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Dynamics of a viscous incompressible flow in presence of a rigid body and of an inviscid incompressible flow in presence of a source and a sink

Marco Bravin

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Marco Bravin. Dynamics of a viscous incompressible flow in presence of a rigid body and of an inviscid incompressible flow in presence of a source and a sink. Fluid mechanics [physics.class-ph]. Université de Bordeaux, 2019. English. NNT : 2019BORD0192 . tel-02724747

HAL Id: tel-02724747

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Submitted on 2 Jun 2020

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École doctorale
**Mathématiques
et informatique**

Université
de **BORDEAUX**

Thèse de Doctorat
présentée à
L'université de Bordeaux

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET INFORMATIQUE

par **Marco Bravin**

sous la direction de Franck SUEUR et Marius TUCSNAK

POUR OBTENIR LE GRADE DE DOCTEUR

SPÉCIALITÉ : MATHÉMATIQUES APPLIQUÉES ET CALCUL SCIENTIFIQUE

**Dynamics of a viscous incompressible flow
in presence of a rigid body and of an
inviscid incompressible flow in presence of
a source and a sink**

Soutenue le 24 octobre 2019 à l'Institut de Mathématiques de Bordeaux
devant le jury composé de :

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Title: Dynamics of a viscous incompressible flow in presence of a rigid body and of an inviscid incompressible flow in presence of a source and a sink.

Abstract. In this thesis, we investigate properties of incompressible flows that interact with a rigid body or a source and a sink. In the case of an incompressible viscous fluid that satisfies the Navier Stokes equations in a $2D$ bounded domain well-posedness of Leray-Hopf weak solutions is well-understood. Existence and uniqueness are proved. Moreover solutions are continuous in time with values in $L^2(\Omega)$ and they satisfy the energy equality. Recently the problem of a rigid body moving in a viscous incompressible fluid modelled by the Navier-Stokes equations coupled with the Newton laws that prescribe the motion of the solid, was also tackled in the case where the no-slip boundary conditions were imposed. And the correspondent well-posedness result for Leray-Hopf type weak solutions was proved. In this manuscript we consider the case of the Navier-slip boundary conditions. In this setting, the existence result for the coupled system was proved by Gérard-Varet and Hillairet in 2014. Here, we prove that solutions are continuous in time, that they satisfy the energy equality and that they are unique. Moreover we show an existence result for weak solutions of a viscous incompressible fluid plus rigid body system in the case where the fluid velocity has an orthoradial part of infinite energy.

For an inviscid incompressible fluid modelled by the Euler equations in a $2D$ bounded domain, the case where the fluid is allowed to enter and to exit from the boundary was tackled by Judovič who introduced some conditions which consist in prescribing the normal component of the velocity and the entering vorticity. In this manuscript we consider a bounded domain with two holes, one of them is a source which means that the fluid is allowed to enter in the domain and the other is a sink from where the fluid can exit. In particular we find the limiting equations satisfied by the fluid when the source and the sink shrink to two different points. The limiting system is characterized by a point source/sink and a point vortex in each of the two points where the holes shrunk.

Keywords: Fluid mechanics, Navier-Stokes equations, fluid-structure interaction, source and sink, Euler equations, asymptotic limit

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Titre : Dynamique d'un écoulement incompressible visqueux en présence d'un corps rigide et d'un écoulement incompressible non visqueux en présence d'une source et d'un puits.

Résumé. Dans cette thèse, nous étudions les propriétés des écoulements de fluides qui interagissent avec un corps rigide ou avec une source et un puits. Dans le cas d'un fluide visqueux incompressible qui satisfait les équations de Navier Stokes dans un domaine borné $2D$, les solutions faibles de Leray-Hopf sont bien comprises. L'existence et l'unicité sont prouvées. De plus, les solutions sont continues en temps à valeurs dans $L^2(\Omega)$ et satisfont l'égalité d'énergie classique. Plus récemment, le problème d'un corps rigide en mouvement dans un fluide visqueux incompressible modélisé par les équations de Navier-Stokes couplées aux lois de Newton qui décrivent le mouvement du solide a également été abordé dans le cas où des conditions aux limites sans glissement ont été prescrites. Des résultats analogues concernant les solutions de Leray-Hopf ont également été démontrés dans ce contexte. Dans ce manuscrit, nous étudions le cas de conditions aux limites de Navier-Slip. Dans ce cadre, le résultat d'existence pour le système couplé a été prouvée par Gérard-Varet et Hillairet en 2014. Ici, nous montrons que les solutions sont continues en temps, qu'elles satisfont l'égalité d'énergie et qu'elles sont uniques. De plus, nous montrons un résultat d'existence des solutions faibles dans le cas d'un fluide incompressible visqueux auquel s'ajoute un corps rigide dans le cas où la vitesse du fluide a une partie orthonormale d'énergie infinie.

Pour un fluide incompressible non visqueux modélisé par les équations d'Euler dans un domaine borné $2D$, le cas où le fluide est autorisé à entrer et à sortir de la frontière a été traité par Judovič qui a introduit certaines conditions limites consistant à prescrire la composante normale de la vitesse et de la vorticit  entrante. Dans ce manuscrit, nous considérons un domaine borné qui possède deux trous. L'un d'eux est une source, ce qui signifie que le fluide est autorisé à entrer dans le domaine et l'autre est un puits où le fluide peut sortir. En particulier, nous établissons les équations limites vérifiées par le fluide lorsque la source et le puits se contractent en deux points différents. Le système limite est caractérisé par un point source/puits et un point vortex en chacun des deux points où les trous se sont contractés.

Mots-clés: Mécaniques des fluides, équations de Navier-Stokes, interaction fluide-structure, source et puits, équations d'Euler, limite asymptotique.

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Introduction (Français)

Dans cette brève introduction, nous présentons les principaux résultats qui seront abordés dans la thèse. Pour une discussion plus large sur le sujet, nous renvoyons au chapitre suivant.

Dans ce manuscrit, nous étudions l'interaction des écoulements qui interagissent avec un corps rigide ou avec une source et un puits. Ces problèmes font partie de la branche des mathématiques appelée mécanique des fluides, et qui a pour objectif l'étude des propriétés d'un fluide. Nous nous limiterons ici au cas des fluides incompressibles à densité constante. Ceux-ci sont divisés en deux classes principales: les fluides parfaits qui satisfont les équations de type Euler et les fluides visqueux qui satisfont les équations de Navier-Stokes. Le système incompressible d'Euler avec densité constante ρ a été introduit par Euler dans [Eul57] en 1757 et est composé des équations suivantes

$$\begin{aligned}\rho(\partial_t u + (u \cdot \nabla)u) + \nabla p &= 0 \\ \operatorname{div} u &= 0\end{aligned}$$

où u est un champ de vitesse et p est une fonction à valeurs dans \mathbb{R} qui désigne la pression. Les équations de Navier-Stokes ont été introduites par Navier et Stokes dans [Sto80] et modélisent le mouvement d'un fluide visqueux incompressible avec densité constante. Le système prend la forme suivante

$$\begin{aligned}\rho(\partial_t u + u \cdot \nabla u) - \nu \Delta u + \nabla p &= 0 \\ \operatorname{div} u &= 0\end{aligned}$$

où ρ et ν sont deux constantes positives décrivant la densité et la viscosité du fluide respectivement. Ces systèmes ont été largement étudiés au siècle dernier et les principaux résultats d'existence et d'unicité seront traités dans la version anglaise de l'introduction.

Cette thèse est divisée en deux parties: dans la première partie, nous étudions le problème d'un objet rigide se déplaçant dans un fluide visqueux. Ce problème est modélisé par un système couplant les équations incompressibles de Navier-Stokes aux lois de Newton décrivant

la dynamique de l'objet. Nous étudions en particulier les propriétés des solutions faibles de Hopf-Leray en dimension deux et prouvons que ces solutions sont continues en temps à valeurs dans L^2 , satisfont l'égalité d'énergie et sont uniques tant que le solide ne touche pas le bord. De plus, dans le cas où le système composé du fluide et du solide occupe tout \mathbb{R}^2 , nous prouvons l'existence du système couplé dans le cas où l'énergie du fluide peut être infinie.

Dans la deuxième partie, nous considérons l'écoulement d'un fluide parfait incompressible dans un domaine troué. Nous considérons en particulier le cas où le domaine a exactement deux trous. Du premier, le liquide ne peut qu'entrer (nous l'appellerons la source), tandis que de l'autre, le liquide ne peut que sortir (nous l'appellerons le puits). Les équations qui modélisent le fluide sont les équations d'Euler. Dans cette thèse, nous établissons l'équation limite lorsque la taille de la source et celle du puits tend vers zéro. Nous démontrons également que les solutions du système d'Euler avec une source et un puits macroscopiques convergent vers une solution du système limite, pour des normes appropriées, lorsque les trous se contractent en deux points distincts.

Dans les deux sections suivantes, nous présenterons les modèles mathématiques et décrirons les principaux résultats de la thèse.

1.1 Modèle mathématique d'un solide immergé dans un fluide visqueux

Pour les problèmes d'interaction fluide-structure, la dimension de l'espace physique est égale à trois. Les modèles 2D décrivent des situations dans lesquelles le mouvement est homogène dans une direction. Nous présentons maintenant le modèle mathématique satisfait par un solide dans un fluide visqueux soumis aux forces de Newton. Soit Ω est égal à \mathbb{R}^2 , soit à un sous-ensemble borné, simplement connexe et connexe, avec un bord lisse. Soit \mathcal{S}_0 un sous-ensemble de Ω fermé, connexe et simplement connexe, avec une bord lisse occupée par le solide. Soit $\mathcal{S}(t)$ la position du corps rigide au temps t et $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$ le domaine du fluide. Le système, ayant pour inconnues (\mathcal{S}, u, p) , prend la forme

$$\begin{aligned}
\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) - \nu \Delta u + \nabla p &= 0 && \text{pour } x \in \mathcal{F}(t), \\
\operatorname{div} u &= 0 && \text{pour } x \in \mathcal{F}(t), \\
u \cdot n &= u_{\mathcal{S}} \cdot n && \text{pour } x \in \partial \mathcal{S}(t), \\
(D(u)n) \cdot \tau &= -\alpha(u - u_{\mathcal{S}}) \cdot \tau \\
u \cdot n &= 0 && \text{pour } x \in \partial \Omega, \text{ si } \Omega \text{ borné,} \\
(D(u)n) \cdot \tau &= -\alpha u \cdot \tau \\
|u| &\longrightarrow 0 && |x| \longrightarrow \infty, \text{ si } \Omega = \mathbb{R}^2, \\
mh''(t) &= - \int_{\partial \mathcal{S}(t)} \Sigma n \, ds, \\
\mathcal{J}r'(t) &= - \int_{\partial \mathcal{S}(t)} (x - h(t))^\perp \cdot \Sigma n \, ds, \\
u|_{t=0} &= u_0 && \text{pour } x \in \mathcal{F}_0, \\
h(0) = h_0, h'(0) = \ell_0, &&& r(0) = r_0.
\end{aligned} \tag{1.1.1}$$

Ici, $u = (u_1, u_2)$ et p sont respectivement le champ de vitesse et la pression, ρ et ν sont deux constantes positives décrivent la densité et la viscosité du fluide respectivement, n et τ sont respectivement le vecteur normal quittant le domaine fluide et le vecteur tangent indirect, α est une constante positive qui quantifie le glissement. De plus, m et \mathcal{J} désignent la masse et le moment d'inertie du corps rigide, tandis que le fluide a une densité constante de 1 et, sans perte de généralité, on suppose que la viscosité est égale à 1. Le tenseur de contraintes de Cauchy est défini par

$$\Sigma = -p \text{Id}_2 + 2\nu D(u),$$

où $D(u)$ est le tenseur de déformation défini par

$$2D(u) = ((\partial_j u_i + \partial_i u_j))_{1 \leq i, j \leq 2}.$$

Pour $x = (x_1, x_2)$ on note x^\perp le vecteur $x^\perp = (-x_2, x_1)$, $h'(t)$ est la vitesse du centre de masse $h(t)$ du corps rigide et $r(t)$ est la vitesse angulaire. Nous indiquons la vitesse du solide par:

$$u_S(t, x) = h'(t) + r(t)(x - h(t))^\perp.$$

1.1.1 Contribution

Prenons le cas où Ω est borné. L'existence de solutions faibles de Hopf-Leray pour le système (1.1.1) a été démontrée dans [GVH14] en dimension trois. Dans cette thèse, nous montrons que la même démonstration peut être faite en dimension deux pour une définition plus restrictive des solutions faibles. Nous prouvons que ces solutions sont continues en temps à valeurs dans L^2 , satisfont l'égalité d'énergie et sont uniques. En particulier, pour obtenir l'égalité d'énergie, nous avons utilisé une procédure d'approximation et une version du théorème de Magenes-Lions dans les domaines temporels. Ces idées ont également été utilisées dans [MNR19] pour démontrer le caractère unique des solutions tridimensionnelles de (1.1.1) avec les conditions de Prodi-Serrin. Pour montrer l'unicité, nous prouvons un résultat de régularité maximale dans les espaces de Banach pour un système étroitement lié à (1.1.1), voir Théorème 3.1.5.

Dans le cas où Ω est le plan nous étendons le résultat d'existence [PS14] au cas où l'énergie cinétique du fluide peut être infinie avec une structure très précise. En particulier, si le solide est un disque, le champ de vitesse se comporte comme

$$u = \beta H(x - h(t))^\perp + \tilde{u} \quad \text{où} \quad H(x) = \frac{|x|^\perp}{2\pi|x|^2}, \quad \tilde{u} \in L^2(\mathcal{F})$$

et β est une constante en temps. Nous nous intéressons à ce problème en vue d'étudier la limite lorsque l'objet se contracte en un point et que la viscosité tend vers zéro. L'idée est d'arriver à la limite à un système similaire à celui étudié dans [GLS16]. Le système limite est caractérisé par l'équation d'Euler avec un vortex qui évolue sous l'effet d'une force de type Kutta-Joukowski. Une description plus détaillée sera présentée dans le chapitre suivant.

1.2 Modèle mathématique d'un source et d'un puits

Commençons par présenter le système. Soit Ω un ouvert borné, connexe, simplement connexe du plan avec un contour lisse. Soit \mathcal{S}^+ et \mathcal{S}^- deux sous-ensembles connexes, simplement connexes et non vides de \mathbb{R}^2 contenus de manière compacte dans Ω avec un contour lisse et

soit $\mathcal{F} = \Omega \setminus \overline{(\mathcal{S}^+ \cup \mathcal{S}^-)}$ le domaine du fluide. Nous appelons \mathcal{S}^+ la source et \mathcal{S}^- le puits. Les équations qui modélisent la dynamique pour les inconnues (u, p) prennent la forme

$$\begin{aligned}
\rho(\partial_t u + u \cdot \nabla u) + \nabla p &= 0 && \text{pour } (t, x) \in \mathbb{R}^+ \times \mathcal{F}, \\
\operatorname{div} u &= 0 && \text{pour } (t, x) \in \mathbb{R}^+ \times \mathcal{F}, \\
u \cdot n &= g && \text{pour } (t, x) \in \mathbb{R}^+ \times \partial\mathcal{F}, \\
\operatorname{curl} u &= \omega^+ && \text{pour } (t, x) \in \mathbb{R}^+ \times \partial\mathcal{S}^+, \\
u(0, \cdot) &= u^{in} && \text{pour } x \in \mathcal{F},
\end{aligned} \tag{1.2.1}$$

où $u : \mathbb{R}^+ \times \mathcal{F} \rightarrow \mathbb{R}^2$ est un champ de vitesse, $p : \mathbb{R}^+ \times \mathcal{F} \rightarrow \mathbb{R}$ est une pression, ρ est une constante positive décrivant la densité, n le vecteur normal sortant du domaine \mathcal{F} , g est la composante normale de la vitesse sur le bord telle qu'elle ait une moyenne nulle et satisfasse $g \leq -c < 0$ sur $\mathbb{R}^+ \times \partial\mathcal{S}^+$, $g \geq c > 0$ sur $\mathbb{R}^+ \times \partial\mathcal{S}^-$, $g = 0$ sur $\mathbb{R}^+ \times \partial\Omega$, ω^+ est la vortacité entrant et u^{in} telle que $\operatorname{div} u^{in} = 0$ est la donnée initiale.

Le système (1.2.1) a été introduit par Judovič dans [Jud63]. Dans le même article, il a démontré l'existence et le caractère unique de solutions régulières pour des données initiales relativement régulier. L'existence de solutions moins régulières de (2.2.4) dans le cas où la vortacité est borné a été prouvée dans [Ale76]. L'auteur a utilisé une méthode de viscosité évanescence. Dans ce manuscrit, nous présentons une démonstration alternative via un théorème de point fixe de Schauder.

Le choix de compléter le système (1.2.1) en spécifiant la vortacité entrante n'est pas le seul possible. Une autre possibilité consiste à prescrire des conditions sur la pression. Pour une étude plus complète du sujet, nous suggérons de lire [Mam10], où l'auteur présente une description exhaustive des travaux relatifs au système (2.2.4).

1.2.1 Contribution

Dans l'article [Jud63], Judovič pose le problème de la dynamique du système (1.2.1) lorsque les diamètres de la source et du puits tendent vers zéro. Cette question est ouverte et a été rappelée dans [CS02], où les auteurs ont prouvé l'existence de solutions faibles d'un système comportant un nombre fini de sources. Dans ce manuscrit, nous établissons la limite du système (1.2.1) quand \mathcal{S}^+ et \mathcal{S}^- sont contractés respectivement en les points x_+ et x_- , avec $x_+ \neq x_-$. Pour décrire le système limite, il est plus facile de considérer la formulation en ω . On peut alors reconstituer la vitesse à partir de la vortacité via l'opérateur de Biot-Savart $K_\Omega[\cdot]$ dans Ω qui transforme une fonction ω assez régulière en un champ de vitesse $K_\Omega[\cdot]$ comme solution à

$$\begin{aligned}
\operatorname{div} K_\Omega[\omega] &= 0 && \text{pour } x \in \Omega, \\
\operatorname{curl} K_\Omega[\omega] &= \omega && \text{pour } x \in \Omega, \\
K_\Omega[\omega] \cdot n &= 0 && \text{pour } x \in \partial\Omega.
\end{aligned}$$

Nous avons également besoin de la contrepartie de l'opérateur de Biot-Savart pour résoudre la partie du champ de vitesse associée à la source et au puits avec une divergence non nulle.

Pour chaque fonction assez régulière m dans Ω , nous définissons $\mathcal{J}_\Omega[m]$ la solution de

$$\begin{aligned} \operatorname{div} \mathcal{J}_\Omega[m] &= m && \text{pour } x \in \Omega, \\ \operatorname{curl} \mathcal{J}_\Omega[m] &= 0 && \text{pour } x \in \Omega, \\ \mathcal{J}_\Omega[m] \cdot n &= 0 && \text{pour } x \in \partial\Omega. \end{aligned}$$

Une hypothèse cruciale pour l'établissement du système limite de (1.2.1) lorsque le diamètre de la source et du puits tend vers zéro est le fait que les intégrales de g et de $g\omega^+$ sur $\partial\mathcal{S}^+$ convergent vers les quantités $\mu = \mu(t)$ et $j = j(t)$ respectivement. Le système limite obtenu est

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= j \delta_{x_+} - \left(j + \frac{d}{dt} \int_{\mathcal{F}} \omega \right) \delta_{x_-} && \text{pour } (t, x) \in \mathbb{R}^+ \times \Omega, \\ u &= \mathcal{J}_\Omega[\mu \delta_{x_+} - \mu \delta_{x_-}] + K_\Omega[\omega + \mathcal{C}_+ \delta_{x_+} + \mathcal{C}_- \delta_{x_-}] && \text{pour } (t, x) \in \mathbb{R}^+ \times \Omega, \\ \mathcal{C}_+(t) &= \mathcal{C}_+^{in} - \int_0^t j && \text{pour } t \in \mathbb{R}^+, \\ \mathcal{C}_-(t) &= \mathcal{C}_-^{in} + \int_{\mathcal{F}} \omega(t, \cdot) - \int_{\mathcal{F}} \omega^{in} + \int_0^t j && \text{pour } t \in \mathbb{R}^+. \end{aligned}$$

Ici δ_x désigne la mesure de Dirac au point x . Nous voulons souligner que le système (1.2.1) est caractérisé par la présence d'un point source et d'un vortex dans chacun des deux points x_+ et x_- . De plus, leur intensité dépend en temps de m , j et ω . Les quantités \mathcal{C}_+ et \mathcal{C}_- décrivent les circulations respectives autour de \mathcal{S}^+ et \mathcal{S}^- .

Introduction (English)

Dans cette introduction, nous présentons les résultats classiques d'existence et d'unicité pour les équations d'Euler et de Navier-Stokes incompressibles. Nous expliquons également les principaux résultats de nos recherches.

Fluid dynamics is a branch of mathematics that aims to describe the behaviours of fluids. There exists a wide literature on the subject and in this introduction we will focus on two classical systems the Euler equations and the Navier-Stokes equations.

Let d a natural number greater or equal to two and let Ω or the full space \mathbb{R}^d or an open and bounded subset of \mathbb{R}^d with smooth boundary. Then the homogeneous incompressible Euler equations with constant density ρ read as

$$\begin{aligned}
 \rho \left(\frac{\partial}{\partial t} u + u \cdot \nabla u \right) + \nabla p &= 0 && \text{in } \mathbb{R}^+ \times \Omega, \\
 \operatorname{div} u &= 0 && \text{in } \mathbb{R}^+ \times \Omega, \\
 u \cdot n &= 0 && \text{on } \mathbb{R}^+ \times \partial\Omega, \quad \text{if } \Omega \text{ is bounded,} \\
 |u| &\longrightarrow 0 && \text{as } |x| \longrightarrow \infty, \quad \text{if } \Omega = \mathbb{R}^d, \\
 u(0, \cdot) &= u^{in} && \text{in } \Omega,
 \end{aligned} \tag{2.0.1}$$

where

$$u : \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}^d$$

is a vector field that describes the velocity of the fluid, more precisely the particle at position x at time t has velocity $u(x, t)$. The function

$$p : \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}$$

is a scalar quantity and it describes the pressure. In the case where Ω is bounded, n denotes the normal vector to the boundary $\partial\Omega$ exiting from the domain Ω . Finally $u^{in} : \Omega \longrightarrow \mathbb{R}^d$ is a divergence vector field tangent to the boundary and it is the initial condition. Moreover ∇ denotes the gradient, the i -th component of the convective term is

$$(u \cdot \nabla u)_i = \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j}$$

XXI. Nous n'avons donc qu'à équaler ces forces accélératrices avec les accélérations actuelles que nous venons de trouver, & nous obtiendrons les trois équations suivantes :

$$P - \frac{1}{q} \left(\frac{dp}{dx} \right) = \left(\frac{du}{dt} \right) + u \left(\frac{du}{dx} \right) + v \left(\frac{du}{dy} \right) + w \left(\frac{du}{dz} \right)$$

$$Q - \frac{1}{q} \left(\frac{dp}{dy} \right) = \left(\frac{dv}{dt} \right) + u \left(\frac{dv}{dx} \right) + v \left(\frac{dv}{dy} \right) + w \left(\frac{dv}{dz} \right)$$

$$R - \frac{1}{q} \left(\frac{dp}{dz} \right) = \left(\frac{dw}{dt} \right) + u \left(\frac{dw}{dx} \right) + v \left(\frac{dw}{dy} \right) + w \left(\frac{dw}{dz} \right)$$

Si nous ajoutons à ces trois équations premièrement celle, que nous a fournie la considération de la continuité du fluide :

$$\left(\frac{dq}{dt} \right)$$

$$\left(\frac{dq}{dt} \right) + \left(\frac{d.qu}{dx} \right) + \left(\frac{d.qv}{dy} \right) + \left(\frac{d.qw}{dz} \right) = 0,$$

& ensuite celle que donne le rapport entre l'élasticité p , la densité q , & l'autre qualité r , qui influë sur l'élasticité p , outre la densité q , nous aurons cinq équations qui renferment toute la Théorie du mouvement des fluides.

Figure 2.1: Euler equations from [Eul57]

and the divergence operator is defined by

$$\operatorname{div} u = \sum_{j=1}^d \frac{\partial u_j}{\partial x_j}.$$

The system (2.0.1) in dimension three was introduced in [Eul57] by L. Euler in 1757 and it is one of the first example of PDE written in a modern style. Actually the system (2.0.1) is a special version of the one presented by Euler in [Eul57], see in Figure 2.1, in fact we assume the fluid to be incompressible, of constant density ρ and the external forces P , Q and R to be zero. Physically the Euler system describes the behaviour of an inviscid fluid.

Later, C. L. M. H. Navier proposed a set of equations to model the motion of a homogeneous viscous fluid that were clarified by G. G. Stokes in [Sto80]. This system is called the incompressible Navier-Stokes equations, and it reads as

$$\begin{aligned} \rho \left(\frac{\partial}{\partial t} u + u \cdot \nabla u \right) - \nu \Delta u + \nabla p &= 0 && \text{in } \mathbb{R}^+ \times \Omega, \\ \operatorname{div} u &= 0 && \text{in } \mathbb{R}^+ \times \Omega, \\ u &= 0 && \text{on } \mathbb{R}^+ \times \partial\Omega, \quad \text{if } \Omega \text{ is bounded,} \\ |u| &\longrightarrow 0 && \text{as } |x| \longrightarrow \infty, \quad \text{if } \Omega = \mathbb{R}^d, \\ u^{in}(0, \cdot) &= u_0 && \text{in } \Omega, \end{aligned} \tag{2.0.2}$$

where as before $u : \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}^d$ is a velocity field, $p : \mathbb{R}^+ \times \Omega \longrightarrow \mathbb{R}$ is a pressure, the positive constants ρ and ν are respectively the density and the viscosity, the initial datum is $u^{in} : \Omega \longrightarrow \mathbb{R}^d$ which is a divergence free vector fields tangent to the boundary. Moreover the Laplace operator is defined by

$$\Delta u = \sum_{j=1}^d \frac{\partial^2 u_j}{\partial x_j^2}.$$

The system (2.0.1) and (2.0.2) have been widely studied in the last century, in this introduction we recall the classical result of well-posedness of Judovič from [Jud63] relative to (2.0.1), the existence of Leray-Hopf weak solutions and of strong solutions for the system (2.0.2). Moreover we present the motivations and the main results of this thesis. Although we presented the Euler system first we prefer to start by considering the results on Navier-Stokes equations and then on the system (2.0.1).

2.1 Viscous fluids systems

This section is entirely dedicated to study some systems involving viscous fluids satisfying the Navier-Stokes equations. We will start by recalling the classical result of existence of Leray-Hopf weak solutions and of strong solutions in Hilbert space setting of (2.0.2). Then we present some fluid-structure interaction problems. These types of systems couple the equations of Navier-Stokes with the ones of some structures. The general difficulty that occurs in the study of these systems is that the domain, where the equations are satisfied, is time dependent and actually even depends on the solution itself. Then we will discuss about alternative boundary conditions. Classically to couple the Navier-Stokes equations and the fluid-structure systems, people use the no-slip boundary condition. Actually this is not the only one that gives rise to a well-posedness theory. We will present the so called Navier-slip boundary condition and we will explain the advantages and disadvantages in choosing this conditions. Finally we will state our contributions concerning the study of viscous fluids.

2.1.1 Well-posedness for the Navier Stokes equations

To study well-posedness of the system (2.0.2), it is first necessary to choose an appropriate definition of solution. In this subsection we consider two different definitions. The first one was proposed by Leray and Hopf and deals with weak solutions and the second one establish the concept of strong solutions in Hilbert space setting.

We start by giving the definition of Leray-Hopf weak solutions. We denote by $C_c^\infty(\Omega; \mathbb{R}^d)$ the set of smooth vector valued functions with compact support in Ω , by $L_\sigma^2(\Omega)$ the closure in $L^2(\Omega)$ of the divergence free functions in $C_c^\infty(\Omega; \mathbb{R}^d)$ and finally $H_{\sigma,0}^1 = H_0^1 \cap L_\sigma^2$.

Definition 2.1.1 (Leray-Hopf weak solutions). Let $u^{in} \in L_\sigma^2(\Omega)$ the initial datum, then we say that $u \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(\mathbb{R}^+; H_{\sigma,0}^1(\Omega))$ is a Leray-Hopf weak solution of (2.0.2) with initial data u^{in} if

$$\int_\Omega \rho u^{in} \cdot \varphi(0, \cdot) dx + \int_{\mathbb{R}^+} \int_\Omega \rho u \cdot \partial_t \varphi + \int_{\mathbb{R}^+} \int_\Omega \rho (u \cdot \nabla) \varphi \cdot u - \nu \int_{\mathbb{R}^+} \int_\Omega \nabla u : \nabla \varphi = 0 \quad (2.1.1)$$

for any $\varphi \in C_c^\infty(\Omega; \mathbb{R}^d)$ such that $\operatorname{div} \varphi = 0$ in $\mathbb{R}^+ \times \Omega$. Moreover it satisfies the following energy inequality:

$$\frac{1}{2} \int_\Omega \rho |u|^2(t, \cdot) dx + \nu \int_0^t \int_\Omega |\nabla u|^2 dx dt \leq \frac{1}{2} \int_\Omega \rho |u^{in}|^2 dx. \quad (2.1.2)$$

for almost any $t \in \mathbb{R}^+$.

In the special case of dimension two, it is not necessary to impose in the definition above the condition (2.1.2). In fact any vector field $u \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2_\sigma(\mathbb{R}^+; H^1_{\sigma,0}(\Omega))$, that satisfies (2.1.1), will verify also (2.1.2).

We now recall the definition of strong solutions in Hilbert space setting.

Definition 2.1.2. Let $u^{in} \in H^1_{\sigma,0}(\Omega)$ the initial datum, then we say that $u \in L^2(0, T; H^1_{\sigma,0}(\Omega) \cap H^2(\Omega)) \cap H^1(0, T; L^2_\sigma(\Omega))$ is a strong solution in the Hilbert space setting of (2.0.2) with initial datum u^{in} if u satisfies almost everywhere (2.0.2) and $u(0, \cdot) = u^{in}$.

The system (2.0.2) admits solutions in the sense of Definition 2.1.1 and of Definition 2.1.2. Moreover strong solutions are unique. The situation is more complicated for weak solutions and depends on the dimension. We postpone this discussion in Section 2.1.1.2. In what follow, we first present the results of existence and then we treat the uniqueness issue for Leray-Hopf solutions.

2.1.1.1 Existence

This subsection is dedicated to the study of existence of solution for the Navier-Stokes equations in the sense of Definition 2.1.1 and of Definition 2.1.2. We start by presenting the classical existence result for Leray-Hopf weak solutions.

Theorem 2.1.1. *Let $u^{in} \in L^2_\sigma(\Omega)$, then there exists u a weak solution of (2.0.2) in the sense of Definition 2.1.1 with initial datum u^{in} .*

One of the possible proofs of this result makes use of a Faedo-Galerkin approximation and the Aubin-Lions' compactness lemma to pass to the limit in the non-linear term. For the original proof see the works [Ler34] of Leray and [Hop50] of Hopf. About the existence of strong solutions we can recall the classical result.

Theorem 2.1.2. *Let Ω bounded and let $u^{in} \in H^1_{\sigma,0}(\Omega)$, then for a short time $T > 0$ there exists a unique strong solution in the Hilbert space setting with initial datum u^{in} in the sense of Definition 2.1.2.*

The proof of the above theorem can be done via a fixed point argument and the fact that the Stokes operator associated with the linearisation of (2.0.2) generates an analytic semigroup. The above result also provides uniqueness, moreover in dimension two solutions are global, in other words T can be chosen equal to $+\infty$. In dimension three the solution is global if we assume the initial data small enough in the H^1 norm.

Before moving to the discussion on uniqueness of weak solutions we note that if u is a strong solution in the Hilbert space setting of (2.0.2), then u is a Leray-Hopf weak solution in the sense of Definition 2.1.1 and satisfies the energy inequality (2.1.2).

2.1.1.2 Uniqueness

As presented before strong solutions of (2.0.2) are unique. The situation is different in the case of Leray-Hopf weak solutions, in fact up to now uniqueness has been proved only in dimension two by Ladyzhenskaya in [Lad67] and in dimension three it is an open problem.

We conclude this section by presenting the uniqueness result of Ladyzhenskaya and two classical partial results of uniqueness in dimension three.

Theorem 2.1.3. *Let $d = 2$ the spatial dimension, let $u^{in} \in L^2_\sigma(\Omega)$ an initial datum and let u a Leray-Hopf weak solution with initial datum u^{in} . Then u is continuous in time with values in $L^2_\sigma(\Omega)$ and satisfies the energy equality*

$$\frac{1}{2} \int_\Omega \rho |u|^2(t, \cdot) dx + \nu \int_0^t \int_\Omega |\nabla u|^2 dx dt = \frac{1}{2} \int_\Omega \rho |u^{in}|^2 dx,$$

for any $t \in \mathbb{R}^+$. Moreover u is the unique Leray-Hopf weak solution with initial datum u^{in} .

The proof of this result is based on the interpolation inequality

$$\int_\Omega |u|^4 \leq C \int_\Omega |u|^2 \int_\Omega |\nabla u|^2 \quad (\text{for } \Omega \subset \mathbb{R}^2) \quad (2.1.3)$$

and a Grönwall estimate. The original proof is contained in [Lad67]. This proof can not be trivially generalized in dimension three because the estimate (2.1.3) reads as

$$\int_\Omega |u|^4 \leq C \left(\int_\Omega |u|^2 \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u|^2 \right)^{\frac{3}{2}} \quad (\text{for } \Omega \subset \mathbb{R}^3),$$

when Ω is a subset of \mathbb{R}^3 . As said before uniqueness of Leray-Hopf weak solutions is still open but in the literature there are many partial results. Here we present two of them. The first one is a strong-weak result that states that if there exists a strong solution for an initial datum u^{in} than any Leray-Hopf weak solution with initial datum u^{in} is identically equal to the strong one. And the second one is due to Prodi and states that if a Leray-Hopf solution satisfies some better integrability property then it is unique in the class of solutions from Definition 2.1.1. We start by stating the weak-strong uniqueness result.

Theorem 2.1.4. *Let $d = 3$ the space dimension, let $u^{in} \in L^2_\sigma(\Omega)$ an initial datum and let \tilde{u} a strong solution of (2.0.2) in sense of Definition 2.1.2 with initial datum u^{in} . Then any Leray-Hopf weak solution with initial datum u^{in} is equal to \tilde{u} .*

We move to the result of Prodi. We say that a vector field u satisfies the Prodi-Serrin condition if

$$u \in L^{\frac{2p}{3-p}}(0, T; L^p(\Omega)) \quad \text{for some } p \in (3, +\infty]. \quad (2.1.4)$$

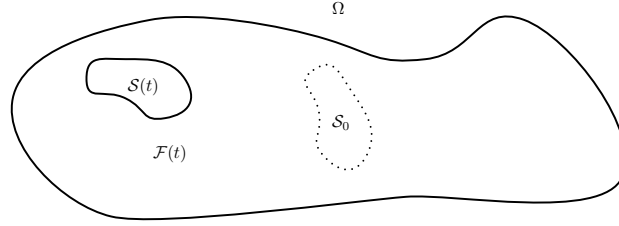
We now state the uniqueness theorem.

Theorem 2.1.5. *Let $d = 3$ the space dimension, let u^{in} an initial datum and let u a Leray-Hopf weak solution with initial datum u^{in} such that it satisfies the Prodi-Serrin condition 2.1.4. Then u is the unique Leray-Hopf weak solution of (2.0.2).*

The original proof can be founded in [Pro59] and it is based on the fact that under the condition (2.1.4), it is possible to close a Grönwall estimate similar to the one in [Lad67].

2.1.2 Fluid-structure interaction

The fluid-structure interaction problems are systems of equations coming from physical models that couple the fluid equations with the ones that describe the structure. The study of these problems is becoming more and more important in real life applications. Examples are available in different scientific areas. For instance a boat sailing on the ocean, where the

Figure 2.2: The domain $\Omega = \mathcal{F}(t) \cup \mathcal{S}(t)$.

water is the fluid and the structure is the boat. A submarine, in this case the structure is totally immersed in the liquid and a swimming fish, which is a more complicated situation. Analogously a flying plane is an example where the fluid is supposed to be compressible (for an appropriate range of velocities). Another case is the veins, in this situation the blood can be modelled by a non-Newtonian fluid and the veins by an elastic structure. Some more recent applications are the wave-energy converters which are systems composed by oscillating structures that transform the energy received by the waves in clean electric energy.

For fluid-structure problems the physical dimension is three. In this manuscript we consider only $2D$ models that predict well motions that are independent of one specific direction. In particular the problem that we will study in this thesis models the movement of a rigid body in a viscous incompressible fluid. The system is contained in the plane and the unknown are the velocity of the fluid u , the pressure p and the position of the solid \mathcal{S} . The evolution of the velocity field of the fluid and of the pressure is modelled by the Navier-Stokes equations in the fluid domains and to recover the position of the rigid object it is enough to know the velocity of the center of mass l and the angular velocity r of the solid. These two quantities satisfy the Newton laws

$$\begin{aligned} ml'(t) &= F(t, u, p) \\ \mathcal{J}r'(t) &= G(t, u, p), \end{aligned}$$

where m and \mathcal{J} are respectively the mass and the momentum of inertia and F and G are respectively the force and the torque. In our setting the force F and the torque G are modelled by the linear hydrodynamics force and torque and they read as

$$\begin{aligned} F(t, u, p) &= - \int_{\partial\mathcal{S}(t)} \Sigma(u, p)n ds, \\ G(t, u, p) &= - \int_{\partial\mathcal{S}(t)} (x - h(t))^\perp \cdot \Sigma(u, p) ds, \end{aligned}$$

where $\Sigma = -p\text{Id}_2 + 2\nu Du$ is the stress tensor with $2Du = \nabla u + \nabla u^T$ the symmetric gradient, n is the normal on the boundary of the solid entering in the solid, h is the position of the center of mass and finally $x^\perp = (x_1, x_2)^\perp = (-x_2, x_1)$ is the counterclockwise perpendicular vector to x .

We are now able to write down the equations that describe the dynamics of the system. At a mathematical level we have a domain Ω which is or bounded with smooth boundary of the all space \mathbb{R}^2 , independent in time, which is the union of two time-dependent domains $\mathcal{F}(t)$ and $\mathcal{S}(t)$, i.e. $\Omega = \mathcal{F}(t) \cup \mathcal{S}(t)$ as in Figure 2.2. Here $\mathcal{F}(t)$ is the part of the domain fulfilled by

an incompressible viscous fluid, which satisfies Navier-Stokes equations and $\mathcal{S}(t)$ the part of the domain which is occupied by the body which rigidly moves from the initial configuration \mathcal{S}_0 following Newton's laws. The system then reads as:

$$\begin{aligned}
\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) - \nu \Delta u + \nabla p &= 0 && \text{for } x \in \mathcal{F}(t), \\
\operatorname{div} u &= 0 && \text{for } x \in \mathcal{F}(t), \\
u &= u_s && \text{for } x \in \partial\mathcal{S}(t), \\
u &= 0 && \text{for } x \in \partial\Omega, \text{ if } \Omega \text{ bounded,} \\
|u| &\longrightarrow 0 && \text{as } |x| \longrightarrow \infty, \text{ if } \Omega = \mathbb{R}^2, \\
ml'(t) &= - \int_{\partial\mathcal{S}(t)} \Sigma(u, p)n \, ds, \\
\mathcal{J}r'(t) &= - \int_{\partial\mathcal{S}(t)} (x - h(t))^\perp \cdot \Sigma(u, p)n \, ds, \\
u|_{t=0} &= u_0 && \text{for } x \in \mathcal{F}_0, \\
h(0) = h_0, \, l(0) = l_0, &&& r(0) = r_0.
\end{aligned} \tag{2.1.5}$$

where $u_s = l + (x - h(t))^\perp r$ and the position of the solid $\mathcal{S}(t)$ is recover by the velocity l of its center of mass and its angular velocity r . As you can easily see the above system consists of the Navier-Stokes equations, the first two lines, the boundary conditions, the next three lines, the Newton laws that are coupled with the fluid equations and finally the initial conditions.

The system (2.1.5) has been widely studied and all the corresponding well-posedness results presented in Section 2.1 were proved. The two first existence results for Leray-Hopf weak solutions can be found in [Jud74] and in [Ser87], where the authors assume that the domain Ω is \mathbb{R}^3 . Around 2000, the case where Ω is bounded was tackled, in particular existence of weak solutions was shown in [GLS00] and [CSMT00]. Moreover in the case of finitely many bodies existence was exhibit in [DE99], [Fei02], and [SMST02]. Regarding well-posedness of strong solution we refer to [DE00]-[GM00]-[Tak03] and [TT04].

Later in dimension two the uniqueness of Leray-Hopf type weak solutions of (2.1.5) was proved in [GS15] and in [CNM19] the weak-strong uniqueness result in dimension three was shown for slightly different boundary conditions that will be described in the next section. Finally this year using also two ideas from [Bra19], that will be presented in Section 4.1.1-4.1.2, in [MNR19] was proved a uniqueness result in three dimension under some Prodi-Serrin type conditions (see 2.1.4).

2.1.2.1 Other fluid structure models

The study of the motion of a rigid body in a viscous fluid is only an example of fluid-structure problem. This area of research is broad and in fast evolution. We now recall some of the fluid structure-model that have been studied in the last decade. Here we are not hoping to present an exhaustive list but we chose some of them to contextualized our coupled-system. Moreover we chose in most of the case the literature associated with the weak well-posedness.

A dual system to (2.1.5) is the one that describes a viscous incompressible fluid in a cavity. The well-posedness of the system was proved in [ST12] and the long time behaviour

was considered in [Gal17], where the author shows a stabilizing effect of the fluid on the motion of the rigid body.

Regarding the coupled system (2.1.5), it is possible to consider similar model where the fluid behaves differently, in this case the first two equations of (2.1.5) are different. In the works [FHN08], [Fei03] and [OT18], the authors studied the motion of some rigid body respectively in a viscous incompressible non-Newtonian, in a viscous compressible and in a viscous-plastic fluid, and in particular they proved existence of weak solutions.

Another possibility is to change the behaviour of the solid by considering an elastic body. In this situation an existence result of strong solutions is proved in [CS05] via a fixed point argument. Regarding elastic structure, it is possible to consider system where the structure is on the boundary of the fluid domain. The literature related to this problem is wide and we give as example the works [BadV04], [Bou07], [Leq11] and [DT19].

Let us also mention [SMSTT08] where the authors studied well-posedness for strong solutions of a simplify model that describes a swimming fish. In particular they consider a body that deforms in a prescribed way and they study its interaction with a viscous fluid.

The last type of fluid-structure problem that we recall is when a rigid body is floating on the boundary of the fluid. Few is known about the problem and the only work we are aware of is [MSMTT18], where the authors consider the case of a cylinder floating on a liquid modelled by the viscous water-waves. Actually in the case the fluid is inviscid more is known and the well-posedness of the water-waves plus floating body is treated in [IL18], [Boc18] and [Ben17].

2.1.3 Navier-slip boundary condition

Classically the no-slip boundary conditions

$$u = 0 \quad \text{on} \quad \partial\Omega$$

are used to close the Navier-Stokes equations to obtain a well-posedness theory. Physically they model the fact the fluid is stuck to the boundary so it has the same speed as the boundary. In the case where these conditions are used in the fluid-structure system (2.1.5), they bring to the non-physical paradox that a sphere, immersed in the half upper three dimensional space and subjected only to the gravity as external force, will not touch the boundary in finite time. See for instance [Hil07] and [HT09].

In the literature also other types of boundary conditions are considered, for instance the so called Navier-slip boundary conditions that read as

$$\begin{aligned} u \cdot n &= 0 \\ (Dun) \cdot \tau &= -\alpha u \cdot \tau \end{aligned} \quad \text{on} \quad \partial\Omega, \quad (\Omega \subset \mathbb{R}^2),$$

where τ is the counterclockwise tangent to n to the boundary $\partial\Omega$ and $\alpha > 0$ is a positive constant. These conditions allow the fluid to slip on $\partial\Omega$ and they appear naturally for example in the study of microfluidics see [LBS07], [Hoc73] and in the study of the rugosity limit, see [BFNW08]. Moreover they allow collision between the body and the boundary, see for example [GVHW15], which is in contrast with the paradox of the no-slip conditions.

The advantages of the Navier-slip boundary conditions are at a physical level that they do not give rise to the no-collision paradox in the fluid plus rigid body model and at a mathematical level the inviscid limit is proved whereas it is an open problem in the no-slip

case unless imposing some additional assumptions. The disadvantage is that a more careful analysis has to be done to prove well-posedness of the fluid-structure problem (2.1.5) with Navier-slip boundary conditions that will be written down only later in (3.0.1)-(3.0.9). The difficulties arise from the fact that the extension of the fluid velocity by the rigid one inside the body does not give an H^1 vector field.

In the case of the fluid alone all the well-posedness results of Section 2.1 can be quite easily carried out. In the case of the coupled system (2.1.5) with Navier-slip boundary condition the existence result of Leray-Hopf type weak solutions was proved in [GVH14] and existence of strong solutions in Hilbert space setting in [Wan14].

Let us also mention [Kol18] where the author proves the small-time global controllability of the solid motion (position and velocity) in the case where $\Omega = \mathbb{R}^2$ and Navier slip-with-friction boundary condition are prescribed on the solid boundary.

Before discuss the hypothesis that the slip α is a positive constant, we notice that in case Ω is bounded the boundary of the fluid domain $\mathcal{F}(t)$ is composed by two disconnected sets $\partial\Omega$ and $\partial\mathcal{S}(t)$, on which is possible to prescribe two different boundary conditions. One of the choice is the so called mixed boundary condition, which corresponds to impose no-slip on $\partial\Omega$ and Navier-slip on $\partial\mathcal{S}(t)$ and it is used in [CN17] and [ABCNM18] where well-posedness of respectively Leray-Hopf type weak solutions and of strong one in Hilbert space setting were proved. Moreover the work [CNM19], that was mentioned before, deals with weak strong uniqueness for mixed boundary conditions.

At the beginning of the section we present the Navier-slip boundary condition with α a positive constant. Actually it is possible to consider space dependent α , even non positive. The first well-posedness result of (2.0.2) in two dimension with Navier-slip conditions with positive $\alpha \in C^2(\partial\Omega)$ was considered in [CMR98], in particular in this case existence and uniqueness of strong solutions was proved. Later in [Kel06] existence of Leray-Hopf type weak solutions was proved for $\alpha \in L^\infty(\partial\Omega)$ without any assumption on the sign. Finally in [IS11] existence of weak solution was tacked for $\alpha \in C^2(\partial\Omega)$ where Ω is a bounded three dimensional domain.

In the case of the stationary Stokes and Navier-Stokes equations, recently in [AACG19] was proved existence of weak and strong solutions with the friction α in $L^2(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ respectively. Moreover the authors consider also the non-Hilbert space setting.

2.1.4 Contributions

In this thesis we will study the system (2.1.5) with Navier-Slip boundary condition that will be presented only later in (3.0.1)-(3.0.9). In the case where Ω is a bounded subset of the two dimensional plane, the main contributions of the manuscript come from [Bra19] and deal with the proof of the energy equality and of the uniqueness for Leray-Hopf type weak solutions. Moreover we will prove an existence result for strong solutions in Banach space setting which is the counterpart of the works [GGH13]-[MT18] in the case of Navier-slip boundary conditions.

In particular to prove the energy equality, see Theorem 4.1.1, we show an approximation procedure and a Magenes-Lions type theorem in time depending domains, see Section 4.1.1-4.1.2. These two ideas were then used in [MNR19] to extend the uniqueness result in three dimensions under some Prodi-Serrin type conditions of the fluid-structure problem (2.1.5) with both no-slip and Navier-slip boundary conditions. About the proof of uniqueness we follow the strategy presented in [GS15], where we use a special version of the existence result

of strong solutions in Banach space setting, see Theorem 3.1.5.

In the case where Ω is equal to \mathbb{R}^2 , we consider the problem (2.1.5) with Navier-slip boundary conditions and we extend the existence result [PS14] to the case where the fluid kinetic energy is infinite and has a strong structure, more precisely in the case where the rigid body is a homogeneous disk the fluid velocity u is assumed to behave like

$$u = \beta H(x - h(t)) + \tilde{u}, \quad \text{where} \quad H(x) = \frac{|x|^\perp}{2\pi|x|^2},$$

β is a fixed constant and \tilde{u} is an L^2 function in the space variables. The motivation that drives us to study this special infinite-energy solutions is to study the “inviscid+shrinking-body” limit. For the inviscid limit we recall the result from [PS14], where the authors proved that as ν goes to zero, the solutions u_ν converge to the solution of the corresponding Euler system. In [PS14], the “rigid+body” system occupies all the space \mathbb{R}^3 , in the case of \mathbb{R}^2 the situation is a bit more tricky and the argument of [PS14] holds at least in the case where the solid is a disk, see Appendix C. Moreover by the work [GLS16], we know that as the size of the object goes to zero (and the mass remains constant) the system converges to a variant of the vortex-wave system where the vortex, placed in the point occupied by the shrunk body is accelerated by a Kutta-Joukowski-type lift force. In the massless case, i.e. the density of the object is constant respect to the scale of the object, similar results are available when the fluid satisfies incompressible Navier-Stokes equation and no-slip boundary conditions are prescribe on the boundary of the solid, for example in [LT17] it is proven that for a fixed viscosity the “fluid+disk” system converges to the Navier-Stokes system in all \mathbb{R}^2 when the object shrinks to a point. The goal of further studies is to understand the limiting equations when both the viscosity and the size of the object go to zero at the same time (in both mass and massless cases) and to find in the limit a similar system of the one in [GLS16]. We expect that the appearance of a Kutta-Joukowski-type lift force in the limiting system is strictly related to the presence of the circulation due to βH , i.e. in the absence of this term we do not expect to see any force on the point mass in the limit. Indeed in the case where the vorticity is integrable, β describes the circulation at infinity.

2.2 Ideal fluids systems

We go back to the Euler system (2.0.1) which describes the motion of an incompressible perfect fluid of density 1. In this section we fix the space dimension $d = 2$. The equations (2.0.1) are well studied and a huge literature is available on the subject. In this short introduction we will recall only a classical result due to Judovič from [Jud63] and then will move to a similar system introduced in [Jud64], which describes the flow of an incompressible inviscid fluid through a given region. The existence of regular solutions was proved in [Jud64] and existence of solutions with bounded vorticity was proved using a vanishing viscosity method in [Ale76]. Judovič in [Jud64] left open the question of well-posedness for equations satisfied by the flow when the domain has some holes that shrink to points. In this Thesis we will propose an alternative proof of the result presented in [Ale76], moreover we will answer to the question proposed by Judovič in the special case of source and a sink in a two dimensional bounded domain. For a more exhaustive presentation of the subject we refer to [Mam10], where a detailed description of the literature is made.

In what follows we will first recall the existence result from [Jud63], we will write down the system studied in [Jud64] and finally we will introduce our contributions.

2.2.1 Existence and uniqueness

For simplicity we assume that the domain $\Omega \subset \mathbb{R}^2$ is a bounded simply connected with smooth boundary. Due to the fact that the space dimension is two, the curl operator is defined by

$$\operatorname{curl} : C^1(\Omega; \mathbb{R}^2) \longrightarrow C^0(\Omega; \mathbb{R}) \quad \text{where } \operatorname{curl} v = \partial_1 v_2 - \partial_2 v_1.$$

If we apply the curl operator to the Euler system (2.0.1) the equations become

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= 0 & \text{in } \mathbb{R}^+ \times \Omega \\ \omega(0, \cdot) &= \omega_0 & \text{in } \Omega \end{aligned} \quad (2.2.1)$$

where $\mathbb{R}^+ = [0, \infty)$ and u is recovered by the elliptic system

$$\begin{aligned} \operatorname{div} u &= 0 & \text{in } \mathbb{R}^+ \times \Omega \\ \operatorname{curl} u &= \omega & \text{in } \mathbb{R}^+ \times \Omega \\ u \cdot n &= 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \end{aligned} \quad (2.2.2)$$

with n the normal to the boundary exiting from Ω . Judovič defined in [Jud63] weak solutions of (2.2.1)-(2.2.2) in the vorticity form as follow.

Definition 2.2.1 (Weak solutions in the vorticity form). Let $\omega_0 \in L^\infty(\Omega)$, then we say that a couple $(\omega, u) \in L^\infty(\mathbb{R}^+; L^\infty(\Omega; \mathbb{R})) \times L^\infty(\mathbb{R}^+; \mathcal{LL}(\Omega; \mathbb{R}^2))$ is a weak solution of (2.2.1)-(2.2.2) if for any $\varphi \in C_c^\infty(\mathbb{R}^+ \times \overline{\Omega})$, it holds

$$\int_{\Omega} \omega_0 \varphi(0, \cdot) + \int_{\mathbb{R}^+} \int_{\Omega} \omega \partial_t \varphi + \int_{\mathbb{R}^+} \int_{\Omega} \omega u \cdot \nabla \varphi = 0 \quad (2.2.3)$$

and u satisfies the elliptic problem (2.2.2) almost everywhere.

In the above definition $\mathcal{LL}(\Omega)$ denotes the space of Log-Lipschitz functions, which is the set of functions $f \in L^\infty(\Omega)$ such that the following norm is finite.

$$\|f\|_{\mathcal{LL}(\Omega)} = \|f\|_{L^\infty(\Omega)} + \sup_{x \neq y, x, y \in \Omega} \frac{|f(x) - f(y)|}{L(|x - y|)}$$

where

$$L(r) = \begin{cases} r(1 - \ln r) & \text{for } r \in [0, 1] \\ 1 & \text{for } r > 1. \end{cases}$$

We note that equation (2.2.1) is the equivalent of (3.2) in [Jud63] and (2.2.3) of (3.6) in the same paper. We state the classical result of Judovič from [Jud63].

Theorem 2.2.1. *Let $\omega_0 \in L^\infty(\Omega)$, then there exists a unique weak solution (ω, u) of (2.2.1)-(2.2.2) in the vorticity form with initial datum ω_0 . Moreover for any $p \in [1, \infty)$, it holds $\partial_t u \in L_{loc}^\infty(\mathbb{R}^+; L^p(\Omega))$ and there exists $\nabla p \in L_{loc}^\infty(\mathbb{R}^+; L^p(\Omega))$ such that (u, p) satisfies almost everywhere the Euler equations (2.0.1). Finally for $q \in [1, \infty]$ the norm $\|\omega(t, \cdot)\|_{L^q}$ and $\int_{\Omega} \omega(t, \cdot)$ is preserved in time.*

Following [MB02], the proof of the theorem above can be made by an iterative schemes as follow. Let ϕ_0 the steady flow. Then at step n we have the flow ϕ_n , we define the n -th vorticity ω_n as the flow of the initial datum ω_0 through ϕ_n and we recover the velocity field u_n as the solution of the elliptic system (2.2.2). Finally ϕ_{n+1} is the flow associated with u_n . The above scheme converges and provides a weak solution of (2.2.1)-(2.2.2). To prove uniqueness it is possible to perform a Grönwall estimates for the different of two solutions in the velocity form (2.0.1).

2.2.2 A source plus a sink system

We start by presenting the system. Let Ω an open bounded connected simply-connected non-empty subset of \mathbb{R}^2 with smooth boundary. Let \mathcal{S}^+ and \mathcal{S}^- two open connected simply-connected non-empty disjoint subsets of \mathbb{R}^2 compactly contained in Ω with smooth boundary and let $\mathcal{F} = \Omega \setminus (\overline{\mathcal{S}^+} \cup \overline{\mathcal{S}^-})$ the fluid domain. We will call \mathcal{S}^+ the source and \mathcal{S}^- the sink. The equations in the unknown (u, p) that model the dynamics read as

$$\begin{aligned} \rho(\partial_t u + u \cdot \nabla u) + \nabla p &= 0 && \text{in } \mathbb{R}^+ \times \mathcal{F}, \\ \operatorname{div} u &= 0 && \text{in } \mathbb{R}^+ \times \mathcal{F}, \\ u \cdot n &= g && \text{on } \mathbb{R}^+ \times \partial\mathcal{F}, \\ \operatorname{curl} u &= \omega^+ && \text{on } \mathbb{R}^+ \times \partial\mathcal{S}^+, \\ u(0, \cdot) &= u^{in} && \text{in } \mathcal{F}, \end{aligned} \tag{2.2.4}$$

where $u : \mathbb{R}^+ \times \mathcal{F} \rightarrow \mathbb{R}^2$ is a velocity field, $p : \mathbb{R}^+ \times \mathcal{F} \rightarrow \mathbb{R}$ is a pressure, ρ is a positive constant, n is the normal exiting from the domain \mathcal{F} , g is the normal component of the velocity on the boundary such that it has average zero and satisfies $g \leq -c < 0$ on $\mathbb{R}^+ \times \partial\mathcal{S}^+$, $g \geq c > 0$ on $\mathbb{R}^+ \times \partial\mathcal{S}^-$, $g = 0$ on $\mathbb{R}^+ \times \partial\Omega$, ω^+ is the entering vorticity and u^{in} such that $\operatorname{div} u^{in} = 0$ is the initial datum.

The system (2.2.4) was introduced by Judovič in [Jud64] where he proved existence and uniqueness of regular solutions for smooth enough data. Existence of less regular solutions of (2.2.4) was proved in [Ale76], where the authors showed the existence in the case where the vorticity is bounded via a vanishing viscosity method. In this manuscript we present an alternative approach by a Schauder fixed point theorem. Let us mention that the choice of completing the system (2.2.4) by prescribing the entering vorticity is not the only one. Another possibility is to prescribe a condition on the pressure. To have a more complete discussion on the subject we suggest [Mam10], where the author presents a complete description of the works on system (2.2.4).

2.2.3 Contributions

In [Jud64] Judovič raised the issue of the dynamics of the system (2.2.4) when the diameters of the source and of the sink tend to zero. This open question was recalled in [CS02], where the authors proved existence for weak solutions of a system in the presence of finitely many point sources. In this manuscript we establish the limit of the system (2.2.4) when the sets \mathcal{S}^+ and \mathcal{S}^- homothetically shrink to x_+ and x_- respectively, with $x_+ \neq x_-$. Indeed the limit system is easier to describe thanks to the fluid vorticity ω . To recover the velocity from the

vorticity we will make use of the Biot-Savart operator $K_\Omega[\cdot]$ on Ω which maps any reasonable function ω to the vector field $K_\Omega[\omega]$ solution to

$$\begin{aligned} \operatorname{div} K_\Omega[\omega] &= 0 && \text{in } \Omega, \\ \operatorname{curl} K_\Omega[\omega] &= \omega && \text{in } \Omega, \\ K_\Omega[\omega] \cdot n &= 0 && \text{on } \partial\Omega. \end{aligned}$$

To tackle the part of the velocity due to the source and to the sink we will also use the following counterpart of the Biot-Savart operator for a nonzero divergence: for any reasonable function m with zero integral over Ω , the vector field $\mathcal{J}_\Omega[m]$ is defined by

$$\begin{aligned} \operatorname{div} \mathcal{J}_\Omega[m] &= m && \text{in } \Omega, \\ \operatorname{curl} \mathcal{J}_\Omega[m] &= 0 && \text{in } \Omega, \\ \mathcal{J}_\Omega[m] \cdot n &= 0 && \text{in } \partial\Omega. \end{aligned}$$

A crucial assumption in our asymptotic analysis of the system (2.2.4) is that the integrals of g and $g\omega^+$ on the boundary $\partial\mathcal{S}^+$ of the shrinking source have some limits respectively denoted by $\mu = \mu(t)$ and $j = j(t)$ when the diameters of the source and of the sink tend to zero. The limit system obtained in this manuscript then reads:

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= j \delta_{x_+} - \left(j + \frac{d}{dt} \int_{\mathcal{F}} \omega \right) \delta_{x_-} && \text{in } \mathbb{R}^+ \times \Omega, \\ u &= \mathcal{J}_\Omega[\mu \delta_{x_+} - \mu \delta_{x_-}] + K_\Omega[\omega + \mathcal{C}_+ \delta_{x_+} + \mathcal{C}_- \delta_{x_-}] && \text{in } \mathbb{R}^+ \times \Omega, \\ \mathcal{C}_+(t) &= \mathcal{C}_+^{in} - \int_0^t j && \text{in } \mathbb{R}^+, \\ \mathcal{C}_-(t) &= \mathcal{C}_-^{in} + \int_{\mathcal{F}} \omega(t, \cdot) - \int_{\mathcal{F}} \omega^{in} + \int_0^t j && \text{in } \mathbb{R}^+. \end{aligned} \tag{2.2.5}$$

Above the notation δ_x stands for the Delta measure in the point x . Let us highlight that the system (2.2.4) exhibits the presence of a point source in x_+ , of a point sink x_- and of two point vortices in x_+ and x_- whose strength depends on time through μ , j and ω . The letter c in the notations \mathcal{C}_+ and \mathcal{C}_- refers to the circulations respectively around \mathcal{S}^+ and \mathcal{S}^- .

2.3 Plan of the thesis

This manuscript is divided in two parts. The first one deals with a fluid-structure model where the fluid is viscous and incompressible and the structure is a rigid body that moves due to Newton laws. In particular

- ▷ Chapter 3 is dedicated to the prove of existence of solutions of the system (2.1.5) with Navier-slip boundary conditions, in three different settings. Namely, the case of Leray-Hopf weak solutions where we adapt the work [GVH14] to the 2D case for a more strict definition of weak solutions. The case of strong solutions in Banach space setting. Finally we consider the situation of weak solutions with possibly infinite fluid kinetic energy.
- ▷ Chapter 4 deals with the system (2.1.5) where Ω is a bounded domain and Navier-slip boundary conditions are prescribe on the boundary of the fluid. More precisely we prove

that Leray-Hopf weak solutions are continuous in time with values in L^2 , they satisfy the energy equality and they are unique. All these three results holds until the first contact of the solid with the boundary occurs.

In the second part we study the flow of an inviscid incompressible fluid in a domain with holes, on which the fluid can enter or exit. In particular

- ▷ Chapter 5 is dedicated to study the behaviour of a fluid in a domain with two holes. One of them is called source and the fluid can enter from the boundary of it, the other is the sink from where the fluid can exit. More precisely, we will derived the limiting system when the source and the sink shrink to two different points and we will show that solutions of (2.2.4) converge to solutions of (2.2.5) in appropriate norms.

Finally Chapter 3 is based on [Bra19] and [Bra18] and Chapter 4 is inspired by [Bra19].

Note to the reader. In what follows, we will consider all the equations in the dimensionless variables. Moreover we will suppose the density and the viscosity of the fluid equal to 1, unless in Appendix C where we study a vanishing viscosity limit.

Existence of solutions for a fluid structure problem

Dans ce chapitre, nous étudions l'existence de solutions fortes et de solutions faibles pour un système d'interaction fluide-structure.

This chapter is devoted to the study of existence of solutions for a fluid-structure problem consisting of a rigid body immersed in a viscous fluid, in particular we will concentrate our attention on two different situations. The first one is the case where the system of fluid plus rigid body occupies a bounded domain and the fluid kinetic energy is finite (in all the Thesis the kinetic energy of the solid is always finite). In this setting we will present a result of existence [GVH14] due to Gérard-Varet and Hillairet where the authors proved existence of weak solutions in dimension three and we will conclude the subsection explaining how their proof can be reproduced in dimension two and for a more restrictive class of weak solutions. Then we will present an existence result for strong solutions in Banach space setting. It is based on the study of the linearised problem and on a fixed point argument to deal with the non-linear coupled equations. The second situation is the case where the system of fluid plus rigid body occupies the plane and the fluid kinetic energy is supposed to be infinite. In this setting the velocity field that gives rise to the presence of infinite energy has a fixed structure. In the same situation in Appendix C, we prove, in the special case the solid is a disk, that weak solutions are continuous in time with values in an appropriate Hilbert space, they satisfy an energy equality and they are unique. Moreover we tackle the vanishing viscosity limit. In particular as the viscosity tends to zero the solutions converge to the one of a Euler with disk system studied in [ORT07].

Before considering the two different cases, let us present the equations which govern the system at stake. Let Ω equal to the two dimensional Euclidean space or $\Omega \subset \mathbb{R}^2$ an open and bounded set with smooth boundary and consider \mathcal{S}_0 a closed, bounded, connected and simply connected subset of the plane compactly contained in Ω with smooth boundary. We assume that the body initially occupies the domain \mathcal{S}_0 , has density $\rho_{\mathcal{S}_0}$ and rigidly moves so that at time t it occupies an isometric domain denoted by $\mathcal{S}(t) \subset \Omega$. We set $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$ the domain occupied by the fluid at time t starting from the initial domain $\mathcal{F}_0 = \Omega \setminus \mathcal{S}_0$.

The equations modelling the dynamics of the system then read

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \Delta u + \nabla p = 0 \quad \text{for } x \in \mathcal{F}(t), \quad (3.0.1)$$

$$\operatorname{div} u = 0 \quad \text{for } x \in \mathcal{F}(t), \quad (3.0.2)$$

$$\begin{aligned} u \cdot n &= u_{\mathcal{S}} \cdot n \\ (D(u)n) \cdot \tau &= -\alpha(u - u_{\mathcal{S}}) \cdot \tau \end{aligned} \quad \text{for } x \in \partial\mathcal{S}(t), \quad (3.0.3)$$

$$\begin{aligned} u \cdot n &= 0 \\ (D(u)n) \cdot \tau &= -\alpha u \cdot \tau \end{aligned} \quad \text{for } x \in \partial\Omega, \text{ if } \Omega \text{ bounded}, \quad (3.0.4)$$

$$|u| \longrightarrow 0 \quad \text{as } |x| \longrightarrow \infty, \text{ if } \Omega = \mathbb{R}^2, \quad (3.0.5)$$

$$mh''(t) = - \int_{\partial\mathcal{S}(t)} \Sigma n \, ds, \quad (3.0.6)$$

$$\mathcal{J}r'(t) = - \int_{\partial\mathcal{S}(t)} (x - h(t))^\perp \cdot \Sigma n \, ds, \quad (3.0.7)$$

$$u|_{t=0} = u_0 \quad \text{for } x \in \mathcal{F}_0, \quad (3.0.8)$$

$$h(0) = h_0, \quad h'(0) = \ell_0, \quad r(0) = r_0. \quad (3.0.9)$$

Here $u = (u_1, u_2)$ and p denote the velocity and pressure fields, n and τ are respectively the unit outwards normal and counterclockwise tangent vectors to the boundary of the fluid domain, $\alpha > 0$ is a material constant (the friction coefficient). On the other hand m and \mathcal{J} denote respectively the mass and the moment of inertia of the body while the fluid is supposed to be homogeneous of density 1 and the viscosity coefficient of the fluid is set equal to 1, to simplify the notations. The Cauchy stress tensor is defined by

$$\Sigma = -p \operatorname{Id}_2 + 2D(u),$$

where $D(u)$ is the deformation tensor defined by

$$2D(u) = ((\partial_j u_i + \partial_i u_j))_{1 \leq i, j \leq 2}.$$

When $x = (x_1, x_2)$ the notation x^\perp stands for $x^\perp = (-x_2, x_1)$, $h'(t)$ is the velocity of the center of mass $h(t)$ of the body and $r(t)$ denotes the angular velocity of the rigid body. We denote by $u_{\mathcal{S}}$ the velocity of the body:

$$u_{\mathcal{S}}(t, x) = h'(t) + r(t)(x - h(t))^\perp.$$

We assume from now on that $h_0 = 0$. Since $\mathcal{S}(t)$ is obtained from \mathcal{S}_0 by a rigid motion, there exists a rotation matrix

$$Q(t) = \begin{bmatrix} \cos \theta(t) & -\sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) \end{bmatrix}, \quad (3.0.10)$$

such that the position $\eta(t, x) \in \mathcal{S}(t)$ at the time t of the point fixed to the body with an initial position x is $h(t) + Q(t)x$. The angle θ satisfies $\theta'(t) = r(t)$, and we choose $\theta(t)$ such that

$\theta(0) = 0$. We note also that given $h'(t)$ and $\theta'(t)$, we can reconstruct the position of the body through the formula

$$\begin{aligned} \mathcal{S}^{h',\theta'}(t) &= \{x = Q(t)y + h(t) \quad \text{for } y \in \mathcal{S}_0\}, \text{ where} \\ h(t) &= \int_0^t h'(t)dt \quad \text{and} \quad \theta(t) = \int_0^t \theta'(t)dt, \end{aligned}$$

and $Q(t)$ is obtain by θ via (3.0.10). In the same spirit if the motion of the body is described by $h'(t)$ and $\theta'(t)$, then

$$\rho_{\mathcal{S}(t)}(t, x) = \rho_{\mathcal{S}_0}(Q^T(t)(x - h(t))) \quad \text{for any } x \in \mathcal{S}(t).$$

3.1 Bounded domain and finite energy setting

In what follow, we suppose Ω to be a bounded subset of \mathbb{R}^2 , therefore we study the system (3.0.1)-(3.0.9) where we choose the boundary condition (3.0.4) and not (3.0.5). In this setting we first prove existence of weak solutions with bounded energy and secondly we prove existence of strong solutions in Banach space setting.

3.1.1 Weak solutions

The goal of this section is to introduce an appropriate definition of weak solutions and to present the correspondent existence result. This work is an adaptation of the result in [GVH14] in dimension two. In particular we will prove that the solutions constructed via the method used in [GVH14] satisfy a more restrict definition of weak solution that the corresponding introduced in [GVH14].

In what follow we start by stating the definition of weak solutions. Then we present the existence theorem. We discuss the differences between this result and the corresponding one in [GVH14]. We conclude the section with the proof of existence where we emphasize only the part of the proof of [GVH14] that we adapt.

3.1.1.1 Definition of weak solutions

We present the definition of weak solution and the existence result from [GVH14]. Let \mathcal{O} be an open subset of \mathbb{R}^2 with Lipschitz boundary then we define

$$\begin{aligned} \mathcal{D}_\sigma(\mathcal{O}) &= \{\varphi \in \mathcal{D}(\mathcal{O}), \operatorname{div} \varphi = 0\}, \\ L_\sigma^2(\mathcal{O}) &= \text{the closure of } \mathcal{D}_\sigma(\mathcal{O}) \text{ in } L^2(\mathcal{O}), \quad H_\sigma^1(\mathcal{O}) = H^1(\mathcal{O}) \cap L_\sigma^2(\mathcal{O}). \end{aligned}$$

We also define the finite dimensional space of rigid vector fields in \mathbb{R}^2

$$\mathcal{R} = \{\varphi_S, \quad \varphi_S(x) = V + \omega x^\perp, \quad \text{for some } V \in \mathbb{R}^2, \omega \in \mathbb{R}\},$$

and the space of initial data

$$\begin{aligned} \mathcal{H}_{S_0} &= \{v \in L_\sigma^2(\Omega), \text{ there exists } v_F \in L_\sigma^2(\Omega) \text{ and there exists} \\ &\quad v_S \in \mathcal{R} \text{ such that } v = v_F \text{ on } \mathcal{F}_0, v = v_S \text{ on } \mathcal{S}_0\}, \end{aligned} \tag{3.1.1}$$

with norm

$$\|v\|_{\mathcal{H}_{S_0}}^2 = \int_{\mathcal{F}_0} v_F^2 dx + \int_{\mathcal{S}_0} \rho_{S_0} v_S^2 dx = \|v\|_{L^2(\mathcal{F}_0)} + m|l_v|^2 + \mathcal{J}r_v^2,$$

where l_v and r_v are related to v_S via $v_S(t, x) = l_v(t) + r_v(t)(x - h_0)^\perp$, with $h_0 = 0$ the center of mass of \mathcal{S}_0 . We define for any $T > 0$ the space of solutions

$$\begin{aligned} \mathcal{V}_T = \{ & u \in L^\infty(0, T; L^2_\sigma(\Omega)), \text{ there exists } u_F \in L^2(0, T; H^1_\sigma(\Omega)), \\ & u_S \in L^\infty(0, T; \mathcal{R}) \text{ such that } u(t, \cdot) = u_F(t, \cdot) \text{ on } \mathcal{F}(t), \\ & u(t, \cdot) = u_S(t, \cdot) \text{ on } \mathcal{S}(t), \text{ for a.e. } t \in [0, T)\}, \end{aligned}$$

Note that for any $\varphi \in \mathcal{V}_T$ we have $\varphi_f(t, \cdot) \cdot n = \varphi_S(t, \cdot) \cdot n$ on $\partial\mathcal{S}(t)$; analogously we define

$$\begin{aligned} \mathcal{W}_T = \{ & u \in L^\infty(0, T; L^2_\sigma(\Omega)), \text{ there exists } u_F \in L^\infty(0, T; H^1_\sigma(\Omega)) \cap W^{1,\infty}(0, T; L^2_\sigma(\Omega)), \\ & u_S \in W^{1,\infty}(0, T; \mathcal{R}) \text{ such that } u(t, \cdot) = u_F(t, \cdot) \text{ on } \mathcal{F}(t), \\ & u(t, \cdot) = u_S(t, \cdot) \text{ on } \mathcal{S}(t), \text{ for a.e. } t \in [0, T]\}. \end{aligned}$$

Moreover we denote by $\mathcal{W}_{0,T}$ the set of φ in \mathcal{W}_T such that $\varphi \equiv 0$ in a neighbourhood of T . We are now able to give the definition of weak solution.

Definition 3.1.1. Let $\Omega \subset \mathbb{R}^2$ an open set with smooth boundary, \mathcal{S}_0 a closed, bounded, connected and simply connected subset of Ω , with smooth boundary and $u_0 \in \mathcal{H}_{S_0}$. A weak solution of (3.0.1)-(3.0.9) on $[0, T)$, associated with the initial data (\mathcal{S}_0, u_{S_0}) is a couple (\mathcal{S}, u) satisfying

- ▷ $\overline{\mathcal{S}(t)} \subset \Omega$ is a bounded domain of \mathbb{R}^2 for all $t \in [0, T)$, such that $\chi_{\mathcal{S}(t)}(x) = 1_{\mathcal{S}(t)}(x) \in L^\infty((0, T) \times \Omega)$,
- ▷ u belongs to the space \mathcal{V}_T where $\mathcal{F}(t) = \Omega \setminus \overline{\mathcal{S}(t)}$ for all $t \in [0, T)$,
- ▷ for any $\varphi \in \mathcal{W}_{0,T}$, it holds

$$\begin{aligned} & - \int_0^T \int_{\mathcal{F}(t)} u_F \cdot \partial_t \varphi_F dx dt - \int_0^T \int_{\mathcal{S}(t)} \rho_S u_S \cdot \partial_t \varphi_S dx dt \\ & - \int_0^T \int_{\mathcal{F}(t)} u_F \otimes u_F : \nabla \varphi_F dx dt + 2 \int_0^T \int_{\mathcal{F}(t)} Du_F : D\varphi_F dx dt \\ & + 2\alpha \int_0^t \int_{\partial\Omega} u_F \cdot \varphi_F ds dt + 2\alpha \int_0^T \int_{\partial\mathcal{S}(t)} (u_F - u_S) \cdot (\varphi_F - \varphi_S) ds dt \\ & = \int_{\mathcal{F}(0)} u_{F,0} \cdot \varphi_F|_{t=0} dx + \int_{\mathcal{S}(0)} \rho_S u_{S,0} \cdot \varphi_S|_{t=0} dx. \end{aligned} \tag{3.1.2}$$

In what follow we sometimes do not write explicitly the variables in which the integrations are made to shorten the notation.

- ▷ \mathcal{S} is transported by the rigid vector fields u_S , i.e. it holds

$$\int_0^T \int_{\mathcal{S}(t)} \partial_t \psi + \int_0^T \int_{\mathcal{S}(t)} u_S \cdot \nabla \psi = - \int_{\mathcal{S}_0} \psi|_{t=0}, \tag{3.1.3}$$

for any $\psi \in C_c^\infty([0, T); \mathcal{D}(\overline{\Omega}))$.

The formal derivation of equations (3.1.2)-(3.1.3) from (3.0.1)-(3.0.9) is presented in Section 1 of [GVH14]. Equation (3.1.3) ensures that the solid is transported via the rigid vector field v_S . Equation (3.1.2) is a weak version of (3.0.1)-(3.0.9), in fact the sum of the first and the third term of (3.1.2) correspond to the convective derivative in the equation (3.0.1), the sum of the second and the fourth term of (3.1.2) corresponds to the pressure and the viscous term in (3.0.1) together with the Newton equations (3.0.6)-(3.0.7) associated with the solid motion, the fifth and the sixth term correspond respectively to the boundary condition (3.0.4) and (3.0.3), and finally the last line corresponds to the initial condition (3.0.8)-(3.0.9).

3.1.1.2 An existence result

Let us conclude this section with recalling the existence result from [GVH14].

Theorem 3.1.1 (Theorem 1 of [GVH14]). *Let $\Omega \subset \mathbb{R}^2$ an open, bounded, connected set with smooth boundary, \mathcal{S}_0 a closed, bounded, connected and simply connected subset of Ω with smooth boundary and $u_0 \in \mathcal{H}_{S_0}$. There exists a weak solution (\mathcal{S}, u) to the problem (3.0.1)-(3.0.9) with initial data (\mathcal{S}_0, u_0) for some $T > 0$. Moreover either $T = +\infty$ and $\mathcal{S}(t) \Subset \Omega$ for any $t \geq 0$ or $T < +\infty$ and it holds $\mathcal{S}(t) \Subset \Omega$ for $t \in [0, T)$ and $\text{dist}(\mathcal{S}(t), \partial\Omega) \rightarrow 0$ as $t \rightarrow T^-$.*

The theorem above states that weak solutions exist up to collision, in fact by Definition 3.1.1 we have that the position of the solid is continuous in time and the condition $\mathcal{S}(t) \Subset \Omega$ implies that $\text{dist}(\mathcal{S}(t), \partial\Omega) > 0$, this means that the solid never touches the boundary until the final time $t = T$, when $\text{dist}(\mathcal{S}(T), \partial\Omega) = 0$.

Theorem 3.1.1 differs from Theorem 1 of [GVH14] in two points. The first one is that Theorem 3.1.1 deals with the 2D case whereas Theorem 1 of [GVH14] deals with the 3D case, but the proof of Theorem 1 of [GVH14] can be straightforwardly adapted to the 2D case. The second difference is the set of test functions used in (3.1.2), in fact in (3.1.2) we substitute the space

$$\begin{aligned} \mathcal{T}_{0,T} = \{ & \varphi \in C_c^0([0, T]; L_\sigma^2(\Omega)), \text{ there exists } \varphi_F \in C^\infty([0, T]; \mathcal{D}_\sigma(\bar{\Omega})), \\ & \varphi_S \in C^\infty([0, T]; \mathcal{R}) \text{ such that } \varphi(t, \cdot) = \varphi_F(t, \cdot) \text{ on } \mathcal{F}(t), \\ & \varphi(t, \cdot) = \varphi_S(t, \cdot) \text{ on } \mathcal{S}(t), \text{ for all } t \in [0, T)\}, \end{aligned} \quad (3.1.4)$$

which is the space of test functions mentioned in Definition 1 of [GVH14], by $\mathcal{W}_{0,T}$. But the weak solutions constructed in [GVH14] satisfy (3.1.2) for any test function in $\mathcal{W}_{0,T}$ in the 2D case. Moreover observe that there is no energy inequality in Definition 3.1.1. Indeed in Theorem 4.1.1 we are going to prove that any solution satisfies an energy equality.

3.1.1.3 Proof of Theorem 3.1.1

As pointed out previously it is possible to follow the proof presented in [GVH14] and to prove that there exists a weak solution which satisfies (3.1.2) for any test function in $\mathcal{T}_{0,T}$ (recall the definition in (3.1.4)).

We prove that the solution satisfies (3.1.2) for any test functions in $\mathcal{W}_{0,T}$. The proof in [GVH14] is based on a local-in-time existence which leads to concatenate solutions up to collision, see last paragraph of Section 5.7 of [GVH14]. Therefore it is enough to prove that the local-in-time existence result holds also for the restriction in time of the element of $\mathcal{W}_{0,T}$. We state the local-in-time existence result.

Theorem 3.1.2. *Let $\Omega \subset \mathbb{R}^2$ an open, bounded, connected set with smooth boundary, \mathcal{S}_0 a closed, bounded, connected and simply connected subset of Ω with smooth boundary, $u_0 \in \mathcal{H}_{\mathcal{S}_0}$ and $\delta > 0$ such that $\text{dist}(\partial\Omega, \mathcal{S}_0) > 2\delta$. There exists $\tau > 0$ and a couple (\mathcal{S}, u) satisfying*

- ▷ $\mathcal{S}(t) \subset \Omega$ is a bounded domain of \mathbb{R}^2 for all $t \in [0, \tau]$, such that $\chi_{\mathcal{S}}(t, x) = 1_{\mathcal{S}(t)}(x) \in L^\infty((0, \tau) \times \Omega)$,
- ▷ u belongs to the space \mathcal{V}_τ where $\mathcal{F}(t) = \Omega \setminus \overline{\mathcal{S}(t)}$ for all $t \in [0, \tau]$,
- ▷ for any $\varphi \in \mathcal{W}_\tau$, it holds

$$\begin{aligned}
& - \int_0^\tau \int_{\mathcal{F}(t)} u_F \cdot \partial_t \varphi_F dx dt - \int_0^\tau \int_{\mathcal{S}(t)} \rho_S u_S \cdot \partial_t \varphi_S dx dt \\
& - \int_0^\tau \int_{\mathcal{F}(t)} u_F \otimes u_F : \nabla \varphi_F dx dt + 2 \int_0^\tau \int_{\mathcal{F}(t)} Du_F : D\varphi_F dx dt \\
& + 2\alpha \int_0^\tau \int_{\partial\Omega} u_F \cdot \varphi_F ds dt + 2\alpha \int_0^\tau \int_{\partial\mathcal{S}(t)} (u_F - u_S) \cdot (\varphi_F - \varphi_S) ds dt \\
& = \int_{\mathcal{F}(0)} u_{F,0} \cdot \varphi_F|_{t=0} dx + \int_{\mathcal{S}(0)} \rho_S u_{S,0} \cdot \varphi_S|_{t=0} dx \\
& - \int_{\mathcal{F}(\tau)} u_F \cdot \varphi_F|_{t=\tau} dx - \int_{\mathcal{S}(\tau)} \rho_S u_S \cdot \varphi_S|_{t=\tau} dx.
\end{aligned} \tag{3.1.5}$$

- ▷ \mathcal{S} is transported by the rigid vector fields u_S , i.e. it holds

$$\int_0^\tau \int_{\mathcal{S}(t)} \partial_t \psi + \int_0^\tau \int_{\mathcal{S}(t)} u_S \cdot \nabla \psi = \int_{\mathcal{S}(\tau)} \psi|_{t=\tau} - \int_{\mathcal{S}_0} \psi|_{t=0},$$

for any $\psi \in C^\infty([0, \tau]; \mathcal{D}(\overline{\Omega}))$.

- ▷ $\text{dist}(\partial\Omega, \mathcal{S}(t)) \geq \delta$ for almost any $t \in [0, \tau]$.

Proof. By the proof in [GVH14] we already know that this theorem holds with test functions in \mathcal{T}_τ , which is the set of $\varphi|_{[0, \tau]}$ where $\varphi \in \mathcal{T}_{0, T}$. We prove that (3.1.5) holds for any test function in \mathcal{W}_τ . To do so we approximate the test functions in \mathcal{W}_τ by admissible test functions of the approximate problem defined in [GVH14, Section 2]. To do so we need an equivalent of Proposition 12 in [GVH14], i.e. we prove the following claim.

Claim 1. Let $\alpha > 0$ and let $\varphi \in \mathcal{W}_\tau$. Then there exists a sequence $\varphi^n \in W^{1, \infty}(0, \tau; L^2_\sigma(\Omega)) \cap L^\infty(0, \tau; H^1_\sigma(\Omega))$ of the form

$$\varphi^n = (1 - \chi_S^n) \varphi_F + \chi_S^n \varphi_S^n,$$

that satisfies

- ▷ $\|\sqrt{\chi_S^n}(\varphi_S^n - \varphi_S)\|_{L^\infty(0, \tau; L^p(\Omega))} = O(n^{-\alpha/(p+\varepsilon)})$ for any $p \in (1, \infty)$ and for any $\varepsilon > 0$,
- ▷ $\varphi^n \rightarrow \varphi$ strongly in $L^\infty(0, \tau; L^p)$ for $p \in (1, \infty)$,
- ▷ $\|\varphi^n\|_{L^\infty(0, \tau; H^1(\Omega))} = O(n^{\alpha(1-1/p)})$ for any $p \in (1, \infty)$,
- ▷ $\|\chi_S^n(\partial_t + P_S^n u^n \cdot \nabla)(\varphi_S^n - \varphi_S)\|_{L^q(0, \tau; L^p(\Omega))} = O(n^{-\alpha(2-p)/2p})$ for any $p \in (1, 2]$ and $q \in (1, \infty)$, in particular $\chi_S^n(\partial_t + P_S^n u^n \cdot \nabla)(\varphi_S^n - \varphi_S)$ converges to 0 in $L^2(0, \tau; L^2(\Omega))$.

$\triangleright (\partial_t + P_S^n u^n \cdot \nabla) \varphi^n \rightarrow (\partial_t + P_S u \cdot \nabla) \varphi$ weakly* in $L^\infty(0, \tau; L^p(\Omega))$ for any $p \in (1, 2]$.

Proof of the claim. To prove this claim we make the same construction as [GVH14]. The main difficulty is not the lack of regularity in time but the lack of regularity in space. By the fact that the construction of this approximation is quite technical and involved, we present here quite rapidly the construction and we refer to [GVH14, Section 5.3] for more details.

Recall that a weak solution (\mathcal{S}, u) , constructed in [GVH14] comes as a limit of solutions (\mathcal{S}_n, u_n) of some approximate problems and recall the flow $\phi_n \in W^{1,\infty}(0, \tau; C^\infty)$ from [GVH14], i.e. $\phi_n(t) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a C^∞ -diffeomorphism and it is the flow associated with $P_S^n u_n$. We define an approximation φ_n of $\varphi \in \mathcal{W}_\tau$ using the flow ϕ_n .

The idea of Gérard-Varet and Hillairet is to use ϕ_n to translate the problem to a “fixed” domain and then approximate. Define $\tilde{\Phi}_S^n$ and $\tilde{\Phi}_F^n$ via

$$\varphi_S(t, \phi_n(t, y)) = d\phi_n(t, y)(\tilde{\Phi}_S^n(t, y)), \quad \varphi_F(t, \phi_n(t, y)) = d\phi_n(t, y)(\tilde{\Phi}_F^n(t, y)).$$

$\tilde{\Phi}_S^n$ and $\tilde{\Phi}_F^n$ are defined in a fixed solid domain in the sense that the solid part is fixed, i.e. $\phi_n|_{\mathcal{S}_0} : \mathcal{S}_0 \rightarrow \mathcal{S}_n(t)$. In the approximation we do not change φ in the fluid part so we define $\varphi_n|_{(0,\tau) \times \mathcal{F}^n(t)} = \varphi|_{(0,\tau) \times \mathcal{F}_n(t)}$ and φ_n in the solid part such that it is closed to a solid rotation and such that it makes φ_n an $L^2(0, \tau; H_\sigma^1(\Omega))$ function. To do so we approximate $\tilde{\Phi}_S^n$ by $\Phi_S^n = \Phi_{1,S}^n + \Phi_{2,S}^n$, where

$$\Phi_{1,S}^n = \tilde{\Phi}_S^n + \chi(n^\alpha z)((\tilde{\Phi}_F^n - \tilde{\Phi}_S^n) - [(\tilde{\Phi}_S^n - \tilde{\Phi}_F^n) \cdot e_z]e_z),$$

and $\Phi_{2,S}^n$ is zero on the boundary and it is defined in such a way to make Φ_S^n divergence free following the construction in [Gal11] Chapter III. This leads us to define

$$\varphi_S^n(t, \phi_n(t, y)) = d\phi_n(t, y)(\Phi_S^n(t, y)).$$

Note that

$$\begin{aligned} & \|\Phi_{1,S}^n - \tilde{\Phi}_S\|_{L^\infty(0,\tau;L^p(\mathcal{S}_0))} \\ &= \|\chi(n^\alpha z)((\tilde{\Phi}_F^n - \tilde{\Phi}_S^n) - [(\tilde{\Phi}_S^n - \tilde{\Phi}_F^n) \cdot e_z]e_z)\|_{L^\infty(0,\tau;L^p(\mathcal{S}_0))} \\ &\leq \|\chi(n^\alpha z)\|_{L^q(\mathcal{S}_0)} \|(\tilde{\Phi}_F^n - \tilde{\Phi}_S^n) - [(\tilde{\Phi}_S^n - \tilde{\Phi}_F^n) \cdot e_z]e_z\|_{L^\infty(0,\tau;L^r(\mathcal{S}_0))} \\ &\leq C(r) \|\chi(n^\alpha z)\|_{L^q(\mathcal{S}_0)} \|(\tilde{\Phi}_F^n - \tilde{\Phi}_S^n) - [(\tilde{\Phi}_S^n - \tilde{\Phi}_F^n) \cdot e_z]e_z\|_{L^\infty(0,\tau;H^1(\mathcal{S}_0))} \\ &\leq C(r)n^{-\alpha/q}, \end{aligned}$$

where $1/q + 1/r = 1/p$. In a similar way

$$\begin{aligned} & \|\Phi_{1,S}^n - \tilde{\Phi}_S\|_{L^\infty(0,\tau;H^1(\mathcal{S}_0))} \\ &= \|\chi(n^\alpha z)((\tilde{\Phi}_F^n - \tilde{\Phi}_S^n) - [(\tilde{\Phi}_S^n - \tilde{\Phi}_F^n) \cdot e_z]e_z)\|_{L^\infty(0,\tau;H^1(\mathcal{S}_0))} \\ &\leq C + \|n^\alpha \nabla \chi(n^\alpha z)\|_{L^q(\mathcal{S}_0)} \\ &\quad \|(\tilde{\Phi}_F^n - \tilde{\Phi}_S^n) - [(\tilde{\Phi}_S^n - \tilde{\Phi}_F^n) \cdot e_z]e_z\|_{L^\infty(0,\tau;L^r(\mathcal{S}_0))} \\ &\leq C + C(r)n^{\alpha(1-1/q)} \|(\tilde{\Phi}_F^n - \tilde{\Phi}_S^n) - [(\tilde{\Phi}_S^n - \tilde{\Phi}_F^n) \cdot e_z]e_z\|_{L^\infty(0,\tau;H^1(\mathcal{S}_0))} \\ &\leq C + C(r)n^{\alpha(1-1/q)}. \end{aligned}$$

Moreover

$$\begin{aligned}
\|\Phi_{2,S}^n\|_{L^\infty(0,\tau;L^p(\mathcal{S}_0))} &\leq C(r)\|\Phi_{2,S}^n\|_{L^\infty(0,\tau;W^{1,r}(\mathcal{S}_0))} \\
&\leq C(r)\|\operatorname{div}\Phi_{1,S}^n\|_{L^\infty(0,\tau;L^r(\mathcal{S}_0))} \\
&= C(r)\|\chi(n^\alpha z)\mathcal{J}^n\|_{L^\infty(0,\tau;L^r(\mathcal{S}_0))} \\
&\leq C(r)\|\chi(n^\alpha z)\|_{L^p(\mathcal{S}_0)}\|\mathcal{J}^n\|_{L^\infty(0,\tau;L^2(\mathcal{S}_0))} \\
&\leq C(r)n^{-\alpha/p}.
\end{aligned}$$

where $\mathcal{J}^n = \operatorname{div}(x \mapsto ((\tilde{\Phi}_F^n - \tilde{\Phi}_S^n) - [(\tilde{\Phi}_S^n - \tilde{\Phi}_F^n) \cdot e_z]e_z))$, $1 - 2/r = -2/p$ i.e. $r = 2p/(p+2)$ and it holds $1/p + 1/2 = 1/r$. In a similar way

$$\|\Phi_{2,S}^n\|_{L^\infty(0,\tau;H^1(\mathcal{S}_0))} \leq \|\operatorname{div}\Phi_{1,S}^n\|_{L^\infty(0,\tau;L^2(\mathcal{S}_0))} \leq C.$$

The estimates above prove the first three points of the claim. For the last two points we follow the computation of [GVH14], namely

$$\begin{aligned}
&\|\chi_S^n(\partial_t + P_S^n u^n \cdot \nabla)(\varphi_S^n - \varphi_S)\|_{L^\infty(0,\tau;L^p(\Omega))} \\
&\leq \left\| \frac{\partial}{\partial t} d\phi^n(t, y)(\Phi_S^n - \Phi_S) \right\|_{L^\infty(0,\tau;L^p(\Omega))} \\
&\leq C\|d\phi^n(t, y)\|_{W^{1,\infty}(0,T;L^\infty(\Omega))} \left(\|\chi(n^\alpha z)\|_{L^\infty(0,T;L^{2p/(2-p)}(\Omega))} \|\Phi_{1,S}^n - \tilde{\Phi}_S\|_{W^{1,q}(0,T;L^2(\Omega))} \right) \\
&\leq Cn^{-\alpha p/(2-p)},
\end{aligned}$$

where we use the estimate $\|\partial_t \Phi_{2,S}^n\|_{L^q(0,T;L^2(\Omega))} \leq C\|\partial_t(\Phi_{1,S}^n - \tilde{\Phi}_S)\|_{L^q(0,T;L^2(\Omega))}$. For the last point we compute

$$\begin{aligned}
&(\partial_t + P_S^n u^n \cdot \nabla)\varphi^n - (\partial_t + P_S u \cdot \nabla)\varphi = (1 - \chi_S^n)(\partial_t + P_S^n u^n \cdot \nabla)(\varphi_F^n - \varphi_F) \\
&\quad + \chi_S^n(\partial_t + P_S^n u^n \cdot \nabla)(\varphi_S^n - \varphi_S) + (1 - \chi_S^n)(\partial_t + P_S^n u^n \cdot \nabla)\varphi_F \\
&\quad + \chi_S^n(\partial_t + P_S^n u^n \cdot \nabla)\varphi_S,
\end{aligned}$$

which converges weakly* to 0 by the strong convergence of χ_S^n and the weak convergence of $P_S^n u^n$. \square

The above claim proves that there exists a good approximation φ_n , for $\varphi \in \mathcal{W}_\tau$. Passing to the limit in the approximate problem, we deduce that the weak formulation holds for any test function in \mathcal{W}_τ . \square

3.1.2 Strong solutions

This section is dedicated to the proof of existence and uniqueness of solutions of the problem (3.0.1)-(3.0.9) in the Banach space setting $L^p - L^q$ in the case where Ω is a bounded domain.

To obtain this result we follow the approach presented in [MT18], where the authors studied the system (3.0.1)-(3.0.9) in both two and three dimensions and no-slip boundary conditions are prescribed on the boundary of the fluid domain.

This section is divided in six subsections. In the first one we present the functional space setting and the theorem of existence and uniqueness. In the second one we recall a well-known

change of variables that will be used along all the Thesis. In the third one we present the equations (3.0.1)-(3.0.9) in the new variables. In the fourth we study the linearised system associated with (3.0.1)-(3.0.9) in the new variables. In the fifth we prove the existence and uniqueness result via a fixed point argument and finally we present an existence and uniqueness result for a special system, which will appear in the Section 4.2.1.

3.1.2.1 Functional spaces and result

We start by specifying the notation that we will use for all this section. Let \mathcal{O} be an open, bounded subset of \mathbb{R}^2 with smooth boundary, then we denote by $W^{k,q}(\mathcal{O})$ the usual Sobolev spaces with $k \in \mathbb{N}^+$ and $q \in [1, \infty]$ and we add tilde to denote the restriction to the functions with zero average, i.e. we define

$$\tilde{L}^q(\mathcal{O}) = \left\{ f \in L^q(\mathcal{O}) \text{ such that } \int_{\mathcal{O}} f = 0 \right\} \quad \text{and} \quad \tilde{W}^{k,q}(\mathcal{O}) = \tilde{L}^q(\mathcal{O}) \cap W^{k,q}(\mathcal{O}).$$

For $k \in \mathbb{N}^+$, $0 < s < k$ and $1 \leq p, q < +\infty$, we denote by

$$B_{p,q}^s(\mathcal{O}) = \left(L^q(\mathcal{O}), W^{k,q}(\mathcal{O}) \right)_{s/k,p}$$

the Besov spaces obtained by real interpolation of Sobolev spaces.

Fix $T > 0$ and let $\mathcal{O}_T : [0, T] \rightarrow \{\text{open, bounded subset of } \mathbb{R}^2 \text{ with smooth boundary}\}$ a time-dependent domain such that there exists a C^1 -diffeomorphism $\Lambda(t, \cdot)$ from $\mathcal{O}(0)$ to $\mathcal{O}(t)$ such that it is continuous up to the first derivative in time and the second derivatives in space. For function $v(t, \cdot) : \mathcal{O}(t) \rightarrow \mathbb{R}$ we denote $\hat{v}(t, y) = v(t, \Lambda(t, y))$ and we define for $1 < p, q < \infty$ and $0 < s < 2$ the time-dependent spaces

$$\begin{aligned} L^p(0, T; L^q(\mathcal{O}(\cdot))) &= \{v \mid \hat{v} \in L^p(0, T; L^q(\mathcal{O}(0)))\}, \\ L^p(0, T; W^{2,q}(\mathcal{O}(\cdot))) &= \{v \mid \hat{v} \in L^p(0, T; W^{2,q}(\mathcal{O}(0)))\}, \\ W^{1,p}(0, T; L^q(\mathcal{O}(\cdot))) &= \{v \mid \hat{v} \in W^{1,p}(0, T; L^q(\mathcal{O}(0)))\}, \\ C([0, T]; B_{q,p}^s(\mathcal{O}(\cdot))) &= \{v \mid \hat{v} \in C([0, T]; B_{q,p}^s(\mathcal{O}(0)))\}. \end{aligned}$$

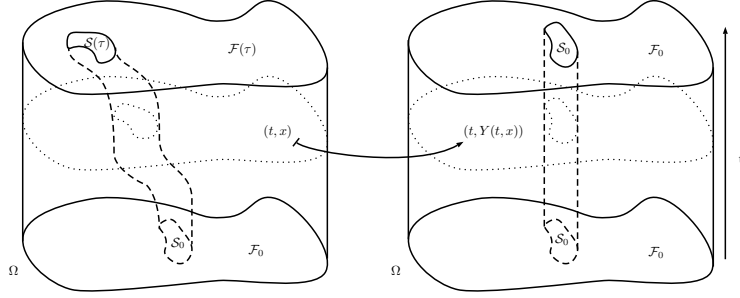
Finally we present the set of initial datum. Let \mathcal{S}_0 and $\mathcal{F}_0 = \Omega \setminus \mathcal{S}_0$ the initial position of the solid and the fluid such that the origin is the center of mass of the solid. We define the space

$$\mathcal{X} = L_\sigma^q(\mathcal{F}_0) \times \mathbb{R}^2 \times \mathbb{R},$$

the domain $\mathcal{D}(\mathcal{A}_{FS}^q)$ of the operator will be defined in (3.1.9) and finally

$$\mathfrak{D}_{q,p}(\Omega) = (\mathcal{X}, \mathcal{D}(\mathcal{A}_{FS}^q))_{1-1/p,p}$$

where for $0 < \theta < 1$ we denote $(\cdot, \cdot)_{\theta,p}$ the real interpolation space. We recall from [Shi07] that $\mathfrak{D}_{q,p} \subset B_{q,p}^{2(1-1/p)} \times \mathbb{R}^2 \times \mathbb{R}$ and for $2/p + 1/q \neq 2$, $2/p + 1/q \neq 1$ the norm $\mathfrak{D}_{q,p}(\Omega)$ of (u, l, r) is equivalent to the sum of the norm $B_{q,p}^{2(1-1/p)}$ of u and Euclidian norms for l and r . We are now able to state the main theorem of the section.

Figure 3.1: Change of variables Y_T , which is the inverse of X_T .

Theorem 3.1.1. *Let $1 < p, q < \infty$ satisfying the conditions $1/p + 1/q \leq 3/2$, $2/p + 1/q \neq 2$ and $2/p + 1/q \neq 1$ and let $(u_0, l_0, r_0) \in B_{q,p}^{2(1-1/p)} \times \mathbb{R}^2 \times \mathbb{R}$ satisfying the compatibility conditions*

$$\operatorname{div} u_0 = 0 \quad \text{in } \mathcal{F}_0, \quad u_0 \cdot n = 0 \quad \text{in } \partial\Omega \quad \text{and} \quad u \cdot n = (l_0 + r_0 x^\perp) \cdot n \quad \text{in } \partial\mathcal{S}_0.$$

Moreover if $1 > 2/p + 1/q$, we also assume that

$$(D(u_0)n) \cdot \tau = -\alpha u_0 \cdot \tau \quad \text{on } \partial\Omega \quad \text{and} \quad (D(u_0)n) \cdot \tau = -\alpha(u_0 - l_0 - r_0 x^\perp) \cdot \tau \quad \text{on } \partial\mathcal{S}_0.$$

Then for some $T > 0$ there exists a unique strong solution (u, p, l, r) of (3.0.1)-(3.0.9) such that

$$\begin{aligned} & \|u\|_{L^p(0,T;W^{2,q}(\mathcal{F}(\cdot)))^2} + \|u\|_{W^{1,p}(0,T;L^q(\mathcal{F}(\cdot)))^2} + \|u\|_{L^\infty(0,T;B_{q,p}^{2(1-1/p)}(\mathcal{F}(\cdot)))^2} \\ & \|p\|_{L^p(0,T;\tilde{W}^{1,q}(\mathcal{F}(\cdot)))} + \|l\|_{W^{1,p}(0,T;\mathbb{R}^2)} + \|r\|_{W^{1,p}(0,T;\mathbb{R})} \\ & \leq C \left(\|u_0\|_{B_{p,q}^{2(1-p)}(\mathcal{F}_0)} + |l_0| + |r_0| \right). \end{aligned}$$

The next four subsections are devoted to the proof of the above theorem. The idea of the proof is to use a change of variables that moves the problem in a time-independent domain, to study the linearised problem in the time-independent setting and then conclude by a fixed point argument. In the next subsection we will present the change of variables.

3.1.2.2 Change of variables

Let $\mathcal{S}(t) \subset \Omega$ and $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$ respectively the domain occupied by the rigid body and the fluid domain. We will now present a well-known geometric change of variables, see for instance [IW77, Proposition 2.1], [CT08, Lemma 2.1 and Lemma 2.2], that fixes the position of the solid and is the identity on a neighbourhood of $\partial\Omega$, see also Figure 3.1. Moreover it is rigid in a neighbourhood of the solid. This change of variables will be used many times in the thesis so we will detail the construction. Let $h \in W^{1,\infty}([0, T], \mathbb{R}^2)$ and $r \in L^\infty([0, T], \mathbb{R})$ such that $\mathcal{S}(t) = \mathcal{S}^{h',r}(t)$ and such that there exists a δ for which

$$\min_{t \in [0, T]} (\operatorname{dist}(\mathcal{S}(t), \partial\Omega)) \geq \delta.$$

Finally let $\psi \in C_c^\infty(\Omega; \mathbb{R})$ be a cut-off, such that $\psi \equiv 1$ for any x with $\operatorname{dist}(x, \partial\Omega) \geq \delta/2$ and $\psi \equiv 0$ for any x with $\operatorname{dist}(x, \partial\Omega) \leq \delta/4$. We define w in $[0, T] \times \Omega$ by

$$w(t, x) = (x - h(t))^\perp \cdot h'(t) + \frac{|x - h(t)|^2}{2} r(t),$$

and we define $\Lambda : [0, T] \times \Omega \rightarrow \mathbb{R}^2$ by

$$\Lambda(t, x) = \begin{pmatrix} -\frac{\partial \psi}{\partial x_2} w + \psi u_{S,1} \\ \frac{\partial \psi}{\partial x_1} w + \psi u_{S,2} \end{pmatrix}, \quad (3.1.6)$$

where $u_{S,i}$ is the i -th component of $u_S(t, x) = h'(t) + r(t)(x - h(t))^\perp$. Then $\Lambda \in L^\infty(0, T; C^k(\Omega))$ for any $k \in \mathbb{N}$, $\Lambda(t, x) = 0$ for all t in $(0, T)$ when $\text{dist}(x, \partial\Omega) < \delta/4$, $\text{div} \Lambda(t, x) = 0$ for any (t, x) and $\Lambda(t, x) = h'(t) + r(t)(x - h(t))^\perp$ for any t in $(0, T)$ and for any x in $\mathcal{S}(t)$.

We define the change of variables associated with the flow Λ .

Proposition 3.1.1. *Let Λ defined in (3.1.6). Then there exists a unique solution $X : [0, T] \times \Omega \rightarrow \Omega$ with $X \in W^{1,\infty}(0; T; C^k(\Omega))$ for any $k \in \mathbb{N}$ of the equation*

$$\begin{cases} \partial_t X(t, y) = \Lambda(t, X(t, y)) & \text{in } (0, T) \times \Omega, \\ X(0, y) = y & \text{in } \Omega. \end{cases}$$

Moreover it holds

- ▷ $X(t, \cdot)$ is a C^∞ -diffeomorphism for any $t \in [0, T]$,
- ▷ $\det \nabla X(t, \cdot) = 1$ for any $t \in [0, T]$,
- ▷ $Y(t, \cdot) = [X(t, \cdot)]^{-1}$ is the inverse of $X(t, \cdot)$ for any $t \in [0, T]$.

3.1.2.3 Equation in the time-independent frame

In this subsection we present the system (3.0.1)-(3.0.9) in the time-independent frame associate with the change of variables X defined in Proposition 3.1.1. We use a classical idea introduced in [IW77] and we follow the notation introduced in [GGH13]. Define

$$\begin{aligned} v(t, y) &= \nabla Y(t, X(t, y))u(t, X(t, y)), \\ \pi(t, y) &= p(t, X(t, y)), \\ \omega(t) &= r(t), \\ \ell(t) &= Q^T(t)l(t), \\ \mathcal{T}(v(t, y), \pi(t, y)) &= Q^T(t)\Sigma(Q(t)u(t, y), p(t, y)Q(t)). \end{aligned}$$

In this new variables the equations become

$$\begin{aligned}
\frac{\partial v}{\partial t} + (\mathcal{M} - \mathcal{L})v + \mathcal{N}(v) + \mathcal{G}\pi &= 0 & \text{for } x \in \mathcal{F}_0, \\
\operatorname{div} v &= 0 & \text{for } x \in \mathcal{F}_0, \\
v \cdot n &= v_S \cdot n & \text{for } x \in \partial\mathcal{S}_0, \\
(D(v)n) \cdot \tau &= -\alpha(v - v_S) \cdot \tau & \text{for } x \in \partial\mathcal{S}_0, \\
v \cdot n &= 0 & \text{for } x \in \partial\Omega, \\
(D(v)n) \cdot \tau &= -\alpha v \cdot \tau & \text{for } x \in \partial\Omega, \\
m\ell'(t) &= -\int_{\partial\mathcal{S}_0} \mathcal{T}(v, \pi)n \, ds - m\omega\ell^\perp, \\
\mathcal{J}\omega'(t) &= -\int_{\partial\mathcal{S}_0} y^\perp \cdot \mathcal{T}(v, \pi)n \, ds, \\
v|_{t=0} &= u_0 & \text{for } x \in \mathcal{F}_0, \\
\ell(0) &= l_0, & \omega(0) = r_0.
\end{aligned} \tag{3.1.7}$$

where $\mathcal{M}, \mathcal{L}, \mathcal{N}$ and \mathcal{G} are defined as follows

$$\begin{aligned}
(\mathcal{L}w)_i &= \sum_{j,k}^2 \partial_j (g^{ij} \partial_k w_i) + 2 \sum_{j,k,l=1}^2 g^{kl} \gamma_{jk}^i \partial_l w_j \\
&\quad + \sum_{j,k,l}^2 \left(\partial_k (g^{kl} \Gamma_{jl}^i) + \sum_{m=1}^2 g^{kl} \Gamma_{jl}^m \Gamma_{km}^i \right) w_j,
\end{aligned}$$

$$(\mathcal{N}w)_i = \sum_{j=1}^2 w_j \partial_j w_i + \sum_{j,k=1}^2 \Gamma_{j,k=1}^i \Gamma_{jk}^i w_j w_k,$$

and

$$(\mathcal{M}w)_i = \sum_{j=1}^2 Y_j \partial_j w_i + \sum_{j,k=1}^2 \left(\Gamma_{jk}^i \dot{Y}_k + (\partial_k Y_i) (\partial_j \dot{X}_k) \right) w_j, \quad (\mathcal{G}q)_i = \sum_{j=1}^2 g^{ij} \partial_j q,$$

where

$$g^{ij} = \sum_{k=1}^2 (\partial_k Y_i) (\partial_k Y_j), \quad g_{ij} = \sum_{k=1}^2 (\partial_i X_k) (\partial_j X_k),$$

$$\Gamma_{jk}^i = \frac{1}{2} \sum_{l=1}^2 g^{jk} (\partial_i g_{jl} + \partial_j g_{lk} - \partial_k g_{ij}).$$

3.1.2.4 Study of the linearised problem

In this subsection we prove existence and uniqueness for solutions of the linearised problem of (3.1.7), which read as

$$\begin{aligned}
\frac{\partial v}{\partial t} - \Delta v + \nabla \pi &= f && \text{for } x \in \mathcal{F}_0, \\
\operatorname{div} v &= 0 && \text{for } x \in \mathcal{F}_0, \\
v \cdot n &= v_S \cdot n && \text{for } x \in \partial \mathcal{S}_0, \\
(D(v)n) \cdot \tau &= -\alpha(v - v_S) \cdot \tau && \text{for } x \in \partial \mathcal{S}_0, \\
v \cdot n &= 0 && \text{for } x \in \partial \Omega, \\
(D(v)n) \cdot \tau &= -\alpha v \cdot \tau && \text{for } x \in \partial \Omega, \\
m\ell'(t) &= - \int_{\partial \mathcal{S}_0} \Sigma(v, \pi) n \, ds + g_1, \\
\mathcal{J}\omega'(t) &= - \int_{\partial \mathcal{S}_0} (x - h(t))^\perp \cdot \Sigma(v, \pi) n \, ds + g_2, \\
v|_{t=0} &= v_0 && \text{for } x \in \mathcal{F}_0, \\
\ell(0) &= \ell_0, && \omega(0) = \omega_0.
\end{aligned} \tag{3.1.8}$$

where we introduce the source terms $f \in L^p(0, T; L^q(\Omega))$, $g_1 \in L^p(0, T; \mathbb{R}^2)$ and $L^p(0, T; \mathbb{R})$.

We are now able to present the existence and uniqueness result.

Theorem 3.1.2. *Let $1 < p, q < \infty$ satisfying the conditions $1/p + 1/q \leq 3/2$, $2/p + 1/q \neq 2$ and $2/p + 1/q \neq 1$, let $f \in L^p(0, T; L^q(\mathcal{F}_0))$, $g_1 \in L^p(0, T; \mathbb{R}^2)$ and $g_2 \in L^p(0, T; \mathbb{R})$ and let $(u_0, l_0, r_0) \in B_{q,p}^{2(1-1/p)} \times \mathbb{R}^2 \times \mathbb{R}$ satisfying the compatibility conditions*

$$\operatorname{div} u_0 = 0 \quad \text{in } \mathcal{F}_0, \quad u_0 \cdot n = 0 \quad \text{in } \partial \Omega \quad \text{and} \quad u \cdot n = (l_0 + r_0 x^\perp) \cdot n \quad \text{in } \partial \mathcal{S}_0.$$

Moreover if $1 > 2/p + 1/q$, we also assume that

$$(D(u_0)n) \cdot \tau = -\alpha u_0 \cdot \tau \quad \text{on } \partial \Omega \quad \text{and} \quad (D(u_0)n) \cdot \tau = -\alpha(u_0 - l_0 - r_0 x^\perp) \cdot \tau \quad \text{on } \partial \mathcal{S}_0.$$

Then there exists a unique solution (v, π, ℓ, ω) of (3.1.8) such that it satisfies

$$\begin{aligned}
&\|v\|_{L^p(0, T; W^{2,q}(\mathcal{F}_0))} + \|v\|_{W^{1,p}(0, T; L^q(\mathcal{F}_0))} + \|\pi\|_{L^p(0, T; \tilde{W}^{1,q}(\mathcal{F}_0))} \\
&+ \|\ell\|_{W^{1,p}(0, T; \mathbb{R}^2)} + \|\omega\|_{W^{1,p}(0, T; \mathbb{R})} \leq C(\|v_0\|_{B_{q,p}^{2(1-1/p)}(\mathcal{F}_0)} + |l_0| + |\omega_0| \\
&+ \|f\|_{L^p(0, T; L^q(\omega))} + \|g_1\|_{L^p(0, T; \mathbb{R}^2)} + \|g_2\|_{L^p(0, T; \mathbb{R})}).
\end{aligned}$$

To prove the result we will prove that we can rewrite the above system in the abstract form

$$z'(t) = \mathfrak{A}z(t) + \mathfrak{f}, \quad z(t) = 0,$$

and we prove that \mathfrak{A} is a \mathcal{R} -sectorial operator. This will implies Theorem 3.1.2 by the well-known result of Weis [Wei01]. We recall the definition and the basic properties of \mathcal{R} -sectorial operators in the Appendix B.

3.1.2.5 Reformulation of (3.1.8)

To reformulate problem (3.1.8), we need some notation. Let \mathcal{P} the projection

$$\mathcal{P}_{q, \mathcal{F}_0} : L^q(\mathcal{F}_0; \mathbb{R}^2) \rightarrow L^q_\sigma(\mathcal{F}_0; \mathbb{R}^2).$$

Then we define the operator $A^q : \mathcal{D}(A^q) \rightarrow L^q_\sigma(\mathcal{F}_0)$ such that for any $u \in \mathcal{D}(A^q)$,

$$A^q u = \mathcal{P}_{q, \mathcal{F}_0} \Delta u,$$

and

$$\mathcal{D}(A^q) = L^q_\sigma(\mathcal{F}_0) \cap \left\{ u \in W^{2,q}(\mathcal{F}_0) \text{ s.t. } D(u) \cdot \tau = -\alpha u \cdot \tau \text{ on } \partial \mathcal{F}_0 \right\}.$$

We define $(S(\mathbf{l}, \mathbf{r}), S_{pr}(\mathbf{l}, \mathbf{r})) \in \mathcal{L}(\mathbb{R}^2 \times \mathbb{R}; W^{2,q}(\mathcal{F}_0) \times \tilde{W}^{1,q}(\mathcal{F}_0))$ to be the solution (v, p) of

$$\begin{aligned} -\Delta v + \nabla p &= 0 & \text{for } x \in \mathcal{F}_0, \\ \operatorname{div} v &= 0 & \text{for } x \in \mathcal{F}_0, \\ v \cdot n &= (\mathbf{l} + \mathbf{r}x^\perp) \cdot n & \text{for } x \in \partial \mathcal{S}_0, \\ (D(v)n) \cdot \tau &= -\alpha(v - (\mathbf{l} + \mathbf{r}x^\perp)) \cdot \tau & \text{for } x \in \partial \mathcal{S}_0, \\ v \cdot n &= 0 & \text{for } x \in \partial \Omega, \\ (D(v)n) \cdot \tau &= -\alpha v \cdot \tau & \text{for } x \in \partial \Omega. \end{aligned}$$

We define the operator $\mathcal{A}_{FS} : \mathcal{D}(\mathcal{A}_{FS}) \rightarrow \mathcal{X}$ with

$$\mathcal{X} = L^q_\sigma(\mathcal{F}_0) \times \mathbb{R}^2 \times \mathbb{R},$$

$$\mathcal{D}(\mathcal{A}_{FS}) = \{(\mathcal{P}u, \mathbf{l}, \mathbf{r}) \in X \mid \mathcal{P}u - \mathcal{P}S_S(\mathbf{l}, \mathbf{r}) \in \mathcal{D}(A_q)\}, \quad (3.1.9)$$

and

$$\mathcal{A}_{FS} = \begin{pmatrix} A_q & -A_q \mathcal{P}S \\ \mathbb{K}^{-1} \mathcal{C}_1 & \mathbb{K}^{-1} \mathcal{C}_2 \end{pmatrix},$$

where \mathbb{K} is the mass plus the added mass matrix, i.e.

$$\mathbb{K} = \begin{bmatrix} m \operatorname{Id}_2 & 0 \\ 0 & \mathcal{J} \end{bmatrix} + \left(\int_{\mathcal{F}_0} \nabla \phi_i \cdot \nabla \phi_j dx \right)_{i,j=1,2,3}$$

with ϕ_i the Kirchhoff potentials defined as

$$\begin{cases} -\Delta \phi_i = 0 & \text{on } \mathcal{F}_0, \\ \nabla \phi_i \cdot n = K_i & \text{in } \partial \mathcal{S}_0, \\ \nabla \phi_i \cdot n = 0 & \text{in } \partial \Omega, \end{cases}$$

where $\phi_i : \mathcal{F}_0 \rightarrow \mathbb{R}$ and $K_i = e_i \cdot n$ for $i = 1, 2$ and $K_3 = x^\perp \cdot n$.

$$\mathcal{C}_1(\mathcal{P}v) = \begin{pmatrix} -2 \int_{\partial \mathcal{S}_0} D(\mathcal{P}v) n d\gamma + \int_{\partial \mathcal{S}_0} N(\Delta \mathcal{P}v \cdot n) n d\gamma \\ -2 \int_{\partial \mathcal{S}_0} y^\perp \cdot D(\mathcal{P}v) n d\gamma + \int_{\partial \mathcal{S}_0} y^\perp \cdot N(\Delta \mathcal{P}v \cdot n) n d\gamma \end{pmatrix},$$

$$\mathcal{C}_2 \begin{pmatrix} \mathfrak{l} \\ \mathfrak{r} \end{pmatrix} = \begin{pmatrix} -2 \int_{\partial \mathcal{S}_0} D((\text{Id} - \mathcal{P})S(\mathfrak{l}, \mathfrak{r}))nd\gamma \\ -2 \int_{\partial \mathcal{S}_0} y^\perp \cdot D((\text{Id} - \mathcal{P})S(\mathfrak{l}, \mathfrak{r}))nd\gamma, \end{pmatrix}$$

where $Nh = \phi$ is defined by $\Delta\phi = 0$ in \mathcal{F}_0 , $\frac{\partial\phi}{\partial n} = h$ on $\partial\mathcal{F}_0$.

Moreover we define $N_S(h) = N(\mathbf{1}_{\mathcal{S}_0}h)$ and $(\tilde{g}_1, \tilde{g}_2)^T = (\mathbb{K}^{-1}(g_1, g_2)^T)^T$.

Theorem 3.1.3. *Let $f \in L^p(0, T; L_\sigma^q(\mathcal{F}_0))$, $g_1 \in L^p(0, T; \mathbb{R}^2)$ and $g_2 \in L^p(0, T; \mathbb{R})$. Then $(v, \pi, \ell, \omega) \in L^p(0, T; W^{2,p}(\mathcal{F}_0)) \cap W^{1,p}(0, T; L_\sigma^q(\mathcal{F}_0)) \times L^p(0, T; \tilde{W}^{1,q}(\mathcal{F}_0)) \times L^p(0, T; \mathbb{R}^2) \times L^p(0, T; \mathbb{R})$ is a solution to the system (3.1.8) if and only if it satisfies*

$$\begin{aligned} \partial_t \begin{pmatrix} \mathcal{P}v \\ \ell \\ \omega \end{pmatrix} &= \mathcal{A}_{FS} \begin{pmatrix} \mathcal{P}v \\ \ell \\ \omega \end{pmatrix} + \begin{pmatrix} \mathcal{P}f \\ \tilde{g}_1 \\ \tilde{g}_2 \end{pmatrix}, & \begin{pmatrix} \mathcal{P}u(0) \\ \mathfrak{l}(0) \\ \omega(0) \end{pmatrix} &= \begin{pmatrix} \mathcal{P}v_0 \\ \ell_0 \\ \omega_0 \end{pmatrix}, \\ (I - \mathcal{P})v &= (I - \mathcal{P})S(\ell, \omega), \\ \pi &= N(\Delta\mathcal{P}v \cdot n) - N_S((\ell' + x^\perp\omega') \cdot n), \end{aligned}$$

where $(\tilde{g}_1, \tilde{g}_2)^T = \mathbb{K}^{-1}(g_1, g_2)^T$.

Proof. The proof is contained in [MT18, Section 3.1], in fact the only boundary condition that they use is $v \cdot n = (\ell + \omega x^\perp) \cdot n$, and the second one is not relevant. \square

Before moving to the proof of \mathcal{R} -sectoriality for the operator \mathcal{A}_{FS} we note that the map $(\mathcal{P}u, \ell, \omega) \mapsto \|\mathcal{P}u\|_{W^{2,q}(\mathcal{F}_0)} + \|\ell\|_{\mathbb{R}^2} + \|\omega\|_{\mathbb{R}}$ is a norm on $\mathcal{D}(\mathcal{A}_{FS}^q)$ equivalent to the graph norm. See for instance [Ray10][Proposition 3.3]. And the fact that $(S(\mathfrak{l}, \mathfrak{r}), S_{pr}(\mathfrak{l}, \mathfrak{r})) \in \mathcal{L}(\mathbb{R}^2 \times \mathbb{R}; W^{2,q}(\mathcal{F}_0) \times \tilde{W}^{1,q}(\mathcal{F}_0))$ is proved in [Bra19][Proposition 7.3].

3.1.2.6 \mathcal{R} -sectoriality of the operator \mathcal{A}_{FS}

To prove $L^p - L^q$ regularity we prove \mathcal{R} -sectoriality of the operator $-\mathcal{A}_{FS}$.

Theorem 3.1.4. *Let $1 < q < \infty$. Then $-\mathcal{A}_{FS} = -\mathcal{A}_{FS}^q$ is a \mathcal{R} -sectorial operator of angle $\theta < \pi/2$.*

Proof. To prove this theorem we just show that in some sense the operator \mathcal{A}_{FS} is a small perturbation of the operator A_q . To do so we write

$$\mathcal{A}_{FS} = \tilde{\mathcal{A}}_{FS} + \mathcal{B}_{FS}$$

where

$$\tilde{\mathcal{A}}_{FS} = \begin{pmatrix} A_q & -A_q\mathcal{P}S \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B}_{FS} = \begin{pmatrix} 0 & 0 \\ \mathbb{K}^{-1}\mathcal{C}_1 & \mathbb{K}^{-1}\mathcal{C}_2 \end{pmatrix}.$$

$-\tilde{\mathcal{A}}_{FS}$ is an \mathcal{R} -sectorial operator of angle $\theta < \pi/2$ on the same domain of \mathcal{A}_{FS} , in fact

$$\lambda(\lambda \text{Id} - \tilde{\mathcal{A}}_{FS})^{-1} = \begin{pmatrix} \lambda(\lambda \text{Id} - A_q)^{-1} & -\lambda(\lambda \text{Id} - A_q)^{-1}\mathcal{P}S + \mathcal{P}S \\ 0 & \text{Id} \end{pmatrix}$$

and the desired resolvent estimates follow by the \mathcal{R} -boundedness of the resolvent of A_q and the continuity of $\mathcal{P}S$. Finally \mathcal{C}_1 and \mathcal{C}_2 are linear and continuous operators with finite dimension

codomain. The proof is exactly the same as [MT18, Theorem 3.11], in fact the estimates are only based on the normal boundary condition and on the interior regularity (i.e. the fact that $u \in W^{2,q}(\mathcal{F}_0)$ or the fact that $\operatorname{div} u = 0$). This prove that \mathcal{B}_{FS} is a finite rank operator on $\mathcal{D}(\mathcal{A}_{FS})$, which implies that \mathcal{B}_{FS} is a $\tilde{\mathcal{A}}_{FS}$ -bounded operator with bound zero. To conclude the proof we apply Proposition B.0.1. \square

3.1.2.7 Fixed point argument

To perform the fixed point is now standard and can be done following exactly the same estimates of Section 5 and 6 of [GGH13]. Note that in [GGH13] they study the problem in dimension three so we have just to change the coefficients of the embedding.

We choose to present the main ingredients of the fixed point argument for the sake of completeness. Let start by introducing the following spaces to short the notation. We set

$$X_{p,q}^T = W^{1,p}(0, T; L^q(\mathcal{F}_0)) \cap L^p(0, T; W^{2,q}(\mathcal{F}_0)), \quad \text{and} \quad Y_{p,q}^T = L^p(0, T; \tilde{W}^{1,q}(\mathcal{F}_0)),$$

with norms respectively

$$\|\cdot\|_{X_{p,q}^T} = \|\cdot\|_{W^{1,p}(0,T;L^q(\mathcal{F}_0))} + \|\cdot\|_{L^p(0,T;W^{2,q}(\mathcal{F}_0))} \quad \text{and} \quad \|\cdot\|_{Y_{p,q}^T} = \|\cdot\|_{L^p(0,T;\tilde{W}^{1,q}(\mathcal{F}_0))}.$$

Moreover, the subset of functions of respectively $X_{p,q}^T$, $Y_{p,q}^T$ and $W^{1,p}(0, T)$ that are zero at initial time is denoted by

$$X_{p,q,0}^T, \quad Y_{p,q,0}^T \quad \text{and} \quad W^{1,p,0}(0, T).$$

To define the contractive map, we lift the initial condition as follow. Let $(v^*, \pi^*, \ell^*, \omega^*)$ the solution of

$$\begin{aligned} \frac{\partial v^*}{\partial t} - \Delta v^* + \nabla \pi^* &= 0 & \text{for } x \in \mathcal{F}_0, \\ \operatorname{div} v^* &= 0 & \text{for } x \in \mathcal{F}_0, \\ v^* \cdot n &= v_S^* \cdot n & \text{for } x \in \partial \mathcal{S}_0, \\ (D(v^*)n) \cdot \tau &= -\alpha(v^* - v_S^*) \cdot \tau & \text{for } x \in \partial \mathcal{S}_0, \\ v^* \cdot n &= 0 & \text{for } x \in \partial \Omega, \\ (D(v^*)n) \cdot \tau &= -\alpha v^* \cdot \tau & \text{for } x \in \partial \Omega, \\ m(\ell^*)'(t) &= - \int_{\partial \mathcal{S}_0} \Sigma(v^*, \pi^*) n \, ds, \\ \mathcal{J}(\omega^*)'(t) &= - \int_{\partial \mathcal{S}_0} (x - h(t))^\perp \cdot \Sigma(v^*, \pi^*) n \, ds, \\ v^*|_{t=0} &= v_0 & \text{for } x \in \mathcal{F}_0, \\ \ell^*(0) &= \ell_0, \quad \omega^*(0) = \omega_0. \end{aligned}$$

Moreover, from Theorem 3.1.2, it holds

$$\|v^*\|_{X_{p,q}^T} + \|\pi^*\|_{Y_{p,q}^T} + \|\ell^*\|_{W^{1,p}(0,T)} + \|\omega^*\|_{W^{1,q}(0,T)} \leq C \left(\|v_0\|_{B_{q,p}^{2-2/p}(\mathcal{F}_0)} + |\ell_0| + |\omega_0| \right).$$

Let $\hat{v} = v - v^*$, $\hat{\pi} = \pi - \pi^*$, $\hat{\ell} = \ell - \ell^*$ and $\hat{\omega} = \omega - \omega^*$, then the quadruple $(\hat{v}, \hat{\pi}, \hat{\ell}, \hat{\omega})$ satisfies

$$\begin{aligned}
\frac{\partial \hat{v}}{\partial t} - \Delta \hat{v} + \nabla \hat{\pi} &= F_0(\hat{v}, \hat{\pi}, \hat{\ell}, \hat{\omega}) && \text{for } x \in \mathcal{F}_0, \\
\operatorname{div} \hat{v} &= 0 && \text{for } x \in \mathcal{F}_0, \\
\hat{v} \cdot n &= \hat{v}_S \cdot n && \text{for } x \in \partial \mathcal{S}_0, \\
(D(\hat{v})n) \cdot \tau &= -\alpha(\hat{v} - \hat{v}_S) \cdot \tau && \text{for } x \in \partial \mathcal{S}_0, \\
\hat{v} \cdot n &= 0 && \text{for } x \in \partial \Omega, \\
(D(\hat{v})n) \cdot \tau &= -\alpha \hat{v} \cdot \tau && \text{for } x \in \partial \Omega, \\
m\hat{\ell}'(t) + \int_{\partial \mathcal{S}_0} \Sigma(\hat{v}, \hat{\pi})n \, ds &= F_1(\hat{v}, \hat{\pi}, \hat{\ell}, \hat{\omega}), \\
\mathcal{J}\hat{\omega}'(t) + \int_{\partial \mathcal{S}_0} (x - h(t))^\perp \cdot \Sigma(\hat{v}, \hat{\pi})n \, ds &= F_2(\hat{v}, \hat{\pi}), \\
\hat{v}|_{t=0} &= 0 && \text{for } x \in \mathcal{F}_0, \\
\hat{\ell}(0) &= 0, && \hat{\omega}(0) = 0.
\end{aligned} \tag{3.1.10}$$

where

$$\begin{aligned}
F_0(\hat{v}, \hat{\pi}, \hat{\ell}, \hat{\omega}) &= -\mathcal{N}(\hat{v} + v^*) - \mathcal{M}(\hat{v} + v^*) + (\mathcal{L} - \Delta)(\hat{v} + v^*) + (\mathcal{G} - \nabla)(\hat{\pi} - \pi^*) - \Delta v^*, \\
F_1(\hat{v}, \hat{\pi}, \hat{\ell}, \hat{\omega}) &= -m(\hat{\omega} - \omega^*)(\hat{\ell} + \ell^*)^\perp + \int_{\partial \mathcal{S}_0} (\Sigma - \mathcal{T})(\hat{v}, \hat{\pi})n, \\
F_2(\hat{v}, \hat{\pi}) &= \int_{\partial \mathcal{S}_0} x^\perp \cdot (\Sigma - \mathcal{T})(\hat{v}, \hat{\pi})n.
\end{aligned}$$

Note that the operators \mathcal{N} , \mathcal{M} , \mathcal{L} , \mathcal{G} and \mathcal{T} depends on the change of variables therefore on $\hat{\ell}$, ℓ^* , $\hat{\omega}$ and ω^* . To find the solution of (3.1.10), we perform a fixed point argument. Consider the set

$$\begin{aligned}
\mathcal{K}_{p,q}^T &= \{(v, \pi, \ell, \omega) \in X_{p,q}^T \times Y_{p,q}^T \times W^{1,p,0}(0, T) \text{ such that} \\
&\|v\|_{X_{p,q,0}^T} + \|\pi\|_{Y_{p,q,0}^T} + \|(\ell, \omega)\|_{W^{1,p,0}(0, T)} \leq R\},
\end{aligned}$$

where p, q satisfy $1/p + 1/q \leq 3/2$, $2/p + 1/q \neq 2$ and $2/p + 1/q \neq 1$. Let define the map $\phi_R^T : \mathcal{K}_{p,q}^T \rightarrow \mathcal{K}_{p,q}^T$ as follow

$$\phi_R^T : \begin{pmatrix} \tilde{v} \\ \tilde{\pi} \\ \tilde{\ell} \\ \tilde{\omega} \end{pmatrix} \mapsto \begin{pmatrix} F_0(\tilde{v}, \tilde{\pi}, \tilde{\ell}, \tilde{\omega}) \\ F_1(\tilde{v}, \tilde{\pi}, \tilde{\ell}, \tilde{\omega}) \\ F_2(\tilde{v}, \tilde{\pi}) \end{pmatrix} \mapsto \begin{pmatrix} v \\ \pi \\ \ell \\ \omega \end{pmatrix}$$

where (v, π, ℓ, ω) is the solution of problem (3.1.8) with right hand side F_0 , F_1 and F_2 and zero initial datum. In the remaining part of the subsection we prove, for small enough $T, R > 0$, that $\phi_R^T(\mathcal{K}_{p,q}^T) \subset \mathcal{K}_{p,q}^T$, and ϕ_R^T is a contraction.

The first ingredient of the proof are some estimates on the change of variables. This correspond to Lemma 6.5 of [GGH13].

Lemma 3.1.1. *For $T > 0$ and for $h = 1, 2$, we consider two triples $(\ell_h, \omega_h) \in W^{1,p,0}(T)$ such that $\|(\ell_h, \omega_h)\|_{W^{1,p,0}(T)} \leq R$ and X_h, Y_h the change of variables associated with $(\ell_h + \ell^*, \omega_h + \omega^*)$. Then it holds*

$$\|\partial_\alpha g_h^{ij}\|_{L^\infty(0,T) \times \mathcal{F}_0} + \|\partial_\alpha g_h^{ij}\|_{L^\infty(0,T) \times \mathcal{F}_0} + \|\partial^\alpha (\Gamma_h)_{jk}^i\|_{L^\infty(0,T) \times \mathcal{F}_0} \leq C$$

and

$$\begin{aligned} \|\partial_\alpha (g_1^{ij} - g_2^{ij})\|_{L^\infty(0,T) \times \mathcal{F}_0} + \|\partial_\alpha (g_1^{ij} - g_2^{ij})\|_{L^\infty(0,T) \times \mathcal{F}_0} + \|\partial^\alpha ((\Gamma_h)_{jk}^1 - (\Gamma_h)_{jk}^2)\|_{L^\infty(0,T) \times \mathcal{F}_0} \\ \leq CT \|(\ell_1 - \ell_2, \omega_1 - \omega_2)\|_{L^\infty(0,T)} \end{aligned}$$

for all multi-indices alpha with $0 \leq |\alpha| \leq 1$ and the constant C depends on R and the norm of the initial datum.

The second ingredient is the embedding theorem Proposition 4.3 of [GGH13].

Theorem 3.1.3. *Let $\mathcal{O} \subset \mathbb{R}^2$ an open, connected, bounded set with smooth boundary. Let $p, q \in (1, \infty)$, $\alpha \in (0, 1)$ and $T_0 > 0$. Then*

$$X_{p,q}^{T_0} \hookrightarrow W^{\alpha,p}(0, T_0; W^{2(1-\alpha),q}(\mathcal{O})).$$

In particular, if $r, s \in (1, \infty) \cup \{\infty\}$, $\mu \in \{0, 1\}$ and

$$\frac{2-\mu}{2} + \frac{1}{r} - \frac{1}{q} \geq \frac{1}{p} - \frac{1}{s},$$

then for $T \in (0, T_0)$

$$X_{p,q}^{T_0} \hookrightarrow L^s(0, T_0; W^{\mu,r}(\mathcal{O})).$$

Moreover, there exists a constant $C_0 = C(T_0)$, independent of $T \in (0, T_0)$, such that the estimate

$$\|v\|_{L^s(0,T_0;W^{\mu,r}(\mathcal{O}))} \leq C \|v\|_{X_{p,q}^{T_0}}$$

holds true for all $u \in X_{p,q,0}^T$.

With the help of these last two ingredients we will provide estimates for \mathcal{M} , \mathcal{N} , $\mathcal{L} - \Delta$ and $\mathcal{J}(\Sigma - \mathcal{T})$ where

$$\mathcal{J} : W^{1,p}(\mathcal{F}_0; \mathbb{R}^{2 \times 2}) \rightarrow \mathbb{R}^3, \quad h \mapsto \begin{pmatrix} \int_{\partial S_0} hn \\ \int_{\partial S_0} x^\perp \cdot hn \end{pmatrix},$$

in particular it holds $|\mathcal{J}(u)| \leq C \|g\|_{W^{1,p}(\mathcal{F}_0; \mathbb{R}^{2 \times 2})}$. The inequalities correspond to the one in Lemma 6.6 of [GGH13].

Lemma 3.1.2. *Let T_0 , let $1/p + 1/q \leq 3/2$, let $s = 3/p$, $s' = 3p/2$, $r = 3q$, $r' = 3q/2$, finally for $0 \leq T \leq T_0$ we denote by*

$$C_*(T) = \|\nabla v^*\|_{L^{s'}(0,T;L^{r'}(\mathcal{F}_0))} + \|v^*\|_{L^s(0,T;L^r\mathcal{F}_0)}.$$

Then the following holds

$$\begin{aligned}\|\mathcal{M}v\|_{L^p(0,T;L^q(\mathcal{F}_0))} &\leq C(T^{1/2} + T^{1/p})\|v\|_{X_{p,q}^T}, \\ \|\mathcal{N}v\|_{L^p(0,T;L^q(\mathcal{F}_0))} &\leq C[(R + C_*(T))^2 + T], \\ \|(\mathcal{L} - \Delta)v\|_{L^p(0,T;L^q(\mathcal{F}_0))} &\leq C(T + T^{1/2} + T^{1/p})\|v\|_{X_{p,q}^T}, \\ \|(\mathcal{G} - \nabla)\pi\|_{L^p(0,T;L^q(\mathcal{F}_0))} &\leq CT\|p\|_{Y_{p,q}^T}.\end{aligned}$$

Moreover, we obtain

$$\|\mathcal{J}((\Sigma - \mathcal{T})(v, \pi))\|_{L^p(0,T)} \leq CT\|v\|_{X_{p,q}^T}.$$

Proof. The proof is based on the estimates for the change of variables from Lemma 3.1.1 and the embedding from Theorem 3.1.3. In particular if we apply Lemma 3.1.1 to (ℓ_1, ω_1) and $(\ell_2, \omega_2) = (-\ell^*, -\omega^*)$, we have

$$\|g_1^{ij} - \delta_{ij}\|_{L^\infty(0,T;L^\infty(\mathcal{F}_0))} \leq CT\|(\ell_1 + \ell^*, \omega_1 + \omega^*)\|_{W^{1,p}(0,T)} \leq CT(R + C_0).$$

And from Theorem 3.1.3, we have the embedding

$$\begin{aligned}X_{p,q}^T &\hookrightarrow L^s(0, T; L^r(\mathcal{F}_0)), \quad X_{p,q}^T \hookrightarrow L^{s'}(0, T; W^{1,r'}(\mathcal{F}_0)), \\ &\text{and } X_{p,q}^T \hookrightarrow L^k(0, T; W^{1,q}(\mathcal{F}_0))\end{aligned}$$

where $k = \infty$ if $p \geq 2$ and $1/k = 1/p - 1/2$ otherwise. In the estimates the difficult terms are the one with the higher other derivatives, i.e. $\mathcal{L} - \Delta$ and $\mathcal{G} - \nabla$ and the nonlinear term \mathcal{N} . The proof is exactly the same of the one in [GGH13], so to be more instructive, we treat only the terms $\mathcal{G} - \nabla$ and \mathcal{N} .

$$\|(\mathcal{G} - \nabla)\pi\|_{L^p(0,T;L^q(\mathcal{F}_0))} \leq C \sup_{j,k} \|g^{ij} - \delta_{ij}\|_{L^\infty(0,T;L^\infty(\mathcal{F}_0))} \leq CT\|\pi\|_{Y_{p,q}^T}.$$

and

$$\begin{aligned}\|\mathcal{N}v\|_{L^p(0,T;L^q(\mathcal{F}_0))} &\leq \|(v \cdot \nabla)v\|_{L^p(0,T;L^q(\mathcal{F}_0))} + \sup_{i,j,k} \|\Gamma_{jk}^i\|_{L^\infty(0,T;L^\infty(\mathcal{F}_0))} \|v\|_{L^{2q}(0,T;L^{2q}(\mathcal{F}_0))}^2 \\ &\leq \|v\|_{L^r(0,T;L^s(\mathcal{F}_0))} \|\nabla v\|_{L^{r'}(0,T;L^{s'}(\mathcal{F}_0))} + CT^{2-4p/k} \|v\|_{L^l(0,T;L^{2q}(\mathcal{F}_0))} \\ &\leq C[(R + C_*(T))^2 + T]\end{aligned}$$

where $1/k = 1/p + 1/2q - 1$. With this choice of k , it holds $X_{p,q}^T \hookrightarrow L^k(0, T; L^{2q}(\mathcal{F}_0))$, $k > 2p$ and $2 - 4p/k \geq 1$ by using the hypothesis $1/p + 1/q \leq 3/2$. \square

At this point we have all the instruments to prove that ϕ_R^T maps \mathcal{K}_R^T into itself. The following is Lemma 6.7 of [GGH13].

Lemma 3.1.3. *For T and R enough small we have $\phi_R^T(\mathcal{K}_R^T) \subset \mathcal{K}_R^T$.*

Proof. By Theorem 3.1.2, we know that the norm of $(v, \pi, \ell, \omega) = \phi_R^T(\tilde{v}, \tilde{\pi}, \tilde{\ell}, \tilde{\omega})$ depend on the norm of the initial data, which in our case is zero, and on the source terms F_0, F_1 and F_2 . It

is then enough to prove that F_0 , F_1 and F_2 are enough small. As before we choose only one term as example. Consider

$$\begin{aligned} \|F_0(\tilde{v}, \tilde{\pi}, \tilde{\ell}, \tilde{\omega})\|_{L^p(0,T;L^q(\mathcal{F}_0))} &\leq \|-\mathcal{N}(\hat{v} + v^*) - \mathcal{M}(\hat{v} + v^*) + (\mathcal{L} - \Delta)(\hat{v} + v^*) \\ &\quad + (\mathcal{G} - \nabla)(\hat{\pi} - \pi^*) - \Delta v^*\|_{L^p(0,T;L^q(\mathcal{F}_0))} \\ &\leq C[T^{1/2} + T^{1/p} + (R + C_*(T))^2]. \end{aligned}$$

After performing similar estimates on F_1 and F_2 , we obtain

$$\begin{aligned} \|\phi_R^T(\tilde{v}, \tilde{\pi}, \tilde{\ell}, \tilde{\omega})\|_{\mathcal{K}_R^T} &\leq C(\|F_0(\tilde{v}, \tilde{\pi}, \tilde{\ell}, \tilde{\omega})\|_{L^p(0,T;L^q(\mathcal{F}_0))} + \|F_1(\tilde{v}, \tilde{\pi}, \tilde{\ell}, \tilde{\omega})\|_{L^p(0,T)} \\ &\quad + \|F_2(\tilde{v}, \tilde{\pi})\|_{L^p(0,T)}) \\ &\leq C[T^{1/2} + T^{1/p} + (R + C_*(T))^2] \\ &\leq R, \end{aligned}$$

when we choose R and T small enough. Note that $C_*(T)$ converge to zero as T goes to zero. \square

The last step of the proof is to show that the map ϕ_R^T is a contraction, which correspond to Lemma 6.8 of [GGH13]

Lemma 3.1.4. *For R and T enough small the map ϕ_R^T is contractive.*

Proof. The system (3.1.10) is linear which implies that is enough to estimates the difference of the source term. To do that we notice that the operator \mathcal{M} , \mathcal{N} , \mathcal{L} and \mathcal{G} depends on the solution itself. Let $(\tilde{v}_1, \tilde{\pi}_1, \tilde{\ell}_1, \tilde{\omega}_1)$ and $(\tilde{v}_2, \tilde{\pi}_2, \tilde{\ell}_2, \tilde{\omega}_2)$ in \mathcal{K}_R^T . One of the term to estimate is

$$\begin{aligned} \|(\mathcal{G}_1 - \nabla)\tilde{\pi}_1 - (\mathcal{G}_2 - \nabla)\tilde{\pi}_2\|_{L^p(0,T;L^q(\mathcal{F}_0))} &\leq \|(\mathcal{G}_1 - \nabla)(\tilde{\pi}_1 - \tilde{\pi}_2)\|_{L^p(0,T;L^q(\mathcal{F}_0))} \\ &\quad + \|(\mathcal{G}_1 - \mathcal{G}_2)\tilde{\pi}_1\|_{L^p(0,T;L^q(\mathcal{F}_0))} \\ &\leq CT\|\tilde{\pi}_1 - \tilde{\pi}_2\|_{Y_R^T} \\ &\quad + CT\|(\tilde{\ell}_1 - \tilde{\ell}_2, \tilde{\omega}_1 - \tilde{\omega}_2)\|_{W^{1,p}(0,T)}. \end{aligned}$$

Where for the term $\mathcal{G}_1 - \mathcal{G}_2$ we use Lemma 3.1.1. The other difficult term is \mathcal{N} , but in this case we prefer not to treat it. To conclude, following Lemma 6.8 of [GGH13] we prove

$$\begin{aligned} \|\phi_R^T(\tilde{v}_1, \tilde{\pi}_1, \tilde{\ell}_1, \tilde{\omega}_1) - \phi_R^T(\tilde{v}_2, \tilde{\pi}_2, \tilde{\ell}_2, \tilde{\omega}_2)\|_{\mathcal{K}_R^T} &\leq \\ &\leq L_{R,T}\|(\tilde{v}_1, \tilde{\pi}_1, \tilde{\ell}_1, \tilde{\omega}_1) - (\tilde{v}_2, \tilde{\pi}_2, \tilde{\ell}_2, \tilde{\omega}_2)\|_{\mathcal{K}_R^T}, \end{aligned}$$

where $L_{R,T}$ converge to zero as R and T converges to zero. Which implies ϕ_R^T is a contraction. \square

Theorem 3.1.1 is proved. We move to the existence result for a similar system.

3.1.2.8 Special system

In this subsection we study an auxiliary system that we will use in the next Chapter. Let $l \in C(0, T; \mathbb{R}^2)$, let $r \in C(0, T; \mathbb{R})$ such that $\mathcal{S}(t) = \mathcal{S}^{l,r}(t) \Subset \Omega$ and let $\mathcal{F}(t) = \Omega \setminus \mathcal{S}(t)$. Consider the system in the unknowns $(v, p, \mathfrak{l}, \mathfrak{r})$

$$\begin{aligned}
\frac{\partial v}{\partial t} - \Delta v + \nabla p &= f && \text{for } x \in \mathcal{F}(t), \\
\operatorname{div} v &= 0 && \text{for } x \in \mathcal{F}(t), \\
v \cdot n &= v_S \cdot n && \text{for } x \in \partial \mathcal{S}(t), \\
(D(v)n) \cdot \tau &= -\alpha(v - v_S) \cdot \tau && \text{for } x \in \partial \mathcal{S}(t), \\
v \cdot n &= 0 && \text{for } x \in \partial \Omega, \\
(D(v)n) \cdot \tau &= -\alpha v \cdot \tau && \text{for } x \in \partial \Omega, \\
m\mathfrak{l}'(t) &= - \int_{\partial \mathcal{S}(t)} \Sigma(v, p)n \, ds + ml(t), \\
\mathcal{J}\mathfrak{r}'(t) - \mathcal{J}r(t) &= - \int_{\partial \mathcal{S}(t)} (x - h(t))^\perp \cdot \Sigma(v, p)n \, ds, \\
v|_{t=0} &= 0 && \text{for } x \in \mathcal{F}_0, \\
\mathfrak{l}(0) &= 0, && \mathfrak{r}(0) = 0.
\end{aligned} \tag{3.1.11}$$

The above system is characterized by a time dependent domain which, in contrast to (3.0.1)-(3.0.9), does not depend on the solution itself. Moreover we will prove short in time existence with 0 initial datum. These makes the analysis easier.

Theorem 3.1.5. *Let $f \in L^{4/3}(0, T; L^{4/3}(\Omega))$. Then for some $\tilde{T} \in (0, T)$ there exists a unique solution of (3.1.11) such that*

$$\begin{aligned}
&\|v\|_{L^{4/3}(0, \tilde{T}; W^{2,4/3}(\mathcal{F}(\cdot)))} + \|v\|_{W^{1,4/3}(0, \tilde{T}; L^{4/3}(\mathcal{F}(\cdot)))} + \|p\|_{L^{4/3}(0, \tilde{T}; \tilde{W}^{1,4/3}(\mathcal{F}(\cdot)))} \\
&\quad + \|\mathfrak{l}\|_{W^{1,4/3}(0, \tilde{T}; \mathbb{R}^2)} + \|\mathfrak{r}\|_{W^{1,4/3}(0, \tilde{T}; \mathbb{R})} \leq C(\|f\|_{L^{4/3}(0, \tilde{T}; L^{4/3}(\Omega))} \\
&\quad + \|l\|_{L^{4/3}(0, \tilde{T}; \mathbb{R}^2)} + \|r\|_{L^{4/3}(0, \tilde{T}; \mathbb{R})})
\end{aligned}$$

The proof of the theorem follow the same strategy of the Theorem 3.1.1. Moreover the fixed point argument is easier due to the fact that the domain does not depend on the solution.

3.2 Full space and infinite energy setting

We study the Cauchy problem (3.0.1)-(3.0.9) when we allow the fluid kinetic energy to be infinite. In the case where the fluid is alone and satisfies the two dimensional Navier-Stokes equations (2.0.2), well-posedness of (2.0.2) for solutions with possibly infinite energy has been studied a lot in the past years. We recall the work [GMO88], where the authors prove existence for initial data which have measure vorticity and the corresponding uniqueness result is available in [GG05]. Other interesting works are [LR02] and [MMP19], where the authors prove existence of weak solutions in loc-uniform Lebesgue spaces. The first result deals with solutions defined in the three dimensional space, the second one defined in the half space \mathbb{R}_+^3 . For exterior domains, where no slip boundary condition are prescribed on the boundary, it was

proved in [KY95] an existence result for initial data in the weak- L^2 space with some restriction on the concentration of the initial energy. These solutions will remain uniformly bounded in weak- L^2 norm for almost every time and bounded in the K_4 norm which is the Kato norm for $p = 4$.

Regarding the fluid structure problem (3.0.1)-(3.0.9), in [PS14], the existence of global weak solutions with finite energy were established in the case where the whole system occupies all \mathbb{R}^3 . Moreover, several properties of these solutions were exhibited. We consider here the 2D case, for which our analysis can be carried out for initial data corresponding to unbounded fluid kinetic energy. More precisely we will study the Cauchy problem (3.0.1)-(3.0.9) when Ω is the two dimensional space \mathbb{R}^2 . The system is characterized by condition (3.0.5) and not (3.0.4) and the fluid velocity is not in general square integrable.

To study the system we use a change of variables to move it in the body frame. Let $h(t)$ the position of the center of mass of the solid $\mathcal{S}(t)$ and $Q(t)$ the rotation matrix and let u a smooth solution of (3.0.1)-(3.0.9) without (3.0.4). Then the functions

$$\begin{aligned} v(t, x) &= Q^T(t)u(t, h(t) + Q^T(t)x), & q(t, x) &= p(t, h(t) + Q^T(t)x), \\ \ell(t) &= Q^T(t)h'(t) & \text{and} & \quad r(t) = r(t) \end{aligned}$$

satisfy the system

$$\begin{aligned} \frac{\partial v}{\partial t} + [(v - \ell - rx^\perp) \cdot \nabla] v + rv^\perp + \nabla q &= \Delta v & \text{for } x \in \mathcal{F}_0, & \\ \operatorname{div} v &= 0 & \text{for } x \in \mathcal{F}_0, & \end{aligned} \quad (3.2.1)$$

$$\begin{aligned} v \cdot n &= (\ell + rx^\perp) \cdot n & \text{for } x \in \partial\mathcal{S}_0, & \\ (D(v)n) \cdot \tau &= -\alpha(v - \ell - rx^\perp) \cdot \tau & & \\ |v| &\longrightarrow 0 & \text{as } |x| \longrightarrow \infty, & \end{aligned} \quad (3.2.2)$$

$$m\ell'(t) = - \int_{\partial\mathcal{S}_0} \sigma n \, ds - mr\ell^\perp,$$

$$\mathcal{J}r'(t) = - \int_{\partial\mathcal{S}_0} x^\perp \cdot \sigma n \, ds,$$

$$\begin{aligned} v(0, x) &= v_0(x) & x \in \mathcal{F}_0, & \\ \ell(0) &= \ell_0, \quad r(0) = r_0, & & \end{aligned} \quad (3.2.3)$$

where n and τ are the unit outwards normal and counterclockwise tangent vectors to the boundary of \mathcal{F}_0 and σ is the stress tensor equal to $-q\operatorname{Id}_2 + 2D(v)$.

3.2.1 Leray-type solutions with infinite energy

We are interested in solution with initial data (v_0, ℓ_0, r_0) with fluid velocity of the form

$$v_0 = \tilde{v}_0 + \beta H_{\mathcal{S}_0} \in L^2(\mathcal{F}_0) \oplus \mathbb{R}H_{\mathcal{S}_0},$$

where v_0 is divergence free in a distributional sense and $H_{\mathcal{S}_0}$ is the unique solution of

$$\begin{aligned} \operatorname{div} H_{\mathcal{S}_0} &= 0 & \text{for } x \in \mathcal{F}_0, \\ \operatorname{curl} H_{\mathcal{S}_0} &= 0 & \text{for } x \in \mathcal{F}_0, \\ H_{\mathcal{S}_0} \cdot n &= 0 & \text{for } x \in \partial\mathcal{S}_0, \\ \int_{\partial\mathcal{S}_0} H_{\mathcal{S}_0} \cdot \tau \, ds &= 1, \\ |H_{\mathcal{S}_0}| &\longrightarrow 0 & \text{as } |x| \longrightarrow \infty. \end{aligned}$$

See, for instance, [Kik83]. This solution is smooth and decays like $1/|x|$ at infinity. For any x_0 in the interior of \mathcal{S}_0 , we also have

$$\begin{aligned} H_{\mathcal{S}_0}, \nabla H_{\mathcal{S}_0} &\in L^\infty(\mathcal{F}_0) \quad \text{and} \tag{3.2.4} \\ H_{\mathcal{S}_0} - \frac{(x-x_0)^\perp}{2\pi|x-x_0|^2}, \nabla H_{\mathcal{S}_0}, H_{\mathcal{S}_0}^\perp - (x-x_0)^\perp \cdot \nabla H_{\mathcal{S}_0} &\in L^2(\mathcal{F}_0), \end{aligned}$$

but $H_{\mathcal{S}_0}$ is not a L^2 function. In the case of regular solutions to the Euler equations this vector field is useful to take the velocity circulation around the body into account, which is a conserved quantity according to Kelvin's theorem.

First of all we note that the couple $(u, q) = (H_{\mathcal{S}_0}, -|H_{\mathcal{S}_0}|^2/2)$, satisfies the equations (3.2.1)-(3.2.2) unless the second boundary condition (3.2.2) for ℓ and r equal to zero. This leads us to expect that a solution (v, ℓ, r, q) of (3.2.1)-(3.2.3) with initial data $(\tilde{v}_0 + \beta H_{\mathcal{S}_0}, \ell_0, r_0)$ has fluid velocity v of the form

$$v = \tilde{v} + \beta H_{\mathcal{S}_0}, \quad \text{with } \tilde{v} \in L^2(\mathcal{F}_0)$$

and β is independent of time.

We now introduce a definition of Leray-type solutions for these initial data. First of all in the literature, for example in [PS14], there is already a definition of weak solutions of Leray-type with finite energy, i.e. with $\beta = 0$, so we want to be coherent with this definition. In the next subsection we recall the definition of weak solution with finite energy coming from [PS14] and then we notice that we can extend this definition in a straightforward way to our setting.

3.2.2 A weak formulation with finite energy

We use the notation \mathcal{H} for the space $\mathcal{H}_{\mathcal{S}_0}$ defined in (3.1.1) and we denote by

$$\begin{aligned} \underline{\mathcal{V}} &= \left\{ \phi \in \mathcal{H} \mid \int_{\mathcal{F}_0} |\nabla \phi(y)|^2 dy < +\infty \right\}, \\ \mathcal{V} &= \left\{ \phi \in \mathcal{H} \mid \int_{\mathcal{F}_0} |\nabla \phi(y)|^2 (1 + |y|^2) dy < +\infty \right\}, \\ \widehat{\mathcal{V}} &= \left\{ \phi \in \mathcal{V} \mid \phi|_{\mathcal{F}_0} \in \operatorname{Lip}(\overline{\mathcal{F}_0}) \right\}, \end{aligned}$$

with norms respectively

$$\begin{aligned} \|\phi\|_{\underline{\mathcal{V}}} &= \|\phi\|_{\mathcal{H}} + \|\nabla \phi\|_{L^2(\mathcal{F}_0, dy)}, & \|\phi\|_{\mathcal{V}} &= \|\phi\|_{\mathcal{H}} + \|\nabla \phi\|_{L^2(\mathcal{F}_0, (1+|y|^2)^{\frac{1}{2}} dy)}, \\ \text{and } \|\phi\|_{\widehat{\mathcal{V}}} &= \|\phi\|_{\mathcal{V}} + \|\phi\|_{\operatorname{Lip}(\overline{\mathcal{F}_0})}. \end{aligned}$$

Let emphasize that $\widehat{\mathcal{V}} \subset \mathcal{V} \subset \underline{\mathcal{V}}$. We define formally for appropriate u and v ,

$$\begin{aligned} a(u, v) &= -\alpha \int_{\partial \mathcal{S}_0} (u - u_{\mathcal{S}}) \cdot (v - v_{\mathcal{S}}) - \int_{\mathcal{F}_0} D(u) : D(v) \\ b(u, v, w) &= \int_{\mathcal{F}_0} \left([(u - u_{\mathcal{S}}) \cdot \nabla w] \cdot v - r_u v^\perp \cdot w \right) - m r_u \ell_v^\perp \cdot \ell_w. \end{aligned}$$

The next proposition clarify in which spaces a and b are defined.

Proposition 3.2.1. *The following holds true:*

- i. b is a trilinear continuous map from $\underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \mathcal{V}$ to \mathbb{R} , i.e. there exists a constant $C > 0$ such that for any $(u, v, w) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \mathcal{V}$,*

$$|b(u, v, w)| \leq C \|u\|_{\underline{\mathcal{V}}} \|v\|_{\underline{\mathcal{V}}} \|w\|_{\mathcal{V}}.$$

Moreover if $v \in \underline{\mathcal{V}}$ it holds $b(u, v, v) = 0$ and if $v, w \in \mathcal{V}$, it holds $b(u, v, w) = -b(u, w, v)$.

- ii. b can be extended to a continuous map from $\mathcal{H} \times \mathcal{H} \times \widehat{\mathcal{V}}$ to \mathbb{R} , i.e. there exists a constant $C > 0$ such that for any $(u, v, w) \in \mathcal{H} \times \mathcal{H} \times \widehat{\mathcal{V}}$,*

$$|b(u, v, w)| \leq C \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \|w\|_{\widehat{\mathcal{V}}}.$$

- iii. $a(\cdot, \cdot)$ is a continuous map from $\underline{\mathcal{V}} \times \underline{\mathcal{V}}$ to \mathbb{R} , i.e. for any u, v in $\underline{\mathcal{V}}$,*

$$|a(u, v)| \leq C \|u\|_{\underline{\mathcal{V}}} \|v\|_{\underline{\mathcal{V}}}.$$

Proof. We present here only the proof of point *i.*

$$\begin{aligned} |b(u, v, w)| &= \left| \int_{\mathcal{F}_0} \left([(u - u_{\mathcal{S}}) \cdot \nabla w] \cdot v - r_u v^\perp \cdot w \right) - m r_u \ell_v^\perp \cdot \ell_w \right| \\ &\leq \left| \int_{\mathcal{F}_0} \left([(u \cdot \nabla) w] \cdot v - r_u v^\perp \cdot w \right) - m r_u \ell_v^\perp \cdot \ell_w \right| \end{aligned} \quad (3.2.5)$$

$$+ \left| \int_{\mathcal{F}_0} [(u_{\mathcal{S}} \cdot \nabla) w] \cdot v \right|. \quad (3.2.6)$$

We separately estimates (3.2.5) and (3.2.6). By Hölder and interpolation inequalities, it holds

$$(3.2.5) \leq C \|u\|_{\underline{\mathcal{V}}} \|v\|_{\underline{\mathcal{V}}} \|w\|_{\underline{\mathcal{V}}} \leq C \|u\|_{\underline{\mathcal{V}}} \|v\|_{\underline{\mathcal{V}}} \|w\|_{\mathcal{V}},$$

and by the fact that $u_{\mathcal{S}} = \ell_u + r_u x^\perp$, we have

$$\begin{aligned} (3.2.6) &= \left| \int_{\mathcal{F}_0} [((\ell_u + r_u x^\perp) \cdot \nabla) w] \cdot v \right| \\ &\leq C |\ell_u| \|\nabla w\|_{L^2(\mathcal{F}_0)} \|v\|_{L^2(\mathcal{F}_0)} + C |r_u| \| |x| \nabla w \|_{L^2(\mathcal{F}_0)} \|v\|_{L^2(\mathcal{F}_0)} \\ &\leq C \|u\|_{\underline{\mathcal{V}}} \|v\|_{\underline{\mathcal{V}}} \|w\|_{\underline{\mathcal{V}}} + C \|u\|_{\underline{\mathcal{V}}} \|v\|_{\underline{\mathcal{V}}} \|w\|_{\mathcal{V}}. \end{aligned}$$

□

We are now able to state the definition of weak solution defined in [PS14].

Definition 3.2.1. Let $v_0 \in \mathcal{H}$, we say that $v \in C(0, T; \mathcal{H}) \cap L^2(0, T; \underline{\mathcal{V}})$ is a weak solution of (3.2.1)-(3.2.3) with finite energy if for all $\varphi \in C_w^\infty([0, T]; \mathcal{H})$ such that $\varphi|_{\mathcal{F}_0} \in C^\infty(0, T; C_c^\infty(\overline{\mathcal{F}_0}))$ and for a.e. $t \in [0, T]$ it holds

$$(v, \varphi)_{\mathcal{H}}(t) - (v_0, \varphi|_{t=0})_{\mathcal{H}} = \int_0^t \left[(v, \partial_t \varphi)_{\mathcal{H}} + 2a(v, \varphi) - b(v, \varphi, v) \right].$$

3.2.3 A weak formulation with infinite energy

To extend the definition of weak solution in the case of unbounded energy we start with noticing that we can continuously extend the map a and b in our new setting. First of all for X one of the spaces \mathcal{H} , $\underline{\mathcal{V}}$, \mathcal{V} or $\widehat{\mathcal{V}}$, the space $X \oplus \mathbb{R}H$ is endowed with the norm $\|u\|_{X \oplus \mathbb{R}H} = \|\tilde{u} + \beta H\|_{X \oplus \mathbb{R}H} = \|\tilde{u}\|_X + |\beta|$, moreover we use the convention that $u_S = \tilde{u}_S$, $l_u = l_{\tilde{u}}$ and $r_u = r_{\tilde{u}}$, i.e. we extend the function H by 0 inside the solid \mathcal{S}_0 .

Proposition 3.2.2. *The map a and b can be linearly extended as follow:*

- i. *the map b can be continuously extended to a trilinear map on $(\underline{\mathcal{V}} \oplus \mathbb{R}H) \times \underline{\mathcal{V}} \times (\mathcal{V} \oplus \mathbb{R}H)$ by*

$$b(u, \tilde{v}, w) = \int_{\mathcal{F}_0} \left([(u - u_S) \cdot \nabla w] \cdot \tilde{v} - r_u \tilde{v}^\perp \cdot w \right) - m r_u \ell_{\tilde{v}}^\perp \cdot \ell_w.$$

The continuity assumption is equivalent to the following inequality : there exists a constant $C > 0$ such that for any $(u = \tilde{u} + \beta_1 H, \tilde{v}, w = \tilde{w} + \beta_3 H) \in (\underline{\mathcal{V}} \oplus \mathbb{R}H) \times \underline{\mathcal{V}} \times (\mathcal{V} \oplus \mathbb{R}H)$,

$$|b(u, \tilde{v}, w)| \leq C(\|\tilde{u}\|_{\underline{\mathcal{V}}} + |\beta_1|) \|\tilde{v}\|_{\underline{\mathcal{V}}} (\|\tilde{w}\|_{\mathcal{V}} + |\beta_3|).$$

- ii. *The map $b(H, \cdot, \cdot)$, $b(\cdot, \cdot, H)$ are continuous bilinear map from $\underline{\mathcal{V}} \times \underline{\mathcal{V}}$ to \mathbb{R} and by Blasius lemma $b(H, \tilde{v}, H) = 0$ for any $\tilde{v} \in \underline{\mathcal{V}}$ (we refer to [GLS16, Lemma A.1] for the Blasius lemma).*

- iii. *For $u \in \underline{\mathcal{V}} \oplus \mathbb{R}H$ and $\tilde{v} \in \underline{\mathcal{V}}$, we have $b(u, \tilde{v}, \tilde{v}) = 0$. Moreover if $\tilde{v}, \tilde{w} \in \mathcal{V}$, it holds $b(u, \tilde{v}, \tilde{w}) = -b(u, \tilde{w}, \tilde{v})$.*

- iv. *The trilinear map b can be extended in a unique way on $(\mathcal{H} \oplus \mathbb{R}H) \times \mathcal{H} \times (\widehat{\mathcal{V}} \oplus \mathbb{R}H)$ in a continuous way, i.e. there exists a constant $C > 0$ such that for any $(u, \tilde{v}, w) = (\tilde{u} + \beta_1 H, \tilde{v}, \tilde{w} + \beta_3 H)$,*

$$|b(u, \tilde{v}, w)| \leq C(\|\tilde{u}\|_{\mathcal{H}} + |\beta_1|) \|\tilde{v}\|_{\mathcal{H}} (\|\tilde{w}\|_{\widehat{\mathcal{V}}} + |\beta_3|).$$

- v. *$a(\cdot, \cdot)$ can be extended to a continuous bilinear map from $(\underline{\mathcal{V}} \oplus \mathbb{R}H) \times \underline{\mathcal{V}}$ to \mathbb{R} , where for any (u, \tilde{v})*

$$a(u, \tilde{v}) = -\alpha \int_{\partial \mathcal{S}_0} (u - u_S) \cdot (\tilde{v} - \tilde{v}_S) - \int_{\mathcal{F}_0} D(u) : D(\tilde{v}).$$

Proof. Point i. is direct consequence of point ii. so we begin by ii. For $(\tilde{u}, \tilde{v}) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}}$,

$$b(\tilde{u}, \tilde{v}, H) = \int_{\mathcal{F}_0} [\tilde{u} \cdot \nabla H] \cdot \tilde{v} - \int_{\mathcal{F}_0} [\ell_{\tilde{u}} \cdot \nabla H] \cdot \tilde{v} - r_{\tilde{u}} \int_{\mathcal{F}_0} (x^\perp \cdot \nabla H - H^\perp) \cdot \tilde{v}$$

is well defined thanks to the properties (3.2.4), moreover there exists $C > 0$ such that $|b(\tilde{u}, \tilde{v}, H)| \leq C\|\tilde{u}\|_{\underline{\mathcal{V}}}\|\tilde{v}\|_{\underline{\mathcal{V}}}$. For $(\tilde{v}, \tilde{w}) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}}$, $b(H, \tilde{v}, \tilde{w}) = \int_{\mathcal{F}_0} [H \cdot \nabla] \tilde{w} \cdot \tilde{v}$. Thanks to (3.2.4), it is clear that $|b(H, \tilde{v}, \tilde{w})| \leq C\|\tilde{v}\|_{\underline{\mathcal{V}}}\|\tilde{w}\|_{\underline{\mathcal{V}}}$. Moreover, for any $\tilde{v} \in \underline{\mathcal{V}}$

$$b(H, \tilde{v}, H) = \int_{\partial \mathcal{S}_0} |H|^2 (\ell_{\tilde{v}} + r_{\tilde{v}} x^\perp) \cdot n = 0,$$

where in the first step we use the fact $\operatorname{curl} H = 0$ and an integration by part. In the second step we use the Blasius lemma and the Cauchy's Residue theorem as for the terms $C_{i,c}^\varepsilon$ in [GLS16]. This concludes the proof of point *i.* and *ii.*

For *iii.*, we use an integration by parts to see that for any $\tilde{v} \in \mathcal{V}$ we have $b(H, \tilde{v}, \tilde{v}) = 0$, which implies, together with point *i.* of Proposition 3.2.1, that it holds $b(u, \tilde{v}, \tilde{v}) = 0$ for any $u = \tilde{u} + \beta H \in \underline{\mathcal{V}} \oplus \mathbb{R}H$. Integrating by part we have also that for any $u \in \underline{\mathcal{V}} \oplus \mathbb{R}H$, for any $\tilde{v}, \tilde{w} \in \mathcal{V}$, $b(u, \tilde{v}, \tilde{w}) = -b(u, \tilde{w}, \tilde{v})$.

Point *iv.* is trivial after notice that $\nabla H \in L^\infty$ and recall *ii.* of Proposition 3.2.1.

Finally to prove *v.* we use the same procedure of point *iii.* of Proposition 3.2.1. \square

We now introduce the definition of weak solution, with possibly unbounded energy, of the system (3.2.1)-(3.2.3).

Definition 3.2.2. Let $v_0 = \tilde{v}_0 + \beta H \in \mathcal{H} \oplus \mathbb{R}H$ and $T > 0$. We say that $v = \tilde{v} + \beta H$ where

$$\tilde{v} \in C_w([0, T]; \mathcal{H}) \cap L^2(0, T; \underline{\mathcal{V}})$$

is a weak solution of (3.2.1)-(3.2.3) if for any test function $\varphi \in C^1([0, T]; \mathcal{H})$ such that $\varphi|_{\overline{\mathcal{F}_0}} \in C^1([0, T]; C_c^\infty(\overline{\mathcal{F}_0}))$

$$(\tilde{v}, \varphi)_{\mathcal{H}}(t) - (\tilde{v}_0, \varphi|_{t=0})_{\mathcal{H}} = \int_0^t \left[(\tilde{v}, \partial_t \varphi)_{\mathcal{H}} + 2a(v, \varphi) - b(v, \varphi, v) \right].$$

Observe that we took into account here that H is time independent, and β as well. For our convenience we give an equivalent but more explicit definition of weak formulation of the system (3.2.1)-(3.2.3).

Definition 3.2.3 (Weak solution with β circulation at infinity). Let $\tilde{v}_0 \in \mathcal{H}$ and $T > 0$ given. We say that

$$\tilde{v} \in C_w([0, T]; \mathcal{H}) \cap L^2((0, T); \underline{\mathcal{V}})$$

is a weak solution for 2D Navier-Stokes with β circulation at infinity if for every test function $\varphi \in C^1([0, T]; \mathcal{H})$ with $\varphi|_{\overline{\mathcal{F}_0}} \in C^1([0, T]; C_c^\infty(\overline{\mathcal{F}_0}))$, it holds

$$\begin{aligned} (\tilde{v}(t), \varphi(t))_{\mathcal{H}} - (\tilde{v}_0, \varphi(0))_{\mathcal{H}} = \int_0^t \left[(\tilde{v}, \partial_t \varphi)_{\mathcal{H}} + 2a(\tilde{v}, \varphi) + 2\beta a(H, \varphi) \right. \\ \left. - b(\tilde{v}, \varphi, \tilde{v}) - \beta b(H, \varphi, \tilde{v}) - \beta b(\tilde{v}, \varphi, H) \right] dt. \end{aligned}$$

To conclude this section, we observe that any smooth solution of (3.2.1)-(3.2.3) with infinite energy is also a weak solution.

Proposition 3.2.3. *Let $v = \tilde{v} + \beta H$ a smooth solution of (3.2.1)-(3.2.3) with initial data $v_0 = \tilde{v}_0 + \beta H$, then \tilde{v} is a weak solution for 2D Navier-Stokes with β circulation at infinity.*

Proof. Multiply the equation (3.2.1) by the test function φ , integrate in all \mathcal{F}_0 , integrate by parts and use the boundary condition. \square

3.2.4 Existence result: infinite energy case

The following result establishes the existence of global weak solutions of the system (3.2.1)-(3.2.3).

Theorem 3.2.1. *Let $\tilde{v}_0 \in \mathcal{H}$ and let $T > 0$. Then there exists a weak solution $\tilde{v} \in \mathcal{H}$ of 2D Navier-Stokes with β circulation at infinity in $C_w([0, T]; \mathcal{H}) \cap L^2(0, T; \underline{\mathcal{V}})$ such that satisfies the following energy inequality: for almost every $t \in [0, T]$ we have*

$$\frac{1}{2} \|\tilde{v}(t, \cdot)\|_{\mathcal{H}}^2 + 2 \int_{(0,t) \times \mathcal{F}_0} |D(\tilde{v})|^2 + 2\alpha \int_0^t \int_{\partial \mathcal{S}_0} |\tilde{v} - \tilde{v}_S|^2 \leq C(1 + \|\tilde{v}_0\|_{\mathcal{H}}^2),$$

where C depends on T, \mathcal{S}_0 and β . Moreover $(\ell, r) \in H^1(0, T; \mathbb{R}^2 \times \mathbb{R})$.

Proof. The proof of this theorem follows the proof of Theorem 1 in [PS14]. The main difficulty is to deal with the fact that the function H is not an $L^2(\mathcal{F}_0)$ function. In this work we emphasize only the changes in the proof in [PS14], for this reason we divide the proof in several steps as in the paper mentioned above.

The idea of the proof is to use an energy estimate to prove that the Galerkin approximation converges. To get the energy estimate at a formal level is enough to test the equation with \tilde{u} , but this does not work because b is not bounded in $\underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \underline{\mathcal{V}}$ but only in $\underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \mathcal{V}$. The idea is to use a truncation of the solid velocity far from the solid. This procedure was introduced by [ORT07] in a slightly different setting.

For simplicity in the proof we consider the case $\beta = 1$. Dealing with $\beta \neq 1$ is not an issue.

► *Truncation.* As said in the beginning we refer to [PS14] for more details. Let R_0 such that $\mathcal{S}_0 \subset B(0, R_0/2)$. For $R > R_0$, let $\chi_R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the map such that

$$\chi_R(x) = \begin{cases} \chi_R(x) = x^\perp & \text{for } x \text{ in } B(0, R/2); \\ \chi_R(x) = \frac{R}{|x|} x^\perp & \text{for } x \text{ in } \mathbb{R}^2 \setminus B(0, R/2). \end{cases}$$

Note that for $w \in \mathcal{V}$ we have that

$$\chi_R \cdot \nabla w \rightarrow x^\perp \cdot \nabla w \quad \text{in } L^2(\mathbb{R}^2) \text{ as } R \rightarrow +\infty.$$

We can use the functions χ_R to truncate the solid velocity in the following way: we define

$$v_{\mathcal{S}, R}(t, x) = \ell(t) + r(t)\chi_R(x),$$

and the forms

$$b_R(u, v, w) = mr_u l_u^\perp \cdot l_v^\perp + \mathcal{J}_0 r_u r_v r_w + \int_{\mathcal{F}_0} [((u - u_{\mathcal{S}, R}) \cdot \nabla) w] \cdot v - r_u v^\perp \cdot w dx.$$

The advantage of b_R is that it is a continuous form from $\underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \underline{\mathcal{V}}$ to \mathbb{R} . Moreover there exists a constant C independent from R such that for any $(u, v, w) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \underline{\mathcal{V}}$, $|b_R(u, v, w)| \leq C \|u\|_{\underline{\mathcal{V}}} \|v\|_{\underline{\mathcal{V}}} \|w\|_{\underline{\mathcal{V}}}$ and for any $(u, v) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}}$, $|b_R(u, u, v)| \leq C (\|u\|_{L^4(\mathcal{F}_0)}^2 + \|u\|_{\mathcal{H}}^2) \|v\|_{\underline{\mathcal{V}}}$. The cancellation property still hold, in fact for any $(u, v) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}}$, $b_R(u, v, v) = 0$. Finally we note that for any $(u, v, w) \in \underline{\mathcal{V}} \times \underline{\mathcal{V}} \times \mathcal{V}$ $b_R(u, v, w) \rightarrow b(u, v, w)$ when R goes $+\infty$.

► *Existence for the truncated system.* In this step we present the existence of a solution for the truncated system. We claim that for any $\tilde{v}_0 \in \mathcal{H}$ and $T > 0$, there exists $\tilde{v}_R \in C([0, T]; \mathcal{H}) \cap L^2([0, T]; \mathcal{V})$ such that for all $\varphi \in C^\infty([0, T]; \mathcal{H})$ and $\varphi|_{\overline{\mathcal{F}_0}} \in C^1([0, T]; C_c^\infty(\overline{\mathcal{F}_0}))$, and for all $t \in [0, T]$, it holds

$$\begin{aligned} (\tilde{v}_R(t), \varphi(t))_{\mathcal{H}} - (\tilde{v}_{R,0}, \varphi(0))_{\mathcal{H}} &= \int_0^t \left[(\tilde{v}_R, \partial_t \varphi)_{\mathcal{H}} + 2a(\tilde{v}_R, \varphi) + 2a(H, \varphi) \right. \\ &\quad \left. - b_R(\tilde{v}_R, \varphi, \tilde{v}_R) - b(H, \varphi, \tilde{v}_R) - b(\tilde{v}_R, \varphi, H) \right] dt. \end{aligned}$$

Moreover \tilde{v}_R satisfies for almost every $t \in [0, T]$ the energy inequality

$$\begin{aligned} \frac{1}{2} \|\tilde{v}_R(t)\|_{\mathcal{H}}^2 + \int_0^t \int_{\partial \mathcal{S}_0} |\tilde{v}_R - \tilde{v}_{R,S}|^2 ds dt + \int_0^t \int_{\mathcal{F}_0} |D(\tilde{v}_R)|^2 dx dt \\ \leq C \int_0^t (\|\tilde{v}_R\|_{\mathcal{H}}^2 + 1) dt. \end{aligned}$$

The idea of the proof is based on the Galerkin method. We consider the set

$$\begin{aligned} \mathcal{Y} = \{u \in L_\sigma^2(\mathbb{R}^2) \mid \text{there exist } u_F \in C_{\sigma,c}^\infty(\mathbb{R}^2) \text{ and } u_R \in \mathcal{R} \\ \text{such that } u|_{\mathcal{F}} = u_F|_{\mathcal{F}} \text{ and } u|_{\mathcal{S}_0} = u_S|_{\mathcal{S}_0}\}, \end{aligned}$$

which is dense in \mathcal{V} , see Lemma A.0.2 in the appendix. Therefore there exists a base $\{w_i\}_{i \in \mathbb{N}}$ of the Hilbert space \mathcal{V} such that $w_i \in \mathcal{Y}$ for all i . We consider the approximate solution

$$\tilde{v}_N(t, x) = \tilde{v}_{N,R}(t, x) = \sum_{i=1}^N g_{i,N}(t) w_i(x),$$

where we forgot R for simplicity. The function \tilde{v}_N satisfies

$$\begin{aligned} (\partial_t \tilde{v}_N, w_j)_{\mathcal{H}} &= 2a(\tilde{v}_N, w_j) + 2a(H, w_j) + b_R(\tilde{v}_N, \tilde{v}_N, w_j) \\ &\quad - b(H, w_j, \tilde{v}_N) - b(\tilde{v}_N, w_j, H), \\ \tilde{v}_N|_{t=0} &= \tilde{v}_{N0}, \end{aligned} \tag{3.2.7}$$

where \tilde{v}_{N0} is the orthogonal projection in \mathcal{H} of \tilde{v}_0 onto the space spanned by w_1, \dots, w_N . The existence of such $g_{i,N}$ is due to the Cauchy-Lipschitz theorem applied to the system of ODE:

$$\begin{aligned} \mathcal{G}'_N &= \mathcal{M}_N^{-1} [2\mathcal{A}_N \mathcal{G}_N + 2\mathcal{A}_{N,H} - \mathcal{B}_{N,H_1}(\mathcal{G}_N) - \mathcal{B}_{N,H_3}(\mathcal{G}_N) + \mathcal{B}_N(\mathcal{G}_N, \mathcal{G}_N)], \\ \mathcal{G}_N(0) &= \mathcal{G}_{N,0}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{M}_N &= [(w_i, w_j)_{\mathcal{H}}]_{1 \leq i, j \leq N}, \quad \mathcal{G}_N = [g_{1,N} \dots g_{N,N}]^T, \\ \mathcal{A}_N &= [a(w_i, w_j)]_{1 \leq i, j \leq N}, \quad [\mathcal{B}_{N,H_1}(u)]_j = \sum_{k=1}^N u_k b(H, w_j, w_k), \\ [\mathcal{B}_{N,H_3}(u)]_j &= \sum_{i=1}^N u_i b(w_i, w_j, H), \quad [\mathcal{B}_N(u, v)]_j = \sum_{i,k=1}^N u_i v_k b_R(w_i, w_j, w_k) \\ \text{and } [\mathcal{A}_{N,H}]_j &= a(H, w_j). \end{aligned}$$

Note that \mathcal{M}_N is invertible because $\{w_i\}_{i \in \mathbb{N}}$ are linear independent in \mathcal{H} .

The Cauchy-Lipschitz theorem ensures a local in time existence for the functions $g_{i,N}$. To prove that the existence is in all the interval $[0, T]$ we need an estimate that leads us to conclude that the function $g_{i,N}$ are defined in all $[0, T]$. To do that we multiply (3.2.7) by $g_{j,N}$ and we sum over j to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{v}_N\|_{\mathcal{H}}^2 + 2 \int_{\mathcal{F}_0} |D(\tilde{v}_N)|^2 dx + 2\alpha \int_{\partial S_0} |\tilde{v}_N - \tilde{v}_{N,S}|^2 ds \\ = 2a(H, \tilde{v}_N) - b(\tilde{v}_N, \tilde{v}_N, H). \end{aligned} \quad (3.2.8)$$

We now estimate the right hand side of the last equality. Note that for any ε there exists C_ε such that

$$\begin{aligned} |a(H, \tilde{v}_N)| &\leq C_\varepsilon + \varepsilon \left(\int_{\mathcal{F}_0} |D(\tilde{v}_N)|^2 dx + \int_{\partial S_0} |\tilde{v}_N - \tilde{v}_{N,S}|^2 ds \right) \\ \text{and that } |b(\tilde{v}_N, \tilde{v}_N, H)| &\leq C \|\tilde{v}_N\|_{\mathcal{H}}^2, \end{aligned}$$

where C and C_ε do not depend on N and R . If we integrate (3.2.8) in $(0, t)$, we use the two inequality above and we bring on the left the terms multiplied by ε we get

$$\begin{aligned} \|\tilde{v}_N\|_{\mathcal{H}}^2 + \int_0^t \int_{\mathcal{F}_0} |D(\tilde{v}_N)|^2 dx + \int_0^t \int_{\partial S_0} |\tilde{v}_N - \tilde{v}_{N,S}|^2 ds \\ \leq \int_0^t \left(C + \|\tilde{v}_N\|_{\mathcal{H}}^2 \right) dt + \|\tilde{v}_{N0}\|_{\mathcal{H}}^2. \end{aligned}$$

Using the Grönwall lemma we obtain the estimate

$$\|\tilde{v}_N\|_{\mathcal{H}}^2 \leq te^{tC} \left(C \frac{t}{2} + \|\tilde{v}_{N0}\|_{\mathcal{H}}^2 \right) + Ct + \|\tilde{v}_{N0}\|_{\mathcal{H}}^2,$$

which leads us to conclude that the function $g_{i,N}$ can be extended in all $[0, T]$.

Moreover, by the fact that $\|\tilde{v}_{N0}\|_{\mathcal{H}} \leq \|\tilde{v}_0\|_{\mathcal{H}}$ and by the Korn inequality, we conclude that

$$\begin{aligned} \tilde{v}_N &\in L^\infty((0, T); \mathcal{H}) \\ \tilde{v}_N &\in L^2((0, T); \underline{\mathcal{V}}) \end{aligned}$$

are uniformly bounded in both the spaces. This leads us to conclude that there exists $\tilde{v} \in L^\infty((0, T); \mathcal{H}) \cap L^2((0, T); \underline{\mathcal{V}})$ such that \tilde{v}_N converges to \tilde{v} weakly in $L^2((0, T); \underline{\mathcal{V}})$ and *-weakly in $L^\infty((0, T); \mathcal{H})$ as N goes to $+\infty$.

We pass to the limit in (3.2.7). The only not triviality is to prove the convergence of the non-linear term, i.e. $b_R(\tilde{v}_N, \tilde{v}_N, w_j)$ converges to $b_R(\tilde{v}, \tilde{v}, w_j)$. The idea is to notice that \tilde{v}_N is relatively compact in $L^2((0, T); L^2_{loc}(\mathbb{R}^2))$, in fact this follows from the proof of Theorem 1 in [PS14], where the only difference is the estimate

$$\|f_N\|_{\underline{\mathcal{V}}'} \leq C(1 + \|\tilde{v}_N\|_{\underline{\mathcal{V}}} + \|\tilde{v}_N\|_{\underline{\mathcal{V}}}^2),$$

with f_N defined by

$$\langle f_N, w \rangle = 2a(\tilde{v}_N, w) + 2a(H, w) + b_R(\tilde{v}_N, \tilde{v}_N, w) - b(H, w, \tilde{v}_N) - b(\tilde{v}_N, w, H).$$

At this point we are able to pass to the limit in

$$\begin{aligned} (\tilde{v}_N(t), \varphi(t))_{\mathcal{H}} - (\tilde{v}_{N0}, \varphi(0))_{\mathcal{H}} &= \int_0^t \left[(\tilde{v}_N, \partial_t \varphi)_{\mathcal{H}} + 2a(\tilde{v}_N, \varphi) + 2a(H, \varphi) \right. \\ &\quad \left. - b_R(\tilde{v}_N, \varphi, \tilde{v}_N) - b(H, \varphi, \tilde{v}_N) - b(\tilde{v}_N, \varphi, H) \right] dt. \end{aligned}$$

which means that $\tilde{v} = \tilde{v}_R$ satisfies

$$\begin{aligned} (\tilde{v}_R(t), \varphi(t))_{\mathcal{H}} - (\tilde{v}_{R0}, \varphi(0))_{\mathcal{H}} &= \int_0^t \left[(\tilde{v}_R, \partial_t \varphi)_{\mathcal{H}} + 2a(\tilde{v}_R, \varphi) + 2a(H, \varphi) \right. \\ &\quad \left. - b_R(\tilde{v}_R, \varphi, \tilde{v}_R) - b(H, \varphi, \tilde{v}_R) - b(\tilde{v}_R, \varphi, H) \right] dt. \end{aligned}$$

► *Limit of the solutions of the truncated problems.* We note that the energy estimate do not depend on R , so there exists sequence \tilde{v}_{k,R_k} converging to $\tilde{v} \in C((0, T); \mathcal{H}) \cap L^2((0, T); \underline{\mathcal{V}})$ *-weakly in $L^\infty((0, T); \mathcal{H})$ and weakly in $L^2((0, T); \underline{\mathcal{V}})$ as k goes to $+\infty$.

This convergence does not lead us to pass directly to the limit because of the non-linearity of b_R , in other words we have to find an argument to prove that

$$\int_0^t b_{R_k}(\tilde{v}_{k,R_k}, \tilde{v}_{k,R_k}, \varphi) \rightarrow \int_0^t b(\tilde{v}, \tilde{v}, \varphi) dx, \quad \text{as } k \text{ goes to } +\infty.$$

As presented in the paper [PS14], it is enough to prove that \tilde{v}_{k,R_k} is relatively compact in $L^2((0, T); L^2_{loc}(\mathbb{R}^2))$. We have already presented this compactness property for $\tilde{v}_{N,R}$, but the estimates are R depending so we cannot directly conclude.

The idea is to apply the Aubin-Lions lemma to get the compactness result. First of all we note that \tilde{v}_{k,R_k} are uniformly bounded in $L^4(0, T; L^4(\mathcal{F}_0))$, in fact

$$\begin{aligned} \|\tilde{v}_{k,R_k}\|_{L^4(0,T;L^4(\mathcal{F}_0))}^4 &\leq \int_0^t \|\tilde{v}_{k,R_k}\|_{L^4(\mathcal{F}_0)}^4 dt \\ &\leq \int_0^t \|\tilde{v}_{k,R_k}\|_{L^2(\mathcal{F}_0)}^2 \|\nabla \tilde{v}_{k,R_k}\|_{L^2(\mathcal{F}_0)}^2 dt \\ &\leq \|\tilde{v}_{k,R_k}\|_{L^\infty(0,T;L^2(\mathcal{F}_0))} \|\nabla \tilde{v}_{k,R_k}\|_{L^2(0,T;L^2(\mathcal{F}_0))}. \end{aligned}$$

This leads us to prove that $\partial_t \tilde{v}_{k,R_k}$ is uniformly bounded in $L^2((0, T); \mathcal{V}')$, in fact the only non-linear term that can be an issue is

$$\int_{\mathcal{F}_0} [(\tilde{v}_{k,R_k} \cdot \nabla)g] \cdot \tilde{v}_{k,R_k} dx,$$

where $g \in L^2(0, T; \mathcal{V})$, but it can be bound by

$$\begin{aligned} \left| \int_0^T \int_{\mathcal{F}_0} [(\tilde{v}_{k,R_k} \cdot \nabla)g] \cdot \tilde{v}_{k,R_k} dx dt \right| &\leq C \int_0^T \|\tilde{v}_{k,R_k}\|_{L^4(\mathcal{F}_0)}^2 \|\nabla g\|_{L^2(\mathcal{F}_0)} dt \\ &\leq C \|\tilde{v}_{k,R_k}\|_{L^4(0,T;L^4(\mathcal{F}_0))}^2 \|g\|_{L^2(0,T;\mathcal{V})}. \end{aligned}$$

For every ball $B_r(0)$ of radius $r \in \mathbb{N}$ such that $\mathcal{S}_0 \subset B_r(0)$, the Aubin-Lions lemma applies and implies that $\{\tilde{v}_{k,R_k}|_{B_r(0)}\}_{k \in \mathbb{N}}$ is relatively compact in $L^2(0, T; L^2(B_r(0)))$. By extracting a diagonal subsequence we get that $\{\tilde{v}_{k,R_k}\}_{k \in \mathbb{N}}$ is relatively compact in $L^2(0, T; L^2_{loc}(\mathbb{R}^2))$.

We can now pass to the limit in the weak formulation to get the desired result.

► *Improved regularity for (l, r) .* In two dimensions the Kirchhoff potentials are the solutions $\Phi = (\Phi_i)_{i=1,2,3}$ of the following problems:

$$\begin{aligned} -\Delta\Phi_i &= 0 \quad \text{for } x \in \mathcal{F}_0, \\ \Phi_i &\longrightarrow 0 \quad \text{for } |x| \rightarrow \infty, \\ \frac{\partial\Phi_i}{\partial n} &= K_i \quad \text{for } x \in \partial\mathcal{F}_0, \end{aligned}$$

where

$$(K_1, K_2, K_3) = (n_1, n_2, x^\perp \cdot n).$$

These functions are smooth and decay at infinity as follows:

$$\nabla\Phi_i = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{and} \quad \nabla^2\Phi_i = \mathcal{O}\left(\frac{1}{|x|^3}\right) \quad \text{as } x \rightarrow \infty.$$

We now define three functions u_i , for $i = 1, 2, 3$, defined by

$$u_i = \nabla\Phi_i \text{ in } \mathcal{F}_0 \text{ and } u_i = \begin{cases} e_i & \text{if } i = 1, 2, \\ x^\perp & \text{if } i = 3, \end{cases} \quad \text{in } \mathcal{S}_0, \quad (3.2.9)$$

and which are $\widehat{\mathcal{V}}$. The body's equations can then be rephrased as follows:

$$\mathcal{M} \begin{bmatrix} \ell \\ r \end{bmatrix}' = (2a(\tilde{v}, u_i) + 2a(H, u_i) + b(\tilde{v}, \tilde{v}, u_i) + b(H, \tilde{v}, u_i) - b(\tilde{v}, u_i, H))_{i \in \{1, \dots, 3\}},$$

where

$$\mathcal{M} = \begin{bmatrix} m \text{Id}_2 & 0 \\ 0 & \mathcal{J} \end{bmatrix} + \left[\int_{\mathcal{F}_0} \nabla\Phi_a \cdot \nabla\Phi_b dx \right]_{a,b \in \{1,2,3\}}.$$

Since the matrix \mathcal{M} is symmetric and positive definite, applying Proposition 3.2.2 yields that (ℓ, r) is in $H^1(0, T; \mathbb{R}^2 \times \mathbb{R})$. \square

3.3 Future directions and open problems

In this section we discuss some open problems related to Chapter 3.

- The first question is if *it is possible to extend the existence results with finite fluid kinetic energy also in the case where many objects are considered.* We believe that the answer is yes and only some technicality are missing. The methods that are presented here should apply also in that setting.
- The second question is if *it is possible to extend the proof of existence after that the solid touches the boundary.* We believe that in the case of strong solution this is not possible because the norm of the second derivatives in space blows up. Regarding the case of weak solutions, it is not clear that the method of [GVH14] can be applied and the existence after contact is an open problem unless in the case where some artificial source terms are added, see for instance [Sta03].

- ▷ The third question is if *there exists strong solutions in Banach space setting with mixed boundary conditions*. The answer is yes but a complete proof is missing. The idea is that to prove \mathcal{R} -sectoriality of the operator associated only with the fluid, it is enough to consider the problem in the entire space and in the half space. And then use a localization argument to deal with bounded domains. In the case of a solid immersed in a viscous fluid where contact does not occur, the boundary of the fluid is composed by two connected components with empty intersection. While performing the localization argument it is possible to translate the problems into one that contains at maximum one type of boundary conditions. Which implies that is possible to prove \mathcal{R} -sectoriality for the operator associated with the mixed boundary conditions.
- ▷ The fourth question is if *it is possible to deal with viscous coefficient α non positive and non-constant in the space variables*. To prove existence of weak solutions with infinite kinetic energy, it is enough to assume $\alpha \in L^\infty$ and also non-positive. The demonstration can be adapted as in [Kel06]. In the case of weak solutions in bounded domain, the proof in [GVH14] is more involved and is not clear if it can be perform with only L^∞ friction coefficient. Regarding existence of strong solutions in Banach space setting $\alpha \in L^\infty$ is probably too weak. Regarding the case of the fluid alone the weakest assumption that we are aware are presented in [AACG19], where they show existence of weak solutions for $\alpha \in L^2$ and of strong solutions for $\alpha \in W^{1/2,2}$. Moreover they treat also the case in the non-Hilbert space setting.
- ▷ The fifth question is if *it is possible to prove uniqueness for the fluid-structure problem with infinite energy*. We believe that uniqueness holds but for technical reasons we are not able. The difficulty lies in the term involving x^\perp . In the case the solid is a disk, to move the problem to a fixed domain is enough a translation, and the situation become easier. In the Appendix C, we show existence and uniqueness of weak solution with infinite energy of the type studied in Section 3.2. Moreover we prove that as the viscosity tend to zero the solutions converge to one of an Euler plus disk system studied in [ORT07].

Continuity, energy equality and uniqueness

Dans ce chapitre, nous étudions la continuité, l'égalité d'énergie et le caractère unique des solutions faibles d'un système d'interaction fluide-structure.

This chapter is completely dedicated to the proof of the continuity in time with values in $L^2_\sigma(\Omega)$, the energy equality and uniqueness for weak solutions of the system (3.0.1)-(3.0.9) in the case where Ω is bounded. In the first section we prove the continuity in time and the energy equality. At a formal level the equality is obtained by testing the weak formulation (3.1.2) with the solution itself and by some integrations by parts. To perform a rigorous proof, we construct smooth enough approximations and we pass to the limit in the weak formulation tested with these approximations of the solution. To be able to conclude we finally need a Lions-Magenes type lemma in time dependent domain.

The second part of the chapter is devoted to the proof of uniqueness and follow the ideas from [GS15]. The difficulty arises from the fact that the domain of definition of the weak solution is time dependent and we cannot naively take the difference of two weak solutions. To solve this issue we use a change of variables to have the same geometry for the two weak solutions and we use the existence result of strong solutions from Section 3.1.2 to perform a Grönwall estimate that leads us to the proof uniqueness.

4.1 Energy equality and continuity

As announced before in this section we will present and prove the continuity in time and the energy equality for weak solutions of the system (3.0.1)-(3.0.9). Let us state the result.

Theorem 4.1.1. *Let $\Omega \subset \mathbb{R}^2$ an open, bounded set with smooth boundary, \mathcal{S}_0 a closed, bounded, connected and simply connected subset of Ω with smooth boundary, $u_0 \in \mathcal{H}_{\mathcal{S}_0}$, and (\mathcal{S}, u) a weak solution of (3.0.1)-(3.0.9) with initial data (\mathcal{S}_0, u_0) for some $T > 0$. Then*

$$u \in C^0([0, T]; L^2_\sigma(\Omega)). \quad (4.1.1)$$

Moreover, for every $\tau \in [0, T)$, the following energy equality holds:

$$\begin{aligned} & \frac{1}{2} \int_{\mathcal{F}(\tau)} |u_F(\tau, \cdot)|^2 + \frac{1}{2} \int_{\mathcal{S}(\tau)} \rho_S |u_S(\tau, \cdot)|^2 + 2 \int_0^\tau \int_{\mathcal{F}(t)} |Du_F|^2 \\ & \quad + 2\alpha \int_0^\tau \int_{\partial\Omega} |u_F|^2 + 2\alpha \int_0^\tau \int_{\partial\mathcal{S}(t)} |u_F - u_S|^2 \\ & \quad = \frac{1}{2} \int_{\mathcal{F}_0} |u_{F,0}|^2 + \frac{1}{2} \int_{\mathcal{S}_0} \rho_S |u_{0,S}|^2. \end{aligned} \quad (4.1.2)$$

Before going straight in the proof of the above theorem, we will present two results. The first one is a way to construct a smoother in time approximation of the solution and the second one is a Lions-Magenes type Lemma in time dependent domain. We emphasize that these two results are one of the main novelty in this work and they have been used in [MNR19] to prove uniqueness of weak solution of the the system (3.0.1)-(3.0.9) with both no-slip or Navier boundary conditions in dimension three under a Prodi-Serrin type hypothesis on fluid velocity.

4.1.1 Construction of an approximation

The goal of this section is to present a Hilbert space \mathcal{E}_τ with the two following properties:

- ▷ \mathcal{E}_τ contains \mathcal{V}_τ , the set where weak solutions are searched,
- ▷ \mathcal{E}_τ is the closure of \mathcal{W}_τ , the set of test functions, in the norm of \mathcal{E}_τ ,

Moreover the norm on \mathcal{E}_τ has to be strong enough to be able to pass to the limit in the weak formulation.

We consider the space

$$\begin{aligned} \mathcal{E}_\tau = \{ & v \in L^2(0, \tau; L^2_\sigma(\Omega)), \text{ there exists } v_F \in L^2(0, \tau; H^1_\sigma(\Omega)), \\ & v_S \in L^2(0, \tau; \mathcal{R}) \text{ such that } v(t, \cdot) = v_F(t, \cdot) \text{ on } \mathcal{F}(t), \\ & v(t, \cdot) = v_S(t, \cdot) \text{ on } \mathcal{S}(t), \text{ for a.e. } t \in [0, \tau]\}, \end{aligned}$$

with norm $\|v\|_{\mathcal{E}_\tau}$ given by

$$\|v\|_{\mathcal{E}_\tau}^2 = \int_0^\tau \|v_F\|_{H^1(\mathcal{F}(t))}^2 dt + m \int_0^\tau |l_v|^2(t) dt + \mathcal{J} \int_0^\tau r_v^2(t) dt,$$

where v_S is decomposed into $v_S(t) = l_v(t) + (x - h(t))^\perp r_v(t)$.

It is immediate to see that \mathcal{E}_τ is an Hilbert space and satisfies point one. To verify point two we will construct explicitly, for any element u in \mathcal{E}_τ , a sequence u_ε in \mathcal{W}_τ that converges to u in the norm of \mathcal{E}_τ . This construction is made in the lemma that follow.

Lemma 4.1.1. *The space \mathcal{W}_τ is dense in \mathcal{E}_τ .*

Proof. To prove this lemma we show for any element f of \mathcal{E}_τ there exists an approximating sequence in \mathcal{W}_τ that converges to f in \mathcal{E}_τ . We present all the details of this construction because we use its special properties to prove Theorem 4.1.1. Let f an element of \mathcal{E}_τ , this element is not in the space \mathcal{W}_τ because is not regular enough in time. To regularize f in time and preserve the rigidity of the motion inside $\mathcal{S}(t)$ we use the geometric change of variables X

from Claim 3.1.1 that fixes the position of the solid. We make a convolution in time in these variables. Finally we go back to the original variables.

Let $\eta \in C_c^\infty(-1, 1)$ be an even function such that $\eta = 1$ in a open neighbourhood of 0, $0 \leq \eta \leq 1$ and $\int \eta = 1$ and let $\eta_\varepsilon = \eta(\cdot/\varepsilon)/\varepsilon$. Let $\psi \in C_c^\infty(\mathbb{R})$ such that $0 \leq \psi \leq 1$ and such that $\psi \equiv 1$ in an open neighbourhood of $[0, \tau]$. Finally let X_τ be the extension in $(-\infty, +\infty)$ of X defined in Claim 3.1.1, i.e.

$$X_\tau(t, y) = \begin{cases} X(0, y) & \text{for } t \leq 0 \text{ and any } y \in \Omega, \\ X(t, y) & \text{for } t \in (0, \tau) \text{ and any } y \in \Omega, \\ X(\tau, y) & \text{for } t \geq \tau \text{ and any } y \in \Omega. \end{cases} \quad (4.1.3)$$

And in analogous way we extend the inverse Y_τ , h_τ and Q_τ . In what follows we do not write the index τ for simplicity.

We introduce the functions

$$\begin{aligned} \tilde{v}(t, y) &= \nabla Y(t, X(t, y))f(t, X(t, y)), \\ \tilde{v}_S(t, y) &= Q^T(t)f_S(t, h(t) + Q(t)y), \\ \tilde{v}_F(t, y) &= \nabla Y(t, X(t, y))f_F(t, X(t, y)), \end{aligned} \quad (4.1.4)$$

that satisfy $\tilde{v} \in L^\infty(0, \tau; L_\sigma^2(\Omega))$, $\tilde{v}_S \in L^2(0, \tau; \mathcal{R})$ and $\tilde{v}_F \in L^2(0, \tau; H_\sigma^1(\Omega))$. Let v , v_S and v_F the following extension of \tilde{v} , \tilde{v}_S and \tilde{v}_F in $(-\infty, +\infty)$, i.e.

$$v(t, \cdot) = \begin{cases} \psi(t)\tilde{v}(0, \cdot) & \text{for } t \leq 0, \\ \psi(t)\tilde{v}(t, \cdot) & \text{for } t \in (0, \tau), \\ \psi(t)\tilde{v}(\tau, \cdot) & \text{for } t \geq \tau, \end{cases}$$

then we define $v_\varepsilon = \eta_\varepsilon * v$ and in an analogous way $v_{S,\varepsilon} = \eta_\varepsilon * v_S$ and $v_{F,\varepsilon} = \eta_\varepsilon * v_F$. We notice that when we convolute in time we average velocities associated or only with the fluid, in the case $y \in \mathcal{F}_0$ or only with the body, in the case $y \in \mathcal{S}_0$, see Figure 3.1. We are now able to define

$$\begin{aligned} f_\varepsilon(t, x) &= \nabla X(t, Y(t, x))v_\varepsilon(t, Y(t, x)), \\ f_{S,\varepsilon}(t, x) &= Q(t)v_{S,\varepsilon}(t, Q^T(x - h(t))), \\ f_{F,\varepsilon}(t, x) &= \nabla X(t, Y(t, x))v_{F,\varepsilon}(t, Y(t, x)). \end{aligned} \quad (4.1.5)$$

Note that

$$f_{S,\varepsilon}(t, x) = Q(t) \left(\eta_\varepsilon * (Q^T l_f) \right) (t) + (x - h(t))^\perp \eta_\varepsilon * r_f(t).$$

Then it holds that $f_\varepsilon \in \mathcal{W}_\tau$ (observe that $X(t, y) = h(t) + Q(t)y$ in a neighbourhood of $\mathcal{S}(t)$) and that $f_\varepsilon \rightarrow f$ in $L^2(0, \tau; L_\sigma^2(\Omega))$, $f_{F,\varepsilon} \rightarrow f_F$ in $L^2(0, \tau; H_\sigma^1(\Omega))$ and $f_{S,\varepsilon} \rightarrow f_S$ in $L^2(0, \tau; \mathcal{R})$. \square

4.1.2 A Lions-Magenes type lemma

In this section we present a Lions-Magenes Lemma for time-dependent domain. One of the more general version can be found in [LM68][Chapter I, Theorem 3.1]. Here we are interested in an application so we prefer to present this result on a concrete example. The proposition that follows is Theorem 3 of Chapter 5 Section 9 of [Eva10] and is the toy model of the result that we are going to prove in time-dependent domain.

Proposition 4.1.1. *Let $w \in L^2(0, \tau; H_0^1(\Omega))$, with $w' \in L^2(0, T; H^{-1}(\Omega))$.*

1. *Then $w \in C^0([0, \tau]; L^2(\Omega))$ after possibly being redefined on a set of measure zero.*
2. *The mapping $t \mapsto \|w(t)\|_{L^2(\Omega)}^2$ is absolutely continuous, with*

$$\frac{d}{dt} \|w(t)\|_{L^2(\Omega)}^2 = 2\langle w'(t), w(t) \rangle \quad \text{for almost any } 0 \leq t \leq \tau.$$

To state our result we need two ingredients. The first one is \mathcal{E}_τ^* the dual space of \mathcal{E}_τ and we embed \mathcal{E}_τ into \mathcal{E}_τ^* through the inner product of \mathcal{H}_τ .

The second ingredient is the convective derivative. Let (\mathcal{S}, u) a weak solution for some $T > 0$, for any function $f \in \mathcal{W}_\tau$ we define the convective derivative associated with u via

$$\frac{D_u}{dt} f = \begin{cases} \partial_t f_F + u_F \cdot \nabla f_F & \text{for a.e. } (t, x) \in \bigcup_{t \in [0, \tau]} \{t\} \times \mathcal{F}(t), \\ \partial_t f_S + u_S \cdot \nabla f_S & \text{for a.e. } (t, x) \in \bigcup_{t \in [0, \tau]} \{t\} \times \mathcal{S}(t), \end{cases}$$

In what follows we will not write the dependence on u of the convective derivative. Moreover note that the second line of the convective derivative can be rewritten in the following way:

$$\frac{D}{dt} f(t, x) = l'_f(t) + (x - h(t))^\perp r'_f(t) - (x - h(t)) r_u(t) r_f(t),$$

where f_S is decomposed into $f_S(t, x) = l_f(t) + (x - h(t))^\perp r_f(t)$.

Definition 4.1.1. Given $w \in \mathcal{E}_\tau$, we say that w admits a convective derivative

$$\frac{D}{dt} w \in \mathcal{E}_\tau^*$$

if there exists a representative w and $F \in \mathcal{E}_\tau^*$ such that for almost every $t_1 < t_2 \in [0, \tau]$, it holds

$$\begin{aligned} \langle F, 1_{(t_1, t_2)} \varphi \rangle_{\mathcal{E}_\tau^*, \mathcal{E}_\tau} &= (w(t_2), \varphi(t_2))_{\mathcal{H}_S(t_2)} - (w(t_1), \varphi(t_1))_{\mathcal{H}_S(t_1)} \\ &\quad - \int_{t_1}^{t_2} \int_{\mathcal{F}(t)} w \cdot \frac{D}{dt} \varphi dx dt - \int_{t_1}^{t_2} \int_{\mathcal{S}(t)} \rho_S w \cdot \frac{D}{dt} \varphi dx dt, \end{aligned}$$

for any $\varphi \in \mathcal{W}_\tau$, where $1_{(t_1, t_2)}$ is the characteristic function on (t_1, t_2) . In this case we will denote

$$F = \frac{D}{dt} w.$$

Note that the above definition is an extension of the classical definition for the space \mathcal{E}_τ and in what follows we will denote by $\langle \cdot, \cdot \rangle$ the pairing $\langle \cdot, \cdot \rangle_{\mathcal{E}_\tau^*, \mathcal{E}_\tau}$.

We are now able to state rigorously the main theorem of the section.

Theorem 4.1.2. *Let $w \in \mathcal{V}_\tau$, with $\frac{D}{dt} w \in \mathcal{E}_\tau^*$.*

1. *Then $w \in C^0([0, \tau]; L_\sigma^2(\Omega))$ after possibly being redefined on a set of measure zero.*

2. It holds

$$\left\langle \frac{D}{dt}w, w \right\rangle_{\mathcal{E}_\tau^*, \mathcal{E}_\tau} = \frac{1}{2} \|w\|_{\mathcal{H}_{S(\tau)}}^2(\tau) - \frac{1}{2} \|w\|_{\mathcal{H}_{S(0)}}^2(0).$$

Proof. The proof of the two points of the theorem are based on the same idea and on similar computation. To avoid to perform this computation two times, we will assume point one and prove the second point. At the end we will show point one.

From point one we have $w \in C^0([0, \tau]; L_\sigma^2(\Omega))$. Let w_ε be the approximation of w as in Lemma 4.1.1, in other words let w_ε defined as in (4.1.5) where we replace f by w . We are going to prove

$$\left\langle \frac{D}{dt}w_\varepsilon, w_\varepsilon \right\rangle = \frac{1}{2} \|w_\varepsilon\|_{\mathcal{H}_{S(\tau)}}^2(\tau) - \frac{1}{2} \|w_\varepsilon\|_{\mathcal{H}_{S(0)}}^2(0) \quad (4.1.6)$$

and

$$\left\langle \frac{D}{dt}w_\varepsilon, w_\varepsilon \right\rangle = \left\langle \frac{D}{dt}w, W_\varepsilon \right\rangle + o_\varepsilon(1), \quad (4.1.7)$$

where $W_\varepsilon \in \mathcal{W}_\tau$ converges to w in \mathcal{E}_τ and $o_\varepsilon(1)$ is a quantity that tends to zero as ε converges to zero (we will use the same notation also in the sequel). The proof of the second point of the theorem follows from the assumption on the regularity of w , (4.1.6) and (4.1.7), in fact

$$\begin{aligned} \frac{1}{2} \|w_\varepsilon\|_{\mathcal{H}_{S(\tau)}}^2(\tau) - \frac{1}{2} \|w_\varepsilon\|_{\mathcal{H}_{S(0)}}^2(0) &\rightarrow \frac{1}{2} \|w\|_{\mathcal{H}_{S(\tau)}}^2(\tau) - \frac{1}{2} \|w\|_{\mathcal{H}_{S(0)}}^2(0), \\ \frac{1}{2} \|w_\varepsilon\|_{\mathcal{H}_{S(\tau)}}^2(\tau) - \frac{1}{2} \|w_\varepsilon\|_{\mathcal{H}_{S(0)}}^2(0) &= \left\langle \frac{D}{dt}w, W_\varepsilon \right\rangle + o_\varepsilon(1), \\ \left\langle \frac{D}{dt}w, W_\varepsilon \right\rangle + o_\varepsilon(1) &\rightarrow \left\langle \frac{D}{dt}w, w \right\rangle, \end{aligned}$$

as ε goes to 0.

To prove (4.1.6), we use the fact that $\frac{D}{dt}w_\varepsilon \in \mathcal{E}_\tau$, the identification of \mathcal{E}_τ in \mathcal{E}_τ^* through the scalar product in \mathcal{H}_S and Reynold's transport Theorem, see for instance [CNM19, Lemma 2.1].

Let now tackle (4.1.7). We define W_ε as follows:

$$\begin{aligned} l_{W_\varepsilon}(s) &= \int_{-\infty}^{+\infty} \eta_\varepsilon(s-t) Q(s) Q^T(t) l_\varepsilon(t) dt, \quad r_{W_\varepsilon}(s) = \int_{-\infty}^{+\infty} \eta(s-t) r_\varepsilon(t) dt, \\ W_{S,\varepsilon} &= l_{W_\varepsilon} + (x - h(t))^\perp r_{W_\varepsilon}, \end{aligned} \quad (4.1.8)$$

where for simplicity we wrote l_ε instead of l_{w_ε} and r_ε instead of r_{w_ε} and $w_{S,\varepsilon}(t, x) = l_\varepsilon(t) + (x - h(t))^\perp r_\varepsilon(t)$,

$$\begin{aligned} W_{F,\varepsilon}(t, x) &= \sum_{m,l,i} \int_{-\infty}^{\infty} \eta_\varepsilon(s-t) \nabla Y_m(t, x) \partial_m X_l(s, Y(t, x)) \partial_i X_l(s, Y(t, x)) \\ &\quad v_{F,\varepsilon,i}(s, Y(t, x)) ds, \\ W_\varepsilon(t, x) &= \sum_{m,l,i} \int_{-\infty}^{\infty} \eta_\varepsilon(s-t) \nabla Y_m(t, x) \partial_m X_l(s, Y(t, x)) \\ &\quad \partial_i X_l(s, Y(t, x)) v_{\varepsilon,i}(s, Y(t, x)) ds, \end{aligned} \quad (4.1.9)$$

where X and Y are defined in (4.1.3) and v_ε and $v_{F,\varepsilon}$ are defined in (4.1.5), if we replace f_ε and $f_{F,\varepsilon}$ by w_ε and $w_{F,\varepsilon}$.

Observe that $W_\varepsilon \in \mathcal{W}_\tau$ and W_ε converges to w in \mathcal{E}_τ . To prove (4.1.7), it is sufficient to prove

$$\int_0^\tau r'_\varepsilon r_\varepsilon = - \int_0^\tau r_w r'_{W_\varepsilon}(0) + r_w(\tau) r_{W_\varepsilon}(\tau) - r_w(0) r_{W_\varepsilon} + o_\varepsilon(1), \quad (4.1.10)$$

$$\int_0^\tau l'_\varepsilon \cdot l_\varepsilon = - \int_0^\tau l_w \cdot l'_{W_\varepsilon} + l_w(\tau) \cdot l_{W_\varepsilon}(\tau) - l_w(0) \cdot l_{W_\varepsilon}(0) + o_\varepsilon(1), \quad (4.1.11)$$

and

$$\begin{aligned} \int_0^\tau \int_{\mathcal{F}(t)} (\partial_t w_\varepsilon \cdot w_\varepsilon + (u \cdot \nabla) w_\varepsilon \cdot w_\varepsilon) dx dt = \\ - \int_0^\tau \int_{\mathcal{F}(t)} (w \cdot \partial_t W_\varepsilon + (u \cdot \nabla) W_\varepsilon \cdot w) dx dt \\ + \frac{1}{2} \int_{\mathcal{F}(\tau)} w(\tau, \cdot) \cdot W_\varepsilon(\tau, \cdot) dx - \frac{1}{2} \int_{\mathcal{F}_0} w(0, \cdot) \cdot W_\varepsilon(0, \cdot) dx + o_\varepsilon(1). \end{aligned} \quad (4.1.12)$$

We start with the proof of (4.1.10). From (4.1.5), we have that $r_\varepsilon = \eta_\varepsilon * r_w$. The following computation holds

$$\begin{aligned} \int_0^\tau r'_\varepsilon r_\varepsilon &= \int_{-\infty}^{+\infty} r'_\varepsilon(t) r_\varepsilon(t) dt - \int_{-\infty}^0 r'_\varepsilon r_\varepsilon - \int_\tau^{+\infty} r'_\varepsilon r_\varepsilon \\ &= \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \eta'(t-s) r_w(s) ds \right) r_\varepsilon(t) dt - \int_{-\infty}^0 r'_\varepsilon r_\varepsilon - \int_\tau^{+\infty} r'_\varepsilon r_\varepsilon \\ &= - \int_{-\infty}^{+\infty} r_w(s) \left(\int_{-\infty}^{+\infty} \eta'(s-t) r_\varepsilon(t) dt \right) ds - \int_{-\infty}^0 r'_\varepsilon r_\varepsilon - \int_\tau^{+\infty} r'_\varepsilon r_\varepsilon \\ &= - \int_0^\tau r_w(s) \left(\int_{-\infty}^{+\infty} \eta'(s-t) r_\varepsilon(t) dt \right) ds + r_w(\tau) r_{W_\varepsilon}(\tau) \\ &\quad - r_w(0) r_{W_\varepsilon}(0) + o_\varepsilon(1) \\ &= - \int_0^\tau r_w r'_{W_\varepsilon} + r_w(\tau) r_{W_\varepsilon}(\tau) - r_w(0) r_{W_\varepsilon}(0) + o_\varepsilon(1). \end{aligned}$$

where to go from line 2 to line 3 we use the fact that η' is odd and in the last line we use (4.1.8).

We perform similar computation to prove (4.1.11). Clearly we have that

$$\int_0^\tau l'_\varepsilon \cdot l_\varepsilon = \int_{-\infty}^{+\infty} l'_\varepsilon \cdot l_\varepsilon - \int_{-\infty}^0 l'_\varepsilon \cdot l_\varepsilon - \int_\tau^{+\infty} l'_\varepsilon \cdot l_\varepsilon \quad (4.1.13)$$

and that

$$\int_{-\infty}^0 l'_\varepsilon \cdot l_\varepsilon dt \rightarrow \frac{1}{2} |l_w|^2(0) \quad \text{and} \quad \int_\tau^{+\infty} l'_\varepsilon \cdot l_\varepsilon dt \rightarrow -\frac{1}{2} |l_w|^2(\tau) \quad \text{as } \varepsilon \rightarrow 0. \quad (4.1.14)$$

We recall that by definition (4.1.5) of $w_{S,\varepsilon}$ we have

$$l_\varepsilon(t) = Q(t) \int_{-\infty}^{+\infty} \eta_\varepsilon(t-s) Q^T(s) l_w(s) ds.$$

Using this definition we have

$$\int_{-\infty}^{+\infty} l'_\varepsilon(t) \cdot l_\varepsilon(t) dt = \int_{-\infty}^{+\infty} Q'(t) \int_{-\infty}^{+\infty} \eta_\varepsilon(t-s) Q^T(s) l_w(s) ds \cdot l_\varepsilon(t) dt \quad (4.1.15)$$

$$+ \int_{-\infty}^{+\infty} Q(t) \int_{-\infty}^{+\infty} \eta'_\varepsilon(t-s) Q^T(s) l_w(s) ds \cdot l_\varepsilon(t) dt. \quad (4.1.16)$$

We use the fact that η' is odd and we invert the integration in s and t to arrive at

$$\begin{aligned} (4.1.16) &= - \int_{-\infty}^{+\infty} Q^T(s) l_w(s) \cdot \int_{-\infty}^{+\infty} \eta'_\varepsilon(s-t) Q^T(t) l_\varepsilon(t) dt ds \\ &= - \int_{-\infty}^{+\infty} Q^T(s) l_w(s) \cdot \partial_s \int_{-\infty}^{+\infty} \eta_\varepsilon(s-t) Q^T(t) l_\varepsilon(t) dt ds \\ &= \int_{-\infty}^{+\infty} l_w(s) \cdot \int_{-\infty}^{+\infty} \eta_\varepsilon(s-t) Q'(s) Q^T(t) l_\varepsilon(t) dt ds \end{aligned} \quad (4.1.17)$$

$$- \int_{-\infty}^{+\infty} l_w(s) \cdot \partial_s \int_{-\infty}^{+\infty} \eta_\varepsilon(s-t) Q(s) Q^T(t) l_\varepsilon(t) dt ds. \quad (4.1.18)$$

We summarize the last computations to arrive at

$$\int_{-\infty}^{+\infty} l'_\varepsilon(t) \cdot l_\varepsilon(t) dt = (4.1.15) + (4.1.17) + (4.1.18). \quad (4.1.19)$$

Moreover by the fact that $\partial_t(Q(t)Q^T(t)) = Q'(t)Q^T(t) + Q(t)(Q^T)'(t) = 0$ we have

$$(4.1.15) + (4.1.17) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (4.1.20)$$

Gathering (4.1.13), (4.1.14), (4.1.19), (4.1.20) and using that

$$- \int_{-\infty}^{+\infty} l_w \cdot l'_{W_\varepsilon} = - \int_0^\tau l_w \cdot l'_{W_\varepsilon} - \frac{1}{2} l_w(0) \cdot l'_{W_\varepsilon}(0) + \frac{1}{2} l_w(\tau) \cdot l'_{W_\varepsilon}(\tau) + o_\varepsilon(1),$$

we obtain

$$\begin{aligned} \int_0^\tau l'_\varepsilon \cdot l_\varepsilon dt &= - \int_0^\tau l_w(s) \cdot \partial_s \int_{-\infty}^{+\infty} \eta_\varepsilon(s-t) Q(s) Q^T(t) l_\varepsilon(t) dt ds \\ &\quad + l_w(\tau) \cdot l_{W_\varepsilon}(\tau) - l_w(0) \cdot l_{W_\varepsilon}(0) + o_\varepsilon(1) \\ &= - \int_0^\tau l_w(s) \cdot l'_{W_\varepsilon} ds + l_w(\tau) \cdot l_{W_\varepsilon}(\tau) - l_w(0) \cdot l_{W_\varepsilon}(0) + o_\varepsilon(1). \end{aligned}$$

where we use (4.1.8).

We are left with the proof of (4.1.12). We start with the term

$$\begin{aligned} \int_0^\tau \int_{\mathcal{F}(t)} \partial_t w_\varepsilon \cdot w_\varepsilon dx dt &= \int_{-\infty}^{+\infty} \int_{\mathcal{F}(t)} \partial_t w_\varepsilon \cdot w_\varepsilon dx dt \\ &\quad - \frac{1}{2} \int_{\mathcal{F}_0} w^2(0, \cdot) dx + \frac{1}{2} \int_{\mathcal{F}(\tau)} w^2(\tau, \cdot) dx + o_\varepsilon(1). \end{aligned}$$

As before we start the computation by the definition (4.1.5) of the approximate sequence w_ε (we exchange f with w) and we compute the derivative in time. We recall the definition (4.1.4) and (4.1.5):

$$\begin{aligned} v_m(t, y) &= \sum_l \partial_l Y_m(t, X(t, y)) w_l(t, X(t, y)), \\ w_{\varepsilon, l}(t, x) &= \sum_m \partial_m X_l(t, Y(t, x)) v_{\varepsilon, m}(t, Y(t, x)). \end{aligned}$$

If we compute explicitly the derivative in time we get

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{\mathcal{F}(t)} \partial_t w_\varepsilon \cdot w_\varepsilon dx dt &= \\ \sum_{l, m} \int_{-\infty}^{\infty} \int_{\mathcal{F}(t)} \partial_t \partial_m X_l(t, Y(t, x)) v_{\varepsilon, m}(t, Y(t, x)) w_{\varepsilon, l}(t, x) dx dt & \quad (4.1.21) \end{aligned}$$

$$\begin{aligned} + \sum_{l, m, k} \int_{-\infty}^{\infty} \int_{\mathcal{F}(t)} \partial_t Y_k(t, x) \partial_k \partial_m X_l(t, Y(t, x)) \\ v_{\varepsilon, m}(t, Y(t, x)) w_{\varepsilon, l}(t, x) dx dt & \quad (4.1.22) \end{aligned}$$

$$\begin{aligned} + \sum_{l, m} \int_{-\infty}^{\infty} \int_{\mathcal{F}(t)} \partial_m X_l(t, Y(t, x)) \partial_t v_{\varepsilon, m}(t, Y(t, x)) w_{\varepsilon, l}(t, x) dx dt & \quad (4.1.23) \end{aligned}$$

$$\begin{aligned} + \sum_{l, m, k} \int_{-\infty}^{\infty} \int_{\mathcal{F}(t)} \partial_m X_l(t, Y(t, x)) \partial_t Y_k(t, x) \\ \partial_k v_{\varepsilon, m}(t, Y(t, x)) w_{\varepsilon, l}(t, x) dx dt. & \quad (4.1.24) \end{aligned}$$

Using the change of variables and the fact that the determinant of the Jacobian of the change of variables is 1 we have

$$\begin{aligned} (4.1.23) &= \sum_{l, m, i} \int_{-\infty}^{\infty} \int_{\mathcal{F}_0} \partial_m X_l(t, y) \partial_t v_{\varepsilon, m}(t, y) \partial_i X_l(t, y) v_{\varepsilon, i}(t, y) dy dt \\ &= \sum_{l, m, i} \int_{-\infty}^{\infty} \int_{\mathcal{F}_0} \int_{-\infty}^{\infty} \eta'_\varepsilon(t-s) v_m(s, y) ds \partial_m X_l(t, y) \\ &\quad \partial_i X_l(t, y) v_{\varepsilon, i}(t, y) dy dt \\ &= - \sum_{l, m, i} \int_{-\infty}^{\infty} \int_{\mathcal{F}_0} v_m(s, y) \int_{-\infty}^{\infty} \eta'_\varepsilon(s-t) \partial_m X_l(t, y) \\ &\quad \partial_i X_l(t, y) v_{\varepsilon, i}(t, y) dt dy ds \\ &= - \sum_{l, m, i} \int_{-\infty}^{\infty} \int_{\mathcal{F}_0} v_m(s, y) \partial_s \left(\int_{-\infty}^{\infty} \eta_\varepsilon(s-t) \partial_m X_l(t, y) \right. \\ &\quad \left. \partial_i X_l(t, y) v_{\varepsilon, i}(t, y) dt \right) dy ds \\ &= - \sum_{l, m, i, n} \int_{-\infty}^{\infty} \int_{\mathcal{F}_0} \partial_n Y_m(s, X(s, y)) w_n(s, X(s, y)) \\ &\quad \partial_s \left(\int_{-\infty}^{\infty} \eta_\varepsilon(s-t) \partial_m X_l(t, y) \partial_i X_l(t, y) v_{\varepsilon, i}(t, y) dt \right) dy ds, \end{aligned}$$

where in the second line we use the definition of convolution, in the third line we exchange the integration in t with the integration in s and we use the fact that η is even which implies that η' is odd, in the fourth line we use a property of the derivative of a convolution and in the last one we use the relation (4.1.4) between w and v .

Going back to the original variables we get that the last line is equal to minus

$$\sum_{l,m,i,n} \int_{-\infty}^{\infty} \int_{\mathcal{F}(s)} \partial_n Y_m(s, x) w_n(s, x) \partial_s \left(\int_{-\infty}^{\infty} \eta_\varepsilon(t-s) \partial_m X_l(t, Y(s, x)) \partial_i X_l(t, Y(s, x)) v_{\varepsilon,i}(t, Y(s, x)) dt \right) dx ds \quad (4.1.25)$$

plus the following three terms

$$\sum_{l,m,i,n,e} \int_{-\infty}^{\infty} \int_{\mathcal{F}(s)} \partial_n Y_m(s, x) w_n(s, x) \int_{-\infty}^{\infty} \eta_\varepsilon(t-s) \partial_s Y_e(s, x) \partial_e \partial_m X_l(t, Y(s, x)) \partial_i X_l(t, Y(s, x)) v_{\varepsilon,i}(t, Y(s, x)) dt dx ds, \quad (4.1.26)$$

$$\sum_{l,m,i,n,e} \int_{-\infty}^{\infty} \int_{\mathcal{F}(s)} \partial_n Y_m(s, x) w_n(s, x) \int_{-\infty}^{\infty} \eta_\varepsilon(t-s) \partial_m X_l(t, Y(s, x)) \partial_s Y_e(s, x) \partial_e \partial_i X_l(t, Y(s, x)) v_{\varepsilon,i}(t, Y(s, x)) dt dx ds, \quad (4.1.27)$$

$$\sum_{l,m,i,n,e} \int_{-\infty}^{\infty} \int_{\mathcal{F}(s)} \partial_n Y_m(s, x) w_n(s, x) \int_{-\infty}^{\infty} \eta_\varepsilon(t-s) \partial_m X_l(t, Y(s, x)) \partial_i X_l(t, Y(s, x)) \partial_s Y_e(s, x) \partial_e v_{\varepsilon,i}(t, Y(s, x)) dt dx ds. \quad (4.1.28)$$

We isolate $w_n(s, x)$. To do this we note that (4.1.25) is equal to the difference of the following two terms

$$\sum_{l,m,i,n} \int_{-\infty}^{\infty} \int_{\mathcal{F}(s)} w_n(s, x) \partial_s \left(\int_{-\infty}^{\infty} \eta_\varepsilon(t-s) \partial_n Y_m(s, x) \partial_m X_l(t, Y(s, x)) \partial_i X_l(t, Y(s, x)) v_{\varepsilon,i}(t, Y(s, x)) dt \right) dx ds \quad (4.1.29)$$

$$\sum_{l,m,i,n} \int_{-\infty}^{\infty} \int_{\mathcal{F}(s)} w_n(s, x) \int_{-\infty}^{\infty} \eta_\varepsilon(t-s) \partial_s \partial_n Y_m(s, x) \partial_m X_l(t, Y(s, x)) \partial_i X_l(t, Y(s, x)) v_{\varepsilon,i}(t, Y(s, x)) dt dx ds. \quad (4.1.30)$$

We arrive at

$$\int_{-\infty}^{\infty} \int_{\mathcal{F}(t)} \partial_t w_\varepsilon \cdot w_\varepsilon dx dt = (4.1.21) + (4.1.22) - (4.1.29) + (4.1.30) + (4.1.26) + (4.1.27) + (4.1.28) + (4.1.24).$$

Notice that as ε goes to 0 we have

$$(4.1.21) \rightarrow \sum_{l,m,n} \int_{-\infty}^{\infty} \int_{\mathcal{F}(t)} \partial_t \partial_m X_l(t, Y(t, x)) \partial_n Y_m(t, x) w_n(t, x) w_l(t, x) dx dt,$$

$$(4.1.30) \rightarrow \sum_{l,m,n} \int_{-\infty}^{\infty} \int_{\mathcal{F}(s)} w_n(t, x) \partial_t \partial_n Y_m(t, x) \partial_m X_l(t, Y(t, x)) w_l(t, x) dx dt,$$

$$(4.1.26) \rightarrow \sum_{l,m,n,e} \int_{-\infty}^{\infty} \int_{\mathcal{F}(s)} \partial_n Y_m(t, x) w_n(t, x) \partial_t Y_e(t, x) \partial_e \partial_m X_l(t, Y(t, x)) w_l(t, x) dx dt.$$

Moreover using that $\partial_t(\nabla X(t, Y(t, x)) \nabla Y(t, x)) = 0$, we arrive at

$$(4.1.21) + (4.1.30) + (4.1.26) \rightarrow 0.$$

We study the terms (4.1.22), (4.1.27), (4.1.28), (4.1.24). As ε goes to 0 we have

$$(4.1.22), (4.1.27) \rightarrow \sum_{l,m,k} \int_{-\infty}^{\infty} \int_{\mathcal{F}(t)} \partial_t Y_k(t, x) \partial_k \partial_m X_l(t, Y(t, x)) v_m(t, Y(t, x)) w_l(t, x) dx dt,$$

$$(4.1.28), (4.1.24) \rightarrow \sum_{l,m,k} \int_{-\infty}^{\infty} \int_{\mathcal{F}(t)} \partial_m X_l(t, Y(t, x)) \partial_t Y_k(t, x) \partial_k v_m(t, Y(t, x)) w_l(t, x) dx dt.$$

Moreover it holds

$$\begin{aligned} & \sum_m \partial_k \partial_m X_l(t, Y(t, x)) v_m(t, Y(t, x)) + \partial_m X_l(t, Y(t, x)) \partial_k v_m(t, Y(t, x)) \\ &= \sum_{m,i,j} \partial_k X_j(t, Y(t, x)) \partial_j Y_i(t, x) \partial_i \partial_m X_l(t, Y(t, x)) v_m(t, Y(t, x)) \\ & \quad + \sum_{m,i,j} \partial_k X_j(t, Y(t, x)) \partial_j Y_i(t, x) \partial_m X_l(t, Y(t, x)) \partial_i v_m(t, Y(t, x)) \\ &= \sum_j \partial_k X_j(t, Y(t, x)) \partial_j w_l(t, x), \end{aligned}$$

where we multiply by the identity matrix $\sum_j \partial_k X_j(t, Y(t, x)) \partial_j Y_i(t, x) = \delta_{ki}$ and by the fact that Y is the inverse of X , it holds $X(t, Y(t, x)) = x$, which implies that

$$0 = \partial_t (X_j(t, Y(t, x))) = \partial_t X_j(t, Y(t, x)) + \sum_k \partial_k X_j(t, Y(t, x)) \partial_t Y_k(t, x).$$

Combining the two last equalities, we have that

$$\begin{aligned} & \sum_{m,k} \partial_t Y_k(t, x) \partial_k \partial_m X_l(t, Y(t, x)) v_m(t, Y(t, x)) \\ & \quad + \partial_t Y_k(t, x) \partial_m X_l(t, Y(t, x)) \partial_k v_m(t, Y(t, x)) \\ &= \sum_{j,k} \partial_t Y_k(t, x) \partial_k X_j(t, Y(t, x)) \partial_j w_l(t, x) \\ &= - \sum_j \partial_t X_j(t, Y(t, x)) \partial_j w_l(t, x), \end{aligned}$$

which implies that

$$(4.1.22) + (4.1.27) + (4.1.28) + (4.1.24) \rightarrow \\ - 2 \sum_{j,l} \int_{-\infty}^{+\infty} \int_{\mathcal{F}(t)} \partial_t X_j(t, Y(t, x)) \partial_j w_l(t, x) w_l(t, x) dx dt.$$

We note that X (defined in (4.1.3)) does not change in time respectively in the intervals $(-\infty, 0]$ and in $[\tau, +\infty)$ and by an integration by parts we have

$$(4.1.22) + (4.1.27) + (4.1.28) + (4.1.24) \rightarrow \\ - 2 \int_0^\tau \int_{\mathcal{F}(t)} [(u(t, x) \cdot \nabla) w(t, x)] \cdot w(t, x) dx dt.$$

Recall the definition of W_ε from (4.1.9) and let $W_{\varepsilon,n}$ the n -th component of W_ε , with this notation, it holds

$$(4.1.29) = \int_{-\infty}^{+\infty} \int_{\mathcal{F}(t)} w \cdot \partial_t W_\varepsilon dx dt \\ = \int_0^\tau \int_{\mathcal{F}(t)} w \cdot \partial_t W_\varepsilon dx dt \\ + \frac{1}{2} \int_{\mathcal{F}_0} w(0, \cdot) \cdot W_\varepsilon(0, \cdot) dx - \frac{1}{2} \int_{\mathcal{F}(\tau)} w(\tau, \cdot) \cdot W_\varepsilon(\tau, \cdot) dx + o_\varepsilon(1).$$

To conclude the proof of (4.1.12) we note that

$$\int_0^\tau \int_{\mathcal{F}(t)} (\partial_t w_\varepsilon \cdot w_\varepsilon + (u \cdot \nabla) w_\varepsilon \cdot w_\varepsilon) dx dt \\ = \int_{-\infty}^{+\infty} \int_{\mathcal{F}(t)} \partial_t w_\varepsilon \cdot w_\varepsilon dx dt + \frac{1}{2} \int_{\mathcal{F}_0} |w(0, \cdot)|^2 dx - \frac{1}{2} \int_{\mathcal{F}(\tau)} |w(\tau, \cdot)|^2 dx + o_\varepsilon(1) \\ + \int_0^\tau \int_{\mathcal{F}(t)} (u \cdot \nabla) w_\varepsilon \cdot w_\varepsilon dx dt \\ = (4.1.22) + (4.1.27) + (4.1.28) + (4.1.24) - (4.1.29) - \int_{\mathcal{F}_0} |w(0, \cdot)|^2 dx \\ + \int_{\mathcal{F}(\tau)} |w(\tau, \cdot)|^2 dx + o_\varepsilon(1) + \int_0^\tau \int_{\mathcal{F}(t)} (u \cdot \nabla) w_\varepsilon \cdot w_\varepsilon dx dt \\ = - \int_0^\tau \int_{\mathcal{F}(t)} (w \cdot \partial_t W_\varepsilon + (u \cdot \nabla) W_\varepsilon \cdot w) dx dt - \frac{1}{2} \int_{\mathcal{F}_0} w(0, \cdot) \cdot W_\varepsilon(0, \cdot) dx \\ + \frac{1}{2} \int_{\mathcal{F}(\tau)} w(\tau, \cdot) \cdot W_\varepsilon(\tau, \cdot) dx + o_\varepsilon(1).$$

We go back to the proof of continuity. Let w_ε be the approximation of w as in Lemma 4.1.1, in other words let w_ε defined as in (4.1.5) where we replace f by w . To show the continuity (4.1.1) in time of the solution we follow a standard technique. The idea is to consider the approximation sequence w_ε defined in (4.1.5) and to prove that the sequence is a Cauchy sequence in $C^0([0, \tau]; L_\sigma^2(\Omega))$. To do so, we suppose without loss of generality that $w_\varepsilon(0) \rightarrow w(0)$ in \mathcal{H}_S , then

$$\|w_\varepsilon(t, \cdot) - w_\delta(t, \cdot)\|_{\mathcal{H}_S(t)}^2 = \|w_\varepsilon(0, \cdot) - w_\delta(0, \cdot)\|_{\mathcal{H}_S}^2 + 2 \left\langle \frac{D}{dt}(w_\varepsilon - w_\delta), w_\varepsilon - w_\delta \right\rangle.$$

If we show that the convective derivative of w_ε is uniformly bounded in \mathcal{E}_τ^* , the equality above proves that w_ε is a Cauchy in the space $C^0([0, \tau]; L_\sigma^2(\Omega))$, in fact L_σ^2 and $\mathcal{H}_{S(t)}$ norms are equivalent as long as $\sup_{t \in [0, \tau]} \text{dist}(S(t), \partial\Omega) > 0$.

In what follow we prove that the convective derivative of w_ε is uniformly bounded in \mathcal{E}_τ^* which is equivalent to show that for any $\varphi \in \mathcal{W}_\tau$ such that φ is identically zero in a neighbourhood of 0 and τ , there exists a constant independent of ε such that

$$\left| \left\langle \frac{D}{dt} w_\varepsilon, \varphi \right\rangle \right| \leq C \|\varphi\|_{\mathcal{E}_\tau}.$$

But this is clear from the computations performed to prove point two, in fact

$$\left\langle \frac{D}{dt} w_\varepsilon, \varphi \right\rangle = \left\langle \frac{D}{dt} w, \Phi_\varepsilon \right\rangle + \langle F_\varepsilon, \varphi \rangle$$

where $\|\Phi_\varepsilon\|_{\mathcal{E}_\tau} \leq C \|\varphi\|_{\mathcal{E}_\tau}$, F_ε is uniformly bounded in \mathcal{E}_τ^* and Φ_ε and F_ε are defined as follow.

$$l_{\Phi_\varepsilon}(s) = \int_{-\infty}^{+\infty} \eta_\varepsilon(s-t) Q(s) Q^T(t) l_\varphi(t) dt, \quad r_{\Phi_\varepsilon}(s) = \int_{-\infty}^{+\infty} \eta(s-t) r_\varphi(t) dt,$$

$$\Phi_{S,\varepsilon} = l_{\Phi_\varepsilon} + (x - h(t))^\perp r_{\Phi_\varepsilon},$$

where $\varphi_{S,\varepsilon}(t, x) = l_\varphi(t) + (x - h(t))^\perp r_\varphi(t)$,

$$\Phi_{F,\varepsilon}(t, x) = \int_{-\infty}^{\infty} \eta_\varepsilon(t-s) (\nabla X(s, Y(t, x)) \nabla Y(t, x))^T \varphi_F(s, X(s, Y(t, x))) ds,$$

$$\Phi_\varepsilon(t, x) = \int_{-\infty}^{\infty} \eta_\varepsilon(t-s) (\nabla X(s, Y(t, x)) \nabla Y(t, x))^T \varphi(s, X(s, Y(t, x))) ds,$$

where X and Y are defined in (4.1.3) and the map F is defined as

$$\begin{aligned} \langle F, \varphi \rangle &= \int_{-\infty}^{+\infty} Q'(t) \int_{-\infty}^{+\infty} \eta_\varepsilon(t-s) Q(s)^T(s) l_w(s) ds \cdot l_\varphi(t) dt \\ &+ \int_{-\infty}^{+\infty} l_w \cdot Q'(s) \int_{-\infty}^{+\infty} \eta(t-s) Q^T(t) l_\varphi(t) dt \\ &+ \int_{-\infty}^{+\infty} \int_{\mathcal{F}(t)} (\partial_t \nabla X(t, Y(t, x)) \cdot \nabla \varphi \cdot w_\varepsilon) dx dt \\ &+ \int_{-\infty}^{+\infty} \int_{\mathcal{F}(t)} w \cdot (\nabla Y(s, x))^T \int_{-\infty}^{+\infty} \eta_\varepsilon(s-t) \times \\ &(\partial_s Y(s, x) \cdot \nabla) (\nabla X(t, Y(s, x)))^T \varphi(t, X(t, Y(s, x))) dt dx ds \\ &+ \int_{-\infty}^{+\infty} \int_{\mathcal{F}(t)} w \cdot (\nabla Y(s, x))^T \int_{-\infty}^{+\infty} \eta_\varepsilon(s-t) (\nabla X(t, Y(s, x)))^T \times \\ &(\partial_s Y(s, x) \cdot \nabla X(t, Y(s, x)) \nabla) \varphi(t, X(t, Y(s, x))) dt dx ds \\ &+ \int_{-\infty}^{+\infty} w \cdot \partial_s (\nabla Y(s, x))^T \int_{-\infty}^{+\infty} \eta_\varepsilon(s-t) (\nabla X(t, Y(s, x)))^T \times \\ &\varphi(t, X(t, Y(s, x))) dt dx ds. \end{aligned}$$

□

4.1.3 Proof of the energy equality

In this subsection we conclude the proof of Theorem 4.1.1. Let (\mathcal{S}, u) a weak solution with initial data (\mathcal{S}_0, u_0) for some $T > 0$. Fix a representative of u . For almost every $\tau \in [0, T)$ it holds:

$$\begin{aligned}
& - \int_0^\tau \int_{\mathcal{F}(t)} u_F \cdot \partial_t \varphi_F dx dt - \int_0^\tau \int_{\mathcal{S}(t)} \rho_S u_S \cdot \partial_t \varphi_S dx dt \\
& - \int_0^\tau \int_{\mathcal{F}(t)} u_F \otimes u_F : \nabla \varphi_F dx dt + 2 \int_0^\tau \int_{\mathcal{F}(t)} Du_F : D\varphi_F dx dt \\
& + 2\alpha \int_0^\tau \int_{\partial\Omega} u_F \cdot \varphi_F ds dt + 2\alpha \int_0^\tau \int_{\partial\mathcal{S}(t)} (u_F - u_S) \cdot (\varphi_F - \varphi_S) ds dt \quad (4.1.31) \\
& = \int_{\mathcal{F}(0)} u_{F,0} \cdot \varphi_F|_{t=0} dx + \int_{\mathcal{S}(0)} \rho_S u_{S,0} \cdot \varphi_S|_{t=0} dx \\
& - \int_{\mathcal{F}(\tau)} (u_F \cdot \varphi_F)|_{t=\tau} dx - \int_{\mathcal{S}(\tau)} (\rho_S u_{S,0} \cdot \varphi_S)|_{t=\tau} dx,
\end{aligned}$$

for all test functions $\varphi \in \mathcal{W}_\tau$. Formally the energy equality is a direct consequence of (4.1.31), in fact the equation (4.1.31) tested by the solution u itself is the energy equality. To do this in a rigorous way we reformulate (4.1.31) in such a way that we can test with less regular in time functions. We notice that

$$\frac{D}{dt} u \in \mathcal{E}_\tau^*.$$

Indeed (4.1.31) tells us that for almost every $t_1 < t_2 \in [0, \tau]$ it holds

$$\begin{aligned}
& - \int_{t_1}^{t_2} \int_{\mathcal{F}(t)} u_F \cdot \partial_t \varphi_F - \int_{t_1}^{t_2} \int_{\mathcal{S}(t)} \rho_S u_S \cdot \partial_t \varphi_S + \int_{\mathcal{F}(t_2)} (u_F \cdot \varphi_F)|_{t=t_2} \\
& + \int_{\mathcal{S}(t_2)} (\rho_S u_S \cdot \varphi_S)|_{t=t_2} - \int_{\mathcal{F}(t_1)} u_F \cdot \varphi_F|_{t=t_1} - \int_{\mathcal{S}(t_1)} \rho_S u_S \cdot \varphi_S|_{t=t_1} \\
& - \int_{t_1}^{t_2} \int_{\mathcal{F}(t)} u_F \otimes u_F : \nabla \varphi_F = -2 \int_{t_1}^{t_2} \int_{\mathcal{F}(t)} Du_F : D\varphi_F \\
& - 2\alpha \int_{t_1}^{t_2} \int_{\partial\Omega} u_F \cdot \varphi_F - 2\alpha \int_{t_1}^{t_2} \int_{\partial\mathcal{S}(t)} (u_F - u_S) \cdot (\varphi_F - \varphi_S)
\end{aligned}$$

and the following estimate holds

$$\begin{aligned}
& \left| 2 \int_0^\tau \int_{\mathcal{F}(t)} Du_F : D\varphi_F + 2\alpha \int_0^\tau \int_{\partial\Omega} u_F \cdot \varphi_F \right. \\
& \quad \left. + 2\alpha \int_0^\tau \int_{\partial\mathcal{S}(t)} (u_F - u_S) \cdot (\varphi_F - \varphi_S) \right| \leq C \|\varphi\|_{\mathcal{E}_\tau}.
\end{aligned}$$

This implies that $u \in C^0([0, \tau]; L_\sigma^2(\Omega))$ from Theorem 4.1.2, moreover we can write the weak formulation (4.1.31) in the following way

$$\begin{aligned}
\left\langle \frac{D}{dt} u, \varphi \right\rangle & = -2 \int_0^\tau \int_{\mathcal{F}(t)} Du_F : D\varphi_F - 2\alpha \int_0^\tau \int_{\partial\Omega} u_F \cdot \varphi_F \\
& - 2\alpha \int_0^\tau \int_{\partial\mathcal{S}(t)} (u_F - u_S) \cdot (\varphi_F - \varphi_S).
\end{aligned}$$

The advantage of this formulation is that we can test it with any function in \mathcal{E}_τ . In fact \mathcal{W}_τ is dense in \mathcal{E}_τ and we can pass to the limit in norm of \mathcal{E}_τ .

If we test the equation with u , we obtain

$$\begin{aligned} \left\langle \frac{D}{dt}u, u \right\rangle &= -2 \int_0^\tau \int_{\mathcal{F}(t)} Du_F : Du_F - 2\alpha \int_0^t \int_{\partial\Omega} u_F \cdot u_F \\ &\quad - 2\alpha \int_0^\tau \int_{\partial\mathcal{S}(t)} (u_F - u_S) \cdot (u_F - u_S). \end{aligned}$$

For almost every $\tau \in (0, T)$ the proof of the energy equality (4.1.2) therefore follows from Theorem 4.1.2.

4.2 Uniqueness

In this section we present a result of uniqueness for weak solutions of the system (3.0.1)-(3.0.9) in the case where Ω is bounded. More precisely, we will prove that given a weak solution (\mathcal{S}, u) such that at initial time the distance of the solid from the boundary is a positive quantity then (\mathcal{S}, u) is the unique weak solution of (3.0.1)-(3.0.9) up to the first contact.

This result is rigorously stated in the following theorem.

Theorem 4.2.1. *Let $\Omega \subset \mathbb{R}^2$ an open set with smooth boundary, \mathcal{S}_0 be a closed, bounded, connected and simply connected subset of Ω with smooth boundary, $u_0 \in \mathcal{H}_{\mathcal{S}_0}$. Let (\mathcal{S}, u) a weak solution of (3.0.1)-(3.0.9) with initial data (\mathcal{S}_0, u_0) for some $T > 0$. Then (\mathcal{S}, u) is the unique weak solution to (3.0.1)-(3.0.9) with initial data (\mathcal{S}_0, u_0) in $[0, T)$.*

To prove the uniqueness result we will show a Grönwall type estimate for the difference of two solutions. In the case of parabolic equations with a time independent domain, this estimate is obtained by considering the difference of the weak formulations satisfied by the two weak solutions tested with the difference of the two solutions. This strategy cannot directly apply in our setting so the idea is to use a change of variables such that the solutions are defined on the same domain and then we perform the standard strategy. To deal with the extra terms coming from the change of variables, we need some improved regularity result.

The rest of the section is divided in three parts. In the first one we show the regularity results, in the second one we present the change of variables and in the third one we prove the Grönwall estimate that implies uniqueness.

4.2.1 Regularity in time for the solutions

We present some estimates that are the keypoint for the proof of uniqueness. Fix now (\mathcal{S}, u) a weak solution of (3.0.1)-(3.0.9) in a time interval $[0, T)$. We define:

$$\mathcal{F}_T = \cup_{t \in (0, T)} \{t\} \times \mathcal{F}(t).$$

The first estimates are the following.

Lemma 4.2.1. *The following holds true*

$$((u \cdot \nabla)u, u) \in L^{\frac{4}{3}}(\mathcal{F}_T; \mathbb{R}^4).$$

Proof. The estimates follow by interpolation inequality and Hölder inequality. \square

The second estimates are related to the regularization result due to viscosity.

Lemma 4.2.2. *There exists a strictly positive $\tilde{T} \leq T$ such that the following hold true*

$$tu \in L^{\frac{4}{3}}((0, \tilde{T}); W^{2, \frac{4}{3}}(\mathcal{F}(t))), \quad (t\partial_t u, t\nabla p) \in L^{\frac{4}{3}}(\mathcal{F}_{\tilde{T}}; \mathbb{R}^4).$$

Proof. The proof is based on the study of an auxiliary system. Let l and r such that $u_{\mathcal{S}}(x, t) = l(t) + (x - h(t))^{\perp} r(t)$. Then consider the system in the unknowns $(\mathbf{l}, \mathbf{r}, v, p)$

$$\begin{aligned} \frac{\partial v}{\partial t} - \Delta v + \nabla p &= u - tu \cdot \nabla u && \text{for } x \in \mathcal{F}(t), \\ \operatorname{div} v &= 0 && \text{for } x \in \mathcal{F}(t), \\ v \cdot n &= v_{\mathcal{S}} \cdot n && \text{for } x \in \partial\mathcal{S}(t), \\ (D(v)n) \cdot \tau &= -\alpha(v - v_{\mathcal{S}}) \cdot \tau && \text{for } x \in \partial\mathcal{S}(t), \\ v \cdot n &= 0 && \text{for } x \in \partial\Omega, \\ (D(v)n) \cdot \tau &= -\alpha v \cdot \tau && \text{for } x \in \partial\Omega, \\ m'(t) &= - \int_{\partial\mathcal{S}(t)} \Sigma(v, p)n \, ds + ml(t), \\ \mathcal{J}\mathbf{r}'(t) - \mathcal{J}r(t) &= - \int_{\partial\mathcal{S}(t)} (x - h(t))^{\perp} \cdot \Sigma(v, p)nds, \\ v|_{t=0} &= 0 && \text{for } x \in \mathcal{F}_0, \\ \mathbf{l}(0) &= 0, \quad \mathbf{r}(0) = 0. \end{aligned} \tag{4.2.1}$$

The following holds true:

1. Weak solution of (4.2.1) are unique (we can test the equation with the difference of two solutions because the domain is not an unknown of the problem);
2. There exists a unique strong solution $(\mathbf{l}, \mathbf{r}, v, p)$ of (4.2.1) in $L^{4/3} - L^{4/3}$ for a short time see Theorem 3.1.5.
3. Any strong solution of (4.2.1) is a weak solution (by some integrations by parts);
4. tu is a weak solution of (4.2.1), so it is strong for short time.

\square

4.2.2 Solution in the new frame

Let (\mathcal{S}_1, u_1) and (\mathcal{S}_2, u_2) two weak solutions of (3.0.1)-(3.0.9) on some common time interval $[0, T)$, with $T > 0$. In this subsection we present a change of variables that moves \mathcal{S}_2 on \mathcal{S}_1 and the equations satisfy by \tilde{u}_2 , which is u_2 after the change of variables.

We recall that if (\mathcal{S}_i, u_i) is one of the two weak solutions, then for any $\tau \in (0, T)$ there exist $l_i \in C([0, \tau]; \mathbb{R}^2)$ and $r_i \in C([0, \tau]; \mathbb{R})$ such that $\mathcal{S}_i = \mathcal{S}^{l_i, r_i}$ and there exists $\delta > 0$ such that $\operatorname{dist}(\mathcal{S}_i, \partial\Omega) > \delta$ for any $i = 1, 2$ and for any $t \in [0, \tau]$.

We define X_i as in Claim 3.1.1, where in addition we ask that X_i coincides with the solid motion associated with \mathcal{S}_i for any (t, x) such that $\operatorname{dist}(x, \partial\Omega) \geq \delta/2$, X_i is the identity in a $\delta/4$

neighbourhood of $\delta\Omega$, i.e. $X_i(t, x) = x$ for any (t, x) such that $\text{dist}(x, \partial\Omega) \leq \delta/4$, and we define the change of variables $\varphi : [0, \tau] \times \Omega \rightarrow \Omega$ and its inverse $\psi : [0, \tau] \times \Omega \rightarrow \Omega$ as follow $\varphi(t, x) = X_2(t, X_1^{-1}(t, x))$ and $\psi(t, y) = X_1(t, X_2^{-1}(t, y))$. We easily see that $\varphi, \psi \in C^1(0, \tau; C^\infty(\Omega))$.

We can define

$$\begin{aligned}\tilde{u}_2(t, x) &= \nabla\psi(t, \varphi(t, x))u_2(t, \varphi(t, x)), \\ \tilde{u}_{F,2}(t, x) &= \nabla\psi(t, \varphi(t, x))u_{F,2}(t, \varphi(t, x)), \\ \tilde{u}_{S,2}(t, x) &= Q_1(t)Q_2^T(t)u_{S,2}(t, h_2(t) + Q_2(t)Q_1^T(t)(x - h_1(t))).\end{aligned}$$

Note that $u_{S,2}(t, \bar{x}) = l_2(t) + (\bar{x} - h_2(t))^\perp r_2(t)$, then we have

$$\begin{aligned}\tilde{u}_{S,2}(t, x) &= Q_1(t)Q_2^T(t) \left(l_2(t) + Q_2(t)Q_1^T(t)(x - h_1(t))^\perp r_2(t) \right) \\ &= Q_1(t)Q_2^T(t)l_2(t) + (x - h_1(t))^\perp r_2(t),\end{aligned}$$

so we define $\tilde{l}_2(t) = Q_1(t)Q_2^T(t)l_2(t)$ and $\tilde{r}_2(t) = r_2(t)$. Finally by Lemma 4.2.2 in the previous section we have proved that for a short time we have improved regularity that leads us to define the pressure $p_2(t, \bar{x})$, so we define $\tilde{p}_2(t, x) = p_2(t, \varphi(t, x))$. We are now able to write the equations satisfied by $\tilde{u}_2, \tilde{l}_2, \tilde{r}_2$ and \tilde{p}_2 . The computations were done in Section 3.1.2.3. The first equation of (3.1.7) reads as

$$\partial_t \tilde{u}_2 + (\mathcal{M} - \mathcal{L})\tilde{u}_2 + \mathcal{N}\tilde{u}_2 + \mathcal{G}\tilde{p}_2 = 0$$

If we multiply on the left the above equality by $\nabla\varphi$, we sum and subtract the expression

$$\partial_t \tilde{u}_2 + (\tilde{u}_2 \cdot \nabla)\tilde{u}_2 - \Delta\tilde{u}_2 + \nabla\tilde{p}_2,$$

we write explicitly $\mathcal{M}, \mathcal{L}, \mathcal{N}$ and \mathcal{G} and we use Einstein's summation convention, the i -th component of the equation reads as

$$\begin{aligned}0 &= \partial_t \tilde{u}_2^i + \tilde{u}_2^j \partial_j \tilde{u}_2^i + \partial_i \tilde{p}_2 - \Delta \tilde{u}_2^i \\ &+ (\partial_k \varphi^i - \delta_{ik}) \partial_t \tilde{u}_2^k + \partial_k \varphi^i \partial_l \tilde{u}_2^k (\partial_l \psi^l) + (\partial_k \partial_t \varphi^i) \tilde{u}_2^k + (\partial_{kl}^2 \varphi^i) (\partial_t \psi^l) \tilde{u}_2^k \\ &+ \tilde{u}_2^l \partial_l \tilde{u}_2^k (\partial_k \varphi^i - \delta_{ik}) + (\partial_{lk}^2 \varphi^i) \tilde{u}_2^l \tilde{u}_2^k + \partial_k \tilde{p}_2 (\partial_i \psi^k - \delta_{ik}) \\ &- \partial_j \psi^m (\partial_{mk}^2 \varphi^i) \partial_l \tilde{u}_2^k \partial_j \psi^l - (\partial_k \varphi^i \partial_j \psi^m \partial_j \psi^l - \delta_{ik} \delta_{jm} \delta_{jl}) \partial_{ml}^2 \tilde{u}_2^k \\ &- \partial_k \varphi^i \partial_l \tilde{u}_2^k (\partial_{jj}^2 \psi^l) \\ &- \partial_j \psi^m (\partial_{mlk}^3 \varphi^i) \partial_j \psi^l \tilde{u}_2^k - (\partial_{lk}^2 \varphi^i) \partial_{jj}^2 \psi^l \tilde{u}_2^k - (\partial_{lk}^2 \varphi^i) \partial_j \psi^l \partial_j \psi^m \partial_m \tilde{u}_2^k.\end{aligned}\tag{4.2.2}$$

The equation above is true almost everywhere if we restrict the time interval where the estimates of Lemma 4.2.2 hold.

4.2.3 Grönwall estimate

In this subsection we present the Grönwall estimate for the $u_1 - \tilde{u}_2$, which implies the proof of Theorem 4.2.1.

Recall that equation (4.2.2) is true almost everywhere if we restrict the time interval where the estimates of Lemma 4.2.2 hold. Moreover the change of variables conserves the Navier-slip boundary conditions, i.e. it holds $(D\tilde{u}_2 n) \cdot \tau = -\alpha(\tilde{u}_2 - \tilde{u}_{S,2})$ and $\tilde{u}_2 \cdot n = \tilde{u}_{S,2} \cdot n$ on $\partial\mathcal{S}(t)$,

because it is rigid in an open neighbourhood of $\mathcal{S}(t)$ and of $\partial\Omega$. We multiply the equation (4.2.2) with a test function $\varphi \in \mathcal{W}_\tau$ associated with the motion of \mathcal{S}_1 to arrive at

$$\begin{aligned}
& - \int_0^\tau \int_{\mathcal{F}_1(t)} \tilde{u}_{F,2} \cdot \partial_t \varphi_F - \int_0^\tau \int_{\mathcal{S}_1(t)} \rho_S \tilde{u}_{S,2} \cdot \partial_t \varphi_S \\
& + \int_0^\tau \int_{\mathcal{F}_1(t)} \tilde{u}_{F,2} \otimes \tilde{u}_{F,2} : \nabla \varphi_F + 2 \int_0^\tau \int_{\mathcal{F}_1(t)} D\tilde{u}_{F,2} : D\varphi_F \\
& + 2\alpha \int_0^\tau \int_{\partial\Omega} \tilde{u}_{F,2} \cdot \varphi_F + 2\alpha \int_0^\tau \int_{\partial\mathcal{S}_1(t)} (\tilde{u}_{F,2} - \tilde{u}_{S,2}) \cdot (\varphi_F - \varphi_S) \\
& - \int_{\mathcal{F}_1(0)} \tilde{u}_{F,2} \cdot \varphi_F|_{t=0} - \int_{\mathcal{S}_1(0)} \rho_S \tilde{u}_{S,2} \cdot \varphi_S|_{t=0} \\
& + \int_{\mathcal{F}_1(\tau)} (\tilde{u}_{F,2} \cdot \varphi_F)|_{t=\tau} + \int_{\mathcal{S}_1(\tau)} (\rho_S \tilde{u}_{S,2} \cdot \varphi_S)|_{t=\tau} = - \int_0^\tau \int_{\mathcal{F}_1(t)} \tilde{f} \cdot \varphi_F dt dx,
\end{aligned}$$

where \tilde{f} is just the last five lines of (4.2.2). We denote by $\hat{u} = u_1 - \tilde{u}_2$, and we take the difference of the weak formulations satisfied by u_1 and \tilde{u}_2 to obtain

$$\begin{aligned}
& - \int_0^\tau \int_{\mathcal{F}_1(t)} \hat{u}_F \cdot \partial_t \varphi_F - \int_0^\tau \int_{\mathcal{S}_1(t)} \rho_S \hat{u}_S \cdot \partial_t \varphi_S - \int_0^\tau \int_{\mathcal{F}_1(t)} u_{F,1} \otimes \hat{u}_F : \nabla \varphi_F \\
& + 2 \int_0^\tau \int_{\mathcal{F}_1(t)} D\hat{u}_F : D\varphi_F + 2\alpha \int_0^\tau \int_{\partial\Omega} \hat{u}_F \cdot \varphi_F \\
& + 2\alpha \int_0^\tau \int_{\partial\mathcal{S}_1(t)} (\hat{u}_F - \hat{u}_S) \cdot (\varphi_F - \varphi_S) - \int_{\mathcal{F}_1(0)} \hat{u}_F \cdot \varphi_F|_{t=0} \\
& - \int_{\mathcal{S}_1(0)} \rho_S \hat{u}_S \cdot \varphi_S|_{t=0} + \int_{\mathcal{F}_1(\tau)} (\hat{u}_F \cdot \varphi_F)|_{t=\tau} + \int_{\mathcal{S}_1(\tau)} (\rho_S \hat{u}_S \cdot \varphi_S)|_{t=\tau} \\
& = - \int_0^\tau \int_{\mathcal{F}_1(t)} \hat{u}_F \otimes u_{F,2} : \nabla \varphi_F dx dt - \int_0^\tau \int_{\mathcal{F}_1(t)} \tilde{f} \cdot \varphi_F dt dx,
\end{aligned}$$

for any $\varphi \in \mathcal{W}_\tau$. The vector field fulfil the hypothesis of Theorem 4.1.2, in fact $\hat{u} \in \mathcal{V}_\tau$ and $\frac{D}{dt}\hat{u} \in \mathcal{E}_\tau^*$, which implies that we can test the previous equation with \hat{u} to obtain

$$\begin{aligned}
& \frac{1}{2} \int_{\mathcal{F}_1(\tau)} |\hat{u}_F|^2(\tau, \cdot) dx + \frac{1}{2} \int_{\mathcal{S}_1(\tau)} \rho_S |\hat{u}_S|^2(\tau, \cdot) dx + 2 \int_0^\tau \int_{\mathcal{F}_1(t)} |D\hat{u}_F|^2 dx dt \\
& + 2\alpha \int_0^\tau \int_{\partial\Omega} |\hat{u}_F|^2 ds dt + 2\alpha \int_0^\tau \int_{\partial\mathcal{S}_1(t)} |\hat{u}_F - \hat{u}_S|^2 ds dt \\
& = - \int_0^\tau \int_{\mathcal{F}_1(t)} \hat{u}_F \otimes u_{F,2} : \nabla \hat{u}_F dx dt - \int_0^\tau \int_{\mathcal{F}_1(t)} \tilde{f} \cdot \hat{u}_F dt dx.
\end{aligned}$$

We have to estimate the right hand side of the above inequality. The first of the two terms

can be estimated via a standard technique i.e.

$$\begin{aligned}
& \left| \int_0^\tau \int_{\mathcal{F}_1(t)} \hat{u}_F \otimes u_{F,2} : \nabla \hat{u}_F dx dt \right| \\
& \leq \varepsilon \int_0^\tau \int_{\mathcal{F}_1(t)} |\nabla \hat{u}_F|^2 dx dt \\
& \quad + \frac{1}{\varepsilon} \int_0^\tau \left(\int_{\mathcal{F}_1(t)} |\hat{u}_F|^4 dx \right)^{1/2} \left(\int_{\mathcal{F}_1(t)} |u_{F,2}|^4 dx \right)^{1/2} dt \\
& \leq 2\varepsilon \int_0^\tau \int_{\mathcal{F}_1(t)} |\nabla \hat{u}_F|^2 dx dt \\
& \quad + \frac{C}{\varepsilon} \|u_2\|_{L^\infty(0,\tau;L^2(\Omega))} \int_0^\tau \int_{\mathcal{F}_1(t)} |\hat{u}_F|^2 dx \int_{\mathcal{F}_1(t)} |\nabla u_{F,2}|^2 dx dt.
\end{aligned}$$

To absorb on the r.h.s. the norm of the gradient of \hat{u}_F , we use the Korn inequality (Theorem 3.1 of [DL76]). For the second one we follow the estimate of [GS15], in fact these estimates do not depend on the boundary condition of our problem, if we take as example the first term of \tilde{f} we have the estimates

$$\begin{aligned}
& \left| \int_0^\tau \int_{\mathcal{F}_1(t)} (\partial_k \varphi^i - \delta_{ik}) \partial_t \tilde{u}_2^k \hat{u}_F dx dt \right| \\
& \leq \int_0^\tau \left\| \frac{1}{t} (\partial_k \varphi^i - \delta_{ik}) \right\|_{L^\infty(\Omega)} \|t \partial_t \tilde{u}_2^k\|_{L^{4/3}(\mathcal{F}_1(t))} \|\hat{u}_F\|_{L^4(\mathcal{F}_1(t))} dt \\
& \leq C \int_0^\tau \|(\hat{l}, \hat{r})\|_{L^\infty(0,\tau)} \|t \partial_t \tilde{u}_2^k\|_{L^{4/3}(\mathcal{F}_1(t))} \|\hat{u}_F\|_{L^4(\mathcal{F}_1(t))} dt \\
& \leq C \int_0^\tau \|(\hat{l}, \hat{r})\|_{L^\infty(0,\tau)}^2 \|t \partial_t \tilde{u}_2^k\|_{L^{4/3}(\mathcal{F}_1(t))}^{4/3} dt \\
& \quad + C \int_0^\tau \|t \partial_t \tilde{u}_2^k\|_{L^{4/3}(\mathcal{F}_1(t))}^{2/3} \|\hat{u}\|_{L^2(\mathcal{F}_1(t))} \|\nabla \hat{u}\|_{L^2(\mathcal{F}_1(t))} dt \\
& \leq C \int_0^\tau \|(\hat{l}, \hat{r})\|_{L^\infty(0,\tau)}^2 \|t \partial_t \tilde{u}_2^k\|_{L^{4/3}(\mathcal{F}_1(t))}^{4/3} dt \\
& \quad + C \frac{1}{\varepsilon} \int_0^\tau \|t \partial_t \tilde{u}_2^k\|_{L^{4/3}(\mathcal{F}_1(t))}^{4/3} \|\hat{u}\|_{L^2(\mathcal{F}_1(t))}^2 dt \\
& \quad + C\varepsilon \int_0^\tau \|\nabla \hat{u}\|_{L^2(\mathcal{F}_1(t))}^2 dt.
\end{aligned}$$

In an analogous way we can obtain the following Grönwall estimate

$$\begin{aligned}
& \|\hat{u}(\tau, \cdot)\|_{L^2(\mathcal{F}_1(t))}^2 + m |l_{\hat{u}}(\tau)|^2 + \mathcal{J} |r_{\hat{u}}(\tau)|^2 \\
& \leq \int_0^\tau C \mathcal{B}(t) \left[\max_{s \in [0,t]} \|\hat{u}(s, \cdot)\|_{L^2(\mathcal{F}_1(s))}^2 + \max_{s \in [0,t]} |(\hat{h}, \hat{\theta}, \hat{l}, \hat{r})|^2 \right] dt,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{B}(t) = & \|\tilde{u}_2\|_{L^\infty(0,T;L^2(\mathcal{F}_1(t)))} \left(1 + \|\nabla \tilde{u}_2(t, \cdot)\|_{L^2(\mathcal{F}_1(t))} \right) \\
& + \|\tilde{u}_2\|_{L^\infty(0,T;L^2(\mathcal{F}_1(t)))}^{1/2} \|\nabla \tilde{u}_2(t)\|_{L^2(\mathcal{F}_1(t))}^{1/2} \|t \nabla \tilde{u}_2(t)\|_{L^4(\mathcal{F}_1(t))} \\
& + \left(\|t \partial_t \tilde{u}_2\|_{L^{4/3}(\mathcal{F}_1(t))} + \|t \tilde{u}_2\|_{W^{2,4/3}(\mathcal{F}_1(t))} + \|t \nabla \tilde{p}_2\|_{L^{4/3}(\mathcal{F}_1(t))} \right)^{4/3}.
\end{aligned}$$

Moreover we have

$$\frac{d}{dt} (|\hat{h}|^2 + |\hat{\theta}|^2) \leq C(|\hat{l}|^2 + |\hat{r}|^2 + |\hat{h}|^2 + |\hat{\theta}|^2)$$

and we have $\mathcal{B} \in L^1(0, \tau)$. The Grönwall lemma leads us to conclude that uniqueness holds locally in time.

Moreover, by a continuation argument, we deduce that uniqueness holds on the whole time interval $[0, T)$.

4.3 Future directions and open problems

In this section we discuss some open problems related to Chapter 4.

- ▷ The first question is if *it possible to prove continuity in time with values in L^2 , the energy equality and uniqueness in the case of many objects*. We believe that this is possible and we believe that our approach is enough robust.
- ▷ The second question is if *uniqueness holds after contact*. We believe that the answer is negative but a proof is missing. In the case where no-slip boundary conditions are prescribed on the boundary of the fluid, non uniqueness was showed in [Sta03] with the help of an artificial source term. The key idea of the demonstration is that when the solid touches the boundary and no-slip boundary condition are prescribed, the admissible test functions for the weak formulation are identically zero in the solid domain. For Navier-slip conditions this is not true and is the main issue to proceed as in [Sta03].
- ▷ The third question is if *the proof of uniqueness can be adapted to different boundary conditions*. The answer is yes an for example the proof can be carry out also in the case of mixed boundary condition. Moreover it has been adapted for example in [MNR19] to prove uniqueness of fluid-structure problem in three dimension under Prodi type boundary conditions.

Asymptotic limit of a shrinking source and sink

Dans ce chapitre, nous établissons le système limite d'un écoulement dans un domaine caractérisé par une source et un puits macroscopiques qui se contractent en deux points distincts.

This Chapter is dedicated to the study of a system that describes the flow of an incompressible inviscid fluid in a region characterized by the presence of a source and a sink, in particular we will study the limiting equations as the size of the source and of the sink tends to zero. We briefly recall the system (2.2.4) that was introduced in Section 2.2.2. Let Ω an open bounded connected simply-connected non-empty subset of \mathbb{R}^2 with smooth boundary, let \mathcal{S}^+ and \mathcal{S}^- two open connected simply-connected non-empty disjoint subsets of \mathbb{R}^2 compactly contained in Ω with smooth boundary and let $\mathcal{F} = \Omega \setminus (\overline{\mathcal{S}^+} \cup \overline{\mathcal{S}^-})$ the fluid domain. We will call \mathcal{S}^+ the source and \mathcal{S}^- the sink. The equations in the unknown (u, p) that model the dynamics read as

$$\begin{aligned}
 \partial_t u + u \cdot \nabla u + \nabla p &= 0 && \text{in } \mathbb{R}^+ \times \mathcal{F}, \\
 \operatorname{div} u &= 0 && \text{in } \mathbb{R}^+ \times \mathcal{F}, \\
 u \cdot n &= g && \text{on } \mathbb{R}^+ \times \partial\mathcal{F}, \\
 \operatorname{curl} u &= \omega^+ && \text{on } \mathbb{R}^+ \times \partial\mathcal{S}^+, \\
 u(0, \cdot) &= u^{in} && \text{in } \mathcal{F},
 \end{aligned} \tag{5.0.1}$$

where $u : \mathbb{R}^+ \times \mathcal{F} \rightarrow \mathbb{R}^2$ is a velocity field, $p : \mathbb{R}^+ \times \mathcal{F} \rightarrow \mathbb{R}$ is a pressure, n is the normal exiting from the domain \mathcal{F} , g is the normal component of the velocity on the boundary such that it has average zero and satisfies $g \leq -c < 0$ on $\mathbb{R}^+ \times \partial\mathcal{S}^+$, $g \geq c > 0$ on $\mathbb{R}^+ \times \partial\mathcal{S}^-$, $g = 0$ on $\mathbb{R}^+ \times \partial\Omega$, ω^+ is the entering vorticity and u^{in} such that $\operatorname{div} u^{in} = 0$ is the initial datum.

The target of this chapter is to show that when the source \mathcal{S}^+ and the sink \mathcal{S}^- shrink homothetically to x_+ and x_- respectively, with $x_+ \neq x_-$, the system (5.0.1) converges to the

coupled equations:

$$\begin{aligned}
\partial_t \omega + u \cdot \nabla \omega &= j \delta_{x_+} - \left(j + \frac{d}{dt} \int_{\mathcal{F}} \omega \right) \delta_{x_-} && \text{in } \mathbb{R}^+ \times \Omega, \\
u &= \mathcal{J}_\Omega[\mu \delta_{x_+} - \mu \delta_{x_-}] + K_\Omega[\omega + \mathcal{C}_+ \delta_{x_+} + \mathcal{C}_- \delta_{x_-}] && \text{in } \mathbb{R}^+ \times \Omega, \\
\mathcal{C}_+(t) &= \mathcal{C}_+^{in} - \int_0^t j && \text{in } \mathbb{R}^+, \\
\mathcal{C}_-(t) &= \mathcal{C}_-^{in} + \int_{\mathcal{F}} \omega(t, \cdot) - \int_{\mathcal{F}} \omega^{in} + \int_0^t j && \text{in } \mathbb{R}^+,
\end{aligned} \tag{5.0.2}$$

where μ and j are the limit of the integrals on $\partial\mathcal{S}^+$ of the quantities respectively g and ω^+g . Above the notation δ_x stands for the Delta measure in the point x and K_Ω and \mathcal{J}_Ω are respectively the Biot-Savart operator in Ω and the counterpart of the Biot-Savart operator for nonzero divergence. For the definitions of K_Ω and \mathcal{J}_Ω , see Section 2.2.3 of the introduction. Let us highlight that the system (2.2.4) exhibits the presence of a point source in x_+ , of a point sink in x_- and of two point vortices in x_+ and x_- whose strength depends on time through μ , j and ω . The letter c in the notations \mathcal{C}_+ and \mathcal{C}_- refers to the circulations respectively around \mathcal{S}^+ and \mathcal{S}^- .

Because of the singular terms in the system (5.0.2) it is natural to consider a weak formulation. Indeed the solutions that we will obtain in this chapter are weak solutions of the system (5.0.2) in the following sense.

Definition 5.0.1. Let x_+ and x_- two different points in Ω , let $\omega^{in} \in L^\infty(\Omega)$, let \mathcal{C}_+^{in} and \mathcal{C}_-^{in} two real constants, let $\mu \in L^1_{loc}(\mathbb{R}^+)$ such that $\mu \leq 0$, let $j \in L^1_{loc}(\mathbb{R}^+)$ such that $\text{supp}(j) \subset \text{supp}(\mu)$ and let $q \in (1, 2)$. We say that a couple $(\omega, u) \in L^\infty_{loc}(\mathbb{R}^+; L^\infty(\Omega)) \cap L^1_{loc}(\mathbb{R}^+; L^q(\Omega))$ is a weak solution of (5.0.2) if

▷ for any $\varphi \in C_c^\infty(\mathbb{R}^+ \times \overline{\Omega})$,

$$\begin{aligned}
\int_{\Omega} \omega^{in} \varphi(0, \cdot) dx + \int_{\mathbb{R}^+} \int_{\Omega} \omega \partial_t \varphi dx dt + \int_{\mathbb{R}^+} \int_{\Omega} u \cdot \nabla \varphi \omega dx dt &= \int_{\mathbb{R}^+} j \varphi(\cdot, x_+) dt \\
&- \int_{\mathbb{R}^+} j \varphi(\cdot, x_-) dt + \int_{\Omega} \omega^{in} dx \varphi(0, x_-) + \int_{\mathbb{R}^+} \left(\int_{\Omega} \omega(t, \cdot) dx \right) \partial_t \varphi(\cdot, x_-) dt,
\end{aligned} \tag{5.0.3}$$

▷ for any $\xi \in C^\infty(\overline{\Omega})$ and for almost any time,

$$\int_{\Omega} u \cdot \nabla \xi dx = \mu \xi(x_+) - \mu \xi(x_-), \tag{5.0.4}$$

▷ for any $\zeta \in C_c^\infty(\Omega)$ and for almost any time,

$$\begin{aligned}
\int_{\Omega} u \cdot \nabla^\perp \zeta dx &= - \int_{\Omega} \omega(t, \cdot) \zeta dx + \left[\mathcal{C}_+^{in} - \int_0^t j dt \right] \zeta(x_+) \\
&+ \left[\mathcal{C}_-^{in} + \int_{\Omega} \omega dx - \int_{\Omega} \omega^{in} + \int_0^t j dt \right] \zeta(x_-).
\end{aligned} \tag{5.0.5}$$

The plan of the chapter is to rewrite the system (5.0.1) in the vorticity form, to give a definition of weak solutions for the vorticity form of (5.0.1) and recall an existence result from [Ale76]. Then we will state the main result, followed by a section dedicated to the reflection method and finally we prove the main theorem. We conclude the chapter with an existence result based on a Schauder fixed point method whereas the proof in [Ale76] is based on a vanishing viscosity method.

5.1 Weak vorticity formulation and existence result for Judovič's system

Weak solutions of the limit system (5.0.2) will be obtained by passing to the limit in a weak formulation of the vorticity form of (5.0.1). Let us therefore recall first the strong formulation of the vorticity form of (5.0.1). We apply the curl to the first equation of (5.0.1) to obtain a transport equation for the vorticity ω and we recover the velocity field by solving a div-curl system associated with the vorticity, the normal component of the velocity on $\partial\mathcal{F}$ and the circulations around $\partial\mathcal{S}^+$ and $\partial\mathcal{S}^-$. In the case where $v \cdot n = 0$ on $\partial\mathcal{F}$, Kelvin's theorem ensures that the circulation around $\partial\mathcal{S}^+$, $\partial\mathcal{S}^-$ and $\partial\Omega$ remains constant. In our setting this is not true anymore. Indeed the circulation around $\partial\mathcal{S}^i$ evolves according to the equations

$$\frac{d}{dt} \oint_{\partial\mathcal{S}^i} v(t, \cdot) \cdot \tau ds = - \oint_{\partial\mathcal{S}^i} \omega(t, \cdot) g(t, \cdot) ds, \quad \oint_{\partial\mathcal{S}^i} v(0, \cdot) \cdot \tau ds = \oint_{\partial\mathcal{S}^i} v^{in} \cdot \tau ds$$

for $i \in \{-, +\}$ and it is constant around $\partial\Omega$, where we denoted by τ the counterclockwise tangent vectors to the boundary. The equations above were obtained for example in Lemma 1.2 of [Jud63] by explicit computations assuming that the solution is regular enough. Moreover if we denote the circulations by

$$\mathcal{C}_\Omega = \oint_{\partial\Omega} v(t, \cdot) \cdot \tau ds, \quad \mathcal{C}_+(t) = \oint_{\partial\mathcal{S}^+} v(t, \cdot) \cdot \tau ds \quad \text{and} \quad \mathcal{C}_-(t) = \oint_{\partial\mathcal{S}^-} v(t, \cdot) \cdot \tau ds,$$

it holds

$$\int_{\mathcal{F}} \omega(t, \cdot) dx = \mathcal{C}_\Omega + \mathcal{C}_+(t) + \mathcal{C}_-(t).$$

With this preliminary, the system (5.0.1) in the vorticity form reads as

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= 0 && \text{in } \mathbb{R}^+ \times \mathcal{F}, \\ \omega &= \omega^+ && \text{on } \mathbb{R}^+ \times \partial\mathcal{S}^+, \\ \omega(0, \cdot) &= \omega^{in} && \text{in } \mathcal{F}, \end{aligned} \tag{5.1.1}$$

where u is recovered by the elliptic system:

$$\begin{aligned} \operatorname{div} u &= 0 && \text{in } \mathcal{F}, \\ \operatorname{curl} u &= \omega && \text{in } \mathcal{F}, \\ u \cdot n &= g && \text{on } \partial\mathcal{F}, \end{aligned} \tag{5.1.2}$$

$$\begin{aligned} \mathcal{C}_+(t) &= \mathcal{C}_+^{in} - \int_0^t \oint_{\partial\mathcal{S}^+} \omega^+ g ds dt, \\ \mathcal{C}_-(t) &= \mathcal{C}_-^{in} - \int_0^t \oint_{\partial\mathcal{S}^-} \omega g ds dt. \end{aligned}$$

We now turn to the weak vorticity formulation of the system (5.1.1)-(5.1.2).

Definition 5.1.1. Let $\omega^{in} \in L^\infty(\mathcal{F})$ the initial datum, let \mathcal{C}_+^{in} and \mathcal{C}_-^{in} real constants, let $g \in L^\infty_{loc}(\mathbb{R}^+; W^{1,\infty}(\partial\mathcal{F}))$ such that $g \leq -c < 0$ in $\partial\mathcal{S}^+$, $g \geq c > 0$ in $\partial\mathcal{S}^-$, $g = 0$ on $\partial\Omega$ and such that $\oint_{\partial\mathcal{S}^+} g + \oint_{\partial\mathcal{S}^-} g = 0$ and let $\omega^+ \in L^\infty_{loc}(\mathbb{R}^+; L^\infty(\partial\mathcal{S}^+))$. Then a triple

$(\omega, \omega^-, u) \in L_{loc}^\infty(\mathbb{R}^+; L^\infty(\mathcal{F})) \times L_{loc}^\infty(\mathbb{R}^+ \times \partial\mathcal{S}^-) \times L_{loc}^\infty(\mathbb{R}^+; \mathcal{L}\mathcal{L}(\mathcal{F}))$ is a weak solution of (5.1.1)-(5.1.2) if for any $\varphi \in C_c^\infty(\mathbb{R}^+ \times \overline{\mathcal{F}})$,

$$\begin{aligned} \int_{\mathcal{F}} \omega^{in} \varphi(0, \cdot) dx + \int_{\mathbb{R}^+} \int_{\mathcal{F}} \omega \partial_t \varphi dx dt + \int_{\mathbb{R}^+} \int_{\mathcal{F}} u \cdot \nabla \varphi \omega dx dt \\ = \int_{\mathbb{R}^+} \oint_{\partial\mathcal{S}^+} g \omega^+ \varphi ds dt + \int_{\mathbb{R}^+} \oint_{\partial\mathcal{S}^-} g \omega^- \varphi ds dt, \end{aligned} \quad (5.1.3)$$

for any $\xi \in C^\infty(\overline{\mathcal{F}})$ and for almost any time,

$$\int_{\mathcal{F}} u \cdot \nabla \xi dx = \oint_{\partial\mathcal{S}^+} g \xi ds + \oint_{\partial\mathcal{S}^-} g \xi ds, \quad (5.1.4)$$

for any $\zeta \in C^\infty(\overline{\mathcal{F}})$ such that $\zeta = 0$ on $\partial\Omega$ and constant on $\partial\mathcal{S}^-$ and on $\partial\mathcal{S}^+$ with some real values respectively denoted by $\zeta(\partial\mathcal{S}^+)$ and $\zeta(\partial\mathcal{S}^-)$ and for almost any time,

$$\begin{aligned} \int_{\mathcal{F}} u \cdot \nabla^\perp \zeta dx = - \int_{\mathcal{F}} \omega(t, \cdot) \zeta dx + \left[\mathcal{C}_+^{in} - \int_0^t \oint_{\partial\mathcal{S}^+} g \omega^+ ds dt \right] \zeta(\partial\mathcal{S}^+) \\ + \left[\mathcal{C}_-^{in} - \int_0^t \oint_{\partial\mathcal{S}^-} g \omega^- ds dt \right] \zeta(\partial\mathcal{S}^-). \end{aligned} \quad (5.1.5)$$

Moreover

$$\begin{aligned} \|\omega(t)\|_{L^\infty(\mathcal{F})} \leq \max \left\{ \|\omega^{in}\|_{L^\infty(\mathcal{F})}, \|\omega^+\|_{L^\infty(0,t;L^\infty(\partial\mathcal{S}^+))} \right\} \quad \text{and} \\ \|\omega^-\|_{L^\infty((0,t) \times \partial\mathcal{S}^-)} \leq \max \left\{ \|\omega^{in}\|_{L^\infty(\mathcal{F})}, \|\omega^+\|_{L^\infty(0,t;L^\infty(\partial\mathcal{S}^+))} \right\}. \end{aligned}$$

Let us now recall the following existence theorem for the system (5.1.1)-(5.1.2).

Theorem 5.1.1. *Let $\omega^{in} \in L^\infty(\mathcal{F})$, let \mathcal{C}_+^{in} and \mathcal{C}_-^{in} real constants, let $g \in L_{loc}^\infty(\mathbb{R}^+; W^{1,\infty}(\partial\mathcal{F}))$ such that $g \leq -c < 0$ in $\partial\mathcal{S}^+$, $g \geq c > 0$ in $\partial\mathcal{S}^-$, $g = 0$ on $\partial\Omega$ and such that $\oint_{\partial\mathcal{S}^+} g + \oint_{\partial\mathcal{S}^-} g = 0$ and let $\omega^+ \in L_{loc}^\infty(\mathbb{R}^+; L^\infty(\partial\mathcal{S}^+))$. Then there exists a weak solution (ω, ω^-, u) of (5.1.1)-(5.1.2) in the sense of Definition 5.1.1.*

As already mentioned, the above Theorem was proved in [Ale76] by a vanishing viscosity limit. Let us also refer to [CC13] where the same method is used in a slightly different setting. In Section 5.5 we present an alternative approach by a Schauder fixed point theorem.

5.2 Main result: the source and sink shrinking limit

In this section we present the asymptotic result. Let Ω , \mathcal{S}^+ and \mathcal{S}^- as before, let $x_i \in \mathcal{S}^i$ for $i = +, -$, then we denote by

$$\mathcal{S}_\varepsilon^i = \left\{ x \in \mathbb{R}^2 \quad \text{s.t.} \quad x_i + \frac{x - x_i}{\varepsilon} \in \mathcal{S}^i \right\},$$

for $\varepsilon \leq 1$, moreover we assume that for any fixed ε the closure of $\mathcal{S}_\varepsilon^i$ are contained in Ω and are disjoint (these properties are clearly satisfied if \mathcal{S}^i are convex). We denote the fluid domain by $\mathcal{F}_\varepsilon = \Omega \setminus \cup \overline{\mathcal{S}_\varepsilon^i}$.

Before stating the main result we note that the limit velocity u defined in (5.0.2) is the sum of a regular vector field $K_\Omega[\omega]$ and a singular one $\mathcal{J}_\Omega[\mu\delta_{x_+} - \mu\delta_{x_-}] + K_\Omega[\mathcal{C}_+\delta_{x_+} + \mathcal{C}_-\delta_{x_-}]$. Moreover the singular part has a rather explicit expression as the sum of a singular vector field L which is associated with $\mu, j, \mathcal{C}_+^{in}$ and \mathcal{C}_-^{in} and defined by

$$\begin{aligned} L(t, x) = & \mu(t) \frac{x - x_+}{2\pi|x - x_+|^2} + \left(\mathcal{C}_+^{in} - \int_0^t j(s)ds \right) \frac{(x - x_+)^\perp}{2\pi|x - x_+|^2} \\ & - \mu(t) \frac{x - x_-}{2\pi|x - x_-|^2} + \left(\mathcal{C}_-^{in} + \int_\Omega \omega dx - \int_\Omega \omega^{in} + \int_0^t j(s)ds \right) \frac{(x - x_-)^\perp}{2\pi|x - x_-|^2}, \end{aligned} \quad (5.2.1)$$

and a regular part denoted by R which corrects the boundary condition of L . The vector field R is the unique solution of the system

$$\begin{aligned} \operatorname{div} R &= 0 && \text{in } \Omega, \\ \operatorname{curl} R &= 0 && \text{in } \Omega, \\ \nabla R \cdot n &= -L \cdot n && \text{on } \partial\Omega. \end{aligned} \quad (5.2.2)$$

The decomposition of u as the sum of $L + R + K_\Omega[\omega]$ will be useful in the proof of Theorem 5.2.1. We are now able to state the main theorem.

Theorem 5.2.1. *Let $\omega^{in} \in L^\infty(\Omega)$ the initial vorticity, let $\mathcal{C}_{+, \varepsilon}^{in}$ and $\mathcal{C}_{-, \varepsilon}^{in}$ real constants, let $g_\varepsilon \in L_{loc}^\infty(\mathbb{R}^+; W^{1, \infty}(\partial\mathcal{F}_\varepsilon))$ such that $g_\varepsilon \leq -c_\varepsilon < 0$ on $\partial\mathcal{S}_\varepsilon^+$, $g_\varepsilon \geq c_\varepsilon > 0$ on $\partial\mathcal{S}_\varepsilon^-$, $g_\varepsilon = 0$ on $\partial\Omega$ and such that $\oint_{\partial\mathcal{S}_\varepsilon^+} g_\varepsilon + \oint_{\partial\mathcal{S}_\varepsilon^-} g_\varepsilon = 0$ and let $\omega_\varepsilon^+ \in L_{loc}^\infty(\mathbb{R}^+; L^\infty(\partial\mathcal{S}_\varepsilon^+))$ such that the following conditions hold: $\|\omega_\varepsilon^+\|_{L^\infty(0, T; L^\infty(\partial\mathcal{S}_\varepsilon^+))} \leq C_T$ for any $T \in \mathbb{R}^+$ with C_T independent of ε , $\mathcal{C}_{i, \varepsilon}^{in} \rightarrow \mathcal{C}_i^{in}$ for $i = +, -$,*

$$\oint_{\partial\mathcal{S}_\varepsilon^+} g_\varepsilon(t, \cdot) ds \rightarrow \mu(t), \quad \oint_{\partial\mathcal{S}_\varepsilon^+} g_\varepsilon(t, \cdot) \omega_\varepsilon^+(t, \cdot) ds \rightarrow j(t), \quad \text{and} \quad \varepsilon^{1/q} \|g_\varepsilon(t, \cdot)\|_{L^q(\partial\mathcal{S}_\varepsilon^i)} \rightarrow 0, \quad (5.2.3)$$

in $L_{loc}^r(\mathbb{R}^+)$ with $r > 1$ as ε converges to zero, with $q \in (1, 2)$ and $i = +, -$. Moreover let $(\omega_\varepsilon, u_\varepsilon)$ weak solutions associated with the data $\omega^{in}|_{\mathcal{F}_\varepsilon}$, $\mathcal{C}_{i, \varepsilon}^{in}$, g_ε and ω_ε^+ . Then there exists a sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$, $\varepsilon_n \rightarrow 0$ such that

- ▷ $\omega_{\varepsilon_n} \xrightarrow{w^*} \omega$ in $L_{loc}^\infty(\mathbb{R}^+; L^\infty(\Omega))$ as n converges to infinity where we extended ω_{ε_n} by zero in $\mathcal{S}_\varepsilon^i$,
- ▷ $u_{\varepsilon_n} \rightarrow u = L + R + K_\Omega[\omega]$ in $L_{loc}^r(\mathbb{R}^+, L^q(\Omega))$ as n converges to infinity where we extended u_{ε_n} by zero in $\mathcal{S}_\varepsilon^i$ and L and R are defined respectively in (5.2.1)-(5.2.2),
- ▷ the couple (ω, u) is a weak solution of (5.0.2) in the sense of Definition 5.0.1.

The above result states that if a source and a sink shrink to two different points x_+ and x_- , the limiting system is characterized by a point source/sink and a point vortex in each of x_+ and x_- . We conclude the section with an example of g_ε and ω_ε^+ that give rise to a point source/sink and a point vortex in the limit in x_+ and in x_- . Let μ, j such that μ is non-positive,

$$\mu \in L_{loc}^r(\mathbb{R}^+), \quad j \in L_{loc}^r(\mathbb{R}^+) \quad \text{s.t.} \quad \operatorname{supp}(j) \subset \operatorname{supp}(\mu) \quad \text{and} \quad j/\mu \in L_{loc}^\infty(\mathbb{R}^+), \quad (5.2.4)$$

where we define $j/\mu = 0$ in $\mathbb{R}^+ \setminus \text{supp}(\mu)$. Consider

$$g_\varepsilon = \frac{\mu_\varepsilon}{\varepsilon} \left(\int_{\partial\mathcal{S}_\varepsilon^+} 1 ds \right)^{-1} \quad \text{in } \partial\mathcal{S}_\varepsilon^+, \quad g_\varepsilon = -\frac{\mu_\varepsilon}{\varepsilon} \left(\int_{\partial\mathcal{S}_\varepsilon^-} 1 ds \right)^{-1} \quad \text{in } \partial\mathcal{S}_\varepsilon^- \quad (5.2.5)$$

$$\text{and } \omega_\varepsilon^+ = \frac{j}{\mu} \quad \text{in } \partial\mathcal{S}_\varepsilon^+, \quad (5.2.6)$$

where

$$\mu_\varepsilon(t) = \begin{cases} -\varepsilon & \text{if } 0 \geq \mu(t) \geq -\varepsilon, \\ \mu(t) & \text{if } -\varepsilon \geq \mu(t) \geq -\frac{1}{\varepsilon}, \\ -\frac{1}{\varepsilon} & \text{if } \mu(t) \leq -\frac{1}{\varepsilon}. \end{cases}$$

With this choice

$$\oint_{\partial\mathcal{S}_\varepsilon^+} g_\varepsilon(t, \cdot) ds \rightarrow \mu(t) \quad \text{and} \quad \oint_{\partial\mathcal{S}_\varepsilon^+} g_\varepsilon(t, \cdot) \omega_\varepsilon^+(t, \cdot) ds \rightarrow j(t) \quad \text{in } L^1_{loc}(\mathbb{R}^+).$$

Moreover g_ε satisfies the third hypothesis of (5.2.3), in fact $\mu_\varepsilon \varepsilon^{\frac{1}{q}-1+\frac{1}{q}} \rightarrow 0$ for $q < 2$ as ε converges to zero.

In particular given $\omega^{in} \in L^\infty(\Omega)$, \mathcal{C}_+^{in} , \mathcal{C}_-^{in} , μ and j satisfying (5.2.4), Theorem 5.2.1, applied to ω^{in} , $\mathcal{C}_{i,\varepsilon}^{in} = \mathcal{C}_i^{in}$, g_ε and ω_ε^+ defined as in (5.2.5)-(5.2.6), shows existence of weak solutions of (5.0.2) in the sense of Definition 5.0.1 associated with the initial data ω , \mathcal{C}_i^{in} , μ and j .

In the rest of the chapter we prove Theorem 5.1.1 and 5.2.1. Before going directly in the proof of these results, we present the reflection method which is a key ingredient in the proof of Theorem 5.2.1.

5.3 An introduction to the reflection method

This subsection is entirely dedicated to present the reflection method which aims to translate the Laplace problems in \mathcal{F}_ε in others where scale estimates can be performed. Let $[\beta_\varepsilon] = [\beta_\varepsilon^+, \beta_\varepsilon^-, \beta_\varepsilon^\Omega] \in \tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)$ where \tilde{L}^q is the set of L^q with average zero, moreover we use the notation $\beta_\varepsilon \in \tilde{L}^q(\partial\mathcal{F}_\varepsilon)$ to denote the functions that restricted respectively to $\partial\mathcal{S}_\varepsilon^+$, $\partial\mathcal{S}_\varepsilon^-$ and $\partial\Omega$ are equal to β_ε^+ , β_ε^- and β_ε^Ω . Then we define $h[\beta_\varepsilon]$ to be the solution of

$$\begin{aligned} -\Delta h[\beta_\varepsilon] &= 0 & \text{in } \Omega, \\ \nabla h[\beta_\varepsilon] \cdot n &= \beta_\varepsilon^\Omega & \text{on } \partial\Omega, \end{aligned} \quad (5.3.1)$$

and for $i \in \{-, +\}$, we define $\hat{f}_\varepsilon^i[\beta_\varepsilon^i]$ to be the solution of

$$\begin{aligned} -\Delta \hat{f}_\varepsilon^i[\beta_\varepsilon^i] &= 0 & \text{in } \mathbb{R}^2 \setminus \overline{\mathcal{S}_\varepsilon^i}, \\ \nabla \hat{f}_\varepsilon^i[\beta_\varepsilon^i] \cdot n &= \beta_\varepsilon^i & \text{on } \partial\mathcal{S}_\varepsilon^i, \\ |\hat{f}_\varepsilon^i[\beta_\varepsilon^i]| &\rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{aligned} \quad (5.3.2)$$

finally we define

$$\mathbf{n}_\varepsilon[\beta_\varepsilon] = h[\beta_\varepsilon] + \sum_{i=-,+} \hat{f}_i \left[\beta_\varepsilon^i - \nabla h[\beta_\varepsilon] |_{\partial\mathcal{S}_\varepsilon^i} \cdot n \right],$$

which satisfies

$$\begin{aligned}
-\Delta \mathbf{n}_\varepsilon &= 0 && \text{in } \mathcal{F}_\varepsilon, \\
\nabla \mathbf{n}_\varepsilon[\beta_\varepsilon] \cdot \mathbf{n} &= \beta_\varepsilon^\Omega + \nabla \left(\sum_{i=-,+} \hat{f}_i [\beta_\varepsilon^i - \nabla h[\beta_\varepsilon]|_{\partial \mathcal{S}_\varepsilon^i} \cdot \mathbf{n}] \right) \cdot \mathbf{n} && \text{on } \partial \Omega, \\
\nabla \mathbf{n}_\varepsilon[\beta_\varepsilon] \cdot \mathbf{n} &= \beta_\varepsilon^i + \nabla \left(\hat{f}_j [\beta_\varepsilon^j - \nabla h[\beta_\varepsilon]|_{\partial \mathcal{S}_\varepsilon^j} \cdot \mathbf{n}] \right) \cdot \mathbf{n} && \text{on } \partial \mathcal{S}_\varepsilon^i, \text{ with } j \in \{+, -\} \setminus \{i\}.
\end{aligned} \tag{5.3.3}$$

From the above system we deduce that on the boundary $\partial \mathcal{F}_\varepsilon$

$$\nabla \mathbf{n}_\varepsilon[\beta_\varepsilon] \cdot \mathbf{n} = (\text{Id} + \mathcal{T}_\varepsilon)[\beta_\varepsilon]$$

where

$$\mathcal{T}_\varepsilon[\beta_\varepsilon] = \begin{cases} \nabla \left(\sum_{i=-,+} \hat{f}_i [\beta_\varepsilon^i - \nabla h[\beta_\varepsilon]|_{\partial \mathcal{S}_\varepsilon^i} \cdot \mathbf{n}] \right) \cdot \mathbf{n} & \text{on } \partial \Omega, \\ \nabla \left(\hat{f}_j [\beta_\varepsilon^j - \nabla h[\beta_\varepsilon]|_{\partial \mathcal{S}_\varepsilon^j} \cdot \mathbf{n}] \right) \cdot \mathbf{n} & \text{on } \partial \mathcal{S}_\varepsilon^i, \text{ with } j \in \{+, -\} \setminus \{i\}. \end{cases} \tag{5.3.4}$$

Let now $\alpha_\varepsilon = [\alpha_\varepsilon^+, \alpha_\varepsilon^-, \alpha_\varepsilon^\Omega] \in \tilde{L}^q(\partial \mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial \mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial \Omega)$ and consider the problem

$$\begin{aligned}
-\Delta \mathfrak{U}_\varepsilon[\alpha_\varepsilon] &= 0 && \text{in } \mathcal{F}_\varepsilon, \\
\nabla \mathfrak{U}_\varepsilon[\alpha_\varepsilon] \cdot \mathbf{n} &= \alpha_\varepsilon && \text{on } \partial \mathcal{F}_\varepsilon.
\end{aligned} \tag{5.3.5}$$

From (5.3.3) and (5.3.4), if $(I + \mathcal{T})$ is invertible then

$$\mathfrak{U}_\varepsilon[\alpha_\varepsilon] = \mathbf{n}_\varepsilon[(I + \mathcal{T})^{-1}[\alpha_\varepsilon]]. \tag{5.3.6}$$

This implies that to perform estimates independent of ε for the problem (5.3.5), it is enough to study the operator \mathcal{T} and the ε dependence of the solutions of the systems (5.3.1) and (5.3.2).

In what follow we will prove that $\text{Id} + \mathcal{T}$ is invertible for ε small enough and that (5.3.6) holds. To do that we start by recalling well-posedness results for (5.3.1), (5.3.2), (5.3.3) and finally we prove that $\text{Id} + \mathcal{T}$ is invertible.

Lemma 5.3.1. *For any $[\beta_\varepsilon] \in \tilde{L}^q(\partial \mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial \mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial \Omega)$ the system (5.3.1) has a unique solution $h[\beta_\varepsilon]$ in $L^q(\Omega)$. Moreover in any compact subset of Ω , $h[\beta_\varepsilon]$ is of class C^k and the norm is controlled by the L^q norm of β_ε^Ω .*

Lemma 5.3.2. *For any $\beta_\varepsilon^i \in \tilde{L}^q(\partial \mathcal{S}_\varepsilon^i)$, there exists a unique solution $\hat{f}_\varepsilon^i[\beta_\varepsilon^i]$ to (5.3.2), moreover for $q \in (1, 2)$ it satisfies*

$$\|\nabla \hat{f}_\varepsilon^i[\beta_\varepsilon^i]\|_{L^q(\mathcal{F}_\varepsilon)} \leq C\varepsilon^{1/q} \|\beta_\varepsilon^i\|_{L^q(\partial \mathcal{S}_\varepsilon^i)} \quad \text{and} \quad \left| \nabla \hat{f}_\varepsilon^i[\beta_\varepsilon^i] \right| \leq \frac{C\varepsilon^{2-1/q}}{|x - x_i|^2} \|\beta_\varepsilon^i\|_{L^q(\partial \mathcal{S}_\varepsilon^i)} \quad \text{for } |x| \geq \bar{C}\varepsilon, \tag{5.3.7}$$

where C and \bar{C} are constant independent of ε .

Proof. The proof of the first estimates is due to scaling. The second one follow from (D.0.1) which is proved in the Appendix D and the Hölder estimate

$$\|\beta_\varepsilon\|_{L^1(\partial \mathcal{S}_\varepsilon^i)} \leq C\varepsilon^{1-\frac{1}{q}} \|\beta_\varepsilon\|_{L^q(\partial \mathcal{S}_\varepsilon^i)}$$

□

Lemma 5.3.3. *There exist $\bar{\varepsilon} > 0$ such that for any $0 < \varepsilon < \bar{\varepsilon}$, the operator \mathcal{T}_ε , defined in (5.3.4), is a 1/2 contraction on $\tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)$.*

Proof. Let $[\alpha_\varepsilon^+, \alpha_\varepsilon^-, \alpha_\varepsilon^\Omega] \in \tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)$. From the previous lemma we have

$$\begin{aligned} \|\mathcal{T}(\alpha_\varepsilon)\|_{L^q(\partial\Omega)} &\leq \sum_{i=+,-} C\varepsilon^{2-1/q} \|\alpha_\varepsilon^i - \nabla h[\alpha_\varepsilon]|_{\partial\mathcal{S}_\varepsilon^i} \cdot n\|_{L^q(\partial\mathcal{S}_\varepsilon^i)} \left\| \frac{1}{x - x_i} \right\|_{L^q(\partial\Omega)} \\ &\leq C\varepsilon^{2-1/q} \left(\sum_{i=+,-} \|\alpha_\varepsilon^i\|_{L^q(\partial\mathcal{S}_\varepsilon^i)} + \|\alpha_\varepsilon^\Omega\|_{L^q(\partial\Omega)} \right), \end{aligned} \quad (5.3.8)$$

where C is ε independent. Similarly

$$\begin{aligned} \|\mathcal{T}(\alpha_\varepsilon)\|_{L^q(\partial\mathcal{S}^i)} &\leq C\varepsilon^{2-1/q} \|\alpha_\varepsilon^j - \nabla h[\alpha_\varepsilon]|_{\partial\mathcal{S}_\varepsilon^j} \cdot n\|_{L^q(\partial\mathcal{S}_\varepsilon^j)} \left\| \frac{1}{x - x_j} \right\|_{L^q(\partial\Omega)} \\ &\leq C\varepsilon^{2-1/q} \left(\|\alpha_\varepsilon^j\|_{L^q(\partial\mathcal{S}_\varepsilon^j)} + \|\alpha_\varepsilon^\Omega\|_{L^q(\partial\Omega)} \right). \end{aligned} \quad (5.3.9)$$

If we choose ε small enough \mathcal{T} is an 1/2 contraction. \square

Theorem 5.3.1. *There exists $\bar{\varepsilon} > 0$ such that for positive $\varepsilon \leq \bar{\varepsilon}$ and for any $[\alpha_\varepsilon^+, \alpha_\varepsilon^-, \alpha_\varepsilon^\Omega] \in \tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)$, it holds*

$$\mathfrak{U}_\varepsilon[\alpha_\varepsilon^+, \alpha_\varepsilon^-, \alpha_\varepsilon^\Omega] = \mathfrak{n}_\varepsilon[\beta_\varepsilon^+, \beta_\varepsilon^-, \beta_\varepsilon^\Omega],$$

where $[\beta_\varepsilon^+, \beta_\varepsilon^-, \beta_\varepsilon^\Omega] = (\text{Id} + \mathcal{T})^{-1}[\alpha_\varepsilon^+, \alpha_\varepsilon^-, \alpha_\varepsilon^\Omega]$

Proof. The proof follows from uniqueness of solutions of system (5.3.3) and the fact that $\text{Id} + \mathcal{T}$ is invertible for ε smaller or equal to $\bar{\varepsilon}$ follows from Lemma 5.3.3. \square

For now on we assume always $0 < \varepsilon < \bar{\varepsilon}$. We are now able to state one of the key lemma in the proof of the asymptotic convergence.

Lemma 5.3.4. *Let $\alpha_\varepsilon^i \in \tilde{L}^q(\partial\mathcal{S}_\varepsilon^i)$ and let $[\bar{\alpha}_\varepsilon^i]$ the extension by zero of α_ε^i in $\tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)$ then*

$$\left\| \nabla \mathfrak{U}_\varepsilon[\bar{\alpha}_\varepsilon^i] - \nabla \hat{f}_\varepsilon^i[\alpha_\varepsilon^i] \right\|_{L^q(\mathcal{F}_\varepsilon)} \leq C\varepsilon^{2-1/q} \|\alpha_\varepsilon^i\|_{L^q(\partial\mathcal{S}_\varepsilon^i)}. \quad (5.3.10)$$

Proof. We are in the situation where $\text{Id} + \mathcal{T}$ is invertible. Define $[\beta_\varepsilon] = (\text{Id} + \mathcal{T})^{-1}([\bar{\alpha}_\varepsilon^i])$, by Theorem 5.3.1 we have $\mathfrak{U}_\varepsilon[\bar{\alpha}_\varepsilon^i] = \mathfrak{n}_\varepsilon[\beta_\varepsilon]$ which leads us to rewrite

$$\mathfrak{U}_\varepsilon[\bar{\alpha}_\varepsilon^i] - \hat{f}_\varepsilon^i[\alpha_\varepsilon^i] = h[\beta_\varepsilon] + \sum_{i=+,-} \hat{f}_\varepsilon^i[\beta_\varepsilon], \quad (5.3.11)$$

where for $k \in \{\Omega, +, -\}$

$$\tilde{\beta}_\varepsilon^k = \begin{cases} \beta_\varepsilon^k & \text{if } k = \Omega, \\ \beta_\varepsilon^k - \nabla h[\beta_\varepsilon] \cdot n|_{\partial\mathcal{S}^k} & \text{if } k \neq i, \\ \beta_\varepsilon^k - \nabla h[\beta_\varepsilon] \cdot n|_{\partial\mathcal{S}^k} - \alpha_\varepsilon^i & \text{if } k = i. \end{cases}$$

We want to estimate $[\beta_\varepsilon] - [\bar{\alpha}_\varepsilon^i]$. To do that we note that $[\beta_\varepsilon] - [\bar{\alpha}_\varepsilon^i] = -\mathcal{T}(\text{Id} + \mathcal{T})^{-1}[\bar{\alpha}_\varepsilon^i]$ and

$$\left\| [\beta_\varepsilon] - [\bar{\alpha}_\varepsilon^i] \right\|_{\tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)} \leq 2 \left\| \mathcal{T}([\bar{\alpha}_\varepsilon^i]) \right\|_{\tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)}$$

due to the fact that \mathcal{T} and $(\text{Id} - \mathcal{T})^{-1}$ commute. With the help of (5.3.8)-(5.3.9), it holds

$$\left\| [\beta_\varepsilon] - [\bar{\alpha}_\varepsilon^i] \right\|_{\tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)} \leq C\varepsilon^{2-\frac{1}{q}} \|\bar{\alpha}_\varepsilon^i\|_{\tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)}.$$

It is straightforward to see that it also holds

$$\left\| [\tilde{\beta}_\varepsilon] \right\|_{\tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)} \leq C\varepsilon^{2-\frac{1}{q}} \|\bar{\alpha}_\varepsilon^i\|_{\tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)}.$$

To conclude the proof it is enough to estimate the gradient of the right hand side of (5.3.11). The worst term is the one related to $h[\tilde{\beta}_\varepsilon]$ for which it holds

$$\|\nabla h[\tilde{\beta}_\varepsilon]\|_{L^q(\mathcal{F}_\varepsilon)} \leq C\|\tilde{\beta}_\varepsilon^\Omega\|_{L^q(\partial\Omega)} \leq C\varepsilon^{2-\frac{1}{q}} \|\alpha_\varepsilon^i\|_{L^q(\partial\mathcal{S}_\varepsilon^i)}.$$

□

Corollary 5.3.1. *Let $[\alpha_\varepsilon] = [\alpha_\varepsilon^+, \alpha_\varepsilon^-, \alpha_\varepsilon^\Omega] \in \tilde{L}^q(\partial\mathcal{S}_\varepsilon^+) \times \tilde{L}^q(\partial\mathcal{S}_\varepsilon^-) \times \tilde{L}^q(\partial\Omega)$, and let w_ε the solution of the div-curl system*

$$\begin{aligned} \operatorname{div} w_\varepsilon &= 0 && \text{in } \mathcal{F}_\varepsilon, \\ \operatorname{curl} w_\varepsilon &= 0 && \text{in } \mathcal{F}_\varepsilon, \\ w_\varepsilon \cdot n &= \alpha_\varepsilon && \text{on } \partial\mathcal{F}_\varepsilon, \\ \oint_{\partial\mathcal{S}^i} w_\varepsilon \cdot \tau &= 0, \end{aligned}$$

then it holds

$$\|w_\varepsilon - \nabla h[\alpha_\varepsilon]\|_{L^q(\mathcal{F}_\varepsilon)} \leq C\varepsilon^{\frac{1}{q}} \|\alpha_\varepsilon\|_{L^q(\partial\mathcal{S}^+) \times L^q(\partial\mathcal{S}^-) \times L^q(\partial\Omega)}.$$

Proof. We can write the solution $w_\varepsilon = \nabla \mathfrak{U}_\varepsilon[\alpha_\varepsilon] = \nabla \mathfrak{U}_\varepsilon[\bar{\alpha}_\varepsilon^\Omega] + \nabla \mathfrak{U}_\varepsilon[\bar{\alpha}_\varepsilon^+] + \nabla \mathfrak{U}_\varepsilon[\bar{\alpha}_\varepsilon^-]$ by linearity. If we recall (5.3.7) and (5.3.10), we have for $i = +, -$ that

$$\|\nabla \mathfrak{U}_\varepsilon[\bar{\alpha}_\varepsilon^i]\|_{L^q(\mathcal{F}_\varepsilon)} \leq \|\nabla \mathfrak{U}_\varepsilon[\bar{\alpha}_\varepsilon^i] - \nabla \hat{f}_\varepsilon^i[\alpha_\varepsilon^i]\|_{L^q(\mathcal{F}_\varepsilon)} + \|\nabla \hat{f}_\varepsilon^i[\alpha_\varepsilon^i]\|_{L^q(\mathcal{F}_\varepsilon)} \leq C\varepsilon^{\frac{1}{q}} \|\alpha_\varepsilon^i\|_{L^q(\partial\mathcal{S}^i)}, \quad (5.3.12)$$

where we use $2 - 1/q > 1/q$ for $q > 1$. To deal with $\nabla \mathfrak{U}_\varepsilon[\bar{\alpha}_\varepsilon^\Omega]$ we note that

$$\nabla \mathfrak{U}_\varepsilon[\bar{\alpha}_\varepsilon^\Omega] - \nabla h[\alpha_\varepsilon] = \nabla \mathfrak{U}_\varepsilon[\beta_\varepsilon],$$

with

$$\beta_\varepsilon^k = \begin{cases} 0 & \text{if } k = \Omega \\ \nabla h[\alpha_\varepsilon] \cdot n|_{\partial\mathcal{S}^i} & \text{if } k = +, -. \end{cases}$$

Using (5.3.12) we have

$$\|\nabla \mathfrak{U}_\varepsilon[\bar{\alpha}_\varepsilon^\Omega] - \nabla h[\alpha_\varepsilon]\|_{L^q(\mathcal{F}_\varepsilon)} \leq C\varepsilon^{\frac{1}{q}} \sum_{i=+,-} \|\beta_\varepsilon^i\|_{L^q(\partial\mathcal{S}^i)} \leq C\varepsilon^{\frac{1}{q}} \|\bar{\alpha}_\varepsilon^\Omega\|_{L^q(\partial\Omega)}.$$

□

5.4 Proof of Theorem 5.2.1

In this section we prove Theorem 5.2.1. Before attaching the main result, we prove some bounds independent of ε for weak solutions $(\omega_\varepsilon, \omega_\varepsilon^-, u_\varepsilon)$ that satisfies the hypothesis of Theorem 5.2.1.

Lemma 5.4.1. *Let $(\omega_\varepsilon, \omega_\varepsilon^-, u_\varepsilon)$ be solutions of (5.1.1)-(5.1.2) in the sense of Definition 5.0.1, that satisfy the hypothesis of Theorem 5.2.1 and let $T > 0$ a time. Then it holds*

- ▷ $\|\omega_\varepsilon\|_{L^\infty((0,T)\times\Omega)} \leq \max\{\|\omega^{in}\|_{L^\infty(\Omega)}, C_T\}$, where $\|\omega_\varepsilon^+\|_{L^\infty(0,T;L^\infty(\partial\mathcal{S}^+))} \leq C_T$,
- ▷ $\|\omega_\varepsilon^-\|_{L^\infty((0,T)\times\partial\mathcal{S}^-)} \leq \max\{\|\omega^{in}\|_{L^\infty(\Omega)}, C_T\}$, where $\|\omega_\varepsilon^+\|_{L^\infty(0,T;L^\infty(\partial\mathcal{S}^+))} \leq C_T$,
- ▷ $\|u_\varepsilon\|_{L^r(0,T;L^q(\Omega))} \leq C(T)$ where $q \in (1, 2)$.

Proof. The first two points are direct consequence of the definition of weak solutions and the hypothesis of Theorem 5.2.1. We now move to prove the third point. By assumption u_ε is Log-Lipschitz which implies that it satisfies the div-curl system (5.1.4)-(5.1.5) in a strong sense. We decompose u_ε in the same fashion as the limiting velocity field u . To do that we define $L_\varepsilon : \mathcal{F}_\varepsilon \rightarrow \mathbb{R}^2$ as

$$L_\varepsilon(t, x) = \left(\oint_{\partial\mathcal{S}_\varepsilon^+} g_\varepsilon \right) \frac{x - x_+}{2\pi|x - x_+|^2} - \left[\mathcal{C}_{+, \varepsilon}^{in} - \int_0^t \oint_{\partial\mathcal{S}_\varepsilon^+} \omega_\varepsilon^+ g_\varepsilon \right] \frac{(x - x_+)^\perp}{2\pi|x - x_+|^2} \\ - \left(\oint_{\partial\mathcal{S}_\varepsilon^+} g_\varepsilon \right) \frac{x - x_-}{2\pi|x - x_-|^2} + \left[\mathcal{C}_{-, \varepsilon}^{in} + \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon(t) - \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon^{in} + \int_0^t \oint_{\partial\mathcal{S}_\varepsilon^+} \omega_\varepsilon^+ g_\varepsilon \right] \frac{(x - x_-)^\perp}{2\pi|x - x_-|^2}$$

and R_ε the unique solution of

$$\begin{aligned} \operatorname{div} R_\varepsilon &= 0 && \text{in } \Omega, \\ \operatorname{curl} R_\varepsilon &= 0 && \text{in } \Omega, \\ \nabla R_\varepsilon \cdot n &= -L_\varepsilon \cdot n && \text{on } \partial\Omega. \end{aligned} \tag{5.4.1}$$

The vector field $u_\varepsilon = L_\varepsilon + R_\varepsilon + K_\Omega[\omega_\varepsilon] + w_\varepsilon$ where w_ε satisfies the div-curl system

$$\begin{aligned} \operatorname{div} w_\varepsilon &= 0 && \text{in } \mathcal{F}_\varepsilon, \\ \operatorname{curl} w_\varepsilon &= 0 && \text{in } \mathcal{F}_\varepsilon, \\ w_\varepsilon \cdot n &= g_\varepsilon - (L_\varepsilon + R_\varepsilon + K_\Omega[\omega_\varepsilon]) \cdot n && \text{on } \partial\mathcal{F}_\varepsilon, \\ \oint_{\partial\mathcal{S}^i} w_\varepsilon \cdot \tau &= 0 && \text{for } i = +, -. \end{aligned}$$

The vector fields $K_\Omega[\omega_\varepsilon]$, L_ε and R_ε are uniformly bounded in $L^r(0, T; L^q(\Omega))$ for $q \in (1, 2)$ from the hypothesis of Theorem 5.2.1, in fact

$$\|K_\Omega[\omega_\varepsilon]\|_{L^\infty(0,T;L^2(\Omega))} \leq C\|\omega_\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega))},$$

which is uniformly bounded by point one. Regarding L_ε , it is sum of four terms of the type $\ell_k(t)\mathfrak{l}_k(x)$ where \mathfrak{l}_k depends only on the space variables and it is of the form

$$\frac{x - x_k}{2\pi|x - x_k|^2} \quad \text{or} \quad \frac{(x - x_k)^\perp}{2\pi|x - x_k|^2}$$

and ℓ_k depends only in time and it is of the type

$$\oint_{\partial\mathcal{S}_\varepsilon^+} g_\varepsilon \quad \text{or} \quad \mathcal{C}_+^{in} - \int_0^t \oint_{\partial\mathcal{S}_\varepsilon^+} \omega_\varepsilon^+ g_\varepsilon \quad \text{or} \quad \mathcal{C}_-^{in} + \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon(t) - \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon^{in} + \int_0^t \oint_{\partial\mathcal{S}_\varepsilon^+} \omega_\varepsilon^+ g_\varepsilon.$$

The ℓ_k are uniformly bounded in $L^q(\Omega)$ for $q < 2$ and the ℓ_k are uniformly bounded in $L^r(0, T)$ due to the first point of the lemma and for the hypothesis

$$\oint_{\partial\mathcal{S}_\varepsilon^+} g_\varepsilon \longrightarrow \mu, \quad \mathcal{C}_{i,\varepsilon}^{in} \longrightarrow \mathcal{C}_i^{in} \quad \text{and} \quad \oint_{\partial\mathcal{S}_\varepsilon^+} \omega_\varepsilon^+ g_\varepsilon \longrightarrow j.$$

Regarding R_ε , it is enough to have a uniform estimates of $L_\varepsilon \cdot n$ in $L^r(0, T; L^q(\partial\Omega))$ due to the inequality

$$\|R_\varepsilon\|_{L^r(0,T;L^q(\Omega))} \leq C \|L_\varepsilon \cdot n\|_{L^r(0,T;L^q(\partial\Omega))}.$$

The above estimate follows from the fact that the unique solution of the div-curl system (5.4.1) is the solution of a Laplace problem with Neumann boundary conditions, see for instance [GLS16] systems (76)-(79). To estimates $L_\varepsilon \cdot n$ on $(0, T) \times \partial\Omega$ we use the same strategy as for L_ε , in fact $L_\varepsilon \cdot n$ is the sum of four terms $\ell_k(t) \mathbb{V}_k^r$, where the ℓ_k are as before and the only difference is that \mathbb{V}_k^r is of the form

$$\frac{x - x_k}{2\pi|x - x_k|^2} \cdot n \quad \text{or} \quad \frac{(x - x_k)^\perp}{2\pi|x - x_k|^2} \cdot n$$

and they are uniformly bounded in $L^q(\partial\Omega)$.

Finally, it remains to show that w_ε is bounded. Actually we will prove that w_ε converges to zero in $L^r(0, T; L^q(\Omega))$. Note that $w_\varepsilon \cdot n = 0$ on $\partial\Omega$ and $w_\varepsilon \cdot n = g_\varepsilon - (L_\varepsilon - R_\varepsilon - K_\Omega[\omega]) \cdot n \in \tilde{L}^q(\partial\mathcal{S}^i)$ for $i = +, -$. Moreover $\varepsilon^{1/q} \|w_\varepsilon \cdot n\|_{L^q(\partial\mathcal{S}_\varepsilon^i)}$ converges to zero as ε tend to zero, due to the hypothesis on g_ε , the fact the R_ε and $K_\Omega[\omega_\varepsilon]$ are uniformly bounded in $C^0(\Omega)$ and the scaling estimates

$$\varepsilon^{\frac{1}{q}} \left\| \frac{x - x_i}{2\pi|x - x_i|^2} \right\|_{L^q(\partial\mathcal{S}_\varepsilon^i)} = \varepsilon^{\frac{1}{q} \varepsilon^{\frac{1}{q} - 1}} \left\| \frac{x - x_i}{2\pi|x - x_i|^2} \right\|_{L^q(\partial\mathcal{S}_1^i)} \longrightarrow 0,$$

where we use the assumption $q \in (1, 2)$. If we apply Corollary 5.3.1, we have

$$\|w_\varepsilon\|_{L^q(\mathcal{F}_\varepsilon)} \leq C \varepsilon^{\frac{1}{q}} \sum_{i=+,-} \|w_\varepsilon \cdot n\|_{L^q(\partial\mathcal{S}^i)} \rightarrow 0 \quad \text{in } L^r(0, T).$$

□

Lemma 5.4.2. *Let $(\omega_\varepsilon, \omega_\varepsilon^-, u_\varepsilon)$ be solutions of (5.1.1)-(5.1.2) in the sense of Definition 5.0.1, that satisfy the hypothesis of Theorem 5.2.1. Then it holds*

$$\|\partial_t \omega_\varepsilon\|_{L^r((0,T);W^{-1,p}(\Omega))} \leq C(p, T)$$

where $C(T)$ does not depend on ε and $p \in (2, \infty)$ is the dual exponent of q .

Proof. The proof is a consequence of the hypothesis on the data g_ε and ω_ε^+ and of Lemma 5.4.1, in fact from the weak formulation (5.1.3) we have

$$\begin{aligned} |\langle \partial_t \omega_\varepsilon, \varphi \rangle| &= \left| \int_0^T \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon \partial_t \varphi dx dt - \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon(T, \cdot) \varphi(T, \cdot) dx + \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon^{in} \varphi(0, \cdot) dx \right| \\ &\leq \left| \int_0^T \int_{\mathcal{F}_\varepsilon} u_\varepsilon \cdot \nabla \varphi \omega_\varepsilon dx dt \right| + \left| \int_0^T \oint_{\partial \mathcal{S}_\varepsilon^+} g_\varepsilon \omega_\varepsilon^+ \varphi ds dt \right| + \left| \int_0^T \oint_{\partial \mathcal{S}_\varepsilon^-} g_\varepsilon \omega_\varepsilon^- \varphi ds dt \right|, \\ \left| \int_0^T \int_{\mathcal{F}_\varepsilon} u_\varepsilon \cdot \nabla \varphi \omega_\varepsilon \right| &\leq \|\omega_\varepsilon\|_{L^\infty([0,T] \times \mathcal{F}_\varepsilon)} \int_0^T \int_{\mathcal{F}_\varepsilon} |u_\varepsilon \cdot \nabla \varphi| dx dt \\ &\leq \|\omega_\varepsilon\|_{L^\infty([0,T] \times \mathcal{F}_\varepsilon)} \|u_\varepsilon\|_{L^r(0,T;L^q(\mathcal{F}_\varepsilon))} \|\nabla \varphi\|_{L^{r'}(0,T;L^p(\mathcal{F}_\varepsilon))} \\ &\leq C(T) \|\varphi\|_{L^{r'}(0,T;W^{1,p}(\Omega))}, \\ \left| \int_0^T \oint_{\partial \mathcal{S}_\varepsilon^+} g_\varepsilon \omega_\varepsilon^+ \varphi \right| &\leq \|\omega_\varepsilon^+\|_{L^\infty([0,T] \times \partial \mathcal{S}_\varepsilon)} \|g_\varepsilon\|_{L^r(0,T;L^1(\partial \mathcal{S}_\varepsilon))} \|\varphi\|_{L^{r'}(0,T;L^\infty(\Omega))} \\ &\leq C(T) \|\varphi\|_{L^{r'}(0,T;W^{1,p}(\Omega))}, \end{aligned}$$

and similarly

$$\begin{aligned} \left| \int_0^T \oint_{\partial \mathcal{S}_\varepsilon^-} g_\varepsilon \omega_\varepsilon^- \varphi \right| &\leq \|\omega_\varepsilon^-\|_{L^\infty([0,T] \times \partial \mathcal{S}_\varepsilon)} \|g_\varepsilon\|_{L^r(0,T;L^1(\partial \mathcal{S}_\varepsilon))} \|\varphi\|_{L^{r'}(0,T;L^\infty(\Omega))} \\ &\leq C(T) \|\varphi\|_{L^{r'}(0,T;W^{1,p}(\Omega))}, \end{aligned}$$

□

Proof of Theorem 5.2.1. The idea of the proof is to use some a priori estimates to pass to the limit in the weak formulations (5.1.3)-(5.1.4)-(5.1.5) to reach (5.0.3)-(5.0.4)-(5.0.5). Let $(\omega_\varepsilon, \omega_\varepsilon^-, u_\varepsilon)$ a weak solutions of (5.1.1)-(5.1.2) in the sense of Definition 5.0.1. From Lemma 5.4.1, we have the a priori estimate

$$\|\omega_\varepsilon\|_{L^\infty(0,T;L^\infty(\mathcal{F}_\varepsilon))} \leq \max \left\{ \|\omega_\varepsilon^{in}\|_{L^\infty(\mathcal{F}_\varepsilon)}, \|\omega_\varepsilon^+\|_{L^\infty([0,T] \times \partial \mathcal{F}_\varepsilon)} \right\} \leq \max \{ \|\omega^{in}\|_{L^\infty(\Omega)}, C_T \}.$$

If we extend ω_ε by 0 in $\mathcal{S}_\varepsilon^i$ then there exists a decreasing subsequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ that converges to 0 such that

$$\omega_{\varepsilon_n} \xrightarrow{w^*} \omega \quad \text{in } L^\infty((0,T) \times \Omega) \quad \text{as } n \rightarrow +\infty.$$

To pass to the limit in the non linear term of (5.1.3), we prove strong convergence of the velocity. To do that we decompose the velocity field $u_\varepsilon = L_\varepsilon + R_\varepsilon + K_\Omega[\omega_\varepsilon] + w_\varepsilon$ as in Lemma 5.4.1. We have already proved in Lemma 5.4.1 that w_ε converges to zero in $L_{loc}^r(\mathbb{R}^+; L^q(\Omega))$. Now we show that

$$L_\varepsilon \rightarrow L, \quad R_\varepsilon \rightarrow R \quad \text{and} \quad K_\Omega[\omega_\varepsilon] \rightarrow K_\Omega[\omega] \quad \text{in } L_{loc}^r(\mathbb{R}^+; L^q(\Omega)). \quad (5.4.2)$$

About the first convergence of (5.4.2) the difficulty is to prove

$$\left(\int_{\mathcal{F}_\varepsilon} \omega_\varepsilon(t, \cdot) \right) \frac{(x - x_-)^\perp}{2\pi|x - x_-|^2} \longrightarrow \left(\int_\Omega \omega(t, \cdot) \right) \frac{(x - x_-)^\perp}{2\pi|x - x_-|^2}.$$

It suffices to show by using Arzela-Ascoli that

$$A_\varepsilon(t) = \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon(t, \cdot) \longrightarrow \int_{\Omega} \omega(t, \cdot) \quad \text{in } C_{loc}^0(\mathbb{R}^+).$$

We need to verify the hypothesis of Arzela-Ascoli, in particular we have to show that A_ε is uniformly bounded and equicontinuous in $C^0(0, T)$ for any $T > 0$. We start by proving that A_ε are equicontinuous. If we test (5.1.3) with the characteristic function on $[s, t]$, we obtain adding the absolute values on both side

$$\begin{aligned} \left| \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon(t, \cdot) - \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon(s, \cdot) \right| &= \left| \int_s^t \oint_{\partial S_\varepsilon^+} \omega_\varepsilon^+ g_\varepsilon + \int_s^t \oint_{\partial S_\varepsilon^-} g_\varepsilon \omega_\varepsilon^- \right| \\ &\leq -2 \max \left\{ \|\omega^{in}\|_{L^\infty(\Omega)}, C_T \right\} \int_s^t \oint_{\partial S_\varepsilon^+} g_\varepsilon, \end{aligned}$$

the equicontinuity follows from the fact that $\int_s^t \oint_{\partial S_\varepsilon^+} g_\varepsilon$ converges to μ in $L_{loc}^r(\mathbb{R}^+)$. The uniform boundedness in $C^0(0, T)$ follows from the equicontinuity and point one of Lemma 5.4.1. For the moment we only proved that A_ε converges to a function $A(t)$ in $C^0(0, T)$. It remains to show $A(t) = \int_{\Omega} \omega(t, \cdot)$. Let $\psi \in C_c^\infty(0, T)$ then by the weak-star convergence in L^∞ , we have

$$\int_0^T \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon \psi \longrightarrow \int_0^T \int_{\Omega} \omega \psi, \quad \text{at the same time} \quad \int_0^T \int_{\mathcal{F}_\varepsilon} \omega_\varepsilon \psi = \int_0^T A(t)_\varepsilon \psi \longrightarrow \int_0^T A(t) \psi,$$

which imply $A(t) = \int_{\Omega} \omega(t, \cdot)$ for almost any time.

The convergence of R_ε to R follow from the fact that $L_\varepsilon \cdot n \rightarrow L \cdot n$ in $L^r(0, T; L^q(\partial\Omega))$ as shown in the proof of point three of Lemma 5.4.1. The prove of the convergence of $L_\varepsilon \cdot n$ is the same as the one of L_ε . We concentrate now to $K_\Omega[\omega_\varepsilon] \rightarrow K_\omega[\omega]$ in $L_{loc}^r(\mathbb{R}^+, L^q(\Omega))$. As before we prove via Arzela-Ascoli that the sequence $K_\Omega[\omega_\varepsilon] \rightarrow \bar{K}$ in $C_{loc}^0(\mathbb{R}^+; L^q(\Omega))$ for $q \in (1, 2)$. Moreover by the fact that $K_\Omega[\omega_\varepsilon]$ satisfies the div-curl system in a weak sense, we can pass to the limit in the weak formulation and conclude that $\bar{K} = K_\Omega[\omega]$. To do that let $T > 0$, to apply Arzela-Ascoli we need to show

- ▷ $K_\Omega[\omega_\varepsilon]$ are uniformly bounded in $C^0(0, T; L^q(\Omega))$,
- ▷ $K_\Omega[\omega_\varepsilon]$ are equicontinuous in $C^0(0, T; L^q(\Omega))$,

We prove the equicontinuity. To do that let $0 \leq t < s \leq T$ and let G the Green function on Ω with 0 Dirichlet boundary condition on $\partial\Omega$, we have to estimate

$$\begin{aligned} \|K_\Omega[\omega_\varepsilon](t) - K_\Omega[\omega_\varepsilon](s)\|_{L_\sigma^q(\Omega)} &= \sup_{\|\eta\|_{L_\sigma^p(\Omega)} \leq 1} \left| \int_{\Omega} (K_\Omega[\omega_\varepsilon](t) - K_\Omega[\omega_\varepsilon](s)) \cdot \eta dx \right| \\ &= \sup_{\|\eta\|_{L_\sigma^p(\Omega)} \leq 1} \left| \int_{\Omega} \int_{\Omega} (\omega_\varepsilon(t, y) - \omega_\varepsilon(s, y)) \nabla^\perp G(x, y) \cdot \eta(x) dy dx \right| \\ &= \sup_{\|\eta\|_{L_\sigma^p(\Omega)} \leq 1} \left| \int_{\Omega} (\omega_\varepsilon(t, y) - \omega_\varepsilon(s, y)) \left(\int_{\Omega} \nabla^\perp G(x, y) \cdot \eta(x) dx \right) dy \right| \\ &\leq \sup_{\|\eta\|_{L_\sigma^p(\Omega)} \leq 1} \left(\|\partial_t \omega_\varepsilon\|_{L^r(t, s; W^{-1, p}(\Omega))} \right) \left\| \int_{\Omega} \nabla^\perp G(x, y) \cdot \eta(x) dx \right\|_{L^{r'}(t, s; W_0^{1, p}(\Omega))} \\ &\leq C \sup_{\|\eta\|_{L_\sigma^p(\Omega)} \leq 1} \left(\|\partial_t \omega_\varepsilon\|_{L^r(t, s; W^{-1, p}(\Omega))} \right) \|\eta\|_{L^p(\Omega)} (s - t)^{1/r'}, \end{aligned}$$

where we use Fubini and the equation satisfied by ω_ε . Note that to use the equation we have to verify that

$$\varphi(t, y) = \int_{\Omega} \nabla^\perp G(x, y) \eta(x) dx,$$

is an admissible test function. To check that after an integration by parts and using the zero boundary values for G , it is straightforward to see that

$$\begin{cases} \Delta \varphi = \operatorname{curl} \eta & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

The function φ is an admissible test function, moreover $\|\varphi\|_{W_0^{1,p}(\Omega)} \leq \|\operatorname{curl} \eta\|_{W^{-1,p}(\Omega)} \leq \|\eta\|_{L^p(\Omega)}$. The equicontinuity follows by the second point of Lemma 5.4.2 and the estimates $\|\varphi\|_{W_0^{1,p}(\Omega)} \leq \|\eta\|_{L^p(\Omega)}$.

At this point we know that

$$\omega_{\varepsilon_n} \xrightarrow{w^*} \omega \quad \text{in } L^\infty((0, T) \times \mathbb{R}^2) \quad \text{and} \quad u_{\varepsilon_n} \longrightarrow u = L + R + K_{\mathbb{R}^2}[\omega] \quad \text{in } L_{loc}^r(\mathbb{R}^+; L_{loc}^q(\mathbb{R}^2)),$$

as n converges to infinity with $q \in (1, 2)$.

It remains to show that (ω, u) is a solution of (5.0.2) in the sense of Definition 5.1.1. To do that we first pass to the limit in ε in the weak formulation (5.1.3) to obtain (5.0.3) and by the explicit expression of u , we check a posteriori that (5.0.4)-(5.0.5) are satisfied. The difficulty arises while passing to the limit in the term involving ω_ε^- . To avoid it, we note that by testing (5.1.3) with a constant in space function $\varphi(t, x) = \psi(t) \in C_c^\infty(\mathbb{R}^+ \times \overline{\mathcal{F}})$ we obtain

$$\left(\int_{\mathcal{F}_\varepsilon} \omega_\varepsilon^{in} \right) \psi(0) + \int_{\mathbb{R}^+} \left(\int_{\mathcal{F}_\varepsilon} \omega_\varepsilon \right) \partial_t \psi = \int_{\mathbb{R}^+} \left(\oint_{\partial S_\varepsilon^+} \omega_\varepsilon^+ g_\varepsilon \right) \psi + \int_{\mathbb{R}^+} \left(\oint_{\partial S_\varepsilon^-} \omega_\varepsilon^- g_\varepsilon \right) \psi. \quad (5.4.3)$$

Moreover, if we test (5.1.3) with functions $\varphi \in C_c^\infty((0, T) \times \overline{\mathcal{F}})$ for some $T > 0$, the right hand side converges to the right hand side of (5.0.3) as ε goes to zero by using the weak-star convergence of ω_ε in $L^\infty((0, T) \times \Omega)$ and the strong one of u_ε in $L^r(0, T; L^q(\Omega))$. The left hand side reads as

$$\begin{aligned} & \int_{\mathbb{R}^+} \oint_{\partial S_\varepsilon^+} g_\varepsilon \omega_\varepsilon^+ \varphi ds dt + \int_{\mathbb{R}^+} \oint_{\partial S_\varepsilon^-} g_\varepsilon \omega_\varepsilon^- \varphi ds dt \\ &= \int_{\mathbb{R}^+} \left(\oint_{\partial S_\varepsilon^+} g_\varepsilon \omega_\varepsilon^+ ds \right) \varphi(\cdot, x_+) dt + \int_{\mathbb{R}^+} \left(\oint_{\partial S_\varepsilon^-} g_\varepsilon \omega_\varepsilon^- ds \right) \varphi(\cdot, x_-) dt \\ & \quad + \int_{\mathbb{R}^+} \oint_{\partial S_\varepsilon^+} g_\varepsilon \omega_\varepsilon^+ (\varphi - \varphi(\cdot, x_+)) ds dt + \int_{\mathbb{R}^+} \oint_{\partial S_\varepsilon^-} g_\varepsilon \omega_\varepsilon^- (\varphi - \varphi(\cdot, x_-)) ds dt \\ &= \int_{\mathbb{R}^+} \left(\oint_{\partial S_\varepsilon^+} g_\varepsilon \omega_\varepsilon^+ ds \right) \varphi(\cdot, x_+) dt + \left(\int_{\mathcal{F}_\varepsilon} \omega_\varepsilon^{in} \right) \varphi(0, x_-) \end{aligned} \quad (5.4.4)$$

$$+ \int_{\mathbb{R}^+} \left(\int_{\mathcal{F}_\varepsilon} \omega_\varepsilon \right) \partial_t \varphi(\cdot, x_-) - \int_{\mathbb{R}^+} \left(\oint_{\partial S_\varepsilon^+} \omega_\varepsilon^+ g_\varepsilon \right) \varphi(\cdot, x_-) \quad (5.4.5)$$

$$+ \int_{\mathbb{R}^+} \oint_{\partial S_\varepsilon^+} g_\varepsilon \omega_\varepsilon^+ (\varphi - \varphi(\cdot, x_+)) ds dt + \int_{\mathbb{R}^+} \oint_{\partial S_\varepsilon^-} g_\varepsilon \omega_\varepsilon^- (\varphi - \varphi(\cdot, x_-)) ds dt. \quad (5.4.6)$$

By the strong convergence of $\oint_{\partial\mathcal{S}^+} g\omega_\varepsilon^+$ to j in $L^r(0, T)$ and the weak-star convergence of ω_ε to ω in $L^\infty((0, T) \times \Omega)$, we have

$$(5.4.4) - (5.4.5) \longrightarrow \int_{\mathbb{R}^+} j\varphi(\cdot, x_+) dt + \left(\int_{\Omega} \omega^{in} dx \right) \varphi(0, x_-) + \int_{\mathbb{R}^+} \left(\int_{\Omega} \omega dx \right) \partial_t \varphi(\cdot, x_-) dt - \int_{\mathbb{R}^+} j\varphi(\cdot, x_-) dt,$$

which is the right hand side of (5.0.3). It remains to show that (5.4.6) converges to zero. The term

$$\begin{aligned} & \left| \int_{\mathbb{R}^+} \oint_{\partial\mathcal{S}_\varepsilon^+} g_\varepsilon \omega_\varepsilon^+ (\varphi - \varphi(\cdot, x_+)) ds dt \right| \\ & \leq \|\omega_\varepsilon^+\|_{L^\infty(0, T; L^\infty(\partial\mathcal{S}_\varepsilon^+))} \|g_\varepsilon^i\|_{L^r(0, T; L^1(\partial\mathcal{S}_\varepsilon^+))} \|\varphi - \varphi(\cdot, x_+)\|_{L^\infty(0, T; L^\infty(\partial\mathcal{S}_\varepsilon^+))} \\ & \leq C_T (1 + \|\mu\|_{L^r(0, T)}) L \varepsilon D T^{1/r'} \longrightarrow 0, \end{aligned}$$

where C_T is the one of the hypothesis of the theorem, L is the Lipschitz norm of φ and D is the diameter of $\partial\mathcal{S}_1^+$. The second term of (5.4.6) converges also to zero and the only difference is that we will use point two of Lemma 5.4.1 to estimate the L^∞ -norm of ω_ε^- .

Finally to verify that u satisfies (5.0.4)-(5.0.5), we consider $\varphi \in C_c^\infty(\Omega)$ and we integrate by part

$$\int_{\Omega} u \cdot \nabla \varphi dx.$$

The function u can be explicitly written as $u = L + R + K_\Omega[\omega]$, the difficulty is to integrate by parts the not smooth enough terms, that are the one of the form

$$\frac{x - x_i}{2\pi|x - x_i|^2} \quad \text{and} \quad \frac{(x - x_i)^\perp}{2\pi|x - x_i|^2}.$$

Denote by $B_\varepsilon(x) \subset \mathbb{R}^2$ the open ball centred in x of radius ε . Then

$$\int_{\Omega \setminus B_\varepsilon(x_i)} \frac{x - x_i}{2\pi|x - x_i|^2} \cdot \nabla \varphi = \int_{\partial B_\varepsilon(x_i)} \frac{x - x_i}{2\pi|x - x_i|^2} \cdot n \varphi + \int_{\partial\Omega} \frac{x - x_i}{2\pi|x - x_i|^2} \cdot n \varphi.$$

The first term converges to $\varphi(x_i)$ and the second term will cancel with one from R .

□

5.5 Proof of Theorem 5.1.1

In this section we propose an alternative proof of Theorem 5.1.1, based on the Schauder fixed point theorem, to the one presented in [Ale76]. For small enough $T > 0$, that will be fixed

later, consider the space

$$\begin{aligned} \mathcal{Z} = & \left\{ (\omega, \omega^-) \in L^\infty(0, T; L^\infty(\mathcal{F})) \times L^\infty(0, T; L^\infty(\partial\mathcal{S}^-)) \text{ such that} \right. \\ & i. \quad \|\omega\|_{L^\infty(0, T; L^1(\mathcal{F}))} \leq \|\omega^{in}\|_{L^1(\mathcal{F})} - \int_0^T \int_{\partial\mathcal{S}^+} g|\omega^+| ds dt \\ & \quad \|\omega\|_{L^\infty(0, T; L^\infty(\mathcal{F}))} \leq \max \left\{ \|\omega^{in}\|_{L^\infty(\mathcal{F})}, \|\omega^+\|_{L^\infty(0, T; L^\infty(\partial\mathcal{S}^+))} \right\} \\ & \quad \|\omega^-\|_{L^\infty(0, T; L^\infty(\partial\mathcal{S}^-))} \leq \max \left\{ \|\omega^{in}\|_{L^\infty(\mathcal{F})}, \|\omega^+\|_{L^\infty(0, T; L^\infty(\partial\mathcal{S}^+))} \right\} \\ & ii. \quad \|\partial_t \omega\|_{L^2(0, T; W^{-1,3}(\mathcal{F}))} \leq 1 \left. \right\}. \end{aligned}$$

Note that \mathcal{Z} is non empty.

Consider the map $\mathcal{V} : \mathcal{Z} \rightarrow \mathcal{Z}$ such that (ω, ω^-) is send to $(\tilde{\omega}, \tilde{\omega}^-)$ defined as follow. First we consider v the solution to

$$\begin{aligned} \operatorname{div} v &= 0 && \text{in } \mathbb{R}^+ \times \mathcal{F}, \\ \operatorname{curl} v &= \omega && \text{in } \mathbb{R}^+ \times \mathcal{F}, \\ v \cdot n &= g && \text{on } \mathbb{R}^+ \times \partial\mathcal{F}, \\ \oint_{\partial\mathcal{S}^+} v \cdot \tau &= C_+^{in} - \int_0^t \oint_{\partial\mathcal{S}^+} g\omega^+ && \text{in } \mathbb{R}^+, \\ \oint_{\partial\mathcal{S}^-} v \cdot \tau &= C_-^{in} - \int_0^t \oint_{\partial\mathcal{S}^-} g\omega^- && \text{in } \mathbb{R}^+. \end{aligned} \tag{5.5.1}$$

The vector field v satisfies for $p > 2$

$$\begin{aligned} \|v\|_{L^p(0, T; W^{1,p}(\mathcal{F}))} &\leq C_p (\|\omega\|_{L^p(0, T; L^p(\mathcal{F}))} + |C_+^{in}| + |C_-^{in}| \\ &\quad + (1 + \|\omega^+\|_{L^p(0, T; L^p(\partial\mathcal{S}))} + \|\omega^-\|_{L^p(0, T; L^p(\partial\mathcal{S}))}) \|g\|_{L^\infty(0, T; W^{1,\infty}(\partial\mathcal{S}))}). \end{aligned} \tag{5.5.2}$$

From Theorem 4.1 of [Boy05], there exist a unique renormalized solution $(\tilde{\omega}, \tilde{\omega}^-)$ to the transport equation

$$\begin{aligned} \partial \tilde{\omega} + v \cdot \nabla \tilde{\omega} &= 0 && \text{in } [0, T] \times \mathcal{F}, \\ \tilde{\omega} &= \omega^+ && \text{on } [0, T] \times \partial\mathcal{S}^+, \\ \tilde{\omega} &= \omega^{in} && \text{in } \mathcal{F}. \end{aligned}$$

We define $\mathcal{V}(\omega, \omega^-) = (\tilde{\omega}, \tilde{\omega}^-)$.

Let us prove that $\mathcal{V}(\mathcal{Z}) \subset \mathcal{Z}$. Point *i.* follows from Theorem 4.1. in [Boy05] and the second point is a direct consequence of the weak formulation of $(\tilde{\omega}, \tilde{\omega}^-)$ and a priori bounds point *i.* of the definition of \mathcal{Z} for T small enough . Note that the a priori estimates in *i.* are uniform, these allow us to choose T uniform respect to the element of \mathcal{Z} .

Now we endow the space \mathcal{Z} with the norm $L^p(0, T; L^p(\mathcal{F}))$ for some $p > 2$ and we observe that \mathcal{Z} is a closed convex subset of $L^p(0, T; L^p(\mathcal{F}))$.

Let us prove that \mathcal{V} is relatively compact. Let (ω_n, ω_n^-) a sequence of elements of \mathcal{Z} , we denote by $(\tilde{\omega}, \tilde{\omega}^-) = \mathcal{V}(\omega, \omega^-)$ and v_n the velocity field defined in (5.5.1) associated with ω_n . Consider v_g the solution of (5.5.1) with the vorticity ω and the circulations around $\partial\mathcal{S}^+$ and

$\partial\mathcal{S}^-$ identically zero. First we deduce from estimate (5.5.2) that $v_n - v_g$ is uniformly bounded in $L^p(0, T; W^{1,p}(\mathcal{F}))$. Then we recall that any element φ in $C_{\sigma, c}^\infty((0, T) \times \mathcal{F})$ can be written as $\varphi = \nabla^\perp \zeta$ where ζ satisfies $\Delta \zeta = \text{curl } \varphi$ in \mathcal{F} , $\zeta = 0$ on $\partial\mathcal{F}$, $\zeta = c_i$ on $\partial\mathcal{S}^i$, for $i = +, -$, for appropriate choices of the c_i . Recall that v satisfies a div-curl system so

$$\int_{\mathcal{F}} v \cdot \nabla^\perp \zeta = - \int_{\mathcal{F}} \omega \zeta + \left(c_+^{in} - \int_0^t \oint_{\partial\mathcal{S}^+} g\omega^+ \right) c_+ + \left(c_-^{in} - \int_0^t \oint_{\partial\mathcal{S}^-} g\omega^- \right) c_-.$$

We can compute

$$\begin{aligned} \int_{\mathbb{R}^+} \int_{\mathcal{F}} v \cdot \partial_t \varphi &= \int_{\mathbb{R}^+} \int_{\mathcal{F}} v \cdot \partial_t \nabla^\perp \zeta = - \int_{\mathbb{R}^+} \int_{\mathcal{F}} \omega \partial_t \zeta + \int_{\mathbb{R}^+} \left(\oint_{\partial\mathcal{S}^+} g\omega^+ \right) \zeta(\partial\mathcal{S}^+) \\ &\quad + \int_{\mathbb{R}^+} \left(\oint_{\partial\mathcal{S}^-} g\omega^- \right) \zeta(\partial\mathcal{S}^-), \end{aligned}$$

If we take the modulus, we use point *ii.* of \mathcal{Z} , we obtain

$$\left| \int_{\mathbb{R}^+} \int_{\mathcal{F}} v \cdot \partial_t \varphi \right| \leq \|\zeta\|_{L^2(0, T; W^{1,3}(\mathcal{F}))} \leq C \|\varphi\|_{L^2(0, T; L^3(\mathcal{F}))}.$$

By Aubin-Lions' Lemma the sequence $v_n - v_g$ converges up to subsequence to a vector field v_ω in $L^p(0, T; L^p(\mathcal{F}))$. As a consequence v_{n_k} converges to $v = v_g + v_\omega$ in $L^p(0, T; L^p(\mathcal{F}))$. Thanks to [BF07, Theorem 4] $(\tilde{\omega}_{n_k}, \tilde{\omega}_{n_k}^-)$ converges in $L^p([0, T]; L^p(\mathcal{F})) \times L^p([0, T]; L^p(\partial\mathcal{S}^-))$ to the solution $(\tilde{\omega}, \tilde{\omega}^-)$ of the transport equation associated with v .

Finally we show that \mathcal{V} is continuous. To do that let (ω_n, ω_n^-) a sequence in \mathcal{Z} that converges to (ω, ω^-) . Due to estimates (5.5.2) the velocity v_n associated with (ω_n, ω_n^-) through (5.5.1) converges to v associated with (ω, ω^-) , and as before by the stability Theorem 4 from [BF07], it follows that $\mathcal{V}(\omega_n, \omega_n^-) = (\tilde{\omega}_n, \tilde{\omega}_n^-)$ converges to $\mathcal{V}(\omega, \omega^-) = (\tilde{\omega}, \tilde{\omega}^-)$.

We proved that \mathcal{V} is continuous, relatively compact map from two bounded closed convex subset of a Banach space. By the Schauder fixed point \mathcal{V} admits a fixed point. The couple (ω, ω^-, v) , where ω is the fixed point and v is recover by (ω, ω^-) through (5.5.1) is a solution of (5.1.1)-(5.1.2) in the sense of Definition 5.1.1.

5.6 Future directions and open problems

In this section we discuss some open problems related to Chapter 5.

- ▷ The first question is if *it is possible to extend the result in the case of multiple sources and sinks*. If one sink is present and there are many sources the above demonstration can be carry out with no difficulties. In the case of more than one sink, it is not possible to use (5.4.3) to pass to the limit in the terms of (5.1.3) involving the exiting vorticity ω^- . At this stage we believe that we can treat this difficulty by using some auxiliary functions to isolate the different sinks.
- ▷ The second question is *the enquiry of Judovič in [Jud63]*. We recall that in [Jud63], the author studied the motion of a inviscid fluid in a time dependent domain with holes and the fluid was allowed to enter and exit from the domain in a prescribed way. Judovič asked if there is a theory that can deal with the case when holes can at a certain time become points. We believe that the setting of solutions with bounded vorticity is the right one to obtain an existence theory in the case of holes with possible empty interior.

-
- ▷ The third question is if *it is possible to prove uniqueness for solutions in the sense of Definition 5.0.1 of the limiting system (5.0.2)*. Uniqueness for similar equations was proved in [LM09], where the authors studied the wave-vortex system and more recently in [LM19], where a similar system was tackled. In both the works a crucial hypothesis is that the vorticity is constant around the moving point vortices. In our setting this assumption is not natural and the question is open. Moreover they use some energy estimates typical of the impermeable setting.

Appendix A

Some density arguments

In this appendix we present two density results. For the first one we do not claim originality but we are not able to find a reference in the literature. The second result is used in the proof of Theorem 3.2.1. Lemma A.0.2 is also essential in [PS14], where we propose to change the set $\mathcal{T} = \{\varphi \in C_{c,\sigma}^\infty(\mathbb{R}^2) \mid D(\varphi) = 0 \text{ in } \mathcal{S}_0\}$ with the set defined in (A.0.1) in the proof of Theorem 1. The set \mathcal{T} is not dense in $\underline{\mathcal{V}}$ neither in \mathcal{V} . On the other hand we will introduce below, cf. (A.0.1), a set \mathcal{Y} which is dense and has all the property to make the proof of Theorem 1 of [PS14] working. To see that \mathcal{T} is not dense in \mathcal{V} , it is enough to consider $\mathcal{S}_0 = B_1(0)$ and the function

$$f(x) = \begin{cases} 0 & \text{in } B_1(0), \\ \nabla^\perp(x^2\chi) & \text{elsewhere,} \end{cases}$$

where χ is a smooth cut off such that $\chi \equiv 1$ in $B_2(0)$ and $\chi \equiv 0$ outside $B_4(0)$. It is clear that $f \in \mathcal{V} \subset \underline{\mathcal{V}}$. Suppose by contradiction that there exist approximations $f_\varepsilon \in \mathcal{T}$ such that $f_\varepsilon \rightarrow f$ in \mathcal{V} , then $l_{f_\varepsilon} \rightarrow l_f = 0$ and $r_{f_\varepsilon} \rightarrow r_f = 0$. By continuity, $f_\varepsilon = l_{f_\varepsilon} + r_{f_\varepsilon}x^\perp$ in $\overline{B_1(0)}$ which implies $f_\varepsilon|_{\partial B_1(0)} = l_{f_\varepsilon} + r_{f_\varepsilon}x^\perp \rightarrow l_f + r_f x^\perp = 0$ in $L^2(\partial B_1(0))$. Moreover $f_\varepsilon|_{\mathbb{R}^2 \setminus B_1(0)} \rightarrow f|_{\mathbb{R}^2 \setminus B_1(0)}$ in $H^1(\mathbb{R}^2 \setminus B_1(0))$, then by trace theorem $f_\varepsilon|_{\partial B_1(0)} \rightarrow f|_{\partial B_1(0)}$, but $f|_{\partial B_1(0)} = 2x^\perp \neq 0$ which is a contradiction.

We start by presenting the first density result.

Lemma A.0.1. *Let Ω an open, bounded subset of \mathbb{R}^2 with smooth boundary such that $\partial\Omega = \cup \Gamma_i$ where Γ_i for $i = 0, \dots, n$ are open connected components of the boundary with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, then the set $C_\sigma^\infty(\Omega) \cap L_\sigma^2(\Omega)$ of smooth divergence-free functions with 0 normal component on the boundary $\partial\Omega$ is dense in $H^1(\Omega) \cap L_\sigma^2(\Omega)$.*

Proof. Let $v \in H^1(\Omega) \cap L_\sigma^2(\Omega)$, then by [GR12, Corollary 3.3] there exists a stream function ψ such that $\nabla^\perp \psi = v$ and $\psi \in H^2(\Omega)$. Using the condition $v \cdot n = 0$ on $\partial\Omega$, ψ satisfies w.l.o.g.

$$\begin{cases} -\Delta\psi = -\operatorname{curl} v & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma_0, \\ \psi = c_i & \text{on } \Gamma_i, \end{cases}$$

for some constant c_i . Consider η_ε a symmetric convolution kernel of mass 1 with support in $B_\varepsilon(0)$ and consider χ_ε the characteristic function such that $\chi_\varepsilon(x) = 1$ if $\operatorname{dist}(x, \partial\Omega) > \varepsilon$ and 0 else. We define

$$\begin{cases} -\Delta\psi_\varepsilon = -(\chi_{3\varepsilon} \operatorname{curl} v) * \eta_\varepsilon & \text{in } \Omega, \\ \psi_\varepsilon = 0 & \text{on } \Gamma_0, \\ \psi_\varepsilon = c_i & \text{on } \Gamma_i. \end{cases}$$

The functions $v_\varepsilon = \nabla^\perp \psi_\varepsilon$ are the desired approximations of v . First of all we prove that $v_\varepsilon \in C_c^\infty(\Omega) \cap L_\sigma^2(\Omega)$. This is clear by elliptic regularity and $v_\varepsilon \cdot n = \nabla^\perp \psi_\varepsilon \cdot n = \nabla \psi \cdot \tau = 0$ on $\partial\Omega$ (ψ_ε is constant in any Γ_i). To prove the convergence we use the elliptic regularity from [Eva10, Theorem 4, Chapter 6] (in particular the remark that follow Theorem 4), we have

$$\|v_\varepsilon - v\|_{H^1(\Omega)} \leq \|\psi_\varepsilon - \psi\|_{H^2(\Omega)} \leq C \|(\chi_{3\varepsilon} \operatorname{curl} v) * \eta_\varepsilon - \operatorname{curl} v\|_{L^2(\Omega)} \rightarrow 0.$$

□

Lemma A.0.2. *The set*

$$\mathcal{Y} = \{u \in L_\sigma^2(\mathbb{R}^2) \mid \text{there exist } u_F \in C_{\sigma,c}^\infty(\mathbb{R}^2) \text{ and } u_R \in \mathcal{R} \quad (\text{A.0.1}) \\ \text{such that } u|_{\mathcal{F}} = u_F|_{\mathcal{F}} \text{ and } u|_{\mathcal{S}_0} = u_S|_{\mathcal{S}_0}\},$$

is dense in \mathcal{V} , $\underline{\mathcal{V}}$ and \mathcal{H} .

Proof. The proof is easy in the case of \mathcal{H} . We turn to the case of \mathcal{V} and $\underline{\mathcal{V}}$. The difference between the two spaces is the integrability at $+\infty$ but this will not change much the proof so we will do it only for $\underline{\mathcal{V}}$.

Let $v \in \underline{\mathcal{V}}$ and let l and r such that $v_S = l + x^\perp r$. For $\rho > 0$ such that $\rho > \operatorname{diam}(\mathcal{S}_0)$, we define χ_ρ to be a smooth cut off function such that $0 \leq \chi_\rho \leq 1$, $\chi_\rho = 1$ in $B_\rho(0)$, $\chi_\rho = 0$ outside $B_{2\rho}(0)$ and $|\nabla \chi_\rho| \leq C/\rho$. Fix $R > 0$ such that $R/4 > \operatorname{diam}(\mathcal{S}_0)$, we decompose $v = u + v_1$, where $u = \nabla^\perp(\chi_{R/4}(-l^\perp \cdot x + r/2|x|^2))$. The function $u \in C_{\sigma,c}^\infty(\mathbb{R}^2)$ and $v_1|_{\mathcal{F}_0} \in H^1(\mathbb{R}^2) \cap L_\sigma^2(\mathcal{F}_0)$ and $v_1|_{\mathcal{S}_0} = 0$. By Theorem 3.3 of [GR12] there exists $\varphi \in H^2(B_{2R}(0) \setminus \mathcal{S}_0)$ such that $v_1 = \nabla^\perp \varphi$. We decompose $v_1 = w + z$ where $w = \nabla^\perp(\chi_R \varphi)$. The function z is such that $z|_{B_R(0)} = 0$ and $z|_{\mathbb{R}^2 \setminus B_R(0)} \in H_0^1(\mathbb{R}^2 \setminus \overline{B_R(0)}) \cap L_\sigma^2(\mathbb{R}^2 \setminus B_R(0)) = E$, where

$$E = \overline{\{C_{\sigma,c}^\infty(\mathbb{R}^2 \setminus \overline{B_R(0)})\}}^{\|\cdot\|_{H^1}},$$

see for example [Gal11, Section III.4.2]. This provides the existence of a sequence $\tilde{z}_\varepsilon \in C_{\sigma,c}^\infty(\mathbb{R}^2 \setminus \overline{B_R(0)})$ such that $\tilde{z}_\varepsilon \rightarrow z|_{\mathbb{R}^2 \setminus B_R(0)}$ in $H^1(\mathbb{R}^2 \setminus B_R(0))$. Let z_ε to be the extension by 0 of \tilde{z}_ε inside $B_R(0)$, then $z_\varepsilon \rightarrow z$ in $\underline{\mathcal{V}}$. We now study w . The function $w \in H^1(B_{4R}(0) \setminus \overline{\mathcal{S}_0}) \cap L_\sigma^2(B_{4R}(0) \setminus \overline{\mathcal{S}_0})$. By Lemma A.0.1 there exist $\tilde{w}_\varepsilon \in C^\infty(B_{4R}(0) \setminus \mathcal{S}_0) \cap L_\sigma^2((B_{4R}(0) \setminus \overline{\mathcal{S}_0}))$ such that $\tilde{w}_\varepsilon \rightarrow w|_{B_{4R}(0) \setminus \overline{\mathcal{S}_0}}$ in $H^1(B_{4R}(0) \setminus \overline{\mathcal{S}_0})$. Let $\psi_\varepsilon \in H^2(B_{4R}(0) \setminus \overline{\mathcal{S}_0})$ such that $\tilde{w}_\varepsilon = \nabla^\perp \psi_\varepsilon$. The function ψ_ε is unique up to a constant, so we choose the unique ψ_ε such that $\int_{B_{4R}(0) \setminus \overline{B_{2R}(0)}} \psi_\varepsilon = 0$. Define $\bar{w}_\varepsilon = \nabla^\perp(\chi_{2R} \psi_\varepsilon)$ and denote by $\bar{w} = w|_{B_{4R}(0) \setminus \overline{\mathcal{S}_0}}$. We have

$$\begin{aligned} \|\bar{w} - \bar{w}_\varepsilon\|_{H^1(B_{4R}(0) \setminus \overline{\mathcal{S}_0})} &\leq \|\bar{w} - \tilde{w}_\varepsilon\|_{H^1(B_{4R}(0) \setminus \overline{\mathcal{S}_0})} + C \|\tilde{w}_\varepsilon\|_{H^1(B_{4R}(0) \setminus B_{2R}(0))} \\ &\quad + \|(\nabla^\perp \chi_{2R}) \psi_\varepsilon\|_{H^1(B_{4R}(0) \setminus \overline{\mathcal{S}_0})} \\ &\leq o(\varepsilon) + C \|\tilde{w}_\varepsilon\|_{H^1(B_{4R}(0) \setminus B_{2R}(0))} + C \|\psi_\varepsilon\|_{L^2(B_{4R}(0) \setminus B_{2R}(0))} \\ &\leq o(\varepsilon) + C \|\tilde{w}_\varepsilon\|_{H^1(B_{4R}(0) \setminus B_{2R}(0))} = o(\varepsilon), \end{aligned}$$

in fact we can use the Poincaré inequality on the ψ_ε and $\|\tilde{w}_\varepsilon\|_{H^1(B_{4R}(0)\setminus B_{2R}(0))} = o(\varepsilon)$ because $w = 0$ in $B_{4R}(0) \setminus B_{2R}(0)$ (C is a constant that change from line to line). Let w_ε be the extension by 0 of \tilde{w}_ε . The functions

$$v_\varepsilon = u + w_\varepsilon + z_\varepsilon \rightarrow v \quad \text{in } \mathcal{Y},$$

Moreover $v_\varepsilon, u, w_\varepsilon$ and z_ε are element of \mathcal{Y} (where we extend w_ε by 0 in the interior of \mathcal{S}_0). \square

Appendix B

On \mathcal{R} -sectorial operators

In this section we briefly present the relation between \mathcal{R} -sectoriality of an operator A and maximal regularity. For a complete treatment of the subject we refer to [DHP03]. Consider the abstract Cauchy problem

$$\begin{cases} \dot{z}(t) = Az(t) + f(t), \\ z(0) = 0, \end{cases} \quad (\text{B.0.1})$$

where $A : \mathcal{D}(A) \subset H \rightarrow H$ is a closed densely-defined unbounded linear operator and H is an Hilbert space. It is a classical result that for any $f \in L^2(0, T; H)$ there exists a unique solution $z \in W^{1,2}(0, T; H)$ with $Az \in L^2(0, T; H)$ if and only if there exists $w \in \mathbb{R}$ such that $-A + w$ is a sectorial operator of angle $\theta < \pi/2$.

If we want to prove an analogous result in the Banach space setting, i.e. $A : \mathcal{D}(A) \subset E \rightarrow E$ with E a Banach space, the sectoriality of A is not enough. To solve this problem Weiss proved in [Wei01] that given E a UMD Banach space and given the abstract Cauchy problem (B.0.1) where $A : \mathcal{D}(A) \subset E \rightarrow E$ is a closed densely-defined unbounded linear operator and given $1 < p < \infty$ it holds that for any $f \in L^p(0, T; E)$ there exists a unique solution to (B.0.1) with $z \in W^{1,p}(0, T; E)$ and $Az \in L^p(0, T; E)$ if and only if there exists $w \in \mathbb{R}$ such that $-A + w$ is \mathcal{R} -sectorial of angle $\theta < \pi/2$.

Recall that E is a UMD-Banach space if the Hilbert transform extends to a bounded operator on $L^p(\mathbb{R}; E)$ for $1 < p < +\infty$, or equivalently the function $m(t) = \text{sign}(t)$ is a Fourier multiplier on $L^p(\mathbb{R}; E)$. Note that any closed subspace of L^p is a UMD space. We now recall the definition of \mathcal{R} -bounded family of operators and of \mathcal{R} -sectoriality for an unbounded operator.

Definition B.0.1 (\mathcal{R} -bounded family of operators). Let X and Y Banach spaces. A family of linear operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called \mathcal{R} -bounded on $\mathcal{L}(X, Y)$, if there exist constant $C > 0$ and $p \in [1, \infty)$ such that for every $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{x_j\}_{j=1}^n \subset X$ and for all sequences $\{r_j(\cdot)\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$ valued random variables on $[0, 1]$, we have

$$\left\| \sum_{j=1}^n r_j(\cdot) T_j x_j \right\|_{L^p([0,1]; Y)} \leq C \left\| \sum_{j=1}^n r_j(\cdot) x_j \right\|_{L^p([0,1]; X)}.$$

Definition B.0.2 (\mathcal{R} -sectorial operator). Let A be a densely defined closed linear operator on a Banach space X with domain $\mathcal{D}(A)$. Then A is said \mathcal{R} -sectorial of angle $\theta \in (0, \pi)$ if

the spectrum $\sigma(A) \subset \overline{\{\lambda \in \mathbb{C} \setminus 0 \mid |\arg(\lambda)| \leq \theta\}}$ and for any $\theta_1 > \theta$ the set $\{\lambda(\lambda I - A)^{-1} \mid \theta_1 \leq |\arg(\lambda)| \leq \pi\}$ is an \mathcal{R} -bounded family of operators.

We conclude the section with an important property of stability of \mathcal{R} -sectorial operator (which is used in Section 3.1.2.6).

Definition B.0.3 (*A*-bounded operator of bound zero). Let X a Banach space, let A and B two closed densely-defined unbounded linear operator with domains $\mathcal{D}(A) \subset \mathcal{D}(B) \subset X$, then B is called *A*-bounded operator of bound zero if for any $\delta > 0$ there exists a constant $C(\delta)$ such that

$$\|Bz\| \leq \delta \|Az\| + C(\delta)\|z\| \quad \text{for any } z \in \mathcal{D}(A).$$

Proposition B.0.1. *Let A a closed densely-defined unbounded linear operator with domain $\mathcal{D}(A) \subset X$, such that $-A + w$ is \mathcal{R} -sectorial for some $w \in \mathbb{R}$ and let B an *A*-bounded operator of bound zero, then there exists $\tilde{w} \geq w$ such that $-A + B + \tilde{w}$ is \mathcal{R} -sectorial with the same angle as $-A + w$.*

Special case where the solid is a disk

In this appendix we consider the special case of a disk immersed in a viscous fluid with infinite energy as in Section 3.2. In this setting we show that weak solutions are continuous in time with values in $L^2_\sigma(\mathbb{R}^2)$, they satisfy an energy equality and they are unique. Moreover we prove that as the viscosity ν vanish, the solutions converge to an Euler plus rigid body system.

We start by presenting the system and the well-posedness result. We denote by \mathcal{B} the solid which is a disk of radius 1, in other words

$$\mathcal{B} = \left\{ x \in \mathbb{R}^2 \text{ such that } |x| \leq 1 \right\},$$

and the fluid domain is $\mathcal{F} = \mathbb{R}^2 \setminus \mathcal{B}$. Under this simplification, to translate the problem (3.0.1)-(3.0.9) on a fix time independent domain, it is enough to use a translation as change of variables. Let consider the new variables

$$v(t, y) = u(t, y + h(t)), \quad q(t, y) = p(t, y + h(t)) \quad l(t) = h'(t), \quad \text{and} \quad r(t) = r(t).$$

The equations read as

$$\begin{aligned} \partial_t v + [(v - l(t)) \cdot \nabla]v - \nu \Delta v + \nabla q &= 0 && \text{for } x \in \mathcal{F} \\ \operatorname{div} v &= 0 && \text{for } x \in \mathcal{F} \\ v \cdot n &= v_S \cdot n && \text{for } x \in \partial \mathcal{B} \\ D(v)n \cdot \tau &= -\alpha(u - u_S) \cdot \tau && \\ |v| &\longrightarrow 0 && \text{as } |x| \longrightarrow 0, \end{aligned} \tag{C.0.1}$$

$$\begin{aligned} ml'(t) &= - \int_{\partial \mathcal{B}} \Sigma n ds \\ \mathcal{J}r'(t) &= - \int_{\partial \mathcal{B}} x^\perp \cdot \Sigma n ds \\ v(0, y) &= v_0(y) && \text{on } \mathcal{F} \\ l'(0) &= l_0, \quad r(0) = r_0, \end{aligned}$$

as usual n and τ are the unit outwards normal and counterclockwise tangent vectors to the boundary of \mathcal{F} , α is the friction coefficient and Σ is the stress tensor equal to $-q \operatorname{Id}_2 + 2\nu D(v)$. For this system we define weak solutions with β circulation at infinity as in Section 3.2.3. As before we introduce the maps a and b and the definition of weak solutions as follow

$$a(u, v) = -\alpha \int_{\partial \mathcal{S}_0} (u - u_S) \cdot (v - v_S) - \int_{\mathcal{F}_0} Du : Dv,$$

$$b(u, v, w) = \int_{\mathcal{F}_0} [(u - l_u) \cdot \nabla] w \cdot v.$$

Note that in (C.0.1) r does not play a role in the dynamics due to the special shape of \mathcal{B} and it can be recover via

$$\mathcal{J}r'(t) = -\alpha\nu \int_{\partial\mathcal{B}} (v - l_v) \cdot \tau, \quad r(0) = r_0.$$

For this reason we introduce the spaces

$$\mathcal{H}_{\mathcal{B}} = \{v \in L^2_{\sigma}(\mathbb{R}^2) \mid \text{there exists } l \in \mathbb{R}^2 \text{ such that } v = l \text{ in } \mathcal{B}\},$$

and we endowed it with the norm

$$\|v\|_{\mathcal{H}_{\mathcal{B}}} = \|v\|_{L^2(\mathcal{F})} + m|l|.$$

In a similar fashion, we define

$$\underline{\mathcal{V}}_{\mathcal{B}} = \{v \in \mathcal{H}_{\mathcal{B}} \mid \text{there exists } v_F \in H^1(\mathbb{R}^2) \text{ such that } v = v_F \text{ in } \mathcal{F}\},$$

and we endowed it with the norm

$$\|v\|_{\underline{\mathcal{V}}_{\mathcal{B}}} = \|v\|_{H^1(\mathcal{F})} + m|l| = \|\nabla v\|_{L^2(\mathcal{F})} + \|v\|_{\mathcal{H}_{\mathcal{B}}}.$$

We present the definition of weak solution with β circulation at infinity for the system (C.0.1).

Definition C.0.1 (Weak solution with β circulation at infinity). Let $\tilde{v}_0 \in \mathcal{H}_{\mathcal{B}}$ and $T > 0$ given. We say that

$$\tilde{v} \in C_w([0, T]; \mathcal{H}_{\mathcal{B}}) \cap L^2((0, T); \underline{\mathcal{V}}_{\mathcal{B}})$$

is a weak solution for (C.0.1) with β circulation at infinity if for every test function $\varphi \in C^1([0, T]; \mathcal{H}_{\mathcal{B}})$ with $\varphi|_{\overline{\mathcal{F}_0}} \in C^1([0, T]; C_c^\infty(\overline{\mathcal{F}_0}))$, it holds

$$\begin{aligned} (\tilde{v}(t), \varphi(t))_{\mathcal{H}_{\mathcal{B}}} - (\tilde{v}_0, \varphi(0))_{\mathcal{H}_{\mathcal{B}}} &= \int_0^t \left[(\tilde{v}, \partial_t \varphi)_{\mathcal{H}_{\mathcal{B}}} + 2\nu a(\tilde{v}, \varphi) + 2\beta\nu a(H, \varphi) \right. \\ &\quad \left. - b(\tilde{v}, \varphi, \tilde{v}) - \beta b(H, \varphi, \tilde{v}) - \beta b(\tilde{v}, \varphi, H) \right] dt. \end{aligned}$$

The next section deals with the well-posedness result for the system (C.0.1), in particular we show continuity in time of the solutions, the fact that they satisfy an energy equality and uniqueness.

C.1 Well-posedness for a moving rigid disk in a viscous fluid with infinite kinetic energy

We tackle the well-posedness of weak solutions with β circulation at infinity of the system (C.0.1), in particular we prove the following theorem.

Theorem C.1.1. *Let $\tilde{v}_0 \in \mathcal{H}_{\mathcal{B}}$ given and let $T > 0$ be a fixed time. Then there exists a unique weak solution $\tilde{v} \in C([0, T]; \mathcal{H}_{\mathcal{B}}) \cap L^2([0, T]; \underline{\mathcal{V}}_{\mathcal{B}})$ with β circulation at infinity and with \tilde{v}_0 as initial datum. Moreover it satisfies the following energy equality*

$$\begin{aligned} \frac{1}{2} \|\tilde{v}(t)\|_{\mathcal{H}_{\mathcal{B}}}^2 - \frac{1}{2} \|\tilde{v}_0\|_{\mathcal{H}_{\mathcal{B}}}^2 + 2\nu \int_0^t \int_{\mathcal{F}} |D\tilde{v}|^2 dx dt + 2\alpha\nu \int_0^t \int_{\mathcal{F}} |\tilde{v} - l_{\tilde{v}}|^2 ds dt \\ = - \int_0^t 2\nu\beta a(H, \tilde{v}) + b(\tilde{v}, H, \tilde{v}). \end{aligned}$$

Proof. The proof of the theorem follows from the fact that the trilinear map b is bounded in $(\underline{\mathcal{V}}_{\mathcal{B}} \oplus \mathbb{R}H) \times \underline{\mathcal{V}}_{\mathcal{B}} \times (\underline{\mathcal{V}}_{\mathcal{B}} \oplus \mathbb{R}H)$ and the classical Faedo-Galerking methods applies. In particular we do not have to consider the truncated trilinear map b_R . The other three results, the continuity in time, the energy equality and the uniqueness follow from the fact that $\partial_t \tilde{v} \in L^2(0, T; \underline{\mathcal{V}}_{\mathcal{B}}^*)$. Here $\underline{\mathcal{V}}_{\mathcal{B}}^*$ is the dual of $\underline{\mathcal{V}}_{\mathcal{B}}$ where we identify the element of $\underline{\mathcal{V}}_{\mathcal{B}}$ through the scalar product in $\mathcal{H}_{\mathcal{B}}$. We prove $\partial_t \tilde{v} \in L^2(0, T; \underline{\mathcal{V}}_{\mathcal{B}}^*)$. From the weak formulation we have that

$$\begin{aligned} (\tilde{v}(t), \varphi(t))_{\mathcal{H}_{\mathcal{B}}} - (\tilde{v}_0, \varphi(0))_{\mathcal{H}_{\mathcal{B}}} - \int_0^t (\tilde{v}, \partial_t \varphi)_{\mathcal{H}_{\mathcal{B}}} = \int_0^t \left[2\nu a(\tilde{v}, \varphi) + 2\beta\nu a(H, \varphi) \right. \\ \left. - b(\tilde{v}, \varphi, \tilde{v}) - \beta b(H, \varphi, \tilde{v}) - \beta b(\tilde{v}, \varphi, H) \right] dt. \end{aligned}$$

It is enough to estimates the right hand side. The proof is classical and it is possible to proceed as in the case where the fluid is alone. \square

C.2 Well-posedness for a disk in a perfect fluid

In this section we study the well-posedness for a model that describes the motion of a disk in an incompressible inviscid fluid. In particular we consider the well-posedness for classical solutions. The system in the unknown (v^E, l^E, q^E) reads as

$$\begin{aligned} \partial_t v^E + [(v^E - l^E) \cdot \nabla] v^E + \nabla q^E &= 0 & \text{for } x \in \mathcal{F} \\ \operatorname{div} v^E &= 0 & \text{for } x \in \mathcal{F} \\ v^E \cdot n &= v_{\mathcal{B}}^E \cdot n & \text{for } x \in \partial\mathcal{B} \\ |v^E| &\longrightarrow 0 & \text{as } |x| \longrightarrow 0, \\ m(l^E)'(t) &= - \int_{\partial\mathcal{B}} q^E n ds \\ \mathcal{J}(r^E)' &= - \int_{\partial\mathcal{B}} x^\perp \cdot q^E n ds \\ v^E(0, x) &= v_0^E(x) & \text{on } \mathcal{F} \\ l^E(0) &= l_0^E \text{ and } r^E(0) = r_0^E. \end{aligned} \tag{C.2.1}$$

Note that as before the angular velocity does not play a role in the dynamics due to the shape of \mathcal{B} . In particular $r^E(t) = r_0^E$ is constant in time, in fact $x^\perp \cdot n = 0$ on $\partial\mathcal{B}$. Moreover, the above system admits a unique classical solution for Hölder continuous initial data with compactly supported vorticity.

Theorem C.2.1. *Let $\lambda \in (0, 1)$, let $T > 0$, let $v_0^E \in C^{1,\lambda}(\mathcal{F})$ and $l_0^E \in \mathbb{R}^2$ such that*

$$\operatorname{div} v_0^E = 0 \text{ in } \mathcal{F}, \quad v_0^E \cdot n = l_0^E \cdot n \text{ on } \partial\mathcal{B} \quad \text{and} \quad \lim_{x \rightarrow +\infty} v_0^E = 0,$$

and such that $\omega_0^E = \operatorname{curl} v_0^E$ is compactly supported. Then there exists a unique solution $(v^E, l^E) \in C^0(0, T; L^p(\mathcal{F})) \times C^1(0, T)$ for $p \in (2, +\infty)$, such that $v^E, \partial_t v^E, \nabla v^E, \nabla p^E$ are $C^0(\overline{\mathcal{F}})$ and they satisfy the system (C.2.1) in a classical sense.

We are not aware of a proof of the above result in the literature but it can be deduced using the strategy in [ORT07] and [GS12]. To be more precise this result is not contained in [ORT07] because the authors require the initial vector field v_0 to be H^1 , which is not the case if

$$\int_{\mathcal{F}} \omega_0^E + \int_{\partial\mathcal{B}} v_0^E \cdot \tau \neq 0.$$

Proof. In [ORT07] the authors ask the initial datum to be in H^1 to perform a vanishing viscosity method to prove existence of solution with bounded vorticity. Our idea is to use the existence result of [GS12] to show existence and uniqueness of solution with L^∞ vorticity. In particular this result does not required the bound on the H^1 norm of the initial data. Then we conclude using the idea of [ORT07] to improve the regularity of the solutions.

More precisely $v_0 \in C^{1,\lambda}(\mathcal{F})$ and $\omega_0 = \operatorname{curl} v_0$ is compactly supported. This implies that $\omega_0 \in L_c^\infty(\mathcal{F})$. We are in the hypothesis of Theorem 1 of [GS12]. There exists a unique solution $(v^E, l^E) \in L^\infty(0, T; \mathcal{LL}(\mathcal{F})) \times C^1(0, T)$. To show that (v^E, l^E) solution satisfies the regularity property of the theorem, we use the following idea of [ORT07]. The vector field $v^E \in L^\infty(0, T; \mathcal{LL}(\mathcal{F}))$, which implies that it admits a flow $\phi(t, x)$ and its inverse and $\psi(t, x)$ Lipschitz in time and Hölder continuous in space. See Theorem 3.7 of [BCD11]. The vorticity $\omega^E = \operatorname{curl} v^E$ can be define by the inverse flow as follow

$$\omega^E(t, x) = \omega_0^E(\psi(t, x)).$$

The Hölder continuity of the initial datum and of the inverse flow ψ , implies that ω^E is Hölder continuous in space and time. To recover v we solve a div-curl system with Hölder continuous source terms. By Schauder estimates v^E is differentiable in space variables with Hölder continuous differential. The other assertions follow as in [ORT07]. □

Before moving to the proof of the vanishing viscosity limit we notice that the solution v^E can be decomposed as

$$v^E = \tilde{v}^E + \left[\int_{\mathcal{F}} \omega_0^E + \int_{\partial\mathcal{B}} v_0^E \cdot \tau \right] H, \quad \text{where } \tilde{v}^E \in L^2(\mathcal{F}_0).$$

Note that the quantity in front of H is constant in time. Moreover with the use of this decomposition the following energy equality holds true

$$\frac{1}{2} \|\tilde{v}^E(t)\|_{\mathcal{H}}^2 - \frac{1}{2} \|\tilde{v}_0^E\|_{\mathcal{H}}^2 = - \int_0^t \beta b(\tilde{v}^E, \tilde{v}^E, H) dx,$$

with $\beta = \int_{\mathcal{F}} \omega_0^E + \int_{\partial\mathcal{B}} v_0^E \cdot \tau$.

This equality follow from the fact that v^E satisfies (C.2.1) in a classical sense. We multiply the first equation of (C.2.1) by \tilde{v}^E and we integrate in \mathcal{F} . Some integrations by parts give the equality.

C.3 Vanishing viscosity limit

In this section we state and prove the vanishing viscosity result.

Theorem C.3.1. *Let $\lambda \in (0, 1)$, let $T > 0$, let $v_0 \in C^{1,\lambda}(\mathcal{F})$ and $l_0 \in \mathbb{R}^2$ such that*

$$\operatorname{div} v_0 = 0 \text{ in } \mathcal{F}, \quad v_0 \cdot n = l_0 \cdot n \text{ on } \partial\mathcal{B} \quad \text{and} \quad \lim_{x \rightarrow +\infty} v_0 = 0,$$

and such that $\omega_0 = \operatorname{curl} v_0$ is compactly supported. Denote by

$$\beta = \int_{\mathcal{F}} \omega_0 + \int_{\partial\mathcal{B}} v_0 \cdot \tau \quad \text{and} \quad \tilde{v}_0 = v_0 - \beta H.$$

Moreover let (\tilde{v}^ν, l^ν) the weak solution of (C.0.1) in the sense of Definition C.0.1, with β circulation at infinity, with viscosity ν and with initial datum (\tilde{v}_0, l_0) . Analogously let $(\tilde{v}^E + \beta H, l^E)$ the classical solution of (C.2.1) with initial datum (v_0, l_0) . Then it holds

▷ If $\alpha = \alpha^\nu$ satisfies $\alpha\nu$ converges to zero, then

$$\begin{aligned} \tilde{v}^\nu &\longrightarrow \tilde{v}^E \quad \text{in} \quad L^\infty(0, T; \mathcal{H}) \\ \sqrt{\nu} \|\tilde{v}^\nu\|_{L^2(0, T; H^1(\mathcal{F}))} &\longrightarrow 0 \quad \text{and} \quad \sqrt{\alpha\nu} \|\tilde{v}^\nu - l^\nu\|_{L^2((0, T) \times \partial\mathcal{B})} \longrightarrow 0. \end{aligned}$$

▷ Moreover $l^\nu \longrightarrow l^E$ in $H^1(0, T)$.

Proof. Consider $w = \tilde{v}^\nu - \tilde{v}^E$. We estimate

$$\begin{aligned} \|w(t)\|_{\mathcal{H}_B}^2 &= (\tilde{v}^\nu - \tilde{v}^E, \tilde{v}^\nu - \tilde{v}^E)_{\mathcal{H}_B} \\ &= \|\tilde{v}^\nu(t)\|_{\mathcal{H}_B}^2 + \|\tilde{v}^E(t)\|_{\mathcal{H}_B}^2 - 2(\tilde{v}^\nu, \tilde{v}^E)_{\mathcal{H}_B} \\ &= \|\tilde{v}_0\|_{\mathcal{H}_B}^2 + \|\tilde{v}_0^E\|_{\mathcal{H}_B}^2 - 2(\tilde{v}^\nu, \tilde{v}^E)_{\mathcal{H}_B} + 4 \int_0^t \nu a(\tilde{v}^\nu, \tilde{v}^\nu) + \nu a(H, \tilde{v}^\nu) dt \\ &\quad - 2 \int_0^t b(\tilde{v}^\nu, \tilde{v}^\nu, H) + b(\tilde{v}^E, \tilde{v}^E, H) dt, \end{aligned} \quad (\text{C.3.1})$$

where we use the energy equality for the \tilde{v}^ν and \tilde{v}^E . If we test the weak formulation of \tilde{v}^ν with \tilde{v}^E , we get

$$\begin{aligned} (\tilde{v}^\nu, \tilde{v}^E)_{\mathcal{H}_B}(t) &= (\tilde{v}_0, \tilde{v}_0^E)_{\mathcal{H}_B} + \int_0^t (\tilde{v}^\nu, \partial_t \tilde{v}^E)_{\mathcal{H}_B} + b(\tilde{v}^\nu, \tilde{v}^\nu, \tilde{v}^E) + b(H, \tilde{v}^\nu, \tilde{v}^E) \\ &\quad - b(\tilde{v}^\nu, \tilde{v}^E, H) + 2\nu a(\tilde{v}^\nu, \tilde{v}^E) + 2\nu a(H, \tilde{v}^E) dt. \end{aligned} \quad (\text{C.3.2})$$

We recall that \tilde{v}^E satisfies

$$(\partial_t \tilde{v}^E, \varphi)_{\mathcal{H}_B} = b(\tilde{v}^E, \tilde{v}^E, \varphi) + b(H, \tilde{v}^E, \varphi) - b(\tilde{v}^E, \varphi, H)$$

for any $\varphi \in C^1([0, T]; \mathcal{H}_B)$ such that $\varphi|_{\overline{\mathcal{F}_0}} \in C^1([0, T]; C_c^\infty(\overline{\mathcal{F}_0}))$, because is a classical solution. If we tested with $\varphi = \tilde{v}^\nu$ we get

$$(\partial_t \tilde{v}^E, \tilde{v}^\nu)_{\mathcal{H}_B} = b(\tilde{v}^E, \tilde{v}^E, \tilde{v}^\nu) + b(H, \tilde{v}^E, \tilde{v}^\nu) - b(\tilde{v}^E, \tilde{v}^\nu, H).$$

If we inject the last equality in (C.3.2), we get

$$\begin{aligned}
(\tilde{v}^\nu, \tilde{v}^E)_{\mathcal{H}_B}(t) &= (\tilde{v}_0, \tilde{v}_0^E)_{\mathcal{H}_B} + \int_0^t b(\tilde{v}^E, \tilde{v}^E, \tilde{v}^\nu) + b(H, \tilde{v}^E, \tilde{v}^\nu) \\
&\quad - b(\tilde{v}^E, \tilde{v}^\nu, H) + b(\tilde{v}^\nu, \tilde{v}^\nu, \tilde{v}^E) + b(H, \tilde{v}^\nu, \tilde{v}^E) \\
&\quad - b(\tilde{v}^\nu, \tilde{v}^E, H) + 2\nu a(\tilde{v}^\nu, \tilde{v}^E) + 2\nu a(H, \tilde{v}^E) dt.
\end{aligned} \tag{C.3.3}$$

If we plug (C.3.3) into (C.3.1), we have

$$\begin{aligned}
\|w\|_{\mathcal{H}_B}^2 &= \|w_0\|_{\mathcal{H}_B}^2 - 2 \int_0^t b(\tilde{v}^E, \tilde{v}^E, \tilde{v}^\nu) - b(\tilde{v}^E, \tilde{v}^\nu, H) + b(\tilde{v}^\nu, \tilde{v}^\nu, \tilde{v}^E) - b(\tilde{v}^\nu, \tilde{v}^E, H) \\
&\quad + 2\nu a(\tilde{v}^\nu, \tilde{v}^E) + 2\nu a(H, \tilde{v}^E) - 2\nu a(\tilde{v}^\nu, \tilde{v}^\nu) - 2\nu a(H, \tilde{v}^\nu) \\
&\quad + b(\tilde{v}^\nu, \tilde{v}^\nu, H) + b(\tilde{v}^E, \tilde{v}^E, H) dt.
\end{aligned}$$

Note that

$$\begin{aligned}
&b(\tilde{v}^E, \tilde{v}^E, \tilde{v}^\nu) - b(\tilde{v}^E, \tilde{v}^\nu, H) + b(\tilde{v}^\nu, \tilde{v}^\nu, \tilde{v}^E) - b(\tilde{v}^\nu, \tilde{v}^E, H) + b(\tilde{v}^\nu, \tilde{v}^\nu, H) + b(\tilde{v}^E, \tilde{v}^E, H) \\
&= b(w, \tilde{v}^\nu, \tilde{v}^E) - b(\tilde{v}^E, w, H) + b(w, \tilde{v}^\nu, H) \\
&= b(w, w, \tilde{v}^E) - b(\tilde{v}^E, w, H) + b(\tilde{v}^\nu, w, H) \\
&= b(w, w, \tilde{v}^E) + b(w, w, H).
\end{aligned}$$

Which implies

$$\begin{aligned}
\|w\|_{\mathcal{H}_B}^2 &= \|w_0\|_{\mathcal{H}_B}^2 - 2 \int_0^t b(w, w, \tilde{v}^E) + b(w, w, H) - 2\nu a(\tilde{v}^\nu, w) - 2\nu a(H, w) dt \\
&= \|w_0\|_{\mathcal{H}_B}^2 - 2 \int_0^t b(w, w, \tilde{v}^E) + b(w, w, H) - 2\nu a(w, w) - 2\nu a(H, w) \\
&\quad - 2\nu a(\tilde{v}^E, w) dt.
\end{aligned}$$

We can conclude that

$$\begin{aligned}
\|w\|_{\mathcal{H}_B}^2 - \|w_0\|_{\mathcal{H}_B}^2 - 2\nu \int_0^t a(w, w) &\leq \int_0^t C(\|\tilde{v}^E\|_{Lip} + \|H\|_{Lip}) \|w\|_{\mathcal{H}_B}^2 \\
&\quad + 3\nu(a(\tilde{v}^E, \tilde{v}^E) + a(H, H)) dt.
\end{aligned}$$

The difference of the initial datum w_0 is identically zero. By Grönwall and letting ν to zero we prove point one. Regarding the second point, it is obtained by the formulas

$$\begin{aligned}
(\mathcal{M}(l^\nu)'(t))_i &= 2\nu a(\tilde{v}^\nu, u_i) + 2\nu a(H, u_i) + b(\tilde{v}^\nu, \tilde{v}^\nu, u_i) + b(H, \tilde{v}^\nu, u_i) - b(\tilde{v}^\nu, u_i, H), \\
(\mathcal{M}(l^E)'(t))_i &= b(\tilde{v}_i^E, \tilde{v}^E, u_i) + b(H, \tilde{v}^E, u_i) - b(\tilde{v}^E, u_i, H),
\end{aligned}$$

where u_i are the gradients of the Kirchhoff potentials defined in (3.2.9) and the added mass is defined by

$$\mathcal{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} + \left[\int_{\mathcal{F}} u_i \cdot u_j dx \right]_{i,j \in \{1,2\}}.$$

The matrix \mathcal{M} is symmetric and positive definite. The scalar product $\langle x, y \rangle = x \cdot \mathcal{M}y$ is then equivalent to the Euclidean one. It holds

$$\begin{aligned} \langle (l^\nu)' - (l^E)', (l^\nu)' - (l^E)' \rangle &\leq C \sum_{i=1,2} \left(\nu^2 (a(\tilde{v}^\nu, u_i))^2 + \nu^2 (a(H, u_i))^2 + (b(\tilde{v}^\nu - \tilde{v}^E, \tilde{v}^\nu, u_i))^2 \right. \\ &\quad \left. + (b(\tilde{v}^E, \tilde{v}^\nu - \tilde{v}^E, u_i))^2 + (b(H, \tilde{v}^\nu - \tilde{v}^E, u_i))^2 + (b(\tilde{v}^\nu - \tilde{v}^E, u_i, H))^2 \right). \end{aligned} \quad (\text{C.3.4})$$

We estimate the right hand side.

$$\begin{aligned} a(\tilde{v}^\nu, u_i)^2 &\leq 2a(\tilde{v}^\nu, \tilde{v}^\nu)a(u_i, u_i), \\ (b(\tilde{v}^\nu - \tilde{v}^E, \tilde{v}^\nu, u_i))^2 &\leq C \|\tilde{v}^\nu - \tilde{v}^E\|_{\mathcal{H}}^2 \|\tilde{v}^\nu\|_{\mathcal{H}}^2 \|u_i\|_{\text{Lip}(\mathcal{F})}^2 \\ &\leq C \|\tilde{v}^\nu - \tilde{v}^E\|_{\mathcal{H}}^2 \left(\|\tilde{v}^E\|_{\mathcal{H}}^2 + \|\tilde{v}^\nu - \tilde{v}^E\|_{\mathcal{H}}^2 \right) \|u_i\|_{\text{Lip}(\mathcal{F})}^2, \\ (b(\tilde{v}^E, \tilde{v}^\nu - \tilde{v}^E, u_i))^2 &\leq C \|\tilde{v}^E\|_{\mathcal{H}}^2 \|\tilde{v}^\nu - \tilde{v}^E\|_{\mathcal{H}}^2 \|u_i\|_{\text{Lip}(\mathcal{F})}^2, \\ (b(H, \tilde{v}^\nu - \tilde{v}^E, u_i))^2 &\leq C \|H\|_{L^\infty} \|\tilde{v}^\nu - \tilde{v}^E\|_{\mathcal{H}}^2 \|\nabla u_i\|_{L^2(\mathcal{F})}^2 \quad \text{and} \\ (b(\tilde{v}^\nu - \tilde{v}^E, u_i, H))^2 &\leq C \|\tilde{v}^\nu - \tilde{v}^E\|_{\mathcal{H}}^2 \|u_i\|_{L^2(\mathcal{F})}^2 \|H\|_{\text{Lip}(\mathcal{F})}. \end{aligned}$$

We integrate (C.3.4) in time and we use the above estimates. From the first point of the theorem we conclude that also the second points holds true. \square

Before concluding this appendix, we briefly discuss the asymptotic of the solid rotation. Suppose that $r_0^\nu = r_0^E = r_0$. Then r^ν and r^E associated with respectively (v^ν, l^ν) and (v^E, l^E) evolve as

$$(r^\nu)' = \frac{\alpha\nu}{\mathcal{J}} \int_{\partial\mathcal{B}} (v^\nu - l^\nu) \cdot \tau \quad \text{and} \quad r^E(t) = r_0.$$

Which imply that also $r^\nu \rightarrow r^E = r_0$ in $H^1(0, T)$.

Appendix D

Estimates for Neumann Laplacian

This appendix is devoted to the study of solutions of the Laplace problem in exterior domain with non-homogeneous Neumann boundary condition. Let \mathcal{E} an exterior domain such that $\mathbb{R}^2 \setminus \mathcal{E}$ is closed, connect, simply-connected, bounded subset of \mathbb{R}^2 with smooth boundary and without loss of generality suppose that the origin is contained in $\mathbb{R}^2 \setminus \mathcal{E}$. The system reads as

$$\begin{cases} -\Delta f_\varepsilon = s_\varepsilon & \text{in } \mathcal{E}_\varepsilon, \\ \nabla f_\varepsilon \cdot n = \alpha_\varepsilon & \text{in } \partial\mathcal{E}_\varepsilon, \\ |f_\varepsilon| \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases}$$

We will look for uniform estimates independent of the parameter ε . To study the system we use a change of variables associated with the biholomorphic map $T : \mathcal{E}_1 \rightarrow \mathbb{R}^2 \setminus B_1(0)$ and we denote $T_\varepsilon = T(x/\varepsilon)$.

Proposition D.0.1. *Let f_ε , s_ε and α_ε smooth functions such that*

$$\begin{cases} -\Delta f_\varepsilon = s_\varepsilon & \text{in } \mathcal{E}_\varepsilon, \\ \nabla f_\varepsilon \cdot n = \alpha_\varepsilon & \text{on } \partial\mathcal{E}_\varepsilon, \\ |f_\varepsilon| \rightarrow 0 & \text{as } |x| \rightarrow \infty, \end{cases}$$

and let

$$\tilde{f}_\varepsilon(T_\varepsilon(x, y)) = f_\varepsilon(x, y), \quad \tilde{s}_\varepsilon(T_\varepsilon(x, y)) = \frac{s_\varepsilon(x, y)}{|\nabla T_\varepsilon(x, y)|^2} \quad \text{and} \quad \tilde{\alpha}_\varepsilon(T_\varepsilon(x, y)) = \frac{\alpha_\varepsilon(x, y)}{|\nabla T_\varepsilon(x, y)|}.$$

Then \tilde{f}_ε satisfies the system

$$\begin{cases} -\Delta \tilde{f}_\varepsilon = \tilde{s}_\varepsilon & \text{in } \mathbb{R}^2 \setminus B_1(0), \\ \nabla \tilde{f}_\varepsilon \cdot n = \tilde{\alpha}_\varepsilon & \text{on } \partial B_1(0), \\ |\tilde{f}_\varepsilon| \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases}$$

Lemma D.0.1. *Let $\alpha_\varepsilon \in L^1(\partial\mathcal{F}_\varepsilon)$ then*

$$\|\tilde{\alpha}_\varepsilon\|_{L^1(\partial B_1(0))} = \|\alpha_\varepsilon\|_{L^1(\partial\mathcal{F}_\varepsilon)}.$$

Proof. The prove follow by a change of variables. □

Proposition D.0.2. *Suppose $s_\varepsilon = 0$ for any ε and let $\alpha_\varepsilon \in L^1(\partial\mathcal{F}_\varepsilon)$ of average zero. Then there exists R independent of ε such that*

$$|\nabla f_\varepsilon(x)| \leq \varepsilon \frac{C \|\alpha_\varepsilon\|_{L^1(\partial\mathcal{F}_\varepsilon)}}{|x|^2} \quad \text{for } x \in \mathbb{R}^2 \setminus B_{\varepsilon R}(0) \quad (\text{D.0.1})$$

Proof. We use the change of variables T_ε to move the problem in the fixed geometry $\mathbb{R}^2 \setminus B_1(0)$. Recall that $\|\tilde{\alpha}_\varepsilon\|_{L^1(\partial B_1(0))} = \|\alpha_\varepsilon\|_{L^1(\partial\mathcal{F}_\varepsilon)}$ and $\tilde{\alpha}_\varepsilon$ has average zero. Using the representation of the Green's function

$$G_{N, B_1(0)^c}(x, y) = \frac{1}{4\pi} \left[\log \frac{|x||y|}{|x-y|^2} + \log \frac{|x||y|}{|x|y| - \frac{y}{|y|}|^2} \right]$$

from [STT17], we have that

$$|\nabla \tilde{f}_\varepsilon(\tilde{x})| \leq \frac{C \|\tilde{\alpha}_\varepsilon\|_{L^1(\partial B_1(0))}}{|\tilde{x}|^2} \quad \text{for } \tilde{x} \in \mathbb{R}^2 \setminus B_{\tilde{R}}(0).$$

After changing back the variables we obtain the desired result

$$|\nabla f_\varepsilon(x)| \leq \varepsilon \frac{C \|\alpha_\varepsilon\|_{L^1(\mathcal{F}_\varepsilon)}}{|x|^2} \quad \text{for } x \in \mathbb{R}^2 \setminus B_{\varepsilon R}(0).$$

□

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