On the constructions of supercuspidal representations

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Titre: Sur les constructions des représentations supercuspidales

Résumé: Nous commençons par comparer les constructions des représentations supercuspidales de Bushnell-Kutzko [13] et Yu [41]. Nous associons de manière explicite, sous une hypothèse nécessaire de modération, à chaque étape de la construction de Bushnell-Kutzko une partie d'une donnée de Yu. Nous obtenons ainsi finalement un lien entre les deux constructions dans le cas où les constructions sont toutes les deux définies: GL_N dans une situation modérée. Dans une seconde partie, G désigne un groupe réductif connexe défini sur un corps p-adique k, nous définissons pour chaque point rationnel x dans l'immeuble de Bruhat-Tits de G et chaque nombre rationnel positif r, un sous-groupe k-affinoïde $G_{x,r}$ de l'analytifié (au sens de Bekovich) G^{an} de G. Le bord de Shilov de $G_{x,r}$ est un singleton remarquable dans G^{an}. Nous obtenons alors un cône dans l'analytifié G^{an} de G paramétrisant les groupes k-affinoides $G_{x,r}$. Nous définissons aussi des filtrations pour l'algèbre de Lie de G. Nous énonçons et prouvons plusieurs propriétés des filtrations analytiques et produisons une comparaison avec les filtrations de Moy-Prasad.

Mots clefs: Représentations des groupes réductifs p-adiques, théorie des types, comparaison des constructions de représentations supercuspidales de Bushnell-Kutzko et J.-K. Yu, filtrations de Moy-Prasad, profondeur, espaces de Berkovich, immeubles de Bruhat-Tits, analytifié d'un schéma en groupe réductif p-adique, filtrations analytiques, plongement canonique de Rémy-Thuillier-Werner, cône, groupe k-affinoïde, bord de Shilov.

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Abstract: In a first part, we compare Bushnell-Kutzko's [13] and Yu's [41] constructions of supercuspidal representations. In a tame situation, at each step of Bushnell-Kutzko's construction, we associated a part of a Yu datum. We finally get a link between these constructions when they are both defined: GL_N in the tame case. In a second part we define analytic filtrations. For any rational point x in the reduced Bruhat-Tits building of G and any positive rational number r, we introduce a k-affinoid group $G_{x,r}$ contained in the Berkovich analytification G^{an} of G. The Shilov boundary of $G_{x,r}$ is a singleton. In this way we obtain a topological cone, whose basis is the reduced Bruhat-Tits building and vertex the neutral element, inside G^{an} parametrizing the k-affinoid groups $G_{x,r}$. We also define filtrations for the Lie algebra. We state and prove various properties of analytic filtrations and compare them with Moy-Prasad ones.

Keywords: Representations of reductive p-adic groups, types theory, comparison of Bushnell-Kutzko and J.-K. Yu's construction of supercuspidal representations, Moy-Prasad filtrations, depth, Berkovich k-analytic spaces, Bruhat-Tits buildings, analytification of a p-adic reductive group scheme, analytic filtrations, canonical Rémy-Thuillier-Werner embedding, cone, k-affinoid group, Shilov boundary.

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Contents

Introduction

This thesis consists of two chapters. The goal of the first one is to produce an explicit link between Bushnell-Kutzko's construction of supercuspidal representations for $GL_N(F)$ and Yu's construction of tamely ramified supercuspidal representations of the F-points of an arbitrary connected reductive group G. Here F is a non archimedean local field. In both Bushnell-Kutzko's and Yu's constructions, the authors construct a compact modulo the center subgroup K of $G(F)$, and a certain irreducible representation ρ of K. The compactly induced representation $\mathrm{c}-\mathrm{ind}_K^{G(F)}(\rho)$ is irreducible and supercuspidal. Given a collection of objects called a Yu datum, Yu constructs one supercuspidal representation. In the first chapter of this thesis, assuming a tameness hypothesis, we associate at various steps of the construction of Bushnell-Kutzko, parts of a Yu datum. At the end, we get a complete Yu datum. Moreover, the supercuspidal representation obtained at the end of Bushnell-Kutzko's construction is equal to the supercuspidal representation associated to the obtained Yu datum. Let us describe this process. Let V be an F-vector space of dimension N, $A = \mathrm{End}_F(V)$ and $\mathrm{G} = \mathrm{Aut}_F(V)$. Bushnell and Kutzko introduce the notion of a simple stratum. This consists in a 4-uple $[\mathfrak{A}, n, r, \beta]$ where $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in A, n and r are integers and β is an element in A. This 4-uple is submitted to strong conditions, in particular the algebra generated by F and β in A has to be a field; we denote this field by E. To a simple stratum are attached two compact open subgroups $H^1 \subset J^0$ of G and a set of characters of H^1, called the simple characters. Let θ be a simple character, a β-extension of θ is a certain representation κ of J^0 whose restriction to H^1 contains θ. Fix such a κ. To $[\mathfrak{A}, n, r, \beta]$ is attached an $\mathfrak{o}_E$-order $\mathfrak{B}_\beta$, it is equal to $\mathfrak{A} \cap B$ where B is the centralizer of E in A. We assume that this $\mathfrak{o}_E$-order is maximal. Let σ be an irreducible cuspidal representation of $GL_{\frac{N}{[E:F]}}(k_E)$, where k_E denotes the residual field of E. The representation σ extends to J^0 by inflation (see section 1.2), we still denote σ this inflation. Let Λ be an extension to $E^\times J^0$ of $\sigma \otimes \kappa$. Then the representation $\mathrm{c}-\mathrm{ind}_{E^\times J^0}^{\mathrm{G}}\Lambda$ of $A^\times = \mathrm{G}$ obtained by compact induction is irreducible and supercuspidal. Moreover all the irreducible supercuspidal representations of G are obtained in this way. In this thesis we say that $([\mathfrak{A}, n, r, \beta], \theta, \kappa, \sigma, \Lambda)$ is a Bushnell-Kutzko datum. A Yu datum for a connected reductive group G defined over F consists in a 5-tuple $(\overrightarrow{G}, y, \overrightarrow{r}, \rho, \overrightarrow{\mathbf{\Phi}})$. Let us explain roughly what is such a 5-uple (a precise definition will be given in section 1.3). First, $\overrightarrow{G}$ is a strictly increasing tower of reductive F-group schemes $\overrightarrow{G} = (G^0 \subset G^1 \subset \dots G^d = G)$ defined over F such that their exists a finite Galois tamely ramified extension E/F such that

$$(G^0 \times_F E \subset G^1 \times_F E \subset \dots \subset G^d \times_F E)$$

is a split Levi sequence. Secondly, y is a vertex in the Bruhat-Tits building ([8], [9]) of G^0. Thirdly, $\overrightarrow{r}$ is an increasing sequence $(r_0, \dots, r_d)$ of real numbers. Fourthly, ρ is an irreducible representation of $G^0(F)_{[y]}$ such that its compact induction to $G^0(F)$ is irreducible supercuspidal of depth zero. Here $G^0(F)_{[y]}$ is the stabilizer in $G^0(F)$ of the image of y in the reduced Bruhat-Tits building of G^0, it is an open subgroup of $G^0(F)$ compact modulo the center. Fifthly, $\overrightarrow{\mathbf{\Phi}}$ is a sequence $\mathbf{\Phi}_0, \dots, \mathbf{\Phi}_d$ of characters such that $\mathbf{\Phi}_i$ is a character of $G^i(F)$ which is G^{i+1}-generic of depth r_i. Here, the depth is the notion introduced by Moy and Prasad [29]. The notion of generic characters will be recalled in section 1.8. To each Yu datum, Yu has associated a representation ρ_d of a subgroup K^d of $G(F)$ such that the compactly induced representation $\mathrm{c-ind}_{K^d}^{G(F)} \rho_d$ is irreducible and supercuspidal. We explain this construction in the section 1.3. In this text, we start with a Bushnell-Kutzko datum $([\mathfrak{A}, n, r, \beta], \theta, \kappa, \sigma, \Lambda)$ satisfying that the field extension $F[\beta]/F$ is tamely ramified. We then explain that we can find a *defining sequence* $\{[\mathfrak{A}, n, r_i, \beta_i],\ 0 \leq i \leq s\}$ $(\beta_0 = \beta)$ such that $F[\beta_{i+1}] \subset F[\beta_i]$ for all $0 \leq i \leq s-1$; this result is due to Bushnell-Henniart. We then show that this implies an other important property (see Proposition 1.4.3 and Proposition 1.4.4). As we will explain in section 1.2 a *defining sequence* is needed to define the simple characters attached to a simple stratum. In the previous tame situation, the properties of the choosen defining sequence imply that a simple character θ attached to the simple stratum $[\mathfrak{A}, n, r, \beta]$ factors as a product of s characters θ_i, $0 \leq i \leq s$. We introduce an integer d depending on s and on the condition $\beta_s \in F$ or $\beta_s \notin F$. We introduce a strictly increasing tower of reductive algebraic group $\overrightarrow{G}$, using the defining sequence and putting $G^i = \mathrm{Res}_{F[\beta_i]/F} \underline{\mathrm{Aut}}_{F[\beta_i]}(V)$. We explain that the sequence $\overrightarrow{G}$ satisfies Yu's conditions. Thanks to the work of Bruhat-Tits [10] and Broussous-Lemaire [7], we show that $\mathfrak{B}_\beta$ induces a point y in the building of G^0. We also introduce in this context an increasing sequence $\overrightarrow{\mathbf{r}}$ of real numbers. Moreover, we can attach to each θ_i a character $\mathbf{\Phi}_i$ of $G^i(F)$, we prove that theses characters satisfy Yu's condition. Then, using κ, σ and Λ, we introduce a representation ρ of $G^0(F)_{[y]}$. Finally, the 5-tuple $(\overrightarrow{G}, y, \overrightarrow{r}, \rho, \overrightarrow{\mathbf{\Phi}})$ forms a Yu datum. Moreover the representation ρ_d associated to this Yu datum is isomorphic to Λ, in particular $K^d = F[\beta]^\times J^0$.

This implies that the associated supercuspidal representations $\mathrm{c}-\mathrm{ind}(\Lambda)$ and $\mathrm{c}-\mathrm{ind}(\rho_d)$ are isomorphic.

Let us describe the structure of the first chapter. The section 1.1 presents the definition of a supercuspidal representation. It also presents a basic result which is at the root of these two constructions. Given an open subgroup K of $G(F)$ compact modulo the center, and an irreducible representation ρ of K, it gives a criterion for the compactly induced representation from ρ to $G(F)$ to be irreducible and supercuspidal. The section 1.2 presents the construction of Bushnell-Kutzko [13]. The section 1.3 presents the construction of Yu [41]. The section 1.4 contains the definition of tame pure strata and tame simple strata. It contains the Bushnell-Henniart result which allows to choose an approximation γ of a tame pure stratum $[\mathfrak{A}, n, r, \beta]$ inside the field $F[\beta]$. In section 1.4, we also prove a technical result (proposition 1.4.4) which is crucial in the proof that the characters Φ_i, $0 \leq i \leq s$ are G^{i+1}-generic. In section 1.5 we recall the notion of a standard representative introduced by Howe [25] and prove a proposition which links tame minimal elements of Bushnell-Kutzko and the notion of standard representative of Howe (proposition 1.5.8). The proposition 1.5.8 is also crucial in our proof that the characters Φ_i, $0 \leq i \leq s$ are G^{i+1}-generic. In section 1.6, we associate to each tame minimal element a generic element. In section 1.7 we show that a tame simple character factors as a product of s characters, where s is the length of a defining sequence. In section 1.8, we construct generic characters $\mathbf{\Phi}_i$, $0 \leq i \leq s$. In section 1.9, we complete the Yu datum and state the final result of our comparison. Readers are advised to read Theorem 1.9.3 and others results mentioned in Theorem 1.9.3 before reading all the details of chapter 1.

Before explaining the content of chapter 2, let us explain one motivation. In chapter 1 we have compared two developements wich can be regarded as formalisms, theories or constructions. One conclusion of chapter 1 is that these theories are compatible where they are both defined. One can naturally ask if there exists an other construction of supercuspidal representations containing both Yu's construction and Bushnell-Kutzko's construction. As chapter 1 shows, one needs firstly a formalism for some filtrations by compact open subgroups.

The goal of chapter 2 of this thesis is to define a filtration, natural after the work [33]. These filtrations are defined and studied using Berkovich's k-analytic spaces [3] and Berkovich's point of view on Bruhat-Tits buildings [3, chapter 5] [33]. V. Berkovich in the split case [3, Chapter 5], and B. Rémy, A. Thuillier, and A. Werner (RTW) [33] have proved that the reduced Bruhat-Tits Building of G embbeds canonically and continuously in G^{an}. To each rational[1] point $x \in \mathrm{BT}^R(G,k)$, and to each positive number r we define a k-affinoid groups $G_{x,r}$. The Shilov boundary of $G_{x,r}$ is a singleton $\theta(x,r)$ in

[1]See Definition 2.3.1 for our definition of rational points.

G^{an}. Finally we get a continous and injective map

$$\theta : \mathrm{BT}^R_{rat}(G,k) \times \mathbb{Q}_{\geq 0} \to G^{an}.$$

Let us explain these constructions. Let x be a rational point in the reduced Bruhat-Tits building of G and r be a positive rational number, there exists a finite Galois extension K/k satisfying the following three conditions. Firstly, G is split over K. Secondly, the image of x in the reduced Bruhat-Tits building of G over K is special. Thirdly, the rational number r is contained in $\mathrm{ord}_k(K)$ where ord_k is the unique valuation on finite extensions extending the valuation on k. By the two first conditions, we obtain a K°-Demazure group scheme $\mathfrak{G}$. Since $\mathrm{ord}_k(K) = \frac{1}{e(K,k)}\mathbb{Z}$ (where $e(K,k)$ is the ramification index), the third condition implies that the number $e(K,k) \times r$ is a positive integer. We consider $\Gamma_{e(K,k)r}(\mathfrak{G})$, the $e(K,k)r$-th congruence K°-subgroup of $\mathfrak{G}$ defined by J.-K. Yu [43]. It is a smooth K°-group scheme satisfying $\Gamma_{e(K,k)r}(\mathfrak{G})(K^\circ) = \ker(\mathfrak{G}(K^\circ) \to \mathfrak{G}(K^\circ/{\pi_K}^{e(K,k)r}))$. Now we can consider $\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta$ the generic fiber of the formal completion of $\Gamma_{e(K,k)r}(\mathfrak{G})$ along its special fiber. Finally we define $G_{x,r}$ to be the projection $\mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta)$, we explain that it is a k-affinoid subgroup of G^{an}. We show that $G_{x,r}$ is well-defined, i.e. that it does not depend on the choice of K. In chapter 2, we prove the following result:

Theorem. 1. *The Shilov boundary of $G_{x,r}$ is a singleton denoted $\theta(x,r)$, it is a norm on* $\mathrm{Hopf}(G)$ *(see Proposition 2.5.3).*

2. *If $r = 0$, then $G_{x,r} = G_x$ where G_x is Rémy-Thuillier-Werners's k-affinoid group [33]. (see Proposition 2.5.3)*

3. *The holomorphically convex envelope of $\theta(x,r)$ is egal to $G_{x,r}$ (see Proposition 2.5.3).*

4. *If we can choose the extension K/k tamely ramified in order to define $G_{x,r}$, then $G_{x,r}(k)$ is egal to the coresponding normalized Moy-Prasad groups (see Proposition 2.5.9).*

5. *The map θ is injective and continuous (see Proposition 2.5.7).*

We also prove, among others things, that compatibility by base change holds (see Proposition 2.5.7).

The image of θ union the neutral element of G^{an} forms a topological cone in G^{an}. If $G = GL_1$, $\mathrm{BT}^R(G,k) = \{x\}$ is a singleton and G^{an} embbeds in $(\mathbb{A}^1_k)^{an}$ and corresponds to $(\mathbb{A}^1_k)^{an} \setminus 0$. In this case $\theta(x,r)$ is the norm $|\,|_{1,e^{-r}} \in (\mathbb{A}^1_k)^{an}$. In this case, if $r = 0$, $\theta(x,r)$ corresponds to the Gauss point and to the reduced Bruhat-Tits building via [33]. In the case $G = GL_1$, the topological cone is a segment (see 2.5.6).

In this text we also define filtrations for the Lie algebra (see 2.4.3).

Let us describe the structure of the second chapter. In section 2.1, we recall some results about schemes, we also introduce schematic congruence groups following [43], [32] and [6]. In section 2.2, we introduce Berkovich's theory of k-analytic spaces following closely main steps of [3]. In section 2.3, we recall some facts about Bruhat-Tits buildings and Moy-Prasad filtrations. In section 2.4, we define analytic filtrations, in a natural and general context of potentially Demazure objects (see 2.4), and prove various properties about them. In section 2.5, we apply the results obtained in section 2.4 in special cases: we obtain analytic filtrations for points in the Bruhat-Tits building and properties about them.

At the end of the second chapter, the appendix A is part of a work in progress about Moy-Prasad isomorphism for analytic filtrations. Appendix B is a discussion about notions of rational points in Bruhat-Tits buildings: we compare there the notion introduced by Broussous-Lemaire with the notion introduced in the chapter 2 of this text, we show that both notions are equivalent for GL_N.

Chapter 1

Comparison of constructions of supercuspidal representations: from Bushnell-Kutzko's construction to Yu's construction

Notations and conventions for chapter 1

$$
\begin{aligned}
F &= \text{a fixed non archimedean local field}\\
\mathfrak{o}_F &= \text{ring of integer of } F\\
\mathfrak{p}_F &= \text{maximal ideal of } \mathfrak{o}_F\\
k_F &= \text{residual field of } F\\
\pi_F &= \text{a fixed uniformizer of } F\\
e(E \mid F) &= \text{ramification index of a finite extension } E/F\\
\pi_E &= \text{a uniformizer of an extension } E \text{ of } F\\
\nu_E &= \text{unique valuation on a finite}\\
&\quad\ \text{extension } E/F \text{ such that } \nu_E(\pi_E) = 1\\
\mathrm{ord} &= \text{unique valuation on algebraic}\\
&\quad \text{extensions of } F \text{ such that } \mathrm{ord}(\pi_F) = 1
\end{aligned}
$$

If k is a field and if G is a k-group scheme, we denote by $\underline{\mathrm{Lie}}(G)$ the Lie algebra functor and $\mathrm{Lie}(G)$ the usual Lie algebra $\underline{\mathrm{Lie}}(G)(k)$. The Lie algebra functor, of a k-group scheme denoted with a big capital letter G, is denoted by the same small gothic letter $\mathfrak{g}$. If G is a connected reductive group defined over F, we denote by $\mathrm{BT}^E(G,F)$ and $\mathrm{BT}^r(G,F)$ the enlarged and reduced Bruhat-Tits buildings of G over F [8], [9]. In this situation, if y is a point of $\mathrm{BT}^E(G,F)$, we denote $[y]$ the image of y via the canonical projection $\mathrm{BT}^E(G,F) \to \mathrm{BT}^R(G,F)$. The group $G(F)$ acts on $\mathrm{BT}^E(G,F)$ and $\mathrm{BT}^R(G,F)$. We denote by $G(F)_y$ and $G(F)_{[y]}$ the stabilizers in $G(F)$ of y and $[y]$. If G splits over a tamely ramified extension, we consider the so called Moy-Prasad filtration[1] defined by Moy and Prasad [29] [30]. This is the filtration used by Yu [41]. We use Yu's notations. So for each real number $r \geq 0$ and each y in $\mathrm{BT}^E(G,F)$, we have some groups $G(F)_{y,r}$ and $G(F)_{y,r+}$. As in [29] and [41], we have a filtration of the Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}(F)$ and of the dual of the Lie algebra $\mathfrak{g}^*(F)$. So for each y in $\mathrm{BT}^E(G,F)$ and each real number $y \geq 0$, the notations $\mathfrak{g}(F)_{y,r}$, $\mathfrak{g}(F)_{y,r+}$, $\mathfrak{g}^*(F)_{y,r}$ and $\mathfrak{g}^*(F)_{y,r+}$ are well defined. Let us recall here the definition of $\mathfrak{g}^*(F)_{y,r}$ and $\mathfrak{g}^*(F)_{y,r+}$, due to Moy-Prasad [29, page 400]. We have

$$\mathfrak{g}^*(F)_{y,-r} = \{X \in \mathfrak{g}^*(F) \mid X(Y) \in \mathfrak{p}_F \text{ for all } Y \in \mathfrak{g}(F)_{y,r+}\},$$

and

$$\mathfrak{g}^*(F)_{y,(-r)+} = \bigcup_{s<r} \mathfrak{g}^*(F)_{y,-s}.$$

If $s < r$, we denote by $G(F)_{y,s:r}$ the quotient $G(F)_{y,s}/G(F)_{y,r}$. If G is a torus we can avoid the symbol y, we write for examples $G(F)_r$ and $\mathrm{Lie}^*(G)_{-r}$. If $H \subset G$ are groups and ρ is a representation of H, we denote by $I_G(\rho)$ the intertwining of ρ in G, i.e. the set

$$I_G(\rho) = \{g \in G \mid \mathrm{Hom}_{{}^gH\cap H}({}^g\rho, \rho) \neq 0\}$$

1.1 Intertwining, compact induction and supercuspidal representations

Let G be a connected reductive group defined over F and let $P = MN$ be a parabolic subgroup of G. As usual in the litterature, the notation $P = MN$ means that M is a Levi subgroup of P and N is the unipotent radical of P. Let r_P^G denote the normalized parabolic restriction functor from the category $\mathcal{M}(G)$ of smooth representations of $G(F)$ to the category $\mathcal{M}(M)$of smooth representations of $M(F)$.

Let recall the definition of a supercuspidal representation.

Definition 1.1.1. *A representation $\pi \in \mathcal{M}(G)$ is supercuspidal if $r_P^G(\pi) = 0$ for all proper parabolic subgroups P of G.*

The following lemma is an important characterization of supercuspidal representations.

[1] This filtration is defined without the tameness hypothesis.

Lemma 1.1.2. *[34] A representation $\pi \in \mathcal{M}(G)$ is supercuspidal if and only if its matrix coefficients are compactly supported modulo the center of $G(F)$.*

If K is an open subgroup of $G(F)$, we denote by the symbol $\mathrm{c}-\mathrm{ind}_K^G$ the compact induction functor. The lemma 1.1.2 allows one to prove the following proposition.

Proposition 1.1.3. *[14] Let K be an open subgroup of $G(F)$ which is compact modulo the center of $G(F)$. Let ρ be a smooth irreducible representation of K and let $\pi = \mathrm{c}-\mathrm{ind}_K^G(\rho)$ be the compactly induced representation of ρ on $G(F)$. The following assertions are equivalent.*

(i) The intertwining $I_G(\rho)$ of ρ is reduced to K.

(ii) The representation π is irreducible and supercuspidal.

This observation (proposition 1.1.3) is absolutely fundamental and both constructions of supercuspidal representations studied in this paper are based on this fact.

1.2 Bushnell-Kutzko's construction of supercuspidal representations for GL_N

Bushnell and Kutzko [13] have constructed for each irreducible supercuspidal representation π of $GL_N(F)$, an open subgroup K, compact modulo the center of $GL_N(F)$, and a smooth irreducible representation Λ of K such that $\pi = \mathrm{c}-\mathrm{ind}_K^{GL_N(F)}(\Lambda)$. There are several texts which resume this construction (for example see [11]). In this section we give an other overview of this construction.

In the following we describe the construction of Bushnell and Kutzko, as in their book [13]. We follow very closely Bushnell and Kutzko and most parts of this section are copies of the original book [13]. We give almost all the definitions and recall the main step of the construction, we add some comments to help the reader. We want to insist that almost everything in this section is extracted from Bushnell-Kutzko's book. The reader is welcome to read at the same time [13].

1.2.1 Simple strata

Let V be an F-vector space of dimension N. Let A be the algebra $\mathrm{End}_F(V)$. If $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in A, we denote by $\mathfrak{P}$ its Jacobson radical and by $\nu_{\mathfrak{A}}$ the valuation on $\mathfrak{A}$ given by $\nu_{\mathfrak{A}}(x) = \max\{k \in \mathbb{Z} \mid x \in \mathfrak{P}^k\}$. A stratum in A is a quadruple $[\mathfrak{A}, n, r, \beta]$ where $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order, $n > r$ are integers and β is an element in A such that $\nu_{\mathfrak{A}}(\beta) \geq -n$. Let $e(\mathfrak{A} \mid \mathfrak{o}_F)$ denote the period of an $\mathfrak{o}_F$-lattice chain associated to $\mathfrak{A}$. Let $\mathfrak{K}(\mathfrak{A})$ be the normalizer of $\mathfrak{A}$ in $\mathrm{G} = A^\times$.

Before giving the definition of a pure stratum let us prove an elementary lemma which will be used often in others sections of this paper.

Lemma 1.2.1. *Let $\mathfrak{A}$ be an hereditary $\mathfrak{o}_F$-order in A, and let E be a field in A such that $E^\times \subset \mathfrak{K}(\mathfrak{A})$. Let β be an element in E, then*

$$\nu_{\mathfrak{A}}(\beta)e(E \mid F) = e(\mathfrak{A} \mid \mathfrak{o}_F)\nu_E(\beta). \tag{1.1}$$

Proof. Let π_E denote a uniformizer element in E. Since $E^\times \subset \mathfrak{K}(\mathfrak{A})$, the elements π_E, π_F and β are in $\mathfrak{K}(\mathfrak{A})$. Thus the equality [13, 1.1.3] is valid for these elements. We use it in the following equalities.

On the one hand

$$\beta^{e(E|F)}\mathfrak{A} = \pi_E^{\nu_E(\beta)e(E|F)}\mathfrak{A} = \pi_F^{\nu_E(\beta)}\mathfrak{A}. \tag{1.2}$$

On the other hand

$$\beta^{e(E|F)}\mathfrak{A} = \mathfrak{P}^{\nu_{\mathfrak{A}}(\beta)e(E|F)}. \tag{1.3}$$

Moreover by definition of $e(\mathfrak{A} \mid \mathfrak{o}_F)$ (see [13, 1.1.2]), we have

$$\pi_F^{\nu_E(\beta)}\mathfrak{A} = \mathfrak{P}^{e(\mathfrak{A}|\mathfrak{o}_F)\nu_E(\beta)}. \tag{1.4}$$

The equalities 1.2 , 1.3 and 1.4 show that

$$\mathfrak{P}^{\nu_{\mathfrak{A}}(\beta)e(E|F)} = \mathfrak{P}^{e(\mathfrak{A}|\mathfrak{o}_F)\nu_E(\beta)}. \tag{1.5}$$

Consequently $\nu_{\mathfrak{A}}(\beta)e(E \mid F) = e(\mathfrak{A} \mid \mathfrak{o}_F)\nu_E(\beta)$ and the equality 1.1 holds as required.

□

Definition 1.2.2. *[13, 1.5.5] A stratum is pure if the following conditions hold.*

(i) The F-algebra $E = F[\beta]$, generated by F and β in A, is a field.

(ii) $E^\times$ is included in $\mathfrak{K}(\mathfrak{A})$.

(iii) The equality $\nu_{\mathfrak{A}}(\beta) = -n$ holds.

Let $[\mathfrak{A}, n, r, \beta]$ be a pure stratum, for each $k \in \mathbb{Z}$ let $\mathfrak{N}_k(\beta, \mathfrak{A})$ be the set [13, 1.4.3]

$$\mathfrak{N}_k(\beta, \mathfrak{A}) := \{x \in \mathfrak{A} \mid \beta x - x\beta \in \mathfrak{P}^k\}.$$

Put $B = \mathrm{End}_{F[\beta]}(V)$ and $\mathfrak{B} = B \cap \mathfrak{A}$. We can define the following *critical exponent* $k_0(\beta, \mathfrak{A})$ [13, 1.4.5]:

$$k_0(\beta, \mathfrak{A}) := \begin{cases} -\infty \text{ if } E = F \\ \max\{k \in \mathbb{Z} \mid \mathfrak{N}_k(\beta, \mathfrak{A}) \not\subset \mathfrak{B} + \mathfrak{P}\} \text{ if } E \neq F. \end{cases}$$

Definition 1.2.3. *[13, 1.5.5] A stratum $[\mathfrak{A}, n, r, \beta]$ is simple if it is pure and $r < -k_0(\beta, \mathfrak{A})$.*

The simple stratum are constructed inductively from minimal elements, through a process which is the object of the section 2.2 of Bushnell-Kutzko 's work [13, 2.2]. The following is the definition of a minimal element giving birth to a stratum with just one iteration.

Definition 1.2.4. *[13, 1.4.14] Let E/F be a finite extension. An element $\beta \in E$ is minimal relatively to E/F if the following three conditions are satisfied.*

(i) The field $F[\beta]$ is equal to the field E.

(ii) The integer $\gcd(\nu_E(\beta), e(E \mid F))$ is equal to 1.

(iii) The element $\pi_F^{-\nu_E(\beta)}\beta^{e(E|F)} + \mathfrak{p}_E$ generates the residual field k_E over k_F.

An element β in $\overline{F}$ is minimal over F if it is minimal relatively to the extension $F[\beta]/F$.

Proposition 1.2.5. *Let $[\mathfrak{A}, n, n-1, \beta]$ be a pure stratum in the algebra $\mathrm{End}_F(V)$. The following assertions are equivalent.*

(i) The element β is minimal over F.

(ii) The critical exponent $k_0(\beta, \mathfrak{A})$ is equal to $-n$ or is equal to $-\infty$.

(iii) The stratum $[\mathfrak{A}, n, n-1, \beta]$ is simple.

Proof. This is a direct consequence of [13, 1.4.15]. Indeed, assume that $\beta \in F$, then β is clearly minimal over F, moreover $k_0(\beta, \mathfrak{A}) = -\infty$ by definition, and thus $k_0(\beta, \mathfrak{A}) < -(n-1)$, so the stratum $[\mathfrak{A}, n, n-1, \beta]$ is simple. The three properties, being always satisfied in this case, are equivalent. Assume now that $\beta \notin F$, by [13, 1.4.15] (i) and (ii) are equivalent, moreover it is clear that (ii) implies (iii). If (iii) is true then $k_0(\beta, \mathfrak{A}) < -(n-1)$ by definition of a simple stratum, moreover [13, 1.4.15] shows that $-n \leq k_0(\beta, \mathfrak{A})$. So $k_0(\beta, \mathfrak{A}) = -n$ and the assertion (ii) holds.

□

We need, for the rest of the paper, to define the notion of a tame corestriction [13, 1.3]. Let E/F be a finite extension of F contained in A. Let B denote $\mathrm{End}_E(V)$, the centralizer of E in A.

Definition 1.2.6. *[13, 1.3.3] A tame corestriction on A relatively to E/F is a (B,B)-bimodule homomorphism $s : A \to B$ such that $s(\mathfrak{A}) = \mathfrak{A} \cap B$ for every hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ normalized by $E^\times$.*

The following proposition shows that such maps exist.

Proposition 1.2.7. *[13, 1.3.4, 1.3.8 (ii)] With the same notations as before, the following holds.*

(i) Let ψ_E, ψ_F be complex, smooth, additive characters of E, F with conductor $\mathfrak{p}_E$, $\mathfrak{p}_F$ respectively. Let ψ_B and ψ_A the additive characters defined by $\psi_B = \psi_E \circ \mathrm{Tr}_{B/E}$ and $\psi_A = \psi_F \circ \mathrm{Tr}_{A/F}$. There exists a unique map $s : A \to B$ such that $\psi_A(ab) = \psi_B(s(a)b)$, $a \in A$, $b \in B$. The map s is a tame corestriction on A relatively to E/F.

(ii) If the field extension E/F is tamely ramified, there exists a tame corestriction s such that $s\mid_B = \mathrm{Id}_B$.

1.2.2 Simple characters

To each simple stratum $[\mathfrak{A}, n, r, \beta]$ is associated a group $H^1(\beta, \mathfrak{A})$ and a set of characters $\mathcal{C}(\beta, 0, \mathfrak{A})$ of $H^1(\beta, \mathfrak{A})$ whose intertwining in G is remarkable. This is the object of this section.

Definition 1.2.8. *Two strata $[\mathfrak{A}, n, r, \beta_1]$ and $[\mathfrak{A}, n, r, \beta_2]$ are equivalent if $\beta_1 - \beta_2 \in \mathfrak{P}^{-r}$. The notation $[\mathfrak{A}, n, r, \beta_1] \sim [\mathfrak{A}, n, r, \beta_2]$ means that $[\mathfrak{A}, n, r, \beta_1]$ and $[\mathfrak{A}, n, r, \beta_2]$ are equivalent.*

The following theorem is fundamental for the construction of the group $H^1(\beta, \mathfrak{A})$.

Theorem 1.2.9. *[13, 2.4.1]*

(i) Let $[\mathfrak{A}, n, r, \beta]$ be a pure stratum in the algebra A. There exists a simple stratum $[\mathfrak{A}, n, r, \gamma]$ in A equivalent to $[\mathfrak{A}, n, r, \beta]$, i.e. such that

$$[\mathfrak{A}, n, r, \gamma] \sim [\mathfrak{A}, n, r, \beta].$$

Moreover, for any simple stratum $[\mathfrak{A}, n, r, \gamma]$ satisfying this condition, $e(F[\gamma] \mid F)$ divides $e(F[\beta] \mid F)$ and $f(F[\gamma] \mid F)$ divides $f(F[\beta] \mid F)$. Moreover, among all pure strata $[\mathfrak{A}, n, r, \beta']$ equivalent to the given pure stratum $[\mathfrak{A}, n, r, \beta]$, the simple ones are precisely those for which the field extension $F[\beta']/F$ has minimal degree.

(ii) Let $[\mathfrak{A}, n, r, \beta]$ be a pure stratum in A with $r = -k_0(\beta, \mathfrak{A})$. Let $[\mathfrak{A}, n, r, \gamma]$ be a simple stratum in A which is equivalent to $[\mathfrak{A}, n, r, \beta]$, let s_γ be a tame corestriction on A relative to $F[\gamma]/F$, let B_γ be the A-centralizer of γ, i.e $B_\gamma = \mathrm{End}_{F[\gamma]}(V)$, and $\mathfrak{B}_\gamma = \mathfrak{A} \cap B_\gamma$. Then $[\mathfrak{B}_\gamma, r, r-1, s_\gamma(\beta - \gamma)]$ is equivalent to a simple stratum in B_γ.

Remark 1.2.10. *Let $[\mathfrak{A}, n, r, \beta]$ be a pure stratum which is not simple and let $[\mathfrak{A}, n, r, \gamma]$ be a simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$, by 1.2.9 (i) the degree $[F[\beta] : F]$ is strictly bigger than the degree $[F[\gamma] : F]$.*

Corollary 1.2.11. *[13, 2.4.2] Given a pure stratum $[\mathfrak{A}, n, r, \beta]$, the previous theorem and remark allow us to associate an integer s and a family $\{[\mathfrak{A}, n, r_i, \beta_i],\ 0 \leq i \leq s\}$ such that*

(i) $[\mathfrak{A}, n, r_i, \beta_i]$ *is a simple stratum for* $0 \leq i \leq s$,

(ii) $[\mathfrak{A}, n, r_0, \beta_0] \sim [\mathfrak{A}, n, r, \beta]$,

(iii) $r = r_0 < r_1 < \ldots < r_s < n$ *and* $[F[\beta_0] : F] > [F[\beta_1] : F] > \ldots > [F[\beta_s] : F]$,

(iv) $r_{i+1} = -k_0(\beta_i, \mathfrak{A})$, *and* $[\mathfrak{A}, n, r_{i+1}, \beta_{i+1}]$ *is equivalent to* $[\mathfrak{A}, n, r_{i+1}, \beta_i]$ *for* $0 \leq i \leq s-1$,

(v) $k_0(\beta_s, \mathfrak{A}) = -n$ *or* $-\infty$,

(vi) Let $\mathfrak{B}_{\beta_i}$ be the centralizer of β_i in $\mathfrak{A}$ and s_i a tame corestrition on A relativelty to $F[\beta_i]/F$. The derived stratum $[\mathfrak{B}_{\beta_{i+1}}, r_{i+1}, r_{i+1}-1, s_{i+1}(\beta_i - \beta_{i+1})]$ is equivalent to a simple stratum for $0 \leq i \leq s-1$.

This family is not unique and is called a ***defining sequence*** *for $[\mathfrak{A}, n, r, \beta]$.*

In order to help the reader, we give an explanation for this corollary.

Proof. • If $[\mathfrak{A}, n, r, \beta]$ is a simple stratum, put $[\mathfrak{A}, n, r_0, \beta_0] = [\mathfrak{A}, n, r, \beta]$ (remark that $r_0 < -k_0(\beta_0, \mathfrak{A})$). We now have an algorithm. If β_0 is minimal over F, put $s = 0$. Then (i) and (ii) are obviously satified, $r = r_0 < n$ is satisfied by definition of a simple stratum and because the rest of condition (iii) is empty. Condition (iv) is empty in this case so is satisfied. Condition (v) is satisfied by proposition 1.2.9. The condition (vi) is empty in this case and so is satisfied. If β_0 is not minimal, consider the stratum $[\mathfrak{A}, n, -k_0(\beta_0, \mathfrak{A}), \beta_0]$, it is pure but not simple. We now have a general process: the theorem 1.2.9 shows that there exists a simple stratum $[\mathfrak{A}, n, -k_0(\beta_0, \mathfrak{A}), \beta_1]$ equivalent to $[\mathfrak{A}, n, -k_0(\beta_0, \mathfrak{A}), \beta_0]$ (remark that $[F[\beta_0] : F] > [F[\beta_1] : F]$ by 1.2.10) such that for any tame corestriction s_{β_1} the stratum $[\mathfrak{B}_{\beta_0}, r, r-1, s_{\beta_1}(\beta_0 - \beta_1)]$ is simple. Put $r_1 = -k_0(\beta_0, \mathfrak{A})$. If β_1 is minimal over F, put $s = 1$. The condition (i), (ii), (iii), (iv) are now obviously satisfied. The condition (v) is also satisfied by proposition 1.2.5 and because β_1 is minimal over F. The condition (vi) is now obviously satisfied. If β_1 is not minimal over F. Consider the stratum $[\mathfrak{A}, n, -k_0(\beta_1, \mathfrak{A}), \beta_1]$, it is pure but not simple. As before, we apply the process to get a stratum $[\mathfrak{A}, n, -k_0(\beta_1, \mathfrak{A}), \beta_2]$ equivalent to $[\mathfrak{A}, n, -k_0(\beta_1, \mathfrak{A}), \beta_1]$. Put $r_2 = -k_0(\beta_1, \mathfrak{A})$. If β_2 is minimal, put $s = 2$. As before, the conditions (i) to (vi) are easily satisfied. If β_2 is not minimal, we can apply the process and get a simple stratum $[\mathfrak{A}, n, -k_0(\beta_2, \mathfrak{A}), \beta_3]$, if

β_3 is minimal we put $s = 3$ and $r_3 = -k_0(\beta_2, \mathfrak{A})$. If β_3 is not minimal, we apply the process and get a new stratum and an element β_4 and so on. We claim that there exists an integer s such that this algorithm stops, i.e β_s is minimal. Assume the contrary, then we have an infinite strictly increasing sequence of numbers between r and n

$$r = r_0 < r_1 = -k_0(\beta_0, \mathfrak{A}) < r_2 = -k_0(\beta_1, \mathfrak{A}) < \ldots < r_{i+1} = -k_0(\beta_i, \mathfrak{A}) < \ldots < n$$

this is a contradiction. This concludes the proposition in this case.

- If $[\mathfrak{A}, n, r, \beta]$ is pure but not simple, there exists a simple stratum $[\mathfrak{A}, n, r, \beta_0]$ equivalent to it and the previous case complete the proof.

□

Fix a simple stratum $[\mathfrak{A}, n, r, \beta]$, and let r be the integer $-k_0(\beta, \mathfrak{A})$. The following is the definition of various groups and orders associated to $[\mathfrak{A}, n, r, \beta]$. Choose and fix a defining sequence $\{[\mathfrak{A}, n, r_i, \beta_i], 0 \leq i \leq s\}$ of $[\mathfrak{A}, n, r, \beta]$ (we thus have $\beta = \beta_0$). If $s > 0$, the element β_1 is often denoted γ. We now define by induction on the length of the defining sequence various objects.

Definition 1.2.12. *[13, 3.1.7 ,3.1.8, 3.1.14]*

(i) Suppose that β is minimal over F. Put

  (a) $\mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{B}_\beta + \mathfrak{P}^{[\frac{n}{2}]+1}$,

  (b) $\mathfrak{J}(\beta, \mathfrak{A}) = \mathfrak{B}_\beta + \mathfrak{P}^{[\frac{n+1}{2}]}$.

(ii) Suppose that $r < n$, and let $[\mathfrak{A}, n, r, \gamma]$ be the simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$ chosen in the previously fixed defining sequence. Put

  (a) $\mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{B}_\beta + \mathfrak{H}(\gamma, \mathfrak{A}) \cap \mathfrak{P}^{[\frac{r}{2}]+1}$,

  (b) $\mathfrak{J}(\beta, \mathfrak{A}) = \mathfrak{B}_\beta + \mathfrak{J}(\gamma, \mathfrak{A}) \cap \mathfrak{P}^{[\frac{r+1}{2}]}$.

(iii) For $k \geq 0$, put

  (a) $\mathfrak{H}^k(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) \cap \mathfrak{P}^k$,

  (b) $\mathfrak{J}^k(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A}) \cap \mathfrak{P}^k$.

(iv) Finally, put $U^m(\mathfrak{A}) = (1 + \mathfrak{P}^m)$ if $m > 0$ and $U^m(\mathfrak{A}) = \mathfrak{A}^\times$ if $m = 0$ and put

  (a) $H^m(\beta, \mathfrak{A}) = \mathfrak{H}(\beta, \mathfrak{A}) \cap U^m(\mathfrak{A})$,

  (b) $J^m(\beta, \mathfrak{A}) = \mathfrak{J}(\beta, \mathfrak{A}) \cap U^m(\mathfrak{A})$.
  The set $H^m(\beta, \mathfrak{A})$ and $J^m(\beta, \mathfrak{A})$ are groups. The group $J^0(\beta, \mathfrak{A})$ is also denoted $J(\beta, \mathfrak{A})$.

Remark 1.2.13. *In the case $r < n$, $\mathfrak{H}(\beta, \mathfrak{A})$ is defined inductively: the order $\mathfrak{H}(\beta_s, \mathfrak{A})$ is well-defined since β_s is minimal, then $\mathfrak{H}(\beta_{s-1}, \mathfrak{A})$ is well defined and so on. The same remark occurs for $\mathfrak{J}(\beta, \mathfrak{A})$.*

Remark 1.2.14. *By [13, 3.1.7, 3.1.9 (v)], $\mathfrak{J}^k(\beta, \mathfrak{A})$ and $\mathfrak{H}^k(\beta, \mathfrak{A})$ are well-defined, they do not depend on the choice of a defining sequence. So the same is true for $H^m(\beta, \mathfrak{A})$ and $J^m(\beta, \mathfrak{A})$.*

Proposition 1.2.15. *[13, 3.1.15] Let $m \geq 0$ be an integer then the following assertions hold.*

(i) The groups $H^m(\beta, \mathfrak{A})$ and $J^m(\beta, \mathfrak{A})$ are normalized by $\mathfrak{K}(\mathfrak{B}_\beta)$, so in particular by $F[\beta]^\times$.

(ii) The group $H^m(\beta, \mathfrak{A})$ is included in $J^m(\beta, \mathfrak{A})$.

(iii) The group $H^{m+1}(\beta, \mathfrak{A})$ is a normal subgroup of $J^0(\beta, \mathfrak{A})$.

The following is devoted to the definition of the so called simple characters. Let Ψ be an additive character of F with conductor $\mathfrak{p}_F$. Let ψ_A be the function on A defined by $\psi_A(x) = \psi \circ \mathrm{Tr}_{A/F}(x)$. To any $b \in A$ is associated a function ψ_b on A given by

$$\psi_b(x) = \psi_A(b(x-1)).$$

Definition 1.2.16. *(i) Suppose that β is minimal over F.*

For $0 \leq m \leq n-1$, let $\mathcal{C}(\mathfrak{A}, m, \beta)$ denote the set of characters θ of $H^{m+1}(\beta)$ such that:

(a) $\theta \mid_{H^{m+1}(\beta) \cap U^{[\frac{n}{2}]+1}(\mathfrak{A})} = \psi_\beta$,

(b) $\theta \mid_{H^{m+1}(\beta) \cap B_\beta^\times}$ factors through $\det_{B_\beta} : B_\beta^\times \to F[\beta]^\times$.

(ii) Suppose that $r < n$. For $0 \leq m \leq r-1$, let $\mathcal{C}(\mathfrak{A}, m, \beta)$ be the set of characters θ of $H^{m+1}(\beta)$ such that the following conditions hold.

(a) $\theta \mid H^{m+1}(\beta) \cap B_\beta^\times$ factors through $\det_{B_\beta}$

(b) θ is normalised by $\mathfrak{K}(\mathfrak{B}_\beta)$

(c) if $m' = \max\{m, [\frac{r}{2}]\}$, the restriction $\theta \mid H^{m'+1}(\beta)$ is of the form $\theta_0 \psi_c$ for some $\theta_0 \in \mathcal{C}(\mathfrak{A}, m', \gamma)$ where $c = \beta - \gamma$ and γ is the first element of the fixed defining sequence.

Remark 1.2.17. *In the second case, $\mathcal{C}(\mathfrak{A}, m, \beta)$ is defined by induction: recall that we have fixed a defining sequence $\{[\mathfrak{A}, n, r_i, \beta_i],\ 0 \leq i \leq s\}$ of $[\mathfrak{A}, n, 0, \beta]$, the last term of the defining sequence is such that β_s is minimal over F and by the first case, there is a set of character attached. Then, those attached to $[\mathfrak{A}, n, r_{s-1}, \beta_{s-1}]$ are defined, and by iteration the set $\mathcal{C}(\mathfrak{A}, m, \beta)$ is defined.*

Remark 1.2.18. *[13, 3.2] The set $\mathcal{C}(\mathfrak{A}, m, \beta)$ defined above is independent of the choice of the defining sequence.*

Proposition 1.2.19. *[13, 3.3.2] Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum in the algebra A. Put $r = -k_0(\beta, \mathfrak{A})$. For $0 \leq m \leq [\frac{r}{2}]$ and $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$, the intertwining of θ in G is given by*

$$I_G(\theta) = J^{[\frac{r+1}{2}]}(\beta, \mathfrak{A}) B_\beta^\times J^{[\frac{r+1}{2}]}(\beta, \mathfrak{A}).$$

1.2.3 Simple types and representations

This section is devoted to the definition of simple types and to one of the main theorems of Bushnell-Kutzko's theory.

Let $[\mathfrak{A}, n, 0, \beta]$ be a simple stratum and let $\theta \in \mathcal{C}(\beta, 0, \mathfrak{A})$ be a simple character attached to this stratum. There exists a unique, up to isomorphism, irreducible representation η of $J^1(\beta, \mathfrak{A})$ containing θ [13, 5.1.1]. The dimension of η is equal to $[J^1(\beta, \mathfrak{A}) : H^1(\beta, \mathfrak{A})]^{\frac{1}{2}}$.

Definition 1.2.20. *[13, 5.2.1] A β-extension of η is a representation κ of $J^0(\beta, \mathfrak{A})$ such that the following conditions hold.*

(i) $\kappa \mid_{J^1(\beta,\mathfrak{A})} = \eta$

(ii) *κ is intertwined by the whole of $B^\times$.*

We say that κ is a β-extension of θ if there exists an irreducible representation η of $J^1(\beta, \mathfrak{A})$ containing θ such that κ is a β-extension of η.

Proposition 1.2.21. *Let κ be an irreducible representation of $J^0(\beta, \mathfrak{A})$. The following assertions are equivalent.*

(i) The representation κ is a $\beta-$extension of θ.

(ii) The representation κ satisfies the following three conditions.

- *(a) κ contains θ*
- *(b) κ is intertwined by the whole of $B^\times$*
- *(c)* $\dim(\kappa) = [J^1(\beta, \mathfrak{A}) : H^1(\beta, \mathfrak{A})]^{\frac{1}{2}}$.

Proof. If κ is a $\beta - extension$, κ satisfies $(a), (b), (c)$. Indeed, by definition κ restricted to $J^1(\beta, \mathfrak{A})$ is equal to an irreducible representation η which contains θ, thus κ contains θ and $\dim(\kappa) = \dim(\eta) = [J^1(\beta, \mathfrak{A}) : H^1(\beta, \mathfrak{A})]^{\frac{1}{2}}$. By definition, κ is intertwined by the whole of $B^\times$. Reciprocally, if κ satisfies $(a), (b), (c)$ then $(\kappa \mid_{J^1(\beta,\mathfrak{A})}) \mid_{H^1(\beta,\mathfrak{A})}$ contains θ, so $\kappa \mid_{J^1(\beta,\mathfrak{A})}$ contains an irreducible representation η which contains θ, and the equality on dimension thus shows $\kappa \mid_{J^1(\beta,\mathfrak{A})} = \eta$. Thus κ is a $\beta-$extension as required. □

Proposition 1.2.22. *Let κ_1 and κ_2 be two β-extension of θ. There exists a character $\chi : U^0(\mathfrak{o}_E)/U^1(\mathfrak{o}_E) \to \mathbb{C}^\times$ such that κ_1 is isomorphic to $\kappa_2 \otimes \chi \circ \det_B$.*

Proof. There exists η_1 and η_2, irreducible representations containing θ, such that κ_1 is a β-extension of η_1 and κ_2 is a β-extension of η_2. The representation η_1 is isomorphic to η_2. The proposition 1.2.22 is now a consequence of [13, 5.2.2].

□

Definition 1.2.23. *A simple type in* G *is one of the following (a) or (b).*

(a) An irreducible representation $\lambda = \kappa \otimes \sigma$ of $J(\beta, \mathfrak{A})$ where:

- *(i) $\mathfrak{A}$ is a principal $\mathfrak{o}_F$-order in $\mathfrak{A}$ and $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum;*
- *(ii) κ is a $\beta-$extension of a character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$;*
- *(iii) if we write $E = F[\beta], \mathfrak{B} = \mathfrak{A} \cap \mathrm{End}_E(V)$, so that*

$$J(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \simeq U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \mathrm{GL}_f(k_E)^e$$

for certain integers e, f, then σ is the inflation of a representation $\sigma_0 \otimes \cdots \otimes \sigma_0$ where σ_0 is an irreducible cuspidal representation of $\mathrm{GL}_f(k_E)$,

(b) An irreducible representation σ of $U(\mathfrak{A})$ where:

- *(i) $\mathfrak{A}$ is a principal $\mathfrak{o}_F-$order in A,*
- *(ii) if we write $U(\mathfrak{A})/U^1(\mathfrak{A}) \simeq \mathrm{GL}_f(k_F)^e$, for certain integers e, f, then σ is the inflation of a representation $\sigma_0 \otimes \cdots \otimes \sigma_0$, where σ_0 is an irreducible cuspidal representation of $\mathrm{GL}_f(k_F)$.*

The following theorem is one of the main theorem of Bushnell-Kutzko theory [13].

Theorem 1.2.24. *[13, 8.4.1] Let π be an irreducible supercuspidal representation of $\mathrm{G} = \mathrm{Aut}_F(V) \simeq \mathrm{GL}_N(F)$. There exists a simple type (J, λ) in* G *such that $\pi \mid J$ contains λ. Further,*

- *(i) the simple type (J, λ) is uniquely determined up to* G*-conjugacy,*
- *(ii) if (J, λ) is given by a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A = \mathrm{End}_F(V)$ with $E = F[\beta]$, there is a a uniquely determined representation Λ of $E^\times J$ such that $\Lambda \mid_J = \lambda$ and $\pi = \mathrm{c-ind}(\Lambda)$, in this case $\mathfrak{A} \cap \mathrm{End}_E(V)$ is a maximal $\mathfrak{o}_E$-order $\mathrm{End}_E(V)$.*
- *(iii) if (J, λ) is of the form (b) , i.e if $J = U(\mathfrak{A})$ for some maximal $\mathfrak{o}_F$-order $\mathfrak{A}$ and λ is trivial on $U^1(\mathfrak{A})$, then there is a uniquely determined representation Λ of $F^\times U(\mathfrak{A})$ such that $\Lambda \mid_{U(\mathfrak{A})} = \lambda$ and $\pi = \mathrm{c-ind}(\Lambda)$.*

Let us now introduce a terminology specific to the purpose of this text.

Definition 1.2.25. *A Bushnell-Kutzko datum in A is one of the following sequence.*

(a) A uple of the form $([\mathfrak{A}, n, 0, \beta], \theta, \kappa, \sigma, \Lambda)$ such that:

(i) $[\mathfrak{A}, n, 0, \beta]$ is a simple stratum in A such that $\mathfrak{B}_\beta$ is a maximal $\mathfrak{o}_E$-order,

(ii) $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$ is a simple character attached to $[\mathfrak{A}, n, 0, \beta]$,

(iii) κ is a β-extension of θ,

(iv) σ is an irreducible cuspidal representation of $U^0(\mathfrak{B}_\beta)/U^1(\mathfrak{B}_\beta)$,

(v) Λ is an extension to $E^\times J^0(\beta, \mathfrak{A})$ of $\kappa \otimes \sigma$.

(b) A uple of the form $(\mathfrak{A}, \sigma, \Lambda)$ where $\mathfrak{A}$ is a maximal $\mathfrak{o}_F$-order in A, σ is a cuspidal representation of $U^0(\mathfrak{A})/U^1(\mathfrak{A})$ and Λ is an extension to $F^\times U^0(\mathfrak{A})$ of σ.

Remark 1.2.26. *As in definition [13, 5.5.10], this distinction* (*a*) *and* (*b*) *is quite superficial (see the remark after [13, 5.5.10]).*

Remark 1.2.27. *As we have explained in this section, in order to construct one supercuspidal representation, Bushnell and Kutzko do some choices of objects at various steps of the construction. These choices of objects may depend on previously considered and choosen other objects. The "notion" of Bushnell-Kutzko datum takes into account this. In the Bushnell-Kutzko datum $([\mathfrak{A}, n, 0, \beta], \theta, \kappa, \sigma, \Lambda)$, θ depends on $[\mathfrak{A}, n, 0, \beta]$, κ depends on θ, and Λ depends on κ and σ. In Yu's construction, as we will see in the next section, all the choices are done at the beginning.*

In this chapter we are going to associate to each Bushnell-Kutzko datum satisfying a tameness condition a Yu datum. The following is the definition of a tame Bushnell-Kutzko datum.

Definition 1.2.28. *A tame Bushnell-Kutzko datum is a Bushnell-Kutzko datum $([\mathfrak{A}, n, 0, \beta], \theta, \kappa, \sigma, \Lambda)$ of type* (*a*) *such that $[\mathfrak{A}, n, 0, \beta]$ is a tame simple stratum (see 1.4.1 for the definition of a tame simple stratum) or a Bushnell-Kutzko datum of type* (*b*).

1.3 Yu's construction of tame supercuspidal representations

Given a connected reductive algebraic F-group G, Yu [41] constructs irreducible supercuspidal representations of $G(F)$, these representations are said to be tame. Adler's work [1] has inspired parts of Yu's construction. Kim [27] has proved that when the residual characteristic of F is sufficiently big, the construction of Yu is exhaustive. Fintzen has recently posted online a better exhaustion result [20].

In the following we describe the construction of Yu, as in Yu's paper [41]. We follow very closely Yu and most parts of this section are copies of original Yu's paper. We give almost all definitions and recall the main steps of the construction, we add some comments to help the reader. Chapter 3 of Hakim-Murnaghan's paper [24] should also be helpful for this section. We use some of Hakim-Murnaghan's notations, in particular we use the notations π_{-1}, and κ_i. We want to insist that almost everything in this section is extracted from Yu's article [41]. The reader is welcome to read at the same time [41]. In particular, the reader who knows Yu's construction does not have to read this part except for notations.

We start by recalling some facts on tame twisted Levi sequences (1.3.1). We then introduce the definition of generic characters (1.3.2). This allows us to introduce the definition of a generic supercuspidal Yu datum. We also use the simpler expression "Yu datum" in this text. The notion of (non-necessary supercuspidal) generic Yu datum exists [28] and generalize the notion of supercuspidal Yu datum. Now in this text Yu datum will always mean supercuspidal generic Yu datum.

1.3.1 Tamely ramified twisted Levi sequences and groups

In this section we introduce some notations and facts relative to them used in Yu's construction. We refer to the sections 1 and 2 of [41] for proofs.

We refer the reader to [8, 6.4.1] for the definition of the totally ordered commutative monoid $\tilde{\mathbb{R}} = \mathbb{R} \sqcup \ \mathbb{R}+ \sqcup \infty$.

Definition 1.3.1. *A tame twisted Levi sequence $\overrightarrow{G}$ in G is a sequence*

$$(G^0 \subset G^1 \subset \ldots \subset G^d = G)$$

of reductive F-subgroups of G such that there exists a tamely ramified finite Galois extension E/F such that $G^i \times_{\mathrm{spec}(F)} \mathrm{spec}(E)$ is a split Levi subgroup of $G \times_{\mathrm{spec}(F)} \mathrm{spec}(E)$, for $0 \leq i \leq d$.

Let $\overrightarrow{G}$ be a tame twited Levi sequence, there exists a maximal torus $T \subset G^0$ defined over F such that $T \times_{\mathrm{spec}(F)} \mathrm{spec}(E)$ is split. For each $0 \leq i \leq d$, let Φ_i be the union of the set of roots $\Phi(G^i, T, E)$ and $\{0\}$, i.e $\Phi_i = \Phi(G^i, T, E) \cup \{0\}$. For each $a \in \Phi_d \setminus \{0\}$, let $G_a \subset G = G^d$ the root subgroup corresponding to a, and let G_a be T if $a = 0$. Let $\mathfrak{g}(E)$ be the Lie algebra of G over E, and and let $\mathfrak{g}^*(E)$ be the dual of $\mathfrak{g}(E)$. For each $a \in \Phi_d$ let $\mathfrak{g}_a(E)$ (resp $\mathfrak{g}^*_a(E)$) be the a-eigenspace of $\mathfrak{g}(E)$ (resp $\mathfrak{g}^*(E)$) as a rational representation of T. Then $\mathfrak{g}_a(E)$ is the Lie algebra of G_a, and $\mathfrak{g}^*_a(E)$ is the dual of $\mathfrak{g}_{-a}(E)$.

If $0 \leq i \leq j \leq d$, we have a natural inclusion of roots: $\Phi_i \subset \Phi_j$.

Let $\overrightarrow{r} = (r_0, \ldots, r_i, \ldots, r_d)$ be a sequence of numbers in $\tilde{\mathbb{R}}$, we introduce a function $f_{\overrightarrow{r}}$ from $\Phi(G^d, T, E)$ to $\tilde{\mathbb{R}}$ as follows: $f(a) = r_0$ if $a \in \Phi_0$, $f(a) = r_k$ if $a \in \Phi_k \setminus \Phi_{k-1}$.

By definition, a sequence $\overrightarrow{r} = (r_0, r_1, \ldots, r_d)$ of numbers in $\tilde{\mathbb{R}}$ is admissible if there exists $\nu \in \mathbb{Z}$ such that $0 \leq \nu \leq d$ and

$$0 \leq r_0 = \ldots = r_\nu, \tfrac{1}{2} r_\nu \leq r_{\nu+1} \leq \ldots \leq r_d.$$

Let y be in the appartement $A(G, T, E) \subset \mathrm{BT}^E(G, E)$.

The point y determines filtration subgroups $\{G_a(E)_{y,r}\}_{r \in \tilde{R}, r \geq 0}$ of $G_a(E)$, lattices $\{\mathfrak{g}_a(E)_{y,r}\}_{r \in \tilde{R}}$ and latttices $\{\mathfrak{g}^*_a(E)_{y,r}\}_{r \in \tilde{R}}$ in $\mathfrak{g}^*_a(E)_{y,r}$, for each $a \in \Phi_d$. If $a \neq 0$, the filtration of $G_a(E)$ can be extended to a filtration $\{G_a(E)_{y,r}\}_{r \in \tilde{R}}$ indexed by the whole of $\tilde{\mathbb{R}}$. For any $\tilde{\mathbb{R}}$-valued function f on Φ_d such that $f(0) \geq 0$, let $G(E)_{y,f}$ be the subgroup generated by $G_a(E)_{y,f(a)}$ for all $a \in \Phi_d$, and let $\mathfrak{g}(E)_{y,f}$ (resp $\mathfrak{g}^*(E)_{y,f}$) be the lattice generated by $\mathfrak{g}_a(E)_{y,f(a)}$ (resp $\mathfrak{g}^*_a(E)_{y,f(a)}$) for all $a \in \Phi_d$. We will denote $G(E)_{y,f_{\overrightarrow{r}}}$ by $\overrightarrow{G}(E)_{y,\overrightarrow{r}}$, and $\mathfrak{g}(E)_{y,f_{\overrightarrow{r}}}$ (resp $\mathfrak{g}^*(E)_{y,f_{\overrightarrow{r}}}$) by $\overrightarrow{\mathfrak{g}}(E)_{y,\overrightarrow{r}}$ (resp $\overrightarrow{\mathfrak{g}}^*(E)_{y,\overrightarrow{r}}$). Let $\overrightarrow{r}, \overrightarrow{s}$ be two admissible sequences of elements in $\tilde{\mathbb{R}}$. We write $\overrightarrow{r} < \overrightarrow{s}$ (resp $\overrightarrow{r} \leq \overrightarrow{s}$) if $r_i < s_i$ (resp $r_i \leq s_i$) for $0 \leq i \leq d$. If $\overrightarrow{r} < \overrightarrow{s}$, to simplify the notation, we put

$$\overrightarrow{G}(E)_{y,\overrightarrow{r}:\overrightarrow{s}} = \overrightarrow{G}(E)_{y,\overrightarrow{r}} / \overrightarrow{G}(E)_{y,\overrightarrow{s}} \text{ and } \overrightarrow{\mathfrak{g}}(E)_{y,\overrightarrow{r}:\overrightarrow{s}} = \overrightarrow{\mathfrak{g}}(E)_{y,\overrightarrow{r}} / \overrightarrow{\mathfrak{g}}(E)_{y,\overrightarrow{s}}.$$

We have assumed that $y \in A(G, T, E) \subset \mathrm{BT}^E(G, E)$. Therefore, y determines a valuation of the root datum of (G, T, E) in the sense of [8]. This valuation restricted on the root datum of (G^i, T, E), is a valuation there. Therefore, it determines a point y_i in $A(G^i, T, E)$ modulo the action of $X_*(Z(G^i), E) \otimes_{\mathbb{Z}} \mathbb{R}$. A choice of y_i determines an embedding $j_i : \mathrm{BT}^E(G^i, E) \to \mathrm{BT}^E(G, E)$, which is $G^i(E)$-equivariant and maps y_i to y. We now fix y_i for $0 \leq i \leq d$ and identify $\mathrm{BT}^E(G^i, E)$ with its image in $\mathrm{BT}^E(G, E)$ under j_i. We thus identify y_i with y.

The following is an important proposition.

Proposition 1.3.2. *[41] The following assertions hold.*

(i) $\overrightarrow{G}(E)_{y,\overrightarrow{r}}$, $\overrightarrow{\mathfrak{g}}(E)_{y,\overrightarrow{r}}$ *and* $\overrightarrow{\mathfrak{g}}(E)_{y,\overrightarrow{r}}$ *are independent of the choice of* T.

(ii) If $\overrightarrow{r}, \overrightarrow{s}$ *are two admissible sequences such that*

$$0 < r_i \leq s_i \leq \min(r_i, \ldots, r_d) + \min(\overrightarrow{r}) \text{ for } 0 \leq i \leq d$$

then $\overrightarrow{G}(E)_{y,\overrightarrow{r}:\overrightarrow{s}}$ *is abelian and isomorphic to* $\overrightarrow{\mathfrak{g}}(E)_{y,\overrightarrow{r}:\overrightarrow{s}}$.

(iii) If $\overrightarrow{r}$ *is an admissible increasing sequence, we have*

$$\overrightarrow{G}(E)_{y,\overrightarrow{r}} = G^0(E)_{y,r_0}G^1(E)_{y,r_1}\dots G^d(E)_{y,r_d}$$

where $G^i(E)_{y,r_i}$, $0 \le i \le d$, are Moy-Prasad's groups (see Notation).

The sets $A(G,T,E)$ and $\mathrm{BT}^E(G,F)$ are both subsets of $\mathrm{BT}^E(G,E)$. We put $A(G,T,F) = A(G,T,E) \cap \mathrm{BT}^E(G,F)$, it does not depend on the choice of the splitting field E. Since T (hence $\overrightarrow{G}$) has a tamely ramified Galois splitting field E, $\mathrm{Gal}(E/F)$ acts on $A(G,T,E)$ by affine automorphisms. The center of mass of a $\mathrm{Gal}(E/F)$-orbit in $A(G,T,E)$ is fixed by $\mathrm{Gal}(E/F)$, and is a point of $A(G,T,F)$ by a result of Rousseau. This observation has been used by Adler in [1]. Let $y \in A(G,T,F) \subset A(G,T,E)$, and let $\overrightarrow{r}$ be an ($\tilde{\mathbb{R}}$-valued) admissible sequence of length $d+1$. We define $\overrightarrow{G}(F)_{y,\overrightarrow{r}}$ to be $\overrightarrow{G}(E)_{y,\overrightarrow{r}} \cap G(F)$, it does not depend on the choice of E. Recall that we have assumed E/F to be a Galois extension. The group $\overrightarrow{G}(E)_{y,\overrightarrow{r}}$ is Galois stable and $\overrightarrow{G}(F)_{y,\overrightarrow{r}} = \overrightarrow{G}(E)_{y,\overrightarrow{r}}{}^{\mathrm{Gal}(E/F)}$. The lattices $\overrightarrow{\mathfrak{g}}(F)_{y,\overrightarrow{r}}$ and $\overrightarrow{\mathfrak{g}}^*(F)_{y,\overrightarrow{r}}$ are defined in the same fashion. Again we define $\overrightarrow{G}(F)_{y,\overrightarrow{r}:\overrightarrow{s}} = \overrightarrow{G}(F)_{y,\overrightarrow{r}}/\overrightarrow{G}(F)_{y,\overrightarrow{s}}$ and define $\overrightarrow{\mathfrak{g}}(F)_{y,\overrightarrow{r}:\overrightarrow{s}}$ and $\overrightarrow{\mathfrak{g}}^*(F)_{y,\overrightarrow{r}:\overrightarrow{s}}$ similarly.

The following is an important proposition.

Proposition 1.3.3. *Let $0 \le \overrightarrow{r} \le \overrightarrow{s}$ and $\overrightarrow{s} > 0$. Then*

(i) The natural morphisms of groups

$$\overrightarrow{G}(F)_{y,\overrightarrow{r}:\overrightarrow{s}} \to \overrightarrow{G}(E)_{y,\overrightarrow{r}:\overrightarrow{s}}{}^{\mathrm{Gal}(E/F)}$$

and

$$\overrightarrow{\mathfrak{g}}(F)_{y,\overrightarrow{r}:\overrightarrow{s}} \to \overrightarrow{\mathfrak{g}}(E)_{y,\overrightarrow{r}:\overrightarrow{s}}{}^{\mathrm{Gal}(E/F)}$$

are surjective

(ii) If $0 < \overrightarrow{r} < \overrightarrow{s}, s_i \le \min(r_i,\dots,r_d) + \min(\overrightarrow{r})$ for all i, and E/F is a splitting field of $\overrightarrow{G}$ which is Galois and tamely ramified, then the isomorphism $\overrightarrow{G}(E)_{y,\overrightarrow{r}:\overrightarrow{s}} \to \overrightarrow{\mathfrak{g}}(E)_{y,\overrightarrow{r}:\overrightarrow{s}}$ induces an isomorphism

$$\overrightarrow{G}(F)_{y,\overrightarrow{r}:\overrightarrow{s}} \to \overrightarrow{\mathfrak{g}}(F)_{y,\overrightarrow{r}:\overrightarrow{s}}.$$

We have assumed that $y \in \mathrm{BT}^E(G,E) \cap A(G,T,E)$. We may assume that y_i is fixed by $\mathrm{Gal}(E/F)$. Then y_i is a point in $\mathrm{BT}^E(G^i,F)$ by a result of Rousseau. The embedding $j_i : \mathrm{BT}^E(G^i,E) \to \mathrm{BT}^E(G,E)$ is Galois equivariant, hence induces an embeddings $\mathrm{BT}^E(G^i,F) \to \mathrm{BT}^E(G,F)$ by an other result of Rousseau. We identify $\mathrm{BT}^E(G^i,F)$ with its image in $\mathrm{BT}^E(G,F)$. Therefore, we identify y_i with y.

We now have an other important proposition

Proposition 1.3.4. *[41, 2.10] If $\overrightarrow{r}$ is increasing with $r_0 > 0$, we have*

$$\overrightarrow{G}(F)_{y,\overrightarrow{r}} = G^0(F)_{y,r_0} G^1(F)_{y,r_1} \dots G^d(F)_{y,r_d}$$

where $G^i(F)_{y,r_i}$, $0 \leq i \leq d$, are Moy-Prasad's groups (see Notation).

1.3.2 Generic elements and generic characters

Recall that if L is a lattice in an F-vector space V, the dual lattice L^* is defined to be

$$\{x \in V^* \mid x(L) \subset \mathfrak{o}_F\}.$$

Put $L^\bullet = L^* \otimes_{\mathfrak{o}_F} \mathfrak{p}_F$. If $L \subset M$ are lattices in V, then the Pontrjagin dual of M/L can be identified with $L^\bullet/M^\bullet$ via an additive character ψ_F of conductor $\mathfrak{p}_F$. Explicitly, every element $a \in L^\bullet$ defines a character $\chi = \chi_a$ on M by $\chi_a(m) = \psi_F(a(m))$. Clearly, χ_a factors through $M \to M/L$ and χ_a depends on a mod $M^\bullet$ only. We say that a realizes the character χ.

If $\overrightarrow{r} = (r_0, \dots, r_d)$ is an $\mathbb{R}$-valued sequence, we define $\overrightarrow{r}+$ to be the sequence $(r_0+, \dots, r_d+)$. Then $\mathfrak{g}^*(F)_{y,\overrightarrow{r}}$ is equal to $\mathfrak{g}(F)^*_{y,(-\overrightarrow{r})+} \otimes_{\mathfrak{o}_F} \mathfrak{p}_F$ and $\mathfrak{g}^*(F)_{y,\overrightarrow{r}+}$ is equal to $\mathfrak{g}(F)^*_{y,-\overrightarrow{r}} \otimes_{\mathfrak{o}_F} \mathfrak{p}_F$.

Let $r > 0$ and let S be any group lying between $G(F)_{y,(r/2)+}$ and $G(F)_{y,r}$. Then $S/G(F)_{y,r+} \simeq \mathfrak{s}/\mathfrak{g}(F)_{y,r+}$, where $\mathfrak{s}$ is a lattice between $\mathfrak{g}(F)_{y,(r/2)+}$ and $\mathfrak{g}(F)_{y,r}$.

Definition 1.3.5. *A character of $S/G(F)_{y,r+}$ is said to be realized by an element $a \in \mathfrak{g}^*(F)_{y,-r} = (\mathfrak{g}(F)_{y,r+})^\bullet$ if it is egal to the composition*

$$S/G(F)_{y,r+} \xrightarrow{\sim} \mathfrak{s}/\mathfrak{g}(F)_{y,r+} \xrightarrow{\chi_a} \mathbb{C}^\times.$$

We now introduce the notion of generic element, a generic character will be defined as certain characters whose restrictions are realized by generic elements. Let $G' \subset G$ be a tamely ramified twisted Levi sequence. Let Z' de the center of G', and let T be a maximal torus of G'. The space $\mathrm{Lie}^*((Z')^\circ)$ can be regard as a subspace of $\mathrm{Lie}^*(G')$ in a canonical way: let V be the subspace of $\mathrm{Lie}^*(G')$ fixed by the coadjoint action of G'. Each element of V induces a linear function on $\mathrm{Lie}((Z')^\circ) \subset \mathrm{Lie}(G')$ by restriction. This gives a linear bijection from V to $\mathrm{Lie}^*((Z')^\circ)$. We identify $\mathrm{Lie}^*((Z')^\circ)$ with $V \subset \mathrm{Lie}^*(G')$. The space $\mathrm{Lie}^*(G')$ can also be regarded as a subspace of $\mathrm{Lie}^*(G)$ in a canonical way: if we consider the action of $(Z')^\circ$ on $\mathrm{Lie}^*(G)$, then the subspace fixed by $(Z')^\circ$ can be identified with $\mathrm{Lie}^*(G')$. The connected center $(Z')^\circ$ is a torus which split over a tamely ramified extension, so the set $(Z')^\circ(F), \mathrm{Lie}((Z')^\circ$ and $\mathrm{Lie}^*((Z')^\circ)$ carry canonical filtrations.

An element X^* of $(\mathrm{Lie}^*(Z')^\circ)_{-r}$ is called G-generic of depth $r \in \mathbb{R}$ if two conditions **GE1** and **GE2** hold. Let us explain **GE1**. Let a denote a root in $\Phi(G, T, \overline{F})$, let $a^\vee$ be the coroot of a, and let $\mathrm{d}a^\vee$ denote the differential of $a^\vee$. Let H_a denote the element $\mathrm{d}a^\vee(1)$.

Remark 1.3.6. *In the following definition of Yu, it is implicit that we see* X^* *canonically as an element in* $\mathrm{Lie}^*(Z'^{\circ} \times_{\mathrm{spec}(F)} \mathrm{spec}(\overline{F}))$*. This is done remarking two elementary facts valid for every reductive* F*-group scheme* G*. First,* $\mathrm{Lie}(\mathrm{G}^{\circ} \times_{\mathrm{spec}(F)} \mathrm{spec}(\overline{F}))$ *is canonically isomorphic to* $\mathrm{Lie}(\mathrm{G}^{\circ}) \otimes_F \overline{F}$ *since* F *is a field, and theirs duals are thus canonically isomorphic. The canonical injective map*

$$\begin{aligned} \mathrm{Lie}^*(\mathrm{G}^{\circ}) &\to (\mathrm{Lie}(\mathrm{G}^{\circ}) \otimes_F \overline{F})^* \\ f &\mapsto (z \otimes \lambda \mapsto f(z)\lambda) \end{aligned}$$

ends this remark.

Definition 1.3.7. *An element* X^* *of* $(\mathrm{Lie}^*(Z')^{\circ})_{-r}$ *satisfies* ***GE1*** *with depth* r *if* $\mathrm{ord}(X^*(H_a)) = -r$ *for all root* $a \in \Phi(G, T, \overline{F}) \setminus \Phi(G, T, F)$*.*

We refer to section 8 of [41] for the definition of the condition **GE2**. In general, the condition **GE2** is implied by the condition **GE1** in most cases. In particular, in this paper the condition **GE2** will always hold as soon as the condition **GE1** will hold thank to the following propositions. We refer to the section 7 of [41], or [37] for the notion of torsion prime for a root datum.

Proposition 1.3.8. *[41, 8.1] If the residual characteristic of* F *is not a torsion prime for the root datum* $(X, \Phi(G, T, \overline{F}), X^{\vee}, \Phi^{\vee}(G, T, \overline{F})$*, then* ***GE1*** *implies* ***GE2****.*

Proposition 1.3.9. *[37] Let* $(X, \Phi, X^{\vee}, \Phi^{\vee})$ *be a root datum of type A. Then, the set of torsion prime for* $(X, \Phi, X^{\vee}, \Phi^{\vee})$ *is empty.*

As announced before, the definition of a generic element is the following.

Definition 1.3.10. *An element* X^* *of* $(\mathrm{Lie}^*(Z')^{\circ})_{-r}$ *is called G-generic of depth* $r \in \mathbb{R}$ *if the conditions* ***GE1*** *and* ***GE2*** *hold.*

We can now give Yu's definition of generic characters.

Definition 1.3.11. *(i) A character* χ *of* $G'(F)$ *is called G-generic if it is realized (in the sense of definition 1.3.5) by an element* X^* *in* $(\mathrm{Lie}^*(Z')^{\circ})_{-r} \subset (\mathrm{Lie}^* G')_{y,-r}$ *which is G-generic of depth* r*.*

(ii) A character $\mathbf{\Phi}$ *of* $G'(F)$ *is called G-generic (relative to y) of depth* r *if* $\mathbf{\Phi}$ *is trivial on* $G'(F)_{y,r+}$*, non-trivial on* $G'(F)_{y,r}$ *and* $\mathbf{\Phi}$ *restricted to* $G'(F)_{y,r:r+}$ *is G-generic of depth* r *in the sense of* (i)*.*

1.3.3 Yu data

The following is the list of objects in a Yu datum.

Definition 1.3.12. *A Yu datum consists in the following objects.*

($\overrightarrow{G}$) *An anisotropic tame twisted Levi sequence in G, i.e*

$$G^0 \subset \cdots \subset G^i \subset \cdots \subset G^d = G$$

such that

(a) there exists a finite tamely ramified Galois extension E/F such that $G^i \times_{\mathrm{spec}(F)} \mathrm{spec}(E)$ is a split Levi subgroup of $G \times_{\mathrm{spec}(F)} \mathrm{spec}(E)$,

(b) $Z(G^0)/Z(G)$ is anisotropic.

(y) A point $y \in \mathrm{BT}^E(G^0, F) \cap A(G, T, E)$ where T is a maximal torus of G^0, such that $T \times_{\mathrm{spec}(F)} \mathrm{spec}(E)$ is split and $A(G, T, E)$ denotes the appartement associated to T over E,

($\overrightarrow{r}$) *A sequence of real numbers $0 < \mathbf{r}_0 < \mathbf{r}_1 < ... < \mathbf{r}_{d-1} \leq \mathbf{r}_d$ if $d > 0$, $0 \leq \mathbf{r}_0$ if $d = 0$,*

(ρ) An irreducible representation ρ of $K^0 = G^0_{[y]}$ such that $\rho\mid_{G^0(F)_{y,0+}} = 1$ and such that $\pi_{-1} := \mathrm{c} - \mathrm{ind}_{K^0}^{G^0(F)}(\rho)$ is irreducible and supercuspidal.

($\overrightarrow{\mathbf{\Phi}}$) *A sequence $\mathbf{\Phi}_0, \ldots, \mathbf{\Phi}_d$ of characters of $G^0(F), \ldots, G^d(F)$. We assume that $\mathbf{\Phi}_i$ is trivial on $G^i(F)_{y,\mathbf{r}_i+}$ but not on $G^i(F)_{y,\mathbf{r}_i}$ for $0 \leq i \leq d-1$. If $\mathbf{r}_{d-1} < \mathbf{r}_d$, we assume $\mathbf{\Phi}_d$ is trivial on $G^d(F)_{y,\mathbf{r}_d+}$ but not on $G^d(F)_{y,\mathbf{r}_d}$. If $\mathbf{r}_{d-1} = \mathbf{r}_d$, we assume that $\mathbf{\Phi}_d = 1$. The characters are assumed to satisfy the generic condition of Yu: $\mathbf{\Phi}_i$ is G^{i+1}-generic of depth $\mathbf{r}_i$ for $0 \leq i \leq d-1$.*

1.3.4 Yu's construction

We fix in the rest of this section a generic Yu datum. The three first objects $(\overrightarrow{G}, y, \overrightarrow{r})$ allow to define various groups. The point y can be seen as a point in the enlarged Bruhat-Tits Building of G^i for each i using embeddings $\mathrm{BT}^E(G^0, F) \hookrightarrow \mathrm{BT}^E(G^1, F) \hookrightarrow \ldots \hookrightarrow \mathrm{BT}^E(G^d, F)$ as explained in the section 2 of Yu's paper [41, §2 , page 589 line 5]. We fix, for the rest of this section, such embeddings. The following is the definition of three groups.

Definition 1.3.13. *[41, §3, 15.3] Put $\mathbf{s}_i = \frac{\mathbf{r}_i}{2}$ for $0 \leq i \leq d$.*
For $i = 0$, put

(i) $K^0_+ = G^0(F)_{y,0+}$

(ii) $^\circ K^0 = G^0(F)_y$

(iii) $K^0 = G^0(F)_{[y]}$.

For $1 \le i \le d$, put

(i)

$$\begin{aligned} K^i_+ &= G^0(F)_{y,0+} G^1(F)_{y,\mathbf{s}_0+} \cdots G^i(F)_{y,\mathbf{s}_{i-1}+} \\ &= (G^0, G^1, \ldots, G^i)(F)_{y,(0+,s_0+,\ldots,s_{i-1}+)} \end{aligned}$$

(ii)

$$\begin{aligned} ^\circ K^i &= G^0(F)_y G^1(F)_{y,\mathbf{s}_0} \cdots G^i(F)_{y,\mathbf{s}_{i-1}} \\ &= G^0(F)_y (G^0, G^1, \ldots, G^i)(F)_{y,(0,s_0,\ldots,s_{i-1})} \end{aligned}$$

(iii)

$$\begin{aligned} K^i &= G^0(F)_{[y]} G^1(F)_{y,\mathbf{s}_0} \cdots G^i(F)_{y,\mathbf{s}_{i-1}} \\ &= G^0(F)_{[y]} (G^0, G^1, \ldots, G^i)(F)_{y,(0,s_0,\ldots,s_{i-1})}. \end{aligned}$$

Proposition 1.3.14. *[41] Let $0 \le i \le d$.*

(i) The three objects K^i_+, $^\circ K^i$, K^i defined precedently are groups.

(ii) They do not depend on the choice of the embeddings

$$\mathrm{BT}^E(G^0, F) \hookrightarrow \mathrm{BT}^E(G^1, F) \hookrightarrow \ldots \hookrightarrow \mathrm{BT}^E(G^i, F).$$

(iii) There are inclusions $K^i_+ \ \subset\ ^\circ K^i \ \subset\ K^i$.

(iv) The groups K^i_+ and $^\circ K^i$ are compact and K^i is compact modulo the center. Moreover $^\circ K^i$ is the maximal compact subgroup of K^i.

Yu also define groups J^i and J^i_+ for $1 \le i \le d$ as follows. For $1 \le i \le d$, (r_{i-1}, s_{i-1}) and $(r_{i-1}, s_{i-1}+)$ are admissible sequence

Definition 1.3.15. *Let J^i be the group $(G^{i-1}, G^i)(F)_{(r_{i-1}, s_{i-1})}$ and J^i_+ be the group $(G^{i-1}, G^i)(F)_{(r_{i-1}, s_{i-1}+)}$.*

Proposition 1.3.16. *Let $0 \le i \le d-1$. The following equalities of groups hold:*

(i) $K^{i-1} J^i = K^i$

(ii) $K^{i-1}_+ J^i_+ = K^i_+$.

Thanks to $\overrightarrow{\mathbf{\Phi}}$, Yu defines a character $\prod_{i=1}^{d} \hat{\mathbf{\Phi}}_i$ on K_+^d. Then, he constructs a representation $\rho_d = \rho_d(\overrightarrow{G}, y, \overrightarrow{\mathbf{r}}, \rho, \overrightarrow{\mathbf{\Phi}})$ on K^d [41, §4]. Let us explain the construction of these objects.

Let $0 \leq i \leq d-1$.

Put $T^i = (Z(G^i))^\circ$, let us consider the adjoint action of T^i on $\mathfrak{g}$, the space $\mathfrak{g}^i = \mathrm{Lie}(G^i)$ is the maximal subspace on which T^i acts trivialy. Let $\mathfrak{n}^i$ be the sum of the remaining isotypic subspaces. Let $\mathbf{s} \geq 0 \in \tilde{\mathbb{R}}$, then $\mathfrak{g}(F)_s = \mathfrak{g}^i(F)_s \oplus \mathfrak{n}^i(F)_s$ where $\mathfrak{n}^i(F)_s \subset \mathfrak{n}^i(F)$. There exists a sequence of morphisms as follows (see [41, section 4]).

$$G^i(F)_{\mathbf{s}_{i+}:\mathbf{r}_{i+}} \simeq \mathfrak{g}^i(F)_{\mathbf{s}_{i+}:\mathbf{r}_{i+}} \subset \mathfrak{g}^i(F)_{\mathbf{s}_{i+}:\mathbf{r}_{i+}} \oplus \mathfrak{n}^i(F)_{\mathbf{s}_{i+}:\mathbf{r}_{i+}} \simeq G(F)_{\mathbf{s}_{i+}:\mathbf{r}_{i+}} \tag{1.6}$$

The character $\mathbf{\Phi}_i$ of $G^i(F)$ is of depth $\mathbf{r}_i$. Thus it induces, thanks to the isomorphism (1.6) , a character on $\mathfrak{g}^i(F)_{\mathbf{s}_{i+}:\mathbf{r}_{i+}}$. We extend the latter to $\mathfrak{g}^i(F)_{\mathbf{s}_{i+}:\mathbf{r}_{i+}} \oplus \mathfrak{n}^i(F)_{\mathbf{s}_{i+}:\mathbf{r}_{i+}}$ by decreting that it is 1 on $\mathfrak{n}^i(F)_{\mathbf{s}_{i+}:\mathbf{r}_{i+}}$. We obtain thanks to the last isomorphim in 1.6 a character on $G(F)_{\mathbf{s}_{i+}}$ that Yu denotes by $\hat{\mathbf{\Phi}}_i$. By construction, the following equality holds $\hat{\mathbf{\Phi}}_i \mid_{G^i(F)_{\mathbf{s}_{i+}}} = \mathbf{\Phi}_i \mid_{G^i(F)_{\mathbf{s}_{i+}}}$. There exists a unique character on $G^0(F)_{[y]}G^i(F)_0G(F)_{\mathbf{s}_{i+}}$ which extends $\mathbf{\Phi}_i$ and $\hat{\mathbf{\Phi}}_i$. Yu denote this character also by the symbol $\hat{\mathbf{\Phi}}_i$. Remark that $K_+^d \subset G^0(F)_{[y]}G^i(F)_0G(F)_{\mathbf{s}_{i+}}$, in particular we have defined a character $\hat{\mathbf{\Phi}}_i$ on K_+^d. The character $\hat{\mathbf{\Phi}}_i$ depends only on $(\overrightarrow{G}, y, \overrightarrow{\mathbf{r}}, \mathbf{\Phi}_i)$, we sometimes denote it $\hat{\mathbf{\Phi}}_i = \hat{\mathbf{\Phi}}_i(\overrightarrow{G}, y, \overrightarrow{\mathbf{r}}, \mathbf{\Phi}_i)$. Let $\theta(\overrightarrow{G}, y, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{\Phi}})$ be the character $\prod_{i=0}^{d} \hat{\mathbf{\Phi}}_i \mid_{K_+^d}$. We put $\hat{\mathbf{\Phi}}_d = \mathbf{\Phi}_d$.

Then Yu constructs for $0 \leq j \leq d$ a representation ρ_j of K^j. The compactly induced representation $\mathrm{c}-\mathrm{ind}_{K^j}^{G^j(F)}(\rho_j)$ is an irreducible and supercuspidal representation of $G^j(F)$. However, we are mainly interested in the case $j = d$, i.e in the representation ρ_d, since ρ_d depends on $\overrightarrow{G}, y, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{\Phi}}, \rho$, we also write $\rho_d = \rho_d(\overrightarrow{G}, y, \overrightarrow{\mathbf{r}}, \overrightarrow{\mathbf{\Phi}}, \rho)$. We will use similar notations in the following. For each j, the representation ρ_j of K^j is naturally expressed as a tensor product of representations.

Lemma 1.3.17. *[41, §4]Let $0 \leq i \leq d-1$, there exists a canonical irreducible representation $\tilde{\mathbf{\Phi}}_i$ of $K^i \rtimes J^{i+1}$ such that the following conditions hold.*

(i) The restriction of $\tilde{\mathbf{\Phi}}_i$ to $1 \ltimes J_+^{i+1}$ is $(\hat{\mathbf{\Phi}}_i \mid_{J_+^{i+1}})-$isotypic.

(ii) The restriction of $\tilde{\mathbf{\Phi}}_i$ to $K_+^i \ltimes 1$ is $1-$isotypic.

Lemma 1.3.18. *Let $0 \leq i \leq d-1$. Let $\inf(\mathbf{\Phi}_i)$ be the inflation of $\mathbf{\Phi}_i \mid_{K^i}$ to $K^i \ltimes J^{i+1}$. Let $\tilde{\mathbf{\Phi}}_i$ be the canonical irreducible representation introduced in lemma 1.3.17. Then $\inf(\mathbf{\Phi}_i) \otimes \tilde{\mathbf{\Phi}}_i$ factors through the map*

$$K^i \ltimes J^{i+1} \to K^i J^{i+1} = K^{i+1}.$$

Proof. This is easy and proved in section 4 of [41]. □

Definition 1.3.19. *Let us denote by $\mathbf{\Phi}'_i$ the representation of K^{i+1} whose inflation to $K^i \ltimes J^{i+1}$ is $\inf(\mathbf{\Phi}_i) \otimes \tilde{\mathbf{\Phi}}_i$.*

Lemma 1.3.20. *[24, page 50] The following assertions hold.*

(i) If μ is a representation of K^i which is 1-isotypic on $K^i \cap J^{i+1} = G^i(F)_{y,\mathbf{r}_i}$ then there is a unique extension of μ to a representation, denoted $\inf_{K^i}^{K^{i+1}}(\mu)$, of K^{i+1} which is 1-isotypic on J^{i+1}. If $i < d-1$, this inflated representation is 1-isotypic on $K^{i+1} \cap J^{i+2}$.

(ii) We may repeatedly inflate μ. More precisely, if $0 \le i \le j \le d$ then we may define $\inf_{K^i}^{K^j}(\mu) = \inf_{K^{j-1}}^{K^j} \circ \ldots \circ \inf_{K^i}^{K^{i+1}}(\mu)$.

Definition 1.3.21. *Let $0 \le j \le d$. Let $0 \le i < j$. Let κ_i^j be the inflation of $\mathbf{\Phi}'_i$ to K^j, i.e $\kappa_i^j = \inf_{K^{i+1}}^{K^j}(\mathbf{\Phi}'_i)$. Let κ_j^j be $\mathbf{\Phi}_j \mid_{K^j}$. Let κ_{-1}^j be the inflation of ρ to K^j, i.e $\kappa_{-1}^j = \inf_{K^0}^{K^j}(\rho)$.*

If $j = d$ and $-1 \le i \le d$, we also denote κ_i^d by κ_i. This notation and the statement of the following proposition is due to Hakim-Murnaghan.

Proposition 1.3.22. *The representation ρ_j constructed by Yu is isomorphic to*

$$\kappa_{-1}^j \otimes \kappa_0^j \otimes \ldots \otimes \kappa_j^j.$$

In particular, the representation ρ_d constructed by Yu is isomorphic to

$$\kappa_{-1} \otimes \kappa_0 \otimes \ldots \otimes \kappa_d.$$

Proof. The representation ρ_j is constructed in [41] at page 592. Yu constructs inductively two representations ρ_j and ${\rho_j}'$.

Let us show by induction on j that ${\rho_j}' = \kappa_{-1}^j \otimes \kappa_0^j \otimes \ldots \otimes \kappa_{j-1}^j$ and

$\rho_j = \kappa_{-1}^j \otimes \kappa_0^j \otimes \ldots \otimes \kappa_j^j$ If $j = 0$, then by definition, the representation ρ'_0 constructed by Yu is ρ and ρ_0 is $\rho'_0 \otimes (\mathbf{\Phi}_0 \mid_{K^0})$. We have $\kappa_{-1}^0 = \rho$ and $\kappa_0^0 = \mathbf{\Phi}_0 \mid_{K^0}$. So the case $j = 0$ is complete. Assume that $\rho'_{j-1} = \kappa_{-1}^{j-1} \otimes \kappa_0^{j-1} \ldots \otimes \kappa_{j-2}^{j-1}$ and $\rho_{j-1} = \kappa_{-1}^{j-1} \otimes \kappa_0^{j-1} \otimes \ldots \otimes \kappa_{j-1}^{j-1}$. Then by definition ρ'_j is equal to $\inf_{K^{j-1}}^{K^j}(\rho'_{j-1}) \otimes \mathbf{\Phi}'_{j-1}$. By definition $\mathbf{\Phi}'_{j-1}$ is equal to κ_{j-1}^j. Moreover

$$\begin{aligned}
\inf_{K^{j-1}}^{K^j}(\rho'_{j-1}) &= \inf_{K^{j-1}}^{K^j}(\kappa_{-1}^{j-1} \otimes \kappa_0^{j-1} \otimes \ldots \otimes \kappa_{j-2}^{j-1}) \\
&= \inf_{K^{j-1}}^{K^j}(\kappa_{-1}^{j-1}) \otimes \inf_{K^{j-1}}^{K^j}(\kappa_0^{j-1}) \otimes \ldots \inf_{K^{j-1}}^{K^j}(\kappa_{j-2}^{j-1}) \\
&= \kappa_{-1}^j \otimes \kappa_0^j \otimes \ldots \otimes \kappa_{j-2}^j
\end{aligned}$$

Consequently $\rho_j{}' = \kappa_{-1}^j \otimes \kappa_0^j \otimes \ldots \otimes \kappa_i^j \otimes \ldots \otimes \kappa_{j-1}^j$. Finally, by Yu's definition, ρ_j is equal to $\rho_j' \otimes \mathbf{\Phi}_j \mid_{K^j}$, and thus $\rho_j = \kappa_{-1}^j \otimes \kappa_0^j \otimes \ldots \otimes \kappa_j^j$, as required.

□

Proposition 1.3.23. *Let $0 \leq j \leq d$. Let $0 \leq i < j$. The dimension of κ_i^j is equal to the dimension of $\mathbf{\Phi}_i'$. The dimension of $\mathbf{\Phi}_i'$ is equal to $[J^{i+1} : J_+^{i+1}]^{\frac{1}{2}}$.*

Proof. By definition κ_i^j is an inflation of $\mathbf{\Phi}_i$, consequently theses representations have equal dimensions. The representation $\mathbf{\Phi}_i'$ is the unique representation of $K^i + 1$ whose inflation to $K^i \ltimes J^{i+1}$ is $\tilde{\mathbf{\Phi}}_i$. Thus, the dimension of $\mathbf{\Phi}_i'$ is equal to $\tilde{\mathbf{\Phi}}_i$. The representation $\tilde{\mathbf{\Phi}}_i$ is constructed in [41, 11.5] and is the pull back of the Weil representation of $Sp(J^{i+1}/J_+^{i+1}) \ltimes (J^{i+1}/N_i)$ where $N_i = \ker(\hat{\mathbf{\Phi}}_i)$ (see [41]). Thus, the dimension of $\tilde{\mathbf{\Phi}}_i$ is $[J^{i+1} : J_+^{i+1}]^{\frac{1}{2}}$.

□

Theorem 1.3.24. *(Yu) [41, 4.6 , §15] The representation $\mathrm{c}-\mathrm{ind}_{K^d}^{G(F)} \rho_d$ is irreducible and supercuspidal.*

We now introduce some notations that we will use later in chapter 1. Put ${}^\circ\rho_d = {}^\circ\rho_d(\overrightarrow{G}, y, \overrightarrow{\mathbf{r}}, \rho, \overrightarrow{\mathbf{\Phi}}) = \rho_d \mid_{{}^\circ K^d}$. Put also ${}^\circ\kappa_i = \kappa_i \mid_{{}^\circ K_d}$ and $\lambda^\circ = {}^\circ\kappa_0 \otimes \ldots \otimes {}^\circ\kappa_d$.

The following theorem shows that the construction of Yu is exhaustive when the residual characteristic is sufficiently large.

Theorem 1.3.25. *(Kim) [27] Let G be a connected reductive F group, if the residue characteristic p of F is sufficiently large, for each irreducible supercuspidal representation π of $G(F)$, there exists $(\overrightarrow{G}, y, \overrightarrow{\mathbf{r}}, \rho, \overrightarrow{\mathbf{\Phi}})$, such that $\pi = \mathrm{c}-\mathrm{ind}_{K^d}^{G(F)} \rho_d(\overrightarrow{G}, y, \overrightarrow{\mathbf{r}}, \rho, \overrightarrow{\mathbf{\Phi}})$.*

Fintzen has recently ameliorated this exhaustion result [20].

1.4 Tame simple strata

In this section, the main object of study is the approximation process for simple strata $[\mathfrak{A}, n, r, \beta]$ described previously in section 1.2, when the field extension $F[\beta]/F$ is tamely ramified. It is a well-known result that in this situation, an approximation element γ can be chosen inside the field $F[\beta]$. We will refer to Bushnell-Henniart for this fact which will be recalled as proposition 1.4.3 in this section. The main new result in this section is proposition 1.4.4, the proposition 1.4.2 is used to prove proposition 1.4.4.

Definition 1.4.1. *A pure (resp simple) stratum $[\mathfrak{A}, n, r, \beta]$ is a tame pure (resp tame simple) stratum if the field extension $F[\beta]/F$ is tamely ramified.*

Let $[\mathfrak{A}, n, r, \beta]$ be a tame pure stratum in the algebra $A = \mathrm{End}_F(V)$, set $E = F[\beta]$. Set also $B_E = \mathrm{End}_E(V)$. Let $s : A \to B_E$ be the tame corestriction which is the identity on B_E, we recall that such map exists by 1.2.7. The element $s(b)$ is denote by "b" when b is in B_E. Let $\mathfrak{P}$ be the Jacobson radical of $\mathfrak{A}$. Set $\mathfrak{B}_E = \mathfrak{A} \cap B_E$ and $\mathfrak{Q}_E = \mathfrak{P} \cap \mathfrak{B}_E$. Thus $\mathfrak{B}_E$ is an $\mathfrak{o}_E$-hereditary order in B_E and $\mathfrak{Q}_E$ is the Jacobson radical of $\mathfrak{B}_E$.

The following is a analogous to [13, 2.2.3], the difference is that the tameness condition is supposed and a maximality one removed.

Proposition 1.4.2. *Let $[\mathfrak{A}, n, r, \beta]$ be a tame simple stratum. Let $b \in \mathfrak{Q}_E^{-r}$, and suppose that the stratum $[\mathfrak{B}_E, r, r-1, b]$ is simple. Then*

(i) The stratum $[\mathfrak{A}, n, r-1, \beta + b]$ is simple

(ii) The field $F[\beta + b]$ is equal to the field $F[\beta, b]$

(iii) We have

$$k_0(\beta + b, \mathfrak{A}) = \begin{cases} -r = k_0(b, \mathfrak{B}_E) \text{ if } b \notin E \\ k_0(\beta, \mathfrak{A}) \text{ if } b \in E \end{cases}$$

Proof. Let $\mathcal{L} = \{L_i\}_{i \in \mathbb{Z}}$ be an $\mathfrak{o}_F$-lattice chain such that

$$\mathfrak{A} = \{x \in A \mid x(L_i) \subset L_i, i \in \mathbb{Z}\}.$$

By definition [13, 2.2.1],

$$\mathfrak{K}(\mathfrak{A}) = \{x \in G \mid x(Li) \in \mathcal{L}, i \in \mathbb{Z}\}$$

and

$$\mathfrak{K}(\mathfrak{B}_E) = \{x \in G_E \mid x(L_i) \in \mathcal{L}, i \in \mathbb{Z}\}.$$

Thus

$$\mathfrak{K}(\mathfrak{B}_E) \subset \mathfrak{K}(\mathfrak{A}). \tag{1.7}$$

The stratum $[\mathfrak{B}_E, r, r-1, b]$ is simple, thus the definition of a simple stratum shows that

$$E[b]^\times \subset \mathfrak{K}(\mathfrak{B}_E). \tag{1.8}$$

Put $E_1 = E[b] = F[\beta, b]$. Equations 1.7 and 1.8 imply that $E_1^\times \subset \mathfrak{K}(\mathfrak{A})$. This allows us to use the machinery of [13, 1.2] for $\mathfrak{A}$ and E_1.

Set $B_{E_1} = \mathrm{End}_{E_1}(V)$ and $\mathfrak{B}_{E_1} = \mathfrak{A} \cap \mathrm{End}_{E_1}(V)$. The proposition [13, 1.2.4] implies that $\mathfrak{B}_{E_1}$ is an $\mathfrak{o}_{E_1}$-hereditary order in B_{E_1}. Let $A(E_1)$ be the algebra $\mathrm{End}_F(E_1)$ and let $\mathfrak{A}(E_1)$ be the $\mathfrak{o}_F$-hereditary order in $A(E_1)$ defined by $\mathfrak{A}(E_1) = \{x \in \mathrm{End}_F(E_1) \mid x(\mathfrak{p}_{E_1}^i) \subset \mathfrak{p}_{E_1}^i, i \in \mathbb{Z}\}$. Let W be the F-span of an $\mathfrak{o}_{E_1}$-basis of the $\mathfrak{o}_{E_1}$-lattice chain $\mathcal{L}$. The proposition [13, 1.2.8] shows that the (W, E_1)-decomposition of A restricts to an isomorphism $\mathfrak{A} \simeq$

$\mathfrak{A}(E_1)\otimes_{\mathfrak{o}_{E_1}}\mathfrak{B}$ of $(\mathfrak{A}(E_1),\mathfrak{B}_{E_1})$-bimodules. Similarly we have a decomposition $\mathfrak{B}_E \simeq \mathfrak{B}_E(E_1)\otimes_{\mathfrak{o}_{E_1}}\mathfrak{B}_{E_1}$.

Set $B_E(E_1) = \mathrm{End}_E(E_1)$ and $\mathfrak{B}_E(E_1) = B_E(E_1)\cap\mathfrak{A}(E_1)$. Set also $n(E_1) = \dfrac{n}{e(\mathfrak{B}_{E_1}\mid\mathfrak{o}_{E_1})}$ and $r(E_1) = \dfrac{r}{e(\mathfrak{B}_{E_1}\mid\mathfrak{o}_{E_1})}$. Let us prove that the following two equalities hold.

$$\nu_{\mathfrak{A}(E_1)}(\beta) = -n(E_1) \tag{1.9}$$

$$\nu_{\mathfrak{B}_E(E_1)}(b) = -r(E_1) \tag{1.10}$$

Let us prove that the equation 1.9 holds. By definition of E_1, the element β is inside E_1 and thus $\nu_{\mathfrak{A}(E_1)}(\beta) = \nu_{E_1}(\beta)$. The lemma 1.2.1 thus shows that

$$\nu_{\mathfrak{A}}(\beta)e(E_1\mid F) = e(\mathfrak{A}\mid\mathfrak{o}_F)\nu_{\mathfrak{A}(E_1)}(\beta). \tag{1.11}$$

The proposition [13, 1.2.4] give us the equality

$$e(\mathfrak{B}_{E_1}\mid\mathfrak{o}_{E_1}) = \frac{e(\mathfrak{A}\mid\mathfrak{o}_F)}{e(E_1\mid F)}. \tag{1.12}$$

Since $[\mathfrak{A},n,r,\beta]$ is a simple stratum, n is equal to $-\nu_{\mathfrak{A}}(\beta)$, consequently using equations 1.11 and 1.12, the following sequence of equality holds.

$$\nu_{\mathfrak{A}(E_1)}(\beta) = \frac{\nu_{\mathfrak{A}}(\beta)e(E_1\mid F)}{e(\mathfrak{A}\mid\mathfrak{o}_F)} = \frac{\nu_{\mathfrak{A}}(\beta)}{e(\mathfrak{B}_{E_1}\mid\mathfrak{o}_{E_1})} = \frac{-n}{e(\mathfrak{B}_{E_1}\mid\mathfrak{o}_{E_1})} = -n(E_1)$$

This concludes the proof of the equality 1.9 and the equality 1.10 is easily proved in the same way.

The proposition [13, 1.4.13] gives

$$\begin{cases} k_0(\beta,\mathfrak{A}(E_1)) = \dfrac{k_0(\beta,\mathfrak{A})}{e(\mathfrak{B}_{E_1}\mid\mathfrak{o}_{E_1})} \\ k_0(b,\mathfrak{B}_E(E_1)) = \dfrac{k_0(b,\mathfrak{B}_E)}{e(\mathfrak{B}_{E_1}\mid\mathfrak{o}_{E_1})} \end{cases}.$$

Consequently $[\mathfrak{A}(E_1),n(E_1),r(E_1),\beta]$ and $[\mathfrak{B}_E(E_1),r(E_1),r(E_1)-1,b]$ are simple strata and satisfy the hypothesis of the proposition [13, 2.2.3]. Consequently $[\mathfrak{A}(E_1),n,r-1,\beta+b]$ is simple and the field $F[\beta+b]$ is equal to the field $F[\beta,b]$. Moreover [13, 2.2.3] implies that

$$k_0(\beta+b,\mathfrak{A}(E_1)) = \begin{cases} -r(E_1) = k_0(b,\mathfrak{B}_E(E_1)) \text{ if } b\notin E \\ k_0(\beta,\mathfrak{A}(E_1)) \text{ if } b\in E \end{cases}$$

The valuation $\nu_{\mathfrak{A}(E_1)}(\beta+b)$ is equal to $-n(E_1)$ and the same argument as before shows that $\nu_{\mathfrak{A}}(\beta+b) = -n$. The proposition [13, 1.4.13] shows that $k_0(\beta+b,\mathfrak{A}) = k_0(\beta+b,\mathfrak{A}(E_1))e(\mathfrak{B}_{E_1}\mid\mathfrak{o}_{E_1})$.

Thus

$$k_0(\beta + b, \mathfrak{A}) = \begin{cases} -r = k_0(b, \mathfrak{B}_E) \text{ if } b \notin E \\ k_0(\beta, \mathfrak{A}) \text{ if } b \in E \end{cases}$$

This completes the proof.

□

Given a, non necessary tame, pure stratum $[\mathfrak{A}, n, r, \beta]$, the existence of a simple stratum $[\mathfrak{A}, n, r, \gamma]$ equivalent to $[\mathfrak{A}, n, r, \beta]$ is a fundamental theorem in Bushnell-Kutzko's theory. Given such $[\mathfrak{A}, n, r, \beta]$ and $[\mathfrak{A}, n, r, \gamma]$, there is no, in general, inclusion between the field $F[\beta]$ and $F[\gamma]$, however the following arithmetical properties are always true.

$$e(F[\gamma] \mid F) \mid e(F[\beta] \mid F) \tag{1.13}$$

$$f(F[\gamma] \mid F) \mid f(F[\beta] \mid F) \tag{1.14}$$

Moreover if $[\mathfrak{A}, n, r, \beta]$ is not simple, then the degree $[F[\beta] : F]$ is strictly bigger than $[F[\gamma] : F]$ by 1.2.9.

In the tame situation, a new property is always true. Given a tame pure stratum $[\mathfrak{A}, n, r, \beta]$ such that $r = -k_0(\beta, \mathfrak{A})$, there is an equivalent tame simple stratum $[\mathfrak{A}, n, r, \gamma]$ such that the field $F[\gamma]$ is included in the field $F[\beta]$. We refer to Bushnell-Henniart for the proof of this fact. This property is the following proposition.

Proposition 1.4.3. *[12, 3.1 Corollary] Let $[\mathfrak{A}, n, r, \beta]$ be a tame pure stratum in the algebra $A = \mathrm{End}_F(V)$ such that $r = -k_0(\beta, \mathfrak{A})$. There is an element γ in the field $F[\beta]$ such that the stratum $[\mathfrak{A}, n, r, \gamma]$ is simple and equivalent to $[\mathfrak{A}, n, r, \beta]$*

In order to make an explicit link between Bushnell-Kutzko and Yu's formalisms, the following proposition is used crucialy in the section 1.8 of this paper.

Proposition 1.4.4. *Let $[\mathfrak{A}, n, r, \beta]$ be a tame pure stratum such that*

$$r = -k_0(\beta, \mathfrak{A}).$$

For all elements γ in the field $F[\beta]$ such that $[\mathfrak{A}, n, r, \gamma]$ is a simple stratum equivalent to $[\mathfrak{A}, n, r, \beta]$, the stratum $[\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$ is simple, here $\mathfrak{B}_\gamma = \mathrm{End}_{F[\gamma]}(V) \cap \mathfrak{A}$.

Proof. Using a similar argument than in the proposition 1.4.2, it is enough to prove the proposition in the case where $F[\beta]$ is a maximal subfield of the algebra $A = \mathrm{End}_F(V)$. So let $[\mathfrak{A}, n, r, \beta]$ be a tame pure stratum such that $F[\beta]$ is a maximal subfield of A and $k_0(\beta, \mathfrak{A}) = -r$. Let γ be in $F[\beta]$ such that $[\mathfrak{A}, n, r, \gamma]$ is simple. The stratum $[\mathfrak{B}_\gamma, r, r-1, \beta - \gamma]$ is pure in the algebra

$\mathrm{End}_{F[\gamma]}(V)$, because it is equivalent to a simple one by [13, 2.4.1]. Moreover $[\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$ is tame pure so the proposition 1.4.3 shows that there exists a simple stratum $[\mathfrak{B}_\gamma, r, r-1, \alpha]$ equivalent to $[\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$, such that $F[\gamma][\alpha] \subset F[\gamma][\beta-\gamma]$. By proposition 1.4.2, $[\mathfrak{A}, n, r-1, \gamma+\alpha]$ is simple and $F[\gamma+\alpha]$ is equal to the field $F[\gamma, \alpha]$. Set $\mathfrak{Q}_\gamma = \mathrm{rad}(\mathfrak{B}_\gamma) = \mathfrak{B}_\gamma \cap \mathfrak{P}$. The equivalence $[\mathfrak{B}_\gamma, r, r-1, \alpha] \sim [\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$ shows that $\alpha \equiv \beta - \gamma$ $\pmod{\mathfrak{Q}_\gamma^{-(r-1)}}$. This implies $\gamma + \alpha \equiv \beta \pmod{\mathfrak{P}^{-(r-1)}}$. We deduce that $[\mathfrak{A}, n, r-1, \gamma+\alpha]$ and $[\mathfrak{A}, n, r-1, \beta]$ are two simple strata equivalent. Indeed, the first is simple by construction, and the second by hypothesis, since $k_0(\beta, \mathfrak{A}) = -r$. The definitions shows that $F[\gamma+\alpha] \subset F[\beta]$, and 1.2.9 shows that $[F[\gamma+\alpha] : F] = [F[\beta] : F]$. Thus $F[\gamma+\alpha] = F[\beta]$. The trivial inclusions $F[\gamma+\alpha] \subset F[\gamma, \alpha] \subset F[\beta]$ then shows that $F[\gamma+\alpha] = F[\gamma, \alpha] = F[\beta]$.

We have thus obtained that the three assertions hold.

- The stratum $[\mathfrak{B}_\gamma, r, r-1, \alpha]$ is a simple stratum in $\mathrm{End}_{F[\gamma]}(V)$.
- The field $F[\gamma][\alpha]$ is a maximal subfield of the $F[\gamma]$-algebra $\mathrm{End}_{F[\gamma]}(V)$.
-$[\mathfrak{B}_\gamma, r, r-1, \alpha] \sim [\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$

Consequently, by [13, 2.2.2], $[\mathfrak{B}_\gamma, r, r-1, \beta-\gamma]$ is simple as required.

$\square$

1.5 Minimal elements and standard representatives

Recall that we have fixed a non-archimedean local field F and a uniformizer π_F of F. In this section we prove some properties relying minimal elements of Bushnell-Kutzko and standard representative elements introduced by Howe [25]. We recall that Howe's construction of supercuspidal representations should be considered as the common ancestor of [13] and [41] and Moy's presentation of Howe's construction has been an hint in our work. The main result of this section is the proposition 1.5.8.

The following describes the multiplicative group of a non archimedean local field.

Proposition 1.5.1. *[31, Chapter 2 Proposition 5.7]*

Let K be a non archimedean local field and $q = p^f$ the number of elements in the residue field of K. Let μ_{q-1} denote the group of $(q-1)$-th roots of unity in K. Let π_K be a uniformizer in K. Then the following hold.

(i) If K has characteristic 0, then one has the following isomorphisms of topological groups

$$K^\times \simeq \pi_K^{\mathbb{Z}} \times \mathfrak{o}_K^\times \simeq \pi_K^{\mathbb{Z}} \times \mu_{q-1} \times (1+\mathfrak{p}_K) \simeq \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}_p^d$$

where $a \geq 0$ and $d = [K : \mathbb{Q}_p]$.

The first three groups are denoted multipticatively and the last one additively.

(ii) If K has characteristic p, then one has the following isomorphisms of topological groups:

$$K^\times \simeq \pi_K^{\mathbb{Z}} \times \mathfrak{o}_K^\times \simeq \pi_K^{\mathbb{Z}} \times \mu_{q-1} \times 1 + \mathfrak{p}_K \simeq \mathbb{Z} \times \mathbb{Z}/(q-1)\mathbb{Z} \times \mathbb{Z}_p^{\mathbf{N}}.$$

The first three groups are denoted multipticatively and the last one additively.

The previous proposition allows us to deduce the following corollary which is a well-know result. Recall that we have a fixed uniformizer π_F.

Corollary 1.5.2. *Let E denote a tamely ramified extension of F. There exists a uniformizer π_E of E and a root of unity $z \in E$, of order prime to p, such that $\pi_E^e z = \pi_F$.*

Proof. Let π be a uniformizer of E. The proposition 1.5.1 shows that there exist an isomorphism $f : E^\times \simeq \pi^{\mathbb{Z}} \times \mu_{q-1} \times G'$ where $G' = 1 + \mathfrak{p}_E$ is a multiplicatively denoted group. Each element of G' have an e-th root. Indeed, the proposition 1.5.1 shows that $1 + \mathfrak{p}_E$ is isomorphic to the additive group $\mathbb{Z}/p^a\mathbb{Z} \times \mathbb{Z}_p^d$ or to the additive group $\mathbb{Z}_p^{\mathbf{N}}$. The image of π_F by f is (e, z, g) where $(e, z, g) \in \pi_E^{\mathbb{Z}} \times \mu_{q-1} \times G'$, i.e $\pi_F = \pi^e z g$. Let r be in G' such that $r^e = g$. Then $r\pi$ is a uniformizer of E and $\pi_F = (r\pi)^e z$. So $\pi_E = r\pi$ has the required property.

□

Definition 1.5.3. *Let E/F and π_E as in the previous corollary, i.e such that $\pi_F = \pi_E^e z$ with z a root of unity of order prime to p. Let C_E be the group generated by π_E and the roots of unity of order prime to p in $E^\times$.*

Proposition 1.5.4. *The group C_E is independent of the choice of π_E used in 1.5.3 to define it.*

Proof. Let π_1 and π_2 be two uniformizers of E and z_1 , z_2 be two roots of unity of order prime to p such that $\pi^e z_1 = \pi_F$ and $\pi_2^e z_2 = \pi_F$. Let C^1 be the group generated by π_1 and the root of unity of order prime to p. Let C^2 be the group generated by π_2 and the root of unity of order prime to p. By symmetry, it is enough show that $C^1 \subset C^2$. It is also enough to show that $\pi_1 \in C_2$. The equation $\pi_1^e z_1 = \pi_F$ implies that $\pi_1^e \in C_2$, thus there exists a root of unity z of order prime to p such that $\pi_1^e = \pi_2^e z$. We have $(\pi_1 \pi_2^{-1})^e = z$. Let o_z be the order of z, it is an integer prime to p. We have $(\pi_1 \pi_2^{-1})^{e o_z} = 1$. The integer $e o_z$ is prime to p, indeed $e = e(E \mid F)$ is prime to p since E/F is a tamely ramified extension and o_z is prime to p. Consequently $\pi_1 \pi_2^{-1}$ is a root of unity of order prime to p. This implies that $\pi_1 \in C_2$ as required.

□

We have fixed at the beginning of the text a uniformizer π_F. So to each tamely ramified extension E/F, the group C_E is well-defined and does not depend on any choice.

Proposition 1.5.5. *Let E/F be a tamely ramified extension. Let c be an element in $E^\times$. The following holds.*

(i) There exists a unique element $sr(c) \in C_E$, called the standard representative of c and a unique element $x \in 1+\mathfrak{p}_E$ such that $c = sr(c) \times x$.

(ii) The element $sr(c)$ is the unique element in C_E such that $\nu_E(sr(c)-c) > \nu_E(c)$

Proof. (i) The proposition 1.5.1 shows that $E^\times \simeq C_E \times (1+\mathfrak{p}_E)$ and (i) is a consequence.

(ii) The element $sr(c)$ is the unique element in C_E such that $c = sr(c) \times (1+y)$ with $y \in \mathfrak{p}_E$. Thus $sr(c)$ is the unique element in C_E such that $c - sr(c) \in sr(c)\mathfrak{p}_E$. Thus (ii) holds remarking that $sr(c)$ and c have the same valuation.

□

Proposition 1.5.6. *Let $E'/E/F$ be a tower of finite tamely ramified extensions. The following assertions hold.*

(i) The group C_E is included in the group $C_{E'}$.

(ii) If E/F is a Galois extension, then C_E is stable under the Galois action of $\mathrm{Gal}(E/F)$ on E. Moreover, if σ_1 and σ_2 are elements in $\mathrm{Gal}(E/F)$ and s is an element in C_E such that $\sigma_1(s) \neq \sigma_2(s)$, then

$$\nu_E(\sigma_1(s) - \sigma_2(s)) = \nu_E(s) \ .$$

Proof. (i) Recall that the group C_E and $C_{E'}$ are independent of the choices of uniformizers used to define them by 1.5.4. Let π_E be a uniformizer of E and z a root of unity of order prime to p in E such that $\pi_E^{e(E|F)} z = \pi_F$. Since E'/E is tamely ramified, there exists a uniformizer $\pi_{E'} \in E'$ and a root of unity w of order prime to p in E' such that $\pi_{E'}^{e(E'|E)} w = \pi_E$. Elevating to the power $e(E \mid F)$ we have $\pi_{E'}^{e(E'|E)e(E|F)} w^{e(E|F)} = \pi_E^{e(E|F)}$. We thus get $\pi_{E'}^{e(E|F)} w^{e(E|F)} z = \pi_F$. The element $w^{e(E|F)} z$ is a root of unity of order prime to p. Consequently $C_{E'}$ is the group generated by $\pi_{E'}$ and the roots of unity of order prime to p in E'. The equation $\pi_{E'}^{e(E'|E)} w = \pi_E$ shows that π_E is inside $C_{E'}$. Trivially, the roots of unity of order prime to p in E are inside the roots of unity of order prime to p in E'. Consequently C_E is inside $C_{E'}$ as required.

(ii) Let $\sigma \in Gal(E/F)$, and let π_E be an element such that $\pi_E^e z = \pi_F$ for z a root of unity in E of order prime to p. Let o_z the order of z. It is enough to show that z and π_E are mapped in C_E by σ. The equality $(\sigma(z))^{o_z} = 1$ shows that $\sigma(z)$ is a root of unity of order prime to p and thus inside C_E. The equality $\sigma(\pi_E)^e \sigma(z) = \pi_F$ together with 1.5.4 show that we can use $\sigma(\pi_E)$ to define C_E, and thus $\sigma(\pi_E)$ is inside C_E. This proves the first part of the assertion.

The element $\sigma_1(s)$ is in C_E so $sr(\sigma_1(s)) = \sigma_1(s)$. Consequently $\nu_E(\sigma_1(s) - \sigma_2(s)) = \nu_E(\sigma_1(s))$, indeed assume $\nu_E(\sigma_1(s) - \sigma_2(s)) \neq \nu_E(\sigma_1(s))$, then $\nu_E(\sigma_1(s) - \sigma_2(s)) > \sigma_1(s)$, and so $\sigma_2(s) = sr(\sigma_1(s)) = \sigma_1(s)$ by 1.5.5, this is a contradiction. This completes the second part of the assertion and the proof of the proposition.

□

We need to remark an elementary lemma in order to prove the proposition 1.5.8 which is the main result of this section.

Lemma 1.5.7. *Let E/F be a finite unramified extension. Let $z \in E$ be a root of unity of order prime to p. Then z generates E/F if and only if $z + \mathfrak{p}_E$ generates the residual field extension k_E/k_F.*

Proof. If z generates E over F, then z generates $\mathfrak{o}_E$ over $\mathfrak{o}_F$ by [31, 7.12]. Thus z generates the residual field extension k_E/k_F. Let us check the reverse implication. Assume $z + \mathfrak{p}_E$ generates k_E/k_F. The field extension E/F is unramified, so $[k_E : k_F] = [E : F]$. Let $P_z \in F[X]$ be the minimal polynomial of z and d its degree, clearly P_z is in $\mathfrak{o}_F[X]$. It is enough to show that $d = [E : F]$. We have $d \leq [E : F]$. The reduction $\mod \mathfrak{p}_E$ of P_z is of degree d and annihilates $z + \mathfrak{p}_E$, a generator of k_E/k_F, and thus $[k_E : k_F] \leq d$. So $[k_E : k_F] \leq d \leq [E : F]$. So $d = [E : F]$, and this concludes the proof.

□

Proposition 1.5.8. *Let E/F be a finite tamely ramified extension, let β be an element in E such that $E = F[\beta]$, the following assertions are equivalent.*

(i) The element β is minimal over F.

(ii) The standard representative element of β generates the field extension E/F, i.e. $F[sr(\beta)] = E$.

Proof. Let us prove that (i) implies (ii). Assume β is minimal over F. Let us remark that the definition of $sr(\beta)$ implies trivially that $F[sr(\beta)] \subset E$. Let E^{nr} denote the maximal unramified extension contained in E. In order to prove the opposite inclusion $E \subset F[sr(\beta)]$, it is enough to show that $E^{\mathrm{nr}} \subset F[sr(\beta)]$ and $E \subset E^{\mathrm{nr}}[sr(\beta)]$. Put $\nu = \nu_E(\beta)$, $e = e(E \mid F)$. The valuation of $\pi_F^{-\nu}\beta^e$ is equal to 0, consequently by 1.5.5 we have $\nu_E(sr(\pi_F^{-\nu}\beta^e) - \pi_F^{-\nu}\beta^e) > 0$, and so $sr(\pi_F^{-\nu}\beta^e) + \mathfrak{p}_E = \pi_F^{-\nu}\beta^e + \mathfrak{p}_E$. We have $sr(\pi_F^{-\nu}\beta^e) =$

$\pi_F^{-\nu} sr(\beta)^e$, and this is a root of unity of order prime to p. The definition of being minimal implies that $\pi_F^{-\nu} sr(\beta)^e + \mathfrak{p}_E$ generates k_E/k_F. So $\pi_F^{-\nu} sr(\beta)^e$ generates E^{nr} by 1.5.7. So $E^{\mathrm{nr}} \subset F[sr(\beta)]$. We have $\nu_E(\beta) = \nu_E(sr(\beta))$, so $\gcd(\nu_E(sr(\beta)), e) = 1$. Let a and b be integers such that $a\nu_E(sr(\beta)) + be = 1$. Thus $\nu_E(sr(\beta)^a \pi_F^b) = 1$ and so $E^{\mathrm{nr}}[sr(\beta)^a \pi_F^b] = E$ since a finite totaly ramified extension is generated by an arbitrary uniformizer. So $E^{\mathrm{nr}}[sr(\beta)] = E$ and (i) hold. We have thus show that $E^{\mathrm{nr}} \subset F[sr(\beta)]$ and $E \subset E^{\mathrm{nr}}[sr(\beta)]$ and so (i) implies (ii).

Let us prove that (ii) implies (i). Assume $F[sr(\beta)] = E$. We start by showing that e is prime to ν. The field E^{nr} is generated over F by the roots of unity of order prime to p contained in E. Let $d = \gcd(\nu, e)$ and $b = \frac{e}{d}$. Let π_E be a uniformizer in E such that $\pi_E^e z = \pi_F$ with z a root of unity of order prime to p. The element $sr(\beta)$ is in C_E and so $sr(\beta) = \pi_E^\nu w$ with w a root of unity of order prime to p in E. The equalities $sr(\beta)^b = (\pi_E^e)^{\frac{\nu}{pgcd(\nu,e)}} w^b = (\pi_F z^{-1})^{\frac{\nu}{pgcd(\nu,e)}} w^b$ shows that $sr(\beta)^b$ is contained in E^{nr}. By hypothesis, the element $sr(\beta)$ generates E over F and so generates E over E^{nr}. Consequently the field E is generated by an element whose b-th power is in E^{nr}. Consequently, the inequality $[E : E^{\mathrm{nr}}] \leq b$ holds. The extension E^{nr} is the maximal unramified extension contained in E, so $[E : E^{\mathrm{nr}}] = e$. Thus the inequality $e \leq b \leq \frac{e}{d}$ holds. This implies $d = 1$ and so ν is prime to e. Let us prove that $\pi_F^{-\nu}\beta^e + \mathfrak{p}_E$ generates the residue field extension k_E over k_F. Since $\pi_F^{-\nu}\beta^e + \mathfrak{p}_E = \pi_F^{-\nu} sr(\beta)^e + \mathfrak{p}_E$, it is equivalent to show that $x + \mathfrak{p}_E$ generates k_E over k_F , where $x = \pi_F^{-\nu} sr(\beta)^e$. The element $sr(\beta)$ generates E over F by hypothesis, i.e $E = F[sr(\beta)]$. So the inequality $[E : F[x]] \leq e$ holds, indeed E is generated over $F[x]$ by the element $sr(\beta)$ whose e-th power is in $F[x]$. Since x is a root of unity of order prime to p, the field $F[x]$ is include in E^{nr}, so $[E : E^{\mathrm{nr}}] \leq [E : F[x]]$. Consequently, the identity $e = [E : E^{\mathrm{nr}}] \leq [E : F[x]] \leq e$ holds. Since $F[x] \subset E^{\mathrm{nr}}$, the previous identity implies that $F[x] = E^{\mathrm{nr}}$. Thus by 1.5.7 the element $x + \mathfrak{p}_E$ generates k_E over k_F. So β is minimal over F.

This finish the proof of the proposition 1.5.8.

$\square$

Remark 1.5.9. *The implication (ii) implies (i) is analogous to [39, page 11].*

1.6 Twisted Levi sequences in GL_N and generic elements associated to minimal elements

In this section, we give an example of tamely ramified twisted Levi sequence and an example of generic element. This generic element comes from a minimal element relatively to a finite tamely ramified field extension. More precisely, let $E'/E/F$ be a tower of tamely ramified field extensions and let

V be an E'-vector space of dimension d. We are going to define and describe explicitly the groups scheme $H' = \mathrm{Res}_{E'/F}\underline{\mathrm{Aut}}_{E'}(V)$, $H = \mathrm{Res}_{E/F}\underline{\mathrm{Aut}}_{E}(V)$ and $G = \underline{\mathrm{Aut}}_F(V)$. We will show that the sequence (H', H, G) forms a tamely ramified twisted Levi sequence in G. The choice of an E'-maximal decomposition D, $V = (V_1 \oplus \ldots \oplus V_d)$, of V in 1-dimensional E'-vector spaces gives birth to a maximal torus T_D of $\underline{\mathrm{Aut}}_{E'}(V)$. By restriction of scalar, we get a maximal torus $T = \mathrm{Res}_{E'/E}(T_D)$ of H'. We are going to describe the set over $\overline{F}$ of roots of H' and H with respect to T. Moreover we will describe the condition **GE1** in this situation. Finally, given $c \in E'$ minimal over E, we will introduce an element $X^*_{sr(c)} \in \mathrm{Lie}^*(Z(H'))$ and prove that it satisfies **GE1** and is H-generic.

1.6.1 The group schemes of automorphisms of a free A-module of finite rank

Let A be a commutative ring and M be a free A-module of rank r. The functor

$$\begin{aligned} \{\mathrm{A-algebra}\} &\to \mathbf{Gp} \\ \mathrm{B} &\mapsto \mathrm{Aut}_{\mathrm{B}}(\mathrm{M} \otimes_A B) \end{aligned}$$

is representable by and affine A-scheme that we denote $\underline{\mathrm{Aut}}_{\mathrm{A}}(\mathrm{M})$. This scheme is isomorphic to the group scheme GL_N over A, with $N = r$. Let D be a decomposition $M = \mathrm{M}_1 \oplus \ldots \oplus \mathrm{M}_r$ of M in submodule of rank 1. Let us define a maximal split torus of $\underline{\mathrm{Aut}}_{\mathrm{A}}(M)$. The functor

$$\begin{aligned} \{\mathrm{A}-algebra\} \to &\quad \mathbf{Gp} \\ \mathrm{B} \ \mapsto &\left\{ x \in \mathrm{Aut}_{\mathrm{B}}(\mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}) \;\middle\|\; \begin{array}{l} \text{For all } i \in \{1, \ldots, r\}, \text{ there exists } \ \lambda^i(x) \in B^\times \\ \text{such that } x(v_i \otimes 1) = \lambda^i(x)(v_i \otimes 1) \text{ for all } v_i \in M_i \end{array} \right\} \end{aligned}$$

is representable by and affine A-scheme that we denote T_D, this is a closed affine subscheme of $\underline{\mathrm{Aut}}_{\mathrm{A}}(\mathrm{M})$. The A-scheme T_D is canonicaly isomorphic to $\prod_{i=1}^{r} \underline{\mathrm{Aut}}_{\mathrm{A}}(M_i)$. Let us give an explicit expression of the set of roots $\Phi(\underline{\mathrm{Aut}}_{\mathrm{A}}(\mathrm{M}), T_D)$ in this functorial point of view. The notation $0 \leq i \neq i' \leq r$ means that $1 \leq i \leq r$, $1 \leq i' \leq r$ and that $i \neq i'$. The set of root of $\underline{\mathrm{Aut}}_{\mathrm{A}}(\mathrm{M})$ relatively to T_D is the set

$$\Phi(\underline{\mathrm{Aut}}_{\mathrm{A}}(\mathrm{M}), T_D) = \{\alpha_{ii'} \mid 1 \leq i \neq i' \leq r\}$$

where $\alpha_{ii'}$ is the morphism of algebraic group $T_D \to \mathbb{G}_m$ characterized by the formula, for all A-algebras B, for all $x \in T_D(\mathrm{B})$, $\alpha_{ii'}(x) = \lambda^i(x)(\lambda^{i'}(x))^{-1}$.

For each root α, let $\alpha^\vee : \mathbb{G}_m \to T_D$ be the coroot of α and let $\mathrm{d}\alpha^\vee$ be the derivative of α. Finally let H_α be the element $\mathrm{d}\alpha^\vee(1) \in \mathrm{Lie}(T_D)(A)$.

Let us make these objects explicit in our functorial point of view.

Let $1 \leq i \neq i' \leq r$, the coroot $\alpha^\vee_{ii'}$ is the morphism of algebraic group $\mathbb{G}_m \to T_D$ characterized by the formula,

$$\text{for all A-algebras B, for all } \lambda \in B^\times , \begin{cases} \alpha^\vee_{ii'}(\lambda)(v_i \otimes 1) = \lambda(v_i \otimes 1) \quad \forall v_i \in M_i \\ \alpha^\vee_{ii'}(\lambda)(v_{i'} \otimes 1) = \lambda^{-1}(v_{i'} \otimes 1) \quad \forall v_{i'} \in M_{i'} \\ \alpha^\vee_{ii'}(\lambda)(v_k \otimes 1) = (v_k \otimes 1) \; \forall v_k \in M_k, k \neq i, i' \end{cases}$$

The derivative of $\alpha^\vee_{ii'}$ is the differential morphism $\mathrm{d}\alpha^\vee_{ii'} : \mathbb{G}_a \to \underline{\mathrm{Lie}}(T_D)$, it is characterized by the formula (see [2, 3.9.4]),

$$\text{for all A-algebra B, for all } h \in B , \begin{cases} \mathrm{d}\alpha^\vee_{ii'}(h)(v_i \otimes 1) = h(v_i \otimes 1) \quad \forall v_i \in M_i \\ \mathrm{d}\alpha^\vee_{ii'}(h)(v_{i'} \otimes 1) = -h(v_{i'} \otimes 1) \quad \forall v_{i'} \in M_{i'} \\ \mathrm{d}\alpha^\vee_{ii'}(h)(v_k \otimes 1) = 0 \quad \forall v_k \in M_k \; \forall k \neq i, i'. \end{cases}$$

Consequently the element $H_{\alpha_{ii'}}$ which is by definition $\mathrm{d}\alpha^\vee_{ii'}(1) \in \underline{\mathrm{Lie}}(T_D)(A) = \mathrm{End}_A(M)$ is the element sending each element $v_i \in M_i$ to v_i, each element $v_{i'} \in M_{i'}$ to $-v_{i'}$ and, for all k different of i, i', each element $v_k \in M_k$ to 0.

1.6.2 Trace of endomorphisms and base change

In this paragraph we give the intrinsic definition of the trace and give a formula.

Let A be a commutative ring and let M be a free A-module of rank N.

As usual let $\mathrm{End_A(M)}$ be the A-algebra of A-linear maps $\mathrm{Hom_A(M, M)}$.

The A-linear map

$$\begin{aligned} \mathrm{M} \otimes_{\mathrm{A}} \mathrm{Hom_A(M, A)} &\xrightarrow{\sim} \mathrm{End_A(M)} \\ m \otimes f &\longmapsto (m' \mapsto f(m').m) \end{aligned}$$

is a canonical isomorphism.

The A-linear map

$$\begin{aligned} \mathrm{M} \otimes_{\mathrm{A}} \mathrm{Hom_A(M, A)} &\longrightarrow \mathrm{A} \\ m \otimes f &\longmapsto f(m) \end{aligned}$$

induces a A-linear map $\mathrm{End_A(M)} \to A$, this map is called the trace map and is usually denoted Tr or Tr_A or $\mathrm{Tr}_{\mathrm{End_A(M)}}$ or $\mathrm{Tr}_{\mathrm{End_A(M)/A}}$.

Let B be a commutative A-algebra. The B-linear map

$$\begin{aligned} G : \qquad \mathrm{End_A(M)} \otimes_{\mathrm{A}} \mathrm{B} &\xrightarrow{\sim} \mathrm{End_B(M \otimes_A B)} \\ F \otimes b &\longmapsto ((m \otimes c) \mapsto F(m) \otimes bc) \end{aligned}$$

is a canonical isomorphism.

The following is a lemma which give a compatibility of Tr under base change.

Lemma 1.6.1. *The following triangle of* B*-linear maps is commutative.*

$$\begin{tikzcd}
\mathrm{End_A(M)} \otimes_{\mathrm{A}} \mathrm{B} \arrow[d, "G"'] \arrow[dr, "\mathrm{Tr_A} \otimes Id_{\mathrm{B}}"] & \\
& B \\
\mathrm{End_B}(M \otimes_{\mathrm{A}} \mathrm{B}) \arrow[ur, "\mathrm{Tr_B}"'] &
\end{tikzcd}$$

Proof. It is equivalent to prove that the following triangle of B-linear map is commutative.

$$\begin{tikzcd}
(\mathrm{M} \otimes_{\mathrm{A}} \mathrm{Hom_A(M, A)}) \otimes_{\mathrm{A}} \mathrm{B} \arrow[d, "G"'] \arrow[dr, "\mathrm{Tr_A} \otimes Id_{\mathrm{B}}"] & \\
& B \\
(M \otimes_{\mathrm{A}} \mathrm{B}) \otimes_{\mathrm{B}} \mathrm{Hom_B}(M \otimes_{\mathrm{A}} \mathrm{B}, \mathrm{B}) \arrow[ur, "\mathrm{Tr_B}"'] &
\end{tikzcd}$$

It is enough to compute the image of $(m \otimes f) \otimes b \in (\mathrm{M} \otimes_{\mathrm{A}} \mathrm{Hom_A(M, A)}) \otimes_{\mathrm{A}} \mathrm{B}$ by the map $\mathrm{Tr_A} \otimes Id_{\mathrm{B}}$ and by $\mathrm{Tr_B} \circ G$ and to show that they are equal. By definition of $\mathrm{Tr_A}$, $\mathrm{Tr_A} \otimes Id_{\mathrm{B}}((m \otimes f) \otimes b) = f(m)b$. The map G is explicitely given by $((m \otimes f) \otimes b) \mapsto (m \otimes b) \otimes (m' \otimes c \mapsto f(m')c)$. Consequently $\mathrm{Tr_B} \circ G((m \otimes f) \otimes b) = \mathrm{Tr_B}((m \otimes b) \otimes (m' \otimes c \mapsto f(m')c) = f(m)b$. This concludes the proof of lemma 1.6.1. □

1.6.3 Abstract twisted Levi sequences

In this subsection, we prove algebraic facts that will be applied to the following subsections. We start with a very easy and well-known lemma. Let f be commutative ring and B be a commutative f-algebra, C be an B-algebra. Let A be an f-algebra. In this situation $A \otimes_f B$ is an B-algebra and C is naturally an f-algebra.

Lemma 1.6.2. *With the previous notations, the C-algebra $(A \otimes_f B) \otimes_B C$ is canonically isomorphic to $A \otimes_f C$. Explicitly, the isomorphism is given by*

$$\begin{aligned}(A \otimes_f B) \otimes_B C &\to A \otimes_f C \\ (a \otimes b) \otimes c &\mapsto a \otimes bc.\end{aligned}$$

The inverse is explicitly given by

$$\begin{aligned}A \otimes_f C &\to (A \otimes_f B) \otimes_B C \\ a \otimes c &\mapsto (a \otimes 1) \otimes c.\end{aligned}$$

Proof. The two maps are morphisms of C-algebras and one composed with the other is equal to the identity map. $\square$

We now fix in the rest of this subsection a tower of finite separable extensions of fields $l'/l/f$. In the next subsection, we will apply this to $l' = E', l = E$ and $f = F$, where $E'/E/F$ is a tower of finite tamely ramified extensions. Let V be an l'-vector space of dimension d. Let $D{:}V = (D_1 \oplus \ldots \oplus D_{d)})$, be an l'-decomposition of V in subspaces of dimension 1.

In a previous subsection we have introduced an l'-group scheme $\underline{\mathrm{Aut}}_{l'}(V)$, and a maximal split torus T_D of $\underline{\mathrm{Aut}}_{l'}(V)$. Let H' be the restriction of scalar from l' to f of $\underline{\mathrm{Aut}}_{l'}(V)$. Also, let T be $\mathrm{Res}_{l'/f}(T_D)$.

Thus H' represents the functor

$$\begin{array}{ccc} \{f - algebra\} & \to & \mathbf{Gp} \\ A & \mapsto & \underline{\mathrm{Aut}}_{l'}(V)(A \otimes_f l'). \end{array}$$

For each f-algebra A the group $H'(A)$ is thus equal to the group

$$\mathrm{Aut}_{A\otimes_f l'}\left(V \otimes_{l'} (A \otimes_f l')\right).$$

Since $l \subset l'$, V is an $l-$space and, we have a group $\underline{\mathrm{Aut}}_l(V)$ and its restriction of scalar H. So that for each f-algebra A the group $H(A)$ is equal to the group $\mathrm{Aut}_{A\otimes_f l}\left(V \otimes_l (A \otimes_f l)\right)$. Let also G be $\underline{\mathrm{Aut}}_f(V)$.

For each f-algebra A, the canonical morphism $A \otimes_f l \to A \otimes_f l'$ induces a canonical morphism of groups

$$\mathrm{Aut}_{A\otimes_f l'}\left(V \otimes_{l'} (A \otimes_f l')\right) \to \mathrm{Aut}_{A\otimes_f l}\left(V \otimes_l (A \otimes_f l)\right)$$

, which is functorial in A. We thus get a canonical morphism of f-group scheme $H' \to H$. This morphism is a closed immersion. We also have a canonical morphism of F-group schemes $H \to G$.

We are interested in Condition **GE1**, it is related to the extension of scalar from f to $\overline{f}$, the algebraic closure of f. So let us compute

$$T \times_{\mathrm{spec}(f)} \mathrm{spec}(\overline{f}),\ H' \times_{\mathrm{spec}(f)} \mathrm{spec}(\overline{f}) \text{ and } H \times_{\mathrm{spec}(f)} \mathrm{spec}(\overline{f}).$$

Let A be an $\overline{f}$-algebra, by definition $H \times_{\mathrm{spec}(f)} \mathrm{spec}(\overline{f})(A) = H(A)$. We have seen that it is equal to $\mathrm{Aut}_{A\otimes_f l}\left(V \otimes_l (A \otimes_f l)\right)$. We need to study the algebra $A \otimes_f l$.

We know that there exists $\sigma_1, \ldots, \sigma_i, \ldots, \sigma_{[l:f]}$, distincts morphisms of f-algebra from l to the Galois closure of l. We also know that for $0 \leq i \leq [l : f]$, there exists $[l' : l]$ morphisms of f-algebra from l' to the Galois closure of l' extending σ_i, we denote them $\sigma_{i1}, \ldots, \sigma_{ij}, \ldots, \sigma_{i[l':l]}$. We write $\prod_i$ instead of $\prod_{i=1}^{[l:f]}$ and $\bigoplus_i \bigoplus_j$ instead of $\bigoplus_{i=1}^{[l:f]} \bigoplus_{j=1}^{[l':l]}$, we use others "abuses of notation" of this nature.

Proposition 1.6.3. *Let f be a field, let $l'/l/f$ be a tower of finite separable extensions. Let K' be the Galois closure of l' and let K be the Galois closure of l. Let $\sigma_1, \ldots, \sigma_i, \ldots, \sigma_{[l:f]}$ be the distinct morphisms of f-algebra from l to K. For $1 \leq i \leq [l:f]$, let $\sigma_{i1}, \ldots, \sigma_{ij}, \ldots, \sigma_{i[l':l]}$ be the distinct morphisms of f-algebra from l' to K' which extend σ_i. Let A be a K'-algebra. Let $A \to B$ be a morphism of K'-algebra. The following assertions holds.*

(i) The A-algebra $A \otimes_f l$ is canonicaly isomorphic to $\prod_i A_i$, where $A_i = A$ for each i. Moreover this isomorphism is explicitely given as follow.

$$\begin{aligned} A \otimes_f l &\xrightarrow{\sim} \prod_i A_i \\ a \otimes e &\longmapsto \prod_i a\sigma_i(e) \end{aligned}$$

(ii) The A-algebra $A \otimes_f l'$ is canonicaly isomorphic to $\prod_i \prod_j A_{ij}$, where $A_{ij} = A$ for each i, j. Moreover this isomorphism is explicitely given as follow.

$$\begin{aligned} A \otimes_f l' &\xrightarrow{\sim} \prod_i \prod_j A_{ij} \\ a \otimes e &\longmapsto \prod_i \prod_j a\sigma_{ij}(e) \end{aligned}$$

(iii) The A-algebra $A \otimes_f l'$ is canonicaly an $A \otimes_f l$-algebra. The ring $\prod_i \prod_j A_{ij}$ is canonicaly an $\prod_i A_i$-algebra and the structure is given by

$$(\prod_i \lambda_i).(\prod_i \prod_j a_{ij}) = \prod_i \prod_j \lambda_i a_{ij}.$$

(iv) There is a canonical commutative diagram of A-algebras

$$\begin{tikzcd}
 & A \otimes_f l' \arrow[rr] \arrow[dl] & & B \otimes_f l' \arrow[dl] \\
\prod_i \prod_j A_{ij} \arrow[rr] & & \prod_i \prod_j B_{ij} & \\
 & A \otimes_f l \arrow[rr] \arrow[dl] \arrow[uu] & & B \otimes_f l \arrow[dl] \arrow[uu] \\
\prod_i A_i \arrow[rr] \arrow[uu] & & \prod_i B_i . \arrow[uu] &
\end{tikzcd}$$

Proof. (i) The field l is a finite separable extension of f and thus there exists an element $\alpha \in l$ such that $l = f[\alpha]$. Thus l is isomorphic to the quotient ring $f[X]/(P)$ where P is the minimal polynomial of α. Since K is the Galois closure of l, the polynomial $P(X)$ split over l and the formula $P = \prod_i (X - \sigma_i(\alpha))$ holds. We have some elementary isomorphisms f_1, f_2, f_3, f_4, f_5 of A-algebras.

$$\begin{tikzcd}
A \otimes_f l \arrow[r, "\sim", "f_1"'] & A \otimes_f f[X]/(P) \arrow[rr, "\sim", "f_2"'] & & A[X]/(P) \arrow[d, "f_3"'] \\
\prod_i A & \prod_i A[X]/(X - \sigma_i(\alpha)) \arrow[l, "\sim", "f_5"'] & & A[X]/\prod_i (X - \sigma_i(\alpha)). \arrow[ll, "\sim", "f_4"']
\end{tikzcd}$$

The map f_1 is the isomorphism associating $a \otimes e$ to $a \otimes e(X)$ where $e(X)$ is a polynomial such that $e(\alpha) = e$. The map f_2 is the one which associate to $a \otimes Q$ the polynomial aQ. The map f_3 is obvious. The map f_4 is the product of projection maps and is an isomorphism by the chinese remainder theorem. The map f_5 is the product of the map sending X to $\sigma_i(\alpha)$. The required map is the map $f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$. This map does not depend on the choice of α, and so it is canonical.

(ii) This is a direct consequence of (i).

(iii) Since l'/l is an extension of field, l' is canonically an l-algebra and thus there is a canonical morphism of rings g from $A \otimes_f l$ to $A \otimes_f l'$. So $A \otimes_f l'$ is canonically an $A \otimes_f l$-algebra.

It is enough to show that the square
$$\begin{tikzcd} A \otimes_f l' \arrow[r, "S'"] & \prod_i \prod_j A_{ij} \\ A \otimes_f l \arrow[u, "g"] \arrow[r, "S"] & \prod_i A_i \arrow[u, "h"] \end{tikzcd}$$
is commutative, where g is the canonical map introduced above, S and S' are the maps introduced in (i) and (ii), and h is the map sending $(\prod_i \lambda_i)$ to $(\prod_i \prod_j \lambda_{ij})$, where $\lambda_{ij} = \lambda_i$ for all i, j. Let $a \otimes e \in A \otimes_f l$. We have

$$S' \circ g(a \otimes e) = S'(a \otimes e) = \prod_i \prod_j \sigma_{ij}(e)a.$$

We have

$$h \circ S(a \otimes e) = h(\prod_i \sigma_i(e)a) = \prod_i \prod_j \sigma_i(e)a.$$

We have $\sigma_{ij}(e) = \sigma_i(e)$, since by definition the restriction to l of σ_{ij} is equal to σ_i. This concludes the proof of (iii).

(iv) The square relative to A on the left is introduced in the proof of (iii), the square relative to B on the right is the analogue for B, the horizontal arrow are canonicaly induced by the morphism $A \to B$. It is easy to prove that this is commutative.

□

Let A be a commutative ring, let A_1 and A_2 be two commutative A-algebras. Let B_1 be an A_1-algebra and let B_2 be an A_2-algebra. Let M be a free A-module of rank r.

The canonical projections and injections

$$B_1 \times B_2 \to B_1$$
$$B_1 \times B_2 \to B_2$$
$$B_1 \to B_1 \times B_2$$
$$B_2 \to B_1 \times B_2$$

induce canonical maps

$$\begin{aligned}
p_1 &: M \otimes_A (B_1 \times B_2) \to M \otimes_A B_1 \\
p_2 &: M \otimes_A (B_1 \times B_2) \to M \otimes_A B_2 \\
i_1 &: M \otimes_A B_1 \to M \otimes_A (B_1 \times B_2) \\
i_2 &: M \otimes_A B_2 \to M \otimes_A (B_1 \times B_2).
\end{aligned}$$

Theses maps satisfy various relations, for example, we have

$$\begin{aligned}
p_1 \circ i_1 &= \mathrm{Id} \\
p_2 \circ i_2 &= \mathrm{Id} \\
p_2 \circ i_1 &= 0 \\
p_1 \circ i_2 &= 0
\end{aligned}$$

We have canonical and well-defined maps

$$\begin{aligned}
F : \mathrm{End}_{A_1 \times A_2}(M \otimes_A (B_1 \times B_2)) \to \mathrm{End}_{A_1}(M \otimes_A B_1) \times \mathrm{End}_{A_2}(M \otimes_A B_2) \\
L \mapsto (p_1 \circ L \circ i_1), (p_2 \circ L \circ i_2)
\end{aligned}$$

and

$$\begin{aligned}
G : \mathrm{End}_{A_1}(M \otimes_A B_1) \times \mathrm{End}_{A_2}(M \otimes_A B_2) \to \mathrm{End}_{A_1 \times A_2}(M \otimes_A (B_1 \times B_2)) \\
L_1, L_2 \mapsto (i_1 \circ L_1 \circ p_1 + i_2 \circ L_2 \circ p_2),
\end{aligned}$$

the previously mentioned relations shows that F and G are groups homomorphisms. It is easy to show that $F \circ G = \mathrm{Id}$ and $G \circ F = \mathrm{Id}$ by direct computations. Moreover F and G induce by restriction a canonical isomorphism between $\mathrm{Aut}_{\mathrm{A}_1 \times \mathrm{A}_2}(\mathrm{M} \otimes_A (\mathrm{B}_1 \times \mathrm{B}_2))$ and $\mathrm{Aut}_{\mathrm{A}_1}(\mathrm{M} \otimes_A \mathrm{B}_1) \times \mathrm{Aut}_{\mathrm{A}_2}(\mathrm{M} \otimes_A \mathrm{B}_2)$.

We thus get an explicit and canonical isomorphism of groups

$$\mathrm{Aut}_{\mathrm{A}_1 \times \mathrm{A}_2}(\mathrm{M} \otimes_A (\mathrm{B}_1 \times \mathrm{B}_2)) \simeq \mathrm{Aut}_{\mathrm{A}_1}(\mathrm{M} \otimes_A \mathrm{B}_1) \times \mathrm{Aut}_{\mathrm{A}_2}(\mathrm{M} \otimes_A \mathrm{B}_2). \quad (1.15)$$

The isomorphism (1.15) induces the following lemma.

Lemma 1.6.4. *Let* A *be a commutative ring, let* A_i , $0 \leq i \leq d$, *be some commutative* A*-algebras. For* $0 \leq i \leq d$, *let* B_i *be an* A_i*-algebra. Let* M *be a free* A*-module of finite rank. Then we have a canonical and explicit isomorphism of groups*

$$\mathrm{Aut}_{\prod_{i=1}^d \mathrm{A}_i}(\mathrm{M} \otimes_{\mathrm{A}} \textstyle\prod_{i=1}^d \mathrm{B}_i) \simeq \displaystyle\prod_{i=1}^d \mathrm{Aut}_{\mathrm{A}_i}(\mathrm{M} \otimes_{\mathrm{A}} \mathrm{B}_i).$$

Let us use the notation of proposition 1.6.3. Let i, j, k be integers as above and let C/K' be a field extension (K' is the Galois closure of l'). We put $V_{ij} = V \otimes_{l'} C_{ij}$ ($C_{ij} = C$ is introduced in proposition 1.6.3). We put also $D_{ijk} = D_k \otimes_{l'} C_{ij}$.

Proposition 1.6.5. *With the previously introduced notations, the following assertions hold.*

(i) There is a canonical commutative diagram of f-schemes

$$\begin{tikzcd}
T \times_f C \arrow[r, "h_1"] \arrow[d, "v_1"] & H' \times_f C \arrow[r, "h_2"] \arrow[d, "v_2"] & H \times_f C \arrow[r, "h_3"] \arrow[d, "v_3"] & G \times_f C \arrow[d, "v_4"] \\
\prod_i \prod_j \prod_k \underline{\mathrm{Aut}}_C(D_{ijk}) \arrow[r, "f_1"] & \prod_i \prod_j \underline{\mathrm{Aut}}_C(V_{ij}) \arrow[r, "f_2"] & \prod_i \underline{\mathrm{Aut}}_C(\bigoplus_j V_{ij}) \arrow[r, "f_3"] & \underline{\mathrm{Aut}}_C(\bigoplus_i \bigoplus_j V_{ij}).
\end{tikzcd}$$

(ii) There is a canonical commutative diagram of k-spaces

$$\begin{tikzcd}
\mathrm{Lie}(T) \arrow[r] \arrow[d] & \mathrm{Lie}(H') \arrow[r] \arrow[d] & \mathrm{Lie}(H) \arrow[r] \arrow[d] & \mathrm{Lie}(G) \arrow[d] \\
\mathrm{Lie}(T \times_f C) \arrow[r] \arrow[d] & \mathrm{Lie}(H' \times_f C) \arrow[r] \arrow[d] & \mathrm{Lie}(H \times_f C) \arrow[r] \arrow[d] & \mathrm{Lie}(G \times_f C) \arrow[d] \\
\prod_i \prod_j \prod_k \mathrm{End}_C(D_{ijk}) \arrow[r] & \prod_i \prod_j \mathrm{End}_C(V_{ij}) \arrow[r] & \prod_i \mathrm{End}_C(\bigoplus_j V_{ij}) \arrow[r] & \mathrm{End}_C(\bigoplus_i \bigoplus_j V_{ij}).
\end{tikzcd}$$

(iii) Let s be an element in l'. Let m_s be the element of $\mathrm{Lie}(T)$ *which send an element h to sh. Let $m_{s,C}$ be the element in* $\mathrm{End}_C(\bigoplus_i \bigoplus_j V_{ij})$ *characterized by the formula,*

$$\textit{for all } i, j, \textit{ for all } v_{ij} \in V_{ij},\ m_{s,C}(v_{ij}) = \sigma_{ij}(s) v_{ij}.$$

Then the image of m_s in $\mathrm{End}_C(\bigoplus_i \bigoplus_j V_{ij})$ *through the diagram introduced in (ii) is $m_{s,C}$.*

Remark 1.6.6. *In the next sections we will apply this proposition with $C = \overline{F}$ or C a finite extension of K'.*

Proof. (i) The upper horizontal line is induced by the previously introduced morphisms $T \to H' \to H \to G$. We thus get some maps h_1, h_2

and h_3. Let A be a C-algebra. In the rest of this proof, we still denote $h_1(A)$ by h_1, we do the same for h_2 and h_3. We have

$$\begin{aligned}
\left(T \times_{\mathrm{spec}(f)} \mathrm{spec}(C)\right)(A) &\simeq \left(\mathrm{Res}_{l'/f} \prod_k \underline{\mathrm{Aut}}_{l'}(D_k)\right)(A) \\
\text{By properties of Res} \quad &\simeq \left(\prod_k \mathrm{Res}_{l'/f} \underline{\mathrm{Aut}}_{l'}(D_k)\right)(A) \\
&\simeq \prod_k \left(\mathrm{Res}_{l'/f} \underline{\mathrm{Aut}}_{l'}(D_k)(A)\right) \\
\text{By definition of Res} \quad &\simeq \prod_k \underline{\mathrm{Aut}}_{l'}(D_k)(A \otimes_f l') \\
\text{By definition of } \underline{\mathrm{Aut}} \quad &\simeq \prod_k \mathrm{Aut}_{A \otimes_f l'}(D_k \otimes_{l'} (A \otimes_f l')) \\
\text{By proposition 1.6.3} \quad &\simeq \prod_k \mathrm{Aut}_{\prod_i \prod_j A_{ij}}(D_k \otimes_{l'} (\prod_i \prod_j A_{ij})) \\
\text{By proposition 1.6.4} \quad &\simeq \prod_k \prod_i \prod_j \mathrm{Aut}_{A_{ij}}(D_k \otimes_{l'} A_{ij}) \\
&\simeq \prod_i \prod_j \prod_k \mathrm{Aut}_{A_{ij}}(D_k \otimes_{l'} A_{ij}) \\
&\simeq \prod_i \prod_j \prod_k \mathrm{Aut}_{A_{ij}}(D_{ijk})
\end{aligned}$$

We thus get an isomorphism

$$\left(T \times_{\mathrm{spec}(f)} \mathrm{spec}(C)\right)(A) \to \prod_i \prod_j \prod_k \mathrm{Aut}_{A_{ij}}(D_{ijk}),$$

let us denote it v_1. We have

$$\begin{aligned}
\left(H' \times_{\mathrm{spec}(f)} \mathrm{spec}(C)\right)(A) \simeq& \left(\mathrm{Res}_{l'/f} \underline{\mathrm{Aut}}_{l'}(V)\right)(A) \\
\simeq& \underline{\mathrm{Aut}}_{l'}(V)(A \otimes_f l') \\
\simeq& \mathrm{Aut}_{A \otimes_f l'}(V \otimes_{l'} (A \otimes_f l')) \\
\simeq& \mathrm{Aut}_{\prod_i \prod_j A_{ij}}(V \otimes_f (\prod_i \prod_j A_{ij})) \\
\simeq& \prod_i \prod_j \mathrm{Aut}_{A_{ij}}(V \otimes A_{ij}) \\
\simeq& \prod_i \prod_j \mathrm{Aut}_{A_{ij}}(V_{ij}).
\end{aligned}$$

We thus get an isomorphism

$$\left(H' \times_{\mathrm{spec}(F)} \mathrm{spec}(C)\right)(A) \to \prod_i \prod_j \mathrm{Aut}_{A_{ij}}(V_{ij}),$$

let us denote it v_2. We have

$$\begin{aligned}\left(H \times_{\mathrm{spec}(f)} \mathrm{spec}(C)\right)(A) \simeq & \left(\mathrm{Res}_{l/f} \underline{\mathrm{Aut}}_l(V)\right)(A)\\ \simeq & \underline{\mathrm{Aut}}_l(V)(A \otimes_f l)\\ \simeq & \mathrm{Aut}_{A \otimes_f l}(V \otimes_l (A \otimes_f l)).\end{aligned}$$

As an $A \otimes_f l$-module, $V \otimes_l (A \otimes_f l)$ is isomorphic to $V \otimes_{l'} (A \otimes_f l')$. So

$$\begin{aligned}\left(H \times_{\mathrm{spec}(f)} \mathrm{spec}(C)\right)(A) \simeq & \mathrm{Aut}_{A \otimes_f l}(V \otimes_{l'} (A \otimes_f l'))\\ \simeq & \mathrm{Aut}_{\prod_i A_i}(V \otimes_{l'} (\prod_i \prod_j A_{ij}))\\ \text{By proposition 1.6.4} \qquad \simeq & \prod_i \mathrm{Aut}_{A_i}(V \otimes_{l'} (\prod_j A_{ij})\\ \simeq & \prod_i \mathrm{Aut}_{A_i}(\bigoplus_j V \otimes_{l'} A_{ij})\\ \simeq & \prod_i \mathrm{Aut}_{A_i}(\bigoplus_j V_{ij}).\end{aligned}$$

We thus get an isomorphism

$$\left(H \times_{\mathrm{spec}(f)} \mathrm{spec}(C)\right)(A) \to \textstyle\prod_i \mathrm{Aut}_{A_i}(\bigoplus_j V_{ij}),$$

let us denote it v_3. We have

$$\begin{aligned}\left(G \times_{\mathrm{spec}(f)} \mathrm{spec}(C)\right)(A) \simeq & (\underline{\mathrm{Aut}}_f(V))(A)\\ \simeq & \mathrm{Aut}_A(V \otimes_f A)\\ \simeq & \mathrm{Aut}_A(V \otimes_{l'} (l' \otimes_f A))\\ \simeq & \mathrm{Aut}_A(V \otimes_{l'} (\prod_i \prod_j A_{ij}))\\ \simeq & \mathrm{Aut}_A(\bigoplus_i \bigoplus_j V \otimes_{l'} A_{ij})\\ \simeq & \mathrm{Aut}_A(\bigoplus_i \bigoplus_j V_{ij})\end{aligned}$$

We thus get an isomorphism

$$\left(G \times_{\mathrm{spec}(f)} \mathrm{spec}(C)\right)(A) \simeq (\underline{\mathrm{Aut}}_f(V))(A) \to \mathrm{Aut}_A(\bigoplus_i V_{ij}),$$

let us denote it v_4.

Let us recall that for all i, j, $V_{ij} = \bigoplus_k D_{ijk}$. In the following v_{ijk} denotes an arbitrary vector in D_{ijk}, and v_{ij} denote an arbitrary vector in V_{ij}.

Let f_1 be the canonical morphism

$$\prod_i \prod_j \prod_k \mathrm{Aut}_{A_{ij}}(D_{ijk}) \to \prod_i \prod_j \mathrm{Aut}_{A_{ij}}(\bigoplus_k D_{ijk})$$

sending $\prod_i \prod_j \prod_k (L_{ijk})$ to $\prod_i \prod_j (\sum_k v_{ijk} \mapsto \sum_k L_{ijk}(v_{ijk}))$. It is a formal computation to verify that the morphism $v_2 \circ h_1$ is equal to $f_1 \circ v_1$. Let f_2 be the canonical morphism

$$\prod_i \prod_j \mathrm{Aut}_{A_{ij}}(V_{ij}) \to \prod_i \mathrm{Aut}_{A_i}(\bigoplus_j V_{ij})$$

sending $\prod_i \prod_j L_{ij}$ to $\prod_i \left(\sum_j v_{ij} \mapsto \prod_i \sum_j L_{ij}(v_{ij})\right)$. It is a formal computation to verify that the morphism $v_3 \circ h_2$ is equal to $f_2 \circ v_2$.

Let f_3 be the canonical morphism

$$\prod_i \mathrm{Aut}_{A_i}(\bigoplus_j V_{ij}) \to \mathrm{Aut}_A(\bigoplus_i \bigoplus_j v_{ij})$$

sending $\prod_i L_i$ to $\left(\sum_i \sum_j v_{ij} \mapsto \sum_i L_i(\sum_j v_{ij})\right)$. It is a formal computation to verify that $v_4 \circ h_3$ is equal to $f_3 \circ v_3$.

The previous isomorphisms are functorial in A and form a canonical diagram, thus induce the required diagram at the level of C-algebraic groups. This concludes the proof of (i)

(ii) This is a consequence of (i), taking the Lie algebra of all objects.

(iii) The image of m_s in $\mathrm{Lie}(G) = \mathrm{End}_f(V)$ is the map sending v to sv. The map $\mathrm{Lie}(G) \to \mathrm{Lie}(G \times_{\mathrm{spec}(f)} \mathrm{spec}(C))$ is the map

$$\mathrm{End}_f(V) \to \mathrm{End}_C(V \otimes_f C)$$

sending a f-linear map L to the C-linear map $(v \otimes \lambda \mapsto L(v) \otimes \lambda)$ so the image of m_s in $\mathrm{Lie}(G \times_{\mathrm{spec}(F)} \mathrm{spec}(\overline{F})$ is the map $(v \otimes \lambda \mapsto$

$sv \otimes \lambda)$, let still denote it m_s. Consider the diagram of C-linear maps

$$\begin{tikzcd}
V \otimes_f C \arrow[r, "m_s"] \arrow[d, "c"] & V \otimes_f C \arrow[d, "c"] \\
V \otimes_{l'} (l' \otimes_f C) \arrow[d, "i"] & V \otimes_{l'} (l' \otimes_f C_{ij}) \arrow[d, "i"] \\
V \otimes \prod_i \prod_j C_{ij} \arrow[d, "b"] & V \otimes \prod_i \prod_j C_{ij} \arrow[d, "b"] \\
\bigoplus_i \bigoplus_j V_{ij} & \bigoplus_i \bigoplus_j V_{ij}
\end{tikzcd}$$

where c is the canonical map, i is the map induced by the map introduced in proposition 1.6.3, and b is the canonical map induced by the definition of V_{ij}. The image of m_s in $\mathrm{End}_{\overline{F}}\left(\bigoplus_i \bigoplus_j V_{ij}\right)$ is the composition $b \circ i \circ c \circ m_s \circ c^{-1} \circ i^{-1} \circ b^{-1}$.

Let us show that it is equal to $m_{s,C}$. The equality $b \circ i \circ c \circ m_s \circ c^{-1} \circ i^{-1} \circ b^{-1} = m_{s,C}$ is equivalent to the equality $b \circ i \circ c \circ m_s = m_{s,C} \circ b \circ i \circ c$. Let us prove this last equality by calculation. Let $v \otimes \lambda \in V \otimes_F C$, we have

$$\begin{aligned}
b \circ i \circ c \circ m_s(v \otimes \lambda) &= b \circ i \circ c(sv \otimes \lambda) \\
&= b \circ i(sv \otimes (1 \otimes \lambda)) \\
&= b \circ i(v \otimes (s \otimes \lambda)) \\
&= b(v \otimes \prod_i \prod_j \sigma_{ij}(s)\lambda) \\
&= \sum_i \sum_j v \otimes \sigma_{ij}(s)\lambda
\end{aligned}$$

and

$$\begin{aligned} m_{s,C} \circ b \circ i \circ c(v \otimes \lambda) =& m_{s,C} \circ b \circ i(v \otimes (1 \otimes \lambda)) \\ =& m_{s,C} \circ b(v \otimes \prod_i \prod_j \lambda) \\ =& m_{s,C}(\sum_i \sum_j v \otimes \lambda) \\ =& \sum_i \sum_j v \otimes \sigma_{ij}(s)\lambda. \end{aligned}$$

This concludes the proof of (iii).

□

So, the torus $T \times_{spec(f)} \mathrm{spec}(C)$ is a maximal split torus of $H' \times_{\mathrm{spec}(f)} \mathrm{spec}(C)$, $H \times_{\mathrm{spec}(f)} \mathrm{spec}(C)$ and $G \times_{\mathrm{spec}(f)} \mathrm{spec}(C)$. Moreover, $H' \times_{\mathrm{spec}(f)} \mathrm{spec}(C)$ is a Levi subgroup of $H \times_{\mathrm{spec}(f)} \mathrm{spec}(C)$, and $H \times_{\mathrm{spec}(f)} \mathrm{spec}(C)$ is a Levi subgroup of $G \times_{\mathrm{spec}(f)} \mathrm{spec}(C)$. We thus have inclusion of the corresponding set of roots.

$$\Phi(H', T, C) \subset \Phi(H, T, C) \subset \Phi(G, T, C)$$

Let us identify, using 1.6.5,

$$\begin{array}{lll} T \times_{\mathrm{spec}(f)} \mathrm{spec}(C) & \text{with} & \prod_i \prod_j \prod_k \underline{\mathrm{Aut}}_C(D_{ijk}), \\ H' \times_{\mathrm{spec}(f)} \mathrm{spec}(C) & \text{with} & \prod_i \prod_j \underline{\mathrm{Aut}}_C(V_{ij}), \\ H \times_{\mathrm{spec}(f)} \mathrm{spec}(C) & \text{with} & \prod_i \underline{\mathrm{Aut}}_C(\bigoplus_j V_{ij}), \text{ and} \\ G \times_{\mathrm{spec}(f)} \mathrm{spec}(C) & \text{with} & \underline{\mathrm{Aut}}_C(\bigoplus_i \bigoplus_j V_{ij}). \end{array}$$

Since $\bigoplus_i \bigoplus_j V_{ij}$ is equal to $\bigoplus_i \bigoplus_j \bigoplus_k D_{ijk}$, we can apply 1.6.1 to describe the set of roots $\Phi(G, T, C)$. Putting

$$\begin{aligned} I &= \{1, \ldots, i, \ldots, [l : f]\} \\ J &= \{1, \ldots, j, \ldots, [l' : l]\} \\ K &= \{1, \ldots, k, \ldots, d\}, \end{aligned}$$

we obtain the following equality.

$$\Phi(G,T,C) = \{\alpha_{ijk,i'j'k'} \mid (i,j,k),(i',j',k') \in (I \times J \times K), (i,j,k) \neq (i',j',k')\}$$

The set of roots $\Phi(H,T,C)$ is the following subset of $\Phi(G,T,C)$

$$\Phi(H,T,C) = \{\alpha_{ijk,i'j'k'} \in \Phi(G,T,\overline{F}) \mid i = i'\}.$$

The set of roots $\Phi(H',T,C)$ is the following subset of $\Phi(G,T,C)$

$$\Phi(H',T,C) = \{\alpha_{ijk,i'j'k'} \in \Phi(G,T,C) \mid i = i' \text{ and } j = j'\}.$$

The condition **GE1** is relative to the set $\Phi(H,T,C) \setminus \Phi(H',T,C)$. The following is a description of this set:

$$\Phi(H,T,C) \setminus \Phi(H',T,C) = \{\alpha_{ijk,i'j'k'} \in \Phi(G,T,C) \mid i = i' \text{ and } j \neq j'\}.$$

The condition **GE1** involves the element H_α for α in $\Phi(H,T,C) \setminus \Phi(H',T,C)$. Let us recall the description given in 1.6.1. Let $\alpha_{ijk,i'j'k'} \in \Phi(G,T,C)$, the element H_α which is by defintion $\mathrm{d}\alpha^\vee_{ijk,i'j'k'}(1)$ is the element sending each element $v \in D_{ijk}$ to v, and sending each element $v \in D_{i'j'k'}$ to $-v$ and, for all $i''j''k''$ different of $ijk, i'j'k'$, sending each element $v \in D_{i''j''k''}$ to 0.

1.6.4 Tame twisted Levi sequences

Let $E'/E/F$ be a tower of finite tamely ramified extensions. Let V be an E'-vector space of dimension d and D be a decomposition $V = (D_1 \oplus \ldots \oplus D_k \oplus \ldots \oplus D_d)$ of V in one dimensional E'-vector spaces.

In the previous subsection, we have introduced $H' = \mathrm{Res}_{E'/F}\underline{\mathrm{Aut}}_{E'}(V)$, $H = \mathrm{Res}_{E/F}\underline{\mathrm{Aut}}_E(V)$, and $G = \underline{\mathrm{Aut}}_F(V)$. We have also associated a torus $T = \mathrm{Res}_{E'/F}(T_D)$ to the decomposition D.

In proposition 1.6.5, we have computed the extension of scalar of these F-groups scheme to an extension containing the Galois closure of E'. We deduce the following corollary.

Corollary 1.6.7. *The sequence $H' \subset H \subset G$ is a tamely ramified twisted Levi sequence in G, moreover $Z(H')/Z(G)$ is anisotropic.*

Proof. We have to verify that the definition given in the beginning of section 1.3 is satisfied. Firstly, we need to show that there exists a finite tamely ramified Galois extension L of F such that $H' \times_{\mathrm{spec}(F)} \mathrm{spec}(L)$ and $H' \times_{\mathrm{spec}(F)}$

$\mathrm{spec}(L)$ are Levi subgroups of $G\times_{\mathrm{spec}(F)}\mathrm{spec}(L)$. This is a direct consequence of 1.6.5.

Secondly, the isomorphism of topological groups $(Z(H')/Z(G))(F) \simeq E'^{\times}/F^{\times}$ holds. The explicit description of the topological multiplicative group of a non archimedean local field given in proposition 1.5.1 implies that $E'^{\times}/F^{\times}$ is compact. This implies that $Z(H')/Z(G)$ is anisotropic. This concludes the proof of the corollary.

$\square$

1.6.5 Generic elements associated to minimal elements

We use in this subsection the notations of the previous subsection. The center Z' of H' is isomorphic to $\mathrm{Res}_{E'/F}(\mathbb{G}_m)$. Thus it is connected, i.e. $Z'^{\circ} = Z'$.

The inclusions $Z' \to H' \to H \to G$ induces a canonical diagram

$$\begin{tikzcd}
\mathrm{Lie}(Z') \arrow[r] \arrow[d] & \mathrm{Lie}(G) \arrow[d] \\
\mathrm{Lie}(Z') \otimes_F \overline{F} \arrow[r] \arrow[d, phantom, "\wr\|"] & \mathrm{Lie}(G) \otimes_F \overline{F} \arrow[d, phantom, "\wr\|"] \\
\mathrm{Lie}(Z' \times_{\mathrm{spec}(F)} \mathrm{spec}(\overline{F})) \arrow[r] & \mathrm{Lie}(G \times_{\mathrm{spec}(F)} \mathrm{spec}(\overline{F})
\end{tikzcd}$$

.

As explained after Definition 1.3.5, we have canonical inclusions

$$\mathrm{Lie}^*(Z') \to \mathrm{Lie}^*(H') \to \mathrm{Lie}^*(H) \to \mathrm{Lie}^*(G),$$

inducing a canonical inclusion $\mathrm{Lie}^*(Z') \to \mathrm{Lie}^*(G)$ and a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{Lie}^*(Z') \arrow[r] \arrow[d] & \mathrm{Lie}^*(G) \arrow[d] \\
\mathrm{Lie}^*(Z') \otimes_F \overline{F} \arrow[r] \arrow[d, phantom, "\wr\|"] & \mathrm{Lie}^*(G) \otimes_F \overline{F} \arrow[d, phantom, "\wr\|"] \\
\mathrm{Lie}^*(Z' \times_{\mathrm{spec}(F)} \mathrm{spec}(\overline{F})) \arrow[r] & \mathrm{Lie}^*(G \times_{\mathrm{spec}(F)} \mathrm{spec}(\overline{F}).
\end{tikzcd}$$

Recall that an element $X^* \in \mathrm{Lie}^*(Z')$ is H-generic of depth r if and only if $X^* \in \mathrm{Lie}^*(Z')_{-r}$ and if Conditions **GE1** and **GE2** hold. Since H' and H are of type A, Condition **GE1** implies Condition **GE2** by 1.3.8. Given $X^* \in \mathrm{Lie}^*(Z')$ we denote by $X^*_{\overline{F}}$ the image of X^* in $\mathrm{Lie}^*(Z' \times_{\mathrm{spec}(F)} \mathrm{spec}(\overline{F}))$ via the previous commutative diagram. Let recall that Condition **GE1** holds for X^* if $X^*_{\overline{F}}(H_\alpha) = -r$ for all root $\alpha \in \Phi(H, T, \overline{F}) \setminus \Phi(H', T, \overline{F})$.

Definition 1.6.8. *Let $s \in E'$. Let X_s^* be the element in $\mathrm{Lie}^*(Z')$ sending an element $h \in Lie(Z')$ to $\mathrm{Tr}_{\mathrm{End}_F(V)/F}(m_s \circ i(h))$ where i is the map $Lie(Z') \to \mathrm{Lie}(G)$, and $m_s \in End_F(V)$ is the map sending $v \in V$ to sv, i.e m_s is the multiplication by s.*

Proposition 1.6.9. *Let $s \in E'$. Let $X_s^* \in Lie^*(Z')$ be the element introduced in definition 1.6.8. Let $X_{s,\overline{F}}^*$ be the corresponding element in $\mathrm{Lie}^*(Z' \times_{\mathrm{spec}(F)} \mathrm{spec}(\overline{F}))$. Then*

(i) $X_{s,\overline{F}}^*(H_{\alpha_{i_1j_1k_1,i_2j_2k_2}}) = \sigma_{i_1j_1}(s) - \sigma_{i_2j_2}(s)$ for all roots $\alpha_{i_1j_1k_1,i_2j_2k_2} \in \Phi(G,T,\overline{F})$.

(ii) The element X_s^* is in $Lie^*(Z')_{-r}$ where $r = -\mathrm{ord}(s)$.

Proof. (i) Consider the diagram

$$\begin{tikzcd}
\mathrm{Lie}(Z') \arrow[r, "i"] \arrow[d, "Id\otimes 1"] & \mathrm{End}_F(V) \arrow[r, "m_s\circ"] \arrow[d, "Id\otimes 1"] & \mathrm{End}_F(V)F \arrow[r, "\mathrm{Tr}_F"] \arrow[d, "Id\otimes 1"] & F \arrow[d, "Id\otimes 1"] \\
\mathrm{Lie}(Z')\otimes_F \overline{F} \arrow[r, "i\otimes Id"] \arrow[d, "g"] & \mathrm{End}_F(V)\otimes_F \overline{F} \arrow[r, "m_s\circ\otimes Id"] \arrow[d, "f"] & \mathrm{End}_F(V)\otimes_F \overline{F} \arrow[r, "\mathrm{Tr}_F\otimes Id"] \arrow[d, "f"] & \overline{F} \arrow[d, "Id"] \\
\mathrm{Lie}(Z'\times_{\mathrm{spec}(F)} \mathrm{spec}(\overline{F})) \arrow[r, "i_{\overline{F}}"] & \mathrm{End}_{\overline{F}}(V\otimes_F \overline{F}) \arrow[r, "m_{s,\overline{F}}\circ"] & \mathrm{End}_{\overline{F}}(V\otimes_F \overline{F}) \arrow[r, "\mathrm{Tr}_{\overline{F}}"] & \overline{F}
\end{tikzcd}$$

where i is the canonical inclusion, $m_s\circ$ is the composition by m_s, and $m_{s,\overline{F}}\circ$ is the composition by the image $m_{s,\overline{F}}$ of m_s in $\mathrm{End}_{\overline{F}}(V \otimes_F \overline{F})$. Let us prove that it is commutative. The left part of the diagram was introduced before and is the canonical diagram induced by $Z' \to G$. The upper middle and right square are trivialy commutative. The right lower square is commutative by Lemma 1.6.1. Let us prove that the middle lower square is commutative. Let $L \otimes \lambda \in \mathrm{End}_F(V) \otimes_F \overline{F}$, then

$$\begin{aligned}
\Big((m_{s,\overline{F}}\circ)\circ f)\Big)(L\otimes\lambda) =&(m_{s,\overline{F}}\circ)\big(v\otimes\lambda' \mapsto L(v)\otimes\lambda\lambda'\big) \\
&= \big(v\otimes\lambda' \mapsto sL(v)\otimes\lambda\lambda'\big)
\end{aligned}$$

and

$$\begin{aligned}
(f\circ(m_s\otimes Id))(L\otimes\lambda) &= m_s\circ L\otimes\lambda \\
&= (v\otimes\lambda' \mapsto cL(v)\otimes\lambda\lambda'.
\end{aligned}$$

This concludes the proof of the commutativity of the diagram. By definition, we have

$$X_s^* = \mathrm{Tr}_F \circ (m_s \circ) \circ i$$

and

$$X_{s,\overline{F}}^* = ((\mathrm{Tr}_F \circ (m_s \circ) \circ i) \otimes Id) \circ g^{-1}.$$

We thus get

$$X_{s,\overline{F}}^* = (\mathrm{Tr}_F \otimes Id) \circ ((m_s \circ) \otimes Id) \circ (i \otimes Id) \circ g^{-1}.$$

The commutativity of the previous diagram implies thus

$$X_{s,\overline{F}}^* = \mathrm{Tr}_{\overline{F}} \circ (m_{s,\overline{F}} \circ) \circ i_{\overline{F}}.$$

Consequently for all roots $\alpha \in \Phi(G, T, \overline{F})$, we have

$$X_{s,\overline{F}}^*(H_\alpha) = \mathrm{Tr}_{\overline{F}}(m_{s,\overline{F}} \circ H_\alpha) \tag{1.16}$$

We have already computed $m_{s,\overline{F}}$ and H_α in terms of the decomposition $V \otimes_F \overline{F} = \bigoplus_{i,j,k} D_{ijk}$. Let us recall this. By proposition 1.6.5, $m_{s,\overline{F}}$ is the map

$$\begin{aligned} m_{s,\overline{F}} : \bigoplus_{i,j,k} D_{ijk} &\to \bigoplus_{i,j,k} D_{ijk} \\ \sum_{i,j,k} v_{ijk} &\mapsto \sum_{i,j,k} \sigma_{ij}(s) v_{ijk} \end{aligned}$$

Let $\alpha_{i_1j_1k_1,i_2j_2k_2} \in \Phi(G, T, \overline{F})$. By the calculation done in the end of the subsection 1.6.3, $H_{\alpha_{i_1j_1k_1,i_2j_2k_2}}$ is the map

$$\begin{aligned} H_{\alpha_{i_1j_1k_1,i_2j_2k_2}} : \bigoplus_{i,j,k} D_{ijk} &\to \bigoplus_{i,j,k} D_{ijk} \\ \sum_{i,j,k} v_{ijk} &\mapsto v_{i_1j_1k_1} - v_{i_2j_2k_2}. \end{aligned}$$

Consequently the maps $m_{s,\overline{F}} \circ H_{\alpha_{i_1j_1k_1,i_2j_2k_2}}$ is the map

$$\begin{aligned} m_{s,\overline{F}} \circ H_{\alpha_{i_1j_1k_1,i_2j_2k_2}} : \bigoplus_{i,j,k} D_{ijk} &\to \bigoplus_{i,j,k} D_{ijk} \\ \sum_{i,j,k} v_{ijk} &\mapsto \sigma_{i_1j_1}(s) v_{i_1j_1k_1} - \sigma_{i_2j_2}(s) v_{i_2j_2k_2}. \end{aligned}$$

This implies that

$$Tr_{\overline{F}}(m_{s,\overline{F}} \circ H_{\alpha_{i_1j_1k_1,i_2j_2k_2}}) = \sigma_{i_1j_1}(s) - \sigma_{i_2j_2}(s). \tag{1.17}$$

The proposition is now a consequence of the equations (1.16) and (1.17).

(ii) Recall that we put $r = -\mathrm{ord}(s)$. By definition (see the notation at the beginning of the document)

$$\mathrm{Lie}^*(Z')_{-r} = \{X \in \mathrm{Lie}^*(Z') \mid X(\mathrm{Lie}((Z')_{r+}) \subset \mathfrak{p}_F\}.$$

We have $s\mathrm{Lie}(Z')_{r+} = \mathrm{Lie}(Z')_{0+}$ and thus $\mathrm{Tr}_{End_F(V)/F}(s\mathrm{Lie}(Z')) \subset \mathfrak{p}_F$. So $X_s^* \in \mathrm{Lie}^*(Z')_{-r}$.

□

Proposition 1.6.10. *Let $s \in C_{E'}$ such that $E[s] = E'$.*

*Then the element $X^*_{s,\overline{F}}$ satisfies Condition **GE1**, more precisely, for all roots $\alpha \in \Phi(H,T,\overline{F}) \setminus \Phi(H',T,\overline{F})$, we have*

$$\mathrm{ord}(X^*_{s,\overline{F}}(H_\alpha)) = \mathrm{ord}(s).$$

Proof. Let $\alpha_{i_1j_1k_1,i_2j_2k_2} \in \Phi(H,T,\overline{F}) \setminus \Phi(H',T,\overline{F})$, by 1.6.9,

$$X^*_{s,\overline{F}}(H_{\alpha_{i_1j_1k_1,i_2j_2k_2}}) = \sigma_{i_1j_1}(s) - \sigma_{i_2j_2}(s).$$

We have $i_1 = i_2$ and $j_1 \neq j_2$ (see subsection 1.6.3). Consequently $\sigma_{i_1j_1}$ and $\sigma_{i_2j_2}$ are two distinct morphisms of F-algebras from E' to the Galois closure K' of E' whose restrictions to E are equal. Since s generates E' over E, $\sigma_{i_1j_1}(s)$ is not equal to $\sigma_{i_2j_2}(s)$. Let $\tau_{i_1j_1}$ and $\tau_{i_2j_2}$ be two morphisms of F-algebras from K' to K' extending $\sigma_{i_1j_1}$ and $\sigma_{i_2j_2}$, then $\tau_{i_1j_1}(s) \neq \tau_{i_2j_2}(s)$ and thus $\nu_{K'}(\tau_{i_1j_1}(s) - \tau_{i_2j_2}(s)) = \nu_{K'}(s)$ by 1.5.6. So $\mathrm{ord}(\sigma_{i_1j_1}(s) - \sigma_{i_2j_2}(s)) = \mathrm{ord}(s)$. Consequently for all roots $\alpha \in \Phi(H,T,\overline{F}) \setminus \Phi(H',T,\overline{F})$, we have $\mathrm{ord}(X^*_{s,\overline{F}}) = \mathrm{ord}(s)$, as required.

□

Corollary 1.6.11. *Let $c \in E'$ be minimal relatively to the extension E'/E (see 1.2.4, in particular $E[c] = E'$). Let r be $-\mathrm{ord}(c)$. Let $sr(c)$ be the standard representative of c. Then, the element $X^*_{sr(c)}$ is an element of $\mathrm{Lie}^*(Z')_{-r}$ and is H-generic of depth r*

Proof. Since $\mathrm{ord}(c) = \mathrm{ord}(sr(c))$, the proposition 1.6.9 (ii) implies that the element $X^*_{sr(c)}$ is in $Lie^*(Z')_{-r}$. By 1.5.8 the element $sr(c) \in C_{E'}$ generates E'/E, thus by 1.6.10 the element $X^*_{sr(c)} \in Lie^*(Z')$ satisfies **GE1** with depth $-\mathrm{ord}(sr(c))$. As explained before, Condition **GE2** is also satisfied. So $X^*_{sr(c)}$ is H-generic of depth r, since $\mathrm{ord}(c) = \mathrm{ord}(sr(c))$. □

1.7 Factorization of tame simple characters

Let $[\mathfrak{A}, n, r, \beta]$ be a tame simple stratum. In this section, we choose and fix a defining sequence $\{[\mathfrak{A}, n, r_i, \beta_i],\ 0 \leq i \leq s\}$ and a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$, we show that $\theta = \prod_{i=0}^{s} \theta^i$ where θ^i satisfies some conditions. We then introduce two cases depending on the condition that $\beta_s \in F$ or $\beta_s \notin F$.

1.7.1 Abstract factorizations of tame simple characters

Fix a tame simple stratum $[\mathfrak{A}, n, r, \beta]$ in the algebra $A = \mathrm{End}_F(V)$. Propositions 1.4.3 and 1.4.4 allow us to choose a defining sequence $\{[\mathfrak{A}, n, r_i, \beta_i],$ $0 \leq i \leq s\}$ (see corollary 1.2.11) such that, putting $\mathfrak{B}_{\beta_i} := \mathfrak{A} \cap \mathrm{End}_{F[\beta_i]}(V)$ and $r_0 = 0$, $\beta_0 = \beta$ the following holds.

(vii) $F[\beta_{i+1}] \subsetneq F[\beta_i]$ for $0 \leq i \leq s-1$

(vi') The stratum $[\mathfrak{B}_{\beta_{i+1}}, r_{i+1}, r_{i+1}-1, \beta_i - \beta_{i+1}]$ is simple in the algebra $\mathrm{End}_{F[\beta_{i+1}]}(V)$ for $0 \leq i \leq s-1$.

We fix such a defining sequence in the rest of this section 1.7, this includes the following subsection 1.7.2.

The elements β_i , $0 \leq i \leq s$ are all included in $F[\beta]$. Put $E_i := F[\beta_i]$ for $0 \leq i \leq s$.

Let us define elements c_i , $0 \leq i \leq s$, thanks to the following formulas.

- $c_i = \beta_i - \beta_{i+1}$ if $0 \leq i \leq s-1$
- $c_s = \beta_s$

The following proposition is the factorisation of tame simple characters as anounced before.

Theorem 1.7.1. *Let $\theta \in \mathcal{C}(\mathfrak{A}, m, \beta)$ be a simple character. There exists smooth characters $\phi_0, \ldots, \phi_s$ of $E_0^\times, \ldots, E_s^\times$ such that*

$$\theta = \prod_{i=0}^{s} \theta^i$$

where $\theta^i, 0 \leq i \leq s$, is the character defined by the following conditions.

(i) $\theta^i \mid_{H^{m+1}(\beta,\mathfrak{A}) \cap B_{\beta_i}} = \phi_i \circ \det_{B_{\beta_i}}$

(ii) $\theta^i \mid_{H^{m_i+1}(\beta,\mathfrak{A})} = \psi_{c_i}$ *where* $m_i = \max\{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}], m\}$.

Proof. Let us prove the proposition by induction. Suppose first that $s = 0$ i.e that β is minimal over F. Put $\theta^0 = \theta$. Then the condition (i) is trivially satisfied thanks to the definition of simple character in the minimal case (see [13, 3.2.1] or 1.2.16). The integer s is equal to 0, thus $\beta = \beta_0 = c_0$. So $-\nu_{\mathfrak{A}(c_0)} = -\nu_{\mathfrak{A}}(\beta) = n$. By the definition of simple characters in the minimal case, the restriction $\theta\mid_{H^{m+1}(\beta,\mathfrak{A})\cap U^{[\frac{n}{2}]+1}(\mathfrak{A})}$ is equal to ψ_β. So it is enough to verify that $H^{m+1}(\beta,\mathfrak{A}) \cap U^{[\frac{n}{2}]+1}(\mathfrak{A}) = H^{m_0'+1}(\beta,\mathfrak{A})$ where $m_0' = \max\{[\frac{n}{2}], m\}$ which is a consequence of the definition of $H^{m+1}(\beta,\mathfrak{A})$. Suppose now that $s > 0$. Let us remark that $-k_0(\beta,\mathfrak{A}) = -\nu_{\mathfrak{A}}(c_0)$, indeed the stratum $[\mathfrak{B}_{\beta_1}, -k_0(\beta,\mathfrak{A}), -k_0(\beta,\mathfrak{A}) - 1, \beta_0 - \beta_1]$ is simple. Thus the definition of simple characters implies that $\theta\mid_{H^{m_0+1}(\beta,\mathfrak{A})} = \theta'\psi_{c_0}$ where $\theta' \in \mathcal{C}(\mathfrak{A}, m_0, \beta_1)$. Thanks to the induction hypothesis there exists characters $\phi_1, \dots \phi_s$ of $E_1^\times, \dots, E_s^\times$ such that $\theta' = \prod_{i=1}^{s} \theta'^i$ where the θ'^i are the characters defined by the following conditions.

(i') $\theta'^i\mid_{H^{m_0+1}(\beta,\mathfrak{A})\cap B_{\beta_i}} = \phi_i \circ \det_{A_i}\mid_{H^{m'+1}(\beta,\mathfrak{A})\cap B_{\beta_i}}$

(ii') $\theta'^i\mid_{H^{m_i+1}(\beta,\mathfrak{A})} = \psi_{c_i}$

Identity (ii') is a consequence of the induction hypothesis (ii) and the fact that $\max([\frac{-\nu_{\mathfrak{A}}(c_i)}{2}], m_0) = \max([\frac{-\nu_{\mathfrak{A}}(c_i)}{2}], [\frac{-\nu_{\mathfrak{A}}(c_0)}{2}], m) = m_i$, because $-\nu_{\mathfrak{A}}(c_0) < -\nu_{\mathfrak{A}}(c_i)$.

For $1 \leq i \leq s$, the character θ'^i is defined on $H^{m_0+1}(\beta,\mathfrak{A})$ and we can extend θ'^i to $H^{m+1}(\beta,\mathfrak{A})$ thanks to the character ϕ_i as follows. The group $H^{m+1}(\beta,\mathfrak{A})$ is equal to $U^{m+1}(\mathfrak{B}_{\beta_0})H^{m_0+1}(\beta,\mathfrak{A})$, we extend θ'^i to a function θ^i of $H^{m+1}(\beta,\mathfrak{A})$ by puting $\theta^i(x) = \phi_i \circ \det_{A_0}(x)$ for $x \in U^{m+1}(\mathfrak{B}_{\beta_0})$. The function θ^i is a character. The character θ^i satisfies the required conditions (i) and (ii) by construction.

Finaly, put $\theta^0 = \theta \times \prod_{i=1}^{s}(\theta^i)^{-1}$. The restriction θ^0 to $H^{m+1}(\beta,\mathfrak{A}) \cap B_{\beta_i}$ is equal to the product of the restriction of θ to $H^{m+1}(\beta,\mathfrak{A}) \cap B_{\beta_i}$ by the restriction of θ_i^{-1} for $1 \leq i \leq s$. Let us show that each factor factors through $\det_{B_{\beta_0}}$. By definition of a simple character, this is the case for θ^0. Let $1 \leq i \leq s$, because of $H^{m+1}(\beta,\mathfrak{A}) \cap B_{\beta_i} \subset H^{m+1}(\beta,\mathfrak{A}) \cap B_{\beta_0}$, the restriction of θ^i to $H^{m+1}(\beta,\mathfrak{A}) \cap B_{\beta_i}$ is equal to $\phi_i \circ \det_{B_{\beta_i}}\mid_{H^{m+1}(\beta,\mathfrak{A})\cap B_{\beta_i}}$. However, a basic fact of algebraic number theory shows that $\det_{B_{\beta_i}}\mid_{B_{\beta_0}} = \det_{B_{\beta_0}} \circ N_{E_0/E_i}$, where$N_{E_0/E_i}$ is the norm map. Thus each factor factors through $\det_{B_{\beta_0}}$. Consequently there exists a smooth character ϕ_0 of $E_0^\times$ such that the condition (i) is satisfied. Let us prove that (ii) holds.

$$
\begin{aligned}
\theta^0 \mid_{H^{m_0+1}(\beta,\mathfrak{A})} &= \left(\theta \mid_{H^{m_0+1}(\beta,\mathfrak{A})} \times \prod_{i=1}^{s} (\theta^i)^{-1} \mid_{H^{m_0+1}(\beta,\mathfrak{A})}\right) \\
&= \left(\theta \mid_{H^{m_0+1}(\beta,\mathfrak{A})} \times (\theta')^{-1}\right) \\
&= \left(\psi_{c_0} \times \theta' \times (\theta')^{-1}\right) \\
&= \psi_{c_0}
\end{aligned}
$$

This completes the proof of the theorem, indeed we have found the required characters ϕ_i , $0 \leq i \leq s$ such that Conditions (i) and (ii) are satisfied. $\square$

1.7.2 Explicit factorizations of tame simple characters

In order to associate to each Bushnell-Kutzko datum a generic Yu datum, we need to introduce two cases. The two cases are denoted like this: $(Case\ A)$ or $(Case\ B)$. In the rest of this paper we write $(Case\ A)$ at the begining of a paragraph or in a sentence to signify that we work under the $(Case\ A)$ hypothesis. We will introduce particular notations in the $(Case\ A)$. The same holds for $(Case\ B)$. The $(Case\ A)$ is by definition when the last element β_s of the fixed choosen defining sequence is inside the field F, i.e $\beta_s \in F$. The $(Case\ B)$ is the other case, i.e when $\beta_s \notin F$.

Explicit factorizations of tame simple characters in $(Case\ A)$

Recall that in this case $\beta_s \in F$. In this case we put $d = s$. Let us give an explicit description of the group $H^1(\beta, \mathfrak{A})$ in this case. This explicit description is written in a convenient manner in order to compare with Yu's construction.

Proposition 1.7.2. $(Case\ A)$ *The group $H^1(\beta, \mathfrak{A})$ is equal to the following group*

$$
U^1(\mathfrak{B}_{\beta_0})U^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1}(\mathfrak{B}_{\beta_1})\ldots U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i})\ldots U^{[\frac{-\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}(\mathfrak{B}_{\beta_s}) \tag{1.18}
$$

Proof. Recall that $\beta = \beta_0$. By [13, 3.1.14,3.1.15], it is enough to show that

$$
\mathfrak{H}(\beta, \mathfrak{A}) = \mathfrak{B}_{\beta_0} + \mathfrak{Q}_{\beta_1}^{[-\frac{\nu_{\mathfrak{A}}(c_o)}{2}]+1} + \ldots + \mathfrak{Q}_{\beta_s}^{[-\frac{\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}. \tag{1.19}
$$

Let us prove (1.19) by induction on s. If $s=0$, by definition, $\mathfrak{H}(\beta,\mathfrak{A})=\mathfrak{B}_{\beta_0}+\mathfrak{P}^{[\frac{n}{2}]+1}$. The element β_0 is in F, thus $\mathfrak{B}_{\beta_0}=\mathfrak{A}$. Consequently $\mathfrak{H}(\beta,\mathfrak{A})=\mathfrak{B}_{\beta_0}$. If $s>0$, by induction hypothesis we have $\mathfrak{H}(\beta_1,\mathfrak{A})=\mathfrak{B}_{\beta_1}+\mathfrak{Q}_{\beta_2}^{[-\frac{\nu_{\mathfrak{A}}(c_1)}{2}]+1}+\ldots+\mathfrak{Q}_{\beta_s}^{[-\frac{\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}$.
By definition $\mathfrak{H}(\beta,\mathfrak{A})=\mathfrak{B}_{\beta_0}+\mathfrak{H}(\beta_1,\mathfrak{A})\cap\mathfrak{P}^{[\frac{-k_0(\beta_0,\mathfrak{A})}{2}]+1}$. Let us remark that since the stratum $[\mathfrak{B}_{\beta_1},-k_0(\beta_0,\mathfrak{A}),-k_0(\beta_0,\mathfrak{A})+1,\beta_0-\beta_1]$ is simple by the condition (vi'), the equality $\nu_{\mathfrak{B}_{\beta_1}}(\beta_0-\beta_1)=k_0(\beta_0,\mathfrak{A})$ holds. We have $\nu_{\mathfrak{B}_{\beta_1}}(\beta_0-\beta_1)=\nu_{\mathfrak{A}}(\beta_0-\beta_1)=\nu_{\mathfrak{A}}(c_0)$. So $k_0(\beta_0,\mathfrak{A})=\nu_{\mathfrak{A}}(c_0)$. Consequently

$$\begin{aligned}\mathfrak{H}(\beta,\mathfrak{A})&=\mathfrak{B}_{\beta_0}+\mathfrak{H}(\beta_1,\mathfrak{A})\cap\mathfrak{P}^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1}\\&=\mathfrak{B}_{\beta_0}+\mathfrak{Q}_{\beta_1}^{[-\frac{\nu_{\mathfrak{A}}(c_o)}{2}]+1}+\ldots+\mathfrak{Q}_{\beta_s}^{[-\frac{\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1},\end{aligned}$$

as required. □

We now reformulate Theorem 1.7.1 in (Case A) for simple characters in $\mathcal{C}(\mathfrak{A},0,\beta)$. This will be useful in order to associate generic characters in this case.

Corollary 1.7.3. *(Case A) Let $\theta\in\mathcal{C}(\mathfrak{A},0,\beta)$, let $\phi_0,\phi_1,\ldots,\phi_s$ be the characters introduced in theorem 1.7.1, then $\theta=\prod_{i=0}^{s}\theta^i$ where θ^i is the character defined as follows.*

If $0\le i\le s-1$, the character θ_i is defined by the following two conditions.

(i) $\theta^i\mid_{U^1(\mathfrak{B}_{\beta_0})U^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1}(\mathfrak{B}_{\beta_1})\ldots U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i})}=\phi_i\circ\det_{B_{\beta_i}}$

(ii) $\theta^i\mid_{U^{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]+1}(\mathfrak{B}_{\beta_{i+1}})\ldots U^{[\frac{-\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}(\mathfrak{B}_{\beta_s})}=\psi_{c_i}$.

If $i=s$, θ^i is defined by $\theta^i\mid_{H^1(\beta,\mathfrak{A})}=\phi_i\circ\det_A$.

Proof. The proof consists in applying Theorem 1.7.1 using the explicit description of $H^1(\beta,\mathfrak{A})$ given in the lemma 1.7.2. In Theorem 1.7.1, we have introduced smooth characters $\phi_0,\ldots\phi_s$ of $E_0^\times,\ldots E_s^\times$ such that $\theta=\prod_{i=0}^{s}\theta^i$ where θ^i is defined by the following two conditions.

(i) $\theta^i\mid_{H^1(\beta,\mathfrak{A})\cap B_{\beta_i}}=\phi_i\circ\det_{B_{\beta_i}}$

(ii) $\theta^i\mid_{H^{m_i+1}(\beta,\mathfrak{A})}=\psi_{c_i}$ where $m_i=\max\{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}],0\}$.

Let $0\le i\le s-1$, then Lemma 1.7.2 shows that $H^1(\beta,\mathfrak{A})\cap B_{\beta_i}=U^1(\mathfrak{B}_{\beta_0})U^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1}(\mathfrak{B}_{\beta_1})\ldots U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i})$. Consequently the condition (i) of the corollary 1.7.3 is satisfy for θ^i.

Trivially $m_i = [\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]$, moreover the lemma 1.7.2 shows that $H^{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]+1}(\beta,\mathfrak{A}) = U^{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]+1}(\mathfrak{B}_{\beta_{i+1}})\dots U^{[\frac{-\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}(\mathfrak{B}_{\beta_s})$. Thus Condition (ii) of Corollary 1.7.3 is satisfy for θ^i.

Finally, for $i = s$, we have $\theta^i \mid_{H^1(\beta,\mathfrak{A})\cap B_{\beta_i}} = \phi_i \circ \det_{B_{\beta_i}}$ by the theorem and the condition of the corollary is satisfied remarking that $B_{\beta_s} = A$ since $\beta_s \in F$.

□

Explicit factorizations of tame simple characters in $(Case\ B)$

Recall that in this case $\beta_s \notin F$. In this case we put $d = s+1$. Let us give an explicit description of the group $H^1(\beta,\mathfrak{A})$ in this case. This explicit description is written in a convenient way in order to compare with Yu's construction.

Proposition 1.7.4. $(Case\ B)$ *The group $H^1(\beta,\mathfrak{A})$ is equal to the following group:*

$$U^1(\mathfrak{B}_{\beta_0})U^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1}(\mathfrak{B}_{\beta_1})\dots U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i})\dots U^{[\frac{-\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}(\mathfrak{B}_{\beta_s})U^{[\frac{-\nu_{\mathfrak{A}}(c_s)}{2}]+1}(\mathfrak{A}). \tag{1.20}$$

Remark 1.7.5. *The difference with $(Case\ A)$ is that there is "one more term" in this multiplicative expression of $H^1(\beta,\mathfrak{A})$. This is due to the definition of $\mathfrak{H}(\beta,\mathfrak{A})$ in the minimal case, as explained in the following proof.*

Proof. By [13, 3.1.14,3.1.15], it is enough to show that

$$\mathfrak{H}(\beta,\mathfrak{A}) = \mathfrak{B}_{\beta_0} + \mathfrak{Q}_{\beta_1}^{[-\frac{\nu_{\mathfrak{A}}(c_0)}{2}]+1} + \dots + \mathfrak{Q}_{\beta_s}^{[-\frac{\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1} + \mathfrak{P}^{[\frac{-\nu_{\mathfrak{A}}(c_s)}{2}]+1}. \tag{1.21}$$

Let us prove (1.21) by induction on s. If $s = 0$, by definition, $\mathfrak{H}(\beta,\mathfrak{A}) = \mathfrak{B}_{\beta_0} + \mathfrak{P}^{[\frac{n}{2}]+1}$, where by definition $n = -\nu_{\mathfrak{A}}(\beta,\mathfrak{A})$. Since $s = 0$, the equality $\beta = c_s = c_0$ hold. Thus $\mathfrak{H}(\beta,\mathfrak{A}) = \mathfrak{B}_{\beta_0} + \mathfrak{P}^{[\frac{-\nu_{\mathfrak{A}}(c_s)}{2}]+1}$ as required.

If $s > 0$, by induction hypothesis we have

$$\mathfrak{H}(\beta_1,\mathfrak{A}) = \mathfrak{B}_{\beta_1} + \mathfrak{Q}_{\beta_2}^{[-\frac{\nu_{\mathfrak{A}}(c_1)}{2}]+1} + \dots + \mathfrak{Q}_{\beta_s}^{[-\frac{\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1} + \mathfrak{P}^{[\frac{-\nu_{\mathfrak{A}}(c_s)}{2}]+1}.$$

By definition $\mathfrak{H}(\beta,\mathfrak{A}) = \mathfrak{B}_{\beta_0} + \mathfrak{H}(\beta_1,\mathfrak{A}) \cap \mathfrak{P}^{[\frac{-k_0(\beta_0,\mathfrak{A})}{2}]+1}$. Let us remark that since the stratum $[\mathfrak{B}_{\beta_1}, -k_0(\beta_0,\mathfrak{A}), -k_0(\beta_0,\mathfrak{A})+1, \beta_0-\beta_1]$ is simple by the condition (vi'), the equality $\nu_{\mathfrak{B}_{\beta_1}}(\beta_0-\beta_1) = k_0(\beta_0,\mathfrak{A})$ holds. We have $\nu_{\mathfrak{B}_{\beta_1}}(\beta_0-\beta_1) = \nu_{\mathfrak{A}}(\beta_0-\beta_1) = \nu_{\mathfrak{A}}(c_0)$. So $\nu_{\mathfrak{A}}(c_0) = k_0(\beta_0,\mathfrak{A})$. Consequently

$$\begin{aligned}\mathfrak{H}(\beta,\mathfrak{A}) &= \mathfrak{B}_{\beta_0} + \mathfrak{H}(\beta_1,\mathfrak{A}) \cap \mathfrak{P}^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1} \\ &= \mathfrak{B}_{\beta_0} + \mathfrak{Q}_{\beta_1}^{[-\frac{\nu_{\mathfrak{A}}(c_0)}{2}]+1} + \dots + \mathfrak{Q}_{\beta_s}^{[-\frac{\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1} + \mathfrak{P}^{[\frac{-\nu_{\mathfrak{A}}(c_s)}{2}]+1},\end{aligned}$$

as required.

$\square$

We now reformulate Theorem 1.7.1 in (*Case B*) for the simple characters in $\mathcal{C}(\mathfrak{A}, 0, \beta)$. This will be useful in order to associate generic characters in this case.

Corollary 1.7.6. (*Case B*) *Let* $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$*, there exists* $\phi_0, \phi_1, \ldots, \phi_s$ *such that* $\theta = \prod_{i=0}^{s} \theta^i$ *where the* θ^i *are the characters defined by the following conditions.*

For $0 \leq i \leq s$*, the character* θ_i *is defined as follows.*

(i) $\theta^i \mid_{U^1(\mathfrak{B}_{\beta_0})U^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1}(\mathfrak{B}_{\beta_1})\ldots U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i})} = \phi_i \circ \det_{B_{\beta_i}}$

(ii) $\theta^i \mid_{U^{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]+1}(\mathfrak{B}_{\beta_{i+1}})\ldots U^{[\frac{-\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}(\mathfrak{B}_{\beta_s})U^{[\frac{-\nu_{\mathfrak{A}}(c_s)}{2}]+1}(\mathfrak{A})} = \psi_{c_i}$.

Proof. The proof consists in applying Theorem 1.7.1 using the explicit description of $H^1(\beta, \mathfrak{A})$ given in Lemma 1.7.4. By Theorem 1.7.1, there exist smooth characters $\phi_0, \ldots, \phi_s$ of $E_0^\times, \ldots, E_s^\times$ such that $\theta = \prod_{i=0}^{s} \theta^i$, where θ^i, $0 \leq i \leq s$, is defined by the following two conditions.

(i) $\theta^i \mid_{H^1(\beta,\mathfrak{A}) \cap B_{\beta_i}} = \phi_i \circ \det_{B_{\beta_i}}$

(ii) $\theta^i \mid_{H^{m_i+1}(\beta,\mathfrak{A})} = \psi_{c_i}$ where $m_i = \max\{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}], 0\}$.

Let $0 \leq i \leq s$. Then Lemma 1.7.4 shows that

$$H^1(\beta, \mathfrak{A}) \cap B_{\beta_i} = U^1(\mathfrak{B}_{\beta_0})U^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1}(\mathfrak{B}_{\beta_1}) \ldots U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i}).$$

Thus condition (i) of the corollary 1.7.6 is satisfied for θ^i.

Trivialy we have $m_i = [\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]$. Moreover Lemma 1.7.4 shows that $H^{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]+1}(\beta, \mathfrak{A}) = U^{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]+1}(\mathfrak{B}_{\beta_{i+1}}) \ldots U^{[\frac{-\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}(\mathfrak{B}_{\beta_s})U^{[\frac{-\nu_{\mathfrak{A}}(c_s)}{2}]+1}(\mathfrak{A})$. Thus Condition (ii) of Corollary 1.7.6 is satisfied for θ^i.

$\square$

1.8 Generic characters associated to tame simple characters

We continue with the same notations as in section 1.7. Thus we have a fixed tame simple stratum $[\mathfrak{A}, n, 0, \beta]$ and various objects and notations relative to it. In particular we have a defining sequence and a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta$. We have also distinguished two cases. In both (*Case A*) and (*Case B*), we have introduced various objects and notations and have established results relative to them. In this section we are going to introduce a 4-uple $(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}})$ which will be part of a complete Yu datum.

1.8.1 The characters $\mathbf{\Phi}_i$ associated to a factorization of a tame simple character

We start with (*Case A*).

The characters $\mathbf{\Phi}_i$ in the (*Case A*)

In section 1.7, we have introduced a sequence of fields

$$E_0 \supsetneq E_1 \supsetneq \dots \supsetneq E_i \supsetneq \dots \supsetneq E_s.$$

Recall that in this case $d = s$ and $E_s = F$, since $\beta_s \in F$ and $E_s = F[\beta_s]$. For each i, the field E_i is included in the algebra $A = \mathrm{End}_F(V)$ i.e V is an E_i-vector space.

For $0 \leq i \leq s$, put $G^i = \mathrm{Res}_{E_i/F}\underline{\mathrm{Aut}}_{E_i}(V)$. If $0 \leq i \leq j \leq d$ then G^i is canonically a closed subgroup scheme of G^j.

Let $\overrightarrow{G}$ be the sequence $G^0 \subsetneq G^1 \subsetneq \dots \subsetneq G^s$.

Proposition 1.8.1. (*Case A*) *The sequence $\overrightarrow{G}$ is a tamely ramified twisted Levi sequence in G.*

Proof. This is a consequence of 1.6.7.

□

We now introduce some real numbers $\mathbf{r}_i$ for $0 \leq i \leq s$. Put $\mathbf{r}_i := -\mathrm{ord}(c_i)$ for $0 \leq i \leq s$. Put also $\overrightarrow{\mathbf{r}} = (\mathbf{r}_0, \mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_s)$.

Proposition 1.8.2. (*Case A*) *For $0 \leq i \leq s$, the real number $\mathbf{r_i}$ satisfies the following formula:*

$$\mathbf{r_i} = \tfrac{-\nu_{\mathfrak{A}}(c_i)}{e(\mathfrak{A}|\mathfrak{o}_F)}.$$

Proof. By definition, $\mathbf{r}_i = -\mathrm{ord}(c_i)$. By definition of ord we know that

$$\mathrm{ord}(c_i) = \frac{\nu_{E_i}(c_i)}{e(E_i \mid F)}. \tag{1.22}$$

Lemma 1.2.1 shows that

$$\frac{\nu_{\mathfrak{A}}(c_i)}{e(\mathfrak{A} \mid \mathfrak{o}_F)} = \frac{\nu_{E_i}(c_i)}{e(E_i \mathfrak{o}_F F)}. \tag{1.23}$$

Equations 1.22 and 1.23 together finish the proof of the proposition.

□

Proposition 1.8.3. (*Case A*) *There exists a point y in $\mathrm{BT}^E(G^0, F)$ such that the following properties hold.*

(I) The following equalities hold.

  (i) $U^0(\mathfrak{B}_{\beta_0}) = G^0(F)_{y,0}$

  (ii) $U^1(\mathfrak{B}_{\beta_0}) = G^0(F)_{y,0+}$

  (iii) $\mathfrak{Q}_{\beta_0} = \mathfrak{g}^0(F)_{y,0+}$

  (iv) $\mathfrak{B}_{\beta_0} = \mathfrak{g}^0(F)_{y,0}$

  (v) $F[\beta]^\times U^0(\mathfrak{B}_{\beta_0}) = G^0(F)_{[y]}$

(II) There exist continous, affine and $G^{i-1}(F)$-equivariant maps $\iota_i : \mathrm{BT}^E(G^{i-1}, F) \xhookrightarrow{\iota_i} \mathrm{BT}^E(G^i, F)$, for $1 \leq i \leq s$, such that, denoting ι^i the composition $\iota_i \circ \iota_{\iota_{i-1}} \circ \ldots \circ \iota_1$, the following equalities hold.

  (i) $U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \frac{\mathbf{r}_{i-1}}{2}+}$

  (ii) $U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})+1}{2}]}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \frac{\mathbf{r}_{i-1}}{2}}$

  (iii) $U^{-\nu_{\mathfrak{A}}(c_{i-1})+1}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \mathbf{r}_{i-1}+}$

  (iv) $U^{-\nu_{\mathfrak{A}}(c_{i-1})}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \mathbf{r}_{i-1}}$

  (v) $\mathfrak{Q}_{\beta_i}^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1} = \mathfrak{g}^i(F)_{\iota^i(y), \frac{\mathbf{r}_{i-1}}{2}+}$

  (vi) $\mathfrak{Q}_{\beta_i}^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})+1}{2}]} = \mathfrak{g}^i(F)_{\iota^i(y), \frac{\mathbf{r}_{i-1}}{2}}$

  (vii) $\mathfrak{Q}_{\beta_i}^{-\nu_{\mathfrak{A}}(c_{i-1})+1} = \mathfrak{g}^i(F)_{\iota^i(y), \mathbf{r}_{i-1}+}$

  (viii) $\mathfrak{Q}_{\beta_i}^{-\nu_{\mathfrak{A}}(c_{i-1})} = \mathfrak{g}^i(F)_{\iota^i(y), \mathbf{r}_{i-1}}$ *and moreover,*

  (ix) $U^{-\nu_{\mathfrak{A}}(c_i)}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \mathbf{r}_i}$

  (x) $U^{-\nu_{\mathfrak{A}}(c_i)+1}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \mathbf{r}_i+}$

In the rest of this paper, we identify $\iota^i(y)$ and y.

Proof. In [7], the authors construct an explicit bijection between the set $Latt^1(V)$ of all lattices functions in V (see [7, Definition I.2.1] for the definition of a lattice function) and the enlarged Bruhat-Tits building of $\underline{\mathrm{Aut}}_F(V)$ (combine [7, Prop I.1.4] and [7, Prop I.2.4]). The group $\mathbb{R}$ acts on $Latt^1(V)$ and the previous bijection induces a bijection between $Latt(V) := Latt^1(V)/\mathbb{R}$ and the reduced Bruhat-Tits building $\mathrm{BT}^R(\underline{\mathrm{Aut}}_F(V), F)$. The authors show [7, Theorem II.1.1] that if $E/F \subset A$ is a separable extension of fields, there is a canonical affine and continuous emdedding from $\mathrm{BT}^R(\mathrm{Res}_{E/F}(\underline{\mathrm{Aut}}_E(V), F)$

to $\mathrm{BT}^R(\underline{\mathrm{Aut}}_F, F)$. Using the general fact that if G is a connected reductive k'-group and k'/k is a separable finite extension of non archimedean local field then $\mathrm{BT}^R(\mathrm{Res}_{k'/k}(G), k') = \mathrm{BT}^R(G, k)$; we deduce canonical maps $\mathrm{BT}^R(G^{i-1}, F) \to \mathrm{BT}^R(G^i, F)$ for $1 \leq i \leq d$. Recall that $\mathrm{BT}^E(G, k)$ is defined as $\mathrm{BT}^R(G, k) \times X_*(Z(G), F) \otimes_{\mathbb{Z}} \mathbb{R}$. Since $Z(G^0)/Z(G)$ is anisotropic, $X_*(Z(G^{i-1}), F) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X_*(Z(G^i, F)$ are isomorphic for $1 \leq i \leq d$. Fix such isomorphisms. They induce continous, affine and $G^{i-1}(F)$-equivariant embeddings

$$\mathrm{BT}^E(G^{i-1}, F) \to \mathrm{BT}^E(G^i, F).$$

In [7, I §7], the authors explain that there are injective maps

$$\{\text{Lattices chains in } V\} \to \{\text{Lattices sequences in } V\} \to \{\text{Lattices functions in } V\}.$$

Let $\Lambda \in \mathit{Latt}^1(V)$. To the class $\overline{\Lambda}$ of Λ, Broussous-Lemaire attach a filtration $a_r(\overline{\Lambda})$ of A and a filtration $U_r(\overline{\Lambda})$ of $A^\times = G$, they are indexed by $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$. If Λ comes from a lattices chain $\mathcal{L}$, then the filtration of A of Broussous-Lemaire is compatible with the filtration, indexed by $\mathbb{Z}$, given by powers of the radical of the hereditary order associated to $\mathcal{L}$.

Let $\mathcal{L}$ be an $\mathfrak{o}_E$-lattices chain associated to $\mathfrak{B}$. We thus get a point in $\mathrm{BT}^E(G^0, F)$ by the previous considerations. The rest of the proposition is a consequence of [7][Theorem II.1.1] and [7][Appendix A], up to contemporary normalization of Moy-Prasad filtrations.

$\square$

Let us introduce some character $\mathbf{\Phi}_i$, $0 \leq i \leq s$.

Definition 1.8.4. (*Case A*) *Let $0 \leq i \leq s$, and let $\mathbf{\Phi}_i$ be the smooth complex character of $G^i(F)$ defined by $\mathbf{\Phi}_i := \phi_i \circ \det_{B_{\beta_i}}$ where ϕ_i is the character introduced in 1.7.1 ,1.7.3.*

Proposition 1.8.5. (*Case A*) *The following assertions hold.*

(i) For $0 \leq i \leq s-1$, the character $\mathbf{\Phi}_i$ is G^{i+1}-generic of depth $\mathbf{r}_i$ relatively to y.

(ii) The character $\mathbf{\Phi}_s$ is of depth $\mathbf{r}_s$ relatively to y.

Proof. (i) Let us first prove that $\mathbf{\Phi}_i$ is of depth $\mathbf{r}_i$ relatively to y for $0 \leq i \leq s-1$. The restriction $\mathbf{\Phi}_i \mid_{G^i(F)_{\mathbf{r_i}}}$ is equal to the restriction $\mathbf{\Phi}_i \mid_{U^{-\nu_{\mathfrak{A}}(c_i)}(\mathfrak{B}_{\beta_i})}$ by proposition 1.8.3.

Let us prove that the two inclusions

$$U^{-\nu_{\mathfrak{A}}(c_i)}(\mathfrak{B}_{\beta_i}) \subset U^1(\mathfrak{B}_{\beta_0}) U^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1}(\mathfrak{B}_{\beta_1}) \dots U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i}) \tag{1.24}$$

and

$$U^{-\nu_{\mathfrak{A}}(c_i)}(\mathfrak{B}_{\beta_i}) \subset U^{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]+1}(\mathfrak{B}_{\beta_{i+1}}) \dots U^{[\frac{-\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}(\mathfrak{B}_{\beta_s}) \quad (1.25)$$

hold.

If $i = 0$, the first inclusion is trivial. Assume now $i > 0$. In order to prove the first inclusion in this case, remark that the inequality of integers $-\nu_{\mathfrak{A}}(c_{i-1}) < -\nu_{\mathfrak{A}}(c_i)$ holds.

We deduce easily and successively the inequalities

$$\begin{aligned} -\nu_{\mathfrak{A}}(c_{i-1}) &< -\nu_{\mathfrak{A}}(c_i) \\ \frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2} &< -\nu_{\mathfrak{A}}(c_i) \\ [\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}] + 1 &\leq -\nu_{\mathfrak{A}}(c_i). \end{aligned}$$

So $U^{-\nu_{\mathfrak{A}}(c_i)}(\mathfrak{B}_{\beta_i}) \subset U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i})$, and the first equality holds.

In order to prove the inclusion (1.25), remark that the integer $-\nu_{\mathfrak{A}}(c_i)$ is strictly bigger than 0. We deduce easily successively that

$$\begin{aligned} -\nu_{\mathfrak{A}}(c_i) &> \frac{-\nu_{\mathfrak{A}}(c_i)}{2} \\ -\nu_{\mathfrak{A}}(c_i) &\geq [\frac{-\nu_{\mathfrak{A}}(c_i)}{2}] + 1. \end{aligned}$$

Thus, since $\mathfrak{B}_{\beta_i} \subset \mathfrak{B}_{\beta_{i+1}}$, we get

$$U^{-\nu_{\mathfrak{A}}(c_i)}(\mathfrak{B}_{\beta_i}) \subset U^{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]+1}(\mathfrak{B}_{\beta_{i+1}})$$

and the second inequality follows. The inclusions (1.24) and (1.25) together with 1.7.3 imply that

$$\begin{gathered} \mathbf{\Phi}_i \mid_{G^i(F)_{y,\mathbf{r}_i}} = \phi_i \circ \det \mid_{U^{-\nu_{\mathfrak{A}}(c_i)}(\mathfrak{B}_{\beta_i})} = \theta^i \mid_{U^{-\nu_{\mathfrak{A}}(c_i)}(\mathfrak{B}_{\beta_i})} = \\ \psi_{c_i} \mid_{U^{-\nu_{\mathfrak{A}}(c_i)}(\mathfrak{B}_{\beta_i})}. \end{gathered}$$

We know that ψ_{c_i} is trivial on $U^{-\nu_{\mathfrak{A}}(c_i)+1}(\mathfrak{B}_{\beta_i})$ and non-trivial on $U^{-\nu_{\mathfrak{A}}(c_i)}(\mathfrak{B}_{\beta_i})$. Consequently, since $U^{-\nu_{\mathfrak{A}}(c_i)+1}(\mathfrak{B}_{\beta_i}) = G^i(F)_{y,\mathbf{r}_i+}$ by 1.8.3, the character $\mathbf{\Phi}_i$ is of depth $\mathbf{r}_i$ relatively to y.

We have to show that $\boldsymbol{\Phi_i}$ is G^{i+1}-generic of depth $\mathbf{r}_i$ for $0 \leq i \leq s-1$. By definition, $\psi_{c_i}(1+x) = \psi \circ \mathrm{Tr}_{A/F}(c_i x)$. We have thus obtained that

$$\mathbf{\Phi}_i \mid_{G^i(F)_{\mathbf{r_i}:\mathbf{r_i}+}} (1+x) = \psi \circ \mathrm{Tr}_{A/F}(c_i x) \tag{1.26}$$

As explained in section 1.3, the characters of $\mathfrak{g}^i(F)_{\mathbf{r_i}:\mathbf{r_i}+} \simeq \mathfrak{g}^i(F)_{\mathbf{r_i}}/\mathfrak{g}^i(F)_{\mathbf{r_i}+}$ are in bijection via ψ with $\mathfrak{g}^i(F)^{\bullet}_{\mathbf{r_i}+}/\mathfrak{g}^i(F)^{\bullet}_{\mathbf{r_i}}$ where

$\mathfrak{g}^i(F)^{\bullet}_{\mathbf{r_i}+} = \{x \in \mathfrak{g}^{i^*}(F) \mid x(\mathfrak{g}^i(F)_{\mathbf{r_i}+}) \subset \mathfrak{o}_F\} \otimes_{\mathfrak{o}_F} \mathfrak{p}_F = \mathfrak{g}^{i^*}(F)_{-\mathbf{r_i}}$

and

$\mathfrak{g}^i(F)^{\bullet}_{\mathbf{r_i}} = \{x \in \mathfrak{g}^{i^*}(F) \mid x(\mathfrak{g}^i(F)_{\mathbf{r_i}}) \subset \mathfrak{o}_F\} \otimes_{\mathfrak{o}_F} \mathfrak{p}_F = \mathfrak{g}^{i^*}(F)_{(-\mathbf{r_i})+}$

The isomorphism $G^i(F)_{\mathbf{r_i}:\mathbf{r_i}+} \simeq \mathfrak{g}^i(F)_{\mathbf{r_i}:\mathbf{r_i}+}$ used by Yu [41], is the same as the one used by Adler in [1] , and it is given in our case by the map $((1+x) \mapsto x)$. The element $X^*_{c_i} = (x \mapsto \mathrm{Tr}_{A/F}(c_i x))$ is an element in $\mathrm{Lie}^*(Z(G^i))_{-\mathbf{r_i}} \subset \mathfrak{g}^{i^*}(F)_{y,-\mathbf{r_i}}$. The equation (1.26) shows that $X^*_{c_i}$ realizes $\mathbf{\Phi}_i \mid_{G^i(F)_{\mathbf{r_i}:\mathbf{r_i}+}}$. In order to verify **GE1**, we want to show that the element $X^*_{sr(c_i)}$ realizes also $\mathbf{\Phi}_i \mid_{G^i(F)_{\mathbf{r_i}}}$.

The element $X^*_{sr(c_i)}$ is in $\mathrm{Lie}^*(Z(G^i))_{-\mathbf{r_i}} \subset \mathfrak{g}^{i^*}(F)_{y,-\mathbf{r_i}} = \mathfrak{g}^i(F)^{\bullet}_{\mathbf{r_i}+}$ by 1.6.9 (ii). So it is enough to prove that $(X^*_{sr(c_i)} - X^*_{c_i}) \in \mathfrak{g}^i(F)^{\bullet}_{\mathbf{r_i}}$. Let us remark that the equalities

$$\begin{aligned}\mathfrak{g}^i(F)^{\bullet}_{\mathbf{r_i}} &= \{x \in \mathfrak{g}^{i^*}(F) \mid x(\mathfrak{g}^i(F)_{\mathbf{r_i}}) \subset \mathfrak{o}_F\} \otimes_{\mathfrak{o}_F} \mathfrak{p}_F \subset \mathfrak{g}^{i^*}(F) \\ &= \{x \in \mathfrak{g}^{i^*}(F) \mid x(\mathfrak{g}^i(F)_{\mathbf{r_i}}) \subset \mathfrak{p}_F\}\end{aligned}$$

hold. Let us prove that $(X^*_{c_i} - X^*_{sr(c_i)}) \in \mathfrak{g}^i(F)^{\bullet}_{\mathbf{r_i}}$. Let $y \in \mathfrak{g}^i(F)_{y,\mathbf{r_i}}$, we have

$$\begin{aligned}(X^*_{c_i} - X^*_{sr(c_i)})(y) &= X^*_{c_i}(y) - X^*_{sr(c)}(y) \\ &= \mathrm{Tr}_{A/F}(c_i y) - \mathrm{Tr}_{A/F}(sr(c_i)y) \\ &= \mathrm{Tr}_{A/F}(c_i y - sr(c_i)y) \\ &= \mathrm{Tr}_{A/F}((c_i - sr(c_i)y)\end{aligned}$$

By 1.5.5, $\mathrm{ord}(c_i - sr(c_i)) > \mathrm{ord}(c_i) = \mathrm{ord}(sr(c_i))$. So $(c_i - sr(c_i))y \in \mathfrak{g}^i(F)_{0+}$. This finally implies that $\mathrm{Tr}_{A/F}((c_i - sr(c_i))y) \in \mathfrak{p}_F$. Thus the character $\mathbf{\Phi}_i \mid_{G^i(F)_{\mathbf{r_i}:\mathbf{r_i}+}}$ is realized by the element $X^*_{sr(c_i)}$. This element is G^{i+1}-generic of depth $-\mathrm{ord}(c_i)$ by 1.6.11. Thus $\mathbf{\Phi}_i$ is G^{i+1}-generic of depth $\mathbf{r}_i$.

(ii) Let us show that $\mathbf{\Phi}_s$ is of depth $\mathbf{r}_s$ relatively to y. This is easier than (i). By 1.8.3, we have $G(F)_{y,\mathbf{r}_s} = U^{-\nu_{\mathfrak{A}}(c_s)}(\mathfrak{B}_{\beta_s})$ and $G(F)_{y,\mathbf{r}_s+} = U^{-\nu_{\mathfrak{A}}(c_s)+1}(\mathfrak{B}_{\beta_s})$.

Thus, using 1.7.1, we get

$$\mathbf{\Phi_s} \mid_{G(F)_{y,\mathbf{r}_s}} = \phi_s \circ \det \mid_{U^{-\nu_{\mathfrak{A}}(c_s)}(\mathfrak{B}_{\beta_s})} = \theta^s \mid_{U^{-\nu_{\mathfrak{A}}(c_s)}(\mathfrak{B}_{\beta_s})} = \psi_{c_s}.$$

The character ψ_{c_s} is trivial on $G(F)_{y,\mathbf{r}_s+} = U^{-\nu_{\mathfrak{A}}(c_s)+1}(\mathfrak{B}_{\beta_s})$ and non trivial on $G(F)_{y,\mathbf{r}_s} = U^{-\nu_{\mathfrak{A}}(c_s)}(\mathfrak{B}_{\beta_s})$. This ends the proof of (ii)

□

The characters $\mathbf{\Phi}_i$ in $(Case\ B)$

We have already introduced a sequence of fields

$$E_0 \supsetneq E_1 \supsetneq \ldots \supsetneq E_i \supsetneq \ldots \supsetneq E_s.$$

Recall that in this case $d = s+1$ and $E_d = F$ by definition. For each i, the field E_i is contained in the algebra $A = \mathrm{End}_F(V)$ i.e V is an E_i-vector space.

For $0 \leq i \leq d$, put $G^i = \mathrm{Res}_{E_i/F}\underline{\mathrm{Aut}}_{E_i}(V)$. If $0 \leq i \leq j \leq d$ then G^i is canonically a closed group subscheme of G^j.

Let $\overrightarrow{G}$ be the sequence $G^0 \subset G^1 \subset \ldots \subset G^d$.

Proposition 1.8.6. $(Case\ B)$ *The sequence* $\overrightarrow{G}$ *is a tamely ramified twisted Levi sequence in* G.

Proof. The $(Case\ A)$ proof adapts to $(Case\ B)$ without change. □

We now introduce some real numbers $\mathbf{r}_i$ for $0 \leq i \leq d$. Put $\mathbf{r}_i := -\mathrm{ord}(c_i)$ for $0 \leq i \leq s$. Put $\mathbf{r}_d = \mathbf{r}_s$. Put also $\overrightarrow{\mathbf{r}} = (\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_i, \ldots, \mathbf{r}_s, \mathbf{r}_d)$.

Proposition 1.8.7. $(Case\ B)$ *For* $0 \leq i \leq s$, *the real number* $\mathbf{r_i}$ *satisfy the formula*

$$\mathbf{r_i} = \tfrac{-\nu_{\mathfrak{A}}(c_i)}{e(\mathfrak{A}|\mathfrak{o}_F)}.$$

Proof. The $(Case\ A)$ proof adapts to $(Case\ B)$ without change. □

Proposition 1.8.8. *(Case B) There exists a point* y *in* $\mathrm{BT}^E(G^0, F)$ *such that the following properties hold.*

(I) The following equalities hold.

(i) $U^0(\mathfrak{B}_{\beta_0}) = G^0(F)_{y,0}$

(ii) $U^1(\mathfrak{B}_{\beta_0}) = G^0(F)_{y,0+}$

(iii) $\mathfrak{Q}_{\beta_0} = \mathfrak{g}^0(F)_{y,0+}$

(iv) $\mathfrak{B}_{\beta_0} = \mathfrak{g}^0(F)_{y,0}$

(v) $F[\beta]^\times U^0(\mathfrak{B}_{\beta_0}) = G^0(F)_{[y]}$

(II) There exist continous, affine and $G^{i-1}(F)$-equivariant maps $\iota_i : \mathrm{BT}^E(G^{i-1}, F) \xhookrightarrow{\iota_i} \mathrm{BT}^E(G^i, F)$ for $1 \leq i \leq s$, such that, denoting ι^i the composition $\iota_i \circ \iota_{\iota_{i-1}} \circ \ldots \circ \iota_1$, the following equalities hold.

(i) $U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \frac{\mathbf{r}_{i-1}}{2}+}$

(ii) $U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})+1}{2}]}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \frac{\mathbf{r}_{i-1}}{2}}$

(iii) $U^{-\nu_{\mathfrak{A}}(c_{i-1})+1}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \mathbf{r}_{i-1}+}$

(iv) $U^{-\nu_{\mathfrak{A}}(c_{i-1})}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \mathbf{r}_{i-1}}$

(v) $\mathfrak{Q}_{\beta_i}^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1} = \mathfrak{g}^i(F)_{\iota^i(y), \frac{\mathbf{r}_{i-1}}{2}+}$

(vi) $\mathfrak{Q}_{\beta_i}^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})+1}{2}]} = \mathfrak{g}^i(F)_{\iota^i(y), \frac{\mathbf{r}_{i-1}}{2}}$

(vii) $\mathfrak{Q}_{\beta_i}^{-\nu_{\mathfrak{A}}(c_{i-1})+1} = \mathfrak{g}^i(F)_{\iota^i(y), \mathbf{r}_{i-1}+}$

(viii) $\mathfrak{Q}_{\beta_i}^{-\nu_{\mathfrak{A}}(c_{i-1})} = \mathfrak{g}^i(F)_{\iota^i(y), \mathbf{r}_{i-1}}$ *and moreover,*

(ix) $U^{-\nu_{\mathfrak{A}}(c_i)}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \mathbf{r}_i}$

(x) $U^{-\nu_{\mathfrak{A}}(c_i)+1}(\mathfrak{B}_{\beta_i}) = G^i(F)_{\iota^i(y), \mathbf{r}_i+}$

(III) There exists a continous, affine and $G^s(F)$-equivariant map $\iota_d : \mathrm{BT}^E(G^s, F) \xhookrightarrow{\iota_i} \mathrm{BT}^E(G^d, F)$ such that, denoting ι^d the composition $\iota_d \circ \iota_{\iota_d} \circ \ldots \circ \iota_1$, the following equalities hold.

(i) $U^{[\frac{-\nu_{\mathfrak{A}}(c_s)}{2}]+1}(\mathfrak{A}) = G^d(F)_{\iota^i(y), \frac{\mathbf{r}_s}{2}+}$

(ii) $U^{[\frac{-\nu_{\mathfrak{A}}(c_s)+1}{2}]}(\mathfrak{A}) = G^d(F)_{\iota^i(y), \frac{\mathbf{r}_s}{2}}$

(iii) $U^{-\nu_{\mathfrak{A}}(c_s)+1}(\mathfrak{A}) = G^d(F)_{\iota^i(y), \mathbf{r}_s+}$

(iv) $U^{-\nu_{\mathfrak{A}}(c_s)}(\mathfrak{A}) = G^d(F)_{\iota^i(y), \mathbf{r}_s}$

(v) $\mathfrak{P}^{[\frac{-\nu_{\mathfrak{A}}(c_s)}{2}]+1} = \mathfrak{g}^d(F)_{\iota^i(y), \frac{\mathbf{r}_s}{2}+}$

(vi) $\mathfrak{P}^{[\frac{-\nu_{\mathfrak{A}}(c_s)+1}{2}]} = \mathfrak{g}^d(F)_{\iota^i(y), \frac{\mathbf{r}_s}{2}}$

(vii) $\mathfrak{P}^{-\nu_{\mathfrak{A}}(c_s)+1} = \mathfrak{g}^d(F)_{\iota^i(y), \mathbf{r}_s+}$

(viii) $\mathfrak{P}^{-\nu_{\mathfrak{A}}(c_s)} = \mathfrak{g}^d(F)_{\iota^i(y), \mathbf{r}_s}$

In the rest of this paper, we identify $\iota^i(y)$ and y.

Proof. The $(Case\ A)$ proof adapts to $(Case\ B)$ without change for (I) and (II), the proof of (II) adapts to (III) without effort. □

Let us introduce certain characters $\mathbf{\Phi}_i$, $0 \leq i \leq d$.

Definition 1.8.9. *($Case\ B$) Let $0 \leq i \leq s$, and let $\mathbf{\Phi}_i$ be the smooth complex character of $G^i(F)$ defined by $\mathbf{\Phi}_i := \phi_i \circ \det_{B_{\beta_i}}$, where ϕ_i is the character introduced in 1.7.1, 1.7.6. Let also $\mathbf{\Phi}_d$ be the trivial character 1 of $G^d(F)$.*

Proposition 1.8.10. *($Case\ B$) For $0 \leq i \leq s$, the character $\mathbf{\Phi}_i$ is G^{i+1}-generic of depth $\mathbf{r}_i$.*

Proof. The ($Case\ A$) proof adapts to ($Case\ B$) without change.

□

1.8.2 The characters $\hat{\mathbf{\Phi}}_i$

In both ($Case\ A$) and ($Case\ B$), we have obtained part of a Yu datum $(\overrightarrow{G}, y, r, \overrightarrow{\mathbf{\Phi}})$. To $(\overrightarrow{G}, y, r, \overrightarrow{\mathbf{\Phi}})$ is attached by Yu various objects. In the rest of this section we shows that the characters $\hat{\mathbf{\Phi}}_i$ (see section 1.3) are equal to the factors θ_i of θ.

Proposition 1.8.11. *In both ($Case\ A$) and ($Case\ B$), let $K_+^d = K_+^d(\overrightarrow{G}, y, r, \overrightarrow{\mathbf{\Phi}})$ be the group attached to $(\overrightarrow{G}, y, r, \overrightarrow{\mathbf{\Phi}})$ (see section 1.3). Then $H^1(\beta, \mathfrak{A}) = K_+^d$.*

Proof. ($Case\ A$) By proposition 1.7.2, we have the equality

$$H^1(\beta,\mathfrak{A}) = U^1(\mathfrak{B}_{\beta_0})U^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1}(\mathfrak{B}_{\beta_1})\dots U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i})\dots U^{[\frac{-\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}(\mathfrak{B}_{\beta_s}).$$

By definition of $K_+^d(\overrightarrow{G}, y, r, \overrightarrow{\phi})$, and because of $d = s$, we have the equality

$$K_+^d(\overrightarrow{G}, y, r, \overrightarrow{\phi}) = G^0(F)_{y,0+}G^1(F)_{y,\mathbf{s}_0+}\cdots G^i(F)_{y,\mathbf{s}_{i-1}+}\dots G^s(F)_{y,\mathbf{s}_{s-1}+}.$$

The required statement is now a formal consequence of 1.8.3.

($Case\ B$) By proposition 1.7.4, we have the equality

$$H^1(\beta,\mathfrak{A}) = U^1(\mathfrak{B}_{\beta_0})U^{[\frac{-\nu_{\mathfrak{A}}(c_0)}{2}]+1}(\mathfrak{B}_{\beta_1})\dots U^{[\frac{-\nu_{\mathfrak{A}}(c_{i-1})}{2}]+1}(\mathfrak{B}_{\beta_i})\dots U^{[\frac{-\nu_{\mathfrak{A}}(c_{s-1})}{2}]+1}(\mathfrak{B}_{\beta_s})U^{[\frac{-\nu_{\mathfrak{A}}(c_s)}{2}]+1}(\mathfrak{A}).$$

By definition of $K_+^d(\overrightarrow{G}, y, r, \overrightarrow{\phi})$, and because of $d = s+1$, we have the equality

$$K_+^d(\overrightarrow{G}, y, r, \overrightarrow{\phi}) = G^0(F)_{y,0+}G^1(F)_{y,\mathbf{s}_0+}\cdots G^i(F)_{y,\mathbf{s}_{i-1}}\dots G^s(F)_{y,\mathbf{s}_{s-1}+}G^d(F)_{y,\mathbf{s}_s+}.$$

The required statement is now a formal consequence of 1.8.8.

□

Proposition 1.8.12. *In both ($Case\ A$) and ($Case\ B$), let $0 \leq i \leq d$ and let $\hat{\mathbf{\Phi}}_i$ be the character attached to $\mathbf{\Phi}_i$ (see section 1.3). Then*

(i) $\hat{\mathbf{\Phi}}_i = \theta^i$ *for* $0 \leq i \leq s$

(ii) $\prod_{i=0}^{d} \hat{\mathbf{\Phi}}_i = \theta$

Proof. Recall that $\hat{\mathbf{\Phi}}_i$ is defined in [41, section 4] and also in the section 1.3 of this text. In order to prove (i), we need first to study the decomposition $\mathfrak{g} = \mathfrak{g}^i \oplus \mathfrak{n}^i$. In our situation where $G = \underline{\mathrm{Aut}}_F(V)$, the Lie algebra $\mathfrak{g}$ is $\mathrm{End}_F(V)$ and the Lie algebra of G^i denoted $\mathfrak{g}^i$ is $End_{F[\beta_i]}(V)$. The space $\mathfrak{g}^i$ is characterized by the fact that it is the maximal subspace of $\mathfrak{g}$ such that the adjoint action of the center $Z(G^i(F))$ of $G^i(F)$ is trivial. By definition, $\mathfrak{n}^i$ is the sum of the other isotypic spaces for the adjoint action of $T^i(F)$ on $\mathfrak{g}$. This implies that there is an integer R_i such that each $n \in \mathfrak{n}^i$ is a finite sum

$$n = \sum_{k=0}^{R_i} n_k$$

such that for each $0 \leq k \leq R_i$, there is an element $t_k \in Z(G^i(F))$ and $\lambda_k \neq 1$ such that $\mathrm{ad}_{t_k}(n_k) = \lambda_k n_k$. We are now able to prove (i) of the proposition 1.8.12.

If $x \in \mathfrak{g}$, let $x = \pi_{\mathfrak{g}^i}(x) + \pi_{\mathfrak{n}^i}(x)$ denote the decomposition of x relatively to the decomposition $\mathfrak{g} = \mathfrak{g}^i \oplus \mathfrak{n}^i$.

Let $0 \leq i \leq s$. By definition (see section 1.3) $\hat{\mathbf{\Phi}}_i$ is the character of K_+^d defined by

- $\hat{\mathbf{\Phi}}_i \mid_{G^i(F)\cap K_+^d} = \mathbf{\Phi}_i \mid_{G^i(F)\cap K_+^d}$
- $\hat{\mathbf{\Phi}}_i \mid_{G(F)_{y,\mathbf{s}_i+}\cap K_+^d} (1+x) = \mathbf{\Phi}_i(1+\pi_{\mathfrak{g}^i}(x))$.

Let us verify that it is equal to the character θ^i defined in proposition 1.7.1.

First, note that the group K_+^d is equal to the group $H^1(\beta, \mathfrak{A})$ by proposition 1.8.11, so it makes sense to compare $\hat{\mathbf{\Phi}}_i$ and θ^i. The group $G^i(F) \cap K_+^d$ is equal to $B_{\beta_i} \cap H^1(\beta, \mathfrak{A})$. Thus, the definitions of θ^i given in proposition 1.7.1 shows that

$$\hat{\mathbf{\Phi}}_i \mid_{G^i(F)\cap K_+^d} = \mathbf{\Phi}_i \mid_{G^i(F)\cap K_+^d} = \phi_i \circ \det_{B_{\beta_i}} \mid_{G^i(F)\cap K_+^d} = \theta^i \mid_{G^i(F)\cap K_+^d}. \quad (1.27)$$

It is enough to show that $\hat{\mathbf{\Phi}}_i \mid_{G(F)_{y,\mathbf{s}_i+}\cap K_+^d}$ is equal to $\theta^i \mid_{G(F)_{y,\mathbf{s}_i+}\cap K_+^d}$. The group $G(F)_{y,\mathbf{s}_i+}$ is equal to $U^{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]+1}(\mathfrak{A})$. Consequently

$$\begin{aligned}
\hat{\mathbf{\Phi}}_i \mid_{G(F)_{y,\mathbf{s}_i+}\cap K_+^d} (1+x) &= \mathbf{\Phi}_i \mid_{G(F)_{y,\mathbf{s}_i+}\cap K_+^d} (1+\pi_{\mathfrak{g}^i}(x)) \\
{\scriptstyle (Because\ 1+\pi_{\mathfrak{g}^i}(x) \in G^i(F))} &= \mathbf{\Phi}_i \mid_{G(F)_{y,\mathbf{s}_i+}\cap K_+^d \cap G^i(F)} (1+\pi_{\mathfrak{g}^i}(x)) \\
{\scriptstyle (By\ eq.\ (1.27)\ and\ equality\ of\ groups)} &= \theta^i \mid_{H^1(\beta,\mathfrak{A})\cap B_{\beta_i}\cap U^{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]+1}(\mathfrak{A})} (1+\pi_{\mathfrak{g}^i}(x)) \\
{\scriptstyle (By\ def.\ of\ \theta^i\ on\ H^1(\beta,\mathfrak{A})\cap U^{[\frac{-\nu_{\mathfrak{A}}(c_i)}{2}]+1})} &= \psi \circ \mathrm{Tr}_{A/F}(c_i \pi_{\mathfrak{g}^i}(x)).
\end{aligned}$$

Let us now compute $\mathrm{Tr}_{A/F}(c_i\pi_{\mathfrak{g}^i}(x))$. We have the equalities

$$\mathrm{Tr}(c_ix) = \mathrm{Tr}(c_i(\pi_{\mathfrak{g}^i}(x) + \pi_{\mathfrak{n}^i}(x))) = \mathrm{Tr}(c_i\pi_{\mathfrak{g}^i}(x)) + \mathrm{Tr}(c_i\pi_{\mathfrak{n}^i}(x)).$$

Let us compute $\mathrm{Tr}(c_i\pi_{\mathfrak{n}^i}(x))$. Since $\pi_{\mathfrak{n}^i}(x) \in \mathfrak{n}^i$, there is an integer R_i such that $\pi_{\mathfrak{n}^i}(x)$ is a finite sum

$$\pi_{\mathfrak{n}^i}(x) = \sum_{k=0}^{R_i} n_k$$

such that for each $0 \leq k \leq R_i$, there is an element $t_k \in Z(G^i(F))$ and $\lambda_k \neq 1$ such that $\mathrm{ad}_{t_k}(n_k) = \lambda_k n_k$. We have

$$\mathrm{Tr}(c_i\pi_{\mathfrak{n}^i}(x)) = \mathrm{Tr}(c_i\sum_{k=0}^{R_i} n_k) = \sum_{k=0}^{R_i}\mathrm{Tr}(c_in_k).$$

Fix $0 \leq k \leq R_i$. The element t_k commutes with c_i. Consequently $tc_in_kt^{-1} = c_itn_kt^{-1} = c_i\lambda n_k$. So

$$\mathrm{Tr}(c_in_k) = \mathrm{Tr}(tc_in_kt^{-1}) = \lambda\mathrm{Tr}(c_in_k)$$

This implies that

$$\mathrm{Tr}(c_in_k) = 0.$$

And so

$$\mathrm{Tr}(c_i\pi_{\mathfrak{n}^i}(x)) = 0.$$

Thus the equality

$$\mathrm{Tr}_{A/F}(c_i\pi_{\mathfrak{g}^i}(x)) = \mathrm{Tr}_{A/F}(c_ix)$$

holds.

Consequently

$$\hat{\mathbf{\Phi}}_i \mid_{G(F)_{y,\mathbf{s}_i+}\cap K_+^d} (1+x) = \psi \circ \mathrm{Tr}_{A/F}(c_ix) = \psi_{c_i} = \theta^i \mid_{G(F)_{y,\mathbf{s}_i+}\cap K_+^d},$$

as required. This concludes the proof of (i) of Proposition 1.8.12.

The proof of (ii) is now easy because $\theta = \prod_{i=0}^{s}\theta^i$ and because in (*Case A*), $d = s$, and in (*Case B*), $d = s+1$ and $\hat{\mathbf{\Phi}}_d = 1$. This ends the proof of Proposition 1.8.12. $\square$

1.9 Extensions and main theorem of the comparison: from Bushnell-Kutzko's construction to Yu's construction

In this section, we keep notations of the sections 1.7 and 1.8. In particular, we have fixed a tame simple stratum $[\mathfrak{A}, n, r, \beta]$ and a choosen defining sequence $\{[\mathfrak{A}, n, r_i, \beta_i]\ , 0 \leq i \leq s\}$, such that $F[\beta_{i+1}] \subsetneq F[\beta_i]$ for all $0 \leq i \leq s-1$. We have also fixed a simple character $\theta \in \mathcal{C}(\mathfrak{A}, 0, \beta)$. We have distinguished two cases, (*Case A*) occurs when $\beta_s \in F$. In this case we have put $d = s$. In the opposite (*Case B*), we have put $d = s+1$. In both case we have introduced part of a Yu datum $(\overrightarrow{G}, y, r, \overrightarrow{\phi})$. We have also proved some results relative to these objects. In this section we are going to show that the representation ${}^{\circ}\lambda(\overrightarrow{G}, y, r, \overrightarrow{\phi})$ is a β-extension of θ. Then, given a cuspidal representation σ of $U^0(\mathfrak{B}_{\beta_0})/U^1(\mathfrak{B}_{\beta_0})$ and Λ an extension to $E^{\times}J^0(\beta, \mathfrak{A})$ of $\kappa \otimes \sigma$, we are going to show that there exists ρ such that $\Lambda = \rho_d(\overrightarrow{G}, y, r, \overrightarrow{\phi}, \rho)$.

Proposition 1.9.1. *In both (Case A) and (Case B) the group* ${}^{\circ}K^d(\overrightarrow{G}, y, r, \overrightarrow{\phi})$ *is equal to* $J^0(\beta, \mathfrak{A})$.

Proof. This proposition is similar to that of Proposition 1.8.11 and the proof adapts trivially. $\square$

Proposition 1.9.2. *In both (Case A) and (Case B), the representation*

$${}^{\circ}\lambda(\overrightarrow{G}, y, r, \overrightarrow{\phi})$$

of ${}^{\circ}K^d$ *is a* β*-extension of* θ.

Proof. Let us verify that ${}^{\circ}\lambda ={}^{\circ} \lambda(\overrightarrow{G}, y, r, \overrightarrow{\phi})$ satisfies the criterion given in Proposition 1.2.22.

(a) The representation ${}^{\circ}\lambda$ is equal to ${}^{\circ}\kappa_0 \otimes \ldots \otimes {}^{\circ}\kappa_d$ (see section 1.3). By construction of $\kappa_i, 0 \leq i \leq d$, the representation ${}^{\circ}\kappa_i$ contains $\hat{\mathbf{\Phi}}_i$ (see [24, 3.27]). Consequently ${}^{\circ}\lambda$ contains $\hat{\mathbf{\Phi}}_0 \otimes \ldots \otimes \hat{\mathbf{\Phi}}_d$. Thus ${}^{\circ}\lambda$ contains θ by 1.8.12.

(b) Again, ${}^{\circ}\lambda ={}^{\circ} \kappa_0 \otimes \ldots \otimes^{\circ} \kappa_d$. Thus, it is enough to show that $G^0(F)$ is contained in $I_{G(F)}({}^{\circ}\kappa_i)$ for $0 \leq i \leq d$. Theorem 14.2 of [41], which is satisfied here, implies that $G^0(F)$ is contained in $I_{G(F)}(\mathbf{\Phi}_i{}' \mid_{{}^{\circ}K^i})$. However, ${}^{\circ}\kappa_i$ is an inflation of $\mathbf{\Phi}_i{}' \mid_{{}^{\circ}K^i}$ (see definition 1.3.21). Consequently $I_{G(F)}(\mathbf{\Phi}_i{}' \mid_{{}^{\circ}K^i}) \subset I_{G(F}({}^{\circ}\kappa_i)$. Consequently $G^0(F) \subset I_{G(F)}({}^{\circ}\kappa_i)$ as required.

(c) The representation ${}^{\circ}\lambda$ is equal to ${}^{\circ}\kappa_0 \otimes \ldots \otimes {}^{\circ}\kappa_i \otimes \ldots \otimes {}^{\circ}\kappa_d$. For $0 \leq i \leq d-1$ the dimension of ${}^{\circ}\kappa_i$ is $[J^{i+1} : J_+^{i+1}]^{\frac{1}{2}}$.

The representation ${}^\circ\kappa_d$ is one dimensional. So it is enough to show that $\prod_{i=1}^{d}[J^{i+1} : J_+^{i+1}] = [J^1(\beta,\mathfrak{A}) : H^1(\beta,\mathfrak{A})]$. The group $J^1(\beta,\mathfrak{A})$ is equal to $G^0(F)_{y,0+}G^1(F)_{y,\mathbf{s}_0}\dots G^d(F)_{y,\mathbf{s}_{d-1}}$, this is thus also equal to $G^0(F)_{y,0+}J^1\dots J^d$. The group $H^1(\beta,\mathfrak{A})$ is equal to $G^0(F)_{y,0+}G^1(F)_{y,\mathbf{s}_0+}\dots G^d(F)_{y,\mathbf{s}_{d-1}+}$, this is thus also equal to $G^0(F)_{y,0+}J_+^1\dots J_+^d$. Since $G^0(F)_{y,0+}J^1\dots J^d/G^0(F)_{y,0+}J_+^1\dots J_+^d \simeq J^1\dots J^d/J_+^1\dots J_+^d$ it is enough to show that $\prod_{i=1}^{d}[J^i : J_+^i] = [J^1\dots J^d : J_+^1\dots J_+^d]$. Let us prove this by induction on d. If $d=1$, this is trivial. Let us assume this is true for $d-1$. It is now enough to show that $[J^d : J_+^d] = \frac{[J^1\dots J^d : J_+^1\dots J_+^d]}{[J^1\dots J^{d-1} : J_+^1\dots J_+^{d-1}]}$.

The following fact will be useful.

Fact: Let $G' \subset G$ be groups and let H be a normal subgroup of G. Let ι be the injective morphism of group $G'/(G'\cap H) \hookrightarrow G/H$. As G-set, G/HG' and $(G/H)/\iota(G'/(G'\cap H))$ are isomorphic.

Since $J_+^1\dots J_+^d$ is a normal subgroup of $J^1\dots J^d$, we can apply the previous fact to $G = J^1\dots J^d$, $G' = J^1\dots J^{d-1}$, $H = J_+^1\dots J_+^d$. Using the fact that $H\cap G' = J_+^1\dots J_+^{d-1}$, we deduce that, as $J^1\dots J^d$-sets, $J^1\dots J^d/J^1\dots J^{d-1}J_+^d$ and $(J^1\dots J^d/J_+^1\dots J_+^d)/\iota(J^1\dots J^{d-1}/J_+^1\dots J_+^{d-1})$ are isomorphic. Let X be this $J^1\dots J^d$-set. The set X is a fortiori a J^d-set. The group J^d acts transitively on $X = J^1\dots J^d/J^1\dots J^{d-1}J_+^d$, and the stabilizer of $(J^1\dots J^{d-1}J_+^d) \in J^1\dots J^d/J^1\dots J^{d-1}J_+^d$ is $J^1\dots J^{d-1}J_+^d \cap J^d$. The group $J^1\dots J^{d-1}J_+^d\cap J^d$ is equal to J_+^d. Consequently

$$[J^d : J_+^d] = \#(X) = \frac{[J^1\dots J^d : J_+^1\dots J_+^d]}{[J^1\dots J^{d-1} : J_+^1\dots J_+^{d-1}]},$$

as required. This ends the proof of the proposition.

□

The following theorem is the outcome of Sections 1.7 and 1.8. It shows that given a Bushnell-Kutzko datum, there exists a Yu datum $(\overrightarrow{G}, y, r, \overrightarrow{\boldsymbol{\Phi}}, \rho)$, such that $\Lambda = \rho_d(\overrightarrow{G}, y, r, \overrightarrow{\boldsymbol{\Phi}}, \rho)$. The objects $(\overrightarrow{G}, y, r, \overrightarrow{\boldsymbol{\Phi}}, \rho)$ are given explicitely in terms of the Bushnell-Kutzko datum.

Theorem 1.9.3. *Let V be a N-dimensional F-vector space. Let A denote $\mathrm{End}_F(V)$ and let G denote $A^\times \simeq GL_N(F)$. The following assertions hold.*

(I) Let $([\mathfrak{A}, n, r, \beta], \theta, \sigma, \kappa, \Lambda)$ be a tame Bushnell-Kutzko datum of type (a) in A. Let $\{[\mathfrak{A}, n, r, \beta_i],\ 0 \leq i \leq s\}$ be a defining sequence such that $F[\beta_i] \subsetneq F[\beta_{i+1}]$ for $0 \leq i \leq s-1$.

- *(Case A) If β_s is in F, put $d = s$, and $G^i = \mathrm{Res}_{F[\beta_i]/F}\underline{\mathrm{Aut}}_{F[\beta_i]}(V)$ for $0 \leq i \leq s$. Put $\overrightarrow{G} = (G^0, \ldots, G^s)$. Choose a factorization $\theta = \prod_{i=0}^{s} \theta^i$ as in Theorem 1.7.1, Corollary 1.7.3. Let $\mathbf{\Phi}_i$, $0 \leq i \leq s$, be the associated characters as in Definition 1.8.4. Put $\overrightarrow{\mathbf{\Phi}} = (\mathbf{\Phi}_0, \ldots, \mathbf{\Phi}_s)$. Let $y \in \mathrm{BT}^E(G^0, F)$ and $\overrightarrow{\mathbf{r}}$ as in Proposition 1.8.2. Then, there exists a representation ρ of $G^0_{[y]}$ such that $(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho)$ is a Yu datum and $\rho_d(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho)$ is isomorphic to Λ (see section 1.3).*
- *(Case B) If $\beta_s \notin F$, put $d = s+1$, and $G^i = \mathrm{Res}_{F[\beta_i]/F}\underline{\mathrm{Aut}}_{F[\beta_i]}(V)$ for $0 \leq i \leq s$. Put also $G^d = \underline{\mathrm{Aut}}_F(V)$. Put $\overrightarrow{G} = (G^0, \ldots, G^s, G^d)$. Choose a factorization $\theta = \prod_{i=0}^{s} \theta^i$ as in 1.7.1, 1.7.6. Let $\mathbf{\Phi}_i$, $0 \leq i \leq s$ be the associated characters and let $\mathbf{\Phi}_d$ be the trivial character as in 1.8.9. Put $\overrightarrow{\mathbf{\Phi}} = (\mathbf{\Phi}_0, \ldots, \mathbf{\Phi}_s, \mathbf{\Phi}_d)$. Let $y \in BT^e(G^0, F)$ and $\overrightarrow{\mathbf{r}}$ as in Proposition 1.8.7. Then there exists a representation ρ of $G^0_{[y]}$ such that $(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho)$ is a Yu datum and $\rho_d(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho)$ is isomorphic to Λ (see section 1.3).*

(II) Let $(\mathfrak{A}, \sigma, \Lambda)$ be a Bushnell-Kutzko datum of type (b). Put $d = 0$, $G^0 = \underline{\mathrm{Aut}}_F(V)$ and $\overrightarrow{G} = (G^0)$. Put $\mathbf{r}_0 = 0$ and $\overrightarrow{\mathbf{r}} = (\mathbf{r}_0)$. Let $y \in BT^e(G^0, F)$ such that $\mathfrak{A}^\times = G^0(F)_y$. Put $\mathbf{\Phi}_0 = 1$ and $\overrightarrow{\mathbf{\Phi}} = (\mathbf{\Phi}_0)$. Let ρ be Λ. Then $(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho)$ is a Yu datum and $\rho_d(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho)$ is isomorphic to Λ.

Proof. (I) As usual, put $E = F[\beta]$. Let ρ' be an arbitrary extension of σ to $G^0(F)_{[y]}$. Then the compact induction of ρ' to $G^0(F)$ is irreducible and supercuspidal and so $(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho')$ is a Yu datum. We are going to show that there exists a character χ of $G^0(F)_{[y]}$ such that $(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho' \otimes \chi)$ is a Yu datum such that $\rho_d(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho \otimes \chi)$ is isomorphic to Λ.

The representation ${}^\circ\lambda(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}})$ is a β-extension of θ by Proposition 1.9.2. Consequently, by 1.2.22, there exists a character

$$\xi' : U^0(\mathfrak{B}_{\beta_0})/U^1(\mathfrak{B}_{\beta_1}) \simeq J^0(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \to \mathbb{C}^\times$$

of the form $\alpha' \circ \det$ with $\alpha' : U^0(\mathfrak{o}_E)/U^1(\mathfrak{o}_E) \to \mathbb{C}^\times$ and such that κ is isomorphic to ${}^\circ\lambda \otimes \xi'$. Let χ' be an extension of ξ' to $E^\times U^0(\mathfrak{B}_{\beta_0}) =$

$G^0(F)_{[y]}$. The compact induction of $\rho'\otimes\chi'$ to $G^0(F)$ is irreducible and supercuspidal and so $(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho'\otimes\chi')$ is a Yu datum. The representation ${}^\circ\rho_d(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho'\otimes\chi')$ is equal to $\sigma\otimes\xi'\otimes{}^\circ\lambda(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}})$. Thus it is isomorphic to $\sigma\otimes\kappa$. Consequently $\rho_d(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho'\otimes\chi')$ and Λ are two extensions of $\sigma\otimes\kappa$. This implies that there exists a character

$$\chi'' : E^\times J^0(\beta,\mathfrak{A}) \to E^\times J^0(\beta,\mathfrak{A})/J^0(\beta,\mathfrak{A}) \simeq G^0(F)_{[y]}/G^0(F)_y \to \mathbb{C}^\times,$$

such that $\rho_d(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho'\otimes\chi')\otimes\chi''$ is isomorphic to Λ. Seeing χ'' as a character of $G^0(F)_{[y]}$, the compact induction of the representation $\rho'\otimes\chi'\otimes\chi''$ to $G^0(F)$ is irreducible and supercuspidal, and $\rho_d(\overrightarrow{G}, y, \overrightarrow{r}, \overrightarrow{\mathbf{\Phi}}, \rho'\otimes\chi')\otimes\chi''$ is isomorphic to λ.

The assertion (I) follows putting $\rho = \rho'\otimes\chi'\otimes\chi''$.

(II) In this case the representation ρ_d is ρ, and there is nothing to prove. $\square$

Chapter 2

Analytic filtrations

In this chapter, as announced in the introduction, we define some k-analytic filtrations using Berkovich spaces theory. Taking rational points, we obtain filtrations comparable to Moy-Prasad filtrations.

Notations and conventions for chapter 2

$$\begin{aligned}
p &: \text{a prime number} \\
k/\mathbb{Q}_p &: \text{a finite extension} \\
\pi_k &: \text{a uniformizer of } k \\
\mathrm{ord} = \mathrm{ord}_k &: \text{valuation on algebraic extensions of } k \text{ such that } \mathrm{ord}(\pi_k) = 1 \\
e > 1 &: \ \text{a real number strictly bigger than 1} \\
|\bullet| &= e^{-\mathrm{ord}(\bullet)} \quad (\text{norm on } \ \overline{k}) \\
k^\circ &= \{x \in k \mid \quad |x| \leq 1 \quad \} = \mathfrak{o}_k \\
k^{\circ\circ} &= \{x \in k \mid \quad |x| < 1 \quad \} = \mathfrak{p}_k \\
\tilde{k} &= k^\circ / k^{\circ\circ} \text{ residual field} \\
G &: \text{connected reductive } k\text{-group scheme} \\
\mathrm{BT}^R(G,k) &: \text{reduced Bruhat-Tits building} \\
\mathrm{BT}^E(G,k) &: \text{enlarged Bruhat-Tits building} \\
G^{an} &: \text{analytification of } G \quad (\text{Berkovich } k\text{-analytic space})
\end{aligned}$$

If H is an S-group scheme with $S = \mathrm{spec}(k)$ or $S = \mathrm{spec}(k^\circ)$, then $\mathrm{Lie}(H)$ denote the Lie algebra functor (it was denoted $\underline{\mathrm{Lie}}(H)$ in chapter 1.

2.1 Schemes

2.1.1 Generalities

There is a stripping functor **Sch** $\to$ **Top** associating to a scheme its underlying topological space, a scheme X is called connected or irreducible if and only if its underlying topological space has the same property. If S is a scheme we note $S-$ **Sch** the category of scheme over S, this is a category whose objects are pairs (X, f) where X is a scheme and $f: X \to S$ is a morphism of scheme. There is a stripping functor $S-$ **Sch** $\to$ **Sch**. If $S = \mathrm{spec}(A)$ is affine we sometimes call a S-scheme a A-scheme, and write $A-$ **Sch** instead of $S-$ **Sch**. If S is a scheme and X, Y are two schemes over S we note $X \times_S Y$ the product of X and Y in the category $S-$ **Sch**, if moreover $S = \mathrm{spec}(A)$ is affine, we sometimes denote $X \times_S Y$ by $X \times_A Y$, and if $Y = \mathrm{spec}(B)$ is also affine we denote $X \times_S Y$ by $X \times_A B$.

Let B be an A-algebra B. Let X be an A-scheme. Then $Aut_{A-alg}(B)$ acts canonically on the right of $X \times_{\mathrm{spec}(A)} \mathrm{spec}(B)$ by A-scheme automorphisms.

A group scheme is a group **Sch**-objet, the connected component containing the unit element is a group scheme called the neutral component. A S-group scheme is a group $S-$ **Sch**-objet.

Proposition 2.1.1. *[40, Theorem 6.6] Let* k *be a field and let* $G = \mathrm{spec}(A)$ *be an affine* k*-group scheme such that A is a finitely generated algebra over k, the following are equivalent:*

1. $\mathrm{spec}(A)$ *is connected*
2. $\mathrm{spec}(A)$ *is irreducible.*

If $f: X \to S$ is an S-scheme, and $s \in S$ is a point, let $k(s)$ be the residue field of s and $\mathrm{spec}(k(s)) \to S$ the canonical morphism. The fibre of the morphism f over the point s is the scheme $X_s = X \times_S \mathrm{spec}(k(s))$.

Recall that k denote a finite extension of $\mathbb{Q}_p$. The scheme $\mathrm{spec}(k^\circ)$ is reduced to two points, the first is the prime ideal 0 and the second is the maximal ideal $k^{\circ\circ}$. Let $\mathfrak{X}$ be a k°-scheme, the fibre over 0 is called the generic fibre and the fibre over $k^{\circ\circ}$ is called the special fibre. Explicitely they are given by $\mathfrak{X} \times_{k^\circ} k$ and $\mathfrak{X} \times_{k^\circ} \tilde{k}$. We say that a k°-scheme $\mathfrak{X}$ is connected if its special and generic fibres are connected, as elements of **Sch**. A non connected k°-scheme $\mathfrak{X}$ can have a underlying connected scheme. A connected k°-scheme always have a underlying connected scheme. If $\mathfrak{G}$ is a k°-group scheme, we define the neutral component of the k°-group scheme $\mathfrak{G}$ as the images of the neutral components, as group scheme, of the special and generic fibres, under the natural morphism to $\mathfrak{G}$. The neutral component of a k°-group scheme $\mathfrak{G}$ is denoted by $\mathfrak{G}^\circ$. We have the following result which is useful to have in mind in this text (see [22] for a general statements and proofs)

Proposition 2.1.2. *Let $\mathfrak{X} = \mathrm{spec}(\mathfrak{A})$ be a smooth k°-scheme. Then*

1. *$\mathfrak{X}$ is a flat k°-scheme*
2. *The algebra $\mathfrak{A} \otimes_{k^\circ} \tilde{k}$ is a reduced $\tilde{k}$-algebra.*

2.1.2 Higher dilatations and congruence subgroups

In this section we recall some results about dilatations, higher dilatations, and congruence subgroups for schemes and group schemes over k° where k is a finite extension of $\mathbb{Q}_p$. The references are [6], [43] and [32]. The dilatation is a process which produces, from a flat k°-scheme $\mathfrak{X}$ of finite type and a closed subscheme of the special fiber of $\mathfrak{X}$, a flat closed k°-subscheme of $\mathfrak{X}$. It preserves group schemes structures. Higher dilatation is an iteration of dilatations. It preserves group schemes structures. A congruence subgroup in this setting is obtained by higher dilatation of a k°-group scheme relatively to the neutral element. We start by the definition of dilatation following [6].

Definition/Proposition 2.1.3. *[6, §3.2] Let $\mathfrak{X}$ be a flat k°-scheme of finite type, let $\mathfrak{Y}_{\tilde{k}}$ be a closed subscheme of the special fibre $\mathfrak{X} \times_{k^\circ} \tilde{k}$ of $\mathfrak{X}$, let $\mathcal{J}$ be the sheaf of ideals of $\mathcal{O}_{\mathfrak{X}}$ defining $\mathfrak{Y}_{\tilde{k}}$. Let $\mathfrak{X}' \to \mathfrak{X}$ be the blowing-up of $\mathfrak{Y}_{\tilde{k}}$ on $\mathfrak{X}$, and let $u : \mathfrak{X}'_\pi \to \mathfrak{X}$ denote its restriction to the open subscheme of $\mathfrak{X}'$ where $\mathcal{J}.\mathcal{O}_{\mathfrak{X}}$ is generated by π. Then:*

(a) $\mathfrak{X}'_\pi$ is a flat k°-scheme, and $u_{\tilde{k}} : \mathfrak{X}'_\pi \times_{k^\circ} \tilde{k} \to \mathfrak{X} \times_{k^\circ} \tilde{k}$ factors throught $\mathfrak{Y}_{\tilde{k}}$.

(b) For any flat k°-scheme $\mathfrak{Z}$ and for any k°-morphism $v : \mathfrak{Z} \to \mathfrak{X}$ such that $v_{\tilde{k}} : \mathfrak{Z} \times_{k^\circ} \tilde{k} \to \mathfrak{X} \times_{k^\circ} \tilde{k}$ factor through $\mathfrak{Y}_{\tilde{k}}$, there exists a unique k°-morphism $v' : \mathfrak{Z} \to \mathfrak{X}'_\pi$ such that $v = u \circ v'$.

Moreover $(\mathfrak{X}'_\pi, u)$ is the only couple satisfying (a) and (b) up to canonical isomorphism, we denote it by $Dil(\mathfrak{X}, \mathfrak{Y}_{\tilde{k}})$. If $\mathfrak{Y}_{\tilde{k}}$ is realized as the special fiber $\mathfrak{Y} \times_{k^\circ} \tilde{k}$ of a closed subscheme $\mathfrak{Y}$ of $\mathfrak{X}$, then we also denote $Dil(\mathfrak{X}, \mathfrak{Y} \times_{k^\circ} \tilde{k})$ by $Dil(\mathfrak{X}, \mathfrak{Y})$.

Remark 2.1.4. *Let $\mathfrak{X}$ be a flat k°-scheme of finite type, then $Dil(\mathfrak{X}, \mathfrak{X}) = \mathfrak{X}$ since it satisfies (a) and (b).*

The following functorial compatibility property holds.

Proposition 2.1.5. *[6, §3.2 Proposition 2 (c)] Let $\mathfrak{X}_2$ be a closed subscheme of a flat k°-scheme of finite type $\mathfrak{X}_1$ and let $\mathfrak{Y}_{\tilde{k}}$ be a closed subscheme of the special fibre $\mathfrak{X}_2 \times_{k^\circ} \tilde{k}$. Then there is a natural closed immersion $Dil(\mathfrak{X}_2, \mathfrak{Y}_{\tilde{k}}) \to Dil(\mathfrak{X}_1, \mathfrak{Y}_{\tilde{k}})$.*

Dilatation preserves products and group structures as follows.

Proposition 2.1.6. *[6, §3.1 Proposition 2 (d)] Let $\mathfrak{X}^i$ be flat k°-schemes of finite type and let $\mathfrak{Y}^i_{\tilde{k}}$ be closed subschemes of $\mathfrak{X}^i \times_{k^\circ} \tilde{k}$, for $i = 1, 2$. There is a canonical isomorphism of k°-schemes*

$$Dil(\mathfrak{X}^1 \times_{k^\circ} \mathfrak{X}^2, \mathfrak{Y}^1_{\tilde{k}} \times_{\tilde{k}} \mathfrak{Y}^2_{\tilde{k}}) \simeq Dil(\mathfrak{X}^1, \mathfrak{Y}^1_{\tilde{k}}) \times_{k^\circ} Dil(\mathfrak{X}^2, \mathfrak{Y}^2_{\tilde{k}}).$$

In particular, if $\mathfrak{X}$ is a k°-group scheme, and if $\mathfrak{Y}_{\tilde{k}}$ is a subgroup scheme of $\mathfrak{X} \times_{k^\circ} \tilde{k}$, then $Dil(\mathfrak{X}, \mathfrak{Y}_{\tilde{k}})$ is a k°-group scheme and the canonical map $Dil(\mathfrak{X}, \mathfrak{Y}_{\tilde{k}}) \to \mathfrak{X}$ is a k°-group scheme morphism.

We now introduce the J.-K. Yu and G. Prasad notion of higher dilatation.

Definition 2.1.7. *[32, §7.2] Let $\mathfrak{X}$ be a flat k°-scheme of finite, and $i_0 : \mathfrak{Y} \to \mathfrak{X}$ be a flat closed k°-subscheme. Let us define by induction a sequence of flat k°-scheme $\Gamma_n(\mathfrak{X}, \mathfrak{Y})$ together with closed immersion $i_n : \mathfrak{Y} \to \Gamma_n(\mathfrak{X}, \mathfrak{Y})$. Let $\Gamma_0(\mathfrak{X}, \mathfrak{Y}) = \mathfrak{X}$ and $i_0 : \mathfrak{Y} \to \mathfrak{X} = \Gamma_0(\mathfrak{X}, \mathfrak{Y})$. After $\Gamma_n(\mathfrak{X}, \mathfrak{Y})$ and i_n have been defined, we let $\Gamma_{n+1}(\mathfrak{X}, \mathfrak{Y})$ be $Dil(\Gamma_n(\mathfrak{X}, \mathfrak{Y}), i_n(\mathfrak{Y}))$. Thanks to 2.1.5 we have a closed immersion*

$$i_{n+1} : \mathfrak{Y} = Dil(\mathfrak{Y}, \mathfrak{Y}) \to Dil(\Gamma_n(\mathfrak{X}, \mathfrak{Y}), i_n(\mathfrak{Y})) = \Gamma_{n+1}(\mathfrak{X}, \mathfrak{Y}).$$

Remark 2.1.8. *With the same notations as 2.1.7, the generic fibres of $\mathfrak{X}$ and $\Gamma_n(\mathfrak{X}, \mathfrak{Y})$ are canonically isomorphic.*

Construction of higher dilatations and preservation of groups structure for dilatations imply that higher dilatations preserve groups structure as follows.

Proposition 2.1.9. *[32, §7.4] With the same notations as 2.1.7, suppose $\mathfrak{X}$ is a k°-group scheme and $\mathfrak{Y}$ a closed k°-group scheme. Then $\Gamma_n(\mathfrak{X}, \mathfrak{Y})$ is naturally a k°-group scheme.*

We now give an explicit description of higher dilatations in the affine case. It will be important for us.

Proposition 2.1.10. *[32, Proof of Proposition 7.3] Let $\mathfrak{X}$ be an affine and flat k°-scheme of finite type, and $\mathfrak{Y}$ be a closed k°-subscheme of $\mathfrak{X}$. Let $\mathfrak{A}$ and J such that $\mathfrak{X} = \mathrm{spec}(\mathfrak{A})$ and $\mathfrak{Y} = \mathrm{spec}(\mathfrak{A}/J)$. Then $\Gamma_n(\mathfrak{X}, \mathfrak{Y}) = \mathrm{spec}(\mathfrak{A}_n)$ where*

$$\mathfrak{A}_n = \mathfrak{A}[\pi_k^{-n} J] = \mathfrak{A} + \sum_{i \geq 1} \pi_k^{-in} J^i \subset \mathfrak{A} \otimes_{k^\circ} k$$

We now introduce the notion of congruence subgroups.

Definition 2.1.11. *Let $\mathfrak{G}$ be a flat k°-group scheme of finite type and $e_{\mathfrak{G}}$ be the neutral element, this is a closed k°-group scheme in $\mathfrak{G}$. Then $\Gamma_n(\mathfrak{G}, e_{\mathfrak{G}})$ is called the n-th congruence subgroup of $\mathfrak{G}$, and is denoted by $\Gamma_n(\mathfrak{G})$, this is a flat k°-group scheme together with a closed immersion $\Gamma_n(\mathfrak{G}) \to \mathfrak{G}$.*

Let $X = \mathrm{spec}(A)$ be an affine k-scheme of finite type. Let K/k be a finite Galois extension. Let $\mathfrak{X} = \mathrm{spec}(\mathfrak{A})$ be an affine flat K°-scheme of finite type such that $\mathfrak{X} \times_{K^\circ} K = X \times_k K$. We thus have $\mathfrak{A} \otimes_{K^\circ} K = A \otimes_k K$. The action by k-scheme automorphism on the right of $X \times_k K$ corresponds to a left action by k-algebras automorphisms on $A \otimes_k K$. In this situation, we say that $\mathfrak{X}$ is $\mathrm{Gal}(K/k)$-stable if $\mathfrak{A} \otimes_{K^\circ} 1$ is $\mathrm{Gal}(K/k)$-stable in $\mathfrak{A} \otimes_{K^\circ} K = A \otimes_k K$.

In order to prove preservations of Galois stabilities under the operations of taking congruence subgroups, we need the following lemma.

Lemma 2.1.12. *Let K/k be a finite Galois field extension. Let A be a k-algebra and $A_K = A \otimes_k K$. The action of the Galois group $\mathrm{Gal}(K/k)$ on A_K is given by $\gamma.(a \otimes x) = a \otimes \gamma(x)$ ($\gamma \in \mathrm{Gal}(K/k)$, $a \in A$, $x \in K$). Let $\mathfrak{A}$ be a K°-sub-algebra of A_K and assume $\mathfrak{A} \otimes_{K^\circ} K \to A_K$, $a \otimes x \mapsto ax$ is an isomorphism and identify these rings. Let J be an ideal of $\mathfrak{A}$. Assume $\mathfrak{A}$ and J are $\mathrm{Gal}(K/k)$-stable, then for all positive integer n, the algebra $\mathfrak{A}_n = \mathfrak{A}[\pi_K^{-n} J] = \mathfrak{A} + \sum_{i \geq 0} \pi_K^{-in} J^i \subset A_K$ is $\mathrm{Gal}(K/k)$-stable (π_K is a uniformizer of K).*

Proof. Put $\pi = \pi_K$. The valuation on K is invariant under the action of $\mathrm{Gal}(K/k)$ on K, and so each $\pi^{-in} J^i$ is $\mathrm{Gal}(K/k)$-stable. Indeed, let $y \in \pi^{-in} J^i$, then there is an element $j \in J^i$ such that $y = \pi^{-in} j$, let $\gamma \in \mathrm{Gal}(K/k)$, then $\gamma(\pi^{-in}) = o \times \pi^{-in}$ with o an element of valuation zero in K°. Evidently, $\gamma(j) \in J^i$ because J is $\mathrm{Gal}(K/k)$-stable by hypothesis, so $o \times \gamma(j) \in J^i$ because J^i is an ideal in the ring $\mathfrak{A}$ and $o \in \mathfrak{A}$. So $\gamma(y) \in \pi^{-in} J^i$. Moreover $\mathfrak{A}$ is $\mathrm{Gal}(K/k)$-stable by hypothesis, so $\mathfrak{A}_n$ is $\mathrm{Gal}(K/k)$-stable. □

Lemma 2.1.13. *Let K/k be a finite Galois extension. Let A be a Hopf algebra over k. In particular we have the augmentation $\varepsilon_A : A \to k$. Let $A_K = A \otimes_k K$, it is naturally a Hopf algebra, the augmentation ε_{A_K} is $\varepsilon_A \otimes Id$. Then $\ker(\varepsilon_{A_K}) = \ker(\varepsilon_A) \otimes_k K$ and it is $\mathrm{Gal}(K/k)$-stable in A_K.*

Proof. We have an exact sequence $0 \to \ker(\varepsilon_A) \to A \to k \to 0$, and so, because of K is flat over k, $0 \to \ker(\varepsilon_A) \otimes_k K \to A \otimes_k K \to k \otimes K \to 0$, and so $\ker(\varepsilon_{A_K}) = \ker(\varepsilon_A) \otimes_k K$. The last assertion follows from it. □

Lemma 2.1.14. *Let $G = \mathrm{spec}(A)$ be an affine k-group scheme of finite type and $\mathfrak{A}$ be a flat sub-Hopf K°-algebra of finite type of the Hopf K-algebra $A_K = A \otimes_k K$, put $\mathfrak{G} = \mathrm{spec}(\mathfrak{A})$ and assume that*

1. *$\mathfrak{A} \otimes_{K^\circ} K \underset{x \otimes \lambda \mapsto \lambda x}{\to} A_K$ is an isomorphism,*
2. *$\mathfrak{A} \otimes 1$ is $\mathrm{Gal}(K/k)$-stable in A_K.*

Then for any positive integer n, the congruence subgroup $\Gamma_n(\mathfrak{G}) = \mathrm{spec}(\mathfrak{A}_n)$ is $\mathrm{Gal}(K/k)$-stable.

Proof. Let $\varepsilon_{\mathfrak{A}} : \mathfrak{A} \to K^\circ$ be the augmentation, then by lemma 2.1.10 $\mathfrak{A}_n = \mathfrak{A} + \sum_{i\geq 1} \pi_K^{-in} J^i$ where $J = \ker(\varepsilon_{\mathfrak{A}})$. Let's remark that $\varepsilon_{\mathfrak{A}}$ is the restriction to $\mathfrak{A}$ of the augmentation $\varepsilon_A \otimes Id : A \otimes_k K \to K$ of A_K. So $J = \ker(\varepsilon_A \otimes Id) \cap \mathfrak{A}$. The set $\ker(\varepsilon_A \otimes Id)$ is $\mathrm{Gal}(K/k)$-stable thank to proposition 2.1.13, and $\mathfrak{A}$ is stable by hypothesis, so J is $\mathrm{Gal}(K/k)$-stable as the intersection of two $\mathrm{Gal}(K/k)$-stable subsets of $A \otimes_k K$, the proposition now follows from lemma 2.1.12.

□

We also have a compatibility between extension of scalars and taking congruence subgroups (up to ramification index).

Lemma 2.1.15. *Let K/k be a finite extension, let $e(K,k)$ be the ramification index, let π_k be a uniformizer of k and π_K be a uniformizer of K. Let $\mathfrak{G} = \mathrm{spec}(\mathfrak{A})$ be an affine flat k°-group scheme of finite type. Because of the flatness hypothesis, $\mathfrak{A}$ embeds in $\mathfrak{A} \otimes_{k^\circ} k$ and we identify $\mathfrak{A}$ with $\mathfrak{A} \otimes 1$. We also have an embedding $\mathfrak{A} \otimes_{k^\circ} k \to (\mathfrak{A} \otimes_{k^\circ} k) \otimes_k K$, we identify $\mathfrak{A} \otimes_{k^\circ} k$ with $(\mathfrak{A} \otimes_{k^\circ} k) \otimes 1$. Then the Hopf algebras of $\Gamma_n(\mathfrak{G}) \times_{k^\circ} K^\circ$ and $\Gamma_{e(K,k)n}(\mathfrak{G} \times_{k^\circ} K^\circ)$ are egal in $(\mathfrak{A} \otimes_{k^\circ} k) \otimes_k K$.*

Proof. Let $\varepsilon_{\mathfrak{A}} : \mathfrak{A} \to k^\circ$ be the augmentation and $J = \ker(\varepsilon_{\mathfrak{A}})$. Let $\mathfrak{A}_n$ be the Hopf k°-algebra such that $\Gamma_n(\mathfrak{G}) = \mathrm{spec}(\mathfrak{A}_n)$, then $\mathfrak{A}_n = \mathfrak{A} + \displaystyle\sum_{i\geq 1} \pi_k^{-in} J^i$ by 2.1.10. So, we have

$$\mathrm{Hopf}(\Gamma_n(\mathfrak{G}) \times_{k^\circ} K^\circ) = \mathfrak{A}_n \otimes_{k^\circ} K^\circ = \mathfrak{A} \otimes_{k^\circ} K^\circ + \left(\sum_{i\geq 1} \pi_k^{-in} J^i \right) \otimes_{k^\circ} K^\circ. \quad (2.1)$$

Let $\varepsilon_{\mathfrak{A}} \otimes Id : \mathfrak{A} \otimes_{k^\circ} K^\circ \to K^\circ$ be the augmentation of $\mathfrak{A} \otimes_{k^\circ} K^\circ$ and $J_{K^\circ} = \ker(\mathfrak{A} \otimes_{k^\circ} K^\circ)$. By 2.1.10, the Hopf K°-algebra of $\Gamma_{e(K,k)n}(\mathfrak{G} \times_{k^\circ} K^\circ)$ is

$$\mathrm{Hopf}(\Gamma_{e(K,k)n}(\mathfrak{G} \times_{k^\circ} K^\circ)) = \mathfrak{A} \otimes_{k^\circ} K^\circ + \sum_{i\geq 1} \pi_K^{-e(K,k)in} (J_{K^\circ})^i. \quad (2.2)$$

Because of K° is flat over k°, $J_{K^\circ} = J \otimes_{k^\circ} K^\circ$ (see the proof of Lemma 2.1.13).

We claim and remark that if A is a k°-algebra, J is an ideal of A, and n is a positive integer, then J $\otimes_{k^\circ} K^\circ$ is an ideal of $A \otimes_{k^\circ} K^\circ$ and we have the following equality of ideal $\mathrm{J}^n \otimes_{k^\circ} K^\circ = (\mathrm{J} \otimes_{k^\circ} K^\circ)^n$ in A $\otimes_{k^\circ} K^\circ$ (we report the proof of this claim after deducing the required result).

So finally we can deduce easily the equality

$$
\begin{aligned}
\mathrm{Hopf}(\Gamma_n(\mathfrak{G}) \times_{k^\circ} K^\circ) &= \\
\text{By equation (2.1)} &= \mathfrak{A} \otimes_{k^\circ} K^\circ + \left(\sum_{i\geq 1} \pi_k^{-in} J^i\right) \otimes_{k^\circ} K^\circ \\
\text{By properties of sum and tensor product} &= \mathfrak{A} \otimes_{k^\circ} K^\circ + \sum_{i\geq 1} \left(\left(\pi_k^{-in} J^i\right) \otimes_{k^\circ} K^\circ\right) \\
&= \mathfrak{A} \otimes_{k^\circ} K^\circ + \sum_{i\geq 1} \pi_k^{-in} \left(J^i \otimes_{k^\circ} K^\circ\right) \\
\text{By the claim} &= \mathfrak{A} \otimes_{k^\circ} K^\circ + \sum_{i\geq 1} \pi_k^{-in} \left(J \otimes_{k^\circ} K^\circ\right)^i \\
\pi_k K^\circ = \pi_K^{e(K,k)} K^\circ \text{ and see before the claim} &= \mathfrak{A} \otimes_{k^\circ} K^\circ + \sum_{i\geq 1} \pi_K^{-e(K,k)in} (J_{K^\circ})^i \\
\text{By equation (2.2)} &= \mathrm{Hopf}(\Gamma_{e(K,k)n}(\mathfrak{G} \times_{k^\circ} K^\circ))
\end{aligned}
$$

as required.

Let us now prove the claim i.e. that we have the egality of ideal $(\mathrm{J} \otimes_{k^\circ} K^\circ)^n = (\mathrm{J}^n \otimes_{k^\circ} K^\circ)$. Let us first prove the inclusion $\supset$. Since J^n consists in sums of n-products of elements in J and since $(\mathrm{J} \otimes_{k^\circ} K^\circ)^n$ is stable by addition, it is enough to show that any element of the form $x = (j_1 \dots j_n \otimes \lambda) \in (\mathrm{J}^n \otimes_{k^\circ} K^\circ)$ is contained in $(\mathrm{J} \otimes_{k^\circ} K^\circ)^n$. This is obvious, writting $(j_1 \dots j_n \otimes \lambda) = (j_1 \otimes \lambda)(j_2 \otimes 1) \dots (j_n \otimes 1)$. Now let us prove the inclusion $\subset$. Since $(\mathrm{J} \otimes_{k^\circ} K^\circ)^n$ consists of sums of n-products of elements in $\mathrm{J} \otimes_{k^\circ} K^\circ$ and since $(\mathrm{J} \otimes_{k^\circ} K^\circ)$ consists in sums of pure tensors and since $(\mathrm{J}^n \otimes_{k^\circ} K^\circ)$ is stable by addition, it is enough to show that any element of the form $x = (j_1 \otimes \lambda_1) \dots (j_n \otimes \lambda_n) \in (\mathrm{J} \otimes_{k^\circ} K^\circ)^n$ is contained in $(\mathrm{J}^n \otimes_{k^\circ} K^\circ)$. This is obvious, writting $(j_1 \otimes \lambda_1) \dots (j_n \otimes \lambda_n) = (j_1 \dots j_n \otimes \lambda_1 \dots \lambda_n)$. This ends the proof of the claim and so the lemma is proved.

$\square$

We finish this section with important facts about congruence subgroups.

Proposition 2.1.16. *[43, 2.8] Let $\mathfrak{G}$ be a smooth (thus flat by 2.1.2) k°-group scheme affine and of finite type. Let $n \in \mathbb{N}$, then*

1. *$\Gamma_n(\mathfrak{G})(k^\circ) = \ker(\mathfrak{G}(k^\circ) \to \mathfrak{G}(k^\circ/\pi^n k^\circ))$*
2. *The special fibre of $\Gamma_n(\mathfrak{G})$ is a vector $\tilde{k}$-group scheme. In particular it is connected and irreducible. Moreover since $\Gamma_n(\mathfrak{G})$ is smooth over k°, if $\mathfrak{A}_n$ denotes the k°-Hopf algebra of $\Gamma_n(\mathfrak{G})$, then $\mathfrak{A}_n \otimes_{k^\circ} \tilde{k}$ is reduced (by 2.1.2).*

If $\mathfrak{G}$ is a k°-group scheme, we denote by $\mathrm{Lie}(\mathfrak{G})$ its Lie algebra functor, it is a k°-scheme. We denote by $\mathrm{Lie}(\mathfrak{G})(k^\circ)$ the k°-points.

2.2 Berkovich k-analytic spaces

In this section we recall Berkovich's definitions of k-affinoid algebras and spaces. We follow very closely [3] and most parts of this section are copies of [3]. The reader is welcome to read at the same time [3]. We then give references for definitions of general Berkovich spaces. A general Berkovich analytic space is a locally ringed space obtained by gluing k-affinoid spaces having certain compatibility conditions. The notion of Berkovich k-analytic spaces exists for a larger class of field k than extension of $\mathbb{Q}_p$ (see [3]). The reference for the definition of general Berkovich analytic spaces is [4, §1]. The spaces defined in [3] correspond to good spaces in [4] (see [18, 1.3]). In general, Berkovich k-analytic space are equiped with a Grothendieck topology (see [4, §1.3]). V. Berkovich's k-affinoid theory relies on S. Bosch, U. Güntzer and R. Remmert's book "Non archimedean Analysis" [5]. I. Gelfand, D. Raikov and G. Shilov's book "Commutative normed rings" [21] seems to has fournished important ideas in the Berkovich's approach. For a more complete historical approach of Berkovich's space, we refer the reader to the introduction of Berkovich's book [3]. The litterature on Berkovich's space is wide and applications are abundant. A list of some applications can be fund in [18] and [19].

2.2.1 k-affinoid algebras

We refer to [3, §1.1] for usual definitions concerning Banach rings, we freely use the following notions:

• non-Archimedean seminorms and norms on an abelian group, equivalence of seminorms, residue seminorms, bounded and admissible morphims of seminormed groups,

• seminormed rings, normed rings, Banach rings, non-Archimedean fields,

• seminormed $\mathcal{A}$-modules, normed $\mathcal{A}$-modules, Banach $\mathcal{A}$-modules, complete tensor products $M\hat{\otimes}N$.

Definition 2.2.1. *[3, §2.1] For real numbers $r_1,\ldots,r_n>0$, we set:*

$$k\{r_1^{-1}T_1,\ldots,r_n^{-1}T_n\}=\{f=\sum_{\nu\in\mathbb{N}^n}a_\nu T^\nu \mid a_\nu\in k \text{ and } |a_\nu|r^\nu\to 0 \text{ as } |\nu|\to\infty\}$$

(Here $\nu=(\nu_1,\ldots,\nu_n)$, $|\nu|=\nu_1+\ldots+\nu_n, T^\nu=T_1^{\nu_1}\ldots T_n^{\nu_n}$ and $r^\nu=r_1^{\nu_1}\ldots r_n^{\nu_n}$). This is a commutative Banach k-algebra with respect to the multiplicative norm $||f||=\max\limits_{\nu}|a_\nu|r^\nu$. For brevity this algebra will also be denoted by $k\{r^{-1}T\}$.

• *A k-affinoid algebra is a commutative Banach k-algebra $\mathcal{A}$ such that there exists an admissible epimorphism $k\{r^{-1}T\}\to\mathcal{A}$. If such an epimorphism can be found with $r=1$, $\mathcal{A}$ is said to be strictly k-affinoid.*

• *An affinoid k-algebra is a K-affinoid algebra for some non archimedean field K over k.*

The following proposition characterizes strictly k-affinoid algebras among k-affinoid algebras.

Proposition 2.2.2. *[5, §6.1] Let $r = (r_1, \ldots, r_i, \ldots, r_n) > 0$. The k-affinoid algebra $k\{r^{-1}T\}$ is strictly k-affinoid if and only if, for all i*

$$r_i \in \{\alpha \in \mathbb{R}_{\geq 0} \mid \alpha^m \in |k^*| \textit{ for some integer } m \geq 1\}.$$

Proposition 2.2.3. *[3, 2.1.3] Let $\mathcal{A}$ be a k-affinoid algebra and I be an ideal of $\mathcal{A}$. Then*

1. *$\mathcal{A}$ is a Noetherian ring,*
2. *I is a closed ideal of $\mathcal{A}$.*

We refer to [3, §2.1]for many others interesting propositions on k-affinoid algebras.

2.2.2 k-affinoid spaces

In this section, we introduce the spectrum $\mathcal{M}(\mathcal{A})$ of a k-Banach algebra $\mathcal{A}$, it is a compact topological space. If $\mathcal{A}$ is a k-affinoid algebra, $\mathcal{M}(\mathcal{A})$ is called a k-affinoid space, it is provided with a locally ringed space structure.

Spectrum of a k-Banach algebra

We start with general definitions.

Definition 2.2.4. *[3, 1.2] Let $\mathcal{A}$ be a commutative Banach ring with identity. The spectrum $\mathcal{M}(\mathcal{A})$ is the set of all bounded multiplicative seminorms on $\mathcal{A}$ provided with the weakest topology with respect to which all real valued functions on $\mathcal{M}(\mathcal{A})$ of the form $| \ | \mapsto |f|$, $f \in \mathcal{A}$, are continous.*

Remark 2.2.5. *Let $\mathcal{A}$ be a commutative Banach ring with identity. An element in the "space" $\mathcal{M}(\mathcal{A})$ is generically denoted x, it is a map from $\mathcal{A}$ to $\mathbb{R}_{\geq 0}$. More precisely, the element x is a bounded multiplicative seminorm on $\mathcal{A}$ and we also denote x by $| \ |_x$. An element in $\mathcal{A}$ is genericaly denoted f. If $x \in \mathcal{M}(\mathcal{A})$ and $f \in \mathcal{A}$, the real number $x(f) = | \ |_x(f)$ is also denoted $|f|_x$.*

Proposition 2.2.6. *[3, 1.2.1] Let $\mathcal{A}$ be a non-zero commutative Banach ring with identity. The spectrum $\mathcal{M}(\mathcal{A})$ is a nonempty, compact Hausdorff space.*

Following Berkovich, let us introduce the valuation field associated to a point of $\mathcal{M}(\mathcal{A})$.

Definition 2.2.7. *[3, 1.2.2 (i)] Let $\mathcal{A}$ be a commutative Banach ring with identity. Let $x \in \mathcal{M}(\mathcal{A})$. The kernel $\mathbf{p}_x$ of $|\ |_x$ is a closed prime ideal of $\mathcal{A}$. The value $|f|_x$ depends only on the residue class of f in $\mathcal{A}/\mathbf{p}_x$. The resulting valuation on the integral domain $\mathcal{A}/\mathbf{p}_x$ extends to a valuation $|\ |_x$ on its fraction field F. The closure of F with respect to the valuation is a valuation field denoted by $\mathcal{H}(x)$. The image of an element $f \in \mathcal{A}$ in $\mathcal{H}(x)$ will be denoted by $f(x)$.*

Remark 2.2.8. *Let $\mathcal{A}$ be a commutative Banach ring with identity. Let $x \in \mathcal{M}(\mathcal{A})$. Remark that $|f(x)|_x = |f|_x$. Berkovich does not write the subscript x and therefore denote $|f(x)|_x$ by $|f(x)|$. Thus $x(f)$, $|\ |_x(f)$, $|f|_x$, $|f(x)|_x$ and $|f(x)|$ are well defined notations denoting the same real number (see 2.2.5). Berkovich's notation $|f(x)|$ seems to be the best notation to use and is the most used in the literature. In this text, we also use the notation $|f|_x$.*

The following is an other description of the spectrum $\mathcal{M}(\mathcal{A})$.

Fact 2.2.9. *[3, 1.2.2 (ii)] Let K' and K'' two valuation fields. Two nonzero bounded morphisms $\chi' : \mathcal{A} \to K'$ and $\chi'' : \mathcal{A} \to K''$ are said to be equivalent if there exist a valuation field K and a non zero bounded morphism $\chi : \mathcal{A} \to K$ and embeddings $K \to K'$ and $K \to K''$ such that the diagram*

$$\begin{tikzcd} & \mathcal{A} \arrow[dl] \arrow[d] \arrow[dr] & \\ K' & K \arrow[l] \arrow[r] & K'' \end{tikzcd}$$

is commutative. The set $\mathcal{M}(\mathcal{A})$ coincides with the set of equivalence classes of nonzero bounded morphism from $\mathcal{A}$ to a valuation field.

We have the following functorial fact.

Fact 2.2.10. *[3, 1.2.2 (iii)]Any bounded morphism of commutative Banach rings $\phi : \mathcal{A} \to \mathcal{B}$ sending the identity to the identity induces a continuous map $\phi^* : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$.*

Let us introduce the notion of spectral radius of an element f in a Banach ring $\mathcal{A}$.

Definition 2.2.11. *[3, 1.3] Let $\mathcal{A}$ be a Banach ring and let $f \in \mathcal{A}$. The numbers $\lim\limits_{n\to\infty} ||f^n||^{\frac{1}{n}}$ and $\inf\limits_{n} ||f^n||^{\frac{1}{n}}$ exist and are equal. This number is called the spectral radius of f and is denoted by $\rho(f)$.*

We have he following proposition.

Proposition 2.2.12. *[3, 1.3.3] Let $\mathcal{A}$ be a Banach ring. The function $f \mapsto \rho(f)$, from $\mathcal{A}$ to $\mathbb{R}_{\geq 0}$, is a bounded seminorm called the spectral norm.*

Let us finish this section with the following proposition.

Proposition 2.2.13. *[3, 1.3.5] Suppose that $\mathcal{A}$ is a commutative Banach algebra over a valuation field k, and let K be a finite Galois extension. The group* $\mathrm{Gal}(K/k)$ *naturally acts on the right of $\mathcal{M}(\mathcal{A} \otimes K)$. Moreover we have a bijection*

$$\mathcal{M}(\mathcal{A}\hat{\otimes}_k K)/\mathrm{Gal}(K/k) \simeq \mathcal{M}(\mathcal{A})$$

k-affinoid spaces

Affinoid domains We fix a k-affinoid algebra $\mathcal{A}$ and we put $X = \mathcal{M}(\mathcal{A})$.

Definition 2.2.14. *[3, 2.2.1] A closed subset $V \subset X$ is said to be an affinoid domain in X if there exists a bounded homomorphism of k-affinoid algebras $\phi : \mathcal{A} \to \mathcal{A}_V$ satisfying the following universal property. Given a bounded homomorphism of affinoid k-algebras $\mathcal{A} \to \mathcal{B}$ such that the image of $\mathcal{M}(\mathcal{B})$ in X lies in V, there exits a unique bounded homomorphism $\mathcal{A}_V \to \mathcal{B}$ making the diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\phi"] \arrow[dr] & & \mathcal{A}_V \arrow[dl] \\
& \mathcal{B} &
\end{tikzcd}$$

commutative.

A closed subset of X which is finite union of affinoid domains is called a special subsets of X.

We have the following proposition.

Proposition 2.2.15. *[3, 2.2.4] Let V be an affinoid domain in X. Then*

1. *$\mathcal{M}(\mathcal{A}_V) \simeq V$; in particular, the homomorphism $\mathcal{A} \to \mathcal{A}_V$ is uniquely determined by V;*

2. *$\mathcal{A}_V$ is a flat $\mathcal{A}$-algebra.*

We can now introduce k-affinoid spaces.

Definition/Proposition 2.2.16. *[3, 2.3] For an open set $\mathcal{U} \subset X$, we set*

$$\Gamma(\mathcal{U}, \mathcal{O}_X) = \lim_{\leftarrow} \mathcal{A}_V,$$

where the limit is taken over all special subsets $V \subset \mathcal{U}$.

This is a sheaf of ring on X and the stalk $\mathcal{O}_X, x$ at a point $x \in X$ is a local ring. The locally ringed space X obtained is called a k-affinoid space. If $\mathcal{A}$ is strictly k-affinoid, X is called a strictly k-affinoid space.

The following is the definition of a morphism of k-affinoid spaces.

Definition 2.2.17. *A morphism of k-affinoid spaces $X = \mathcal{M}(\mathcal{A}) \to Y = \mathcal{M}(\mathcal{B})$ is a morphism of locally ringed spaces which comes from a bounded morphism $\mathcal{B} \to \mathcal{A}$*

The category of k-affinoid spaces is antiequivalent to the category of k-affinoid algebras. For any non Archimedean field K over k, we have a ground field extension functor $\mathcal{M}(\mathcal{A}) \mapsto \mathcal{M}(\mathcal{A}\hat{\otimes}K)$.

We refer to [3, 2.3] for many interesting results on k-affinoid spaces.

Shilov boundaries

We start with the definition of the Shilov boundary of a commutative Banach k-algebra.

Definition/Proposition 2.2.18. *[3, page 36] A closed subset Γ of the spectrum of a commutative Banack k-algebra $\mathcal{A}$ is called a boundary if every element of $\mathcal{A}$ has its maximum in Γ. The set of all boundaries is partially ordered by inclusion, and it satisfies the conditions of Zorn's Lemma. Hence, there exist minimal boundaries. If there exists a unique minimal boundary, it is said to be the Shilov boundary of $\mathcal{A}$, and it is denoted by $\Gamma(\mathcal{A})$.*

We are going to explain that the Shilov boundary of a strictly k-affinoid algebra exists. That's why we introduce the reduction map [3, 2.4] in the following. Given a commutative Banach algebra $\mathcal{A}$, the set

$$\mathcal{A}^{\circ} = \{f \in \mathcal{A} \mid \rho(f) \leq 1\}$$

is a ring and

$$\mathcal{A}^{\circ\circ} = \{f \in \mathcal{A} \mid \rho(f) < 1\}$$

is an ideal in it. The residue ring $\mathcal{A}^{\circ}/\mathcal{A}^{\circ\circ}$ is denoted by $\tilde{\mathcal{A}}$. Every morphism of commutative Banach algebras $\phi : \mathcal{A} \to \mathcal{B}$ induces ring morphisms $\phi^{\circ}; \mathcal{A}^{\circ} \to \mathcal{B}^{\circ}$ and $\tilde{\phi} : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$. In particular, for any point $x \in \mathcal{M}(\mathcal{A})$ there is a morphism $\tilde{\chi}_x : \tilde{\mathcal{A}} \to \widetilde{\mathcal{H}(x)}$. Because $\widetilde{\mathcal{H}(x)}$ is a field, $\ker(\tilde{\chi}_x)$ is a prime ideal of $\tilde{\mathcal{A}}$. Letting $\tilde{k}(\tilde{x})$ denote the fraction field of the ring $\tilde{A}/\ker(\tilde{\chi}_x)$, we obtain an embedding of fields $\tilde{k}(\tilde{x}) \to \widetilde{\mathcal{H}(x)}$ and the following so-called reduction map:

$$\begin{aligned} \pi : \mathcal{M}(\mathcal{A}) &\to \operatorname{spec}(\tilde{\mathcal{A}}) \\ x \quad &\mapsto \ker(\tilde{\chi}_x) \end{aligned}$$

We can now state the following important proposition.

Proposition 2.2.19. *[3, 2.4.4] Let $\mathcal{A}$ be a strictly k-affinoid algebra. Set $X = \mathcal{M}(\mathcal{A})$, $\tilde{X} = \operatorname{spec}(\tilde{\mathcal{A}})$ and denote by $\tilde{X}_{gen}$ the set of generic points of the irreducible components of $\tilde{X}$. The following holds.*

1. *The reduction map $\pi\ : X \to \tilde{X}$ is surjective.*

2. *For any $\tilde{x} \in \tilde{X}_{gen}$, there exists a unique point $x \in X$ with $\pi(x) = \tilde{x}$.*

3. *The set $\pi^{-1}(\tilde{X}_{gen})$ is the Shilov boundary of $\mathcal{A}$ (so by the previous assertion, it is in bijection with $\tilde{X}_{gen}$).*

Holomorphically convex envelopes

We recall Berkovich's notion of holomorphically convex envelope following [3, 2.6].

Definition 2.2.20. *Let Σ be a closed subset in a k-affinoid space $X = \mathcal{M}(\mathcal{A})$. Let $||f||_\Sigma = \max_{x\in\Sigma} |f|_x$. The set*

$$\mathrm{Hol}(\Sigma) = \{x \in X \mid \quad |f|_x \leq ||f||_\Sigma \textit{ for all } f \in \mathcal{A}\}$$

is called the holomorphically convex envelope of Σ in X.

If Σ is a singleton $\{\sigma\}$ we simply write $\mathrm{Hol}(\sigma)$ instead of $\mathrm{Hol}(\{\sigma\})$.
We refer to [3, 2.6] for results on this notion.

2.2.3 k-analytic spaces

The category $\mathbf{k-an}$ of k-analytic space is defined by Berkovich in [3]. An enlarged category is introduced in [4, §1]. In [4], analytic spaces corresponding to ones defined in [3] are called good (see [18, §1.3]).

• A k-analytic space is a particular locally ringed space obtained by gluing k-affinoid spaces. By [4, §1], these spaces are equipped with a Grothendieck topology [23]. The category of k-analytic spaces is denoted $\mathbf{k-an}$.

• An analytic space over k is a K-analytic space for some non-Archimedean field K over k. The corresponding category is denoted $\mathbf{An_k}$.

The notion of k-affinoid domains, k-analytic domains, open immersions and closed immersions are defined in [4].

The category of k-affinoid spaces is a full subcategory of the category of k-analytic spaces.

Proposition 2.2.21. *The category of k-analytic spaces admits fibre products and a final object: $\mathcal{M}(k)$.*

Definition 2.2.22. *A k-analytic group is a group $\mathbf{k-an}$-object (see notations). A k-affinoid group is a k-analytic group whose underlying k-analytic space is k-affinoid.*

Let us now introduce a certain class of k-analytic space obtained from schemes over k.

Definition/Proposition 2.2.23. *[3, §3.4] let X be a scheme of locally finite type over k. Let Φ be the functor from the category $\mathbf{An_k}$ of analytic spaces over k to the category of sets which associates to every analytic space $\mathcal{X}$ the set of morphisms of k-ringed spaces $Hom_k(\mathcal{X}, X)$.*

The functor Φ is represented by k-analytic space X^{an} and a morphism $\pi : X^{an} \to X$. Moreover π is surjective and for any non-Archimedean field K/k, there is a bijection $X^{an}(K) \simeq X(K)$.

The k-analytic space X^{an} is called the analytification of X.

Let us describe the analytifaction explicitly.

Proposition 2.2.24. *[3, 3.4.2] If $X = \operatorname{spec}(A)$, where A is a finitely generated ring over k, then the underlying topological space X^{an} coincides with the set of all multiplicative seminorms on A whose restriction to k is the norm on k.*

If X is arbitrary, X^{an} (as a set) can be described as follows. The set $\bigcup\limits_{K/k} X(K)$, where the union is over all non-Archimedean extension of k, is endowed with the following equivalence relation. If $x' \in X(K')$ and $x'' \in X(K'')$, then $x' \sim x''$ if there is a non-Archimedean field K and embeddings $K \to K'$ and $K \to K''$ such that the points x' and x'' come from the same point of $X(K)$. Then X^{an} coincides with the set of such equivalence classes.

We want to make a remark about k-analytic spaces.

Remark 2.2.25. *In the beginning of the section we have written that a **general** k-analytic space is a particular locally ringed space obtained by **gluing** k-**affin**oid spaces. One could try to do a parallel with the definition of a **general** scheme by **gluing affin**e schemes. This parallel could not be deeper than the previous semantic comparison: the analytifcation of an affine k-scheme is absolutely not in general a k-affinoid space. However, the analytification functor enjoys many properties [3, 3.4.3 , 3.4.6].*

If K/k is an affinoid extension, and X is a k-analytic space, we denote by $\mathrm{pr}_{K/k}$ the canonical morphism $X \times_{k-an} \mathcal{M}(K) \to X$ coming from the cartesian square

$$\begin{tikzcd} X \times_{k-an} \mathcal{M}(K) \arrow[r] \arrow[d] & \mathcal{M}(K) \arrow[d] \\ X \arrow[r] & \mathcal{M}(k) \end{tikzcd}$$

where $\mathcal{M}(K) \to \mathcal{M}(k)$ is the map induced by the morphism of k-affinoid algebra $k \to K$, and $X \to \mathcal{M}(k)$ is the canonical morphism of k-analytic spaces $X \to \mathcal{M}(k)$ (recall that $\mathcal{M}(k)$ is the final object in $\mathbf{k-an}$). The map $\mathrm{pr}_{K/k}$ between underlying set is surjective.

If K/k is a finite Galois extension and X is a k-analytic space, the group $\mathrm{Gal}(K/k)$ acts naturally on the right of $X \times_{\mathcal{M}(k)} \mathcal{M}(K)$ as follows.

Let $\gamma \in \mathrm{Gal}(K/k)$, γ is a morphism of k-algebras from K to K. It is a morphism of k-affinoid algebras, so it induces a morphism of k-affinoid spaces $\gamma : \mathcal{M}(K) \to \mathcal{M}(K)$. Let Id_X denote the identity of X. We get a canonical automorphism of k-analytic spaces

$$Id_X \times_{\mathcal{M}(k)} \gamma : X \times_{\mathcal{M}(k)} \mathcal{M}(K) \to X \times_{\mathcal{M}(k)} \mathcal{M}(K).$$

This is a right action.

Proposition 2.2.26. *[3] Let X be a k-analytic space and let K/k be a finite Galois extension, let $\mathrm{Gal}(K/k)$ act on $X \times_{k-an} \mathcal{M}(K)$. Then $\mathrm{pr}_{K/k}$ induces an isomorphism $(X \times_{k-an} \mathcal{M}(K))/\mathrm{Gal}(K/k) \simeq X$.*

We deduce easily the following corollary.

Corollary 2.2.27. *With the same notations as 2.2.26, let D_K be a subset of $X \times_{k-an} \mathcal{M}(K)$ then D_K is $\mathrm{Gal}(K/k)$-stable if and only if $\mathrm{pr}_{K/k}^{-1} \circ \mathrm{pr}_{K/k}(D_K) = D_K$.*

We now get a very important descent theorem, this is due to Rémy-Thuillier-Werner.

Theorem 2.2.28. *[33, Appendix A] Let X be a k-affinoid space. Let K be a k-affinoid extension. Let D be a subset of X, then D is a k-affinoid domain of X if and only if the subset $\mathrm{pr}_{K/k}^{-1}(D)$ is a K-affinoid domain in $X \times_{k-an} \mathcal{M}(K)$.*

In this text, we are going to construct k-affinoid groups by descent of K-affinoid groups, where K/k is a certain finite extension. The K-affinoid groups are constructed from K°-group scheme by the process of taking "the generic fiber of the formal completion along the special fiber". The following is precisely what we need, it is extracted from Rémy-Thuillier-Werner's work [33, 1.2.4] and Thuillier's thesis [38, 2.1.1] (see also [3, 5.3.2]).

Definition/Proposition 2.2.29. *Let $\mathfrak{A}$ be a flat topologically finitely presented k°-algebra whose spectrum we denote $\mathfrak{X}$. Let $X = \mathrm{spec}(\mathfrak{A} \otimes_{k^\circ} k)$ be the generic fibre of $\mathfrak{X}$. The map*

$$|.|_{\mathfrak{A}} : \mathfrak{A} \otimes_{k^\circ} k \to \mathbb{R}_{\geq 0},\ a \mapsto \inf\{|\lambda| \mid \ \lambda \in k^\times \text{ and } a \in \lambda(\mathfrak{A} \otimes 1)\}$$

is a norm on $\mathfrak{A} \otimes_{k^\circ} k$. The Banach algebra $\mathcal{A}$ obtained by completion is a strictly k-affinoid algebra whose spectrum is denoted by $\widehat{\mathfrak{X}}_\eta$ and is called the generic fibre of the formal completion of $\mathfrak{X}$ along its special fibre. This affinoid space is naturally an affinoid domain in X^{an} (whose points are multiplicative seminorms on $\mathfrak{A} \otimes_{k^\circ} k$ which are bounded with respect to the seminorm $|.|_{\mathfrak{A}}$).

Moreover, there is a reduction map $\tau : \hat{\mathfrak{X}}_\eta \to \mathfrak{X} \times_{k^\circ} \tilde{k}$ defined as follows: a point x in $\hat{\mathfrak{X}}_\eta$ gives a sequence of ring homomorphisms:

$$\mathfrak{A} \to \mathcal{H}(x)^\circ \to \widetilde{\mathcal{H}(x)}$$

whose kernel $\tau(x)$ defines a prime ideal of $\mathfrak{A} \otimes_{k^\circ} \tilde{k}$, i.e a point in $\mathfrak{X} \times_{k^\circ} \tilde{k}$.

If the scheme $\mathfrak{X}$ is integrally closed in its generic fibre — in particular if $\mathfrak{X}$ is smooth — then τ is the reduction map of Berkovich (see 2.2.2 or [3, 2.4]). And so the Shilov Boundary of $\hat{\mathfrak{X}}_\eta$ is in bijection with the irreducible components of the special fibre $\mathfrak{X} \times_{k^\circ} \tilde{k}$. Moreover, the spectral norm ρ (see 2.2.12) on $\mathcal{A}$ is egal to $|\ |_{\mathfrak{A}}$ if and only if the algebra $\mathfrak{A} \otimes_{k^\circ} k$ is reduced [38, Proposition 2.1.1].

Let us state an other result in this area.

Lemma 2.2.30. *Let $\mathfrak{A}$ be a flat k°-algebra of finite type such that*

1. $\mathrm{spec}(\mathfrak{A})$ *is a smooth k°-scheme*
2. $\mathrm{spec}(\mathfrak{A}) \times_{k^\circ} \tilde{k}$ *is irreducible*

Then the Shilov boundary of $\widehat{\mathrm{spec}(\mathfrak{A})}_\eta$ is egal to the norm $|\ |_{\mathfrak{A}}$ (see 2.2.29).

Proof. By 2.2.29, $\mathrm{Shi}(\widehat{\mathrm{spec}(\mathfrak{A})}_\eta)$ is a singleton. By 2.1.2, $\mathfrak{A} \otimes_{k^\circ} \tilde{k}$ is reduced, thus by 2.2.29, $|\ |_{\mathfrak{A}}$ is the spectral norm. This implies that $\mathrm{Shi}(\widehat{\mathrm{spec}(\mathfrak{A})}_\eta) = |\ |_{\mathfrak{A}}$ (see [3, page 26], see also [33, proof of 2.4(ii)]). □

We now show that being Galois stable is preserved by taking the generic fiber of the formal completion along the special fibre. We prove it under somes conditions.

Proposition 2.2.31. *Let K/k be a finite Galois extension. Let $X = \mathrm{spec}(A)$ be an affine k-scheme of finite type and let $\mathfrak{X} = \mathrm{spec}(\mathfrak{A})$ be a smooth, flat K°-scheme of finite type such that $\mathfrak{X} \times_{K^\circ} K = X \times_k K$ and such that $\mathfrak{X} \times_{K^\circ} \tilde{K}$ is irreducible with a reduced $\tilde{K}$-algebra. Suppose $\mathfrak{A}$ is a stable $\mathrm{Gal}(K/k)$-stable subalgebra of $A \otimes_k K$. Then the generic fibre of the formal completion of $\mathfrak{X}$ along its special fibre is a $\mathrm{Gal}(K/k)$-stable K-affinoid domain $\hat{\mathfrak{X}}_\eta$ of $X \times_{k-an} \mathcal{M}(K)$.*

Proof. Let $|\ |_x \in \hat{\mathfrak{X}}_\eta \subset X \times_{k-an} \mathcal{M}(K)$, it is a seminorm on $A \otimes_k K$ bounded by $|\ |_{\mathfrak{A}}$. Recall that $\mathrm{Gal}(K/k)$ acts on the right of $X \times_{k-an} \mathcal{M}(K)$. Let $\gamma \in \mathrm{Gal}(K/k)$, we need to show that $|\ |_x.\gamma$ stay in $\hat{\mathfrak{X}}_\eta$. Let $f \in A \otimes_k K$, then $(|\ |_x.\gamma)(f) = |\gamma.f|_x$. By definition of $|\ |_x$, we have $|\gamma.f|_x \leq |\gamma.f|_{\mathfrak{A}}$. Since $\mathfrak{A}$ is $\mathrm{Gal}(K/k)$ stable in $A \otimes_k K$, we have $\gamma.\mathfrak{A} = \mathfrak{A}$ for all $\gamma \in \mathrm{Gal}(K/k)$ and we deduce the following.

$$
\begin{aligned}
|\gamma.f|_{\mathfrak{A}} &= \inf_{\lambda \in K^\times} \{|\lambda| \quad | \quad \gamma.f \in \mathfrak{A} \subset \mathfrak{A} \otimes_{K^\circ} K\} \\
&= \inf_{\lambda \in K^\times} \{|\lambda| \quad | \quad f \in \gamma^{-1}\mathfrak{A} \subset \mathfrak{A} \otimes_{K^\circ} K\} \\
&= \inf_{\lambda \in K^\times} \{|\lambda| \quad | \quad f \in \mathfrak{A} \subset \mathfrak{A} \otimes_{K^\circ} K\} \\
&= |f|_{\mathfrak{A}}
\end{aligned}
$$

Consequently, we have $(|\ |_x.\gamma)(f) = |\gamma.f|_x \leq |\gamma.f|_{\mathfrak{A}} = |f|_{\mathfrak{A}}$. Thus $|\ |_x.\gamma \leq |\ |_{\mathfrak{A}}$, and so $(|\ |_x.\gamma) \in \hat{\mathfrak{X}}_\eta$ as required.

□

We will also need the following proposition, to ensure that certain k-affinoid spaces are k-affinoid groups.

Proposition 2.2.32. *Let G be a k-analytic group, let K/k be an affinoid extension, let H_K be a k-affinoid subgroup of $G \times_{k-an} \mathcal{M}(K)$, let $H = \mathrm{pr}_{K/k}(H_K)$, if it is a k-affinoid domain of G then it is a k-affinoid subgroup of G.*

Proof. Let $m : G \times_{k-an} G \to G$ be the multiplication map and $inv : G \to G$ the inversion map comming from the group-structure on G. We have to show that the restriction maps $m : H \times_{k-an} H \to G$ and $inv : H \to G$ factor through H. Consider the following diagram whose four squares are commutative:

$$
\begin{tikzcd}
H_K \times H_K \arrow[rr, "m"] \arrow[dr, hook, "i"] \arrow[dd, two heads, "p"] & & H_K \arrow[dr, hook, "i"] \arrow[dd, two heads, "p"] & \\
& G_K \times G_K \arrow[rr, "m"] \arrow[dd, two heads, "p"] & & G_K \arrow[dd, two heads, "p"] \\
H \times H \arrow[dr, hook, "i"] & & H \arrow[dr, hook, "i"] & \\
& G \times G \arrow[rr, "m"] & & G
\end{tikzcd}
$$

Let x be in $H \times H$, it is enough to show that there is y in H such that $m \circ i(x) = i(y)$. Let z in $H_K \times H_K$ such that $p(z) = x$, then

$$m \circ i(x) = m \circ p \circ i(z) = p \circ m \circ i(z) = p \circ i \circ m(z) = i \circ p \circ m(z)$$

So $y = p \circ m(z)$ works. The same argument works for inv.

□

2.3 Bruhat-Tits buildings and Moy-Prasad filtrations

Let G be a connected reductive group over a complete non archimedean field k. Bruhat and Tits defined a combinatoric structure called the reduced Bruhat-Tits building $\mathrm{BT}^R(G, \mathrm{k})$. It is an euclidean building in the sense of Rousseau [36], in particular it is a topological space with a metric and facets, walls and vertices are defined, moreover we have a notion of special points. We do not recall these definitions here. If k is discretly valued, $\mathrm{BT}^R(G, k)$ is a polysimplicial complex. In this situation a facet is a polysimplex. Bruhat and Tits also defined the enlarged Bruhat-Tits building $\mathrm{BT}^E(G, k)$ of G. The enlarged building $\mathrm{BT}^E(G, k)$ is the direct product of $\mathrm{BT}^R(G, k)$ by a real affine space of dimension depending on the split rank of the center of G. There is a natural projection $\mathrm{BT}^E(G, k) \to \mathrm{BT}^R(G, k)$. The group $G(k)$ of rational points of G acts on $\mathrm{BT}^R(G, k)$ and $\mathrm{BT}^E(G, k)$, and the natural projection is $G(k)$-equivariant. To certain subsets Ω of $\mathrm{BT}^R(G, k)$, Bruhat-Tits associated a canonical smooth group scheme $\mathfrak{G}_\Omega$ over k°, $\mathfrak{G}_\Omega$ has the property that $\mathfrak{G}_\Omega(k^\circ)$ is the stabilizer of the preimage of Ω under the projection $\mathrm{BT}^E(G, k) \to \mathrm{BT}^R(G, k)$. In this paper we only consider the case where $\Omega = \{x\}$ is a singleton, in this case $\mathfrak{G}_\Omega$ is well-defined and is denoted $\mathfrak{G}_x$. If G is defined over a non archimedean local field k, Rousseau [35] proved that for each extension K/k of non archimedean local fields there is a canonical injective map $\mathrm{BT}^R(G, k) \to \mathrm{BT}^R(G, K)$ which is continous and $G(k)$-equivariant. This induces the same property for enlarged buildings.

Definition 2.3.1. *A point $x \in \mathrm{BT}^R(G, k)$ is called rational if there exists a finite extension k'/k such that*

1. *$i_{k'/k}(x)$ is a special point of $\mathrm{BT}^R(G, k')$,*
2. *G is split over k'.*

The set of rational points is denoted $\mathrm{BT}^R_{rat}(G, k)$.

Proposition 2.3.2. *The set $\mathrm{BT}^R_{rat}(G, k)$ is a dense subset of $\mathrm{BT}^R(G, k)$.*

Proof. Remark first that if G is split over k, it is obvious that $\mathrm{BT}^R_{rat}(G, k)$ is dense in $\mathrm{BT}^R(G, k)$, since for any maximal split torus S over k and any finite extension K/k, the appartement $A^R(G, S)/K$ is obtained from $A^R(G, S)/k$ adding regularly $e(K, k)$ times more walls. Let us now prove the proposition. It is enough to show that for any maximal split torus S of G over k, $A^R_{rat}(G, S)$ is dense in $A^R(G, S)$. Let L be a finite Galois extension such that G is split over L. By [9, 4.1.1,4.1.2,5.1.12], there exists a torus $T \supset S$ defined over k such that $T \times_k L$ is a maximal split torus of $G \times_k L$. There exists a facet F in $A^R(G, T)/L$ which is $\mathrm{Gal}(L/k)$-stable. The barycentre x of F is $\mathrm{Gal}(L/k)$-stable and so $x \in A(G, S)/k$

(since $(A^R(G,T)/L)^{\mathrm{Gal}(L/k)} = A^R(G,S)/k$). By [16, §6.3.4, lines 8-9], the point x becomes special over a finite extension K/L. So we have proved that there exists one rational point x in $A^R(G,S)$. Now the set of points $\{g.x \mid g \in S(\overline{k})\}$ consists in a dense subset of $A^R(G,S)$ constitued of rational points. Indeed, let us first show that this set consists in rational points. So let $g \in S(\overline{k})$, there exists a finite extension K/L such that $g \in S(K)$. The point x is special in the building $\mathrm{BT}^R(G,K)$ (since G is split over L and x is special in the building $\mathrm{BT}^R(G,L)$), so $g.x$ is special in $BT^R(G,K)$. By definition $T(\overline{k})$ acts on $A^R(G,T)/L$ by translation (the translation vector v associated to $t \in T(\overline{k})$ is given by the usual formula "$< v, \alpha >= -\mathrm{ord}(\alpha(t)) \; \forall \alpha$", see [9, 4.2.3(I)]) and for any $g \in S(\overline{k}) \subset T(\overline{k})$, we have $g.x \in A^R(G,S)/k$, so $g.x$ is a rational point in $BT^R(G,k)$. Since $\mathrm{ord}(\overline{k})$ is dense in $\mathbb{R}$, $\{g.x \mid g \in S(\overline{k})\}$ is dense in $A^R(G,S)$. The propositon follows.

□

In an appendix at the end of this document, we produce a discussion on the notion of rational points.

Following [33, 1.1] we refer to [17, *Exposés* XIX to XXVI] for group schemes and theirs properties. A Demazure k°- group scheme is a connected and split reductive k°-group scheme (see [3, 1.1.2]).

Proposition 2.3.3. *[33, end of page 15] [9, 4.6.22] If G is split over k and x is a special point, then $\mathfrak{G}_x$ is a Demazure group scheme and $\mathfrak{G}_x \times_{k^\circ} k = G$. Moreover $\mathfrak{G}_x$ is smooth and its special fibre is irreducible (Thus by 2.1.2, it is flat over k° and the $\tilde{k}$-algebra of its special fibre is reduced).*

To any point $x \in \mathrm{BT}^R(G,k)$ and any $r \in \mathbb{R}_{\geq 0}$ A. Moy and G. Prasad attached a compact subgroup $G(k)^{MP}_{x,r} \subset G(k)$, they also introduced a subgroup $\mathfrak{g}(k)^{MP}_{x,r}$ of the Lie algebra $\mathfrak{g}(k)$. If $r' \geq r$ then $G(k)^{MP}_{x,r'} \subset G(k)^{MP}_{x,r}$, we thus get filtrations. We refer to Moy-Prasad original articles [29] [30] for the original definition of Moy-Prasad filtrations in the general case. We refer to [41] for a current and contemporary definition of these filtrations, with suitable normalizations, they are defined there only if G split over a tamely ramified extension. See also [43, 0.4] and [42] for important commentaries, informations and works that one should know about Moy-Prasad filtrations.

Fact 2.3.4. *[41, line 36 page 588] [27, line 15 page 278] Let $r > 0$ and $x \in \mathrm{BT}^R(G,k)$ and assume G split over a tamely ramified extension, then for any finite tamely ramified extension E/k, $G^{MP}(E)_{x,r} \cap G(k) = G^{MP}(k)_{x,r}$.*

2.4 Definitions and first properties of analytic filtrations

Recall that k is a finite extension of $\mathbb{Q}_p$.

2.4.1 Notions of potentially Demazure objects

Let $G = \mathrm{spec}(A)$ be a connected reductive k-group scheme. Let G^{an} be the k-analytic group associated to G by analytification. B. Rémy, A. Thuillier and A. Werner [33] have introduced the notion of potentially k-affinoid Demazure subgroup of G^{an}. We also introduce a related notion of *rational* potentially k-affinoid Demazure subgroup of G^{an}.

Definition 2.4.1. *A k-affinoid subgroup H of G^{an} is called a k-affinoid Demazure subgroup of G^{an} if there is a Demazure k°-group scheme $\mathfrak{G}$ with generic fibre G and such that H is the generic fibre of the formal completion of $\mathfrak{G}$ along its special fibre, i.e. $H = \widehat{\mathfrak{G}}_\eta$. A k-affinoid subgroup H is called potentially of Demazure type if there is a k-affinoid extension K such that $H \times_{k-an} \mathcal{M}(K)$ is a K-affinoid Demazure subgroup of $G^{an} \times_{k-an} \mathcal{M}(K)$. A potentially k-affinoid Demazure subgroup of G^{an} is called a rational potentially k-affinoid Demazure subgroup if the extension K/k can be choosen finite.*

Proposition 2.4.2. *[33] [19] Let H be a potentially k-affinoid Demazure subgroup of G^{an}. Then*

1. *The Shilov Boundary of H is reduced to a point σ_H.*
2. *The underlying k-affinoid domain of H is the holomorphically convex envelope of σ_H.*

Definition 2.4.1 and Proposition 2.4.2 give birth naturally to the following notions.

Definition 2.4.3. *Let x be a point in G^{an}.*

- *It is a Demazure point if its holomorphically convex envelope in G^{an} is a k-affinoid Demazure subgroup of G^{an}.*
- *It is a potentially Demazure point if its holomorphically convex envelope in G^{an} is a potentially k-affinoid Demazure subgroup of G^{an}.*
- *It is a rational potentially Demazure point if its holomorphically convex envelope in G^{an} is a rational potentially k-affinoid Demazure subgroup of G^{an}.*

We denote by $Dem(G), \widehat{Dem}(G), \overline{Dem}(G)$ the corresponding subsets of G^{an}, of course the following inclusions hold

$$Dem(G) \subset \overline{Dem}(G) \subset \widehat{Dem}(G) \subset G^{an}.$$

As we are going to explain in the following, Rémy-Thuillier-Werner [33] (sometimes following certain ideas of Berkovich [3, Chapter 5]) proved that the reduced Bruhat-Tits building $\mathrm{BT}^R(G,k)$ of a conneted reductive group over a non archimedean local field k canonically embeds in $\widehat{Dem}(G)$. Thuillier gave a non published characterization of $\mathrm{BT}^R(G,k)$ inside $\widehat{Dem}(G)$.

For each $x \in \overline{Dem}(G)$ and each positive real rational number r in $\mathbb{Q}_{\geq 0}$, using the notion of congruence subgroup, we are going to introduce a point $\theta(x,r) \in G^{an}$, whose holomorphically convex envelope is a subanalytic group of the holomorphically convex envelope of x. For each $x \in \overline{Dem}(G)$, the map $\mathbb{Q}_{\geq 0} \to G^{an}$, $x \mapsto \theta(x,r)$ is continous. We have $\mathrm{BT}^R(G,k) \cap \overline{Dem}(G) = \mathrm{BT}^R_{rat}(G,k)$ and we will prove that the otained map $(\mathrm{BT}^R(G,k) \cap \overline{Dem}(G)) \times \mathbb{Q}_{\geq 0} \to G^{an}$ is continous and injective. By density, we get a continous and injective map $\mathrm{BT}^R(G,k) \times \mathbb{R}_{\geq 0} \to G^{an}$. The image of this map forms a cone in G^{an} whose basis is $\mathrm{BT}^R(G,k)$ and vertex is the neutral element of G.

2.4.2 Filtrations of rational potentially Demazure k-affinoid groups

Let k denote a finite extension of $\mathbb{Q}_p$ and G be a connected reductive k-group scheme. For each $x \in \overline{Dem}(G)$ and each positive real rational number in $\mathbb{Q}_{\geq 0}$, using the notion of congruence subgroup, we are going to introduce a point $\theta(x,r) \in G^{an}$ whose holomorphically convex envelope is a k-affinoid subgroup of the holomorphicaly convex envelope of x. We start by a particular case.

The split rational case

Assume G is split and let x be a Demazure point in G^{an}. Let $\mathfrak{G}$ be the k°-Demazure group scheme such that $H := \mathrm{Hol}(x) = \widehat{\mathfrak{G}}_\eta$. Let $\mathfrak{T}$ be a maximal k°-split torus of $\mathfrak{G}$ and Φ be the corresponding set of roots. Let $\mathfrak{B}$ be a Borel subgroup such that $\mathfrak{T}$ is a Levi component of $\mathfrak{B}$. Let Φ^-, Φ^+ be the corresponding sets of negative and positive roots. For each $\alpha \in \Phi$, we have a canonical k°-root subgroup $\mathfrak{U}_\alpha \subset \mathfrak{G}$. Choose an ordering on Φ^-, Φ^+, then the multiplication morphism of k°-schemes

$$\prod_{\alpha \in \Phi^-} \mathfrak{U}_\alpha \times_{k^\circ} \mathfrak{T} \times_{k^\circ} \prod_{\alpha \in \Phi^+} \mathfrak{U}_\alpha \to \mathfrak{G} \tag{2.3}$$

is an open immersion. Its image, which does not depend on the choice of the ordering, is denoted $\underline{\Omega}$ and is called the *grosse cellule* of $\mathfrak{G}$. Taking generic fibres, we obtain similar objects for G. The objects

$$\begin{aligned} T &:= \mathfrak{T} \times_{\mathrm{spec}(k^\circ)} \mathrm{spec}(k) \\ U_\alpha &:= \mathfrak{U}_\alpha \times_{\mathrm{spec}(k^\circ)} \mathrm{spec}(k) \\ B &:= \mathfrak{B} \times_{\mathrm{spec}(k^\circ)} \mathrm{spec}(k) \end{aligned}$$

are respectively a maximal split torus, a roots subgroup, and a Borel subgroup of $G = \mathfrak{G} \times_{\mathrm{spec}(k^\circ)} \mathrm{spec}(k)$. We can identify canonically Φ with the set of roots associated to G, T. Moreover (2.3) induces an open immersion

$$\prod_{\alpha\in\Phi^-} U_\alpha \times_k T \times_k \prod_{\alpha\in\Phi^+} U_\alpha \to G$$

whose image, independent of the ordering, is denoted Ω and is called the *grosse cellule* of G. We can identify Ω and $\underline{\Omega} \times_{\mathrm{spec}(k^\circ)} \mathrm{spec}(k)$. The *grosse cellule* Ω is affine and the open immersion $\Omega \to G$ corresponds to an injective morphism of Hopf algebras from $\mathrm{Hopf}(G)$ to $\mathrm{Hopf}(\Omega)$ (see [3, line 24 page 103]. We are going to construct k-affinoid subgroups H_r of $\mathrm{Hol}(x)$ satisfying that $\mathrm{Shi}(H_r) \in G^{an}$ is a singleton. So $\mathrm{Shi}(H_r)$ will appear as a function on $\mathrm{Hopf}(G)$, the Hopf algebra of G. We will show that $\mathrm{Shi}(H_r)$ can be seen as a function on the Hopf algebra of Ω. This leads us to study firstly Hopf algebras of various affine group schemes.

The torus $\mathfrak{T}$ is split so it is isomorphic to $(\mathbb{G}_m/k^\circ)^s$ for some integer s. Fix an isomorphism

$$\mathfrak{T} \simeq \mathrm{spec}(k^\circ[X_1, \dots, X_s, Y_1, \dots, Y_s]/(X_iY_i = 1 \text{ for } 1 \leq i \leq s)).$$

Fix an integral Chevalley basis of $\mathrm{Lie}(\mathfrak{G}, k^\circ)$, it induces, for each root $\alpha \in \Phi$, a k°-isomorphism $\mathfrak{U}_\alpha \simeq \mathbb{G}_{add}$, where $\mathbb{G}_{add}$ is the additive group over k°. Thus we have fixed an isomorphism $\mathfrak{U}_\alpha \simeq \mathrm{spec}(k^\circ[Z_\alpha])$, i.e. we have fixed an isomorphism $\mathrm{Hopf}(\mathfrak{U}_\alpha) \simeq k^\circ[Z_\alpha]$, for any root α.

Recall that ord is a valuation on $\overline{k}$ such that $\mathrm{ord}(\pi_k) = 1$ for any uniformizer π_k of k (see notations). Let $r \in \mathbb{Z}_{\geq 0}$, and consider the r-th congruence k°-group scheme $\Gamma_r(\mathfrak{G})$ (see 2.1.11). By [43] we have an open immersion

$$\prod_{\alpha\in\Phi^-} \Gamma_r(\mathfrak{U}_\alpha) \times_{k^\circ} \Gamma_r(\mathfrak{T}) \times_{k^\circ} \prod_{\alpha\in\Phi^+} \Gamma_r(\mathfrak{U}_\alpha) \to \Gamma_r(\mathfrak{G}), \tag{2.4}$$

its image does not depend on the ordering and is $\Gamma_r(\underline{\Omega})$.

Definition/Proposition 2.4.4. *Using the process given in 2.2.29, let H_r be $\widehat{\Gamma_r(\mathfrak{G})}_\eta$, the generic fiber of the formal completion of $\Gamma_r(\mathfrak{G})$ along its special fiber. We have*

1. *$\widehat{\Gamma_r(\mathfrak{G})}_\eta$ is a k-affinoid subgroup of H*
2. *Its Shilov Boundary $\mathrm{Shi}(H_r)$ is reduced to a point.*

Proof. If $r = 0$, $\widehat{\Gamma_r(\mathfrak{G})}_\eta$ is just H and the proposition follows from Proposition 2.4.2. If $r > 0$, by 2.1.16, $\Gamma_r(\mathfrak{G})$ is a smooth (and thus flat by 2.1.2) k°-scheme of finite type. Moreover its special fibre $\Gamma_r(\mathfrak{G}) \times_{k^\circ} \tilde{k}$ is irreducible. So by 2.2.29, $\widehat{\Gamma_r(\mathfrak{G})}_\eta$ is a k-affinoid group and the Shilov Boundary of $\widehat{\Gamma_r(\mathfrak{G})}_\eta$ is in bijection with the irreductible component of the special fiber of $\Gamma_r(\mathfrak{G})$. So $\mathrm{Shi}(\widehat{\Gamma_r(\mathfrak{G})}_\eta)$ is a singleton. $\square$

Proposition 2.4.5. *Let $\mathfrak{A}$, $\mathfrak{A}'$ be two k°-subalgebra of* $\mathrm{Hopf}(G)$ *such that* $\mathfrak{G} = \mathrm{spec}(\mathfrak{A})$ *and* $\mathfrak{G}' = \mathrm{spec}(\mathfrak{A}')$ *are two Demazure k°-group scheme with generic fibers G (recall that G is split).*

If $\widehat{\mathfrak{G}}_\eta = \widehat{\mathfrak{G}'}_\eta$ (equality in G^{an}), then $\mathfrak{A} = \mathfrak{A}'$.

Proof. By 2.4.4, $\widehat{\mathfrak{G}}_\eta$ and $\widehat{\mathfrak{G}'}_\eta$ are two k-affinoid domain in G^{an} whose Shilov boundaries are singletons. By 2.2.30, we thus have $\mathrm{Shi}(\widehat{\mathfrak{G}'}_\eta) = \mathrm{Shi}(\widehat{\mathfrak{G}}_\eta) = |\ |_{\mathfrak{A}} = |\ |_{\mathfrak{A}'}$. By definition, $|\ |_{\mathfrak{A}}$ is a norm on $\mathrm{Hopf}(G)$ given by the formula $|f|_{\mathfrak{A}} = \inf_{\lambda \in k^\times} \{|\lambda| \mid f \in \lambda(\mathfrak{A} \otimes 1)\}$. The valuation of k is discrete, so we have

$$f \in \mathfrak{A} \Leftrightarrow 1 \in \{\lambda \in k^\times \mid f \in \lambda(\mathfrak{A} \otimes 1)\} \Leftrightarrow \inf_{\lambda \in k^\times} \{|\lambda| \mid f \in \lambda(\mathfrak{A} \otimes 1)\} \leq 1 \Leftrightarrow |f|_{\mathfrak{A}} \leq 1.$$

Similarly we have $f \in \mathfrak{A}' \Leftrightarrow |f|_{\mathfrak{A}'} \leq 1$. So finally $f \in \mathfrak{A} \Leftrightarrow f \in \mathfrak{A}'$, as required.

□

In order to give an explicit description of $\mathrm{Shi}(H_r)$, we need to study the Hopf algebra of $\Gamma_r(\underline{\Omega})$. We start by studying the Hopf algebra of Ω.

Since

$$\Omega = \prod_{\alpha \in \Phi^-} U_\alpha \times_k T \times_k \prod_{\alpha \in \Phi^+} U_\alpha,$$

we obtain

$$\mathrm{Hopf}(\Omega) = \bigotimes_{\alpha \in \Phi^-} \mathrm{Hopf}(U_\alpha) \otimes_k \mathrm{Hopf}(T) \otimes_k \bigotimes_{\alpha \in \Phi^+} \mathrm{Hopf}(U_\alpha).$$

The torus T is egal to $\mathfrak{T} \times_{k^\circ} k$. The previously fixed isomorphism

$$\mathfrak{T} \simeq \mathrm{spec}(k^\circ[X_1, \ldots, X_s, Y_1, \ldots, Y_s]/(X_i Y_i = 1 \text{ for } 1 \leq i \leq s)).$$

induces a similar isomorphism over k for T. The set[1]

$$\{X^k Y^l \mid k, l \in \mathbb{N}; k \neq 0 \Rightarrow l = 0\}$$

is a basis of the k-vector space $k[X,Y]/XY - 1$. We need an other basis of $\mathrm{Hopf}(\mathbb{G}_m)$, "centered at unity". The set

$$\{(X-1)^k (Y-1)^l \mid k, l \in \mathbb{N}; k \neq 0 \Rightarrow l = 0\}$$

[1]Remark that the condition ($k \neq 0 \Rightarrow l = 0$) is equivalent to the condition ($k = 0$ or $l = 0$), it is also equivalent to the condition ($l \neq 0 \Rightarrow k = 0$) and to the condition ($\neg(k \neq 0$ and $l \neq 0)$). So it is a symmetric condition. The algebra $k[X,Y]/XY - 1$ is sometimes written $k[X, X^{-1}]$ and $X^{\mathbb{Z}}$ is a k-basis of the underlying vector space. Similar remarks about this kind of conditions apply in the following. Remark also that k denote a field and also an integer, it is not a problem.

is a basis of the k-vector space $k[X,Y]/XY-1$.

The previously fixed isomorphisms $\{\mathrm{Hopf}(\mathfrak{U}_\alpha) \simeq k^\circ[Z_\alpha]\}_{\alpha\in\Phi}$ induce isomorphisms $\mathrm{Hopf}(U_\alpha) \simeq k[Z_\alpha]$. We identify the corresponding objects. The set $\{{Z_\alpha}^{m_\alpha} \mid m_\alpha \in \mathbb{Z}_{\geq 0}\}$ is a basis of the k-vector space $\mathrm{Hopf}(U_\alpha)$. These considerations allow us to fix an isomorphism

$$\begin{aligned}\mathrm{Hopf}(\Omega) &\simeq \left(\bigotimes_{\alpha\in\Phi^-} k[Z_\alpha]\right) \otimes_k \left(\bigotimes_{i=1}^{s} k[X_i,Y_i]/X_iY_i-1\right) \otimes_k \left(\bigotimes_{\alpha\in\Phi^+} k[Z_\alpha])\right) \\ &\simeq k[X_1,\ldots,X_s,Y_1,\ldots,Y_s,\{Z_\alpha\}_{\alpha\in\Phi}]/(X_iY_i-1,\ 1\leq i\leq s)\end{aligned}$$

Moreover the set

$$\{\prod_{i=1}^{s}(X_i-1)^{k_i}(Y_i-1)^{l_i}\prod_{\alpha\in\Phi}{Z_\alpha}^{m_\alpha} \mid k_i,l_i,m_\alpha\in\mathbb{N};\forall 1\leq i\leq s, k_i\neq 0 \Rightarrow l_i=0\}$$

is a k-basis of the k-vector space $\mathrm{Hopf}(\Omega)$. So given $f\in\mathrm{Hopf}(\Omega)$, f can be written uniquely as

$$f = \sum_{k_1,\ldots,k_s,l_1,\ldots,l_s,m_\alpha\alpha\in\Phi} a_{k_1\ldots k_s l_1\ldots l_s, m_\alpha\alpha\in\Phi}\prod_{i=1}^{s}(X_i-1)^{k_i}(Y_i-1)^{l_i}\prod_{\alpha\in\Phi}Z_\alpha^{m_\alpha}\ .$$

In order to simplify the notation, we denote a parameter $k_1,\ldots,k_s,l_1,\ldots,l_s,m_\alpha,\alpha\in\Phi$ with $k_i,l_i,m_\alpha\in\mathbb{N}; k_i\neq 0\Rightarrow l_i=0$ by the symbol u, and U the set of all such parameters. Moreover, the element $\prod_{i=1}^{s}(X_i-1)^{k_i}(Y_i-1)^{l_i}\prod_{\alpha\in\Phi}Z_\alpha^{m_\alpha}$ is denoted by the symbol $((X-1)(Y-1)Z)^u$. With these conventions, an element $f\in\mathrm{Hopf}(\Omega)$ is written uniquely as

$$f = \sum_{u\in U} a_u\,((X-1)(Y-1)Z)^u.$$

Since

$$\Gamma_r(\underline{\Omega}) = \prod_{\alpha\in\Phi^-}\Gamma_r(\mathfrak{U}_\alpha)\times_{k^\circ}\Gamma_r(\mathfrak{T})\times_{k^\circ}\prod_{\alpha\in\Phi^+}\Gamma_r(\mathfrak{U}_\alpha),$$

we obtain

$$\mathrm{Hopf}(\Gamma_r(\underline{\Omega})) = \bigotimes_{\alpha\in\Phi^-}\mathrm{Hopf}(\Gamma_r(\mathfrak{U}_\alpha))\otimes_{k^\circ}\mathrm{Hopf}(\Gamma_r(\mathfrak{T}))\otimes_{k^\circ}\bigotimes_{\alpha\in\Phi^+}\mathrm{Hopf}(\Gamma_r(\mathfrak{U}_\alpha))\ .$$

Using 2.1.10 , we have

$$\mathrm{Hopf}(\Gamma_r(\mathfrak{U}_\alpha)) = k^\circ[\pi_k^{-r}Z_\alpha]$$

and

$$\mathrm{Hopf}(\Gamma_r(\mathfrak{T})) = k^\circ[\pi_k^{-r}(X_1-1),\ldots,\pi_k^{-r}(X_s-1),\pi_k^{-r}(Y_1-1),\ldots,\pi_k^{-r}(Y_s-1)] \subset \mathrm{Hopf}(T).$$

Finally, we get the formula

$$\mathrm{Hopf}(\Gamma_r(\underline{\Omega})) = k^\circ[\{\pi_k^{-r}Z_\alpha\}_{\alpha\in\Phi},\{\pi_k^{-r}(X_i-1),\pi_k^{-r}(Y_i-1)\}_{1\le i\le s}] \subset \mathrm{Hopf}(\Omega).$$

Proposition 2.4.6. *With the same notations as in 2.4.4,* $\mathrm{Shi}(H_r)$ *is a norm on* $\mathrm{Hopf}(G)$ *inside* G^{an}*. The point* $\mathrm{Shi}(H_r)$ *belongs to* Ω^{an} *and corresponds to a norm on* $\mathrm{Hopf}(\Omega)$*. The norm* $\mathrm{Shi}(H_r)$ *factorizes trough the canonical injective morphism of Hopf algebras* $\mathrm{Hopf}(G)\to\mathrm{Hopf}(\Omega)$*. The corresponding norm on* $\mathrm{Hopf}(\Omega)$ *is explicitely given, using the notations introduced previously, by the following formula*

$$\begin{array}{rcl} \mathrm{Hopf}(\Omega) & \longrightarrow & \mathbb{R}_{\geq 0} \\ \displaystyle\sum_{u\in U} a_u\,((X-1)(Y-1)Z)^u & \mapsto & \displaystyle\max_{u\in U}|a_u|e^{-r|u|} \end{array}$$

where $|u|$ *is egal to* $k_1+\ldots+k_s+l_1+\ldots+l_s+\displaystyle\sum_{\alpha\in\Phi}m_\alpha$*.*

Proof. By 2.4.4, $\mathrm{Shi}(\widehat{\Gamma_r(\mathfrak{G})}) \in \widehat{\Gamma_r(\mathfrak{G})}$ is the unique point such that the reduction map sends to the generic point of $\Gamma_r(\mathfrak{G})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k})$. Let x denote the generic point of $\Gamma_r(\mathfrak{G})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k})$. The closure $\overline{x}$ of x is egal to $\Gamma_r(\mathfrak{G})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k})$. The special fibre $\Gamma_r(\underline{\Omega})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k})$ is open in $\Gamma_r(\mathfrak{G})$ (and non empty), consequently x is contained in $\Gamma_r(\underline{\Omega})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k})$. Indeed, assume $x\notin\Gamma_r(\underline{\Omega})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k})$, then x is contained in the closed subset $\Gamma_r(\mathfrak{G})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k})\setminus\Gamma_r(\underline{\Omega})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k})$, and so $\overline{x}\neq\Gamma_r(\mathfrak{G})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k})$, this is a contradiction. So x is contained in $\Gamma_r(\underline{\Omega})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k})$. The commutative diagram

$$\begin{array}{ccc} \widehat{\Gamma_r(\underline{\Omega})}_\eta & \xrightarrow{\pi_{\underline{\Omega}}} & \Gamma_r(\underline{\Omega})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k}) \ni x \\ \big\downarrow & & \big\downarrow \\ \widehat{\Gamma_r(\mathfrak{G})}_\eta & \xrightarrow{\pi_{\mathfrak{G}}} & \Gamma_r(\mathfrak{G})\times_{\mathrm{spec}(k^\circ)}\mathrm{spec}(\tilde{k}) \end{array}$$

whose vertical arrows are inclusions shows that

$$\mathrm{Shi}(\widehat{\Gamma_r(\mathfrak{G})}) = \pi_{\underline{\Omega}}^{-1}(x) \in \widehat{\Gamma_r(\underline{\Omega})}_\eta.$$

.

So $\mathrm{Shi}((\widehat{\Gamma_r(\mathfrak{G})}) = \mathrm{Shi}(\widehat{\Gamma_r(\underline{\Omega})}_\eta)$.

By 2.2.29 and 2.2.30, $\mathrm{Shi}(\widehat{\Gamma_r(\underline{\Omega})}_\eta)$ is the norm $|\ |_{\mathrm{Hopf}(\Gamma_r(\underline{\Omega}))}$ on $\mathrm{Hopf}(\Omega)$ given as follows.

For $f \in \mathrm{Hopf}(\Omega)$, write $f = \sum_{u \in U} a_u \left((X-1)(Y-1)Z\right)^u$.

$$\begin{aligned}
|f|_{\mathrm{Hopf}(\Gamma_r(\underline{\Omega}))} &= \inf\{|\lambda| \mid \lambda \in k \text{ and } f \in \lambda(\mathrm{Hopf}(\Gamma_r(\underline{\Omega}) \otimes 1)\} \\
&= \inf\{|\lambda| \mid \lambda \in k \text{ and } a_u \in \lambda(\pi_k^{-r})^{|u|}k^\circ \quad \forall u \in U\} \\
&= \inf\{|\lambda| \mid \lambda \in k \text{ and } |a_u| \leq |\lambda||\pi_k^{-r}|^{|u|} \quad \forall u \in U\} \\
&= \inf\{|\lambda| \mid \lambda \in k \text{ and } |a_u||\pi_k^{r}|^{|u|} \leq |\lambda| \quad \forall u \in U\} \\
&= \max_{u \in U} |a_u||\pi_k^r|^{|u|} \\
&= \max_{u \in U} |a_u| e^{-r|u|}
\end{aligned}$$

This ends the proof. □

Let's show that H_r is determined by its Shilov boundary point.

Proposition 2.4.7. *With the previously introduced notations, the k-affinoid group H_r is the holomorphically convex envelope of* $\mathrm{Shi}(H_r)$.

Proof. Put $\sigma_{H_r} = \mathrm{Shi}(H_r)$. The point σ_{H_r} is a norm on $\mathrm{Hopf}(G)$ that we also denote $| \ |_{\sigma_{H_r}}$. Recall that the holomorphically convex envelope of σ_{H_r} is

$$\mathrm{Hol}(\sigma_{H_r}) = \{x \in G^{an} \mid \quad |f|_x \leq |f|_{\sigma_{H_r}} \quad \forall f \in \mathrm{Hopf}(G) \ \}.$$

By 2.2.29 and 2.2.30, the k-affinoid algebra $\mathcal{A}_r$ of H_r is the completion of $\mathrm{Hopf}(G)$ relatively to the norm $| \ |_{\sigma_{H_r}}$. Let i denote the natural corresponding injective k-algebras morphism $\mathrm{Hopf}(G) \to \mathcal{A}_r$. The inclusion $H_r = \mathcal{M}(\mathcal{A}_{\sigma_{H_r}}) \subset G^{an}$ is given by

$$\begin{aligned}
\iota : \mathcal{M}(\mathcal{A}_{\sigma_{H_r}}) &\to G^{an} \\
| \ |_x &\mapsto | \ |_x \circ i \ .
\end{aligned}$$

Since $\mathcal{M}(\mathcal{A}_{\sigma_{H_r}})$ is the set of all multiplicative bounded seminorm on $\mathcal{A}_{\sigma_{H_r}}$, $\iota(\mathcal{M}(\mathcal{A}_{\sigma_{H_r}}))$ is contained in the holomorphically convex envelope of σ_{H_r}.

Reciprocally, let $x \in \mathrm{Hol}(\sigma_{H_r})$, $x = | \ |_x$ is a multiplicative seminorm $\mathrm{Hopf}(G) \to \mathbb{R}_{\geq 0}$ such that $|f|_x \leq |f|_{\sigma_{H_r}} \quad \forall f \in \mathrm{Hopf}(G)$. Since $\mathcal{A}_r$ is the completion of $\mathrm{Hopf}(G)$, x induces a multiplicative seminorm on $\mathcal{A}_r$ bounded by σ_{H_r}. This ends the proof. □

The general case

Lemma 2.4.8. *Let $\bar{k}$ be an algebraic closure of k. Let $r \in \mathbb{Q}_{\geq 0}$. Let H be a rational potentially k-affinoid Demazure subgroup of G^{an}. There exists a finite Galois extension K/k in $\bar{k}$ such that:*

- $r \in \mathrm{ord}(K)$
- *$H \times_{k-an} \mathcal{M}(K)$ is a $K-$affinoid Demazure subgroup of $G^{an} \times_{\boldsymbol{k\text{-}an}} \mathcal{M}(K)$.*

Proof. By definition, there exists a finite extension L/k, such that $H \times_{\mathcal{M}(k)} \mathcal{M}(L)$ is a L-affinoid Demazure subgroup of $G^{an} \times_{\mathcal{M}(k)} \mathcal{M}(L)$. There exists a finite extension E/k, such that $r \in \mathrm{ord}(E)$. Let K be a finite Galois extension of k such that $L, E \subset K$, it obviously exists. Then K satisfies the required properties since being potentially k-affinoid Demazure subgroup is stable by finite base change. $\square$

Definition 2.4.9. *Let H be a rational potentially k-affinoid Demazure subgroup of G^{an} and $r \in \mathbb{Q}_{\geq 0}$. Let K be a finite extension as in the previous lemma. Let $\mathfrak{G}$ be the Demazure K°-group scheme such that $H \times_{\boldsymbol{k\text{-}an}} \mathcal{M}(K) = \widehat{\mathfrak{G}}_\eta$. We assume that the Hopf K°-algebra $\mathfrak{A}$ of $\mathfrak{G}$ is $\mathrm{Gal}(K/k)$-stable.*

Then, pose $H_r = \mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta)$, the projection of the generic fibre of the formal completion along the special fibre of the $e(K,k)r$-th congruence subgroup of $\mathfrak{G}$.

We have the following proposition.

Proposition 2.4.10. *We have*

1. *In definition 2.4.9, H_r is independant of the choice of K.*
2. *$H_0 = H$*
3. *H_r is a k-affinoid subgroup of G^{an}, it is a k-affinoid subgroup of H.*

Proof. We first prove (3), then (1) and then (2). By 2.1.12, 2.1.10 and 2.2.31, $\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta$ is $\mathrm{Gal}(K/k)$-stable in $G^{an} \times_{\mathcal{M}(k)} \mathcal{M}(K)$. Consequently, 2.2.27 shows that $\mathrm{pr}^{-1}_{K/k}(\mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta)) = \widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta$. So by Theorem 2.2.28, $\mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})})$ is a k-affinoid domain in G^{an}. By Proposition 2.2.32, $\mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})})$ is a k-affinoid group. This finishes (3). Let us now show that $\mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})})$ does not depend on the choice of the extension. So let K and K' be two extensions satisfying the conditions of Lemma 2.4.8. Let $\mathfrak{G}\ /K^\circ$ and $\mathfrak{G}'\ /K'^\circ$ be the integral Demazure group schemes such that $H \times_{\mathcal{M}(k)} \mathcal{M}(K) = \widehat{\mathfrak{G}}_\eta$ and $H \times_{\mathcal{M}(k)} \mathcal{M}(K') = \widehat{\mathfrak{G}'}_\eta$. Let K'' be a finite Galois extension such that $K, K' \subset K''$. We have equalities (in $(G^{an} \times_{\mathcal{M}(k)} \mathcal{M}(K'')))$

$$\begin{aligned} H \times_{\mathcal{M}(k)} \mathcal{M}(K'') &= (H \times_{\mathcal{M}(k)} \mathcal{M}(K)) \times_{\mathcal{M}(K)} \mathcal{M}(K'') = \widehat{\mathfrak{G}}_\eta \times_{\mathcal{M}(K)} \mathcal{M}(K'') \\ &= (H \times_{\mathcal{M}(k)} \mathcal{M}(K')) \times_{\mathcal{M}(K')} \mathcal{M}(K'') = \widehat{\mathfrak{G}'}_\eta \times_{\mathcal{M}(K')} \mathcal{M}(K'') \end{aligned}$$

and

$$\widehat{\mathfrak{G}}_\eta \times_{\mathcal{M}(K)} \mathcal{M}(K'') = (\widehat{\mathfrak{G} \times_{K^\circ} K''^\circ})_\eta$$
$$\widehat{\mathfrak{G}'}_\eta \times_{\mathcal{M}(K')} \mathcal{M}(K'') = (\widehat{\mathfrak{G}' \times_{K'^\circ} K''^\circ})_\eta.$$

We thus get an equality $(\widehat{\mathfrak{G} \times_{K^\circ} K''^\circ})_\eta \simeq (\widehat{\mathfrak{G}' \times_{K'^\circ} K''^\circ})_\eta$. Using 2.4.5, we deduce an equality $\mathfrak{G} \times_{K^\circ} K''^\circ = \mathfrak{G}' \times_{K'^\circ} K''^\circ$, let $\mathfrak{G}''$ denote this K°-Demazure-group scheme.

By Proposition 2.1.15, $\Gamma_{e(K'',k)r}(\mathfrak{G} \times_{K^\circ} K''^\circ) = \Gamma_{e(K,k)r}(\mathfrak{G}) \times_{K^\circ} K''^\circ$. So $\widehat{\Gamma_{e(K'',k)r}(\mathfrak{G}'')}_\eta = \widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta \times_{\mathcal{M}(K)} \mathcal{M}(K'')$. We deduce that

$$\mathrm{pr}_{K''/K}(\widehat{\Gamma_{e(K'',k)r}(\mathfrak{G}'')}_\eta) = \widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta.$$

However, $\mathrm{pr}_{K''/k}(\widehat{\Gamma_{e(K'',k)r}(\mathfrak{G}'')}_\eta) = \mathrm{pr}_{K/k}(\mathrm{pr}_{K''/K}(\widehat{\Gamma_{e(K'',k)r}(\mathfrak{G}'')}_\eta))$.

So $\mathrm{pr}_{K''/k}(\widehat{\Gamma_{e(K'',k)r}(\mathfrak{G}'')}_\eta) = \mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta)$. By symmetry, we get $\mathrm{pr}_{K''/k}(\widehat{\Gamma_{e(K'',k)r}(\mathfrak{G}'')}_\eta) = \mathrm{pr}_{K'/k}(\widehat{\Gamma_{e(K',k)r}(\mathfrak{G}')}_\eta)$. So $\mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta) = \mathrm{pr}_{K'/k}(\widehat{\Gamma_{e(K',k)r}(\mathfrak{G}')}_\eta)$, and (1) is proved. Let K/k be a finite Galois extension such that $H \times_{\mathcal{M}(k)} \mathcal{M}(K)$ is a Demazure K°-affinoid group scheme $\widehat{\mathfrak{G}}_\eta$. Then

$$\begin{aligned} H_0 &= \mathrm{pr}_{K/k}(\widehat{\Gamma_0(\mathfrak{G})}_\eta) \\ &= \mathrm{pr}_{K/k}(\widehat{\mathfrak{G}}_\eta) \\ &= H \end{aligned}$$

So (2) is proved and the proof ends here.

□

We now have a fundamental result.

Proposition 2.4.11. *Let H be a rational potentially k-affinoid Demazure group. Let K/k and $\mathfrak{G}$ be objects such that $H \times_{\mathcal{M}(k)} \mathcal{M}(K) \simeq \widehat{\mathfrak{G}}_\eta$ (see definition 2.4.9). Let $\mathfrak{A}^K_{e(K,k)r}$ be the K°-algebra of $\Gamma_{e(K,k)r}(\mathfrak{G})$.*

1. *The Shilov boundary of H_r is reduced to a point σ_{H_r}.*

2. *The map*

$$\begin{aligned} |\ |_{\mathfrak{A}^K_{e(K,k)r}} :& \mathrm{Hopf}(G \times_k K) \to \mathbb{R}_{\geq 0} \\ & f \mapsto \inf_{\lambda \in K^\times} \{|\lambda| \mid f \in \lambda(\mathfrak{A}^K_{e(K,k)r} \otimes 1) \subset \mathrm{Hopf}(G \times_k K)\}. \end{aligned}$$

is a norm on $\mathrm{Hopf}(G \times_k K)$*, moreover*

$$| \ |_{\mathfrak{A}^K_{e(K,k)r}} |_{\mathrm{Hopf}(G)} = \mathrm{Shi}(H_r).$$

3. *The k-affinoid algebra of H_r is the completion of* $\mathrm{Hopf}(G)$ *relatively to the norm* $| \ |_{\mathfrak{A}^K_{e(K,k)r}} |_{\mathrm{Hopf}(G)}$.

4. *H_r is the holomorphically convex envelope of σ_{H_r}.*

5. $\mathrm{Shi}(\mathrm{Hol}(\sigma_{H_r})) = \sigma_{H_r}$ *and* $\mathrm{Hol}(\mathrm{Shi}(H_r)) = H_r$.

Proof. 1. The Shilov boundary of $\Gamma_{\widehat{e(K,k)r}}(\mathfrak{G})_\eta$ is a singleton by the split rational case (see 2.4.4). The Shilov boundary of $\Gamma_{\widehat{e(K,k)r}}(\mathfrak{G})_\eta$ surjects onto the Shilov boundary of H_r by [3, 1.4.5 proof], and so the Shilov boundary of H_r is a singleton.

2. By 2.4.6 the map $| \ |_{\mathfrak{A}^K_{e(K,k)r}}$ is a norm on $\mathrm{Hopf}(G \times_k K)$. By 2.2.29 and 2.2.30, $| \ |_{\mathfrak{A}^K_{e(K,k)r}}$ is the Shilov boundary of $\Gamma_{\widehat{e(K,k)r}}(\mathfrak{G})_\eta$. The Shilov boundary of H_r is egal to $\mathrm{pr}_{K/k}(\mathrm{Shi}(\Gamma_{\widehat{e(K,k)r}}(\mathfrak{G})_\eta))$ and $\mathrm{pr}_{K/k}$ is realized by the restriction map of functions from $\mathrm{Hopf}(G \times_{\mathrm{spec}(k)} \mathrm{spec}(K))$ to $\mathrm{Hopf}(G)$. This explains both assertions.

3. We have already prove it in the "split rational case". We adapt the argument given by [33, proof of 2.4 (ii)] to descent this result. Let $\mathcal{A}_{H_r}$ be the k-affinoid algebra of H_r. Since $\mathcal{A}_{H_r}$ is reduced, the norm of $\mathcal{A}_{H_r}$ coincides with the spectral norm [3, 2.1.4] of $\mathcal{A}_{H_r}$, and so it is egal to $| \ |_{\sigma_{H_r}}$ since $\sigma_{H_r} = \mathrm{Shi}(H_r)$. Let $\overline{\mathrm{Hopf}(G)}^{| \ |_{\sigma_{H_r}}}$ be the completion of $\mathrm{Hopf}(G)$ relatively to the norm $| \ |_{\sigma_{H_r}}$. The injective morphism of k-algebras $i : \mathrm{Hopf}(G) \to \mathcal{A}_{H_r}$ (corresponding to $H_r \subset G^{an}$), extends to an isometric embedding $i : \overline{\mathrm{Hopf}(G)}^{| \ |_{\sigma_{H_r}}} \to \mathcal{A}_{H_r}$. Let $\mathcal{A}_{H_r \times_{\mathcal{M}(k)} \mathcal{M}(K)}$ be the K-affinoid algebra of $H_r \times_{\mathcal{M}(k)} \mathcal{M}(K)$. By definition $H_r \times_{\mathcal{M}(k)} \mathcal{M}(K)$ is egal to $\Gamma_{\widehat{e(k,k)r}}(\mathfrak{G})_\eta$ ($\mathfrak{G}$ is the K°-Demazure group scheme used to define H_r). So, by the rational split case,

$$\mathcal{A}_{H_r \times_{\mathcal{M}(k)} \mathcal{M}(K)} = \overline{\mathrm{Hopf}(G \times_k K)}^{| \ |_{\sigma_{H_r \times_{\mathcal{M}(k)} \mathcal{M}(K)}}}.$$

In particular $\mathrm{Hopf}(G \times_k K)$ is dense in $\mathcal{A}_{H_r \times_{\mathcal{M}(k)} \mathcal{M}(K)} = \mathcal{A}_{H_r}$. In other words $\mathrm{Hopf}(G) \otimes_k K$ is dense in $\mathcal{A}_{H_r} \hat{\otimes}_k K$. It follows that $i \hat{\otimes}_k K : \overline{\mathrm{Hopf}(G)}^{| \ |_{\sigma_{H_r}}} \hat{\otimes}_k K \to \mathcal{A}_{H_r} \hat{\otimes}_k K$ is an isomorphism of Banach algebras, hence $\overline{\mathrm{Hopf}(G)}^{| \ |_{\sigma_{H_r}}} = \mathcal{A}_{H_r}$ by [3, Lemma A.5].

4. Following the "split rational case " (see 2.4.7), this is a consequence of the previous assertion. Let us write it. By the previous assertion, the k-affinoid algebra $\mathcal{A}_r$ of H_r is the completion of $\mathrm{Hopf}(G)$ relatively

to the norm $| \ |_{\sigma_{H_r}}$. Let i denote the natural corresponding inclusion $\mathrm{Hopf}(G) \to \mathcal{A}_{H_r}$. The inclusion $H_r = \mathcal{M}(\mathcal{A}_{\sigma_{H_r}}) \subset G^{an}$ is given by

$$\begin{aligned} \iota : \mathcal{M}(\mathcal{A}_{\sigma_{H_r}}) &\to G^{an} \\ | \ |_x \quad &\mapsto | \ |_x \circ i \ . \end{aligned}$$

Since $\mathcal{M}(\mathcal{A}_{\sigma_{H_r}})$ is the set of all multiplicative bounded seminorms on $\mathcal{A}_{\sigma_{H_r}}$, $\iota(\mathcal{M}(\mathcal{A}_{\sigma_{H_r}}))$ is contained in the holomorphically convex envelope of σ_{H_r}. Reciprocally, let $x \in \mathrm{Hol}(\sigma_{H_r})$, x is a multiplicative seminorm $\mathrm{Hopf}(G) \to \mathbb{R}_{\geq 0}$ such that $|f|_x \leq |f|_{\sigma_{H_r}}$ $\forall f \in \mathrm{Hopf}(G)$. Since $\mathcal{A}_r$ is the completion of $\mathrm{Hopf}(G)$, x induces a multiplicative seminorm on $\mathcal{A}_r$ bounded by σ_{H_r}. This ends the proof.

5. These are obvious consequences of the previous assertions.

□

Remark 2.4.12. *If $r > s \in \mathbb{Q}_{\geq 0}$, $H_r \underset{\neq}{\subset} H_s$.*

Proof. This is an easy consequence of the definition taking the K-points for any sufficiently big extension K/k. □

Proposition 2.4.13. *The map $\mathbb{Q}_{\geq 0} \to G^{an}$, $r \mapsto \sigma_{H_r}$ is continous.*

Proof. One can adapt the proof of 2.5.7.

□

2.4.3 Filtrations of Lie algebra

Let $\mathfrak{g}$ be the k-Lie algebra of G it is a a k-scheme. In this section we define k-affinoid groups $\mathfrak{h}_r \subset \mathfrak{g}^{an}$, for any rational potentially Demazure k-affinoid subgroup H and any $r \in \mathbb{Q}_{\geq 0}$. So let H be a rational potentially Demazure k-affinoid subgroup of G^{an} and $r \in \mathbb{Q}_{\geq 0}$. In 2.4.9, we have defined an analytic group H_r. In order to define H_r, we have choosen a certain extension K/k (see 2.4.9). Let $\mathfrak{G}$ the K°-Demazure group scheme such that $H \times_{\mathcal{M}(k)} \mathcal{M}(K) = \widehat{\mathfrak{G}}_\eta$. Let $\Gamma_{e(K,k)r}(\mathfrak{G})$ be the $e(K,k)r$-th congruence subgroup of Γ. Let $\mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))$ be its K°-Lie algebra, it is in particular a K°-group scheme, it is a smooth (and thus flat by 2.1.2) group scheme over K°, its special fibre is irreducible with reduced $\tilde{K}$-algebra. We denote by $\mathrm{pr}_{K/k}$ the canonical map $\mathfrak{g}^{an} \times_{\textbf{k-an}} \mathcal{M}(K) \to \mathfrak{g}^{an}$.

Definition 2.4.14. *With the previously introduced notations, we put*

$$\mathfrak{h}_r = \mathrm{pr}_{K/k}\left(\widehat{\mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))}_\eta\right),$$

the projection of the generic fibre of the formal completion along its special fibre of $\mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))$.

Proposition 2.4.15. *With the previously introduced notations, $\mathfrak{h}_r$ is a k-affinoid domain of $\mathfrak{g}^{an}$, it is a k-affinoid group, moreover*

1. *The Shilov boundary of $\mathfrak{h}_r$ is reduced to a point $\sigma_{\mathfrak{h}_r}$ and is egal to $| \ |_{\mathrm{Hopf}(\mathrm{Lie}(\Gamma_{e(K,k)r}))} \ |_{\mathrm{Hopf}(\mathfrak{g})}$.*

2. $\mathrm{Hol}(\sigma_{\mathfrak{h}_r}) = \mathfrak{h}_r$

3. *The k-affinoid algebra of $\mathfrak{h}_r$ is the completion of $\mathrm{Hopf}(\mathfrak{g})$ relatively to the norm $| \ |_{\mathrm{Hopf}(\mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G})))} \ |_{\mathrm{Hopf}(\mathfrak{g})}$.*

Proof. The proof is similar to that of Proposition 2.4.11. □

2.5 Filtrations associated to points in the Bruhat-Tits building

2.5.1 Definitions and properties of $G_{x,r}$ and θ

Let G be a connected reductive k-group scheme, let $x \in \mathrm{BT}^R(G,k)$ be a rational point in the reduced Bruhat-Tits building of G and let r be a positive rational number.

Proposition 2.5.1. *There exists a finite Galois extension K/k such that*

1. *$i_{K/k}(x)$ is a special point in $\mathrm{BT}^R(G,K)$,*

2. *G is split over K*

3. *r is in $\mathrm{ord}(K^\times)$*

Proof. Since x is rational, by Definition 2.3.1 there is a finite Galois extension K_1/k such that (1) and (2) are satified. It is obvious that there exists a finite Galois extension K_2/k such that (3) is satisfied. The proposition follows taking a finite Galois extension K/k containing K_1 and K_2. It is easy to check that K satisfies the three properties (recall that if G is split over K_1 and y is special over K_1, then $i_{K/K_1}(y)$ is special over any finite extension K of K_1). □

Let K be an extension of k as in Proposition 2.5.1. Let $\mathfrak{G} = \mathfrak{G}_x$ be the canonical K°-Demazure group scheme attached to $x \in \mathrm{BT}^R(G,K)$ characterized by the fact that its K°-points form the stabilizer of a preimage of $i_{K/k}(x)$ in the enlarged Bruhat-Tits building (see section 2.3). In these conditions, the K°-Hopf algebra of $\mathfrak{G}$ is $\mathrm{Gal}(K/k)$-stable in $A \otimes_k K$. As usual, let $\mathrm{pr}_{K/k}$ denote the projection $G^{an} \times_{\mathbf{k\text{-}an}} \mathcal{M}(K) \to G^{an}$.

Definition/Proposition 2.5.2. • *Let G_x be $\mathrm{pr}_{K/k}(\widehat{\mathfrak{G}}_\eta)$, it is a rational potentially Demazure k-affinoid subgroup of G^{an} equal to the k-affinoid group defined and considered in [33, theorem 2.1]. It is characterized by the fact that for any non archimedean extension k'/k,*

$$G_x(k') = \mathrm{stab}(i_{k'/k}(\tilde{x})) \subset G(k').$$

where $\tilde{x} \in \mathrm{BT}^E(G,k)$ is a preimage of x under the projection (see section 2.3).

• *Let $r \in \mathbb{Q}_{\geq 0}$, using 2.4.9, we obtain a k-affinoid subgroup $(G_x)_r$ of G^{an}, it is equal to $\mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta)$. We simply write $G_{x,r}$ instead of $(G_x)_r$.*

Proof. The fact that $\mathrm{pr}_{K/k}(\widehat{\mathfrak{G}}_\eta)$ is a rational potentially Demazure k-affinoid subgroup of G^{an} equal to the k-affinoid group G_x defined and considered in [33, definition 2.1] is explained during the proof of [33, 2.1]. The last part of the proposition is a direct consequence of 2.4.9.

□

The previous section 2.4 gives us the following properties of $G_{x,r}$.

Proposition 2.5.3. *We have:*

1. *$G_{x,r}$ is a k-affinoid subgroup of G^{an}.*
2. *The Shilov boundary of $G_{x,r}$ is reduced to a point that we denote $\theta(x,r)$. The point $\theta(x,r) \in G^{an}$ is a norm on $\mathrm{Hopf}(G)$ egal to $|\ |_{\mathrm{Hopf}(\Gamma_{e(K,k)r}(\mathfrak{G}))}\ |_{\mathrm{Hopf}(G)}$.*
3. *$G_{x,r}$ is the holomorphically convex envelope of $\theta(x,r)$.*
4. *If $r = 0$, $G_{x,r} = G_x$ where G_x is the k-analytic group defined in [33, 2.1].*
5. *The k-affinoid algebra of $G_{x,r}$ is the completion of $\mathrm{Hopf}(G)$ relatively to the norm $|\ |_{\mathrm{Hopf}(\Gamma_{e(K,k)r}(\mathfrak{G}))}\ |_{\mathrm{Hopf}(G)}$, i.e. by a previous assertion, the completion of $\mathrm{Hopf}(G)$ relatively to $\theta(x,r)$.*

Proof. These are corollaries of 2.4.10 and 2.4.11 . □

Proposition 2.5.4. *Let x be a rational point in the reduced Bruhat-Tits building of G, let r be a positive rational number and let $g \in G(k)$, then*

1. *$G_{g.x,r} = gG_{x,r}g^{-1}$*
2. *$\theta(g.x,r) = g\theta(x,r)g^{-1}$*

Proof. By 2.5.3, the two assertions are equivalent. Let us prove the first assertion. The case $r = 0$ is proved in [33, 2.5]. For $r \geq 0$ rational, choose an extension K/k as in the definition of $G_{x,r}$ and let $\mathfrak{G}$ be the K°-Demazure group scheme attached to the special point $i_{K/k}(x) \in \mathrm{BT}^{r}(G, K)$. The point $g.i_{K/k}(x) \in \mathrm{BT}^{R}(G, K)$ is special and the K°-Demazure group scheme attached to $g.i_{K/k}(x) \in \mathrm{BT}^{R}(G, K)$ is $\mathfrak{G}_{g.x} = g\mathfrak{G}_x g^{-1}$. We then deduce the equality

$$\begin{aligned}
G_{g.x,r} &= (G_{g.x})_r \\
&= \mathrm{pr}_{K/k}\left(\Gamma_{e(k,k)r}\widehat{(\mathfrak{G}_{g.x})}_{\eta}\right) \\
&= \mathrm{pr}_{K/k}\left(\Gamma_{e(k,k)r}\widehat{(g\mathfrak{G}_x g^{-1})}_{\eta}\right) \\
&= \mathrm{pr}_{K/k}\left((g\Gamma_{e(k,k)r}\widehat{(\mathfrak{G}_x)}g^{-1})_{\eta}\right) \\
&= \mathrm{pr}_{K/k}\left(g\ \Gamma_{e(k,k)r}\widehat{(\mathfrak{G}_x)}_{\eta}\ g^{-1}\right) \\
&= g\ \mathrm{pr}_{K/k}\left(\Gamma_{e(k,k)r}\widehat{(\mathfrak{G}_x)}_{\eta}\right)g^{-1} \\
&= gG_{x,r}g^{-1}.
\end{aligned}$$

□

We now introduce a natural map.

Definition 2.5.5. *Let $\mathbb{Q}_{\geq 0}$ denote the semi-field of positive real rational numbers. Let $\mathrm{BT}^{R}_{rat}(G, k)$ be the set of rational points of the reduced Bruhat-Tits building of G. Let*

$$\theta : \mathrm{BT}^{R}_{rat}(G, k) \times \mathbb{Q}_{\geq 0} \to G^{an}$$

be the map sending (x, r) to the Shilov boundary of the previously defined k-affinoid group $G_{x,r}$.

Remark 2.5.6. *Let k'/k be a finite extension of k. Let $x \in \mathrm{BT}^{R}_{rat}(G, k')$, let $r \in \mathbb{Q}_{\geq 0}$, we define a k'-affinoid group as follows. Let K/k' be a finite Galois extension such that G is split over K, $i_{K/k'}(x)$ is special in $\mathrm{BT}^{R}_{rat}(G, K)$ and $r \in \mathrm{ord}_k(K)$. Let $\mathfrak{G}$ be the K°-Demazure group scheme attached to $i_{K/k'}(x)$. We put $G'_{x,r} = \mathrm{pr}_{K/k'}(\Gamma_{e(K,k)r}\widehat{(\mathfrak{G})}_{\eta})$, this is a k'-affinoid subgroup of $(G \times_k k')^{an}$. Let $\theta_{k'}$ be the corresponding map $\mathrm{BT}^{R}(G, k') \times \mathbb{Q}_{\geq 0} \to (G \times_k k')^{an}$, sending (x, r) to $\mathrm{Shi}(G'_{x,r})$. If $x \in \mathrm{BT}^{R}_{rat}(G, k')$ comes from k, i.e. $x = i_{k'/k}(\mathrm{x})$ for a point $\mathrm{x} \in \mathrm{BT}^{R}_{rat}(G, k)$, we also denote naturally the k'-affinoid group $G'_{x,r}$ by $G_{i_{k'/k}(\mathrm{x}),r}$. Remark that we have used in the definition the ramification index $e(K, k)$ and not $e(K, k')$, this reflects the fact that*

k is a "reference" object in this work, indeed we work with the valuation $\mathrm{ord} = \mathrm{ord}_k$. *These choices allow us to state the following proposition.*

Proposition 2.5.7. *1. The map $\theta : \mathrm{BT}^R_{rat}(G,k) \times \mathbb{Q}_{\geq 0} \to G^{an}$ is $G(k)$-equivariant relatively to the actions:*

- *$g.(x,r) = (g.x,r)$ for all $(x,r) \in \mathrm{BT}^R_{rat}(G,k) \times \mathbb{Q}_{\geq 0}$*
- *$g.x = gxg^{-1}$ for all $x \in G^{an}$*

2. For any finite extension k'/k, the diagram

$$\begin{tikzcd} \mathrm{BT}^R_{rat}(G,k') \times \mathbb{Q}_{\geq 0} \arrow[r, "\theta_{k'}"] & (G \times_k k')^{an} \arrow[d, "\mathrm{pr}_{k'/k}"] \\ \mathrm{BT}^R_{rat}(G,k) \times \mathbb{Q}_{\geq 0} \arrow[u, "i_{k'/k} \times Id"] \arrow[r, "\theta"] & G^{an} \end{tikzcd}$$

is commutative, where the map $\theta_{k'}$ is defined in the previous remark 2.5.6. Moreover, for any rational point $x \in \mathrm{BT}^R_{rat}(G,k)$ and any $r \in \mathbb{Q}_{\geq 0}$, the equality of k'-affinoid subgroups of $G^{an} \times_{\mathcal{M}(k)} \mathcal{M}(k')$ holds:

$$G_{i_{k'/k}(x),r} = G_{x,r} \times_{\mathcal{M}(k)} \mathcal{M}(k').$$

3. For any finite extension k'/k,

$$G_{i_{k'/k}(x),r}(k') \cap G(k) = G_{x,r}(k)$$

4. The map $\theta : \mathrm{BT}^R_{rat}(G,k) \times \mathbb{Q}_{\geq 0} \to G^{an}$ is continuous and injective.

Proof. 1. We have to show that $g.\theta(x,r) = \theta(g.(x,r))$. This is a direct consequence of 2.5.4, indeed

$$\theta(g.(x,r)) = \theta(g.x,r) = g\theta(x,r)g^{-1} = g.\theta(x,r)$$

2. We use the notation of remark 2.5.6. Let K/k be the extension used to define $G_{x,r}$ as $G_{x,r} = \mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta)$, we can assume that $k' \subset K$. We have $G'_{x,r} = \mathrm{pr}_{K/k'}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta)$, thus $G_{x,r} = \mathrm{pr}_{k'/k}(G'_{x,r})$. By definition $\theta(x,r) = \mathrm{Shi}(G_{x,r})$, by the previous sentence and properties of Shilov boundaries, this is egal to $\mathrm{pr}_{k'/k}(\mathrm{Shi}(G'_{x,r}))$. The commutativity of the diagram follows. We have $\mathrm{pr}^{-1}_{K/k}(G_{x,r}) = G_{i_{K/k}(x),r}$ by definition of $G_{x,r}$ and since K/k is a Galois extension. We thus get $\mathrm{pr}^{-1}_{K/k'}(\mathrm{pr}^{-1}_{k'/k}(G_{x,r})) = G_{i_{K/k}(x),r}$. We also have $\mathrm{pr}^{-1}_{K/k'}(G'_{x,r}) = G_{i_{K/k}(x),r}$ by definition of $G'_{x,r}$ and since K/k' is a Galois extension. We thus obtain

$$\mathrm{pr}^{-1}_{K/k'}(\mathrm{pr}^{-1}_{k'/k}(G_{x,r})) = \mathrm{pr}^{-1}_{K/k'}(G'_{x,r}).$$

This implies $\mathrm{pr}_{k'/k}^{-1}(G_{x,r}) = G'_{x,r}$, since $\mathrm{pr}_{K/k'}$ is surjective. Now since $G'_{x,r}$ is a k'-affinoid domain of $G^{an} \times_{\mathcal{M}(k)} \mathcal{M}(k')$ and $G_{x,r}$ is a k-affinoid domain of G^{an}, we have $G'_{x,r} = G_{x,r} \times_{\mathcal{M}(k)} \mathcal{M}(k')$.

3. This is a direct consequence of the previous assertion and the fact that $G_{x,r}$ is a k-affinoid domain of G^{an}.

4. We follow [33, Proposition 2.6 (ii) and Proposition 2.8 (iii)] for the continuity. Let k'/k be a finite extension such that G is split over k'. Since the maps $i_{k'/k} \times Id : \mathrm{BT}_{rat}^{R}(G,k) \times \mathbb{Q}_{\geq 0} \to \mathrm{BT}_{rat}^{R}(G,k') \times \mathbb{Q}_{\geq 0}$ and $\mathrm{pr}_{k'/k} : (G \times_k k')^{an} \to G^{an}$ are continous, it is enough to show that $\mathrm{BT}_{rat}^{R}(G,k') \times \mathbb{Q}_{\geq 0} \to (G \times_k k')^{an}$ is continous. In other words, we can assume G is split over k. So assume G is split over k and choose a special point $x \in \mathrm{BT}_{rat}^{R}(G,k)$. Let $\mathfrak{G}$ be the k°-Demazure group scheme attached to x. Let $\mathfrak{T}$ be a maximal split k°-torus of $\mathfrak{G}$ and let $\mathfrak{B}$ be a k°-Borel such that $\mathfrak{T}$ is a Levi subgroup of $\mathfrak{B}$. Let Φ, Φ^{-}, Φ^{+} be the corresponding set of roots. Choose a Chevalley basis of the k°-Lie algebra of $\mathfrak{G}$. We are in a similar situation as in 2.4.2, and we use the same notations as 2.4.2 in the following. We can use x to identify the appartement $A(T,k)$ with $V(T) = Hom_{Ab}(X^*(T), \mathbb{R})$. It is enough to show that the restriction map $A_{rat}(T,k) \times \mathbb{Q}_{\geq 0} \to (G)^{an}$ is continous. We claim that for any rational point y in $A(T,k) = V(T)$ and any $r \in \mathbb{Q}_{\geq 0}$, the point $\theta(y,r)$ belong to Ω^{an} and corresponds to the norm

$$\begin{gathered}\mathrm{Hopf}(\Omega) \to \mathbb{R}_{\geq 0} \\ \sum_{u \in U} a_u ((X-1)(Y-1)Z_{\alpha})^u \mapsto \max_{u \in U} |a_u| e^{-r|u|} \prod_{\alpha \in \Phi} e^{m_{\alpha} < y, \alpha >}\end{gathered}$$

where $< .,. >$ is the map $V(T) \times X^*(T) \to \mathbb{R}$, $(y,\alpha) \mapsto < y, \alpha > = y(\alpha)$. This claim implies the continuity since this formula is continous in[2] the variable (y,r) . We now prove the claim following closely [33, Proposition 2.6 (ii)]. Since y is a rational point, there exists a finite extension K/k such that $y = t.x$ with $t \in T(K)$. Let U_{α}^{K} be $U_{\alpha} \times_k K$, Ω^K be $\Omega \times_k K$ and T^K be $T \times_k K$. For any $t \in T(K)$, and any root $\alpha \in \Phi$, the element t normalizes the root group U_{α}^{K} and conjuguation by t induces an automorphism of U_{α}^{K} which is just the homothety of ratio $\alpha(t) \in K^{\times}$. If we read it through the isomorphisms $\mathbb{G}_{add} \to U_{\alpha}^{K}$, we have a commutative diagram

[2] Be carefull that what is denoted by u in [33] is here denoted by y and u here is a parameter for a basis of $\mathrm{Hopf}(\Omega)$ (see 2.4.2)

$$\begin{array}{ccc} \mathrm{spec}(\mathrm{Hopf}(T^K)[\{Z_\alpha\}_{\alpha\in\Phi}]) & \longrightarrow & \Omega^K \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle int(t)} \\ \mathrm{spec}(\mathrm{Hopf}(T^K)[\{Z_\alpha\}_{\alpha\in\Phi}]) & \longrightarrow & \Omega^K \end{array}$$

where τ is induced by the $\mathrm{Hopf}(T^K)$-automorphism τ^* of $\mathrm{Hopf}(T^K)[\{Z_\alpha\}_{\alpha\in\Phi}$ mapping Z_α to $\alpha(t)Z_\alpha$ for any $\alpha\in\Phi$. It follows that, over K, $\theta(t.x,r) = t\theta(x,r)t^{-1}$ is the point of $(G\times_k K)^{an}$ defined by the multiplicative norm on $\mathrm{Hopf}(\Omega^K)$ mapping $f=\sum_{u\in U} a_u((X-1)(Y-1)Z_\alpha)^u$ to

$$\begin{aligned} |\tau^*(f)|_{\theta(x,r)} &= \Big|\sum_{u\in U}\left(a_u\prod_{\alpha\in\Phi}\alpha(t)^{m_\alpha}\right)((X-1)(Y-1)Z_\alpha)^u\Big|_{\theta(x,r)} \\ &= \max_{u\in U}|a_u|e^{-r|u|}\prod_{\alpha\in\Phi}|\alpha(t)|^{m_\alpha} \\ &= \max_{u\in U}|a_u|e^{-r|u|}\prod_{\alpha\in\Phi}e^{m_\alpha<y,\alpha>} \end{aligned}$$

Since G is assumed to be split over k and by properties of Shilov boundaries, we get the claim by restriction from K to k. This ends the proof of the continuity.

Let us explain the injectivity. Let (x_1,r_1) and (x_2,r_2) be in $\mathrm{BT}^R_{rat}(G,k)\times\mathbb{Q}_{\geq 0}$ such that $\theta(x_1,r_1)=\theta(x_2,r_2)$. Let us first explain, by the absurd, that necessarily we have $r_1=r_2$. So assume by the absurd that there exists (x_1,r_1) and (x_2,r_2) with $r_1\neq r_2$ such that $\theta(x_1,r_1)=\theta(x_2,r_2)$. Assume $r_1>r_2$ (the other case can be treated in a similar way). Since $\theta(x_1,r_1)=\theta(x_2,r_2)$, taking holomorphically convexe envelope, we have $G_{x_1,r_1}=G_{x_2,r_2}$ by 2.5.3. Since x_1 and x_2 are rational points, there exists a finite extension K/k such that $\exists g\in G(K)$ such that $g.i_{K/k}(x_1)=i_{K/k}(x_2)$. By 2.5.4, we thus get

$$gG_{i_{K/k}(x_1),r_1}g^{-1}=G_{g.i_{K/k}(x_1),r_1}=G_{i_{K/k}(x_2),r_1}.$$

By 2.4.12, we thus obtain

$$gG_{i_{K/k}(x_2),r_2}g^{-1}=gG_{i_{K/k}(x_1),r_1}g^{-1}\subsetneq G_{i_{K/k}(x_2),r_2}.$$

We have thus deduced the existence of a k-affinoid group $G_{absurd}:=G_{i_{K/k}(x_2),r_2}$ satisfying

$$gG_{absurd}g^{-1}\subsetneq G_{absurd},$$

this is absurd. So we have proved that $\theta(x_1, r_1) = \theta(x_2, r_2) \Rightarrow r_1 = r_2$. Let us now prove that we also necessarily have $x_1 = x_2$. Assume first G is split over k. Let (x_1, r) and (x_2, r) be in $\mathrm{BT}^R_{rat}(G, k) \times \mathbb{Q}_{\geq 0}$. We know, by properties of Bruhat-Tits buildings, that there exists an appartement $A(T, k)$ such that x_1 and x_2 belongs to this appartement. The choice of a special point in $A(T, k)$ induces, as in the proof of the continuity, an explicit map $A_{rat}(T, k) \times \mathbb{Q}_{\geq 0} \to G^{an}$ which factorizes through Ω^{an}. Let $(y, r) \in A_{rat}(T, k) \times \mathbb{Q}_{\geq 0}$, the explicit formula for $\theta(y, r)$

$$\begin{gathered}\mathrm{Hopf}(\Omega) \to \mathbb{R}_{\geq 0} \\ \sum_{u \in U} a_u((X-1)(Y-1)Z_\alpha)^u \mapsto \max_{u \in U} |a_u| e^{-r|u|} \prod_{\alpha \in \Phi} e^{m_\alpha <y, \alpha>}\end{gathered}$$

claimed and proved before (during the proof of the continuity) shows that $\theta(x_1, r) = \theta(x_2, r) \Rightarrow x_1 = x_2$. Indeed, the formula give us

$$\begin{aligned}\theta(x_1, r)(Z_\alpha) &= e^{-r} e^{<x_1, \alpha>} \text{ for any root } \alpha \\ \theta(x_2, r)(Z_\alpha) &= e^{-r} e^{<x_2, \alpha>} \text{ for any root } \alpha,\end{aligned}$$

consequently,

$$\begin{aligned}\theta(x_1, r) = \theta(x_2, r) &\Rightarrow < x_1, \alpha > = < x_2, \alpha > \text{ for all roots } \alpha \\ &\Rightarrow x_1 = x_2\end{aligned}$$

as required.

In general, if G is not split, we prove injectivity using a finite Galois extension k'/k such that G is split over k' and using the diagram

$$\begin{tikzcd} \mathrm{BT}^R_{rat}(G, k') \times \mathbb{Q}_{\geq 0} \arrow[r, "\theta_{k'}"] & (G \times_k k')^{an} \arrow[d, "\mathrm{pr}_{k'/k}"] \\ \mathrm{BT}^R_{rat}(G, k) \times \mathbb{Q}_{\geq 0} \arrow[u, "i_{k'/k} \times Id"] \arrow[r, "\theta"] & G^{an}. \end{tikzcd}$$

By the split case, the map $\theta_{k'}$ is injective. The map $i_{k'/k} \times Id$ is injective. So it is enough to show the restriction of $\mathrm{pr}_{k'/k} : (G \times_k k')^{an} \to G^{an}$ to the image of $\theta_{k'} \circ i_{k'/k} \times Id$ is injective. This is a consequence of the fact that $\theta_{k'}(i_{k'/k}(x), r)$ is $\mathrm{Gal}(k'/k)$-stable for any $(x, r) \in \mathrm{BT}^R_{rat}(G, k) \times \mathbb{Q}_{\geq 0}$.

□

2.5.2 A cone

We have defined a continous and injective map $\theta : \mathrm{BT}^R_{rat}(G, k) \times \mathbb{Q}_{\geq 0} \to G^{an}$. By completion, we get a continous and injective map $\theta : \mathrm{BT}^R(G, k) \times \mathbb{R}_{\geq 0} \to G^{an}$. For all $x \in \mathrm{BT}^R(G, k)$, we put $\theta(x, +\infty) = e_G$, where $e_G \in G^{an}$ is the neutral element.

Definition/Proposition 2.5.8. *The set* $\{\theta(\mathrm{BT}^R(G,k),\mathbb{R}_{\geq 0})\cup e_G\}\subset G^{an}$ *is a topological cone in* G^{an}*. Its base is the reduced Bruhat-Tits building and its vertex is the neutral element. If* $p=\theta(x,r)\in G^{an}$ *is in this cone, the depth of* p *is by definition the number* r*. The subset* $\theta(\mathrm{BT}^R_{rat}(G,k),\mathbb{Q}_{\geq 0})\cup e_G$ *is called the rational cone.*

Proof. For any $x\in\mathrm{BT}^R_{rat}(G,k)$, the point $\theta(x,r)$ approaches e_G as r approaches $+\infty$. This makes clear 2.5.8. □

2.5.3 Comparison with Moy-Prasad filtrations in the tame case

Let G be a connected reductive k-group scheme that split over a tamely ramified extension. Recall that $G(k)^{MP}_{x,r}$ denote the normalized Moy-Prasad filtration (see section 2.3). The well known results

• if G is split and x is special, then Moy-Prasad filtrations are obtained by taking set-theoretic congruence subgroups of the integral points of the attached integral Demazure group $\mathfrak{G}_x$;
• Moy-Prasad filtrations are compatible relatively to field extensions in the tame case;

together with the definitions of $G_{x,r}$ imply the following proposition.

Proposition 2.5.9. *Assume we can choose the extension* K/k *tamely ramified in order to define* $G_{x,r}$ *(see Definitions 2.5.2 and 2.4.9), then* $G_{x,r}(k)=G(k)^{MP}_{x,r}$*.*

Proof. Let K/k be a finite tamely ramified extension such that we can write $G_{x,r}=\mathrm{pr}_{K/k}\left(\Gamma_{\widehat{e(K,k)r}}(\mathfrak{G})_\eta\right)$. The following equality hold.

$$\begin{aligned}
G_{x,r}(k) &= G_{x,r}(K)\cap G(k)\\
&= \Gamma_{\widehat{e(K,k)r}}(\mathfrak{G})_\eta(K)\cap G(k)\\
&= \Gamma_{e(k,k)}(K^\circ)\cap G(k)\\
[43,8.8] &= \ker(\mathfrak{G}(K^\circ)\to\mathfrak{G}(K^\circ/\pi_K^{e(K,k)r}K^\circ))\cap G(k)\\
[43,8.8] &= \ker(\mathfrak{G}(K^\circ)\to\mathfrak{G}(K^\circ/\pi_k^{r}K^\circ)\cap G(k)\\
[43,8.8] &= G(K)^{MP}_{x,r}\cap G(k)\\
[27,\text{line 5 page 6}] &= G(k)^{MP}_{x,r}
\end{aligned}$$

This ends the proof. □

2.5.4 Filtrations of the Lie algebra

By section 2.4.3 and Definition 2.5.2, we obtain analytic filtrations of the Lie algebra $\mathfrak{g}_{x,r} := (\mathfrak{g}_x)_r$, for each $x \in \mathrm{BT}^R_{rat}(G,k)$ and each $r \in \mathbb{Q}_{\geq 0}$. We recall the formal definition in the following definition.

Definition 2.5.10. *Let $x \in \mathrm{BT}^R_{rat}(G,k)$ and $r \in \mathbb{Q}_{\geq 0}$. Let K/k as in 2.5.1, then $\mathfrak{g}_{x,r} = \mathrm{pr}_{K/k}\left(\mathrm{Lie}(\widehat{\Gamma_{e(K,k)r}}(\mathfrak{G}))_\eta\right)$ where $\mathfrak{G}$ is the K°-Demazure group scheme attached to $i_{K/k}(x) \in \mathrm{BT}^R_{rat}(G,K)$.*

If K/k can be choosen tamely ramified in order to define $\mathfrak{g}_{x,r}$, then $\mathfrak{g}_{x,r}(k) = \mathfrak{g}(k)^{MP}_{x,r}$ for $x \in \mathrm{BT}^R_{rat}(G,k)$ and $r \in \mathbb{Q}_{>0}$ (the proof of the G case, using [27] and [43], can be easily adapted).

2.5.5 Moy-Prasad isomorphism

Let $x \in \mathrm{BT}^R_{rat}(G,k)$ and let $r,s \in \mathbb{Q}_{\geq 0}$ be rational numbers such that

$$0 < \frac{r}{2} \leq s \leq r.$$

Question 2.5.11. *Do we have an isomorphism*

$$G_{x,s}(k)/G_{x,r}(k) \xrightarrow{\sim} \mathfrak{g}_{x,s}(k)/\mathfrak{g}_{x,r}(k) \quad ?$$

If such an isomorphism exists we say that the filtrations $\{G_{x,r}(k)\}$ and $\{\mathfrak{g}_{x,r}(k)\}$ introduced in Definition 2.5.2 and Definition 2.5.10 satisfy Moy-Prasad isomorphism.

The question can also be posed for general stable rational potentially k-affinoid groups. In Appendix A, we present a partial answer.

2.5.6 Examples and pictures

In this section we give some examples and pictures of the previously introduced objects.

The split torus of rank one

Let R be a commutative ring. The R-algebra $R[X,Y]/(XY-1)$ is naturally a Hopf R-algebra. Recall that its augmentation map is

$$\begin{aligned} R[X,Y]/XY-1 &\to R \\ X &\mapsto 1 \\ Y &\mapsto 1 \end{aligned}$$

and its kernel is generated by $X-1$ and $Y-1$. Now let A be $k[X,Y]/XY-1$ (k is our fixed p-adic field). Let G be $\mathrm{spec}(A)$, it is a split torus of rank one over k. The morphism of k-algebra $k[X] \to k[X,Y]/XY-1$ induces a morphism of affine scheme $G \to \mathbb{A}^1_k$. It also induces an inclusion $G^{an} \subset (\mathbb{A}^1_k)^{an}$, it is injective and $G^{an} = (\mathbb{A}^1_k)^{an} \setminus 0$. The reduced Bruhat-Tits building of G is a singleton $\{x\}$. The point x is special and G is split over k. The grosse cellule of G is G. The k°-Demazure group scheme attached to x is $\mathfrak{G} = \mathrm{spec}(k^\circ[X,Y]/XY-1)$. Let make explicit the definition of the k-affinoid group $G_{x,r}$ for $r \geq 0$. If $r = 0$, $G_{x,r} = G_{x,0} = \widehat{\mathfrak{G}}_\eta$ and $\widehat{\mathfrak{G}}_\eta = \mathcal{M}(k\{X,Y\}/XY-1)$. Assume now $r > 0$, we have to choose a finite Galois extension K/k such that $r \in \mathrm{ord}(K)$. Let $\mathfrak{G}$ be the K°-Demazure group scheme attached to $i_{K/k}(x)$. It is egal to $\mathrm{spec}(K^\circ[X,Y]/XY-1)$. By definition $G_{x,r}$ is egal to $\mathrm{pr}_{K/k}\left(\widehat{\Gamma_{re(K,k)}(\mathfrak{G})}_\eta\right)$. The K°-scheme $\Gamma_{e(k,k)r}(\mathfrak{G})$ is the $e(k,k)r-th$ congruence subgroup of $\mathfrak{G}$. By 2.1.10 , $\mathrm{Hopf}(\Gamma_{e(k,k)r}(\mathfrak{G})$ is egal to $K^\circ[\pi_K^{-e(K,k)r}(X-1), \pi_K^{-e(K,k)r}(Y-1)] \subset K[X,Y]/XY-1$, since the kernel of the augmentation is generated by $X-1$ and $Y-1$. The K-affinoid group $\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta$ is the Berkovich spectrum of the K-affinoid algebra obtained by completion of $K[X,Y]/XY-1$ relatively to the norm $\|\ \|_{\mathrm{Hopf}(\Gamma_{e(K,k)r}(\mathfrak{G}))}$. Writting $f \in K[X,Y]/XY-1$ as $\sum_{(k_1,k_2)\in U} a_{k_1k_2}(X-1)^{k_1}(Y-1)^{k_2}$ (U is the set of parameter for the basis of $K[X,Y]/XY-1$ "centered at unity", see 2.4.2), the norm $\|\ \|_{\mathrm{Hopf}(\Gamma_{e(K,k)r}(\mathfrak{G}))}$ is explicitely given by the map

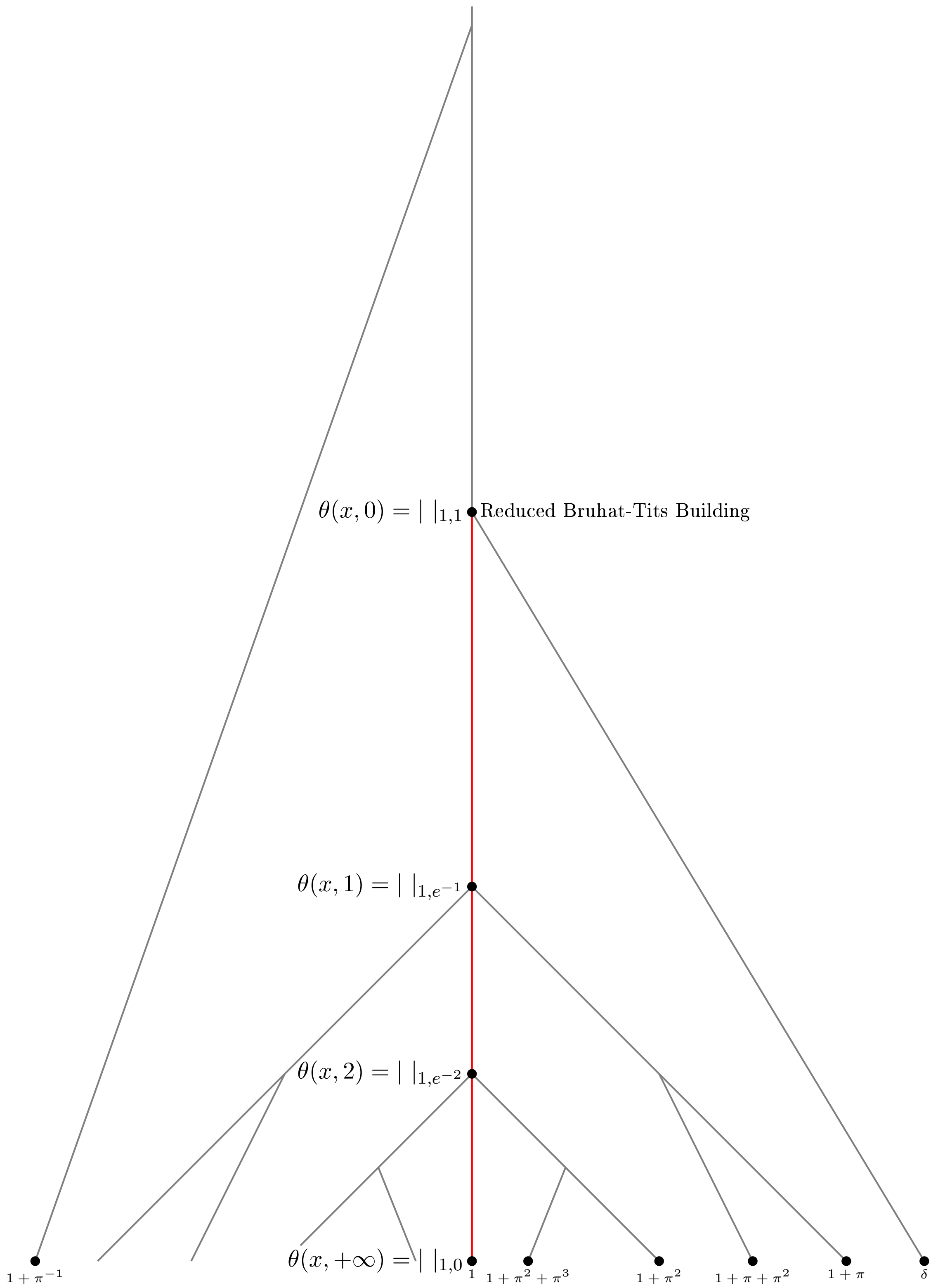
$$
\begin{aligned}
&K[X,Y]/XY-1 \to \mathbb{R}_{\geq 0} \\
f \mapsto &\inf_{\lambda\in K^\times}\{|\lambda| \mid f \in \lambda(K^\circ[\pi_K^{-e(K,k)r}(X-1), \pi_K^{-e(K,k)r}(Y-1)]) \subset K[X,Y]/XY-1\} \\
&= \inf_{\lambda\in K^\times}\{|\lambda| \mid a_{k_1k_2}(X-1)^{k_1}(Y-1)^{k_2} \in \lambda(K^\circ[\pi_K^{-e(K,k)r}(X-1), \pi_K^{-e(K,k)r}(Y-1)])\forall(k_1,k_2)\in U\} \\
&= \inf_{\lambda\in K^\times}\{|\lambda| \mid a_{k_1k_2} \in \lambda\pi_K^{-e(K,k)r(k_1+k_2)}K^\circ \quad \forall(k_1,k_2)\in U\} \\
&= \inf_{\lambda\in K^\times}\{|\lambda| \mid |a_{k_1k_2}| \leq |\lambda\pi_K^{-re(K,k)(k_1+k_2)}| \quad \forall(k_1,k_2)\in U\} \\
&= \inf_{\lambda\in K^\times}\{|\lambda| \mid |a_{k_1k_2}|e^{-r(k_1+k_2)} \leq |\lambda| \quad \forall(k_1,k_2)\in U\} \\
&= \max_{(k_1,k_2)\in U}|a_{k_1k_2}|e^{-r(k_1+k_2)} .
\end{aligned}
$$

Completing $K[X,Y]/XY-1$, we deduce that the K-affinoid algebra of $\widehat{\Gamma_{re(K,k)}(\mathfrak{G})}_\eta$ is

$$\{\sum_{(k_1,k_2)\in U} a_{k_1k_2}(X-1)^{k_1}(Y-1)^{k_2} \mid a_{k_1k_2} \in k \text{ and } |a_{k_1k_2}|(e^{-r})^{|u|} \to 0 \text{ as } |u| \to \infty\} \subset K[[X,Y]]/XY-1$$

We denote it as $K\{e^r(X-1), e^r(Y-1)\}/XY-1$. The Shilov boundary of $\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta$ is $\|\ \|_{\mathrm{Hopf}(\Gamma_{e(K,k)r}(\mathfrak{G}))}$. The Shilov boundary $\theta(x,r)$ of $\mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta)$ is $\|\ \|_{\mathrm{Hopf}(\Gamma_{e(K,k)r}(\mathfrak{G}))}$ restricted to the k-algebra $\mathrm{Hopf}(G)$. The point $\theta(x,r) \in G^{an}$ is thus egal to the norm on $k[X,Y]/XY-1$ which map $\sum_{(k_1,k_2)\in U} a_{k_1k_2}(X-1)^{k_1}(Y-1)^{k_2}$ to $\max_{(k_1,k_2)\in U}|a_{k_1k_2}|e^{-r(k_1+k_2)}$. It

corresponds via the embedding $G^{an} \to (\mathbb{A}_k^1)^{an}\backslash 0$ to the norm usually denoted $|\ |_{1,e^{-r}}$ inside $(\mathbb{A}_k^1)^{an}$.
We have the picture



$\theta(x,0) = |\ |_{1,1}$
Reduced Bruhat-Tits Building
$\theta(x,1) = |\ |_{1,e^{-1}}$
$\theta(x,2) = |\ |_{1,e^{-2}}$
$\theta(x,+\infty) = |\ |_{1,0}$
$1+\pi^{-1}$
1
$1+\pi^2+\pi^3$
$1+\pi^2$
$1+\pi+\pi^2$
$1+\pi$
δ

giving some points (of course it is not exhaustive) of G^{an} inside $(\mathbb{A}^1_k)^{an}$. Here δ is an element in $(k^\circ)^\times \setminus 1 + k^{\circ\circ}$. The point $\theta(x,0)$ is mapped to the so-called Gauss point, and corresponds to the reduced Bruhat-Tits building. When $r \geq 0$ is increasing the point $\theta(x,r)$ is getting closer to 1, the neutral element of G^{an}. The holomorphically convex envelope $G_{x,r}$ of $\theta(x,r)$ should be thought as all the points under (attainable by going only down) $\theta(x,r)$ and the k-rational points of $G_{x,r}$ as certain lower extremities. In this situation the cone is the red line, it is homeomorphic to the segment $[0,1]$ (Note that $[0,+\infty] \overset{r\mapsto e^{-r}}{\simeq} [1,0]$).

A computation of $G_{x,0}$ in the case of a wild torus of norm one elements in a quadratic extension

In this section $k = \mathbb{Q}_2$. The polynomial X^2-2 does not have any solution in k. Let $\pi_l \in \overline{k}$ be a root of this polynomial and let l be the field $k(\pi_l) \subset \overline{k}$. The extension l/k is a widely ramified Galois extension. We have $[l:k] = e(f:k) = 2$. The element π_l is a uniformizer of l. The k-vector space l is 2-dimensional and $\{1,\pi_l\}$ is a k-basis. So each element in l can be written as $x+\pi_l y$ with $x,y \in k$. The norm of $x+\pi_l y$ is egal to $(x+\pi_l y)(x-\pi_l y) = x^2-2y^2$. The set of norm 1 elements is an algebraic group. Let us write the Hopf algebra of the corresponding affine k-group scheme G. The Hopf k-algebra of G is $k[X,Y]/X^2-2Y^2-1$, moreover the comultiplication Δ, the antipode τ and the augmentation ε are

$$\begin{aligned}\Delta : k[X,Y]/X^2-2Y^2-1 &\to k[X,Y]/X^2-2Y^2-1 \otimes k[X,Y]/X^2-2Y^2-1\\ X &\mapsto X\otimes X + 2Y\otimes Y\\ Y &\mapsto X\otimes Y + Y \otimes X\end{aligned}$$

$$\begin{aligned}\tau : k[X,Y]/X^2+2Y^2-1 &\to k[X,Y]/X^2+2Y^2-1\\ X &\mapsto X\\ Y &\mapsto -Y\end{aligned}$$

$$\begin{aligned}\varepsilon : k[X,Y]/X^2+2Y^2-1 &\to k\\ X &\mapsto 1\\ Y &\mapsto 0\ .\end{aligned}$$

The k-group G is a torus, indeed the equation

$$\begin{aligned}k[X,Y]/X^2-2Y^2-1\otimes_k l &\simeq l[X,Y]/X^2-2Y^2-1\\ &\simeq l[X,Y]/(X+\pi_l Y)(X-\pi_l Y)-1\\ &\simeq l[U,V]/UV-1\end{aligned}$$

shows that $G \times_{\mathrm{spec}(k)} \mathrm{spec}(l) \simeq \mathbb{G}_m/l$. The reduced Bruhat-Tits building $\mathrm{BT}^R(G,k)$ is a singleton $\{x\}$. The point x is a (rational) special point of $\mathrm{BT}^R(G,k)$ and $i_{K/k}(x) \in \mathrm{BT}^R(G,K)$ is special for any finite extension K/k. The group G is not split over k, it is split over l. Let us make explicit the group $G_{x,0}$. We need to find an extension K/k such that G is split over K, $i_{K/k}(x)$ is special, and $r = 0 \in \mathrm{ord}(K)$. The field $K = l$ works. By definition the k-analytic group $G_{x,0}$ is egal to $\mathrm{pr}_{l/k}(\widehat{\mathfrak{G}}_\eta)$, where $\mathfrak{G}$ is the l°-Demazure group scheme attached to $i_{l/k}(x)$. By the previous example 2.5.6, in the coordinate U, V, $\mathfrak{G} = \mathrm{spec}(l^\circ[U,V]/UV-1)$. Thus in the coordinate X, Y, $\mathrm{Hopf}(\mathfrak{G})$ is egal to the l°-subalgebra of $l[X,Y]/X^2-2Y^2-1$ generated by $l^\circ, X+\pi_l Y, X-\pi_l Y$. By 2.5.3, the k-affinoid algebra of $G_{x,0}$ is the completion of $\mathrm{Hopf}(G) = k[X,Y]/X^2-2Y^2-1$ relatively to the norm $|\ |_{\mathrm{Hopf}(\mathfrak{G})}\ |_{\mathrm{Hopf}(G)}$ (recall that $|\ |_{\mathrm{Hopf}(\mathfrak{G})}$ is a norm on $\mathrm{Hopf}(G \times_k l)$). So let us make as explicit as possible the norm $|\ |_{\mathrm{Hopf}(\mathfrak{G})}\ |_{\mathrm{Hopf}(G)}$. By definition, we have

$$\begin{aligned} |\ |_{\mathrm{Hopf}(\mathfrak{G})} : \mathrm{Hopf}(G \times_k l) &\to \mathbb{R}_{\geq 0} \\ f &\mapsto \inf_{\lambda \in l^\times} \{|\lambda| \mid f \in \lambda(\mathrm{Hopf}(\mathfrak{G}) \otimes 1) \subset \mathrm{Hopf}(G \times_k l)\}. \end{aligned}$$

We deduce that

$$\begin{aligned} |\ |_{\mathrm{Hopf}(\mathfrak{G})} : l[X,Y]/X^2-2Y^2-1 &\to \mathbb{R}_{\geq 0} \\ f &\mapsto \inf_{\lambda \in l^\times} \{|\lambda| \mid f \in \lambda(< l^\circ, X-\pi_l Y, X+\pi_l Y >) \subset \mathrm{Hopf}(G \times_k l)\}. \end{aligned}$$

And so, by restriction

$$\begin{aligned} |\ |_{\mathrm{Hopf}(\mathfrak{G})}\ |_{\mathrm{Hopf}(G)} : k[X,Y]/X^2-2Y^2-1 &\to \mathbb{R}_{\geq 0} \\ f &\mapsto \inf_{\lambda \in l^\times} \{|\lambda| \mid f \in \lambda(< l^\circ, X-\pi_l Y, X+\pi_l Y >) \subset \mathrm{Hopf}(G \times_k l)\}. \end{aligned}$$

We have to complete $k[X,Y]/X^2-2Y^2-1$ relatively to this norm, in order to simplify notation let us put $||\ || = |\ |_{\mathrm{Hopf}(\mathfrak{G})}\ |_{\mathrm{Hopf}(G)}$.

Let us compute the value $||X||$. By definition it is egal to

$$\inf_{\lambda \in l^\times} \{|\lambda| \mid X \in \lambda(< l^\circ, X-\pi_l Y, X+\pi_l Y >) \subset \mathrm{Hopf}(G \times_k l)\}.$$

Since

$$\begin{aligned} X &\notin (< l^\circ, X-\pi_l Y, X+\pi_l Y >) \subset \mathrm{Hopf}(G \times_k l) \\ \pi_l X &\notin (< l^\circ, X-\pi_l Y, X+\pi_l Y >) \subset \mathrm{Hopf}(G \times_k l) \\ 2X = \pi_l^2 X &\in (< l^\circ, X-\pi_l Y, X+\pi_l Y >) \subset \mathrm{Hopf}(G \times_k l), \end{aligned}$$

we deduce that $||X|| = |2^{-1}| = e$ (2 is a uniformizer of k). Let us now compute the value $||Y||$. By definition it is egal to

$$\inf_{\lambda \in l^\times} \{|\lambda| \mid Y \in \lambda(< l^\circ, X - \pi_l Y, X + \pi_l Y >) \subset \mathrm{Hopf}(G \times_k l)\}.$$

Since

$$\begin{aligned} Y &\notin (< l^\circ, X - \pi_l Y, X + \pi_l Y >) \subset \mathrm{Hopf}(G \times_k l) \\ \pi_l Y &\notin (< l^\circ, X - \pi_l Y, X + \pi_l Y >) \subset \mathrm{Hopf}(G \times_k l) \\ 2X = \pi_l^2 Y &\notin (< l^\circ, X - \pi_l Y, X + \pi_l Y >) \subset \mathrm{Hopf}(G \times_k l) \\ 2\pi_l X = \pi_l^3 Y &\in (< l^\circ, X - \pi_l Y, X + \pi_l Y >) \subset \mathrm{Hopf}(G \times_k l), \end{aligned}$$

we deduce that $||Y|| = |\pi_l^{-3}| = e^{\frac{3}{2}}$.
By completion, the k-Banach algebra of $G_{x,0}$ is egal to

$$k\{e^{-1}X, (e^{\frac{3}{2}})^{-1}Y\}/X^2 - 2Y^2 - 1 \quad , \quad || \; ||$$

where $k\{e^{-1}X, (e^{\frac{3}{2}})^{-1}Y\}$ is the k-algebra

$$\{\sum_{k_1,k_2} a_{k_1k_2} X^{k_1} Y^{k_2} \mid \; |a_{k_1k_2}| e^{k_1} (e^{\frac{3}{2}})^{k_2} \to 0 \text{ as } k_1 + k_2 \to \infty\} \subset k[[X,Y]].$$

Let us check directly that the k-affinoid algebra of $G_{x,0}$ is $k\{e^{-1}X, (e^{\frac{3}{2}})^{-1}Y\}/X^2-2Y^2 - 1 \;\; , \;\; || \; ||$.

We need to check that $(k\{e^{-1}X, (e^{\frac{3}{2}})^{-1}Y\}/X^2 - 2Y^2 - 1)\hat{\otimes}_k l$ is isomorphic to the l-affinoid algebra of $\widehat{\mathfrak{G}}_\eta$. In the coordinates U, V, the l-affinoid algebra of $\widehat{\mathfrak{G}}_\eta$ is $l\{U, V\}/UV - 1$. The l-algebra $(k\{e^{-1}X, (e^{\frac{3}{2}})^{-1}Y\}/X^2 - 2Y^2 - 1)\hat{\otimes}_k l$ is isomorphic to $l\{e^{-1}X, (e^{\frac{3}{2}})^{-1}Y\}/X^2 - 2Y^2 - 1)$.

The isomorphism previously considered $l[X,Y]/X^2-2Y^2-1 \simeq l[U,V]/UV-1$ induces maps

$$\begin{array}{ccc} l\{e^{-1}X, (e^{\frac{3}{2}})^{-1}Y\}/X^2 - 2Y^2 - 1 & \leftrightarrow & l\{U,V\}/UV - 1 \\ X + \pi_l Y & \leftarrow\!\shortmid & U \\ X - \pi_l Y & \leftarrow\!\shortmid & V \\ X & \mapsto & \dfrac{U+V}{2} \\ Y & \mapsto & \dfrac{U-V}{2\pi_l}. \end{array}$$

These maps are mutual inverse k-Banach algebras isometries.

APPENDIX A: About Moy-Prasad isomorphism (part of a work in progress)

In this Appendix we discuss Question 2.5.11. We work in order to answer if there exists an isomorphism

$$H_s(k)/H_r(k) \simeq \mathfrak{h}_s(k)/\mathfrak{h}_r(k).$$

for some rational numbers $0 < \frac{r}{2} \leq s \leq r$ and any Galois stable k-affinoid rational potentially Demazure subgroup H of G^{an}.

Recall that the filtration H_r is defined as the projection of $\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta$ where $\mathfrak{G}$ and K are as in 2.4.9 and 2.4.10 ($H_r = \mathrm{pr}_{K/k}(\widehat{\Gamma_{e(K,k)r}(\mathfrak{G})}_\eta)$. The filtration on Lie algebra is obtained by a similar process taking the K°-Lie algebra of $\mathfrak{G}$ (see section 2.4.3). The K-rational points of H_r are $\Gamma_{e(K,k)r}(\mathfrak{G})(K^\circ)$ and the k-rational points of H_r are $\Gamma_{e(K,k)r}(\mathfrak{G})(K^\circ) \cap G(k)$ (see in the beginning of the second part of the proof of 2.5.25 below for more details). Similarly the K-rational points of $\mathfrak{h}_r$ are $\mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))(K^\circ)$ and the k-rational points of $\mathfrak{h}_r$ are $\mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))(K^\circ) \cap \mathfrak{g}(k)$.

In this appendix, the idea is to use the identity written in [43, §2.8, proof of Lemma]. In [43], Yu writes (we translate here with our notations) in §2.8, in the second line of the proof of Lemma, that given a K°-smooth affine scheme $\mathfrak{G}$ and integers $0 < a \leq b \leq 2a$, there is a functorial isomorphism

$$\Gamma_b(\mathfrak{G})(K^\circ)/\Gamma_a(\mathfrak{G})(K^\circ) \simeq \mathrm{Lie}(\Gamma_b(\mathfrak{G}))(K^\circ)/\mathrm{Lie}(\Gamma_a(\mathfrak{G}))(K^\circ).$$

There is no proof of this fact in [43], and we did not find a proof in the litterature. In this appendix we construct explicitely an injective morphism of groups for integers r, s such that $0 < \frac{r}{2} \leq s \leq r$

$$\Psi : \Gamma_s(\mathfrak{G})(K^\circ)/\Gamma_r(\mathfrak{G})(K^\circ \simeq \mathrm{Lie}(\Gamma_s(\mathfrak{G}))(K^\circ)/\mathrm{Lie}(\Gamma_r(\mathfrak{G}))(K^\circ) \quad ,$$

and we conjecture that it is surjective (we work under certains hypothesis as explained after).

In the litterature such isomorphism (see for example [1, §1.5] , [41, Lemma 1.3]) is constructed in the situation of reductive group and it is constructed using a maximal torus, roots groups and splitting the reductive group. Here we do not use a torus, the approach is algebraic. We use the explicit description of Hopf algebras of congruence groups of $\mathfrak{G}$. For any k°-group scheme $\mathfrak{G}$, we use that $\mathrm{Lie}(\mathfrak{G})$ is explicitely given by $\mathrm{Hom}_{k^{\circ}-\mathrm{mod}}(I/I^2, k^{\circ})$ where I is the augmentation ideal of $\mathrm{Hopf}(\mathfrak{G})$.

About proof of [43, Proof of Lemma 2.8]

Let k be a non arch. local field, and let π , k°, the usual associated notations.

Lemma 2.5.12. *Let $\mathfrak{G} = \mathrm{spec}(\mathfrak{A})$ be an affine smooth (thus flat) k°-group scheme. Let $\mathfrak{A}$ be the k°-Hopf algebra of $\mathfrak{G}$. Let $\varepsilon : \mathfrak{A} \to k^{\circ}$ be the counit. Let $I := \ker(\varepsilon)$ be the augmentation ideal. Let I^2 be the ideal II, it is a submodule of I. Then*

1. *I/I^2 is a free k°-module.*

2. *There exists a section $\mathfrak{s}$ of the projection $I \xrightarrow{p} I/I^2$. It is a morphism of k°-modules*

$$\mathfrak{s} : I/I^2 \to I$$

such that $p \circ \mathfrak{s} = \mathrm{Id}$.

Proof. 1. By [15, remark6.7] I/I^2 is projective, thus by [26], it is free.

2. It is a direct consequence of the previous assertion. Indeed, choose a basis $g_1, \dots, g_n$ of I/I^2; and choose also $\tilde{g_1}, \dots, \tilde{g_n}$, preimages of $g_1, \dots, g_n$ under p. Theses choices induce a section $\mathfrak{s}$ of p, sending g_i to $\tilde{g_i}$.

□

Lemma 2.5.13. *Let $\mathfrak{G} = \mathrm{spec}(\mathfrak{A})$ be a flat affine k°-scheme satisfying the hypothesis of [32, Lemma 5.1]. Then*

1. *$\mathfrak{A}$ contains no non-zero k°-divisble element.*

2. *The ideal of augmentation I and its square power I^2 contain no non-zero k°-divisible element.*

3. *$\mathfrak{A}$, I and I^2 are free k°-modules.*

Proof. The first assertion is the conclusion of [32, Lemma 5.1]. The second assertion is implied by the first one since I and I^2 are contained in $\mathfrak{A}$. The third assertion is a consequence of the assertion "Let V be a vector space of at most countable dimension over K and L an k°-submodule of V such that L contains no non-zero k°-divisible elements. Then L is a free k°-module." written and proved in [32, proof of Lemma 5.2]

$\square$

Remark 2.5.14. *If $\mathfrak{G}$ is a k°-Demazure group scheme, $\mathfrak{G}$ satisfies hypothesis of [32, Lemma 5.1]*

Lemma 2.5.15. *Let* R *be a ring and let A be a* R*-Hopf algebra. Let I be the augmentation ideal of A. Let $\Delta : A \to A \otimes A$ be the comultiplication map. Then*

$$\forall g \in I \qquad \Delta(g) = g \otimes 1 + 1 \otimes g \mod I \otimes I \quad .$$

Proof. It is a well-know fact which is a direct consequence of the axioms "$(\mathrm{Id} \otimes \varepsilon)\Delta = \mathrm{Id}$ " and " $(\varepsilon \otimes \mathrm{Id})\Delta = \mathrm{Id}$ " of Hopf algebras, writing $\Delta(g)$ as a sum of tensors and using that $\varepsilon(g) = 0$. $\square$

Let us fix from now on a smooth k°-scheme $\mathfrak{G} = \mathrm{spec}(\mathfrak{A})$ satisfying the hypothesis of Lemmas 2.5.12 and 2.5.13. Let ε be its counit and I the augmentation ideal of $\mathfrak{A}$. Let $n \geq 0$ be an integer. We recall that the n-th congruence subgroup of $\mathfrak{G}$ is an affine k°-scheme with Hopf algebra $\mathfrak{A}_n := \mathfrak{A}[\pi^{-n}I] = \mathfrak{A} + \sum_{k\geq 1} \pi^{-kn}I^k \subset \mathfrak{A} \otimes_{k^\circ} k$ (see Proposition 2.1.10).

Lemma 2.5.16. *Let I_n be the augmentation ideal of $\mathfrak{A}_n$. Then*

1. *The ideal I_n is egal to $(\pi^{-n}I)$, the ideal of the ring $\mathfrak{A}_n$ generated by the k°-module $\pi^{-n}I \subset \mathfrak{A}_n$.*
2. *The ideal I_n is egal to $\sum_{k\geq 1} \pi^{-nk}I^k \subset \mathfrak{A} \otimes_{k^\circ} k$.*

Proof. 1. The counit $\varepsilon_{\mathfrak{A}_n}$ is the restriction to $\mathfrak{A}_n$ of the counit of $\mathfrak{A} \otimes_{k^\circ} k$, and the counit of $\mathfrak{A} \otimes_{k^\circ} k$ is $\varepsilon \otimes \mathrm{Id}$. Let $x \in \mathfrak{A}_n$. Since $\mathfrak{A}_n = \mathfrak{A}[\pi^{-n}I]$, we can write x as a finite sum

$$x = a + \sum_{\substack{\nu=\nu_1\ldots\nu_j\ldots\nu_{k_\nu} \\ k_\nu \geq 1}} a_\nu \pi^{-n} i_{\nu_1} \ldots \pi^{-n} i_{\nu_{k_\nu}} \qquad a \in \mathfrak{A}, i_{\nu_j} \in I$$

Assume $x \in I_n$. So $\varepsilon_{\mathfrak{A}_n}(x) = 0$, thus

$$0 = \varepsilon(a) + \sum_{\substack{\nu=\nu_1\ldots\nu_j\ldots\nu_{k_\nu} \\ k_\nu \geq 1}} \varepsilon(a_\nu)\pi^{-n}\varepsilon(i_{\nu_1}) \ldots \pi^{-n}\varepsilon(i_{\nu_{k_\nu}}) \qquad a \in \mathfrak{A}, i_{\nu_j} \in I.$$

This implies $\varepsilon(a) = 0$. So $a \in I$. Thus $a \in \pi^{-n}I$ and so $x \in (\pi^{-n}I)$.

So we have proved that $I_n \subset (\pi^{-n}I)$. It is obvious that the reverse inclusion holds, indeed if $x \in (\pi^{-n}I)$, then

$$x = \sum_{\nu} a_\nu \pi^{-n} i_\nu \qquad a \in \mathfrak{A}_n, i_\nu \in I$$

and applying $\varepsilon_{\mathfrak{A}_n}$ gives zero. This finishes the first assertion.

2. Let us prove the formula $(\pi^{-n}I) = \sum_{k \geq 1} \pi^{-kn} I^k$. The k°-module $\sum_{k \geq 1} \pi^{-kn} I^k$ is stable by multiplication by element of $\mathfrak{A}_n$, so it is an ideal. Moreover $\pi^{-n}I$ is contained in this ideal, so $(\pi^{-n}I) \subset \sum_{k \geq 1} \pi^{-kn} I^k$.

   Let us now show that $\sum_{k \geq 1} \pi^{-kn} I^k \subset (\pi^{-n}I)$. It is enough to show that for any $k \geq 1$, we have $\pi^{-kn} I^k \subset (\pi^{-n}I)$. So let $x \in \pi^{-kn} I^k$. We have

$$\begin{aligned} x = \pi^{-kn} & \sum_{\nu = \nu_1 \ldots \nu_j \ldots \nu_k} a_\nu i_{\nu_1} \ldots i_{\nu_k} \qquad a_\nu \in \mathfrak{A} \;\; i_{\nu_j} \in I \\ &= \sum_{\nu = \nu_1 \ldots \nu_j \ldots \nu_k} a_\nu \pi^{-n} i_{\nu_1} \ldots \pi^{-n} i_{\nu_k} \qquad a_\nu \in \mathfrak{A} \;\; i_{\nu_j} \in I \end{aligned}$$

   So $x \in (\pi^{-n}I)$, and this ends the proof.

□

Lemma 2.5.17. *Let $p \geq 0$, then $\pi^p I \cap I^2 = \pi^p I^2$.*

Proof. Recall that $\mathfrak{A}$, I, I^2 and I/I^2 are free k°-modules by 2.5.13 and 2.5.12. We will use in this proof that I/I^2 and I^2 are free. Choose a k°-basis $\{e_k\}_{k \in T}$ of I^2. Choose preimages $\{e_k\}_{k \in S}$ under the projection $I \xrightarrow{p} I/I^2$ of a k°-basis $\{\underline{e_k}\}_{k \in S}$ of I/I^2 ($S \cap T = \emptyset$). Let us prove that $\{e_k\}_{k \in S \cup T}$ is a k°-basis of I. Let us prove that this is generator. Let $x \in I$. Write the image $[x]$ of x under p as $\sum_{k \in S} \lambda_k \underline{e_k}$. Then $x - \sum_{k \in S} \lambda_k e_k$ is contained in I^2. So $x - \sum_{k \in S} \lambda_k e_k = \sum_{k \in T} \lambda_k e_k$. This shows that $\{e_k\}_{k \in S \cup T}$ is generator. Let us show that this is a free family. So assume $\sum_{k \in S \cup T} \lambda_k e_k = 0$. Then $\sum_{k \in S} \lambda_k \underline{e_k} = 0$. So $\lambda_k = 0$ for all $k \in S$. So $\sum_{k \in T} \lambda_k e_k = 0$. Thus $\lambda_k = 0$ for all $k \in T$. So the family if free. Consequently the family is a basis.

Now let $x \in \pi^p I \cap I^2$. Write $x = \sum_{k \in S \cup T} \lambda_k e_k$. Since $x \in I^2$ we have $\lambda_k = 0$ for all $k \in S$. Since $x \in \pi^p I$ we have $\lambda_k \in \pi^p k^\circ$ for all $k \in S \cup T$.

So we conclude that $x = \sum_{k\in T} \lambda_k e_k$ with $\lambda_k \in \pi^p k^\circ$ for all $k \in T$. This implies that $x \in \pi^p I^2$. So $\pi^p I \cap I^2 \subset \pi^p I^2$.

The reverse inclusion $\pi^p I \cap I^2 \supset \pi^p I^2$ is obvious since $\pi^p I^2 \subset I^2$ and $\pi^p I^2 \subset \pi^p I$. The lemma is proved.

$\square$

Lemma 2.5.18. *Let $n \geq 0$. Then ${I_n}^2 = \sum_{k\geq 2} \pi^{-kn} I^k$.*

Proof. Let us show first that ${I_n}^2 \supset \sum_{k\geq 2} \pi^{-kn} I^k$. It is enough to show that for any $k \geq 2$, $\pi^{-kn} I^k \subset I_n^2$. So let $x \in \pi^{-kn} I^k$. We write

$$x = \pi^{-kn} \sum_{\nu=\nu_1\ldots\nu_j\ldots\nu_k} a_\nu i_{\nu_1} \ldots i_{\nu_k} \quad a_\nu \in \mathfrak{A} \;\; i_{\nu_j} \in I.$$

So

$$x = \sum_{\nu=\nu_1\ldots\nu_k} a_\nu \pi^{-n} i_{\nu_1} \ldots \pi^{-n} i_{\nu_k} \quad a_\nu \in \mathfrak{A} \;\; i_{\nu_j} \in I.$$

So $x \in (\pi^{-n} I)^k$. Thus $x \in {I_n}^k$. Consequently $x \in {I_n}^2$ since $k \geq 2$.

Let us now show that ${I_n}^2 \subset \sum_{k\geq 2} \pi^{-kn} I^k$. Let $x \in {I_n}^2$, it can be written as

$$x = \sum_{\beta=\beta_1,\beta_2} a_\beta i_{\beta_1} i_{\beta_2} \quad a_\beta \in \mathfrak{A}_n \quad i_{\beta_1} \in I_n \quad i_{\beta_2} \in I_n.$$

So it is enough to show that for each β, we have $a_\beta i_{\beta_1} i_{\beta_2} \in \sum_{k\geq 2} \pi^{-kn} I^k$. Since $\mathfrak{A}_n = \mathfrak{A} + \sum_{k\geq 1} \pi^{-kn} I^k$, the element $a_\beta \in \mathfrak{A}_n$ can be written as

$$a_\beta = a + \sum_{k\geq 1} \pi^{-kn} i_{\beta k} \quad a \in \mathfrak{A} \quad i_{\beta k} \in I^k.$$

Similarly by 2.5.16, for $j = 1, 2$ we can write

$$i_{\beta_j} = \sum_{k\geq 1} \pi^{-kn} i_{\beta_j k} \;\; i_{\beta_j k} \in I^k.$$

Now by distributivity the element $a_\beta i_{\beta_1} i_{\beta_2}$ is a sum of terms of the form $a\, \pi^{-kn} i_{\beta_1 k}\, \pi^{-k'n} i_{\beta_2 k'}$ $k, k' \geq 1$ or of the form $\pi^{-k''n} i_{\beta k''}\, \pi^{-kn} i_{\beta_1 k}\, \pi^{-k'n} i_{\beta_2 k'}$ $k'', k, k' \geq 1$. Thus each term is included in $\sum_{k\geq 2} \pi^{-kn} I^k$. So $a_\beta i_{\beta_1} i_{\beta_2}$ is included in $\sum_{k\geq 2} \pi^{-kn} I^k$. Consequently x is included in $\sum_{k\geq 2} \pi^{-kn} I^k$ as required.

$\square$

Lemma 2.5.19. *Let $a \geq b \geq 0$. Then $I_b \cap {I_a}^2 = {I_b}^2$.*

Proof. Recall that by 2.5.16 and 2.5.18 we have

$$I_b = \pi^{-b}I + \pi^{-2b}I^2 + \pi^{-3b}I^3 + \pi^{-4b}I^4 + \dots$$

and that

$${I_a}^2 = \pi^{-2a}I^2 + \pi^{-3a}I^3 + \pi^{-4a}I^4 + \dots \quad .$$

Put $M = \pi^{-b}I$ and $N = \pi^{-2b}I^2 + \pi^{-3b}I^3 + \pi^{-4b}I^4 + \dots$ so that $I_b = M + N$; put also $P = {I_a}^2$. Thus we want to prove that $(M+N) \cap P = N$, since by 2.5.18 we have $N = {I_b}^2$. We have $N \subset P$. So the inclusion $N \subset (M+N) \cap P$ is obvious.

Let us prove the reverse inclusion. We have

$$(M+N) \cap P = (M \cap P) + N.$$

Indeed $(M \cap P) \subset (M+N) \cap P$ and $N \subset (M+N) \cap P$ and so $(M \cap P) + N \subset (M+N) \cap P$. Reciprocally let $x \in (M+N) \cap P$ thus $x = m+n$ with $m \in M$ and $n \in N$. The element $m+n$ and n are contained in P, so m is in p so $x = m+n$ is in $(M \cap P) + N$.

We are thus reduced to prove that $(M \cap P) \subset N$. Let $x \in M \cap P$. We have $P = \pi^{-2a}I^2 + \pi^{-3a}I^3 + \pi^{-4a}I^4 + \dots$. There is an integer $D \geq 2$ such that $x \in \sum_{k=2}^{D} \pi^{-2ka}I^k$. So $\pi^{2Da}x \in \sum_{k=2}^{D} \pi^{2a(D-k)}I^k$. For all $2 \leq k \leq D$, $\pi^{2a(D-k)}I^k \subset I^2$. So $\pi^{2Da}x \in I^2$. But by hypothesis, $x \in M = \pi^{-b}I$. So $\pi^{2Da} \in \pi^{2Da-b}I$. Thus $\pi^{2Da}x \in \pi^{2Da-b}I \cap I^2$. So by 2.5.17 $\pi^{2Da}x \in \pi^{2Da-b}I^2$. So $x \in \pi^{-b}I^2$. So $x \in N$ and this ends the proof.

$\square$

Maps Φ , Θ

Let $0 < s \leq r$. We have $I_s \subset I_r$, the kernel of the composed morphism

$$I_s \subset I_r \to I_r/{I_r}^2$$

is $I_s \cap {I_r}^2 = I_s^2$ by 2.5.19. So we get an injective morphism of k°-modules $I_s/{I_s}^2 \overset{\iota_{s,r}}{\to} I_r/{I_r}^2$, we sometimes write $I_s/{I_s}^2 \subset I_r/{I_r}^2$. It induces a morphism of k°-modules

$$\begin{array}{rccc} \Phi \ : & \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_r/{I_r}^2, k^\circ) & \to & \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_s/{I_s}^2, k^\circ) \\ & g & \mapsto & g \circ \iota_{s,r} \end{array} .$$

We have an inclusion morphism of k°-algebras $\mathfrak{A}_s \subset \mathfrak{A}_r$, it induces a map

$$\begin{array}{rccc} \Theta \ : & \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_r, k^\circ) & \to & \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_s, k^\circ) \\ & g & \mapsto & g\,|_{\mathfrak{A}_s} \end{array} .$$

Lemma 2.5.20. *Let $r \geq s$, then*

1. *Let $x \in \mathfrak{A}_r$, then there exists a positive integer N such that $\pi^N x \in \mathfrak{A}_s$.*
2. *Let $x \in I_r$, then there exists a positive integer N such that $\pi^N x \in I_s$*
3. *Let $x \in I_r/{I_r}^2$, then $\pi^{r-s}x \in I_s/{I_s}^2$.*

Proof. The first two assertions are direct consequences of the fact that for any positive integer n, we have

$$\mathfrak{A}_n = \mathfrak{A} + \sum_{k \geq 1} \pi^{-kn} I^k \subset \mathfrak{A} \otimes_{k^\circ} k$$

and

$$I_n = \sum_{k \geq 1} \pi^{-nk} I^k \subset \mathfrak{A} \otimes_{k^\circ} k.$$

Let us prove the third assertion. Let $x \in I_r/{I_r}^2$. Consider the commutative diagram

$$\begin{tikzcd} I_s \arrow[r, "\subset"] \arrow[d, "p_s"] & I_r \arrow[d, "p_r"] \\ I_s/{I_s}^2 \arrow[r, "\subset"] & I_r/{I_r}^2 \end{tikzcd} .$$

We have $I_r = \pi^{-r}I + {I_r}^2$ by 2.5.18. So we can choose a preimage $\tilde{x}$ of x under p_r in $\pi^{-r}I$. Then $\pi^{r-s}\tilde{x} \in \pi^{-s}I \subset I_s$. The projection $p_s(\pi^{r-s}\tilde{x}) \in I_s/{I_s}^2$ is egal to $\pi^{r-s}x$. So $\pi^{r-s}x \in I_s/{I_s}^2$.

□

Lemma 2.5.21. 1. *An element $f \in \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_s/{I_s}^2, k^\circ)$ is in the image of Φ if and only if for all $i \in I$, the image of i under the composed morphism*

$$I \subset I_s \xrightarrow{p_s} I_s/{I_s}^2 \xrightarrow{f} k^\circ$$

is inside $\pi^r k^\circ$.

2. *An element $f \in \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_s, k^\circ)$ is in the image of Θ if and only if for all $i \in I$, the image of i under the the composed morphism*

$$I \subset \mathfrak{A}_s \xrightarrow{f} k^\circ$$

is inside $\pi^r k^\circ$.

3. *The morphism Φ is injective.*
4. *The morphism Θ is injective.*

Proof. 1. Let $f \in \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_s/{I_s}^2, k^\circ)$. Assume it is in the image of Φ. So there is $g \in \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_r/{I_r}^2, k^\circ)$ such that f is the composed morphism

$$I_s/{I_s}^2 \overset{\iota_{s,r}}{\to} I_r/{I_r}^2 \overset{g}{\to} k^\circ.$$

Let $i \in I$, and let p_r be the morphism $I_r \overset{p_r}{\to} I_r/{I_r}^2$. We have

$$\begin{aligned} f \circ p_s(i) &= g \circ p_r(i) \\ &= g(p_r(i)) \\ &= g(p_r(\pi^r \pi^{-r} i)) \\ &= \pi^r g(p_r(\pi^{-r} i)) \in \pi^r k^\circ, \end{aligned}$$

as required. Reciprocally, assume for all $i \in I$ we have $f \circ p_s(i) \in \pi^r k^\circ$. The restriction of p_s to $\pi^{-s}I$, $\pi^{-s}I \to I_s/{I_s}^2$, is surjective; since $I_s = \pi^{-s}I + {I_s}^2$ by 2.5.18. So we deduce that for any $x \in I_s/{I_s}^2$ we have $f(x) \in \pi^{r-s}k^\circ$ (indeed let $x \in I_s/{I_s}^2$, then $x = p_s(\pi^{-s}i)$ so $\pi^s f(x) = \pi^s(f(p_s(i))) = f(p_s(i) \in \pi^r k^\circ)$. Now for any $x \in I_r/{I_r}^2$, $\pi^{r-s}x \in I_s/{I_s}^2$ by 2.5.20 and we put $g(x) := \pi^{-(r-s)} f(\pi^{r-s}x)$. This defines a morphism of k°-module $g : I_r/{I_r}^2 \to k^\circ$, such that $\Phi(g) = f$.

2. Let $f \in \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_s, k^\circ)$. Assume it is in the image of Θ. So there is $g \in \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_r, k^\circ)$ such that $f = g\mid_{\mathfrak{A}_s}$. Then for any $i \in I$, $f(i) = f(\pi^r\pi^{-r}i) = \pi^r f(\pi^{-r}i) \in \pi^r k^\circ$. Reciprocally, assume that for all $i \in I$, $f(i) \in \pi^r k^\circ$. We are going to construct a morphism $g : \mathfrak{A}_r \to k^\circ$ whose restriction to $\mathfrak{A}_s$ is f. We have that $\mathfrak{A}_r \subset \mathfrak{A}_s \otimes_{k^\circ} k$ and f induces a morphism of k-algebras $\mathfrak{A}_s \otimes_{k^\circ} k \overset{f\otimes \mathrm{Id}}{\to} k$. By restriction we obtain a morphism of ring $g : \mathfrak{A}_r \to k$.

   Recall that $\mathfrak{A}_r = \mathfrak{A}[\pi^{-r}I]$, and write and $x \in \mathfrak{A}_r$ as a finite sum

   $$x = \sum_{\nu=\nu_1\ldots\nu_{k_\nu}} a_\nu \pi^{-r} i_{\nu_1} \ldots \pi^{-r} i_{\nu_{k_\nu}} \quad a_\nu \in \mathfrak{A} \quad i_{\nu_j} \in I.$$

   The map g sends $\mathfrak{A}_r \ni \sum_{\nu=\nu_1\ldots\nu_{k_\nu}} a_\nu \pi^{-r} i_{\nu_1} \ldots \pi^{-r} i_{\nu_{k_\nu}}$ to

   $\sum_{\nu=\nu_1\ldots\nu_{k_\nu}} f(a_\nu)\pi^{-r} f(i_{\nu_1}) \ldots \pi^{-r} f(i_{\nu_{k_\nu}})$ it is in k°. So g is a morphism of k°-algebras. We have $g\mid_{\mathfrak{A}_s} = f$. This ends the proof of the assertion.

3. The map is injective, indeed let $g \in \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_r/{I_r}^2, k^\circ)$, assume $\Phi(g) = 0$. Let $x \in I_r/{I_r}^2$, then $\pi^{r-s}x \in I_s/{I_s}^2$ and so $0 = g(\pi^{r-s}x) = \pi^{r-s}g(x)$ so $g(x) = 0$, so $g = 0$. Thus $\ker(\Phi) = 0$.

4. Let $g \in \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_r, k^\circ)$, assume $\Theta(g) = 0$. Let $x \in \mathfrak{A}_r$, by 2.5.20, there is an N such that $\pi^N x \in \mathfrak{A}_s$. We have $0 = g(\pi^N x) = \pi^N g(x)$, so $g(x) = 0$. Thus $g = 0$. Consequently $\ker(\Theta) = 0$.

$\square$

Lemma 2.5.22. *Let $n \in \mathbb{N}$. Recall that we have a canonical injective morphism of k°-modules $I/I^2 \subset I_n/{I_n}^2$. Let $\mathfrak{s} : I/I^2 \to I$ be a section of $p : I \to I/I^2$.*

Then $\mathfrak{s}$ induces a section $I_n/{I_n}^2 \overset{\mathfrak{s}_n}{\to} I_n$ of the projection $I_n \overset{p_n}{\to} I_n/{I_n}^2$. This induces an explicit bijection

$$Z : section_p(I/I^2, I) \underset{\mathfrak{s} \mapsto \mathfrak{s}_n}{\simeq} section_{p_n}(I_n/{I_n}^2, I_n)$$

such that $\mathfrak{s}_n \mid_{I/I^2} = \mathfrak{s}$ (here $section_p(I/I^2, I)$ means all the section of $p : I \to I/I^2$, and similarly for p_n).

Proof. By 2.5.18, $I_n = \pi^{-n}I + {I_n}^2$. So the natural composed morphism

$$\pi^{-n}I \to I_n \to I_n/{I_n}^2$$

is surjective. The kernel is $\pi^{-n}I \cap {I_n}^2 = \pi^{-n}{I_n}^2$ by 2.5.19. So there is a canonical isomorphism

$$\pi^{-n}I/\pi^{-n}I^2 \simeq I_n/{I_n}^2.$$

So for any $x \in I_n/{I_n}^2$, $\pi^n x \in I/I^2$ by 2.5.20. Now let $\mathfrak{s} \in \text{section}_p(I/I^2, I)$. Let us define a map $I_n/{I_n}^2 \overset{\mathfrak{s}_n}{\to} I_n$ by

$$\mathfrak{s}_n(x) = \pi^{-n}(\mathfrak{s}(\pi^n x)) \quad \forall x \in I_n/{I_n}^2.$$

The map $\mathfrak{s}_n$ is a morphism of k°-modules. Moreover we have, for any $x \in I_n/{I_n}^2$

$$\pi^n p_n(\mathfrak{s}_n(x)) = \pi^n p_n(\pi^{-n}\mathfrak{s}(\pi^n x)) = p_n(\mathfrak{s}(\pi^n x)) = p(\mathfrak{s}(\pi^n x)) = \pi^n x$$

So $p_n(\mathfrak{s}_n(x)) = x$. Thus $\mathfrak{s}_n$ is a section of p_n. So we have introduced a map $Z : \mathfrak{s} \mapsto \mathfrak{s}_n$. Let $\mathfrak{s}$ a section $I/I^2 \to I$ and let $x \in I/I^2$ then $\mathfrak{s}_n(x) = \pi^{-n}\mathfrak{s}(\pi^n x) = \mathfrak{s}(x)$, so $\mathfrak{s} \mid_{I/I^2} = \mathfrak{s}$. This immediately implies that the previously introduced map Z is injective. The map Z is surjective, indeed for any section $\mathfrak{s}_n : I_n/{I_n}^2 \to I_n$, we have $\mathfrak{s}_n = Z(\mathfrak{s}_n \mid_{I/I^2})$ (indeed let $x \in I_n/{I_n}^2$, we have the identity $\mathfrak{s}(x) = \pi^{-n}\pi^n\mathfrak{s}_n(x) = \pi^{-n}\mathfrak{s}_n(\pi^n x) = \pi^{-n}\mathfrak{s}_n \mid_{I/I^2} (\pi^n x) = Z(\mathfrak{s}_n \mid_{I/I^2})(x)$).

□

Let us now state the theorem

Theorem 2.5.23. *Let r, s be integers such that $0 < \frac{r}{2} \leq s \leq r$. There is an explicit injective morphism of groups*

$$\Gamma_s(\mathfrak{G})(k^\circ)/\Gamma_r(\mathfrak{G})(k^\circ) \simeq \text{Lie}(\Gamma_s(\mathfrak{G}))(k^\circ)/\text{Lie}(\Gamma_r(\mathfrak{G}))(k^\circ).$$

Proof. Recall that $\Gamma_n(\mathfrak{G})(k^\circ) = \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_n, k^\circ)$ and $\mathrm{Lie}(\Gamma_s(\mathfrak{G}))(k^\circ) = \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_n/{I_n}^2, k^\circ)$ for $n \geq 0$.

Let $\mathfrak{s} : I_s/{I_s}^2 \to I_s$ be a section of $p_s : I_s \to I_s/{I_s}^2$. Let

$$\begin{array}{rcl} \Psi_{\mathfrak{s}} & : & \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_s, k^\circ) \to \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_s/{I_s}^2, k^\circ) \\ & & \qquad x \quad \mapsto \quad x\,|_{I_s} \circ \mathfrak{s} \end{array}.$$

Let us prove that the composed map Ψ

$$\mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_s, k^\circ) \xrightarrow{\Psi_{\mathfrak{s}}} \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_s/{I_s}^2, k^\circ) \to \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_s/{I_s}^2, k^\circ)/\mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_r/{I_r}^2, k^\circ)$$

does not depend on $\mathfrak{s}$. So let $\mathfrak{s}'$ be another section $I_s/{I_s}^2 \xrightarrow{\mathfrak{s}'} I_s$. We need to show that

$$x\,|_{I_s} \circ \mathfrak{s} - x\,|_{I_s} \circ \mathfrak{s}' \in \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_r/{I_r}^2, k^\circ),$$

thus by 2.5.21 we need to show that

$$(x\,|_{I_s} \circ \mathfrak{s} - x\,|_{I_s} \circ \mathfrak{s}')(p_s(i)) \in \pi^r k^\circ \quad \forall i \in I.$$

Put $a = p_s(i)$ and let us study $\mathfrak{s}(a) - \mathfrak{s}'(a)$. We have $p_s(\mathfrak{s}(a) - \mathfrak{s}'(a)) = a - a = 0$ so $\mathfrak{s}(a) - \mathfrak{s}(a') \in {I_s}^2$. Moreover $\mathfrak{s}(a) = \mathfrak{s}(p_s(i)) = \mathfrak{s}(p_s(\pi^s\pi^{-s}i)) = \pi^s\mathfrak{s}(p_s(\pi^{-s}i)) \in I$. Similarly $\mathfrak{s}'(a) \in I$. So $\mathfrak{s}(a) - \mathfrak{s}'(a) \in I$. Consequently $\mathfrak{s}(a) - \mathfrak{s}'(a) \in {I_s}^2 \cap I$. By 2.5.19, we deduce that $\mathfrak{s}(a) - \mathfrak{s}'(a) \in I^2$. So we have

$$\begin{aligned} (x\,|_{I_s} \circ \mathfrak{s} - x\,|_{I_s} \circ \mathfrak{s}')(p_s(i)) &= x(\mathfrak{s}(p_s(i)) - \mathfrak{s}'(p_s(i))) \\ &= x(\mathfrak{s}(a) - \mathfrak{s}'(a)) \\ &= x(\gamma) \text{ with } \gamma \in I^2 \end{aligned}$$

Recall that $x \in \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_s, k^\circ)$; the algebra $\mathfrak{A}_s$ is egal to $\mathfrak{A}[\pi^{-s}I]$, so for any $i \in I$, we have $x(i) \in \pi^s k^\circ$. We deduce that $x(\gamma) \in \pi^{2s}k^\circ$. Since $0 < \frac{r}{2} \leq s \leq r$, we deduce $\pi^{2s}k^\circ \subset \pi^r k^\circ$. So $x(\gamma) \in \pi^r k^\circ$. So we have finished to prove that Ψ does not depend on the section $\mathfrak{s}$. So we get a well-defined map

$$\begin{array}{rcl} \Psi & : & \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_s, k^\circ) \to \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_s/{I_s}^2, k^\circ)/\mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_r/{I_r}^2, k^\circ) \\ & & \qquad x \quad \mapsto \quad [x\,|_{I_s} \circ \mathfrak{s}] \end{array},$$

which does not depend on $\mathfrak{s}$.

Let us now show that Ψ is a morhism of groups. The source is denoted multiplicatively and the target additively. So let $x, y \in \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_s, k^\circ)$. Take a section $\mathfrak{s} : I_s/I_s^2 \to I_s$. We need to show that $\Psi_{\mathfrak{s}}(xy) = \Psi_{\mathfrak{s}}(x) + \Psi_{\mathfrak{s}}(y)$ mod $\mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_r/{I_r}^2, k^\circ)$. By 2.5.21, it is enough to show that for an $i \in I$, $\Psi_{\mathfrak{s}}(xy)(p_s(i)) - \Psi_{\mathfrak{s}}(x)(p_s(i)) - \Psi_{\mathfrak{s}}(y)(p_s(i)) \in \pi^r k^\circ$. We have $\Psi_{\mathfrak{s}}(xy)(p_s(i)) - \Psi_{\mathfrak{s}}(x)(p_s(i)) - \Psi_{\mathfrak{s}}(y)(p_s(i)) = xy(\mathfrak{s}(p_s(i)) - x(\mathfrak{s}(p_s(i))) - y(\mathfrak{s}(p_s(i)))$. Put $a = \mathfrak{s}(p_s(i))$, as we have already explained before in a similar situation, it is in I. By definition xy is the following composed morphism

$$\mathfrak{A}_s \overset{\Delta}{\to} \mathfrak{A}_s \otimes_{k^\circ} \mathfrak{A}_s \overset{x \otimes y}{\to} k^\circ \otimes_{k^\circ} k^\circ \simeq k^\circ.$$

Thus $xy(a) = (x \otimes y)(\Delta(a))$. By Lemma 2.5.15 applied to $\mathrm{R} = k^\circ$, $A = \mathfrak{A}$, we obtain $\Delta(a) = a \otimes 1 + 1 \otimes a \mod I \otimes I$. Thus $xy(a) = x(a) + y(a) \mod \pi^{2s} k^\circ$. So Ψ is a morphism of groups.

Let us now prove that $\ker(\Psi) = \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_r, k^\circ)$. Let us first prove the inclusion $\ker(\Psi) \subset \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_r, k^\circ)$. So let $x \in \ker(\Psi)$. By 2.5.21, it is enough to show that $x(i) \in \pi^r k^\circ$ for all $i \in I$. As in the proof of 2.5.17, choose a basis $\{e_k\}_{k \in T}$ of I^2 and complete it by $\{e_k\}_{s \in S}$ in order to obtain a basis $\{e_k\}_{S \cup T}$ of I. The family $\{e_k\}_{k \in S}$ induces a section $\mathfrak{s} : I/I^2 \to I$, which send $p(e_k)$ to e_k for any $k \in S$. By 2.5.22, we obtain a section of p_s whose restriction to I/I^2 is $\mathfrak{s}$. We denote it also by $\mathfrak{s}$. The element x is in $\ker(\Psi)$, this implies that $\Psi_{\mathfrak{s}}(x) \in \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_r/{I_r}^2, k^\circ)$. Let us fix an $i \in I$. Write $i = \sum_{k \in S \cup T} \lambda_k e_k \quad \lambda_k \in k^\circ$. Then $x(i) = \sum_{k \in S \cup T} \lambda_k x(e_k)$. Let us study $x(e_k)$ for any $k \in S \cup T$. If $k \in T$, then $e_k \in I^2$, and $x(e_k) \in \pi^{2s} k^\circ$ (by 2.5.21). Now if $k \in S$. Then by 2.5.21 $\Psi_{\mathfrak{s}}(x)(p_s(e_k)) \in \pi^r k^\circ$. Now $\Psi_{\mathfrak{s}}(x)(p_s(e_k)) = x(\mathfrak{s}(p_s(e_k))) = x(\mathfrak{s}(p(e_k))) = x(e_k)$. So $x(e_k) \in \pi^r k^\circ$. So $x(i) \in \pi^r k^\circ$. Consequently $x \in \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_r, k^\circ)$. So $\ker(\Psi) \subset \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_r, k^\circ)$. Let us show now the reverse inclusion. Let $x \in \mathrm{Hom}_{k^\circ-\mathrm{alg}}(\mathfrak{A}_r, k^\circ)$. Let $\mathfrak{s}$ be a section $I_s/{I_s}^2 \to I_s$. It is enough to show that $\Psi_{\mathfrak{s}}(x) \in \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_r/{I_r}^2, k^\circ)$. Let $i \in I$. By 2.5.21, it is enough to show that $\Psi_{\mathfrak{s}}(x)(p_s(i)) \in \pi^r k^\circ$. We have $\Psi_{\mathfrak{s}}(x)(p_s(i)) = x(\mathfrak{s}(p_s(i)))$. We have $\mathfrak{s}(p_s(i)) \in I$ (for example by 2.5.22). So $x(\mathfrak{s}(p_s(i))) \in \pi^r k^\circ$. This ends the proof of the injectivity.

□

Remark 2.5.24. *Let us now give a comment about surjectivity. Let* $x \in \mathrm{Hom}_{k^\circ-\mathrm{mod}}(I_s/I_s, k^\circ)$. *By construction of* Ψ, *it is enough to find* $g_1, \ldots, g_n \in I_s$, *such that*

1. *The class* $[g_1], \ldots, [g_n] \in I_s/{I_s}^2$ *of* $g_1, \ldots, g_n \in I_s$ *is a basis of* $I_s/{I_s}^2$ *(so that* $g_1, \ldots, g_n \in I_s$ *induce a section* $I_s/{I_s}^2 \to I_s$*).*
2. *There is a morphism* f *of* k°*-algebra* $\mathfrak{A}_s \to k^\circ$ *such that* $f(g_i) = x([g_i])$ *for* $1 \leq i \leq n$.

We are thus interested in finding $g_1, \ldots, g_n \in I_s$ *such that the first assertion holds and such that* $g_1, \ldots, g_n$ *have essentially no algebraic relations. This should be a consequence of smoothness.*

About Moy-Prasad isomorphism for analytic filtrations

In this section we write a partial answer to the question 2.5.11. This is done using the morphism 2.5.23, at level of congruence groups, written by Yu in [43, §2.8] and studied in the previous section.

Proposition 2.5.25. *Let H be a stable rational potentially Demazure k-affinoid subgroup of G^{an}. Let $r \in \mathbb{Q}_{>0}$ and $s \in \mathbb{Q}_{>0}$ be rational numbers such that $0 < \frac{r}{2} \leq s \leq r$. Let K/k be a finite Galois extension and $\mathfrak{G}$ be a K°-Demazure group scheme such that $H_r = \mathrm{pr}_{K/k}\left(\Gamma_{\widehat{e(K,k)r}(\mathfrak{G})_\eta}\right)$ and $H_s = \mathrm{pr}_{K/k}\left(\Gamma_{\widehat{e(K,k)s}(\mathfrak{G})_\eta}\right)$. Assume that*

1. *The morphism of groups Ψ of Theorem 2.5.23 is surjective,*
2. *$H^1(\mathrm{Gal}(K/k), \Gamma_{e(K,k)s}(\mathfrak{G})(K^\circ)) = 0$,*
3. *$H^1(\mathrm{Gal}(K/k), \mathrm{Lie}(\Gamma_{e(K,k)s}(\mathfrak{G}))(K^\circ)) = 0$.*

Then we have

$$H_s(k)/H_r(k) \simeq \mathfrak{h}_s(k)/\mathfrak{h}_r(k). \tag{2.5}$$

Proof. Let us first prove it in the split rational case. Thus assume first that H is a Demazure k-affinoid group and $r \in \mathrm{ord}(K)$. Let $\mathfrak{G}$ be the k°-Demazure group scheme such that $H = \widehat{\mathfrak{G}}_\eta$. Then by definitions

$$\begin{aligned} H_r &= \widehat{\Gamma_r(\mathfrak{G})}_\eta \\ H_s &= \widehat{\Gamma_s(\mathfrak{G})}_\eta \\ \mathfrak{h}_r &= \mathrm{Lie}\widehat{(\Gamma_r(\mathfrak{G}))}_\eta \\ \mathfrak{h}_s &= \mathrm{Lie}\widehat{(\Gamma_s(\mathfrak{G}))}_\eta. \end{aligned}$$

So we have

$$\begin{aligned} H_r(k) &= \Gamma_r(\mathfrak{G})(k^\circ) \\ H_s(k) &= \Gamma_s(\mathfrak{G})(k^\circ) \\ \mathfrak{h}_r(k) &= \mathrm{Lie}(\Gamma_r(\mathfrak{G}))(k^\circ) \\ \mathfrak{h}_s(k) &= \mathrm{Lie}(\Gamma_s(\mathfrak{G}))(k^\circ). \end{aligned}$$

The isomorphism (2.5) is now a consequence of Theorem 2.5.23 and the first hypothesis.

Let us prove now the general case. We have

$$\begin{array}{lll} H_r = \mathrm{pr}_{K/k}\left(\Gamma_{\widehat{e(K,k)r}(\mathfrak{G})_\eta}\right) & H_r \times_{\mathcal{M}(k)} \mathcal{M}(K) = \Gamma_{\widehat{e(K,k)r}(\mathfrak{G})_\eta} & H_r(K) = \Gamma_{e(K,k)r}(\mathfrak{G})(K^\circ) \\ H_s = \mathrm{pr}_{K/k}\left(\Gamma_{\widehat{e(K,k)s}(\mathfrak{G})_\eta}\right) & H_s \times_{\mathcal{M}(k)} \mathcal{M}(K) = \Gamma_{\widehat{e(K,k)r}(\mathfrak{G})_\eta} & H_s(K) = \Gamma_{e(K,k)s}(\mathfrak{G})(K^\circ) \\ \mathfrak{h}_r = \mathrm{pr}_{K/k}\left(\mathrm{Lie}(\Gamma_{\widehat{e(K,k)r}}(\mathfrak{G}))_\eta\right) & \mathfrak{h}_r \times_{\mathcal{M}(k)} \mathcal{M}(K) = \mathrm{Lie}(\Gamma_{\widehat{e(K,k)r}}(\mathfrak{G}))_\eta & \mathfrak{h}_r(K) = \mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))(K^\circ) \\ \mathfrak{h}_s = \mathrm{pr}_{K/k}\left(\mathrm{Lie}(\Gamma_{\widehat{e(K,k)s}}(\mathfrak{G}))_\eta\right) & \mathfrak{h}_s \times_{\mathcal{M}(k)} \mathcal{M}(K) = \mathrm{Lie}(\Gamma_{\widehat{e(K,k)r}}(\mathfrak{G}))_\eta & \mathfrak{h}_s(K) = \mathrm{Lie}(\Gamma_{e(K,k)s}(\mathfrak{G}))(K^\circ), \end{array}$$

left equalities are definitions, middle ones are formal consequences of left ones and right ones are direct consequences of middle ones. Since $H_r, H_s, \mathfrak{h}_r$ and $\mathfrak{h}_s$ are k-affinoid spaces, we have

$$\begin{aligned}
H_r(k) &= H_r(K)^{\mathrm{Gal}(K/k)} \\
H_s(k) &= H_s(K)^{\mathrm{Gal}(K/k)} \\
\mathfrak{h}_r(k) &= \mathfrak{h}_s(K)^{\mathrm{Gal}(K/k)} \\
\mathfrak{h}_s(k) &= \mathfrak{h}_s(K)^{\mathrm{Gal}(K/k)}.
\end{aligned}$$

So all together we have $(*)$

$$\begin{aligned}
H_r(k) &= \Gamma_{e(K,k)r}(\mathfrak{G})(K^\circ)^{\mathrm{Gal}(K/k)} = \Gamma_{e(K,k)r}(\mathfrak{G})(K^\circ) \cap G(k) \\
H_s(k) &= \Gamma_{e(K,k)s}(\mathfrak{G})(K^\circ)^{\mathrm{Gal}(K/k)} = \Gamma_{e(K,k)s}(\mathfrak{G})(K^\circ) \cap (G(k) \\
\mathfrak{h}_r(k) &= \mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))(K^\circ)^{\mathrm{Gal}(K/k)} = \mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))(K^\circ) \cap \mathfrak{g}(k) \\
\mathfrak{h}_s(k) &= \mathrm{Lie}(\Gamma_{e(K,k)s}(\mathfrak{G}))(K^\circ)^{\mathrm{Gal}(K/k)} = \mathrm{Lie}(\Gamma_{e(K,k)s}(\mathfrak{G}))(K^\circ) \cap \mathfrak{g}(k).
\end{aligned}$$

Since $0 < \frac{e(K,k)r}{2} \leq e(K,k)s \leq e(K,k)r$, by the previous split rational case, we have

$$\Gamma_{e(K,k)s}(\mathfrak{G})(K^\circ)/\Gamma_{e(K,k)r}(\mathfrak{G})(K^\circ) \simeq \mathrm{Lie}(\Gamma_{e(K,k)s}(\mathfrak{G}))(K^\circ)/\mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))(K^\circ). \quad (2.6)$$

The group $\mathrm{Gal}(K/k)$ acts canonically on $\Gamma_{e(K,k)s}(\mathfrak{G})(K^\circ)/\Gamma_{e(K,k)r}(\mathfrak{G})(K^\circ)$ and on $\mathrm{Lie}(\Gamma_{e(K,k)s}(\mathfrak{G}))(K^\circ)/\mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))(K^\circ)$, these actions are equivariant relatively to the isomorphism (2.6). We thus get

$$\left(\Gamma_{e(K,k)s}(\mathfrak{G})(K^\circ)/\Gamma_{e(K,k)r}(\mathfrak{G})(K^\circ)\right)^{\mathrm{Gal}(K/k)} \simeq \left(\mathrm{Lie}(\Gamma_{e(K,k)s}(\mathfrak{G}))(K^\circ)/\mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))(K^\circ)\right)^{\mathrm{Gal}(K/k)}.$$

Conditions on H^1 implies now that

$$\Gamma_{e(K,k)s}(\mathfrak{G})(K^\circ)^{\mathrm{Gal}(K)}/\Gamma_{e(K,k)r}(\mathfrak{G})(K^\circ)^{\mathrm{Gal}(K)} \simeq \mathrm{Lie}(\Gamma_{e(K,k)s}(\mathfrak{G}))(K^\circ)^{\mathrm{Gal}(K)}/\mathrm{Lie}(\Gamma_{e(K,k)r}(\mathfrak{G}))(K^\circ)^{\mathrm{Gal}(K)}$$

where $\mathrm{Gal}(K) := \mathrm{Gal}(K/k)$. We deduce now the desired isomorphism (2.5) using equations $(*)$.

□

We now state and prove a Lemma which ensure that hypothesis of the previous proposition holds.

Let K/k be a finite Galois extension. Let $\mathfrak{G}$ be a $\mathrm{Gal}(K/k)$-stable K°-Demazure group scheme. Let $N \in \mathbb{Z}_{>0}$ be a strictly positive integer. Let $\Gamma_N(\mathfrak{G})$ be the N-th congruence K°-scheme of $\mathfrak{G}$. Write $\Gamma_N := \Gamma_N(\mathfrak{G})(K^0)$. It is $\mathrm{Gal}(K/k)$-stable by 2.1.14. Let t be a positive integer, Γ_t and Γ_{t+1} are $\mathrm{Gal}(K/k)$-stable, so $\mathrm{Gal}(K/k)$ acts on Γ_t/Γ_{t+1}.

Lemma 2.5.26. *Assume* $H^1(\mathrm{Gal}(K/k), \Gamma_t/\Gamma_{t+1}) = 0$ *for all positive integer* t. *Then, for any* $N > 0$,

$$H^1(\mathrm{Gal}(K/k), \Gamma_N(\mathfrak{G})(K^\circ)) = 0.$$

Proof. By [41, Lemma 2.8], it is enough to prove that

$$(*i) \quad H^1\left(\mathrm{Gal}(K/k)\ , \Gamma_N(\mathfrak{G})(K^\circ)/\Gamma_{N+i}(\mathfrak{G})(K^\circ)\right) = 0 \quad \text{for all } i \in \mathbb{Z}_{>0}.$$

Let us prove it by induction on i. The initialisation $(i = 1)$ is a direct consequence of the hypothesis. Let us do the heredity. Assume the relation $(*i)$ is satisfied for an $i > 0$ and let us show that this implies that $(*i+1)$ is also satisfied. We have an exact sequence of $\mathrm{Gal}(K/k)$-groups

$$\begin{tikzcd} 0 \arrow[r] & \Gamma_{N+i}/\Gamma_{N+i+1} \arrow[r] & \Gamma_N/\Gamma_{N+i+1} \arrow[r] & (\Gamma_N/\Gamma_{N+i+1})/(\Gamma_{N+i}/\Gamma_{N+i+1}) \arrow[r] \arrow[d, "\simeq"] & 0 \\ & & & \Gamma_N/\Gamma_{N+i}. & \end{tikzcd}$$

By hypothesis we have $H^1(\mathrm{Gal}(K/k), \Gamma_{N+i}/\Gamma_{N+i+1}) = 0$. By induction hypothesis $H^1(\mathrm{Gal}(K/k), \Gamma_N/\Gamma_{N+i}) = 0$. Thus by [41, Lemma 2.5], we deduce $H^1(\mathrm{Gal}(K/k), \Gamma_N/\Gamma_{N+i+1}) = 0$. This ends the proof of the heredity. We have finished the induction and the proof ends here.

□

APPENDIX B: On notions of rational points in the reduced Bruhat-Tits building

Let k be a non archimedean local field and G be a connected reductive k-group scheme. We have two natural notions of rational points in the reduced Bruhat-Tits building $\mathrm{BT}^R(G,k)$.

1. (Here $G = GL_N$ in the original definition of Broussous-Lemaire [7]) A point $x \in \mathrm{BT}^R(G,k)$ is called barycentrically rational if it is the barycentre of vertex in a chamber with rational weights (this definition is natural after Broussous-Lemaire work, see their work on comparison of filtrations [7]). We denote by $\mathrm{BT}^R_{rat_{bar}}(G,k)$ the associated subset of $\mathrm{BT}^R(G,k)$.

2. A point $x \in \mathrm{BT}^R(G,k)$ is called specially rational if there exists K/k finite such that

   (a) $i_{K/k}(x) \in \mathrm{BT}^R(G,K)$ is a special point ($i_{K/k}$ is the canonical map between buildings, this notion of rational point is introduced in this text (see section 2.3))

   (b) G is split over K (this condition is always satisfied in this appendix).

   We denote $\mathrm{BT}^R_{rat_{spe}}(G,k)$ (it was denoted $\mathrm{BT}^R_{rat}(G,k)$ in section 2.3) the associated subset of $\mathrm{BT}^R(G,k)$.

In this appendix we prove that they are equivalent in the case $G = GL_N$, i.e $\mathrm{BT}^R_{rat_{spe}}(GL_N,k) = \mathrm{BT}^R_{rat_{bar}}(GL_N,k)$. We then illustrate the proof in the GL_3 case with an example and a picture.

Proof that the two notions are equivalent for $G = GL_N$

Here $G = GL_N$, it is split $/k$ and the reduced building is a simplicial complex. That last condition means that any facet F is a simplex. Let F be a maximal facet in an appartement A, and fix it. Let $S_1, \dots, S_i, \dots, S_N$ be the vertex of the facet F. Put $I = \{1, \dots, N\}$.

• Since F is a maximal simplex, for all $i \in I$, the set

$$R_i = \{\overrightarrow{S_i S_j} \mid j \in I \text{ and } i \neq j\}$$

is a *repère* of A. That means that for any $i \in I$ and each $P \in A$, there exists[3] unique real numbers $x_1, \dots, \widehat{x_i}, \dots x_N$ such that $\overrightarrow{S_i P} = \sum_{\substack{j \in I \\ j \neq i}} x_j \overrightarrow{S_i S_j}$.
The numbers $x_1, \dots, \widehat{x_i}, \dots x_N$ are called the coordinates of P in the *repère* R_i.

• (Since $G = GL_N$) The directions [4] of the walls in A are in bijection with the vertex of the maximal simplex F as follows

$$\begin{aligned} &\{\text{Vertex of F }\} \leftrightarrow \{\text{direction of the walls in A}\} \\ &\qquad\qquad S_i \mapsto D_i = \{\text{direction of the wall containing } S_1, \dots, \widehat{S_i}, \dots, S_N\} \end{aligned}$$

• Let K/k be a finite extension, since G is split, for any maximal split torus S, the simplicial structure on the associated appartement $A^R(G, S)$ satisfies the following: The appartement $A^R(G, S)/K$ is obtained from $A^R(G, S)/k$ adding regularly e times more walls for each direction. Fix a vertex S_i, we thus get a direction D_i , and we put:

$$Walls_{S_i} = \{\text{The set of walls having direction } D_i\text{, and coming from finite extensions }\}$$

Let $P \in A$ (think $P \in F$). Write the coordinates of P in the *repère* R_i: $\overrightarrow{S_i P} = \sum_{j \in I \setminus i} x_j \overrightarrow{S_i S_j}$. By Thalès Theorem, we deduce ($j \in I \setminus i$):
$P \in Walls_{S_j} \Leftrightarrow$ The j-th coordin. x_j of P
in the *repère* R_i is a rational numb.

Recall that a point P is special over an extension K/k if for every direction D_i, there exists a wall of $\mathrm{BT}^R(G, K)$ such that P is contained in this wall.

We deduce that

[3]The symbol hat over a symbol in a list of symbol means that we ommit it.
[4]By the direction of a wall we mean the vectorial part.

$$
\begin{aligned}
P \in \mathrm{BT}^R_{rat_{spe}}(G,k) &\Leftrightarrow \forall j \in I, P \in Wall_{S_j} \\
&\Leftrightarrow \forall i,j \in I; i \neq j;\ \text{the } j\text{-th coordinate of } P \\
&\quad\ \text{in the } \textit{repère } R_i \text{ is rational.} \\
&\Leftrightarrow \forall i \in I,\ \text{the coordinates of } P \text{ in the} \\
&\quad\ \textit{repère } R_i \text{ are rational numbers}
\end{aligned}
$$

• Let us now prove that $\mathrm{BT}^R_{rat_{spe}} = \mathrm{BT}^R_{rat_{bar}}$.

We start by the inclusion $\subset$. Let $x \in \mathrm{BT}^R_{rat_{spe}}$. Let $i \in I$, we write P in the *repère* R_i

$$\overrightarrow{S_iP} = \sum_{j \in I \setminus i} x_j \overrightarrow{S_iS_j}$$

with x_j rational numbers.

We deduce the relation, using Chasles

$$\overrightarrow{S_iP} = \sum_{j \in I \setminus i} x_j (\overrightarrow{S_iP} + \overrightarrow{PS_j}).$$

This allows us to write

$$0 = ((\sum_{j \in I \setminus i} x_j) - 1)\overrightarrow{S_iP} + \sum_{j \in I \setminus i} x_j \overrightarrow{PS_j}$$

This makes clear that $P \in \mathrm{BT}^R_{rat_{bar}}(G,k)$ by definition of barycentres.

Let us prove the reverse inclusion $\supset$. Let $P \in \mathrm{BT}^R_{rat_{bar}}(G,k)$. We have to show that for each i, the coordinates of P in the *repère* R_i are rational numbers. By definition of $\mathrm{BT}^R_{rat_{bar}}(G,k)$, there exists rational numbers c_j such that

$$\sum_{j \in I} c_j \overrightarrow{PS_j} = 0.$$

We thus get

$$\sum_{j \in I \setminus i} c_j (\overrightarrow{PS_i} + \overrightarrow{S_iS_j}) + c_i \overrightarrow{PS_i} = 0.$$

So we obtain

$$\sum_{j \in I \setminus i} c_j \overrightarrow{S_iS_j} + (\sum_{j \in I} c_j) \overrightarrow{PS_i} = 0.$$

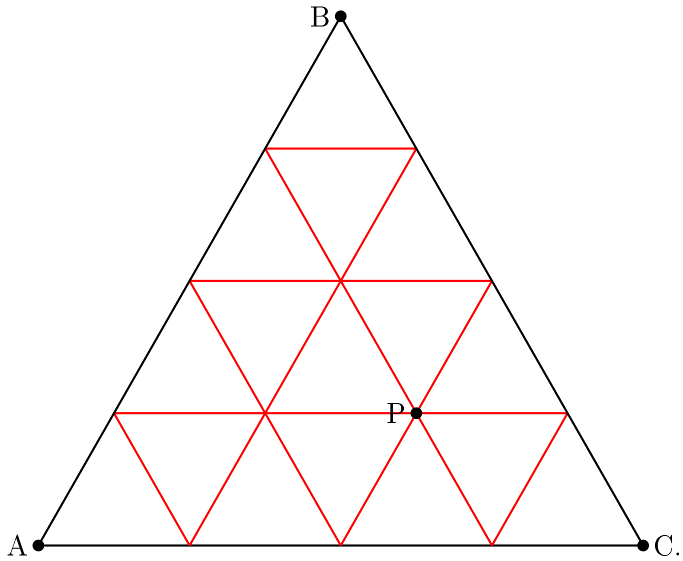
Putting $k = \sum_{j \in I} c_j$ (it is $\neq 0$), we get

$$\overrightarrow{S_iP} = \sum_{j \in I \setminus i} \frac{c_j}{k} \overrightarrow{S_iS_j}.$$

This shows that the coordinates of P in the *repère* R_i are rational numbers. So $P \in \mathrm{BT}^R_{rat_{spe}}(G, k)$, as required.

Illustration of the proof in GL_3

Take $G = GL_3$. A maximal simplex of an appartement in the reduced building look like this:



In black are represented walls over k and in red walls over an extension K/k of ramification index 4. There are three directions, here one horizontal realized by AC, an other oblic realized by AB and an other BC. With the notations introduced before, the direction AC is D_B, AB is D_C and BC is D_A. Consider the point P, it is a point in $\mathrm{BT}^R_{rat_{spe}}(G, k)$, since for each direction, a red line realizing this direction pass by P. In the *repère* R_A, the coordinates of P are $(\frac{1}{4}, \frac{1}{2})$, i.e. $\overrightarrow{AP} = \frac{1}{4}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{AC}$.

We succesively deduce the relations

$$\overrightarrow{AP} = \tfrac{1}{4}\overrightarrow{AP} + \tfrac{1}{4}\overrightarrow{PB} + \tfrac{1}{2}\overrightarrow{AP} + \tfrac{1}{2}\overrightarrow{PC}$$

$$\tfrac{1}{4}\overrightarrow{AP} + \tfrac{1}{4}\overrightarrow{BP} + \tfrac{1}{2}\overrightarrow{CP} = 0$$

So P is a rational special point in the sense of Broussous-Lemaire, i.e. $P \in \mathrm{BT}^R_{rat_{bar}}(G, k)$.

Now take a point $P \in \mathrm{BT}^R_{rat_{bar}}(G,k)$, and we research an extension K/k such that P is special in $\mathrm{BT}^R(G,K)$. Assume for example that P is the barycentre $((A,2),(B,3),(C,5))$, we thus get

$$2\overrightarrow{AP} + 3\overrightarrow{BP} + 5\overrightarrow{CP} = 0$$

We deduce, using Chasles, that

$$10\overrightarrow{AP} + 3\overrightarrow{BA} + 5\overrightarrow{CA} = 0$$

So $\overrightarrow{AP} = \frac{3}{10}\overrightarrow{AB} + \frac{5}{10}\overrightarrow{AC}$, and the coordinates of P in R_A are rational. Analogously, one can see that the coordinates of P in the *repère* R_B and R_C are rational. We deduce that P becomes special over an extension with ramification index multiple of 10.

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