_{k',k}^{\text{der}}$	157
3.1.1.4	An horizontal relation in $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / K_\tau]$	158
3.1.1.5	Vertical relation in $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbf{I}_\tau^{(n)}]$, $n \geq 1$	158

3.1.2	The global Iwahori filtration $(\mathbf{I}^{(n)})_{n \geq 1}$	159
3.1.3	The <i>norm-compatible</i> family $(\mathcal{Z}(n, \tau^m \cdot \mathfrak{c}))$	165
3.1.3.1	Links with geometric Hecke correspondences.	170

Chapter 0

Introduction (french version)

Quelques points de motivation

Cette thèse se situe quelque part dans le vaste monde que représente l'étude des fonctions L associées aux représentations automorphes d'un groupe réductif, et les diverses conjectures portant sur leurs valeurs dites *spéciales* aux points entiers, dont la plus célèbre est sans doute la conjecture de Birch et Swinnerton-Dyer. Le présent travail est principalement motivé par deux idées, elle-mêmes encore conjecturales à bien des aspects.

La première idée provient de l'étude des représentations p -adiques du groupe de Galois absolu d'un corps de nombres K , et plus précisément de celles qui se réalisent dans la cohomologie étale p -adique d'une variété algébrique sur K . A une telle représentation ρ_p est conjecturalement associée une fonction L - une fonction de la variable complexe s - définie pour $\text{Re}(s) \gg 0$ par un produit Eulérien

$$L(\rho_p, s) = \prod_{\ell} \frac{1}{\text{Eul}_{\ell}(\rho_p, s)}.$$

Les conjectures de Tate, Beilinson, Bloch-Kato et Kato ([47], [1], [5], [28]) suggèrent l'existence d'un système de classes dans la cohomologie galoisienne en degré 1 des extensions abéliennes de K vérifiant des relations de distribution dites *horizontales*, c'est-à-dire des relations de compatibilité pour la trace (ou corestriction) faisant intervenir les facteurs d'Euler du produit ci-dessus : ce qu'on appelle un *Système d'Euler* (cf. [30]).

La première application frappante, à la genèse du développement de la théorie des systèmes d'Euler, est l'exploitation lumineuse par Kolyvagin ([30]) des relations horizontales satisfaites par le système dit des *points d'Heegner* sur la courbe modulaire, qui a permis une avancée considérable dans la démonstration de la conjecture de Birch et Swinnerton-Dyer dans le cas d'une courbe elliptique rationnelle de rang analytique $r \leq 1$. En théorie d'Iwasawa, l'existence d'un système d'Euler fournit bien souvent une borne supérieure sur la taille des modules de Selmer de ρ_p , et joue donc un rôle essentiel dans les preuves des cas connus de la conjecture

principale de la théorie d'Iwasawa ([44, 29, 24, 2]).

Les exemples effectivement connus à ce jour de systèmes d'Euler proviennent du cas où la variété algébrique sous-jacente est une variété de Shimura pour un certain groupe réductif G et la preuve des relations de distribution horizontales satisfaites par ce système d'Euler fait intervenir l'algèbre de Hecke de G . Plus précisément, une source possiblement abondante de tels systèmes semble provenir de la construction de collections convenables de cycles algébriques *spéciaux* sur une variété de Shimura. Lorsque celle-ci est une courbe modulaire ou, plus généralement, la variété associée au groupe des unités d'une algèbre de quaternions déployée en au plus une place archimédienne d'un corps de nombres totalement réel, ces classes proviennent des points dits « CM », que l'on sait reliées aux valeurs spéciales par la formule de Gross-Zagier et ses généralisations (voir [21, 51, 3]).

La seconde idée, essentielle pour ce manuscrit, est qu'un système d'Euler de cycles spéciaux sur une variété de Shimura Sh_U de niveau donné U doit satisfaire, en sus des relations de distribution horizontales, des relations de distribution dites *verticales* dans la tour de variétés de Shimura $(\text{Sh}_{U'})_{U' \subset U}$. Que ces relations supplémentaires puissent exister est motivé par la théorie des familles p -adiques de représentations automorphes. En effet, on sait depuis les travaux de Hida et Coleman interpoler une forme modulaire (parabolique, propre) dans une famille paramétrée par un espace de poids p -adiques (sous des hypothèses « d'ordinarité » ou plus généralement de « pente finie » en p) et la construction de cette famille repose sur la géométrie de la courbe modulaire $X_1(Np^r)$ lorsque r tend vers l'infini ([23, 10, 15]). Si plus généralement on peut construire une famille p -adique de représentations automorphes à partir d'une tour de variétés de Shimura $(\text{Sh}_{U'})_{U' \subset U}$ avec U' de plus en plus *petit* en p , si les fonctions L des *points classiques* de cette famille s'interpolent également en une *fonction L p -adique* L_p et si, enfin, les systèmes d'Euler associés à ces fonctions L s'interpolent en une famille p -adique de systèmes d'Euler correspondant à L_p , alors il est naturel de penser que les systèmes d'Euler sur les variétés de Shimura $\text{Sh}_{U'}$ satisfont des relations de distribution verticales. Autrement dit, la propension qu'a un système d'Euler de cycles spéciaux sur une variété de Shimura à s'interpoler p -adiquement *doit* provenir sur la variété de départ d'une compatibilité sur le niveau en p , i.e. vis-à-vis des morphismes naturels entre variétés de niveaux $U'' = U''_p U^p$ et $U' = U'_p U^p$, où $U''_p \subset U'_p$ (resp. U^p) sont des sous-groupes ouverts compacts des points p -adiques (resp. des points adéliques finis hors de p) de G . Ces relations de distribution verticales, lorsqu'elles sont connues, permettent en retour de construire par interpolation de nouveaux systèmes d'Euler, même dans des cas où une construction géométrique « directe » n'est pas encore connue ([25, 16]).

Cette thèse démontre que ces deux espoirs sont fondés pour le système d'Euler des cycles spéciaux sur la variété de groupe $U(2, 1) \times U(1, 1)$ construit dans [27] par D.Jetchev.

Le système d'Euler de Jetchev

Récemment, Jetchev a construit dans [27] une famille de cycles spéciaux sur la variété de Shimura unitaire associée au groupe $U(2, 1) \times U(1, 1)$ - comprendre, le produit des groupes unitaires associés à une paire d'espaces hermitiens de signature respective $(2, 1)$ et $(1, 1)$ en une place archimédienne distinguée d'une extension CM E/F - en exploitant le plongement de Gan-Gross-Prasad $U(1, 1) \hookrightarrow U(2, 1) \times U(1, 1)$. Il a montré que cette famille vérifiait des relations locales horizontales en des places inertes « acceptables » de F . Les travaux ultérieurs de Boumasmoud-Brooks-Jetchev et la thèse de Boumasmoud - cette dernière dans le cadre du groupe $U(n - 1, 1) \times U(n - 2, 1)$ pour $n \geq 3$ quelconque - ont permis d'étendre ces relations dans une direction anticyclotomique ([8, 7]) et de formaliser plus précisément et profondément les relations Hecke-Galois satisfaites par ces cycles spéciaux. L'argument principal de ces travaux réside en l'étude d'opérateurs bien choisis sur les immeubles de Bruhat-Tits en les places finies considérées. Le résultat principal de ce manuscrit consiste, en vertu de la *seconde idée* ci-dessus, à établir la compatibilité sur le niveau *en p* (ou plus précisément, en une certaine place $\tau|p$ inerte dans E/F) de la famille de cycles spéciaux étudiée par Jetchev, Boumasmoud-Brooks-Jetchev et Boumasmoud.

Déroulé du manuscrit.

Nous expliquons maintenant le contenu du présent travail, dont le cadre géométrique est celui des conjectures de W-T.Gan, B.Gross et D.Prasad dans le cas des groupes unitaires (cf. [18], Conjectures 24.1 et 27.1).

- **Notations.**

Chap. 1

Dans cette thèse, F désigne un corps de nombres totalement réel de degré $d \geq 1$ dont on désigne par $I_{F,f}$ l'ensemble des places finies, et E/F est une extension quadratique totalement imaginaire. On a fixé une fois pour toutes un plongement $\overline{\mathbb{Q}} \subset \mathbb{C}$ et, pour tout nombre premier p , des plongements $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ ainsi que des isomorphismes abstraits $\eta_p : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$. L'ensemble $I_{F,\infty}$ des places archimédiennes de F est identifié à l'ensemble $\text{Hom}(F, \overline{\mathbb{Q}}) = \{\rho_1, \dots, \rho_d\}$ des plongements réels de F , et $\Phi = \{\tilde{\rho}_1, \dots, \tilde{\rho}_d\}$ désigne un *type CM* pour E/F , c'est à dire un système de représentants pour le quotient $\text{Gal}(E/F) \backslash \text{Hom}(E, \overline{\mathbb{Q}})$. On se fixe $(V, \langle \cdot, \cdot \rangle)$, un E/F -espace hermitien de dimension 3 et de signature $(2, 1)$ en $\tilde{\rho}_1$ (resp. de signature $(3, 0)$ en $\tilde{\rho}_i$, pour $i \geq 2$). On fixe également un vecteur anisotrope $e_D \in V$ tel que $\langle e_D, e_D \rangle = 1^{(1)}$ et l'on pose $D := E e_D$ et $W := D^\perp$. Ce dernier est un hyperplan hermitien de signature $(1, 1)$ en $\tilde{\rho}_1$ et $(2, 0)$ en les autres places. Associés à V et W sont les groupes unitaires $U(V)$ et $U(W)$: ce sont des F -groupes réductifs, qui vérifient $U(V)_E \cong \text{GL}(V)_E \simeq \text{GL}_{3,E}$ et $U(W)_E \cong \text{GL}(W)_E \simeq \text{GL}_{2,E}$, et la donnée de $W \subset V$ induit un plongement

(1). Un tel e_D existe toujours, quitte à re-normaliser le produit hermitien par un certain $\lambda \in F_{\gg 0}$, ce qui ne modifiera pas les données de Shimura étudiées.

$\iota : U(W) \hookrightarrow U(V)$ entre groupes unitaires, dont le premier s'identifie au sous-groupe $\{g \in U(V); g \cdot e_D = e_D\}$ du second.

- **Groupes et données de Shimura.**

L'application du foncteur $\text{Res}_{F/\mathbb{Q}}$ de restriction des scalaires (dû à Weil) aux groupes unitaires ci-dessus nous fournit les \mathbb{Q} -groupes réductifs $\mathbf{G}_V := \text{Res}_{F/\mathbb{Q}}U(V)$, $\mathbf{G}_W := \text{Res}_{F/\mathbb{Q}}\mathbf{G}_W$, et l'on pose $\mathbf{G} := \mathbf{G}_V \times \mathbf{G}_W$ ainsi que $\mathbf{H} := \Delta(\mathbf{G}_W) \subset \mathbf{G}$, où Δ est le plongement diagonal $(\iota_{\mathbb{Q}}, \text{Id}) : \mathbf{G}_W \hookrightarrow \mathbf{G}_V \times \mathbf{G}_W$, avec $\iota_{\mathbb{Q}} := \text{Res}_{F/\mathbb{Q}}(\iota)$. Le plongement Δ se prolonge en un plongement entre deux variétés de Shimura $\text{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y) \hookrightarrow \text{Sh}_K(\mathbf{G}, X)$ - ce sont des variétés algébriques quasi-projectives sur \mathbb{C} , propres lorsque $F \neq \mathbb{Q}$, de dimensions respectives 1 et 3 - associées à \mathbf{H} et \mathbf{G} . Ce plongement induit une immersion fermée entre leurs modèles canoniques respectifs, définis sur leur corps réflexe commun égal à E . Les sous-groupes compacts ouverts $K \subset \mathbf{G}(\mathbb{A}_f)$ et $K_{\mathbf{H}} := \mathbf{H}(\mathbb{A}_f) \cap K$ ci-dessus peuvent être quelconques, et l'on imposera seulement la condition qu'ils soient *nets* (cf. Définition 1.1.1), ce qui implique que les variétés de Shimura et leurs modèles canoniques sont lisses.

- **Cycles spéciaux, actions de Galois et de Hecke.**

Le plongement $\text{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y) \hookrightarrow \text{Sh}_K(\mathbf{G}, X)$ - qui correspond au cycle diagonal étudié dans les conjectures Gan-Gross-Prasad - donne naissance à une famille $\mathcal{Z}_K(\mathbf{G}, \mathbf{H}) \subset \mathbf{Z}_1(\text{Sh}_K(\mathbf{G}, X))$ de 1-cycles algébriques *\mathbf{H} -spéciaux* sur $\text{Sh}_K(\mathbf{G}, X)$, paramétrés par les éléments de $\mathbf{G}(\mathbb{A}_f)$. Par construction, ces cycles sont définis sur des extensions finies abéliennes de E et, plus précisément, sur le *corps de transfert* $E(\infty)/E$, sur lequel nous revenons plus bas. L'ensemble $\mathcal{Z}_K(\mathbf{G}, \mathbf{H})$ se trouve être en bijection avec le double quotient $\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cdot \mathbf{H}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f)/K$, via l'application qui à $g \in \mathbf{G}(\mathbb{A}_f)$ associe le cycle $\mathcal{Z}_K(g)$ défini comme l'image de la composante connexe neutre de $\text{Sh}_{K_{g, \mathbf{H}}}(\mathbf{H}, Y)$ ⁽²⁾ par la suite de E -morphisms :

$$\text{Sh}_{K_{g, \mathbf{H}}}(\mathbf{H}, Y) \hookrightarrow \text{Sh}_{gKg^{-1}}(\mathbf{G}, X) \xrightarrow{[g]} \text{Sh}_K(\mathbf{G}, X),$$

où $K_{g, \mathbf{H}} := \mathbf{H}(\mathbb{A}_f) \cap gKg^{-1}$. L'action galoisienne de $\text{Gal}(E(\infty)/E)$ sur le \mathbb{Z} -module $\mathbb{Z}[\mathcal{Z}_K(\mathbf{G}, \mathbf{H})]$, qui provient de l'action sur les composantes connexes de $\text{Sh}(\mathbf{H}, Y)$, s'exprime de manière simple via la *réciprocité de Shimura* suivante : si $\sigma \in \text{Gal}(E(\infty)/E)$ et si $h_{\sigma} \in \mathbf{H}(\mathbb{A}_f)$ vérifie $\text{Art}_E^1(\det(h_{\sigma})) = \sigma$, on montre à la Proposition 1.42 que

$$\sigma \cdot \mathcal{Z}_K(g) = \mathcal{Z}_K(h_{\sigma} g), \text{ pour tout } g \in \mathbf{G}(\mathbb{A}_f).$$

Le tore $\mathbf{T}^1 := \text{Res}_{F/\mathbb{Q}}U(1)_{E/F}$ ci-dessus s'identifie avec le centre des groupes \mathbf{G}_V et \mathbf{G}_W , et l'isomorphisme $\text{Art}_E^1 : \frac{\mathbf{T}^1(\mathbb{A}_f)}{\mathbf{T}^1(\mathbb{Q})} \xrightarrow{\sim} \text{Gal}(E(\infty)/E)$ provient de la théorie

(2). C'est à dire, correspondant sur les \mathbb{C} -points à $\mathbf{H}(\mathbb{Q}) \cdot (Y \times K_{g, \mathbf{H}}) \subset \text{Sh}_{K_{g, \mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C})$.

du corps de classes globale (voir plus bas). En particulier, l'action de Galois commute avec l'action des éléments de l'algèbre de Hecke $\mathcal{H}_K := \mathcal{H}_{\mathbb{Z}}(\mathbf{G}(\mathbb{A}_f) // K)$, qui opèrent sur $\mathbb{Z}[\mathcal{Z}_K(\mathbf{G}, \mathbf{H})]$ de la manière usuelle suivante : pour tout $g, g' \in \mathbf{G}(\mathbb{A}_f)$ l'opérateur de double classe $[K g K] = \rho(\mathbf{1}_{K g K}) \in \mathcal{H}_K$ agit sur $\mathcal{Z}_K(g')$ par la formule $\rho(\mathbf{1}_{K g K}) \cdot \mathcal{Z}_K(g') := \sum_{i=1}^r \mathcal{Z}_K(g' g_i)$, avec $K g K = \sqcup_{i=1}^r g_i K$, $g_i \in \mathbf{G}(\mathbb{A}_f)$.

- **Réseaux globaux, modèles entiers et compact de base.**

Un choix particulier pour le sous-groupe compact ouvert $K \subset \mathbf{G}(\mathbb{A}_f)$ est donné au § 1.2.3.1 : celui-ci, que l'on note provisoirement K_0 , s'écrit sous la forme $K_V \times K_W$, où K_W et K_V sont définis comme les stabilisateurs, dans $U(W)(\mathbb{A}_{F,f})$ et $U(V)(\mathbb{A}_{F,f})$ respectivement, des $\mathcal{O}_E \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ -réseaux $L_W \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ et $L_V \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, où L_W et $L_V := L_W \oplus \mathcal{O}_E e_D$ sont des \mathcal{O}_E -réseaux dans W et V sur lesquels le produit hermitien prend des valeurs entières. De manière équivalente, le compact K_0 s'identifie aux $\widehat{\mathcal{O}_F}$ -points d'un modèle \mathcal{O}_F -entier $\underline{U}_V \times \underline{U}_W$ de $U(V) \times U(W)$, lisse et à fibres réductives en dehors d'un ensemble fini de places $S^1 \subset I_{F,f}$. On introduit la notion de place *inerte acceptable* de F : celles-ci consistent en les places τ de F , inertes dans E/F et en dehors de l'ensemble fini de places $S^2 \supset S^1$. L'ensemble S^2 correspond aux places finies v pour lesquelles les réseaux locaux $L_{W,v}$ et $L_{V,v}$ ne sont pas autoduaux, ou bien où le déterminant de l'espace hermitien local $(W_v, \langle \cdot, \cdot \rangle_v)$ - et donc celui de $(V_v, \langle \cdot, \cdot \rangle_v)$ - n'est pas trivial dans le quotient $\frac{F_v^\times}{N_{E_v/F_v}(E_v^\times)}$. Le choix du compact K_0 implique qu'il s'écrit, en chaque place finie τ de F , inerte et acceptable, comme un produit $K_0 = K_\tau \times K^\tau$, où K_τ est le stabilisateur dans $G_\tau := (U(V) \times U(W))(F_\tau)$ de la paire $(L_{V,\tau}, L_{W,\tau})$ de réseaux *autoduaux* - pour la dualité induite par le produit hermitien local $\langle \cdot, \cdot \rangle_\tau$ - dans $V_\tau \oplus W_\tau$. De manière équivalente, les places inertes acceptables τ sont telles que l'espace hermitien local V_τ admet des bases particulières appelées *bases de Witt* (cf. § 1.2.3.2). Sauf mention du contraire, la notation K désignera toujours par la suite le compact particulier K_0 .

- **Application(s) d'Artin et \mathcal{K} -corps de transfert.**

On fixe enfin arbitrairement un élément $g_0 \in \mathbf{G}(\mathbb{A}_f)$: par définition des points adéliques, celui-ci vérifie $g_{0,v} \in K_v$ pour presque toute place finie v , ce qui induit la définition de notre ensemble de *mauvaises places* $\Sigma \supset S^2$: sans perte de généralité vis-à-vis des cycles spéciaux, on supposera que $g_{0,v} = 1$, $\forall v \notin \Sigma$. On termine ce premier chapitre par une discussion sur la théorie du corps de classes globale. Après quelques généralités sur l'application d'Artin globale Art_L associée à un corps de nombres L quelconque, et sa restriction aux idèles finis de L , on s'intéresse ensuite au cas de l'extension particulière E/F . On introduit le *corps de transfert* (en anglais, *transfer field*) $E(\infty) \subset E^{ab}$, que l'on définit comme l'extension abélienne maximale de E fixée par l'image de l'application de transfert $\text{Ver}_{E/F} : \text{Gal}(F^{ab}/F) \rightarrow \text{Gal}(E^{ab}/E)$, et l'on montre l'existence d'un

isomorphisme noté $\text{Art}_E^1 : \frac{\mathbf{T}^1(\mathbb{A}_f)}{\mathbf{T}^1(\mathbb{Q})} \xrightarrow{\sim} \text{Gal}(E(\infty)/E)$ rendant le carré suivant

$$\begin{array}{ccc} \frac{\mathbb{A}_{E,f}^\times}{E^\times} & \xrightarrow{\text{Art}_E} & \text{Gal}(E^{ab}/E) \\ \downarrow \begin{smallmatrix} z \mapsto \bar{z} \\ z \end{smallmatrix} & & \downarrow \sigma \mapsto \sigma|_{E(\infty)} \\ \frac{\mathbf{T}^1(\mathbb{A}_f)}{\mathbf{T}^1(\mathbb{Q})} & \xrightarrow[\text{Art}_E^1]{\sim} & \text{Gal}(E(\infty)/E) \end{array}$$

commutatif.

On exprime ensuite le corps de transfert comme une limite inductive d'extensions finies abéliennes de E , diédrales sur F , appelées *corps de transfert de conducteur \mathfrak{c}* . Ce sont des variantes des *corps de classes d'anneaux* (ou *ring class fields* en anglais) prenant en compte l'inclusion $\mathbb{A}_{F,f}^\times \subset \mathbb{A}_{E,f}^\times$, dont les *conducteurs* parcourent l'ensemble des idéaux entiers $\mathfrak{c} \subset \mathcal{O}_F$. On définit enfin, à la manière de ([7], VII), une extension finie abélienne particulière \mathcal{K}/E , vu comme le corps de définition minimal du *cycle de base $\mathcal{Z}_K(g_0)$* , et l'on introduit la notion de *\mathcal{K} -corps de transfert* (en anglais, *\mathcal{K} -transfer fields*) de conducteurs variables $\mathfrak{f} \subset \mathcal{O}_F$, où \mathfrak{f} parcourt tous sauf un nombre fini d'idéaux de \mathcal{O}_F . Un résultat dû à Nekovář (Lemma 1.3.17) entraîne que les extensions intermédiaires entre \mathcal{K} -corps de transferts de conducteurs convenables se découpent naturellement en produits de composantes locales (cf. Corollaire 1.3.18). Ceci motive l'utilisation de ces corps comme corps de définition naturels pour la famille de cycles spéciaux que l'on introduira au chapitre 3.

Chap. 2 • Les immeubles \mathbf{B}_V et \mathbf{B}_W .

Au chapitre 2, une place inerte acceptable τ étant fixée - dont on note q le cardinal du corps résiduel $\mathcal{O}_{F_\tau}/\varpi$, de caractéristique p - on étudie la composante τ -locale de l'ensemble $\mathcal{Z}_K(\mathbf{G}, \mathbf{H})$ des cycles \mathbf{H} -spéciaux de niveau K , en introduisant l'immeuble de Bruhat-Tits \mathbf{B}_V . L'immeuble \mathbf{B}_V , attaché au groupe unitaire local $G_{V,\tau} = \text{U}(V)(F_\tau)$, est un complexe poly-simplicial obtenu par recollement d'une famille distinguée d'espaces affines de dimension réelle 1 appelés *appartements*. On montre qu'il possède une structure d'arbre *bi-couleur* dont les sommets se divisent en deux types : *hyperspéciaux* (ou noirs) et *spéciaux* (ou blancs). Les sommets d'un même type sont permutés par l'action transitive de $G_{V,\tau}$ et sont en bijection avec les \mathcal{O}_{E_τ} -réseaux autoduaux, pour les sommets hyperspéciaux, ou avec les \mathcal{O}_{E_τ} -réseaux *strictement presque auto-duaux* de V_τ , pour les sommets spéciaux (cf. Définition 1.2.4). Leurs stabilisateurs dans $G_{V,\tau}$ s'identifient, de cette manière, aux sous-groupes portant la même dénomination. On normalise la distance dist dans l'immeuble de sorte que deux sommets hyperspéciaux partageant un voisin spécial soient à distance 1. On peut également reproduire la même construction à partir du groupe $G_{W,\tau} = \text{U}(W)(F_\tau)$ et obtenir l'immeuble \mathbf{B}_W , qui se

plonge isométriquement comme un sous-immeuble de \mathbf{B}_V dont les sommets noirs (resp. blancs) correspondent aux réseaux auto-duaux (resp. strictement presque auto-duaux) L de V_τ s'écrivant $L = L' \oplus \mathcal{O}_{E_\tau} e_D$, pour un certain réseau $L' \subset W_\tau$ auto-dual (resp. strictement presque auto-dual), que l'on identifiera donc à L dans l'immeuble \mathbf{B}_V .

- **Combinatoire de l'immeuble.**

Les observations précédentes impliquent la combinatoire suivante : les sommets noirs (resp. blancs) de \mathbf{B}_V possèdent $q^3 + 1$ voisins blancs (resp. $q + 1$ voisins noirs), et les sommets (noirs ou blancs) de \mathbf{B}_W possèdent $q + 1$ voisins blancs ou noirs.

- **Un appartement *spécial* \mathcal{A}' .**

On se concentre désormais sur les sommets hyperspéciaux de l'immeuble, davantage liés aux cycles. En notant Hyp_V l'ensemble de sommets hyperspéciaux de \mathbf{B}_V et Hyp_W son sous-ensemble $\text{Hyp}_V \cap \mathbf{B}_W$, on fixe arbitrairement l'origine de l'immeuble en $x := L_{V,\tau} = L_{W,\tau} \in \text{Hyp}_W$, de sorte que le quotient G_τ/K_τ s'identifie naturellement à l'ensemble $\text{Hyp} = \text{Hyp}_V \times \text{Hyp}_W$ via l'action sur la paire de sommets $(x, x) \in \text{Hyp}$. On choisit un appartement \mathcal{A}' de \mathbf{B}_V , associé à une base de Witt $\mathcal{B}' = \{e'_+, e'_0, e'_-\}$ engendrant le réseau x , et l'on suppose que c'est un appartement *spécial* de base x , i.e., qu'il intersecte le sous-immeuble \mathbf{B}_W en une demi-droite $(\mathcal{A}')^+$ d'extrémité x (cf. Définition 2.2.2). Nous choisissons également une base de Witt $\mathcal{B} = \{e_+, e_0, e_-\}$ de V_τ telle que $e_+, e_- \in W_\tau$ et $e_0 = e_D \in D_\tau$, dont l'appartement associé $\mathcal{A} := \mathcal{A}_\mathcal{B}$ - inclus dans \mathbf{B}_W - intersecte \mathcal{A}' en $(\mathcal{A}')^+$.

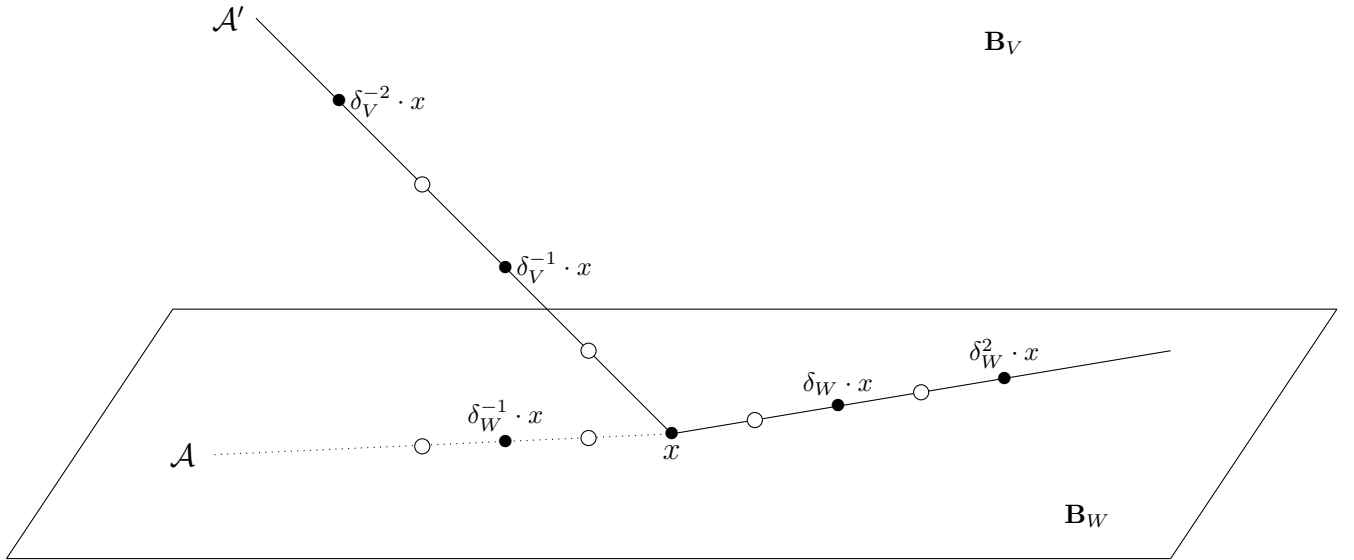


Figure 1 – On représente graphiquement le sous-immeuble \mathbf{B}_W sous la forme d'un hyperplan affine dans \mathbf{B}_V . On a représenté l'appartement *spécial* \mathcal{A}' sous la forme d'une droite, dont les points noirs correspondent aux sommets hyperspéciaux et les points blancs aux sommets spéciaux, et qui intersecte \mathbf{B}_W en le demi-appartement partagé avec $\mathcal{A} \subset \mathbf{B}_W$. Les segments $\Xi_{V,0}^{(2)} = \llbracket x, \delta_V^{-1} \cdot x, \delta_V^{-2} \cdot x \rrbracket$ et $\Xi_{W,0}^{(2)} = \llbracket x, \delta_W \cdot x, \delta_W^2 \cdot x \rrbracket$ sont admissibles de longueur 2.

- **Segments et segments admissibles.**

Nous rappelons en 2.3.4, pour tout $n \geq 0$, la notion de *segment* d'appartenance de longueur n dans \mathbf{B}_V et \mathbf{B}_W , et introduisons la variante de *segment admissible*, i.e., de segment *qui s'éloigne* de l'origine x (cf. Définition 2.3.4). Pour $\star \in \{V, W\}$, on désigne par $\underline{\text{Hyp}}_\star^{(n)} \subset \text{Hyp}_\star^{(n)}$ les ensembles formés respectivement des segments admissibles et des segments de longueur n , et l'on forme les produits $\underline{\text{Hyp}}^{(n)} \stackrel{\text{def}}{=} \underline{\text{Hyp}}_V^{(n)} \times \underline{\text{Hyp}}_W^{(n)} \subset \text{Hyp}_V^{(n)} \times \text{Hyp}_W^{(n)} \stackrel{\text{def}}{=} \text{Hyp}^{(n)}$. Pour tout $n \geq 0$, on définit en 2.26 les segments admissibles standard $\Xi_{V,0}^{(n)} \in \underline{\text{Hyp}}_V^{(n)}$ et $\Xi_{W,0}^{(n)} \in \underline{\text{Hyp}}_W^{(n)}$ obtenus en parcourant, partant de x , les $n+1$ premiers sommets hyperspéciaux de \mathcal{A}' *sortant* du sous-immeuble \mathbf{B}_W (resp. *restant* dans la demi-droite $(\mathcal{A}')^+ \subset \mathbf{B}_W$). On pose enfin $\Xi_0^{(n)} := \Xi_{V,0}^{(n)} \otimes \Xi_{W,0}^{(n)} \in \underline{\text{Hyp}}^{(n)}$.

- **La filtration d'Iwahori sur G_τ .**

Pour $n \geq 0$ toujours, les sous-groupes *de type Iwahori* $I_{V,\tau}^{(n)} \subset K_{V,\tau}$ et $I_{W,\tau}^{(n)} \subset K_{W,\tau}$ sont introduits à la Définition 2.3.6 comme étant les stabilisateurs, dans $G_{V,\tau}$ et $G_{W,\tau}$ respectivement, de $\Xi_{V,0}^{(n)}$ et $\Xi_{W,0}^{(n)}$. La famille $\left(I_\tau^{(n)} := I_{V,\tau}^{(n)} \times I_{W,\tau}^{(n)} \right)_{n \geq 0}$, dont chaque terme contient le centre \mathbf{Z}_{G_τ} de G_τ , définit ainsi une filtration décroissante de K_τ par des sous-groupes compacts ouverts.

- **Opérateurs sur les segments.**

Nous reprenons la notion d'opérateurs *successeurs* \mathcal{U} , initialement introduite par Cornut-Vatsal dans ([13], §6.3) via les opérateurs T_P^u , et étendue par Boumasmoud-Brooks-Jetchev ([8], 3.2) comme un opérateur sur le module $\mathbb{Z}_{(p)}[\text{Hyp}]^{(3)}$. Nous l'étendons aux segments admissibles en un opérateur $\mathcal{U}^{(n)} : \mathbb{Z}[\underline{\text{Hyp}}^{(n)}] \rightarrow \mathbb{Z}[\underline{\text{Hyp}}^{(n+1)}]$, l'omission de la localisation en (p) étant valable si $n \geq 1$ (cf. Définition 2.3.5). Lorsque $n \geq 1$, l'opérateur $\mathcal{U}^{(n)}$ est « comparable » à l'élément $t^{(n)} = \rho(\mathbf{1}_{I_\tau^{(n)}} \delta_{I_\tau^{(n)}})$ de *l'algèbre de Hecke-Iwahori de niveau n* , $\mathcal{H}_\tau^{(n)} := \mathcal{H}_\mathbb{Z}(G_\tau // I_\tau^{(n)})$, comme le montre le Lemma 2.31. Ici, $\delta \in G_\tau$ désigne l'élément $(\delta_V^{-1}, \delta_W) \in G_\tau$, où $\delta_V^{-1} \stackrel{\mathcal{B}'}{=} \text{diag}(\varpi, 1, \varpi^{-1})$ (resp. $\delta_W \stackrel{\mathcal{B}}{=} \text{diag}(\varpi^{-1}, 1, \varpi)$) correspondent, dans leur action sur \mathcal{A}' , à des *shifts* dans la direction *sortante* (resp. *rentrante*) par rapport à \mathbf{B}_W . En particulier, la filtration d'Iwahori vérifie $I_\tau^{(n+1)} = I_\tau^{(n)} \cap \delta I_\tau^{(n)} \delta^{-1}$, pour tout $n \geq 0$.

- **La filtration $(H_n)_{n \geq 0}$ et la relation verticale dans l'immeuble.**

La filtration d'Iwahori $(I_\tau^{(n)})_{n \geq 0}$ induit une filtration $\left(H_n := H_\tau \cap I_\tau^{(n)} \right)_{n \geq 0}$ sur le sous-groupe diagonal $H_\tau := \Delta(G_{W,\tau}) \subset G_\tau$. L'injection naturelle de H_n dans $I_\tau^{(n)}$ induit

(3). La localisation en l'idéal (p) provient seulement de la définition de \mathcal{U} en x . Ces opérateurs sont redéfinis dans un degré de généralité supérieur par Boumasmoud dans [7], pour des espaces hermitiens de dimension quelconque et dans le cas où τ est également scindé dans E/F , sans référence au point base x . Nous avons cependant préféré conserver dans ce manuscrit les opérateurs de [8], au prix d'un degré de généralité moindre et d'une définition un peu *ad-hoc*.

une bijection $H_n / H_{n+1} \simeq \mathbb{I}_\tau^{(n)} / \mathbb{I}_\tau^{(n+1)}$ pour tout $n \geq 1$, et ce dernier quotient est en bijection avec l'orbite de l'opérateur \mathcal{U} agissant sur un élément bien choisi de Hyp (cf. Lemma 2.3.7). Ceci induit :

$$\#(H_n / H_{n+1}) = \#(\mathbb{I}_\tau^{(n)} / \mathbb{I}_\tau^{(n+1)}) = q^6, \quad \forall n \geq 1.$$

Les groupes $H_\tau \subset G_\tau$ agissent naturellement sur $\mathbb{Z}[\text{Hyp}^{(n)}]$, pour tout $n \geq 0$. Pour tout $k \geq 0$, on définit le sous- \mathbb{Z} -module $\mathbb{Z}[\text{Hyp}^{(n)}]_k \subset (\mathbb{Z}[\text{Hyp}^{(n)}])^{H_k}$ comme étant le produit tensoriel $(\mathbb{Z}[\text{Hyp}_V^{(n)}])^{H_k} \otimes_{\mathbb{Z}} (\mathbb{Z}[\text{Hyp}_W^{(n)}])^{H_k}$, induisant une filtration croissante

$$\mathbb{Z}[\text{Hyp}^{(n)}]_0 \subset \mathbb{Z}[\text{Hyp}^{(n)}]_1 \subset \dots \subset \mathbb{Z}[\text{Hyp}^{(n)}]_k \subset \mathbb{Z}[\text{Hyp}^{(n)}]_{k+1} \subset \dots$$

sur $\mathbb{Z}[\text{Hyp}^{(n)}]$. Pour tout $k' \geq k \geq 0$, les *opérateurs trace*

$$\text{Tr}^{k',k} : \mathbb{Z}[\text{Hyp}^{(n)}]_{k'} \longrightarrow \mathbb{Z}[\text{Hyp}^{(n)}]_k$$

sont définis par

$$\mathfrak{X}_V \otimes \mathfrak{X}_W \longmapsto \sum_{h \in H_k / H_{k'}} (h \cdot \mathfrak{X}_V) \otimes (h \cdot \mathfrak{X}_W),$$

avec $\mathfrak{X}_\star \in \mathbb{Z}[\text{Hyp}_\star^{(n)}]_{k'}$, pour $\star \in \{V, W\}$. Nous obtenons finalement à la Proposition 2.3.14 la *relation de distribution verticale sur le conducteur*, formulée localement en τ dans l'immeuble \mathbf{B}_V comme une relation entre l'action de l'opérateur successeur $\mathcal{U}^{(n)}$ agissant sur $\Xi_m^{(n)} := \delta^m \cdot \Xi_0^{(n)}$ et l'opérateur trace $\text{Tr}^{n+m+1, n+m}$, pour tout $m \geq 0$:

$$\text{Tr}^{n+m+1, n+m}(\Xi_m^{(n+1)}) = \mathcal{U}^{(n)}(\Xi_m^{(n)}).$$

La formulation (2.40) est légèrement plus commode pour les applications ultérieures :

$$(\mathbf{s}_n \otimes \mathbf{s}_n) \circ (\mathbf{v}_{n+1} \circ \mathbf{v}_{n+1}) \cdot \text{Tr}^{n+m+1, n+m}(\Xi_m^{(n+1)}) = t^{(n)} \cdot \Xi_m^{(n)},$$

où l'opérateur $\mathbf{s}_n \circ \mathbf{v}_{n+1}$ agit sur un segment admissible de longueur $n+1$ dans \mathbf{B}_V en retirant son premier sommet (i.e., le plus proche de x).

- **Relation horizontale dans l'immeuble.**

En fin de chapitre 2, on applique les constructions précédentes au cas $n = 0$ (i.e., au cas des sommets de l'immeuble) pour démontrer, sous une forme légèrement différente de ([27], Theorem 1.6), la *relation de distribution horizontale* en τ . Celle-ci relie l'action du polynôme de Hecke He_τ - un polynôme à coefficients dans l'algèbre de Hecke *sphérique* $\mathcal{H}_\tau = \mathcal{H}_{\mathbb{Z}}(G_\tau // K_\tau)$, provenant de la donnée de Shimura en τ et dont les coefficients peuvent être exprimés simplement à partir des opérateurs « d'adjacence » $t_{1,0}$, $t_{0,1}$ et $t = t_{1,0} \otimes t_{0,1}$ (cf. Théorème 2.3.2) - à l'opérateur trace $\text{Tr}^{1,0}$ agissant sur $\mathbb{Z}[\text{Hyp}]_1 \subset \mathbb{Z}[\text{Hyp}]$. Nous formulons la relation de distribution horizontale de la manière suivante à la Proposition 2.3.16 :

$$\text{He}_\tau(\mathbf{1}) \cdot (x \otimes x) \in \frac{1}{q^2(q^2 - 1)} \text{Tr}^{1,0}(\mathbb{Z}[\text{Hyp}]_1).$$

- **La filtration** $(\mathcal{O}_c^1)_{c \geq 0}$ **sur** E_τ^1 . Introduite au paragraphe 2.2.3, la filtration $(\mathcal{O}_c^\times)_{c \geq 0}$ sur $\mathcal{O}_{E_\tau}^\times$ induite par les ordres locaux $\mathcal{O}_c := \mathcal{O}_{F_\tau} + \varpi^c \mathcal{O}_{E_\tau}$, $c \geq 0$, induit à son tour une filtration $(\mathcal{O}_c^1 := \nu(\mathcal{O}_c^\times))_{c \geq 0}$ sur $E_\tau^1 := \mathrm{U}(1)(F_\tau)^{(4)}$, où $\nu : E_\tau^\times \rightarrow \mathrm{U}(1)(F_\tau)$ est l'application $z \mapsto \frac{\bar{z}}{z}$. C'est une conséquence importante de la formule du conducteur local (Proposition 2.2.7) que le déterminant induit une surjection $\det : H_n \twoheadrightarrow \mathcal{O}_n^1$, pour tout $n \geq 0$.

Chap. 3

- **Relations horizontales et verticales dans** $\mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]$.

En posant $H_\tau^{\mathrm{der}} := \ker(\det : H_\tau \rightarrow \mathrm{U}(1)(F_\tau))$, on traduit au chapitre 3 les relations de distribution ci-dessus - obtenues via et exprimées dans l'immeuble - en des variantes dans le \mathbb{Z} -module $\mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]$. On montre au Lemme 3.1.3 que les fibres de l'application déterminant $\det^\# : H_n / H_{n+1} \twoheadrightarrow \mathcal{O}_n^1 / \mathcal{O}_{n+1}^1$ ont toutes même cardinal, égal à q^5 . En définissant, pour tout $n \geq 0$, la flèche naturelle $\pi^{\mathrm{der}} : \mathbb{Z}[\mathrm{Hyp}^{(n)}] = \mathbb{Z}[G_\tau / \mathbb{I}_\tau^{(n)}] \rightarrow \mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]$, on obtient les relations suivantes : si $k \geq 0$ et si $\mathfrak{X} \in \mathbb{Z}[\mathrm{Hyp}^{(n)}]_{k+1}$, alors :

$$\pi^{\mathrm{der}}(\mathrm{Tr}^{k+1,k}(\mathfrak{X})) = \begin{cases} q^3(q^2 - 1) \mathrm{Tr}_{k+1,k}^{\mathrm{der}}(\pi^{\mathrm{der}}(\mathfrak{X})), & \text{si } k = 0. \\ q^5 \mathrm{Tr}_{k+1,k}^{\mathrm{der}}(\pi^{\mathrm{der}}(\mathfrak{X})), & \text{si } k \geq 1. \end{cases} \quad (1)$$

où, pour tous entiers $k' \geq k \geq 0$, l'opérateur trace $\mathrm{Tr}_{k',k}^{\mathrm{der}}$ est défini par

$$\begin{aligned} \mathrm{Tr}_{k',k}^{\mathrm{der}} : (\mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}])^{H_{k'}} &\longrightarrow (\mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}])^{H_k}, \\ \mathfrak{X} &\longmapsto \sum_{h \in H_k' / H_{k'}} h \cdot \mathfrak{X}, \end{aligned}$$

avec $H_k' := (H_k \cap H_\tau^{\mathrm{der}}) \backslash H_k$.

Dans $\mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]$, les relations de distribution précédentes deviennent donc, en $n = 0$ (relation horizontale) :

$$\mathrm{Tr}_{1,0}^{\mathrm{der}}(\tilde{y}) = \mathrm{He}_\tau(\mathbf{1}) \cdot \pi^{\mathrm{der}}(x \otimes x), \quad (2)$$

pour un certain $\tilde{y} \in q(\mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / K_\tau])^{H_1}$ et, si $n \geq 1$ (relation verticale) :

$$\mathrm{Tr}_{n+m+1,n+m}^{\mathrm{der}}\left(q^5 \pi^{\mathrm{der}}(\Xi_{m+1}^{(n)})\right) = t^{(n)} \cdot \pi^{\mathrm{der}}(\Xi_m^{(n)}). \quad (3)$$

- **La tour de variétés** $(\mathrm{Sh}_{\mathbb{I}^{(n)}}(\mathbf{G}, X))$ **en** τ .

On fixe désormais une place inerte acceptable particulière τ , que l'on appellera *place verticale*. De retour au contexte *global*, on déduit de la filtration d'Iwahori locale en τ , une filtration globale décroissante $(\mathbb{I}^{(n)})_{n \geq 0}$ de K obtenue en posant $\mathbb{I}^{(n)} := \mathbb{I}_\tau^{(n)} \times K^\tau$, $\forall n \geq 0$, et donc une tour de variétés de Shimura $(\mathrm{Sh}_{\mathbb{I}^{(n)}}(\mathbf{G}, X))_{n \geq 1}$, où les morphismes

(4). Lequel est égal à $\det(H_\tau)$.

de transition $\pi^{(n)} : \mathrm{Sh}_{\mathbf{I}(n+1)}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_{\mathbf{I}(n)}(\mathbf{G}, X)$ sont finis-étales. Utilisant la compatibilité entre applications d'Artin globale et locales de la théorie du corps de classe, nous exprimons à la Remarque 3.1.1 l'action des groupes de décomposition locaux sur les cycles spéciaux de la manière suivante : si $v \in I_{F,f}$, si $\sigma \in \mathrm{Art}_v(E_v^\times) \subset \mathrm{Gal}(E_v^{ab}/E_v)$ vérifie $\sigma|_{E(\infty)_v} = \mathrm{Art}_v^1(h)$ pour un certain $h \in H_v$, alors

$$\mathcal{Z}_{\mathbf{I}(n)}(\phi_v(h)g) = \sigma \cdot \mathcal{Z}_{\mathbf{I}(n)}(g), \quad \forall g \in \mathbf{G}(\mathbb{A}_f),$$

où $\phi_v : H_v \hookrightarrow \mathbf{H}(\mathbb{A}_f)$, $h \mapsto (h, \mathbf{1}^v)$ est le plongement à la place v de H_v dans $\mathbf{H}(\mathbb{A}_f)$.

- **La famille de cycles** $(\mathfrak{z}(n, \tau^m \cdot \mathbf{c}))_{n,m,\mathbf{c}}$.

On désigne par \mathcal{P}^τ l'ensemble des produits finis de places inertes acceptables deux à deux distinctes, en dehors de τ et d'un nombre fini de « mauvaises » places supplémentaires issues des hypothèses de (1.59). Les relations horizontales et verticales sur $\mathbb{Z}[H_\tau^{\mathrm{der}} \setminus G_\tau / \mathbf{I}_\tau^{(n)}]$ nous permettent maintenant de définir une famille globale de cycles spéciaux $(\mathfrak{z}(n, \tau^m \cdot \mathbf{c}))_{n \geq 1}$ sur la tour $(\mathrm{Sh}_{\mathbf{I}(n)}(\mathbf{G}, X))_{n \geq 1}$, engendrés par des éléments bien choisis de $\mathbf{G}(\mathbb{A}_f)$. Ces cycles sont paramétrés, outre leur *niveau* n , par leur *conducteur vertical* $m \geq 0$ et leur *conducteur modéré* $\mathbf{c} \in \mathcal{P}^\tau$. Leur construction consiste essentiellement à rajouter place par place à l'élément g_0 , des composantes locales issues du terme de gauche de la relation horizontale (2) (en $\tau' \in \mathrm{Supp} \mathbf{c}$) ou bien le *shift* δ^m d'amplitude m (en τ , cf. Définition 3.1.4). Ces cycles spéciaux vérifient $\mathfrak{z}(n, \tau^m \cdot \mathbf{c}) \in \mathbb{Z}[\mathcal{Z}_{\mathbf{I}(n)}(\mathbf{G}, \mathbf{H})]$, $\forall n \geq 1$ et, par construction, $\mathfrak{z}(n, \tau^m \cdot \mathbf{c})$ est rationnel sur le \mathcal{K} -corps de transfert $\mathcal{K}(\tau^{n+m} \cdot \mathbf{c})$.

Résultat principal

Pour tout $n \geq 1$, le morphisme fini étale $\pi_\delta^{(n)} : \mathrm{Sh}_{\mathbf{I}(n+1)}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_{\mathbf{I}(n)}(\mathbf{G}, X)$ est défini par la composition $\pi_\delta^{(n)} := \pi_{\mathbf{I}(n) \cap \delta^{-1}\mathbf{I}(n)\delta / \mathbf{I}(n)} \circ [\cdot\delta]$, où $\pi_{\mathbf{I}(n) \cap \delta^{-1}\mathbf{I}(n)\delta / \mathbf{I}(n)}$ est donné dans le diagramme suivant :

$$\begin{array}{ccc} \mathrm{Sh}_{\mathbf{I}(n+1)}(\mathbf{G}, X) & \xrightarrow{[\cdot\delta]} & \mathrm{Sh}_{\mathbf{I}(n) \cap \delta^{-1}\mathbf{I}(n)\delta}(\mathbf{G}, X) \\ \downarrow [\cdot 1] \pi^{(n)} & & \downarrow [\cdot 1] \pi_{\mathbf{I}(n) \cap \delta^{-1}\mathbf{I}(n)\delta / \mathbf{I}(n)} \\ \mathrm{Sh}_{\mathbf{I}(n)}(\mathbf{G}, X) & & \mathrm{Sh}_{\mathbf{I}(n)}(\mathbf{G}, X) \end{array}$$

On désigne par $T(\delta)$ la correspondance algébrique *de Hecke* associée au diagramme précédent, et l'on pose $T(\delta)_* = (\pi_\delta^{(n)})_* \circ (\pi^{(n)})^*$.

Comme mentionné au début de cette introduction, le fait que les fonctions L associées aux cycles spéciaux puissent s'écrire sous la forme d'un produit eulérien devrait être la traduction, dans notre cas, d'une compatibilité de la famille $(\mathfrak{z}(n, \tau^m \cdot \mathbf{c}))$ avec le changement de conducteur en toute place inerte acceptable $\tau \neq \tau'$, qui s'exprime ci-dessous

au moyen du polynôme de Hecke $\text{He}_{\tau'}$ attaché à la donnée de Shimura locale en τ' . Par ailleurs, en vertu de la *seconde idée* motivant ce travail, la propension qu'ont les représentations automorphes associées (du moins conjecturalement) aux cycles spéciaux à s'interpoler en familles p -adiques devrait être manifestation de l'existence relations verticales sur le niveau en τ . La structure particulièrement commode des extensions entre \mathcal{K} -corps de transfert et les relations locales horizontales et verticales (2) et (3) induisent finalement le théorème principal de cette thèse (cf. Théorème 3.1.8) :

Théorème. *La famille $(\mathfrak{z}(n, \tau^m \cdot \mathfrak{c}))$ vérifie les compatibilités suivantes :*

- (Relation horizontale) Pour tous $n \geq 1$, $m \geq 0$ et tout produit d'idéaux $\mathfrak{c} \cdot \tau' \in \mathcal{P}^\tau$, on a :

$$\text{Tr}_{\mathcal{K}(\tau^{n+m} \cdot \mathfrak{c} \cdot \tau')/\mathcal{K}(\tau^{n+m} \cdot \mathfrak{c})}(\mathfrak{z}(n, \tau^m \cdot \mathfrak{c} \cdot \tau')) = \text{He}_{\tau'}(\text{Frob}_{\tau'}) \cdot \mathfrak{z}(n, \tau^m \cdot \mathfrak{c}),$$

où $\text{Frob}_{\tau'} \in \text{Gal}(E_{\tau'}^{ab}/E_{\tau'})$ est un relevé du Frobenius géométrique associé à l'idéal premier $\tau' \mathcal{O}_E$ de \mathcal{O}_E .

- (Relation verticale sur le niveau) Pour tous $n \geq 1$, $m \geq 0$ et $\mathfrak{c} \in \mathcal{P}^\tau$, on a :

$$q^5 \pi_\delta^{(n)} \left(\text{Tr}_{\mathcal{K}(\tau^{n+m+1} \cdot \mathfrak{c})/\mathcal{K}(\tau^{n+m} \cdot \mathfrak{c})}(\mathfrak{z}(n+1, \tau^m \cdot \mathfrak{c})) \right) = t^{(n)} \cdot \mathfrak{z}(n, \tau^m \cdot \mathfrak{c})$$

En supposant $g_0 = \mathbf{1}$ pour simplifier, la relation verticale sur le niveau admet la reformulation suivante :

$$(\pi_\delta^{(n)})_* \left(\text{Tr}_{\mathcal{K}(\tau^{n+m+1} \cdot \mathfrak{c})/\mathcal{K}(\tau^{n+m} \cdot \mathfrak{c})}(\mathfrak{z}(n+1, \tau^m \cdot \mathfrak{c})) \right) = T(\delta)_* (\mathfrak{z}(n, \tau^m \cdot \mathfrak{c})). \quad (4)$$

Remarquons l'analogie formelle avec la Proposition 4.5 de [16], formulée dans le cas d'une famille $(\mathbf{x}(\mathfrak{c}, S))$ de points CM sur une tour de courbes de Shimura, associée au groupe des unités d'une algèbre de quaternions (de niveaux variables S) : la relation verticale de *loc. cit.*

$$T(S' - S) \cdot \mathbf{x}(\mathfrak{c}, S) = \pi_{S'/S} \left(\sum_{\sigma \in G_{S'/S}} \sigma \cdot \mathbf{x}(\mathfrak{c}, S') \right)$$

coïncide formellement avec (4) dans le cas particulier où τ est l'unique place de F au-dessus de p . La relation horizontale de *loc. cit.* :

$$T(l) \cdot \mathbf{x}(\mathfrak{c}, S) = \sum_{\sigma \in \text{Gal}(K(\mathfrak{c}l, S)/K(\mathfrak{c}, S))} \sigma \cdot \mathbf{x}(\mathfrak{c}l, S),$$

où l est une place inerte dans l'extension K/F (dans les notations de *loc. cit.*) admet quant à elle la reformulation suivante :

$$\sum_{\sigma \in \text{Gal}(K(\mathfrak{c}l, S)/K(\mathfrak{c}, S))} \sigma \cdot [\mathbf{x}(\mathfrak{c}, S) - \mathbf{x}(\mathfrak{c}l, S)] = \text{H}_l(\text{Frob}_\lambda) \cdot \mathbf{x}(\mathfrak{c}, S),$$

où l'idéal premier $\lambda = l\mathcal{O}_K$ est scindé dans $K(\mathfrak{c})$, et où $H_l := X^2 - T(l)X + l$ coïncide avec le polynôme de Hecke en l de la donnée de Shimura considérée par O.Fouquet.

Vers un système d'Euler en familles p -adiques.

Rappelons que p désigne la caractéristique résiduelle de la place verticale τ . On a donc construit un *proto* système d'Euler en familles p -adiques de nature géométrique, satisfaisant deux types de relations : horizontales et verticales sur le niveau. Il est tentant d'écrire que la prochaine étape serait désormais d'utiliser l'application d'Abel-Jacobi p -adique pour traduire ces relations de distribution en des relations entre classes de cohomologie, dans un certain H^1 galoisien d'une représentation p -adique de $\text{Gal}(\bar{E}/E)$.

Rappelons que, dans le cas où $F \neq \mathbb{Q}$ (ce que l'on supposera jusqu'à la fin de cette introduction, pour simplifier) la variété de Shimura $\text{Sh}_{\mathbb{I}(n)}(\mathbf{G}, X)$, tacitement identifiée à son modèle entier sur $\text{Spec } E$, est propre et lisse pour tout $n \geq 1$. On dispose (cf. [37]) d'une application de *classe de cycle* p -adique commutant à l'action de Galois et des correspondances finies (dont les correspondances de Hecke) :

$$\text{cl}_0 : \text{CH}^2(\text{Sh}_{\mathbb{I}(n)}(\mathbf{G}, X))_{\mathcal{L}} \longrightarrow H_{\text{ét}}^4(\text{Sh}_{\mathbb{I}(n)}(\mathbf{G}, X) \times_{\mathcal{L}} \bar{\mathcal{L}}, \mathbb{Q}_p(2)),$$

où \mathcal{L} est une extension finie abélienne quelconque de $\mathcal{K}(\tau)$ (le corps de définition du cycle *de base* $\mathfrak{z}(1, 1)$) que l'on fera varier. Les groupes de cohomologie ci-dessus correspondent à la cohomologie étale continue au sens de [26]. Le morphisme précédent induit à son tour, par dégénérescence de la suite spectrale de Hochschild-Serre, une flèche :

$$\text{cl}_1 : \text{CH}^2(\text{Sh}_{\mathbb{I}(n)}(\mathbf{G}, X))_{\mathcal{L}, 0} \longrightarrow H^1(\text{Gal}(\bar{\mathcal{L}}/\mathcal{L}), H_{\text{ét}}^3(\text{Sh}_{\mathbb{I}(n)}(\mathbf{G}, X) \times_{\mathcal{L}} \bar{\mathcal{L}}, \mathbb{Q}_p(2))),$$

où $\text{CH}^2(\text{Sh}_{\mathbb{I}(n)}(\mathbf{G}, X))_{\mathcal{L}, 0}$ désigne le noyau de cl_0 , que l'on appelle communément application d'Abel-Jacobi p -adique, notée AJ_p .

Quelques obstacles demeurent cependant à ce stade. Le premier consiste, en vertu de ce qui précède, à trouver un moyen de « trivialisier » nos cycles, i.e. de construire une série de projections

$$\text{CH}^2(\text{Sh}_{\mathbb{I}(n)}(\mathbf{G}, X))_{\mathcal{L}} \rightarrow \text{CH}^2(\text{Sh}_{\mathbb{I}(n)}(\mathbf{G}, X))_{\mathcal{L}, 0}, \quad n \geq 1$$

compatibles aux morphismes $\pi^{(n)}$ et $\pi_{\delta}^{(n)}$ ainsi qu'à l'action de Galois. Un moyen raisonnable de procéder pourrait, semble-t-il, résider dans l'utilisation d'un projecteur de type *Hecke-Künneth* à la manière de Morel-Suh ([36]) agissant par projection sur la cohomologie en degrés impairs. L'existence de ce dernier est conjecturale en toute généralité, mais une définition inconditionnelle pourrait être rendue possible par l'utilisation d'une très proche variante $\text{Sh}(\tilde{\mathbf{G}}, \tilde{X})$ de notre variété de Shimura $\text{Sh}(\mathbf{G}, X)$, étudiée par Rapoport-Smithling-Zhang. Cette variante a notamment l'avantage d'être de type PEL - contrairement aux

variétés de Gan-Gross-Prasad qu'on sait être seulement de type *abélien* - et de posséder de « meilleurs » modèles entiers ([43], §6), au prix seulement d'être possiblement définie sur un corps réflexe légèrement plus gros. Qu'on puisse étendre notre famille compatible de cycles spéciaux à la variante de Rapoport-Smithling-Zhang par des procédés similaires nous semble raisonnable, le groupe $\tilde{\mathbf{G}}$ de *loc. cit.* étant le produit direct de notre \mathbf{G} avec un tore $Z^{\mathbb{Q}}$.

A condition, enfin, de savoir ensuite rendre inversible l'opérateur $T(\delta)_*$ agissant sur le groupe $H_{\acute{e}t}^3(\mathrm{Sh}_{\Gamma(n)}(\mathbf{G}, X) \times_E \overline{E}, \mathbb{Q}_p(2))$ - ou plus précisément, de projeter nos classes de cycles sur une partie dite *de pente finie* où cet opérateur serait, par définition, inversible - la relation (4) induirait donc l'existence d'un système de classes

$$T(\delta)_*^{-n} (\mathrm{AJ}_p(\mathfrak{z}(n, \tau^m \cdot \mathfrak{c}))_{n,m,\mathfrak{c}})$$

compatibles pour la corestriction dans la direction de la pro-extension p -adique formée par les \mathcal{K} -corps de transfert.

Chapter 1

Special diagonal cycles on $\mathrm{Sh}_K(\mathbf{G}, X)$.

This chapter introduces the geometric settings in which the family of so-called *special cycles* is defined. We start by introducing the *Gan-Gross-Prasad like* embedding between the unitary Shimura varieties $\mathrm{Sh}(\mathbf{H}, Y) \hookrightarrow \mathrm{Sh}(\mathbf{G}, X)$, which gives rise to the special cycles, and then study the Galois action of the abelian group $\mathrm{Gal}(E^{ab}/E)$ (which acts through its quotient $\mathrm{Gal}(E(\infty)/E)$) on the set of special cycles. The end of this chapter, which is of global nature, is devoted to the study of the fields of definition of these special cycles and to the structure of the Galois groups obtained from the various extensions attached to them, using global class field theory.

1.1 The unitary Shimura varieties $\mathrm{Sh}(\mathbf{G}, X)$ and $\mathrm{Sh}(\mathbf{H}, Y)$.

We start by fixing an algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} embedded inside \mathbb{C} and, for any rational prime p , we fix an algebraic closure $\overline{\mathbb{Q}_p}$ of \mathbb{Q}_p together with a fixed embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$. This amounts to fixing a compatible system $(\mathfrak{p}_L)_{L/\mathbb{Q}}$, indexed by the finite field extensions L/\mathbb{Q} , where \mathfrak{p}_L is a prime ideal of \mathcal{O}_L above p . We also fix for convenience an abstract isomorphism $\eta_p : \overline{\mathbb{Q}_p} \simeq \mathbb{C}$. For any number field L , we denote by \mathbb{A}_L (resp. $\mathbb{A}_{L,f}$) the ring of adèles of L (resp. the ring of finite adèles), and we lighten notations by setting $\mathbb{A} := \mathbb{A}_{\mathbb{Q}}$ and $\mathbb{A}_f := \mathbb{A}_{\mathbb{Q},f}$.

1.1.1 Vector spaces V and W .

We let E/F be a CM extension of number fields, by which we mean that the number field E is a totally imaginary quadratic extension of its maximal totally real subfield F , and set $d := [F : \mathbb{Q}]$. We denote by $c : x \mapsto \bar{x}$ the non-trivial element of $\mathrm{Gal}(E/F)$, whose action will often - slightly abusively - be referred to as *complex conjugation*. We let $I_{F,\infty}$ and $I_{E,\infty}$ denote the finite sets of archimedean places for F and E respectively. We identify $I_{F,\infty}$ with the set $\mathrm{Spec} F(\mathbb{R}) = \mathrm{Spec} F(\overline{\mathbb{Q}}) := \{\rho_1, \dots, \rho_d\}$ of real embeddings of F , and we identify $I_{E,\infty}$ with the quotient set $\mathrm{Spec} E(\mathbb{C})/\mathrm{Gal}(E/F)$, where $c \in \mathrm{Gal}(E/F)$ acts on $\mathrm{Spec} E(\mathbb{C}) = \mathrm{Spec} E(\overline{\mathbb{Q}})$

via $c \cdot \phi(x) = \phi(\bar{x})$, for all $\phi : E \hookrightarrow \mathbb{C}$ and $x \in E$. We let $\Phi := \{\tilde{\rho}_1, \dots, \tilde{\rho}_d\} \subset \mathrm{Spec} E(\overline{\mathbb{Q}})$ be a CM-type for E/F - that is, a family of coset representatives for $\mathrm{Spec} E(\mathbb{C})/\mathrm{Gal}(E/F)$ - such that $\tilde{\rho}_i : E \hookrightarrow \overline{\mathbb{Q}}$ extends ρ_i for all $i = 1, \dots, d$. Accordingly, we will write $I_{E,\infty} = \{\tilde{\rho}_1, \dots, \tilde{\rho}_d\}$ and, every time we will identify E with a subfield of \mathbb{C} with no more precision, it will be done through the distinguished embedding $\tilde{\rho}_1$.

We let $I_{F,f}$ denote the set of finite (or non-archimedean) places of F , so that $I_F := I_{F,\infty} \sqcup I_{F,f}$ is the set of all places of F . For all $v \in I_{F,f}$, we denote by $\mathfrak{p}_v \in |\mathrm{Spec} \mathcal{O}_F| \setminus \{0\}$ its corresponding non-zero prime ideal in \mathcal{O}_F , and we set $F_v := F_{\mathfrak{p}_v}$ to be the \mathfrak{p}_v -adic completion of F , whose ring of integers is denoted \mathcal{O}_{F_v} and whose residue field \mathbb{F}_v has cardinality q_v . We shall say that v is *ramified* (resp. *inert*, *split*) in the extension E/F if the ideal \mathfrak{p}_v is. For all $v|p$, we identify the algebraic closures $\overline{F}_v \simeq \overline{\mathbb{Q}_p}$ and we let $I_{F,v}$ denote the set of embeddings $F_v \hookrightarrow \overline{F}_v$. We mention that the sets $I_{F,\infty}$ and $I_{F,p} := \bigsqcup_{v|p} I_{F,v}$ may be identified (and are) via η_p .

Let $(V, \langle \cdot, \cdot \rangle)$ be a 3-dimensional⁽¹⁾ hermitian space relatively to the involution $c \in \mathrm{Gal}(E/F)$. In our convention, this means that V is a 3-dimensional E -vector space and $\langle \cdot, \cdot \rangle : V \times V \rightarrow E$ is a F -bilinear form satisfying the following relations:

- $\langle v, w \rangle = \overline{\langle w, v \rangle}$, $\forall v, w \in V$.
- $\langle \lambda v, \mu w \rangle = \overline{\lambda} \mu \langle v, w \rangle$, $\forall v, w \in V$, $\forall \lambda, \mu \in E$.

Let $\alpha \in E \setminus F$ be an element satisfying $E = F[\alpha]$ and $\alpha^2 \in F^\times$ ⁽²⁾. For any $v \in I_F$, we set $E_v := E \otimes_F F_v$. More precisely:

- if $v = \rho_i \in I_{F,\infty}$ for some $i \in \{1, \dots, d\}$ then, by assumption on E , one has $\tilde{\rho}_i(\alpha) \notin \mathbb{R}$, hence $\rho_i(\alpha^2) = \tilde{\rho}_i(\alpha)^2 \in \mathbb{R}_{<0}$. This gives

$$E_v = E \otimes_{F,\rho_i} \mathbb{R} \simeq \frac{F[X]}{X^2 - \alpha^2} \otimes_{F,\rho_i} \mathbb{R} \simeq \frac{\mathbb{R}[X]}{X^2 - \rho_i(\alpha)^2} \xrightarrow{X \mapsto \tilde{\rho}_i(\alpha)} \underset{\simeq}{\mathbb{C}}_{\tilde{\rho}_i},$$

where $\mathbb{C}_{\tilde{\rho}_i}$ stands for the copy of \mathbb{C} endowed with the E -algebra structure given by $\tilde{\rho}_i$.

- if $v \in I_{F,f}$ then E_v is the quadratic unramified (resp. totally ramified) extension of F_v if v is inert (resp. if v is ramified) in E/F , in which case one may identify the local Galois group $\mathrm{Gal}(E_v/F_v)$ with $\mathrm{Gal}(E/F)$, and $c \in \mathrm{Gal}(E/F)$ acts naturally on E_v . If $\mathfrak{p}_v \mathcal{O}_E$ splits as $\mathfrak{q}\bar{\mathfrak{q}}$, with \mathfrak{q} and $\bar{\mathfrak{q}}$ respectively attached to places w and \bar{w} of E , then E_v is a quadratic étale algebra over F_v which is isomorphic to $F_v \times F_v$, and where the element $c \in \mathrm{Gal}(E/F)$ acts by permutation of the factors⁽³⁾.

(1). For simplicity, we will stick to the $n = 3$ case throughout the whole thesis, mainly because our approach of level-wise vertical distribution relations is specific to the Bruhat-Tits building of $U(3)$. In this first chapter though, the setting can be easily generalized to any $n \geq 3$ by obvious modifications of the appropriate parts.

(2). Such an α always exists and is well-defined up to multiplication by elements of F^\times .

(3). We shall come back to this later on.

For all $v \in I_F$, one may thus define an 3-dimensional E_v/F_v -hermitian space $V_v := V \otimes_E E_v$, i.e., a free E_v -module of rank 3 endowed with the extended hermitian pairing

$$\langle \cdot, \cdot \rangle_v : V_v \times V_v \rightarrow E_v$$

defined as follows: for families $(\lambda_i)_{i=1}^r$ and $(\mu_j)_{j=1}^r$ of elements of E_v , and vectors $(v_i), (w_j)$ in V , one sets:

$$\left\langle \sum_{i=1}^r \lambda_i v_i, \sum_{j=1}^r \mu_j w_j \right\rangle_v := \sum_{1 \leq i, j \leq r} \bar{\lambda}_i \mu_j (\langle v_i, w_j \rangle \otimes 1) \in E_v. \quad (1.1)$$

If $v = \rho_i$ is a real place, the pair $(V_{\rho_i}, \langle \cdot, \cdot \rangle_{\rho_i})$ defines an actual complex hermitian space, for all $i \in \{1, \dots, d\}$.

Assumption 1.1.1. *We will assume throughout the following that V_{ρ_1} has signature $(2, 1)$ and that V_{ρ_j} has signature $(3, 0)$ for all $j \geq 2$.*

Let us choose a vector $e_D \in V$ which is anisotropic and such that $\langle e_D, e_D \rangle = 1$. Such a vector e_D may always be assumed to exist, up to rescale the hermitian pairing by a totally positive scalar $\lambda \in F_{\gg 0}$, which will not affect the upcoming Shimura data. Indeed, as the hermitian space $V_{\rho_1} := V \otimes_{E, \tilde{\rho}_1} \mathbb{C}$ has signature $(2, 1)$ and by density of E in \mathbb{C} (embedded via $\tilde{\rho}_1$), one may always choose some vector $e \in V$ satisfying $\langle e, e \rangle_{\rho_1} = \rho_1(\langle e, e \rangle) > 0$. Therefore $\langle e, e \rangle_{\rho_i} > 0$ for all $i \in \{1, \dots, d\}$, i.e., $\langle e, e \rangle \in F_{\gg 0}$. If $e_D := e$, the modified hermitian space $(V, \langle \cdot, \cdot \rangle')$, with $\langle \cdot, \cdot \rangle' := \frac{1}{\langle e, e \rangle} \langle \cdot, \cdot \rangle$ is such that $\langle e_D, e_D \rangle' = 1$.

Consequently, the hyperplane $W := (E e_D)^\perp \subset V$ endowed with the restricted hermitian pairing, is a 2-dimensional hermitian space with signature $(1, 1)$ at ρ_1 , and signature $(2, 0)$ at ρ_j , $j \geq 2$.

1.1.2 The F -algebraic groups $\mathrm{U}(V)$ and $\mathrm{U}(W)$.

If S is an F -algebra then one may, as above, extend naturally the hermitian pairing $\langle \cdot, \cdot \rangle$ to the 3-dimensional $E \otimes_F S$ -module

$$V \otimes_F S = V \otimes_E (E \otimes_F S),$$

with values in $E \otimes_F S$, by letting $\mathrm{Gal}(E/F)$ act on $E \otimes_F S$ on the leftmost component only (i.e., through $c \cdot (x \otimes s) := \bar{x} \otimes s$), and thus setting

$$\left\langle \sum_i \lambda_i v_i, \sum_j \mu_j w_j \right\rangle_{V \otimes_F S} := \sum_{i,j} \bar{\lambda}_i \mu_j (\langle v_i, w_j \rangle \otimes 1) \in E \otimes_F S,$$

with scalars $\lambda_i, \mu_j \in E \otimes_F S$ and vectors $v_i, w_j \in V$.

We let $\mathrm{GL}(V_E)$ denote the algebraic group (over $\mathrm{Spec} E$) of linear automorphisms of V , whose R -points are simply given by $\mathrm{GL}(V \otimes_E R)$ for all E -algebra R . We denote by V_F

the set V seen as an F -vector space, and we set $U(V) \subset \mathrm{Res}_{E/F}\mathrm{GL}(V_E)$ to be the group of unitary isometries of V_F : this is a reductive linear algebraic group over $\mathrm{Spec} F$, whose S -points are given - for any F -algebra S - by

$$U(V)(S) := \{g \in \mathrm{GL}(V \otimes_F S); \langle g \cdot x, g \cdot y \rangle_{V \otimes_F S} = \langle x, y \rangle_{V \otimes_F S}, \text{ for all } x, y \in V \otimes_F S\}.$$

We define the F -algebraic group $U(W)$ in the same way. We let \mathbf{G}_V and \mathbf{G}_W be the \mathbb{Q} -reductive groups obtained by Weil restrictions of scalars,

$$\mathbf{G}_V := \mathrm{Res}_{F/\mathbb{Q}}U(V) \text{ and } \mathbf{G}_W = \mathrm{Res}_{F/\mathbb{Q}}U(W),$$

and we set $\mathbf{G} := \mathbf{G}_V \times \mathbf{G}_W$. The orthogonal decomposition $V = W \perp D$ yields a natural embedding $\iota : U(W) \hookrightarrow U(V)$, induced on R -points by

$$\begin{aligned} \mathrm{GL}(W \otimes_F R) &\hookrightarrow \mathrm{GL}(V \otimes_F R), \\ g &\longmapsto \begin{bmatrix} g & 0 \\ 0 & \mathrm{Id}_D \end{bmatrix}. \end{aligned}$$

This gives - by left-exactness of the Weil restriction - a closed immersion $\iota_{\mathbb{Q}} : \mathbf{G}_W \hookrightarrow \mathbf{G}_V$ of algebraic groups over \mathbb{Q} , thus a diagonal embedding:

$$\Delta := (\iota_{\mathbb{Q}}, \mathrm{Id}_{\mathbf{G}_W}) : \mathbf{G}_W \hookrightarrow \mathbf{G}_V \times \mathbf{G}_W = \mathbf{G}.$$

We denote by $\mathbf{H} := \Delta(\mathbf{G}_W)$ the diagonal image of \mathbf{G}_W inside \mathbf{G} , which we shall often identify with \mathbf{G}_W .

1.1.2.1 The groups $U(V)$ and $U(W)$ split over E .

If S is now an E -algebra, we get an isomorphism

$$\begin{aligned} \phi : E \otimes_F S &\xrightarrow{\sim} S \times S, \\ e \otimes z &\longmapsto (\bar{e}z, ez) \end{aligned} \tag{1.2}$$

Indeed, recall that we have $E = F[\alpha]$, where $\alpha \in E \setminus F$ has minimal polynomial $f = X^2 - \alpha^2 \in F[X]$. Then

$$E \otimes_F S \simeq \frac{F[X]}{f} \otimes_F S \simeq \frac{S[X]}{(X + \alpha)(X - \alpha)} \simeq \frac{S[X]}{X + \alpha} \times \frac{S[X]}{X - \alpha} \simeq S \times S,$$

where we see α as an element of S via $E \rightarrow S$, and where the last two isomorphisms follow from the Chinese remainder theorem:

$$sX + t \pmod{f} \mapsto (sX + t \pmod{X + \alpha}, sX + t \pmod{X - \alpha}) = (-s\alpha + t, s\alpha + t) \in S \times S,$$

for any $s, t \in S$. Thus for any $e = a\alpha + b \in E$, $a, b \in F$ - corresponding to the element $aX + b \in \frac{F[X]}{f}$ - and for all $z \in S$, the element $e \otimes_F z$ corresponds to $azX + bz \in S[X]$, thus

maps to $(-az\alpha + bz, az\alpha + bz) = (\overline{(a\alpha + b)z}, (a\alpha + b)z) = (\bar{e}z, ez)$, as $\bar{\alpha} = -\alpha$. The inverse morphism $\phi^{-1} : S \times S \rightarrow E \otimes_F S$ is now given by

$$(s, t) \mapsto \alpha \otimes \frac{t-s}{2\alpha} + 1 \otimes \frac{s+t}{2} \in E \otimes_F S.$$

The element α being well defined up to multiplication by F^\times , one gets that the preceding isomorphism does not depend on any choice.

Let us recall that the Galois group $\mathrm{Gal}(E/F)$ acts on $E \otimes_F S$ by complex conjugation on the leftmost component. Therefore, if $\sum_i e_i \otimes z_i \in E \otimes_F S$ maps to $(s, t) := (\sum_i \bar{e}_i z_i, \sum_i e_i z_i) \in S \times S$, then $\overline{\sum_i e_i \otimes z_i} := \sum_i \bar{e}_i \otimes z_i$ maps to $(\sum_i e_i z_i, \sum_i \bar{e}_i z_i) = (t, s) \in S \times S$. In other words, the complex conjugation $c \in \mathrm{Gal}(E/F)$ acts on $S \times S$ by permuting the two variables. We set V_S to be the free S -module $V \otimes_E S$ of rank 3. The isomorphism ϕ induces an isomorphism of S -modules:

$$V \otimes_F S = V \otimes_E (E \otimes_F S) \xrightarrow{\phi} V \otimes_E (S \oplus S) \simeq V_S \oplus V_S.$$

For each $v \in V \otimes_F S$, we denote by $(v_s, v_t) \in V_S \oplus V_S$ the image of v via the above isomorphisms. If $g \in \mathrm{GL}(V \otimes_F S)$, we let $(g_s, g_t) \in \mathrm{GL}(V_S) \times \mathrm{GL}(V_S)$ be the induced pair of automorphisms: g_s and g_t are S -linear automorphisms of V_S such that, for each $v \in V \otimes_F S$, the image of $g \cdot v \in V \otimes_F S$ via the above identification is $(g_s \cdot v_s, g_t \cdot v_t) \in V_S \oplus V_S$ ⁽⁴⁾.

Let \mathcal{B} be an S -basis for V_S . By extension, \mathcal{B} can also be seen as a $E \otimes_F S$ -basis of $V \otimes_E (E \otimes_F S) = V \otimes_F S$, via the map $S \hookrightarrow E \otimes_F S$, $s \mapsto 1 \otimes s$. Let v and w be elements of $V \otimes_F S$, and let $X_v, X_w \in (E \otimes_F S)^3$ denote their respective coordinate vectors with respect to \mathcal{B} . Let $\varphi : (E \otimes_F S)^3 \xrightarrow{\sim} (S \times S)^3 = S^3 \times S^3$ be the isomorphism induced by ϕ on column vectors, which we decompose as $\varphi = (\varphi_s, \varphi_t)$. Accordingly, the vectors v_s, v_t, w_s and w_t of V_S have respective coordinate vectors $\varphi_s(X_v), \varphi_t(X_v), \varphi_s(X_w)$ and $\varphi_t(X_w)$ with respect to \mathcal{B} . Finally, let us also denote by $\varphi = (\varphi_s, \varphi_t)$ the map induced by ϕ on matrices, $M_3(E \otimes_F S) \xrightarrow{\phi} M_3(S) \times M_3(S)$. We define two S -valued, non-degenerate, S -bilinear forms $\langle \cdot, \cdot \rangle_{\mathcal{B},s}$ and $\langle \cdot, \cdot \rangle_{\mathcal{B},t}$ on $V_S \times V_S$ by setting, for all $y, z \in V_S$:

$$\langle y, z \rangle_{\mathcal{B},s} := {}^t X_y \varphi_s(H) X_z,$$

$$\langle y, z \rangle_{\mathcal{B},t} := {}^t X_y \varphi_t(H) X_z$$

where $X_y, X_z \in S^3$ are the coordinates of y and z in the basis \mathcal{B} , and where $H := (\langle e_i, e_j \rangle)_{1 \leq i, j \leq 3} \in M_3(E \otimes_F S)$ is the matrix of the hermitian pairing $\langle \cdot, \cdot \rangle_{V \otimes_F S}$ with respect

(4). One checks that g_s (resp. g_t) are defined, for all $w \in V_S$, by

$$g_s \cdot w := (g \cdot (w \oplus 0))_s \text{ (resp. } g_t \cdot w := (g \cdot (0 \oplus w))_t),$$

where $w \oplus 0 \in V_S \oplus V_S$ (resp. $0 \oplus w \in V_S \oplus V_S$) is seen as an element of $V \otimes_F S$ via the above identification.

to \mathcal{B} . Let $X \mapsto \overline{X}$ denote the component-wise action of c on $(E \otimes_F S)^3$ induced by the action on E^3 . The above considerations imply that:

$$\begin{aligned} \phi(\langle v, w \rangle) &= \phi({}^t \overline{X}_v H X_w) = \varphi({}^t \overline{X}_v) \varphi(H) \varphi(X_w) \\ &= ({}^t \varphi_t(X_v), {}^t \varphi_s(X_v)) (\varphi_s(H), \varphi_t(H)) (\varphi_s(X_w), \varphi_t(X_w)) \\ &= ({}^t \varphi_t(X_v) \varphi_s(H) \varphi_s(X_w), {}^t \varphi_s(X_v) \varphi_t(H) \varphi_t(X_w)). \end{aligned}$$

In other words, one has:

$$\phi(\langle v, w \rangle_{V \otimes_F S} V \otimes_F S) = (\langle v_t, w_s \rangle_{\mathcal{B}, s}, \langle v_s, w_t \rangle_{\mathcal{B}, t}) \in S \times S \quad (1.3)$$

One deduces that, if $g \in \mathrm{GL}(V \otimes_F S)$ and $v, w \in V \otimes_F S$, then

$$\phi(\langle g \cdot v, g \cdot w \rangle_{V \otimes_F S}) = (\langle g_t \cdot v_t, g_s \cdot w_s \rangle_{\mathcal{B}, s}, \langle g_s \cdot v_s, g_t \cdot w_t \rangle_{\mathcal{B}, t}) \in S \times S. \quad (1.4)$$

From (1.3) and (1.4), one deduces that the element g lies in $\mathrm{U}(V)(S)$ if and only if $\langle g_s \cdot v_s, g_t \cdot w_t \rangle_{\mathcal{B}, t} = \langle v_s, w_t \rangle_{\mathcal{B}, t}$ for all vectors $v_s, w_t \in V_S$ ⁽⁵⁾. Consequently, the element g_t determines completely g_s , and the mapping $g \mapsto g_t$ defines an isomorphism between the groups $\mathrm{U}(V)(S)$ and $\mathrm{GL}(V_S) = \mathrm{GL}(V_E)(S)$, which is functorial in S , for the map ϕ and the identification $V \otimes_F S \simeq V_S \oplus V_S$ are. In other words, we showed that there is an isomorphism $\mathrm{U}(V)_E \simeq \mathrm{GL}(V_E)$ between reductive groups over E , the latter being isomorphic to $\mathrm{GL}_{3,E}$ after fixing an E -basis of V . By applying the same argument to W , we get that $\mathrm{U}(W)_E \simeq \mathrm{GL}(W_E) \simeq \mathrm{GL}_{2,E}$.

Remark 1.1.1. — *The equality (1.3) - whose left-hand side clearly does not depend on any choice of basis - remains consistent, as the right-hand side does not depend on the chosen basis as well. Indeed, if v and w are vectors of $V \otimes_F S$ with columns vectors X_v and X_w in \mathcal{B} , then*

$$\langle v_s, w_t \rangle_{\mathcal{B}, t} := {}^t X_{v_s} \varphi_t(H) X_{w_t} = {}^t (\varphi_s(X_v)) \varphi_t(H) \varphi_t(X_w).$$

If \mathcal{B}' is another S -basis of V_S with corresponding transition matrix $P \in \mathrm{GL}_3(S) \hookrightarrow \mathrm{GL}_3(E \otimes_F S)$ then, in this new basis, H is changed into ${}^t \overline{P}^{-1} H P^{-1} = {}^t P^{-1} H P^{-1}$. Thus $\varphi_t(H)$ is changed into ${}^t \varphi_t(P^{-1}) \varphi_t(H) \varphi_t(P^{-1}) = {}^t P^{-1} \varphi_t(H) P^{-1}$, $\varphi_s(X_v)$ is changed into $\varphi_s(P) \varphi_s(X_v) = P \varphi_s(X_v)$ and $\varphi_t(X_w)$ is changed into $P \varphi_t(X_w)$. Consequently, the quantity ${}^t (\varphi_s(X_v)) \varphi_t(H) \varphi_t(X_w)$ is changed into ${}^t (P \varphi_s(X_v)) ({}^t P^{-1} H P^{-1}) P \varphi_t(X_w)_t$, i.e., remains invariant. Similar computations show that $\langle v_t, w_s \rangle_{\mathcal{B}, c}$ also remains unchanged, which makes (1.3) consistent.

(5). As ${}^t H = \overline{H} \in M_3(E \otimes_F S)$, one has ${}^t (\varphi_s(H)) = \varphi_s({}^t H) = \varphi_s(\overline{H}) = \varphi_t(H)$, thus one checks that the preceding condition alone also implies that $\langle g_t \cdot v_t, g_s \cdot w_s \rangle_{\mathcal{B}, s} = \langle v_t, w_s \rangle_{\mathcal{B}, s}$ for all $v_t, w_s \in V_S$.

- If S is a general E -algebra, the arithmetic action of $\mathrm{Gal}(E/F)$ on $E \otimes_F S$ through $e \otimes z \mapsto \bar{e} \otimes z$, which enables us to extend the hermitian product to $V \otimes_F S$ and to define the S -points of $\mathrm{U}(\star)$, does not induce an algebraic Galois action on the groups of S -points $\mathrm{U}(\star)(S) \subset \mathrm{GL}(\star \otimes_F S)$, for $\star \in \{V, W\}$. However, when S admits itself a $\mathrm{Gal}(E/F)$ -action which is compatible with the canonical one on $E \subset S$, one may then define a natural $\mathrm{Gal}(E/F)$ -action on the groups $\mathrm{U}(\star)(S) \simeq \mathrm{GL}_3(S)$ in a functorial way, as we shall see at § 1.1.5.

1.1.3 Hermitian symmetric domains X_V and X_W .

Let us define the hermitian symmetric domains X and Y attached to the reductive groups \mathbf{G} and \mathbf{H} . We start by fixing an E -basis $\mathcal{B} = \{e_1, e_2, e_3\}$ of V , such that $e_1 \in D$ and such that $\{e_2, e_3\}$ is an orthogonal E -basis of W . The extension of scalars from E to $\bar{\mathbb{Q}}$ - via $\tilde{\rho}_1$ - makes \mathcal{B} into an orthogonal $\bar{\mathbb{Q}}$ -basis of $V \otimes_{E, \tilde{\rho}_1} \bar{\mathbb{Q}}$. By signature property of V_{ρ_1} and by construction of D , one may assume without loss of generality that $\langle e_1, e_1 \rangle_{\rho_1} > 0$, $\langle e_2, e_2 \rangle_{\rho_1} > 0$, and $\langle e_3, e_3 \rangle_{\rho_1} < 0$, which we do. Consequently, the modified basis

$$\mathcal{B}_1 = \{e'_1, e'_2, e'_3\} := \left\{ \frac{1}{\sqrt{\langle e_1, e_1 \rangle}} e_1, \frac{1}{\sqrt{\langle e_2, e_2 \rangle}} e_2, \frac{1}{\sqrt{-\langle e_3, e_3 \rangle}} e_3 \right\}$$

is an orthogonal $\bar{\mathbb{Q}}$ -basis of $V \otimes_{E, \tilde{\rho}_1} \bar{\mathbb{Q}}$ in which the hermitian pairing $\langle \cdot, \cdot \rangle_{\rho_1}$ has matrix $J := \mathrm{diag}(1, 1, -1)$.

We denote by $\mathbf{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{C}}$ the so-called *Deligne torus*: this is an algebraic torus over \mathbb{R} , whose \mathbb{R} -points are \mathbb{C}^\times and which splits over \mathbb{C} as $\mathbf{S}_{\mathbb{C}} \simeq \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}}$, due to the following isomorphism:

$$\begin{aligned} (\mathbb{C} \otimes_{\mathbb{R}} S)^\times &\xrightarrow{\sim} S^\times \times S^\times, \\ x \otimes s &\longmapsto (\bar{x}s, xs) \end{aligned}$$

for all \mathbb{C} -algebra S . Via the above isomorphism, the natural embedding $\mathbb{C}^\times = \mathbf{S}(\mathbb{R}) \hookrightarrow \mathbf{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ given by $\mathbb{C}^\times \ni z \mapsto z \otimes 1 \in (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times$ corresponds to $z \mapsto (\bar{z}, z) \in \mathbb{C}^\times \times \mathbb{C}^\times$.

As F is totally real of degree d , one has a decomposition $F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{i=1}^d F_{\rho_i}$ ⁽⁶⁾ $\simeq \mathbb{R}^d$. For any \mathbb{R} -algebra S , with structure morphism $\pi : \mathbb{R} \rightarrow S$, one gets

$$\begin{aligned} \mathbf{G}_V(S) &\stackrel{\mathrm{def}}{=} \mathrm{U}(V)(F \otimes_{\mathbb{Q}} S) = \mathrm{U}(V)((F \otimes_{\mathbb{Q}} \mathbb{R}) \otimes_{\mathbb{R}} S) \\ &\simeq \mathrm{U}(V) \left(\prod_{i=1}^d F_{\rho_i} \otimes_{\mathbb{R}} S \right) = \prod_{i=1}^d \mathrm{U}(V)(S_{\rho_i}), \end{aligned}$$

(6). Recall that F_{ρ_i} denotes the archimedean completion of F with respect to the place ρ_i , i.e., the copy of \mathbb{R} endowed with the F -algebra structure induced by $\rho_i : F \hookrightarrow \mathbb{R}$, for $i \in \{1, \dots, d\}$.

with $S_{\rho_i} := S \otimes_{\mathbb{R}} F_{\rho_i} = S \otimes_{F, \pi \circ \rho_i} F$. Similarly, one has $\mathbf{G}_W(S) = \prod_{i=1}^d \mathrm{U}(W)(S_{\rho_i})$. In other words, one has isomorphisms of \mathbb{R} -algebraic groups:

$$\mathbf{G}_{V, \mathbb{R}} = \prod_{i=1}^d \mathrm{U}(V)_{\rho_i, \mathbb{R}} \quad \text{and} \quad \mathbf{G}_{W, \mathbb{R}} = \prod_{i=1}^d \mathrm{U}(W)_{\rho_i, \mathbb{R}}, \quad (1.5)$$

where $\mathrm{U}(V)_{\rho_i, \mathbb{R}}$ (resp. $\mathrm{U}(W)_{\rho_i, \mathbb{R}}$) denotes the base change of $\mathrm{U}(V)$ (resp. $\mathrm{U}(W)$) from F to \mathbb{R} with respect to ρ_i . We let X_V be the $\mathbf{G}_V(\mathbb{R})$ -conjugacy class of the morphism $h_V : \mathbf{S} \rightarrow \mathbf{G}_{V, \mathbb{R}}$, expressed in the basis \mathcal{B}_1 by:

$$\begin{aligned} \mathbf{S} &\longrightarrow \prod_{i=1}^d \mathrm{U}(V)_{\rho_i, \mathbb{R}}, \\ z &\longmapsto \left(\mathrm{diag} \left(1, 1, \frac{z}{\bar{z}} \right), \mathbf{1}_2, \dots, \mathbf{1}_d \right), \end{aligned} \quad (1.6)$$

$\mathbf{1}_i$ being the identity element of $\mathrm{U}(V)_{\rho_i, \mathbb{R}}$, for all i . The above definition makes sense as, for any \mathbb{R} -algebra S , the term $\mathrm{U}(V)_{\rho_i, \mathbb{R}}(S)$ appears as the subgroup of $\mathrm{GL}(V \otimes_{F, \rho_i} S) = \mathrm{GL}(V \otimes_E (E \otimes_{F, \rho_i} \mathbb{R}) \otimes_{\mathbb{R}} S) \simeq \mathrm{GL}(V_{\rho_i} \otimes_{\mathbb{R}} S) \stackrel{\mathcal{B}, \rho_i}{\simeq} \mathrm{GL}_3(\mathbb{C} \otimes_{\mathbb{R}} S)$, made of those matrices $G \in \mathrm{GL}_3(\mathbb{C} \otimes_{\mathbb{R}} S)$ such that ${}^t \bar{G} J G = J$ (here, \bar{G} denotes the image of G by the involution of $\mathrm{GL}_3(\mathbb{C} \otimes_{\mathbb{R}} S)$ induced coordinate-wise by $\mathbb{C} \otimes_{\mathbb{R}} S \rightarrow \mathbb{C} \otimes_{\mathbb{R}} S$, $z \otimes s \mapsto \bar{z} \otimes s$.) Similarly, we define X_W to be the $\mathbf{G}_W(\mathbb{R})$ -conjugacy class of the morphism

$$\begin{aligned} h_W : \mathbf{S} &\longrightarrow \prod_{i=1}^d \mathrm{U}(W)_{\rho_i, \mathbb{R}}, \\ z &\longmapsto \left(\mathrm{diag} \left(1, \frac{z}{\bar{z}} \right), \mathbf{1}_2, \dots, \mathbf{1}_d \right). \end{aligned}$$

For all $i \in \{1, \dots, d\}$, the commutativity of the following diagram

$$\begin{array}{ccc} F & \xleftarrow{\rho_i} & \mathbb{R} \\ \cap & & \cap \\ E & \xleftarrow{\tilde{\rho}_i} & \mathbb{C} \end{array}$$

implies that the base change $(\mathrm{U}(V)_{\rho_i, \mathbb{R}})_{\mathbb{C}}$ of the \mathbb{R} -group $\mathrm{U}(V)_{\rho_i, \mathbb{R}}$ to \mathbb{C} is isomorphic to $(\mathrm{U}(V)_E)_{\tilde{\rho}_i, \mathbb{C}}$ which is, by the preceding paragraph, isomorphic to $\mathrm{GL}(V_E)_{\tilde{\rho}_i, \mathbb{C}} \simeq \mathrm{GL}_{3, \mathbb{C}, \rho_i}$, via the map

$$\begin{aligned} \psi : \mathrm{U}(V)(S) &\xrightarrow{\sim} \mathrm{GL}(V_S), \\ g &\longmapsto g_t \end{aligned}$$

for any \mathbb{C} -algebra S , with $V_S := V \otimes_{E, \tilde{\rho}_i} S$. Notice that, by definition, the map $h_V : \mathbf{S} \rightarrow \mathbf{G}_{V, \mathbb{R}}$ factors through the product $(\mathfrak{T}_{\mathcal{B}})_{\rho_1, \mathbb{R}} \times \dots \times (\mathfrak{T}_{\mathcal{B}})_{\rho_d, \mathbb{R}}$, where the torus $\mathfrak{T}_{\mathcal{B}}$ is defined as

$$\mathfrak{T}_{\mathcal{B}} := \mathrm{U}(E e_1) \times \mathrm{U}(E e_2) \times \mathrm{U}(E e_3) \subset \mathrm{U}(V).$$

Indeed, one has $(E \otimes_{F, \rho_i} \mathbb{R})e_j = \mathbb{C}_{\rho_i}e_j = \mathbb{C}_{\rho_i}e'_j$, for all $i \in \{1, \dots, d\}$ and $j = 1, 2, 3$, by construction of \mathcal{B}_1 , hence

$$\begin{aligned} (U(Ee_1) \times U(Ee_2) \times U(Ee_3))_{\rho_i, \mathbb{R}} &= U(\mathbb{C}_{\rho_i}e_1) \times U(\mathbb{C}_{\rho_i}e_2) \times U(\mathbb{C}_{\rho_i}e_3) \\ &= U(\mathbb{C}_{\rho_i}e'_1) \times U(\mathbb{C}_{\rho_i}e'_2) \times U(\mathbb{C}_{\rho_i}e'_3) \subset U(V)_{\rho_i, \mathbb{R}}, \end{aligned}$$

where $U(\mathbb{C}_{\rho_i}e_j) \subset \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}(\mathbb{C}_{\rho_i}e_j)$ denotes the one dimensional unitary group defined over \mathbb{R} in the obvious way.

The restriction $\psi|_{(\mathfrak{T}_{\mathcal{B}})_{\tilde{\rho}_i, \mathbb{C}}}$ induces - for all $i \in \{1, \dots, d\}$ - a splitting $(\mathfrak{T}_{\mathcal{B}})_{\tilde{\rho}_i, \mathbb{C}} \xrightarrow{\psi} \mathbb{G}_{m, \mathbb{C}_{\rho_i}} \times \mathbb{G}_{m, \mathbb{C}_{\rho_i}} \times \mathbb{G}_{m, \mathbb{C}_{\rho_i}} \subset U(V)_{\tilde{\rho}_i, \mathbb{C}}$, given for any \mathbb{C} -algebra S by:

$$((x_{1,s}, x_{1,t}), (x_{2,s}, x_{2,t}), (x_{3,s}, x_{3,t})) \mapsto (x_{1,t}, x_{2,t}, x_{3,t}),$$

with $x_j = (x_{j,s}, x_{j,t}) \in U(Ee'_j)(S) \subset S^\times \times S^\times$, for all $j = 1, 2, 3$. For all $i \in \{1, \dots, d\}$ and all \mathbb{C} -algebra S , one has

$$\begin{aligned} (\mathrm{Res}_{E/F}\mathbb{G}_{m,E})_{\tilde{\rho}_i, \mathbb{C}}(S) &:= (E \otimes_{F, \rho_i} S)^\times \\ &= ((E \otimes_{F, \rho_i} \mathbb{R}) \otimes_{\mathbb{R}} S)^\times \xrightarrow{\rho_i} (\mathbb{C} \otimes_{\mathbb{R}} S)^\times = \mathbf{S}(S), \end{aligned}$$

which gives an identification between $(\mathrm{Res}_{E/F}\mathbb{G}_{m,E})_{\tilde{\rho}_i, \mathbb{C}}$ and $\mathbf{S}_{\mathbb{C}} \simeq \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}}$. One checks that, via the above identification, the complex conjugation $c \in \mathrm{Gal}(E/F)$ acts on $(E \otimes_{F, \rho_i} S)^\times \simeq S^\times \times S^\times$ via $(z_1, z_2) := (z_2, z_1) \in S^\times \times S^\times$, for all \mathbb{C} -algebra S . Consequently, one has

$$\begin{aligned} \psi((x_1, x_2, x_3) \cdot \overline{(x_1, x_2, x_3)}^{-1}) &= \psi(((x_{1,s}, x_{1,t}), (x_{2,s}, x_{2,t}), (x_{3,s}, x_{3,t})) \cdot ((x_{1,t}^{-1}, x_{1,s}^{-1}), (x_{2,t}^{-1}, x_{2,s}^{-1}), (x_{3,t}^{-1}, x_{3,s}^{-1}))) \\ &= \psi(((x_{1,s}x_{1,t}^{-1}, x_{1,t}x_{1,s}^{-1}), (x_{2,s}x_{2,t}^{-1}, x_{2,t}x_{2,s}^{-1}), (x_{3,s}x_{3,t}^{-1}, x_{3,t}x_{3,s}^{-1}))) = \left(\frac{x_{1,t}}{x_{1,s}}, \frac{x_{2,t}}{x_{2,s}}, \frac{x_{3,t}}{x_{3,s}} \right), \end{aligned}$$

for all $(x_1, x_2, x_3) \in (\mathfrak{T}_{\mathcal{B}})_{\rho_i, \mathbb{R}}(S)$, with S a \mathbb{C} -algebra. Accordingly, the natural extension of scalars of h_V to \mathbb{C} corresponds, via the above identifications, to the map:

$$\begin{aligned} h_{V, \mathbb{C}} : \mathbf{S}_{\mathbb{C}} &\longrightarrow \prod_{i=1}^d \mathrm{GL}_{3, \mathbb{C}_{\rho_i}} \\ (z_s, z_t) &\longmapsto \left(\mathrm{diag} \left(1, 1, \frac{z_t}{z_s} \right), \mathbf{1}_2, \dots, \mathbf{1}_d \right), \end{aligned}$$

and we extend similarly h_W to $h_{W, \mathbb{C}} : \mathbf{S}_{\mathbb{C}} \rightarrow G_{V, \mathbb{C}}$ by $(z_s, z_t) \mapsto \left(\mathrm{diag} \left(1, \frac{z_t}{z_s}, \mathbf{1}_2, \dots, \mathbf{1}_d \right) \right)$. We check that both complex forms $h_{V, \mathbb{C}}$ and $h_{W, \mathbb{C}}$ are consistent with the inclusion $\mathbf{S}(\mathbb{R}) \hookrightarrow \mathbf{S}(\mathbb{C})$, $z \mapsto (\bar{z}, z)$.

We finally define X to be the product $X := X_V \times X_W$. By definition, the embedding $\iota_{\mathbb{Q}} : \mathbf{G}_W \hookrightarrow \mathbf{G}_V$ satisfies $\iota_{\mathbb{Q}} \circ h_W = h_V$. This induces an embedding $\iota_{\mathrm{therm}} : X_W \hookrightarrow X_V$,

therefore the diagonal embedding $\Delta = (\iota_{\mathbb{Q}}, \mathrm{Id}_{\mathbf{G}_W}) : \mathbf{G}_W \hookrightarrow \mathbf{G}_V \times \mathbf{G}_W$ induces an embedding $\Delta_{\mathrm{herm}} : X_W \hookrightarrow X$. We denote by $Y := \Delta_{\mathrm{herm}}(X_W) \subset X$ the diagonal image of X_W . One checks that the set Y admits the following alternative description:

$$Y = \{(h : \mathbf{S} \rightarrow \mathbf{G}_{\mathbb{R}}) \in X; h \text{ factors through } \Delta_{\mathbb{R}} : \mathbf{H}_{\mathbb{R}} \hookrightarrow \mathbf{G}_{\mathbb{R}}\}. \quad (1.7)$$

Remark 1.1.2 (Another description of X_V and X_W). *One may divide the \mathbb{C} -vector space V_{ρ_1} into the following partition*

$$V_{\rho_1} := V_+ \sqcup V_0 \sqcup V_-,$$

with $V_+ := \{v \in V_{\rho_1}; \langle x, x \rangle_{\rho_1} \in \mathbb{R}_{>0}\}$, $V_0 := \{v \in V_{\rho_1}; \langle x, x \rangle_{\rho_1} = 0\}$ and $V_- := \{v \in V_{\rho_1}; \langle x, x \rangle_{\rho_1} \in \mathbb{R}_{<0}\}$. The above partition is stable by the action of $(E \otimes_{F, \rho_1} \mathbb{R})^{\times} \simeq \mathbb{C}_{\rho_1}^{\times}$, hence the sets $V_+ \sqcup \{0\}$ (resp. $V_0, V_- \sqcup \{0\}$) can be written as disjoint unions of positive (resp. null, negative) \mathbb{C}_{ρ_1} -lines. If $v = \sum_{i=1}^3 x_i e'_i \in V_{\rho_1}$, with $x_1, x_2, x_3 \in \mathbb{C}$, then

$$\langle v, v \rangle_{\rho_1} = |x_1|^2 + |x_2|^2 - |x_3|^2,$$

hence $V_- = \{\sum_{i=1}^3 x_i e'_i; \frac{|x_1|^2}{|x_3|^2} + \frac{|x_2|^2}{|x_3|^2} < 1\}$. The map (depending on the choice of \mathcal{B}_1):

$$\begin{aligned} \Theta_{\mathcal{B}_1} : V_- &\longrightarrow \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 < 1\}, \\ \sum_{i=1}^3 x_i e'_i &\longmapsto \begin{pmatrix} x_1 & x_2 \\ x_3 & x_3 \end{pmatrix} \end{aligned}$$

induces a bijection between $\mathcal{X}_V := \mathbb{C}_{\rho_1}^{\times} \backslash V_-$ and the open complex 2-dimensional ball $\mathbb{B}_2 := \{(z_1, z_2) \in \mathbb{C}; |z_1|^2 + |z_2|^2 < 1\}$. Let us show that \mathcal{X}_V surjects onto the set X_V defined previously. Indeed, to any negative line $\ell \in \mathcal{X}_V$ corresponds a representative $e \in \ell$ such that $\langle e, e \rangle_{\rho_1} = -1$ (such an e is well-defined up to multiplication by some element of $\mathbb{S}^1 \subset \mathbb{C}^{\times}$). One may thus extend e into an orthogonal \mathbb{C}_{ρ_1} -basis $\mathcal{B}_{\ell} = \{c, d, e\}$, such that $\{c, d\}$ is an orthogonal basis of ℓ^{\perp} satisfying $\langle c, c \rangle = \langle d, d \rangle = 1$. The morphism $h_{V, \ell} : \mathbf{S} \rightarrow \mathbf{G}_{V, \mathbb{R}}$ defined, for all \mathbb{R} -algebra S , by mapping $z \in \mathbf{S}(S)$ to the element $(\mathrm{diag}(1, 1, \frac{z}{\bar{z}}), \mathbf{1}_2, \dots, \mathbf{1}_d) \in \mathbf{G}_{V, \mathbb{R}}$ (expressed in \mathcal{B}_{ℓ}) does not depend on the choice of c and d . By construction, one has $h_V = h_{V, \mathbb{C}_{\rho_1}^{\times} e'_3}$ and, if $g \in \mathrm{U}(V)_{\rho_1, \mathbb{R}}(\mathbb{R})$, one checks that $h_{V, g \cdot \ell} = g h_{V, \ell} g^{-1}$: indeed, if the suitable $e \in \ell$ is extended into $\{c, d, e\}$, then one may decide without loss of generality to extend the suitable $g \cdot e \in g \cdot \ell$ into the basis $\mathcal{B}_{g \cdot \ell} := \{g \cdot c, g \cdot d, g \cdot e\} = g \cdot \mathcal{B}_{\ell}$, as g is unitary.

That the correspondence $\ell \mapsto h_{V, \ell}$ induces a bijection between \mathcal{X}_V and the $\mathbf{G}_V(\mathbb{R})$ -conjugacy class of h_V - denoted as X_V - follows from Witt's theorem: indeed, the group $\mathbf{G}_V(\mathbb{R})$ acts on the set \mathcal{X}_V of negative \mathbb{C}_{ρ_1} -lines via the projection $\mathbf{G}_V(\mathbb{R}) \twoheadrightarrow \mathrm{U}(V)_{\rho_1, \mathbb{R}}(\mathbb{R})$, and the action of the latter is transitive by Witt's theorem, i.e., both sets \mathcal{X}_V and X_V are formed of a unique $\mathbf{G}_V(\mathbb{R})$ -orbit. Finally, notice that the choice of \mathcal{B}_1 induces an isomorphism between $\mathrm{U}(V)_{\rho_1, \mathbb{R}}$ and $\mathrm{U}(2, 1) \subset \mathrm{GL}_3(\mathbb{C})$. If $\pi : \mathbb{C}^2 \rightarrow \mathbb{P}^2(\mathbb{C})$ is the map defined by sending $(x, y) \in \mathbb{C}^2$ to $[x : y : 1] \in \mathbb{P}^2(\mathbb{C})$, then one checks easily that the composite map $\pi \circ \Theta_{\mathcal{B}_1} : \mathcal{X}_V \rightarrow \mathbb{P}^2(\mathbb{C})$ is

$U(V)_{\rho_1, \mathbb{R}} \simeq U(2, 1)$ -equivariant, and that the stabilizer of $[0 : 0 : 1] = \pi \circ \Theta_{\mathcal{B}_1}(e'_3) \in \mathbb{P}^2(\mathbb{C})$ is the compact subgroup $U(2) \times U(1) \subset U(2, 1)$. Accordingly, the preceding implies that the map $\ell \mapsto h_{V, \ell}$ induces a sequence of bijections:

$$\begin{aligned} \mathbb{B}_2 &\simeq (U(2) \times U(1)) \backslash U(2, 1) \\ &\simeq \mathrm{Stab}_{U(V)_{\rho_1, \mathbb{R}}(\mathbb{R})}(\mathbb{C}_{\rho_1} e'_3) \backslash U(V)_{\rho_1, \mathbb{R}}(\mathbb{R}) \\ &\simeq \mathcal{X}_V \\ &\simeq X_V, \end{aligned}$$

which identifies X_V with the 2-dimensional complex homogeneous space $(U(2) \times U(1)) \backslash U(2, 1)$.

Similarly, one may replace V_{ρ_1} with W_{ρ_1} and obtain a bijection between $\mathcal{X}_W := \mathbb{C}_{\rho_1}^\times \backslash W_- \simeq \mathbb{B}_1$ and the set X_W defined previously, both of which are isomorphic to the homogeneous one-dimensional quotient $\mathrm{Stab}_{U(W)_{\rho_1, \mathbb{R}}(\mathbb{R})}(\mathbb{C}_{\rho_1} e'_3) \backslash U(W)_{\rho_1, \mathbb{R}}(\mathbb{R}) \simeq (U(1) \times U(1)) \backslash U(1, 1)$. One checks easily that the above identifications commute with the natural injection $\mathcal{X}_W \hookrightarrow \mathcal{X}_V$ induced by $W \subset V$ and with the embedding $\iota_{\mathrm{herm}} : X_W \hookrightarrow X_V$.

1.1.4 The Shimura varieties $\mathrm{Sh}_K(\mathbf{G}, X)$ and $\mathrm{Sh}_{K_H}(\mathbf{H}, Y)$.

1.1.4.1 The Deligne axioms SV1 - SV6.

Let us check, as a warm-up exercise, that the pairs (\mathbf{G}_V, X_V) , (\mathbf{G}_W, X_W) , (\mathbf{G}, X) and (\mathbf{H}, Y) introduced in the preceding paragraph are *Shimura data*, i.e., that they satisfy the set of properties SV1 - SV3 - as well as the additional properties SV4 - SV6 - such as defined in ([32] §5) and originally introduced by Deligne in order to axiomatize the theory of Shimura varieties via the theory of reductive groups. One checks that these axioms are compatible with products, therefore it will be enough to show that these hold for both pairs (\mathbf{G}_V, X_V) and (\mathbf{G}_W, X_W) .

For all $\star \in \{V, W\}$, let us denote by $\mathbf{Z}_{\mathbf{G}_\star}$ the center of \mathbf{G}_\star . According to ([12], Proposition A.5.15), one has $\mathbf{Z}_{\mathbf{G}_\star} = \mathrm{Res}_{F/\mathbb{Q}} \mathbf{Z}_{U(\star)} = \mathrm{Res}_{F/\mathbb{Q}} U(1)_{E/F}$, where $U(1)_{E/F} \subset U(\star)$ is the F -torus whose points are unitary homotheties inside $U(\star)$, i.e., $U(1)_{E/F}$ is isomorphic to the unitary group over F attached to the one-dimensional E/F -hermitian space $(E, \langle z, z' \rangle := \bar{z}z')$, and the identification between $U(1)_{E/F}$ and $\mathbf{Z}_{U(\star)}$ corresponds to $z \mapsto z \cdot \mathbf{1}_\star$. Accordingly, one has $\mathbf{Z}_{\mathbf{G}_\star}(\mathbb{R}) \simeq U(1)_{E/F}(F \otimes_{\mathbb{Q}} \mathbb{R}) = \prod_{i=1}^d U(1)_{E/F}(\mathbb{R}_{\rho_i}) \simeq \prod_{i=1}^d \mathbb{S}_{\rho_i}^1$, where $\mathbb{S}_{\rho_i}^1$ is the usual unit sphere $\{z \in \mathbb{C}_{\rho_i}^\times; z\bar{z} = 1\}$. From now on, we shall denote by \mathbf{T}^1 the \mathbb{Q} -torus $\mathrm{Res}_{F/\mathbb{Q}} U(1)_{E/F}$. We define the *adjoint group* of \mathbf{G}_\star as the connected, reductive \mathbb{Q} -group

$$\mathbf{G}_\star^{\mathrm{ad}} := \mathbf{G}_\star / \mathbf{Z}_{\mathbf{G}_\star},$$

We denote the quotient map by $\pi : \mathbf{G}_\star \rightarrow \mathbf{G}_\star^{\mathrm{ad}}$. We also set $\mathbf{G}_\star^{\mathrm{der}}$ to be the *derived group* of \mathbf{G}_\star (see [33], Definition 6.16). As \mathbf{G}_\star is a smooth \mathbb{Q} -group, Proposition 6.18 of

[33] ensures that $\mathbf{G}_\star^{\mathrm{der}}$ coincides with the commutator subgroup of \mathbf{G}_\star , which implies that $\mathbf{G}_\star^{\mathrm{der}} = \ker(\det : \mathbf{G}_\star \rightarrow \mathbf{T}^1) = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{SU}(\star)$, with $\mathrm{SU}(\star) = \mathrm{Res}_{E/F}\mathrm{SL}(\star_E) \cap \mathrm{U}(\star)$.

We shall give the details for the axioms SV1 - SV6 in the case of (\mathbf{G}_V, X_V) , the case of (\mathbf{G}_W, X_W) being an obvious adaptation of it, obtained by adjusting dimensions in the appropriate parts.

— SV1: For all $h \in X_V$, the Hodge structure defined by $\mathrm{Ad} \circ h$ on $\mathrm{Lie}(\mathbf{G}_V(\mathbb{R}))$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$.

It is enough to consider the case $h = h_V$ as for all $h, h' \in X_V$, the adjoint actions $\mathrm{Ad} \circ h$ and $\mathrm{Ad} \circ h'$ on $\mathrm{Lie}(\mathbf{G}_V(\mathbb{R}))$ induce isomorphic Hodge structures (as h and h' are conjugated by some element of $\mathbf{G}_V(\mathbb{R})$). The morphism $h_V : \mathbf{S} \rightarrow \mathbf{G}_{V,\mathbb{R}}$ induces a representation of the Deligne torus obtained by composing with the adjoint map $\mathrm{Ad} : \mathbf{G}_{V,\mathbb{R}} \rightarrow \mathrm{GL}(\mathrm{Lie}(\mathbf{G}_{V,\mathbb{R}}))$. According to ([32], §2), this amounts to giving a real Hodge structure on the \mathbb{R} -vector space $\mathcal{V} := \mathrm{Lie}(\mathbf{G}_{V,\mathbb{R}}) \simeq M_3(\mathbb{R})^d$. Indeed, the space $\mathcal{V}_{\mathbb{C}} = \mathrm{Lie}(\mathbf{G}_{V,\mathbb{R}})_{\mathbb{C}}$ decomposes as a direct sum of eigenspaces of the form $\mathcal{V}^{p,q}$, $(p, q) \in \mathbb{Z}^2$, such that $\mathrm{Ad} \circ h_V$ acts on $\mathcal{V}^{p,q}$ by the character $z^{-p}\bar{z}^{-q}$, and such that $\mathcal{V}^{p,q} = \overline{\mathcal{V}^{q,p}}$ (as h_V is defined on \mathbb{R}). It is a straightforward consequence of the construction of h_V that the only characters occurring in $\mathrm{Ad} \circ h_V$ are $\frac{z}{\bar{z}}$, 1 and $\frac{\bar{z}}{z}$, i.e., the Hodge structure attached to h_V has weights $\{(-1, 1), (0, 0), (1, -1)\}$.

— SV2: For all $h \in X_V$, the inner automorphism $\mathrm{ad}(h(i))$ of $\mathbf{G}_{V,\mathbb{R}}$ is a Cartan involution. We recall (see [32], §1 p15) that by a *Cartan involution*, we mean an involution θ of $\mathbf{G}_{V,\mathbb{R}}^{\mathrm{ad}}$ (seen as a morphism of \mathbb{R} -groups) such that the group

$$\mathbf{G}_{V,\mathbb{R}}^{\mathrm{ad}}(\theta) := \{g \in \mathbf{G}_V^{\mathrm{ad}}(\mathbb{C}); g = \theta(c \cdot g)\} \subset \mathbf{G}_V^{\mathrm{ad}}(\mathbb{C})$$

is compact, where $g \mapsto c \cdot g$ denotes the action of the complex conjugation on $\mathbf{G}_V^{\mathrm{ad}}(\mathbb{C})$. A direct computation shows that, if $g_0 \in \mathbf{G}_{V,\mathbb{R}}^{\mathrm{ad}}(\mathbb{R})$ and if θ is an involution $\mathbf{G}_{V,\mathbb{R}}^{\mathrm{ad}}$, then $\mathbf{G}_{V,\mathbb{R}}^{\mathrm{ad}}(g_0 \theta g_0^{-1}) = g_0 \cdot \mathbf{G}_{V,\mathbb{R}}^{\mathrm{ad}}(\theta)$: accordingly, it is enough to show that $\mathrm{ad}(h_V(i))$ is a Cartan involution of $\mathbf{G}_{V,\mathbb{R}}^{\mathrm{ad}}$ (which will imply that all Cartan involutions of $\mathbf{G}_{V,\mathbb{R}}^{\mathrm{ad}}$ arise as $\mathbf{G}_{V,\mathbb{R}}^{\mathrm{ad}}(\mathbb{R})$ -conjugates of $\mathrm{ad}(h_V(i))$, according to [32], Theorem 1.16). Notice that one has

$$\mathbf{G}_{V,\mathbb{R}}^{\mathrm{ad}} \simeq \prod_{i=1}^d \mathrm{U}(V)_{\rho_i, \mathbb{R}} / (\mathbf{Z}_{\mathrm{U}(V)})_{\rho_i, \mathbb{R}},$$

and we denote by π_i the projection map $\mathbf{G}_{V,\mathbb{R}} \twoheadrightarrow \mathrm{U}(V)_{\rho_i, \mathbb{R}} / (\mathbf{Z}_{\mathrm{U}(V)})_{\rho_i, \mathbb{R}}$, for all $i \in \{1, \dots, d\}$.

For all $i \in \{1, \dots, d\}$ - as we shall see below at § 1.1.5 - the identification $g \mapsto g_t$ which we make between the E -groups $\mathrm{U}(V)_E$ and $\mathrm{GL}(V_E)$, induces a *twisted* algebraic action of the complex conjugation $c \in \mathrm{Gal}(E/F)$ on S -points, where $S = E \otimes_F R$ is an E -algebra endowed with a $\mathrm{Gal}(E/F)$ -action. This action is explicitly described via the choice of an S -basis \mathcal{B} of $V \otimes_F R = V \otimes_E S$ in which the hermitian product (extended

to $V \otimes_F S$) has matrix $H \in \mathrm{GL}_3(E \otimes_F S)$. In the case $R = \mathbb{R}_{\rho_i}$, $S = E \otimes_F R = \mathbb{C}_{\rho_i}$ and $H = J$ (if $i = 1$) or $H = \mathbf{1}$ (if $i \geq 2$), the equality (1.13) rewrites in the following way: if $g \in \mathrm{U}(V)(\mathbb{C}_{\rho_i})$ is identified to $g_t \in \mathrm{GL}(V_{\rho_i}) \stackrel{\mathcal{B}}{\simeq} \mathrm{GL}_3(\mathbb{C})$, then $c \cdot g \in \mathrm{U}(V)(\mathbb{C}_{\rho_i})$ is now identified to $\bar{J}^{-1} {}^t(\bar{g}_t)^{-1} \bar{J}^{-1} = J {}^t(\bar{g}_t)^{-1} J \in \mathrm{GL}_3(\mathbb{C})$ if $i = 1$ (resp. to ${}^t(\bar{g}_t)^{-1}$ if $i \geq 2$). By definition, one has $h_V(i) = (\mathrm{diag}(1, 1, -1), \mathbf{1}_2, \dots, \mathbf{1}_d) = (J, \mathbf{1}_2, \dots, \mathbf{1}_d) \in \mathbf{G}_V(\mathbb{C})$. One gets:

$$\begin{aligned}
\mathbf{G}_{V, \mathbb{R}}^{\mathrm{ad}}(\mathrm{ad}(h_V(i))) &= \pi(\{g \in \mathbf{G}_V^{\mathrm{ad}}(\mathbb{C}); \exists \lambda \in \mathbf{Z}_{\mathbf{G}_V}(\mathbb{C}), g = \lambda \mathrm{ad}(h_V(i))(c \cdot g)\}) \\
&= \pi_1(\{g \in \mathrm{U}(V)(\mathbb{C}_{\rho_1}); \exists \lambda \in \mathbf{Z}_{\mathrm{U}(V)}(\mathbb{C}_{\rho_1}), g = \lambda J(c \cdot g)J\}) \\
&\quad \times \prod_{i=2}^d \pi_i(\{g \in \mathrm{U}(V)(\mathbb{C}_{\rho_i}); \exists \lambda \in \mathbf{Z}_{\mathrm{U}(V)}(\mathbb{C}_{\rho_i}), g = \lambda(c \cdot g)\}) \\
&\simeq \pi_1(\{g \in \mathrm{GL}_3(\mathbb{C}); \exists \lambda \in \mathbb{C}^\times, g = \lambda J(J {}^t(\bar{g})^{-1} J)J\}) \\
&\quad \times \prod_{i=2}^d \pi_i(\{g \in \mathrm{GL}_3(\mathbb{C}); \exists \lambda \in \mathbb{C}^\times, g = \lambda {}^t(\bar{g})^{-1}\}) \\
&\stackrel{\diamond}{=} \prod_{i=1}^d \pi_i(\{g \in \mathrm{GL}_3(\mathbb{C}); \exists \mu \in \mathbb{C}^\times, ((\mu g) {}^t \overline{\mu g})^3 = \mathbf{1}\}) \\
&= (\mathbb{C}^\times r^{-1}(\mathrm{U}(3))/\mathbb{C}^\times) \times \dots \times (\mathbb{C}^\times r^{-1}(\mathrm{U}(3))/\mathbb{C}^\times) = r^{-1}(\mathrm{U}(3)) \times \dots \times r^{-1}(\mathrm{U}(3)),
\end{aligned}$$

where $r : \mathrm{GL}_3(\mathbb{C}) \rightarrow \mathrm{GL}_3(\mathbb{C})$ is the map $g \mapsto g^3$, and where the equality \diamond follows from the following fact: if $g \in \mathrm{GL}_3(\mathbb{C})$ is such that $g = \lambda {}^t(\bar{g})^{-1}$ then $\det(g)\overline{\det(g)} = \det(\lambda \cdot \mathbf{1}) = \lambda^3 \in \mathbb{R}_{>0}$. Then λ can be written as $\lambda = \nu \bar{\nu} \zeta$ for some complex number ν and some $\zeta \in \mu_3$, i.e., the element $\nu^{-1}g \in \mathrm{GL}_3(\mathbb{C})$ satisfies $(\nu^{-1}g) {}^t \overline{\nu^{-1}g} = \zeta \cdot \mathbf{1}$. As g and ${}^t \bar{g}$ commute, this gives $r(\nu^{-1}g) {}^t \overline{r(\nu^{-1}g)} = \nu^{-3} \bar{\nu}^{-3} (g {}^t \bar{g})^3 = \mathbf{1}$, hence $\nu^{-1}g \in r^{-1}(\mathrm{U}(3))$. Accordingly, $\mathbf{G}_{V, \mathbb{R}}^{\mathrm{ad}}(\theta)$ is a closed and bounded subgroup of $\mathrm{GL}_3(\mathbb{C})^d$, therefore a compact group.

— SV3: The group $\mathbf{G}_V^{\mathrm{ad}}$ has no \mathbb{Q} -factor on which the projection of h_V is trivial.

Let us start by showing that the group $\mathbf{G}_V^{\mathrm{ad}}$ is \mathbb{Q} -simple. For all $i \in \{1, \dots, d\}$, we set $\mathrm{U}(V)_{\rho_i, \mathbb{R}}^{\mathrm{ad}} := \mathrm{U}(V)_{\rho_i, \mathbb{R}} / (\mathbf{Z}_{\mathrm{U}(V)})_{\rho_i, \mathbb{R}} = (\mathrm{U}(V)^{\mathrm{ad}})_{\rho_i, \mathbb{R}}$, where $\mathrm{U}(V)^{\mathrm{ad}}$ is the F -group $\mathrm{U}(V)/\mathbf{Z}_{\mathrm{U}(V)}$. Consequently, one has $\mathbf{G}_{V, \mathbb{R}}^{\mathrm{ad}} = \prod_{i=1}^d \mathrm{U}(V)_{\rho_i, \mathbb{R}}^{\mathrm{ad}}$, and for all i , an isomorphism:

$$\mathrm{U}(V)_{\rho_i, \mathbb{R}}^{\mathrm{ad}}(\mathbb{R}) \simeq \{g \in \mathrm{GL}_3(\mathbb{C}_{\rho_i}); {}^t \bar{g} H_i g = H_i\} / \mathbb{C}_{\rho_i}^\times,$$

with $H_i = J$ if $i = 1$, and $\mathbf{1}$ otherwise. Accordingly, one has

$$\mathbf{G}_V^{\mathrm{ad}}(\mathbb{R}) = \mathrm{PU}(2, 1) \times \mathrm{PU}(3) \times \dots \times \mathrm{PU}(3) \simeq \mathrm{PSU}(2, 1) \times \mathrm{PSU}(3) \dots \times \mathrm{PSU}(3),$$

which is a product of simple compact Lie groups. Let $G' \triangleleft \mathbf{G}_V^{\mathrm{ad}}$ be a \mathbb{Q} -factor of $\mathbf{G}_V^{\mathrm{ad}}$ (i.e., a connected normal algebraic subgroup of $\mathbf{G}_V^{\mathrm{ad}}$ defined over \mathbb{Q}) and denote by $f : G' \hookrightarrow \mathbf{G}_V^{\mathrm{ad}}$ the corresponding closed immersion. The group $G'_{\mathbb{R}} = (G'_F)_{\rho_i, \mathbb{R}} \triangleleft \mathbf{G}_{V, \mathbb{R}}^{\mathrm{ad}}$ is a normal \mathbb{R} -algebraic subgroup - the equality $G'_{\mathbb{R}} = (G'_F)_{\rho_i, \mathbb{R}}$ coming from the fact

that G' is defined over \mathbb{Q} - hence the image of $G'_{\mathbb{R}}$ under the projections $\pi_i : \mathbf{G}_{V, \mathbb{R}}^{\mathrm{ad}} \rightarrow \mathrm{U}(V)_{\rho_i, \mathbb{R}} / (\mathbf{Z}_{\mathrm{U}(V)})_{\rho_i, \mathbb{R}}$ is again a normal subgroup. As connected groups over perfect fields k are determined by their k -points (which are Zariski dense) and as $\mathrm{U}(V)_{\rho_i, \mathbb{R}}^{\mathrm{ad}}(\mathbb{R})$ is a simple Lie group, one gets that $\pi_i(G'_{\mathbb{R}})$ is either the trivial subgroup or is equal to $\mathrm{U}(V)_{\rho_i, \mathbb{R}}^{\mathrm{ad}}(\mathbb{R})$, for all i .

On the other hand, one checks easily that $\mathbf{G}_V^{\mathrm{ad}} = (\mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(V))^{\mathrm{ad}} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(V)^{\mathrm{ad}}$: by the universal property of the Weil restriction - namely, that the functor $\mathrm{Res}_{F/\mathbb{Q}}$ is right-adjoint to the functor which extends scalars to F - one has a group isomorphism

$$\mathrm{Hom}_{\mathbb{Q}}(G', \mathbf{G}_V^{\mathrm{ad}}) \simeq \mathrm{Hom}_F(G'_F, \mathrm{U}(V)^{\mathrm{ad}}).$$

Accordingly, the canonical immersion $f : G' \hookrightarrow \mathbf{G}_V^{\mathrm{ad}}$ induces an F -morphism $G'_F \rightarrow \mathrm{U}(V)^{\mathrm{ad}}$, which factors through the map $(\mathbf{G}_V^{\mathrm{ad}})_F = (\mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(V)^{\mathrm{ad}})_F \rightarrow \mathrm{U}(V)^{\mathrm{ad}}$ (the latter is induced by the ring morphism $R \otimes_{\mathbb{Q}} F \rightarrow R$, for all F -algebra R). This gives a sequence of maps (over $\mathrm{Spec} \mathbb{R}$):

$$G'_{\mathbb{R}} = (G'_F)_{\rho_i, \mathbb{R}} \xrightarrow{f_{\mathbb{R}}} (\mathbf{G}_{V, F}^{\mathrm{ad}})_{\rho_i, \mathbb{R}} = (\mathbf{G}_V^{\mathrm{ad}})_{\mathbb{R}} \xrightarrow{\pi_i} \mathrm{U}(V)_{\rho_i, \mathbb{R}}^{\mathrm{ad}}.$$

If $\pi_i(G'_{\mathbb{R}}) \subset (\mathrm{U}(V)^{\mathrm{ad}})_{\rho_i, \mathbb{R}}$ is trivial for some $i \in \{1, \dots, d\}$, then the map $G'_F \rightarrow \mathrm{U}(V)^{\mathrm{ad}}$ is already trivial over $\mathrm{Spec} F$ which implies (again by functoriality properties of the Weil restriction) that the map $f_{\mathbb{R}}$ is itself trivial, hence $G' = 1$. Otherwise, one gets that $\pi_i(G'_{\mathbb{R}}) = (\mathrm{U}(V)^{\mathrm{ad}})_{\rho_i, \mathbb{R}}$ for all i , hence $f_{\mathbb{R}}$ is surjective, which implies that $G' = \mathbf{G}_V^{\mathrm{ad}}$. This shows that $\mathbf{G}_V^{\mathrm{ad}}$ is \mathbb{Q} -simple. It now amounts to showing that the composite of h_V with the quotient map $\mathbf{G}_{V, \mathbb{R}} \rightarrow \mathbf{G}_{V, \mathbb{R}}^{\mathrm{ad}}$ is not trivial, i.e., that h_V does not factor through the center $\mathbf{Z}_{\mathbf{G}_{V, \mathbb{R}}} = \mathbf{Z}_{\mathrm{U}(V)_{\rho_1, \mathbb{R}}} \times \dots \times \mathbf{Z}_{\mathrm{U}(V)_{\rho_d, \mathbb{R}}}$. But $\mathrm{diag}(1, 1, \frac{z}{2})$ does not lie in $\mathbf{Z}_{\mathrm{U}(V)_{\rho_1, \mathbb{R}}}(\mathbb{R})$, which concludes.

The additional axioms SV4 - SV6 are presented in ([32], p63) and will induce later one some useful simplifications in the theory of the attached Shimura variety.

— SV4: *The weight homomorphism is defined over \mathbb{Q} .*

The weight homomorphism is defined as

$$\begin{array}{ccc} w_{X_V} : \mathbb{G}_{m, \mathbb{R}} & \hookrightarrow & \mathbf{S} \xrightarrow{h_V} \mathbf{G}_{V, \mathbb{R}} \\ & & r \longmapsto h_V(r^{-1}) \end{array}$$

for all \mathbb{R} -algebra S and all $r \in S^\times$. By construction of h_V the weight homomorphism is in fact trivial, hence clearly defined over \mathbb{Q} . Such a condition ensures that Hodge structure on \mathcal{V} given by morphism $h_V : \mathbf{S} \rightarrow \mathbf{G}_{V, \mathbb{R}}$ is in fact a \mathbb{Q} -Hodge structure.

— SV5: *The group $\mathbf{Z}_{\mathbf{G}_V}(\mathbb{Q})$ is discrete in $\mathbf{Z}_{\mathbf{G}_V}(\mathbb{A}_f)$.*

Recall that we identify the center $\mathbf{Z}_{\mathbf{G}_V}$ with the torus $\mathbf{T}^1 = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{U}(1)_{E/F})$. One has $\mathbf{T}^1(\mathbb{R}) = \mathrm{U}(1)(F \otimes_{\mathbb{Q}} \mathbb{R}) \simeq \mathrm{U}(1)(\mathbb{R}^d) = \mathrm{U}(1)(\mathbb{R})^d = (\mathbb{S}^1)^d$, the isomorphism $F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^d$ coming from the fact that F is totally real. Therefore $\mathbf{T}^1(\mathbb{R})$ is compact, which induces (see [32], Remark 5.27) that $\mathbf{T}_{\mathbb{R}}^1$ contains no split \mathbb{R} -torus, i.e., \mathbf{T}^1 is anisotropic over \mathbb{Q} and remains anisotropic when extended to \mathbb{R} . By ([32], Remark 5.27 and Theorem 5.26), this gives that $\mathbf{T}^1(\mathbb{Q}) \subset \mathbf{T}^1(\mathbb{A}_f)$ is discrete.

— SV6: *The identity component $\mathbf{Z}_{\mathbf{G}_V}^0$ of $\mathbf{Z}_{\mathbf{G}_V}$ is split over a CM field.*

The center $\mathbf{Z}_{\mathbf{G}_V}$ is already split over a CM field. Indeed, let F'/F denote the Galois closure of F inside $\overline{\mathbb{Q}}$: by assumption on F , F' is a totally real number field, and the quadratic extension $E' := F' \cdot E = F'[\alpha]$ of F is totally complex by, hence E' is again a CM field. One now has:

$$(\mathbf{G}_V)_{E'} = (\mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(V))_{E'} = ((\mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(V))_{F'})_{E'} \stackrel{\natural}{\simeq} (\mathrm{U}(V)^d)_E \simeq \mathrm{GL}_{3,E}^d,$$

the isomorphism \natural coming from $F \otimes_{\mathbb{Q}} S \simeq S^{[F:\mathbb{Q}]}$ for all F' -algebra S . This implies that the center \mathbf{T}^1 of \mathbf{G}_V satisfies $\mathbf{T}_{E'}^1 \simeq \mathbb{G}_{m,E'}^{3d}$.

1.1.4.2 The Shimura varieties.

Let \star belong to $\{V, W\}$. Notice that, as \mathbf{G}_{\star} is an affine algebraic group over \mathbb{Q} then the subgroup $\mathbf{G}_{\star}(\mathbb{Q}) \subset \mathbf{G}_{\star}(\mathbb{A}) = \mathbf{G}_{\star}(\mathbb{R}) \times \mathbf{G}_{\star}(\mathbb{A}_f)$ is discrete⁽⁷⁾ hence a closed subgroup. To any compact open subgroup K_{\star} of $\mathbf{G}_{\star}(\mathbb{A}_f)$ is thus attached a complex manifold, denoted as $\mathrm{Sh}_{K_{\star}}(\mathbf{G}_{\star}, X_{\star})(\mathbb{C})$, as the following double quotient:

$$\mathrm{Sh}_{K_{\star}}(\mathbf{G}_{\star}, X_{\star})(\mathbb{C}) := \mathbf{G}_{\star}(\mathbb{Q}) \backslash (X_{\star} \times (\mathbf{G}_{\star}(\mathbb{A}_f)/K_{\star})) = \mathrm{U}(\star)(F) \backslash (X_{\star} \times (\mathrm{U}(\star)(\mathbb{A}_{F,f})/K_{\star})),$$

where the left-hand term $\mathbf{G}_{\star}(\mathbb{Q})$ acts diagonally on $X_{\star} \times \mathbf{G}_{\star}(\mathbb{A}_f) \simeq \mathbf{G}_{\star}(\mathbb{A})/K_{\star,\infty}$, $K_{\star,\infty}$ being the compact open subgroup of $\mathbf{G}_{\star}(\mathbb{R})$ defined by

$$K_{\star,\infty} := \mathrm{Stab}_{\mathrm{U}(\star)(\mathbb{R}_{\rho_1})}(\mathbb{C}_{\rho_1} e'_3) \times \mathrm{U}(\star)(\mathbb{R}_{\rho_2}) \times \dots \times \mathrm{U}(\star)(\mathbb{R}_{\rho_d}).$$

Let K_V and K_W be open compact subgroups of $\mathbf{G}_V(\mathbb{A}_f)$ and $\mathbf{G}_W(\mathbb{A}_f)$ respectively, and we assume throughout the following that $\iota_{\mathbb{Q}}(K_W) \subset K_V$. The embeddings $\iota_{\mathbb{Q}} : \mathbf{G}_W \hookrightarrow \mathbf{G}_V$ and $\iota_{\mathrm{herm}} : X_W \hookrightarrow X_V$ induce an embedding $\iota : \mathrm{Sh}_{K_W}(\mathbf{G}_W, X_W) \hookrightarrow \mathrm{Sh}_{K_V}(\mathbf{G}_V, X_V)$. Set $K := K_V \times K_W \subset \mathbf{G}(\mathbb{A}_f)$ and $K_{\mathbf{H}} := \Delta(K_W) = K \cap \mathbf{H}(\mathbb{A}_f) \subset \mathbf{H}(\mathbb{A}_f)$. Similarly, one may

(7). This follows from the fact that $\mathbb{Q} \hookrightarrow \mathbb{A}_f$ is a discrete subgroup, and from the definition of the adelic topology induced on \mathbf{G}_{\star} .

define the following double quotients:

$$\mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C}) := \mathbf{G}(\mathbb{Q}) \backslash (X \times (\mathbf{G}(\mathbb{A}_f)/K)) = \mathrm{Sh}_{K_V}(\mathbf{G}_V, X_V)(\mathbb{C}) \times \mathrm{Sh}_{K_W}(\mathbf{G}_W, X_W)(\mathbb{C}) \quad (1.8)$$

$$\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y) := \mathbf{H}(\mathbb{Q}) \backslash (Y \times (\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}})) = \Delta(\mathrm{Sh}_{K_W}(\mathbf{G}_W, X_W)(\mathbb{C})), \quad (1.9)$$

with $\Delta = (\iota, \mathrm{Id}) : \mathrm{Sh}_{K_W}(\mathbf{G}_W, X_W)(\mathbb{C}) \hookrightarrow \mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C})$. For all $x \in X$ (resp. $y \in Y$) and all $g \in \mathbf{G}(\mathbb{A}_f)$ (resp. $h \in \mathbf{H}(\mathbb{A}_f)$), we shall denote by $[x, g]_K$ the point of $\mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C})$ defined by the double coset $\mathbf{G}(\mathbb{Q}) \cdot (x, gK)$ (resp. by $[y, h]_{K_{\mathbf{H}}}$ the point of $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C})$ attached to the double coset $\mathbf{H}(\mathbb{Q}) \cdot (y, hK_{\mathbf{H}})$). The sets X_V and X_W being connected as topological spaces, then so are X and Y and one obtains by ([32], Lemma 5.13) that the manifolds $\mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C})$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C})$ decompose into the following disjoint union of connected components:

$$\mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C}) = \bigsqcup_{i=1}^{r_X} \Gamma_{\mathbf{G}, i} \backslash X \quad (1.10)$$

$$\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C}) = \bigsqcup_{j=1}^{r_Y} \Gamma_{\mathbf{H}, j} \backslash Y \quad (1.11)$$

where for all i, j , $\Gamma_{\mathbf{G}, i}$ and $\Gamma_{\mathbf{H}, j}$ are the *arithmetic* subgroups of $\mathbf{G}(\mathbb{Q})$ and $\mathbf{H}(\mathbb{Q})$ defined respectively by $\Gamma_{\mathbf{G}, i} := g_i K g_i^{-1} \cap \mathbf{G}(\mathbb{Q})$ and $\Gamma_{\mathbf{H}, j} = h_j K_{\mathbf{H}} h_j^{-1} \cap \mathbf{H}(\mathbb{Q})$, where the families $\mathcal{C}_{\mathbf{G}} := \{g_1, \dots, g_{r_X}\} \subset \mathbf{G}(\mathbb{A}_f)$ and $\mathcal{C}_{\mathbf{H}} := \{h_1, \dots, h_{r_Y}\} \subset \mathbf{H}(\mathbb{A}_f)$ are coset representatives for the finite⁽⁸⁾ double quotients $\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f)/K$ and $\mathbf{H}(\mathbb{Q}) \backslash \mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}}$. The arithmetic quotient $\Gamma_{\mathbf{G}, i} \backslash X$ (resp. $\Gamma_{\mathbf{H}, j} \backslash Y$) injects in $\mathrm{Sh}_K(\mathbf{G}, X)$ (resp. $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$) via $x \mapsto [x, g_i]_K$ (resp. $y \mapsto [y, h_j]_{K_{\mathbf{H}}}$).

In the following, we shall restrict to compact subgroups K_V and K_W which are small in a certain sense, so that the manifolds $\mathrm{Sh}_K(\mathbf{G}, X)$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$ are "nice" enough. Namely, we will require that K_V and K_W be *neat*, following [41]:

Definition 1.1.1 (Neat arithmetic subgroups). *Let G/\mathbb{Q} be an algebraic group endowed with a faithful representation $\rho : G \hookrightarrow \mathrm{GL}_n$ for some $n \geq 1$. If $g = (g_p)_p \in \mathrm{GL}_n(\mathbb{A}_f)$, where p runs among the positive rational prime numbers, we denote by $\Lambda_p \subset \overline{\mathbb{Q}}_p^\times$ the subgroup generated by the eigenvalues of $g_p \in \mathrm{GL}_n(\mathbb{Q}_p)$. The element g is called neat if one has*

$$\bigcap_{p \text{ prime}} \left(\iota_p(\overline{\mathbb{Q}}^\times) \cap \Lambda_p \right)_{\mathrm{tors}} = \{1\}.$$

An element $g \in G(\mathbb{A}_f)$ is called neat if $\rho(g) \in \mathrm{GL}_n(\mathbb{A}_f)$ is neat, and one checks that this does not depend on the choice of ρ . Finally, a subgroup of K of $G(\mathbb{A}_f)$ is called neat if all its elements are neat.

(8). See [32], Lemma 5.12.

The neatness assumption will not be so restrictive in our case, as every compact open subgroup contains a neat subgroup of finite index (e.g., defined by imposing congruence conditions on the image $\rho(K) \subset \mathrm{GL}_n(\mathbb{A}_f)$). If $K_\star \subset \mathbf{G}_\star(\mathbb{A}_f)$ is neat, for all $\star \in \{V, W\}$, then for all $i \in \{1, \dots, r_X\}$ and $j \in \{1, \dots, r_Y\}$, the arithmetic subgroups $\Gamma_{\mathbf{G}, i} \subset \mathbf{G}(\mathbb{Q})$ and $\Gamma_{\mathbf{H}, j} \subset \mathbf{H}(\mathbb{Q})$ will be neat, therefore will act without torsion on X and Y respectively, which makes the quotients $\Gamma_{\mathbf{G}, i} \backslash X$ and $\Gamma_{\mathbf{H}, j} \backslash Y$ smooth hermitian manifolds.

By the work of Baily-Borel, Borel, Satake (among others), there exists unique structures of quasi-projective complex algebraic varieties $\mathrm{Sh}_K(\mathbf{G}, X)$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$ of respective dimensions $\dim \mathrm{Sh}_K(\mathbf{G}, X) = \dim_{\mathbb{C}} X = \dim_{\mathbb{C}} X_V + \dim_{\mathbb{C}} X_W = 3$, $\dim \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y) = \dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X_W = 1$, and such that

$$\mathrm{Sh}_K(\mathbf{G}, X)^{an} = \mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C}) \quad \text{and} \quad \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)^{an} = \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C}).$$

This motivates the notational choice for the complex manifolds $\mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C})$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C})$, which are indeed identified to the complex points of the algebraic varieties $\mathrm{Sh}_K(\mathbf{G}, X)$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$ respectively. The morphism of Shimura data $\Delta : (\mathbf{G}, X) \hookrightarrow (\mathbf{H}, Y)$ induces a complex analytic map

$$\Delta : \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C}) \hookrightarrow \mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C})$$

which corresponds - assuming that $K_{\mathbf{H}}$ is sufficiently small, e.g., when $K_{\mathbf{H}} = K \cap \mathbf{H}(\mathbb{A}_f)$ - to a closed immersion at the level of schemes, still denoted as $\Delta : \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y) \hookrightarrow \mathrm{Sh}_K(\mathbf{G}, X)$. We refer to ([32], Theorem 3.12, Theorem 3.14 and Theorem 5.15) for details.

The theory of canonical models developed by Deligne, Milne, Piatetski-Shapiro, Shimura (among others)⁽⁹⁾ enables us to significantly improve the preceding statements. For all neat compact open subgroups $K \subset \mathbf{G}(\mathbb{A}_f)$, $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_f)$, both varieties $\mathrm{Sh}_K(\mathbf{G}, X)$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$ turn out to be defined over $\overline{\mathbb{Q}}$, and admit canonical models defined over their respective reflex fields $E(\mathbf{G}, X)$ and $E(\mathbf{H}, Y)$, which happen (see § 1.1.5) to be both equal to E . These canonical models are smooth $\mathrm{Spec} E$ -schemes $M_K(\mathbf{G}, X)$ and $M_{K_{\mathbf{H}}}(\mathbf{H}, Y)$ such that $M_K(\mathbf{G}, X) \otimes_{\mathrm{Spec} E} \mathrm{Spec} \mathbb{C} \simeq \mathrm{Sh}_K(\mathbf{G}, X)$ and $M_{K_{\mathbf{H}}}(\mathbf{H}, Y) \otimes_{\mathrm{Spec} E} \mathrm{Spec} \mathbb{C} \simeq \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$, which are uniquely determined by the expression of the action of $\mathrm{Gal}(E^{ab}/E)$ on the sets of so-called *special points* of $\mathrm{Sh}_K(\mathbf{G}, X)$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$ respectively, as we shall see at §1.1.5.

According to ([32], Remark 13.8) the closed immersions $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y) \hookrightarrow \mathrm{Sh}_K(\mathbf{G}, X)$ are defined over $E(\mathbf{G}, X) \cdot E(\mathbf{H}, Y) = E$ as well. Similarly, for all neat $K' \subset K \subset \mathbf{G}(\mathbb{A}_f)$ and

(9). Our varieties of interest $\mathrm{Sh}_K(\mathbf{G}, X)$ and $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$ turn out to be of *abelian type*, for which class of Shimura varieties the existence (and uniqueness) of canonical models is in fact proved in [14].

$K'_{\mathbf{H}} \subset K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_f)$), the canonical maps induced by inclusions

$$\mathbf{G}(\mathbb{Q}) \backslash (X \times (\mathbf{G}(\mathbb{A}_f)/K')) \longrightarrow \mathbf{G}(\mathbb{Q}) \backslash (X \times (\mathbf{G}(\mathbb{A}_f)/K))$$

$$[x, g]_{K'} \longmapsto [x, g]_K$$

and

$$\mathbf{H}(\mathbb{Q}) \backslash (Y \times (\mathbf{H}(\mathbb{A}_f)/K'_{\mathbf{H}})) \longrightarrow \mathbf{H}(\mathbb{Q}) \backslash (Y \times (\mathbf{H}(\mathbb{A}_f)/K_{\mathbf{H}}))$$

$$[y, h]_{K'} \longmapsto [y, h]_K$$

are algebraic, and induce morphisms of schemes

$$\pi_{K'/K} : \mathrm{Sh}_{K'}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_K(\mathbf{G}, X) \quad \text{and} \quad \pi_{K'_{\mathbf{H}}/K_{\mathbf{H}}} : \mathrm{Sh}_{K'_{\mathbf{H}}}(\mathbf{H}, Y) \rightarrow \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y),$$

which are finite étale morphisms defined over $\mathrm{Spec} E$, of degree $\deg(\pi_{K'/K}) = \#(K/K')$. We may introduce the following towers of varieties:

Definition 1.1.2 (The Shimura varieties $\mathrm{Sh}(\mathbf{G}, X)$ and $\mathrm{Sh}(\mathbf{H}, Y)$). *The Shimura varieties $\mathrm{Sh}(\mathbf{G}, X)$ and $\mathrm{Sh}(\mathbf{H}, Y)$ are the projective systems of complex algebraic varieties:*

$$\mathrm{Sh}(\mathbf{G}, X) := \varprojlim_{K \subset \mathbf{G}(\mathbb{A}_f) \text{ neat}} \mathrm{Sh}_K(\mathbf{G}, X),$$

$$\mathrm{Sh}(\mathbf{H}, Y) := \varprojlim_{K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_f) \text{ neat}} \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y).$$

For all neat $K \subset \mathbf{G}(\mathbb{A}_f)$ and $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_f)$, we denote by $\pi_K : \mathrm{Sh}(\mathbf{G}, X) \rightarrow \mathrm{Sh}_K(\mathbf{G}, X)$ (resp. $\pi_{K_{\mathbf{H}}} : \mathrm{Sh}(\mathbf{H}, Y) \rightarrow \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$) the canonical projection map, and we refer to $\mathrm{Sh}_K(\mathbf{G}, X)$ as the *Shimura variety of level K* attached to the datum (\mathbf{G}, X) (and similarly for $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$). For all neat $K \subset \mathbf{G}(\mathbb{A}_f)$ and $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_f)$ and for all $g_0 \in \mathbf{G}(\mathbb{A}_f)$, $h_0 \in \mathbf{H}(\mathbb{A}_f)$, the maps

$$\mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C}) \xrightarrow{[\cdot g_0]} \mathrm{Sh}_{g_0^{-1}K g_0}(\mathbf{G}, X)(\mathbb{C})$$

$$[x, g]_K \longmapsto [x, g g_0]_{g_0^{-1}K g_0}$$

and

$$\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C}) \xrightarrow{[\cdot h_0]} \mathrm{Sh}_{h_0^{-1}K_{\mathbf{H}} h_0}(\mathbf{H}, Y)(\mathbb{C})$$

$$[y, h]_{K_{\mathbf{H}}} \longmapsto [y, h h_0]_{h_0^{-1}K_{\mathbf{H}} h_0}$$

are also algebraic and defined over $\mathrm{Spec} E$ ([32], Theorem 13.6). They induce actions of $\mathbf{G}(\mathbb{A}_f)$ and $\mathbf{H}(\mathbb{A}_f)$ on the respective towers $\mathrm{Sh}(\mathbf{G}, X)$ and $\mathrm{Sh}(\mathbf{H}, Y)$, by so-called *Hecke*

correspondences. If $K \subset \mathbf{G}(\mathbb{A}_f)$, and $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_f)$ are neat, the action of the Hecke correspondences $T(g_0)$ and $T(h_0)$ can be respectively described via the following diagrams:

$$\begin{array}{ccc} \mathrm{Sh}_{g_0 K g_0^{-1} \cap K}(\mathbf{G}, X) & \xrightarrow{[g_0]} & \mathrm{Sh}_{K \cap g_0^{-1} K g_0}(\mathbf{G}, X) \\ \downarrow \pi_{K, g_0} & & \downarrow \pi_{K, g_0^{-1}} \\ \mathrm{Sh}_K(\mathbf{G}, X) & \cdots \cdots \cdots & \mathrm{Sh}_K(\mathbf{G}, X) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Sh}_{h_0 K_{\mathbf{H}} h_0^{-1} \cap K_{\mathbf{H}}}(\mathbf{H}, Y) & \xrightarrow{[h_0]} & \mathrm{Sh}_{K_{\mathbf{H}} \cap h_0^{-1} K_{\mathbf{H}} h_0}(\mathbf{H}, Y) \\ \downarrow \pi_{K_{\mathbf{H}}, h_0} & & \downarrow \pi_{K_{\mathbf{H}}, h_0^{-1}} \\ \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y) & \cdots \cdots \cdots & \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y) \end{array}$$

We shall come back to these Hecke correspondences later on, at §3.1.3.1.

Remark 1.1.3 (A compactness criterion for $\mathrm{Sh}_{K_{\star}}(\mathbf{G}_{\star}, X_{\star})$). *Given a field k and a reductive group G over $\mathrm{Spec} k$, we define the k -rank of G as the rank of (any) maximal k -split torus inside G , i.e., the maximal integer $r := \mathrm{rk}_k G$ such that $T \simeq \mathbb{G}_{m, k}^r$ for some torus $T \subset G$.*

Let $n := a + b$, with a, b non-negative integers. The subgroup $\mathrm{SU}(a, b) \subset \mathrm{U}(a, b)$ - both seen as \mathbb{R} -reductive groups - consisting determinant 1 isometries of the \mathbb{C}/\mathbb{R} -hermitian space \mathbb{C}^n endowed with the canonical hermitian pairing of signature (a, b) , can be shown to have real rank equal to the dimension any maximal totally isotropic subspace of \mathbb{C}^n , which equals $\min(a, b)$ (see [50], §8.1). Accordingly, one has $\mathrm{rk}_{\mathbb{R}} \mathrm{SU}(2, 1) = 1$ and $\mathrm{rk}_{\mathbb{R}} \mathrm{SU}(3, 0) = 0$. The same argument works mutatis mutandis when the hermitian form is defined over \mathbb{Q} , i.e., when applied to the special unitary group attached to an E/\mathbb{Q} -hermitian space, with E a quadratic imaginary field. Recall that, for $\star \in \{V, W\}$, the derived group $\mathbf{G}_{\star}^{\mathrm{der}} \subset \mathbf{G}_{\star}$ is equal to the kernel $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SU}(\star)$ of the determinant map $\det : \mathbf{G}_{\star} \rightarrow \mathbf{T}^1$. In the case of $d = 1$, i.e., $F = \mathbb{Q}$, the above discussion together with Assumption 1.1.1 give:

$$\mathrm{rk}_{\mathbb{Q}} \mathbf{G}_{\star}^{\mathrm{der}} = \mathrm{rk}_{\mathbb{Q}} \mathrm{SU}(\star) = \mathrm{rk}_{\mathbb{R}} \mathrm{SU}(\star)_{\rho_1, \mathbb{R}} = \mathrm{rk}_{\mathbb{R}} \mathrm{SU}(\dim(\star) - 1, 1) = 1.$$

However, if $F \neq \mathbb{Q}$ then one gets $\mathrm{rk}_{\mathbb{Q}} \mathbf{G}_{\star}^{\mathrm{der}} = 0$. Indeed, if $T \subset \mathbf{G}_{\star}^{\mathrm{der}} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SU}(\star)$ was a non-trivial split torus then, by ([12], Proposition A.5.15) there would exist a (unique) non-trivial F -split torus $T' \subset \mathrm{SU}(\star)_F$ such that $T \subset \mathrm{Res}_{F/\mathbb{Q}} T'$. Accordingly, for all $i \in \{2, \dots, d\}$, the torus $T'_{\rho_i, \mathbb{R}}$ would be a non-trivial \mathbb{R} -split torus inside $\mathrm{SU}(\star)_{\rho_i, \mathbb{R}} \simeq \mathrm{SU}(3, 0)$, which would contradict $\mathrm{rk}_{\mathbb{R}} \mathrm{SU}(3, 0) = 0$. Therefore $\mathrm{rk}_{\mathbb{Q}} \mathbf{G}_{\star}^{\mathrm{der}} = 0$ if and only if $F \neq \mathbb{Q}$.

Let $K_\star \subset \mathbf{G}_\star(\mathbb{A}_f)$ be a neat compact open subgroup. Recall that the Shimura variety $\mathrm{Sh}_{K_\star}(\mathbf{G}_\star, X_\star)$ splits as a disjoint union of connected components

$$\mathrm{Sh}_{K_\star}(\mathbf{G}_\star, X_\star) = \bigsqcup_{i=1}^{r_\star} \Gamma_i \backslash X_\star,$$

with $\Gamma_i := \mathbf{G}_\star(\mathbb{Q}) \cap g_i K_\star g_i^{-1}$, and where $g_1, \dots, g_{r_\star} \in \mathbf{G}_\star(\mathbb{A}_f)$ is a family of representatives for the double quotient $\mathbf{G}_\star(\mathbb{Q}) \backslash \mathbf{G}_\star(\mathbb{A}_f) / K_\star$. If $i, j \in \{1, \dots, r_\star\}$ and if $g_{i,j} \in \mathbf{G}_\star(\mathbb{Q}) \cap g_j K_\star g_i^{-1}$, then the map $X_\star \rightarrow X_\star$, $x \mapsto g_{i,j} \cdot x$ induces an homeomorphism between $\Gamma_i \backslash X_\star$ and $\Gamma_j \backslash X_\star$, i.e., the connected components of $\mathrm{Sh}_{K_\star}(\mathbf{G}_\star, X_\star)$ are pairwise homeomorphic.

As $\mathbf{G}_\star^{\mathrm{der}}$ is simply connected - being isomorphic to $\mathrm{SL}_{\dim(\star)-1}$ over $\overline{\mathbb{Q}}$ - and as X_\star is connected, one obtains by ([32], Theorem 5.17) that the identity connected component $\mathbf{G}_\star(\mathbb{Q}) \cap K_\star \backslash X_\star$ of $\mathrm{Sh}_{K_\star}(\mathbf{G}_\star, X_\star)(\mathbb{C})$, is canonically isomorphic to some arithmetic quotient $\Gamma \backslash X_\star$, where $\Gamma \supset \mathbf{G}_\star^{\mathrm{der}}(\mathbb{Q}) \cap K_\star$ is an arithmetic subgroup of $\mathbf{G}_\star^{\mathrm{der}}(\mathbb{Q})$. On the other hand, a consequence of a theorem of Borel and Harrish-Chandra (simultaneously proven by Mostow-Tamagawa, see [6] Theorem 3) implies that the quotient $\Gamma \backslash X_\star$ is compact if and only if $\mathrm{rk}_{\mathbb{Q}} \mathbf{G}_\star^{\mathrm{der}} = 0$. Putting every together finally implies that, for $\star \in \{V, W\}$, the Shimura varieties $\mathrm{Sh}_{K_\star}(\mathbf{G}_\star, X_\star)$ are compact (for the complex-analytic topology), hence proper, if and only if $F \neq \mathbb{Q}$.

1.1.5 Reflex fields, Galois descent and reflex norm-map.

1.1.5.1 The reflex fields $E(\mathbf{G}_\star, X_\star)$, $\star \in \{V, W, \emptyset\}$.

Let $\mu : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{S}_{\mathbb{C}} \simeq \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}}$ be the co-character given on S -points (for all \mathbb{C} -algebra S) by $z \mapsto (z, 1)$, for all $z \in S^\times$. For all $\star \in \{V, W\}$, we set $\mu_{\star, \mathbb{C}} := h_{\star, \mathbb{C}} \circ \mu$. For any subfield k of \mathbb{C} , we let $\mathcal{C}(k) := \mathbf{G}_V(k) \backslash \mathrm{Hom}_k(\mathbb{G}_{m, k}, \mathbf{G}_{V, k})$ denote the set of $\mathbf{G}_V(k)$ -conjugacy classes of k -morphisms $\mathbb{G}_{m, k} \rightarrow \mathbf{G}_{V, k}$.

Let $c(X_\star) \in \mathcal{C}(\mathbb{C})$ be the conjugacy class of $\mu_{\star, \mathbb{C}}$. As \mathbf{G}_\star splits over E as $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}(\star_E)$, one has $(\mathbf{G}_\star)_{\overline{\mathbb{Q}}} \simeq \mathrm{GL}(\star)_{\overline{\mathbb{Q}}}^d$. According to ([32], discussion after Lemma 12.1), one obtains that $\mathcal{C}(\mathbb{C})$ is equal to $\mathcal{C}(\overline{\mathbb{Q}})$, hence $c(X_\star)$ can be seen as the $\mathbf{G}_\star(\overline{\mathbb{Q}})$ -conjugacy class of the map μ_\star , seen as a $\overline{\mathbb{Q}}$ -morphism between $\mathbb{G}_{m, \overline{\mathbb{Q}}}$ and $\mathbf{G}_{\star, \overline{\mathbb{Q}}}$ ⁽¹⁰⁾. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathcal{C}_\star(\overline{\mathbb{Q}})$ via the following action on $\overline{\mathbb{Q}}$ -points: if $f \in \mathrm{Hom}_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}}^\times, \mathbf{G}_\star(\overline{\mathbb{Q}}))$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then

$$(\sigma \cdot f)(r) := f(\sigma^{-1} \cdot r), \quad \forall r \in \overline{\mathbb{Q}}^\times.$$

Definition 1.1.3 (Reflex field). *The reflex field $E(\mathbf{G}_\star, X_\star)$ of the Shimura datum $(\mathbf{G}_\star, X_\star)$ is the subfield of $\overline{\mathbb{Q}}$ fixed by the stabilizer of $c(X_\star)$ (as a set) inside $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.*

(10). The conjugacy class c_\star does not depend on the choice of $h_V \in X_\star$, i.e., on the choice of torus \mathfrak{T}_B .

One shows that the reflex fields $E(\mathbf{G}_\star, X_\star)$, $\star \in \{V, W\}$ are both equal to E , unless when $\star = W$ and $F = \mathbb{Q}$, where $E(\mathbf{G}_W, X_W) = \mathbb{Q}$. We refer to ([7], VI.5) for a detailed computation. Accordingly, one obtains that $E(\mathbf{H}, Y) = E$ (or \mathbb{Q} if $F = \mathbb{Q}$) and $E(\mathbf{G}, X) = E(\mathbf{G}_V, X_V) \cdot E(\mathbf{G}_W, X_W) = E$.

1.1.5.2 Interlude on Galois descent.

We recall the following basic result of Galois descent for algebras over fields. We begin with a definition:

Definition 1.1.4. *Let L/K be a finite Galois extension of fields and let S be an L -algebra. A $\mathrm{Gal}(L/K)$ -structure on S is a morphism of groups $\mathrm{Gal}(L/K) \rightarrow \mathrm{Aut}_{\mathrm{rings}}(S)$ which induces the natural action of $\mathrm{Gal}(L/K)$ on the subring $L \hookrightarrow S$.*

Proposition 1.1.1. *Let L/K be a finite Galois extension of fields in characteristic 0. Let \mathcal{C} be the category of L -algebras endowed with a $\mathrm{Gal}(L/K)$ -structure - with morphisms being the $\mathrm{Gal}(L/K)$ -equivariant morphisms of L -algebras - and let \mathcal{D} be the category of K -algebras. The functor*

$$\begin{aligned} F : \mathcal{C} &\longrightarrow \mathcal{D} \\ S &\longmapsto S^{\mathrm{Gal}(L/K)} \end{aligned}$$

is an equivalence of categories whose inverse is given by

$$\begin{aligned} G : \mathcal{D} &\longrightarrow \mathcal{C} \\ R &\longmapsto L \otimes_K R \end{aligned}$$

Proof. For simplicity we will only treat the case $[L : K] = 2$ (the one we are interested in), the general case being just as conceptual. Let $c : z \mapsto \bar{z}$ be the non-trivial element of $\mathrm{Gal}(L/K)$, and let $\alpha \in L \setminus K$ be an element such that $L = K[\alpha]$, $\alpha^2 \in K$ (hence $\bar{\alpha} = -\alpha$). Let S be an L -algebra endowed with a $\mathrm{Gal}(L/K)$ -structure and set $R := F(S) = S^{\mathrm{Gal}(L/K)}$. Let us show that the morphism of rings

$$\begin{aligned} \phi : L \otimes_K R &\rightarrow S, \\ z \otimes r &\longmapsto zr \end{aligned}$$

is an isomorphism. Indeed, if r, r' are elements of R such that $r + \alpha r' = 0$ in S , then $c \cdot (r + \alpha r') = r + \bar{\alpha} r' = r - \alpha r' = 0$, i.e., $r = r' = 0$. If $t = \sum_i z_i \otimes r_i$ is any element of $L \otimes_K R$, with $z_i = x_i + \alpha y_i$ ($x_i, y_i \in R$), one has

$$\phi(t) = \sum_i z_i r_i = \sum_i x_i r_i + \alpha \sum_i y_i r_i \in R + \alpha R.$$

Thus $\phi(t) = 0$ if and only if $\sum_i x_i r_i = \sum_i y_i r_i = 0$, i.e., $t = 1 \otimes (\sum_i x_i r_i) + \alpha \otimes (\sum_i y_i r_i) = 0$, which shows the injectivity of ϕ . On the other hand if s is any element of S then one has

$s = r + \alpha r'$, where $r = \frac{s+c \cdot s}{2}$ and $r' = \frac{s-c \cdot s}{2\alpha}$ are both fixed by c . Thus $s = \phi(1 \otimes r + \alpha \otimes r')$ which shows the surjectivity of ϕ .

Let R be a K -algebra and set $S := G(R) = L \otimes_K R$, endowed with the natural $\mathrm{Gal}(L/K)$ -structure obtained by letting $\mathrm{Gal}(L/K)$ act on the leftmost component. The morphism of rings $\psi : R \rightarrow S$, $r \mapsto 1 \otimes r$ has image in $S^{\mathrm{Gal}(L/K)}$ by assumption. Let us show that ψ induces an isomorphism between these two rings. The injectivity of ψ is just the flatness of R as a K -module (K is a field). If $s = \sum_j z_j \otimes r_j$ is any element of S , with $z_j \in L$, $r_j \in R$, we set $\mathrm{Tr}(z_j) := z_j + \bar{z}_j \in K$, for all j . Then $\frac{s+c \cdot s}{2} = \sum_j \frac{z_j + \bar{z}_j}{2} \otimes r_j = \sum_j 1 \otimes (\frac{1}{2} \mathrm{Tr}(z_j) r_j) = \psi\left(\frac{1}{2} \sum_j \mathrm{Tr}(z_j) r_j\right) \in \psi(R)$. If s belongs to $S^{\mathrm{Gal}(L/K)}$ then $s = \frac{s+c \cdot s}{2}$, which shows that $\psi : R \rightarrow S^{\mathrm{Gal}(L/K)}$ is surjective. This concludes. \square

We go back to the case $L/K = E/F$. Let \star belong to $\{V, W, D\}$ and let $\dim(\star) \in \{1, 2, 3\}$ denote the corresponding dimension (as E -vector spaces). Let us choose an S -basis \mathcal{B} for the free S -module $\star \otimes_E S$ of rank $\dim(\star)$, which can also be seen as an $E \otimes_F S$ -basis for $\star \otimes_F S$, via $S \hookrightarrow E \otimes_F S$. Recall that, back in § 1.1.2.1 we identified the groups $\mathrm{GL}(\star \otimes_F S)$ and $\mathrm{GL}(\star \otimes_E S) \oplus \mathrm{GL}(\star \otimes_E S)$ via the map $g \mapsto (g_s, g_t)$, which admits the following description with respect to the basis \mathcal{B} : if $G \in \mathrm{GL}_{\dim(\star)}(E \otimes_F S)$ is the matrix attached to $g \in \mathrm{GL}(\star \otimes_F S)$, then $(g_s, g_t) \in \mathrm{GL}(\star \otimes_E S) \oplus \mathrm{GL}(\star \otimes_E S)$ is attached to the pair $\varphi(G) = (\varphi_s(G), \varphi_t(G)) =: (G_s, G_t) \in \mathrm{GL}_3(S) \times \mathrm{GL}_3(S)$, where φ was the map defined at § 1.1.2.1, induced on each coordinate by the isomorphism

$$\begin{aligned} \phi : E \otimes_F S &\xrightarrow{\sim} S \times S \\ e \otimes s &\mapsto (\bar{e}s, es) \end{aligned}$$

Recall that, if $H \in M_3(E \otimes_F S)$ denotes the matrix of the hermitian form $\langle \cdot, \cdot \rangle_{V \otimes_F S}$ with respect to \mathcal{B} , then the equalities (1.3) and (1.4) implied that the elements g of $\mathrm{U}(V)(S) \subset \mathrm{GL}(V \otimes_F S)$ correspond to those matrices $G \in \mathrm{GL}_{\dim(\star)}(E \otimes_F S)$ which satisfy ${}^t G_s \varphi_t(H) G_t = \varphi_t(H)$, i.e.,

$$G_s = {}^t(\varphi_t(H) (G_t)^{-1} \varphi_t(H)^{-1}) = \varphi_s(H)^{-1} {}^t(G_t)^{-1} \varphi_s(H). \quad (1.12)$$

Let us now assume that S is an E -algebra endowed with an action of the complex conjugation $c \in \mathrm{Gal}(E/F)$ - or equivalently, when $S = E \otimes_F R$ for some F -algebra R , by the above proposition - and let us restrict ourselves to S -bases \mathcal{B} of $\star \otimes_E S$ which are already E -bases of \star or, a bit more generally, to bases $\mathcal{B} = \{e_i\}_{i=1}^{\dim(\star)}$ of $\star_S := \star \otimes_F R$ such that $e_i = e'_i r_i$, for some E -basis $\{e'_i\}_{i=1}^{\dim(\star)}$ of \star , and some elements $r_i \in R^\times$ ⁽¹¹⁾. We fix one such \mathcal{B} . One may then define functorial algebraic $\mathrm{Gal}(E/F)$ -actions on the groups of S -points $\mathrm{U}(\star)(S) \subset \mathrm{GL}(\star \otimes_F S)$, in the following way: if $g \in \mathrm{GL}(\star \otimes_F S)$ corresponds in \mathcal{B} to the matrix $G \in \mathrm{GL}_{\dim(\star)}(E \otimes_F S) = \mathrm{GL}_{\dim(\star)}(E \otimes_F E \otimes_F R)$, we define $c \cdot g \in \mathrm{GL}(\star \otimes_F S)$ to be the

(11). This is in fact the case which appears in the proof of axiom SV2.

element attached to the matrix $G^c \in \mathrm{GL}_{\dim(\star)}(E \otimes_F S)$, obtained by applying coordinate-wise the involution ι_c of $E \otimes_F S$ induced by the action of c on S , namely:

$$\begin{aligned} \iota_c : E \otimes_F E \otimes_F R &\xrightarrow{\sim} E \otimes_F E \otimes_F R \\ e_1 \otimes e_2 \otimes r &\longmapsto e_1 \otimes \overline{e_2} \otimes r \end{aligned}$$

This definition does not depend on the chosen \mathcal{B} : indeed, choosing a different E -basis \mathcal{B}' would yield a transition matrix $P \in \mathrm{GL}_{\dim(\star)}(E) \subset \mathrm{GL}_{\dim(\star)}(E \otimes_F S)$ such that $g \in \mathrm{GL}(V \otimes_F S)$ has matrix $P G P^{-1} \in \mathrm{GL}_{\dim(\star)}(E \otimes_F S)$ with respect to \mathcal{B}' , hence $c \cdot g$ has matrix $\iota_c(P G P^{-1}) = \iota_c(P) \iota_c(G) \iota_c(P)^{-1} = P G^c P^{-1}$, as $P \in \mathrm{GL}_{\dim(\star)}(E)$, which makes the action of c consistent. The matrix $G^c \in \mathrm{GL}_{\dim(\star)}(E \otimes_F S)$ satisfies $\varphi(G^c) = (\varphi_t(G)^c, \varphi_s(G)^c)$, where $\varphi_t(G)^c$ (resp. $\varphi_s(G)^c$) denotes the image of $\varphi_t(G) \in \mathrm{GL}_{\dim(\star)}(S)$ (resp. $\varphi_s(G) \in \mathrm{GL}_{\dim(\star)}(S)$) under the action of c (acting coordinate-wise on $\mathrm{GL}_{\dim(\star)}(S)$). Consequently, the identification between $\mathrm{U}(V)(S)$ and $\mathrm{GL}(V_S)$ given by $g \mapsto g_t$ yields an equality:

$$\begin{aligned} (c \cdot g)_t = g_s^c &= (\varphi_s(H)^{-1} {}^t(g_t)^{-1} \varphi_s(H))^c \\ &= (\varphi_s(H)^{-1})^c ({}^t g_t^{-1})^c (\varphi_s(H))^c. \end{aligned} \tag{1.13}$$

As a useful application, let us make explicit the case $\star = D$, $\dim(\star) = 1$. In this case, if S is any E -algebra endowed with a $\mathrm{Gal}(E/F)$ -action, one may choose $\mathcal{B} = \{e_D \otimes 1\} \in V_S$, hence $H = (\langle e_D, e_D \rangle_{D \otimes_F S}) = (1) \in S$. If $z \in \mathrm{U}(D)(S) \simeq \mathrm{U}(1)_{E/F}(S)$ is identified with $z_t \in \mathrm{GL}_1(S) = S^\times$, one gets the equality:

$$(c \cdot z)_t = (z_t^{-1})^c. \tag{1.14}$$

1.1.5.3 Reflex map and canonical models.

Let us review some of the theory of canonical models. If $(\mathbf{G}, \mathcal{X})$ is a general Shimura datum with reflex field $E(\mathbf{G}, \mathcal{X})$, and if $x \in \mathcal{X}$ corresponds to the morphism $h_x : \mathbf{S} \rightarrow \mathbf{G}_{\mathbb{R}}$, then the conjugacy class $c(\mu_x) \subset \mathrm{Hom}_{\mathbb{Q}}(\mathbb{G}_{m, \mathbb{Q}}, \mathbb{G}_{\mathbb{Q}})$ of the co-character

$$\mu_x : \mathbb{G}_{m, \mathbb{C}} \xrightarrow{z \mapsto (z, 1)} \mathbf{S}_{\mathbb{C}} \xrightarrow{h_{x, \mathbb{C}}} \mathbf{G}_{\mathbb{C}}$$

does not depend on the choice of $x \in \mathcal{X}$ and is defined (as a set) over $E(\mathbf{G}, \mathcal{X})$. We give the definition of a *special point* x of \mathcal{X} , following ([32], Definition 12.5):

Definition 1.1.5 (Special points and special pairs). *A point $x \in \mathcal{X}$ is called special if there exists a torus $\mathbf{T} \subset \mathbf{G}$ (defined over \mathbb{Q}) such that $h_x : \mathbf{S} \rightarrow \mathbf{G}_{\mathbb{R}}$ factors through the inclusion $\mathbf{T}_{\mathbb{R}} \hookrightarrow \mathbf{G}_{\mathbb{R}}$ (or equivalently, such that x is fixed by the adjoint action of $\mathbf{T}(\mathbb{R})$). The pair (\mathbf{T}, x) is then called a special pair.*

We have already encountered examples of special pairs. Indeed recall that, by construction, the point $h_V \in X_V$ (resp. $h_W \in X_W$) defined at § 1.1.3 factors through the

torus $(\mathfrak{T}_{\mathcal{B}})_{\mathbb{R}}$ (resp. through $(\mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(E e_2) \times \mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(E e_3))_{\mathbb{R}}$), with $\mathcal{B} = \{e_1, e_2, e_3\}$ being the orthogonal E -basis of V defined at § 1.1.3. Accordingly, the point $h_{\star} \in X_{\star}$ is special, for $\star \in \{V, W\}$. We recall the definition of the norm map attached to a finite separable extension of fields:

Definition 1.1.6 (Norm map). *Let L/K be a finite separable extension of fields and let K^{sep} be a separable closure of K . Let \mathbf{G} be a commutative group scheme defined over $\mathrm{Spec} K$ and set $\mathbf{G}_L = \mathbf{G} \times_{\mathrm{Spec} K} \mathrm{Spec} L$. The norm map $N_{L/K} : \mathrm{Res}_{L/K} \mathbf{G}_L \rightarrow \mathbf{G}$, is defined for any K -algebra S as follows*

$$\begin{aligned} \mathrm{Res}_{L/K} \mathbf{G}(S) &:= \mathbf{G}(L \otimes_K S) \longrightarrow \mathbf{G}(S) \\ P &\longmapsto \prod_{\sigma \in \mathrm{Hom}_K(L, K^{\mathrm{sep}})} \sigma(P), \end{aligned}$$

where $\sigma(P) \in \mathbf{G}(\sigma(L) \otimes_K S) \subset \mathbf{G}(K^{\mathrm{sep}} \otimes_K S)$ is defined by $\mathrm{Spec}(\sigma(L) \otimes_K S) \xrightarrow{\sigma} \mathrm{Spec}(L \otimes_K S) \xrightarrow{P} \mathbf{G}$, for all $\sigma \in \mathrm{Hom}_K(L, K^{\mathrm{sep}})$. The right-hand side is $\mathrm{Aut}_K(K^{\mathrm{sep}})$ -invariant, hence belongs to $\mathbf{G}(K \otimes_K S) = \mathbf{G}(S)$.

One may now define the reflex map attached to a special pair (\mathbf{T}, x) :

Definition 1.1.7. *Let $(\mathbf{G}, \mathcal{X})$ be a Shimura datum and let $(\mathbf{T}, x) \subset (\mathbf{G}, \mathcal{X})$ be a special pair. Let $E(x) \supset E(\mathbf{G}, \mathcal{X})$ be the field of definition of the co-character $\mu_x : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{T}_{\mathbb{C}} \subset \mathbf{G}_{\mathbb{C}}$ defined above. We define the reflex map (or reflex norm-map) $r(\mathbf{T}, \mu_x)$ as follows:*

$$r(\mathbf{T}, \mu_x) : \mathrm{Res}_{E(x)/\mathbb{Q}} \mathbb{G}_{m, E(x)} \xrightarrow{\mathrm{Res}_{E(x)/\mathbb{Q}} \mu_x} \mathrm{Res}_{E(x)/\mathbb{Q}} \mathbf{T} \xrightarrow{N_{E(x)/\mathbb{Q}}} \mathbf{T} \quad (1.15)$$

The above reflex map induces, on adelic points, the following map:

$$r_{x, \mathbb{A}} : \mathbb{A}_{E(x)}^{\times} \xrightarrow{\mathrm{Res}_{E(x)/\mathbb{Q}} \mu_x(\mathbb{A})} \mathbf{T}(\mathbb{A}_{E(x)}) \xrightarrow{N_{E(x)/\mathbb{Q}}} \mathbf{T}(\mathbb{A}) \xrightarrow{\mathrm{proj}_f} \mathbf{T}(\mathbb{A}_f) \quad (1.16)$$

The adelic reflex map (1.16) characterizes uniquely the *canonical* models, via the following rule sometimes called *Shimura reciprocity law* ([32], Definition 12.8):

Definition 1.1.8. *Let $(\mathbf{G}, \mathcal{X})$ be a Shimura datum with reflex field $E(\mathbf{G}, \mathcal{X})$, and let $K \subset \mathbf{G}(\mathbb{A}_f)$ be a compact open subgroup. A model $M_K(\mathbf{G}, \mathcal{X})$ of the Shimura variety $\mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$, defined over $E(\mathbf{G}, \mathcal{X})$, is called a canonical model if, for any special pair $(\mathbf{T}, x) \subset (\mathbf{G}, \mathcal{X})$, for all $g \in \mathbf{G}(\mathbb{A}_f)$ and all $\sigma \in \mathrm{Gal}(E(x)^{ab}/E(x))$, the following are true:*

- (i) $[x, g]_K \in \mathrm{Sh}_K(\mathbf{G}, \mathcal{X})$ is defined over $E(x)^{ab}$,
- (ii) $\sigma \cdot [x, g]_K = [x, r_{x, \mathbb{A}}(s_{\sigma}) g]_K$, where $s_{\sigma} \in \mathbb{A}_{E(x)}^{\times}$ is any idèle such that $\mathrm{Art}_{E(x)}^{(12)}(s_{\sigma}) = \sigma$.

(12). $\mathrm{Art}_{E(x)} : \mathbb{A}_{E(x)}^{\times} \rightarrow \mathrm{Gal}(E(x)^{ab}/E)$ is the Artin map attached to the number field $E(x)$, which we will study in details at § 1.3.1. As we shall see at § 1.3.1, one might also directly work with finite idèles and choose some $s_{\sigma} \in \mathbb{A}_{E(x), f}^{\times}$.

1.2 The set of special cycles on $\mathrm{Sh}_K(\mathbf{G}, X)$.

1.2.1 The normalizer of \mathbf{H} in \mathbf{G} .

Recall that we denote by $\mathbf{Z}_{\mathbf{G}}$ the center of \mathbf{G} , which satisfies $\mathbf{Z}_{\mathbf{G}} = \mathbf{Z}_{\mathbf{G}_V} \times \mathbf{Z}_{\mathbf{G}_W}$. Both groups $\mathbf{Z}_{\mathbf{G}_V}$ and $\mathbf{Z}_{\mathbf{G}_W}$ are isomorphic to the torus $\mathbf{T}^1 := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(1)_{E/F}$, via the map $\lambda \mapsto \lambda \cdot \mathbf{1}$. Let $\mathrm{N}_{\mathbf{G}}(\mathbf{H})$ be the normalizer of \mathbf{H} into \mathbf{G} : this is a \mathbb{Q} -algebraic subgroup of \mathbf{G} whose points are given, for any \mathbb{Q} -algebra R , by

$$\mathrm{N}_{\mathbf{G}}(\mathbf{H})(R) := \{g \in \mathbf{G}(R); g^{-1}\mathbf{H}_R g = \mathbf{H}_R\}.$$

The normalizer $\mathrm{N}_{\mathbf{G}}(\mathbf{H})$ is the only algebraic \mathbb{Q} -subgroup of \mathbf{G} such that $\mathrm{N}_{\mathbf{G}}(\mathbf{H})(\overline{\mathbb{Q}}) = \mathrm{N}_{\mathbf{G}(\overline{\mathbb{Q}})}(\mathbf{H}(\overline{\mathbb{Q}}))$. For any field extension k/\mathbb{Q} , one has $\mathrm{N}_{\mathbf{G}}(\mathbf{H})(k) = \mathrm{N}_{\mathbf{G}(k)}(\mathbf{H}(k))$, which follows from ([33], Proposition 1.84), using the fact that \mathbf{H} is a connected reductive group, hence $\mathbf{H}(\mathbb{Q})$ is dense in \mathbf{H} for the Zariski topology. The normalizer of \mathbf{H} in \mathbf{G} admits the following description:

Lemma 1.2.1. *One has $\mathrm{N}_{\mathbf{G}}(\mathbf{H}) = \mathbf{H} \cdot (\mathbf{Z}_{\mathbf{G}_V} \cdot \iota(\mathbf{Z}_{\mathbf{G}_W}) \times 1) = \mathbf{H} \cdot \mathbf{Z}_{\mathbf{G}}$.*

Proof.

Let us start by the second equality. One has, for any $g_W \in \mathbf{G}_W$, $\mu \in \mathbf{Z}_{\mathbf{G}_V}$ and $\lambda \in \mathbf{Z}_{\mathbf{G}_W}$,

$$(\iota(g_W), g_W) \cdot (\mu \cdot \iota(\lambda), 1) = (\iota(g_W), g_W) \cdot (\iota(\lambda), \lambda) \cdot (\mu, \lambda^{-1}) = (\iota(g_W \lambda), g_W \lambda) \cdot (\mu, \lambda^{-1}) \in \mathbf{H} \cdot \mathbf{Z}_{\mathbf{G}},$$

and

$$(\iota(g_W), g_W) \cdot (\mu, \lambda) = (\iota(g_W \lambda), g_W \lambda) \cdot (\mu \cdot \iota(\lambda^{-1}), 1) \in \mathbf{H} \cdot (\mathbf{Z}_{\mathbf{G}_V} \cdot \iota(\mathbf{Z}_{\mathbf{G}_W}) \times 1),$$

hence $\mathbf{H} \cdot (\mathbf{Z}_{\mathbf{G}_V} \cdot \iota(\mathbf{Z}_{\mathbf{G}_W}) \times 1) = \mathbf{H} \cdot \mathbf{Z}_{\mathbf{G}}$.

The inclusion $\mathbf{H} \cdot \mathbf{Z}_{\mathbf{G}} \subset \mathrm{N}_{\mathbf{G}}(\mathbf{H})$ is immediate. On the other hand, let k be a \mathbb{Q} -algebra and set $V_k := V \otimes_{\mathbb{Q}} k$, $W_k := W \otimes_{\mathbb{Q}} k$ and $D_k := D \otimes_{\mathbb{Q}} k$. By the Zariski density argument above, one may assume that k is a field without loss of generality. Accordingly, one has $W_k = D_k^{\perp}$. Let $g = (g_1, g_2) \in \mathbf{G}_V(k) \times \mathbf{G}_W(k) = \mathbf{G}(k)$. If $g \in \mathrm{N}_{\mathbf{G}}(\mathbf{H})(k)$ and if $h = \Delta(g_W) \in \mathbf{H}(k)$, with $g_W \in \mathbf{G}_W(k)$, then $ghg^{-1} \in \mathbf{H}(k)$ fixes the line $D_k \oplus 0 \subset V_k \oplus W_k$ point-wise, i.e., the group $\mathbf{G}_W(k) \simeq \iota(\mathbf{G}_W)(k)$ fixes the line $g_1^{-1} \cdot D_k \subset V_k$ point-wise. The latter implies that $g_1^{-1} \cdot D_k = D_k$, as the only line in V_k being fixed point-wise by $\mathbf{G}_W(k)$ is D_k itself⁽¹³⁾. By assumption on $g \in \mathrm{N}_{\mathbf{G}(k)}(\mathbf{H}(k))$ then, for all $g_W \in \mathbf{G}_W(k)$, $h := \Delta(g_W) \in \mathbf{H}(k)$, there exists some $g'_W \in \mathbf{G}_W(k)$ such that $ghg^{-1} \stackrel{\text{def}}{=} (g_1 \iota(g_W) g_1^{-1}, g_2 g_W g_2^{-1}) = (\iota(g'_W), g'_W)$. This gives $\iota(g_2 g_W g_2^{-1}) = g_1 \iota(g_W) g_1^{-1}$, hence

$$(\iota(g_2)^{-1} g_1) \cdot \iota(g_W) \cdot (\iota(g_2)^{-1} g_1)^{-1} = \iota(g_W),$$

(13). Indeed, let $v \in V_k$ and write $v = v_W \oplus v_D \in W_k \perp D_k$. If $g_W \in \mathbf{G}_W(k)$, one has $\iota(g_W) \cdot v = g_W \cdot v_W \oplus v_D$, therefore the equality $v = \iota(g_W) \cdot v$, for all $g_W \in \mathbf{G}_W(k)$, implies $v_W = 0$ (take $g_W = -\mathbf{1}_W \in \mathbf{G}_W$), hence $v \in D_k$.

for all $g_W \in \mathbf{G}_W(k)$. Accordingly, $\iota(g_2)^{-1}g_1 \in \mathbf{G}_V(k)$ lies in the centralizer of $\mathbf{G}_W(k)$. As g_1 stabilizes D_k as a set, then the same is also true for $\iota(g_2)^{-1}g_1$, for $\iota(g_2)$ acts trivially on D_k . Being an isometry, $\iota(g_2)^{-1}g_1$ also stabilizes $D_k^\perp = W_k$. If $\lambda \in \mathbf{T}^1(k)$ is the scalar by which $\iota(g_2)^{-1}g_1$ acts on D_k , then

$$\iota(g_2)^{-1}g_1 = (\lambda \cdot \mathbf{1}_V) \iota \left((\lambda^{-1} \cdot \mathbf{1}_W) (\iota(g_2)^{-1}g_1)|_{W_k} \right),$$

and the above discussion implies that the element $(\iota(g_2)^{-1}g_1)|_{W_k}$ of $\mathbf{G}_W(k)$ lies in fact in $\mathbf{Z}_{\mathbf{G}_W}(k)$. Therefore $\iota(g_2)^{-1}g_1 \in \mathbf{Z}_{\mathbf{G}_V} \cdot \iota(\mathbf{Z}_{\mathbf{G}_W})(k)$, i.e., $g = (\iota(g_2), g_2) \cdot (\iota(g_2)^{-1}g_1, 1) \in \mathbf{H}(k) \cdot (\mathbf{Z}_{\mathbf{G}_V} \cdot \iota(\mathbf{Z}_{\mathbf{G}_W}) \times 1)(k)$, which finishes the proof. \square

1.2.2 The set $\mathcal{Z}_K(\mathbf{G}, \mathbf{H})$ of \mathbf{H} -special cycles.

We may now introduce a collection of *special* algebraic cycles on the 3-dimensional Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$. From now on, and unless the contrary is explicitly mentioned, the compact open subgroups will always be assumed to be neat. Fix one such $K \subset \mathbf{G}(\mathbb{A}_f)$.

Definition 1.2.1 (\mathbf{H} -special cycle of $\mathrm{Sh}_K(\mathbf{G}, X)$). *A closed, one-dimensional, irreducible subvariety $\mathcal{Z} \subset \mathrm{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ is called a \mathbf{H} -special cycle of level K (or simply a special cycle, when both \mathbf{H} and K are understood) of $\mathrm{Sh}_K(\mathbf{G}, X)$ if \mathcal{Z} is an irreducible component of the image of the map:*

$$\mathrm{Sh}(\mathbf{H}, Y) \hookrightarrow \mathrm{Sh}(\mathbf{G}, X) \xrightarrow{[g]} \mathrm{Sh}(\mathbf{G}, X) \xrightarrow{\pi_K} \mathrm{Sh}_K(\mathbf{G}, X),$$

for some $g \in \mathbf{G}(\mathbb{A}_f)$.

According to ([35], Remark 2.6), as $Y \simeq \mathbb{B}_1$ is a connected hermitian symmetric domain which satisfies the characterization (1.7), there is an equivalence between the preceding description - which Moonen calls *algebraic cycle of Shimura type* - and the following one, which Moonen refers to as *cycles of Hodge type* (cf [35], Definition 2.5): a closed irreducible subvariety $\mathcal{Z} \subset \mathrm{Sh}_K(\mathbf{G}, X)$ is a \mathbf{H} -special cycle of level K if and only if \mathcal{Z} is the image $[Y \times gK] \subset \mathrm{Sh}_K(\mathbf{G}, X)$ of the product $Y \times gK \subset X \times \mathbf{G}(\mathbb{A}_f)$ in the double quotient $\mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C}) = \mathbf{G}(\mathbb{Q}) \backslash (X \times (\mathbf{G}(\mathbb{A}_f)/K))$, i.e.,

$$\mathcal{Z} = \{[y, g]_K; y \in Y\} \subset \mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C}). \quad (1.17)$$

For all $g \in \mathbf{G}(\mathbb{A}_f)$, we shall denote by $\mathcal{Z}_K(g)$ the one-dimensional special cycle defined by $[Y \times gK] \subset \mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C})$. The cycles $\mathcal{Z}_K(g)$ can be seen as the right $\mathbf{G}(\mathbb{A}_f)$ -translates of the image in $\mathrm{Sh}_K(\mathbf{G}, X)$ of a distinguished connected component of $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$, such as described below:

Lemma 1.2.2. *The special cycle $\mathcal{Z}_K(g) \subset \mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C})$ is the image of the connected component $(\mathbf{H}(\mathbb{Q}) \cap K_{g,\mathbf{H}}) \backslash Y$ of $\mathrm{Sh}_{K_{g,\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C})$ under the finite morphisms:*

$$\mathrm{Sh}_{K_{g,\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C}) \hookrightarrow \mathrm{Sh}_{gKg^{-1}}(\mathbf{G}, X)(\mathbb{C}) \xrightarrow{[\cdot g]} \mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C}).$$

with $K_{g,\mathbf{H}} := \mathbf{H}(\mathbb{A}_f) \cap gKg^{-1} \subset \mathbf{H}(\mathbb{A}_f)$.

Proof. Recall that the decomposition

$$\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C}) = \bigsqcup_{j=1}^{r_Y} \Gamma_{\mathbf{H},j} \backslash Y$$

into a disjoint union of connected components is induced by the map $[y, h]_{K_{\mathbf{H}}} \mapsto y \bmod \Gamma_{\mathbf{H},j}$, where $j \in \{1, \dots, r_Y\}$ is such that $h \in \mathbf{H}(\mathbb{Q})h_jK_{\mathbf{H}}$. One may assume without loss of generality that $\Gamma_{\mathbf{H},1} = \mathbf{H}(\mathbb{Q}) \cap K_{g,\mathbf{H}}$, i.e., that $h_1 \in \mathbf{H}(\mathbb{Q})K_{\mathbf{H}}$: accordingly, the connected component $(\mathbf{H}(\mathbb{Q}) \cap K_{g,\mathbf{H}}) \backslash Y$ corresponds to the double coset $\mathbf{H}(\mathbb{Q}) \cdot (Y \times K_{\mathbf{H}}) \subset \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C})$. Let $y \in Y$ and $h \in K_{g,\mathbf{H}}$: by definition, one may find some $k \in K$ such that $gkg^{-1} = h \in \mathbf{H}(\mathbb{A}_f)$. The element $[y, h]_{K_{g,\mathbf{H}}}$ lies in $\mathbf{H}(\mathbb{Q})(Y \times K_{g,\mathbf{H}})$, and one has $[\cdot g]([y, h]_{gKg^{-1}}) = [y, hg]_K = [y, gk]_K = [y, g]_K \in \mathcal{Z}_K(g)$. Conversely, if $[y, g]_K$ is an element $\mathcal{Z}_K(g)$, with $y \in Y$, $[y, g]_K = [y, gk]_K$ for any $k \in K \cap g^{-1}\mathbf{H}(\mathbb{A}_f)g \subset K$, thus $[y, g]_K = [\cdot g]([y, gkg^{-1}]_{gKg^{-1}})$ is in the image of the map, for $gkg^{-1} \in K_{g,\mathbf{H}}$ \square

Notice that, as the above maps are finite morphisms defined over $\mathrm{Spec} E$, the special cycles $\mathcal{Z}_K(g)$ are defined over finite extensions of E , for all $g \in \mathbf{G}(\mathbb{A}_f)$. As we shall see later, the *minimal* field of definition of $\mathcal{Z}_K(g)$ - seen as a closed subvariety of $\mathrm{Sh}_K(\mathbf{G}, X)$ - relates to the minimal field of definition of the corresponding connected component $\mathbf{H}(\mathbb{Q}) \cap K_{g,\mathbf{H}} \backslash Y \subset \mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$. The latter is a finite abelian extension of E characterized by the canonical model $M_{K_{\mathbf{H}}}(\mathbf{H}, Y)$ of $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$ and the so-called *Shimura reciprocity* law.

Definition 1.2.2. *We denote by $\mathcal{Z}_K(\mathbf{G}, \mathbf{H}) := \{\mathcal{Z}_K(g); g \in \mathbf{G}(\mathbb{A}_f)\}$ the set of \mathbf{H} -special cycles of level K .*

As $h_{\mathbb{Q}} \cdot Y = Y$ for all $h_{\mathbb{Q}} \in \mathbf{H}(\mathbb{Q})$, one has for all $k \in K$ that $\mathcal{Z}_K(h_{\mathbb{Q}}gk) = \{[y, h_{\mathbb{Q}}gk]_K; y \in Y\} = \{[h_{\mathbb{Q}}^{-1}y, g]_K; y \in Y\} = \{[y', g]_K; y' \in Y\} = \mathcal{Z}_K(g)$. Accordingly, the map

$$\begin{aligned} \mathcal{Z}_K(\bullet) : \mathbf{G}(\mathbb{A}_f) &\rightarrow \mathcal{Z}_K(\mathbf{G}, \mathbf{H}) \\ g &\mapsto \mathcal{Z}_K(g) \end{aligned}$$

factors through the map $\mathbf{G}(\mathbb{A}_f) \rightarrow \mathbf{H}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f) / K$. One can be a bit more precise and give the following description, due to Jetchev ([27], Lemma 2.3, (ii)):

Lemma 1.2.3. *The map $\mathcal{Z}_K(\bullet) : \mathbf{G}(\mathbb{A}_f) \rightarrow \mathcal{Z}_K(\mathbf{G}, \mathbf{H})$ induces a bijection*

$$\mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f) / K \simeq \mathcal{Z}_K(\mathbf{G}, \mathbf{H})$$

Proof.

The proof decomposes in the following two steps:

- 1. For all $g, g' \in \mathbf{G}(\mathbb{A}_f)$, one has $\mathcal{Z}_K(g) = \mathcal{Z}_K(g')$ if and only if $g' \in \mathrm{Stab}_{\mathbf{G}(\mathbb{Q})}(Y)gK$. Indeed, if $g' = g_{\mathbb{Q}}gk$ with $k \in K$ and $g_{\mathbb{Q}} \in \mathrm{Stab}_{\mathbf{G}(\mathbb{Q})}(Y) \subset \mathbf{G}(\mathbb{Q})$ then $[y, g]_K = [g_{\mathbb{Q}} \cdot y, g_{\mathbb{Q}}gk]_K = [g_{\mathbb{Q}} \cdot y, g']_K \in \mathcal{Z}_K(g')$, for all $y \in Y$, and $[y', g']_K = [g_{\mathbb{Q}}^{-1} \cdot y', g_{\mathbb{Q}}^{-1}g'k^{-1}]_K = [g_{\mathbb{Q}}^{-1} \cdot y', g]_K \in \mathcal{Z}_K(g)$, for all $y' \in Y$, thus $\mathcal{Z}_K(g) = \mathcal{Z}_K(g')$. Conversely, assume that we have an inclusion $\mathcal{Z}_K(g) \subset \mathcal{Z}_K(g')$ as subsets of $\mathrm{Sh}_K(\mathbf{G}, X)(\mathbb{C})$: it is equivalent to asking that for all $y \in Y$ there is some $y' \in Y$ such that $[y, g]_K = [y', g']_K$, i.e., that there exists $g_{\mathbb{Q}} \in \mathbf{G}(\mathbb{Q})$ and $k \in K$ such that $y = g_{\mathbb{Q}} \cdot y'$ and $g = g_{\mathbb{Q}}g'k$. This rewrites as $Y = \bigcup_{g_{\mathbb{Q}} \in \mathbf{G}(\mathbb{Q}) \cap gK(g')^{-1}} (g_{\mathbb{Q}}Y \cap Y)$. Baire's category theorem implies the existence of some non-empty open subset U of Y and some $g_{\mathbb{Q}} \in \mathbf{G}(\mathbb{Q}) \cap gK(g')^{-1}$ such that $U \subset g_{\mathbb{Q}}Y \cap Y$. Using the fact that the Riemannian manifolds Y and $g_{\mathbb{Q}}Y$ are connected totally geodesic submanifolds of X , one obtains (see [27], Lemma 2.3, (ii) for details) that $g_{\mathbb{Q}}Y \cap Y = Y$, hence $Y \subset g_{\mathbb{Q}}Y$. As $g_{\mathbb{Q}}^{-1} \cdot U \subset Y$ is an open subset, one gets that $U = g_{\mathbb{Q}} \cdot (g_{\mathbb{Q}}^{-1} \cdot U)$ is also an open subset of $g_{\mathbb{Q}} \cdot Y$ and the same argument now implies $g_{\mathbb{Q}} \cdot Y \subset Y$ i.e., $g_{\mathbb{Q}} \in \mathrm{Stab}_{\mathbf{G}(\mathbb{Q})}(Y) \cap gK(g')^{-1}$. Therefore, one has $g' \in g_{\mathbb{Q}}^{-1}gK \subset \mathrm{Stab}_{\mathbf{G}(\mathbb{Q})}(Y)gK$.

- 2. $\mathrm{Stab}_{\mathbf{G}(\mathbb{Q})}(Y) = \mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q})$.

Recall that the hermitian symmetric domain $Y \subset X$ is defined as $\Delta_{\mathrm{herm}}(X_W)$, with $\Delta_{\mathrm{herm}} = (\iota_{\mathrm{herm}}, \mathrm{Id}) : X_W \hookrightarrow X_V \times X_W = X$. Let $g = (g_1, g_2) \in \mathbf{G}_V(\mathbb{Q}) \times \mathbf{G}_W(\mathbb{Q}) = \mathbf{G}(\mathbb{Q})$. If $\mathcal{D} \in X_W$ corresponds to a negative definite line of W_{ρ_1} (see the discussion in Remark 1.1.2), then one may find another negative definite line $\mathcal{D}' \in X_W$ such that $(g_1, g_2) \cdot (\iota_{\mathrm{herm}}(\mathcal{D}), \mathcal{D}) = (\iota_{\mathrm{herm}}(\mathcal{D}'), \mathcal{D}')$ if and only if $\mathcal{D}' = g_2 \cdot \mathcal{D}$ and $\iota_{\mathrm{herm}}(\mathcal{D}') = g_1 \cdot \iota_{\mathrm{herm}}(\mathcal{D})$. The latter is equivalent to asking that the element $\iota(g_2)^{-1}g_1 \in \mathbf{G}_V(\mathbb{Q})$ fixes $\iota(\mathcal{D})$, for all $\mathcal{D} \in X_W$, i.e., fixes every negative definite line of W_{ρ_1} . One obtains that $\iota(g_2)^{-1}g_1 \in \mathbf{G}_V(\mathbb{Q})$ must act by homothety on W , which implies that $\iota(g_2)^{-1}g_1$ also stabilizes $W^{\perp} = D$ (as a set), hence $\iota(g_2)^{-1}g_1$ lies in $\iota(\mathbf{Z}_{\mathbf{G}_W})\mathbf{Z}_{\mathbf{G}_V}$. Accordingly, $(g_1, g_2) = (\iota(g_2), g_2)(\iota(g_2)^{-1}g_1, 1) \in \mathbf{H}(\mathbb{Q}) \cdot (\iota(\mathbf{Z}_{\mathbf{G}_W})\mathbf{Z}_{\mathbf{G}_V} \times 1) = \mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q})$, the last equality coming from Lemma 1.2.1.

1.2.3 Smooth integral models, and a choice for K .

We start this paragraph by recalling the notion of an hyperspecial subgroup over a non-archimedean local field.

Definition 1.2.3 (Hyperspecial maximal compact subgroup). *Let G be a connected reductive group over a local non-archimedean field k , whose ring of integers \mathcal{O} has residue field \mathbb{F} . A subgroup K of $G(k)$ is called a hyperspecial maximal compact subgroup if the following conditions hold:*

- (i) K is a maximal compact subgroup of $G(k)$,

(ii) There exists a group scheme \mathcal{G} defined over \mathcal{O} such that $\mathcal{G} \times_{\mathrm{Spec} \mathcal{O}} \mathrm{Spec} k \simeq G$, $\mathcal{G}(\mathcal{O}) = K$ and such that $\mathcal{G}_{\mathbb{F}}$ is a connected reductive group scheme over \mathbb{F} .

As we shall see below, hyperspecial maximal compact subgroups of $K_{\star} \subset \mathbf{G}_{\star}(F_v)$, for $v \in I_{F,f}$, arise as stabilizers of self-dual local lattices in \star_v , for all $\star \in \{V, W\}$. First, we need to introduce the notion of duality for hermitian (global or local) lattices:

Definition 1.2.4 (Dual lattice). *Let $v \in I_{F,v}$ be a finite place of F , set E_v to be the quadratic étale F_v -algebra $E \otimes_F F_v$, with ring of integers $\mathcal{O}_{E_v} := \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}$. Let $m \geq 1$ be a positive integer and let $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}})$ be either a m -dimensional non-degenerate E/F -hermitian space, or a free E_v -module of rank m endowed with a non-degenerate E_v/F_v -hermitian pairing (again, with respect to the involution $c \in \mathrm{Gal}(E/F)$ acting on E_v). Let $L \subset \mathcal{V}$ be a global (resp. local) lattice, by which we mean a free \mathcal{O}_E -submodule (resp. a free \mathcal{O}_{E_v} -submodule) of maximal rank m .*

— If L is global, we define its \vee -dual (or simply dual) lattice by setting:

$$L^{\vee} := \{v \in \mathcal{V}; \langle v, L \rangle_{\mathcal{V}} \subset \mathcal{O}_E\}. \quad (1.18)$$

— Similarly, if L is local we define its dual lattice by setting

$$L^{\vee} := \{v \in \mathcal{V}; \langle v, L \rangle_{\mathcal{V}} \subset \mathcal{O}_{E_v}\} \quad (1.19)$$

— We shall say that L is self-dual if $L = L^{\vee}$. In the local case, if $\varpi \in F_v$ is any uniformizer, then a local \mathcal{O}_{E_v} -lattice L is said to be almost self-dual if one has inclusions

$$\varpi L^{\vee} \subset L \subset L^{\vee},$$

and to be strictly almost self-dual if the inclusion $L \subset L^{\vee}$ is strict.

Remark 1.2.1. — Let v be a finite place of F and let \mathcal{V} be as in the above definition. In this setting, let L be either a global or local lattice and let the symbol $? \in \{\emptyset, v\}$ encode whether L is global ($? = \emptyset$) or local ($? = v$). Let $\mathcal{B} = \{e_1, \dots, e_m\}$ be an $\mathcal{O}_{E_?}$ -basis of L . The non-degeneracy hypothesis on \mathcal{V} implies that the map

$$\begin{aligned} \psi : \mathcal{V} &\longrightarrow \mathcal{V}^{\vee} := \mathrm{Hom}_{E_?}(\mathcal{V}, E_?) \\ v &\longmapsto (w \mapsto \langle v, w \rangle_{\mathcal{V}}) \end{aligned}$$

is an injective $E_?$ -linear (hence $F_?$ -linear) morphism between free $E_?$ -modules of rank m (hence $2m$ -dimensional $F_?$ -vector spaces), i.e., ψ is an isomorphism between \mathcal{V} and \mathcal{V}^{\vee} . Accordingly, the family $\psi(\mathcal{B}) = \{\psi(e_1), \dots, \psi(e_m)\}$ forms an $E_?$ -basis of \mathcal{V}^{\vee} and the dual lattice L^{\vee} is generated by the ψ -dual basis $\psi(\mathcal{B})^{\vee} := \{\psi(e_1)^{\vee}, \dots, \psi(e_m)^{\vee}\}$, which is characterized by the equality

$$\psi(e_i)(\psi(e_j)^{\vee}) := \langle e_i, \psi(e_j)^{\vee} \rangle_{\mathcal{V}} = \delta_{i,j}, \quad \forall i, j \in \{1, \dots, m\},$$

where $\delta_{i,j}$ is the Kronecker symbol. The above characterization implies that one has $e_i = \psi(\psi(e_i)^{\vee})^{\vee}$ for all $i \in \{1, \dots, m\}$, hence $(L^{\vee})^{\vee} = L$ for all global or local lattice L .

- The global and local dualities are compatible in the following sense: if V is a m -dimensional non-degenerate hermitian E/F -space and if $L \subset V$ is a global lattice then, for all $v \in I_{F,v}$, the above considerations imply that the corresponding local \mathcal{O}_{E_v} -lattices $L_v := L \otimes_{\mathcal{O}_E} \mathcal{O}_{E_v}$ and $(L^\vee)_v := L^\vee \otimes_{\mathcal{O}_E} \mathcal{O}_{E_v}$ satisfy

$$(L_v)^\vee = (L^\vee)_v,$$

which we will denote by L_v^\vee as there is no ambiguity.

Let us go back to our case $m \in \{2, 3\}$, $\mathcal{V} \in \{V, W\}$. We fix an \mathcal{O}_E -lattice $L_W \subset W$ such that $L_W \subset L_W^\vee$ ⁽¹⁴⁾. We call L_D the self-dual lattice \mathcal{O}_{Ee_D} , where $e_D \in D$ was defined immediately after Assumption 1.1.1, and we set $L_V := L_W \oplus L_D \subset V$, so that $L_V \subset L_V^\vee$. There are also adelic versions of L_W and L_V , given by:

$$\widehat{L}_V := L_V \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \subset V \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}} = V \otimes_E \mathbb{A}_{E,f},$$

$$\widehat{L}_W := L_W \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \subset W \otimes_{\mathbb{Q}} \widehat{\mathbb{Q}} = W \otimes_E \mathbb{A}_{E,f},$$

with $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ being the profinite completion of \mathbb{Z} and $\widehat{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \simeq \mathbb{A}_f$. Both \widehat{L}_V and \widehat{L}_W are free $\widehat{\mathcal{O}}_E$ -modules, with $\widehat{\mathcal{O}}_E = \mathcal{O}_E \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \prod_{v \nmid \infty} \mathcal{O}_{E_v}$, and will be also referred to as $\widehat{\mathcal{O}}_E$ -lattices.

By construction, the lattices L_\star satisfy $L_\star \subset L_\star^\vee$, for $\star \in \{V, W\}$. As $L_D^\vee = L_D$, one has $L_V^\vee = L_W^\vee \oplus L_D$, hence $[L_V^\vee : L_V] = [L_W^\vee : L_W]$. We set $N := [L_V^\vee : L_V]$, and we let S^1 be the set of finite places of F which divide $N\mathcal{O}_F$. From the inclusion $L_\star^\vee \subset N^{-1}L_\star$, one deduces that $(N^{-1}L_\star)^\vee = NL_\star^\vee \subset L_\star$, hence $NL_{\star,v}^\vee \subset L_{\star,v}^\vee$, for all finite places $v \in I_{F,f}$. If $v \notin S^1$ then $N \in \mathcal{O}_{F_v}^\times$ and the previous inclusion induces $L_{\star,v}^\vee \subset L_{\star,v}^\vee$, i.e., the local lattices $L_{\star,v}$ are self-dual \mathcal{O}_{E_v} -lattices, for all $\star \in \{V, W\}$.

Let us denote by $\mathcal{O}_F^{S^1} := \{x \in F; \mathrm{ord}_v(x) \geq 0, \forall v \notin S^1\} \subset F$ the ring of S^1 -integers, and set $\mathcal{O}_E^{S^1} := \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_F^{S^1} = \{x \in E; \mathrm{ord}_w(x) \geq 0, \forall w \in I_{E,f} \text{ not above } S^1\} \subset E$. Define the $\mathcal{O}_E^{S^1}$ -module $L_\star^{S^1} := L_\star \otimes_{\mathcal{O}_F} \mathcal{O}_F^{S^1} = L_\star \otimes_{\mathcal{O}_E} \mathcal{O}_E^{S^1} \subset \star$, for all $\star \in \{V, W\}$. The restriction $\langle \cdot, \cdot \rangle_{L_\star^{S^1}}$ of $\langle \cdot, \cdot \rangle_\star$ to $L_\star^{S^1} \times L_\star^{S^1}$, induces a bijection

$$L_\star^{S^1} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_E^{S^1}}(L_\star^{S^1}, \mathcal{O}_E^{S^1}),$$

i.e., is a perfect hermitian $\mathcal{O}_E^{S^1}/\mathcal{O}_F^{S^1}$ -pairing. The functor $\mathrm{GL}(L_\star)_{\mathcal{O}_F^{S^1}}$, from the category $\mathcal{O}_F^{S^1}$ -alg of $\mathcal{O}_F^{S^1}$ -algebras to the category Grp of groups, that maps any R to the group $\mathrm{GL}(L_\star \otimes_{\mathcal{O}_F^{S^1}} R)$, is representable by a smooth $\mathcal{O}_F^{S^1}$ -group scheme - still denoted $\mathrm{GL}(L_\star)$ -

(14). Such an assumption is not that restrictive: indeed, for any lattice $L_W \subset W$ generated by some basis $\{e_1, e_2\}$, one may choose some $N \in \mathbb{N}$ such that $N\langle e_i, e_j \rangle \in \mathcal{O}_E$ for all $i, j \in \{1, 2\}$. Thus NL_W satisfies $\langle NL_W, NL_W \rangle_{\mathcal{O}_E}$, i.e., $NL_W \subset (NL_W)^\vee$.

which is an S^1 -integral model of the reductive group $\mathrm{GL}(\star)_F = \mathrm{Res}_{E/F}\mathrm{GL}(\star)$ over F . It admits the following subfunctor:

$$\begin{aligned} \underline{U}_\star : \mathcal{O}_F^{S^1}\text{-alg} &\longrightarrow \mathrm{Grp} \\ R &\longmapsto \{g \in \mathrm{GL}(\mathbf{L}_\star \otimes_{\mathcal{O}_F^{S^1}} R); \langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle, \forall v, w \in \mathbf{L}_\star^{S^1} \otimes_{\mathcal{O}_F^{S^1}} R\}. \end{aligned}$$

By fixing an \mathcal{O}_E -basis for \mathbf{L}_\star , one might rewrite the invariance condition with respect to the hermitian pairing satisfied by the elements of $\underline{U}_\star(R)$, into a finite set of polynomial equations satisfied by their matrix coefficients. We deduce that the functor \underline{U}_\star is representable by a closed subgroup scheme of $\mathrm{GL}(\mathbf{L}_\star)$ - still denoted by \underline{U}_\star - which is smooth over $\mathrm{Spec} \mathcal{O}_F^{S^1}$ and whose base change to $\mathrm{Spec} F$ is nothing but $\mathrm{U}(\star)$. By definition of S^1 , if $v \notin S^1$ then the fiber $(\underline{U}_\star)_v$ of \underline{U}_\star represents the unitary group of a perfect hermitian pairing defined over the residue field \mathbb{F}_v of \mathcal{O}_F at v , hence is a reductive \mathbb{F}_v -group scheme. One obtains, for all finite places v of F outside S^1 , a hyperspecial maximal compact subgroup $K_{\star,v} := \underline{U}_\star(\mathcal{O}_{F_v}) = \mathrm{U}_\star(F_v) \cap \mathrm{GL}(\mathbf{L}_\star)(\mathcal{O}_{F_v})$, i.e., $K_{\star,v}$ is equal to the stabilizer in $\mathrm{U}_\star(F_v)$ of the self-dual local lattice $\mathbf{L}_{\star,v}$.

As a consequence of the definition of the finite-adelic points of the \mathbb{Q} -reductive group \mathbf{G}_\star , one has an identification between topological groups

$$\begin{aligned} \mathbf{G}_\star(\mathbb{A}_f) &:= \mathrm{U}(\star)(\mathbb{A}_{F,f}) \simeq \prod'_{v \in I_{f,F}} \underline{U}_\star(F_v) \\ &= \mathrm{U}(V)(F_{S^1}) \times \prod'_{v \notin S^1} \mathrm{U}(\star)(F_v), \end{aligned}$$

with $F_{S^1} := \prod_{v \in S^1} F_v$, the restricted product $\prod'_{v \notin S^1}$ being understood with respect to the hyperspecial subgroups $\underline{U}_\star(\mathcal{O}_{F_v}) = \mathrm{U}(\star)(F_v) \cap \mathrm{GL}(\mathbf{L}_\star)_{\mathcal{O}_F^{S^1}}(\mathcal{O}_{F_v})$, for all $v \notin S^1$ and $\star \in \{V, W\}$.

Remark 1.2.2. *The above construction admits the following variant. A slightly more general construction related to general reductive groups is described, for instance, in Chapter II of [19], and the outline of the argument is the following. The choice of \mathbf{L}_\star defines an integral model $\mathrm{GL}(\mathbf{L}_\star)_{\mathcal{O}_F}$ for $\mathrm{GL}(\star)$ over $\mathrm{Spec} \mathcal{O}_F$, which gives a sequence of closed immersions*

$$\mathrm{U}(\star) \hookrightarrow \mathrm{GL}(\star)_F \hookrightarrow \mathrm{GL}(\star)_{\mathcal{O}_F}, \quad \forall \star \in \{V, W\}.$$

By setting $\underline{U}_\star \subset \mathrm{GL}(\star)_{\mathcal{O}_F}$ to be the Zariski closure of the image of $\mathrm{U}(\star)$ in $\mathrm{GL}(\star)_{\mathcal{O}_F}$ by the above sequence, then by ([19], Lemma 2.4.2) there exists a finite set S' of finite places of F , such that $(\underline{U}_\star)_{\mathcal{O}_F^{S'}}$ is a smooth $\mathcal{O}_F^{S'}$ -group scheme which is a model for $\mathrm{U}(\star)$. Then, for the generic fiber $\mathrm{U}(\star)$ of \underline{U}_\star is already reductive, one may apply ([11], Proposition 3.1.9) to deduce that the fibers of $(\underline{U}_V)_{\mathcal{O}_F^{S'}}$ are reductive outside a finite number of places, i.e., that there exists a finite set S'' of finite places of F containing S' and such that $(\underline{U}_V)_{\mathcal{O}_F^{S''}}$ has reductive fibers.

1.2.3.1 The level K_0 .

The upshot of the above remark is that it constructs an \mathcal{O}_F -model of $U(\star)$ (with $\star \in \{V, W\}$), which is smooth with reductive fibers outside the set S^1 (up to possibly ⁽¹⁵⁾enlarging S^1), which enables us to consider \mathcal{O}_{F_v} -points of \underline{U}_\star even if $v \in S^1$. By construction, one has $\underline{U}_\star(\mathcal{O}_{F_v}) \subset U(\star)(F_v) \cap \mathrm{GL}(L_\star \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v})$, for such a v . We set $K_0 \subset \mathbf{G}(\mathbb{A}_f)$ to be the open compact subgroup given by

$$K_0 := (\underline{U}_V \times \underline{U}_W)(\widehat{\mathcal{O}}_F) = (\underline{U}_V \times \underline{U}_W) \left(\prod_{v \in I_{F,f}} \mathcal{O}_{F_v} \right) = (K_0)_{S^1} \times K_0^{S^1},$$

with

$$(K_0)_{S^1} := (\underline{U}_V \times \underline{U}_W) \left(\prod_{v \in S^1} \mathcal{O}_{F_v} \right) \subset (U(V) \times U(W))(F_{S^1}) \cap \prod_{v \in S^1} \mathrm{GL}(L_V \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}) \times \mathrm{GL}(L_W \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}),$$

with $F_{S^1} := \prod_{v \in S^1} F_v$, and

$$K_0^{S^1} := \prod_{v \notin S} (\underline{U}_V \times \underline{U}_W)(\mathcal{O}_{F_v}) = \mathrm{Stab}_{(U(V) \times U(W))(\mathbb{A}_{F,f}^S)} \left(L_V \otimes_{\mathcal{O}_F} \widehat{\mathcal{O}}_F^S, L_W \otimes_{\mathcal{O}_F} \widehat{\mathcal{O}}_F^S \right).$$

We set $(K_0)_{\star,v} := \underline{U}_\star(\mathcal{O}_{F_v})$ for $\star \in \{V, W\}$, and $(K_0)_v := (K_0)_{V,v} \times (K_0)_{W,v}$, for all $v \notin S^1$.

Notice that the compact open subgroup K_0 may be assumed to be neat. Indeed, otherwise one might shrink $(K_0)_{S^1}$ to some *small* enough compact open subgroup $(K_0)'_{S^1} \subset (K_0)_{S^1}$ - by adding, for instance, some congruence conditions with respect to the inclusion $(K_0)_{S^1} \subset \mathrm{GL}(L_V \otimes_{\mathcal{O}_F} \mathcal{O}_{F_{S^1}}) \times \mathrm{GL}(L_W \otimes_{\mathcal{O}_F} \mathcal{O}_{F_{S^1}})$ - and thus replace (K_0) by its finite-index subgroup $K'_0 := (K_0)'_{S^1} \times K_0^{S^1} \subset K_0$. We finally set $(K_0)_{\mathbf{H}} := \mathbf{H}(\mathbb{A}_f) \cap K_0 = \Delta((K_0)_W)$.

1.2.3.2 Allowable inert places τ .

We assume in this paragraph that $K = K_0$. Let τ be a finite place of F . We set $G_{V,\tau} := U(V)(F_\tau)$, $G_{W,\tau} := U(W)(F_\tau)$, $G_\tau := G_{V,\tau} \times G_{W,\tau}$, and we let $H_\tau := \Delta(G_{W,\tau}) \subset G_\tau$ be the diagonal image of $G_{W,\tau}$. We assume that τ is inert in the extension E/F and set, as usual, $E_\tau := E \otimes_F F_\tau$ to be the quadratic unramified extension of F_τ with ring of integers $\mathcal{O}_{E_\tau} = \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{F_\tau}$.

Let $\varpi \in F_\tau$ be a chosen uniformizer for both F_τ and E_τ . We let $m \geq 1$ be a positive integer and $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}})$ be a non-degenerate m -dimensional E_τ/F_τ -hermitian space. If $\mathcal{B}_{\mathcal{V}} = \{v_1, \dots, v_m\}$ is an E_τ -basis of \mathcal{V} , we set $H_{\mathcal{B}_{\mathcal{V}}} := (\langle v_i, v_j \rangle_{\mathcal{V}})_{1 \leq i, j \leq m} \in M_{m \times m}(E_\tau)$ to be the matrix of the hermitian product with respect to $\mathcal{B}_{\mathcal{V}}$. The equality

$$c \cdot \det(H_{\mathcal{B}_{\mathcal{V}}}) = \det(c \cdot \langle v_i, v_j \rangle_{\mathcal{V}})_{1 \leq i, j \leq m} = \det(\langle v_j, v_i \rangle_{\mathcal{V}})_{1 \leq i, j \leq m} = \det({}^t H_{\mathcal{B}_{\mathcal{V}}}) = \det(H_{\mathcal{B}_{\mathcal{V}}})$$

(15). We believe that S^1 - such as defined in the previous paragraph - does actually not need to be enlarged, but this point is still not totally clear to us.

implies that $\det(H_{\mathcal{B}_{\mathcal{V}}})$ lies in F_{τ}^{\times} . If $\mathcal{B}'_{\mathcal{V}} = \{v'_1, \dots, v'_m\}$ is a different E_{τ} -basis for \mathcal{V} , with transition matrix $P \in \mathrm{GL}_m(E_{\tau})$ with respect to $\mathcal{B}_{\mathcal{V}}$, we set $H_{\mathcal{B}'_{\mathcal{V}}} := (\langle v'_i, v'_j \rangle_{\mathcal{V}})_{1 \leq i, j \leq m} \in M_{m \times m}(E_{\tau})$. Then $H_{\mathcal{B}'_{\mathcal{V}}} = {}^t P H_{\mathcal{B}_{\mathcal{V}}} \bar{P}$, which gives $\det H_{\mathcal{B}'_{\mathcal{V}}} = N_{E_{\tau}/F_{\tau}}(\det(P)) \cdot \det(H_{\mathcal{B}_{\mathcal{V}}})$. Accordingly, we call *determinant* of the hermitian space $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\tau \mathcal{V}})$, the quantity:

$$\det(\langle \cdot, \cdot \rangle_{\mathcal{V}}) := \det(H_{\mathcal{B}_{\mathcal{V}}}) N_{E_{\tau}/F_{\tau}}(E_{\tau}^{\times}) \in F_{\tau}^{\times} / N_{E_{\tau}/F_{\tau}}(E_{\tau}^{\times}),$$

which is well-defined by the above discussion. According to ([45], Remark 10.1.4 and Theorem 2.14.5), non-degenerate hermitian forms over p -adic local fields are classified by their dimension and determinant: by local class-field theory, one has $|\frac{F_{\tau}^{\times}}{N_{E_{\tau}/F_{\tau}}(E_{\tau}^{\times})}| = [E_{\tau}^{\times} : F_{\tau}^{\times}]$, so there are only two isomorphism classes of m -dimensional hermitian E_{τ}/F_{τ} -spaces. If $m = 2k$ is even, we say that $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}})$ is a *split hermitian space* if \mathcal{V} is the sum of k pairwise orthogonal hyperbolic spaces, i.e., if \mathcal{V} admits an E_{τ} -basis of the form $\{v_1, v_{-1}, \dots, v_k, v_{-k}\}$, with $\langle v_i, v_j \rangle_{\mathcal{V}} = 0$ if $i \neq -j$, and $\langle v_i, v_{-i} \rangle_{\mathcal{V}} = 1$, for all $i \in \{1, \dots, k\}$. Such a basis will be referred to as a *Witt basis* in the next chapter. If m is odd, we say that $(\mathcal{V}, \langle \cdot, \cdot \rangle_{\mathcal{V}})$ is split if \mathcal{V} is the orthogonal sum of a split hermitian hyperplane and an anisotropic line. Accordingly, the determinant of a split E_{τ}/F_{τ} -hermitian space of even dimension $m = 2k$ is $(-1)^k \in F_{\tau}^{\times} / N_{E_{\tau}/F_{\tau}}(E_{\tau}^{\times})$.

Going back to our setting $m \in \{2, 3\}$, $\mathcal{V} \in \{V, W\}$, let us fix an \mathcal{O}_E -basis $\{e_1, e_2\}$ of L_W . The condition $L_W \subset L_W^{\vee}$ implies that $\det(\langle e_i, e_j \rangle)_{1 \leq i, j \leq 2} \in \mathcal{O}_E$, hence $\det(\langle e_i, e_j \rangle_{\tau})_{1 \leq i, j \leq 2} \in \mathcal{O}_{E_{\tau}}$ for all inert place τ . Set $L_{W, \tau} := L_W \otimes_{\mathcal{O}_E} \mathcal{O}_{E_{\tau}} \subset W_{\tau}$ and $L_{V, \tau} := L_V \otimes_{\mathcal{O}_E} \mathcal{O}_{E_{\tau}} = L_{W, \tau} \oplus L_{D, \tau} \subset V_{\tau}$ be the corresponding local $\mathcal{O}_{E_{\tau}}$ -lattices, with $L_{D, \tau} = \mathcal{O}_{E_{\tau}} e_D$. The preceding discussion implies that W_{τ} is split - hence $V_{\tau} = W_{\tau} \perp D_{\tau}$ is split - if and only if the E_{τ} -basis $\{e_1, e_2\}$ of W_{τ} satisfies $\det(\langle e_i, e_j \rangle_{\tau})_{1 \leq i, j \leq 2} = -1 \in E_{\tau}^{\times} / N_{E_{\tau}/F_{\tau}}(E_{\tau}^{\times})$. This also implies that $W_{\tau} = E_{\tau} e_+ \oplus E_{\tau} e_-$ is a hyperbolic plane generated by some Witt basis $\{e_+, e_-\}$. The preceding conditions are *almost* always satisfied, i.e., for all but finitely many inert places τ :

Lemma 1.2.4. *There exists a finite set $S^2 \subset I_{F, f}$ of finite places of F containing S^1 and the ramification places $\mathrm{Ram}(E/F) := \{v \in I_{F, f}; v \text{ is ramified in } E/F\}$, such that every inert place $\tau \notin S^2$ satisfies the following properties:*

- The compact open subgroups $K_{V, \tau} \subset G_{V, \tau}$ and $K_{W, \tau} \subset G_{W, \tau}$ are both hyperspecial,
- The $\mathcal{O}_{E_{\tau}}$ -lattices $L_{W, \tau}$ and $L_{V, \tau}$ are both self-dual,
- The local hermitian spaces W_{τ} and V_{τ} are split.

Proof. The construction of S^1 ensures that the first two properties are automatically satisfied for any finite set $S^2 \supset S^1$. Notice that, if τ is unramified in E/F , then the norm map $N_{E/F} : \mathcal{O}_{E_{\tau}}^{\times} \rightarrow \mathcal{O}_{F_{\tau}}^{\times}$ is surjective (see [46], §2, Proposition 3 (b)). As $\det(\langle e_i, e_j \rangle) \in \mathcal{O}_E$ by assumption on W , one may set

$$S^2 := S^1 \cup \mathrm{Ram}(E/F) \cup \{\tau \in I_{F, f} \text{ is inert; } \det(\langle e_1, e_2 \rangle_{\tau}) \notin \mathcal{O}_{E_{\tau}}^{\times}\}$$

If $\tau \notin S^2$ is inert, then both $\det(\langle e_1, e_2 \rangle_\tau)$ and -1 lie in $\mathcal{O}_{E_\tau}^\times N_{E_\tau/F_\tau}(\mathcal{O}_{E_\tau}^\times) \subset N_{E_\tau/F_\tau}(E_\tau^\times)$, i.e., $(W_\tau, \langle \cdot, \cdot \rangle_\tau)$ is split, and so is $V_\tau = W_\tau \perp D_\tau$. \square

Definition 1.2.5 (Allowable inert place of F). *We call allowable inert place of F , any inert place of $I_{F,f} \setminus S^2$.*

Consequently if τ is an allowable inert place, the lattice $L_{V,\tau} = L_{W,\tau} \oplus L_{D,\tau}$ is self-dual and the hermitian space V_τ admits Witt bases, i.e., bases of the form $\{e_+, e_0, e_-\}$ with $\langle e_+, e_0 \rangle = \langle e_-, e_0 \rangle = 0$ and $\langle e_0, e_0 \rangle = \langle e_+, e_- \rangle = 1$.

Remark 1.2.3. *We use the same terminology as [27], Definition 1.1, although our definition of allowable inert places is slightly more restrictive (namely, we require that the hermitian space V_τ and W_τ be split).*

1.2.4 A base cycle $\mathcal{Z}_{K_0}(g_0)$.

We still assume in this paragraph that $K = K_0$. Let $g_0 \in \mathbf{G}(\mathbb{A}_f)$ be any element. By definition of the (finite-) adelic points of \mathbf{G} , there exists a finite set $\Sigma = \Sigma(g_0)$ of finite places of F , containing the set S^2 introduced at Lemma 1.2.4 and such that

$$g_0 = (g_{0,\Sigma}, g_0^\Sigma) \in (\mathrm{U}(V) \times \mathrm{U}(W))(F_\Sigma) \times \prod_{v \notin \Sigma} (\underline{\mathrm{U}}_V \times \underline{\mathrm{U}}_W)(\mathcal{O}_{F_v}).$$

In particular, as $S^1 \subset S^2 \subset \Sigma$, one has $g_0^\Sigma \in K^\Sigma \subset \prod_{v \notin S^1} (\underline{\mathrm{U}}_V \times \underline{\mathrm{U}}_W)(\mathcal{O}_{F_v}) =: K$. Accordingly, when dealing with special cycles one may replace g_0 by $\tilde{g}_0 := (g_{0,\Sigma}, 1) \in \mathbf{G}(\mathbb{A}_f)$, as $\mathcal{Z}_K(g_0) = \mathcal{Z}_K(\tilde{g}_0)$. Consequently, we may and will assume that $g_0^\Sigma = 1 \in K^\Sigma$ throughout the following. As we shall see later on, the action of the Galois group $\mathrm{Gal}(E(\infty)/E)$ on the set of special cycles, such as described at Proposition 1.42, implies that

$$\mathrm{Stab}_{\mathbf{H}(\mathbb{A}_f)}(\mathcal{Z}_K(g_0)) = \mathbf{H}(\mathbb{A}_f) \cap (\mathbf{Z}_\mathbf{G}(\mathbb{Q}) \cdot \mathbf{H}(\mathbb{Q}) g_0 K g_0^{-1}).$$

The preceding may be rewritten into the following useful form, using ([7], Lemma 65):

Proposition 1.2.5. *Set $K_{\mathbf{H}}^\Sigma := \Delta(K_W^\Sigma)$, and set*

$$K_{\mathbf{H},g_0,\Sigma}^Z := ((\mathbf{Z}_\mathbf{G}(\mathbb{Q}) \cap K^\Sigma) g_{0,\Sigma} K_\Sigma g_{0,\Sigma}^{-1}) \cap \mathbf{H}(F_\Sigma).$$

Then the stabilizer of $\mathcal{Z}_K(g_0)$ in $\mathbf{H}(\mathbb{A}_f)$ - with respect to the action described later on in Proposition 1.42 - is equal to

$$\mathbf{H}(\mathbb{Q}) \cdot (K_{\mathbf{H},g_0,\Sigma}^Z \times K_{\mathbf{H}}^\Sigma) \subset \mathbf{H}(\mathbb{A}_f).$$

Notice that, by axiom SV5 satisfied by the Shimura datum (\mathbf{G}, X) , the group $\mathbf{Z}_\mathbf{G}(\mathbb{Q}) \simeq \mathbf{T}^1(\mathbb{Q}) \times \mathbf{T}^1(\mathbb{Q})$ is discrete in $\mathbf{Z}_\mathbf{G}(\mathbb{A}_f) \simeq \mathbf{T}^1(\mathbb{A}_f) \times \mathbf{T}^1(\mathbb{A}_f) \subset \mathbf{G}(\mathbb{A}_f)$, hence the intersection $\mathbf{Z}_\mathbf{G}(\mathbb{Q}) \cap K^\Sigma$ is at most finite.

1.2.5 Hecke action on $\mathbb{Z}[\mathcal{Z}_K(\mathbf{G}, \mathbf{H})]$.

In this paragraph, we let G be a totally disconnected group which is either the adelic group $\mathbf{G}(\mathbb{A}_f)$ or one of its local components $G_{\star, v} = \mathrm{U}(\star)(F_v)$ ($v \in I_{F, f}$, $\star \in \{V, W\}$) or $G_v = (\mathrm{U}(V)) \times \mathrm{U}(W)(F_v)$, endowed with the corresponding topology. We let U be a compact open subgroup of G . For any commutative ring R , we let $\mathcal{C}_c^\infty(G, R)$ be the set of R -valued, locally constant, compactly supported functions $f : G \rightarrow R$. This set is endowed with a natural structure of R -module, induced by addition and R -multiplication on the target. If $R = \mathbb{Q}$, the \mathbb{Q} -vector space $\mathcal{C}_c^\infty(G, \mathbb{Q})$ is also endowed with a ring structure (without unit), whose product law is given by the convolution product of functions:

$$(f_1 * f_2)(g) := \int_{h \in G} f_1(h) f_2(h^{-1}g) d\mu_U(h) = \int_{h \in G} f_1(gh) f_2(h^{-1}) d\mu_U(h), \quad (1.20)$$

where $\mu_U : \mathcal{C}_c^\infty(G, \mathbb{Q}) \rightarrow \mathbb{Q}$ is the unique left-invariant Haar measure on G which is U -normalized, i.e., such that $\mu_U(\mathbf{1}_U) =: \int_{h \in G} \mathbf{1}_U(h) d\mu_U(h) = 1$ (see [49], §2.4). The substitution $h \mapsto g^{-1}h$ shows that both integrals above are indeed equal, and one gets from the above equality that $f_1 * f_2$ remains H_1 left-invariant (resp. H_2 -right invariant) whenever f_1 is H_1 left-invariant (resp. f_2 is H_2 right-invariant), for all subgroups $H_1, H_2 \subset G$.

We set $\mathcal{C}_c^\infty(G/U, \mathbb{Z}) \subset \mathcal{C}_c^\infty(G, \mathbb{Q})$ to be the \mathbb{Z} -submodule formed by those functions $f : G \rightarrow \mathbb{Z}$ that are \mathbb{Z} -valued and right U -invariant. We check that this module is generated (over \mathbb{Z}) by the family of indicator functions $\mathbf{1}_{gU}$, for g running among the quotient G/U . We let $\mathcal{C}_c^\infty(G // U, \mathbb{Z}) \subset \mathcal{C}_c^\infty(G/U, \mathbb{Z})$ be the \mathbb{Z} -submodule formed by those functions that are also left U -invariant, i.e., that are \mathbb{Z} -valued and U -bi-invariant. The generating family $\mathbf{1}_{gU}$, $g \in G/U$, being stable under convolution (see [7], §III.5, Example 3), one gets that $\mathcal{C}_c^\infty(G/U, \mathbb{Z})$ and $\mathcal{C}_c^\infty(G // U, \mathbb{Z})$ remain stable by convolution. Finally, for any commutative ring A , we may extend the scalars to A and define the ring

$$\mathcal{H}_A(G // U) := \mathcal{C}_c^\infty(G // U, \mathbb{Z}) \otimes_{\mathbb{Z}} A,$$

which we call the *Hecke algebra* attached to the pair (G, U) , with coefficients in A .

The ring $\mathcal{H}_A(G // U)$ is generated over A by finite sums of indicator functions $\mathbf{1}_{UgU}$, with $g \in U \backslash G/U$. Accordingly, one has an isomorphism (of A -modules)

$$\mathcal{H}_A(G // U) \simeq A[U \backslash G/U].$$

For all $g \in G$, the set $UgU \subset G$ is endowed with a right action from U , acting by right-multiplication: accordingly, UgU splits as a disjoint union $\bigsqcup_{i \in I} g_i U$, with $g_i \in UgU$, which is finite as U is both compact and open. Consider the A -linear map

$$\rho : \mathcal{H}_A(G // U) \rightarrow \mathrm{End}_{A[G]}(A[G/U]),$$

given by mapping the indicator function $\mathbf{1}_{UgU}$, $g \in G$, to the unique $A[G]$ -equivariant endomorphism of $A[G/U]$ which maps the element $[1 \cdot U] \in A[G/U]$ to the sum $\sum_{i=1}^{r_g} [g_i U]$, where $UgU = \sqcup_{i=1}^{r_g} g_i U$, or equivalently, defined by:

$$\rho(\mathbf{1}_{UgU}) \cdot [g'U] = \sum_{i=1}^{r_g} [g'g_i U] \in \mathbb{Z}[G/U], \quad \forall g' \in G. \quad (1.21)$$

One checks that ρ is a bijection, and satisfies $\rho(f_1 * f_2) = \rho(f_2) \circ \rho(f_1)$ for all $f_1, f_2 \in \mathcal{H}_A(G//U)$, i.e., that ρ is an isomorphism between the rings $\mathcal{H}_A(G//U)$ and $\mathrm{End}_{A[G]}(A[G/U])^{\mathrm{op}}$ (see [7], Proposition III.5.1).

We now assume that $K = K_0 \subset \mathbf{G}(\mathbb{A}_f)$. We set $\mathcal{H}_K := \mathcal{H}_{\mathbb{Z}}(\mathbf{G}(\mathbb{A}_f) // K)$ and $\mathcal{H}_v := \mathcal{H}_{\mathbb{Z}}(G_v // K_v) \simeq \mathcal{H}_{\mathbb{Z}}(\mathbf{G}_{V,v} // K_{V,v}) \otimes \mathcal{H}_{\mathbb{Z}}(\mathbf{G}_{W,v} // K_{W,v})$, for all $v \in I_{F,f}$. The *global* Hecke algebra \mathcal{H}_K admits the following factorization (as rings):

$$\begin{aligned} \mathcal{H}_K &\simeq \mathbb{Z}[K \backslash \mathbf{G}(\mathbb{A}_f) / K] \simeq \mathbb{Z}[K_{\Sigma} \backslash \mathbf{G}(F_{\Sigma}) / K_{\Sigma}] \otimes \bigotimes_{v \notin \Sigma}^{\prime} \mathbb{Z}[K_v \backslash G_v / K_v] \\ &= \mathcal{H}_{\mathbb{Z}}(\mathbf{G}(F_{\Sigma}) // K_{\Sigma}) \otimes \bigotimes_{v \notin \Sigma}^{\prime} \mathcal{H}_v, \end{aligned} \quad (1.22)$$

where the restricted tensor product \bigotimes^{\prime} is understood with respect to the elements $[\mathbf{1}_{G_v}] = [K_v] \in \mathbb{Z}[K_v \backslash G_v / K_v]$. Accordingly, the action of \mathcal{H}_K on $\mathbb{Z}[\mathbf{G}(\mathbb{A}_f) / K]$ can be extended to actions on $\mathbb{Z}[\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f) / K]$ and on $\mathbb{Z}[\mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \backslash \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f) / K] \stackrel{(1.2.3)}{=} \mathbb{Z}[\mathcal{Z}_K(\mathbf{G}, \mathbf{H})]$, by the same rule as (1.21). One has:

$$\mathbf{1}_{KgK} \cdot \mathcal{Z}_K(g') = \sum_{i=1}^{r_g} \mathcal{Z}_K(g'g_i), \quad \forall g' \in \mathbf{G}(\mathbb{A}_f), \quad \text{with } KgK = \sqcup_{i=1}^{r_g} g_i K. \quad (1.23)$$

In particular, one checks immediately that the Hecke action above and the Galois action - such as expressed e.g. in Proposition 1.42 - commute with each other. Notice that the \mathbb{Z} -module $\mathbb{Z}[\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f) / K]$ is also factorizable, as follows:

$$\mathbb{Z}[\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f) / K] \simeq \mathbb{Z}[\mathbf{H}^{\mathrm{der}}(F_{\Sigma}) \backslash \mathbf{G}(F_{\Sigma}) / K_{\Sigma}] \otimes \bigotimes_{v \notin \Sigma}^{\prime} \mathbb{Z}[H_v^{\mathrm{der}} \backslash G_v / K_v],$$

with $H_v^{\mathrm{der}} := \mathbf{H}^{\mathrm{der}}(F_v)$ and where the restricted tensor product is understood with respect to the elements $\bigotimes_{v \notin \Sigma} [H_v^{\mathrm{der}} \mathbf{1}_{G_v} K_v]$. One checks the action of \mathcal{H}_K on $\mathbb{Z}[\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f) / K]$ intertwines the local actions of the local Hecke algebras \mathcal{H}_v on $\mathbb{Z}[H_v^{\mathrm{der}} \backslash G_v / K_v]$.

1.3 Transfer fields, Galois traces and distribution relations.

The aim of this section is multiple. The first goal is to provide, for all neat compact open subgroups $K \subset \mathbf{G}(\mathbb{A}_f)$, an explicit and simple description for the action of the abelian group

$\text{Gal}(E^{ab}/E)$ - in fact, of his quotient $\text{Gal}(E(\infty)/E)$ - on the set $\mathcal{Z}_K(\mathbf{G}, \mathbf{H})$ of \mathbf{H} -special cycles of level K , via global class-field theory. The second is to discuss the notion of *ring class fields*, *transfer fields* and of \mathcal{K} -transfer fields - and in particular, the structure and ramification of the intermediate extensions between the latter - which also arise from global class-field theory and will appear later on as natural choices for *fields of definition*, when exhibiting a particular family of special cycles in Chapter 3. Most of the content of the next three subsections is mainly considered as *standard* in the literature. Anyway, we found relevant to recall some general constructions and to provide details for some non-trivial facts, in the same vein as what is done in [7].

1.3.1 The global Artin map.

We shall review some properties of the global Artin map in the case of number fields, in view towards the computation of the *transfer field* $E(\infty) \subset E^{ab}$ with respect to the CM extension E/F . This review follows relatively closely Boumasmoud's exposition made in ([7], §VI.9 and §VI.14), exception made of some slight notational differences. First, let L be a general number field, and let $I_{L,\infty}$ (resp. $I_{L,\mathbb{R}}$ and $I_{L,\mathbb{C}}$) denote the set of infinite places (resp. of real and complex places) of L . For any $\sigma \in I_{L,\mathbb{R}}$ (resp. $I_{L,\mathbb{C}}$), we let L_σ denote the copy of \mathbb{R} (resp. of \mathbb{C}) endowed with the L -algebra structure induced by v . We let $I_{L,f}$ denote the set of finite places of L , which we identify with the set $|\text{Spec } \mathcal{O}_L| \setminus \{0\}$ of non-zero prime ideals of \mathcal{O}_L .

We set, mainly for notational convenience and after ([7], §VI.9), X_L to be the algebraic torus over \mathbb{Q} defined by $X_L := \text{Res}_{L/\mathbb{Q}} \mathbb{G}_m$. Accordingly, one has $X_L(\mathbb{Q}) = L^\times$ and $X_L(\mathbb{R}) = \prod_{\sigma \in I_{L,\infty}} L_\sigma^\times$. The group of finite idèles $X_L(\mathbb{A}_f) = \mathbb{A}_{L,f}^\times$ and the group of idèles $X_L(\mathbb{A}_\mathbb{Q}) = \mathbb{A}_L^\times \simeq X_L(\mathbb{R}) \times X_L(\mathbb{A}_f)$ are endowed with the restricted product topology: the group $X_L(\mathbb{A}_\mathbb{Q})$ (resp. $X_L(\mathbb{A}_f)$) admits a basis of neighbourhoods of the identity formed by subgroups of the form

$$\prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_{L_v}^\times,$$

where S runs among the finite sets of places (resp. of finite places) of L , with $U_v \subset L_v^\times$ an open subgroup for the corresponding (archimedean or non-archimedean) topology, and where we set $\mathcal{O}_{L_\sigma} := L_\sigma, \forall \sigma \in I_{L,\infty}$. The above topology makes the group $X_L(\mathbb{A}_f)$ into a totally disconnected topological group.

We let $\iota_\mathbb{A}, \iota_f$ and ι_∞ denote the diagonal embeddings of $X_L(\mathbb{Q})$ into $X_L(\mathbb{A}_\mathbb{Q}), X_L(\mathbb{A}_f)$ and $X_L(\mathbb{R})$ respectively. We shall sometimes, and when the context is clear, omit to precise the chosen embedding while identifying $X_L(\mathbb{Q})$ with a subgroup of the preceding three groups. ⁽¹⁶⁾

(16). One should remain careful that these embeddings do not commute with the inclusions of $X_L(\mathbb{A}_f)$ (resp. $X_L(\mathbb{R})$) into $X_L(\mathbb{A}_\mathbb{Q})$ obtained by adding the identity at the archimedean places (resp. at the finite places),

By ([39], Proposition 2.5), the group $X_L(\mathbb{Q})$ is discrete (hence closed) in $X_L(\mathbb{A}_{\mathbb{Q}})$, thus the *idèle class group* $C_L := X_L(\mathbb{A}_{\mathbb{Q}}) / X_L(\mathbb{Q}) = \mathbb{A}_L^{\times} / L^{\times}$, endowed with the quotient topology, is a locally compact topological group. We let $\pi : X_L(\mathbb{A}_{\mathbb{Q}}) \twoheadrightarrow C_L$ denote the canonical projection map.

By the global *Artin map* - sometimes called *global reciprocity map* - we mean the surjective, continuous morphism of topological groups

$$\mathrm{Art}_L : C_L \twoheadrightarrow \mathrm{Gal}(L^{ab}/L),$$

which satisfies a various set of nice properties, among which:

- Functoriality with respect to finite abelian extensions M/L , with respect to the norm maps $N_{M/L} : M^{\times} \rightarrow L^{\times}$ on the left-hand side, and restriction maps $\mathrm{res} : \mathrm{Gal}(M^{ab}/M) \twoheadrightarrow \mathrm{Gal}(L^{ab}/L)$ on the right-hand side,
- Compatibility with the local reciprocity maps $\mathrm{Art}_{L_v} : L_v^{\times} \hookrightarrow \mathrm{Gal}(L_v^{ab}/L_v)$, with respect to the inclusions $L_v^{\times} \hookrightarrow \mathbb{A}_L^{\times}$ and $\mathrm{Gal}(L_v^{ab}/L_v) \simeq \mathcal{D}_v \hookrightarrow \mathrm{Gal}(L^{ab}/L)$, where \mathcal{D}_v stands for the decomposition subgroup at v .

According to ([40], Corollary 8.2.2) the kernel of Art_L is equal to the connected component D_L of the identity in C_L . We let $X_L(\mathbb{R})_+ = \prod_{\sigma \in I_{L,\mathbb{R}}} (L_{\sigma}^{\times})_{>0} \times \prod_{\sigma \in I_{L,\mathbb{C}}} L_{\sigma}^{\times}$ denote the connected component of the identity in $X_L(\mathbb{R})$ - with respect to the archimedean product topology - which can also be identified with the connected component of the identity in $X_L(\mathbb{A}_{\mathbb{Q}})$.⁽¹⁷⁾ By ([40], Theorem 8.2.1, (i)), D_L is equal to the topological closure of the subgroup $\pi(X_L(\mathbb{R})_+) \subset C_L$. If Y is a subset of either $X_L(\mathbb{A}_{\mathbb{Q}})$ or C_L , we denote by \bar{Y} its closure with respect to the corresponding topology. Let us start with the following observation:

Lemma 1.3.1. *The kernel of $\mathrm{Art}_L \circ \pi : X_L(\mathbb{A}_{\mathbb{Q}}) \twoheadrightarrow \mathrm{Gal}(L^{ab}/L)$ is $\overline{X_L(\mathbb{R})_+ X_L(\mathbb{Q})}$.*

Proof. The proof amounts to showing that $\overline{X_L(\mathbb{R})_+ X_L(\mathbb{Q})} = \pi^{-1}(\ker \mathrm{Art}_L) = \pi^{-1}(D_L) = \pi^{-1}\left(\overline{\pi(X_L(\mathbb{R})_+)}\right)$, which uses only basic properties of quotient maps between topological groups. More precisely, the inclusion $\overline{X_L(\mathbb{R})_+ X_L(\mathbb{Q})} \subset \pi^{-1}\left(\overline{\pi(X_L(\mathbb{R})_+)}\right)$ is clear, for $\pi^{-1}\left(\overline{\pi(X_L(\mathbb{R})_+)}\right)$ is a closed subset of $X_L(\mathbb{A}_{\mathbb{Q}})$ containing $\pi^{-1}\left(\pi(X_L(\mathbb{R})_+)\right) = X_L(\mathbb{R})_+ X_L(\mathbb{Q})$. Conversely, if z is any element of $\pi^{-1}\left(\overline{\pi(X_L(\mathbb{R})_+)}\right)$ and if U is any open subgroup of $X_L(\mathbb{A}_{\mathbb{Q}})$, then $\pi(z)\pi(U) \cap \pi(X_L(\mathbb{R})_+) = \pi(zU) \cap \pi(X_L(\mathbb{R})_+) \neq \emptyset$ - as $\pi(U)$ is an open subgroup of C_L - thus

$$\emptyset \neq \pi^{-1}\left(\pi(zU) \cap \pi(X_L(\mathbb{R})_+)\right) = zUX_L(\mathbb{Q}) \cap X_L(\mathbb{R})_+ X_L(\mathbb{Q}),$$

but rather commute with the projections $\mathrm{proj}_f : X_L(\mathbb{A}_{\mathbb{Q}}) \rightarrow X_L(\mathbb{A}_f)$ (resp. $\mathrm{proj}_{\infty} : X_L(\mathbb{A}_{\mathbb{Q}}) \rightarrow X_L(\mathbb{R})$) on to the finite (resp. infinite) places.

(17). Indeed, $X_L(\mathbb{A}_f)$ is totally disconnected and $X_L(\mathbb{A}_{\mathbb{Q}})$ is homeomorphic to $X_L(\mathbb{R}) \times X_L(\mathbb{A}_f)$.

i.e., $zU \cap X_L(\mathbb{R})_+ X_L(\mathbb{Q}) \neq \emptyset$ and z belongs to $\overline{X_L(\mathbb{R})_+ X_L(\mathbb{Q})}$. \square

By a slight abuse of notation and when the context is clear, we shall still denote by Art_L the composition $\text{Art}_L \circ \pi : X_L(\mathbb{A}_{\mathbb{Q}}) \rightarrow \text{Gal}(L^{ab}/L)$.

1.3.2 Restriction to finite idèles.

Our first aim will be to provide an alternative description of the Artin map and its kernel in terms of $X_L(\mathbb{A}_f)$ instead of $X_L(\mathbb{A}_{\mathbb{Q}})$, which will enable us to better describe the Galois action on the various geometric objects we are dealing with, via Shimura reciprocity for instance. We set $X_L(\mathbb{Q})_+ := \iota_{\infty}^{-1}(X_L(\mathbb{R})_+) = \{z \in L^{\times}; \sigma(z) > 0, \forall \sigma \in I_{F, \mathbb{R}}\}$, which we refer to as the group of *totally positive scalars*. We also set $\mathcal{O}_{L,+}^{\times} := \mathcal{O}_L^{\times} \cap X_L(\mathbb{Q})_+$, which we call the group of *totally positive units*. If $Y \subset X_L(\mathbb{Q})$ is any subset, we shall temporarily enforce notation and denote by \overline{Y}^f its closure in $X_L(\mathbb{A}_f)$ with respect to the embedding ι_f , by contrast with its closure \overline{Y} in $X_L(\mathbb{A}_{\mathbb{Q}})$ (with respect to $\iota_{\mathbb{A}}$).

Lemma 1.3.2. *The following equalities stand:*

(i) $X_L(\mathbb{A}_{\mathbb{Q}}) = X_L(\mathbb{A}_f) \overline{X_L(\mathbb{R})_+ X_L(\mathbb{Q})}$.

(ii) In $X_L(\mathbb{A}_f)$, one has $\overline{X_L(\mathbb{Q})}^f = X_L(\mathbb{Q}) \cdot \overline{\mathcal{O}_L^{\times}}^f$ and $\overline{X_L(\mathbb{Q})_+}^f = X_L(\mathbb{Q})_+ \cdot \overline{\mathcal{O}_{L,+}^{\times}}^f$.

Proof. Notice that, for each of the three statements, the inclusions from the right-hand sides into the left-hand sides are clear.

(i) Let $z = z_{\infty} z_f$ be any element of $X_L(\mathbb{A}_{\mathbb{Q}})$, with $z_{\infty} \in X_L(\mathbb{R})$ and $z_f \in X_L(\mathbb{A}_f)$. By the so-called *weak approximation* for number fields (see for instance [42], Theorem 1.4) the rational idèles $X_L(\mathbb{Q})$ are dense in $X_L(\mathbb{R})$. In particular, the map

$$\iota_{\infty}^{\circ} : X_L(\mathbb{Q}) \xrightarrow{\iota_{\infty}} X_L(\mathbb{R}) \rightarrow X_L(\mathbb{R})/X_L(\mathbb{R})_+ \simeq \{\pm\}^{r_1},$$

with $r_1 = \#(I_{L, \mathbb{R}})$, is surjective. Accordingly, one may find a rational idèle $\lambda \in X_L(\mathbb{Q})$ such that $\iota_{\infty}(\lambda)^{-1} z_{\infty} \in X_L(\mathbb{R})_+$. As $\lambda = \iota_{\infty}(\lambda) \iota_f(\lambda)$, this gives

$$z = \lambda(\iota_{\infty}(\lambda^{-1}) z_{\infty})(\iota_f(\lambda^{-1}) z_f) \in X_L(\mathbb{Q}) X_L(\mathbb{R})_+ X_L(\mathbb{A}_f) \subset X_L(\mathbb{A}_f) \overline{X_L(\mathbb{Q}) X_L(\mathbb{R})_+},$$

hence the result.

(ii) It remains to show the inclusions $\overline{X_L(\mathbb{Q})}^f \subset X_L(\mathbb{Q}) \overline{\mathcal{O}_L^{\times}}^f$ and $\overline{X_L(\mathbb{Q})_+}^f \subset X_L(\mathbb{Q})_+ \overline{\mathcal{O}_{L,+}^{\times}}^f$.

Let $z_f = (z_v)_v \in X_L(\mathbb{A}_f)$ belong to the closure $\overline{X_L(\mathbb{Q})}^f$ of $X_L(\mathbb{Q}) = L^{\times}$. For all finite places v of L , we let $\varpi_v \in L_v^{\times}$ be a choice of uniformizer and we set $k_v := \text{ord}_{\varpi_v}(z_v) \in \mathbb{Z}$. Recall that, for all v , if $y_v \in L_v^{\times}$ is such that $\text{ord}_{\varpi_v}(y_v - z_v) > k_v$, then the ultrametric inequality implies that $\text{ord}_{\varpi_v}(y_v) = k_v$. There exists a finite set S of finite places of L such that $z_v \in \mathcal{O}_{L_v}^{\times}$, i.e., $k_v = 0$, for all $v \notin S$. We define an open subset of $L_S^{\times} := \prod_{v \in S} L_v^{\times}$, by setting

$$U_{S,z} := \prod_{v \in S} z_v (1 + \varpi_v^{k_v+1} \mathcal{O}_{L_v}),$$

and we let $U_z := U_{S,z} \times \prod_{v \notin S} \mathcal{O}_{L_v}^\times$ be the corresponding open subset of $X_L(\mathbb{A}_f)$ containing z . Let $(z^{(n)})_{n \geq 0} \in (L^\times)^\mathbb{N}$ be a sequence in L^\times which converges to z_f in $X_L(\mathbb{A}_f)$, and let $N \geq 0$ be some integer satisfying

$$z^{(n)} \in U_z, \forall n \geq N.$$

Thus $\mathrm{ord}_{\varpi_v}(z_v^{(n)}) = k_v$ for all v , i.e., the sequence $((z^{(N)})^{-1}z^{(n)})_{n \geq N}$ in L^\times is made of global units and converges to $(z^{(N)})^{-1}z_f$ in $X_L(\mathbb{A}_f)$. This gives $z_f = z^{(N)}(z^{(N)})^{-1}z_f \in L^\times \overline{\mathcal{O}_{L,+}^\times}^f$, as wanted.

The last inclusion $\overline{X_L(\mathbb{Q})_+} \subset X_L(\mathbb{Q})_+ \cdot \overline{\mathcal{O}_{L,+}^\times}^f$ follows directly from the previous one: if $z \in \overline{X_L(\mathbb{Q})_+}^f$ then the scalars $z^{(n)}$ may be chosen to be totally positive for all $n \geq N$, making $(z^{(N)})^{-1}z^{(n)}$ a totally positive unit for all $n \geq N$, hence $z = z^{(N)} \lim_{n \rightarrow \infty} (z^{(N)})^{-1}z^{(n)} \in X_L(\mathbb{Q})_+ \overline{\mathcal{O}_{L,+}^\times}^f$. \square

An immediate consequence of Lemma 1.3.2, (i), is that the restriction $\mathrm{Art}_L|_{X(\mathbb{A}_f)}$ of Art_L to the finite idèles $X_L(\mathbb{A}_f)$ remains surjective.

Lemma 1.3.3. *The kernel of $\mathrm{Art}_L|_{X_L(\mathbb{A}_f)} : X_L(\mathbb{A}_f) \rightarrow \mathrm{Gal}(L^{ab}/L)$ is equal to*

$$\overline{X_L(\mathbb{Q})_+}^f = X_L(\mathbb{Q})_+ \overline{\mathcal{O}_{L,+}^\times}^f$$

Proof. The last equality being just Lemma 1.3.2, (iii), the proof amounts to showing that $X_L(\mathbb{A}_f) \cap \overline{X_L(\mathbb{Q})X_L(\mathbb{R})_+} = \overline{X_L(\mathbb{Q})}^f$. Indeed, if $z_f \in X_L(\mathbb{A}_f)$ belongs to $\overline{X_L(\mathbb{Q})X_L(\mathbb{R})_+} \subset X_L(\mathbb{A}_\mathbb{Q})$, then one may find sequences $(x^{(n)})_{n \geq 0} \in (X_L(\mathbb{R})_+)^\mathbb{N}$ and $(y^{(n)})_{n \geq 0} \in (X_L(\mathbb{Q}))^\mathbb{N}$ such that $x^{(n)}y^{(n)}$ converges to z_f in $X_L(\mathbb{A}_\mathbb{Q})$, i.e., such that $x^{(n)}\iota_\infty(y^{(n)})$ converges to $1 \in X_L(\mathbb{R})$ and $\iota_f(y^{(n)})$ converges to z_f in $X_L(\mathbb{A}_f)$. In particular, the element $x^{(n)}\iota_\infty(y^{(n)})$ lies in $X_L(\mathbb{R})_+$ for $n \gg 0$, which means that $\iota_\infty(y^{(n)}) \in X_L(\mathbb{R})_+$, i.e., that $x^{(n)}$ lies in $X_L(\mathbb{Q})_+$ for $n \gg 0$. In other words, one has $z_f \in \overline{X_L(\mathbb{Q})_+}^f$.

Conversely, if $z_f \in X_L(\mathbb{A}_f)$ belongs to $\overline{X_L(\mathbb{Q})_+}^f$ then one may find a sequence $(y^{(n)})_{n \geq 0} \in (X_L(\mathbb{Q})_+)^\mathbb{N}$ which converges to z_f . In other words, the sequence

$$(\iota_\infty(y^{(n)})^{-1}y^{(n)})_{n \geq 0} \in (X_L(\mathbb{R})_+X_L(\mathbb{Q}))^\mathbb{N}$$

converges to z_f in $X_L(\mathbb{A}_\mathbb{Q})$, i.e., $z_f \in X_L(\mathbb{A}_f) \cap \overline{X_L(\mathbb{R})_+X_L(\mathbb{Q})}$. \square

From now on - as we will mainly be dealing with finite idèles - we shall drop the superscript \square^f from the notation of the topological closure (in $X_L(\mathbb{A}_f)$) of subsets of $Y \subset X_L(\mathbb{Q})$. We shall also slightly abuse notations and still denote by $\mathrm{Art}_L : X_L(\mathbb{A}_f) \twoheadrightarrow \mathrm{Gal}(L^{ab}/L)$ the above restricted Artin map. The previous lemma provides us with the following isomorphism:

$$\mathrm{Art}_L : X_L(\mathbb{A}_f) / X_L(\mathbb{Q})_+ \overline{\mathcal{O}_{L,+}^\times} \simeq \mathrm{Gal}(L^{ab}/L). \quad (1.24)$$

1.3.3 The case of F and E and the transfer map.

The non-trivial element $c : (z \mapsto \bar{z})$ of $\text{Gal}(E/F)$ induces an automorphism of the F -torus $\text{Res}_{E/F}\mathbb{G}_{m,E}$, by

$$c : \text{Res}_{E/F}\mathbb{G}_{m,E} \rightarrow \text{Res}_{E/F}\mathbb{G}_{m,E},$$

$$z \longmapsto \bar{z}$$

defined, for any F -algebra R , by letting c act on the left-most component of $(E \otimes_F R)^\times$ only. One may then define the following map:

$$\nu : \text{Res}_{E/F}\mathbb{G}_{m,E} \longrightarrow \text{U}(1)_{E/F},$$

$$z \longmapsto \frac{\bar{z}}{z} \tag{1.25}$$

The following sequence of F -groups

$$1 \longrightarrow \mathbb{G}_{m,F} \longrightarrow \text{Res}_{E/F}\mathbb{G}_{m,E} \xrightarrow{\nu} \text{U}(1) \longrightarrow 1$$

is exact (see [7], §II.1.3.3 for details). The Weil restriction functor from F to \mathbb{Q} enables us to define the following \mathbb{Q} -tori:

$$\mathbf{Z} := X_F = \text{Res}_{F/\mathbb{Q}}\mathbb{G}_{m,F} \text{ and } \mathbf{T} := X_E = \text{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E} = \text{Res}_{F/\mathbb{Q}}(\text{Res}_{E/F}\mathbb{G}_{m,E}).$$

Recall that, by definition, the torus $\mathbf{T}^1 = \text{Res}_{F/\mathbb{Q}}\text{U}(1)_{E/F} \subset \mathbf{T}$ may be identified with the kernel of the *norm map* $N_{E/F} : \mathbf{T} \rightarrow \mathbf{Z}$. Accordingly, for each $\star \in \{V, W\}$, the diagonal inclusion $z \mapsto z \cdot \mathbf{1}$ identifies \mathbf{T}^1 with the center $\mathbf{Z}_{\mathbf{G}_\star}$ of \mathbf{G}_\star .

According to ([12], A.5), the Weil-restriction is a left-exact functor and preserves surjectivity between smooth morphisms ([12], Corollary A.5.4, (1)). Consequently, the following sequence of \mathbb{Q} -tori

$$1 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{T} \longrightarrow \mathbf{T}^1 \longrightarrow 1 \tag{1.26}$$

remains exact. By assumptions on E and F one has $I_{F,\infty} = I_{F,\mathbb{R}}$ whereas $I_{E,\infty} = I_{E,\mathbb{C}}$, i.e., $\mathbf{Z}(\mathbb{R})_+ \simeq (\mathbb{R}_{>0})^d$ and $\mathbf{T}(\mathbb{R})_+ = \mathbf{T}(\mathbb{R}) \simeq (\mathbb{C}^\times)^d$. Equality (1.24) translates into the following isomorphisms:

$$\text{Art}_F : \mathbf{Z}(\mathbb{A}_f) / \mathbf{Z}(\mathbb{Q})_+ \overline{\mathcal{O}_{F,+}^\times} \xrightarrow{\sim} \text{Gal}(F^{ab}/F) \tag{1.27}$$

$$\text{Art}_E : \mathbf{T}(\mathbb{A}_f) / \mathbf{T}(\mathbb{Q}) \overline{\mathcal{O}_E^\times} \xrightarrow{\sim} \text{Gal}(E^{ab}/E) \tag{1.28}$$

The Artin map satisfies one last functoriality, with respect to the group-theoretic transfer map between $\text{Gal}(F^{ab}/F)$ and $\text{Gal}(E^{ab}/E)$, which is a continuous group homomorphism

$$\text{Ver}_{E/F} : \text{Gal}(F^{ab}/F) \rightarrow \text{Gal}(E^{ab}/E). \tag{18}$$

Let us denote by $\iota_{E/F} : \mathbf{Z} \hookrightarrow \mathbf{T}$ the inclusion induced by $F \hookrightarrow E$. By taking the inverse limit of the commutative diagrams provided in ([39], Proposition 2.9), one obtains that the following diagram

$$\begin{array}{ccc} \mathbf{Z}(\mathbb{A}_f) & \xrightarrow{\mathrm{Art}_F} & \mathrm{Gal}(F^{ab}/F) \\ \iota_{E/F} \downarrow & & \downarrow \mathrm{Ver}_{E/F} \\ \mathbf{T}(\mathbb{A}_f) & \xrightarrow{\mathrm{Art}_E} & \mathrm{Gal}(E^{ab}/E) \end{array} \quad (1.29)$$

is commutative. We denote by $F_{\gg 0} := \mathbf{Z}(\mathbb{Q})_+$ the subgroup of totally positive elements in F^\times . The following Proposition is a very slight modification of [7], Lemma 64:

Proposition 1.3.4 (Kernel of $\mathrm{Ver}_{E/F}$). *The kernel of $\mathrm{Ver}_{E/F}$ is isomorphic to the quotient*

$$\mathbf{Z}(\mathbb{Q}) / \mathbf{Z}(\mathbb{Q})_+ = F^\times \overline{\mathcal{O}_{F,+}^\times} / F_{\gg 0} \overline{\mathcal{O}_{F,+}^\times},$$

which receives a surjection from $F^\times / F_{\gg 0} \simeq \{\pm\}^d$ (in particular, $\ker(\mathrm{Ver}_{E/F})$ is finite).

Proof. By chasing through the diagram 1.29 one obtains that

$$\begin{aligned} \ker(\mathrm{Ver}_{E/F}) &= \mathrm{Art}_F \left(\iota_{E/F}^{-1}(\ker \mathrm{Art}_E) \right) \\ &= \mathrm{Art}_F \left(\mathbf{Z}(\mathbb{A}_f) \cap \mathbf{T}(\mathbb{Q}) \overline{\mathcal{O}_E^\times} \right) \end{aligned}$$

By Dirichlet's unit theorem, both groups $\mathcal{O}_F^\times \subset \mathcal{O}_E^\times$ are free \mathbb{Z} -modules of rank $d-1$, hence $[\mathcal{O}_E^\times : \mathcal{O}_F^\times] < \infty$. The choice of a family $t_1, \dots, t_r \in \mathcal{O}_E^\times$ of coset representatives for $\mathcal{O}_E^\times / \mathcal{O}_F^\times$ enables us to write

$$\overline{\mathcal{O}_E^\times} = \overline{\bigsqcup_{i=1}^r t_i \mathcal{O}_F^\times} = \bigcup_{i=1}^r t_i \overline{\mathcal{O}_F^\times},$$

therefore $E^\times \overline{\mathcal{O}_E^\times} = E^\times \overline{\mathcal{O}_F^\times}$. Similarly, one has $[\mathcal{O}_F^\times : \mathcal{O}_{F,+}^\times] < \infty$, hence $F^\times \overline{\mathcal{O}_F^\times} = F^\times \overline{\mathcal{O}_{F,+}^\times}$. This gives

$$\begin{aligned} \ker(\mathrm{Ver}_{E/F}) &= \mathrm{Art}_F \left(\mathbb{A}_{F,f}^\times \cap E^\times \overline{\mathcal{O}_F^\times} \right) \\ &= \mathrm{Art}_F \left(F^\times \overline{\mathcal{O}_{F,+}^\times} \right), \end{aligned}$$

as $\overline{\mathcal{O}_F^\times} \subset A_{F,f}^\times$ and $E^\times \cap \iota_{E/F}(\mathbb{A}_{F,f}^\times) = F^\times$. One obtains that

$$\ker(\mathrm{Ver}_{E/F}) \simeq \frac{F^\times \overline{\mathcal{O}_{F,+}^\times}}{F_{\gg 0} \overline{\mathcal{O}_{F,+}^\times}},$$

(18). The notation Ver stands for the german *Verlagerung* (shift, or transfer). The group-theoretic transfer map $\mathrm{Ver} : G^{ab} \rightarrow H^{ab}$ exists whenever H is a finite-index subgroup of G , and we refer to [39], chapter II for a precise definition.

which receives a surjection from $F^\times/F_{\gg 0}$ ⁽¹⁹⁾. Finally, the isomorphism between $F^\times/F_{\gg 0}^\times$ and $\{\pm\}^d \simeq (\mathbb{Z}/2\mathbb{Z})^d$ is given by the map

$$F^\times \longrightarrow \{\pm\}^d$$

$$z \mapsto (\text{sgn}(\sigma(z)))_{\sigma \in I_{F, \mathbb{R}}},$$

which is surjective (by weak approximation for F) and whose kernel is precisely $F^{\gg 0 \times}$. \square

1.3.4 Galois action on $\pi_0(\text{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y))$.

We shall switch back for a moment to the world of Shimura varieties, in order to compute the Galois action on the set $\pi_0(\text{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y))$ of connected components of the Shimura variety $\text{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$, via global class field theory. Throughout this subsection, K might refer to any neat compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$, and we have set $K_{\mathbf{H}} = \mathbf{H}(\mathbb{A}_f) \cap K$.

1.3.4.1 The determinant map.

Let us start by the following important result:

Lemma 1.3.5. *For each $\star \in \{V, W\}$, the map $\det : \mathbf{G}_\star(\mathbb{A}_f) \rightarrow \mathbf{T}^1(\mathbb{A}_f)$ is surjective.*

Proof. We adapt here the proof of ([32], Lemma 5.21) to our situation. We will show that the map $\det : \mathbf{G}_\star(\mathbb{Q}_\ell) \rightarrow \mathbf{T}^1(\mathbb{Q}_\ell)$ is surjective for all rational prime ℓ , and that the corresponding map between the respective smooth $\mathcal{O}_F^{S^1}$ -models for $\underline{\mathbf{U}}_\star$ and $\mathbf{U}(1)_{E/F} = \mathbf{Z}_{\mathbf{U}(\star)}$,

$$\underline{\det} : \underline{\mathbf{U}}_\star \rightarrow \underline{\mathbf{T}}^1,$$

is surjective on (\mathcal{O}_{F_v}) points, for all $v \notin S^1$. This will imply, for all rational prime ℓ not divided by elements of S^1 , that one has a surjection:

$$\det : \mathbf{T}(\mathbb{Z}_\ell) := \prod_{v|\ell} \underline{\mathbf{U}}_\star(\mathcal{O}_{F_v}) \twoheadrightarrow \prod_{v|\ell} \underline{\mathbf{T}}^1(\mathcal{O}_{F_v}) =: \mathbf{T}^1(\mathbb{Z}_\ell).$$

For each rational prime ℓ , one has an exact sequence:

$$1 \longrightarrow \mathbf{H}^{\text{der}}(\overline{\mathbb{Q}}_\ell) \longrightarrow \mathbf{H}(\overline{\mathbb{Q}}_\ell) \xrightarrow{\underline{\det}} \underline{\mathbf{T}}^1(\overline{\mathbb{Q}}_\ell) \longrightarrow 1$$

which induces the following long exact sequence in Galois cohomology,

$$1 \longrightarrow \mathbf{H}^{\text{der}}(\mathbb{Q}_\ell) \longrightarrow \mathbf{H}(\mathbb{Q}_\ell) \xrightarrow{\det} \mathbf{T}^1(\mathbb{Q}_\ell) \longrightarrow H^1(\mathbb{Q}_\ell, \mathbf{H})$$

(19). To our knowledge, one should refrain from expecting it to be an isomorphism in general, i.e., one could still expect to find some number field F such that $F_{\gg 0} \overline{\mathcal{O}_{F,+}^\times} \subsetneq F^\times \overline{\mathcal{O}_{F,+}^\times}$, i.e., where $\overline{\mathcal{O}_{F,+}^\times} \cap \mathcal{O}_F^\times \subset \widehat{\mathcal{O}_F^\times}$ is strictly larger than $\mathcal{O}_{F,+}^\times$.

As \mathbf{H} is isomorphic (over $\overline{\mathbb{Q}}$) to GL_2 , then $\mathbf{H}^{\mathrm{der}} = \mathrm{SL}_2$ is semisimple, simply connected and one has $H^1(\mathbb{Q}_\ell, \mathbf{H}) = 0$ by ([32], lemma 5.19-a), thus $\det : \mathbf{H}(\mathbb{Q}_\ell) \rightarrow \mathbf{T}^1(\mathbb{Q}_\ell)$ is surjective.

Now, let $v \notin S^1$ be a finite place of F . The smooth $\mathcal{O}_F^{S^1}$ -models $\underline{\mathbf{U}}_\star^{\mathrm{der}}$ and $\underline{\mathbf{T}}^1$, for $\mathrm{U}(\star)$ and $\mathrm{U}(1)_{E/F}$ respectively - recall that we identify $\mathrm{U}(1)_{E/F}$ and $\underline{\mathbf{T}}^1$ with the respective centers of $\mathrm{U}(\star)$ and $\underline{\mathbf{U}}_\star$ - have connected reductive fibers, hence so does the kernel $\underline{\mathbf{U}}_\star^{\mathrm{der}} := \ker(\underline{\det} : \underline{\mathbf{U}}_\star \rightarrow \underline{\mathbf{T}}^1)$. After base change to $\mathrm{Spec} \mathcal{O}_{F_v}$, one gets an exact sequence

$$1 \longrightarrow (\underline{\mathbf{U}}_\star^{\mathrm{der}})_{\mathcal{O}_{F_v}} \longrightarrow \underline{\mathbf{U}}_{\star \mathcal{O}_{F_v}} \xrightarrow{\underline{\det}} (\underline{\mathbf{T}}^1)_{\mathcal{O}_{F_v}} \longrightarrow 1$$

of group schemes over \mathcal{O}_{F_v} , which gives another exact sequence

$$1 \longrightarrow (\underline{\mathbf{U}}_\star^{\mathrm{der}})_{\mathbb{F}_v} \longrightarrow \underline{\mathbf{U}}_{\star \mathbb{F}_v} \xrightarrow{\underline{\det}} (\underline{\mathbf{T}}^1)_{\mathbb{F}_v} \longrightarrow 1,$$

by base change to $\mathrm{Spec} \mathbb{F}_v$. Let p belong to $\underline{\mathbf{T}}^1(\mathcal{O}_{F_v})$ and set $Y \subset \underline{\mathbf{U}}_{\star \mathcal{O}_{F_v}}$ to be the smooth scheme over $\mathrm{Spec} \mathcal{O}_{F_v}$ defined by $Y := \underline{\det}^{-1}(p)$. Let us show that $Y(\mathcal{O}_{F_v})$ is non-empty. The preceding exact sequences gives rise to a long exact sequence in Galois cohomology:

$$1 \longrightarrow (\underline{\mathbf{U}}_\star^{\mathrm{der}})_{\mathbb{F}_v}(\mathbb{F}_v) \longrightarrow (\underline{\mathbf{U}}_\star)_{\mathbb{F}_v}(\mathbb{F}_v) \xrightarrow{\underline{\det}} (\underline{\mathbf{T}}^1)_{\mathbb{F}_v}(\mathbb{F}_v) \longrightarrow H^1(\mathbb{F}_v, \underline{\mathbf{U}}_\star^{\mathrm{der}}(\overline{\mathbb{F}_v}))$$

By a theorem of Lang - a short proof of which can be found in [34] - one has $H^1(\mathbb{F}_v, \underline{\mathbf{U}}_\star^{\mathrm{der}}(\overline{\mathbb{F}_v})) = 0$, hence $\underline{\det} : (\underline{\mathbf{U}}_\star)_{\mathbb{F}_v} \rightarrow (\underline{\mathbf{T}}^1)_{\mathbb{F}_v}$ is surjective, thus $Y_{\mathbb{F}_v}(\mathbb{F}_v)$ is non-empty. The non-emptiness of $Y(\mathcal{O}_{F_v})$ - i.e., the surjectivity of $\underline{\det} : \underline{\mathbf{U}}_\star(\mathcal{O}_{F_v}) \rightarrow \underline{\mathbf{T}}^1(\mathcal{O}_{F_v})$ - then follows from a version of Hensel's lemma applied the smooth scheme Y over the henselian ring \mathcal{O}_{F_v} , i.e., from the surjectivity of the map $Y(\mathcal{O}_{F_v}) \rightarrow Y_{\mathbb{F}_v}(\mathbb{F}_v)$ (see [22], Théorème 18.5.17). \square

Lemma 1.2.1 shows that the elements of $\mathrm{N}_{\mathbf{G}}(\mathbf{H}) \subset \mathbf{G}$ are completely determined by their action on D , i.e., that for all \mathbb{Q} -algebra R , the map:

$$\lambda_D : \mathrm{N}_{\mathbf{G}}(\mathbf{H})(R) = \mathbf{H}(R) \cdot \mathbf{Z}_{\mathbf{G}}(R) \longrightarrow \mathbf{G}_V(R)$$

$$\Delta(h) \cdot (z_1, z_2) \longmapsto z_1 \iota(z_2)^{-1}$$

is injective. This enables us to see both groups $\mathbf{H} \stackrel{\iota_{\mathbb{Q}} \circ \Delta^{-1}}{\simeq} \iota(\mathbf{G}_W) \subset \mathbf{G}_V$ and $\mathrm{N}_{\mathbf{G}}(\mathbf{H})$ as subgroups of \mathbf{G}_V , hence to see the product group $\mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \mathbf{H}(\mathbb{A}_f)$ as a subgroup of \mathbf{G}_V containing $\iota_{\mathbb{Q}}(\mathbf{G}_W)$. One may thus extend the determinant map into a sequence of surjective map $\det : \mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \mathbf{H}(\mathbb{A}_f) \twoheadrightarrow \mathbf{T}^1(\mathbb{A}_f)$. We denote by \det^* the map obtained by composing the following row:

$$\mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \mathbf{H}(\mathbb{A}_f) \xrightarrow{\det} \mathbf{T}^1(\mathbb{A}_f) \twoheadrightarrow \frac{\mathbf{T}^1(\mathbb{A}_f)}{\mathbf{T}^1(\mathbb{Q})}$$

Lemma 1.3.6. *The map \det^* induces an isomorphism*

$$\det^* : \frac{\mathbf{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q})\mathbf{H}(\mathbb{A}_f)}{\mathbf{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q})\mathbf{H}^{\text{der}}(\mathbb{A}_f)} \xrightarrow{\sim} \frac{\mathbf{T}^1(\mathbb{A}_f)}{\mathbf{T}^1(\mathbb{Q})}.$$

Proof. In other words, one has to show that $\det^{-1}(\mathbf{T}^1(\mathbb{Q})) = \mathbf{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q})\mathbf{H}^{\text{der}}(\mathbb{A}_f)$, i.e., that $\det^{-1}(\mathbf{T}^1(\mathbb{Q})) \subset \mathbf{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q})\mathbf{H}^{\text{der}}(\mathbb{A}_f)$. Let us first show that the map $\mathbf{H}(\mathbb{Q}) \xrightarrow{\det} \mathbf{T}^1(\mathbb{Q})$ is surjective. Recall that the derived group \mathbf{H}^{der} is simply connected, as it is isomorphic (over $\overline{\mathbb{Q}}$) to $\text{SL}_{2,\overline{\mathbb{Q}}}$ and that the center of \mathbf{H} , $\mathbf{Z}_{\mathbf{H}} \subset \mathbf{H}$ is isomorphic to $\mathbf{Z}_{\mathbf{G}_W} \simeq \mathbf{T}^1$. The composite map $\gamma : \mathbf{Z}_{\mathbf{H}} \hookrightarrow \mathbf{H} \xrightarrow{\det} \mathbf{T}^1$ is given by $z \mapsto z^{\dim_E W} = z^2$. Recall that $\mathbf{Z}_{\mathbf{H}}(\mathbb{R}) = \text{U}(1)(F \otimes_{\mathbb{Q}} \mathbb{R}) = \prod_{i=1}^d \text{U}(1)_{E/F}(\mathbb{R}_{\rho_i}) \simeq (\mathbb{S}^1)^d$, thus γ induces on \mathbb{R} -points the morphism:

$$\begin{aligned} (\mathbb{S}^1)^d &\longrightarrow (\mathbb{S}^1)^d \\ (z_1, \dots, z_d) &\longmapsto (z_1^2, \dots, z_d^2), \end{aligned}$$

which is certainly surjective. From the exact sequence of \mathbb{Q} -morphisms

$$1 \longrightarrow \mathbf{H}^{\text{der}} \longrightarrow \mathbf{H} \xrightarrow{\det} \mathbf{T}^1 \longrightarrow 1$$

one obtains the following commutative diagram, whose rows are exact ⁽²⁰⁾:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{H}^{\text{der}}(\mathbb{Q}) & \longrightarrow & \mathbf{H}(\mathbb{Q}) & \xrightarrow{\det} & \mathbf{T}^1(\mathbb{Q}) \longrightarrow H^1(\mathbb{Q}, \mathbf{H}^{\text{der}}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{H}^{\text{der}}(\mathbb{R}) & \longrightarrow & \mathbf{H}(\mathbb{R}) & \xrightarrow{\det} & \mathbf{T}^1(\mathbb{R}) \longrightarrow H^1(\mathbb{R}, \mathbf{H}^{\text{der}}) \end{array}$$

The natural maps $H^1(\mathbb{Q}, \mathbf{H}^{\text{der}}) \rightarrow H^1(\mathbb{Q}_{\ell}, \mathbf{H}^{\text{der}})$ - for ℓ running among all the places of \mathbb{Q} - satisfy the following *Hasse principle* ([42], Theorem 6.6):

$$H^1(\mathbb{Q}, \mathbf{H}^{\text{der}}) \hookrightarrow \prod_{\ell \leq \infty} H^1(\mathbb{Q}_{\ell}, \mathbf{H}^{\text{der}}).$$

As $H^1(\mathbb{Q}_{\ell}, \mathbf{H}^{\text{der}}) = 0$ for all finite prime ℓ (this is [42], Theorem 6.4), one gets that the right-hand vertical arrow $H^1(\mathbb{Q}, \mathbf{H}^{\text{der}}) \rightarrow H^1(\mathbb{R}, \mathbf{H}^{\text{der}})$ is in fact injective. Let t be an element of $\mathbf{T}^1(\mathbb{Q}) \subset \mathbf{T}^1(\mathbb{R})$: the surjectivity of $\gamma(\mathbb{R})$ ensures that $t = \det(z)$ for some $z \in \mathbf{Z}_{\mathbf{H}}(\mathbb{R}) \subset \mathbf{H}(\mathbb{R})$, hence $t \in \mathbf{T}^1(\mathbb{R})$ maps to the trivial class of $H^1(\mathbb{R}, \mathbf{H}^{\text{der}})$. By injectivity of $H^1(\mathbb{Q}, \mathbf{H}^{\text{der}}) \rightarrow H^1(\mathbb{R}, \mathbf{H}^{\text{der}})$ and exactness of the top row, one gets that t lies in $\det(\mathbf{H}(\mathbb{Q}))$, i.e., that the map $\mathbf{H}(\mathbb{Q}) \xrightarrow{\det} \mathbf{T}^1(\mathbb{Q})$ is surjective.

Now, let $g = n_{\mathbb{Q}}h$ be an element of $\mathbf{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q})\mathbf{H}(\mathbb{A}_f)$ satisfying $\det^*(g) = \det(n_{\mathbb{Q}})\det(h) \in \mathbf{T}^1(\mathbb{Q})$. As $\det(n_{\mathbb{Q}})$ is, by definition, equal to $\det(z_{1,\mathbb{Q}}z_{2,\mathbb{Q}}^{-1}) \in \mathbf{T}^1(\mathbb{Q})$, with $n_{\mathbb{Q}} = \Delta(h_{\mathbb{Q}})(z_{1,\mathbb{Q}}, z_{2,\mathbb{Q}}) \in$

(20). The two rows of the diagram are obtained from taking the long exact sequences in group cohomology attached to the short exact sequences $1 \rightarrow \mathbf{H}^{\text{der}}(\overline{\mathbb{Q}}) \rightarrow \mathbf{H}(\overline{\mathbb{Q}}) \xrightarrow{\det} \mathbf{T}^1(\overline{\mathbb{Q}}) \rightarrow 1$ and $1 \rightarrow \mathbf{H}^{\text{der}}(\mathbb{C}) \rightarrow \mathbf{H}(\mathbb{C}) \xrightarrow{\det} \mathbf{T}^1(\mathbb{C}) \rightarrow 1$, with respect to the groups $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\text{Gal}(\mathbb{C}/\mathbb{R})$.

$\mathbf{H}(\mathbb{Q}) \cdot \mathbf{Z}_{\mathbf{G}}(\mathbb{Q})$, one obtains that $\det(h) \in \mathbf{T}^1(\mathbb{Q})$. Accordingly, one may find some $h_t \in \mathbf{H}(\mathbb{Q})$ with $\det(h_t^{-1}h) = 1$, hence $h_t^{-1}h \in \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)$. Thus, $g = n_{\mathbb{Q}}h_t(h_t^{-1}h) \in N_{\mathbf{G}}(\mathbf{H})(\mathbb{Q})\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)$, as wanted. \square

1.3.4.2 The zero-dimensional Shimura variety $\pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y))$.

Recall that we saw at (1.11) that $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C})$ splits as a disjoint union of connected components

$$\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C}) = \bigsqcup_{j=1}^{r_Y} \Gamma_{\mathbf{H},j} \backslash Y,$$

where for all j , $\Gamma_{\mathbf{H},j}$ is the arithmetic subgroup $h_j K_{\mathbf{H}} h_j^{-1} \cap \mathbf{H}(\mathbb{Q})$ of $\mathbf{H}(\mathbb{Q})$, and where $\{h_1, \dots, h_{r_Y}\} \subset \mathbf{H}(\mathbb{A}_f)$ is a system of representatives for the double quotient $\mathbf{H}(\mathbb{Q}) \backslash \mathbf{H}(\mathbb{A}_f) / K_{\mathbf{H}}$. The induced bijection between $\pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C}))$ and $\mathbf{H}(\mathbb{Q}) \backslash \mathbf{H}(\mathbb{A}_f) / K_{\mathbf{H}}$ turns out to have some additional structure.

More precisely, we recall that the composite map $\gamma : \mathbf{T}^1 \simeq \mathbf{Z}_{\mathbf{H}} \hookrightarrow \mathbf{H} \xrightarrow{\det} \mathbf{T}^1$ is surjective on \mathbb{R} -points (see e.g., the proof of Lemma 1.3.6), therefore one has $\mathbf{T}^1(\mathbb{Q})^\dagger := \mathbf{T}^1(\mathbb{Q}) \cap (\mathrm{Im}(\mathbf{T}^1(\mathbb{R}) \xrightarrow{\gamma} \mathbf{T}^1(\mathbb{R}))) = \mathbf{T}^1(\mathbb{Q})$, in the notations of ([32], §5). As $\mathbf{H}_{\overline{\mathbb{Q}}}^{\mathrm{der}} \simeq \mathrm{SL}_{2,\overline{\mathbb{Q}}}$ is a simply connected reductive group, one may apply a theorem of Deligne - a simplified version of which is given in ([32], Theorem 5.17) - and get the following isomorphisms:

$$\begin{aligned} \pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)) &= \pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C})) \\ &\simeq \mathbf{H}(\mathbb{Q}) \backslash \mathbf{H}(\mathbb{A}_f) / K_{\mathbf{H}} \xrightarrow{\det} \mathbf{T}^1(\mathbb{A}_f) / \mathbf{T}^1(\mathbb{Q}) \det(K_{\mathbf{H}}). \end{aligned} \quad (1.30)$$

The latter is endowed with a structure of a *zero-dimensional Shimura variety* $\mathrm{Sh}_{\det(K_{\mathbf{H}})}(\mathbf{T}^1, \{\det(\mu_{W,\mathbb{C}})\})$, such as defined in [32], §5, whose points are (by definition) all special, and the canonical model $M_K(\mathbf{H}, Y)$ for $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$ over E induces a canonical model for the zero-dimensional Shimura variety $\mathrm{Sh}_{\det(K_{\mathbf{H}})}(\mathbf{T}^1, \{\det(\mu_{W,\mathbb{C}})\})$ (see [32], §13). Moreover, the co-character $\det \circ \mu_{W,\mathbb{C}} : \mathbb{G}_{m,\mathbb{C}} \rightarrow \mathbf{T}_{\mathbb{C}}^1$ is certainly defined over E (as the conjugacy-class of $\mu_{W,\mathbb{C}}$ is, and \mathbf{T}^1 is commutative).

1.3.4.3 The adelic reflex map $r_{\det(\mu_{W,\mathbb{C}}),\mathbb{A}}$.

Recall that we keep on identifying the abstract number field E with its image $\tilde{\rho}_1(E) \subset \mathbb{C}$. One may express the *adelic reflex map* $r_{\det(\mu_{W,\mathbb{C}}),\mathbb{A}}$, appearing in the Shimura reciprocity, in the following way:

Proposition 1.3.7. *The adelic reflex map $r_{\det(\mu_{W,\mathbb{C}}),\mathbb{A}}$ is given by:*

$$\begin{aligned} r_{\det(\mu_{W,\mathbb{C}}),\mathbb{A}} : \mathbb{A}_E^\times &\longrightarrow \mathbf{T}^1(\mathbb{A}_f), \\ z = (z_\infty, z_f) &\longmapsto \mathrm{proj}_f\left(\frac{\bar{z}}{z}\right) = \frac{\bar{z}_f}{z_f}, \end{aligned}$$

where $z \mapsto \bar{z}$ (resp. $z_f \mapsto \bar{z}_f$) denotes the action of the complex conjugation $c \in \text{Gal}(E/F)$ on $\mathbb{A}_E = E \otimes_F \mathbb{A}_F$ (resp. on $\mathbb{A}_{E,f} = E \otimes_F \mathbb{A}_{F,f}$).

Proof.

Recall that we showed in § 1.1.3 that $h_{W,\mathbb{C}} : \mathbf{S}_{\mathbb{C}} \rightarrow \mathbf{G}_{W,\mathbb{C}}$ is given, for all \mathbb{C} -algebra S and all $(z_s, z_t) \in \mathbf{S}(S) \simeq S^\times \times S^\times$, by

$$(z_s, z_t) \mapsto \left(\text{diag}\left(1, \frac{z_t}{z_s}, \mathbf{1}_2, \dots, \mathbf{1}_d\right), \mathbf{1}_2, \dots, \mathbf{1}_d \right) \in \mathbf{G}_{W,\mathbb{C}}(S) = \prod_{i=1}^d \mathbf{U}(W)_{\tilde{\rho}_i, \mathbb{C}}(S),$$

where we implicitly identified $\mathbf{U}(W)_{\tilde{\rho}_i, \mathbb{C}}$ with $\text{GL}_{2, \mathbb{C}_{\rho_i}}$ for all $i \in \{1, \dots, d\}$. Accordingly, for all $z \in \mathbb{G}_{m, \mathbb{C}}(S)$ one has:

$$\det(\mu_{W,\mathbb{C}})(z) = (\det(1, z^{-1}), \det(\mathbf{1}_2), \dots, \det(\mathbf{1}_d)) = (z^{-1}, 1, \dots, 1) \in \mathbf{T}_{\mathbb{C}}^1(S),$$

still, with the implicit identification $(\mathbf{U}(1)_{E/F})_{\tilde{\rho}_i, \mathbb{C}}(S) \simeq \text{GL}_1(S_{\rho_i})$, for all i ⁽²¹⁾. When applying the morphism $\det(\mu_{W,\mathbb{C}})$ to E -algebras, one *may not* properly use the above expression, which only makes sense for \mathbb{C} -algebras. However, we recall that for all $i \in \{1, \dots, d\}$, the field $\mathbb{R}_{\rho_i} \otimes_F E = \mathbb{C}_{\rho_i}$ is the copy of \mathbb{C} endowed with the E -algebra structure given by $\tilde{\rho}_i$. Accordingly, if R is any \mathbb{Q} -algebra then one has:

$$\mathbb{R} \otimes_{\mathbb{Q}} E \otimes_{\mathbb{Q}} R \simeq \mathbb{R} \otimes_{\mathbb{Q}} F \otimes_F E \otimes_{\mathbb{Q}} R = \prod_{i=1}^r \mathbb{C}_{\rho_i} \otimes_{\mathbb{Q}} R.$$

That $\det(\mu_{W,\mathbb{C}})$ is a morphism of group schemes over E implies that the following diagram

$$\begin{array}{ccc} \mathbb{G}_m(E \otimes_{\mathbb{Q}} R) & \xrightarrow{\det(\mu_{W,\mathbb{C}})} & \mathbf{T}^1(E \otimes_{\mathbb{Q}} R) \\ \downarrow z \mapsto (\tilde{\rho}_i(z))_{i=1}^d & & \downarrow t \mapsto (\tilde{\rho}_i(t))_{i=1}^d \\ \mathbb{G}_m(\mathbb{R} \otimes_{\mathbb{Q}} E \otimes_{\mathbb{Q}} R) = \mathbb{G}_m\left(\prod_{i=1}^r \mathbb{C}_{\rho_i} \otimes_{\mathbb{Q}} R\right) & \xrightarrow[s \mapsto (s^{-1}, 1, \dots, 1)]{\det(\mu_{W,\mathbb{C}})} & \mathbf{T}^1(\mathbb{R} \otimes_{\mathbb{Q}} E \otimes_{\mathbb{Q}} R) \end{array}$$

is commutative. For the same reasons is the diagram

$$\begin{array}{ccc} \mathbb{G}_m(E \otimes_{\mathbb{Q}} R) & \xrightarrow{\det(\mu_{W,\mathbb{C}})} & \mathbf{T}^1(E \otimes_{\mathbb{Q}} R) \\ \downarrow c & & \downarrow c \\ \mathbb{G}_m(E \otimes_{\mathbb{Q}} R) & \xrightarrow{\det(\mu_{W,\mathbb{C}})} & \mathbf{T}^1(E \otimes_{\mathbb{Q}} R) \end{array}$$

commutative. The equality $\overline{\tilde{\rho}_i} = \tilde{\rho}_i \circ c$, for all $i \in \{1, \dots, d\}$, enables us to write the following: for all $z \in (E \otimes_{\mathbb{Q}} R)^\times$, one has

$$\tilde{\rho}_1 \circ (\det(\mu_{W,\mathbb{C}})(z)) = \tilde{\rho}_1(z)^{-1} = \tilde{\rho}_1(z^{-1}), \quad \overline{\tilde{\rho}_1} \circ (\det(\mu_{W,\mathbb{C}})(z)) = \tilde{\rho}_1((c \cdot z)^{-1}) = \tilde{\rho}_1(c \cdot z^{-1}), \quad (1.31)$$

$$\text{and } \tilde{\rho}_i(\det(\mu_{W,\mathbb{C}})(z)) = \overline{\tilde{\rho}_i}(\det(\mu_{W,\mathbb{C}})(z)) = 1, \quad \text{for } i \geq 2. \quad (1.32)$$

(21). This is alright, since the identifications $\mathbf{U}(\star)_E \simeq \text{GL}(\star_E)$, $\star \in \{V, W\}$, and $(\mathbf{U}(1)_{E/F})_E \simeq \text{GL}_{1,E}$ commute with the determinant maps, as one checks easily.

Consequently, one has for all $z \in (E \otimes_{\mathbb{Q}} R)^{\times}$:

$$\begin{aligned} N_{E/\mathbb{Q}}(\det(\mu_{W,\mathbb{C}})(z)) &\stackrel{\text{def}}{=} \prod_{i=1}^d \tilde{\rho}_i(\det(\mu_{W,\mathbb{C}})(z)) \times \overline{\tilde{\rho}_i}(\det(\mu_{W,\mathbb{C}})(z)) \\ &\stackrel{(1.31)+(1.32)}{=} \tilde{\rho}_1(z^{-1}) \times \tilde{\rho}_1(c \cdot z^{-1}) \\ &\stackrel{E \leftrightarrow \tilde{\rho}_1(E)}{=} z^{-1}(c \cdot z^{-1}) \end{aligned} \quad (1.33)$$

We may now apply the preceding to the case $R = \mathbb{A}$. This gives, for all $z \in (E \otimes_{\mathbb{Q}} \mathbb{A})^{\times} = \mathbb{A}_E^{\times}$, that:

$$\begin{aligned} r_{\det(\mu_{W,\mathbb{C}}),\mathbb{A}}(z) &\stackrel{\text{def}}{=} \mathrm{proj}_f(N_{E/\mathbb{Q}}(\det(\mu_{W,\mathbb{C}})(z))) \\ &\stackrel{(1.33)}{=} \mathrm{proj}_f(z^{-1}(c \cdot z^{-1})) \\ &= z_f^{-1}(c \cdot z_f^{-1}) \\ &\stackrel{\#}{=} \frac{\overline{z_f}}{z_f} \end{aligned}$$

Equality $\#$ follows from (1.13) and from our implicit identification $\mathfrak{z} \mapsto \mathfrak{z}_t$ between $\mathrm{U}(1)(\mathbb{A}_{E,f})$ and $\mathrm{GL}_1(\mathbb{A}_{E,f})$. Namely, using the notations of § 1.1.5, we check that, if $z_f \in \mathrm{GL}_1(\mathbb{A}_{E,f})$ is seen as the \mathfrak{z}_t of some element $\mathfrak{z} \in \mathrm{U}(1)(\mathbb{A}_{E,f})$, then our notation $c \cdot z_f$ actually stands for $(c \cdot \mathfrak{z})_t = (\mathfrak{z}_t^{-1})^c =: \overline{\mathfrak{z}_t}^{-1} \in \mathrm{GL}_1(\mathbb{A}_{E,f})$, according to (1.13). Therefore, the quantity $z_f^{-1}(c \cdot z_f)^{-1}$ corresponds to

$$(\mathfrak{z}^{-1}(c \cdot \mathfrak{z}^{-1}))_t = \mathfrak{z}_t^{-1} \cdot (c \cdot \mathfrak{z}^{-1})_t = \frac{\overline{z_f}}{z_f}.$$

□

We set ν to be the morphism between \mathbb{Q} -groups $\mathbf{T} = \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$ and $\mathbf{T}^1 = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(1)_{E/F} \subset \mathbf{T}$, which corresponds to the Weil restriction (from F to \mathbb{Q}) of (1.25):

$$\begin{aligned} \nu : \mathbf{T} &\longrightarrow \mathbf{T}^1 \\ z &\longmapsto \frac{\overline{z}}{z} \end{aligned} \quad (1.34)$$

This enables us to express the Galois action on $\pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y))$ in the following way:

Corollary 1.3.8 (Galois action on $\pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y))$). *Let $\sigma \in \mathrm{Gal}(E^{ab}/E)$, let $s_{\sigma} \in \mathbb{A}_{E,f}^{\times}$ be such that $\mathrm{Art}_E(s) = \sigma$, let t be an element of $\mathbf{T}^1(\mathbb{A}_f)$ and denote by $\mathcal{C}(t)$ the connected component in $\pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y))$ corresponding via (1.30) to the coset*

$$t \mathbf{T}^1(\mathbb{Q}) \det(K_{\mathbf{H}}) \in \mathbf{T}^1(\mathbb{A}_f) / \mathbf{T}^1(\mathbb{Q}) \det(K_{\mathbf{H}}) \quad (22).$$

One has:

$$\sigma \cdot \mathcal{C}(t) = \mathcal{C}(\nu(s_{\sigma}) t). \quad (1.35)$$

(22). Or equivalently, to the point $[\{\det(\mu_{W,\mathbb{C}})\}, t]_{\det(K_{\mathbf{H}})} \in \mathrm{Sh}_{\det(K_{\mathbf{H}})}(\mathbf{T}^1, \{\det(\mu_{W,\mathbb{C}})\})$.

1.3.5 The field $E(\infty)$ and the map Art_E^1 .

We shall reformulate the Galois action on the set $\pi_0(\text{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y))$ given above in a slightly different way, using the field $E(\infty)$ and the map Art_E^1 , defined as follows. From (1.24) applied to $L = E$, one may form the following map

$$r_{\text{fin}} : \text{Gal}(E^{ab}/E) \twoheadrightarrow \frac{\mathbf{T}^1(\mathbb{A}_f)}{\mathbf{T}^1(\mathbb{Q})} \quad (1.36)$$

obtained by composing the inverse of the (restricted) Artin map

$$\text{Art}_E^{-1} : \text{Gal}(E^{ab}/E) \rightarrow \mathbf{T}(\mathbb{A}_f) / \mathbf{T}(\mathbb{Q}) \overline{\mathcal{O}_E^\times} = \mathbf{T}(\mathbb{A}_f) / \overline{\mathbf{T}(\mathbb{Q})}$$

together with the map $\nu : \mathbf{T}(\mathbb{A}_f) \twoheadrightarrow \mathbf{T}^1(\mathbb{A}_f)$ ⁽²³⁾. That ν indeed maps $\overline{\mathbf{T}(\mathbb{Q})}$ into $\mathbf{T}^1(\mathbb{Q})$ is a consequence of the axiom SV5 being satisfied by the Shimura datum (\mathbf{G}, X) , namely, that $\mathbf{T}^1(\mathbb{Q})$ is discrete in $\mathbf{T}^1(\mathbb{A}_f)$, hence $\mathbf{T}^1(\mathbb{Q}) = \overline{\mathbf{T}^1(\mathbb{Q})}$. By definition, the map r_{fin} makes the following diagram

$$\begin{array}{ccc} \text{Gal}(E^{ab}/E) & \xrightarrow{r_{\text{fin}}} & \mathbf{T}^1(\mathbb{A}_f) / \mathbf{T}^1(\mathbb{Q}) \\ \text{Art}_E \uparrow & & \uparrow \\ \mathbf{T}(\mathbb{A}_f) & \xrightarrow{\nu} & \mathbf{T}^1(\mathbb{A}_f) \end{array}$$

commute. The following Proposition is due to [7] (Proposition VI.14.1):

Proposition 1.3.9. *The following sequence*

$$1 \longrightarrow \ker(\text{Ver}_{E/F}) \longrightarrow \text{Gal}(F^{ab}/F) \xrightarrow{\text{Ver}_{E/F}} \text{Gal}(E^{ab}/E) \xrightarrow{r_{\text{fin}}} \mathbf{T}^1(\mathbb{A}_f) / \mathbf{T}^1(\mathbb{Q}) \longrightarrow 1 \quad (1.37)$$

is exact.

Definition 1.3.1 (The field $E(\infty)$). *The let $E(\infty) \subset E^{ab}$ denote the abelian extension of E fixed by the image of $\text{Ver}_{E/F} : \text{Gal}(F^{ab}/F) \rightarrow \text{Gal}(E^{ab}/E)$.*

By definition of $E(\infty)$, and by Proposition 1.3.9, the map r_{fin} induces an isomorphism between $\text{Gal}(E^{ab}/E) / \text{Ver}_{E/F}(\text{Gal}(F^{ab}/F)) = \text{Gal}(E(\infty)/E)$ and $\mathbf{T}^1(\mathbb{A}_f) / \mathbf{T}^1(\mathbb{Q})$. We set

$$\text{Art}_E^1 : \mathbf{T}^1(\mathbb{A}_f) / \mathbf{T}^1(\mathbb{Q}) \xrightarrow{\sim} \text{Gal}(E(\infty)/E) \quad (1.38)$$

to be the inverse isomorphism. By construction, the map Art_E^1 makes the following diagram

$$\begin{array}{ccc} \mathbf{T}(\mathbb{A}_f) / \mathbf{T}(\mathbb{Q}) & \xrightarrow{\text{Art}_E} & \text{Gal}(E^{ab}/E) \\ \nu \downarrow & & \text{res}_{|E(\infty)} \downarrow \\ \mathbf{T}^1(\mathbb{A}_f) / \mathbf{T}^1(\mathbb{Q}) & \xrightarrow[\text{Art}_E^1]{\sim} & \text{Gal}(E(\infty)/E) \end{array}$$

(23). The surjectivity of $\nu : \mathbf{T}(\mathbb{A}_f) \twoheadrightarrow \mathbf{T}^1(\mathbb{A}_f)$ follows from the surjectivity of $\nu : E_v^\times \rightarrow \text{U}(1)_{E/F}(F_v)$ for all $v \in I_{F,f}$ (which is Hilbert's Theorem 90 when v is inert or ramified in E/F), and from the surjectivity of $\nu : \mathcal{O}_{E_v}^\times \rightarrow \text{U}(1)_{E/F}(\mathcal{O}_{F_v})$ at every unramified v (see the beginning of §1.3.8.1).

commute. By Corollary 1.3.8, one obtains that the action of $\sigma \in \mathrm{Gal}(E^{ab}/E)$ on $\pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y))$ depends only on the restriction $\sigma|_{E(\infty)}$, i.e., that the elements of $\pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y))$ are all defined over the transfer field $E(\infty)$. The above diagram enables us to rephrase Corollary 1.3.8 into the following:

Corollary 1.3.10. *Let $\sigma \in \mathrm{Gal}(E(\infty)/E)$ and let $t_\sigma \in \mathbf{T}^1(\mathbb{A}_f)$ be such that $\mathrm{Art}_E^1(t_\sigma) = \sigma$. For all $t \in \mathbf{T}^1(\mathbb{A}_f)$, we let $\mathcal{C}(t) \in \pi_0(\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y))$ be the connected component of $\mathrm{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$ corresponding⁽²⁴⁾ to the coset*

$$t \mathbf{T}^1(\mathbb{Q}) \det(K_{\mathbf{H}}) \in \mathbf{T}^1(\mathbb{A}_f) / \mathbf{T}^1(\mathbb{Q}) \det(K_{\mathbf{H}}).$$

One has:

$$\sigma \cdot \mathcal{C}(t) = \mathcal{C}(t_\sigma t), \quad \forall t \in \mathbf{T}^1(\mathbb{A}_f) \quad (1.39)$$

1.3.6 The Galois action on $\mathcal{Z}_K(\mathbf{G}, \mathbf{H})$.

We now wish to translate the above into an action on the set $\mathcal{Z}_K(\mathbf{G}, \mathbf{H})$ of \mathbf{H} -special cycles of level K . Let us start with the following useful lemma:

Lemma 1.3.11. *Let $K_{\mathbf{H}}$ be a compact open subgroup of $\mathbf{H}(\mathbb{A}_f)$, K be a compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$, and let $\overline{\mathbf{H}(\mathbb{Q})}$ be the closure of $\mathbf{H}(\mathbb{Q})$ in $\mathbf{H}(\mathbb{A}_f)$ for the finite-adelic topology. One has*

$$\mathbf{H}(\mathbb{Q}) \backslash \mathbf{H}(\mathbb{A}_f) / K_{\mathbf{H}} = \overline{\mathbf{H}(\mathbb{Q})} \backslash \mathbf{H}(\mathbb{A}_f) / K_{\mathbf{H}} = \mathbf{H}(\mathbb{Q}) \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f) \backslash \mathbf{H}(\mathbb{A}_f) / K_{\mathbf{H}} \quad (1.40)$$

and

$$N_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f) / K = \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \overline{\mathbf{H}(\mathbb{Q})} \backslash \mathbf{G}(\mathbb{A}_f) / K = N_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f) \backslash \mathbf{H}(\mathbb{A}_f) / K. \quad (1.41)$$

Proof. The first equality of (1.40), $\mathbf{H}(\mathbb{Q}) \backslash \mathbf{H}(\mathbb{A}_f) / K_{\mathbf{H}} = \overline{\mathbf{H}(\mathbb{Q})} \backslash \mathbf{H}(\mathbb{A}_f) / K_{\mathbf{H}}$, follows from the fact that, for all $h \in \mathbf{H}(\mathbb{A}_f)$, one has $\overline{\mathbf{H}(\mathbb{Q})} h = \overline{\mathbf{H}(\mathbb{Q})} h$, hence $\overline{\mathbf{H}(\mathbb{Q})} h K_{\mathbf{H}} = \overline{\mathbf{H}(\mathbb{Q})} h K_{\mathbf{H}} = \mathbf{H}(\mathbb{Q}) h K_{\mathbf{H}}$, as $K_{\mathbf{H}}$ is an open subgroup. The *strong approximation* theorem ([32], Theorem 4.16) applied to the semisimple, simply connected group $\mathbf{H}^{\mathrm{der}}$ implies that $\mathbf{H}^{\mathrm{der}}(\mathbb{Q})$ is dense in $\mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)$. By axiom SV6 the subgroup $\mathbf{T}^1(\mathbb{Q}) \subset \mathbf{T}^1(\mathbb{A}_f)$ is discrete, hence closed, thus $\det^{-1}(\mathbf{T}^1(\mathbb{Q}))$ is a closed subgroup of $\mathbf{H}(\mathbb{A}_f)$. This gives:

$$\mathbf{H}(\mathbb{Q}) \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f) \subset \overline{\mathbf{H}(\mathbb{Q})} \subset \overline{\det^{-1}(\mathbf{T}^1(\mathbb{Q}))} = \det^{-1}(\mathbf{T}^1(\mathbb{Q})) = \mathbf{H}(\mathbb{Q}) \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f),$$

the last equality coming from the surjectivity of $\mathbf{H}(\mathbb{Q}) \xrightarrow{\det} \mathbf{T}^1(\mathbb{Q})$ shown in the preceding lemma. This shows that $\mathbf{H}(\mathbb{Q}) \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f) = \overline{\mathbf{H}(\mathbb{Q})}$ and concludes for (1.40). The equalities of

(24). Still via (1.30).

(1.41) follow directly after noticing that $\mathbf{H}(\mathbb{A}_f) \subset \mathbf{G}(\mathbb{A}_f)$ is a closed subgroup, hence $\overline{\mathbf{H}(\mathbb{Q})}$ coincides with the closure of $\mathbf{H}(\mathbb{Q})$ in $\mathbf{G}(\mathbb{A}_f)$, which gives:

$$\begin{aligned} N_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) g K &= \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \mathbf{H}(\mathbb{Q}) g K = \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \overline{\mathbf{H}(\mathbb{Q})} g K = \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \overline{\mathbf{H}(\mathbb{Q})} g K \\ &= \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \mathbf{H}(\mathbb{Q}) \mathbf{H}^{\text{der}}(\mathbb{A}_f) g K = N_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \mathbf{H}^{\text{der}} g K, \end{aligned}$$

for all $g \in \mathbf{G}(\mathbb{A}_f)$. □

Let us now translate the Galois action of $\text{Gal}(E(\infty)/E)$, defined for any level $K_{\mathbf{H}} \subset \mathbf{H}(\mathbb{A}_f)$ on the set of connected components of the *smaller* variety $\text{Sh}_{K_{\mathbf{H}}}(\mathbf{H}, Y)$, into a Galois action on the whole set $\mathcal{Z}_K(\mathbf{G}, \mathbf{H})$. As characterized in Lemma 1.2.2, the cycle $\mathcal{Z}_K(g)$ arises as the image of the connected component $\mathbf{H}(\mathbb{Q}) \cdot (Y \times K_{g, \mathbf{H}}) \in \pi_0(\text{Sh}_{K_{g, \mathbf{H}}}(\mathbf{H}, Y))$ via the sequence of morphisms

$$\text{Sh}_{K_{g, \mathbf{H}}}(\mathbf{H}, Y) \hookrightarrow \text{Sh}_{g K g^{-1}}(\mathbf{G}, X) \xrightarrow{[\cdot g]} \text{Sh}_K(\mathbf{G}, X),$$

with $K_{g, \mathbf{H}} := \mathbf{H}(\mathbb{A}_f) \cap g K g^{-1}$. Let us abuse notations and still denote by $[\cdot g]$ the above composite map.

Let us fix a particular $g_0 \in \mathbf{G}(\mathbb{A}_f)$ until the end of this paragraph. Notice that, if h_0 is any element of $\mathbf{H}(\mathbb{A}_f)$, then the compact open subgroups $K_{h_0 g_0, \mathbf{H}}$ and $h_0 K_{g_0, \mathbf{H}} h_0^{-1}$ are equal. Accordingly, the Hecke morphism $[\cdot h_0]$ induces an isomorphism

$$\begin{aligned} \text{Sh}_{K_{h_0 g_0, \mathbf{H}}}(\mathbf{H}, Y) &\xrightarrow{[\cdot h_0]} \text{Sh}_{K_{g_0, \mathbf{H}}}(\mathbf{H}, Y), \\ [y, h]_{K_{h_0 g_0, \mathbf{H}}} &\longmapsto [y, h h_0]_{K_{g_0, \mathbf{H}}}, \end{aligned}$$

which itself induces an isomorphism, still denoted $[\cdot h_0]$, between their corresponding zero-dimensional Shimura varieties $\text{Sh}_{\det K_{h_0 g_0, \mathbf{H}}}(\mathbf{T}^1, \det(\mu_{W, \mathbb{C}})) \simeq \text{Sh}_{\det K_{g_0, \mathbf{H}}}(\mathbf{T}^1, \det(\mu_{W, \mathbb{C}}))$, induced on $\mathbf{T}^1(\mathbb{A}_f)$ by multiplication by $\det(h_0)$. Recall that connected component

$$\mathcal{C}_{1, g_0} := \mathbf{H}(\mathbb{Q}) \cdot (Y \times K_{g_0, \mathbf{H}}) \in \pi_0(\text{Sh}_{K_{g_0, \mathbf{H}}}(\mathbf{H}, Y))$$

is the pre-image of the double coset $\mathbf{H}(\mathbb{Q}) \mathbf{H}^{\text{der}}(\mathbb{A}_f) K_{g_0, \mathbf{H}} \in \mathbf{H}(\mathbb{Q}) \mathbf{H}^{\text{der}}(\mathbb{A}_f) \backslash \mathbf{H}(\mathbb{A}_f) / K_{g_0, \mathbf{H}}$, via the map:

$$\begin{aligned} \Pi_{g_0} : \text{Sh}_{K_{g_0, \mathbf{H}}}(\mathbf{H}, Y)(\mathbb{C}) &\longrightarrow \mathbf{H}(\mathbb{Q}) \backslash \mathbf{H}(\mathbb{A}_f) / K_{g_0, \mathbf{H}} \stackrel{(1.40)}{=} \mathbf{H}(\mathbb{Q}) \mathbf{H}^{\text{der}}(\mathbb{A}_f) \backslash \mathbf{H}(\mathbb{A}_f) / K_{g_0, \mathbf{H}} \\ [y, h]_{K_{g_0, \mathbf{H}}} &\longmapsto \mathbf{H}(\mathbb{Q}) \mathbf{H}^{\text{der}}(\mathbb{A}_f) h K_{g_0, \mathbf{H}}, \end{aligned}$$

hence the pre-image of $1 = \mathbf{T}^1(\mathbb{Q}) \det(K_{g_0, \mathbf{H}}) \in \mathbf{T}^1(\mathbb{A}_f) / \mathbf{T}^1(\mathbb{Q}) \det(K_{g_0, \mathbf{H}})$ via the map $\det \circ \Pi_{g_0}$. Similarly, we set $\mathcal{C}_{h_0, g_0} := \mathbf{H}(\mathbb{Q}) \cdot (Y \times h_0 K_{g_0, \mathbf{H}}) \in \pi_0(\text{Sh}_{K_{g_0, \mathbf{H}}}(\mathbf{H}, Y))$. Set $t_0 := \det(h_0)$ and let σ_0 be the element $\text{Art}_E^1(t_0) \in \text{Gal}(E(\infty)/E)$: as $\det \circ \Pi_{g_0}(\mathcal{C}_{h_0, g_0}) = t_0 \mathbf{T}^1(\mathbb{Q}) \det(K_{g_0, \mathbf{H}})$, one gets by Corollary 1.3.10 that:

$$\sigma_0 \cdot \mathcal{C}_{1, g_0} = \mathcal{C}_{h_0, g_0} \in \pi_0(\text{Sh}_{K_{g_0, \mathbf{H}}}(\mathbf{H}, Y)).$$

On the other hand, the connected component $\mathcal{C}_{1,h_0g_0} := \mathbf{H}(\mathbb{Q}) \cdot (Y \times K_{h_0g_0, \mathbf{H}}) \in \pi_0 \left(\mathrm{Sh}_{K_{h_0g_0, \mathbf{H}}}(\mathbf{H}, Y) \right)$ maps to $\mathbf{H}(\mathbb{Q}) \cdot (Y \times h_0K_{g_0, \mathbf{H}}) = \mathcal{C}_{h_0, g_0}$ via $[\cdot h_0]$. That the following diagram, whose arrows are defined over E :

$$\begin{array}{ccc} \mathrm{Sh}_K(\mathbf{G}, X) & \xrightarrow[\sim]{\mathrm{id}} & \mathrm{Sh}_K(\mathbf{G}, X) \\ \uparrow [\cdot h_0g_0] & & \uparrow [\cdot g_0] \\ \mathrm{Sh}_{K_{h_0g_0, \mathbf{H}}}(\mathbf{H}, Y) & \xrightarrow[\sim]{[\cdot h_0]} & \mathrm{Sh}_{K_{g_0, \mathbf{H}}}(\mathbf{H}, Y) \end{array}$$

is commutative, implies that

$$\sigma_0 \cdot \mathcal{Z}_K(g_0) = [\cdot g_0](\sigma_0 \cdot \mathcal{C}_{1, g_0}) = [\cdot h_0g_0](\mathcal{C}_{1, h_0g_0}) = \mathcal{Z}_K(h_0g_0).$$

As $\det : \mathbf{H}(\mathbb{A}_f) \longrightarrow \mathbf{T}^1(\mathbb{A}_f)$ is surjective, we just showed the following:

Proposition 1.3.12 (Galois action on $\mathcal{Z}_K(\mathbf{G}, X)$). *Let $g \in \mathbf{G}(\mathbb{A}_f)$. Let $\sigma \in \mathrm{Gal}(E(\infty)/E)$ and let h_s be any element of $\mathbf{H}(\mathbb{A}_f)$ which satisfies $\mathrm{Art}_E^1 \circ \det(h_s) = \sigma$. Then*

$$\sigma \cdot \mathcal{Z}_K(g) = \mathcal{Z}_K(h_s g). \quad (1.42)$$

Remark 1.3.1. *For all $h_0 \in \mathbf{H}(\mathbb{A}_f)$, the correspondence $\mathcal{Z}_K(g) \mapsto \mathcal{Z}_K(h_0g)$ is well-defined, i.e., does not depend on the choice of $g \in \mathbf{G}(\mathbb{A}_f)$. Indeed, by (1.41) one has*

$$\mathcal{Z}_K(\mathbf{G}, \mathbf{H}) \simeq \mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}_f) / K = \mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f) / K.$$

Accordingly, if g and g' are two elements of $\mathbf{G}(\mathbb{A}_f)$ such that $\mathcal{Z}_K(g) = \mathcal{Z}_K(g')$, then one may find some $n_{\mathbb{Q}} \in \mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q})$, $h \in \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)$ and $k \in K$ such that $g' = n_{\mathbb{Q}} h g k$. Thus $h_0 g' = h_0 n_{\mathbb{Q}} h g k = (h_0 n_{\mathbb{Q}} h h_0^{-1}) h_0 g k$, with $h_0 n_{\mathbb{Q}} h h_0^{-1} \in \mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \mathbf{H}(\mathbb{A}_f)$. But $\det(h_0 n_{\mathbb{Q}} h h_0^{-1}) = \det(n_{\mathbb{Q}}) \det(h) = \det(n_{\mathbb{Q}}) \in \mathbf{T}^1(\mathbb{Q})$, hence $h_0 n_{\mathbb{Q}} h h_0^{-1} \in \mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) \mathbf{H}^{\mathrm{der}}(\mathbb{A}_f)$ (by Lemma 1.3.6), i.e.,

$h_0 g' \in \mathrm{N}_{\mathbf{G}}(\mathbf{H})(\mathbb{Q}) h_0 g K$, thus $\mathcal{Z}_K(h_0 g') = \mathcal{Z}_K(h_0 g)$.

1.3.7 Orders, ring class fields and transfer fields.

We end up this section (and chapter) in a discussion about *ring class fields*, their variants called *transfer fields*, which enable us to describe nicely the field $E(\infty)$, and then *K-transfer fields* (using the terminology of [7], VII.) which will be the natural candidates on which to define our particular family of special cycles in Chapter 3.

Recall that, if L is a number field, by an *order* \mathcal{O} of L we mean a (unitary) subring of L such that $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} = L$. Every order is a free \mathbb{Z} -module of maximal rank and is contained inside the *maximal order* \mathcal{O}_L . We may also define orders in a relative situation: if L/K is an extension of number fields, we define \mathcal{O}_K -orders of L to be those orders \mathcal{O} of L which are \mathcal{O}_K -algebras - or equivalently, \mathcal{O}_K -modules - and therefore satisfy

$$\mathcal{O} \otimes_{\mathcal{O}_K} K = \mathcal{O} \otimes_{\mathcal{O}_K} \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Q} = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} = L.$$

Going back to our CM-extension E/F , if \mathcal{O} is any \mathcal{O}_F -order then one may define its *conductor* by setting

$$\mathfrak{c}_{\mathcal{O}} := \{x \in E; x\mathcal{O}_E \subset \mathcal{O}\}.$$

One checks easily that $\mathfrak{c}_{\mathcal{O}}$ is the greatest element⁽²⁵⁾ among those ideals of \mathcal{O}_E which are contained in \mathcal{O} . Accordingly, every \mathcal{O}_F -order \mathcal{O} contains a maximal order of the form $\mathcal{O}_{\mathfrak{c}} := \mathcal{O}_F + \mathfrak{c}\mathcal{O}_E$, where $\mathfrak{c} \subset \mathcal{O}_F$ is a non-zero ideal, and one may choose \mathfrak{c} to be $\mathfrak{c}_{\mathcal{O}} \cap \mathcal{O}_F$.⁽²⁶⁾

If \mathcal{O} is an \mathcal{O}_F -order, we define $\widehat{\mathcal{O}} := \mathcal{O} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \simeq \mathcal{O} \otimes_{\mathcal{O}_F} \widehat{\mathcal{O}}_F \subset \widehat{\mathcal{O}}_E$ to be its profinite completion. For all finite places v of F and all integers $c \geq 0$, we set

$$\mathcal{O}_{v,c} := \mathcal{O}_{F_v} + \varpi^c \mathcal{O}_{E_v}, \quad (1.43)$$

to be the *local order of conductor c* , where $\mathcal{O}_{E_v} := \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}$ is the ring of integers of the étale algebra $E_v := E \otimes_F F_v$, and $\varpi \in \mathcal{O}_{F_v}$ is a fixed uniformizer. Accordingly, one has

$$\widehat{\mathcal{O}}_{\mathfrak{c}}^{\times} = \left((\mathcal{O}_F + \mathfrak{c}\mathcal{O}_E) \otimes_{\mathcal{O}_F} \prod_{v \in I_{F,f}} \mathcal{O}_v \right)^{\times} = \prod_{v \in I_{F,f}} \mathcal{O}_{v, \text{ord}_v(\mathfrak{c})}^{\times}, \quad (1.44)$$

for all ideal $\mathfrak{c} \subset \mathcal{O}_F$, where $\text{ord}_v(\mathfrak{c})$ stands for valuation of \mathfrak{c} at the prime ideal \mathfrak{p}_v of \mathcal{O}_F which corresponds to v . By (1.44), the unit group $\widehat{\mathcal{O}}_{\mathfrak{c}}^{\times}$ is a compact open subgroup of $\mathbb{A}_{E,f}^{\times}$. One has the following useful result:

Lemma 1.3.13. *If $v \in I_{F,f}$, then the following equality stands:*

$$\mathcal{O}_{F_v}^{\times} = \bigcap_{c \geq 0} \mathcal{O}_{c,v}^{\times}.$$

Proof. The inclusion from the left-hand side to the right-hand side is clear. For the converse inclusion, we start by assuming that $v = w\bar{w}$ is split in E/F , where $w \neq \bar{w}$ are places of E . In this case, one has an isomorphism $\mathcal{O}_{E_v} := \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v} \simeq \mathcal{O}_{E_w} \times \mathcal{O}_{E_{\bar{w}}}$. One may identify the field E_w (resp. $E_{\bar{w}}$) with F_v , which gives identifications between \mathcal{O}_{E_w} (resp. $\mathcal{O}_{E_{\bar{w}}}$) with \mathcal{O}_{F_v} . The preceding isomorphism $\mathcal{O}_{E_v} \simeq \mathcal{O}_{F_v} \times \mathcal{O}_{F_v}$ can therefore be written as

$$\phi : x \otimes y \mapsto (\bar{x}y, xy),$$

with $y \in \mathcal{O}_{F_v}$ and $x, \bar{x} \in \mathcal{O}_E \hookrightarrow \mathcal{O}_{E_w} \simeq F_v$. In particular, the ring \mathcal{O}_{F_v} embeds diagonally into $\mathcal{O}_{F_v} \times \mathcal{O}_{F_v}$ via ϕ . Let $z \in \mathcal{O}_{E_v}^{\times}$, and set $\phi(z) = (x, y) \in \mathcal{O}_{F_v}^{\times} \times \mathcal{O}_{F_v}^{\times}$ to be its image in our

(25). With respect to the inclusion relation, i.e., the smallest element with respect to divisibility.

(26). Nekovář claims in ([38], §2.6.2) that the inclusion $\mathcal{O}_{\mathfrak{c}} \subset \mathcal{O}$ is in fact an equality, for every \mathcal{O}_F -order \mathcal{O} . The reason on this claim is still unclear to the author without any additional hypothesis on the extension E/F (for instance, that \mathcal{O}_E admits an \mathcal{O}_F -basis of the form $\{1, \alpha\}$ for some $\alpha \in \mathcal{O}_E$).

chosen identification. If (x, y) belongs to $\phi(\mathcal{O}_{v,c})$ then one may find $u, s, t \in \mathcal{O}_{F_v}$ such that $(x, y) = (u + \varpi^c s, u + \varpi^c t)$, i.e., $x - y = \varpi^c(s - t) \in \varpi^c \mathcal{O}_{F_v}$. Consequently, the condition $z \in \cap_{c \geq 0} \mathcal{O}_{v,c}$ implies $x = y$, hence $\phi(z) \in \phi(\mathcal{O}_{F_v})$, i.e., $z \in \mathcal{O}_{F_v}^\times$, which concludes.

On the other hand if v is inert (resp. ramified) then the extension E_v/F_v is an unramified (resp. a totally ramified) quadratic extension, and \mathcal{O}_{E_v} always admits an \mathcal{O}_{F_v} -basis of the form $\{1, \alpha\}$, where $\alpha \in \mathcal{O}_{E_v}$ can be chosen to be a root of unity (resp. to be a uniformizer of E_v ⁽²⁷⁾). If we fix such a basis $\{1, \alpha\}$, then $\mathcal{O}_{v,c} = \mathcal{O}_{F_v} + \varpi^c(\mathcal{O}_{F_v} + \mathcal{O}_{F_v}\alpha) = \mathcal{O}_{F_v} + \varpi^c\alpha\mathcal{O}_{F_v}$. Accordingly, if $z \in \mathcal{O}_{E_v}^\times$ belongs to the intersection $\cap_{c \geq 0} \mathcal{O}_{v,c}^\times$ then its α -coordinate has to be 0, i.e., $z \in \mathcal{O}_{F_v} \cap \mathcal{O}_{E_v}^\times = \mathcal{O}_{F_v}^\times$. \square

Definition 1.3.2 (Ring class fields and transfer fields). *To each ideal \mathfrak{c} of \mathcal{O}_F are attached two abelian extensions $E(\mathfrak{c}) \subset E[\mathfrak{c}]$ of E , defined as follows:*

- *The transfer field of conductor \mathfrak{c} is the subfield $E(\mathfrak{c})$ of E^{ab} fixed by the subgroup $\mathrm{Art}_E \left(\mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathfrak{c}}^\times \right) \subset \mathrm{Gal}(E^{ab}/E)$.*
- *The ring class field of conductor \mathfrak{c} is the subfield $E[\mathfrak{c}]$ of E^{ab} fixed by the subgroup $\mathrm{Art}_E \left(\widehat{\mathcal{O}}_{\mathfrak{c}}^\times \right) \subset \mathrm{Gal}(E^{ab}/E)$.*

Accordingly, one has

$$\mathrm{Gal}(E^{ab}/E(\mathfrak{c})) \supset \mathrm{Art}_E \left({}_{\iota_{E/F}} \mathbb{A}_{F,f}^\times \right) = \mathrm{Ver}_{E/F}(\mathrm{Gal}(F^{ab}/F)) = \mathrm{Gal}(E^{ab}/E(\infty)),$$

hence $E(\mathfrak{c}) \subset E(\infty) \cap E[\mathfrak{c}]$ for all \mathfrak{c} . As $\widehat{\mathcal{O}}_{\mathfrak{c}}^\times$ is an open subgroup of $\mathbb{A}_{E,f}^\times$, one has $\overline{E^\times} \subset E^\times \widehat{\mathcal{O}}_{\mathfrak{c}}^\times$ for all \mathfrak{c} , hence

$$\frac{\mathbb{A}_{E,f}^\times}{E^\times \widehat{\mathcal{O}}_{\mathfrak{c}}^\times} = \frac{\mathbb{A}_{E,f}^\times}{E^\times \widehat{\mathcal{O}}_{\mathfrak{c}}^\times} \simeq \mathrm{Gal}(E[\mathfrak{c}]/E) \quad (1.45)$$

$$\frac{\mathbb{A}_{E,f}^\times}{E^\times \mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathfrak{c}}^\times} = \frac{\mathbb{A}_{E,f}^\times}{E^\times \mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathfrak{c}}^\times} \simeq \mathrm{Gal}(E(\mathfrak{c})/E) \quad (1.46)$$

$$\text{and } \frac{E^\times \mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathfrak{c}}^\times}{E^\times \widehat{\mathcal{O}}_{\mathfrak{c}}^\times} \simeq \mathrm{Gal}(E[\mathfrak{c}]/E(\mathfrak{c})). \quad (1.47)$$

Remark 1.3.2. — *One checks that, for all non-zero ideal \mathfrak{c} of \mathcal{O}_F , the groups $\mathrm{Art}_E \left(\widehat{\mathcal{O}}_{\mathfrak{c}}^\times \right)$ and $\mathrm{Art}_E \left(\mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathfrak{c}}^\times \right)$ are both closed (or equivalently, compact) subgroups of $\mathrm{Gal}(E^{ab}/E)$. Indeed, the former arises as the image of a compact subgroup of $\mathbb{A}_{E,f}^\times$ by the continuous map Art_E . The latter is equal to $\mathrm{Art}_E(\mathbb{A}_{F,f}^\times) \cdot \mathrm{Art}_E(\widehat{\mathcal{O}}_{\mathfrak{c}}^\times)$, which is still compact as a product of compact subgroups (indeed, one has $\mathrm{Art}_E(\mathbb{A}_{F,f}^\times) = \mathrm{Ver}_{E/F}(\mathrm{Gal}(F^{ab}/F)) = \ker(r_{\mathrm{fin}})$, which closed - hence compact - in $\mathrm{Gal}(E^{ab}/E)$). Consequently, the fields $E(\mathfrak{c})$ and $E[\mathfrak{c}]$ are both well-defined via infinite Galois correspondence.*

(27). We refer to ([46], I.6) for proofs of these properties.

- The left-hand side of (1.45) is isomorphic to the Picard group $\text{Pic}(\mathcal{O}_{\mathfrak{c}})$ of the order $\mathcal{O}_{\mathfrak{c}}$, which is known to be finite: therefore the extensions $E[\mathfrak{c}] \supset E(\mathfrak{c}) \supset E$ are finite for all \mathfrak{c} .
- The left-hand side of (1.47) is a quotient of $\frac{\mathbb{A}_{F,f}^{\times}}{F^{\times}\mathcal{O}_F} \simeq \text{Pic}(\mathcal{O}_F)$. Consequently, if F has class number equal to 1 then $E[\mathfrak{c}] = E(\mathfrak{c})$ for all \mathfrak{c} .

For all ideals $\mathfrak{c}, \mathfrak{c}'$ of \mathcal{O}_F , the equality (1.44) induces the following relations:

$$\begin{aligned} \widehat{\mathcal{O}}_{\mathfrak{c}}^{\times} \cdot \widehat{\mathcal{O}}_{\mathfrak{c}'}^{\times} &= \widehat{\mathcal{O}}_{\text{gcd}(\mathfrak{c}, \mathfrak{c}')}^{\times}, \\ E^{\times}\widehat{\mathcal{O}}_{\mathfrak{c}}^{\times} \cap E^{\times}\widehat{\mathcal{O}}_{\mathfrak{c}'}^{\times} &\supset E^{\times}\widehat{\mathcal{O}}_{\text{lcm}(\mathfrak{c}, \mathfrak{c}')}^{\times} \text{ and} \\ E^{\times}\mathbb{A}_{F,f}^{\times}\widehat{\mathcal{O}}_{\mathfrak{c}}^{\times} \cap E^{\times}\mathbb{A}_{F,f}^{\times}\widehat{\mathcal{O}}_{\mathfrak{c}'}^{\times} &\supset E^{\times}\mathbb{A}_{F,f}^{\times}\widehat{\mathcal{O}}_{\text{lcm}(\mathfrak{c}, \mathfrak{c}')}^{\times}, \end{aligned}$$

therefore one has

$$E(\mathfrak{c}) \cap E(\mathfrak{c}') = E(\text{gcd}(\mathfrak{c}, \mathfrak{c}')), \quad E[\mathfrak{c}] \cap E[\mathfrak{c}'] = E[\text{gcd}(\mathfrak{c}, \mathfrak{c}')], \quad (1.48)$$

$$E(\mathfrak{c}) \cdot E(\mathfrak{c}') \subset E(\text{lcm}(\mathfrak{c}, \mathfrak{c}')) \text{ and } E[\mathfrak{c}] \cdot E[\mathfrak{c}'] \subset E[\text{lcm}(\mathfrak{c}, \mathfrak{c}')] \quad (1.49)$$

The previous relations enable us to define $E[\infty] := \bigcup_{\mathfrak{c} \subset \mathcal{O}_F \text{ ideal} \neq 0} E[\mathfrak{c}] \subset E^{ab}$ as the infinite compositum of all the ring class fields. This is an abelian extension of E associated, via infinite Galois correspondence, with the closed subgroup

$$\text{Gal}(E^{ab}/E[\infty]) = \bigcap_{\mathfrak{c} \neq 0} \text{Art}_E \left(\widehat{\mathcal{O}}_{\mathfrak{c}}^{\times} \right) \subset \text{Gal}(E^{ab}/E).$$

Similarly we let $\widetilde{E}(\infty) := \bigcup_{\mathfrak{c} \subset \mathcal{O}_F \text{ ideal} \neq 0} E(\mathfrak{c}) \subset E[\infty]$ denote the infinite compositum of all the transfer fields, which corresponds to the closed subgroup

$$\text{Gal}(E^{ab}/\widetilde{E}(\infty)) = \bigcap_{\mathfrak{c} \neq 0} \text{Art}_E \left(\mathbb{A}_{F,f}^{\times} \widehat{\mathcal{O}}_{\mathfrak{c}}^{\times} \right) \subset \text{Gal}(E^{ab}/E).$$

One may describe the above fields in a refined way:

Lemma 1.3.14. *There exists a increasing sequence $(\mathfrak{c}_i)_{i \geq 0}$ - with respect to divisibility - of non-zero ideals of \mathcal{O}_F , such that*

$$\widehat{\mathcal{O}}_F^{\times} = \bigcap_{i \geq 0} \widehat{\mathcal{O}}_{\mathfrak{c}_i}^{\times} \quad (1.50)$$

Proof. The monoid formed by non-zero integral ideals of \mathcal{O}_F is a countable set, as a subset of the group of non-zero fractional \mathcal{O}_F -ideals (which is itself countable, for it is generated by the countable set of non-zero prime ideals of \mathcal{O}_F). Accordingly, let $(\widetilde{\mathfrak{c}}_j)_{j \geq 0}$ be a choice of sequence exhausting all the non-zero integral ideals of \mathcal{O}_F , and set

$$\mathfrak{c}_i := \prod_{j=0}^i \widetilde{\mathfrak{c}}_j, \quad \forall i \geq 0.$$

By construction, the sequence $(\mathbf{c}_i)_{i \geq 0}$ is increasing and, for all $n \geq 0$ and all $\mathfrak{p} \in |\mathrm{Spec} \mathcal{O}_F| \setminus \{0\}$, there is some constant $i(n, \mathfrak{p}) \geq 0$ such that

$$\mathrm{ord}_{\mathfrak{p}}(\mathbf{c}_i) \geq n, \forall i \geq i(n, \mathfrak{p}).$$

The inclusion $\widehat{\mathcal{O}}_F^\times \subset \bigcap_{i \geq 0} \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times$ is clear. Conversely, let $x \in \widehat{\mathcal{O}}_E^\times$ belong to the intersection $\bigcap_{i \geq 0} \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times$ and denote by $x_v \in \mathcal{O}_{E_v}^\times$ its v -component (the map $\widehat{\mathcal{O}}_E = \mathcal{O}_E \otimes_{\mathcal{O}_F} \widehat{\mathcal{O}}_F \rightarrow \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}$ is induced by the usual projection $\widehat{\mathcal{O}}_F^\times \rightarrow \mathcal{O}_{F_v}^\times$), for all $v \in I_{F,v}$. Denote by \mathfrak{p}_v the prime ideal of \mathcal{O}_F corresponding to v , and let $n \geq 0$: the relation $x \in \widehat{\mathcal{O}}_{\mathbf{c}_i(n, \mathfrak{p}_v)}^\times$ implies $x_v \in \mathcal{O}_{v, \mathrm{ord}_{\mathfrak{p}_v}(\mathbf{c}_i(n, \mathfrak{p}_v))} \subset \mathcal{O}_{v,n}^\times$, by construction of $i(n, \mathfrak{p}_v)$. In other words, one has $x_v \in \bigcap_{n \geq 0} \mathcal{O}_{v,n}^\times = \mathcal{O}_{F_v}^\times$, the last equality being Lemma 1.3.13, which gives

$$x = (x_v)_v \in \prod_{v \in I_{F,f}} \mathcal{O}_v^\times = \widehat{\mathcal{O}}_F^\times,$$

as desired. \square

We state here a simple (but very useful) topological argument:

Lemma 1.3.15. *Let G and G' be (Hausdorff) topological groups, let $f : G \rightarrow G'$ be a continuous group homomorphism and let $H \subset G$ be a subgroup. Assume that the subgroup $f(H) \subset G'$ is closed, and let $(C_i)_{i \geq 0}$ be a decreasing sequence of compact subgroups of G . Then*

$$f\left(\bigcap_{i \geq 0} HC_i\right) = \bigcap_{i \geq 0} f(HC_i).$$

Proof. The inclusion from the left-hand side to the right-hand side is obvious. If $z \in G'$ belongs to $\bigcap_{i \geq 0} f(HC_i) = \bigcap_{i \geq 0} f(H)f(C_i)$ then, for all $i \geq 0$, the set $D_i := f^{-1}(f(H)z) \cap C_i$ is a non-empty subset of G . As $f(H)$ is closed in G' then so is $f(H)z$, hence D_i is a closed subset of the compact subgroup C_i , i.e., D_i is a non-empty compact subset of G . It follows that the intersection $\bigcap_{i \geq 0} D_i$ is also non-empty (this result is sometimes called *Cantor's intersection theorem*). Accordingly, one may find some $x \in \bigcap_{i \geq 0} C_i$ such that $f(x) \in f(H)z$. In other words, z belongs to $f(H)f\left(\bigcap_{i \geq 0} C_i\right) = f(H \bigcap_{i \geq 0} C_i) \subset f\left(\bigcap_{i \geq 0} HC_i\right)$. \square

The preceding lemma has the following applications: if $G = \mathbb{A}_{E,f}^\times$, $G' = \mathrm{Gal}(E^{ab}/E)$, $f = \mathrm{Art}_E$, $C_i = \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times$ and $H = \{1\} \subset G$, then one gets

$$\mathrm{Art}_E \left(\bigcap_{i \geq 0} \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times \right) = \bigcap_{i \geq 0} \mathrm{Art}_E(\widehat{\mathcal{O}}_{\mathbf{c}_i}^\times). \quad (1.51)$$

The map $r_{\mathrm{fin}} : \mathbf{T}(\mathbb{A}_f) \rightarrow \mathbf{T}^1(\mathbb{A}_f)$ being continuous, one may also apply the preceding lemma (with G' now equal to $\mathbf{T}^1(\mathbb{A}_f)$ and $f = r_{\mathrm{fin}}$) to deduce that, if $x \in \mathbb{A}_{E,f}^\times$ belongs to $\bigcap_{i \geq 0} \mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times$, then

$$r_{\text{fin}}(x) \in \bigcap_{i \geq 0} r_{\text{fin}}(\widehat{\mathcal{O}}_{\mathbf{c}_i}^\times) = r_{\text{fin}}\left(\bigcap_{i \geq 0} \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times\right) \subset \mathbf{T}^1(\mathbb{A}_f),$$

hence

$$z \in \ker(r_{\text{fin}}) \bigcap_{i \geq 0} \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times = \mathbb{A}_{F,f}^\times \bigcap_{i \geq 0} \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times.$$

By Lemma 1.3.14, one finally gets

$$\bigcap_{i \geq 0} \mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times = \mathbb{A}_{F,f}^\times \bigcap_{i \geq 0} \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times = \mathbb{A}_{F,f}^\times. \quad (1.52)$$

Putting up the preceding together gives us:

$$\begin{aligned} (E^{ab})^{\text{Art}_E(\widehat{\mathcal{O}}_F^\times)} &= (E^{ab})^{\text{Art}_E(\bigcap_{i \geq 0} \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times)} = (E^{ab})^{\bigcap_{i \geq 0} \text{Art}_E(\widehat{\mathcal{O}}_{\mathbf{c}_i}^\times)} \\ &\stackrel{\spadesuit}{=} \bigcup_{i \geq 0} E[\mathbf{c}_i] \subset E[\infty] = (E^{ab})^{\bigcap_{\mathbf{c} \neq 0} \text{Art}_E(\widehat{\mathcal{O}}_{\mathbf{c}}^\times)} \\ &\subset (E^{ab})^{\text{Art}_E(\bigcap_{\mathbf{c} \neq 0} \widehat{\mathcal{O}}_{\mathbf{c}}^\times)} \subset (E^{ab})^{\text{Art}_E(\widehat{\mathcal{O}}_F^\times)}, \end{aligned}$$

i.e., $E[\infty] = \bigcup_{i \geq 0} E[\mathbf{c}_i] = (E^{ab})^{\text{Art}_E(\widehat{\mathcal{O}}_F^\times)}$. Equality (\spadesuit) above follows from infinite Galois correspondence, using that $\bigcap_{i \geq 0} \text{Art}_E(\widehat{\mathcal{O}}_{\mathbf{c}_i}^\times)$ is the closed subgroup of $\text{Gal}(E^{ab}/E)$ consisting of elements which fix the abelian extension $\bigcup_{i \geq 0} E[\mathbf{c}_i]$ pointwise. Similarly, one has

$$\begin{aligned} \widetilde{E}(\infty) &= (E^{ab})^{\bigcap_{\mathbf{c} \neq 0} \text{Art}_E(\mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathbf{c}}^\times)} \subset (E^{ab})^{\text{Art}_E(\bigcap_{\mathbf{c} \neq 0} \mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathbf{c}}^\times)} \\ &\subset (E^{ab})^{\text{Art}_E(\mathbb{A}_{F,f}^\times)} = (E^{ab})^{\text{Ver}_{E/F}(\text{Gal}(F^{ab}/F))} =: E(\infty) \\ &= (E^{ab})^{\text{Art}_E(\bigcap_{i \geq 0} \mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times)} \stackrel{\clubsuit}{=} (E^{ab})^{\bigcap_{i \geq 0} \text{Art}_E(\mathbb{A}_{F,f}^\times \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times)} \\ &= \bigcup_{i \geq 0} E(\mathbf{c}_i) \subset \widetilde{E}(\infty), \end{aligned}$$

i.e., $\widetilde{E}(\infty) = E(\infty) = \bigcup_{i \geq 0} E(\mathbf{c}_i) = (E^{ab})^{\text{Ver}_{E/F}(\text{Gal}(F^{ab}/F))}$. The equality (\clubsuit) is another direct application of Lemma 1.3.15, this time with $G = \mathbb{A}_{E,f}^\times$, $G' = \text{Gal}(E^{ab}/E)$, $f = \text{Art}_E$, $H = \mathbb{A}_{F,f}^\times$ and $C_i = \widehat{\mathcal{O}}_{\mathbf{c}_i}^\times$.

The above equalities imply that $E[\infty]$ is a Galois extension of $E(\infty)$, whose Galois group is

$$\text{Gal}(E[\infty]/E(\infty)) \simeq \frac{\mathbb{A}_{F,f}^\times}{(\overline{E}^\times \cap \mathbb{A}_{F,f}^\times) \widehat{\mathcal{O}}_F^\times} = \frac{\mathbb{A}_{F,f}^\times}{F^\times \widehat{\mathcal{O}}_F^\times} \simeq \text{Pic}(\mathcal{O}_F),$$

the equality $(\overline{E}^\times \cap \mathbb{A}_{F,f}^\times) \widehat{\mathcal{O}}_F^\times = F^\times \widehat{\mathcal{O}}_F^\times$ coming from $\overline{E}^\times \cap \mathbb{A}_{F,f}^\times = \overline{F}^\times$ (see the proof of Proposition 1.3.4) and from the inclusion $\overline{F}^\times \subset F^\times \widehat{\mathcal{O}}_F^\times$ ($\widehat{\mathcal{O}}_F^\times \subset \mathbb{A}_{F,f}^\times$ is an open subgroup). As was already implied by the last item of the preceding remark, one finds that $E[\infty] = E(\infty)$

if F has class number equal to 1. One may show that $E(\infty)/F$ is also Galois and that its Galois group fits inside the following exact sequence

$$1 \longrightarrow \mathrm{Gal}(E(\infty)/E) \longrightarrow \mathrm{Gal}(E(\infty)/F) \longrightarrow \mathrm{Gal}(E/F) \longrightarrow 1$$

which splits as a semi-direct product

$$\mathrm{Gal}(E(\infty)/F) \simeq \mathrm{Gal}(E(\infty)/E) \rtimes \mathbb{Z}/2\mathbb{Z},$$

i.e., $E(\infty)/F$ is a dihedral extension.

Lemma 1.3.16 (Ramification in $E(\mathfrak{c})$ and $E[\mathfrak{c}]$). *Let \mathfrak{c} be a non-zero ideal of \mathcal{O}_F . Any prime ideal \mathfrak{p} of \mathcal{O}_E which does not divide $\mathfrak{c}\mathcal{O}_E$ is unramified in $E[\mathfrak{c}]$ (hence also in $E(\mathfrak{c})$).*

Proof. Let v be the place of E corresponding to \mathfrak{p} , and let $p \in \mathbb{Z}$ be the rational prime below \mathfrak{p} . A choice of identification between \overline{E}_v and $\overline{\mathbb{Q}_p}$ induces, via our fixed embedding $\eta_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p} \simeq \overline{E}_v$, an identification between the Galois group $\mathrm{Gal}(E_v^{ab}/E_v)$ and the decomposition group $\mathcal{D}_v \subset \mathrm{Gal}(\overline{E}/E)$ at v , hence an identification between their abelianized versions $\mathrm{Gal}(E_v^{ab}/E_v)$ and $\mathcal{D}_v^{ab} \subset \mathrm{Gal}(E^{ab}/E)$. Let w stand for the finite place of $E[\mathfrak{c}]$ defined by η_p and corresponding to some prime ideal \mathfrak{q} of $\mathcal{O}_{E[\mathfrak{c}]}$. As $E[\mathfrak{c}]/E$ is Galois then it is enough to show that the inertia index $e(\mathfrak{q}|\mathfrak{p})$ is equal to 1, i.e., that the inertia group $I(\mathfrak{q}|\mathfrak{p}) \subset \mathcal{D}(\mathfrak{q}|\mathfrak{p})$ is trivial. As $\mathfrak{p} \nmid \mathfrak{c}\mathcal{O}_E$ then $\mathrm{ord}_{\varpi}(\mathfrak{c}) = 0$ (ϖ being any uniformizer of the field $F_{v \cap \mathcal{O}_F} \subset E_v$) thus the embedding at v , $\phi_v : E_v^\times \hookrightarrow \mathbb{A}_{F,f}^\times$, induces an inclusion $\mathcal{O}_{E_v}^\times \hookrightarrow \widehat{\mathcal{O}}_{\mathfrak{c}}^\times$.

The functoriality properties of the Artin map, together with the above identifications, make the following diagram

$$\begin{array}{ccccc} E_v^\times & \longleftarrow & \mathcal{O}_{E_v}^\times & \longleftarrow & \widehat{\mathcal{O}}_{\mathfrak{c}}^\times \\ \downarrow \mathrm{Art}_v & & \downarrow \wr & & \downarrow \mathrm{Art}_E \\ \mathrm{Gal}(E_v^{ab}/E_v) \simeq \mathcal{D}_v^{ab} & \supset & I_v^{ab} & \subset & \mathrm{Gal}(E^{ab}/E) \\ \downarrow \mathrm{res}_{|E[\mathfrak{c}]_w} & & \downarrow \mathrm{res}_{|E[\mathfrak{c}]_w} & & \downarrow \mathrm{res}_{|E[\mathfrak{c}]} \\ \mathrm{Gal}(E[\mathfrak{c}]_w/E_v) = \mathcal{D}(\mathfrak{q}|\mathfrak{p}) & \supset & I(\mathfrak{q}|\mathfrak{p}) & \subset & \mathrm{Gal}(E[\mathfrak{c}]/E) \end{array}$$

commute. The composition of the right vertical arrows being trivial - by definition of $E[\mathfrak{c}]$ - one obtains that the surjection $\mathcal{O}_{E_v}^\times \twoheadrightarrow I(\mathfrak{q}|\mathfrak{p})$ - obtained by composing the middle vertical arrows - is also trivial, i.e., that $I(\mathfrak{q}|\mathfrak{p}) = 1$. \square

1.3.8 The field \mathcal{K} .

From this subsection onwards, and unless the contrary is explicitly mentioned, we will always assume that the notation K refers to the base compact open subgroup $K_0 \subset \mathbf{G}(\mathbb{A}_f)$.

Recall that we defined a base cycle $\mathcal{Z}_K(g_0) \in \mathcal{Z}_K(\mathbf{G}, \mathbf{H})$, where $g_0 \in \mathbf{G}(\mathbb{A}_f)$ was some arbitrarily fixed element. The choice of g_0 , together with our particular choice of compact open subgroup K - defined as $K := (\underline{U}_V \times \underline{U}_W)(\widehat{\mathcal{O}}_F) \subset \mathbf{G}(\mathbb{A}_f)$, where the models \underline{U}_V and \underline{U}_W depended on the chosen lattices L_W and L_V - provided us with a finite set $\Sigma \subset I_{F,f}$ containing the ramification places $\text{Ram}(E/F)$ ⁽²⁸⁾. By definition, one had $g_0 = (g_{0,\Sigma}, g_0^\Sigma)$ with $g_0^\Sigma \in (\underline{U}_V \times \underline{U}_W)(\widehat{\mathcal{O}}_F^\Sigma) =: K^\Sigma \subset K$. Accordingly, when dealing with special cycles one may still assume without loss of generality that $g_0^\Sigma = 1$, which we do.

Lemma 65 of [7] enabled us to describe the stabilizer of $\mathcal{Z}_K(g_0)$ in $\mathbf{H}(\mathbb{A}_f)$ in the following useful way:

$$\text{Stab}_{\mathbf{H}(\mathbb{A}_f)}(\mathcal{Z}_K(g)) = \mathbf{H}(\mathbb{Q}) \cdot (K_{\mathbf{H},g_0,\Sigma}^{\mathbf{Z}} \times K_{\mathbf{H}}^{\Sigma}),$$

with $K_{\mathbf{H},g_0,\Sigma}^{\mathbf{Z}} := ((\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap K^\Sigma) \cdot g_{0,\Sigma} K_\Sigma g_{0,\Sigma}^{-1}) \cap \mathbf{H}(F_\Sigma)$ and $K_{\mathbf{H}}^{\Sigma} = \Delta(K_W^\Sigma) := \Delta(\underline{U}_W(\widehat{\mathcal{O}}_F^\Sigma))$.

1.3.8.1 Smooth models for \mathbf{Z} , \mathbf{T} and \mathbf{T}^1 over \mathcal{O}_F^Σ .

The morphism between Dedekind rings $\mathcal{O}_F \rightarrow \mathcal{O}_E$ is finite and flat. We set $\mathcal{O}_E^\Sigma := \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_F^\Sigma$, which is still a finite and flat \mathcal{O}_F^Σ -algebra. One may thus define the following algebraic tori over $\text{Spec } \mathcal{O}_F^\Sigma$:

$$\begin{aligned} \underline{\mathbf{Z}} &:= \mathbb{G}_{m, \mathcal{O}_F^\Sigma}, \\ \underline{\mathbf{T}} &:= \text{Res}_{\mathcal{O}_E^\Sigma / \mathcal{O}_F^\Sigma} \mathbb{G}_m, \\ \text{and } \underline{\mathbf{T}}^1 &:= \ker(\text{N}_{\mathcal{O}_E^\Sigma / \mathcal{O}_F^\Sigma} : \underline{\mathbf{T}} \rightarrow \underline{\mathbf{Z}}) \subset \underline{\mathbf{T}}. \end{aligned}$$

These are reductive groups over $\text{Spec } \mathcal{O}_F^\Sigma$ which are models for the corresponding F -groups $\mathbb{G}_{m,F}$, $\text{Res}_{E/F} \mathbb{G}_{m,E}$ and $\text{U}(1)$. By construction, the set Σ contains the ramification $\text{Ram}(E/F)$: accordingly, the surjective morphism $\underline{\nu} : \underline{\mathbf{T}} \twoheadrightarrow \underline{\mathbf{T}}^1$, still defined by $z \mapsto \frac{\bar{z}}{z}$, remains surjective on $\widehat{\mathcal{O}}_F^\Sigma$ -points. Indeed:

- if v is split then $\underline{\mathbf{T}}^1(\mathcal{O}_{F_v})$ is isomorphic to the group $\{(s, s^{-1}); s \in \mathcal{O}_{F_v}^\times\} \subset \mathcal{O}_{F_v}^\times \times \mathcal{O}_{F_v} \simeq \mathcal{O}_{E_v}^\times$, and for all $s, t \in \mathcal{O}_{F_v}^\times$, the map $\underline{\nu}$ sends $(s, t) \in \mathcal{O}_{E_v}^\times$ to (ts^{-1}, st^{-1}) . Accordingly, one has $(s, s^{-1}) = \underline{\nu}(1, s)$, for all $s \in \mathcal{O}_{F_v}^\times$.

- if v is inert then one has $E_v^\times = \mathcal{O}_{E_v}^\times F_v^\times = \mathcal{O}_{E_v}^\times \ker(\nu)$, and the surjectivity of $\underline{\nu} : \underline{\mathbf{T}}(\mathcal{O}_{F_v}) \rightarrow \underline{\mathbf{T}}^1(\mathcal{O}_{F_v})$ follows directly from the surjectivity of $\nu : E_v^\times \rightarrow \mathbf{T}^1(F_v)$ (which is Hilbert's Theorem 90).

Recall that we showed at Lemma 1.3.5 that the map $\underline{\det} : \text{U}_W(\mathcal{O}_{F_v}) \rightarrow \underline{\mathbf{T}}^1(\mathcal{O}_{F_v})$, is surjective for all $v \notin \Sigma$. Consequently, one has:

(28). Notice that, mainly for convenience, we actually assumed that Σ contains the set S^2 (which itself contains $\text{Ram}(E/F)$, see Lemma 1.2.4). However, this additional condition defining places of S^2 (plainly, that the local hermitian spaces V_τ and W_τ are split, enabling up to pick Witt bases of V_τ and to study the attached Bruhat-Tits building) is superfluous in the present subsection and could well be ignored.

$$\begin{aligned}
U^\Sigma &:= \det(K_{\mathbf{H}}^\Sigma) = \det\left(\underline{U}_W(\widehat{\mathcal{O}}_F^\Sigma)\right) = \prod_{v \notin \Sigma} \det(\underline{U}_W(\mathcal{O}_{F_v})) \\
&= \prod_{v \notin \Sigma} \mathbf{T}^1(\mathcal{O}_{F_v}) = \mathbf{T}^1\left(\widehat{\mathcal{O}}_F^\Sigma\right) = \nu\left(\mathbf{T}(\widehat{\mathcal{O}}_F^\Sigma)\right) =: \prod_{v \notin \Sigma} \mathcal{O}_v^1
\end{aligned} \tag{1.53}$$

with $\mathcal{O}_v^1 = \nu(\mathcal{O}_{E_v}^\times)$, for all v . We set

$$U_{g_0, \Sigma} := \det(K_{\mathbf{H}, g_0, \Sigma}^{\mathbf{Z}}) \subset \mathrm{U}(1)(F_\Sigma).$$

As the map \det is open and continuous then $U_{g_0, \Sigma}$ is an open compact subgroup of $\mathrm{U}(1)(F_\Sigma)$, hence $U_{g_0, \Sigma} \times U^\Sigma$ is an open compact subgroup of $\mathbf{T}^1(\mathbb{A}_f)$.

Definition 1.3.3. *The field \mathcal{K} is the finite abelian extension of E defined as the subfield of $E(\infty)$ fixed by the closed subgroup*

$$\mathrm{Art}_E^1(\det(\mathrm{Stab}_{\mathbf{H}(\mathbb{A}_f)}(\mathcal{Z}_K(g_0)))) = \mathrm{Art}_E^1(\mathbf{T}^1(\mathbb{Q}) \cdot (U_{g_0, \Sigma} \times U^\Sigma)) \subset \mathrm{Gal}(E(\infty)/E).$$

By construction, the field \mathcal{K} is the *minimal* field of definition of the cycle $\mathcal{Z}_K(g_0)$. We set $\mathbf{T}_\Sigma^1 := \prod_{v \in \Sigma} \mathbf{T}^1(\mathcal{O}_{F_v}) \subset \prod_{v \in \Sigma} \mathcal{O}_{E_v}^\times$. We define $\mathcal{O}_{g_0, \Sigma}^\times$ to be the open subgroup of $E_\Sigma^\times := (E \otimes_F F_\Sigma)^\times$ given by

$$\mathcal{O}_{g_0, \Sigma}^\times := \nu^{-1}(U_{g_0, \Sigma} \cap \mathbf{T}_\Sigma^1) \tag{1.54}$$

1.3.8.2 The \mathcal{K} -transfer fields.

Definition 1.3.4. *Let $\mathfrak{f} \subset \mathcal{O}_F$ be a non-zero ideal, relatively prime to Σ . The open subgroup $\mathfrak{D}_\mathfrak{f}^\times \subset \mathbf{T}(\mathbb{A}_f)$ is defined as*

$$\mathfrak{D}_\mathfrak{f}^\times := \mathcal{O}_{g_0, \Sigma}^\times \times \left(\mathcal{O}_\mathfrak{f} \otimes_{\mathcal{O}_F} \widehat{\mathcal{O}}_F^\Sigma\right)^\times = \mathcal{O}_{g_0, \Sigma}^\times \times \prod_{v \notin \Sigma} \mathcal{O}_{v, \mathrm{ord}_v(\mathfrak{f})}^\times.$$

One has

$$U_\mathfrak{f} := \nu(\mathfrak{D}_\mathfrak{f}^\times) = (U_{g_0, \Sigma} \cap \mathbf{T}_\Sigma^1) \times U_\mathfrak{f}^\Sigma \subset \mathbf{T}^1(\widehat{\mathcal{O}}_F) \subset \mathbf{T}^1(\mathbb{A}_f),$$

with $U_\mathfrak{f}^\Sigma := \prod_{v \notin \Sigma} \nu(\mathcal{O}_{v, \mathrm{ord}_v(\mathfrak{f})}^\times)$.

Definition 1.3.5 (Transfer field $\mathcal{K}(\mathfrak{f})$). *The transfer field $\mathcal{K}(\mathfrak{f})$ is the subfield of $E(\infty)$ fixed by the subgroup $\mathrm{Art}_E^1(\nu(\mathfrak{D}_\mathfrak{f}^\times)) \subset \mathrm{Gal}(E(\infty)/E)$.*

Notice that, though we use the same symbols and terminology as ([7], VII.1.5), *our* \mathcal{K} -transfer fields might in general be slightly different from Boumasmoud's (namely, a little bit bigger) as we chose, for convenience, to shrink the group $\mathcal{O}_{g_0, \Sigma}^\times$ so that $\nu(\mathcal{O}_{g_0, \Sigma}^\times) \subset \mathbf{T}_\Sigma^1$, which Boumasmoud doesn't. In particular, the inclusion $\mathcal{K} \subset \mathcal{K}(1)$ could be strict.

By definition, one has $\nu(\mathfrak{D}_f^\times) \subset U_{g_0, \Sigma} \times U^\Sigma$, hence $\text{Gal}(E(\infty)/K(\mathfrak{f})) = \text{Art}_E^1(\nu(\mathfrak{D}_f^\times) \subset \text{Art}_E^1(U_{g_0, \Sigma} \times U^\Sigma) = \text{Gal}(E(\infty)/\mathcal{K})$, i.e., $\mathcal{K} \subset \mathcal{K}(\mathfrak{f})$, for all \mathfrak{f} prime to Σ . Let \mathfrak{f} and \mathfrak{g} be (non-necessarily coprime) prime-to- Σ non-zero ideals of \mathcal{O}_F . By definition, one has $\mathcal{K}(\mathfrak{f}) \subset \mathcal{K}(\mathfrak{f} \cdot \mathfrak{g})$, as well as isomorphisms:

$$\frac{E^\times \mathbb{A}_{F,f}^\times \mathfrak{D}_f^\times}{E^\times \mathbb{A}_{F,f}^\times \mathfrak{D}_{f \cdot \mathfrak{g}}^\times} \xrightarrow[\sim]{\nu} \frac{\mathbf{T}^1(\mathbb{Q}) U_f}{\mathbf{T}^1(\mathbb{Q}) U_{f \cdot \mathfrak{g}}} \xrightarrow[\sim]{\text{Art}_E^1} \text{Gal}(\mathcal{K}(\mathfrak{f} \cdot \mathfrak{g})/\mathcal{K}(\mathfrak{f})) \quad (1.55)$$

Let us recall the following basic result of group theory: if $A, B \supset C$ are subgroups of a commutative group G , then natural inclusions induce the following exact sequence:

$$1 \longrightarrow \frac{A \cap B}{A \cap C} \longrightarrow \frac{B}{C} \longrightarrow \frac{AB}{AC} \longrightarrow 1 \quad (1.56)$$

By taking $G = \mathbb{A}_{E,f}^\times$, $A = E^\times$, $B = \mathbb{A}_{F,f}^\times \mathfrak{D}_f^\times$ and $C = \mathbb{A}_{F,f}^\times \mathfrak{D}_{f \cdot \mathfrak{g}}^\times$, we obtain by (1.55) the following exact sequence:

$$1 \longrightarrow \frac{E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{D}_f^\times}{E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{D}_{f \cdot \mathfrak{g}}^\times} \longrightarrow \frac{\mathbb{A}_{F,f}^\times \mathfrak{D}_f^\times}{\mathbb{A}_{F,f}^\times \mathfrak{D}_{f \cdot \mathfrak{g}}^\times} \longrightarrow \text{Gal}(\mathcal{K}(\mathfrak{f} \cdot \mathfrak{g})/\mathcal{K}(\mathfrak{f})) \longrightarrow 1 \quad (1.57)$$

Notice that $\mathbb{A}_{F,f}^\times$ can be rewritten as $F_\Sigma^\times \times (\mathbb{A}_{F,f}^\Sigma)^\times := F_\Sigma^\times \times \prod_{v \in I_{F,f}} {}^{F_v^\times}$, the restricted product being, as usual, with respect to the groups $\mathcal{O}_{F_v}^\times$ for $v \notin \Sigma$. For every prime-to- Σ , non-zero ideal $\mathfrak{f} \subset \mathcal{O}_F$, one has:

$$\left(\mathcal{O}_f \otimes_{\mathcal{O}_F} \widehat{\mathcal{O}_F^\Sigma} \right)^\times \cap (\mathbb{A}_{F,f}^\Sigma)^\times = \prod_{v \notin \Sigma} \mathcal{O}_{v, \text{ord}_v(\mathfrak{f})}^\times \cap (\mathbb{A}_{F,f}^\Sigma)^\times = \widehat{\mathcal{O}_F^\Sigma}^\times,$$

hence

$$\mathbb{A}_{F,f}^\times \cap \mathfrak{D}_f^\times = (F_\Sigma^\times \cap \mathcal{O}_{g_0, \Sigma}^\times) \times \widehat{\mathcal{O}_F^\Sigma}^\times.$$

Consequently, for all \mathfrak{f} and \mathfrak{g} prime to Σ , the quotient $\frac{\mathbb{A}_{F,f}^\times \cap \mathfrak{D}_f^\times}{\mathbb{A}_{F,f}^\times \cap \mathfrak{D}_{f \cdot \mathfrak{g}}^\times}$ is trivial, which gives by (1.56) an isomorphism

$$\frac{\mathbb{A}_{F,f}^\times \mathfrak{D}_f^\times}{\mathbb{A}_{F,f}^\times \mathfrak{D}_{f \cdot \mathfrak{g}}^\times} \simeq \frac{\mathfrak{D}_f^\times}{\mathfrak{D}_{f \cdot \mathfrak{g}}^\times} \simeq \prod_{v \notin \Sigma} \frac{\mathcal{O}_{v, \text{ord}_v(\mathfrak{f})}^\times}{\mathcal{O}_{v, \text{ord}_v(\mathfrak{f}) + \text{ord}_v(\mathfrak{g})}^\times}. \quad (1.58)$$

One may control the first term appearing in the exact sequence (1.57) by using the following result, due to Nekovář:

Lemma 1.3.17. *Let $\mathfrak{f} \subset \mathcal{O}_F$ be a non-zero, prime-to- Σ ideal, and assume that*

$$\mathfrak{f} \cdot \mathcal{O}_E \nmid I_0 := \text{lcm}\{(u-1); u \in (\mathcal{O}_E^\times)_{\text{tors}} \setminus \{1\}\}.$$

Then one has

$$E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{D}_f^\times = F^\times.$$

Proof. We adapt the proof of [38], Proposition 2.10 in our case⁽²⁹⁾, to show the only non-trivial inclusion $E^\times \cap \mathbb{A}_{F,f}^\times \mathfrak{D}_f^\times \subset F^\times$. Let $x \in \mathbb{A}_{F,f}^\times$, $y \in \mathfrak{D}_f^\times$ and assume that $z = xy \in E^\times$. Then $u := \nu(z) = \nu(y)$ lies in $\nu(\mathfrak{D}_f^\times) \cap E^\times = U_f \cap E^\times \subset \widehat{\mathcal{O}}_E^\times \cap E^\times = \mathcal{O}_E^\times$, and $u \in \ker(N_{E/F} : \mathcal{O}_E^\times \rightarrow \mathcal{O}_F^\times) = (\mathcal{O}_E^\times)_{\mathrm{tors}}$ ⁽³⁰⁾. On the other hand, $u_v \in \mathcal{O}_{E_v}^\times$ is congruent to 1 modulo $\varpi^{\mathrm{ord}_v(f)}$ for all $v \notin \Sigma$, where $\varpi \in \mathcal{O}_{F_v}$ is a fixed uniformizer. As $\mathfrak{f} \cdot \mathcal{O}_E$ is prime to Σ , this gives $u \equiv 1 \pmod{\mathfrak{f}\mathcal{O}_E}$. The hypothesis on \mathfrak{f} implies that $u = 1$, hence $z \in F^\times$. \square

As an immediate consequence, one has:

Corollary 1.3.18. *Let \mathfrak{f} and \mathfrak{g} be prime-to- Σ ideals of \mathcal{O}_F , and assume that $\mathfrak{f}\mathcal{O}_E \nmid I_0$. Then*

$$\mathrm{Gal}(\mathcal{K}(\mathfrak{f} \cdot \mathfrak{g})/\mathcal{K}(\mathfrak{f})) \simeq \frac{\mathfrak{D}_f^\times}{\mathfrak{D}_{\mathfrak{f}\mathfrak{g}}^\times} \simeq \prod_{v \notin \Sigma, \mathfrak{p}_v | \mathfrak{g}} \frac{\mathcal{O}_{v, \mathrm{ord}_v(\mathfrak{f})}^\times}{\mathcal{O}_{v, \mathrm{ord}_v(\mathfrak{f}) + \mathrm{ord}_v(\mathfrak{g})}^\times}. \quad (1.59)$$

The above corollary is very important as it will enable us to work locally (outside Σ) and to treat global traces as local traces. By definition, one has $U_f = \nu(\mathfrak{D}_f^\times) \subset \nu(\widehat{\mathcal{O}}_f^\times)$, hence $E(\mathfrak{f}) \subset \mathcal{K}(\mathfrak{f})$ for all prime-to- Σ ideal \mathfrak{f} . On the other hand, as $\mathfrak{D}_f^\times \subset \mathbf{T}(\mathbb{A}_f)$ is an open subgroup, one may find some ideal $\mathfrak{c} \subset \mathcal{O}_F$ such that $\widehat{\mathcal{O}}_{\mathfrak{c}}^\times \subset \mathfrak{D}_f^\times$, i.e., such that $\mathcal{K}(\mathfrak{f}) \subset E(\mathfrak{c})$. Set $\mathfrak{c}_f := \mathrm{gcd}\{\mathfrak{c} \subset \mathcal{O}_F; \widehat{\mathcal{O}}_{\mathfrak{c}}^\times \subset \mathfrak{D}_f^\times\}$. By (1.48), one has

$$E(\mathfrak{c}_f) = \bigcap_{\widehat{\mathcal{O}}_{\mathfrak{c}}^\times \subset \mathfrak{D}_f^\times} E(\mathfrak{c}),$$

i.e., $E(\mathfrak{c}_f)$ is the smallest transfer field containing $\mathcal{K}(\mathfrak{f})$. For all prime-to- Σ ideal \mathfrak{f} , the equality $\mathfrak{D}_f^\times = \mathfrak{D}_1^\times \cap \widehat{\mathcal{O}}_f^\times$ induces inclusions

$$\widehat{\mathcal{O}}_{\mathfrak{c}_1 \cdot \mathfrak{f}}^\times \subset \widehat{\mathcal{O}}_{\mathrm{lcm}(\mathfrak{c}_1, \mathfrak{f})}^\times = \widehat{\mathcal{O}}_{\mathfrak{c}_1}^\times \cap \widehat{\mathcal{O}}_f^\times \subset \mathfrak{D}_f^\times,$$

hence $\mathfrak{c}_f \subset \mathfrak{c}_1 \cdot \mathfrak{f}$ and

$$K(\mathfrak{f}) \subset E(\mathfrak{c}_f) \subset E(\mathfrak{c}_1 \cdot \mathfrak{f}) \quad (1.60)$$

for all \mathfrak{f} relatively prime to Σ .

Remark 1.3.3. *The above discussion has the following consequence: if W admits a global self-dual \mathcal{O}_E lattice L_W (which implies, as we saw, that V admits a global self-dual lattice*

(29). One could apply Nekovár's Proposition directly if one had $\mathfrak{D}_f^\times \subset \widehat{\mathcal{O}}_f^\times$, which we do not assume to be true.

(30). Indeed, as E/F is a CM extension then by Dirichlet's unit theorem, the following three groups

$$(\mathcal{O}_F^\times)^2 = N_{E/F}(\mathcal{O}_F^\times) \subset \mathrm{Im}(N_{E/F} : \mathcal{O}_E^\times \rightarrow \mathcal{O}_F^\times) \subset \mathcal{O}_F^\times$$

all have the same rank $d-1 = \mathrm{rk}_{\mathbb{Z}} \mathcal{O}_E^\times$. Thus $\ker(N_{E/F} : \mathcal{O}_E^\times \rightarrow \mathcal{O}_F^\times)$ has rank 0 (i.e., is a finite group), hence is contained in the roots of unity. Conversely, for any root of unity $\zeta \in \mathcal{O}_E^\times$ then its norm $N_{E/F}(\zeta) \in \mathcal{O}_F^\times$ is a root of unity, satisfying $\rho_i(N_{E/F}(\zeta)) = \tilde{\rho}_i(\zeta) \tilde{\rho}_i(\bar{\zeta}) \in \mathbb{R}_{>0} \cap \mu_\infty$, for all $i \in \{1, \dots, d\}$, i.e., $N_{E/F}(\zeta) = 1$.

$L_V := L_W \oplus L_D$) and if the extension E/F is unramified at every finite place, then all the above constructions imply that one might simply choose $S^1 = \emptyset$ and $g_0 = 1 \in \mathbf{G}(\mathbb{A}_f)$, in which case Σ might also be chosen to be \emptyset . Therefore one has $\mathfrak{D}_{\mathfrak{f}}^\times = \widehat{\mathcal{O}_{\mathfrak{f}}}^\times$, hence $\mathcal{K}(\mathfrak{f}) = E(\mathfrak{f})$ for all non-zero ideal \mathfrak{f} of \mathcal{O}_F . The cycle $\mathcal{Z}_K(1)$ is then defined over $\mathcal{K} \subset \mathcal{K}(1) = E(1) \subset E[1]$, i.e., $\mathcal{Z}_K(1)$ is defined over the Hilbert class field of E .

Chapter 2

Bruhat-Tits building at allowable inert places, and local distribution relations.

2.1 The Bruhat-Tits building of $U(3)$.

The aim of the present section is to give an introduction - which will be very far from a general treatment and will use only very little of the monumental theory of Bruhat and Tits - to the Bruhat-Tits building \mathbf{B}_V attached to the unitary group $U(V)$ at any *allowable* inert place τ of F (in the sense of Definition 1.2.5). Very briefly speaking, given a general connected reductive group G over a non-archimedean local-field F , the building of G is a metric space equipped with a structure of a poly-simplicial complex, obtained by glueing together a family of distinguished subsets called *apartments*, and on which G acts isometrically. Notice that these apartments arise as affine spaces relatively to some fixed real vector space. In our case, we will be mainly interested in studying the combinatorial properties of \mathbf{B}_V - which turns out to be a tree - together with its sub-building \mathbf{B}_W , attached to the subgroup $U(W) \subset U(V)$. This introduction adopts the point of view of *self-dual ultrametric norms*, such as initially introduced by Goldman-Iwahori in [20] and formalized later on by Bruhat and Tits (see [9]) in the case of unitary groups. The beginning of this section follows closely Koskivirta's thesis ([31], chapter 4), exception made of notational changes.

Let us fix notations. We recall that, for all finite *inert* place $\tau \in I_{F,f}$, the 3-dimensional E_τ/F_τ -hermitian space $(V_\tau, \langle \cdot, \cdot \rangle_\tau)$ is obtained from our initial E/F -hermitian space $(V, \langle \cdot, \cdot \rangle)$ by setting $V_\tau := V \otimes_E E_\tau$ and by extending the E -valued pairing $\langle \cdot, \cdot \rangle$ into an E_τ -valued pairing $\langle \cdot, \cdot \rangle_\tau$, as explained in (1.1). We recall that $G_{V,\tau} := U(V)(F_\tau) = \{g \in \mathrm{GL}(V_\tau), \langle g \cdot v, g \cdot w \rangle_\tau = \langle v, w \rangle_\tau, \text{ for all } v, w \in V_\tau\}$ is the local unitary group of V , and that we implicitly identify the local unitary group $G_{W,\tau} := U(W)(F_\tau)$ with the subgroup $\iota(G_{W,\tau}) = \{g \in G_{V,\tau}, g \cdot z = z \text{ for all } z \in D_\tau\} \subset G_{V,\tau}$, via $\iota: U(W) \hookrightarrow U(V)$.

We let p be the rational prime number lying under τ , let $\varpi \in F_\tau$ be a fixed uniformizer. We let $\mathbb{F}_0 := \mathcal{O}_{F_\tau}/\varpi$ and $\mathbb{F} := \mathcal{O}_{E_\tau}/\varpi$ be the respective residue fields of F_τ and E_τ , and we set $q := q_\tau = |\mathbb{F}_0|$, so that $\mathbb{F} \simeq \mathbb{F}_{q^2}$. We normalize, as usual, the p -adic valuation v_p on $\overline{\mathbb{Q}_p}$ so that $v_p(p) = 1$, and we let $|\cdot| = p^{-v_p(\cdot)}$ be the usual p -adic norm on $\overline{\mathbb{Q}_p}$. We denote by ord_ϖ the ϖ -adic valuation on E_τ , such that $|x| = |\varpi|^{\text{ord}_\varpi(x)}$ for all $x \in E_\tau$.

Throughout the whole section, we will assume that τ is *allowable*, in the sense of Definition 1.2.5. This implies that $\det(\langle \cdot, \cdot \rangle_\tau) = 1 \in F_\tau^\times / N_{E_\tau/F_\tau}(E_\tau^\times)$, i.e., that the hermitian space $(V_\tau, \langle \cdot, \cdot \rangle_\tau)$ is split. As explained in § 1.2.3.2, this allows us to choose Witt bases $\mathcal{B} = \{e_+, e_0, e_-\}$, which are E_τ -bases of V_τ in which the hermitian product $\langle \cdot, \cdot \rangle_\tau$ has matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

2.1.1 The buildings $\tilde{\mathbf{B}}_V$ and \mathbf{B}_V via self-dual ultrametric norms.

Let $\star \in \{V, W\}$. In a similar way as $\mathbf{U}(\star)$, one may define the F -reductive group $\mathbf{GU}(\star) \subset \text{Res}_{E/F}\text{GL}(\star_E)$ of *unitary similitudes*, whose functor of points is given, for any F -algebra R , by

$$\mathbf{GU}(\star)(R) := \{g \in \text{GL}(\star \otimes_F R); \exists \kappa(g) \in R^\times, \langle g \cdot v, g \cdot w \rangle = \kappa(g) \langle v, w \rangle, \forall v, w \in \star \otimes_F R\}.$$

The similitude factor $\kappa : \mathbf{GU}(V) \rightarrow \mathbb{G}_{m,F}$ and the determinant map $\det : \mathbf{GU}(V) \rightarrow \text{Res}_{E/F}(\mathbb{G}_{m,E})$ are both defined over $\text{Spec } F$, and the relation $\langle g \cdot v, g \cdot w \rangle = \kappa(g) \langle v, w \rangle$ for all $g \in \mathbf{GU}(\star)(R)$ and all $v, w \in \star \otimes_F R$ implies that

$$\det(g) (c \cdot \det(g)) =: \det(g) \overline{\det(g)} = \kappa(g)^3 \in R^\times, \quad \forall g \in \mathbf{GU}(V)(R).$$

Let $R = F_\tau$ and set $\tilde{G}_{V,\tau} := \mathbf{GU}(V)(F_\tau)$. Let g belong to $\tilde{G}_{V,\tau}$. By local class-field theory, one has

$$2 = [E_\tau : F_\tau] = \left| \frac{F_\tau^\times}{N_{E_\tau/F_\tau}(E_\tau^\times)} \right|,$$

hence the relation $N_{E_\tau/F_\tau}(\det(g)) = \kappa(g)^3$ implies that $\kappa(g) = 1 \in \frac{F_\tau^\times}{N_{E_\tau/F_\tau}(E_\tau^\times)}$. Accordingly, the sequence

$$1 \longrightarrow G_{V,\tau} \longrightarrow \tilde{G}_{V,\tau} \xrightarrow{g \mapsto \kappa(g)} N_{E_\tau/F_\tau}(E_\tau^\times) \longrightarrow 1$$

is exact, and induces a decomposition $\tilde{G}_{V,\tau} = E_\tau^\times \cdot G_{V,\tau}$.

The rest of the present section being purely local, we shall simplify the notations and now use the symbol $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_\tau$, when there is no possible ambiguity.

Definition 2.1.1 (Almost self-dual and self-dual ultrametric norms). *An almost self-dual ultrametric norm on V_τ is a function $\alpha : V_\tau \rightarrow \mathbb{R}_{\geq 0}$ such that :*

- $\alpha(V_\tau) \subset \mathbb{R}_{\geq 0}$ is discrete,
- $\alpha(\lambda v) = |\lambda|\alpha(v)$, for all $\lambda \in E_\tau$ and $v \in V_\tau$,
- $\alpha(v) \neq 0$ if $v \neq 0$,
- $\alpha(v + w) \leq \max(\alpha(v), \alpha(w))$,
- The dual norm α^\vee , defined by

$$\alpha^\vee(v) := \sup_{w \in V_\tau, w \neq 0} \frac{|\langle v, w \rangle|}{\alpha(w)} \in \mathbb{R}_+, \quad \forall v \in V_\tau$$

satisfies $\alpha^\vee = r(\alpha)\alpha$, for some $r(\alpha) \in \mathbb{R}_{>0}$. The norm α is called a self-dual ultrametric norm if $r(\alpha) = 1$.

Notice that $(\lambda\alpha)^\vee = \lambda^{-1}\alpha^\vee$ for all $\lambda \in \mathbb{R}_{>0}$, thus $\lambda\alpha$ is an almost self-dual ultrametric norm whenever α is, and one has $r(\lambda\alpha) = \lambda^{-2}r(\alpha)$. Accordingly, if α is an almost self-dual ultrametric norm, then $(\alpha^\vee)^\vee = (r(\alpha)\alpha)^\vee = r(\alpha)^{-1}\alpha^\vee = \alpha$: one obtains that α^\vee is also an almost self-dual ultrametric norm, with $r(\alpha^\vee) = r(\alpha)^{-1}$.

Definition 2.1.2. We let $\tilde{\mathbf{B}}_V$ be the set of almost self-dual ultrametric norms on V_τ , and we call it the building of $\tilde{G}_{V,\tau}$. We define \mathbf{B}_V as the set of self-dual ultrametric norms on V_τ , and we call it the building of $G_{V,\tau}$.

The group $\tilde{G}_{V,\tau}$ acts on $\tilde{\mathbf{B}}_V$ via $(g \cdot \alpha)(v) := \alpha(g^{-1} \cdot v)$, for all $\alpha \in \tilde{\mathbf{B}}_V$ and all $v \in V_\tau$. A small computation gives $(g \cdot \alpha)^\vee = |\kappa(g)|(g \cdot \alpha^\vee) = |\kappa(g)|r(\alpha)(g \cdot \alpha)$, hence $r(g \cdot \alpha) = |\kappa(g)|r(\alpha)$. Accordingly, one obtains - by restriction to $G_{V,\tau}$ - an action of $G_{V,\tau}$ on the building \mathbf{B}_V of $G_{V,\tau}$. On the other hand, one gets a map:

$$\begin{aligned} \mathbb{R} \times \mathbf{B}_V &\longrightarrow \tilde{\mathbf{B}}_V, \\ (t, \alpha) &\longmapsto |\varpi|^{-t}\alpha \end{aligned}$$

If α, α' are self-dual ultrametric norms and if $t, t' \in \mathbb{R}$ are such that $|\varpi|^{-t}\alpha = |\varpi|^{-t'}\alpha'$, then $r(|\varpi|^{-t}\alpha) = r(|\varpi|^{-t'}\alpha')$, i.e., $|\varpi|^{2t} = |\varpi|^{2t'}$ thus $t = t'$ and $\alpha = \alpha'$. On the other hand, if α is an almost self-dual ultrametric norm, then $r(\alpha)^{\frac{1}{2}} = |\varpi|^t$ for some $t \in \mathbb{R}$, hence $\alpha = |\varpi|^{-t}(r(\alpha)^{\frac{1}{2}}\alpha)$ and $r(r(\alpha)^{\frac{1}{2}}\alpha) = 1$, i.e., $r(\alpha)^{\frac{1}{2}}\alpha$ is self-dual and the above map is in fact a bijection.

Under the identifications $\tilde{G}_{V,\tau} = E_\tau^\times \cdot G_{V,\tau}$ and $\tilde{\mathbf{B}}_V = \mathbb{R} \times \mathbf{B}_V$, the action of $G_{V,\tau}$ on $\tilde{\mathbf{B}}_V$ can be interpreted as an action on the second factor \mathbf{B}_V only, whereas the action of E_τ^\times on $\tilde{\mathbf{B}}_V$ consists in letting the element $e \in E_\tau^\times$ act by translation of length $\text{ord}_\varpi(e)$ on the first factor \mathbb{R} . **From now on, we will consider only the building \mathbf{B}_V .**

Definition 2.1.3. Let $\alpha \in \mathbf{B}_V$ be a self-dual ultrametric norm. A decomposition basis for α is a Witt-basis $\mathcal{B} = \{e_+, e_0, e_-\}$ of V_τ satisfying the following conditions:

- $\alpha(e_0) = 1$, and there exists $\lambda \in \mathbb{R}$ such that $\alpha(e_+) = |\varpi|^\lambda$ and $\alpha(e_-) = |\varpi|^{-\lambda}$.

— For all $v \in V_\tau$ with coordinate vector $\begin{bmatrix} a_+ \\ a_0 \\ a_- \end{bmatrix} \in E_\tau^3$ with respect to \mathcal{B} , one has

$$\alpha(v) = \max_{* \in \{+, 0, -\}} (|a_*| \cdot \alpha(e_*)).$$

The quantity $\lambda \in \mathbb{R}$ is called the *parameter* of α with respect to the basis \mathcal{B} .

Definition 2.1.4 (Apartment attached to a Witt basis). *Let \mathcal{B} be a Witt basis of V_τ . The apartment attached to \mathcal{B} , denoted $\mathcal{A}_\mathcal{B}$, is the subset of \mathbf{B}_V defined by:*

$$\mathcal{A}_\mathcal{B} := \{\alpha \in \mathbf{B}_V; \mathcal{B} \text{ is a decomposition basis for } \alpha\}.$$

If \mathcal{B} is a Witt basis, then an element $\alpha \in \mathcal{A}_\mathcal{B}$ is uniquely determined by its parameter λ , which implies that any apartment of \mathbf{B}_V is in bijection (as a set) with the real line \mathbb{R} . As we shall see in the following - and as a general feature of Bruhat-Tits buildings of rank 1 (which is the rank of the maximal split torus of $U(V)_{F_\tau}$) - this bijection is in fact an *isometry*, with respect to some distance δ in the building. An important fact is the following: any self-dual ultrametric norm $\alpha \in \mathbf{B}_V$ is contained in some apartment (i.e., admits a decomposition basis), and any two elements α, α' of \mathbf{B}_V are contained in (at least) one common apartment. These features are proved in [20] [Proposition 1.1 and Proposition 1.3 (following an idea of A. Weil)].

We shall denote by \mathfrak{B} the set of Witt bases of V_τ , and by \mathfrak{A} the set of apartments of \mathbf{B}_V . One has an obvious surjection $\mathfrak{a} : \mathfrak{B} \rightarrow \mathfrak{A}$ given by $\mathfrak{a}(\mathcal{B}) = \mathcal{A}_\mathcal{B}$. This can be refined as follows:

Proposition 2.1.1. *Let $V^{\text{can}} := (E^{\oplus 3}, \langle \cdot, \cdot \rangle_{\text{can}})$ be the "canonical" non-degenerate 3-dimensional E/F -hermitian space, endowed with the hermitian product $\langle v, w \rangle_{\text{can}} := \bar{v}_1 w_3 + \bar{v}_2 w_2 + \bar{v}_3 w_1$, for all vectors $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ of $E^{\oplus 3}$. Let $\mathfrak{T}_V := \{t = \begin{bmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & \bar{u}^{-1} \end{bmatrix}; u, v \in E_\tau^\times, v\bar{v} = 1\}$ be the maximal torus of $U(E_{\text{can}}^{\oplus 3})(F_\tau)$ formed by diagonal matrices.*

(i) *The group \mathfrak{T}_V acts on \mathfrak{B} by $t \cdot \{e_+, e_0, e_-\} := \{ue_+, ve_0, \bar{u}^{-1}e_-\}$. If \mathcal{B} is a Witt basis, one has $\mathcal{A}_\mathcal{B} = \mathcal{A}_{t \cdot \mathcal{B}}$ for all $t \in \mathfrak{T}_V$. The element $p = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in U(E_{\text{can}}^{\oplus 3})(F_\tau)$ acts on \mathfrak{B} by $p \cdot \{e_+, e_0, e_-\} = \{e_-, e_0, e_+\}$, and the equality $\mathcal{A}_\mathcal{B} = \mathcal{A}_{p \cdot \mathcal{B}}$ for all $\mathcal{B} \in \mathfrak{B}$, induces a sequence of surjections:*

$$\mathfrak{a} : \mathfrak{B} \rightarrow \mathfrak{T}_V \backslash \mathfrak{B} \rightarrow \langle \mathfrak{T}_V, p \rangle \backslash \mathfrak{B} \rightarrow \mathfrak{A}.$$

(ii) *Let $t = \begin{bmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & \bar{u}^{-1} \end{bmatrix} \in \mathfrak{T}_V$. If α belongs to $\mathcal{A}_\mathcal{B} = \mathcal{A}_{t \cdot \mathcal{B}} = \mathcal{A}_{p \cdot \mathcal{B}}$ and has parameter λ with respect to \mathcal{B} , then α has parameter $\lambda + \text{ord}_\varpi(u)$ with respect to $t \cdot \mathcal{B}$, and has parameter $-\lambda$ with respect to $p \cdot \mathcal{B}$.*

Both (i) and (ii) are straightforward computations. □

The group $G_{V,\tau}$ acts on \mathfrak{B} in a simply transitive way, as any pair $\mathcal{B} = \{e_+, e_0, e_-\}$, $\mathcal{B}' = \{e'_+, e'_0, e'_-\}$ of Witt bases defines the element $g_{\mathcal{B},\mathcal{B}'}$ of $G_{V,\tau}$, such that $g_{\mathcal{B},\mathcal{B}'} \cdot e_\star = e'_\star$, $\star \in \{+, 0, -\}$. On the other hand, the action of $G_{V,\tau}$ on \mathbf{B}_V induces an action on \mathfrak{A} , and one checks that the map $\mathfrak{a} : \mathfrak{B} \rightarrow \mathfrak{A}$ is $G_{V,\tau}$ -equivariant⁽¹⁾. Consequently, the action of $G_{V,\tau}$ on \mathfrak{A} is transitive.

We attach, to each self-dual ultrametric norm α , a « flag of balls » given by

$$B^*(\alpha) := \{B(\alpha \leq |\varpi|^{-\theta})\}_{\theta \in \mathbb{R}}, \quad (2.1)$$

with $B(\alpha \leq |\varpi|^{-\theta}) := \{v \in V_\tau, \alpha(v) \leq |\varpi|^{-\theta}\}$ being the ball of radius $|\varpi|^{-\theta}$ attached to α . We also define the open ball $B(\alpha < |\varpi|^{-\theta}) := \{v \in V_\tau; \alpha(v) < |\varpi|^{-\theta}\}$, for all $\theta \in \mathbb{R}$.

Lemma 2.1.2. *For all $\theta \in \mathbb{R}$, the balls $B(\alpha \leq |\varpi|^{-\theta})$ and $B(\alpha < |\varpi|^{-\theta})$ are \mathcal{O}_{E_τ} -lattices inside V_τ .*

Proof. Let $\theta \in \mathbb{R}$. As α has discrete image in \mathbb{R}_+ , one may always find some $\theta_0 < \theta$ such that $B(\alpha < |\varpi|^{-\theta}) = B(\alpha \leq |\varpi|^{-\theta_0})$. Accordingly, it is enough to show that $B(\alpha \leq |\varpi|^{-\theta})$ is a lattice, for all $\theta \in \mathbb{R}$.

Let choose a decomposition basis $\mathcal{B} = \{e_+, e_0, e_-\}$ for α , in which α has parameter $\lambda \in \mathbb{R}$. By definition, one has $\alpha(v) \leq |\varpi|^{-\theta}$ if and only if $|a_0| \leq |\varpi|^{-\theta}$, $|a_+| \leq |\varpi|^{-\theta-\lambda}$ and $|a_-| \leq |\varpi|^{-\theta+\lambda}$, for all $v = a_+e_+ + a_0e_0 + a_-e_- \in V_\tau$. Set $m := \lfloor \theta + \lambda \rfloor$, $n := \lfloor \theta \rfloor$ and $r := \lfloor \theta - \lambda \rfloor$ to be the respective integral parts of $\theta + \lambda$, θ and $\theta - \lambda$. Then v belongs to $B(\alpha \leq |\varpi|^{-\theta})$ if and only if $\text{ord}_\varpi(a_+) \geq -m$, $\text{ord}_\varpi(a_0) \geq -n$ and $\text{ord}_\varpi(a_-) \geq -r$. In other words, one has $B(\alpha \leq |\varpi|^{-\theta}) = \mathcal{O}_{E_\tau} \varpi^{-m} e_+ \oplus \mathcal{O}_{E_\tau} \varpi^{-n} e_0 \oplus \mathcal{O}_{E_\tau} \varpi^{-r} e_- =: \langle \varpi^{-m} e_+, \varpi^{-n} e_0, \varpi^{-r} e_- \rangle$ is the lattice generated by the basis $\{\varpi^{-m} e_+, \varpi^{-n} e_0, \varpi^{-r} e_-\}$. \square

2.1.2 The graph structure on \mathbf{B}_V / \sim .

Let \sim be the equivalence relation on \mathbf{B}_V , defined by $\alpha \sim \alpha'$ if and only if $B^*(\alpha) = B^*(\alpha')$, for which we denote by $\text{cl}(\alpha)$ the class of α . Notice that, for all $\alpha \in \mathbf{B}_V$, $\theta \in \mathbb{R}$ and $g \in G_{V,\tau}$, one has

$$B(g \cdot \alpha \leq |\varpi|^{-\theta}) = \{v \in V_\tau; \alpha(g^{-1}v) \leq |\varpi|^{-\theta}\} = g \cdot B(\alpha \leq |\varpi|^{-\theta}).$$

This gives an equality at the level of flags

$$g \cdot B^*(\alpha) = B^*(g \cdot \alpha),$$

(1). Indeed, if $g \in G_{V,\tau}$ and $\mathcal{B} = \{e_+, e_0, e_-\} \in \mathfrak{B}$, then $\alpha \in g \cdot \mathcal{A}_\mathcal{B}$ if and only if $g^{-1} \cdot \alpha(v)$ belongs to $\mathcal{A}_\mathcal{B}$ and has some parameter $\lambda \in \mathbb{R}$ with respect to \mathcal{B} , i.e., $(g^{-1} \cdot \alpha)(v) = \alpha(g \cdot v) = \max(|a_+||\varpi|^\lambda, |a_0|, |a_-||\varpi|^{-\lambda})$ for all $v = a_+e_+ + a_0e_0 + a_-e_- \in V_\tau$ (i.e., $g \cdot v = a_+(g \cdot e_+) + a_0(g \cdot e_0) + a_-(g \cdot e_-)$). In other words, α belongs to $\mathcal{A}_{g \cdot \mathcal{B}}$, with same parameter λ , hence $g \cdot \mathcal{A}_\mathcal{B} = \mathcal{A}_{g \cdot \mathcal{B}}$.

which shows that the relation \sim is compatible with the action of $G_{V,\tau}$. One may thus define a (simple, undirected) graph structure on the set \mathbf{B}_V/\sim , endowed with an action of $G_{V,\tau}$, as follows:

Definition 2.1.5. — *A vertex of \mathbf{B}_V/\sim is an equivalence class of self-dual ultrametric norms which is a singleton. By extension, we say that $\alpha \in \mathbf{B}_V$ is a vertex if $\text{cl}(\alpha) = \{\alpha\}$, and we denote by $|\mathbf{B}_V| \subset \mathbf{B}_V$ the subset of vertices.*

- *We call edges of \mathbf{B}_V/\sim , the equivalence classes of elements in $\mathbf{B}_V \setminus |\mathbf{B}_V|$. For any edge X , we denote by $B^*(X)$ the common flag of balls of elements in X .*
- *We say that two vertices $\alpha \neq \alpha' \in |\mathbf{B}_V|$ are neighbours, if there exists an edge X such that $B^*(\alpha)$ and $B^*(\alpha')$ are contained in $B^*(X)$, in which case α and α' are called extremities of the edge X .*

The induced graph structure $(|\mathbf{B}_V|, (\mathbf{B}_V \setminus |\mathbf{B}_V|)/\sim)$ will - slightly abusively - be still denoted by \mathbf{B}_V/\sim . The action of $G_{V,\tau}$ on \mathbf{B}_V induces corresponding actions on the sets of vertices and edges of \mathbf{B}_V/\sim . Recall that, if $L \subset V_\tau$ is an \mathcal{O}_{E_τ} -lattice, we defined in Definition 1.2.4 the *dual* of L to be the lattice $L^\vee := \{v \in V_\tau; \langle v, L \rangle \subset \mathcal{O}_{E_\tau}\}$.

Lemma 2.1.3. *Let $\alpha \in \mathbf{B}_V$ be a self-dual ultrametric norm and let $\theta \in \mathbb{R}$. One has:*

- (i) $B(\alpha \leq |\varpi|^{1-\theta}) = \varpi B(\alpha \leq |\varpi|^{-\theta})$,
- (ii) $B(\alpha \leq |\varpi|^{-\theta-1}) = B(\alpha < |\varpi|^\theta)^\vee$.

Proof.

(i) If $v \in V_\tau$ is such that $\alpha(v) \leq |\varpi|^{1-\theta}$, then $\alpha(\varpi^{-1}v) = |\varpi|^{-1}\alpha(v) \leq |\varpi|^{-\theta}$, i.e., $v \in \varpi B(\alpha \leq |\varpi|^{-\theta})$.

(ii) Notice that every non-zero vector $w \in V_\tau$ can be scaled by some (unique) integral power ϖ^{n_w} of the uniformizer, such that $\alpha(\varpi^{n_w}w) = |\varpi^{n_w}\alpha(w) \in [|\varpi|^{\theta+1}, |\varpi|^\theta[$. This gives

$$\begin{aligned} \alpha(v) = \alpha^\vee(v) &= \sup_{w \in V_\tau, w \neq 0} \frac{|\langle v, w \rangle|}{\alpha(w)} \\ &= \sup_{w \in V_\tau, w \neq 0} \frac{|\langle v, \varpi^{n_w}w \rangle|}{\alpha(\varpi^{n_w}w)} = \sup_{w \in V_\tau, \alpha(w) \in [|\varpi|^{\theta+1}, |\varpi|^\theta[} \frac{|\langle v, w \rangle|}{\alpha(w)}. \end{aligned}$$

Now if v belongs to $B(\alpha^\vee < |\varpi|^\theta)^\vee$, one has $|\langle v, w \rangle| \leq 1$ for all $w \in V_\tau$ such that $\alpha(w) < |\varpi|^\theta$. Thus $\frac{|\langle v, w \rangle|}{\alpha(w)} \leq |\varpi|^{-\theta-1}$, for all $w \in V_\tau$ such that $\alpha(w) \in [|\varpi|^{\theta+1}, |\varpi|^\theta[$, hence $\alpha^\vee(v) \leq |\varpi|^{-\theta-1}$. Conversely, if $v \in B(\alpha^\vee \leq |\varpi|^{-\theta-1})$ then $|\langle v, w \rangle| \leq \alpha(w)|\varpi|^{-\theta-1}$ for all non-zero $w \in V_\tau$. By restricting to those w 's such that $\alpha(w) < |\varpi|^\theta$, one gets $|\langle v, w \rangle| < |\varpi|^{-1}$, hence $|\langle v, w \rangle| \leq 1$, for ϖ is a uniformizer of E_τ . Thus $v \in B(\alpha < |\varpi|^\theta)^\vee$. \square

Let $\mathcal{L}(V_\tau)$ denote the set of lattices in V_τ and let $\mathcal{F}(\mathbb{R}, \mathcal{L}(V_\tau))$ denote the set of functions $f : \mathbb{R} \rightarrow \mathcal{L}(V_\tau)$. The map $\gamma : \mathbf{B}_V \rightarrow \mathcal{F}(\mathbb{R}, \mathcal{L}(V_\tau))$ which sends a self-dual norm α , to the function

$$\gamma(\alpha) := (\theta \mapsto B(\alpha \leq |\varpi|^{-\theta})) \in \mathcal{F}(\mathbb{R}, \mathcal{L}(V_\tau)),$$

is $G_{V,\tau}$ -equivariant. If α and $\alpha' \in \mathbf{B}_V$ have respective parameters λ and λ' in a common decomposition basis $\mathcal{B} = \{e_+, e_0, e_-\}$, we showed in the proof of Lemma 2.1.2 that:

$$\gamma(\alpha)(\theta) = \langle \varpi^{-[\theta+\lambda]}e_+, \varpi^{-[\theta]}e_0, \varpi^{-[\theta-\lambda]}e_- \rangle \text{ and } \gamma(\alpha')(\theta) = \langle \varpi^{-[\theta+\lambda']}e_+, \varpi^{-[\theta]}e_0, \varpi^{-[\theta-\lambda']}e_- \rangle,$$

for all $\theta \in \mathbb{R}$. If both functions $\gamma(\alpha) = \gamma(\alpha')$ are equal, then one gets $[\theta + \lambda] = [\theta + \lambda']$ and $[\theta - \lambda] = [\theta - \lambda']$, for all $\theta \in \mathbb{R}$ (see next page's footnote), which gives $\lambda = \lambda'$ ⁽²⁾. As elements of $\mathcal{A}_{\mathcal{B}}$ are determined by their parameter with respect to \mathcal{B} , one gets $\alpha = \alpha'$, hence γ is injective.

We let $\mathcal{N} \subset \mathcal{F}(\mathbb{R}, \mathcal{L}(V_\tau))$ be the set of those functions f which satisfy the following properties:

- (N1) : f is increasing,
- (N2) : $f(\theta - 1) = \varpi f(\theta)$, for all $\theta \in \mathbb{R}$,
- (N3) : $f(1 - \theta) = \left(\lim_{\theta' < \theta} f(\theta') \right)^\vee$, for all $\theta \in \mathbb{R}$.

Elements of \mathcal{N} will be referred to as *norm functions*. Property (N3) implies that such function are *right-continuous* in the sense that, if $(\theta_n)_{n \geq 0}$ is a decreasing sequence which converges to θ , then one has $f(\theta_n) = f(\theta)$ for $n \gg 0$. Properties (N2)-(N3) ensure that norm functions are fully determined by their restriction to $[0, \frac{1}{2}[$. Lemma 2.1.3 implies that γ has image in \mathcal{N} . Moreover, if $f \in \mathcal{N}$, one checks that the function $\alpha_f : V_\tau \rightarrow \mathbb{R}$, $v \mapsto |\varpi|^{-\sup\{\theta; v \in f(\theta)\}}$ is the only self-dual ultrametric norm on V_τ such that $\gamma(\alpha_f) = f$. In other words, γ is a bijection between \mathbf{B}_V and \mathcal{N} .

We say that a Witt basis $\mathcal{B} = \{e_+, e_0, e_-\}$ is *adapted* to a subset $S \subset \mathcal{L}(V_\tau)$ if, for all $L \in S$, one has $L = \langle \varpi^m e_+, \varpi^n e_0, \varpi^r e_- \rangle$ for some triple $(m, n, r) \in \mathbb{Z}^3$, which is necessarily unique ⁽³⁾. One checks easily that, given a Witt basis $\mathcal{B} = \{e_+, e_0, e_-\}$, one has $\langle \varpi^m e_+, \varpi^n e_0, \varpi^r e_- \rangle^\vee = \langle \varpi^{-r} e_+, \varpi^{-n} e_0, \varpi^{-m} e_- \rangle$, for all $(m, n, r) \in \mathbb{Z}$. Accordingly, a Witt basis \mathcal{B} is adapted to S if and only if it is adapted to $S^\vee := \{L^\vee; L \in S\}$.

By Lemma 2.1.2, any Witt basis \mathcal{B} is adapted to the set $B^*(\alpha) \subset \mathcal{L}(V_\tau)$ if α belongs to $\mathcal{A}_{\mathcal{B}}$. The converse is also true, as stated in ([31], Lemme 53):

Lemma 2.1.4. *Let $\alpha \in \mathbf{B}_V$. The Witt bases \mathcal{B} such that $\alpha \in \mathcal{A}_{\mathcal{B}}$ are exactly the Witt bases adapted to the flag $B^*(\alpha)$.*

(2). Indeed, if one had $\lambda \neq \lambda'$ then one might always assume $\lambda' > \lambda$, in which case taking $\theta = -\lambda'$ would induce $0 > [\theta + \lambda] = [\theta + \lambda'] = 0$, which is not.

(3). Indeed, if $(m, n, r), (m', n', r') \in \mathbb{Z}^3$ are such that $\langle \varpi^m e_+, \varpi^n e_0, \varpi^r e_- \rangle = \langle \varpi^{m'} e_+, \varpi^{n'} e_0, \varpi^{r'} e_- \rangle$, then the diagonal matrix $\begin{bmatrix} \varpi^{m'-m} & 0 & 0 \\ 0 & \varpi^{n'-n} & 0 \\ 0 & 0 & \varpi^{r'-r} \end{bmatrix}$ has to belong to $\text{GL}_3(\mathcal{O}_{E_\tau})$, i.e., $\varpi^{m'-m}, \varpi^{n'-n}, \varpi^{r'-r} \in \mathcal{O}_{E_\tau}^\times$, hence $m' - m = n' - n = r' - r = 0$.

Proof. Set $f := \gamma(\alpha)$ and assume that the Witt basis $\mathcal{B} = \{e_+, e_0, e_-\}$ is adapted to $S = B^*(\alpha)$. By the above discussion, one gets three well-defined functions $\underline{m}, \underline{n}, \underline{r} : \mathbb{R} \rightarrow \mathbb{Z}$ such that

$$f(\theta) = B(\alpha \leq |\varpi|^{-\theta}) = \langle \varpi^{\underline{m}(\theta)} e_+, \varpi^{\underline{n}(\theta)} e_0, \varpi^{\underline{r}(\theta)} e_- \rangle,$$

for all $\theta \in \mathbb{R}$. By (N1) and (N2), the functions $\underline{m}, \underline{n}$ and \underline{r} are decreasing and satisfy $\underline{m}(\theta + 1) = \underline{m}(\theta) - 1$, $\underline{n}(\theta + 1) = \underline{n}(\theta) - 1$ and $\underline{r}(\theta + 1) = \underline{r}(\theta) - 1$, for all $\theta \in \mathbb{R}$. The right-continuity of f implies that the functions $\underline{m}, \underline{n}$ and \underline{r} have to be also right-continuous, hence piecewise constant on intervals of the form $[z, z + 1[\subset \mathbb{R}$. Moreover, property (N3) and the above discussion imply that, outside a discrete subset of \mathbb{R} , the functions $\underline{m}, \underline{n}$ and \underline{r} satisfy the following symmetries:

$$\begin{cases} \underline{m}(1 - \theta) = -\underline{r}(\theta) \\ \underline{n}(1 - \theta) = -\underline{n}(\theta) \\ \underline{r}(1 - \theta) = -\underline{m}(\theta) \end{cases} \quad (2.2)$$

If $k \in \mathbb{Z}$ was such that $\underline{n}(\theta) = k$ for all $\theta \in [-\frac{1}{2}, \frac{1}{2}[$, and $\underline{n}(\theta) = k - 1$ for all $\theta \in [\frac{1}{2}, \frac{3}{2}[$, then the equality

$$f\left(\frac{1}{2}\right) = \left(\lim_{\theta' \rightarrow \frac{1}{2}^-} f(\theta') \right)^\vee$$

would give $k - 1 = \underline{n}(\frac{1}{2}) = -\lim_{\theta' \rightarrow \frac{1}{2}^-, \theta' < \frac{1}{2}} \underline{n}(\theta') = -k$, i.e., $2k - 1 = 0$, which would contradict $k \in \mathbb{Z}$. On the other hand, if $z \in]0, 1[$, $z \neq \frac{1}{2}$ and $k \in \mathbb{Z}$ were such that $\underline{n}(\theta) = k$ for all $\theta \in [z - 1, z[$ and $\underline{n}(\theta) = k - 1$ for all $\theta \in [z, z + 1[$ then, choosing some $\theta \in [z - 1, z[$ such that $1 - \theta \in [z, z + 1[$ and such that $\underline{n}(1 - \theta) = -\underline{n}(\theta)$, would also give $2k - 1 = 0$, which is not. Consequently, \underline{n} is constant with value $k = -k = 0$ on $[0, 1[$, i.e., $\underline{n}(\theta) = -\lfloor \theta \rfloor$, for all $\theta \in \mathbb{R}$.

Set $\theta_0 := \min\{\theta \in \mathbb{R}; \underline{r}(\theta) = 0\}$, so that $\underline{r}(\theta) = -\lfloor \theta - \theta_0 \rfloor$, for all $\theta \in \mathbb{R}$. Set $\alpha_{\theta_0} \in \mathcal{A}_{\mathcal{B}}$ to be the norm of parameter θ_0 with respect to \mathcal{B} , and let $f_{\theta_0} = \gamma(\alpha_{\theta_0})$ be its corresponding norm function. By (2.2), one has $\underline{m}(\theta) = -\underline{r}(1 - \theta) = \lfloor 1 - (\theta + \theta_0) \rfloor$, the latter being equal to $-\lfloor \theta + \theta_0 \rfloor$ outside a discrete subset of \mathbb{R} . As both piecewise constant functions \underline{m} and $\theta \mapsto -\lfloor \theta + \theta_0 \rfloor$ are right-continuous, we finally get $\underline{m}(\theta) = -\lfloor \theta + \theta_0 \rfloor$, for all $\theta \in \mathbb{R}$. In other words, one has $f(\theta) = \langle \varpi^{-\lfloor \theta + \theta_0 \rfloor} e_+, \varpi^{-\lfloor \theta \rfloor} e_0, \varpi^{-\lfloor \theta - \theta_0 \rfloor} \rangle = f_{\theta_0}(\theta)$ for all $\theta \in \mathbb{R}$, i.e., $\alpha = \alpha_{\theta_0} \in \mathcal{A}_{\mathcal{B}}$. \square

In particular, two equivalent norms belong to the same apartments. Accordingly, if \mathcal{B} is a Witt basis and if $\alpha_0 \notin |\mathbf{B}_V|$ belongs to $\mathcal{A}_{\mathcal{B}}$, then the edge $X := \text{cl}(\alpha_0)$ is entirely contained in $\mathcal{A}_{\mathcal{B}}$, and so are the extremities of X . Given a Witt basis $\mathcal{B} = \{e_+, e_0, e_-\}$, we denote by $\alpha_\lambda \in \mathcal{A}_{\mathcal{B}}$ the self-dual ultrametric norm which has parameter λ with respect to \mathcal{B} . By

Lemma 2.1.3, (i), the group E_τ^\times acts on the flag $B^*(\alpha_\lambda)$ by homothety. The structure of the quotient set $E_\tau^\times \backslash B^*(\alpha_\lambda)$ depends on λ , as described below.

Lemma 2.1.5. *Let $\mathcal{B} = \{e_+, e_0, e_-\}$ be a Witt basis and let $\lambda \in \mathbb{R}$. The flag $B^*(\alpha_\lambda)$ decomposes as the following disjoint union of E_τ^\times -homothety classes:*

— If $\lambda \in \mathbb{Z}$, then $B^*(\alpha_\lambda)$ is the single homothety class

$$B^*(\alpha_\lambda) = \{\varpi^n \langle \varpi^{-\lambda} e_+, e_0, \varpi^\lambda e_- \rangle; n \in \mathbb{Z}\}.$$

— If $\lambda \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then $B^*(\alpha_\lambda)$ is the disjoint union of two homothety classes:

$$B^*(\alpha_\lambda) = \left\{ \varpi^n \langle \varpi^{-(\lambda-\frac{1}{2})} e_+, e_0, \varpi^{\lambda+\frac{1}{2}} e_- \rangle; n \in \mathbb{Z} \right\} \sqcup \left\{ \varpi^n \langle \varpi^{-\lambda-\frac{1}{2}} e_+, e_0, \varpi^{\lambda-\frac{1}{2}} e_- \rangle; n \in \mathbb{Z} \right\}.$$

— If $\lambda \notin \frac{1}{2}\mathbb{Z}$, then $B^*(\alpha_\lambda)$ is the disjoint union of three homothety classes. More precisely:

— If $\lambda \in]\lfloor \lambda \rfloor, \lfloor \lambda \rfloor + \frac{1}{2}[$, then one has:

$$B^*(\alpha_\lambda) = \left\{ \varpi^n \langle \varpi^{-\lfloor \lambda \rfloor} e_+, e_0, \varpi^{\lfloor \lambda \rfloor} e_- \rangle; n \in \mathbb{Z} \right\} \sqcup \left\{ \varpi^n \langle \varpi^{-\lfloor \lambda \rfloor} e_+, e_0, \varpi^{\lfloor \lambda \rfloor + 1} e_- \rangle; n \in \mathbb{Z} \right\} \\ \sqcup \left\{ \varpi^n \langle \varpi^{-\lfloor \lambda \rfloor - 1} e_+, e_0, \varpi^{\lfloor \lambda \rfloor} e_- \rangle; n \in \mathbb{Z} \right\}.$$

— If $\lambda \in]\lfloor \lambda \rfloor + \frac{1}{2}, \lfloor \lambda \rfloor + 1[$, then one has:

$$B^*(\alpha_\lambda) = \left\{ \varpi^n \langle \varpi^{-\lfloor \lambda \rfloor - 1} e_+, e_0, \varpi^{\lfloor \lambda \rfloor + 1} e_- \rangle; n \in \mathbb{Z} \right\} \sqcup \left\{ \varpi^n \langle \varpi^{-\lfloor \lambda \rfloor} e_+, e_0, \varpi^{\lfloor \lambda \rfloor + 1} e_- \rangle; n \in \mathbb{Z} \right\} \\ \sqcup \left\{ \varpi^n \langle \varpi^{-\lfloor \lambda \rfloor - 1} e_+, e_0, \varpi^{\lfloor \lambda \rfloor} e_- \rangle; n \in \mathbb{Z} \right\}.$$

Proof. We showed in Lemma 2.1.2 that $B(\alpha_\lambda \leq |\varpi|^{-\theta}) = \langle \varpi^{-[\theta+\lambda]} e_+, \varpi^{-[\theta]} e_0, \varpi^{-[\theta-\lambda]} e_- \rangle$, for all $\theta \in \mathbb{R}$.

- If $\lambda \in \mathbb{Z}$, this gives $B(\alpha_\lambda \leq |\varpi|^{-\theta}) = \varpi^{-[\theta]} \langle \varpi^{-\lambda} e_+, e_0, \varpi^\lambda e_- \rangle$, for all $\theta \in \mathbb{R}$. Hence the result.

- If $\lambda \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, one gets $\lfloor \lambda \rfloor = \lambda - \frac{1}{2}$. Let $\theta \in \mathbb{R}$ and set $k := \lfloor \theta \rfloor$. If $\theta \in [k, k + \frac{1}{2}[$, one has $\lfloor \theta + \lambda \rfloor = k + \lambda - \frac{1}{2}$ and $\lfloor \theta - \lambda \rfloor = k - \lambda - \frac{1}{2}$, thus

$$B(\alpha_\lambda \leq |\varpi|^{-\theta}) = \varpi^{-k} \langle \varpi^{-(\lambda-\frac{1}{2})} e_+, e_0, \varpi^{\lambda+\frac{1}{2}} e_- \rangle.$$

If $\theta \in [k + \frac{1}{2}, k + 1[$, one has $\lfloor \theta + \lambda \rfloor = k + \lambda + \frac{1}{2}$ and $\lfloor \theta - \lambda \rfloor = k - \lambda + \frac{1}{2}$, hence

$$B(\alpha_\lambda \leq |\varpi|^{-\theta}) = \varpi^{-k} \langle \varpi^{-(\lambda+\frac{1}{2})} e_+, e_0, \varpi^{\lambda-\frac{1}{2}} e_- \rangle.$$

One checks that the lattices $\langle \varpi^{-(\lambda-\frac{1}{2})} e_+, e_0, \varpi^{\lambda+\frac{1}{2}} e_- \rangle \subset \langle \varpi^{-(\lambda+\frac{1}{2})} e_+, e_0, \varpi^{\lambda-\frac{1}{2}} e_- \rangle$ are not homothetic, as the former has index $q^2 \notin q^{3\mathbb{N}}$ in the latter, hence the result.

- If $\lambda \notin \frac{1}{2}\mathbb{Z}$, write $\lambda = \lfloor \lambda \rfloor + \epsilon$ and set $\eta := \min(\epsilon, 1 - \epsilon)$, $\eta' := \max(\epsilon, 1 - \epsilon)$, so that $0 < \eta < \eta' < 1$. We shall treat only the case $\eta = \epsilon$, the case $\eta = 1 - \epsilon$ being similar. Let

$\theta \in \mathbb{R}$ and set $k := \lfloor \theta \rfloor$. If $\theta \in [k, k + \epsilon[$, one has $\lfloor \theta + \lambda \rfloor = k + \lfloor \lambda \rfloor$ and $\lfloor \theta - \lambda \rfloor = k - \lfloor \lambda \rfloor - 1$, hence

$$B(\alpha_\lambda \leq |\varpi|^{-\theta}) = \varpi^{-k} \langle \varpi^{-\lfloor \lambda \rfloor} e_+, e_0, \varpi^{\lfloor \lambda \rfloor + 1} e_- \rangle.$$

If $\theta \in [k + \epsilon, k + 1 - \epsilon[$, one has $\lfloor \theta + \lambda \rfloor = k + \lfloor \lambda \rfloor$ and $\lfloor \theta - \lambda \rfloor = k - \lfloor \lambda \rfloor$, hence

$$B(\alpha_\lambda \leq |\varpi|^{-\theta}) = \varpi^{-k} \langle \varpi^{-\lfloor \lambda \rfloor} e_+, e_0, \varpi^{\lfloor \lambda \rfloor} e_- \rangle.$$

If $\theta \in [k + 1 - \epsilon, k + 1[$, one has $\lfloor \theta + \lambda \rfloor = k + \lfloor \lambda \rfloor + 1$ and $\lfloor \theta - \lambda \rfloor = k - \lfloor \lambda \rfloor$, hence

$$B(\alpha_\lambda \leq |\varpi|^{-\theta}) = \varpi^{-k} \langle \varpi^{-\lfloor \lambda \rfloor - 1} e_+, e_0, \varpi^{\lfloor \lambda \rfloor} e_- \rangle.$$

One checks, finally, that the tree lattices $\langle \varpi^{-\lfloor \lambda \rfloor} e_+, e_0, \varpi^{\lfloor \lambda \rfloor + 1} e_- \rangle \subset \langle \varpi^{-\lfloor \lambda \rfloor} e_+, e_0, \varpi^{\lfloor \lambda \rfloor} e_- \rangle \subset \langle \varpi^{-\lfloor \lambda \rfloor - 1} e_+, e_0, \varpi^{\lfloor \lambda \rfloor} e_- \rangle$ are pairwise non-homothetic, since they have index q or q^2 in each other. This gives the result. \square

Accordingly, if $\alpha \in \mathbf{B}_V$ is a self-dual ultrametric norm, one may say that α has *integral* (resp. *half-integral* but not integral) *parameter*, irrespective of any choice of apartment that contains α ; and if two norms α and α' are such that $\alpha \sim \alpha'$, then their respective parameters must have the same type. Notice that, if a, b, a' and b' are integers, the lattices $\langle \varpi^{-a} e_+, e_0, \varpi^b e_- \rangle$ and $\langle \varpi^{-a'} e_+, e_0, \varpi^{b'} e_- \rangle$ are homothetic if and only if $a = a'$ and $b = b'$. Indeed, if both lattices are homothetic then there exists some $n \in \mathbb{Z}$ such that

$$\langle \varpi^{-a+n} e_+, \varpi^n e_0, \varpi^{b+n} e_- \rangle = \langle \varpi^{-a'} e_+, e_0, \varpi^{b'} e_- \rangle,$$

hence $n + a' - a = n + b - b' = n = 0$, thus $a = a'$ and $b = b'$.

Consequently, if $\lambda, \lambda' \in \mathbb{Z}$, then $B^*(\alpha_\lambda) \neq B^*(\alpha_{\lambda'})$ unless $\lambda = \lambda'$. Similarly, if $\lambda, \lambda' \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then $B^*(\alpha_\lambda) \neq B^*(\alpha_{\lambda'})$ if $\lambda \neq \lambda'$. Finally, if $\lambda, \lambda' \notin \frac{1}{2}\mathbb{Z}$, then $B^*(\alpha_\lambda) = B^*(\alpha_{\lambda'})$ if and only if one has $\lambda, \lambda' \in]m, m + \frac{1}{2}[$ for some $m \in \frac{1}{2}\mathbb{Z}$. As any two equivalent norms lie in the same apartments, one deduces the following classification:

Lemma 2.1.6. *Let $\alpha \in \mathbf{B}_V$, with parameter λ with respect to the Witt basis $\mathcal{B} = \{e_+, e_0, e_-\}$.*

Then:

(i) α is a vertex if and only if $\lambda \in \frac{1}{2}\mathbb{Z}$.

(ii) Edges that are contained in $\mathcal{A}_{\mathcal{B}}$ are the sets $X_{\lambda, \lambda + \frac{1}{2}} := \{\alpha_\mu; \mu \in]\lambda, \lambda + \frac{1}{2}[\} \subset \mathcal{A}_{\mathcal{B}}$, with $\lambda \in \frac{1}{2}\mathbb{Z}$. The extremities of $X_{\lambda, \lambda + \frac{1}{2}}$ are the two vertices α_λ and $\alpha_{\lambda + \frac{1}{2}}$.

Accordingly, any apartment \mathcal{A} , endowed with the sub-graph structure $(\mathcal{A} \cap |\mathbf{B}_V|, (\mathcal{A} \setminus |\mathbf{B}_V|) / \sim)$ is a connected sub-graph of \mathbf{B}_V / \sim (with respect to the adjacency relation of Definition 2.1.5). As any two self-dual ultrametric norms α, α' belong to some common apartment, we deduce that \mathbf{B}_V / \sim is also a connected graph, which recovers the original idea of \mathbf{B}_V being obtained by glueing apartments together.

Definition 2.1.6. Vertices α of \mathbf{B}_V such that $B^*(\alpha)$ is a single E_τ^\times -homothety class are called hyperspecial vertices. Other vertices of \mathbf{B}_V are called special vertices. The set of hyperspecial vertices (resp. of special vertices) of \mathbf{B}_V is denoted by Hyp_V (resp. Sp_V). Those correspond to self-dual ultrametric norms which have integral (resp. half-integral but not integral) parameters in all the apartments in which they are contained.

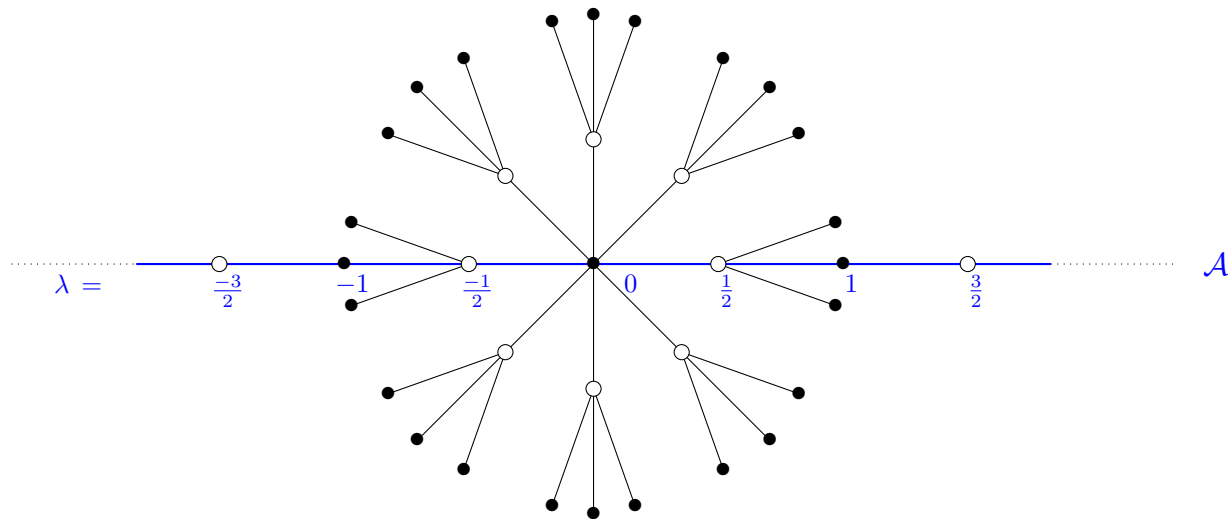


Figure 2.1 – Representation of the building \mathbf{B}_V as a bi-colored graph. Black points correspond to hyperspecial vertices, whereas white points correspond to special vertices. The blue line corresponds to an apartment \mathcal{A} , an affine real line which we identified with \mathbb{R} on the figure. In \mathcal{A} , hyperspecial (resp. special) vertices have an integral (resp. half-integral) parameter λ . Notice that every point is not drawn here, and that the number of neighbours represented is not necessarily realistic.

Proposition 2.1.7. The action of $G_{V,\tau}$ on \mathbf{B}_V induces transitive actions of $G_{V,\tau}$ on the sets Hyp_V and Sp_V .

Proof. Let $\alpha \neq \alpha'$ be two vertices of the same type. Let \mathcal{B} and \mathcal{B}' be decomposition bases in which α and α' have respective parameters λ and λ' . As $\lambda - \lambda' \in \mathbb{Z}$, one may assume without loss of generality that $\lambda = \lambda'$. Indeed, if not, one may switch the basis $\mathcal{B}' = \{e'_+, e'_0, e'_-\}$ with $\{\varpi^{\lambda-\lambda'} e'_+, e'_0, \varpi^{\lambda'-\lambda} e'_-\}$, which is still a decomposition basis for α' , in which α' now has parameter λ . This gives an E_τ -linear map $g_{\alpha,\alpha'} : V_\tau \rightarrow V_\tau$, defined by $g \cdot e_i = e'_i$, for $i \in \{+, 0, -\}$, which is clearly unitary. Moreover, one has $g_{\alpha,\alpha'} \cdot \alpha \in \mathcal{A}_{g_{\alpha,\alpha'} \cdot \mathcal{B}} = \mathcal{A}_{\mathcal{B}'}$, and satisfies $(g_{\alpha,\alpha'} \cdot \alpha)(e'_\star) = \alpha(e_\star) = |\varpi|^{-\star\lambda} = \alpha'(e'_\star)$, for all $\star \in \{+, 0, -\}$, thus $g_{\alpha,\alpha'} \cdot \alpha$ and α' have the same parameter in \mathcal{B}' , i.e., $g_{\alpha,\alpha'} \cdot \alpha = \alpha'$. \square

Let $\alpha \in |\mathbf{B}_V|$ be a vertex with parameter $\lambda \in \frac{1}{2}\mathbb{Z}$ in the Witt basis $\mathcal{B} = \{e_+, e_0, e_-\}$. The above description of the norm function $f = \gamma(\alpha)$ implies that f is constant on the interval $[0, \frac{1}{2}[$. One gets an injection:

$$\gamma_0 : |\mathbf{B}_V| \longleftarrow \longrightarrow \mathcal{L}(V_\tau), \quad (2.3)$$

$$\alpha \longmapsto \gamma(\alpha)(0) = B(\alpha \leq 1)$$

We recall that a lattice $L \subset V_\tau$ is said to be *self-dual* if $L = L^\vee$, to be *almost self-dual* if one has inclusions $\varpi L^\vee \subset L \subset L^\vee$, and to be *strictly almost self-dual* if the last inclusion is strict.

Remark 2.1.1. *If L and L' are self-dual lattices such that $L \subset L'$, then $L' = (L')^\vee \subset L^\vee = L$, i.e., $L = L'$. If L and L' are almost self-dual lattices which are homothetic, then they are equal. Indeed, if not, then up to permuting L and L' one may write $L = \varpi^n L'$ for some $n \geq 1$. Inclusions $\varpi L^\vee \subset L$ and $L' \subset (L')^\vee$ rewrite as $\varpi^{1-n}(L')^\vee \subset \varpi^n L' \subset \varpi^n (L')^\vee$, hence $(L')^\vee \subset \varpi^{2n-1} L' \subset \varpi^{2n-1} (L')^\vee \subset \varpi (L')^\vee$, which is not.*

A fairly direct consequence of (N1)-(N2)-(N3) is the following:

Lemma 2.1.8. *The image of γ_0 is the set of almost self-dual lattices of V_τ . The image of $\gamma_0|_{\text{Hyp}_V}$ is the set of self-dual lattices, whereas the image of $\gamma_0|_{\text{Sp}_V}$ is the set of strictly almost self-dual lattices.*

Proof. Let $\alpha \in |\mathbf{B}_V|$ and let $\mathcal{B} = \{e_+, e_0, e_-\}$ be any decomposition basis such that α has parameter 0 or $\frac{1}{2}$ (which is always possible, by Proposition 2.1.1, (ii)), and set $f := \gamma(\alpha)$. If $\alpha \in \text{Hyp}_V$, one has

$$f(0) = \langle \varpi^{-[0]} e_+, \varpi^{-[0]} e_0, \varpi^{-[0]} e_- \rangle = \langle e_+, e_0, e_- \rangle,$$

which is a self-dual lattice, as \mathcal{B} is a Witt basis. If $\alpha \in \text{Sp}_V$, one has

$$f(0) = \langle \varpi^{-[\frac{1}{2}]} e_+, \varpi^{-[0]} e_0, \varpi^{-[-\frac{1}{2}]} e_- \rangle = \langle e_+, e_0, \varpi e_- \rangle,$$

hence $f(0)^\vee = \langle \varpi^{-1} e_+, e_0, e_- \rangle$, thus $\varpi f(0)^\vee \subset f(0) \subset f(0)^\vee$. On the other hand, if L is an almost self-dual lattice of V_τ , one may define a norm function $f_L : \mathbb{R} \rightarrow \mathcal{L}(V_\tau)$ which is constant on $[0, \frac{1}{2}[$ and such that $f(0) = L$. That f indeed satisfies (N1) (i.e., that f is increasing) is a consequence of the inclusions $\varpi L^\vee \subset L \subset L^\vee$. This completes the proof. \square

If $\alpha \in \mathbf{B}_V$ and $g \in G_{V,\tau}$, one has by injectivity of γ that $g \cdot \alpha = \alpha$ if and only if $\gamma(g \cdot \alpha) = g \cdot (\gamma(\alpha)) = \gamma(\alpha)$, i.e., $B(g \cdot \alpha \leq |\varpi|^\theta) = B(\alpha \leq |\varpi|^\theta)$, for all $\theta \in \mathbb{R}$. If $\alpha \in |\mathbf{B}_V|$, one just showed that $\gamma(\alpha)$ is fully determined by $L := \gamma_0(\alpha)$, hence $\gamma \cdot \alpha = \alpha$ if and only if $g \cdot L = L$. This shows that stabilizers in $G_{V,\tau}$ of hyperspecial (resp. special) vertices of \mathbf{B}_V correspond to stabilizers of self-dual lattices (resp. of strictly almost self-dual lattices), and Proposition 2.1.7 ensures that these are all $G_{V,\tau}$ -conjugated. The use of the term hyperspecial is now justified by the fact that, as seen in § 1.2.3, the stabilizers of self-dual local lattices arise as hyperspecial maximal compact subgroups of $G_{V,\tau}$.

2.1.3 The distance in $|\mathbf{B}_V|$.

Definition 2.1.7 (Distance between lattices). *The lattice distance in $\mathcal{L}(V_\tau)$ is the map*

$$\delta : \mathcal{L}(V_\tau) \times \mathcal{L}(V_\tau) \rightarrow \mathbb{N}$$

given by $\delta(L, L') := \inf\{k \geq 0; \exists n \in \mathbb{Z}, \varpi^{n+k}L \subset L' \subset \varpi^n L\}$, for all $L, L' \in \mathcal{L}(V_\tau)$.

Lemma 2.1.9. *The lattice distance δ satisfies the following properties:*

- (i) L and L' are homothetic if and only if $\delta(L, L') = 0$.
- (ii) For all $L, L', L'' \in \mathcal{L}(V_\tau)$, one has $\delta(L, L'') \leq \delta(L, L') + \delta(L', L'')$.
- (iii) If $\delta(L, L') = 1$, then either $L \subset L'$ or $L' \subset L$.

Proof.

(i) is an immediate consequence of the definition.

(ii) If k and l are minimal non-negative integers such that $\varpi^{n+k}L \subset L' \subset \varpi^n L$ and $\varpi^{m+l}L' \subset L'' \subset \varpi^m L'$ for some $n, m \in \mathbb{Z}$, one has $\varpi^{m+n+k+l}L \subset \varpi^{m+l}L' \subset L'' \subset \varpi^m L' \subset \varpi^{n+m}L$, hence $\delta(L, L'') \leq k + l = \delta(L, L') + \delta(L', L'')$.

(iii) Let $n \in \mathbb{Z}$ be such that $\varpi^{n+1}L \subset L' \subset \varpi^n L$. If $n \geq 0$, this gives $L' \subset L$. If $n < 0$, this gives $L \subset \varpi^{-n-1}L' \subset L'$. \square

By embedding the vertices of \mathbf{B}_V inside $\mathcal{L}(V_\tau)$ via $\gamma_0 : \alpha \mapsto B(\alpha \leq 1)$, one may consider δ as a function on $|\mathbf{B}_V| \times |\mathbf{B}_V|$. That γ_0 is $G_{V,\tau}$ -equivariant implies, directly from the preceding definition, that

$$\delta(\gamma_0(g \cdot \alpha), \gamma_0(g \cdot \alpha')) = \delta(\gamma_0(\alpha), \gamma_0(\alpha')), \quad \forall \alpha, \alpha' \in |\mathbf{B}_V|.$$

Lemma 2.1.10. *Two vertices $\alpha \neq \alpha' \in |\mathbf{B}_V|$ are neighbours if and only if $\delta(\gamma_0(\alpha), \gamma_0(\alpha')) = 1$.*

Proof.

Assume first that α and α' are neighbours. Set $L = \gamma_0(\alpha) = B(\alpha \leq 1)$, $L' = \gamma_0(\alpha') = B(\alpha' \leq 1)$ and let $\alpha_0 \in \mathbf{B}_V \setminus |\mathbf{B}_V|$ be a self-dual norm such that α and α' are common extremities of the edge $X = \text{cl}(\alpha_0)$, i.e., such that the flags $B^*(\alpha)$ and $B^*(\alpha')$ are included in $B^*(\alpha_0)$. This gives the existence of some $\theta, \theta' \in \mathbb{R}$ such that $L = B(\alpha_0 \leq |\varpi|^{-\theta})$ and $L' = B(\alpha_0 \leq |\varpi|^{-\theta'})$. By setting $n := \lceil \theta - \theta' \rceil$, the inequalities $\theta' \leq \theta - n \leq \theta' + 1$ induce

$$\begin{aligned} \varpi^n L &= B(\alpha_0 \leq |\varpi|^{-(\theta-n)}) \supset B(\alpha \leq |\varpi|^{-\theta'}) = L', \\ \varpi^{n+1} L &= B(\alpha_0 \leq |\varpi|^{-(\theta-n-1)}) \subset B(\alpha \leq |\varpi|^{-\theta'}) = L', \end{aligned}$$

thus $\delta(L, L') \leq 1$, hence $\delta(L, L') = 1$, for otherwise L and L' would be homothetic, hence equal (by Remark 2.1.1), which would contradict the assumption $\alpha \neq \alpha'$.

Conversely, let $\mathcal{B} = \{e_+, e_0, e_-\}$ be a common decomposition basis for α and α' , and let $\lambda, \lambda' \in \frac{1}{2}\mathbb{Z}$ be their respective parameters in \mathcal{B} . One has $L = \langle \varpi^{-\lfloor \lambda \rfloor} e_+, e_0, \varpi^{\lfloor -\lambda \rfloor} e_- \rangle$,

$L' = \langle \varpi^{-\lfloor \lambda' \rfloor} e_+, e_0, \varpi^{\lfloor -\lambda' \rfloor} e_- \rangle$. If $\delta(L, L') = 1$, then by Lemma 2.1.9, (iii), one may assume that $L \subset L'$ without loss of generality. This gives $\lfloor \lambda \rfloor - \lfloor \lambda' \rfloor \geq 0$ and $\lfloor -\lambda' \rfloor - \lfloor -\lambda \rfloor \geq 0$. If α and α' were both hyperspecial (resp. special), one would have $\lfloor \lambda \rfloor - \lfloor \lambda' \rfloor = \lambda - \lambda'$ and $\lfloor -\lambda' \rfloor - \lfloor -\lambda \rfloor = \lambda' - \lambda$, hence $\lambda = \lambda'$ and $\alpha = \alpha'$, which is not. Thus α and α' are not of the same type, and we may assume α to be hyperspecial and α' to be special without loss of generality. Therefore

$$\lfloor \lambda \rfloor - \lfloor \lambda' \rfloor = \lambda - \lambda' + \frac{1}{2},$$

and

$$\lfloor -\lambda' \rfloor - \lfloor -\lambda \rfloor = \lambda - \lambda' - \frac{1}{2},$$

hence $\lambda' = \lambda + \frac{1}{2}$, i.e., α and α' are common extremities of the edge $]\lambda, \lambda + \frac{1}{2}[\subset \mathbf{B}_V \setminus |\mathbf{B}_V|$. \square

By applying similar arguments as above, one shows that if α, α' are vertices of \mathbf{B}_V with respective parameters λ, λ' in a common decomposition basis, one has

$$\begin{aligned} \delta(\gamma_0(\alpha), \gamma_0(\alpha')) &= |\lfloor -\lambda \rfloor - \lfloor -\lambda' \rfloor| + |\lfloor \lambda' \rfloor - \lfloor \lambda \rfloor| \\ &= 2|\lambda - \lambda'|. \end{aligned} \tag{2.4}$$

By Lemma 2.1.8, one obtains that δ satisfies $\delta(L, L') = \delta(L', L)$ for all almost self-dual lattices L, L' , i.e., that δ is an actual distance when restricted to vertices of \mathbf{B}_V . Another distance - namely, the naive distance from graph theory - can be defined on $|\mathbf{B}_V|$ as follows :

Definition 2.1.8 (Geodesic distance on $|\mathbf{B}_V|$). *Define the following terms:*

- We call path of length $l \geq 0$ a finite sequence $[x_0, x_1, \dots, x_l]$ of vertices of \mathbf{B}_V such that $x_i \neq x_j$ if $i \neq j \in \{0, \dots, l-1\}$, and such that x_i and x_{i+1} are neighbours for all $i \in \{0, \dots, l-1\}$ (if $l \geq 1$). A cyclic path is a path of length $l \geq 3$ such that $x_0 = x_l$.
- We call line (or infinite path) an infinite injective sequence $\ell = (\dots, y_i, y_{i+1}, \dots)$ of vertices of \mathbf{B}_V indexed by \mathbb{Z} and such that y_i and y_{i+1} are neighbours, for all $i \in \mathbb{Z}$.
- Two lines $\ell = (\dots, y_i, y_{i+1}, \dots)$ and $\ell' = (\dots, y'_i, y'_{i+1}, \dots)$ are called equivalent if there exists $m_0 \in \mathbb{Z}$ such that $y'_i = y_{m_0+i}$, for all $i \in \mathbb{Z}$.
- A path $[x_0, \dots, x_l]$ is said to be contained in a line $\ell = (\dots, y_i, y_{i+1}, \dots)$ if there exists $n_0 \in \mathbb{Z}$ such that $x_i = y_{n_0+i}$, for all $i \in \{0, \dots, l-1\}$.
- A path $[x_0, \dots, x_l]$ is called geodesic if its length l is minimal among paths which share the same extremities x_0 and x_l (in particular, $x_i \neq x_j$ for all $i \neq j$).
- A line ℓ is called geodesic if any path contained in ℓ is geodesic.

If α and α' lie in $|\mathbf{B}_V|$, we define the geodesic distance between α and α' to be the common length $\delta_{\text{gd}}(\alpha, \alpha')$ of geodesic paths of the form $[\alpha = x_0, x_1, \dots, x_l = \alpha']$ in $|\mathbf{B}_V|$.

The two distances we have defined on $|\mathbf{B}_V| \times |\mathbf{B}_V|$ turn out to be the same :

Proposition 2.1.11. *One has $\delta|_{|\mathbf{B}_V|} = \delta_{\text{gd}}$.*

Proof. Let $\alpha, \alpha' \in |\mathbf{B}_V|$, set $L = \gamma_0(\alpha)$, $L' = \gamma_0(\alpha')$ and let $[x_0 = \alpha, x_1, \dots, x_n = \alpha']$ be a path of minimal length n . By Lemma 2.1.9, (ii), one has $\delta(\alpha, \alpha') \leq \delta(\alpha, x_1) + \dots + \delta(x_{n-1}, \alpha') = n$, i.e., $\delta(\alpha, \alpha') \leq \delta_{\text{gd}}(\alpha, \alpha')$. Conversely, if $\mathcal{B} = \{e_+, e_0, e_-\}$ is a common decomposition basis such that α and α' have respective parameters λ and λ' , then one may assume without loss of generality that $\lambda \leq \lambda'$. Then the path $[\alpha = \alpha_\lambda, \alpha_{\lambda+\frac{1}{2}}, \dots, \alpha_{\lambda'-\frac{1}{2}}, \alpha_{\lambda'} = \alpha']$ has length $2|\lambda' - \lambda| = \delta(\alpha, \alpha')$ (by (2.4)), i.e., $\delta_{\text{gd}}(\alpha, \alpha') \leq \delta(\alpha, \alpha')$, hence the equality. \square

Remark 2.1.2. *The preceding implies that the apartments of \mathbf{B}_V are fully determined - as sets - by their subsets of vertices. Indeed, let $\mathcal{B} = \{e_+, e_0, e_-\}$ and $\mathcal{B}' = \{e'_+, e'_0, e'_-\}$ be Witt bases, and denote by $|\mathcal{A}_{\mathcal{B}}| := \mathcal{A}_{\mathcal{B}} \cap |\mathbf{B}_V|$ (resp. by $|\mathcal{A}_{\mathcal{B}'}| := \mathcal{A}_{\mathcal{B}'} \cap |\mathbf{B}_V|$) the corresponding subsets of vertices of the apartments $\mathcal{A}_{\mathcal{B}}$ and $\mathcal{A}_{\mathcal{B}'}$. By Lemma 2.1.6, the sets $|\mathcal{A}_{\mathcal{B}}|$ and $|\mathcal{A}_{\mathcal{B}'}|$ are in bijection with $\frac{1}{2}\mathbb{Z}$, via*

$$\begin{aligned} \frac{1}{2}\mathbb{Z} &\longrightarrow |\mathcal{A}_{\mathcal{B}}| && (\text{resp. } |\mathcal{A}_{\mathcal{B}'}|), \\ \lambda &\longmapsto \alpha_\lambda && (\text{resp. } \alpha'_\lambda). \end{aligned}$$

If $|\mathcal{A}_{\mathcal{B}}| = |\mathcal{A}_{\mathcal{B}'}|$ then there exists a bijection $\mu : \frac{1}{2}\mathbb{Z} \longrightarrow \frac{1}{2}\mathbb{Z}$ such that, for all $\lambda \in \frac{1}{2}\mathbb{Z}$, the vertex $\alpha_\lambda \in |\mathcal{A}_{\mathcal{B}}|$ is equal to $\alpha'_{\mu(\lambda)} \in |\mathcal{A}_{\mathcal{B}'}|$. If λ and λ' are half-integers, then by (2.4), the distance $\delta(\alpha_\lambda, \alpha_{\lambda'}) = \delta(\alpha'_{\mu(\lambda)}, \alpha'_{\mu(\lambda')})$ is equal to $2|\lambda - \lambda'| = 2|\mu(\lambda) - \mu(\lambda')|$. Therefore the permutation μ preserves the archimedean distance on $\frac{1}{2}\mathbb{Z} \subset \mathbb{R}$, which gives the existence of $n_0 \in \mathbb{Z}$ and $\epsilon \in \mathbb{Z}/2\mathbb{Z}$ such that $\mu(\lambda) = (-1)^\epsilon \cdot (\lambda + n_0)$, for all $\lambda \in \frac{1}{2}\mathbb{Z}$. By Proposition 2.1.1, (ii), one may assume, up to replacing \mathcal{B} by $p^\epsilon t \cdot \mathcal{B}$ (with $t := \text{diag}(\varpi^{-n_0}, \epsilon, \varpi^{n_0}) \in \mathfrak{T}_V$) that $n_0 = 0$ and $\epsilon = 1$, i.e., that $\alpha_\lambda = \alpha'_\lambda$ for all $\lambda \in \frac{1}{2}\mathbb{Z}$. If $\lambda \in \mathbb{Z}$, the equalities $\gamma_0(\alpha_\lambda) = \gamma_0(\alpha'_\lambda)$ and $\gamma_0(\alpha_{\lambda+\frac{1}{2}}) = \gamma_0(\alpha'_{\lambda+\frac{1}{2}})$ rewrite as

$$\langle \varpi^{-\lambda} e_+, e_0, \varpi^\lambda e_- \rangle = \langle \varpi^{-\lambda} e'_+, e'_0, \varpi^\lambda e'_- \rangle \text{ and } \langle \varpi^{-\lambda} e_+, e_0, \varpi^{\lambda+1} e_- \rangle = \langle \varpi^{-\lambda} e'_+, e'_0, \varpi^{\lambda+1} e'_- \rangle$$

If $\mu \in]\lambda, \lambda + \frac{1}{2}[$ and if $\theta \in [0, \frac{1}{2}[$, one has

$$\gamma(\alpha_\mu)(\theta) = \langle \varpi^{-[\theta+\mu]} e_+, \varpi^{-[\theta]} e_0, \varpi^{-[\theta-\mu]} e_- \rangle = \begin{cases} \langle \varpi^{-\lambda} e_+, e_0, \varpi^{\lambda+1} e_- \rangle & \text{if } \theta \in [0, \mu - \lambda[, \\ \langle \varpi^{-\lambda} e_+, e_0, \varpi^\lambda e_- \rangle & \text{if } \theta \in [\mu - \lambda, \frac{1}{2}[\end{cases}$$

Similarly, one has

$$\gamma(\alpha'_\mu)(\theta) = \langle \varpi^{-[\theta+\mu]} e'_+, \varpi^{-[\theta]} e'_0, \varpi^{-[\theta-\mu]} e'_- \rangle = \begin{cases} \langle \varpi^{-\lambda} e'_+, e'_0, \varpi^{\lambda+1} e'_- \rangle & \text{if } \theta \in [0, \mu - \lambda[, \\ \langle \varpi^{-\lambda} e'_+, e'_0, \varpi^\lambda e'_- \rangle & \text{if } \theta \in [\mu - \lambda, \frac{1}{2}[\end{cases}$$

Accordingly, the norm functions $\gamma(\alpha_\mu)$ and $\gamma(\alpha'_\mu)$ coincide on $[0, \frac{1}{2}[$, hence are equal, which gives $\alpha_\mu = \alpha'_\mu$ for all $\mu \in]\lambda, \lambda + \frac{1}{2}[$. By similar computations, one shows that $\alpha_\mu = \alpha'_\mu$ for all $\mu \in]\lambda + \frac{1}{2}, \lambda + 1[$: as λ is arbitrary, this shows that $\mathcal{A}_{\mathcal{B}} = \mathcal{A}_{\mathcal{B}'}$.

We shall therefore, from now on, make the confusion between apartments and their subsets of vertices (which encode the local information about special cycles).

2.1.4 Apartments are geodesic lines.

Proposition 2.1.12. *Let α , α' and α'' be vertices of $|\mathbf{B}_V|$ be such that $\delta(\alpha, \alpha'') + \delta(\alpha'', \alpha') = \delta(\alpha, \alpha')$. Then any apartment containing α and α' contains also α'' .*

Proof. Set $d_1 = \delta(\alpha, \alpha'')$ and $d_2 = \delta(\alpha'', \alpha')$, let $\mathcal{B} = \{e_+, e_0, e_-\}$ be a Witt basis such that $\alpha, \alpha' \in \mathcal{A}_{\mathcal{B}}$ and let, as usual, λ and λ' denote the respective parameters of α and α' . Up to permuting α and α' , one may always assume $\lambda \leq \lambda'$. One wants to show that α'' is the norm of parameter λ'' in $\mathcal{A}_{\mathcal{B}}$, where $\lambda'' \in \frac{1}{2}\mathbb{Z}$ is such that $|\lambda - \lambda''| = \frac{d_1}{2}$ and $|\lambda' - \lambda''| = \frac{d_2}{2}$, i.e., $\lambda' = \lambda + \frac{d_1}{2} = \lambda'' - \frac{d_2}{2}$. By the injection $\gamma_0 : |\mathbf{B}_V| \hookrightarrow \mathcal{L}_{V_r}$, it is enough to show the equality on unit balls, i.e., that $\gamma_0(\alpha'') =: L'' = \langle \varpi^{-\lfloor \lambda'' \rfloor} e_+, e_0, \varpi^{-\lfloor -\lambda'' \rfloor} e_- \rangle$.

— First assume that λ, λ' and λ'' all belong to \mathbb{Z} . Thus $d_1, d_2 \in 2\mathbb{Z}$ and, up to replacing $\{e_+, e_0, e_-\}$ by $\{\varpi^{-\lambda} e_+, e_0, \varpi^{\lambda} e_-\}$, one may assume $\lambda = 0 \leq \lambda'' = \frac{d_1}{2} \leq \lambda' = \frac{d_1+d_2}{2}$. By assumption, there exists integers n_1, n_2 such that

$$\varpi^{d_1+n_1} \langle e_+, e_0, e_- \rangle \subset L'' \subset \varpi^{n_1} \langle e_+, e_0, e_- \rangle \quad (2.5)$$

and

$$\varpi^{d_2+n_2} \langle \varpi^{-\frac{d_1+d_2}{2}} e_+, e_0, \varpi^{\frac{d_1+d_2}{2}} e_- \rangle \subset L'' \subset \varpi^{n_2} \langle \varpi^{-\frac{d_1+d_2}{2}} e_+, e_0, \varpi^{\frac{d_1+d_2}{2}} e_- \rangle \quad (2.6)$$

Taking the dual of these inclusions gives, by self-duality:

$$\varpi^{-n_1} \langle e_+, e_0, e_- \rangle \subset L'' \subset \varpi^{-(d_1+n_1)} \langle e_+, e_0, e_- \rangle \quad (2.7)$$

and

$$\varpi^{-n_2} \langle \varpi^{-\frac{d_1+d_2}{2}} e_+, e_0, \varpi^{\frac{d_1+d_2}{2}} e_- \rangle \subset \varpi^{-(d_2+n_2)} \langle \varpi^{-\frac{d_1+d_2}{2}} e_+, e_0, \varpi^{\frac{d_1+d_2}{2}} e_- \rangle. \quad (2.8)$$

If one had $n_1 + d_1 \leq -n_1 - 1$, this would give inclusions

$$\varpi^{n_1+d_1} \langle e_+, e_0, e_- \rangle = \varpi^{(d_1-1)+(n_1+1)} \langle e_+, e_0, e_- \rangle \subset L'' \subset \varpi^{-(n_1+d_1)} \langle e_+, e_0, e_- \rangle \subset \varpi^{n_1+1} \langle e_+, e_0, e_- \rangle,$$

which would give $\delta(\alpha, \alpha'') \leq d_1 - 1$, which is not. On the other hand, if one had $n_1 + d_1 \geq -n_1 + 1$, then $-(n_1 + d_1 - 1) \leq n_1$ and one would have inclusions

$$\varpi^{-n_1} \langle e_+, e_0, e_- \rangle = \varpi^{(d_1-1)-(d_1+n_1-1)} \langle e_+, e_0, e_- \rangle \subset L'' \subset \varpi^{n_1} \langle e_+, e_0, e_- \rangle \subset \varpi^{-(d_1+n_1-1)} \langle e_+, e_0, e_- \rangle,$$

which would also give $\delta(\alpha, \alpha'') \leq d_1 - 1$. Hence $n_1 + d_1 = -n_1$, i.e., $n_1 = -\frac{d_1}{2}$, and the same arguments show that $n_2 = -\frac{d_2}{2}$. Inclusions (2.5) and (2.6) now become:

$$\langle \varpi^{\frac{d_1}{2}} e_+, \varpi^{\frac{d_1}{2}} e_0, \varpi^{\frac{d_1}{2}} e_- \rangle \subset L'' \subset \langle \varpi^{-\frac{d_1}{2}} e_+, \varpi^{-\frac{d_1}{2}} e_0, \varpi^{-\frac{d_1}{2}} e_- \rangle$$

and

$$\langle \varpi^{-\frac{d_1}{2}} e_+, \varpi^{\frac{d_2}{2}} e_0, \varpi^{d_2+\frac{d_1}{2}} e_- \rangle \subset L'' \subset \langle \varpi^{-d_2-\frac{d_1}{2}} e_+, \varpi^{-\frac{d_2}{2}} e_0, \varpi^{\frac{d_1}{2}} e_- \rangle.$$

This gives

$$\langle \varpi^{-\frac{d_1}{2}} e_+, \varpi^{\frac{d_1}{2}} e_0, \varpi^{\frac{d_1}{2}} e_- \rangle \subset L'' \subset \langle \varpi^{-\frac{d_1}{2}} e_+, \varpi^{-\frac{d_1}{2}} e_0, \varpi^{\frac{d_1}{2}} e_- \rangle.$$

Notice that, if $v = a_+ \varpi^{-\frac{d_1}{2}} e_+ + a_0 \varpi^{-\frac{d_1}{2}} e_0 + a_- \varpi^{\frac{d_1}{2}} e_-$ is a vector of L'' (with $a_+, a_0, a_- \in \mathcal{O}_{E_\tau}$) then, as $L'' = (L'')^\vee$, one has $\langle v, v \rangle = a_+ \bar{a}_- + a_- \bar{a}_+ + a_0 \bar{a}_0 \varpi^{-d_1} \in \mathcal{O}_{E_\tau}$, i.e., $\text{ord}_\varpi(a_0 \bar{a}_0) = 2 \text{ord}_\varpi(a_0) \geq d_1$, thus $a_0 \varpi^{-\frac{d_1}{2}} \in \mathcal{O}_{E_\tau}$ and $L'' \subset \langle \varpi^{-\frac{d_1}{2}} e_+, e_0, \varpi^{\frac{d_1}{2}} e_- \rangle$. Both lattices are self-dual, therefore $L'' = \tilde{L} = \langle \varpi^{-\frac{d_1}{2}} e_+, e_0, \varpi^{\frac{d_1}{2}} e_- \rangle$, as expected.

— If λ and λ' both belong to \mathbb{Z} and $\lambda'' \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then inclusion (2.5) induce, after dualizing:

$$\varpi^{d_1+n_1} \langle e_+, e_0, e_- \rangle \subset L'' \subset (L'')^\vee \subset \varpi^{-(d_1+n_1)} \langle e_+, e_0, e_- \rangle$$

and

$$\varpi^{-n_1} \langle e_+, e_0, e_- \rangle \subset (L'')^\vee \subset \varpi^{-1} L'' \subset \varpi^{n_1-1} \langle e_+, e_0, e_- \rangle,$$

and similarly for (2.6). Similar considerations on distances now imply that $n_1 = -\frac{d_1-1}{2}$ and $n_2 = -\frac{d_2-1}{2}$. This gives inclusions

$$\langle \varpi^{\frac{d_1+1}{2}} e_+, \varpi^{\frac{d_1+1}{2}} e_0, \varpi^{\frac{d_1+1}{2}} e_- \rangle \subset L'' \subset \langle \varpi^{-\frac{d_1-1}{2}} e_+, \varpi^{-\frac{d_1-1}{2}} e_0, \varpi^{-\frac{d_1-1}{2}} e_- \rangle$$

and

$$\langle \varpi^{-\frac{d_1-1}{2}} e_+, \varpi^{\frac{d_2+1}{2}} e_0, \varpi^{d_2+\frac{d_1+1}{2}} e_- \rangle \subset L'' \subset \langle \varpi^{-d_2-\frac{d_1-1}{2}} e_+, \varpi^{-\frac{d_2-1}{2}} e_0, \varpi^{\frac{d_1+1}{2}} e_- \rangle,$$

thus

$$\langle \varpi^{-\frac{d_1-1}{2}} e_+, \varpi^{\frac{d_1+1}{2}} e_0, \varpi^{\frac{d_1+1}{2}} e_- \rangle \subset L'' \subset \langle \varpi^{-\frac{d_1-1}{2}} e_+, \varpi^{-\frac{d_1-1}{2}} e_0, \varpi^{\frac{d_1+1}{2}} e_- \rangle$$

But $L'' \subset (L'')^\vee$, therefore the last inclusion induces in fact (by the same argument as in the hyperspecial case), an inclusion

$$L'' \subset \langle \varpi^{-\frac{d_1-1}{2}} e_+, e_0, \varpi^{\frac{d_1+1}{2}} e_- \rangle =: \tilde{L},$$

the latter being a strictly almost self-dual lattice. This gives the following inclusions :

$$\varpi \tilde{L} \subset \varpi \tilde{L}^\vee \subset \varpi (L'')^\vee \subset L'' \subset \tilde{L},$$

hence $\delta(L'', \tilde{L}) \leq 1$. But L'' and \tilde{L} have the same type, thus $\delta(L'', \tilde{L}) = 0$ and $L'' = \tilde{L} = \langle \varpi^{-\frac{d_1-1}{2}} e_+, e_0, \varpi^{\frac{d_1+1}{2}} e_- \rangle = \langle \varpi^{-\lfloor \frac{d_1}{2} \rfloor} e_+, e_0, \varpi^{-\lfloor -\frac{d_1}{2} \rfloor} e_- \rangle$, as expected.

That the other cases follow directly from the previous two cases is left to the reader. \square

A consequence is the following result:

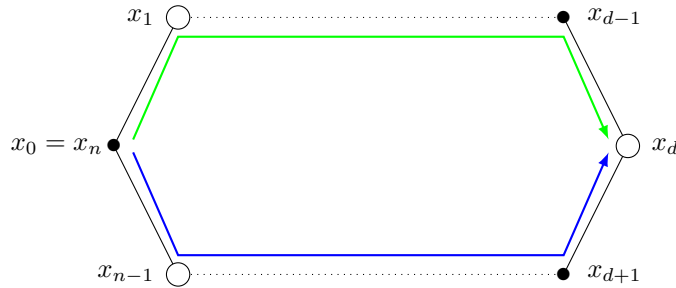
Corollary 2.1.13. *Given two vertices $\alpha, \alpha' \in |\mathbf{B}_V|$, there exists exactly one geodesic path in $|\mathbf{B}_V|$ joining α and α' .*

Proof. Let $\mathcal{A}_{\mathcal{B}}$ be an apartment containing α and α' , and let λ, λ' be their respective parameters. Up to permuting α and α' , one may assume $\lambda \leq \lambda'$. The proof of Proposition 2.1.11 shows, using the same notations, that the path $\mathfrak{p} := [\alpha, \alpha_{\lambda+\frac{1}{2}}, \dots, \alpha_{\lambda'-\frac{1}{2}}, \alpha'] \subset \mathcal{A}_{\mathcal{B}}$ is a geodesic path joining α and α' . Conversely, Proposition 2.1.12 shows that any point α'' of a geodesic path connecting α and α' also belongs to $\mathcal{A}_{\mathcal{B}}$, and has parameter $\lambda'' \in [\lambda, \lambda']$, hence belongs to \mathfrak{p} . \square

This allows us to compute the structure of the graph $(|\mathbf{B}_V|, (\mathbf{B}_V \setminus |\mathbf{B}_V|) / \sim)$:

Corollary 2.1.14. *The graph $(|\mathbf{B}_V|, (\mathbf{B}_V \setminus |\mathbf{B}_V|) / \sim)$ is connected and has no cyclic path, i.e., is a tree.*

Proof. The graph was already shown to be connected. Assume that $\mathfrak{p} = [x_0, x_1, \dots, x_n = x_0]$ is a cyclic path of minimal length $n \geq 3$, i.e., such that $x_i \neq x_j$ for all $i \neq j \in \{0, \dots, n-1\}$. As any two neighbours must have different types, then $n = 2d$ has to be even (with $d \geq 2$). Set $\alpha := x_0$ and $\alpha' := x_d$. By minimality of \mathfrak{p} , both paths $[x_0, x_1, \dots, x_d]$ and $[x_n, x_{n-1}, \dots, x_d]$, which are distinct, have to be geodesic. As $x_n = x_0$, this contradicts the previous corollary, hence $|\mathbf{B}_V|$ is indeed a tree. \square



As a consequence of the last two corollaries, any path in $|\mathbf{B}_V|$ (of length ≥ 2) is geodesic. One also has the following important result:

Corollary 2.1.15. *Every geodesic line of $|\mathbf{B}_V|$ is contained in an apartment. Therefore, apartments of $|\mathbf{B}_V|$ coincide with geodesic lines.*

Proof. That apartments coincide with geodesic lines is an important feature of Euclidean buildings of dimension 1. Let $\ell = (\dots, y_i, y_{i+1}, \dots)$ be a geodesic line, and denote by $L_i = \gamma_0(y_i)$ be the almost self-dual lattice attached to y_i , for all $i \in \mathbb{Z}$.

By the previous proposition, for all $k \geq 1$ one may find some Witt basis $\mathcal{B}_k := \{e_+^{(k)}, e_0^{(k)}, e_-^{(k)}\}$ such that the path $[y_{-2k}, \dots, y_0, \dots, y_{2k}] \subset \ell$ is entirely contained in the apartment $\mathcal{A}_k := \mathcal{A}_{\mathcal{B}_k}$ (indeed, it is enough that y_{-2k} and $y_{2k} \in \mathcal{A}_k$). Without loss of generality, one may assume that y_i has parameter $\frac{i}{2}$ in \mathcal{B}_k , for all $i \in \{-2k, \dots, 2k\}$ i.e., that hyperspecial vertices of ℓ satisfy

$$L_{2i} = \langle \varpi^{-i} e_+^{(k)}, e_0^{(k)}, \varpi^i e_-^{(k)} \rangle, \quad \forall i \in \{-k, \dots, k\}, k \geq 1.$$

In particular, \mathcal{B}_k is a basis of L_0 , for all $k \geq 1$. The stabilizer $G_0 := \text{Stab}(L_0) \subset G_{V,\tau}$ is homeomorphic, via the basis \mathcal{B}_1 (for instance) to the closed subgroup $\{M \in \text{GL}_3(\mathcal{O}_{E_\tau}); {}^t M J \overline{M} = J\}$ of $\text{GL}_3(\mathcal{O}_{E_\tau})$, with $J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. As $\text{GL}_3(\mathcal{O}_{E_\tau})$ is compact, then so is G_0 .

Let $g_k \in G_{V,\tau}$ be such that $g_k \cdot e_\star^{(k)} = e_\star^{(k+1)}$, for $\star \in \{+, 0, -\}$. Then $g_k \cdot \mathcal{A}_k = \mathcal{A}_{k+1}$ and g_k fixes the whole path $[y_{-2k}, \dots, y_0, \dots, y_{2k}] \subset \mathcal{A}_k \cap \mathcal{A}_{k+1}$, or equivalently, g_k fixes L_{-2k} and L_{2k} . For $k \geq 1$, the subgroup $G_k := \text{Stab}(L_{-2k}, L_{2k}) = \text{Stab}([y_{-2k}, \dots, y_{2k}])$ of G_0 is isomorphic, via the basis \mathcal{B}_k , to the closed subgroup

$$\left\{ M \in \text{GL}_3(\mathcal{O}_{E_\tau}) \cap \begin{bmatrix} \mathcal{O}_{E_\tau} & \varpi^k \mathcal{O}_{E_\tau} & \varpi^{2k} \mathcal{O}_{E_\tau} \\ \varpi^k \mathcal{O}_{E_\tau} & \mathcal{O}_{E_\tau} & \varpi^k \mathcal{O}_{E_\tau} \\ \varpi^{2k} \mathcal{O}_{E_\tau} & \varpi^k \mathcal{O}_{E_\tau} & \mathcal{O}_{E_\tau} \end{bmatrix}; {}^t M J \overline{M} = J \right\} \subset \text{GL}_3(\mathcal{O}_{E_\tau}).$$

Therefore, the matrix of g_k in \mathcal{B}_k has the form $M_k = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \in M_3(\mathcal{O}_{E_\tau})$, with $\text{ord}_\varpi(x_{12}), \text{ord}_\varpi(x_{23}), \text{ord}_\varpi(x_{21}), \text{ord}_\varpi(x_{32}) \geq k$ and $\text{ord}_\varpi(x_{13}), \text{ord}_\varpi(x_{31}) \geq 2k$. The equality ${}^t M_k J \overline{M}_k = J$ induces

$$\begin{cases} x_{11}\bar{x}_{33} + x_{21}\bar{x}_{23} + x_{31}\bar{x}_{13} = 1, \\ x_{12}\bar{x}_{32} + x_{22}\bar{x}_{22} + x_{32}\bar{x}_{12} = 1. \end{cases}$$

This gives $\text{ord}_\varpi(x_{11}\bar{x}_{33} - 1), \text{ord}_\varpi(x_{22}\bar{x}_{22} - 1) \geq 2k$, hence $\max(\text{ord}_\varpi(x_{22} - 1), \text{ord}_\varpi(\bar{x}_{22} - 1)) \geq 2k$.⁽⁴⁾

Set $u_k = x_{11} \in \mathcal{O}_{E_\tau}^\times$ and let $v_k \in \mathcal{O}_{E_\tau}$ be either 1, if $\text{ord}_\varpi(x_{22} - 1) \geq 2k$, or $\frac{x_{22}}{\bar{x}_{22}}$ otherwise. Then v_k satisfies $v_k \bar{v}_k = 1$ and, according to Proposition 2.1.1, (ii), one may replace $\mathcal{B}_k = \{e_+^{(k)}, e_0^{(k)}, e_-^{(k)}\}$ by $\{u_k e_+^{(k)}, v_k e_0^{(k)}, \bar{u}_k^{-1} e_-^{(k)}\}$ and still get a Witt basis attached to the apartment \mathcal{A}_k , in which all points keep the same parameters. This ends up replacing g_k by $g_k t_k$, where $t_k \in G_k$ has matrix $\begin{bmatrix} u_k^{-1} & 0 & 0 \\ 0 & v_k^{-1} & 0 \\ 0 & 0 & \bar{u}_k \end{bmatrix}$ in the basis \mathcal{B}_k . Therefore, the matrix of $g_k t_k$ in \mathcal{B}_k has diagonal entries congruent to 1 mod ϖ^{2k} , and other entries congruent to 0 mod ϖ^k , for all $k \geq 1$. As all the \mathcal{B}_k 's are bases of the same lattice L_0 , one may always assume - up to replacing g_k by $g_k t_k$ - that the sequence $(g_k)_k$ converges to 1 in G_0 .

For all $n \geq 1$, set $\tilde{g}_n = g_n g_{n-1} \dots g_1 \in G_0$. The sequence $(\tilde{g}_n)_{n \geq 0}$ is clearly a Cauchy sequence : as G_0 is compact, hence complete, the sequence $(\tilde{g}_n)_n$ converges to some $g_\infty \in G_0$. Let \mathcal{B}_∞ be the Witt basis defined by $\{g_\infty \cdot e_+^{(1)}, g_\infty \cdot e_0^{(1)}, g_\infty \cdot e_-^{(1)}\}$. One has, for all $k \geq 1$:

$$g_\infty (g_{k-1} g_{k-2} \dots g_1)^{-1} = \lim_{n \rightarrow \infty} (g_n g_{n-1} \dots g_k) \in G_k,$$

hence

$$L_{2k} = g_\infty (g_{k-1} \dots g_1)^{-1} L_{2k} = g_\infty (g_{k-1} \dots g_1)^{-1} g_{k-1} \dots g_1 \cdot \langle \varpi^{-k} e_+^{(1)}, e_0^{(1)}, \varpi^k e_-^{(1)} \rangle$$

(4). Indeed, one has $x_{22}\bar{x}_{22} - 1 = x_{22}(\bar{x}_{22} - 1) + (x_{22} - 1)$. If $\text{ord}_\varpi(x_{22} - 1) < 2k$ then, as $\text{ord}_\varpi x_{22} = 0$, one gets $\text{ord}_\varpi(\bar{x}_{22} - 1) = \text{ord}_\varpi(x_{22}\bar{x}_{22} - 1) \geq 2k$.

$$= g_\infty \cdot \langle \varpi^{-k} e_+^{(1)}, e_0^{(1)}, \varpi^k e_-^{(1)} \rangle = \langle \varpi^{-k} g_\infty \cdot e_+^{(1)}, g_\infty \cdot e_0^{(1)}, \varpi^k g_\infty \cdot e_-^{(1)} \rangle,$$

and also $L_{-2k} = g_\infty \cdot \langle \varpi^k e_+^{(1)}, e_0^{(1)}, \varpi^{-k} e_-^{(1)} \rangle = \langle \varpi^k g_\infty \cdot e_+^{(1)}, g_\infty \cdot e_0^{(1)}, \varpi^{-k} g_\infty \cdot e_-^{(1)} \rangle.$

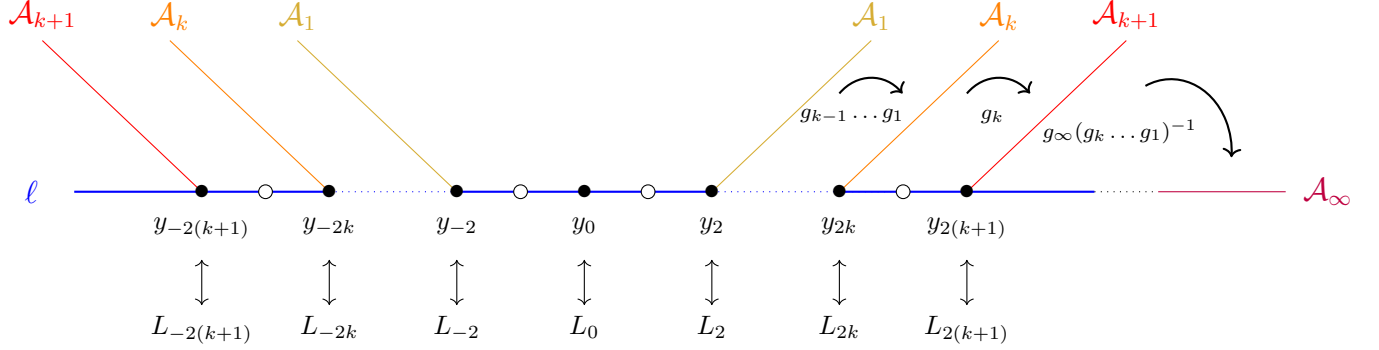


Figure 2.2 – The line ℓ is the "limit apartment" $\mathcal{A}_\infty = g_\infty \cdot \mathcal{A}_1$.

In other words, for all $k \geq 1$, the vertices y_{-2k} and y_{2k} belong to the apartment $\mathcal{A}_\infty := \mathcal{A}_{\mathcal{B}_\infty}$ with respective parameters $-k$ and k in \mathcal{B}_∞ , i.e., $\ell \cap \text{Hyp}_V \subset \mathcal{A}_{\mathcal{B}_\infty}$. As every special vertex is stuck between two hyperspecial vertices and by Proposition 2.1.12, one gets $\ell \subset \mathcal{A}_{\mathcal{B}_\infty}$. \square

2.1.5 Counting the number of neighbours.

One shall now compute the number of neighbours in the tree \mathbf{B}_V / \sim . We showed that hyperspecial (resp. special) vertices have only special (resp. hyperspecial) neighbours, and the number of neighbours of a given vertex depends only on its type, since hyperspecial (resp. special) vertices are all conjugated by $G_{V,\tau}$, whose action preserves adjacency relations.

Lemma 2.1.16. *Let α be an hyperspecial vertex and α' be a special vertex, and set $L = \gamma_0(\alpha)$, $L' = \gamma_0(\alpha')$. Then L and L' are neighbours if and only if one has the sequence of inclusions:*

$$\cdots \subsetneq \varpi L^\vee \subsetneq \varpi(L')^\vee \subsetneq L' \subsetneq L \subsetneq (L')^\vee \subsetneq \cdots \quad (2.9)$$

Proof. Clearly if (2.9) is satisfied then $\delta(L, L') = 1$, i.e., L and L' are neighbours. Conversely, if L and L' are neighbours then, by Lemma 2.1.9, (iii), one has either $L \subset L'$ or $L' \subset L$. If $L \subset L'$, then by taking duals one gets $L' \subset (L')^\vee \subset L^\vee = L$ thus $L = L'$, which would contradict $\delta(L, L') = 1$. Therefore $L' \subset L$, hence after dualizing $L = L^\vee \subset (L')^\vee$, which gives $\varpi L \subset \varpi(L')^\vee \subset L' \subset L \subset (L')^\vee$. This is exactly (2.9). \square

Let $\alpha \in |\mathbf{B}_V|$ and set $L := \gamma_0(\alpha)$ if α is hyperspecial, and $L' := \gamma_0(\alpha)$ if α is special. The hermitian pairing $\langle \cdot, \cdot \rangle$ on V_τ induces two pairings:

$$\psi_0 : \frac{L}{\varpi L} \times \frac{L^\vee}{\varpi L^\vee} \rightarrow \mathbb{F},$$

$$\psi_1 : \frac{(L')^\vee}{L'} \times \frac{(L')^\vee}{L'} \rightarrow \frac{\varpi^{-1}\mathcal{O}_{E_\tau}}{\mathcal{O}_{E_\tau}} \simeq \mathbb{F},$$

which can both easily be shown to be perfect pairings.

Proposition 2.1.17. *Let $\alpha \in |\mathbf{B}_V|$ and set $L := \gamma_0(\alpha)$ if α is hyperspecial, and $L' := \gamma_0(\alpha)$ if α is special.*

(i) *If α is hyperspecial, then the set of neighbours of α in $|\mathbf{B}_V|$ is in bijection with the set of totally isotropic lines contained in the 3-dimensional hermitian \mathbb{F} -space $(\frac{L}{\varpi L}, \psi_0)$.*

(ii) *If α is special, then the set of neighbours of α in $|\mathbf{B}_V|$ is in bijection with the set of isotropic lines contained in the 2-dimensional hermitian $\mathbb{F}^{(5)}$ -space $(\frac{(L')^\vee}{L'}, \psi_1)$.*

Proof.

(i) By the preceding lemma, neighbours of α in $|\mathbf{B}_V|$ are in bijection with strictly almost self-dual lattices L' such that (L, L') satisfies (2.9). Let $N(L)$ be the set of those L' , let $\text{Iso}(\frac{L}{\varpi L})$ denote the set of isotropic lines in the hermitian space $(\frac{L}{\varpi L}, \psi_0)$, and consider the map :

$$\begin{aligned} \mathbf{n} : N(L) &\longrightarrow \text{Iso}\left(\frac{L}{\varpi L}\right), \\ L' &\longmapsto \frac{\varpi(L')^\vee}{\varpi L} \subset \frac{L}{\varpi L}. \end{aligned}$$

Let α' be a neighbour of α and set $L' \in N(L)$ to be its corresponding strictly almost self-dual lattice. Let $\mathcal{B} = \{e_+, e_0, e_-\}$ be a common decomposition basis for α and α' : by Proposition 2.1.1, (ii), one may always assume that α and α' have respective parameters 0 and $\frac{1}{2}$ with respect to \mathcal{B} , that is to say, $L = \langle e_+, e_0, e_- \rangle$ and $L' = \langle e_+, e_0, \varpi e_- \rangle$. Therefore $\varpi(L')^\vee = \langle e_+, \varpi e_0, \varpi e_- \rangle$, thus $\frac{\varpi(L')^\vee}{\varpi L} = \mathbb{F}e_-$ is indeed an isotropic line contained in $\frac{L}{\varpi L}$, i.e., \mathbf{n} is well-defined.

Let $\ell \in \text{Iso}(\frac{L}{\varpi L})$. Let $\pi : L \rightarrow L/\varpi L$ be the reduction map, and set $L' := \varpi(\pi^{-1}(\ell))^\vee$. Clearly, one has $\frac{\varpi(L')^\vee}{\varpi L} = \frac{\pi^{-1}(\ell)}{\varpi L} = \ell$. Let us show that L' is an almost self-dual lattice contained in L . One may write $\ell = \mathbb{F}\pi(x)$, for some $x \in L \setminus \varpi L$, hence $\pi^{-1}(\ell) = \varpi L + \mathcal{O}_{E_\tau}x$. As $\psi_0(\pi(x), \pi(x)) = 0 \in \mathbb{F}$, one has $\langle x, x \rangle \in \varpi\mathcal{O}_{E_\tau}$. If t belongs to L' , then one has $\langle t, s \rangle \in \varpi\mathcal{O}_{E_\tau}$ for all $s \in \pi^{-1}(\ell)$, hence $\varpi\langle t, z \rangle = \langle t, \varpi z \rangle \in \varpi\mathcal{O}_{E_\tau}$ for all $z \in L$, thus $t \in L^\vee = L$. Therefore $L' \subset L$, which gives $L' \subset L^\vee \subset (L')^\vee$, as L is self-dual. Let s belong to $\varpi(L')^\vee = \pi^{-1}(\ell) = \varpi L + \mathcal{O}_{E_\tau}x$: if t belongs to $\varpi^{-1}\pi^{-1}(\ell) = L + \varpi^{-1}\mathcal{O}_{E_\tau}x$, one has $\langle s, t \rangle \in \mathcal{O}_{E_\tau}$ as $\langle x, x \rangle \in \varpi\mathcal{O}_{E_\tau}$. This gives $\varpi(L')^\vee \subset (\varpi^{-1}\pi^{-1}(\ell))^\vee = \varpi(\pi^{-1}(\ell))^\vee = L'$, hence L' is a (strictly) almost self-dual lattice contained in L , i.e., $L' \in N(L)$, $\mathbf{n}(L') = \ell$ and \mathbf{n} is surjective. Notice that if $L'' \in N(L)$ was another lattice such that $\mathbf{n}(L'') = \ell$, then one would have $\varpi(L'')^\vee \subset \pi^{-1}(\ell) = \varpi(L')^\vee$, hence $L' \subset L''$. This would give inclusions $\varpi L'' \subset \varpi(L'')^\vee \varpi(L')^\vee \subset L' \subset L''$, therefore $\delta(L', L'') \leq 1$. If $L' \neq L''$, one would get a cyclic

(5). Here, \mathbb{F} is identified with the quotient $\frac{\varpi^{-1}\mathcal{O}_{E_\tau}}{\mathcal{O}_{E_\tau}}$.

length-3 path $[L, L', L'', L]$, which would contradict Corollary 2.1.14. Accordingly, one has $L' = L''$ and \mathbf{n} is injective.

(ii) Still by the preceding lemma, neighbours of α are in bijection with self-dual lattices L such that (L, L') satisfies (2.9). Let $N(L')$ be the set of those L , let $\text{Iso}(\frac{(L')^\vee}{L'})$ denote the set of isotropic lines in the hermitian space $(\frac{(L')^\vee}{L'}, \psi_1)$, and consider the map

$$\begin{aligned} \mathbf{m} : N(L') &\longrightarrow \text{Iso}\left(\frac{(L')^\vee}{L'}\right), \\ L &\longmapsto \frac{L}{L'} \subset \frac{(L')^\vee}{L'}. \end{aligned}$$

Again, the proof of the preceding lemma shows that \mathbf{m} is well-defined. Let $\ell \in \text{Iso}(\frac{(L')^\vee}{L'})$. Let $\pi : (L')^\vee \rightarrow (L')^\vee/L'$ be the reduction map, and set $L := \pi^{-1}(\ell) \subset (L')^\vee$. Clearly, one has $L' \subset L$ and $\frac{L}{L'} = \ell$. Let us show that L is a self-dual lattice. Let $x \in (L')^\vee \setminus L'$ be such that $\ell = \mathbb{F}\pi(x)$: then $L = L' + \mathcal{O}_{E_\tau}x$ and $\langle x, x \rangle \in \mathcal{O}_{E_\tau}$, as $\psi_1(\pi(x), \pi(x)) = 0 \in \frac{\varpi^{-1}\mathcal{O}_{E_\tau}}{\mathcal{O}_{E_\tau}}$. Accordingly, if $t, s \in L$, then one has $\langle t, s \rangle \in \mathcal{O}_{E_\tau}$, hence $L \subset L^\vee$.

Conversely, let t belong to $L^\vee \subset (L')^\vee$. As $(\frac{(L')^\vee}{L'}, \psi_1)$ is a non-degenerate 2-dimensional hermitian space, one may find a vector $y \in (L')^\vee$ such that $\{\pi(x), \pi(y)\}$ is an \mathbb{F} -basis of $\frac{(L')^\vee}{L'}$, hence $\psi_1(\pi(x), \pi(y)) \neq 0$ as $\pi(x)$ is a non-zero isotropic vector. Write $\pi(t) = a\pi(x) + b\pi(y)$, with $a, b \in \mathbb{F}$: that $\langle t, x \rangle$ belongs to \mathcal{O}_{E_τ} implies $0 = \psi_1(\pi(t), \pi(x)) = a\psi_1(\pi(x), \pi(x)) + b\psi_1(\pi(y), \pi(x)) = b\psi_1(\pi(y), \pi(x))$, hence $b = 0$ and $t \in L = \pi^{-1}(\ell)$. This shows that $L^\vee \subset L$, which means that L is a self-dual lattice such that $L' \subset L$, i.e., $L \in N(L')$, $\mathbf{m}(L) = \ell$ and \mathbf{m} is surjective.

On the other hand, if \tilde{L} was another self-dual lattice in $N(L')$ such that $\mathbf{m}(\tilde{L}) = \ell$, then $\tilde{L} \subset \pi^{-1}(\ell) = L$, thus $\tilde{L} = L$ as both \tilde{L} and L are self-dual. This shows the injectivity of \mathbf{m} and completes the proof. \square

Facts: If \mathcal{V} is an n -dimensional non-degenerate hermitian \mathbb{F}_{q^2} vector space, then the number of (non-zero) isotropic vectors in \mathcal{V} is $(q^{n-1} - (-1)^{n-1})(q^n - (-1)^n)$. If $n = 3$, this gives $\frac{(q^2-1)(q^3+1)}{q^2-1} = q^3 + 1$ distinct isotropic lines in \mathcal{V} . If $n = 2$, this gives $\frac{(q+1)(q^2-1)}{q^2-1} = q + 1$ distinct isotropic lines in \mathcal{V} . The proof is given in ([48], Lemma 10.4).

Proposition 2.1.17 has the following consequence: any hyperspecial point of $|\mathbf{B}_V|$ has exactly $q^3 + 1$ special neighbours, and any special point has exactly $q + 1$ hyperspecial neighbours.

2.1.6 The sub-building \mathbf{B}_W .

In the Witt bases we have considered so far, the anisotropic element e_0 hardly intervenes in all the considerations and definitions. One may therefore apply *mutatis mutandis* all the above constructions to the 2-dimensional non-degenerate hermitian space W_τ ⁽⁶⁾ instead of

(6). And more generally to any *hyperbolic plane*, as we shall see later.

V_τ . One obtains a building \mathbf{B}_W called the *building of $G_{W,\tau}$* , which is endowed with an action of $G_{W,\tau}$, whose points correspond to self-dual ultrametric norms on W_τ and whose vertices are now identified with almost self-dual rank-2 \mathcal{O}_{E_τ} -lattices inside W_τ . One should now consider apartments attached to decomposition bases of the form $\{e_+, e_-\}$ in which self-dual ultrametric norms α satisfy $\alpha(a_+e_+ + a_-e_-) = \max(|a_+||\varpi|^\lambda, |a_-||\varpi|^{-\lambda})$ for some parameter $\lambda \in \mathbb{R}$, where e_+, e_- are isotropic vectors such that $\langle e_+, e_- \rangle = 1$. Accordingly, one may define a geodesic distance on $|\mathbf{B}_W|$ in a similar way as the one we defined on $|\mathbf{B}_V|$. The orthogonal decomposition $V = W \perp D$ yields an embedding $\iota : \mathbf{B}_W \hookrightarrow \mathbf{B}_V$. Namely, ι maps any self-dual ultrametric norm α on W_τ , to the norm

$$\iota(\alpha) : V_\tau \ni v = w + \mu e_D \mapsto \max(\alpha(w), |\mu|),$$

with $w \in W_\tau$, $\mu \in E_\tau$ and $e_D \in D_\tau$ such that $\langle e_D, e_D \rangle = 1$ as fixed in the beginning. One checks that ι is compatible with the action of $G_{W,\tau}$ and with the identification of $G_{W,\tau}$ with the subgroup $\iota(G_{W,\tau}) \subset G_{V,\tau}$ of $G_{V,\tau}$, i.e., that $\iota(h \cdot \alpha) = \iota(h) \cdot \iota(\alpha)$, for all $\alpha \in \mathbf{B}_W$ and $h \in G_{W,\tau}$. If $\{e_+, e_-\}$ is a Witt basis of W_τ in which α is decomposed with parameter λ , then $\{e_+, e_D, e_-\}$ is again a decomposition basis for $\iota(\alpha)$ and the parameter of $\iota(\alpha)$ remains unchanged. In particular, ι preserves distances. Let $L_D = \mathcal{O}_{E_\tau}e_D$ be the unique self-dual lattice inside D_τ : via ι , apartments of \mathbf{B}_W are identified with apartments of \mathbf{B}_V attached to Witt bases $\{e_+, e_0, e_-\}$ such that $\mathcal{O}_{E_\tau}e_0 = L_D$. Consequently, given two vertices α and $\alpha' \in \mathbf{B}_W$, the unique geodesic segment (in \mathbf{B}_V) connecting α with α' is entirely contained in \mathbf{B}_W , hence \mathbf{B}_W is a convex subspace of \mathbf{B}_V .

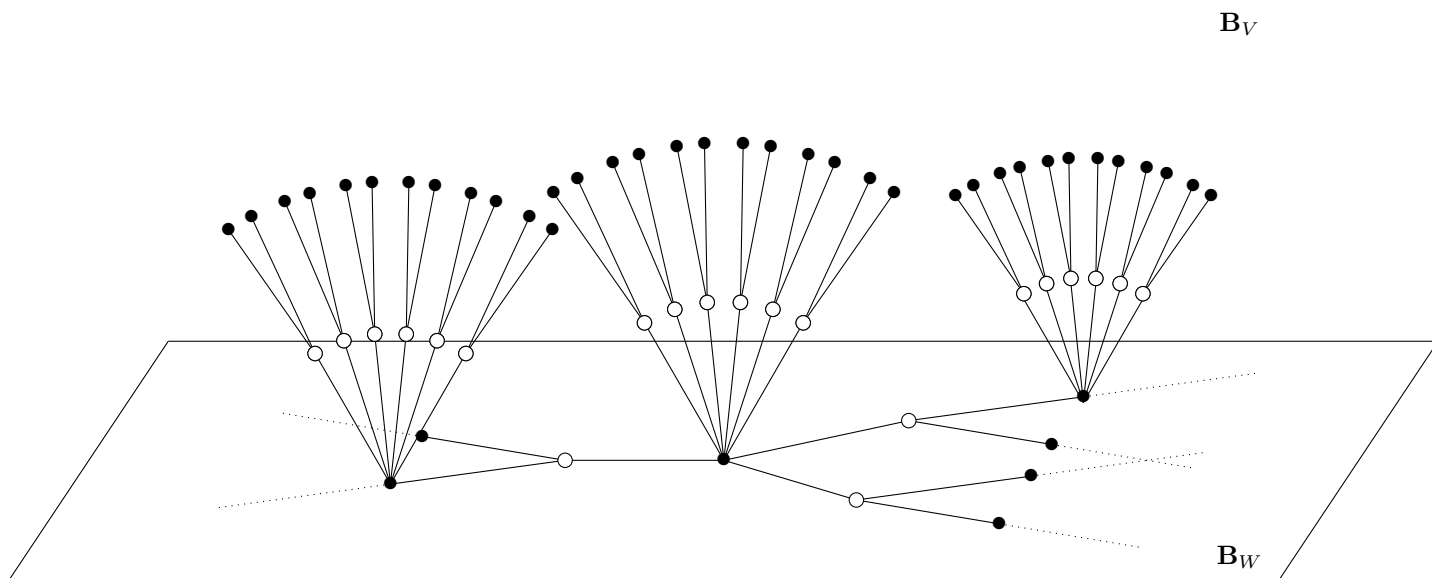


Figure 2.3 – The building \mathbf{B}_W can be embedded as a sub-building of \mathbf{B}_V . The number of neighbours represented here corresponds to the case $q = 2$.

At the level of almost self-dual lattices, ι corresponds to the map $L \mapsto \iota(L) := L \oplus L_D$. If L

is an almost self-dual lattice inside W_τ , the quotient lattices $\frac{L}{\varpi L}$ and $\frac{L^\vee}{L}$ (if L is strictly almost self-dual) are both 2-dimensional hermitian \mathbb{F} -spaces. The previous proposition thus admits the following variant: in \mathbf{B}_W , hyperspecial vertices admit exactly $q + 1$ special neighbours, and special vertices admit exactly $q + 1$ hyperspecial neighbours. In particular, special points inside $|\mathbf{B}_W|$ have their set of hyperspecial neighbours (in $|\mathbf{B}_V|$) entirely contained in $|\mathbf{B}_W|$. We shall denote by $\mathrm{Sp}_W = \mathrm{Sp}_V \cap \mathbf{B}_W$ and $\mathrm{Hyp}_W = \mathrm{Hyp}_V \cap \mathbf{B}_W$ the sets of special and hyperspecial vertices of the sub-building \mathbf{B}_W .

We will be mainly interested in *hyperspecial* vertices in the following. Accordingly, **we re-normalize the distance in $|\mathbf{B}_V| \supset |\mathbf{B}_W|$ so that any two hyperspecial vertices which share a common special neighbour (respectively, any two special vertices which share a common hyperspecial neighbour) are at distance 1.** This corresponds to rescaling δ by a factor $\frac{1}{2}$, i.e., to set $\mathrm{dist}(\alpha, \alpha') := |\lambda - \lambda'|$ if $\alpha, \alpha' \in |\mathbf{B}_V|$ have respective parameters λ and λ' in a common apartment. We shall constantly make the confusion between vertices of \mathbf{B}_V (resp. of \mathbf{B}_W) and almost self-dual lattices inside V_τ (resp. inside W_τ): henceforth, we will now denote vertices of \mathbf{B}_V by the letter L instead of the previously used α .

2.1.6.1 Projection onto \mathbf{B}_W and *invariants map*.

Proposition-Definition 2.1.18 (Projection onto \mathbf{B}_W). *Let L be a vertex of \mathbf{B}_V .*

(i) *There exists a unique vertex of \mathbf{B}_W , denoted $\mathrm{pr}_W(L)$ and called the projection of L on \mathbf{B}_W , such that*

$$\mathrm{dist}(L, \mathrm{pr}_W(L)) = \min\{\mathrm{dist}(L, L'); L' \in |\mathbf{B}_W|\}.$$

(ii) *If \mathcal{A} is any apartment of \mathbf{B}_V which contains L and such that $\mathcal{A} \cap \mathbf{B}_W \neq \emptyset$, then \mathcal{A} also contains $\mathrm{pr}_W(L)$.*

(iii) *If $g \in G_{W,\tau}$, then $\mathrm{pr}_W(g \cdot L) = g \cdot \mathrm{pr}_W(L)$, for all $L \in |\mathbf{B}_V|$.*

(iv) *Moreover, if L does not belong to \mathbf{B}_W then $\mathrm{pr}_W(L) \in \mathrm{Hyp}_W$ is hyperspecial.*

Proof.

(i) The distance dist has image in the discrete set $\frac{1}{2}\mathbb{Z}_{\geq 0}$, thus the minimal distance is attained at least once. Set $n := \min\{\mathrm{dist}(L, L'); L' \in |\mathbf{B}_W|\} \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and assume, by absurd, that $L_1 \neq L_2$ are two distinct vertices of \mathbf{B}_W such that $\mathrm{dist}(L, L_1) = \mathrm{dist}(L, L_2) = n$. Set $d = \mathrm{dist}(L_1, L_2) \geq \frac{1}{2}$, and let $[x_0^{(1)} := L, x_1^{(1)}, \dots, x_{2n}^{(1)} := L_1]$ (resp. $[x_0^{(2)} := L, x_1^{(2)}, \dots, x_{2n}^{(2)} := L_2]$) be geodesic paths in $|\mathbf{B}_V|$ connecting L with L_1 (resp. connecting L with L_2). As $L_1 \neq L_2$ and for $|\mathbf{B}_V|$ contains no cyclic paths, there would exist a unique integer $k \in \{0, \dots, 2n - 1\}$ such that $x_k^{(1)} = x_k^{(2)}$ and $x_i^{(1)} \neq x_j^{(2)}$, for all $i, j \geq k + 1$ (in particular, $x_k^{(1)} = x_k^{(2)} \notin \mathbf{B}_W$ by minimality of n). On the other hand, one could find a path $[L_1, x_{2n+1}, \dots, x_{2n+2d} = L_2]$ inside $|\mathbf{B}_W|$ which connects L_1 and L_2 (take any apartment \mathcal{A}_W inside \mathbf{B}_W which contains both L_1 and L_2). This would make $[x_k^{(1)}, x_{k+1}^{(1)}, \dots, x_{2n}^{(1)} = L_1, x_{2n+1}, \dots, x_{2n+2d} = L_2, x_{2n-1}^{(2)}, \dots, x_{k+1}^{(2)}, x_k^{(2)}]$ a cyclic path inside $|\mathbf{B}_V|$, which would contradict Corollary 2.1.14. This forces $L_1 = L_2$.

(ii) Assume $L \notin \mathbf{B}_W$ (otherwise $L = \text{pr}_W(L)$ and the statement is trivial). If \mathcal{A} is an apartment of \mathbf{B}_V containing L together with some $L' \in |\mathbf{B}_W|$, then the assumption $\text{pr}_W(L) \in |\mathbf{B}_W| \setminus \mathcal{A}$ would enable us to construct - in a way very similar to the preceding statement - a cyclic path inside \mathbf{B}_V connecting L , L' and $\text{pr}_W(L)$. This contradiction implies $\text{pr}_W(L) \in \mathcal{A}$.

(iii) Let $g \in G_{W,\tau}$ and $L \in |\mathbf{B}_V|$, and set $n = \text{dist}(L, \text{pr}_W(L))$. Recall that g , seen as $\iota(g) \in G_{V,\tau}$, stabilizes $|\mathbf{B}_W| \simeq \iota(|\mathbf{B}_W|) \subset |\mathbf{B}_V|$. Accordingly, one has $g \cdot \text{pr}_W(L) \in |\mathbf{B}_W|$ and $g^{-1} \cdot \text{pr}_W(g \cdot L) \in |\mathbf{B}_W|$, thus

$$\begin{aligned} n = \text{dist}(L, \text{pr}_W(L)) &= \text{dist}(g \cdot L, g \cdot \text{pr}_W(L)) \geq \text{dist}(g \cdot L, \text{pr}_W(g \cdot L)) \\ &= \text{dist}(L, g^{-1} \cdot \text{pr}_W(g \cdot L)) \geq \text{dist}(L, \text{pr}_W(L)) = n, \end{aligned}$$

hence $\text{dist}(g \cdot L, g \cdot \text{pr}_W(L)) = \text{dist}(g \cdot L, \text{pr}_W(g \cdot L))$ thus $g \cdot \text{pr}_W(L) = \text{pr}_W(g \cdot L)$, by (i).

(iv) Assume that $L \notin |\mathbf{B}_W|$, i.e., that $\text{dist}(L, \text{pr}_W(L)) \geq \frac{1}{2}$. By Remark 2, if $\text{pr}_W(L)$ was special then its set of neighbours would be entirely contained in \mathbf{B}_W . Take any path connecting L with $\text{pr}_W(L)$: such a path would have positive length, hence would also contain a (hyperspecial) neighbour L' of $\text{pr}_W(L)$, hence a vertex of \mathbf{B}_W such that $\text{dist}(L, L') < \text{dist}(L, \text{pr}_W(L))$. This contradicts the minimality of $\text{dist}(L, \text{pr}_W(L))$, thus $\text{pr}_W(L) \in \text{Hyp}_V$ whenever $L \notin |\mathbf{B}_W|$. \square

In the following, we will call *segment of length n* in $|\mathbf{B}_V|$, the full subset $[[x_0, \dots, x_n]]$ formed by the hyperspecial vertices contained in a path $[x_0, x_{\frac{1}{2}}, x_1, \dots, x_{n-\frac{1}{2}}, x_n]$ of length $2n$. Equivalently, any segment of length n is defined by $n+1$ hyperspecial vertices x_0, \dots, x_n , lying in a same apartment and such that $\text{dist}(x_i, x_j) = |j-i|$, for all $i, j \in \{0, \dots, n-1\}$.⁽⁷⁾ This motivates the following definition :

Definition 2.1.9 (Invariants attached to a pair of vertices). *Let $(x, y) \in |\mathbf{B}_V| \times |\mathbf{B}_W|$ be a pair of vertices, and let $\text{pr}_W(x) \in \text{Hyp}_W$ be the projection of x on \mathbf{B}_W . We call invariants of (x, y) , and denote by $\text{inv}_\tau(x, y)$, the pair of half-integers*

$$\text{inv}_\tau(x, y) := (\text{dist}(x, \text{pr}_W(x)), \text{dist}(\text{pr}_W(x), y)) \in \frac{1}{2}\mathbb{Z}_{\geq 0} \times \frac{1}{2}\mathbb{Z}_{\geq 0}.$$

We call

$$\begin{aligned} \text{inv}_\tau : |\mathbf{B}_V| \times |\mathbf{B}_W| &\longrightarrow \frac{1}{2}\mathbb{Z}_{\geq 0} \times \frac{1}{2}\mathbb{Z}_{\geq 0}, \\ (x, y) &\longmapsto \text{inv}_\tau(x, y), \end{aligned}$$

the *invariants map*.

(7). One should remain careful that, in general, an injective sequence $\{x_0, \dots, x_n\}$ of hyperspecial vertices satisfying $\text{dist}(x_i, x_{i+1}) = 1$, $\forall i \in \{0, \dots, n-1\}$, is not always a segment of length n .

2.2 Computing the local conductor of special cycles.

Definition 2.2.1 (Half-apartments of \mathbf{B}_V). *Let $\mathcal{B} = \{e_+, e_0, e_-\} \in \mathfrak{B}$ be a Witt basis of V_τ . The half-apartment attached to \mathcal{B} is the set $\mathcal{A}_\mathcal{B}^+ := \{\langle \varpi^{-n}e_+, e_0, \varpi^n e_-\rangle; n \geq 0\} \subset \text{Hyp}_V$ which corresponds to hyperspecial vertices in $\mathcal{A}_\mathcal{B}$ whose parameter is non-negative (with respect to \mathcal{B}). If $\mathcal{A} \in \mathfrak{A}$ is an apartment of \mathbf{B}_V , we say that $\mathcal{A}^+ \subset \mathcal{A}$ is a half-apartment of \mathcal{A} , if $\mathcal{A}^+ = \mathcal{A}_\mathcal{B}^+$ for some Witt basis \mathcal{B} such that $\mathcal{A} = \mathcal{A}_\mathcal{B}$.*

Accordingly, half-apartments of \mathbf{B}_V correspond to half-lines in the naive sense. We call *half-apartment of \mathbf{B}_W* , a half-apartment contained inside \mathbf{B}_W : such a half-apartment is thus of the form $\mathcal{A}_\mathcal{B}^+$, where $\mathcal{B} = \{e_+, e_0, e_-\} \in \mathfrak{B}$ is such that $e_0 \in D_\tau$.

2.2.1 The *standard situation*.

Let $L_V \in \text{Hyp}_V$ and $L_W \in \text{Hyp}_W$ be hyperspecial vertices, and set $a := \text{dist}(L_V, \text{pr}_W L_V)$ and $b := \text{dist}(\text{pr}_W L_V, L_W)$, so that $a + b = \text{dist}(L_V, L_W)$. Let \mathcal{A} be an apartment of \mathbf{B}_W containing $\text{pr}_W(L_V)$ and L_W , and let $\mathcal{A}^+ \subset \mathcal{A}$ be the half-apartment containing both vertices and starting from $\text{pr}_W(L_V)$. Let ℓ be a geodesic line in \mathbf{B}_V containing L_V and such that $\ell \cap \mathbf{B}_W$ equals \mathcal{A}^+ (8). Let us fix such an ℓ : by Corollary 2.1.15, ℓ defines a unique apartment \mathcal{A}' of \mathbf{B}_V .

Let $\mathcal{B} = \{e_+, e_0, e_-\}$ and $\mathcal{B}' = \{e'_+, e'_0, e'_-\}$ be Witt bases attached to \mathcal{A} and \mathcal{A}' respectively, such that $\text{pr}_W(L_V)$ and L_W have respective parameter 0 and b in both bases (hence, L_V has parameter $-a$ in \mathcal{B}'). As noticed earlier, one may also assume that $e_0 = e_D$. This gives $\text{pr}_W(L_V) = \langle e_+, e_0, e_- \rangle = \langle e'_+, e'_0, e'_- \rangle$, $L_W = \langle \varpi^{-b}e_+, e_0, \varpi^b e_- \rangle = \langle \varpi^{-b}e'_+, e'_0, \varpi^b e'_- \rangle$, and $L_V = \langle \varpi^a e'_+, e_0, \varpi^{-a} e'_- \rangle$.

Let $u : V_\tau \rightarrow V_\tau$ be the E_τ -linear automorphism defined by $u(e_\star) = e'_\star$, $\star \in \{+, 0, -\}$. Then $u \in G_{V,\tau}$ and we set $S_{\mathcal{B},\mathcal{B}'} \in \text{GL}_3(E_\tau)$ to be the matrix of u with respect to the basis \mathcal{B} . Equivalently, $S_{\mathcal{B},\mathcal{B}'}$ is the transition matrix from \mathcal{B} to \mathcal{B}' , whose columns are the coordinates of elements of \mathcal{B}' , expressed in the basis \mathcal{B} . The following lemma is due to D.Jetchev [27] [Lemma 3.3] :

Lemma 2.2.1 (Transition matrices between Witt bases). *Let \mathcal{B} and \mathcal{B}' be as before. The transition matrix from \mathcal{B} to \mathcal{B}' is of the form*

$$S_{\mathcal{B},\mathcal{B}'} = \begin{bmatrix} 1 & \beta & \gamma \\ 0 & 1 & -\bar{\beta} \\ 0 & 0 & 1 \end{bmatrix} \in \text{GL}_3(\mathcal{O}_{E_\tau}), \text{ with } \beta, \gamma \in \mathcal{O}_{E_\tau}^\times \text{ and } \beta\bar{\beta} + \gamma + \bar{\gamma} = 0.$$

(8). Such an ℓ does exist. Indeed, choose any apartment $\tilde{\mathcal{A}}$ of \mathbf{B}_V that contains L_V and $\text{pr}_W(L_V)$: its half-apartment $\tilde{\mathcal{A}}^+$ starting from $\text{pr}_W(L_V)$ and containing L_V intersects \mathbf{B}_W in $\{\text{pr}_W(L_V)\}$. One may then then "glue" $\tilde{\mathcal{A}}^+$ with \mathcal{A}^+ at $\text{pr}_W(L_V)$, which gives a suitable ℓ .

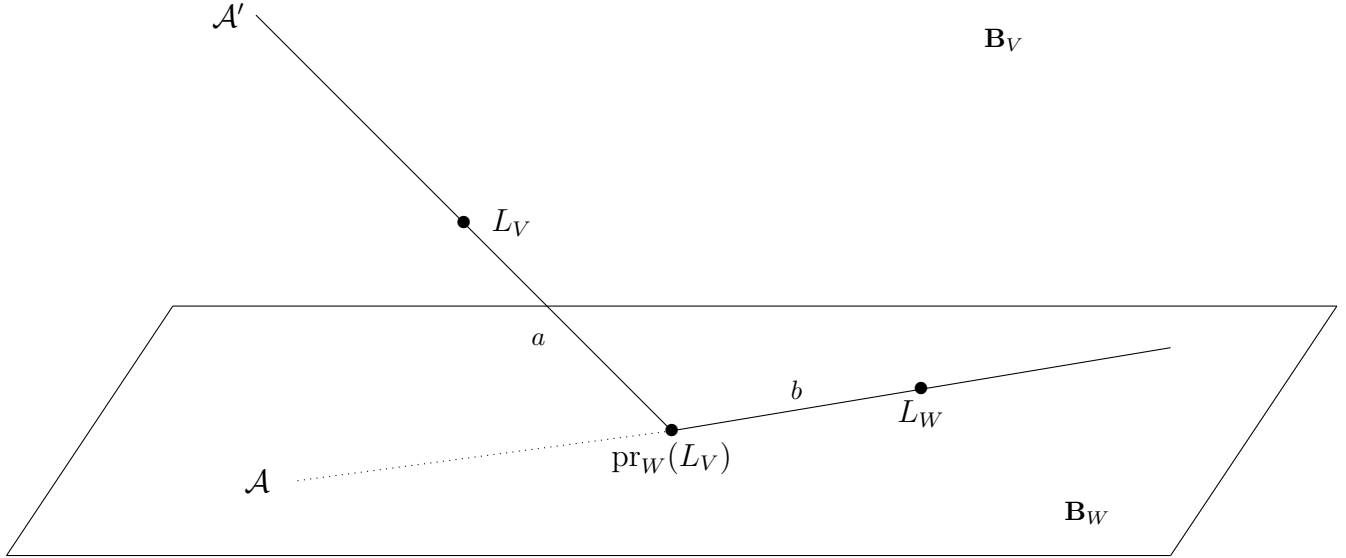


Figure 2.4 – The apartments \mathcal{A}' and \mathcal{A} (the latter is inside \mathbf{B}_W) share the half-line starting at $\text{pr}_W(L_V)$ and passing through L_W .

Proof. By assumption, both lattices $\langle e_+, e_0, e_- \rangle$ and $\langle e'_+, e'_0, e'_- \rangle$ are equal to the hyperspecial vertex $\text{pr}_W(L_V)$, thus the transition matrix $S_{\mathcal{B}, \mathcal{B}'}$ lies in $\text{GL}_3(\mathcal{O}_{E_\tau})$. By construction, one has $(\mathcal{A}')^+ := \mathcal{A}_{\mathcal{B}'}^+ = \mathcal{A}_{\mathcal{B}}^+ = \mathcal{A}^+$, hence $\langle \varpi^{-m}e_+, e_0, \varpi^m e_- \rangle = \langle \varpi^{-m}e'_+, e'_0, \varpi^m e'_- \rangle$, for all $m \geq 0$. The transition matrix between $\mathcal{B}_m := \{\varpi^{-m}e_+, e_0, \varpi^m e_-\}$ and $\mathcal{B}'_m := \{\varpi^{-m}e'_+, e'_0, \varpi^m e'_-\}$ is

$$S_m := \text{diag}(\varpi^m, 1, \varpi^{-m}) S_{\mathcal{B}, \mathcal{B}'} \text{diag}(\varpi^{-m}, 1, \varpi^m).$$

The preceding implies that S_m belongs to $\text{GL}_3(\mathcal{O}_{E_\tau})$ for all $m \geq 0$, which forces $S_{\mathcal{B}, \mathcal{B}'}$ to be upper-triangular. By equalities $\langle e'_+, e'_- \rangle = \langle e'_0, e'_0 \rangle = 1$, one has

$$S_{\mathcal{B}, \mathcal{B}'} = \begin{bmatrix} u & \beta & \gamma \\ 0 & v & \delta \\ 0 & 0 & \bar{u}^{-1} \end{bmatrix}, \text{ with } u \in \mathcal{O}_{E_\tau}^\times, v\bar{v} = 1 \text{ and } \beta, \gamma, \delta \in \mathcal{O}_{E_\tau}.$$

Accordingly, up to replacing e'_+ by $u^{-1}e'_+$, e'_- by $\bar{u}e'_-$ and e'_0 by $v^{-1}e'_0$ ⁽⁹⁾ one may assume that $u = v = 1$. The equality $\langle e'_-, e'_- \rangle = 0$ induces $\gamma + \bar{\gamma} + \delta\bar{\delta} = 0$, whereas $\langle e'_0, e'_- \rangle = 0$ induces $\beta + \bar{\delta} = 0$, hence $\delta = -\bar{\beta}$ and $\gamma + \bar{\gamma} + \beta\bar{\beta} = 0$. It remains to show that β and γ are units. As $\beta\bar{\beta} = -(\gamma + \bar{\gamma})$, one has $2 \text{ord}_\varpi(\beta) \geq \text{ord}_\varpi(\gamma)$, hence γ belongs to $\mathcal{O}_{E_\tau}^\times$ whenever β does.

Let $L_{-1} := \langle \varpi e'_+, e'_0, \varpi^{-1}e'_- \rangle$ be the hyperspecial vertex of parameter -1 with respect to \mathcal{B}' (then $L_{-1} \in \mathcal{A}' \setminus (\mathcal{A}')^+$) and assume that $\text{ord}_\varpi(\beta) \geq 1$. As $e_0 = e'_0 - \beta e_+ \in L_{-1}$

(9). As u is a unit and $v\bar{v} = 1$, then Proposition 2.1.1, (ii) ensures that such a replacement neither modify the apartment \mathcal{A}' nor its half-apartment $\mathcal{A}' = \mathcal{A}_{\mathcal{B}'}^+$.

then $\mathcal{O}_{E_\tau} e_0 = L_D \subset L_{-1}$, and the equality $\varpi^{-1} e'_- = \varpi^{-1}(\gamma e_+ + e_-) + (\varpi^{-1} \beta) e_0$ implies that $L_{-1} \subset W_\tau + L_D$, hence $L_{-1} = (L_{-1} \cap W_\tau) \oplus L_D$. Notice that the rank-2 lattice $L_{-1,W} := L_{-1} \cap W_\tau$ inside W_τ is self-dual. Indeed, one has $\langle L_{-1,W}, L_{-1,W} \rangle \subset \langle L_{-1}, L_{-1} \rangle \subset \mathcal{O}_{E_\tau}$, which gives $L_{-1,W} \subset L_{-1,W}^\vee$, and conversely if $w \in W_\tau$ is such that $\langle w, L_{-1,W} \rangle \subset \mathcal{O}_{E_\tau}$, then $\langle w, L_{-1,W} \oplus L_D \rangle \subset \mathcal{O}_{E_\tau}$, i.e., $w \in (L_{-1,W} \oplus L_D)^\vee \cap W_\tau = L_{-1,W}$, since $(L_{-1,W} \oplus L_D)^\vee = L_{-1,W}^\vee = L_{-1}$. This makes $L_{-1} = L_{-1,W} \oplus L_D$ a hyperspecial vertex of \mathbf{B}_W , which contradicts $L_{-1} \notin (\mathcal{A}')^+ = \mathcal{A}' \cap \mathbf{B}_W$. One finally obtains that $\beta, \gamma \in \mathcal{O}_{E_\tau}^\times$, which finishes the proof. \square

2.2.2 Special apartments.

The preceding "standard situation" will appear later on in the expression of local conductors of special cycles, and motivates the introduction of the so-called *special apartments*⁽¹⁰⁾ of \mathbf{B}_V .

Definition 2.2.2 (Special apartments). *An apartment \mathcal{A} of \mathbf{B}_V is called special if its intersection with \mathbf{B}_W is an half-apartment of \mathbf{B}_W . We denote by $\mathfrak{S} \subset \mathfrak{A}$ the set of special apartments.*

The apartment \mathcal{A}' of § 2.2.1 belongs to \mathfrak{S} . If \mathcal{A} is a special apartment, we call *base point of \mathcal{A}* the starting point of the half-line $\mathcal{A} \cap \mathbf{B}_W$. If $\mathcal{B} = \{e_+, e_0, e_-\}$ is any Witt basis such that $\mathcal{A}_\mathcal{B}^+ = \mathcal{A} \cap \mathbf{B}_W$, then the base point of \mathcal{A} is just the hyperspecial vertex $\langle e_+, e_0, e_- \rangle \in |\mathbf{B}_W|$.

Definition 2.2.3 (Hyperbolic pairs). *An ordered pair of vectors (e, f) is called a hyperbolic pair of V_τ , if e and f are isotropic vectors of V_τ such that $\langle e, f \rangle = 1$.*

We denote by \mathfrak{H} the set of hyperbolic pairs of V_τ , and by $\mathfrak{H}^{(W)} \subset \mathfrak{H}$ the subset formed by those hyperbolic pairs (e, f) such that $e \in W_\tau$ and $f \notin W_\tau$. The group $G_{V,\tau}$ acts diagonally on \mathfrak{H} , and the subgroup $G_{W,\tau} \subset G_{V,\tau}$ acts diagonally on $\mathfrak{H}^{(W)}$. According to ([48], [Lemma 7.3]), every isotropic vector $e \in W_\tau$ may be extended into a hyperbolic pair $(e, e') \in \mathfrak{H}$, with $e' \in W_\tau$ (i.e., e may be extended into a Witt basis $\{e, e'\}$ of W_τ). Consequently, the group $G_{W,\tau}$ acts transitively on the set of isotropic vectors inside W_τ , since it acts transitively on Witt bases of W_τ .

On the other hand, the group E_τ^\times acts on \mathfrak{H} , by $u \cdot (e, f) := (ue, \bar{u}^{-1}f)$, for all $u \in E_\tau^\times$ and (e, f) in \mathfrak{H} , and this action clearly stabilizes $\mathfrak{H}^{(W)}$. Accordingly, the groups $G_{V,\tau}$ and $G_{W,\tau}$ act respectively on the quotient sets $E_\tau^\times \backslash \mathfrak{H}$ and $E_\tau^\times \backslash \mathfrak{H}^{(W)}$. If $(e, f) \in \mathfrak{H}$ (resp. $(e, f) \in \mathfrak{H}^{(W)}$) is a hyperbolic pair, we denote by $[(e, f)]$ its corresponding orbit in $E_\tau^\times \backslash \mathfrak{H}$ (resp. in $E_\tau^\times \backslash \mathfrak{H}^{(W)}$).

Lemma 2.2.2. *The group $G_{W,\tau}$ acts transitively on the quotient set $E_\tau^\times \backslash \mathfrak{H}^{(W)}$.*

(10). We keep the same term as ([27], §3.3).

Proof. Let (e_1, f_1) and (e_2, f_2) belong to $\mathfrak{H}^{(W)}$, and let $g \in G_{W,\tau}$ be such that $g \cdot e_1 = e_2$. It is enough to show that there exists some $h \in G_{W,\tau}$, which stabilizes (as a set) the line $E_\tau e_1$ and such that $h \cdot f_1 \in E_\tau^\times(g^{-1} \cdot f_2)$. Indeed, then there would exist some $\mu, \nu \in E_\tau^\times$ such that $h \cdot e_1 = \mu e_1$ and $h \cdot f_1 = \nu g^{-1} \cdot f_2$, hence $(gh) \cdot (e_1, f_1) = (\mu e_2, \nu f_2)$. But $1 = \langle e_1, f_1 \rangle = \langle (gh) \cdot e_1, (gh) \cdot f_1 \rangle = \mu \bar{\nu} \langle e_2, f_2 \rangle = \mu \bar{\nu}$, hence $\nu = \bar{\mu}^{-1}$ and $(gh) \cdot (e_1, f_1) = (\mu e_2, \bar{\mu}^{-1} f_2)$, i.e., $(gh) \cdot [(e_1, f_1)] = [(e_2, f_2)]$.

We let $\ell_1 := E_\tau f_1$ and $\ell_2 := E_\tau(g^{-1} \cdot f_2)$ denote the isotropic lines generated by f_1 and $g^{-1} f_2$ (these do not lie inside W_τ), and let $e'_1 \in W_\tau$ be an isotropic vector such that (e_1, e'_1) forms an hyperbolic pair of W_τ : this gives a Witt basis $\mathcal{B} := \{e_1, e_D, e'_1\}$ of V_τ . We set $B_{W,\tau}$ (resp., $B_{V,\tau}$) to be the Borel subgroup of $G_{W,\tau}$ (resp. of $G_{V,\tau}$) which stabilizes the line $E_\tau e_1$ ⁽¹¹⁾. The line ℓ_1 is of the form $E_\tau(ue_1 + e_D + ve'_1)$ where $u, v \in E_\tau$ are such that $u\bar{v} + v\bar{u} + 1 = 0$ (which implies $u, v \neq 0$). Let t_1 denote the element of $B_{W,\tau}$ such that $\iota(t) \in B_{V,\tau}$ has matrix $\begin{bmatrix} \bar{v} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{bmatrix}$ with respect to \mathcal{B} . Then, up to replacing ℓ_1 by $t_1 \cdot \ell_1$, one may assume the line ℓ_1 to be of the form $E_\tau(\mathbf{u}e_1 + e_D + e'_1)$, with $\mathbf{u} + \bar{\mathbf{u}} + 1 = 0$. Accordingly, the line ℓ_2 can also be assumed to be of the form $E_\tau((\mathbf{u} + \epsilon)e_1 + e_D + e'_1)$, with $\epsilon \in E_\tau$ satisfying $\epsilon + \bar{\epsilon} = 0$. By setting s to be the element of $B_{W,\tau}$ such that $\iota(s) \in B_{V,\tau}$ has matrix $\begin{bmatrix} 1 & 0 & \epsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with respect to \mathcal{B} , one obtains $\ell_2 = s \cdot \ell_1$, which finishes the proof. \square

Notice that any hyperbolic pair of V_τ can be completed into a Witt basis of V_τ . Indeed, let $(e, f) \in \mathfrak{H}$: the hermitian space $(V_\tau, \langle \cdot, \cdot \rangle)$ being split with determinant $1 \in F_\tau^\times / N_{E_\tau/F_\tau}(E_\tau^\times)$, the line $(E_\tau e \oplus E_\tau f)^\perp$ is then anisotropic and contains some vector d satisfying $\langle d, d \rangle = 1$. The choice of such a d is only well-defined up to scalars $v \in U(1)_{E/F}(F_\tau) \subset E_\tau^\times$, i.e., such that $v\bar{v} = 1$. If $\mathcal{B} \in \mathfrak{B}$, we denote by $[\mathcal{B}]$ the orbit of \mathcal{B} in $\mathfrak{T}_V \backslash \mathfrak{B}$. The preceding discussion gives a map

$$\begin{aligned} \mathfrak{b} : E_\tau^\times \backslash \mathfrak{H} &\longrightarrow \mathfrak{T}_V \backslash \mathfrak{B}, \\ [(e, f)] &\longmapsto [(e, d, f)], \end{aligned}$$

which is well-defined and clearly $G_{V,\tau}$ -equivariant and surjective. Recall that the map $\mathfrak{a} : \mathfrak{T}_V \backslash \mathfrak{B} \rightarrow \mathfrak{A}$ given by $[\mathcal{B}] \mapsto \mathcal{A}_{\mathcal{B}}$ is also $G_{V,\tau}$ -equivariant and surjective. Therefore, one obtains a surjection

$$\begin{aligned} \mathfrak{a} \circ \mathfrak{b} : E_\tau^\times \backslash \mathfrak{H} &\longrightarrow \mathfrak{A}, \\ [(e, f)] &\longmapsto \mathcal{A}_{\{e,d,f\}} \end{aligned}$$

which remains $G_{V,\tau}$ -equivariant.

Proposition 2.2.3. *The restriction of $\mathfrak{a} \circ \mathfrak{b}$ to the subset $E_\tau^\times \backslash \mathfrak{H}^{(W)} \subset E_\tau^\times \backslash \mathfrak{H}$ induces a $G_{V,\tau}$ -equivariant surjection*

$$\mathfrak{a} \circ \mathfrak{b}|_{\mathfrak{H}^{(W)}} : E_\tau^\times \backslash \mathfrak{H}^{(W)} \twoheadrightarrow \mathfrak{S} \subset \mathfrak{A}$$

(11). Or equivalently, which stabilizes the flag $E_\tau e_1 \subset E_\tau e_1 \oplus E_\tau e'_1 \subset V_\tau$.

on to the set of special apartments.

Proof.

- Let us first show that the restriction of $\mathfrak{a} \circ \mathfrak{b}$ to $E_\tau^\times \setminus \mathfrak{H}^{(W)}$ factors through $\mathfrak{S} \hookrightarrow \mathfrak{A}$. Let $(e, f) \in \mathfrak{H}^{(W)}$ and set $\mathcal{A}' := \mathcal{A}_{\{e, d, f\}}$, where d is any vector of $(E_\tau e \oplus E_\tau f)^\perp$ such that $\langle d, d \rangle = 1$. As $e \in W_\tau$ is isotropic, one may find some isotropic vector $e' \in W_\tau$ such that (e, e') is a hyperbolic pair of W_τ , hence $\mathcal{B} := \{e, e_D, e'\}$ is a Witt basis of V_τ . Set $\mathcal{A} := \mathcal{A}_{\mathcal{B}}$ and let $M = \begin{bmatrix} 1 & x & t \\ 0 & y & u \\ 0 & z & v \end{bmatrix} \in \mathrm{GL}_3(E_\tau)$ be the transition matrix from \mathcal{B} to $\{e, f, d\}$ (i.e., columns of M are the coordinates of elements of $\{e, f, d\}$ expressed in the basis \mathcal{B}).

Equalities $\langle d, e \rangle = 0$ and $\langle d, d \rangle = 1$ induce $z = 0$ and $y\bar{y} = 1$. Thus, up to switching d with $y^{-1}d \in (E_\tau e \oplus E_\tau f)^\perp$, one may assume $y = 1$. Equalities $\langle d, f \rangle = \langle f, f \rangle = 0$ imply that $x = -\bar{u}$, and that $u\bar{u} + t + \bar{t} = 0$, whereas equality $\langle e, f \rangle = 1$ induces $\bar{v} = 1$, hence

$$M = \begin{bmatrix} 1 & -\bar{u} & t \\ 0 & 1 & u \\ 0 & 0 & 1 \end{bmatrix}.$$

Set $n_1 = -\mathrm{ord}_\varpi(u) \in \mathbb{Z}$ and $n_2 = -\mathrm{ord}_\varpi(t) \in \mathbb{Z}$. As $u\bar{u} = -(t + \bar{t})$, one has $-2n_1 \geq -n_2$, i.e., $n_1 \leq \frac{n_2}{2}$. If $n \geq \frac{n_2}{2}$, then $\varpi^{2n}t$, $\varpi^n u$ and $\varpi^n \bar{u}$ all belong to \mathcal{O}_{E_τ} , hence

$$\mathrm{diag}(\varpi^n, 1, \varpi^{-n}) M \mathrm{diag}(\varpi^{-n}, 1, \varpi^n) \in \mathrm{GL}_3(\mathcal{O}_{E_\tau}).$$

In other words, one has an equality between lattices $\langle \varpi^{-n}e, d, \varpi^n f \rangle = \langle \varpi^{-n}e, e_D, \varpi^n e' \rangle$, which means that the half-apartment $(\mathcal{A}')^{\geq \frac{n_2}{2}} := \{\langle \varpi^{-n}e, d, \varpi^n f \rangle, n \geq \frac{n_2}{2}\}$ of \mathcal{A}' is equal to the half-apartment $\mathcal{A}^{\geq \frac{n_2}{2}} := \{\langle \varpi^{-n}e, e_D, \varpi^n e' \rangle, n \geq \frac{n_2}{2}\}$ of $\mathcal{A} \subset \mathbf{B}_W$.

On the other hand, if $n < n_1$ then the hyperspecial point $\langle \varpi^{-n}e, d, \varpi^n f \rangle$ of \mathcal{A}' does not belong to \mathbf{B}_W . Indeed, if such were true then there would exist some self-dual lattice $L_W \subset W_\tau$ such that $\langle \varpi^{-n}e, d, \varpi^n f \rangle = L_W \oplus L_D$, with $L_D = \mathcal{O}_{E_\tau} e_D$. This would give $\varpi^n f = \varpi^n t e + \varpi^n u e_D + \varpi^n e' \in L_W \oplus L_D$, hence $\varpi^n u e_D \in L_D$ (because $\varpi^n t e + \varpi^n e' \in W_\tau$), i.e., $\varpi^n u \in \mathcal{O}_{E_\tau}$, which contradicts $n < n_1$.

As geodesics (in \mathbf{B}_W) coincide with apartments of \mathbf{B}_W , and by the unicity of paths (in \mathbf{B}_V) which connect a given pair of points of \mathbf{B}_V , one deduces that there exists an integer $n_0 \in [n_1, \frac{n_2}{2}]$ such that $\langle \varpi^{-n}e, d, \varpi^n f \rangle$ belongs to \mathbf{B}_W if $n \geq n_0$, and belongs to $\mathbf{B}_V \setminus \mathbf{B}_W$ if $n < n_0$. This allows one to choose an apartment $\tilde{\mathcal{A}}$ inside \mathbf{B}_W such that $\mathcal{A}' \cap \mathbf{B}_W$ is the half-apartment $\tilde{\mathcal{A}}^{n \geq n_0}$ of $\tilde{\mathcal{A}}$ corresponding to the half-line obtained by glueing the segment $[\langle \varpi^{-n_0}e, d, \varpi^{n_0} f \rangle, \dots, \langle \varpi^{-\lfloor \frac{n_2}{2} \rfloor} e, d, \varpi^{-\lfloor \frac{n_2}{2} \rfloor} f \rangle]$ together with $\mathcal{A}^{\geq \frac{n_2}{2}}$, i.e., that \mathcal{A}' is a special apartment.

- Conversely, let $\mathcal{A}' \in \mathfrak{S}$ be a special apartment of \mathbf{B}_V , and let $\mathcal{B}' := \{e'_+, e'_0, e'_-\}$ be a Witt basis such that $\mathcal{A}' = \mathcal{A}_{\mathcal{B}'}$, i.e., such that $\mathcal{A}' = \mathfrak{a} \circ \mathfrak{b}([\langle e'_+, e'_-\rangle])$. By assumption on

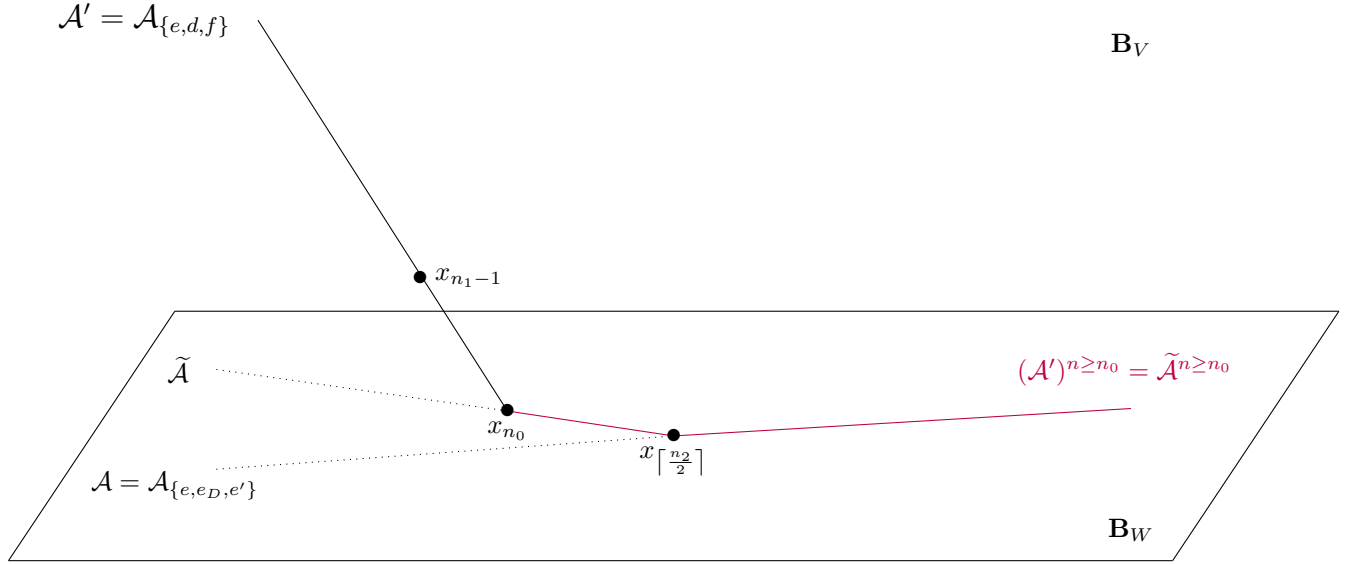


Figure 2.5 – The point x_{n_1} may or may not lie in \mathbf{B}_V . The purple line is the half-apartment $(\mathcal{A}')^{\geq n_0}$, consisting in the union of the half-apartment $\mathcal{A}^{\geq \frac{n_2}{2}}$ with the segment $[[\langle \varpi^{-n_0} e, d, \varpi^{n_0} f \rangle, \dots, \langle \varpi^{-\lfloor \frac{n_2}{2} \rfloor} e, d, \varpi^{-\lfloor \frac{n_2}{2} \rfloor} f \rangle]]$

\mathcal{A}' , there exists some $\mathcal{B} = \{e_+, e_0, e_-\} \in \mathfrak{B}$ such that $\mathcal{A}' \cap \mathbf{B}_W = \mathcal{A}_{\mathcal{B}}^+$ (in particular, $e_+, e_- \in W_\tau$ and one may assume $e_0 = e_D$). Let L_V be any hyperspecial point of $\mathcal{A} \setminus \mathbf{B}_W$ and let L_W be any hyperspecial point of $\mathcal{A} \cap \mathbf{B}_W$, and set $\text{pr}_W(L_V) \in \mathcal{A}' \cap \mathbf{B}_W$ to be the projection of L_V on \mathbf{B}_W . We find ourselves in the *standard situation* of the preceding paragraph: by Lemma 2.2.1, \mathcal{B} and \mathcal{B}' can be chosen such that the transition matrix between \mathcal{B} and \mathcal{B}' has the form

$$S_{\mathcal{B}, \mathcal{B}'} = \begin{bmatrix} 1 & \beta & \gamma \\ 0 & 1 & -\bar{\beta} \\ 0 & 0 & 1 \end{bmatrix}, \text{ with } \beta, \gamma \in \mathcal{O}_{E_\tau}^\times.$$

This gives $e'_+ = e_+ \in W_\tau$, and $e'_- = (\gamma e_+ + e_-) - \bar{\beta} e_0 \notin W_\tau$, as $\beta \neq 0$. In other words, one has $(e'_+, e'_-) \in \mathfrak{H}^{(W)}$, thus $\mathcal{A}' = \mathbf{a} \circ \mathbf{b}([(e'_+, e'_-)]) \in \mathbf{a} \circ \mathbf{b}(E_\tau^\times \setminus \mathfrak{H}^{(W)})$, which shows the surjectivity and finishes the proof. \square

An immediate consequence of the previous two lemmas is the following:

Corollary 2.2.4. *The group $G_{W,\tau}$ acts transitively on the set \mathfrak{S} of special apartments.*

Recall that H_τ is the diagonal image $\Delta(G_{W,\tau}) \subset G_\tau$ of $G_{W,\tau}$. The group G_τ acts on the product building $\mathbf{B}_V \times \mathbf{B}_W$ and this action preserves distances and stabilizes the product $\text{Hyp} \stackrel{\text{def}}{=} \text{Hyp}_V \times \text{Hyp}_W$. If $h \in G_{W,\tau}$ and if $(x, y) \in \text{Hyp}$, then one has $\text{pr}_W(h \cdot x) = h \cdot \text{pr}_W(x)$ (this is Proposition-Definition 2.1.18, (iii)), hence

$$(\text{dist}(h \cdot x, \text{pr}_W(h \cdot x)), \text{dist}(\text{pr}_W(h \cdot x), h \cdot y)) = (\text{dist}(h \cdot x, h \cdot \text{pr}_W(x)), \text{dist}(h \cdot \text{pr}_W(x), h \cdot y))$$

$$= (\text{dist}(x, \text{pr}_W(x)), \text{dist}(\text{pr}_W(x), y)).$$

Consequently, the restricted *invariants map* $\text{inv}_\tau|_{\text{Hyp}}$ factors through the map $\text{Hyp} \rightarrow H_\tau \backslash \text{Hyp}$. The following result, due to Jetchev [27] [Theorem 1.3, (i)], is part of the *local conductor formula* which we will prove later on.

Proposition 2.2.5. *The restriction of inv_τ to hyperspecial vertices induces a bijection*

$$\text{inv}_\tau : H_\tau \backslash \text{Hyp} \xrightarrow{\sim} \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}.$$

Proof. That inv_τ is surjective is an easy consequence of the existence of special apartments. Indeed, let $(a, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, take any special apartment $\mathcal{A}' \in \mathfrak{S}$ and let $\mathcal{B}' = \{e'_+, e'_0, e'_-\} \in \mathfrak{B}$ be a Witt basis such that $\mathcal{A}' = \mathcal{A}_{\mathcal{B}'}$ and $\mathcal{A}' \cap \mathbf{B}_W = \mathcal{A}_{\mathcal{B}'}^+$. The hyperspecial point $x := \langle \varpi^a e'_+, e'_0, \varpi^{-a} e'_- \rangle \in \mathcal{A}' \setminus \mathbf{B}_W$ is such that $\text{pr}_W(x) = \langle e'_+, e'_0, e'_- \rangle \in \mathcal{A}'$: if we set $y := \langle \varpi^{-b} e'_+, e'_0, \varpi^b e'_- \rangle \in \mathcal{A}' \cap \mathbf{B}_W$, one obtains $\text{dist}(x, \text{pr}_W(x)) = a$ and $\delta(\text{pr}_W(x), y) = b$, hence $\text{inv}_\tau(x, y) = (a, b)$.

On the other hand, assume that two pairs $(x, y), (x', y') \in \text{Hyp}$ have the same invariants, and set $(a, b) := \text{inv}_\tau(x, y) = \text{inv}_\tau(x', y')$:

- If $x \in \mathbf{B}_W$ then $x = \text{pr}_W(x)$, thus $a = 0$ and $x' = \text{pr}_W(x') \in \mathbf{B}_W$. Accordingly, there exist apartments \mathcal{A} and \mathcal{A}' of \mathbf{B}_W which contain respectively $\{x, y\}$ and $\{x', y'\}$. Without loss of generality, one may take Witt bases $\mathcal{B} = \{e_+, e_0, e_-\}$ and $\mathcal{B}' = \{e'_+, e'_0, e'_-\}$ such that $\mathcal{A} = \mathcal{A}_{\mathcal{B}}$ and $\mathcal{A}' = \mathcal{A}_{\mathcal{B}'}$, where vectors e_+, e'_+, e_-, e'_- lie in W_τ , such that $e_0 = e'_0 = e_D$ and such that

$$x = \langle e_+, e_0, e_- \rangle, y = \langle \varpi^{-b} e_+, e_0, \varpi^b e_- \rangle, x' = \langle e'_+, e'_0, e'_- \rangle \text{ and } y' = \langle \varpi^{-b} e'_+, e'_0, \varpi^b e'_- \rangle.$$

Therefore the element $h \in G_{W,\tau}$ given by $h \cdot e_+ = e'_+$ and $h \cdot e_- = e'_-$ satisfies $\Delta(h) \cdot (x, y) = (h \cdot x, h \cdot y) = (x', y')$.

- If $x \notin \mathbf{B}_W$, then $x' \notin \mathbf{B}_W$ and we may choose two special apartments \mathcal{A} and \mathcal{A}' which contain $\{x, \text{pr}_W(x), y\}$ and $\{x', \text{pr}_W(x'), y'\}$ respectively. By Corollary 2.2.4, one may find some $h \in G_{W,\tau}$ such that $h \cdot \mathcal{A} = \mathcal{A}'$. As h preserves dist , one has $a = \text{dist}(x, \text{pr}_W(x)) = \text{dist}(h \cdot x, h \cdot \text{pr}_W(x)) = \text{dist}(h \cdot x, \text{pr}_W(h \cdot x))$, i.e., $h \cdot x$ is the only point of $\mathcal{A}' \setminus \mathbf{B}_W$ such that $\text{dist}(x, \mathbf{B}_W) = a$: in other words, $h \cdot x = x'$. Accordingly, $h \cdot y$ is the only point of $\mathcal{A}' \cap \mathbf{B}_W$ such that $a + b = \text{dist}(x, y) = \text{dist}(h \cdot x, h \cdot y) = \text{dist}(x', h \cdot y)$, i.e., $h \cdot y = y'$. v In any case, we showed that $(x, y) = (x', y') \in H_\tau \backslash \text{Hyp}$. \square

2.2.3 Interlude: filtration of \mathcal{O}_{E_τ} by local orders.

Let $c \geq 0$ be an integer. Recall that we defined in section 1.43 the *local order of conductor* c of \mathcal{O}_{E_τ} to be the ring $\mathcal{O}_{\tau,c} := \mathcal{O}_{F_\tau} + \varpi^c \mathcal{O}_{E_\tau} \subset \mathcal{O}_{E_\tau}$, where $\varpi \in F_\tau^\times$ is a common uniformizer for F_τ and E_τ . This section being purely local at τ , we shall drop the subscript τ and simply

denote this order by \mathcal{O}_c . The family $(\mathcal{O}_c)_{c \geq 0}$ provides us with a decreasing open filtration on \mathcal{O}_{E_τ} :

$$\mathcal{O}_{E_\tau} = \mathcal{O}_0 \supset \mathcal{O}_1 \supset \cdots \supset \mathcal{O}_c \supset \mathcal{O}_{c+1} \supset \cdots$$

For all $c \geq 1$, one has $\mathcal{O}_c^\times = \mathcal{O}_{F_\tau}^\times (1 + \varpi^c \mathcal{O}_{E_\tau}) \subset \mathcal{O}_{E_\tau}^\times$. This gives a fundamental system of open subgroups in $\mathcal{O}_{E_\tau}^\times$:

$$\mathcal{O}_{E_\tau}^\times = \mathcal{O}_0^\times \supset \mathcal{O}_1^\times \supset \cdots \supset \mathcal{O}_c^\times \supset \mathcal{O}_{c+1}^\times \supset \cdots \quad (2.10)$$

We set E_τ^1 to be the group $U(1)(F_\tau) = \{z \in E_\tau^\times; z\bar{z} = 1\} \subset \mathcal{O}_{E_\tau}^\times$. Recall that we defined at (1.25) the morphism $\nu : \text{Res}_{E/F} \mathbb{G}_m \rightarrow U(1)$, by setting

$$\begin{aligned} \nu : (E \otimes_F R)^\times &\longrightarrow U(1)(R), \\ s &\longmapsto \frac{\bar{s}}{s}, \end{aligned}$$

for all F -algebra R . According to Hilbert's Theorem 90, ν is surjective on F_τ -points, i.e., $E_\tau^1 = \nu(E_\tau^\times) = \nu(\mathcal{O}_{E_\tau}^\times)$, the last equality coming from $\nu(x) = 1$ for all $x \in F_\tau^\times$, hence $\nu(\varpi^n) = 1$ for all $n \in \mathbb{Z}$. We set $\mathcal{O}_c^1 := \nu(\mathcal{O}_c^\times) \subset \mathcal{O}_{E_\tau}^\times$, for all $c \geq 0$. By the preceding observation, one has $\mathcal{O}^1 := \mathcal{O}_0^1 = E_\tau^1$, and taking the image of (2.10) by ν induces a decreasing open filtration on \mathcal{O}_0^1 :

$$\mathcal{O}^1 \supset \mathcal{O}_1^1 \supset \cdots \supset \mathcal{O}_c^1 \supset \mathcal{O}_{c+1}^1 \supset \cdots \quad (2.11)$$

Let $n \in \mathbb{Z}$ and $c \geq 1$ be integers. If $s = x(1 + \varpi^c z) \in \mathcal{O}_c^\times$, with $x \in \mathcal{O}_{F_\tau}^\times$ and $z \in \mathcal{O}_{E_\tau}$, then one has $\nu(s) = \nu(x^{-1}s) = \frac{1+\varpi^c \bar{z}}{1+\varpi^c z} \in 1 + \varpi^c \mathcal{O}_{E_\tau}$, i.e., $\nu(\varpi^{\mathbb{Z}} \mathcal{O}_c^\times) = \nu(\mathcal{O}_c^\times) \subset 1 + \varpi^c \mathcal{O}_{E_\tau}$, i.e., $\mathcal{O}_c^1 \subset E_\tau^1 \cap (1 + \varpi^c \mathcal{O}_{E_\tau})$.

The following lemma is due to Jechev⁽¹²⁾ ([27], Lemma 3.1):

Lemma 2.2.6. *For all $c \geq 0$, one has $\nu^{-1}(1 + \varpi^c \mathcal{O}_{E_\tau}) = \varpi^{\mathbb{Z}} \mathcal{O}_c^\times \subset E_\tau^\times$. Therefore, $\mathcal{O}_c^1 = E_\tau^1 \cap (1 + \varpi^c \mathcal{O}_{E_\tau})$.*

Proof. If $c = 0$, then $\nu^{-1}(\mathcal{O}_{E_\tau}) = \nu^{-1}(E_\tau^1) = E_\tau^\times = \varpi^{\mathbb{Z}} \mathcal{O}_{E_\tau}^\times = \varpi^{\mathbb{Z}} \mathcal{O}_0^\times$, which is what we wanted. Now assume that $c \geq 1$. We already showed that $\varpi^{\mathbb{Z}} \mathcal{O}_c^\times \subset \nu^{-1}(1 + \varpi^c \mathcal{O}_{E_\tau})$. As E_τ/F_τ is unramified, one may find some element $\zeta \in \mathcal{O}_{E_\tau}^\times$ such that $\mathcal{O}_{E_\tau} = \mathcal{O}_{F_\tau}[\zeta]$ and $\bar{\zeta} - \zeta \in \mathcal{O}_{E_\tau}^\times$ ⁽¹³⁾. Let $s \in E_\tau^\times$ be such that $\nu(s) \in 1 + \varpi^c \mathcal{O}_{E_\tau}$. As $\nu(s) = \nu(\varpi^n s)$ for all $n \in \mathbb{Z}$, one may always assume that $s \in \mathcal{O}_{E_\tau}^\times$. Write $s = x + y\zeta$, with $x, y \in \mathcal{O}_{F_\tau}$. Then

(12). Up to a small correction, removing the unnecessary assumption $\text{char}(\mathbb{F}_0) \neq 2$.

(13). Indeed, the extension E_τ/F_τ is an unramified quadratic extension, hence is isomorphic (up to unique isomorphism) to the extension $F_\tau[\zeta]$, for some $(q^2 - 1)$ -th root of unity $\zeta \in E_\tau$. In particular, one has $\zeta \in \mathcal{O}_{E_\tau}^\times$ and the image $\tilde{\zeta}$ of ζ in \mathbb{F} generates the extension \mathbb{F}/\mathbb{F}_0 . The latter follows from the fact that the minimal polynomial P of ζ over F_τ - which is separable and has coefficients in \mathcal{O}_{F_τ} - reduces to the minimal polynomial \tilde{P} of $\tilde{\zeta}$ over \mathbb{F}_0 (this is a consequence of Hensel's Lemma), hence $\tilde{\zeta} \notin \mathbb{F}_0$. It is now a direct consequence of Nakayama's Lemma that the equality $\mathbb{F} = \mathbb{F}_0[\tilde{\zeta}]$ induces the equality $\mathcal{O}_{E_\tau} = \mathcal{O}_{F_\tau}[\zeta]$. Finally, as both ζ and $\bar{\zeta}$

$\nu(s) = \frac{x+y\bar{\zeta}}{x+y\zeta} = 1 + \varpi^c z$ for some $z \in \mathcal{O}_{E_\tau}$, hence $x + y\bar{\zeta} = x + y\zeta + \varpi^c z s$, i.e., $y(\bar{\zeta} - \zeta) = \varpi^c z s$ thus $y = \varpi^c \frac{zs}{\bar{\zeta} - \zeta} \in \varpi^c \mathcal{O}_{E_\tau}$ and $s = x + y\zeta \in \mathcal{O}_c \cap \mathcal{O}_{E_\tau}^\times = \mathcal{O}_c^\times$. Accordingly, one has $\nu^{-1}(1 + \varpi^c \mathcal{O}_{E_\tau}) = \varpi^{\mathbb{Z}} \mathcal{O}_c^\times$ and the last equality $E_\tau^1 \cap (1 + \varpi^c \mathcal{O}_{E_\tau})$ follows directly from the surjectivity of $\nu : E_\tau^\times \rightarrow E_\tau^1$. \square

2.2.4 The local conductor formula.

We are now ready to prove the *local conductor formula*, due to Jetchev:

Proposition 2.2.7 ([27], Theorem 1.3, (ii)). *Let $(L_V, L_W) \in \text{Hyp}$, and set $(a, b) := \text{inv}_\tau(L_V, L_W)$ and $c := \min(a, 2b)$. One has*

$$\det(\text{Stab}_{H_\tau}(L_V, L_W)) = \mathcal{O}_c^1.$$

Proof. As in the *standard situation*, we may fix a special apartment \mathcal{A}' together with an apartment \mathcal{A} of \mathbf{B}_W , attached respectively to Witt bases $\mathcal{B}' = \{e'_+, e'_0, e'_-\}$ and $\mathcal{B} = \{e_+, e_0, e_-\}$, containing respectively $\{L_V, \text{pr}_W(L_V), L_W\}$ and $\{\text{pr}_W(L_V), L_W\}$ and such that $\mathcal{A}' \cap \mathbf{B}_W$ is the half-line $\ell = \mathcal{A}'_+ \subset \mathcal{A}'$ starting from $\text{pr}_W(L_V)$. By Lemma 2.2.1, the transition matrix $S := S_{\mathcal{B}, \mathcal{B}'}$ between \mathcal{B} and \mathcal{B}' has the form $\begin{bmatrix} 1 & \beta & \gamma \\ 0 & 1 & -\bar{\beta} \\ 0 & 0 & 1 \end{bmatrix}$, with $\beta, \gamma \in \mathcal{O}_{E_\tau}^\times$ and $\beta\bar{\beta} + \gamma + \bar{\gamma} = 0$.

Let $g \in G_{W, \tau}$ and set $h = \Delta(g) \in H_\tau$. Let M and N be the matrices of g in the bases \mathcal{B} and \mathcal{B}' respectively. Then M is of the form $\begin{bmatrix} x & 0 & y \\ 0 & 1 & 0 \\ z & 0 & t \end{bmatrix} \in \text{GL}_3(E_\tau)$, with $x, y, z, t \in E_\tau$, and one has $N = S^{-1}MS$. As $S^{-1} = \begin{bmatrix} 1 & -\beta & \bar{\gamma} \\ 0 & 1 & \bar{\beta} \\ 0 & 0 & 1 \end{bmatrix}$, one gets

$$N = \begin{bmatrix} x + \bar{\gamma}z & \beta(x - 1 + \bar{\gamma}z) & y + \gamma\bar{\gamma}z + \gamma(x - 1) + \bar{\gamma}(t - 1) \\ \bar{\beta}z & 1 + \beta\bar{\beta}z & \bar{\beta}(\gamma z + t - 1) \\ z & \beta z & \gamma z + t \end{bmatrix}, \quad (2.12)$$

where we used the relation $\beta\bar{\beta} + \gamma + \bar{\gamma} = 0$ to simplify the (1, 3) component. Assume that h lies in $\text{Stab}_{H_\tau}(L_V, L_W)$. As g fixes both L_V and L_W , it fixes the whole segment between these two vertices: in particular, one has $g \cdot \text{pr}_W(L_V) = \text{pr}_W(L_V)$.

We shall work in the basis \mathcal{B}' . In this basis, the equality $g \cdot \text{pr}_W(L_V) = \text{pr}_W(L_V)$ expresses as $N \in \text{GL}_3(\mathcal{O}_{E_\tau})$. The equality $g \cdot L_V = L_V$ corresponds to

$$\text{diag}(\varpi^{-a}, 1, \varpi^a) N \text{diag}(\varpi^a, 1, \varpi^{-a}) \in \text{GL}_3(\mathcal{O}_{E_\tau}),$$

are $(q^2 - 1)$ -th roots of unity, the element $\frac{\bar{\zeta}}{\zeta} - 1 \neq 0$ is a root of the polynomial $\frac{(X+1)^{q^2-1}-1}{X}$, which rewrites as $(X+1)^{q^2-2} + (X+1)^{q^2-3} + \dots + (X+1) + 1$. The latter has a constant term equal to $q^2 - 1 \in \mathcal{O}_{E_\tau}^\times$, which shows that $\frac{\bar{\zeta}}{\zeta} - 1 \in \mathcal{O}_{E_\tau}^\times$, hence $\bar{\zeta} - \zeta = \zeta(\frac{\bar{\zeta}}{\zeta} - 1) \in \mathcal{O}_{E_\tau}^\times$.

whereas $g \cdot L_W = L_W$ expresses as

$$\text{diag}(\varpi^b, 1, \varpi^{-b}) N \text{diag}(\varpi^{-b}, 1, \varpi^b) \in \text{GL}_3(\mathcal{O}_{E_\tau}).$$

This induces $N \in \text{GL}_3(\mathcal{O}_{E_\tau}) \cap (\text{diag}(\varpi^a, 1, \varpi^{-a}) \text{GL}_3(\mathcal{O}_{E_\tau}) \text{diag}(\varpi^{-a}, 1, \varpi^a))$

$$= \text{GL}_3(\mathcal{O}_{E_\tau}) \cap \begin{bmatrix} \mathcal{O}_{E_\tau} & \varpi^a \mathcal{O}_{E_\tau} & \varpi^{2a} \mathcal{O}_{E_\tau} \\ \varpi^{-a} \mathcal{O}_{E_\tau} & \mathcal{O}_{E_\tau} & \varpi^a \mathcal{O}_{E_\tau} \\ \varpi^{-2a} \mathcal{O}_{E_\tau} & \varpi^{-a} \mathcal{O}_{E_\tau} & \mathcal{O}_{E_\tau} \end{bmatrix}. \quad (2.13)$$

Similarly, one has $N \in \text{GL}_3(\mathcal{O}_{E_\tau}) \cap (\text{diag}(\varpi^{-b}, 1, \varpi^b) \text{GL}_3(\mathcal{O}_{E_\tau}) \text{diag}(\varpi^b, 1, \varpi^{-b}))$

$$= \text{GL}_3(\mathcal{O}_{E_\tau}) \cap \begin{bmatrix} \mathcal{O}_{E_\tau} & \varpi^{-b} \mathcal{O}_{E_\tau} & \varpi^{-2b} \mathcal{O}_{E_\tau} \\ \varpi^b \mathcal{O}_{E_\tau} & \mathcal{O}_{E_\tau} & \varpi^{-b} \mathcal{O}_{E_\tau} \\ \varpi^{2b} \mathcal{O}_{E_\tau} & \varpi^b \mathcal{O}_{E_\tau} & \mathcal{O}_{E_\tau} \end{bmatrix}. \quad (2.14)$$

Combining (2.12) with (2.13) and (2.14) gives us, by analyzing component-wise:

$$\left\{ \begin{array}{ll} x, y, z, t \in \mathcal{O}_{E_\tau}, & xt - yz \in \mathcal{O}_{E_\tau}^\times & (M \in \text{GL}_3(\mathcal{O}_{E_\tau})) \\ & \text{ord}_\varpi(z) \geq 2b, & (3, 1) \\ \text{ord}_\varpi(\beta(x-1)) = \text{ord}_\varpi(x-1) \geq \min(a, 2b) = c, & & (1, 2) \\ \text{ord}_\varpi(\bar{\beta}(t-1)) = \text{ord}_\varpi(t-1) \geq \min(a, 2b) = c, & & (3, 2) \\ & \text{ord}_\varpi(y) \geq c. & (1, 3) \end{array} \right.$$

Consequently, one obtains $\text{ord}_\varpi(\det(g) - 1) = \text{ord}_\varpi(\det M - 1) = \text{ord}_\varpi(xt - yz - 1) = \text{ord}_\varpi(x(t-1) - yz + (x-1)) \geq \min(\text{ord}_\varpi(t-1), \text{ord}_\varpi(yz), \text{ord}_\varpi(x-1)) \geq c$. According to Lemma 2.2.6, this gives

$$\det(\text{Stab}_{H_\tau}(L_V, L_W)) \subset E_\tau^1 \cap (1 + \varpi^c \mathcal{O}_{E_\tau}) = \mathcal{O}_c^1.$$

We still need to show the surjectivity of $\det : \text{Stab}_{H_\tau}(L_V, L_W) \rightarrow \mathcal{O}_c^1$. Let $s \in \mathcal{O}_c^1$, and let us construct some $g \in G_{W, \tau}$ such that $\Delta(g) \in \text{Stab}_{H_\tau}(L_V, L_W)$ and $\det(g) = s$.

- Assume first that $c = a$, and let $\lambda \in \mathcal{O}_c^\times = \mathcal{O}_{F_\tau}^\times (1 + \varpi^c \mathcal{O}_{E_\tau})$ be such that $s = \nu(\lambda) = \frac{\bar{\lambda}}{\lambda}$. As ν is trivial on F_τ^\times , one may assume that $\lambda \in 1 + \varpi^c \mathcal{O}_{E_\tau}$ without loss of generality. Accordingly, one may find $u, v \in \mathcal{O}_{E_\tau}$ such that $\bar{\lambda} = 1 + \varpi^c u$ and $\lambda^{-1} = 1 + \varpi^c v$. Let g be the element of $\text{GL}(V_\tau)$ whose matrix in \mathcal{B} is

$$M := \begin{bmatrix} \bar{\lambda} & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \in \text{GL}_3(E_\tau),$$

with $y := -\varpi^c \bar{\lambda}(\gamma u - \bar{\gamma} u) \in \varpi^c \mathcal{O}_{E_\tau}$.

Clearly, one has $\det(M) = \bar{\lambda}\lambda^{-1} = s$ and, still denoting by J the matrix $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, we have

$$\begin{aligned} {}^t M J \bar{M} &= \begin{bmatrix} \bar{\lambda} & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & \lambda^{-1} \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \lambda & 0 & \bar{y} \\ 0 & 1 & 0 \\ 0 & 0 & \bar{\lambda}^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \bar{\lambda}\bar{\lambda}^{-1} = 1 \\ 0 & 1 & 0 \\ \lambda\lambda^{-1} = 1 & 0 & \bar{\lambda}^{-1}y + \lambda^{-1}\bar{y} \end{bmatrix} \end{aligned}$$

As $\bar{\lambda}^{-1}y = -\varpi^c(\gamma u - \bar{\gamma}\bar{u})$, one gets $\bar{\lambda}^{-1}y + \bar{\lambda}^{-1}\bar{y} = -\varpi^c(\gamma u - \bar{\gamma}\bar{u} + \bar{\gamma}\bar{u} - \gamma u) = 0$, hence ${}^t M J \bar{M} = J$. This shows that g fixes e_0 and belongs to $G_{V,\tau}$, i.e., that $g \in G_{W,\tau}$. Set N to be the matrix of g in the basis \mathcal{B}' . By (2.12), one has

$$\begin{aligned} N = S^{-1}MS &= \begin{bmatrix} \bar{\lambda} & \beta(\bar{\lambda} - 1) & y + \gamma(\bar{\lambda} - 1) + \bar{\gamma}(\lambda^{-1} - 1) \\ 0 & 1 & \bar{\beta}(\lambda^{-1} - 1) \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 1 + \varpi^c u & \varpi^c \beta u & y + \varpi^c(\gamma u + \bar{\gamma}v) \\ 0 & 1 & \varpi^c \bar{\beta}v \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \end{aligned}$$

As $c = a$, the preceding matrix has entries (1,2) and (2,3) in $\varpi^a \mathcal{O}_{E_\tau}$. As $1 = \lambda\lambda^{-1} = (1 + \varpi^c \bar{u})(1 + \varpi^c v) = 1 + \varpi^c(\bar{u} + v) + \varpi^{2c}\bar{u}v$, one has $\bar{u} + v \equiv 0 \pmod{\varpi^c \mathcal{O}_{E_\tau}}$, hence $\gamma u + \bar{\gamma}v \equiv \gamma u - \bar{\gamma}\bar{u} \pmod{\varpi^c \mathcal{O}_{E_\tau}}$. But $\bar{\lambda} = 1 + \varpi^c u \equiv 1 \pmod{\varpi^c \mathcal{O}_{E_\tau}}$, thus $\gamma u + \bar{\gamma}v \equiv \bar{\lambda}(\gamma u - \bar{\gamma}\bar{u}) \pmod{\varpi^c \mathcal{O}_{E_\tau}}$, i.e.,

$$\varpi^c(\gamma u + \bar{\gamma}v) \equiv \varpi^c \bar{\lambda}(\gamma u - \bar{\gamma}\bar{u}) = -y \pmod{\varpi^{2c} \mathcal{O}_{E_\tau}},$$

which gives finally $y + \varpi^c(\gamma u + \bar{\gamma}v) \in \varpi^{2c} \mathcal{O}_{E_\tau} = \varpi^{2a} \mathcal{O}_{E_\tau}$. In other words, the unitary matrix N has the shape required by both (2.13) and (2.14), which is equivalent to saying that $\Delta(g)$ is indeed an element of $\text{Stab}_{H_\tau}(L_V, L_W)$ which satisfies $\det(g) = s$.

— If $c = 2b$, we set $g \in \text{GL}(V_\tau)$ to be the element whose matrix in \mathcal{B} is

$$M := \begin{bmatrix} 1 - \bar{\gamma}z & 0 & \gamma\bar{\gamma}z \\ 0 & 1 & 0 \\ z & 0 & 1 - \gamma z \end{bmatrix} \in \text{GL}_3(E_\tau),$$

where we set $z := \frac{1-s}{\gamma+\bar{\gamma}} \in \varpi^c \mathcal{O}_{E_\tau}$ (which is well-defined, for $\gamma + \bar{\gamma} = -\beta\bar{\beta}$ belongs to $\mathcal{O}_{E_\tau}^\times$). As a first check, one has $\det(g) = (1 - \bar{\gamma}z)(1 - \gamma z) - \gamma\bar{\gamma}z^2 = 1 - z(\gamma + \bar{\gamma}) = s$.

Notice that, as $s\bar{s} = 1$, one has

$$z\bar{z}(\gamma + \bar{\gamma}) = \frac{(1-s)(1-\bar{s})}{\gamma + \bar{\gamma}} = \frac{(1-s) + (1-\bar{s})}{\gamma + \bar{\gamma}} = z + \bar{z}. \quad (2.15)$$

We have

$$\begin{aligned} {}^t M J \overline{M} &= \begin{bmatrix} 1 - \overline{\gamma}z & 0 & z \\ 0 & 1 & 0 \\ \gamma\overline{\gamma}z & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \overline{z} & 0 & 1 - \overline{\gamma}z \\ 0 & 1 & 0 \\ 1 - \gamma\overline{z} & 0 & \gamma\overline{\gamma}z \end{bmatrix} \\ &= \begin{bmatrix} \overline{z}(1 - \gamma z) + z(1 - \overline{\gamma}z) & 0 & (1 - \gamma z)(1 - \overline{\gamma}z) + \gamma\overline{\gamma}z\overline{z} \\ 0 & 1 & 0 \\ \gamma\overline{\gamma}z\overline{z} + (1 - \overline{\gamma}z)(1 - \gamma z) & 0 & \gamma\overline{\gamma}z(1 - \overline{\gamma}z) + \gamma\overline{\gamma}z(1 - \gamma z) \end{bmatrix}. \end{aligned}$$

By (2.15), one has $\gamma^2 z \overline{z} = \gamma(z + \overline{z}) - \gamma\overline{\gamma}z\overline{z}$ and $\overline{\gamma}^2 z = \overline{\gamma}(z + \overline{z}) - \gamma\overline{\gamma}z\overline{z}$, which shows that the preceding matrix has entries (1, 3) and (3, 1) both equal to 1. Applying again (2.15) to the entries (1, 1) and (3, 3) finally gives ${}^t M J \overline{M} = J$. As g fixes e_0 , one gets $g \in G_{W,\tau}$. Finally, let N be the matrix of g with respect to the basis \mathcal{B}' . By (2.12), and for $\beta\overline{\beta} + \gamma + \overline{\gamma} = 0$, one has

$$N = S^{-1} M S = \begin{bmatrix} 1 & 0 & 0 \\ \overline{\beta}z & 1 + \beta\overline{\beta}z & 0 \\ z & \beta z & 1 \end{bmatrix}.$$

Accordingly, N is unitary and, as $z \in \varpi^c \mathcal{O}_{E_\tau} = \varpi^{2b} \mathcal{O}_{E_\tau}$, its entries satisfy the conditions required by (2.13) and by (2.14), which also amounts to saying that $\Delta(g)$ is an element of $\text{Stab}_{H_\tau}(L_V, L_W)$. This finishes the proof. \square

For all $g_\tau \in G_\tau$, the group $\det(H_\tau \cap g_\tau K_\tau g_\tau^{-1})$ is a compact open subgroup of $U(1)_{E/F}(F_\tau) = \mathcal{O}_0^1$, thus contains some term \mathcal{O}_c^1 ($c \geq 0$) from the decreasing filtration (2.11).

Definition 2.2.4. *Let $g_\tau \in G_\tau$. The local conductor of g_τ , denoted $c_\tau(g_\tau)$, is defined as the minimal integer $c \geq 0$ such that*

$$\mathcal{O}_c^1 \subset \det(H_\tau \cap g_\tau K_\tau g_\tau^{-1}).$$

As seen in § 1.2.3, the hyperspecial maximal compact subgroup $K_\tau \subset G_\tau$ arises as the stabilizer of the pair $(L_{V,\tau}, L_{W,\tau}) := (L_V \otimes_{\mathcal{O}_E} \mathcal{O}_{E_\tau}, L_W \otimes_{\mathcal{O}_E} \mathcal{O}_{E_\tau}) \in \text{Hyp}$ of self-dual lattices, where the lattices $L_V := L_W \oplus L_D$ are part of the construction of the integral models $\underline{\mathbf{G}}_V$ and $\underline{\mathbf{G}}_W$ ⁽¹⁴⁾. One may reformulate Proposition 2.2.7 in the following way:

Corollary 2.2.8. *For all $g_\tau = (g_{V,\tau}, g_{W,\tau}) \in G_{V,\tau} \times G_{W,\tau} = G_\tau$, one has*

$$c_\tau(g_\tau) = \min(a, 2b), \quad \text{with } (a, b) := \text{inv}_\tau(g_\tau \cdot L_{V,\tau}, g_\tau \cdot L_{W,\tau}).$$

(14). Accordingly, the vertices $L_{V,\tau} = L_{W,\tau} \oplus L_{D,\tau} =: \iota(L_{W,\tau})$ coincide in \mathbf{B}_V .

2.3 The Iwahori filtration and the *level-wise* vertical relations.

As since the beginning of this chapter, we continue to fix an allowable inert place τ of F together with the induced buildings $\mathbf{B}_W \subset \mathbf{B}_V$ attached to the local unitary groups $G_{W,\tau} \subset G_{V,\tau}$. We recall that the choice of global lattices $L_V := L_W \oplus L_D$ (with $L_D := \mathcal{O}_E e_D$ being the only global self-dual \mathcal{O}_E -lattice in D) made at § 1.2.3, and the construction of the base compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$, ensure that the local component $K_\tau = K_{V,\tau} \times K_{W,\tau}$ is equal to the stabilizer $\text{Stab}_{G_\tau}(L_{V,\tau}, L_{W,\tau})$, where $L_{\star,\tau} = L_\star \otimes_{\mathcal{O}_E} \mathcal{O}_{E_\tau}$ is a self-dual \mathcal{O}_{E_τ} -lattice in \star_τ , for $\star \in \{V, W\}$. As mentioned in the footnote above, the equality $L_{V,\tau} = L_{W,\tau} \oplus L_{D,\tau} =: \iota(L_{W,\tau})$ imply that the local lattices $L_{V,\tau}$ and $L_{W,\tau}$ define the same hyperspecial vertex of the building, which we denote by $x \in \text{Hyp}_W \subset \text{Hyp}_V$.

We fix, once and for all, a special apartment \mathcal{A}' of \mathbf{B}_V with base point x . The base vertex x shall sometimes be referred to as *the origin* of the buildings \mathbf{B}_V and \mathbf{B}_W . According to Corollary 2.2.4, the H_τ -orbit of \mathcal{A}' is the whole set \mathfrak{S} of special apartments: this means that, in terms the local component of special cycles at τ , choosing a different apartment $\mathcal{A}'' = h \cdot \mathcal{A}'$, for some $h \in G_{W,\tau}$, will just correspond to shifting special cycles by the action of $\text{Art}_\tau^1(\det h) \in \text{Gal}(E(\infty)_\tau/E_\tau)$, by Proposition 1.42⁽¹⁵⁾. The Galois groups we are dealing with being all abelian - and because Galois and Hecke actions commute with each other - we deduce that every kind of *distribution relations* which will be established via the choice of \mathcal{A}' will remain unchanged in \mathcal{A}'' . We place ourselves in the *standard situation* of § 2.2.1, and let $\mathcal{B}' = \{e'_+, e'_0, e'_-\}$ and $\mathcal{B} = \{e_+, e_0, e_-\}$, with $e_0 = e_D \in D_\tau$, be Witt bases of V_τ such that $\mathcal{A}' = \mathcal{A}_{\mathcal{B}'}$ and such that the apartment $\mathcal{A} := \mathcal{A}_{\mathcal{B}}$ of \mathbf{B}_W satisfies $\mathcal{A}_{\mathcal{B}}^+ = \mathcal{A}_{\mathcal{B}'}^+ = \mathcal{A}' \cap \mathbf{B}_W$. Accordingly, the base vertex x satisfies $x = \langle e_+, e_0, e_- \rangle = \langle e'_+, e'_0, e'_- \rangle$.

Recall that, if A is any ring and if X, Y are sets, the free A -modules $A[X]$, $A[Y]$ and $A[X \times Y]$ generated by X, Y and $X \times Y$ respectively, satisfy $A[X \times Y] \simeq A[X] \otimes_A A[Y]$ and $\text{End}_A(A[X]) \otimes_A \text{End}_A(A[Y]) \hookrightarrow \text{End}_A(A[X] \otimes_A A[Y]) = \text{End}_A(A[X \times Y])$. Let p be the residue characteristic of τ . We let $\mathbb{Z}_{(p)}$ denote the localization of \mathbb{Z} at the prime ideal (p) , and let $\mathbb{Z}[\text{Hyp}_V]$, $\mathbb{Z}[\text{Hyp}_W]$ and $\mathbb{Z}[\text{Hyp}] \simeq \mathbb{Z}[\text{Hyp}_V] \otimes_{\mathbb{Z}} \mathbb{Z}[\text{Hyp}_W]$ (resp. $\mathbb{Z}_{(p)}[\text{Hyp}_V] = \mathbb{Z}[\text{Hyp}_V] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, $\mathbb{Z}_{(p)}[\text{Hyp}_W] = \mathbb{Z}[\text{Hyp}_W] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(p)}[\text{Hyp}] = \mathbb{Z}_{(p)}[\text{Hyp}_V] \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}[\text{Hyp}_W] = \mathbb{Z}[\text{Hyp}] \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$) denote the free abelian groups (resp. the free $\mathbb{Z}_{(p)}$ -modules) generated by the sets of hyperspecial vertices of \mathbf{B}_V , of \mathbf{B}_W and of $\mathbf{B}_V \times \mathbf{B}_W$ respectively. Following ([8], 3.2) we set

$$R_V := \text{End}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}[\text{Hyp}_V]), \quad R_W := \text{End}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}[\text{Hyp}_W])$$

and

$$R := R_V \otimes_{\mathbb{Z}} R_W \subset \text{End}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}[\text{Hyp}]).$$

(15). See Remark 3.1.1 of the next chapter for the definition of Art_τ^1

The natural left-action of $G_{\star,\tau}$ by isometries on Hyp_\star induces maps

$$G_{\star,\tau} \longrightarrow \text{End}_{\mathbb{Z}}(\mathbb{Z}[\text{Hyp}_\star]) \hookrightarrow R_\star, \quad \forall \star \in \{V, W\},$$

hence a map

$$G_\tau \longrightarrow \text{End}_{\mathbb{Z}}(\mathbb{Z}[\text{Hyp}_V]) \otimes \text{End}_{\mathbb{Z}}(\mathbb{Z}[\text{Hyp}_W]) \subset R_V \otimes R_W = R. \quad (2.16)$$

If $\mathfrak{X} = \sum_{L \in \text{Hyp}_V} a(L)L$ belongs to $\mathbb{Z}_{(p)}[\text{Hyp}_V]$, we define the support of \mathfrak{X} to be the finite set $\text{Supp} \mathfrak{X} := \{L \in \text{Hyp}_V; a(L) \neq 0\}$.

Remark 2.3.1. *The importance of the localization at the prime (p) - which is required by the definition of the successor and predecessor operators at the origin x , in a view towards Lemma 2.3.3 below - is rather limited. It can (and will) be ignored whenever dealing with segments of length $n \geq 1$, where the origin x will not intervene.*

2.3.1 Global and local Hecke algebras.

Recall that we defined in § 1.2.5 the local Hecke algebras $\mathcal{H}_{\star,\tau} := \mathcal{H}_{\mathbb{Z}}(\mathbf{G}_{\star,\tau} // K_{\star,\tau})$, for $\star \in \{V, W, \emptyset\}$, to be the rings $\mathcal{C}_c^\infty(G_{\star,\tau} // K_{\star,\tau}, \mathbb{Z})$ of continuous (for the discrete topology on \mathbb{Z}), compactly supported, $K_{\star,\tau}$ -bi-invariant \mathbb{Z} -valued functions on $G_{\star,\tau}$. The local Hecke algebra $\mathcal{H}_{\star,\tau}$ of hyperspecial level $K_{\star,\tau}$ is sometimes called the *spherical Hecke algebra*, and is isomorphic to the ring $\mathbb{Z}[K_{\star,\tau} \backslash G_{\star,\tau} / K_{\star,\tau}]$, endowed with the convolution product $*$ of functions⁽¹⁶⁾. The latter turns out to be isomorphic to the endomorphism ring $\text{End}_{\mathbb{Z}[G_{\star,\tau}]}(\mathbb{Z}[G_{\star,\tau} / K_{\star,\tau}])^{\text{op}}$.

We recall that this last isomorphism is obtained by mapping, for all $g \in G_{\star,\tau}$, the indicator function $\mathbf{1}_{K_{\star,\tau}gK_{\star,\tau}} \in \mathcal{H}_{\star,\tau}$ to the $\mathbb{Z}[G_{\star,\tau}]$ -equivariant endomorphism $\rho(\mathbf{1}_{K_{\star,\tau}gK_{\star,\tau}})$ of $\mathbb{Z}[G_{\star,\tau} / K_{\star,\tau}]$ obtained by mapping the unit coset $[K_{\star,\tau}] \in \mathbb{Z}[G_{\star,\tau} / K_{\star,\tau}]$ to the sum $\sum_{i=1}^{r_g} g_i K_{\star,\tau}$, where $K_{\star,\tau}gK_{\star,\tau}$ splits as the disjoint union $\bigsqcup_{i=1}^{r_g} g_i K_{\star,\tau}$. By Lemma 2.1.7, the map $G_{\star,\tau} \rightarrow \text{Hyp}_\star$, $g \mapsto g \cdot x$ is surjective for all $\star \in \{V, W\}$. This allows us to identify Hyp_\star with the quotient $G_{\star,\tau} / K_{\star,\tau}$ for all $\star \in \{V, W, \emptyset\}$. Consequently, the local algebra $\mathcal{H}_{\star,\tau}$ acts on $\mathbb{Z}[\text{Hyp}_\star]$ (for $\star \in \{V, W\}$), via

$$\mathbf{1}_{K_{\star,\tau}gK_{\star,\tau}} \cdot (g' \cdot x) := \sum_{i=1}^{r_h} (g' g_i) \cdot x, \quad (2.17)$$

for all $g' \in G_{\star,\tau}$ and all $g \in G_{\star,\tau}$, with $K_{\star,\tau}gK_{\star,\tau} = \bigsqcup_{i=1}^{r_h} g_i K_{\star,\tau}$. Accordingly, one obtains embeddings

(16). Notice that the spherical Hecke algebra \mathcal{H}_τ is a commutative ring, thanks to the so-called *Satake isomorphism*. Roughly speaking, one may see the Satake isomorphism as establishing a link between \mathcal{H}_τ and the commutative Hecke algebra $\mathcal{H}_{\mathbb{Z}}(T_{\mathcal{B}} // T_{\mathcal{B},K})$, where $T_{\mathcal{B}} = T_{\mathcal{B},V} \times T_{\mathcal{B},W} \subset G_{V,\tau} \times G_{W,\tau}$ is the unitary torus attached the Witt basis $\mathcal{B} = \{e_+, e_0, e_-\}$ of V_τ (recall that $\{e_+, e_-\}$ is a Witt basis of W_τ , and $e_0 = e_D$ satisfies $\langle e_0, e_0 \rangle = 1$), and where we set $T_{\mathcal{B},K} := T_{\mathcal{B}} \cap K$ (or more precisely, the fixed part $\mathcal{H}_{\mathbb{Z}}(T_{\mathcal{B}}, T_{\mathcal{B},K})^W$ of $\mathcal{H}_{\mathbb{Z}}(T_{\mathcal{B}}, T_{\mathcal{B},K})$ under the action of the Weyl group $W := N_{G_\tau}(T_{\mathcal{B}}) / T_{\mathcal{B}}$). We refer to ([19], §7.5) for some details.

$$\mathcal{H}_{\star,\tau} \simeq \text{End}_{\mathbb{Z}[G_{\star,\tau}]}(\mathbb{Z}[\text{Hyp}_{\star}]) \hookrightarrow \text{End}_{\mathbb{Z}}(\mathbb{Z}[\text{Hyp}_{\star}]) \hookrightarrow R_{\star}, \quad (2.18)$$

$$\mathbf{1}_{K_{\star,\tau}gK_{\star,\tau}} \longmapsto \rho(\mathbf{1}_{K_{\star,\tau}gK_{\star,\tau}}),$$

for all $\star \in \{V, W\}$, hence an embedding $\mathcal{H}_{\tau} \simeq \mathcal{H}_{V,\tau} \otimes_{\mathbb{Z}} \mathcal{H}_{W,\tau} \xrightarrow{(2.18)} R_V \otimes R_W = R$.

Definition 2.3.1 (Elements δ_V , δ_W and δ). *We let $\delta_V \in G_{V,\tau}$ be the element whose matrix is $\text{diag}(\varpi^{-1}, 1, \varpi)$ with respect to the basis \mathcal{B}' of V_{τ} . We set $\delta_W \in G_{W,\tau}$ to be the element whose matrix is $\text{diag}(\varpi^{-1}, \varpi)$ with respect to the basis $\{e_+, e_-\}$ of W_{τ} (i.e., such that $\iota(\delta_W) \in G_{V,\tau}$ has matrix $\text{diag}(\varpi^{-1}, 1, \varpi)$ with respect to \mathcal{B}). We finally set $\delta := (\delta_V^{-1}, \delta_W) \in G_{\tau}$.*

Accordingly, the vertex $\delta_V^k \cdot x \in \mathcal{A}'$ (resp. $\iota(\delta_W)^k \cdot x \in \mathcal{A}$) has parameter k with respect to \mathcal{B}' (resp. \mathcal{B}), for all $k \in \mathbb{Z}$. This gives $\mathcal{A}' = \{\delta_V^k \cdot x; k \in \mathbb{Z}\}$, $\mathcal{A} = \{\iota(\delta_W)^k \cdot x; k \in \mathbb{Z}\}$ and $\mathcal{A}' \cap \mathbf{B}_W = \{\delta_V^k \cdot x; k \geq 0\} = \{\iota(\delta_W)^k \cdot x; k \geq 0\}$. Let us define the following operators:

Definition 2.3.2 (Hecke operators $t_{1,0}$, $t_{0,1}$ and t). *We set $t_{1,0}$ to be the element $\rho(\mathbf{1}_{K_{V,\tau}\delta_V^{-1}K_{V,\tau}}) = \rho(\mathbf{1}_{K_{V,\tau}\delta_V K_{V,\tau}})^{(17)}$ of $\mathcal{H}_{V,\tau} \simeq \text{End}_{\mathbb{Z}[G_{V,\tau}]}(\mathbb{Z}[\text{Hyp}_V])$. We set $t_{0,1}$ to be the element $\rho(\mathbf{1}_{K_{W,\tau}\delta_W K_{W,\tau}})$ of $\mathcal{H}_{W,\tau} \simeq \text{End}_{\mathbb{Z}[G_{W,\tau}]}(\mathbb{Z}[\text{Hyp}_W])$. In the following, we shall slightly abuse notations and see both operators as elements of \mathcal{H}_{τ} , i.e., we identify $t_{1,0}$ and $t_{0,1}$ with the elements $t_{1,0} \otimes 1$ and $1 \otimes t_{0,1}$ of $\text{End}_{\mathbb{Z}[G_{\tau}]}(\mathbb{Z}[\text{Hyp}]) = \mathcal{H}_{\tau}$. We finally set $t := t_{1,0}t_{0,1} = t_{1,0} \otimes t_{0,1} = \rho(\mathbf{1}_{K_{\tau}\delta K_{\tau}}) \in \mathcal{H}_{\tau}$.*

By construction, for all $\star \in \{V, W\}$ and all $g, g' \in G_{\star,\tau}$, the double coset operator $\rho(\mathbf{1}_{K_{\star,\tau}gK_{\star,\tau}}) \in \text{End}_{\mathbb{Z}[G_{\star,\tau}]}(\mathbb{Z}[\text{Hyp}_{\star}])$ - which acts by a finite sum of right multiplications - commutes with the natural action of g' (by left-multiplication) on $\mathbb{Z}[\text{Hyp}_{\star}]$ described in (2.16). In the special case of $t_{1,0}$ and $t_{0,1}$, this commutation relation can also be obtained by expressing $t_{1,0}$ and $t_{0,1}$ as *adjacency operators*. Namely:

Lemma 2.3.1. (i) *The operator $t_{1,0}$ maps any element L of Hyp_V to the sum:*

$$\sum_{\substack{L' \in \text{Hyp}_V \\ \text{dist}(L, L')=1}} L' \in \mathbb{Z}[\text{Hyp}_V]$$

(ii) *The operator $t_{0,1}$ maps any element L of Hyp_W to the sum*

$$\sum_{\substack{L' \in \text{Hyp}_W \\ \text{dist}(L, L')=1}} L' \in \mathbb{Z}[\text{Hyp}_W].$$

(17). Indeed, one has $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{diag}(\varpi^{-1}, 1, \varpi) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \text{diag}(\varpi, 1, \varpi^{-1})$, hence $K_{V,\tau}\delta_V K_{V,\tau} = K_{V,\tau}\delta_V^{-1} K_{V,\tau}$.

Proof. Let us show the point (i) (the point (ii) being an exact analogue). If $L \in \text{Hyp}_V$ is a hyperspecial vertex, one may choose some $g_0 \in G_{V,\tau}$ such that $L = g_0 \cdot x$. It suffices to show that the map:

$$\begin{aligned} \nu^{(0)} : K_{V,\tau} \delta_V K_{V,\tau} &\longrightarrow \text{Hyp}_V, \\ g &\longmapsto (g_0 g) \cdot x \end{aligned}$$

induces a bijection between the quotient set $K_{V,\tau} \delta_V K_{V,\tau} / K_{V,\tau}$ and the set of hyperspecial neighbours $\mathcal{N}(L) := \{L' \in \text{Hyp}_V; \text{dist}(L, L') = 1\}$. That $\nu^{(0)}$ is right $K_{V,\tau}$ -invariant is an immediate consequence of the equality $K_{V,\tau} = \text{Stab}_{G_{V,\tau}}(x)$, which implies straight away that $\nu^{(0)} : K_{V,\tau} \delta_V K_{V,\tau} / K_{V,\tau} \rightarrow \text{Hyp}_V$ is also injective. That $\nu^{(0)}$ indeed takes values in $\mathcal{N}(L)$ follows from the equalities:

$$\text{dist}(g_0(k\delta_V k') \cdot x, L) = \text{dist}(g_0(k\delta_V k') \cdot x, g_0 \cdot x) = \text{dist}((k\delta_V k') \cdot x, x)$$

$$\stackrel{k' \cdot x = x}{=} \text{dist}(k\delta_V \cdot x, x) = \text{dist}(\delta_V \cdot x, k^{-1} \cdot x) \stackrel{k^{-1} \cdot x = x}{=} \text{dist}(\delta_V \cdot x, x) = 1,$$

for all $k, k' \in K_{V,\tau}$.

Now, let $L' \in \text{Hyp}_V$ be such that $\text{dist}(L, L') = 1$, i.e., $\text{dist}(g_0 \cdot x, L') = \text{dist}(x, g_0^{-1} \cdot L') = 1$: one may then find some apartment \tilde{A} containing x and $g_0^{-1} \cdot L'$, attached to some Witt basis $\tilde{\mathcal{B}} = \{\tilde{e}_+, \tilde{e}_0, \tilde{e}_-\}$ such that x and $g_0^{-1} \cdot L'$ have respective parameters 0 and 1 with respect to $\tilde{\mathcal{B}}$. This gives $x = \langle \tilde{e}_+, \tilde{e}_0, \tilde{e}_- \rangle$ and $g_0^{-1} \cdot L' = \langle \varpi^{-1} \tilde{e}_+, \tilde{e}_0, \varpi \tilde{e}_- \rangle$. The element $g \in G_{V,\tau}$ defined by mapping \tilde{e}_\star to e'_\star , for all $\star \in \{+, 0, -\}$, satisfies $g \cdot x = x$ and $g \cdot (g_0^{-1} \cdot L') = \langle \varpi^{-1} e_+, e_0, \varpi e_- \rangle = \delta_V \cdot x$. The former equality ensures that $g \in K_{V,\tau}$, i.e., $g^{-1} \delta_V \in K_{V,\tau} \delta_V K_{V,\tau}$, and the latter gives $L' = (g_0 g^{-1} \delta_V) \cdot x = \nu^{(0)}(g^{-1} \delta_V)$. This shows the surjectivity of $\nu^{(0)}$ and finishes the proof. \square

2.3.2 The Hecke polynomial.

The Shimura datum (\mathbf{G}, X) and the induced co-character $\mu : \mathbb{G}_{m,\mathbb{C}} \hookrightarrow \mathbf{G}_{\mathbb{C}}$, introduced at § 1.1.5, give rise to a polynomial He_τ with coefficients in the local Hecke algebra \mathcal{H}_τ , called the *Hecke polynomial*. The Hecke polynomial⁽¹⁸⁾ was originally defined by Langlands and was studied by Blasius and Rogawski: it is conjectured ([4], §6) to be an annihilator of the geometric Frobenius acting on ℓ -adic étale cohomology of $\text{Sh}_K(\mathbf{G}, X)$ (or intersection cohomology when $F = \mathbb{Q}$), in an attempt to generalize the celebrated Eichler-Shimura congruence relation on the modular curve.

The construction of the Hecke polynomial is dealt with in great generality by Boumasmoud in ([7], IV) who proves an annihilation relation (which he calls *seed relation*, Theorem (18)). We prefer to use the notation He_τ to H_τ (as used in [27], [8] or [31]) so as to avoid confusion with the local subgroup $H_\tau = \Delta(G_{W,\tau}) \subset G_\tau$.

IV.1) involving the Hecke polynomial attached to an unramified reductive group \mathcal{G} over a non-archimedean local field k , with hyperspecial level \tilde{K} , and a class of operators acting on the associated Bruhat-Tits building. In a more down-to-earth way, Jetchev gives a detailed computation of He_τ in our precise setting ([27], 4) which provides us with the following explicit formula:

Theorem 2.3.2 ([27], Theorem 4.1.). *The Hecke polynomial $\text{He}_\tau(z) \in \mathcal{H}_\tau[z]$ at the place τ for the Shimura datum (\mathbf{G}, X) , is given by $\text{He}_\tau(z) = \text{He}_\tau^{(2)}(z) \cdot \text{He}_\tau^{(4)}(z)$ with*

$$\text{He}_\tau^{(2)}(z) = z^2 - q^2 [t_{0,1} - (q-1)]z + q^6, \quad (2.19)$$

and

$$\begin{aligned} \text{He}_\tau^{(4)}(z) &= z^4 - \left[t - (q-1)(t_{1,0} + t_{0,1}) + (q-1)^2 \right] z^3 \\ &\quad + q^2 \left[t_{1,0}^2 + q^2 t_{0,1}^2 - 2(q-1)t_{1,0} - 2q^2(q-1)t_{0,1} - q^4 - 2q^3 + 2q^2 - 2q + 1 \right] z^2 \\ &\quad - q^6 \left[t - (q-1)(t_{1,0} + t_{0,1}) + (q-1)^2 \right] z + q^{12}. \end{aligned}$$

The important feature of He_τ is, for us, that its coefficients are \mathbb{Z} -linear combinations of the double coset operators $t_{1,0}$, $t_{0,1}$ and t .

2.3.3 The partial Hecke operators \mathcal{U} , \mathcal{V} and \mathcal{S} .

We define, after Boumasmoud-Brooks-Jetchev ([8],3.2), the following important elements of R :

Definition 2.3.3 (Operators \mathcal{U} , \mathcal{V} and \mathcal{S}). *We define the following operators on vertices, and extend them $\mathbb{Z}_{(p)}$ -linearly:*

— $\mathcal{U}_V \in R_V$, which maps $L \in \text{Hyp}_V \setminus \{x\}$ to the sum

$$\mathcal{U}_V(L) := \sum_{\substack{L' \in \text{Hyp}_V, \\ \text{dist}(L', L) = 1, \\ \text{dist}(x, L') > \text{dist}(x, L)}} L' \in \mathbb{Z}[\text{Hyp}_V] \subset \mathbb{Z}_{(p)}[\text{Hyp}_V],$$

and which maps x to the sum

$$\mathcal{U}_V(x) := \frac{q^3}{q^3 + 1} \sum_{\substack{L' \in \text{Hyp}_V, \\ \text{dist}(x, L') = 1}} L' \in \mathbb{Z}_{(p)}[\text{Hyp}_V].$$

— $\mathcal{V}_V \in R_V$, which maps $L \in \text{Hyp}_V \setminus \{x\}$ to the unique⁽¹⁹⁾ $L' \in \text{Hyp}_V$ such that $\text{dist}(L, L') = 1$ and $\text{dist}(x, L') < \text{dist}(x, L)$, and which maps x to the sum

$$\mathcal{V}_V(x) := (1 - q)x + \frac{1}{q^3 + 1} \sum_{\substack{L' \in \text{Hyp}_V, \\ \text{dist}(x, L') = 1}} L' \in \mathbb{Z}_{(p)}[\text{Hyp}_V].$$

(19). The unicity of such an L' follows from the non-existence of cycles in the graph $|\mathbf{B}_V|$.

— $\mathcal{S}_V \in R_V$, which maps $L \in \text{Hyp}_V \setminus \{x\}$ to the sum

$$\mathcal{S}_V(L) := L + \sum_{\substack{L' \in \text{Hyp}_V, \\ \text{dist}(L', L) = 1, \\ \text{dist}(x, L') = \text{dist}(x, L)}} L' \in \mathbb{Z}[\text{Hyp}_V] \subset \mathbb{Z}_{(p)}[\text{Hyp}_V],$$

and which maps x to $\mathcal{S}_V(x) := qx$.

These admit the following variants on \mathbf{B}_W :

— $\mathcal{U}_W \in R_W$, which maps $L \in \text{Hyp}_W \setminus \{x\}$ to the sum

$$\mathcal{U}_W(L) := \sum_{\substack{L' \in \text{Hyp}_W, \\ \text{dist}(L', L) = 1, \\ \text{dist}(x, L') > \text{dist}(x, L)}} L' \in \mathbb{Z}[\text{Hyp}_W] \subset \mathbb{Z}_{(p)}[\text{Hyp}_W],$$

and which maps x to the sum

$$\mathcal{U}_W(x) := \frac{q}{q+1} \sum_{\substack{L' \in \text{Hyp}_W, \\ \text{dist}(x, L') = 1}} L' \in \mathbb{Z}_{(p)}[\text{Hyp}_W].$$

— $\mathcal{V}_W \in R_W$, which maps $L \in \text{Hyp}_W \setminus \{x\}$ to the unique $L' \in \text{Hyp}_W$ such that $\text{dist}(L, L') = 1$ and $\text{dist}(x, L') < \text{dist}(x, L)$, and which maps x to the sum

$$\mathcal{V}_W(x) := (1-q)x + \frac{1}{q+1} \sum_{\substack{L' \in \text{Hyp}_W, \\ \text{dist}(x, L') = 1}} L' \in \mathbb{Z}_{(p)}[\text{Hyp}_W].$$

— $\mathcal{S}_W \in R_W$, which maps $L \in \text{Hyp}_W \setminus \{x\}$ to the sum

$$\mathcal{S}_W(L) := L + \sum_{\substack{L' \in \text{Hyp}_W, \\ \text{dist}(L', L) = 1, \\ \text{dist}(x, L') = \text{dist}(x, L)}} L' \in \mathbb{Z}[\text{Hyp}_W] \subset \mathbb{Z}_{(p)}[\text{Hyp}_W],$$

and which maps x to $\mathcal{S}_W(x) := qx$

We finally set $\mathcal{U} := \mathcal{U}_V \otimes \mathcal{U}_W \in R$, $\mathcal{V} := \mathcal{V}_V \otimes \mathcal{V}_W \in R$ and $\mathcal{S} := \mathcal{S}_V \otimes \mathcal{S}_W \in R$.

For all $\star \in \{V, W, \emptyset\}$, the operators \mathcal{V}_\star and \mathcal{U}_\star are referred to as the *predecessor* and *successors* operator in [8], the terminology of *predecessor* being first introduced in ([13], §6.3).

Remark 2.3.2. Notice that if $g \in K_{V, \tau} \subset G_{V, \tau}$ fixes x , then $g \cdot \mathcal{S}_V(x) = g \cdot (qx) = qx = \mathcal{S}_V(g \cdot x)$ and, as g acts on the set $\text{Supp} \mathcal{U}_V(x) \setminus \{x\} = \text{Supp} \mathcal{V}_V(x) \setminus \{x\} = \{L' \in \mathbf{B}_V; \text{dist}(x, L') = 1\}$ by permutation, one obtains that $g \cdot \mathcal{U}_V(x) = \mathcal{U}_V(x) = \mathcal{U}_V(g \cdot x)$, and $g \cdot \mathcal{V}_V(x) = \mathcal{V}_V(x) = \mathcal{V}_V(g \cdot x)$. Similarly, if $h \in K_{W, \tau} \subset G_{W, \tau}$ fixes x , then $h \cdot \mathcal{S}_W(x) = \mathcal{S}_W(x) = \mathcal{S}_W(h \cdot x)$ and one also has $h \cdot \mathcal{U}_W(x) = \mathcal{U}_W(h \cdot x)$, and $h \cdot \mathcal{V}_W(x) = \mathcal{V}_W(h \cdot x)$. If $h \in K_{W, \tau}$ and if $y \neq x$

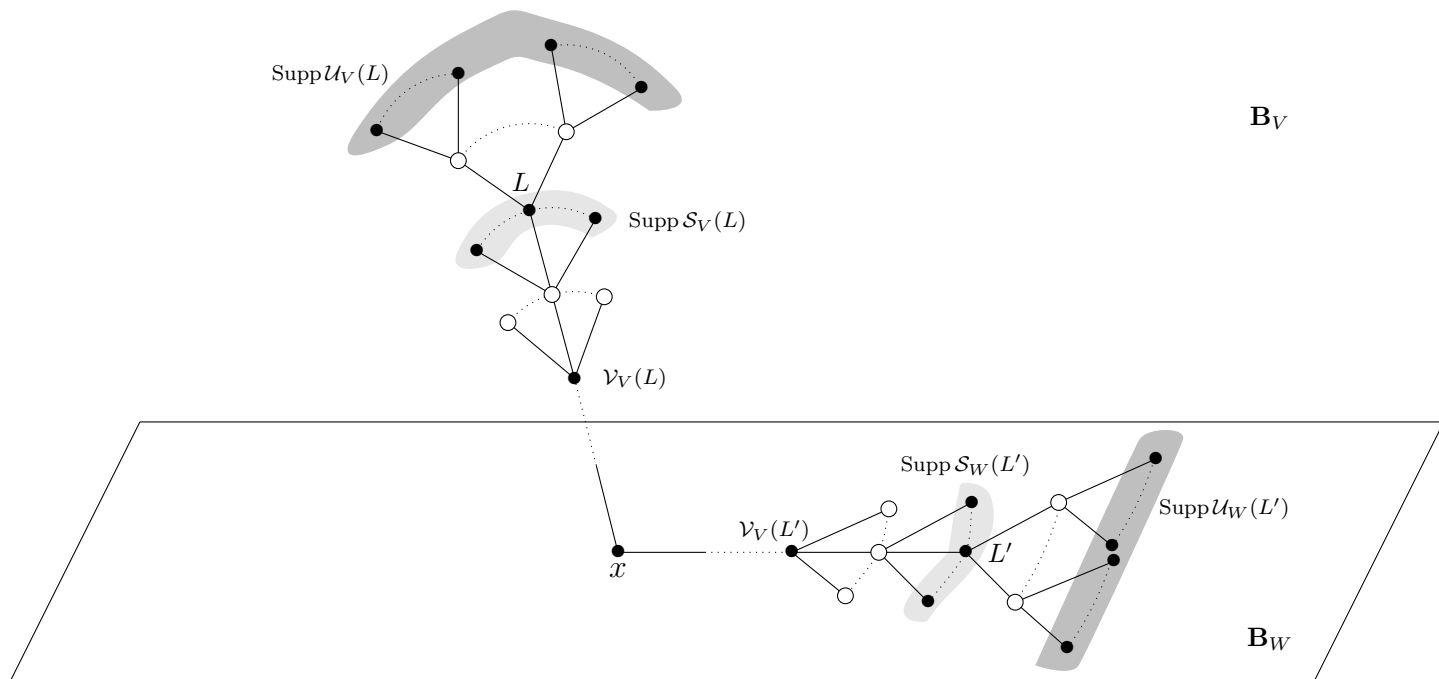


Figure 2.6 – Representation of the operators \mathcal{U}_V , \mathcal{S}_V , \mathcal{V}_V , \mathcal{U}_W , \mathcal{S}_W and \mathcal{V}_W acting on Hyp_V and Hyp_W .

is an hyperspecial vertex of \mathbf{B}_V , then one has $\text{dist}(x, h \cdot y) = \text{dist}(h \cdot x, h \cdot y) = \text{dist}(x, y)$. Accordingly, if $L, L' \in \text{Hyp}_V$ then one has $\text{dist}(x, h \cdot L') > \text{dist}(x, h \cdot L)$ (resp. $\text{dist}(x, h \cdot L') = \text{dist}(x, h \cdot L)$, resp. $\text{dist}(x, h \cdot L') < \text{dist}(x, h \cdot L)$) whenever $\text{dist}(x, L') > \text{dist}(x, L)$ (resp. $\text{dist}(x, L') = \text{dist}(x, L)$, resp. $\text{dist}(x, L') < \text{dist}(x, L)$).

If $L \in \text{Hyp}_V \setminus \{x\}$, this gives:

$$h \cdot \mathcal{U}_V(L) = \sum_{\substack{L' \in \text{Hyp}_V, \\ \text{dist}(L, L')=1, \\ \text{dist}(x, L') > \text{dist}(x, L)}} h \cdot L' \stackrel{L'' := h \cdot L'}{=} \sum_{\substack{L'' \in \text{Hyp}_V, \\ \text{dist}(h \cdot L, L'')=1, \\ \text{dist}(x, L'') > \text{dist}(x, h \cdot L)}} L'' =: \mathcal{U}_V(h \cdot L).$$

If $L \in \text{Hyp}_W \setminus \{x\}$, one gets:

$$h \cdot \mathcal{U}_W(L) = \sum_{\substack{L' \in \text{Hyp}_W, \\ \text{dist}(L, L')=1, \\ \text{dist}(x, L') > \text{dist}(x, L)}} h \cdot L' \stackrel{L'' := h \cdot L'}{=} \sum_{\substack{L'' \in \text{Hyp}_W, \\ \text{dist}(h \cdot L, L'')=1, \\ \text{dist}(x, L'') > \text{dist}(x, h \cdot L)}} L'' =: \mathcal{U}_W(h \cdot L),$$

and similarly for \mathcal{V}_V , \mathcal{S}_V , \mathcal{V}_W and \mathcal{S}_W . In other words, the action of $K_{\star, \tau}$ on $\mathbb{Z}_{(p)}[\text{Hyp}_{\star}]$ by left-multiplication commutes with the operators \mathcal{U}_{\star} , \mathcal{V}_{\star} and \mathcal{S}_{\star} , for $\star \in \{V, W\}$. Therefore the action of K_{τ} on $\mathbb{Z}_{(p)}[\text{Hyp}]$ commutes with \mathcal{U} , \mathcal{V} and \mathcal{S} .

The definition of the preceding operators at the origin x may seem a little bit *ad-hoc* at first sight. Notice that the operators \mathcal{U} , \mathcal{V} and \mathcal{S} do not commute with each other. Rather,

they satisfy the following relations, as stated in ([8], Lemma 3.2).⁽²⁰⁾

Lemma 2.3.3 ([8], Lemma 3.2). *In R , the following relations hold:*

- $\mathcal{V}_V \mathcal{U}_V = q^4$ and $\mathcal{V}_W \mathcal{U}_W = q^2$.
- $\mathcal{U}_V + \mathcal{V}_V + \mathcal{S}_V = \text{Id} + t_{1,0}$, and $\mathcal{U}_W + \mathcal{V}_W + \mathcal{S}_W = \text{Id} + t_{0,1}$.
- $\mathcal{V}_V \mathcal{S}_V = q \mathcal{V}_V$ and $\mathcal{V}_W \mathcal{S}_W = q \mathcal{V}_W$.
- $\mathcal{S}_V \mathcal{U}_V = q \mathcal{U}_V$ and $\mathcal{S}_W \mathcal{U}_W = q \mathcal{U}_W$.
- $\mathcal{S}_V^2 = q \mathcal{S}_V$ and $\mathcal{S}_W^2 = q \mathcal{S}_W$.

The proof is a simple counting argument away from the origin, and is a consequence of the suitable definition of \mathcal{U} , \mathcal{V} and \mathcal{S} at the origin.

Let $\mathcal{R} \subset R$ be the (non-commutative) sub-ring generated by the image of \mathcal{H}_τ via (2.18), and by the operators \mathcal{U}_V , \mathcal{U}_W , \mathcal{V}_V , \mathcal{V}_W , \mathcal{S}_V and \mathcal{S}_W . We shall recall, after [8] section 3, some language of non-commutative algebra. If S is a non-commutative ring and if $f = \sum_{i=1}^r a_i z^i \in S[z]$ is a polynomial⁽²¹⁾, we say that an element $z_0 \in S$ is a *right-root* of f , which we denote by " $f(z_0) = 0$ ", if $\sum_{i=1}^r a_i z_0^i = 0$. Notice that the evaluation map at z_0 ,

$$\begin{aligned} \text{ev}_{z_0} : S[z] &\rightarrow S, \\ \sum_{i=1}^r a_i z^i &\mapsto \sum_{i=1}^r a_i z_0^i, \end{aligned}$$

is *not* a ring homomorphism if S is non-commutative. However one still shows that, if $z_0 \in S$ and $f \in S[z]$, then z_0 is a right root of f if and only if f admits a factorization $f = g \cdot (z - z_0)$, for some polynomial $g \in S[z]$.

Via (2.18), the polynomial $\text{He}_\tau \in \mathcal{H}_\tau[z]$ may be seen as a polynomial - still denoted He_τ - with coefficients in \mathcal{R} . The following important result is due, in this form, to Boumasmoud, Brooks and Jetchev:

Proposition 2.3.4 ([8], Lemma 3.3). *In \mathcal{R} , one has $\text{He}_\tau(\mathcal{U}) = 0$.*

The integrality of the successor operator \mathcal{U} over the spherical Hecke algebra \mathcal{H}_τ (both terms being embedded in the non-commutative ring \mathcal{R}) is a key tool used by Boumasmoud et al. in establishing the *vertical (conductor-wise) distribution relation* - relating cycles in $\mathbb{Z}[\mathcal{Z}_K(\mathbf{G}, \mathbf{H})]$ of increasing *vertical* conductor at τ - in a view towards the existence of a *universal norm* in the Galois cohomology of a Galois stable lattice attached to an automorphic representation appearing in the cohomology of $\text{Sh}_K(\mathbf{G}, X)$ (see [8], Proposition 5.1). The

(20). Notice that the definition of \mathcal{U} , \mathcal{V} and \mathcal{S} requires to localize at (p) at the origin x only - in order to satisfy the above relations - but away from x these relations still hold naturally in $\text{End}_{\mathbb{Z}}(\mathbb{Z}[\text{Hyp}])$.

(21). Recall that the indeterminate z is central in $S[z]$.

proof given in [8] amounts to showing that $\text{He}_\tau^{(4)}(\mathcal{U}) = 0$ - hence $\text{He}_\tau(\mathcal{U}) = 0$, according to the above criterion - by developing each coefficient of $\text{He}_\tau^{(4)}(\mathcal{U})$ and by making full use of the relations between \mathcal{U} , \mathcal{V} and \mathcal{S} stated in Lemma 2.3.3 to show that this coefficient vanishes.

Remark 2.3.3. *The above annihilation relation, which seems like a computational coincidence at first sight, can in fact be realized as a consequence of a much broader class of relations, which Boumasmoud calls seed relation ([7], Theorem IV.1). In his thesis, Boumasmoud studies a class of so-called \mathbb{U} -operators attached to an unramified reductive group over a non-archimedean local field, which are of group-theoretic nature but admit a geometric realization in terms of the corresponding Bruhat-Tits building. Boumasmoud's seed relation is, roughly speaking, a statement of integrality of the \mathbb{U} -operators over the spherical Hecke algebra, both terms being embedded in a wider (non-commutative) ring. We refer the interested reader to e.g. ([7], Remark 71) for explanations about the link between Boumasmoud's annihilation relations (op. cit., Theorem IV.1 and Theorem V.3) and Proposition 2.3.4.*

2.3.4 Allowable segments.

Recall that, by a segment of length $n \geq 0$ in Hyp_V , we mean the full subset $\llbracket x_0, \dots, x_n \rrbracket$ formed by the hyperspecial vertices contained inside a path $[x_0, x_{1/2}, x_1, \dots, x_{n-\frac{1}{2}}, x_n]$ of length $2n$ in $|\mathbf{B}_V|$. Such a segment satisfies $\text{dist}(x_i, x_j) = |i - j|$, for all $i, j \in \{0, \dots, n\}$. Similarly, we define segments of length n in Hyp_W by requiring that the above x_i lie in $|\mathbf{B}_W|$, for $i = 0, \dots, 2n$. For $\star \in \{V, W\}$, we let $\text{Hyp}_\star^{(n)}$ denote the set of segments of length n in Hyp_\star and we set $\text{Hyp}^{(n)} := \text{Hyp}_V^{(n)} \times \text{Hyp}_W^{(n)}$. Accordingly, one has $\text{Hyp}_V^{(0)} = \text{Hyp}_V$, $\text{Hyp}_W^{(0)} = \text{Hyp}_W$, $\text{Hyp}_V^{(0)} = \text{Hyp}$, and

$$\mathbb{Z}[\text{Hyp}^{(n)}] \simeq \mathbb{Z}[\text{Hyp}_V^{(n)}] \otimes_{\mathbb{Z}} \mathbb{Z}[\text{Hyp}_W^{(n)}], \quad \forall n \geq 0.$$

The transitive action of $G_{\star, \tau}$ on the set of apartments of \mathbf{B}_\star induces a transitive action of $G_{\star, \tau}$ on $\text{Hyp}_\star^{(n)}$, for $\star \in \{V, W\}$, hence a transitive action of G_τ on $\text{Hyp}^{(n)}$.

Definition 2.3.4 (Allowable segments). *Let $n \geq 0$. We denote by $\underline{\text{Hyp}}_V^{(n)} \subset \text{Hyp}_V^{(n)}$ the subset formed by those segments of length n , $\llbracket x_0, \dots, x_n \rrbracket$, which satisfy:*

$$\text{dist}(x, x_0) < \text{dist}(x, x_1) < \dots < \text{dist}(x, x_n).$$

Elements of $\underline{\text{Hyp}}_V^{(n)}$ are called x -allowable (or simply allowable) segments. We set $\underline{\text{Hyp}}_W^{(n)} := \underline{\text{Hyp}}_V^{(n)} \cap \text{Hyp}_W^{(n)}$, and we set $\underline{\text{Hyp}}^{(n)} := \underline{\text{Hyp}}_V^{(n)} \times \underline{\text{Hyp}}_W^{(n)}$.

The segment $\llbracket x_0, \dots, x_n \rrbracket \in \text{Hyp}_V^{(n)}$ is allowable if and only if $x_i = \mathcal{V}_V(x_{i+1})$, for all $i \in \{0, \dots, n-1\}$. If $\llbracket x_0, \dots, x_n \rrbracket \in \underline{\text{Hyp}}_V^{(n)}$ is allowable, then one has $\text{dist}(x, x_i) = \text{dist}(x, x_0) + i$, for all $i \in \{0, \dots, n\}$, hence the vertices x, x_0, \dots, x_n all belong to a common apartment (by Proposition 2.1.12) of \mathbf{B}_V and are encountered in this order. The groups $K_{V, \tau}$ and $K_{W, \tau}$ act on $\underline{\text{Hyp}}^{(n)}$ and $\underline{\text{Hyp}}_W^{(n)}$ (but this action is by no means transitive).

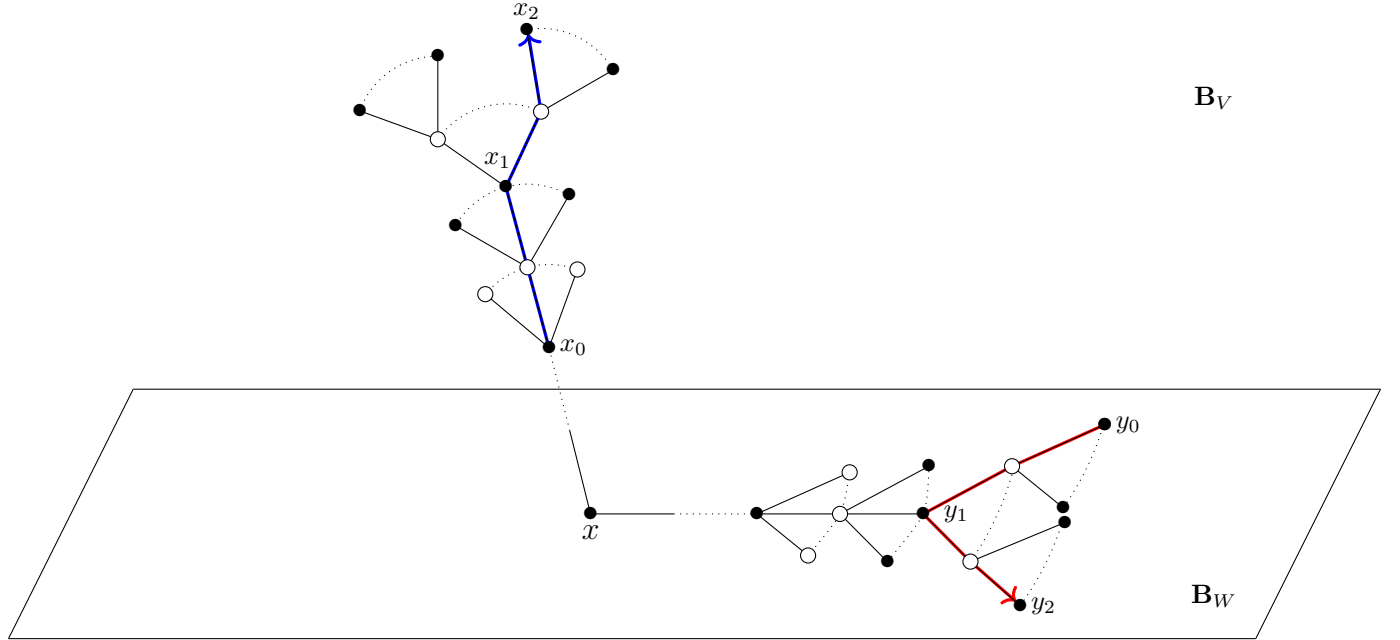


Figure 2.7 – The blue line defines an allowable segment $\llbracket x_0, x_1, x_2 \rrbracket \in \underline{\text{Hyp}}_V^{(2)}$. The red line defines a segment $\llbracket y_0, y_1, y_2 \rrbracket \in \text{Hyp}_W^{(2)}$ which is not allowable.

One has the following useful property linking allowable segments with operators \mathcal{U}_V and \mathcal{V}_V :

Lemma 2.3.5. *Let $L \in \text{Hyp}_V$. If $L' \in \text{Hyp}_V$, the following assertions are equivalent:*

- (i) $L' \in \text{Supp} \mathcal{U}_V(L)$,
- (ii) For all $m \geq 0$ and for every allowable segment $\llbracket x_0, \dots, x_{m-1}, L \rrbracket \in \underline{\text{Hyp}}_V^{(m)}$, the segment $\llbracket x_0, \dots, x_{m-1}, L, L' \rrbracket \in \underline{\text{Hyp}}_V^{(m+1)}$ remains allowable.
- (iii) $\text{dist}(L', L) = 1$ and $\text{dist}(L', \mathcal{V}_V(L)) = 2$.

Proof. The equivalence between (i) and (ii) is immediate.

[(ii) \Rightarrow (iii)] Assume (ii). As $\llbracket \mathcal{V}_V(L), L \rrbracket \in \underline{\text{Hyp}}_V^{(1)}$ is allowable, we get that $\llbracket \mathcal{V}_V(L), L, L' \rrbracket$ is allowable, hence $\text{dist}(L, L') = 1$ and $\text{dist}(x, L') = \text{dist}(x, \mathcal{V}_V(L)) + 2$, thus

$$2 = \text{dist}(x, L') - \text{dist}(x, \mathcal{V}_V(L)) \leq \text{dist}(\mathcal{V}_V(L), L') \leq \text{dist}(\mathcal{V}_V(L), L) + \text{dist}(L, L') = 2.$$

[(iii) \Rightarrow (i)] Assume that $\text{dist}(L', L) = 1$ and $\text{dist}(L', \mathcal{V}_V(L)) = 2$. Then $\text{dist}(x, L') \in \{\text{dist}(x, L) - 1, \text{dist}(x, L), \text{dist}(x, L) + 1\}$. As $L' \neq \mathcal{V}_V(L)$, then $\text{dist}(x, L') \neq \text{dist}(x, L) - 1$, hence one has either $\text{dist}(x, L') = \text{dist}(x, L)$ or $\text{dist}(x, L') = \text{dist}(x, L) + 1$. In the first case, one would get $L' \in \text{Supp} \mathcal{S}_V(L)$, hence $\mathcal{V}_V(L') \in \text{Supp} \mathcal{V}_V \mathcal{S}_V(L) = \text{Supp}(q^2 \mathcal{V}_V(L)) = \{\mathcal{V}_V(L)\}$ according to Lemma 2.3.3. This would contradict $\text{dist}(L', \mathcal{V}_V(L)) = 2$. Accordingly, one has $\text{dist}(x, L') = \text{dist}(x, L) + 1$, thus $L' \in \text{Supp} \mathcal{U}_V(L)$. \square

We set:

$$\underline{\text{Hyp}}_V^\infty := \bigsqcup_{n \geq 0} \underline{\text{Hyp}}_V^{(n)}, \quad \underline{\text{Hyp}}_W^\infty := \bigsqcup_{n \geq 0} \underline{\text{Hyp}}_W^{(n)}$$

and $\underline{\text{Hyp}}^\infty := \bigsqcup_{n \geq 0} \underline{\text{Hyp}}^{(n)}$. For all $\star \in \{V, W, \emptyset\}$, we shall denote by $\underline{\mathfrak{X}}_\star$ (resp. \mathfrak{X}_\star) a generic element of $\mathbb{Z}_{(p)}[\underline{\text{Hyp}}_\star^{(n)}]$ when $n \geq 1$ (resp. when $n = 0$, in which case $\underline{\text{Hyp}}_\star^{(0)} = \text{Hyp}_\star$).

Let $n \geq 0$. We set:

$$\text{Hyp}_{V, \geq n} := \{L \in \text{Hyp}_V; \text{dist}(x, L) \geq n\} \subset \text{Hyp}_V, \quad \text{Hyp}_{W, \geq n} := \text{Hyp}_{V, \geq n} \cap \text{Hyp}_W$$

and $\text{Hyp}_{\geq n} := \text{Hyp}_{V, \geq n} \times \text{Hyp}_{W, \geq n}$. The correspondence between allowable segments and vertices is the following:

- For all $L \in \text{Hyp}_{V, \geq n}$, we set $n_L := \text{dist}(x, L)$. By unicity of the segment $\llbracket x, x_1, \dots, x_{n_L} = L \rrbracket \in \text{Hyp}_V^{(n_L)}$ which connects x and L , one has a map

$$\mathbf{s}_n : \mathbb{Z}[\text{Hyp}_{V, \geq n}] \rightarrow \mathbb{Z}[\underline{\text{Hyp}}_V^{(n)}],$$

obtained by mapping any $L \in \text{Hyp}_{V, \geq n}$ to the truncated segment $\llbracket x_{n_L-n}, \dots, x_{n_L} = L \rrbracket$. If $L \in \text{Hyp}_{W, \geq n}$, the above segment $\llbracket x, x_1, \dots, x_{n_L} = L \rrbracket$ lies in $\underline{\text{Hyp}}_W^{(n_L)}$, therefore the restriction of \mathbf{s}_n to $\mathbb{Z}[\text{Hyp}_{W, \geq n}]$ has image in $\mathbb{Z}[\underline{\text{Hyp}}_W^{(n)}]$.

- On the other hand, one has a map $\mathbf{v} : \mathbb{Z}[\underline{\text{Hyp}}_V^\infty] := \bigoplus_{n \geq 0} \mathbb{Z}[\underline{\text{Hyp}}_V^{(n)}] \rightarrow \mathbb{Z}[\text{Hyp}_V]$ given by $\mathbf{v}(\llbracket x_0, \dots, x_n \rrbracket) = x_n$, for all $n \geq 0$ and all $\llbracket x_0, \dots, x_n \rrbracket \in \underline{\text{Hyp}}_V^{(n)}$. For all $n \geq 0$, the restriction \mathbf{v}_n of \mathbf{v} to $\mathbb{Z}[\underline{\text{Hyp}}_V^{(n)}]$ has image in $\mathbb{Z}[\text{Hyp}_{V, \geq n}]$, and one checks that

$$\mathbf{s}_n \circ \mathbf{v}_n = \text{Id}_{\mathbb{Z}[\underline{\text{Hyp}}_V^{(n)}]} \quad \text{and} \quad \mathbf{v} \circ \mathbf{s}_n = \text{Id}_{\mathbb{Z}[\text{Hyp}_{V, \geq n}]} \quad (2.20)$$

The restriction of \mathbf{v} to $\underline{\text{Hyp}}_W^{(n)}$ has image in $\text{Hyp}_{W, \geq n} \subset \text{Hyp}_W$, hence

$$\mathbf{s}_n \circ \mathbf{v}_n|_{\underline{\text{Hyp}}_W^{(n)}} = \text{Id}_{\mathbb{Z}[\underline{\text{Hyp}}_W^{(n)}]} \quad \text{and} \quad \mathbf{v} \circ (\mathbf{s}_n|_{\text{Hyp}_{W, \geq n}}) = \text{Id}_{\mathbb{Z}[\text{Hyp}_{W, \geq n}]} \quad (2.21)$$

If $\mathfrak{X}_V = \sum_{L \in \text{Hyp}_V} a_L L \in \mathbb{Z}_{(p)}[\text{Hyp}_V]$ and if $\llbracket x_0, \dots, x_n \rrbracket \in \text{Hyp}_V^{(n)}$ is a segment of length n such that $\llbracket x_0, \dots, x_n, L \rrbracket \in \text{Hyp}_V^{(n+1)}$ for all $L \in \text{Supp } \mathfrak{X}_V$, then we set:

$$\llbracket x_0, \dots, x_n, \mathfrak{X}_V \rrbracket := \sum_{L \in \text{Hyp}_V} a_L \llbracket x_0, \dots, x_n, L \rrbracket \in \mathbb{Z}[\text{Hyp}_V^{(n+1)}].$$

Definition 2.3.5 (Operators $\mathcal{U}_V^{(n)}$, $\mathcal{U}_W^{(n)}$ and $\mathcal{U}^{(n)}$). *Let $n \geq 0$. Let us define the following operators:*

- We define $\mathcal{U}_V^{(n)} : \mathbb{Z}_{(p)}[\underline{\text{Hyp}}_V^{(n)}] \rightarrow \mathbb{Z}_{(p)}[\underline{\text{Hyp}}_V^{(n+1)}]$, by

$$\llbracket x_0, \dots, x_n \rrbracket \longmapsto \llbracket x_0, \dots, x_n, \mathcal{U}_V(x_n) \rrbracket.$$

In other words, one has $\mathcal{U}_V^{(n)} := \mathbf{s}_{n+1} \circ \mathcal{U}_V \circ \mathbf{v}_n$.

— We define $\mathcal{U}_W^{(n)} : \mathbb{Z}_{(p)}[\underline{\text{Hyp}}_W^{(n)}] \longrightarrow \mathbb{Z}_{(p)}[\underline{\text{Hyp}}_W^{(n)}]$, by

$$\llbracket x_0, \dots, x_n \rrbracket \longmapsto \llbracket x_0, \dots, x_n, \mathcal{U}_W(x_n) \rrbracket$$

In other words, one has $\mathcal{U}_W^{(n)} := \mathbf{s}_{n+1} \circ \mathcal{U}_W \circ \mathbf{v}_n$

We finally set:

$$\mathcal{U}^{(n)} := \mathcal{U}_V^{(n)} \otimes \mathcal{U}_W^{(n)} : \mathbb{Z}_{(p)}[\underline{\text{Hyp}}^{(n)}] \longrightarrow \mathbb{Z}_{(p)}[\underline{\text{Hyp}}^{(n+1)}],$$

which satisfies $\mathcal{U}^{(n)} = (\mathbf{s}_{n+1} \otimes \mathbf{s}_{n+1}) \circ \mathcal{U} \circ (\mathbf{v}_n \otimes \mathbf{v}_n)$.

2.3.5 The Iwahori filtration $(\mathbf{I}_\tau^{(n)})_{n \geq 0}$

Recall that, by Lemma 2.2.1, the transition matrix between \mathcal{B} and \mathcal{B}' has the form $u := \begin{bmatrix} 1 & \beta & \gamma \\ 0 & 1 & -\bar{\beta} \\ 0 & 0 & 1 \end{bmatrix} \in \text{GL}_3(\mathcal{O}_{E_\tau})$, with $\beta, \gamma \in \mathcal{O}_{E_\tau}^\times$ and $\gamma + \bar{\gamma} + \beta\bar{\beta} = 0$. If $g \in G_{V,\tau}$, then g belongs to $K_{V,\tau}$ if and only if its matrix with respect to \mathcal{B} or \mathcal{B}' belongs to $\text{GL}_3(\mathcal{O}_{E_\tau})$.

Lemma 2.3.6. *For all $m \geq 0$, one has*

$$\iota(\delta_W^m) \in \delta_V^m K_{V,\tau} \text{ and } u^{-1} \text{diag}(\varpi^{-m}, 1, \varpi^m) u \in \text{diag}(\varpi^{-m}, 1, \varpi^m) \text{GL}_3(\mathcal{O}_{E_\tau}).$$

Proof. This can be done without any explicit computation. As both half-apartments $\mathcal{A}_\mathcal{B}^+$ and $\mathcal{A}_{\mathcal{B}'}^+$ are equal then, if $m \geq 0$, one has

$$\delta_V^m \cdot x = \langle \varpi^{-m} e'_+, e'_0, \varpi^m e'_- \rangle = \langle \varpi^{-m} e_+, e_0, \varpi^m e_- \rangle = \iota(\delta_W^m) \cdot x,$$

hence $\delta_V^{-m} \iota(\delta_W^m) \in \text{Stab}_{G_{V,\tau}}(x) = K_{V,\tau}$. On the other hand, as u is the matrix - with respect to \mathcal{B} - of the element $g_u \in K_{V,\tau}$ that maps e_\star to e'_\star , for $\star \in \{+, 0, -\}$, one gets $g_u^{-1} \delta_V g_u = \iota(\delta_W)$, thus $g_u^{-1} \delta_V^m g_u \in \delta_V^m K_{V,\tau}$. Expressing the last equality in the basis \mathcal{B}' - where g_u has matrix u - yields $u^{-1} \text{diag}(\varpi^{-m}, 1, \varpi^m) u \in \text{diag}(\varpi^{-m}, 1, \varpi^m) \text{GL}_3(\mathcal{O}_{E_\tau})$, as wanted. \square

We shall make the confusion between δ_W and $\iota(\delta_W) \in \iota(G_{W,\tau}) \subset G_{V,\tau}$ in the following. As a consequence of the preceding lemma, one has:

$$\delta_V^m K_{V,\tau} \delta_V^{-m} = \delta_W^m K_{V,\tau} \delta_W^{-m},$$

and

$$K_{W,\tau} \cap \delta_W^m K_{W,\tau} \delta_W^{-m} = K_{W,\tau} \cap \delta_V^m K_{W,\tau} \delta_V^{-m} = K_{W,\tau} \cap \delta_V^m K_{V,\tau} \delta_V^{-m},$$

for all $m \geq 0$.

Definition 2.3.6 (Iwahori subgroups of level $n \geq 0$). *Let $n \geq 0$. The Iwahori subgroups of level n , $\mathbf{I}_{V,\tau}^{(n)} \subset K_{V,\tau}$ and $\mathbf{I}_{W,\tau}^{(n)} \subset K_{W,\tau}$ are defined in the following way:*

— We set $\mathbf{I}_{V,\tau}^{(n)} := K_{V,\tau} \cap \delta_V^{-n} K_{V,\tau} \delta_V^n$. In other words, $\mathbf{I}_{V,\tau}^{(n)}$ is the compact open subgroup of $G_{V,\tau}$ which is the stabilizer (in $G_{V,\tau}$) of both vertices x and $\delta_V^{-n} \cdot x$, i.e., the stabilizer of the allowable segment $\llbracket x, \delta_V^{-1} \cdot x, \dots, \delta_V^{-n} \cdot x \rrbracket$.

- We set $I_{W,\tau}^{(n)} := K_{W,\tau} \cap \delta_W^n K_{W,\tau} \delta_W^{-n} = K_{W,\tau} \cap \delta_V^n K_{V,\tau} \delta_V^{-n}$. In other words, $I_{W,\tau}^{(n)}$ is the compact open subgroup of $G_{W,\tau}$ which is the stabilizer (in $G_{W,\tau}$) of both vertices x and $\delta_W^n \cdot x$, i.e., the stabilizer of the allowable segment $\llbracket x, \delta_W^n \cdot x, \dots, \delta_W^n \cdot x \rrbracket$.

We finally set $I_\tau^{(n)} := I_{V,\tau}^{(n)} \times I_{W,\tau}^{(n)} = K_\tau \cap \delta^n K_\tau \delta^{-n} \subset G_\tau$, which we call the Iwahori subgroup of level n of G_τ . The family $(I_\tau^{(n)})_{n \geq 0}$ provides us with a decreasing filtration of G_τ by compact open subgroups.

Remark 2.3.4. — The terminology of Iwahori subgroup which we have used to denote the above groups $I_{\star,\tau}^{(n)}$, $\star \in \{V, W\}$, $n \geq 0$, though slightly abusive, remains motivated by the fact that a generic Iwahori subgroup of a reductive group G (over a non-archimedean local field) arises, roughly speaking, as the (pointwise) stabilizer of an alcove in the Bruhat-Tits building of G (see [7], Definition 9 and Definition 17). In our case, equivalence classes in \mathbf{B}_V / \sim are either vertices (zero-dimensional) or edges (one-dimensional), therefore alcoves of \mathbf{B}_V can be identified with edges. In the apartment \mathcal{A}' , the group $I_{V,\tau}^{(n)}$ arises as the intersection of the stabilizers of the alcoves defined by the segments $\llbracket x, \delta_V^{-1} \cdot x \rrbracket, \dots, \llbracket \delta_V^{-(n-2)} \cdot x, \delta_V^{-(n-1)} \cdot x \rrbracket$ and $\llbracket \delta_V^{-(n-1)} \cdot x, \delta_V^{-n} \cdot x \rrbracket$ (and similarly for $I_{W,\tau}^{(n)}$), which justifies our chosen denomination.

- Notice that, unlike $I_{V,\tau}^{(0)} = K_{V,\tau}$ and $I_{W,\tau}^{(0)} := K_{W,\tau}$, the Iwahori subgroups $I_{V,\tau}^{(n)}$ and $I_{W,\tau}^{(n)}$ do not satisfy the inclusion $\iota(I_{W,\tau}^{(n)}) \subset I_{V,\tau}^{(n)}$ whenever $n \geq 1$, but this slight discrepancy shall not be much of a problem for applications. By formula (2.13) the elements of $I_{V,\tau}^{(n)}$ correspond, in the basis \mathcal{B}' , to those matrices $N = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \in \mathrm{GL}_3(\mathcal{O}_{E_\tau})$ such that $\mathrm{ord}_\varpi(b), \mathrm{ord}_\varpi(f) \geq n$ and $\mathrm{ord}_\varpi(c) \geq 2n$. Similarly, by formulas (2.12) and (2.14), the elements of $I_{W,\tau}^{(n)}$ correspond, in the basis \mathcal{B} , to those matrices $N = \begin{bmatrix} x & 0 & y \\ 0 & 1 & 0 \\ z & 0 & t \end{bmatrix} \in \mathrm{GL}_3(\mathcal{O}_{E_\tau})$ such that $\mathrm{ord}_\varpi(z) \geq 2n$. In particular, an element z_\star of the center $\mathbf{Z}_{G_\star,\tau}$ corresponds to some unitary scalar $\lambda_\star \in \mathrm{U}(1)_{E/F}(F_\tau) = \mathcal{O}_\mathcal{O}^1 \subset \mathcal{O}_{E_\tau}^\times$. Consequently, the homothety $\lambda_\star \cdot \mathbf{1}$ lies in $I_{\star,\tau}^{(n)}$, for all $n \geq 0$ - in fact, fixes every (special or hyperspecial) vertex of \mathbf{B}_\star - which shows the following:

$$\mathbf{Z}_{G_\tau} \subset \bigcap_{n \geq 0} I_\tau^{(n)}. \quad (2.22)$$

By construction, one has $I_\tau^{(n+1)} = I_\tau^{(n)} \cap \delta I_\tau^{(n)} \delta^{-1}$, for all $n \geq 0$. The following lemma enables us to compute the size of the quotients $I_\tau^{(n)} / I_\tau^{(n+1)}$, for all $n \geq 1$:

Lemma 2.3.7. *Let $n \geq 1$. Let $m \geq 0$, and set $L_1 := \delta_V^{-(n+m)} \cdot x \in \mathrm{Hyp}_V$, and $L_2 := \delta_W^{n+m} \cdot x = \delta_V^{n+m} \cdot x \in \mathrm{Hyp}_W$.*

(i) *The map*

$$\begin{aligned} \nu_m^{(n)} : I_{V,\tau}^{(n)} &\longrightarrow \mathrm{Hyp}_V, \\ i &\longmapsto (\delta_V^{-m} i \delta_V^{-(n+1)}) \cdot x \end{aligned}$$

induces a bijection between the quotient set $\mathbb{I}_{V,\tau}^{(n)} / \mathbb{I}_{V,\tau}^{(n+1)}$ and $\text{Supp } \mathcal{U}_V(L_1) = \{L' \in \text{Hyp}_V; \text{dist}(L_1, L') = 1, \text{dist}(x, L') > \text{dist}(x, L_1)\}$.

(ii) The map

$$\begin{aligned} \mu_m^{(n)} : \mathbb{I}_{W,\tau}^{(n)} &\longrightarrow \text{Hyp}_W, \\ i &\longmapsto (\delta_W^m i \delta_W^{n+1}) \cdot x \end{aligned}$$

induces a bijection between the quotient set $\mathbb{I}_{W,\tau}^{(n)} / \mathbb{I}_{W,\tau}^{(n+1)}$ and $\text{Supp } \mathcal{U}_W(L_2) = \{L'' \in \text{Hyp}_W; \text{dist}(L_2, L'') = 1, \text{dist}(x, L'') > \text{dist}(x, L_2)\}$.

Proof. Both statements being very similar, we shall prove only (i) and leave (ii) to the reader. That $\nu_m^{(n)}$ is right $\mathbb{I}_{V,\tau}^{(n+1)}$ -invariant is clear. Also, if $i, i' \in \mathbb{I}_{V,\tau}^{(n)}$ are such that $\delta_V^{-m} i \delta_V^{-(n+1)} \cdot x = \delta_V^{-m} i' \delta_V^{-(n+1)} \cdot x$, then $i^{-1} i' \cdot (\delta_V^{-(n+1)} \cdot x) = \delta_V^{-(n+1)} \cdot x$, hence $i^{-1} i' \in \mathbb{I}_{V,\tau}^{(n)} \cap \text{Stab}_{G_{V,\tau}}(\delta_V^{-(n+1)} \cdot x) = \mathbb{I}_{V,\tau}^{(n+1)}$, which gives $i \in i' \mathbb{I}_{V,\tau}^{(n+1)}$ and shows that $\nu_m^{(n)}$ is injective.

Let us show that $\nu_m^{(n)}$ has image in $\text{Supp } \mathcal{U}_V(L_1)$. First, notice that $\mathcal{V}_V(L_1) = \delta_V^{-(n+m-1)} \cdot x$. If $i \in \mathbb{I}_{V,\tau}^{(n)}$ then, by definition, i fixes the segment $\llbracket x, \dots, \delta_V^{-n} \cdot x \rrbracket$. Accordingly, for all $k \in \{0, \dots, n\}$, one has

$$\begin{aligned} \text{dist}\left((i \delta_V^{-(n+1)}) \cdot x, \delta_V^{-(n-k)} \cdot x\right) &= \text{dist}\left((i \delta_V^{-(n+1)}) \cdot x, i \cdot (\delta_V^{-(n-k)} \cdot x)\right) \\ &= \text{dist}(\delta_V^{-(n+1)} \cdot x, \delta_V^{-(n-k)} \cdot x) \\ &= k + 1. \end{aligned}$$

Taking $k = 0, 1$ gives us:

$$\begin{aligned} \text{dist}(\nu_m^{(n)}(i), L_1) &= \text{dist}\left(\delta_V^{-m} \cdot ((i \delta_V^{-(n+1)}) \cdot x), \delta_V^{-m} \cdot (\delta_V^{-n} \cdot x)\right) \\ &= \text{dist}\left((i \delta_V^{-(n+1)}) \cdot x, \delta_V^{-n} \cdot x\right) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} \text{dist}(\nu_m^{(n)}(i), \mathcal{V}_V(L_1)) &= \text{dist}\left(\delta_V^{-m} \cdot ((i \delta_V^{-(n+1)}) \cdot x), \delta_V^{-m} \cdot (\delta_V^{-(n-1)} \cdot x)\right) \\ &= \text{dist}\left((i \delta_V^{-(n+1)}) \cdot x, \delta_V^{-(n-1)} \cdot x\right) \\ &= 2. \end{aligned}$$

By Lemma 2.3.5, (iii), this implies that $\nu_m^{(n)}(i) \in \text{Supp } \mathcal{U}_V(L_1)$.

On the other hand, let L' belong to $\text{Supp } \mathcal{U}_V(L_1)$. As the segment $[\delta_V^{-m} \cdot x, \dots, \delta_V^{-(n+m)} \cdot x = L] \in \underline{\text{Hyp}}_V^{(n)}$ is allowable, then so is the segment $[\delta_V^{-m} \cdot x, \dots, \delta_V^{-(n+m)} \cdot x, L'] \in \underline{\text{Hyp}}_V^{(n+1)}$, according to Lemma 2.3.5, (ii). Therefore one has

$$\text{dist}(\delta_V^{-(n+m-k)} \cdot x, L') = \text{dist}(\delta_V^{-(n-k)} \cdot x, \delta_V^m \cdot L') = k + 1, \quad \forall k \in \{0, \dots, n\},$$

i.e., the segment $\delta_V^m \cdot [\delta_V^{-m} \cdot x, \dots, L, L'] = [x, \dots, \delta_V^{-n} \cdot x, \delta_V^m \cdot L']$ is again allowable. Accordingly, one may find a Witt basis $\tilde{\mathcal{B}} = \{\tilde{e}_+, \tilde{e}_0, \tilde{e}_-\}$ such that the vertices $x, \dots, \delta_V^{-n} \cdot x$ and $\delta_V^m \cdot L'$ all belong to the apartment $\mathcal{A}_{\tilde{\mathcal{B}}}$ and have respective parameters $0, \dots, -n$ and $-(n+1)$ with respect to $\tilde{\mathcal{B}}$. Accordingly, one has:

$$\begin{aligned} \delta_V^{-j} \cdot x &= \langle \varpi^j \tilde{e}_+, e_0, \varpi^{-j} \tilde{e}_- \rangle = \langle \varpi^j e_+, e_0, \varpi^{-j} e_- \rangle, \quad \forall j \in \{0, \dots, n\}, \\ \delta_V^m \cdot L' &= \langle \varpi^{n+1} \tilde{e}_+, e_0, \varpi^{-(n+1)} \tilde{e}_- \rangle. \end{aligned}$$

Consequently, if $i \in G_{V,\tau}$ is defined by $i(e'_\star) = \tilde{e}_\star$, for $\star \in \{+, 0, -\}$, then i fixes the whole segment $[x, \dots, \delta_V^{-n} \cdot x]$ - i.e., i belongs to $\mathbf{I}_{V,\tau}^{(n)}$ - and satisfies $i \cdot (\delta_V^{-(n+1)} \cdot x) = \delta_V^m \cdot L'$. In other words, one has $L' = (\delta_V^{-m} i \delta_V^{-(n+1)}) \cdot x = \nu_m^{(n)}(i)$, which finishes the proof. \square

Notice that, for all $n \geq 0$, the map $i_V \mapsto i_V \delta_V^{-1}$ induces a bijection between $\mathbf{I}_{V,\tau}^{(n)} / \mathbf{I}_{V,\tau}^{(n+1)} = \mathbf{I}_{V,\tau}^{(n)} / \mathbf{I}_{V,\tau}^{(n)} \cap \delta_V^{-1} \mathbf{I}_{V,\tau}^{(n)} \delta_V$ and $\mathbf{I}_{V,\tau}^{(n)} \delta_V^{-1} \mathbf{I}_{V,\tau}^{(n)} / \mathbf{I}_{V,\tau}^{(n)}$. If $n = 0$, the latter is also equal to $K_{V,\tau} \delta_V K_{V,\tau} / K_{V,\tau}$. Similarly, if $n \geq 0$, the map $i_W \mapsto i_W \delta_W$ induces a bijection between $\mathbf{I}_{W,\tau}^{(n)} / \mathbf{I}_{W,\tau}^{(n+1)} = \mathbf{I}_{W,\tau}^{(n)} / \mathbf{I}_{W,\tau}^{(n)} \cap \delta_W \mathbf{I}_{W,\tau}^{(n)} \delta_W^{-1}$ and $\mathbf{I}_{W,\tau}^{(n)} \delta_W \mathbf{I}_{W,\tau}^{(n)} / \mathbf{I}_{W,\tau}^{(n)}$. The above considerations, together with the preceding lemma, admit the following consequence:

Corollary 2.3.8. — *One has*

$$\# \left(\mathbf{I}_{V,\tau}^{(0)} / \mathbf{I}_{V,\tau}^{(1)} \right) = \# \left(K_{V,\tau} \delta_V K_{V,\tau} / K_{V,\tau} \right) \stackrel{\text{Lemma 2.3.1,(i)}}{=} q(q^3 + 1)$$

and

$$\# \left(\mathbf{I}_{W,\tau}^{(0)} / \mathbf{I}_{W,\tau}^{(1)} \right) = \# \left(K_{W,\tau} \delta_W K_{W,\tau} / K_{W,\tau} \right) \stackrel{\text{Lemma 2.3.1,(ii)}}{=} q(q + 1),$$

thus

$$\# \left(\mathbf{I}_\tau^{(0)} / \mathbf{I}_\tau^{(1)} \right) = (q^3 + 1)q^2(q + 1). \quad (2.23)$$

— For all $n \geq 1$, one has $\# \left(\mathbf{I}_{V,\tau}^{(n)} / \mathbf{I}_{V,\tau}^{(n+1)} \right) = q^4$ and $\# \left(\mathbf{I}_{W,\tau}^{(n)} / \mathbf{I}_{W,\tau}^{(n+1)} \right) = q^2$, hence

$$\# \left(\mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)} \right) = q^6. \quad (2.24)$$

For all $n, m \geq 0$ and $\star \in \{V, W\}$, we define the *standard* allowable segment $\Xi_{\star,m}^{(n)} \in \underline{\text{Hyp}}_\star^{(n)}$ as follows:

$$\Xi_{V,m}^{(n)} := \delta_V^{-m} \cdot [x, \delta_V^{-1} \cdot x, \dots, \delta_V^{-n} \cdot x] \in \underline{\text{Hyp}}_V^{(n)}, \quad (2.25)$$

$$\Xi_{W,m}^{(n)} := \delta_V^m \cdot [x, \delta_V \cdot x, \dots, \delta_V^n \cdot x] \in \underline{\text{Hyp}}_W^{(n)}, \quad (2.26)$$

and

$$\Xi_m^{(n)} := \Xi_{V,m}^{(n)} \otimes \Xi_{W,m}^{(n)} \in \underline{\text{Hyp}}^{(n)}. \quad (2.27)$$

Accordingly, one has $\mathbf{I}_{\star,\tau}^{(n)} = \text{Stab}_{G_{\star,\tau}}(\Xi_{\star,0}^{(n)})$, for all $n \geq 0$ and all $\star \in \{V, W\}$. This gives a bijection

$$\begin{aligned} G_\tau / \mathbf{I}_\tau^{(n)} &\xrightarrow{\sim} \text{Hyp}_V^{(n)}, \\ g_\tau \mathbf{I}_\tau^{(n)} &\longmapsto g_\tau \cdot \Xi_0^{(n)} \end{aligned} \quad (2.28)$$

The above Lemma 2.3.7 can also be rewritten as follows:

Corollary 2.3.9. *Let $n \geq 1$ and $m \geq 0$. One has*

$$\sum_{i \in \mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)}} \delta^m i \cdot \Xi_0^{(n+1)} = \sum_{i \in \mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)}} (\delta^m i \delta^{-m}) \cdot \Xi_m^{(n+1)} = \mathcal{U}^{(n)}(\Xi_m^{(n)}),$$

where $\sum_{i \in \mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)}}$ denotes a summation over a fixed system of representatives for the quotient $\mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)}$. □

2.3.5.1 Expression of $\mathcal{U}^{(n)}$ in terms of the n -Iwahori-Hecke algebra.

Let $\star \in \{V, W, \emptyset\}$. In a similar way as we defined the local *spherical* Hecke algebra $\mathcal{H}_{\star,\tau}$ attached to the hyperspecial maximal compact open subgroup $K_{\star,\tau} \subset G_{\star,\tau}$ (see § 2.3.1)- one may define a Hecke algebra of n -Iwahori level (or *n -Iwahori-Hecke algebra*) $\mathcal{H}_{\star,\tau}^{(n)}$, by replacing the hyperspecial level $K_{\star,\tau}$ by its n -th Iwahori subgroup $\mathbf{I}_{\star,\tau}^{(n)}$. Accordingly, the ring

$$\mathcal{H}_{\star,\tau}^{(n)} := \mathcal{C}_c^\infty(G_{\star,\tau} // \mathbf{I}_{\star,\tau}^{(n)}, \mathbb{Z}) \xrightarrow{\rho} \text{End}_{\mathbb{Z}[G_{\star,\tau}]}(\mathbb{Z}[G_{\star,\tau} / \mathbf{I}_{\star,\tau}^{(n)}]) \simeq \text{End}_{\mathbb{Z}[G_{\star,\tau}]}(\mathbb{Z}[\text{Hyp}_\star^{(n)}])$$

is generated by indicator functions of the form $\mathbf{1}_{\mathbf{I}_{\star,\tau}^{(n)} g \mathbf{I}_{\star,\tau}^{(n)}}$, for elements $g \in G_{\star,\tau}$.

Definition 2.3.7 (Hecke operators $t_{1,0}^{(n)}$, $t_{0,1}^{(n)}$ and $t^{(n)}$). *Let $n \geq 1$. We define the following double coset operators of $\text{End}_{\mathbb{Z}[G_{\star,\tau}]}(\mathbb{Z}[\text{Hyp}_{\star,\tau}^{(n)}])$, $\star \in \{V, W, \emptyset\}$:*

- $t_{1,0}^{(n)} := \rho(\mathbf{1}_{\mathbf{I}_{V,\tau}^{(n)} \delta_V^{-1} \mathbf{I}_{V,\tau}^{(n)}})^{(22)} \in \text{End}_{\mathbb{Z}[G_{V,\tau}]}(\mathbb{Z}[\text{Hyp}_{V,\tau}^{(n)}])$.
- $t_{0,1}^{(n)} := \rho(\mathbf{1}_{\mathbf{I}_{W,\tau}^{(n)} \delta_W \mathbf{I}_{W,\tau}^{(n)}}) \in \text{End}_{\mathbb{Z}[G_{W,\tau}]}(\mathbb{Z}[\text{Hyp}_{W,\tau}^{(n)}])$.
- $t^{(n)} := \rho(\mathbf{1}_{\mathbf{I}_\tau^{(n)} \delta \mathbf{I}_\tau^{(n)}}) = t_{1,0}^{(n)} \otimes t_{0,1}^{(n)} \in \text{End}_{\mathbb{Z}[G_\tau]}(\mathbb{Z}[\text{Hyp}_\tau^{(n)}])$.

The operators $t_{1,0}^{(n)}$, $t_{0,1}^{(n)}$ and $t^{(n)}$, when restricted to *allowable* n -segments, turn out to be related to the previously defined operators $\mathcal{U}_V^{(n)}$, $\mathcal{U}_W^{(n)}$ and $\mathcal{U}^{(n)}$ respectively. More precisely:

(22). Notice that this is no longer the same operator as $\rho(\mathbf{1}_{\mathbf{I}_{V,\tau}^{(n)} \delta_V \mathbf{I}_{V,\tau}^{(n)}})$, whenever $n \geq 1$.

Lemma 2.3.10. *Let $n \geq 1$. One has*

$$t_{1,0}^{(n)} \Big|_{\mathbb{Z}[\underline{\text{HYP}}_V^{(n)}]} = \mathbf{s}_n \circ \mathbf{v}_{n+1} \circ \mathcal{U}_V^{(n)}, \quad (2.29)$$

$$t_{0,1}^{(n)} \Big|_{\mathbb{Z}[\underline{\text{HYP}}_W^{(n)}]} = \mathbf{s}_n \circ \mathbf{v}_{n+1} \circ \mathcal{U}_W^{(n)}, \quad (2.30)$$

$$\text{hence } t_{\Big|_{\mathbb{Z}[\underline{\text{HYP}}^{(n)}]}}^{(n)} = (\mathbf{s}_n \otimes \mathbf{s}_n) \circ (\mathbf{v}_{n+1} \otimes \mathbf{v}_{n+1}) \circ \mathcal{U}^{(n)}. \quad (2.31)$$

Proof. We shall, as usual, give the details for the first statement regarding $t_{1,0}^{(n)}$ and leave the (similar) one about $t_{0,1}^{(n)}$ to the reader. Let $g_1, \dots, g_r \in \mathbb{I}_{V,\tau}^{(n)} \delta_V^{-1} \mathbb{I}_{V,\tau}^{(n)}$ be elements satisfying

$$\mathbb{I}_{V,\tau}^{(n)} \delta_V^{-1} \mathbb{I}_{V,\tau}^{(n)} = \bigsqcup_{j=1}^r g_j \mathbb{I}_{V,\tau}^{(n)}.$$

Without loss of generality, one may assume that $g_j = i_j \delta_V^{-1}$, for some family i_1, \dots, i_j of representatives for the quotients $\mathbb{I}_{V,\tau}^{(n)} / \mathbb{I}_{V,\tau}^{(n)} \cap \delta_V^{-1} \mathbb{I}_{V,\tau}^{(n)} \delta_V = \mathbb{I}_{V,\tau}^{(n)} / \mathbb{I}_{V,\tau}^{(n+1)}$.

Let $\underline{\mathfrak{X}} := \llbracket x_0, x_1, \dots, x_n \rrbracket \in \underline{\text{HYP}}_V^{(n)}$ be an allowable segment of length n , which may be written as $\underline{\mathfrak{X}} = g \cdot \Xi_{V,0}^{(n)}$, for some $g \in G_{V,\tau}$, well defined in $G_{V,\tau} / \mathbb{I}_{V,\tau}^{(n)}$. One has $\mathbf{s}_n \circ \mathbf{v}_{n+1} \circ \mathcal{U}_V^{(n)}(\underline{\mathfrak{X}}) = \sum_{L \in \text{Supp} \mathcal{U}_V(x_n)} \llbracket x_1, \dots, x_n, L \rrbracket$. On the other hand, one has:

$$t_{1,0}^{(n)}(\underline{\mathfrak{X}}) := \sum_{j=1}^r g g_j \cdot \Xi_{V,0}^{(n)} = \sum_{j=1}^r g i_j \delta_V^{-1} \cdot \Xi_{V,0}^{(n)}$$

One has $i_j \cdot (\delta_V^{-k} \cdot x) = \delta_V^{-k} \cdot x$, for all $k \in \{1, \dots, n\}$ and, by Lemma 2.3.7 (case $m = 0$), one has $i_j \cdot (\delta_V^{-(n+1)} \cdot x) \in \text{Supp} \mathcal{U}_V(\delta_V^{-n} \cdot x)$. This implies, by Lemma 2.3.5, that the segment $i_j \delta_V^{-1} \cdot \Xi_{V,0}^{(n)} = \llbracket \delta_V^{-1} \cdot x, \dots, \delta_V^{-n} \cdot x, i_j \delta_V^{-(n+1)} \cdot x \rrbracket$ is allowable. Accordingly, the segment

$$g g_j \cdot \Xi_{V,0}^{(n)} = g \cdot \llbracket \delta_V^{-1} x, \dots, \delta_V^{-n} \cdot x, i_j \delta_V^{-(n+1)} \cdot x \rrbracket = \llbracket x_1, \dots, x_n, g i_j \delta_V^{-(n+1)} \cdot x \rrbracket$$

remains allowable, i.e., $g i_j \delta_V^{-(n+1)} \cdot x \in \text{Supp} \mathcal{U}_V(x_n)$.

Indeed, assume that $g i_j \delta_V^{-(n+1)} \cdot x \notin \text{Supp} \mathcal{U}_V(x_n)$: as $\text{dist}(x_n, g i_j \delta_V^{-(n+1)} \cdot x) = \text{dist}(\delta_V^{-n} \cdot x, i_j \delta_V^{-(n+1)} \cdot x) = 1$, then one would have either $\text{dist}(x, g i_j \delta_V^{-(n+1)} \cdot x) = \text{dist}(x, x_n) - 1$ or $\text{dist}(x, g i_j \delta_V^{-(n+1)} \cdot x) = \text{dist}(x, x_n)$. The former case would mean that $g i_j \delta_V^{-(n+1)} \cdot x = \mathcal{V}_V(x_n) \stackrel{\text{Lemma 2.3.5}}{=} x_{n-1}$, which would already contradict the fact that $g g_j \cdot \Xi_{V,0}^{(n)}$ is a segment. The former case would imply that $g i_j \delta_V^{-(n+1)} \cdot x \in \text{Supp} \mathcal{S}_V(x_n)$, i.e., that

$$\text{dist}(x_{n-1}, g i_j \delta_V^{-(n+1)} \cdot x) \stackrel{x_{n-1} = \mathcal{V}_V(x_n)}{=} 1 < \text{dist}(x_{n-1}, x_n) + \text{dist}(x_n, g i_j \delta_V^{-(n+1)} \cdot x),$$

thus also contradicting the fact that $\llbracket x_{n-1}, x_n, g i_j \delta_V^{-(n+1)} \cdot x \rrbracket \subset g g_j \cdot \Xi_{V,0}^{(n)}$ is a segment. Accordingly, one has

$$g i_j \delta_V^{-1} \cdot \Xi_{V,0}^{(n)} \in \text{Supp} \left(\mathbf{s}_n \circ \mathbf{v}_{n+1} \circ \mathcal{U}_V^{(n)}(\underline{\mathfrak{X}}) \right), \quad \forall j \in \{1, \dots, r\}.$$

By Lemma 2.3.7, the mapping $j \mapsto i_j \delta_V^{-1} \cdot \Xi_{V,0}^{(n)}$ is injective, hence so is the map $j \mapsto gg_j \cdot \Xi_{V,0}^{(n)}$. Finally, the following equalities

$$r := \# \left(\mathbf{I}_{V,\tau}^{(n)} / \mathbf{I}_{V,\tau}^{(n+1)} \right) = q^4 = \# \text{Supp} \mathcal{U}_V(x_n) = \# \text{Supp} \left(\mathbf{s}_n \circ \mathbf{v}_{n+1} \circ \mathcal{U}_V^{(n)}(\mathbf{\Xi}) \right)$$

show that $j \mapsto gg_j \cdot \Xi_{V,0}^{(n)}$ is a bijection between $\{1, \dots, r\}$ and $\text{Supp} \left(\mathbf{s}_n \circ \mathbf{v}_{n+1} \circ \mathcal{U}_V^{(n)} \right)$, which finishes the proof. \square

The preceding Lemma together with Definition 2.3.5 give:

$$\begin{aligned} t_{\mathbb{Z}(p)[\underline{\text{HYP}}^{(n)}]}^{(n)} &= (\mathbf{s}_n \otimes \mathbf{s}_n) \circ (\mathbf{v}_{n+1} \otimes \mathbf{v}_{n+1}) \circ ((\mathbf{s}_{n+1} \otimes \mathbf{s}_{n+1}) \circ \mathcal{U} \circ (\mathbf{v}_n \otimes \mathbf{v}_n)) \\ &= (\mathbf{s}_n \otimes \mathbf{s}_n) \circ \mathcal{U} \circ (\mathbf{v}_n \otimes \mathbf{v}_n). \end{aligned}$$

By the relation $\mathbf{v}_n \circ \mathbf{s}_n = \text{Id}$ on $\text{Hyp}_{\star, \geq n}$, $\star \in \{V, W\}$, one obtains that:

$$(t^{(n)})^i = (\mathbf{s}_n \otimes \mathbf{s}_n) \circ \mathcal{U}^i \circ (\mathbf{v}_n \otimes \mathbf{v}_n) \quad \forall i \geq 0. \quad (2.32)$$

2.3.5.2 The filtration $(H_n)_{n \geq 0}$ of H_τ .

The decreasing filtration $(\mathbf{I}_\tau^{(n)})_{n \geq 0}$ of G_τ induces the following decreasing filtration on H_τ :

Definition 2.3.8. *Let $n \geq 0$. We set H_n to be the subgroup of H_τ defined by*

$$\begin{aligned} H_n &:= H_\tau \cap \mathbf{I}_\tau^{(n)} = \text{Stab}_{H_\tau}(\Xi_{V,0}^{(n)}, \Xi_{W,0}^{(n)}) \\ &= \text{Stab}_{H_\tau}((\delta_V^{-n} \cdot x, \delta_V^n \cdot x)) = \Delta \left(\text{Stab}_{G_{W,\tau}}(\llbracket \delta_V^{-n} \cdot x, \dots, x, \dots, \delta_V^n \cdot x \rrbracket) \right). \end{aligned}$$

The above characterizations imply that $H_{n+1} \subset H_n$, for all $n \geq 0$. Also, for all $m \geq 0$, one has

$$\begin{aligned} H_\tau \cap \delta^m \mathbf{I}_\tau^{(n)} \delta^{-m} &= \text{Stab}_{H_\tau}(\Xi_{V,m}^{(n)}, \Xi_{W,m}^{(n)}) \\ &= \text{Stab}_{H_\tau}(\delta_V^{-(n+m)} \cdot x, \delta_V^{n+m} \cdot x) \\ &= H_{n+m} \end{aligned} \quad (2.33)$$

We shall sometimes slightly abuse notations and make the confusion between H_n and $\text{Stab}_{G_{W,\tau}}(\llbracket \delta_V^{-n} \cdot x, \dots, x, \dots, \delta_V^n \cdot x \rrbracket)$. According to Corollary 2.2.8, and for $\text{inv}_\tau((\delta_V^{-n} \cdot x, \delta_V^n \cdot x)) = (n, n)$, one has

$$\det(H_n) = \mathcal{O}_n^1, \quad \text{for all } n \geq 0. \quad (2.34)$$

Notice that, by Remark 2.3.2, the action of $H_0 \subset K_\tau$ on Hyp by left-multiplication commutes with the operators \mathcal{U} , \mathcal{V} and \mathcal{S} . Accordingly, the group H_0 acts on $\text{Supp} \mathcal{U}(y)$ (resp. $\text{Supp} \mathcal{V}(y)$, $\text{Supp} \mathcal{S}(y)$) by permutation, for all $y \in \text{Hyp}$. The inclusion induces a natural injective map:

$$H_n / H_{n+1} \hookrightarrow \mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)}. \quad (2.35)$$

We define the following subsets of Hyp:

$$\text{We set } \mathcal{N}_0 := (\text{Supp } t_{1,0}(x) \setminus \text{Supp } t_{0,1}(x)) \times (\text{Supp } t_{0,1}(x))$$

and, if $n \geq 1$,

$$\mathcal{N}_n := \text{Supp } \mathcal{U}(\delta^n \cdot (x \otimes x)) = (\text{Supp } \mathcal{U}_V(\delta_V^{-n} \cdot x)) \times (\text{Supp } \mathcal{U}_W(\delta_V^n \cdot x))$$

Proposition 2.3.11. *For all $n \geq 0$, the map:*

$$\begin{aligned} (\nu_0^{(n)}, \mu_0^{(n)}) : \mathbf{I}_\tau^{(n)} &\longrightarrow \text{Hyp}, \\ i &\longmapsto i \cdot (\delta_V^{-(n+1)} \cdot x, \delta_V^{n+1} \cdot x) \end{aligned}$$

induces a bijection between the subset $H_n / H_{n+1} \subset \mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)}$ and $\mathcal{N}_n \subset \text{Hyp}$.

Proof. That $(\nu_0^{(n)}, \mu_0^{(n)}) : H_n / H_{n+1} \longrightarrow \text{Hyp}$ is well-defined and injective is an immediate consequence of the characterization

$$H_k = \text{Stab}_{H_\tau}(\delta_V^{-k} \cdot x, \delta_V^k \cdot x), \text{ for all } k \geq 0.$$

That the restriction $(\nu_0^{(n)}, \mu_0^{(n)})|_{H_n}$ has indeed image in \mathcal{N}_n is a consequence of Lemma 2.3.7 (in the case $m = 0$) if $n \geq 1$. If $n = 0$, this boils down to the facts that, for all $h \in K_{W,\tau}$, so that $\Delta(h) = (h, h) \in H_0$, one has:

$$\text{dist}(h\delta_V^{-1} \cdot x, x) = \text{dist}(\delta_V^{-1} \cdot x, h^{-1} \cdot x) = \text{dist}(\delta_V^{-1} \cdot x, x) = 1,$$

$$\text{dist}(h\delta_W \cdot x, x) = \text{dist}(\delta_W \cdot x, h^{-1} \cdot x) = \text{dist}(\delta_W \cdot x, x) = 1,$$

and $h\delta_V^{-1} \cdot x \notin \mathbf{B}_W$ (for otherwise $\delta_V^{-1} \cdot x = h \cdot (h^{-1}\delta_V^{-1} \cdot x) \in \mathbf{B}_W$, which is not), hence $\Delta(h) \cdot (\delta_V^{-1} \cdot x, \delta_W \cdot x) \in \mathcal{N}_0$.

For the surjectivity: let $y = (y_V, y_W)$ be an element of \mathcal{N}_n . As $y_V \in \mathbf{B}_V \setminus \mathbf{B}_W$ and $y_W \in \mathbf{B}_W$, one may form a segment s_y of length $2n$ in \mathbf{B}_V by setting

$$s_y := \llbracket y_V, \delta_V^{-n} \cdot x, \dots, x, \dots, \delta_V^n \cdot x, y_W \rrbracket \in \text{Hyp}_V^{(n)}.$$

Let us choose a special apartment $\mathcal{A}'' \in \mathfrak{S}$ which contains s_y : thanks to Corollary 2.2.4, one may then find some $h \in G_{W,\tau}$ such that $h \cdot \mathcal{A}' = \mathcal{A}''$. First notice that, as \mathcal{A}' and \mathcal{A}'' share the same base point x , then x must be fixed by h .

Indeed, one may consider the vertex $\delta_V^{-(n+1)} \cdot x \in \mathcal{A}' \setminus \mathbf{B}_W$: then $h \cdot (\delta_V^{-(n+1)} \cdot x) \in \mathcal{A}'' \setminus \mathbf{B}_W$, hence $\text{pr}_W(h\delta_V^{-(n+1)} \cdot x) = x$. According to Proposition-Definition 2.1.18, (iii), one gets:

$$h \cdot x = h \cdot \text{pr}_W(\delta_V^{-(n+1)} \cdot x) = \text{pr}_W(h\delta_V^{-(n+1)} \cdot x) = x.$$

Accordingly, one has

$$\text{dist}(\delta_V^{-k} \cdot x, x) = \text{dist}(h \cdot (\delta_V^{-k} \cdot x), h \cdot x) = \text{dist}(h\delta_V^{-k} \cdot x, x) = k,$$

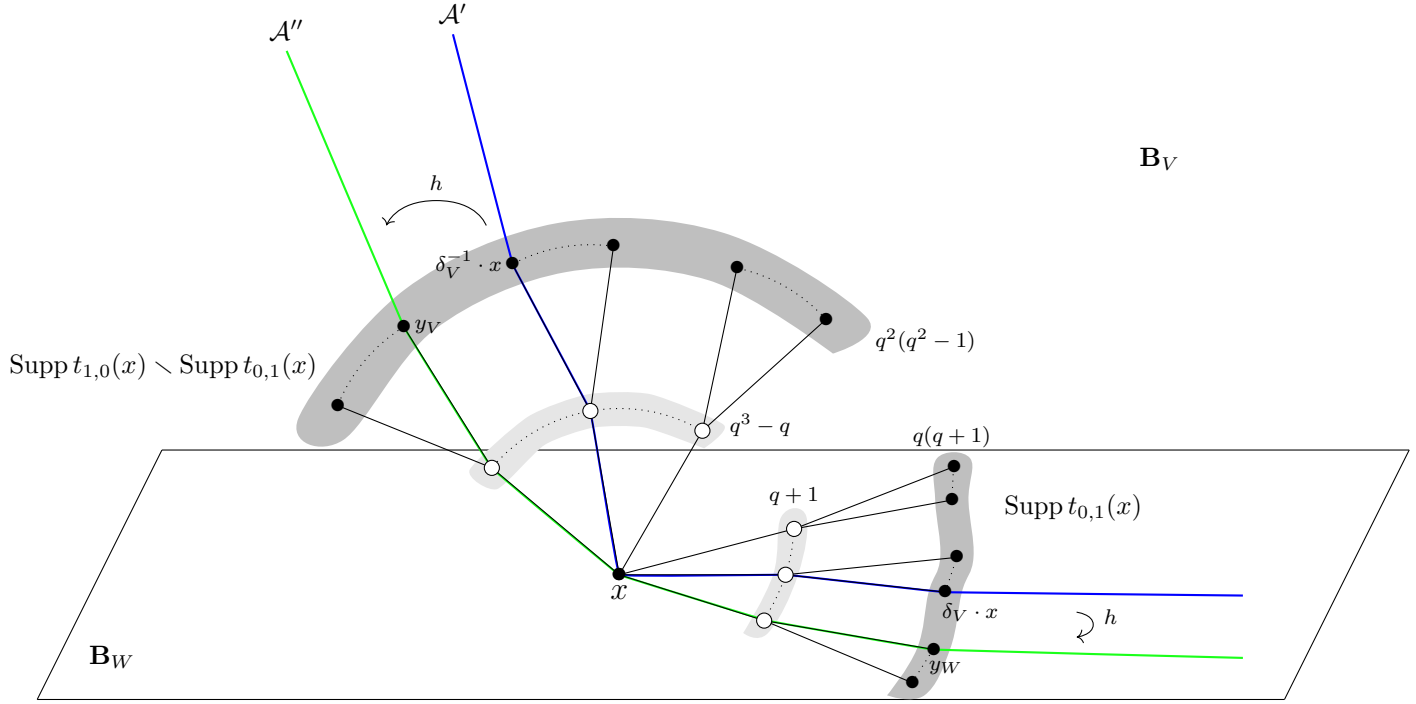


Figure 2.8 – Illustration in the case $n = 0$. The element $h \in H_0$ maps the special apartment \mathcal{A}' (in blue) to \mathcal{A}'' (in green), hence maps $(\delta_V^{-1} \cdot x, \delta_V \cdot x)$ to $(y_V, y_W) \in \mathcal{N}_0$.

$$\text{dist}(\delta_W^k \cdot x, x) = \text{dist}(h \cdot (\delta_W^k \cdot x), h \cdot x) = \text{dist}(h \cdot (\delta_W^k \cdot x), x) = k,$$

for all $k \geq 0$. As h stabilizes both \mathbf{B}_W and $\mathbf{B}_V \setminus \mathbf{B}_W$, there is no other possibility than to have $h \cdot (\delta_V^{-k} \cdot x) = \delta_V^{-k} \cdot x$ and $h \cdot (\delta_W^k \cdot x) = \delta_W^k \cdot x$, for $k \in \{0, \dots, n\}$ - therefore $h \in H_n$ - and:

$$h \cdot (\delta_V^{-(n+1)} \cdot x) = y_V, \quad h \cdot (\delta_V^{n+1} \cdot x) = y_W.$$

This shows the surjectivity of $(\nu_0^{(n)}, \mu_0^{(n)})|_{H_n/H_{n+1}} \rightarrow \mathcal{N}_n$ and finishes the proof. \square

Consequently, one has

$$\#(H_0 / H_1) = \#(\mathcal{N}_0) = q^3(q^2 - 1)(q + 1), \quad (2.36)$$

$$\text{and } \#(H_n / H_{n+1}) = \#(\mathcal{N}_n) = q^6, \quad \text{for all } n \geq 1. \quad (2.37)$$

The last equality admits the following immediate consequence:

Corollary 2.3.12. *The map (2.35) is a bijection, for all $n \geq 1$.*

For all $m \geq 0$, we let $\text{ad}(\delta_V^m)$ and $\text{ad}(\delta_W^m)$ be the morphisms:

$$\begin{aligned} \text{ad}(\delta_V^m) : G_{W,\tau} &\hookrightarrow G_{V,\tau}, \\ h &\longmapsto \delta_V^m h \delta_V^{-m} \end{aligned}$$

and

$$\begin{aligned} \text{ad}(\delta_W^{-m}) : G_{W,\tau} &\hookrightarrow G_{W,\tau}, \\ h &\longmapsto \delta_W^{-m} h \delta_W^m \end{aligned}$$

Lemma 2.3.13. *For all $n \geq 1$, $m \geq 0$ one has:*

$$\text{ad}(\delta_V^m)(H_{n+m}) \subset \mathbf{I}_{V,\tau}^{(n)}, \quad \text{ad}(\delta_W^{-m})(H_{n+m}) \subset \mathbf{I}_{W,\tau}^{(n)} \quad (23)$$

and the map

$$\text{ad}(\delta^{-m}) := (\text{ad}(\delta_V^m), \text{ad}(\delta_W^{-m})) : H_{n+m} \longrightarrow \mathbf{I}_\tau^{(n)}$$

induces a bijection between the quotient sets H_{n+m} / H_{n+m+1} and $\mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)}$.

Notice that, though $\text{ad}(\delta^{-m})$ is a bijection between H_{n+m} / H_{n+m+1} and $\mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)} = H_n / H_{n+1}$, it does not map H_{n+m} to H_n whenever $m > 0$.

Proof. Let $h \in H_{n+m}$. As h fixes the whole segment $\llbracket x, \dots, \delta_V^{-(n+m)} \cdot x \rrbracket$, one has

$$\begin{aligned} \text{ad}(\delta_V^m)(h) \cdot \Xi_{V,0}^{(n)} &:= (\delta_V^m h \delta_V^{-m}) \cdot \llbracket x, \dots, \delta_V^{-n} \cdot x \rrbracket \\ &= \delta_V^m h \cdot \llbracket \delta_V^{-m} \cdot x, \dots, \delta_V^{-(n+m)} \cdot x \rrbracket \\ &= \delta_V^m \llbracket \delta_V^{-m} \cdot x, \dots, \delta_V^{-(n+m)} \cdot x \rrbracket \\ &= \llbracket x, \dots, \delta_V^{-n} \cdot x \rrbracket, \end{aligned}$$

hence $\text{ad}(\delta_V^m)(h) \in \text{Stab}_{G_{V,\tau}}(\Xi_{V,0}^{(n)}) = \mathbf{I}_{V,\tau}^{(n)}$. Similarly, as $h \in H_{n+m}$ fixes the whole segment $\llbracket x, \dots, \delta_W^{n+m} \cdot x \rrbracket = \llbracket x, \dots, \delta_V^{n+m} \cdot x \rrbracket$, one has

$$\begin{aligned} \text{ad}(\delta_W^{-m})(h) \cdot \Xi_{W,0}^{(n)} &= (\delta_W^{-m} h \delta_W^m) \cdot \llbracket x, \dots, \delta_W^n \cdot x \rrbracket \\ &= \delta_W^{-m} h \cdot \llbracket \delta_W^m \cdot x, \dots, \delta_W^{n+m} \cdot x \rrbracket \\ &= \delta_W^{-m} \llbracket \delta_W^m \cdot x, \dots, \delta_W^{n+m} \cdot x \rrbracket \\ &= \llbracket x, \dots, \delta_W^n \cdot x \rrbracket, \end{aligned}$$

hence $\text{ad}(\delta_W^{-m})(h) \in \text{Stab}_{G_{W,\tau}}(\Xi_{W,0}^{(n)}) = \mathbf{I}_{W,\tau}^{(n)}$.

As $\#(H_{n+m} / H_{n+m+1}) = q^6 = \#(\mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)})$ by (2.37), it is enough to check that $\text{ad}(\delta^{-m}) : H_{n+m} / H_{n+m+1} \longrightarrow \mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)}$ is injective, i.e., that $H_\tau \cap \text{ad}(\delta^{-m})^{-1}(\mathbf{I}_\tau^{(n+1)}) \subset H_{n+m+1}$. The latter follows from the fact that, if $h \in G_{W,\tau}$ then $\Delta(h) = (h, h) \in H_\tau$ belongs to $\text{ad}(\delta^{-m})^{-1}(\mathbf{I}_\tau^{(n+1)})$ if and only if

$$\delta_V^m h \delta_V^{-m} \cdot \Xi_{V,0}^{(n+1)} = \Xi_{V,0}^{(n+1)} \quad \text{and} \quad \delta_V^{-m} h \delta_V^m \cdot \Xi_{W,0}^{(n+1)} = \Xi_{W,0}^{(n+1)},$$

(23). Here, we see H_{n+m} as the subgroup $\Delta^{-1}(H_{n+m}) \subset G_{W,\tau}$.

hence

$$h \cdot \Xi_{V,m}^{(n+1)} = \Xi_{V,m}^{(n+1)} \quad \text{and} \quad h \cdot \Xi_{W,m}^{(n)} = \Xi_{W,m}^{(n)}.$$

The latter is equivalent to requiring that h fixes both $\delta_V^{-(n+m)} \cdot x$ and $\delta_W^{n+m} \cdot x$, i.e., that $\Delta(h)$ lie in H_{n+m+1} . \square

2.3.5.3 Trace operators in $\mathbb{Z}[\text{Hyp}^{(n)}]$.

For all $n, k \geq 0$ and for $\star \in \{V, W\}$, we define $\mathbb{Z}[\text{Hyp}_\star^{(n)}]_k \subset \mathbb{Z}[\text{Hyp}_\star^{(n)}]$ to be the submodule formed by linear combinations of n -segments which are fixed (as sums) by the action of the group H_k (seen as a subgroup of $G_{\star,\tau}$) via the natural left-action of $G_{\star,\tau}$ on $\text{Hyp}_\star^{(n)}$. Plainly:

$$\mathbb{Z}[\text{Hyp}_\star^{(n)}]_k := \left(\mathbb{Z}[\text{Hyp}_\star^{(n)}] \right)^{H_k}.$$

This gives an increasing filtration on $\mathbb{Z}[\text{Hyp}_\star^{(n)}]$:

$$\mathbb{Z}[\text{Hyp}_\star^{(n)}]_0 \subsetneq \cdots \subsetneq \mathbb{Z}[\text{Hyp}_\star^{(n)}]_k \subsetneq \mathbb{Z}[\text{Hyp}_\star^{(n)}]_{k+1} \subsetneq \cdots \quad (2.38)$$

For all $n, k \geq 0$, we set $\mathbb{Z}[\text{Hyp}^{(n)}]_k := \mathbb{Z}[\text{Hyp}_V^{(n)}]_k \otimes_{\mathbb{Z}} \mathbb{Z}[\text{Hyp}_W^{(n)}]_k \subset \left(\mathbb{Z}[\text{Hyp}^{(n)}] \right)^{H_k}$, where the last term H_k is now seen as a subgroup of G_τ acting diagonally on $\mathbb{Z}[\text{Hyp}^{(n)}] = \mathbb{Z}[\text{Hyp}_V^{(n)}] \otimes_{\mathbb{Z}} \mathbb{Z}[\text{Hyp}_W^{(n)}]$. Accordingly, we set

$$\mathbb{Z}[\underline{\text{Hyp}}_\star^{(n)}]_k := \mathbb{Z}[\underline{\text{Hyp}}_\star^{(n)}] \cap \mathbb{Z}[\text{Hyp}_\star^{(n)}]_k, \quad \forall \star \in \{V, W\},$$

and

$$\mathbb{Z}[\underline{\text{Hyp}}^{(n)}]_k := \mathbb{Z}[\underline{\text{Hyp}}_V^{(n)}]_k \otimes \mathbb{Z}[\underline{\text{Hyp}}_W^{(n)}]_k = \mathbb{Z}[\underline{\text{Hyp}}^{(n)}] \cap \mathbb{Z}[\text{Hyp}^{(n)}]_k.$$

By construction, one has $\Xi_{\star,m}^{(n)} \in \mathbb{Z}[\underline{\text{Hyp}}_\star^{(n)}]_{n+m}$, for all $\star \in \{V, W\}$, hence $\Xi_m^{(n)} \in \mathbb{Z}[\underline{\text{Hyp}}^{(n)}]_{n+m}$, for all $n, m \geq 0$.

Definition 2.3.9 (The trace operators $\text{Tr}_V^{k',k}$, $\text{Tr}_W^{k',k}$ and $\text{Tr}^{k',k}$). *Let $n \geq 0$ and let $k' \geq k$ be positive integers.*

- For $\star \in \{V, W\}$, we define the trace operator $\text{Tr}_\star^{k',k}$ as the element of $\text{End} \left(\mathbb{Z}[\text{Hyp}_\star^{(n)}]_{k'} \right)$ given by:

$$\begin{aligned} \text{Tr}_\star^{k',k} : \mathbb{Z}[\text{Hyp}_\star^{(n)}]_{k'} &\longrightarrow \mathbb{Z}[\text{Hyp}_\star^{(n)}]_k \subset \mathbb{Z}[\text{Hyp}_\star^{(n)}]_{k'}, \\ \mathfrak{X}_\star &\longmapsto \sum_{h \in H_k/H_{k'}} h \cdot \mathfrak{X}_\star. \end{aligned}$$

- We finally define the diagonal trace operator $\text{Tr}^{k',k} \in \text{End} \left(\mathbb{Z}[\text{Hyp}^{(n)}]_{k'} \right)$, by setting:

$$\begin{aligned} \text{Tr}^{k',k} (\mathfrak{X}_V \otimes \mathfrak{X}_W) &:= \sum_{h \in H_k/H_{k'}} (h \cdot \mathfrak{X}_V) \otimes (h \cdot \mathfrak{X}_W) \in \mathbb{Z}[\text{Hyp}^{(n)}]_k \subset \mathbb{Z}[\text{Hyp}^{(n)}]_{k'}, \\ &\text{for all } \mathfrak{X}_V \in \mathbb{Z}[\text{Hyp}_V^{(n)}]_{k'}, \mathfrak{X}_W \in \mathbb{Z}[\text{Hyp}_W^{(n)}]_{k'}. \end{aligned}$$

A direct computation shows that, for all $\star \in \{V, W, \emptyset\}$ and all $n \geq 0$, one has $\mathrm{Tr}^{k'',k} = \mathrm{Tr}^{k',k} \circ \mathrm{Tr}^{k'',k'}$ whenever $k'' \geq k' \geq k \geq 0$.

For all $n \geq 0$, $m \geq 0$, one has

$$\mathrm{Tr}^{n+m+1,n+m}(\Xi_m^{(n+1)}) := \sum_{h \in H_{n+m}/H_{n+m+1}} (h \cdot \Xi_{V,m}^{(n+1)}) \otimes (h \cdot \Xi_{W,m}^{(n)}).$$

Given any system $(h_i)_{i=1}^{q^6} \subset H_{n+m}$ of representatives for the quotient set H_{n+m}/H_{n+m+1} , one gets by Lemma 2.3.13 that the family $(\mathrm{ad}(\delta^{-m})(h_i))_{i=1}^{q^6} = (\delta^{-m} h_i \delta^m)_{i=1}^{q^6}$ forms a system of representatives for the quotient $I_\tau^{(n)}/I_\tau^{(n+1)}$. A straightforward consequence of Corollary 2.3.9 is now the following:

Proposition 2.3.14 (Vertical distribution relation, I.). *Let $n \geq 1$ and $m \geq 0$. One has:*

$$\mathrm{Tr}^{n+m+1,n+m}(\Xi_m^{(n+1)}) = \mathcal{U}^{(n)}(\Xi_m^{(n)}). \quad (2.39)$$

By Lemma 2.31, the vertical distribution relation (2.39) - an equality in $\mathbb{Z}[\mathrm{Hyp}^{(n+1)}]$ - can be rephrased in terms of n -segments as:

$$\begin{aligned} (\mathbf{s}_n \otimes \mathbf{s}_n) \circ (\mathbf{v}_n \circ \mathbf{v}_n) \circ \mathrm{Tr}^{n+m+1,n+m}(\Xi_m^{(n+1)}) &= \mathrm{Tr}^{n+m+1,n+m}(\Xi_{m+1}^{(n)}) \\ &= t^{(n)}(\Xi_m^{(n)}). \end{aligned} \quad (2.40)$$

As we shall see, the last equality can be interpreted in terms of Shimura varieties as a *twisted* pushforward $(\pi_\delta^{(n)})_*$ between $\mathrm{Sh}_{I^{(n+1)}}(\mathbf{G}, X)$ and $\mathrm{Sh}_{I^{(n)}}(\mathbf{G}, X)$.

2.3.6 An horizontal distribution relation in $|\mathbf{B}_V|$.

In this paragraph, we go back to the case of vertices (i.e., $n = 0$) in order to give some ideas about the proof of Jetchev's *horizontal distribution relation* ([27], Theorem 1.6), although we shall use a slightly different approach⁽²⁴⁾. For all $k \geq 0$, we still set:

$$\mathbb{Z}[\mathrm{Hyp}_V]_k := (\mathbb{Z}[\mathrm{Hyp}_V])^{H_k}, \quad \mathbb{Z}[\mathrm{Hyp}_W]_k := (\mathbb{Z}[\mathrm{Hyp}_W])^{H_k},$$

$$\text{and } \mathbb{Z}[\mathrm{Hyp}]_k := \mathbb{Z}[\mathrm{Hyp}_V]_k \otimes \mathbb{Z}[\mathrm{Hyp}_W]_k \subset (\mathbb{Z}[\mathrm{Hyp}])^{H_k}.$$

Accordingly, one now has trace operators:

$$\mathrm{Tr}_V^{1,0} : \mathbb{Z}[\mathrm{Hyp}_V]_1 \longrightarrow \mathbb{Z}[\mathrm{Hyp}_V]_0 \subset \mathbb{Z}[\mathrm{Hyp}_V]_1,$$

(24). Indeed, Jetchev's Theorem 1.6 is formulated slightly differently - namely, in terms of the trace $\mathrm{Tr}^{2,0} = \mathrm{Tr}^{1,0} \circ \mathrm{Tr}^{2,1}$ - and his proof makes strong use of the distribution of invariants appearing in the sum $\mathrm{He}_\tau(\mathbf{1}) \cdot (0, 0)$ in order to deduce the relation, which we don't.

$$\mathrm{Tr}_W^{1,0} : \mathbb{Z}[\mathrm{Hyp}_W]_1 \longrightarrow \mathbb{Z}[\mathrm{Hyp}_W]_0 \subset \mathbb{Z}[\mathrm{Hyp}_W]_1,$$

and

$$\mathrm{Tr}^{1,0} : \mathbb{Z}[\mathrm{Hyp}]_1 \longrightarrow \mathbb{Z}[\mathrm{Hyp}]_0 \subset \mathbb{Z}[\mathrm{Hyp}]_1,$$

who satisfy - as x is fixed by H_0 - the following compatibilities:

$$\mathrm{Tr}^{1,0}(\mathfrak{X}_V \otimes x) = \mathrm{Tr}_V^{1,0}(\mathfrak{X}_V) \otimes x, \quad \text{and} \quad \mathrm{Tr}^{1,0}(x \otimes \mathfrak{X}_W) = x \otimes \mathrm{Tr}_W^{1,0}(\mathfrak{X}_W), \quad (2.41)$$

for all $\mathfrak{X}_V \in \mathbb{Z}[\mathrm{Hyp}_V]_1$, $\mathfrak{X}_W \in \mathbb{Z}[\mathrm{Hyp}_W]_1$. Typical examples for \mathfrak{X}_V (resp. \mathfrak{X}_W) shall be $\delta_V^{-1} \cdot x$, $\delta_V \cdot x$ (resp. $\delta_V \cdot x$, $t_{0,1}(x)$)

We extend all the previously defined \mathbb{Z} -linear operators (namely, Hecke operators and trace operators) into $\mathbb{Z}[\frac{1}{q^2(q^2-1)}]$ -linear operators. By commutation between the adjacency Hecke operator $t_{1,0}$ (resp. $t_{0,1}$) with the left-action action of $G_{V,\tau}$ (resp. of $G_{W,\tau}$) on $\mathbb{Z}[\mathrm{Hyp}_V]$ (resp. on $\mathbb{Z}[\mathrm{Hyp}_W]$), one obtains that $t_{1,0}(x) \in \mathbb{Z}[\mathrm{Hyp}_V]_0$ and $t_{0,1}(x) \in \mathbb{Z}[\mathrm{Hyp}_V]_0$, hence $t(x \otimes x) = t_{1,0}(x) \otimes t_{0,1}(x) \in \mathbb{Z}[\mathrm{Hyp}]_0$. One may even say a bit more:

Lemma 2.3.15. *One has*

$$t_{1,0}(x \otimes x) = \frac{1}{q(q+1)} \mathrm{Tr}^{1,0}(\delta_V^{-1} \cdot x \otimes x) + \frac{1}{q^2(q^2-1)} \mathrm{Tr}^{1,0}(\delta_V \cdot x \otimes x), \quad (2.42)$$

$$t_{0,1}(x \otimes x) = \frac{1}{q^2(q^2-1)} \mathrm{Tr}^{1,0}(x \otimes \delta_V \cdot x), \quad (2.43)$$

$$\text{and } t(x \otimes x) = \mathrm{Tr}^{1,0}((\delta_V^{-1} \cdot x) \otimes (\delta_V \cdot x)) + \frac{1}{q^2(q^2-1)} \mathrm{Tr}^{1,0}((\delta_V \cdot x) \otimes t_{0,1}(x)). \quad (2.44)$$

In other words, $t_{1,0}(x \otimes x)$, $t_{0,1}(x \otimes x)$ and $t(x \otimes x)$ all belong to $\frac{1}{q^2(q^2-1)} \mathrm{Tr}^{1,0}(\mathbb{Z}[\mathrm{Hyp}]_1)$.

Proof. By Proposition 2.3.11, and as illustrated in Figure 2.8, the quotient H_0/H_1 acts simply transitively on the set $\mathcal{N}_0 = (\mathrm{Supp} t_{1,0}(x) \setminus \mathrm{Supp} t_{0,1}(x)) \times (\mathrm{Supp} t_{0,1}(x))$. As $\delta_V \cdot x \in \mathrm{Supp} t_{0,1}(x)$, one gets

$$\sum_{h \in H_0/H_1} h \cdot (\delta_V \cdot x) = \#(\mathrm{Supp} t_{1,0}(x) \setminus \mathrm{Supp} t_{0,1}(x)) t_{0,1}(x) = q^2(q^2-1) t_{0,1}(x),$$

i.e.,

$$t_{0,1}(x) = \frac{1}{q^2(q^2-1)} \mathrm{Tr}_W^{1,0}(\delta_V \cdot x) = \frac{1}{q^2(q^2-1)} \mathrm{Tr}_V^{1,0}(\delta_V \cdot x).$$

As $\delta_V^{-1} \cdot x \in \mathrm{Supp} t_{1,0}(x) \setminus \mathrm{Supp} t_{0,1}(x)$, one obtains

$$\sum_{h \in H_0/H_1} h \cdot (\delta_V^{-1} \cdot x) = \#(\mathrm{Supp} t_{0,1}(x)) \cdot (t_{1,0}(x) - t_{0,1}(x)) = q(q+1) (t_{1,0}(x) - t_{0,1}(x)),$$

i.e.,

$$t_{1,0}(x) = \frac{1}{q(q+1)} \mathrm{Tr}_V^{1,0}(\delta_V^{-1} \cdot x) + \frac{1}{q^2(q^2-1)} \mathrm{Tr}_V^{1,0}(\delta_V \cdot x).$$

This gives

$$t_{0,1}(x \otimes x) = x \otimes t_{0,1}(x) = \frac{1}{q^2(q^2 - 1)} \mathrm{Tr}^{1,0}(x \otimes \delta_V \cdot x)$$

and

$$t_{1,0}(x \otimes x) = t_{1,0}(x) \otimes x = \frac{1}{q(q+1)} \mathrm{Tr}^{1,0}((\delta_V^{-1} \cdot x) \otimes x) + \frac{1}{q^2(q^2 - 1)} \mathrm{Tr}^{1,0}((\delta_V \cdot x) \otimes x).$$

Applying $t_{0,1}$ to the preceding gives, as $t_{0,1}$ commutes with $G_{W,\tau}$:

$$\begin{aligned} t(x \otimes x) &= t_{0,1}(t_{1,0}(x) \otimes x) \\ &= t_{0,1}\left(\frac{1}{q(q+1)} \mathrm{Tr}^{1,0}((\delta_V^{-1} \cdot x) \otimes x) + \frac{1}{q^2(q^2 - 1)} \mathrm{Tr}^{1,0}((\delta_V \cdot x) \otimes x)\right) \\ &= \frac{1}{q(q+1)} \mathrm{Tr}^{1,0}((\delta_V^{-1} \cdot x) \otimes t_{0,1}(x)) + \frac{1}{q^2(q^2 - 1)} \mathrm{Tr}^{1,0}((\delta_V \cdot x) \otimes t_{0,1}(x)). \end{aligned}$$

Notice that, as H_0/H_1 acts simply transitively on \mathcal{N}_0 then, for all $L \in \mathrm{Supp} t_{0,1}(x)$:

$$\begin{aligned} \mathrm{Tr}^{1,0}((\delta_V^{-1} \cdot x) \otimes L) &= \mathrm{Tr}^{1,0}((\delta_V^{-1} \cdot x) \otimes (\delta_V \cdot x)) \\ &= (t_{1,0}(x) - t_{0,1}(x)) \otimes t_{0,1}(x), \end{aligned}$$

hence

$$\begin{aligned} \mathrm{Tr}^{1,0}((\delta_V^{-1} \cdot x) \otimes t_{0,1}(x)) &= \sum_{L \in \mathrm{Supp} t_{0,1}(x)} \mathrm{Tr}^{1,0}((\delta_V^{-1} \cdot x) \otimes L) \\ &= \#(\mathrm{Supp} t_{0,1}(x)) \mathrm{Tr}^{1,0}((\delta_V^{-1} \cdot x) \otimes (\delta_V \cdot x)) \\ &= q(q+1) \mathrm{Tr}^{1,0}((\delta_V^{-1} \cdot x) \otimes (\delta_V \cdot x)) \end{aligned}$$

This yields finally

$$t(x \otimes x) = \mathrm{Tr}^{1,0}((\delta_V^{-1} \cdot x) \otimes (\delta_V \cdot x)) + \frac{1}{q^2(q^2 - 1)} \mathrm{Tr}^{1,0}((\delta_V \cdot x) \otimes t_{0,1}(x)),$$

as claimed. □

Accordingly, one has

$$\begin{aligned} t_{1,0}^2(x \otimes x) &= t_{1,0}(t_{1,0}(x \otimes x)) \\ &= t_{1,0}\left(\frac{1}{q(q+1)} \mathrm{Tr}^{1,0}(\delta_V^{-1} \cdot x \otimes x) + \frac{1}{q^2(q^2 - 1)} \mathrm{Tr}^{1,0}(\delta_V \cdot x \otimes x)\right) \\ &= \frac{1}{q(q+1)} \mathrm{Tr}^{1,0}(t_{1,0}(\delta_V^{-1} \cdot x) \otimes x) + \frac{1}{q^2(q^2 - 1)} \mathrm{Tr}^{1,0}(t_{1,0}(\delta_V \cdot x) \otimes x) \\ &\in \frac{1}{q^2(q^2 - 1)} \mathrm{Tr}^{1,0}(\mathbb{Z}[\mathrm{Hyp}]_1), \end{aligned}$$

and

$$\begin{aligned} t_{0,1}^2(x \otimes x) &= t_{0,1}(t_{0,1}(x \otimes x)) \\ &= t_{0,1}\left(\frac{1}{q^2(q^2-1)}\mathrm{Tr}^{1,0}(x \otimes \delta_V \cdot x)\right) \\ &= \frac{1}{q^2(q^2-1)}\mathrm{Tr}^{1,0}(x \otimes t_{0,1}(\delta_V \cdot x)) \in \frac{1}{q^2(q^2-1)}\mathrm{Tr}^{1,0}(\mathbb{Z}[\mathrm{Hyp}]_1), \end{aligned}$$

the terms $\mathrm{Tr}^{1,0}(t_{1,0}(\delta_V^{-1} \cdot x) \otimes x)$, $\mathrm{Tr}^{1,0}(t_{1,0}(\delta_V \cdot x) \otimes x)$ and $\mathrm{Tr}^{1,0}(x \otimes t_{0,1}(\delta_V \cdot x))$ being justified by the facts that $t_{1,0}(\delta_V^{-1} \cdot x) \in \mathbb{Z}[\mathrm{Hyp}_V]_1$, $t_{1,0}(\delta_V \cdot x) \in \mathbb{Z}[\mathrm{Hyp}_V]_1$ and $t_{0,1}(\delta_V \cdot x) \in \mathbb{Z}[\mathrm{Hyp}_W]_1$. One may therefore apply the preceding Lemma to the coefficients $C_i \in \mathcal{H}_\tau$, $i \in \{0, \dots, 4\}$ defined at § 2.3.5.1, and obtain that:

$$C_i(x \otimes x) \in \frac{1}{q^2(q^2-1)}\mathrm{Tr}^{1,0}(\mathbb{Z}[\mathrm{Hyp}]_1) \subset \frac{1}{q^2(q^2-1)}\mathbb{Z}[\mathrm{Hyp}]_0, \quad \forall i \in \{0, \dots, 4\},$$

hence

$$\mathrm{He}_\tau^{(4)}(\mathbf{1}) \cdot (x) = \sum_{i=0}^4 C_i(x \otimes x) \in \mathrm{Tr}^{1,0}(\mathbb{Z}[\mathrm{Hyp}]_1).$$

By iterating the same process to

$$\begin{aligned} \mathrm{He}_\tau(\mathbf{1}) \cdot (x \otimes x) &= \mathrm{He}_\tau^{(2)}(\mathbf{1}) \mathrm{He}_\tau^{(4)}(\mathbf{1}) \cdot (x \otimes x) \\ &= -q^2 t_{0,1} \cdot \mathrm{He}_\tau^{(4)}(\mathbf{1}) \cdot (x \otimes x) + (q^6 + q^2(q-1) + 1) \mathrm{He}_\tau^{(4)}(\mathbf{1}) \cdot (x \otimes x), \end{aligned}$$

we get the following result, which relates to ([27], Theorem 1.6):

Proposition 2.3.16. *One has:*

$$\mathrm{He}_\tau(\mathbf{1}) \cdot (x \otimes x) \in \mathrm{Tr}^{1,0}(\mathbb{Z}[\mathrm{Hyp}]_1).$$

We may now use the local (*level-wise*) *vertical relation* (stated in the form of (2.40)) and the above horizontal relation in order to build a "compatible" family of cycles on the tower $(\mathrm{Sh}_{\mathbf{I}^{(n)}}(\mathbf{G}, X))_{n \geq 1}$ constructed via the local Iwahori filtration $(\mathbf{I}_\tau^{(n)})_{n \geq 1}$.

Chapter 3

Global family of special cycles and equivariant cohomology classes.

3.1 Going back to cycles.

3.1.1 Local distribution relations.

We now leave gradually the world of buildings to go back to something more related to our main subject of interest: special cycles. In this subsection, as in the previous chapter, we still work with a fixed allowable inert place τ . In order to treat both vertical and horizontal relations, we shall still allow the integer n to be possibly equal to 0. This subsection being purely local at τ , we shall again (temporarily) omit the subscripts \square_τ in various occasions, but these will be reintroduced later on when considering multiple allowable inert places simultaneously.

3.1.1.1 Interlude on local orders, II.

Notice that the assumption *allowable* on τ could be removed throughout this paragraph. In the previous chapter (§ 2.2.3) we studied the local filtration (2.10) on the unit group $\mathcal{O}_{E_\tau}^\times$ induced by the groups of units of local orders of the form $\mathcal{O}_c := \mathcal{O}_{F_\tau} + \varpi^c \mathcal{O}_{E_\tau}$. In order to relate the *distribution relations* of the preceding chapter, obtained in and expressed via the building \mathbf{B}_V , to actual distribution relations involving Galois traces, we are now interested in computing the structure (and the size) of the successive quotients $\mathcal{O}_c^\times / \mathcal{O}_{c+1}^\times$, for all $c \geq 0$.

Recall that we set $\mathbb{F}_0 := \mathcal{O}_{F_\tau} / \varpi \simeq \mathbb{F}_q$ together with its quadratic extension $\mathbb{F} := \mathcal{O}_{E_\tau} / \varpi \simeq \mathbb{F}_{q^2}$, to be the respective residue fields of F_τ and E_τ , where $\varpi \in F_\tau$ is a fixed common uniformizer of F_τ and E_τ . For all $k \geq 0$, we set (after [7], §VII.1.6):

$$\mathbb{F}_0^k := \mathcal{O}_{F_\tau} / \varpi^k,$$

and

$$\mathbb{F}_0^k[\epsilon] := \frac{\mathbb{F}^k[X]}{(X^2)}$$

to be the attached ring of *dual numbers*. Similarly, we set

$$\mathbb{F}^k := \mathcal{O}_{E_\tau} / \varpi^k.$$

We also set $\mathcal{O}_{-1} := \mathcal{O}_0 = \mathcal{O}_{E_\tau}$.

Lemma 3.1.1. *For all $c \geq 0$, the local order \mathcal{O}_c is a local ring with maximal ideal $\mathfrak{P}_c := \varpi \mathcal{O}_{c-1}$.*

Proof. The case $c = 0$ being immediate, we assume $c \geq 1$. In this case, an element $z = x + \varpi^c y \in \mathcal{O}_c$, $x \in \mathcal{O}_{F_\tau}$, $y \in \mathcal{O}_{E_\tau}$, is invertible in \mathcal{O}_{E_τ} if and only if $x \notin \varpi \mathcal{O}_{F_\tau}$. This gives

$$\mathcal{O}_c \cap \mathcal{O}_{E_\tau}^\times = \mathcal{O}_{F_\tau}^\times + \varpi^c \mathcal{O}_{E_\tau} = \mathcal{O}_{F_\tau}^\times (1 + \varpi^c \mathcal{O}_{E_\tau}).$$

As $\mathcal{O}_{F_\tau}^\times$ and $(1 + \varpi^c \mathcal{O}_{E_\tau})$ are already subgroups of $\mathcal{O}_{E_\tau}^\times$ contained in \mathcal{O}_c , one gets $\mathcal{O}_c^\times = \mathcal{O}_c \cap \mathcal{O}_{E_\tau}^\times = \mathcal{O}_{F_\tau}^\times (1 + \varpi^c \mathcal{O}_{E_\tau})$.

As seen e.g. in Lemma 1.3.13, the local ring \mathcal{O}_{E_τ} is a free \mathcal{O}_{F_τ} -module of rank 2, and admits a basis of the form $\{1, \alpha\}$ for some $\alpha \in \mathcal{O}_{E_\tau}^\times$. One deduces that $\mathcal{O}_c = \mathcal{O}_{F_\tau} \oplus \alpha \varpi^c \mathcal{O}_{F_\tau}$. If $z = x' + \alpha \varpi^c y' \in \mathcal{O}_c$, with $x', y' \in \mathcal{O}_{F_\tau}$, the condition $z \in \mathcal{O}_c^\times$ is, by the preceding, equivalent to $\text{ord}_\varpi(x') = 0$, i.e.,

$$z \in \mathcal{O}_c^\times \iff z \notin \varpi \mathcal{O}_{F_\tau} \oplus \alpha \varpi^c \mathcal{O}_{F_\tau} = \varpi (\mathcal{O}_{F_\tau} \oplus \alpha \varpi^{c-1} \mathcal{O}_{F_\tau}) = \varpi \mathcal{O}_{c-1} =: \mathfrak{P}_c,$$

which concludes. \square

The following result is a special case of ([7], Lemmas 70, 71):

Lemma 3.1.2. *Let $c \geq 0$. For $k \geq 1$, one has:*

$$\mathcal{O}_c^\times / \mathcal{O}_{c+k}^\times \simeq \begin{cases} \frac{(\mathbb{F}^k)^\times}{(\mathbb{F}_0^k)^\times}, & \text{if } c = 0, \\ \frac{(\mathbb{F}_0^k[\epsilon])^\times}{(\mathbb{F}_0^k)^\times}, & \text{if } 1 \leq k \leq c. \end{cases} \quad (3.1)$$

Proof. Assume first that $c = 0$. We need to show that $\mathcal{O}_0^\times / \mathcal{O}_k^\times = \mathcal{O}_{E_\tau}^\times / \mathcal{O}_k^\times$ is isomorphic to $(\mathbb{F}^k)^\times / (\mathbb{F}_0^k)^\times := \frac{(\mathcal{O}_{E_\tau} / \varpi^k \mathcal{O}_{E_\tau})^\times}{(\mathcal{O}_{F_\tau} / \varpi^k \mathcal{O}_{F_\tau})^\times}$. This amounts to showing that the following map:

$$\phi_k : \mathcal{O}_0^\times \xrightarrow{z \mapsto z \bmod \varpi^k} (\mathbb{F}^k)^\times \rightarrow \frac{(\mathbb{F}^k)^\times}{(\mathbb{F}_0^k)^\times},$$

which is surjective - as the $\bmod \varpi^k$ reduction map $\mathcal{O}_0^\times \xrightarrow{z \mapsto z \bmod \varpi^k} (\mathbb{F}^k)^\times$ is - has kernel \mathcal{O}_k^\times . Indeed, an element $z \in \mathcal{O}_0^\times$ satisfies $\phi_k(z) = 1$ if and only if $z \bmod \varpi^k \in (\mathcal{O}_{F_\tau} / \varpi^k \mathcal{O}_{F_\tau})^\times$,

i.e., if there exists some $y \in \mathcal{O}_{F_\tau}$ such that $z - y \in \varpi^k \mathcal{O}_{E_\tau}$, hence $z \in \mathcal{O}_{F_\tau} + \varpi^k \mathcal{O}_{E_\tau} = \mathcal{O}_k$. As z is invertible in \mathcal{O}_{E_τ} , the preceding proof implies that $z \in \mathcal{O}_k^\times$, thus:

$$\frac{\mathcal{O}_0^\times}{\mathcal{O}_k^\times} \simeq \frac{(\mathbb{F}^k)^\times}{(\mathbb{F}_0^k)^\times}.$$

Now, assume that $c \geq 1$. Recall that, by the proof of the preceding lemma, there exists some $\alpha \in \mathcal{O}_{E_\tau}$ such that $\mathcal{O}_{c+k} = \mathcal{O}_{F_\tau} \oplus \alpha \varpi^{c+k} \mathcal{O}_{F_\tau}$, for all $k \geq 0$. Let us write $\alpha^2 = u\alpha + v$, with $u, v \in \mathcal{O}_{F_\tau}$. Assume that $k \in \{1, \dots, c\}$. We define the map:

$$\begin{aligned} \phi_k : \mathcal{O}_c &\longrightarrow \mathbb{F}_0^k[\epsilon] \\ z = x + \alpha \varpi^c y &\longmapsto \bar{x} + \bar{y}\epsilon, \end{aligned}$$

where we set $\bar{x} := x \bmod \varpi^k$, $\bar{y} := y \bmod \varpi^k \in (\mathcal{O}_{F_\tau}/\varpi^k \mathcal{O}_{F_\tau}) =: \mathbb{F}_0^k$. That ϕ_k is additive is immediate. Moreover, if $z = x + \alpha \varpi^c y$, $z' = x' + \alpha \varpi^c y' \in \mathcal{O}_c$ then

$$zz' = xx' + \alpha^2 \varpi^{2c} yy' + \alpha \varpi^c (xy' + x'y) = xx' + v \varpi^{2c} yy' + \alpha \varpi^c (xy' + x'y + \varpi^c uyy').$$

This gives:

$$\begin{aligned} \phi_k(zz') &= \overline{xx' + v \varpi^{2c} yy' + xy' + x'y + \varpi^c uyy'} \epsilon \\ &= \overline{xx'} + \overline{xy' + x'y} \epsilon && (c \geq k) \\ &= (\bar{x} + \bar{y}\epsilon) \cdot (\bar{x}' + \bar{y}'\epsilon) && (\epsilon^2 = 0) \\ &= \phi_k(z) \cdot \phi_k(z'), \end{aligned}$$

hence ϕ_k is indeed a morphism of rings, which is clearly surjective by construction. Accordingly, ϕ_k induces a group morphism $\mathcal{O}_c^\times \longrightarrow (\mathbb{F}_0^k[\epsilon])^\times$, which is again surjective for the following reason: an element $a + b\epsilon \in \mathbb{F}_0^k[\epsilon]$ is invertible (with inverse $a^{-1} - a^{-2}b\epsilon$) if and only if $a \in (\mathbb{F}_0^k)^\times$, i.e., $z = x + \alpha \varpi^c y \in \mathcal{O}_c$ satisfies $\phi_k(z) \in (\mathbb{F}_0^k[\epsilon])^\times$ if and only if $x \bmod \varpi^k \in (\mathcal{O}_F/\varpi^k \mathcal{O}_F)^\times$, hence $\text{ord}_\varpi(x) = 0$, hence $z \in \mathcal{O}_c^\times$. Finally, if $z = x + \alpha \varpi^c y \in \mathcal{O}_c$ then one has $\phi_k(z) \in (\mathbb{F}_0^k)^\times \subset (\mathbb{F}_0^k[\epsilon])^\times$ if and only if $y \bmod \varpi^k = 0 \in \mathbb{F}_0^k$, which means that z can be written as

$$z = x + \alpha \varpi^{c+k} y', \text{ for some } y' \in \mathcal{O}_{F_\tau},$$

i.e., $z \in \mathcal{O}_{F_\tau} \oplus \alpha \varpi^{c+k} \mathcal{O}_{F_\tau} = \mathcal{O}_{c+k}$. This gives $\phi_k^{-1}((\mathbb{F}_0^k)^\times) = \mathcal{O}_{c+k}^\times$, hence

$$\frac{\mathcal{O}_c^\times}{\mathcal{O}_{c+k}^\times} \simeq \frac{(\mathbb{F}_0^k[\epsilon])^\times}{(\mathbb{F}_0^k)^\times},$$

as claimed. □

As a consequence, one gets that $\mathcal{O}_0^\times / \mathcal{O}_1^\times \simeq \mathbb{F}^\times / \mathbb{F}_0^\times$ is a cyclic group of order $\#(\mathbb{F}^\times) / \#(\mathbb{F}_0^\times) = \frac{q^2-1}{q-1} = q+1$. On the other hand, if $c \geq 1$, one has for all $k \in \{1, \dots, c\}$:

$$\begin{aligned} \#(\mathcal{O}_c^\times / \mathcal{O}_{c+k}^\times) &= \#((\mathbb{F}_0^k[\epsilon])^\times / (\mathbb{F}_0^k)^\times) \\ &= \#((\mathbb{F}_0^k)^\times \cdot (1 + \mathbb{F}_0^k \epsilon) / (\mathbb{F}_0^k)^\times) \\ &= \#(1 + \mathbb{F}_0^k \epsilon) \\ &= \#(\mathbb{F}_0^k) \\ &= q^k. \end{aligned}$$

3.1.1.2 From $\mathbb{Z}[\text{Hyp}^{(n)}]$ to $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]$.

We denote by $H_\tau^{\text{der}} \subset H_\tau$ the kernel of the determinant map $\det : H_\tau \rightarrow \text{U}(1)(F_\tau) = \mathcal{O}_0^1$. For all $n \geq 0$, let us denote by π^{der} the natural projection

$$\pi^{\text{der}} : \text{Hyp}^{(n)} \simeq G_\tau / \mathbb{I}_\tau^{(n)} \rightarrow H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}.$$

This map induces naturally a \mathbb{Z} -linear map, still denoted $\pi^{\text{der}} : \mathbb{Z}[\text{Hyp}^{(n)}] \rightarrow \mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]$. The group H_τ acts on the double quotient $H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}$ on the left, by the rule

$$H_\tau^{\text{der}} g \mathbb{I}_\tau^{(n)} \mapsto H_\tau^{\text{der}} hg \mathbb{I}_\tau^{(n)}, \text{ for all } h \in H_\tau.$$

This action is well-defined, as $H_\tau^{\text{der}} \subset H_\tau$ is a normal subgroup, and makes the map π^{der} a $\mathbb{Z}[H_\tau]$ -equivariant map.

Definition 3.1.1. Let $k \geq 0$. We set $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]_k$ to be the submodule of $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]$ fixed by the action of $H_k \subset H_\tau$, i.e.,

$$\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]_k := (\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}])^{H_k}.$$

Notice that, unlike the previous case of $\mathbb{Z}[\text{Hyp}^{(n)}]_k$ - defined as $\mathbb{Z}[\text{Hyp}_V^{(n)}]_k \otimes \mathbb{Z}[\text{Hyp}_W^{(n)}]_k \subset (\mathbb{Z}[\text{Hyp}^{(n)}])^{H_k}$ - one may not define the \mathbb{Z} -module $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]_k \subset \mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]$ diagonally, for the bijection

$$\text{Hyp}_V^{(n)} \times \text{Hyp}_W^{(n)} \xrightarrow{\sim} \text{Hyp}^{(n)}$$

does *not* induce a map

$$H_\tau^{\text{der}} \backslash G_{V,\tau} / \mathbb{I}_{V,\tau}^{(n)} \times H_\tau^{\text{der}} \backslash G_{W,\tau} / \mathbb{I}_{W,\tau}^{(n)} \rightarrow H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}$$

(where we see H_τ^{der} as a subgroup of $G_{\star,\tau}$, for $\star \in \{V, W\}$), hence there is no natural map

$$\mathbb{Z}[H_\tau^{\text{der}} \backslash G_{V,\tau} / \mathbb{I}_{V,\tau}^{(n)}] \otimes \mathbb{Z}[H_\tau^{\text{der}} \backslash G_{W,\tau} / \mathbb{I}_{W,\tau}^{(n)}] \rightarrow \mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}].$$

Accordingly, π^{der} maps $\mathbb{Z}[\text{Hyp}^{(n)}]_k$ into $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]_k$, for all $k \geq 0$.

3.1.1.3 The map $\det^\#$ and the trace operators $\mathrm{Tr}_{k',k}^{\mathrm{der}}$.

Let $k \geq 0$. According to (2.34), the determinant map yields a surjective map

$$\det^\# : H_k / H_{k+1} \twoheadrightarrow \mathcal{O}_k^1 / \mathcal{O}_{k+1}^1 \quad (3.2)$$

The above map is not a group homomorphism (as the subgroup $H_{k+1} \subset H_k$ is not normal) but one still has the following result:

Lemma 3.1.3. *Let $k \geq 0$. The fibers of the map $\det^\# : H_k / H_{k+1} \twoheadrightarrow \mathcal{O}_k^1 / \mathcal{O}_{k+1}^1$ all have the same cardinality.*

Proof. Let $s \in \mathcal{O}_k^1$ and set $\bar{s} := s \mathcal{O}_{k+1}^1 \in \mathcal{O}_k^1 / \mathcal{O}_{k+1}^1$. The subgroup $H_k^{\mathrm{der}} := H_k \cap H_\tau^{\mathrm{der}} = \ker(\det : H_k \rightarrow \mathcal{O}_k^1)$ of H_k acts on $(\det^\#)^{-1}(\bar{s})$ by left-multiplication, and this action is transitive. Indeed, let h and h' be elements of H_k such that hH_{k+1} and $h'H_{k+1}$ both lie in $(\det^\#)^{-1}(\bar{s})$. Then $\det(h)$ and $\det(h')$ both lie in $s \mathcal{O}_{k+1}^1$, hence $\det(h'h^{-1}) \in \mathcal{O}_{k+1}^1$. Accordingly, there exists some $h'' \in H_{k+1}$ such that $\det(h'h^{-1}) = \det(h'')$, i.e., $\det(h'(h'')^{-1}h^{-1}) = 1$. This gives $h'H_{k+1} = (h'(h'')^{-1}h^{-1}) \cdot (hH_{k+1}) \in H_k^{\mathrm{der}} \cdot (hH_{k+1})$, which shows that $(\det^\#)^{-1}(\bar{s})$ is a single H_k^{der} -orbit, whose cardinality equals $\# \left(H_k^{\mathrm{der}} / \mathrm{Stab}_{H_k^{\mathrm{der}}}(hH_{k+1}) \right)$, for any $h \in H_k$ such that $hH_{k+1} \in (\det^\#)^{-1}(\bar{s})$. Notice that

$$\mathrm{Stab}_{H_k^{\mathrm{der}}}(hH_{k+1}) = H_k^{\mathrm{der}} \cap hH_{k+1}h^{-1} = hH_{k+1}^{\mathrm{der}}h^{-1},$$

therefore the quotient $H_k^{\mathrm{der}} / \mathrm{Stab}_{H_k^{\mathrm{der}}}(hH_{k+1}) = H_k^{\mathrm{der}} / hH_{k+1}^{\mathrm{der}}h^{-1}$ is in bijection with $H_k^{\mathrm{der}} / H_{k+1}^{\mathrm{der}}$ (via $g \mapsto gh$), for all $h \in H_k$. The latter being independent of h - hence of s - we obtain that the fibers of $\det^\#$ are all of the same cardinality. \square

For all $k \geq 0$, the left-action of $H_k \subset H_\tau$ on $\mathrm{Hyp}^{(n)}$ induces a natural action of $H'_k := H_\tau^{\mathrm{der}} \backslash H_k$ on $H_\tau^{\mathrm{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}$. Accordingly, one may extend the notion of trace operators to $\mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / K_\tau]$ in the following way:

Definition 3.1.2 (Trace operators $\mathrm{Tr}_{k',k}^{\mathrm{der}}$). *Let $k' \geq k \geq 0$.*

The trace operator $\mathrm{Tr}_{k',k}^{\mathrm{der}} : \mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]_{k'} \longrightarrow \mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]_k \subset \mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]_{k'}$ is defined by:

$$\mathrm{Tr}_{k',k}^{\mathrm{der}}(\mathfrak{X}) := \sum_{h \in H'_{k'} / H'_k} h \cdot \mathfrak{X},$$

for all $\mathfrak{X} \in \mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau / \mathbb{I}_\tau^{(n)}]_{k'}$.

By Lemmas 3.1.3, 3.1 and equality (2.37), the projection map $H_k / H_{k+1} \twoheadrightarrow H'_k / H'_{k+1} \xrightarrow{\det^\#} \mathcal{O}_k^1 / \mathcal{O}_{k+1}^1$ induces the following equalities: for all $\mathfrak{X} \in \mathbb{Z}[\mathrm{Hyp}^{(n)}]_{k'}$, one has:

$$\pi^{\mathrm{der}}(\mathrm{Tr}^{k+1,k}(\mathfrak{X})) = \begin{cases} q^3(q^2 - 1) \mathrm{Tr}_{k+1,k}^{\mathrm{der}}(\pi^{\mathrm{der}}(\mathfrak{X})), & \text{if } k = 0. \\ q^5 \mathrm{Tr}_{k+1,k}^{\mathrm{der}}(\pi^{\mathrm{der}}(\mathfrak{X})), & \text{if } k \geq 1. \end{cases} \quad (3.3)$$

3.1.1.4 An horizontal relation in $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / K_\tau]$.

We assume here that $n = 0$, i.e., that $I_\tau^{(n)} = K_\tau$. Proposition (2.3.16) gives us the existence of an element $\mathfrak{X}_\tau \in \mathbb{Z}[\text{Hyp}]_1$ - which may be explicitly described in terms of the adjacency operators $t_{1,0}$ and $t_{0,1}$, and of the hyperspecial vertices $\delta_V^{-1} \cdot x$, $\delta_V \cdot x$, thanks to Lemma 2.3.15 and the discussion thereafter - such that

$$\text{He}_\tau(\mathbf{1}) \cdot (x \otimes x) = \frac{1}{q^2(q^2 - 1)} \text{Tr}^{1,0}(\mathfrak{X}_\tau). \quad (3.4)$$

Notice that, since the operators $t_{1,0}$, $t_{0,1}$ and $t \in \mathcal{H}_\tau$ act on $\mathbb{Z}[\text{Hyp}] = \mathbb{Z}[G_\tau / K_\tau]$ in a $\mathbb{Z}[G_\tau]$ -equivariant way, they also act naturally on $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / K_\tau]$ by the exact same formulas. This gives sense to the operator $\text{He}_\tau(\mathbf{1}) \in \text{End}_{\mathbb{Z}}(\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / K_\tau])$, and one checks immediately that projection map $\pi^{\text{der}} : \mathbb{Z}[\text{Hyp}] \rightarrow \mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / K_\tau]$ intertwines the respective actions of $\text{He}_\tau(\mathbf{1})$ on $\mathbb{Z}[\text{Hyp}]$ and on $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / K_\tau]$. Set $\tilde{y}_\tau := \pi^{\text{der}}(q\mathfrak{X}_\tau) \in \mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / K_\tau]_1$.

One obtains:

$$\begin{aligned} \text{He}_\tau(\mathbf{1}) \cdot \pi^{\text{der}}((x \otimes x)) &= \pi^{\text{der}}(\text{He}_\tau(\mathbf{1}) \cdot (x \otimes x)) \\ &= \pi^{\text{der}}\left(\frac{1}{q^2(q^2 - 1)} \text{Tr}^{1,0}(\mathfrak{X}_\tau)\right) && \text{by (3.4)} \\ &= q \text{Tr}_{1,0}^{\text{der}}(\pi^{\text{der}}(\mathfrak{X}_\tau)) && \text{by (3.3)} \\ &= \text{Tr}_{1,0}^{\text{der}}(\tilde{y}_\tau). && (3.5) \end{aligned}$$

Consequently, for every allowable inert place τ , we shall fix arbitrarily a lift $\hat{y}_\tau \in \mathbb{Z}[G_\tau]$ of $q\mathfrak{X}_\tau \in \mathbb{Z}[\text{Hyp}]$ - thus a lift of $\tilde{y}_\tau \in \mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / K_\tau]$ - i.e., one has

$$\hat{y}_\tau = \sum_{i \in I} a_i g_{\tau,i} \in \mathbb{Z}[G_\tau], \quad \text{with} \quad \sum_{i \in I} a_i (g_{\tau,i} \cdot x) = q\mathfrak{X}_\tau \in \mathbb{Z}[\text{Hyp}]. \quad (3.6)$$

Notation. We shall adopt, whenever dealing with vertices, the following notational convention: given an allowable inert place τ , a general element of $\mathbb{Z}[\text{Hyp}] \simeq \mathbb{Z}[G_\tau / K_\tau]$ will now be denoted by the symbol y , whereas \hat{y} (resp. \tilde{y}) will be used to denote a general element in $\mathbb{Z}[G_\tau]$ (resp. in $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / K_\tau]$).

3.1.1.5 Vertical relation in $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / I_\tau^{(n)}]$, $n \geq 1$.

We restrict to $n \geq 1$ in this paragraph. In a similar way as above, the double coset operator $t^{(n)} = \mathbf{1}_{I_\tau^{(n)}} \delta_{I_\tau^{(n)}} \in \mathcal{H}_{\mathbb{Z}}(G_\tau // I_\tau^{(n)})$, defined at § 2.3.5.1, acts on $\mathbb{Z}[\text{Hyp}^{(n)}] = \mathbb{Z}[G_\tau / I_\tau^{(n)}]$ in a $\mathbb{Z}[G_\tau]$ -equivariant way, hence also acts naturally on $\mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / I_\tau^{(n)}]$, and these two actions are intertwined by the projection $\pi^{\text{der}} : \mathbb{Z}[\text{Hyp}^{(n)}] \rightarrow \mathbb{Z}[H_\tau^{\text{der}} \backslash G_\tau / I_\tau^{(n)}]$. The equality (3.3) now induces:

$$\begin{aligned} \mathrm{Tr}_{n+m+1, n+m}^{\mathrm{der}} \left(q^5 \pi^{\mathrm{der}}(\Xi_{m+1}^{(n)}) \right) &= \pi^{\mathrm{der}} \left(\mathrm{Tr}^{n+m+1, n+m}(\Xi_{m+1}^{(n)}) \right) \\ &= \pi^{\mathrm{der}} \left(t^{(n)} \cdot \Xi_m^{(n)} \right) \\ &= t^{(n)} \cdot \pi^{\mathrm{der}} \left(\Xi_m^{(n)} \right). \end{aligned} \tag{3.7}$$

Using the isomorphism $\mathbb{Z}[\mathrm{Hyp}^{(n)}] \simeq \mathbb{Z}[G_\tau/I_\tau^{(n)}] = \mathbb{Z}[G_\tau/\mathrm{Stab}_{G_\tau}(\Xi_0^{(n)})]$, and for $\Xi_{m+1}^{(n)} = \delta^{m+1} \cdot \Xi_0^{(n)}$, the above equality may be rewritten in $\mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau/I_\tau^{(n)}]$ in the following useful form:

$$q^5 \mathrm{Tr}_{n+m+1, n+m}^{\mathrm{der}} (\delta^{m+1}) = \sum_{i \in I_\tau^{(n)} \backslash \delta I_\tau^{(n)} / I_\tau^{(n)}} \delta^m i \in \mathbb{Z}[H_\tau^{\mathrm{der}} \backslash G_\tau/I_\tau^{(n)}]. \tag{3.8}$$

3.1.2 The global Iwahori filtration $(I^{(n)})_{n \geq 1}$.

Recall that we fixed in § 1.2.4 a *base* element $g_0 \in \mathbf{G}(\mathbb{A}_f)$ which, together with our choice of *base level* K , enabled us to define a finite set $\Sigma \subset I_{F,f}$ such that $g = (g_{0,v})_{v \in I_{F,f}}$, with $g_{0,v} \in K_v$, for all $v \notin \Sigma$. Accordingly, we made the simplifying assumption that $g_0^\Sigma = \mathbf{1}^\Sigma \in (\mathrm{U}(V) \times \mathrm{U}(W))(\mathbb{A}_{F,f}^\Sigma)$, which didn't modify the base cycle $\mathcal{Z}_K(g_0) \in \mathcal{Z}_K(\mathbf{G}, \mathbf{H})$. We let $\mathcal{I}_{E/F} \subset I_{F,f}$ denote the set of finite places τ which are inert in E/F . By Definition 1.2.5, the set $\mathcal{I}_{E/F}^{S^2} := \mathcal{I}_{E/F} \setminus S^2$ then corresponds to the set of *allowable* inert places of F , and we let $\mathcal{I}_{E/F}^\Sigma := \mathcal{I}_{E/F} \setminus \Sigma \subset \mathcal{I}_{E/F}^{S^2}$ be the subset formed by those allowable inert places τ of F such that $g_{0,\tau} = 1$. Recall that, back in § 1.3.8.1, we defined (after Nekovář) the \mathcal{O}_F -ideal $I_0 := \mathrm{lcm}\{(u-1); u \in (\mathcal{O}_E^\times)_{\mathrm{tors}} \setminus \{1\}\}$ and showed in (1.60) the existence of some \mathcal{O}_F -ideal \mathfrak{c}_1 such that

$$\mathcal{K}(\mathfrak{f}) \subset E(\mathfrak{c}_1 \cdot \mathfrak{f}),$$

for all \mathcal{O}_F -ideal \mathfrak{f} not dividing I_0 . Accordingly, we shall define the following subset of allowable inert primes:

$$\mathcal{I} := \mathcal{I}_{E/F}^{\Sigma, I_0, \mathfrak{c}_1} = \{v \in \mathcal{I}_{E/F}^\Sigma \text{ is an allowable inert place of } F; v \nmid I_0, v \nmid \mathfrak{c}_1\}.$$

Let us fix an arbitrary allowable inert place $\tau \in \mathcal{I}$, which will be referred to as the *vertical place* or *p-adic place*, where the prime number p denotes the residue characteristic of $\tau^{(1)}$. For all $n \geq 1$, recall that the *Iwahori subgroup* $I_\tau^{(n)}$ was defined at § 2.3.5 to be the product $I_{V,\tau}^{(n)} \times I_{W,\tau}^{(n)}$ of the respective stabilizers (in $G_{V,\tau}$ and $G_{W,\tau}$) of the allowable segments $\Xi_{V,0}^{(n)} = \llbracket x, \delta_V^{-1} \cdot x, \dots, \delta_V^{-n} \cdot x \rrbracket$ and $\Xi_{W,0}^{(n)} = \llbracket x, \delta_V \cdot x, \dots, \delta_V^n \cdot x \rrbracket$ of length n in the τ -local Bruhat-Tits building $\mathbf{B}_{V,\tau}$. The τ -local filtration $(I_\tau^{(n)})_{n \geq 1}$ induces the following global filtration on $K \subset \mathbf{G}(\mathbb{A}_f)$:

(1). For instance, if the Galois closure M/\mathbb{Q} of the extension E/\mathbb{Q} is a cyclic extension, then one may start by choosing a rational prime number p which is inert in E/\mathbb{Q} and such that $p \cdot \mathcal{O}_F$ is prime to Σ (there are infinitely many of those prime numbers, by Chebotarev density theorem), and finally set $\tau := p \cdot \mathcal{O}_F$.

Definition 3.1.3. For all $n \geq 1$, we set $I^{(n)} \subset K$ to be the neat compact open subgroup of $\mathbf{G}(\mathbb{A}_f)$ defined by

$$I^{(n)} := I_\tau^{(n)} \times K^\tau = K_\Sigma \times I_\tau^{(n)} \times \prod_{v \notin \Sigma \cup \{\tau\}} K_v.$$

We set $I_\Sigma^{(n)} := K_\Sigma$ and $I^{(n), \Sigma} := I_\tau^{(n)} \times \prod_{v \notin \Sigma \cup \{\tau\}} K_v$, so that $I^{(n)} = I_\Sigma^{(n)} \times I^{(n), \Sigma}$. We let $\mathcal{I}^\tau := \mathcal{I} \setminus \{\tau\}$ be the set of allowable inert primes away from I_0 , \mathfrak{c}_1 and τ . We set \mathcal{P} to be the set of square-free products⁽²⁾ of elements of \mathcal{I} , and \mathcal{P}^τ to be the set of square-free products of elements of \mathcal{I}^τ .

Remark 3.1.1. Let $v \in \mathcal{I}$. The embedding at v ,

$$\begin{aligned} \phi_v : E_v^\times &\hookrightarrow \mathbf{T}(\mathbb{A}_f) = \mathbb{A}_{E,f}^\times, \\ x &\longmapsto (x, (1)_{w \neq v}), \end{aligned}$$

admits the following variants (still denoted in the same way):

$$\begin{aligned} \phi_v : \mathbf{U}(1)_{E/F}(F_v) &\hookrightarrow \mathbf{U}(1)_{E/F}(\mathbb{A}_{F,f}) = \mathbf{T}^1(\mathbb{A}_f), \\ x_v &\longmapsto (x_v, (1)_{w \neq v}), \\ &\text{and} \\ \phi_v : G_v &\hookrightarrow \mathbf{G}(\mathbb{A}_f), \\ g_v &\longmapsto (g_v, \mathbf{1}^v), \end{aligned}$$

with $\mathbf{1}^v$ being the identity of $(\mathbf{U}(V) \times \mathbf{U}(W))(\mathbb{A}_{F,f}^v)$. These satisfy the following compatibilities for \det and ν : if $h \in H_v \subset G_v$ then $\det(\phi_v(h)) = (\det(h), (1)_{w \neq v}) = \phi_v(\det(h)) \in \mathbf{T}^1(\mathbb{A}_f)$. Also, if $x \in E_v^\times$, then $\nu(\phi_v(x)) = (\nu(x), (1)_{w \neq v}) = \phi_v(\nu(x)) \in \mathbf{T}^1(\mathbb{A}_f)$.

Let us denote by $\text{Art}_v^1 : \mathbf{U}(1)_{E/F}(F_v) \hookrightarrow \text{Gal}(E(\infty)/E)$ the map obtained by composing $\text{Art}_E^1 : \mathbf{T}^1(\mathbb{A}_f) \xrightarrow{\sim} \text{Gal}(E(\infty)/E)$ with $\phi_v : \mathbf{U}(1)_{E/F}(F_v) \hookrightarrow \mathbf{T}^1(\mathbb{A}_f)$. The compatibility between local and global Artin maps can be expressed as:

$$\text{Art}_E(\phi_v(x)) = \text{Art}_v(x) \Big|_{E^{ab}} \in \text{Gal}(E^{ab}/E),$$

for all $x \in E_v^\times$, thus by restricting to $E(\infty)$:

$$\begin{aligned} \text{Art}_v(x) \Big|_{E(\infty)} &= \text{Art}_E(\phi_v(x)) \Big|_{E(\infty)} = \text{Art}_E^1(\nu \circ \phi_v(x)) \\ &= \text{Art}_E^1(\phi_v \circ \nu(x)) \stackrel{\text{def}}{=} \text{Art}_v^1(\nu(x)). \end{aligned}$$

(2). Including the product 1, indexed by \emptyset .

Let $\sigma \in W_{E_v}^{ab} \subset \text{Gal}(E_v^{ab}/E_v)$ belong to the image of Art_v , i.e., $\sigma = \text{Art}_v(x)$ for some $x \in E_v^\times$. If $h \in H_v$ is such that $\det(h) = \nu(x) \in \text{U}(1)_{E/F}(F_v)$, the preceding implies that:

$$\text{Art}_E^1 \circ \det(\phi_v(h)) = \text{Art}_E^1 \circ \phi_v(\det(h)) \stackrel{\text{def}}{=} \text{Art}_v^1(\det(h)) = \sigma|_{E(\infty)}.$$

By Proposition 1.42, one then obtains:

$$\mathcal{Z}_{\mathbf{I}^{(n)}}(\phi_v(h)g) = \sigma \cdot \mathcal{Z}_{\mathbf{I}^{(n)}}(g), \quad \forall g \in \mathbf{G}(\mathbb{A}_f). \quad (3.9)$$

Notice that the map $\phi_\tau : G_\tau \hookrightarrow \mathbf{G}(\mathbb{A}_f)$, $g_\tau \mapsto (g_\tau, \mathbf{1}^\tau)$ induces a bijection

$$\mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)} \xrightarrow{\sim} \mathbf{I}^{(n)} / \mathbf{I}^{(n+1)},$$

hence $\#(\mathbf{I}^{(n)} / \mathbf{I}^{(n+1)}) = q^6$, for all $n \geq 1$.

Notations. From now on, we shall switch back to *global* notations and reintroduce - when necessary - some subscripts referring to the various allowable inert places which will be dealt with simultaneously. For instance, whenever $v \in \mathcal{I}$ we shall denote by $\mathcal{O}_{v,c}^\times$ and $\mathcal{O}_{v,c}^1$ ($c \geq 0$) the terms of the local filtrations (2.10) and (2.11) attached to v ; and by $H_{v,k} = H_v \cap \mathbf{I}_v^{(k)}$, $H_{v,k}^{\text{der}} = H_{v,k} \cap H_v^{\text{der}}$, $H'_{v,k} := H_{v,k}^{\text{der}} \setminus H_{v,k}$ ($k \geq 0$) the corresponding terms of the v -local filtration on H_v .

We shall adapt Lemma 65 of [7] in the case of K being replaced by its global Iwahori variant $\mathbf{I}^{(n)}$, $n \geq 1$. First, notice that the inclusion (2.22) has the following immediate consequence:

Lemma 3.1.4. *One has*

$$\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap K^\Sigma = \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{I}^{(n),\Sigma}, \quad \text{for all } n \geq 1.$$

This enables the following description of the stabilizer of $\mathcal{Z}_{\mathbf{I}^{(n)}}(g_0)$ in $\mathbf{H}(\mathbb{A}_f)$:

Lemma 3.1.5. *One has*

$$\text{Stab}_{\mathbf{H}(\mathbb{A}_f)}(\mathcal{Z}_{\mathbf{I}^{(n)}}(g_0)) = \mathbf{H}(\mathbb{Q}) \cdot \left(K_{\mathbf{H},g_0,\Sigma}^{\mathbf{Z}} \times \mathbf{I}_{\mathbf{H}}^{(n),\Sigma} \right), \quad (3.10)$$

with

$$K_{\mathbf{H},g_0,\Sigma}^{\mathbf{Z}} \stackrel{\text{Prop. 1.2.5}}{:=} ((\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap K^\Sigma) \cdot g_{0,\Sigma} K_{\Sigma} g_{0,\Sigma}^{-1}) \cap \mathbf{H}(F_\Sigma) = ((\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{I}^{(n),\Sigma}) \cdot g_{0,\Sigma} K_{\Sigma} g_{0,\Sigma}^{-1}) \cap \mathbf{H}(F_\Sigma),$$

$$\text{and } \mathbf{I}_{\mathbf{H}}^{(n),\Sigma} := (H_\tau \cap \mathbf{I}_\tau^{(n)}) \times K_{\mathbf{H}}^{\Sigma,\tau} = H_n \times \Delta \left(\prod_{v \notin \Sigma \cup \{\tau\}} K_{W,v} \right).$$

Proof. In a similar way as in the proof of ([7], Lemma 65), one may observe that:

$$\begin{aligned} \text{Stab}_{\mathbf{H}(\mathbb{A}_f)}(\mathcal{Z}_{\mathbf{I}^{(n)}}(g_0)) &= (\mathbf{H}(\mathbb{Q})\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cdot g_0 \mathbf{I}^{(n)} g_0^{-1}) \cap \mathbf{H}(\mathbb{A}_f) \\ &= \mathbf{H}(\mathbb{Q}) \cdot ((\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cdot g_0 \mathbf{I}^{(n)} g_0^{-1}) \cap \mathbf{H}(\mathbb{A}_f)) \\ &\subset \mathbf{H}(\mathbb{Q}) \cdot (((\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{Z}_{\mathbf{H}}(\mathbb{Q}) K^\Sigma) \cdot g_0 \mathbf{I}^{(n)} g_0^{-1}) \cap \mathbf{H}(\mathbb{A}_f)) \end{aligned}$$

the last inclusion coming from the fact that $(g_0)_v = 1, \forall v \notin \Sigma$. Indeed, if $z_{\mathbb{Q}} = (z_V, z_W) \in \mathbf{Z}_{\mathbf{G}}(\mathbb{Q})$ satisfies $z_{\mathbb{Q}}(g_{0,\Sigma}, \mathbf{1}^{\Sigma}) i (g_{0,\Sigma}^{-1}, \mathbf{1}^{\Sigma}) \in \mathbf{H}(\mathbb{A}_f)$ for some $i = (k_{\Sigma}, k^{\Sigma}) \in \mathbf{I}^{(n)}$, with $k^{\Sigma} = (k_V^{\Sigma}, k_W^{\Sigma}) \in \mathbf{I}_V^{(n),\Sigma} \times \mathbf{I}_W^{(n),\Sigma} = \mathbf{I}^{(n),\Sigma}$, then

$$z_{\mathbb{Q}} k^{\Sigma} \in \Delta(\underline{\mathbf{U}}_W)(\mathbb{A}_{F,f}^{\Sigma}),$$

hence $z_V k_V^{\Sigma} = \iota(z_W k_W^{\Sigma})$, i.e., $z_V \iota(z_W)^{-1} = \iota(k_W^{\Sigma})(k_V^{\Sigma})^{-1} \in K_{V,\tau}$. This gives $z_{\mathbb{Q}} = (\iota(z_W), z_W) \cdot (z_V \iota(z_W)^{-1}, 1) \in \mathbf{Z}_{\mathbf{H}}(\mathbb{Q}) \cdot K^{\Sigma}$. By equality (1) of the proof of (*loc. cit.*), one has $\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{Z}_{\mathbf{H}}(\mathbb{Q}) K^{\Sigma} = \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap K^{\Sigma}$, which equals $\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{I}^{(n),\Sigma}$ by the preceding Lemma, i.e.,

$$\text{Stab}_{\mathbf{H}(\mathbb{A}_f)}(\mathcal{Z}_{\mathbf{I}^{(n)}}(g_0)) \subset \mathbf{H}(\mathbb{Q}) \cdot ((\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{I}^{(n),\Sigma}) \cdot g_0 \mathbf{I}^{(n)} g_0^{-1}) \cap \mathbf{H}(\mathbb{A}_f).$$

If $z_{\mathbb{Q}} \in \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{I}^{(n),\Sigma}$ and $i = (k_{\Sigma}, i_{\tau}, k^{\Sigma,\tau}) \in K_{\Sigma} \times \mathbf{I}_{\tau}^{(n)} \times K^{\Sigma,\tau} =: \mathbf{I}^{(n)}$ are such that $z_{\mathbb{Q}} g_0 i g_0^{-1} = (z_{\mathbb{Q}} g_{0,\Sigma} k_{\Sigma} g_{0,\Sigma}^{-1}, z_{\mathbb{Q}} i_{\tau}, z_{\mathbb{Q}} k^{\Sigma,\tau}) \in \mathbf{H}(\mathbb{A}_f)$, then $z_{\mathbb{Q}} g_{0,\Sigma} k_{\Sigma} g_{0,\Sigma}^{-1} \in \mathbf{H}(F_{\Sigma})$, $z_{\mathbb{Q}} i_{\tau} \in H^{\tau} \cap \mathbf{I}_{\tau}^{(n)} = H_n$ and $z_{\mathbb{Q}} k^{\Sigma,\tau} \in K^{\Sigma,\tau} \cap \Delta(\mathbf{U}(W)(\mathbb{A}_{F,f}^{\Sigma,\tau})) = K_{\mathbf{H}}^{\Sigma,\tau}$, i.e.,

$$z_{\mathbb{Q}} g_0 i g_0^{-1} \in ((\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{I}^{(n),\Sigma}) \cdot g_{0,\Sigma} K_{\Sigma} g_{0,\Sigma}^{-1}) \cap \mathbf{H}(F_{\Sigma}) \times \mathbf{I}_{\mathbf{H}}^{(n),\Sigma},$$

which shows that

$$\text{Stab}_{\mathbf{H}(\mathbb{A}_f)}(\mathcal{Z}_{\mathbf{I}^{(n)}}(g_0)) \subset \mathbf{H}(\mathbb{Q}) \cdot (K_{\mathbf{H},g_0,\Sigma}^{\mathbf{Z}} \times \mathbf{I}_{\mathbf{H}}^{(n)}).$$

Conversely, if

$$h_{\mathbb{Q}}(z_{\mathbb{Q}} g_{0,\Sigma} k_{\Sigma} g_{0,\Sigma}^{-1}, i, k^{\Sigma,\tau}) \in \mathbf{H}(\mathbb{Q}) \cdot (((\mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{I}^{(n),\Sigma}) \cdot g_{0,\Sigma} K_{\Sigma} g_{0,\Sigma}^{-1}) \cap \mathbf{H}(F_{\Sigma}) \times \mathbf{I}_{\mathbf{H}}^{(n)}),$$

with $h_{\mathbb{Q}} \in \mathbf{H}(\mathbb{Q})$, $z_{\mathbb{Q}} \in \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{I}^{(n),\Sigma}$, $k_{\Sigma} \in K_{\Sigma}$, $i \in H_n$, and $k^{\Sigma,\tau} \in K_{\mathbf{H}}^{\Sigma,\tau}$, then one has:

$$\begin{aligned} h_{\mathbb{Q}}(z_{\mathbb{Q}} g_{0,\Sigma} k_{\Sigma} g_{0,\Sigma}^{-1}, i, k^{\Sigma,\tau}) g_0 &= h_{\mathbb{Q}} z_{\mathbb{Q}} (g_{0,\Sigma} k_{\Sigma}, z_{\mathbb{Q}}^{-1} i, z_{\mathbb{Q}}^{-1} k^{\Sigma,\tau}) \\ &= h_{\mathbb{Q}} z_{\mathbb{Q}} (g_{0,\Sigma}, \mathbf{1}^{\Sigma}) (k_{\Sigma}, z_{\mathbb{Q}}^{-1} i, z_{\mathbb{Q}}^{-1} k^{\Sigma,\tau}) \in \mathbf{H}(\mathbb{Q}) \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) g_0 \mathbf{I}^{(n)}, \end{aligned}$$

as $z_{\mathbb{Q}}^{-1} i \in \mathbf{I}_{\tau}^{(n)}$ and $z_{\mathbb{Q}}^{-1} k^{\Sigma,\tau} \in K^{\Sigma,\tau} = \mathbf{I}^{(n),\Sigma,\tau}$, i.e., $\mathbf{H}(\mathbb{Q}) \cdot (K_{\mathbf{H},g_0,\Sigma}^{\mathbf{Z}} \times \mathbf{I}_{\mathbf{H}}^{(n)}) \subset \text{Stab}_{\mathbf{H}(\mathbb{A}_f)}(\mathcal{Z}_{\mathbf{I}^{(n)}}(g_0))$. \square

Let us introduce some additional useful piece of notation. If $\tilde{K} = \tilde{K}_{\Sigma} \times \prod_{v \notin \Sigma} \tilde{K}_v \subset \mathbf{G}(\mathbb{A}_f)$ is any neat compact open subgroup, if $v_1, \dots, v_r \in I_{F,f}$ ($r \geq 1$) are distinct finite (non-necessarily inert) places of F and if, for all $j \in \{1, \dots, r\}$, one has an element $\hat{y}_j := \sum_{l=1}^{s_j} a_l^{(j)} g_l^{(j)}$ in $\mathbb{Z}[G_v]$ (with $a_l^{(j)} \in \mathbb{Z}$ and $g_l^{(j)} \in G_v$), then for all $\tilde{g} \in (\mathbf{U}(V) \times \mathbf{U}(W))(\mathbb{A}_{F,f}^{(v_j)_{j=1}^r})$, one may define the element $(\hat{y}_1, \dots, \hat{y}_r, \tilde{g}) \in \mathbb{Z}[\mathbf{G}(\mathbb{A}_f)/\tilde{K}]$ to be the linear combination:

$$(\hat{y}_1, \dots, \hat{y}_r, \tilde{g}) := \sum_{1 \leq l_1 \leq s_1, \dots, 1 \leq l_r \leq s_r} a_{l_1}^{(1)} \cdot \dots \cdot a_{l_r}^{(r)} (g_{l_1}^{(1)}, \dots, g_{l_r}^{(r)}, \tilde{g}) \tilde{K} \in \mathbb{Z}[\mathbf{G}(\mathbb{A}_f)/\tilde{K}].$$

This notation is compatible, for all $v \in I_{F,f}$, with the natural left-actions of $\mathbf{H}(\mathbb{A}_f)$ and $H_v = \Delta(\mathbf{U}(W))(F_v)$ on $\mathbb{Z}[\mathbf{G}(\mathbb{A}_f)/\tilde{K}]$ and $\mathbb{Z}[G_v/\tilde{K}_v]$ respectively. Namely, if

$$h = (h_v)_{v \in I_{F,f}} = \left(h_1, \dots, h_r, h^{(v_j)_{j=1}^r} \right) \in \mathbf{H}(\mathbb{A}_f),$$

then

$$h \cdot \left(\hat{y}_1, \dots, \hat{y}_r, \tilde{g} \right) = \left(h_1 \cdot \hat{y}_1, \dots, h_r \cdot \hat{y}_r, h^{(v_j)_{j=1}^r} \cdot \tilde{g} \right) \in \mathbb{Z}[\mathbf{G}(\mathbb{A}_f)/\tilde{K}].$$

Accordingly, given $\tilde{g}, \hat{y}_1, \dots, \hat{y}_r$ as above, we may define the \mathbf{H} -special cycle of level \tilde{K} , $\mathcal{Z}_{\tilde{K}}(\hat{y}_1, \dots, \hat{y}_r, \tilde{g}) \in \mathbb{Z}[\mathcal{Z}_{\tilde{K}}(\mathbf{G}, \mathbf{H})]$ to be the linear combination:

$$\mathcal{Z}_{\tilde{K}}(\hat{y}_1, \dots, \hat{y}_r, \tilde{g}) := \sum_{1 \leq l_1 \leq s_1, \dots, 1 \leq l_r \leq s_r} a_{l_1}^{(1)} \cdot \dots \cdot a_{l_r}^{(r)} \mathcal{Z}_{\tilde{K}}\left(g_{l_1}^{(1)}, \dots, g_{l_r}^{(r)}, \tilde{g}\right),$$

and Proposition 1.42 admits the following immediate extension: for all $\sigma \in \text{Gal}(E(\infty)/E)$, for all $h \in \mathbf{H}(\mathbb{A}_f)$ such that $\text{Art}_E^1(\det(h)) = \sigma$, then

$$\sigma \cdot \mathcal{Z}_{\tilde{K}}(\hat{y}_1, \dots, \hat{y}_r, \tilde{g}) = \mathcal{Z}_{\tilde{K}}\left(h \cdot (\hat{y}_1, \dots, \hat{y}_r, \tilde{g})\right) \in \mathbb{Z}[\mathcal{Z}_{\tilde{K}}(\mathbf{G}, \mathbf{H})] \quad (3.11)$$

As $\mathbf{H}^{\text{der}}(\mathbb{A}_f) \subset \mathbf{H}(\mathbb{A}_f)$ acts trivially on $\mathcal{Z}_{\tilde{K}}(\mathbf{G}, \mathbf{H})$, one may thus go one step further: if $\tilde{y}_1, \dots, \tilde{y}_r$ are now elements, respectively, of $\mathbb{Z}[H_{v_1}^{\text{der}} \backslash G_{v_1}/\tilde{K}_{v_1}]$, \dots , $\mathbb{Z}[H_{v_r}^{\text{der}} \backslash G_{v_r}/\tilde{K}_{v_r}]$ and if $\tilde{g} \in (\mathbf{U}(V) \times \mathbf{U}(W))(\mathbb{A}_{F,f}^{(v_j)_{j=1}^r})$, one may now define the cycle $\mathcal{Z}_{\tilde{K}}(\tilde{y}_1, \dots, \tilde{y}_r, \tilde{g})$ as

$$\mathcal{Z}_{\tilde{K}}(\tilde{y}_1, \dots, \tilde{y}_r, \tilde{g}) := \mathcal{Z}_{\tilde{K}}(\hat{y}_1, \dots, \hat{y}_r, \tilde{g}) \in \mathbb{Z}[\mathcal{Z}_{\tilde{K}}(\mathbf{G}, \mathbf{H})], \quad (3.12)$$

where $\hat{y}_1, \dots, \hat{y}_r$ are arbitrary lifts of $\tilde{y}_1, \dots, \tilde{y}_r$ respectively, in $\mathbb{Z}[G_{v_1}], \dots, \mathbb{Z}[G_{v_r}]$.

We shall apply the preceding discussion to the following situation: let $\tau_1, \dots, \tau_r \in \mathcal{I}^\tau$ be pairwise distinct allowable inert places of F away from \mathfrak{c}_1, I_0 and τ . For all $j \in \{1, \dots, r\}$, we let \hat{y}_{τ_j} be an element of $\mathbb{Z}[G_{\tau_j}]$ and we assume that the image \tilde{y}_{τ_j} of \hat{y}_{τ_j} in $\mathbb{Z}[H_{\tau_j}^{\text{der}} \backslash G_{\tau_j}/K_{\tau_j}]$ lies in $\mathbb{Z}[H_{\tau_j}^{\text{der}} \backslash G_{\tau_j}/K_{\tau_j}]_{c_j}$, for some $c_j \geq 0$.⁽³⁾ Let $m \geq 0$ be an integer (which shall later on be referred to as a *vertical conductor*). We set $\hat{g} := (g_{0,\Sigma}, \hat{g}^\Sigma) \in \mathbb{Z}[\mathbf{G}(\mathbb{A}_f)]$, with:

$$\hat{g}^\Sigma := \left(\delta^m, \hat{y}_{\tau_1}, \dots, \hat{y}_{\tau_r}, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \in \mathbb{Z}[(\mathbf{U}(V) \times \mathbf{U}(W))(\mathbb{A}_{F,f}^\Sigma)]. \quad (3.13)$$

One then has:

Corollary 3.1.6. *The cycle $\mathcal{Z}_{\mathbf{I}^{(n)}}(\hat{g}) \in \mathbb{Z}[\mathcal{Z}_{\mathbf{I}^{(n)}}(\mathbf{G}, \mathbf{H})]$ is defined over $\mathcal{K}(\tau^{n+m} \cdot \tau_1^{c_1} \dots \tau_r^{c_r})$, the \mathcal{K} -transfer field of conductor $\tau^{n+m} \cdot \tau_1^{c_1} \cdot \dots \cdot \tau_r^{c_r} \subset \mathcal{O}_F$.*

(3). In applications, we will deal with the (slightly stronger case) where the image y_{τ_j} of \hat{y}_{τ_j} in $\mathbb{Z}[\text{Hyp}_{\tau_j}]$ lies in $\mathbb{Z}[\text{Hyp}_{\tau_j}]_{c_j}$. If $\hat{y}_{\tau_j} = g_{\tau_j} \in G_{\tau_j}$ is a single element, one may just take $c_j := c_{\tau_j}(g_{\tau_j}) \in \mathbb{Z}_{\geq 0}$ to be the local conductor of g_{τ_j} , such as defined in Definition 2.2.4.

Proof. By definition of $\mathcal{K}(\tau^{n+m} \cdot \tau_1^{c_1} \dots \tau_r^{c_r})$, all we need to show is that the subgroup:

$$\begin{aligned} & \text{Art}_E^1 \left(\nu \left(\mathcal{O}_{g_0, \Sigma}^\times \times \mathcal{O}_{\tau, n+m}^\times \times \mathcal{O}_{\tau_1, c_1}^\times \cdots \times \mathcal{O}_{\tau_r, c_r}^\times \times \prod_{v \notin \Sigma \cup \{\tau\} \cup \{\tau_j\}_{j=1}^r} \mathcal{O}_v^\times \right) \right) \\ &= \text{Art}_E^1 \left(\nu \left(\mathcal{O}_{g_0, \Sigma}^\times \times \mathcal{O}_{\tau, n+m}^1 \times \mathcal{O}_{\tau_1, c_1}^1 \times \cdots \times \mathcal{O}_{\tau_r, c_r}^1 \times \prod_{v \notin \Sigma \cup \{\tau\} \cup \{\tau_j\}_{j=1}^r} \mathcal{O}_{v, 0}^1 \right) \right) \end{aligned}$$

of $\text{Gal}(E(\infty)/E)$ fixes the cycle $\mathcal{Z}_{\mathbf{I}^{(n)}}(\widehat{g})$. By construction, one has $\nu(\mathcal{O}_{g_0, \Sigma}^\times) \subset U_{g_0, \Sigma} = \det(K_{\mathbf{H}, g_0, \Sigma}^{\mathbf{Z}})$ and, by Corollary 2.2.8, one has

$$\begin{aligned} \det(H_\tau \cap \delta^m \mathbf{I}_\tau^{(n)} \delta^{-m}) &\stackrel{(2.33)}{=} \det(H_{\tau, n+m}) = \mathcal{O}_{\tau, n+m}^1, \\ \text{and } \det(H_{\tau_j, c_j}) &= \mathcal{O}_{\tau_j, c_j}^1, \quad j = 1, \dots, r. \end{aligned}$$

By (3.11), it is thus enough to show that the group:

$$\mathbf{H}(\mathbb{Q}) \cdot \left(K_{\mathbf{H}, g_0, \Sigma}^{\mathbf{Z}} \times (H_\tau \cap \delta^m \mathbf{I}_\tau^{(n)} \delta^{-m}) \times H_{\tau_1, c_1} \times \cdots \times H_{\tau_r, c_r} \times K_{\mathbf{H}}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right)$$

is contained in $\text{Stab}_{\mathbf{H}(\mathbb{A}_f)}(\mathcal{Z}_{\mathbf{I}^{(n)}}(\widehat{g}))$.

Let $h_{\mathbb{Q}} \in \mathbf{H}(\mathbb{Q})$ and $k_\Sigma \in K_{\mathbf{H}, g_0, \Sigma}^{\mathbf{Z}}$. Let $i \in \mathbf{I}_\tau^{(n)}$, with $\delta^m i \delta^{-m} \in H_\tau$, let $h_j \in H_{\tau_j, c_j}$, for $j = 1, \dots, r$ and let $k^{\Sigma, \tau, (\tau_j)_{j=1}^r} \in K_{\mathbf{H}}^{\Sigma, \tau, (\tau_j)_{j=1}^r}$. Accordingly, we set

$$\tilde{h} := h_{\mathbb{Q}} \cdot \left(k_\Sigma, \delta^m i \delta^{-m}, h_1, \dots, h_r, k^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right).$$

By assumption on k_Σ , one has $k_\Sigma g_{0, \Sigma} = z_{\mathbb{Q}} g_{0, \Sigma} k'_\Sigma$ for some $z_{\mathbb{Q}} \in \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{I}^{(n), \Sigma}$ and some $k'_\Sigma \in K_\Sigma$. Also, for all $j \in \{1, \dots, r\}$, the assumption $\tilde{y}_{\tau_j} \in \mathbb{Z}[H_{\tau_j}^{\text{der}} \backslash G_{\tau_j} / K_{\tau_j}]_{c_j}$ implies that $h_j \tilde{y}_{\tau_j} = \tilde{y}_{\tau_j}$ in $\mathbb{Z}[H_{\tau_j}^{\text{der}} \backslash G_{\tau_j} / K_{\tau_j}]$. As $z_{\mathbb{Q}}^{-1} \in \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \cap \mathbf{I}^{(n), \Sigma}$, this implies that both elements $h_j \widehat{y}_{\tau_j} z_{\mathbb{Q}}^{-1}$ and \widehat{y}_{τ_j} have the same image \tilde{y}_{τ_j} in $\mathbb{Z}[H_{\tau_j}^{\text{der}} \backslash G_{\tau_j} / \mathbf{I}_{\tau_j}^{(n)}] = \mathbb{Z}[H_{\tau_j}^{\text{der}} \backslash G_{\tau_j} / K_{\tau_j}]$. This gives:

$$\begin{aligned} \tilde{h} \cdot \widehat{g} &\stackrel{\text{def}}{=} h_{\mathbb{Q}} \cdot \left(k_\Sigma, \delta^m i \delta^{-m}, h_1, \dots, h_r, k^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \cdot \left(g_{0, \Sigma}, \delta^m, \widehat{y}_{\tau_1}, \dots, \widehat{y}_{\tau_r}, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \\ &= z_{\mathbb{Q}} h_{\mathbb{Q}} \cdot \left(g_{0, \Sigma} k'_\Sigma, \delta^m i z_{\mathbb{Q}}^{-1}, h_1 \widehat{y}_{\tau_1} z_{\mathbb{Q}}^{-1}, \dots, h_r \widehat{y}_{\tau_r} z_{\mathbb{Q}}^{-1}, k^{\Sigma, \tau, (\tau_j)_{j=1}^r} z_{\mathbb{Q}}^{-1} \right) \end{aligned}$$

The above discussion yields that both elements $\tilde{h} \cdot \widehat{g}$ and

$$z_{\mathbb{Q}} h_{\mathbb{Q}} \cdot \left(g_{0, \Sigma}, \delta^m, \widehat{y}_{\tau_1}, \dots, \widehat{y}_{\tau_r}, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) = z_{\mathbb{Q}} h_{\mathbb{Q}} \cdot \widehat{g}$$

have the same image in $\mathbb{Z}[\mathbf{H}^{\text{der}}(\mathbb{A}_f) \backslash \mathbf{G}(\mathbb{A}_f) / \mathbf{I}^{(n)}]$, hence

$$\begin{aligned} \mathcal{Z}_{\mathbf{I}^{(n)}}(\tilde{h} \cdot \widehat{g}) &= \mathcal{Z}_{\mathbf{I}^{(n)}}(z_{\mathbb{Q}} h_{\mathbb{Q}} \cdot \widehat{g}) \\ &= \mathcal{Z}_{\mathbf{I}^{(n)}}(\widehat{g}) \in \mathbb{Z}[\mathcal{Z}_{\mathbf{I}^{(n)}}(\mathbf{G}, \mathbf{H})], \end{aligned}$$

which finishes the proof. \square

3.1.3 The *norm-compatible* family $(\mathcal{Z}(n, \tau^m \cdot \mathbf{c}))$.

The exact sequences (1.59) of Chapter 1, together with Lemma 3.1, imply the following result (which is part of [7], Proposition VII.1.1):

Lemma 3.1.7. *For all $\mathfrak{f} \in \mathcal{P}$ and $\mathfrak{p} \in \mathcal{I}$ relatively prime to \mathfrak{f} , such that $\#(\mathcal{O}_F/\mathfrak{p}) =: q_{\mathfrak{p}}$, then*

$$\# \left(\text{Gal} \left(K(\mathfrak{f} \cdot \mathfrak{p}) / K(\mathfrak{f}) \right) \right) = \begin{cases} q_{\mathfrak{p}} + 1 & \text{if } \mathfrak{f} \neq 1, \\ [E^\times \cap \mathbb{A}_{F, \mathfrak{f}}^\times \mathfrak{D}_1^\times : F^\times]^{-1} \cdot (q_{\mathfrak{p}} + 1) & \text{if } \mathfrak{f} = 1. \end{cases}$$

Proof. If $\mathfrak{f} \neq 1$ then, by assumption, \mathfrak{f} and $\mathfrak{f} \cdot \mathfrak{p}$ are both relatively prime to I_0 . By Corollary 1.3.18, one has

$$\text{Gal}(\mathcal{K}(\mathfrak{f} \cdot \mathfrak{p})/\mathcal{K}(\mathfrak{f})) \simeq \frac{\mathcal{O}_{\mathfrak{p},0}^\times}{\mathcal{O}_{\mathfrak{p},1}^\times}$$

and the latter has order $q_{\mathfrak{p}} + 1$, by the discussion following Lemma 3.1.

If $\mathfrak{f} = 1$ then the exact sequence (1.57) together with (1.58) imply that

$$\text{Gal}(\mathcal{K}(\mathfrak{f} \cdot \mathfrak{p})/\mathcal{K}(\mathfrak{f})) = \text{Gal}(\mathcal{K}(\mathfrak{p})/\mathcal{K}(1)) \simeq \frac{\mathcal{O}_{\mathfrak{p},0}^\times / \mathcal{O}_{\mathfrak{p},1}^\times}{E^\times \cap \mathbb{A}_{F, \mathfrak{f}}^\times \mathfrak{D}_1^\times / E^\times \cap \mathbb{A}_{F, \mathfrak{f}}^\times \mathfrak{D}_{\mathfrak{p}}^\times}.$$

The assumption $\mathfrak{p} \in \mathcal{I}$ ensures, by Lemma 1.3.17, that $E^\times \cap \mathbb{A}_{F, \mathfrak{f}}^\times \mathfrak{D}_{\mathfrak{p}}^\times = F^\times$. Consequently, one has

$$\begin{aligned} \# \left(\text{Gal}(\mathcal{K}(\mathfrak{p})/\mathcal{K}(1)) \right) &= [E^\times \cap \mathbb{A}_{F, \mathfrak{f}}^\times \mathfrak{D}_1^\times : F^\times]^{-1} \# (\mathcal{O}_{\mathfrak{p},0}^\times / \mathcal{O}_{\mathfrak{p},1}^\times) \\ &= [E^\times \cap \mathbb{A}_{F, \mathfrak{f}}^\times \mathfrak{D}_1^\times : F^\times]^{-1} (q_{\mathfrak{p}} + 1). \end{aligned}$$

□

Set $\epsilon := [E^\times \cap \mathbb{A}_{F, \mathfrak{f}}^\times \mathfrak{D}_1^\times : F^\times] \in \mathbb{N}_{\geq 1}$. One may now apply all the preceding and construct a *global* family of **H**-special cycles on the tower

$$\dots \longrightarrow \text{Sh}_{I^{(n+1)}}(\mathbf{G}, X) \longrightarrow \text{Sh}_{I^{(n)}}(\mathbf{G}, X) \longrightarrow \dots \longrightarrow \text{Sh}_{I^{(1)}}(\mathbf{G}, X),$$

of varying Iwahori levels $I^{(n)} \subset \mathbf{G}(\mathbb{A}_f)$ ($n \geq 1$), vertical conductors $m \geq 0$ and tame conductors $\mathbf{c} \in \mathcal{P}^\tau$.

Definition 3.1.4. *Let $n \geq 1$ be a positive integer, $m \geq 0$ be a vertical conductor and $\mathbf{c} = \tau_1 \cdot \dots \cdot \tau_r \in \mathcal{P}^\tau$ be a tame conductor. Recall that we defined in (3.4) (applied to $\tau = \tau_i$) the element $\widehat{y}_{\tau_i} \in \mathbb{Z}[G_{\tau_j}]$ to be a fixed lift of the element $q \mathfrak{X}_{\tau_j} \in \mathbb{Z}[\text{Hyp}_{\tau_j}]_1$, for all $j \in \{1, \dots, r\}$. We define the cycle $\mathfrak{z}(n, \tau^m \cdot \mathbf{c}) \in \mathbb{Z}[\mathcal{Z}_{I^{(n)}}(\mathbf{G}, \mathbf{H})]$ by:*

$$\mathfrak{z}(n, \tau^m \cdot \mathbf{c}) := r(\mathbf{c}) \mathcal{Z}_{I^{(n)}}(g_{0, \Sigma}, \delta^m, \widehat{y}_{\tau_1}, \dots, \widehat{y}_{\tau_r}, \mathbf{1}^\Sigma), \quad (3.14)$$

where $r(\mathbf{c}) \in \mathbb{N}_{\geq 1}$ is defined as

$$r(\mathbf{c}) = \begin{cases} 1, & \text{if } \mathbf{c} = 1 \\ \epsilon, & \text{if } \mathbf{c} \neq 1. \end{cases} \quad (3.15)$$

According to Corollary 3.1.6, the cycle $\mathfrak{z}(n, \tau^m \cdot \mathfrak{c})$ is defined over the \mathcal{K} -transfer field $\mathcal{K}(\tau^{n+m} \cdot \tau_1 \cdot \dots \cdot \tau_r) = \mathcal{K}(\tau^{n+m} \cdot \mathfrak{c})$, for all $n \geq 1$, $m \geq 0$ and $\mathfrak{c} \in \mathcal{P}^\tau$.

Notice that the equality $I_\tau^{(n+1)} = I_\tau^{(n)} \cap \delta I_\tau^{(n)} \delta^{-1}$ induces

$$I^{(n+1)} = I^{(n)} \cap \delta I^{(n)} \delta^{-1},$$

where we identified δ with $\phi_\tau(\delta) = (\delta, \mathbf{1}^\tau) \in \mathbf{G}(\mathbb{A}_f)$. Consider the following diagram:

$$\begin{array}{ccc} \mathrm{Sh}_{I^{(n+1)}}(\mathbf{G}, X) & \xrightarrow{[\cdot\delta]} & \mathrm{Sh}_{I^{(n)} \cap \delta^{-1} I^{(n)} \delta}(\mathbf{G}, X) \\ \downarrow \pi_{I^{(n+1)}/I^{(n)}} & & \downarrow \pi_{I^{(n)} \cap \delta^{-1} I^{(n)} \delta / I^{(n)}} \\ \mathrm{Sh}_{I^{(n)}}(\mathbf{G}, X) & & \mathrm{Sh}_{I^{(n)}}(\mathbf{G}, X) \end{array}$$

and set $\pi^{(n)} := \pi_{I^{(n+1)}/I^{(n)}}$ and $\pi_\delta^{(n)} := \pi_{I^{(n)} \cap \delta^{-1} I^{(n)} \delta / I^{(n)}} \circ [\cdot\delta]$. These are finite étale morphisms between $\mathrm{Sh}_{I^{(n+1)}}(\mathbf{G}, X)$ and $\mathrm{Sh}_{I^{(n)}}(\mathbf{G}, X)$, defined over $\mathrm{Spec} E$. For all irreducible closed subschemes $\mathcal{Z} \subset \mathrm{Sh}_{I^{(n+1)}}(\mathbf{G}, X)$, we denote by $\pi^{(n)}(\mathcal{Z})$ and $\pi_\delta^{(n)}(\mathcal{Z}) \subset \mathrm{Sh}_{I^{(n)}}(\mathbf{G}, X)$ the corresponding scheme-theoretic images of \mathcal{Z} inside $\mathrm{Sh}_{I^{(n)}}(\mathbf{G}, X)$. For all $g \in \mathbf{G}(\mathbb{A}_f)$, $\pi^{(n)}(\mathcal{Z}_{I^{(n+1)}}(g))$ and $\pi_\delta^{(n)}(\mathcal{Z}_{I^{(n+1)}}(g))$ are, by definition, irreducible closed subschemes of $\mathrm{Sh}_{I^{(n)}}(\mathbf{G}, X)$ and arise as the image of the identity irreducible component of $\mathrm{Sh}(\mathbf{H}, Y)$, by the respective sequences of morphisms

$$\mathrm{Sh}(\mathbf{H}, Y) \hookrightarrow \mathrm{Sh}(\mathbf{G}, X) \xrightarrow{[g]} \mathrm{Sh}(\mathbf{G}, X) \xrightarrow{\pi_{I^{(n)}}} \mathrm{Sh}_{I^{(n)}}(\mathbf{G}, X)$$

and

$$\mathrm{Sh}(\mathbf{H}, Y) \hookrightarrow \mathrm{Sh}(\mathbf{G}, X) \xrightarrow{[g\delta]} \mathrm{Sh}(\mathbf{G}, X) \xrightarrow{\pi_{I^{(n)}}} \mathrm{Sh}_{I^{(n)}}(\mathbf{G}, X)$$

In other words, $\pi^{(n)}(\mathcal{Z}_{I^{(n+1)}}(g))$ and $\pi_\delta^{(n)}(\mathcal{Z}_{I^{(n+1)}}(g))$ are \mathbf{H} -special cycles of level $I^{(n)}$, equal respectively to $\mathcal{Z}_{I^{(n)}}(g)$ and $\mathcal{Z}_{I^{(n)}}(g\delta)$. One may thus extend the previous maps into the following \mathbb{Z} -linear maps:

$$\begin{aligned} \pi^{(n)} &: \mathbb{Z}[\mathcal{Z}_{I^{(n+1)}}(\mathbf{G}, \mathbf{H})] \longrightarrow \mathbb{Z}[\mathcal{Z}_{I^{(n)}}(\mathbf{G}, \mathbf{H})] \\ \pi_\delta^{(n)} &: \mathbb{Z}[\mathcal{Z}_{I^{(n+1)}}(\mathbf{G}, \mathbf{H})] \longrightarrow \mathbb{Z}[\mathcal{Z}_{I^{(n)}}(\mathbf{G}, \mathbf{H})] \end{aligned}$$

Notice that, since $[\cdot\delta]$ is an isomorphism, then one has

$$\deg(\mathcal{Z}_{I^{(n+1)}}(g)/\mathcal{Z}_{I^{(n)}}(g)) = \deg(\mathcal{Z}_{I^{(n)} \cap \delta^{-1} I^{(n)} \delta}(g\delta)/\mathcal{Z}_{I^{(n)}}(g\delta)) = \deg(\mathcal{Z}_{I^{(n+1)}}(g)/\mathcal{Z}_{I^{(n)}}(g\delta)).$$

Recall that, by (1.22), the global Iwahori-Hecke algebra $\mathcal{H}_{I^{(n)}} := \mathcal{H}_{\mathbb{Z}}(\mathbf{G}(\mathbb{A}_f) // I^{(n)})$ of level $n \geq 1$ factors as a restricted tensor product of local components. Accordingly, one may

see the element $t^{(n)} = \rho(\mathbf{1}_{\mathbf{I}_\tau^{(n)}} \delta_{\mathbf{I}_\tau^{(n)}}) \in \mathcal{H}_\tau^{(n)} = \mathcal{H}_{\mathbb{Z}}(G_\tau // \mathbf{I}_\tau^{(n)})$ as an element of $\mathcal{H}_{\mathbf{I}^{(n)}}$, acting on \mathbf{H} -special cycles of level $\mathbf{I}^{(n)}$ by the rule:

$$\begin{aligned} t^{(n)} \cdot \mathcal{Z}_{\mathbf{I}^{(n)}}(g) &:= \mathcal{Z}_{\mathbf{I}^{(n)}}(t^{(n)} \cdot g_\tau, g^\tau) \\ &= \sum_{i \in \mathbf{I}_\tau^{(n)} \delta_{\mathbf{I}_\tau^{(n)}} / \mathbf{I}^{(n)}} \mathcal{Z}_{\mathbf{I}^{(n)}}(g \phi_\tau(i)) \\ &= \sum_{i' \in \mathbf{I}_\tau^{(n)} / \mathbf{I}_\tau^{(n+1)}} \mathcal{Z}_{\mathbf{I}^{(n)}}(g \phi_\tau(i' \delta)), \end{aligned}$$

for all $g = (g_\tau, g^\tau) \in \mathbf{G}(\mathbb{A}_f)$.

Theorem 3.1.8. *The family $(\mathfrak{z}(n, \tau^m \cdot \mathbf{c}))$, indexed by levels $n \geq 1$, vertical conductors $m \geq 0$ and tame conductors $\mathbf{c} \in \mathcal{P}^\tau$, satisfies the following compatibilities:*

— (Horizontal relation) For all $\mathbf{c} \cdot \tau' \in \mathcal{P}^\tau$, one has:

$$\mathrm{Tr}_{\mathcal{K}(\tau^{n+m} \cdot \mathbf{c} \cdot \tau') / \mathcal{K}(\tau^{n+m} \cdot \mathbf{c})}(\mathfrak{z}(n, \tau^m \cdot \mathbf{c} \cdot \tau')) = \mathrm{He}_{\tau'}(\mathrm{Frob}_{\tau'}) \cdot \mathfrak{z}(n, \tau^m \cdot \mathbf{c}),$$

where $\mathrm{Frob}_{\tau'} \in \mathrm{Gal}(E_{\tau'}^{ab} / E_{\tau'})$ is a lift of the geometric Frobenius attached to the prime ideal $\tau' \mathcal{O}_E$ of \mathcal{O}_E .

— (Level-wise vertical relation)

$$q^5 \pi_\delta^{(n)} \left(\mathrm{Tr}_{\mathcal{K}(\tau^{n+m+1} \cdot \mathbf{c}) / \mathcal{K}(\tau^{n+m} \cdot \mathbf{c})}(\mathfrak{z}(n+1, \tau^m \cdot \mathbf{c})) \right) = t^{(n)} \cdot \mathfrak{z}(n, \tau^m \cdot \mathbf{c})$$

Proof.

— (Horizontal relation) Write $\mathbf{c} = \tau_1 \cdot \dots \cdot \tau_r \in \mathcal{P}^\tau$. Notice that, by Lemma 1.3.16, the prime ideal τ' is unramified in $E(\mathbf{c}_1 \cdot \tau^{n+m} \cdot \mathbf{c})$ - hence also in $\mathcal{K}(\tau^{n+m} \cdot \mathbf{c}) \subset E(\mathbf{c}_1 \cdot \tau^{n+m} \cdot \mathbf{c})$, by (1.60) - whenever $\tau' \in \mathcal{I}^\tau$ does not divide \mathbf{c} . The cycle $\mathfrak{z}(n, \tau^m \cdot \mathbf{c})$ being defined over $\mathcal{K}(\tau^{n+m} \cdot \mathbf{c})$, this gives sense to the action of $\mathrm{Frob}_{\tau'}$ on $\mathfrak{z}(n, \tau^m \cdot \mathbf{c})$, and thus to the notation $\mathrm{He}_{\tau'}(\mathrm{Frob}_{\tau'}) \cdot \mathfrak{z}(n, \tau^m \cdot \mathbf{c}) \in \mathbb{Z}[\mathcal{Z}_{\mathbf{I}^{(n)}}(\mathbf{G}, \mathbf{H})]$.

If $x_{\tau'} \in E_{\tau'}^\times$ is any element such that $\mathrm{Art}_{\tau'}(x_{\tau'}) \in \mathrm{Gal}(E_{\tau'}^{ab} / E_{\tau'})$ is a lift of the geometric Frobenius, and if $h_{\mathrm{Frob}_{\tau'}} \in H_{\tau'}$ is such that $\det(h_{\mathrm{Frob}_{\tau'}}) = \nu(x_{\tau'})$, one obtains by Remark 3.1.1 that:

$$\mathrm{Art}_{\tau'}^1(\det(h)) = \mathrm{Frob}_{\tau'}|_{E(\infty)} \in \mathrm{Gal}(E(\infty) / E).$$

Equality (3.9) (applied to $\sigma = \mathrm{Frob}_{\tau'}$) then gives:

$$\begin{aligned} \mathrm{Frob}_{\tau'} \cdot \mathfrak{z}(n, \tau^m \cdot \mathbf{c}) &= r(\mathbf{c}) \mathrm{Frob}_{\tau'} \cdot \mathcal{Z}_{\mathbf{I}^{(n)}}(g_{0,\Sigma}, \delta^m, y_{\tau_1}, \dots, y_{\tau_r}, \mathbf{1}^{\tau, (\tau_i)_{i=1}^r}) \\ &= r(\mathbf{c}) \mathcal{Z}_{\mathbf{I}^{(n)}}(g_{0,\Sigma}, \delta^m, y_{\tau_1}, \dots, y_{\tau_r}, h_{\mathrm{Frob}_{\tau'}}, \mathbf{1}^{\tau, (\tau_i)_{i=1}^r, \tau'}). \end{aligned}$$

According to our chosen convention for the local Artin map $\text{Art}_{\tau'}$ - which maps uniformizers to geometric Frobenii - one may choose, without loss of generality, the element $x_{\tau'} \in E_{\tau'}^\times$ to be the (previously fixed) uniformizer $\varpi_{\tau'} \in F_{\tau'}^\times$. Accordingly, one has $\nu(x_{\tau'}) = \nu(\varpi_{\tau'}) = 1$, which implies that one may well choose $h_{\text{Frob}_{\tau'}}$ to be equal to $\mathbf{1} \in H_{\tau'}$. This gives $\text{Frob}_{\tau'} \cdot \mathfrak{z}(n, \tau^m \cdot \mathbf{c}) = \mathfrak{z}(n, \tau^m \cdot \mathbf{c})$, i.e.,

$$\text{He}_{\tau'}(\text{Frob}_{\tau'}) \cdot \mathfrak{z}(n, \tau^m \cdot \mathbf{c}) = \text{He}_{\tau'}(\mathbf{1}) \cdot \mathfrak{z}(n, \tau^m \cdot \mathbf{c}).$$

Let $[\tilde{\mathbf{1}}] \in \mathbb{Z}[H_{\tau'}^{\text{der}} \backslash G_{\tau'} / K_{\tau'}]$ be the unit coset, i.e. the image of $(x_{\tau'}, x_{\tau'}) \in \mathbb{Z}[\text{Hyp}_{\tau'}]$ by $\pi_{\tau'}^{\text{der}}$, where $x_{\tau'} \in \text{Hyp}_{V, \tau'}$ denotes the corresponding fixed origin of the Bruhat-Tits buildings at τ' . This gives

$$\begin{aligned} \text{He}_{\tau'}(\mathbf{1}) \cdot [\tilde{\mathbf{1}}] &= \text{He}_{\tau'}(\mathbf{1}) \cdot \pi_{\tau'}^{\text{der}}(x_{\tau'} \otimes x_{\tau'}) \\ &= \text{Tr}_{1,0}^{\text{der}}(y_{\tau'}) := \sum_{h \in H'_{\tau',0}/H'_{\tau',1}} h \cdot \tilde{y}_{\tau'} \in \mathbb{Z}[H_{\tau'}^{\text{der}} \backslash G_{\tau'} / K_{\tau'}]. \end{aligned}$$

This implies:

$$\begin{aligned} \text{He}_{\tau'}(\mathbf{1}) \cdot \mathfrak{z}(n, \tau^m \cdot \mathbf{c}) &= r(\mathbf{c}) \mathcal{Z}_{\mathbf{1}(n)}(g_{0,\Sigma}, \delta^m, \tilde{y}_{\tau_1}, \dots, \tilde{y}_{\tau_r}, \text{He}_{\tau'}(\mathbf{1}) \cdot [\tilde{\mathbf{1}}], \mathbf{1}^{\Sigma, \tau, (\tau_i)_{i=1}^r, \tau'}) \\ &= r(\mathbf{c}) \mathcal{Z}_{\mathbf{1}(n)}(g_{0,\Sigma}, \delta^m, \tilde{y}_{\tau_1}, \dots, \tilde{y}_{\tau_r}, \sum_{h \in H'_{\tau',0}/H'_{\tau',1}} h \cdot \tilde{y}_{\tau'}, \mathbf{1}^{\Sigma, \tau, (\tau_i)_{i=1}^r, \tau'}) \\ &\stackrel{\text{a}}{=} r(\mathbf{c}) \sum_{s \in \mathcal{O}_{\tau',0}^1 / \mathcal{O}_{\tau',1}^1} \text{Art}_{\tau'}^1(s) \cdot \mathcal{Z}_{\mathbf{1}(n)}(g_{0,\Sigma}, \delta^m, \hat{y}_{\tau_1}, \dots, \hat{y}_{\tau_r}, \hat{y}_{\tau'}, \mathbf{1}^{\Sigma, \tau, (\tau_i)_{i=1}^r, \tau'}) \\ &\stackrel{\text{b}}{=} \text{Tr}_{\mathcal{K}(\tau^{n+m} \cdot \mathbf{c} \cdot \tau') / \mathcal{K}(\tau^{n+m} \cdot \mathbf{c})} \left(r(\mathbf{c} \cdot \tau') \mathcal{Z}_{\mathbf{1}(n)}(g_{0,\Sigma}, \delta^m, \hat{y}_{\tau_1}, \dots, \hat{y}_{\tau_r}, \hat{y}_{\tau'}, \mathbf{1}^{\Sigma, \tau, (\tau_i)_{i=1}^r, \tau'}) \right) \\ &=: \text{Tr}_{\mathcal{K}(\tau^{n+m} \cdot \mathbf{c} \cdot \tau') / \mathcal{K}(\tau^{n+m} \cdot \mathbf{c})} (\mathfrak{z}(n, \tau^m \cdot \mathbf{c} \cdot \tau')). \end{aligned}$$

The equality (a) above comes from Proposition 1.42, from the definition $\text{Art}_{\tau'}^1 = \text{Art}_E^1 \circ \phi_{\tau'}$ (see Remark 3.1.1) and from the isomorphism $\det^\# : H'_{\tau',0} / H'_{\tau',1} \xrightarrow{\sim} \mathcal{O}_{\tau',0}^1 / \mathcal{O}_{\tau',1}^1$, whereas equality (b) comes from the fact that the surjective map

$$\text{Art}_{\tau'}^1 : \mathcal{O}_{\tau',0}^1 / \mathcal{O}_{\tau',1}^1 \longrightarrow \text{Gal}(\mathcal{K}(\tau^{n+m} \cdot \mathbf{c} \cdot \tau') / \mathcal{K}(\tau^{n+m} \cdot \mathbf{c})) \simeq \text{Gal}(\mathcal{K}(\mathbf{c} \cdot \tau') / \mathcal{K}(\mathbf{c}))$$

has kernel of size $\frac{r(\mathbf{c}\tau')}{r(\mathbf{c})}$ (4)

- (Vertical relation) Notice that, since $n \geq 1$ then $\tau^{n+m} \mathbf{c} \nmid I_0$ for all $m \geq 0$, $\mathbf{c} \in \mathcal{P}^\tau$. Consequently, the map

$$\text{Art}_\tau^1 \circ \nu : \frac{\mathcal{O}_{\tau, n+m+1}^\times}{\mathcal{O}_{\tau, n+m}^\times} \xrightarrow{\sim} \frac{\mathcal{O}_{\tau, n+m}^1}{\mathcal{O}_{\tau, n+m+1}^1} \longrightarrow \text{Gal}(\mathcal{K}(\tau^{n+m+1} \cdot \mathbf{c}) / \mathcal{K}(\tau^n \cdot \mathbf{c}))$$

(4). We recall that $r(\mathbf{c})$ is equal to 1 if $\mathbf{c} \neq 1$, and to $\epsilon = [E^\times \cap \mathbb{A}_{E,f}^\times \mathfrak{D}_1^\times : F^\times]$ otherwise.

is an isomorphism, according to Corollary 1.3.18. This gives:

$$\begin{aligned}
& \mathrm{Tr}_{\mathcal{K}(\tau^{n+m+1} \cdot \mathbf{c}) / \mathcal{K}(\tau^n \cdot \mathbf{c})} (\mathfrak{z}(n+1, \tau^m \cdot \mathbf{c})) \\
&= \sum_{s \in \mathcal{O}_{\tau, n+m}^1 / \mathcal{O}_{\tau, n+m+1}^1} \mathrm{Art}_{\tau}^1(s) \cdot \mathfrak{z}(n+1, \tau^m \cdot \mathbf{c}) \\
&= r(\mathbf{c}) \sum_{s \in \mathcal{O}_{\tau, n+m}^1 / \mathcal{O}_{\tau, n+m+1}^1} \mathrm{Art}_{\tau}^1(s) \cdot \mathcal{Z}_{\mathbf{I}(n+1)} \left(g_{0, \Sigma}, \delta^m, \widehat{y}_{\tau_1}, \dots, \widehat{y}_{\tau_r}, 1^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \\
&= r(\mathbf{c}) \sum_{h' \in H'_{\tau, n+m+1} / H'_{\tau, n+m}} \mathrm{Art}_{\tau}^1(\det h') \cdot \mathcal{Z}_{\mathbf{I}(n+1)} \left(g_{0, \Sigma}, \delta^m, \widehat{y}_{\tau_1}, \dots, \widehat{y}_{\tau_r}, 1^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \\
&= r(\mathbf{c}) \sum_{h' \in H'_{\tau, n+m+1} / H'_{\tau, n+m}} \mathcal{Z}_{\mathbf{I}(n+1)} \left(g_{0, \Sigma}, h' \delta^m, \widehat{y}_{\tau_1}, \dots, \widehat{y}_{\tau_r}, 1^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right)
\end{aligned}$$

Accordingly, one has

$$\begin{aligned}
& q^5 \pi_{\delta}^{(n)} \left(\mathrm{Tr}_{\mathcal{K}(\tau^{n+m+1} \cdot \mathbf{c}) / \mathcal{K}(\tau^n \cdot \mathbf{c})} (\mathfrak{z}(n+1, \tau^m \cdot \mathbf{c})) \right) \\
&= q^5 r(\mathbf{c}) \sum_{h' \in H'_{\tau, n+m+1} / H'_{\tau, n+m}} \mathcal{Z}_{\mathbf{I}(n)} \left(g_{0, \Sigma}, h' \delta^{m+1}, \widehat{y}_{\tau_1}, \dots, \widehat{y}_{\tau_r}, 1^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \\
&\stackrel{(3.8)}{=} r(\mathbf{c}) \sum_{i \in \mathbf{I}_{\tau}^{(n)} \delta \mathbf{I}_{\tau}^{(n)} / \mathbf{I}_{\tau}^{(n)}} \mathcal{Z}_{\mathbf{I}(n)} \left(g_{0, \Sigma}, \delta^m i, \widehat{y}_{\tau_1}, \dots, \widehat{y}_{\tau_r}, 1^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \\
&\stackrel{\mathrm{def}}{=} r(\mathbf{c}) \mathcal{Z}_{\mathbf{I}(n)} \left(g_{0, \Sigma}, t^{(n)}(\delta^m), \widehat{y}_{\tau_1}, \dots, \widehat{y}_{\tau_r}, 1^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \\
&=: r(\mathbf{c}) t^{(n)} \cdot \mathcal{Z}_{\mathbf{I}(n)} \left(g_{0, \Sigma}, \delta^m, \widehat{y}_{\tau_1}, \dots, \widehat{y}_{\tau_r}, 1^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \\
&= t^{(n)} \cdot \mathfrak{z}(n, \tau^m \cdot \mathbf{c}).
\end{aligned}$$

□

Remark 3.1.2 (Compatibility between $t^{(n)}$ and $\pi_{\delta}^{(n)}$). *Let $n \geq 1$. As already mentioned in the previous chapter, a consequence of Lemma 2.3.13 (applied to $m = 1$) is the following: if $(h_j)_{j=1}^{q^6} \subset H_{\tau, n+1}$ is a system of representatives for the quotient $H_{\tau, n+1} / H_{\tau, n+2}$, then the family $(\mathrm{ad}(\delta^{-1})(h_j))_{j=1}^{q^6} = (\delta^{-1} h_j \delta)_{j=1}^{q^6}$ forms a system of representatives for the quotient $\mathbf{I}_{\tau}^{(n)} / \mathbf{I}_{\tau}^{(n+1)}$, hence the family $(\delta^{-1} h_j \delta) \cdot \delta = (\delta^{-1} h_j \delta^2)_{j=1}^{q^6}$ forms a system of representatives for the quotient $\mathbf{I}_{\tau}^{(n)} \delta \mathbf{I}_{\tau}^{(n)} / \mathbf{I}_{\tau}^{(n)}$. The family $(h_j)_{i=1}^{q^6} \subset \mathbf{I}_{\tau}^{(n+1)}$ being already a system of representatives for the quotient $\mathbf{I}_{\tau}^{(n+1)} / \mathbf{I}_{\tau}^{(n+2)}$ (this is Corollary 2.3.12), one gets that $(h_j \delta)_{j=1}^{q^6}$ forms a system of representatives for $\mathbf{I}_{\tau}^{(n+1)} \delta \mathbf{I}_{\tau}^{(n+1)} / \mathbf{I}_{\tau}^{(n+2)}$. This gives, for all $g \in \mathbf{G}(\mathbb{A}_f)$:*

$$\pi_{\delta}^{(n)} \left(t^{(n+1)} \cdot \mathcal{Z}_{\mathbf{I}(n+1)}(g) \right) = t^{(n)} \cdot \pi_{\delta}^{(n)} \left(\mathcal{Z}_{\mathbf{I}(n+1)}(g) \right). \quad (3.16)$$

Indeed, one has $\pi_\delta^{(n)}(\mathcal{Z}_{\mathbf{I}^{(n+1)}}(g)) = \mathcal{Z}_{\mathbf{I}^{(n)}}(g \phi_\tau(\delta))$, for all $g \in \mathbf{G}(\mathbb{A}_f)$. The above discussion implies that $t^{(n)} \cdot \pi_\delta^{(n)}(\mathcal{Z}_{\mathbf{I}^{(n+1)}}(g))$ may be computed as follows:

$$\begin{aligned} t^{(n)} \cdot \pi_\delta^{(n)}(\mathcal{Z}_{\mathbf{I}^{(n+1)}}(g)) &= \sum_{j=1}^{q^6} \mathcal{Z}_{\mathbf{I}^{(n)}}(g \phi_\tau(\delta) \phi_\tau(\delta^{-1} h_j \delta^2)) \\ &= \sum_{j=1}^{q^6} \mathcal{Z}_{\mathbf{I}^{(n)}}(g \phi_\tau(h_j \delta) \phi_\tau(\delta)) \\ &= \pi_\delta^{(n)} \sum_{j=1}^{q^6} \mathcal{Z}_{\mathbf{I}^{(n)}}(g \phi_\tau(h_j \delta)) \\ &= \pi_\delta^{(n)} (t^{(n+1)} \cdot \mathcal{Z}_{\mathbf{I}^{(n+1)}}(g)). \end{aligned}$$

3.1.3.1 Links with geometric Hecke correspondences.

In this paragraph, we let $\tilde{K} \subset \mathbf{G}(\mathbb{A}_f)$ be any neat compact open subgroup. We may translate the action of the global Hecke algebra $\mathcal{H}_{\tilde{K}}$ on the \mathbb{Z} -module $\mathbb{Z}[\mathcal{Z}_{\tilde{K}}(\mathbf{G}, X)]$, defined in § 1.2.5 in a *set-theoretic* way - and which enabled us to define the action of $t^{(n)} = \mathbf{1}_{\mathbf{I}^{(n)} \delta \mathbf{I}^{(n)}}$ on $\mathbb{Z}[\mathcal{Z}_{\mathbf{I}^{(n)}}(\mathbf{G}, \mathbf{H})]$ via the local action of $t^{(n)}$ on $\mathbb{Z}[\text{Hyp}_\tau^{(n)}]$ - into a geometric action on cycles by *Hecke correspondences*.

Recall that, given $g \in \mathbf{G}(\mathbb{A}_f)$, the Hecke correspondence $T(g)$ is given by the pair of morphisms

$$(\pi_{\tilde{K}}, \pi_{\tilde{K},g}) : \text{Sh}_{\tilde{K} \cap g \tilde{K} g^{-1}} \longrightarrow \text{Sh}_{\tilde{K}}(\mathbf{G}, X) \times \text{Sh}_{\tilde{K}}(\mathbf{G}, X),$$

with $\pi_{\tilde{K}} := \pi_{\tilde{K} \cap g \tilde{K} g^{-1}} / \tilde{K}$ and $\pi_{\tilde{K},g} := \pi_{\tilde{K} \cap g^{-1} \tilde{K} g} / \tilde{K} \circ [\cdot g]$, and where $\pi_{\tilde{K} \cap g \tilde{K} g^{-1}} / \tilde{K}$ and $\pi_{\tilde{K} \cap g^{-1} \tilde{K} g} / \tilde{K}$ are defined as follows:

$$\begin{array}{ccc} \text{Sh}_{\tilde{K} \cap g \tilde{K} g^{-1}}(\mathbf{G}, X) & \xrightarrow{[\cdot g]} & \text{Sh}_{\tilde{K} \cap g^{-1} \tilde{K} g}(\mathbf{G}, X) \\ \downarrow \pi_{\tilde{K} \cap g \tilde{K} g^{-1}} / \tilde{K} & & \downarrow \pi_{\tilde{K} \cap g^{-1} \tilde{K} g} / \tilde{K} \\ \text{Sh}_{\tilde{K}}(\mathbf{G}, X) & & \text{Sh}_{\tilde{K}}(\mathbf{G}, X) \end{array}$$

We define (after e.g. [17], 1.4 and 1.7), the *proper push-forward* and *flat pullback* functors as follows:

Definition 3.1.5 (Proper push-forward and flat pull-back of cycles.). *Let $f : X \rightarrow Y$ be a morphism of schemes of relative dimension $d \geq 0$, and let $i \in \{1, \dots, \dim X\}$ and $j \in \{1, \dots, \dim Y\}$ be positive integers.*

(i) If f is proper, we define the push-forward for algebraic cycles as follows:

$$\begin{aligned} f_* : \mathbf{Z}_i(X) &\longrightarrow \mathbf{Z}_i(Y), \\ [V] &\longmapsto \deg(V/W) \cdot [W] \end{aligned}$$

where, for all i -dimensional irreducible closed subscheme $V \subset X$, we set $W := f(V)$ and $\deg(V/W)$ to be the extension degree $[R(V) : R(W)]$ between the fields of rational functions $R(V)$ and $R(W)$ of V and W respectively, if $\dim V = \dim W$ (in which case $[R(V) : R(W)] < \infty$), and to be 0 otherwise.

(ii) If f is flat, we define the pull-back for algebraic cycles as follows:

$$\begin{aligned} f^* : \mathbf{Z}_j(Y) &\longrightarrow \mathbf{Z}_{j+d}(X), \\ [W] &\longmapsto [V] \end{aligned}$$

where we set V to be the $j + d$ dimensional closed subscheme $f^{-1}(W) \subset X$, whose irreducible components (and corresponding generic points) are $(V_k, \eta_k)_{k=1, \dots, s}$, and where $[V]$ is the fundamental cycle $\sum_{k=1}^s \text{len}_{\mathcal{O}_{V_k, \eta_k}}(\mathcal{O}_{V_k, \eta_k}) \cdot [V_k]$ attached to V .

Recall that the transition morphisms $\pi_{\tilde{K}}$ and $\pi_{\tilde{K}, g}$ are finite étale, hence proper, smooth and flat (of relative dimension 0). Accordingly, the Hecke correspondence $T(g)$ yields an operator $T(g)_*$ on $\mathbb{Z}[\mathcal{Z}_{\tilde{K}}(\mathbf{G}, \mathbf{H})] \subset \mathbf{Z}_1(\text{Sh}_{\tilde{K}}(\mathbf{G}, X))$, defined as $T(g)_* := (\pi_{\tilde{K}, g})_* \circ \pi_{\tilde{K}}^*$. That the map $\pi_{\tilde{K}}$ is smooth imply that, for all integral closed subschemes $Z \subset \text{Sh}_{\tilde{K}}(\mathbf{G}, X)$, the induced map $Z \times_{\text{Sh}_{\tilde{K}}(\mathbf{G}, X)} \text{Sh}_{\tilde{K} \cap g\tilde{K}g^{-1}}(\mathbf{G}, X) \longrightarrow Z$ is again smooth, and the source is reduced⁽⁵⁾. Accordingly, if $Z' \subset Z \times_{\text{Sh}_{\tilde{K}}(\mathbf{G}, X)} \text{Sh}_{\tilde{K} \cap g\tilde{K}g^{-1}}(\mathbf{G}, X)$ is an irreducible component with generic point η' , then the ring $\mathcal{O}_{Z', \eta'}$ is integral and has finite length as an $\mathcal{O}_{Z', \eta'}$ -module (i.e., $\mathcal{O}_{Z', \eta'}$ is artinian). Thus $\mathcal{O}_{Z', \eta'}$ is in fact a field and $\text{len}_{\mathcal{O}_{Z', \eta'}}(\mathcal{O}_{Z', \eta'}) = 1$, which shows that the multiplicities appearing in the definition of the flat pullback are all equal to 1 in the case of $\pi_{\tilde{K}}$. By definition of the family of \mathbf{H} -special cycles, one gets the following equality, for all $g' \in \mathbf{G}(\mathbb{A}_f)$:

$$\pi_{\tilde{K}}^*(\mathcal{Z}_{\tilde{K}}(g')) = \sum_{k=1}^s \mathcal{Z}_{\tilde{K} \cap g\tilde{K}g^{-1}}(g'g_k), \tag{3.17}$$

where g_1, \dots, g_s is a minimal family of elements of \tilde{K} such that

$$\pi_{\tilde{K}}^{-1}(\mathcal{Z}_{\tilde{K}}(g')) = \bigcup_{\tilde{g}_i \in \tilde{K}/\tilde{K} \cap g\tilde{K}g^{-1}} \mathcal{Z}_{\tilde{K} \cap g\tilde{K}g^{-1}}(g'\tilde{g}_i) = \bigcup_{k=1}^s \mathcal{Z}_{\tilde{K} \cap g\tilde{K}g^{-1}}(g'g_k).$$

Let us go back to the case $\tilde{K} = \mathbf{I}^{(n)}$, $n \geq 1$, and $g = \delta$. From now on, we assume to simplify that $g_{0, \Sigma} = \mathbf{1}_{\Sigma}$, i.e., that $g_0 = \mathbf{1} \in \mathbf{G}(\mathbb{A}_f)$.

(5). This can be seen as a consequence of smooth morphisms being geometrically regular.

Lemma 3.1.9. *Let $n \geq 1$, $m \geq 0$ and $\mathbf{c} = \tau_1 \cdot \dots \cdot \tau_r \in \mathcal{P}^\tau$. For all $j \in \{1, \dots, r\}$, we let g_j be an element of G_{τ_j} . Let h, h' belong to $H_{\tau, n+m}$. One has*

$$\mathcal{Z}_{\mathbf{I}^{(n+1)}} \left(g_{0, \Sigma}, h \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) = \mathcal{Z}_{\mathbf{I}^{(n+1)}} \left(g_{0, \Sigma}, h' \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right)$$

if and only if

$$h \in H_{\tau, n+m}^{\text{der}} h' H_{n+m+1},$$

i.e., h and h' define the same element of $H'_{\tau, n+m} / H_{\tau, n+m} = \frac{H_{\tau, n+m}^{\text{der}} \backslash H_{\tau, n+m}}{H_{\tau, n+m+1}^{\text{der}} \backslash H_{\tau, n+m+1}}$.

Proof. Assume that h' belongs to $H_{\tau, n+m}^{\text{der}} h H_{\tau, n+m+1}$. In the building $\mathbf{B}_{V, \tau}$, one has $H_{\tau, n+m+1} = \text{Stab}_{H_\tau}(\Xi_{\tau, m}^{(n+1)})$ with, we recall, $\Xi_{\tau, m}^{(n+1)} = \delta^m \cdot \Xi_{\tau, 0}^{(n+1)} = \delta_V^{-m} \cdot \Xi_{\tau, V, 0}^{(n+1)} \otimes \delta_W^m \cdot \Xi_{\tau, W, 0}^{(n+1)} \in \text{Hyp}_\tau^{(n+1)}$. This gives

$$(h' \delta^m) \cdot \Xi_{\tau, 0}^{(n+1)} = h' \cdot \Xi_{\tau, m}^{(n+1)} \in H_\tau^{\text{der}} h \cdot \Xi_{\tau, m}^{(n+1)} = H_\tau^{\text{der}} (h \delta^m) \cdot \Xi_{\tau, 0}^{(n+1)},$$

i.e., $h' \delta^m$ and $h \delta^m$ are equal in $H_\tau^{\text{der}} \backslash G_\tau / \mathbf{I}_\tau^{(n+1)}$. This implies that

$$\mathcal{Z}_{\mathbf{I}^{(n+1)}} \left(g_{0, \Sigma}, h' \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) = \mathcal{Z}_{\mathbf{I}^{(n+1)}} \left(g_{0, \Sigma}, h \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right).$$

Conversely, assume that both elements

$$\mathcal{Z}_{\mathbf{I}^{(n+1)}} \left(g_{0, \Sigma}, h \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right), \quad \mathcal{Z}_{\mathbf{I}^{(n+1)}} \left(g_{0, \Sigma}, h' \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \in \mathcal{Z}_{\mathbf{I}^{(n+1)}}(\mathbf{G}, \mathbf{H})$$

are equal. Then there exists $z_{\mathbb{Q}} = (z_{V, \mathbb{Q}}, z_{W, \mathbb{Q}}) \in \mathbf{Z}_{\mathbf{G}}(\mathbb{Q})$, $h_{\mathbb{Q}} = \Delta(g_{W, \mathbb{Q}}) \in \mathbf{H}(\mathbb{Q})$ and $k = (k_v)_{v \in I_{E, f}} \in \mathbf{I}^{(n+1)}$, with $k_v = (k_{V, v}, k_{W, v})$ for all $v \in I_{E, f}$, such that

$$z_{\mathbb{Q}} h_{\mathbb{Q}} \cdot \left(g_{0, \Sigma}, h \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) = \left(g_{0, \Sigma}, h' \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) k.$$

This gives locally (recall that $g_{0, \Sigma} = \mathbf{1}_\Sigma$):

$$\begin{cases} z_{V, \mathbb{Q}} \iota(g_{W, \mathbb{Q}}) = k_{V, v}, \quad z_{W, \mathbb{Q}} g_{W, \mathbb{Q}} = k_{W, v} & \text{if } v \notin \{\tau, \tau_1, \dots, \tau_r\} & \text{(a)} \\ z_{\mathbb{Q}} h_{\mathbb{Q}} g_j = g_j k_{\tau_j} & \text{for } j = 1, \dots, r. & \text{(b)} \\ z_{\mathbb{Q}} h_{\mathbb{Q}} h \delta^m = h' \delta^m k_\tau & \text{at } v = \tau, & \text{(c)} \end{cases}$$

— Equality (a) gives, by acting on the anisotropic vector $e_D \in D_v$, that $z_{V, \mathbb{Q}} \iota(g_{W, \mathbb{Q}}) \cdot e_D = z_{V, \mathbb{Q}} \cdot e_D = k_{V, v} \cdot e_D$. Accordingly, $k_{V, v} \in K_{V, v} \subset \text{GL}(L_{V, v})$ fixes the line $\mathcal{O}_{E_v} e_D \subset L_{V, v}$, hence stabilizes also $L_{W, v}$. We get that $z_{V, \mathbb{Q}} \in \mathcal{O}_{E_v}$, as $k_{V, v}$ acts integrally on $L_{V, v}$, and even that $\det(k_{V, v}) = z_{V, \mathbb{Q}} \det(k_{V, v}|_{W_v}) \in \mathcal{O}_{E_v}^\times$, thus $z_{V, \mathbb{Q}} \in \mathcal{O}_{E_v}^\times$, i.e., $\det(z_{V, \mathbb{Q}}) = z_{V, \mathbb{Q}}^3 \in \mathcal{O}_{E_v}^\times$. Therefore one has

$$\det(h_{\mathbb{Q}}) := \det(g_{W, \mathbb{Q}}) = \det(\iota(g_{W, \mathbb{Q}})) = \det(k_{V, v}) \det(z_{V, \mathbb{Q}})^{-1} \in \mathcal{O}_{E_v}^\times,$$

i.e., $\det(h_{\mathbb{Q}}) \in \mathcal{O}_{E_v}^\times$, for all $v \notin \{\tau, \tau_1, \dots, \tau_r\}$.

— Equality (b) induces $z_{\mathbb{Q}} h_{\mathbb{Q}} g_j \cdot (x_{\tau_j} \otimes x_{\tau_j}) = g_j k_{\tau_j} \cdot (x_{\tau_j} \otimes x_{\tau_j}) = g_j \cdot (x_{\tau_j} \otimes x_{\tau_j})$, where $(x_{\tau_j} \otimes x_{\tau_j})$ still refers to the origin of the product building $\mathbf{B}_{V,\tau_j} \times \mathbf{B}_{W,\tau_j}$, which corresponds to the pair of self-dual lattices $(L_{V,\tau_j} = L_{W,\tau_j} \oplus L_{D,\tau_j}, L_{W,\tau_j})$. This gives $h_{\mathbb{Q}} g_j \cdot (x_{\tau_j} \otimes x_{\tau_j}) = g_j \cdot (z_{\mathbb{Q}}^{-1} \cdot (x_{\tau_j} \otimes x_{\tau_j})) = g_j \cdot (x_{\tau_j} \otimes x_{\tau_j})$, the last equality coming from (2.22). Accordingly, $h_{\mathbb{Q}}$ fixes the pair $(g_{j,V} \cdot x_{\tau_j}, g_{j,W} \cdot x_{\tau_j}) \in \text{Hyp}_{\tau_j}$ of hyperspecial vertices, i.e., fixes (as a set) at least one $\mathcal{O}_{E_{\tau}}$ -lattice in V_{τ_j} , which implies that $\det(h_{\mathbb{Q}})$ belongs to $\mathcal{O}_{E_{\tau_j}}^{\times}$.

— Finally, equality (c) gives, by acting on $\Xi_{\tau,0}^{(n+1)} = \Xi_{\tau,V,0}^{(n+1)} \otimes \Xi_{\tau,W,0}^{(n+1)} \in \text{Hyp}_{\tau}^{(n+1)}$:

$$z_{\mathbb{Q}} h_{\mathbb{Q}} h \delta^m \cdot \Xi_{\tau,0}^{(n+1)} = h' \delta^m k_{\tau} \cdot \Xi_{\tau,0}^{(n+1)} = h' \delta^m \cdot \Xi_{\tau,0}^{(n+1)},$$

hence $z_{\mathbb{Q}} h_{\mathbb{Q}} h \cdot \Xi_{\tau,m}^{(n+1)} = h' \cdot \Xi_{\tau,m}^{(n+1)}$, thus $z_{\mathbb{Q}} (h')^{-1} h_{\mathbb{Q}} h \in \mathbf{I}_{\tau}^{(n+m+1)}$. As $z_{\mathbb{Q}} \in \mathbf{Z}_{\mathbf{G}}(\mathbb{Q}) \subset \mathbf{I}^{(n+m+1)}$ - still by (2.22) - one obtains that $(h')^{-1} h_{\mathbb{Q}} h \in H_{\tau} \cap \mathbf{I}_{\tau}^{(n+m+1)} = H_{\tau,n+m+1}$, which implies that $h_{\mathbb{Q}} \in H_{\tau,n+m}$. In particular, one has $h_{\mathbb{Q}} \in K_{\tau}$, thus $\det(h_{\mathbb{Q}}) \in \mathcal{O}_{E_{\tau}}^{\times}$.

The three points above imply that the element $\det(h_{\mathbb{Q}}) \in \mathbf{U}(1)_{E/F}(F) \subset E^{\times}$ lies in fact in \mathcal{O}_E^{\times} , therefore is an element of $\ker(N_{E/F} : \mathcal{O}_E^{\times} \rightarrow \mathcal{O}_F^{\times}) = (\mathcal{O}_E^{\times})_{\text{tors}}$. On the other hand, that $h_{\mathbb{Q}} \in H_{n+m}$ implies that $\det(h_{\mathbb{Q}}) \in \mathcal{O}_{\tau,n+m}^1 \subset 1 + \varpi^{n+m} \mathcal{O}_{E_{\tau}}$, i.e., that $\text{ord}_{\tau, \mathcal{O}_E}(\det(h_{\mathbb{Q}}) - 1) \geq 1$. However, the assumption $\tau \cdot \mathcal{O}_E \nmid I_0$ induces that $\det(h_{\mathbb{Q}}) - 1 = 0$, hence $h_{\mathbb{Q}} \in H_{\tau,n+m}^{\text{der}}$. That $(h')^{-1} h_{\mathbb{Q}} h \in H_{\tau,n+m+1}$ finally gives:

$$h \in H_{\tau,n+m}^{\text{der}} h' H_{\tau,n+m+1}.$$

as claimed □

Let $n \geq 1$, $m \geq 0$ and $\mathbf{c} = \tau_1 \cdot \dots \cdot \tau_r \in \mathcal{P}^{\tau}$. Let $g_1 \in G_{\tau_1}, \dots, g_r \in G_{\tau_r}$ be elements in the support of, respectively, $\hat{y}_{\tau_1}, \dots, \hat{y}_{\tau_r}$. A consequence of the preceding lemma is that the terms of the sum

$$\begin{aligned} & \text{Tr}_{\mathcal{K}(\tau^{n+m+1}, \mathbf{c}) / \mathcal{K}(\tau^{n+m}, \mathbf{c})} \left(\mathcal{Z}_{\mathbf{I}^{(n+1)}} \left(g_{0,\Sigma}, \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \right) \\ &= \sum_{h \in H'_{\tau,n+m} / H'_{\tau,n+m+1}} \mathcal{Z}_{\mathbf{I}^{(n+1)}} \left(g_{0,\Sigma}, h \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \end{aligned}$$

are pairwise distinct. On the other hand one has,

$$\begin{aligned} & \text{Supp Tr}_{\mathcal{K}(\tau^{n+m+1}, \mathbf{c}) / \mathcal{K}(\tau^{n+m}, \mathbf{c})} \left(\mathcal{Z}_{\mathbf{I}^{(n)}} \left(g_{0,\Sigma}, \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \right) \\ &= \text{Supp } q^5 \text{Tr}_{\mathcal{K}(\tau^{n+m+1}, \mathbf{c}) / \mathcal{K}(\tau^{n+m}, \mathbf{c})} \left(\mathcal{Z}_{\mathbf{I}^{(n)}} \left(g_{0,\Sigma}, \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \right) \\ &= \left\{ \mathcal{Z}_{\mathbf{I}^{(n+1)}} \left(g_{0,\Sigma}, h \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right); h \in H_{\tau,n+m} / H_{\tau,n+m+1} \right\} \\ &= \left\{ \mathcal{Z}_{\mathbf{I}^{(n+1)}} \left(g_{0,\Sigma}, \delta^m (\delta^{-m} h \delta^m), g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right); h \in H_{\tau,n+m} / H_{\tau,n+m+1} \right\} \\ &= \left\{ \mathcal{Z}_{\mathbf{I}^{(n+1)}} \left(g_{0,\Sigma}, \delta^m i, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right); i \in \mathbf{I}_{\tau}^{(n)} / \mathbf{I}_{\tau}^{(n+1)} \right\} \quad \text{by Lemma 2.3.13} \\ &= \text{Supp } (\pi^{(n)})^* \left(\mathcal{Z}_{\mathbf{I}^{(n)}} \left(g_{0,\Sigma}, \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \right) \quad \text{by (3.17)}. \end{aligned}$$

This gives finally:

$$\begin{aligned} & \mathrm{Tr}_{\mathcal{K}(\tau^{n+m+1} \cdot \mathbf{c}) / \mathcal{K}(\tau^{n+m} \cdot \mathbf{c})} \left(\mathcal{Z}_{\mathbf{I}(n+1)} \left(g_{0,\Sigma}, \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \right) \\ &= (\pi^{(n)})^* \left(\mathcal{Z}_{\mathbf{I}(n)} \left(g_{0,\Sigma}, \delta^m, g_1, \dots, g_r, \mathbf{1}^{\Sigma, \tau, (\tau_j)_{j=1}^r} \right) \right), \end{aligned}$$

hence

$$\mathrm{Tr}_{\mathcal{K}(\tau^{n+m+1} \cdot \mathbf{c}) / \mathcal{K}(\tau^{n+m} \cdot \mathbf{c})} (\mathfrak{z}(n+1, \tau^m \cdot \mathbf{c})) = (\pi^{(n)})^* (\mathfrak{z}(n, \tau^m \cdot \mathbf{c})). \quad (3.18)$$

In other words, the level-wise vertical relation admits the following variant:

$$(\pi_\delta^{(n)})_* \left(\mathrm{Tr}_{\mathcal{K}(\tau^{n+m+1} \cdot \mathbf{c}) / \mathcal{K}(\tau^{n+m} \cdot \mathbf{c})} (\mathfrak{z}(n+1, \tau^m \cdot \mathbf{c})) \right) = T(\delta)_* (\mathfrak{z}(n, \tau^m \cdot \mathbf{c})), \quad (3.19)$$

for all $n \geq 1$, $m \geq 0$ and $\mathbf{c} \in \mathcal{P}^\tau$.

Remark 3.1.3. *The above equality, together with the level-wise vertical relation of Theorem 3.1.8, lead us to the following conjecture:*

Conjecture 3.1.1. *For all $\mathcal{Z} \in \mathrm{Supp} \mathrm{Tr}_{\mathcal{K}(\tau^{n+m+1} \cdot \mathbf{c}) / \mathcal{K}(\tau^{n+m} \cdot \mathbf{c})} (\mathfrak{z}(n+1, \tau^m \cdot \mathbf{c})) = \mathrm{Supp} (\pi^{(n)})^* (\mathfrak{z}(n, \tau^m \cdot \mathbf{c}))$, one has*

$$\mathrm{deg} \left(\mathcal{Z} / \pi_\delta^{(n)}(\mathcal{Z}) \right) = q^5.$$

An immediate consequence of Conjecture 3.1.1 would be that:

$$\begin{aligned} T(\delta)_* (\mathfrak{z}(n, \tau^m \cdot \mathbf{c})) &\stackrel{\mathrm{def}}{=} (\pi_\delta^{(n)})_* \left((\pi^{(n)})^* (\mathfrak{z}(n, \tau^m \cdot \mathbf{c})) \right) \\ &= (\pi_\delta^{(n)})_* \left(\mathrm{Tr}_{\mathcal{K}(\tau^{n+m+1} \cdot \mathbf{c}) / \mathcal{K}(\tau^{n+m} \cdot \mathbf{c})} (\mathfrak{z}(n+1, \tau^m \cdot \mathbf{c})) \right) && \text{by (3.18)} \\ &= q^5 \pi_\delta^{(n)} \left(\mathrm{Tr}_{\mathcal{K}(\tau^{n+m+1} \cdot \mathbf{c}) / \mathcal{K}(\tau^{n+m} \cdot \mathbf{c})} (\mathfrak{z}(n+1, \tau^m \cdot \mathbf{c})) \right) && \text{by Conjecture 3.1.1} \\ &= t^{(n)} \cdot \mathfrak{z}(n, \tau^m \cdot \mathbf{c}), && \text{by Theorem 3.1.8} \end{aligned}$$

i.e., would yield the reasonable claim that the geometric Hecke correspondence $T(\delta)_$ acts on our chosen family of cycles (and, we suppose, on the whole $\mathbb{Z}[\mathcal{Z}_{\mathbf{I}(n)}(\mathbf{G}, \mathbf{H})]$) in the same way as the « set-theoretic » double coset operator $t^{(n)}$.*

Bibliography

- [1] A. A. Beilinson. Higher regulators and values of L -functions. In *Current problems in mathematics, Vol. 24*, Itogi Nauki i Tekhniki, pages 181–238. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984.
- [2] M. Bertolini and H. Darmon. Iwasawa’s main conjecture for elliptic curves over anticyclotomic \mathbb{Z}_p -extensions. *Ann. of Math. (2)*, 162(1):1–64, 2005.
- [3] Massimo Bertolini, Henri Darmon, and Kartik Prasanna. Generalized Heegner cycles and p -adic Rankin L -series. *Duke Math. J.*, 162(6):1033–1148, 2013. With an appendix by Brian Conrad.
- [4] Don Blasius and Jonathan D. Rogawski. Zeta functions of Shimura varieties. *Proceedings of Symposia in Pure Mathematics*, 55, Motives:525–571, 1994.
- [5] Spencer Bloch and Kazuya Kato. L -functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.
- [6] Armand Borel and Harish-Chandra. Arithmetic subgroups of algebraic groups. *Bulletin of the American Mathematical Society*, 67(6):579–583, 1961.
- [7] Reda Boumasmoud. *Generalized Norm-compatible Systems on Unitary Shimura Varieties*. PhD thesis, École polytechnique fédérale de Lausanne, June 2019.
- [8] Reda Boumasmoud, Hunther Brooks, and Dimitar Jetchev. Vertical distribution relations for special cycles on unitary Shimura varieties. *International Mathematics Research Notices*, june 2018. doi:10.1093/imrn/rny119, available at <http://oup.prod.sis.lan/imrn/advance-article-pdf/doi/10.1093/imrn/rny119/25046247/rny119.pdf>.
- [9] François Bruhat and Jacques Tits. Schémas en groupes et immeubles des groupes classiques sur un corps local. II: groupes unitaires. *Bulletin de la S. M. F.*, 115:141–195, 1987.
- [10] Robert Coleman and Barry Mazur. The eigencurve. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 1–113. Cambridge Univ. Press, Cambridge, 1998.

- [11] Brian Conrad. Reductive group schemes, 2011. Course notes for the CIRM summer school "Group schemes: an introduction to the SGA III seminar of Demazure-Grothendieck".
- [12] Brian Conrad, Ofer Gabber, and Gopal Prasad. *Pseudo-reductive groups*, volume 26 of *new mathematical monographs*. Cambridge University Press, 2015. second edition 2015.
- [13] Christophe Cornut and Vinayak Vatsal. *Nontriviality of Rankin-Selberg L-functions and CM points*, pages 121–186. Cambridge University Press, 2007.
- [14] Pierre Deligne. Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques. *Proceedings of Symposia in Pure Mathematics*, 33:247–289, 1979.
- [15] Matthew Emerton. On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. *Invent. Math.*, 164(1):1–84, 2006.
- [16] Olivier Fouquet. Dihedral Iwasawa theory of nearly ordinary quaternionic automorphic forms. *Compositio Mathematica*, 149:356–416, 2013.
- [17] William Fulton. *Intersection Theory*. Springer-Verlag New-York, 1998. second edition.
- [18] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad. Symplectic local root numbers, central critical L -values, and restriction problems in the representation theory of classical groups. *Astérisque*, 346:1–110, 2012.
- [19] Jayce R. Getz and Heekyoung Hahn. *An introduction to Automorphic representations with a view toward Trace formulae*. GTM Series. Springer. To be published in 2019, preliminary version available at <https://services.math.duke.edu/hahn/GTM.pdf>.
- [20] Oscar Goldman and Nagayoshi Iwahori. The space of \mathfrak{p} -adic norms. *Acta Mathematica*, 109:133–177, 1963.
- [21] Benedict Gross and Don Zagier. Heegner points and derivatives of L -series. *Inventiones Mathematicae*, 84(2):225–320, 1986.
- [22] Alexander Grothendieck and Jean Dieudonné. *Éléments de Géométrie Algébrique: IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie*, volume 32 of *Publications mathématiques de l'I.H.É.S.* Institut des Hautes Études Scientifiques, 1967.
- [23] Haruzo Hida. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986.
- [24] Benjamin Howard. The Heegner point Kolyvagin system. *Compositio Mathematica*, 140(6):1439–1472, 2004.
- [25] Benjamin Howard. Variation of Heegner points in Hida families. *Inventiones Mathematicae*, 167(1):91–128, 2007.
- [26] Uwe Jannsen. Continuous étale cohomology. *Mathematische Annalen*, page 207–245, 1988.

- [27] Dimitar Jetchev. Hecke and Galois properties of special cycles on unitary Shimura varieties, 2016. Preprint available at <https://arxiv.org/pdf/1410.6692v3.pdf> (version 3).
- [28] Kazuya Kato. Iwasawa theory and p -adic Hodge theory. *Kodai Math. J.*, 16(1):1–31, 1993.
- [29] Kazuya Kato. p -adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, (295):ix, 117–290, 2004. Cohomologies p -adiques et applications arithmétiques. III.
- [30] Victor Kolyvagin. *Euler systems*, volume II, pages 435–485. Birkhäuser, 1990.
- [31] Jean-Stefan Koskivirta. *Relation de congruence pour les variétés de Shimura associées aux groupes unitaires* $\mathrm{GU}(n-1, 1)$. PhD thesis, Université de Strasbourg, may 2013.
- [32] James S. Milne. *Introduction to Shimura varieties*, volume 4 of *Clay Mathematics Proceedings*, pages 265–378. American Mathematical Society, 2005. version updated in 2017, available on <http://www.jmilne.org/math/>.
- [33] James S. Milne. *Algebraic Groups, The Theory of Group Schemes of Finite Type over a Field*. Cambridge University Press, 2017.
- [34] Peter Müller. Algebraic groups over finite fields, a quick proof of Lang’s Theorem. *Proceedings of the American Mathematical Society*, 131(2):369–370, February 2003.
- [35] Ben Moonen. Linearity properties of Shimura varieties, i. *Journal of Algebraic Geometry*, 3(7):539–567, 1998.
- [36] Sophie Morel and Suh. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 748:139–151, 2019.
- [37] Jan Nekovář. p -adic Abel-Jacobi maps and p -adic heights. In *The Arithmetic and Geometry of Algebraic Cycles*, 2000. 1998 Banff Conference “The Arithmetic and Geometry of Algebraic Cycles”.
- [38] Jan Nekovář. *The Euler system method for CM points on Shimura curves*, volume 320 of *London Mathematical Society Lecture Note Series*, pages 471–547. Cambridge University Press, 2007.
- [39] Jürgen Neukirch. *Class field theory*, volume 280 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 1986.
- [40] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of Number fields*, volume 323 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 2008. second edition, version 2.2, July 2015.
- [41] Richard Pink. *Arithmetical compactification of mixed Shimura varieties*. PhD thesis, Universität Bonn, Bonner Mathematische Schriften, 1989.
- [42] Vladimir Platonov and Andreï Rapinchuk. *Algebraic groups and Number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press Inc., 1994.

- [43] Michael Rapoport, Brian Smithling, and Wei Zhang. Arithmetic diagonal cycles on unitary Shimura varieties. 2017. preprint available at <https://arxiv.org/abs/1710.06962> (version 4).
- [44] Karl Rubin. The “main conjectures” of Iwasawa theory for imaginary quadratic fields. *Invent. Math.*, 103(1):25–68, 1991.
- [45] Winfried Scharlau. *Quadratic and Hermitian Forms*, volume 270 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1985.
- [46] Jean-Pierre Serre. *Local Fields*, volume 67 of *Graduate texts in mathematics*. Springer-Verlag New York, 1979.
- [47] John T. Tate. Algebraic cycles and poles of zeta functions. In *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, pages 93–110. Harper & Row, New York, 1965.
- [48] Donald E. Taylor. *The geometry of the classical groups*, volume 9 of *Sigma series in pure mathematics*. Heldermann Verlag, Berlin, 1992.
- [49] Marie-France Vignéras. *Représentations ℓ -modulaires d’un groupe réductif p -adique, avec $\ell \neq p$.*, volume 137 of *Progress in Mathematics*. Birkhäuser, 1996. ISBN: 0-8176-3929-2.
- [50] Dave Witte Morris. *Introduction to Arithmetic subgroups*. Deductive Press, 2015. ISBN-13 : 978-0986571602, preliminary version available at ISBN-13 : 978-0986571602 (v6).
- [51] Xinyi Yuan, Shou-Wu Zhang, and Wei Zhang. *The Gross-Zagier formula on Shimura curves*, volume 184 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2013.

Titre : Familles p -adiques de cycles spéciaux sur une tour de variétés de Shimura unitaires.

Mots-clefs : conjectures de Gan-Gross-Prasad, cycles algébriques, immeubles de Bruhat-Tits, systèmes d'Euler, valeurs spéciales des fonctions L , variétés de Shimura unitaires.

Résumé : Nous étudions les propriétés p -adiques d'une famille de 1-cycles algébriques *spéciaux* sur une variété de Shimura unitaire de dimension 3 apparaissant dans le cadre des conjectures de Gan-Gross-Prasad. Ces cycles, introduits par D.Jetchev et étudiés également par Boumasmod-Brooks-Jetchev et R.Boumasmod, proviennent du plongement diagonal $\mathbf{U}(1, 1) \hookrightarrow \mathbf{U}(2, 1) \times \mathbf{U}(1, 1)$ associé à une extension CM E/F . Ils satisfont des relations de distribution « horizontales » et « verticales » sur leur conducteur, faisant de cette famille un nouvel exemple de *système d'Euler* géométrique généralisant celui des « points CM » sur la courbe modulaire, dont l'exploitation par V.Kolyvagin permit une avancée conceptuelle majeure dans l'attaque de la conjecture BSD. La preuve de ces relations locales entre action de Galois et celle de l'algèbre de Hecke de $\mathbf{G} = \mathbf{U}(2, 1) \times \mathbf{U}(1, 1)$ exploite les propriétés de certains opérateurs agissant sur l'immeuble de Bruhat-Tits de \mathbf{G} , en les places finies de F correspondantes. Nous construisons, en une place τ inerte de F divisant p , une filtration de \mathbf{G} par des sous-groupes ouverts compacts de type *Iwahori* définis comme les stabilisateurs d'une famille croissante de segments d'un même appartement. Nous adaptons au cas des segments la notion d'opérateurs « successeurs » étudiés par Boumasmod-Brooks-Jetchev et montrons que ceux-ci proviennent de l'algèbre de Hecke-Iwahori locale. Nous démontrons que la tour de variétés de Shimura induite par cette filtration rend compatibles les actions de Galois et Hecke sur les cycles avec les morphismes de changement de niveau. Cette relation *verticale sur le niveau* est un ingrédient en faveur de l'existence d'un système d'Euler en familles p -adiques dans la cohomologie étale en degré médian de la variété de Shimura de groupe \mathbf{G} .

Title : p -adic families of special cycles on a tower of unitary Shimura varieties.

Keywords : algebraic cycles, Bruhat-Tits buildings, Euler systems, Gan-Gross-Prasad conjectures, special values of L -functions, unitary Shimura varieties.

Abstract : We study the p -adic properties of a family of *special* algebraic 1-cycles defined on a 3-dimensional unitary Shimura variety which appears in the setting of the Gan-Gross-Prasad conjectures. These cycles, introduced by Jetchev and also studied by Boumasmod-Brooks-Jetchev and Boumasmod, arise from the diagonal embedding $\mathbf{U}(1, 1) \hookrightarrow \mathbf{U}(2, 1) \times \mathbf{U}(1, 1)$ attached to a CM extension E/F . These satisfy « horizontal » and « vertical » distribution relations for their conductors, making this family a new instance of a geometric *Euler system* generalizing the family of « CM-points » on modular curves, whose use by Kolyvagin provided a major conceptual advance towards the BSD conjecture. The proof of these local relations between the Galois action and the action of the Hecke algebra of $\mathbf{G} = \mathbf{U}(2, 1) \times \mathbf{U}(1, 1)$ make full use of some operators acting on the local Bruhat-Tits building of \mathbf{G} , at the corresponding finite places of F . We construct a τ -local filtration of \mathbf{G} - for some inert place τ of F above p - by *Iwahori*-type compact open subgroups, which are the stabilizers of an increasing family of segments in a same apartment. We adapt to segments the notion of « successor » operators studied by Boumasmod-Brooks-Jetchev and show that these arise from the local Iwahori-Hecke algebra. We show that the tower of varieties induced by this filtration makes the Galois and Hecke actions compatible with the change-of-level maps. This *level-wise vertical relation* is an ingredient towards the existence of a p -adic family of Euler systems in the middle-degree étale cohomology of the Shimura variety attached to \mathbf{G} .

