Transfer operators and horocycle averages on closed manifolds

Alexander Adam

Sorbonne Université

École Doctorale de Sciences Mathématiques de Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Alexander Adam

Opérateurs de transfert et moyennes horocycliques sur les variétés fermées

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dirigée par Viviane Baladi

Soutenue le 10 décembre 2018 devant le jury composé de :

M[me] Viviane Baladi	Sorbonne Université & CNRS	directrice
M. Oscar Bandtlow	Queen Mary University of London	examinateur
M. Yves Coudène	Sorbonne Université	examinateur
M. Giovanni Forni	University of Maryland	rapporteur
M. Colin Guillarmou	Université Paris-Sud & CNRS	rapporteur
M. Carlangelo Liverani	Universita Roma 2	examinateur
M. Frédéric Naud	Université d'Avignon	examinateur
M[me] Barbara Shapira	Université de Rennes 1	examinatrice

Institut de mathématiques de Jussieu-Paris Rive gauche. UMR 7586.
Boîte courrier 247
4 place Jussieu
75 252 Paris Cedex 05

École Doctorale de Sciences Mathématiques de Paris Centre.
Université Sorbonne.
Campus Pierre et Marie Curie.
Boîte courrier 290
4 place Jussieu
75 252 Paris Cedex 05

"To live in an asymmetric world, you should better be symmetric. However if you live in a symmetric world, asymmetry suffices."

L'author

Remerciements

Au cours des trois dernières années, Mme Viviane Baladi a été ma directrice et également un véritable mentor pour moi. Je suis très reconnaissant envers elle pour son énergie et sa persévérance. Elle m'a aidé à rester concentré tout au long de ma thèse et ce particulièrement pendant mes quelques moments d'égarement. Grâce à Mme Viviane Baladi, je suis entré en contact avec M. Frédéric Naud dont une idée mathématique faisant partie de ses travaux que j'ai fait aboutir dans ma première publication scientifique. Toujours grâce à Mme Viviane Baladi, j'ai pu, plusieurs fois, participer à des conférences à différents endroits, ce qui m'a permis de mieux connaître la communauté mathématique dans les domaines des systèmes dynamiques et des résonances.
Avec certaines des personnes que j'ai rencontrées lors de ces conférences, je suis toujours en contact. Je mentionnerai ici M. Giovanni Forni, M. Colin Guillarmou et M. Carlangelo Liverani. La rencontre avec ces personnes avec qui j'ai eu plusieurs longues discussions a contribuée - entre autres - à clarifier mes idées.
Mes déplacements n'auraient certainement pas été possibles sans le financement des organisateurs concernés ou de l'IMJ. Je remercie Le Centre Henri Lebesgue et le CIRM Marseille Luminy pour leur accueil chaleureux et la fondation Knut et Alice Wallenberg pour les invitations à l'université de Lund.
Il n'est pas facile de trouver un logement à Paris. Je suis donc très reconnaissant envers la "Maison d'Italie" (Cité internationale) qui a mis à ma disposition un logement très raisonnable pendant mes trois ans de thèse.
Je me sens également très reconnaissant à Shu Shen, Malo Jézéquel, Colin Guillarmou et Giovanni Forni pour avoir signalé plusieurs incohérences et remarques.
Pour avoir fait preuve de patience et m'avoir apporté son soutien, je suis également redevable envers mon épouse Virginie et la rest de ma famille : je pense à vous !

L'auteur a été soutenu partiellement par la subvention ERC SOS (ERC AdG 787304).

Opérateurs de transfert et moyennes horocycliques sur les variétés fermées

Résumé

Cette thèse de doctorat approfondit l'étude de la dynamique hyperbolique sur les variétés fermées et connexes M et des opérateurs de transfert associés.
Nous étudions deux problèmes : le premier problème concerne les perturbations analytiques réelles des difféomorphismes d'Anosov linéaires sur le tore : une résonance non triviale apparaît-t-elle pour une perturbation génériques d'un difféomorphisme d'Anosov linéaire sur le tore ?
Le second problème concerne une hypothèse sur la moyenne temporelle des flots horocycliques induits par un flot d'Anosov : la moyenne temporelle des flots horocycliques en courbure négative variable converge-t-elle vers la moyenne ergodique en vitesse polynomiale ?
Les opérateurs de transfert associés agissent de façon bornée sur certains espaces de Banach anisotropes par la composition du système dynamique inverse suivie d'une multiplication avec des fonctions de poids spécifiques. Dans notre analyse des problèmes mentionnés ci-dessus, ces opérateurs de transfert représentent le principal intérêt. Nous devons étudier leur spectre bas pour progresser sur nos deux problèmes. Par le *spectre bas*, nous entendons la partie du spectre qui se situe entre le spectre périphérique et le spectre essentiel de ces opérateurs de transfert.
L'approche fonctionnelle de ces opérateurs de transfert se concentre sur les espaces de Banach anisotropes. Nous expliquons l'idée principale derrière cette approche dans le cas des difféomorphismes d'Anosov : des exemples simples de difféomorphismes d'Anosov F sont donnés par les difféomorphismes linéaires d'Anosov sur le tore bidimensionnel. Nous savons que les difféomorphismes d'Anosov transitifs et analytiques ont une unique mesure SRB μ_{SRB} (qui est invariante par le difféomorphisme). Pour les automorphismes linéaires sur le tore, la mesure SRB est la mesure de Lebesgue μ_{Leb}. Notons toutefois que même de petites perturbations analytiques de A ne préservent pas systématiquement μ_{Leb}. Puisque μ_{SRB} est une mesure de Borel, on a $\mu_{\mathrm{SRB}} \in C(M)'$. Nous souhaitons maintenant écrire μ_{SRB} comme l'unique vecteur propre associé à la valeur propre 1 pour un certain opérateur de transfert $\mathcal{L}$ qui apparaît comme l'adjoint

de l'opérateur de composition $\mathcal{K}_F$. Cependant les mesures supportées sur les orbites périodiques de F sont également contenues dans $C(M)'$. Afin de trouver les bonnes propriétés spectrales de l'opérateur $\mathcal{L}$, celui-ci doit être défini sur un espace de Banach *anisotrope* $\mathcal{B}$ et non sur $C(M)'$. La norme de $\mathcal{B}$ prend en compte le comportement dilatant et contractant de l'application F. En particulier, la norme anisotrope de $\mathcal{B}$ traite les éléments de $\mathcal{B}$ comme des fonctions dans les directions dilatantes et comme des distributions dans les directions contractantes de F.
Les valeurs propres discrètes réciproques de $\mathcal{L}$ sont aussi appelées les résonances de F. Si $F = A$, alors il y a seulement les résonances triviales $\{0, 1\}$. Jusque là il n'était pas su qu'il s'agissait d'un comportement attendu si A est perturbé de manière générique.
On entend ici par *perturbation générique* toute application d'un ensemble ouvert et dense dans une boule de difféomorphismes analytiques réels contenant A.
Dans de l'étude du premier problème, nous agissons avec $\mathcal{L}$ sur un espace de Hilbert anisotrope. Nous répondons à la question dans le premier problème par l'affirmative.

Le second problème que nous examinons fait intervenir les *flots d'Anosov*. Ces flots ont été instaurés par Anosov pour étudier le flot géodésique sur le fibré tangent unitaire de variétés fermées à courbure sectionnelle négative variable. De plus, nous avons besoin les flots d'Anosov d'être des *flots de contact*. Des exemples de flots d'Anosov-contact sont donnés par les flots géodésiques. Les flots *horocycliques* associés au flot d'Anosov sont dirigés dans la direction contractant du flot d'Anosov. Nous savons par les travaux de Marcus que pour tout flot horocyclique continu qui correspond à un flot d'Anosov C^2 mélangeant, il existe une unique mesure de probabilité de Borel invariante par le flot horocyclique. Katok et Burns ont démontré que tout flot d'Anosov-contact est mélangeant. Par conséquent, dans notre contexte, la moyenne temporelle de l'horocycle converge vers la moyenne ergodique unique. Mais à quelle vitesse converge la moyenne temporelle ?
Dans le contexte de la courbure négative constante, on sait grâce aux travaux de Flaminio et Forni que cette vitesse est polynomiale. La vitesse est contrôlée par des valeurs propres pour certaines distributions propres du flot géodésique. Un problème analogue dans lequel le flot géodésique est remplacé par un difféomorphisme d'Anosov a été étudié plus tard par Giulietti et Liverani. Ils ont, de plus, dans leurs travaux, supposé que le résultat de Flaminio–Forni devrait s'étendre au flot géodésique dans le contexte de la courbure négative variable.

Dans l'étude du second problème des opérateurs de transfert pondérés $\mathcal{L}_\alpha$, $\alpha > 0$ apparaissent. Suivant l'approche fonctionnelle, il suffit essentiellement de construire un espace de Banach anisotrope $\mathcal{B}$ tel que les opérateurs $\mathcal{L}_\alpha$ agissent sur $\mathcal{B}$, et d'avoir un spectre périphérique consistant en une valeur propre simple isolée. Cependant, la direction d'écoulement du flot d'Anosov n'est ni contractée ni dilatée par le flot d'Anosov, ce qui pose problème dans notre analyse. Nous appliquons donc à la place la stratégie suivante:
Sur un espace de Banach anisotrope $\mathcal{B}$ bien choisi, la famille d'opérateur de transfert $\{\mathcal{L}_\alpha \colon \mathcal{B} \to \mathcal{B} \mid \alpha \geqslant 0\}$ forme un semi-groupe fortement continu et admet donc un générateur bien défini. La quasi-compacité de la résolvante de ce générateur garantit qu'une partie du spectre du générateur par rapport à $\mathcal{B}$ ne contient qu'un spectre discret de multiplicité finie. On trouve alors que le spectre discret contrôle la vitesse de convergence. Cependant, pour montrer que la vitesse de convergence est polynomiale, nous devons imposer une condition supplémentaire à la norme de la résolvante.

Cette thèse est organisée comme suit : dans le premier chapitre, nous étudions les perturbations analytiques réelles et génériques $\widetilde{A}$ d'un difféomorphisme d'Anosov linéaire sur le tore bidimensionnel. Nous appliquons l'approche fonctionnelle aux opérateurs de transfert comme décrit ci-dessus dans le cadre d'un espace de Hilbert anisotrope.
Cet espace de Hilbert est explicitement construit comme la complétion d'un espace de Hardy par rapport à une norme anisotrope. Nous montrons que l'opérateur de composition $\mathcal{K}_{\widetilde{A}}$ est à trace. En fait, nous montrons que $\mathcal{K}_{\widetilde{A}}$ est nucléaire d'ordre 0, ce qui est un résultat plus fort. On obtient la réponse au premier problème par un calcul de la trace de $\mathcal{K}_{\widetilde{A}}$ associé au système perturbé. Nous abordons ensuite des propriétés spectrales de l'opérateur de transfert $\mathcal{L}$.
Le premier chapitre figure tel qu'il a été publié en 2017 dans Nonlinearity 30.3, à l'exception de l'annexe qui est séparée en *Appendix A* et de la liste de références qui est combinée avec celle de cette thèse.

Dans le second chapitre, nous traitons des flots d'Anosov différentiables finis sur des variétés fermées et connexes ainsi que les flots horocycliques stables associés.
La définition de cônes et d'hyperbolicité via des cônes d'une application est développée. Nous introduisons des familles d'opérateurs de transfert $\{\mathcal{L}_\alpha, \alpha > 0\}$ avec des fonctions de poids positives arbitraires. Ceci est suivi par la construction d'espaces de Banach anisotropes. Ces espaces sont analogues aux

espaces construits par Baladi et Tsujii pour étudier les difféomorphismes hyperboliques.
Nous étudions les propriétés des opérateurs $\mathcal{L}_\alpha$, $\alpha > 0$, sur les espaces de Banach anisotropes ainsi que les propriétés de la résolvante des générateurs des familles $\{\mathcal{L}_\alpha, \alpha > 0\}$. Nous établissons une inégalité de Lasota–Yorke pour la résolvante. Nous introduisons et abordons la condition supplémentaire sur les bornes de la résolvante.
Ensuite, nous nous focalisons sur les flots d'Anosov de contact en dimension 3 (nous considérons aussi la co-dimension 1). Nous donnons des bornes locales à l'intégrale de l'horocycle, ce qui nous permet de obtenir notre décomposition de l'intégrale de l'horocycle. Enfin, la condition supplémentaire sur la résolvante est utilisée pour obtenir la vitesse polynomiale de convergence vers la moyenne ergodique.
Le troisième chapitre est également disponible sur arXiv.

Mots-clés

Flot d'Anosov, flot horocyclique, opérateur de transfert, espace de Banach anisotrope, resonances, moyenne ergodique.

Transfer operators and horocycle averages on closed manifolds

Abstract

This doctoral thesis deepens the study of hyperbolic dynamics on connected, closed Riemannian manifolds M and associated transfer operators.
We investigate two problems: The first problem concerns real analytic perturbations of linear toral Anosov diffeomorphisms: Does a non-trivial resonance appear for generic perturbations of a linear toral Anosov diffeomorphism?
The second problem is to make a statement about the time average of horocycle flows with underlying contact Anosov flow: Does the time average of horocycle flows in variable negative curvature converge to the ergodic mean in polynomial time?
The associated transfer operators act boundedly on certain anisotropic Banach spaces by composition of the inverse dynamical system followed by a multiplication with specific weight functions. In our analysis of the beforementioned problems these transfer operators are of central interest. We need to investigate their deeper spectrum to progress on our two problems. By the *deeper spectrum* we mean here the part of the spectrum which lies in between the peripheral and the essential spectrum of these transfer operators.
The functional approach to these transfer operators puts importance on the anisotropic Banach spaces. We explain the principal idea behind this approach in the case of Anosov diffeomorphisms: Simple examples of Anosov diffeomorphisms F are provided by the *linear Anosov diffeomorphisms* A on the two-dimensional torus. Real analytic transitive Anosov diffeomorphisms are known to have associated a so-called unique SRB-measure μ_{SRB} which is invariant by the diffeomorphism. For the linear toral automorphisms the SRB-measure is just the Lebesgue measure μ_{Leb}. Note however that even small real-analytic perturbations of A may not preserve μ_{Leb}. Since μ_{SRB} is a Borel measure it holds $\mu_{\mathrm{SRB}} \in C(M)'$. We wish now to recover μ_{SRB} as the unique 1-eigenvector for a certain transfer operator $\mathcal{L}$ which arises as the adjoint of the composition operator $\mathcal{K}_F$. However measures supported on periodic orbits of F are also contained in $C(M)'$. In order to find good spectral properties of the operator $\mathcal{L}$, it has to be defined on an *anisotropic* Banach space $\mathcal{B}$ and not on $C(M)'$. The norm of

$\mathcal{B}$ takes into account the expansive and contractive behavior of the map F. In particular, the anisotropic norm of $\mathcal{B}$ treats elements in $\mathcal{B}$ as functions along the expanding directions and as distributions along the contracting directions of F. The reciprocal discrete eigenvalues of $\mathcal{L}$ are also called the resonances of F. In case that $F = A$ there are only the trivial resonances $\{0, 1\}$. It was not known before whether this is an expected behavior if A is perturbed generically.
By a *generic perturbation* we mean here any map in an open and dense subset of a ball of real analytic toral diffeomorphisms containing A with respect to the uniform norm.
In the investigation of the first problem, we act with $\mathcal{L}$ on an anisotropic Hilbert space. We answer the question in the first problem in the affirmative.

The second problem that we investigate involves *Anosov flows*. These flows where introduced by Anosov to study the geodesic flow on the unit tangent bundle of closed Riemannian manifolds with variable negative sectional curvature. Moreover, we require the *contact* assumption for the Anosov flow. Examples of contact Anosov flows are provided by geodesic flows on the unit tangent space. The *horocycle flows* associated to the contact Anosov flow point into the contracting direction of the Anosov flow. We know by the work of Marcus that for every continuous horocycle flow which corresponds to a C^2 mixing Anosov flow, there exists exactly one Borel probability measure which is invariant by the horocycle flow. Katok and Burns have shown that every contact Anosov flow is mixing. Hence in our setting, the continuous horocycle time average converges to the unique ergodic mean. But how fast is this convergence?
In the constant negative curvature setting, for the geodesic flow, it is known due to the work of Flaminio and Forni, that this speed is polynomial and is controlled by eigenvalues for certain eigendistributions for the geodesic flow.
An analogous problem where the geodesic flow is replaced by an Anosov diffeomorphism was studied later by Giulietti and Liverani. Moreover, in their work they conjectured that the above result of Flaminio–Forni should extend to the geodesic flow in variable negative curvature.
In the study of the second problem weighted transfer operators $\mathcal{L}_\alpha$, $\alpha > 0$, appear. Following the functional approach, in principal, it is enough to construct an anisotropic Banach space $\mathcal{B}$ such that the operators $\mathcal{L}_\alpha$, acting on $\mathcal{B}$, have a peripheral spectrum consisting of an isolated simple eigenvalue. However, the flow direction of the Anosov flow is neither contracted nor expanded by the Anosov flow which poses a problem in our analysis. We apply instead the following strategy:

With a good choice of an anisotropic Banach space $\mathcal{B}$ the transfer operator family $\{\mathcal{L}_\alpha : \mathcal{B} \to \mathcal{B} \mid \alpha \geqslant 0\}$ forms a strongly continuous semigroup and admits therefore a well-defined generator. Quasi-compactness of the resolvent of this generator ensures that part of the spectrum of the generator with respect to $\mathcal{B}$ contains only discrete spectrum of finite multiplicity. The discrete spectrum is then found to control the speed of convergence. However to show polynomial speed of convergence we need to impose an additional condition on bounds of the resolvent.

Keywords

Anosov flow, horocycle flow, transfer operator, anistropic Banach space, resonances, ergodic mean.

Contents

0 Introduction

This doctoral thesis deepens the study of hyperbolic dynamics on connected, closed Riemannian manifolds M and associated transfer operators.
Two problems are studied: The first problem concerns real analytic perturbations of linear toral Anosov diffeomorphisms. The second problem is to make a detailed statement about the time average of horocycle flows with underlying C^3 contact Anosov flow. The precise problems are formulated in Problem 1 and Problem 2 in the next section below.
The associated transfer operators act boundedly on certain anisotropic Banach spaces by composition of the inverse dynamical system followed by a multiplication with a weight function. In our analysis of the beforementioned problems for specific weight functions these transfer operators are central objects. We need to investigate their deeper spectral properties to progress on our two problems. By the *deeper spectrum* we mean here part of the spectrum which lies in between the peripheral and the essential spectrum of these transfer operators.

The functional approach to these transfer operators puts importance on the anisotropic Banach spaces. It is a more recent method with notably results in the last decades for Anosov diffeomorphisms (e.g. [5], [7], [11], [15], [30], [32], [33], [61]) and flows (e.g. [8], [21]–[24], [27], [31], [48], [64], [65]).
We explain the principal idea behind the functional approach briefly in the case of Anosov diffeomorphisms: A diffeomorphic dynamical system $F\colon M \to M$ being an *Anosov diffeomorphism* means that the tangent space TM of the manifold is split into unstable $E_+ \subset TM$ and stable $E_- \subset TM$ distributions. In particular, one assumes a non-trivial splitting

$$TM = E_- \oplus E_+, \tag{0.1}$$

such that for every norm $\|\cdot\|$ on linear maps $TM \to TM$, for some $0 < \beta < 1$,

$$\sup_{n\in\mathbb{N}} \left\| \beta^{-n} \,\mathrm{D}\, F^n_{|E_-} \right\| < \infty \quad \text{and} \quad \sup_{n\in\mathbb{N}} \left\| \beta^{-n} \,\mathrm{D}\, F^{-n}_{|E_+} \right\| < \infty. \tag{0.2}$$

The arguably simplest example is a toral Anosov diffeomorphism given by Arnold's cat map:

$$A\colon \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2\colon x \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x.$$

Similar examples on the torus $\mathbb{R}^2/\mathbb{Z}^2$ are provided by any hyperbolic unimodular matrix $A \in \mathrm{SL}_2(\mathbb{Z})$, where *hyperbolic matrix* means here that A has one eigenvalue larger than 1 in modulus. Those are the *toral linear Anosov diffeomorphisms*. The map A is also analytic and mixing. By *topological mixing* of the map F we mean that for all non-empty subsets $U, V \subseteq M$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ it holds

$$U \cap F^n(V) \neq \varnothing.$$

By *transitivity* of the map F we mean that there exists a dense orbit of F in M. Real analytic transitive Anosov diffeomorphisms F are known to have associated so-called *SRB-measures* μ_{SRB} [67, Theorem 1]. (A detailed explanation of μ_{SRB} is given in [6], [67].) What is of importance here is that for a given such map F the Borel probability measure μ_{SRB} is *uniquely* characterized by the property that for Lebesgue-almost every $x \in M$ and every continuous function $\varphi \in C(M)$

$$\mu_{\mathrm{SRB}}(\varphi) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} \varphi \circ F^k(x).$$

An example for which μ_{SRB} equals the Lebesgue measure μ_{Leb} is given by the map A. However we note that even small perturbations of A may not preserve μ_{Leb}.

The composition operator is defined by $\mathcal{K}_F\varphi = \varphi \circ F$ for every $\varphi \in C(M)$. Invariance of μ_{SRB} by F yields for every $\varphi \in C(M)$

$$\mu_{\mathrm{SRB}}(\mathcal{K}_F\varphi) = \mu_{\mathrm{SRB}}(\varphi).$$

Hence the dual operator $\mathcal{L} := \mathcal{K}_F'$ acts on $C(M)'$ and fixes μ_{SRB}. It is straight forward to show that $\mathcal{L}$ with respect to μ_{Leb} is given for all $\varphi \in C(M)$ by

$$\mathcal{L}\varphi = \left|\det \mathrm{D}\, F^{-1}\right| \cdot \varphi \circ F^{-1},$$

where $\left|\det \mathrm{D}\, F^{-1}\right|$ is called the *weight function* for $\mathcal{L}$. In the functional approach one wishes now to recover μ_{SRB} as the unique eigenvector to the eigenvalue 1 for $\mathcal{L}$. More precisely, it is the peripheral spectrum of $\mathcal{L}$ which is here of immediate interest. The peripheral spectrum of $\mathcal{L}$ and the properties of the map F should

be related in the following way:

- The peripheral spectrum of $\mathcal{L}$ contains 1 which is a simple eigenvalue.
- The map F is transitive.
- The peripheral spectrum of $\mathcal{L}$ is $\{1\}$ and 1 is a simple eigenvalue.
- The map F is mixing.

However measures supported on periodic orbits are also contained in $C(M)'$ and are therefore eigenvectors to the eigenvalue 1 for $\mathcal{L}$. In order to proceed one changes the domain of the transfer operator $\mathcal{L}$. In particular $\mathcal{L}$ has to be defined on an *anisotropic* Banach space $\mathcal{B}$ and not on $C(M)'$. The space $\mathcal{B}$ contains usually densely $C^{r-1}(M)$ functions for some $r > 1$ or $r \in \{\infty, \omega\}$ if $F \in C^r$[1]. The norm of $\mathcal{B}$ takes into account the expansive and contractive behavior of the map F. In particular, the anisotropic norm of $\mathcal{B}$ treats elements in $\mathcal{B}$ as functions along E_+ and as distributions along E_-.
As we have explained, the construction of $\mathcal{B}$ is constrained by the resulting properties of the spectrum of $\mathcal{L}$ on $\mathcal{B}$. This makes such anisotropic Banach spaces an important part in the functional approach. In the last two decades several constructions of $\mathcal{B}$ have been provided in the differentiable and analytic category of the map F:
The detailed study of anisotropic Banach spaces in the hyperbolic case started in the differentiable setting with the work of Blank, Keller and Liverani [15]. It is now a well established tool (e.g. see the references mentioned above).
Real analytic perturbations of hyperbolic toral automorphisms A were later addressed by Faure and Roy [25]. They considered an anisotropic Hilbert space $\mathcal{H}$, which appeared already briefly in a work of Fried [29, Sect 8, I].
On this Hilbert space $\mathcal{H}$, the operator $\mathcal{L}$ turns out to be compact which implies that the essential spectrum of $\mathcal{L}$ is contained in $\{0\}$ and the deeper spectrum of $\mathcal{L}$ consist of eigenvalues of finite multiplicity. Moreover the spectrum of $\mathcal{L}$ on $\mathcal{H}$ admits a spectral gap between the deeper and the peripheral spectrum which relates to the following:

- The peripheral spectrum of $\mathcal{L}$ is $\{1\}$ and 1 is an isolated simple eigenvalue.
- The correlation function for F admits exponential decay.

[1] If $0 < r < \infty$ is not an integer, C^r means $C^{\lfloor r \rfloor}$ with all partial derivatives of order $\lfloor r \rfloor$ being $(r - \lfloor r \rfloor)$-Hölder continuous.

By *exponential decay* of the correlation function for F we mean that for some $\varphi_1, \varphi_2 \in C(M, \mathbb{C})$ and for some $\lambda > 0$ it holds

$$\sup_{n\in\mathbb{N}} e^{\lambda n} \left|\mu_{\mathrm{SRB}}(\varphi_1 \cdot \varphi_2 \circ F^n) - \mu_{\mathrm{SRB}}(\varphi_1)\,\mu_{\mathrm{SRB}}(\varphi_2)\right| < \infty.$$

Usually λ depends on the regularity of the *observables* φ_1 and φ_2 and *not* on the observables itself.
We comment shortly on the weight function $\left|\det \mathrm{D}\, F^{-1}\right|$ in $\mathcal{L}$: In general the weight function can be any positive C^{r-1} function, depending on the application. Then of course the maximal eigenvalue $\lambda_{\max}$ may change, as well as the associated eigenvector (this is also called the Gibbs state), replacing μ_{SRB}. If one considers then the renormalized transfer operator $\widetilde{\mathcal{L}} := \lambda_{\max}^{-1}\mathcal{L}$, we expect a peripheral spectrum for $\widetilde{\mathcal{L}}$ as discussed, depending on the properties of the map F.

The *dynamical determinant* d_F carries statistical information about the behavior of the map F at periodic orbits of F. It is a holomorphic function defined for all small enough $z \in \mathbb{C}$ by

$$d_F(z) := \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{F^n(x)=x} \left|\det\left(\mathrm{id} - \mathrm{D}_x\, F^n\right)\right|^{-1}.$$

A way to extend the domain of holomorphy of d_F is by using the transfer operator. For example if $\mathcal{K}_F$ is of trace class then the holomorphic extension is given by the Fredholm determinant for all $z \in \mathbb{C}$ by

$$\det(1 - z\mathcal{L}) = \det(1 - z\mathcal{K}_F) = d_F(z). \tag{0.3}$$

The second equality in (0.3) is a consequence of the trace formula for all $n \in \mathbb{N}$

$$\operatorname{tr} \mathcal{K}_F^n = \sum_{F^n(x)=x} \left|\det\left(\mathrm{id} - \mathrm{D}_x\, F^n\right)\right|^{-1}. \tag{0.4}$$

The reciprocal discrete eigenvalues of the transfer operator are also called the (Ruelle-Pollicott) resonances for the map F and the above equality in (0.3) shows a direct relation between the resonances and the zeros of d_F.
In the hyperbolic setting, Rugh proved the holomorphy of the dynamical determinant of real analytic Anosov diffeomorphisms on surfaces [55], [56].

The two problems

A small computation shows the following (e.g. using Lemma A.1):

Lemma 1 (Trivial resonances)**.** *For every hyperbolic matrix $M \in SL_2(\mathbb{Z})$ the associated dynamical determinant satisfies for every $z \in \mathbb{C}$*

$$d_M(z) = 1 - z.$$

Another direct computation shows:

Lemma 2 (Superexponential decay ([6, Chapter 4.4.1]))**.** *For every hyperbolic matrix in $SL_2(\mathbb{Z})$ the dynamical correlation function decays superexponentially on real analytic observables.*

It was not known before whether the above two lemmas show an expected behavior for a generic real analytic toral Anosov diffeomorphisms.
In particular, what happens if we perturb generically a hyperbolic matrix A by a real analytic map?
By a *generic perturbation* we mean here any map in an open and dense subset of a ball of real analytic toral diffeomorphisms containing A with respect to the uniform norm (see directly above Theorem 1.4.3).

Problem 1 (Non-trivial resonances)**.** *Does a non-trivial resonance appear for generic perturbations of a hyperbolic matrix in $SL_2(\mathbb{Z})$?*

If we consider for a moment only an expanding system ($E_- \equiv M \times \{0\}$) then more was already known: The expanding case was initially studied by Ruelle [54]. More recently, Bandtlow, Slipantschuk and Just [13], [59] calculated the resonances of real analytic expanding maps $T\colon S \to S$ on the unit circle S explicitly for Blaschke products. Their transfer operator acts on the Hardy space of holomorphic functions on the annulus. (See also Keller and Rugh [45] in the differentiable category.)
Moreover, Bandtlow and Naud [12] showed that generically expanding circle maps admit infinitely many resonances.

The second problem that we investigate involves *Anosov flows* $g_\alpha \in C^3(M, M)$ where $\dim M = 3$, $\alpha \in \mathbb{R}$ and

$$g_\alpha = g_{\alpha_1} \circ g_{\alpha_2}, \quad \text{for all} \quad \alpha_1 + \alpha_2 = \alpha.$$

These flows where introduced by Anosov to study the geodesic flow on the unit tangent bundle of closed Riemannian manifolds with variable negative sectional curvature [3], [4]. It is required that the splitting of TM contains in addition a neutral (or central) distribution spanned by the bounded vector field X generating the flow:

$$TM = E_+ \oplus E_- \oplus \mathbb{R}X, \tag{0.5}$$

with an analogous condition on the distributions E_- and E_+ as in (0.2). Moreover, we require the *contact* assumption for the flow g_α which means that a certain invariant $3-form$ in $(T^*M)^3$ is never zero (see Section 2.2). Geodesic flows are well-studied examples of contact Anosov flows.

If E_- is orientable we define another flow $h_\rho \colon M \to M$, $\rho \in \mathbb{R}$, which points into E_-. This is the (stable) *horocycle flow* associated to the contact Anosov flow g_α. (The term *horocycle flow* was used originally only in the case of the geodesic flow, e.g. see [49, p.84] or [38].) In general the defining vector field of the horocycle flow is at best $C^{2-\epsilon}$ for all $\epsilon > 0$ [40].

Theorem 1 (Unique ergodicity, [50, Theorem 3.5])**.** *For every continuous horocycle flow which corresponds to a C^2 mixing Anosov flow there exists exactly one Borel probability measure which is invariant by the horocycle flow.*

Theorem 2 (Mixing ([41, Theorem 3.6], [48, Corollary B.6]))**.** *Every contact Anosov flow is mixing.*

Hence in our setting, the continuous time average converges to the unique ergodic mean for all $\varphi \in C(M)$ and for all $x \in M$

$$\mu(\varphi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \varphi \circ h_\rho(x)\, \mathrm{d}\rho,$$

where μ denotes the unique Borel measure given by Theorem 1. But how fast is the convergence to $\mu(\varphi)$? Put otherwise, what can we say about

$$\frac{1}{T} \int_0^T \varphi \circ h_\rho(x)\, \mathrm{d}\rho - \mu(\varphi),$$

for all $T > 0$ for fixed x and fixed φ? Clearly, if $\varphi = \partial_\rho \widetilde{\varphi} \circ h_{\rho|\rho=0}$ for some $\widetilde{\varphi} \in C(M, \mathbb{C})$ then

$$\int_0^T \varphi \circ h_\rho(x)\, \mathrm{d}\rho = \widetilde{\varphi} \circ h_T(x) - \widetilde{\varphi}(x).$$

This poses a generic lower bound on the speed of convergence as $\sim 1/T$, even if $\widetilde{\varphi}$ is very regular such that $\varphi \in C^{2-\epsilon}$ for all $\epsilon > 0$.
Are there any other obstructions which can slow down the speed of convergence even more? The approach to answer this question is again by means of a weighted transfer operator. We find for every $T, \alpha \geqslant 0$ and every $x \in M$

$$\gamma_x(\varphi, T) := \int_0^T \varphi \circ h_\rho(x) \,\mathrm{d}\,\rho = \int_0^{\tau(T,\alpha,x)} \mathcal{L}_\alpha \varphi \circ h_\rho \circ g_\alpha(x) \,\mathrm{d}\,\rho,$$

where for all $\alpha \geqslant 0$, assuming here for simplicity that the flow h_ρ has unit speed,

$$\mathcal{L}_\alpha \varphi = \det \mathrm{D}\, g_{-\alpha|E_-} \cdot \varphi \circ g_{-\alpha}, \tag{0.6}$$

and where for all $x \in M$ and all $\rho, \alpha \in \mathbb{R}$

$$g_\alpha \circ h_\rho(x) = h_{\tau(\rho,\alpha,x)} \circ g_\alpha(x).$$

The function τ is called the *renormalization time*. To find a stronger slowdown than $1/T$ in the speed of convergence, the idea is the following: If on a certain anisotropic Banach space $\mathcal{B}$ the operator $\mathcal{L}_\alpha$ has an eigenvector $\mathcal{D}_\lambda \in \mathcal{B}$ such that $\mathcal{L}_\alpha \mathcal{D}_\lambda = e^{\lambda\alpha}\mathcal{D}_\lambda$ for some $\Re\lambda > 0$ and for all $\alpha \geqslant 0$ then formally

$$\gamma_x(\mathcal{D}_\lambda, T) = \int_0^{\tau(T,\alpha,x)} \mathcal{L}_\alpha \mathcal{D}_\lambda \circ h_\rho \circ g_\alpha(x) \,\mathrm{d}\,\rho = e^{\alpha\lambda} \int_0^{\tau(T,\alpha,x)} \mathcal{D}_\lambda \circ h_\rho \circ g_\alpha(x) \,\mathrm{d}\,\rho.$$

As pointed out by Anosov [3], the topological entropy h_{top} of the time-one map g_1 is positive. It is an important property of the renormalization time τ that $\tau(T, \alpha, x) \leqslant 1$ implies $e^{h_{\mathrm{top}}\alpha} \leqslant CT$ for some constant $C > 0$ independent of every $T \geqslant 1$ and every $x \in M$ (e.g. use Proposition 2.5.13 below).
Hence in our example, the unique ergodic mean is reached only with at most a speed of $\sim T^{\frac{\Re\lambda}{h_{\mathrm{top}}}-1}$.
Let us assume for simplicity that all eigenvalues $e^{\lambda\alpha}$ are simple for all $\alpha > 0$. Then we have associated to each λ a finite rank operator given by $\mathcal{D}_\lambda \otimes \mathcal{O}_\lambda$, where $\mathcal{O}_\lambda \in \mathcal{B}'$. Now we can decompose formally for every $\varphi \in \mathcal{B}$ and for every $\delta \in \mathbb{R}$

$$\varphi = \sum_{\Re\lambda > \delta} \mathcal{O}_\lambda(\varphi)\, \mathcal{D}_\lambda + \varphi_{\mathcal{E}}, \tag{0.7}$$

with some remainder term $\varphi_{\mathcal{E}}$.

In the constant negative curvature setting, for the geodesic flow, the following is known due to the work of Flaminio and Forni, where the flows g_α and h_ρ arise both from constant vector fields:

Theorem 3 (Flaminio–Forni, [28, Theorem 1.5]). *Let M be the unit tangent bundle of a compact hyperbolic Riemannian surface of constant negative curvature. Let* vol *be the canonical volume form on M. Then it holds for all $\varphi \in C^4$, for all $x \in M$ and for all $T > 1$*

$$\int_0^T \varphi \circ h_\rho(x)\, \mathrm{d}\rho = T \operatorname{vol}(\varphi) + \sum_{\lambda \in \Sigma_{\frac{1}{2}} \setminus \{1\}} T^\lambda c_\lambda(T,x)\, \mathcal{O}_\lambda(\varphi) + \mathcal{E}_{T,x}(\varphi),$$

where $\sup_{T>1, x\in M} \frac{|\mathcal{E}_{T,x}(\varphi)|}{T^{\frac{1}{2}} \log T} < \infty$ *and* $\sup_{T>1,x\in M} |c_\lambda(T,x)| < \infty$.

In fact the result of Flaminio–Forni gives a much more detailed expansion, including a summation over all $\Re\lambda > 0$ and lower bounds on the coefficients c_λ.

An analogous problem where the geodesic flow is replaced by an Anosov diffeomorphism was studied later by Giulietti and Liverani [31]. Moreover, in their work they conjectured (see [31, Conjecture 2.14]) that the above result of Flaminio–Forni should extend to the geodesic flow in the variable negative curvature setting.

Problem 2 (Horocycle flows in variable negative curvature). *Does an expansion of the horocycle integral analogous to Theorem 3 hold for the horocycle flow induced by the geodesic flow of a surface of variable negative curvature?*

Following the functional approach, it is in principle enough to construct an anisotropic Banach space $\mathcal{B}$ such that the operator $\mathcal{L}_\alpha$, acting on $\mathcal{B}$, has a peripheral spectrum consisting of an isolated simple eigenvalue at $e^{h_{\mathrm{top}}\alpha}$ for all $\alpha \geqslant 0$ and such that $\gamma_x\left(\mathcal{D}_{h_{\mathrm{top}}}, T\right)$ and $\gamma_x(\varphi\mathcal{E}, T)$ are well-defined. Indeed all this could follow if $\mathcal{L}_{\alpha,\phi_\alpha}$ is quasi-compact on $\mathcal{B}$ for all $\alpha \geqslant 0$. What prevents us in doing so is the flow direction X in the splitting in (0.5) which is neither contracted nor expanded by the geodesic flow. We apply instead the following strategy:
With a good choice of an anisotropic Banach space $\mathcal{B}$ the transfer operator family

$$\{\mathcal{L}_\alpha \colon \mathcal{B} \to \mathcal{B} \mid \alpha \geqslant 0\}$$

forms a strongly continuous semigroup and admits therefore a well-defined generator $X + V$. Quasi-compactness of the resolvent $\mathcal{R}_z$ of $X + V$ for large values

$\Re z > 0$ ensures that part of the spectrum of $X + V$ with respect to $\mathcal{B}$

$$\Sigma_\delta := \sigma\left(X + V\right)|_{\mathcal{B}} \cap \{\lambda \in \mathbb{C} \mid \Re\lambda > \delta\}$$

contains only discrete spectrum of finite multiplicity for some $\delta > 0$.

Note that $\mathcal{L}_\alpha$ is a transfer operator with a different weight than we discussed above in the diffeomorphism case.
In full analogy to our discussion of the peripheral spectrum of $\mathcal{L}$, we expect that the peripheral spectrum of the operator $e^{-h_{\mathrm{top}}\alpha}\mathcal{L}_\alpha$ on $\mathcal{B}$ consists of the simple eigenvalue 1 for all $\alpha > 0$ and that the associated eigenvector coincides with the unique Borel probability measure given by Theorem 1.
Moreover, the weight function for $\mathcal{L}_\alpha$ is $\det \mathrm{D}\, g_{-\alpha|E_-}$ which depends on the regularity of E_-, which we noted is $C^{2-\epsilon}$ for all $\epsilon > 0$. To deal with such irregular weights one can lift the dynamics to the Grassmanian. This has been used with success, e.g. in [30], [33] and more recently in [65]. However, we handle directly the given weight function which allows us to avoid such additional technicalities.

We should add here that the additional conjecture that the distributions $\mathcal{O}_v$ appearing in the right-hand side in (0.7) are fixed by the adjoint of the horocycle flow remains still open. In contrast this was the starting point in [28]. Here, progress has been made by Faure and Guillarmou [24] in dimension 3 for smooth contact Anosov flows.

Although we do not study here the dynamical zeta function for the transfer operator $\mathcal{L}_\alpha$, we believe that the anisotropic Banach space $\mathcal{B}$ constructed in Section 2.3.2 could be a suitable choice to be dealt with.

Statement of results

We present here in a simplified form the main results of this thesis. The first result states that it is quite common for the composition operator to have non-trivial spectrum if the dynamical system is a real analytically perturbed hyperbolic matrix:

Theorem I (Non-trivial resonances (Theorem 1.4.3))**.** *Let $A \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. For a generic real analytic perturbation $\widetilde{A}$ of A there exists an anisotropic*

Hilbert space $\mathcal{H}$ such that

$$\mathcal{K}_{\tilde{A}} \colon \mathcal{H} \to \mathcal{H}$$

is of trace class and it holds

$$\operatorname{sp}\left(\mathcal{K}_{\tilde{A}}\right) \setminus \{0,1\} \neq \varnothing.$$

This result implies that the dynamical determinant $d_{\tilde{A}}$ has at least one non-trivial zero and therefore answers the question in Problem 1 in the affirmative. By construction the Hilbert space $\mathcal{H}$ contains real analytic observables as a dense subset. Hence the existence of a non-trivial resonance poses an obstruction to the superexponential decay of the correlation function for real analytic observables.
In some sense our result is optimal: We cannot hope to replace '*generic perturbations*' with '*for all perturbations*'. In the expanding case there exists Blaschke products arbitrary close to a linear function on the circle and which have trivial spectrum (and which are note C^1 conjugated to a linear function) [59, Example 5.6].
It is reasonable to believe that similar constructions work in the Anosov case, using the generalized Blaschke products in [61]. This means that isospectral perturbations are expected but they are *not* generic.

The second result makes a statement about the time average of a horocycle flow with underlying contact Anosov flow. The full result treats the general case with possible non-trivial Jordan blocks. For simplicity of the statement we assume here that the spectrum is simple:

Proposition II (Horocycle integral (Theorem 2.5.7, Proposition 2.5.10))**.** *There exists an anisotropic Banach space $\mathcal{B}$ and $\lambda_{min} < h_{top}$ such that $\Sigma_{\lambda_{min}}$ consists only of discrete eigenvalues of finite multiplicity. For all $x \in M$ and all $T \geqslant 1$, for every $\lambda_{min} \leqslant \delta < h_{top}$ and for every finite subset $\Lambda_\delta \subseteq \Sigma_\delta$ and for all $\varphi \in C^3$ it holds*

$$\int_0^T \varphi \circ h_\rho(x) \,\mathrm{d}\,\rho = c_{h_{top}}(T,x)\,\mu(\varphi) + \sum_{\substack{\lambda \in \Lambda_\delta \\ \Re\lambda < h_{top}}} T^{\frac{\lambda}{h_{top}}} c_\lambda(T,x)\,\mathcal{O}_\lambda(\varphi) + \mathcal{E}_{T,x,\Lambda_\delta}(\varphi),$$

where μ is the unique Borel measure which is invariant by the horocycle flow h_ρ and

$$\sup_{T\geqslant 1, x\in M} \frac{c_\lambda(T,x)}{T} < \infty \quad \text{and} \quad \lim_{T\to\infty} \frac{\mathcal{E}_{T,x,\Lambda_\delta}(\varphi)}{T} = 0.$$

Note that the expected principal term $T\mu(\varphi)$ is obscured since we ordered the expansion by $\mathcal{O}_\lambda(\varphi)$. One recovers the principal term, using

$$T\mu(\varphi) = \int_0^T 1 \circ h_\rho(x)\, \mathrm{d}\rho.$$

We use this later in Corollary 2.5.9 in Section 2.5 below. Clearly, we only answer partially the question in Problem 2 in the affirmative. The reason is that quasi-compactness of $\mathcal{R}_z$ is not enough to give us e.g. a finite set Σ_δ for some $\delta \geqslant \lambda_{\min}$. We impose the following extra condition on the resolvent to affirm the full question for C^3 contact Anosov flows in dimension 3:

Condition A (Spectral gap with (Dolgopyat) bounds (Condition 2.4.11))**.** *For some $0 < \delta < h_{top}$, $a > 0$, $b > 0$, $C > 0$ and some*

$$\gamma \in (0, 1/\log(1 + (h_{top} - \delta)/a)),$$

and for all $z \in \mathbb{C}$ with $\Re z = a$ and $|\Im z| \geqslant b$, it holds

$$\left\| \mathcal{R}^{\widetilde{n}}_{z+\lambda_{top}} \right\|_{W^{s,t,q}_p} \leqslant C^{\widetilde{n}} |\Re z + (h_{top} - \delta)|^{-\widetilde{n}}, \quad \text{where } \widetilde{n} = \lceil \gamma \log |\Im z| \rceil.$$

Under this additional condition we obtain:

Theorem III (Theorem 2.5.7, Proposition 2.5.10)**.** *Under the assumptions of Proposition II, if in addition Condition A holds with same δ then we can take $\Lambda_\delta = \Sigma_\delta$ and it holds $|\mathcal{E}_{T,x,\Sigma_\delta}(\varphi)| \leqslant CT^{\frac{\delta}{h_{top}}+\epsilon}$ for all $\epsilon > 0$ and some $C = C(\varphi) \geqslant 0$ independent of T and x.*

We shall note a curiosity which we do not discuss further in this thesis: An application of the last theorem which presents itself is the deeper analysis of the renormalization time τ itself! It follows from the construction of the transfer operator $\mathcal{L}_\alpha$ that for all $\rho, \alpha > 0$ and for all $x \in M$ it holds

$$\tau(\rho, -\alpha, x) = \gamma_x(\mathcal{L}_\alpha 1, \rho).$$

Of course, in the setting of constant vector fields (i.e. constant negative curvature) we cannot learn anything new about τ, but other cases might be of further interest (e.g. in studying small perturbations of the constant vector fields).

Organization of the thesis

We investigate Problem 1 in Chapter 1 and Problem 2 in Chapter 2.

In Chapter 1 we study generic real analytic perturbations $\widetilde{A}$ of a linear Anosov diffeomorphism on the two-dimensional torus as introduced before Problem 1. We apply the functional approach to transfer operators as described above in the setting of an anisotropic Hilbert space.
This Hilbert space is explicitly constructed in Section 1.2 as the completion of some Hardy space with respect to an anisotropic norm. The trace class property of the Koopman operator $\mathcal{K}_{\widetilde{A}}$ is shown in Section 1.3. In fact, we show that $\mathcal{K}_{\widetilde{A}}$ is nuclear of order 0 which is a stronger result.
Theorem I which is Theorem 1.4.3 below is finally shown in Section 1.4. This is done essentially by the calculation of the trace of the transfer operator associated to the perturbed system, using the equality in (0.4) which is shown as well.
The spectral properties of the transfer operator $\mathcal{L}$ are discussed in Section 1.5.
Chapter 1 is presented as it was published [1], except that the appendix is moved to Chapter A and the reference list is combined with that of this thesis.

In Chapter 2 we deal with finite differentiable Anosov flows on connected, closed Riemannian manifolds and the associated stable horocycle flows.
The necessary notion of cones and cone-hyperbolicity of a map is introduced in Section 2.2.
In Section **??** we introduce families $\{\mathcal{L}_\alpha \mid \alpha > 0\}$ of transfer operators similar to those in (0.6) but with arbitrary positive weight functions. This is followed by the construction of anisotropic Banach spaces. These spaces are a flow analogue to the spaces constructed by Baladi and Tsujii [10] to study hyperbolic diffeomorphisms.
In Section 2.4 we discuss properties of the operators $\mathcal{L}_\alpha$, $\alpha > 0$, on the constructed anisotropic Banach spaces as well as properties of the resolvent of the generators of the families $\{\mathcal{L}_\alpha \mid \alpha > 0\}$. We show a Lasota–Yorke inequality for the resolvent, which is Theorem 2.4.5 below. We introduce and discuss Condition A which is Condition 2.4.11 below.
In Section 2.5 we specialize to contact Anosov flows in dimension 3 (we consider also the co-dimension 1 case).
We give local bounds on the horocycle integral in Lemma 2.5.14. This lemma enables us to show Theorem III which is Theorem 2.5.7 below.
The additional Condition A is finally needed to obtain a polynomial rate of

convergence to the ergodic mean for horocycle flows induced by contact Anosov flows in dimension 3.
This chapter is also available on arXiv [2].

1 Generic non-trivial resonances for Anosov diffeomorphisms

1.1 Introduction

Let $T\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a real analytic Anosov diffeomorphism. We define the Ruelle resonances of T to be the zeroes of the (holomorphically continued in $z \in \mathbb{C}$) dynamical determinant

$$d_T(z) := \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{T^n(x)=x} |\det(\mathrm{id} - \mathrm{D}_x T^n)|^{-1} . \tag{1.1}$$

It is well-known (e.g. combining (1.1) and Lemma A.1) that 1 is the only resonance if T is a hyperbolic linear toral automorphism M. A subset of the Banach space of $\mathbb{T}^2$-preserving maps, holomorphic and uniformly bounded on some annulus, is called generic if it is open and dense. We show in Theorem 1.4.3, using an idea of Naud [51], that there is such a set $\mathcal{G}$ so that for all $\psi \in \mathcal{G}$, appropriately scaled, the Anosov diffeomorphism $M + \psi$ admits non-trivial Ruelle resonances. For this, we construct a Hilbert space of anisotropic generalized functions on which the transfer operator $\mathcal{L}_T f := (f/|\det \mathrm{D}\, T|) \circ T^{-1}$ is nuclear with its Fredholm determinant equal to d_T. Moreover, we prove that some of those generic perturbations preserve the volume while some do not.

The expanding case is easier and was initially studied by Ruelle [54]. More recently, Bandtlow et. al [13], [59] calculated the resonances of real analytic expanding maps $T\colon S \to S$ on the unit circle S explicitly for Blaschke products. Their transfer operator acts on the Hardy space of holomorphic functions on the annulus. (See also Keller and Rugh [45] in the differentiable category.)

In the hyperbolic setting, Rugh proved the holomorphy of the dynamical determinant of real analytic Anosov diffeomorphisms on surfaces [55], [56]. The idea was generalized by Fried to hyperbolic flows in all dimensions [29]. The detailed study of anisotropic Banach spaces in the hyperbolic case started with the pioneering work of [15] (in the differentiable setting) and is now a well established

tool, see e.g. [11] and [32].
Faure and Roy [25] later addressed real analytic perturbations of hyperbolic linear toral automorphisms on the two-dimensional torus, considering an anisotropic complex Hilbert space, which had already been briefly discussed by Fried [29, Sect 8, I].
Our approach is based on this construction and strongly relies on an idea suggested by Naud [51]. We put the transfer operator at the center of our analysis. We introduce an anisotropic Hilbert space (Definition 1.2.4) in Section 1.2.
In Section 1.3, we rephrase a result from Faure and Roy [25, Theorem 6] to show that the Koopman operator $\mathcal{K}_T f := f \circ T$ is nuclear of order 0 when acting on our anisotropic Hilbert space.
In Section 1.4, we use this result and an idea of Naud [51] to show that the Koopman operator admits non-trivial Ruelle resonances under a small generic perturbation of the dynamics.
In Section 1.5, we consider the adjoint of the Koopman operator, which is just the transfer operator, acting on the dual Hilbert space and obtain our final results.
In the Appendix, we recall two needed basic properties of integer matrices (seen as linear maps on the torus) and provide a sufficient condition for determinant preserving perturbations of differentiable real maps.

In principal the analogous problem on any higher dimensional torus can be treated with the presented method. However, one has to modify slightly the used space from Section 1.2 if the linear toral automorphism has non-trivial Jordan blocks.

Blaschke products were recently generalized to the hyperbolic setting by Slipantschuk et al. [60] who calculate the entire spectrum of these real analytic Anosov volume preserving diffeomorphisms explicitly.

1.2 An anisotropic Hilbert space

We denote the flat 2-torus by $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$. We embed $\mathbb{T}^2$ into the standard polyannulus in $\mathbb{C}^2$ and set for each $r > 0$

$$\mathbb{A}_r := \mathbb{T}^2 + \mathrm{i}\,(-r, r)^2.$$

We see $\mathbb{A}_r$ as a submanifold of $\mathbb{C}^2$. The Hilbert space $L_2\left(\mathbb{T}^2\right)$ is equipped with the canonical Lebesgue measure on $\mathbb{T}^2$. This space admits an orthonormal Fourier basis given by

$$\varphi_n \colon \mathbb{T}^2 \to \mathbb{C} \colon x \mapsto \exp\left(\mathrm{i}\, 2\pi n^* x\right), \quad n \in \mathbb{Z}^2, \tag{1.2}$$

where n^* is the canonical dual of n. We recall a construction from Faure and Roy [25] for a complex Hilbert space $\mathcal{H}_{A_{M,c}}$. This space also has been described briefly by Fried as an "ad hoc example" [29, Sect. 8, I.] of a generalized function space. The construction will be based on:

Definition 1.2.1 (Hardy space $H_2\left(\mathbb{A}_r\right)$)**.** *For each $r > 0$ and each holomorphic function $f \colon \mathbb{A}_r \to \mathbb{C}$, we define the norm*

$$\|f\|_{H_2(\mathbb{A}_r)} := \sup_{y \in (-r,r)^2} \left(\int_{\mathbb{T}^2} |f\left(x + \mathrm{i}\, y\right)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.$$

Then we set

$$H_2\left(\mathbb{A}_r\right) := \left\{ f \colon \mathbb{A}_r \to \mathbb{C} \;\middle|\; f \text{ holomorphic, } \|f\|_{H_2(\mathbb{A}_r)} < \infty \right\}.$$

The space $H_2\left(\mathbb{A}_r\right)$ is the 2-dimensional analogue of the Hardy space studied in [58, p. 4]. It admits a Fourier basis given by

$$\vartheta_n^r \colon \mathbb{A}_r \to \mathbb{C} \colon x \mapsto \exp\left(-2\pi r \|n\|\right) \varphi_n, \quad n \in \mathbb{Z}^2,$$

where $\|z\| := |z_1| + |z_2|$ for all $(z_1, z_2) =: z \in \mathbb{C}^2$ and $z \in \mathbb{T}^2$. With this choice of norm, the Fourier basis is orthonormal. Under the canonical isomorphism $L_2\left(\mathbb{T}^2\right) \cong L_2\left(\mathbb{T}^2\right)^*$, we have the isomorphism

$$\left(\vartheta_n^r\right)^* \cong \vartheta_n^{-r}. \tag{1.3}$$

A matrix $M \in \mathrm{SL}_2\left(\mathbb{Z}\right)$ is called hyperbolic if its eigenvalues do not lie on the unit circle. We denote by E_M^+ the eigenspace for the eigenvalue of modulus $\lambda_M > 1$ and by E_M^- the eigenspace of the eigenvalue of modulus λ_M^{-1}. We decompose $y \in \mathbb{R}^2$ uniquely as

$$y = y_M^+ + y_M^- \quad \text{with} \quad y_M^+ \in E_{M*}^+ \quad \text{and} \quad y_M^- \in E_{M*}^-. \tag{1.4}$$

We have

$$\left\|M^* y_M^+\right\| = \lambda_M \left\|y_M^+\right\| \quad \text{and} \quad \left\|M^* y_M^-\right\| = \lambda_M^{-1} \left\|y_M^-\right\|. \tag{1.5}$$

Definition 1.2.2 (Scaling map $A_{M,c}$)**.** *Let $c > 0$, and $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. For every $n \in \mathbb{Z}^2$, we set, recalling (1.2),*

$$A_{M,c}\varphi_n := \exp\left(-2\pi c\left(\left\|n_M^+\right\| - \left\|n_M^-\right\|\right)\right)\varphi_n.$$

Lemma 1.2.3 (Continuous embedding of $H_2(\mathbb{A}_r)$)**.** *Let $c > 0$ and let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. Then the map $A_{M,c}$ can be extended by continuity to an injective linear map*

$$A_{M,c}\colon H_2(\mathbb{A}_c) \to L_2\left(\mathbb{T}^2\right),$$

bounded in operator norm by 1.

Proof. By Definition 1.2.2, for each $f \in H_2(\mathbb{A}_c)$ we have

$$\begin{aligned}\left\|A_{M,c}f\right\|_{L_2(\mathbb{T}^2)}^2 &= \sum_{n\in\mathbb{Z}^2}\left|\varphi_n^* A_{M,c}f\right|^2 = \sum_{n\in\mathbb{Z}^2}\exp\left(-4\pi c\left(\left\|n_M^+\right\| - \left\|n_M^-\right\|\right)\right)\left|\varphi_n^* f\right|^2 \\ &= \sum_{n\in\mathbb{Z}^2}\exp\left(-4\pi c\left(\left\|n_M^+\right\| - \left\|n_M^-\right\| + \|n\|\right)\right)\left|\vartheta_n^{c\,*} f\right|^2,\end{aligned}$$

where we used (1.3) in the last step. Using the triangle inequality, we find

$$\left\|n_M^+\right\| - \left\|n_M^-\right\| + \|n\| \geqslant 0.$$

Hence, it holds

$$\sum_{n\in\mathbb{Z}^2}\exp\left(-4\pi c\left(\left\|n_M^+\right\| - \left\|n_M^-\right\| + \|n\|\right)\right)\left|\vartheta_n^{c\,*} f\right|^2 \leqslant \|f\|_{H_2(\mathbb{A}_c)}^2.$$

Injectivity follows since $A_{M,c}$ is invertible on the Fourier basis of $L_2\left(\mathbb{T}^2\right)$. □

The image of $H_2(\mathbb{A}_c)$ under $A_{M,c}$ is dense in $L_2\left(\mathbb{T}^2\right)$ since it contains all Fourier polynomials.

Definition 1.2.4 (Hilbert space $\mathcal{H}_{A_{M,c}}$)**.** *Let $c > 0$ and let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. Let $A_{M,c}$ be the map given by Definition 1.2.2. Then we set*

$$\mathcal{H}_{A_{M,c}} := \textit{closure of } H_2(\mathbb{A}_c) \textit{ with respect to the norm } \left\|A_{M,c}\cdot\right\|_{L_2(\mathbb{T}^2)},$$

and extend $A_{M,c}$ by continuity to a linear map

$$A_{M,c}\colon \mathcal{H}_{A_{M,c}} \to L_2\left(\mathbb{T}^2\right).$$

As a direct consequence of this construction, the scalar product on $\mathcal{H}_{A_{M,c}}$ satisfies

$$\langle \cdot, \cdot \rangle_{\mathcal{H}_{A_{M,c}}} : \mathcal{H}_{A_{M,c}} \times \mathcal{H}_{A_{M,c}} \to \mathbb{C} \colon (f, g) \mapsto \langle A_{M,c} f, A_{M,c} g \rangle_{L_2(\mathbb{T}^2)} .$$

An orthonormal Fourier basis of $\mathcal{H}_{A_{M,c}}$ is given by

$$\varrho_n := A_{M,c}^{-1} \varphi_n, \quad n \in \mathbb{Z}^2. \tag{1.6}$$

Lemma 1.2.5 (Dual space of $\mathcal{H}_{A_{M,c}}$)**.** *Under the canonical isomorphism $L_2\left(\mathbb{T}^2\right) \cong L_2\left(\mathbb{T}^2\right)^*$, the dual space $\mathcal{H}_{A_{M,c}}^*$ is isomorphic to $A_{M,c}^2 \mathcal{H}_{A_{M,c}}$.*

Proof. Under the canonical isomorphism $L_2\left(\mathbb{T}^2\right) \cong L_2\left(\mathbb{T}^2\right)^*$, we have for each $n_1, n_2 \in \mathbb{Z}^2$, using (1.6),

$$\varphi_{n_1}^* \left(\varphi_{n_2}\right) = \varphi_{n_1}^* \left(A_{M,c} \varrho_{n_2}\right) = \left(A_{M,c} \varphi_{n_1}\right)^* \left(\varrho_{n_2}\right) = \left(A_{M,c}^2 \varrho_{n_1}\right)^* \left(\varrho_{n_2}\right).$$

□

Remark 1.2.6. *By Lemma 1.2.5, we associate to every linear functional $f^* \in \mathcal{H}_{A_{M,c}}^*$ a unique vector $f \in A_{M,c}^2 \mathcal{H}_{A_{M,c}}$. Then, for every $g \in \mathcal{H}_{A_{M,c}}$, the product fg is absolutely integrable with respect to the Lebesgue measure on $\mathbb{T}^2$.*

The decomposition in (1.4) defines two cones

$$C_M^+ := \left\{ y \in \mathbb{R}^2 \;\middle|\; \left\|y_M^+\right\| \geqslant \left\|y_M^-\right\| \right\} \quad \text{and} \quad C_M^- := \left\{ y \in \mathbb{R}^2 \;\middle|\; \left\|y_M^+\right\| \leqslant \left\|y_M^-\right\| \right\}.$$

Example 1.2.7. *We let $M = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, then $\lambda_M = 2 + \sqrt{3}$. An eigenvector for λ_M for M^* is $\left(1 + \sqrt{3}, 1\right)$ and an eigenvector for λ_M^{-1} is $\left(1 - \sqrt{3}, 1\right)$. The two subspaces E_{M*}^+ and E_{M*}^- and the two cones C_M^+ and C_M^- are shown in Figure 1.1.*

We set

$$\mathcal{H}_{A_{M,c}}^+ := \left\{ \sum_{n \in C_M^+ \cap \mathbb{Z}^2} \langle \varrho_n, f \rangle_{\mathcal{H}_{A_{M,c}}} \varrho_n \;\middle|\; f \in \mathcal{H}_{A_{M,c}} \right\} \text{ and}$$

$$\mathcal{H}_{A_{M,c}}^- := \left\{ \sum_{n \in C_M^- \cap \mathbb{Z}^2} \langle \varrho_n, f \rangle_{\mathcal{H}_{A_{M,c}}} \varrho_n \;\middle|\; f \in \mathcal{H}_{A_{M,c}} \right\}.$$

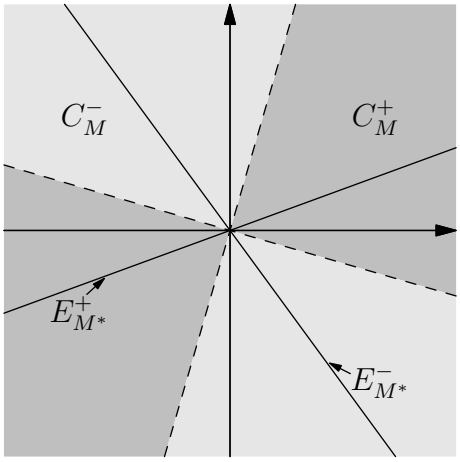

Figure 1.1: The map M is from Example 1.2.7. The dark gray area is the cone C_M^+ which contains the subspace E_{M*}^+. The light gray area is the cone C_M^- and contains E_{M*}^-. A part $y \in \mathbb{R}^2$ belongs to the dashed lines if and only if $\|y_M^+\| = \|y_M^-\|$.

Hence, we have $\mathcal{H}_{A_{M,c}} = \mathcal{H}^+_{A_{M,c}} + \mathcal{H}^-_{A_{M,c}}$. Comparing for each $n \in C_M^-$ the Fourier basis ϱ_n with φ_n, it follows immediately that $\mathcal{H}^-_{A_{M,c}} \subset L_2\left(\mathbb{T}^2\right)$. For each $n \in C_M^+$, comparing the Fourier basis ϱ_n with $\vartheta_n^{c\,*}$, using (1.3), shows $\mathcal{H}^+_{A_{M,c}} \subset H_2\left(\mathbb{A}_c\right)^*$. We conclude therefore that $\mathcal{H}_{A_{M,c}}$ contains linear functionals which do not belong to $L_2\left(\mathbb{T}^2\right)$. By construction, the space $\mathcal{H}_{A_{M,c}}$ is a rigged Hilbert space, i.e.:

$$H_2\left(\mathbb{A}_c\right) \subset \mathcal{H}_{A_{M,c}} \subset H_2\left(\mathbb{A}_c\right)^* . \tag{1.7}$$

Remark 1.2.8. *We note that in the construction of $\mathcal{H}_{A_{M,c}}$, the expanding and contracting directions appear in the dual coordinates $n \in \mathbb{Z}^2$ of the Fourier basis (1.6). This distinguishes $\mathcal{H}_{A_{M,c}}$ from the space of Rugh [56] where expanding and contracting coordinates are spatial. We observe*

$$n^* x = \left(n_M^+ + n_M^-\right)^* \left(x_{M*}^+ + n_{M*}^-\right) = \left(n_M^+\right)^* x_{M*}^+ + \left(n_M^-\right)^* x_{M*}^- .$$

Hence, we can rewrite (1.6) as

$$\begin{aligned} \varrho_n(x) &= \exp\left(2\pi c\left(\left\|n_M^+\right\| - \left\|n_M^-\right\|\right)\right) \exp\left(\mathrm{i}\, 2\pi n^* x\right) \\ &= \exp\left(2\pi c\left\|n_M^+\right\|\right) \exp\left(\mathrm{i}\, 2\pi \left(n_M^+\right)^* x_{M*}^+\right) \\ &\quad \times \exp\left(-2\pi c\left\|n_M^-\right\|\right) \exp\left(\mathrm{i}\, 2\pi \left(n_M^-\right)^* x_{M*}^-\right) . \end{aligned} \tag{1.8}$$

It is tempting to think of the ϱ_n as basis elements for a tensor product space of

a Hardy space on an annulus, with the dual of such a Hardy space. However, we cannot use ϱ_n as such a basis since n_M^+ and n_M^- are not independent of each other. Nevertheless, we can decompose $\mathcal{H}_{A_{M,c}}$ into two generalized Hardy spaces as follows. We define four norms

$$\mu_j(f) := \sup_{y\in A_j}\left(\int_{\mathbb{T}^2}|f(x+\mathrm{i}\,y)|^2\,\mathrm{d}x\right)^{\frac{1}{2}},\ f\in L_2\left(\mathbb{T}^2\right),\ j\in\{1,2,3,4\},\ \textit{where}$$
$$A_1 := \left\{y\in\mathbb{R}^2 \;\middle|\; y_{M*}^-\in(-c,c)^2,\ y_{M*}^+\in(c,\infty)^2\right\},$$
$$A_2 := \left\{y\in\mathbb{R}^2 \;\middle|\; y_{M*}^-\in(-c,c)^2,\ y_{M*}^+\in(-\infty,-c)^2\right\},$$
$$A_3 := \left\{y\in\mathbb{R}^2 \;\middle|\; y_{M*}^-\in(-c,c)^2,\ y_{M*}^+\in(c,\infty)\times(-\infty,-c)\right\},$$
$$A_4 := \left\{y\in\mathbb{R}^2 \;\middle|\; y_{M*}^-\in(-c,c)^2,\ y_{M*}^+\in(-\infty,-c)\times(c,\infty)\right\}.$$

For all $f\in L_2\left(\mathbb{T}^2\right)$ the norms $\mu_j(f)$ cannot be finite but they are so at least for some Fourier polynomials. The spaces H_j, $j\in\{1,2,3,4\}$, are the completions with respect to the norms μ_j above. E.g. using μ_1, it holds for all $f\in H_1$

$$\begin{aligned}\mu_1(f)^2 &= \sup_{y\in A_1}\left(\int_{\mathbb{T}^2}|f(x+\mathrm{i}\,y)|^2\,\mathrm{d}x\right) = \sup_{y\in A_1}\sum_{n\in\mathbb{Z}^2}\exp\left(-4\pi n^* y\right)|\varphi_n^* f|^2\\ &= \sup_{y\in A_1}\sum_{n\in\mathbb{Z}^2}\exp\left(-4\pi\left(n_M^-\right)^* y_{M*}^- - 4\pi\left(n_M^+\right)^* y_{M*}^+\right)|\varphi_n^* f|^2\\ &= \sup_{y_{M*}^+\in(c,\infty)^2}\sum_{n\in\mathbb{Z}^2}\exp\left(4\pi c\left\|n_M^-\right\| - 4\pi\left(n_M^+\right)^* y_{M*}^+\right)|\varphi_n^* f|^2\\ &= \sum_{\substack{n\in\mathbb{Z}^2\\ n_M^+\in[0,\infty)^2}}\exp\left(4\pi c\left\|n_M^-\right\| - 4\pi c\left\|n_M^+\right\|\right)|\varphi_n^* f|^2 = \sum_{\substack{n\in\mathbb{Z}^2\\ n_M^+\in[0,\infty)^2}}|\varphi_n^* A_{M,c} f|^2.\end{aligned}$$

Similar calculations for the other three norms show then that the spaces H_j, $j\in\{1,2,3,4\}$ disjointly partition the space $\mathcal{H}_{A_{M,c}}$ with respect to the dual coordinate up to $n=0$. Since E_M^+ is a one dimensional subspace of $\mathbb{R}^2$, always two of the spaces contain only the constant functions (note that $n_M^+=0$ implies $n=0$), say, H_3 and H_4. Then all vectors in the spaces H_1 and H_2 are holomorphic functions on $\mathbb{T}^2+\mathrm{i}\,A_1$ and on $\mathbb{T}^2+\mathrm{i}\,A_2$, respectively.

1.3 The Koopman operator is nuclear

We set for each $r>0$

$$\mathcal{T}_r := \left\{T\colon\mathbb{T}^2\to\mathbb{T}^2 \;\middle|\; T \text{ extends holomorphically and boundedly on } \mathbb{A}_r\right\}.$$

For every $T \in \mathcal{T}_r$ the Koopman operator

$$\mathcal{K}_T \colon L_2\left(\mathbb{T}^2\right) \to L_2\left(\mathbb{T}^2\right) : f \mapsto f \circ T$$

is well-defined by differentiability of T. It is well-known that the operator $\mathcal{K}_T$ acting on $L_2\left(\mathbb{T}^2\right)$ is not compact. We say that two maps $f, g \in \mathcal{T}_r$ are C^1-close if the distance

$$d(f,g) := \sup_{z \in \mathbb{A}_r} \|f(z) - g(z)\| + \sup_{z \in \mathbb{A}_r} \|\mathrm{D}_z f - \mathrm{D}_z g\|$$

is small. In this section we revisit the proof of Faure and Roy [25]. They showed that $\mathcal{K}_T$, acting on the Hilbert space $\mathcal{H}_{A_{M,c}}$, (see Definition 1.2.4), is nuclear of order 0 if T is sufficiently C^1-close to a hyperbolic matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ for some $c > 0$.

We recall that a linear operator $\mathcal{L} \colon \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ with norm $\|\cdot\|_{\mathcal{H}}$ is called nuclear of order 0 if it can be written as a sum $\mathcal{L} = \sum_{n \in \mathbb{N}} d_n \psi_{1,n} \psi_{2,n}^*$ with $\inf\{p > 0 \mid \sum_{n \in \mathbb{N}} |d_n|^p < \infty\} = 0$ and $\psi_{1,n}, \psi_{2,n} \in \mathcal{H}$, $\|\psi_{1,n}\|_{\mathcal{H}}, \|\psi_{2,n}\|_{\mathcal{H}} \leqslant 1$, $d_n \in \mathbb{C}$, $n \in \mathbb{N}$ [34, II, §1, n°1, p.4]. In particular, such an operator is trace class, hence bounded and admits a trace $\mathrm{tr}\mathcal{L} := \sum_{n \in \mathbb{N}} e_n^* \mathcal{L} e_n$, invariant for any choice of orthonormal basis e_n, $n \in \mathbb{N}$ of $\mathcal{H}$. Moreover, one can show that $\mathrm{tr}\mathcal{L}$ equals the sum, including multiplicity (dimension of corresponding generalized eigenspace), over the spectrum $\mathrm{sp}(\mathcal{L})$ of $\mathcal{L}$. The Fredholm determinant, defined for small enough $z \in \mathbb{C}$ by

$$\det(1 - z\mathcal{L}) := \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{tr}\mathcal{L}^n\right), \tag{1.9}$$

extends to an entire function in z, having zeroes at $z = \lambda^{-1}$, $\lambda \in \mathrm{sp}(\mathcal{L}) \setminus \{0\}$ of same order as the multiplicity of λ.

Theorem 1.3.1 (Nuclearity of $\mathcal{K}_T$)**.** *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and let $r > 0$. Then there exist constants $\delta_M > 0$ and $0 < c_1 < r$ such that for each $T \in \mathcal{T}_r$ with $d(T, M) \leqslant \delta_M$ the map*

$$\mathcal{K}_T \colon \mathcal{H}_{A_{M,c_1}} \to \mathcal{H}_{A_{M,c_1}} : f \mapsto f \circ T$$

defines a nuclear operator of order 0. In particular, there exists $c_2 > 0$ depending

only on c_1, M, *and* $\|\cdot\|$ *so that for each* n_1, $n_2 \in \mathbb{Z}^2$

$$\left|\left\langle \varrho_{n_1}, \mathcal{K}_T \varrho_{n_2} \right\rangle_{\mathcal{H}_{A_{M,c_1}}}\right| \leqslant \exp\left(-2\pi c_2 \left(\|n_1\| + \|n_2\|\right)\right).$$

For every n_1, $n_2 \in \mathbb{Z}^2$, we set

$$I_{n_1,n_2}(T) := \left\langle \varphi_{n_1}, \mathcal{K}_T \varphi_{n_2} \right\rangle_{L_2(\mathbb{T}^2)}. \tag{1.10}$$

Estimating this "oscillatory integral" is central for Theorem 1.3.1. In the case $T = M$, we have simply

$$I_{n_1,n_2}(M) = \begin{cases} 1 & \text{if } M^* n_2 = n_1 \\ 0 & \text{if } M^* n_2 \neq n_1 \end{cases}. \tag{1.11}$$

The strategy of the proof is as follows. We get an upper bound for $|I_{n_1,n_2}(T)|$ in Lemma 1.3.2, taking advantage of the holomorphicity of T. In Lemma 1.3.3, we compare the contribution of n_1 and n_2 in the expanding and contracting directions, using here essentially the hyperbolicity of M. Combining both results, we obtain a weaker bound on $|I_{n_1,n_2}(T)|$ in Proposition 1.3.4, which finally allows for the proof of Theorem 1.3.1.
For every $n \in \mathbb{Z}^2$ and $y \in \mathbb{R}^2$ any solution $x \in \mathbb{T}^2$ so that

$$\exp\left(-2\pi \left(n^* \, \mathrm{D}_x \, Ty\right)\right) = \int_{\mathbb{T}^2} \exp\left(-2\pi \left(n^* \, \mathrm{D}_z \, Ty\right)\right) \mathrm{d}z \tag{1.12}$$

is denoted by $x_n(y)$. Since the integrand is continuous in y such a solution exists by the Mean Value Theorem.

Lemma 1.3.2 (Upper bound on $|I_{n_1,n_2}(T)|$ (I))**.** *Let* $r > 0$. *Then, there exists* $C \geqslant 0$ *so that for each* $T \in \mathcal{T}_r$ *and* n_1, $n_2 \in \mathbb{Z}^2$ *and* $y \in (-r, r)^2$, *recalling (1.10), we have*

$$|I_{n_1,n_2}(T)| \leqslant \exp\left(2\pi \left(-n_2^* \, \mathrm{D}_{x_{n_2}(y)} \, Ty + n_1^* y + C d(T,0) \, \|y\|^3 \, \|n_2\|\right)\right).$$

Proof. By definition

$$I_{n_1,n_2}(T) = \left\langle \varphi_{n_1}, \mathcal{K}_T \varphi_{n_2} \right\rangle_{L_2(\mathbb{T}^2)} = \int_{\mathbb{T}^2} \exp\left(\mathrm{i}2\pi \left(n_2^* T(x) - n_1^* x\right)\right) \mathrm{d}x.$$

Since $T \in \mathcal{T}_r$, the $\mathbb{Z}^2$-invariance of the integrand follows. By holomorphicity of T on $\mathbb{A}_r$, we can change the path of integration to $x \mapsto x + \mathrm{i}y$ for every $y \in (-r, r)^2$.

Therefore for any $y \in (-r, r)^2$

$$|I_{n_1,n_2}(T)| \leqslant \int_{\mathbb{T}^2} \exp\left(2\pi \left(n_1^* y - \Im\left(n_2^* T(x + \mathrm{i}\, y)\right)\right)\right) \mathrm{d}x,$$

where $\Im$ is the imaginary part. We expand T (or rather its lift to $\mathbb{R}^2$) at $x \in \mathbb{T}^2$ in a Taylor series to the second order. This yields

$$T(x + \mathrm{i}\, y) = T(x) + \mathrm{i}\, \mathrm{D}_x T y + P(x + \mathrm{i}\, y) + R_2(x + \mathrm{i}\, y).$$

Here, $P(x + \mathrm{i}\, y)$ is the second order term of the expansion which is $\mathbb{R}^2$-valued, and R_2 is the remainder of the series expansion. We find therefore

$$\Im T(x + \mathrm{i}y) = \mathrm{D}_x T y + \Im R_2(x + \mathrm{i}\, y).$$

Since T is holomorphic we find a constant $C > 0$ independent of T such that

$$|n_2^* R_2(x + \mathrm{i}\, y)| \leqslant C d(T, 0) \|n_2\| \|y\|^3.$$

We are left with the evaluation of

$$\int_{\mathbb{T}^2} \exp\left(-2\pi \left(n^* \mathrm{D}_z T y\right)\right) \mathrm{d}z.$$

Using (1.12) yields the result. □

The following abbreviation is used in the remaining section. We set for each $y \in \mathbb{R}^2$

$$|y|_M := \left\|y_M^+\right\| - \left\|y_M^-\right\|. \tag{1.13}$$

Lemma 1.3.3 (Directional inequality)**.** *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. Let $\epsilon > 0$ and $\kappa \geqslant 0$ and let $R\colon \mathbb{R}^2 \to \mathbb{R}_{\geqslant 0}$ be a map such that for all $z \in \mathbb{R}^2$ with $\|z\| < \epsilon$ it holds*

$$R(z) \leqslant \kappa \|z\|.$$

Then there exists $c_M > 0$ such that if $\kappa < c_M$ there exist $0 < c_2 < c_1 < \epsilon$ such that for all n_1, $n_2 \in \mathbb{Z}^2$ there exists $y_{n_1,n_2} \in \mathbb{R}^2$ independent of R with $\|y_{n_1,n_2}\| < \epsilon$ such that it holds

$$-c_1\left(|n_1|_M - |n_2|_M\right) - \left(n_2^* M - n_1^*\right) y_{n_1,n_2} + \|n_2\| R\left(y_{n_1,n_2}\right) \leqslant -c_2\left(\|n_1\| + \|n_2\|\right).$$

Proof. We assume $0 < c_2 \leqslant c_1$. For $n_1 = n_2 = 0$ there is nothing to prove. For every $(y_1, y_2) \in \mathbb{R}^2$ we set $|(y_1, y_2)| := \sqrt{y_1^2 + y_2^2}$. We let $0 < \tilde{c}_1 \leqslant 1 \leqslant \tilde{c}_2$ such

that

$$\tilde{c}_2^{-1}\left(\left\|y_M^+\right\| + \left\|y_M^-\right\|\right) \leqslant \|y\| \leqslant \tilde{c}_1^{-1}\,|y|\,, \quad \text{for all } y \in \mathbb{R}^2. \tag{1.14}$$

Whenever $n_2 \neq 0$ we find a linear map M_a such that $M_a n_2 = M^* n_2 - n_1$ and whenever $n_1 \neq 0$ we find a linear map M_b such that $M_b n_1 = M^* n_2 - n_1$. For now we let $\tilde{\kappa} > 0$ be a variable which will be fixed later on, independently of n_1 and n_2. We consider the following four cases

(a) $\|n_2\| > 0$ and $\|n_2\| \geqslant \|n_1\|$
  (i) $\|M_a n_2\| \geqslant \tilde{\kappa}\,\|n_2\|$,
  (ii) $\|M_a n_2\| < \tilde{\kappa}\,\|n_2\|$,

(b) $\|n_1\| > 0$ and $\|n_1\| \geqslant \|n_2\|$
  (i) $\|M_b n_1\| \geqslant \tilde{\kappa}\,\|n_1\|\,,$
  (ii) $\|M_b n_1\| < \tilde{\kappa}\,\|n_1\|$.

We assume Case (a)(i). For every $\delta > 0$ we let

$$y = \delta M_a \frac{n_2}{\|n_2\|}.$$

It follows, using (1.14), that

$$-\left(n_2^* M - n_1^*\right) y = -n_2^* M_a^* y \leqslant -\tilde{c}_1^2\,\|M_a n_2\|\,\|y\|\,. \tag{1.15}$$

We recall $|\cdot|_M$ from (1.13). Using that $c_1 + c_2 > 0$ and that (a) holds, we estimate

$$\begin{aligned} -c_1\left(|n_1|_M - |n_2|_M\right) &\leqslant c_1\left(\|n_1\| + \|n_2\|\right) \\ &= -c_2\left(\|n_1\| + \|n_2\|\right) + \left(c_1 + c_2\right)\left(\|n_1\| + \|n_2\|\right) \\ &\leqslant -c_2\left(\|n_1\| + \|n_2\|\right) + 2\left(c_1 + c_2\right)\|n_2\|\,. \end{aligned}$$

Using (a)(i) and the assumed bound on R for $\|y\| < \epsilon$, we have

$$\begin{aligned} -c_1\left(|n_1|_M - |n_2|_M\right) - \|n_2\|\left(\tilde{c}_1^2\left\|M_a \frac{n_2}{\|n_2\|}\right\|\|y\| - R\left(y\right)\right) \leqslant \\ - c_2\left(\|n_1\| + \|n_2\|\right) + \left(2(c_1 + c_2) - \left(\tilde{c}_1^2\tilde{\kappa} - \kappa\right)\|y\|\right)\|n_2\|\,. \end{aligned} \tag{1.16}$$

We put $c_M := \tilde{c}_1^2\tilde{\kappa}$. Any value $\|y\| \in (0, \epsilon)$ can be attained by controlling δ. Assuming that $c_M > \kappa$, it follows from (1.15) and (1.16) that

$$0 < c_1 + c_2 < \frac{c_M - \kappa}{2}\epsilon. \tag{1.17}$$

The reasoning in Case (b)(i) is completely analogous and yields the same bounds

on $c_1 + c_2$.
In Case (a)(ii) and (b)(ii), we take $y = 0$, where $R(0) = 0$ by assumption on R. We assume now Case (a)(ii). We find, using (1.14),

$$\begin{aligned}\left\|(M_a n_2)^+_M\right\| + \left\|(M_a n_2)^-_M\right\| &\leqslant \tilde{c}_2 \left\|M_a n_2\right\| \\ &< \tilde{c}_2\tilde{\kappa} \left\|n_2\right\| \leqslant \tilde{c}_2\tilde{\kappa} \left(\left\|n^+_{2,M}\right\| + \left\|n^-_{2,M}\right\|\right). \end{aligned} \tag{1.18}$$

We have

$$\left\|(M_a n_2)^+_M\right\| = \left\|M^* n^+_{2,M} - n^+_{1,M}\right\| \quad \text{and} \quad \left\|(M_a n_2)^-_M\right\| = \left\|M^* n^-_{2,M} - n^-_{1,M}\right\|.$$

Recalling (1.5), this allows the estimate

$$\begin{aligned}\left\|(M_a n_2)^+_M\right\| + \left\|(M_a n_2)^-_M\right\| &\geqslant \left\|M^* n^+_{2,M}\right\| - \left\|n^+_{1,M}\right\| - \left\|M^* n^-_{2,M}\right\| + \left\|n^-_{1,M}\right\| \\ &\geqslant \lambda_M \left\|n^+_{2,M}\right\| - \lambda_M^{-1}\left\|n^-_{2,M}\right\| - \left\|n^+_{1,M}\right\| + \left\|n^-_{1,M}\right\|.\end{aligned}$$

Together with (1.18) we find therefore

$$-\left|n_1\right|_M = -\left\|n^+_{1,M}\right\| + \left\|n^-_{1,M}\right\| < -\left(\lambda_M - \tilde{\kappa}\tilde{c}_2\right)\left\|n^+_{2,M}\right\| + \left(\lambda_M^{-1} + \tilde{\kappa}\tilde{c}_2\right)\left\|n^-_{2,M}\right\|.$$

We set

$$\kappa_+ := \lambda_M - \tilde{\kappa}\tilde{c}_2 - 1 \quad \text{and} \quad \kappa_- := 1 - \lambda_M^{-1} - \tilde{\kappa}\tilde{c}_2.$$

We finally estimate

$$-c_1\left(\left|n_1\right|_M - \left|n_2\right|_M\right) < -c_1\kappa_+\left\|n^+_{2,M}\right\| - c_1\kappa_-\left\|n^-_{2,M}\right\|.$$

Note that we have $\kappa_+ > \kappa_-$ because $\lambda_M > 1$. Assuming that $c_1\kappa_- \geqslant 2c_2$, we find

$$-c_1\kappa_-\left\|n^+_{2,M}\right\| - c_1\kappa_-\left\|n^-_{2,M}\right\| < -c_1\kappa_-\left\|n_2\right\| \leqslant -2c_2\left\|n_2\right\| \leqslant -c_2\left(\left\|n_1\right\| + \left\|n_2\right\|\right).$$

In Case (b)(ii) we consider the bounds

$$\begin{aligned}\left|n_2\right|_M + \lambda_M\left\|n^-_{1,M}\right\| - \lambda_M^{-1}\left\|n^+_{1,M}\right\| &\leqslant \left\|\left((M^*)^{-1} M_b n_1\right)^+_M\right\| + \left\|\left((M^*)^{-1} M_b n_1\right)^-_M\right\| \\ &\leqslant \tilde{c}_2\left\|(M^*)^{-1} M_b n_1\right\| < \tilde{\kappa}\tilde{c}_2\left\|(M^*)^{-1}\right\|\left\|n_1\right\|.\end{aligned}$$

Therefore κ_- is replaced by $1 - \lambda_M^{-1} - \left\|(M^*)^{-1}\right\|\tilde{\kappa}\tilde{c}_2$ which we require to be

positive. Since $\left\|(M^*)^{-1}\right\| > 1$, this yields the stronger conditions

$$0 < \tilde{\kappa} < \frac{1-\lambda_M^{-1}}{\left\|(M^*)^{-1}\right\|\tilde{c}_2} \quad \text{and} \quad c_2 \leqslant \frac{1-\lambda_M^{-1}-\left\|(M^*)^{-1}\right\|\tilde{\kappa}\tilde{c}_2}{2}c_1. \tag{1.19}$$

Any such choice for $\tilde{\kappa}$ is independent of n_1 and n_2 and fixes c_M. Using (1.19) for an upper bound on c_2 and (1.17), we find the stronger condition

$$0 < c_1 < \frac{c_M - \kappa}{3-\lambda_M^{-1}}\epsilon.$$

Therefore the choices of c_1 and c_2 are valid if $\kappa < c_M$. They depend only on ϵ, M and $\|\cdot\|$ and not on n_1 or n_2. □

Proposition 1.3.4 (Upper bound on $|I_{n_1,n_2}(T)|$ (II))**.** *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and let $r > 0$. Then there exist constants $0 < \delta_M$ and $0 < c_2 < c_1 < r$ such that for each n_1, $n_2 \in \mathbb{Z}^2$ and each $T \in \mathcal{T}_r$ with $d(T, M) \leqslant \delta_M$ it holds that*

$$\exp\left(-2\pi c_1\left(|n_1|_M - |n_2|_M\right)\right)|I_{n_1,n_2}(T)| \leqslant \exp\left(-2\pi c_2\left(\|n_1\| + \|n_2\|\right)\right).$$

Proof. By Lemma 1.3.2 there is a constant $C > 0$ independent of T such that for each $y \in (-r, r)^2$ and n_1, $n_2 \in \mathbb{Z}^2$ it holds that

$$|I_{n_1,n_2}(T)| \leqslant \exp\left(2\pi\left(-n_2^*\,\mathrm{D}_{x_{n_2}(y)}\,Ty + n_1^* y + Cd(T,0)\,\|y\|^3\,\|n_2\|\right)\right). \tag{1.20}$$

We rewrite

$$n_2^*\,\mathrm{D}_{x_{n_2}(y)}\,Ty = n_2^* My + n_2^*\,\mathrm{D}_{x_{n_2}(y)}\,(T-M)\,y,$$

and set

$$R(y) := \begin{cases} \frac{n_2^*}{\|n_2\|}\,\mathrm{D}_{x_{n_2}(y)}\,(M-T)\,y + Cd(T,0)\,\|y\|^3 & \text{if } n_2 \neq 0 \\ 0 & \text{if } n_2 = 0 \end{cases}.$$

Let $\delta_M > 0$ and assume that $d(T, M) \leqslant \delta_M$. We choose $0 < \epsilon \leqslant r$ sufficiently small such that for all $y \in \mathbb{R}^2$ with $\|y\| < \epsilon$ there is $\kappa > 0$ such that

$$|R(y)| \leqslant \kappa\delta_M.$$

Since $d(T, 0) \leqslant d(T, M) + d(M, 0) \leqslant \delta_M + d(M, 0)$ this choice of ϵ is independent

of T. Lemma 1.3.3 applied to M and $|R|$ gives c_1, c_2 and $y_{n_1,n_2} \in \mathbb{R}^2$ for which the right-hand side of (1.20) fulfills the desired inequality. □

Proof of Theorem 1.3.1. Proposition 1.3.4 yields $0 < \delta_M$ and $0 < c_2 < c_1 < r$ such that if $d(T, M) \leqslant \delta_M$ it holds

$$C_{n_1} C_{n_2}^{-1} \left| I_{n_1,n_2}(T) \right| \leqslant \exp\left(-2\pi c_2 \left(\|n_1\| + \|n_2\|\right)\right), \tag{1.21}$$

where

$$C_n := \exp\left(-2\pi c_1 \left(\left\|n_M^+\right\| - \left\|n_M^-\right\|\right)\right), \quad n \in \mathbb{Z}^2.$$

We put $c := c_1$ and M in Definitions 1.2.2 and 1.2.4, giving a linear map A_{M,c_1} and a Hilbert space $\mathcal{H}_{A_{M,c_1}}$. Recalling (1.6), and assuming that $\mathcal{K}_T \colon \mathcal{H}_{A_{M,c_1}} \to \mathcal{H}_{A_{M,c_1}}$ is well-defined, we have

$$\begin{aligned} \left| \left\langle \varrho_{n_1}, \mathcal{K}_T \varrho_{n_2} \right\rangle_{\mathcal{H}_{A_{M,c_1}}} \right| &= \left| \left\langle \varphi_{n_1}, A_{M,c_1} \mathcal{K}_T A_{M,c_1}^{-1} \varphi_{n_2} \right\rangle_{L_2(\mathbb{T}^2)} \right| \\ &= C_{n_1} C_{n_2}^{-1} \left| I_{n_1,n_2}(T) \right|. \end{aligned} \tag{1.22}$$

Using (1.21) to estimate the right-hand side, the bound in Theorem 1.3.1 follows. We next obtain well-definedness and nuclearity of order 0 of $\mathcal{K}_T$. Let $f \in \mathcal{H}_{A_{M,c_1}}$ and put $g := A_{M,c_1} f$. We have then

$$\begin{aligned} \mathcal{K}_T f \in \mathcal{H}_{A_{M,c_1}} &\Leftrightarrow A_{M,c_1} \mathcal{K}_T f \in L_2\left(\mathbb{T}^2\right) \Leftrightarrow \sum_{n \in \mathbb{Z}^2} \left| \varphi_n^* A_{M,c_1} \mathcal{K}_T f \right|^2 < \infty \\ &\Leftrightarrow \sum_{n_1 \in \mathbb{Z}^2} \left| \sum_{n_2 \in \mathbb{Z}^2} \varphi_{n_1}^* A_{M,c_1} \mathcal{K}_T A_{M,c_1}^{-1} \varphi_{n_2} \varphi_{n_2}^* g \right|^2 < \infty \\ &\Leftrightarrow \sum_{n_1 \in \mathbb{Z}^2} \left| \sum_{n_2 \in \mathbb{Z}^2} C_{n_1} C_{n_2}^{-1} I_{n_1,n_2}(T) \, \varphi_{n_2}^* g \right|^2 < \infty. \end{aligned}$$

Using (1.21) and the Cauchy-Schwartz inequality, it follows that

$$\sum_{n_1 \in \mathbb{Z}^2} \left| \sum_{n_2 \in \mathbb{Z}^2} C_{n_1} C_{n_2}^{-1} I_{n_1,n_2}(T) \, \varphi_{n_2}^* g \right|^2 \leqslant \left(\sum_{n \in \mathbb{Z}^2} e^{-4\pi c_2 \|n\|} \right)^2 \|g\|_{L_2(\mathbb{T}^2)}^2 < \infty.$$

This gives the well-definedness of $\mathcal{K}_T$. Now, using the Cauchy-Schwartz inequa-

lity, we have

$$\left|\langle \varrho_n, \mathcal{K}_T f\rangle_{\mathcal{H}_{A_{M,c_1}}}\right|^2 \leqslant \sum_{m\in\mathbb{Z}^2} \left|\langle \varrho_n, \mathcal{K}_T \varrho_m\rangle_{\mathcal{H}_{A_{M,c_1}}}\right|^2 \|f\|^2_{\mathcal{H}_{A_{M,c_1}}} .$$

Using (1.22) and (1.21) to bound $\left|\langle \varrho_n, \mathcal{K}_T \varrho_m\rangle_{\mathcal{H}_{A_{M,c_1}}}\right|$, we find a constant $C > 0$ such that

$$\left|\langle C\exp\left(2\pi c_2 \|n\|\right)\varrho_n, \mathcal{K}_T f\rangle_{\mathcal{H}_{A_{M,c_1}}}\right| \leqslant \|f\|_{\mathcal{H}_{A_{M,c_1}}} .$$

This allows the representation of $\mathcal{K}_T$ as

$$\mathcal{K}_T f = \sum_{n\in\mathbb{Z}^2} C^{-1}\exp\left(-2\pi c_2\|n\|\right)\langle C\exp\left(2\pi c_2\|n\|\right)\varrho_n, \mathcal{K}_T f\rangle_{\mathcal{H}_{A_{M,c_1}}}\varrho_n,$$

from which nuclearity of order 0 follows. Finally, a brief inspection of the proofs for Lemma 1.3.3 and Proposition 1.3.4 gives the statement about the constants. □

1.4 Non-trivial resonances for the Koopman operator

Given any hyperbolic matrix $M \in \mathrm{SL}_2(\mathbb{Z})$, we find by Theorem 1.3.1 constants $0 < \delta_M$ and $c > 0$ such that for each map $T \in \mathcal{T}_r$, satisfying $d(T, M) \leqslant \delta_M$, the operator $\mathcal{K}_T$ acting on the Hilbert space $\mathcal{H}_{A_{M,c}}$ is nuclear of order 0. Therefore it has a well-defined trace

$$\mathrm{tr}\mathcal{K}_T := \sum_{n\in\mathbb{Z}^2}\langle \varrho_n, \mathcal{K}_T\varrho_n\rangle_{\mathcal{H}_{A_{M,c}}} . \tag{1.23}$$

The map T is an Anosov diffeomorphism (for all small enough δ_M), by structural stability [35, Theorem 9.5.8]. Then the map T has the same number $N_M = |\det(\mathrm{id} - M)|$ of fixed points as the matrix M. We recall a well-known result [25, Proposition 9].

Lemma 1.4.1 (Trace formula for $\mathcal{K}_T$)**.** *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and let $r > 0$. Then there exist constants $\delta_M > 0$ and $c > 0$ such that for each $T \in \mathcal{T}_r$ with $d(T, M) \leqslant \delta_M$, letting $\mathcal{K}_T$ act on $\mathcal{H}_{A_{M,c}}$, it holds*

$$\mathrm{tr}\mathcal{K}_T = \sum_{T(x)=x} |\det(\mathrm{id} - \mathrm{D}_x T)|^{-1} .$$

For the convenience of the reader, we give a proof:

Proof. Using Theorem 1.3.1 gives constants $c > 0$ and $\delta_M > 0$ and well-definedness of $\mathcal{K}_T$. For small enough $\delta_M > 0$, by structural stability and Lemma A.1 (ii), the map $\mathrm{id} - T$ can be partitioned into N_M surjective submaps. In particular, there are diffeomorphisms $y_j \colon D_j \to \mathbb{T}^2$, $D_j \subseteq \mathbb{T}^2$, $1 \leqslant j \leqslant N_M$ such that $\mathrm{id} - T = \bigsqcup_{j=1}^{N_M} y_j$. Then, using (1.6), we have for each $n \in \mathbb{Z}^2$

$$\begin{aligned}
\langle \varrho_n, \mathcal{K}_T \varrho_n \rangle_{\mathcal{H}_{A_{M,c}}} &= \left\langle \varphi_n, A_{M,c} \mathcal{K}_T A_{M,c}^{-1} \varphi_n \right\rangle_{L_2(\mathbb{T}^2)} = \int_{\mathbb{T}^2} \exp\left(\mathrm{i}\, 2\pi n^* \left(T - \mathrm{id}\right)(x)\right) \mathrm{d}x \\
&= \sum_{j=1}^{N_M} \int_{y_j^{-1}(\mathbb{T}^2)} \exp\left(\mathrm{i}\, 2\pi n^* y_j(x)\right) \mathrm{d}x \\
&= \sum_{j=1}^{N_M} \int_{\mathbb{T}^2} \frac{\exp\left(\mathrm{i}\, 2\pi n^* z\right)}{\left|\det\left(\mathrm{id} - \mathrm{D}_{y_j^{-1}(z)} T\right)\right|} \mathrm{d}z.
\end{aligned}$$

For $N \in \mathbb{N}$ and $z \in \mathbb{T}^2$ the following sum

$$D_N(z) := \sum_{\substack{n \in \mathbb{Z}^2 \\ \|z\| \leqslant N}} \exp\left(\mathrm{i}\, 2\pi n^* z\right)$$

is the 2-dimensional analogue of the Dirichlet kernel [44, p.13]. Together with (1.23), this yields immediately

$$\operatorname{tr} \mathcal{K}_T = \lim_{N \to \infty} \sum_{\substack{n \in \mathbb{Z}^2 \\ \|n\| \leqslant N}} \langle \varrho_n, \mathcal{K}_T \varrho_n \rangle_{\mathcal{H}_{A_{M,c}}} = \sum_{T(x) = x} \left|\det\left(\mathrm{id} - \mathrm{D}_x T\right)\right|^{-1}.$$

□

Using Lemma 1.4.1, and the definitions (1.1) and (1.9) for the dynamical determinant and Fredholm determinant, respectively, we see directly that

$$\det\left(1 - z\mathcal{K}_T\right) = d_T(z). \tag{1.24}$$

The Ruelle resonances correspond to the zeroes of the Fredholm determinant, hence to the inverses of the non-zero eigenvalues of $\mathcal{K}_T$.

Remark 1.4.2. *In view of Equation 1.24 and the relation of the Ruelle resonances of T to the eigenvalues of $\mathcal{K}_T$, one may ask how the spectrum of $\mathcal{K}_T$ would be affected if we let $\mathcal{K}_T$ act on a different Banach space. The following relates a part of the eigenvalues of two linear operators sharing a common dense subspace and is due to a result of Baladi and Tsujii [11, Appendix A]. Consider two separable Banach spaces $(\mathcal{B}_1, \|\cdot\|_1)$ and $(\mathcal{B}_2, |\cdot|)$. This induces two other*

Banach spaces

$$\left(\mathcal{B}_1 + \mathcal{B}_2, \|\cdot\|_+\right) \text{ and } \left(\mathcal{B}_1 \cap \mathcal{B}_2, \|\cdot\|_\cap\right), \text{ where}$$

$$\|f\|_+ := \inf\left\{\|f_1\|_1 + |f_2| \quad | \quad f_1 \in \mathcal{B}_1,\ f_2 \in \mathcal{B}_2,\ f = f_1 + f_2\right\} \text{ and}$$
$$\|f\|_\cap := \max\left\{\|f\|_1, |f|\right\}.$$

Suppose that $\mathcal{B}_\cap$ is dense in $\mathcal{B}_1$ and $\mathcal{B}_2$. Let $\mathcal{L}\colon \mathcal{B}_+ \to \mathcal{B}_+$ be a linear map which preserves the spaces $\mathcal{B}_\cap$, $\mathcal{B}_1$ and $\mathcal{B}_2$ and is a bounded linear operator on the restrictions $\mathcal{L}_{|\mathcal{B}_1}$ and $\mathcal{L}_{|\mathcal{B}_2}$. Then the part of the spectrum of $\mathcal{L}_{|\mathcal{B}_1}$ and of $\mathcal{L}_{|\mathcal{B}_2}$ which lies outside the closed disc with radius larger to both essential spectral radii of $\mathcal{L}_{|\mathcal{B}_1}$ and $\mathcal{L}_{|\mathcal{B}_2}$ coincide. Moreover, the corresponding generalized eigenspaces of $\mathcal{L}_{|\mathcal{B}_1}$ and $\mathcal{L}_{|\mathcal{B}_2}$ coincide and are contained in $\mathcal{B}_\cap$.
For the applications that we have in mind, the map $\mathcal{L}$ is just the Koopman or transfer operator, defined on $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, extended to the space $\mathcal{B}_+$.

The spectrum $\mathrm{sp}\left(\mathcal{K}_T\right)$ of $\mathcal{K}_T$ on $\mathcal{H}_{A_{M,c}}$ is invariant under complex conjugation since T is real. The constant functions on $\mathbb{T}^2$ are all fixed by $\mathcal{K}_T$. Therefore we have $1 \in \mathrm{sp}\left(\mathcal{K}_T\right)$. If we take $T = M^k$, $k \in \mathbb{N}$ in Lemma 1.4.1, it follows that $\mathrm{tr}\mathcal{K}_T = 1$. Hence, the dynamical determinant is just $d_T(z) = 1 - z$, also noted in [55, p.3]. We find immediately that 1 is the only Ruelle resonance. We show now that this finding is non-generic in the following sense. The rest of this section is devoted to an idea of Naud [51]. We put for every $r > 0$

$$\mathcal{B}_r := \left\{T \in \mathcal{T}_r \quad | \quad \text{The lift of } T \text{ to } \mathbb{R}^2 \text{ is } \mathbb{Z}^2\text{-periodic}\right\}. \tag{1.25}$$

Endowed with the uniform norm this is a Banach space.

Theorem 1.4.3 (Non-trivial Ruelle resonances (I))**.** *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. For each $r > 0$ there exists an open and dense set $\mathcal{G} \subset \mathcal{B}_r$ such that the linear functional*

$$B_M\colon \mathcal{B}_r \to \mathbb{R}\colon \psi \mapsto {N_M}^{-1} \sum_{Mx=x} \mathrm{tr}\left((\mathrm{id} - M)^{-1} \mathrm{D}_x \psi\right)$$

never vanishes on $\mathcal{G}$. For all $\psi \in \mathcal{G}$ there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$

$$\mathrm{tr}\mathcal{K}_{M+\epsilon\psi} = 1 + \epsilon B_M(\psi) + O\left(\epsilon^2\right).$$

In particular, for all sufficiently small $\epsilon > 0$ it holds

$$\operatorname{sp}\left(\mathcal{K}_{M+\epsilon\psi}\right) \setminus \{0,1\} \neq \varnothing.$$

Lemma 1.4.4 (Real analyticity of fixed points)**.** *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and $r > 0$. Then for all $\psi \in \mathcal{B}_r$ the fixed points of the map*

$$M + \delta\psi$$

are real analytic functions of δ where δ lies in a real neighborhood of 0.

Proof. We set for $\delta \in \mathbb{R}$

$$F(\delta, x) := Mx + \delta\psi(x) - x.$$

We fix a point $y_j := (0, x_j)$ where x_j, $1 \leqslant j \leqslant N_M$, is a fixed point of M. By construction, the map F has a holomorphic extension to $\mathbb{C} \times \mathbb{A}_r$. Since M is hyperbolic, we have $\det \mathrm{D}_{x_j}\left(F(0,\cdot)\right) \neq 0$. We apply the Holomorphic Implicit Function Theorem [47, Theorem 1.4.11] on F with $F(y_j) = 0$. This yields a holomorphic function $x_j(\delta)$ such that $x_j(0) = x_j$ and which is obviously real analytic for $\delta \in \mathbb{R}$ in a neighborhood of 0. □

Proof of Theorem 1.4.3. Let $\delta \in \mathbb{R}$ and $\psi \in \mathcal{B}_r$ and set $\tilde{M} := M + \delta\psi$. We choose δ small in Lemma 1.4.4 which gives for each fixed point x of M a real analytic function $\tilde{x}$ with $\tilde{x}(0) = x$. Using a Taylor expansion on $\tilde{x}$ at 0, we have

$$\tilde{x}(\delta) = x + O(\delta).$$

Using real analyticity of the derivative $\mathrm{D}_x\psi$, we have

$$\mathrm{D}_x\psi - \mathrm{D}_{\tilde{x}(\delta)}\psi = O(\delta).$$

We write now for each fixed point x of M

$$\begin{aligned}
\left|\det\left(\mathrm{id} - \mathrm{D}_{\tilde{x}(\delta)}\tilde{M}\right)\right| &= \left|\det\left(\mathrm{id} - M - \delta\,\mathrm{D}_x\psi + \delta\left(\mathrm{D}_x\psi - \mathrm{D}_{\tilde{x}(\delta)}\psi\right)\right)\right| \\
&= N_M\left|\det\left(\mathrm{id} - (\mathrm{id} - M)^{-1}\left(\delta\,\mathrm{D}_x\psi + \left(\delta\,\mathrm{D}_x\psi - \delta\,\mathrm{D}_{\tilde{x}(\delta)}\psi\right)\right)\right)\right| \\
&= N_M\left|\det\left(\mathrm{id} - \delta\,(\mathrm{id} - M)^{-1}\,\mathrm{D}_x\psi + O\left(\delta^2\right)\right)\right| \\
&= N_M\left(1 - \delta\mathrm{tr}\left((\mathrm{id} - M)^{-1}\,\mathrm{D}_x\psi\right) + O\left(\delta^2\right)\right).
\end{aligned}$$

We have by Lemma 1.4.1 for δ small enough

$$\mathrm{tr}\mathcal{K}_{\tilde{M}} = 1 + \frac{\delta}{N_M} \sum_{Mx=x} \mathrm{tr}\left((\mathrm{id} - M)^{-1} \mathrm{D}_x \psi \right) + O\left(\delta^2\right).$$

Now we set

$$B_M \colon \mathcal{B}_r \to \mathbb{R} \colon \psi \mapsto {N_M}^{-1} \sum_{Mx=x} \mathrm{tr}\left((\mathrm{id} - M)^{-1} \mathrm{D}_x \psi \right).$$

We next check that this is a non-trivial linear functional. Note that formally $B_M(\mathrm{id} - M) = 2$. However, no non-zero linear map is in the space of additive perturbations $\mathcal{B}_r$. We denote by v_j, $j \in \{1, 2\}$ the j-th column of the matrix $\left((\mathrm{id} - M)^*\right)^{-1}$ and we fix now j. Let $\psi_0 \colon \mathbb{T} + \mathrm{i}(-r, r) \to \mathbb{C}$ be holomorphic and bounded. For every $(x_1, x_2) =: x \in \mathbb{T}^2$ we put

$$\psi(x) := \psi_0(x_j) v_j.$$

By construction, we have $\psi \in \mathcal{B}_r$ and we evaluate

$$B_M(\psi) = \frac{v_j^* v_j}{N_M} \sum_{Mx=x} \psi_0^{(1)}(x_j).$$

The right-hand side is a finite sum and by taking for ψ_0 a suitable Fourier polynomial (e.g. a shifted sine with sufficiently high frequency), we can establish $B_M(\psi) \neq 0$. We set $\mathcal{G} := B_M^{-1}(\mathbb{R} \setminus \{0\})$. By continuity of B_M, the set $\mathcal{G}$ is open and dense in $\mathcal{B}_r$. □

1.5 Non-trivial resonances for the transfer operator

As before, we consider maps $T \in \mathcal{T}_r$, $r > 0$ which are sufficiently C^1-close to a hyperbolic linear map $M \in \mathrm{SL}_2(\mathbb{R})$. We turn to the adjoint of $\mathcal{K}_T$, acting on the dual space $\mathcal{H}^*_{A_{M,c}}$, which we denote by $\mathcal{L}_T$.

Lemma 1.5.1 (Transfer operator)**.** *Let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic and let $r > 0$. Then there exist constants $0 < \delta_M$ and $c > 0$ such that for each $T \in \mathcal{T}_r$ with $d(T, M) \leqslant \delta_M$ the map*

$$\mathcal{L}_T \colon \mathcal{H}^*_{A_{M,c}} \to \mathcal{H}^*_{A_{M,c}} \colon f \mapsto \frac{f}{|\det \mathrm{D}\, T|} \circ T^{-1}$$

defines a nuclear operator of order 0*, conjugate to* $\mathcal{K}_T$*. In particular,*

$$\operatorname{sp}(\mathcal{L}_T) = \operatorname{sp}(\mathcal{K}_T).$$

Proof. By Theorem 1.3.1 there is $0 < \delta_M$, $c > 0$ and $\mathcal{H}_{A_{M,c}}$ such that $\mathcal{K}_T$ acting on $\mathcal{H}_{A_{M,c}}$ is nuclear of order 0 if $d(T, M) \leqslant \delta_M$. The same can be said about its adjoint, acting on $\mathcal{H}^*_{A_{M,c}}$ (e.g. see [57, p. 77]). The trace of $\mathcal{K}_T$ and $\mathcal{L}_T$ coincide, so does their Fredholm determinant, and hence their resonances. By definition of the adjoint, $\forall f^* \in \mathcal{H}^*_{A_{M,c}}$, $\forall g \in \mathcal{H}_{A_{M,c}}$: $(\mathcal{L}_T f)^*(g) = f^*(\mathcal{K}_T g)$. Using Lemma 1.2.5, it holds

$$\begin{aligned} f^*(\mathcal{K}_T g) &= \left\langle A_{M,c}^{-2} f, \mathcal{K}_T g \right\rangle_{\mathcal{H}_{A_{M,c}}} = \int_{\mathbb{T}^2} \left(A_{M,c}^{-1} \bar{f}\right)(x) \left(A_{M,c} \mathcal{K}_T g\right)(x) \, \mathrm{d}x \\ &= \int_{\mathbb{T}^2} \bar{f}(x) \left(\mathcal{K}_T g\right)(x) \, \mathrm{d}x = \int_{\mathbb{T}^2} \left(\frac{\bar{f}}{|\det \mathrm{D}\, T|} \circ T^{-1}\right)(x) \, g(x) \, \mathrm{d}x \\ &= \left\langle A_{M,c}^{-2} \left(\frac{f}{|\det \mathrm{D}\, T|} \circ T^{-1}\right), g \right\rangle_{\mathcal{H}_{A_{M,c}}} = \left(\frac{f}{|\det \mathrm{D}\, T|} \circ T^{-1}\right)^*(g). \end{aligned}$$

□

By Lemma 1.5.1, recalling (1.6), and Lemma 1.4.1 it holds

$$\operatorname{tr} \mathcal{L}_T = \sum_{n \in \mathbb{Z}^2} \mathcal{L}_T \varrho_n^*(\varrho_n) = \sum_{T(x) = x} |\det(\mathrm{id} - \mathrm{D}\, T)|^{-1}.$$

We have the equality

$$d_T(z) = \det(1 - z\mathcal{K}_T) = \det(1 - z\mathcal{L}_T).$$

We give now analogously to Theorem 1.4.3 a spectral result for the transfer operator (recall $\mathcal{B}_r$ from (1.25)).

Lemma 1.5.2 (Non-trivial Ruelle resonances (II))**.** *Let* $M \in \mathrm{SL}_2(\mathbb{Z})$ *be hyperbolic. For each* $r > 0$ *there exists an open and dense set* $\mathcal{G} \subset \mathcal{B}_r$ *such that for all* $\psi \in \mathcal{G}$ *there exists* $\epsilon_0 > 0$ *such that for all* $0 < \epsilon \leqslant \epsilon_0$

$$\operatorname{sp}(\mathcal{L}_{M+\epsilon\psi}) \setminus \{0, 1\} \neq \varnothing.$$

Proof. By Theorem 1.4.3 we know that under every perturbation $\psi \in \mathcal{G}$ there is $\epsilon_0 > 0$ such that we find for all $0 < \epsilon \leqslant \epsilon_0$ non-trivial Ruelle resonances. Using Lemma 1.5.1 for well-definedness of $\mathcal{L}_{M+\epsilon\psi}$ and for the relation $\operatorname{sp}(\mathcal{L}_T) = \operatorname{sp}(\mathcal{K}_T)$, the result follows. □

Clearly, the Lebesgue measure (by Remark 1.2.6, the constant density 1) is fixed by $\mathcal{L}_M$. This does not persist under a generic perturbation of M. However, the spectral relation in Lemma 1.5.1 implies that $\mathcal{L}_T$ fixes some functionals in $\mathcal{H}^*_{A_{M,c}}$. In particular, using Remark 1.4.2, we can apply [15, Theorem 3] to our transfer operators $\mathcal{L}_M$ and $\mathcal{L}_T$. Hence, the eigenvalue 1 of $\mathcal{L}_T$ is simple and the projector Π_1^* onto the corresponding eigenspace of $\mathcal{L}_T$ gives us the SRB measure

$$\mu_{\mathrm{SRB}} := \Pi_1^* 1^*,$$

in the usual sense. (It is absolutely continuous with respect to Lebesgue measure in the unstable direction.)
We finish this section by showing the existence of non-zero perturbations $\psi \in \mathcal{B}_r$ which allow the determinant $\det(M + \epsilon \mathrm{D}_x \psi)$ to remain constant or to vary for $x \in \mathbb{T}^2$.

Lemma 1.5.3 (Volume under perturbations)**.** *Let $r > 0$ and let $M \in \mathrm{SL}_2(\mathbb{Z})$ be hyperbolic. Then there exist non-zero maps $\psi \in \mathcal{B}_r$ in each of the following cases:*

(i) *For all $\epsilon > 0$ and all $x \in \mathbb{T}^2$ it holds $\det(M + \epsilon \mathrm{D}_x \psi) = 1$.*

(ii) *For all $\epsilon > 0$ and Lebesgue almost all $x \in \mathbb{T}^2$ it holds $|\det(M + \epsilon \mathrm{D}_x \psi)| \neq 1$.*

In particular, the map ψ can be chosen such that for all small $\epsilon > 0$ the corresponding transfer operator

$$\mathcal{L}_{M+\epsilon\psi}$$

admits non-trivial Ruelle resonances.

Proof. We prove first Claim (i), including the statement about the non-trivial Ruelle resonances. We will apply Lemma A.2 (i). We choose $j \in \{1, 2\}$, $r > 0$ and let $\phi \colon \mathbb{T} + \mathrm{i}(-r, r) \to \mathbb{C}$ be a holomorphic and bounded map. For $\alpha \in \mathbb{R}^2$ we set for every $(x_1, x_2) =: x \in \mathbb{T}^2$

$$\psi_{\phi,\alpha}(x) := (\alpha_1 \phi(x_j), \alpha_2 \phi(x_j)).$$

We put $d := 2$, j, $T := M$, ϕ and $T_\phi := \psi_{\phi,\alpha}$ (e.g. as lift to $\mathbb{R}^2$) in Lemma A.2. Since M is a constant matrix, say, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for suitable $a, b, c, d \in \mathbb{Z}$, we can write Condition A.2 (i) as

$$\alpha_1 d = \alpha_2 b \quad \text{if } j = 1 \quad \text{or} \quad \alpha_1 c = \alpha_2 a \quad \text{if } j = 2. \tag{1.26}$$

Hence, we have non-zero solutions in α independent of x. We choose such a solution α and take $\psi = \psi_{\phi,\alpha}$. Then $\psi \in \mathcal{B}_r$ which yields $\det(M + \epsilon \mathrm{D}_x \psi) = 1$ for every $\epsilon > 0$. We are free to choose any suitable ϕ. In particular, Theorem 1.4.3 yields a linear functional B_M and a dense subset $\mathcal{G} \subset \mathcal{B}_r$ on which B_M is non-zero. We have to make sure that $\psi \in \mathcal{G}$. Then for ϵ small $\mathcal{L}_{M+\epsilon\psi}$ admits non-trivial Ruelle resonances by Lemma 1.5.2. To this end, we evaluate B_M at ψ which yields

$$B_M(\psi) = B_M(\psi_{\phi,\alpha}) = {N_M}^{-1} \sum_{Mx=x} \operatorname{tr}\left((\mathrm{id} - M)^{-1} \mathrm{D}_x \psi_{\phi,\alpha}\right) = \frac{v_j^* \alpha}{N_M} \sum_{Mx=x} \phi^{(1)}(x_j),$$

where v_j^* is the j-th row of $(\mathrm{id} - M)^{-1}$. The sum over the fixed points of M can be made non-zero by a suitable Fourier polynomial. Now we have

$$v_1^* \alpha = \frac{(1-d)\alpha_1 + c\alpha_2}{\det(\mathrm{id} - M)} \quad \text{or} \quad v_2^* \alpha = \frac{b\alpha_1 + (1-a)\alpha_2}{\det(\mathrm{id} - M)}.$$

Using (1.26), we find

$$v_1^* \alpha = \frac{\left(c - b + \frac{b}{d}\right)\alpha_2}{\det(\mathrm{id} - M)} \quad \text{or} \quad v_2^* \alpha = \frac{\left(b - c + \frac{c}{a}\right)\alpha_1}{\det(\mathrm{id} - M)}.$$

Both equations can never be zero since M is not diagonal. We prove now Claim (ii) by modifying the map ψ. For $\delta \in \mathbb{R} \setminus \{0\}$ we set $\tilde{\alpha} := \alpha + \delta w_j$, where w_j is the j-th column of M and put $\widetilde{\psi} := \psi_{\phi,\tilde{\alpha}}$. We have

$$\det\left(M + \epsilon \mathrm{D}_x \widetilde{\psi}\right) = \det\left(M + \epsilon \mathrm{D}_x \psi + \epsilon \mathrm{D}_x\left(\widetilde{\psi} - \psi\right)\right) = 1 + \delta\epsilon\phi^{(1)}(x_j).$$

Since ϕ is not constant, the right-hand side differs from 1 (and -1) for Lebesgue almost all x. Since $v_j^* \tilde{\alpha} = v_j^* \alpha + \delta v_j^* w_j \neq 0$ for the right choice of the sign of δ, we have $B_M\left(\widetilde{\psi}\right) \neq 0$.

□

2 Horocycle averages on closed manifolds

2.1 Introduction

Let M be a closed (compact without boundary) orientable Riemannian manifold of arbitrary finite dimension $d \geqslant 3$. On such manifolds Anosov introduced C^2 flows

$$g_{\alpha_1} \circ g_{\alpha_2} = g_{\alpha_1+\alpha_2} = g_\alpha \colon M \to M, \quad \alpha, \alpha_1, \alpha_2 \in \mathbb{R},$$

to study the geodesic flow on the unit tangent bundle of closed Riemannian manifolds with variable negative sectional curvature [3], [4]. As pointed out by Anosov [3], the topological entropy h_{top} of the time-one map g_1 of an Anosov flow is positive.

A special class of such *Anosov* flows are those which preserve a *contact* structure. The geodesic flows are well-studied examples of contact Anosov flows. We give the precise definition of a (contact) Anosov flows in Section 2.2.

Every Anosov flow admits a contracting transversal foliation. The underlying vector bundle E_- is called the strong stable distribution. If the leaves of the contracting foliation are one-dimensional and orientable, one associates with g_α another flow, the *horocycle* flow $h_\rho \colon M \to M$, $\rho \in \mathbb{R}$. (The term *horocycle flow* was used originally only in the case of the geodesic flow, e.g. see [49, p.84] or [38].) For every $x \in M$ the flow trajectory $h_{\mathbb{R}}(x)$ is such a contracting leaf. Statistical properties of contact Anosov flows are nowadays fairly well understood (see [23], [33], [48]). Regarding the horocycle flow one knows by the work of Bowen and Marcus unique ergodicity of and minimality of the horocycle flow (e.g. see [18], [50]). The corresponding invariant probability measure μ will play an important role below. (It is related to but distinct from the measure of maximal entropy of the flow.)

Since the horocycle flow is induced by the Anosov flow the following pointwise

equality for all $x \in M$ holds for a suitable function $\tau(\rho, \alpha, x)$:

$$g_\alpha \circ h_\rho(x) = h_{\tau(\rho,\alpha,x)} \circ g_\alpha(x).$$

We call $\tau(\rho, \alpha, x)$ the *renormalization time*.
This kind of renormalization has been used effectively in the work of Flaminio and Forni [28] to give a precise understanding of the horocycle integral

$$\gamma_x(\varphi, T) \coloneqq \int_0^T \varphi \circ h_\rho(x) \, \mathrm{d}\rho, \quad x \in M, \quad T > 0,$$

in the setting of unit speed geodesic flows on hyperbolic compact (more generally finite volume) Riemannian surfaces with constant negative sectional curvature (i.e. Riemann surfaces), for $\varphi : M \to \mathbb{R}$ in Sobolev spaces of positive order. In this case, $h_{\mathrm{top}} = 1$. Flaminio and Forni found that the speed of convergence of $\gamma_x(\varphi, T)/T$ to $\mu(\varphi)$ as $T \to \infty$ is controlled by *invariant distributions* under the push-forward of the horocyclic vector field. These distributions are also eigendistributions under the push-forward of the geodesic vector field and the eigenvalues give the powers of T appearing in the expansion of $T^{-1}\gamma_x(\varphi, T) - \mu(\varphi)$.
Their approach inspired Giulietti and Liverani [30] to study a toy model, replacing the Anosov flow with a hyperbolic diffeomorphism, using the renormalization dynamics as a key to study $\gamma_x(\varphi, T)$. They show analogously (for the corresponding invariant measure μ) that the speed of convergence to zero of $T^{-1}\gamma_x(\varphi, T) - \mu(\varphi)$ is controlled by eigendistributions for a weighted transfer operator of the hyperbolic diffeomorphism.
Giulietti and Liverani conjectured that a similar behavior holds in the setting of more general Anosov flows, e.g. for the geodesic flow on the unit tangent bundle of a Riemannian manifold with variable negative sectional curvature [30, Conjecture 2.12]. More precisely, we expect for smooth enough observables φ an expansion like

$$\gamma_x(\varphi, T) = T \int \varphi \, \mathrm{d}\mu + \sum_{\delta < \Re\lambda < h_{\mathrm{top}}} T^{\frac{\Re\lambda}{h_{\mathrm{top}}}} c(\lambda, T, x) \mathcal{O}_\lambda(\varphi) + \mathcal{E}_{T,x}(\varphi), \tag{2.1}$$

with $\mathcal{E}_{T,x} = O(T^{\frac{\delta}{h_{\mathrm{top}}}})$, uniformly in x. The $\mathcal{O}_\lambda$ are generalized eigendistributions associated to the eigenvalue λ for the adjoint of the generator $X + V$ of a certain weighted transfer operator $\mathcal{L}_{\alpha,\phi_\alpha}$, acting on an anisotropic Banach space (see below). The real parameter δ is an upper bound on the essential spectral bound

of $X + V$. The complex coefficients $c(\lambda, T, x)$ are bounded from above independently of x by $|\log T|^c$ for some $c = c(\lambda) \geqslant 0$ which depends whether $\Re\lambda < 0$, $\Re\lambda = 0$ or $\Re\lambda > 0$ and if there are non-trivial Jordan blocks for λ. This is analogous to the bounds in [28],[30]. However our methods show no substantial improvement of the error term $\mathcal{E}_{T,x}$ if the summation in λ includes some $\Re\lambda < 0$ (this is seen also in [28],[30]). We restrict ourself therefore to $\delta \geqslant 0$ (i.e. always $\Re\lambda > 0$).

The main result of this work, Theorem 2.5.7, gives conditions under which such an asymptotic expansion indeed holds, for some $\delta > 0$, for codimension one topologically mixing Anosov flows, under an assumption of "spectral gap with (Dolgopyat) bounds" (Condition 2.4.11 below). In Proposition 2.5.10 we specialize to C^3 contact Anosov flows in dimension $d = 3$. For compact Riemann surfaces (recall that this is the constant negative curvature case) Randol [53] proved that there exist eigenvalues arbitrarily close to 1 (his result is for the associated Laplacian). This provides examples with a non-trivial expansion.
Analogous to the work of Giulietti and Liverani [30], the key idea to study $\gamma_x(\varphi, T)$ is to introduce a weighted transfer operator family

$$\mathcal{L}_{\alpha,\phi_\alpha} : W_p^{s,t,q} \to W_p^{s,t,q}, \quad \mathcal{L}_{\alpha,\phi_\alpha}\varphi = \phi_\alpha \cdot \varphi \circ g_{-\alpha}, \quad \alpha \geqslant 0,$$

where the weight is $\phi_\alpha = \partial_\rho \tau\,(0, -\alpha, \cdot)$ and where $W_p^{s,t,q}$ is an anisotropic Banach space with certain real regularity parameters s, t, q and p. In the case of the unit speed parametrization of the flow h_ρ, the weight $\partial_\rho \tau\,(0, -\alpha, \cdot)$ is just the Jacobian along the strong stable distribution evaluated at negative time $-\alpha$.

The paper is organized as follows: After recalling some facts about Anosov flows in Section 2.2, the transfer operator $\mathcal{L}_{\alpha,\phi_\alpha}$ is defined in Section 2.3.1 (for more general weights) and the Banach spaces $W_p^{s,t,q}$ are constructed in Section 2.3.2. These spaces are a flow analogue to the spaces constructed by Baladi and Tsujii [10] to study hyperbolic diffeomorphisms. Anisotropic Banach spaces are now considered a standard tool (yet with still ongoing research) for investigating transfer operators and zeta functions associated to hyperbolic dynamics [7]–[9], [13], [15], [23], [31], [33], [48], [52], [64], [65]. Although we do not study here the dynamical zeta function for the transfer operator $\mathcal{L}_{\alpha,\phi_\alpha}$, we believe that this space could be a suitable choice to be dealt with.
In Section 2.4 we establish properties of the transfer operator, its generator $X + V$ and the resolvent $\mathcal{R}_z$. Most of these results do not require the contact

assumption. Among those are norm estimates which yield a Lasota–York inequality for the resolvent. This is Theorem 2.4.5. Then in Lemma 2.4.10 one obtains a strip in the spectrum of the generator, containing at most countable eigenvalues of finite multiplicity. Those are precisely the eigenvalues λ in the summation over λ in (2.1). Finally, these results are used in Section 2.5 to give the expansion (2.1) of $\gamma_x(\varphi, T)$ in terms of eigendistributions and eigenvalues of $X + V$ under a *spectral gap with bounds* condition, see Condition 2.4.11.

We end this introduction with two remarks about possible further work:

First, the conjecture that the distributions $\mathcal{O}_v$ appearing in the expansion (2.1) are fixed by the (adjoint) of the horocycle flow remains still open. (In contrast this was the starting point in [28]!) Here, progress has been made by Faure and Guillarmou [24] in dimension 3 for smooth contact Anosov flows.

Second, the renormalization time $\tau(\rho, \alpha, x)$ inherits the regularity properties of the underlying Anosov foliation and horocycle flow, i.e. the regularity in x is expected to be no more than Hölder. To deal with such irregular flows one can lift the dynamics to the Grassmanian. This has been used with success, e.g. in [30], [33] and more recently in [65]. However in this work we wish to avoid such technicalities and we will make additional assumptions ensuring that $\tau(\rho, \alpha, x)$ enjoys sufficient regularity.
In particular, if the Anosov flow is C^r we require $\partial_\rho \tau(0, \alpha, \cdot)$ to be C^{r-1} for all $\alpha \geqslant 0$. This is reasonably only if r is small since the regularity of the stable foliation is usually only Hölder. In the setting of C^3 contact Anosov flows in dimension 3 we can take $r = 2 - \epsilon$ for all $\epsilon > 0$ by a result of [40] (see also Remark 2.5.8 in Section 2.5).

The Appendix comprises our computational tools. On the lowest level, we utilize Fourier transform, integration by parts, and Young's inequality [17, Theorem 3.9.4] to estimate convolutions.

2.2 Geometric setting

Let M be a closed, connected, orientable, smooth Riemannian manifold of dimension $d \geqslant 3$. We let $g_\alpha \colon M \to M$, $\alpha \in \mathbb{R}$, be a C^r Anosov flow on M [1]for $r > 1$. That is, there exists a decomposition of the tangent space TM of M as

[1]In this chapter it holds: if $r > 0$, is not an integer, C^r means $C^{\lfloor r \rfloor}$ with all partial derivatives of order $\lfloor r \rfloor$ being $(r - \lfloor r \rfloor)$-Hölder continuous.

a direct sum

$$\mathrm{T}\, M = E_- \oplus E_+ \oplus E_0, \tag{2.2}$$

such that for some constants $C \geqslant 1$, $0 < \theta < 1$ and every $\alpha \geqslant 0$

$$\begin{aligned} \|\mathrm{D}\, g_\alpha v\| &\leqslant C\theta^\alpha \|v\|, \quad \text{for all } v \in E_-, \\ \|\mathrm{D}\, g_{-\alpha} v\| &\leqslant C\theta^\alpha \|v\|, \quad \text{for all } v \in E_+, \end{aligned} \tag{2.3}$$

and $E_0 = \langle X \rangle$ where X is the generator of the Anosov flow

$$X := \partial_\alpha g_{-\alpha|\alpha=0}. \tag{2.4}$$

Note that the conditions in (2.3) are closed. Hence by compactness of M the distributions E_- and E_+ are uniformly continuous and so are the weak-stable $E_- \oplus E_0$ and weak-unstable $E_+ \oplus E_0$ distributions. The restriction of the tangent space to a base point $x \in M$ is denoted by

$$\mathrm{T}_x\, M = E_{-,x} \oplus E_{+,x} \oplus E_{0,x}, \tag{2.5}$$

The dimensions of those vector spaces do not vary with x and we set for some $x \in M$

$$d_- := \dim E_{-,x}. \tag{2.6}$$

The cotangent space T^*M is the dual space of $\mathrm{T}\, M$ and has the canonical splitting

$$\mathrm{T}^*\, M = E_-^* \oplus E_+^* \oplus E_0^* \quad \text{and} \quad \mathrm{T}_x^*\, M = E_{-,x}^* \oplus E_{+,x}^* \oplus E_{0,x}^*, \quad x \in M, \tag{2.7}$$

where $E_-^* \cong (E_+ \oplus E_0)^\perp$, $E_+^* \cong (E_- \oplus E_0)^\perp$, $E_0^* \cong (E_- \oplus E_+)^\perp$. This splitting is $(\mathrm{D}\, g_\alpha)^{\mathrm{tr}}$-invariant and satisfies an analogue of (2.3).
A contact form is a 1-form $\eta \in T^*M$ such that $\eta \wedge \bigwedge_{n=1}^{\frac{d-1}{2}} \mathrm{d}\,\eta$ vanishes nowhere ($\mathrm{d}\,\eta$ is the exterior derivative of η). An Anosov flow is a contact flow if there exists a C^1 contact form η which is preserved by the pullback of g_α. Clearly, a contact form can only exist if d is odd.
We mean by "$\Subset$" for sets $A, B \subseteq T^*M$ (or $\subseteq \mathbb{R}^d$) that

$$A \Subset B \Leftrightarrow \bar{A} \subseteq (\operatorname{int} B \cup \{0\}).$$

Here $\bar{A}$ denotes the closure of A and $\operatorname{int} B$ the interior of B. We say that a cone A is *compactly included* in a cone B if and only if $A \Subset B$. We say that a cone A and a cone B are *transversal* if and only if $A \cap B = \{0\}$.
We introduce two closed convex cone fields on M in the cotangent space:
For every $x \in M$ and for every $v \in \mathrm{T}_x^* M$ we have $v = v^- + v^+ + v^0$, where $v^\sigma \in E^*_{\sigma,x}$, $\sigma \in \{-,+,0\}$. For every $0 < \gamma < 1$ we set

$$\begin{aligned} C^-_\gamma(x) &:= \left\{ v \in \mathrm{T}_x^* M \mid \|v^+\| + \|v^0\| \leqslant \gamma \|v^-\| \right\}, \\ C^+_\gamma(x) &:= \left\{ v \in \mathrm{T}_x^* M \mid \|v^-\| + \|v^0\| \leqslant \gamma \|v^+\| \right\}. \end{aligned} \tag{2.8}$$

If $\gamma' > \gamma$ then we have the compact inclusions

$$C^-_\gamma(x) \Subset C^-_{\gamma'}(x) \quad \text{and} \quad C^+_\gamma(x) \Subset C^+_{\gamma'}(x).$$

Moreover, this construction implies $E^*_{-,x} \subset C^-_\gamma(x)$, and $E^*_{+,x} \subset C^+_\gamma(x)$ and also transversality $E^*_{0,x} \cap \left(C^-_\gamma(x) \cup C^+_\gamma(x)\right) = \{0\}$ and $C^-_\gamma(x) \cap C^+_\gamma(x) = \{0\}$.
We have (see Lemma B.1) for all $\alpha \geqslant 0$ so that $C^2\theta^\alpha\gamma < \gamma' < 1$ and for all $x \in M$ the compact inclusions

$$(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, C^-_\gamma(x) \Subset C^-_{\gamma'}(g_\alpha(x)) \quad \text{and} \quad (\mathrm{D}\, g_\alpha)^{\mathrm{tr}}\, C^+_\gamma(x) \Subset C^+_{\gamma'}(g_{-\alpha}(x)). \tag{2.9}$$

The cones defined in (2.8) are expanding and contracting, respectively (see Lemma B.2). Note that the cones in (2.8) have non-empty interior while [43, Proposition 17.4.4] uses "flat" cones included in $E^*_+ \oplus E^*_-$.
Let $V_\omega \subseteq \Omega$, $\omega \in \Omega$, be an open cover of M, where Ω is a finite index set. We let $\mathcal{A}$ be an atlas for M, containing diffeomorphic C^r-charts $\kappa_\omega \colon V_\omega \to \mathbb{R}^d$, compatible with the splittings (2.2) and (2.7), as we explain now. Fixing coordinates $(x_1, \ldots, x_d) \in \mathbb{R}^d$ and recalling X from (2.4), we may and do require the flowbox condition

$$\mathrm{D}\, \kappa_\omega X_{|V_\omega} = \partial_{x_d}. \tag{2.10}$$

Since g_α is C^r the chart maps κ_ω, $\omega \in \Omega$, are also C^r diffeomorphisms. We set

$$C^\sigma_{\gamma,\omega} := \bigcup_{x \in V_\omega} \left(\mathrm{D}\, \kappa_\omega^{-1}\right)^{\mathrm{tr}} C^\sigma_\gamma(x), \quad \sigma \in \{-,+\}, \omega \in \Omega. \tag{2.11}$$

We require the sets V_ω to be small enough such that for small $0 < \gamma_-, \gamma_+ \leqslant 1$ there exist $0 < \gamma_-^*, \gamma_+^* \leqslant 1$ such that for all $\omega \in \Omega$ and for all $x \in V_\omega$

$$(\mathrm{D}_x \kappa_\omega)^{\mathrm{tr}}\, C^-_{\gamma_-,\omega} \subseteq C^-_{\gamma_-^*}(x) \quad \text{and} \quad (\mathrm{D}_x \kappa_\omega)^{\mathrm{tr}}\, C^+_{\gamma_+,\omega} \subseteq C^+_{\gamma_+^*}(x)\,. \tag{2.12}$$

This is possible by uniform continuity of the weak-stable and weak-unstable distributions and the flowbox condition in (2.10). Note that the cones $C^\sigma_{\gamma,\omega}$ are not necessarily convex. This poses no problem since the differential is linear and hence the convex closure of $C^\sigma_{\gamma_\sigma,\omega}$ is contained in $C^\sigma_{\gamma_\sigma^*}(x)$ (this is already a convex, closed cone) for all $x \in M$. Without loss of generality we identify $C^\sigma_{\gamma_\sigma,\omega}$ with its convex closure.

Definition 2.2.1 (Cone ensemble)**.** *Let $C^-, C^+ \subset \mathbb{R}^d$, $d \geqslant 3$, be transversal, convex, closed cones with non-empty interiors. Let $\Phi_\sigma \colon \mathbb{R}^d \backslash \{0\} \to [0,1]$ be C^∞ maps, $\sigma \in \{-,+,0\}$, such that*

$$\Phi_{-|\operatorname{int} C^-} \equiv 1, \qquad \Phi_{+|\operatorname{int} C^+} \equiv 1, \qquad \Phi_- + \Phi_+ + \Phi_0 \equiv 1 \qquad \textit{and}$$
$$C^- = \mathbb{R}^d \backslash (\operatorname{supp} \Phi_+ \cup \operatorname{supp} \Phi_0)\,,\ C^+ = \mathbb{R}^d \backslash (\operatorname{supp} \Phi_- \cup \operatorname{supp} \Phi_0)\,.$$

We call $\Theta := (\Phi_-, \Phi_+, \Phi_0)$ a cone ensemble.[2]

Definition 2.2.2 (Cone hyperbolicity)**.** *Let $K \subset \mathbb{R}^d$ be open and let $F \colon K \to F(K)$ be a diffeomorphism. Let Θ, Θ° be two cone ensembles. Let*

$$C^- := \mathbb{R}^d \backslash (\operatorname{supp} \Phi_+ \cup \operatorname{supp} \Phi_0)\,.$$

We say that F is (Θ°, Θ)-cone hyperbolic on K if there exists C^∞ maps

$$\widetilde{\Phi}_+, \widetilde{\Phi}^\circ_\sigma \colon \mathbb{R}^d \backslash \{0\} \to [0,1]$$

such that $\widetilde{\Phi}_{+|\operatorname{supp} \Phi_+}, \widetilde{\Phi}^\circ_{\sigma|\operatorname{supp} \Phi^\circ_\sigma} \equiv 1$ for all $\sigma \in \{-,0\}$ such that for all $z \in K$

$$(\mathrm{D}_z F)^{\mathrm{tr}} \operatorname{supp} \widetilde{\Phi}^\circ_- \Subset C^- \ \textit{and}\ (\mathrm{D}_z F)^{\mathrm{tr}} \operatorname{supp} \widetilde{\Phi}^\circ_0 \Subset \mathbb{R}^d \backslash \operatorname{supp} \widetilde{\Phi}_+\,. \tag{2.13}$$

In Section 2.3.2 an anisotropic Banach space is constructed where the cones C^-, C^+ determine the directions of lowest and highest regularity, respectively. The inclusions (2.13) ensure that no parts of higher regularity are mapped to parts of lower regularity.

[2] By the support of a function $f \colon S \to \mathbb{C}$ we mean $\operatorname{supp} := \{x \in S \mid f(x) \neq 0\}$ which can be an open set in the topology of S.

Lemma 2.2.3 (Existence of admissible cones)**.** *Let $\alpha \in \mathbb{R}$ and let $\omega, \omega' \in \Omega$. Set $V_{\alpha,\omega\omega'} := V_\omega \cap g_\alpha(V_{\omega'})$ and set*

$$F_{-\alpha,\omega\omega'} \colon \kappa_\omega\left(V_{\alpha,\omega\omega'}\right) \to \kappa_{\omega'}\left(V_{-\alpha,\omega'\omega}\right) : y \mapsto \kappa_{\omega'} \circ g_{-\alpha} \circ \kappa_\omega^{-1}(y).$$

Then there exists $\alpha_0 > 0$ such that for all $\omega, \omega' \in \Omega$ there exist cone ensembles

$$\Theta_\omega = (\Phi_{-,\omega}, \Phi_{+,\omega}, \Phi_{0,\omega}) \quad \text{and} \quad \Theta^\circ_{\omega'} = \left(\Phi^\circ_{-,\omega'}, \Phi^\circ_{+,\omega'}, \Phi^\circ_{0,\omega'}\right),$$

such that for all $\alpha \geqslant \alpha_0$ the map $F_{-\alpha,\omega\omega'}$ is $\left(\Theta^\circ_{\omega'}, \Theta_\omega\right)$-cone hyperbolic. Moreover, for every $\omega \in \Omega$ it holds

$$\operatorname{supp}\Phi^\circ_{0,\omega} \Subset \operatorname{supp}\Phi_{0,\omega} \cup \operatorname{supp}\Phi_{+,\omega} \quad \text{and} \quad \operatorname{supp}\Phi^\circ_{+,\omega} \Subset \operatorname{supp}\Phi_{+,\omega}. \tag{2.14}$$

Proof. We let $\omega, \omega' \in \Omega$. We assume $V_{\alpha,\omega\omega'} \neq \varnothing$ (otherwise we are done). We let $0 < \gamma_-, \gamma_+ \leqslant 1$ be small such that $\gamma^*_-, \gamma^*_+ > 0$ are the values attained in (2.12) for all cones $C^-_{\gamma_-,\omega}$, $C^+_{\gamma_+,\omega}$, $\omega \in \Omega$. These cones are transversal, convex and closed by construction. We repeat the construction, resulting in values $\widetilde{\gamma}^*_- < \gamma^*_-$ and $\widetilde{\gamma}^*_+ < \gamma^*_+$, using now values

$$\widetilde{\gamma}_- < \gamma_-, \quad \widetilde{\gamma}_+ < \gamma_+,$$

sufficiently small (possibly by passing to a finer open cover) such that for all $\omega \in \Omega$ and all $x \in V_\omega$

$$\left(\mathrm{D}_{\kappa_\omega(x)}\,\kappa_\omega^{-1}\right)^{\mathrm{tr}} C^-_{\widetilde{\gamma}^*_-}(x) \Subset C^-_{\gamma_-,\omega}, \quad \left(\mathrm{D}_{\kappa_\omega(x)}\,\kappa_\omega^{-1}\right)^{\mathrm{tr}} C^+_{\widetilde{\gamma}^*_+}(x) \Subset C^+_{\gamma_+,\omega}. \tag{2.15}$$

We note that the map $F_{\alpha,\omega\omega'}$ is a diffeomorphism by construction. We construct further cones as follows: By the construction of local cones in (2.11) and the compact inclusion given in (2.12) for some $C^2\beta^\alpha\gamma^*_+ \leqslant \gamma'_+ < \widetilde{\gamma}^*_+$ and for all $\alpha \geqslant \alpha_0$ we have for all $x \in V_{\alpha,\omega\omega'}$

$$(\mathrm{D}\,g_\alpha)^{\mathrm{tr}}\,(\mathrm{D}_x\,\kappa_\omega)^{\mathrm{tr}}\,C^+_{\gamma_+,\omega} \subseteq (\mathrm{D}\,g_\alpha)^{\mathrm{tr}}\,C^+_{\gamma^*_+}(x) \Subset C^+_{\gamma'_+}(g_{-\alpha}(x)) \Subset C^+_{\widetilde{\gamma}^*_+}(g_{-\alpha}(x)).$$

Comparing with the compact inclusion in (2.15), there exists a convex, closed cone $\widetilde{C}^+_{\gamma_+,\omega'} \subset \mathbb{R}^d$ such that

$$\left(\mathrm{D}\,F_{\alpha,\omega'\omega}\right)^{\mathrm{tr}} C^+_{\gamma_+,\omega} \Subset \widetilde{C}^+_{\gamma_+,\omega'} \Subset C^+_{\gamma_+,\omega'}. \tag{2.16}$$

Analogously we find

$$\left(\mathrm{D}\, F_{-\alpha,\omega'\omega}\right)^{\mathrm{tr}} C^-_{\gamma_-,\omega} \Subset \widetilde{C}^-_{\gamma_-,\omega'} \Subset C^-_{\gamma_-,\omega'}. \tag{2.17}$$

Recalling Definition 2.2.1, we let

$$\Theta_\omega = (\Phi_{-,\omega}, \Phi_{+,\omega}, \Phi_{0,\omega}) \quad \text{and} \quad \Theta^\circ_{\omega'} = \left(\Phi^\circ_{-,\omega'}, \Phi^\circ_{+,\omega'}, \Phi^\circ_{0,\omega'}\right)$$

be the cone ensembles such that

$$\Phi_{-,\omega|\operatorname{int}\widetilde{C}^-_{\gamma_-,\omega}} \equiv \Phi_{+,\omega|\operatorname{int} C^+_{\gamma_+,\omega}} \equiv \Phi^\circ_{-,\omega'|\operatorname{int} C^-_{\gamma_-,\omega'}} \equiv \Phi^\circ_{+,\omega'|\operatorname{int}\widetilde{C}^+_{\gamma_+,\omega'}} \equiv 1.$$

The supports of $\Phi_{-,\omega}, \Phi_{+,\omega}$ and $\Phi^\circ_{-,\omega'}, \Phi^\circ_{+,\omega'}$ are taken to be disjoint, respectively, considering slightly larger convex cones. We check $\left(\Theta^\circ_{\omega'}, \Theta_\omega\right)$-cone hyperbolicity of $F_{\alpha,\omega\omega'}$, recalling Definition 2.2.2. The supports of $\widetilde{\Phi}^\circ_{-,\omega'}, \widetilde{\Phi}_{+,\omega'}, \widetilde{\Phi}^\circ_{0,\omega'}$ are chosen analogously on corresponding slightly larger cones. The first compact inclusion in (2.13) is a direct consequence of the compact inclusion in (2.17). To see the second compact inclusion in (2.13) note that

$$\left(\left(\mathrm{D}\, F_{-\alpha,\omega\omega'}\right)^{\mathrm{tr}}\right)^{-1}\left(\mathbb{R}^d \setminus \operatorname{supp}\Phi_{+,\omega}\right) = \mathbb{R}^d \setminus \left(\mathrm{D}\, F_{\alpha,\omega'\omega}\right)^{\mathrm{tr}} \operatorname{supp}\Phi_{+,\omega}.$$

Comparing with the compact inclusion in (2.16), we conclude. The claim in (2.14) follows again by comparing with the compact inclusions in (2.16) and (2.17). □

2.3 The transfer operator and the anisotropic Banach space

2.3.1 The transfer operator

We denote by $C^r(M)$ the space of $C^{\lfloor r\rfloor}$ functions whose $\lfloor r\rfloor$-th partial derivatives in charts are $C^{r-\lfloor r\rfloor}$. We let $C^{r-1}_X(M)$[3] be the space of C^{r-1} functions which are C^r in the flow direction X defined by (2.4). Fixing a "potential function" $V \in C^{r-1}(M, \mathbb{R})$, we introduce the ϕ_α-weighted transfer operator family

$$\mathcal{L}_{\alpha,\phi_\alpha} : \varphi \mapsto \phi_\alpha \cdot (\varphi \circ g_{-\alpha}), \quad \alpha \geqslant 0, \tag{2.18}$$

[3] If $\varphi \in C^{r-1}(M)$ then $\varphi_c := \frac{1}{c}\int_0^c \varphi \circ g_{-\alpha}\,\mathrm{d}\,\alpha \in C^{r-1}_X(M)$ for all $c > 0$. In the Banach spaces we construct the limit $\lim_{c\to 0}\varphi_c$ exists.

acting on $\varphi \in C_X^{r-1}(M)$, where

$$\phi_\alpha(x) := \exp\left(\int_0^\alpha V \circ g_{-\alpha'}(x)\,\mathrm{d}\,\alpha'\right).$$

We will construct Banach spaces $W_p^{s,t,q}$ containing $C_X^{r-1}(M)$ as a dense subspace (for suitable choices $p, s, t, q \in \mathbb{R}$) on which the family (2.18) of operators extends continuously to a strongly continuous semigroup (see Lemma 2.4.4 below). Note that

$$V = \partial_\alpha \phi_{\alpha|\alpha=0^+}. \tag{2.19}$$

Our construction will show that for all $\varphi \in C_X^{r-1}(M)$

$$\partial_\alpha \mathcal{L}_{\alpha,\phi_\alpha}\varphi_{|\alpha=0^+} = X\varphi + V\varphi,$$

is well-defined in the sense that $(X+V)\,\varphi \in W_p^{s,t,q}$ if $\varphi \in C_X^{r-1}(M)$. The operator $X+V$ is the generator of the semigroup $\left\{\mathcal{L}_{\alpha,\phi_\alpha} : W_p^{s,t,q} \to W_p^{s,t,q} \mid \alpha \geqslant 0\right\}$. We denote by

$$\sigma(X+V)\,|_{W_p^{s,t,q}}$$

the spectrum of $X+V$ to emphasize the dependency of the domain and hence the spectrum of $X+V$ on $W_p^{s,t,q}$. We show in Theorem 2.4.5 that the resolvent of $X+V$

$$\mathcal{R}_z\varphi := (z-V-X)^{-1}\varphi, \quad z \notin \sigma(X+V)\,|_{W_p^{s,t,q}}, \quad \varphi \in W_p^{s,t,q}, \tag{2.20}$$

admits a Lasota–Yorke inequality for large $\Re z > 0$. This allows us to identify a vertical left-open strip in the complex plane in which $\sigma(X+V)\,|_{W_p^{s,t,q}}$ contains only isolated eigenvalues of finite multiplicity of $X+V$ (see Lemma 2.4.10).

2.3.2 The anisotropic Banach space

We work locally with the atlas $\mathcal{A}$, introduced in Section 2.2. We let $\Psi_n : \mathbb{R}^d \to [0,1]$, $n \in \mathbb{Z}_{\geqslant 0}$, be a Paley–Littlewood decomposition as follows:
Let $\chi : \mathbb{R}_{>0} \to [0,1]$ be a C^∞ map so that $\chi_{|(0,1]} \equiv 1$ and $\operatorname{supp}\chi \subseteq [0,2]$. Let $|\cdot| : \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ be a smooth norm on $\mathbb{R}^d \setminus \{0\}$. Define Ψ_n by setting for all

$\xi \in \mathbb{R}^d \setminus \{0\}$

$$\Psi_0(\xi) := \chi(|\xi|) \quad \text{and} \quad \Psi_n(\xi) := \chi\left(\left|2^{-n}\xi\right|\right) - \chi\left(\left|2^{1-n}\xi\right|\right),\ n \geqslant 1. \tag{2.21}$$

This defines a partition of unity on $\mathbb{R}^d \setminus \{0\}$ since we have

$$\sum_{n=0}^{\infty} \Psi_n(\xi) = \lim_{n\to\infty} \chi\left(\left|2^{-n}\xi\right|\right) = 1.$$

For all $n \geqslant 1$ it holds $\Psi_n(\xi) = \Psi_1(2^{-n+1}\xi)$ from which one finds

$$\operatorname{supp}\Psi_n \subseteq \left\{\xi \in \mathbb{R}^d \mid 2^{n-1} \leqslant |\xi| \leqslant 2^{n+1}\right\}. \tag{2.22}$$

The inverse Fourier transform is given by

$$\mathbb{F}^{-1}\varphi(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{\mathrm{i}\,\xi x}\varphi(\xi)\,\mathrm{d}\,\xi,$$

where

$$\xi x := \langle \xi, x\rangle$$

is the canonical scalar product on $\mathbb{R}^d$. The convolution of two complex valued functions φ_1, φ_2 on $\mathbb{R}^d$ (and extended to distributions) is given by

$$\varphi_1 * \varphi_2(x) := \int_{\mathbb{R}^d} \varphi_1(x-y)\varphi_2(y)\,\mathrm{d}\,y.$$

We will make frequent use (e.g. in the proof of Lemma 2.3.1 below and Lemma 2.4.16 in Section 2.4.4) of a special case of Young's inequality for convolutions,

$$\|\varphi_1 * \varphi_2\|_{L_p} \leqslant \|\varphi_1\|_{L_1} \|\varphi_2\|_{L_p}, \quad \text{for all } p \in [1,\infty].$$

Given a cone ensemble $\Theta = (\Phi_-, \Phi_+, \Phi_0)$, we set for all $\sigma \in \{-,+,0\}, n \in \mathbb{Z}_{\geqslant 0}$,

$$\Psi_{\sigma,n} := \Psi_n \Phi_\sigma \quad \text{and} \quad \Psi^{\mathrm{Op}}_{\sigma,n}\varphi := \left(\mathbb{F}^{-1}\Psi_{\sigma,n}\right) * \varphi. \tag{2.23}$$

We let $\widetilde{\Psi}_0, \widetilde{\Psi}_1 \in C^\infty$ such that $\widetilde{\Psi}_{0|\operatorname{supp}\Psi_0} \equiv 1$ and $\widetilde{\Psi}_{1|\operatorname{supp}\Psi_1} \equiv 1$. We set for every $n \in \mathbb{N}$

$$\widetilde{\Psi}_n := \widetilde{\Psi}_1 \circ 2^{1-n}.$$

(In principal it is enough to require the condition on the support of $\widetilde{\Psi}_n$ for each n individually. Regarding the bounds in (2.25) below our choice here is

reasonable.) Then we set for every $\sigma \in \{-,+,0\}$ and every $n \in \mathbb{Z}_{\geqslant 0}$

$$\widetilde{\Psi}_{\sigma,n} := \widetilde{\Psi}_n \widetilde{\Phi}_\sigma \quad \text{and} \quad \widetilde{\Psi}^{\mathrm{Op}}_{\sigma,n}\varphi := \left(\mathbb{F}^{-1}\widetilde{\Psi}_{\sigma,n}\right) * \varphi, \tag{2.24}$$

where $\widetilde{\Phi}_\sigma \in C^\infty$ and $\operatorname{supp}\widetilde{\Phi}_\sigma$ is a closed convex cone such that $\widetilde{\Phi}_{\sigma|\operatorname{supp}\Phi_\sigma} \equiv 1$ and $\Phi_{\sigma_1}\Phi_{\sigma_2} \equiv 0 \Rightarrow \widetilde{\Phi}_{\sigma_1}\widetilde{\Phi}_{\sigma_2} \equiv 0$ for all $\sigma_1, \sigma_2 \in \{-,+,0\}$. We have the following estimates for all σ and all $n \in \mathbb{N}$:

$$\left\|\mathbb{F}^{-1}\Psi_n\right\|_{L_1} = \left\|\mathbb{F}^{-1}\Psi_1\right\|_{L_1} < \infty, \quad \left\|\mathbb{F}^{-1}\Psi_{\sigma,n}\right\|_{L_1} = \left\|\mathbb{F}^{-1}\Psi_{\sigma,1}\right\|_{L_1} < \infty. \tag{2.25}$$

Analogous estimates hold for $\mathbb{F}^{-1}\Psi_{\sigma,0}$ and $\mathbb{F}^{-1}\Psi_0$ and for the $\sim$-versions as well. If Θ° is another cone ensemble we define $\Psi^\circ_{\sigma,n}, \Psi^{\circ\,\mathrm{Op}}_{\sigma,n}$ and $\widetilde{\Psi}^\circ_{\sigma,n}, \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\sigma,n}$ analogously. We set

$$B := \left\{x \in \mathbb{R}^d \mid |x| < 1\right\} \quad \text{and} \quad B^{\mathrm{c}} := \mathbb{R}^d \backslash B. \tag{2.26}$$

In order to show a continuous embedding of certain spaces we will use very often the following statement about convolution operators (an extension of [62, Theorem 0.3.1] for the case $r = 1$ and $K(x,y) = K(x-y)$ in his notation). In Lemma 2.3.1 below all the occurring L_p-spaces are understood (as Bochner spaces, cf. [16]) such that if $a \in L_p\left(\mathbb{R}^d, \mathcal{B}\right)$ for some complex Banach space $\mathcal{B}$ then the norm of a is given by

$$\|a\|_{L_p(\mathbb{R}^d,\mathcal{B})} := \left\|\|a\|_{\mathcal{B}}\right\|_{L_p(\mathbb{R}^d,\mathbb{R}_{\geqslant 0})}.$$

The following lemma handles the range $p \in [1,\infty]$. (For parameters $p \in (1,\infty)$ one could apply instead the classical Marcinkiewicz theorem quoted e.g. as [10, Theorem 3.1].)

Lemma 2.3.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be (complex) Banach spaces, let $d \in \mathbb{N}$ and let $Q \in C^{d+1}\left(\mathbb{R}^d, \mathcal{L}\left(\mathcal{B}_1, \mathcal{B}_2\right)\right)$ satisfy for its partial derivatives*

$$\left\|\partial_\xi^\beta Q(\xi)\right\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} \leqslant C(\beta)\,|\xi|^{-|\beta|} \quad \text{as} \quad |\xi| \to \infty,$$

for some constants $C(\beta) > 0$ and all multi-indices $\beta \in \{0,\ldots,d+1\}^d$ such that $|\beta| \leqslant d+1$, where $|\beta| := \beta_1 + \ldots + \beta_d$. Then for all $p \in [1,\infty]$ the map

$$Q^{\mathrm{Op}} \colon L_p\left(\mathbb{R}^d, \mathcal{B}_1\right) \to L_p\left(\mathbb{R}^d, \mathcal{B}_2\right) : a \mapsto \int_{\mathbb{R}^d} \left(\mathbb{F}^{-1}Q\right)(x-y)a(y)\,\mathrm{d}\,y, \tag{2.27}$$

defines a bounded linear operator, where for every $b \in \mathcal{B}_1$ and every $x \in \mathbb{R}^d$

$$\mathbb{F}^{-1}Q\,(x)\,b := (2\pi)^{-d}\int_{\mathbb{R}^d} e^{\mathrm{i}\,x\xi}Q\,(\xi)\,b\,\mathrm{d}\,\xi.$$

It holds

$$\left\|Q^{\mathrm{Op}}\right\|_{\mathcal{L}(L_p(\mathbb{R}^d,\mathcal{B}_1),L_p(\mathbb{R}^d,\mathcal{B}_2))} \leqslant \left\|\mathbb{F}^{-1}Q\right\|_{L_1(\mathbb{R}^d,\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2))} < \infty.$$

Proof. Linearity of Q^{Op} follows if Q^{Op} is a bounded operator. Suppose first that $\mathbb{F}^{-1}Q \in L_1\left(\mathbb{R}^d, \mathcal{L}\left(\mathcal{B}_1,\mathcal{B}_2\right)\right)$. We estimate

$$\begin{aligned}
\left\|Q^{\mathrm{Op}}a\right\|_{L_p(\mathbb{R}^d,\mathcal{B}_2)} &= \left\|\left\|\int_{\mathbb{R}^d}\left(\mathbb{F}^{-1}Q\right)(\cdot - y)a(y)\,\mathrm{d}\,y\right\|_{\mathcal{B}_2}\right\|_{L_p} \\
&\leqslant \left\|\int_{\mathbb{R}^d}\left\|\left(\mathbb{F}^{-1}Q\right)(\cdot - y)a(y)\right\|_{\mathcal{B}_2}\mathrm{d}\,y\right\|_{L_p} \\
&\leqslant \left\|\int_{\mathbb{R}^d}\left\|\left(\mathbb{F}^{-1}Q\right)(\cdot - y)\right\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}\|a(y)\|_{\mathcal{B}_1}\,\mathrm{d}\,y\right\|_{L_p} \\
&= \left\|\left\|\mathbb{F}^{-1}Q\right\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} * \|a\|_{\mathcal{B}_1}\right\|_{L_p}.
\end{aligned}$$

Using Young's inequality, we estimate and conclude

$$\begin{aligned}
\left\|\left\|\mathbb{F}^{-1}Q\right\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)} * \|a\|_{\mathcal{B}_1}\right\|_{L_p} &\leqslant \left\|\left\|\mathbb{F}^{-1}Q\right\|_{\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2)}\right\|_{L_1}\left\|\|a\|_{\mathcal{B}_1}\right\|_{L_p} \\
&= \left\|\mathbb{F}^{-1}Q\right\|_{L_1(\mathbb{R}^d,\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2))}\|a\|_{L_p(\mathbb{R}^d,\mathcal{B}_1)}.
\end{aligned}$$

We now show $\left\|\mathbb{F}^{-1}Q\right\|_{L_1(\mathbb{R}^d,\mathcal{L}(\mathcal{B}_1,\mathcal{B}_2))} < \infty$. It remains to show an upper bound for

$$I := \int_{\mathbb{R}^d}\left\|\int_{\mathbb{R}^d} e^{\mathrm{i}\,y\xi}Q\,(\xi)\,a\,\mathrm{d}\,\xi\right\|_{\mathcal{B}_2}\mathrm{d}\,y.$$

Inside I we substitute, whenever $y \neq 0$

$$\xi \mapsto \langle y, y\rangle^{-\frac{1}{2}}\,\xi$$

which yields

$$I = \int_{\mathbb{R}^d}\langle y, y\rangle^{-\frac{d}{2}}\,\|J\,(y)\|_{\mathcal{B}_2}\,\mathrm{d}\,y,$$

where

$$J := J\,(y) = \int_{\mathbb{R}^d} e^{\mathrm{i}\,y\langle y,y\rangle^{-\frac{1}{2}}\xi}Q\left(\langle y, y\rangle^{-\frac{1}{2}}\,\xi\right)a\,\mathrm{d}\,\xi.$$

For every $y \in \mathbb{R}^d \setminus \{0\}$ we set

$$\xi_0 := y \langle y, y \rangle^{-\frac{1}{2}} \pi.$$

Clearly, it holds

$$\langle \xi_0, \xi_0 \rangle = \pi^2.$$

We now repeat the following substitution $(d+1)$-times

$$\xi \mapsto \xi + \xi_0,$$

which yields

$$J = \int_{\mathbb{R}^d} e^{\mathrm{i}\, y \langle y, y \rangle^{-\frac{1}{2}} \xi} \widetilde{Q}(\xi)\, a \,\mathrm{d}\,\xi,$$

where

$$\widetilde{Q}(\xi) = 2^{-d-1} \sum_{n=0}^{d+1} \binom{d+1}{n} (-1)^n Q\left(\langle y, y \rangle^{-\frac{1}{2}} (\xi + n\xi_0) \right).$$

We let $0 < \epsilon < 1$. We split the part in $\widetilde{Q}$ if $y \in B$ for every $0 \leqslant n \leqslant d+1$ as

$$1 \equiv \chi\left((\xi + n\xi_0) \langle y, y \rangle^{\frac{\epsilon}{2}} \right) + (1 - \chi)\left((\xi + n\xi_0) \langle y, y \rangle^{\frac{\epsilon}{2}} \right),$$

for every corresponding summand in $\widetilde{Q}$, respectively. The part in I which corresponds to $\chi\left((\xi + n\xi_0) \langle y, y \rangle^{\frac{\epsilon}{2}} \right)$ is estimated trivially, using boundedness of Q and integrability of $\langle y, y \rangle^{\frac{\epsilon - d}{2}}$ on B. Using the identity

$$\begin{aligned} Q(\xi) - Q(\xi + \xi_0) &= \int_0^1 \partial_t Q\left((\xi + \xi_0 - t\xi_0) \langle y, y \rangle^{-\frac{\epsilon}{2}} \right) \mathrm{d}\, t \\ &= -\langle y, y \rangle^{-\frac{\epsilon}{2}} \int_0^1 (\mathrm{D}\, Q)\left((\xi + \xi_0 - t\xi_0) \langle y, y \rangle^{-\frac{\epsilon}{2}} \right) \xi_0 \,\mathrm{d}\, t, \end{aligned}$$

we now write the remaining part in $\widetilde{Q}$ as

$$\begin{aligned} \widetilde{Q}(\xi) &= 2^{-d-1} \int_{[0,1]^{d+1}} \sum_{n=0}^{d+1} \binom{d+1}{n} \langle y, y \rangle^{\frac{n\epsilon}{2}} \langle y, y \rangle^{-\frac{d+1-n}{2}} \\ &\times \left((\mathrm{D}^n (1 - \chi)) \circ \langle y, y \rangle^{\frac{\epsilon}{2}} \left(\mathrm{D}^{d+1-n} Q \right) \circ \langle y, y \rangle^{-\frac{1}{2}} \right) (\xi(t)) \, (-\xi_0)^{\otimes (d+1)} \,\mathrm{d}\, t, \end{aligned} \tag{2.28}$$

where we put

$$\xi(t) := \xi + (d+1)\,\xi_0 - t\xi_0^{\otimes (d+1)} \quad \text{and} \quad \xi_0^{\otimes (d+1)} := \underbrace{(\xi_0, \ldots, \xi_0)}_{(d+1)-\text{times}}.$$

We observe that the part where derivatives of $\left((1-\chi)\circ\langle y,y\rangle^{\frac{\epsilon}{2}}\right)(\xi)$ contribute implies

$$\xi \in ((2B)\setminus B)\langle y,y\rangle^{-\frac{\epsilon}{2}}.$$

Using the decay condition on all the partial derivatives of Q, recalling that

$$\langle y,y\rangle^{\frac{\epsilon-d}{2}}\log\langle y,y\rangle$$

is integrable on the unit ball, and exchanging the order of integration with respect to t as the outermost (justified by absolute integrability), we find for the corresponding part in I, for some constants $C_1, C_2, C_3 > 0$

$$\begin{aligned} I &\leqslant C_1\frac{\pi^{d+1}}{2^{d+1}}\|a\|_{\mathcal{B}_1}\sum_{n=0}^{d+1}\binom{d+1}{n}\int_B\int_{((2B)\setminus B)\langle y,y\rangle^{-\frac{\epsilon}{2}}}|\xi|^{-d-1+n}\,\mathrm{d}\,\xi\,\langle y,y\rangle^{\frac{n\epsilon-d}{2}}\,\mathrm{d}\,y \\ &\qquad + C_1\pi^{d+1}\|a\|_{\mathcal{B}_1}\int_B\int_{2\langle y,y\rangle^{-\frac{\epsilon}{2}}B^{\mathrm{c}}}|\xi|^{-d-1}\,\mathrm{d}\,\xi\,\langle y,y\rangle^{-\frac{d}{2}}\,\mathrm{d}\,y \\ &\leqslant C_2\pi^{d+1}(\log d)\|a\|_{\mathcal{B}_1}\int_B\left(1+\frac{\epsilon}{2}\left|\log\langle y,y\rangle\right|\right)\langle y,y\rangle^{\frac{\epsilon-d}{2}}\,\mathrm{d}\,y \\ &\leqslant C_3\|a\|_{\mathcal{B}_1}. \end{aligned}$$

In the case $y\in B^{\mathrm{c}}$ we proceed analogously, using the formular for $\widetilde{Q}(\xi)$ given in (2.28), but without splitting the integral with respect to ξ. We have now

$$\widetilde{Q}(\xi) = 2^{-d-1}\int_{[0,1]^{d+1}}\langle y,y\rangle^{-\frac{d+1}{2}}\left(\mathrm{D}^{d+1}Q\right)\left(\langle y,y\rangle^{-\frac{1}{2}}\xi(t)\right)(-\xi_0)^{\otimes(d+1)}\,\mathrm{d}\,t.$$

If $\xi\in B\langle y,y\rangle^{\frac{\epsilon}{2}}$ we bound the corresponding part in I trivially, using boundedness of the $(d+1)$-th partial derivatives of Q and integrability of $\langle y,y\rangle^{\frac{d\epsilon-2d-1}{2}}$ on B^{c}. If $\xi\in B^{\mathrm{c}}\langle y,y\rangle^{\frac{\epsilon}{2}}$ we use the decay condition of the $(d+1)$-th partial derivatives of Q instead and integrability of $\langle y,y\rangle^{-\frac{d+\epsilon}{2}}$ on B^{c}. □

For every open set $K\subseteq\mathbb{R}^d$ with compact closure we let $C_0^{r-1}(K)$ be the space of C^{r-1} functions which vanish at the boundary of K. Since $C_0^{r-1}(K)\subset L_p(K,\mathbb{C})$ for all $p\in[1,\infty]$, the following definition makes sense.

Definition 2.3.2 (Local norm and local Banach space)**.** *Let $p\in[1,\infty]$ and let $s,t,q<r-1$. Let Θ be a cone ensemble from Definition 2.2.1 and let $K\subset\mathbb{R}^d$ be an open set with compact closure. For every $\varphi\in C_0^{r-1}(K)$ we set as the local*

norm

$$\|\varphi\|_{W^{s,t,q}_{p,\Theta,K}} := \left\| \left(\sum_{n=0}^{\infty} 4^{ns} \left| \Psi^{\mathrm{Op}}_{-,n} \varphi \right|^2 + 4^{nt} \left| \Psi^{\mathrm{Op}}_{+,n} \varphi \right|^2 + 4^{nq} \left| \Psi^{\mathrm{Op}}_{0,n} \varphi \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p(\mathbb{R}^d)}.$$

The completion $W^{s,t,q}_{p,\Theta,K}$ of $C_0^{r-1}(K)$ under $\|\cdot\|_{W^{s,t,q}_{p,\Theta,K}}$ is our local anisotropic Banach space.

This is an anisotropic version of a Triebel–Lizorkin space [63, p.45, Definition 2] with a certain inner l_2-norm and an outer L_p-norm. More precisely, we relate the summation in n and σ which appears in the norm of $W^{s,t,q}_{p,\Theta,K}$ to the norm of a Hilbert space of complex valued sequences defined on $\{-,+,0\} \times \mathbb{Z}_{\geqslant 0}$. We set

$$c(-) := s, \quad c(+) := t, \quad c(0) := q. \tag{2.29}$$

Then we denote by ℓ^c_2 the Hilbert space with norm given for all $a \in \ell^c_2$ by

$$\|a\|_{\ell^c_2} := \left(\sum_{\sigma,n} 4^{c(\sigma)n} |a_{\sigma,n}|^2 \right)^{\frac{1}{2}}. \tag{2.30}$$

For $s', t', q' \in \mathbb{R}$ we define c' and ℓ^c_2 analogously.

Lemma 2.3.3 (Multiplication and composition operator)**.** *Let $p \in [1,\infty]$ and let $s', t', q', s, t, q < r-1$. Let $\widetilde{r} > \max\{0,s,t,q\} - \min\{0,s',t',q'\}$ and let $f \in C_0^{\widetilde{r}}(K)$ for some open set $K \in \mathbb{R}^d$ with compact closure and let $F\colon K \to F(K)$ be a $C^{\widetilde{r}}$ diffeomorphism. Let Θ and Θ° be two cone ensembles. Then the linear operator*

$$\mathcal{M}_{F,f}\colon W^{s',t',q'}_{p,\Theta^\circ,K} \to W^{s,t,q}_{p,\Theta,F^{-1}(K)}\colon \varphi \mapsto f \cdot (\varphi \circ F)$$

is bounded if $c(\sigma) \leqslant c'(\tau)$ whenever $\bigcup_{x\in K} \operatorname{supp}\Psi_\sigma \cap \mathrm{D}F(x)^{\mathrm{tr}} \operatorname{supp}\Psi^\circ_\tau \neq \varnothing$. Moreover, if $F = \mathrm{id}$ and $\Theta = \Theta^\circ$ the linear operator $\mathcal{M}_{\mathrm{id},f}$ is bounded if $s \leqslant q \leqslant t$.

Proof. We exclude first the indices for given $\sigma, \tau \in \{-,+,0\}$ such that

$$\bigcup_{x\in K} \operatorname{supp}\Psi_\sigma \cap \mathrm{D}F(x)^{\mathrm{tr}} \operatorname{supp}\Psi^\circ_\tau \neq \varnothing, \tag{2.31}$$

and given $n, \ell \in \mathbb{Z}_{\geqslant 0}$ such that

$$\left| \sup_{x\in F(K), \xi \in (\operatorname{supp}\Psi_\sigma \cap B)} \left| \mathrm{D}F^{-1}(x)^{\mathrm{tr}} \xi \right| \right|^{-1} 2^{-4} \leqslant 2^{n-l} \leqslant 2^4 \sup_{x\in K, \eta \in (\operatorname{supp}\Psi^\circ_\tau \cap B)} \left| \mathrm{D}F(x)^{\mathrm{tr}} \eta \right|.$$

For all remaining $\sigma, \tau \in \{-, +, 0\}$ and $n, \ell \in \mathbb{Z}_{\geqslant 0}$ we bound the local norm for every $\epsilon > 0$ and some constant $C_1 = C_1(\epsilon) > 0$

$$\begin{aligned}\|\mathcal{M}_{F,f}\varphi\|_{W^{s,t,q}_{p,\Theta,K}} &= \left\| \left(\sum_{\sigma,n} 4^{-\epsilon n} 4^{(c(\sigma)+\epsilon)n} \left| \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f}\varphi \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p(\mathbb{R}^d)} \\ &\leqslant C_1 \sup_{\sigma,n} 2^{(c(\sigma)+\epsilon)n} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f}\varphi \right\|_{L_p(\mathbb{R}^d)} . \end{aligned} \tag{2.32}$$

On the excluded indices we estimate as in the proof of Lemma 2.4.13 below, using Lemma 2.3.1 and Cauchy–Schwarz in ℓ and that $n \sim \ell$ and using $c(\sigma) \leqslant c'(\tau)$. We recall the map $\widetilde{\Psi}^{\circ}_{\tau,\ell}$ defined in (2.24). Then we bound for every $n \geqslant 0$ and every $\sigma \in \{-, +, 0\}$

$$\begin{aligned}\left\| \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f}\varphi \right\|_{L_p(\mathbb{R}^d)} &\leqslant \sum_{\tau,\ell} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right\|_{L_p(\mathbb{R}^d)} \\ &= \sum_{\tau,\ell} 2^{-c'(\tau)\ell} 2^{c'(\tau)\ell} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right\|_{L_p(\mathbb{R}^d)} . \end{aligned} \tag{2.33}$$

Note that if $\operatorname{supp} \Psi_\sigma$ and $\mathrm{D}\, F(x)^{\mathrm{tr}} \operatorname{supp} \Psi^{\circ}_{\tau}$ have empty intersection, since the supports are open, we may assume that $\operatorname{supp} \Psi_\sigma$ and $\mathrm{D}\, F(x)^{\mathrm{tr}} \operatorname{supp} \widetilde{\Psi}^{\circ}_{\tau}$ have empty intersection as well. Since we excluded the conditions regarding certain σ, τ and n, ℓ given in (2.31) and below of it then by construction of $\widetilde{\Psi}^{\circ}_{\tau,\ell}$, for some constant $C_2 > 0$ it holds, in the following assuming $n, \ell > 0$,

$$\begin{aligned}\inf_{x \in K} \left| \operatorname{supp} \Psi_{\sigma,n} - \mathrm{D}\, F(x)^{\mathrm{tr}} \operatorname{supp} \widetilde{\Psi}^{\circ}_{\tau,\ell} \right| &\geqslant C_2 2^{\max\{n,\ell\}} \text{ or} \\ \inf_{x \in F(K)} \left| \mathrm{D}\, F^{-1}(x)^{\mathrm{tr}} \operatorname{supp} \Psi_{\sigma,n} - \operatorname{supp} \widetilde{\Psi}^{\circ}_{\tau,\ell} \right| &\geqslant C_2 2^{\max\{n,\ell\}}. \end{aligned} \tag{2.34}$$

In the following we assume the first inequality in (2.34). Otherwise the next estimates are done with the substitution $F(y) \mapsto y$. If $n = 0$ or $\ell = 0$ the following estimate is done analogously, using that either ξ or η is bounded. We set

$$\widetilde{\xi} := 2^{-n}\xi, \quad \widetilde{\eta} := 2^{-\ell}\eta, \quad \text{and} \quad U := \mathbb{R}^d \times \mathbb{R}^d \times K \times \mathbb{R}^d.$$

We write for every $x \in \mathbb{R}$

$$\begin{aligned} I(x) &:= I_{\sigma,n,\tau,\ell}(x) := \frac{(2\pi)^{2d}}{2^{(n+\ell)d}} \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi(x) \\ &= \frac{(2\pi)^{2d}}{2^{(n+\ell)d}} \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi(x) \\ &= \int_U e^{\mathrm{i}\, 2^n \widetilde{\xi}(x-y)} e^{\mathrm{i}\, 2^\ell \widetilde{\eta}(F(y)-z)} \Psi_{\sigma,1}\left(\widetilde{\xi}\right) \widetilde{\Psi}^{\circ}_{\tau,1}(\widetilde{\eta})\, f(y)\, \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi(z)\, \mathrm{d}\, z\, \mathrm{d}\, y\, \mathrm{d}\, \widetilde{\xi}\, \mathrm{d}\, \widetilde{\eta}. \end{aligned}$$

Note that by assumption we have

$$\widetilde{r} > \max\{0, s, t, q\} - \min\left\{0, s', t', q'\right\} \geqslant 0. \tag{2.35}$$

Integrating $\widetilde{r}$-times by parts (see Lemma B.3-Lemma B.5) in y, using the lower bound in (2.33), we arrive at

$$I(x) = \int_U e^{\mathrm{i}\, 2^n \widetilde{\xi}(x-y)} e^{\mathrm{i}\, 2^\ell \widetilde{\eta}(F(y)-z)} \Psi_{\sigma,1}\left(\widetilde{\xi}\right) \widetilde{\Psi}^{\circ}_{\tau,1}(\widetilde{\eta})\, \frac{f_{\widetilde{r}}(y)}{2^{\max\{n,\ell\}\widetilde{r}}} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi(z)\, \mathrm{d}\, z\, \mathrm{d}\, y\, \mathrm{d}\, \widetilde{\xi}\, \mathrm{d}\, \widetilde{\eta},$$

where all derivatives of $f_{\widetilde{r}}(y)$ with respect to $\widetilde{\eta}$ and $\widetilde{\xi}$ are bounded uniformly for all $\left(\widetilde{\xi}, \widetilde{\eta}, y\right) \in \operatorname{supp} \Psi_{\sigma,1} \times \operatorname{supp} \widetilde{\Psi}^{\circ}_{\tau,1} \times K$. We set for every $y \in \mathbb{R}^d$ and for every $n \geqslant 0$

$$u(y) := \begin{cases} 1, & |y| \leqslant 1 \\ |y|^{-d-1}, & \text{otherwise} \end{cases}, \quad u_n := u \circ 2^n.$$

If $|x-y|\, 2^n > 1$ we integrate $(d+1)$-times by parts in $\widetilde{\xi}$ and if $|z - F(y)|\, 2^\ell > 1$ we integrate $(d+1)$-times by parts in $\widetilde{\eta}$. Hence we arrive at

$$I(x) = 2^{-\max\{n,\ell\}\widetilde{r}} \int_U \widetilde{f}_{\widetilde{r}}\left(\widetilde{\xi}, \widetilde{\eta}, y\right) u_n(x-y)\, u_\ell(z - F(y))\, \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi(z)\, \mathrm{d}\, z\, \mathrm{d}\, y\, \mathrm{d}\, \widetilde{\xi}\, \mathrm{d}\, \widetilde{\eta},$$

where $\widetilde{f}_{\widetilde{r}}\left(\widetilde{\xi}, \widetilde{\eta}, y\right)$ is uniformly bounded for all $\left(\widetilde{\xi}, \widetilde{\eta}, y\right) \in \operatorname{supp} \Psi_{\sigma,1} \times \operatorname{supp} \widetilde{\Psi}^{\circ}_{\tau,1} \times K$. Hence we estimate for some constant $C_3 > 0$

$$|I(x)| \leqslant C_3 2^{-\max\{n,\ell\}\widetilde{r}} u_n * (u_\ell \circ F) * \left|\Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi\right|(x). \tag{2.36}$$

We estimate for every $\sigma, \tau \in \{-, +, 0\}$ and every $n, \ell \geqslant 1$, using the equality in (2.35) and assuming $\epsilon > 0$ small enough,

$$2^{(c(\sigma)+\epsilon)n - c'(\tau)\ell - \max\{n,\ell\}\widetilde{r}} \leqslant 2^{(\max\{s,t,q\}+\epsilon)n - \min\{s',t',q'\}\ell - \max\{n,\ell\}\widetilde{r}} \leqslant 2^{-\epsilon\ell}. \tag{2.37}$$

Hence we bound, using the estimates in (2.32), (2.33), (2.36), two times Young's

inequality and the bound in (2.37), for some constants $C_4, \dots, C_6 > 0$

$$\begin{aligned}
\|\mathcal{M}_{F,f}\varphi\|_{W^{s,t,q}_{p,\Theta,K}} &\leqslant C_1 \sup_{\sigma,n} \sum_{\tau,\ell} 2^{(c(\sigma)+\epsilon)n - c'(\tau)\ell} 2^{c'(\tau)\ell} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} \mathcal{M}_{F,f} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right\|_{L_p} \\
&= C_1 (2\pi)^{-2d} 2^{d(n+\ell)} \sup_{\sigma,n} \sum_{\tau,\ell} 2^{(c(\sigma)+\epsilon)n - c'(\tau)\ell} 2^{c'(\tau)\ell} \left\| I_{\sigma,n,\tau,\ell} \right\|_{L_p} \\
&\leqslant C_4 \sup_{\sigma,n} \sum_{\tau,\ell} 2^{(c(\sigma)+\epsilon)n - c'(\tau)\ell - \max\{n,\ell\}\tilde{r}} 2^{(n+\ell)d} 2^{c'(\tau)\ell} \left\| u_n * (u_\ell \circ F) * \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right\|_{L_p} \\
&\leqslant C_5 \sum_{\tau,\ell} 2^{-\ell\epsilon} 2^{c'(\tau)\ell} \left\| \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right\|_{L_p} \leqslant C_6 \sup_{\tau,\ell} 2^{c'(\tau)\ell} \left\| \Psi^{\circ\,\mathrm{Op}}_{\tau,\ell} \varphi \right\|_{L_p} .
\end{aligned}$$

To see the statement if $F = \mathrm{id}$ we estimate the corresponding cases $c'(\tau) < c(\sigma)$ if $\sigma \neq \tau$ and $n \sim \ell$ and $n, \ell \neq 0$ in a different way. We use

$$\widetilde{I}(x) := \int_U e^{\mathrm{i}\, 2^n \tilde{\xi}(x-y)} e^{\mathrm{i}\, 2^\ell \tilde{\eta}(y-z)} \widetilde{\Psi}_{\sigma,1}\left(\widetilde{\xi}\right) \Psi_{\sigma,1}\left(\widetilde{\xi}\right) \widetilde{\Psi}^{\circ}_{\tau,1}(\tilde{\eta})\, f(y)\, \Psi^{\mathrm{Op}}_{\tau,\ell}\varphi(z)\, \mathrm{d}\, z\, \mathrm{d}\, y\, \mathrm{d}\, \widetilde{\xi}\, \mathrm{d}\, \tilde{\eta}.$$

We express $\Psi_{\sigma,1}\left(\widetilde{\xi}\right)$, using the identity

$$\Psi_{\sigma,1}\left(\widetilde{\xi}\right) - \Psi_{\sigma,1}(\tilde{\eta}) = \int_0^1 (\mathrm{D}\,\Psi_{\sigma,1})\left(\widetilde{\xi} + (1-h)\left(\tilde{\eta} - \widetilde{\xi}\right)\right) \mathrm{d}\, h \left(\widetilde{\xi} - \tilde{\eta}\right).$$

We repeat this k-times in the right-hand side of this identity, replacing $\widetilde{\xi}$ and yielding in total $k+1$ terms. The first k terms are linear combinations of

$$\Psi_{\sigma,1}\left(j\tilde{\eta} - (j-1)\widetilde{\xi}\right),$$

where $1 \leqslant j \leqslant k+1$. If $j = 0$ then this is just $\Psi_{\sigma,1}(\tilde{\eta})$. The corresponding part in $\widetilde{I}(x)$ is hence

$$\widetilde{I}_1(x) = \widetilde{\Psi}^{\mathrm{Op}}_{\sigma,n}\left(f \cdot \Psi^{\mathrm{Op}}_{\sigma,n} \Psi^{\mathrm{Op}}_{\tau,\ell}\varphi\right) = \widetilde{\Psi}^{\mathrm{Op}}_{\sigma,n}\left(f \cdot \Psi^{\mathrm{Op}}_{\sigma,n}\left(1 - \sum_{\sigma' \neq \tau} \Psi^{\mathrm{Op}}_{\sigma',\ell}\right)\varphi\right).$$

Note that $\widetilde{\Psi}_{\sigma,n}$ and $\Psi_{\sigma,n}$ satisfy the vanishing conditions in Lemma 2.3.1 as seen as an operator $\ell^c_2 \mapsto \ell^c_2$. Then we bound with some constant $C_5 = C_5(f)$

$$\left\| \widetilde{I}_1 \right\|_{L_p} \leqslant C_5 \left\| \left(\sum_{n=0}^{\infty} \left| 4^{\sigma n} \Psi^{\mathrm{Op}}_{\sigma,n} \varphi \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p},$$

using two times Lemma 2.3.1 and that $c(-) \leqslant c(0) \leqslant c(+)$ and that $\Psi^{\mathrm{Op}}_{+,\ell} \Psi^{\mathrm{Op}}_{-,n} \equiv$

0. The terms where $j > 0$ are dealt with, using first the substitution

$$j\widetilde{\eta} - (j-1)\,\widetilde{\xi} \mapsto \widetilde{\eta},$$

and then $\widetilde{r}$-times integration by parts analogous as before. The last $k+1$ term is

$$\widetilde{\Psi}_k\left(\widetilde{\eta}\right) := \int_{[0,1]^k} \left(\mathrm{D}^k\,\Psi_{\sigma,1}\right)\left(\widetilde{\xi} + \sum_{j=1}^{k}\left(1-t_j\right)\left(\widetilde{\eta} - \widetilde{\xi}\right)\right) \mathrm{d}\,t\left(\widetilde{\xi} - \widetilde{\eta}\right)^{\otimes k}.$$

We split now according to the size $\left|\widetilde{\eta} - \widetilde{\xi}\right|$. We let $\epsilon > 0$. We note that

$$2^{-n\epsilon((d+1)-k)}\chi\left(\left|\widetilde{\eta} - \widetilde{\xi}\right| 2^{n\epsilon}\right)\widetilde{\Psi}_k\left(\widetilde{\eta}\right)$$

satisfies the vanishing conditions in Lemma 2.3.1 uniformly in $\widetilde{\xi}$ as seen as an operator $\ell_2^c \mapsto \ell_2^c$ in $\widetilde{\eta}$. We bound the L_p norm of the corresponding part analogous as in the case $\widetilde{I}_1$. This is bounded appropriately with the choice of k below. On the range $\left(1 - \chi\left(\left|\widetilde{\eta} - \widetilde{\xi}\right| 2^{n\epsilon}\right)\right) > 0$ we integrate $\widetilde{r}$-times by parts in y and then $(d+1)$-times in $\widetilde{\xi}$ and $\widetilde{\eta}$ in the corresponding part of $\widetilde{I}\left(x\right)$. The terms which depend on χ are treated as in the range $\chi\left(\left|\widetilde{\eta} - \widetilde{\xi}\right| 2^{n\epsilon}\right) > 0$. In the remaining part we gained a factor $\sim 2^{(-n+\epsilon)\widetilde{r} + n\epsilon(d+1)}$. We choose ϵ small compatible with the inequality given in (2.35) and then k large enough such that

$$2^{(c(\sigma)-c(\tau))n} \leqslant 2^{-n\epsilon((2d+2)-k)}.$$

□

Lemma 2.3.4 (Continuity and compactness)**.** *Let $p \in [1,\infty]$, let $s' \leqslant s$, $q' \leqslant q$, $t' \leqslant t$, and $s \leqslant q \leqslant t$ and let Θ, Θ° be two cone ensembles, recalling Definition 2.2.1. Suppose the compact inclusions*

$$\operatorname{supp}\Phi_0^\circ \Subset \operatorname{supp}\Phi_0 \cup \operatorname{supp}\Phi_+ \quad \text{and} \quad \operatorname{supp}\Phi_+^\circ \Subset \operatorname{supp}\Phi_+. \tag{2.38}$$

Then the inclusion

$$W_{p,\Theta,K}^{s,t,q} \subseteq W_{p,\Theta^\circ,K}^{s',t',q'}$$

is continuous for every open subset $K \subset \mathbb{R}^d$ with compact closure. Moreover, if $s' < s, t' < t$ and $q' < q$ then the inclusion

$$W_{p,\Theta,K}^{s,t,q} \subseteq W_{p,\Theta,K}^{s',t',q'}$$

is compact.

Proof. We prove first the claim on the continuous inclusion. We set for all $n \in \mathbb{Z}_{\geqslant 0}$

$$\begin{aligned} F_{-,n} &:= 2^{(s'-s)n} \left(\Psi^{\circ}_{-,n} \Big/ \left(2^{(q-s)n} \Psi_{0,n} + 2^{(t-s)n} \Psi_{+,n} + \Psi_{-,n} \right) \right), \\ F_{+,n} &:= 2^{(t'-t)n} \left(\Psi^{\circ}_{+,n} / \Psi_{+,n} \right), \\ F_{0,n} &:= 2^{(q'-q)n} \left(\Psi^{\circ}_{0,n} \Big/ \left(2^{(t-q)n} \Psi_{+,n} + \Psi_{0,n} \right) \right). \end{aligned}$$

We define a map Q on the Hilbert space ℓ^c_2 (with norm as given in (2.30)) by setting for all $\sigma \in \{-,+,0\}$, $n \in \mathbb{Z}_{\geqslant 0}$ and all $a \in \ell^c_2$

$$(Qa)_{\sigma,n} := F_{\sigma,n} a_{\sigma,n}.$$

In Lemma 2.3.1 we take $\mathcal{B}_1 = \mathcal{B}_2 = \ell^c_2$. It follows from the definition of $\Psi_{\sigma,n}$ in (2.23), the compact inclusion assumptions in (2.38) and the assumptions on s, t, q, s', t', q' that Q satisfies the decay conditions on Q in Lemma 2.3.1. It follows that the corresponding operator Q^{Op} in (2.27) is bounded. Let $\varphi \in W^{s,t,q}_{p,\Theta,K}$. We set for all $n \in \mathbb{N}_{\geqslant 0}$

$$\begin{aligned} b_{-,n} &:= 2^{sn} \mathbb{F}^{-1} \left(2^{(q-s)n} \Psi_{0,n} + 2^{(t-s)n} \Psi_{+,n} + \Psi_{-,n} \right) * \varphi, \\ b_{+,n} &:= 2^{tn} \mathbb{F}^{-1} \Psi_{+,n} * \varphi, \\ b_{0,n} &:= 2^{qn} \mathbb{F}^{-1} \left(2^{(t-q)n} \Psi_{+,n} + \Psi_{0,n} \right) * \varphi. \end{aligned}$$

Then $(b_{\sigma,n} \mid \sigma \in \{-,+,0\}, n \in \mathbb{N}_{\geqslant 0}) =: b \in L_p\left(\mathbb{R}^d, \ell^c_2\right)$ by assumption on φ and in particular it holds, for some constant $C \geqslant 1$, $\|b\|_{L_p\left(\mathbb{R}^d, \ell^c_2\right)} \leqslant C \|\varphi\|_{W^{s,t,q}_{p,\Theta,K}}$. We estimate, using Lemma 2.3.1, and conclude

$$\|\varphi\|_{W^{s',t',q'}_{p,\Theta^\circ,K}} = \left\| \left\| Q^{\mathrm{Op}} b \right\|_{\ell^c_2} \right\|_{L_p} \leqslant C \|b\|_{L_p\left(\mathbb{R}^d, \ell^c_2\right)}.$$

We show the claim on the compact inclusion. We let $U \subset W^{s,t,q}_{p,\Theta,K}$ be a bounded set in $W^{s,t,q}_{p,\Theta,K}$ with bound $R > 0$. We set c' with respect to s', t', q' analogous to c. It is enough to find for each $\epsilon > 0$ an open cover of U in $W^{s',t',q'}_{p,\Theta,K}$ where each open set in the cover has size $\sim \epsilon$. (This yields total boundedness of U in $W^{s',t',q'}_{p,\Theta,K}$ and hence compactness.) Now there is $\delta > 0$ such that for all $\sigma \in \{-,+,0\}$

$$c'(\sigma) + \delta - c(\sigma) < 0. \tag{2.39}$$

For all $\varphi \in U$ and all $N \in \mathbb{N}$ we bound

$$\begin{aligned}\left\| \sqrt{\sum_{\sigma,n\geqslant N} 4^{-\delta n} \left| 2^{(c'(\sigma)+\delta)n} \Psi^{\mathrm{Op}}_{\sigma,n}\varphi \right|^2} \right\|_{L_p} &\leqslant C \sup_{\sigma,n\geqslant N} 2^{(c'(\sigma)+\delta)n} \left\| \Psi^{\mathrm{Op}}_{\sigma,n}\varphi \right\|_{L_p} \\ &\leqslant C 2^{(c'(\sigma)+\delta-c(\sigma))N} R, \end{aligned} \tag{2.40}$$

for some $\sigma \in \{-,+,0\}$. Recalling the bound in (2.39), we make the bound in (2.40) smaller than ϵ by taking $N = N(\epsilon, R)$ large enough. Suppose now that the embedding is not compact. Then there are infinitely many $\varphi_m \in U$, $m \in \mathbb{N}$, such that for all $m_1 > m_2$ it holds

$$\left\| \varphi_{m_1} - \varphi_{m_2} \right\|_{W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} > \epsilon. \tag{2.41}$$

Recalling the bound in (2.40), it holds for some $n < N$ and some $\sigma \in \{-,+,0\}$

$$\left\| \varphi_{m_1} - \varphi_{m_2} \right\|_{W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} \leqslant C 2^{(c'(\sigma)+\delta)n} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} \left(\varphi_{m_1} - \varphi_{m_2} \right) \right\|_{L_p}. \tag{2.42}$$

Since $C_0^{r-1}(K)$ is dense in $W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$ we may assume $\varphi_m \in C_0^{r-1}(K)$. We set $S := \cup_{\sigma,n<N} \operatorname{supp} \Psi_{\sigma,n}$. Since all φ_m are uniformly bounded in $W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$-norm and $\operatorname{supp} \varphi_m$ is uniformly bounded in m as well, the Fourier transform of φ_m cannot diverge on a dense subset of S as $m \to \infty$ (this would violate the Paley–Wiener Theorem [39, Theorem 1.7.7]). By passing to a subsequence in m we may split $S = S_1 \sqcup S_2$ such that the family $\left\{ \mathbb{F}\varphi_{m|S_1} \mid m \in \mathbb{N} \right\}$ is uniformly bounded. Then, using again that φ_m has compact support with maximal diameter independent of m, the family $\left\{ \mathbb{F}\varphi_{m|S_1} \mid m \in \mathbb{N} \right\}$ is also uniformly equicontinuous. Hence by the Arzelà–Ascoli Theorem there is a subsequence in m such that $\varphi_{m|S_1}$ is a Cauchy sequence in C^0. Repeating the argument inductively for the part $\mathbb{F}\varphi_{m|S_2}$, then using a diagonal argument, we find a subsequence in m such that $\mathbb{F}\varphi_{m|S}$ is a Cauchy sequence in C^0. Hence the right-hand side in (2.42) can be made arbitrary small which contradicts the lower bound in (2.41) and we conclude. □

Lemma 2.3.5 (Local derivative)**.** *Let $p \in [1,\infty]$, $s,t,q < r-1$ and let $\varphi \in W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$. It holds for some constant $C > 0$, for every $1 \leqslant j \leqslant d$, for every $\sigma \in \{-,+,0\}$ such that*

$$\xi_j \neq 0 \quad \textit{if} \quad (\xi_1,\dots,\xi_d) = \xi \in \operatorname{supp} \Psi_{1,\sigma},$$

and for every $\widetilde{r} \in \mathbb{R}$

$$\left\| \left(\sum_{n=0}^{\infty} 4^{\widetilde{r}n} \left| \Psi_{\sigma,n}^{\mathrm{Op}} \varphi \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \leqslant C \left\| \left(\sum_{n=0}^{\infty} 4^{(\widetilde{r}-1)n} \left| \Psi_{\sigma,n}^{\mathrm{Op}} \partial_{x_j} \varphi \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} . \tag{2.43}$$

Proof. Using the triangle inequality, it is enough to consider only the terms with $n > 0$. For every $\xi \in \mathbb{R}^d \setminus \{0\}$ and $b \in \mathbb{C}$ we put

$$\left(\mathcal{D}(\xi) b\right)_n := \mathrm{i} \, \frac{\xi_j}{2^n} \Psi_{\sigma,n}(\xi) b, \quad n \in \mathbb{N}.$$

We note

$$\Psi_{\sigma,n}^{\mathrm{Op}} \left(\partial_{x_j} \varphi \right) = \left(\mathbb{F}^{-1} \Psi_{\sigma,n} \right) * \partial_{x_j} \varphi = \left(\partial_{x_j} \mathbb{F}^{-1} \Psi_{\sigma,n} \right) * \varphi = 2^n \left(\mathcal{D}^{\mathrm{Op}} \varphi \right)_n .$$

We let ℓ_2 be the space of complex valued sequences space over $\mathbb{N}$. As norm we set $\|a\|_{\ell_2} := \sqrt{\sum_{n=1}^{\infty} 4^{\widetilde{r}n} |a_n|^2}$. For every $\xi \in \mathbb{R}^d \setminus \{0\}$, every $a \in \ell_2$ and every $n \in \mathbb{N}$ we put

$$\left(Q(\xi) a\right)_n := -\mathrm{i} \, \frac{2^n}{\xi_j} \widetilde{\Psi}_{\sigma,n}(\xi) a_n,$$

where $\widetilde{\Psi}_{\sigma,n}$ is defined in (2.24). Note that $\left(Q^{\mathrm{Op}} \mathcal{D}^{\mathrm{Op}} \varphi \right)_n = \Psi_{\sigma,n}^{\mathrm{Op}} \varphi$. Moreover since $\xi_j \neq 0$ by assumption, the map Q satisfies the decay condition on its derivatives as required in Lemma 2.3.1. Hence, using Lemma 2.3.1 with $\mathcal{B}_1 = \mathcal{B}_2 = \ell_2$, the map $Q^{\mathrm{Op}} \colon L_p\left(\mathbb{R}^d, \ell_2\right) \to L_p\left(\mathbb{R}^d, \ell_2\right)$ is a bounded linear operator. We conclude, using the estimate for some constant $C > 0$

$$\left\| \left\| Q^{\mathrm{Op}} \mathcal{D}^{\mathrm{Op}} \varphi \right\|_{\ell_2} \right\|_{L_p} \leqslant C \left\| \left\| \mathcal{D}^{\mathrm{Op}} \varphi \right\|_{\ell_2} \right\|_{L_p} .$$

□

We recall the open cover $V_\omega \subseteq M$ and the chart maps $\kappa_\omega \in \mathcal{A}$, $\omega \in \Omega$, introduced in Section 2.2. Also we recall the vector space $C_X^{r-1}(M)$ from the beginning of Section 2.3.2.

Definition 2.3.6 (Anisotropic Banach space)**.** *Let* $\vartheta_\omega \colon V_\omega \to [0,1]$ *be a* C^r *partition of unity adapted to the chart maps* κ_ω *and let* Θ_ω *be hyperbolic cone ensembles, recalling Definition 2.2.2, where* $\omega \in \Omega$*. Let* $p \in [1, \infty]$*, let* $s, q, t < r - 1$ *and let* $\alpha_0 > 0$*. We put for every* $\varphi \in C_X^{r-1}(M)$ *and every* $p \in [1, \infty]$

$$\|\varphi\|_{W_p^{s,t,q}} := \left(\sum_{\omega \in \Omega} \int_0^{\alpha_0} \left\| \left(\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha,\phi_\alpha} \varphi \right) \circ \kappa_\omega^{-1} \right) \right\|_{W_{p,\Theta_\omega,\kappa_\omega(V_\omega)}^{s,t,q}}^2 \mathrm{d}\, \alpha \right)^{\frac{1}{2}}, \tag{2.44}$$

We denote by $W_p^{s,t,q}$ *the completion of* $C_X^{r-1}(M)$ *under this norm.*

Remark 2.3.7. *Note that* $W_p^{s,t,q}$ *depends on the dynamics,* α_0*, the atlas* $\mathcal{A}$ *and the cone ensembles* Θ_ω*,* $\omega \in \Omega$*. We understand each* $\vartheta_\omega \circ \kappa_\omega^{-1}$ *in (2.44) as extended to* $\mathbb{R}^d$ *by zero. By Lemma 2.3.3 a* C^r *change of the atlas and hence a change of the cone ensemble yields an equivalent norm if* $s \leqslant q \leqslant t < r-1$*. The integration with respect to* α *is a way to "project out" the small times where the flow is not sufficiently hyperbolic. This is similar to [27, Definition 8.1] and also Baladi–Liverani [9, p.705, (3.2)] with the supremum replaced by an integral in the latter case. In turn, for* $p = 2$ *the space* $W_2^{s,t,q}$ *is a Hilbert space because the parallelogram law*

$$\|\varphi_1 + \varphi_2\|^2_{W_p^{s,t,q}} + \|\varphi_1 - \varphi_2\|^2_{W_p^{s,t,q}} = 2\|\varphi_1\|^2_{W_p^{s,t,q}} + 2\|\varphi_2\|^2_{W_p^{s,t,q}}$$

holds [14, Proposition 15.2].

The compact inclusion of the local Banach space in Lemma 2.3.4 carries over to the anisotropic Banach space $W_p^{s,t,q}$.

Lemma 2.3.8 (Compactness)**.** *Let* $p \in [1,\infty]$*, let* $s \leqslant q \leqslant t$ *such that* $\max\{0,t\} - \min\{0,s\} < r-1$ *and let* $s' < s, t' < t$ *and* $q' < q$ *such that* $\max\{0, \min\{s',t',q'\}\} - \min\{0,s',t',q'\} < r-1$*. Then there exist cone ensembles* Θ_ω*,* $\omega \in \Omega$*, such that the inclusion*

$$W_p^{s,t,q} \subseteq W_p^{s',t',q'}$$

is compact.

Proof. We let $s' < s, t' < t, q' < q$. Let $U \subseteq W_p^{s,t,q}$ be a bounded set in the norm of $W_p^{s,t,q}$. In order to show the compact inclusion we proceed analogous to the proof in Lemma 2.3.4. To this end we let $\varphi_m \in U$, $m \in \mathbb{N}$, be a sequence, satisfying the analog bound in (2.41). Suppose now that there has to be some fixed $\omega \in \Omega$ and some fixed $\alpha \geqslant 0$ such that there exists $C_1 > 0$ such that for all $m > 0$

$$\sum_{\omega \in \Omega} \left\| \left(\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha,\phi_\alpha} \varphi_m \circ \kappa_\omega^{-1}\right)\right) \right\|_{W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} \leqslant C_1.$$

and that there exists some $\epsilon > 0$ such that for all $m_1 > m_2$ (up to some subsequence)

$$\left\| \vartheta_\omega \cdot \left(\mathcal{L}_{\alpha,\phi_\alpha}\left(\varphi_{m_1} - \varphi_{m_2}\right)\right) \circ \kappa_\omega^{-1} \right\|_{W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} > \epsilon. \tag{2.45}$$

Since $\left(\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha,\phi_\alpha}\varphi_m\right) \circ \kappa_\omega^{-1}\right) \in W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$ we find a Cauchy subsequence,

using the statement on the compact inclusion in Lemma 2.3.4. Note that by the Mean Value Theorem there exist non-fixed $\alpha = \alpha(m)$ and $\alpha = \alpha(m_1, m_2)$ which satisfy these inequalities. In particular, we wish to find a Cauchy subsequence for the left-hand side in the inequality (2.45) for the choice $\alpha = \alpha(m_1, m_2)$. Suppose $0 \leqslant \alpha' \leqslant \alpha_0$. We have

$$\begin{aligned}&\left\|\vartheta_\omega \cdot \left(\mathcal{L}_{2\alpha_0,\phi_{2\alpha_0}}\varphi_m\right) \circ \kappa_\omega^{-1}\right\|_{W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} \leqslant \\ &\sum_{\omega'\in\Omega}\left\|\vartheta_\omega \cdot \left(\mathcal{L}_{2\alpha_0-\alpha',\phi_{2\alpha_0-\alpha'}}\left(\vartheta_{\omega'}\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi_m\right)\right) \circ \kappa_\omega^{-1}\right\|_{W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}}.\end{aligned}$$

By Lemma 2.2.3 there exists cone ensembles $(\Theta_\omega, \Theta_\omega^\circ)$, $\omega \in \Omega$, satisfying the condition (2.38) in Lemma 2.3.4 such that the local diffeomorphism of $g_{-\alpha}$ with $\alpha \geqslant \alpha_0$ is cone hyperbolic. Then, using Lemma 2.3.3 and $s \leqslant q \leqslant t$, $\max\{0,t\} - \min\{0,s\} < r-1$ and taking $\alpha' = \alpha(m)$, we bound this sequence in m uniformly from above. Let $s'' = \min\{s',t',q'\}$. Then, using Lemma 2.3.3, recalling that it holds $\max\{0,\min\{s',t',q'\}\} - \min\{0,s',t',q'\} < r-1$, we find (abusing the notation $\mathcal{L}_{\alpha,\phi_\alpha}$ with negative α)

$$\begin{aligned}&\left\|\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha',\phi_{\alpha'}}\left(\varphi_{m_1}-\varphi_{m_2}\right)\right) \circ \kappa_\omega^{-1}\right\|_{W^{s'',s'',s''}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}} \leqslant \qquad (2.46)\\ &\sum_{\omega'\in\Omega}\left\|\left(\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha'-2\alpha_0,\phi_{\alpha'-2\alpha_0}}\left(\vartheta_{\omega'}\cdot\mathcal{L}_{2\alpha_0,\phi_{2\alpha_0}}\left(\varphi_{m_1}-\varphi_{m_2}\right)\right)\right) \circ \kappa_\omega^{-1}\right)\right\|_{W^{s'',s'',s''}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}}\\ &\leqslant C_2\sum_{\omega\in\Omega}\left\|\vartheta_\omega \cdot \left(\mathcal{L}_{2\alpha_0,\phi_{2\alpha_0}}\left(\varphi_{m_1}-\varphi_{m_2}\right)\right) \circ \kappa_\omega^{-1}\right\|_{W^{s',t',q'}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}}.\end{aligned}$$

for some constant $C_2 > 0$ independent of the choice of α' and of m_1, m_2. Now we take $\alpha' = \alpha(m_1, m_2)$ and let the right-hand side vanish in (2.46) as $(m_1, m_2) \to \infty$. Then from the left-hand side for all σ, n

$$\left\|\Psi^{\mathrm{Op}}_{\sigma,n}\left(\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha(m_1,m_2),\phi_{\alpha(m_1,m_2)}}\left(\varphi_{m_1}-\varphi_{m_2}\right)\right) \circ \kappa_\omega^{-1}\right)\right\|_{L_p} \to 0.$$

By uniform boundedness in $W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$ (analogous to the proof of Lemma 2.3.4), the lower bound in (2.45) comes from a finite number of terms

$$\left\|\Psi^{\mathrm{Op}}_{\sigma,n}\left(\vartheta_\omega \cdot \left(\mathcal{L}_{\alpha(m_1,m_2),\phi_{\alpha(m_1,m_2)}}\left(\varphi_{m_1}-\varphi_{m_2}\right)\right) \circ \kappa_\omega^{-1}\right)\right\|_{L_p}$$

with $n \leqslant C_3 = C_3(\epsilon)$. Hence we found a Cauchy subsequence for the left-hand side in the inequality (2.45) for the choice $\alpha = \alpha(m_1, m_2)$. □

2.4 Properties of the transfer operator, the generator and its resolvent

2.4.1 Bounds on the transfer operator

We introduce a local transfer operator in (2.47) below and state a local norm estimate for this operator in Lemma 2.4.1. We then give a norm estimate for the transfer operator family (2.18) in Lemma 2.4.2, making use of Lemma 2.4.1. Let $K \subset \mathbb{R}^d$ be an open set. Let $f\colon \mathbb{R}^d \to \mathbb{C}$ be a $C_0^{r-1}(K)$-map and let $F\colon K \to F(K)$ be a (Θ°, Θ)-cone hyperbolic C^r-diffeomorphism on K (recall Definition 2.2.2). The f-weighted local transfer operator is defined by

$$L_{F,f}\colon C_0^{r-1}(F(K)) \to C_0^{r-1}(K) : \varphi \mapsto f \cdot \varphi \circ F. \tag{2.47}$$

Recalling $\widetilde{\Phi}_-^\circ, \widetilde{\Phi}_+^\circ, \widetilde{\Phi}_0^\circ$ from (2.24), we put for every subset $I \subseteq K$

$$\|F\|_{-,I} := \inf_{\substack{y\in I \\ 0\neq\eta\in\operatorname{supp}\widetilde{\Phi}_-^\circ}} \frac{\left|(\mathrm{D}_y F)^{\mathrm{tr}}\,\eta\right|}{|\eta|}, \quad \|F\|_{+,I} := \sup_{\substack{y\in I \\ 0\neq\eta\in\operatorname{supp}\widetilde{\Phi}_+^\circ}} \frac{\left|(\mathrm{D}_y F)^{\mathrm{tr}}\,\eta\right|}{|\eta|},$$

$$\|F\|_{0,I} := \sup_{\substack{y\in I \\ 0\neq\eta\in\operatorname{supp}\widetilde{\Phi}_0^\circ}} \frac{\left|(\mathrm{D}_y F)^{\mathrm{tr}}\,\eta\right|}{|\eta|}.$$

Lemma 2.4.1 (Upper bound for local transfer operator)**.** *Let $\{\mathcal{W}\}$ denote the connected components of* $\operatorname{supp} f$*. Let $p \in [1,\infty]$. Let*

$$s' < s < 0 < q \leqslant t < r - 1 + s', \quad q' < q, \quad t' < t.$$

Then for every $\varphi \in W^{s,t,q}_{p,\Theta,F(K)}$ it holds

$$\|L_{F,f}\varphi\|_{W^{s,t,q}_{p,\Theta,K}} \leqslant C_0 \|\varphi\|_{W^{s',t',q'}_{p,\Theta^\circ,F(K)}} + C_1 \|\varphi\|_{W^{s',t',q}_{p,\Theta^\circ,F(K)}} + C_2 \|\varphi\|_{W^{s,t,q}_{p,\Theta^\circ,F(K)}},$$

where, for some constants $C > 0$ and $k > 0$, it holds
$C_0 \leqslant C \sum_{\mathcal{W}} \max\left\{1, \|F\|_{-,\mathcal{W}}^{1-r}, \|F\|_{0,\mathcal{W}}^{1-r}\right\} \|\mathrm{D}\,F\|^k_{C^{r-1}(\mathcal{W})} \|f\|_{C^{r-1}(\mathcal{W})} \left\||\det \mathrm{D}\,F|^{-\frac{1}{p}}\right\|_{L_\infty(\mathcal{W})}$,
$C_1 \leqslant C \sup_{\mathcal{W}} \left\|f\,|\det \mathrm{D}\,F|^{-\frac{1}{p}}\right\|_{L_\infty(\mathcal{W})} \max\left\{1, \|F\|^q_{0,\mathcal{W}}\right\}$ *and*
$C_2 \leqslant C \sup_{\mathcal{W}} \left\|f\,|\det \mathrm{D}\,F|^{-\frac{1}{p}}\right\|_{L_\infty(\mathcal{W})} \max\left\{\|F\|^t_{+,\mathcal{W}}, \|F\|^s_{-,\mathcal{W}}\right\}$.

Lemma 2.4.1 is proven in Section 2.4.4. For every $s, t, \alpha \in \mathbb{R}$ and every $x \in M$

we set

$$\lambda^{(t,s,\alpha)}(x) := \max\left\{ \left\| (\mathrm{D}\, g_\alpha)^{\mathrm{tr}}_{|E^*_{+,g_{-\alpha}(x)}} \right\|^{-t}, \left\| (\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}_{|E^*_{-,x}} \right\|^{s} \right\}. \tag{2.48}$$

For $s < 0 < t$ this quantity decreases exponentially fast to 0 as $\alpha \to \infty$, which is a consequence of the Anosov property given in (2.3) of the flow g_α.

Lemma 2.4.2 (Bound on the transfer operator)**.** *Let $p \in [1, \infty]$. Let*

$$s' < s < 0 < q \leqslant t < r - 1 + s' \quad \textit{and} \quad t' < t.$$

There exist $\alpha_0 > 0$, cone ensembles Θ_ω, $\omega \in \Omega$, and constants $A > 0$ and $C > 0$, such that for all $\varphi \in W_p^{s,t,q}$ with $\|\varphi\|_{W_p^{s,t,q}} = 1$ and all $\alpha \geqslant 0$ it holds

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|_{W_p^{s,t,q}} \leqslant Ce^{A\alpha}\|\varphi\|_{W_p^{s',t',q}} + C(\alpha+1)\left\| \phi_\alpha \left|\det \mathrm{D}\, g_{-\alpha}\right|^{-\frac{1}{p}} \cdot \lambda^{(t,s,\alpha)} \right\|_{L_\infty}.$$

Proof. We recall the map $F_{-\alpha,\omega\omega'}$ and the set $V_{\alpha,\omega\omega'}$ defined in Lemma 2.2.3 for all $\alpha \geqslant 0$ and all $\omega', \omega \in \Omega$. By Lemma 2.2.3 there exist cone ensembles $\Theta^\circ_{\omega'}, \Theta_\omega$ such that the map $F_{-\alpha,\omega\omega'}$ is $\left(\Theta^\circ_{\omega'}, \Theta_\omega\right)$-cone hyperbolic. We recall the partition of unity ϑ_ω (see Definition 2.3.6). We let

$$V_{\alpha,\omega\omega'} \subseteq \widetilde{V}_{\alpha,\omega\omega'} \subseteq V_\omega$$

such that $F_{-\alpha,\omega\omega'}$ is also $\left(\Theta^\circ_{\omega'}, \Theta_\omega\right)$-cone hyperbolic on $\kappa_\omega\left(\widetilde{V}_{\alpha,\omega\omega'}\right)$. This is possible due to the compact inclusion of cones as required in the cone-hyperbolicity definition. We let

$$\vartheta_{\alpha,\omega\omega'} \colon \widetilde{V}_{\alpha,\omega\omega'} \to [0,1]$$

be a C_0^{r-1} map such that

$$\vartheta_{\alpha,\omega\omega'|V_{\alpha,\omega\omega'}} \equiv \vartheta_{\omega|V_{\alpha,\omega\omega'}}.$$

For all $z \in \kappa_\omega(V_\omega)$ we have

$$\vartheta_\omega \circ \kappa_\omega^{-1}(z) \cdot \vartheta_{\omega'} \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'}(z) = \vartheta_{\alpha,\omega\omega'} \circ \kappa_\omega^{-1}(z) \cdot \vartheta_{\omega'} \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'}(z).$$

Note that $\|\vartheta_{\alpha,\omega\omega'}\|_{C^{r-1}}$ is controlled by the rate of expansion of $F_{-\alpha,\omega\omega'}$. Let $\varphi \in W_p^{s,t,q}$ and put $W_\omega := W^{s,t,q}_{p,\Theta_\omega,\kappa_\omega(V_\omega)}$ and $W_{\omega\omega'} := W^{s,t,q}_{p,\Theta_\omega,\widetilde{V}_{\alpha,\omega\omega'}}$. For all

$\alpha \geqslant \alpha_0$, for some $C \geqslant 1$, we estimate for every $p \in [1, \infty]$

$$\begin{aligned}
\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|^2_{W_p^{s,t,q}} &\leqslant C \max_{\omega\in\Omega} \int_0^{\alpha_0} \left\|(\vartheta_\omega \cdot (\phi_{\alpha'} \cdot \mathcal{L}_{\alpha,\phi_\alpha}\varphi \circ g_{-\alpha'})) \circ \kappa_\omega^{-1}\right\|^2_{W_\omega} \mathrm{d}\,\alpha' \\
&= C \max_{\omega\in\Omega} \int_0^{\alpha_0} \left\|(\vartheta_\omega \cdot \phi_\alpha) \circ \kappa_\omega^{-1} \cdot \sum_{\omega'\in\Omega} \left(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right) \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'}\right\|^2_{W_\omega} \mathrm{d}\,\alpha' \\
&= C \max_{\omega\in\Omega} \int_0^{\alpha_0} \left\|\sum_{\omega'\in\Omega} \left(\vartheta_{\alpha,\omega\omega'} \cdot \phi_\alpha\right) \circ \kappa_\omega^{-1} \cdot \left(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right) \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'}\right\|^2_{W_\omega} \mathrm{d}\,\alpha' \\
&\leqslant C^2 \max_{\omega,\omega'\in\Omega} \int_0^{\alpha_0} \left\|\left(\vartheta_{\alpha,\omega\omega'} \cdot \phi_\alpha\right) \circ \kappa_\omega^{-1} \cdot \left(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right) \circ \kappa_{\omega'}^{-1} \circ F_{-\alpha,\omega\omega'}\right\|^2_{W_\omega} \mathrm{d}\,\alpha' \\
&= C^2 \max_{\omega,\omega'\in\Omega} \int_0^{\alpha_0} \left\|L_{F_{-\alpha,\omega\omega'},(\vartheta_{\alpha,\omega\omega'}\cdot\phi_\alpha)\circ\kappa_\omega^{-1}} \left(\left(\vartheta_{\omega'} \cdot \mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right) \circ \kappa_{\omega'}^{-1}\right)\right\|^2_{W_{\omega\omega'}} \mathrm{d}\,\alpha'.
\end{aligned} \tag{2.49}$$

We used in the last step the definition of the weighted local transfer operator (see (2.47)) in which we take $F := F_{-\alpha,\omega\omega'}$ and as the C_0^{r-1}-weight $f := \left(\vartheta_{\alpha,\omega\omega'} \cdot \phi_\alpha\right) \circ \kappa_\omega^{-1}$.

We now show the claimed upper bound for $\mathcal{L}_{\alpha,\phi_\alpha}$. We recall that

$$\operatorname{supp} f = \kappa_\omega\left(\widetilde{V}_{\alpha,\omega\omega'}\right) = \bigsqcup \kappa_\omega\left(\mathcal{W}\right),$$

where the disjoint union is over all the finitely many connected components $\mathcal{W}$ of $\widetilde{V}_{\alpha,\omega\omega'}$. The inclusion $W^{s,t,q}_{p,\Theta^\circ,\kappa_{\omega'}(V_{\omega'})} \subseteq W_{\omega'}$ is continuous by Lemma 2.3.4. Together with the bound given by Lemma 2.4.1 this yields the upper bound

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|_{W_p^{s,t,q}} \leqslant \widetilde{C}_1 \|\varphi\|_{W_p^{s',t',q}} + \widetilde{C}_2 \|\varphi\|_{W_p^{s,t,q}},$$

where

$$\widetilde{C}_1 \leqslant C^2 \max_{\omega,\omega'\in\Omega} C_0\left(F_{-\alpha,\omega\omega'}, f\right) + C_1\left(F_{-\alpha,\omega\omega'}, f\right), \quad \widetilde{C}_2 \leqslant C^2 \max_{\omega,\omega'\in\Omega} C_2\left(F_{-\alpha,\omega\omega'}, f\right),$$

and C_0, C_1, C_2 are the constants from Lemma 2.4.1. We claim for some constant $C_4 > 0$ the following bound

$$\begin{aligned}
\left\|\phi_\alpha \left|\det \mathrm{D}\, g_{-\alpha}\right|^{-\frac{1}{p}}\right\|_{L_\infty(\mathcal{W})} &\leqslant C_4 \inf_{x\in\mathcal{W}} \left|\phi_\alpha \left|\det \mathrm{D}\, g_{-\alpha}\right|^{-\frac{1}{p}}\right|(x) \\
&= C_4 \left\|\left(\phi_{-\alpha} \left|\det \mathrm{D}\, g_\alpha\right|^{-\frac{1}{p}}\right) \circ g_{-\alpha}\right\|_{L_\infty(\mathcal{W})}.
\end{aligned} \tag{2.50}$$

Due to the construction of $\widetilde{V}_{\alpha,\omega\omega'}$, all points in a connected component $\mathcal{W}$ stay close under iterates by $g_{\alpha'}$ for all $0 \leqslant \alpha' \leqslant \alpha$. Then in the case of hyperbolic maps

the bound in (2.50) follows, using [43, Proposition 20.2.6.]. However for Anosov flows the distance between two points $x_1, x_2 \in \mathcal{W}$ may never be sufficiently contracted under iterates by $g_{\alpha'}$, e.g. if x_1, x_2 belong to a same orbit of $g_{\alpha'}$. We split (along the flow direction X in charts) each $\mathcal{W}$ into parts $\mathcal{W}_j$, $1 \leqslant j \leqslant \lfloor \alpha \rfloor + 1$, in which now two points are no more than $\sim (\lfloor \alpha \rfloor + 1)^{-1}$ apart. We set $\widetilde{\mathcal{W}} := \{\mathcal{W}_j\}$ for all $1 \leqslant j \leqslant \lfloor \alpha \rfloor + 1$. Then it holds the bound in (2.50) with $\mathcal{W}$ replaced by $\mathcal{W}_j$. We modify $\vartheta_{\alpha,\omega\omega'}$, taking a sufficiently small neighborhood U_j containing $\mathcal{W}_j$, such that $\vartheta_{\alpha,\omega\omega'|U_j}$ is C_0^r. Then passing to this new weights $\vartheta_{\alpha,\omega\omega'|U_j}$ and summing over j we obtain an additional factor $\sim (\alpha + 1)$ in the right-hand side in (2.49). We recall $\lambda^{(t,s,\alpha)}(x)$ from (2.48) and $\|F\|_{-,I}$, $\|F\|_{+,I}$ introduced below (2.47) in which we take $I = \mathcal{W}_j$ and $F = F_{-\alpha,\omega\omega'}$. In addition note $F^{-1}_{-\alpha,\omega\omega'} = F_{\alpha,\omega'\omega}$. Then we write

$$\left\|F_{-\alpha,\omega\omega'}\right\|_{+,\kappa_\omega(\mathcal{W}_j)} = \left(\inf_{\substack{y \in \kappa_\omega(\mathcal{W}_j) \\ 0 \neq \eta \in \left(\mathrm{D}_y F_{-\alpha,\omega\omega'}\right)^{\mathrm{tr}} \operatorname{supp} \widetilde{\Phi}^\circ_{+,\omega'}}} \frac{\left| \left(\mathrm{D}_{F_{-\alpha,\omega\omega'}(y)} F_{\alpha,\omega'\omega}\right)^{\mathrm{tr}} \eta \right|}{|\eta|} \right)^{-1} .$$

We recall the construction in (2.16) of the C^+-cones in the proof of Lemma 2.2.3. We find a compactly embedded cone

$$C^+_{\gamma_+,\omega} \Subset \left(\mathrm{D}_y F_{-\alpha,\omega\omega'}\right)^{\mathrm{tr}} \operatorname{supp} \widetilde{\Phi}^\circ_{+,\omega'},$$

which is transversal to another cone $C^-_{\gamma_-,\omega}$. Hence the unstable distribution E_- (in charts) stays away from $\left(\mathrm{D}_y F_{-\alpha,\omega\omega'}\right)^{\mathrm{tr}} \operatorname{supp} \widetilde{\Phi}^\circ_{+,\omega'}$ by some positive angle. Replacing the inf with the sup, it holds for some constant $C_5 > 0$

$$\left\|F_{-\alpha,\omega\omega'}\right\|_{+,\kappa_\omega(\mathcal{W}_j)} \leqslant C_5 \left(\sup_{x \in \mathcal{W}_j} \left\| (\mathrm{D}\, g_\alpha)^{\mathrm{tr}}_{|E^*_{+,g_{-\alpha}(x)}} \right\| \right)^{-1} . \tag{2.51}$$

By analogous reasoning we conclude similar for $\left\|F_{-\alpha,\omega\omega'}\right\|_{-,\kappa_\omega(\mathcal{W}_j)}$. We estimate

for some constants $C_6, \dots, C_9 > 0$, using the bounds in (2.51) and (2.50),

$$\begin{aligned}
\widetilde{C}_2 &\leqslant C_6 \max_{\widetilde{\mathcal{W}}, \mathcal{W}_j \in \widetilde{\mathcal{W}}} \left\| \phi_\alpha \left| \det \mathrm{D}\, g_{-\alpha} \right|^{-\frac{1}{p}} \right\|_{L_\infty(\mathcal{W}_j)} \left\| \lambda^{(t,s,\alpha)} \right\|_{L_\infty(\mathcal{W}_j)} \\
&\leqslant \alpha C_7 \max_{\widetilde{\mathcal{W}}, \mathcal{W}_j \in \widetilde{\mathcal{W}}} \left\| \phi_\alpha \left| \det \mathrm{D}\, g_{-\alpha} \right|^{-\frac{1}{p}} \right\|_{L_\infty(\mathcal{W}_j)} \left\| \lambda^{(t,s,\alpha)} \right\|_{L_\infty(\mathcal{W}_j)} \\
&\leqslant \alpha C_8 \max_{\widetilde{\mathcal{W}}, \mathcal{W}_j \in \widetilde{\mathcal{W}}} \left\| \left(\phi_{-\alpha} \left| \det \mathrm{D}\, g_\alpha \right|^{-\frac{1}{p}} \right) \circ g_{-\alpha} \right\|_{L_\infty(\mathcal{W}_j)}^{-1} \left\| \lambda^{(t,s,\alpha)} \right\|_{L_\infty(\mathcal{W}_j)} \\
&\leqslant \alpha C_9 \max_{\widetilde{\mathcal{W}}, \mathcal{W}_j \in \widetilde{\mathcal{W}}} \left\| \phi_\alpha \left| \det \mathrm{D}\, g_{-\alpha} \right|^{-\frac{1}{p}} \lambda^{(t,s,\alpha)} \right\|_{L_\infty(\mathcal{W}_j)} .
\end{aligned}$$

Inspecting the constant $\widetilde{C}_1$, all terms depending on F and f are bounded by the maximal expansion of $F_{-\alpha,\omega\omega'}$ and ϕ_α, respectively, which grow at most exponentially in α. Hence, there is $A > 0$ and $C_{10} \geqslant 1$ such that $\widetilde{C}_1 \leqslant C_{11} e^{A\alpha}$. If $\alpha < \alpha_0$ we split $\int_0^{\alpha_0} = \int_0^{\alpha_0 - \alpha} + \int_{\alpha_0 - \alpha}^{\alpha_0}$. Hence it holds $\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|_{W_p^{s,t,q}} \leqslant \|\varphi\|_{W_p^{s,t,q}} + \left\|\mathcal{L}_{\alpha_0,\phi_{\alpha_0}}\varphi\right\|_{W_p^{s,t,q}}$. The latter term is estimated as in the case $\alpha \geqslant \alpha_0$. Since $\alpha \leqslant \alpha_0$, we combine here the upper bound of $\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|_{W_p^{s,t,q}}$ with the second term of our desired estimate, increasing the constant C_{11}. □

Remark 2.4.3. *A weaker upper bound for the transfer operator, e.g.*

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\|_{W_p^{s,t,q} \to W_p^{s,t,q}} \leqslant C_1 \exp\left(C_2 \alpha\right)$$

for all $\alpha \geqslant 0$ and for some constants $C_1, C_2 \geqslant 1$ independent of α, can be obtained for a wider choice of s, t, q, e.g. for some $s > 0$ (and this carries over to Lemma 2.4.4 below as well). However, we are interested in the parameter range as assumed in Lemma 2.4.2 which allows us to show the Lasota–Yorke inequality for the resolvent given in Theorem 2.4.5 below. See also Lemma 2.5.17 in the next section below for such a bound in the case of a special weight.

We recall that the family $\left\{\mathcal{L}_{\alpha,\phi_\alpha} : W_p^{s,t,q} \to W_p^{s,t,q} \mid \alpha \geqslant 0\right\}$ forms a strongly continuous semigroup if and only if $\lim_{\alpha \to 0^+} \|\mathcal{L}_{\alpha,\phi_\alpha}\varphi - \varphi\|_{W_p^{s,t,q}} = 0$ for all $\varphi \in W_p^{s,t,q}$ (e.g. see [46, Proposition I.1.3]).

Lemma 2.4.4 (Strongly continuous semigroup)**.** *Let $p \in [1, \infty]$ and let $s < 0 < q \leqslant t < r - 1 + s$. Then the transfer operator family*

$$\left\{\mathcal{L}_{\alpha,\phi_\alpha} : W_p^{s,t,q} \to W_p^{s,t,q} \mid \alpha \geqslant 0\right\}$$

forms a strongly continuous semigroup.

Proof. Let $\varphi \in W_p^{s,t,q}$. For fixed $s < 0 < q \leqslant t$ such that $t - s < r - 1$ there is $\delta > 0$ such that $t - s < r - 1 - \delta$. We set $s' := s - \delta$ and let $t' < t$. Then s, t, q, s' and t' satisfy the assumptions of Lemma 2.4.2. Using Lemma 2.4.2, we bound the transfer operator for all small $\alpha \geqslant 0$

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi\|_{W_p^{s,t,q}} \leqslant C_1 \|\varphi\|_{W_p^{s',t',q}} + C_2 \|\varphi\|_{W_p^{s,t,q}} \leqslant (C_1 + C_2) \|\varphi\|_{W_p^{s,t,q}}, \tag{2.52}$$

for some constants $C_1, C_2 > 0$ independent of α. By density, for every $\epsilon > 0$ there is $\widetilde{\varphi} \in C_X^{r-1}(M)$ such that

$$\|\varphi - \widetilde{\varphi}\|_{W_p^{s,t,q}} \leqslant \epsilon. \tag{2.53}$$

Using first the triangle inequality and then the bounds (2.52)-(2.53), we estimate

$$\begin{aligned}\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi - \varphi\|_{W_p^{s,t,q}} &\leqslant \|\mathcal{L}_{\alpha,\phi_\alpha}(\varphi - \widetilde{\varphi})\|_{W_p^{s,t,q}} + \|\varphi - \widetilde{\varphi}\|_{W_p^{s,t,q}} + \|\mathcal{L}_{\alpha,\phi_\alpha}\widetilde{\varphi} - \widetilde{\varphi}\|_{W_p^{s,t,q}} \\ &\leqslant C_3\epsilon + \|\mathcal{L}_{\alpha,\phi_\alpha}\widetilde{\varphi} - \widetilde{\varphi}\|_{W_p^{s,t,q}}, \end{aligned} \tag{2.54}$$

for some constant $C_3 > 0$ independent of ϵ and α. Since $\varphi \in C_X^{r-1}(M)$ we have

$$\mathcal{L}_{\alpha,\phi_\alpha}\widetilde{\varphi} - \widetilde{\varphi} = \alpha \int_0^1 \left(\partial_{\alpha'}\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)_{|\alpha'=h\alpha} \mathrm{d}\, h.$$

Since $\left(\partial_{\alpha'}\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)_{|\alpha'=h\alpha} \in C^{(r-1)}(M)$ the norm $\left\|\left(\partial_{\alpha'}\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)_{|\alpha'=h\alpha}\right\|_{W_p^{s,t,q}}$ is finite for all $0 \leqslant h \leqslant 1$. Hence for some constant $C_4(\varphi) = C_4 > 0$ we bound

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\widetilde{\varphi} - \widetilde{\varphi}\|_{W_p^{s,t,q}} \leqslant \alpha \sup_{0\leqslant h\leqslant 1} \left\|\left(\partial_{\alpha'}\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)_{|\alpha'=h\alpha}\right\|_{W_p^{s,t,q}} \leqslant C_4\alpha. \tag{2.55}$$

We conclude by a combination of the estimates (2.54)-(2.55). □

2.4.2 Lasota–Yorke inequality for the resolvent

We use Lemma 2.4.2 to prove Theorem 2.4.5 below. We use in addition that the resolvent improves regularity in the flow direction. We set, recalling $\lambda^{(t,s,\alpha)}$ in (2.48),

$$\lambda_{\min} = \lambda_{\min}(s,t,p) := \lim_{\alpha\to\infty} \frac{1}{\alpha} \log \left\|\phi_\alpha \left|\det \mathrm{D}\, g_{-\alpha}\right|^{-\frac{1}{p}} \lambda^{(t,s,\alpha)}\right\|_{L_\infty(M)}. \tag{2.56}$$

The following theorem will allow us to show that $\lambda_{\min}(s,t)$ plays the role of the essential spectral bound of $X + V$:

Theorem 2.4.5 (Lasota–Yorke inequality for the resolvent)**.** *Let $p \in [1,\infty]$ and let*

$$s' < s < 0 < q \leqslant t < r - 1 + s', \quad q - 1 \leqslant q' < q, \quad t' < t.$$

There exist $\alpha_0 > 0$, $A_0 > \lambda_{min}$, cone ensembles Θ_ω, $\omega \in \Omega$, and a constant $C > 0$ such that for every $\varphi \in W_p^{s,t,q}$ with $\|\varphi\|_{W_p^{s,t,q}} = 1$, for every $z \in \mathbb{C}$ with $\Re z > A_0$ and for every $n \in \mathbb{N}$ it holds

$$\left\|\mathcal{R}_z^{n+1}\varphi\right\|_{W_p^{s,t,q}} \leqslant C\frac{|z| + 1 + (\Re z - A_0)}{(\Re z - A_0)^{n+1}} \|\varphi\|_{W_p^{s',t',q'}} + \frac{Cn\,(\Re z - \lambda_{min})^{-1} + C}{(\Re z - A_0)\,(\Re z - \lambda_{min})^n}.$$

Proof. Since $\lambda^{(s,t,\alpha)}$ grows at most exponentially as $\alpha \to \infty$, the constant $\lambda_{\min}$ is finite by a result on superadditive functions [37, Theorem 7.6.1]. We let $A_0 > \lambda_{\min}$. By Lemma 2.4.4 the transfer operator family (2.18) forms a strongly continuous semigroup with a well-defined generator $X + V$. We estimate powers of the resolvent $\mathcal{R}_z$ defined in (2.20). To this end we work with the integral representation of powers of the resolvent defined in (2.57) below (see [46, Corollary II.1.11]). We recall the constant A given in Lemma 2.4.2 and let $A_0 > A$. We set for every $z \in \mathbb{C}$ such that $\Re z > A_0$ and every $n \in \mathbb{N}$

$$\mathcal{R}_z^n\varphi := \int_0^\infty \frac{\alpha^{n-1}e^{-z\alpha}}{(n-1)!}\mathcal{L}_{\alpha,\phi_\alpha}\varphi\,\mathrm{d}\,\alpha, \quad \varphi \in W_p^{s,t,q}. \tag{2.57}$$

We have directly from (2.57) for all $\alpha \geqslant 0$

$$\mathcal{R}_z^n\mathcal{L}_{\alpha,\phi_\alpha}\varphi = \mathcal{L}_{\alpha,\phi_\alpha}\mathcal{R}_z^n\varphi. \tag{2.58}$$

Using Lemma 2.4.2, we estimate for some constant $C_1 > 0$

$$\begin{aligned}\left\|\mathcal{R}_z^{n+1}\varphi\right\|_{W_p^{s,t,q}} &\leqslant \int_0^\infty \frac{\alpha^{n-1}e^{-\Re z\alpha}}{(n-1)!}\left\|\mathcal{L}_{\alpha,\phi_\alpha}\mathcal{R}_z\varphi\right\|_{W_p^{s,t,q}}\mathrm{d}\,\alpha \\ &\leqslant \frac{C_1}{(\Re z - A_0)^n}\left\|\mathcal{R}_z\varphi\right\|_{W_p^{s',t',q}} + \frac{C_1\,(n + (\Re z - \lambda_{\min}))}{(\Re z - \lambda_{\min})^{n+1}}\left\|\mathcal{R}_z\varphi\right\|_{W_p^{s,t,q}}.\end{aligned} \tag{2.59}$$

Using Lemma 2.4.2, we get boundedness for some constant $C_2 > 0$

$$\left\|\mathcal{R}_z\varphi\right\|_{W_p^{s,t,q}} \leqslant \frac{C_2}{\Re z - A_0}\|\varphi\|_{W_p^{s,t,q}}. \tag{2.60}$$

Therefore the second term in the right-hand side in (2.59) is bounded as claimed. We bound now the first term in the right-hand side in (2.59). Inverting the

flowbox condition (2.10), we find $\mathrm{D}\,\kappa_\omega^{-1}\partial_{x_d} = X_{|V_\omega}$. Hence it holds

$$\begin{aligned}\partial_{x_d}\left(\vartheta_\omega\cdot\varphi\right)\circ\kappa_\omega^{-1}(x) &= \mathrm{D}\left(\vartheta_\omega\cdot\varphi\right)\circ\kappa_\omega^{-1}(x)\,\mathrm{D}_x\,\kappa_\omega^{-1}\partial_{x_d}\\ = \left(\mathrm{D}\left(\vartheta_\omega\cdot\varphi\right)X_{|V_\omega}\right)\circ\kappa_\omega^{-1}(x) &= \left(\left(X\vartheta_\omega\right)\cdot\varphi+\vartheta_\omega\cdot\left(X\varphi\right)\right)\circ\kappa_\omega^{-1}(x)\,. \end{aligned}\tag{2.61}$$

We set $W_\omega^q := W_{p,\Theta_\omega,\kappa_\omega(V_\omega)}^{s',t',q}$, $\omega\in\Omega$. We estimate the local norms inside the norm $\|\mathcal{R}_z\varphi\|_{W_p^{s',t',q}}$, using the equality in (2.61), then Lemma 2.3.5 and the equality in (2.58), for some constant $C_3>0$:

$$\left\|\left(\vartheta_\omega\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^q}\leqslant C_3\left\|\left(\vartheta_\omega\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}\tag{2.62}$$

$$+C_3\left\|\left(\left(X\vartheta_\omega\right)\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}+C_3\left\|\left(\vartheta_\omega\cdot X\mathcal{R}_z\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}.$$

We note that $\left(X\vartheta_\omega\right)\circ\kappa_\omega^{-1}\in C_0^{r-1}\left(\kappa_\omega\left(V_\omega\right)\right)$ and $t-s<r-1$. Using Lemma 2.3.3, we bound for some constant $C_4(X)=C_4>0$

$$\left\|\left(\left(X\vartheta_\omega\right)\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}\leqslant C_4\sup_{\omega\in\Omega}\left\|\left(\vartheta_\omega\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}.\tag{2.63}$$

Using the equality

$$X\mathcal{R}_z\varphi = z\mathcal{R}_z\varphi - V\mathcal{R}_z\varphi-\varphi,$$

together with the equality in (2.58), we find

$$\begin{aligned}\left\|\left(\vartheta_\omega\cdot X\mathcal{R}_z\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}} &\leqslant |z|\left\|\left(\vartheta_\omega\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}\\ +\left\|\left(\vartheta_\omega\cdot V\mathcal{L}_{\alpha',\phi_{\alpha'}}\mathcal{R}_z\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}&+\left\|\left(\vartheta_\omega\cdot\mathcal{L}_{\alpha',\phi_{\alpha'}}\varphi\right)\circ\kappa_\omega^{-1}\right\|_{W_\omega^{q-1}}.\end{aligned}\tag{2.64}$$

Recalling that $V\in C^{r-1}(M)$, we bound the term which contains the factor $(\vartheta_\omega\cdot V)\circ\kappa_\omega^{-1}$ in the right-hand side in (2.64) analogous as in the estimate in (2.63). The final estimate follows by a combination of the bounds (2.59)-(2.60) and (2.62)-(2.64), together with the trivial continuous inclusion $W_\omega^{q'}\subseteq W_\omega^{q-1}$. □

A direct consequence of Theorem 2.4.5 is the bound on the essential spectral radius of the resolvent:

Corollary 2.4.6 (Essential spectral radius)**.** *Under the assumptions of Theorem 2.4.5 (including the choices for $p,s,t,q\in\mathbb{R}$), letting A_0 and $\lambda_{min}=\lambda_{min}(s,t,p)$*

be the constants from that theorem, the essential spectral radius of the resolvent $\mathcal{R}_z \colon W_p^{s,t,q} \to W_p^{s,t,q}$ *is bounded by* $|\Re z - \lambda_{min}|^{-1}$ *for all* $z \in \mathbb{C}$ *with* $\Re z > A_0$.

Proof. Let $s' < s, t' < t$ and $q' < q$. The inclusion $W_p^{s,t,q} \subseteq W_p^{s',t',q'}$ is compact by Lemma 2.3.8. Then, together with a result of Hennion [36, Corollaire 1] and Theorem 2.4.5 we find the claimed bound on the essential spectral radius of the resolvent. □

We recall $\lambda_{\min}$ defined in (2.56) and ϕ_α in (2.18).

Lemma 2.4.7. *Let* $d = 3$ *and let* $|\det \mathrm{D}\, g_\alpha| \equiv 1$. *Set* $\widetilde{t} := \min\{-t, s\}$. *Then it holds*

$$\lambda_{min} = \lim_{\alpha \to \infty} \frac{1}{\alpha} \log \left\| \phi_\alpha \left| \det \left(\mathrm{D}\, g_{-\alpha} \right)^{\mathrm{tr}}{}_{|E_-^*} \right|^{\widetilde{t}} \right\|_{L_\infty(M)}.$$

Proof. Since the flow is volume preserving, we have

$$\left| \det \left(\mathrm{D}\, g_{-\alpha} \right)^{\mathrm{tr}}{}_{|E_+^*} \right|^{-1} = \left| \det \left(\mathrm{D}_{g_{-\alpha}}\, g_\alpha \right)^{\mathrm{tr}}{}_{|E_0^*} \right| \left| \det \left(\mathrm{D}_{g_{-\alpha}}\, g_\alpha \right)^{\mathrm{tr}}{}_{|E_-^*} \right|.$$

Since $d_- = 1 = d-2$ we can replace $\left\| (\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}_{|E^*_{-,x}} \right\|$ in $\lambda^{(s,t,\alpha)}$ by $\left| \det \left(\mathrm{D}\, g_{-\alpha} \right)^{\mathrm{tr}}{}_{|E^*_{-,x}} \right|$ and $\left\| (\mathrm{D}\, g_\alpha)^{\mathrm{tr}}_{|E^*_{+,g_{-\alpha}(x)}} \right\|$ by $\left| \det \left(\mathrm{D}\, g_{-\alpha} \right)^{\mathrm{tr}}{}_{|E^*_{+,x}} \right|^{-1}$. Moreover $\left| \det \left(\mathrm{D}_{g_{-\alpha}}\, g_\alpha \right)^{\mathrm{tr}}{}_{|E_0^*} \right|$ is bounded from above and below and we conclude. □

Remark 2.4.8. *Note that Lemma 2.4.7 holds in the particular case of a contact Anosov flow if* $d = 3$. *Clearly, if* $|\phi_\alpha| \leqslant \left| \det \left(\mathrm{D}\, g_{-\alpha} \right)^{\mathrm{tr}}{}_{|E_-^*} \right|^{-\widetilde{t}}$ *for all* $\alpha > 0$ *then* $\lambda_{min} \leqslant 0 < h_{top}$.

2.4.3 Spectral properties of the generator

All spectral properties of the generator $X + V$ are with respect to its domain $D(X+V) = D(X+V)_{|W_p^{s,t,q}}$ for admissible choices $p, s, t, q \in \mathbb{R}$ which is discussed in the following lemma.

Lemma 2.4.9 (Domain of the generator)**.** *Let* $p, s, q, t \in \mathbb{R}$ *satisfy the assumptions of Lemma 2.4.4. Then the family* $\left\{ \mathcal{L}_{\alpha,\phi_\alpha} \colon W_p^{s,t,q} \to W_p^{s,t,q} \mid \alpha \geqslant 0 \right\}$ *admits a generator*

$$X + V \colon D(X+V) \to W_p^{s,t,q},$$

which is a closed operator on its domain $D(X+V)$. *Moreover, the inclusion*

$$D(X+V) \subseteq W_p^{s,t,q}$$

is dense and the inclusion

$$C_X^{r-1}(M) \subseteq D(X+V)$$

is dense for the graph norm $\|\cdot\|_{W_p^{s,t,q}} + \|(X+V)(\cdot)\|_{W_p^{s,t,q}}$.

Proof. Using Lemma 2.4.4, the statement about $X+V$ being a densely (in $W_p^{s,t,q}$) defined closed operator is [46, Theorem II.1.4]. Suppose now $\mathcal{L}_{\alpha,\phi_\alpha}\left(C_X^{r-1}(M)\right) \subseteq C_X^{r-1}(M)$. Then the inclusion statement $C_X^{r-1}(M) \subseteq D(X+V)$ is [46, Proposition II.1.7], using [46, Definition II.1.6]. We let $\varphi \in C_X^{r-1}(M)$. It holds $X\varphi, \mathcal{L}_{\alpha,\phi_\alpha}\varphi \in C^{r-1}(M)$ since the flow is C^r. Recalling the weight ϕ_α of the transfer operator in (2.18), with generating function $f \in C^{r-1}(M)$, we calculate and conclude:

$$X\mathcal{L}_{\alpha,\phi_\alpha}\varphi = (X\phi_\alpha)\cdot\varphi\circ g_{-\alpha} + \phi_\alpha\cdot(X\varphi)\circ g_{-\alpha} = (f\circ g_{-\alpha} - f)\cdot\mathcal{L}_{\alpha,\phi_\alpha}\varphi + \phi_\alpha\cdot(X\varphi)\circ g_{-\alpha}.$$

□

We set as the maximal spectral bound of the generator

$$\lambda_{\max} = \lambda_{\max}(s,t,q,p) := \sup \Re\sigma(X+V)|_{W_p^{s,t,q}}. \tag{2.65}$$

Lemma 2.4.10 (Discrete spectrum)**.** *Under the assumptions of Theorem 2.4.5 (including the choices for $p,s,t,q \in \mathbb{R}$), the set*

$$\left\{\lambda \in \sigma(X+V)|_{W_p^{s,t,q}} \mid \Re\lambda > \lambda_{min}\right\}$$

consists of isolated eigenvalues of finite multiplicity.

The discrete spectrum described in the previous lemma if $\lambda_{\max} > \lambda_{\min}$, is sometimes referred to as (Ruelle-Pollicott) resonances of $X+V$. In principle, the resonances depend on the choices p,s,t, and q of the space $W_p^{s,t,q}$. We shall not enter into details here, but note that our main result in the next section shows that this dependence is mild, in particular, for the choice of V there, $\lambda_{\max}$ is independent of p,t,s, and q.

Proof. Using Corollary 2.4.6, spectral radius of the resolvent is bounded from above by $|\Re z - \lambda_{\min}|^{-1}$. Assume $\lambda \in \sigma(X+V)|_{W_p^{s,t,q}}$ such that $\Re\lambda > \lambda_{\min}$. It follows from the Spectral Theorem for the Resolvent [46, Theorem V.1.13] that there exists $z \in \mathbb{C}$ (e.g. with $\Im z = \Im\lambda$) in the resolvent set of $X+V$ such that

the spectral radius of the resolvent $\mathcal{R}_z$ has a lower bound given by

$$|z-\lambda|^{-1} = (\Re z - \Re\lambda)^{-1} > (\Re z - \lambda_{\min})^{-1}.$$

Since $\Im\lambda$ was arbitrary we conclude. □

The following notation associated to the eigenvalue spectrum is needed in Section 2.5 for the statement and proof of Theorem 2.5.7. We assume for the rest of this subsection

$$\lambda_{\max} = \lambda_{\max}(s,t,q,p) > \lambda_{\min}(s,t,p) = \lambda_{\min},$$

for any fixed choice $p \in [1,\infty]$ and $-s < 0 < q \leqslant t < r-1+s$. By Lemma 2.4.10 each $\lambda \in \sigma(X+V)|_{W_p^{s,t,q}}$ such that $\Re\lambda > \lambda_{\min}$ has a finite geometric multiplicity $n_\lambda \in \mathbb{N}$ and finite algebraic multiplicities $m_{\lambda,i} \in \mathbb{N}$, $1 \leqslant i \leqslant n_\lambda$, with generalized eigenstates

$$\mathcal{D}_{(\lambda,i,j)} \in D(X+V), \quad 1 \leqslant j \leqslant m_{\lambda,i},$$

satisfying

$$(X+V-\lambda)^j \mathcal{D}_{(\lambda,i,j)} = 0 \quad \text{and} \quad \text{if } j>1\text{: } (X+V-\lambda)^{j-1}\mathcal{D}_{(\lambda,i,j)} \neq 0.$$

Moreover, to each geometric eigenvector there is associated a projector $\Pi_{\lambda,i}$ and a nil-potent operator $\mathcal{N}_{\lambda,i}$ of finite ranks such that

$$\Pi_{\lambda_1,i_1}\Pi_{\lambda_2,i_2} \equiv 0, \quad \mathcal{N}_{\lambda_1,i_1}\mathcal{N}_{\lambda_2,i_2} \equiv 0 \quad \text{if} \quad \lambda_1 \neq \lambda_2 \quad \text{or} \quad i_1 \neq i_2, \tag{2.66}$$

$$\Pi_{\lambda_1,i_1}\mathcal{N}_{\lambda_2,i_2} = \mathcal{N}_{\lambda_2,i_2}\Pi_{\lambda_1,i_1} = \begin{cases} \mathcal{N}_{\lambda_2,i_2} & \text{if } \lambda_1=\lambda_2 \text{ and } i_1=i_2 \\ 0 & \text{if } \lambda_1\neq\lambda_2 \text{ or } i_1\neq i_2 \end{cases},$$
$$\mathcal{N}_{\lambda,i}^{m_{\lambda,i}-1} \equiv 0.$$

Note that the projector $\Pi_{\lambda,i}$ can be written as a finite rank operator

$$\Pi_{\lambda,i} = \sum_{j=1}^{n_{\lambda,i}} \mathcal{D}_{(\lambda,i,j)} \otimes \mathcal{O}_{(\lambda,i,j)}, \tag{2.67}$$

where the dual vectors $\mathcal{O}_{(\lambda,i,j)} \in D\left(X+V\right)'$ satisfy

$$\mathcal{O}_{(\lambda_1,i_1,j_1)}\left(\mathcal{D}_{(\lambda_2,i_2,j_2)}\right) = \begin{cases} 1, & \text{if } (\lambda_1,i_1,j_1) = (\lambda_2,i_2,j_2) \\ 0, & \text{otherwise.} \end{cases}$$

We shall use the following Dolgopyat-type condition, adapted from [19, Assumption 3A], on the resolvent $\mathcal{R}_z = (z - X - V)^{-1}$, to control the remainder term $\mathcal{E}_{T,x}$ in (2.1) in Theorem 2.5.7 (to reduce to the case studied by Butterley, consider the renormalized semi-group $e^{-\lambda_{\max}\alpha}\mathcal{L}_{\alpha,\phi_\alpha}$ with generator $X + V - \lambda_{\max}$ and resolvent $\mathcal{R}_{z+\lambda_{\max}}$)[4]:

Condition 2.4.11 (Spectral gap with (Dolgopyat) bounds)**.** *There exists*

$$\delta \in (\lambda_{min}(s,t,p), \lambda_{max}(s,t,q,p))$$

so that the following holds: For some $a > 0$, $b > 0$, $C > 0$, some

$$\gamma \in \left(0, 1/\log(1 + (\lambda_{max} - \delta)/a)\right),$$

and for all $z \in \mathbb{C}$ with $\Re z = a$ and $|\Im z| \geqslant b$, we have

$$\left\|\mathcal{R}^{\tilde{n}}_{z+\lambda_{max}}\right\|_{W^{s,t,q}_p} \leqslant C^{\tilde{n}}\left|\Re z + (\lambda_{max} - \delta)\right|^{-\tilde{n}}, \quad \textit{where } \tilde{n} = \left\lceil \gamma \log|\Im z| \right\rceil.$$

It is well known that if $\|\mathcal{L}_{\alpha,\phi_\alpha}\|_{W^{s,t,q}_p \to W^{s,t,q}_p} \leqslant Ce^{\lambda_{\max}\alpha}$ for all α and if $\mathcal{R}_z$ enjoys Lasota–Yorke estimates for $\lambda_{\min}(s,t)$ on $W^{s,t,q}_p$, in the sense of Theorem 2.4.5, then Condition 2.4.11 for some constant δ implies a *spectral gap* for the same δ, in the sense that

$$\sigma\left(X+V\right)|_{W^{s,t,q}_p} \cap \{\Re\lambda > \delta\} \text{ is a finite set,} \tag{2.68}$$

see e.g. [19, Theorem 1]. (Note that [19, Assumption 1] follows from the facts that $W^{s,t,q}_p \subset W^{s,t,q-1}_p$, $\|(X+V)\varphi\|_{W^{s,t,q-1}_p} \leqslant C\|\varphi\|_{W^{s,t,q}_p}$ for some constant $C > 0$, using Lemma 2.3.3 and Lemma 2.3.5, and

$$e^{-\lambda_{\max}\alpha}\mathcal{L}_{\tilde{\alpha},\phi_{\tilde{\alpha}}}\varphi - \varphi = (X + V - \lambda_{\max})\int_0^\alpha e^{-\lambda_{\max}\tilde{\alpha}}\mathcal{L}_{\tilde{\alpha},\phi_{\tilde{\alpha}}}\varphi \,\mathrm{d}\,\tilde{\alpha} \tag{2.69}$$

for all $\varphi \in W^{s,t,q}_p$.)

[4]Note the iterated constant $C^{\tilde{n}}$ contrary to C in [19, Assumption 3A]. This change was made to avoid a conflict in the proof of [19, Lemma 4.4], involving in there the constant C_6, and was communicated with Butterley [20].

Beware that even when $W_p^{s,t,q}$ is a Hilbert space, the operator $X+V$ is not self-adjoint a priori, so the existence of a spectral gap for $X+V$ with δ does *n*ot imply a spectral gap with bounds on the resolvent in general. (In the self-adjoint case, classical bounds on the iterated resolvent $\mathcal{R}_z^n$ in terms of the distance between z and the spectrum give bounds stronger than Condition 2.4.11.)
See also Remark 2.5.11 for a further discussion of Condition 2.4.11.

2.4.4 Proof of Lemma 2.4.1

We need some preparations. We recall the quantities $\|F\|_{-,I}, \|F\|_{+,I}, \|F\|_{0,I}$ given below (2.47). We introduce an arrow relation as used by Baladi and Tsujii in [10, p.16].

Definition 2.4.12 (Arrow relation)**.** *Let $n, \ell \in \mathbb{Z}_{\geqslant 0}$ and $\sigma, \tau \in \{-,+,0\}$. We write*

$$(\tau,\ell) \hookrightarrow_I (\sigma,n) \Leftrightarrow \begin{cases} \tau = + \text{ and } 2^{n-\ell} \leqslant 2^4 \|F\|_{+,I} \\ \tau = \sigma = - \text{ and } 2^{n-\ell} \geqslant 2^{-4} \|F\|_{-,I} \end{cases},$$

$$\ell \hookrightarrow_I (\sigma,n) \Leftrightarrow \begin{cases} \tau = \sigma = 0 \text{ and } 2^{n-\ell} \leqslant 2^4 \|F\|_{0,I} \\ \sigma = - \text{ and } \tau = 0 \end{cases},$$

and $(\tau,\ell) \nhookrightarrow_I (\sigma,n)$ in the other cases.

We recall the function c defined in (2.29). We let c' be analogously defined for $s' \leqslant s$, $t' \leqslant t$, $q' \leqslant q$. We have for some constant $C > 0$, for all fixed $\tau \in \{-,+,0\}, \ell \in \mathbb{Z}_{\geqslant 0}$

$$\begin{aligned} \sum_{(\tau,\ell)\hookrightarrow_I(\sigma,n)} 2^{c(\sigma)n - c(\tau)\ell} &= \sum_{(\tau,\ell)\hookrightarrow_I(\sigma,n)} 2^{(c(\sigma)-c(\tau))n + c(\tau)(n-\ell)} \leqslant \sum_{(\tau,\ell)\hookrightarrow_I(\sigma,n)} 2^{c(\tau)(n-\ell)} \\ &\leqslant C \max\left\{ \|F\|_{+,I}^t, \|F\|_{-,I}^s \right\}. \end{aligned} \tag{2.70}$$

An analogous estimate holds for all fixed σ, n. Similarly, we find either for all fixed ℓ or for all fixed σ, n

$$\sum_{\ell \hookrightarrow_I (\sigma,n)} 2^{c(\sigma)n - q\ell} \leqslant C \max\left\{1, \|F\|_{0,I}^q\right\}. \tag{2.71}$$

We recall the norm of the Hilbert space ℓ_2^c (and analogously $\ell_2^{c'}$) given in (2.30). Clearly, we have the inclusion $\ell_2^c \subseteq \ell_2^{c'}$. We recall the definitions of $\Psi_{\sigma,n}^{\mathrm{Op}}$ in (2.23).

We let given a family of pairwise disjoint sets

$$\mathcal{I} := \{I \subseteq K\}.$$

For every $(a_{\tau,\ell}) = a \in L_p\left(\mathbb{R}^d, \ell_2^{c'}\right)$ we set

$$\begin{aligned} \left(Q^{\mathrm{Op}}_{\hookrightarrow_{\mathcal{I}}} a\right)_{\sigma,n} &:= \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{I\in\mathcal{I}} \sum_{(\tau,\ell)\hookrightarrow_I(\sigma,n)} 1_{|I} a_{\tau,\ell}, \\ \left(Q^{\mathrm{Op}}_{\hookrightarrow_{0,\mathcal{I}}} a\right)_{\sigma,n} &:= \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{I\in\mathcal{I}} \sum_{l\hookrightarrow_I(\sigma,n)} 1_{|I} a_{0,l}. \end{aligned} \tag{2.72}$$

Lemma 2.4.13 (Boundedness I)**.** *For all $p \in [1,\infty]$ the map*

$$Q^{\mathrm{Op}}_{\hookrightarrow_{\mathcal{I}}} \colon L_p\left(\mathbb{R}^d, \ell_2^c\right) \to L_p\left(\mathbb{R}^d, \ell_2^c\right)$$

is a bounded linear operator. Moreover, for some constant $C \geqslant 0$, for every $f \in L_\infty\left(\mathbb{R}^d, \mathbb{R}\setminus\{0\}\right)$ and every $a \in L_p\left(\mathbb{R}^d, \ell_2^c\right)$, it holds

$$\left\|Q^{\mathrm{Op}}_{\hookrightarrow_{\mathcal{I}}} a\right\|_{L_p\left(\mathbb{R}^d,\ell_2^c\right)} \leqslant C \sup_{I\in\mathcal{I}} \max\left\{\|F\|^t_{+,I}, \|F\|^s_{-,I}\right\} \left\|f_{|I}\right\|_{L_\infty} \left\|\sum_{I\in\mathcal{I}} \frac{1}{f_{|I}} \|a\|_{\ell_2^c}\right\|_{L_p}.$$

Let $c'(0) = c(0)$. Then for all $p \in [1,\infty]$ the map

$$Q^{\mathrm{Op}}_{\hookrightarrow_{0,\mathcal{I}}} \colon L_p\left(\mathbb{R}^d, \ell_2^{c'}\right) \to L_p\left(\mathbb{R}^d, \ell_2^c\right)$$

is a bounded linear operator. Moreover, for every $a \in L_p\left(\mathbb{R}^d, \ell_2^{c'}\right)$ it holds

$$\left\|Q^{\mathrm{Op}}_{\hookrightarrow_{0,\mathcal{I}}} a\right\|_{L_p\left(\mathbb{R}^d,\ell_2^c\right)} \leqslant C \sup_{I\in\mathcal{I}} \max\left\{1, \|F\|^q_{0,I}\right\} \left\|f_{|I}\right\|_{L_\infty} \left\|\sum_{I\in\mathcal{I}} \frac{1}{f_{|I}} \|a\|_{\ell_2^{c'}}\right\|_{L_p}.$$

Proof. For every $b \in \ell_2^c$ we set

$$(Qb)_{\sigma,n} := \Psi_{\sigma,n} b_{\sigma,n},$$

and for every $a \in L_p\left(\mathbb{R}^d, \ell_2^c\right)$ we set

$$(\mathcal{K}a)_{\sigma,n} := \sum_{I\in\mathcal{I}} \sum_{(\tau,\ell)\hookrightarrow_I(\sigma,n)} 1_{|I} a_{\tau,\ell}.$$

We let Q^{Op} be the operator in (2.27) associated to Q. We note that

$$Q^{\mathrm{Op}}_{\hookrightarrow_I} = Q^{\mathrm{Op}} \circ \mathcal{K}.$$

Using Lemma 2.3.1, we bound for some constant $C_1 > 0$

$$\left\| Q^{\mathrm{Op}} \mathcal{K} a \right\|_{L_p\left(\mathbb{R}^d, \ell^c_2\right)} \leqslant C_1 \left\| \mathcal{K} a \right\|_{L_p\left(\mathbb{R}^d, \ell^c_2\right)}.$$

We estimate with constants $C_2, C_3 > 0$, using pairwise disjointness of elements $I \in \mathcal{I}$, Cauchy–Schwarz and the bound in (2.70),

$$\begin{aligned}
\left\| \mathcal{K} a \right\|_{L_p\left(\mathbb{R}^d, \ell^c_2\right)} &= \left\| \left(\sum_{\sigma,n} 4^{c(\sigma)n} \left(\sum_{I \in \mathcal{I}} \sum_{(\tau,\ell) \hookrightarrow_I (\sigma,n)} 1_{|I} a_{\tau,\ell} \right)^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\
&\leqslant \left\| \left(\sum_{\sigma,n} \sum_{I \in \mathcal{I}} \left(\sum_{(\tau,\ell) \hookrightarrow_I (\sigma,n)} 2^{c(\sigma)n - c(\tau)\ell} \right) \sum_{(\tau,\ell) \hookrightarrow_I (\sigma,n)} 2^{c(\sigma)n + c(\tau)\ell} \left| 1_{|I} a_{\tau,\ell} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\
&\leqslant C_2 \left\| \left(\sum_{I \in \mathcal{I}} \max\left\{ \|F\|^t_{+,I}, \|F\|^s_{-,I} \right\} \sum_{\sigma,n} \sum_{(\tau,\ell) \hookrightarrow_I (\sigma,n)} 2^{c(\sigma)n + c(\tau)\ell} \left| 1_{|I} a_{\tau,\ell} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\
&= C_2 \left\| \left(\sum_{I \in \mathcal{I}} \max\left\{ \|F\|^t_{+,I}, \|F\|^s_{-,I} \right\} \sum_{\tau,\ell} 2^{2c(\tau)\ell} \left| 1_{|I} a_{\tau,\ell} \right|^2 \sum_{(\tau,\ell) \hookrightarrow_I (\sigma,n)} 2^{c(\sigma)n - c(\tau)\ell} \right)^{\frac{1}{2}} \right\|_{L_p} \\
&\leqslant C_3 \left\| \left(\sum_{I \in \mathcal{I}} \max\left\{ \|F\|^{2t}_{+,I}, \|F\|^{2s}_{-,I} \right\} \left\| f_{|I} \right\|^2_{L_\infty} \sum_{\tau,\ell} 2^{2c(\tau)\ell} \left| \frac{1}{f_{|I}} a_{\tau,\ell} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_\infty} \\
&\leqslant C_3 \sup_{I \in \mathcal{I}} \max\left\{ \|F\|^t_{+,I}, \|F\|^s_{-,I} \right\} \left\| f_{|I} \right\|_{L_\infty} \left\| \left(\sum_{I \in \mathcal{I}} \sum_{\tau,\ell} 2^{2c(\tau)\ell} \left| \frac{1}{f_{|I}} a_{\tau,\ell} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p}.
\end{aligned}$$

The statement about $Q^{\mathrm{Op}}_{\hookrightarrow_{0,\mathcal{I}}}$ follows analogously, using (2.71). □

We recall (see above (2.47)) that F is assumed to be (Θ°, Θ)-hyperbolic on K and recall the maps $\widetilde{\Phi}^\circ_-, \widetilde{\Phi}^\circ_0$ assumed in Definition 2.2.2 which we use to construct

$\widetilde{\Psi}^{\circ}_{\sigma,n}$ defined in (2.24). We set

$$\mathcal{J} := \begin{array}{c} \{(-,0,0,0),(-,0,+,0),(0,0,+,0),(+,0,-,0)\} \cup \\ \{(+,\ell,+,0),(+,\ell,0,0),(0,\ell,0,0) \mid \ell \geqslant 0\} \cup \{(-,0,-,n) \mid n \geqslant 0\} \end{array}. \tag{2.73}$$

We recall the arrow notation f from Definition 2.4.12.

Lemma 2.4.14 (Directional inequality)**.** *Let* $(\tau,\ell) \nhookrightarrow_I (n,\sigma)$ *and* $(\tau,\ell,\sigma,n) \notin \mathcal{J}$*. Let* $\eta \in \operatorname{supp} \widetilde{\Psi}^{\circ}_{\tau,\ell}$ *and* $\xi \in \operatorname{supp} \Psi_{\sigma,n}$*. Set*

$$m(\tau) := \begin{cases} \max\{n,\ell\}, & \text{if } \tau \in \{-,0\} \\ n, & \text{if } \tau = + \end{cases}. \tag{2.74}$$

Then, for some $C > 0$ *and for all* $y \in I$ *it holds*

$$\left|(\mathrm{D}_y F)^{\mathrm{tr}}\,\eta - \xi\right| \geqslant C 2^{m(\tau)} \min\left\{1, \|F\|_{-,I}, \|F\|_{0,I}\right\}.$$

Proof. This can be seen case-by-case for admissible σ,τ as follows. We recall the set $\mathcal{J}$ defined in (2.73). We let $(\tau,\ell) \nhookrightarrow_I (n,\sigma)$ such that $(\tau,\ell,\sigma,n) \notin \mathcal{J}$. Due to the construction of $\Psi_{\sigma,n}$ and $\widetilde{\Psi}^{\circ}_{\tau,\ell}$, respectively, if $n \geqslant 1$ then $2^{n-1} \leqslant |\xi| \leqslant 2^{n+1}$ and if $\ell \geqslant 1$ then $2^{\ell-2} \leqslant |\eta| \leqslant 2^{\ell+2}$. We assume first $c(\sigma) \leqslant c(\tau)$. Let $\tau = +$. Then $2^{n-\ell} > 2^4 \|F\|_{+,I}$ and moreover, the exclusion of $\mathcal{J}$ implies $n \geqslant 1$. Using the triangle inequality, we find

$$\left|(\mathrm{D}_y F)^{\mathrm{tr}}\,\eta - \xi\right| \geqslant |\xi| - \left|(\mathrm{D}_y F)^{\mathrm{tr}}\,\eta\right| \geqslant 2^{n-1} - \|F\|_{+,I}\, 2^{\ell+2} \geqslant 2^{n-1} - 2^{n-2} \geqslant 2^{n-2}.$$

The case $\tau = 0$ is analogous. Just note that we have also the estimate

$$2^{n-1} - \|F\|_{0,I}\, 2^{\ell+2} \geqslant \|F\|_{0,I}\, 2^{l+3} - \|F\|_{0,I}\, 2^{\ell+2}.$$

If $\tau = -$ it holds $2^{n-\ell} < 2^{-4} \|F\|_{-,I}$. The exclusion of $\mathcal{J}$ implies $l \geqslant 1$. Using the triangle inequality, we find

$$\left|(\mathrm{D}_y F)^{\mathrm{tr}}\,\eta - \xi\right| \geqslant \left|(\mathrm{D}_y F)^{\mathrm{tr}}\,\eta\right| - |\xi| \geqslant \|F\|_{-,I}\, 2^{\ell-2} - 2^{n+1} > 2^{n+2} - 2^{n+1} \geqslant 2^{n+1}.$$

On the other hand we have also the estimate

$$\|F\|_{-,I}\, 2^{\ell-2} - 2^{n+1} > \|F\|_{-,I}\, 2^{\ell-2} - \|F\|_{-,I}\, 2^{l-3}.$$

Now we assume $c(\sigma) > c(\tau)$. We assume first $\tau = -$. Then $\sigma \in \{0,+\}$. We recall

that F is cone-hyperbolic (see Definition 2.2.2). The exclusion of $\mathcal{J}$ implies $n \neq 0$ or $l \neq 0$. Together with the first compact inclusion in (2.13) we conclude that the angle between $(\mathrm{D}_y F)^{\mathrm{tr}} \eta$ and ξ is bounded from below. This implies a lower bound $\geqslant C 2^{\max\{n,l\}}$ for the distance in both cases where $C > 0$ is some constant. We assume now $\tau = 0$ which implies $\sigma = +$. The reasoning is analogous as for $\tau = -$, using the second compact inclusion in (2.13) to bound the angle between $(\mathrm{D}_y F)^{\mathrm{tr}} \eta$ and ξ from below. □

Lemma 2.4.15. *Let $p \in [1, \infty]$, $b \in L_p$ and let $(\tau, \ell) \nhookrightarrow_I (\sigma, n)$ and $(\tau, n, \sigma, \ell) \neq \mathcal{J}$. It holds for the local transfer operator $L_{F, f_{|I}}$*

$$\left\| \Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f_{|I}} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} b \right\|_{L_p} \leqslant C_3(F, f_{|I}) 2^{-(r-1)m(\tau)} \|b\|_{L_p},$$

where for some $C \geqslant 1$ it holds
$C_3(F, f_{|I}) \leqslant C \max\left\{1, \|F\|_{-,I}^{1-r}, \|F\|_{0,I}^{1-r}\right\} \|\mathrm{D}\, F\|_{C^{r-1}}^{k} \|f\|_{C^{r-1}} \sup_{y \in K} |\det \mathrm{D}_y F|^{-\frac{1}{p}}$.

Proof. This is analogous to the proof of Lemma 2.3.3, except that we have to deal with the additional composition operation by the map F. We set $f := f_{|I}$. We expand the convolution and inverse Fourier transform

$$\begin{aligned}\Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} b(x) &= C \int_{\mathbb{R}^{4d}} e^{\mathrm{i}\,\eta(F(z)-y)} e^{\mathrm{i}\,\xi(x-z)} f(z) b(y) \Psi_{\sigma,n}(\xi) \widetilde{\Psi}^{\circ}_{\tau,\ell}(\eta) \mathrm{d}\eta \mathrm{d}\xi \mathrm{d}z \mathrm{d}y \\ &= C \int_{\mathbb{R}^d} V^{\tau,\ell}_{\sigma,n}(x, y) b(F(y)) \left|\det \mathrm{D}\, F(y)\right| \mathrm{d}y,\end{aligned}$$

for some constant $C > 0$ and where we set

$$V^{\tau,\ell}_{\sigma,n}(x, y) := \int_{\mathbb{R}^{3d}} e^{-\mathrm{i}\,\eta F(y)} e^{\mathrm{i}\,\xi x} \Psi_{\sigma,n}(\xi) \widetilde{\Psi}^{\circ}_{\tau,\ell}(\eta) e^{\mathrm{i}(\eta F(z) - \xi z)} f(z) \mathrm{d}z \mathrm{d}\eta \mathrm{d}\xi. \tag{2.75}$$

We transform (2.75), first integrating by parts $\lfloor r \rfloor - 1$-times in z (see Lemma B.3 with function $G(z) := \eta F(z) - \xi z$ which has a gradient bounded from below by Lemma 2.4.14). Therefore we replace $f(z)$ in (2.75) with another function $V_{\lfloor r \rfloor - 1}(z, \eta, \xi)$ which satisfies the iterative construction given in Lemma B.3 (B.1). Using Lemma 2.4.14 and Lemma B.3 (B.2), we estimate for some constant $C \geqslant 1$

$$\left\| V_{\lfloor r-1 \rfloor} \right\|_{C^0} \leqslant C \widetilde{C}_1 2^{-m(\tau)\lfloor r-1 \rfloor} \max\left\{1, \|F\|_{-,I}^{-\lfloor r-1 \rfloor}, \|F\|_{0,I}^{-\lfloor r-1 \rfloor}\right\} \|f\|_{C^{\lfloor r-1 \rfloor}},$$

where $\widetilde{C}_1 := \sup_{(z,\eta,\xi) \in \operatorname{supp} f} \max_{0 \leqslant |\gamma| \leqslant \lfloor r-1 \rfloor} \left| \left| (\mathrm{D}_z F)^{\mathrm{tr}} \eta - \xi \right| \partial_z^{\gamma} \frac{(\mathrm{D}_z F)^{\mathrm{tr}} \eta - \xi}{\left|(\mathrm{D}_z F)^{\mathrm{tr}} \eta - \xi\right|^2} \right|^{\lfloor r-1 \rfloor}$.
Moreover, this function is a $C^{\widetilde{r}}$-map for $\widetilde{r} := r - \lfloor r \rfloor$. Using Lemma B.5 (in there we take $\epsilon = L^{-1} = 2^{-m(\tau)}$), we proceed with a regularized integration by parts

in z. This yields

$$V_{\sigma,n}^{\tau,\ell}(x,y) = \int_{\mathbb{R}^{3d}} e^{\mathrm{i}\,\eta(F(z)-F(y))} e^{\mathrm{i}\,\xi(x-z)} \Psi_{\sigma,n}(\xi) \widetilde{\Psi}_{\tau,\ell}^{\circ}(\eta) V_{r-1}(z,\eta,\xi) \mathrm{d}z \mathrm{d}\eta \mathrm{d}\xi, \tag{2.76}$$

where V_{r-1} is given in (B.4) in Lemma B.5 with bound

$$\|V_{r-1}\|_{C^0} \leqslant C\widetilde{C}_2 2^{-m(\tau)\tilde{r}} \max\left\{1, \|F\|_{-,I}^{-\tilde{r}}, \|F\|_{0,I}^{-\tilde{r}}\right\}, \tag{2.77}$$

where $\widetilde{C}_2 := \sup_{(z,\eta,\xi)\in \mathrm{supp}\, f} \left(1 + \left|\left((\mathrm{D}_z F)^{\mathrm{tr}}\,\eta - \xi\right)\right|\right) \left\| \frac{\left((\mathrm{D}_z F)^{\mathrm{tr}}\eta - \xi\right) V_{\lfloor r-1 \rfloor}(\cdot,\eta,\xi)}{\left|(\mathrm{D}_z F)^{\mathrm{tr}}\eta - \xi\right|^2} \right\|_{C^{\tilde{r}}}$.
We now substitute $\xi \to 2^{\ell}\xi'$ and $\eta \to 2^{n}\eta'$ in (2.76). By construction the function $V_{r-1}\left(z, 2^n\eta', 2^\ell\xi'\right)$ is uniformly bounded in n and ℓ in the C^∞-norm with respect to η' and ξ'. We transform (2.76), integrating by parts $d+1$-times in ξ' if $|2^n(z-x)| > 1$, and $d+1$-times in η' if $\left|2^\ell(F(z)-F(y))\right| > 1$, which yields for some constant $C_1 > 0$

$$V_{\sigma,n}^{\tau,\ell}(x,y) = C_1 \int_{\mathbb{R}^{3d}} \frac{e^{\mathrm{i}\,2^\ell\eta'(F(z)-F(y))}}{u_\ell(F(z)-F(y))} \frac{e^{\mathrm{i}\,2^n\xi'(x-z)}}{u_n(x-z)} 2^{d(\ell+n)} \widetilde{V}_{\sigma,n}^{\tau,\ell}(z,\eta',\xi') \mathrm{d}z \mathrm{d}\eta' \mathrm{d}\xi',$$

where $\widetilde{V}_{\sigma,n}^{\tau,\ell}(z,\eta',\xi')$ together with $u_n \colon \mathbb{R}^d \to (0,1] : x \mapsto \begin{cases} 1 & , \text{ if } |2^n x| \leqslant 1 \\ |2^n x|^{d+1} & , \text{ else} \end{cases}$
replaces $\Psi_{\sigma,n}(2^\ell\xi')\widetilde{\Psi}_{\tau,\ell}^{\circ}(2^n\eta')V_{r-1}\left(z, 2^n\eta', 2^\ell\xi'\right)$ in (2.76). Since we only derived V_{r-1} with respect to η' and ξ', respectively, the C^0-norm of $\widetilde{V}_{\sigma,n}^{\tau,\ell}(z,\eta',\xi')$ is controlled by the upper bound given in (2.77). We recall that ξ', η' are uniformly bounded. We estimate trivially for some constant $C_2 \geqslant 1$

$$\begin{aligned} \left\|\Psi_{\sigma,n}^{\mathrm{Op}} L_{F,f} \widetilde{\Psi}_{\tau,\ell}^{\circ\,\mathrm{Op}} b\right\|_{L_p} &\leqslant C \left\|\widetilde{V}_{\sigma,n}^{\tau,\ell}\right\|_{C^0} 2^{d(n+\ell)} \left\| \frac{1}{u_n} * \left((b * \frac{1}{u_\ell}) \circ F \right) \right\|_{L_p} \\ &\leqslant C_2 C_3(F,f) 2^{-m(\tau)(r-1)} \|b\|_{L_p}, \end{aligned}$$

where we used twice Young's inequality in the last step.

□

We set for all $n \in \mathbb{Z}_{\geqslant 0}$, for all $\sigma \in \{-,+,0\}$ and for all $I \in \mathcal{I}$

$$\left(Q_{\mapsto,I}^{\mathrm{Op}} a\right)_{\sigma,n} := \Psi_{\sigma,n}^{\mathrm{Op}} \sum_{(\tau,\ell)\mapsto_I(\sigma,n)} a_{\tau,\ell}, \quad \left(Q^{\mathrm{Op}} a\right)_{\sigma,n} := \widetilde{\Psi}_{\sigma,n}^{\circ\,\mathrm{Op}} a_{\sigma,n}. \tag{2.78}$$

Lemma 2.4.16 (Boundedness II)**.** *Let $c(+) - c'(-) < r - 1$. Then for all $p \in [1, \infty]$ the map $Q^{\mathrm{Op}}_{\dashv\!\rightarrow,I} L_{F,f_{|I}} Q^{\mathrm{Op}} \colon L_p\left(\mathbb{R}^d, \ell_2^{c'}\right) \to L_p\left(\mathbb{R}^d, \ell_2^{c}\right)$ is a bounded linear operator. In particular, it holds*

$$\left\| Q^{\mathrm{Op}}_{\dashv\!\rightarrow,I} L_{F,f_{|I}} Q^{\mathrm{Op}} \right\|_{L_p\left(\mathbb{R}^d,\ell_2^{c'}\right)\to L_p\left(\mathbb{R}^d,\ell_2^{c}\right)} \leqslant C C_4(F, f_{|I}),$$

where for some $C \geqslant 1$ and some $k \geqslant 0$

$$C_4(F,f) \leqslant C \max\left\{1, \|F\|_{-,I}^{1-r}, \|F\|_{0,I}^{1-r}\right\} \max\left\{1, \|\mathrm{D}\, F\|_{C^{r-1}}^{k}\right\} \|f\|_{C^{r-1}} \sup_{y\in K} |\det \mathrm{D}_y\, F|^{-\frac{1}{p}}$$

Proof. Let $(a_{\tau,\ell}) = a \in L_p\left(\mathbb{R}^d, \ell_2^{c'}\right)$. We have

$$\begin{aligned}\left\| Q^{\mathrm{Op}}_{\dashv\!\rightarrow,I} L_{F,f_{|I}} Q^{\mathrm{Op}} a \right\|_{L_p\left(\mathbb{R}^d,\ell_2^{c}\right)} &= \left\| \left(\sum_{\sigma,n} 4^{c(\sigma)n} \left| \sum_{(\tau,\ell)\dashv\!\rightarrow_I(\sigma,n)} \Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f_{|I}} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} a_{\tau,\ell} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\leqslant \sum_{(\tau,\ell)\dashv\!\rightarrow_I(\sigma,n)} 2^{c(\sigma)n} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f_{|I}} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} a_{\tau,\ell} \right\|_{L_p}. \quad (2.79)\end{aligned}$$

We recall the set of indices $\mathcal{J}$ in (2.73). We assume $(\tau, l, \sigma, n) \in \mathcal{J}$. Now we make three distinctions in the estimate of the corresponding part of the sum in (2.79). If $\tau \in \{+, 0\}$ then $n = 0$ and $l \geqslant 0$. Then, using Young's inequality, for some $C \geqslant 1$

$$\begin{aligned}\sum_{l=0}^{\infty} \left\| \Psi^{\mathrm{Op}}_{\sigma,0} L_{F,f_{|I}} \widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell} a_{\tau,\ell} \right\|_{L_p} &\leqslant C \sup_{z\in I} |f(z)|\, |\det \mathrm{D}_z\, F|^{-\frac{1}{p}} \sum_{l=0}^{\infty} 2^{-c'(\tau)l} \sup_{l} 2^{c'(\tau)l} \|a_{\tau,\ell}\|_{L_p} \\ &\leqslant C \sup_{z\in I} |f(z)|\, |\det \mathrm{D}_z\, F|^{-\frac{1}{p}} \|a\|_{L_p\left(\mathbb{R}^d,\ell_2^{c'}\right)}.\end{aligned}$$

If $\tau = \sigma = -$ then $n \geqslant 0$ and $l = 0$. Recall that $s < 0$. Then, using Young's inequality,

$$\sum_{n=0}^{\infty} 2^{sn} \left\| \Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f_{|I}} \widetilde{\Psi}'^{\mathrm{Op}}_{-,0} a_{-,0} \right\|_{L_p} \leqslant C \sup_{z\in I} |f(z)|\, |\det \mathrm{D}_z\, F|^{-\frac{1}{p}} \|a\|_{L_p\left(\mathbb{R}^d,\ell_2^{c'}\right)}.$$

In the three remaining cases $n = l = 0$ we estimate analogously, using Young's inequality. Now we assume $(\tau, l, \sigma, n) \notin \mathcal{J}$. We recall $m(\tau)$ defined in (2.74) in Lemma 2.4.14 and the constant $C_3(F, f)$ in Lemma 2.4.15. Using Lemma 2.4.15,

we estimate the remaining part of the sum in (2.79)

$$\begin{aligned}\sum_{\substack{(\tau,\ell)\nmapsto_I(\sigma,n)\\(\tau,l,\sigma,n)\notin\mathcal{J}}} 2^{c(\sigma)n}\left\|\Psi^{\mathrm{Op}}_{\sigma,n}L_{F,f_{|I}}\widetilde{\Psi}^{\circ\,\mathrm{Op}}_{\tau,\ell}a_{\tau,\ell}\right\|_{L_p}&\\ \leqslant CC_3(F,f)\sum_{\substack{(\tau,\ell)\nmapsto_I(\sigma,n)\\(\tau,l,\sigma,n)\notin\mathcal{J}}} 2^{c(\sigma)n-c'(\tau)l-m(\tau)(r-1)}\sup_{\tau,\ell}2^{c'(\tau)l}\left\|a_{\tau,\ell}\right\|_{L_p}&\\ \leqslant CC_3(F,f)\left\|a\right\|_{L_p\left(\mathbb{R}^d,\ell_2^{c'}\right)},&\end{aligned}\tag{2.80}$$

where the sums in n,l, respectively, in the right-hand side in (2.80) are bounded by geometric sums, using the assumption $c'(-)>r-1-c(+)$. In particular, we find for (the worst-case since $0<c(0)\leqslant c(+)<r-1$) $\tau=-$, $\sigma=+$, if $l\geqslant n$ for all small enough $\epsilon>0$

$$\begin{aligned}(c(+)+\epsilon)\,n-c'(-)l-m\,(-)\,(r-1)&\leqslant\left(c(+)+\epsilon-c'(-)\right)l-m\,(-)\,(r-1)\\&=\left(c(+)+\epsilon-c'(-)-r+1\right)l<0,\end{aligned}$$

and an analogous estimate holds for $l<n$. We note

$$\sup_{z\in I}\left|f(z)\right|\left|\det \mathrm{D}_z\,F\right|^{-\frac{1}{p}}\leqslant C\left\|f\right\|_{C^{r-1}}\sup_{z\in I}\left|\det \mathrm{D}_z\,F\right|^{-\frac{1}{p}}.$$

We set $C_4(F,f_{|I}):=CC_3(F,f_{|I})$. Combining the estimates for all the parts of the sum (2.79), we conclude. □

Proof of Lemma 2.4.1. Let s,q,t,p satisfy the hypotheses in Lemma 2.4.1. That is $s'<s<0<q\leqslant t<r-1+s'$, $q'<q$, $t'<t$ and $p\in[1,\infty]$. We put $c(-):=s$, $c(+):=t$, $c(0):=c'(0):=q$ and $c'(-):=c''(-):=s'$, $c'(+):=c''(+):=t'$ and $c''(0):=q'$. Then c,c',c'' satisfy (2.29), respectively, while c, c' satisfy the hypotheses in Lemma 2.4.13, and c, c'' that of Lemma 2.4.16. Let $\varphi\in W^{s,t,q}_{p,\Theta,F(K)}$. We set

$$a_{\tau,\ell}:=L_{F,f}\Psi^{\circ\,\mathrm{Op}}_{\tau,\ell}\varphi.$$

We have $a \in L_p\left(\mathbb{R}^d, \ell_2^c\right) \subseteq L_p\left(\mathbb{R}^d, \ell_2^{c'}\right)$ because

$$\begin{aligned}\|a\|_{L_p\left(\mathbb{R}^d,\ell_2^c\right)} &= \left\| \left(\sum_{\tau,\ell} 4^{c(\tau)\ell} \left|a_{\tau,\ell}\right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} = \left\| \left(\sum_{\tau,\ell} 4^{c(\tau)\ell} \left| f \cdot \left(\Psi_{\tau,\ell}^{\circ\,\mathrm{Op}}\varphi\right) \circ F \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\leqslant \left\| f \left|\det \mathrm{D}\, F\right|^{-\frac{1}{p}} \right\|_{L_\infty(K)} \left\| \left(\sum_{\tau,\ell} 4^{c(\tau)\ell} \left| \Psi_{\tau,\ell}^{\circ\,\mathrm{Op}}\varphi \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p(F(K))} \qquad (2.81) \\ &\leqslant \left\| f \left|\det \mathrm{D}\, F\right|^{-\frac{1}{p}} \right\|_{L_\infty} \|\varphi\|_{W^{s,t,q}_{p,\Theta^\circ,F(K)}}.\end{aligned}$$

We set $b_{\tau,\ell} := \Psi_{\tau,\ell}^{\circ\,\mathrm{Op}}\varphi$. By the first statement in Lemma 2.3.4, it holds $\varphi \in W^{s',t',q'}_{p,\Theta^\circ,F(K)}$ hence $(b_{\tau,\ell} \mid \tau \in \{-,+,0\}, \ell \in \mathbb{Z}_{\geqslant 0}) =: b \in L_p\left(\mathbb{R}^d, \ell_2^{c''}\right)$. By assumption on K, we can decompose $K = \sqcup \mathcal{W}$ into finitely many open sets $\mathcal{W}$. For each component $\mathcal{W}$ we set

$$a_{|\mathcal{W},\tau,\ell} := L_{F,f_{|\mathcal{W}}} \Psi_{\tau,\ell}^{\circ\,\mathrm{Op}}\varphi \quad \text{and} \quad a := \sum_{\mathcal{W}} a_{|\mathcal{W}}.$$

By construction (see above (2.78)), it holds $\widetilde{\Psi}^\circ_{\tau,\ell|\operatorname{supp}\Psi'_{\tau,\ell}} \equiv 1$ hence $\Psi_{\tau,\ell}^{\circ\,\mathrm{Op}}\varphi = \widetilde{\Psi}_{\tau,\ell}^{\circ\,\mathrm{Op}}\Psi_{\tau,\ell}^{\circ\,\mathrm{Op}}\varphi$. For each $\mathcal{W}$ there is a corresponding arrow relation given by Definition 2.4.12 and the restriction $f_{|\mathcal{W}}$ is also C^{r-1}. We rewrite

$$\begin{aligned}\Psi^{\mathrm{Op}}_{\sigma,n} L_{F,f_{|\mathcal{W}}}\varphi = \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{\tau,\ell} a_{|\mathcal{W},\tau,\ell} = &\Psi^{\mathrm{Op}}_{\sigma,n} \sum_{(\tau,\ell)\hookrightarrow\mathcal{W}(\sigma,n)} a_{|\mathcal{W},\tau,\ell} + \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{l\hookrightarrow\mathcal{W}(\sigma,n)} a_{|\mathcal{W},0,\ell} \\ &+ \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{(\tau,\ell)\nhookrightarrow\mathcal{W}(\sigma,n)} L_{F,f_{|\mathcal{W}}} \widetilde{\Psi}_{\tau,\ell}^{\circ\,\mathrm{Op}} \Psi_{\tau,\ell}^{\circ\,\mathrm{Op}}\varphi. \qquad (2.82)\end{aligned}$$

We recall the definitions of the operators $Q^{\mathrm{Op}}_{\hookrightarrow,\{\mathcal{W}\}}$, $Q^{\mathrm{Op}}_{\hookrightarrow_0,\{\mathcal{W}\}}$, $Q^{\mathrm{Op}}_{\nhookrightarrow,\mathcal{W}}$ given in (2.72) and in (2.78), respectively (in which we take $\mathcal{I} = \{\mathcal{W}\}$ and $I = \mathcal{W}$). We estimate, using the decomposition given in (2.82),

$$\begin{aligned}\|L_{F,f}\varphi\|_{W^{s,t,q}_{p,\Theta,K}} &= \left\| \sum_{\mathcal{W}} L_{F,f_{|\mathcal{W}}}\varphi \right\|_{W^{s,t,q}_{p,\Theta,K}} = \left\| \left(\sum_{\sigma,n} 4^{c(\sigma)n} \left| \Psi^{\mathrm{Op}}_{\sigma,n} \sum_{\mathcal{W}} L_{F,f_{|\mathcal{W}}}\varphi \right|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\leqslant 5 \left\| Q^{\mathrm{Op}}_{\hookrightarrow,\{\mathcal{W}\}} a \right\|_{L_p\left(\mathbb{R},\ell_2^c\right)} + 5 \left\| Q^{\mathrm{Op}}_{\hookrightarrow_0,\{\mathcal{W}\}} a \right\|_{L_p\left(\mathbb{R},\ell_2^c\right)} + 5 \sum_{\mathcal{W}} \left\| Q^{\mathrm{Op}}_{\nhookrightarrow,\mathcal{W}} L_{F,f_{|\mathcal{W}}} Q^{\mathrm{Op}} b \right\|_{L_p\left(\mathbb{R},\ell_2^c\right)}.\end{aligned}$$

We conclude, using Lemma 2.4.13 and Lemma 2.4.16 together with the estimate given in (2.81). □

2.5 Asymptotics of horocycle averages

In this section, we assume $r \geqslant 2$ and topological mixing of the Anosov flow g_α. (Contact Anosov flows are topologically mixing [41, Theorem 3.6] and hence serve as examples for such Anosov flows g_α in the case $d = 3$.) In order to define the horocycle flow in Definition 2.5.1 below we assume that the stable dimension $d_- = 1$ and that the strong-stable distribution E_- is orientable. The stable manifolds of M with respect to the flow g_α are those (non-compact) Riemannian submanifolds which are tangent to E_-. As consequence of topological mixing, each of those stable manifolds is dense in M [49, p. 84].

2.5.1 Horocycle flows and integrals and main results (Theorem 2.5.7)

Definition 2.5.1 (Horocycle flow)**.** *A flow $h_\rho \colon M \to M$ in $\rho \in \mathbb{R}$ is called a stable horocycle flow if and only if for all $\rho \in \mathbb{R}$*

$$\partial_\rho h_\rho \in E_- \setminus \{0\} .$$

Remark 2.5.2 (Unit speed parametrization)**.** *By the Stable Manifold Theorem (see e.g. [42, Theorem 8.12]), there exists a parametrization of stable manifolds by the arc-length induced by the Riemannian metric on M. Since we assumed that E_- is orientable, this yields the unit speed parametrization of the horocycle flow (i.e. $|\partial_\rho h_\rho| \equiv 1$).*

Our main result, Theorem 2.5.7 provides a decomposition giving the T-asymptotics of the following horocycle integral:

Definition 2.5.3 (Horocycle integral)**.** *For all $\varphi \in C_X^{r-1}(M)$, for all $x \in M$ let*

$$\gamma_x(\varphi, T) := \int_0^T \varphi \circ h_\rho(x) \, \mathrm{d}\, \rho. \tag{2.83}$$

denote the horocycle integral of the horocycle flow h_ρ for the observable φ at base point x.

In Theorem 2.5.7 we reveal its connection to the eigendistributions of a weighted transfer operator for the Anosov flow $g_{-\alpha}$ introduced in Section 2.3.1, using renormalization dynamics to connect the stable flow with the Anosov flow. Results can be obtained for an unstable horocycle flow in an analogous way.

Definition 2.5.4 (Pointwise renormalization time)**.** *A map* $\tau\colon \mathbb{R}^2 \times M \to \mathbb{R}$ *which satisfies*

$$g_\alpha \circ h_\rho(x) = h_{\tau(\rho,\alpha,x)} \circ g_\alpha(x)\,, \qquad \forall \rho, \alpha \in \mathbb{R}\,, \forall x \in M, \tag{2.84}$$

is called a pointwise renormalization time for the stable flow h_ρ*.*

Remark 2.5.5. *This definition of the renormalization time* τ *is the same as used by Marcus (denoted by* s^* *in his notation) in [49, p.83] to study ergodic properties of the horocycle flow.*

Lemma 2.5.6 (Existence and uniqueness)**.** *A pointwise renormalization time exists and is unique.*

Proof. For every $x \in M$ and for every $\rho, \alpha \in \mathbb{R}$ we set $h_{\alpha,\rho}(x) := g_\alpha \circ h_\rho \circ g_{-\alpha}(x)$. By Definition 2.5.1 and the invariant splitting (2.2), we find $\partial_\rho h_{\alpha,\rho} \in E_{-,x} \setminus \{0\}$. Hence $h_{\alpha,\rho}(x)$ parametrizes the same stable manifold with respect to ρ as $h_\rho(x)$. If there were two different pointwise renormalization times τ, there would be $\rho_1 < \rho_2 \in \mathbb{R}$ such that $h_{\alpha,\rho}(x) = h_{\rho_1}(x) = h_{\rho_2}(x)$. By density of stable leaves and non-singularity of the flow h_ρ, there are no periodic points of h_ρ hence $\rho_1 = \rho_2$. □

Further properties of the renormalization time τ are given in Proposition 2.5.13 below. Assuming $\partial_\rho \tau(0, -\alpha, \cdot) \in C^{r-1}(M)$ for all $\alpha \geqslant 0$, we will consider the potential V defined by

$$V \equiv -\partial_\alpha \partial_\rho \tau(0, 0, \cdot)\,. \tag{2.85}$$

Then ϕ_α defined in (2.18) is just

$$\phi_\alpha := \partial_\rho \tau(0, -\alpha, \cdot). \tag{2.86}$$

It follows from (iv) in Lemma 2.5.18 below that for any $p \in [1, \infty]$, $t - s < r - 1$ and $s < 0 < q \leqslant t$ the spectral bound $\lambda_{\max} = \sup \Re\sigma(X + V)|_{W_p^{s,t,q}}$ for the generator satisfies

$$\lambda_{\max} = h_{\mathrm{top}}. \tag{2.87}$$

In the special case of unit speed horocycle flow (see Remark 2.5.2) it holds (using Proposition 2.5.13 (viii) below)

$$\phi_\alpha = \det \mathrm{D}\, g_{-\alpha|E^-}.$$

Hence if the strong stable distribution E_- is C^1 (see Proposition 2.5.10 where this holds true if $d = 3$ under the contact assumption) and $r \geqslant 2$ then we find $\partial_\rho \tau(0, -\alpha, \cdot) \in C^1_X(M)$. In particular, our results apply to all C^1 time reparametrizations of the unit speed horocycle flow h_ρ (this is analogous to [30, Remark 2.4]).
The following theorem will be proved at the end of Section 2.5.3:

Theorem 2.5.7 (Expansion of horocycle integrals)**.** *Let g_α be a topologically mixing C^r-Anosov flow, with $r \geqslant 2$, such that E_- is orientable and $d_- = 1$. Let μ be the unique Borel measure which is invariant by the horocycle flow h_ρ. Assume for all $\alpha \geqslant 0$*

$$\phi_\alpha := \partial_\rho \tau(0, -\alpha, \cdot) \in C^{r-1}(M).$$

Assume further that there exist $p \in [1, \infty]$, and $s < 0 < q \leqslant t$ with $t - s < r - 1$ such that, for the corresponding anisotropic space $W_p^{s,t,q}$ it holds $\lambda_{min} < \lambda_{max} = h_{top}$, with $\lambda_{min} = \lambda_{min}(t, s, p)$ from (2.56). Then, for all $x \in M$ and $T \geqslant 0$ there exist, for each $\lambda \in \sigma(X + V)|_{W_p^{s,t,q}}$ with $\Re\lambda > \lambda_{min}$, constants $c_{(\lambda,i,j)}(T, x) \in \mathbb{C}$ with

$$\sup_{T>0,\, x\in M} |c_{(\lambda,i,j)}(T, x)| < \infty\,,\ \forall 1 \leqslant i \leqslant n_\lambda\,,\ 1 \leqslant j \leqslant m_{\lambda,i},$$

such that, for any $\delta \in \mathbb{R}$ with

$$\max\{\lambda_{min}, 0\} \leqslant \delta < h_{top}$$

and any finite [5] subset Λ_δ of

$$\Sigma_\delta := \sigma(X + V)|_{W_p^{s,t,q}} \cap \{\lambda \in \mathbb{C} \mid \Re\lambda > \delta\}\,,$$

such that for all $\varphi \in C^{r-1}_X(M)$ and all $T \geqslant e$

$$\int_0^T \varphi \circ h_\rho(x)\, \mathrm{d}\rho = \gamma_x\left(\mathcal{D}_{(h_{top},1,1)}, T\right)\mu(\varphi) + \sum_{\substack{\lambda\in\Lambda_\delta \\ \Re\lambda<h_{top}}} \sum_{\substack{1\leqslant i\leqslant n_\lambda \\ 1\leqslant j\leqslant m_{\lambda,i}}} T^{\frac{\lambda}{h_{top}}} (\log T)^{j-1} c_{(\lambda,i,j)}(T, x)\, \mathcal{O}_{(\lambda,i,j)}(\varphi) + \mathcal{E}_{T,x,\Lambda_\delta}(\varphi)\,,$$

where the dual eigendistributions $\mathcal{O}_{(\lambda,i,j)} \in D(X + V)'$ are associated to the

[5] Note that Lemma 2.4.10 and our choice of δ ensure that for any finite $b > 0$ the spectral box $\Lambda_\sigma(b) = \sigma(X + V)|_{W_p^{s,t,q}} \cap \{\Re\lambda > \delta,\ |\Im\lambda| \leqslant b\}$ is a finite set.

eigenvalue λ by Lemma 2.4.10 (see (2.67)), and where

$$\lim_{T\to\infty}\frac{\gamma_x\left(\mathcal{D}_{(h_{top},1,1)},T\right)}{T}=1\quad\text{and}\quad\lim_{T\to\infty}\frac{\mathcal{E}_{T,x,\Lambda_\delta}(\varphi)}{T}=0.$$

Moreover, if for some $c>0$ and some constant $C_1=C_1(\varphi,c,\Lambda_\delta)\geqslant 0$ for all $\alpha\geqslant 0$

$$\left\|\mathcal{L}_{\alpha,\phi_\alpha}\left(\sum_{\lambda\in\Lambda_\delta}\sum_{i=1}^{n_\lambda}\Pi_{\lambda,i}\varphi-\varphi\right)\right\|_{W_p^{s,t,q}}\leqslant C_1e^{c\alpha},\tag{2.88}$$

then there exists $C_2>0$ such that

$$\sup_{x\in M}|\mathcal{E}_{T,x,\Lambda_\delta}(\varphi)|\leqslant C_2\left(C_1T^{\frac{c}{h_{top}}}+\|\varphi\|_{C^0}+1\right).$$

If, in addition, Condition 2.4.11 holds for δ, then Σ_δ is finite and, taking $\Lambda_\delta=\Sigma_\delta$ and assuming

$$t-r+2\leqslant 0<r-2,$$

it holds $c=\delta+\epsilon$ in (2.88) for all $\epsilon>0$ and all $\varphi\in W_p^{s,t,q}$.

Recall that if $\mathcal{D}_{(\lambda,i,j)}\in W_p^{s,t,q}$, for some λ with $\Re\lambda>\delta$, is a generalized eigenvector of the generator $X+V$ then for all $\Re\tilde{\lambda}>\delta$ we have that $\mathcal{O}_{\tilde{\lambda},\tilde{\imath},\tilde{\jmath}}(\mathcal{D}_{\lambda,i,j})=1$ if $\lambda=\tilde{\lambda}$, $i=\tilde{\imath}$, and $j=\tilde{\jmath}$, while $\mathcal{O}_{\tilde{\lambda},\tilde{\imath},\tilde{\jmath}}(\mathcal{D}_{\lambda,i,j})$ vanishes otherwise.

Remark 2.5.8. *The condition $\lambda_{max}=h_{top}$ is superficial although we show only $\lambda_{max}=h_{top}$ and unique simplicity under an additional vanishing assumption in Section 2.5.3. The proof of Theorem 2.5.7 however shows that the horocycle expansion sees only the part of the spectrum with real part below h_{top} and the eigendistribution μ which is associated to h_{top}.*
Recalling Remark 2.4.8, we find always $\lambda_{min}<h_{top}$ if $-s$ and t can be taken to be $1-\epsilon$ for all $\epsilon>0$. This is the case if the geodesic flow is $C^{3-\epsilon}$ for all $\epsilon>0$ (e.g. the flow is of Zygmund type). If one knows then that the weight is $C^{2-\epsilon}$ the basic assumptions of Theorem 2.5.7 are all satisfied (an example is given in Proposition 2.5.10 below for C^3 contact Anosov flows when $d=3$).
Note that $\gamma_x\left(\mathcal{D}_{(h_{top},1,1)},T\right)$ is well-defined in the sense of distributions is part of the theorem. By unique ergodicity the expected principal term $T\mu(\varphi)$ is hidden by the term $\gamma_x\left(\mathcal{D}_{(h_{top},1,1)},T\right)\mu(\varphi)$ as we ordered the expansion by the distributions $\mathcal{O}_{...}(\varphi)$. We can always write

$$T\mu(\varphi)=\gamma_x(1,T)\,\mu(\varphi)$$

and use the expansion result on $\gamma_x(1,T)$ *again which shows that the leading order term is indeed what we expect. The other terms are modified by the contributions of* $\mathcal{O}_{...}(1)\,\mu(\varphi)$*. We make use of this in the following corollary.*

Assuming all conditions in the above theorem, this gives polynomial convergence of horocycle averages to the ergodic mean:

Corollary 2.5.9 (Polynomial convergence)**.** *Under the assumptions of Theorem 2.5.7 (including Condition 2.4.11 for* δ *and* $t-r+2\leqslant 0<r-2$*) then there exists* $\epsilon>0$ *such that for all* $\varphi\in C_X^{r-1}(M)$ *there exists* $C>0$ *such that*

$$\left|\frac{1}{T}\int_0^T \varphi\circ h_\rho(x)\,\mathrm{d}\,\rho-\mu(\varphi)\right|\leqslant CT^{-\epsilon},$$

where μ *is the unique Borel measure which is invariant by the horocycle flow* h_ρ*.*

Proof. We apply Theorem 2.5.7, using the assumption that Condition 2.4.11 holds for δ and that $t-r+2\leqslant 0<r-2$. Then there are only finitely many eigenvalues $\lambda\in\sigma(X+V)$ such that $\Re\lambda>\delta$ and the remainder term $\mathcal{E}_{T,x,\Lambda_\delta}(\varphi)$ is bounded from above by $T^{\frac{\delta}{h_{\mathrm{top}}}+\epsilon}$ for all $\epsilon>0$. Hence all but one term in the expansion of the ergodic average decay like $T^{-\epsilon}$ for some $\epsilon>0$. We finally bound the leading term in the expansion

$$\frac{1}{T}\gamma_x\left(\mathcal{D}_{(h_{\mathrm{top}},1,1)},T\right)\mu(\varphi)-\mu(\varphi)=\frac{1}{T}\gamma_x\left(\mathcal{D}_{(h_{\mathrm{top}},1,1)}-1,T\right)\mu(\varphi),$$

using again Theorem 2.5.7 as before, noting that $\mu(1)=\mu\left(\mathcal{D}_{h_{\mathrm{top}},1,1}\right)=1$. □

We next discuss the assumptions of our main theorem and the corollary above. We first give sufficient conditions ensuring that $\partial_\rho\tau(0,-\alpha,\cdot)\in C^{r-1}$ and that there exist parameters in our anisotropic space giving $\lambda_{\min}(s,t,p)<h_{\mathrm{top}}$:

Proposition 2.5.10. *Let* g_α *be a* C^3 *contact Anosov flow on a closed Riemannian manifold* M *of dimension* $d=3$ *preserving a* C^1 *contact form and let the strong-stable distribution* E_- *be orientable. Then there exists a horocycle flow* h_ρ *such that* $\partial_\rho\tau(0,-\alpha,\cdot)\in C^{r-1}$ *for every* $\alpha\geqslant 0$ *and for any* $r\in[2,3)$*.*
Setting $-s=t=\frac{r-1}{2}-\frac{\epsilon}{2}$ *for suitable* $0<\epsilon<\frac{r-1}{2}$*, the constant* $\lambda_{min}(s,t,p)$ *is independent of* p *and can be taken arbitrary close to* 0^+ *while* $t-r+2\leqslant 0<r-2$*.*

Proof. The contact assumption means that there is an invariant 1-form $\eta\in T^*M$ such that $\mu:=\eta\wedge\mathrm{d}\,\eta\neq 0$ everywhere. By assumption η is C^1. Moreover η is annihilated on $E_+ + E_-$ and $\mu\in\wedge^3 T^*M$ is preserved by the flow. We use [40, Theorem 3.1] together with the comment on the relation between Zygmund

and Hölder regularity to infer that the strong-stable distribution is C^{r-1} for all $r \in [2,3)$ if $d = 3$. Hence for the horocycle flow given by the unit speed parametrization (and more general all of its C^{r-1} reparametrizations) we find $\partial_\rho \tau(0, -\alpha, \cdot) \in C^{r-1}$. By assumption the flow g_α preserves volume and $d = 3$.
To see a gap between $\lambda_{\min} = \lambda_{\min}(s,t,p)$ and h_{top}, we may assume the unit speed parametrization of the horocycle flow h_ρ. It follows by Proposition 2.5.13 (viii) and Lemma 2.3.3 that for all C^{r-1} reparametrizations the resulting transfer operators are conjugate to each other.
Then it follows from Lemma 2.4.7 together with Proposition 2.5.13 (viii) that $\lambda_{\min}$ is independent of p and is arbitrary close to 0^+ for a suitable choice of s, t and r. Moreover, if we assume $0 < t \leqslant \frac{r-3}{2} + \epsilon$ we satisfy $t - r + 2 \leqslant 0 < r - 2$ since $\epsilon < \frac{r-1}{2}$. □

Second, we discuss Condition 2.4.11:

Remark 2.5.11. *Condition 2.4.11 was inspired by estimates of Dolgopyat [23], who was working with operators acting on symbolic spaces. This condition, replacing however our $W_p^{s,t,q}$ by other anisotropic Banach spaces, was proved by several authors [9], [31], [48], [66] for the generator $X + V$, associated to contact Anosov flows and $V = 0$ the trivial potential, for which they also obtained the additional condition in Corollary 2.5.9.*

In the case of geodesic flows on compact surfaces of constant negative curvature, we find that V is a constant, but the fact that our Banach space is different makes it difficult to apply the results of [9], [31], [48], [66] directly in order to establish Condition 2.4.11. We expect however that the condition holds and (as pointed out by Liverani and Butterley) can be obtained by exploiting e.g. [19, Remark 2.6].

For non-constant potential V, since Dolgopyat [23] obtained exponential decay of correlations for Gibbs measures with Hölder potentials, we expect that Condition 2.4.11 indeed holds also in our setting, in particular for compact surfaces of variable negative curvature (e.g. using an argument similar as for the proof in [24, Proposition 3.4]). (We warn the reader that the value of δ given by Dolgopyat-type arguments is usually very close to λ_{max}.)

We end this subsection by a comparison of our main theorem and the results of Flaminio and Forni [28]: Let M be the unit tangent bundle of a compact hyperbolic Riemann surface. Let g_α be its unit speed geodesic flow and let vol be the canonical (invariant) volume form on M (which is also a measure of

maximal entropy) and consider the unit speed horocycle flow which leaves vol invariant as well (hence $\mu = \mathrm{vol}$). Then $h_{\mathrm{top}} = 1$ because $\tau(\rho, \alpha, x) = \rho \exp(-\alpha)$ and $\mathcal{D}_{(1,1,1)} \equiv 1$ (hence $\gamma_x\left(\mathcal{D}_{(1,1,1)}, T\right) = T$). In the setting of Riemann surfaces, the possible Jordan blocks are known [28, Theorem 1.5]. In particular, the eigenvalue $h_{\mathrm{top}} = 1$ is simple, there are no other eigenvalues of real part equal to one, all eigenvalues with $\Re\lambda > 0$ are semi-simple, and there are only finitely many eigenvalues with $\Re\lambda > \frac{1}{2}$. Moreover, since the vector fields are constant, the regularity parameters $-s, t$ can be taken large enough such that $\lambda_{\min} < 0$. Hence we can take any $\delta \geqslant 0$ in Theorem 2.5.7, and we find, for any finite subset of Σ_δ containing 1,

$$\int_0^T \varphi \circ h_\rho(x)\, \mathrm{d}\rho = T \,\mathrm{vol}(\varphi) + \sum_{\lambda \in \Lambda_\delta \setminus \{1\}} \sum_{i=1}^{n_\lambda} T^\lambda c_{(\lambda,i,1)}(T, x)\, \mathcal{O}_{(\lambda,i,1)}(\varphi) + \mathcal{E}_{T,x,\Lambda_\delta}(\varphi),$$

where we can take $\Lambda_\delta = \Sigma_\delta$ if $\delta \geqslant \frac{1}{2}$, and where $c_{(\lambda,i,1)}$ and $\mathcal{E}_{T,x,\Lambda_\delta}$ satisfy the claims of Theorem 2.5.7 (with an additional $\log T$-factor if $\lambda = \frac{1}{2}$). In particular, if Condition 2.4.11 holds for some $\delta > \frac{1}{2}$ (see Remark 2.5.11) there exists $C > 0$ such that for all $\epsilon > 0$

$$|\mathcal{E}_{T,x,\Sigma_\delta}(\varphi)| \leqslant C T^{\delta+\epsilon}.$$

Note that we required $\delta \geqslant 0$ because for $\delta < 0$ we find no improvement of the remainder term (this comes the local bounds in Lemma 2.5.14). An analogous behavior is seen in the corresponding expansion of Flaminio–Forni in [28, Theorem 1.5]. However they are not limited to finite sets Λ_δ of eigenvalues (Faure–Tsujii do not seem to be limited either in [26]). Our methods, however, do not seem to allow to go beyond the first vertical line with infinitely many resonances in $\sigma(X + V)|_{W_p^{s,t,q}}$ in the expansion of the horocycle integral. (This could be a natural limitation, as discussed in [66, p.1497, below Theorem 1.1].)

2.5.2 Weighted horocycle integrals, properties of τ, local bounds

In order to use a smooth cutoff trick of Giulietti–Liverani to decompose $\gamma_x(\cdot, T)$ in Lemma 2.5.14 below, we need to consider weighted horocycle integrals: For all $\varphi \in C_X^{r-1}(M)$, for all compactly supported $w \in C(\mathbb{R}, \mathbb{C})$ and for all $x \in M$, let

$$\gamma_{w,x}(\varphi) := \int_{\mathbb{R}} w(\rho) \cdot (\varphi \circ h_\rho(x))\, \mathrm{d}\rho. \tag{2.89}$$

denote the horocycle integral of the horocycle flow h_ρ for the observable φ at base point x with weight w.
For further purposes, it is useful to view $\gamma_{w,x}$ as a functional in the topological dual space of $W_p^{s,t,q}$ for weights w with compact support and sufficient differentiability:

Lemma 2.5.12. *Let $p \in [1,\infty]$ and let $0 < q \leqslant t < r-1$ and let $-r < s < 0$. Let $x \in M$. Then for some $C > 0$, for all C^{-s} maps $w\colon \mathbb{R} \to \mathbb{C}$ with compact support it holds*

$$\|\gamma_{w,x}\|_{W_p^{s,t,q}\to\mathbb{C}} \leqslant C\,|\operatorname{supp} w|\,\|w\|_{C^{-s}}\,.$$

Proof. We recall the partition of unity ϑ_ϖ and chart maps κ_ϖ, $\varpi \in \Omega$ (see Definition 2.3.6). We set for all $x \in M$, for all $\alpha \geqslant 0$ and for all $\varphi \in C_X^{r-1}(M)$

$$\begin{aligned} y_{x,\varpi_1,\alpha}(\rho) &:= \kappa_{\varpi_1} \circ g_\alpha \circ h_\rho(x), \\ \varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z) &:= (\vartheta_{\varpi_2} \cdot \phi_{-\alpha} \circ g_{-\alpha}) \circ \kappa_{\varpi_1}^{-1}(z) \cdot \int_{-\infty}^{\infty} w(\rho)\,\delta(z - y_{x,\varpi,\alpha}(\rho))\,\mathrm{d}\,\rho, \\ \varphi_{\varpi_1,\alpha}(z) &:= (\vartheta_{\varpi_1} \cdot \mathcal{L}_{\alpha,\phi_\alpha}\varphi_1) \circ \kappa_{\varpi_1}^{-1}(z)\,. \end{aligned} \tag{2.90}$$

With this notation, recalling the weighted horocycle integral associated to Definition 2.5.1, we express for all $\alpha \geqslant 0$

$$\gamma_{w,x}(\varphi) = \sum_{\varpi_1,\varpi_2\in\Omega} \int_{\mathbb{R}^d} \varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z) \cdot \varphi_{\varpi_1,\alpha}(z)\,\mathrm{d}\,z. \tag{2.91}$$

We set

$$c'(+) := -s, \quad c'(0) := -t, \quad c'(-) := -t.$$

We recall $\widetilde{\Psi}_{\sigma,n}$ defined in (2.24). We bound, using Plancherel's Theorem, Cauchy–Schwarz for the sum in σ and n, and twice Hölder's inequality with respect to z and α, respectively, for some constant $C > 0$

$$\begin{aligned} \alpha_0\,|\gamma_{w,x}(\varphi)| &= \int_0^{\alpha_0} \left| \sum_{\varpi_1,\varpi_2} \int_{\mathbb{R}^d} \varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z) \cdot \varphi_{\varpi_1,\alpha}(z)\,\mathrm{d}\,z \right| \mathrm{d}\,\alpha \\ &\leqslant \int_0^{\alpha_0} \sum_{\varpi_1,\varpi_2} \left| \int_{\mathbb{R}^d} \sum_{\sigma,n} 2^{-c(\sigma)n} \widetilde{\Psi}_{\sigma,n}^{\mathrm{Op}} \varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z)\, 2^{c(\sigma)n} \Psi_{\sigma,n}^{\mathrm{Op}} \varphi_{\varpi_1,\alpha}(z)\,\mathrm{d}\,z \right| \mathrm{d}\,\alpha \\ &\leqslant C \sup_{\alpha,\varpi_1,\varpi_2} \left\| \left(\sum_{\sigma,n} 4^{c'(\sigma)n} \left| \widetilde{\Psi}_{\sigma,n}^{\mathrm{Op}} \varphi_{w,x,\varpi_1,\varpi_2,\alpha} \right|^2 \right)^{\frac{1}{2}} \right\|_{L_{p*}} \|\varphi\|_{W_p^{s,t,q}}\,, \end{aligned} \tag{2.92}$$

where $p^* := 1 - \frac{1}{p}$ is the Hölder conjugate of p. To conclude, it is enough to establish the following upper bounds for $\left\|\widetilde{\Psi}^{\mathrm{Op}}_{\sigma,n}\varphi_{w,x,\varpi_1,\varpi_2,\alpha}\right\|_{L_{p^*}}$:

(i) There exists a constant $C_1 > 0$ such that for every C^0 map $w\colon \mathbb{R} \to \mathbb{R}$, every $p \in [1,\infty]$, every $\sigma \in \{-,+,0\}$, $n \in \mathbb{N}$, every $\varpi_1, \varpi_2 \in \Omega$ it holds

$$\left\|\widetilde{\Psi}^{\mathrm{Op}}_{\sigma,n}\varphi_{w,x,\varpi_1,\varpi_2,\alpha}\right\|_{L_p} \leqslant C_1 \left|\operatorname{supp} w\right| \|w\|_{L_\infty}, \ \forall x \in M, \ \forall 0 \leqslant \alpha \leqslant \alpha_0.$$

(ii) There exists a constant $C_2 > 0$ such that for every $-r < s < 0$, for every $C^{|s|}$ map $w\colon \mathbb{R} \to \mathbb{R}$ with compact support, every $p \in [1,\infty]$, every $n \in \mathbb{N}$, every $\varpi_1, \varpi_2 \in \Omega$ it holds

$$\left\|\widetilde{\Psi}^{\mathrm{Op}}_{-,n}\varphi_{w,x,\varpi_1,\varpi_2,\alpha}\right\|_{L_p} \leqslant C_2 2^{sn} \left|\operatorname{supp} w\right| \|w\|_{C^{-s}}, \ \forall x \in M, \ \forall 0 \leqslant \alpha \leqslant \alpha_0.$$

We first show claim (i). We fix $w, \sigma, n, \varpi_1, \varpi_2, x$ and α. We let $J \subseteq \operatorname{supp} w$ be the maximal subset such that $y_{x,\varpi_1,\alpha|J}$ is well-defined. We note that J decomposes into a finite disjoint union, e.g. $J = \bigsqcup_{k=1}^{N} I_k$ for some $N \in \mathbb{N}$ and some real intervals I_k. In particular, since the flow h_ρ is non-singular and, in addition the manifold M is compact and each stable leaf is dense in M and $0 \leqslant \alpha \leqslant \alpha_0$, for some constant $C_1 > 0$, we have $|I_k| \leqslant C_1 \operatorname{diam} V_{\varpi_1}$ and $N \leqslant C_1 \frac{|\operatorname{supp} w|}{\operatorname{diam} V_{\varpi_1}}$. For every $z \in \mathbb{R}^d$ we estimate for some constants $C_2, \ldots, C_4 > 0$

$$\begin{aligned}
&\left|\varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z)\right| \leqslant C_2 \left|\int_J w(\rho)\, \delta\left(z - y_{x,\varpi_1,\alpha}(\rho)\right) \mathrm{d}\rho\right| \\
&= \left|\sum_{k=1}^{N} \int_{I_k} w(\rho)\, \delta\left(z - y_{x,\varpi_1,\alpha}(\rho)\right) \mathrm{d}\rho\right| = \left|\sum_{k=1}^{N} \sum_{\rho \in y^{-1}(z) \cap I_k} w(\rho) \left|\partial_\rho y_{x,\varpi_1,\alpha}(\rho)\right|^{-1}\right| \\
&\leqslant C_3 N \max_{\rho \in \operatorname{supp} w} \left|w(\rho) \left|\partial_\rho y_{x,\varpi_1,\alpha}(\rho)\right|^{-1}\right| \leqslant C_4 \left|\operatorname{supp} w\right| \|w\|_{L_\infty}, \qquad (2.93)
\end{aligned}$$

where we used in the last step non-singularity of h_ρ and $0 \leqslant \alpha \leqslant \alpha_0$. We conclude, using Young's inequality on $\left\|\widetilde{\Psi}^{\mathrm{Op}}_{\sigma,n}\varphi_{w,x,\varpi_1,\varpi_2,\alpha}\right\|_{L_p}$ together with the bound in (2.93).

We now show claim (ii). Again we fix $w, \sigma, n, \varpi_1, \varpi_2, x$ and α and set $y := y_{x,\varpi_1,\alpha}$. Analogously as in the proof of (i), we let $I_k \subseteq \mathbb{R}$, $1 \leqslant k \leqslant N$, be the N connected components of J for some $N \in \mathbb{N}$. For every $z \in \mathbb{R}^d$ we expand

$$\widetilde{\Psi}^{\mathrm{Op}}_{-,n}\varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z) = \frac{2^{dn}}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \widetilde{\Psi}_{-,1}(\xi)\, e^{\mathrm{i}\, 2^n \xi(z-\widetilde{y})} \varphi(\widetilde{y})\, \mathrm{d}\xi\, \mathrm{d}\widetilde{y},$$

where we set

$$\varphi(\widetilde{y}) := \varphi_{w,x,\varpi_1,\varpi_2,\alpha}(\widetilde{y}).$$

We note that $\operatorname{supp}\varphi \subseteq y(J)$. In particular, we reparametrize $\widetilde{y} \in \operatorname{supp}\varphi$ by $\widetilde{y} = \widetilde{z}(\widetilde{\rho})$ for some diffeomorphism $z \in C^r$ and $\widetilde{\rho} \in \mathbb{R}$. We set $\mathrm{D}_{\widetilde{\rho}}(\cdot) := \mathrm{i}\,\partial_{\widetilde{\rho}}\frac{(\cdot)}{\partial_{\widetilde{\rho}}\xi\widetilde{z}}$. Since $\widetilde{z}(\mathbb{R})$ is a piece of a stable manifold in charts there exists a constant $C_3 > 0$ such that we have $\left|\partial_{\widetilde{\rho}}\xi\widetilde{z}(\widetilde{\rho})\right| \geqslant C_1 2^n$ for all ξ in $\operatorname{supp}\widetilde{\Psi}_{-,n}$ is essentially part of an unstable cone in charts by construction. We note that $\varphi\circ z$ is C^r. Using $\lfloor -s\rfloor$-times integration by parts (see Lemma B.3), followed by a regularized integration by parts with respect to $\widetilde{\rho}$ if $-s \notin \mathbb{N}$, respectively (see Lemma B.5 in which we take $d = 1$, $G = y$ and $L^{-1} = \epsilon = 2^{-n}$), this yields

$$\widetilde{\Psi}^{\mathrm{Op}}_{-,n}\varphi_{w,x,\varpi_1,\varpi_2,\alpha}(z) = \frac{2^{((s+d)n)}}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\widetilde{\Psi}_{+,1}(\xi)\,e^{\mathrm{i}\,2^n\xi(z-\widetilde{z}(\widetilde{\rho}))}\widetilde{\mathrm{D}}^{-s}_{\widetilde{\rho}}\widetilde{\varphi}(\widetilde{\rho})\,\mathrm{d}\,\xi\,\mathrm{d}\,\widetilde{\rho},$$

where

$$\widetilde{\mathrm{D}}^{-s}_{\widetilde{\rho}}\widetilde{\varphi} := \begin{cases}\mathrm{D}^{-s}_{\widetilde{\rho}}\left(\widetilde{\varphi}\circ\widetilde{z}\cdot\partial_{\widetilde{\rho}}\widetilde{z}\right), & \text{if } -s\in\mathbb{N}\\ 2^{(-s+\lfloor -s\rfloor)n}\left(\partial_{\widetilde{\rho}}\left(\frac{1}{2^n\xi\widetilde{z}}\widetilde{\widetilde{\varphi}}\right)_\epsilon + \partial_{\widetilde{\rho}}\xi\widetilde{z}\cdot\left(\frac{1}{\xi\widetilde{z}}\widetilde{\widetilde{\varphi}} - \left(\frac{1}{\xi\widetilde{z}}\widetilde{\widetilde{\varphi}}\right)_\epsilon\right)\right), & \text{if } -s\notin\mathbb{N}\end{cases},$$
$$\widetilde{\widetilde{\varphi}} := \mathrm{D}^{\lfloor -s\rfloor}_{\widetilde{\rho}}\left(\widetilde{\varphi}\circ\widetilde{z}\cdot\partial_{\widetilde{\rho}}\widetilde{z}\right),$$

and the ϵ-term is just the convolution $\left(\frac{1}{\xi y}\,\mathrm{D}^{\lfloor -s\rfloor}_{\widetilde{\rho}}\,\widetilde{\varphi}\right) * \nu_\epsilon$ with a C^∞ map ν_ϵ with $\operatorname{supp}\nu_\epsilon \subseteq (-\epsilon,\epsilon)$. Note that all derivatives of $\widetilde{\mathrm{D}}^{-s}_{\widetilde{\rho}}\widetilde{\varphi}$ in ξ are bounded in n, using Lemma B.5 and non-singularity of h_ρ and $0 \leqslant \alpha \leqslant \alpha_0$. We proceed analogously as in the proof of Lemma 2.3.3, integrating $(d+1)$-times by parts in ξ if $2^n\left|z - \widetilde{z}(\widetilde{\rho})\right| > 1$ and conclude, using that $\operatorname{supp}\widetilde{\mathrm{D}}^{-s}_{\widetilde{\rho}}\widetilde{\varphi} \subseteq \operatorname{supp}\widetilde{\varphi}$ is bounded. □

We group below some properties of the pointwise renormalization. (Note in particular that Claim (xi) in Proposition 2.5.13, which will follow from [31, Remark C.4] of Giulietti–Liverani–Pollicott, will play a key part to estimate the spectral bound of $X + V$. Also, Claim (viii) in Proposition 2.5.13 shows that $\phi_\alpha = \partial_\rho\tau(0,-\alpha,\cdot)$ differs from the unit speed parametrization function by a multiplicative 1-coboundary.)

Proposition 2.5.13 (Properties of pointwise renormalization)**.** *Let τ be the renormalization time of a stable horocycle flow. For all $\rho,\alpha \in \mathbb{R}$ and for all $x \in M$ it holds:*

(i) $\tau(0,\alpha,x) = 0$,

(ii) $\tau(\rho,0,x) = \rho$,

(iii) $\tau(\rho,\alpha_1+\alpha_2,x) = \tau(\tau(\rho,\alpha_2,x),\alpha_1,g_{\alpha_2}(x))$, *for all* $\alpha_1,\alpha_2 \in \mathbb{R}$,

(iv) $\tau(\rho_1+\rho_2,\alpha,x)=\tau\left(\rho_1,\alpha,h_{\rho_2}(x)\right)+\tau(\rho_2,\alpha,x)$, *for all* $\rho_1,\rho_2\in\mathbb{R}$,

(v) $\partial_\rho\tau(\rho,\alpha,x)=\partial_\rho\tau(0,\alpha,h_\rho(x))$,

(vi) $\partial_\rho\tau(0,\alpha_1,g_{\alpha_2}(x))\partial_\rho\tau(0,\alpha_2,x)=\partial_\rho\tau(0,\alpha_1+\alpha_2,x)$, *for all* $\alpha_1,\alpha_2\in\mathbb{R}$,

(vii) $\tau(\rho,\alpha,x)=\gamma_x\left(\partial_\rho\tau(0,\alpha,\cdot),\rho\right)$,

(viii) $\partial_\rho\tau(0,\alpha,\cdot)=\det \mathrm{D}\, g_{\alpha|E_-}\frac{(\partial_\rho h_0)^*(\partial_\rho h_0)}{(\partial_\rho h_0\circ g_\alpha)^*(\partial_\rho h_0\circ g_\alpha)}$ *where* $\partial_\rho h_0(x):=\partial_\rho h_{\rho|\rho=0}(x)$,

(ix) $0<\partial_\rho\tau(0,\alpha,x)<\infty$,

(x) *if* $\alpha\geqslant 0$ *there exist* $C_1>0$ *and* $0<\theta<1$ *both independent of* α *and* x *such that* $\left\|\partial_\rho\tau\left(\cdot,\alpha,x\right)\right\|_{C^{r-1}}\leqslant C_1\theta^\alpha$,

(xi) *if* $|\rho|\geqslant 1$ *and* $\alpha\geqslant 0$ *there exists* $C_2\geqslant 1$ *independent of* ρ,α *and* x *such that*

$$C_2^{-1}\left|\rho\right|e^{h_{top}\alpha}\leqslant\left|\tau\left(\rho,-\alpha,x\right)\right|\leqslant C_2\left|\rho\right|e^{h_{top}\alpha}.$$

(xii) *if* $\alpha\geqslant 0$ *such that* $\left|\tau\left(\rho,\alpha,x\right)\right|=c$ *for some* $c\geqslant 1$ *then there exists* $C_3\geqslant 1$ *independent of* ρ,α *and* x *such that*

$$C_3^{-1}ce^{h_{top}\alpha}\leqslant|\rho|\leqslant C_3ce^{h_{top}\alpha}.$$

Proof. We note that by Definition 2.5.1, Definition 2.5.4 and Lemma 2.5.6 the renormalization time is differentiable in ρ. Every stable leaf is dense in M hence together with non-singularity of the flow h_ρ it follows $h_{\rho_1}(x)=h_{\rho_2}(x)\Rightarrow\rho_1=\rho_2$. Then Claim (i)-(ii) follow directly from (2.84). We deduce from (2.84)

$$\begin{aligned}h_{\tau(\rho,\alpha_1+\alpha_2,x)}\circ g_{\alpha_1+\alpha_2}(x)&=g_{\alpha_1+\alpha_2}\circ h_\rho(x)=g_{\alpha_1}\circ h_{\tau(\rho,\alpha_2,x)}\circ g_{\alpha_2}(x)\\&=h_{\tau\left(\tau(\rho,\alpha_2,x),\alpha_1,g_{\alpha_2}(x)\right)}\circ g_{\alpha_1+\alpha_2}(x).\end{aligned}$$

This yields Claim (iii). Also from (2.84) we find

$$\begin{aligned}h_{\tau(\rho_1+\rho_2,\alpha,x)}\circ g_\alpha(x)&=g_\alpha\circ h_{\rho_1+\rho_2}(x)=g_\alpha\circ h_{\rho_1}\circ g_{-\alpha}\circ g_\alpha\circ h_{\rho_2}\circ g_{-\alpha}\circ g_\alpha(x)\\&=h_{\tau(\rho_1,\alpha,h_{\rho_2}(x))}\circ h_{\tau(\rho_2,\alpha,x)}\circ g_\alpha(x).\end{aligned}$$

This yields Claim (iv). Claim (v) and (vi), using Claim (i), follow by differentiating both sides in (iv) and (iii) at $\rho_1=0$ and $\rho=0$, respectively.
Claim (vii) follows from (2.83) and (v).
To show Claim (viii), we take derivatives on both sides of (2.84) with respect to ρ

$$\mathrm{D}\, g_\alpha\partial_\rho h_\rho(x)=\partial_\rho\tau(\rho,\alpha,x)\cdot(\partial_\rho h_0)\circ h_{\tau(\rho,\alpha,x)}\circ g_\alpha(x).\tag{2.94}$$

Now we let $(\partial_\rho h_0)^* \in E_-^*$ be the canonical dual of $\partial_\rho h_0$. We calculate

$$\begin{aligned}(\partial_\rho h_0 \circ g_\alpha)^* (\mathrm{D}\, g_\alpha \partial_\rho h_0) &= (\partial_\rho h_0 \circ g_\alpha)^* ((g_\alpha)_* \partial_\rho h_0) = (g_\alpha)^* (\partial_\rho h_0 \circ g_\alpha)^* (\partial_\rho h_0) \\ &= \det \left(\mathrm{D}\, g_{\alpha|E_-}\right)^* (\partial_\rho h_0)^* (\partial_\rho h_0) = \det \mathrm{D}\, g_{\alpha|E_-} (\partial_\rho h_0)^* (\partial_\rho h_0)\,.\end{aligned} \tag{2.95}$$

We set $\rho = 0$ in (2.94) and conclude, using (2.95) and non-singularity of the horocycle flow.

Claim (ix) follows from (viii) together with the fact $\lim_{\alpha\to 0} \det \mathrm{D}\, g_{\alpha|E_-} = 1$ and compactness of M.

In order to show (x), we note first, since $r \geqslant 2$, using Claim (v) and the cocycle property (vi),

$$\partial_\rho \tau(\rho, \alpha, x) = \partial_\rho \tau(0, \alpha, h_\rho(x)) = \exp - \int_0^\alpha V \circ g_{\tilde{\alpha}} \circ h_\rho(x)\, \mathrm{d}\,\tilde{\alpha},$$

where $V := -\partial_\alpha \partial_\rho \tau(0, 0, \cdot) \in C^{r-1}$. Therefore it holds, using the equality in (2.84),

$$\begin{aligned}\partial_\rho^2 \tau(\rho, \alpha, x) &= -\partial_\rho \tau(\rho, \alpha, x) \cdot \partial_\rho \int_0^\alpha V \circ g_{\tilde{\alpha}} \circ h_\rho(x)\, \mathrm{d}\,\tilde{\alpha} \\ &= -\partial_\rho \tau(\rho, \alpha, x) \cdot \int_0^\alpha \partial_\rho \tau(\rho, \tilde{\alpha}, x) \cdot (\mathrm{D}\, V \partial_\rho h_0) \circ g_{\tilde{\alpha}} \circ h_\rho(x)\, \mathrm{d}\,\tilde{\alpha},\end{aligned} \tag{2.96}$$

where $|\partial_\rho \tau(\rho, \alpha, x)| \leqslant C\theta^\alpha$ for some $0 < \theta < 1$ and $C_1 > 0$ both independent of α, ρ and x by (viii). Hence there is $C_2 = C_2(V) > 0$ such that $\left|\partial_\rho^2 \tau(\rho, \alpha, x)\right| \leqslant C_2 \theta^\alpha$. By induction, all derivatives $\partial_\rho^k \tau(\rho, \alpha, x)$, where $k \in \mathbb{N}$, depend only on $\partial_\rho \tau(\rho, \alpha, x)$ (and k and derivatives of V which are independent of α) and so does the Hoelder norm $\|\partial_\rho \tau(\rho, \alpha, x)\|_{C^{r-1}}$. Since $r \geqslant 2$ the Hoelder coefficicent of $\partial_\rho \tau(\cdot, \alpha, x)$ is bounded by $\left\|\partial_\rho^2 \tau(\cdot, \alpha, x)\right\|_{C^0}$ and we conclude.

Claim (xi) for $\rho \geqslant 1$ and $\alpha \leqslant 0$ follows from [31, Lemma C.1] and [31, Remark C.4] (recall that g_α is transitive) in which we replace W with a manifold which contracts in forward time. To this end we set $W_x := h_{[0,1]}(x)$ for every $x \in M$. Since the stable flow is non-singular, the stable manifold W_x is of bounded length (from above and below) for all $x \in M$. We estimate, using Proposition 2.5.13 (viii) for the first and [31, Remark C.4] for the last inequality, with constants

$C_3, \dots, C_6 > 0$ independent of ρ, α, x

$$\begin{aligned}\tau(\rho, -\alpha, x) &\leqslant C_3 \int_0^\rho \det \mathrm{D}\, g_{-\alpha|E_-} \circ h_\rho(x)\, \mathrm{d}\rho \leqslant C_4 \rho \int \det \mathrm{D}\, g_{-\alpha|E_-}\, \mathrm{d}W_x \\ &\leqslant C_5 \rho \operatorname{vol}(g_{-\alpha}(W_x)) \leqslant C_6 \rho e^{h_{\mathrm{top}}\alpha}.\end{aligned}$$

A lower bound for $\tau(\rho, -\alpha, x)$ is obtained in an analogous way, using the last statement in [31, Lemma C.1]. We conclude for all $|\rho| \geqslant 1$, noting that $\tau(-\rho, \alpha, x) = -\tau(\rho, \alpha, h_{-\rho}(x))$, using Claims (iv) and (i).
Claim (xii) follows from Proposition 2.5.13 (viii), and the following equality which follows from Claim (iii)

$$\rho = \tau(\tau(\rho, \alpha, x), -\alpha, g_\alpha(x)) = \tau(c, -\alpha, g_\alpha(x)).$$

□

We shall use in the next two lemmas the following key identity for the horocyle integral (2.89)

$$\gamma_{w,x}(\varphi) = \gamma_{w\circ\tau(\cdot,-\alpha,g_\alpha(x)),g_\alpha(x)}\left(\mathcal{L}_{\alpha,\partial_\rho\tau(0,-\alpha,\cdot)}\varphi\right), \ \forall \alpha \geqslant 0. \tag{2.97}$$

To check the above identity, using (2.84) and Proposition 2.5.13 (iii), (v)-(vi), just notice that for all $\alpha \in \mathbb{R}$

$$\begin{aligned}\gamma_{w,x}(\varphi) &= \int_{-\infty}^{\infty} w(\rho) \cdot \varphi \circ g_{-\alpha} \circ g_\alpha \circ h_\rho(x)\, \mathrm{d}\rho \\ &= \int_{-\infty}^{\infty} w(\rho) \cdot \varphi \circ g_{-\alpha} \circ h_{\tau(\rho,\alpha,x)} \circ g_\alpha(x)\, \mathrm{d}\rho \\ &= \int_{-\infty}^{\infty} w(\tau(\rho, -\alpha, g_\alpha(x))) \cdot \varphi \circ g_{-\alpha} \circ h_\rho \circ g_\alpha(x) \cdot \partial_\rho\tau(\rho, -\alpha, g_\alpha(x))\, \mathrm{d}\rho \\ &= \int_{-\infty}^{\infty} w(\tau(\rho, -\alpha, g_\alpha(x))) \cdot (\partial_\rho\tau(0, -\alpha, \cdot) \cdot \varphi \circ g_{-\alpha}) \circ h_\rho \circ g_\alpha(x)\, \mathrm{d}\rho \\ &= \gamma_{w\circ\tau(\cdot,-\alpha,g_\alpha(x)),g_\alpha(x)}(\partial_\rho\tau(0, -\alpha, \cdot) \cdot \varphi \circ g_{-\alpha}) \qquad (2.98) \\ &= \gamma_{w\circ\tau(\cdot,-\alpha,g_\alpha(x)),g_\alpha(x)}\left(\mathcal{L}_{\alpha,\partial_\rho\tau(0,-\alpha,\cdot)}\varphi\right), \quad \text{if } \alpha \geqslant 0.\end{aligned}$$

We now state upper bounds for $|\gamma_x(\varphi, T)|$ similar to the results in [28, Lemma 5.16]. The prove uses the analogue of the smooth cutoff used by Giulietti–Liverani [30] but uses a different construction of the local decomposition of $\gamma_x(\varphi, T)$.

Lemma 2.5.14 (Local bounds)**.** *For every $T > 0$ and for every $x \in M$ there exists $w \in C^{-s}(\mathbb{R}, [0,1])$ such that for every $\varphi \in W_p^{s,t,q}$, where $p \in [1, \infty]$ and*

$s < 0 < q \leqslant t < r - 1 + s$, the following holds:

(i) There exists $C_1 > 0$ independent of T, x and φ such that

$$|\gamma_x(\varphi, T) - \gamma_{w,x}(\varphi)| \leqslant C_1 \|\varphi\|_{C^0}.$$

Moreover, if $-s < \frac{\theta_{min}}{\theta_{max}}$, where for some $0 < \theta_{min} < \theta_{max}$, for some $C_0 \geqslant 1$ and for all $\alpha \geqslant 0$ it holds

$$C_0^{-1} e^{-\theta_{max}\alpha} \leqslant \inf_{x \in M} \partial_\rho \tau(\cdot, \alpha, x) \leqslant \sup_{x \in M} \partial_\rho \tau(\cdot, \alpha, x) \leqslant C_0 e^{-\theta_{min}\alpha}, \tag{2.99}$$

then for some $C_2 > 0$ independent of T, x and φ it holds

$$|\gamma_x(\varphi, T)| \leqslant C_2 \max\left\{T, T^{\frac{\theta_{min}}{\theta_{max}} + s\frac{\theta_{max}}{\theta_{min}}}\right\} \|\varphi\|_{W_p^{s,t,q}}.$$

(ii) If for some $\widetilde{\varphi} \in W_p^{s,t,q}$ it holds for all $\alpha \geqslant 0$, for some $\lambda \in \mathbb{R}$, $c \geqslant 0$ and some $C = C(\lambda, c, \widetilde{\varphi}) > 0$

$$\|\partial_\rho \tau(0, -\alpha, \cdot) \cdot \widetilde{\varphi} \circ g_{-\alpha}\|_{W_p^{s,t,q}} \leqslant C e^{\lambda\alpha} \max\{1, |\alpha|^c\}, \tag{2.100}$$

then there exists $C_3 = C_3(\lambda, c) > 0$ independent of T, x and $\widetilde{\varphi}$ such that

$$|\gamma_{w,x}(\widetilde{\varphi})| \leqslant C C_3 \begin{cases} T^{\frac{\lambda}{h_{top}}} (\max\{1, \log T\})^c, & \text{if } \lambda > 0 \\ \min\{1, T\} (\max\{1, \log T\})^{c+1}, & \text{if } \lambda = 0 \\ \min\{1, T\} (\max\{1, \log T\})^c, & \text{if } \lambda < 0 \end{cases}.$$

Moreover, if the bound in (2.100) holds for all $\alpha \in \mathbb{R}$ with $\lambda > 0$ then

$$|\gamma_x(\widetilde{\varphi}, T) - \gamma_{w,x}(\widetilde{\varphi})| \leqslant C C_3.$$

Proof. Let $x \in M$, $T > 0$, $0 < \epsilon \leqslant \frac{1}{4}$. We define $\beta_k^+, \beta_k^- \in \mathbb{R}$ for every $k \in \mathbb{N}$ by

$$\begin{aligned} \tau(T, \beta_0^+, x) = \frac{1}{\epsilon} \quad \text{and} \quad \tau\left(\tau\left(\frac{1}{\epsilon}, -\beta_k^+, g_{\beta_k^+}(x)\right), \beta_{k-1}^+, x\right) = 1, \\ \beta_0^- := \beta_0^+ \quad \text{and} \quad \tau\left(\tau\left(-\frac{1}{\epsilon}, -\beta_k^-, g_{\beta_k^-} \circ h_T(x)\right), \beta_{k-1}^-, h_T(x)\right) = -1. \end{aligned} \tag{2.101}$$

If $T > 1$ we assume $\beta_0^+ > 0$, if $T = 1$ we assume $\beta_0^+ = 0$ and if $T < 1$ we assume $\beta_0^+ < 0$. This is justified since $\tau(T, 0, x) = T$ and by Proposition 2.5.13 (xi). Since $\epsilon < 1$ we may assume without loss of generality for all $k \in \mathbb{N}$

$$\beta_k^+ < \beta_{k-1}^+ \quad \text{and} \quad \beta_k^- < \beta_{k-1}^-.$$

Combining the definitions in (2.101) with (iii) and (xi) in Proposition 2.5.13, we find $C_1 \geqslant 1$ independent of ϵ, x, k and T such that for all $k \in \mathbb{N}$ it holds

$$C_1^{-1}\epsilon \leqslant e^{h_{\mathrm{top}}\left(\beta_k^+ - \beta_{k-1}^+\right)}, e^{h_{\mathrm{top}}\left(\beta_k^- - \beta_{k-1}^-\right)} \leqslant C_1\epsilon. \tag{2.102}$$

If $\beta_0^+ \geqslant 0$ it follows for all $k \in \mathbb{Z}_{\geqslant 0}$, using the upper bounds in (2.102) and Proposition 2.5.13 (xii) on $\tau\left(T, \beta_0^+, x\right) = \frac{1}{\epsilon}$,

$$T\left(C_1^{-1}\epsilon\right)^{k+1} \leqslant e^{h_{\mathrm{top}}\beta_k^+} \leqslant T\left(C_1\epsilon\right)^{k+1}. \tag{2.103}$$

If $\beta_0^+ < 0$ it holds for all $k \in \mathbb{Z}_{\geqslant 0}$

$$\left(C_0^{-1}T\right)^{\frac{h_{\mathrm{top}}}{\theta_{\min}}}\left(C_1^{-1}\epsilon\right)^k \leqslant e^{h_{\mathrm{top}}\beta_k^+} \leqslant \left(C_0T\right)^{\frac{h_{\mathrm{top}}}{\theta_{\max}}}\left(C_1\epsilon\right)^k, \tag{2.104}$$

where C_0, $\theta_{\min}$ and $\theta_{\max}$ are from the assumptions in (2.99). By symmetry we obtain analogous bounds for β_k^-. We let $w^+, w^- \in C^\infty\left(\mathbb{R}, [0,1]\right)$ such that

$$w^- = w^+ \circ (T - \cdot), \quad w^+_{|\left(\frac{1}{2\epsilon}, \infty\right)} \equiv 1 \quad \text{and} \quad w^+_{|\left(-\infty, \frac{1}{4\epsilon}\right)} \equiv 0.$$

We set

$$w_0 := w^+ \circ \tau\left(\cdot, \beta_0^+, x\right) \cdot w^- \circ \left(T + \tau\left(\cdot - T, \beta_0^-, h_T(x)\right)\right),$$

and we set for all $k \in \mathbb{N}$

$$\begin{aligned}
w_k^+ &:= w^+ \circ \tau\left(\cdot, \beta_k^+, x\right) - w^+ \circ \tau\left(\cdot, \beta_{k-1}^+, x\right), \\
w_k^- &:= w^- \circ \left(T + \tau\left(\cdot - T, \beta_k^-, h_T(x)\right)\right) - w^- \circ \left(T + \tau\left(\cdot - T, \beta_{k-1}^-, h_T(x)\right)\right), \\
w_k &:= w_k^+ + w_k^-.
\end{aligned}$$

We let $N \in \mathbb{Z}$ for now be arbitrary. If $N \geqslant 0$ we set

$$\begin{aligned}
w := \sum_{k=0}^{N} w_k &= w_0 + w^+ \circ \tau\left(\cdot, \beta_N^+, x\right) - w^+ \circ \tau\left(\cdot, \beta_0^+, x\right) \\
&\quad + w^- \circ \left(T + \tau\left(\cdot - T, \beta_N^-, h_T(x)\right)\right) - w^- \circ \left(T + \tau\left(\cdot - T, \beta_0^-, h_T(x)\right)\right).
\end{aligned}$$

If $N < 0$ we put $w \equiv 0$. Since $\epsilon \leqslant \frac{1}{4}$, it follows directly from the definitions of

β_k^+ and β_k^- in (2.101) that for all $k \in \mathbb{N}$

$$\tau\left(\frac{1}{2\epsilon}, -\beta_k^+, g_{\beta_k^+}(x)\right) \leqslant \tau\left(\frac{1}{4\epsilon}, -\beta_{k-1}^+, g_{\beta_{k-1}^+}(x)\right) \quad \text{and}$$
$$\tau\left(\frac{1}{-2\epsilon}, -\beta_k^-, g_{\beta_k^-} \circ h_T(x)\right) \geqslant \tau\left(-\frac{1}{4\epsilon}, -\beta_{k-1}^-, g_{\beta_{k-1}^+} \circ h_T(x)\right).$$

Together with the assumptions on the supports of w^+ and w^-, we find if $N \geqslant 0$

$$\operatorname{supp}\left(1_{|[0,T]} - w\right) \subseteq \left(0, \tau\left(\frac{1}{2\epsilon}, -\beta_N^+, g_{\beta_N^+}(x)\right)\right) \cup \left(T + \tau\left(-\frac{1}{2\epsilon}, -\beta_N^-, g_{\beta_N^-} \circ h_T(x)\right), T\right).$$

We put

$$N := \left\lfloor -\log\left(C_1^{-1}T\right) / \log\left(C_1\epsilon\right)\right\rfloor.$$

Hence if $N < 0$ then T is bounded and if $N \geqslant 0$ then $\beta_N^+, \beta_N^- \geqslant 0$. The latter follows from the lower bounds in (2.103). Therefore the first statement in Claim (i) follows immediately, using in addition Proposition 2.5.13 (xii) and the upper bounds in (2.103).
Using Proposition 2.5.13 (iii) and also Proposition 2.5.13 (iv) in the last equality for $\widetilde{w}_0$, we find for all $k \in \mathbb{N}$

$$\begin{aligned}
\widetilde{w}_k^+ :=& w_k^+ \circ \tau\left(\cdot, -\beta_k^+, g_{\beta_k^+}(x)\right) = w^+ - w^+ \circ \tau\left(\cdot, \beta_{k-1}^+ - \beta_k^+, g_{\beta_k^+}(x)\right), \\
\widetilde{w}_k^- :=& w_k^-(\cdot + T) \circ \tau\left(\cdot, -\beta_k^-, g_{\beta_k^-} \circ h_T(x)\right) \\
=& w^-(T + \cdot) - w^- \circ \left(T + \tau\left(\cdot, \beta_{k-1}^- - \beta_k^-, g_{\beta_k^-} \circ h_T(x)\right)\right), \\
\widetilde{w}_0 :=& w_0 \circ \tau\left(\cdot, -\beta_0^+, g_{\beta_0^+}(x)\right) \\
=& w^+ \cdot w^- \circ \left(T + \tau\left(\tau\left(\cdot, -\beta_0^+, g_{\beta_0^+}(x)\right) - T, \beta_0^+, h_T(x)\right)\right) \\
=& w^+ \cdot w^- \circ \left(T - \frac{1}{\epsilon} + \cdot\right).
\end{aligned}$$

For this construction it holds for all $k \in \mathbb{N}$

$$\operatorname{supp} \widetilde{w}_k^+, -\operatorname{supp} \widetilde{w}_k^-, \operatorname{supp} \widetilde{w}_0 \subseteq \left[0, \frac{C_1^2}{2} 1/\epsilon^2\right]. \tag{2.105}$$

Since $\partial_\rho \tau(\cdot, \alpha, x) \in C^{r-1}$ for all $\alpha \in \mathbb{R}$ and all $x \in M$ it follows for some constant $C_3 > 0$ for all $k \in \mathbb{N}$, all $x \in M$ and all $T > 0$, using Proposition 2.5.13 (viii) and

the bounds in (2.102),

$$\left\|\widetilde{w}_k^+\right\|_{C^r}, \left\|\widetilde{w}_k^-\right\|_{C^r}, \left\|\widetilde{w}_0\right\|_{C^r} \leqslant C_3. \tag{2.106}$$

We note

$$\gamma_{w,x}(\varphi) = \gamma_{w\circ(\cdot+T),h_T(x)}(\varphi).$$

Assuming $N \geqslant 0$, together with the equality in (2.97), we find the local decomposition[6] for all $\varphi \in W_p^{s,t,q}$

$$\begin{aligned}\gamma_{w,x}(\varphi) =& \gamma_{\widetilde{w}_0,g_{\beta_0^+}(x)}\left(\mathcal{L}_{\beta_0^+,\phi_{\beta_0^+}}\varphi\right) \\ &+ \sum_{k=1}^{N}\gamma_{\widetilde{w}_k^+,g_{\beta_k^+(x)}}\left(\mathcal{L}_{\beta_k^+,\phi_{\beta_k^+}}\varphi\right) + \gamma_{\widetilde{w}_k^-,g_{\beta_k^-}\circ h_T(x)}\left(\mathcal{L}_{\beta_k^-,\phi_{\beta_k^-}}\varphi\right).\end{aligned} \tag{2.107}$$

Using the bound in Lemma 2.5.12 with the bounds in (2.106), and using the assumption in (2.100) for some $\widetilde{\varphi} \in W_p^{s,t,q}$

$$\left\|\mathcal{L}_{\alpha,\phi_\alpha}\widetilde{\varphi}\right\|_{W_p^{s,t,q}} = \left\|\partial_\rho\tau(0,-\alpha,\cdot)\cdot\widetilde{\varphi}\circ g_{-\alpha}\right\|_{W_p^{s,t,q}} \leqslant Ce^{\lambda\alpha}\max\left\{1,|\alpha|^c\right\}, \tag{2.108}$$

and the bounds in (2.103), we estimate the right-hand side in the decomposition in (2.107) for some constant $C_4 = C_4(c) > 0$, recalling that T is uniformly bounded from below if $N \geqslant 0$, and conclude the first statement in Claim (ii):

$$\left|\gamma_{w,x}(\varphi)\right| \leqslant CC_4T^{\frac{\lambda}{h_{\text{top}}}}\sum_{k=0}^{N}\left(C_1\epsilon\right)^{k\frac{\lambda}{h_{\text{top}}}}\left(\max\left\{1,(k+1)\left|\log\left(C_1\epsilon\right)\right|,\log T\right\}\right)^c. \tag{2.109}$$

If $N < 0$ then $w \equiv 0$ and T is uniformly bounded from above and we conclude as well. To see the second statements in Claims (i)-(ii), we recall that the construction of the functions w_k is valid for every $T > 0$ and hence

$$\sum_{k=0}^{\infty} w_k = 1_{|(0,T)}.$$

Since for all $\varphi \in C_X^{r-1}(M)$ it holds

$$\gamma_x(\varphi,T) - \gamma_{\sum_{k=0}^{\infty}w_k,x}(\varphi) = 0,$$

[6] This is analogous to the decomposition in [31, Lemma 3.1]. The main difference to our decomposition is that we use a more explicit construction of the smoothing functions.

we find by density for all $\varphi \in W_p^{s,t,q}$

$$\gamma_x(\varphi, T) = \gamma_{\sum_{k=0}^{\infty} w_k, x}(\varphi).$$

It holds

$$\operatorname{supp} w_0 \subseteq [0, T].$$

Comparing with the supports in (2.105), together with the bounds in (2.103) and (2.104), we find some $C_5 > 0$ independent of k, T, x and φ such that for all $k \in \mathbb{N}$, if $\beta_k^+ \geqslant 0$ and $\beta_k^- \geqslant 0$,

$$\left(T - \operatorname{supp} w_k^-\right), \operatorname{supp} w_k^+ \subseteq \left[0, C_5 (C_1\epsilon)^k T\right],$$

respectively, and if $\beta_k^+ < 0$ and $\beta_k^- < 0$,

$$\left(T - \operatorname{supp} w_k^-\right), \operatorname{supp} w_k^+ \subseteq \left[0, C_5 T^{\frac{\theta_{\min}}{\theta_{\max}}} \left((C_1\epsilon)^k\right)^{\frac{\theta_{\min}}{h_{\text{top}}}}\right].$$

Moreover, we find for some constant $C_5 \geqslant 1$ for all $\rho_1, \rho_2 \in \mathbb{R}$, all $\alpha, -s \geqslant 0$ and all $x \in M$, using Proposition 2.5.13 (iv) and the assumption of the upper bound for $\partial_\rho \tau$ in (2.99),

$$(\tau(\rho_1, \alpha, x) - \tau(\rho_2, \alpha, x))^{-s} = \tau(\rho_1 - \rho_2, \alpha, h_{\rho_2}(x))^{-s} \leqslant C_5^{-s} (\rho_1 - \rho_2)^{-s} e^{s\theta_{\min}\alpha}.$$

If $\alpha \leqslant 0$ it holds analogously, now using the lower bound for $\partial_\rho \tau$ in (2.99),

$$(\tau(\rho_1, \alpha, x) - \tau(\rho_2, \alpha, x))^{-s} = \tau(\rho_1 - \rho_2, \alpha, h_{\rho_2}(x))^{-s} \leqslant C_5^{-s} (\rho_1 - \rho_2)^{-s} e^{s\theta_{\max}\alpha}.$$

Since $0 < -s < 1$ and $0 < \theta_{\min} \leqslant \theta_{\max}$ it holds for some constant $C_6 > 0$ independent of T, x and φ, using the lower bounds in (2.103)-(2.104), for all $k \in \mathbb{N}$ and for all $T > 0$,

$$\|w_k\|_{C^{-s}} \leqslant C_6 \max\left\{1, T^{s\frac{\theta_{\max}}{\theta_{\min}}} \left((C_1\epsilon)^k\right)^{s\frac{\theta_{\max}}{h_{\text{top}}}}\right\}.$$

Then we estimate for every $\varphi \in W_p^{s,t,q}$, using Lemma 2.5.12 and $-s < \frac{\theta_{\min}}{\theta_{\max}}$, for

some constants $C_7, C_8 > 0$ independent of T, x, w and φ

$$
\begin{aligned}
&|\gamma_x(\varphi, T)| \leqslant \lim_{n\to\infty} \left| \sum_{k=0}^{n} \gamma_{w_k,x}(\varphi) \right| \\
&\leqslant C_7 \sum_{k=0}^{\infty} \max\left\{ T, T^{\frac{\theta_{\min}}{\theta_{\max}} + s\frac{\theta_{\max}}{\theta_{\min}}} \right\} \left((C_1\epsilon)^k \right)^{\frac{\theta_{\min}}{h_{\text{top}}}} \max\left\{ 1, \left((C_1\epsilon)^k \right)^{s\frac{\theta_{\max}}{h_{\text{top}}}} \right\} \|\varphi\|_{W_p^{s,t,q}} \\
&\leqslant C_8 \max\left\{ T, T^{\frac{\theta_{\min}}{\theta_{\max}} + s\frac{\theta_{\max}}{\theta_{\min}}} \right\} \|\varphi\|_{W_p^{s,t,q}} .
\end{aligned}
$$

This yields the second statement in Claim (i). On the other hand, using the equality in (2.98) and assuming $N > 0$, we find,

$$
\begin{aligned}
\gamma_x(\widetilde{\varphi}, T) - \sum_{k=0}^{N} \gamma_{w_k,x}(\widetilde{\varphi}) &= \sum_{k=N+1}^{\infty} \gamma_{\widetilde{w}_k^+, g_{\beta_k^+(x)}} \left(\partial_\rho \tau\left(0, -\beta_k^+, \cdot\right) \cdot \widetilde{\varphi} \circ g_{-\beta_k^+} \right) \\
&+ \sum_{N+1}^{\infty} \gamma_{\widetilde{w}_k^-, g_{\beta_k^-} \circ h_T(x)} \left(\partial_\rho \tau\left(0, -\beta_k^-, \cdot\right) \cdot \widetilde{\varphi} \circ g_{-\beta_k^-} \right).
\end{aligned}
$$

Then we proceed analogously as for the bound in (2.109), now using the upper bounds in (2.103) and the assumption in (2.108) for all $\alpha \in \mathbb{R}$ and some $\lambda > 0$, $c \geqslant 0$ (recall that $T(C_1\epsilon)^N$ is bounded from above). If $N \leqslant 1$ then $T^\lambda |\log T|^c$ is bounded from above and we conclude as well, now using the upper bounds in (2.104). □

Remark 2.5.15. *The second statement in Lemma 2.5.14 (i) can be used to avoid the $\|\varphi\|_{C^0}$-term in the bound of the error term in Theorem 2.5.7. However the required range for s may not be very large (except in the case of constant vector fields). The second statement in Lemma 2.5.14 (ii) is free from an additional condition on s. We use it in the following subsection in the proof of Lemma 2.5.18 (v) and Theorem 2.5.7. Both statements give also bounds for all values $T > 0$ which seems to be new.*

2.5.3 Showing $\lambda_{\max} = h_{\text{top}}$ and Theorem 2.5.7

In this subsection we shall prove Theorem 2.5.7. First, we state and prove two lemmas which will imply that $\lambda_{max} = h_{\text{top}}$, assuming $\lambda_{\min} < \lambda_{\max}$, is a simple eigenvalue and that λ_{max} is uniquely attained.

We remind the reader that uniqueness and simplicity of the spectral bound is known to hold (see [22, Lemma 5.1], [21]) for the spectrum of mixing Anosov flows (which are not necessarily contact), but for different anisotropic spaces, and only for the potential V given by the Jacobian of the flow (and associated

to the SRB measure).
For the sake of the next two lemmas we have to introduce the following condition[7]:

Condition 2.5.16 (Strong vanishing). *Let $0 < t, q, -s < r-1$ and let $p \in [1, \infty]$. Let $\varphi_\alpha \in W_p^{s,t,q}$ for all $\alpha \geqslant 0$ such that $\|\varphi_\alpha\|_{W_p^{s,t,q}} = 1$ and*

$$\limsup_{\alpha\to\infty} e^{-h_{top}\alpha} \|\mathcal{L}_{\alpha,\phi_\alpha}\varphi_\alpha\|_{W_p^{s,t,q}} > 0.$$

If for some $-s \leqslant -s'$ for all $x \in M$ and all $w \in C_0^{-s'}(\mathbb{R})$

$$\lim_{\alpha\to\infty} \gamma_{w,x}\left(\frac{\mathcal{L}_{\alpha,\phi_\alpha}\varphi_\alpha}{\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi_\alpha\|_{W_p^{s,t,q}}}\right) = 0$$

then $\lim_{\alpha\to\infty} \|\varphi_\alpha\|_{W_p^{s,t,q}} = 0$.

We give the upper bound on the spectral radius:

Lemma 2.5.17 (Upper bound on the spectral radius). *Let $0 < t, q, -s < r-1$ and let $p \in [1, \infty]$. For all $x \in M$ and all $\alpha \geqslant 0$ let $\partial_\rho\tau(\cdot, -\alpha, x) \in C^{r-1}(\mathbb{R}, M)$. Under Condition 2.5.16, With the choice $\phi_\alpha = \partial_\rho\tau(0, -\alpha, \cdot)$ for some constant $C > 0$ it holds for all $\alpha \geqslant 0$*

$$\|\mathcal{L}_{\alpha,\phi_\alpha}\|_{W_p^{s,t,q}\to W_p^{s,t,q}} \leqslant Ce^{h_{top}\alpha}.$$

Proof. We show the claim on $\|\mathcal{L}_{\alpha,\phi_\alpha}\|_{W_p^{s,t,q}\to W_p^{s,t,q}}$ by contradiction. Suppose

$$e^{-h_{\mathrm{top}}\alpha} \|\mathcal{L}_{\alpha,\phi_\alpha}\|_{W_p^{s,t,q}\to W_p^{s,t,q}} \to \infty \quad \text{as} \quad \alpha \to \infty.$$

Then there exists $\varphi_\alpha \in W_p^{s,t,q}$ such that $\|\varphi_\alpha\|_{W_p^{s,t,q}} = 1$ and

$$\left\|e^{-h_{\mathrm{top}}\alpha}\mathcal{L}_{\alpha,\phi_\alpha}\varphi_\alpha\right\|_{W_p^{s,t,q}} \to \infty \quad \text{as} \quad \alpha \to \infty. \tag{2.110}$$

We assume for some $w \in C_0^s(\mathbb{R})$ and some $x \in M$

$$\liminf_{\alpha\to\infty} \left|\gamma_{w,x}\left(\frac{\mathcal{L}_{\alpha,\phi_\alpha}\varphi_\alpha}{\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi_\alpha\|_{W_p^{s,t,q}}}\right)\right| > 0. \tag{2.111}$$

[7] This is introduced ad hoc as it was pointed out by Colin Guillarmou and Giovanni Forni that for the weak-vanishing to imply strong vanishing is not obvious here. In some sense one would expect even a stronger statement. Namely that for every eigendistribution $\mathcal{D}$ in the expansion of Theorem 2.5.7 at least for one piece of horocycle orbit w around $x \in M$ one has $|\gamma_{w,x}(\mathcal{D})| > 0$.

This assumption is justified, assuming Condition 2.5.16. We choose $T \geqslant 1$ and $\alpha \geqslant 0$ such that

$$\tau(T, \alpha, x) = 1.$$

Then, using Proposition 2.5.13 (xii), we find for some constant $C \geqslant 1$

$$C^{-1} e^{h_{\mathrm{top}}\alpha} \leqslant T \leqslant C e^{h_{\mathrm{top}}\alpha}. \tag{2.112}$$

We have, using the equality given in (2.97),

$$\gamma_{w\circ\tau(\cdot,\alpha,x),x}(\varphi_\alpha) = \gamma_{w,g_\alpha(x)}(\mathcal{L}_{\alpha,\phi_\alpha}\varphi_\alpha). \tag{2.113}$$

We recall $-s < r - 1$. Therefore the norm $\|w \circ \tau(\cdot, \alpha, x)\|_{C^{-s}}$ is bounded as $\alpha \to \infty$, using Proposition 2.5.13 (x). By Lemma 2.5.12, the linear functionals $\gamma_{w\circ\tau(\cdot,\alpha,x),x}$ and $\gamma_{w,g_\alpha(x)}$ which appear in (2.113) are continuous on $W_p^{s,t,q}$. Hence the left-hand side in (2.113) grows at most by T as $\alpha \to \infty$ uniformly in x. Then, comparing with the estimates for T in (2.112), using the assumption in (2.111), this contradicts the assumption in (2.110) and we conclude. □

We next show the lower bound (and uniqueness and simplicity of the spectral bound $\lambda_{\max}$):

Lemma 2.5.18 (Invariant measure and spectral bound)**.** *Let μ be the unique Borel probability measure which is invariant by the horocycle flow h_ρ. Let $p \in [1, \infty]$ and let $s < 0 < q \leqslant t$ such that $t - s < r - 1$. It holds:*

(i) $\mu \in \left(W_p^{s,t,q}\right)'$,

(ii) $\mathcal{L}'_{\alpha,\phi_\alpha}\mu = e^{h_{top}\alpha}\mu$ *($\mathcal{L}'_{\alpha,\phi_\alpha}$ denotes the adjoint operator of $\mathcal{L}_{\alpha,\phi_\alpha}$),*

(iii) $h_{top} \in \sigma(X + V)|_{W_p^{s,t,q}}$.
Moreover, assuming Condition 2.5.16, it holds:

(iv) $\lambda_{max} = h_{top}$.

(v) The spectral bound λ_{max} is uniquely attained by the simple eigenvalue h_{top}, assuming $\lambda_{min} < \lambda_{max}$.

The vector μ is also invariant by the adjoint horocycle flow since the time average converges to the (unique) ergodic mean (a result by Marcus [50]). This is in analogy to [31, Lemma 2.11].

Proof. We note for every $\varphi \in C_X^{r-1}(M)$, using [49, Theorem 2.1] for the first, the equality in (2.97) for the second and [49, Lemma 3.1] for the third equality, for some $\lambda > 0$, for every $\alpha \geqslant 0$

$$\begin{aligned}\mu(\varphi) &= \lim_{T\to\infty} \frac{1}{T}\gamma_x(\varphi, T) = \lim_{T\to\infty} \frac{\tau(T,\alpha,x)}{T}\frac{1}{\tau(T,\alpha,x)}\gamma_{g_{\alpha(x)}}(\mathcal{L}_{\alpha,\phi_\alpha}\varphi, \tau(T,\alpha,x)) \\ &= \lambda^{-\alpha}\mu(\mathcal{L}_{\alpha,\phi_\alpha}\varphi). \end{aligned} \tag{2.114}$$

To see $\lambda = e^{h_{\text{top}}}$ we refer to [49, p.84] (alternatively use Proposition 2.5.13 (xi)). Using Claims (i)-(ii) with $\lambda = h_{\text{top}}$ in Lemma 2.5.14 together with the bound given by Lemma 2.5.17, there is $w \in C^r(\mathbb{R})$ and a constant $C_1 > 0$ such that for all $\varphi \in C_X^{r-1}(M)$

$$|\mu(\varphi)| \leqslant \lim_{T\to\infty}\left|\frac{1}{T}\gamma_{w,x}(\varphi)\right| \leqslant C_1 \|\varphi\|_{W_p^{s,t,q}}. \tag{2.115}$$

Claim (iii) follows from $\sigma\left((X+V)'\right)|_{\left(W_p^{s,t,q}\right)'} = \sigma(X+V)|_{W_p^{s,t,q}}$, using [46, Section II.2.5]. Claim (iv) follows from (iii) together with Lemma 2.5.17. To see Claim (v), first we note that all $\lambda \in \sigma(X+V)$ such that $\Re\lambda = h_{\text{top}}$ are eigenvalues, using Lemma 2.4.10 together with the assumption $\lambda_{\min} < \lambda_{\max}$. Using Claim (iii), there exists $\mathcal{D}_1 \in W_p^{s,t,q}$ such that $\mathcal{L}_{\alpha,\phi_\alpha}\mathcal{D}_1 = e^{h_{\text{top}}\alpha}\mathcal{D}_1$ for all $\alpha \geqslant 0$. We let $\mathcal{D}_1 \neq \mathcal{D}_2 \in W_p^{s,t,q}\setminus\{0\}$ such that $\mathcal{L}_{\alpha,\phi_\alpha}\mathcal{D}_2 = e^{\lambda\alpha}\mathcal{D}_2$ for all $\alpha \geqslant 0$, where $\lambda \in \mathbb{C}$ and $\Re\lambda = h_{\text{top}}$. Then it holds, using Claim (ii) for the last equality,

$$e^{\lambda\alpha}\mu(\mathcal{D}_2) = \mu(\mathcal{L}_{\alpha,\phi_\alpha}\mathcal{D}_2) = e^{h_{\text{top}}\alpha}\mu(\mathcal{D}_2).$$

Since $\lambda \neq h_{\text{top}}$ it holds $\mu(\mathcal{D}_2) = 0$. In fact, by same reasoning we can always assume $\mu(\mathcal{D}_2) = 0$ if $\lambda \neq h_{\text{top}}$. And if $\lambda = h_{\text{top}}$ there are only finitely many such $\mathcal{D}_2$ and we can again assume $\mu(\mathcal{D}_2) = 0$ by a change of basis. The upshot is that the following reasoning works always if $\Re\lambda \geqslant h_{\text{top}}$ and $\mu(\mathcal{D}_2) = 0$.
Then, using Claim (i) and the equality in (2.114), for every $\epsilon > 0$ there is $\varphi \in C_X^{r-1}(M)$ such that for all $\alpha \in \mathbb{R}$ and for all $x \in M$

$$\lim_{T\to\infty}\left|\frac{1}{T}\gamma_{g_{-\alpha(x)}}(\varphi, T)\right| = |\mu(\varphi)| \leqslant \epsilon. \tag{2.116}$$

Using Lemma 2.5.14 (i), for all $T > 0$, for all $x \in M$ and for all $\alpha \in \mathbb{R}$ there exists $w \in C^r$ such that

$$\lim_{T\to\infty}\left|\frac{1}{T}\gamma_{g_{-\alpha(x)}}(\varphi, T)\right| = \lim_{T\to\infty}\left|\frac{1}{T}\gamma_{w,g_{-\alpha(x)}}(\varphi)\right|. \tag{2.117}$$

Since $\partial_\rho \tau(0,-\alpha,\cdot) \in C^{r-1}$ for all $\alpha \in \mathbb{R}$, using Lemma 2.3.3, we find $\mathcal{L}_{\alpha,\phi_\alpha}\mathcal{D}_2 = e^{\lambda\alpha}\mathcal{D}_2$ for all $\alpha \in \mathbb{R}$ which matches the condition (2.100) in Lemma 2.5.14 (ii). Then, using Lemma 2.5.14 (ii) for the upper bound and the equality in (2.97) for the last step, we find for some constant $C_2 > 0$ independent of x, α and φ, for all $\alpha \in \mathbb{R}$

$$\begin{aligned}\lim_{T\to\infty}\left|\frac{1}{T}\gamma_{w,g_{-\alpha}(x)}(\varphi)\right| &= \lim_{T\to\infty}\left|\frac{1}{T}\gamma_{w,g_{-\alpha}(x)}(\mathcal{D}_2)+\frac{1}{T}\gamma_{w,g_{-\alpha}(x)}(\varphi-\mathcal{D}_2)\right| \\ &\geqslant \lim_{T\to\infty}\left|\frac{1}{T}\gamma_{g_{-\alpha}(x)}(\mathcal{D}_2,T)\right| - C_2\left\|\varphi-\mathcal{D}_2\right\|_{W_p^{s,t,q}} \\ &= \lim_{T\to\infty}\left|\frac{e^{h_{\mathrm{top}}\alpha}}{T}\gamma_x(\mathcal{D}_2,\tau(T,\alpha,x))\right| - C_2\left\|\varphi-\mathcal{D}_2\right\|_{W_p^{s,t,q}}.\end{aligned} \tag{2.118}$$

By density of $C_X^{r-1}(M)$ in $W_p^{s,t,q}$ we assume

$$\left\|\varphi-\mathcal{D}_2\right\|_{W_p^{s,t,q}} \leqslant \epsilon.$$

For every $T \geqslant T_0 \geqslant 1$ we let $\alpha \geqslant 0$ such that $\tau(T,\alpha,x) = T_0$. By Proposition 2.5.13 there is $C_3 \geqslant 1$ independent of T and x such that $e^{h_{\mathrm{top}}\alpha} \geqslant C_3^{-1}\frac{T}{T_0}$. Since $\epsilon > 0$ was arbitrary we conclude for all $T_0 \geqslant 1$ and all $x \in M$, using the estimates (2.116)-(2.118),

$$\gamma_x(\mathcal{D}_2,T_0) = 0.$$

On the other hand we find for all $T_1, T_0 \geqslant 1$

$$\gamma_x(\mathcal{D}_2,T_0-T_1) = \gamma_{h_{-T_1}(x)}(\mathcal{D}_2,T_0) - \gamma_{h_{-T_1}(x)}(\mathcal{D}_2,T_1) = 0.$$

Hence it holds $\gamma_x(\mathcal{D}_2,T) = 0$ for every $T \in \mathbb{R}$ and every $x \in M$. Then for every $w \in C_0^{s+1}$ we find, using integration by parts,

$$\gamma_{w,x}(\mathcal{D}_2) = -\int_{\mathbb{R}} (\partial_\rho w)(\rho)\cdot\gamma_x(\mathcal{D}_2,\rho)\,\mathrm{d}\,\rho.$$

Since $\gamma_x(\mathcal{D}_2,\rho) = 0$ for all $\rho \geqslant 0$ we conclude $\gamma_{w,x}(\mathcal{D}_2) = 0$. Then, using Condition 2.5.16, we find $\mathcal{D}_2 \equiv 0$ but we assumed $\mathcal{D}_2 \not\equiv 0$. □

Proof of Theorem 2.5.7. By assumption

$$\max\{\lambda_{\min},0\} < \delta \leqslant \lambda_{\max} = h_{\mathrm{top}}.$$

We note that we have always $\lambda_{\max} = h_{\mathrm{top}}$ and uniqueness and simplicity of $\lambda_{\max}$

under Condition 2.5.16. Using the equality in (2.67) for the projectors $\Pi_{\lambda,i}$, we have for all $1 \leqslant i \leqslant n_\lambda$

$$\Pi_{\lambda,i}\varphi = \sum_{j=1}^{m_{\lambda,i}} \mathcal{O}_{(\lambda,i,j)}(\varphi)\, \mathcal{D}_{(\lambda,i,j)}.$$

Recalling the nil-potent operators $\mathcal{N}_{\lambda,i}$ of finite rank (e.g. see in (2.66)), using the formula for the matrix action $\mathcal{L}_{\alpha,\phi_\alpha}\Pi_{\lambda,i} = \exp(\lambda\alpha)\exp(\mathcal{N}_{\lambda,i}\alpha)\,\Pi_{\lambda,i}$ for all $\alpha \geqslant 0$ and

$$\exp(-\lambda\alpha)\exp(-\mathcal{N}_{\lambda,i}\alpha)\,\mathcal{L}_{\alpha,\phi_\alpha}\Pi_{\lambda,i} = \Pi_{\lambda,i},$$

we find for some constant $C_1 = C_1(\lambda,i,j)$ for all $\alpha \in \mathbb{R}$

$$\left\|\partial_\rho(0,-\alpha,\cdot)\cdot\mathcal{D}_{(\lambda,i,j)}\circ g_{-\alpha}\right\|_{W_p^{s,t,q}} \leqslant C_1\exp(\Re\lambda\alpha)\max\left\{1,|\alpha|^{j-1}\right\}.$$

Hence $\mathcal{D}_{(\lambda,i,j)}$ satisfies the upper bound in (2.100) for all $\alpha \in \mathbb{R}$ if $\Re\lambda > 0$. Inspecting the end of the proof of Lemma 2.5.18, we notice that all eigendistributions $\mathcal{D}_{(\lambda,i,j)}$ associated to some eigenvalue λ with $\Re\lambda \geqslant h_{\mathrm{top}}$ do not contribute to the expansion except $\mathcal{D}_{(h_{\mathrm{top}},1,1)}$. This follows, if $j = 1$ using that $\gamma_x\left(\mathcal{D}_{(\lambda,i,1)},T\right) = 0$ for all $T \geqslant 0$ and all $x \in M$. If $j > 1$ we arrive at the same conclusion, using in the estimate in (2.118) for all $\alpha \geqslant 0$

$$\mathcal{L}_{\alpha,\phi_\alpha}\mathcal{D}_{(\lambda,i,j)} = \exp(\lambda\alpha)\exp(\mathcal{N}_{\lambda,i}\alpha)\,\mathcal{D}_{(\lambda,i,j)}.$$

Let $\lambda \in \Sigma_\delta = \sigma(X+V)|_{W_p^{s,t,q}} \cap \{z \in \mathbb{C} \mid \Re z \geqslant \delta\}$. For every $T \geqslant 0$ and every $x \in M$ we set, using $w \in C^r$ given in Lemma 2.5.14,

$$c_{(\lambda,i,j)} = c_{(\lambda,i,j)}(T,x) := T^{-\frac{\lambda}{h_{\mathrm{top}}}}\max\left\{1,|\log T|^{1-j}\right\}\gamma_{w,x}\left(\mathcal{D}_{(\lambda,i,j)}\right).$$

Then, using the first statement in Lemma 2.5.14 (ii), the coefficients $c_{(\lambda,i,j)}$ are bounded independently of T and x. It holds

$$\begin{aligned}\gamma_{w,x}(\Pi_{\lambda,i}\varphi) &= \sum_{j=1}^{m_{\lambda,i}} \mathcal{O}_{(\lambda,i,j)}(\varphi)\,\gamma_{w,x}\left(\mathcal{D}_{(\lambda,i,j)}\right)\\ &= \sum_{j=1}^{m_{\lambda,i}} c_{(\lambda,i,j)}T^{\frac{\lambda}{h_{\mathrm{top}}}}\max\left\{1,|\log T|^{j-1}\right\}\mathcal{O}_{(\lambda,i,j)}(\varphi).\end{aligned}$$

We let μ as given in Lemma 2.5.18. Using Lemma 2.5.18 (v), and assuming

$T \geqslant e$, we find for every finite subset $\Lambda_\delta \subseteq \Sigma_\delta$

$$\gamma_x(\cdot, T) = \gamma_x\left(\mathcal{D}_{(h_{\mathrm{top}},1,1)}, T\right)\mu + \sum_{\substack{\lambda\in\Lambda_\delta \\ \Re\lambda<h_{\mathrm{top}}}} \sum_{i=1}^{n_\lambda}\sum_{j=1}^{m_{\lambda,i}} c_{(\lambda,i,j)} T^{\frac{\lambda}{h_{\mathrm{top}}}} (\log T)^{j-1}\, \mathcal{O}_{(\lambda,i,j)} + \mathcal{E}_{T,x,\Lambda_\delta},$$

where the remainder term is

$$\begin{aligned}\mathcal{E}_{T,x,\Lambda_\delta} &:= \left(\gamma_{w,x}\left(\mathcal{D}_{h_{\mathrm{top}},1,1}\right) - \gamma_x\left(\mathcal{D}_{(h_{\mathrm{top}},1,1)}, T\right)\right)\mu \\ &\quad + \gamma_{w,x}\left(\mathrm{id} - \sum_{\lambda\in\Lambda_\delta}\sum_{i=1}^{n_\lambda}\Pi_{\lambda,i}\right) + \left(\gamma_x(\cdot, T) - \gamma_{w,x}\right). \end{aligned} \tag{2.119}$$

The existence of the limit $\lim_{T\to\infty} T^{-1}\gamma_x\left(\mathcal{D}_{(h_{\mathrm{top}},1,1)}, T\right)$ is shown by analogue estimates (2.116)-(2.118). Then the statement on the limit $\lim_{T\to\infty} T^{-1}\mathcal{E}_{T,x,\Lambda_\delta}(\varphi)$ follows, using unique ergodicity of the horocycle flow [49, Theorem 2.1] and finiteness of Λ_δ. We bound $|\mathcal{E}_{T,x,\Lambda_\delta}(\varphi)|$ as required, using the first statement in Lemma 2.5.14 (i) and the full statement in Lemma 2.5.14 (ii) together with the assumed upper bound in (2.88).

The additional claims under Condition 2.4.11 can be seen as follows (see also the remarks above and below Condition 2.4.11): The finiteness of Σ_δ follows from [19, Theorem 1]. To this end we have to show that [19, Assumption 1-3A] are satisfied for the renormalized semigroup $e^{-h_{\mathrm{top}}\alpha}\mathcal{L}_{\alpha,\phi_\alpha} : W_p^{s,t,q} \to W_p^{s,t,q}$. In fact Condition 2.4.11 yields just a reformulation of [19, Assumption 3A] for the resolvent of the generator $X + V - h_{\mathrm{top}}$. Now [19, Assumption 1] states that for some Banach space $W_p^{s,t,q} \subset \mathcal{B}$ it holds

$$\sup_{\alpha\geqslant 0}\frac{1}{\alpha}\left\|\mathrm{id} - e^{-h_{\mathrm{top}}\alpha}\mathcal{L}_{\alpha,\phi_\alpha}\right\|_{W_p^{s,t,q}\to\mathcal{B}} < \infty. \tag{2.120}$$

We set $\mathcal{B} := W_p^{s,t,q-1}$. We bound the left-hand side in (2.120), using the equality in (2.69) together with Lemma 2.3.3, Lemma 2.3.5 and Lemma 2.5.17. Now [19, Assumption 2] just states that the essential spectral bound of $X + V - h_{\mathrm{top}}$ is bounded by some $\lambda < 0$, where $V = -\partial_\alpha\partial_\rho\tau(0, 0, \cdot)$. By assumption it holds $\lambda \leqslant \lambda_{\min} - h_{\mathrm{top}} < 0$. Finally, the claimed choice $c = \delta + \epsilon$ for all $\epsilon > 0$ follows from [19, Theorem 1] as well. In particular, this choice for c follows if for all $\alpha \geqslant 0$ and for all $\epsilon > 0$ there exists $C_2 = C_2(\delta, \epsilon, \varphi)$ such that

$$\left\|\mathcal{L}_{\alpha,\phi_\alpha}\left(\mathrm{id} - \sum_{\lambda\in\Sigma_\delta}\sum_{i=1}^{n_\lambda}\Pi_{\lambda,i}\right)\varphi\right\|_{W_p^{s,t,q}} \leqslant C_2 e^{(\delta+\epsilon)\alpha}.$$

We set $\varphi_\delta := \varphi - \sum_{\lambda \in \Sigma_\delta} \sum_{i=1}^{n_\lambda} \Pi_{\lambda,i}\varphi$. If $t - q + 1 < r - 1$ it follows, using Lemma 2.3.3, Lemma 2.3.5 and [19, Theorem 1], for some constants $C_3, C_4 = C_4(\epsilon) > 0$

$$\begin{aligned}
\|\mathcal{L}_{\alpha,\phi_\alpha}\varphi_\delta\|_{W_p^{s,t,q}} &\leqslant C_3 \|\mathcal{L}_{\alpha,\phi_\alpha}(X+V)\varphi_\delta\|_{W_p^{s,t,q-1}} + C_3 \|\mathcal{L}_{\alpha,\phi_\alpha}\varphi_\delta\|_{W_p^{s,t,q-1}} \\
&\leqslant C_3 \|\mathcal{L}_{\alpha,\phi_\alpha}(X+V-h_{\mathrm{top}})\varphi_\delta\|_{W_p^{s,t,q-1}} + (C_3 + h_{\mathrm{top}}) \|\mathcal{L}_{\alpha,\phi_\alpha}\varphi_\delta\|_{W_p^{s,t,q-1}} \\
&\leqslant C_4 e^{(\delta+\epsilon)\alpha} \left(\left\| (X+V-h_{\mathrm{top}})^2 \varphi \right\|_{W_p^{s,t,q}} + \|(X+V-h_{\mathrm{top}})\varphi\|_{W_p^{s,t,q}} \right).
\end{aligned}$$

Boundedness of the last estimate follows if $q < r-2$ because then $\|X(V\varphi)\|_{W_p^{s,t,q}}$ and $\|X^2\varphi\|_{W_p^{s,t,q}}$ are bounded, recalling $\varphi \in C_X^{r-1}(M)$ and $V \in C^{r-1}$. Combining the required bounds for q yields

$$t - r + 2 < q < r - 2.$$

Since we required $q > 0$ it is enough to require $t - r + 2 \leqslant 0$ and $0 < r - 2$ which yields the additional condition on t and r. □

Appendix A

For the readers convenience we give a proof of a well-known result:

Lemma A.1 (Fixed points)**.** *Let M be 2×2 integer matrix acting on $\mathbb{T}^2$. Assume that $\det(\mathrm{id} - M) \neq 0$. Then the following holds:*

(i) *The number N_M of fixed points of M is given by $N_M = |\det(\mathrm{id} - M)|$.*

(ii) *There exists a disjoint partition $D_j \subseteq \mathbb{T}^2$, $1 \leqslant j \leqslant N_M$ of $\mathbb{T}^2$ such that the maps $y_j \colon D_j \to \mathbb{T}^2 \colon x \mapsto (\mathrm{id} - M)\, x$ are bijections.*

Proof. We let $\mathrm{id} - M$ act on the cover $\mathbb{R}^2$. The linear map $\mathrm{id} - M$ sends a fundamental region of $\mathbb{T}^2$, e.g. $[0,1)^2$, to a convex polytope having a non-zero volume given by $|\det(\mathrm{id} - M)|$. Each fixed point of M on $\mathbb{T}^2$ is mapped by $\mathrm{id} - M$ to an element of $\mathbb{Z}^2$, and the number of integer points contained in the polytope is just given by its volume. Claim (i) follows.
Let $v_1, v_2 \in \mathbb{Z}^2$ be two different such integer points in the polytope. Now assume that there are $f_1,\ f_2 \in [0,1)^2$ such that

$$(\mathrm{id} - M)^{-1}(f_1 - f_2) \equiv (\mathrm{id} - M)^{-1}(v_1 - v_2) \quad \left(\mathrm{mod}\ [0,1)^2\right).$$

The right-hand side is mapped to a fixed point of M on $\mathbb{T}^2$, implying that $f_1 - f_2$ is an integer point, which is only possible if $f_1 = f_2$. Therefore, $v_1 = v_2$, which contradicts the assumption, and Claim (ii) follows. □

For $d \in \mathbb{N}$ and every real $d \times d$ matrix M we denote by $\square_{i,j}(M)$, $1 \leqslant i, j \leqslant d$ the submatrix arising by removing the i-th row and j-th column from M.

Lemma A.2 (Determinant preserving transformation)**.** *Let $d \in \mathbb{N}$, and let $T \colon \mathbb{R}^d \to \mathbb{R}^d$ and $\phi \colon \mathbb{R} \to \mathbb{R}$ be differentiable maps. Fix $1 \leqslant j \leqslant d$ and $\alpha \in \mathbb{R}^d$ and set*

$$T_\phi \colon \mathbb{R}^d \to \mathbb{R}^d \colon x \mapsto (\alpha_i \phi(x_j) \quad | \quad 1 \leqslant i \leqslant d).$$

Then for $x \in \mathbb{R}^d$ it holds

$$\det \mathrm{D}_x (T + T_\phi) - \det \mathrm{D}_x (T) = 0$$

if and only if at least one of the conditions holds:

(i) $\sum_{i=1}^{d} (-1)^i \alpha_i \det \square_{i,j} (\mathrm{D}_x T) = 0$ *or*

(ii) $\phi^{(1)} (x_j) = 0$.

Proof. We develop the determinant of $\mathrm{D}_x (T + T_\phi)$ with respect to the j-th column. Since T_ϕ depends only on x_j this gives

$$\det \mathrm{D}_x (T + T_\phi) = (-1)^j \sum_{i=1}^{d} (-1)^i \partial_j (T + T_\phi)_i (x) \det \square_{i,j} (\mathrm{D}_x T) .$$

Hence, it holds

$$\begin{aligned} \det \mathrm{D}_x (T + T_\phi) - \det (\mathrm{D}_x T) &= (-1)^j \sum_{i=1}^{d} (-1)^i \det \square_{i,j} (\mathrm{D}_x T) \, \partial_j (T_\phi)_i (x) \\ &= (-1)^j \phi^{(1)} (x_j) \sum_{i=1}^{d} (-1)^i \alpha_i \det \square_{i,j} (\mathrm{D}_x T) . \end{aligned}$$

One deduces Claim (i) and (ii) directly from the right-hand side. □

Dolgopyat pointed out that one can use some sort of bootstrapping by looking at even powers of traces to show that there are at least two non-trivial resonances. Suppose that there exists only one non-trivial $\lambda \in \operatorname{sp} \mathcal{K}_T$. Then we find

$$\begin{aligned} \operatorname{Tr} \mathcal{K}_T^2 &= 1 + \lambda^2 = 1 + \epsilon^2 B_M(\psi)^2 + O(2\epsilon^3 B_M(\psi)) + O(\epsilon^4) \\ &= \operatorname{Tr} \mathcal{K}_{T^2} = 1 + \epsilon B_{M^2}(\psi_\epsilon) + O(\epsilon^2), \end{aligned}$$

where $\psi_\epsilon := M\psi + \psi \circ (M + \epsilon\psi)$. Then if $\psi_0 \not\equiv 0$ would imply the existence of a further resonance (for smaller ϵ_0). We could extend the reasoning by using

$$\operatorname{Tr} \mathcal{K}_{T^{2^k}} = \operatorname{Tr} \mathcal{K}_T^{2^k} \leqslant 1 + \left(\operatorname{Tr} \mathcal{K}_T^{2^{k-1}} - 1 \right)^2 ,$$

to show additional resonances for even smaller ϵ_0.

Appendix B

We check the expansion and contraction properties of the cones claimed in Section 2.2:

Lemma B.1. *Let C and θ be the constants from (2.2). Let $x \in M$ and $0 < \gamma < 1$ and recall the cones $C^-_\gamma(x)$ and $C^+_\gamma(x)$ defined in (2.8). Let $\alpha > 0$ and $\gamma' > 0$ such that $C^2\theta^\alpha\gamma < \gamma' \leqslant 1$. Then it holds:*

(i) $(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, C^-_\gamma(x) \Subset C^-_{\gamma'}(g_{-\alpha}(x))$,

(ii) $(\mathrm{D}\, g_{\alpha})^{\mathrm{tr}}\, C^+_\gamma(x) \Subset C^+_{\gamma'}(g_{\alpha}(x))$.

In particular, there exists $\gamma' > 0$ such that for all large enough $\alpha > 0$ it holds $\gamma' < \gamma$.

Proof. First we note that a fixed choice $\gamma' < \gamma$ is possible for all large $\alpha > 0$ because $\theta < 0$. We show claim (i). Claim (ii) is shown analogously. We let $v^- + v^+ + v^0 = v \in C^-_\gamma(x)$. We estimate (assuming $\frac{1}{C}\|v^0\| \leqslant \|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v^0\| \leqslant C\|v^0\|$)

$$\begin{aligned}\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v^+\| + \|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v^0\| &\leqslant C\left(\|v^+\| + \|v^0\|\right) \leqslant C\gamma\|v^-\| \\ &\leqslant C^2\theta^\alpha\gamma\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v^-\|.\end{aligned}$$

It follows that $(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v \in C^-_{\gamma'}(g_{-\alpha}(x))$ if $\gamma' \geqslant C^2\theta^\alpha\gamma$. Since $C^-_{\gamma'}(g_{-\alpha}(x)) \Subset C^-_{\gamma'+\epsilon}(g_{-\alpha}(x))$ for all $\epsilon > 0$ we conclude. □

Lemma B.2. *Assuming the constants C and θ from (2.2), let $\gamma > 0$, $x \in M$ and suppose that $C^2\theta^\alpha\gamma < 1$. Then for all $C^2\theta^\alpha\gamma < \gamma' < 1$ it holds:*

(i) If $v \in C^-_\gamma(x)$ then $\frac{\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}} v\|}{\|v\|} \geqslant C\frac{1+\gamma}{1+\gamma'}\theta^{-\alpha}$.

(ii) If $v \in (\mathrm{D}\, g_\alpha)^{\mathrm{tr}}\, C^+_\gamma(x)$ then $\frac{\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}} v\|}{\|v\|} \leqslant C\frac{1+\gamma}{1-\gamma'}\theta^{\alpha}$.

Proof. Let $v \in T^*_x M$. We recall

$$v = v^- + v^+ + v^0, \quad v^\sigma \in E_{\sigma,x}, \sigma \in \{-,+,0\}.$$

If $v \in C_\gamma^-(x)$ then by (2.9), for all $\lambda \geqslant 0$ it holds

$$\begin{aligned}\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v\right\| &\geqslant \left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v_-\right\| - \left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v_+\right\| - \left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v_0\right\| \\ &\geqslant \frac{1}{C}\theta^{-\alpha}\left\|v_-\right\| + C\left(\left\|v_+\right\| + \left\|v_0\right\|\right) \\ &\geqslant C\theta^{-\alpha}\left(1-\gamma'\right)\left\|v^-\right\| = C\theta^{-\alpha}\left(\left(1-\gamma'-\lambda\right)\left\|v^-\right\| + \lambda\left\|v^-\right\|\right) \\ &\geqslant C\theta^{-\alpha}\left(\left(1+\gamma-\lambda\right)\left\|v^-\right\| + \frac{\lambda}{\gamma'}\left(\left\|v^0\right\| + \left\|v^+\right\|\right)\right).\end{aligned}$$

The choice $\lambda = \frac{1+\gamma}{\frac{1}{\gamma'}+1}$ yields $\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v\right\| \geqslant C\frac{1+\gamma}{1+\gamma'}\theta^{-\alpha}\left\|v\right\|$.
If $v \in (\mathrm{D}\, g_\alpha)^{\mathrm{tr}}\, C_\gamma^+(x)$ then by (2.9), for all $\lambda \geqslant 0$ it holds

$$\begin{aligned}\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v\right\| &\leqslant C\theta^{\alpha}\left(1+\gamma\right)\left\|v^+\right\| = C\theta^{\alpha}\left(\left(1+\gamma+\lambda\right)\left\|v^+\right\| - \lambda\left\|v^+\right\|\right) \\ &\leqslant C\theta^{\alpha}\left(\left(1+\gamma+\lambda\right)\left\|v^+\right\| - \frac{\lambda}{\gamma'}\left(\left\|v^0\right\| + \left\|v^-\right\|\right)\right).\end{aligned}$$

The choice $\lambda = \frac{1+\gamma}{\frac{1}{\gamma'}-1}$ yields $\left\|(\mathrm{D}\, g_{-\alpha})^{\mathrm{tr}}\, v\right\| \leqslant C\frac{1+\gamma}{1-\gamma'}\theta^{\alpha}\left\|v\right\|$. □

We let ∇_z be the gradient and ∇_z^{tr} the divergence with respect to $z \in \mathbb{R}^d$.

Lemma B.3 (Integration by parts (cf. [10, p.10]))**.** *Let $\mathcal{B}$ be a Banach space and let $f\colon \mathbb{R}^d \to \mathcal{B}$ be C^1 such that*

$$\left\|f\left(z\right)\right\|_{\mathcal{B}} \to 0 \quad \text{as} \quad |z| \to \infty.$$

Let $G\colon \mathbb{R}^d \to \mathbb{R}$ be C^2 and assume that $|\nabla_z G(z)| > 0$ for every $z \in \operatorname{supp} f$. Then it holds

$$\int_{\mathbb{R}^d} e^{\mathrm{i}\, G(z)} f(z)\, \mathrm{d}\, z = \mathrm{i}\int_{\mathbb{R}^d} e^{\mathrm{i}\, G(z)} \nabla_z^{\mathrm{tr}} \frac{\nabla_z G(z) f(z)}{\left|\nabla_z G(z)\right|^2}\, \mathrm{d}\, z.$$

We understand the above transformation as integration by parts. Repeated application leads to the following iteration pattern.

Lemma B.4. *Let $f(z,\eta,\xi)$ and $\nabla_z G\left(z,\eta,\xi\right)$ be complex and real valued functions, respectively, both C^{r_1}, C^{r_2}, C^{r_3} in $z,\eta,\xi \in \mathbb{R}^d$ for some $r_1, r_2, r_3 > 0$, respectively. Let $V_0(z,\eta,\xi) := f(z,\eta,\xi)$ and*

$$V_k(z,\eta,\xi) \coloneqq \nabla_z^{\mathrm{tr}} \frac{\nabla_z G(z,\eta,\xi) V_{k-1}(z,\eta,\xi)}{\left|\nabla_z G(z,\eta,\xi)\right|^2}, \text{ where } k = 1, \ldots, \lfloor r_1 \rfloor. \tag{B.1}$$

If $|\nabla_z G(x,\eta,\xi)| > 0$ *then it holds*

$$V_k(z,\eta,\xi) = |\nabla_z G(z,\eta,\xi)|^{-k} f_k(z,\eta,\xi),$$

where $f_k(z,\eta,\xi)$ *is* C^{r_1-k} *in* z, C^{r_2}, C^{r_3} *in* η,ξ, *respectively and* $\operatorname{supp} f = \operatorname{supp} f_k$. *Moreover, it holds for some constant* $C \geqslant 1$

$$\|f_k\|_{C^0} \leqslant C \sup_{(z,\eta,\xi)\in \operatorname{supp} f} \max_{0\leqslant|\gamma|\leqslant k} \left| |\nabla_z G(z,\eta,\xi)|\, \partial_z^\gamma \frac{\nabla_z G(z,\eta,\xi)}{|\nabla_z G(z,\eta,\xi)|^2} \right|^k \|f(\cdot,\eta,\xi)\|_{C^k}\,. \tag{B.2}$$

Proof. We prove this by induction. For $V_0 = V_0(z,\eta,\xi)$ the hypothesis holds. We assume the hypothesis to hold for $V_k = V_k(z,\eta,\xi)$ up to some $0 \leqslant k \leqslant \lfloor r_1 \rfloor - 1$. We have therefore

$$V_{k+1} = \nabla_z^{\mathrm{tr}} \frac{\frac{\nabla_z G}{|\nabla_z G|} f_k}{|\nabla_z G|^{k+1}} = \frac{\nabla_z^{\mathrm{tr}} \left(\frac{\nabla_z G}{|\nabla_z G|} f_k \right)}{|\nabla_z G|^{k+1}} - (k+1) \frac{\frac{\nabla_z^{\mathrm{tr}} G}{|\nabla_z G|^2} f_k \nabla_z |\nabla_z G|}{|\nabla_z G|^{k+1}}. \tag{B.3}$$

Hence we can write $V_{k+1} = |\nabla_z G|^{-k-1} f_{k+1}$, where $f_{k+1} = f_{k+1}(z,\eta,\xi)$ is regular as required by the lower bound on $|\nabla_z G|$. In (B.3) one sees that $\operatorname{supp} f_{k+1} \subseteq \operatorname{supp} f_k$. From (B.3) one finds

$$f_{k+1} = |\nabla_z G|^{k+1} \nabla_z^{\mathrm{tr}} \left(\frac{\nabla_z G}{|\nabla_z G|^2} \frac{f_k}{|\nabla_z G|^k} \right).$$

We recursively expand f_k into this equation and estimate by the worst term which yields the upper bound (B.2). □

A regularized version of integration by parts is used if the involved maps are only Hölder continuous. A form of Lemma B.5 below appeared in a work of Baladi-Tsujii [10, p.12, Equation 3.4]. We let $\phi\colon \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ be C^∞, supported on the unit ball such that $\int_{\mathbb{R}^d} \phi(x)\,\mathrm{d}\,x = 1$. For every $\epsilon > 0$ we set $\phi_\epsilon(x) = \frac{1}{\epsilon^d}\phi\left(\frac{x}{\epsilon}\right)$.

Lemma B.5 (Regularized integration by parts)**.** *Let* $0 < \delta < 1$. *Let* $f\colon \mathbb{R}^d \to \mathbb{C}$ *be a compactly supported* C^δ*-map and let* $G\colon \mathbb{R}^d \to \mathbb{R}$ *be* $C^{1+\delta}$ *and assume that* $|\nabla_z G| > 0$ *for every* $z \in \operatorname{supp} f$. *Set* $h(z) := \frac{\nabla_z G(z) f(z)}{|\nabla_z G(z)|^2}$ *and* $h_\epsilon := h * \phi_\epsilon$. *For every* $L \geqslant 1$ *it holds*

$$\begin{aligned} \int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} f(z)\,\mathrm{d}\,z = &\frac{\mathrm{i}}{L} \int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} \nabla_z^{\mathrm{tr}} h_\epsilon(z)\,\mathrm{d}\,z \\ &+ \int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} \nabla_z^{\mathrm{tr}} G(z)\,(h(z) - h_\epsilon(z))\,\mathrm{d}\,z. \end{aligned} \tag{B.4}$$

In particular, for some constant $C \geqslant 1$, it holds $\|\nabla_z h_\epsilon\|_{L_\infty} \leqslant C \|h\|_{C^\delta} \epsilon^{\delta-1}$ and $\|h - h_\epsilon\|_{L_\infty} \leqslant C \|h\|_{C^\delta} \epsilon^{\delta}$.

Proof. Since G is $C^{1+\delta}$ and $|\nabla_z G| > 0$, the map h is C^δ. We have $\nabla_z^{\mathrm{tr}} G(z) h(z) = f(z)$ and we write

$$\int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} f(z)\, \mathrm{d}\, z = \int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} \left(\nabla_z^{\mathrm{tr}} G(z) h_\epsilon(z) + \nabla_z^{\mathrm{tr}} G(z)\, (h(z) - h_\epsilon(z)) \right) \mathrm{d}\, z.$$

And since h_ϵ is compactly supported we have, using integration by parts,

$$\int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} \nabla_z^{\mathrm{tr}} G(z) h_\epsilon(z)\, \mathrm{d}\, z = -\frac{1}{\mathrm{i}\, L} \int_{\mathbb{R}^d} e^{\mathrm{i}\, LG(z)} \nabla_z^{\mathrm{tr}} h_\epsilon(z)\, \mathrm{d}\, z.$$

To see the norm estimates, we have

$$\begin{aligned} |h(z) - h_\epsilon(z)| &= \left| \epsilon^{-d} \int_{\mathbb{R}^d} \left(h(z) - h(z - z') \right) \nu \left(\frac{z'}{\epsilon} \right) \mathrm{d}\, z' \right| \\ &= \left| \int_{\mathbb{R}^d} \left(h(z) - h(z - \epsilon z') \right) \nu \left(z' \right) \mathrm{d}\, z' \right| \leqslant \|h\|_{C^\delta} \epsilon^{\delta}. \end{aligned}$$

Since $\operatorname{supp} h$ is compact, for every $z \in \mathbb{R}^d$ there exists $\bar{z} \in \mathbb{R}^d$ such that $h(z - \bar{z}) = 0$. We estimate, for some constant $C \geqslant 1$, using 1-Lipschitz continuity of the norm,

$$\begin{aligned} \left| \int_{\mathbb{R}^d} h(z') \left(\nabla_z \nu \right) \left(\frac{z - z'}{\epsilon} \right) \mathrm{d}\, z' \right| &= \left| \int_{\mathbb{R}^d} \left(h(z - z') - h(z - \bar{z}) \right) \left(\nabla_z \nu \right) \left(\frac{z'}{\epsilon} \right) \mathrm{d}\, z' \right| \\ &\leqslant C \|h\|_{C^\delta} \int_{\mathbb{R}^d} \left| z' - \bar{z} \right|^\delta \left| \left(\nabla_z \nu \right) \left(\frac{z'}{\epsilon} \right) \right| \mathrm{d}\, z' \leqslant C \|h\|_{C^\delta} \epsilon^{\delta + d}. \end{aligned}$$

□

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