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# THÈSE DE DOCTORAT DE L'UNIVERSITÉ PIERRE ET MARIE CURIE

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**Thomas DUPIC**

pour obtenir le grade de :

**DOCTEUR DE L'UNIVERSITÉ PIERRE ET MARIE CURIE**

Sujet de la thèse :

**Application des théories conformes étendues à des  
problèmes de physique statistique**

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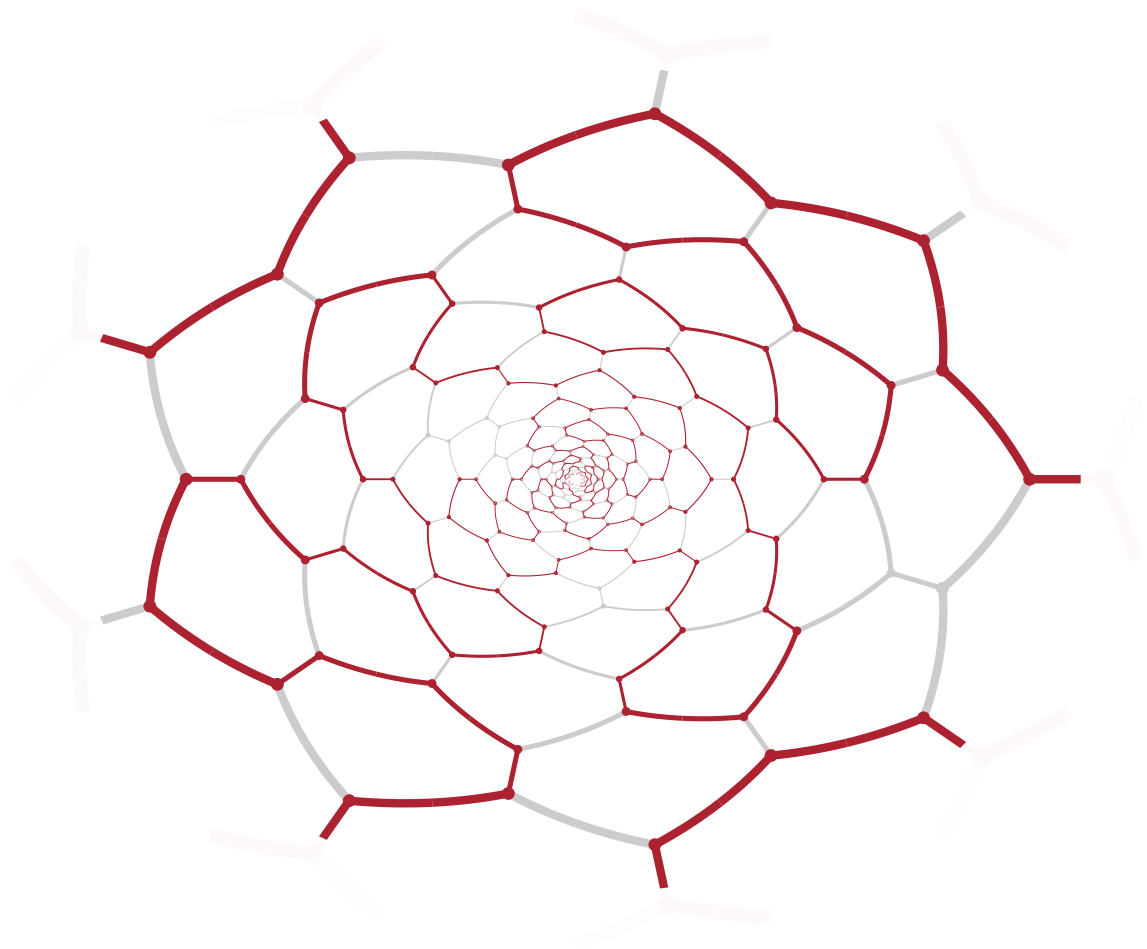
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APPLICATION DES THÉORIES CONFORMES ÉTENDUES À DES  
PROBLÈMES DE PHYSIQUE STATISTIQUE

THOMAS DUPIC







*Il allait falloir écrire et je ne pourrais pas : Je m'étais  
mis au pied du mur, et n'étais pas maçon.  
— Pierre Michon, Vies minuscules*

*À ma famille*



## ABSTRACT

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The study of critical phenomena in two-dimensional statistical physics is mainly performed with the help of conformal field theory and integrable models. The relationship between these two formalisms is an active field of research, particularly in the framework of the so-called non-rational theories. This thesis is focused on certain critical systems described by an extended conformal theory, i.e. a theory that presents additional symmetries. The first problem studied is the fully packed loop model (FPL). Loop models are non-local statistical models based on the description of assembly of polymers. In particular, they represent the interfaces formed within a particular spin model. Their continuous limit is a non-rational conformal theory. The FPL model is integrable and its spectrum reflects an underlying symmetry  $U_q(\mathfrak{sl}(3))$ . The link between this model and the  $\mathcal{W}_3$  symmetry, a conformal symmetry extended by a three-dimensional field, is studied in detail, numerically (by exact diagonalization) and analytically. The relationship with loop models naturally leads to the study of the non-scalar operator content of the  $\mathcal{W}_3$  theory. The second problem concerns the calculation of entanglement in unidimensional quantum systems. In this context, the preferred object of study is the entropy of entanglement between a subsystem and its complement. For the fundamental state of a spin chain, the behaviour of this entropy as a function of the size of the subsystem is a clear marker of the criticality of the chain. In this manuscript, a new way of calculating these entropies in critical models is presented. It is based on conformal theories extended by a symmetry called orbifold. This method is particularly applicable to entropies of excited states or disjointed subsystems. Here the Yang-Lee and Ising models serve as examples, entropy is calculated analytically and compared to a numerical study.

## RÉSUMÉ

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L'étude des phénomènes critiques en physique statistique bidimensionnelle a pour outils privilégiés la théorie conforme et les modèles intégrables. La relation entre ces deux formalismes est un domaine de recherche actif, notamment dans le cadre des théories dites non-rationnelles. Cette thèse s'intéresse à certains systèmes critiques décrits par une théorie conforme étendue, c'est-à-dire présentant des symétries supplémentaires. Le premier problème étudié est le modèle de boucles entièrement compactes (fully packed loop model, FPL). Les modèles de boucles sont des modèles de physique statistique non locaux, s'inspirant de la description des polymères. Ils représentent notamment les interfaces formées au sein d'un modèle de spin. Leur limite continue est une théorie conformes non-rationnelle. Le modèle FPL est intégrable et son spectre reflète une symétrie sous-jacente  $U_q(\mathfrak{sl}(3))$ . Le lien entre ce modèle et la symétrie  $\mathcal{W}_3$ , une symétrie conforme étendue par un champ de dimension trois, est étudié en détail, numériquement (par diagonalisation exacte) et analytiquement. La relation avec les modèles de boucles mène naturellement à l'étude du contenu non-scalaire de la théorie  $\mathcal{W}_3$ . Le second problème concerne le calcul de l'intrication dans des systèmes quantiques unidimensionnels. Dans ce cadre, l'objet d'étude privilégié est l'entropie d'intrication entre un sous-système et son complément. Pour l'état fondamental d'une chaîne de spin, le comportement de cette entropie en fonction de la taille du sous-système est un marqueur clair de la criticalité de la chaîne. Dans ce manuscrit, une nouvelle manière de calculer ces entropies dans le cadre des modèles critiques est présentée. Elle s'appuie sur des théories conformes étendues par une symétrie dite d'orbifold. Cette méthode est particulièrement applicable aux entropies d'états excités ou de sous-systèmes disjoints. Ici les modèles de Yang-Lee et d'Ising servent d'exemples, l'entropie est calculée analytiquement et comparée à une étude numérique.

## PUBLICATIONS

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Some ideas and figures have appeared previously in the following publications:

- [1] T. Dupic, B. Estienne, and Y. Ikhlef. “The fully packed loop model as a non-rational  $W_3$  conformal field theory.” In: *Journal of Physics A Mathematical General* 49, 505202 (Dec. 2016), p. 505202. DOI: [10.1088/1751-8113/49/50/505202](https://doi.org/10.1088/1751-8113/49/50/505202). arXiv: [1606.05376](https://arxiv.org/abs/1606.05376) [[cond-mat.stat-mech](#)].
- [2] T. Dupic, B. Estienne, and Y. Ikhlef. “The imaginary Toda field theory.” In: *ArXiv e-prints* (Sept. 2018). arXiv: [1809.05568](https://arxiv.org/abs/1809.05568) [[math-ph](#)].
- [3] Thomas Dupic, Benoit Estienne, and Yacine Ikhlef. “Entanglement entropies of minimal models from null-vectors.” In: *SciPost Phys.* 4 (2018), p. 031. DOI: [10.21468/SciPostPhys.4.6.031](https://doi.org/10.21468/SciPostPhys.4.6.031). arXiv: [1709.09270](https://arxiv.org/abs/1709.09270) [[math-ph](#)].



« - *Qu'est-ce que vous faites dans la vie, vous ? demanda le professeur.*  
- *J'apprends des choses* »

*Boris Vian, l'Écume des jours*

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---

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# CONTENTS

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1	INTRODUCTION	1
1.1	Historical considerations	1
1.2	Conformal field theory in two dimensions	5
1.2.1	Conformal symmetry	5
1.2.2	Conformal field theory	10
1.2.3	Degenerate operators and rational models	15
1.2.4	Bootstrap	17
1.3	Conformal field theory (extended)	23
1.3.1	Extended symmetries	23
1.3.2	The $\mathbb{Z}_2$ orbifold, algebraic perspective	24
1.3.3	The $\mathbb{Z}_2$ orbifold, geometrical perspective	26
1.4	Critical lattice models	30
1.4.1	Discrete statistical model	30
1.4.2	The Temperley-Lieb loop model	32
1.4.3	Transfer matrix exact diagonalisation	36
1.5	Brief thesis outline	40
2	THE FULLY PACKED LOOP MODEL	41
2.1	Fully-Packed Loop model and related lattice models	42
2.1.1	The loop model	42
2.1.2	The fifteen-vertex model	44
2.1.3	The RSOS model	45
2.1.4	Hecke algebra and Markov trace	47
2.2	Continuum limit of the FPL model	48
2.2.1	Coulomb gas description	49
2.2.2	Liouville term	52
2.2.3	The FPL model on the cylinder: non-contractible loops	54
2.2.4	The FPL model on the torus, modular invariance	55
2.2.5	Operators and spectrum of the loop model	57
2.2.6	Generalized electric three-point functions	59
2.3	Numerical study	61
2.3.1	Hamiltonian, Hilbert space and quantum numbers	61
2.3.2	$\mathcal{W}_3$ symmetry of the FPL model at $\tilde{n} = 2 \cos(2\lambda)$	63
2.3.3	Three-point functions and Liouville theory	68
2.4	Conclusion	73
	Appendices	75
2.A	Resummation of the loop partition function	75
2.A.1	Zero-strand sectors	75
2.A.2	Other magnetic sectors	75
2.B	Some interesting particular cases	77

2.B.1	Combinatorial point	77
2.B.2	The model at $n = 1$ : symplectic fermions and Dimer model	79
2.C	Models with $U_q(\mathfrak{sl}_3)$ symmetry	81
2.C.1	The Hecke relations	81
2.C.2	The Spider model	82
3	CONFORMAL TODA FIELD THEORY	85
3.1	Introduction	86
3.2	The $W_3$ conformal symmetry	86
3.2.1	Definition	86
3.2.2	Free fields realization	89
3.2.3	Coulomb gas theory	91
3.2.4	Minimal models	91
3.2.5	Toda field theory	92
3.2.6	Semi-degenerate and fully-degenerate representations of $W_3$	92
3.2.7	Fusion in $W_3$	93
3.2.8	$W_n$ algebra and conventions for $\mathfrak{sl}_n$	95
3.3	Structure constants of scalar fields	95
3.4	Non-scalar fields in the imaginary Toda field theory	99
3.4.1	Non-degenerate non-scalar operators	100
3.4.2	Semi-degenerate non-scalar operators	103
3.4.3	Structure constants	104
3.5	Conclusion	107
	Appendices	109
3.A	Conventions for $\mathfrak{sl}_n$ : roots, weights and Weyl group	109
3.A.1	Conventions for $\mathfrak{sl}_3$	109
3.A.2	Conventions for $\mathfrak{sl}_n$	111
3.B	Hypergeometric conformal blocks for $\mathfrak{sl}_3$	112
3.B.1	Hypergeometric conformal blocks for $\mathfrak{sl}_3$	112
3.B.2	Hypergeometric conformal blocks for $\mathfrak{sl}_n$	114
3.C	Upsilon and double Gamma functions	115
3.D	Detailed analysis of non-diagonal fields in $\mathfrak{sl}_n$ Toda theory	116
3.E	A different path toward the differential equation	118
4	ENTANGLEMENT ENTROPY AND ORBIFOLDS	123
4.1	Introduction	124
4.2	General background	126
4.2.1	Entanglement entropy and conformal mappings	126
4.2.2	Correlation functions of twisted operators	127
4.2.3	Non-unitary models	128
4.3	The cyclic orbifold	132
4.3.1	The orbifold Virasoro algebra	132
4.3.2	Operator content of the $\mathbb{Z}_N$ orbifold	133
4.3.3	Induction procedure	135

4.3.4	Null-vector equations for untwisted and twisted operators	136
4.4	First examples	138
4.4.1	Yang-Lee two-interval correlation function	138
4.4.2	Yang-Lee one-interval correlation function	140
4.5	Twist operators with a fractional null vector	141
4.5.1	Orbifold Ward identities	141
4.5.2	Ising two-interval ground state entropy	142
4.5.3	Yang-Lee one-interval ground state entropy	144
4.5.4	Excited state entropy for minimal models at $N = 2$	148
4.5.5	Excited state entropy for minimal models at $N = 3$	150
4.6	Twist operators in critical RSOS models	152
4.6.1	The critical RSOS model	153
4.6.2	Partition function in the presence of branch points	155
4.6.3	Rényi entropies of the RSOS model	156
4.6.4	Numerical computations	157
4.7	Conclusion	161
Appendices 163		
4.A	Properties of hypergeometric functions	163
4.B	Four-point function satisfying a second-order differential equation	164
4.B.1	Differential equation	164
4.B.2	Determination of the four-point function	165
4.C	Direct computation of OPE coefficients of twist operators	167
4.D	Direct computation of the function in Section 4.4	168
4.E	Quantum Ising chain in an imaginary magnetic field	169
Conclusion 173		
BIBLIOGRAPHY 175		

## ACRONYMS

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FPL	Fully Packed Loops
QFT	Quantum Field Theory
CFT	Conformal Field Theory
OPE	Operator Product Expansion
EE	Entanglement Entropy
SOS	Solid On Solid
RSOS	Restricted Solid On Solid
TL	Temperley-Lieb
DOZZ	Dorn-Otto-Zamolodchikov-Zamolodchikov

## INTRODUCTION

---

Critical phenomena appear in a wide range of physical systems. Classical examples include the critical points of water, or the ferromagnetic transition in metals. A modern example list would also include turbulence, polymer physics, and multiple condensed matter examples (critical quantum chain, quantum Hall effect, . . .). These systems are extremely constrained by their symmetries, to the extent that, in specific circumstances, their behaviour can be fully determined by these symmetries alone. The aim of this thesis is to apply the methods of conformal field theories to study various statistical systems at their critical point. This introduction is structured as follows: after a short historical recapitulation, basic concepts of conformal field theory will be reviewed, with a focus on the bootstrap method. Then we will describe extended conformal field theories, using the  $\mathbb{Z}_2$  orbifold as an example. Lastly we will rapidly describe some relevant discrete statistical systems, and common numerical methods.

### 1.1 HISTORICAL CONSIDERATIONS

The scientific study of criticality started in earnest at the beginning of the 19th century. One of the main question at that time involved phase transition in liquids. Charles Cagniard de la Tour, discovered in 1823 that above a certain temperature and pressure, the clear phase transition between liquid and gas disappears and the transformation becomes continuous, in modern term the fluid enters the supercritical phase. Scientists such as Faraday and Mendeleev took up the subject but it is the work of Thomas Andrew, which proved to be critical (no pun intended).

In 1869, he studied the phase transition in carbon dioxide by measuring both pressure and volume along isotherms. He carefully characterized the position of the point where liquid and gas become indistinguishable (located at a reasonable 31 °C for CO<sub>2</sub>). At that exact point, the liquid starts to get cloudy and diffuse. Andrew christened the word “critical” to talk about this precise point (which rolled more off the tongue than “disliquefying” that Faraday used).

A theoretical framework was developed subsequently by Gibbs, Maxwell and Van der Waals, forming the basis of modern thermodynamic theory. In Van der Waals analysis [142] the density of the

*On the Continuity  
of the Gaseous and  
Liquid States of  
Matter [6]*

two coexistent phases, near the critical point, should approach each other like the square root of the temperature.

$$\rho_g(T) - \rho_l(T) \propto (T - T_c)^\beta \text{ with } \beta = \frac{1}{2}$$

This behaviour is now considered a staple of critical system.  $\beta = 1/2$  was the first *critical exponent* defined. Amusingly, it happens to be way off marks for this system, in which the critical exponent is closer to  $1/3$ .

On the theoretical side, things went relatively quiet for more or less twenty years. Precise measure of critical exponents were (and still are) difficult to do, and the deviations from the theory of Van der Waals were few. On the experimental side however, many new critical points were discovered in various systems. One of the most important in terms of the theoretical development that followed was the discovery, by Pierre Curie, of a critical point in ferromagnetic materials. Ferromagnetic iron, heated up to the Curie temperature (1043 K), suddenly lost any magnetization it could have had. The various attempts to simplify the complex magnetization models and to explain this behaviour lead to the definition of a very simplified magnetization model by Lenz in 1920, the Ising model [30, 87].

In the Ising model, the magnetic spins can only take two values  $\sigma = \pm 1$ , and the interaction happens between direct neighbours. The energy of a configuration is hence:

$$E = -J \sum_{i,j \text{ neighbours}} \sigma_i \sigma_j$$

Depending on the sign of  $J$ , the model is either ferromagnetic ( $J > 0$ ) or anti-ferromagnetic ( $J < 0$ )

The model was going to become one of the, if not the most, studied model of statistical physics, but the lack of a phase transition in one dimension disheartened its early investigators. Ising, which was Lenz student at that time, conclude in his dissertation in 1928 that the model does not have a phase transition in one dimension. Heisenberg built on Ising legacy to develop his own model, giving a more realistic spin interaction to the atoms. His hope was that the additional sophistication would be enough to obtain a phase transition. In his own words : "Ising succeeded in showing that also the assumption of directed sufficiently great forces between two neighbouring atoms of a chain is not sufficient to explain ferromagnetism."

Bethe found in 1935 the generic form of the eigenvectors of the Heisenberg model [21]. The *Bethe ansatz* became one of the main tool in the analysis of integrable models, but, like the Ising model, not before going through a period of relative obscurity. The Ising model was revived one year after, by Rudolf Peierls [119] (helped by Bethe) who showed with a simple argument (nicknamed Peierls droplets) that the two-dimensional Ising model did exhibit a phase transition.

This led to a flurry of work to determine this critical point. In 1941 Krammers and Wannier [102] determined the exact critical temperature using a duality between high and low energy behaviour. And, in a seminal presentation in 1942, Lars Onsager (a polymath, who later won a Nobel Prize for chemistry) announced a solution for the Ising model in two dimensions [113]. In a large part his result can be considered as the starting point of the study of integrable models in statistical physics. Many physicists tried to simplify his result, or to express it in a more elegant fashion, and these attempts later give way to a complete set of tools at the heart of modern integrable theory. Those impressive developments only happened in two dimensions. Despite high hopes the three-dimensional case remained closed.

At the same period, Lev Landau formed a comprehensive field-theoretical description of generic phase transitions. His main contribution to the subject was to associate a phase transition to a broken symmetry in the theory [105]. Landau approach is strictly mean-field. In most real critical systems (at least in dimension lower than four) local fluctuations keep the critical point away from the Landau solution. However, his approach was pioneering in the sense that it unified every critical theory behind an identical formalism.

A new set of ideas emerged in the sixties, with, at the center, scale covariance and *renormalization*. Renormalization emerged as a way to deal with perturbative expansions in quantum field theory. It consisted in adding a dependence on the scale in the parameters of a Lagrangian. A deeper understanding of what was at that point a relatively empirical method was obtained through statistical physics by Leo P. Kadanoff who, in 1966, introduced the idea of the *renormalization group* [94–96]. The works of Kenneth Wilson and Michael Fisher [69, 147] gave substance to these ideas and introduced important new concepts.

In the case of the ferromagnetic Ising model, to which it was first applied, the statistical renormalization transformation consists in a form of coarse-graining, where close spins are averaged to form a new model at a different scale. If one starts at a temperature below the critical point, under repeated application of this transformation, the effective temperature of the model will decrease until it reaches the zero temperature point where all spins are oriented in the same direction. Similarly, if we start above the critical temperature, the model will converge toward the high temperature point. If the temperature is exactly the critical temperature, the effective temperature does not change under the renormalization group. Hence, the Ising model has three fixed point, two stable ones at zero and infinite temperature respectively, and an unstable one at the critical point. Wilson remarked that during the renormalization process, a lot of the details of the interaction were lost. This created a new justification for the univer-

*Lars Onsager was, it seems, not known for his pedagogy*

*For example the Ising phase transition correspond to the broken  $\mathbb{Z}_2$  symmetry between + and - spins*

*One of the first application of renormalization to the Ising model comes from the well-known paper "Critical exponents in 3.99 dimensions" [148]*



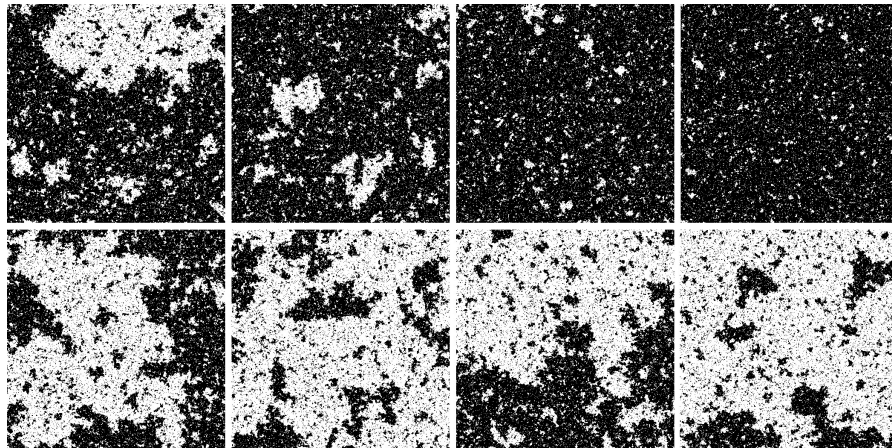


Figure 1.1: Renormalization flow for the Ising model, left to right, starting at two different temperatures, slightly below criticality (top), and at criticality (bottom).

sality of critical points: they are the end-points in a generic flow of parametrised Hamiltonians.

Influenced by works on current algebras (Gell-Mann [79]), Wilson also put forward the concept, in 1969, of *Operator product expansions* [146]. The idea consists in re-expressing two fields close to each other as a sum of fields, effectively giving an algebra structure to the operators in the theory. Similar concepts were also, seemingly independently, developed by Kadanoff and Polyakov. While Wilson himself did not venture much in that direction [124], multiple groups (Polyakov 1974 [125], Mack 1977 [110], Gatto et al. 1973 [68]) realized that if the system was also supposed invariant under conformal transformation – a generalization of scale transformations – the operator product expansions simplified.

Belavin, Polyakov and Zamolodchikov understood in 1984 [14], how this conformal symmetry, in two dimensions, opened the possibility of completing the OPE program analytically. One of the first application of this formalism to statistical physics is due to Cardy [37], and it led to the classification of many critical point. These progress were paralleled with the development of the theory of quantum integrable models. The Bethe Ansatz underwent a renewal in 1963, with the work of Lieb and Liniger [109]. Later, the Yang-Baxter relation (1968 [151] and 1971 [12]) gave a common formalism to the field. It led to deep mathematical development at the interface, such as the notion of quantum groups [55]. The relation between integrable models and conformal field theory in two dimension has been extremely fruitful, yet many questions remain open.

## 1.2 CONFORMAL FIELD THEORY IN TWO DIMENSIONS

The aim of this section is to provide a very basic introduction to Conformal Field Theory (CFT), and to fix notations for the rest of the thesis. More comprehensive references include the very well-known yellow book [73] by P. Di Francesco, P. Mathieu and D. Senechal, conformal invariance and critical phenomena by M. Henkel [82], the more mathematical courses by M. Schottenloher [133] or K. Gawedski [78] the recent CFT review by S. Ribault [129] and many others.

## 1.2.1 Conformal symmetry

Let  $\Omega$  be a two-dimensional smooth manifold, equipped with a metric  $g$ , with the associated distance measure:

$$d^2s = g_{\mu\nu} dr^\mu dr^\nu .$$

We will consider theories covariant under *Weyl transformations*, transformations of the metric of the form:

$$g_{\mu\nu}(\mathbf{r}) \rightarrow g'_{\mu\nu}(\mathbf{r}) = \Lambda(\mathbf{r})g_{\mu\nu}(\mathbf{r}) .$$

Where  $\Lambda$  is a smooth, positive, real function.

The metric  $g_{\mu\nu}(\mathbf{r})$  is the bilinear form associated with the local scalar product between any two tangent vectors at a position  $\mathbf{r}$ . Hence, in geometric terms, a Weyl transformation is a local scale transformation : lengths are changed, but not angles. The equivalence class of metrics by all possible Weyl transformations is named a conformal structure or conformal class.

Conformal transformations are angle-preserving maps. More formally a diffeomorphism between two Riemannian manifolds is conformal if the pulled back metric is the same as the original one up to a Weyl transformation.

A canonical example of a conformal transformation is the stereographic projection. The sphere  $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$  is usually represented by the stereographic coordinates  $\zeta$  and  $\bar{\zeta}$ :

$$\begin{aligned} \zeta &= \frac{x + iy}{1 - z} & (x, y, z) \neq (0, 0, 1), \\ \bar{\zeta} &= \frac{x - iy}{1 + z} & (x, y, z) \neq (0, 0, -1). \end{aligned}$$

The metric inherited from this embedding in  $\mathbb{R}^3$  is the so-called round metric:

$$ds^2 = \frac{4}{(1 + \zeta\bar{\zeta})^2} d\zeta d\bar{\zeta}. \quad (1.1)$$

Like every conformal metric in two dimensions, the round metric is locally conformally flat. It's obvious from the expression 1.1 that there exists a Weyl transformation that can, locally, map the metric to the flat metric.

Generically, for a metric of the form  $e^{\sigma(z)} dz d\bar{z}$  ("conformal"), the curvature is given by  $R = -\frac{1}{e^{\sigma}} \Delta\sigma$ , where  $\Delta$  designs the Laplacian, hence:

*The two different stereographic projections each cover the whole sphere except for one of the pole. Both coordinates live in the complex plane.*

$$R = \frac{(1 + \zeta\bar{\zeta})^2}{2} \Delta \log(1 + \zeta\bar{\zeta}) = 2.$$

As expected for a round sphere of radius one, the curvature is constant and the total curvature (the integral of the scalar curvature) is equal to  $8\pi$ .

The total curvature does not vanish, hence if the round metric can be made *locally* flat after a Weyl transformation, it cannot be made *globally* flat. Case in point, the Weyl transformation  $g \rightarrow \frac{1}{4}(1 + \zeta\bar{\zeta})^2 g$  flattens the metric on the open set  $\mathcal{S}^2 \setminus \{(x, y, z) = (0, 0, 1)\}$ . The rest of the curvature is then localized at the north pole  $(x, y, z) = (0, 0, 1)$ . To see this explicitly consider the second stereographic chart  $\bar{\zeta}$ . Outside of  $\zeta = 0$  and  $\bar{\zeta} = 0$ , the two coordinate are mapped to each other through  $\zeta = \bar{\zeta}^{-1}$ , hence the metric and the curvature on  $\mathcal{S}^2 \setminus \{z = -1\}$  can be written:

$$d^2s = \frac{1}{(\bar{\zeta}\zeta)^2} d\bar{\zeta} d\zeta, \quad R = 4(\bar{\zeta}\zeta)^2 \delta(\bar{\zeta}).$$

The scalar curvature is zero everywhere except at the north pole,  $\bar{\zeta} = 0$  ( $\zeta \rightarrow \infty$ ).

Another commonly used Weyl transformation is  $d^2s = e^{2\pi(w+\bar{w})/L} dw d\bar{w}$ , which deforms the punctured plane at the origin – *i.e.* the twice punctured sphere – into a cylinder. Similarly, the cylinder has vanishing curvature everywhere. But if one compactifies it by adding the two points at infinity, the curvature will be localized at these points.

Any two-dimensional oriented riemannian manifold is naturally a Riemann surface, *i.e.* it admits an atlas whose transition functions are holomorphic.

This stems from the fact that two-dimensional riemannian manifolds are locally conformally flat: there exists isothermal coordinates around any point, that is coordinates in which the metric is of the form  $g = \Lambda(x, y)(dx^2 + dy^2)$ .

Since transition functions between isothermal coordinates preserve the angles, they are either holomorphic or anti-holomorphic. Upon adding a global orientation, one can make all the transition functions holomorphic. Moreover, orientation preserving conformal transformations coincide with holomorphic maps.

Riemann surfaces can be classified by the sign of their total Gaussian curvatures, which does not change under Weyl transformations. This separates them in three families:

- elliptic surfaces, with a positive curvature, for which the unique representative is the Riemann sphere that we described previously,
- parabolic surfaces, with vanishing curvature, like the plane and the various tori,
- hyperbolic surfaces, with negative curvature, the largest family. It contains for example the open disk and higher genus surfaces (with more than one “hole”).

More precisely, the uniformization theorem states that every simply connected Riemann surface is conformally equivalent to one of three simply connected

*For a detailed  
reference on  
Riemann surfaces  
see [71]*

surfaces, the Riemann sphere (for surfaces with positive total curvature), the complex plane (zero curvature) or the open disk (negative curvature). For non simply connected surfaces, the surface is the quotient of its universal cover (which is a simply connected Riemann surface) by a discrete subgroup.

The Riemann sphere  $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$  is the simplest of the three simply connected Riemann surfaces. Up to conformal transformation, it is the only Riemann surface of positive curvature. For physical applications of CFT one rarely works with the sphere equipped with the round metric, but rather with the compactified plane with the metric  $ds^2 = dz d\bar{z}$ , in which all the curvature is at infinity. Note that these two metrics are in the same conformal class (up to a slight technical issue : the Weyl factor vanishing at infinity).

The flat metric (around 0) on the compactified plane :  $ds^2 = dz d\bar{z}$  is a natural choice. While there will be some foray on surfaces with no or negative total curvature, most of this manuscript will take place on the Riemann sphere, equipped with this metric.

A particular type of Weyl transformation on the sphere is given by:

$$d^2s = dz d\bar{z} \rightarrow d^2s = \frac{dz d\bar{z}}{(cz + d)^2(\bar{c}\bar{z} + \bar{d})^2}, \quad \text{with } c, d \in \mathbb{C}, cd \neq 0.$$

Contrary to the transformation from the round to the flat metric, these transformations leave the surface invariant, in the sense that there exist bi-holomorphic diffeomorphism from  $\mathbb{C}P^1 = \mathbb{C} \cup \infty$  to itself, such that the pull-back of the new metric gives back the flat metric. Those are called the Möbius (or projective) transformations. They form a group (under composition):

$$f : z \rightarrow \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1. \quad (1.2)$$

The elements of this group, named *global conformal transformations* include the translations ( $z \rightarrow z + b$ ), rotations ( $z \rightarrow e^{i\omega}z$ ,  $\omega \in \mathbb{R}$ ) and scale transformations ( $z \rightarrow \lambda z$ ). The full group also contains an additional transformation of the form:

$$z \rightarrow \frac{1}{cz + d}, \quad c \neq 0.$$

This last type of transformations is called *special conformal transformation*. Special conformal transformations are harder to represent on the plane than the three other types of transformations (see figure 1.2). On the cylinder (or the sphere) however, the inversion  $z \rightarrow \frac{1}{z}$  is simply a reflection with respect to the  $x$ -axis.

An important property of the global conformal transformations is that there is always one (and only one) of them which send the triplet  $\{z_1, z_2, z_3\} \in (\mathbb{C}P^1)^3$  to  $\{0, 1, \infty\}$ . The subgroup which does not contain the special transformations can only fix the position of two points.

*On the sphere, the rotations  $z \rightarrow e^{i\omega}z$  correspond to azimuthal rotation (around  $z$ ), scale transformations correspond to polar rotation and translations are more difficult to visualize.*

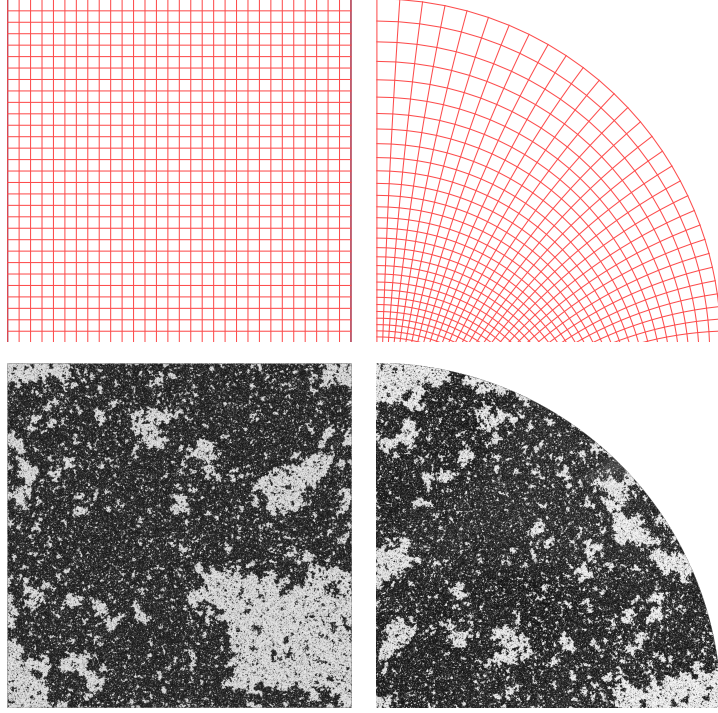


Figure 1.2: Effect of the conformal transformation  $z \rightarrow z/(z+1)$  on a square lattice and a typical configuration of the Ising model at its critical temperature.

The group of global conformal transformation 1.2, known as the Möbius group, is in bijection with  $PSL(2, \mathbb{C})$ .

$$\varphi : PSL(2, \mathbb{C}) \rightarrow G$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \left\{ z \rightarrow \frac{az + b}{cz + d} \right\}.$$

This bijection implies that the set of global transformations is a Lie group of (complex) dimension three. The associated Lie algebra is in bijection with  $\mathfrak{sl}_2$ . Instead of the traditional Pauli base  $\{\sigma_x, \sigma_y, \sigma_z\}$ , a more adapted basis for this algebra is:

$$l_0 = \frac{\sigma_z}{2} \quad l_{\pm 1} = \frac{\sigma_x \pm i\sigma_y}{2}.$$

For  $\epsilon_a$  infinitesimal real numbers, we see that:

$$\begin{aligned} & \varphi(\mathbb{1} + \epsilon_0 l_0 + \epsilon_{-1} l_{-1} + \epsilon_{+1} l_1)(z) \\ &= \frac{(1 + \epsilon_0/2)z + \epsilon_{-1}}{\epsilon_{+1}z + (1 - \epsilon_0/2)} \approx z + \epsilon_{-1} + \epsilon_0 z + \epsilon_1 z^2. \end{aligned}$$

Hence  $l_{-1}$  action on the group of conformal transformation corresponds to an infinitesimal translation (associated to the Lie generator  $\partial_z$ ),  $l_0$  to a dilatation ( $z\partial_z$ ) and  $l_1$  to one of the special conformal transformations ( $z^2\partial_z$ ). The Lie group formed by rotation, translation and scaling only have an associated algebra of dimension two.



These Weyl transformations are the only one which do not modify the surface. Now, we can slightly relax this constraint and look for Weyl transformations that keep a surface invariant locally : on an open set around every point there exists coordinates  $y, \bar{y}$  in which the metric has the same expression as before the Weyl transformation.

The transformation  $r \rightarrow r'$ , of Jacobian matrix  $J$ , will be conformal if and only if:

$$J^t g J = \Lambda g$$

With  $\Lambda$  a smooth, positive, function. If the original metric is flat, this implies:

$$J^t J = \Lambda \mathbb{1} \quad \Rightarrow \quad \det(J) = \pm \Lambda \text{ and } \det(J) J^{-1} = \pm J^T$$

Re-expressed in term of derivatives, this relation gives the Cauchy-Riemann equations (or the Riemann-Cauchy equations, their anti-holomorphic equivalent):

$$\frac{\partial x'}{\partial x} = \pm \frac{\partial y'}{\partial y} \quad \frac{\partial x'}{\partial y} = \mp \frac{\partial y'}{\partial x}$$

Hence, in two dimensions, the conformal transformations are the holomorphic and anti-holomorphic functions. Only the former preserves the orientation of the surface. These transformations are *local conformal transformations*. They are not diffeomorphism in general, the initial surface is only conserved locally.

An interesting surface (that will come in handy in chapter 3) is formed by the disjoint union of two planes  $\mathbb{C}$ , both with a cut along the semi-infinite real line  $[0, \infty]$  on which they are joined. This object is a Riemann surface, flat nearly everywhere. If the two planes have coordinates  $z_1$  and  $z_2$ , one can define a continuous coordinate  $z$ , on the whole surface, with  $z = \sqrt{z_1}$  on the first surface and  $z = -\sqrt{z_2}$  on the second one. In term of  $z$ , the metric can be written:

$$d^2s = \frac{1}{4|z|} dz d\bar{z}.$$

Of course, outside of 0 and  $\infty$ , the change of variable  $y = z^2$  will give back a flat metric.  $z \rightarrow z^2$  is a transformation which is locally conformal, but is not a diffeomorphism.

Following on the footsteps of the global transformations, we can write local generators for the Lie algebra associated with these transformations:

$$l_n = -z^{n+1} \frac{\partial}{\partial z}.$$

This algebra is called the Witt algebra and the commutation relations take the form:

$$\begin{aligned} [l_n, l_m] &= z^{n+1} \partial_z (z^{m+1} \partial_z) - z^{m+1} \partial_z (z^{n+1} \partial_z) \\ &= z^{n+m+1} (m+1 - n+1) \partial_z \\ &= (n-m) l_{n+m}. \end{aligned}$$

*The large number of conformal transformations – holomorphic functions form an infinite dimensional vector space – is a particularity of the two-dimensional case. A classic theorem of Liouville proves that in larger dimension conformal transformations are far less common.*

*The quotient of a manifold by a discrete subgroup is not always a manifold and is generically named orbifold. The surfaces we consider here are nonetheless always manifolds.*

*The Witt algebra was first defined by Cartan [39].*

## 1.2.2 Conformal field theory

A conformal field theory is composed of operators  $\{\mathcal{O}_a\}$ . Operators are functions from the Riemann surface  $\Omega$  to a complex vector space. The full set of operators is called the spectrum of the theory.

The operators covariant under global conformal transformations are called quasi-primary. Here this takes the following meaning, if  $f$  is a global conformal transformation:

$$\mathcal{O}_a(z) \rightarrow f'(z)^{-h_a} \overline{f'(z)}^{-\bar{h}_a} \mathcal{O}_a(z), \quad (1.3)$$

$h_a$  and  $\bar{h}_a$  are called the chiral dimensions of the operator  $\mathcal{O}_a$ .  $\Delta_a = h_a + \bar{h}_a$  and  $s_a = h_a - \bar{h}_a$  are respectively the conformal dimension and the conformal spin of the operator. A *scalar* operator satisfies  $s_a = 0$ . An operator present in all theories is the identity operator  $\mathbb{1}$ , with trivial action. It has  $\Delta_{\mathbb{1}} = 0$  and  $s_{\mathbb{1}} = 0$ .

States can be defined naturally by radial quantification. Usually one assumes the existence of a state  $|\mathbb{1}\rangle$ , every other states are then defined as:

$$\mathcal{O}(0)|\mathbb{1}\rangle = |\mathcal{O}\rangle,$$

This relation defines the *state-operator correspondence*.

The symmetry algebra acting on the spectrum is not exactly the Witt algebra. Because states in the spectrum are only defined up to a multiplicative constant, hence the action of the symmetry group only translates to a projective action on the spectrum. This gives an additional degree of freedom to the Lie algebra acting on the operators. In full generality, the algebra is a *central extension* of the Witt algebra, called the *Virasoro algebra*:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n,-m}\mathbb{1}, \quad \forall n, m \in \mathbb{Z}. \quad (1.4)$$

The factor  $c$  is called the *central charge* of the theory. The presence of the central charge does not modify the space of global conformal transformations, generated by  $L_0, L_1$  and  $L_{-1}$ , and all three keep the same interpretations. Notably  $L_0$  is associated with scale transformations and  $L_{-1}$  with translations, which permits us to write:

$$(L_0\mathcal{O}_a)(z) = h_a\mathcal{O}_a(z) \quad (L_{-1}\mathcal{O}_a)(z) = \partial_z\mathcal{O}_a(z).$$

The algebra is not complete yet, as we allowed the operators to behave differently in  $z$  and  $\bar{z}$ . This means that we need to consider another algebra, commuting with the previous one, and which will verify exactly the same relations:

$$\begin{aligned} [\bar{L}_n, \bar{L}_m] &= (n - m)\bar{L}_{n+m} + \frac{\bar{c}}{12}n(n+1)(n-1)\delta_{n,-m}\mathbb{1} \\ \text{and } [\bar{L}_n, L_m] &= 0, \quad \forall n, m \in \mathbb{Z}. \end{aligned} \quad (1.5)$$

In two dimensions, the distinction is made between operators covariant under global or local conformal transformations. This lead to the notion of quasi-primary and primary field, respectively. In dimension  $d \geq 3$  however there are no local conformal transformations, and quasi-primary fields are simply called primary in this context.

For the same reasons, the symmetry group of a rotationally invariant quantum system is  $SU(2)$  rather than  $SO(3)$ . If the initial Lie algebra is finite dimensional though, the commutation relations are not modified.

In all the models we consider, the central charge is also real,  $c \in \mathbb{R}$ . For unitary models this comes from the "unitarity of  $L_0$ " (and in that case  $c$  is positive). It is also generally true for statistical models with real weights.

And for a given operator  $\mathcal{O}_a$ ,  $\bar{L}_0 \mathcal{O}_a = \bar{h}_a \mathcal{O}_a$ . This means that, purely in algebraic terms, the holomorphic and anti-holomorphic parts of the operators have been dissociated. The two sectors are called chiral, with a left-moving one (holomorphic) and a right-moving one (anti-holomorphic). In a full theory, the two chiral sectors are rarely independent: this translates into constraints on the spins of operators. An example of the kind of restriction imposed will be given in chapter 3. The equality between the holomorphic and anti-holomorphic central charge,  $c = \bar{c}$  usually holds, in statistical systems at least, and will always be supposed true thereafter.

To prove that this algebra is indeed the only one possible, see that if  $\rho$  is a projective representation of the conformal group, then, by definition:

$$\rho(g_1)\rho(g_2) = \omega(g_1, g_2)\rho(g_1g_2).$$

Elements of the associated Lie algebra are given by  $P = \left. \frac{d}{dt}\rho(g(t)) \right|_{t=0}$ , where  $g(t)$  is a differentiable path in the group of conformal transformation equal to the identity in  $t = 0$ . The Lie bracket can be computed:

$$[P_1, P_2] = \left. \frac{d}{dt}\rho(g_1(t)) \rho(g_2(t)) \rho(g_1(t))^{-1} \rho(g_2(t))^{-1} \right|_{t=0}.$$

Hence:

$$[P_1, P_2] = \left. \frac{d}{dt}C(g_1(t), g_2(t))\rho(g_1(t)g_2(t)g_1(t)^{-1}g_2(t)^{-1}) \right|_{t=0}.$$

With  $C$  a complex-valued function. We can use the product rule on the derivative (remembering that  $g_i(0) = \mathbb{1}$ ), and obtain:

$$[P_1, P_2] = \left. \frac{d}{dt}C(g_1(t), g_2(t)) \right|_{t=0} \mathbb{1} + \left. \frac{d}{dt}\rho(g_1(t)g_2(t)g_1(t)^{-1}g_2(t)^{-1}) \right|_{t=0}.$$

The second term is what one would obtain in a regular representation, but the first one is new. In our case, it means that:

$$[L_n, L_m] = (n - m)L_{n+m} + c_{n,m}\mathbb{1}, \quad (1.6)$$

$c_{n,m}$  is anti-symmetric in its two variables and  $c_{n,n} = 0$ . The transformation:

$$\tilde{L}_n = \frac{1}{n}[L_0, L_n] \quad \forall n \neq 0 \quad \text{and} \quad \tilde{L}_0 = L_0,$$

is a Lie algebra homomorphism. Hence, up to equivalence, we can always replace the  $L_n$  by the  $\tilde{L}_n$ , and choose  $c_{0,n} = 0$ .

Applying the Jacobi relation to 1.6 gives the following equation on  $c$ :

$$0 = (n - m)c_{k,n+m} + (k - n)c_{m,k+n} + (m - k)c_{n,m+k}.$$

With  $n = 0$ :

$$(m + k)c_{k,m} = (m - k)c_{0,m+k} = 0.$$

Hence,  $c_{k,m}$  is only non-zero if  $k + m = 0$ ,  $c_{k,m} = c_k \delta_{k,-m}$ . And  $c_m$  verify the recursions relation:

$$\frac{c_{k+1}}{c_k} = \frac{2 + k}{k - 1}.$$

Which finally implies  $c_k \propto (k - 1)k(k + 1)$ , and the equation 1.4 is obtained. The factor  $\frac{1}{12}$  is chosen to give to the free boson theory a central charge equal to 1.



Operators in a conformal theory belong in a representation of the Virasoro algebra. In a given representation, consider the eigenstate of the operator  $L_0$  with the lowest (non-zero) eigenvalue  $\Delta : |\mathcal{O}\rangle$ . Because  $L_0 L_n |\mathcal{O}\rangle = (\Delta - n) L_n |\mathcal{O}\rangle$ , this implies that  $L_n |\mathcal{O}\rangle = 0, \forall n > 0$ . This type of state is called primary.  $\Delta = h + \bar{h}$  is the conformal weight of the state, it controls its behaviour under a scaling transformation. The states  $L_{-r_1} \cdots L_{-r_n} \bar{L}_{-s_1} \cdots \bar{L}_{-s_m} \mathcal{O}$  are called the descendant states and have dimensions  $h + \sum_{i=1}^m r_i$  and  $\bar{h} + \sum_{i=1}^m s_i$ . The order of the descendant is given by  $\sum_{i=1}^n r_i + \sum_{i=1}^m s_i$ . The vector space formed by the primary field and its descendants is the *Verma module*  $V(c, \Delta)$ . Primary operators  $\mathcal{O}$  have the particularity to satisfy the relation 1.3 for any holomorphic function  $f$ :

$$\mathcal{O}_{\text{prim.}}(z) \rightarrow f'(z)^{-h_{\text{prim.}}} \overline{f'(z)}^{-\bar{h}_{\text{prim.}}} \mathcal{O}_{\text{prim.}}(z). \quad (1.7)$$

To fully define a CFT we need the spectrum but also the way different operators interact. In a general Quantum Field Theory (QFT), it would be necessary to determine all possible correlation functions.. In a CFT, thanks to conformal invariance, we can restrict ourselves to study the behaviour of two operators infinitesimally close from one another, a *fusion*. The fusion creates one unique local operator which can be decomposed into operators of the spectrum (or else the CFT would not be closed under this operation). This decomposition is called an *operator product expansion* (OPE). Formally, scale invariance imposes the following form:

$$\mathcal{O}_1(z) \mathcal{O}_2(0) \underset{z \rightarrow 0}{\approx} \sum_{\mathcal{O}_p \in \text{spectr.}} C(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_p) z^{h_p - h_1 - h_2} \bar{z}^{\bar{h}_p - \bar{h}_1 - \bar{h}_2} \mathcal{O}_p(0). \quad (1.8)$$

The coefficients  $C(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$  are the *structure constants* of the theory. With the spectrum, they fully determine correlation functions on the sphere. Because the operators are defined up to a multiplicative constant, the coefficients are not well-defined yet. A choice often made in statistical physics is to normalize the two-point correlation function of primary fields as follows:

$$\langle \mathcal{O}_1(z) \mathcal{O}_1(0) \rangle = \frac{1}{(z_1 - z_2)^{2h} (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \Leftrightarrow C(\mathcal{O}_1, \mathcal{O}_1, \mathbb{1}) = 1. \quad (1.9)$$

This relation allows to define conjugate states:

$$\langle \mathcal{O} | = \lim_{\substack{\bar{z} \rightarrow \infty \\ z \rightarrow \infty}} z^{2h} \bar{z}^{2\bar{h}} \langle \mathbb{1} | \mathcal{O}(z, \bar{z}) \quad \text{such that} \quad \langle \mathcal{O} | \mathcal{O} \rangle = 1.$$

The three-point correlation function of primary fields have a simple expression:

$$\begin{aligned} & \langle \mathcal{O}_1(z_1) \mathcal{O}_2(z_2) \mathcal{O}_3(z_3) \rangle \\ &= \frac{C(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_2}}. \end{aligned} \quad (1.10)$$

A tacit assumption of the convention 1.9 is that the identity always appears in the OPE of two identical fields. Many counter-examples exist, in extended CFTs or in LogCFTs for example. In some of these cases, the dual of an operator  $\mathcal{O}^*$  can be defined as the only field such that  $\langle \mathcal{O}(0) \mathcal{O}^*(z) \rangle$ .

Stronger still, the three-point functions involving descendants can all be computed in term of the three-point function of their primaries and their conformal dimensions. Hence, only the structure constants and dimensions of *primary* fields are needed to fully build the theory. The primary fields are the building blocks of two-dimensional CFTs.

Two operators of particular importance are the formal power series of the  $L_{-n}$  and of the  $\bar{L}_{-n}$ :

$$T(z) = \sum_{n \in \mathbb{Z}} L_{-n} z^{n-2}, \quad \bar{T}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{L}_{-n} \bar{z}^{n-2}.$$

Those two fields are local: they are simply the components of the *stress energy tensor* of the theory :  $T_{z,z} = T(z)$ ,  $T_{z,\bar{z}} = T_{\bar{z},z} = 0$ ,  $T_{\bar{z},\bar{z}} = \bar{T}(\bar{z})$ . The stress energy tensor itself is not primary, using the correspondence between states and operators, we can see it as a descendant of the identity operator.

$$T(z)|\mathbb{1}\rangle = \sum_{n \leq -2} z^{-n-2} L_n |\mathbb{1}\rangle, \quad \lim_{z \rightarrow 0} T(z)|\mathbb{1}\rangle = L_{-2} |\mathbb{1}\rangle.$$

OPEs between the stress-energy operator and primary operators are quite simple:

$$T(z)\mathcal{O}_a(0) \approx \frac{h_a \mathcal{O}_a(0)}{z^2} + \frac{\partial \mathcal{O}_a(0)}{z} + \text{regular terms}.$$

And the OPE between the stress-energy tensor and itself gives the result:

$$T(z)T(0) = \frac{c/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \text{regular terms}.$$

Using the relation 1.7 for individual primaries, the transformation of a correlation function of primary operators can be written:

$$\begin{aligned} & \langle \mathcal{O}_{a_1}(z_1) \cdots \mathcal{O}_{a_n}(z_n) \rangle_{\Omega} \\ & \rightarrow \prod_i f'(z_i)^{-h_i} \overline{f'(z_i)^{-\bar{h}_i}} \langle \mathcal{O}_{a_1}(z_1) \cdots \mathcal{O}_{a_n}(z_n) \rangle_{f(\Omega)}. \end{aligned}$$

Where  $\Omega$  and  $f(\Omega)$  are respectively the original Riemann surface and its image under the transformation  $f$ . While this relation is valid for any holomorphic function, if the transformation is not global the surface on which the correlation is computed is modified. In the event where  $f$  is a global conformal transformation, these relations can be used to find back the expressions 1.9 and 1.10 for the two and three-point correlation functions. In contrast, a four-points function cannot be entirely determined from the global conformal transformations alone. However, they depend on only one parameter instead

*The stress-energy tensor has the same meaning as in traditional QFT: when the theory has a well-defined action,  $T_{\mu\nu}$  is the functional derivative of the action with respect to the metric.*

*Because the stress-energy tensor is not a primary, its transformation under a local conformal transformation is more complicated, and depends on  $c$ . This gives another possible interpretation of the central charge as an anomaly. i.e. a breaking of the conformal symmetry at the quantum level. For this reason  $c$  is sometimes referred to as the conformal anomaly. As a consequence the central charge appears whenever a scale is introduced in the system (when working in finite volume or on a curved background for instance).*

of four, as there is always a transformation sending the quadruplet  $\{z_1, z_2, z_3, z_4\}$  to  $\{0, 1, z = \frac{(z_3 - z_1)(z_2 - z_4)}{(z_3 - z_4)(z_2 - z_1)}, \infty\}$ .

Inserting the OPE relation 1.8 in a four-points function gives:

$$\begin{aligned} & \langle \mathcal{O}_1 | \mathcal{O}_2(1) \mathcal{O}_3(z) | \mathcal{O}_4 \rangle \\ &= \sum_{\mathcal{O}_p \in \text{spect.}} C(\mathcal{O}_3 \mathcal{O}_4 \mathcal{O}_p) z^{h_p - h_3 - h_4} \bar{z}^{\bar{h}_p - \bar{h}_3 - \bar{h}_4} \langle \mathcal{O}_1 | \mathcal{O}_2(1) | \mathcal{O}_p \rangle. \end{aligned}$$

The sum on the right-hand side can be split into sums over the different Verma modules of each primary. A Verma module  $V(c, \Delta)$  is a graded algebra, the set of descendants at a given level  $(n, \bar{n})$  forms a vector space  $V_{c, \Delta}^{(n, \bar{n})}$ , and  $V(c, \Delta) = \bigoplus_{n, \bar{n}} V_{c, \Delta}^{(n, \bar{n})}$ . These vector spaces have finite dimensions, and we can extract an orthogonal basis from them.

*Orthogonal, and not orthonormal since the inner product need not be positive definite for non-unitary theories.*

For example, an orthogonal basis for the descendants at level  $(2, 0)$  of a state  $|\mathcal{O}\rangle$ , with dimension  $h$ , is formed by the two states:

$$K_0^{(2,0)} = L_{-2} |\mathcal{O}\rangle \quad \text{and} \quad K_1^{(2,0)} = \left( L_{-2} - \frac{8h + c}{12h} L_{-1}^2 \right) \mathcal{O}.$$

Given these bases we can use the completeness relation:

$$\mathbb{1} = \sum_{\mathcal{O} \text{ primary}} \sum_{n, \bar{n} \in \mathbb{N}} \sum_{j=0}^{\dim(V_{c, \Delta \mathcal{O}}^{(n, \bar{n})})} \frac{1}{\langle K_j^{(n, \bar{n})} | K_j^{(n, \bar{n})} \rangle} |K_j^{(n, \bar{n})} \mathcal{O}\rangle \langle K_j^{(n, \bar{n})} \mathcal{O}|.$$

And fully rewrite the four-point function as follows:

*If one of the descendant has norm zero this formula fails. We will see what this entails in the next section.*

$$\begin{aligned} & \langle \mathcal{O}_1 | \mathcal{O}_2(1) \mathcal{O}_3(z) | \mathcal{O}_4 \rangle \\ &= \sum_{\mathcal{O}_p \text{ primary}} z^{h_p - h_3 - h_4} \bar{z}^{\bar{h}_p - \bar{h}_3 - \bar{h}_4} \sum_{n, \bar{n} \in \mathbb{N}} \frac{z^n \bar{z}^{\bar{n}}}{\langle K_j^{(n, \bar{n})} | K_j^{(n, \bar{n})} \rangle} \\ & \times \sum_{j=0}^{\dim(V_{c, \Delta \mathcal{O}}^{(n, \bar{n})})} C(\mathcal{O}_3, \mathcal{O}_4, K_j^{(n, \bar{n})} \mathcal{O}_p) \langle \mathcal{O}_1 | \mathcal{O}_2(1) | K_j^{(n, \bar{n})} \mathcal{O}_p \rangle. \end{aligned} \quad (1.11)$$

The structure constants of descendants are proportional to the three-point function of their associated primaries, hence in full generality:

$$\langle \mathcal{O}_1 | \mathcal{O}_2(1) \mathcal{O}_3(z) | \mathcal{O}_4 \rangle = \sum_{\mathcal{O}_p \text{ primary}} C_{34p} C_{12p} \mathcal{F}_{1,2 \rightarrow 3,4}(z) \bar{\mathcal{F}}_{1,2 \rightarrow 3,4}(\bar{z}). \quad (1.12)$$

the functions  $\mathcal{F}$  isolated in 1.12 are entirely defined by the algebra and only depends on the conformal dimensions of the operators. These two functions are named *conformal blocks*, and have the form:

$$\mathcal{F} : z \rightarrow z^{h_p - h_3 - h_4} (1 + \#z + \#z^2 + \dots).$$

We will use the term generic to describe a situation where operators  $\mathcal{O}_p$ , appearing in the fusion, have different dimensions modulo 1. In this case all the conformal blocks appearing in a given fusion are linearly independent. While possible in theory, computing and using directly these blocks in analytic computations is difficult.

### 1.2.3 Degenerate operators and rational models

An operator is *degenerate* if one of its descendant state vanishes. The operator is degenerate at order  $n$  if the first descendant to vanish is of order  $n$ :

$$\mathcal{L}_{-n}\mathcal{O} = 0,$$

where  $\mathcal{L}_{-n}\mathcal{O}$  is a linear combination of order  $n$  descendants. A necessary condition for degeneracy is that  $\mathcal{L}_{-n}\mathcal{O}$  is a primary operator. This already constrains the dimension of the primary  $\mathcal{O}$  as a function of the central charge  $c$ , as the operator will have to verify:

$$L_p(\mathcal{L}_{-n}\mathcal{O}) = 0 \quad \forall p \geq 1. \quad (1.13)$$

For example:

- If the operator  $\mathcal{O}$  is degenerate at order  $n = 1$ , the condition  $L_{-1}\mathcal{O} = 0$  must be verified. Hence:

$$L_1L_{-1}\mathcal{O} = 2L_0\mathcal{O} = 2\Delta_{\mathcal{O}} = 0$$

which boils down to  $\Delta = 0$ . Note that  $L_{-1}$  is also the generator of the translations, so  $\mathcal{O}$  is invariant under translation, so  $\mathcal{O}$  can only be the identity in this scenario.

- If the operator  $\mathcal{O}$  is degenerate at order  $n = 2$ , the condition is of the form  $(L_{-2} - \alpha L_{-1}^2)\mathcal{O} = 0$ . Which gives two equations:

$$L_2(L_{-2} - \alpha L_{-1}^2)\mathcal{O} = 4L_0 + \frac{c}{2} - 6\alpha L_0 = 0,$$

and:

$$L_1(L_{-2} - \alpha L_{-1}^2)\mathcal{O} = (3 - 2\alpha - 4\alpha\Delta)L_{-1}\mathcal{O} = 0,$$

which fixes the expression of the degenerate descendant but also add a constraint on the conformal dimension of  $\mathcal{O}$ .

$$h = \frac{(5 - c) \pm \sqrt{(c - 1)(c - 25)}}{16}.$$

Before introducing the general form of all the degenerate states, let's introduce more workable notations:

$$c = 1 - 6Q^2 \quad Q = b^{-1} - b \quad b = \sqrt{\frac{1 - c}{24}} + \sqrt{\frac{25 - c}{24}}.$$

These notations are coming from Liouville field theory.  $b$  is usually called the *coupling constant*, and  $Q$  the *background charge*. If the central charge is lower than one (or larger than 25),  $b^2$  is real. Additionally, instead of defining an operator through its conformal dimensions  $(h, \bar{h})$ ,

*Such a state would be of norm 0, the representation to consider in this case are the Verma module with the sub-module generated by the null state quotiented out.*

*Due to the Virasoro algebra it is sufficient that  $L_1(\mathcal{L}_{-n}\mathcal{O}) = 0$  and  $L_2(\mathcal{L}_{-n}\mathcal{O}) = 0$ . This in turns imply the other constraints since  $L_p \propto [L_1^{p-2}, L_2]$ .*

*The origin of these notations will be made clear later 1.2.4.*

we will often use the *conformal charge*  $\alpha$ , with its anti-holomorphic counterpart  $\bar{\alpha}$ . Which are related to  $h$  and  $\bar{h}$  through:

$$h = \alpha(\alpha - Q) \quad \bar{h} = \bar{\alpha}(\bar{\alpha} - Q).$$

This expression is defined up to the reflection  $\alpha \rightarrow Q - \alpha$ . Another common notation is:

$$P = \alpha - \frac{Q}{2} \quad h = P^2 - \frac{Q^2}{4}. \quad (1.14)$$

In general, the classification of all degenerate operators can be done through the study of the modular structure of the theory (the consistency of the theory on the torus) [137], or through the Coulomb-gas analysis [54]. Here we will just list them without proof. The conformal charge of a degenerate operator can be written:

$$\alpha_{r,s} = \frac{Q}{2} + \frac{r}{2}b - \frac{s}{2}b^{-1} \quad r, s \in \mathbb{Z}.$$

And its fusion rules with a given operator  $\mathcal{O}_\alpha$  takes the form:

$$\mathcal{O}_{\alpha_{r,s}} \otimes \mathcal{O}_\alpha = \sum_{\substack{k=1-r \\ k+r \equiv 1[2]}}^{r-1} \sum_{\substack{l=1-s \\ l+s \equiv 1[2]}}^{s-1} \mathcal{O}_{\alpha + \frac{k}{2}b + \frac{l}{2}b^{-1}}. \quad (1.15)$$

If  $b^2$  is a rational number, there exist two integers  $p$  and  $p'$  such that  $pb - p'b^{-1} = 0$ . In these circumstances, it is possible to close the operator algebra with a finite number of primary fields. The fusion relations between these operators are also well-known, they mimic the fusion rules of the representation of the  $\mathfrak{su}(2)$  Lie algebra:

$$\mathcal{O}_{\alpha_{r_1, s_1}} \otimes \mathcal{O}_{\alpha_{r_2, s_2}} = \sum_{\substack{r=1+|r_1-r_2| \\ r+r_1+r_2 \equiv 1[2]}}^{r_1+r_2-1} \sum_{\substack{s=1+|s_1-s_2| \\ s+s_1+s_2 \equiv 1[2]}}^{s_1+s_2-1} \mathcal{O}_{\alpha_{r,s}}.$$

With these fusion rules at hand, we can define *minimal conformal field theories*, which have the minimal number of fields necessary to be closed under the OPE. The minimal conformal field theory  $\mathcal{M}(p, p')$  is formed by the primary  $\{\mathcal{O}_{\alpha_{r,s}, \alpha_{r,s}}\}_{1 \leq r < p, 1 \leq s < p'}$  and their descendants. Additionally, if  $p = p' + 1$ , the theory does not contain any operator of negative dimension, and the theory is *unitary*. Examples of minimal models include:

- $\mathcal{M}_{1,1} = \{\mathbb{1}\}$ , the trivial CFT.
- $\mathcal{M}_{5,2} = \{\mathbb{1}, \Phi\}$  the simplest non-trivial theory.  $\Phi$  is a field of dimension  $-2/5$  and  $\Phi \otimes \Phi = \mathbb{1} \oplus \Phi$ . Because one of the field has negative dimension, the model is also not unitary. This minimal model corresponds to the Yang-Lee edge singularity, and its central charge is  $c = -22/5$ .

*These operators form the Kac table of the CFT. Kac was one of the first to study the representation theory of Virasoro [92].*

*Unitarity is a sine qua non condition in quantum system, less so in statistical systems.*

*In the Ising model, for  $T > T_c$  and specific imaginary values of the magnetization  $ih$ , the partition function can be equal to zero. These zeros are all localized below a critical value  $ih_c$ . The density of zeros near  $h_c$  behaves, in the continuum limit, like  $(h - h_c)^{-1/6}$ . This power-law betrays the presence of a CFT, corresponding to the Yang-Lee edge singularity [106].*

- $\mathcal{M}_{4,3} = \{\mathbb{1}, \sigma, \epsilon\}$ , whose fields have dimensions  $\{0, 1/8, 1\}$ . Two important operators of the critical Ising model are contained in  $\mathcal{M}_{4,3}$ ,  $\sigma$  is the spin operator and  $\epsilon$  the energy operator. Fusion rules give  $\sigma \otimes \sigma = \epsilon \oplus \mathbb{1}$ ,  $\sigma \otimes \epsilon = \sigma$  and  $\epsilon \otimes \epsilon = \mathbb{1}$ . The model has central charge  $c = \frac{1}{2}$ .

Another type of model that can be constructed easily (but whose consistency is harder to infer) are the CFTs formed by the reunion of all the degenerate operators  $\{\mathcal{O}_{\alpha_{r,s}, \alpha_{r,s}}\}_{r,s \in \mathbb{Z}}$ . These CFTs are called *generalized minimal models*.

The condition 1.13 :  $\mathcal{L}_{-n}\mathcal{O}$  is a primary operator is called the null-vector condition and is a very strong constraint.  $L_{-n}\mathcal{O}$  and  $\mathcal{O}$  do not have the same dimension, hence  $\langle L_{-n}\mathcal{O} | \mathcal{O} \rangle = 0$ . This relation stays true for all the descendants of  $\mathcal{O}$ , hence  $\langle L_{-n}\mathcal{O} | L_{-p}\mathcal{O} \rangle = 0$ , and in particular the norm of  $L_{-n}\mathcal{O}$  is 0. This does not mean  $\mathcal{O} = 0$  however, in general the inner product is not positive definite.

#### 1.2.4 Bootstrap

In a nutshell, the bootstrap approach consists in a method to *iteratively* build entire theories from their underlying symmetries and potential additional constraints.

The starting point is a four-point function of scalar primary operators. The conformal block decomposition that was presented previously 1.12 gives:

$$\begin{aligned} \langle \prod_{i=1}^4 \mathcal{O}_i(z_i) \rangle &= \sum_p C_{12p} C_{p34} \left| \mathcal{F}_{1,2 \rightarrow 3,4}^p(z_i) \right|^2 \\ &= \sum_p C_{12p} C_{p34} \begin{array}{c} \mathcal{O}_1 \quad \quad \mathcal{O}_3 \\ \quad \diagdown \quad \diagup \\ \quad \mathcal{O}_p \\ \quad \diagup \quad \diagdown \\ \mathcal{O}_2 \quad \quad \mathcal{O}_4 \end{array} . \end{aligned}$$

The previous equation was obtained by expanding the operator product between  $\mathcal{O}_1$  and  $\mathcal{O}_2$ . Naturally, the same result can be obtained by starting with the expansion of  $\mathcal{O}_1 \otimes \mathcal{O}_3$  instead, the order in which the products in the correlation function are taken should not matter. This means that the following equation between conformal blocks should be valid:

$$\begin{aligned} \sum_p C_{12p} C_{p34} \mathcal{F}_{1,2 \rightarrow 3,4}^p(z_i) \overline{\mathcal{F}}_{\bar{1},\bar{2} \rightarrow \bar{3},\bar{4}}^p(\bar{z}_i) \\ = \sum_q C_{13q} C_{q24} \mathcal{F}_{1,3 \rightarrow 2,4}^q(z_i) \overline{\mathcal{F}}_{\bar{1},\bar{3} \rightarrow \bar{2},\bar{4}}^q(\bar{z}_i) . \end{aligned} \tag{1.16}$$

This equation is known as the *bootstrap equation*. The underlying symmetry, called *crossing symmetry* is very natural and should be verified in every CFT. The expansion on the left-hand side is sometimes called t-channel expansion by analogy with particle scattering, while the right-hand side is the s-channel expansion.

In diagrammatic form:

$$\sum_p C_{12p} C_{p34} \begin{array}{c} \mathcal{O}_1 \\ \diagdown \\ \mathcal{O}_p \\ \diagup \\ \mathcal{O}_2 \end{array} \begin{array}{c} \mathcal{O}_3 \\ \diagup \\ \mathcal{O}_p \\ \diagdown \\ \mathcal{O}_4 \end{array} = \sum_q C_{13q} C_{q24} \begin{array}{c} \mathcal{O}_1 \\ \diagdown \\ \mathcal{O}_q \\ \diagup \\ \mathcal{O}_2 \end{array} \begin{array}{c} \mathcal{O}_3 \\ \diagup \\ \mathcal{O}_q \\ \diagdown \\ \mathcal{O}_4 \end{array} \quad (1.17)$$

The conformal blocks only depend on the conformal dimensions of the operators of the theory’s spectrum. If this spectrum is known, the quadratic equations 1.16 are enough to entirely determine the three-point functions (if the spectrum does not contain any degeneracies). In turn, we saw that the fusion rules and the spectrum can, at least theoretically, determine all the correlation functions of the theory. The theory is building itself from its most basic bloc (the spectrum), effectively pulling itself up by its own bootstraps.

The bootstrap equations also imply that the conformal blocks in the s and t channels are linearly related. Again assuming generic dimensions, the number of operators appearing in the sum should be the same on both sides of the equality.

In concrete examples, these equations are complicated to use directly. As explained before, conformal blocks are hard to compute, and the OPE between generic operators creates a large, if not infinite, number of fields. But the recursion relations have been used as a consistency check of various CFTs [130].

If one assumes unitarity, the three-point functions must be real, and the bootstrap equation for a correlation function consisting of four copies of the same fields  $\langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_1 \rangle$  can be written:

$$\sum_p C_{11p}^2 |\mathcal{F}_{11 \rightarrow 11}^{(s)}(z_i)|^2 = \sum_q C_{11q}^2 |\mathcal{F}_{11 \rightarrow 11}^{(t)}(z_i)|^2 \quad (1.18)$$

Because the three-point functions are real, the equation is convex, which restrict the possible spectrum. The unitary minimal models happens to all lie on the extremal boundary of the convex domain of the solution [122].

There is one interesting case where these equations simplify drastically and the bootstrap becomes possible to solve analytically : if one of the operator of the four-points function is degenerate. Suppose that one of the operators in the correlation function is one of the two operators degenerate at level 2 seen in the previous section,  $\mathcal{O}_{2,1}$  ( $\alpha_{2,1} = b/2$ ) for example. The OPE of  $\mathcal{O}_{2,1}$  with another operator  $\mathcal{O}_\alpha$  is a special case of 1.15:

$$\mathcal{O}_{2,1} \otimes \mathcal{O}_\alpha = \mathcal{O}_{\alpha+\frac{b}{2}} \oplus \mathcal{O}_{\alpha-\frac{b}{2}}.$$

Consider a four-points function of the form:

$$\langle \mathcal{O}_{\alpha_3} | \mathcal{O}_{\alpha_2}(1) \mathcal{O}_{2,1}(z) | \mathcal{O}_{\alpha_1} \rangle,$$

*The “bootstrap” image is used in many domains and supposedly originates from the German Baron Munchausen’s book: “Here I would have undoubtedly died, if not the strength of my own arm, grabbing my own pigtail, had pulled me, including my horse —which I squeezed tightly between my legs—out of it.” A similar argument can be made in larger dimensions, and can be used to numerically estimate critical exponents [136]*



the sum in the bootstrap relations 1.16 will only have two terms. The conformal blocks appearing in the fusions  $z \rightarrow 0$ ,  $z \rightarrow \infty$  and  $z \rightarrow 1$  form a basis of the same function space. They are holomorphic nearly everywhere except at the positions 0, 1 and  $\infty$  where they have, respectively, singularities of the form  $z^{h_p-h_{2,1}-h_1}$ ,  $(1-z)^{h_p-h_{2,1}-h_1}$  and  $z^{-(h_p+h_{2,1}-h_1)}$ , which translates as:

- $z^{\alpha_1 b}$  and  $z^{(Q-\alpha_1)b}$  for  $z \rightarrow 0$ ,
- $(z-1)^{\alpha_2 b}$ ,  $(z-1)^{(Q-\alpha_2)b}$  for  $z \rightarrow 1$ ,
- $\left(\frac{1}{z}\right)^{(\alpha_3+b/2-Q)b}$ ,  $\left(\frac{1}{z}\right)^{-(\alpha_3-b/2)b}$  for  $z \rightarrow \infty$ .

Notice that the sum of these six exponents is equal to one, an indirect consequence of the residue theorem. A result due to Riemann [43] states that the only solutions possible are in the vector space generated by the two functions:

$$I_1(z) = z^{\alpha_1 b} (z-1)^{b\alpha_2} {}_2F_1 \left[ \begin{matrix} bP_t, b(P_t-2P_3) \\ 1+2P_1b \end{matrix}; z \right],$$

$$I_2(z) = z^{(Q-\alpha_1)b} (z-1)^{b\alpha_2} {}_2F_1 \left[ \begin{matrix} b(P_t-2P_1), b(P_t-2P_1-2P_3) \\ 1-2P_1b \end{matrix}; z \right].$$

Where  ${}_2F_1$  is the hypergeometric function, and  $P_t = \frac{1}{2b} + \sum_i P_i$ , with the notation  $P_i = (\alpha_i - Q/2)$  defined previously 1.14.

Generically one writes a Riemann scheme (or Riemann P-symbol) describing the behaviour of the expected function(s) near singularities.

$$\left\{ \begin{matrix} 0 & 1 & \infty \\ t_1 & u_1 & v_1 \\ t_2 & u_2 & v_2 \end{matrix} \right\}$$

This reads as follows, the different functions should have singularities of the form:

- Near 0,  $\#z^{t_1} + \#z^{t_2}$ ,
- Near 1,  $\#(1-z)^{u_1} + \#(1-z)^{u_2}$ ,
- Near  $\infty$ ,  $\#z^{-v_1} + \#z^{-v_2}$ .

The relation  $\sum_i t_i + u_i + v_i = 1$  is imposed by the residue theorem as the functions are holomorphic outside of these three points. The only vector space verifying these conditions is generated by:

$$I_1(z) = z^{t_1} (z-1)^{u_1} {}_2F_1 \left[ \begin{matrix} t_1 + u_1 + v_1, t_1 + u_1 + v_2 \\ 1 + t_1 - t_2 \end{matrix}; z \right],$$

$$I_2(z) = z^{t_2} (z-1)^{u_1} {}_2F_1 \left[ \begin{matrix} t_2 + u_1 + v_1, t_2 + u_1 + v_2 \\ 1 + t_2 - t_1 \end{matrix}; z \right].$$

The presence of two solutions is made obvious by the exchange  $t_1 \leftrightarrow t_2$ .

*Riemann's point of view was that an analytic function, instead of being defined through an obscure series expression, could just as well be defined by its singularities. In the case of hypergeometric functions this is particularly notable, many seemingly obscure change of variable formulas become quite trivial when considering the "singularity" definition. See the appendix 4.A for details on the hypergeometric function.*



${}_2F_1$  has an entire expansion for small  $z$ , hence this specific basis has a diagonal *monodromy* around 0, when the variable winds around 0  $I_1$  and  $I_2$  stay invariant up to a scalar:

$$I_1(z) \xrightarrow{z \circlearrowleft 0} e^{2i\alpha_1 b} I_1(z) \quad I_2(z) \xrightarrow{z \circlearrowleft 0} e^{2iQ(Q-\alpha_1)b} I_2(z).$$

However around 1 (or  $\infty$ ), their monodromy matrix is not diagonal. The basis with a diagonal monodromy around 1 is given by:

$$J_1(z) = z^{b\alpha_1} (1-z)^{b\alpha_2} {}_2F_1 \left[ \begin{matrix} bP_t, b(P_t - 2P_3) \\ 1 + 2bP_2 \end{matrix}; 1-z \right],$$

$$J_2(z) = z^{b\alpha_1} (1-z)^{b(Q-\alpha_2)} {}_2F_1 \left[ \begin{matrix} bP_t, b(P_t - 2P_3) \\ 1 - 2bP_2 \end{matrix}; 1-z \right].$$

And the two basis are related through an invertible matrix  $M$ ,  $I_i = M_{i,k} J_k$ , with:

$$M = \frac{\pi}{\sin(2\pi bP_2)} \begin{pmatrix} \frac{1}{\Gamma(\frac{1}{2}+\rho_{+--})\Gamma(\frac{1}{2}+\rho_{+-+})} & \frac{1}{\Gamma(\frac{1}{2}+\rho_{+++})\Gamma(\frac{1}{2}+\rho_{+-+})} \\ \frac{1}{\Gamma(\frac{1}{2}+\rho_{-+-})\Gamma(\frac{1}{2}+\rho_{--+})} & \frac{1}{\Gamma(\frac{1}{2}+\rho_{-++})\Gamma(\frac{1}{2}+\rho_{--+})} \end{pmatrix}. \quad (1.19)$$

$$\rho_{\epsilon_1, \epsilon_2, \epsilon_3} = b \sum_{i=1} (-1)^{\epsilon_i} P_i$$

Another way, more canonical, to obtain the hypergeometric solutions would be to use the null-vector equation of the operator  $\mathcal{O}_{2,1}$ :

$$(L_{-2} - \alpha L_{-1}^2) \mathcal{O}_{2,1} = 0.$$

Generically, all the Virasoro generators  $L_{-n}$ , applied to an operator in a correlation function, have a differential expression. We already saw the example of  $L_{-1}\mathcal{O}(z) = \partial_z \mathcal{O}(z)$ , but in general:

$$\begin{aligned} & \langle (L_{-n}\mathcal{O})(z) \mathcal{O}(z_2) \cdots \mathcal{O}(z_p) \rangle \\ &= \prod_i \left( \frac{h_i}{(z-z_i)^2} + \frac{\partial_z}{(z-z_i)} \right) \langle \mathcal{O}(z_1) \mathcal{O}(z_2) \cdots \mathcal{O}(z_p) \rangle. \end{aligned}$$

Hence, the relation  $\langle \mathcal{O}_{\alpha_3} | (L_{-2} - \alpha L_{-1}^2) \mathcal{O}_{2,1} \mathcal{O}_{\alpha_2}(1) | \mathcal{O}_{\alpha_1} \rangle = 0$ , will lead to a differential equation of order 2, which happens to be a hypergeometric equation.

*Some correlation functions do not have a trivial monodromy, for example in theories involving (para)fermions, or non-local operators in general.*

*Additionally, if non-scalar operators can appear in the theory, imposing the monodromy can be trickier [61], and see chapter 3 in this thesis.*

Identical arguments apply to the anti-holomorphic conformal blocks. Hence, the full correlation function (with the two chiralities) can be written:

$$\mathcal{G}(z) = \sum_{i,j} X_{i,j} I_i(z) \overline{I_j(z)} = \sum_{i,j} Y_{i,j} J_i(z) \overline{J_j(z)}.$$

At this point the two chiral sectors are still independent and up to 4 fields can appear in each fusions :  $(h_{\alpha+\epsilon_1 b/2}, h_{\alpha+\epsilon_2 b/2})$ ,  $\epsilon_1, \epsilon_2 \in \{-1, +1\}$ . Only two of them are scalar, the non-scalar operators will not appear if we impose that the correlation function has a trivial monodromy at every point:

$$G(z \circlearrowleft a) = G(z) \text{ for } a \in \{0, 1, \infty\}$$

This will be true for  $G$  if and only if  $X_{i,j}$  and  $Y_{i,j}$  are diagonal – In the generic case again.

$$X_{i,j} = \delta_{i,j} X_{i,i} \quad Y_{i,j} = \delta_{i,j} Y_{i,i}$$

$X_{i,i}$  and  $Y_{i,i}$  can be written in terms of the structure constants appearing in the bootstrap equation:

$$\begin{aligned} X_{1,1} &= C\left(\alpha_1 + \frac{b}{2}, \alpha_1, \frac{b}{2}\right) C\left(\alpha_1 + \frac{b}{2}, \alpha_2, \alpha_3\right), \\ X_{2,2} &= C\left(\alpha_1 - \frac{b}{2}, \alpha_1, \frac{b}{2}\right) C\left(\alpha_1 - \frac{b}{2}, \alpha_2, \alpha_3\right). \end{aligned}$$

Using the change of basis 1.19,

$$Y_{k,l} = 0 \quad \forall k \neq l \Rightarrow \sum_i X_{i,i} M_{i,k} \overline{M}_{i,l} = 0 \quad \forall k \neq l$$

this reduces to:

$$\frac{X_{11}}{X_{22}} = \frac{M_{2,1} \overline{M}_{2,2}}{M_{1,1} \overline{M}_{1,2}} = \frac{C(\alpha_1 + \frac{b}{2}, \alpha_1, \frac{b}{2}) C(\alpha_1 + \frac{b}{2}, \alpha_2, \alpha_3)}{C(\alpha_1 - \frac{b}{2}, \alpha_1, \frac{b}{2}) C(\alpha_1 - \frac{b}{2}, \alpha_2, \alpha_3)}.$$

These relations are valid for every  $\alpha_1, \alpha_2, \alpha_3$ , and give recursion relations for all the structure constants.

The operator  $\mathcal{O}_{1,2}$  and  $\mathcal{O}_{2,1}$  can be exchanged under the symmetry  $b \rightarrow -b^{-1}$ . If both operators are present in the theory, another recursion relation, involving  $-b^{-1}$  rather than  $b$  can be written through the same method:

$$\frac{C\left(\alpha_1 - \frac{b^{-1}}{2}, \alpha_2, \alpha_3\right) C\left(\alpha_1 - \frac{b^{-1}}{2}, \alpha_1, -\frac{1}{2b}\right)}{C\left(\alpha_1 + \frac{b^{-1}}{2}, \alpha_2, \alpha_3\right) C\left(\alpha_1 + \frac{b^{-1}}{2}, \alpha_1, -\frac{1}{2b}\right)} = \frac{N_{2,1} \overline{N}_{2,2}}{N_{1,1} \overline{N}_{1,2}}.$$

If  $b^2$  is not a rational number, these two relations entirely fix the three-point correlation functions ( $b\mathbb{Z} + b^{-1}\mathbb{Z}$  is dense in  $\mathbb{R}$ ). The solution admits an analytic form:

$$C(\alpha_1, \alpha_2, \alpha_3) = \frac{Y_b(-P_1 - P_2 - P_3) \prod_{\{i,j,k\} \in \text{Cycl}_3} Y_b(P_i - P_j - P_k)}{\sqrt{\prod_{i=1}^3 Y_b(b + 2P_i) Y_b(b^{-1} + 2P_i)}}, \quad (1.20)$$

where the product in the numerator is over the 3 cyclic permutations of  $\{1, 2, 3\}$ .

This is the Dorn-Otto-Zamolodchikov-Zamolodchikov (DOZZ) formula initially found by [53, 153]. The development we followed is closer to the one of Tschner [139]. The normalization is fixed by imposing  $C(\alpha, \alpha, 0) = 1$ .

*Initially, the result was found for  $c > 1$ . The extension to  $c \leq 1$  is not just an analytic continuation and took some time to be understood [132, 156], numerically and conceptually.*

The function  $Y_b$  is defined on the real interval  $]0, b + b^{-1}[$  by:

$$Y_b \left( \frac{b + b^{-1}}{2} - z \right) = \exp \left( \int \frac{dt}{t} \left[ z^2 e^{-t} - \frac{\sinh \left( \frac{zt}{2} \right)^2}{\sinh \left( \frac{bt}{2} \right) \sinh \left( \frac{t}{2b} \right)} \right] \right).$$

The function can be extended outside of this interval by the relations (valid on the full definition space  $\mathbb{C}$ ):

$$\begin{aligned} Y_b(z + b) &= \gamma(bz) b^{1-2bz} Y_b(z), \\ Y_b(z + b^{-1}) &= \gamma(b^{-1}z) b^{-1+2zb^{-1}} Y_b(z), \\ Y_b(b + b^{-1} - z) &= Y_b(z). \end{aligned}$$

where  $\gamma(z) = \frac{\Gamma(z)}{\Gamma(1-z)}$  with  $\Gamma$  the Euler gamma function.

$Y_b(\frac{b+b^{-1}}{2}) = 1$  and  $Y_b$  is entire analytic everywhere, with 0 at:

$$\begin{aligned} z &= -kb^{-1} - lb \\ z &= (k+1)b^{-1} + (l+1)b \end{aligned} \quad k, l \in \mathbb{N}.$$

This expression is valid for every scalar theory without degeneracies, but its interpretation can depend upon the exact spectrum of primaries considered. For operators in the Kac table, appearing for example in generalized minimal models, the expression 1.20 needs to be regularized. The bootstrap method can also be applied to non-scalar fields, with additional complications (as will be shown in the third chapter). Remarkably, this expression can also be used to fully define a theory with a continuous spectrum  $\{\mathcal{O}_\alpha\}_{\alpha \in \mathbb{R}}$  and the structure constants 1.20: the *Liouville conformal field theory*. The consistency of such a theory is hard to prove, short of obtaining an analytic expression for every four-points functions. Numerical tests have so far not found any inconsistency [130].

*A theory is consistent if every four-points functions verify the bootstrap equations*

We avoided until now the traditional representation of a QFT, with fields and Lagrangians, and the Liouville CFT is entirely defined by its spectrum and structure constants. Still, most CFT can have a Lagrangian formulation, and Liouville is not an exception:

$$\mathcal{A} = \int d^2x \frac{\sqrt{g}}{4\pi} \left[ g^{a,b} \partial_a \varphi \partial_b \varphi + iQR\varphi + 4\pi\mu e^{-2ib\varphi} \right].$$

*Taking the operator  $\mathcal{O}_{\alpha_1=0}$ , one can notice that  $C(0, \alpha_2, \alpha_3) \neq 0$  even if  $\alpha_2 \neq \alpha_3$ . This implies that  $\mathbb{1}$  is not the only operator of conformal dimension 0. The theory is not completely free of degeneracy, in fact every dimension belonging in the Kac table is represented twice.*

$R$  is the curvature associated with the metric  $g_{\mu\nu}$  and the following normalization for the bosonic field  $\varphi$  will be used:

$$\langle \varphi(x)\varphi(y) \rangle = -\log|x-y|.$$

The equation of motion is given by the Liouville equation (from which the theory takes its name):

$$\Delta(i\varphi) = \mu b e^{-2ib\varphi},$$

while the stress-energy tensor takes the form:

$$T(z) = - : (\partial\varphi)^2 : + iQ\partial^2\varphi.$$

In this setting the primary operators can be expressed as *Vertex operators*, exponential of the bosonic field. If an operator is designated by its conformal charge  $\alpha$ , the associated vertex operator is:

$$\mathcal{O}_\alpha \equiv \mathcal{V}_\alpha(x) = e^{i\alpha\varphi(x)}, \text{ with } \Delta_\alpha = \alpha \cdot (\alpha - Q).$$

## 1.3 CONFORMAL FIELD THEORY (EXTENDED)

1.3.1 *Extended symmetries*

In addition to conformal invariance, some field theories present a symmetry under a larger group of transformations. In a field theory these underlying symmetries can be obvious, such as the  $\mathbb{Z}_3$  symmetry of the three-Potts model or not so obvious in the case of the super-symmetry of the Ising model at its tricritical point.

The description of conformal field theories made in the previous sections still apply to extended CFTs. However, in the spectrum, the presence of an additional symmetry leads to multiple degeneracies. In the case of the Pott's model with three states (the minimal model  $\mathcal{M}(6,5)$ ) for example; one of the primary state is of dimension 3. The solution often consists in classifying operators not only in terms of their dimensions, but also in term of their behaviour under the new symmetries. These new internal symmetries are also responsible for the apparition of additional currents in the theory, which themselves lead to new conserved charges. The modes generated by the current will, together with the modes  $L_n$ , yields a new, "extended", algebra, containing Virasoro. The notion of primary field can then be refined: only few primary fields are still primary under the large symmetry, the rest will be re-expressed as descendants in the extended algebra.

For example, the spectrum of the three-Pott's model, as a Virasoro representation, contains the operators  $\{\mathbb{1}, \epsilon, X, Y, \sigma, Z\}$  of dimensions  $\{0, 2/5, 7/5, 3, 1/15, 2/3\}$ . The fields  $(\mathbb{1}, Y)$ , and  $(\epsilon, X)$  have integer differences between their dimensions, which betrays the presence of an additional symmetry: the three-Pott's model has a  $\mathcal{W}_3$  symmetry. The conserved current is the operator  $Y \equiv W(z)$ . The extended primaries are  $\{\mathbb{1}, \epsilon, \sigma, Z\}$ , and the two Virasoro primaries  $X$  and  $Y$  can be interpreted as extended descendants of, respectively,  $\epsilon$  (at order 1) and  $\mathbb{1}$  (at order 3).

Restricting the spectrum of the theory is not just interesting for computational purposes. Minimal models are one of the major accomplishments of conformal field theory, they permit to restrict the space of all possible unitary models with  $c \leq 1$ . However, this classification fails for  $c$  greater than 1, where every model (with no primary of negative dimension) is unitary. Additional symmetries allow to go beyond the  $c = 1$  frontier and introduce a partial classification in the space of possible theories for larger  $c$ .

A standard method to build extended CFTs consists in including in the spectrum an additional operator of integer (or half-integer) conformal dimension. Common examples include:

- $W_n$  algebras with an additional current of dimension  $n$  [27, 66, 67].

*One of the generalizations of the Ising model includes the possibility of introducing vacancies in the lattice. The model then depends both on the temperature and on the chemical potential. At a unique point of its phase diagram, three phases met. This point is the tricritical Ising model.*

- WZW models are associated with a specific Lie algebra  $\mathfrak{g}$ . They have  $\dim(\mathfrak{g})$  additional one-dimensional currents which behave like the basis of  $\mathfrak{g}$  [144, 149].
- Orbifolds with additional currents of dimension 2 [52].
- Super-conformal theories, such as the tricritical Ising model, contain a current of half-integer dimension [20].

### 1.3.2 The $\mathbb{Z}_2$ orbifold, algebraic perspective

One of the simplest non-trivial example can be found by adding a unique primary operator of chiral dimensions  $(2, 0)$ , temporarily nicknamed  $\tilde{T}(z)$ . We will only work in the holomorphic sector, but the same reasoning can be applied in both sectors. Because  $\tilde{T}$  is primary, its OPE with the stress-energy tensor  $T$  should verify:

$$T(z)\tilde{T}(0) = \frac{2\tilde{T}}{z^2} + \frac{\partial\tilde{T}(0)}{z} + \text{regular terms}.$$

Additionally, the fusion between two operators  $\tilde{T}$  will only create the identity and its descendants,  $\tilde{T} \otimes \tilde{T} = \mathbb{1}$ .

To see this, consider the following four-points function:

$$\mathcal{G} : z \rightarrow \langle T|\tilde{T}(z)\tilde{T}(1)|T \rangle.$$

Its conformal blocks in the limits  $z \rightarrow 0$  and  $z \rightarrow \infty$  are constrained by the OPE  $T \otimes \tilde{T}$ , and should verify:

- $z \rightarrow 0, \mathcal{G}(z) \underset{z \rightarrow 0}{\propto} \frac{1}{z^2},$
- $z \rightarrow \infty, \mathcal{G}(z) \underset{z \rightarrow 0}{\propto} \frac{1}{z^2}.$

Hence, in the limit  $z \rightarrow 1, \mathcal{G}(z) \propto 1$ , which implies  $\tilde{T} \otimes \tilde{T} = \mathbb{1}$ . This is a consequence of a more general result: operators with integer dimensions form a closed sub-algebra under fusion.

The standard normalization for the operator  $\tilde{T}$  is not the most appropriate in this context, and the convention  $\langle \tilde{T}(z)\tilde{T}(0) \rangle = \frac{c/2}{z^4}$  will be preferred.

This gives the following OPE for the operator  $\tilde{T}$  with himself:

$$\begin{aligned} \tilde{T}(z) \cdot \tilde{T}(0) &= \frac{c}{2z^4} + \frac{C(\tilde{T}\tilde{T}L_{-2}\mathbb{1})}{\langle L_{-2}\mathbb{1}|L_{-2}\mathbb{1} \rangle} \frac{T(0)}{z^2} + \frac{C(\tilde{T}\tilde{T}L_{-3}\mathbb{1})}{\langle L_{-3}\mathbb{1}|L_{-3}\mathbb{1} \rangle} \frac{\partial T(0)}{z} + \dots \\ &= \frac{c}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \text{regular terms}. \end{aligned}$$

The operators  $T^{(0)} = \frac{1}{2}(T + \tilde{T})$  and  $T^{(1)} = \frac{1}{2}(T - \tilde{T})$  satisfy the OPEs:

$$T^{(i)}(z)T^{(j)}(0) = \frac{c\delta_{i,j}}{4} \frac{\delta_{i,j}}{z^4} + \frac{2T^{(i)}}{z^2} + \frac{\partial T^{(i)}}{z} + \text{regular terms} \quad \forall i, j \in \{0, 1\}.$$

Hence, both  $T^{(0)}$  and  $T^{(1)}$  behave like stress-energy tensors in two decoupled copies of a theory with central charge  $c/2$ . In this setup,  $T$  and  $\tilde{T}$  become the symmetric and anti-symmetric modes associated with the operators  $T^{(i)}$ . In accordance, we will modify our conventions and rename  $T = T^{(0)} + T^{(1)} \rightarrow \hat{T}^{(0)}$  and  $\tilde{T} = T^{(0)} - T^{(1)} \rightarrow \hat{T}^{(1)}$ .

Crucially, the OPE of  $\tilde{T} = \hat{T}^{(1)}$  with a generic primary operator  $\mathcal{O}$  is also constrained. The four-point correlation function  $\langle \mathcal{O} | \hat{T}^{(1)}(1) \hat{T}^{(1)}(z) | \mathcal{O} \rangle$  only have one canal in  $z \rightarrow 1$ . This means that only one operator appears in the two other possible fusions, corresponding to the limits  $z \rightarrow 0$  and  $z \rightarrow \infty$ .

$$\hat{T}^{(1)}(z)\mathcal{O}(0) = \frac{1}{z^{h_{\mathcal{O}_p} - h_{\mathcal{O}} - 2}} \mathcal{O}_p(0).$$

With:

$$h_{\mathcal{O}_p} - h_{\mathcal{O}} \in \frac{1}{2}\mathbb{Z}.$$

The function  $f(z) = \langle \mathcal{O}(z_1) \hat{T}^{(1)}(z) \hat{T}^{(1)}(0) \mathcal{O}(z_2) \rangle$  is holomorphic everywhere except in  $0, z_1$  and  $z_2$ . Near  $z = z_i$ , the fusion rules impose the form  $f(z) = (z - z_i)^{h_{\mathcal{O}_p} - h_{\mathcal{O}} - 2} g_i(z)$  and  $f(z) = z^{-4} g_0(z)$  near  $z = 0$ , where  $g_a, a \in \{0, 1, 2\}$  are holomorphic functions. The function  $\frac{\partial_z f}{f} = \partial_z (\log(z))$  can only have integer residues outside of the three points  $0, z_1$  and  $z_2$ . Hence, the sum of the residues in these three points should be an integer:

$$-4 + 2(h_{\mathcal{O}_p} - h_{\mathcal{O}} - 2) \in \mathbb{Z} \quad \rightarrow \quad h_{\mathcal{O}_p} - h_{\mathcal{O}} \in \frac{1}{2}\mathbb{Z}$$

We can then separate the space of primary states between:

- *Untwisted* states, for which the states appearing in the fusion  $T^{(1)} \otimes \mathcal{O}$  have the same dimension as  $\mathcal{O}$  up to an integer,
- *Twisted* states, where the same fusion leads to states with dimension  $h_{\mathcal{O}} + n + 1/2$  with  $n \in \mathbb{Z}$ .

For the two operators  $T^{(0)}$  and  $T^{(1)}$ , untwisted states behave like primary states. Hence, if only the untwisted sector is taken into account, the CFT we are considering is the tensor product of two CFTs of central charges  $c/2$  each. However a different modular invariant theory can be built using the full spectrum – with both twisted and untwisted states – is considered, this CFT is called a  $\mathbb{Z}_2$  orbifold conformal field theory.

Both  $\hat{T}^{(0)}$  and  $\hat{T}^{(1)}$  admit a mode decomposition,

$$\hat{T}^{(0)}(z) = \sum_n \hat{L}_n^{(0)} z^{-n-2} \quad \hat{T}^{(1)}(z) = \sum_{n \in \mathbb{Z}/2} \hat{L}_n^{(1)} z^{-n-2}.$$

For  $\hat{T}^{(1)}$ , twisted states impose the presence of half-integer modes. It is easy to check that the commutation relations of the different modes are similar to the Virasoro ones:

$$\begin{aligned} \left[ \hat{L}_n^{(i)}, \hat{L}_m^{(j)} \right] &= (n - m) \hat{L}_{n+m}^{(i+j)} + \frac{c}{12} n(n+1)(n-1) \delta_{n,-m} \delta_{i,j} \mathbb{1} \\ n, m &\in \frac{1}{2}\mathbb{Z} \quad i, j \in \mathbb{Z}_2. \end{aligned} \quad (1.21)$$

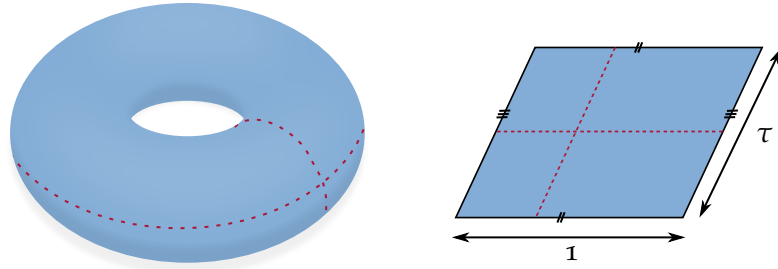


Figure 1.3: Two representations of the torus, embedded in a three-dimensional space and as a quotient of the plane by the lattice  $\mathbb{Z} + \tau\mathbb{Z}$ .  $\tau$  is the ratio of the two period of the torus. The Gauss-Bonnet theorem can be used to prove that the genus of the torus is 1.

### 1.3.3 The $\mathbb{Z}_2$ orbifold, geometrical perspective

While this construction gives an idea of the different types of restrictions applied on an extended CFT, it misses the remarkable interpretation of orbifold CFTs. Indeed, these CFTs have a deeply geometric construction. This is far from being the case for all extended CFTs, and, for most the only possible construction is a variation on the method exposed previously (the  $\mathcal{W}_3$  CFT, studied in chapter 3, gives a good example of a CFT whose extended symmetry does not have any obvious interpretation).

We will present this construction on a torus. Contrary to the surface of genus zero, all tori are not conformally equivalent. The set of conformally distinct tori are labelled by a complex parameter called  $\tau$ , which is the ratio of the two periods in the flat representation of the torus (see figure 1.3). Hence, on the torus, the partition (or generating) function  $Z$  depends non-trivially on  $\tau$ , and becomes an interesting quantity to study. It is invariant under the modular transformations on the torus, a group in bijection with  $PSL(2, \mathbb{Z})$ , which consists in the functions:

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1 \quad a, b, c, d \in \mathbb{Z}. \quad (1.22)$$

In particular, it contains the two transformations  $\mathcal{T}$  and  $\mathcal{S}$ .

$$\mathcal{T} : \tau \rightarrow \tau + 1 \quad \text{and} \quad \mathcal{S} : \tau \rightarrow \frac{-1}{\tau}$$

which together, generate the full group.

Now, consider a CFT of central charge  $c_M$ , where  $M$  stands for *Mother theory*. To create the orbifold theory we will associate two completely identical copies of this CFT, labelled 0 and 1. On the torus, they can be coupled through the periodic conditions, and the full partition

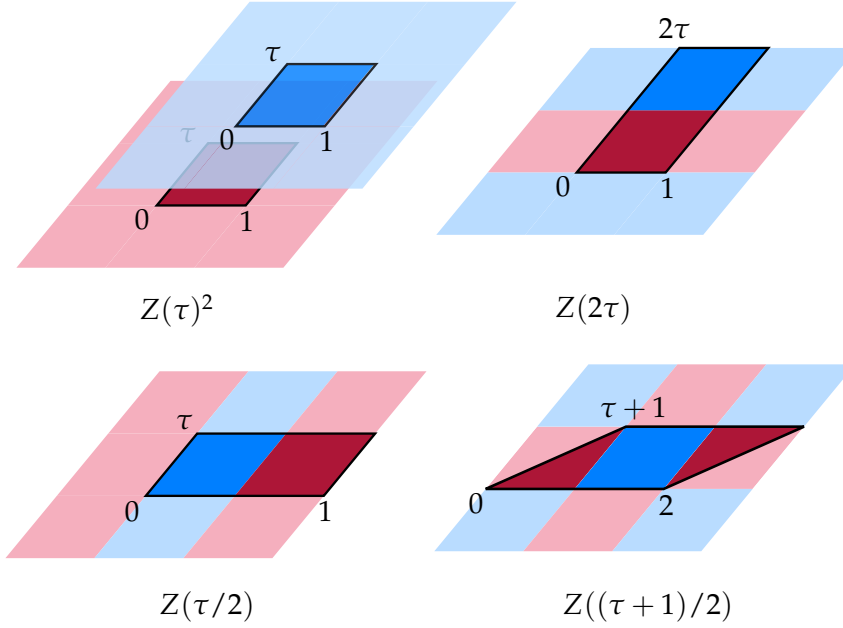


Figure 1.4: The four possible periodic conditions on the torus for the orbifold conformal field theory. The two colors represent the two independent CFTs. The parts surrounded with black correspond to the primitive cells of each sector. Except for the top-left case, twisted boundary conditions in one or more of the torus directions will couple both sectors together, in this case the new partition functions are equal to the original one, but on a larger torus.

function is separated in four sectors  $Z_{m,n}$ ,  $m, n \in \{0, 1\}$  depending on its behaviour under a period shift (see figure 1.4).  $Z_{0,0}$  has periodic boundary conditions in both direction while  $Z_{0,1}$ ,  $Z_{1,0}$  and  $Z_{1,1}$  have “twisted” boundary conditions – which exchanges copies – respectively vertically, horizontally and in both directions.

$$Z_{m,n}(\tau + 1) = Z_{m+n,n}(\tau) \quad Z_{m,n}(-\tau^{-1}) = Z_{n,m}(\tau) \quad n, m \in \mathbb{Z}_2$$

$$Z_{\text{orb}} = \frac{1}{2} \sum_{m,n=0}^1 Z_{m,n}.$$

On one hand, in the partition function  $Z_{0,0}$ , the two CFTs are completely disconnected. Hence, because the two CFTs are identical:

$$Z_{0,0}(\tau) = Z(\tau)^2,$$

where  $Z$  is the torus partition function of the mother theory.

On the other hand, it is easy to see on figure 1.4 that the coupled sectors will have the expression:

$$Z_{1,0}(\tau) = Z(2\tau) \quad Z_{0,1}(\tau) = Z(\tau/2) \quad Z_{1,1}(\tau) = Z((\tau+1)/2).$$



The full partition function is therefore given by:

$$\begin{aligned} Z_{\mathcal{O}rb}(\tau) &= \frac{1}{2} (Z_{0,0}(\tau) + Z_{0,1}(\tau) + Z_{1,0}(\tau) + Z_{1,1}(\tau)) \\ &= \frac{1}{2} \left( Z(\tau)^2 + Z(2\tau) + Z\left(\frac{\tau}{2}\right) + Z\left(\frac{\tau+1}{2}\right) \right). \end{aligned}$$

The central charge of the orbifold theory will be twice the one of the mother theory,  $c = 2c_M$ .

The partition function of a CFT is often said to “contain the full spectrum” of the theory. More specifically, for a theory which only contains scalar operators, the following is true:

$$Z(\tau) = \sum_{\mathcal{O} \in \text{primaries}} |\chi_{\mathcal{O}}(\tau)|^2.$$

Where  $\chi_{\mathcal{O}}$  are the *characters* of the theory. They are defined as the trace over the Virasoro representation associated with  $\mathcal{O}$ .

$$\chi_{\mathcal{O}}(\tau) = \text{Tr}_{\mathcal{O} \text{ module}} \left[ e^{2i\pi\tau(L_0 - c/24)} \right].$$

The characters can be viewed as the conformal blocks of the zero-point correlation function on the torus. They transform diagonally under the action of the operator  $\mathcal{T}$ :

$$\chi_{\mathcal{O}}(\tau + 1) = e^{2i\pi\tau(h_{\mathcal{O}} - c/24)} \chi_{\mathcal{O}}(\tau).$$

The partition function itself associates the two sectors (holomorphic and anti-holomorphic) of the theory to create an object invariant under modular transformations. When  $\tau \rightarrow i\infty$ , the characters take the form:

$$\chi_{\mathcal{O}}(\tau) \underset{\tau \rightarrow i\infty}{\approx} \exp(2i\pi\tau(h_{\mathcal{O}} - c/24)).$$

Assuming that the spectrum of the mother CFT is known and non-degenerate, we find back the dichotomy between untwisted and twisted states:

$$Z_{\mathcal{O}rb}(\tau) = \frac{1}{2} \left[ \underbrace{Z(\tau)^2 + Z(2\tau)}_{\text{untwisted}} + \underbrace{Z\left(\frac{\tau}{2}\right) + Z\left(\frac{\tau+1}{2}\right)}_{\text{twisted}} \right]. \quad (1.23)$$

If we decompose the partition function  $Z$  in characters, we get:

$$\begin{aligned} &\sum_{i < j} |\chi_i(\tau)\chi_j(\tau)|^2 + \frac{1}{2} \sum_i (|\chi_i(\tau)|^4 + |\chi_i(2\tau)|^2) \\ &+ \frac{1}{2} \sum_i (|\chi_i(\tau/2)|^2 + |\chi_i((\tau+1)/2)|^2). \end{aligned}$$

This expression allows us to find the characters of the orbifold CFT (and hence its complete spectrum).

*In general, the action of  $\mathcal{S}$  mixes the character. Knowing how the character transform under both  $\mathcal{T}$  and  $\mathcal{S}$  entirely fixes the theory.*

More precisely, the object we consider are the highest-weight representations of the algebra  $\mathcal{A}$ , generated by the operator  $\Pi$  – which exchanges the two copies – and the “neutral” combinations of descendant operators  $\hat{L}_{m_1}^{(r_1)} \dots \hat{L}_{m_p}^{(r_p)}$  (all invariant under  $\Pi$ ). The extended algebra of the orbifold contains the Virasoro algebra but is strictly smaller than the simple product of two Virasoro algebras.

To extract the characters we can use the fact that they diagonalize the operator  $\mathcal{T}$ .

- The first terms,  $|\chi_i(\tau)\chi_j(\tau)|^2$ ,  $i \neq j$ , lead to operators of dimensions  $h_{\phi_i} + h_{\phi_j}$ . The states associated are simply the tensor product of the two state  $|\phi_i\rangle$  and  $|\phi_j\rangle$  from the mother theory:  $\phi_i \otimes \phi_j$ . They are part of the untwisted sector.
- The next term can be decomposed as:

$$\begin{aligned} & \frac{1}{2} |\chi_i(\tau)|^4 + |\chi_i(2\tau)|^2 \\ &= \left| \frac{1}{2} \chi_i(\tau)^2 + \frac{1}{2} \chi_i(2\tau) \right|^2 + \left| \frac{1}{2} \chi_i(\tau)^2 - \frac{1}{2} \chi_i(2\tau) \right|^2 \\ &= \left| \chi_{(\phi_i \otimes \phi_i)^{(0)}} \right|^2 + \left| \chi_{(\phi_i \otimes \phi_i)^{(1)}} \right|^2. \end{aligned}$$

The two states  $(\phi_i \otimes \phi_i)^{(0)}$  and  $(\phi_i \otimes \phi_i)^{(1)}$  are respectively of dimension  $2h_{\phi_i}$  and  $2h_{\phi_i} + 1$ . They correspond to the splitting between anti-symmetric and symmetric representation of the tensor product  $\phi_i \otimes \phi_i$ . Both of them are also part of the untwisted sector.

- The two last terms form the twisted sector. Under the operator  $\mathcal{T}$ , the two characters  $\chi_i(\tau/2)$  and  $\chi_i((\tau+1)/2)$  transform as:

$$\begin{pmatrix} \chi_i\left(\frac{\tau+1}{2}\right) \\ \chi_i\left(\frac{\tau}{2}\right) \end{pmatrix} \rightarrow \begin{pmatrix} \chi_i\left(\frac{\tau+1}{2}\right) \\ e^{2i\pi\tau(h_{\phi_i}-c_M/24)} \chi_i\left(\frac{\tau}{2}\right) \end{pmatrix}$$

Hence, the decomposition which is diagonal under  $\mathcal{T}$  will be written:

$$\begin{aligned} & \frac{1}{2} |\chi_i(\tau/2)|^2 + |\chi_i((\tau+1)/2)|^2 \\ &= \left| \frac{1}{2} \chi_i(\tau/2) + e^{-i\pi(h_{\phi_i}-c_M/24)} \chi_i\left(\frac{\tau+1}{2}\right) \right|^2 \\ &+ \left| \frac{1}{2} \chi_i(\tau/2) - e^{-i\pi(h_{\phi_i}-c_M/24)} \chi_i\left(\frac{\tau+1}{2}\right) \right|^2 \\ &= \left| \chi_{\tau_{\phi_i}^{(0)}} \right|^2 + \left| \chi_{\tau_{\phi_i}^{(1)}} \right|^2. \end{aligned}$$

In the limit  $\tau \rightarrow i\infty$ , these two terms have the form:

$$\begin{aligned} \chi_{\tau_{\phi_i}^{(0)}} \Big|_{i\tau \rightarrow \infty} &\approx e^{i\pi\tau(h_i-c_M/24)} = e^{2i\pi\tau(h_i/2+c_M/16-c/24)}, \\ \chi_{\tau_{\phi_i}^{(1)}} \Big|_{i\tau \rightarrow \infty} &\approx e^{i\pi\tau(h_i-c_M/24+1)} = e^{2i\pi\tau(h_i/2+c_M/16+1/2-c/24)}. \end{aligned}$$

*The two representations (symmetric and anti-symmetric) generated by  $\phi_i \otimes \phi_j$  with  $i \neq j$  are isomorphic. In the orbifold theory we only conserve one copy.*

Again two types of fields appear,  $\tau_{\phi_i} \equiv \tau_{\phi_i}^{(0)}$ , is of dimension  $h_{\phi_i}/2 + c_M/16$ , while  $\tau_{\phi_i}^{(1)}$  is of dimension  $h_{\phi_i}/2 + 1/2 + c_M/16$ .

*The operators  $\tau_{\mathbb{1}}$  are particularly interesting, as they can be interpreted as “plane-switching” operators. Starting from the plane, they can be used to build up more complicated surfaces (see chapter 4 for details).*

In every extended CFTs, extended rational models – containing only a finite number of operators *under the extended algebra* and degeneracy in descendants – can be constructed, like in the standard Virasoro setup. These rational extended CFTs can sometimes even coincide with (Virasoro) minimal models. For example the orbifold formed by the Yang-Lee model will have operators with dimensions  $\{0, -1/5, -2/5, -11/40, -3/8\}$  and is included in the minimal model  $\mathcal{M}_{3,10}$ .

Extended symmetries also permit to define extended conformal blocks, which sum over the descendants under the extended symmetry. In theory, they could be used to apply a bootstrap process to extended CFTs. However, in contrast with the standard Virasoro case, these conformal blocks are not, in general, computable through the algebra alone. Some three-point functions of extended descendants for example, can only be determined through bootstrap.

#### 1.4 CRITICAL LATTICE MODELS

Let’s put continuum field theories aside for a moment. Most two-dimensional statistical models are defined on a discrete lattice. Critical systems were first studied through these discrete models, and they never got out of fashion. The exact nature of the link between discrete critical systems and conformal theories has been explored since the 80s, starting with the seminal paper by Cardy [37] and followed by, for example and without a specific order, the link between the Restricted Solid On Solid (RSOS) model and minimal models [116], the description of quantum group symmetries [55], the link between percolation and boundaries CFT [38]. One of the most overlooked contribution of lattice model to the study of critical system is the access they give to numerical computations and verifications.

##### 1.4.1 Discrete statistical model

The Ising model is one of the simplest, non-trivial, two-dimensional system, and it will provide a common thread for this section. Let’s consider the Ising model on a square lattice, with  $M$  rows and  $N$  columns. The spins  $\{s_{i,j}\}_{i,j}$  at each vertex of the lattice can take the values  $\pm 1$ . They satisfy, in this section at least, periodic boundary conditions horizontally:

$$s_{i,j+N} = s_{i,j}.$$

The energy of a given spin configuration  $\{s_{i,j}\}_{i,j}$  takes the usual form:

$$E(\{s_{i,j}\}_{i,j}) = -J \sum_{i,j} (s_{i,j}s_{i+1,j} + s_{i,j}s_{i,j+1}) .$$

The partition function can be written:

$$Z = \sum_{\{s_{i,j}=\pm 1\}_{i,j}} e^{-\beta E(\{s_{i,j}\}_{i,j})} . \quad (1.24)$$

Calling  $\mu_a = \{s_{i,a}\}$  the set of spins belonging to the row  $a$ , the energy can be decomposed into the energy between rows and the internal energy of each row:

$$E(\mu_a) = -J \sum_{k=1}^N s_{k,a} s_{k+1,a} ,$$

$$E(\mu_a, \mu_{a+1}) = -J \sum_{k=1}^N s_{k,a} s_{k,a+1} .$$

And the total energy of the system is written:

$$E(\{\mu_a\}_a) = \sum_{a=1}^N E(\mu_a) + E(\mu_a, \mu_{a+1}) .$$

The transfer matrix method consists in re-expressing the partition function 1.24 in term of a *transfer matrix*,  $T$ , which act on rows of spins. In the case of the Ising model it is a  $2^N \times 2^N$  square matrix defined by:

$$\langle \mu_a | T | \mu_{a+1} \rangle = \exp(-\beta [(E(\mu_a) + E(\mu_{a+1})) / 2 + E(\mu_a, \mu_{a+1})]) .$$

In this formalism, the Ising model partition function can be rewritten:

$$\begin{aligned} Z &= \sum_{\mu_1} \cdots \sum_{\mu_M} \exp(-\beta E(\{\mu_a\}_a)) \\ &= \sum_{\mu_1} \cdots \sum_{\mu_M} \prod_a \exp(-\beta (E(\mu_a) + E(\mu_a, \mu_{a+1}))) \\ &= \sum_{\mu_1} \cdots \sum_{\mu_M} \prod_a \langle \mu_a | T | \mu_{a+1} \rangle \\ &= \langle \psi_{\text{down}} | T^M | \psi_{\text{up}} \rangle . \end{aligned} \quad (1.25)$$

With  $|\psi_{\text{down}}\rangle$  and  $|\psi_{\text{up}}\rangle$  two row vectors specifying the boundary conditions at the two vertical ends of the cylinder. If  $\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_M$  are the eigenvalue of  $T$  and  $|\Lambda_i\rangle$  the associated eigenvectors, the partition function takes the form:

$$Z = \sum_i \Lambda_i^M \langle \psi_{\text{down}} | \Lambda_i \rangle \langle \Lambda_i | \psi_{\text{up}} \rangle .$$

*As is typically the case with unitary models, the Ising transfer matrix  $T$  can be chosen positive symmetric. This ensures a real spectrum and a unique ground state.*

In the infinite cylinder limit  $M \rightarrow \infty$ ,  $Z \approx \Lambda_1^M \langle \psi_{\text{down}} | \Lambda_1 \rangle \langle \Lambda_1 | \psi_{\text{up}} \rangle$  (if the boundary conditions are generic) and the free-energy per unit surface  $f = -\frac{1}{\beta MN} \log(Z)$  will depend only on the largest eigenvalue:

$$f \approx_{M \rightarrow \infty} \frac{1}{N\beta} \Lambda_1.$$

The vector space associated with the matrix  $T$  is a  $2^N$ -dimensional space, consisting of  $N$  copies of a two-dimensional space  $V$ :

$$\begin{aligned} \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_y &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

$$\underbrace{V \otimes \cdots \otimes V}_{N \text{ times}}.$$

It can be re-expressed in a simpler form, by introducing the Pauli matrices  $\sigma_x, \sigma_y$  and  $\sigma_z$ :

$$T = \prod_k e^{\beta J \sigma_{z,a} \sigma_{z,a+1}} e^{\beta J \sigma_{x,a}}.$$

Where  $\sigma_{u,a}$  represent the  $\sigma_u$  matrix acting on site  $a$ , leaving the rest of the row unchanged:

$$\sigma_{u,a} = \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma_u \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{N \text{ times}}.$$

If the constant  $\beta J$  is taken to be imaginary, the transfer matrix  $T$  can also be understood as the time-evolution operator of a quantum system, consisting of two-by-two interacting spin  $1/2$  particles, with a time-step  $\Delta\tau = -i\beta J$ . Within this framework, the cylinder limit of the partition function  $\langle \psi_{\text{down}} | T^M | \psi_{\text{up}} \rangle$  gives the amplitude of obtaining a state  $|\psi_{\text{down}}\rangle$  after an evolution during a time  $M\Delta\tau$  with an initial state  $|\psi_{\text{up}}\rangle$ .

This quantum system will have for Hamiltonian the operator  $H$ , such that  $T = \exp(\Delta\tau H)$ . If the time-steps  $\Delta\tau$  are infinitesimally small,  $T \underset{\Delta\tau \rightarrow 0}{\approx} \mathbb{1} + \Delta\tau H$  (the limit  $\Delta\tau \rightarrow 0$  is the Hamiltonian limit). In this case, the Hamiltonian takes the form:

$$H \propto \sum_a [\sigma_{x,a} + \sigma_{z,a} \cdot \sigma_{z+1,a}].$$

This is a first example of a quantum chain Hamiltonian, the quantum Ising model. The quantum Ising model itself is a special case of the XY spin chain. But other examples exist, including the Heisenberg model (XXX, XXZ or XYZ chain) that we already mentioned, models with higher order symmetry (the  $\mathfrak{sl}_3$  spin chain will be considered again in the next chapter) or models with higher spins.

#### 1.4.2 The Temperley-Lieb loop model

Going back to statistical physics, a common description of the Ising model is the “low-temperature expansion”.

*A classic example of a Wick transformation.*

1D quantum system	2D statistical system
Generating function	Partition function
Time evolution operator	Transfer matrix
Classical Hamiltonian $H$	Classical action $\mathcal{A}$
Ground state	Equilibrium state
Propagators	Correlation functions
inverse correlation length	energy gaps
Time	Complex temperature
Quantum Ising model	Ising model
Heisenberg model	8-vertex model
$SU(3)$ spin chain	15-vertex models

Table 1.1: Correspondence between one-dimensional quantum systems and two-dimensional statistical systems.

$$\begin{aligned}
 Z &= \sum_{s_x} \prod_{|x-y|=1} e^{\beta J s_x s_y} \\
 &= e^{-2\beta J MN} \sum_{s_x} \prod_{|x-y|=1} \left( 1 + 2 \sinh(\beta J) e^{\beta J} \delta(s_x, s_y) \right),
 \end{aligned}$$

where  $\delta$  is the Kronecker delta. For this section, the periodicity at the edge of the lattice is removed and the model is defined on a finite rectangular lattice, of size  $M \times N$  with open boundary conditions.

The product can be expanded, leaving a sum over  $2^{|E|}$  terms where  $|E|$  is the number of edges, each term corresponding to a subset  $F$  of all the edges  $E$ . Using the notation  $v = 2 \sinh(\beta J) e^{\beta J}$ , the sum becomes:

$$Z = e^{-\beta J |E|} \sum_{F \subset E} v^{|F|} 2^{k(F)},$$

$k(F)$  is the number of connected components in the graph  $(V, F)$ , where  $V$  is the set of all vertices.

This graph-theory inspired form can be generalized to arbitrary graphs and has an interesting consequence for planar graphs. Consider the planar dual  $E^*$  of the graph  $E$  – the faces of  $E$  are the vertices of  $E^*$  and vice-versa. In the case where  $E$  is a regular square lattice, its dual is also a square lattice, simply translated by half a diagonal. If we define the “dual”  $F^*$  of a subset  $F \subset E$  by the set of all edges that do not intersect edges that belongs to  $F$  (see figure 1.5), another Ising model, “dual” to the first one, whose spins lie on the faces rather than on the vertices, can be constructed. Its partition function is given by:

$$Z_{\text{dual}} = e^{2\beta J |E|} \sum_{F^* \subset E^*} v^{|F^*|} 2^{k(F^*)}.$$

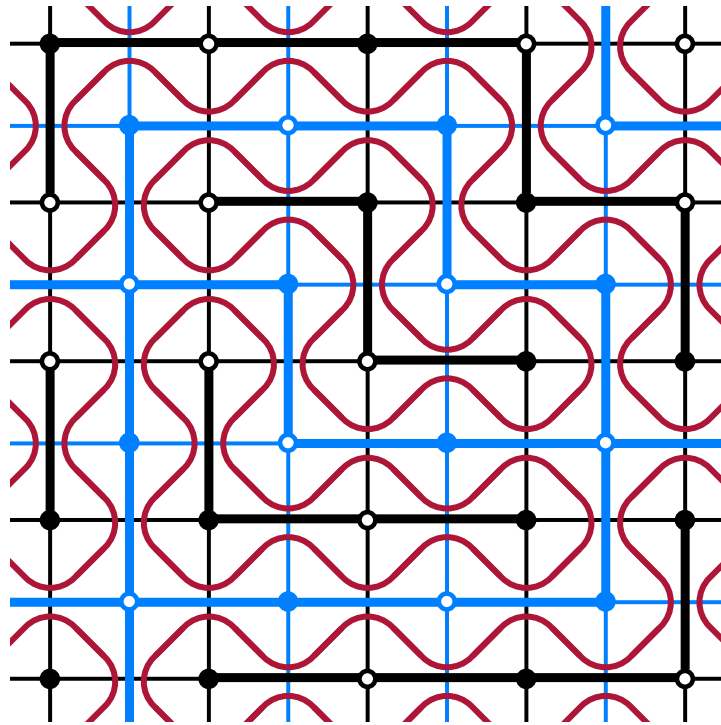
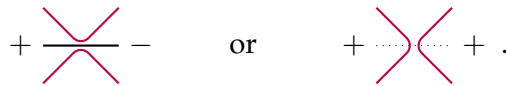


Figure 1.5: The graph  $E$  associated to the initial Ising model is in black, and its dual in blue. The thick edges correspond to a specific configuration  $F$  (black) and  $F^*$  (blue). The red lines form the loop structure associated with this configuration.

Starting from a given configuration  $F \subset E$ , one can draw the loops separating the spins clusters  $F$  of the model from those of its dual  $F^*$ . In practice, they can be defined by adding a decoration on each edge of the initial graph:



*Non-exhaustive list of other models that can be mapped to the Ising model : XX model ( $\approx Ising^2$ ), lattice free fermions,  $A_3$  RSOS model, ...*

The number of such loops  $l(F)$  in an edge configuration  $F$  is equal to:

$$l(F) = 2k(F) - |V| + |F|.$$

Where  $k(F)$  was defined previously as the number of connected components in the configuration, and  $|V|$  and  $|F|$  are respectively the number of vertices and occupied edges in the graph.

This can be proved by induction:

- $F = \emptyset : l(\emptyset) = |V|, k(\emptyset) = |V|.$
- If we add an edge to  $F, |F| \rightarrow |F| + 1$ , two cases are possible:
  - Either the new edge links two clusters which were not connected, in which case both clusters merge, and so do the loops surrounding them. On the dual graph though the number of clusters does not change.  $k(F) \rightarrow k(F) - 1$  and  $l(F) \rightarrow l(F) - 1.$

- Or the new edge links two already existing clusters. This move does not change the number of clusters in the original graph, but splits one of the dual cluster in two, which increases the number of loops by one.  $k(F) \rightarrow k(F)$  and  $l(F) \rightarrow l(F) + 1$ .

In these conditions the partition function can be rewritten:

$$Z \propto \sum_{F \subseteq E} \left( \frac{v}{\sqrt{2}} \right)^{|F|} (\sqrt{2})^{l(F)}.$$

And the dual partition function will verify, noting that  $|F^*| = |E| - |F|$  and that, by definition,  $l(F) = l(F^*)$ :

$$Z_{\text{dual}} \propto \sum_{F^* \subseteq E^*} \left( \frac{v}{\sqrt{2}} \right)^{-|F|} (\sqrt{2})^{l(F)}.$$

Both the model and its dual should have the same critical point (assuming that the Ising critical point is unique), hence the value of the critical point is reached for  $v = \sqrt{2}$  i.e.  $\beta J = \frac{1}{2} \log(1 + \sqrt{2})$ .

The partition function at the critical point will take the form:

$$Z = \sum_{\text{loop configurations}} (\sqrt{2})^{\# \text{ of loops}}.$$

This form of partition function can be generalized by replacing  $\sqrt{2}$  by a generic real  $n$  (called the *fugacity*):

$$Z = \sum_{\text{loop configurations}} n^{\# \text{ of loops}}.$$

This is a first example of a loop model. For generic values of  $n$ , this class of loop models are called Temperley-Lieb (TL) models (or dense  $\mathcal{O}(n)$  models). They are critical for real values of the fugacity  $n$ , from  $-2$  to  $2$ . The model is a generalization of various well-known critical models. For  $n = \sqrt{Q}$ , with  $Q \in \mathbb{Z}$ , the loop model is directly equivalent to the  $Q$ -Potts model. The  $n \rightarrow 1$  limit is linked to a percolation system, while the  $n \rightarrow 0$  limit is equivalent to the self-avoiding walks problem.

Because they are critical, the low-energy effective field theory of these models is expected to be a conformal field theory, but the exact equivalence class should depend on the value of  $n$  (the Ising and percolation model for example are known to live in very different CFTs). At least part of the operators of the loop model can be described by a Liouville CFT, parametrized by  $b$  such that  $n = -2 \cos(\pi b^2)$ . The continuum limit of a particular loop model will be studied in section 2. This is the so-called Fully Packed loop model, a variant of the  $\mathcal{O}(n)$  model.

The different representations of the Ising model we have described (the classical two-dimensional spin model, the quantum spin chain,

*The  $\mathcal{O}(n)$  model has a loop representation valid for every real  $n$ . If every edge has an occupation probability  $x$ , the model has two non-trivial critical points. A diluted one, for  $x = x_c = (2 + (2 - n)^{1/2})^{1/2}$ , and a dense one for  $x < x_c$ . This last one being in the universality class of the TL model. Mathematicians have developed a model of random growing planar curves, named SLE (Schramm-Loewner evolution) [134] which, in many ways, is related to the loop models considered here – with more rigorous if slightly less powerful tools. Specifically, the notion of conformal loop ensemble ( $\text{CLE}_\kappa$ , “soup” of SLE)[135], is conjectured to correspond to the universality class of the  $\mathcal{O}(n)$  model, with  $n = -2 \cos(\pi \kappa / 4)$ .*



Critical statistical system	CFTs
RSOS models	Minimal models
Solid On Solid (SOS) models	Generalized minimal models
XXX Spin Chain	Free compact boson
loop models	Non-rational CFTs, discrete spectrum

Table 1.2: Correspondence between statistical models and CFTs.

and the loop model) are all described by a conformal field theory with  $c = 1/2$ . Another important representative of a  $c = 1/2$  CFT is the one-dimensional Majorana fermion. However, all these models do not have *the same* CFT in the continuum limit. For example, the  $TL(\sqrt{2})$  loop model converges toward a Liouville theory with a continuous spectrum, while the spectrum of the quantum chain Ising model is discrete. Similarly, lattice fermions converges toward a free-field theory, with an infinite spectrum, but the transformation between Ising and the lattice fermions is non-local, and they do not share the same spectrum. Claims of universality must be considered with caution, even the existence of an exact, discrete, mapping between two models is not enough to ensure that they converge toward the exact same CFT.

#### 1.4.3 Transfer matrix exact diagonalisation

Discrete models give a useful tool to study conformally invariant theory. And, while for integrable models an analytic solution is sometimes at hand, most methods are numerical. The most common tools used to study a spin-chain or a statistical model include:

- Monte Carlo methods. They rely on a large number of random sampling, and are a very natural tool in the study of statistical models. While the critical exponents can be hard to reach, Monte Carlo methods are well adapted to the measure of correlation functions.
- Density Matrix Renormalization Group (DMRG). A relatively recent method [145], which gives extremely good results for one-dimensional quantum systems with short range interactions. It works by reducing the effective degrees of freedom to those responsible for the quantum entanglement in a given state. In a more modern language, DMRG is now understood as a variational approach within the space of matrix-product states. The method is slightly less effective for critical systems where correlation lengths are larger. It is also not adapted to statistical

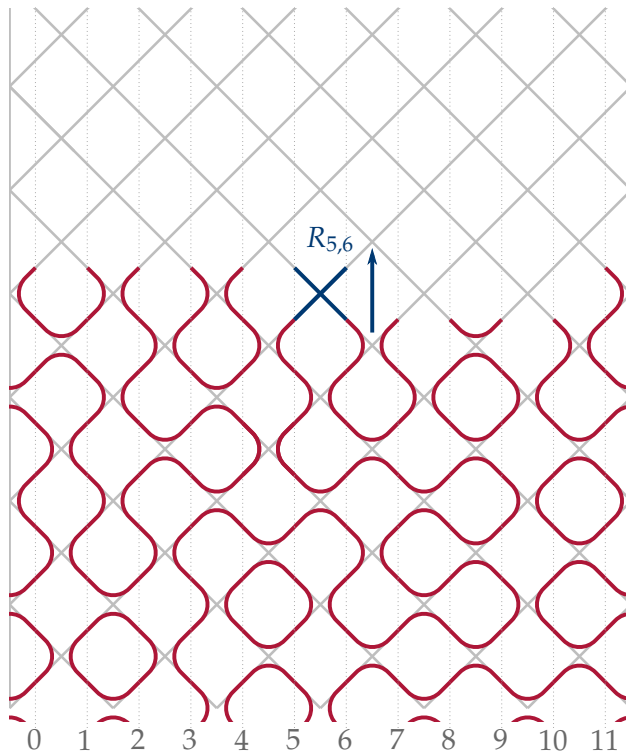


Figure 1.6: Representation of the action of the transfer matrix on the cylinder. The  $R$  matrix acts only on two sites.

models which do not have a local quantum spin chain equivalent (such as loop models).

- Exact diagonalisation. The method consists in extracting the highest eigenvalues of either the transfer matrix or the Hamiltonian. It does not depend on the criticality of the system, and gives direct access to the critical exponents. However, it can only be used for very small sizes (up to 30 sites for a simple quantum chain), leading to important finite-size effects. And while computing correlation functions is certainly possible it can be slightly tricky. Nonetheless, this is the method employed in this thesis.

The transfer matrix on the cylinder is related with the dilatation operator of the CFT. When the transfer matrix is defined on a periodic system, the numerical results should be compared with the conformal theory on a cylinder. Translation along the axis of the cylinder correspond to dilatation on the plane. Hence, we expect the transfer matrix to have, at least in the large  $L$  limit, the same eigenvalues as the  $L_0 + \bar{L}_0$  operator. If the cylinder perimeter is  $L$ , the eigenvalues of the transfer matrix take the general form (at the isotropic point):

$$-\log(\Lambda_i) = f_0 L - \frac{2\pi(h_i + \bar{h}_i - \frac{c}{12})}{L}.$$

*For integrable models, Bethe ansatz methods can sometimes give access to the spectrum and eigenvalues, but exact computations of correlation functions are still complicated.*

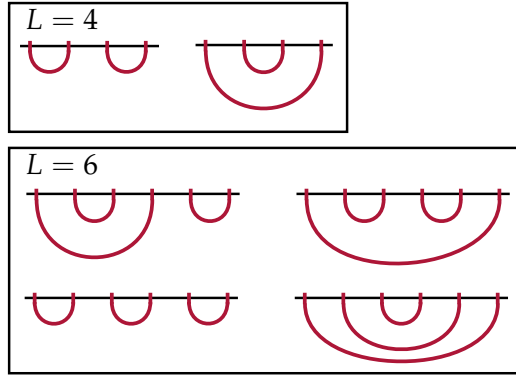


Figure 1.7: Loop states (up to translation) for a given perimeter  $L$ .

$h_i$  is the dimension of a quasi-primary field in the spectrum, and  $f_0$  is a constant. In a unitary theory, where the operator of the lowest dimension is the identity, both the central charge  $c$  and the constant  $f_0$  can be extracted by fitting the highest eigenvalue as a function of  $L$ . The Arnoldi iteration algorithm was used to diagonalize the matrices, with help from the PETSC/SLEPC libraries [8–10, 83, 84, 131].

*The Arnoldi algorithm is a more efficient variant on the simple power iteration method  $A^n|x\rangle \approx \lambda_{A,max}|x\rangle$ . The transfer matrix is not sparse, sadly*

The transfer matrix associated with the Temperley-Lieb loop model gives an interesting non-trivial example. We will consider the transfer matrix on the square lattice rotated by  $\pi/4$  (see figure 1.6). In this setup, the transfer matrix for the TL model can be written:

$$T = \prod_{\substack{i=1,\dots,L \\ i \text{ odd}}} R_{i,i+1} \prod_{\substack{i=1,\dots,L \\ i \text{ even}}} R_{i,i+1}, \tag{1.26}$$

$$R_{i,i+1} = \sin(u) \begin{array}{c} \diagup \\ \diagdown \end{array} + \sin(\lambda - u) \begin{array}{c} \diagdown \\ \diagup \end{array} \quad R_{L,L+1} = R_{L,1}.$$

The matrix  $R_{i,j}$  acts only on two neighbouring sites at a time. It corresponds to the  $R$ -matrix of the system, which plays a large role in integrable theories.

The space on which the transfer matrix acts is a “loop space”, formed by all the valid half-loop configurations that can be generated by the  $R$ -matrix. For example, for  $L = 4$ , there are 2 possible states up to translation, 4 for  $L = 6$  (see figure 1.7).

The loops are in bijection with the binary words with the same number of 0 and 1, in total they are  $\binom{L}{L/2}$  loop states. If we consider only states up to translation (circular shift), the sequence becomes A003239, whose large  $L$  asymptotic is, as expected, roughly equal to  $\frac{1}{L} \binom{L}{L/2}$ .

*For  $L = 30$ ,  $\binom{30}{15}^2$  octets  $\approx 10^4$  Teraoctets*

In general the size of the full matrix is then of order  $2^L \times 2^L$ . At the sizes we consider (up to 30 sites for TL), the main limitation is the memory usage of the program. For these reasons, we do not store the full transfer matrix, but code directly its action on a given vector. The diagonalisation algorithm only need to know the product operation, and this allows to keep only of the order of  $2^L$  variables.

A seemingly simpler method would be to directly look at the eigenvalues of the Hamiltonian. The Hamiltonian takes the form:

$$H \propto \sum_i U_{i,i+1},$$

where  $U_{i,i+1}$  is an operator acting only on the two sites  $i, i+1$ , related to the  $R$ -matrix. We can define the translation operator  $\tau$ , which operates a rotation of  $e^{2i\pi/L}$  around the axis of the cylinder.  $\tau$  sends the site  $i$  toward the site  $i+1$ . Both the Hamiltonian and the transfer matrix commute with this operator, but not the individual  $R_{i,i+1}/U_{i,i+1}$  matrices.

Critical models are invariant under rotation around the axis of the cylinder, hence every eigenstate can be decomposed on the sectors created by the action of  $\tau$ . States without (conformal) spin – a property shared by most primary states – will be found in the sector of momentum 0. When diagonalizing the Hamiltonian we can stay in this sector during the whole process, but this is not the case for the transfer matrix. This invariance can be taken into account to divide the space of states by a factor  $L$ .

The drawback is that the Hamiltonian is defined up to a multiplicative constant,  $v_F$ , called the Fermi velocity of the system. Hence, the critical exponents are only defined up to this multiplicative constant. Nonetheless, finding the right constant is sometimes possible, especially for integrable model.

While the eigenvalues give a lot of information about the spectrum of the theory, to study the full theory we also need its structure constants. The eigenvectors of the transfer matrix can be used to compute correlation function and more precisely three-point correlation functions, hence the structure constants.

A (normalized) correlation function between three primary fields can be written:

$$\langle \mathcal{O}_1 | \mathcal{O}_2(1) | \mathcal{O}_3 \rangle_{\text{cyl}} = C(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3).$$

If  $\Lambda_1$  and  $\Lambda_3$  are the eigenvectors associated respectively with the primaries  $\mathcal{O}_1$  and  $\mathcal{O}_3$ , the structure constants have the expression:

$$C(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3) = \frac{\langle \Lambda_1 | \mathcal{O}_2^{\text{discrete}} \Lambda_3 \rangle}{\langle \Lambda_2 | \mathcal{O}_2^{\text{discrete}} \Lambda_1 \rangle} \sqrt{\frac{\langle \Lambda_2 | \Lambda_2 \rangle \langle \Lambda_1 | \Lambda_1 \rangle}{\langle \Lambda_1 | \Lambda_1 \rangle \langle \Lambda_3 | \Lambda_3 \rangle}}.$$

Two difficulties still need to be overcome to compute numerically the structure constants. The first is to find a discrete equivalent of the  $\mathcal{O}_2$  operator and the second is to find the “right” scalar product, that is, the scalar product which makes the Hamiltonian density self-adjoint. For unitary models, only the first point is problematic, the scalar product is the standard one, associated with the Hermitian Hamiltonian. Finding the right operator  $\mathcal{O}_2^{\text{discrete}}$  can still be tricky though.

For non-local models, like the loop model, or non-unitary theories in general, the choice of the scalar product itself is not trivial. In the case of the Temperley-Lieb model (and in general, of loop models), the right scalar product is the one associated with the loop algebra. Some example of discrete operators in loop models will be studied in detail in the next chapter.

## 1.5 BRIEF THESIS OUTLINE

This thesis is divided in three mostly independent chapters. They can be read in any order.

The first chapter is principally based on the publication [57]. It seeks to clarify the continuum limit of the Fully Packed Loops (FPL) model. The FPL model is a variant of the dense  $\mathcal{O}(n)$  model allowing some vacancies. On the cylinder it is related to vertex models with  $U_q(\mathfrak{sl}_3)$  symmetry. After a description of the FPL model and its relation with other integrable models, the chapter contains an in-depth analysis both of the spectrum and of the structure constants of the FPL model. Numerical simulations are used to confirm these results. Particular care is given to the exact link between the fully packed loop model,  $\mathcal{W}_3$  and the Liouville field theory.

The second chapter, based on the paper [] is more technical and focuses on the computation of structure constants in the imaginary Toda field theory, a generalization of Liouville field theory, associated with a description of spin constraints in generic  $\mathcal{W}_n$  model. These spin constraints permit to obtain a classification of the different non-scalar operators that can be obtained in a generic  $\mathcal{W}_n$  theory.

The last chapter, based on the article [56], presents a new method to compute Rényi entropies – a measure of entanglement entropy– of minimal models. This method uses as key ingredients the differential equations associated with the null-vectors of an orbifold theory. The paper starts with a description of the orbifold CFTs, and of their link with entanglement entropy. The method is then illustrated through the Yang-Lee and the Ising model.

## SUMMARY

The **FPL** model is defined on the hexagonal lattice. Its configurations are reunion of non-crossing loops, such that every vertex of the lattice are occupied. Each loop contributes a weight  $n$  to the configuration. If  $n$  is real and  $|n| \leq 2$ , the model is critical, and in the continuum will be described by a **CFT**.

The model shows sign of a  $\mathbb{Z}_3$  symmetry. For  $n = 2$ , the model is equivalent to a three-colour model. Additionally, on the cylinder, the **FPL** model can be exactly mapped to a  $SU(3)$  vertex model, known to renormalize, up to a twist, toward a CFT with  $\mathcal{W}_3$  symmetry.

This chapter, using both numeric and analytic arguments, explains why this symmetry appears in a specific situation, and clarifies why the loop model in general does not verify it. Main results include:

- The full description of the low-energy field theory of the **FPL** model, through Coulomb-Gas arguments. The theory renormalizes toward the tensor product of a compact boson and a Liouville field theory.
- The expression of the toroidal partition function of the model (2.2.4).
- The complete description of the **FPL** model spectrum on the cylinder (2.2.5), compared to numerical simulations (2.3.2). In the twisted case the  $\mathcal{W}_3$  symmetry can be observed.
- A conjecture, checked numerically, for a certain type of three-point correlation functions. Operators that give a certain weight to loops around them can be associated with primary fields of the CFT, with generic conformal weights. We verify that their three-point functions are given by the Liouville structure constants (2.2.6).

In this chapter, we consider a variant of the **TL** loop model, namely the **FPL** model on the hexagonal lattice. This model was introduced by Reshetikhin [128], and is related to the fifteen-vertex model with  $U_q(\widehat{\mathfrak{sl}}_3)$  symmetry. The spectrum of the corresponding quantum spin

chain has been studied numerically [3] and analytically [157]. On one hand, when appropriate twisted periodic boundary conditions (PBC) are applied, it was shown that the  $U_q(\widehat{\mathfrak{sl}}_3)$  spin chain scales to the  $W_3$  algebra [65], an extension of the Virasoro algebra including a spin-3 conserved current  $W(z)$ , additionally to the stress-energy tensor  $T(z)$ . On the other hand, the Coulomb Gas formalism for the FPL model was developed in [101]. In both approaches, the scaling theory was found to be a CFT with central charge in the range  $-\infty < c \leq 2$ . In this chapter, we aim to clarify the exact nature of the Fully Packed Loop model, by making a connection between the Coulomb Gas construction of [101] and the Liouville conformal field theory. The consequence of the  $U_q(\widehat{\mathfrak{sl}}_3)$  symmetry and the link with the  $W_3$  algebra [65] are also discussed in detail. This opens the way to a complete description of the excitation spectrum of the FPL model, and allows us to exhibit an interesting class of modular invariant partition function.

The structure of this chapter is the following. In Sec. 2.1, we recall the definition of the FPL model, and its relation with the integrable  $U_q(\widehat{\mathfrak{sl}}_3)$  vertex and face models. In Sec. 2.2, we study the continuum limit of the FPL model with potentially different loop fugacities for contractible and non-contractible loops. We then classify the spectrum of the loop model and discuss three-point correlation functions. These results are verified numerically by an exact diagonalisation procedure, explained in Sec. 2.3. In Sec. 2.2.4, we turn to the study of toroidal partition functions: by applying the steps of [72], we derive the full spectrum of conformal dimensions of the FPL model, and express the FPL model partition function on the torus in terms of Coulombic partition functions. Particular values of the loop fugacity  $n = \sqrt{2}$  and  $n = 1$  are examined in more detail. The Appendix contains some technical calculations needed in Sec. 2.2.4 as well as a discussion over diagrammatic representations of  $U_q(\widehat{\mathfrak{sl}}_3)$ .

## 2.1 FULLY-PACKED LOOP MODEL AND RELATED LATTICE MODELS

### 2.1.1 The loop model

The  $A_2^{(1)}$  loop model on the square lattice is defined in [143] as follows. The allowed configurations are those represented in Fig. 2.1 a, their local weights are given in terms of the crossing parameter  $\lambda$  and the spectral parameter  $u$  :

$$a_1, \dots, a_7 = \sin(\lambda - u), e^{+iu} \sin \lambda, e^{-iu} \sin \lambda, \sin u, \sin u, \sin(\lambda - u), \sin u, \quad (2.1)$$

and each closed loop gets a weight  $n = 2 \cos \lambda$ . On a surface with cycles (cylinder, torus ...), the non-trivial loops (those which wind

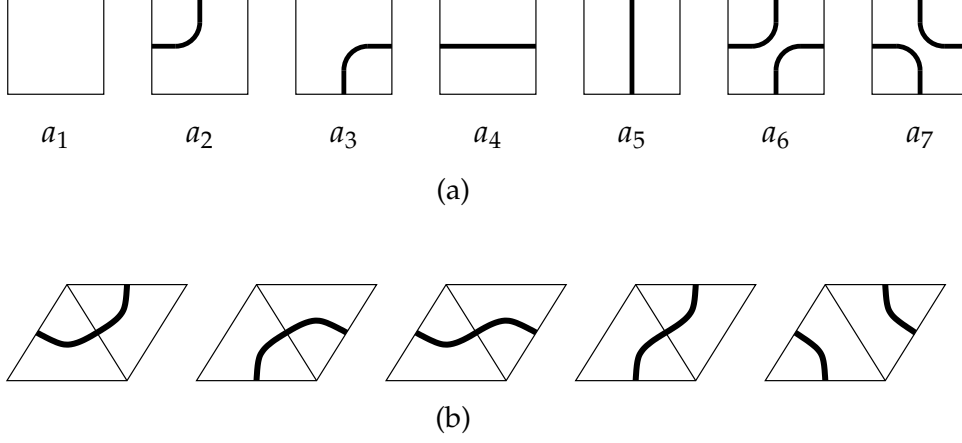


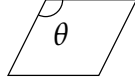
Figure 2.1: (a) The plaquette configurations of the  $A_2^{(1)}$  loop model on the square lattice. (b) The double-plaquette configurations of the Fully Packed Loop model on the hexagonal lattice, obtained by setting  $a_1 = a_6 = 0$  and  $a_2 = a_3 = a_4 = a_5 = a_7$ .

around the cycles of the surface) get a different weight  $\tilde{n}$ . Hence, the partition function reads:

$$Z_{\text{loop}}(n, \tilde{n}, u) = \sum_{\text{loop config. } C} a_i^{N_i(C)} n^{L(C)} \tilde{n}^{\tilde{L}(C)}, \quad (2.2)$$

where the sum is over every loop configuration  $C$  on the square lattice obtained by combining the plaquettes  $\{a_1, \dots, a_7\}$ ,  $N_i(C)$  is the number of plaquettes of type  $a_i$  appearing in  $C$ , and  $L(C)$  [resp.  $\tilde{L}(C)$ ] is the number of trivial (resp. non-trivial) closed loops in  $C$ .

For a generic value of  $u$ , the natural embedding of the square lattice is given by rhombi of the form



with opening angle  $\theta = \frac{4}{3}\pi u / \lambda$ . The ‘‘isotropic’’ point sits at the value  $u = \lambda$ : for this value, one has, after dropping the irrelevant factors  $e^{\pm iu}$ :

$$a_1 = a_6 = 0, \quad a_2 = a_3 = a_4 = a_5 = a_7 = \sin \lambda. \quad (2.3)$$

At this value  $u = \lambda$ , the  $A_2^{(1)}$  loop model reduces [128] to the FPL model on the hexagonal lattice (see Fig. 2.1 b). The partition function becomes

$$\begin{aligned} Z_{\text{loop}}(n, \tilde{n}, u = \lambda) &= (\sin \lambda)^{\mathcal{N}/2} \sum_{\text{FPL loop config. } C} n^{L(C)} \tilde{n}^{\tilde{L}(C)} \\ &\equiv (\sin \lambda)^{\mathcal{N}/2} \times Z_{\text{FPL}}(n, \tilde{n}), \end{aligned} \quad (2.4)$$

where the sum is over every loop configuration  $C$  on the hexagonal lattice visiting each vertex exactly once, and  $\mathcal{N}$  is the total number of vertices.

*Modifying the spectral parameter  $u$  modify the form of the lattice.*



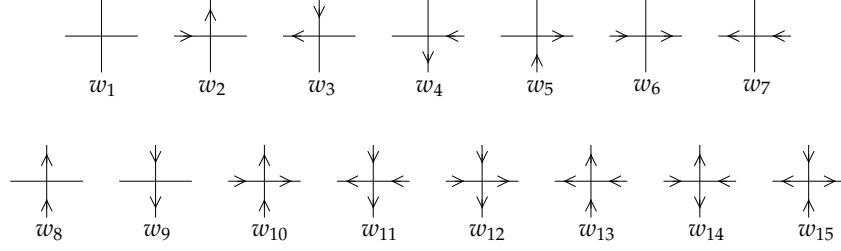


Figure 2.2: Configurations of the 15-vertex model

Note that  $Z_{\text{FPL}}(n = \tilde{n} = 2)$  is equal to the number of three-colouring of the hexagonal lattice [101]: if the colours are called  $(A, B, C)$ , each empty edge is labelled  $A$ , and the edges along a closed loop are labelled  $BCBCBC\dots$  or  $CBCBCB\dots$ , resulting in loop fugacities  $n = \tilde{n} = 2$  ( $\lambda = 0$ ).

2.1.2 The fifteen-vertex model

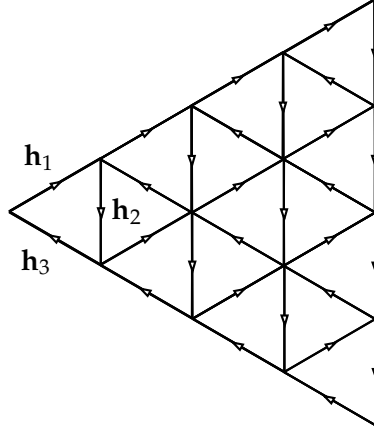
The 15-vertex model is defined by the  $R$ -matrix acting on two fundamental representations of the  $U_q(\widehat{\mathfrak{sl}}_3)$  quantum affine algebra, whose basis is taken as  $(|\uparrow\rangle, |\cdot\rangle, |\downarrow\rangle)$ . The corresponding vertex configurations are shown in Fig. 2.2, and their weights are given by [89]:

$$\begin{aligned}
 w_1 &= w_{10} = w_{11} = \sin(\lambda - u), \\
 w_2 &= w_4 = w_{14} = e^{+iu} \sin \lambda, \\
 w_3 &= w_5 = w_{15} = e^{-iu} \sin \lambda, \\
 w_6 &= w_7 = w_8 = w_9 = w_{12} = w_{13} = \sin u.
 \end{aligned}
 \tag{2.5}$$

When spectral parameters are attached to the lines of the vertex model, we use the following graphical conventions:

$$R(u_1 - u_2) = \begin{array}{c} \nearrow \\ u_1 \text{ --- } \rightarrow \\ \searrow \\ u_2 \end{array} .$$

The 15-vertex model can be obtained directly from the loop model, by using the trick of [13]. Let us describe this equivalence on the plane first, where no non-trivial loop can occur. One gives an orientation to each loop (independently of the others), and assigns a weight  $e^{i\lambda}$  (resp.  $e^{-i\lambda}$ ) to the anti-clockwise (resp. clockwise) oriented loops, so that the total loop weight is indeed  $n = 2 \cos \lambda$ . Then, the phase factor for each loop is distributed locally, by attributing a factor  $\exp[i\alpha\lambda/(2\pi)]$  to each oriented loop segment in Fig. 2.1 which turns by an angle  $\alpha$  (we take the convention that the loop segments cross the edges of the rhombic plaquettes orthogonally). Then, on each edge carrying the spectral parameter  $u$ , one inserts the operator  $\mathbb{1} = e^{+iu\eta} \times e^{-iu\eta}$ , where  $\eta = \text{diag}(\frac{1}{3}, 1, -\frac{1}{3})$ , so that the factor

Figure 2.3: The height graph  $A_{k=4}$ .

$e^{+iu\eta}$  (resp.  $e^{-iu\eta}$ ) acts on the incoming (resp. outgoing) vector space of the adjacent R-matrix. This last step may be viewed as a change of gauge. Through the whole procedure, the loop model with weights (2.1) maps to the 15-vertex model with weights (2.5).

On surfaces with cycles, the non-trivial loops have a total winding equal to zero, and must be treated separately. On a cylinder of width  $N$  sites and circumference  $M$  sites, the partition function of the loop model is obtained by introducing twisted boundary conditions:

$$Z_{\text{cyl}}(n, \tilde{n}) = \text{tr} \left[ (K_\mu \otimes \cdots \otimes K_\mu) (t_N)^M \right], \quad \tilde{n} = 2 \cos \mu, \quad (2.6)$$

where  $t_N$  is the row-to-row transfer matrix,  $\text{tr}$  stands for the conventional trace, and  $K_\mu = \text{diag}(e^{i\mu}, 1, e^{-i\mu})$ .

On the torus, the weight of non-trivial loops cannot be distributed locally into the vertex model: rather, to give non-trivial loops a weight  $\tilde{n} = 2 \cos \mu$ , each vertex configuration with arrow fluxes  $m$  and  $m'$  through the two cycles of the torus must be given the non-local phase factor  $\exp[i\mu(m \wedge m')]$ , where  $m \wedge m'$  denotes the greatest common divisor (gcd) of  $m$  and  $m'$ .

### 2.1.3 The RSOS model

The third model of interest is the RSOS model based on the Weyl alcove  $A_k$  of  $\mathfrak{sl}_3$  [90]. Let us first rapidly set up our conventions for  $\mathfrak{sl}_3$ , which shall be used throughout the thesis. A more complete description of general  $\mathfrak{sl}_n$  Lie algebra can be found in the appendix 3.A. The simple roots ( $e_1, e_2$ ) are two-dimensional vectors with scalar products:

$$e_1 \cdot e_1 = e_2 \cdot e_2 = 2, \quad e_1 \cdot e_2 = -1. \quad (2.7)$$

The fundamental weights ( $\omega_1, \omega_2$ ) are given by:

$$\omega_1 = \frac{1}{3}(2e_1 + e_2), \quad \omega_2 = \frac{1}{3}(2e_2 + e_1) \quad \Leftrightarrow \quad \omega_i \cdot e_j = \delta_{ij}. \quad (2.8)$$

This gives  $\omega_1 \cdot \omega_1 = \omega_2 \cdot \omega_2 = 2/3$  and  $\omega_1 \cdot \omega_2 = 1/3$ . The root and weight lattices are respectively:

$$\mathcal{R} = \mathbb{Z}e_1 + \mathbb{Z}e_2, \quad \mathcal{R}^* = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2. \quad (2.9)$$

The weights of the fundamental representation are

$$h_1 = \omega_1, \quad h_2 = \omega_2 - \omega_1, \quad h_3 = -\omega_2. \quad (2.10)$$

The Weyl vector is  $\rho = e_1 + e_2 = \omega_1 + \omega_2$ . The oriented graph  $A_k$  is defined as follows (see also Fig. 2.3). The set of vertices of  $A_k$  is given by the dominant integral weights of  $\widehat{\mathfrak{su}}(3)_k$ , namely:

$$A_k = \{\lambda_1\omega_1 + \lambda_2\omega_2, \lambda_i \in \mathbb{N}, \lambda_1 + \lambda_2 \leq k\}, \quad (2.11)$$

and the edges of  $A_k$  are oriented along the three vectors  $(h_1, h_2, h_3)$ . In the following, we will refer to  $A_k$  as *the height graph*. In the RSOS model, each vertex of the square lattice carries a height variable which is a vertex of  $A_k$ , and the Boltzmann weight of a face is denoted by:

$$W \left( \begin{array}{cc|c} a & b & u \\ d & c & \end{array} \right) = \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \square \quad u \\ \diagup \quad \diagdown \\ d \quad c \end{array} \quad (2.12)$$

$$= \begin{array}{c} a \quad \kappa \\ \diagdown \quad \diagup \\ \square \quad u \\ \diagup \quad \diagdown \\ \mu \quad v \end{array} \sigma = W \left( \begin{array}{cc|c} a & a + h_\kappa & u \\ a + h_\mu & a + h_\mu + h_\nu & \end{array} \right),$$

where the labels  $1 \leq \kappa, \sigma, \nu, \mu \leq 3$ , and must satisfy  $h_\mu + h_\nu = h_\kappa + h_\sigma$ . Setting  $\mu \neq \nu$ , the face weights are given by:

$$\begin{aligned} \begin{array}{c} a \quad \mu \\ \diagdown \quad \diagup \\ \square \quad u \\ \diagup \quad \diagdown \\ \mu \quad \mu \end{array} &= \sin(\lambda - u), \\ \begin{array}{c} a \quad \mu \\ \diagdown \quad \diagup \\ \square \quad u \\ \diagup \quad \diagdown \\ \mu \quad v \end{array} &= \frac{\sin \lambda \sin(\lambda a_{\mu\nu} + u)}{\sin \lambda a_{\mu\nu}}, \\ \begin{array}{c} a \quad v \\ \diagdown \quad \diagup \\ \square \quad u \\ \diagup \quad \diagdown \\ \mu \quad v \end{array} &= \frac{\sin u \sin \lambda(a_{\mu\nu} + 1)}{\sin \lambda a_{\mu\nu}}, \end{aligned} \quad (2.13)$$

where  $\lambda \equiv \pi/(k+3)$  and  $a_{\mu\nu} \equiv (a + \rho) \cdot (h_\mu - h_\nu)$ . On simply connected domains, the RSOS model is related to the 15-vertex model

Classic RSOS models are defined on a ADE graph. In the continuum limit they correspond to Virasoro minimal models. The correspondence between RSOS graphs and the modular invariant of  $\mathcal{W}_3$  CFTs was done by Di Francesco and Zuber [49]. These graphs also classify the modular invariant of  $SU(3)$ . For other type of extended algebras, the question is still partially open.

through the vertex-face correspondence [91]. On the cylinder, the relation between these two models (along with the FPL model) uses the Hecke algebra symmetry, combined with the Markov trace: see next section.

#### 2.1.4 Hecke algebra and Markov trace

The FPL, 15-vertex and RSOS models are three realizations of the  $\mathfrak{su}(3)$  Hecke algebra. In all three cases, the Boltzmann weight of a square plaquette at position  $j$  on the lattice can be written as

$$\sin(\lambda - u) \mathbb{1} + \sin u U_j, \quad (2.14)$$

where  $U_1, U_2 \dots U_{N-1}$  are the Hecke generators, obeying the algebraic relations:

$$U_j^2 = 2 \cos \lambda \times U_j, \quad (2.15)$$

$$U_j U_{j+1} U_j - U_j = U_{j+1} U_j U_{j+1} - U_{j+1}, \quad (2.16)$$

$$U_{j'} U_j = U_j U_{j'} \quad \text{if } |j - j'| > 1, \quad (2.17)$$

$$(U_{j-1} - U_{j+1} U_j U_{j-1} + U_j)(U_j U_{j+1} U_j - U_j) = 0. \quad (2.18)$$

With  $j$  lives in  $\mathbb{Z}/N\mathbb{Z}$ ,  $N$  being the length of the periodic system.

Equations (2.15–2.17) are the defining relations of the Hecke algebra, and the additional relation (2.18) defines the  $\mathfrak{su}(3)$  quotient of the Hecke algebra, which we denote as  $\mathcal{H}_N$  in the following.

When the model is defined on a cylinder of  $N \times M$  sites (where  $M$  is the circumference), the partition function is defined as

$$Z_{\text{cyl}} = \text{Tr} \left[ (t_N)^M \right], \quad (2.19)$$

where  $t_N$  is the row-to-row transfer matrix with open boundary conditions, and  $\text{Tr}$  is a linear form on  $\mathcal{H}_N$ , which obeys the *Markov property*:

$$\forall j, \quad \forall x \in \mathcal{H}_j, \quad \text{Tr}(U_j x) = \frac{\sin 2\lambda}{\sin 3\lambda} \times \text{Tr} x \quad (2.20)$$

where  $\mathcal{H}_j$  is the sub-algebra generated by  $\{U_1, \dots, U_{j-1}\}$ . A linear form obeying this property is called a Markov trace. Let us give the explicit form of the generators  $U_j$  and the Markov trace  $\text{Tr}$  for the three lattice models of interest.

- In the FPL model, the Hecke generator takes the form

$$U = e^{i\lambda} \begin{array}{|c|} \hline \text{┌} \\ \text{└} \\ \hline \end{array} + e^{-i\lambda} \begin{array}{|c|} \hline \text{└} \\ \text{┌} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{—} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{||} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{┌} \\ \text{└} \\ \hline \end{array} \quad (2.21)$$

The Markov trace is  $\text{Tr} x = (2 \cos 2\lambda)^{\tilde{L}(x)}$ , where  $\tilde{L}(x)$  is the number of closed loops appearing when the top and bottom part of the diagram  $x$  are identified. In other words, the Markov property is obeyed when  $\tilde{n} = 2 \cos 2\lambda$ .

See appendix 2.C.1 for details.

- In the 15V model, the R-matrix  $\check{R}(u) \equiv PR(u)$ , where  $P$  is the permutation operator, has the form  $\check{R}_{j,j+1}(u) = \sin(\lambda - u) \mathbb{1} + \sin u U_j$ , with

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{+i\lambda} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{+i\lambda} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & e^{-i\lambda} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{+i\lambda} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & e^{-i\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & e^{-i\lambda} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.22)$$

expressed in the basis  $(|\uparrow\rangle, |\cdot\rangle, |\downarrow\rangle) \otimes (|\uparrow\rangle, |\cdot\rangle, |\downarrow\rangle)$ . The Markov trace is given by

$$\text{Tr } x = \text{tr}[K^{\otimes N} x], \quad (2.23)$$

where  $\text{tr}$  denotes the conventional trace, and  $K = \text{diag}(e^{2i\lambda}, 1, e^{-2i\lambda})$ .

- In the RSOS model, the face weights can be written

$$W \left( \begin{array}{cc|c} a & b & u \\ d & c & \end{array} \right) = \sin(\lambda - u) \delta_{b,d} + \sin u U \left( \begin{array}{cc} a & b \\ d & c \end{array} \right), \quad (2.24)$$

where the Hecke generators read:

$$U \left( \begin{array}{cc} a & a + h_\kappa \\ a + h_\mu & a + h_\mu + h_\nu \end{array} \right) = (1 - \delta_{\mu\nu}) \frac{\sin \lambda (a_{\mu\nu} + 1)}{\sin \lambda a_{\mu\nu}}. \quad (2.25)$$

The Markov trace in this model is given in the Appendix of [117].

As it is argued in [117], the algebraic relations (2.15–2.17) and the Markov property (2.20) determine completely the value of  $\text{Tr } x$  for any  $x \in \mathcal{H}_N$ . Hence, the FPL, 15-vertex and RSOS models have the same partition function on the cylinder.

## 2.2 CONTINUUM LIMIT OF THE FPL MODEL

The continuum limit of loop models is rather well understood and typically yields a bosonic action coupled to the background curvature [70]. The scaling limit of the  $A_2^{(1)}$  loop model in the case  $n = \tilde{n}$  has been studied extensively in [101]. Through a mapping to Coulomb gas, (part of) the spectrum and conformal dimensions were obtained. We extend those results and discuss the continuum limit of the FPL. While the Coulomb gas obtained for the FPL loop model is not exactly equivalent to the  $W_3$  Coulomb gas on generic Riemann surfaces, we argue that these two theories are indistinguishable on the flat cylinder.

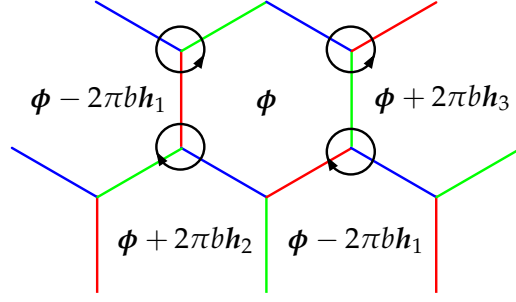


Figure 2.4: Mapping from the three-colour model to a height model.

### 2.2.1 Coulomb gas description

Later in this section we will focus on the flat cylinder, but for now we consider a generic (i.e. non flat) Riemann surface of genus zero. As pointed out in [70], this is extremely instructive in order to understand the coupling to the curvature. The flat torus will be considered in Sec. 2.2.4. Before extending to  $n < 2$ , let us first consider the case  $n = \tilde{n} = 2$  on the hexagonal lattice, in which the partition function simply counts the number of three-colouring (see Sec. 2.1.1). Let  $\phi$  be a two-components discrete height field living on the dual lattice, for which the loops are level lines :  $\phi$  varies by  $\pm 2\pi b h_i$  when crossing an edge.  $b$  is a non-zero real constant and the value of  $i = \{1, 2, 3\}$  depends on the colour crossed, lastly the sign depends on the orientation of the edge, defined by the bi-partition of the lattice, see Fig. 2.4.  $h_1 + h_2 + h_3 = 0$  hence the height is well-defined. In the scaling limit, this height field renormalizes towards a two-component compactified free boson with compactification lattice  $2\pi b \mathcal{R}$  (2.9).

The compactification radius can be extracted easily by looking at the effect on the field of the different defects possible. A defect in this model correspond to the non-respect of the three-colours rule. In term of the field, the insertion of a defect modify the value of  $\phi$  by a vector of the lattice  $\mathcal{R}$ . A different explanation, which does not involve defects, can be given by considering the height differences between different types of “most flat” configurations [101].

The action of the three-colour model is then given by:

$$\frac{1}{8\pi} \int d^2x \sqrt{|g|} g^{\mu\nu} \partial_\mu \phi \cdot \partial^\mu \phi, \quad (2.26)$$

$$\phi \equiv \phi + 2\pi b \mathcal{R},$$

where  $g$  is the metric of the underlying Riemann surface. The normalization of the field  $\phi$  is chosen in such a way that:

$$\langle \phi_i(z, \bar{z}) \phi_j(0, 0) \rangle = -\delta_{ij} \log |z|^2.$$

The loop model is obtained from the colour model by drawing a loop on the edges of colours  $h_1$  or  $h_3$  and letting empty the ones

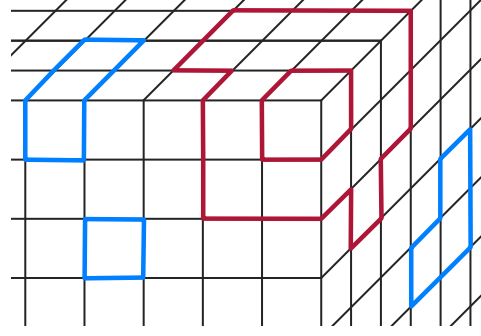


Figure 2.5: Lattice with a “cubic corner” singularity. The sums of the corner angles of loops around the singularity (in red) are equal to  $\frac{3\pi}{2}$  versus  $2\pi$  for the other loops (in blue).

of colour  $h_2$ . We saw in the last section that a local way to give a weight  $n$  to the loops consisted in giving an orientation to each loop associating a weight  $e^{-i\lambda}$  (resp.  $e^{+i\lambda}$ ) to left-turning loops (resp. right-turning loops). This method only works on the plane, for example loops circling a cylinder, or following the great circle of a sphere will only get weight 1. These curves do not get the right weight because they separate two regions of non-zero curvature. In the Lagrangian, this effect can be compensated by the presence of a coupling to the curvature:

$$\mathcal{A}_0 = \frac{1}{8\pi} \int d^2x \sqrt{|g|} (g^{\mu\nu} \partial_\mu \phi(x) \cdot \partial_\nu \phi(x) + 2i\mathbf{Q} \cdot \phi(x) R(x)) ,$$

Empty edges should not be affected by the curvature, hence  $\mathbf{Q} \cdot h_2 = 0$ , which implies  $\mathbf{Q} = Q\rho$ . To link the value of  $Q$  to the weight of the loops we can, for example, consider a “cube corner” singularity, located at  $x = 0$  on an otherwise flat surface (see figure 2.5).

On one hand, loops surrounding such a singularity will have a total angle  $\frac{3\pi}{2}$ , hence the weight of a counter-clockwise loop will be  $e^{i3\lambda/4}$ , instead of  $e^{i\lambda}$  for a loop on flat space.

On the other hand, the Ricci scalar around zero will have the form  $R(x) = \pi\delta(x)$  (the integral of the Ricci scalar on a full cube is  $8\pi$ ). The curvature term is then equal to  $e^{i\mathbf{Q} \cdot \phi(0)/4}$ . Adding a (counter-clockwise) loop around 0 increases the value of  $\phi(0) \cdot \rho$  by  $2\pi b$ , hence add a factor  $e^{i(2\pi bQ)/4}$ .

Therefore, the curvature term will compensate for the angular defect if and only if  $\lambda = 2\pi bQ$ .

$$n = e^{2i\pi bQ} + e^{-2i\pi bQ} = 2 \cos(2\pi bQ) . \quad (2.27)$$

The presence of a curvature coupling, in the action, can be shown more generally. Consider the model on a simply connected surface  $\Omega$ , with no boundaries but with a marked point  $x_0$ , in order to be able to differentiate between the interior and the exterior of a loop (the exterior contains  $x_0$ ). Call  $h(x) = \rho \cdot \phi(x)$ , the projection of  $\phi$  over the direction of the loops. Suppose that a clockwise loop  $\gamma$  encloses a surface  $\text{int}(\gamma)$ . The Gauss-Bonnet theorem implies that:

$$\iint_{\text{int}(\gamma)} \frac{1}{2} R(x) d^2x + \int_\gamma k_g(s) ds = 4\pi .$$

where  $R(x)$  is the scalar curvature of the surface and  $k_g$  is the geodesic curvature of  $\partial M$ . If the loop is anti-clockwise we can write:

$$-\iint_{\text{int}(\gamma)} \frac{1}{2} R(x) d^2x + \int_{\gamma} k_g(s) ds = -4\pi.$$

Summing these two expressions over every loop (clockwise or otherwise) surrounding  $x$ , and then summing over  $x$ :

$$\begin{aligned} & \iint_{\Omega} \frac{1}{2} R(x) [(\# \text{ loops } \circlearrowleft x) - (\# \text{ loops } \circlearrowright x)] d^2x + \sum_{\mathcal{L} \text{ loop}} \int_{\mathcal{L}} k_g(s) ds \\ & = 4\pi(\# \text{ clockwise loops} - \# \text{ anti-clockwise loops}). \end{aligned} \quad (2.28)$$

The number of loops around a point  $x$  is controlled by the height  $h(x)$ . Explicitly:

$$[(\# \text{ loops } \circlearrowleft x) - (\# \text{ loops } \circlearrowright x)] = \frac{h(x) - h(x_0)}{2\pi b}.$$

For a geodesic loop – for example one of the great circle on the sphere –  $k_g(s) = 0$  and only the first term matter. Inserting in the action a term of the form:

$$\frac{ibQ}{2} \iint_{\Omega} \sqrt{|g|} d^2x R(x) \frac{(h(x) - h(x_0))}{4\pi b},$$

would give straight loops a weight  $e^{2i\pi bQ} + e^{-2i\pi bQ} = 2\cos(2\pi bQ)$ , as expected.

The second term, which depends on  $k_g$  is local: on the lattice, it would be implemented by local weights, added at each loop turn. What exactly becomes of this term after the renormalization flow is difficult to analyse. We will ignore the exact form of this term in the following, and replace it with an effective term. It is nonetheless possible to re-express it in term of the height field  $h$ .  $k_g$  is the geodesic curvature of the curve implicitly defined by  $h(x, y) = H$ , where  $H$  is a constant. Along this curve,  $k_g$  has the expression [80]:

$$k_g = \frac{(\nabla h)^T \cdot (\nabla \nabla h) \cdot (\nabla h)}{|\nabla h|^3} - \frac{\Delta h}{|\nabla h|}$$

Where  $\nabla h$ ,  $\nabla \nabla h$  and  $\Delta h$  are respectively the gradient, the Hessian matrix and the Laplacian of  $h$ .

The line integral on  $k_g$  can be rewritten:

$$\int_{\gamma} k_g(s) ds = \iint_{\text{int}(\gamma)} \sqrt{|g|} d^2x \delta(h - H) k_g |\nabla h|.$$

Finally, by summing over all possible value of  $H$ , we get an integral over the full space  $\Omega$ , and a term proportional to  $k_g |\nabla h|$  can be added to the Lagrangian.

The coupling to the curvature modifies the stress-energy tensor, lowering the central charge :

$$T(z) = -\frac{1}{2} : \partial \varphi \cdot \partial \varphi : + Q \cdot \partial^2 \varphi, \quad c = 2 - 12 Q \cdot Q = 2 - 24Q^2, \quad (2.29)$$

where we have used the convention  $\varphi(z, \bar{z}) = \varphi(z) + \bar{\varphi}(\bar{z})$  for the holomorphic and anti-holomorphic parts of the free field.  $Q$  is called the *background charge*.

*The term  $\frac{R(x)h(x_0)}{4\pi b}$  can be integrated over to gives a vertex operator at the point  $x_0$ , of the form  $e^{-2iQ \cdot \phi(x_0)}$ . This operator has weight zero, but it does have an effect on the theory, and is not the identity.*

*Without the compactification condition, this is simply the product between a free boson theory and a one-dimensional coulomb-gas theory, in the direction of  $\rho$ .*



This is a *Coulomb-gas theory*, where the vertex operators defined by:

$$V_\alpha = :e^{i\alpha\cdot\phi}:, \quad (2.30)$$

have conformal dimensions:

$$h_\alpha = \bar{h}_\alpha = \frac{1}{2}\alpha \cdot (\alpha - 2Q). \quad (2.31)$$

### 2.2.2 Liouville term

In order to give the right weight to contractile loops, we need to add to the action an additional potential which will, at the lattice level, give a weight  $n$  to the loops.

$$\mathcal{A} = \mathcal{A}_0 + \frac{1}{8\pi} \int d^2x V[\phi(x)].$$

On the hexagonal lattice, adding a weight for the (contractile) loop is equivalent to adding a local weight  $e^{\pm i\pi\lambda/6}$  to every turn, with a positive sign when the loop turns right and a negative one when the loop turns left. This means that the potential is local, and, at a point, it should not be modified by the presence of additional loops surrounding this point. Hence, the potential should be invariant under  $\phi \rightarrow \phi + 2\pi b h_i$ ,  $i = 1, 3$ . Hence,  $V$  is periodic on the sub-lattice  $2\pi b \mathcal{R}^* = 2\pi b \sum_i \mathbb{Z} h_i$  of the lattice  $2\pi b \mathcal{R}$ .

This periodicity imposes a specific form under Fourier transformation:

$$V(\phi) = \sum_{e \in \mathcal{R}} \tilde{V}_e e^{ib^{-1}\phi \cdot e}. \quad (2.32)$$

Since the loop model is critical, the potential  $V$  cannot be relevant in the  $R$  sense. Furthermore, there must be at least one term in the sum (2.32) that is marginal, lest this potential be entirely washed away in the infra-red. Therefore, we demand:

$$\min_{e \in \mathcal{R}} (h_{b^{-1}e} + \overline{h_{b^{-1}e}}) = 2.$$

Because the dimensions depend directly on the value of  $Q$ , this equation will fix the value of the background charge. But the lowest-dimensional term in the potential 2.32 can vary depending on the value of  $Q$ . Therefore, after obtaining the value of  $Q$  we must verify, in retrospect, that the operator chosen is still the most relevant.

This analysis give two possible values for  $Q$ ,  $Q = \pm \frac{1}{2}(\frac{1}{b} - b)$ , corresponding to two different choice of relevant operator:  $V_{\pm b^{-1}\rho} = e^{\pm ib^{-1}\phi \cdot \rho}$ . This dichotomy corresponds to a choice of sign  $\pm \phi$ , and do not change the physic of the system. We choose the convention:

$$\mathcal{A} = \int \frac{d^2x}{8\pi} \sqrt{|g|} \left\{ \partial_\mu \phi \cdot \partial^\mu \phi + iR(x) \left( \frac{1}{b} - b \right) \rho \cdot \phi + \mu e^{ib^{-1}\phi \cdot \rho} \right\},$$

$$\phi \equiv \phi + 2\pi b \mathcal{R}.$$

*The notation  $\mathcal{R}^*$  comes from the fact that this graph is also the dual of  $\mathcal{R}$ .*

(2.33)

Where  $\mu$  is a constant (nicknamed cosmological constant in QFT applications).

To see that  $b^2$  must be less or equal to 1, one can notice that the relation:

$$2 \geq h_{b^{-1}e_1} + h_{-b^{-1}e_1} = \frac{2}{b^2}$$

implies that  $b^2 \leq 1$ .

To see that the most relevant terms in 2.32 are necessarily of the form  $V_{\pm b^{-1}\rho}$ , one can do as follows. The most relevant term is generically of the form  $V_{n_1 b^{-1}e_1 + n_2 b^{-1}e_2}$ , with  $n_1, n_2 \in \mathbb{Z}$ . Since we want this term to be exactly marginal, we have  $h_{n_1 b^{-1}e_1 + n_2 b^{-1}e_2} = 1$  should be verified. This fixes the value of which yields the following value for  $Q = Q\rho$ :

$$Q = b \frac{b^{-2} (n_1^2 - n_1 n_2 + n_2^2) - 1}{n_1 + n_2}.$$

But that means that the weight of the fields  $V_{\pm b^{-1}\rho}$  are now:

$$h_{b^{-1}\rho} = \frac{n_1 + n_2 - n_1^2 - n_2^2 - (n_1 - n_2)^2}{b^2(n_1 + n_2)} + \frac{2}{n_1 + n_2},$$

$$h_{-b^{-1}\rho} = \frac{n_1 + n_2 + n_1^2 + n_2^2 + (n_1 - n_2)^2}{b^2(n_1 + n_2)} - \frac{2}{n_1 + n_2}.$$

Both terms should be larger than 1 if we want  $V_{\pm b^{-1}\rho}$  to be a sub-dominant term. On one hand, if  $|n_1 + n_2| \geq 2$  (and  $(n_1, n_2) \neq (1, 1)$ ), the term  $n_1 + n_2 - n_1^2 - n_2^2 - (n_1 - n_2)^2$  in the first line is strictly negative and  $h_{-b^{-1}\rho} < 2$ . On the other hand:

$$h_{b^{-1}(n_1+n_2)e_1} = 1 + \frac{3n_1 n_2}{b^2}.$$

Hence,  $n_1 n_2 \geq 0$ . This leaves only two possibilities for  $(n_1, n_2)$  – outside of  $(n_1, n_2) = (1, 1)$ . Either  $(n_1, n_2) = (0, 1)$  or  $(1, 0)$ , and in this case:

$$h_{b^{-1}\rho} = 3b^{-2} - 2 \quad h_{-b^{-1}\rho} = 2 - b^{-2}.$$

At least one of these terms is smaller than one, for any real value of  $b$ .

If we decompose the field according to its components along  $\rho$  and  $h_2$ ,  $\phi = h\rho + \phi h_2$ , the action can be split into two terms:

$$\mathcal{A} = \int \frac{d^2x}{8\pi} \sqrt{|g|} \left\{ 2\partial_\mu h \partial^\mu h + 2iR(x)(b^{-1} - b)h + \mu e^{i2b^{-1}h} + \frac{2}{3} \partial_\mu \phi \partial^\mu \phi \right\}.$$

In a nutshell, the theory is the tensor product of a Liouville field theory for the loops (along the direction  $\rho$ ) and of a free boson along the direction  $h_2$ , with a compactification lattice mixing the two directions. Since  $n = 2 \cos \pi b Q$ , and  $Q = \frac{1}{2}(b^{-1} - b)$  we have:

$$n = -2 \cos \pi b^2, \quad b^2 \leq 1 \tag{2.34}$$

*Note that this argument is only valid for  $b^2 \neq 1$ , if  $b = 1$ , there are six marginal operators  $V_{\pm b^{-1}e_1}$ ,  $V_{\pm b^{-1}e_2}$  and  $V_{\pm b^{-1}\rho}$ . We recover the bosonic description of  $SU(3)_1$ , together with its 6 currents.*

A confirmation of this value of  $n$ , which does not assume that all loops have the same weight consists in going back to the relation between the FPL model and the Hecke algebra, detailed in the first section 2.1. The Hecke algebra, with  $\lambda = \pi - \pi b^2$  is associated with a CFT (with  $\mathcal{W}_3$  symmetry) of central charge  $c = 2 - 24 \frac{\lambda^2}{(\pi - \lambda)^2}$  [117]. Which gives back the relation 2.34. The weight of the contractible loops is “local”, in the sense that it does not depend on the genus of the surface. So even if the correspondence between the Hecke algebra and the loop model is not valid on curved surfaces, the relation 2.34 is true.

If for some reason the term  $e^{ib\phi} \rho$  happens to be absent or forbidden, the next leading term in the potential  $V(\phi)$  would be of the form  $\mu_1 e^{ib^{-1}\phi \cdot e_1} + \mu_2 e^{ib^{-1}\phi \cdot e_2}$ , and the value of  $Q$  matching a conformal field theory would be modified. This theory is the Toda field theory associated with  $\mathfrak{sl}(3)$  and will be described in much more details in the next chapter.

### 2.2.3 The FPL model on the cylinder: non-contractible loops

On a compact surface of genus 0, all closed loops are in the same homotopy class except if one marks two points on the surface, say  $x_0$  and  $y_0$ . Then, those closed loops which separates the points  $x_0$  and  $y_0$  become non-contractible, and we want to assign them a weight  $n_1 = 2 \cos(\pi e_1)$ . One may think of changing the amplitude of the curvature term in the action (2.33), but this would modify the loop fugacities as soon as loops enclose a non-zero curvature, so this procedure is incorrect for generic surfaces of genus zero.

Rather, we keep the action (2.33), and insert the vertex operators  $V_{Q-\beta}(y_0)$  and  $V_{Q+\beta}(x_0)$ , with  $\beta_0 = \frac{e_1}{2b} \rho$ . Both of these operators have conformal dimensions  $h = \bar{h} = (\beta_0^2 - Q^2)/2$ .

On the flat cylinder, the curvature is localized at both infinities:

$$R(x) = 4\pi (\delta(x, \infty) + \delta(x, -\infty)) ,$$

with  $x \in \mathbb{R}$  being the non-compact coordinate. Hence, on the cylinder, adding two operators at  $x_0 = +\infty$ ,  $y_0 = -\infty$ , is equivalent to modifying the coupling to the curvature.

$$\frac{Z_{\tilde{n}=n_1}}{Z_{\tilde{n}=n}} = \left\langle V_{\left(\frac{e_1}{2}-Q\right)\rho}(\infty) V_{\left(\frac{e_1}{2}-Q\right)\rho}(-\infty) \right\rangle_{\tilde{n}=n}$$

For the FPL model on the flat cylinder, changing the weight  $\tilde{n}$  of non-contractible loops can be interpreted in two ways. One can change the coupling to curvature  $Q$ , or one can add electric charges at infinity. While indistinguishable on the flat cylinder, these two theories become different when we deform the cylinder. In particular changing the coupling to curvature modifies the central charge  $c \rightarrow \tilde{c}$ , while inserting electric field does not : it changes the ground-state, and  $\tilde{c}$  is now an effective central-charge. When dealing with the FPL

theory on the flat cylinder, we will stick to the later interpretation. In particular the effective central charge is

$$c_{\text{eff}} = c - 24h_{Q-\beta_0} = 2 - 12\beta_0^2 = 2 - \frac{6(e_1/\pi^2)^2}{b^2}. \quad (2.35)$$

In this theory, the effective conformal dimensions of  $V_\alpha$  are found by shifting the vertex charge by  $Q - \beta_0$ :

$$h_\alpha^{\text{eff}} = \bar{h}_\alpha^{\text{eff}} = h_{\alpha+Q-\beta_0} - h_{Q-\beta_0} = \frac{1}{2}(\alpha^2 - 2\beta_0 \cdot \alpha). \quad (2.36)$$

A particularly interesting case to study, is the twist prescribed by the Hecke algebra and the Markov trace (see Sec. 2.1.4):

$$n = 2 \cos \lambda, \quad \tilde{n} = 2 \cos 2\lambda, \quad 0 < \lambda < \pi. \quad (2.37)$$

The effective theory will then have:

$$b = \sqrt{\frac{\pi - \lambda}{\pi}}, \quad \beta_0 = \left(\frac{1}{b} - b\right) \rho, \quad (2.38)$$

The central charge in this case reads:

$$c_{\text{eff}} = 2 - \frac{24(1 - b^2)^2}{b^2}. \quad (2.39)$$

This is the central charge of the  $\mathcal{W}_3$  theory, and we expect the spectrum to reflect this symmetry.

#### 2.2.4 The FPL model on the torus, modular invariance

The method that we used in the previous sections only works on simply connected surfaces. For example on the flat torus, the Ricci scalar is 0, and the weight of straight loops must be imposed differently. The partition function on the torus will give the complete spectrum of the theory. As usual, we parametrize the torus by a complex number  $\tau$ , with  $\text{Im}(\tau) > 0$  such that  $\mathbb{T} = \mathbb{C} / (\tau\mathbb{Z} + \mathbb{Z})$ . The partition function will be expressed as a function of  $q = e^{2i\pi\tau}$ .

##### 2.2.4.1 Partition function of the compact boson

The partition function of the loop model can be obtained starting from the partition function of a compact boson, following [49]. If  $\phi$  is a boson compactified on a given lattice  $2\pi\mathcal{R}$ , a configuration of the field will verify:

$$\begin{aligned} \phi(z+1, \bar{z}+1) &= \phi(z, \bar{z}) + 2\pi b q, \\ \phi(z+\tau, \bar{z}+\bar{\tau}) &= \phi(z, \bar{z}) + 2\pi b q', \end{aligned} \quad (2.40)$$

where  $q, q' \in \mathcal{R}^2$ . The calculation of the torus partition function in the sector  $(q, q')$  is a standard exercise. Separating the quantum and classical degrees of freedom of  $\phi$ , one finds

$$Z_{q,q'}(b, \tau) = \frac{1}{2b^2 \operatorname{Im} \tau |\eta(\tau)|^4} \exp\left(-\frac{\pi b^2}{2 \operatorname{Im} \tau} |q\tau - q'|^2\right). \quad (2.41)$$

The full partition function is expected to be of the form

$$Z(\tau) = \sum_{(q,q') \in \mathcal{R}^2} c_{q,q'} Z_{q,q'}(b, \tau), \quad (2.42)$$

where the  $c_{q,q'}$  are some constant coefficients. For the compact boson theory (2.26), all the sectors contribute with the same weight, and by convention  $c_{q,q'} = \operatorname{Vol}(\mathcal{R})$ , where  $\operatorname{Vol}(\mathcal{R}) = \sqrt{3}$  is the area of the unit cell of  $\mathcal{R}$ :

$$Z_c[b, \mathcal{R}, \tau] = \operatorname{Vol}(\mathcal{R}) \sum_{(q,q') \in \mathcal{R}^2} Z_{q,q'}(b, \tau). \quad (2.43)$$

The global normalization has been chosen to recover the usual behaviour  $Z \sim (q\bar{q})^{-c/24}$  as  $q \rightarrow 0$  of the full partition function. Indeed, using a Poisson summation to express the result in term of magnetic and electric charges yields

$$Z_c[b, \mathcal{R}, \tau] = \frac{1}{|\eta(\tau)|^4} \sum_{e \in \mathcal{R}^*} \sum_{q \in \mathcal{R}} q^{\delta(e,q)} \bar{q}^{\bar{\delta}(e,q)}. \quad (2.44)$$

By analogy with the  $\mathfrak{su}(2)$  case, we will call this model of compact boson the  $f$  model [115]. The weights  $\delta, \bar{\delta}$  correspond to the spectrum of the free compact boson:

$$\delta(e, q) = \frac{1}{2} \left( \frac{e}{b} - \frac{1}{2} b q \right)^2 \quad \bar{\delta}(e, q) = \frac{1}{2} \left( \frac{e}{b} + \frac{1}{2} b q \right)^2. \quad (2.45)$$

We also recall the duality relation

$$Z_c[b, \mathcal{R}, \tau] = Z_c[2/b, \mathcal{R}^*, \tau]. \quad (2.46)$$

#### 2.2.4.2 Partition function of the loop model

In the loop model the coefficients  $c_{q,q'}$  in (2.42) depend on the weight of the non-contractible loops, as was first established in the case of the  $O(n)$  model [49]. The same kind of arguments apply to the FPL model. Let  $\tilde{n} = 2 \cos 2\pi e_0$  be the weight of non-contractible loops, then the partition function on the torus is

$$\begin{aligned} Z_{\text{loop}}[b, e_0, \mathcal{R}] &= \operatorname{Vol}(\mathcal{R}) \\ &\times \sum_{(q,q') \in \mathcal{R}^2} \exp[2i\pi e_0(q \cdot \rho) \wedge (q' \cdot \rho)] Z_{q,q'}(b) \end{aligned} \quad (2.47)$$

where  $a \wedge b$  stands for the greatest common divisor (gcd) of  $a$  and  $b$  (note that for any  $q \in \mathcal{R}$ , one has  $q \cdot \rho \in \mathbb{Z}$ ). The spectrum can be extracted using Poisson summation and a careful treatment of the gcd factor, as was done for the  $O(n)$  model [49, 127]. The details of this derivation for the FPL model can be found in Appendix 2.A. The partition function takes the form

$$Z_{\text{loop}}[b, e_0, \mathcal{R}] = \frac{(q\bar{q})^{(2-c)/24}}{|\eta(\tau)|^4} \left[ \sum_{\substack{q \in \mathcal{R} \\ m=0}} \sum_{e \in \mathcal{R}^*} q^{h(e-e_0, \rho, q)} \bar{q}^{\bar{h}(e-e_0, \rho, q)} + \sum_{\substack{q \in \mathcal{R} \\ m \neq 0}} \sum_{k|m} \sum_{e \in \tilde{\mathcal{R}}_k^*} \Lambda(k, m, e_0) q^{h(e, q)} \bar{q}^{\bar{h}(e, q)} \right], \quad (2.48)$$

where we have introduced  $m = q \cdot \rho$  for compactness. The lattice  $\tilde{\mathcal{R}}_k^*$  is defined as

$$\tilde{\mathcal{R}}_k^* = \left\{ a\omega_1 + \frac{b\rho}{k}, \quad (a, b) \in \mathbb{Z}^2, \quad b \wedge k = 1 \right\}, \quad (2.49)$$

and the central charge and the exponents are

$$\begin{aligned} h(e, q) &= \delta(e, q) - \delta(e_0, 0), \\ \bar{h}(e, q) &= \bar{\delta}(e, q) - \bar{\delta}(e_0, 0), \\ c &= 2 - 24h(e_0, 0). \end{aligned} \quad (2.50)$$

The function  $\Lambda(k, m, e_0)$  can be found in Appendix 2.A.

The expression (2.48) can be used to extract the full spectrum of the theory. Indeed, for a generic module of conformal weights  $h$ , the character reads:

$$\text{Tr}_h \left[ q^{L_0 - c/24} \right] = \frac{q^{h+(2-c)/24}}{\eta(\tau)^2}. \quad (2.51)$$

### 2.2.5 Operators and spectrum of the loop model

We saw in the previous section that the bulk spectrum of the FPL model with fugacities (2.37) could be described by considering separately the sectors of given ‘‘magnetic charge’’  $q \in \mathcal{R}$ , where  $2\pi b q$  is the defect picked by the field  $\phi$  along the circumference of the cylinder.

In this section, we describe the spectrum of the theory, in the case  $\tilde{n} = 2 \cos(2\lambda)$  to take advantage of the  $U_q(\widehat{\mathfrak{sl}}_3)$  symmetry, and highlight the link with  $\mathcal{W}_3$ . Additionally, we adopt the notation, inspired by the  $\mathcal{W}_3$  symmetry (see next chapter 3):

$$\begin{aligned} \alpha \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} &= b^{-1} [(n_1 - 1)\omega_1 + (n_2 - 1)\omega_2] \\ &\quad - b [(m_1 - 1)\omega_1 + (m_2 - 1)\omega_2]. \end{aligned} \quad (2.52)$$

- In the “purely electric” sector, i.e. the sector for  $q = 0$ , non-contractible loops may occur. The primary operators are all scalar vertex operators  $V_\alpha$  allowed by the compactification condition  $\phi \equiv \phi + 2\pi b\mathcal{R}$ . For the vertex operator  $V_\alpha = :e^{i\alpha\cdot\phi}$ : to be single-valued, one needs to impose  $\alpha \in b^{-1}\mathcal{R}^*$ . We end up with vertex charges of the form

$$\alpha = b^{-1}(n_1\omega_1 + n_2\omega_2) = \alpha \begin{pmatrix} 1 + n_1 & 1 \\ 1 + n_2 & 1 \end{pmatrix}, \quad (n_1, n_2) \in \mathbb{Z}^2. \quad (2.53)$$

This “purely electric” sector, which includes the ground state, appears for lattices of width  $N$  multiple of three. Due to the conservation laws,  $N/3$  lines of empty edges propagate along the axis of the cylinder.

- The sector where an additional  $2k$  lines of empty edges propagate has a magnetic charge  $q = k(2h_2 - h_1 - h_3) = k(e_2 - e_1)$ . Note that  $(e_2 - e_1)\mathbb{Z}$  is the set of vectors  $q \in \mathcal{R}$  such that  $q \cdot \rho = 0$ . Like in the  $q = 0$  sector, non-contractible loops are allowed. The magnetic charge  $q = k(e_2 - e_1)$  may be combined with a vertex operator of charge  $\beta \in b^{-1}\mathcal{R}^*$ . The corresponding eigenvalues of  $(L_0, \bar{L}_0)$  may be computed from the Gaussian action by standard methods, and one gets

$$h = h_\alpha, \quad \bar{h} = h_{\bar{\alpha}}, \quad (2.54)$$

where

$$\begin{aligned} \alpha &= \alpha \begin{pmatrix} 1 + n_1 & 1 - \frac{3k}{2} \\ 1 + n_2 & 1 + \frac{3k}{2} \end{pmatrix} \\ \bar{\alpha} &= \alpha \begin{pmatrix} 1 + n_1 & 1 + \frac{3k}{2} \\ 1 + n_2 & 1 - \frac{3k}{2} \end{pmatrix} \end{aligned} \quad (n_1, n_2) \in \mathbb{Z}^2. \quad (2.55)$$

The conformal spin is

$$s = h_\alpha - h_{\bar{\alpha}} = \frac{1}{2}(\alpha + \bar{\alpha} - 4Q) \cdot (\alpha - \bar{\alpha}), \quad (2.56)$$

which yields, for the above values of the vertex charges,  $s = 3k(n_1 - n_2)$ . Hence, this sector contains an infinity of scalar operators, including the most relevant one, obtained by setting  $n_1 = n_2 = 0$ .

- Any sector with one or more loop strands propagating has a magnetic charge  $q$  with  $q \cdot \rho \neq 0$ . For instance, the combination of  $2\ell$  strands and  $2k$  extra empty lines gives a magnetic charge  $q = \ell(e_1 + e_2) + k(e_2 - e_1)$ . A generic charge  $q \in \mathcal{R}$  satisfying  $q \cdot \rho \neq 0$  can be written:

$$q = q_1 e_1 + q_2 e_2, \quad \text{where } (q_1, q_2) \in \mathbb{Z}^2, \text{ and } q_1 + q_2 \neq 0. \quad (2.57)$$

In this case, because of the non-intersecting nature of loops, there are no non-contractible loops. Moreover, the oriented loop strands may wind around the cylinder, which produces unwanted factors due to the vertex charges at  $\pm\infty$ . Hence, the magnetic defects in these sectors must be combined with “electric charges”  $\pm Q$  to compensate this effect. One ends up with eigenvalues (2.54), parametrized by the charges:

$$\begin{aligned} \alpha &= \alpha \begin{pmatrix} n_1 & q_1 - \frac{q_2}{2} \\ n_2 & q_2 - \frac{q_1}{2} \end{pmatrix}, \\ \bar{\alpha} &= \alpha \begin{pmatrix} n_1 & -q_1 + \frac{q_2}{2} \\ n_2 & -q_2 + \frac{q_1}{2} \end{pmatrix}, \end{aligned} \quad (n_1\omega_1 + n_2\omega_2) \in \mathcal{R}_q^*. \quad (2.58)$$

In this expression,  $\mathcal{R}_q = \mathbb{Z}q + \mathbb{Z}(e_1 - e_2)$  is the lattice of allowed defects along the axis of the cylinder, and  $\mathcal{R}_q^*$  is its reciprocal lattice:

$$\mathcal{R}_q^* = \left\{ n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \frac{\mathbb{Z}}{q_1 + q_2}, n_1 + n_2 \in \mathbb{Z} \right\}. \quad (2.59)$$

This includes operators with rational, non-integer Kac indices  $n_1$  and  $n_2$  in (2.58). However, the conformal spin remains an integer:

$$s = -(n_1 + n_2)(q_1 + q_2) - (n_1 - n_2)(q_1 - q_2). \quad (2.60)$$

### 2.2.6 Generalized electric three-point functions

Let us go back to the untwisted,  $n = \tilde{n}$ , FPL model. On the cylinder, we saw that modifying the weight of loops separating  $x_1$  and  $x_2$  from  $n$  to  $n_1$  was equivalent to the insertion of two operators of charges:

$$\alpha_1 = Q + \frac{e_1\rho}{2b}.$$

where  $e_1$  is defined by  $n_1 = 2 \cos(\pi e_1)$ .

Similarly to the results obtained in [86], it is possible to generalize this result to three points functions.

The partition function  $Z_{n_1, n_2, n_3}(x_1, x_2, x_3) \equiv Z_{123}$  is defined as the partition function of the loop model, on the plane, where loops separating  $x_i$  from the two other points have a weight  $n_i$  (related to the vertex charge  $\alpha_i$ ); while other types of loops keep their weight  $n$  (see figure 2.6). We use the abbreviation  $Z_{ijk} \equiv Z_{n_i, n_j, n_k}$ , with  $i, j, k \in \{0, 1, 2, 3\}$  and  $n_0 = n$ .

These partition functions can be interpreted as (non-normalized) correlation functions of three operators. As the FPL model flow toward a Liouville field theory (in the direction of  $\rho$ ) we expect to find, like in the paper [86] where the  $\mathcal{O}(n)$  was discussed, that the modified partition functions  $Z_{123}$  behave, with the correct normalisation,



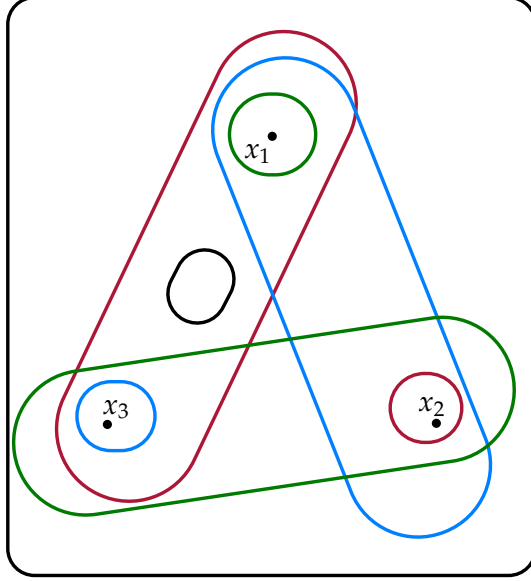


Figure 2.6: The four possible types of loops. In black, contractible loops, of weight  $n$ , red, blue and green loops have respectively weights  $n_1$ ,  $n_2$  and  $n_3$ .

like Liouville structure constant  $C_L$  defined in the introduction 1.20. To make this relation exact, the partition function  $Z_{123}$  needs to be normalized by “two-point functions” of the form  $Z_{0ii}$ ,  $Z_{i0i}$  and  $Z_{ii0}$ :

$$C_L \left( \alpha_1 \cdot \frac{\rho}{2}, \alpha_2 \cdot \frac{\rho}{2}, \alpha_3 \cdot \frac{\rho}{2} \right) = \frac{Z_{123}}{Z_{000}} \sqrt{\frac{Z_{011}Z_{000}}{Z_{101}Z_{110}} \frac{Z_{202}Z_{000}}{Z_{220}Z_{022}} \frac{Z_{330}Z_{000}}{Z_{033}Z_{303}}}. \quad (2.61)$$

Where  $C_L$  are the structure constants of the Liouville conformal theory, defined in the previous chapter 1.20. Explicitly:

$$C_L(a_1, a_2, a_3) = \frac{Y_b(b - 2Q + a_1 + a_2 + a_3) \prod_{i=1}^3 Y_b(b + a_1 + a_2 + a_3 - 2a_i)}{\sqrt{Y_b(b)^3 Y_b(b - 2Q)^{-1} \prod_{i=1}^3 Y_b(b + 2a_i) Y_b(b - 2Q + 2a_i)}}.$$

Where  $Y_b$  is defined like in the introduction (1.2.4).

It seems tempting to explain this behaviour by going back to the height model, and modelling the weight change by the insertion of vertex operators. This is the method we employed for two insertions, and it is at the heart of the Coulomb-gas methods applied to loop models. But, three weights operators of the form  $e^{i\alpha_i \cdot \phi}$  can only be inserted consistently if the sum of their charges,  $\sum \alpha_i$  is equal to 0. This leaves the freedom of fixing two of the loops weights, but not the third one. The operators changing the weights of the loops are non-local, which marks the difference between the loop model and the vertex/height models.

On the cylinder, with two points sent to the two extremities  $\pm\infty$ , the expression 2.61 simplifies to:

$$C_L(a_1, a_2, a_3) = \frac{Z_{123}^{\text{cyl}}}{Z_{202}^{\text{cyl}}} \sqrt{\frac{Z_{202}^{\text{cyl}} Z_{000}^{\text{cyl}}}{Z_{101}^{\text{cyl}} Z_{303}^{\text{cyl}}}}, \quad (2.62)$$

$$a_i = Q + \frac{e_i}{2b} \quad n_i = 2 \cos(\pi e_i).$$

Where  $Z_{ijk}^{\text{cyl}}$  is the partition function, on the cylinder, with loop weights  $n_i$  (resp.  $n_k$ ) for non-contractible loops below (resp. above) 0,  $n_j$  for loops around 0 and  $n = n_0$  elsewhere.

While the spectrum of the two theories are markedly different, these ratios of partition function verify the same formula as in the TL and  $\mathcal{O}(n)$  model (see [86] for these two cases). We show numerically that this is also true for the (non-integrable) fully packed loop model on the square lattice (see Figure 2.22).

The empty lines ( $h_2$ ) do not form loops. Hence, like on the vertex model, generic bulk operators cannot be defined. But it is possible to insert a “twist line” along the cylinder, corresponding to the insertion of two operators of the form  $e^{\pm iah_2 \cdot \phi}$  at the two extremities of the surface. In the loop model, this twist line adds a weight  $e^{i\mu}$  (resp.  $e^{-i\mu}$ ) each time one empty line crosses the line from left to right (resp. right to left). This modification only affects the free boson part, and it leaves the ratio of partition functions invariant.

## 2.3 NUMERICAL STUDY

### 2.3.1 Hamiltonian, Hilbert space and quantum numbers

In order to check the previous results we use exact diagonalisation methods on systems ranging from  $N = 6$  sites to  $N = 18$ . For simplicity, we restrict ourselves to the case when  $N$  is multiple of 3. We consider the Hamiltonian of the loop model transfer matrix with periodic boundary conditions:

$$H_N \propto \left. \frac{d \log t_N(u)}{du} \right|_{u=0} = - \sum_{j=1}^N U_j, \quad (2.63)$$

where  $U_j$  is given by (2.21), and we restrict specifically to the case  $\tilde{n} = n^2 - 2$ . The Hilbert space for the Hamiltonian is generated by non-intersecting link patterns which allow vacancies: see next section.

In the scaling limit, one expects from conformal invariance the following form of the energy and momentum:

$$E_N \sim N e_{\text{bulk}} + \frac{2\pi v_f}{N} \left( h + \bar{h} - \frac{c}{12} \right), \quad (2.64)$$

$$P_N \sim \text{const} + \frac{2\pi}{N} (h - \bar{h}), \quad (2.65)$$

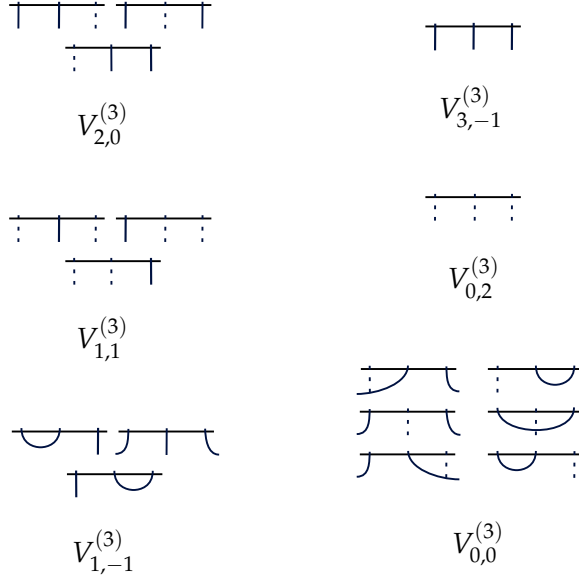


Figure 2.7: The sectors  $V_{d,v}^{(N)}$  of the loop model for  $N = 3$ .

where  $e_{\text{bulk}}$  is the non-universal bulk energy density. The Fermi velocity  $v_f$  in (2.64) may be inferred from the expression of the embedding angle  $\theta = \frac{4}{3}\pi u/\lambda$  (see Sec. 2.1.1):

$$v_f = \frac{2\pi \sin \lambda}{3\lambda}. \quad (2.66)$$

By analogy with the standard modules in the representation theory of the (periodic) TL algebra, we consider the representation of the Hamiltonian  $H_N$  on vector spaces generated by link patterns. In this work, we are not treating mathematically the representation theory of the periodic version of the  $\mathfrak{sl}(3)$  Hecke algebra. Rather, we choose empirically a family of representations (which we call *loop sectors*), and show numerically that the associated conformal weights match the predictions from the Coulomb-Gas approach.

We fix the system size  $N$ , multiple of 3. For any  $d \in \{0, 1, \dots, N\}$  and  $v \in \{-\frac{N}{3}, -\frac{N}{3} + 1, \dots, \frac{2N}{3} - d\}$  such that  $d \equiv v \pmod{2}$ , let  $V_{d,v}^{(N)}$  be the vector space generated by all link patterns with  $d$  strands connected to infinity, and  $(N/3 + v)$  vacant sites: see Fig. 2.7 The action of the generators  $U_j$  on  $V_{d,v}^{(N)}$  is analogous to that of the periodic Temperley-Lieb on its standard modules: each plaquette of (2.21) (considered as acting from SW to NE), evolves the link pattern according to the graphical prescription, and introduces a factor  $n$  or  $\tilde{n}$  for every closed contractible or non-contractible loop. Note that two strands connected at infinity cannot get contracted under the action of the  $U_j$ 's.

In the continuum limit, i.e. in the two-component boson theory of Sec. 2.2.3, we expect the low-energy part of  $V_{d,v}^{(N)}$  to be described by the sectors of magnetic charges  $\mathbf{q}$  which correspond to  $d$  propagating strands and  $N/3 + v$  vacancies. For instance, for  $d$  and  $v$  even, the

lowest-energy state of  $V_{d,v}^{(N)}$  has magnetic charge  $q = \frac{1}{2}[d(\mathbf{e}_1 + \mathbf{e}_2) + v(\mathbf{e}_2 - \mathbf{e}_1)]$  (see Sec. 2.2.5). The ground state is located in the sector  $V_{0,0}^{(N)}$ .

Translation invariance is also respected on the cylinder, the different sectors can be further decomposed along momentum sectors, defined by the eigenspaces of the translation operator. A non-zero conformal spin ( $h \neq \bar{h}$ ) in the continuum limit will correspond to a non-zero momenta on the lattice, however an additional lattice momentum, independent of the conformal spin can appear.

### 2.3.2 $\mathcal{W}_3$ symmetry of the FPL model at $\tilde{n} = 2 \cos(2\lambda)$

When the non-contractible loop of the FPL model on the cylinder have weight  $\tilde{n} = 2 \cos(2\lambda)$ , then the cylinder partition function of the model coincides with models known for their  $U_q(\mathfrak{sl}(3))$  symmetries. As a priori, the FPL model does not have this symmetry, this value of  $\tilde{n}$  is particularly intriguing. Plotting the first weights as a function of the momentum (for every sector) shows well the underlying  $\mathbb{Z}_3$  structure of the model (see figure 2.9). In this section we focus on both confirming numerically the result of the previous sections, and identifying the trace of the  $\mathcal{W}_3$  symmetry appearing in the spectrum.

We compute the energies of  $H_N$  in various sub-sectors of fixed momentum within the sectors  $V_{d,v}^{(N)}$ , and extract the scaling dimensions ( $h + \bar{h}$ ) using (2.64). We extrapolate the data from system sizes  $L = 6 \dots 18$  using Shank's method (see [11]). The sizes obtainable are limited by the memory needed by the Hilbert space (of size  $\sim 10^6$  for  $L = 18$ ). The noticeable results we have obtained are:

- The expression (2.39) for the central charge is confirmed.
- The predictions from Sec. 2.2.5 on the scaling dimensions of primary operators in various sectors are confirmed. Fractional Kac indices, as predicted by 2.59 appear for non-zero magnetic charges. Contrary to the  $O(n)$  case, they can appear even in sectors with zero conformal spin.
- In addition to Virasoro descendants, we identify additional states, whose conformal dimensions match with descendants under the modes  $W_{n<0}$  of the  $W$  operator, the current associated with the  $\mathcal{W}_3$  symmetry.
- Unlike in the  $O(n)$  model, some electro-magnetic excitations [see (2.55)] can have vanishing conformal spin, even though their electric and magnetic charges are non-zero.
- The finite-size effects can be quite important especially near  $c = 2$ , due to the increased presence of logarithmic corrections, or when different states cross. Far from the free boson point ( $n =$

2), descendant states are extremely sensible to finite-size effects compared to primaries.

- Similarly to the  $O(n)$  model [61], [118], the descendants  $L_{-1}|0\rangle$  and  $W_{-1}|0\rangle$ , although they have zero norm, do appear in the spectrum of the lattice Hamiltonian  $H_N$ . This suggests the existence of logarithmic CFT features of the FPL model for any value of  $c$ .
- States with no conformal spin in the continuum limit can appear at momenta  $0, \frac{N}{3}$  and  $\frac{2N}{3}$ . This decoupling seems related to the charges of the vertex in the continuum limit, if  $e(n_1, n_2) = n_1\omega_1 + n_2\omega_2$  the charge  $n_1 + n_2$  [3] determine the sector in the example we observed. If  $n_1 + n_2$  is a multiple of three the state appear in the zero momentum sector, otherwise it appears in the  $\frac{2\pi}{3} / \frac{4\pi}{3}$  momentum sector.
- Descendants stay in the same “sector” as the primary (meaning for example that if a field  $\Phi$  is in a sector of momentum  $p$ , its descendant  $L_{-1}\Phi$  will be in a sector with momentum  $p + 2\pi/N$ ).

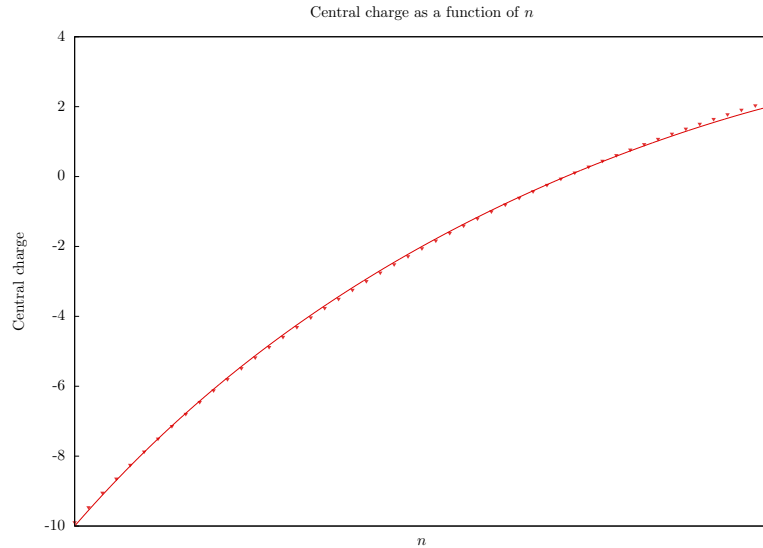


Figure 2.8: Central charge of the FPL model for  $\tilde{n} = n^2 - 2$ , as a function of  $n$ . The numerical data for sizes  $N = 6, 9, 12, 15, 18$  are compared to the analytic prediction (2.39).

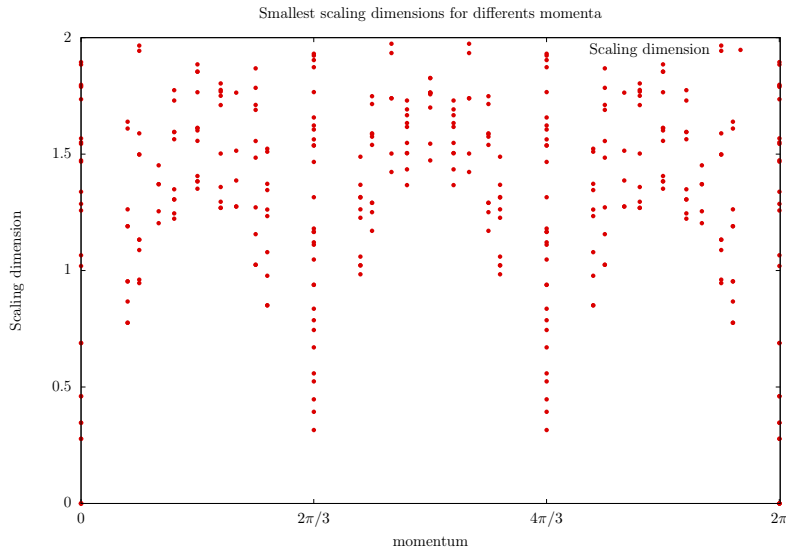


Figure 2.9: The smallest scaling dimensions of the Hamiltonian as a function of the momentum, for  $n = 1.7$ . Multiple size are represented, for  $N = 6, 9, 12, 15$

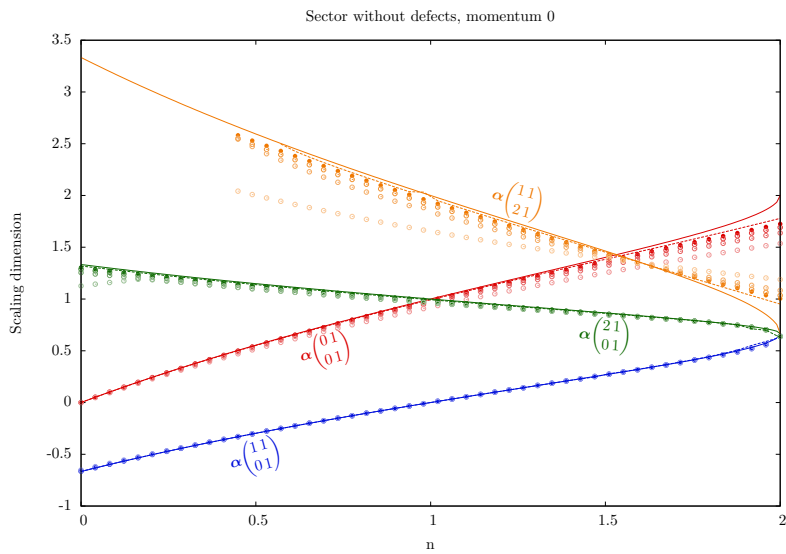


Figure 2.10: The purely electric sector,  $V_{0,0}^{(N)}$ , momentum  $P = 0$ . Dots are numerical results, darker dots corresponds to larger size, from 6 to 15, dotted lines their Shank's transform and the full lines represent the theory. The labels on the curve represent vertex charges and correspond to (2.53).

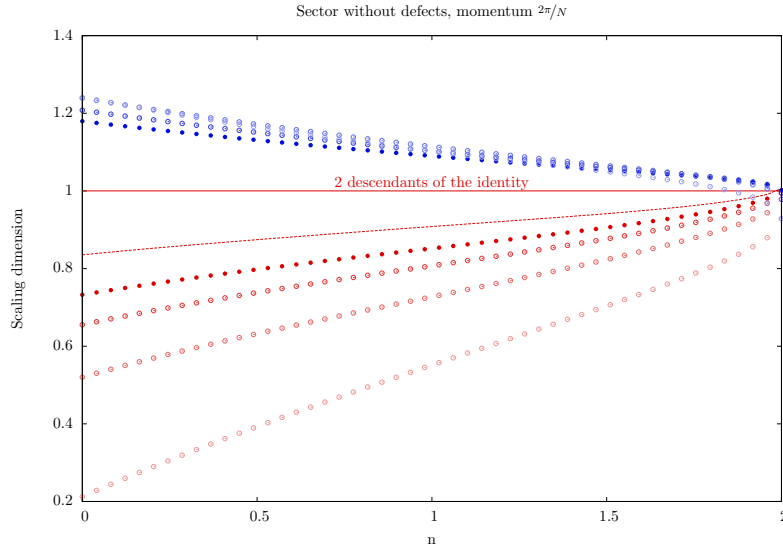


Figure 2.11: The purely electric sector,  $V_{0,0}^{(N)}$ , momentum  $P = 2\pi/N$ . Dots are numerical results (lighter  $N = 6$  up to darker  $N = 15$  in step of 3), dotted lines their Shank's transform and the full lines give the theory. While the convergence for descendant states is pretty bad, their position in the spectrum and their momentum label those two states as  $L_{-1}|0\rangle$  and  $W_{-1}|0\rangle$ . The Shank's transform of the blue descendant do not converge well due to eigenvalues crossing and is not shown here.

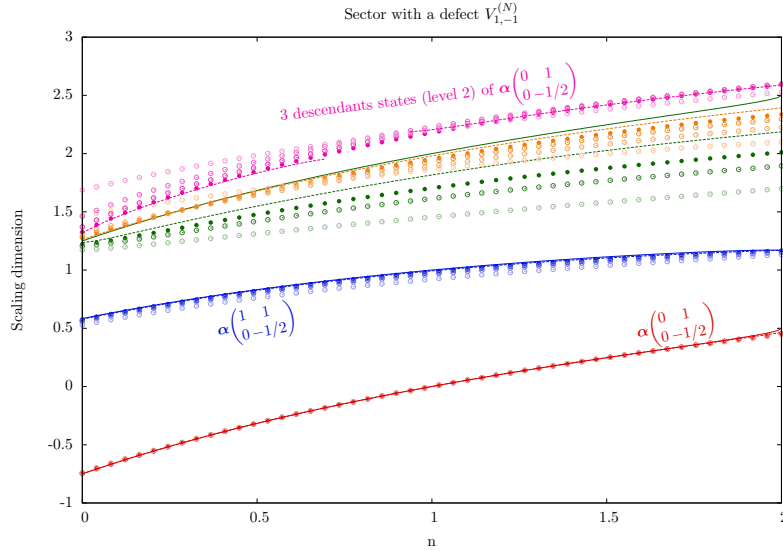


Figure 2.12: Sector  $V_{1,-1}^{(N)}$ , momenta  $P = 0$  and  $P = 2\pi/3$ , same conventions as in Fig. 2.10. The label on the curve have to be understood as the holomorphic part of a defect plus an electric charge, corresponding to the expression (2.58) with  $q_1 = 1$  and  $q_2 = 0$ . As per (2.59), no fractional electric charge appears in this sector.

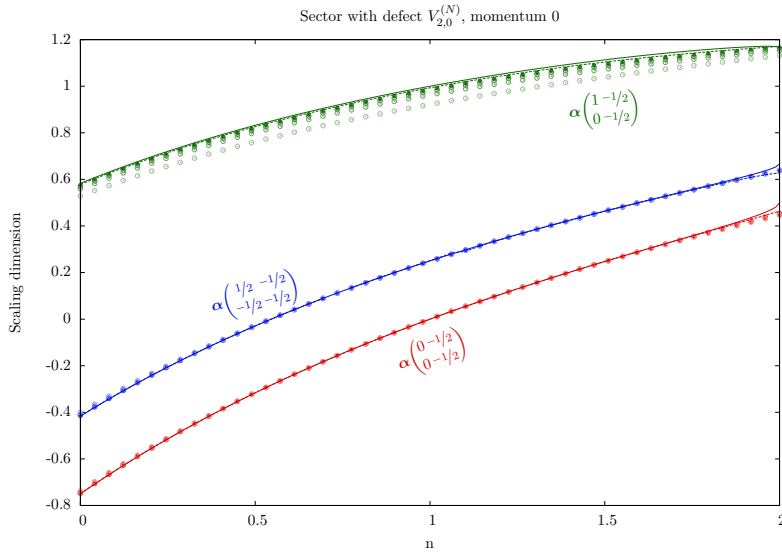


Figure 2.13: Sector  $V_{2,0}^{(N)}$  (two strands) momentum 0, same conventions as Fig. 2.10. Again the label have to be understood through 2.58 with  $q_1 = q_2 = 1$ . This allows for half-integer Kac indices  $n_1$  and  $n_2$ .

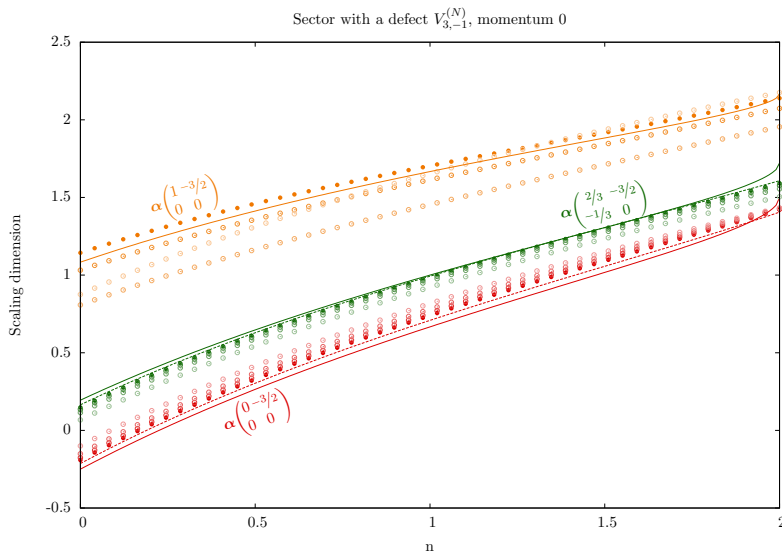


Figure 2.14: Sector  $V_{3,-1}^{(N)}$  (three strands added and one hole removed), momentum 0, same conventions as Fig. 2.10. The labels correspond to (2.58) with  $q_1 = 2$  and  $q_2 = 1$ . This time electric charges live in  $\mathbb{Z}/3$ .



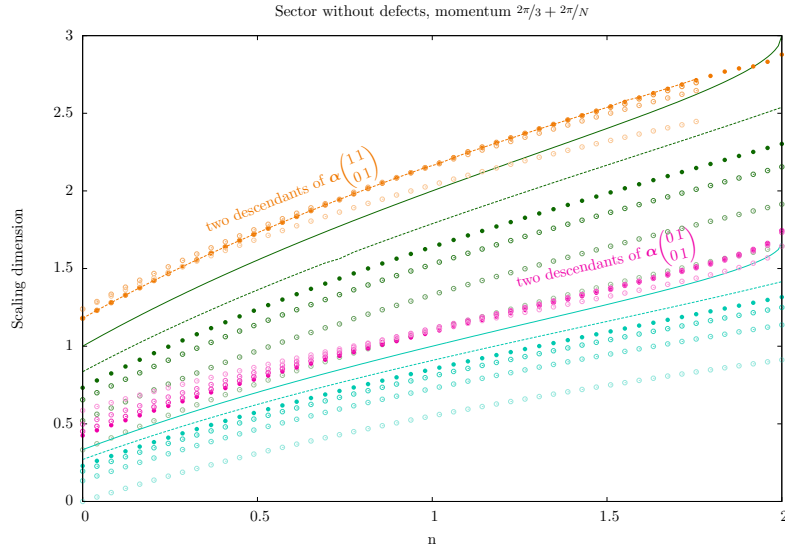


Figure 2.15: The purely electric sector,  $V_{0,0}^{(N)}$ , momentum  $P = 2\pi/3 + 2\pi/N$ , same conventions as Fig. 2.10. Again, descendants states tend to converge badly, but their degeneracy is consistent with the theory.

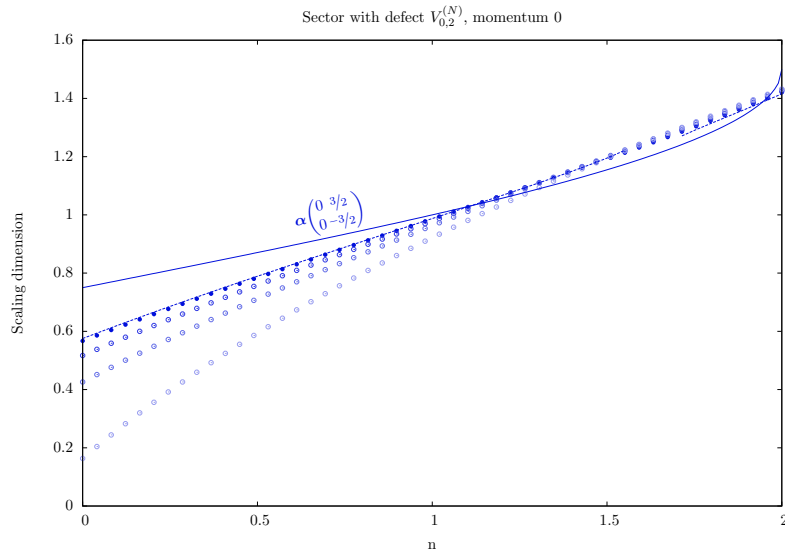


Figure 2.16: Two hole sector  $V_{0,2}^{(N)}$ , same conventions as Fig. 2.10. It is a magnetic sector with charge  $q$  perpendicular to  $\rho$ , hence the vertex charges correspond to the case (2.55). In the case  $n = \tilde{n} = 2$ , the model enjoys an additional  $\mathbb{Z}_3$  symmetry (cyclic permutations of the colours – see Fig. 2.4), and the sectors  $V_{0,2}^{(N)}$  and  $V_{3,-1}^{(N)}$  become isomorphic.

### 2.3.3 Three-point functions and Liouville theory

In this section we come back to the original FPL model,  $n = \tilde{n}$ . While the spectrum of the twisted model on the cylinder does show sign of a  $\mathcal{W}_3$  symmetry, however the analysis done in section 2.2 invalidates

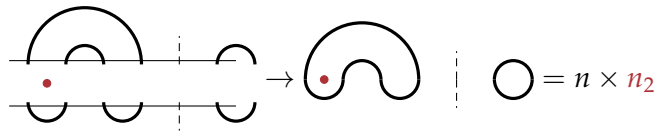
this hypothesis. This motivates us to look at the behaviour of three-point functions.

We compute the “modified partition function” defined in the section 2.2.6:  $Z_{n_1, n_2, n_3}^{\text{cyl}}$ . On the cylinder the loops can have four different fugacities:

- $n_1$  if the loop is non-contractible and passes above 0,
- $n_2$  if the loop is contractible and separates 0 from the two extremities of the cylinder,
- $n_3$  if the loop is non-contractible and passes below 0,
- $n$  otherwise.

To obtain this partition function, we compute the transfer matrix in the sector where the contractible loops have fugacity  $\tilde{n} = n_i$ ,  $n_i = n_1, n_3$ . The eigenvector associated with the highest eigenvalue of the transfer matrix,  $|n_i\rangle$ , is then kept.

The (fully packed) loop scalar product is defined on the loop model Hilbert space. In principle, it is similar to the Temperley-Lieb scalar product. Starting with two half-loop states, we give a weight corresponding to the number of loops they create when associated. If the empty edges do not match, the scalar product is zero. To insert an operator  $\mathcal{O}_2$ , we modify, in the scalar product, the weight of loops around the vertex at position 0. For example, in size  $L = 7$ :



Like before, the computations are made for cylinders with perimeter sizes multiple of three,  $L = 6, 9, 12, 15$ . The results are compared to the theoretical values computed in 2.2.6. In figure 2.17, 2.18 and 2.19, operators of dimension 0 are considered. Their non-triviality is made obvious:  $\langle n_1 | \mathcal{O}_{h=0} | n_3 \rangle$  is non-zero. In Figure 2.22, we compare different types of loop models. We use the definitions of [143] for FPL square lattice, TL and diluted  $\mathcal{O}(n)$ .

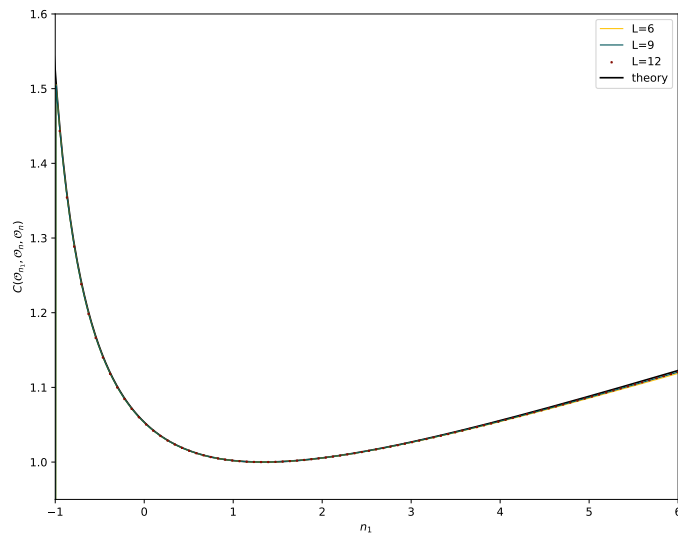


Figure 2.17: Correlation function  $C(n_1, n, n)$  as a function of  $n_1$ . Only one operator gives weight  $n_1$  to the loops at one end of the cylinder. The weight of contractible loops is fixed, equal to  $n = 2 \cos(0.84)$ .

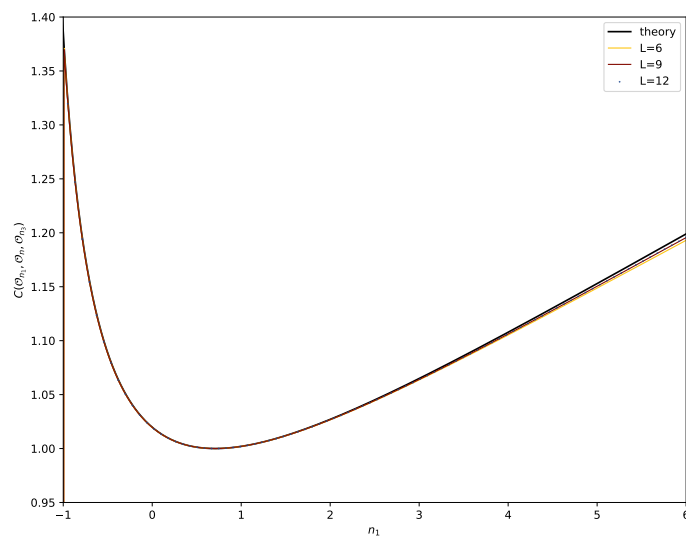


Figure 2.18: Correlation function  $C(n_1, n, n_3)$  as a function of  $n_1$ . Two operators modify the weight of the loops at both ends of the cylinder. The weight of contractible loops is fixed, equal to  $n = 2 \cos(0.84)$  and the weight of the non-contractible loops below 0 is equal to  $n_3 = 2 \cos(1.21)$  (those above 0 get weight  $n_1$ ).

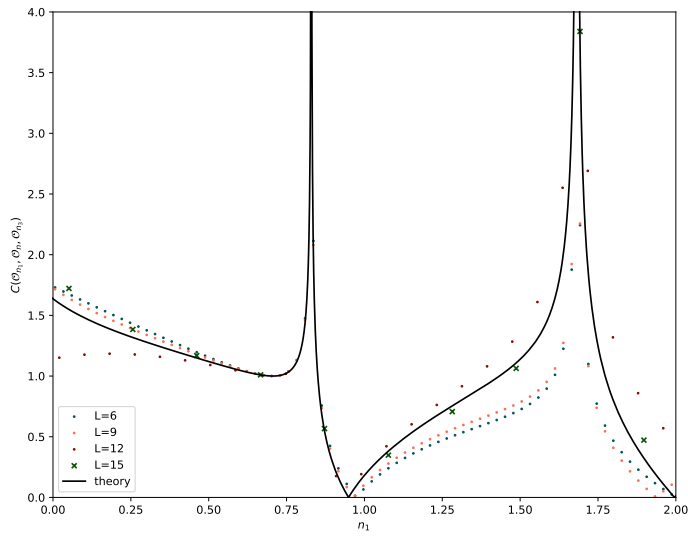


Figure 2.19: Correlation function  $C(n_1, n_1, n_3)$  as a function of  $n_1$ . Two operators modifying the weight of the loops at both ends of the cylinder. The weight of contractible loops is fixed, equal to  $n = 2 \cos(2.57)$  and the weight of the non-contractible loops below 0 is equal to  $n_3 = 2 \cos(1.21)$  (those above 0 get weight  $n_1$ ).

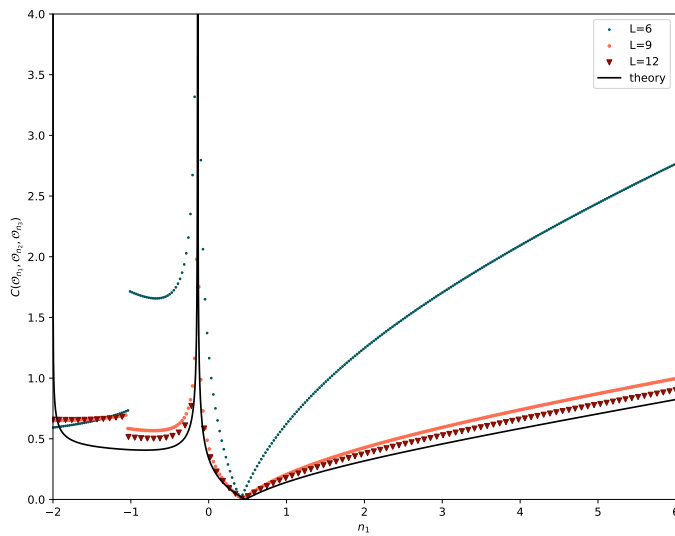


Figure 2.20: Correlation function  $C(n_1, n_2, n_3)$  as a function of  $n_1$ . Three operators modify the weight of the loops at both ends of the cylinder and around 0. The weight of contractible loops is fixed, equal to  $n = 2 \cos(1.5)$ , except for loops around 0 which get the weight  $n_2 = 2 \cos(2.24)$  and the weight of the non-contractible loops below 0 (resp. above 0) is equal to  $n_3 = 2 \cos(1.21)$  (resp.  $n_1$ ).

*Note that in this figure 2.21,  $n$  goes beyond the points where the model is "critical"  $n \in [-2, 2]$ . The relation 2.61 is still a match (with a complex value of  $b$ ). This fact, already remarked for the  $\mathcal{O}(n)$  model [86], seems to indicate that a part of the structure of the CFT is conserved for the (integrable) FPL model with  $n$  larger than 2.*

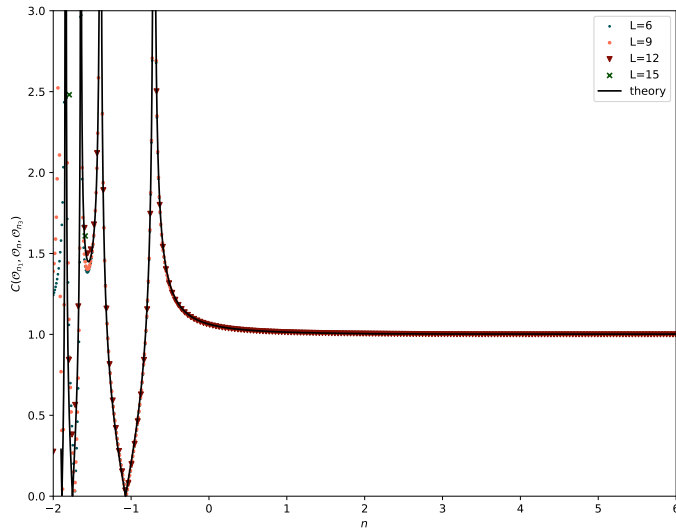


Figure 2.21: Correlation function  $C(n_1, n, n_3)$  as a function of  $n$  this time. Three operators modify the weight of the loops at both ends of the cylinder and around 0. The weight of the non-contractible loops below 0 (resp. above 0) is equal to  $n_3 = 2 \cos(1.21)$  (resp.  $n_1 = 2 \cos(0.8)$ ).

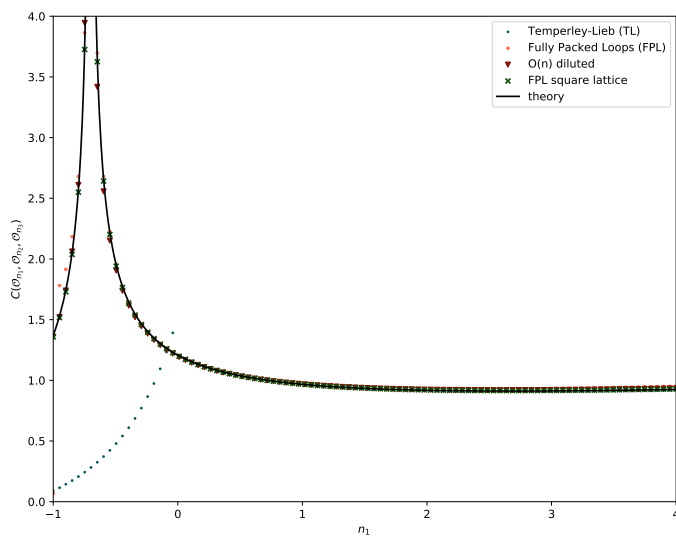


Figure 2.22: Correlation function  $C(n_1, n, n)$  as a function of  $n_1$  for different models. Three operators modify the weight of the loops at both ends of the cylinder and around 0. The weight of contractible loops is fixed, equal to  $n = 0.7$ , except for loops around 0 which get the weight  $n_2 = 2 \cos(\pi \cdot 0.12)$  and the weight of the non-contractible loops below (resp. above) 0 is equal to  $n_3 = 2 \cos(\pi \cdot 0.22)$  (resp.  $n_1$ ). All models are defined on a cylinder with size  $L = 12$ .

## 2.4 CONCLUSION

In this chapter we considered the FPL model, we gave a particular emphasis to the case  $\tilde{n} = n^2 - 2$  where an extended  $W_3$  symmetry is expected in the continuum.

Through the exact calculation of the partition function on the torus in the continuum limit, the full spectrum was obtained for generic values of  $n$  and  $\tilde{n}$ . These results are consistent with the Coulomb gas approach. In the case  $\tilde{n} = n^2 - 2$ , the spectrum has been checked numerically by an exact diagonalisation procedure, and an excellent agreement was obtained.

On the cylinder, the integrable structure underlying the FPL model is closely related to a  $\mathfrak{su}(3)$  quotient of the Hecke algebra. We showed that this symmetry was not a feature of the FPL model in general, and that, in its continuum limit, the model could be mapped to the product of a Liouville CFT with a free boson, compactified on a two-dimensional lattice with a symmetry of order three. This hypothesis was confirmed through the numerical computations of ratio of partition functions, which match the three-point correlation functions of Liouville field theory.

Experience in the  $O(n)$  model suggests that the FPL model has a rich logarithmic CFT structure in the continuum limit, and understanding the Operator Product Expansions of this theory remains a challenge. The structure constants associated with magnetic operators notably are, like in the  $O(n)$  case, still unknown.



## APPENDICES

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### 2.A RESUMMATION OF THE LOOP PARTITION FUNCTION

In this appendix we give the derivation of (2.48). The goal is to resum the expression:

$$\begin{aligned} Z_{\text{loop}} [b, e_0, \mathcal{R}] &= \text{Vol}(\mathcal{R}) \\ &\times \sum_{(q, q') \in \mathcal{R}^2} \exp [2i\pi e_0 (q \cdot \rho) \wedge (q' \cdot \rho)] Z_{q, q'}(b) \end{aligned} \quad (2.67)$$

by Poisson summation and a careful treatment of the gcd factor, as was done for the  $O(n)$  model [49, 127].

#### 2.A.1 Zero-strand sectors

Let us start with the sectors where  $q \cdot \rho = 0$  (including the purely electric sector where  $q = 0$ ). In this case the gcd boils down to  $(q \cdot \rho) \wedge (q' \cdot \rho) = |q' \cdot \rho|$  and the sum over  $q'$  can be Poisson transformed directly. For fixed  $q$  one has

$$\begin{aligned} &\text{Vol}(\mathcal{R}) \times \sum_{q' \in \mathcal{R}} Z_{q, q'}(b) \exp(2i\pi e_0 q' \cdot \rho) \\ &= \sqrt{3} \sum_{q' \in \mathcal{R}} Z_{q, q'}(b) \exp(2i\pi e_0 q' \cdot \rho) \\ &= \frac{b^2 \sqrt{3}}{2 \text{Im } \tau |\eta(\tau)|^4} \sum_{q' \in \mathcal{R}} \exp \left( -\frac{\pi b^2}{2 \text{Im } \tau} |q\tau - q'|^2 - 2i\pi e_0 q' \cdot \rho \right) \\ &= \frac{1}{|\eta(\tau)|^4} \sum_{e \in \mathcal{R}^*} q^{\delta(e - e_0 \rho, q)} \bar{q}^{\bar{\delta}(e - e_0 \rho, q)}. \end{aligned}$$

#### 2.A.2 Other magnetic sectors

In the magnetic sectors where  $q \cdot \rho \neq 0$ , things are a bit more complicated. Following [127], we define the function  $f_q : \mathbb{N} \rightarrow \mathbb{R}$  as

$$f_q(d) = \text{Vol}(\mathcal{R}) \sum_{q' \in \mathcal{R}: m \wedge m' = d} Z_{q, q'}(b), \quad (2.68)$$

where the integer  $m$  is defined as  $m = q \cdot \rho$  (and  $m' = q' \cdot \rho$ ). Note that  $f_q(d) = 0$  unless  $d|m$  ( $d$  is a divisor of  $m$ ). We have:

$$Z_{\text{loop}} [b, e_0, \mathcal{R}] = \sum_{q \in \mathcal{R}} \sum_{d > 0, d|m} \cos(2\pi e_0 d) f_q(d). \quad (2.69)$$



We introduce an auxiliary function  $g_q(d)$  defined as

$$g_q(d) = \sum_{d' > 0: d|d', d'|m} f_q(d'). \quad (2.70)$$

Notice that this sum is finite since  $m \neq 0$ , and that  $f_q(d)$  can be recovered using a variant of the Möbius inversion formula

$$f_q(d) = \sum_{d' > 0: d|d', d'|m} g_q(d') \mu(d'/d), \quad (2.71)$$

where  $\mu$  is the Möbius function. The quantity  $f_q(d)$  is not easy to calculate directly, because the sum is not over a regular lattice. The auxiliary  $g_q(d)$  on the other hand is better behaved. Indeed,

$$g_q(d) = \text{Vol}(\mathcal{R}) \sum_{q' \in \mathcal{R}: d|m'} Z_{q,q'} = \text{Vol}(\mathcal{R}) \sum_{q' \in \mathcal{R}_d} Z_{q,q'},$$

$$\mathcal{R}_d = \mathbb{Z}d\mathbf{e}_1 + \mathbb{Z}(\mathbf{e}_2 - \mathbf{e}_1).$$

It is now straightforward to transform this expression, using the Poisson summation formula, yielding

$$g_q(d) = \frac{\text{Vol}(\mathcal{R})}{|\eta(\tau)|^4 \text{Vol}(\mathcal{R}_d)} \sum_{e \in \mathcal{R}_d^*} q^{\delta(e,q)} \bar{q}^{\bar{\delta}(e,q)}, \mathcal{R}_d^* = \mathbb{Z}\omega_1 + \mathbb{Z}\frac{\omega_1 + \omega_2}{d}.$$

where the volume of the unit cell is  $\text{Vol}(\mathcal{R}_d) = |d\mathbf{e}_1 \wedge (\mathbf{e}_2 - \mathbf{e}_1)| = d\sqrt{3}$ . Going back to  $f_q(d)$  we have

$$f_q(d) = \frac{1}{|\eta(\tau)|^4} \sum_{d' > 0: d|d', d'|m} \frac{\mu(d'/d)}{d'} \sum_{e \in \mathcal{R}_{d'}^*} q^{\delta(e,q)} \bar{q}^{\bar{\delta}(e,q)} \quad (2.72)$$

There are duplicates in the sums over  $q'$ , because  $\mathcal{R}_{d_1}^* \cap \mathcal{R}_{d_2}^* = \mathcal{R}_{d_1 \wedge d_2}^*$ . We can decompose

$$\tilde{\mathcal{R}}_d^* = \mathcal{R}_d^* \setminus \cup_{d' < d: d'|d} \mathcal{R}_{d'}^* = \left\{ a\omega_1 + \frac{b}{d}\rho, a, b \in \mathbb{Z}, \text{gcd}(b, d) = 1 \right\}$$

and

$$\mathcal{R}_{d'}^* = \sqcup_{k > 0: k|d'} \tilde{\mathcal{R}}_k^*$$

Therefore

$$f_q(d) = \frac{1}{|\eta(\tau)|^4} \sum_{d' > 0: d|d', d'|m} \frac{\mu(d'/d)}{d'} \sum_{k: k|d'} \sum_{e \in \tilde{\mathcal{R}}_k^*} q^{\delta(e,q)} \bar{q}^{\bar{\delta}(e,q)} \quad (2.73)$$

$$= \frac{1}{|\eta(\tau)|^4} \sum_{k: k|m} \frac{\mu\left(\frac{k}{d \wedge k}\right)}{m} \frac{\varphi\left(\frac{m}{d}\right)}{\varphi\left(\frac{k}{d \wedge k}\right)} \sum_{e \in \tilde{\mathcal{R}}_k^*} q^{\delta(e,q)} \bar{q}^{\bar{\delta}(e,q)} \quad (2.74)$$

where we used the following identity [127]

$$\begin{aligned} \sum_{d' > 0: d|d', k|d', d'|m} \frac{\mu(d'/d)}{d'} &= \frac{1}{m} \sum_{d'' > 0: \frac{k}{d \wedge k} |d'', d'' | \frac{m}{d}} \mu(d'') \frac{m/d}{d''} \\ &= \frac{\mu\left(\frac{k}{d \wedge k}\right)}{m} \frac{\varphi\left(\frac{m}{d}\right)}{\varphi\left(\frac{k}{d \wedge k}\right)}, \end{aligned} \quad (2.75)$$

and  $\varphi(n)$  is Euler's totient function. We end up with

$$\begin{aligned} Z_{\text{loop}}[b, e_0, \mathcal{R}] &= \frac{1}{|\eta(\tau)|^4} \left[ \sum_{q \in \mathcal{R}} \sum_{e \in \mathcal{R}^*} q^{\delta(e-e_0, \rho, q)} \bar{q}^{\delta(e-e_0, \rho, q)} \right. \\ &\quad \left. + \sum_{\substack{q \in \mathcal{R} \\ m \neq 0}} \sum_{k|m} \sum_{e \in \mathcal{R}_k^*} \Lambda(k, m, e_0) q^{\delta(e, q)} \bar{q}^{\delta(e, q)} \right], \end{aligned} \quad (2.76)$$

where

$$\Lambda(k, m, e_0) = \sum_{d|m} \frac{\mu\left(\frac{k}{d \wedge k}\right)}{m} \frac{\varphi\left(\frac{m}{d}\right)}{\varphi\left(\frac{k}{d \wedge k}\right)} \cos(2\pi d e_0). \quad (2.77)$$













## 2.B SOME INTERESTING PARTICULAR CASES

### 2.B.1 Combinatorial point

The FPL model with  $n = \sqrt{2}$  and  $\tilde{n} = n^2 - 2 = 0$  has a vanishing central charge.

This is a combinatorial point [50], and like in the  $O(n)$  model at  $n = 1$ , the coefficients of the eigenvector associated to the ground state have a particular structure, in this case they are all either integers or multiple of  $\sqrt{2}$ . For example, for  $L = 6$  (taking into account

translation invariance and mirror symmetry) the ground eigenstate is:

	2	
	$12\sqrt{2}$	
	$18\sqrt{2}$	
	12	
	8	
	$5\sqrt{2}$	
	40	
	$15\sqrt{2}$	
	20	
	$50\sqrt{2}$	
	$20\sqrt{2}$	
	10	

(2.78)

Let us now compute the torus partition function. Setting  $\tilde{n} = 0$  means  $e_0 = 1/4$ , and the partition function can be simplified to

$$Z_{\text{loop}}[b, 1/4, \mathcal{R}] = \text{Vol}(\mathcal{R}) \sum_{q, q' \in \mathcal{R}} \cos \left[ \frac{\pi}{2} (q \cdot \rho) \wedge (q' \cdot \rho) \right] Z_{q, q'}(b)$$

This boils down to

$$\begin{aligned} Z_{\text{loop}}[b, 1/4, \mathcal{R}] &= \text{Vol}(\mathcal{R}) \left( \sum_{\substack{q, q' \in \mathcal{R} \\ m \wedge m' = 0 \pmod{4}}} Z_{q, q'} - \sum_{\substack{q, q' \in \mathcal{R} \\ m \wedge m' = 2 \pmod{4}}} Z_{q, q'} \right) \\ &= \text{Vol}(\mathcal{R}) \left( 2 \sum_{\substack{q, q' \in \mathcal{R} \\ m \wedge m' = 0 \pmod{4}}} Z_{q, q'} - \sum_{\substack{q, q' \in \mathcal{R} \\ m \wedge m' = 0 \pmod{2}}} Z_{q, q'} \right) \\ &= \text{Vol}(\mathcal{R}) \left( 2 \sum_{q, q' \in \mathcal{R}_4} Z_{q, q'}(b) - \sum_{q, q' \in \mathcal{R}_2} Z_{q, q'}(b) \right) \end{aligned}$$

This means

$$Z_{\text{loop}}[b, 1/4, \mathcal{R}] = \frac{1}{2} (Z_c[b, \mathcal{R}_4] - Z_c[b, \mathcal{R}_2])$$

This can be further simplified by observing that  $\mathcal{R}_d$  is a square lattice when  $d = 2k$  is even. As a consequence the partition function of the two-component compact boson  $Z_c[b, \mathcal{R}_{2k}]$  factorizes into

$$Z_c[b, \mathcal{R}_{2k}] = Z[\sqrt{6}b] Z[k\sqrt{2}b] \quad (2.79)$$

where  $Z[R]$  stands for the partition function of the one-component compact boson at radius  $R$  :

$$Z[R] = Z[2/R] = \frac{1}{|\eta(\tau)|^2} \sum_{e,q \in \mathbb{Z}} q^{\frac{1}{2}(\frac{e}{R} - \frac{qR}{2})^2} \bar{q}^{\frac{1}{2}(\frac{e}{R} + \frac{qR}{2})^2} \quad (2.80)$$

Setting then  $n = \sqrt{2}$ , which corresponds to  $b = \frac{\sqrt{3}}{2}$ , we get

$$Z_{\text{loop}} \left[ \sqrt{3}/2, 1/4, \mathcal{R} \right] = Z \left[ 3\sqrt{2}/2 \right] \frac{Z \left[ \sqrt{6} \right] - Z \left[ \sqrt{3}/2 \right]}{2}$$

As expected the contributions from the identity in both sums cancel out, since the cylinder partition function is trivial. While this theory has a vanishing central charge, the field content is not trivial as illustrated by the partition function on the torus.

## 2.B.2 The model at $n = 1$ : symplectic fermions and Dimer model

### 2.B.2.1 The case $\tilde{n} = 1$

For  $n = \tilde{n} = 1$  the FPL model can be mapped to the dimer model, and their partition function must be equal in the continuum. This can be seen from the three-colour model, we can choose one of the colour to act as a dimer. Then we obtain the usual partition function of the dimer model on the hexagonal lattice. We now have  $e_0 = \frac{1}{6}$ , and therefore:

$$\begin{aligned} Z_{\text{loop}} [b, 1/6, \mathcal{R}] &= \text{Vol}(\mathcal{R}) \sum_{q,q' \in \mathcal{R}} \cos \left[ \frac{\pi}{3} (\mathbf{q} \cdot \boldsymbol{\rho}) \wedge (\mathbf{q}' \cdot \boldsymbol{\rho}) \right] Z_{q,q'}(b) \\ &= \text{Vol}(\mathcal{R}) \left( \sum_{\substack{q,q' \in \mathcal{R} \\ m \wedge m' = 0 \pmod{6}}} Z_{q,q'} + \frac{1}{2} \sum_{\substack{q,q' \in \mathcal{R} \\ m \wedge m' = \pm 1 \pmod{6}}} Z_{q,q'} \right. \\ &\quad \left. - \frac{1}{2} \sum_{\substack{q,q' \in \mathcal{R} \\ m \wedge m' = \pm 2 \pmod{6}}} Z_{q,q'} - \sum_{\substack{q,q' \in \mathcal{R} \\ m \wedge m' = 3 \pmod{6}}} Z_{q,q'} \right) \\ &= \text{Vol}(\mathcal{R}) \left( 3 \sum_{\substack{q,q' \in \mathcal{R} \\ m \wedge m' = 0 \pmod{6}}} Z_{q,q'} - \frac{3}{2} \sum_{\substack{q,q' \in \mathcal{R} \\ m \wedge m' = 0 \pmod{3}}} Z_{q,q'} \right. \\ &\quad \left. - \sum_{\substack{q,q' \in \mathcal{R} \\ m \wedge m' = 0 \pmod{2}}} Z_{q,q'} + \frac{1}{2} \sum_{q,q' \in \mathcal{R}} Z_{q,q'} \right) \\ &= \frac{1}{2} (Z_c[b, \mathcal{R}_6] - Z_c[b, \mathcal{R}_3] - Z_c[b, \mathcal{R}_2] + Z_c[b, \mathcal{R}]) \end{aligned}$$

For  $b = \sqrt{2/3}$  we again get the cancellation from  $Z_c[\sqrt{2/3}, \mathcal{R}_3] = Z_c[\sqrt{2/3}, \mathcal{R}]$ , and we have

$$Z_{\text{loop}} \left[ \sqrt{2/3}, 1/6, \mathcal{R} \right] = \frac{1}{2} \left( Z_c[\sqrt{2/3}, \mathcal{R}_6] - Z_c[\sqrt{2/3}, \mathcal{R}_2] \right) \quad (2.81)$$

Since  $\mathcal{R}_6$  and  $\mathcal{R}_2$  are square lattices, we get a factorization

$$Z_{\text{loop}} \left[ \sqrt{2/3}, 1/6, \mathcal{R} \right] = Z[1] \frac{\left( Z[\sqrt{12}] - Z[\sqrt{3}] \right)}{2} \quad (2.82)$$

Finally the term  $(Z[\sqrt{12}] - Z[\sqrt{3}])/2 = 1$  since it is the partition function of the  $c = 0$  minimal model  $M(2, 3)$ , and we have

$$Z_{\text{loop}} \left[ \sqrt{2/3}, 1/6, \mathcal{R} \right] = Z[1] \quad (2.83)$$

which is the partition function of the one-component compact boson at radius  $R = 1$ , or equivalently the one of the Dirac fermions. This is also the partition function of the Dimer model.

$$Z_{\text{loop}} \left[ \sqrt{2/3}, 1/6, \mathcal{R} \right] = Z_{\text{Dimer}} = \frac{1}{2} \left( \left| \frac{\theta_2(0|\tau)}{\eta(\tau)} \right|^2 + \left| \frac{\theta_3(0|\tau)}{\eta(\tau)} \right|^2 + \left| \frac{\theta_4(0|\tau)}{\eta(\tau)} \right|^2 \right)$$

### 2.B.2.2 The case $\tilde{n} = -1$

When  $n = 1$  the  $W_3$  symmetry is expected for  $\tilde{n} = n^2 - 2 = -1$ . This means  $c = -2$  ( $b = \sqrt{2/3}$  and  $e_0 = 1/3$ ). The central charge  $c = -2$  does not correspond to the dimer model ( $c = 1$ ), because of the behaviour of the non-contractible loop. It's the central charge of the symplectic fermions model [75] (interestingly the  $\mathcal{W}_3$  algebra is a sub-algebra of the symplectic fermions ones). It's a different take on the well known correspondence between dimers and symplectic fermions [112]. The partition function vanishes identically in this case.

$$\begin{aligned} Z_{\text{loop}} [b, 1/3, \mathcal{R}] &= \text{Vol}(\mathcal{R}) \sum_{q, q' \in \mathcal{R}} \cos \left[ \frac{2\pi}{3} (q \cdot \rho) \wedge (q' \cdot \rho) \right] Z_{q, q'}(b) \\ &= \text{Vol}(\mathcal{R}) \left( \sum_{\substack{q, q' \in \mathcal{R} \\ m \wedge m' = 0 \pmod{3}}} Z_{q, q'} - \frac{1}{2} \sum_{\substack{q, q' \in \mathcal{R} \\ m \wedge m' = \pm 1 \pmod{3}}} Z_{q, q'} \right) \\ &= \text{Vol}(\mathcal{R}) \left( \frac{3}{2} \sum_{\substack{q, q' \in \mathcal{R} \\ m \wedge m' = 0 \pmod{3}}} Z_{q, q'} - \frac{1}{2} \sum_{q, q' \in \mathcal{R}} Z_{q, q'} \right) \\ &= \frac{1}{2} (Z_c[b, \mathcal{R}_3] - Z_c[b, \mathcal{R}]) \end{aligned}$$

However for  $b = \sqrt{2/3}$ , one finds that  $Z_c[\sqrt{2/3}, \mathcal{R}_3] = Z_c[\sqrt{2/3}, \mathcal{R}]$ , so we get

$$Z_{\text{loop}} \left[ b = \sqrt{2/3}, e_0 = 1/3, \mathcal{R} \right] = 0$$

2.C MODELS WITH  $U_q(\mathfrak{sl}_3)$  SYMMETRY

## 2.C.1 The Hecke relations

The aim of this section is to justify the relation between the  $\mathfrak{sl}_3$  group and the Hecke algebra 2.18.

Start with an integrable spin chain, with  $L$  sites. The integrability of the theory expresses itself in the  $R$ -matrix. If we assume that the  $R$ -matrix has the following ansatz (trigonometric):

$$R_i(\lambda, u) = \sin(\lambda - u)\mathbb{1} + \sin(u)U_i.$$

The  $R$ -matrix verifies the Yang-Baxter equation:

$$\begin{aligned} (R(u) \otimes \mathbb{1})(\mathbb{1} \otimes R(u+v))(R(v) \otimes \mathbb{1}) \\ = (\mathbb{1} \otimes R(v))(R(u+v) \otimes \mathbb{1})(\mathbb{1} \otimes R(u)). \end{aligned} \quad (2.84)$$

Which can be rewritten as algebraic relations on the  $U_i$  operators:

$$U_i^2 = [2]_q U_i, \quad (2.85)$$

$$U_i U_j = U_j U_i \text{ if } |i - j| > 2, \quad (2.86)$$

$$U_i U_{i+1} U_i - U_i = U_{i+1} U_i U_{i+1} - U_{i+1}. \quad (2.87)$$

Where quantum numbers, such as  $[2]_q$ , are defined by  $[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}$ .

The second relation is trivially verified, while the other two can be obtained through simple algebraic manipulation. With  $u = -v = \lambda$  we find:

$$U_i^2 - 2 \cos(\lambda) U_i = U_{i+1}^2 - 2 \cos(\lambda) U_{i+1}.$$

Hence,  $U_i^2 = 2 \cos(\lambda) U_i$ .

While, with  $u = v = \lambda$ ,

$$\begin{aligned} -\sin(\lambda) U_i^2 + \sin(2\lambda) U_i U_{i+1} U_i &= +\sin(\lambda) U_{i+1}^2 + \sin(2\lambda) U_{i+1} U_i U_{i+1} \\ U_i - U_i U_{i+1} U_i &= U_{i+1} - U_{i+1} U_i U_{i+1}. \end{aligned}$$

In the limit where  $q = 1$ , the quantum group will behave like their associated classical group. Notably, the matrices  $\mathbb{1} - \tau_{i,j}$ , where  $\tau_{i,j}$  is the transposition matrix exchanging  $i$  and  $j$ , verify the Hecke relations 2.85. Hence, the spin chain is simply a reunion of  $L$  independent particles. If we additionally consider that every particle has a  $SU(N)$  symmetry, the tensor space decomposes into a direct sum of tensor products of irreducible modules (Schur-Weyl duality).

This condition has a corollary in the group algebra  $\mathbb{C}\mathcal{S}_L$ , the Young anti-symmetrizer associated with the alternating tensor product space of dimension  $N + 1$ ,  $\bigwedge^{N+1} \mathbb{C}^{\otimes N}$ , should be 0:

$$A_{N+1} = \sum_{\sigma \in \mathcal{S}_{N+1}} \epsilon(\sigma) \tau_\sigma = 0.$$

Where  $\tau_\sigma = \tau_{i_1, i_1+1} \cdots \tau_{i_p, i_p+1}$  if  $\sigma = (i_1 \ i_1 + 1)(i_2 \ i_2 + 1) \cdots (i_p \ i_p + 1)$ , and  $\epsilon(\sigma)$  is the signature of the permutation. And, in terms of  $U_i$ :

*The conventions for the definition of the Hecke algebra can vary, the sign of  $U$  is different in most of the mathematical sources.*

*If  $U_i = \tau_i + q^{-1}$ , the two last relations 2.86, 2.87 written in terms of  $\tau_i$  form the Artin Braid group [98].*

- For  $\mathfrak{sl}_2$ , we find back the Temperley-Lieb relation:

$$U_i U_{i+1} U_i = U_i. \tag{2.88}$$

- For  $\mathfrak{sl}_3$ :

$$(U_{i-1} - U_{i+1} U_i U_{i-1} + U_i) (U_i U_{i+1} U_i - U_i) = 0. \tag{2.89}$$



For  $q < 1$ , the permutation group is replaced by the (Artin) Braid group, but most of the representation structure stays in place. The anti-symmetrizer  $A_{N+1}$  can be defined (up to a scalar multiple) by the relation:

$$U_i A_{N+1} = A_{N+1} U_i = 0 \quad \forall i \in \{1, \dots, N-1\}.$$

And the previous relations 2.88 and 2.89 are still valid.

### 2.c.2 The Spider model

The TL loop model is a diagrammatic representation of the TL algebra, which itself is the Hecke algebra associated with  $U_q(\widehat{\mathfrak{sl}}_2)$ . This opens up the question of finding an equivalent diagrammatic representation for the  $U_q(\mathfrak{sl}_3)$  Hecke algebra. For the reasons developed in this chapter, the fully packed loops do not fulfil this role. A good diagrammatic representation was found by Kuperberg [103]. In turn this diagrammatic expansion can naturally be transformed into a statistical model with an  $U_q(\widehat{\mathfrak{sl}}_3)$  symmetry. Describing this model is the aim of this appendix.

The Spider model is defined on a finite square lattice of height  $M$  and width  $N$ . Each face of the lattice can be occupied by either one of the two plaquettes  or . They have a specific orientation: the model is not invariant under a  $\pi/2$  rotation. The set of all plaquette configurations forms the configuration space.

A given finite configuration of plaquettes form a planar bicubic graph : a bipartite planar graph, with three edges at every vertices.

The weight of a configuration is given by the following rules:

*A more honest name may be Web model to follow the convention of mathematicians, but "spiders" sounds better*

$$\begin{aligned}
 \bigcirc &= [3]_q, \\
 \text{---} \bullet \bigcirc \bullet \text{---} &= [2]_q \text{---}, \\
 \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} &= \left. \right) \left( + \begin{array}{c} \text{---} \quad \text{---} \\ \text{---} \quad \text{---} \end{array} .
 \end{aligned} \tag{2.90}$$

Those rules are enough to reduce any bicubic graph to a scalar weight. Suppose that there exists a connected graph that cannot to be simplified, this graph

cannot have any faces of size 2 or 4, and because it is bipartite every face are of even length. Being planar, and connected, it verifies the Euler relation, if  $V$  is the number of vertices,  $E$  the number of edges and  $F$  the number of faces:

$$V - E + F = 2.$$

Every edge is attached to three vertices hence  $2E = 3V$ . And, if  $F_n$  is the number of faces with  $n$  sides:

$$F = \sum_{n>0} F_{2n} \quad E = \sum_{n>0} \frac{2}{2n} F_{2n}.$$

Reinserted in the Euler relation, we find:

$$\sum_{n>0} \left(1 - \frac{n}{3}\right) F_{2n} = 2.$$

Hence, if  $2F_2 + F_4 < 6$ , there is no such graph. So in particular if  $F_2 = 0$  and  $F_4 = 0$  the graph simplifies to a scalar weight.

This model is integrable. On a strip of length  $L$ , the transfer matrix in the top-right direction is given by:

$$T = \prod_{j \text{ even } \geq 0} R_{j,j+1} \prod_{j \text{ odd } > 0} R_{j,j+1}.$$

Where  $R_{j,j+1}$  is the  $R$ -matrix of the system:

$$R_{j,j+1}(\lambda, u) = \sin(\lambda - u) \begin{array}{|c|} \hline \square \\ \hline \end{array}_j + \sin(u) \begin{array}{|c|} \hline \square \\ \hline \end{array}_j \quad q = e^{i\lambda}.$$

The operator  $U_j = \begin{array}{|c|} \hline \square \\ \hline \end{array}_j$  plays the role of the Hecke generator, and verifies the algebraic relations:

$$U_j^2 = (q + q^{-1})U_j, \quad (2.91)$$

$$U_j U_{j+1} U_j - U_j = U_{j+1} U_j U_{j+1} - U_{j+1}, \quad (2.92)$$

$$U_j U_k = U_k U_j \quad \text{if } |j - k| \geq 2, \quad (2.93)$$

$$(U_{j-1} - U_{j+1} U_j U_{j-1} + U_j)(U_j U_{j+1} U_j - U_j) = 0. \quad (2.94)$$

Those relations can be found by applying the rules 2.90. The Markov trace (noted  $\text{Tr}$ ) associated with this model consist simply in linking together the incoming and outgoing links, then applying the rules 2.90 on the resulting graph. The trace verifies the relation:

$$\text{Tr}[U_j x] = \frac{q^2 - q^{-2}}{q^3 - q^{-3}} \text{Tr}[x]. \quad (2.95)$$

If  $x$  belongs to the algebra generated by  $\{U_0, U_1, \dots, U_{j-1}\}$ .

The relations 2.90 can be found by diagrammatically manipulating the graphs. For example:

$$U_j^2 = \begin{array}{|c|} \hline j \\ \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline j+1 \\ \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline j \\ \hline \bullet \\ \hline \end{array} = [2]_q \begin{array}{|c|} \hline j \\ \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline j+1 \\ \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline j \\ \hline \bullet \\ \hline \end{array} = (q + q^{-1}) U_j.$$

The equation 2.95 can also be verified through similar methods. For example:

$$\begin{aligned} \text{Tr}[U_1] &= \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = [2]_q \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = [2]_q [3]_q \\ \text{Tr}[1] &= \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array} = [3]_q^2 = \frac{q^3 - q^{-3}}{q^2 - q^{-2}} \text{Tr}[U_1] \end{aligned}$$



The Hilbert space of the system on a cylinder of perimeter  $L$  is formed of all possible configurations that can be obtained from letting the R-matrix iterate over a simple half-web configuration such as  $\frown \cdots \smile$ , and simplified using the relations 2.90.

A natural bilinear two-form can be defined over the half-webs, by mirroring a half-web then simply concatenating it to another, through the outgoing edges. The resulting object is a complete web and hence can be simplified to a scalar. The resulting form is generically not positive (hence not a scalar product), except if  $q$  is real positive.

The Hilbert space is finite, the proof is not trivial and details can be found in [62].

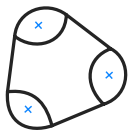
It is possible to be more precise and link the number of states to the combinatoric of Young tableaux. The number of states in the space of half-webs is equal to the number of standard Young tableaux of size  $\left(\frac{L}{3}, \frac{L}{3}, \frac{L}{3}\right)$  (a standard Young tableau is a Young diagram filled with strictly increasing integers in both directions). The short explanation of the bijection that follows can be found in more details within [120]. Start with a half-web, defined on the half plane, and associate to every face its depth with respect to the exterior face (the only infinite face). Every edge can then be associated to a number in  $\{-1, 0, 1\}$  equal to the difference of depth between its two adjacent faces (left minus right, the orientation of the edge can be defined uniquely). The  $L$  outgoing edges are then associated to a word of size  $L$  with alphabet  $\{-1, 0, 1\}$ ,  $x_1 x_2 \cdots x_L$ . The Young tableau is then constructed by putting in the row number  $1 - x_i$  the number  $i$ .

*It is also possible to define operators, which would modify the weight of the loop surrounding them (only pure loops, faces with only one edge), similarly to the loop model, and to compute their weight. An equivalent of the "three-point" partition functions of the loop model however is more complicated to find, due to the existence of non-trivial spider diagrams on the sphere with three punctures. For example:*

While there is no exact mapping toward a vertex model (yet ?), some specific value of  $q$  are of interest on their own:

- $q = 1$ :  $[3]_q = 3$  and  $[2]_q = 2$ . The model maps directly to the three-colors model.
- $q = e^{i\pi/3}$ :  $[3]_q = 0$  and  $[2]_q = 1$ . Corresponds to  $c = -2$ . Only fully connected configurations are authorized, and they all have the same weight.
- $q = e^{i\pi/4}$ : Corresponds to  $c = 0$ .  $[3]_q = 1$  and  $[2]_q = \sqrt{2}$ .
- $q = e^{i\pi/5}$ : Corresponds to  $c = 4/5$ .  $[3]_q = [2]_q = \frac{1+\sqrt{5}}{2}$ .

These models are expected to map to the first rational models  $(p + 1, p)$   $p = 1, \dots, 4$  with  $\mathcal{W}_3$  conformal symmetry.



## SUMMARY

The Toda theory is a generalization of the Liouville CFT with an additional  $\mathcal{W}_n$  symmetry. We study the  $c \leq n - 1$  theory, which interpolates between the different  $\mathcal{W}_3$  minimal models. Like in the  $c \leq 2$  Liouville theory the spectrum is continuous on the real line, operators are labelled by  $(n - 1)$  dimensional vertex charges. We are interested in both the three-point constants of the theory and in the restrictions imposed on the conformal spin of the operators. Main results include:

- The expression of the structure constants (3.36) of three scalar operators when one of them is semi-degenerate. Up to operator normalization, the structure constant of three operators  $\mathcal{O}_{\alpha_1}$ ,  $\mathcal{O}_{\alpha_2}$  and  $\mathcal{O}_{\alpha_3}$  takes the form, if  $\alpha_3$  is semi-degenerate:

$$C(\alpha_1, \alpha_2, \alpha_3) = \prod_{k,\ell=1}^n Y_b [b + (\alpha_1 - \mathbf{Q}) \cdot \mathbf{h}_k + (\alpha_2 - \mathbf{Q}) \cdot \mathbf{h}_\ell + 2\mu],$$

$$\mu = \begin{cases} \frac{1}{2}(b^{-1} - b) - \frac{\kappa}{2n} & \text{if } \alpha_3 = \kappa\omega_1, \\ \frac{\kappa}{2n} & \text{if } \alpha_3 = \kappa\omega_{n-1}. \end{cases}$$

Where  $\mathbf{h}_i$  are the weights of the first representation of  $SU(n)$ ,  $\mathbf{Q} = (b^{-1} - b)(\mathbf{h}_1 + \mathbf{h}_3)$  and  $c = (n - 1) - 12\mathbf{Q}^2$ .

- The classification of the possible non-scalar operators in a general Toda field theory. Non-scalar operators can be labelled by their vertex charges  $\alpha$  and  $\bar{\alpha}$  and a permutation  $\sigma \in \mathcal{S}_N$ , which fixes the fusion rules between the operator and degenerate fields. We find the constraints:

$$\alpha - \bar{\alpha} \in \mathcal{R}/b, \quad \text{and} \quad \alpha - \sigma \star \bar{\alpha} \in b\mathcal{R},$$

where  $\mathcal{R}$  is the root lattice of  $SU(3)$ .

- The shift equations obeyed by the structure constants of non-scalar operators, with their solution (3.77) when one of the operators is scalar.

### 3.1 INTRODUCTION

In the context of Conformal Field Theory (CFT) applied to critical models of statistical mechanics, the understanding of the operator algebra, i.e. the spectrum of primary operators, their fusion rules and the structure constants appearing in the Operator Product Expansion (OPE), are of central importance. While the conformal dimensions determine the critical exponents of a given universality class, the structure constants (together with the conformal blocks) are the building blocks for  $N$ -point correlation functions.

In this chapter, we consider a model with  $W_n$  algebra symmetry, namely the  $\mathfrak{sl}_n$  Toda field theory. Like for the Virasoro algebra, a whole class of critical integrable lattice models – vertex and face models – scale to  $W_n$ -symmetric CFTs in the continuum limit [117], and their central charge is  $c \leq n - 1$ . For  $n = 3$ , we saw in the previous chapter that the FPL model on the honeycomb lattice [101, 128] displays some features related to  $\mathcal{W}_3$ . The  $W_n$  analogue of the DOZZ formula (see introduction 1.20) [53, 153], i.e. the structure constant of three vertex operators for the  $\mathfrak{sl}_n$  Toda field theory with a real background charge  $Q$ , was obtained by the conformal bootstrap procedure in [63, 64]. The results are restricted to the case when one of the operators is “semi-degenerate” with respect to the  $W_n$  algebra.

This chapter is mostly centered around the  $\mathcal{W}_3$  model, which serves as the main example for all the results. It will be described in details in the first part. Then, we calculate the structure constants of vertex operators in the  $\mathfrak{sl}_n$  Toda CFT with imaginary background charge  $Q$ . Like in the Liouville case [156], the structure constant *does not* follow an analytic continuation, and the analytic expressions for real and imaginary  $Q$  are different.

We then turn our attention to non-scalar primary fields. By demanding well-defined monodromies with the fully degenerate fields, a classification of the possible non-scalar primary fields is obtained. These fields are found to be in one-to-one correspondence with conjugacy classes of  $\mathfrak{S}_n$ , with the particular case of the identity being the usual scalar fields.

Finally, we consider the structure constants in the presence of non-scalar operators, which are found to be given by a geometrical mean relation as was the case for  $n = 2$  (Liouville) [61, 111].

### 3.2 THE $\mathcal{W}_3$ CONFORMAL SYMMETRY

#### 3.2.1 Definition

The  $W_3$  algebra is a chiral symmetry algebra generated by two fields [65, 152]: the usual spin-2 stress energy tensor, ensuring conformal invariance, and an additional spin-3 current  $W(z)$ . The operator al-

gebra generated by these two operators is completely defined by the singularities in the Operator Product Expansion (OPE):

$$\begin{aligned}
 T(z)T(0) &= \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) + \left[ \frac{3}{10}\partial^2 T(0) + \Lambda(0) \right] + \mathcal{O}(z) \\
 T(z)W(0) &= \frac{3}{z^2}W(0) + \frac{1}{z}\partial W(0) + \text{reg.} \\
 W(z)W(0) &= \frac{c}{3z^6} + \frac{2T(0)}{z^4} + \frac{\partial T(0)}{z^3} \\
 &\quad + \frac{1}{z^2} \left[ \frac{3}{10}\partial^2 T(0) + \frac{32}{22+5c}\Lambda(0) \right] \\
 &\quad + \frac{1}{z} \left[ \frac{1}{15}\partial^3 T(0) + \frac{16}{22+5c}\partial\Lambda(0) \right] + \text{reg.} .
 \end{aligned} \tag{3.1}$$

The first relation defines the usual Virasoro algebra (central charge  $c$ ), the second one expresses that  $W$  is a primary field of dimension 3, while the third one comes from the closure of the operator algebra.

To develop the origin of this last equation, the closure of the algebra imposes the relation  $W \otimes W \rightarrow \mathbb{1} \oplus W$ , but if the structure constant  $C_{W,W}^W$  was non-zero, the correlation function  $\xi \rightarrow \langle W(z_1)T(\xi)W(z_2)W(z_3) \rangle$  would define a non-zero meromorphic function with only three poles, of degree two, in  $z_i$ , and which behaves at infinity like  $\xi^{-4}$ , which is prohibited by the residue theorem. Hence,  $W \otimes W \rightarrow \mathbb{1}$ .

And because  $W(z)$  is a primary field of weight 3,

$$W(z)W(0) = \langle W|W \rangle \left( \frac{1}{z^6} + \frac{6}{c} \frac{L_{-2}}{z^4} + \frac{L_{-1}L_{-2}}{z^3} + \dots \right) \mathbb{1}(0),$$

where the whole development is fixed by conformal invariance. The norm  $\langle W|W \rangle$  is arbitrarily chosen to make the developments of  $T \cdot T$  and  $W \cdot W$  somewhat similar.

The modes of these two operators are defined by:

$$\begin{aligned}
 L_n \Phi(\xi) &= \oint d\xi (u - \xi)^{n+1} T(u) \Phi(\xi), \\
 W_n \Phi(\xi) &= \oint d\xi (u - \xi)^{n+2} W(u) \Phi(\xi).
 \end{aligned} \tag{3.2}$$

The OPEs defined previously are equivalent to the following commutations relations between the modes:

$$\begin{aligned}
 [L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\
 [L_n, W_m] &= (2n - m)W_{n+m} \\
 [W_n, W_m] &= \frac{c}{360}(n^2 - 4)(n^2 - 1)n\delta_{n+m,0} + \frac{16}{22+5c}(n - m)\Lambda_{n+m} \\
 &\quad + \frac{1}{30}(n - m)(2m^2 + 2n^2 - mn - 8)L_{n+m},
 \end{aligned} \tag{3.3}$$

where  $\Lambda_n$  are the modes of the (quasi-primary) composite field  $\Lambda(z) = :T^2(z): - \frac{3}{10}\partial^2 T(z)$ , namely:

$$\Lambda_n = \sum_{k < n/2} L_k L_{n-k} + \sum_{k \geq n/2} L_{n-k} L_k + \frac{(1 + \lceil \frac{n}{2} \rceil)(1 + \lfloor \frac{n}{2} \rfloor)}{5} L_n \tag{3.4}$$

Contrary to the case of the Virasoro algebra, the vector space defined by the modes is not a Lie algebra because of the quadratic term in  $L$  appearing in  $[W_n, W_m]$ . Nevertheless, the full algebra formed by the modes (with modes multiplication allowed) is a Lie algebra, hence the Jacobi identity is obviously verified by the bracket.

A primary field for  $W_3$  is a field  $\Phi$  which is primary with respect to both operators,

$$W_n \Phi = 0 \quad \forall n \in \mathbb{N}^*, \quad L_n \Phi = 0 \quad \forall n \in \mathbb{N}^*,$$

$$W_0 \Phi = w_\Phi \Phi, \quad L_0 \Phi = h_\Phi \Phi,$$

$h_\Phi$  is the conformal dimension of the field, the couple  $(h_\Phi, w_\Phi)$  identifies a field in a  $W_3$ -conformal theory. Based on these conditions, the OPE of the fields  $T$  and  $W$  with the primary fields can be written:

$$W(z)\Phi(0) = \frac{w_\Phi \Phi(0)}{z^3} + \frac{W_{-1}\Phi(0)}{z^2} + \frac{W_{-2}\Phi(0)}{z} + \text{reg.}$$

$$T(z)\Phi(0) = \frac{h_\Phi \Phi(0)}{z^2} + \frac{W_{-1}\Phi(0)}{z} + \text{reg.}$$

As a consequence,  $T(z)$  and  $W(z)$  behaves, when  $z \rightarrow \infty$ , like  $\frac{1}{z^4}$  and  $\frac{1}{z^6}$  respectively.

The conjugate of a field  $\Phi^*$  is defined by the relation  $|\Phi\rangle^\dagger = \langle \Phi^*|$ . It's easy to see that,  $W$  being of odd dimension, the conjugate  $\Phi^*$  of a field  $\Phi$  have the opposite weight under  $W$ :  $w_\Phi = -w_{\Phi^*}$ , and of course  $h_\Phi = h_{\Phi^*}$ .

If  $\Phi_1$  and  $\Phi_2$  are two operators, both primary for  $W_3$ , and the fusion  $\Phi_1 \otimes \Phi_2 \rightarrow \Phi_p \oplus \dots$  is allowed, then the (holomorphic part of the) OPE of  $\Phi_1$  with  $\Phi_2$  will also involve all the descendants of  $\Phi_p$ , with respect to  $T$  and  $W$ ,

$$\Phi_1(z)\Phi_2(0) = \sum_{n,m} C_{\Phi_1, \Phi_2}^{W_{-n}L_{-n}\Phi_p} W_{-n}L_{-m}\Phi_p,$$

where, for  $\mathbf{n}$  a multi-indices vector  $X_{-\mathbf{n}} = X_{-n_1}X_{-n_2}\dots$

Contrary to the  $L_n$  operators, the modes of  $W$  cannot be expressed as differential operators. If in Virasoro, every three-point functions involving descendants operator can be expressed in terms of the three-point function of primaries, it is not the case in  $W_3$ . Not all generic three-point functions involving descendants of  $W$  can be computed through the algebra alone.

To be more precise, every three-point functions involving descendants of primary fields, of the form  $\langle W_{-n_1, -n_2, \dots} \Phi_1 W_{-m_1, -m_2, \dots} \Phi_2 W_{-p_1, -p_2, \dots} \Phi_3 \rangle$ , can be expressed in terms of the structure constants  $\langle W_{-1}^n \Phi_1 \Phi_2 \Phi_3 \rangle$ , with  $n \in \mathbb{N}$ , which themselves cannot be simplified more (using the sole algebra, bootstrap methods can of course be applied). This was proven in [97]. For completeness,

we present a shortened version of their argument. If  $\Psi_1$ ,  $\Psi_2$  and  $\Psi_3$  are primaries or descendants of primaries, the correlation function:

$$z \rightarrow \langle W(z)\Psi_1(z_1)\Psi_2(z_2)\Psi_3(z_3) \rangle \frac{(z-z_2)^2(z-z_3)^2}{(z-z_1)^n}$$

behaves like  $1/z^{n+2}$  at infinity. So for  $n \in \mathbb{N}$ , we can apply the residue theorem and say that:

$$\sum_i \text{Res} \left[ \langle W(z)\Psi_1(z_1)\Psi_2(z_2)\Psi_3(z_3) \rangle \frac{(z-z_2)^2(z-z_3)^2}{(z-z_1)^n} \right] \Big|_{z=z_i} = 0.$$

In  $z_2$  and  $z_3$ , the residue is only a function of  $\langle \Psi_1(z_1)\Psi_2(z_2)\Psi_3(z_3) \rangle$  and  $w_i$ , while in  $z_1$  it involves all the three-point functions  $\langle W_{-k}\Psi_1(z_1)\Psi_2(z_2)\Psi_3(z_3) \rangle$  with  $k \in \{1, \dots, n+2\}$ . By recurrence every correlation functions can be computed in terms of  $\langle \Psi_1(z_1)\Psi_2(z_2)\Psi_3(z_3) \rangle$  and  $\langle W_{-1}\Psi_1(z_1)\Psi_2(z_2)\Psi_3(z_3) \rangle$ . Because the commutation of the  $W$  modes only involves modes of  $T$ , the generalization to multiple descendants is trivial.

### 3.2.2 Free fields realization

The free-field theory composed of two bosons has a  $\mathcal{W}_3$  symmetry, with  $c = 2$ . The chiral part of its stress-energy tensor takes the form:

$$T(z) = -\frac{1}{2} :(\partial\boldsymbol{\varphi} \cdot \partial\boldsymbol{\varphi}): .$$

With  $\boldsymbol{\varphi}$  the chiral part of a two-dimensional bosonic field. And, in order to match the OPE 3.1,  $W$  must take the form:

$$W(z) \propto : \prod_{i=1}^3 (\partial\boldsymbol{\varphi} - \mathbf{h}_i) : .$$

We take the convention  $\varphi_i(z)\varphi_j(w) \sim -\delta_{ij} \log(z-w)$   $i, j \in \{1, 2\}$  for the two components of the chiral part of the boson. The vectors  $\mathbf{h}_i$  are the weights of the first fundamental representation of  $\mathfrak{su}(3)$  (see the appendix 3.A).

Another possible representation of the  $\mathcal{W}_3$  symmetry is the WZW (Wess-Zumino-Witten) model associated with  $\hat{A}_2$ . The relation between the currents of the  $\hat{A}_2$  WZW and the  $T$  and  $W$  operators is interesting to establish, as it underlines clearly the link between  $\mathcal{W}_3$  and the  $SU(3)$  theory. The Lie symmetry in the action of the WZW theory is reflected, at the quantum level, in a *current algebra*, whose basis is a set of primary fields  $J_a$  of dimension 1, closed under OPE:

$$J^a(z)J^b(w) \sim \frac{\delta_{ab}}{(z-w)^2} + \sum_c \mathbf{i}f_{abc} \frac{J^c(w)}{(z-w)} .$$

Of course these operators  $J^a$  are in correspondence with the standard basis of the Lie algebra:  $J^a(z)J^b(w) - J^b(w)J^a(z) \sim 2\sum_c \mathbf{i}f_{abc} \frac{J^c(w)}{(z-w)}$ . As often when working with Lie algebras it is easier to consider the operators  $E^{\pm i}(z)$   $i \in \{1, 2, 3\}$  and  $H^j(z)$   $j \in \{1, 2\}$ , that are associated with the Cartan-Weyl basis (see appendix 3.A). WZW theories have

a well-known vertex representation. Specializing to the  $SU(3)$  case, it is simple to check that, given a two-dimensional bosonic field, the bosonic operators:

$$H^j = i\partial\phi^j \text{ and } E^{\pm i} \propto e^{\pm ie_i \cdot \phi}$$

satisfy the correct OPEs. All these operators are current of dimension 1.

The stress-energy tensor of the theory is obtained by forming the quadratic Casimir of the Lie algebra with these operators:

$$T \propto \sum_a :J^a J^a: \propto \sum_i :H^i H^i: + \sum_i :E^i E^{-i}: + :E^{-i} E^i: . \quad (3.5)$$

And, maybe less commonly known, the  $W$  operator comes from the Casimir of order 3 [28]:

$$W \propto \sum_{a,b,c} d_{a,b,c} :J^a J^b J^c: \quad (3.6)$$

Where  $d_{a,b,c}$  are the completely symmetric coefficient constants, defined in 3.A. This operator is of course of conformal dimension 3, hence verifies the OPE with  $T$  and by extension the OPE with itself.

Coming back to the bosonic realization, for  $c < 2$ , the stress-energy tensor can be modified through a background charge  $Q$ :

$$T(z) = -\frac{1}{2} : \partial\varphi \cdot \partial\varphi : + iQ\partial^2\varphi, \quad c = 2 - 12Q \cdot Q$$

In order to keep the OPEs 3.1 valid, this implies:

$$W(z) \propto :(\partial\varphi_2)^3: - 3 :(\partial\varphi_1)^2\partial\varphi_2: + 3iQ : \partial^2\varphi_1\partial\varphi_2 : \\ + 9iQ : \partial\varphi_1\partial^2\varphi_2 : + 6Q^2 : \partial^3\varphi_2 : .$$

The primary operators are given by vertex operators  $V_\alpha = : e^{i\alpha \cdot \varphi} :$ , which have weights:

$$\Delta_\alpha = \frac{1}{2}\alpha \cdot (\alpha - 2Q) \text{ and } w_\alpha = \sqrt{\frac{48}{22+5c}} \prod_i (\alpha - Q) \cdot h_i .$$

Where  $h_i$  are the weights of the first representation of  $\mathfrak{su}(3)$ , defined in the appendix 3.A. From these expressions it is easy to extract the following relations:

$$V_\alpha^* = V_{\alpha^*} \quad V_{2Q-\alpha^*} = V_\alpha .$$

Or, more generically:

$$V_{Q+R_i(\alpha-Q)} = V_\alpha \quad \forall i .$$

With the dual of an operator  $\Phi$  defined as the operator  $\Phi^*$  such that  $\langle \Phi(0)\Phi^*(z) \rangle \neq 0$ .

*The operator  $E^{\pm i}$  is equal to  $e^{\pm ie_i \cdot \phi}$  up to a sign, this sign is necessary in order to verify the correct OPEs*

### 3.2.3 Coulomb gas theory

From the stress-energy tensor, we extract the action:

$$\mathcal{A} = \frac{1}{8\pi} \int d^2x \sqrt{g(x)} [g^{\mu\nu} \partial_\mu \boldsymbol{\varphi}(x) \cdot \partial_\nu \boldsymbol{\varphi}(x) + 2iR(x) \mathbf{Q} \cdot \boldsymbol{\varphi}(x)] .$$

Where  $g$  and  $R$  are respectively the metric and the scalar curvature of the manifold on which the model is defined. This is the action of the Coulomb-gas theory. The vector  $\mathbf{Q}$ , which creates a coupling between the field and the curvature, is called *background charge*.

All correlation functions of vertex operators,  $\langle V_{\alpha_1} V_{\alpha_2} \cdots V_{\alpha_p} \rangle$  must respect the neutrality condition, wherein the sum of their charges must be equal to the background charge:

$$\sum_i \alpha_i = \mathbf{Q}$$

The primary fields of this Coulomb-gas theory are vertex operators  $V_\alpha$ , with  $\alpha$  verifying:

$$\begin{aligned} \alpha &= \alpha \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} \\ &= b^{-1} [(n_1 - 1)\omega_1 + (n_2 - 1)\omega_2] - b [(m_1 - 1)\omega_1 + (m_2 - 1)\omega_2] \end{aligned}$$

If  $b^2$  is rational, this leads to a finite number of fields, a minimal model.

### 3.2.4 Minimal models

The minimal models  $W_3(p, q)$  – with  $p$  and  $q$  coprime integers,  $p, q \geq 3$  – are rational conformal field theories with  $W_3$  symmetry and central charges:

$$c(p, q) = 2 \left( 1 - 12 \frac{(q-p)^2}{pq} \right) \quad (3.7)$$

They have finitely many scalar primary fields  $\Phi_{n,m}$ , with  $\mathbf{n} \in \boldsymbol{\rho} + A_{p-3}$  and  $\mathbf{m} \in \boldsymbol{\rho} + A_{q-3}$ , i.e.

$$\begin{aligned} \mathbf{n} &\in \{n_1\omega_1 + n_2\omega_2, n_i \geq 1, n_1 + n_2 \leq p-1\}, \\ \mathbf{m} &\in \{m_1\omega_1 + m_2\omega_2, m_i \geq 1, m_1 + m_2 \leq q-1\}. \end{aligned} \quad (3.8)$$

The conformal dimension of  $\Phi_{n,m}$  is

$$\Delta_{n,m} = \bar{\Delta}_{n,m} = \frac{(q\mathbf{n} - p\mathbf{m})^2 - 2(q-p)^2}{2pq}. \quad (3.9)$$

We are concerned with the simplest  $W_3$  minimal models, i.e. those with diagonal modular invariant [17].

$p$  and  $q$  are coprime integers, so a and  $b$  exists such that  $ap + bq = 1$ , take  $\mathbf{n} = a\boldsymbol{\rho}$  and  $\mathbf{m} = b\boldsymbol{\rho}$ . If  $(p-q)^2 \neq 1$  the weight  $\Delta_{n,m}$  is negative.



While rational, the minimal models  $W_3(p, q)$  are only unitary for  $q = p + 1$  and  $p \geq 4$ , and in this case they are equivalent to the following GKO coset[121]

$$W_3(p, p + 1) = \frac{\widehat{\mathfrak{su}}(3)_k \otimes \widehat{\mathfrak{su}}(3)_1}{\widehat{\mathfrak{su}}(3)_{k+1}}, \quad p = k + 3. \quad (3.10)$$

The minimal model  $W_3(4, 5)$  is equivalent to the Virasoro minimal model  $(6, 5)$  (3-Potts), but models  $W_3(p, p + 1)$  with  $p \geq 5$  verify  $c \geq 1$ , and hence are not minimal models for Virasoro.

### 3.2.5 Toda field theory

The model we are studying in this chapter is the analogue of the Liouville model for the  $\mathcal{W}_n$  symmetry. The action associated with the Toda field theory is:

$$\mathcal{A} = \frac{1}{8\pi} \int d^2x g^{\mu\nu} \partial_\mu \boldsymbol{\varphi} \cdot \partial_\nu \boldsymbol{\varphi} + 2i\mathcal{R}(x) \mathbf{Q} \cdot \boldsymbol{\varphi} + \mu \sum_{k=1}^{n-1} e^{b(e_k \cdot \boldsymbol{\varphi})}$$

Where  $\boldsymbol{\varphi}$  is a  $n - 1$  dimensional field normalized by:

$$\langle \varphi_i(z, \bar{z}) \varphi_j(0, 0) \rangle = -\delta_{ij} \log |z|^2$$

In the case of  $\mathcal{W}_3$ , this model interpolates between the minimal model described previously. It also contains the chiral part of the Coulomb gas.

### 3.2.6 Semi-degenerate and fully-degenerate representations of $W_3$

Some particular  $\mathcal{W}_3$  primary fields have null vectors (i.e. some of their descendants are linearly dependent). These primary fields are called degenerate. Two types of degenerate representations must be distinguished:

- semi-degenerate representations, satisfying only one null-vector equation. The conformal dimension and the  $W$ -charge of the semi-degenerate primaries are related by a polynomial equation.
- fully degenerate representations, with two or more null-vector equations, for which both the conformal dimension and the  $W$ -charge are fixed (for a given central charge).

For semi-degenerate representations, there is only one primitive linear relation between descendants. The simplest example is the representation generated by an operator with a null vector at level one.

$$L_{-1} \Phi_\alpha \propto W_{-1} \Phi_\alpha, \quad (3.11)$$

which requires the relation

$$9w_\alpha^2 = \frac{2\Delta_\alpha^2(32\Delta_\alpha + 2 - c)}{22 + 5c}. \quad (3.12)$$

In terms of charge vectors, this means (up to Weyl transformations)

$$\alpha = \kappa\omega_1 \quad \text{or} \quad \alpha = \kappa\omega_2, \quad (3.13)$$

where  $\kappa \in \mathbb{C}$ . The explicit relation between descendants is:

$$W_{-1}\Phi_\alpha = \frac{3w}{2\Delta}L_{-1}\Phi_\alpha. \quad (3.14)$$

Fully degenerate states possess additional null vectors. Focusing on the simplest type of degenerate representation, consider a field  $\Phi$  satisfies the semi-degenerate condition (3.14), together with a linear relation at level two, between  $\{W_{-2}\Phi_\alpha, L_{-2}\Phi_\alpha, L_{-1}^2\Phi_\alpha\}$ . The algebra (3.3) then yields four possible values  $\alpha \in \{b\omega_1, b\omega_2, -b^{-1}\omega_1, -b^{-1}\omega_2\}$ , where  $b$  is related to the background charge  $Q$  by

$$Q = \frac{1}{b} - b, \quad b = \frac{1}{2} \left( -Q + \sqrt{4 + Q^2} \right).$$

The precise form of the additional null-vector conditions is then (see [29]):

$$\begin{aligned} \left( W_{-2} - \frac{12w}{\Delta(5\Delta + 1)}L_{-1}^2 + \frac{6w(\Delta + 1)}{\Delta(5\Delta + 1)}L_{-2} \right) \Phi_\alpha &= 0, \\ \left( W_{-3} - \frac{16w}{\Delta(\Delta + 1)(5\Delta + 1)}L_{-1}^3 + \frac{12w}{\Delta(5\Delta + 1)}L_{-1}L_{-2} \right. & \\ \left. + \frac{3w}{2\Delta} \frac{(\Delta - 3)}{(5\Delta + 1)}L_{-3} \right) \Phi_\alpha &= 0, \end{aligned} \quad (3.15)$$

where  $\Delta$  and  $w$  are given by (2.54).

### 3.2.7 Fusion in $\mathcal{W}_3$

The above null-vector equations impose constraints on the fusion rules and on the structure constants involving degenerate fields. In particular, we can extract from them the fusion rules of the fully degenerate states. For example, one can show that the correlation function  $\langle \Phi_\alpha \Phi_{b\omega_1} \Phi_{\beta^*} \rangle$  vanishes unless  $\beta = \alpha + bh_j$  for some  $j \in \{1, 2, 3\}$ . Fusion rules match the behaviour of tensor product of representation in  $\mathfrak{su}(3)$ :

$$\Phi_{b\omega_1} \otimes \Phi_\alpha \rightarrow \bigoplus_{j=1}^3 \Phi_{\alpha + bh_j}, \quad \Phi_{b\omega_2} \otimes \Phi_\alpha \rightarrow \bigoplus_{j=1}^3 \Phi_{\alpha - bh_j}. \quad (3.16)$$

This result can be found in [63], it can also be extracted from the result in the Appendix 3.B.1, by setting  $\kappa = 0$  in the differential equation.

Note however that generically there will be non-trivial multiplicities in these fusion rules. This is why a generic four-point function involving a fully degenerate field  $\Phi_{b\omega_1}$  does not obey a differential equation of order three. If one of the other fields is semi-degenerate though, a differential equation can be found, but its order depends on the semi-degenerate field [15, 16].

For semi-degenerate operators, similar (if less strict) restrictions exist. Let  $\Phi_1$  and  $\Phi_2$  be two semi-degenerate fields at level 1. The residue theorem applied to the contour integral:

$$\oint_C dz (z-1)^2 z \langle \Phi_2 | W(z) \Phi_\alpha(1) | \Phi_1 \rangle,$$

where  $C$  is a contour enclosing 0 and 1, yields

$$\begin{aligned} \langle \Phi_2 | W_1 \Phi_\alpha(1) | \Phi_1 \rangle - \langle \Phi_2 | \Phi_\alpha(1) W_{-1} | \Phi_1 \rangle \\ + (2(w_1 - w_2) - w_\alpha) \langle \Phi_2 | \Phi_\alpha(1) | \Phi_1 \rangle = 0. \end{aligned}$$

Now using the null-vector condition  $W_{-1} \Phi_j = \frac{3w_j}{2\Delta_j} L_{-1} \Phi_j$ , and

$$\begin{aligned} \langle \Phi_2 | \Phi_\alpha(1) L_{-1} | \Phi_1 \rangle &= (\Delta_1 + \Delta_\alpha - \Delta_2) \langle \Phi_2 | \Phi_\alpha(1) | \Phi_1 \rangle, \\ \langle \Phi_2 | L_1 \Phi_\alpha(1) | \Phi_1 \rangle &= (\Delta_2 + \Delta_\alpha - \Delta_1) \langle \Phi_2 | \Phi_\alpha(1) | \Phi_1 \rangle \end{aligned}$$

we get the condition for  $\langle \Phi_2 | \Phi_\alpha(1) | \Phi_1 \rangle$  to be non-zero:

$$w_\alpha = \frac{3w_2}{2\Delta_2} (\Delta_2 + \Delta_\alpha - \Delta_1) - \frac{3w_1}{2\Delta_1} (\Delta_1 + \Delta_\alpha - \Delta_2) + 2(w_1 - w_2).$$

This restricts the fusion rules (up to Weyl transformations) as follows:

$$\begin{aligned} \langle \Phi_{\kappa\omega_1} \Phi_{\kappa'\omega_1} \Phi_{\beta^*} \rangle &\neq 0 \quad \text{for } \beta \in \mathbb{R}e_1 + (\kappa + \kappa')\omega_1, \\ \langle \Phi_{\kappa\omega_2} \Phi_{\kappa'\omega_2} \Phi_{\beta^*} \rangle &\neq 0 \quad \text{for } \beta \in \mathbb{R}e_2 + (\kappa + \kappa')\omega_2, \\ \langle \Phi_{\kappa\omega_1} \Phi_{\kappa'\omega_2} \Phi_{\beta^*} \rangle &\neq 0 \quad \text{for } \beta \in \mathbb{R}\rho + \frac{\kappa' - \kappa}{2} h_2. \end{aligned} \tag{3.17}$$

A special case is the fusion between a semi- and a fully degenerate field. It is possible to explicitly describe the allowed operators in the fusion  $\Phi_{b\omega_1} \otimes \Phi_{\kappa\omega_i}$ . Putting together the previous fusion rules we get

$$\begin{aligned} \Phi_{b\omega_1} \otimes \Phi_{\kappa\omega_1} &\rightarrow \Phi_{\kappa\omega_1 + bh_1} \oplus \Phi_{\kappa\omega_1 + bh_2}, \\ \Phi_{b\omega_1} \otimes \Phi_{\kappa\omega_2} &\rightarrow \Phi_{\kappa\omega_2 + bh_1} \oplus \Phi_{\kappa\omega_2 + bh_3}. \end{aligned} \tag{3.18}$$

The semi- and fully degenerate representations of higher level can be described explicitly by studying level- $N$  null vectors. The fields  $\Phi_{\kappa\omega_2 + bh_1}$  and  $\Phi_{\kappa\omega_2 + bh_3}$ , which appear in the previous fusions are also semi-degenerate, see for example [16] for an explicit derivation.

### 3.2.8 $W_n$ algebra and conventions for $\mathfrak{sl}_n$

The  $W_n$  algebra is a generalization of the  $W_3$  algebra, in which there are  $n - 1$  holomorphic currents  $W^{(k)}(z)$  with spin  $k = 2, 3, \dots, n$ . The central charge is parametrized as:

$$c = (n - 1) - 12 \mathbf{Q} \cdot \mathbf{Q} = (n - 1) (1 - n(n + 1)Q^2), \quad (3.19)$$

where  $\mathbf{Q} = Q\rho$  and  $\rho$  is the Weyl vector of  $\mathfrak{sl}_n$  (see Appendix 3.A). Primary fields are characterized by their zero modes  $w^{(k)}$

$$W^{(k)}(z)\Phi(0) = \frac{w^{(k)}}{z^k}\Phi(0) + \dots \quad (3.20)$$

and the quantum numbers  $w^{(k)} = w_\alpha^{(k)}$  are naturally parametrized by an  $(n - 1)$ -dimensional vector  $\alpha$ . Furthermore, they are invariant under action of the  $\mathfrak{sl}_n$  Weyl group

$$\forall \sigma \in \mathcal{W}, \quad w_{\sigma^*\alpha}^{(k)} = w_\alpha^{(k)}. \quad (3.21)$$

The dual of a charge  $\alpha$  is  $\alpha^* = -s_0(\alpha)$ , and is equivalent to  $2\mathbf{Q} - \alpha$ . Indeed, it follows from  $s_0^2 = 1$  and  $s_0(\rho) = -\rho$  that

$$2\mathbf{Q} - \alpha = s_0 \star (\alpha^*) \quad (3.22)$$

Of particular importance are the four completely degenerate operators  $\{\Phi_{b\omega_1}, \Phi_{b\omega_{n-1}}, \Phi_{-b^{-1}\omega_1}, \Phi_{-b^{-1}\omega_{n-1}}\}$ . Their (chiral) fusion rules with a generic primary field  $\Phi_\alpha$  are given by

$$\Phi_{b\omega_1} \otimes \Phi_\alpha \rightarrow \bigoplus_{j=1}^n \Phi_{\alpha+bh_j}, \quad \Phi_{b\omega_{n-1}} \otimes \Phi_\alpha \rightarrow \bigoplus_{j=1}^3 \Phi_{\alpha-bh_j}, \quad (3.23)$$

$$\Phi_{-b^{-1}\omega_1} \otimes \Phi_\alpha \rightarrow \bigoplus_{j=1}^n \Phi_{\alpha-b^{-1}h_j}, \quad \Phi_{-b^{-1}\omega_{n-1}} \otimes \Phi_\alpha \rightarrow \bigoplus_{j=1}^3 \Phi_{\alpha+b^{-1}h_j}. \quad (3.24)$$

A particularly simple class of semi-degenerate field is given by Wyl-lard fields

$$\Phi_{\kappa\omega_1} \quad \text{and} \quad \Phi_{\kappa\omega_{n-1}}, \quad \kappa \in \mathbb{C}. \quad (3.25)$$

These fields have  $(n - 2)$  null-vectors at level one:

$$W_{-1}^{(k)}\Phi_{\kappa\omega_j} \propto L_{-1}\Phi_{\kappa\omega_j}, \quad k = 3, \dots, n, \quad (3.26)$$

but they lack an extra null-vector to be fully degenerate.

## 3.3 STRUCTURE CONSTANTS OF SCALAR FIELDS

So far we have given a purely chiral description of the  $W_n$  theory. In order to build a consistent conformal field theory the holomorphic

and anti-holomorphic sectors have to be glued appropriately. On the torus the constraint is modular invariance of the partition function, while on the sphere the constraint is crossing-symmetry of four-point functions. Here we consider the latter, and we first discuss scalar fields in the  $W_n$  Toda field theory with  $c \leq n - 1$ . Generically a field is scalar as long as its conformal spin is zero:  $s = \Delta - \bar{\Delta} = 0$ . In the context of an extended symmetry such as the  $W_n$  theory, we take a more constraining definition, namely that all left and right quantum numbers coincide:

$$\forall k \in \{2, \dots, n\}, \quad w^{(k)} = \bar{w}^{(k)}. \quad (3.27)$$

Primary fields are labelled by their holomorphic and anti-holomorphic charge vectors as  $\Phi_{\alpha, \bar{\alpha}}(z, \bar{z})$ , and scalar primary fields are those which have  $\alpha = \bar{\alpha}$  (up to Weyl group action): we shall simply denote them as  $\Phi_{\alpha}(z, \bar{z})$ . Our purpose is to compute the structure constants of the operator algebra between scalar primary fields. These are related to the three-point correlation functions:

$$C(\alpha_1, \alpha_2, \alpha_3) = \langle \Phi_{\alpha_1}(0) \Phi_{\alpha_2}(1) \Phi_{\alpha_3}(\infty) \rangle. \quad (3.28)$$

When one of the vertex charges is semi-degenerate, say

$$\alpha_3 = \kappa \omega_1 \quad \text{or} \quad \alpha_3 = \kappa \omega_{n-1}, \quad \text{with} \quad \kappa \in \mathbb{R}, \quad (3.29)$$

and  $\alpha_1, \alpha_2$  are generic, the conformal bootstrap approach developed in [63, 64] can be adapted to the imaginary Toda case. It is important to detail this calculation for the  $\mathfrak{sl}_n$  Toda field theory with  $c \leq n - 1$ . Indeed, in the case of Liouville ( $n = 2$ ) it is known that the three-point structure constants for  $c \leq 1$  are *not* given by the analytic continuation of the ones obtained for  $c \notin ] - \infty, 1 ]$  [130, 156].

In order to compute these structure constants, following [63, 64], we impose crossing symmetry on the correlation function

$$\mathcal{G}(z, \bar{z}) = \langle \Phi_{\alpha_2}(\infty) \Phi_{b\omega_1}(z, \bar{z}) \Phi_{\alpha_3}(1) \Phi_{\alpha_1}(0) \rangle.$$

*In fact one formally treats the variables  $z$  and  $\bar{z}$  as being independent, and there are two differential equations: one with respect to  $z$ , and one with respect to  $\bar{z}$ .*

This correlation function obeys a Fuchsian differential equation of order  $n$  (see Appendix 3.B). The solutions of this differential equation form a representation of the fundamental group

$$\rho : \pi_1 \left( \mathbb{CP}^1 \setminus \{0, 1, \infty\} \right) \rightarrow \text{GL}(n, \mathbb{C})$$

The fundamental group of the punctured sphere  $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$  is generated by the loops  $\gamma_0$  and  $\gamma_{\infty}$  winding around 0 and  $\infty$  respectively (in the positive direction). Accordingly, we introduce two bases of solutions for the differential equation:  $\{F_i(z), i = 1, \dots, n\}$  with Abelian monodromies around 0, and  $\{G_i(z), i = 1, \dots, n\}$  with Abelian

monodromies around  $\infty$ . Explicit formulas in terms of hypergeometric functions can be found in Appendix 3.B. These two bases are related through:

$$F_i(z) = \sum_{j=1}^n M_{ij} G_j(z), \quad (3.30)$$

where the matrix  $M$  is given by (3.93) in Appendix 3.B. The correlation function  $\mathcal{G}(z, \bar{z})$  is built by gluing left and right conformal blocks

$$\mathcal{G}(z, \bar{z}) = \sum_{i,j=1}^n X_{ij} F_i(z) F_j(\bar{z})$$

in such a way as to obtain a single-valued function of  $z$ , *i.e.* a function with trivial monodromies. This means that for any  $\lambda \in \pi_1(\mathbb{CP}^1 \setminus \{0, 1, \infty\})$ , one must have

$$\rho(\lambda)^T X \rho(\lambda) = X$$

It is sufficient to impose this condition for  $\lambda = \gamma_0$  and  $\lambda = \gamma_\infty$ . In the generic case the blocks  $F_i(z)$  have distinct monodromies, and trivial monodromy of  $\mathcal{G}$  around  $z = 0$  is equivalent to the matrix  $X$  being diagonal. Similarly, around  $\infty$  one can decompose

$$\mathcal{G}(z, \bar{z}) = \sum_{i,j=1}^n Y_{ij} G_i(z) G_j(\bar{z}),$$

and the matrix  $Y = M^T X M$  has to be diagonal as well. Since  $X$  and  $Y$  are diagonal, one has

$$\sum_j X_j M_{jk} M_{jl} = Y_k \delta_{k,l}.$$

This over-complete set of linear equations has a unique solution (up to a global pre-factor)

$$\frac{X_i}{X_j} = \frac{M_{jm} (M^{-1})_{mi}}{M_{im} (M^{-1})_{mj}}.$$

provided the r.h.s. does not depend on  $m$ . For scalar fields this is indeed the case, since

$$\frac{M_{jm} (M^{-1})_{mi}}{M_{im} (M^{-1})_{mj}} = \frac{\prod_{k \neq i} \gamma(A_k - A_i)}{\prod_{k \neq j} \gamma(A_k - A_j)} \prod_{k=1}^n \frac{\gamma(A_i + B_k)}{\gamma(A_j + B_k)},$$

where  $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ ,

$$A_i = \Delta_{\alpha_1 + bh_i} - \Delta_{\alpha_1} + b\mu, \quad B_i = \Delta_{\alpha_2 + bh_i} - \Delta_{\alpha_2} + b\mu, \quad (3.31)$$

and

$$2\mu = \begin{cases} (1/b - b) - \frac{\kappa}{n} & \text{if } \alpha_3 = \kappa\omega_1, \\ \frac{\kappa}{n} & \text{if } \alpha_3 = \kappa\omega_{n-1}. \end{cases} \quad (3.32)$$

The coefficients  $X_i$  are related to the structure constants as follows

$$\frac{X_i}{X_j} = \frac{C(\alpha_1, b\omega_1, 2Q - \alpha_1 - bh_i)}{C(\alpha_1, b\omega_1, 2Q - \alpha_1 - bh_j)} \frac{C(\alpha_1 + bh_i, \alpha_2, \alpha_3)}{C(\alpha_1 + bh_j, \alpha_2, \alpha_3)},$$

and crossing symmetry boils down to

$$\frac{C(\alpha_1 + bh_i, \alpha_2, \alpha_3)}{C(\alpha_1 + bh_j, \alpha_2, \alpha_3)} = \frac{K_i(\alpha_1)}{K_j(\alpha_1)} \times \prod_{k=1}^n \frac{\gamma(A_i + B_k)}{\gamma(A_j + B_k)}, \quad (3.33)$$

where  $K_i(\alpha_1)$  and  $K_j(\alpha_1)$  are some normalizing factors. Repeating the same steps with the fully degenerate field  $\Phi_{-b^{-1}\omega_1}$  yields the same relation with  $b \rightarrow -b^{-1}$ :

$$\frac{C(\alpha_1 - h_i/b, \alpha_2, \alpha_3)}{C(\alpha_1 - h_j/b, \alpha_2, \alpha_3)} = \frac{\widehat{K}_i(\alpha_1)}{\widehat{K}_j(\alpha_1)} \times \prod_{k=1}^n \frac{\gamma(C_i + D_k)}{\gamma(C_j + D_k)}, \quad (3.34)$$

where  $\widehat{K}_i(\alpha_1)$  and  $\widehat{K}_j(\alpha_1)$  are some normalizing factors, and

$$C_i = \Delta_{\alpha_1 - h_i/b} - \Delta_{\alpha_1} - \mu/b, \quad D_i = \Delta_{\alpha_2 - h_i/b} - \Delta_{\alpha_2} - \mu/b. \quad (3.35)$$

If  $b$  is real and  $b^2$  is irrational then these two relations determine uniquely the three point function up to a  $\kappa$  dependent multiplicative factor. It can be expressed in terms of the  $Y_b$  function, whose definition and main properties we recall in appendix 3.C:

$$C(\alpha_1, \alpha_2, \alpha_3) = M(\kappa) \times \frac{\prod_{k,\ell=1}^n Y_b [b + (\alpha_1 - Q) \cdot h_k + (\alpha_2 - Q) \cdot h_\ell + 2\mu]}{\sqrt{\prod_{i=1}^2 \prod_{e>0} Y_b [b + (\alpha_i - Q) \cdot e] Y_b [b - (\alpha_i - Q) \cdot e]}}, \quad (3.36)$$

where  $\alpha_3$  must be semi-degenerate and  $\mu$  is given in (3.32). In the denominator the product is over all positive roots of  $\mathfrak{sl}_n$ . The full bootstrap argument above is only valid when  $\alpha_3$  is semi-degenerate, and  $\alpha_1, \alpha_2$  are non-degenerate, i.e. they are not of the form (3.29). Indeed, if for instance  $\alpha_1$  is semi-degenerate (say  $\alpha_1 = \lambda\omega_1$ ), then in the channel  $z \rightarrow 0$  only two conformal blocks are allowed for  $\mathcal{G}$ , due to the fusion rule

$$\Phi_{b\omega_1} \otimes \Phi_{\lambda\omega_1} \rightarrow \Phi_{\lambda\omega_1 + bh_1} \oplus \Phi_{\lambda\omega_1 + bh_2},$$

and then the above derivation of structure constants is no longer valid. An exception is when  $\alpha_2 = 0$ : in this case one considers the OPE of a semi-degenerate field  $\Phi_{\alpha_3}$  with the identity.

Note that it is not necessary to keep track of the normalizing factors in (3.33–3.34) to derive (3.36). Any function of the form

$$C(\alpha_1, \alpha_2, \alpha_3) = \frac{\prod_{k,\ell=1}^n Y_b [b + (\alpha_1 - Q) \cdot h_k + (\alpha_2 - Q) \cdot h_\ell + 2\mu]}{\phi(\alpha_1)\phi(\alpha_2)}$$

is a solution of (3.33–3.34) with some functions  $\{K_i(\alpha)\}$  and  $\{\widehat{K}_i(\alpha)\}$  given in terms of  $\phi(\alpha)$ , and the precise form of  $\phi(\alpha)$  is completely determined by imposing:

$$C(\alpha_1, \alpha_1^*, 0) = 1 \quad (3.37)$$

$$C(\alpha_1, \alpha_2, \kappa\omega_1) = C(\alpha_1^*, \alpha_2^*, \kappa\omega_{n-1}). \quad (3.38)$$

The factor  $M(\kappa)$  can be then be found by demanding that  $C(\kappa\omega_1, 0, \kappa\omega_{n-1}) = 1$ , which yields:

$$M(\kappa) = \frac{1}{Y_b(b)^n} \sqrt{\frac{Y_b(b)Y_b(b+nQ)}{Y_b(b+\kappa)Y_b(b-\kappa+nQ)}}. \quad (3.39)$$

The three point function(3.36) enjoys the following properties:

$$C(\alpha_1, \alpha_2, \alpha_3) = C(\alpha_2, \alpha_1, \alpha_3), \quad (3.40)$$

$$\forall \sigma, \sigma' \in \mathfrak{S}_n^2, \quad C(\alpha_1, \alpha_2, \alpha_3) = C(\sigma \star \alpha_1, \sigma' \star \alpha_2, \alpha_3). \quad (3.41)$$

Finally, to compare, the three-point function found by Fateev and Litvinov [63, 64] in the case of real Toda with central charge  $c = (n-1) + 12\widehat{Q}^2$  is of the form:

$$C_{\text{FL}}(\alpha_1, \alpha_2, \kappa\omega_{n-1}) = \frac{A(\alpha_1)A(\alpha_2)B(\kappa)}{\prod_{i,j} Y_{\widehat{b}}\left(\kappa/3 + (\alpha_1 - \widehat{Q}) \cdot h_i + (\alpha_2 - \widehat{Q}) \cdot h_j\right)} \quad (3.42)$$

where  $\widehat{Q} = (\widehat{b} + 1/\widehat{b})\rho$ , and  $A(\alpha)$  and  $B(\kappa)$  are some normalizing factors. We see that, like for the Liouville theory, the structure constants of the imaginary Toda theory are *not* the analytic continuation of the real Toda ones.

### 3.4 NON-SCALAR FIELDS IN THE IMAGINARY TODA FIELD THEORY

In the previous section we considered scalar fields parametrized by the same charges  $\alpha = \bar{\alpha}$  in the holomorphic and anti-holomorphic sectors, in a  $W_n$ -conformal field theory with a generic central charge  $c \leq n-1$  (i.e.  $b^2$  non rational). Scalar fields are mutually local, and this leads to monodromy invariant correlation function. But it is also possible for a physical correlation function to acquire a non-trivial phase as a field winds around another. For instance this is typically the case for spin and disorder operators in the  $\mathbb{Z}_n$  parafermion model, in which the phase is then a  $n^{\text{th}}$  root of unity.

This motivates an investigation of non-scalar primary fields  $\Phi_{\alpha, \bar{\alpha}}$  parametrized by two vector charges  $\alpha$  and  $\bar{\alpha}$ .



## 3.4.1 Non-degenerate non-scalar operators

The fully degenerate fields  $\{\Phi_{b\omega_1}, \Phi_{b\omega_{n-1}}, \Phi_{-b^{-1}\omega_1}, \Phi_{-b^{-1}\omega_{n-1}}\}$  are assumed to be part of the spectrum of the theory. By demanding a well-defined monodromy between  $\Phi_{\alpha, \bar{\alpha}}$  and these fully degenerate fields, some constraints are obtained on the possible values of  $\alpha$  and  $\bar{\alpha}$ . Consider for instance the OPE between  $\Phi_{b\omega_1}$  and  $\Phi_{\alpha, \bar{\alpha}}$ . Since we know the (chiral) fusion rules of  $\Phi_{b\omega_1}$  with any field, this OPE has to be of the form

$$\begin{aligned} & \Phi_{b\omega_1}(z, \bar{z}) \Phi_{\alpha, \bar{\alpha}}(0) \\ &= \sum_{i,j=1}^n C_{ij} z^{\Delta_{\alpha+bh_i} - \Delta_{b\omega_1} - \Delta_{\alpha}} \bar{z}^{\Delta_{\bar{\alpha}+bh_j} - \Delta_{b\omega_1} - \Delta_{\bar{\alpha}}} \Phi_{\alpha+bh_i, \bar{\alpha}+bh_j}(0) + \dots \end{aligned}$$

We impose that every term in the right-hand side has the same monodromy  $e^{2i\pi\eta}$  when  $z$  goes around zero, so that:

$$\Phi_{b\omega_1}\left(e^{2i\pi}z, e^{-2i\pi}\bar{z}\right) \Phi_{\alpha, \bar{\alpha}}(0) = e^{2i\pi\eta} \Phi_{b\omega_1}(z, \bar{z}) \Phi_{\alpha, \bar{\alpha}}(0). \quad (3.43)$$

The monodromy exponent for the term  $(i, j)$  in the above sum is

$$\eta_{ij} = (\Delta_{\alpha+bh_i} - \Delta_{\bar{\alpha}+bh_j}) - (\Delta_{\alpha} - \Delta_{\bar{\alpha}}),$$

and so we impose that  $(\eta_{ij} - \eta_{ik})$  is an integer for any non-zero terms  $(i, j)$  and  $(i, k)$  in the sum. After simple manipulations, this is equivalent to  $(\bar{\alpha} - \mathbf{Q}) \cdot (\mathbf{h}_j - \mathbf{h}_k) \in \mathbb{Z}/b$ . If  $\bar{\alpha}$  is generic, this means that  $j = k$ . Hence, the coefficient matrix must be of the form:

$$C_{ij} = \delta_{i, \tau(j)} C_j,$$

where  $\tau \in \mathfrak{S}_n$  is a permutation. This permutation encodes the fusion rules

$$\Phi_{b\omega_1} \times \Phi_{\alpha, \bar{\alpha}} = \sum_{j=1}^n \Phi_{\alpha+bh_{\tau(j)}, \bar{\alpha}+bh_j}. \quad (3.44)$$

Note that, using (3.21), we are free to relabel  $\alpha \rightarrow \alpha' = \mu \star \alpha$ , and  $\bar{\alpha} \rightarrow \bar{\alpha}' = \bar{\mu} \star \bar{\alpha}$ , to get:

$$\begin{aligned} \Phi_{b\omega_1} \times \Phi_{\alpha', \bar{\alpha}'} &\equiv \Phi_{b\omega_1} \times \Phi_{\alpha, \bar{\alpha}} = \sum_{j=1}^n \Phi_{\alpha+bh_{\tau(j)}, \bar{\alpha}+bh_j} \\ &\equiv \sum_{j=1}^n \Phi_{\mu \star (\alpha+bh_{\tau(j)}), \bar{\mu} \star (\bar{\alpha}+bh_j)} = \sum_{j=1}^n \Phi_{\alpha'+bh_{\mu\tau(j)}, \bar{\alpha}'+bh_{\bar{\mu}(j)}} \\ &= \sum_{k=1}^n \Phi_{\alpha'+bh_{\mu\tau\bar{\mu}^{-1}(k)}, \bar{\alpha}'+bh_k}. \end{aligned} \quad (3.45)$$

In the OPE, which we now write as

$$\begin{aligned} & \Phi_{b\omega_1}(z, \bar{z}) \Phi_{\alpha, \bar{\alpha}}(0) \\ &= \sum_{j=1}^n C_j z^{\Delta_{\alpha+bh_{\tau(j)}} - \Delta_{b\omega_1} - \Delta_{\alpha}} \bar{z}^{\Delta_{\bar{\alpha}+bh_j} - \Delta_{b\omega_1} - \Delta_{\bar{\alpha}}} \Phi_{\alpha+bh_{\tau(j)}, \bar{\alpha}+bh_j} + \dots \end{aligned}$$

the monodromy exponent for the  $j^{\text{th}}$  term in the sum is

$$\eta_j = (\Delta_{\alpha+bh_{\tau(j)}} - \Delta_{\bar{\alpha}+bh_j}) - (\Delta_{\alpha} - \Delta_{\bar{\alpha}}) = bh_j \cdot (\tau^{-1} \star \alpha - \bar{\alpha}).$$

In the generic case, the  $n$  possible values of  $j$  are allowed (*i.e.* none of the  $C_j$  vanishes) and all the terms must yield the same monodromy  $e^{2i\pi\eta}$ , leading to the condition

$$\forall (j, k), \quad \eta_j - \eta_k \in \mathbb{Z}.$$

This boils down to  $b(\mathbf{h}_j - \mathbf{h}_k) \cdot (\tau^{-1} \star \alpha - \bar{\alpha}) \in \mathbb{Z}$ , which is equivalent to

$$\tau^{-1} \star \alpha - \bar{\alpha} \in \mathcal{R}/b \quad \Leftrightarrow \quad \alpha - \tau \star \bar{\alpha} \in \mathcal{R}/b, \quad (3.46)$$

where  $\mathcal{R}$  is the weight lattice defined in Appendix 3.A. An interesting consequence is that

$$\eta \in \frac{\mathbb{Z}}{n},$$

*i.e.* the overall monodromy around  $z = 0$  of the OPE  $\Phi_{b\omega_1}(z, \bar{z})\Phi_{\alpha, \bar{\alpha}}(0)$  can only be a  $n^{\text{th}}$  root of unity.

One can repeat this argument with the fully-degenerate field  $\Phi_{b\omega_1}$  replaced by its dual  $\Phi_{b\omega_{n-1}}$ . The fusion rules can be written:

$$\Phi_{b\omega_{n-1}} \times \Phi_{\alpha, \bar{\alpha}} = \sum_{j=1}^n \Phi_{\alpha-bh_{\mu(j)}, \bar{\alpha}-bh_j}, \quad (3.47)$$

with the constraint:

$$\mu^{-1} \star \alpha - \bar{\alpha} \in \mathcal{R}/b \quad \Leftrightarrow \quad \alpha - \mu \star \bar{\alpha} \in \mathcal{R}/b, \quad (3.48)$$

for some permutation  $\mu \in \mathfrak{S}_n$ . If  $\mu \neq \tau$ , the constraints (3.46) and (3.48) will together yield:

$$(\mu\tau^{-1}) \star \alpha - \alpha \in \mathcal{R}/b, \quad \text{and} \quad (\mu^{-1}\tau) \star \bar{\alpha} - \bar{\alpha} \in \mathcal{R}/b,$$

which forces both  $\alpha$  and  $\bar{\alpha}$  to live in some determined lattices. But since we want to reason on generic values of  $\alpha$  and  $\bar{\alpha}$ , we discard this case, and in the following we consider only the case when  $\mu = \tau$ .

Let us now examine the fusion of our  $\Phi_{\alpha, \bar{\alpha}}$  with the fields  $\Phi_{-\omega_1/b}$  and  $\Phi_{-\omega_{n-1}/b}$ , and suppose again that they are both determined by some permutation  $\sigma \in \mathfrak{S}_n$ . The same line of reasoning as above yields:

$$\sigma^{-1} \star \alpha - \bar{\alpha} \in b\mathcal{R} \quad \Leftrightarrow \quad \alpha - \sigma \star \bar{\alpha} \in b\mathcal{R}, \quad (3.49)$$

corresponding to the fusion rules

$$\Phi_{-\omega_1/b} \times \Phi_{\alpha, \bar{\alpha}} = \sum_{j=1}^n \Phi_{\alpha-h_{\sigma(j)/b}, \bar{\alpha}-h_j/b}, \quad (3.50)$$

$$\Phi_{-\omega_{n-1}/b} \times \Phi_{\alpha, \bar{\alpha}} = \sum_{j=1}^n \Phi_{\alpha+h_{\sigma(j)/b}, \bar{\alpha}+h_j/b}. \quad (3.51)$$

Therefore, it appears that a generic primary field is labelled by two charges  $(\alpha, \bar{\alpha})$  and two permutations  $(\tau, \sigma)$ . But we still have to discuss the effect of charge reparametrization by the Weyl group (3.21). As seen in (3.45), if we relabel  $(\alpha, \bar{\alpha}) \rightarrow (\alpha', \bar{\alpha}') = (\mu \star \alpha, \bar{\mu} \star \bar{\alpha})$ , the permutations  $\tau$  and  $\sigma$  are changed to:

$$\tau \rightarrow \mu\tau\bar{\mu}^{-1}, \quad \sigma \rightarrow \mu\sigma\bar{\mu}^{-1}. \quad (3.52)$$

Taking a generic permutation  $\mu$  and setting  $\bar{\mu} = \mu\tau$ , we get

$$\tau \rightarrow \text{id}, \quad \sigma \rightarrow \mu\sigma'\mu^{-1}, \quad (3.53)$$

where  $\sigma' = \sigma\tau^{-1}$ . Hence, without loss of generality,  $\tau$  can always be set to the identity, whereas  $\sigma$  is defined modulo conjugation by any permutation  $\mu$ .

One may ask what permutations are attached to the primary fields appearing in the right-hand side of fusion (3.44). This is easy to see from the constraints (3.46) and (3.49) on vertex charges. We set  $\tau$  to the identity, so that (3.46) becomes  $\alpha - \bar{\alpha} \in \mathcal{R}/b$ . This condition is obviously satisfied by  $(\alpha + bh_j, \bar{\alpha} + bh_j)$ . The second condition (3.49) for this pair of charges reads:

$$(\alpha - \sigma \star \bar{\alpha}) + b(\mathbf{h}_j - \mathbf{h}_{\sigma(j)}) \in b\mathcal{R},$$

which is also satisfied, because every  $\mathbf{h}_k$  belongs to  $\mathcal{R}$ . Similar arguments hold for the other fusions considered above, and for the dual field  $\Phi_{2Q-\alpha, 2Q-\bar{\alpha}}$ .

Overall, we get the following characterization of generic primary fields:

- A primary field is labelled by a pair of vertex charges  $(\alpha, \bar{\alpha})$ , and a permutation  $\sigma \in \mathfrak{S}_n$ . We denote it as  $\Phi_{\alpha, \bar{\alpha}}^{(\sigma)}$ , and the vertex charges must satisfy:

$$\alpha - \bar{\alpha} \in \mathcal{R}/b, \quad \text{and} \quad \alpha - \sigma \star \bar{\alpha} \in b\mathcal{R}. \quad (3.54)$$

Note that these conditions, as well as the quantum numbers  $w_\alpha^{(k)}, w_{\bar{\alpha}}^{(k)}$ , are invariant under reparametrization  $(\alpha, \bar{\alpha}, \sigma) \rightarrow (\mu \star \alpha, \mu \star \bar{\alpha}, \mu\sigma\mu^{-1})$ . Hence, the behaviour of  $\Phi_{\alpha, \bar{\alpha}}^{(\sigma)}$  under fusion is really determined by the *conjugacy class* of  $\sigma$ .

- The fusion rules with the fully degenerate fields are:

$$\begin{aligned} \Phi_{b\omega_1} \times \Phi_{\alpha, \bar{\alpha}}^{(\sigma)} &= \sum_{j=1}^n \Phi_{\alpha+b\mathbf{h}_j, \bar{\alpha}+b\mathbf{h}_j}^{(\sigma)}, \\ \Phi_{b\omega_{n-1}} \times \Phi_{\alpha, \bar{\alpha}}^{(\sigma)} &= \sum_{j=1}^n \Phi_{\alpha-b\mathbf{h}_j, \bar{\alpha}-b\mathbf{h}_j}^{(\sigma)}, \end{aligned} \quad (3.55)$$

$$\begin{aligned}\Phi_{-\omega_1/b} \times \Phi_{\alpha, \bar{\alpha}}^{(\sigma)} &= \sum_{j=1}^n \Phi_{\alpha - h_{\sigma(j)}/b, \bar{\alpha} - h_j/b}^{(\sigma)}, \\ \Phi_{-\omega_{n-1}/b} \times \Phi_{\alpha, \bar{\alpha}}^{(\sigma)} &= \sum_{j=1}^n \Phi_{\alpha + h_{\sigma(j)}/b, \bar{\alpha} + h_j/b}^{(\sigma)}.\end{aligned}\tag{3.56}$$

The monodromy exponents (defined up to the addition of an integer) of the corresponding OPEs are :

$$\eta(\alpha, \bar{\alpha}) = bh_1 \cdot (\alpha - \bar{\alpha}), \quad \hat{\eta}(\alpha, \bar{\alpha}) = -\frac{1}{b}h_1 \cdot (\alpha - \sigma \star \bar{\alpha}), \tag{3.57}$$

for the fusion of  $\Phi_{\alpha, \bar{\alpha}}^{(\sigma)}$  with  $\Phi_{b\omega_1}$  and  $\Phi_{-\omega_1/b}$ , respectively. These exponents belong to  $\mathbb{Z}/n$ , and the monodromy factors  $e^{2i\pi\eta(\alpha, \bar{\alpha})}$  and  $e^{2i\pi\hat{\eta}(\alpha, \bar{\alpha})}$  can be considered as two  $\mathbb{Z}_n$  charges associated to the field  $\Phi_{\alpha, \bar{\alpha}}^{(\sigma)}$ .

- The dual of the field  $\Phi_{\alpha, \bar{\alpha}}^{(\sigma)}$  is :  $(\Phi_{\alpha, \bar{\alpha}}^{(\sigma)})^* = \Phi_{2Q-\alpha, 2Q-\bar{\alpha}}^{(\sigma)}$ .
- The particular case  $\sigma = \text{id}$  corresponds to *scalar* primary fields. Indeed, as long as  $b^2$  is not rational, the conditions (3.54) yield  $\alpha = \bar{\alpha}$ .

Note that this characterization is valid for *generic* (i.e. non-degenerate) values of the vertex charges  $(\alpha, \bar{\alpha})$ . Indeed, the fusion of a degenerate operator with  $\Phi_{b\omega_1}, \Phi_{b\omega_{n-1}}, \Phi_{\omega_1/b}$  and  $\Phi_{\omega_{n-1}/b}$  is different from the generic one (3.55–3.56), and the whole argument of this section is then invalid.

### 3.4.2 Semi-degenerate non-scalar operators

For a semi-degenerate field, which we shall denote  $\tilde{\Phi}_{\alpha, \bar{\alpha}}$ , we have to use the fusion rules (3.92). For instance, for  $(\alpha, \bar{\alpha}) = (\kappa\omega_1, \bar{\kappa}\omega_1)$ , one has the chiral fusion rule:

$$\Phi_{b\omega_1} \otimes \Phi_{\kappa\omega_1} \rightarrow \Phi_{\kappa\omega_1 + bh_1} + (n-1)\Phi_{\kappa\omega_1 + bh_2},$$

where the coefficient on the second term means that there are  $(n-1)$  independent conformal blocks corresponding to this internal field in the fusion channel  $z \rightarrow 1$  for any four-point function of the form (3.59). For generic values of  $\kappa$ , the monodromy exponents corresponding to the first term and the next  $(n-1)$  do not differ by an integer. Hence, in order to get a well-defined monodromy for the four-point function, one has to select the fusion rule:

$$\Phi_{b\omega_1} \otimes \tilde{\Phi}_{\kappa\omega_1, \bar{\kappa}\omega_1} \rightarrow \Phi_{\kappa\omega_1 + bh_1, \bar{\kappa}\omega_1 + bh_1} + (n-1)\Phi_{\kappa\omega_1 + bh_2, \bar{\kappa}\omega_1 + bh_2}.$$

From there, using a similar computation as for the case of generic non-scalar operators (see previous section), one gets the constraint:

$$\kappa - \bar{\kappa} \in \mathbb{Z}/b. \tag{3.58}$$

## 3.4.3 Structure constants

3.4.3.1 Shift equation from the null descendant of  $\Phi_{b\omega_1}$ 

In order to compute the structure constants, we turn to the four point function

$$\mathcal{G}(z, \bar{z}) = \left\langle \Phi_{\alpha_2^*, \bar{\alpha}_2^*}^{(\sigma_2)}(\infty) \Phi_{b\omega_1}(z, \bar{z}) \tilde{\Phi}_{\alpha_3, \bar{\alpha}_3}(1) \Phi_{\alpha_1, \bar{\alpha}_1}^{(\sigma_1)}(0) \right\rangle, \quad (3.59)$$

where

$$(\alpha_3, \bar{\alpha}_3) = (\kappa\omega_1, \bar{\kappa}\omega_1) \quad \text{or} \quad (\alpha_3, \bar{\alpha}_3) = (\kappa\omega_{n-1}, \bar{\kappa}\omega_{n-1}), \quad \text{with } (\kappa, \bar{\kappa}) \in \mathbb{R}^2.$$

For this correlation function to be non-trivial, one needs to impose the constraint on monodromy exponents (3.57):

$$e^{2i\pi[\eta(\alpha_1, \bar{\alpha}_1) + \eta(\alpha_2, \bar{\alpha}_2) + \eta(\alpha_3, \bar{\alpha}_3)]} = 1, \quad (3.60)$$

which can be viewed as a  $\mathbb{Z}_n$  charge neutrality condition on  $\mathcal{G}$ .

The correlation function (3.59) is built by gluing the left and right conformal blocks in such a way as to ensure well-defined global monodromies :

$$\mathcal{G}(z, \bar{z}) = \sum_{i,j=1}^n X_{ij} F_i(z) \bar{F}_j(\bar{z}) = \sum_{i,j=1}^n Y_{ij} G_i(z) \bar{G}_j(\bar{z}),$$

where the conformal blocks are the same as in the scalar case, and are given in appendix 3.B. For the reader's convenience, we recall that the holomorphic blocks  $F_i$  and  $G_i$  are expressed in terms of

$$\begin{aligned} A_i &= \Delta_{\alpha_1 + bh_i} - \Delta_{\alpha_1} + b\mu, & \bar{A}_i &= \Delta_{\bar{\alpha}_1 + bh_i} - \Delta_{\bar{\alpha}_1} + b\bar{\mu}, \\ B_i &= \Delta_{\alpha_2 + bh_i} - \Delta_{\alpha_2} + b\mu, & \bar{B}_i &= \Delta_{\bar{\alpha}_2 + bh_i} - \Delta_{\bar{\alpha}_2} + b\bar{\mu}, \end{aligned} \quad (3.61)$$

where

$$(2\mu, 2\bar{\mu}) = \begin{cases} [(b^{-1} - b) - \frac{\kappa}{n}, (b^{-1} - b) - \frac{\bar{\kappa}}{n}] & \text{if } (\alpha_3, \bar{\alpha}_3) = (\kappa\omega_1, \bar{\kappa}\omega_1), \\ (\frac{\kappa}{n}, \frac{\bar{\kappa}}{n}) & \text{if } (\alpha_3, \bar{\alpha}_3) = (\kappa\omega_{n-1}, \bar{\kappa}\omega_{n-1}). \end{cases}$$

From the fusion rules of non-scalar fields with  $\Phi_{b\omega_1}$  one must have  $X_{ij} = X_i \delta_{ij}$  and  $Y_{ij} = Y_i \delta_{ij}$ . At this point it is interesting to compute the differences:

$$\begin{aligned} A_i - \bar{A}_i &= bh_i \cdot (\alpha_1 - \bar{\alpha}_1) + b(\mu - \bar{\mu}), \\ B_i - \bar{B}_i &= bh_i \cdot (\alpha_2 - \bar{\alpha}_2) + b(\mu - \bar{\mu}). \end{aligned} \quad (3.62)$$

Since  $(\alpha_k - \bar{\alpha}_k) \in \mathcal{R}/b$ , these quantities are independent of  $i$  (up to an integer), and, due to (3.60), one has:

$$(A_i - \bar{A}_i) + (B_j - \bar{B}_j) \in \mathbb{Z}. \quad (3.63)$$

Now if we apply the change of bases between 0 and  $\infty$  to  $\sum_j Y_j |J_j(z)|^2$ , we find

$$Y = M^t X \bar{M},$$

where the matrices  $M = M(\{A_i\}, \{B_j\})$  and  $\bar{M} = M(\{\bar{A}_i\}, \{\bar{B}_j\})$  are given by (3.93). Since  $X$  and  $Y$  are diagonal, one has for all  $k$  and  $\ell$ :

$$\sum_j M_{jk} X_j \bar{M}_{j\ell} = Y_k \delta_{k\ell}. \quad (3.64)$$

The subsystem of equations corresponding to fixed  $\ell$  and  $k = 1, \dots, n$  yields:

$$X_j \propto \frac{(M^{-1})_{\ell j}}{\bar{M}_{j\ell}} \Rightarrow \forall(i, j), \quad \frac{X_i}{X_j} = \frac{\bar{M}_{j\ell} (M^{-1})_{\ell i}}{\bar{M}_{i\ell} (M^{-1})_{\ell j}}. \quad (3.65)$$

For a non-trivial solution to exist, the ratio  $X_i/X_j$  should be independent of  $\ell$ :

$$\forall(i, j, \ell, m), \quad \frac{\bar{M}_{j\ell} (M^{-1})_{\ell i}}{\bar{M}_{i\ell} (M^{-1})_{\ell j}} = \frac{\bar{M}_{jm} (M^{-1})_{mi}}{\bar{M}_{im} (M^{-1})_{mj}},$$

which boils down to the consistency condition:

$$\begin{aligned} & \frac{\sin \pi(\bar{A}_i + \bar{B}_\ell) \sin \pi(A_j + B_\ell)}{\sin \pi(A_i + B_\ell) \sin \pi(\bar{A}_j + \bar{B}_\ell)} \\ &= \frac{\sin \pi(\bar{A}_i + \bar{B}_m) \sin \pi(A_j + B_m)}{\sin \pi(A_i + B_m) \sin \pi(\bar{A}_j + \bar{B}_m)}. \end{aligned} \quad (3.66)$$

This is the  $\mathfrak{sl}_n$  generalization of the constraint obtained in [61] in the  $\mathfrak{sl}_2$  case. The property (3.63) deriving from the single constraint (3.60) is actually a sufficient condition for (3.66) to be satisfied.

Let us now turn to (3.65). This translates into the following shift equation:

$$\begin{aligned} & \frac{C(\Phi_{\alpha_1, \bar{\alpha}_1}^{(\sigma_1)}, \Phi_{b\omega_1}, \Phi_{\alpha_1 + bh_i, \bar{\alpha}_1 + bh_i}^{(\sigma_1)*})}{C(\Phi_{\alpha_1, \bar{\alpha}_1}^{(\sigma_1)}, \Phi_{b\omega_1}, \Phi_{\alpha_1 + bh_j, \bar{\alpha}_1 + bh_j}^{(\sigma_1)*})} \times \frac{C(\Phi_{\alpha_1 + bh_i, \bar{\alpha}_1 + bh_i}^{(\sigma_1)}, \Phi_{\alpha_2, \bar{\alpha}_2}^{(\sigma_2)}, \tilde{\Phi}_{\alpha_3, \bar{\alpha}_3})}{C(\Phi_{\alpha_1 + bh_j, \bar{\alpha}_1 + bh_j}^{(\sigma_1)}, \Phi_{\alpha_2, \bar{\alpha}_2}^{(\sigma_2)}, \tilde{\Phi}_{\alpha_3, \bar{\alpha}_3})} \\ &= \left[ \prod_{k \neq i} \frac{\Gamma(A_k - A_i)}{\Gamma(1 - \bar{A}_k + \bar{A}_i)} \prod_{k \neq j} \frac{\Gamma(1 - \bar{A}_k + \bar{A}_j)}{\Gamma(A_k - A_j)} \right] \\ & \times \left[ \frac{\Gamma(A_i + B_\ell) \Gamma(1 - \bar{A}_j - \bar{B}_\ell)}{\Gamma(A_j + B_\ell) \Gamma(1 - \bar{A}_i - \bar{B}_\ell)} \prod_{k \neq \ell} \frac{\Gamma(\bar{A}_i + \bar{B}_k) \Gamma(1 - A_j - B_k)}{\Gamma(\bar{A}_j + \bar{B}_k) \Gamma(1 - A_i - B_k)} \right]. \end{aligned} \quad (3.67)$$

The first bracket in the right-hand side only depends on  $\alpha_1$  and  $\bar{\alpha}_1$ . Let us rewrite the second bracket as:

$$\frac{\sin \pi(\bar{A}_i + \bar{B}_\ell) \sin \pi(A_j + B_\ell)}{\sin \pi(A_i + B_\ell) \sin \pi(\bar{A}_j + \bar{B}_\ell)} \prod_{k=1}^n \frac{\Gamma(\bar{A}_i + \bar{B}_k) \Gamma(1 - A_j - B_k)}{\Gamma(\bar{A}_j + \bar{B}_k) \Gamma(1 - A_i - B_k)}.$$

We note that (3.64) is invariant under the exchange  $(A_i, B_i) \leftrightarrow (\bar{A}_i, \bar{B}_i)$ . Hence, up to a sign, one can replace the right-hand side of (3.67) by the geometric mean:

$$\begin{aligned} & \frac{C(\Phi_{\alpha_1+bh_i, \bar{\alpha}_1+bh_i}^{(\sigma_1)}, \Phi_{\alpha_2, \bar{\alpha}_2}^{(\sigma_2)}, \tilde{\Phi}_{\alpha_3, \bar{\alpha}_3})}{C(\Phi_{\alpha_1+bh_j, \bar{\alpha}_1+bh_j}^{(\sigma_1)}, \Phi_{\alpha_2, \bar{\alpha}_2}^{(\sigma_2)}, \tilde{\Phi}_{\alpha_3, \bar{\alpha}_3})} \\ &= \frac{K_i^{(\sigma_1)}(\alpha_1, \bar{\alpha}_1)}{K_j^{(\sigma_1)}(\alpha_1, \bar{\alpha}_1)} \times \sqrt{\prod_{k=1}^n \frac{\gamma(A_i + B_k) \gamma(\bar{A}_i + \bar{B}_k)}{\gamma(A_j + B_k) \gamma(\bar{A}_j + \bar{B}_k)}}, \end{aligned} \quad (3.68)$$

where  $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ , and  $K_i^{(\sigma_1)}(\alpha_1, \bar{\alpha}_1)$  and  $K_j^{(\sigma_1)}(\alpha_1, \bar{\alpha}_1)$  are some normalization factors.

### 3.4.3.2 Shift equation from the null descendant of $\Phi_{-\omega_1/b}$

Let us replace the degenerate field  $b\omega_1$  by  $-\omega_1/b$  in the four-point function (3.59):

$$\hat{\mathcal{G}}(z, \bar{z}) = \left\langle \Phi_{\alpha_2^*, \bar{\alpha}_2^*}^{(\sigma_2)}(\infty) \Phi_{-\omega_1/b}(z, \bar{z}) \tilde{\Phi}_{\alpha_3, \bar{\alpha}_3}(1) \Phi_{\alpha_1, \bar{\alpha}_1}^{(\sigma_1)}(0) \right\rangle, \quad (3.69)$$

and demand that the monodromy exponents satisfy:

$$\hat{\eta}(\alpha_1, \bar{\alpha}_1) + \hat{\eta}(\alpha_2, \bar{\alpha}_2) + \hat{\eta}(\alpha_3, \bar{\alpha}_3) \in \mathbb{Z}. \quad (3.70)$$

One can write the decomposition:

$$\hat{\mathcal{G}}(z, \bar{z}) = \sum_{i,j} \hat{X}_{ij} \hat{F}_i(z) \hat{F}_j(\bar{z}) = \sum_{k,\ell} \hat{Y}_{k\ell} \hat{G}_k(z) \hat{G}_\ell(\bar{z}), \quad (3.71)$$

where  $(\hat{F}_i, \hat{F}_j, \hat{G}_k, \hat{G}_\ell)$  are the analogues of  $(F_i, \bar{F}_j, G_k, \bar{G}_\ell)$  with the quadruplet  $(A_i, \bar{A}_j, B_k, \bar{B}_\ell)$  replaced by:

$$\begin{aligned} C_i &= \Delta_{\alpha_1-h_i/b} - \Delta_{\alpha_1} - \mu/b, & \bar{C}_j &= \Delta_{\bar{\alpha}_1-h_j/b} - \Delta_{\bar{\alpha}_1} - \bar{\mu}/b, \\ D_k &= \Delta_{\alpha_2-h_k/b} - \Delta_{\alpha_2} - \mu/b, & \bar{D}_\ell &= \Delta_{\bar{\alpha}_2-h_\ell/b} - \Delta_{\bar{\alpha}_2} - \bar{\mu}/b. \end{aligned}$$

Using the fusion rules (3.56), the coefficient matrices must be of the form:

$$\hat{X}_{ij} = \delta_{i, \sigma_1(j)} \hat{X}_j, \quad \hat{Y}_{k\ell} = \delta_{k, \sigma_2(\ell)} \hat{Y}_\ell. \quad (3.72)$$

Moreover, these matrices are related by

$$\hat{Y} = N^t \hat{X} \bar{N}, \quad (3.73)$$

where  $N = M(\{C_i\}, \{D_j\})$  and  $\bar{N} = M(\{\bar{C}_i\}, \{\bar{D}_j\})$  in (3.93). We get a relation similar to (3.64):

$$\sum_j N'_{jk} \hat{X}_j \bar{N}_{j\ell} = \delta_{k\ell} \hat{Y}_\ell, \quad (3.74)$$

where we have defined the matrix elements  $N'_{jk} = N_{\sigma_1(j), \sigma_2(k)}$ . This corresponds to the matrix in (3.93):

$$N' = M(\{C'_i\}, \{D'_j\}), \quad C'_i = C_{\sigma_1(i)}, \quad D'_j = D_{\sigma_2(j)}. \quad (3.75)$$

Reasoning as above, we obtain a shift equation analogous to (3.68):

$$\begin{aligned} & \frac{C(\Phi_{\alpha_1 - h_{\sigma_1(i)}/b, \bar{\alpha}_1 - h_i/b'}^{(\sigma_1)}, \Phi_{\alpha_2, \bar{\alpha}_2}^{(\sigma_2)}, \tilde{\Phi}_{\alpha_3, \bar{\alpha}_3})}{C(\Phi_{\alpha_1 - h_{\sigma_1(j)}/b, \bar{\alpha}_1 - h_j/b'}^{(\sigma_1)}, \Phi_{\alpha_2, \bar{\alpha}_2}^{(\sigma_2)}, \tilde{\Phi}_{\alpha_3, \bar{\alpha}_3})} \\ &= \frac{\widehat{K}_i^{(\sigma_1)}(\alpha_1, \bar{\alpha}_1)}{\widehat{K}_j^{(\sigma_1)}(\alpha_1, \bar{\alpha}_1)} \times \sqrt{\prod_{k=1}^n \frac{\gamma(C_{\sigma_1(i)} + D_k) \gamma(\bar{C}_i + \bar{D}_k)}{\gamma(C_{\sigma_1(j)} + D_k) \gamma(\bar{C}_j + \bar{D}_k)}}, \end{aligned} \quad (3.76)$$

where  $\widehat{K}_i^{(\sigma_1)}(\alpha_1, \bar{\alpha}_1)$  and  $\widehat{K}_j^{(\sigma_1)}(\alpha_1, \bar{\alpha}_1)$  are some normalizing factors.

### 3.4.3.3 Solution of the shift equations

The shift equations (3.68–3.76) have a form very close to the one for scalar operators (3.33). Up to normalizing factors, the right-hand side of these equations is simply the geometric mean of the right-hand side of (3.33), with charges  $(\alpha_1, \alpha_2, \alpha_3)$  and  $(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3)$ , respectively. The major difference with scalar operators is the fact that the constraints (3.54) impose a quantisation of the vertex charges  $\alpha$  and  $\bar{\alpha}$ .

However, if one of the operators is scalar (say if  $\sigma_1 = \text{id}$ ) then its vertex charge can take continuous values, and the solution takes the form:

$$C(\Phi_{\alpha_1}, \Phi_{\alpha_2, \bar{\alpha}_2}^{(\sigma_2)}, \tilde{\Phi}_{\alpha_3, \bar{\alpha}_3}) = \sqrt{C(\alpha_1, \alpha_2, \alpha_3) C(\alpha_1, \bar{\alpha}_2, \bar{\alpha}_3)}, \quad (3.77)$$

where  $C(\alpha_1, \alpha_2, \alpha_3)$  is the structure constant of scalar operators, given in (3.36–3.39). Note that this result is valid only when  $\Phi_{\alpha_1, \bar{\alpha}_1}^{(\sigma_1)}$  and  $\Phi_{\alpha_2, \bar{\alpha}_2}^{(\sigma_2)}$  are non-degenerate,  $\tilde{\Phi}_{\alpha_3, \bar{\alpha}_3}$  is semi-degenerate, and the  $\mathbb{Z}_n$  charge neutrality conditions are satisfied:

$$e^{2i\pi[\eta(\alpha_1, \bar{\alpha}_1) + \eta(\alpha_2, \bar{\alpha}_2) + \eta(\alpha_3, \bar{\alpha}_3)]} = 1, \quad e^{2i\pi[\hat{\eta}(\alpha_1, \bar{\alpha}_1) + \hat{\eta}(\alpha_2, \bar{\alpha}_2) + \hat{\eta}(\alpha_3, \bar{\alpha}_3)]} = 1,$$

where  $\eta$  and  $\hat{\eta}$  are defined in (3.57).

In the case of generic, non-scalar, operators, the vertex charges obey the quantisation conditions (3.54), and the structure constants  $C(\Phi_{\alpha_1, \bar{\alpha}_1}^{(\sigma_1)}, \Phi_{\alpha_2, \bar{\alpha}_2}^{(\sigma_2)}, \tilde{\Phi}_{\alpha_3, \bar{\alpha}_3})$  are determined by the shift equations (3.68–3.76), up to an overall factor.

## 3.5 CONCLUSION

In this chapter, we have defined and studied the “timelike” analogue of the conformal  $\mathfrak{sl}_3$  Toda field theory, with central charge  $c = 2 - 12(b - b^{-1})^2$ . Our main results are (i) the structure constant (3.36) of



three scalar operators when one of them is semi-degenerate, (ii) the classification of non-scalar operators (see Sec. 3.4.1), and (iii) the shift equations (3.68–3.76) that the structure constants  $C(\Phi_{\alpha_1, \bar{\alpha}_1}^{(\sigma_1)}, \Phi_{\alpha_2, \bar{\alpha}_2}^{(\sigma_2)}, \tilde{\Phi}_{\alpha_3, \bar{\alpha}_3})$  obey, and their solution (3.77) when one of the operators is scalar.

## APPENDICES

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### 3.A CONVENTIONS FOR $\mathfrak{sl}_n$ : ROOTS, WEIGHTS AND WEYL GROUP

Many features of the Toda field theory are related to the  $\mathfrak{sl}_n$  Lie algebra. We recall basic facts and notations in this appendix.

#### 3.A.1 Conventions for $\mathfrak{sl}_3$

The Lie algebra  $\mathfrak{sl}_3$  has two simple roots  $e_1$  and  $e_2$ , its Cartan matrix, defined by the scalar product  $K_{i,j} = e_i \cdot e_j$ , takes the form:

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

The weight  $\omega_i$  of the Lie algebra are dual to its roots,  $e_i \cdot \omega_j = \delta_{i,j}$ . They can be written:

$$\begin{aligned} \omega_1 &= \frac{1}{3}(2e_1 + e_2), & \omega_2 &= \frac{1}{3}(2e_2 + e_1) \\ \Rightarrow \quad \omega_1 \cdot \omega_1 &= \omega_2 \cdot \omega_2 = \frac{2}{3} & \text{and } \omega_1 \cdot \omega_2 &= \frac{1}{3} \end{aligned}$$

The weight of the first fundamental representation are defined as:

$$h_1 = \omega_1, \quad h_2 = \omega_1 - e_1 = \omega_2 - \omega_1, \quad h_3 = \omega_1 - e_1 - e_2 = -\omega_2$$

The weight lattice is  $\mathcal{R} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ , and its dual  $\mathcal{R}^*$  is the root lattice. The Weyl vector can be written both in terms of  $e_i$  and  $\omega_j$ :  $\rho = \omega_1 + \omega_2 = e_1 + e_2 = h_1 - h_3$ .

The Weyl group  $W$  is generated by the reflections  $s_1$  and  $s_2$

$$s_i(v) = v - (v \cdot e_i)e_i.$$

It has 6 elements, three reflections  $s_j$  and three rotations  $R_j$  (see Fig. 3.A.1),  $j = 1, 2, 3$ . In terms of the generators  $s_1$  and  $s_2$  one has:

$$R_1 = 1, \quad R_2 = s_1s_2, \quad R_3 = s_2s_1, \quad s_3 = s_1s_2s_1 = s_2s_1s_2$$

The Weyl group of  $\mathfrak{sl}_3$  can be identified with  $\mathfrak{S}_3$ , the group of permutations of three elements. In terms of the  $h_i$ , the Weyl group acts by permutations  $h_i \rightarrow h_{\sigma(i)}$ . In the following we will denote an element of the Weyl group by the corresponding permutation  $\sigma$ .

Finally, the duality operation  $\alpha \mapsto \alpha^*$  is the reflection with respect to  $\rho$  (this is the unique linear involution exchanging  $\omega_1$  and  $\omega_2$ ). This reflection does not belong to the Weyl group.

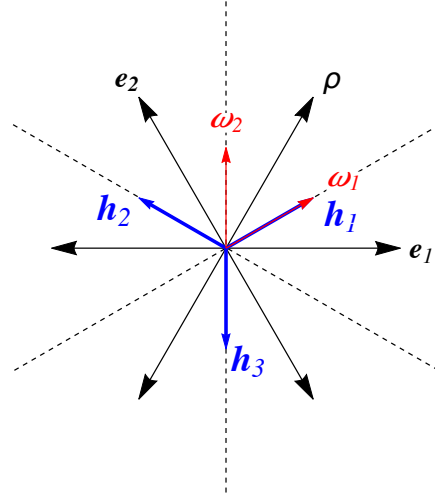


Figure 3.A.1: The generators of the lattices  $\mathcal{R}$  (in blue/red) and  $\mathcal{R}^*$  (in black) for  $\mathfrak{sl}_3$ . The reflections of the Weyl group are the reflections *w.r.t.* the dashed lines, while the rotation  $R_j$  ( $j = 1, 2, 3$ ) sends  $h_1$  to  $h_j$ .

An explicit basis in the fundamental representation (of dimension 8) of  $\mathfrak{su}(3)$  can be written in terms of the Gell-Mann matrices:

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

They verify the relations:

$$\text{Tr}(\lambda_i \lambda_j) = 2\delta_{ij} \forall i, j.$$

The two relations:

$$[\lambda_j, \lambda_k] = 2i f_{jkl} \lambda_l, \quad \{\lambda_j, \lambda_k\} = \frac{4}{3} \delta_{j,k} + 2d_{jkl} \lambda_l \tag{3.78}$$

define respectively the completely anti-symmetric structure constants of the theory,  $f_{jkl}$ , and the completely symmetric coefficient constants,

$d_{jkl}$ . Another possible basis, standard for all Lie algebras, is the Cartan-Weyl basis:

$$\begin{aligned} E^{\pm e_1} &= \frac{\lambda_4 \pm i\lambda_5}{\sqrt{2}} & E^{\pm e_2} &= \frac{\lambda_6 \pm i\lambda_7}{\sqrt{2}} & E^{\pm e_3} &= \frac{\lambda_1 \mp i\lambda_2}{\sqrt{2}} \\ H_2 &= \frac{\lambda_8}{\sqrt{2}} & H_1 &= \frac{\lambda_3}{\sqrt{2}} \end{aligned}$$

This normalization gives the relations  $[H_i, E^\alpha] = \alpha_i E^\alpha$  for  $i \in 1, 2$  and  $\alpha$  a root.

Lastly the universal enveloping algebra of  $\mathfrak{su}(3)$  has a center that can be generated by two elements, the Casimir operators:

$$C_2 = \sum_k \lambda_k \lambda_k \quad C_3 = \sum_{jkl} d_{jkl} \lambda_j \lambda_k \lambda_l$$

Where the  $\lambda_k$  have to be understood as elements of the enveloping algebra rather than as matrices.

### 3.A.2 Conventions for $\mathfrak{sl}_n$

The generalization to  $\mathfrak{sl}_n$  is straightforward. Let  $\{\mathbf{h}_i, i = 1, \dots, n\}$  be the first fundamental representation of  $\mathfrak{sl}_n$ , normalized as

$$\mathbf{h}_i \cdot \mathbf{h}_j = \delta_{ij} - \frac{1}{n}. \quad (3.79)$$

The weight and root lattices are

$$\mathcal{R} = \sum_{i=1}^n \mathbb{Z} \omega_i, \quad \mathcal{R}^* = \sum_{i=1}^n \mathbb{Z} e_i \quad (3.80)$$

where the fundamental weights  $\omega_i$  and the simple roots  $e_i$  can be expressed as

$$\begin{aligned} \omega_i &= \sum_{k=1}^i \mathbf{h}_k \\ e_i &= \mathbf{h}_i - \mathbf{h}_{i+1} \end{aligned}$$

for  $i = 1, \dots, n-1$ . The Weyl vector is

$$\boldsymbol{\rho} = \sum_{i=1}^{n-1} \omega_i = - \sum_{i=1}^n i \mathbf{h}_i \quad \boldsymbol{\rho} \cdot \boldsymbol{\rho} = \frac{n(n^2 - 1)}{12}. \quad (3.81)$$

The Weyl group is isomorphic to  $\mathfrak{S}_n$ , and acts as

$$\sigma(\mathbf{h}_i) = \mathbf{h}_{\sigma(i)}. \quad (3.82)$$

In particular the reflection  $s_i : \mathbf{x} \rightarrow \mathbf{x} - (\mathbf{e}_i \cdot \mathbf{x}) \mathbf{e}_i$  is mapped to the transposition  $\tau_{i,i+1}$ , and the longest element of Weyl group (denoted by  $s_0$ ), which for  $\mathfrak{sl}_n$  reads

$$s_0 = s_1 s_2 \cdots s_{n-1} s_1 s_2 \cdots s_{n-2} \cdots s_1 s_2 s_1 \quad (3.83)$$

corresponds to the permutation

$$\rho(i) = n - i. \quad (3.84)$$

Finally, the dual operation is defined as

$$\mathbf{x}^* = -s_0(\mathbf{x}). \quad (3.85)$$

### 3.B HYPERGEOMETRIC CONFORMAL BLOCKS FOR $\mathfrak{sl}_3$

#### 3.B.1 Hypergeometric conformal blocks for $\mathfrak{sl}_3$

Event though the fusion of the form  $\Phi_{b\omega_1} \otimes \Phi_\alpha$  only gives rise to three primary operators as in (3.16), it is known that a generic four point function  $\langle \Phi_{b\omega_1} \Phi_{\alpha_1} \Phi_{\alpha_2} \Phi_{\alpha_3} \rangle$  has more than three Virasoro conformal blocks. However, if one of the fields  $\alpha_i$  is semi-degenerate, a third-order differential equation can be obtained for this function [15, 16, 63, 64]. The simplest semi-degenerate fields correspond to  $\alpha_3 = \kappa\omega_j$  ( $j = 1, 2$ ), with a null-vector at level 1 as in (3.14). It was found in Fateev and Litvinov for the real Toda theory [63, 64] that the correlation function

$$\mathcal{G}(z) = \langle \Phi_{\beta^*} | \Phi_{b\omega_1}(z) \Phi_{\kappa\omega_j}(1) | \Phi_\alpha \rangle$$

obeys a Fuchsian differential equation of order 3, whose solutions are given in terms of generalized hypergeometric functions. In this section we adapt these results in the case of imaginary Toda.

The residue theorem applied to the function

$$w \rightarrow \langle \Phi_{\beta^*} | W(w) \Phi_{b\omega_1}(z) \Phi_{\kappa\omega_j}(1) | \Phi_\alpha \rangle \frac{(w-1)w^2}{w-z}$$

yields

$$\begin{aligned} & \left( w_\beta + w_{b\omega_1} + \frac{w_\alpha}{z} + \frac{(1-2z)}{(z-1)^2} w_{\kappa\omega_j} \right) \langle \Phi_{\beta^*} | \Phi_{b\omega_1}(z) \Phi_{\kappa\omega_j}(1) | \Phi_\alpha \rangle \\ & + \frac{1}{1-z} \langle \Phi_{\beta^*} | \Phi_{b\omega_1}(z) (W_{-1} \Phi_{\kappa\omega_j})(1) | \Phi_\alpha \rangle \\ & + (3z-1) \langle \Phi_{\beta^*} | (W_{-1} \Phi_{b\omega_1})(z) \Phi_{\kappa\omega_j}(1) | \Phi_\alpha \rangle \\ & + z(3z-2) \langle \Phi_{\beta^*} | (W_{-2} \Phi_{b\omega_1})(z) \Phi_{\kappa\omega_j}(1) | \Phi_\alpha \rangle \\ & + z^2(z-1) \langle \Phi_{\beta^*} | (W_{-3} \Phi_{b\omega_1})(z) \Phi_{\kappa\omega_j}(1) | \Phi_\alpha \rangle = 0. \end{aligned}$$

By using the null-vector equations (3.15), it is possible to rewrite this equation in terms Virasoro modes, leading to a differential equation for the correlation function. This differential equation takes a very simple form in term of the function

$$f(z) := \mathcal{G}(z)(1-z)^{-2b\mu} z^{b\mu + \Delta_{b\omega_1}},$$

where

$$\mu = \begin{cases} \frac{1}{2}\kappa\omega_2 \cdot h_1 & \text{if } j = 2 \\ \frac{1}{2}(\mathbf{Q} - \kappa\omega_2) \cdot h_1 & \text{if } j = 1 \end{cases}$$

The function  $f(z)$  obeys the following Fuchsian differential equation

$$z \prod_i (\vartheta + B_i) f(z) = \prod_i (\vartheta - A_i) f(z). \quad (3.86)$$

where  $\vartheta = z\partial_z$  and

$$A_i = \Delta_{\alpha+bh_i} - \Delta_{\alpha} + b\mu, \quad B_i = \Delta_{\beta+bh_i} - \Delta_{\beta} + b\mu. \quad (3.87)$$

Note that the equation is invariant under the simultaneous change  $z \rightarrow z^{-1}$  and  $A_i \leftrightarrow B_i$ . This simply reflects the fact that

$$\langle \Phi_{\beta^*} | \Phi_{b\omega_1}(z) \Phi_{\kappa\omega_j}(1) | \Phi_{\alpha} \rangle = \langle \Phi_{\alpha^*} | \Phi_{b\omega_1}(1/z) \Phi_{\kappa\omega_j}(1) | \Phi_{\beta} \rangle z^{-2\Delta_{b\omega_1}}.$$

The Riemann scheme of this Fuchsian differential equation is

$$\left( \begin{array}{ccc} z = 0 & z = 1 & z = \infty \\ A_1 & 0 & B_1 \\ A_2 & 1 & B_2 \\ A_3 & 2 - \sum_i (A_i + B_i) & B_3 \end{array} \right) \quad (3.88)$$

and the sum of all exponents is 3, as it should, according to the Fuchs relation. The exponents as  $z \rightarrow 1$  are compatible with the fusion rules (3.18).

A basis of solutions with Abelian monodromies around  $z = 0$  is given by

$$f_i(z) = (-z)^{A_i} {}_3F_2 \left[ \begin{array}{c} B_1 + A_i, \dots, B_3 + A_i \\ 1 - A_1 + A_i, \dots, 1 - A_3 + A_i \end{array}; z \right],$$

where  $\dots$  denotes suppression of the term  $1 - A_i + A_i$ . Likewise, the solutions with Abelian monodromies around  $\infty$  are simply obtained through  $A_i \leftrightarrow B_i$  and  $z \rightarrow z^{-1}$ :

$$g_i(z) = (-z)^{-B_i} {}_3F_2 \left[ \begin{array}{c} A_1 + B_i, \dots, A_3 + B_i \\ 1 - B_1 + B_i, \dots, 1 - B_3 + B_i \end{array}; \frac{1}{z} \right].$$

Going back to the function  $\mathcal{G}$ , we have the following conformal blocks

$$\begin{aligned} F_i(z) &= (1-z)^{2b\mu} (-z)^{\eta_i} {}_3F_2 \left[ \begin{array}{c} B_1 + A_i, \dots, B_3 + A_i \\ 1 - A_1 + A_i, \dots, 1 - A_3 + A_i \end{array}; z \right], \\ G_i(z) &= \left(1 - \frac{1}{z}\right)^{2b\mu} \left(-\frac{1}{z}\right)^{\zeta_i} {}_3F_2 \left[ \begin{array}{c} A_1 + B_i, \dots, A_3 + B_i \\ 1 - B_1 + B_i, \dots, 1 - B_3 + B_i \end{array}; \frac{1}{z} \right], \end{aligned}$$

where

$$\begin{aligned}\eta_i &= A_i - b\mu - \Delta_{b\omega_1} = \Delta_{\alpha+bh_i} - \Delta_{\alpha} - \Delta_{b\omega_1} \\ \zeta_i &= B_i - b\mu + \Delta_{b\omega_1} = \Delta_{\beta+bh_i} - \Delta_{\beta} + \Delta_{b\omega_1}\end{aligned}$$

are the fusion exponents as  $z \rightarrow 0$  and  $z \rightarrow \infty$ .

Acting with the Weyl group on  $\alpha$  simply permutes the blocks as follows

$$F_i \rightarrow F_{\sigma(i)}, \quad G_i \rightarrow G_i$$

while reparametrization of  $\beta$  yields

$$F_i \rightarrow F_i, \quad G_i \rightarrow G_{\sigma(i)}$$

The two bases are related through  $F_i(z) = M_{ij}G_j(z)$  :

$$\begin{aligned}M_{ij} &= \prod_{k \neq i} \frac{\Gamma(1 + A_i - A_k)}{\Gamma(1 - B_j - A_k)} \prod_{\ell \neq j} \frac{\Gamma(B_\ell - B_j)}{\Gamma(B_\ell + A_i)} \\ &= \prod_{k \neq i} \frac{\Gamma(1 + \eta_i - \eta_k)}{\Gamma(1 - 2b\mu - \zeta_j - \eta_k)} \prod_{\ell \neq j} \frac{\Gamma(\zeta_\ell - \zeta_j)}{\Gamma(2b\mu + \zeta_\ell + \eta_i)}.\end{aligned}$$

The coefficients of  $M^{-1}$  are obtained by exchanging  $A_i \leftrightarrow B_i$ .

### 3.B.2 Hypergeometric conformal blocks for $\mathfrak{sl}_n$

The generalization to  $\mathfrak{sl}_n$  is as follows. Consider the correlation function

$$f(z) := \langle \Phi_{\beta^*} | \Phi_{b\omega_1}(z) \Phi_{\kappa\omega_j}(1) | \Phi_{\alpha} \rangle (1-z)^{-2b\mu} z^{b\mu + \Delta_{b\omega_1}},$$

where  $j = 1$  or  $j = n - 1$ , and

$$2b\mu = \begin{cases} \Delta_{\kappa\omega_2 + bh_1} - \Delta_{\kappa\omega_2} - \Delta_{b\omega_1} = \frac{b\kappa}{n} & \text{if } j = n - 1 \\ \Delta_{\kappa\omega_1 + bh_2} - \Delta_{\kappa\omega_1} - \Delta_{b\omega_1} = -\frac{b\kappa}{n} + (1 - b^2) & \text{if } j = 1 \end{cases}$$

The function  $f(z)$  obeys the following Fuchsian differential equation

$$z(\vartheta + B_1) \cdots (\vartheta + B_n) f(z) = (\vartheta - A_1) \cdots (\vartheta - A_n) f(z) \quad (3.89)$$

where

$$A_i = \Delta_{\alpha + bh_i} - \Delta_{\alpha} + b\mu, \quad B_i = \Delta_{\beta + bh_i} - \Delta_{\beta} + b\mu. \quad (3.90)$$

The Riemann scheme is

$$\left\{ \begin{array}{ccc} z = 0 & z = 1 & z = \infty \\ \hline A_1 & 0 & B_1 \\ A_2 & 1 & B_2 \\ \vdots & \vdots & \vdots \\ A_{n-1} & n-2 & B_{n-1} \\ A_n & n-1 - \sum_i (A_i + B_i) & B_n \end{array} \right\} \quad (3.91)$$

and the exponents as  $z \rightarrow 1$  give the fusion rules

$$\begin{aligned} \Phi_{b\omega_1} \otimes \Phi_{\kappa\omega_{n-1}} &\rightarrow \Phi_{\kappa\omega_{n-1}+bh_1} \oplus \Phi_{\kappa\omega_{n-1}+bh_n}, \\ \Phi_{b\omega_1} \otimes \Phi_{\kappa\omega_1} &\rightarrow \Phi_{\kappa\omega_1+bh_1} \oplus \Phi_{\kappa\omega_1+bh_2}. \end{aligned} \tag{3.92}$$

A basis of solutions can be obtained by series expansion around  $z = 0$ , namely

$$f_i(z) = (-z)^{A_i} {}_nF_{n-1} \left[ \begin{matrix} B_1 + A_i, \dots, B_n + A_i \\ 1 - A_1 + A_i, \dots, 1 - A_n + A_i \end{matrix}; z \right],$$

where  $\dots$  denotes suppression of the term  $1 - A_i + A_i$ . The above series is convergent for  $|z| < 1$ , and it can be analytically continued. The Weyl group (reparametrization of  $\alpha$ ) acts by permutations on these  $n$  conformal blocks.

Likewise, around  $z = \infty$  the solutions with Abelian monodromies around  $\infty$  are simply obtained through  $A_i \leftrightarrow B_i$  and  $z \rightarrow z^{-1}$ :

$$g_i(z) = (-z)^{-B_i} {}_nF_{n-1} \left[ \begin{matrix} A_1 + B_i, \dots, A_n + B_i \\ 1 - B_1 + B_i, \dots, 1 - B_n + B_i \end{matrix}; \frac{1}{z} \right].$$

The change of bases

$$f_i(z) = \sum_j M_{ij} g_j(z)$$

can be obtained using contour deformation of the following Mellin-Barnes integral

$$I(z) = \frac{1}{2\pi i} \int ds \Gamma(B_1 + s) \cdots \Gamma(B_n + s) \Gamma(A_1 - s) \cdots \Gamma(A_n - s) (\epsilon z)^s,$$

where  $\epsilon = (-1)^n$  and the integration contour goes from  $i\infty$  to  $-i\infty$  while keeping all the poles  $\{A_i + k, k \in \mathbb{N}\}$  to the left and the poles  $\{-B_i - k, k \in \mathbb{N}\}$  to the right. One finds

$$M_{ij} = \prod_{k \neq i} \frac{\Gamma(1 + A_i - A_k)}{\Gamma(1 - B_j - A_k)} \prod_{\ell \neq j} \frac{\Gamma(B_\ell - B_j)}{\Gamma(B_\ell + A_i)}, \tag{3.93}$$

and the coefficients of  $M^{-1}$  are obtained by exchanging  $A_i \leftrightarrow B_i$ .

### 3.C UPSILON AND DOUBLE GAMMA FUNCTIONS

For  $0 < \text{Re}(x) < b + b^{-1}$ , the function  $x \mapsto Y_b(x)$  is given by:

$$\ln Y_b(x) \equiv \int_0^\infty \frac{dt}{t} \left[ \left( \frac{b + b^{-1}}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left[ \left( \frac{b + b^{-1}}{2} - x \right) \frac{t}{2} \right]}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right].$$

Outside of this interval, the function can be computed using the recursion formulas:

$$\begin{aligned} Y_b(x + b) &= \gamma(bx) b^{1-2bx} Y_b(x), \\ Y_b(x + b^{-1}) &= \gamma(xb^{-1}) b^{-1+2xb^{-1}} Y_b(x). \end{aligned} \tag{3.94}$$



Moreover, it is clear from the integral definition that

$$\begin{aligned} Y_b(x) &= Y_{b^{-1}}(x), \\ Y_b(x) &= Y_b(b + b^{-1} - x). \end{aligned} \quad (3.95)$$

$\Gamma_b = \Gamma_{b^{-1}}$  is a double Gamma function with periods  $b$  and  $b^{-1}$ . It satisfies:

$$\begin{aligned} \Gamma_b(x + b) &= \sqrt{2\pi} \frac{b^{bx-1/2}}{\Gamma(bx)} \Gamma_b(x), \\ \Gamma_b(x + b^{-1}) &= \sqrt{2\pi} \frac{b^{-x/b-1/2}}{\Gamma(x/b)} \Gamma_b(x). \end{aligned} \quad (3.96)$$

These two functions are related through

$$Y_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(b + b^{-1} - x)}. \quad (3.97)$$

### 3.D DETAILED ANALYSIS OF NON-DIAGONAL FIELDS IN $\mathfrak{sl}_n$ TODA THEORY

A non-diagonal field is characterized by two charges  $(\alpha, \bar{\alpha})$  and a permutation  $\sigma$  such that:

$$\alpha - \bar{\alpha} \in b^{-1}\mathcal{R}, \quad \alpha - \sigma \star \bar{\alpha} \in b\mathcal{R}, \quad (3.98)$$

And up to the identification

$$\Phi_{\alpha, \bar{\alpha}}^{(\sigma)} = \Phi_{\mu \star \alpha, \mu \star \bar{\alpha}}^{(\mu \sigma \mu^{-1})} \quad \forall \mu \in \mathfrak{S}_n. \quad (3.99)$$

Therefore, to each conjugacy class of  $\sigma$  corresponds a different type of non-diagonal field. We can choose a canonical representative of the conjugacy class, with a simple cycle decomposition:

$$\sigma = (1, 2, \dots, c_1)(c_1 + 1, c_1 + 2, \dots, c_2) \cdots (c_{p-1} + 1, \dots, c_p). \quad (3.100)$$

We define  $c_0 = 0$  and  $\{c_i\}_{i=0 \dots p}$  are an increasing sequence of integers. Like previously let  $\alpha = \mathbf{Q} + \mathbf{P}$ . The previous constraints, rewritten for  $\mathbf{P}$  implies:

$$\mathbf{P} - \sigma \mathbf{P} \in b\mathcal{R} + b^{-1}\mathcal{R}.$$

Which is equivalent to

$$P_{\sigma(i)} - P_i - (P_{\sigma(j)} - P_j) \in b\mathbb{Z} + b^{-1}\mathbb{Z} \quad \forall i, j. \quad (3.101)$$

where  $P_i = \mathbf{P} \cdot \mathbf{h}_i$ . For a given  $i$  we can sum  $j$  over one orbit of  $\sigma$ , with a representative  $a$ ,  $\mathcal{O}_a(\sigma)$ , which leads to:

$$\sum_{j \in \mathcal{O}_a(\sigma)} P_j - P_{\sigma(j)} = |\mathcal{O}_a(\sigma)| (P_i - P_{\sigma(i)}) \in b\mathbb{Z} + b^{-1}\mathbb{Z},$$

where  $|\mathcal{O}_a(\sigma)|$  is the cardinal of the orbit. This relation is valid for any value of  $j$  hence:

$$P_i - P_{\sigma(i)} \in \frac{1}{\text{GCD}(\{|\mathcal{O}_a(\sigma)|\}_a)} \left( b\mathbb{Z} + b^{-1}\mathbb{Z} \right). \quad (3.102)$$

Where  $\text{GCD}(\{|\mathcal{O}_a(\sigma)|\}_a)$  is the greatest common denominator of the integer set  $\{|\mathcal{O}_j(\sigma)|\}_j$ .

Let's call  $\mathcal{R}_{\text{nf}}$  the lattice generated by the vectors  $\{\mathbf{h}_i - \mathbf{h}_{\sigma(i)}\}_{i \in 1 \dots n}$ , and  $V_{\text{nf}}$  the vector space generated by the same vectors:  $\mathbb{R}^{n-1} = V_{\text{nf}} \oplus \text{Vect}[\mathbf{h}_i]_{i=\sigma(i)}$ . The vector  $\boldsymbol{\rho}_{\text{nf}} = -\sum_{i, i \neq \sigma(i)} i \mathbf{h}_i$  belongs to  $V_{\text{nf}}$  and is defined so that  $\boldsymbol{\rho}_{\text{nf}} \cdot (\mathbf{h}_j - \mathbf{h}_{\sigma(j)}) \equiv 1 \pmod{|\mathcal{O}_j(\sigma)|}$  (for this to work, the canonical representation of  $\sigma$  3.100 must be used).

The two constraints 3.101 and 3.102 imply that, on the subspace  $V_{\text{nf}}$ ,  $P$  is quantized:

$$P|_{V_{\text{nf}}} = \delta \boldsymbol{\rho}_{\text{nf}} + bE + b^{-1}M$$

$$\text{with } \delta \in \frac{1}{\text{GCD}(\{|\mathcal{O}_j(\sigma)|\}_j)} \left( b\mathbb{Z} + b^{-1}\mathbb{Z} \right) \text{ and } E, M \in \mathcal{R}_{\text{nf}}$$

In contrast, along the direction associated with fixed points of  $\sigma$ ,  $P|_{\text{Vect}[\mathbf{h}_i]_{i=\sigma(i)}}$  is free. Of course  $\bar{P}$  obeys the same restrictions, with  $\bar{\delta}$ ,  $\bar{E}$  and  $\bar{M}$  replacing  $\delta$ ,  $E$  and  $M$ .

For the non-fixed points of the permutation, the constraint  $P - \bar{P} \in b^{-1}\mathcal{R}$  implies that

$$\delta = \bar{\delta} \quad \text{and} \quad E = \bar{E} \quad (3.103)$$

While  $P - \sigma\bar{P} \in b\mathcal{R}$  means that  $M = \sigma\bar{M}$ .

And along the direction of the fixed points  $i = \sigma(i)$ ,  $P \cdot \mathbf{h}_i = \bar{P} \cdot \mathbf{h}_i$ .

The two extremes cases are especially interesting:

- Scalar fields correspond to  $\sigma = 1$ , and the two constraints (3.54) boil down to  $\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}}$ .
- In the case where  $\sigma$  is a maximal cycle,  $(1 - \sigma)$  is invertible, and the charge  $\boldsymbol{\alpha}$  and  $\bar{\boldsymbol{\alpha}}$  are completely quantized:

$$\boldsymbol{\alpha} = \frac{1}{n} \left( bE + b^{-1}M \right) \quad (3.104)$$

$$\bar{\boldsymbol{\alpha}} = \frac{1}{n} \left( bE + b^{-1}\bar{M} \right) \quad (3.105)$$

where  $M = \sigma \star \bar{M}$  and  $E, M \in \mathcal{R}$ . The fields generically have a non-zero conformal spin. This quantification appears naturally in the vertex and RSOS model associated with  $A_n$  [49].

In the main text, we also considered the two fusions involving  $b\omega_1$  and  $b\omega_{n-1}$ . We obtained constraints of the form:

$$\alpha - \bar{\alpha} \in b\mathcal{R}, \quad \alpha - \sigma \star \bar{\alpha} \in b\mathcal{R}, \quad (3.106)$$

In this case one also ends up with a quantification in the direction of the non-fixed points of the permutation:

$$\mathbf{P}|_{V_{\text{nf}}} = \delta \rho_{\text{nf}} + bE \quad \text{with } \delta \in \frac{1}{\text{GCD}(\{|\mathcal{O}_j(\sigma)|\}_j)} b\mathbb{Z} \text{ and } E \in \mathcal{R}_{\text{nf}}$$

Contrary to the previous case, taking  $\sigma$  different from the identity does not relax the constraints on the conformal spin, and only limit the number of allowed fields.

### 3.E A DIFFERENT PATH TOWARD THE DIFFERENTIAL EQUATION

Another method, which only involves the fusion rules of the  $\mathcal{W}_3$  algebra, can be used to find the differential equation 3.89 for the conformal blocks. Its interest lies in the fact that it does not rely as much on null-vectors equations.

Consider the following correlation function of scalar operators:

$$\mathcal{G}(z) = \langle \mathcal{O}_{\alpha_\infty} | \mathcal{O}_{b\omega_1}(z) \mathcal{O}_{\alpha_1}(1) | \mathcal{O}_{\alpha_0} \rangle.$$

We will make the assumption that the field  $\mathcal{O}_{\alpha_1}$  is semi-degenerate. The fusion rules 3.23 (and conformal invariance in general) fix the way the correlation function should behave. Given  $a \in \{0, 1, \infty\}$ :

$$\mathcal{G}(z) = \sum_{i=1}^p X_{a,i} |f_{a,i}(z)|^2, \quad (3.107)$$

with  $p$  an integer, larger than  $n$ ,  $X_i \in \mathbb{C}$ , and  $f_{a,i}(z)$  is the product of an analytic function around  $a$  and a power-law term. Explicitly:

$$\begin{aligned} f_{0,i} &= z^{a_{0,i}} (\# + \#z + \#z^2 + \dots), \\ f_{1,i} &= (1-z)^{a_{1,i}} (\# + \#(1-z) + \#(1-z)^2 + \dots), \\ f_{\infty,i} &= z^{a_{\infty,i}} (\# + \#z^{-1} + \#z^{-2} + \dots). \end{aligned} \quad (3.108)$$

In general degeneracies are expected, but we impose  $a_{u,i} \neq a_{u,j}$  if  $i \neq j$ . This is a convention rather than a restriction, a set of  $m$  identical exponents  $a_{0,i}, a_{0,i} \dots, a_{0,i}$  can always be replaced by  $a_{0,i}, a_{0,i} - 1 \dots, a_{0,i} - m$ . The values that the exponents  $a_{u,i}$  can take are fixed by the fusion rules:

$$\begin{aligned} a_{0,i} &\in \{\Delta(\alpha_0 + b\mathbf{h}_k) - \Delta(\alpha_0) - \Delta(b\omega_1) \mid k \in \{1, \dots, n\}\} + \mathbb{N}, \\ a_{1,i} &\in \{\Delta(\alpha_1 + b\mathbf{h}_k) - \Delta(\alpha_1) - \Delta(b\omega_1) \mid k \in \{1, \dots, n\}\} + \mathbb{N}, \\ a_{\infty,i} &\in \{-\Delta(\alpha_\infty + b\mathbf{h}_k) + \Delta(\alpha_\infty) - \Delta(b\omega_1) \mid k \in \{1, \dots, n\}\} + \mathbb{N}. \end{aligned}$$

We aim to show that the functions  $f_{a,i}$  are constrained by their singularities. To do so, we will consider a collection  $\mathcal{F}$  of function elements  $(f, D)$  – of functions  $f$  defined on a domain  $D$  – such that:

- $\mathcal{F}$  is complete: every analytic continuation of an element in  $\mathcal{F}$  also belongs to  $\mathcal{F}$ .
- $\mathcal{F}$  is linear, and for a given domain  $D$ , the subset  $\{(f, D)\} \in \mathcal{F}$  should be at most  $p$  dimensional.
- Functions in  $\mathcal{F}$  are analytic everywhere but in  $0, 1$  and  $\infty$ .
- At one of the singular points  $a \in \{0, 1, \infty\}$ , the functions of  $\mathcal{F}$ , if they are non-zero, must have the same singularities as the functions  $f_{a,i}$ .

The Riemann symbol

$$P \left\{ \begin{array}{ccc|c} 0 & 1 & \infty & \\ a_{0,1} & a_{1,1} & a_{\infty,1} & \\ a_{0,2} & a_{1,2} & a_{\infty,2} & z \\ \vdots & \vdots & \vdots & \\ a_{0,p} & a_{1,p} & a_{\infty,p} & \end{array} \right\}$$

can describe either the full set  $\mathcal{F}$  or any function of the set depending on the context.

Consider a given domain  $D$ , and a basis of function  $f_i$  in this domain. For example,  $D$  can be a small enough ball containing  $0$  and  $f_i = f_{0,i}$ . If  $(f, D)$  is a function in  $\mathcal{F}$ , it will verify:

$$f(z) + c_1 f_1(z) + c_2 f_2(z) + \dots + c_p f_p(z) = 0 \quad \forall z \in D.$$

For a given set of complex coefficients  $c_i$ . This identity can be derived any number of times, and we can write, with  $\vartheta = z\partial_z$ :

$$\begin{vmatrix} f & f_1 & f_2 & \dots & f_p \\ \vartheta f & \vartheta f_1 & \vartheta f_2 & \dots & \vartheta f_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vartheta^{(p)} f & \vartheta^{(p)} f_1 & \vartheta^{(p)} f_2 & \dots & \vartheta^{(p)} f_p \end{vmatrix} = 0. \tag{3.109}$$

*This determinant is a variation on the Wronskian of the system. Using  $\vartheta$  instead of simply  $\partial_z$  allows us to conserve the form of the function in  $0$  and  $\infty$ .*

The determinant can be expanded over the first column, and transformed in a differential equation over  $f$ :

$$\sum_r \left( \vartheta^{(r)} f \right) Q_{p-r}(\{f_i\}_{i,j}) = 0.$$

Where the  $Q$  are the first minor of the determinant, and hence have polynomial expressions in the function  $f_i$  and their derivatives. Rather

than working directly with the functions  $Q_r$ , we will consider the functions:

$$p_r(z) = \frac{Q_r(f_i)}{Q_0(f_i)}.$$

They have the nice property of being invariant under any linear transformation of the functions  $f_i$ . If we replace  $f_i$  by  $\sum_j c_{i,j} f_j$  the expression of the functions  $p_r$  is not modified. This means that  $p_r$  is entire and well-defined on the whole plane, except at the three points 0,1 and  $\infty$  where we can expect singularities. To compute these, we can choose any basis of function  $f_i$  ( $p_r$  do not depend on a specific choice). Near  $z = 0$  for example, if we choose  $f_i = f_{0,i}$ , the coefficients of the determinant have the approximate form  $\vartheta^{(i)} f_j \underset{z \rightarrow 0}{\approx} a_{0,j}^i z^{a_{0,j}}$ . In this limit the determinant 3.109 can be approximated, at first order in  $z$ , through the Vandermonde formula:

$$z^{\sum_j a_{0,j}} \begin{vmatrix} f & 1 & \cdots & 1 \\ \vartheta f & a_{0,1} & \cdots & a_{0,p} \\ \vdots & \vdots & \ddots & \vdots \\ \vartheta^{(p)} f & a_{0,1}^p & \cdots & a_{0,p}^p \end{vmatrix} = z^{\sum_j a_{0,j}} \prod_{1 \leq i < j \leq p} (a_{0,i} - a_{0,j}) \prod_{1 \leq j \leq p} (\vartheta - a_{0,j}) f.$$

This expression can be used to obtain the behaviour of the functions  $p_r$  near 0:

$$p_r(z) \underset{z \rightarrow 0}{\approx} (-1)^{p-r-1} \sum_{1 \leq i_1 < \cdots < i_r \leq p} \left( \prod_{j=1}^r a_{0,i_j} \right) + \mathcal{O}(z)$$

And for the exact same reasons, near  $z = \infty$ :

$$p_r(z) \underset{z \rightarrow \infty}{\approx} (-1)^{p-r-1} \sum_{1 \leq i_1 < \cdots < i_r \leq p} \left( \prod_{j=1}^r a_{\infty,i_j} \right) + \mathcal{O}(z)$$

Near 1, the behaviour is in general more complicated. Generically,  $p_r$  will be proportional to  $(1-z)^{-r-1}$  near  $z = 1$ . In particular, the function  $p_1 = \frac{Q_1}{Q_0} = \frac{\vartheta Q_0}{Q_0} = z \partial_z \log(Q_0)$  verifies:

$$p_1(z) \underset{z \rightarrow 1}{\approx} \frac{C_1}{z-1} + \text{regular terms}.$$

$C_1$  is a constant, that can be obtained by expanding  $Q_0$  near  $z = 1$ :

$$Q_0 = \det \left[ \left\{ \vartheta^{(j-1)} f_i \right\}_{\substack{1 \leq i \leq p \\ 1 \leq j \leq p}} \right] \propto (z-1)^{\sum_i a_{1,i} - p(p-1)/2},$$

which implies  $C_1 = \sum_i a_{1,i} - \frac{p(p-1)}{2}$ .

Depending on the choice of exponents  $p_1$  can be either non-existent or entirely determined.  $p_1/z$  is the derivative of the logarithm of  $Q_0$ , which is an entire function except in 0, 1 and  $\infty$ . As a result,  $p_1$  only

has integer residues, with potential exception in 0, 1 and  $\infty$ . Summing the residues implies the relation:

$$\sum_i a_{0,i} + \sum_i a_{1,i} - \frac{p(p-1)}{2} = \sum_i a_{\infty,i} + n_r \text{ with } n_r \in \mathbb{Z}. \quad (3.110)$$

If  $n_r = 0$ ,  $p_1$  does not have any pole other than 1, and this completely fixes the value of  $p_1 = \frac{\sum_i a_{\infty,i} z^{-a_{0,i}}}{z-1}$ . Additionally, it proves that  $Q_0$  does not have any zero.

Now let us consider the more specific case of the correlation function with a semi-degenerate field of the form  $\alpha_1 = \kappa\omega_1$ :

$$(z-1)^{\Delta(\alpha_1) + \Delta(b\omega_1) - \Delta(\alpha_1 + bh_2)} \langle \mathcal{O}_{\alpha_\infty^*} | \mathcal{O}_{b\omega_1} \mathcal{O}_{\kappa\omega_1} | \mathcal{O}_{\alpha_0} \rangle$$

The fusion rules (3.23) apply, and in particular:

$$a_{1,i} = \begin{cases} i-1 & \text{for } i < p \\ \Delta(\alpha_1 + bh_1) - \Delta(\alpha_1 + bh_2) & \text{if } i = p \end{cases} .$$

It is not difficult to check that the relation 3.110 is verified with  $n_r = 0$ . Additionally, in this particular case:

$$p_i \underset{z \rightarrow 1}{\approx} \frac{C_i}{z-1} + \text{regular terms} .$$

Where  $C_i$  are constants. Because  $Q_0$  does not have any 0 in  $\mathbb{C} \setminus \{0, 1, \infty\}$ , the functions  $p_i$  do not have any other poles than 0, 1 and  $\infty$ . They are entirely determined by their behaviour near 0 and  $\infty$ :

$$p_r(z) = \frac{(-1)^{p-r-1}}{z-1} \left[ \sum_{1 \leq i_1 < \dots < i_r \leq p} \left( \prod_{j=1}^r a_{\infty, i_j} \right) z - \sum_{1 \leq i_1 < \dots < i_r \leq p} \left( \prod_{j=1}^r a_{0, i_j} \right) \right]$$

In consequence, The differential equation verified by the function  $f$  is fixed:

$$z \prod_{i=1}^p (\vartheta - a_{\infty, i}) f = \prod_{i=1}^p (\vartheta - a_{0, i}) f$$

And of course, we find back the generalized hypergeometric differential equation (see 3.89).

Theoretically at least this method can be applied to any situation where the number of conformal blocks is finite. The constraint on  $p_1$  can be used to conjecture the fusion rules knowing the spectrum. For example, we can make the conjecture that in  $\mathcal{W}_3$ , the correlation function  $\langle \mathcal{O}_{\alpha_\infty^*} | \mathcal{O}_{b\omega_1}(z) \mathcal{O}_{\alpha_1}(1) | \mathcal{O}_{\alpha_0} \rangle$  with a semi-degenerate field  $\alpha_1 = \kappa\omega_1 + (bq_2 - b^{-1}q_1)\omega_2$ ,  $q_1, q_2 \in \mathbb{N}$  will have  $p = 3(1 + q_1 + q_2 + q_1q_2)$  conformal blocks. Additionally, the fusion rule in the  $z \rightarrow 1$  canal will verify (with multiplicities):

$$\mathcal{O}_{b\omega_1} \otimes \mathcal{O}_{\alpha_1} = \frac{p}{3} \mathcal{O}_{\alpha_1 + bh_1} \oplus \frac{p}{3} \left( \frac{2 + q_2}{1 + q_2} \right) \mathcal{O}_{\alpha_1 + bh_2} \oplus \frac{p}{3} \left( \frac{q_2}{1 + q_2} \right) \mathcal{O}_{\alpha_1 + bh_3} .$$



## SUMMARY

This chapter presents a new method to compute Rényi entanglement entropies in one-dimensional critical systems described by a minimal CFT. Entanglement entropy is defined with respect to a state  $\psi$  and a bipartition of space into two regions,  $\mathcal{A}$  and its complement  $\mathcal{B}$ :

$$S_N(\mathcal{A}, \psi) = \frac{1}{1-N} \log \text{Tr}_{\mathcal{A}} \left[ \left( \text{Tr}_{\mathcal{B}} (|\psi\rangle\langle\psi|) \right)^N \right].$$

The standard method consists in using the replica trick to link the entropy to the partition function of the CFT on a Riemann surface formed by  $N$  sheets, linked together along  $\mathcal{A}$ . This surface can grow quite complicated, especially if  $\mathcal{A}$  is not connected. Another approach consists instead in complicating the field theory, rather than the surface, by considering an extended orbifold CFT formed from the initial theory. In this context the “twist fields” can be understood as the lowest dimensional operators of the twisted sector of the orbifold. The null-vector conditions on the twist fields of the cyclic orbifold can be used to obtain differential equations on their correlation functions. The correlation functions are then determined by standard bootstrap techniques. This chapter contains:

- The computations of excited state Rényi entropies ( $N = 2$  and  $N = 3$ ) for generic minimal models (4.5.4, 4.5.5).
- The computations of the  $N = 2$  Rényi entropy of the Ising and Yang-Lee minimal models, on two intervals (4.4.1, 4.5.2).
- The lattice implementation of the twist fields in critical RSOS model. This is also used to numerically corroborate our results (4.6).
- A discussion of “entanglement entropy” in non-unitary CFT, focused around the Yang-Lee case. While the object we consider is not the standard entanglement entropy in the non-unitary case, it shares feature with it, notably as a marker of criticality (4.2.3, 4.E).



## 4.1 INTRODUCTION

Ideas coming from quantum information theory have provided invaluable insights and powerful tools for quantum many-body systems. One of the most basic tools in the arsenal of quantum information theory is (entanglement) entropy [18]. Upon partitioning a system into two subsystems,  $\mathcal{A}$  and  $\mathcal{B}$ , the entanglement entropy is defined as the von Neumann entropy  $S(\mathcal{A}) = -\text{Tr} \rho_{\mathcal{A}} \log \rho_{\mathcal{A}}$ , with  $\rho_{\mathcal{A}}$  being the reduced density matrix of subsystem  $\mathcal{A}$ .

Entanglement Entropy (EE) is a versatile tool. For a gapped system in any dimension, the entanglement entropy behaves similarly to the black hole entropy : its leading term grows like the area of the boundary between two subsystems instead of their volume, in a behavior known as the *area law* [59, 104, 107, 108, 123, 138, 150] :

$$S(\mathcal{A}) \simeq \alpha \text{Vol}(\partial\mathcal{A}),$$

*These methods rely on the fact that the ground state is far from random, its correlations tend to be local thanks to the area law.*

where  $\alpha$  is a non-universal quantity. Quantum entanglement – and in particular the area law – has led in recent years to a breakthrough in our understanding of quantum systems, and to the development of remarkably efficient analytical and numerical tools. These methods, dubbed *tensor network* methods, have just begun to be applied to strongly correlated systems with unprecedented success [7, 42, 140].

For critical systems, a striking result is the universal scaling of the EE in one-dimension [35, 85, 141]. For an infinite system, with the subsystem  $\mathcal{A}$  being a single interval of length  $\ell$ , one has

$$S([0, \ell]) \simeq \frac{c}{3} \log \ell,$$

where  $c$  is the central charge of the underlying Conformal Field Theory (CFT). This result is based on a CFT approach to entanglement entropy combined with the replica trick, which maps the (Rényi) EE to the partition function on an  $N$ -sheeted Riemann surface with conical singularities. In some particular cases – essentially for free theories – it is possible to directly calculate this partition function [1, 2, 31, 32, 44, 74] using the general results from the 1980's for free bosonic partition functions on Riemann surfaces [5, 51, 52, 155]. In most cases however this is very difficult. An alternative approach is to replicate the CFT rather than the underlying Riemann surface. Within this scheme one ends up with the tensor product of  $N$  copies of the original CFT modded out by cyclic permutations, and the conical singularities are mapped to twist fields, denoted as  $\tau$ . These theories are known as *cyclic orbifolds* [25, 47, 99, 100]. Within this framework the Rényi EE boils down to a correlation function of twist fields in the cyclic orbifold [34, 36]. The case of a single interval is particularly simple as it maps to a two-twist correlation function. When the subsystem  $\mathcal{A}$

is the union of  $m > 1$  disjoint intervals most results are restricted to free theories [1, 2, 31, 32, 44, 45], and much less is known in general [126]. In the orbifold framework, this maps to a  $2m$ -twist correlation function, which is of course much more involved to compute than a simple two-point function.

In this chapter we report on a new method to compute twist fields correlation functions. The key ingredients are (i) the null-vector conditions obeyed by the twist fields under the extended algebra of the cyclic orbifold and (ii) the Ward identities obeyed by the currents in this extended algebra. Note that the null-vector conditions for twist fields were already detected in [47], but until now then they have only been exploited to determine their conformal dimension. Our method is quite generic, the only requirement being that the underlying CFT be rational (which in turn ensures that the induced cyclic orbifold is rational). This approach provides a rather versatile and powerful tool to compute the EE that is applicable to a variety of situations, such as non-unitary CFT, EE of multiple intervals, EE at finite temperature and finite size, and/or EE in an excited state.

We illustrate this method with the most basic minimal model of CFT: the Yang-Lee model. This model has only two primary fields: the identity  $\mathbb{1}$  and the field  $\phi$ . However, the simplicity of this situation – in particular, the nice form of the null-vector conditions obeyed by the identity operator – comes with a slight complication: the model is not unitary, and  $\phi$  has a negative dimension. Hence, the vacuum  $|0\rangle$  and the ground state  $|\phi\rangle$  are distinct (i.e. the vacuum is not the state with the lowest energy), which implies that the ground state breaks conformal invariance. This leads to an important modification in the path integral description used in the replica trick: the boundary conditions at every puncture must reflect the insertion of the field  $\phi$  (and not the identity operator). In practice, this means that the twist field  $\tau$  must be replaced by  $\tau_\phi \propto \tau\phi$ : as noted in [23], but also that the correlation functions of these twist fields must be evaluated in the ground state  $|\phi\rangle$  rather than in the vacuum  $|0\rangle$ .

Hence, we see that, for the Yang-Lee model at finite size, even the single-interval entropy requires the computation of a four-point function, and this is where the full power of null-vector equations can be brought to bear.

The plan of the chapter is as follows. In Sec. 4.2, we review the cyclic orbifold construction of [25, 47, 99], and its relation to Rényi entropies. In Sec. 4.4, we describe a basic example where the null-vector conditions on the twist field only involve the usual Virasoro modes, and thus yield straightforwardly a differential equation for the twist correlation function. In Sec. 4.5, we turn to more generic situations, where the null-vector conditions involve fractional modes of

the orbifold Virasoro algebra: we first introduce the Ward identities for the conserved currents  $\widehat{T}^{(r)}(z)$  in the cyclic orbifold, and use them to derive differential equations for a number of new twist correlation functions. Finally, in Sec. 2.1.3 we describe a lattice implementation of the twist fields in the lattice discretisation of the minimal models, namely the critical Restricted Solid-On-Solid (RSOS) models. We conclude with a numerical check of our analytical results for various EEs in the Yang-Lee model.

## 4.2 GENERAL BACKGROUND

### 4.2.1 Entanglement entropy and conformal mappings

Consider a critical one-dimensional quantum system (a spin chain for example), described by a conformal field theory (CFT). Suppose that the system is separated into two parts :  $\mathcal{A}$  and its complement  $\mathcal{B}$ . The *amount of entanglement* between  $\mathcal{A}$  and  $\mathcal{B}$  is usually measured through the Von Neumann entropy. If the system is in a normalized pure state  $|\psi\rangle$ , with density matrix  $\rho = |\psi\rangle\langle\psi|$ , its Von Neumann entropy is defined as:

$$S(\mathcal{A}, \psi) = -\text{Tr}_{\mathcal{A}}[\rho_{\mathcal{A}} \log(\rho_{\mathcal{A}})], \quad \text{where } \rho_{\mathcal{A}} = \text{Tr}_{\mathcal{B}} |\psi\rangle\langle\psi|. \quad (4.1)$$

The Rényi entropy is a slight generalization, which depends on a real parameter  $N$ :

$$S_N(\mathcal{A}, \psi) = \frac{1}{1-N} \log \text{Tr}_{\mathcal{A}}(\rho_{\mathcal{A}}^N). \quad (4.2)$$

In the limit  $N \rightarrow 1$ , one recovers the von Neumann entropy:  $S_{N \rightarrow 1}(\mathcal{A}, \psi) = S(\mathcal{A}, \psi)$ .

For integer  $N$ , a replica method to compute this entropy was developed in [35] (see [34] for a recent review). The main idea consists in re-expressing geometrically the problem. The partial trace  $\rho_{\mathcal{A}}$  acts on states living in  $\mathcal{A}$  and propagates them, while tracing over the states in  $\mathcal{B}$ . It can be seen as the density matrix of a “sewn” system kept open along  $\mathcal{A}$  but closed on itself elsewhere.

When  $\mathcal{A}$  is a single interval ( $\mathcal{A} = [u, v]$ ), the resulting Riemann surface is conformally equivalent to the sphere. It can be *unfolded* (mapped to the sphere) using a change of variable of the form:

$$w = \left( \frac{z-u}{z-v} \right)^{1/N}. \quad (4.3)$$

When  $|\psi\rangle = |\text{vac}\rangle$  is the vacuum state of the CFT, this change of coordinates allows [34] to compute the entropy of a single interval in an infinite system, with the well-known result:

$$S_N([u, v], \text{vac}) = \frac{c}{6} \frac{N+1}{N} \log |u-v|. \quad (4.4)$$

Throughout this chapter, we shall rather consider the case of an interval  $\mathcal{A} = [0, \ell]$  in a finite system of length  $L$  with periodic boundary conditions. In this case, one has [34]:

$$S_N(\ell/L, \text{vac}) = \frac{c}{6} \frac{N+1}{N} \log \left[ L \sin \left( \frac{\pi \ell}{L} \right) \right], \quad (4.5)$$

where we have slightly changed the notation to indicate that the total system is of finite size  $L$ .

This type of calculations becomes more complicated for the entropy of other states than the vacuum: two operators then need to be added on each of the sheets of  $\Sigma_N$ . This is one of the main limitations of the method based on conformal mapping : a lot of the structure of the initial problem disappears after the conformal map. In this case, a one-variable problem (the size of the interval) becomes a  $2N$ -variable problem. These complicated correlation functions have only been computed for free theories [4, 19], and have been used in various contexts since then [33, 60, 114]. Moreover, if  $\Sigma$  is the initial surface where the system lives, then the genus of  $\Sigma_N$  is  $g(\Sigma_N) = Ng(\Sigma) + (N-1)(p-1)$ , where  $p$  is the number of connected components of  $\mathcal{A}$ . Hence, if  $\mathcal{A}$  is not connected or if the initial surface is not the Riemann sphere, one has to deal with CFT on higher-genus surfaces.

#### 4.2.2 Correlation functions of twisted operators

The Rényi entropies can alternatively be interpreted as correlation functions of twist operators. We consider a system of finite length  $L$  with periodic boundary conditions, in the quantum state  $|\psi\rangle$ . In the scaling limit, this corresponds to a CFT on the infinite cylinder of circumference  $L$ , with boundary conditions specified by the state  $\psi$  on both ends of the cylinder. After the conformal mapping  $z \mapsto \exp \frac{2\pi z}{L}$ , one recovers the plane geometry. The Rényi entropy of a single interval  $\mathcal{A} = [0, \ell]$  in the pure state  $|\psi\rangle$  is given as a correlation function in the  $\mathbb{Z}_N$  orbifold CFT (see below):

$$S_N([0, \ell], \psi) = \frac{1}{1-N} \log \langle \Psi | \tau(1) \tilde{\tau}(x, \bar{x}) | \Psi \rangle, \quad x = \exp(2i\pi\ell/L), \quad (4.6)$$

where  $\Psi = \psi^{\otimes N}$  corresponds to  $N$  replicas of the operator  $\psi$  at a given point. The twist operators  $\tau$  and  $\tilde{\tau}$  implement the branch points at the ends of the interval. Since these branch points introduce singularities in the metric, one has to choose a particular regularisation of the theory at each branch point: each choice of regularisation corresponds to a choice of primary twist operator  $\tau$ . The classification of primary twist operators is obtained by the induction procedure (see Sec. 4.3.3), which uniquely associated any primary operator  $\phi$  of the mother CFT

to a twist operator  $\tau_\phi$ , with dimension  $\widehat{h}_\phi = (N - 1/N)c/24 + h_\phi/N$ . In a unitary CFT, the most relevant operator is the identity (i.e. the conformally invariant operator), and the correct choice for the twist operator in (4.6) is  $\tau = \tau_1$ . In Sec. 2.1.3 we introduce the construction of a lattice regularisation scaling to *any* given primary twist operator  $\tau_\phi$  in a minimal model of CFT.

More generally, if  $\mathcal{A}$  is a union of  $p \geq 1$  disjoint intervals:

$$\mathcal{A} = [u_1, v_1] \cup [u_2, v_2] \cup \dots [u_p, v_p],$$

then one may define the  $p$ -interval correlation function:

$$\langle \Psi | \tau_1(y_1, \bar{y}_1) \tilde{\tau}_1(x_1, \bar{x}_1) \dots \tau_p(y_p, \bar{y}_p) \tilde{\tau}_p(x_p, \bar{x}_p) | \Psi \rangle, \quad (4.7)$$

with

$$x_j = \exp(2i\pi v_j/L), \quad y_j = \exp(2i\pi u_j/L),$$

and any choice of twist operators  $(\tau_1, \dots, \tau_p)$  and  $(\tilde{\tau}_1, \dots, \tilde{\tau}_p)$ .

#### 4.2.3 Non-unitary models

Although the goal of the present chapter is not to study specifically entanglement in non-unitary models, some emphasis is put on the Yang-Lee singularity model. The reason for this is that the corresponding minimal model has the simplest operator algebra (it has only two primary fields), which makes calculations more tractable and easy to present. However, it should be stressed that what we are computing are partition functions on  $N$ -sheeted surfaces. For a unitary model this corresponds to Rényi entropies, and for that reason we chose to refer to these partition functions as "entropies" even in the non-unitary case. This is just a matter of terminology, and we do not claim that they provide a good measure of the amount of entanglement.

The problem of entanglement entropy in non-unitary models has already been addressed in various contexts [22–24, 46]. For comparison with the existing literature on the subject, we clarify in this section the specific choices and observations that we made for non-unitary models. We refrain from using the bra/ket notations to avoid any possible source of confusion.

Consider a Hamiltonian  $H$  acting on a vector space  $E$ . The transpose operator  ${}^tH$  acts in the dual space (consisting of all linear forms)  $E^*$  as

$${}^tH(w) = w \circ H, \quad (4.8)$$

for any linear form  $w$ . We assume that  $H$  is diagonalizable with a discrete spectrum and eigenbasis  $\{r_j\}$

$$Hr_j = E_j r_j. \quad (4.9)$$

The dual basis  $\{w_j\}$ , which is defined by  $w_i(r_j) = \delta_{ij}$ , is an eigenbasis of  ${}^tH$

$$w_j \circ H = E_j w_j, \quad (4.10)$$

and the Hamiltonian can be written as

$$H = \sum_j E_j r_j w_j. \quad (4.11)$$

A possible definition for the density matrix of the system at inverse temperature  $\beta$  is

$$\rho = \frac{1}{Z} e^{-\beta H} = \frac{1}{Z} \sum_j e^{-\beta E_j} r_j w_j, \quad Z = \sum_j e^{-\beta E_j}. \quad (4.12)$$

In particular at zero temperature this yields

$$\rho = r_0 w_0, \quad (4.13)$$

where  $r_0$  denotes the ground state of  $H$ . Assuming a decomposition  $E = E_{\mathcal{A}} \otimes E_{\mathcal{B}}$  one can then trace over  $\mathcal{B}$  to define  $\rho_{\mathcal{A}}$ . Let  $\{f_j\}$  be a basis of  $E_{\mathcal{B}}$  and  $\{f_j^*\}$  the dual basis, the trace over  $\mathcal{B}$  is defined as

$$\rho_{\mathcal{A}} = \text{Tr}_{\mathcal{B}}(\rho) = \sum_j (\mathbb{1}_{\mathcal{A}} \otimes f_j^*) \rho (\mathbb{1}_{\mathcal{A}} \otimes f_j). \quad (4.14)$$

Note that tracing over  $\mathcal{B}$  is independent of the basis  $\{f_j\}$  chosen, and does not require any inner product. With  $\rho_{\mathcal{A}}$  at hand one then defines the Von Neumann and Rényi entropies in the usual way.

The main advantage of this construction is that the corresponding (Rényi) entropy  $\text{Tr}(\rho_{\mathcal{A}}^N)$  maps within the path-integral approach to an Euclidean partition function on an  $N$ -sheeted Riemann surface. Underlying this result is the identification of the reduced matrix  $\rho_{\mathcal{A}}$  with the partition function on a surface leaving open a slit along  $\mathcal{A}$ . Note that such a partition function can be computed purely in terms of matrix elements of the transfer matrix, and therefore it does not involve any inner product structure.

The disadvantages of this construction are twofold. The main one is that the reduced density matrix (and hence the entanglement entropy) may not be positive. While this may seem like a pathological property, loss of positivity in a non-unitary system might be acceptable depending on the context and motivations. The other one is that this definition only applies to eigenstates of  $H$  (and statistical superposition thereof). This stems for the fact that there is no canonical (i.e. basis independent) isometry between  $E$  and  $E^*$ . In practice this means that knowing the ground state  $r_0$  is not enough to compute the entanglement entropy, one also needs to know  $H$  to characterise  $w_0$ .

Consider now an inner-product structure on  $E$ , *i.e.* a non-degenerate hermitian form  $\langle \cdot, \cdot \rangle$ . By virtue of being non-degenerate, this inner product induces a canonical isometry between linear forms and vectors. For every vector  $v \in E$ , denote by  $v^\dagger$  the linear form defined by

$$v^\dagger(x) = \langle v, x \rangle, \quad x \in E.$$

Every element in  $E^*$  can be written in this form, and the map  $I : v \rightarrow v^\dagger$  is an antilinear isometry from  $E$  to  $E^*$ . In particular one can associate a vector  $l_j$  to every linear form  $w_j$  such that  $w_j = l_j^\dagger$ . The vectors  $l_j$  are what is commonly referred to as left eigenvectors of  $H$ . These are nothing but the eigenvectors of  $H^\dagger$ , the hermitian adjoint of  $H$ , which is characterised by the following property

$$\langle H^\dagger v_1, v_2 \rangle = \langle v_1, H v_2 \rangle$$

for any vectors  $v_1, v_2$  in  $E$ . The transpose  ${}^t H$  and the Hermitian adjoint  $H^\dagger$  are closely related :

$$H^\dagger = I^{-1} \circ {}^t H \circ I.$$

In particular the relation  ${}^t H(w_j) = E_j w_j$  becomes

$$H^\dagger l_j = E_j^* l_j. \quad (4.15)$$

While the previous prescription for the density matrix amounts in this context to

$$\rho = r_0 l_0^\dagger, \quad (4.16)$$

an alternative prescription is

$$\tilde{\rho} = r_0 r_0^\dagger. \quad (4.17)$$

For many non-unitary models there exist a natural notion of inner product that makes the Hamiltonian self-adjoint and is compatible with locality, at the cost of not being definite-positive. For such an inner-product left and right eigenvectors coincide and it follows that

$$\rho = \tilde{\rho}.$$

*Compatible with locality in the sense that the Hamiltonian density is self-adjoint.*

This is typically the case within the CFT framework : the standard CFT inner product is such that  $L_n^\dagger = L_{-n}$ , and in particular the Hamiltonian is self-adjoint:  $L_0 = L_0^\dagger$ . This is also the case for the loop model based on the TL algebra[46] or for the Yang-Lee spin chain (see appendix 4.E).

Let us now assume that the inner product is definite-positive. For a unitary system  $H = H^\dagger$  : left and right eigenvectors coincide, and

both prescriptions yield the usual notion of density matrix and entanglement. For a non-Hermitian Hamiltonian operator  $H$  in general  $l_j$  and  $r_j$  are different (even if  $E_j$  is real). If  $H$  is symmetric but not real (in some orthonormal basis), then the eigenvectors have non-real components, and are related through complex conjugation :

$$l_j = r_j^*$$

On a more fundamental level this illustrates the fact that the canonical map between linear forms and vectors is antilinear.

The prescription  $\tilde{\rho}$ , together with a positive-definite inner product, seems to be physically more natural than  $\rho$  as it yields a positive entanglement entropy (as can be seen from the Schmidt decomposition). Moreover, it does not depend on  $H$ , only on the state considered and on the inner product. However, this quantity is very much sensitive to the inner product chosen, and for a non-Hermitian Hamiltonian there is no canonical choice of a positive-definite inner product.

On a more technical side, when computing any quantity involving  $\tilde{\rho}$  in the path-integral formalism one needs to implement explicitly the (inner-product dependent) time-reversal operation  $r_0 \rightarrow l_0$  (e.g.  $r_0 \rightarrow r_0^*$  for symmetric  $H$ ) in order to get a consistent Euclidean description. Such a time-reversal defect can be thought of as a specific boundary condition in the tensor product  $CFT^{\otimes 2}$ , which is typically a difficult problem.

It has been argued in [23] that for  $PT$ -symmetric Hamiltonian, the left and right ground states  $r_0$  and  $l_0$  coincide (while working with a definite positive inner product). This would circumvent this difficulty. However, we found that this is not the case for the Yang-Lee model in finite size. Moreover, assuming  $r_0 = l_0$  immediately yields positive entanglement entropies, which again we found is not the case (both within our numerical and analytical calculations, see Figure 4.3).

In the following, when the model considered is non-unitary, we will choose (4.13) as the density matrix so that the Euclidean path-integral formalism described in Sec. 4.2.1, i.e. the interpretation of Rényi entropies as partition functions on a replicated surface with branch points, can be used straightforwardly. This is also the choice made in [24, 46].

Within the Euclidean path-integral formalism an additional fact to take into account when studying non-unitary models is the existence of a primary state  $\phi$  with a conformal dimension lower than the CFT vacuum  $h = 0$  (i.e. the conformally invariant state). As was first pointed out in [23], this has a dramatic effect on the twist field : the most relevant twist operator is no longer  $\tau_{\perp}$ , but rather  $\tau_{\phi}$ . Repeating the steps of section 4.2.2, the one-interval Rényi entropy in the ground state  $|\phi\rangle$  is mapped within the orbifold approach to

$$\mathrm{Tr}(\rho_{\mathcal{A}}^N) = \langle \Phi | \tau_{\phi}(u) \tilde{\tau}_{\phi}(v) | \Phi \rangle \quad (4.18)$$



where  $\Phi = \phi^{\otimes N}$ . In [23] it was further claimed that the entanglement entropy in a non-unitary model behaves as

$$S_N \sim \frac{c_{\text{eff}}}{6} \frac{N+1}{N} \log |u-v|, \tag{4.19}$$

where  $c_{\text{eff}} = c - 24h_\phi$  is the effective central charge. However, this result was based on an incorrect mapping to an Euclidean partition function, namely

$$\text{Tr}(\rho_{\mathcal{A}}^N) = \frac{\langle \tau_\phi(u) \tilde{\tau}_\phi(v) \rangle}{\langle \Phi(u) \Phi(v) \rangle} \tag{4.20}$$

instead of (4.18). We claim that the behaviour (4.19) is incorrect, and the Cardy-Calabrese formulas (4.4-4.5) for the entanglement entropy cannot be applied, even with the substitution  $c \rightarrow c_{\text{eff}}$ .

*When discussing the result of [23] the distinction between  $\rho = r_0 l_0^\dagger$  and  $\rho = r_0 r_0^\dagger$  is irrelevant since they argue that  $l_0 = r_0$  at criticality.*

### 4.3 THE CYCLIC ORBIFOLD

The expression (4.4) suggests that the partition function on  $\Sigma_N$  can be considered as the two-point function of a “twist operator” of dimension

$$h_\tau = \frac{c}{24} \left( N - \frac{1}{N} \right). \tag{4.21}$$

Indeed, this point of view corresponds to the construction of the cyclic orbifold. We described the case  $N = 2$  in the introduction, in this section we consider the general case. Mathematically, one starts with  $N$  copies of the original CFT model (called the *mother CFT*, with central charge  $c$ , living on the original surface  $\Sigma$ ) then mod out the  $\mathbb{Z}_N$  symmetry (the cyclic permutations of the copies). This cyclic orbifold theory was studied extensively in [25, 47, 99]. We give an overview of the relevant concepts that we shall use.

*For simplicity, in the following, we consider only the case when  $N$  is a prime integer.*

#### 4.3.1 The orbifold Virasoro algebra

All the copies of the mother CFT have their own energy-momentum tensor  $T_j(z)$  [and  $\bar{T}_j(\bar{z})$  for the anti-holomorphic part]. Their discrete Fourier transforms (in replica space) are called  $\hat{T}^{(r)}(z)$ ,  $r \in \{0, \dots, N-1\}$  and are defined by :

$$\hat{T}^{(r)}(z) = \sum_{j=1}^N e^{2i\pi rj/N} T_j(z), \quad T_j(z) = \frac{1}{N} \sum_{r=0}^{N-1} e^{-2i\pi rj/N} \hat{T}^{(r)}(z). \tag{4.22}$$

The currents  $T_j$  are all energy-momentum tensors of a conformal field theory, so their Operator Product Expansion (OPE) with themselves is:

$$T_j(z) T_k(0) = \delta_{j,k} \left[ \frac{c/2}{z^4} + \frac{2T_j(z)}{z^2} + \frac{\partial T_j(z)}{z} \right] + \text{regular terms.} \tag{4.23}$$

For two distinct copies,  $T_j(z_1) T_k(z_2)$  is regular ; on the unfolded surface, even when  $z_1 \rightarrow z_2$  the two currents are at different points. With that in mind, the OPE between the Fourier transforms of these currents can be written:

$$\begin{aligned} \widehat{T}^{(r)}(z) \widehat{T}^{(s)}(0) &= \frac{(Nc/2) \delta_{r+s,0}}{z^4} + \frac{2 \widehat{T}^{(r+s)}(z)}{z^2} \\ &+ \frac{\partial \widehat{T}^{(r+s)}(z)}{z} + \text{regular terms,} \end{aligned} \quad (4.24)$$

where the indices  $r$  and  $s$  are considered modulo  $N$ . The modes of the currents are defined as:

$$\widehat{L}_m^{(r)} = \frac{1}{2i\pi} \oint dz z^{m+1} \widehat{T}^{(r)}(z). \quad (4.25)$$

In the untwisted sector of the theory the mode indices  $m$  have to be integers since the operators  $\widehat{T}^{(r)}(z)$  are single valued when winding around the origin. In the twisted sector however the operators  $\widehat{T}^{(r)}(z)$  are no longer single-valued, and the mode indices  $m$  can be fractional. Generically in the cyclic  $\mathbb{Z}_N$  orbifold we have

$$m \in \mathbb{Z}/N. \quad (4.26)$$

The actual values of  $m$  appearing in the mode decomposition are detailed below: see (4.35) and (4.38). From the OPE (4.24) one obtains the commutation relations:

$$\left[ \widehat{L}_m^{(r)}, \widehat{L}_n^{(s)} \right] = (m-n) \widehat{L}_{m+n}^{(r+s)} + \frac{Nc}{12} m(m^2-1) \delta_{m+n,0} \delta_{r+s,0}, \quad (4.27)$$

where  $(m, n) \in (\mathbb{Z}/N)^2$ . The actual energy-momentum tensor in the orbifold theory is  $T_{\text{orb}}(z) = \widehat{T}^{(0)}(z)$ . It generates transformations affecting all the sheets in the same way, so in the orbifold it has the usual interpretation (derivative of the action with respect to the metric). Correspondingly, the integer modes  $\widehat{L}_{m \in \mathbb{Z}}^{(0)}$  form a Virasoro subalgebra. The  $\widehat{T}^{(r)}(z)$  for  $r \neq 0$  also have conformal dimension 2, and play the role of additional currents of an extended CFT with internal  $\mathbb{Z}_N$  symmetry.

#### 4.3.2 Operator content of the $\mathbb{Z}_N$ orbifold

**THE UNTWISTED SECTOR** Let  $z$  be a regular point of the surface  $\Sigma_N$ . A generic primary operator at such a regular point, which we shall call an *untwisted* primary operator, is simply given by the tensor product of  $N$  primary operators  $\phi_1, \dots, \phi_N$  of dimensions  $h_1, \dots, h_N$  in the mother CFT, each sitting on a different copy of the model:

$$\Phi(z) = \phi_1(z) \otimes \phi_2(z) \otimes \dots \phi_N(z). \quad (4.28)$$

In the case when  $\Phi$  includes at least one pair of distinct operators  $\phi_i \neq \phi_j$ , it is also convenient to define the discrete Fourier modes

$$\widehat{\Phi}^{(r)}(z) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} e^{2i\pi rj/N} \phi_{1-j}(z) \otimes \phi_{2-j}(z) \otimes \cdots \otimes \phi_{N-j}(z), \quad (4.29)$$

where the indices are understood modulo  $N$ . The normalisation of  $\widehat{\Phi}^{(r)}(z)$  is chosen to ensure a correct normalisation of the two-point function:

$$\left\langle \widehat{\Phi}^{(r)}(z_1) \widehat{\Phi}^{(-r)}(z_2) \right\rangle = (z_1 - z_2)^{-2h_\Phi}, \quad (4.30)$$

where  $h_\Phi = \sum_{j=1}^N h_j$ . In particular, for a primary operator  $\phi$  in the mother CFT, if one sets  $\phi_1 = \phi_h$  and  $\phi_2 = \cdots = \phi_N = \mathbb{1}$ , one obtains the *principal primary fields* of dimension  $h$ :

$$\widehat{\phi}_h^{(r)}(z) = \frac{1}{\sqrt{N}} \sum_{j=1}^N e^{2i\pi rj/N} \mathbb{1}(z) \otimes \cdots \otimes \phi_h(z) \otimes \cdots \otimes \mathbb{1}(z). \quad (4.31)$$

(j-th)

The OPE of the currents with generic primary operators are:

$$\widehat{T}^{(r)}(z) \widehat{\Phi}^{(s)}(0) = \frac{\widehat{h}_\Phi^{(r)} \widehat{\Phi}^{(r+s)}(0)}{z^2} + \frac{\widehat{\partial}^{(r)} \widehat{\Phi}^{(s)}(0)}{z} + \text{regular terms}, \quad (4.32)$$

where we have introduced the notations

$$\widehat{h}_\Phi^{(r)} = \sum_{j=1}^N e^{2i\pi rj/N} h_j, \quad (4.33)$$

and  $\widehat{\partial}^{(r)} = \sum_{j=1}^N e^{2i\pi rj/N} (1 \otimes \cdots \otimes 1 \otimes \frac{\partial}{(j\text{-th})} \otimes 1 \otimes \cdots \otimes 1).$

This expression reduces to a simple form in the case of a principal primary operator:

$$\widehat{T}^{(r)}(z) \widehat{\phi}_h^{(s)}(0) = \frac{h \widehat{\phi}_h^{(r+s)}(0)}{z^2} + \frac{\partial \widehat{\phi}_h^{(r+s)}(0)}{z} + \text{regular terms}. \quad (4.34)$$

From the expression (4.32), the product of  $\widehat{T}^{(r)}(z)$  with an untwisted primary operator is single-valued, and hence, only integer modes appear in the OPE:

$$\widehat{T}^{(r)}(z) \widehat{\Phi}^{(s)}(0) = \sum_{m \in \mathbb{Z}} z^{-m-2} \widehat{L}_m^{(r)} \widehat{\Phi}^{(s)}(0). \quad (4.35)$$

**THE TWISTED SECTORS** The conical singularities of the surface  $\Sigma_N$  are represented by twist operators in the orbifold theory. A twist operator of charge  $k \neq 0$  is generically denoted as  $\tau^{[k]}$ , and corresponds to the end-point of a branch cut connecting the copies  $j$  and  $j+k$ . If

$A_j$  denotes the  $j$ -th copy of a given operator  $A$  of dimension  $h_A$ , one has:

$$A_j(e^{2i\pi z}) \tau^{[k]}(0) = e^{-2i\pi h_A} A_{j+k}(z) \tau^{[k]}(0). \quad (4.36)$$

This relation can be considered as a characterisation of an operator  $\tau^{[k]}$  of the  $k$ -twisted sector.

As a consequence, the Fourier components  $\widehat{T}^{(r)}$  have a simple monodromy around  $\tau^{[k]}$ :

$$\widehat{T}^{(r)}(e^{2i\pi z}) \tau^{[k]}(0) = e^{-2i\pi r k/N} \times \widehat{T}^{(r)}(z) \tau^{[k]}(0), \quad (4.37)$$

and similarly for the primary operators  $\widehat{\Phi}^{(r)}$  and  $\widehat{\phi}_h^{(r)}$ . Hence, the OPE of  $\widehat{T}^{(r)}(z)$  with a twist operator can only include the modes consistent with this monodromy:

$$\widehat{T}^{(r)}(z) \tau^{[k]}(0) = \sum_{m \in \mathbb{Z} + kr/N} z^{-m-2} \widehat{L}_m^{(r)} \tau^{[k]}(0). \quad (4.38)$$

If one supposes that there exists a “vacuum” operator  $\tau_{\mathbb{1}}^{[k]}$  in the  $k$ -twisted sector, one can construct the other primary operators in this sector through the OPE:

$$\tau_{\phi}^{[k]}(z) := N^{h_{\phi}} \lim_{\epsilon \rightarrow 0} \left[ e^{(1-1/N)h_{\phi}} \tau_{\mathbb{1}}^{[k]}(z) (\phi(z+\epsilon) \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}) \right]. \quad (4.39)$$

For convenience, in the following, we shall use the short-hand notations:

$$\tau_{\phi} := \tau_{\phi}^{[k=1]}, \quad \widetilde{\tau}_{\phi} := \tau_{\phi}^{[k=-1]}. \quad (4.40)$$

In a sector of given twist, most fractional descendant act trivially:

$$L_{l/N}^{(r)} \tau^{[k]} = 0 \text{ if } l \notin N\mathbb{Z} + kr$$

Hence the short-hand notation:

$$L_{l/N} \tau^{[k]} = 0 \equiv L_{l/N}^{(l \bmod N)/k} \tau^{[k]} = 0$$

### 4.3.3 Induction procedure

Suppose one quantises the theory around a branch point of charge  $k \neq 0$  at  $z = 0$ . After applying the conformal map  $z \mapsto w = z^{1/N}$  from  $\Sigma_N$  to a surface where  $w = 0$  is a regular point, the currents  $\widehat{T}^{(r)}(z)$  transform as:

$$\widehat{T}^{(r)}(z) \mapsto w^{2-2N} \sum_{j=0}^{N-1} e^{2i\pi j(r\ell+2)/N} T(e^{2i\pi j/N} w) + \frac{(N^2-1)c \delta_{r,0}}{24Nz^2}. \quad (4.41)$$

Accordingly, one gets for the generators:

$$\widehat{L}_m^{(r)} \mapsto \frac{1}{N} L_{Nm} + \frac{c}{24} \left( N - \frac{1}{N} \right) \delta_{r,0} \delta_{m,0}, \quad \text{for } m \in \mathbb{Z} + \frac{rk}{N}, \quad (4.42)$$

where the  $L_n$ 's are the ordinary Virasoro generators of the mother CFT. It is straightforward to check that these operators indeed obey the commutation relations (4.27). Similarly, the twisted operator  $\tau_\phi^{[k]}$  (4.39) maps to the primary operator  $\phi_h$  of the mother CFT:

$$\tau_\phi^{[k]}(z) \mapsto w^{(1-N)h_\phi} \phi(w). \tag{4.43}$$

The relations (4.42–4.43) are called the “induction procedure” in [25]. Using (4.42–4.43) for  $r = m = 0$ , the dimension of  $\tau_\phi^{[k]}$  for any  $k \neq 0$  is

$$\widehat{h}_\phi = \frac{h_\phi}{N} + \frac{c}{24} \left( N - \frac{1}{N} \right). \tag{4.44}$$

This formula first appeared in [26, 93], and, in the context of entanglement entropy, in [40]. In particular, when  $\phi$  is the identity operator  $\mathbb{1}$  with dimension  $h_{\mathbb{1}} = 0$ , the expression (4.44) coincides with the dimension  $h_\tau = \widehat{h}_{\mathbb{1}}$  (4.21) for  $\tau_{\mathbb{1}}$ .

#### 4.3.4 Null-vector equations for untwisted and twisted operators

We intend to fully use the algebraic structure of the orbifold. If the mother theory is rational (i.e. it has a finite number of primary operators), then so is the orbifold theory. Also, from the induction procedure, we shall find null states for the twisted operators in the orbifold. In our approach, these null states are important, as they are the starting point of a conformal bootstrap approach.

**NON-TWISTED OPERATORS.** In the non-twisted sector, the null states are easy to compute. A state  $\Phi$  in the non-twisted sector is a product of states of the mother theory. If one of these states (say, on the  $j^{\text{th}}$  copy), has a null vector descendant in the mother theory, then the modes of  $T_j(z)$  generate a null descendant. After an inverse discrete Fourier transform, these modes are easily expressed in terms of the orbifold Virasoro generators  $\widehat{L}_n^{(r)}$ .

For instance, take the mother CFT of central charge  $c$ , and consider the degenerate operator  $\phi_{12}$ . We can parametrise the central charge and degenerate conformal dimension as

$$c = 1 - \frac{6(1-g)^2}{g}, \quad h_{12} = \frac{3g-2}{4}, \quad 0 < g < 1, \tag{4.45}$$

and the null vector condition then reads:

$$\left( L_{-2} - \frac{1}{g} L_{-1}^2 \right) \phi_{12} \equiv 0. \tag{4.46}$$

For a generic number of copies  $N$  we have

$$\widehat{L}_n^{(r)} = \sum_{j=1}^N e^{2i\pi r j/N} (1 \otimes \dots \otimes 1 \otimes \underset{(j\text{-th})}{L_n} \otimes 1 \otimes \dots \otimes 1), \tag{4.47}$$

and hence for  $N = 2$ , and  $\Phi = \phi_{12} \otimes \phi_{12}$ , we obtain

$$\left[ \widehat{L}_{-2}^{(0)} - \frac{1}{2g} \left( \widehat{L}_{-1}^{(0)} \right)^2 - \frac{1}{2g} \left( \widehat{L}_{-1}^{(1)} \right)^2 \right] \Phi \equiv 0, \quad (4.48)$$

$$\left[ \widehat{L}_{-2}^{(1)} - \frac{1}{g} \widehat{L}_{-1}^{(0)} \widehat{L}_{-1}^{(1)} \right] \Phi \equiv 0. \quad (4.49)$$

More generally, any product of degenerate operators from the mother CFT is itself degenerate under the orbifold Virasoro algebra.

**TWISTED OPERATORS.** The twisted sectors also contain degenerate states, which are of great interest in the following. For example, let us take  $k = 1$ , and characterise the degenerate states at level  $1/N$ . A primary state  $\tau$  obeys  $\widehat{L}_m^{(r)} \tau = 0$  for any  $r$  and positive  $m \in \mathbb{Z} + r/N$ . For  $\tau$  to be degenerated at level  $1/N$ , one needs to impose the additional constraint:  $\widehat{L}_{1/N}^{(1)} \widehat{L}_{-1/N}^{(-1)} \tau = 0$ . Using the commutation relations (4.27), we get:

$$\widehat{L}_{1/N}^{(1)} \widehat{L}_{-1/N}^{(-1)} \tau = \frac{2}{N} \left[ \widehat{L}_0^{(0)} - \frac{c}{24} \left( N - \frac{1}{N} \right) \right] \tau. \quad (4.50)$$

Thus, the operator  $\tau$  is degenerate at level  $1/N$  if and only if it has conformal dimension  $h_\tau = \frac{c}{24} \left( N - \frac{1}{N} \right)$ . This is nothing but the conformal dimension (4.21) of the vacuum twist operator  $\tau_{\mathbb{1}}$ . Hence, one always has

$$\widehat{L}_{-1/N}^{(-1)} \tau_{\mathbb{1}} \equiv 0, \quad \widehat{L}_{-1/N}^{(1)} \tilde{\tau}_{\mathbb{1}} \equiv 0. \quad (4.51)$$

A more generic method consists in using the induction procedure. First, (4.51) can be recovered by applying (4.42–4.43):

$$L_{-1} \mathbb{1} \equiv 0 \quad \Rightarrow \quad N \widehat{L}_{-1/N}^{(-1)} \tau_{\mathbb{1}} \equiv 0. \quad (4.52)$$

One can obtain the other twisted null-vector equations by the same induction principle. Let us give one more example in the  $k = 1$  sector: the case of  $\tau_{\phi_{12}}$ . The relations (4.42–4.43) give:

$$\begin{aligned} \left( L_{-2} + g^{-1} L_{-1}^2 \right) \phi_{12} &\equiv 0 \\ \Rightarrow \left[ N \widehat{L}_{-2/N}^{(-2)} + \frac{N^2}{g} \left( \widehat{L}_{-1/N}^{(-1)} \right)^2 \right] \tau_{\phi_{12}} &\equiv 0. \end{aligned} \quad (4.53)$$

and hence  $\tau_{\phi_{12}}$  is degenerate at level  $2/N$ . Note the insertion of some factors  $N$  in the null-vector of  $\tau_{\phi_{12}}$  as compared to the null-vector equation (4.46) of the mother CFT.

## 4.4 FIRST EXAMPLES

## 4.4.1 Yang-Lee two-interval correlation function

We consider the CFT of the Yang-Lee (YL) singularity of central charge  $c = -22/5$ , where the primary operators are the identity  $\mathbb{1} = \phi_{11} = \phi_{14}$  with conformal dimension  $h_{\mathbb{1}} = 0$ , and  $\phi = \phi_{12} = \phi_{13}$  with  $h_{\phi} = -1/5$ . We shall compute the following four-point function in the  $\mathbb{Z}_2$  orbifold of the YL model:

$$G(x, \bar{x}) = \langle \tau_{\mathbb{1}}(\infty) \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) \tau_{\mathbb{1}}(0) \rangle. \quad (4.54)$$

In the  $N = 2$  cyclic orbifold of the YL model, the untwisted primary operators are  $\mathbb{1}$ ,  $\Phi = \phi \otimes \phi$  and the principal primary fields  $\hat{\phi}^{(r)}$  with  $r = 0, 1$ . They have conformal dimensions, respectively,  $h_{\mathbb{1}} = 0$ ,  $h_{\Phi} = -2/5$  and  $h_{\phi} = -1/5$ . Note that for  $N = 2$  the only twisted sector has  $\ell = 1$ , and hence  $\tilde{\tau} \equiv \tau$ . For the same reason, we shall sometimes omit the superscripts on the generators  $\hat{L}_n^{(r)}$ , as  $r = 0, 1$  are the only possible values. The twisted primary operators are  $\tau_{\mathbb{1}}$  and  $\tau_{\phi}$ , with conformal dimensions  $\hat{h}_{\mathbb{1}} = -11/40$  and  $\hat{h}_{\phi} = -3/8$ . Here we have used the standard convention  $\langle \psi(\infty) \dots \rangle := \lim_{R \rightarrow \infty} [R^{4h_{\psi}} \langle \psi(R) \dots \rangle]$ . Geometrically, this correlation function correspond to the partition function of the Yang-Lee model on a twice branched sphere, which can be mapped to the torus.

The identity operator of the YL model satisfies two null-vector equations:

$$L_{-1} \mathbb{1} = 0, \quad \left( L_{-4} - \frac{5}{3} L_{-2}^2 \right) \mathbb{1} = 0. \quad (4.55)$$

Through the induction procedure, this yields null-vector equations for the twist operator  $\tau_{\mathbb{1}}$ :

$$\hat{L}_{-1/2} \tau_{\mathbb{1}} = 0, \quad \left( \hat{L}_{-2} - \frac{10}{3} \hat{L}_{-1}^2 \right) \tau_{\mathbb{1}} = 0 \quad (4.56)$$

where the Fourier modes are  $r = 1$  and  $r = 0$  respectively, as required from (4.38). The first equation of (4.56) is generic for all  $N = 2$  orbifolds, and determines the conformal dimension of the  $\tau_{\mathbb{1}}$  operator. In contrast, the second equation is specific to the YL model. It only involves the integer modes, which all have the usual differential action when inserted into a correlation function. Hence, due to the second equation of (4.56), the derivation of  $G(x, \bar{x})$  is very similar to the standard case of a four-point function involving the degenerate operator  $\phi_{12}$  (see appendix 4.B). The conformal block in  $z \rightarrow 0$  have the expression:

$$x^{11/20} (1-x)^{11/20} I_1(x) \text{ with } I_1(x) = {}_2F_1(7/10, 11/10; 7/5|x), \quad (4.57)$$

$$x^{11/20} (1-x)^{11/20} I_2(x) \text{ with } I_2(x) = x^{-\frac{2}{5}} {}_2F_1(7/10, 3/10; 3/5|x),$$

And the total correlation function can be written:

$$G(x, \bar{x}) = |x|^{11/10} |1-x|^{11/10} \times \left[ \left| {}_2F_1(7/10, 11/10; 7/5|x) \right|^2 + 2^{16/5} \left| x^{-2/5} {}_2F_1(7/10, 3/10; 3/5|x) \right|^2 \right], \quad (4.58)$$

**OPE COEFFICIENTS** The coefficients  $X_j$  and  $Y_j$  give access to the OPE coefficients in the  $\mathbb{Z}_2$  orbifold of the YL model:

$$C(\tau_{\mathbb{1}}, \tau_{\mathbb{1}}, \Phi) = \sqrt{X_2} = 2^{8/5}, \quad (4.59)$$

Recalling  $h_\phi = -1/5$ , we see that (4.180) is consistent with the expression  $C(\Phi, \tau_{\mathbb{1}}, \tau_{\mathbb{1}}) = 2^{-8h_\phi}$  (see Appendix 4.C).

**MAPPING TO THE TORUS** The mapping from the torus (with coordinates  $t$ ) to the branched sphere (with coordinates  $z$ ) is:

$$z(t) = \frac{\wp(t) - \wp(1/2)}{\wp(\tau/2) - \wp(1/2)} \quad (4.60)$$

This maps  $0, x, 1, \infty \leftarrow \frac{1}{2}, \frac{1}{2}(1+\tau), \frac{\tau}{2}, 0$ . The relation between  $x$  and the nome  $q = e^{2i\pi\tau}$  is given by:

$$x = 16\sqrt{q} \prod_{n=1}^{\infty} \left( \frac{1+q^n}{1+q^{n-1/2}} \right)^8 \quad (4.61)$$

Mapping the torus to the branched sphere, the partition function transforms as:

$$Z(\tau) = 4^{c/3} |x|^{c/12} |1-x|^{c/12} \langle \tau_{\mathbb{1}} | \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) | \tau_{\mathbb{1}} \rangle \quad (4.62)$$

The torus partition function of the Yang-Lee model involves two characters,  $\chi_{1,1}(\tau)$  ( $\mathbb{1}$ ), and  $\chi_{1,2}(\tau)$  ( $\phi$ ).

$$Z = |\chi_{3,1}(\tau)|^2 + |\chi_{4,1}(\tau)|^2 \quad (4.63)$$

The characters of minimal models have the well-known expression  $\chi_{r,s}(\tau) = K_{r,s}(\tau) - K_{r,-s}(\tau)$ , with:

$$K_{r,s}(\tau) = \frac{1}{\eta(\tau)} \sum_{n=-\infty}^{\infty} \exp\left( i\pi\tau \frac{(20n+2r-5s)^2}{20} \right) \quad (4.64)$$

Where  $\eta$  is the Dedekind eta function.

We expect the following relation between the conformal blocks of the correlation function and the characters of the theory:

$$\begin{aligned} \chi_{1,1}(\tau) &= 2^{-22/15} x^{11/30} (1-x)^{11/30} I_1(x) \\ \chi_{1,2}(\tau) &= 2^{2/15} x^{11/30} (1-x)^{11/30} I_2(x) \end{aligned}$$

Those two relations are not trivial, and the simplest way to prove them seems to be by showing that the right-hand terms are vector



modular form, with the same modular transformations as  $\chi$  (see by example [77] for details). An easy check consist in expanding the right-hand side in power of  $q$ , confirming the equality for first orders:

$$2^{-22/15} x^{11/30} (1-x)^{11/30} I_1(x) \approx 1 + q^2 + q^3 + q^4 + \dots$$

$$2^{2/15} x^{11/30} (1-x)^{11/30} I_2(x) \approx 1 + q + q^2 + q^3 + 2q^4 + \dots$$

The relation 4.62 between the partition on the torus and  $G(x, \bar{x})$  is verified:

$$Z(\tau) = 2^{-44/15} |x|^{-11/30} |1-x|^{-11/30} \langle \tau_{\mathbb{1}} | \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) | \tau_{\mathbb{1}} \rangle \quad (4.65)$$

#### 4.4.2 Yang-Lee one-interval correlation function

With minimal modifications to the previous argument, we can also compute the following four-point function :

$$G(x, \bar{x}) = \langle \Phi(\infty) \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) \Phi(0) \rangle . \quad (4.66)$$

Which, physically, is related to the generalised Rényi entropy  $S_{N=2}(x, \phi, \tau_{\mathbb{1}})$ .

Technically, it is more convenient to work with twist operators located at 0 and  $\infty$ , so we introduce

$$F(x, \bar{x}) := \langle \tau_{\mathbb{1}}(\infty) \Phi(1) \Phi(x, \bar{x}) \tau_{\mathbb{1}}(0) \rangle . \quad (4.67)$$

Using the suitable projective mapping, one has the relation:

$$G(x, \bar{x}) = |1-x|^{4(h_{\Phi} - \hat{h}_{\mathbb{1}})} F(x, \bar{x}) .$$

Then, through the null-vector of  $\tau_{\mathbb{1}}$ , we can obtain a differential equation of order two for this correlation function:

$$\left[ 10x^2(1-x)^2 \partial_x^2 + x(1-x)(3-x) \partial_x + \frac{2}{5}(5x^2 + 3) \right] F(x, \bar{x}) = 0. \quad (4.68)$$

The Riemann scheme of this equation is:

$$\left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \frac{2}{5} & \frac{4}{5} & -\frac{2}{5} \\ \frac{3}{10} & \frac{2}{5} & -\frac{1}{2} \end{array} \right\} \quad (4.69)$$

Which is consistent with the OPEs:

$$\Phi \times \tau_{\mathbb{1}} \rightarrow \tau_{\mathbb{1}} + \tau_{\phi} , \quad (4.70)$$

$$\Phi \times \Phi \rightarrow \mathbb{1} + \Phi + (1 \otimes \phi) , \quad (4.71)$$

$$\tau_{\mathbb{1}} \times \tau_{\mathbb{1}} \rightarrow \mathbb{1} + \Phi . \quad (4.72)$$

By appropriately shifting  $F$ ,  $F(x, \bar{x}) = |x|^{4/5} |1 - x|^{8/5} f(x, \bar{x})$ , we can turn 4.68 in 4.166. At that point we can simply re-use the results of the last sections with parameters:

$$a = \frac{4}{5}, \quad b = \frac{7}{10}, \quad c = \frac{11}{10}. \quad (4.73)$$

The final result for the four-point function  $G(x, \bar{x})$  (4.66) is

$$G(x, \bar{x}) = |x|^{4/5} |1 - x|^{11/5} \times \left[ X_1 \left| {}_2F_1(4/5, 7/10; 11/10|x) \right|^2 + X_2 \left| x^{-1/10} {}_2F_1(3/5, 7/10; 9/10|x) \right|^2 \right], \quad (4.74)$$

where  $X_1, X_2$  are given in (4.180), and the parameters  $a, b, c, d$  are given in (4.73). Using the identity (4.157) on hypergeometric functions, we see that the solution (4.74) for  $G(x, \bar{x})$  agrees with the direct computation given in Appendix 4.D.

## 4.5 TWIST OPERATORS WITH A FRACTIONAL NULL VECTOR

### 4.5.1 Orbifold Ward identities

Generically the null vectors for a twist operator can involve some generators  $L_m^{(r)}$ , with  $r \neq 1$  and fractional indices  $m \in \mathbb{Z} + r/N$  [see (4.38)], which do not have a differential action on the correlation function. In this situation, we shall use the extended Ward identities to turn the null-vector conditions into a differential equation for the correlation function.

Let us consider the correlation function:

$$\mathcal{G}^{(r)}(x, \bar{x}, z) = \langle \mathcal{O}_1 | \mathcal{O}_2(1) \mathcal{O}_3(x, \bar{x}) \widehat{T}^{(r)}(z) | \mathcal{O}_4 \rangle, \quad (4.75)$$

where  $(\mathcal{O}_2, \mathcal{O}_3)$  are any two operators and  $(|\mathcal{O}_1\rangle, |\mathcal{O}_4\rangle)$  are any two states of the cyclic orbifold. Each operator  $\mathcal{O}_j$  or state  $|\mathcal{O}_j\rangle$  can be in a twisted sector  $[k_j]$  with  $k_j \neq 0 \pmod{N}$ , or in the untwisted sector ( $k_j \equiv 0 \pmod{N}$ ). The overall  $\mathbb{Z}_N$  symmetry imposes a neutrality condition:  $k_1 + k_2 + k_3 + k_4 \equiv 0 \pmod{N}$ . Let  $C$  be a contour enclosing the points  $\{0, x, 1\}$ . Then this is a closed contour for the following integral:

$$\frac{1}{2i\pi} \oint_C dz (z-1)^{m_2+1} (z-x)^{m_3+1} z^{m_4+1} \mathcal{G}^{(r)}(x, \bar{x}, z), \quad (4.76)$$

where  $m_j \in \mathbb{Z} + rk_j/N$  for  $j = 2, 3, 4$ . Then, by deforming the integration contour to infinity, we obtain the following identity:

$$\begin{aligned}
 & \sum_{p=0}^{\infty} a_p \langle \mathcal{O}_1 | \widehat{L}_{-m_1-p}^{(r)} \mathcal{O}_2(1) \mathcal{O}_3(x, \bar{x}) | \mathcal{O}_4 \rangle \\
 &= \sum_{p=0}^{\infty} b_p \langle \mathcal{O}_1 | [\widehat{L}_{m_2+p}^{(r)} \mathcal{O}_2](1) \mathcal{O}_3(x, \bar{x}) | \mathcal{O}_4 \rangle \\
 &+ \sum_{p=0}^{\infty} c_p \langle \mathcal{O}_1 | \mathcal{O}_2(1) [\widehat{L}_{m_3+p}^{(r)} \mathcal{O}_3](x, \bar{x}) | \mathcal{O}_4 \rangle \\
 &+ \sum_{p=0}^{\infty} d_p \langle \mathcal{O}_1 | \mathcal{O}_2(1) \mathcal{O}_3(x, \bar{x}) \widehat{L}_{m_4+p}^{(r)} | \mathcal{O}_4 \rangle, \tag{4.77}
 \end{aligned}$$

where

$$m_1 = -m_2 - m_3 - m_4 - 2 \quad \Rightarrow \quad m_1 \in \mathbb{Z} + rk_1/N, \tag{4.78}$$

and the coefficients  $a_p, b_p, c_p, d_p$  are defined by the Taylor expansions:

$$\begin{aligned}
 (1-z)^{m_2+1} (1-xz)^{m_3+1} &= \sum_{p=0}^{\infty} a_p z^p, \quad (z-x)^{m_3+1} z^{m_4+1} = \sum_{p=0}^{\infty} b_p (z-1)^p, \\
 (z-1)^{m_2+1} z^{m_4+1} &= \sum_{p=0}^{\infty} c_p (z-x)^p, \quad (z-1)^{m_2+1} (z-x)^{m_3+1} = \sum_{p=0}^{\infty} d_p z^p.
 \end{aligned}$$

In (4.77) we have used the notation:

$$[\widehat{L}_m^{(r)} \mathcal{O}_j](x, \bar{x}) := \frac{1}{2i\pi} \oint_{C_x} dz (z-x)^{m+1} \widehat{T}^{(r)}(z) \mathcal{O}_j(x, \bar{x}), \tag{4.79}$$

where  $C_x$  is a contour enclosing the point  $x$ . If all the  $\mathcal{O}_j$ 's are chosen among the primary operators or their descendants under the orbifold Virasoro algebra, then the sums in (4.77) become finite. By choosing appropriately the indices  $m_1, \dots, m_4$  (with the constraint  $\sum_j m_j = -2$ ), the relations (4.77) can be used to express, say,  $\langle \widehat{L}_{r/N}^{(r)} \tau \dots \rangle$  in terms of correlation functions involving only descendants with integer indices.

#### 4.5.2 Ising two-interval ground state entropy

The Ising model is the smallest non-trivial unitary model, with three fields  $\{\mathbb{1}, \sigma, \epsilon\}$ , of conformal charges  $0, 1/16, 1/2$ . Its central charge is  $\frac{1}{2}$ . By using the correspondence between the Ising model and free fermions, the two-interval case was computed in [1], for any value of  $N$ . Here, as an example, we compute the  $N = 2$  two-interval entropy, with our method, and using only the null vector of the model. The  $N = 2$  orbifold of the Ising model will have central charge 1. The first field in the twisted sector is  $\tau_{\mathbb{1}}$ , the identity twist, of charge  $h_{\tau_{\mathbb{1}}} = 1/32$ .

The two null vector of the identity in the mother theory are:

$$L_{-1} \mathbb{1} = 0, \quad (108L_{-6} + 264L_{-4}L_{-2} - 93L_{-3}^2 - 64L_{-2}^3) \mathbb{1} = 0. \tag{4.80}$$

And, through the induction process, we obtain null vector equations for  $\tau_{\mathbb{1}}$ .

$$\begin{aligned} L_{-1/2}\tau_{\mathbb{1}} &= 0 \text{ and} \\ (54 L_{-3} + 264 L_{-2}L_{-1} - 93 L_{-3/2}^2 - 128 L_{-1}^3) \tau_{\mathbb{1}} &= 0. \end{aligned} \quad (4.81)$$

We aim to obtain a differential equation for the correlation function

$$G(x, \bar{x}) = \langle \tau_{\mathbb{1}} | \tau_{\mathbb{1}}(x, \bar{x}) \tau_{\mathbb{1}}(1) | \tau_{\mathbb{1}} \rangle,$$

We need to compute the term  $\langle (L_{-3/2}^2 \tau_{\mathbb{1}}(0)) \tau_{\mathbb{1}}(x, \bar{x}) \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(\infty) \rangle$  to be able to use the null-vector. By using the Ward identity 4.77 with  $\{m_1, m_2, m_3, m_4\} = \{1/2, -1/2, -1/2, -3/2\}$  and  $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\} = \{\tau_{\mathbb{1}}, \tau_{\mathbb{1}}, \tau_{\mathbb{1}}, L_{-3/2}\tau_{\mathbb{1}}\}$ , we obtain the relation:

$$\sum_{p=0}^3 d_p \langle \tau_{\mathbb{1}} | \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) L_{-3/2+p} | L_{-3/2} \tau_{\mathbb{1}} \rangle = 0$$

Which, using the orbifold algebra is equivalent to:

$$\begin{aligned} &\langle \tau_{\mathbb{1}} | \tau_{\mathbb{1}}(x, \bar{x}) \tau_{\mathbb{1}}(1) | L_{-3/2}^2 \tau_{\mathbb{1}} \rangle = \\ &\frac{1}{64x^3} \left( 32x^2(1+x) \langle \tau_{\mathbb{1}} | \tau_{\mathbb{1}}(x, \bar{x}) \tau_{\mathbb{1}}(1) | L_{-2} \tau_{\mathbb{1}} \rangle + \right. \\ &16x(x-1)^2 \langle \tau_{\mathbb{1}} | \tau_{\mathbb{1}}(x, \bar{x}) \tau_{\mathbb{1}}(1) | L_{-1} \tau_{\mathbb{1}} \rangle + \\ &\left. (x-1)^2(1+x) \langle \tau_{\mathbb{1}} | \tau_{\mathbb{1}}(x, \bar{x}) \tau_{\mathbb{1}}(1) | \tau_{\mathbb{1}} \rangle \right) \end{aligned}$$

Using this relation and the null vector 4.81, we obtain the following differential equation:

$$\begin{aligned} &\left\{ 15(2x-1) + 48x(x-1)(192x^2 - 192x + 5) \partial_x \right. \\ &\left. + 8448(x-1)^2x^2(2x-1)\partial_x^2 + 4096(x-1)^3x^3\partial_x^3 \right\} G(x, \bar{x}) = 0 \end{aligned} \quad (4.82)$$

The Riemann scheme of this equation is:

$$\left( \begin{array}{ccc} 0 & 1 & \infty \\ \frac{-1}{16} & \frac{1}{16} & \frac{15}{16} \\ \frac{-1}{16} & \frac{1}{16} & \frac{15}{16} \\ 0 & -\frac{1}{8} & -1 \end{array} \right) \quad (4.83)$$

Which is compatible with the OPE:

$$\tau_{\mathbb{1}} \otimes \tau_{\mathbb{1}} \rightarrow \mathbb{1}^{\otimes 2} \oplus \sigma^{\otimes 2} \oplus \epsilon^{\otimes 2}.$$

Under the change of variables  $C(x, \bar{x}) = |x|^{-1/24}|1-x|^{-1/24}G(x, \bar{x})$ , the differential equation 4.82 becomes:

$$\begin{aligned} &\left( \partial_x^3 - \frac{(2(2x-1))\partial_x^2}{x(1-x)} + \frac{(391(x^2-x)+7)\partial_x}{192x^2(1-x)^2} \right. \\ &\left. - \frac{23(2-x)(x+1)(2x-1)}{24^3x^3(1-x)^3} \right) C(x, \bar{x}) = 0 \end{aligned} \quad (4.84)$$

The solution of this differential equation are the characters of the Ising model on the torus, as demonstrated in [58], by directly applying the null vectors on the torus. Hence, we recover that the two-interval  $N = 2$  Rényi entropy maps to the torus partition function.

#### 4.5.3 Yang-Lee one-interval ground state entropy

DEFINITIONS. As argued above, the  $N = 2$  ground state entropy of the YL model is related to the correlation function:

$$G(x, \bar{x}) = \langle \Phi(\infty) \tau_\phi(1) \tau_\phi(x, \bar{x}) \Phi(0) \rangle, \quad (4.85)$$

where  $\Phi = \phi \otimes \phi$ . It will be convenient to work rather with the related function:

$$F(x, \bar{x}) = \langle \tau_\phi(\infty) \Phi(1) \Phi(x, \bar{x}) \tau_\phi(0) \rangle, \quad (4.86)$$

with  $G(x, \bar{x}) = |1 - x|^{4(h_\Phi - \hat{h}_\phi)} F(x, \bar{x})$ .

NULL VECTORS AND INDEPENDENT DESCENDANTS. In the mother theory, the primary field  $\phi$  has two null-descendants:

$$\left( L_{-2} - \frac{5}{2} L_{-1}^2 \right) \phi = 0, \quad \left( L_{-3} - \frac{25}{12} \hat{L}_{-1}^3 \right) \phi = 0. \quad (4.87)$$

Through the induction procedure, this implies the following null vectors for the twist field  $\tau_\phi$ :

$$\left( \hat{L}_{-1} - 5 \hat{L}_{-1/2}^2 \right) \tau_\phi = 0, \quad \left( \hat{L}_{-3/2} - \frac{25}{4} \hat{L}_{-1/2}^3 \right) \tau_\phi = 0. \quad (4.88)$$

The Ward identities (4.77) will give relations between descendants at different levels. In order to get an idea of the descendants we need to compute, we need to know the number of independent descendants at each level. The number of Virasoro descendants at a given level  $k$  is equal to the number of integer partitions of  $k$ . In a minimal model not all those descendants are independent. The number of (linearly) independent descendants at level  $k$  of a primary field  $\phi$  is given by the coefficients of the series expansion of the character associated with  $\phi$ . Explicitly, if  $\tau$  is the modular parameter on the torus, the character of a field  $\phi_{rs}$  in the minimal model  $\mathcal{M}_{p,p'}$  is given by:

$$\begin{aligned} \chi_{rs}(\tau) &= K_{pr-p's}(\tau) - K_{pr+p's}(\tau), \\ K_\lambda(\tau) &= \frac{1}{\eta(\tau)} \sum_{n \in \mathbb{Z}} q^{(2pp'n + \lambda)^2 / 2pp'}, \quad q = e^{2i\pi\tau}. \end{aligned}$$

The coefficient of order  $k$  in the series expansion with parameter  $q$  of the character gives the number of independent fields at level  $k$ . Through the induction procedure (4.42–4.43), the module of  $\tau_{\phi_{rs}}$  under the orbifold algebra has the same structure as the module of  $\phi_{rs}$

under the Virasoro algebra. Hence, this coefficient also gives the number of descendants at level  $k/N$  in the module of  $\tau_{\phi_{rs}}$ . For example, for  $N = 2$  and  $(p, p') = (5, 2)$  (Yang-Lee), the numbers of independent descendants for the field  $\tau_\phi = \tau_{\phi_{12}}$  are given in the following table.

level	1/2	1	3/2	2	5/2	3
descendants	1	1	1	2	2	3
integer descendants	0	1	0	2	0	3

So up to level 3, all descendants at an integer level (even formed of non-integer descendants) can be re-expressed in terms of integer modes. One gets explicitly:

$$\begin{aligned}
\widehat{L}_{-5/2}\widehat{L}_{-1/2}\tau_\phi &= \left(\frac{1}{2}\widehat{L}_{-3} + \frac{3}{2}\widehat{L}_{-1}\widehat{L}_{-2} - \frac{5}{3}\widehat{L}_{-1}^3\right)\tau_\phi, \\
\widehat{L}_{-3/2}\widehat{L}_{-1/2}\tau_\phi &= \left(\frac{1}{4}\widehat{L}_{-2} - \frac{1}{2}\widehat{L}_{-1}^2\right)\tau_\phi, \\
\widehat{L}_{-1/2}^2\tau_\phi &= \frac{1}{5}\widehat{L}_{-1}\tau_\phi, \quad \widehat{L}_{1/2}\widehat{L}_{-1/2}\tau_\phi = \left(\widehat{L}_0 - \frac{c}{16}\right)\tau_\phi.
\end{aligned} \tag{4.89}$$

**WARD IDENTITY.** We now use (4.77) with the indices  $(m_1, \dots, m_4) = (1/2, 0, 0, -5/2)$  and :

$$\mathcal{G}(x, \bar{x}, z) = \langle \tau_\phi | \Phi(1)\Phi(x, \bar{x})\widehat{T}^{(1)}(z)\widehat{L}_{-1/2}^{(1)} | \tau_\phi \rangle.$$

Recalling that  $\widehat{L}_0^{(1)}\Phi = 0$ , the only surviving terms are:

$$0 = \sum_{p=0}^{\infty} d_p \langle \tau_\phi | \Phi(1)\Phi(x, \bar{x})\widehat{L}_{-5/2+p}^{(1)}\widehat{L}_{-1/2}^{(1)} | \tau_\phi \rangle.$$

Inserting the explicit expressions for the coefficients  $d_p$ , we get:

$$\langle \tau_\phi | \Phi(1)\Phi(x) \left[ x\widehat{L}_{-5/2} - (x+1)\widehat{L}_{-3/2} + \widehat{L}_{-1/2} \right] \widehat{L}_{-1/2} | \tau_\phi \rangle = 0. \tag{4.90}$$

**DIFFERENTIAL EQUATION.** Combining (4.89) and (4.90), one gets a linear relation involving only the integer modes  $\widehat{L}_m^{(0)}$ . Using (4.161), this leads to the third-order differential equation:

$$\begin{aligned}
&\left[ \frac{5}{3}x^3(1-x)^3\partial_x^3 + 2x^2(1-x)^2(1-2x)\partial_x^2 \right. \\
&\quad + \frac{1}{20}x(1-x)(15x^2 - 14x + 7)\partial_x \\
&\quad \left. - \frac{1}{50}(x^3 - 3x^2 - 29x + 15) \right] F(x, \bar{x}) = 0.
\end{aligned} \tag{4.91}$$

The Riemann scheme of this equation is:

$$\left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \frac{1}{2} & \frac{4}{5} & \frac{1}{10} \\ \frac{2}{5} & \frac{2}{5} & -\frac{3}{10} \\ \frac{9}{10} & \frac{3}{5} & -\frac{2}{5} \end{array} \right\} \tag{4.92}$$

INTERPRETATION IN TERMS OF OPES. We consider the conformal blocks under the invariant subalgebra  $\mathcal{A}_0$  generated by the monomials  $L_{m_1}^{(r_1)} \dots L_{m_k}^{(r_k)}$  with  $r_1 + \dots + r_k \equiv 0 \pmod N$ . With this choice, the toroidal partition function of the orbifold has a diagonal form  $Z = \sum_j |\chi_j|^2$  in terms of the characters (see [25, 99]). The OPEs under the invariant subalgebra  $\mathcal{A}_0$  are then:

$$\Phi \times \tau_\phi \rightarrow \mathbb{1} + \tau_\phi + \widehat{L}_{-1/2}^{(1)} \tau_\phi, \tag{4.93}$$

$$\Phi \times \Phi \rightarrow \mathbb{1} + \Phi + (1 \otimes \phi), \tag{4.94}$$

$$\tau_\phi \times \tau_\phi \rightarrow \mathbb{1} + \Phi + (1 \otimes \phi). \tag{4.95}$$

Note that  $\widehat{L}_{-1/2}^{(1)} \tau_\phi$  is a primary operator with respect to  $\mathcal{A}_0$ . The local exponents (4.92) are consistent with these OPEs.

HOLOMORPHIC SOLUTIONS. To express the solutions in terms of power series around  $x = 0$ , it is convenient to rewrite (4.91) using the differential operator  $\theta = x\partial_x$ . One has, for any  $k \in \mathbb{N}$ :

$$x^k \partial_x^k = \theta(\theta - 1) \dots (\theta - k + 1).$$

This yields the new form for (4.91):

$$[P_0(\theta) + x P_1(\theta) + x^2 P_2(\theta) + x^3 P_3(\theta)] F(x, \bar{x}) = 0, \tag{4.96}$$

where:

$$\begin{aligned} P_0(\theta) &= \frac{1}{3}(2\theta - 1)(5\theta - 2)(10\theta - 9), \\ P_1(\theta) &= -\frac{1}{5}(500\theta^3 - 700\theta^2 + 305\theta - 58), \\ P_2(\theta) &= \frac{1}{5}(500\theta^3 - 500\theta^2 + 145\theta + 6), \\ P_3(\theta) &= -\frac{1}{15}(5\theta - 2)(10\theta - 3)(10\theta + 1). \end{aligned} \tag{4.97}$$

The key identity satisfied by the operator  $\theta$  is, for any polynomial  $P$  and any real  $\alpha$ :

$$P(\theta) \cdot x^\alpha = P(\alpha) x^\alpha. \tag{4.98}$$

Hence, if we choose  $\alpha$  to be a root of  $P_0$  (i.e. one of the local exponents at  $x = 0$ ), there exists exactly one solution of the form:

$$I(x) = x^\alpha \times \sum_{n=0}^{\infty} a_n x^n, \quad \text{with } a_n = 1. \tag{4.99}$$

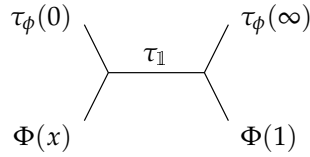
The coefficients  $a_n$  are given by the linear recursion relation:

$$a_n = -\frac{P_1(\alpha+n)a_{n-1} + P_2(\alpha+n)a_{n-2} + P_3(\alpha+n)a_{n-3}}{P_0(\alpha+n)}, \quad (4.100)$$

with the initial conditions:

$$\begin{aligned} a_0 &= 1, \quad a_1 = -\frac{P_1(\alpha+1)}{P_0(\alpha+1)}, \\ a_2 &= \frac{P_1(\alpha+1)P_1(\alpha+2) - P_0(\alpha+1)P_2(\alpha+2)}{P_0(\alpha+1)P_0(\alpha+2)}. \end{aligned} \quad (4.101)$$

For example, the conformal block corresponding to  $\alpha = 1/2$ :



is given by

$$I_1(x) = x^{1/2} \left( 1 + \frac{256}{55}x + \frac{24446}{1925}x^2 + \dots \right). \quad (4.102)$$

The series converges for  $|x| < 1$ , and may be evaluated numerically to arbitrary precision using (4.100).

**NUMERICAL SOLUTION** To determine the physical correlation function  $F(x, \bar{x})$  (4.86), we need to solve the monodromy problem, i. e. find the coefficients for the decompositions:

$$F(x, \bar{x}) = \sum_{i=1}^3 X_i |I_i(x)|^2 = \sum_{j=1}^3 Y_j |J_j(x)|^2, \quad (4.103)$$

where  $(I_1, I_2, I_3)$  are the holomorphic solutions of (4.91) with exponents  $(1/2, 2/5, 9/10)$  around  $x = 0$ , and  $(J_1, J_2, J_3)$  are the holomorphic solutions with exponents  $(4/5, 2/5, 3/5)$  around  $x = 1$ . In the present case, we do not know the analytic form of the  $3 \times 3$  matrix  $A$  for the change of basis:

$$I_i(x) = \sum_{j=1}^3 A_{ij} J_j(x). \quad (4.104)$$

However, this matrix can be evaluated numerically with arbitrary precision, by replacing the  $I_i$ 's and  $J_j$ 's in (4.104) by their series expansions of the form (4.102), and generating a linear system for  $\{A_{ij}\}$  by choosing any set of points  $\{x_1, \dots, x_9\}$  on the interval  $0 < x < 1$  where both sets of series  $I_i(x)$  and  $J_j(x)$  converge. We obtain the matrix:

$$A = \begin{pmatrix} 0.46872 & 2.98127 & -2.61803 \\ 0.292217 & 2.43298 & -1.82483 \\ 3.52145 & 6.92136 & -9.83452 \end{pmatrix}. \quad (4.105)$$



The coefficients giving a monodromy-invariant  $F(x, \bar{x})$  consistent with the change of basis (4.104) are:

$$\begin{aligned} X_1 &= 30.6594, \\ X_2 &= -19.2813, \\ X_3 &= 0.211121, \\ Y_1 &= 1, \\ Y_2 &= 20.2276, \\ Y_3 &= -9.64063. \end{aligned} \tag{4.106}$$

We have set  $Y_1$  to one because it corresponds to the conformal block with the identity as the internal operator.

**OPE COEFFICIENTS.** The coefficients  $X_i$  and  $Y_j$  are related to the OPE coefficients in the  $N = 2$  orbifold of the YL model as follows:

$$X_1 = C(\Phi, \tau_\phi, \tau_{\mathbb{1}})^2, \tag{4.107}$$

$$X_2 = C(\Phi, \tau_\phi, \tau_\phi)^2, \tag{4.108}$$

$$X_3 = C(\Phi, \tau_\phi, \widehat{L}_{-1/2}^{(1)} \tau_\phi)^2, \tag{4.109}$$

$$Y_1 = C(\Phi, \Phi, \mathbb{1}) C(\mathbb{1}, \tau_\phi, \tau_\phi) = 1, \tag{4.110}$$

$$Y_2 = C(\Phi, \Phi, \Phi) C(\Phi, \tau_\phi, \tau_\phi), \tag{4.111}$$

$$Y_3 = C[\Phi, \Phi, (\phi \otimes \mathbb{1})] C[(\phi \otimes \mathbb{1}), \tau_\phi, \tau_\phi]. \tag{4.112}$$

Our numerical procedure can be checked by comparing some of the numerical values (4.106) to a direct calculation of the OPE coefficients done in Appendix 4.C. They match up to machine precision.

#### 4.5.4 Excited state entropy for minimal models at $N = 2$

The same type of strategy can be used to compute excited state entropies in other minimal models, for some specific degenerate states, and small values of  $N$ . In this section we solve explicitly the case of  $N = 2$  for an operator degenerate at level 2.

**DEFINITIONS.** We consider the minimal model  $\mathcal{M}(p, p')$  with central charge  $c = 1 - \frac{6(p-p')^2}{pp'}$ , where  $p, p'$  are coprime integers. Using the Coulomb-gas notation,  $\phi_{21}$  is one of the states of the mother CFT which possess a null vector at level 2. In terms of the Coulomb-gas parameter  $g = p/p'$ , the central charge and conformal dimensions of the Kac table read:

$$c = 1 - \frac{6(1-g)^2}{g}, \tag{4.113}$$

$$h_{rs} = \frac{(rg-s)^2 - (1-g)^2}{4g}, \quad \text{for} \quad \begin{cases} r = 1, \dots, p' - 1 \\ s = 1, \dots, p - 1. \end{cases} \tag{4.114}$$

In particular, we have  $h_{21} = (3g - 2)/4$ . The null-vector condition for  $\phi_{21}$  reads:

$$\left( L_{-2} - \frac{1}{g} L_{-1}^2 \right) \phi_{21} = 0. \quad (4.115)$$

As shown before, in a unitary model (when  $q = p - 1$ ), the entanglement entropy of an interval in the state  $|\phi_{21}\rangle$  is expressed in terms of the correlation function

$$G(x, \bar{x}) = \langle \Phi(\infty) \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) \Phi(0) \rangle, \quad (4.116)$$

where  $\Phi = \phi_{21} \otimes \phi_{21}$ .

**NULL VECTORS.** The null vectors of  $\tau_{\mathbb{1}}$  and  $\Phi$  can be obtained through the induction procedure:

$$\begin{aligned} \widehat{L}_{-1/2}^{(1)} \tau_{\mathbb{1}} &= 0, & \left( \widehat{L}_{-2}^{(1)} - \frac{1}{g} \widehat{L}_{-1}^{(0)} \widehat{L}_{-1}^{(1)} \right) \Phi &= 0, \\ \left[ \widehat{L}_{-2}^{(0)} - \frac{1}{2g} \left( (\widehat{L}_{-1}^{(0)})^2 + (\widehat{L}_{-1}^{(1)})^2 \right) \right] \Phi &= 0. \end{aligned} \quad (4.117)$$

**WARD IDENTITY.** Using the Ward identity (4.77) with the choice of indices  $(m_1, \dots, m_4) = (0, -1/2, -1/2, -1)$  for the function  $\mathcal{G}(x, \bar{x}, z) = \langle \Phi | \tau_{\phi}(1) \tau_{\phi}(x, \bar{x}) \widehat{T}^{(1)}(z) \widehat{L}_{-1}^{(1)} | \Phi \rangle$ , and the null-vector condition (4.117), we obtain the relation:

$$0 = \langle \Phi | \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) \left( d_0 \widehat{L}_{-1}^{(1)} + d_1 \widehat{L}_0^{(1)} + d_2 \widehat{L}_1^{(1)} \right) \widehat{L}_{-1}^{(1)} | \Phi \rangle. \quad (4.118)$$

Using the orbifold Virasoro algebra (4.27) and the explicit expression for the  $d_p$ 's, we get:

$$\begin{aligned} \langle \Phi | \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) (\widehat{L}_{-1}^{(1)})^2 | \Phi \rangle &= \frac{x+1}{2x} \langle \Phi | \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) \widehat{L}_{-1}^{(0)} | \Phi \rangle \\ &+ \frac{h_{21}(x-1)^2}{2x^2} \langle \Phi | \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) | \Phi \rangle. \end{aligned} \quad (4.119)$$

**DIFFERENTIAL EQUATION.** By inserting the second null-vector condition of (4.117) into  $G(x, \bar{x})$ , we get:

$$\langle \Phi | \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) \left[ \widehat{L}_{-2}^{(0)} - \frac{1}{2g} \left( (\widehat{L}_{-1}^{(0)})^2 + (\widehat{L}_{-1}^{(1)})^2 \right) \right] | \Phi \rangle = 0. \quad (4.120)$$

Then, the substitution of the  $(\widehat{L}_{-1}^{(1)})^2$  term by 4.119 gives a linear relation involving only the modes  $\widehat{L}_m^{(0)}$ , which have a differential action on  $G(x, \bar{x})$ . This yields the differential equation for  $G(x, \bar{x})$ :

$$\begin{aligned} &\left[ 64g^2(x-1)^2 x^2 \partial_x^2 \right. \\ &+ (3g-2) (16(g-1)g^2 + 12(1-2g)gx + 3(5g-6)(1-2g)^2 x^2) \\ &\left. + 16g(x-1)x(-2g^2(7x+4) + g(23x+2) - 6x) \partial_x \right] G(x, \bar{x}) = 0. \end{aligned} \quad (4.121)$$

Its Riemann scheme is:

$$\left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \hline \frac{2-3g}{2} & \frac{6g^2-13g+6}{8g} & \frac{-6+17g-10g^2}{8g} \\ \frac{1-g}{2} & \frac{38g^2-29g+6}{8g} & \frac{-18g^2+21g-6}{8g} \end{array} \right\} \quad (4.122)$$

These exponents correspond to the fusion rules

$$\tau_{\mathbb{1}} \times \tau_{\mathbb{1}} \rightarrow \mathbb{1} + \phi_{31} \otimes \phi_{31} + \dots \quad \text{and} \quad \tau_{\mathbb{1}} \times \Phi \rightarrow \tau_{\mathbb{1}} + \tau_{\phi_{31}} + \dots \quad (4.123)$$

in the channels  $x \rightarrow 1$  and  $x \rightarrow 0$ , respectively.

**SOLUTION.** The problem can be solved as in Sec. 4.B.1. If we multiply the correlation function by the appropriate factor, equation (4.121) becomes the hypergeometric differential equation:

$$x(x-1)\partial_x^2 f + [(a+b+1)x-c]\partial_x f + abf = 0,$$

$$\text{where : } f(x, \bar{x}) = |1-x|^{4\hat{h}_{\mathbb{1}}} |x|^{4h_{21}} G(x, \bar{x}), \quad (4.124)$$

$$\text{and : } a = 2-3g, \quad b = \frac{3}{2} - 2g, \quad c = \frac{3}{2} - g.$$

Following exactly the reasoning in Sec. 4.B.2, we find the correlation function:

$$G(x, \bar{x}) = |x|^{-4h_{21}} \left[ \left| (1-x)^{-4\hat{h}_{\mathbb{1}}} {}_2F_1(a, b; 1-d | 1-x) \right|^2 + X \left| (1-x)^{-4\hat{h}_{\mathbb{1}}+4g-2} {}_2F_1(c-b, c-a; 1+d | 1-x) \right|^2 \right],$$

where  $d = c - a - b$  and

$$X = \frac{2^5(1-2g)^2 \gamma(2g-\frac{1}{2})^3 \gamma(2-4g)^2}{(1-4g)^3 \gamma(2-3g)}. \quad (4.125)$$

Of course, we may compare this solution to the one obtained by a conformal mapping of the four-point function  $\langle \phi_{21}\phi_{21}\phi_{21}\phi_{21} \rangle$ . The latter also satisfies a hypergeometric equation, and using the transformation 4.157, the two solutions can be shown to match.

#### 4.5.5 Excited state entropy for minimal models at $N = 3$

We now consider the correlation function in the  $N = 3$  orbifold of the minimal model  $\mathcal{M}(p, p')$ :

$$G(x, \bar{x}) = \langle \Phi(\infty)\tau_{\mathbb{1}}(1)\tau_{\mathbb{1}}(x, \bar{x})\Phi(0) \rangle, \quad (4.126)$$

where  $\Phi = \phi_{21} \otimes \phi_{21} \otimes \phi_{21}$ . The conformal mapping method would result in a much more complicated 6-point function, which is not the solution of an ordinary differential equation. But through the orbifold Virasoro structure, we can obtain such an equation. The method is similar in spirit to what was done for  $N = 2$ , finding the null vector conditions on the field  $\Phi$ , then using the Ward identities.

NULL VECTORS. We will need the null vectors of  $\Phi_3$  up to level 4:

$$\begin{aligned} \widehat{L}_{-1}^{(1)}\widehat{L}_{-1}^{(2)}\Phi_3 &= \frac{1}{2} \left( 3g\widehat{L}_{-2}^{(0)} - \widehat{L}_{-1}^{(0)}\widehat{L}_{-1}^{(0)} \right) \Phi_3 \\ \left[ \widehat{L}_{-1}^{(2)}\widehat{L}_{-1}^{(2)}\widehat{L}_{-1}^{(2)} - 3g \left( \widehat{L}_{-2}^{(1)}\widehat{L}_{-1}^{(2)} - \widehat{L}_{-1}^{(0)}\widehat{L}_{-2}^{(0)} \right) + \widehat{L}_{-1}^{(0)}\widehat{L}_{-1}^{(0)}\widehat{L}_{-1}^{(0)} \right] \Phi_3 &= 0 \\ \left[ 3L_{-4}^{(0)} + L_{-3}^{(0)}L_{-1}^{(0)} + 3L_{-3}^{(1)}L_{-2}^{(2)} - L_{-3}^{(2)}L_{-1}^{(1)} - \frac{3g}{2} \left( L_{-2}^{(0)} \right)^2 + L_{-2}^{(0)} \left( L_{-1}^{(0)} \right)^2 \right. \\ &\quad \left. + 2L_{-2}^{(2)} \left( L_{-1}^{(2)} \right)^2 - \frac{1}{6g} \left( L_{-1}^{(0)} \right)^4 - \frac{1}{3g} L_{-1}^{(0)} \left( L_{-1}^{(2)} \right)^3 \right] \Phi_3 = 0 \end{aligned}$$

WARD IDENTITIES. The descendants involving  $\widehat{L}_{-1}^{(r)}$ , with  $r \neq 0$  are eliminated through the Ward identities. For example, using 4.77, with indices  $(m_1, m_2, m_3, m_4) = (0, 1/3, -1/3, -2)$ , and the function  $\mathcal{G}(x, \bar{x}, z) = \langle \Phi_3 | \tilde{\tau}(1) \tau(x, \bar{x}) T^{(2)}(z) \widehat{L}_{-1}^{(2)} \widehat{L}_{-1}^{(2)} | \Phi_3 \rangle$ , we obtain the relation:

$$\begin{aligned} \sum_{m=-2}^2 Q_m(x) \langle \Phi | \tilde{\tau}(1) \tau(x, \bar{x}) \widehat{L}_m^{(2)} \widehat{L}_{-1}^{(2)} \widehat{L}_{-1}^{(2)} | \Phi \rangle &= 0 \\ Q_0(x) &= 27x^2(x^2 - 8x - 2) \quad Q_{-1}(x) = 2 \cdot 3^4 x^3(2 + x) \\ Q_1(x) &= 12x(x - 1)^3 \quad Q_{-2}(x) = 3^5 x^4 \quad Q_2(x) = (x - 1)^3(5 + 7x) \end{aligned}$$

A similar relation can be found for the correlation functions of the form

$$\langle \Phi | \tilde{\tau}(1) \tau(x, \bar{x}) \widehat{L}_{-m}^{(1)} \widehat{L}_{-1}^{(2)} | \Phi \rangle \quad \text{and} \quad \langle \Phi | \tilde{\tau}(1) \tau(x, \bar{x}) \widehat{L}_{-m}^{(2)} \widehat{L}_{-1}^{(1)} | \Phi \rangle$$

by using other Ward identities.

DIFFERENTIAL EQUATION. Putting everything together we find the following differential equation:

$$\left[ P_0(\theta) + x P_1(\theta) + x^2 P_2(\theta) + x^3 P_3(\theta) + x^4 P_4(\theta) \right] G(x, \bar{x}) = 0, \quad (4.127)$$

where  $\theta = x\partial_x$  and:

$$\begin{aligned} P_0(\theta) &= 16(1 - 2g + g\theta)(1 - g + g\theta)(6 - 5g + 3g\theta)(6 - 4g + 3g\theta) \\ P_1(\theta) &= -16(1 - g + g\theta) (486 - 963g + 666g^2 - 160g^3 \\ &\quad + (567g - 810g^2 + 296g^3)\theta + (234g^2 - 180g^3)\theta^2 + 36g^3\theta^3) \\ P_2(\theta) &= 28980 - 68076g + 60344g^2 - 24320g^3 + 3840g^4 \\ &\quad + (46008g - 84456g^2 + 52272g^3 - 10944g^4)\theta \\ &\quad + (27720g^2 - 35280g^3 + 11424g^4)\theta^2 + (7776g^3 - 5184g^4)\theta^3 + 864g^4\theta^4 \\ P_3(\theta) &= -4(7 - 4g + 2g\theta) (1215 - 1962g + 1008g^2 - 160g^3 \\ &\quad + (1296g - 1404g^2 + 376g^3)\theta + (504g^2 - 288g^3)\theta^2 + 72g^3\theta^3) \\ P_4(\theta) &= (7 - 4g + 2g\theta)(7 - 2g + 2g\theta)(15 + -10g + 6g\theta)(15 - 8g + 6g\theta) \end{aligned}$$

The Riemann scheme is given by:

$$\begin{array}{ccc}
 0 & 1 & \infty \\
 \hline
 \frac{-1+g}{g} & \frac{3}{2g} & \frac{15-8g}{6g} \\
 \frac{-6+4g}{3g} & \frac{-9+6g}{2g} & \frac{7-4g}{2g} \\
 \frac{-1+2g}{g} & \frac{-1+2g}{2g} & \frac{7-2g}{2g} \\
 \frac{-6+5g}{3g} & \frac{-5+4g}{2g} & \frac{15-10g}{6g}
 \end{array} \tag{4.128}$$

INTERPRETATION IN TERMS OF OPES. The conformal dimensions of the internal field in the channels  $x \rightarrow 1$  and  $x \rightarrow 0$  are respectively:

$$\{0, h_{31}, 2h_{31}, 3h_{31}\} \quad \text{and} \quad \{\widehat{h}_{\phi_{21}}, \widehat{h}_{\phi_{21}} + 1/3, \widehat{h}_{\phi_{41}}, \widehat{h}_{\phi_{41}} + 1\}. \tag{4.129}$$

Since  $\langle \tau_1 \cdot \widetilde{\tau}_1 \cdot (\phi_{31} \otimes \mathbb{1} \otimes \mathbb{1}) \rangle \propto \langle \phi_{31} \rangle_{\mathbb{C}} = 0$ , the conformal block with internal dimension  $h_{31}$  is in fact not present in the physical correlation function. In the channel  $x \rightarrow 0$ , this is mirrored by the presence of two fields separated by an integer value : there is a degeneracy for the field  $\tau_{\phi_{41}}$ .

THIS CAN BE VERIFIED BY BOOTSTRAP. The equation has 4 linearly independent solutions, which can be computed by series expansion around the three singularities. Like in 4.5.3, the monodromy problem can be solved by comparing the expansions in their domain of convergence. Up to machine precision the coefficient corresponding to  $h_{\phi_{3,1}}$  vanishes for all central charges. Nevertheless, the other structure constants converge, and we can still compute the full correlation function through bootstrap. For the non-zero structure constants, we checked that they were matching their theoretical expressions for simple minimal models (Yang-Lee and Ising).

The effective presence of only three conformal blocks also seem to imply that we should have been able to find a degree three differential equation, instead of four, for this correlation function. However, we have not managed to derive such a differential equation.

#### 4.6 TWIST OPERATORS IN CRITICAL RSOS MODELS

In this section we describe a lattice implementation of the twist fields in the lattice discretisation of the minimal models, namely the critical Restricted Solid-On-Solid (RSOS) models. Entanglement entropy in RSOS models has already been considered in [48] for unitary models and in [24] for non-unitary models, but for a semi-infinite interval and away from criticality.

4.6.1 The critical RSOS model

Let us define the critical RSOS model with parameters  $(m, k)$ , where  $m$  and  $k$  are coprime integers and  $k < m$ . Each site  $\vec{r}$  of the square lattice carries a height variable  $a_{\vec{r}} \in \{1, 2, \dots, m\}$ , and two variables  $a$  and  $b$  sitting on neighbouring sites should differ by one :  $|a - b| = 1$ . The Boltzmann weight of a height configuration is given by the product of face weights:

$$W \left( \begin{array}{cc|c} a & b & \\ \hline d & c & u \end{array} \right) = \begin{array}{c} \begin{array}{ccc} a & & b \\ \bullet & & \bullet \\ \diagdown & & / \\ & u & \\ \diagup & & \diagdown \\ d & & c \\ \bullet & & \bullet \end{array} \end{array} = \sin(\lambda - u) \delta_{bd} + \sin u \delta_{ac} \frac{\sin \lambda b}{\sin \lambda a}, \quad (4.130)$$

where the crossing parameter  $\lambda$  is

$$\lambda = \frac{\pi k}{m + 1}. \quad (4.131)$$

The quantum model associated to the critical RSOS model is obtained by taking the very anisotropic limit  $u \rightarrow 0$  of the transfer matrix. For periodic boundary conditions, one obtains a spin chain with basis states  $|a_1, a_2, \dots, a_L\rangle$ , where  $a_i \in \{1, \dots, m\}$ , and  $|a_i - a_{i+1}| = 1$ , and the Hamiltonian is

$$H_{\text{RSOS}} = - \sum_{i=1}^L e_i, \quad (4.132)$$

where  $e_i$  only acts non-trivially on the heights  $a_{i-1}, a_i, a_{i+1}$ :

$$e_i |\dots, a_{i-1}, a_i, a_{i+1}, \dots\rangle = \delta_{a_{i-1}, a_{i+1}} \sum_{a'_i} \frac{\sin \lambda a'_i}{\sin \lambda a_i} |\dots, a_{i-1}, a'_i, a_{i+1}, \dots\rangle,$$

$|a'_i - a_{i-1}| = 1$

and the indices  $i \pm 1$  are considered modulo  $L$ .

For simplicity, we now consider the RSOS model on a planar domain. The lattice partition function  $Z_{\text{RSOS}}$  and the correlation functions admit a graphical expansion [115] in terms of non-intersecting, space-filling, closed loops on the dual lattice. The expansion of  $Z_{\text{RSOS}}$  is obtained by associating a loop plaquette to each term in the face weight (4.130) as follows:

$$W \left( \begin{array}{cc|c} a & b & \\ \hline d & c & u \end{array} \right) = \sin(\lambda - u) \begin{array}{c} \begin{array}{ccc} a & & b \\ \bullet & & \bullet \\ \diagdown & & / \\ & & \\ \diagup & & \diagdown \\ d & & c \\ \bullet & & \bullet \end{array} \end{array} + \sin u \begin{array}{c} \begin{array}{ccc} a & & b \\ \bullet & & \bullet \\ \diagdown & & / \\ & & \\ \diagup & & \diagdown \\ d & & c \\ \bullet & & \bullet \end{array} \end{array}. \quad (4.133)$$

Then, after summing on the height variables, each closed loop gets a weight  $\beta = 2 \cos \lambda$ . Furthermore, following [116], correlation functions of the local variables

$$\varphi_q(a) = \frac{\sin \frac{\pi q a}{m+1}}{\sin \lambda a}, \quad q \in \{1, \dots, m\}, \quad (4.134)$$

also fit well in this graphical expansion. Let us recall, for example, the expansion of the one-point function  $\langle \varphi_q(a_{\vec{r}}) \rangle$ . For any loop which does not enclose  $\vec{r}$ , the height-dependent factors from (4.130) end up to  $\sin \lambda b / \sin \lambda a$ , where  $a$  (resp.  $b$ ) is the outer (resp. inner) height adjacent to the loop. Thus, summing on the inner height  $b$  gives the loop weight:

$$\sum_b A_{ab} \times \frac{\sin \lambda b}{\sin \lambda a} = 2 \cos \lambda = \beta, \tag{4.135}$$

where we have introduced the adjacency matrix  $A_{ab} = 1$  if  $|a - b| = 1$ , and  $A_{ab} = 0$  otherwise. For the loop enclosing  $\vec{r}$  and adjacent to it, the factor  $\varphi_q(b)$  should be inserted into the above sum, which gives:

$$\sum_b A_{ab} \times \varphi_q(b) \times \frac{\sin \lambda b}{\sin \lambda a} = \beta_q \times \varphi_q(a), \tag{4.136}$$

where

$$\beta_q = 2 \cos \left( \frac{\pi q}{m + 1} \right). \tag{4.137}$$

Repeating this argument recursively, in the graphical expansion of  $\langle \varphi_q(a_{\vec{r}}) \rangle$ , one gets a loop weight  $\beta_q$  for each loop enclosing the point  $\vec{r}$ . The  $N$ -point functions of the  $\varphi_q$ 's are treated similarly, through the use of a lattice Operator Product Expansion (OPE) [116].

This critical RSOS model provides a discretisation of the minimal model  $\mathcal{M}(p, p')$ , with central charge and conformal dimensions:

$$c = 1 - \frac{6(p - p')^2}{pp'}, \tag{4.138}$$

$$h_{rs} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'} \quad \begin{aligned} r \in \{1, \dots, p' - 1\}, \\ s \in \{1, \dots, p - 1\}, \end{aligned} \tag{4.139}$$

with the identification of parameters:

$$p = m + 1, \quad p' = m + 1 - k. \tag{4.140}$$

*Since  $p$  and  $p'$  are coprime, using the Bézout theorem, there exist two integers  $u$  and  $v$  such that  $pu - p'v = 1$ , and then it is possible to find an integer  $\ell$  such that  $(r, s) = (qu + \ell p', qv + \ell p)$  belongs to the range (4.139), whereas  $pr - p's = q$ .*

The operator  $\varphi_q$  changes the loop weight to  $\beta_q$ : thus, in this sector in the scaling limit, the dominant primary operator, which we denote  $\phi_q$ , has conformal dimension

$$h_{\phi_q} = \frac{q^2 - (p - p')^2}{4pp'}. \tag{4.141}$$

It is then easy to show that  $h_{\phi_q}$  is one of the dimensions of the Kac table (4.139). Note that  $h_{\phi_{q=k}} = 0$  corresponds to the identity operator.

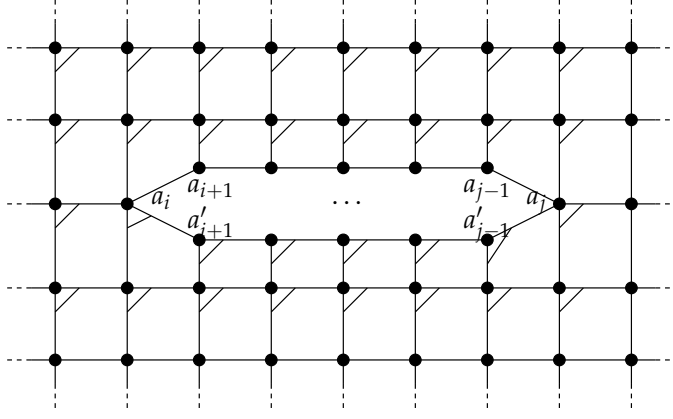


Figure 4.1: A generic reduced density matrix element  $\langle a_i, a_{i+1}, \dots, a_j | \rho_{\mathcal{A}} | a'_i, a'_{i+1}, \dots, a'_j \rangle$  for the interval  $\mathcal{A} = [i, j]$ . This matrix element is set to zero if  $a_i \neq a'_i$  or  $a_j \neq a'_j$ .

#### 4.6.2 Partition function in the presence of branch points

We consider the reduced density matrix  $\rho_{\mathcal{A}}$  for the interval  $\mathcal{A} = [i, j]$ . A generic matrix element  $\langle a_i, a_{i+1}, \dots, a_j | \rho_{\mathcal{A}} | a'_i, a'_{i+1}, \dots, a'_j \rangle$  corresponds to the partition function of the lattice shown in Fig. 4.1, with the heights  $a_i, \dots, a_j$  and  $a'_i, \dots, a'_j$  fixed, and the other heights summed over. In this convention, a branch point (or twist operator) sits on a site  $\vec{r}$  of the square lattice, and is denoted  $t(\vec{r})$ . Computing the  $n$ -th Rényi entropy (4.2) amounts to determining the partition function  $Z_{\text{RSOS}}^{(n)}$  on the surface obtained by “sewing” cyclically  $n$  copies of the diagram in Fig. 4.1 along the cut going from  $a_i$  to  $a_j$ .

The graphical expansion of the partition function on this surface with two branch points is very similar to the case of a planar domain. The only difference concerns the loops which surround one branch point. Since such a loop has a total winding  $\pm 2\pi n$  instead of  $\pm 2\pi$ , the height-dependent factors from (4.130) end up to  $(\sin \lambda b / \sin \lambda a)^n$ , where  $a$  (resp.  $b$ ) is the external (resp. internal) height adjacent to the loop. Since  $(\sin \lambda b)^n$  is not an eigenvector of the adjacency matrix  $A$ , the sum over  $b$  does not give a well-defined loop weight.

For this reason, we introduce a family of modified lattice twist operators:

$$t_q(\vec{r}) = \hat{\varphi}_q(a_{\vec{r}}) \times t(\vec{r}), \quad \text{where} \quad \hat{\varphi}_q(b) = \frac{\sin \frac{\pi q b}{m+1}}{(\sin \lambda b)^n}. \quad (4.142)$$

With this insertion of  $\hat{\varphi}_q$  at the position of the twist, the sum over the internal height gives:

$$\sum_b A_{ab} \left( \frac{\sin \lambda b}{\sin \lambda a} \right)^n \times \hat{\varphi}_q(b) = \beta_q \times \hat{\varphi}_q(a), \quad (4.143)$$



and hence any loop surrounding  $t_q$  gets a weight  $\beta_q$  (4.137). Thus, the scaling limit of  $t_q$  is the primary operator  $\tau_{\phi_q}$ , which belongs to the twisted sector, and has conformal dimension

$$\widehat{h}_{\phi_q} = \frac{c}{24} \left( n - \frac{1}{n} \right) + \frac{h_{\phi_q}}{n}. \tag{4.144}$$

In particular, since  $\beta_k = \beta = 2 \cos \lambda$ , one has  $\phi_k = \mathbb{1}$ , and the lattice operator  $t_k$  corresponds to  $\tau_{\mathbb{1}}$  in the scaling limit. Note that the “bare” twist operator  $t$  is itself a linear combination of the  $t_q$ ’s:

$$t(\vec{r}) = \sum_{q=1}^m x_q t_q(\vec{r}), \text{ with } x_q = \frac{2}{m} \sum_{a=1}^m (\sin \lambda a)^n \sin \left( \frac{\pi q a}{m+1} \right). \tag{4.145}$$

The scaling limit of  $t$  is thus always determined by the term  $t_1$ , since it has the lowest conformal dimension. In the case of a unitary minimal model ( $k = 1$ ), this corresponds to  $\tau_{\mathbb{1}}$ . In contrast, for a non-unitary minimal model  $\mathcal{M}(p, p')$ ,  $t$  scales to the twist operator  $\tau_{\phi_1}$ , where  $\phi_1$  is the primary operator with the lowest (negative) conformal dimension in the Kac table :  $h_{\phi_1} = -[(p - p')^2 - 1]/(4pp')$ .

### 4.6.3 Rényi entropies of the RSOS model

When defining a zero-temperature Rényi entropy, two distinct parameters must be specified:

1. The state  $|\psi\rangle_R$  in which the entropy is measured (or equivalently the density matrix  $\rho = |\psi\rangle_R \langle \psi|_L$ ).
2. The local state  $\phi_q$  of the system in the vicinity of branch points. This determines which twist operator  $t_q$  should be inserted. In the case of the physical Rényi entropy defined as (4.2), one inserts the linear combination  $t(\vec{r}) = \sum_{q=1}^m x_q t_q(\vec{r})$ .

In the following, we will be interested in the Rényi entropy of an interval of length  $\ell$  in the spin chain  $H_{\text{RSOS}}$  (4.132) of length  $L$  with periodic boundary conditions. This corresponds to the lattice average value:

$$\frac{1}{1-N} \log \langle \Psi | t_q(0) t_q(\ell) | \Psi \rangle, \tag{4.146}$$

where  $|\Psi\rangle = |\psi\rangle^{\otimes N}$ . The “physical” Rényi entropy (4.2) is related to:

$$\frac{1}{1-N} \log \langle \Psi | t(0) t(\ell) | \Psi \rangle. \tag{4.147}$$

The average values (4.146–4.147) scale to correlation functions on the cylinder  $\{z | 0 \leq \text{Im } z < L\}$ :

$$\langle \Psi | t_q(0) t_q(\bar{u}) | \Phi \rangle \propto \langle \Phi(-\infty) \tau_{\phi_q}(0) \tau_{\phi_q}(u, \bar{u}) \Psi(+\infty) \rangle_{\text{cyl}}, \tag{4.148}$$

where  $u = i\ell$ , and similarly for  $t_q$  replaced by  $t$ . Using the conformal map  $x = \exp(2\pi u/L)$ , these are related to the correlation functions on the complex plane:

$$\langle \Psi(-\infty) \tau_{\phi_q}(0) \tau_{\phi_q}(u) \Psi(+\infty) \rangle_{\text{cyl}} = \left( \frac{2\pi}{L} \right)^{4\hat{h}_{\phi_q}} \langle \Psi(0) \tau_{\phi_q}(1) \tau_{\phi_q}(x) \Psi(\infty) \rangle_{\text{pl}}.$$

In the case of the (generalised) Rényi entropy in the vacuum, we have  $\psi = \mathbb{1}$ , and this becomes a two-point function, which is easily evaluated:

$$S_N(x, \mathbb{1}, \tau_{\phi_q}) = \frac{4\hat{h}_{\phi_q}}{N-1} \log \left( \frac{L}{\pi} \sin \frac{\pi\ell}{L} \right) + \text{const.} \quad (4.149)$$

In particular, when  $\phi_q = \mathbb{1}$ , one recovers the result from [35]:

$$S_N(x, \mathbb{1}, \tau_{\mathbb{1}}) = \frac{(N+1)c}{6N} \log \left( \frac{L}{\pi} \sin \frac{\pi\ell}{L} \right) + \text{const.} \quad (4.150)$$

For a generic state  $|\psi\rangle$  however, the entropy  $S_N(x, \psi, \tau_{\phi_q})$  remains a non-trivial function of  $\ell$ , and does not reduce to the simple form (4.149).

#### 4.6.4 Numerical computations

##### 4.6.4.1 Numerical setup

We have computed some Rényi entropies (4.146) and (4.147) in the RSOS model with parameters  $m = 4$  and  $k = 3$ , corresponding to the Yang-Lee singularity  $\mathcal{M}(5, 2)$  with central charge  $c = -22/5$ . The primary fields are  $\mathbb{1} = \phi_{11} \equiv \phi_{14}$  and  $\phi = \phi_{12} \equiv \phi_{13}$ , with conformal dimensions  $h_{\mathbb{1}} = 0$  and  $h_{\phi} = -1/5$ . They correspond respectively to  $\mathbb{1} \propto \varphi_3$  and  $\phi \propto \varphi_1$ .

A lattice eigenvector (scaling either to  $|\mathbb{1}\rangle$ , or to  $|\phi\rangle$ ) of  $H_{\text{RSOS}}$  with periodic boundary conditions is obtained by exact diagonalisation with the QR or Arnoldi method, and then used to construct the reduced density matrix  $\rho_{\mathcal{A}}$ , where  $\mathcal{A} = [0, \ell]$ . For the computation of  $S_N(x, \mathbb{1}, t_q)$  and  $S_N(x, \psi, t_q)$ , the factor  $\hat{\varphi}_q(a_0) \hat{\varphi}_q(a_\ell)$  [see (4.142)] is inserted into the trace (4.2) of  $\rho_{\mathcal{A}}^N$ . For the computation of  $S_N(x, \phi, t)$ , no additional factor is inserted. From the above discussion, we expect  $S_N(x, \phi, t)$  to be described by the insertion of the dominant twist operators  $\tau_{\phi}$ .

##### 4.6.4.2 Results for entropies at $N = 2$ in the Yang-Lee model

Here we present our numerical results obtained with the procedure described above. In all the cases considered the cylinder correlation functions have been rescaled with the factor  $L^{4h}$ , where  $h$  is the appropriate twist field conformal dimension. The collapse of various finite size data further confirms the correct identification of the twist field ( $\tau_{\phi}$  or  $\tau_{\mathbb{1}}$ ).

The results obtained are in excellent agreement with our CFT interpretation (4.18) and with our analytical results. Moreover, they are clearly not compatible (see Fig. 4.3) with the claim

$$S_N \sim \frac{c_{\text{eff}}}{6} \frac{N+1}{N} \log |u-v|, \tag{4.151}$$

which can be found in the literature [23, 46] (the effective central charge of the Yang-Lee model is  $c_{\text{eff}} = 2/5$ ).

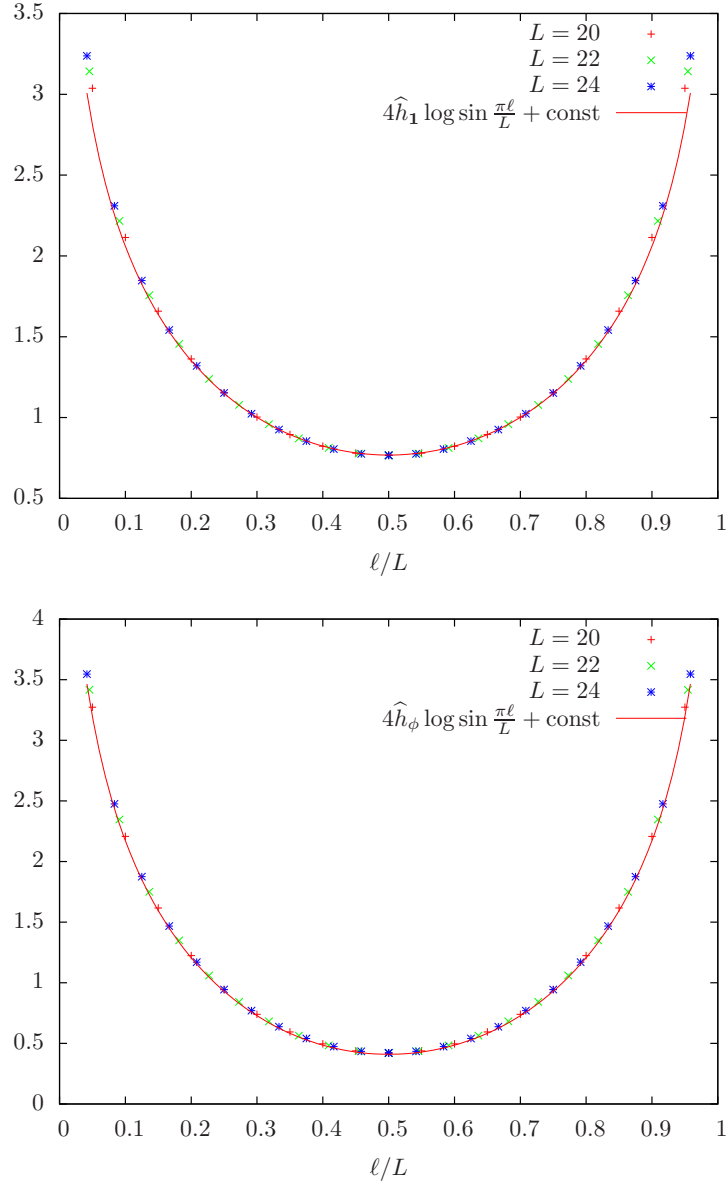


Figure 4.2:  $N = 2$  Rényi entropy of the Yang-Lee model in the vacuum  $|0\rangle$  with various twist fields. In the left panel we consider the twist  $t_3$  as in eq. (4.146), which corresponds in the continuum to  $\tau_\perp$  (with  $\hat{h}_\perp = -\frac{11}{40}$ ). In the right panel the bare twist  $t$  is considered (eq. (4.147)), corresponding to  $\tau_\phi$  (with  $\hat{h}_\phi = -\frac{3}{8}$ ).

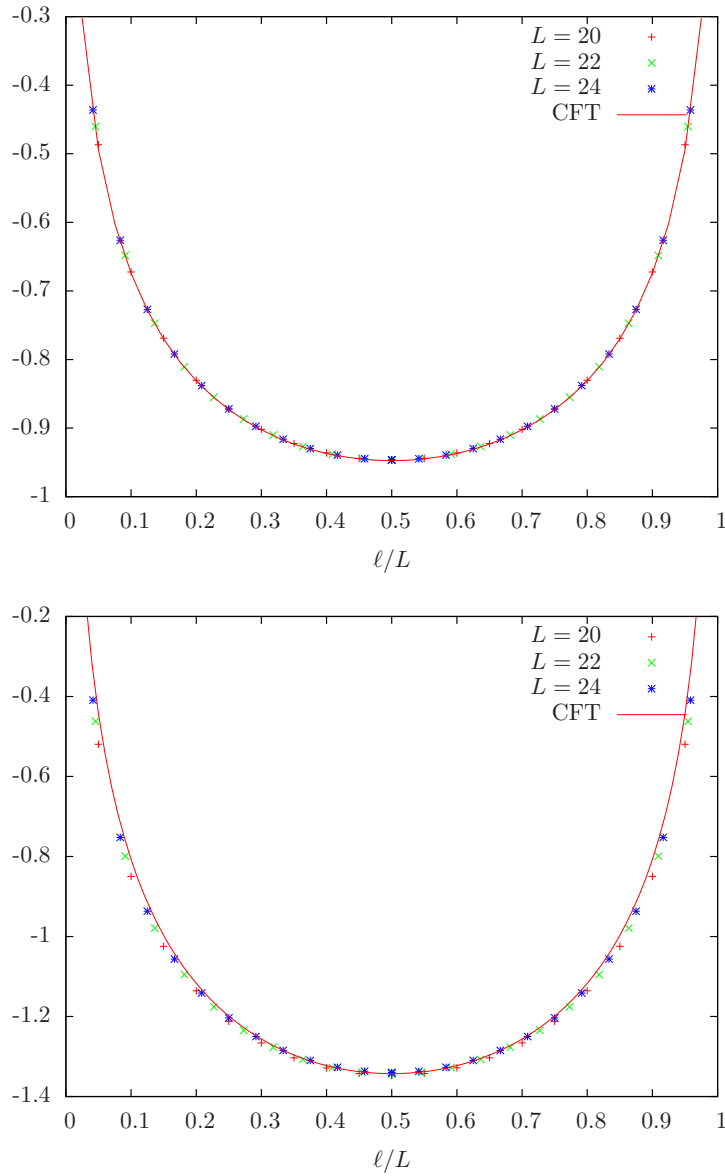


Figure 4.3:  $N = 2$  Rényi entropy of the Yang-Lee model in the ground state  $|\phi\rangle$  with various twist fields. In the left panel we consider the twist  $t_3$  as in eq. (4.146), which corresponds in the continuum to  $\tau_{\perp}$ . Exact diagonalisation results are compared to the CFT prediction (4.181) for the function  $\langle \Phi(0)\tau_{\perp}(1)\tau_{\perp}(x,\bar{x})\Phi(\infty) \rangle$ . In the right panel the bare twist  $t$  is considered (eq. (4.147)). Exact diagonalisation results are compared to the CFT prediction (4.91) for the function  $\langle \Phi(0)\tau_{\phi}(1)\tau_{\phi}(x,\bar{x})\Phi(\infty) \rangle$ .

#### 4.6.4.3 Results for entropies at $N = 3$ in the Yang-Lee model

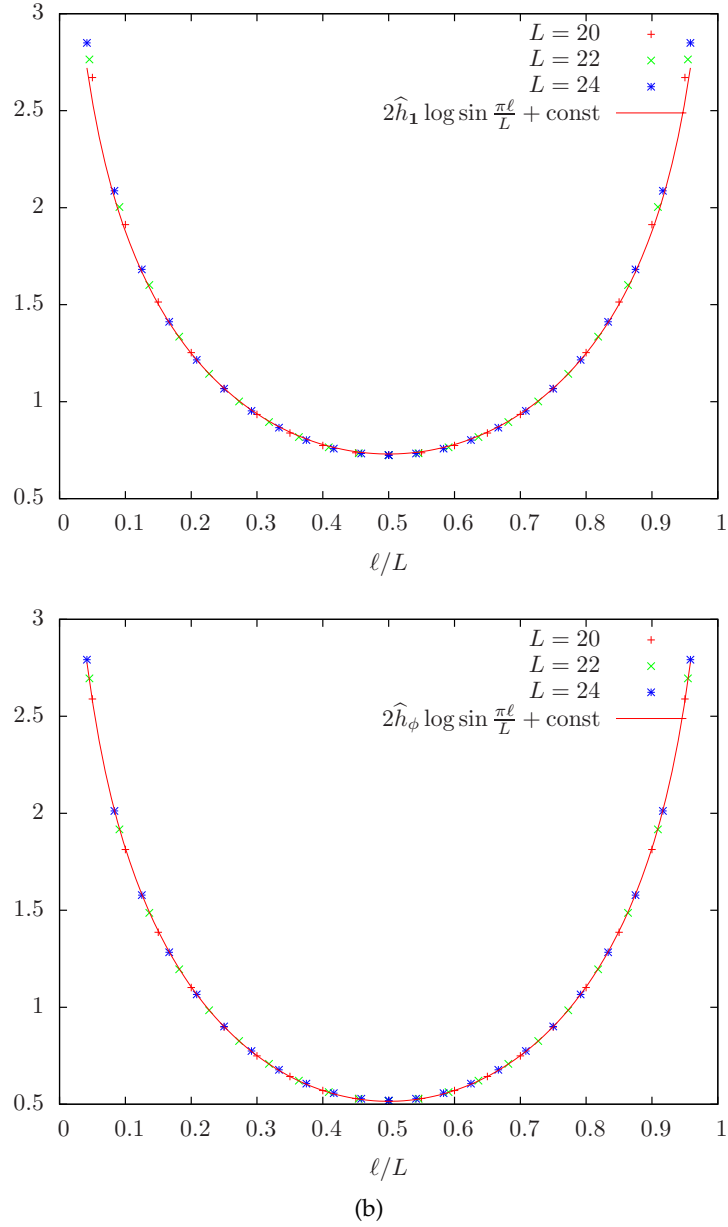


Figure 4.4:  $N = 3$  Rényi entropy of the Yang-Lee model in the vacuum  $|0\rangle$  with various twist fields. In the left panel we consider the twist  $t_3$  as in eq. (4.146), which corresponds in the continuum to  $\tau_{\mathbb{1}}$  (with  $\hat{h}_{\mathbb{1}} = -\frac{22}{45}$ ). In the right panel the bare twist  $t$  is considered (eq. (4.147)), corresponding to  $\tau_{\phi}$  (with  $\hat{h}_{\phi} = -\frac{5}{9}$ ).

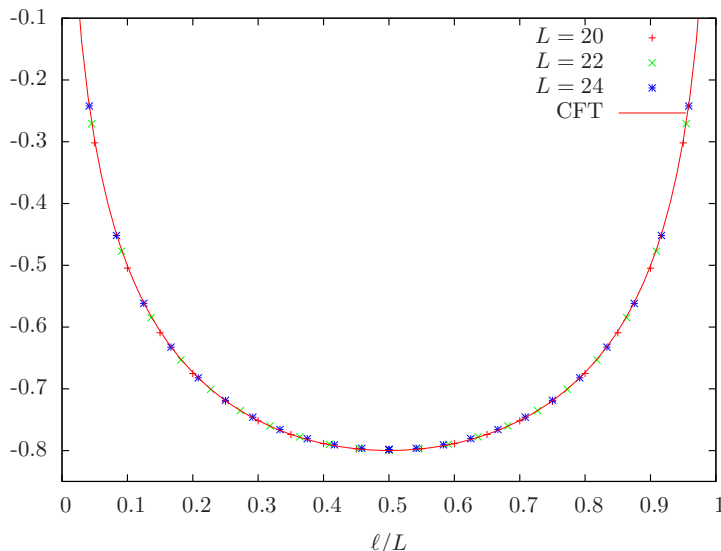


Figure 4.5: The  $N = 3$  Rényi entropy (4.146) of the Yang-Lee model in the state  $|\phi\rangle$ , with the twist  $t_3$  corresponding to  $\tau_{\perp}$ . Exact diagonalisation results are compared to the CFT prediction from Sec. 4.5.5 for the function  $\langle \Phi(0)\tau_{\perp}(1)\tau_{\perp}(x,\bar{x})\Phi(\infty) \rangle$ .

#### 4.7 CONCLUSION

In this chapter we have studied the Rényi entropies of one-dimensional critical systems, using the mapping of the  $N^{\text{th}}$  Rényi entropy to a correlation function involving twist fields in a  $\mathbb{Z}_N$  cyclic orbifold. When the CFT describing the universality class of the critical system is rational, so is the corresponding cyclic orbifold. It follows that the twist fields are degenerate: they have null vectors. From these null vectors a Fuchsian differential equation is derived, although this step can be rather involved since the null-vector conditions generically involve fractional modes of the orbifold algebra. The last step is to solve this differential equation and build a monodromy invariant correlation function, which is done using standard bootstrap methods. We have exemplified this method with the calculation of various one-interval Rényi entropies in the Yang-Lee model, a two-interval entropy in the Ising model and some one-interval entropies computed in specific excited states for all minimal models.

We have also described a lattice implementation of the twist fields in the lattice discretisation of the minimal models, namely the critical Restricted Solid-On-Solid (RSOS) models. This allows us to check numerically our analytical results obtained in the Yang-Lee model. Excellent agreement is found.

The main limitation of our method is that it gets more involved as  $N$  increases, and as the minimal model  $\mathcal{M}(p, p')$  under consideration

gets more complicated (*i.e.* as  $p$  and/or  $p'$  increases). For this reason, we have limited our study to  $N = 2$  and  $N = 3$  in the Yang-Lee model  $\mathcal{M}(5, 2)$ . However, this method is applicable in a variety of situations where no other method is available, for instance when the subsystem  $\mathcal{A}$  is not connected (*e.g.* two-intervals [EE](#)).

Another interesting research direction would be to develop a Coulomb-Gas formalism for the cyclic orbifold, as it would provide an efficient tool to solve the twist-field differential equations *à la* Dotsenko-Fateev. Indeed, the Coulomb Gas yields a very natural way to write down conformal blocks (in the form of closed contour integrals of screening operators), to compute their monodromies, and from there to solve the bootstrap.

## APPENDICES

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### 4.A PROPERTIES OF HYPERGEOMETRIC FUNCTIONS

- The Gauss hypergeometric function  ${}_2F_1$  is defined as:

$${}_2F_1(a, b; c|x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} x^n, \quad (4.152)$$

where  $(a)_n = a(a+1)\dots(a+n-1)$  is the Pochhammer symbol.

- Under the transformation  $x \mapsto 1-x$  of the complex variable, it satisfies the relations:

$$\begin{aligned} \frac{1}{\Gamma(c)} {}_2F_1(a, b; c|x) &= \frac{\Gamma(d)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, 1-d|1-x) \\ + \frac{\Gamma(-d)}{\Gamma(a)\Gamma(b)} (1-x)^d {}_2F_1(c-a, c-b; 1+d|1-x), \end{aligned} \quad (4.153)$$

$$\begin{aligned} &{}_2F_1(a, b; 1-d|1-x) = \\ &x^{1-c} {}_2F_1(a-c+1, b-c+1; 1-d|1-x), \end{aligned} \quad (4.154)$$

for  $d = c - a - b$ .

- The matrix relating the bases of solutions to the hypergeometric differential equation  $\{I_1, I_2\}$  (4.57) and  $\{J_1, J_2\}$  (4.170) as

$$I_i(x) = \sum_{j=1}^2 A_{ij} J_j(x) \quad (4.155)$$

is given by:

$$\begin{aligned} A &= \begin{bmatrix} \frac{\Gamma(c)\Gamma(d)}{\Gamma(c-a)\Gamma(c-b)} & \frac{\Gamma(c)\Gamma(-d)}{\Gamma(a)\Gamma(b)} \\ \frac{\Gamma(2-c)\Gamma(d)}{\Gamma(1-a)\Gamma(1-b)} & \frac{\Gamma(2-c)\Gamma(-d)}{\Gamma(1-c+a)\Gamma(1-c+b)} \end{bmatrix}, \\ A^{-1} &= \begin{bmatrix} \frac{\Gamma(1-c)\Gamma(1-d)}{\Gamma(1-c+a)\Gamma(1-c+b)} & \frac{\Gamma(c-1)\Gamma(1-d)}{\Gamma(a)\Gamma(b)} \\ \frac{\Gamma(1-c)\Gamma(1+d)}{\Gamma(1-a)\Gamma(1-b)} & \frac{\Gamma(c-1)\Gamma(1+d)}{\Gamma(c-a)\Gamma(c-b)} \end{bmatrix}. \end{aligned} \quad (4.156)$$

- Under the transformation  $x \mapsto 4\sqrt{x}/(1+\sqrt{x})^2$ , we have:

$$\begin{aligned} &{}_2F_1(a, b; a-b+1|x) = (1+\sqrt{x})^{-2a} \\ &{}_2F_1\left(a, a-b+\frac{1}{2}; 2a-2b+1 \left| \frac{4\sqrt{x}}{(1+\sqrt{x})^2} \right. \right). \end{aligned} \quad (4.157)$$



#### 4.B FOUR-POINT FUNCTION SATISFYING A SECOND-ORDER DIFFERENTIAL EQUATION

In this appendix we compute the correlation function

$$G(x, \bar{x}) = \langle \tau_{\mathbb{1}}(\infty) \tau_{\mathbb{1}}(1) \tau_{\mathbb{1}}(x, \bar{x}) \tau_{\mathbb{1}}(0) \rangle . \quad (4.158)$$

in the  $\mathbb{Z}_2$  orbifold of the YL model. It follows from the null-vector

$$\left( \widehat{L}_{-2} - \frac{10}{3} \widehat{L}_{-1}^2 \right) \tau_{\mathbb{1}} = 0 \quad (4.159)$$

This is the standard form of a null-vector at level 2, which yields in the usual way a second order differential equation whose solutions are hypergeometric functions. For completeness we recall the key steps in computing  $G(x, \bar{x})$ .

##### 4.B.1 Differential equation

A standard CFT argument yields, for any  $n \in \mathbb{Z}$ , any primary operators  $(\mathcal{O}_2, \mathcal{O}_3)$ , and any states  $(|\mathcal{O}_1\rangle, |\mathcal{O}_4\rangle)$ :

$$\langle \mathcal{O}_1 | \mathcal{O}_2(1) \mathcal{O}_3(x, \bar{x}) L_n | \mathcal{O}_4 \rangle - \langle \mathcal{O}_1 | L_n \mathcal{O}_2(1) \mathcal{O}_3(x, \bar{x}) | \mathcal{O}_4 \rangle = \{ (1 - x^n) [x \partial_x + (n+1)h_3] + (h_4 - h_1) - n(h_2 + h_3) \} \langle \mathcal{O}_1 | \mathcal{O}_2(1) \mathcal{O}_3(x, \bar{x}) | \mathcal{O}_4 \rangle . \quad (4.160)$$

$$\langle \mathcal{O}_1 | \mathcal{O}_2(1) \mathcal{O}_3(x, \bar{x}) | \mathcal{O}_4 \rangle . \quad (4.161)$$

Then (4.56) translates into the ordinary differential equation for  $G(x, \bar{x})$ :

$$[400(x-1)^2 x^2 \partial_x^2 + 40(x-1)x(6x-3)\partial_x + 33] G(x) = 0 \quad (4.162)$$

This equation has the following Riemann scheme, giving the local exponents, i.e. the allowed power-law behaviours at the three singular points 0, 1 and  $\infty$ :

$$\begin{array}{c} 0 \quad 1 \quad \infty \\ \hline \left\{ \begin{array}{ccc} \frac{3}{20} & \frac{3}{20} & 0 \\ \frac{11}{20} & \frac{11}{20} & -\frac{2}{5} \end{array} \right\} \end{array} \quad (4.163)$$

In the limits  $x \rightarrow 0, 1, \infty$ , we have the OPE:

$$\tau_{\mathbb{1}} \times \tau_{\mathbb{1}} \rightarrow \mathbb{1} + \Phi . \quad (4.164)$$

So the local exponents (4.163) are consistent with the internal states  $\{\mathbb{1}, \Phi\}$  of the conformal blocks for the channels. If we perform the appropriate change of function to shift two of these local exponents to zero:

$$G(x, \bar{x}) = |x|^{11/10} |1-x|^{11/10} f(x, \bar{x}) , \quad (4.165)$$

then (4.162) turns into the hypergeometric differential equation:

$$x(x-1)\partial_x^2 f + [(a+b+1)x-c]\partial_x f + abf = 0, \quad (4.166)$$

with parameters:

$$a = \frac{7}{10}, \quad b = \frac{11}{10}, \quad c = \frac{7}{5}. \quad (4.167)$$

It is also convenient to introduce the parameter  $d = c - a - b$ . If one repeats the argument with the anti-holomorphic generators  $\bar{L}_n$ , one obtains the same equation as (4.166), with  $(x, \partial_x)$  replaced by  $(\bar{x}, \bar{\partial}_x)$ .

#### 4.B.2 Determination of the four-point function

A basis of holomorphic solutions to (4.166) is given by:

$$\begin{aligned} I_1(x) &= {}_2F_1(a, b; c|x), \\ I_2(x) &= x^{1-c} {}_2F_1(b-c+1, a-c+1; 2-c|x), \end{aligned} \quad (4.168)$$

where  ${}_2F_1$  is Gauss's hypergeometric function (4.152). The basis  $I_j$  has a diagonal monodromy around  $x = 0$ :

$$\begin{pmatrix} I_1(x) \\ I_2(x) \end{pmatrix} \xrightarrow{x \mapsto e^{2i\pi}x} \begin{pmatrix} 1 & 0 \\ 0 & e^{-2i\pi c} \end{pmatrix} \begin{pmatrix} I_1(x) \\ I_2(x) \end{pmatrix}. \quad (4.169)$$

Similarly, by the change of variable  $x \mapsto 1-x$ , one obtains a basis of solutions

$$\begin{aligned} J_1(x) &= {}_2F_1(a, b; a+b-c+1|1-x), \\ J_2(x) &= (1-x)^{c-a-b} {}_2F_1(c-b, c-a; c-a-b+1|1-x), \end{aligned} \quad (4.170)$$

with a diagonal monodromy around  $x = 1$ :

$$\begin{pmatrix} J_1(x) \\ J_2(x) \end{pmatrix} \xrightarrow{(1-x) \mapsto e^{2i\pi}(1-x)} \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\pi(c-a-b)} \end{pmatrix} \begin{pmatrix} J_1(x) \\ J_2(x) \end{pmatrix}. \quad (4.171)$$

We shall construct a solution of the form

$$G(x, \bar{x}) = |x|^{11/10} |1-x|^{11/10} \sum_{i,j=1}^2 X_{ij} \overline{I_i(x)} I_j(x). \quad (4.172)$$

From the properties of the operators  $\tau_{\parallel}$  and  $\Phi$  (see Sec. 4.2),  $G(x, \bar{x})$  should be single-valued, which imposes the form  $X_{ij} = \delta_{ij} X_i$  for the coefficients in (4.172). The solution should also admit a decomposition of the form:

$$G(x, \bar{x}) = |x|^{11/10} |1-x|^{11/10} \sum_{k,\ell=1}^2 Y_{k\ell} \overline{J_k(x)} J_\ell(x). \quad (4.173)$$

Again, single-valuedness of  $G(x, \bar{x})$  imposes the form  $Y_{k\ell} = \delta_{k\ell} Y_k$ . The key ingredient to determine the coefficients  $X_j$  and  $Y_j$  is the matrix for the change of basis between  $\{I_1(x), I_2(x)\}$  and  $\{J_1(x), J_2(x)\}$ . Using the properties (4.153–4.154) of hypergeometric functions, one obtains:

$$I_i(x) = \sum_{j=1}^2 A_{ij} J_j(x), \tag{4.174}$$

where  $A$  is given in (4.156). Comparing (4.172) and (4.173), we get the matrix relations:

$$Y = A^\dagger X A, \quad X = (A^{-1})^\dagger Y A^{-1}. \tag{4.175}$$

Imposing a diagonal form for  $X$  and  $Y$  yields two linear equations on  $(X_1, X_2)$ :

$$\begin{aligned} \bar{A}_{11} X_1 A_{12} + \bar{A}_{21} X_2 A_{22} &= 0, \\ \bar{A}_{12} X_1 A_{11} + \bar{A}_{22} X_2 A_{21} &= 0. \end{aligned} \tag{4.176}$$

Since the entries of  $A$  are real, these two relations are equivalent. Similarly, one gets a linear relation between  $Y_1$  and  $Y_2$ . Finally, one gets the ratios:

$$\frac{X_2}{X_1} = - \left[ \frac{\Gamma(c)}{\Gamma(2-c)} \right]^2 \gamma(1-a)\gamma(1-b)\gamma(1-c+a)\gamma(1-c+b), \tag{4.177}$$

$$\frac{Y_2}{Y_1} = - \left[ \frac{\Gamma(1-d)}{\Gamma(1+d)} \right]^2 \gamma(1-a)\gamma(1-b)\gamma(c-a)\gamma(c-b). \tag{4.178}$$

The symbol  $\Gamma$  denotes Euler’s Gamma function, and we also introduced the short-hand notation:

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{4.179}$$

Moreover, the term  $|J_1(x)|^2$  in (4.173) corresponds to the OPE  $\tau_{\mathbb{1}} \times \tau_{\mathbb{1}} \rightarrow \mathbb{1}$ , which fixes  $Y_1 = 1$ . We thus get:

$$\begin{aligned} X_1 &= \gamma(1-c)\gamma(1-d)\gamma(c-a)\gamma(c-b) = 1, \\ X_2 &= -\frac{\gamma(c)}{(1-c)^2} \gamma(1-d)\gamma(1-a)\gamma(1-b) = 2^{16/5}, \end{aligned} \tag{4.180}$$

and

$$Y_1 = 1, \quad Y_2 = - \left[ \frac{\Gamma(1-d)}{\Gamma(1+d)} \right]^2 \gamma(1-a)\gamma(1-b)\gamma(c-a)\gamma(c-b) = 2^{16/5}.$$

The final result for the four-point function  $G(x, \bar{x})$  (4.54) is

$$\begin{aligned} G(x, \bar{x}) &= |x|^{11/10} |1-x|^{11/10} \times \left[ \left| {}_2F_1(7/10, 11/10; 7/5|x) \right|^2 \right. \\ &\quad \left. + 2^{16/5} |x^{-2/5} {}_2F_1(7/10, 3/10; 3/5|x) \right|^2 \right], \end{aligned} \tag{4.181}$$

## 4.C DIRECT COMPUTATION OF OPE COEFFICIENTS OF TWIST OPERATORS

In this appendix, we perform the computation of the structure constants appearing in the Yang-Lee model on the  $N = 2$  orbifold. They provide a non-trivial check of the validity of our method based on solving the differential equation for conformal blocks. In the following,  $\langle \dots \rangle$  denotes the average in the orbifold theory, whereas  $\langle \dots \rangle_{\Sigma_2}$  (resp.  $\langle \dots \rangle_{\mathbb{C}}$ ) denotes the average in the mother theory on the two-sheeted Riemann surface (resp. on the Riemann sphere). Some of those results were already obtained in [23], a generic way of computing those three-point functions can be found in [81]. In the specific case of the Ising model similar three-point functions were found in [32] and [126].

For three-point functions involving only untwisted operators, the correlation function decouples between the  $N$  copies. For instance:

$$C(\Phi, \Phi, \Phi) = C(\phi, \phi, \phi)^2, \quad (4.182)$$

where  $C(\phi, \phi, \phi)$  is the structure constant in the mother theory (Yang-Lee):

$$C(\phi, \phi, \phi) = \frac{i\sqrt{\frac{1}{2}(3\sqrt{5}-5)}\Gamma\left(\frac{1}{5}\right)^3}{10\pi\Gamma\left(\frac{3}{5}\right)}.$$

The structure constants involving twist operators can be computed by unfolding through the conformal map  $z \mapsto z^{1/2}$ .

- Let us start with  $C(\Phi, \tau_{\mathbb{1}}, \tau_{\mathbb{1}})$ , which unfolds to a two-point function:

$$\begin{aligned} C(\Phi, \tau_{\mathbb{1}}, \tau_{\mathbb{1}}) &= \langle \tau_{\mathbb{1}}(\infty)\Phi(1)\tau_{\mathbb{1}}(0) \rangle \\ &= \left\langle \phi(1)\phi(e^{2i\pi}) \right\rangle_{\Sigma_2} \\ &= \left\langle 2^{-2h_\phi}\phi(1) \times 2^{-2h_\phi}\phi(-1) \right\rangle_{\mathbb{C}} \\ &= 2^{-8h_\phi}. \end{aligned}$$

- The constant  $C[(\mathbb{1} \otimes \phi)^{(0)}, \tau_\phi, \tau_\phi]$  which involves the untwisted operator  $(\mathbb{1} \otimes \phi)^{(0)} = \frac{1}{\sqrt{2}}(\mathbb{1} \otimes \phi + \phi \otimes \mathbb{1})$ :

$$\begin{aligned} C[(\mathbb{1} \otimes \phi)^{(0)}, \tau_\phi, \tau_\phi] &= \lim_{z_\infty \rightarrow \infty} z_\infty^{4h_\phi} \langle \tau_\phi(0)\tau_\phi(1)(\mathbb{1} \otimes \phi)^{(0)}(z_\infty) \rangle \\ &= \sqrt{2} \lim_{z_\infty \rightarrow \infty} z_\infty^{4h_\phi} \langle \tau_\phi(0)\tau_\phi(1)(\mathbb{1} \otimes \phi)(z_\infty) \rangle \\ &= \frac{\sqrt{2}}{2^{2h_\phi}} \lim_{z_\infty \rightarrow \infty} z_\infty^{4h_\phi} \langle \phi(0)\phi(1)\phi(z_\infty) \rangle_{\mathbb{C}} \\ &= \frac{\sqrt{2}C(\phi, \phi, \phi)}{2^{2h_\phi}} \approx 3.56664i \end{aligned}$$

- $C(\Phi, \tau_\phi, \tau_\phi)$  unfolds to a four-point function, computed in 4.D:

$$\begin{aligned} C(\Phi, \tau_\phi, \tau_\phi) &= \langle \tau_\phi | \tau_\phi(1) | \Phi \rangle \\ &= \frac{1}{2^{4h_\phi}} \langle \phi | \phi(1) \phi(-1) | \phi \rangle_{\mathbb{C}} \\ &= \frac{(\sqrt{5}-1) \Gamma(\frac{1}{5})^6 \Gamma(\frac{2}{5})^2}{80 \cdot 2^{2/5} \pi^4} \approx -5.53709 \end{aligned}$$

- $C(\tau_\phi, \Phi, \tau_{\mathbb{1}})$ :

$$\begin{aligned} C(\tau_\phi, \Phi, \tau_{\mathbb{1}}) &= \langle \tau_\phi | \tau_{\mathbb{1}}(1) | \Phi \rangle \\ &= \frac{\langle \phi(0) \phi(1) \phi(-1) \rangle_{\mathbb{C}}}{2^{4h_\phi}} = \frac{C(\phi, \phi, \phi)}{2^{6h_\phi}} \approx 4.39104i \end{aligned}$$

- $C(\tau_\phi, \Phi, \widehat{L}_{-1/2} \widehat{L}_{-1/2} \tau_\phi)$ : we also need this structure constant which involves a descendant state. The behaviour of the descendant states during the unfolding is given by the induction procedure 4.42:

$$\widehat{L}_{-1/2} \rightarrow \frac{1}{2} L_{-1}$$

Hence, for the three-point function:

$$\langle [\widehat{L}_{-1/2} \widehat{L}_{-1/2} \tau_\phi](0) \tau_\phi(1) \Phi(\infty) \rangle = \frac{1}{2^{4h_\phi+2}} \langle \phi | \phi(1) \phi(-1) L_{-1} \bar{L}_{-1} | \phi \rangle$$

This four-point function is computed in 4.D. To compute the structure constant, we also need to normalize the descendant state:

$$\begin{aligned} C(\tau_\phi, \Phi, \widehat{L}_{-1/2} \widehat{L}_{-1/2} \tau_\phi) &= \frac{\langle [\widehat{L}_{-1/2} \widehat{L}_{-1/2} \tau_\phi](0) \tau_\phi(1) \Phi(\infty) \rangle}{\sqrt{\langle \tau_\phi | \widehat{L}_{1/2} \widehat{L}_{1/2} \widehat{L}_{-1/2} \widehat{L}_{-1/2} | \tau_\phi \rangle}} \\ &= \frac{10}{2^{4h_\phi+2}} [\partial_z \partial_{\bar{z}} \langle \phi(z) \phi(1) \phi(-1) \phi(\infty) \rangle]_{|z=\bar{z}=0} \\ &= \frac{2^{4h_\phi+2}}{5} \approx 0.459479 \end{aligned} \quad (4.183)$$

#### 4.D DIRECT COMPUTATION OF THE FUNCTION IN SECTION 4.4

The correlation function  $F(x, \bar{x})$  (4.54) can be computed using a direct approach, by relating it to the four-point function  $\langle \phi(\infty) \phi(1) \phi(u) \phi(0) \rangle$  through an appropriate conformal mapping from the two-sheeted Riemann surface  $\Sigma_2$  to the Riemann sphere. Indeed, let  $y \in \mathbb{C}$ , and consider the mapping:

$$z \mapsto w = \frac{2y(\sqrt{z}-1)}{(1+y)(\sqrt{z}-y)}, \quad \frac{dw}{dz} = \frac{y(1-y)}{(1+y)(\sqrt{z}-y)^2 \sqrt{z}}. \quad (4.184)$$

The function  $F(x, \bar{x})$  can be written:

$$\begin{aligned} F(x, \bar{x}) &= \langle \tau(\infty) \Phi(1) \Phi(x, \bar{x}) \tau(0) \rangle \\ &= \langle \tau(\infty) \tau(0) \rangle \times \left\langle \phi(1) \phi(e^{2i\pi}) \phi(x, \bar{x}) \phi(e^{2i\pi} x, e^{-2i\pi} \bar{x}) \right\rangle_{\Sigma_2} \end{aligned} \quad (4.185)$$

The four points of this correlation function are mapped as follows under (4.184):

$$\begin{aligned} 1 \mapsto 0, \quad e^{2i\pi} \mapsto \frac{4y}{(1+y)^2}, \quad x \mapsto R = \frac{2y(1-\sqrt{x})}{(1+y)(y-\sqrt{x})}, \\ e^{2i\pi} x \mapsto \frac{2y(1+\sqrt{x})}{(1+y)(y+\sqrt{x})}. \end{aligned} \quad (4.186)$$

If we let  $y \rightarrow \sqrt{x}$ , we have  $R \rightarrow \infty$ , and we get

$$\begin{aligned} F(x, \bar{x}) &= |1 + \sqrt{x}|^{-8h_\phi} |16x|^{-2h_\phi} \times \langle \phi(\infty) \phi(1) \phi(u, \bar{u}) \phi(0) \rangle_{\mathbb{C}}, \\ u &= \frac{4\sqrt{x}}{(1+\sqrt{x})^2}. \end{aligned} \quad (4.187)$$

Since  $\phi \equiv \phi_{12}$  is degenerate at level 2 (see (4.46)), the function  $\langle \phi(\infty) \phi(1) \phi(u, \bar{u}) \phi(0) \rangle$  satisfies a second-order equation, which can be turned into a hypergeometric equation of the form (4.166) with parameters

$$\tilde{a} = \frac{3}{5}, \quad \tilde{b} = \frac{4}{5}, \quad \tilde{c} = \frac{6}{5}, \quad (4.188)$$

for the function  $g(u, \bar{u})$  defined as

$$\langle \phi(\infty) \phi(1) \phi(u, \bar{u}) \phi(0) \rangle = |u|^{-4h_\phi} |1-u|^{-4h_\phi} g(u, \bar{u}). \quad (4.189)$$

One obtains a solution of the form

$$\begin{aligned} F(x, \bar{x}) &= |16x|^{4/5} \left| \frac{1-\sqrt{x}}{1+\sqrt{x}} \right|^{8/5} \left[ \tilde{X}_1 \left| {}_2F_1(3/5, 4/5; 6/5|u) \right|^2 \right. \\ &\quad \left. + \tilde{X}_2 \left| u^{-1/5} {}_2F_1(3/5, 2/5; 4/5|u) \right|^2 \right]. \end{aligned} \quad (4.190)$$

#### 4.E QUANTUM ISING CHAIN IN AN IMAGINARY MAGNETIC FIELD

We consider the Hamiltonian:

$$H = -\frac{1}{2} \sum_{j=1}^L \left( \lambda \sigma_j^x \sigma_{j+1}^x + \sigma_j^z + ih \sigma_j^x \right), \quad (4.191)$$

with periodic boundary conditions, with  $h$  and  $\lambda$  real, in the regime  $0 < \lambda < 1$ .

Within the usual inner product all operators  $\sigma_j^a$  are self-adjoint, and it is clear that  $H$  is not (the matrix representing  $H$  in the usual basis is

symmetric but not real). Alternatively one can work with a different hermitian form, namely

$$\langle \Phi, \Psi \rangle = \langle \Phi | P | \Psi \rangle, \quad P = \prod_{j=1}^L \sigma_j^z \quad (4.192)$$

According to this hermitian form - which is not definite positive - the Hamiltonian density  $\sigma_j^x \sigma_{j+1}^x + \sigma_j^z + ih\sigma_j^x$  (and therefore  $H$  itself) is self-adjoint.

With the usual inner product, note that  $PHP = H^\dagger$ , so  $P$  maps right eigenvectors to left eigenvectors. In particular in the  $\mathcal{PT}$ -unbroken phase, we have

$$H|r_0\rangle = E_0|r_0\rangle, \quad H^\dagger|l_0\rangle = E_0|l_0\rangle, \quad P|r_0\rangle \propto |l_0\rangle \quad (4.193)$$

where  $|r_0\rangle$  is the ground state. Then  $|r_0\rangle = |l_0\rangle$  iff  $|r_0\rangle$  is an eigenstate of  $P$ . For small systems ( $L = 1, 2$ ) one can check analytically that this is not the case. We have observed numerically that this trend persists for larger systems. A curious observation is that for a single site ( $L = 1$ ), the Hamiltonian is not diagonalizable at the transition : the two lowest eigenvalues  $E_0$  and  $E_1$  merge into a non-trivial Jordan block. It would be interesting to study whether this is also the case for larger systems, as it would suggest some logarithmic behaviour in the continuum.

Despite being non-hermitian, the eigenvalues of  $H$  are either real, or they appear in pairs of complex conjugates ( $E, E^*$ ). This can be seen using  $\mathcal{PT}$  symmetry [41], or simply by noting that after the unitary similarity transformation  $\tilde{H} = UH U^\dagger$  with  $U = \prod_{j=1}^L \exp(i\pi\sigma_j^z/4)$ , one gets a real, non-symmetric operator  $\tilde{H}$ .

For  $h = 0$ , the Hamiltonian is Hermitian, and thus its spectrum is real. The regime  $0 < \lambda < 1$  and  $h = 0$  corresponds to the (anisotropic limit of) the 2d Ising model in the high-temperature phase, where the correlation length  $\zeta$  is finite. When  $h$  is increased while  $\lambda$  is kept constant, the ground state and first excited energies remains finite, up to a threshold value  $h_c(\lambda, L)$ , where they “merge” into a complex conjugate pair: see Fig. 4.E.1. The point  $h_c$  corresponds to the vanishing of the partition function in the 2d Ising model. In the scaling limit, it converges to a critical point  $h_c(\lambda, L) \rightarrow h_c(\lambda)$ , called the Yang-Lee edge singularity.

The finite-size study of the Yang-Lee edge singularity through the model (4.191) is rather subtle: for a given system size of  $L$  sites, one should first determine the threshold value  $h_c(\lambda, L)$ , and then approach this value from below. We have computed numerically the one-interval ground state  $N = 2$  Rényi entropy  $S_2$  for the model (4.191) at  $\lambda = 0.8$  and system sizes  $L = 12, 14, 16, 18$  sites. The density matrix is defined as in the rest of the paper as  $\rho = r_0 l_0^\dagger$ , so the quantity

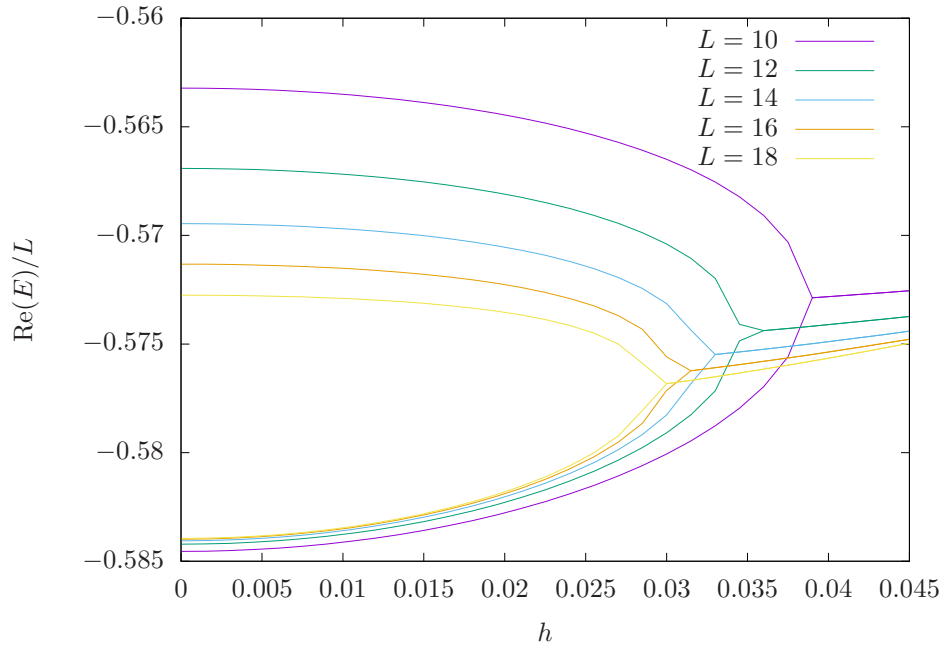


Figure 4.E.1: The two lowest energies in the Yang-Lee model (4.191) as a function of  $h$ , for  $\lambda = 0.8$ .

we compute corresponds at criticality to (4.18). These numerical calculations lead to the following observations, depicted in Fig. 4.E.2. In the off-critical regime  $h \ll h_c(\lambda, L)$ , the entropy  $S_2$  has a concave form. Then, when increasing the value of  $h$  and approaching  $h_c(\lambda, L)$  from below, the function undergoes a crossover to the convex form predicted by CFT (4.91). While not being positive, the entanglement entropy defined using  $\rho = r_0 l_0^\dagger$  is surprisingly effective at detecting the phase transition.



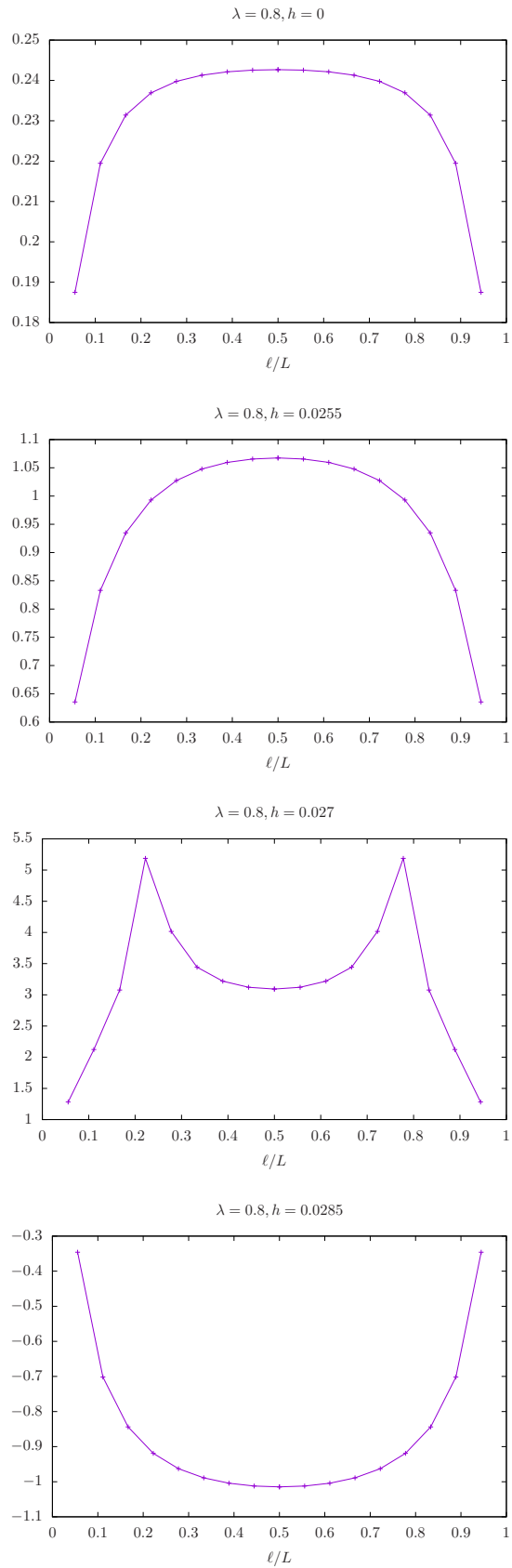


Figure 4.E.2: Crossover of the  $N = 2$  ground state one-interval Rényi entropy in the Yang-Lee model (4.191) for  $L = 18$  sites and  $\lambda = 0.8$ .

## CONCLUSION

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The three subjects developed in this thesis are quite different, both in subject and scope. A summary for each of them can already be found in the local conclusions at the end of each chapter. In this global conclusion I will try to develop the possible links between the different subjects, tying the different chapters together more directly than through the fuzzy concept of extended CFTs, while drawing potential direction for future study. Disclaimer : As is common in thesis conclusions, some of the remarks below are of a rather speculative nature.

**LOOP MODELS AND  $\mathcal{W}_3$**  The first two chapters are the easiest to mix. They do have a common ground, we spent some time, in the first chapter describing how the twisted fully packed loop model, on the cylinder, maps to the 15-vertex model, which possesses a  $U_q(\mathfrak{sl}_3)$  symmetry (its R-matrix verifies the Hecke algebra) and seemingly corresponds, in the continuum limit, to a CFT with  $\mathcal{W}_3$  symmetry. The non-twisted FPL model however, does not have a  $\mathcal{W}_3$  symmetry and does not match the Toda field theory described in the second chapter.

Finding a non-local model corresponding to  $\mathcal{W}_3$  (like  $\mathcal{O}(n)$  corresponds to Virasoro) is still an open question. Some leads do exist, the Spider model (developed in appendix 2.C.2), seems to fit part of the bill. Another option would consist in a more complicated loop model, in particular a loop model with three types of loops also seems to verify the right Hecke algebra. Such a critical model would give a method to compute numerically the  $\mathcal{W}_3$  correlation constants – which is all the more important as we do not know how to compute them otherwise. Hypothetically, it could also help define the equivalent of the operator  $W$  on the lattice, similarly to what have been done for  $T$  through the TL algebra.

**$\mathcal{W}_3$  AND ORBIFOLDS** A possible way to mix the two last chapters would be to look at the entropy of a CFT with  $\mathcal{W}_3$  symmetry. This is not as ad hoc as it seems, the main limitations of the method presented in the last chapter are the difficulties posed by the generalization of the construction to even relatively small minimal models. These difficulties are created by the fast growing rate of the orbifold's spectrum. Using an "extended minimal model" would decrease the spectrum size and the conformal blocks' differential equations would

be of lower degree. Supersymmetric CFTs may be a better choice than  $\mathcal{W}_3$  for a first test of this idea.

The characters of  $\mathcal{W}_3$  minimal models are known [76], but with a rather complicated form, and it would be interesting to check if, like in the Virasoro case, we can find differential equations associated with some of these. In the other direction, the numerical bootstrap that we employed in orbifold CFT could be useful to compute numerically a subset of  $\mathcal{W}_3$  structure constants.

**ORBIFOLDS AND LOOP MODELS** A very natural question would be “what kind of meaning can be given to the concept of entropy in a loop model”. We are not the first to ask this question [88]. While it is certainly possible to define an object similar to the Rényi entropy on a given surface – with the geometric interpretation – giving it a meaning in term of loop seems difficult. The discussion in the last chapter about the issues caused by non-unitarity is probably relevant in this context. While this thesis focused on surfaces without boundary, the methods developed in the last chapter can be applied easily to boundary CFTs, a context where loop models are well understood. This makes it relatively easy to construct multi-boundaries surfaces, which could be useful to work out some structure constants associated with loop models.

In conclusion, it seems important to keep studying in parallel the discrete and the continuum, critical integrable models on one side and conformal field theory on the other. The link between the two topics is promising, in both directions. Integrability often provides an insight on the meaning of CFTs operators and symmetries – or at the very least a stable mathematical setting. While conformal field theory has proven itself a very effective tool that can be used in many, sometime unexpected, ways. This thesis, at its level, tried to develop these ideas, both linking extended CFTs to their realizations with loop models and using conformal field theory to compute the entropy of critical spin chains.

## BIBLIOGRAPHY

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- [1] V. Alba, L. Tagliacozzo, and P. Calabrese. “Entanglement Entropy of Two Disjoint Blocks in Critical Ising Models.” In: *Phys. Rev. B* 81 (2010), 060411(R). DOI: [10.1103/physrevb.81.060411](https://doi.org/10.1103/physrevb.81.060411).
- [2] V. Alba, L. Tagliacozzo, and P. Calabrese. “Entanglement Entropy of Two Disjoint Intervals in  $C=1$  Theories.” In: *J. Stat. Mech.* 2011 (2011), P06012. DOI: [10.1088/1742-5468/2011/06/P06012](https://doi.org/10.1088/1742-5468/2011/06/P06012).
- [3] F. C. Alcaraz and M. J. Martins. “The Operator Content of the Exactly Integrable  $SU(N)$  Magnets.” In: *J. Phys. A: Math. Gen.* 23 (1990), p. L1079.
- [4] Francisco Castilho Alcaraz, Miguel Ibanez Berganza, and German Sierra. “Entanglement of Low-Energy Excitations in Conformal Field Theory.” In: *Phys. Rev. Lett.* 106 (2011), p. 201601. DOI: [10.1103/PhysRevLett.106.201601](https://doi.org/10.1103/PhysRevLett.106.201601). arXiv: [1101.2881](https://arxiv.org/abs/1101.2881) [[cond-mat.stat-mech](https://arxiv.org/archive/cond)].
- [5] L. Alvarez-Gaumé, J-B. Bost, G. Moore, P. Nelson, and C. Vafa. “Bosonization on Higher Genus Riemann Surfaces.” In: *Commun. Math. Phys.* 112 (1987), pp. 503–552. DOI: [10.1142/9789812812650\\_0027](https://doi.org/10.1142/9789812812650_0027).
- [6] Thomas Andrews. “XVIII. The Bakerian Lecture.—On the Continuity of the Gaseous and Liquid States of Matter.” en. In: *Philosophical Transactions of the Royal Society of London* 159 (Jan. 1869), pp. 575–590. ISSN: 0261-0523, DOI: [10.1098/rstl.1869.0021](https://doi.org/10.1098/rstl.1869.0021).
- [7] R. Augusiak, F. M. Cucchietti, and M. Lewenstein. “Many-Body Physics from a Quantum Information Perspective.” In: *Modern Theories of Many-Particle Systems in Condensed Matter Physics*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2012, pp. 245–294. ISBN: 978-3-642-10449-7. DOI: [10.1007/978-3-642-10449-7\\_6](https://doi.org/10.1007/978-3-642-10449-7_6).
- [8] Satish Balay, William D. Gropp, Lois Curfman McInnes, and Barry F. Smith. “Efficient Management of Parallelism in Object Oriented Numerical Software Libraries.” In: *Modern Software Tools in Scientific Computing*. Ed. by E. Arge, A. M. Bruaset, and H. P. Langtangen. Birkhäuser Press, 1997, pp. 163–202.
- [9] Satish Balay et al. “PETSc Web Page.” In: ().
- [10] Satish Balay et al. *PETSc Users Manual*. Tech. rep. ANL-95/11 - Revision 3.9. Argonne National Laboratory, 2018.

- [11] M. N. Barber. "Finite-Size Scaling." In: *Phase Transitions and Critical Phenomena*. Ed. by C. Domb, M.S. Green, and J.L. Lebowitz. Vol. 8. Academic Press, 1983, pp. 145–266.
- [12] R. J. Baxter. "Eight-Vertex Model in Lattice Statistics." In: *Physical Review Letters* 26.14 (Apr. 1971), pp. 832–833. DOI: [10.1103/PhysRevLett.26.832](https://doi.org/10.1103/PhysRevLett.26.832).
- [13] R. J. Baxter. *Exactly Solved Models in Statistical Mechanics*. Dover Publications (Mineola, New York), 1982.
- [14] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. "Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory." In: *Nuclear Physics B* 241.2 (July 1984), pp. 333–380. ISSN: 0550-3213. DOI: [10.1016/0550-3213\(84\)90052-X](https://doi.org/10.1016/0550-3213(84)90052-X).
- [15] V. A. Belavin and R. V. Geiko. "Geodesic Description of Heavy-Light Virasoro Blocks." In: *Journal of High Energy Physics* 2017.8 (Aug. 2017). ISSN: 1029-8479. DOI: [10.1007/JHEP08\(2017\)125](https://doi.org/10.1007/JHEP08(2017)125). arXiv: [1705.10950](https://arxiv.org/abs/1705.10950).
- [16] Vladimir Belavin, Benoit Estienne, Omar Foda, and Raoul Santachiara. "Correlation Functions with Fusion-Channel Multiplicity in  $W_3$  Toda Field Theory." In: *JHEP* 06 (2016), p. 137. DOI: [10.1007/JHEP06\(2016\)137](https://doi.org/10.1007/JHEP06(2016)137). arXiv: [1602.03870 \[hep-th\]](https://arxiv.org/abs/1602.03870).
- [17] E. Beltaos and T. Gannon. "The  $W_N$  Minimal Model Classification." In: *Commun. Math. Phys.* 312 (2012), pp. 337–360. ISSN: 1432-0916. DOI: [10.1007/s00220-012-1473-4](https://doi.org/10.1007/s00220-012-1473-4).
- [18] Charles H. Bennett, Herbert J. Bernstein, Sandu Popescu, and Benjamin Schumacher. "Concentrating Partial Entanglement by Local Operations." In: *Phys. Rev. A* 53 (1996), pp. 2046–2052. DOI: [10.1103/PhysRevA.53.2046](https://doi.org/10.1103/PhysRevA.53.2046). arXiv: [quant-ph/9511030 \[quant-ph\]](https://arxiv.org/abs/quant-ph/9511030).
- [19] Miguel Ibanez Berganza, Francisco Castilho Alcaraz, and German Sierra. "Entanglement of Excited States in Critical Spin Chains." In: *J. Stat. Mech.* 1201 (2012), P01016. DOI: [10.1088/1742-5468/2012/01/P01016](https://doi.org/10.1088/1742-5468/2012/01/P01016). arXiv: [1109.5673 \[cond-mat.stat-mech\]](https://arxiv.org/abs/1109.5673).
- [20] M. A. Bershadsky, V. G. Knizhnik, and M. G. Teitelman. "Superconformal Symmetry in Two Dimensions." In: *Physics Letters B* 151.1 (Jan. 1985), pp. 31–36. ISSN: 0370-2693. DOI: [10.1016/0370-2693\(85\)90818-4](https://doi.org/10.1016/0370-2693(85)90818-4).
- [21] H. Bethe. "Zur Theorie der Metalle." de. In: *Zeitschrift für Physik* 71.3-4 (Mar. 1931), pp. 205–226. ISSN: 0044-3328. DOI: [10.1007/BF01341708](https://doi.org/10.1007/BF01341708).

- [22] Davide Bianchini, Olalla A. Castro-Alvaredo, and Benjamin Doyon. “Entanglement Entropy of Non-Unitary Integrable Quantum Field Theory.” In: *Nuclear Physics B* 896 (2015), pp. 835–880. ISSN: 0550-3213. DOI: <https://doi.org/10.1016/j.nuclphysb.2015.05.013>.
- [23] Davide Bianchini, Olalla A. Castro-Alvaredo, Benjamin Doyon, Emanuele Levi, and Francesco Ravanini. “Entanglement Entropy of Non Unitary Conformal Field Theory.” In: *J. Phys.* A48.4 (2015), 04FT01. DOI: [10.1088/1751-8113/48/4/04FT01](https://doi.org/10.1088/1751-8113/48/4/04FT01). arXiv: [1405.2804](https://arxiv.org/abs/1405.2804) [hep-th].
- [24] Davide Bianchini and Francesco Ravanini. “Entanglement Entropy from Corner Transfer Matrix in Forrester–Baxter Non-Unitary RSOS Models.” In: *Journal of Physics A: Mathematical and Theoretical* 49.15 (2016), p. 154005. DOI: [10.1088/1751-8113/49/15/154005](https://doi.org/10.1088/1751-8113/49/15/154005).
- [25] L. Borisov, M. B. Halpern, and C. Schweigert. “Systematic Approach to Cyclic Orbifolds.” In: *International Journal of Modern Physics A* 13.01 (Jan. 1998), pp. 125–168. ISSN: 0217-751X, 1793-656X. DOI: [10.1142/S0217751X98000044](https://doi.org/10.1142/S0217751X98000044). arXiv: [hep-th/9701061](https://arxiv.org/abs/hep-th/9701061).
- [26] P. Bouwknegt. “Coset Construction for Winding Subalgebras and Applications.” In: *Eprint arXiv:Q-Alg/9610013*. Oct. 1996.
- [27] P. Bouwknegt and K. Schoutens. *W-Symmetry*. en. World Scientific, 1995. ISBN: 978-981-02-1762-4.
- [28] Peter Bouwknegt and Kareljan Schoutens. “W Symmetry in Conformal Field Theory.” In: *Physics Reports* 223.4 (Feb. 1993), pp. 183–276. ISSN: 0370-1573. DOI: [10.1016/0370-1573\(93\)90111-P](https://doi.org/10.1016/0370-1573(93)90111-P).
- [29] P. Bowcock and G. M. T. Watts. “Null Vectors of the  $W(3)$  Algebra.” In: *Phys. Lett.* B297 (1992), pp. 282–288. DOI: [10.1016/0370-2693\(92\)91263-9](https://doi.org/10.1016/0370-2693(92)91263-9). arXiv: [hep-th/9209105](https://arxiv.org/abs/hep-th/9209105) [hep-th].
- [30] Stephen G. Brush. “History of the Lenz-Ising Model.” In: *Reviews of Modern Physics* 39.4 (Oct. 1967), pp. 883–893. DOI: [10.1103/RevModPhys.39.883](https://doi.org/10.1103/RevModPhys.39.883).
- [31] P. Calabrese, J. Cardy, and E. Tonni. “Entanglement Entropy of Two Disjoint Intervals in Conformal Field Theory.” In: *J. Stat. Mech.* 2009 (2009), P11001. DOI: [10.1088/1742-5468/2009/11/p11001](https://doi.org/10.1088/1742-5468/2009/11/p11001).
- [32] P. Calabrese, J. Cardy, and E. Tonni. “Entanglement Entropy of Two Disjoint Intervals in Conformal Field Theory: II.” In: *J. Stat. Mech.* 2011 (2011), P01021. DOI: [10.1088/1742-5468/2011/01/p01021](https://doi.org/10.1088/1742-5468/2011/01/p01021).

- [33] P. Calabrese, F. H. L. Essler, and A. M. Läuchli. “Entanglement Entropies of the Quarter Filled Hubbard Model.” In: *Journal of Statistical Mechanics: Theory and Experiment* 9, 09025 (Sept. 2014), p. 09025. DOI: [10.1088/1742-5468/2014/09/P09025](https://doi.org/10.1088/1742-5468/2014/09/P09025). arXiv: [1406.7477](https://arxiv.org/abs/1406.7477) [cond-mat.str-el].
- [34] Pasquale Calabrese and John Cardy. “Entanglement Entropy and Conformal Field Theory.” In: *J. Phys.* A42 (2009), p. 504005. DOI: [10.1088/1751-8113/42/50/504005](https://doi.org/10.1088/1751-8113/42/50/504005). arXiv: [0905.4013](https://arxiv.org/abs/0905.4013) [cond-mat.stat-mech].
- [35] Pasquale Calabrese and John L. Cardy. “Entanglement Entropy and Quantum Field Theory.” In: *J. Stat. Mech.* 0406 (2004), P06002. DOI: [10.1088/1742-5468/2004/06/P06002](https://doi.org/10.1088/1742-5468/2004/06/P06002). arXiv: [hep-th/0405152](https://arxiv.org/abs/hep-th/0405152) [hep-th].
- [36] J. L. Cardy, O. A. Castro-Alvaredo, and B. Doyon. “Form Factors of Branch-Point Twist Fields in Quantum Integrable Models and Entanglement Entropy.” In: *J. Statist. Phys.* 130 (2008), pp. 129–168. DOI: [10.1007/s10955-007-9422-x](https://doi.org/10.1007/s10955-007-9422-x). arXiv: [0706.3384](https://arxiv.org/abs/0706.3384) [hep-th].
- [37] John L. Cardy. “Operator Content of Two-Dimensional Conformally Invariant Theories.” In: *Nuclear Physics B* 270 (Jan. 1986), pp. 186–204. ISSN: 0550-3213. DOI: [10.1016/0550-3213\(86\)90552-3](https://doi.org/10.1016/0550-3213(86)90552-3).
- [38] John L. Cardy. “Boundary Conditions, Fusion Rules and the Verlinde Formula.” In: *Nuclear Physics B* 324.3 (Oct. 1989), pp. 581–596. ISSN: 0550-3213. DOI: [10.1016/0550-3213\(89\)90521-X](https://doi.org/10.1016/0550-3213(89)90521-X).
- [39] Elie Cartan. “Les groupes de transformations continus, infinis, simples.” fra. In: *Annales scientifiques de l’École Normale Supérieure* 26 (1909), pp. 93–161. ISSN: 0012-9593.
- [40] Olalla A. Castro-Alvaredo, Benjamin Doyon, and Emanuele Levi. “Arguments towards a C-Theorem from Branch-Point Twist Fields.” In: *J. Phys.* A44 (2011), p. 492003. DOI: [10.1088/1751-8113/44/49/492003](https://doi.org/10.1088/1751-8113/44/49/492003). arXiv: [1107.4280](https://arxiv.org/abs/1107.4280) [hep-th].
- [41] Olalla A Castro-Alvaredo and Andreas Fring. “A Spin Chain Model with Non-Hermitian Interaction: The Ising Quantum Spin Chain in an Imaginary Field.” In: *Journal of Physics A: Mathematical and Theoretical* 42.46 (2009), p. 465211. DOI: [10.1088/1751-8113/42/46/465211](https://doi.org/10.1088/1751-8113/42/46/465211).
- [42] J. I. Cirac and F. Verstraete. “Renormalization and Tensor Product States in Spin Chains and Lattices.” In: *J. Phys. A: Math. Theor.* 42 (2009), p. 504004. DOI: [10.1088/1751-8113/42/50/504004](https://doi.org/10.1088/1751-8113/42/50/504004).
- [43] *Complex Analysis*. en. Jan. 1979.

- [44] Andrea Coser, Luca Tagliacozzo, and Erik Tonni. “On Rényi Entropies of Disjoint Intervals in Conformal Field Theory.” In: *Journal of Statistical Mechanics: Theory and Experiment* 2014.1 (2014), Po1008. DOI: [10.1088/1742-5468/2014/01/p01008](https://doi.org/10.1088/1742-5468/2014/01/p01008).
- [45] Andrea Coser, Erik Tonni, and Pasquale Calabrese. “Towards the Entanglement Negativity of Two Disjoint Intervals for a One Dimensional Free Fermion.” In: *Journal of Statistical Mechanics: Theory and Experiment* 2016.3 (2016), p. 033116. DOI: [10.1088/1742-5468/2016/03/033116](https://doi.org/10.1088/1742-5468/2016/03/033116).
- [46] Romain Couvreur, Jesper Lykke Jacobsen, and Hubert Saleur. “Entanglement in Nonunitary Quantum Critical Spin Chains.” In: *Physical Review Letters* 119.4 (July 2017). DOI: [10.1103/physrevlett.119.040601](https://doi.org/10.1103/physrevlett.119.040601).
- [47] Č. Crnković, G. M. Sotkov, and M. Stanishkov. “Minimal Models on Hyperelliptic Surfaces.” In: *Physics Letters B* 220.3 (Apr. 1989), pp. 397–405. ISSN: 0370-2693. DOI: [10.1016/0370-2693\(89\)90894-0](https://doi.org/10.1016/0370-2693(89)90894-0).
- [48] A. De Luca and F. Franchini. “Approaching the Restricted Solid-on-Solid Critical Points through Entanglement: One Model for Many Universalities.” In: *Phys. Rev. B* 87.4 (2013), p. 045118. DOI: [10.1103/physrevb.87.045118](https://doi.org/10.1103/physrevb.87.045118).
- [49] P. Di Francesco and J. -B. Zuber. “SU(N) Lattice Integrable Models Associated with Graphs.” In: *Nuclear Physics B* 338.3 (July 1990), pp. 602–646. ISSN: 0550-3213. DOI: [10.1016/0550-3213\(90\)90645-T](https://doi.org/10.1016/0550-3213(90)90645-T).
- [50] Ph. Di Francesco and P. Zinn-Justin. “Quantum Knizhnik-Zamolodchikov Equation, Generalized Razumov-Stroganov Sum Rules and Extended Joseph Polynomials.” In: *J. Phys. A: Math. Gen.* 38 (2005), pp. L815–L822.
- [51] R. Dijkgraaf, E. Verlinde, and H. Verlinde. “ $C=1$  Conformal Field Theories on Riemann Surfaces.” In: *Commun. Math. Phys.* 115 (1988), pp. 649–690. DOI: [10.1007/bf01224132](https://doi.org/10.1007/bf01224132).
- [52] Lance Dixon, Daniel Friedan, Emil Martinec, and Stephen Shenker. “The Conformal Field Theory of Orbifolds.” In: *Nuclear Physics B* 282 (Jan. 1987), pp. 13–73. ISSN: 0550-3213. DOI: [10.1016/0550-3213\(87\)90676-6](https://doi.org/10.1016/0550-3213(87)90676-6).
- [53] H. Dorn and H. J. Otto. “On Correlation Functions for Non-critical Strings with  $c \leq 1$  but  $d \geq 1$ .” In: *Phys. Lett.* B291 (1992), pp. 39–43. DOI: [10.1016/0370-2693\(92\)90116-L](https://doi.org/10.1016/0370-2693(92)90116-L). arXiv: [hep-th/9206053](https://arxiv.org/abs/hep-th/9206053) [hep-th].



- [54] Vl. S. Dotsenko and V. A. Fateev. “Four-Point Correlation Functions and the Operator Algebra in 2D Conformal Invariant Theories with Central Charge  $C \leq 1$ .” In: *Nuclear Physics B* 251 (Jan. 1985), pp. 691–734. ISSN: 0550-3213. DOI: [10.1016/S0550-3213\(85\)80004-3](https://doi.org/10.1016/S0550-3213(85)80004-3).
- [55] V. G. Drinfeld. “Hopf Algebras and the Quantum Yang-Baxter Equation.” In: *Sov.Math.Dokl.* 32 (1985), pp. 254–258.
- [56] Thomas Dupic, Benoit Estienne, and Yacine Ikhlef. “Entanglement Entropies of Minimal Models from Null-Vectors.” In: *SciPost Physics* 4.6 (June 2018). ISSN: 2542-4653. DOI: [10.21468/SciPostPhys.4.6.031](https://doi.org/10.21468/SciPostPhys.4.6.031). arXiv: [1709.09270](https://arxiv.org/abs/1709.09270).
- [57] Thomas Dupic, Benoît Estienne, and Yacine Ikhlef. “The Fully Packed Loop Model as a Non-Rational  $W_3$  Conformal Field Theory.” In: *Journal of Physics A: Mathematical and Theoretical* 49.50 (Dec. 2016), p. 505202. ISSN: 1751-8113, 1751-8121. DOI: [10.1088/1751-8113/49/50/505202](https://doi.org/10.1088/1751-8113/49/50/505202). arXiv: [1606.05376](https://arxiv.org/abs/1606.05376).
- [58] Tohru Eguchi and Hiroshi Ooguri. “Differential Equations for Characters of Virasoro and Affine Lie Algebras.” In: *Nuclear Physics B* 313.2 (1989), pp. 492–508. ISSN: 0550-3213. DOI: [https://doi.org/10.1016/0550-3213\(89\)90330-1](https://doi.org/10.1016/0550-3213(89)90330-1).
- [59] J. Eisert, M. Cramer, and M. B. Plenio. “Area Laws for the Entanglement Entropy - a Review.” In: *Rev. Mod. Phys.* 82 (2010), pp. 277–306. DOI: [10.1103/RevModPhys.82.277](https://doi.org/10.1103/RevModPhys.82.277). arXiv: [0808.3773](https://arxiv.org/abs/0808.3773) [quant-ph].
- [60] F. H. L. Essler, A. M. Läuchli, and P. Calabrese. “Shell-Filling Effect in the Entanglement Entropies of Spinful Fermions.” In: *Physical Review Letters* 110.11, 115701 (Mar. 2013), p. 115701. DOI: [10.1103/PhysRevLett.110.115701](https://doi.org/10.1103/PhysRevLett.110.115701). arXiv: [1211.2474](https://arxiv.org/abs/1211.2474) [cond-mat.str-el].
- [61] Benoit Estienne and Yacine Ikhlef. “Correlation Functions in Loop Models.” In: *arXiv:1505.00585 [cond-mat, physics:hep-th, physics:math-ph]* (May 2015). arXiv: [1505.00585](https://arxiv.org/abs/1505.00585) [cond-mat, physics:hep-th, physics:math-ph].
- [62] David E. Evans and Mathew Pugh. “ $A_2$ -Planar Algebras I.” In: *arXiv:0906.4225 [hep-th, physics:math-ph]* (June 2009). arXiv: [0906.4225](https://arxiv.org/abs/0906.4225) [hep-th, physics:math-ph].
- [63] V. A. Fateev and A. V. Litvinov. “Correlation Functions in Conformal Toda Field Theory I.” In: *JHEP* 11 (2007), p. 002. DOI: [10.1088/1126-6708/2007/11/002](https://doi.org/10.1088/1126-6708/2007/11/002). arXiv: [0709.3806](https://arxiv.org/abs/0709.3806) [hep-th].
- [64] V. A. Fateev and A. V. Litvinov. “Correlation Functions in Conformal Toda Field Theory II.” In: *JHEP* 01 (2009), p. 033.

- [65] V. A. Fateev and A. B. Zamolodchikov. "Conformal Quantum Field Theory Models in Two Dimensions Having  $\mathbb{Z}_3$  Symmetry." In: *Nucl. Phys. B* 280 (1987), pp. 644–660. ISSN: 0550-3213. DOI: [http://dx.doi.org/10.1016/0550-3213\(87\)90166-0](http://dx.doi.org/10.1016/0550-3213(87)90166-0).
- [66] V. A. Fateev and A. B. Zamolodchikov. "Conformal Quantum Field Theory Models in Two Dimensions Having  $\mathbb{Z}_3$  Symmetry." In: *Nuclear Physics B* 280 (Jan. 1987), pp. 644–660. ISSN: 0550-3213. DOI: [10.1016/0550-3213\(87\)90166-0](https://doi.org/10.1016/0550-3213(87)90166-0).
- [67] V.a. Fateev and S.l. Lykhanov. "The Models of Two-Dimensional Conformal Quantum Field Theory with  $\mathbb{Z}_n$  Symmetry." In: *International Journal of Modern Physics A* 03.02 (Feb. 1988), pp. 507–520. ISSN: 0217-751X. DOI: [10.1142/S0217751X88000205](https://doi.org/10.1142/S0217751X88000205).
- [68] S. Ferrara, R. Gatto, and A. F. Grillo. "Conformal Invariance on the Light Cone and Canonical Dimensions." In: *Nuclear Physics B* 34.2 (Nov. 1971), pp. 349–366. ISSN: 0550-3213. DOI: [10.1016/0550-3213\(71\)90333-6](https://doi.org/10.1016/0550-3213(71)90333-6).
- [69] Michael E. Fisher. "The Renormalization Group in the Theory of Critical Behavior." In: *Reviews of Modern Physics* 46.4 (Oct. 1974), pp. 597–616. DOI: [10.1103/RevModPhys.46.597](https://doi.org/10.1103/RevModPhys.46.597).
- [70] O. Foda and B. Nienhuis. "The Coulomb Gas Representation of Critical RSOS Models on the Sphere and the Torus." In: *Nucl. Phys. B* 324 (1989), pp. 643–683. ISSN: 0550-3213. DOI: [http://dx.doi.org/10.1016/0550-3213\(89\)90525-7](http://dx.doi.org/10.1016/0550-3213(89)90525-7).
- [71] Otto Forster. *Lectures on Riemann Surfaces*. en. Graduate Texts in Mathematics. New York: Springer-Verlag, 1981. ISBN: 978-0-387-90617-1.
- [72] P. di Francesco, H. Saleur, and J. B. Zuber. "Relations between the Coulomb Gas Picture and Conformal Invariance of Two-Dimensional Critical Models." en. In: *Journal of Statistical Physics* 49.1-2 (Oct. 1987), pp. 57–79. ISSN: 0022-4715, 1572-9613. DOI: [10.1007/BF01009954](https://doi.org/10.1007/BF01009954).
- [73] Philippe Francesco, Pierre Mathieu, and David Sénéchal. *Conformal Field Theory*. en. Graduate Texts in Contemporary Physics. New York: Springer-Verlag, 1997. ISBN: 978-0-387-94785-3.
- [74] S. Furukawa, V. Pasquier, and J. Shiraishi. "Mutual Information and Boson Radius in  $C=1$  Critical Systems in One Dimension." In: *Phys. Rev. Lett.* 102 (2009), p. 170602. DOI: [10.1103/physrevlett.102.170602](https://doi.org/10.1103/physrevlett.102.170602).
- [75] M. R. Gaberdiel and H. G. Kausch. "A Rational Logarithmic Conformal Field Theory." In: *Phys. Lett. B* 386 (1996), pp. 131–137. DOI: [10.1016/0370-2693\(96\)00949-5](https://doi.org/10.1016/0370-2693(96)00949-5). eprint: [hep-th/9606050](https://arxiv.org/abs/hep-th/9606050).

- [76] T. Gannon. “The Classification of Affine  $SU(3)$  Modular Invariant Partition Functions.” In: *Commun. Math. Phys.* 161 (1994), pp. 233–264. DOI: [10.1007/BF02099776](https://doi.org/10.1007/BF02099776). arXiv: [hep-th/9212060](https://arxiv.org/abs/hep-th/9212060) [hep-th].
- [77] Terry Gannon. “The Theory of Vector-Modular Forms for the Modular Group.” In: *Contrib. Math. Comput. Sci.* 8 (2014), pp. 247–286. DOI: [10.1007/978-3-662-43831-2\\_9](https://doi.org/10.1007/978-3-662-43831-2_9). arXiv: [1310.4458](https://arxiv.org/abs/1310.4458) [math.NT].
- [78] Krzysztof Gawedzki. “Lectures on Conformal Field Theory.” en. In: (), p. 77.
- [79] M. Gell-Mann and F. E. Low. “Quantum Electrodynamics at Small Distances.” In: *Physical Review* 95.5 (Sept. 1954), pp. 1300–1312. DOI: [10.1103/PhysRev.95.1300](https://doi.org/10.1103/PhysRev.95.1300).
- [80] Ron Goldman. “Curvature Formulas for Implicit Curves and Surfaces.” In: *Computer Aided Geometric Design. Geometric Modelling and Differential Geometry* 22.7 (Oct. 2005), pp. 632–658. ISSN: 0167-8396. DOI: [10.1016/j.cagd.2005.06.005](https://doi.org/10.1016/j.cagd.2005.06.005).
- [81] Matthew Headrick. “Entanglement Renyi Entropies in Holographic Theories.” In: *Phys. Rev. D* 82 (2010), p. 126010. DOI: [10.1103/PhysRevD.82.126010](https://doi.org/10.1103/PhysRevD.82.126010). arXiv: [1006.0047](https://arxiv.org/abs/1006.0047) [hep-th].
- [82] Malte Henkel. *Conformal Invariance and Critical Phenomena*. en. Theoretical and Mathematical Physics, 1+2-Pferdchen. Berlin Heidelberg: Springer-Verlag, 1999. ISBN: 978-3-540-65321-9.
- [83] Vicente Hernandez, Jose E. Roman, and Vicente Vidal. “SLEPC: A Scalable and Flexible Toolkit for the Solution of Eigenvalue Problems.” In: *ACM Trans. Math. Softw.* 31.3 (Sept. 2005), pp. 351–362. ISSN: 0098-3500. DOI: [10.1145/1089014.1089019](https://doi.org/10.1145/1089014.1089019).
- [84] Vicente Hernández, Jose E. Román, and Vicente Vidal. “SLEPC: Scalable Library for Eigenvalue Problem Computations.” en. In: *High Performance Computing for Computational Science — VECPAR 2002. Lecture Notes in Computer Science*. Springer, Berlin, Heidelberg, June 2002, pp. 377–391. ISBN: 978-3-540-36569-3. DOI: [10.1007/3-540-36569-9\\_25](https://doi.org/10.1007/3-540-36569-9_25).
- [85] C. Holzhey, F. Larsen, and F. Wilczek. “Geometric and Renormalized Entropy in Conformal Field Theory.” In: *Nucl. Phys. B* 424 (1994), p. 443. DOI: [10.1016/0550-3213\(94\)90402-2](https://doi.org/10.1016/0550-3213(94)90402-2).
- [86] Y. Ikhlef, J. L. Jacobsen, and H. Saleur. “Three-Point Functions in  $c = 1$  Liouville Theory and Conformal Loop Ensembles.” In: *Phys. Rev. Lett.* 116 (2016), p. 130601. arXiv: [1509.03538](https://arxiv.org/abs/1509.03538) [cond-mat.stat-mech].
- [87] Ernst Ising. “Beitrag zur Theorie des Ferromagnetismus.” de. In: *Zeitschrift für Physik* 31.1 (Feb. 1925), pp. 253–258. ISSN: 0044-3328. DOI: [10.1007/BF02980577](https://doi.org/10.1007/BF02980577).

- [88] J. L. Jacobsen and H. Saleur. "Exact Valence Bond Entanglement Entropy and Probability Distribution in the  $XXX$  Spin Chain and the Potts Model." In: *Physical Review Letters* 100.8 (Feb. 2008), p. 087205. DOI: [10.1103/PhysRevLett.100.087205](https://doi.org/10.1103/PhysRevLett.100.087205).
- [89] M. Jimbo. "Quantum R-Matrix for the Generalized Toda System." In: *Commun. Math. Phys.* 102 (1986), pp. 537–547.
- [90] M. Jimbo, T. Miwa, and M. Okado. "Solvable Lattice Models Whose States Are Dominant Integral Weights of  $A_{n-1}^{(1)}$ ." In: *Lett. Math. Phys.* 14 (1987), pp. 123–131. ISSN: 1573-0530. DOI: [10.1007/BF00420302](https://doi.org/10.1007/BF00420302).
- [91] M. Jimbo, T. Miwa, and M. Okado. "Local State Probabilities of Solvable Lattice Models: An  $A_{n-1}^{(1)}$  Family." In: *Nucl. Phys. B* 300 (1988), pp. 74–108.
- [92] V. Kac. *Highest Weight Representations Of Infinite Dimensional Lie Algebra*. English. Singapore: Wspc, Apr. 1988. ISBN: 978-9971-5-0396-3.
- [93] V. G. Kac and M. Wakimoto. "Branching Functions for Winding Subalgebras and Tensor Products." In: *Acta Applicandae Mathematica* 21.1 (Oct. 1990), pp. 3–39. ISSN: 1572-9036. DOI: [10.1007/BF00053290](https://doi.org/10.1007/BF00053290).
- [94] Leo P. Kadanoff. "Scaling Laws for Ising Models near  $T_c$ ." In: *Physics Physique Fizika* 2.6 (June 1966), pp. 263–272. DOI: [10.1103/PhysicsPhysiqueFizika.2.263](https://doi.org/10.1103/PhysicsPhysiqueFizika.2.263).
- [95] Leo P. Kadanoff. "Operator Algebra and the Determination of Critical Indices." In: *Physical Review Letters* 23.25 (Dec. 1969), pp. 1430–1433. DOI: [10.1103/PhysRevLett.23.1430](https://doi.org/10.1103/PhysRevLett.23.1430).
- [96] Leo P. Kadanoff. "More Is the Same; Phase Transitions and Mean Field Theories." In: *Journal of Statistical Physics* 137.5-6 (Dec. 2009), pp. 777–797. ISSN: 0022-4715, 1572-9613. DOI: [10.1007/s10955-009-9814-1](https://doi.org/10.1007/s10955-009-9814-1). arXiv: [0906.0653](https://arxiv.org/abs/0906.0653).
- [97] Shoichi Kanno, Yutaka Matsuo, and Shotaro Shiba. "Analysis of Correlation Functions in Toda Theory and the Alday-Gaiotto-Tachikawa-Wyllard Relation for  $SU(3)$  Quiver." In: *Physical Review D* 82.6 (Sept. 2010), p. 066009. DOI: [10.1103/PhysRevD.82.066009](https://doi.org/10.1103/PhysRevD.82.066009).
- [98] Christian Kassel, Marc Rosso, and Vladimir Turaev. *Quantum Groups and Knot Invariants*. Société Mathématique de France, 1997.
- [99] Albrecht Klemm and Michael G. Schmidt. "Orbifolds by Cyclic Permutations of Tensor Product Conformal Field Theories." In: *Physics Letters B* 245.1 (Aug. 1990), pp. 53–58. ISSN: 0370-2693. DOI: [10.1016/0370-2693\(90\)90164-2](https://doi.org/10.1016/0370-2693(90)90164-2).

- [100] V. G. Knizhnik. “Analytic Fields on Riemann Surfaces. II.” EN. In: *Communications in Mathematical Physics* 112.4 (1987), pp. 567–590. ISSN: 0010-3616, 1432-0916.
- [101] J. Kondev, J. de Gier, and B. Nienhuis. “Operator Spectrum and Exact Exponents of the Fully Packed Loop Model.” In: *J. Phys. A: Math. Gen.* 29 (1996), p. 6489.
- [102] H. A. Kramers and G. H. Wannier. “Statistics of the Two-Dimensional Ferromagnet. Part I.” In: *Physical Review* 60.3 (Aug. 1941), pp. 252–262. DOI: [10.1103/PhysRev.60.252](https://doi.org/10.1103/PhysRev.60.252).
- [103] Greg Kuperberg. “Spiders for Rank 2 Lie Algebras.” In: *Communications in Mathematical Physics* 180.1 (Sept. 1996), pp. 109–151. ISSN: 0010-3616, 1432-0916. DOI: [10.1007/BF02101184](https://doi.org/10.1007/BF02101184). arXiv: [q-alg/9712003](https://arxiv.org/abs/q-alg/9712003).
- [104] Nicolas Laflorencie. “Quantum Entanglement in Condensed Matter Systems.” In: *Physics Reports* 646.Supplement C (2016). Quantum entanglement in condensed matter systems, pp. 1–59. ISSN: 0370-1573. DOI: <https://doi.org/10.1016/j.physrep.2016.06.008>.
- [105] L. Landau. “The Theory of Phase Transitions.” en. In: *Nature* 138.3498 (Nov. 1936), pp. 840–841. ISSN: 1476-4687. DOI: [10.1038/138840a0](https://doi.org/10.1038/138840a0).
- [106] T. D. Lee and C. N. Yang. “Statistical Theory of Equations of State and Phase Transitions. II. Lattice Gas and Ising Model.” In: *Physical Review* 87.3 (Aug. 1952), pp. 410–419. DOI: [10.1103/PhysRev.87.410](https://doi.org/10.1103/PhysRev.87.410).
- [107] Ö Legeza and J. Sólyom. “Optimizing Momentum Space DMRG Using Quantum Information Entropy.” 2003.
- [108] Ö Legeza and J. Sólyom. “Quantum Data Compression, Quantum Information Generation, and the Density-Matrix Renormalization-Group Method.” In: *Phys. Rev. B* 70 (2004), p. 205118. DOI: [10.1103/physrevb.70.205118](https://doi.org/10.1103/physrevb.70.205118).
- [109] Elliott H. Lieb and Werner Liniger. “Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State.” In: *Physical Review* 130.4 (May 1963), pp. 1605–1616. DOI: [10.1103/PhysRev.130.1605](https://doi.org/10.1103/PhysRev.130.1605).
- [110] G. Mack and K. Symanzik. “Currents, Stress Tensor and Generalized Unitarity in Conformal Invariant Quantum Field Theory.” en. In: *Communications in Mathematical Physics* 27.4 (Dec. 1972), pp. 247–281. ISSN: 0010-3616, 1432-0916. DOI: [10.1007/BF01645514](https://doi.org/10.1007/BF01645514).
- [111] Santiago Migliaccio and Sylvain Ribault. “The Analytic Bootstrap Equations of Non-Diagonal Two-Dimensional CFT.” In: *arXiv:1711.08916 [hep-th]* (Nov. 2017). arXiv: [1711.08916 \[hep-th\]](https://arxiv.org/abs/1711.08916).

- [112] Alexi Morin-Duchesne, Jørgen Rasmussen, and Philippe Ruelle. “Integrability and Conformal Data of the Dimer Model.” In: *Journal of Physics A: Mathematical and Theoretical* 49.17 (2016), p. 174002.
- [113] Lars Onsager. “Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition.” In: *Physical Review* 65.3-4 (Feb. 1944), pp. 117–149. DOI: [10.1103/PhysRev.65.117](https://doi.org/10.1103/PhysRev.65.117).
- [114] T. Palmai. “Entanglement Entropy from the Truncated Conformal Space.” In: *Phys. Lett. B* 759 (2016), pp. 439–445. DOI: [10.1016/j.physletb.2016.06.012](https://doi.org/10.1016/j.physletb.2016.06.012). arXiv: [1605.00444](https://arxiv.org/abs/1605.00444) [[cond-mat.stat-mech](https://arxiv.org/archive/cond-mat)].
- [115] V. Pasquier. “Lattice Derivation of Modular Invariant Partition Functions on the Torus.” In: *J. Phys. A: Math. Gen.* 20 (1987), pp. L1229–L1237. DOI: [10.1088/0305-4470/20/18/003](https://doi.org/10.1088/0305-4470/20/18/003).
- [116] V. Pasquier. “Operator Content of the ADE Lattice Models.” en. In: *Journal of Physics A: Mathematical and General* 20.16 (1987), p. 5707. ISSN: 0305-4470. DOI: [10.1088/0305-4470/20/16/043](https://doi.org/10.1088/0305-4470/20/16/043).
- [117] V. Pasquier. “Continuum Limit of Lattice Models Built on Quantum Groups.” In: *Nucl. Phys. B* 295 (1988), pp. 491–510. ISSN: 0550-3213. DOI: [http://dx.doi.org/10.1016/0550-3213\(88\)90532-9](http://dx.doi.org/10.1016/0550-3213(88)90532-9).
- [118] Paul A. Pearce, Jørgen Rasmussen, and Jean-Bernard Zuber. “Logarithmic Minimal Models.” In: *J. Stat. Mech.* 0611 (2006), P11017. DOI: [10.1088/1742-5468/2006/11/P11017](https://doi.org/10.1088/1742-5468/2006/11/P11017). arXiv: [hep-th/0607232](https://arxiv.org/abs/hep-th/0607232) [[hep-th](https://arxiv.org/archive/hep)].
- [119] R. Peierls. “On Ising’s Model of Ferromagnetism.” en. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 32.3 (Oct. 1936), pp. 477–481. ISSN: 1469-8064, 0305-0041. DOI: [10.1017/S0305004100019174](https://doi.org/10.1017/S0305004100019174).
- [120] T. Kyle Petersen, Pavlo Pylyavskyy, and Brendon Rhoades. “Promotion and Cyclic Sieving via Webs.” In: *arXiv:0804.3375* [[math](https://arxiv.org/archive/math)] (Apr. 2008). arXiv: [0804.3375](https://arxiv.org/abs/0804.3375) [[math](https://arxiv.org/archive/math)].
- [121] V. B. Petkova and J.-B. Zuber. “From CFT to Graphs.” In: *Nucl. Phys. B* 463 (1996), pp. 161–193. DOI: [10.1016/0550-3213\(95\)00670-2](https://doi.org/10.1016/0550-3213(95)00670-2). eprint: [hep-th/9510175](https://arxiv.org/abs/hep-th/9510175).
- [122] Marco Picco, Sylvain Ribault, and Raoul Santachiara. “A Conformal Bootstrap Approach to Critical Percolation in Two Dimensions.” In: *SciPost Physics* 1.1 (Oct. 2016). ISSN: 2542-4653. DOI: [10.21468/SciPostPhys.1.1.009](https://doi.org/10.21468/SciPostPhys.1.1.009). arXiv: [1607.07224](https://arxiv.org/abs/1607.07224).
- [123] M. Plenio, J. Eisert, J. Dreissig, and M. Cramer. “Entropy, Entanglement, and Area: Analytical Results for Harmonic Lattice Systems.” In: *Phys. Rev. Lett.* 94 (2005), p. 060503. DOI: [10.1103/physrevlett.94.060503](https://doi.org/10.1103/physrevlett.94.060503).



- [124] A. M. Polyakov. “Kenneth Wilson in Moscow.” In: *arXiv:1502.03502 [hep-th, physics:physics]* (Feb. 2015). arXiv: [1502.03502 \[hep-th, physics:physics\]](https://arxiv.org/abs/1502.03502).
- [125] Alexander M. Polyakov. “Conformal Symmetry of Critical Fluctuations.” In: *JETP Lett.* 12 (1970), pp. 381–383.
- [126] M. A. Rajabpour and F. Gliozzi. “Entanglement Entropy of Two Disjoint Intervals from Fusion Algebra of Twist Fields.” In: *J. Stat. Mech.* 1202 (2012), P02016. DOI: [10.1088/1742-5468/2012/02/P02016](https://doi.org/10.1088/1742-5468/2012/02/P02016). arXiv: [1112.1225 \[hep-th\]](https://arxiv.org/abs/1112.1225).
- [127] N. Read and H. Saleur. “Exact Spectra of Conformal Supersymmetric Nonlinear Sigma Models in Two Dimensions.” In: *Nucl. Phys. B* 613 (2001), pp. 409–444. DOI: [10.1016/S0550-3213\(01\)00395-9](https://doi.org/10.1016/S0550-3213(01)00395-9). eprint: [hep-th/0106124](https://arxiv.org/abs/hep-th/0106124).
- [128] N. Y. Reshetikhin. “A New Exactly Solvable Case of an  $O(n)$  Model on a Hexagonal Lattice.” In: *J. Phys. A: Math. Gen.* 24 (1991), p. 2387.
- [129] Sylvain Ribault. “Conformal Field Theory on the Plane.” In: *arXiv:1406.4290 [hep-th, physics:math-ph]* (June 2014). arXiv: [1406.4290 \[hep-th, physics:math-ph\]](https://arxiv.org/abs/1406.4290).
- [130] Sylvain Ribault and Raoul Santachiara. “Liouville Theory with a Central Charge Less than One.” In: *Journal of High Energy Physics* 2015.8 (Aug. 2015). ISSN: 1029-8479. DOI: [10.1007/JHEP08\(2015\)109](https://doi.org/10.1007/JHEP08(2015)109). arXiv: [1503.02067](https://arxiv.org/abs/1503.02067).
- [131] J. E. Roman, C. Campos, E. Romero, and A. Tomas. *SLEPc Users Manual*. Tech. rep. DSIC-II/24/02 - Revision 3.9. D. Sistemes Informàtics i Computació, Universitat Politècnica de València, 2018.
- [132] V. Schomerus. “Rolling Tachyons from Liouville Theory.” In: *JHEP* 0311 (2003), p. 043.
- [133] Martin Schottenloher. *A Mathematical Introduction to Conformal Field Theory*. en. 2nd ed. Lecture Notes in Physics. Berlin Heidelberg: Springer-Verlag, 2008. ISBN: 978-3-540-68625-5.
- [134] Oded Schramm. “A Percolation Formula.” en. In: *Electronic Communications in Probability* 6 (2001), pp. 115–120. ISSN: 1083-589X. DOI: [10.1214/ECP.v6-1041](https://doi.org/10.1214/ECP.v6-1041).
- [135] Scott Sheffield. “Exploration Trees and Conformal Loop Ensembles.” EN. In: *Duke Mathematical Journal* 147.1 (Mar. 2009), pp. 79–129. ISSN: 0012-7094, 1547-7398. DOI: [10.1215/00127094-2009-007](https://doi.org/10.1215/00127094-2009-007).

- [136] Sheer El-Showk, Miguel F. Paulos, David Poland, Slava Rychkov, David Simmons-Duffin, and Alessandro Vichi. "Solving the 3D Ising Model with the Conformal Bootstrap." In: *Physical Review D* 86.2 (July 2012). ISSN: 1550-7998, 1550-2368. DOI: [10.1103/PhysRevD.86.025022](https://doi.org/10.1103/PhysRevD.86.025022). arXiv: [1203.6064](https://arxiv.org/abs/1203.6064).
- [137] Hidenori Sonoda. "Sewing Conformal Field Theories II." In: *Nuclear Physics B* 311.2 (Dec. 1988), pp. 417–432. ISSN: 0550-3213. DOI: [10.1016/0550-3213\(88\)90067-3](https://doi.org/10.1016/0550-3213(88)90067-3).
- [138] M. Srednicki. "Entropy and Area." In: *Phys. Rev. Lett.* 71 (1993), p. 666. DOI: [10.1103/physrevlett.71.666](https://doi.org/10.1103/physrevlett.71.666).
- [139] Jorg Teschner. "On the Liouville Three Point Function." In: *Phys. Lett.* B363 (1995), pp. 65–70. DOI: [10.1016/0370-2693\(95\)01200-A](https://doi.org/10.1016/0370-2693(95)01200-A). arXiv: [hep-th/9507109](https://arxiv.org/abs/hep-th/9507109) [hep-th].
- [140] F. Verstraete, J. I. Cirac, and V. Murg. "Matrix Product States, Projected Entangled Pair States, and Variational Renormalization Group Methods for Quantum Spin Systems." In: *Adv. Phys.* 57 (2008), p. 143. DOI: [10.1080/14789940801912366](https://doi.org/10.1080/14789940801912366).
- [141] G. Vidal, J.I. Latorre, E. Rico, and A. Kitaev. "Entanglement in Quantum Critical Phenomena." In: *Phys. Rev. Lett.* 90 (2003), p. 227902. DOI: [10.1103/physrevlett.90.227902](https://doi.org/10.1103/physrevlett.90.227902).
- [142] J. D. van der Waals. "Thermodynamische Theorie Der Kapillarität Unter Voraussetzung Stetiger Dichteänderung." In: *Zeitschrift für Physikalische Chemie* 13U.1 (1894), pp. 657–725. ISSN: 0942-9352. DOI: [10.1515/zpch-1894-1338](https://doi.org/10.1515/zpch-1894-1338).
- [143] S. O. Warnaar and B. Nienhuis. "Solvable Lattice Models Labelled by Dynkin Diagrams." In: *J. Phys. A: Math. Gen.* 26 (1993), pp. 2301–2316. DOI: [10.1088/0305-4470/26/10/005](https://doi.org/10.1088/0305-4470/26/10/005). eprint: [hep-th/9301026](https://arxiv.org/abs/hep-th/9301026).
- [144] J. Wess and B. Zumino. "Consequences of Anomalous Ward Identities." In: *Physics Letters B* 37.1 (Nov. 1971), pp. 95–97. ISSN: 0370-2693. DOI: [10.1016/0370-2693\(71\)90582-X](https://doi.org/10.1016/0370-2693(71)90582-X).
- [145] Steven R. White. "Density Matrix Formulation for Quantum Renormalization Groups." In: *Physical Review Letters* 69.19 (Nov. 1992), pp. 2863–2866. DOI: [10.1103/PhysRevLett.69.2863](https://doi.org/10.1103/PhysRevLett.69.2863).
- [146] Kenneth G. Wilson. "Non-Lagrangian Models of Current Algebra." In: *Physical Review* 179.5 (Mar. 1969), pp. 1499–1512. DOI: [10.1103/PhysRev.179.1499](https://doi.org/10.1103/PhysRev.179.1499).
- [147] Kenneth G. Wilson. "Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture." In: *Physical Review B* 4.9 (Nov. 1971), pp. 3174–3183. DOI: [10.1103/PhysRevB.4.3174](https://doi.org/10.1103/PhysRevB.4.3174).



- [148] Kenneth G. Wilson and Michael E. Fisher. "Critical Exponents in 3.99 Dimensions." In: *Physical Review Letters* 28 (Jan. 1972), pp. 240–243. ISSN: 0031-9007. DOI: [10.1103/PhysRevLett.28.240](https://doi.org/10.1103/PhysRevLett.28.240).
- [149] Edward Witten. "Global Aspects of Current Algebra." In: *Nuclear Physics B* 223.2 (Aug. 1983), pp. 422–432. ISSN: 0550-3213. DOI: [10.1016/0550-3213\(83\)90063-9](https://doi.org/10.1016/0550-3213(83)90063-9).
- [150] T. Xiang, J. Lou, and Z. Su. "Two-Dimensional Algorithm of the Density-Matrix Renormalization Group." In: *Phys. Rev. B* 64 (2001), p. 104414. DOI: [10.1103/physrevb.64.104414](https://doi.org/10.1103/physrevb.64.104414).
- [151] C. N. Yang and C. P. Yang. "Thermodynamics of a One-Dimensional System of Bosons with Repulsive Delta-Function Interaction." In: *Journal of Mathematical Physics* 10.7 (July 1969), pp. 1115–1122. ISSN: 0022-2488. DOI: [10.1063/1.1664947](https://doi.org/10.1063/1.1664947).
- [152] A. B. Zamolodchikov. "Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory." In: *Theor. Math. Phys.* 65 (1985), pp. 1205–1213. ISSN: 1573-9333. DOI: [10.1007/BF01036128](https://doi.org/10.1007/BF01036128).
- [153] A. B. Zamolodchikov and Al B. Zamolodchikov. "Structure Constants and Conformal Bootstrap in Liouville Field Theory." In: *Nuclear Physics B* 477.2 (Oct. 1996), pp. 577–605. ISSN: 05503213. DOI: [10.1016/0550-3213\(96\)00351-3](https://doi.org/10.1016/0550-3213(96)00351-3). arXiv: [hep-th/9506136](https://arxiv.org/abs/hep-th/9506136).
- [154] Al B. Zamolodchikov. "Conformal Symmetry in Two Dimensions: An Explicit Recurrence Formula for the Conformal Partial Wave Amplitude." en. In: *Communications in Mathematical Physics* 96.3 (Sept. 1984), pp. 419–422. ISSN: 0010-3616, 1432-0916. DOI: [10.1007/BF01214585](https://doi.org/10.1007/BF01214585).
- [155] Al. B. Zamolodchikov. "Conformal Scalar Field on the Hyperelliptic Curve and Critical Ashkin-Teller Multipoint Correlation Functions." In: *Nuclear Physics B* 285 (Jan. 1987), pp. 481–503. ISSN: 0550-3213. DOI: [10.1016/0550-3213\(87\)90350-6](https://doi.org/10.1016/0550-3213(87)90350-6).
- [156] Al. B. Zamolodchikov. "On the Three-Point Function in Minimal Liouville Gravity." In: *Theor. Math. Phys.* 142 (2005), p. 183.
- [157] P. Zinn-Justin. "Nonlinear Integral Equations for Complex Affine Toda Models Associated with Simply Laced Lie Algebras." In: *J. Phys. A: Math. Gen.* 31 (1998), p. 6747.

COLOPHON

*"- Yet he would not die lying down; he would find some crag of rock, and there, his eyes fixed on the storm, trying to the end to pierce the darkness, he would die standing. He would never reach R. "*

*Virginia Woolf, To the Lighthouse*