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Chern classes of coherent analytic sheaves: a simplicial approach

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Timothy Hosgood. Chern classes of coherent analytic sheaves: a simplicial approach. Mathematics [math]. Université d'Aix-Marseille (AMU), 2020. English. NNT: . tel-02882140

HAL Id: tel-02882140

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CHERN CLASSES OF COHERENT ANALYTIC SHEAVES

A SIMPLICIAL APPROACH

présentée par
Timothy Hosgood

pour obtenir le grade universitaire
de docteur de l'université d'Aix-Marseille

Soutenue le 25 juin 2020
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Abstract

The aim of this thesis is to review and improve upon an unpublished thesis by Green, whose goal was to construct Chern classes of coherent *analytic* sheaves in de Rham cohomology that respect the Hodge filtration.

The second part of this thesis is dedicated to the construction of a categorical enrichment of the bounded derived category of complexes of coherent sheaves on an arbitrary complex manifold: the objects are ‘simplicial’ vector bundles endowed with a certain type of simplicial connection. This construction uses the theory of twisting cochains, developed in this setting by O’Brian, Toledo, and Tong.

The first part is dedicated to defining a categorical lift of the Chern character in de Rham cohomology that respects the Hodge filtration, and for this we use the categorical model mentioned above. This construction can be undertaken by adapting classical Chern-Weil theory to the simplicial setting, using Dupont’s theory of fibre integration.

Résumé

L’objet de cette thèse est de réinterpréter et de poursuivre un travail non publié de Green, dont l’objet est de construire des classes de Chern pour les faisceaux *analytiques* cohérents à valeurs dans la cohomologie de de Rham en respectant la filtration de Hodge.

La seconde partie de la thèse est consacrée à la construction d’un enrichissement catégorique de la catégorie dérivée bornée des complexes de faisceaux cohérents sur une variété complexe arbitraire: les objets considérés sont des fibrés vectoriels « simpliciaux » munis d’un type spécial de connexions simpliciales. Cette construction repose sur la théorie des cochaînes tordues, développée dans ce cadre par O’Brian, Toledo, et Tong.

La première partie est consacrée à définir un relèvement catégorique via le modèle précédent d’un caractère de Chern en cohomologie de de Rham respectant la filtration de Hodge. Cette construction peut être réalisée en adaptant la théorie de Chern-Weil classique au cadre simplicial, via la théorie de l’intégration fibrée de Dupont.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 768679).

Pre-thanks. Before thanking the people in my life, I would like to thank two people who I have never met: Piper Alexis Harron, and Eugenia Cheng. The former, for writing a thesis that challenged the norm of what a thesis in mathematics 'should be'; the latter, for giving me the language necessary to realise that I might not be terrible at being a mathematician, but just terrible at being an *ingressive* mathematician.

Thanks

A PhD student without their supervisor is like a headless chicken. A PhD student with their supervisor is still like a headless chicken, but now directed by, and under the care of, somebody infinitely more capable of knowing where to go. Unfortunately, not having a head means that the student is still somewhat lacking in the ear department, and so most of the sage advice and directions are not fully appreciated. A good supervisor is somebody talented in the art of herding headless chickens, knowing both the lie of the land and the best way to make themselves understood to the anxious, lost chicken. I was lucky enough to have two such supervisors, and to both of them I give my sincerest thanks, as well as my genuine apologies for being such a terrible student before 2pm on any given day for just over three and a half years. Your patience has honestly been deserving of a medal.

My sincere thanks also go to all the members of the jury, who found the time during such tumultuous weeks to support my thesis defence, and particular thanks are necessary for the reporters, whose suggestions were both thoughtful and constructive.

In my search for a good construction of characteristic classes of coherent sheaves using simplicial methods I realised that the real characteristic classes of coherent sheaves were the friends that I made along the way. To that note, I must extend my deepest and warmest of thanks to Jordan and Benoît, who made my time in Marseille indubitably better than it would have been without them, as well as to (amongst others) Edouard, Juliana, my various flatmates, the many lovely baristas of Marseille and Montpellier (Kasia, Jack, and all the Coffee Club crew), the many lovely avian species of Marseille and Montpellier, the puffins of Skomer, and many others whose names comprise a list too long to fit in my memory, let alone the pages of this thesis. Since, however, this is the one section of the thesis where I can really indulge myself, I will try to name at least a few more names while we're here.

Before naming names though, I would like to say thank you to all of the (extremely nice) people on maths Twitter (especially the actegory theorists), who,

many times, pointed me in the direction of good references, told me which formalism I was trying in vain to describe, and said nice things about a big commutative diagram that took me a long time to draw (which has since been removed from this thesis, sadly). The whole community there has been genuinely supportive, as well as mathematically helpful.

Long before coming to France, I was a student at university in England, where I met some of the most supportive and caring (and amusing) friends that you could ever wish to meet. And long before even that, when at college, and school even, I had other wonderful friends, and, in particular, some incredible teachers. I highly doubt that any of them will ever see these words, but I am indebted beyond belief, so I want to thank them nonetheless. The problem with naming names is that any names not mentioned seem to be intentionally omitted: I promise that this is not the case. Conversely, I hope that people included in this list don't devalue this because of how long the list is: I haven't just included everybody for the sake of it.

So, to name but a few names, thank you to the two Sues who put up with me every Wednesday afternoon for hours, and really ensuring that I would end up studying maths at university; to Brian Edge, for starting me off on my mathematical journey; to Julie Lewis, for preparing me for life abroad; to the two Mr. Whites, for ensuring that I didn't become a mathematical recluse, and Mr. Robilliard, for (unknowingly, I think) starting a passion for jazz that would lead to so many great moments later on in my life; to Diana Percy, for her folk music and calming wisdom; and to Andy Waters, for being the much-needed voice telling me to actually apply for mathematics at university. Then thank you to Alan Lauder, in part for his teaching, but mainly for his brutally honest (but kind) advice and personal reference; to David Seifert for broadening my mathematical interests and being a constant support during low points; to Balázs Szendrői, for supervising my first real project, with more patience than any one person should have to possess; to Kobi Kremnitzer and Christopher Hollings, for helping me to write the thesis that I really wanted to write; and to an unnamed professor or two, who wrote exams hard enough that I was forced to learn how to deal with abysmal failure, as well as an intense frustration at an often deeply anachronistic and uncaring educational system, which made me much more aware of the sort of mathematician that I do not wish to become. Thank you, then, to (and here I work in alphabetical order on an incomplete list) Alice, Alice, Aoife, Aiyan, Ephrem, Evie, Frankie, Frizzy, Jamie, KP, pNaomi, Sean, Simpkin, Tom, Val, all the porters, Jack From The Bar, and many more besides — you truly made those four years some of the best possible. Thank you Alyssa for all your support.

Finally, then, to family. I am so grateful to Chrys and Mike Aitken, for encouraging me to pursue academia, and providing me with much needed poetry whilst growing up. Also to Vera and Michael Saffin, for the countless laughs and stories, and for being responsible adopted godparents (and thank you especially to Michael for putting such heavy rocks in my bag that one time). Thank you to

all my grandparents, even those I never met, but thank you especially to Nan, for being exactly the type of person I want to be when I grow older (but not up).

Thank you to Andrew, for handing me down notes from his programming courses at university, teaching me about physics and the intricate and delicate subtleties of different types of wood, and for making me laugh more than almost anyone I know. I really cannot express in words how much I hope that you one day lose just one of the trouser legs of your zip-off trousers. Thank you to Emma, for letting me crash all of her parties, being a true rock when I most needed it, and for literally keeping me in her room for hours at a time so that she could pretend to be a teacher to me (and, of course, teaching me all that I know about mathematics). There is nobody else with whom I would have preferred to experience the dire pubs of our home town.

To Dad and Deb, for teaching me cryptic crosswords, taking me to see birds (and parts of the country that I would never have otherwise seen), and always making sure that I had a relaxing haven in the middle of home and wherever I currently was. I am also grateful for (and in awe of) my Dad's ability to know exactly which song/town/person/word I am thinking of when I am suffering from lethologica (or just my usual terrible memory), with absolutely no clues needed. To Mum and Rich, for sacrificing so much to get me to where I am today, and for making sure that I always have a home. To Mum, thank you for teaching me about flowers, and for being the embodiment of the hen who dreamed she could fly. To Rich, thank you for taking me to my first gig (and the many that have followed), and for calling me today just to tell me about a funny pigeon that you saw. All together, you are the very best of parents, and have given me more support and love than I could ever have thought possible. I love you all, very dearly.

I must also thank, in a rather bizarre turn of events, the man in Bath who enticed his friend (a pigeon) to sit on my head for a while. It is no exaggeration to say that he changed my life for the better.

At lunchtime I bought a huge orange —
The size of it made us all laugh.
I peeled it and shared it with Robert and Dave —
They got quarters and I got a half.

And that orange, it made me so happy,
As ordinary things often do
Just lately. The shopping. A walk in the park.
This is peace and contentment. It's new.

The rest of the day was quite easy.
I did all of the jobs on my list
And enjoyed them and had some time over.
I love you. I'm glad I exist.

— *Wendy Cope, "The Orange"*.

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Having all the answers just means you've been asking boring questions.
— *Joey Comeau and Emily Horne.*

I. INTRODUCTION

The past and the present

1

I don't wanna talk
About things we've gone through;
Though it's hurting me
Now it's history.

ABBA, "*The Winner Takes It All*".

In 1980, H.I. Green, a student of O'Brian and Eells, wrote his thesis [Gre80] on the subject of Chern classes of coherent sheaves on complex-analytic manifolds. Although the thesis was never published, an exposition was given in [TT86], alongside a sketch of a proof of the Hirzebruch-Riemann-Roch formula for this construction of Chern classes. It combined the theory of twisting cochains, used with great success by Toledo, Tong, and O'Brian in multiple papers ([TT76; TT78; OTT81; OTT85]), with the fibre integration of Dupont ([Dup76]), to construct, from a coherent analytic sheaf, classes in $H^{2k}(X, \Omega_X^{\bullet \geq k})$ that coincide with those given by the classical construction of Chern classes in $H^{2k}(X, \mathbb{Z})$ by Atiyah-Hirzebruch ([AH62]). (It is of historical interest to mention that the approach of expressing characteristic classes in terms of transition functions is very much in line with ideas propounded by Bott; see e.g. the section entitled 'Concluding Remarks' in [BT82, §23].) This construction was considered by Grivaux in his thesis [Gri09], where he constructs unified Chern classes for coherent analytic sheaves (on *compact* analytic manifolds) in Deligne cohomology, and where he states an axiomatisation of Chern classes that ensures uniqueness in any sufficiently nice cohomology theory (of which de Rham cohomology is an example). Although he states that the Grothendieck-Riemann-Roch theorem for closed immersions is not known for Green's construction of Chern classes if X is non-Kähler, this turns out to not be a problem, since it follows from his other axioms by a purely formal, classical argument, involving deformation to the normal cone.

One reason that the study of Chern classes of coherent analytic sheaves is interesting is that it is notably less trivial than the algebraic version. In both the analytic and algebraic settings, Chern classes of locally free sheaves can be constructed by the splitting principle (as explained in e.g. [BT82, §21]) in the 'most general' cohomology theories (Deligne-Beilinson cohomology and Chow rings, respectively); but, although coherent algebraic sheaves admit global locally free resolutions, the same is *not* true of coherent analytic sheaves. In general, complex manifolds have very few holomorphic vector bundles, and there are whole classes of examples of coherent sheaves that do not admit a global locally free resolution ([Voi02, Corollary A.5]). One key insight of [Gre80], however, was that the holomorphic twisting resolutions of Toledo and Tong (whose existence was guaranteed by [TT78, Proposition 2.4]) could be used to construct a global resolution

by ‘*simplicial* locally free sheaves’, or **locally free sheaves on the nerve**: objects that live over the Čech nerve $X_{\bullet}^{\mathcal{U}}$ of a cover \mathcal{U} of X . The existence of such a global resolution, glued together from local pieces, is mentioned in the introduction of [HS01] as a problem that should be amenable to the formal theory of descent. Indeed, these ‘simplicial sheaves’ can be constructed by taking the lax homotopy limit (in the sense of [Ber12, Definition 3.1]) of the diagram of model categories given by the pullback-pushforward Quillen adjunctions along the nerve of a cover of X . One very useful example of such an object is found by pulling back a global (i.e. classical) vector bundle to the nerve: given some $E \rightarrow X$, defining \mathcal{E}^{\bullet} by $\mathcal{E}^p = (X_p^{\mathcal{U}} \rightarrow X)^*E$. This actually satisfies a ‘strongly cartesian’ property: it is given by the ‘strict’ (i.e. not lax) homotopy limit of [Ber12]. The twisting cochains from which Green builds these resolutions are also interesting objects in their own right, having been studied extensively by Toledo, Tong, and O’Brian, as previously mentioned. In fact, they can be seen as specific examples of the twisted complexes of [BK91], which are used to pretriangulate arbitrary dg-categories. This gives a possible moral (yet entirely informal) reason to expect the existence of resolutions such as Green’s: twisted complexes give the ‘smallest’ way of introducing a stable structure on a dg-category, and perfect \mathcal{O}_X -modules can be defined as exactly the objects of the ‘smallest’ stable $(\infty, 1)$ -category that contains \mathcal{O}_X and is closed under retracts. Alternatively, one can appeal to [Wei16], which shows that, under certain restrictions on (X, \mathcal{O}_X) , twisting cochains constitute a dg-enrichment of the derived category of perfect complexes.

Another problem in trying to apply Chern-Weil theory to coherent analytic sheaves is that global holomorphic (Koszul) connections rarely exist: the Atiyah class (whose trace coincides^[1] with the first Chern class in cohomology) measures the obstruction of the existence of such a connection. The other main result of Green’s thesis is the construction of ‘simplicial connections’, which are connections on ‘simplicial sheaves’ pulled back along the projection $X_{\bullet}^{\mathcal{U}} \times \Delta^{\bullet} \rightarrow X_{\bullet}^{\mathcal{U}}$. The idea behind this construction is powerfully simple: given local connections ∇_{α} (which always exist) on a locally-free sheaf E (that is, ∇_{α} is a connection on $E|_{U_{\alpha}}$), on any intersection $U_{\alpha\beta}$ we consider the path $\gamma_{\alpha\beta}(t) = t\nabla_{\beta} + (1-t)\nabla_{\alpha}$ between the two local connections as some type of ‘connection’ on $X_1^{\mathcal{U}} \times \Delta^1$. More generally, on p -fold intersections $U_{\alpha_0 \dots \alpha_p}$, we can consider the ‘connection’ $\sum_{i=0}^p t_i \nabla_{\alpha_i}$ on $X_p^{\mathcal{U}} \times \Delta^p$. These objects then assemble to give us what might deserve to be called a **simplicial connection**. Green shows that we can take the curvature of such things, which consists of $\text{End}(E_p)$ -valued forms on $X_p^{\mathcal{U}} \times \Delta^p$; by certain technical properties of the sheaves in his resolution, Green shows that these forms satisfy the property necessary to define a simplicial differential form (the same property as found in the equivalence relation defining the fat geometric realisation of a simplicial space), which lets us apply Dupont’s fibre integration (after taking the trace, or evaluating under some other invariant polynomial) to recover (Čech

^[1]See [Huy05, Exercise 4.4.8], as well as the comments just below Exercise 4.4.11.

representatives of) classes in de Rham cohomology. One thing that could be considered as missing from Green's thesis is a formal study of simplicial connections, and so this forms one of the key parts of this paper. It is possible to define simplicial connections in a more general setting, and study conditions that ensure that Chern-Weil theory can be applied (these give the notions of **admissibility**, and being **generated in degree zero**). Green's connections do indeed satisfy these formal conditions, and this provides a more rigorous reasoning for their usefulness.

Finally, as $(\infty, 1)$ -categories (presented by homotopical categories), modulo some subtleties in the definitions, complexes of sheaves with coherent cohomology are equivalent to the homotopy colimit of so-called **Green complexes** endowed with simplicial connections generated in degree zero ((10.2.9)). This means that applying Chern-Weil theory to Green complexes does indeed give us a working version of Chern-Weil theory for complexes of sheaves with coherent cohomology.

All I do, is sit down at the typewriter, and start hittin' the keys. Getting them in the right order, that's the trick. That's the trick.

Garth Marengi.

2.1 For mathematicians | Pour les mathématiciens

Chapter 2. This current chapter provides various summaries of this thesis: for mathematicians, for non-mathematicians, and for mathematicians who don't wish to read all the thoughts in my head that made it onto paper.

Chapter 1. In an attempt to place this work into context, we give here a short history of the subjects treated in this thesis.

Chapter 3. Simplicial methods, in particular, are rife with different conventions of notation and nomenclature; we lay out those that we have decided to use.

Chapter 4. The main objects of study are the Atiyah classes of a holomorphic vector bundle; simplicial differential forms; and locally free sheaves on a simplicial space. We define all these, as well as some other constructions that we will later use.

Chapter 5. The trace of the so-called 'exponential' Atiyah classes (which correspond to exponential Chern classes) can be manually lifted to closed elements in the Čech-de Rham bicomplex, and thus correspond to closed classes in de Rham cohomology. We give the construction by hand for the first four cases, but it is messy, and so prompts us to look for a slicker method.

Chapitre 2. Ici on résume cette thèse trois fois: pour les mathématiciens; pour ceux qui ne sont pas des mathématiciens; et pour les mathématiciens qui ne veulent pas lire mon courant de conscience qui accompagne ce que j'écris partout ailleurs.

Chapitre 1. Afin de donner un peu de contexte, on résume brièvement les sujets traités dans cette thèse.

Chapitre 3. Les méthodes simpliciales étant particulièrement lourdes en conventions de notation et nomenclature, on les grave dans le marbre dès le début.

Chapitre 4. Les objets centraux dans cette thèse sont les classes d'Atiyah d'un fibré vectoriel holomorphe, les formes différentielles simpliciales, et les faisceaux localement libres sur un espace simplicial. On donne, entre autres, toutes ces définitions.

Chapitre 5. Le trace des classes d'Atiyah « exponentielles » (qui correspondent aux classes de Chern exponentielles) peuvent être relevées aux éléments fermés du bicomplexe Čech-de Rham et correspondent ainsi aux éléments fermés de la cohomologie de de Rham. On donne une construction à la main pour les quatre premiers cas mais, celle-ci devenant très vite complexe, nous sommes poussés à trouver une méthode plus habile.

Chapter 6. Green's thesis describes a 'barycentric connection', which, via a simplicial version of Chern-Weil, gives us Atiyah classes directly as closed elements in some simplicial Čech-de Rham bicomplex; Dupont's fibre integration then gives us the required quasi-isomorphism to de Rham cohomology. This is mainly just a summary of Green's thesis, but stated in a way that makes it amenable to be placed in a formal framework (which we will introduce in the next chapter).

Chapter 7. This gives the formal framework mentioned above: how to define simplicial connections, how to generalise Chern-Weil theory to simplicial connections and simplicial differential forms, and how to consider Green's barycentric connection as a particular example.

Chapter 8. With the construction of Chern classes in tDR cohomology for vector bundles now under our belt, we recall O'Brian, Toledo, and Tong's twisting cochains, and how they can be thought of as local resolutions of coherent sheaves.

Chapter 9. Green's second main result is that a twisting cochain can be 'locally strictified' somehow, resulting in a resolution by locally free sheaves on the nerve; such resolutions also satisfy various nice properties. We summarise all of this here.

Chapter 10. Finally, then, we fill in the holes and piece together the previous chapters, seeing how to construct Chern classes in tDR cohomology for coherent sheaves. We show that, in the compact case, it agrees with any other construction that you might wish to use.

Part IV. In these appendices we give a bunch of (moderately concise) expository and explanatory notes on various subjects mentioned throughout this thesis.

Chapitre 6. La thèse de Green décrit une « connexion barycentrique » qui donne les classes d'Atiyah comme éléments fermés dans une version simpliciale du bicomplexe Čech-de Rham, via une version simpliciale de Chern-Weil; puis l'intégration fibrée de Dupont donne le quasi-isomorphisme à la cohomologie de de Rham requis. Le but est de donner un résumé de la thèse de Green, exprimé de façon plus propice à sa formalisation (qui sera introduit dans le prochain chapitre).

Chapitre 7. On explique le cadre formel mentionné au-dessus, c'est-à-dire comment définir les connexions simpliciales, comment généraliser la théorie de Chern-Weil aux connexions simpliciales et aux formes différentielles simpliciales, et comment considérer la connexion barycentrique de Green comme un exemple particulier.

Chapitre 8. À l'aide de la construction de classes de Chern dans la cohomologie de tDR pour les fibrés vectoriels, on rappelle à présent les cochaînes tordues de O'Brian, Toledo, et Tong, et le point de vue de résolutions locales des faisceaux cohérents.

Chapitre 9. Le deuxième résultat principal de Green est le fait qu'une cochaîne tordue peut être « localement strictifiée » pour obtenir une résolution par des faisceaux localement libres sur le nerf; ce type de résolutions satisfont quelques bonnes propriétés.

Chapitre 10. Enfin, on combine les résultats des chapitres précédents afin de pouvoir construire les classes de Chern dans la cohomologie de tDR pour les faisceaux cohérents. On montre que, dans le cas compact, cette construction coïncide avec n'importe quelle autre.

Partie IV. Ces appendices servent à expliquer brièvement quelques sujets supplémentaires évoqués au travers de cette thèse.

Part V. In *these* appendices we are often much less coherent and precise (and much more prone to rambling) than we probably should be: there are formal statements of various definitions which we have made use of, as well as mild explanations, but we tend to be content with simply providing the reader with the necessary keywords to find better references than this one. The odd one out, as it were, is Appendix G, which is, for the most part, pretty formal and self-contained. In general, these chapters should be treated with the utmost suspicion and doubt.

Partie V. Ces appendices sont moins précises et tendent à se répéter : malgré la présence de quelques définitions formelles, elles consistent surtout en une liste de mots clés auxquels le lecteur pourra se référer afin de trouver une meilleure source que la présente thèse. L'intrus, pour ainsi dire, est l'appendice G qui est en grande partie assez formel et autocontenu. Dans l'ensemble, ces chapitres devraient être lus avec méfiance et scepticisme.



2.2 For mathematicians in a hurry

Most of the mathematical content of this thesis can be found in a more streamlined format in [Hos20a; Hos20b].

Vector bundles

- The Atiyah class of a holomorphic vector bundle can be iterated to get higher Atiyah classes; these correspond to higher Chern classes. (4.1.21) and (4.1.22)
- The traces of the Atiyah classes admit lifts to the Čech-de Rham bicomplex that result in closed classes in de Rham cohomology. These can be constructed by hand, but the method is 'messy'. (5.1.1), (5.1.2), and (5.1.4)
- Lifting to a simplicial version of the Čech-de Rham bicomplex can be done by considering a simplicial version of the Atiyah classes; Dupont's fibre integration then gives us a quasi-isomorphism into de Rham cohomology. This was originally done in Green's thesis. (6.1.8) and (6.3.1)
- The idea of 'simplicial connections' can be formalised, giving a more general categorical framework, which contains Green's barycentric connection as a particular example. Chapter 7

Coherent sheaves

- The twisting cochains of O'Brian, Toledo, and Tong can be used to construct resolutions of coherent sheaves by 'simplicial vector bundles' (although these are *not* just simplicial objects in the category of vector bundles); these resolutions have particularly nice properties. This is exactly a result of Green. (9.1.2)

- There is an equivalence of categories (indeed, there are many) that prove useful in combining our formalisation of simplicial connections with Green’s resolutions. (10.2.9)

2.3 For non-mathematicians

I get much more respect than I deserve. [...] I’ve developed a great reputation for wisdom by ordering more books than I ever had time to read, and reading more books, by far, than I learned anything useful from, except, of course, that some very tedious gentlemen have written books.

Marilynne Robinson, “Gilead”.

It is very easy to be intimidated by these tens or hundreds of pages of mathematics, given the vast quantity of symbols and words that seem to be entirely unexplained, as if written in a foreign language that seems to only passingly resemble English at times. I’m not going to try to explain my entire thesis using non-mathematical language, for two reasons: (1) other people have already done much better jobs at explaining these sorts of things; and (2) if you really want to know, then ask me! But I will try to give some vague approximation to what’s going on here, as well as some explanation as to what it actually is that *I’ve* done.

For this latter point, the answer is sort of ‘not much’, and sort of ‘quite a bit’. A lot of mathematics relies on building on existing results that other people have discovered; it’s incredibly rare that somebody comes along and just ‘invents’ a whole branch of maths by themselves. What I’ve done is read somebody else’s PhD thesis, written in 1980 by a person named H.I. Green, that was never published, and so never really expanded on by anybody else. The thesis gives a way of constructing certain ‘numbers’ (they’re not really numbers, but sort of a generalisation of numbers) associated to certain mathematical objects (*coherent sheaves*) that works, in that it gives the numbers we want for simple objects, for which we can calculate the numbers in easier ways, and numbers that make sense for more complicated objects, that agree with what we want/expect to be true. There are a *lot* of actual algebraic calculations^[1] in this older thesis, which is great, because it tells us how to actually *do* the construction, but it’s not very easy to understand *what* we’re actually doing, conceptually. Think of it like long division: it’s a really concrete way of actually *calculating* how to divide one number by another, but if you just saw a bunch of long division examples then it wouldn’t be necessarily obvious that we were trying to figure out the simple idea of ‘how many times does this small number go into this bigger number’; complicated algorithms can

^[1]Like the ones I’ve tried to explain a bit in Section 8.3, if you want to see what they look like.

obscure the simpler ideas behind them. So my job was to try to unravel this construction and figure out what might be going on behind the scenes. There were some nice hints in the thesis as to what might be secretly happening, but there were also a bunch of things that were not really explained (sentences like ‘next, we do this’, without any indication as to *why* we would want to, or even think of, doing such a thing). This meant that my supervisors and I had to think of possible explanations, check to see if they made sense, and then to see if they actually gave the same answers. By trying to explain what’s going on here, we managed to come up with a framework that’s a bit more general than what happens in the older thesis, and so maybe in the future somebody can read this thesis and apply some of the results to other examples. This is what mathematicians hope for! Of course, I also had to do some hard calculations myself, and these really are, in essence, not much more than solving algebraic equations: take a look at Section 5.2.

As for what the maths in this thesis is actually about, I think it’s more interesting to describe certain aspects in more details than others. The main result is a construction of *characteristic classes of coherent sheaves in truncated de Rham cohomology*, so let’s see what each of those three main ideas are about.

Coherent sheaves are particularly ‘well-behaved’ sheaves, so the question reduces to ‘what is a sheaf?’. There are many good answers to this, but the one that best fits the ideas in this thesis is that *a sheaf is a space that lives over another space in a nice way*. By ‘space’, I mean something that ‘has some dimensions’, or something that ‘things can move through’. Looking around you, you seem to be living in some three-dimensional space (you can move up/down, forwards/back, and side-to-side); but we could imagine something more abstract, like imagining the current temperature in Bideford, Devon as a small little point that lives in some one-dimensional space (that is, it moves about on a line, exactly how the level of liquid in a thermometer moves). There are even more abstract notions of space than this, but that’s not enough for mathematicians: we want an idea of *generalised spaces* (another name for sheaves). Imagine being in a house, which is sat on top some area of land. I could describe where I am in the house by saying something like ‘in the small cupboard next to the upstairs landing’, but I could also describe where I am by giving you exactly two pieces of information: (1) where I am in relation to the plot of land (i.e. two feet down and four feet left of the top right corner), and (2) what floor of the house I’m in. That is, a house with an upstairs and a downstairs ‘looks exactly like’ (in this mathematical sense) two copies of the plot of land on which it’s built (or even two copies of, say, the downstairs); a house with three floors ‘looks exactly like’ three copies of the land on which it’s built; and so on. This is sort of what a *vector bundle* is (it’s really the most simple kind: a *trivial* vector bundle), and a combined result of my thesis and Green’s thesis, says that coherent sheaves are built up, in a sort of complicated way, from these such things. Anywho, nobody is going to test you on these things, so let’s just move on.

Characteristic classes are pretty easy to vaguely explain: they’re like numbers that we associate to sheaves that tell us things about the sheaf, like how many

holes it has or how boring it is or how much it looks like two other sheaves stuck together. When I say *like* numbers, I mean that they actually were, historically, defined as just numbers, but then people (namely, Emmy Noether) realised that these numbers were actually just the shadows cast by more structured mathematical objects. This becomes a bit easier to understand when we have more of an idea of what *cohomology* is, because characteristic classes are ‘just’ *elements of cohomology*.

So then, what is cohomology? Again, there are a bunch of answers, and many mathematicians study pretty much nothing but cohomology, in its many various disguises, so I won’t try to be at all complete here. In essence though, we can think of it as ‘hole counting’. This is a great area of mathematics, because it starts out so easy and nice, but turns around and says ‘aha, gotcha’, very quickly. Take a piece of paper and, using a hole punch, punch a hole in it. How many holes are there? Well, one, because we just punched one. Punch a few more holes and you have a few more holes. But paper is a bit boring, because it’s flat (basically two-dimensional), so let’s do something a bit more exciting. There are two main types of doughnuts: jam ones, and ring ones.^[1] Imagine a jam doughnut being made in the jam doughnut factory, at the moment just before they put the jam in it.^[2] How many holes are there in the doughnut? Well, there’s one: it’s like a balloon, or a football at this stage, because it just has some air in the middle that’s waiting to be replaced by jam. But now, how many holes are there in a ring doughnut? The answer is also, ‘obviously’, just one: there’s the hole in the middle, just like in a hula hoop, or a ring. But there’s something different about these holes in our doughnuts: one of them we could put our finger through, but the other one is invisible (hidden inside the doughnut itself); you could tie a ring doughnut to a post to stop it from running away, but you’d struggle to do so with a jam doughnut. This is where the idea of ‘hole counting’ needs something more precise than just numbers: it’s not enough to say just *how many* holes there are; we also need to describe *what sort* of holes there are. In fact, you could even imagine filling the not-a-hole part of a ring doughnut with jam, and then sucking out all the jam, to give you some weird shape that has *both* types of holes in it. This is sort of what cohomology helps us to formalise.

(2.3.1) For non-mathematicians currently sat in my thesis defence. If you are some poor family member or friend of mine who has felt that you should come to my thesis defence, you’re probably going to be rather bored after the first few minutes. You will likely spend over an hour having to listen to people talk about maths things without really understanding much, if anything at all, of what they’re saying (bonus points if it’s in French too), but this gives you a chance to

^[1]I think that one is called a donut and the other a doughnut, but I’m not sure if that’s actually true or not.

^[2]N.B. I don’t know how doughnuts are really made, and I imagine that what I’m about to say is wrong, but the mathematical idea behind it is still valid.

experience my life: being completely lost for hours at a time is pretty much what going to seminars often feels like. Try amusing yourself by playing ‘thesis defence bingo’, making a note every time you hear one of the following: *obviously*, *trivially*, or *clearly*^[1]; *by finiteness/compactness/Stein-ness/exactness/functoriality*; *by Cartan’s Theorem B*; anything to do with *Green*; the sentence *these ‘things’ are not just simplicial objects in the category of ‘things’*; any mention of *higher* or *infinity* categories; me saying *oh, I forgot to mention that ...*; some thinly veiled reference or ‘joke’ about birds; something that must be a joke because some of the mathematicians laughed at it but you don’t really know what exactly was so funny; the years 1976 and 1978; an excuse about how *I’m not an algebraic geometer*; an excuse about how *I’m not a category theorist*; slowly saying *that’s a really good question* as a way to give me time to think when somebody asks me a question; or, lastly, but certainly not least-ly, somebody falling asleep (something that happens reasonably often, or so I am told).

Alternatively, read the previous section and play some games with the ideas there. How many holes are there in the earth? How many holes are there in a straw: one big one, or two (one at each end)? An open bottle? In a pair of trousers? A shirt? Given your answers to the above, do they make sense with one another? For example, a bottle looks like a fat straw with one of the holes sealed over, so it should probably have one fewer hole than a straw; a pair of trousers looks like two straws joined together by stapling their top holes together, and then adding a little belt on top, so maybe a pair of trousers should have $2s - 1$ holes, where s is the number of holes in a straw; a shirt looks like a pair of trousers with another hole, or two straws smooshed together in a cross shape, with where they crossed being replaced with a jam-doughnut-type hole. And so on.

Counting holes has helped me while away many a dull hour. Welcome to my life.

^[1]These all mean ‘please don’t ask me for more details because I’m not sure I’ll be able to give them’.

Conventions

3

Clichés can be quite fun. That’s how they got to be clichés.

Alan Bennett, "The History Boys".

(3.0.1) Assumptions. Throughout this entire thesis, X is a complex-analytic paracompact manifold, and any cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ will always be locally finite and Stein.

(3.0.2) Simplicial conventions.

- Δ is the *simplex category*:
 - the objects are the finite ordinals $[p] = [0, 1, \dots, p-1, p]$ for $p \in \mathbb{N}$;
 - the morphisms are the order-preserving maps;
 - we have, for $i \in \{0, \dots, p-1\}$, the *coface*^[1] maps

$$f_p^i: [p-1] \rightarrow [p]$$

$$j \mapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

which are the injections that ‘skip over’ i ;

- we have, for $i \in \{0, \dots, p\}$, the *codegeneracy* maps

$$s_i^p: [p+1] \rightarrow [p]$$

$$j \mapsto \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

which are the surjections that ‘collapse together’ the points i and $i+1$.

- $\Delta[p] = \text{Hom}_\Delta(-, [p])$ is the *abstract p -simplex*; and the collection of all the $\Delta[p]$ forms a *simplicial set*. (This is something that we don’t actually use).
- $\Delta^\bullet: [p] \mapsto \Delta_{\text{Top}}^n$ is the collection of *topological simplices* and is a *cosimplicial space*, and we write $\Delta^p := \Delta^\bullet([p]) = \Delta_{\text{Top}}^p$, where $\Delta_{\text{Top}}^p \subset \mathbb{R}^{p+1}$ is the smooth space consisting of the points (t_0, \dots, t_p) such that all the t_i are non-negative

^[1]Confusingly often called *face* maps, but we opt for *coface* (and, similarly, *codegeneracy*), so that the terminology is consistent when we talk about (co)simplicial spaces. In a (perhaps seemingly odd) turn of events, the simplex category itself is *not* simplicial, but instead *cosimplicial*.

and $\sum_{i=0}^p t_i = 1$. When dealing with coface and codegeneracy maps in this cosimplicial space, we often omit the functor from the notation, and simply write e.g. $f_p^i: \Delta^{p-1} \rightarrow \Delta^p$.

- The coface and codegeneracy maps give us maps on (co)simplicial spaces:
 - simplicial spaces Y_\bullet have *face* maps

$$Y_\bullet f_p^i: Y_p \rightarrow Y_{p-1}$$

and *degeneracy* maps

$$Y_\bullet s_i^p: Y_p \rightarrow Y_{p+1};$$

- cosimplicial spaces Z^\bullet have *coface* maps

$$Z^\bullet f_p^i: Z_{p-1} \rightarrow Z_p$$

and *codegeneracy* maps

$$Z^\bullet s_i^{p+1}: Z_{p+1} \rightarrow Z_p.$$

- Given some topological space Y with a cover $\mathcal{V} = \{V_\beta\}_{\beta \in J}$ we define the *nerve* $Y_\bullet^{\mathcal{V}}$ as the simplicial space given by

$$Y_p^{\mathcal{V}} = \coprod_{\substack{\beta_0, \dots, \beta_p \in J \\ V_{\beta_0 \dots \beta_p} \neq \emptyset}} V_{\beta_0 \dots \beta_p}$$

with face maps acting by

$$Y_p^{\mathcal{V}} f_p^i: V_{\beta_0 \dots \beta_p} \mapsto V_{\beta_0 \dots \hat{\beta}_i \dots \beta_p}$$

and degeneracy maps by

$$Y_p^{\mathcal{V}} s_i^p: V_{\beta_0 \dots \beta_p} \mapsto V_{\beta_0 \dots \beta_i \beta_i \dots \beta_p}.$$

(3.0.3) Notational conventions.

- We write \mathcal{Y} to mean the *Yoneda embedding* $C \rightarrow [C^{\text{op}}, \text{Set}]$ of any category C , given (on objects) by $\mathcal{Y}(x) = \mathcal{Y}_x = \text{Hom}_C(-, x)$.
- If we have some bigraded object $(A^{i,j})_{i,j \in \mathbb{Z}}$ then, instead of writing $A^{\bullet, \bullet}$, we write $A^{\bullet, \star}$, to emphasise that the two gradings are independent from one another. That is, if we do ever write $A^{\bullet, \bullet}$ then the implication is that we are taking the *diagonal*: $A^{\bullet, \bullet} = A^\bullet = (A^{i,i})_{i \in \mathbb{Z}}$. Note that this convention sometimes means that the notation for the grading might switch, e.g. Ω^\bullet might later be written as $\check{C}^\bullet(\Omega^\star)$.

-
- We sometimes write $+$ instead of \oplus .
 - A *vector* is represented by a column vector (an $(n \times 1)$ -matrix) c , with components/coordinates/entries c^i ; a *covector* is represented by a row vector (a $(1 \times m)$ -matrix) r , with components/coordinates/entries r_j . This means that a linear map $f: k\langle u_1, \dots, u_m \rangle \rightarrow k\langle v_1, \dots, v_n \rangle$ between (finite-dimensional) vector spaces (over some field k) is represented by an $(n \times m)$ -matrix A with entries A_j^i , acting^[1] as $v^i = A_j^i u^j$.
 - We denote the symmetric group on n elements by S_n , and the sign of an element $\sigma \in S_n$ by $|\sigma|$.
 - Generally, we use E, F , etc. to denote vector bundles; \mathcal{E}, \mathcal{F} , etc. to denote vector bundles on the nerve; and \mathcal{E}, \mathcal{F} , etc. to denote coherent sheaves.
 - Pullbacks bind higher than restrictions: $f^*\mathcal{F} \mid U$ should be read as $(f^*\mathcal{F}) \mid U$, not $f^*(\mathcal{F} \mid U)$.
 - We write $U_{\alpha_0 \dots \alpha_p} \in \mathcal{U}$ to mean that each U_{α_i} is in \mathcal{U} .
 - Rather than the ‘Bourbaki dangerous bend’ sign to signal a particularly difficult/important/potentially misleading fact (or even just something of which we should be overtly aware), we opt for using a goose. The reasons for this choice should be evident to anybody who has ever met a goose; we refer those who have had the luck to have never done so to [Hou19]. Any reader annoyed by the number of geese signs in this thesis should probably avoid learning about how many real geese there are in the real world.



^[1]Here we use Einstein summation notation, as well as the fact that having a basis gives us a non-degenerate form (via the dual basis), which lets us raise and lower indices.

II. VECTOR BUNDLES AND tDR COHOMOLOGY

Preliminaries

4

I can assure you my credentials are top-notch: I've just graduated from Harvard College Yale. I aced every semester, and I got an 'A'.

Dr. Liz Asher.

(4.0.1) Purpose. *This chapter introduces most of the definitions that form the basis for our study of characteristic classes of coherent sheaves. Much is classical and can be found elsewhere; some things are probably also classical, but didn't come up in any of my reading, most likely because they are already well known to any experts in the field. In particular, the question in (4.1.23), and the fact that one even has to take into account this change of basis (as explained in (4.1.22)), was something that slipped by me and caused problems for a good week or so before one of my supervisors noticed what was going on, and so I make more noise about it than any classical text might. Some of the concepts introduced in Section 4.3 are either 'new' enough or subtle enough to deserve some extra treatment, and I have tried to give them such in Appendices A and F.*

(4.0.2) Throughout, let E be a locally free sheaf of rank r on X , where (X, \mathcal{O}_X) is a paracompact complex-analytic manifold with a locally-finite Stein cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ that trivialises E . We have trivialisation maps $\varphi_\alpha: E|_{U_\alpha} \xrightarrow{\sim} (\mathcal{O}_X|_{U_\alpha})^r$, and transition maps $M_{\alpha\beta}: (\mathcal{O}_X|_{U_{\alpha\beta}})^r \xrightarrow{\sim} (\mathcal{O}_X|_{U_{\alpha\beta}})^r$ given on overlaps by $M_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$. Finally, assume that we have (holomorphic) connections ∇_α on each $E|_{U_\alpha}$ (defined in (4.1.5)).

By picking some basis of sections $\{s_1^\alpha, \dots, s_r^\alpha\}$ of E over U_α we can realise the $M_{\alpha\beta}$ as $(r \times r)$ -matrices that describe the change of basis when we go from $E|_{U_\beta}$ to $E|_{U_\alpha}$, i.e.

$$s_k^\alpha = \sum_{\ell} (M_{\alpha\beta})_k^\ell s_\ell^\beta.$$

4.1 The Atiyah class

(4.1.1) The original reference for most of the material below is [Ati57, §2]; a more modern reference to much of this is [Huy05, 4.2.17 onwards] (although the treatment there of the Atiyah class is somehow backwards from ours here: it defines the Atiyah class as a cocycle and then shows that it corresponds to a splitting of the Atiyah exact sequence).

(4.1.2) Definition. The *Atiyah exact sequence* (or *jet sequence*) of E is the short exact sequence of \mathcal{O}_X -modules

$$0 \rightarrow E \otimes_{\mathcal{O}_X} \Omega_X^1 \rightarrow J^1(E) \rightarrow E \rightarrow 0$$

where $J^1(E) = (E \otimes \Omega_X^1) \oplus E$ as a \mathbb{C}_X -module^[1], but we define the \mathcal{O}_X -action by

$$f(s \otimes \omega, t) = (fs \otimes \omega + t \otimes df, ft).$$

The *Atiyah class* at_E of E is defined by $\text{at}_E = [J^1(E)] \in \text{Ext}_{\mathcal{O}_X}^1(E, E \otimes \Omega_X^1)$.

(4.1.3) We are interested in the Atiyah class because it is ‘equivalent’, in some sense that we make precise below, to the (first) Chern class.

- [Huy05, Proposition 4.3.10] shows us that the trace of the Atiyah class gives the same class in cohomology as the curvature of the Chern connection, which, combined with [Huy05, Example 4.4.8 i)], tells us that the traces of the Atiyah classes (defined in (4.1.21)) and the Chern classes are the same, up to a constant. There are some more general comments about this equivalence on [Huy05, p. 200], just below Example 4.4.11.
- [Huy05, Exercise 4.4.11] tells us that the Chern characters (or exponential Chern classes) are given (up to a constant) by the traces of the exponential Atiyah classes (defined in (4.1.18)).
- The traces of Atiyah classes satisfy an axiomatisation of Chern classes that guarantees uniqueness, *when X is compact*. This is explained in Section 10.4.

(4.1.4) We talk about *connections* in more detail in Appendix B, giving more background motivation and examples, but what follows here is all that is strictly necessary for our purposes.

(4.1.5) Definition. A *holomorphic (Koszul) connection* ∇ on E is exactly a (holomorphic) splitting of the Atiyah exact sequence of E . Recall that, by enforcing the Leibniz rule

$$\nabla(s \otimes \omega) = \nabla s \wedge \omega + s \otimes d\omega,$$

we can extend any connection $\nabla: E \rightarrow E \otimes \Omega_X^1$ to a map $\nabla: E \otimes \Omega_X^r \rightarrow E \otimes \Omega_X^{r+1}$. Using the same symbol ∇ to denote both the connection and any such extension is a common abuse of notation.

Given any continuous map $f: Y \rightarrow X$, we have the *pullback connection* $f^*\nabla$ on the pullback bundle f^*E (as defined in (B.2.3)).

Given some subsheaf $F \subset E$, we *cannot*, a priori, restrict a connection on E to get a connection on F . This is because connections are somehow endomorphism-valued (as we explain in (B.2.5)), and there is no reason for an arbitrary subsheaf to be invariant under the given endomorphism.



^[1]Where \mathbb{C}_X is the constant sheaf of value \mathbb{C} on X .

(4.1.6) Definition. The *curvature* $\kappa(\nabla)$ of a connection ∇ is the map

$$\kappa(\nabla) = \nabla^2: E \rightarrow E \otimes \Omega_X^2,$$

where $\nabla^2 = \nabla \circ \nabla$ is defined as in (4.1.5), by enforcing the Leibniz rule.

(4.1.7) Definition. Given a connection ∇ on E , we say that a section $s \in \Gamma(U, E)$ is *flat* if $\nabla(s) = 0$. We say that ∇ is *flat* if $\kappa(\nabla) = 0$.

(4.1.8) Lemma. ([Gre80, Lemma O.E.3]). By Cartan's Theorem B^[1], any locally free sheaf over a Stein manifold admits a holomorphic connection.

(4.1.9) Lemma. The Atiyah class of E is represented by the cocycle^[2]

$$\{\nabla_\beta - \nabla_\alpha\}_{\alpha, \beta \in I} \in \check{C}_{\mathcal{U}}^1(\mathcal{H}om(E, E \otimes \Omega_X^1)).$$

Proof. First, recall that the difference of any two connections is exactly an \mathcal{O}_X -linear map (see (B.2.4)). Secondly, note that we do indeed have a cocycle:

$$(\nabla_\beta - \nabla_\alpha) + (\nabla_\gamma - \nabla_\beta) = \nabla_\gamma - \nabla_\alpha.$$

Thus $\{\nabla_\beta - \nabla_\alpha\}_{\alpha, \beta} \in \check{C}_{\mathcal{U}}^1(\mathcal{H}om(E, E \otimes \Omega_X^1))$. Then we use the isomorphisms

$$\text{Ext}_{\mathcal{O}_X}^1(E, E \otimes \Omega_X^1) \cong \text{Hom}_{\mathcal{D}(X)}(E, E \otimes \Omega_X^1[1]) \cong H^1(X, \mathcal{H}om(E, E \otimes \Omega_X^1)).$$

Finally, we have to prove that this class in cohomology agrees with that given in our definition of the Atiyah class of E . This fact is true in more generality, and we prove it as so.

Let $0 \rightarrow \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0$ be a short exact sequence in some abelian category \mathcal{A} . The definition of $[\mathcal{B}] \in \text{Ext}_{\mathcal{A}}^1(\mathcal{C}, \mathcal{A})$ is as the class in $\text{Hom}_{\mathcal{D}(\mathcal{A})}(\mathcal{C}, \mathcal{A}[1])$ of a canonical morphism $\mathcal{C} \rightarrow \mathcal{A}[1]$ constructed using \mathcal{B} as follows:

- (i) take the quasi-isomorphism $(\mathcal{A} \rightarrow \mathcal{B}) \xrightarrow{\sim} \mathcal{C}$, where \mathcal{B} is in degree 0;
- (ii) invert it to get a map $\mathcal{C} \xrightarrow{\sim} (\mathcal{A} \rightarrow \mathcal{B})$ such that the composite $\mathcal{C} \xrightarrow{\sim} (\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ is the identity;
- (iii) compose with the identity map $(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{A}[1]$.

When \mathcal{A} is the category of locally free sheaves on X , we can realise the quasi-isomorphism $\mathcal{C} \xrightarrow{\sim} (\mathcal{A} \rightarrow \mathcal{B})$ as a quasi-isomorphism $\mathcal{C} \xrightarrow{\sim} \check{C}^\bullet(\mathcal{A} \rightarrow \mathcal{B})$ by using the Čech complex of a complex:

$$\check{C}^\bullet(\mathcal{A} \rightarrow \mathcal{B}) = \check{C}^0(\mathcal{A}) \xrightarrow{(\check{\delta}, f)} \check{C}^1(\mathcal{A}) \oplus \check{C}^0(\mathcal{B}) \xrightarrow{(\check{\delta}, -f, \check{\delta})} \check{C}^2(\mathcal{A}) \oplus \check{C}^1(\mathcal{B}) \xrightarrow{(\check{\delta}, f, \check{\delta})} \dots$$

^[1]This is the statement that, if \mathcal{E} is a coherent sheaf on a Stein manifold X , then $H^p(X, \mathcal{E}) = 0$ for all $p \geq 1$.

^[2]We omit the restriction from our notation: really we mean $\nabla_\alpha \upharpoonright U_{\alpha\beta} - \nabla_\beta \upharpoonright U_{\alpha\beta}$

where $\check{C}^0(\mathcal{A})$ is in degree -1 . If we have local sections $\sigma_\alpha: C|_{U_\alpha} \rightarrow \mathcal{B}|_{U_\alpha}$ then $\sigma_\beta - \sigma_\alpha$ lies in the kernel $\text{Ker}(\mathcal{B}|_{U_{\alpha\beta}} \rightarrow C|_{U_{\alpha\beta}})$, and so we can lift this difference to \mathcal{A} , giving us the map

$$(\{\sigma_\alpha\}_\alpha, \{\sigma_\beta - \sigma_\alpha\}_{\alpha,\beta}): C \rightarrow \check{C}^0(\mathcal{B}) \oplus \check{C}^1(\mathcal{A}).$$

This map we have constructed is exactly $[\mathcal{B}]$. More precisely,

$$\begin{aligned} \text{Ext}_{\mathcal{O}_X}^1(C, \mathcal{A}) &\cong \text{Hom}_{\mathcal{D}(X)}(C, \mathcal{A}[1]) \cong H^1(X, \mathcal{H}om(C, \mathcal{A})) \\ [B] &\leftrightarrow C \xrightarrow{\sim} (A \rightarrow B) \rightarrow A[1] \leftrightarrow [(\sigma_\beta - \sigma_\alpha)_{\alpha,\beta}]. \end{aligned}$$

□

(4.1.10) We write $\omega_{\alpha\beta} = \nabla_\beta - \nabla_\alpha$, recalling that the ∇_α are local connections.

(4.1.11) **Definition.** When \mathcal{E} , \mathcal{F} , and \mathcal{G} are sheaves of \mathcal{O}_X -modules, with \mathcal{G} locally free, we have the isomorphism

$$\mathcal{H}om(\mathcal{E}, \mathcal{F} \otimes \mathcal{G}) \cong \mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes \mathcal{G}.$$

This means that, taking the trivialisation over U_α , we can consider $\omega_{\alpha\beta} = (\nabla_\beta - \nabla_\alpha)$ as an $(r \times r)$ -matrix of (holomorphic) 1-forms on X , or an *endomorphism-valued form*, since

$$H^1(X, \mathcal{H}om(E, E \otimes \Omega_X^1)) \cong H^1(X, \mathcal{E}nd(E) \otimes \Omega_X^1)$$

where, for a sheaf \mathcal{E} , we define the collection of \mathcal{E} -valued r -forms on X to be $\mathcal{E} \otimes \Omega_X^r$.

(4.1.12) We can calculate $\omega_{\alpha\beta}$ explicitly **in the case where our local sections $\{s_1^\alpha, \dots, s_r^\alpha\}$ are ∇_α -flat:**

$$\omega_{\alpha\beta}(s_k^\alpha) = (\nabla_\beta - \nabla_\alpha)(s_k^\alpha) = \nabla_\beta(s_k^\alpha)$$

But then, using the change-of-basis identity in (4.0.2) followed by the Leibniz rule, we get that

$$\begin{aligned} \nabla_\beta(s_k^\alpha) &= \nabla_\beta \left(\sum_\ell (M_{\alpha\beta})_k^\ell s_\ell^\beta \right) \\ &= \sum_\ell \left[\nabla_\beta(s_\ell^\beta) \wedge (M_{\alpha\beta})_k^\ell + s_\ell^\beta \otimes d(M_{\alpha\beta})_k^\ell \right]. \end{aligned}$$

Since the s_ℓ^β are ∇_β -flat, the first part of each ℓ -term is zero, and, using the inverse change-of-basis identity, we then have that

$$\omega_{\alpha\beta}(s_k^\alpha) = \sum_\ell \left[\left(\sum_m (M_{\alpha\beta}^{-1})_m^\ell s_m^\alpha \right) \otimes d(M_{\alpha\beta})_k^\ell \right]$$

$$= \sum_m s_m^\alpha \otimes (M_{\alpha\beta}^{-1} dM_{\alpha\beta})_k^m$$

where we can move the $M_{\alpha\beta}^{-1}$ across the tensor product because the tensor is over \mathcal{O}_X , and the $M_{\alpha\beta}^m$ are exactly elements of this ring.

Thus, in the U_α trivialisation, $\omega_{\alpha\beta}$ is given by $M_{\alpha\beta}^{-1} dM_{\alpha\beta}$. Seasoned readers might notice that this is exactly the first Chern class $d \log g_{\alpha\beta}$ of the bundle (here $g_{\alpha\beta} = M_{\alpha\beta}$).



(4.1.13) We don't need our local connections ∇_α to be flat in order to *define* these things, but just to *calculate* them in such an explicit manner.

(4.1.14) Lemma. $d\omega_{\alpha\beta} = -\omega_{\alpha\beta}^2$.

Proof. Using the fact^[1] that $d(A^{-1}) = -A^{-1} \cdot dA \cdot A^{-1}$, we see that

$$\begin{aligned} d\omega_{\alpha\beta} &= d(M_{\alpha\beta}^{-1} dM_{\alpha\beta}) \\ &= d(M_{\alpha\beta}^{-1}) dM_{\alpha\beta} \\ &= -M_{\alpha\beta}^{-1} dM_{\alpha\beta} M_{\alpha\beta}^{-1} dM_{\alpha\beta} \\ &= -(M_{\alpha\beta}^{-1} dM_{\alpha\beta})^2 \\ &= -\omega_{\alpha\beta}^2. \end{aligned} \quad \square$$

(4.1.15) Lemma. $d \operatorname{tr} \omega_{\alpha\beta} = 0$.

Proof. Since $d \operatorname{tr} \omega_{\alpha\beta} = \operatorname{tr} d\omega_{\alpha\beta} = -\operatorname{tr} \omega_{\alpha\beta}^2$, by (4.1.14), this lemma is a specific case of the fact that $\operatorname{tr}(A^{2k}) = 0$ for any $k \in \mathbb{N}$, which we prove in (4.4.3). \square

This tells us that $\operatorname{tr} \omega_{\alpha\beta}$ defines a class in de Rham cohomology, since it is closed under the de Rham differential.

(4.1.16) Our next goal is to construct some idea of 'higher' Atiyah classes, i.e. consider the Atiyah class as the *first* Atiyah class, and then define the *second* Atiyah class, the *third* Atiyah class, and so on.

(4.1.17) Recall that, for sheaves \mathcal{F} and \mathcal{G} of \mathcal{O}_X -modules, we have the cup product

$$\smile: H^m(X, \mathcal{F}) \otimes H^n(X, \mathcal{G}) \rightarrow H^{m+n}(X, \mathcal{F} \otimes \mathcal{G})$$

which is given in Čech cohomology by the tensor product:

$$(a \smile b)_{\alpha\beta\gamma} = (a)_{\alpha\beta} \otimes (b)_{\beta\gamma}.$$

^[1] $(M + tH)^{-1} = (M(\operatorname{id} + tM^{-1}H))^{-1} = (\operatorname{id} + tM^{-1}H)^{-1}M^{-1} = (\operatorname{id} - tM^{-1}H + O(t^2))M^{-1} = \operatorname{id} - tM^{-1}HM^{-1} + O(t^2)$.

(4.1.18) Definition. Let us formally construct the second **exponential**^[1] Atiyah class $\text{at}_E^{\circ 2}$. We start with

$$(\text{at}_E \otimes \text{id}_{\Omega_X^1}) \smile \text{at}_E \in H^2\left(X, \mathcal{H}om(E \otimes \Omega_X^1, E \otimes \Omega_X^1 \otimes \Omega_X^1) \otimes \mathcal{H}om(E, E \otimes \Omega_X^1)\right)$$

and apply the composition map

$$H^m\left(X, \mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{H}om(\mathcal{E}, \mathcal{F})\right) \rightarrow H^m\left(X, \mathcal{H}om(\mathcal{E}, \mathcal{G})\right)$$

to obtain

$$\begin{aligned} (\text{at}_E \otimes \text{id}_{\Omega_X^1}) \smile \text{at}_E &\in H^2\left(X, \mathcal{H}om\left(E, E \otimes \Omega_X^1 \otimes \Omega_X^1\right)\right) \\ &\cong H^2\left(X, \mathcal{E}nd(E) \otimes \Omega_X^1 \otimes \Omega_X^1\right). \end{aligned}$$

Finally, applying the wedge product (of forms) gives us

$$\text{at}_E^{\circ 2} = (\text{at}_E \otimes \text{id}_{\Omega_X^1}) \wedge (\text{at}_E) \in H^2\left(X, \mathcal{E}nd(E) \otimes \Omega_X^2\right).$$

In general, the k th **exponential Atiyah class** $\text{at}_E^{\circ k}$ is the class

$$\text{at}_E^{\circ k} = \prod_{i=1}^k \left(\text{at}_E \otimes \text{id}_{\Omega_X^1}^{\otimes(k-i)} \right) \in H^k\left(X, \mathcal{E}nd(E) \otimes \Omega_X^k\right)$$

where the product is given by applying composition and then the wedge product (of forms) as above.

(4.1.19) As a general note on notation, we will omit the wedge symbol \wedge when talking about the wedge product of differential forms (or we will use \cdot if we have to use any symbol at all), and reserve it solely for the wedge product of endomorphisms. In particular then, for endomorphism-valued forms M and N , we write MN (or $M \cdot N$) to mean the object given by composing the endomorphisms (and wedging the forms), and $M \wedge N$ to mean the object given by wedging the endomorphisms (and wedging the forms). In terms of (2×2) -matrices (i.e. taking E to be of rank 2), this looks like

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} &= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \in \Gamma\left(U, \Omega_X^2 \otimes \mathcal{E}nd(E)\right) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \wedge \begin{pmatrix} e & f \\ g & h \end{pmatrix} &= \det \begin{pmatrix} a & f \\ c & h \end{pmatrix} = ah - fc \in \Gamma\left(U, \Omega_X^2 \otimes \mathcal{E}nd(E \wedge E) \cong \Omega_X^2\right). \end{aligned}$$

Note that, if we take the trace, then these two objects will both be 2-forms on U . Similarly, $\text{tr}(M^k)$ and $\text{tr}(\wedge^k M)$ are both just k -forms on U .

^[1]In general, we will be interested in **standard** Atiyah classes, but we consider the manual construction of **exponential** ones since these can be written down very explicitly, as in Chapter 5.



(4.1.20) The classical theory of Chern classes has two important ‘types’ of Chern class: exponential and ‘standard’. For now, we are content with simply saying, as a definition, if you like, that the polynomial that corresponds to the exponential classes is $\text{tr}(M^p)$, and the polynomial that corresponds to the standard classes is $\text{tr}(\wedge^p M)$. We discuss this more in Appendix C. Caution is needed when discussing the k th Atiyah class though: there is **no** trace in the definition; to obtain characteristic classes we **have** to take the trace. The reason for not including the trace in the definition is to do with the simplicial version of Atiyah classes that we will later define.

(4.1.21) **Definition.** The second (**standard**) Atiyah class $\text{at}_E^{\wedge 2}$ can be formally constructed as follows. We start with

$$\begin{aligned} \text{at}_E \smile \text{at}_E &\in \text{H}^2\left(X, \mathcal{H}om(E, E \otimes \Omega_X^1) \otimes \mathcal{H}om(E, E \otimes \Omega_X^1)\right) \\ &\cong \text{H}^2\left(X, \mathcal{E}nd(E) \otimes \mathcal{E}nd(E) \otimes \Omega_X^1 \otimes \Omega_X^1\right) \end{aligned}$$

and then apply the wedge product of endomorphisms and the wedge product of forms to get

$$\text{at}_E^{\wedge 2} \in \text{H}^2\left(X, \mathcal{E}nd(E \wedge E) \otimes \Omega_X^2\right).$$

In general, the k th (**standard**) Atiyah class $\text{at}_E^{\wedge k}$ is the class

$$\text{at}_E^{\wedge k} = \bigwedge_{i=1}^k \text{at}_E \in \text{H}^k\left(X, \mathcal{E}nd(\wedge^k E) \otimes \Omega_X^k\right).$$

(4.1.22) We can find an explicit representative for $\text{at}_E^{\circ 2}$ by using (4.1.11) and (4.1.17):

$$\begin{aligned} \left((\text{at}_E \otimes \text{id}_{\Omega_X^1}) \smile (\text{at}_E)\right)_{\alpha\beta\gamma} &= (\text{at}_E \otimes \text{id}_{\Omega_X^1})_{\alpha\beta} \otimes (\text{at}_E)_{\beta\gamma} \\ &\leftrightarrow \left(\omega_{\alpha\beta} \otimes \text{id}_{\Omega_{U_{\alpha\beta}}^1}\right) \otimes \omega_{\beta\gamma} \in \text{Mat}_{r \times r}\left(\Omega_{U_{\alpha\beta}}^1 \otimes \Omega_{U_{\alpha\beta}}^1\right) \otimes \text{Mat}_{r \times r}\left(\Omega_{U_{\beta\gamma}}^1\right) \end{aligned}$$

where $\text{Mat}_{r \times r}(A)$ is the collection of A -valued $(r \times r)$ -matrices. But before composing these two matrices, as described in (4.1.18), we first have to account for the change of trivialisation from over $U_{\beta\gamma}$ to over $U_{\alpha\beta}$. That is, after applying composition and the wedge product, we have

$$\left(\text{at}_E^{\circ 2}\right)_{\alpha\beta\gamma} = \omega_{\alpha\beta} \wedge M_{\alpha\beta} \omega_{\beta\gamma} M_{\alpha\beta}^{-1}.$$

(4.1.23) We know that $\text{at}_E^{\circ 3}$ is represented locally by $\omega_{\alpha\beta} \omega_{\beta\gamma} \omega_{\gamma\delta}$, but where $\omega_{\beta\gamma}$ and $\omega_{\gamma\delta}$ undergo a base change to become $\Omega_{U_{\alpha\beta}}^1$ -valued. But then, do we

1. base change $\omega_{\gamma\delta}$ to be $\Omega_{U_{\beta\gamma}}^1$ -valued,

2. compose with $\omega_{\beta\gamma}$,
3. then base change this composition to be $\Omega_{U_{\alpha\beta}}^1$ -valued;

or do we instead

1. base change both $\omega_{\gamma\delta}$ and $\omega_{\beta\gamma}$ to be $\Omega_{U_{\alpha\beta}}^1$ -valued, and
2. then perform the triple composition?

That is,

$$\omega_{\alpha\beta} \wedge M_{\alpha\beta}(\omega_{\beta\gamma} \wedge M_{\beta\gamma}\omega_{\gamma\delta}M_{\beta\gamma}^{-1})M_{\alpha\beta}^{-1} \stackrel{?}{=} \omega_{\alpha\beta} \wedge M_{\alpha\beta}\omega_{\beta\gamma}M_{\alpha\beta}^{-1} \wedge M_{\alpha\gamma}\omega_{\gamma\delta}M_{\alpha\gamma}^{-1}.$$

The happy answer is that these two constructions are in fact equal, thanks to the cocycle condition on the $M_{\alpha\beta}$ and some form of associativity^[1], and so we can use whichever one we so please.

4.2 tDR cohomology

(4.2.1) Since \mathcal{U} is Stein (and any finite intersection of Stein open sets is also Stein) and the sheaves Ω_X^r are coherent, we can apply Cartan's Theorem B, which tells us that $\check{H}^k(\mathcal{U}, \Omega_X^\bullet) \cong \check{H}^k(X, \Omega_X^\bullet)$; since X is paracompact and \mathcal{U} is Stein, we know that $\check{H}^k(X, \Omega_X^\bullet) \cong \mathbb{H}^k(X, \Omega_X^\bullet)$; and e.g. [Voi08, Theorem 8.1] tells us that $\mathbb{H}^k(X, \Omega_X^\bullet) \cong \mathbb{H}^k(X, \mathbb{C})$. In summary, we are in a nice enough setting that Čech-de Rham bicomplex lets us calculate singular cohomology:

$$H^r \text{Tot}^\bullet \check{C}^\bullet(\Omega_X^\bullet) \cong H^r(X, \mathbb{C}).$$

(4.2.2) **Definition.** Given the de Rham complex Ω_X^\bullet , we define the *k*th Hodge complex $\Omega_X^{\bullet \geq k}$ as the truncation

$$\Omega_X^{\bullet \geq k} = \left(\Omega_X^k \xrightarrow{d} \Omega_X^{k+1} \xrightarrow{d} \dots \right)[-k]$$

i.e. so that Ω_X^k is in degree k .

(4.2.3) **Definition.** We define the *k*th truncated de Rham (or tDR) cohomology to be $H_{\text{tDR}}^k(X) = \mathbb{H}^k(X, \Omega_X^{\bullet \geq k})$.

(4.2.4) If we have some closed class $c = (c_0, \dots, c_{2k}) \in \text{Tot}^{2k} \check{C}^\bullet(\Omega_X^\bullet)$, where $c_i \in \check{C}^i(\Omega_X^{2k-i})$, such that $c_i = 0$ for $i \geq k+1$, then we can refine the corresponding class in singular cohomology $[c] \in \mathbb{H}^{2k}(X, \mathbb{C})$ to a class $[c] \in H_{\text{tDR}}^{2k}(X)$ in tDR cohomology.

^[1]That is, $A \cdot MB = AM \cdot B$, where M is a matrix of 0-forms.

4.3 Sheaves on simplicial spaces and simplicial forms

(4.3.1) All of our simplicial conventions and notation are listed in (3.0.2).

(4.3.2) **Definition.** A sheaf \mathcal{E}^\bullet on a simplicial space Y_\bullet is a family of sheaves $\{\mathcal{E}^p\}_{p \in \mathbb{N}}$, where \mathcal{E}^p is a sheaf on Y_p , along with, for all $\varphi: [p] \rightarrow [q]$ in Δ , morphisms

$$\mathcal{E}^\bullet(\varphi): (Y_\bullet \varphi)^{-1} \mathcal{E}^p \rightarrow \mathcal{E}^q$$

that are functorial^[1] in φ . The sheaf \mathcal{E}^\bullet is said to be *strongly cartesian* if the $\mathcal{E}^\bullet(\varphi)$ are all isomorphisms. We say that a (cochain) complex $\mathcal{E}^{\bullet, \star}$ of sheaves on Y_\bullet is *cartesian* if the maps $\mathcal{E}^{\bullet, \star}(\varphi): (Y_\bullet \varphi)^{-1} \mathcal{E}^{p, \star} \rightarrow \mathcal{E}^{q, \star}$ are quasi-isomorphisms.

A morphism $\varphi_\bullet: \mathcal{E}^\bullet \rightarrow \mathcal{F}^\bullet$ of sheaves on Y_\bullet is a family of morphisms of sheaves $\varphi_p = \{\varphi_p: \mathcal{E}^p \rightarrow \mathcal{F}^p\}$ such that the ‘obvious’ diagram

$$\begin{array}{ccc} (Y_\bullet \varphi)^{-1} \mathcal{E}^p & \xrightarrow{(Y_\bullet \varphi)^{-1} \varphi_p} & (Y_\bullet \varphi)^{-1} \mathcal{F}^p \\ \mathcal{E}^\bullet(\varphi) \downarrow & & \downarrow \mathcal{F}^\bullet(\varphi) \\ \mathcal{E}^q & \xrightarrow{\varphi_q} & \mathcal{F}^q \end{array}$$

commutes.

(4.3.3) There are a few important warnings to give here:



(i) we do *not* impose the cartesian condition in the definition itself (but we will later see that it comes ‘for free’ in the scenarios that are of interest to us);

(ii) these sheaves are *covariant* functors from Δ , *not* contravariant ones, and are thus more like *cosimplicial* objects — because of this, it makes sense to write \mathcal{E}^\bullet instead of \mathcal{E}_\bullet , but note that [Gre80; TT86] do not follow this convention; and



(iii) these sheaves do *not* really even deserve the name ‘cosimplicial sheaves’, because they are *not* simply cosimplicial objects in some category of sheaves: each \mathcal{E}^p is a sheaf over a different space.

Although it is not necessary for our present purposes, we discuss some of these points a little bit more in Appendix F.

^[1]That is, $\mathcal{E}^\bullet(\beta \circ \varphi) = \mathcal{E}^\bullet(\beta) \circ \mathcal{E}^\bullet(\varphi)$.

(4.3.4) Definition. In the specific case where $Y_\bullet = X_\bullet^{\mathcal{U}}$ and each \mathcal{E}^p is a sheaf of $\mathcal{O}_{X_p^{\mathcal{U}}}$ -modules, we call the sheaf \mathcal{E}^\bullet a *sheaf of $\mathcal{O}_{X_\bullet^{\mathcal{U}}}$ -modules*. We will often play fast and loose with our nomenclature and simply say ‘vector bundles on the nerve’ to mean ‘locally free sheaves of $\mathcal{O}_{X_\bullet^{\mathcal{U}}}$ -modules’, as well as generally using ‘vector bundle’ and ‘locally free sheaf’ somewhat interchangeably.

Note that the morphisms $\mathcal{E}^\bullet(\varphi)$ in (4.3.2) should respect this $\mathcal{O}_{X_\bullet^{\mathcal{U}}}$ -structure: in particular, the pullback $(X_\bullet^{\mathcal{U}}\varphi)^{-1}\mathcal{E}^p$ should be replaced with the algebraic pullback

$$(X_\bullet^{\mathcal{U}}\varphi)^*\mathcal{E}^p = (X_\bullet^{\mathcal{U}}\varphi)^{-1}\mathcal{E}^p \otimes_{(X_\bullet^{\mathcal{U}}\varphi)^{-1}\mathcal{O}_{X_p^{\mathcal{U}}}} \mathcal{O}_{X_q^{\mathcal{U}}}.$$

In fact, throughout this thesis we only work with sheaves of $\mathcal{O}_{X_\bullet^{\mathcal{U}}}$ -modules, and so will only use the algebraic pullback.

It is important to note that there are no conditions on the rank of the bundle between different simplicial levels: it could be the case that \mathcal{E}^p is rank r , but \mathcal{E}^q is of rank s . By definition, however, the rank *is* constant over different open sets of the same simplicial degree: $\mathcal{E}^p|_{U_{\alpha_0\dots\alpha_p}}$ is of the same rank as $\mathcal{E}^p|_{U_{\alpha'_0\dots\alpha'_p}}$.



In the specific case when $\mathcal{E}^\bullet = E^\bullet$ is the pullback of a global vector bundle^[1] then the rank is constant between simplicial degrees. Further, the maps $E^\bullet(\varphi)$ are simply identity maps (and so, in particular, E^\bullet is strongly cartesian).

(4.3.5) As a useful mnemonic, we keep the phrase “Čech complex” in mind when working with complexes $\mathcal{E}^{\bullet,\star}$ of sheaves on the nerve: the first grading \bullet is for the “Čech” part (i.e. the level of the Čech nerve); the second grading \star is for the “complex” part (i.e. the cochain complex degree).

(4.3.6) Definition. Let Y_\bullet be a simplicial complex manifold. Following [Dup76], we define a *simplicial differential r -form* ω_\bullet on Y_\bullet to be a family ω_\bullet of forms, with ω_p a global section of the sheaf

$$\bigoplus_{i+j=r} \pi_{Y_p}^* \Omega_{Y_p}^i \otimes_{\mathcal{O}_{Y_p \times \Delta_{\text{extd}}^p}} \pi_{\Delta_{\text{extd}}^p}^* \Omega_{\Delta_{\text{extd}}^p}^j$$

(where Δ_{extd}^p is the affine subspace of \mathbb{R}^{p+1} given by the vanishing of $1 - \sum_{m=0}^p x_p$; and where Ω_{Y_p} is the sheaf of *holomorphic* forms, and $\Omega_{\Delta_{\text{extd}}^p}$ is the sheaf of *smooth* forms) such that, for all *coface* maps $f_p^i: [p-1] \rightarrow [p]$,

$$(Y_\bullet f_p^i \times \text{id})^* \omega_{p-1} = (\text{id} \times f_p^i)^* \omega_p \in \Omega^r(Y_p \times \Delta^{p-1}). \quad (4.3.6.1)$$

We write $\Omega_{Y_\bullet}(r, \Delta)$ to mean the collection of all *simplicial differential r -forms* on Y_\bullet (distinguishing it from $\Omega^r(Y_\bullet)$, the collection of forms on Y_\bullet). We can describe each ω_p as a form of *type* (i, j) , by writing $\omega_p = \xi_p \otimes \tau_p$, where ξ_p is the Y_p -part

^[1]Given some vector bundle E on X , we define $E_p = (X_p^{\mathcal{U}} \rightarrow X)^*E$.

of ω_p , and τ_p is the Δ^p -part of ω_p ; then $i = |\xi_p|$ and $j = |\tau_p|$. This lets us define a differential

$$d: \Omega^{r,\Delta}(Y_\bullet) \longrightarrow \Omega^{r+1,\Delta}(Y_\bullet)$$

which is given by the Koszul convention with respect to the type of the form:

$$\begin{aligned} d(\xi_p \otimes \tau_p) &= (d_{Y_\bullet} + (-1)^{|\xi_p|} d_{\Delta^\bullet})(\xi_p \otimes \tau_p) \\ &= (d\xi_p \otimes \tau_p) + ((-1)^{|\xi_p|} \xi_p \otimes d\tau_p). \end{aligned}$$

See (A.1.1) and (A.1.2) for more details.

This definition of simplicial differential forms is quite natural when seen side-by-side with the definition of the fat realisation of a simplicial space, since we are exactly asking that these forms on $Y_\bullet \times \Delta^\bullet$ descend to forms on the fat realisation. This point of view is covered in [Dup76], and we discuss some other motivations in (F.6).

(4.3.7) Lemma. ([Dup76, Theorem 2.3]). There is a quasi-isomorphism which, for each fixed r , consists of a map

$$\int_{\Delta^\bullet} : \Omega^{r,\Delta}(Y_\bullet) \rightarrow \bigoplus_{p=0}^r \Omega^{r-p}(Y_p)$$

induced (see Appendix A) by *fibre integration*

$$\int_{\Delta^p} : \Omega^{r,\Delta}(Y_\bullet) \rightarrow \Omega^{r-p}(Y_p)$$

where the latter is given by integrating the type $(r-p, p)$ part of a simplicial form over the geometric realisation of the p -simplex with its canonical orientation (again, see Appendix A).

Proof. The classical proof is [Dup76, Theorem 2.3], and the fact that the morphism is given by integrating over the simplices is mentioned in [Dup76, Remark 1, §2]. This proof is for the smooth case, and although the proof for the holomorphic case is almost identical, we reproduce it (in part) in Appendix A (for the case $Y_\bullet = X_\bullet^{\mathcal{U}}$) so as to clear up the myriad of confusions that can arise from the choices of orientations and signs. \square



(4.3.8) We use a different convention from that of [Dup76], which can lead to even more sign problems — see (A.1.3).

(4.3.9) Taking $Y_\bullet = X_\bullet^{\mathcal{U}}$ gives

$$\int_{\Delta^\bullet} : \Omega^{r,\Delta}(X_\bullet^{\mathcal{U}}) \rightarrow \bigoplus_{p=0}^r \Omega^{r-p}(X_p^{\mathcal{U}}) \cong \text{Tot}^r \check{C}^\bullet(\Omega_X^\bullet).$$

It is interesting to note that the conditions we impose on \mathcal{U} are really only^[1] to ensure that this quasi-isomorphism *calculates* de-Rham cohomology; the existence of the quasi-isomorphism in (4.3.7) does *not* depend on the properties of the cover.

4.4 Trace cycles

(4.4.1) This section contains some useful facts about the trace of matrix-valued forms that we often use to simplify calculations. Throughout, when we say ‘matrix-valued’, we mean $(r \times r)$ -matrix-valued for some fixed r .

(4.4.2) **Lemma.** Let A_1, \dots, A_k be matrix-valued forms. Then

$$\text{tr} A_1 A_2 \cdots A_k = (-1)^{k-1} \text{tr} A_2 \cdots A_k A_1.$$

Proof. This is really just writing down definitions and using the fact that (non-matrix-valued) forms are anti-commutative. Explicitly,

$$\begin{aligned} \text{tr} A_1 A_2 \cdots A_k &= \sum_i (A_1 A_2 \cdots A_k)_i^i \\ &= \sum_i \sum_{i_1, \dots, i_{k-1}} (A_1)_{i_1}^i (A_2)_{i_2}^{i_1} \cdots (A_k)_{i_1}^{i_{k-1}} \\ &= \sum_i \sum_{i_1, \dots, i_{k-1}} -(A_2)_{i_2}^{i_1} (A_1)_{i_1}^i \cdots (A_k)_{i_1}^{i_{k-1}} \\ &= \dots \\ &= \sum_i \sum_{i_1, \dots, i_{k-1}} (-1)^{k-1} (A_2)_{i_2}^{i_1} \cdots (A_k)_{i_1}^{i_{k-1}} (A_1)_{i_1}^i \\ &= (-1)^{k+1} \text{tr} A_2 \cdots A_k A_1. \quad \square \end{aligned}$$

(4.4.3) **Corollary.** Let A be a matrix-valued form. Then $\text{tr} A^{2k} = 0$ for any $k \in \mathbb{N}$.

Proof. By (4.4.2), $\text{tr} A^{2k} = -\text{tr} A^{2k}$, whence $\text{tr} A^{2k} = 0$. □

(4.4.4) **Lemma.** Let A_1, \dots, A_k be matrix-valued forms and P_{ij} some monomial in A_i and A_j . If $P_{ij} = -P_{ji}$ then $\text{tr} \sum_{i,j=1}^k P_{ij} = \text{tr} \sum_{i=1}^k P_{ii}$.

Proof. $\sum_{i,j=1}^k P_{ij} = (\sum_{i<j} + \sum_{j<i} + \sum_{i=j}) P_{ij} = \sum_{i<j} (P_{ij} + P_{ji}) + \sum_i P_{ii} = \sum_i P_{ii}$. □

^[1]As far as fibre integration is concerned.

Manual (exponential) construction

5

Two people are walking along and see a herd of cows in a field. One asks the other how many cows they think there are, to which the other replies, almost instantly, with an answer of 322. The first person asks how the other managed to count them all so quickly, and the other leans in and says: “it’s a simple trick: you just count the number of legs and then divide by four.”

Some joke I once heard.

(5.0.1) Purpose. *The explicit Čech-de Rham lift of the trace of the first (exponential) Atiyah class is very well known classically; the lifts of higher classes are mentioned in the section entitled ‘Concluding Remarks’ in [BT82, §23], but no explicit formulas are given there (although it is mentioned that the components are all zero below the diagonal).*



(5.0.2) We repeat here the fact that it is the *trace* of the Atiyah classes that interests us, not just the Atiyah classes themselves.

(5.0.3) Recalling (4.2.1), we know that we can use the Čech complex to calculate singular cohomology. Now, say we are given some $c_k \in \check{C}^k(\Omega_X^k)$ with $\check{\delta}c_k = 0$ but $dc_k \neq 0$. Then, if we can find $c_i \in \check{C}^i(\Omega_X^{2k-i})$ for $i = 1, \dots, (k-1)$ such that $\check{\delta}c_i = dc_{i-1}$, and define $c_0 = 0 \in \check{C}^0(\Omega_X^{2k})$, then

$$(0, \pm c_1, \dots, \pm c_{k-1}, c_k, 0, \dots, 0) \in \text{Tot}^{2k} \check{C}^\bullet(\Omega_X^\bullet)$$

is $(\check{\delta} + (-1)^k d)$ -closed,^[1] and thus represents a cohomology class in $H^{2k} \text{Tot}^\bullet \check{C}^\bullet(\Omega_X^\bullet)$, and thus a cohomology class in $H^{2k}(X, \mathbb{C})$.

In essence, given some ‘starting element’ in the Čech-de Rham bicomplex, we can try to manually lift it to some *closed* element of the same total degree.

(5.0.4) A few important notes before we start.

- Although we have the isomorphism $H^r \text{Tot}^\bullet \check{C}^\bullet(\Omega_X^\bullet) \cong H^r(X, \mathbb{C})$, we don’t necessarily have an easy way of computing explicitly what a closed class in the Čech-de Rham complex maps to under this isomorphism, *unless* it has a non-zero degree- $(0, r)$ part, in which case it maps to exactly that.

^[1]The signs depend on the parity of k .

- We will construct classes in de Rham cohomology, but note that they can actually all be considered as living in the corresponding tDR cohomology, thanks to (4.2.4).
- Throughout this chapter we adopt the same notation and assumptions as in Chapter 4, but always adopt the further assumption that the local connections ∇_α are **flat**. This is not necessary for the *abstract* theory, but essential for these *explicit* calculations.
- We point out, once more, that the constructions given in this chapter are for the **exponential** Atiyah classes.



5.1 The first three Atiyah classes

(5.1.1) The first Atiyah class. We have already calculated the first exponential Atiyah class in (4.1.12), and we saw, in (4.1.15), that its trace was d-closed. This means that constructing the manual lift is easy: we have nothing left to do. Recalling that we wrote $\omega_{\alpha\beta} = M_{\alpha\beta}^{-1} dM_{\alpha\beta}$, we can draw the closed element in the Čech-de Rham bicomplex as follows:

$$\begin{array}{ccc} & 0 & \\ & \uparrow & \\ & d & \\ & | & \\ \text{tr}(\omega_{\alpha\beta}) & \xrightarrow{\delta} & 0 \end{array}$$

(5.1.2) The second Atiyah class. By (4.1.18), we know that

$$\text{at}_E^{\circ 2} = \left\{ \omega_{\alpha\beta} M_{\alpha\beta} \omega_{\beta\gamma} M_{\alpha\beta}^{-1} \right\}_{\alpha,\beta}$$

and we introduce the notation

$$\begin{aligned} A &= \omega_{\alpha\beta} & M &= M_{\alpha\beta} \\ B &= \omega_{\alpha\gamma} & X &= M \omega_{\beta\gamma} M^{-1} \end{aligned}$$

so that $\text{at}_E^{\circ 2} = AX$, and everything can be thought of as living over U_α .

Thanks to (4.1.14), we know that $dA = -A^2$, and similarly for B and X . Further, by differentiating the cocycle condition $M_{\alpha\beta} M_{\beta\gamma} = M_{\alpha\gamma}$ of the transition maps, and then right-multiplying by $M_{\alpha\gamma}^{-1}$, we see that^[1] $A + X = B$. Hence

$$\text{at}_E^{\circ 2} = A(B - A).$$

^[1]As we already said, $M \omega_{\beta\gamma} M^{-1}$ is the natural way of thinking of X as being a map *into* something lying over U_α , so this equation should be read as a cocycle condition over U_α by thinking of it as $\omega_{\alpha\beta} + \tilde{\omega}_{\beta\gamma} = \omega_{\alpha\gamma}$, where the tilde corresponds to a base change. Note that this is also the result we expect, since $\omega_{\alpha\beta}$ corresponds to $\nabla_\beta - \nabla_\alpha$, and (as we have already noted) this clearly satisfies the additive cocycle condition.

Using the fact that $dA = A^2$ we see that $dA^2 = 0$, whence

$$d \operatorname{tr}(\operatorname{at}_E^{\circ 2}) = d \operatorname{tr}(AB - A^2) = -\operatorname{tr}(A^2B - AB^2).$$

Recalling (5.0.3), we want $f \in \check{C}^1(\Omega_X^3)$ such that $\delta f = -d \operatorname{tr}(\operatorname{at}_E^{\circ 2})$ and $df = 0$. It is clear that we need, at least, f to be (the trace of) a polynomial of homogeneous degree 3 in the one variable $A = \omega_{\alpha\beta}$. But then $f(A) = \operatorname{tr}(A^3)$ is, up to a scalar multiple, our only option. We set $f(A) = \frac{1}{3} \operatorname{tr}(A^3)$ and compute its Čech coboundary:

$$\begin{aligned} (\delta f)_{\alpha\beta\gamma} &= f(\omega_{\beta\gamma}) - f(\omega_{\alpha\gamma}) + f(\omega_{\alpha\beta}) \\ &= f(B - A) - f(B) + f(A) \\ &= \frac{1}{3} \operatorname{tr}((B - A)^3 - B^3 + A^3) \\ &= \frac{1}{3} \operatorname{tr}(-AB^2 - BAB - B^2A + A^2B + ABA + BA^2). \end{aligned}$$

Using (4.4.2) we see that

$$\begin{aligned} \frac{1}{3} \operatorname{tr}(-AB^2 - AB^2 - AB^2 + A^2B + A^2B + A^2B) &= \operatorname{tr}(A^2B - AB^2) \\ &= -d \operatorname{tr}(\operatorname{at}_E^{\circ 2}). \end{aligned}$$

Now we just have to worry about whether or not df is zero. But

$$\begin{aligned} df(A) &= \frac{1}{3} d \operatorname{tr}(A^3) \\ &= \frac{1}{3} \operatorname{tr}(dAA^2 - AdA^2) \\ &= -\frac{1}{3} \operatorname{tr}(A^4), \end{aligned}$$

and we know, by (4.4.3), that this is zero, and so we are done.

This calculation can be summarised by the following diagram:

$$\begin{array}{ccc} & 0 & \\ & \uparrow d & \\ -\frac{1}{3} \operatorname{tr}(A^3) & \xrightarrow{\delta} & \operatorname{tr}(A(B-A)B) \\ & & \uparrow d \\ & & \underbrace{\operatorname{tr}(A(B-A))}_{\operatorname{tr at}_E^{\circ 2}} \xrightarrow{\delta} 0 \end{array}$$

which gives us the closed^[1] element

$$\left(0, \frac{1}{3} \operatorname{tr}(A^3), \operatorname{tr}(AX), 0, 0\right) \in \operatorname{Tot}^4 \check{C}^\bullet(\Omega_X^\star)$$

(5.1.3) We know that $\operatorname{tr} \operatorname{at}_E^{\circ 2}$ is a cocycle by definition, but we can still double-check that it is Čech closed:

$$\begin{aligned} \check{\delta} \operatorname{tr}(\omega_{\alpha\beta}(\omega_{\alpha\gamma} - \omega_{\alpha\beta})) &= \check{\delta} \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) \\ &= \operatorname{tr}(\omega_{\beta\gamma}\omega_{\beta\delta}) - \operatorname{tr}(\omega_{\alpha\gamma}\omega_{\alpha\delta}) + \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\delta}) - \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) \\ &= \operatorname{tr}((\omega_{\alpha\gamma} - \omega_{\alpha\beta})(\omega_{\alpha\delta} - \omega_{\alpha\beta})) - \operatorname{tr}(\omega_{\alpha\gamma}\omega_{\alpha\delta}) + \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\delta}) - \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) \\ &= -\operatorname{tr}(\omega_{\alpha\gamma}\omega_{\alpha\beta}) + \operatorname{tr}(\omega_{\alpha\beta}^2) - \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) \\ &= \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) - \operatorname{tr}(\omega_{\alpha\beta}\omega_{\alpha\gamma}) = 0. \end{aligned}$$

(5.1.4) **The third Atiyah class.** We extend our notation from (5.1.2), writing

$$\begin{aligned} A &= \omega_{\alpha\beta} & M &= M_{\alpha\beta} & X &= M\omega_{\beta\gamma}M^{-1} \\ B &= \omega_{\alpha\gamma} & N &= M_{\alpha\gamma} & Y &= N\omega_{\gamma\delta}N^{-1} \\ C &= \omega_{\alpha\delta} \end{aligned}$$

so that $\operatorname{at}_E^{\circ 3} = AXY = A(B-A)(C-B)$.

It is then relatively simple to calculate that

$$d \operatorname{tr} \operatorname{at}_E^{\circ 3} = -\operatorname{tr}(A(B-A)(C-B)C) \in \check{C}^3(\Omega_X^4).$$

Trying to find some $\varphi \in \check{C}^2(\Omega_X^4)$ such that $\check{\delta}\varphi = d \operatorname{tr} \operatorname{at}_E^{\circ 3}$, however, is slightly harder. The most naive approach is to list all the monomials in $\check{C}^2(\Omega_X^4)$, apply the Čech differential to each one, and then equate coefficients. Using the fact that (up to a sign) we can cyclically permute under the trace, finding all the monomials is the same as finding all degree-2 monomials in non-commutative variables X and Y , modulo equivalence under cyclic permutation, and there are just four of these: X^2Y^2 , $(XY)^2$, X^3Y , and XY^3 . Thus, we find that

$$\varphi = \frac{-1}{4} \operatorname{tr}((A(B-A))^2) + \frac{1}{2} \operatorname{tr}(A^2(B-A)^2) + \frac{-1}{2} \operatorname{tr}(A^3(B-A)) + \frac{-1}{2} \operatorname{tr}(A(B-A)^3)$$

is exactly such that $\check{\delta}\varphi = d \operatorname{tr} \operatorname{at}_E^{\circ 3}$. Factoring $d\varphi$, we see that

$$d\varphi = \frac{1}{10} \operatorname{tr}((B-A)^5 - B^5 + A^5) = \check{\delta} \frac{1}{10} \operatorname{tr}(A^5).$$

^[1]See (5.1.3).

This calculation can be summarised in the following diagram:

$$\begin{array}{c}
 0 \\
 \uparrow \\
 d \\
 \frac{1}{10} \operatorname{tr}(A^5) \xrightarrow{\delta} \frac{1}{10} \operatorname{tr}((B-A)^5 - B^5 + A^5) \\
 \uparrow \\
 d \\
 -\frac{1}{4} \operatorname{tr}((A(B-A))^2) + \frac{1}{2} \operatorname{tr}(A^2(B-A)^2) \\
 -\frac{1}{2} \operatorname{tr}(A^3(B-A)) - \frac{1}{2} \operatorname{tr}(A(B-A)^3) \xrightarrow{\delta} -\operatorname{tr}(A(B-A)(C-B)C) \\
 \uparrow \\
 d \\
 \underbrace{\operatorname{tr}(A(B-A)(C-B))}_{\operatorname{tr}at_E^{\circ 3}} \xrightarrow{\delta} 0
 \end{array}$$

Taking the signs of the total differential into account, this gives us the closed element

$$\left(0, \frac{-1}{10} \operatorname{tr}(A^5), \rho(A, X), \operatorname{tr}(AXY), 0, 0, 0 \right) \in \operatorname{Tot}^6 \check{C}^\bullet(\Omega_X^*)$$

where

$$\rho(A, X) = \frac{-1}{4} \operatorname{tr}(AXAX) + \frac{1}{2} \operatorname{tr}(A^2X^2) + \frac{-1}{2} \operatorname{tr}(A^3X) + \frac{-1}{2} \operatorname{tr}(AX^3).$$

5.2 The fourth Atiyah class and beyond

(5.2.1) Looking at the first three Atiyah classes, there are some patterns that we can begin to see — for example, the Čech 1-cocycle of the k th Atiyah class seems to always be some multiple of $\operatorname{tr}(A^{2k-1})$. But beyond some vague pattern recognition, it becomes increasingly hard to work with the k th Atiyah class as soon as $k \geq 4$, mainly due to the cumbersome amount of monomials that we have to consider in order to equate coefficients in this naive approach. It seems believable that there are two patterns — one for k odd, and one for k even — but even this is hard to verify, since we don't have many explicit cases to study: $k = 0, 1$ are both rather trivial, and $k \geq 4$ is so unwieldy that it is far too hard (for this author) to spot any patterns 'by hand'. For the sake of completeness (and also partly in defence of the

fact that there really are a lot of terms to look at), we give below the lift of $\text{at}_E^{\circ 4}$, which was calculated with a basic Haskell implementation of the method used in (5.1.4): calculating all non-commutative monomials of a certain degree, applying the Čech differential, and then equating coefficients.

(5.2.2) The lift of $\text{at}_E^{\circ 4}$ in the total complex is

$$\left(0, \frac{-1}{35} \text{tr} P^{(1,7)}, \frac{1}{5} \text{tr} P^{(2,6)}, \frac{1}{5} \text{tr} P^{(3,5)}, \text{tr} P^{(4,4)}, 0, 0, 0, 0\right) \in \text{Tot}^8 \check{C}^\bullet(\Omega_X^*)$$

where

$$P^{(4,4)} = A(B-A)(C-B)(D-C)$$

$$\begin{aligned} P^{(3,5)} &= \frac{13}{5} A^5 + 13A^4(B-A) + 5A^3(B-A)^2 + 5A^3(B-A)(C-A) \\ &\quad + 3A^3(C-A)(B-A) + 4A^2(B-A)A(B-A) + 4A^2(B-A)A(C-A) \\ &\quad + 3A^2(B-A)^3 - A^2(B-A)^2(C-A) + 5A^2(B-A)(C-A)^2 \\ &\quad + 5A^2(C-A)A(B-A) + 2A^2(C-A)(B-A)^2 + A^2(C-A)(B-A)(C-A) \\ &\quad + 3A^2(C-A)^2(B-A) - A(B-A)A(C-A)(B-A) + 5A(B-A)A(C-A)^2 \\ &\quad - 5A(B-A)^2(C-A)(B-A) + 5A(B-A)(C-A)A(C-A) + 5A(B-A)(C-A)^3 \\ &\quad + 4(A(C-A))^2(B-A) - 2A(C-A)(B-A)^3 + 4A(C-A)(B-A)^2(C-A) \\ &\quad + A((C-A)(B-A))^2 + 2A(C-A)^2(B-A)^2 + A(C-A)^2(B-A)(C-A) \\ &\quad + 3A(C-A)^3(B-A) \end{aligned}$$

$$\begin{aligned} P^{(2,6)} &= 5A^5(B-A) - 4A^4(B-A)^2 + A^3(B-A)A(B-A) + A^3(B-A)^3 \\ &\quad - 5A^2(B-A)A(B-A)^2 - 4A^2(B-A)^2A(B-A) - 4A^2(B-A)^4 \\ &\quad + \frac{1}{3}(A(B-A))^3 + A(B-A)A(B-A)^3 + A(B-A)^5 \end{aligned}$$

$$P^{(1,7)} = A^7$$

(5.2.3) The moral of the above calculation is the following: we need to find a better way of doing this. Enter Chapter 6.

Simplicial construction

6

A novice was trying to fix a broken Lisp machine by turning the power off and on. Knight, seeing what the student was doing, spoke sternly: “You cannot fix a machine by just power-cycling it with no understanding of what is going wrong.” Knight turned the machine off and on. The machine worked.

<http://catb.org/jargon/html/koans.html>

(6.0.1) Purpose. *In this chapter I introduce Green’s construction of simplicial Atiyah classes, from [Gre80]. The original treatment is rather terse; what I have written here is probably too far in the opposite direction, so I recommend those interested to read both and take some average between them. A lot of the explicit calculations (especially those in Section 6.2) are not found in [Gre80]. This chapter has more hand waving than a Rodgers and Hammerstein musical, but everything should be formally justified in Chapter 7, as well as being generalised from pullbacks (to the nerve) of global vector bundles to arbitrary (well, really, so-called Green) vector bundles on the nerve.*

(6.0.2) The main idea behind this chapter is the following: in the same way that we cannot generally resolve a coherent sheaf by locally free sheaves, but *can* if we pass to similar *simplicial* objects, we cannot generally find a (global holomorphic) connection on a locally free sheaf, but we *can* if we content ourselves with some *simplicial* version. This means that we can take the ‘simplicial curvature’ (whatever this might mean) and apply some invariant polynomial to find a characteristic class, just as we do in the classical (that is, non-simplicial) case. Looking ahead to Chapter 9, one of the main results of [Gre80] is that you can resolve any coherent sheaf (on a sufficiently nice space) by locally free sheaves on the nerve in such a way that you *also* get ‘simplicial connections’ on each bundle that behave ‘very nicely’.



(6.0.3) In this chapter we opt for the approach of blindly calculating examples before formalising why they should even have any sense at all, partially because this can often be a good pedagogical tool, but mostly because that is exactly how the author proceeded when first studying the vast majority of the content here. We formalise everything in Chapter 7.

(6.0.4) We continue to use the notation from (4.0.2). Most of what we say can (probably) be generalised to arbitrary simplicial complex manifolds, but we work throughout with $Y_{\bullet} = X_{\bullet}^U$.

6.1 Global simplicial connections

(6.1.1) We introduce the following notation:

- write $\pi_p: X_p^{\mathcal{U}} \times \Delta^p \rightarrow X_p^{\mathcal{U}}$ to mean the projection map;
- write E^p to mean the pullback of E to $X_p^{\mathcal{U}}$;
- and write $\overline{E^p} = \pi_p^* E^p$, where the pullback here is the *algebraic* one, i.e.

$$\pi_p^* E^p = \pi_p^{-1} E^p \otimes_{\pi_p^{-1} \mathcal{O}_{X_p^{\mathcal{U}}}} \mathcal{O}_{X_p^{\mathcal{U}} \times \Delta^p}$$

where $(\cdot)^{-1}$ is the *set-theoretic* pullback.

Note that the map $\mathcal{O}_{\Delta^p} \rightarrow \mathcal{O}_{X_p^{\mathcal{U}} \times \Delta^p}$ gives an \mathcal{O}_{Δ^p} -action on E^p . Further, we can use π_p to pull back sections^[1] and thus define a connection on E^p from any connection on E^0 (see (6.1.2)), and require that Leibniz's rule holds (see (6.1.3)). As a small abuse of notation, we may simply write ∇_α to mean the pullback connection $\pi_p^* \nabla_\alpha$.

With this definition, ∇_α -flat sections of $\overline{E^p}$ are exactly those of the form $\pi_p^*(s)$ where s is some ∇_α -flat section of $E|_{U_\alpha}$.

(6.1.2) Pullbacks (to the nerve) of (global) vector bundles are *strongly cartesian* (4.3.2). This means that, writing $\zeta_p^i: [0] \rightarrow [p]$ to mean the map of abstract simplices that sends 0 to i , we can identify $(X_\bullet^{\mathcal{U}} \zeta_p^i)^* E^0$ with E^p , by using the isomorphism $E^\bullet \zeta_p^i$ between them. This lets us, in particular, think of any connection ∇_{α_i} on E^0 as a connection on E^p .

(6.1.3) **Definition.** ([Gre80, p. 34]). Define the *barycentric connection* ∇_\bullet^μ on $\overline{E^\bullet}$ by^[2]

$$\nabla_p^\mu = \sum_{i=0}^p t_i \nabla_{\alpha_i}: \overline{E^p} \rightarrow \overline{E^p} \otimes \Omega_{X_p^{\mathcal{U}} \times \Delta^p}^1$$

which acts on a section $s \otimes \varphi$ of $\overline{E^p} = \pi_p^{-1} E^p \otimes \mathcal{O}_{X_p^{\mathcal{U}} \times \Delta^p}$ over $U_{\alpha_0 \dots \alpha_p} \times \Delta^p$ by

$$\begin{aligned} \nabla_p^\mu(s \otimes \varphi) &= \sum_{i=0}^p t_i \nabla_{\alpha_i}(s \otimes \varphi) = \sum_i t_i \nabla_{\alpha_i}(\varphi s \otimes 1) \\ &= \sum_i t_i \left(\varphi \pi_p^*(\nabla_{\alpha_i}(s)) + s \otimes 1 \otimes d\varphi \right) \end{aligned}$$

[1]Note that $\pi_p^*(s^\ell) = s^\ell \otimes 1$ for any section s^ℓ of E^p over $U_{\alpha_0 \dots \alpha_p}$.

[2]Using (6.1.2) to make sense of ∇_{α_i} as a connection on E^p , and recalling, from (6.1.1), that we write ∇_{α_i} when, *really*, we mean $\pi_p^* \nabla_{\alpha_i}$.

$$= \sum_i \pi_p^* (t_i \varphi \otimes \nabla_{\alpha_i}(s)) + s \otimes 1 \otimes t_i d\varphi$$

where we use the \mathcal{O}_{Δ^p} -action on E^p described in (6.1.1) for the second equality.

(6.1.4) Note that, a priori, we do not know if the ∇_p^μ glue to give us a connection with values in simplicial forms, nor if the curvature (defined in (6.1.6)) will give us an endomorphism-valued simplicial 2-form (something which we have yet to define). That is, we haven't justified the fact that we call this map 'simplicial', nor that we call it a 'connection'. This is the subject of Section 7.3, where we introduce the notion of *admissibility*.

(6.1.5) Recalling the notation from (4.1.9), we have an alternative expression for the barycentric connection:

$$\begin{aligned} \nabla_p^\mu &= \nabla_{\alpha_0} + \sum_{i=1}^p t_i (\nabla_{\alpha_i} - \nabla_{\alpha_0}) \\ &= \nabla_{\alpha_0} + \sum_{i=1}^p t_i \omega_{\alpha_0 \alpha_i}. \end{aligned}$$

In some senses this is a more natural expression to work with, since it places the same emphasis on the choice of α_0 in $\alpha_0 \dots \alpha_p$ that twisting cochains do.^[1]

(6.1.6) **Definition.** Defining the *curvature* of the barycentric connection analogously to the classic case^[2], we can see how it acts on a section σ^{α_0} of $\overline{E^p}$ over $U_{\alpha_0 \dots \alpha_p} \times \Delta^p$:

$$\begin{aligned} \kappa(\nabla_p^\mu)(\sigma^{\alpha_0}) &= \left(\nabla_{\alpha_0} + \sum_{i=1}^p t_i \omega_{\alpha_0 \alpha_i} \right)^2 (\sigma^{\alpha_0}) \\ &= \nabla_{\alpha_0}^2 (\sigma^{\alpha_0}) + \sum_{i=1}^p \left[(\nabla_{\alpha_0} (\sigma^{\alpha_0}) \cdot t_i \omega_{\alpha_0 \alpha_i}) + \sigma^{\alpha_0} \otimes d(t_i \omega_{\alpha_0 \alpha_i}) \right] \\ &\quad + \sum_{i=1}^p \left[t_i \omega_{\alpha_0 \alpha_i} \cdot \nabla_{\alpha_0} (\sigma^{\alpha_0}) \right] + \sum_{i,j=1}^p \sigma^{\alpha_0} \otimes (t_j t_i \omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i}). \end{aligned}$$

(6.1.7) If our local sections $\{s_1^{\alpha_0}, \dots, s_r^{\alpha_0}\}$ of $E|_{U_{\alpha_0}}$ happen to be flat (for ∇_{α_0}) then we can simplify the expression in (6.1.6) in the case where $\sigma_k^{\alpha_0} = \pi_p^*(s_k^{\alpha_0})$, since then^[3]

$$\nabla_{\alpha_0} (\sigma_k^{\alpha_0}) = \pi_p^* (\nabla_{\alpha_0} (s_k^{\alpha_0})) = 0$$

^[1]As mentioned in (G.1.4).

^[2]That is, $\kappa(\nabla_\bullet^\mu) = \nabla_\bullet^\mu \circ \nabla_\bullet^\mu$, where we use (4.1.5) to define the composition, as in (4.1.6).

^[3]By the comment at the end of (6.1.1), we are really just saying that the $\sigma_k^{\alpha_0}$ are flat.

whence

$$\begin{aligned}
 & \nabla_{\alpha_0}^2 (\sigma_k^{\alpha_0}) + \sum_{i=1}^p \left[(\nabla_{\alpha_0} (\sigma_k^{\alpha_0}) \cdot t_i \omega_{\alpha_0 \alpha_i}) + \sigma_k^{\alpha_0} \otimes d(t_i \omega_{\alpha_0 \alpha_i}) \right] \\
 & + \sum_{i=1}^p \left[t_i \omega_{\alpha_0 \alpha_i} \cdot \nabla_{\alpha_0} (\sigma_k^{\alpha_0}) \right] + \sum_{i,j=1}^p \sigma_k^{\alpha_0} \otimes (t_j t_i \omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i}) \\
 & = \sum_{i=1}^p \sigma_k^{\alpha_0} \otimes d(t_i \omega_{\alpha_0 \alpha_i}) + \sum_{i,j=1}^p \sigma_k^{\alpha_0} \otimes (t_j t_i \omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i}).
 \end{aligned}$$

In essence, the above calculation tells us that, given a *flat* basis of sections of $\overline{E^p} \big|_{(U_{\alpha_0 \dots \alpha_p} \times \Delta^p)}$ (in the U_{α_0} trivialisation), the curvature simply acts by

$$\kappa(\nabla_p^\mu) = d\overline{\omega}_p + \overline{\omega}_p \cdot \overline{\omega}_p$$

where we define the endomorphism-valued type- $(-1, 0)$ form $\overline{\omega}_\bullet$ by

$$\overline{\omega}_p = \sum_{i=1}^p t_i \omega_{\alpha_0 \alpha_i}.$$

Recall (4.3.6): the differential for simplicial forms is given by the product rule, but with signs given by the Koszul convention according to the type of the form (see (A.1.1) and (A.1.2)). Here this simply means that $d\overline{\omega}_p = (d_X - d_{\Delta^p})\overline{\omega}_p$. There are a few important points to make here.

- 1) There is another (reasonably justified) abuse of notation here: we write $t_i \omega_{\alpha_0 \alpha_i}$ when we really mean $(t_i I_r) \cdot (\omega_{\alpha_0 \alpha_i})$, where I_r is the $(r \times r)$ identity matrix. That is, we (*locally*) identify $\mathcal{E}nd(\overline{E^p})$ -valued forms on $X_p^U \times \Delta^p$ with $(r \times r)$ -matrix-valued forms on X_p^U tensored with forms on Δ^p *times the* $(r \times r)$ *identity matrix*.
- 2) In (6.1.5) we wrote the barycentric connection in the form $d + \overline{\omega}_p$ (since, in the U_{α_0} trivialisation, with flat local sections, ∇_{α_0} ‘equals’ d). This is exactly like the classical case: locally, a connection looks like $d + A$, where A is some matrix-valued 1-form. What we have just shown, then, is that the simplicial curvature is of the form $dA + A \cdot A$, exactly as in the classical case.
- 3) We would hope that the $\overline{\omega}_p$ glue to give us an endomorphism-valued type- $(-1, 0)$ *simplicial* 2-form, but, again, this is something that we need to prove (and first, *define*).
- 4) In the non-simplicial case we had that $d\omega_{\alpha\beta} = -\omega_{\alpha\beta}^2$, but the analogous statement for $\overline{\omega}_p$ is *not* true (i.e. the barycentric connection is not flat). We discuss this further in (6.1.12).

(6.1.8) **Definition.** We define the k th simplicial exponential Atiyah class $\hat{\text{at}}_E^{\text{ok}}$ by

$$\begin{aligned} \hat{\text{at}}_E^{\text{ok}} &= \left\{ \epsilon_k \left(d\bar{\omega}_p + \bar{\omega}_p \cdot \bar{\omega}_p \right)^k \right\}_{p \in \mathbb{N}} \\ &= \left\{ \epsilon_k \left(- \sum_{i=1}^p \omega_{\alpha_0 \alpha_i} \otimes dt_i - \sum_{i=1}^p t_i \omega_{\alpha_0 \alpha_i}^2 + \sum_{i,j=1}^p t_j t_i \omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i} \right)^k \right\}_{p \in \mathbb{N}} \end{aligned}$$

where $\bar{\omega}_p$ is the $\overline{E^p}$ -endomorphism-valued 2-form on $X_p^{\mathcal{U}} \times \Delta^p$ defined as in (6.1.7), and $\epsilon_k = (-1)^{k(k-1)/2}$. The reason for the sign ϵ_k is that we want the product of simplicial forms to respect fibre integration, in some sense. In particular, here, we want the sign of the (k, k) -term of $\text{tr} \int_{\Delta^k} \hat{\text{at}}_E^{\text{ok}}$ to agree with the sign of at_E^{ok} (as Čech k -cocycles of k -forms on X). As shown in (6.3.1), the (k, k) -term involves changing sign T_{k-1} times, where T_n is the n -th triangle number.

(6.1.9) **Definition.** We define the k -th simplicial standard Atiyah class $\hat{\text{at}}_E^{\wedge k}$ by

$$\begin{aligned} \hat{\text{at}}_E^{\wedge k} &= \left\{ \epsilon_k \bigwedge_{i=1}^k \left(d\bar{\omega}_p + \bar{\omega}_p \cdot \bar{\omega}_p \right) \right\}_{p \in \mathbb{N}} \\ &= \left\{ \epsilon_k \bigwedge_{i=1}^k \left(- \sum_{i=1}^p \omega_{\alpha_0 \alpha_i} \otimes dt_i - \sum_{i=1}^p t_i \omega_{\alpha_0 \alpha_i}^2 + \sum_{i,j=1}^p t_j t_i \omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i} \right) \right\}_{p \in \mathbb{N}} \end{aligned}$$

where $\bar{\omega}_p$ and ϵ_k are as in (6.1.8).



(6.1.10) To repeat ourselves once again, we **expect** both $\hat{\text{at}}_E^{\text{ok}}$ and $\hat{\text{at}}_E^{\wedge k}$ to be given (locally) by an endomorphism-valued simplicial differential form (i.e. for them to satisfy the condition in (4.3.6)), but we need to prove this. We also note that there is **no reason** to expect either $\hat{\text{at}}_E^{\text{ok}}$ or $\hat{\text{at}}_E^{\wedge k}$ to be a cocycle *before* applying fibre integration.

(6.1.11) **Lemma.** The trace of the k -th simplicial Atiyah class is d-closed.

Proof. Thanks to the product rule, it suffices to prove the case $k = 1$. For all $p \in \mathbb{N}$,

$$\begin{aligned} d(d\bar{\omega}_p + \bar{\omega}_p \cdot \bar{\omega}_p) &= d^2 \left(\sum_{i=0}^p t_i \omega_{\alpha_0 \alpha_i} \right) + d \left(\sum_{i,j=1}^p t_j t_i \omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i} \right) \\ &= d \left(\sum_{i,j=1}^p t_j t_i \omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i} \right) \\ &= \sum_{i,j=1}^p \left[\omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i} \otimes d_{\Delta}(t_j t_i) + t_j t_i d_X(\omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i}) \right] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i,j=1}^p \left[\omega_{\alpha_0\alpha_j} \omega_{\alpha_0\alpha_i} \otimes t_i dt_j + \omega_{\alpha_0\alpha_j} \omega_{\alpha_0\alpha_i} \otimes t_j dt_i \right. \\
 &\quad \left. + t_j t_i d_X(\omega_{\alpha_0\alpha_j}) \omega_{\alpha_0\alpha_i} - t_j t_i \omega_{\alpha_0\alpha_j} d_X(\omega_{\alpha_0\alpha_i}) \right].
 \end{aligned}$$

But, recalling (4.4.2), we know that $\text{tr}(\omega_{\alpha_0\alpha_j} \omega_{\alpha_0\alpha_i}) = -\text{tr}(\omega_{\alpha_0\alpha_i} \omega_{\alpha_0\alpha_j})$, and (4.1.14) tells us that $d\omega_{\alpha_0\alpha_i} = -\omega_{\alpha_0\alpha_i}^2$, whence

$$\begin{aligned}
 \text{tr } d(\overline{d\omega}_p + \overline{\omega}_p \cdot \overline{\omega}_p) &= \text{tr} \sum_{i,j=1}^p \left[\omega_{\alpha_0\alpha_j} \omega_{\alpha_0\alpha_i} \otimes t_i dt_j + \omega_{\alpha_0\alpha_j} \omega_{\alpha_0\alpha_i} \otimes t_j dt_i \right. \\
 &\quad \left. + t_j t_i d_X(\omega_{\alpha_0\alpha_j}) \omega_{\alpha_0\alpha_i} - t_j t_i \omega_{\alpha_0\alpha_j} d_X(\omega_{\alpha_0\alpha_i}) \right] \\
 &= \text{tr} \sum_{i,j=1}^p \left[\omega_{\alpha_0\alpha_j} \omega_{\alpha_0\alpha_i} \otimes t_i dt_j - \omega_{\alpha_0\alpha_i} \omega_{\alpha_0\alpha_j} \otimes t_j dt_i \right. \\
 &\quad \left. + t_j t_i \omega_{\alpha_0\alpha_j}^2 \omega_{\alpha_0\alpha_i} - t_j t_i \omega_{\alpha_0\alpha_j} \omega_{\alpha_0\alpha_i}^2 \right].
 \end{aligned}$$

For fixed i, j , the first two terms both change sign under $i \leftrightarrow j$, whence, from (4.4.4), they contribute zero to the trace, since they are equal when $i = j$. We also know, from (4.4.2), that $\text{tr}(\omega_{\alpha_0\alpha_j}^2 \omega_{\alpha_0\alpha_i}) = \text{tr}(\omega_{\alpha_0\alpha_i}^2 \omega_{\alpha_0\alpha_j})$, whence the last two terms also both change sign under $i \leftrightarrow j$ and are equal when $i = j$, as above. Thus the trace is zero. \square

This tells us, as in (4.1.15), that the trace of the Atiyah classes will give us classes in de Rham cohomology, after applying fibre integration.^[1]

(6.1.12) Compare (6.1.11) with (4.1.15) and, more generally, the construction of the non-simplicial Atiyah class with that of the simplicial Atiyah class. There is an important difference. *Assuming we have (local) flat connections* ∇_α , then, in the non-simplicial case, the Atiyah class is given by $\omega_{\alpha\beta} = \nabla_\beta - \nabla_\alpha$, and this satisfies $d\omega_{\alpha\beta} + d\omega_{\alpha\beta}^2 = 0$; in the simplicial case, however, we average ('barycentrically') over all the $\omega_{\alpha\beta}$ to give us a connection which is *not* necessarily flat, even if all the local connection ∇_α are, since $d\overline{\omega}_p \neq \pm \overline{\omega}_p^2$ (because, e.g., the former has non-trivial type-(1, 1) terms of the form $\omega_{\alpha_0\alpha_i} \otimes dt_i$, and the latter has only type-(2, 0) terms). In fact, as we have seen, it is exactly the *curvature* of this barycentric average which gives us the simplicial Atiyah class.

There are a few points to note here:

- both the classical and the simplicial Atiyah classes are of total degree 2, but the part of differential-degree 2 of the classical class is zero;

^[1]Since fibre integration is a morphism of complexes, it sends closed elements to closed elements.

- the classical Atiyah class is the *obstruction* to finding a global connection, but the simplicial Atiyah class *is* a ‘global’ connection (where we relax/extend our idea of what ‘global’ should mean).

Because of this, it may be mildly misleading to use the notation ω for both the classical and the simplicial constructions, but it seems to be a reasonably natural choice.

(6.1.13) Recall that we don’t need our local connections ∇_α to be flat in order to *define* these things, but just to *explicitly calculate* them (for example, to obtain the result that the Atiyah class is given (locally) by $M_{\alpha\beta}^{-1}dM_{\alpha\beta}$).

6.2 Semi-explicit examples

(6.2.1) In this section we calculate the first few simplicial (exponential) Atiyah classes for a locally free sheaf E and show that they agree with the manual constructions found in Chapter 5, as well as giving a more general proof (6.3.1) that works for *all* simplicial (exponential) Atiyah classes. For a truly explicit example (i.e. where we take a specific coherent sheaf, calculate some simplicial resolution of it, and then calculate all the relevant barycentric connections and their curvatures), see Section 9.3.

(6.2.2) **Example** ($k = 1$). The first simplicial (exponential) Atiyah class is given by

$$\hat{\text{at}}_E^{\circ 1} = \left\{ - \sum_{i=1}^p \omega_{\alpha_0\alpha_i} \otimes dt_i - \sum_{i=1}^p t_i \omega_{\alpha_0\alpha_i}^2 + \sum_{i,j=1}^p t_j t_i \omega_{\alpha_0\alpha_j} \omega_{\alpha_0\alpha_i} \right\}_{p \in \mathbb{N}}$$

As explained in (A.2.1), the fibre integral of $\hat{\text{at}}_E^{\circ 1}$ depends only on the $(2, 0)$, $(1, 1)$, and $(0, 2)$ parts, but here there is no $(0, 2)$ part (i.e. there is no $dt_j dt_i$ term), and so

$$\begin{aligned} \int_{\Delta^\bullet} \hat{\text{at}}_E^{\circ 1} &= (-1) \int_{\Delta^1} - \sum_{i=1}^{p=1} \omega_{\alpha_0\alpha_i} \otimes dt_i - \int_{\Delta^0} \sum_{i=1}^{p=0} t_i \omega_{\alpha_0\alpha_i}^2 + \int_{\Delta^0} \sum_{i,j=1}^{p=0} t_j t_i \omega_{\alpha_0\alpha_j} \omega_{\alpha_0\alpha_i} \\ &= \int_{\Delta^1} \omega_{\alpha_0\alpha_1} \otimes dt_1. \end{aligned}$$

Continuing the calculation gives us the result that we expect: as in (5.1.1), we get that

$$\text{tr} \int_{\Delta^\bullet} \hat{\text{at}}_E^{\circ 1} = \text{tr} \int_0^1 \omega_{\alpha_0\alpha_1} dt_1 = \underbrace{\text{tr}(\omega_{\alpha_0\alpha_1})}_{p=1} = \text{tr at}_E^{\circ 1},$$

where we write that this term ‘lives in’ $p = 1$, i.e. to remind us that the result is a Čech 1-cocycle.

(6.2.3) In general, as in the manual construction, the (fibre integral of the trace of the) k -th simplicial (exponential) Atiyah class will have terms in $\check{C}^{k-i}(\Omega_X^{k+i})$ for $i = 0, \dots, k-1$, and so we make up for our notational laziness (namely, using $+$ to mean \oplus) by labelling the terms with their Čech degree.

(6.2.4) Note that, although the $(2k, 0)$ part of $\hat{\text{at}}_E^{\circ k}$ is, in general, non-zero, when we fibre integrate we look at it on the 0-simplex (thanks again to (A.2.1)), and there it is zero, since all the sums are trivially zero.

(6.2.5) **Example** ($k = 2$). Not forgetting the sign $\epsilon_2 = -1$, we know that

$$\hat{\text{at}}_E^{\circ 2} = \left\{ - \left(- \sum_{i=1}^p \omega_{\alpha_0 \alpha_i} \otimes dt_i - \sum_{i=1}^p t_i \omega_{\alpha_0 \alpha_i}^2 + \sum_{i,j=1}^p t_j t_i \omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i} \right)^2 \right\}_{p \in \mathbb{N}}$$

but also that the only parts that will be non-zero after fibre integration are the $(2, 2)$ parts on the 2-simplex, and the $(3, 1)$ parts on the 1-simplex.

The only $(2, 2)$ part comes from the first half of the $(d\bar{\omega}_2)^2$ term, which gives us

$$\begin{aligned} \text{tr} \int_{\Delta^2} \hat{\text{at}}_E^{\circ 2} &= \text{tr}(-1)^{2 \cdot 2} \int_{\Delta^2} - \left(- \sum_{i=1}^2 \omega_{\alpha_0 \alpha_i} \otimes dt_i \right)^2 \\ &= \text{tr} \int_{\Delta^2} - \left(\sum_{i,j=1}^2 (\omega_{\alpha_0 \alpha_j} \otimes dt_j) \cdot (\omega_{\alpha_0 \alpha_i} \otimes dt_i) \right) \\ &= \text{tr} \int_{\Delta^2} \sum_{i,j=1}^2 \omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i} \otimes dt_j dt_i \\ &= \text{tr} \int_{\Delta^2} \left(\omega_{\alpha_0 \alpha_1}^2 \otimes (dt_1)^2 + \omega_{\alpha_0 \alpha_1} \omega_{\alpha_0 \alpha_2} \otimes dt_1 dt_2 \right. \\ &\quad \left. + \omega_{\alpha_0 \alpha_2} \omega_{\alpha_0 \alpha_1} \otimes dt_2 dt_1 + \omega_{\alpha_0 \alpha_2}^2 \otimes (dt_2)^2 \right) \\ &= \text{tr} \int_{\Delta^2} \left(\omega_{\alpha_0 \alpha_1} \omega_{\alpha_0 \alpha_2} - \omega_{\alpha_0 \alpha_2} \omega_{\alpha_0 \alpha_1} \right) \otimes dt_1 dt_2 \\ &= \text{tr} \int_0^1 \int_0^{1-t_2} \left(\omega_{\alpha_0 \alpha_1} \omega_{\alpha_0 \alpha_2} - \omega_{\alpha_0 \alpha_2} \omega_{\alpha_0 \alpha_1} \right) \otimes dt_1 dt_2 \\ &= \frac{1}{2} \text{tr} \left(\omega_{\alpha_0 \alpha_1} \omega_{\alpha_0 \alpha_2} - \omega_{\alpha_0 \alpha_2} \omega_{\alpha_0 \alpha_1} \right) \\ &= \frac{1}{2} \cdot 2 \cdot \text{tr} \omega_{\alpha_0 \alpha_1} \omega_{\alpha_0 \alpha_2} \end{aligned}$$

$$\begin{aligned}
 &= \operatorname{tr} \omega_{\alpha_0 \alpha_1} (\omega_{\alpha_0 \alpha_1} + \omega_{\alpha_1 \alpha_2}) \\
 &= \operatorname{tr} \omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2}.
 \end{aligned}$$

This means that, so far, we have

$$\operatorname{tr} \int_{\Delta^\bullet} \hat{\operatorname{at}}_E^{\circ 2} = \underbrace{?}_{p=1} + \underbrace{\operatorname{tr}(\omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2})}_{p=2}.$$

For the (3, 1) part, we work on the 1-simplex and get

$$\begin{aligned}
 \operatorname{tr} \int_{\Delta^1} \hat{\operatorname{at}}_E^{\circ 2} &= \operatorname{tr}(-1)^{3 \cdot 1} \int_{\Delta^1} - \left(- \sum_{i,j=1}^1 (\omega_{\alpha_0 \alpha_j} \otimes dt_j) \cdot (-t_i \omega_{\alpha_0 \alpha_i}^2) \right. \\
 &\quad - \sum_{i,j=1}^1 (-t_j \omega_{\alpha_0 \alpha_j}^2) \cdot (\omega_{\alpha_0 \alpha_i} \otimes dt_i) \\
 &\quad - \sum_{i,j,k=1}^1 (\omega_{\alpha_0 \alpha_k} \otimes dt_k) \cdot (t_j t_i \omega_{\alpha_0 \alpha_j} \omega_{\alpha_0 \alpha_i}) \\
 &\quad \left. - \sum_{i,j,k=1}^1 (t_k t_j \omega_{\alpha_0 \alpha_k} \omega_{\alpha_0 \alpha_j}) \cdot (\omega_{\alpha_0 \alpha_i} \otimes dt_i) \right) \\
 &= \operatorname{tr} \int_0^1 2\omega_{\alpha_0 \alpha_1}^3 (t_1 - t_1^2) dt_1 \\
 &= \frac{1}{3} \operatorname{tr} \omega_{\alpha_0 \alpha_1}^3.
 \end{aligned}$$

Finally then, we have

$$\operatorname{tr} \int_{\Delta^\bullet} \hat{\operatorname{at}}_E^{\circ 2} = \underbrace{\frac{1}{3} \operatorname{tr} \omega_{\alpha_0 \alpha_1}^3}_{p=1} + \underbrace{\operatorname{tr}(\omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2})}_{p=2} = \operatorname{tr} \operatorname{at}_E^{\circ 2}$$

which agrees with the result found in (5.1.2).

(6.2.6) There is a subtlety in the calculations when we reach the third simplicial Atiyah class due to our choice of conventions for Čech cocycles: we don't assume skew-symmetry of cocycles (i.e. that exchanging two indices changes sign), but it is true that skew-symmetrisation of cocycles is a quasi-isomorphism, and so doesn't change the resulting cohomology class^[1]. If we had worked with skew-symmetric Čech cocycles from the start then this calculation would appear in some sense more natural, but we didn't.

^[1]All this says is that you have (at least) two models of Čech cocycles that are quasi-isomorphic: cocycles with arbitrarily-ordered indices, and cocycles with arbitrarily-ordered but skew-symmetric indices.

In particular, the (*traces of the*) first two (exponential) Atiyah classes agree with those that we manually constructed in Chapter 5 on the nose, whereas for $k \geq 3$ we will have equality only in cohomology.



(6.2.7) We write ζ_p to mean the *skew-symmetrisation* of a Čech p -cochain, i.e.

$$(\zeta_p c)_{\alpha_0 \dots \alpha_p} = \frac{1}{(p+1)!} \sum_{\sigma \in S_{p+1}} |\sigma| c_{\alpha_{\sigma(0)} \dots \alpha_{\sigma(p)}}.$$

(6.2.8) **Example** ($k = 3$). Writing μ_i to mean $\omega_{\alpha_0 \alpha_i}$, we start by calculating the (3, 3) part as follows:

$$\begin{aligned} \operatorname{tr} \int_{\Delta^3} \hat{\mathbf{a}}_E^{\circ 3} &= \operatorname{tr}(-1)^{3 \cdot 3} \int_{\Delta^3} - \sum_{i,j,k=1}^3 (-\mu_k \otimes dt_k) \cdot (-\mu_j \otimes dt_j) \cdot (-\mu_i \otimes dt_i) \\ &= \operatorname{tr} \int_{\Delta^3} \sum_{i,j,k=1}^{p=3} \mu_k \mu_j \mu_i \otimes dt_k dt_j dt_i \\ &= \operatorname{tr} \int_{\Delta^3} \sum_{\sigma \in S_3} |\sigma| \mu_{\sigma(1)} \mu_{\sigma(2)} \mu_{\sigma(3)} \otimes dt_1 dt_2 dt_3 \\ &= \frac{1}{6} \operatorname{tr} \sum_{\sigma \in S_3} |\sigma| \mu_{\sigma(1)} \mu_{\sigma(2)} \mu_{\sigma(3)} \\ &= \frac{1}{2} \operatorname{tr}(\mu_1 \mu_2 \mu_3 - \mu_1 \mu_3 \mu_2) \\ &= \frac{1}{2} \operatorname{tr} \left(\omega_{\alpha_0 \alpha_1} (\omega_{\alpha_0 \alpha_1} + \omega_{\alpha_1 \alpha_2}) (\omega_{\alpha_0 \alpha_1} + \omega_{\alpha_1 \alpha_2} + \omega_{\alpha_2 \alpha_3}) \right. \\ &\quad \left. - \omega_{\alpha_0 \alpha_1} (\omega_{\alpha_0 \alpha_1} + \omega_{\alpha_1 \alpha_2} + \omega_{\alpha_2 \alpha_3}) (\omega_{\alpha_0 \alpha_1} + \omega_{\alpha_1 \alpha_2}) \right) \\ &= \frac{1}{2} \operatorname{tr} \left(\omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2} \omega_{\alpha_2 \alpha_3} - \omega_{\alpha_0 \alpha_1} \omega_{\alpha_2 \alpha_3} \omega_{\alpha_1 \alpha_2} \right). \end{aligned}$$

But note that both of the terms in this last expression skew-symmetrise to the same thing, modulo a minus sign:

$$\sum_{\tau \in S_4} |\tau| \omega_{\alpha_{\tau(0)} \alpha_{\tau(1)}} \omega_{\alpha_{\tau(1)} \alpha_{\tau(2)}} \omega_{\alpha_{\tau(2)} \alpha_{\tau(3)}} = - \sum_{\tau \in S_4} |\tau| \omega_{\alpha_{\tau(0)} \alpha_{\tau(1)}} \omega_{\alpha_{\tau(2)} \alpha_{\tau(3)}} \omega_{\alpha_{\tau(1)} \alpha_{\tau(2)}}.$$

Since skew-symmetrisation doesn't change the class in cohomology, we see that

$$\left[\operatorname{tr} \int_{\Delta^3} \hat{\mathbf{a}}_E^{\circ 3} \right] = \left[\underbrace{?}_{p=1} + \underbrace{?}_{p=2} + \underbrace{\operatorname{tr} \omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2} \omega_{\alpha_2 \alpha_3}}_{p=3} \right].$$

The (4, 2) part (including the sign $\epsilon_3 = -1$) is given by

$$X^2 Y + X Y X + Y X^2 - X^2 Z - X Z X - Z X^2$$

where

$$X = - \sum_{i=1}^2 \mu_i \otimes dt_i \quad Y = \sum_{i=1}^2 t_i \mu_i^2 \quad Z = \sum_{i,j=1}^2 t_j t_i \mu_j \mu_i.$$

Using that $\int_{\Delta^2} dt_1 dt_2 = \int_0^1 \int_0^{1-t_2} dt_1 dt_2$, we calculate that

$$(i) \int_{\Delta^2} t_1 dt_1 dt_2 = \int_{\Delta^2} t_2 dt_1 dt_2 = \frac{1}{6};$$

$$(ii) \int_{\Delta^2} t_1^2 dt_1 dt_2 = \int_{\Delta^2} t_2^2 dt_1 dt_2 = \frac{1}{12};$$

$$(iii) \int_{\Delta^2} t_1 t_2 dt_1 dt_2 = \frac{1}{24}.$$

whence

$$\begin{aligned} \text{tr}(-1)^{2 \cdot 2} \int_{\Delta^2} X^2(Y-Z) + X(Y-Z)X + (Y-Z)X^2 &= -\frac{1}{2} \text{tr} \mu_1^3 \mu_2 - \frac{1}{2} \text{tr} \mu_1 \mu_2^3 + \frac{1}{4} \text{tr} \mu_1 \mu_2 \mu_1 \mu_2 \\ &= -\frac{1}{4} \text{tr} \left((\omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2})^2 \right) - \frac{1}{2} \text{tr} \left(\omega_{\alpha_0 \alpha_1}^3 \omega_{\alpha_1 \alpha_2} - \omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2}^3 \right). \end{aligned}$$

Comparing this to (5.1.4), we see that we have the same, except for a missing $+\frac{1}{2} \text{tr}(\omega_{\alpha_0 \alpha_1}^2 \omega_{\alpha_1 \alpha_2}^2)$ term. But this missing term skew-symmetrises to zero, since it is invariant under the permutation that swaps 0 and 2. Thus the (4, 2) part is exactly what we wanted, and

$$\left[\text{tr} \int_{\Delta^\bullet} \hat{\text{at}}_E^{\circ 3} \right] = \left[\underbrace{?}_{p=1} + \underbrace{\begin{aligned} &-\frac{1}{4} \text{tr} \left((\omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2})^2 \right) \\ &-\frac{1}{2} \text{tr} \left(\omega_{\alpha_0 \alpha_1}^3 \omega_{\alpha_1 \alpha_2} \right) \\ &-\frac{1}{2} \text{tr} \left(\omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2}^3 \right) \\ &+\frac{1}{2} \text{tr} \left(\omega_{\alpha_0 \alpha_1}^2 \omega_{\alpha_1 \alpha_2}^2 \right) \end{aligned}}_{p=2} + \underbrace{\text{tr} \omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2} \omega_{\alpha_2 \alpha_3}}_{p=3} \right].$$

Finally, the (5, 1) part is

$$\text{tr}(-1)^{5 \cdot 1} \int_{\Delta^1} \hat{\text{at}}_E^{\circ 3} = \text{tr} \int_0^1 -(3t_1^2 + 3t_1^4 - 6t_1^3) \mu_1^5 \otimes dt_1 = -\frac{1}{10} \omega_{\alpha_0 \alpha_1}^5$$

which agrees exactly with the manual construction from (5.1.4). Thus

$$\left[\text{tr} \int_{\Delta^\bullet} \hat{\text{at}}_E^{\circ 3} \right] = \left[\underbrace{-\frac{1}{10} \omega_{\alpha_0 \alpha_1}^5}_{p=1} + \underbrace{\begin{aligned} &-\frac{1}{4} \text{tr} \left((\omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2})^2 \right) \\ &-\frac{1}{2} \text{tr} \left(\omega_{\alpha_0 \alpha_1}^3 \omega_{\alpha_1 \alpha_2} \right) \\ &-\frac{1}{2} \text{tr} \left(\omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2}^3 \right) \\ &+\frac{1}{2} \text{tr} \left(\omega_{\alpha_0 \alpha_1}^2 \omega_{\alpha_1 \alpha_2}^2 \right) \end{aligned}}_{p=2} + \underbrace{\text{tr} \omega_{\alpha_0 \alpha_1} \omega_{\alpha_1 \alpha_2} \omega_{\alpha_2 \alpha_3}}_{p=3} \right] = \left[\text{tr} \text{at}_E^{\circ 3} \right].$$

6.3 Equivalence with the manual construction

(6.3.1) **Theorem.** The degree- (k, k) term in the trace of fibre integral of the k th simplicial **exponential** Atiyah class agrees with the k th **exponential** Atiyah class, up to skew-symmetrisation. That is,

$$\zeta_k \left(\text{tr} \int_{\Delta^\bullet} \hat{\text{at}}_E^{\circ k} \right)^{(k,k)} = \zeta_k \text{tr} \left(\text{at}_E^{\circ k} \right) \in \check{C}^k_{\mathcal{U}}(\Omega_X^k).$$

Proof. First we rewrite the left-hand side. Generalising (6.2.8), we can write the term coming from fibre integration as

$$\frac{1}{k!} \sum_{\sigma \in S_k} |\sigma| \mu_{\sigma(1)} \cdots \mu_{\sigma(k)} \xrightarrow{\zeta_k} \frac{1}{k!(k+1)!} \sum_{\tau \in S_{k+1}} \sum_{\sigma \in S_k} |\tau\sigma| \omega_{\tau(0)\tau\sigma(1)} \cdots \omega_{\tau(0)\tau\sigma(k)}$$

where $S_k \leq S_{k+1}$ acts on $\{0, 1, \dots, k\}$ by fixing 0. But then, since $\sigma(0) = 0$, we can rewrite this as

$$\frac{1}{k!(k+1)!} \sum_{\tau \in S_{k+1}} \sum_{\sigma \in S_k} |(\tau\sigma)| \omega_{\tau\sigma(0)\tau\sigma(1)} \cdots \omega_{\tau\sigma(0)\tau\sigma(k)}.$$

Now we can use the fact that multiplication by an element of $S_k \leq S_{k+1}$ is an automorphism in order to perform a change of variables, giving us

$$\frac{1}{(k+1)!} \sum_{\eta \in S_{k+1}} |(\eta)| \omega_{\eta(0)\eta(1)} \cdots \omega_{\eta(0)\eta(k)}$$

which is (trivially, since $\omega_{ii} = 0$) equal to

$$\frac{1}{(k+1)!} \sum_{\eta \in S_{k+1}} |(\eta)| \prod_{i=1}^k (\omega_{\eta(0)\eta(i)} - \omega_{\eta(0)\eta(0)}).$$

Next we rewrite the right-hand side. The skew-symmetrisation is simply

$$\begin{aligned} \zeta_k \text{tr} \left(\text{at}_E^{\circ k} \right) &= \frac{1}{(k+1)!} \sum_{\eta \in S_{k+1}} |\eta| \omega_{\eta(0)\eta(1)} \omega_{\eta(1)\eta(2)} \cdots \omega_{\eta(k-1)\eta(k)} \\ &= \frac{1}{(k+1)!} \sum_{\eta \in S_{k+1}} |\eta| \omega_{\eta(0)\eta(1)} (\omega_{\eta(0)\eta(2)} - \omega_{\eta(0)\eta(1)}) \cdots (\omega_{\eta(0)\eta(k)} - \omega_{\eta(0)\eta(k-1)}) \\ &= \frac{1}{(k+1)!} \sum_{\eta \in S_{k+1}} |\eta| \prod_{i=1}^k (\omega_{\eta(0)\eta(i)} - \omega_{\eta(0)\eta(i-1)}). \end{aligned}$$

Now we prove equality. Since, for each fixed η , there are no relations satisfied between the $\omega_{\eta(0)\eta(i)}$, we have two homogeneous (of degree k) polynomials in $(k +$

1) free non-commutative variables. To emphasise the fact the following argument is purely abstract, we write $x_i = \omega_{0i}$ and define an action of S_{k+1} on the x_i by $x_{\eta(i)} = \omega_{\eta(0)\eta(i)}$. So showing that the left- and right-hand sides are equal amounts to showing that

$$A := \sum_{\eta \in S_{k+1}} |\eta| \prod_{i=1}^k (x_{\eta(i)} - x_{\eta(0)}) = \sum_{\eta \in S_{k+1}} |\eta| \prod_{i=1}^k (x_{\eta(i)} - x_{\eta(i-1)}) =: B.$$

Write E to mean the \mathbb{Z} -linear span of degree- k monomials in the $(k+1)$ free non-commutative variables x_0, \dots, x_k , and let $\sigma_{p,q} \in S_{k+1}$ be the transposition that swaps p with q . Then $\sigma_{p,q}$ gives an involution on E by swapping x_p with x_q , and thus $E \cong E_{p,q}(1) \oplus E_{p,q}(-1)$, where $E_{p,q}(\lambda)$ is the eigenspace corresponding to the eigenvalue λ .

Let $H_{p,q}$ be the \mathbb{Z} -linear subspace of E spanned by monomials that contain at least one x_p or x_q . This subspace is stable^[1] under $\sigma_{p,q}$, and so this space also splits as $H_{p,q} \cong H_{p,q}(1) \oplus H_{p,q}(-1)$. Further, we have the inclusion $H_{p,q}(-1) \subseteq E_{p,q}(-1)$. But if $X \in E_{p,q}(-1)$, then, in particular, X must contain at least one^[2] x_p or x_q , so $X \in H_{p,q}$, whence $X \in H_{p,q}(-1)$. Thus $E_{p,q}(-1) = H_{p,q}(-1)$.

The intersection H of the $H_{p,q}$ over all *distinct* pairs $(p,q) \in \{0, \dots, k\} \times \{0, \dots, k\}$ is the \mathbb{Z} -linear span of all monomials containing all but one of the x_i (and, in particular, containing k distinct x_i). But since $H_{p,q}(-1) = E_{p,q}(-1)$, we see that the intersection $E(-1)$ of all the $E_{p,q}(-1)$ is equal to $H(-1)$. Now both A and B are in $E_{p,q}(-1)$ for all p,q (since the sign of $\omega_{p,q}$ is -1), and so $A, B \in E(-1) = H(-1)$. Since the coefficient of, for example, the $x_1 \cdots x_k$ term is the same^[3] (and *non-zero*) in both A and B , it suffices to show that $H(-1)$ is one-dimensional.

So let $X, Y \in H(-1)$ be monomials. Then each one contains k distinct x_i , and so there exists some (unique) $\sigma \in S_{k+1}$ such that $\sigma X = \pm Y$. But, writing $\sigma = \sigma_{p_1, q_1} \cdots \sigma_{p_r, q_r}$, we know that $\sigma X = (-1)^r X$, whence $X = Y$, up to some sign (and so up to some scalar in \mathbb{Z}). \square

^[1]If a monomial X contains, say, one x_p , then $\sigma_{p,q} X$ contains one x_q .

^[2]If not, then the action of $\sigma_{p,q}$ would be trivial and X would lie in $E_{p,q}(1)$.

^[3]It's 1, by taking $\eta = \text{id}$.

Simplicial formalisms

7

I am searching for abstract ways of expressing reality,
abstract forms that will enlighten my own mystery.

Eric Cantona.

(7.0.1) Purpose. *George Orwell wrote, in “Politics and the English Language”, that “[t]he English language [...] becomes ugly and inaccurate because our thoughts are foolish, but the slovenliness of our language makes it easier for us to have foolish thoughts.” Given the explicit calculations of Chapter 6, this chapter aims to formalise and justify the vast amounts of technical details and theory hiding in the background, as well as to generalise the methods so that they can be applied to arbitrary vector bundles on the nerve, instead of just pullbacks of non-simplicial ones. I really only talk about the case where the simplicial space is the nerve of an open cover, and where the simplicial connection is ‘generated in degree zero’, and so a lot of this chapter is really overkill for what’s needed in this thesis. My hope, however, is that somebody can use this framework in an interesting way, and either find a good use for this more general definition of what a simplicial connection “should be”, or find some examples that prove my choice of definition to be somehow not quite right. The important corollary of this section is (7.6.2).*

(7.0.2) Throughout this section, we try to use the notation E^\bullet , F^\bullet , etc. for pullbacks (to the nerve) of (global) vector bundles, and \mathcal{E}^\bullet , \mathcal{F}^\bullet , etc. for arbitrary vector bundles on the nerve.

We slightly modify the notation of (6.1.1) by writing the projections as

$$\pi_{p,q}: X_p^{\mathcal{U}} \times \Delta^q \rightarrow X_p^{\mathcal{U}}.$$

In the case where $p = q$, though, we will simply write π_p to mean $\pi_{p,p}$. So, for example, $\hat{\pi}_p$ is now written as $\pi_{p,p-1}$. We may still write $\overline{E^p}$ to mean $\pi_p^* E^p$.

(7.0.3) As a brief reminder, to hopefully avoid any confusion about the directions of arrows, we recall that $X_\bullet^{\mathcal{U}}$ is *contravariant*, but any sheaf \mathcal{E}^\bullet is *covariant*. That is,



$$\begin{array}{lcl} \alpha: & [p] & \longrightarrow [q] \\ X_\bullet^{\mathcal{U}} \alpha: & X_q^{\mathcal{U}} & \longrightarrow X_p^{\mathcal{U}} \\ \mathcal{E}^\bullet \alpha: & (X_\bullet^{\mathcal{U}} \alpha)^* \mathcal{E}^p & \longrightarrow \mathcal{E}^q \end{array}$$

which, combined with the contravariance of pullbacks, can lead quite easily to careless mistakes.^[1]

^[1]**Proof.** Personal experience of the author. \square

7.1 True morphisms

(7.1.1) Definition. A morphism $f: (A, \nabla_A) \rightarrow (B, \nabla_B)$ of locally free sheaves (on X) with connections is said to be a *true morphism* if $\nabla_B \circ f = (f \otimes \text{id}) \circ \nabla_A$. That is, we want the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \nabla_A \downarrow & & \downarrow \nabla_B \\ A \otimes \Omega_X^1 & \xrightarrow{f \otimes \text{id}} & B \otimes \Omega_X^1 \end{array}$$

to commute.

In some texts, such a morphism is said to be *flat*, but we avoid this terminology here for a few reasons (mainly to avoid confusion with flat connections).

(7.1.2) Lemma. If a morphism of locally free sheaves with connections is a true morphism (7.1.1), then so too is the same morphism of locally free sheaves with the higher iterations of their respective connections. That is,

$$\begin{aligned} \nabla_B \circ f &= f \circ \nabla_A && \in \mathcal{H}om(A, B) \otimes \Omega_X^r \\ \implies \nabla_B \circ (f \otimes \text{id}) &= (f \otimes \text{id}) \circ \nabla_A && \in \mathcal{H}om(A, B) \otimes \Omega_X^{r+1} \end{aligned}$$

where $\nabla_E: E \otimes \Omega_X^r \rightarrow E \otimes \Omega_X^{r+1}$ is given by enforcing the Leibniz rule, as in (4.1.5).

Proof. The proof shows us that the Leibniz rule is exactly what we need to make this calculation work:

$$\begin{aligned} (\nabla_B \circ (f \otimes \text{id}))(s \otimes \omega) &= \nabla_B(f(s) \otimes \omega) \\ &= \omega \wedge \nabla_B(f(s)) + f(s) \otimes d\omega \\ &= \omega \wedge (f \otimes \text{id})(\nabla_A(s)) + f(s) \otimes d\omega \\ &= (f \otimes \text{id})(\omega \wedge \nabla_A(s) + s \otimes d\omega) = ((f \otimes \text{id}) \circ \nabla_A)(s \otimes \omega). \quad \square \end{aligned}$$

(7.1.3) As we mentioned in (6.1.4) (as well as various other places throughout Section 6.1), we have to be slightly more careful when dealing with ‘simplicial connections’ (something that we have still yet to formally define) than we do with their classical counterparts. This is because a (family of) forms needs to earn the name ‘simplicial’ by satisfying the gluing condition in (4.3.6), namely

$$(X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* \omega_{p-1} = (\text{id} \times f_p^i)^* \omega_p$$

as sections of $\Omega_{X_p^{\mathcal{U}} \times \Delta^{p-1}}^r$.

In order to ensure this for some family of connections ∇_p on $\overline{E^p}$ (where E^p is the pullback to the nerve of some global vector bundle^[1]) we claim that we want to ask for the morphism^[2]

$$\mathfrak{C}_p^i: (X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* (\overline{E^{p-1}}, \nabla_{p-1}) \rightarrow (\text{id} \times f_p^i)^* (\overline{E^p}, \nabla_p)$$

of sheaves on $X_p^{\mathcal{U}} \times \Delta^{p-1}$ with connections to be a *true morphism* (7.1.1) with respect to these connections. The only problem with this is that it isn't entirely clear what this morphism *is*. That is, we have the commuting diagram

$$\begin{array}{ccc} & X_p^{\mathcal{U}} \times \Delta^{p-1} & \\ X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id} \swarrow & & \searrow \text{id} \times f_p^i \\ X_{p-1}^{\mathcal{U}} \times \Delta^{p-1} & & X_p^{\mathcal{U}} \times \Delta^p \\ \pi_{p-1} \downarrow & & \downarrow \pi_p \\ X_{p-1}^{\mathcal{U}} & \xleftarrow{X_{\bullet}^{\mathcal{U}} f_p^i} & X_p^{\mathcal{U}} \end{array}$$

which would let us pull back both $\overline{E^{p-1}} = \pi_{p-1}^* E^{p-1}$ and $\overline{E^p} = \pi_p^* E^p$ in exactly the way that we want^[3], as well as the morphism

$$E_{\bullet}^i f_p^i: (X_{\bullet}^{\mathcal{U}} f_p^i)^* E^{p-1} \rightarrow E^p,$$

but how do these give us the morphism that we want?

For now, we forget about the connections, and just work with the bundles. By using the commutativity of the diagram

$$\begin{array}{ccc} X_p^{\mathcal{U}} \times \Delta^{p-1} & \xrightarrow{\pi_{p,p-1}} & X_p^{\mathcal{U}} \\ X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id} \downarrow & & \downarrow X_{\bullet}^{\mathcal{U}} f_p^i \\ X_{p-1}^{\mathcal{U}} \times \Delta^{p-1} & \xrightarrow{\pi_{p-1}} & X_{p-1}^{\mathcal{U}} \end{array}$$

we can write the pull back of $\overline{E^{p-1}}$ via $(X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})$ as

$$(X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* \overline{E^{p-1}} = (X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* \pi_{p-1}^* E^{p-1} = \pi_{p,p-1}^* (X_{\bullet}^{\mathcal{U}} f_p^i)^* E^{p-1}.$$

^[1]We discuss the generalisation to the case of an arbitrary vector bundle on the nerve in (7.1.5).

^[2]**Warning:** the notation here is potentially misleading — the product pullback acts on **both** $\overline{E^p}$ and ∇_p **simultaneously**. That is, $(f \times g)^*(V, \varphi) = ((f \times g)^* V, (f \times g)^* \varphi)$, **not** $(f^* V, g^* \varphi)$.

^[3]That is, to lie over $X_p^{\mathcal{U}} \times \Delta^p$, so that we can impose the simplicial gluing condition.



But we know that $\overline{E^p}$ is defined as $\pi_p^* E^p$, and so is trivial on the simplex part of $X_p^{\mathcal{U}} \times \Delta^p$. This means that

$$(\text{id} \times f_p^i)^* \overline{E^p} = (\text{id} \times f_p^i)^* \pi_p^* E^p = \pi_{p,p-1}^* E^p.$$

So our comparison map \mathfrak{C}_p^i is exactly the pullback of $E^\bullet f_p^i$ along $\pi_{p,p-1}$.

(7.1.4) Definition. Given some $i \in \{0, \dots, p-1\}$, we define the *ith comparison map* $\mathfrak{C}_p^i(E^\bullet)$ of E^\bullet by

$$\begin{array}{ccc} \mathfrak{C}_p^i(E^\bullet): & (X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* \overline{E^{p-1}} & \longrightarrow & (\text{id} \times f_p^i)^* \overline{E^p} \\ \parallel & \parallel & & \parallel \\ \pi_{p,p-1}^*(E^\bullet f_p^i): & \pi_{p,p-1}^*(X_{\bullet}^{\mathcal{U}} f_p^i)^* E^{p-1} & \longrightarrow & \pi_{p,p-1}^* E^p. \end{array}$$

If there is no chance of confusion, we usually omit the dependence on E_\bullet from the notation. We often say ‘the’ comparison map \mathfrak{C}_p when we really mean ‘all’ comparison maps \mathfrak{C}_p^i for $i \in \{0, \dots, p-1\}$.



(7.1.5) Even in the case where E^\bullet is *not* simply the pullback of a locally free sheaf, the pullback $\overline{E^\bullet}$ will *still* be trivial on the simplex part (that is, the pullback is via a projection π_p^* , which entirely forgets the simplex). This means that $(\text{id} \times \psi)^* \overline{E^p}$ will always just be $\pi_{p,p-1}^* E^p$, no matter what ψ is. At first, this seems like a problem, because we cannot impose any conditions on the simplex part that are not trivial, but we have to remember that we are studying sheaves *with connections*.

So even though the *sheaf* itself is trivial over the simplex, the *connections* have no reason to be so. For example, the barycentric connection clearly has non-trivial information over the simplex part (because it has t_i appearing in its definition).

7.2 The motivating example

(7.2.1) In an effort to motivate the definitions in this chapter, we start with a simplified example of what we wish to study: we replace vector bundles with vector spaces, and we replace curvatures of connections with endomorphisms.

(7.2.2) Definition. Let \mathcal{C} be the category whose objects are pairs (V, φ) of finite-dimensional vector spaces V and endomorphisms φ , and whose morphisms $f: (V, \varphi) \rightarrow (W, \psi)$ are the morphisms $f: V \rightarrow W$ of vector spaces such that $f \circ \varphi = \psi \circ f$.

Let $E: \mathcal{C} \rightarrow \mathcal{C}$ be the endofunctor that sends (V, φ) to $(V/\text{Ker } \varphi, \varphi)$. Write $\text{L}_E \mathcal{C}$ to mean the localisation of \mathcal{C} along all morphisms that become isomorphisms after applying E (the wide subcategory of which we denote by \mathcal{W}).

(7.2.3) Recall that the *Grothendieck group* $K(C)$ of C is the group whose elements are isomorphism classes $[A]$ of objects $A \in C$, and where, for each short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in C , we introduce the relation $[A] - [B] + [C] = 0$ in $K(C)$, whence the group operation is given by

$$[(V, \varphi)] + [(W, \psi)] := [(V \oplus W, \varphi \oplus \psi)].$$

(7.2.4) **Definition.** An object $(V, \varphi) \in C$ is *flat* if $\varphi = 0$. We define an equivalence relation \sim on $K(C)$ by saying that flat objects are equivalent to the zero object:

$$[(V, 0)] \sim [(0, 0)].$$

(7.2.5) **Definition.** A morphism $f: (V, \varphi) \rightarrow (W, \psi)$ in C is *admissible* if there exist subspaces $V_1 \hookrightarrow V$ and $W_1 \hookrightarrow W$ such that

1. $V_1 \subseteq \text{Ker } \varphi$ and $W_1 \subseteq \text{Ker } \psi$;
2. f restricts to a morphism $V_1 \rightarrow W_1$;
3. f descends to an isomorphism $V/V_1 \xrightarrow{\sim} W/W_1$.

(7.2.6) **Lemma.** A morphism $f: (V, \varphi) \rightarrow (W, \psi)$ in C is in \mathcal{W} if and only if it is admissible.

Proof. Let $f: (V, \varphi) \rightarrow (W, \psi)$ be a morphism in \mathcal{W} , so that $E(f)$ is an isomorphism. Then we take $V_1 := \text{Ker } \varphi$ and $W_1 := \text{Ker } \psi$.

Conversely, let $f: (V, \varphi) \rightarrow (W, \psi)$ be admissible. Then $V_1 \subseteq \text{Ker } \varphi$, and so, by the third isomorphism theorem, $(V/V_1)/(\text{Ker } \varphi/V_1) \cong V/\text{Ker } \varphi$. It remains then to show that $E(f)$ restricts to an isomorphism $\text{Ker } \varphi/V_1 \xrightarrow{\sim} \text{Ker } \psi/W_1$. But $f \circ \varphi = \psi \circ f$, and so $E(f) \circ E(\varphi) \circ E(f)^{-1} = E(\psi)$, whence

$$E(f): \text{Ker } \varphi/V_1 = \text{Ker } E(\varphi) \xrightarrow{\sim} \text{Ker } E(\psi) = \text{Ker } \psi/W_1. \quad \square$$

(7.2.7) **Lemma.** There is a (canonical) isomorphism $K(L_EC) \xrightarrow{\sim} K(C)/\sim$.

Proof. It suffices to take the ‘identity’ map, as follows. Take two isomorphic objects $(V, \varphi) \cong (W, \psi) \in L_EC$. Then the isomorphism between them is given either by an isomorphism in C or by some morphism in \mathcal{W} . In the former case, we are done; in the latter case, (7.2.6) tells us that there is an admissible decomposition $f: V \cong V_1 \oplus V_2 \rightarrow W_1 \oplus W_2 \cong W$. But then both (V_1, φ) and (W_1, ψ) are equivalent to zero, whence $f: (V_2, \varphi) \xrightarrow{\sim} (W_2, \psi)$ gives us an isomorphism in $K(C)/\sim$.

Conversely, take two isomorphic objects $(V, \varphi) \cong (W, \psi) \in C$. Then they are also isomorphic in L_EC , since all isomorphisms are in \mathcal{W} . Further, if $[(V, \varphi)] \sim [(0, 0)]$ in $K(C)/\sim$, then $\varphi = 0$, whence $V/\text{Ker } \varphi = 0$, and so $[(V, \varphi)] = [(0, 0)] \in K(L_EC)$. \square

(7.2.8) Now assume that we have some ‘invariant polynomial’^[1] P from $C^{\otimes n}$ to some (additive, say) abelian group G (such as \mathbb{C}). If P is additive then it will descend to a well-defined polynomial on $K(C)^{\otimes n}$. If P further sends an n -fold tensor product of flat objects to zero, then it also descends to a well-defined polynomial on $K(C)/\sim$. Thus, by (7.2.7), P is well defined on $K(L_EC)$; thus, by (7.2.6), the resulting characteristic classes (that is, the values of P) are invariant under admissible morphisms.

7.3 Admissibility

(7.3.1) In order to define characteristic classes, we introduce the notion of *admissibility* of simplicial connection. It turns out that this is actually unnecessary for our purposes, because the only simplicial connection that we’re interested in is the barycentric one, and this is ‘generated in degree zero’ (something that we explain in Section 7.7), which makes it very well behaved (in particular, in nice cases, being generated in degree zero implies admissibility, as explained in (7.7.16)). But it might be the case that, at some point, somebody is interested in other simplicial connections, in which case these notions might (hopefully) prove useful.

(7.3.2) **Definition.** An *endomorphism-valued simplicial r -form* on \mathcal{E}^\bullet is a family of forms $\{\omega_p\}_{p \in \mathbb{N}}$, where each ω_p is a global section of

$$\left\{ \omega_p \in \text{End}(\overline{\mathcal{E}^p}) \otimes_{\mathcal{O}_{X_p^u}} \Omega_{X_p^u \times \Delta^p}^r \right\}_{p \in \mathbb{N}}$$

such that

$$(\mathbb{C}_p^i \otimes \text{id}) \circ ((X_\bullet^u f_p^i \times \text{id})^* \otimes (X_\bullet^u f_p^i \times \text{id})^*) \omega_{p-1} = ((\text{id} \times f_p^i)^* \otimes (\text{id} \times f_p^i)^*) \omega_p \circ (\mathbb{C}_p^i \otimes \text{id})$$

as sections of $\mathcal{H}om\left((X_\bullet^u f_p^i \times \text{id})^* \overline{\mathcal{E}^{p-1}}, (\text{id} \times f_p^i)^* \overline{\mathcal{E}^p}\right) \otimes \Omega_{X_p^u \times \Delta^{p-1}}^r$.

(7.3.3) The condition in (7.3.2) is rather daunting as it stands, but appears quite naturally when we think about what exactly we want to be satisfied:

1. *the form part should be simplicial*, in that, if we ‘functorially remove’ the endomorphism part (that is, apply some invariant polynomial, such as the trace), we should be left with a simplicial form — this leads us to ask that the typical condition from (4.3.6) be satisfied;
2. *the endomorphism part should respect the simplicial structure of \mathcal{E}^\bullet* , in that the comparison maps \mathbb{C}_p should be true morphisms with respect to these endomorphisms.

^[1]That is, invariant under a change of basis (the GL_n -action), but with the subtlety that we actually need a *sequence* of such polynomials, indexed by \mathbb{N} : one for each possible dimension of the vector space of an element of C . We describe these things in more detail in Section 7.5.

It is true that the latter of these two points is not explained in such a satisfying manner as the former, and so we give another justification: if we can believe that the definition in (7.4.1) is the good one,^[1] then all we are doing here is thinking of a connection as a form-valued endomorphism and asking that the same condition be satisfied by both parts (the form and the endomorphism).

(7.3.4) Definition. Inspired by (7.2.5), we define an *admissible* endomorphism-valued simplicial form to be an endomorphism-valued simplicial form that is *fibrewise* \mathbb{C}_p -*admissible*. Let's spell out explicitly what we mean by this.

We have the comparison morphism

$$\left(X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id}\right)^* \left(\overline{\mathcal{E}^{p-1}}, \omega_{p-1}\right) \xrightarrow{\mathbb{C}_p^i} \left(\text{id} \times f_p^i\right)^* \left(\overline{\mathcal{E}^p}, \omega_p\right)$$

of sheaves over $X_p^{\mathcal{U}} \times \Delta^{p-1}$ with endomorphism-valued forms, as described in (7.1.3). On each fibre, this gives us a family of morphisms of pairs of vector spaces and endomorphisms, because, over a point in the base space,

(a) the bundle $\overline{\mathcal{E}^p}$ looks like^[2] \mathbb{C}^r ; and

(b) the endomorphism-valued simplicial forms gives us a family of endomorphisms, indexed by tangent vectors (as explained below).

For ω_{\bullet} to be admissible then, we want, for **all** choices of tangent vectors, for the induced morphism of pairs of vector spaces and endomorphisms to be admissible, in the sense of (7.2.5). That is, for all $p \in \mathbb{N}$, all $x \in X$, and all $v_x \in \wedge^r T_x X$ (where r is the differential degree of ω_{\bullet}), the endomorphism

$$\overline{\mathcal{E}^p} \big|_{\{x\}} \xrightarrow{\omega_p(v_x)} \overline{\mathcal{E}^p} \big|_{\{x\}}$$

is such that the map (induced by restriction)

$$\left(X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id}\right)^* \left(\overline{\mathcal{E}^{p-1}} \big|_{\{x\}}, \omega_{p-1}(v_x)\right) \xrightarrow{\mathbb{C}_p^i} \left(\text{id} \times f_p^i\right)^* \left(\overline{\mathcal{E}^p} \big|_{\{x\}}, \omega_p(v_x)\right)$$

is admissible, i.e. the induced map

$$\left(X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id}\right)^* \left(\left(\overline{\mathcal{E}^{p-1}} \big|_{\{x\}}\right) / \text{Ker } \omega_{p-1}(v_x)\right) \xrightarrow{\mathbb{C}_p^i} \left(\text{id} \times f_p^i\right)^* \left(\left(\overline{\mathcal{E}^p} \big|_{\{x\}}\right) / \text{Ker } \omega_p(v_x)\right)$$

is an isomorphism.

^[1]A belief that we hope is sufficiently justified by the rest of this section.

^[2]In general, there are no conditions on the ranks of each simplicial level of a locally free sheaf on the nerve, and so the rank may well jump when passing from one level to another: over a point in the base space, the sheaf would look like $\mathbb{C}^{r(p)}$ for some $r(p)$. In practice though, we often have a tamer situation, as described in (7.3.5). In particular, if \mathcal{E}^{\bullet} is the pullback of a locally free sheaf, the rank is constant.

(7.3.5) **Lemma.** For an endomorphism-valued simplicial form ω_\bullet to be admissible, it is sufficient to ask, for all p , for sub-bundles $L_p, M_p \hookrightarrow \overline{\mathcal{E}^p}$ that lie in the kernel of the endomorphism-part of the ω_p , and such that the comparison map \mathcal{C}_p^i restricts to an isomorphism

$$\mathcal{C}_p^i: (X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* (\overline{\mathcal{E}^{p-1}}/L_{p-1}) \xrightarrow{\sim} (\text{id} \times f_p^i)^* (\overline{\mathcal{E}^p}/M_p).$$

Proof. This is just the statement that “if something is globally true then it is, in particular, fibrewise-ly true”. If L_p lies in the kernel of (the endomorphism part of) ω_p then, in particular, $L_p|_{\{x\}}$ lies in the kernel of $\omega_p(v_x)$ for any $v_x \in \wedge^r T_x X$ (and similarly for M_p). Then we appeal to (7.2.6). \square

7.4 Simplicial connections

(7.4.1) **Definition.** A *simplicial connection* ∇_\bullet on a locally free sheaf \mathcal{E}^\bullet of $\mathcal{O}_{X^{\mathcal{U}}}$ -modules is a family of connections $\nabla_\bullet = \{\nabla_p\}_{p \in \mathbb{N}}$, where ∇_p is a connection on $\pi_p^* \mathcal{E}^p$, such that the comparison maps

$$(X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* (\overline{\mathcal{E}^{p-1}}, \nabla_{p-1}) \xrightarrow{\mathcal{C}_p^i(\mathcal{E}^\bullet)} (\text{id} \times f_p^i)^* (\overline{\mathcal{E}^p}, \nabla_p)$$

are true morphisms.^[1] That is, such that the diagram

$$\begin{array}{ccc} (X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* \overline{\mathcal{E}^{p-1}} & \xrightarrow{\mathcal{C}_p^i} & (\text{id} \times f_p^i)^* \overline{\mathcal{E}^p} \\ \downarrow (X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* \nabla_{p-1} & & \downarrow (\text{id} \times f_p^i)^* \nabla_p \\ (X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* \overline{\mathcal{E}^{p-1}} \otimes \Omega_{X_p^{\mathcal{U}} \times \Delta^p}^1 & \xrightarrow{\mathcal{C}_p^i \otimes \text{id}} & (\text{id} \times f_p^i)^* \overline{\mathcal{E}^p} \otimes \Omega_{X_p^{\mathcal{U}} \times \Delta^p}^1 \end{array}$$

commutes. Note that the ∇_p are connections on $\overline{\mathcal{E}^p} = \pi_p^* \mathcal{E}^p$, and *not* on \mathcal{E}^p itself.

(7.4.2) **Lemma.** The curvature of a simplicial connection is an endomorphism-valued simplicial 2-form.

Proof. This is a direct consequence of the definitions and (7.1.2). \square

(7.4.3) **Definition.** We say that a simplicial connection is *admissible* if its *curvature* (which is an endomorphism-valued simplicial 2-form, by (7.4.2)) is admissible.



^[1]Recalling, again, that the pullback acts on *both* parts of the product *simultaneously*.



When talking about *connections* we often simply write ‘admissible’ instead of ‘admissible simplicial’, under the tacit assumption that it only makes sense to call a connection admissible if it is already simplicial.

The reason that we place this condition on the curvature and not on the 1-form part of the connection is because we really only care about characteristic classes, which are given by evaluating certain polynomials on the *curvature*. So as long as the curvature behaves well, we’re happy.

(7.4.4) Corollary. For a simplicial connection ∇_\bullet to be admissible, it is sufficient to ask for sub-bundles $A_p, B_p \hookrightarrow \overline{\mathcal{E}^p}$ such that

- (i) A_p and B_p are ∇_p -flat;
- (ii) the comparison map

$$(X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* (\overline{\mathcal{E}^{p-1}}, \nabla_{p-1}) \xrightarrow{\mathcal{C}_p^i} (\text{id} \times f_p^i)^* (\overline{\mathcal{E}^p}, \nabla_p).$$

(which is already known to be a *true* morphism, since the connection is assumed to be simplicial) restricts to a morphism

$$(X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* (A_{p-1}, \nabla_{p-1}) \xrightarrow{\mathcal{C}_p^i} (\text{id} \times f_p^i)^* (B_p, \nabla_p);$$

and;

- (iii) the above restriction of the comparison map induces an isomorphism

$$\mathcal{C}_p^i : (X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* (\overline{\mathcal{E}^{p-1}}/A_{p-1}) \xrightarrow{\sim} (\text{id} \times f_p^i)^* (\overline{\mathcal{E}^p}/B_p).$$

Proof. If A_p (resp. B_p) is ∇_p -flat then, in particular, it lies in the kernel of $\kappa(\nabla_p)$. Since $\kappa(\nabla_p)$ is simply $\nabla_p \circ \nabla_p$, (7.1.2) tells us that the (true) morphism

$$(X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* (A_{p-1}, \nabla_{p-1}) \rightarrow (\text{id} \times f_p^i)^* (B_p, \nabla_p)$$

induces a *true* morphism

$$(X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* (A_{p-1}, \kappa(\nabla_{p-1})) \rightarrow (\text{id} \times f_p^i)^* (B_p, \kappa(\nabla_p)).$$

But then, by (7.3.5), we are done. \square

(7.4.5) The difference of two admissible connections has no reason a priori to be an admissible endomorphism-valued simplicial 1-form, because there is no link between the kernels of the curvatures and the kernel of the difference.

This prompts the following definition: a family of admissible connections is said to be *compatible* if the difference of any two is an admissible endomorphism-valued simplicial 1-form.

(7.4.6) As a brief summary of the various definitions proposed so far:

- the **‘true morphism’** condition (7.4.1) of a simplicial connection (or (7.3.2), for endomorphism-valued simplicial forms) ensures that the related forms satisfy the gluing condition needed in order to define a simplicial form, as explained in (7.1.3), and formalised (for the curvature) in (7.4.2);
- the **admissibility** condition (7.4.3) (or (7.3.4), for endomorphism-valued forms) ensures that we can evaluate generalised invariant polynomials on the curvature of an admissible connection and get something that is the same in *all* simplicial degrees, as explained (vaguely) in (7.2.8);
- the **compatibility** condition (7.4.5) ensures that characteristic classes are independent of the choice of connection.

7.5 Generalised invariant polynomials

(7.5.1) Thinking about Section 7.2 (and using its notation), we realise that we need a slightly more general notion of ‘invariant polynomial’ for the following reason: the dimensions of the vector space of an element of C is entirely arbitrary, and so we can’t just expect to get by with some invariant polynomial P on \mathbb{C}^r for some fixed r ; we need an invariant polynomial P_r for each possible dimension r of the vector spaces. But there must be some sort of ‘agreement’ between the invariant polynomials — given some embedding of a low dimensional space into a higher dimensional one, the two invariant polynomials should agree — and we propose the following definition (still using the notation of Section 7.2).

(7.5.2) **Definition.** A sequence $(P_r)_{r \in \mathbb{N}}$ of \mathbb{C} -valued polynomials P_r on $C^{\otimes n}$ is said to be a *generalised invariant polynomial of degree n* if the following conditions are satisfied:

- (i) each P_r is a GL_r -invariant \mathbb{C} -valued \oplus -additive polynomial of degree n on

$$\left\{ \bigotimes_{i=1}^n (V_i, \varphi_i) \in C^{\otimes n} \mid \dim V_i = r \right\};$$

- (ii) the P_r factor through $K(C)/\sim$.

The second condition is equivalent to the following: the P_r all satisfy the ‘extension by zero’ property, i.e. $P_r = P_{r+1} \circ \iota_r$ for all $r \in \mathbb{N}$, where ι_r is the linear embedding of a vector space of dimension r into \mathbb{C}^{r+1} corresponding to $v \mapsto \begin{pmatrix} v \\ 0 \end{pmatrix}$. This extension by zero property basically (when combined with additivity) tells us that P is ‘fully’ \oplus -additive:

$$P_{\dim(V+W)}((V, \varphi) \oplus (W, \psi)) = P_{\dim V}((V, \varphi)) + P_{\dim W}((W, \psi)).$$

Mirroring classical notation, given some $P_\bullet = (P_r)_{r \in \mathbb{N}}$ on $C^{\otimes n}$, we write \widetilde{P}_\bullet to mean the (sequence of (invariant and additive)) polynomial(s) on C given by

$$\widetilde{P}_{\dim V}((V, \varphi)) = P_{\dim V}((V, \varphi)^{\otimes n}).$$

The prototypical example is the degree-1 polynomial given by $P_r = \text{tr}$ for all $r \in \mathbb{N}$, where the trace is of the endomorphism part of a pair, or, more generally, the degree n polynomial given by $P_r^n = \text{tr} \circ \mu^n$, where μ^n is the multiplication map that sends an n -fold tensor product of endomorphisms to the endomorphism given by the composition of all the endomorphisms.^[1] A non-example is $P_r = \frac{1}{r} \text{tr}$.

(7.5.3) Lemma. Using the notation from Section 7.2, let $f: (V, \varphi) \rightarrow (W, \psi)$ be an admissible morphism, and $(P_r)_{r \in \mathbb{N}}$ be a generalised invariant polynomial of degree n that is zero on an n -fold tensor product of flat objects. Then $\widetilde{P}_r((V, \varphi)) = \widetilde{P}_r((W, \psi))$ for all $r \in \mathbb{N}$.

Proof. By definition, there exist decompositions $V \cong V_1 \oplus V_2$ and $W \cong W_1 \oplus W_2$ such that $\varphi(V_1) = \psi(W_1) = 0$ and f restricts to an isomorphism $V/V_1 \xrightarrow{\sim} W/W_1$. Then, necessarily, $\dim V_2 = \dim W_2 = s$ for some $s \in \mathbb{N}$. Since P_r is fully additive,

$$\widetilde{P}_{\dim V}((V, \varphi)) = \widetilde{P}_{\dim V}((V_1, 0) \oplus (V_2, \varphi | V_2)) = \widetilde{P}_{\dim V_1}((V_1, 0)) + \widetilde{P}_s((V_2, \varphi | V_2)).$$

But P_r is assumed to be zero on an n -fold tensor product of flat objects, and so

$$\widetilde{P}_{\dim V_1}((V_1, 0)) + \widetilde{P}_s((V_2, \varphi | V_2)) = \widetilde{P}_s((V_2, \varphi | V_2)).$$

Then the GL_r -invariance tells us that

$$\widetilde{P}_s((V_2, \varphi | V_2)) = \widetilde{P}_s((W_2, \varphi | W_2)),$$

and this is equal to $\widetilde{P}_{\dim W}((W, \psi))$, again by full additivity and being zero on flat objects. \square

(7.5.4) Corollary. Let ω_\bullet be an admissible endomorphism-valued simplicial k -form on \mathcal{E}^\bullet , and $P_\star = (P_r)_{r \in \mathbb{N}}$ a generalised invariant polynomial of degree n that is zero on n -fold tensor products of flat objects. Then $\widetilde{P}_\star(\omega_\bullet)$ is a simplicial k -form.

Proof. First of all, we need to verify that $\widetilde{P}_\star(\omega_\bullet)$ is well defined. Over any $U_{\alpha_0 \dots \alpha_p}$, we have the pair $(\overline{\mathcal{E}^p}, \omega_p) | U_{\alpha_0 \dots \alpha_p} \times \Delta^p$ consisting (after trivialisaton) of an $\mathcal{O}_{U_{\alpha_0 \dots \alpha_p} \times \Delta^p}$ -module of rank $r(\alpha_0 \dots \alpha_p)$ along with a form-valued endomorphism. By (4.3.4), the rank $r(p) = r(\alpha_0 \dots \alpha_p)$ is independent of the open set $U_{\alpha_0 \dots \alpha_p}$. This means that

^[1]The fact that this is well defined relies on being able to view all the V_i in the tensor product as being identical, but we can do this thanks to the GL -invariance of the trace, as well as the ‘extension by zero’ property.

we can define $\widetilde{P}_{r(p)}(\omega_p)$ by applying $\widetilde{P}_{r(p)}$ to the endomorphism part of $\omega_p^{\otimes n}$, and wedging the n -fold tensor product of the form part.

Then we need to show that $(X_{\bullet}^U f_p^i \times \text{id})^* \widetilde{P}_{r(p-1)}(\omega_{p-1}) = (\text{id} \times f_p^i)^* \widetilde{P}_{r(p)}(\omega_p)$. Now we can use the extension by zero property from (7.5.1) to replace $\widetilde{P}_{r(p-1)}$ and $\widetilde{P}_{r(p)}$ with $\widetilde{P}_{\mathfrak{s}}$, where $\mathfrak{s} = \max\{r(p-1), r(p)\}$. But then, by admissibility and (7.5.3), it suffices to show that $\widetilde{P}_{\mathfrak{s}}$ commutes with pullbacks. Because $\widetilde{P}_{\mathfrak{s}}$ is just given by the wedge product of forms on the form part of ω_p , it commutes with pullbacks there. On the endomorphism part of ω_p , since we are working locally, the pullback is simply a change of basis, and so commutes with the $\widetilde{P}_{\mathfrak{s}}$ by GL-invariance. \square

7.6 Chern-Weil theory

(7.6.1) In (7.5.4), we saw how the theory of admissible simplicial connections gave us a simplicial form; we know that we can use fibre integration to obtain a differential form; and so the question remains: how can we get a class in cohomology?

Firstly, to see that the form given by evaluating a generalised invariant polynomial on an admissible endomorphism-valued simplicial form is closed, we can apply the classical argument (e.g. [Huy05, Corollary 4.4.5]) in each simplicial degree. This means that we get some cohomology class.

Next, we could ask about invariance of this class under a different choice of simplicial connection.^[1] To give an example of what we are trying to show here, consider the simplest case, where $P_{\star} = \text{tr}$. Given two admissible simplicial connections $\nabla_{\bullet}^{(1)}$ and $\nabla_{\bullet}^{(2)}$, we want that

$$P_{\star} \left(\kappa \left(\nabla_{\bullet}^{(1)} \right) \right) \sim P_{\star} \left(\kappa \left(\nabla_{\bullet}^{(2)} \right) \right)$$

where \sim here means ‘equal in cohomology’. This is equivalent to asking for their difference to be a *closed* simplicial differential form. But, working *locally*, writing $\nabla_{\bullet}^{(i)} = d + A_{\bullet}^{(i)}$, we see that

$$\begin{aligned} \kappa \left(\nabla_{\bullet}^{(2)} \right) - \kappa \left(\nabla_{\bullet}^{(1)} \right) &= d \left(A_{\bullet}^{(2)} - A_{\bullet}^{(1)} \right) + A_{\bullet}^{(2)} \cdot A_{\bullet}^{(2)} + A_{\bullet}^{(1)} \cdot A_{\bullet}^{(1)} \\ &= d \left(A_{\bullet}^{(2)} - A_{\bullet}^{(1)} \right) + \left(A_{\bullet}^{(2)} - A_{\bullet}^{(1)} \right)^2 + A_{\bullet}^{(2)} \cdot A_{\bullet}^{(1)} + A_{\bullet}^{(1)} \cdot A_{\bullet}^{(2)}. \end{aligned}$$

By skew symmetry, the last two terms will cancel when we apply P_{\star} , and so we simply want for the difference $B_{\bullet} = A_{\bullet}^{(2)} - A_{\bullet}^{(1)}$ to be an *admissible* endomorphism-valued simplicial form, since then $P_{\star}(dB_{\bullet}) = dP_{\star}(B_{\bullet})$ will be the differential applied to a simplicial differential form, and thus closed in the simplicial de Rham complex.

^[1]Here we point out that, since we will eventually be interested in characteristic classes of *coherent* sheaves, there will be another question of independence (that of *twisting cochains*), that we discuss in (9.2.3).

In the general case, we can again use some classical proof *if* we can show that “working simplicial degree by simplicial degree” still results in a valid proof. The statements in [Huy05, §4.4] that are used to prove [Huy05, Lemma 4.4.6] can be adapted to our situation thanks to the following facts^[1] concerning an admissible simplicial connection ∇_\bullet , an admissible endomorphism-valued simplicial 1-form A_\bullet , admissible endomorphism-valued simplicial forms γ_\bullet^j of degree d_j , and a generalised invariant polynomial P_\star :

- (i) the difference of two admissible simplicial connections in a compatible family^[2] (7.4.5) is an admissible endomorphism-valued simplicial 1-form, by definition;
- (ii) $\kappa(\nabla_\bullet + A_\bullet) = \kappa(\nabla_\bullet) + A_\bullet \cdot A_\bullet + \nabla_\bullet(A_\bullet)$;
- (iii) $\widetilde{P}_\star(\kappa(\nabla_\bullet + tA_\bullet)) = \widetilde{P}_\star(\kappa(\nabla_\bullet)) + \kappa(t)P_\star(\kappa(\nabla_\bullet), \dots, \kappa(\nabla_\bullet), \nabla_\bullet(A_\bullet)) + \mathcal{O}(t^2)$;
- (iv) $\nabla_\bullet(\kappa(\nabla_\bullet)) = 0$ (the *Bianchi identity*);
- (v) $dP_\star(\gamma_\bullet^1, \dots, \gamma_\bullet^k) = \sum_{j=1}^k (-1)^{\sum_{\ell=1}^{j-1} d_\ell} P_\star(\gamma_\bullet^1, \dots, \nabla_\bullet(\gamma_\bullet^j), \dots, \gamma_\bullet^k)$.

(7.6.2) The upshot of all of this abstract theory is that the barycentric connection (for the pullback of a locally free sheaf) is an admissible simplicial connection (as we will prove in (7.7.2) and (7.7.3)), and the trace (of the k -th wedge product) is a generalised invariant polynomial, and so all of our calculations in Sections 6.1 and 6.2 are justified — we do indeed end up with well-defined simplicial forms when we take the trace (of wedge products) that give us classes that deserve to be called ‘characteristic classes’.

(7.6.3) **Definition.** Given a locally free sheaf \mathcal{E}^\bullet of \mathcal{O}_{X^u} -modules and an *admissible* simplicial connection ∇_\bullet , we define the *k th simplicial exponential Atiyah class* $\hat{\text{at}}_{\mathcal{E}^\bullet}^{\circ k}$ by

$$\hat{\text{at}}_{\mathcal{E}^\bullet}^{\circ k} = \epsilon_k \kappa(\nabla_\bullet)^k$$

and the *k th simplicial standard Atiyah class* $\hat{\text{at}}_{\mathcal{E}^\bullet}^{\wedge k}$ by

$$\hat{\text{at}}_{\mathcal{E}^\bullet}^{\wedge k} = \epsilon_k \kappa(\nabla_\bullet)^{\wedge k}$$

where $\epsilon_k = (-1)^{k(k-1)/2}$. Note that $\text{tr} \hat{\text{at}}_{\mathcal{E}^\bullet}^{\circ k}$ and $\text{tr} \hat{\text{at}}_{\mathcal{E}^\bullet}^{\wedge k}$ are both generalised invariant polynomials evaluated on the curvature $\kappa(\nabla_\bullet)$.

^[1]All of which are true either by definition, or by working in each simplicial degree and using admissibility of the simplicial connection.

^[2]For why we can assume that we have a compatible family, see (7.7.17).

(7.6.4) **Definition.** Given a *bounded* chain complex

$$\mathcal{X}^\star = \left((\mathcal{E}^{\bullet,\star}, \omega_\bullet^\star), d_{\mathcal{X}}^\star \right)$$

of vector bundles on the nerve with admissible endomorphism-valued simplicial forms, and a generalised invariant polynomial P_\bullet , we define

$$P_\bullet(\mathcal{X}^\star) = \sum_{m \in \mathbb{N}} \tilde{P}_\star(\omega_\bullet)^{(-1)^m}$$

where the sum is necessarily (by the boundedness of \mathcal{X}^\star) finite.

7.7 Simplicial connections generated in degree zero

(7.7.1) We now restrict our attention to a specific type of simplicial connection, which is the generalisation of the barycentric connection, as defined in (6.1.3). The reason for being interested in barycentric connections in the first place is twofold: firstly, they are ‘natural’ ones for our applications;^[1] and secondly, they behave very well, in that they are simplicial and admissible (which we show in (7.7.2) and (7.7.3))

Recall that the barycentric connection was defined for a pullback to the nerve of a *global* vector bundle. We wish to define such a connection for an *arbitrary* vector bundle on the nerve (though, as we shall see, we actually need to impose the Green condition on our ‘arbitrary’ bundles). Before doing so, however, we give a *non-definition*, and explain why it does not suffice.

Non-definition. We say that a simplicial connection ∇_\bullet on a locally free sheaf \mathcal{E}^\bullet of \mathcal{O}_{X^\bullet} -modules is *generated in degree zero* if it is of the form

$$\nabla_p \big|_{U_{\alpha_0 \dots \alpha_p}} = \sum_{i=0}^p t_i \nabla_{\alpha_i}$$

where the ∇_{α_i} are connections on \mathcal{E}^0 over U_{α_i} , pulled back just as in (6.1.3).

This is a **non-definition** because arbitrary locally free sheaves of \mathcal{O}_{X^\bullet} -modules have no reason to be strongly cartesian: they are not necessarily pullbacks of global locally free sheaves. The ‘good’ definition should imply this non-definition whenever the sheaves in question *are* strongly cartesian, but be general enough to work even if not.

Before giving this definition, however, we first prove the nice properties of barycentric connections that we claimed above.

^[1]We have a bunch of local data (e.g. connections) but no global data, and wish to patch together the local data in order to obtain some simplicial version of global data.

(7.7.2) **Lemma.** The barycentric connection is a simplicial connection.

Proof. To prove that ∇_{\bullet}^{μ} is indeed a simplicial connection, we need to show that the comparison map \mathcal{C}_p^i is a true morphism. Since E^{\bullet} is the pullback of a global locally free sheaf, we know that the map $E^{\bullet} f_p^i: (X_{\bullet}^{\mathcal{U}} f_p^i)^* E^{p-1} \rightarrow E^p$ is (over each $U_{\alpha_0 \dots \alpha_p}$) a map

$$(E \mid U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p}) \mid U_{\alpha_0 \dots \alpha_p} \rightarrow E \mid U_{\alpha_0 \dots \alpha_p}$$

which is in fact, by definition, simply the identity. So then the comparison map $\mathcal{C}_p^i = \pi_{p,p-1}^* (E^{\bullet} f_p^i)$ is also just the identity. But, over $U_{\alpha_0 \dots \alpha_p} \times \Delta^{p-1}$, we see that

$$(X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* \nabla_{p-1}^{\mu} = \sum_{\substack{j=0 \\ j \neq i}}^p t_j \nabla_{\alpha_j} = (\text{id} \times f_p^i)^* \nabla_p^{\mu}$$

and so, since the comparison map is the identity and the two connections are identical, the comparison map is a true morphism. \square

(7.7.3) **Lemma.** The barycentric connection is admissible.

Proof. We already know, from (7.7.2), that the barycentric connection really is a simplicial connection. So we take $A_p = B_p = 0$ and $L_p = M_p = \overline{E^p}$, and look towards showing that the conditions in (7.4.4) are satisfied, which reduces to showing that the comparison map is an isomorphism. But, as explained in (7.7.2), the comparison map is really just the identity map, which is an isomorphism. \square

(7.7.4) **Definition.** We say that a complex $\mathcal{E}^{\bullet, \star}$ of vector bundles on the nerve is *Green* if, for all $\gamma \leq \beta \leq \alpha = (\alpha_0 \dots \alpha_p)$, writing \mathcal{E}_{α} to mean $\mathcal{E}^p \mid U_{\alpha}$, the following conditions are satisfied:

(a) $\mathcal{E}_{\alpha}^{\star} \cong \mathcal{E}_{\beta}^{\star} \mid U_{\alpha} \oplus \mathcal{L}_{\alpha, \beta}^{\star}$ for some $(\mathcal{E}_{\alpha_0}^{\star}, \dots, \mathcal{E}_{\alpha_p}^{\star})$ -elementary (9.1.1) sequences $\mathcal{L}_{\alpha, \beta}^{\star}$;

(b) $\mathcal{L}_{\alpha, \gamma}^{\star} \cong \mathcal{L}_{\beta, \gamma}^{\star} \mid U_{\alpha} \oplus \mathcal{L}_{\alpha, \beta}^{\star}$;

(c) over each U_{α} there is an isomorphism of short exact sequences^[1]

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}_{\beta}^{\star} & \longrightarrow & \mathcal{E}_{\alpha}^{\star} & \longrightarrow & \mathcal{L}_{\alpha, \beta}^{\star} \longrightarrow 0 \\ & & \uparrow \wr & & \uparrow \wr & & \text{id} \uparrow \\ 0 & \longrightarrow & \mathcal{E}_{\gamma}^{\star} \oplus \mathcal{L}_{\beta, \gamma}^{\star} & \longrightarrow & \mathcal{E}_{\gamma}^{\star} \oplus \mathcal{L}_{\alpha, \gamma}^{\star} & \longrightarrow & \mathcal{L}_{\alpha, \beta}^{\star} \longrightarrow 0 \end{array}$$

(omitting the restriction notation), where the bottom map is induced by the natural inclusion $\mathcal{L}_{\beta, \gamma}^{\star} \hookrightarrow \mathcal{L}_{\alpha, \gamma}^{\star}$ coming from $\mathcal{L}_{\alpha, \gamma}^{\star} \cong \mathcal{L}_{\beta, \gamma}^{\star} \oplus \mathcal{L}_{\alpha, \beta}^{\star}$.

^[1]In other words, this diagram commutes, its top and bottom rows are short exact sequences, and the vertical arrows are all isomorphisms.



Note that we require *isomorphisms* in conditions (a) and (b) rather than just *quasi-isomorphisms*.

We say that a single vector bundle \mathcal{E}^\bullet is *Green* if the complex $\mathcal{E}^\bullet[0]$ is Green.



(7.7.5) We do not claim that every complex satisfying the conditions of (7.7.4) comes from Green's resolution (9.1.2); because of this, a better name for 'Green complexes' might be *anatine complexes*.^[1]

(7.7.6) **Lemma.** Let $\mathcal{E}^{\bullet,\star}$ be a Green complex of vector bundles on the nerve. Then, in particular,

1. for all coface maps $f_p^i: [p-1] \rightarrow [p]$, the map

$$(X_{\bullet}^{\mathcal{U}} f_p^i)^* \mathcal{E}^{p-1,\star} \xrightarrow{\mathcal{E}^{\bullet,\star} f_p^i} \mathcal{E}^{p,\star}$$

is injective, and $\text{Coker}(\mathcal{E}^{\bullet,\star} f_p^i)$ is an elementary^[2] sequence;

2. for all codegeneracy maps $s_i^p: [p+1] \rightarrow [p]$, the map

$$(X_{\bullet}^{\mathcal{U}} s_i^p)^* \mathcal{E}^{p+1,\star} \xrightarrow{\mathcal{E}^{\bullet,\star} s_i^p} \mathcal{E}^{p,\star}$$

is surjective, and $\text{Ker}(\mathcal{E}^{\bullet,\star} s_i^p)$ is an elementary sequence;

3. the complex $\mathcal{E}^{\bullet,\star}$ is *cartesian* (4.3.2);
4. the comparison maps $\mathcal{C}_p^i(\mathcal{E}^{\bullet,j})$ are injective for all j .

Proof. The first two claims follow from the splittings $\mathcal{E}_\alpha^\star \cong \mathcal{E}_\beta^\star \mid U_\alpha \oplus \mathcal{L}_{\alpha,\beta}^\star$ in the specific case where $\beta = (\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p)$; the third claim (saying that the complex is Cartesian) follows from the fact that adding an elementary sequence gives a quasi-isomorphism of complexes (as explained in (8.3.11)); the fourth claim (that the comparison maps are injective) follows from the fact that they can be written as $\pi_{p,p-1}^*(\mathcal{E}^\bullet f_p^i)$, and, as mentioned a few other times in this section, we know that pullbacks preserve injectivity. \square

(7.7.7) For any simplicial object in a category, its face maps are injections, because the simplicial identities tell us that they have left inverses. With this in mind, Green complexes *actually* look like cosimplicial objects: their coface maps are injective, which is not necessarily the case for arbitrary complexes of vector bundles on the nerve.

^[1]These complexes behave like something coming from Green's resolution, and there is the notion of *duck typing* from computer science: 'if something looks like a duck and quacks like a duck then, for all implementation purposes, it is a duck.' So we might wish for an adjective that combines the idea of ducks with the colour green; mallards are an example of green ducks; mallards come from the Anatinae family, the adjective for which is *anatine*. We refrain, just this once, however, from introducing more ornithological input to this thesis.

^[2]In the $\mathcal{E}_{\alpha_i}^\star$, as in (7.7.4).

(7.7.8) At last, we can *almost* give the definition promised in (7.7.1) of simplicial connections *generated in degree zero*, with an important caveat: we **assume** that our locally free sheaf \mathcal{E}^\bullet of \mathcal{O}_{X^u} -modules satisfies the conditions necessary for it to be a *Green complex* (when viewed as a complex concentrated in degree zero). That is, there are certain splitting between simplicial levels, and these satisfy some cocycle condition, as we will explain *before* giving our definition (since otherwise it makes no sense); the actual definition is given in (7.7.9).



So, for all α , let $\widetilde{\nabla}_\alpha$ be a (local) connection on $\mathcal{E}^0 | U_\alpha$ (**not** on $\overline{\mathcal{E}^0}$), and define $\zeta_p^i: [0] \rightarrow [p]$ to be, for $i \in \{0, \dots, p\}$, the map given by $0 \mapsto i$. Since this is a composition of coface maps, we have, thanks to the Green assumption^[1], that (over each $U_{\alpha_0 \dots \alpha_p}$)

$$\mathcal{E}^p \cong \left((X_\bullet^u \zeta_p^i)^* \mathcal{E}^0 | U_{\alpha_0 \dots \alpha_p} \right) \oplus \mathcal{K}_{0,i,p} \quad (7.7.8.1)$$

with the inclusion map

$$(X_\bullet^u \zeta_p^i)^* \mathcal{E}^0 \hookrightarrow \mathcal{E}^p$$

(in the notation of (7.7.4), we have $\alpha = (\alpha_0, \dots, \alpha_p)$, $\beta = (\alpha_0, \dots, \widehat{\alpha}_i, \dots, \alpha_p)$, and $\mathcal{L}_{\alpha,\beta}^* = \mathcal{K}_{0,i,p}[0]$). Then, since pullbacks preserve direct sum decompositions, we have

$$\pi_p^* \mathcal{E}^p \cong \pi_p^* \left((X_\bullet^u \zeta_p^i)^* \mathcal{E}^0 | U_{\alpha_0 \dots \alpha_p} \right) \oplus \pi_p^* \mathcal{K}_{0,i,p} \quad (7.7.8.2)$$

By the commutativity of the square

$$\begin{array}{ccc} X_p^u & \xrightarrow{X_\bullet^u \zeta_p^i} & X_0^u \\ \uparrow & & \uparrow \\ U_{\alpha_0 \dots \alpha_p} & \hookrightarrow & U_{\alpha_i} \end{array}$$

(and recalling that restricting a sheaf to an open subset is the same as pulling back along the corresponding open inclusion), we see that the two following procedures have the same result when applied to a sheaf on X_0^u :

1. restrict to $U_{\alpha_i} \subseteq X_0^u$ and then to $U_{\alpha_0 \dots \alpha_p} \subseteq U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p}$;
2. pullback by $X_\bullet^u \zeta_p^i$ and then restrict to $U_{\alpha_0 \dots \alpha_p} \subseteq X_p^u$.

^[1]And maybe implicitly using the fact that, as mentioned in (9.1.2), the ‘all coface maps $f_p^i: [p-1] \rightarrow [p]$ ’ statement can be replaced by ‘all maps $\varphi: [p] \rightarrow [q]$ of coface type’.

This means that, over any $U_{\alpha_0 \dots \alpha_p}$, we have an isomorphism^[1] of sheaves

$$(X_{\bullet}^{\mathcal{U}} \zeta_p^i)^* \mathcal{E}^0 |_{U_{\alpha_0 \dots \alpha_p}} = (\mathcal{E}^0 |_{U_{\alpha_i}}) |_{U_{\alpha_0 \dots \alpha_p}}. \quad (7.7.8.3)$$

This means that the connection $\widetilde{\nabla}_{\alpha_i}$ on $\mathcal{E}^0 |_{U_{\alpha_i}}$ can be considered as a connection on $(X_{\bullet}^{\mathcal{U}} \zeta_p^i)^* \mathcal{E}^0$, and this then gives us a connection $\pi_p^* \widetilde{\nabla}_{\alpha_i}$ on $\pi_p^* (X_{\bullet}^{\mathcal{U}} \zeta_p^i)^* \mathcal{E}^0$. Using the direct sum decomposition (7.7.8.2), we can extend this to a connection on $\pi_p^* \mathcal{E}^p$ by simply defining it to be ‘the’ trivial connection on $\pi_p^* \mathcal{K}_{0,i,p}$: this summand is elementary in the $\mathcal{E}_{\alpha_i}^*$, and so we can use a direct sum of the $\widetilde{\nabla}_{\alpha_i}$ to endow it with a connection. We denote this extension again by $\pi_p^* \widetilde{\nabla}_{\alpha_i}$.

(7.7.9) Definition. With the formalities developed in (7.7.8), and using the same notation, we can proceed, as in (6.1.1), by taking a linear combination of connections, and defining

$$\nabla_p |_{U_{\alpha_0 \dots \alpha_p}} = \sum_{i=0}^p t_i \pi_p^* \widetilde{\nabla}_{\alpha_i} \quad (7.7.9.1)$$

and saying that any connection obtained in this way is *generated in degree zero*.



It is important to note that we *cannot* a priori restrict this isomorphism back down to any of the $(X_{\bullet}^{\mathcal{U}} \zeta_p^i)^* \mathcal{E}^0$ because connections are *endomorphism-valued*, and the splittings have no reason to agree for $j \neq i$ (that is, we have no idea how $\mathcal{K}_{0,i,p}$ and $\mathcal{K}_{0,j,p}$ relate to one another, and so no idea if $\widetilde{\nabla}_{\alpha_j}$ can be restricted to $\mathcal{E}^0 |_{U_{\alpha_i}}$).

(7.7.10) Really, Definition (7.7.9) is not that different from that of (6.1.3), but we just take extra care to point out the formal subtleties that follow from \mathcal{E}^{\bullet} not being strongly cartesian, but having, instead, this ‘cocyclic-splitting’ property of being Green.

(7.7.11) If the bundle \mathcal{E}^{\bullet} is strongly cartesian, then the inclusion maps $\mathcal{E}^{\bullet} \zeta_p^i$ in (7.7.8) are isomorphisms. This means, in particular, that the definition of ‘being generated in degree zero’ agrees with that of a barycentric connection for any bundle given by pulling back a global vector bundle to the nerve.

(7.7.12) To reiterate, for clarity: when we use the phrase ‘barycentric connection’, the implicit assumption is that we are working with the pullback (to the nerve) of a *global* locally free sheaf; we say ‘generated in degree zero’ whenever we wish to talk about the more general case of arbitrary vector bundles on the nerve. In an effort to make this even clearer, as mentioned before, we try to use the notation E^{\bullet} , F^{\bullet} , etc. for pullbacks of global locally free sheaves, and \mathcal{E}^{\bullet} , \mathcal{F}^{\bullet} , etc. for arbitrary vector bundles on the nerve.

^[1]Really, an equality, in fact, since the square truly commutes.

(7.7.13) If we wish to study just simplicial connections generated in degree zero then there is a shift of intuition with regards to admissibility: since our simplicial connections are ‘fixed’, the property of being an *admissible* connection is now really a property of the maps $\mathcal{E}^\bullet f_p^i: (X_\bullet^{\mathcal{U}} f_p^i)^* \mathcal{E}^{p-1} \rightarrow \mathcal{E}^p$, which says that admissibility is now really a property of *the bundle itself* (and its *local* connections) instead of the simplicial connection.

The idea is that, rather than working with a category of vector bundles on the nerve and then asking whether an individual object admits an admissible connection or not, we instead work directly with a category of ‘sufficiently nice’ vector bundles on the nerve, where ‘sufficiently nice’ means ‘if we put a simplicial connection generated in degree zero on it then it will be simplicial and admissible’. This is formalised in (7.7.15).

(7.7.14) Maybe somewhat ironically, we have no non-trivial explicit examples of admissible simplicial connections which are *not* generated in degree zero; it is an interesting question to ask whether or not such things exist.

(7.7.15) **Theorem.** Let $\mathcal{E}^{\bullet,\star}$ be a Green complex of vector bundles on the nerve. Then we can endow each $\mathcal{E}^{\bullet,j}$ with an *admissible* connection *generated in degree zero*.

Proof. (Compare the following with [Gre80, Lemma 2.2].) We split the proof into three steps: *defining* connections that are *generated in degree zero*; showing that they are *simplicial*; and then showing that they are *admissible*. For ease of notation, we write \mathcal{E}^\bullet instead of $\mathcal{E}^{\bullet,j}$.

1. Take *arbitrary* local connections $\widetilde{\nabla}_\alpha$ on $\mathcal{E}^0|_{U_\alpha}$ for all α . Since $\mathcal{E}^{\bullet,\star}$ is a *Green* complex, we can use (7.7.9) to define connections ∇_\bullet on $\overline{\mathcal{E}^\bullet}$ that are *generated in degree zero*.

As a small side note, we point out that the direct sums of the ∇_{α_i} give us connections on any $(\mathcal{E}_{\alpha_0}^\star, \dots, \mathcal{E}_{\alpha_p}^\star)$ -elementary sequence, and these (trivially) commute with the differentials (since all differentials are the identity). Such connections on a complex are said to be *compatible*^[1], and an important property of such things is that, when we apply an invariant polynomial to their curvatures, we get 1.^[2]

^[1]Not to be confused with a compatible *family* of connections, as in (7.4.5).

^[2]This can be seen as a corollary to [BB72, Lemma 4.22] applied to the exact sequence $0 \rightarrow M \rightarrow (M \rightarrow M[1]) \rightarrow M[1] \rightarrow 0$, where $0 \rightarrow M \rightarrow M[1] \rightarrow 0$ is the M -elementary sequence.

2. To show that each ∇_\bullet defined above is a *simplicial* connection, we need to show that

$$\begin{array}{ccc}
 (X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* \pi_{p-1}^* \mathcal{E}^{p-1} & \xleftarrow{\mathcal{C}_p^i(\mathcal{E}^\bullet)} & (\text{id} \times f_p^i)^* \pi_p^* \mathcal{E}^p \\
 \downarrow (X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* \nabla_{p-1} & & \downarrow (\text{id} \times f_p^i)^* \nabla_p \\
 (X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* \pi_{p-1}^* \mathcal{E}^{p-1} & \xleftarrow{\mathcal{C}_p^i(\mathcal{E}^\bullet) \otimes \text{id}} & (\text{id} \times f_p^i)^* \pi_p^* \mathcal{E}^p \\
 \otimes \Omega_{X_p^{\mathcal{U}} \times \Delta^p}^1 & & \otimes \Omega_{X_p^{\mathcal{U}} \times \Delta^p}^1
 \end{array} \quad (7.7.15.1)$$

commutes (where we use (7.7.6), which tells us, in particular, that the comparison maps are injective; we also use the fact that tensoring preserves splittings to see that the bottom horizontal arrow is also an injection). The proof of this is somewhat irksome: it is almost trivial, but it might take some convincing to really prove that it is, indeed, almost trivial. We start by making some simplifications (which probably, sadly, make things look a lot more complicated than they really are).^[1]

First of all, all of the sheaves in (7.7.15.1) lie over $X_p^{\mathcal{U}} \times \Delta^{p-1}$, but we make the identification

$$\begin{aligned}
 X_p^{\mathcal{U}} \times \Delta^{p-1} &\simeq X_p^{\mathcal{U}} \times f_p^i(\Delta^{p-1}) \\
 &\subset X_p^{\mathcal{U}} \times \Delta^p
 \end{aligned}$$

so that we can label *both* (i.e. on the nerve and on the simplex) simplicial parts with the same indices: the nerve being labelled with $\{0, 1, \dots, p\}$; the simplex being labelled with $\{0, 1, \dots, \hat{i}, \dots, p\}$ (so that i is now fixed).

Next, similar to how we proved (7.7.8.3), we can use the commutativity of the square

$$\begin{array}{ccc}
 X_p^{\mathcal{U}} & \xrightarrow{X_\bullet^{\mathcal{U}} f_p^i} & X_{p-1}^{\mathcal{U}} \\
 \uparrow & & \uparrow \\
 U_{\alpha_0 \dots \alpha_p} & \hookrightarrow & U_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_p}
 \end{array} \quad (7.7.15.2)$$

^[1]If you think that you understand *why* the square commutes, then we would suggest that you skip the reading of this proof, because it might serve only to confuse.

to see that

$$\begin{aligned} & \left((X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id})^* \overline{\mathcal{E}^{p-1}} \right) | \left(U_{\alpha_0 \dots \alpha_p} \times f_p^i(\Delta^{p-1}) \right) \\ &= \left(\overline{\mathcal{E}^{p-1}} | \left(U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p} \times \Delta^{p-1} \right) \right) | \left(U_{\alpha_0 \dots \alpha_p} \times f_p^i(\Delta^{p-1}) \right) \end{aligned}$$

as sheaves over $U_{\alpha_0 \dots \alpha_p} \times f_p^i(\Delta^{p-1})$.

So, to prove the commutativity of (7.7.15.1), we start by calculating how the two pullbacks of the connections (that is, the vertical arrows in the square) act (after restricting all sheaves to $U_{\alpha_0 \dots \alpha_p}$):

$$\left(X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id} \right)^* \nabla_{p-1} = \left(X_{\bullet}^{\mathcal{U}} f_p^i \times \text{id} \right)^* \sum_{\substack{j=0 \\ j \neq i}}^p t_j \pi_{p-1}^* \widetilde{\nabla}_{\alpha_j} = \sum_{\substack{j=0 \\ j \neq i}}^p t_j \pi_p^* \widetilde{\nabla}_{\alpha_j}$$

where the first equality is simply the definition. The second equality is the tricky one, because we are really writing $\widetilde{\nabla}_{\alpha_j}$ to mean two different things: on the left-hand side, it means the connection $\pi_{p-1}^* \widetilde{\nabla}_{\alpha_j}$ extended by the trivial connection^[1] on $\mathcal{K}_{0,j,p-1}$; on the right-hand side, it means the connection $\pi_p^* \widetilde{\nabla}_{\alpha_j}$ extended by the trivial connection on $\mathcal{K}_{0,j,p}$; all of this using the notation and properties of (7.7.8.1), (7.7.8.2), and (7.7.15.3). So this second equality really follows from the (7.7.15.4), which tells us that extending trivially on $\mathcal{K}_{0,j,p-1}$ and then again to the rest of $\mathcal{K}_{0,j,p}$ (which we do for $\pi_{p-1}^* \widetilde{\nabla}_{\alpha_j}$) is the same as simply extending trivially on $\mathcal{K}_{0,j,p}$ (which we do for $\pi_p^* \widetilde{\nabla}_{\alpha_j}$).

The second pullback is much simpler:

$$\left(\text{id} \times f_p^i \right)^* \nabla_p = \left(\text{id} \times f_p^i \right)^* \sum_{j=0}^p t_j \pi_p^* \widetilde{\nabla}_{\alpha_j} = \sum_{\substack{j=0 \\ j \neq i}}^p t_j \pi_p^* \widetilde{\nabla}_{\alpha_j}$$

which is ‘exactly the same’ as the first pullback — the scare quotes being important, because these connections are on *different sheaves*. But, since the horizontal arrows in (7.7.15.1) are injections, it means that these two connections really are the same when we just follow how they act on the top-left sheaf in the square; i.e., the square commutes.

If the reader is confused by the above proof (which is possible, given the round-about manner in which we show something that follows, almost trivially, from the definitions), we suggest simplifying to the single case where $i = p$.

Looking ahead to (7.7.15.3), since we have extended by something *compatible* on $\mathcal{K}_{p-1,i,p}$, the characteristic class of $\mathcal{K}_{p-1,i,p}$ will be 1, as mentioned

^[1]That is, we extend it in a compatible way, as we mentioned that we could near the start of the proof.

above. This means, by additivity of characteristic classes, that the classes of \mathcal{E}^p and $\mathcal{E}^{p-1} \mid U_{\alpha_0 \dots \alpha_p}$ will agree. This is the content of half of the proof of [Gre80, Lemma 2.2].

3. To show that each ∇_\bullet is an *admissible* simplicial connection, it suffices to show that the conditions in (7.4.4) are satisfied.

Before proceeding with the proof, however, we take some time to look at how the splittings that we have been using respect the simplicial structure. The Green assumption implies that

$$\mathcal{E}^p \mid U_{\alpha_0 \dots \alpha_p} \cong \left(\mathcal{E}^{p-1} \mid U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p} \right) \mid U_{\alpha_0 \dots \alpha_p} \oplus \mathcal{K}_{p-1, i, p} \quad (7.7.15.3)$$

which, combined with (7.7.8.1), says that (for $i \neq j$)^[1]

$$\begin{aligned} \mathcal{E}^p \mid U_{\alpha_0 \dots \alpha_p} &\cong \left((X_\bullet^{\mathcal{U}} \zeta_{p-1}^j)^* \mathcal{E}^0 \mid U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p} \oplus \mathcal{K}_{0, j, p-1} \right) \mid U_{\alpha_0 \dots \alpha_p} \oplus \mathcal{K}_{p-1, i, p} \\ &\cong \left((X_\bullet^{\mathcal{U}} \zeta_{p-1}^j)^* \mathcal{E}^0 \mid U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p} \right) \mid U_{\alpha_0 \dots \alpha_p} \oplus \left(\mathcal{K}_{0, j, p-1} \mid U_{\alpha_0 \dots \alpha_p} \oplus \mathcal{K}_{p-1, i, p} \right) \end{aligned}$$

and (again by the Green assumption) these \mathcal{K} satisfy some cocycle condition:

$$\mathcal{K}_{0, i, p} \cong \mathcal{K}_{0, j, p-1} \mid U_{\alpha_0 \dots \alpha_p} \oplus \mathcal{K}_{p-1, i, p} \quad (7.7.15.4)$$

whence^[2]

$$\mathcal{E}^p \mid U_{\alpha_0 \dots \alpha_p} \cong \left((X_\bullet^{\mathcal{U}} \zeta_{p-1}^j)^* \mathcal{E}^0 \mid U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p} \right) \mid U_{\alpha_0 \dots \alpha_p} \oplus \mathcal{K}_{0, i, p}. \quad (7.7.15.5)$$

Now, for each comparison map \mathcal{C}_p^i , we set

$$\begin{aligned} A_{p-1} &= \pi_p^* \mathcal{K}_{0, j, p-1} \\ B_p &= \pi_p^* \mathcal{K}_{0, i, p} \end{aligned}$$

from the splitting (7.7.8.2), for an arbitrary $j \neq i$.

^[1]And we somehow don't really care about when $i = j$, since we will always be working on the embedding of Δ^{p-1} into Δ^p by f_p^i , which is equivalent to working with the labelling $(\alpha_0, \dots, \widehat{\alpha}_i, \dots, \alpha_p)$ on Δ^{p-1} .

^[2]Here is a **false** conclusion: by the uniqueness of the quotient $\mathcal{E}^p / \mathcal{K}_{0, i, p}$, we see that

$$\left((X_\bullet^{\mathcal{U}} \zeta_{p-1}^j)^* \mathcal{E}^0 \mid U_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p} \right) \mid U_{\alpha_0 \dots \alpha_p} \cong \left(X_\bullet^{\mathcal{U}} \zeta_p^i \right)^* \mathcal{E}^0 \mid U_{\alpha_0 \dots \alpha_p}$$

as sheaves over $U_{\alpha_0 \dots \alpha_p}$. The reason that this argument does **not** work is because we don't know that the isomorphisms preserve the embeddings: $X \cong A \oplus B$ and $X \cong A \oplus C$ does not, in general, imply that $B \cong C$. For groups, this is true whenever X is finite, but here we are working with sheaves of $\mathcal{O}_{X_\bullet^{\mathcal{U}}}$ -modules, which, although locally free of finite rank (in our setting), are still much more complicated objects than groups, and there is no reason for the same statement to be true (indeed, the above actually serves as a counter example). The keywords here are 'stably trivial', or 'stably equivalent'.



- (i) A_p is ∇_p flat by definition: we extended the connection $\pi_p^* \widetilde{\nabla}_{\alpha_i}$ by the trivial connection on this direct summand.
- (ii) The comparison map \mathcal{C}_p^i is simply the pullback (along $\pi_{p,p-1}$) of $\mathcal{E}^\bullet f_p^i$, and so respects the splitting (7.7.8.2) almost by definition, because these splittings come from the Green assumption (7.7.4). (This is sort of what (7.7.15.5) is saying.)
- (iii) We want the comparison map to induce an isomorphism when we take the quotients.

To avoid getting lost in a mire of notation, instead of using (7.7.15.3) to (7.7.15.5), we use the language of (7.7.4). Set

$$\begin{aligned}\alpha &= (\alpha_0, \dots, \alpha_p) \\ \beta &= (\alpha_0, \dots, \widehat{\alpha}_i, \dots, \alpha_p) \\ \gamma &= (\alpha_j)\end{aligned}$$

and note that $A_{p-1} = \pi_{p-1}^* \mathcal{L}_{\beta, \gamma}$ and $B_p = \pi_p^* \mathcal{L}_{\alpha, \gamma}$. Then

$$\begin{aligned}\mathcal{E}_\alpha &\cong \mathcal{E}_\gamma \mid U_\beta \mid U_\alpha \oplus \mathcal{L}_{\alpha, \gamma} \\ \mathcal{E}_\beta &\cong \mathcal{E}_\gamma \mid U_\beta \oplus \mathcal{L}_{\beta, \gamma}\end{aligned}$$

and we are interested in

$$(X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* \pi_{p-1}^* (\mathcal{E}_\gamma \mid U_\beta) \xrightarrow{\pi_{p,p-1}^* (\mathcal{E}^\bullet f_p^i)} (\text{id} \times f_p^i)^* \pi_p^* (\mathcal{E}_\gamma \mid U_\beta \mid U_\alpha).$$

Consider first the source of this map: by (7.1.3), we know that

$$(X_\bullet^{\mathcal{U}} f_p^i \times \text{id})^* \pi_{p-1}^* (\mathcal{E}_\gamma \mid U_\beta) \cong \pi_{p,p-1}^* (X_\bullet^{\mathcal{U}} f_p^i)^* (\mathcal{E}_\gamma \mid U_\beta)$$

but, as in (7.7.15.2), we know that

$$(X_\bullet^{\mathcal{U}} f_p^i)^* (\mathcal{E}_\gamma \mid U_\beta) \mid U_\alpha = (\mathcal{E}_\gamma \mid U_\beta) \mid U_\beta \mid U_\alpha = \mathcal{E}_\gamma \mid U_\beta \mid U_\alpha$$

The target is much simpler, since, as explained in (7.1.3), the pullback along f_p^i on the simplex part changes nothing:

$$(\text{id} \times f_p^i)^* \pi_p^* (\mathcal{E}_\gamma \mid U_\beta \mid U_\alpha) \cong \pi_{p,p-1}^* (\mathcal{E}_\gamma \mid U_\beta \mid U_\alpha).$$

But then we see that the source and target, *when we restrict to U_α* , are identical, so it would suffice to show that the comparison map descends to an injection when we take these quotients. But these quotients are exactly the sheaf with which we started: we added the cokernels and then quotiented them out; and we know, from (7.7.6), that the comparison map is injective here.

□

(7.7.16) Lemma (7.7.15) can really be thought of, in particular, as a proof that, when working with Green complexes, ‘generated in degree zero’ implies admissibility.

(7.7.17) **Lemma.** Let \mathcal{E}^\bullet be a Green vector bundle on the nerve, and let ∇_\bullet and ∇'_\bullet be two simplicial connections on \mathcal{E}^\bullet that are generated in degree zero. Then the difference $\nabla'_\bullet - \nabla_\bullet$ is an admissible endomorphism-valued simplicial form. In other words, the set of generated-in-degree-zero connections on a Green vector bundle on the nerve is a compatible family.

Proof. The fact that the difference of two arbitrary simplicial connections on an arbitrary vector bundle on the nerve is an endomorphism-valued simplicial 1-form follows ‘immediately’ from the definitions, without any extra hypotheses. The content of this lemma is that *generated-in-degree-zero* connections on a Green vector bundle on the nerve have an *admissible* difference.

Write $\nabla_\bullet^{(1)}$ and $\nabla_\bullet^{(2)}$ to mean the two connections, so that $\nabla_\bullet^{(i)}$ is constructed as in (7.7.8), but with a different choice of local connections $\widetilde{\nabla}_\alpha^{(i)}$ on each $\mathcal{E}^0|_{U_\alpha}$. This means that, for each $p \in \mathbb{N}$, we can write

$$\nabla_p^{(i)} = \sum_{j=0}^p t_j \pi_p^* \widetilde{\nabla}_{\alpha_j}^{(i)}$$

which means that the difference is of the form

$$\nabla_p^{(2)} - \nabla_p^{(1)} = \sum_{j=0}^p t_j \pi_p^* (\widetilde{\nabla}_{\alpha_j}^{(2)} - \widetilde{\nabla}_{\alpha_j}^{(1)}) = \sum_{j=0}^p t_j \pi_p^* \eta_{\alpha_j} = \sum_{j=0}^p t_j \overline{\eta_{\alpha_j}}$$

where η_{α_j} is an endomorphism-valued simplicial 1-form on U_{α_j} . The claim is then that $\sum_{j=0}^p t_j \overline{\eta_{\alpha_j}}$ is an *admissible* endomorphism-valued simplicial 1-form, and this follows almost exactly as in the proof of (7.7.15), because the difference $\sum t_j \overline{\eta_{\alpha_j}}$ is somehow also ‘generated in degree zero’, in that it is given by trivially extending on each $(X_\bullet^{\mathcal{U}} f_p^i)^* \mathcal{E}^{p-1} \hookrightarrow \mathcal{E}^p$. \square

(7.7.18) As we mentioned in (7.3.1), the barycentric connection is particularly nice because it is generated in degree zero, in the sense that we define $\nabla_0^\mu = \nabla_{\alpha_0}$ over $U_{\alpha_0} \times \Delta^0$, and every ∇_p^μ (for $p \geq 1$) is just a Δ^p -linear combination of the ∇_0^μ . In some sense, we don’t introduce any ‘new’ connections at higher simplicial levels, and the linear combination that we do take (of those in degree zero) is somehow the ‘most free’ such one. For what it’s worth, we can try to understand the definition of ‘generated in degree zero’ in a more abstract manner, as follows.

A principal GL_n -bundle P (over a field k) on a space X is equivalent to a map $X \rightarrow [*/\mathrm{GL}_n]$, and a principal GL_n -bundle P on X along with a section $\sigma: X \rightarrow$

$P \times_{\text{GL}_n} k^n$ of the associated (k -vector) bundle is equivalent to a map $X \rightarrow [k^n/\text{GL}_n]$. To see the latter, consider the cube

$$\begin{array}{ccccc}
 & P \times k^n & \xrightarrow{\quad} & k^n & \\
 & \swarrow & & \swarrow & \\
 P & \xrightarrow{\quad} & * & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & [(P \times k^n)/\text{GL}_n] & \xrightarrow{\quad} & [k^n/\text{GL}_n] & \\
 \downarrow & & \downarrow & & \downarrow \\
 X & \xrightarrow{\quad} & [*/\text{GL}_n] & &
 \end{array}$$

where all the faces are pullbacks. Using the fact that $[(P \times k^n)/\text{GL}_n] \simeq P \times_{\text{GL}_n} k^n$, we see that having a section of the associated bundle does indeed give us (by composition) a (unique) map $X \rightarrow [k^n/\text{GL}_n]$, and, conversely, having such a map gives us a (unique) map $X \rightarrow P \times_{\text{GL}_n} k^n$ by using the (unique) map $\text{id}: X \rightarrow X$, along with the fact that $P \times_{\text{GL}_n} k^n$ is a pullback.

Now assume that we have some simplicial space X_\bullet along with a principal GL_n -bundle P^p on each simplicial level.^[1] Then, by the above, having $(p + 1)$ sections $\sigma_i: X_0 \rightarrow P^0$ is equivalent to having a map $(X_0)^{p+1} \rightarrow [k^n/\text{GL}_n]^{p+1}$. Our goal is to construct a section in simplicial degree p using these $(p + 1)$ sections in simplicial degree zero.

We have the projection map $[k^n/\text{GL}_n] \rightarrow [*/\text{GL}_n]$; the map $X_p \rightarrow (X_0)^{p+1}$ given by the $(p + 1)$ possible combinations^[2] of face maps; the diagonal map $[*/\text{GL}_n] \rightarrow [*/\text{GL}_n]^{p+1}$; and the classifying map $X_p \rightarrow [*/\text{GL}_n]$ of P_p . Putting these all together, we get the commuting diagram

$$\begin{array}{ccc}
 X_p & \xrightarrow{\quad} & [*/\text{GL}_n] \\
 \downarrow & \searrow & \downarrow \\
 (X_0)^{p+1} & \xrightarrow{\quad} & [*/\text{GL}_n]^{p+1} \\
 & \uparrow & \downarrow \\
 & [(k^n)^{p+1}/\text{GL}_n] & \rightarrow [*/\text{GL}_n]
 \end{array}$$

and so we get a map $X_p \rightarrow [(k^n)^{p+1}/\text{GL}_n]$, which is unique (given the σ_i).

The aim now is to construct a map

$$[(k^n)^{p+1}/\text{GL}_n] \rightarrow [\text{Hom}(\Delta^p, k^n)/\text{GL}_n]$$

^[1]We also want there to be some condition equivalent to the functorial choices of the maps $(X_\bullet, f_p^i)^* E^{p-1} \rightarrow E^p$ for sheaves on a simplicial space, but that doesn't play a role in this story, so we won't worry ourselves about it.

^[2]The maps corresponding to $\{0, \dots, p\} \mapsto \{i\}$ for $i \in \{0, \dots, p\}$.

since, combined with the other map we just constructed, and by currying,^[1] this would give us a map

$$X_p \times \Delta^p \rightarrow [k^n/\mathrm{GL}_n]$$

which is, as we have seen, exactly the information of a section of (the associated vector bundle of) P_p . It suffices to construct a GL_n -equivariant map $(k^n)^{p+1} \rightarrow \mathrm{Hom}(\Delta^p, k^n)$, and here is where we have to make some choice: we pick the map given by

$$(v_0, \dots, v_p) \mapsto \left((t_0, \dots, t_p) \mapsto \sum_{i=0}^p t_i v_i \right).$$

By currying, this is equivalent to the map

$$\begin{aligned} \Delta^p &\rightarrow \mathrm{Hom}\left((k^n)^{p+1}, k^n\right) \\ (t_0, \dots, t_p) &\mapsto \underbrace{\left((0, \dots, v, \dots, 0) \mapsto t_i v \right)}_{\text{zero except in the } i\text{th coordinate}} \end{aligned}$$

which, if we require linear maps that preserve affine structure, as well as making the diagram

$$\begin{array}{ccc} & \mathrm{Hom}(k^n, k^n) & \\ & \nearrow c_{\mathrm{id}} & \uparrow \Delta^* \\ \Delta^p & \longrightarrow & \mathrm{Hom}\left((k^n)^{p+1}, k^n\right) \end{array}$$

(where c_{id} is the constant map to the identity morphism, and Δ^* is pre-composition by the diagonal map) commute, is pretty much the only choice we have.

^[1]That is, the tensor-hom adjunction in a specific setting: $\mathrm{Hom}(A \times B, C) \cong \mathrm{Hom}(A, \mathrm{Hom}(B, C))$.

III. COHERENT SHEAVES AND tDR COHOMOLOGY

Perfect twisting cochains

8

Coo coo.

The pigeon outside as I write this.

(8.0.1) Purpose. *Twisting cochains and twisted complexes appear in many, sometimes seemingly unrelated, settings. This chapter contains bare minimum background needed to be able to understand Chapter 9, and consign everything else to Appendix G. In particular, I try to explain how Green constructs his simplicial resolution using twisting cochains, and how to view it as a sort of ‘local strictification’ argument in Section 8.3.*

8.1 A brief introduction

(8.1.1) Given the multiple, and often only subtly different, possible definitions of twisting cochains, we point out now that all those given in Appendix G are *not* used in this chapter; we use only that which is given below. But, of all of the motivations and approaches discussed in Appendix G, the one that is most relevant to our discussion here is (G.4), which describes the category of twisting cochains as a dg-enhancement of the category of complexes of coherent sheaves, as detailed in [Wei16].

(8.1.2) For this chapter, a good idea to keep in mind is that ‘a twisting cochain is like a complex of coherent sheaves, but not quite as rigidly glued’. That is, rather than having transition maps on overlaps, we have maps that only satisfy the cocycle condition ‘up to homotopy’, and these homotopies only commute ‘up to higher homotopies’, and so on. As we explain in (G.2.12), a complex of coherent sheaves is ‘like’ a truncated twisting cochain, with all homotopies being zero in degrees greater than one.

The other nice thing about twisting cochains is that they let us formalise what we have been doing throughout most of this thesis: thinking of all data above some $U_{\alpha_0 \dots \alpha_p}$ as lying above U_{α_0} , so that we can compare everything on an equal footing.

8.2 Holomorphic twisting resolutions

(8.2.1) Definition. A complex K^\bullet of sheaves on a locally-ringed space (X, \mathcal{O}_X) is said to be *perfect* if, locally, it is quasi-isomorphic to a bounded complex of locally free sheaves. That is, if, for every point $x \in X$, there exists some open

neighbourhood U of x and some bounded^[1] complex E_U^\bullet of locally free sheaves on U such that $K^\bullet|_U$ is quasi-isomorphic to E_U^\bullet .

(8.2.2) By the Hilbert syzygy theorem, the smoothness of X , and the fact that coherent sheaves are defined exactly to ensure that they are locally of finite type over \mathcal{O}_X , with finite-type kernels, we always have **local** resolutions of a coherent sheaf by *finite* complexes of *locally free* sheaves (and similarly for a *complex* of coherent sheaves). We content ourselves with believing that perfect complexes really are the things that interest us if we want to study coherent sheaves, since this is true in the algebraic case (where the derived category of complexes with coherent (internal) cohomology is equivalent to the derived category of coherent complexes). For those interested, however, we delve deeper into the story in Appendix E.

(8.2.3) **Definition.** Let X be some paracompact complex-analytic manifold with a locally-compact Stein open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. Suppose that over each U_α we have a finite-length complex $(V_\alpha^\bullet, d_\alpha)$ of locally free \mathcal{O}_{U_α} -modules^[2], the collection of which we refer to simply as V^\bullet . Define the collection $\text{End}^q(V)$ of *degree- q endomorphisms of V* over each $U_{\alpha_0 \dots \alpha_p}$ by

$$\text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}} = \left\{ (f^i: V_{\alpha_p}^i|_{U_{\alpha_0 \dots \alpha_p}} \rightarrow V_{\alpha_0}^{i+q}|_{U_{\alpha_0 \dots \alpha_p}})_{i \in \mathbb{Z}} \mid d_{\alpha_p} \circ f^i = f^{i+1} \circ d_{\alpha_0} \right\}.$$

That is, an element of $\text{End}^q(V)$ is a ‘true’ (in that it commutes with the differentials) morphism^[3] of degree q of chain complexes $V_{\alpha_p}^\bullet \rightarrow V_{\alpha_0}^\bullet$.

(8.2.4) **Definition.** Following [Gre80, 0.A], we define the *deleted Čech complex*

$$\hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) = \prod_{\substack{(\alpha_0 \dots \alpha_p) \\ \text{s.t. } U_{\alpha_0 \dots \alpha_p} \neq \emptyset}} \text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}}$$

with deleted Čech differential $\hat{\delta}: \hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) \rightarrow \hat{\mathcal{C}}^{p+1}(\mathcal{U}, \text{End}^q(V))$ given by

$$(\hat{\delta}c)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=1}^p (-1)^i c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}|_{U_{\alpha_0 \dots \alpha_{p+1}}}$$

(note that the sum is only over $i \in \{1, \dots, p\}$, missing out both $i = 0$ and $i = p + 1$).



[1]That is, zero in all but finitely-many degrees.

[2]That is, a collection of perfect complexes.

[3]Said differently, $\text{End}^s(V)$ is like Z^0 of the endomorphism module of V in the dg-category of chain complexes (but with this modification of going from V_{α_p} to V_{α_0}).

(8.2.5) There is a natural multiplication on $\hat{\mathcal{C}}^\bullet(\mathcal{U}, V^\bullet)$, given by

$$(c^p \cdot d^q)_{\alpha_0 \dots \alpha_{p+q}} = (-1)^q c_{\alpha_0 \dots \alpha_p}^p d_{\alpha_p \dots \alpha_{p+q}}^q$$

which plays nicely with all the differentials involved. We often omit the \cdot and just write cd .

(8.2.6) **Definition.** A holomorphic twisting cochain for (\mathcal{U}, V^\bullet) , where \mathcal{U} and V^\bullet are as in (8.2.3), is an element

$$\mathfrak{a} = \sum_{k \geq 0} \mathfrak{a}^{k, 1-k} \in \text{Tot}^1 \hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\bullet(V))$$

such that (i) $\mathfrak{a}_{\alpha\alpha}^{1,0} = \text{id}$; (ii) $\mathfrak{a}_{\alpha}^{0,1} = d_\alpha$; and (iii) \mathfrak{a} satisfies the Maurer-Cartan equation:

$$\hat{\delta}\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a} = 0.$$

(8.2.7) Quoting (G.2.11): if we write out explicitly what $\hat{\delta}\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a} = 0$ means at each Čech level k then we can gain some insight into what twisting cochains really look like.

$k = 0 \rightsquigarrow \mathfrak{a}_{\alpha}^{0,1} \mathfrak{a}_{\alpha}^{0,1} = 0$, which says exactly that $\mathfrak{a}_{\alpha}^{0,1}$ gives us a *differential* on V_{α}^\bullet , letting us (finally) think of it as a complex.

$k = 1 \rightsquigarrow \mathfrak{a}_{\alpha\beta}^{0,1} \mathfrak{a}_{\alpha\beta}^{1,0} = \mathfrak{a}_{\alpha\beta}^{1,0} \mathfrak{a}_{\beta}^{0,1}$, which tells us that, over $U_{\alpha\beta}$, we have a *chain map of complexes*

$$\mathfrak{a}_{\alpha\beta}^{1,0}: (V_{\beta}^\bullet | U_{\alpha\beta}, \mathfrak{a}_{\beta}^{0,1}) \rightarrow (V_{\alpha}^\bullet | U_{\alpha\beta}, \mathfrak{a}_{\alpha}^{0,1}).$$

$k = 2 \rightsquigarrow -\mathfrak{a}_{\alpha\gamma}^{1,0} + \mathfrak{a}_{\alpha\beta}^{1,0} \mathfrak{a}_{\beta\gamma}^{1,0} = \mathfrak{a}_{\alpha}^{0,1} \mathfrak{a}_{\alpha\beta\gamma}^{2,-1} + \mathfrak{a}_{\alpha\beta\gamma}^{2,-1} \mathfrak{a}_{\gamma}^{0,1}$, which says that $\mathfrak{a}_{\alpha\gamma}^{1,0}$ and $\mathfrak{a}_{\alpha\beta}^{1,0} \mathfrak{a}_{\beta\gamma}^{1,0}$ are *chain homotopic* via the homotopy $\mathfrak{a}_{\alpha\beta\gamma}^{2,-1}$. Taking the degenerate simplices $\alpha\beta\alpha$ and $\beta\alpha\beta$, this tells us, in particular, that $\mathfrak{a}_{\alpha\beta}^{1,0}$ and $\mathfrak{a}_{\beta\alpha}^{1,0}$ are chain homotopic inverses, i.e. that the chain map $\mathfrak{a}_{\alpha\beta}^{1,0}$ is a *quasi-isomorphism*.

$k \geq 3 \rightsquigarrow$ some sort of ‘higher-order homotopic gluings’, whatever this might mean, formally (which we explain in (8.3.4)).

(8.2.8) **Definition.** Let \mathcal{E} be a sheaf of \mathcal{O}_X -modules on X . Then a *holomorphic twisting resolution* of \mathcal{E} is a triple $(\mathcal{U}, V^\bullet, \mathfrak{a})$ such that the following conditions are satisfied:

- (i) $\mathcal{U} = \{U_\alpha\}$ is a locally-finite Stein open cover of X ;
- (ii) $V^\bullet = (V_\alpha^\bullet, d_\alpha)$ is a collection of local locally free resolutions of \mathcal{E} over each U_α , of **globally**-bounded length^[1];

^[1]That is, each V_α^\bullet is a resolution of $\mathcal{E} | U_\alpha$ by locally free \mathcal{O}_{U_α} -modules. Further, there exists some $B \in \mathbb{N}$ such that every V_α^\bullet is of length no more than B .

(iii) a is a holomorphic twisting cochain for (\mathcal{U}, V^\bullet) over \mathcal{E} — that is, we have the following commutative diagram:

$$\begin{array}{ccc} V^\bullet & \xrightarrow{a^{1,0}} & V^\bullet \\ & \searrow & \swarrow \\ & \mathcal{E} & \end{array}$$

(iv) and, on degenerate simplices of the **specific form** $\alpha = (\alpha_0 \dots \alpha_p)$ with $\alpha_i = \alpha_{i+1}$ for some i , we have that $a^{k,1-k} = 0$ for $k \geq 2$.

(8.2.9) Although (8.2.8) gives a definition of a holomorphic twisting resolution of a single sheaf, we can just as easily define the notion for a *complex* of sheaves and regain this single-sheaf definition by thinking of \mathcal{E} as a complex concentrated in degree zero.

(8.2.10) There are a few existence criteria for holomorphic twisting resolutions. In particular, when \mathcal{E} is coherent, [TT76, Lemma 8.13] and [TT78, Lemma 2.4] both show that a holomorphic twisting cochain exists (and the latter actually shows a stronger result using the Hilbert syzygy theorem, namely that we can ensure that our global-length bound B is no more than the dimension of X).

In the case of resolving *complexes* of coherent sheaves, we can appeal to [Wei16, Proposition 3.20], as we explain further in (10.2.7).

(8.2.11) **Example.** One example to which we will repeatedly refer is the only one found in [Gre80, p. 41], which is the following.^[1] Throughout this example we have $I = \{1, 2\}$, and whenever we write $\alpha, \beta \in I$ we implicitly mean that $\alpha \neq \beta$; we will never write the intersection $U_{\alpha\beta\gamma}$ because there are not three distinct elements of I , and so the only Čech 2-cochains will be those over the degenerate simplices $U_{\alpha\alpha\alpha}, U_{\alpha\alpha\beta}, U_{\alpha\beta\alpha}$, etc.

Let $X = \mathbb{P}^1(\mathbb{C})$, thought of as $\mathbb{C} \cup \{\infty\}$, with the cover \mathcal{U} given by $U_1 = X \setminus \{\infty\}$ and $U_2 = X \setminus \{0\}$. Let \mathcal{F} be the coherent sheaf given by $\mathcal{F} = \mathcal{O}_X/\mathbb{I}(\{0\})$, where $\mathbb{I}(\{0\})$ is the sheaf of ideals corresponding to the subvariety $\{0\} \subset X$. The stalks of \mathcal{F} are easy enough to understand:

$$\mathcal{F}_z = \begin{cases} \mathbb{C} & \text{if } z = 0; \\ 0 & \text{otherwise.} \end{cases}$$

^[1]Interestingly, though, Green doesn't explicitly describe the twisting cochain in this example, but instead just his construction. So in some sense, this example is actually *not* found in [Gre80], but should be.

This makes it easy to find local resolutions by \mathcal{O}_{U_a} -modules: over U_1 we have the resolution

$$\xi_1^\bullet \rightarrow \mathcal{F} \mid U_1 = \underbrace{\left(0 \rightarrow \mathcal{O}_X \mid U_1 \xrightarrow{f \mapsto z \cdot f} \mathcal{O}_X \mid U_1\right)}_{\xi_1^\bullet} \rightarrow \mathcal{F} \mid U_1,$$

and over U_2 we have the resolution

$$\xi_2^\bullet \rightarrow \mathcal{F} \mid U_2 = \underbrace{\left(0 \rightarrow 0 \rightarrow 0\right)}_{\xi_2^\bullet} \rightarrow \mathcal{F} \mid U_2.$$

So we can define some V , with $V_\alpha^\bullet = \xi_\alpha^\bullet$, and also $\mathfrak{a}_\alpha^{0,1} = d_\alpha$, where d_α is the differential of ξ_α^\bullet . Then $k = 0$ in (8.2.7) is satisfied. Note that we have given something much stronger than required: rather than a collection of *graded* vector bundles, or even a collection of *complexes* of vector bundles, we have given a collection of *complexes* of vector bundles *with almost trivial homology* (i.e. zero in all degrees except the last).

We know that we need to define $\mathfrak{a}_{\alpha\alpha}^{1,0} = \text{id}_{\xi_\alpha^\bullet}$, and so we just need to ask what $\mathfrak{a}_{\alpha\beta}$ should be. But ξ_2^\bullet is the zero complex, and so the only possible map (both to and from) ξ_2^\bullet is the zero map, so we have no choice but to define $\mathfrak{a}_{\alpha\beta}^{1,0} = 0$. This is clearly a chain map, and so $k = 1$ in (8.2.7), is satisfied.

Since $0, \infty \notin U_{12}$, the map $f \mapsto z \cdot f$ gives an isomorphism $\mathcal{O}_X \mid U_{12} \xrightarrow{\sim} \mathcal{O}_X \mid U_{12}$ and so the homology of $\xi_1^\bullet \mid U_{12}$ is zero. This means that the zero map is indeed a quasi-isomorphism between $\xi_1^\bullet \mid U_{12}$ and $\xi_2^\bullet \mid U_{12}$, but we need to construct explicit $\mathfrak{a}_{\alpha\beta}^{2,-1}$ and $\mathfrak{a}_{\beta\alpha}^{2,-1}$ that witness this, as well as the other parts of $\mathfrak{a}^{2,-1}$, in order to satisfy $k = 2$ in (8.2.7). We don't do this here, because it turns out to not be necessary, but we claim, however, that they *do* exist, and can be constructed inductively, as explained in (G.2.15).

8.3 Almost strictification

(8.3.1) The rough idea of a *dg-enhancement* is as follows: given some triangulated category \mathcal{T} , we say that a pre-triangulated dg-category \mathcal{C} is a *dg-enhancement* of \mathcal{T} if there exists some equivalence $\text{H}^0(\mathcal{C}) \xrightarrow{\sim} \mathcal{T}$. That is, the triangulated category \mathcal{T} 'looks like' the homotopy category of the dg-category \mathcal{C} , and so the dg-enhancement is like a 'less strict' version of the original category.

A main result^[1] of [Wei16] is that the category of twisting cochains is a dg-enhancement of the derived category of Fréchet^[2] quasi-coherent sheaves. What

^[1]More specifically, [Wei16, Theorem 3.29]. Note, however, that there are some differences of language, e.g. what we call 'twisting cochains' are there called 'twisted perfect complexes'.

^[2]See [Wei16, Appendix 2].

exactly is meant by ‘Fréchet quasi-coherent’ is not exactly important for our purposes: what we care about is that the derived category of such things contains the derived category of coherent sheaves, and so we really can think of holomorphic twisting resolutions (in particular) as ‘complexes of coherent sheaves that are only glued up to higher homotopies’.

We spell out more details in (G.4).

(8.3.2) In [Wei16] the definition of a twisting cochain is slightly weaker: $\alpha_{\alpha\alpha}$ is only required to be *quasi-isomorphic* to the identity on V_{α}^{\bullet} . Morally, this weaker condition is ‘better’, not only because the category of twisting cochains then has mapping cones, but also because it is more ‘in keeping’ with the idea of twisting cochains as weakly-glued objects. In fact, even weaker conditions exist: see [Wei16, §5.2]. We claim, however, that, for our purposes, this difference can be ignored.

If we run all the arguments in this section through, but with this weaker condition, we will end up with morphisms f, g of complexes whose compositions gf and fg are homotopic to something quasi-isomorphic to the identity, rather than homotopic to the identity itself. But since locally free sheaves on Stein sets are projective objects^[1], being quasi-isomorphic to the identity implies being homotopic to the identity. This is like the fact that the derived category of an abelian category is the homotopy category of the projective objects (under certain conditions on the abelian category).

(8.3.3) The problem of *coherence theorems* have been around in higher category theory since almost the birth of the subject. It is often the case that we have a bunch of laws about morphisms that hold ‘up to higher morphisms’, and we want to be able to understand these explicitly. This process of replacing laws that hold ‘up to higher morphisms’ with laws that hold ‘on the nose’ is called *strictification*. A fundamental example is that every bicategory is equivalent to a (strict) 2-category.

What we would **like** to say here, that Green described but didn’t really expound upon, is that holomorphic twisting resolutions can be strictified, but this turns out to **not** be the case, as we will explain in (8.3.12).

(8.3.4) We now spell out the details of what a holomorphic twisting resolution^[2] gives you in simplicial degree two, i.e. over some $U_{\alpha\beta}$, and how we can use this to (almost) strictify, as mentioned in (8.3.3). We will change notation away from that used for twisting cochains, to try to make this explanation slightly clearer than that found in [Gre80, pp. 23–25].

Recalling (8.2.7) and continuing similarly, we have

^[1]This is some version of the *Serre-Swann theorem*.

^[2]*Not* (a priori) an arbitrary twisting cochain, because we will use the fact that a is zero on the degenerate simplices of the specific form $\alpha_0 \dots \alpha_p$ with $\alpha_i = \alpha_{i+1}$ for some i .

- (i) (homologically-graded) complexes: $(K^\bullet, d_K) = (V_\alpha^\bullet, \mathfrak{a}_\alpha^{0,1})$ and $(L^\bullet, d_L) = (V_\beta^\bullet, \mathfrak{a}_\beta^{0,1})$;
- (ii) quasi-inverse morphisms $f = \mathfrak{a}_{\beta\alpha}^{1,0}: K^\bullet \rightarrow L^\bullet$ and $g = \mathfrak{a}_{\alpha\beta}^{1,0}: L^\bullet \rightarrow K^\bullet$;
- (iii) homotopies $p_1 = \mathfrak{a}_{\alpha\beta\alpha}^{2,-1}: K^\bullet \dashrightarrow K^\bullet[1]$ and $q_1 = \mathfrak{a}_{\beta\alpha\beta}^{2,-1}: L^\bullet \dashrightarrow L^\bullet[1]$ (where we write \dashrightarrow to mean a map of complexes that doesn't necessarily commute with the differentials) such that $gf - \text{id}_K = d_K p_1 + p_1 d_K$ and $fg - \text{id}_L = d_L q_1 + q_1 d_L$, i.e. that witness the above quasi-isomorphism;
- (iv) homotopies $p_2 = \mathfrak{a}_{\beta\alpha\beta\alpha}^{3,-2}: K^\bullet \dashrightarrow L^\bullet[2]$ and $q_2 = \mathfrak{a}_{\alpha\beta\alpha\beta}^{3,-2}: L^\bullet \dashrightarrow K^\bullet[2]$ such that $f p_1 - q_1 f = d_L p_2 - p_2 d_K$ and $g q_1 - p_1 g = d_K q_2 - q_2 d_L$ (as explained below), i.e. witnessing the failure of p_1 and q_1 to commute with f and g ;
- (v) generally, for $n \in \mathbb{N}$, we have homotopies

$$p_{2n-1} = \mathfrak{a}_{\alpha\beta\alpha\dots\beta\alpha}^{2n,-2n+1}: K^\bullet \dashrightarrow K^\bullet[2n-1]$$

$$q_{2n-1} = \mathfrak{a}_{\beta\alpha\beta\dots\alpha\beta}^{2n,-2n+1}: L^\bullet \dashrightarrow L^\bullet[2n-1]$$

as well as homotopies

$$p_{2n} = \mathfrak{a}_{\beta\alpha\dots\beta\alpha}^{2n+1,-2n}: K^\bullet \dashrightarrow L^\bullet[2n]$$

$$q_{2n} = \mathfrak{a}_{\alpha\beta\dots\alpha\beta}^{2n+1,-2n}: L^\bullet \dashrightarrow K^\bullet[2n],$$

but the relations that they satisfy aren't immediately clear, but we can find out what they are by evaluating the Maurer-Cartan equation $\hat{\mathfrak{a}} + \mathfrak{a} \cdot \mathfrak{a} = 0$ on the degenerate simplices $\alpha\beta\alpha\dots$ and $\beta\alpha\beta\dots$. For $i \geq 3$, we note that $\hat{\mathfrak{a}} = 0$, because all terms will be of the form $\mathfrak{a}_{\dots\alpha\alpha\dots}$ or $\mathfrak{a}_{\dots\beta\beta\dots}$, which are zero, by the definition of a holomorphic twisting resolution, and so we only have to look at the $\mathfrak{a} \cdot \mathfrak{a}$ term.

For example, on the degenerate simplex $\beta\alpha\beta\alpha$, the Maurer-Cartan equation (remembering what we have just said about \mathfrak{a} being zero on certain degenerate simplices) tells us that

$$\begin{aligned} 0 &= -\mathfrak{a}_\beta \cdot \mathfrak{a}_{\beta\alpha\beta\alpha} + \mathfrak{a}_{\beta\alpha} \cdot \mathfrak{a}_{\alpha\beta\alpha} - \mathfrak{a}_{\beta\alpha\beta} \cdot \mathfrak{a}_{\beta\alpha} + \mathfrak{a}_{\beta\alpha\beta\alpha} \cdot \mathfrak{a}_\alpha \\ &= -d_L p_2 + f p_1 - q_1 f + p_2 d_K \end{aligned}$$

so that the homotopy relation satisfied by p_2 is

$$f p_1 - q_1 f = d_L p_2 - p_2 d_K$$

which has a (maybe surprising) minus sign in the second term on the right-hand side.

To see the relation satisfied by p_3 , we look at what happens on the simplex $\alpha\beta\alpha\beta\alpha$, and we see that

$$g p_2 + q_2 f - p_1 p_1 = d_K p_3 + p_3 d_L.$$

This is maybe not what we would have expected: it says that p_3 witnesses a homotopy between $p_1 p_1$ and $g p_2 + q_2 f$.

Generally, for $i = 2n - 1 \geq 3$, we will have relations of the form

$$\begin{aligned} d_K p_{2n-1} + p_{2n-1} d_K &= g p_{2n-2} - p_1 p_{2n-3} + q_2 p_{2n-4} - p_3 p_{2n-5} + \dots \\ &\dots - p_{2n-3} p_1 + p_{2n-2} f \end{aligned} \quad (8.3.4.1)$$

$$\begin{aligned} d_L q_{2n-1} + q_{2n-1} d_L &= f q_{2n-2} - q_1 q_{2n-3} + p_2 q_{2n-4} - q_3 q_{2n-5} + \dots \\ &\dots - q_{2n-3} q_1 + q_{2n-2} g \end{aligned} \quad (8.3.4.2)$$

and for $i = 2n \geq 4$ we will have

$$\begin{aligned} d_L p_{2n} - p_{2n} d_K &= f p_{2n-1} - q_1 p_{2n-2} + p_2 p_{2n-3} - q_3 p_{2n-4} + \dots \\ &\dots + p_{2n-2} p_1 - q_{2n-1} f \end{aligned} \quad (8.3.4.3)$$

$$\begin{aligned} d_K q_{2n} - q_{2n} d_L &= g q_{2n-1} - p_1 q_{2n-2} + q_2 q_{2n-3} - p_3 q_{2n-4} + \dots \\ &\dots + q_{2n-2} q_1 - p_{2n-1} g. \end{aligned} \quad (8.3.4.4)$$

Both complexes K^\bullet and L^\bullet are bounded on both sides, say $K^j = L^j = 0$ for $j > N$ and $j < 0$, and we trivially have that f^j , g^j , p_i^j , and q_i^j are all zero (for all i) for $j < 0$, where we write e.g. f^j to mean the component of f that has codomain K^j . This will be the first step of our inductive construction.

The idea in [Gre80] is that we can modify the degree-0 and degree-1 terms in K^\bullet and L^\bullet , as well as the degree-0 and degree-1 parts of all the maps and homotopies, in order to make the quasi-isomorphism *strict* in degree 0. That is, to define new complexes \widetilde{K}^\bullet and \widetilde{L}^\bullet that are in some sense^[1] equivalent to K^\bullet and L^\bullet , along with \widetilde{f} , \widetilde{g} , \widetilde{p}_i , and \widetilde{q}_i such that all of the above relations still hold, but \widetilde{p}_i^0 and \widetilde{q}_i^0 are now *zero* for all i . This means that we have replaced our complexes by some equivalent complexes, but where the quasi-isomorphism in degree zero is now a *true* isomorphism of modules, with higher homotopies *all being zero*.^[2] If we then shift the new complexes by one degree, then we can inductively repeat this procedure, until we end up with two complexes that are truly isomorphic in every degree, but still equivalent to the ones that we started with.

In summary, then, we are going to see how Green turns a holomorphic twisting resolution into a resolution by vector bundles on the nerve.

We now describe this construction explicitly.

(8.3.5) Although the following calculations may seem unwieldy, the idea is nowhere near as complex as the notation might suggest. Basically all that we do is force the two complexes to be isomorphic in degree 0 by adding each one to the other, and

^[1]Which we explain in (8.3.11).

^[2]This is stronger than just asking for a true isomorphism of modules in degree zero, since it could be the case that $d p_i + p_i d = 0$ without necessarily having that $p_i = 0$.

modifying the morphisms accordingly, and then modifying things in degree 0 so that everything still commutes. Really we have very little to no choice in how to define the new maps, once we consider what the codomains, domains, and degrees have to be. Then we set all the homotopies in degree 0 to be zero, and adjust the homotopies in degree 1 to ensure that all the relations are still satisfied.

(8.3.6) [Gre80, pp. 23–25]. As we pointed out in (8.3.4), the two complexes that we have are both strictly isomorphic in degree j for $j < 0$, and, further, all homotopies p_i^j and q_i^j are zero for $j < 0$. To give an inductive construction, then, we assume that this is the case, but do *not* assume that K^j and L^j are zero for $j < 0$. This means that, if we can strictify the quasi-isomorphism (*and* kill all homotopies) in degree 0, then we can repeat the procedure indefinitely by just shifting the complex by one degree, since the same hypotheses will then be satisfied.

So we continue with the notation from (8.3.4), but with the small change that K^\bullet and L^\bullet are now complexes from degree N to degree $-M$. We then define the following:

- $\widetilde{K}^j = K^j, \widetilde{L}^j = L^j, \widetilde{f}^j = f^j, \widetilde{g}^j = g^j, \widetilde{p}_i^j = p_i^j$, and $\widetilde{q}_i^j = q_i^j$ (for all i) for all $j \neq 0, 1$;
- $\widetilde{K}^1 \rightarrow \widetilde{K}^0 = (K^1 \rightarrow K^0) \oplus (L^0 \xrightarrow{\text{id}} L^0)$ and $\widetilde{L}^1 \rightarrow \widetilde{L}^0 = (L^1 \rightarrow L^0) \oplus (K^0 \xrightarrow{\text{id}} K^0)$;
- $\widetilde{f}^0 = \begin{pmatrix} f^0 & -d_L q_1^0 \\ \text{id}_K & -g^0 \end{pmatrix}$, $\widetilde{f}^1 = \begin{pmatrix} f^1 & -q_1^0 \\ d_K & -g^0 \end{pmatrix}$, and $\widetilde{g}^0 = \begin{pmatrix} g^0 & -d_K p_1^0 \\ \text{id}_L & -f^0 \end{pmatrix}$, $\widetilde{g}^1 = \begin{pmatrix} g^1 & -p_1^0 \\ d_L & -f^0 \end{pmatrix}$;
- $\widetilde{p}_i^0 = 0$ and $\widetilde{q}_i^0 = 0$ (for all i);
- $\widetilde{p}_i^1 = \begin{pmatrix} p_i^1 & -q_{i+1}^0 \end{pmatrix}$ and $\widetilde{q}_i^1 = \begin{pmatrix} q_i^1 & -p_{i+1}^0 \end{pmatrix}$ (for all i).

That is, the new complexes \widetilde{K}^\bullet and \widetilde{L}^\bullet are unchanged, except in degrees 0 and 1, where we add an elementary sequence^[1], and we only modify the morphisms and all the homotopies in degrees 0 and 1. We summarise the complexes \widetilde{K}^\bullet and \widetilde{L}^\bullet and the morphisms \widetilde{f} and \widetilde{g} in the following diagram:

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{d_K} & K^2 & \xrightarrow{\begin{pmatrix} d_K \\ 0 \end{pmatrix}} & K^1 \oplus L^0 & \xrightarrow{\begin{pmatrix} d_K & 0 \\ 0 & \text{id}_L \end{pmatrix}} & K^0 \oplus L^0 & \xrightarrow{(d_K \ 0)} & K^{-1} & \xrightarrow{d_K} & \dots \\
 & & \uparrow \! \! \! \uparrow f^2 \! \! \! \downarrow g^2 & & \uparrow \! \! \! \uparrow \begin{pmatrix} f^1 & -q_1^0 \\ d_K & -g^0 \end{pmatrix} \! \! \! \downarrow \begin{pmatrix} g^1 & -p_1^0 \\ d_L & -f^0 \end{pmatrix} & & \uparrow \! \! \! \uparrow \begin{pmatrix} f^0 & -d_L q_1^0 \\ \text{id}_K & -g^0 \end{pmatrix} \! \! \! \downarrow \begin{pmatrix} g^0 & -d_K p_1^0 \\ \text{id}_L & -f^0 \end{pmatrix} & & \uparrow \! \! \! \uparrow f^{-1} \! \! \! \downarrow g^{-1} & & \\
 \dots & \xrightarrow{d_L} & L^2 & \xrightarrow{\begin{pmatrix} d_L \\ 0 \end{pmatrix}} & L^1 \oplus K^0 & \xrightarrow{\begin{pmatrix} d_L & 0 \\ 0 & \text{id}_K \end{pmatrix}} & L^0 \oplus K^0 & \xrightarrow{(d_L \ 0)} & L^{-1} & \xrightarrow{d_L} & \dots
 \end{array}$$

There now remain three things to verify:

^[1]See (9.1.1).

(8.3.7) that \tilde{f} and \tilde{g} are morphisms of complexes;

(8.3.8) that $\tilde{g}^0\tilde{f}^0 = \text{id}_K$ and $\tilde{f}^0\tilde{g}^0 = \text{id}_L$ (i.e. we have a strict isomorphism in degree 0);

(8.3.9) that the homotopy relations corresponding to those satisfied by p_i and q_i (given in (8.3.4)) are satisfied by \tilde{p}_i and \tilde{q}_i (so that we can repeat this procedure inductively).

Note that, for all three, since our construction is unchanged if we exchange K with L , f with g , and p with q throughout, it suffices to give the proofs for just e.g. K , f , and p , which saves us from a lot of unnecessary repetition of matrix calculations.

(8.3.7) To check that \tilde{f} is a morphism of complexes we need to check the commutativity of three squares, but the following calculations show that they all commute:

$$\begin{aligned} 1. & \begin{pmatrix} f^1 & -q_1^0 \\ d_K & -g^0 \end{pmatrix} \begin{pmatrix} d_K \\ 0 \end{pmatrix} = \begin{pmatrix} f^1 d_K \\ d_K d_K \end{pmatrix} = \begin{pmatrix} d_L f^2 \\ 0 \end{pmatrix} = \begin{pmatrix} d_L \\ 0 \end{pmatrix} (f^2) \\ 2. & \begin{pmatrix} f^0 & -d_L q_1^0 \\ \text{id}_K & -g^0 \end{pmatrix} \begin{pmatrix} d_K & 0 \\ 0 & \text{id}_L \end{pmatrix} = \begin{pmatrix} f^0 d_K & -d_L q_1^0 \\ d_K & -g^0 \end{pmatrix} = \begin{pmatrix} d_L f^1 & -d_L q_1^0 \\ d_K & -g^0 \end{pmatrix} = \begin{pmatrix} d_L & 0 \\ 0 & \text{id}_K \end{pmatrix} \begin{pmatrix} f^1 & -q_1^0 \\ d_K & -g^0 \end{pmatrix} \\ 3. & (f^{-1}) \begin{pmatrix} d_K & 0 \end{pmatrix} = (f^{-1} d_K \ 0) = (d_L f^0 \ 0) = (d_L \ 0) \begin{pmatrix} f^0 & -d_L q_1^0 \\ \text{id}_K & -g^0 \end{pmatrix} \end{aligned}$$

where we use the fact that f is a morphism of complexes, and so $f^{j-1}d_K = d_L f^j$.

(8.3.8) Calculating $\tilde{g}^0\tilde{f}^0$ is routine:

$$\begin{aligned} \tilde{g}^0\tilde{f}^0 &= \begin{pmatrix} g^0 & -d_K p_1^0 \\ \text{id}_L & -f^0 \end{pmatrix} \begin{pmatrix} f^0 & -d_L q_1^0 \\ \text{id}_K & -g^0 \end{pmatrix} = \begin{pmatrix} g^0 f^0 - d_K p_1^0 & -g^0 d_L q_1^0 + d_K p_1^0 g^0 \\ f^0 - f^0 & -d_L q_1^0 + f^0 g^0 \end{pmatrix} \\ &= \begin{pmatrix} \text{id}_K + p_1^{-1} d_K & g^0 (\text{id}_L - f^0 g^0 + q_1^{-1} d_L) + (g^0 f^0 - \text{id}_K - p_1^{-1} d_K) g^0 \\ 0 & 1 + q_1^{-1} d_L \end{pmatrix} \\ &= \begin{pmatrix} \text{id}_K & g^0 - g^0 f^0 g^0 + g^0 f^0 g^0 - g^0 \\ 0 & \text{id}_L \end{pmatrix} = \begin{pmatrix} \text{id}_K & 0 \\ 0 & \text{id}_L \end{pmatrix} \end{aligned}$$

where we use the fact that q_1^j and p_1^j are both zero for $j < 0$.

(8.3.9) We check the homotopy relations in different ways depending on the (homological complex) degree. Note that the homotopy relations in (homological complex) degree j also involve a homotopy in (homological complex) degree $(j-1)$, e.g. $g^j f^j - \text{id}_K = d_K p_1^j + p_1^{j-1} d_K$.

- (i) In degree $j < 0$, the homotopies are all still zero, and so all the relations are trivially satisfied.

- (ii) In degree $j = 0$, the homotopies are now all defined to be zero, and so all the relations are trivially satisfied.
- (iii) In degree $j > 2$, the homotopies haven't changed, and so the previous relations are still satisfied.
- (iv) In degree 2, we note that the differential $\widetilde{K}^2 \rightarrow \widetilde{K}^1$ is $\begin{pmatrix} d_K \\ 0 \end{pmatrix}$, and the $(1,1)$ -entry of \widetilde{p}_i^1 is just p_i^1 , whence $\widetilde{p}_i^1 d_{\widetilde{K}} = p_i^1 d_K$, and so the relations are satisfied exactly as before, since nothing has really changed.
- (v) In degree $j = 1$, we first consider the cases $i = 1$ and $i = 2$ before generalising to a proof for $i = 2n - 1$ and $i = 2n$.

($i = 1$) We need to show that $\widetilde{g}^1 \widetilde{f}^1 - \begin{pmatrix} id_K & 0 \\ 0 & id_L \end{pmatrix} = d_{\widetilde{K}} \widetilde{p}_1^1 + \widetilde{p}_1^0 d_{\widetilde{K}}$, but $\widetilde{p}_1^0 = 0$ by definition, and $d_{\widetilde{K}} \widetilde{p}_1^1 = \begin{pmatrix} d_K \\ 0 \end{pmatrix} (p_1^1 - q_2^0) = \begin{pmatrix} d_K p_1^1 & -d_K q_2^0 \\ 0 & 0 \end{pmatrix}$. Then

$$\begin{aligned} \widetilde{g}^1 \widetilde{f}^1 &= \begin{pmatrix} g^1 & -p_1^0 \\ d_L & -f^0 \end{pmatrix} \begin{pmatrix} f^1 & -q_1^0 \\ d_K & -g^0 \end{pmatrix} = \begin{pmatrix} g^1 f^1 - p_1^0 d_K & -g^1 q_1^0 + p_1^0 g^0 \\ d_L f^1 - f^0 d_K & d_L q_1^0 + f^0 g^0 \end{pmatrix} \\ &= \begin{pmatrix} id_K + d_K p_1^1 & -d_K q_2^0 + q_2^{-1} d_L \\ 0 & id_L + q_1^{-1} d_L \end{pmatrix} = \begin{pmatrix} id_K + d_K p_1^1 & -d_K q_2^0 \\ 0 & id_L \end{pmatrix} \\ &= \begin{pmatrix} id_K & 0 \\ 0 & id_L \end{pmatrix} + d_{\widetilde{K}} \widetilde{p}_1^1. \end{aligned}$$

($i = 2$) We need to show that $\widetilde{f}^2 \widetilde{p}_1^1 - \widetilde{q}_1^1 \widetilde{f}^1 = d_{\widetilde{L}} \widetilde{p}_2^1 - \widetilde{p}_2^0 d_{\widetilde{K}}$, but $\widetilde{p}_2^0 = 0$ by definition, and $d_{\widetilde{L}} \widetilde{p}_2^1 = \begin{pmatrix} d_L p_2^1 & -d_L q_3^0 \end{pmatrix}$. Then

$$\begin{aligned} \widetilde{f}^2 \widetilde{p}_1^1 - \widetilde{q}_1^1 \widetilde{f}^1 &= (f^2) \begin{pmatrix} p_1^1 & -q_2^0 \end{pmatrix} - \begin{pmatrix} q_1^1 & -p_2^0 \end{pmatrix} \begin{pmatrix} f^1 & -q_1^0 \\ d_K & -g^0 \end{pmatrix} \\ &= (f^2 p_1^1 - f^2 q_2^0) - (q_1^1 f^1 - p_2^0 d_K - q_1^1 q_1^0 + p_2^0 g^0) \\ &= (f^2 p_1^1 - q_1^1 f^1 + p_2^0 d_K - f^2 q_2^0 + q_1^1 q_1^0 - p_2^0 g^0) \\ &= (d_L p_2^1 - d_L q_3^0 - q_3^{-1} d_K) = (d_L p_2^1 - d_L q_3^0) = d_{\widetilde{L}} \widetilde{p}_2^1, \end{aligned}$$

where we use the fact that $f q_2 + p_2 g - q_1 q_1 = d_L q_3 + q_3 d_K$, given by evaluating the Maurer-Cartan equation on the degenerate simplex $\beta \alpha \beta \alpha \beta$, as explained in (8.3.4).

($i = 2n - 1 \geq 3$) The left-hand side of (8.3.4.1) will be $d_{\widetilde{K}} \widetilde{p}_{2n-1}^1 + \widetilde{p}_{2n-1}^0 d_{\widetilde{K}}$, but the latter is zero, by definition. So we are looking at $d_{\widetilde{K}} \widetilde{p}_{2n-1}^1 = \begin{pmatrix} d_K p_{2n-1}^1 & -d_K q_{2n}^0 \end{pmatrix}$. We know that the $(1,1)$ -element of this will satisfy (8.3.4.1), by hypothesis, and we can see that the $(1,2)$ -element will satisfy the same relation, because this is exactly (8.3.4.4).

($i = 2n \geq 4$) This is exactly the same as above: when we replace p_{2n} with $-q_{2n+1}$ in Equation (8.3.4.3), we get exactly Equation (8.3.4.2) (but for q_{2n+1} instead of q_{2n-1}).

(8.3.10) We now want to know what happens in simplicial degree 3, where we have three complexes, say K^\bullet , L^\bullet , and M^\bullet . First we modify K^\bullet and L^\bullet to get $\widetilde{K}^\bullet \cong \widetilde{L}^\bullet$. Then we modify \widetilde{K}^\bullet and M^\bullet to get $\widetilde{\widetilde{K}}^\bullet \cong \widetilde{M}^\bullet$, but perform ‘exactly’ (i.e. up to isomorphism) the same modifications on \widetilde{L}^\bullet (since $\widetilde{K}^\bullet \cong \widetilde{L}^\bullet$) to get $\widetilde{\widetilde{L}}^\bullet \cong \widetilde{M}^\bullet$. Explicitly, what this will consist of is simply adding the elementary sequence $(L^0 \oplus M^0 \rightarrow L^0 \oplus M^0)[0]$ to K^\bullet , the elementary sequence $(K^0 \oplus M^0 \rightarrow K^0 \oplus M^0)[0]$ to L^\bullet , and the elementary sequence $(K^0 \oplus L^0 \rightarrow K^0 \oplus L^0)[0]$ to M^\bullet .

In general, in simplicial degree p over $U_{\alpha_0 \dots \alpha_p}$ we will add

$$(V_{\alpha_0}^0 \oplus \dots \oplus \widehat{V_{\alpha_i}^0} \oplus \dots \oplus V_{\alpha_p}^0 \xrightarrow{\text{id} \oplus \dots \oplus \widehat{\text{id}} \oplus \dots \oplus \text{id}} V_{\alpha_0}^0 \oplus \dots \oplus \widehat{V_{\alpha_i}^0} \oplus \dots \oplus V_{\alpha_p}^0)[0]$$

to $V_{\alpha_i}^\bullet$, where the hat denotes omission, just as in the definition of the Čech differential.

(8.3.11) Adding an elementary sequence to a complex gives a quasi-isomorphism of complexes: one composition (from K^\bullet to itself) is the identity on the nose; the other composition (from $K^\bullet \oplus (M \rightarrow M)[0]$ to itself) is homotopic to the identity, as witnessed by the homotopy that is zero in all degrees except in the degree 0, where we define it to be $(0, -\text{id})$. That is, the homotopy given by

$$\begin{array}{ccccccccc} \dots & \xrightarrow{d_K} & K^2 & \xrightarrow{\begin{pmatrix} d_K & \\ & 0 \end{pmatrix}} & K^1 \oplus M & \xrightarrow{\begin{pmatrix} d_K & 0 \\ 0 & \text{id}_M \end{pmatrix}} & K^0 \oplus M & \xrightarrow{(d_K \ 0)} & K^{-1} & \xrightarrow{d_K} & \dots \\ & \searrow 0 & & \searrow 0 & & \searrow (0 \ -\text{id}) & & \searrow 0 & & \searrow 0 & \\ \dots & \longrightarrow & K^2 & \longrightarrow & K^1 \oplus M & \longrightarrow & K^0 \oplus M & \longrightarrow & K^{-1} & \longrightarrow & \dots \end{array}$$

(8.3.12) A morphism of twisting cochains is a weak equivalence if it is *closed*, of *degree zero*, and its $(0,0)$ -component is a quasi-isomorphism of complexes over each open set, as defined in [Wei16, Definition 2.2.7].

We have already shown, in (8.3.11), that adding elementary sequences gives a quasi-isomorphism, and the above construction is of degree zero, but there is a problem when trying to turn this into a morphism of twisting cochains: namely, it is not *closed* because, in particular, it is not *functorial*. That is, in order for $d\varphi$ to be zero (where we write φ to mean this process of turning K^\bullet into \widetilde{K}^\bullet) we need, in degree zero, to have the diagram

$$\begin{array}{ccc} K^\bullet & \xrightarrow{f} & L^\bullet \\ \varphi \downarrow & & \downarrow \varphi \\ \widetilde{K}^\bullet & \xrightarrow{\widetilde{f}} & \widetilde{L}^\bullet \end{array}$$

commute. But if we take φ to be the ‘obvious’ map (given by inclusion), this is *not* the case. In fact, no matter how we try to define φ , there is no way to make this



so. In other words, this construction is **not functorial**: it *looks* like strictification but is somehow not, because it only strictifies *locally* (i.e. on each $U_{\alpha_0 \dots \alpha_p}$), and twisting cochains are *stronger* than just a disjoint collection of local data.

Surprisingly, and thankfully, however, this is not actually a problem: we use Green's construction only to show essential surjectivity of a functor in (10.2.7), and for this we just need a map defined on objects, which we have. In fact, we will use Green's construction to let us forget almost entirely about twisting cochains and work instead with certain vector bundles on the nerve.

Green's resolution

9

It's not easy [understanding] [G]reen[']s resolution].

Kermit the Frog (almost).

(9.0.1) Purpose. *This chapter is basically a paraphrasing of Green's thesis; the 'novel' part is Section 9.2, which justifies the definitions of Chapter 7 by showing that Green's construction is a specific example of our simplicial framework.*

9.1 The resolution

(9.1.1) Definition. Given a ring R and some R -modules N_1, \dots, N_s , we say that a sequence $0 \rightarrow M_r \rightarrow \dots \rightarrow M_0 \rightarrow 0$ of R -modules is (N_1, \dots, N_s) -*elementary* if it is a (direct) sum of sequences of the form

$$(0 \rightarrow N_i \xrightarrow{\text{id}} N_i \rightarrow 0)[n]$$

for some $i \in \{1, \dots, s\}$ and $n \in \mathbb{Z}$. Given complexes $V_1^\bullet, \dots, V_t^\bullet$ of R -modules, we say that a sequence is $(V_1^\bullet, \dots, V_t^\bullet)$ -*elementary* if it is \mathcal{N} -elementary, where

$$\mathcal{N} = \{V_i^j \mid 1 \leq i \leq t, 1 \leq j \leq s\}.$$

(9.1.2) Green's simplicial resolution. Let \mathcal{F} be a coherent sheaf of \mathcal{O}_X -modules on a paracompact complex-analytic manifold X with locally-finite Stein cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. Let $(\mathcal{U}, V^\bullet, \mathfrak{a})$ be a holomorphic twisted resolution of \mathcal{F} . Denote by \mathcal{F}^\bullet the pullback of \mathcal{F} to the nerve $X_\bullet^\mathcal{U}$. Then there exists a resolution^[1] $\mathcal{E}^{\bullet, \star}$ of \mathcal{F}^\bullet by vector bundles on the nerve:

$$0 \rightarrow \mathcal{E}^{\bullet, n} \rightarrow \dots \rightarrow \mathcal{E}^{\bullet, 0} \rightarrow \mathcal{F}^\bullet$$

where $n = \dim X$. Further, the $\mathcal{E}^{\bullet, j}$ satisfy the following properties:

- (i) $\mathcal{E}^{0, \star} \big|_{U_\alpha} \cong V_\alpha^\star$;
- (ii) for all coface^[2] maps $f_p^i: [p-1] \rightarrow [p]$, the map $\mathcal{E}^{\bullet, \star} f_p^i: (X_\bullet^\mathcal{U} f_p^i)^\star \mathcal{E}^{p-1, \star} \rightarrow \mathcal{E}^{p, \star}$ of complexes of sheaves on $X_p^\mathcal{U}$ is injective, and $\text{Coker}(\mathcal{E}^{\bullet, \star} f_p^i)$ is an elementary sequence;^[3]

^[1]That is, the morphism $\mathcal{E}^{p, \star} \rightarrow \mathcal{F}^p$ is a quasi-isomorphism in for every $p \in \mathbb{N}$.

^[2]In fact, in [Gre80, §1.4], properties (ii) and (iii) are stated for arbitrary compositions of coface (resp. codegeneracy) maps instead of simply for single coface (resp. codegeneracy) maps.

^[3]Here the complex is indexed by \star from 0 to n , and $\mathcal{E}^{\bullet, \star} f_p^i$ is a map of complexes.

- (iii) for all codegeneracy maps $s_i^p : [p+1] \rightarrow [p]$, the map $\mathcal{E}^{\bullet,\star} s_i^p : (X_p^U s_i^p)^* \mathcal{E}^{p+1,\star} \rightarrow \mathcal{E}^{p,\star}$ of complexes of sheaves on X_p^U is surjective, and $\text{Ker}(\mathcal{E}^{\bullet,\star} s_i^p)$ is an elementary sequence.

These follow from the fact that, for all $\gamma \leq \beta \leq \alpha = (\alpha_0 \dots \alpha_p)$, writing \mathcal{E}_α to mean $\mathcal{E}^p \upharpoonright U_\alpha$, we have the following:

- (iv) $\mathcal{E}_\alpha^\star \cong \mathcal{E}_\beta^\star \upharpoonright U_\alpha \oplus \mathcal{L}_{\alpha,\beta}^\star$ for some $(V_{\alpha_0}^\star, \dots, V_{\alpha_p}^\star)$ -elementary sequence $\mathcal{L}_{\alpha,\beta}^\star$;
- (v) $\mathcal{L}_{\alpha,\gamma}^\star \cong \mathcal{L}_{\beta,\gamma}^\star \upharpoonright U_\alpha \oplus \mathcal{L}_{\alpha,\beta}^\star$;
- (vi) over each U_α there is the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{E}_\beta^\star & \longrightarrow & \mathcal{E}_\alpha^\star & \longrightarrow & \mathcal{L}_{\alpha,\beta}^\star \longrightarrow 0 \\
 & & \uparrow \wr & & \uparrow \wr & & \text{id} \uparrow \\
 0 & \longrightarrow & \mathcal{E}_\gamma^\star \oplus \mathcal{L}_{\beta,\gamma}^\star & \longrightarrow & \mathcal{E}_\gamma^\star \oplus \mathcal{L}_{\alpha,\gamma}^\star & \longrightarrow & \mathcal{L}_{\alpha,\beta}^\star \longrightarrow 0
 \end{array}$$

(omitting the restriction notation), where the bottom map is induced by the natural inclusion $\mathcal{L}_{\beta,\gamma}^\star \hookrightarrow \mathcal{L}_{\alpha,\gamma}^\star$ coming from $\mathcal{L}_{\alpha,\gamma}^\star \cong \mathcal{L}_{\beta,\gamma}^\star \oplus \mathcal{L}_{\alpha,\beta}^\star$.

Proof. This is [Gre80, §1.4], although we spell out some of the details (namely the construction of the sheaves, but without the information of the simplicial maps) in Section 8.3. Using the nomenclature of (7.7.4), this says that we can resolve coherent sheaves by Green complexes of vector bundles on the nerve. \square

(9.1.3) As with holomorphic twisting resolutions (8.2.9), Green's resolution (9.1.2) can be generalised to *complexes* of coherent sheaves: the proof is almost identical, following along the lines of (8.3.6).

(9.1.4) Green complexes are, in particular, *cartesian* complexes of locally free sheaves on the nerve.

Proof. Since taking the direct sum with an elementary sequence gives a quasi-isomorphism (8.3.11), this follows from properties (ii) and (iii) of (9.1.2). (See also (7.7.6)). \square

(9.1.5) Something that doesn't appear to be mentioned in [Gre80] is that these elementary sequences $\mathcal{L}_{\alpha,\beta}^\bullet$ satisfy a cocycle condition, and so can be written as elements of degree-zero cohomology of some sheaf. We have no good application of this fact, but mention it here for the sake of mentioning things.

(9.1.6) The key point to make here is that Green's simplicial resolution is **not** just the data of a resolution, but also the properties governing the (co)kernels. This is important, because it tells us, in particular, that we get admissible simplicial connections on each sheaf in the resolution, which lets us define characteristic classes using all the results of Chapter 7. We use the coface-injectivity property when discussing admissibility of connections, but don't seem to need the codegeneracy-surjectivity property anywhere. This might be because the simplicial condition for e.g. simplicial differential forms relies only on (co)face maps, and so we don't care so much about (co)degeneracy maps in our formalism, and should really be talking about Δ - or semi-simplicial-sets.



9.2 Atiyah classes of coherent sheaves

(9.2.1) **Lemma.** Any simplicial connection generated in degree zero on any one of the sheaves in Green's resolution (9.1.2) is admissible.

Proof. This is a direct corollary of (7.7.15) and (9.1.4). \square

(9.2.2) **Definition.** Given some coherent sheaf \mathcal{F} , we define its *standard* (resp. *exponential*) *Atiyah class* to be the fibre integral of the standard (resp. exponential) simplicial Atiyah class of the resolution endowed with simplicial connections generated in degree zero, as defined in Section 7.5.

(9.2.3) The fact that the class defined in (9.2.2) is independent of the choices of local connections and holomorphic twisting resolution is shown in [Gre80, Theorem 2.4], but it seems like the argument could be separated into two disjoint steps:

1. we could claim that two different choices of twisting resolution give complexes of \mathcal{O}_{X^u} -modules that are quasi-isomorphic;
2. we would then claim that two quasi-isomorphic complexes of \mathcal{O}_{X^u} -modules give identical Chern classes.

As mentioned in (8.3.12), however, we know that Green's construction is not functorial, and so it seems like this formalism leads to a dead end. But, once again, we don't really care so much, because Green's resolution lets us pretty much forget entirely about twisting cochains.

9.3 The example

(9.3.1) **Example.** We return to the example in [Gre80, p. 41], already discussed in (8.2.11). Recall that $X = \mathbb{P}^1(\mathbb{C})$ is the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with the (Stein)

cover $U_1 = X \setminus \{\infty\}$ and $U_2 = X \setminus \{0\}$, and we let $\mathcal{F} = \mathbb{I}(\{0\})$ be the sheaf of ideals corresponding to the subvariety $\{0\} \subset X$, so that $\mathcal{F} = \mathcal{O}_X/\mathcal{F}$ is a coherent sheaf.

We already found local resolutions of \mathcal{F} in (8.2.11) — over U_1 we had the resolution

$$\xi_1^\bullet = (0 \rightarrow \mathcal{O}_X|_{U_1} \xrightarrow{f \mapsto z \cdot f} \mathcal{O}_X|_{U_1}) \rightarrow \mathcal{F}|_{U_1},$$

and over U_2 we had the resolution

$$\xi_2^\bullet = (0 \rightarrow 0 \rightarrow 0) \rightarrow \mathcal{F}|_{U_2}.$$

If we want to construct a resolution by locally free sheaves on the nerve then we are going to need a resolution of $\mathcal{F}|_{U_{12}} = 0$ by locally free sheaves over U_{12} , such that the simplicial conditions^[1] are satisfied. This gives us a problem however, since we have two different^[2] choices for such a complex:

$$\begin{aligned} (1) \xi_{12}^\bullet &= (0 \rightarrow \mathcal{O}_X|_{U_{12}} \xrightarrow{f \mapsto z \cdot f} \mathcal{O}_X|_{U_{12}}) \rightarrow \mathcal{F}|_{U_{12}}, \\ (2) \xi_{12}^\bullet &= (0 \rightarrow 0 \rightarrow 0) \rightarrow \mathcal{F}|_{U_{12}}. \end{aligned}$$

But we see that adding the elementary sequence

$$\begin{array}{l} (1) \mathcal{L}^\bullet \\ (2) \mathcal{L}^\bullet \end{array} = (0 \rightarrow \mathcal{O}_X \xrightarrow{\text{id}} \mathcal{O}_X \rightarrow 0)$$

to the latter gives us something isomorphic^[3] to the former:

$$\begin{array}{ccc} (1) \xi_{12}^\bullet & = & (0 \longrightarrow \mathcal{O}_X|_{U_{12}} \xrightarrow{f \mapsto z \cdot f} \mathcal{O}_X|_{U_{12}}) \\ & & \begin{array}{ccc} & f \mapsto z \cdot f \downarrow & \text{id} \downarrow \\ & & \end{array} \\ (2) \xi_{12}^\bullet \oplus (1) \mathcal{L}^\bullet & = & (0 \longrightarrow \mathcal{O}_X|_{U_{12}} \xrightarrow{\text{id}} \mathcal{O}_X|_{U_{12}}) \end{array}$$

and we denote this isomorphism by

$$A_{12}^\bullet: (1) \xi_{12}^\bullet \xrightarrow{\sim} (2) \xi_{12}^\bullet \oplus (1) \mathcal{L}^\bullet$$

i.e. $A_{12}^0 = \text{id}$ and $A_{12}^1 = (z \cdot)$.

We don't actually need this second complex (the codomain of A_{12}^\bullet) to construct the resolution, but we do need it later (in (9.3.4)) in order to be able to pull back the local connection over U_2 to a connection over U_{12} in a non-trivial manner.

^[1]That is, the construction of the maps $\mathcal{E}^\bullet \alpha$ for all $\alpha: [p] \rightarrow [q]$ in the (abstract) simplex category.

^[2]Note, though, that these two complexes are quasi-isomorphic, since $0, \infty \notin U_{12}$, and so $(z \cdot)$ is an isomorphism.

^[3]Since $0, \infty \notin U_{\alpha\beta}$, both $1/z$ and z are well defined and holomorphic.

At this point, the construction stabilises: since our cover consists of only two distinct opens sets, any p -intersection for $p \geq 3$ will be exactly some 2-intersection (either U_{11} , U_{12} , or U_{22}). Thus we have constructed a complex of \mathcal{O}_{X^u} -modules on the nerve of X that give a resolution of \mathcal{F} (pulled back to the nerve):

$$0 \rightarrow \mathcal{E}^{\bullet,1} \rightarrow \mathcal{E}^{\bullet,0} \rightarrow \mathcal{F}^{\bullet}$$

where $\mathcal{E}^{\bullet,1}$ and $\mathcal{E}^{\bullet,0}$ are equal, and defined as

$$\begin{aligned} \mathcal{E}^{0,i} &= \mathcal{O}_X \Big|_{U_1 \sqcup 0} \quad \text{over } U_1 \sqcup U_2 \\ \mathcal{E}^{1,i} &= \mathcal{O}_X \Big|_{U_{12}} \quad \text{over } U_{12} \end{aligned}$$

and the map $\mathcal{E}^{\bullet,1} \rightarrow \mathcal{E}^{\bullet,0}$ is given by

$$\begin{aligned} ((z \cdot) \sqcup 0): \mathcal{E}^{0,1} &\rightarrow \mathcal{E}^{0,0} \quad \text{over } U_1 \sqcup U_2 \\ (z \cdot): \mathcal{E}^{1,1} &\rightarrow \mathcal{E}^{1,0} \quad \text{over } U_{12}. \end{aligned}$$

(9.3.2) Given an isomorphism $f: E \xrightarrow{\sim} F$ of locally free sheaves over X , along with a connection ∇_F on F , we have the *pullback connection* $f^*\nabla_F$ on E given by

$$f^*\nabla_F = (f^{-1} \otimes f^*) \circ \nabla_F \circ f.$$

Locally, this takes the trivial connection d on F to $d + f^{-1}df$ on E .

This is different from the pullback connection of (B.2.3), since there we pull back along some continuous map $Y \rightarrow X$ of the *base spaces*, whereas here we ‘pull back’ along an *isomorphism of bundles* $E \xrightarrow{\sim} F$.



(9.3.3) In general, over each p -intersection $U_{\alpha_0 \dots \alpha_p}$ we have $(p+1)$ connections on \mathcal{E}_p^i , given by the pullback of the trivial connection^[1] on each U_{α_i} for $i \in \{0, \dots, p\}$.

(9.3.4) Example. Continuing the example of (9.3.1), and assuming that we have a basis of local sections over both U_1 and U_2 , we can look at all the local connections that we have: the trivial ones, and the pullbacks of the trivial ones by the isomorphism A_{12}^{\bullet} constructed above. As throughout Chapter 8, we study things on $U_{\alpha_0 \dots \alpha_p}$ by pulling them back to U_{α_0} . We summarise the data of all the local connections in Table 9.1 (which is (at least) 87.5% uninteresting).

Using these local connections, we can form the barycentric connections ∇_{\bullet}^i on $\mathcal{E}^{\bullet,i}$ as follows:

$$\nabla_{\bullet}^0 \text{ on } \mathcal{E}^{\bullet,0} \text{ is given by } \begin{cases} \nabla_0^0 = t_0 d & = d \\ \nabla_1^0 = t_0 d + t_1 d & = d \end{cases}$$

^[1]For this sentence to make sense, in particular we have to assume that we have a basis of local sections over each U_{α} .

bundle	p	p -intersection	local connection
$\mathcal{E}^{p,0}$	0	U_1	d
		U_2	d
	1	U_{12}	$d + (A_{1,2}^0)^{-1} dA_{1,2}^0 = d$
$\mathcal{E}^{p,1}$	0	U_1	d
		U_2	d
	1	U_{12}	$d + (A_{1,2}^1)^{-1} dA_{1,2}^1 = d + \frac{dz}{z}$

Table 9.1. All the local connections for this example.

$$\nabla_{\bullet}^1 \text{ on } \mathcal{E}^{\bullet,1} \text{ is given by } \begin{cases} \nabla_0^1 = t_0 d & = d \\ \nabla_1^1 = t_0 d + t_1 \left(d + \frac{dz}{z} \right) & = d + t_1 \frac{dz}{z}. \end{cases}$$

The only really non-trivial part of the simplicial connections is in (simplicial) degree 1, over U_{12} , so we calculate the curvatures and see that

$$\begin{aligned} \kappa(\nabla_1^0) &= 0, \\ \kappa(\nabla_1^1) &= d \left(t_1 \frac{dz}{z} \right) + \left(t_1 \frac{dz}{z} \right)^2 \\ &= \frac{dz}{z} \otimes dt_1. \end{aligned}$$

Then we can fibre integrate the curvatures to find the^[1] Atiyah classes, recalling that we integrate the $(k-p, p)$ term over the p -simplex (and that here we are taking $k = 1$), and we see that

$$\begin{aligned} \int_{\Delta^1} \kappa(\nabla_1^0) &= 0, \\ \int_{\Delta^1} \kappa(\nabla_1^1) &= \int_0^1 \frac{dz}{z} dt_1 = \frac{dz}{z}. \end{aligned}$$

Note that the square of either curvature is zero, and so we only have a non-trivial first Atiyah class — all higher ones are zero. Using the convention/definition that the zero-th Atiyah class is 1, we have the total Atiyah classes (that is, Chern characters), defined as the sums of all the Atiyah classes:

$$\begin{aligned} \text{at}_{\mathcal{E}^{\bullet,0}}^{\text{tot}} &= 1, \\ \text{at}_{\mathcal{E}^{\bullet,1}}^{\text{tot}} &= 1 + \frac{dz}{z}. \end{aligned}$$

^[1]The first exponential and first standard Atiyah classes agree, by definition.

Finally, justified by (7.6.4), we say that, for a resolution

$$\mathcal{R}^\bullet = (\mathcal{R}^{\bullet,n} \rightarrow \dots \rightarrow \mathcal{R}^{\bullet,1}) \rightarrow \mathcal{R}^{\bullet,0},$$

the total Atiyah class (or Chern character) of $\mathcal{R}^{\bullet,0}$ is given by the alternating sum of the total Atiyah classes (or Chern characters) of the $\mathcal{R}^{\bullet,i}$. Thus

$$\text{ch}(\mathcal{F}) = [\text{at}_{\mathcal{F}^\bullet}^{\text{tot}}] = \left[\frac{dz}{z} \right].$$

This agrees with what one might calculate using a short exact sequence (writing the skyscraper sheaf as a quotient) and the Whitney sum formula, but is stronger, since Green's method gives actual *representatives* of the cohomology class as well.

(Note that we do not have to take the trace of the Atiyah classes here, since we are working with rank-1 bundles, and so end up with (1×1) -matrices as endomorphisms.)

Ic wýsce þæt ic wīsra wære. (*I wish that I were wiser.*)

Peter S. Baker, "Introduction to Old English, 7.10 The subjunctive".

(10.0.1) Purpose. *The main result of this chapter is (10.2.9), which reassures us that we can “do” the theory of characteristic classes by using these ‘vector bundles on the nerve’. Indeed, as mentioned elsewhere in this thesis, these objects seem to have slipped past peoples’ attention somewhat, so my hope is that this equivalence of categories could prove useful to people who work with these sorts of things. Note that there is some subtlety when it comes to working with coherent analytic sheaves that isn’t a problem in the algebraic case, as we point out in (10.1.7), and explain in more detail in Appendix E. It was also never fully explained how the classes constructed in [Gri09] agree with those in [Gre80], but (10.4.1) tidies up that problem by showing that Green’s Chern classes satisfy the properties required to ensure uniqueness in the compact case.*

10.1 Homotopical categories

(10.1.1) Definition. In this chapter we are interested in *relative categories*: pairs (C, \mathcal{W}) where C is a category and \mathcal{W} (whose morphisms we call *weak equivalences*) is a wide subcategory of C . A relative category is said to be a *homotopical category* if its weak equivalences satisfy the *2-out-of-6 property*: if

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

is a sequence of composable morphisms such that the compositions gf and hg are weak equivalences, then f , g , h , and hgf are all weak equivalences too. We often write a relative (or homotopical) category (C, \mathcal{W}) simply as C , omitting the weak equivalences from our notation.

Using the formalism of [Rez00] along with the results of [BK13], we can think of a homotopical category (C, \mathcal{W}) as presenting the $(\infty, 1)$ -category LC , which is the complete Segal space given by taking a Reedy fibrant replacement of the Rezk/simplicial nerve $N(C, \mathcal{W})$. In particular, [BK13, §1.2 (ii)] tells us that any homotopically full relative subcategory of a partial model category is again a partial model category, and all of the categories that we study here are such subcategories of either the category of complexes of sheaves on X or of the category

of complexes of sheaves on $X_{\bullet}^{\mathcal{U}}$, both of which^[1] are model categories, and thus partial model categories.

This is about all the technical information concerning our choice of model for $(\infty, 1)$ -categories that a reader should need for this chapter; more details are given in Appendix D.

(10.1.2) If a relative category has weak equivalences defined to be exactly the morphisms that become isomorphisms under some functor into another (say, abelian) category, then it is a homotopical category, because isomorphisms satisfy the 2-out-of-6 property.^[2]

(10.1.3) Definition. We begin by formally defining some relative categories (all of which are actually *homotopical* categories, by (10.1.2)). All complexes are *cochain* complexes, concentrated in non-negative degrees and bounded above; if we don't mention what the morphisms are then they are simply morphisms of cochain complexes (i.e. degree-wise morphisms that commute with the differentials); and if we don't mention what the weak equivalences are then they are simply quasi-isomorphisms of complexes.

- $\text{Coh}(X)$ is the category of complexes of sheaves of \mathcal{O}_X -modules that are quasi-isomorphic to a complex of coherent sheaves.
- $\text{Coh}_{\mathcal{U}}(X)$ is the category of complexes of sheaves of \mathcal{O}_X -modules whose restriction to any $U \in \mathcal{U}$ is quasi-isomorphic to a complex of coherent sheaves on U .
- $\text{CCoh}(X)$ is the category of complexes of sheaves of \mathcal{O}_X -modules that have coherent (internal) cohomology.
- $\text{Sh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ is the category of *cartesian* complexes of sheaves on the nerve. Note that morphisms between two such complexes are maps such that, in every *internal* degree (i.e. in every degree of the complex), we have a *morphism* of sheaves on the nerve (and such that these commute with the differentials of the complexes); weak equivalences are given by morphisms of complexes such that, in each *simplicial* degree, we have a quasi-isomorphism of complexes.
- $\text{Vect}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ is the full subcategory of $\text{Sh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ consisting of complexes that are *locally (with respect to \mathcal{U}) quasi-isomorphic* to a (cartesian) complex

^[1]The former having, for example, the projective model structure from [Hov01]; the latter having the model structure coming from its construction as a lax homotopy limit via the formalism of [Ber12], as described in (E.5.5).

^[2]If gf is an isomorphism then f must be a monomorphism, and g an epimorphism; if hg is an isomorphism then g must be a monomorphism, and h an epimorphism. But then g is both a monomorphism and an epimorphism, and thus an isomorphism. Then we can use the fact that isomorphisms satisfy 2-out-of-3, for example.

of *locally free* sheaves on the nerve. That is, for $\mathcal{F}^{\bullet,\star} \in \mathrm{Sh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$ to be in $\mathrm{Vect}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$, there must exist, for all $U_{\alpha_0 \dots \alpha_p} \in \mathcal{U}$, a complex $\mathcal{G}_{U_{\alpha_0 \dots \alpha_p}}^{\star}$ of locally free sheaves on $U_{\alpha_0 \dots \alpha_p}$ such that

$$\mathcal{F}_{\alpha_0 \dots \alpha_p}^{p,\star} \simeq \mathcal{G}_{U_{\alpha_0 \dots \alpha_p}}^{\star}.$$

Similarly, $\mathrm{Coh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$ is the full subcategory of $\mathrm{Sh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$ consisting of complexes that are *locally (with respect to \mathcal{U}) quasi-isomorphic* to a (cartesian) complex of coherent sheaves on the nerve.

- $\mathrm{Green}(X_{\bullet}^{\mathcal{U}})$ is the full subcategory of $\mathrm{Vect}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$ spanned by objects that are locally (with respect to \mathcal{U}) quasi-isomorphic to some Green complex. The fact that this actually is a subcategory is justified by (7.7.6). By definition, every object of $\mathrm{Green}(X_{\bullet}^{\mathcal{U}})$ is a Green complex.

(10.1.4) Note that all of these categories in (10.1.3) that depend on \mathcal{U} are natural in the choice of cover: taking a refinement $\mathcal{V} \supset \mathcal{U}$ induces a functor e.g. $\mathrm{Green}(X_{\bullet}^{\mathcal{U}}) \rightarrow \mathrm{Green}(X_{\bullet}^{\mathcal{V}})$. This lets us take homotopy colimits (where

$$\mathrm{hocolim}: [\mathcal{D}^{\mathrm{op}}, (\infty, 1)\text{-Cat}] \rightarrow (\infty, 1)\text{-Cat}$$

(with \mathcal{D} small) is defined as the $(\infty, 1)$ -categorical left adjoint to the constant-diagram functor) over refinements of covers, e.g. $\mathrm{hocolim}_{\mathcal{U}} \mathrm{Green}(X_{\bullet}^{\mathcal{U}})$.

(10.1.5) Our categories are defined as having ‘objects *locally quasi-isomorphic* to a certain class of objects’, and *not* ‘objects in the class of objects’. For example, an object $\mathcal{F}^{\bullet} \in \mathrm{Coh}_{\mathcal{U}}(X)$ is such that, for all $U \in \mathcal{U}$, there exists some complex of coherent sheaves \mathcal{G}_U^{\bullet} such that $\mathcal{G}_U^{\bullet} \simeq \mathcal{F}^{\bullet} \upharpoonright U$. This does *not* a priori imply the existence of some global complex of coherent sheaves \mathcal{G}^{\bullet} such that $\mathcal{G}^{\bullet} \simeq \mathcal{F}^{\bullet}$. This subtlety makes the proofs in this section more technical than they morally are.

(10.1.6) Given a full embedding $(C, \mathcal{W}') \hookrightarrow (\mathcal{D}, \mathcal{W})$ of one homotopical category into another such that $\mathcal{W}' = C \cap \mathcal{W}$, there are *two* $(\infty, 1)$ -categories defined by C that, in general, do *not* agree:

1. $L_{\mathcal{W}'}C$, given by localising C along \mathcal{W}' ;
2. $L_{\mathcal{D}}C$, given by taking the full sub- $(\infty, 1)$ -category of \mathcal{D} spanned by C .

As mentioned in (10.1.5), we define all our subcategories in the same way: given some $(\mathcal{D}, \mathcal{W})$, we construct some (C, \mathcal{W}') by taking the full subcategory of \mathcal{D} of objects that are (locally) connected via \mathcal{W} to objects satisfying some specific property. This means that, when we write LC , we implicitly mean $L_{\mathcal{D}}C$.

(10.1.7) In the algebraic setting, there is an equivalence between the category of complexes of coherent sheaves and the category of complexes of sheaves with coherent (internal) cohomology. In the analytic case, things are more subtle, and so we have to take a slightly longer route to prove our desired result. Indeed, to the best of our knowledge, the question of whether or not $\text{Coh}(X)$ and $\text{CCoh}(X)$ are equivalent is still open, except in low dimensions, where it is known to be true (see [Yu13, §2.2.2]). For more details, see Appendix E.

Note that, given a refinement of our cover $\mathcal{V} \supset \mathcal{U}$, we have full embeddings

$$\text{Coh}(X) \hookrightarrow \text{Coh}_{\mathcal{U}}(X) \hookrightarrow \text{Coh}_{\mathcal{V}}(X) \hookrightarrow \text{CCoh}(X)$$

that preserve and reflect quasi-isomorphisms.

(10.1.8) **Definition.** We define the category $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ via the Grothendieck construction applied to the functor $F: \text{Green}(X_{\bullet}^{\mathcal{U}}) \rightarrow \text{Set}$ given, on an object $\mathcal{E}^{\bullet,*}$, by

$$F(\mathcal{E}^{\bullet,j}) = \{\text{generated-in-degree-zero simplicial connections on } \mathcal{E}^{\bullet,j}\},$$

where (7.7.15) tells us that this set is non-empty. So an object of $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ is a pair $(\mathcal{E}^{\bullet,*}, \nabla_{\bullet}^*)$, where $\mathcal{E}^{\bullet,*}$ is an object of $\text{Green}(X_{\bullet}^{\mathcal{U}})$, and ∇_{\bullet}^j is a simplicial connection generated in degree zero (and thus admissible) on $\mathcal{E}^{\bullet,j}$; the morphisms of $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ are exactly those of $\text{Green}(X_{\bullet}^{\mathcal{U}})$. In particular, $\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ is a homotopical category with the same weak equivalences as $\text{Green}(X_{\bullet}^{\mathcal{U}})$.

(10.1.9) By construction, the forgetful functor

$$\textcircled{4}: \text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}}) \rightarrow \text{Green}(X_{\bullet}^{\mathcal{U}})$$

(that forgets about the connections) is fully faithful and essentially surjective, and thus induces an equivalence of categories. More importantly, though, it also restricts to give an equivalence of the corresponding subcategories of weak equivalences, which will be useful later on.

(10.1.10) In summary, we have the following diagram (of 1-categories), where we write \succrightarrow to denote a fully-faithful functor, and \dashrightarrow to denote an essentially-surjective one:

$$\begin{array}{ccccccc} \text{Green}(X_{\bullet}^{\mathcal{U}}) & \xrightarrow{\textcircled{1}} & \text{Vect}^{\text{cart}}(X_{\bullet}^{\mathcal{U}}) & \xrightarrow{\textcircled{2}} & \text{Coh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}}) & \dashrightarrow^{\textcircled{3}} & \text{Coh}_{\mathcal{U}}(X) \\ & & \uparrow \textcircled{4} & & & & \\ & & \text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}}) & & & & \end{array}$$

(10.1.10.1)

where we will define ③ later, and, even then, only at the level of $(\infty, 1)$ -categories. Note that ① and ② are fully-faithful by definition (as full subcategories), and the forgetful functor ④ is an equivalence by definition (as explained in (10.1.9)).

Our goal for the rest of this chapter is to prove that, when we localise all the categories in (10.1.10.1), all the functors become equivalences of $(\infty, 1)$ -categories.

This is the content of (10.2.9) and (10.2.14).



(10.1.11) We omit the localisation notation from our functors: we will write ① to mean both the functor between homotopical categories and the induced functor between their localisations.

(10.1.12) We delay our study of ③ for a little while, and consider first the rest of (10.1.10.1). Our first goal is to show that this diagram descends to a diagram at the level of localisations. For this we need to know that all our functors ① really are functors of relative categories, in that they preserve weak equivalences.

(10.1.13) Note that ① and ② are inclusions of full subcategories, and so trivially preserve weak equivalences; similarly, ④ is constructed in such a way that it automatically preserves weak equivalences.

In fact, by (10.1.9), ④ actually directly induces an equivalence at the level of localisations:

$$\textcircled{4}: \text{LGreen}_{\nabla, 0}(X_{\bullet}^{\mathcal{U}}) \xrightarrow{\sim} \text{LGreen}(X_{\bullet}^{\mathcal{U}}).$$

This is because an equivalence of relative categories that restricts to an equivalence of the wide subcategories of weak equivalences induces an equivalence of the localisations, as can be shown by using [BK12, Lemma 5.4], as well as the fact that an equivalence of categories induces an equivalence of their nerves.

As a side note, even though ④ gives an equivalence at the level of localisations, it there no longer looks like a Grothendieck construction: there is no reason for two weakly equivalent Green complexes (or even, more simply, quasi-isomorphic complexes of vector bundles) to admit local connections that are in bijective correspondence with one another. This isn't something that poses a problem, but is interesting to note.

(10.1.14) Now we wish to study this mysterious functor ③. We aim to build an adjunction of $(\infty, 1)$ -categories, step by step.

Writing $i: X_{\bullet}^{\mathcal{U}} \rightarrow X$ to mean the map given by inclusion of each open subset into X , we have an adjunction^[1]

$$\mathrm{Sh}(X_{\bullet}^{\mathcal{U}}) : (i_* \vdash i^*) : \mathrm{Sh}(X).$$

Then, recalling that the limit functor can be defined as being the right adjoint to the constant diagram functor, and writing $\mathrm{Sh}(X_{\bullet}^{\mathcal{U}})$ to mean the category of complexes of sheaves of $\mathcal{O}_{X_{\bullet}^{\mathcal{U}}}$ -modules, thought of as a category of $X_{\bullet}^{\mathcal{U}}$ -diagrams, we can compose this adjunction with the above to get a Quillen adjunction

$$\mathrm{Sh}(X_{\bullet}^{\mathcal{U}}) : (\lim \circ i_* \vdash \mathrm{cst} \circ i^*) : \mathrm{Sh}(X)$$

where being Quillen follows from the fact that the pullback/pushforward adjunction and the limit/constant adjunction are both Quillen.

So we are now in the following situation: we have a diagram

$$\begin{array}{ccc} \mathrm{Sh}(X_{\bullet}^{\mathcal{U}}) & \begin{array}{c} \xrightarrow{\lim \circ i_*} \\ \top \\ \xleftarrow{\mathrm{cst} \circ i^*} \end{array} & \mathrm{Sh}(X) \\ \uparrow & & \uparrow \\ \mathrm{Coh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) & & \mathrm{Coh}_{\mathcal{U}}(X) \end{array}$$

where the adjunction is Quillen. Deriving the functors (and localising the categories) then gives us the diagram

$$\begin{array}{ccc} \mathrm{LSh}(X_{\bullet}^{\mathcal{U}}) & \begin{array}{c} \xrightarrow{\mathbb{R}(\lim \circ i_*)} \\ \top \\ \xleftarrow{\mathbb{L}(\mathrm{cst} \circ i^*)} \end{array} & \mathrm{LSh}(X) \\ \uparrow & & \uparrow \\ \mathrm{LCoh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) & & \mathrm{LCoh}_{\mathcal{U}}(X) \end{array}$$

and we wish to know if the adjunction restricts to give an adjunction of the subcategories. Following [TV08b, Lemma 2.2.2.13], we write \int to mean the total right derived functor $\mathbb{R}(\lim \circ i_*)$; we write ι^* to mean the total left derived functor $\mathbb{L}(\mathrm{cst} \circ i^*)$. So we want to show that

$$\begin{array}{l} \int : \mathrm{LCoh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) \rightarrow \mathrm{LCoh}_{\mathcal{U}}(X) \\ \mathrm{LCoh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) \leftarrow \mathrm{LCoh}_{\mathcal{U}}(X) : \iota^* \end{array}$$

(where we omit the restriction of the functors from our notation).

^[1]Note that, to eventually agree with the orientation of our diagram, we write our adjunctions as $(R \vdash L)$ instead of as $(L \vdash R)$.

(10.1.15) Lemma. The image of $i^* : \mathrm{LCoh}_{\mathcal{U}}(X) \rightarrow \mathrm{LSh}(X_{\bullet}^{\mathcal{U}})$ is contained in $\mathrm{LCoh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$.

Proof. Let $\mathcal{F}^{\bullet, \star} \in \mathrm{LCoh}_{\mathcal{U}}(X)$. The pullback functor i^* is exact since it is given by the topological pullback tensored with the structure sheaf, and the topological pullback is exact, and tensoring along a disjoint union of open immersions is also exact. Then, since the constant diagram functor is (trivially) also exact, we see that i^* is just the pullback to the nerve (as with global vector bundles). But, as mentioned there, the simplicial maps are then simply identity maps, which means that the resulting object is indeed cartesian; and being coherent is a local property, so it suffices to check it on each $U_{\alpha_0 \dots \alpha_p}$ in $X_p^{\mathcal{U}}$, but, over such an open set, $i^* \mathcal{F}^j$ is simply $\mathcal{F}^j \upharpoonright U_{\alpha_0 \dots \alpha_p}$, which is coherent by definition. \square

(10.1.16) It is a good idea to fully understand the cosimplicial structure of $i_* \mathcal{F}^{\bullet, \star}$ before proceeding, so we spell out all the details here.

Recall that $i_* = \mathbb{R}i_* = i_*$, where $i : X_{\bullet}^{\mathcal{U}} \rightarrow X$. We want to describe what $(i_* \mathcal{F}^{\bullet, \star})^p$ is for each $p \in \mathbb{N}$, as well as how these ‘fit together’ to give a cosimplicial object. This can be explained by just improving our notation: write $i_p : X_p^{\mathcal{U}} \rightarrow X$ to mean the map given by inclusion of each $U_{\alpha_0 \dots \alpha_p}$ in $X_p^{\mathcal{U}}$ into X , so that i is exactly the data of $(i_p)_{p \in \mathbb{N}}$. Then define^[1]

$$(i_* \mathcal{F}^{\bullet, \star})^p = (i_p)_* \mathcal{F}^{p, \star} = \bigoplus_{(\alpha_0 \dots \alpha_p)} (i_p)_* \mathcal{F}_{\alpha_0 \dots \alpha_p}^{p, \star}.$$

Then, given some $\varphi : [p] \rightarrow [q]$ in Δ , we want to know how to define

$$(i_p)_* \mathcal{F}^{p, \star} \xrightarrow{(i_* \mathcal{F}^{\bullet, \star})^\circ(\varphi)} (i_q)_* \mathcal{F}^{q, \star},$$

but, using the pullback/pushforward adjunction, this is the same as asking for a map

$$(i_q)^*(i_p)_* \mathcal{F}^{p, \star} \longrightarrow \mathcal{F}^{q, \star}.$$

Firstly, we claim, for $p < q$ (dealing with the other case shortly), that there is a natural map $(i_q)^*(i_p)_* \mathcal{F}^{p, \star} \rightarrow (X_{\bullet}^{\mathcal{U}} \varphi)^* \mathcal{F}^{p, \star}$; secondly, we claim that this gives us the map that we want. To see the first claim, we appeal to the geometric nature of pushforwards and pullbacks: since the U_{α} do not necessarily have trivial intersection with one another, it is not necessarily true that $(i^* i_* \mathcal{F}) \upharpoonright U = \mathcal{F} \upharpoonright U$; what is true, however, is that the right-hand side is a direct summand of the left:

$$(i^* i_* \mathcal{F}) \upharpoonright U \cong \mathcal{F} \upharpoonright U \oplus \mathcal{G}.$$

This gives us the first claim (for $p < q$; we deal with the other case shortly), since restriction is the same as pulling back along $X_{\bullet}^{\mathcal{U}} \varphi$. For the second claim,

^[1]It would probably suffice to simply write $i_* \mathcal{F}^{\circ, \star}$ instead of $(i_* \mathcal{F}^{\bullet, \star})^\circ$, since the cosimplicial structure is described by \bullet , but then we should really write $(i_{\bullet})_* \mathcal{F}^{\bullet, \star}$, which is equally unsightly.

by definition of what it means to be an element of $\text{Coh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$, we have maps $(X_{\bullet}^{\mathcal{U}}\varphi)^*\mathcal{F}^{p,\star} \rightarrow \mathcal{F}^{q,\star}$ for every $\varphi: [p] \rightarrow [q]$. Combining all of the above then gives us the desired maps, and thus the cosimplicial structure.

Finally then, when $p > q$ we can do something similar. Here the map $X_{\bullet}^{\mathcal{U}}\varphi$ is given by inserting degenerate intersections,^[1] and so it again isn't necessarily the case that $(i_q)^*(i_p)_*\mathcal{F}^{p,\star} = (X_{\bullet}^{\mathcal{U}}\varphi)^*\mathcal{F}^{p,\star}$, since we have intersections $U_{\alpha_0\dots\alpha_p}$ that can be strictly smaller than any $U_{\beta_0\dots\beta_q}$. But we can still construct some $(i_q)^*(i_p)_*\mathcal{F}^{p,\star} \rightarrow \mathcal{F}^{q,\star}$ by precomposing the $(X_{\bullet}^{\mathcal{U}}\varphi)^*\mathcal{F}^{p,\star} \rightarrow \mathcal{F}^{p,\star}$ with the projection maps

$$(i_q)^*(i_p)_*\mathcal{F}^{p,\star} = (i_p)_*\mathcal{F}^{p,\star} \Big|_{U_{\beta_0\dots\beta_q}} = \bigoplus_{(\alpha_0\dots\alpha_p)} (i_p)_*\mathcal{F}_{\alpha_0\dots\alpha_p}^{p,\star} \Big|_{U_{\beta_0\dots\beta_q}} \twoheadrightarrow (i_p)^*\mathcal{F}_{\beta_0\dots\beta_q\beta_q\dots\beta_q}^{q,\star}$$

where the first 'equality' really means that we work locally over each $U_{\beta_0\dots\beta_q}$, and we write $(\beta_0\dots\beta_q\beta_q\dots\beta_q)$ to mean some degenerate embedding of $(\beta_0\dots\beta_q)$ into $X_p^{\mathcal{U}}$.

(10.1.17) Lemma. The image of $\int: \text{LCoh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}}) \rightarrow \text{LSh}(X)$ is contained in $\text{LCoh}_{\mathcal{U}}(X)$.

Proof. Before giving the proof, we give a short summary of how it will proceed, to save anybody familiar with such arguments the arduous task of following the notation. In particular, this proof is incredibly similar to that of (10.2.3), which is really an analytic version of [TV08b, Lemma 2.2.2.13] in that it follows the same line of argument. We can argue everything locally on some $U \in \mathcal{U}$; by definition, weak equivalences are quasi-isomorphisms; we can use the total complex construction to calculate the homotopy limit in the definition of \int ; the fact that our complexes are cartesian gives us a weak equivalence between this total complex and the (total complex of the) Čech complex of the simplicial-degree-zero part of our original complex; the latter is weakly equivalent to the simplicial-degree-zero part of our original complex (since all covers can be taken to be Stein); a commuting triangle then tells us that the desired quasi-isomorphism is indeed a quasi-isomorphism.

Let $\mathcal{F}^{\bullet,\star} \in \text{LCoh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ and $U \in \mathcal{U}$. Then $\mathcal{F}^{0,\star} \Big|_U$ is a complex of coherent sheaves on U , and so it would suffice to show that there is a quasi-isomorphism

$$\mathcal{F}^{0,\star} \Big|_U \xrightarrow{\sim} \left(\int \mathcal{F}^{\bullet,\star} \right) \Big|_U.$$

Firstly, note that there exists a good candidate morphism: the cosimplicial structure of \mathcal{F} means that we have a morphism

$$i_p^*\mathcal{F}^{0,\star} \rightarrow \mathcal{F}^{p,\star}$$

^[1]Obtaining things that look like $U_{\alpha_0\dots\alpha_i\alpha_i\dots\alpha_p}$.

of sheaves over $X_p^{\mathcal{U}}$ for all $p \in \mathbb{N}$, where $i_p: X_p^{\mathcal{U}} \rightarrow X$; by the pull/push adjunction, this gives us a morphism

$$\mathcal{F}^{0,\star} \rightarrow (i_p)_* \mathcal{F}^{p,\star}$$

and so, by the universal property of the homotopy limit (since $\mathbb{R}\lim = \text{holim}$), we get a morphism

$$\mathcal{F}^{0,\star} \rightarrow \text{holim}_p (i_p)_* \mathcal{F}^{p,\star} = \int \mathcal{F}^{\bullet,\star}$$

which, denoting restriction to U by a subscript U , induces

$$\mathcal{F}_U^{0,\star} \rightarrow \left(\int \mathcal{F}^{\bullet,\star} \right)_U. \quad (10.1.17.1)$$

Now we wish to show that (10.1.17.1) is indeed a quasi-isomorphism. We can (justified by (10.1.20)) calculate the homotopy limit with the *total construction*: writing $F^{\bullet,\star}$ to mean the cosimplicial object $((i_\bullet)_* \mathcal{F}^{\bullet,\star})$, we define

$$\begin{aligned} \text{Tot}(F)^n &= \bigoplus_{\ell \in \mathbb{N}} F^{\ell, n-\ell} \\ d_{\text{Tot}(F)} &= d_F + (-1)^n \check{\delta} \end{aligned}$$

where d_F is the differential $d_{\mathcal{F}^{\bullet,\star}}$ coming from the \star -grading of $\mathcal{F}^{\bullet,\star}$, and where

$$\check{\delta}^n = \sum_{i=0}^{n+1} (-1)^i F^{\bullet,\star}(f_{n+1}^i)$$

is the alternating sum of coface maps, whose action is given by the cosimplicial structure of $F^{\bullet,\star}$. To see that this makes sense in terms of degrees, note that

$$\begin{aligned} \check{\delta}^m: F^{\ell, m} &\rightarrow F^{\ell+1, m} \\ d_F^\ell: F^{\ell, m} &\rightarrow F^{\ell, m+1}. \end{aligned}$$

So (10.1.17.1) becomes

$$\mathcal{F}_U^{0,\star} \rightarrow \text{Tot}(F)_U^\star, \quad (10.1.17.2)$$

and to show that this is a quasi-isomorphism, it is enough to show that

$$\mathcal{F}_U^{0,\star}(V) \rightarrow \text{Tot}(F)_U^\star(V) \quad (10.1.17.3)$$

is a quasi-isomorphism for all $V \in \mathcal{V}$, where \mathcal{V} is some cover of U .

The cartesian condition gives us quasi-isomorphisms

$$\mathcal{F}_{\beta_0 \dots \beta_p}^{0,\star} \xrightarrow{\sim} \mathcal{F}_{\beta_0 \dots \beta_p}^{p,\star}$$

of complexes of (coherent) sheaves over any $V_{\beta_0 \dots \beta_p} \in \mathcal{V}$, and so we can refine \mathcal{V} to a cover (which we can always take to be Stein) $\mathcal{W} = \{W_\gamma\}$ of $V_{\beta_0 \dots \beta_p}$ such that we have quasi-isomorphisms

$$\mathcal{F}_{\beta_0 \dots \beta_p}^{0, \star}(W_{\gamma_0 \dots \gamma_p}) \xrightarrow{\sim} \mathcal{F}_{\beta_0 \dots \beta_p}^{p, \star}(W_{\gamma_0 \dots \gamma_p}) \quad (10.1.17.4)$$

of complexes of abelian groups.

But, as complexes of sheaves over $V_{\beta_0 \dots \beta_p}$, we trivially have that

$$\mathcal{F}_{\beta_0 \dots \beta_p}^{p, \star} = (i_p)_* \mathcal{F}_{\beta_0 \dots \beta_p}^{p, \star} \Big|_{W_{\beta_0 \dots \beta_p}}$$

and so the right-hand side of (10.1.17.4) is exactly $F_U^{\bullet, \star}(V_{\beta_0 \dots \beta_p})$; further, the left-hand side is exactly $\check{C}_{\mathcal{W}}^{\bullet}(\mathcal{F}_U^{0, \star})$. Together, then, this tells us that (10.1.17.4) gives a morphism of bicomplexes

$$\check{C}_{\mathcal{W}}^{\bullet}(\mathcal{F}_U^{0, \star}) \rightarrow F_U^{\bullet, \star}(V_{\beta_0 \dots \beta_p})$$

which is a quasi-isomorphism on each row. By a classical spectral sequence argument^[1] we can show that such a morphism of bicomplexes induces a quasi-isomorphism on the respective total complexes:

$$\text{Tot } \check{C}_{\mathcal{W}}^{\bullet}(\mathcal{F}_U^{0, \star}) \xrightarrow{\sim} \text{Tot } F_U^{\bullet, \star}(V_{\beta_0 \dots \beta_p}).$$

We can see that the triangle

$$\begin{array}{ccc} \mathcal{F}_U^{0, \star}(V_{\beta_0 \dots \beta_p}) & \longrightarrow & \text{Tot } F_U^{\bullet, \star}(V_{\beta_0 \dots \beta_p}) \\ \uparrow & \nearrow & \\ \text{Tot } \check{C}_{\mathcal{W}}^{\bullet}(\mathcal{F}_U^{0, \star}) & & \end{array}$$

commutes, where the horizontal arrow is exactly the morphism (10.1.17.3). But the vertical arrow is a quasi isomorphism (because taking the Čech complex with respect to a Stein cover gives a resolution), and we have already shown that the diagonal arrow is a quasi-isomorphism (by the cartesian condition), and so the horizontal arrow must also be a quasi-isomorphism. \square

(10.1.18) In summary then, we have an adjunction

$$\text{LCoh}^{\text{cart}}(X_{\bullet}^{\mathcal{U}}) : (\int \vdash \iota^*) : \text{LCoh}_{\mathcal{U}}(X)$$

(10.1.19) Definition. $\textcircled{3} = \int$.

^[1]Taking the mapping cone, applying the spectral sequences associated to a bicomplex, and using induction on the number of non-zero rows, combined with the fact that the direct limit functor for complexes of abelian groups is exact.

(10.1.20) The projective model structure on non-negatively-graded cochain complexes gives us a simplicial model category (by the dual of Dold-Kan), and so, if we can show that $F^{\diamond, \star}$ is Reedy fibrant, then we can apply [Hir03, Theorem 19.8.7], which tells us that $\text{holim } F^{\diamond, \star} \simeq \text{Tot}(F^{\diamond, \star})$, for some abstract definition of Tot. The fact that the totalization (in the sense of Hirschhorn) agrees with the totalization of a bicomplex (in the usual homological algebra sense), and that the Bousfield-Kan spectral sequence and the spectral sequence(s) associated to a bicomplex coincide, can be found^[1] in [Fre17, III.1.1.13].

To show Reedy fibrancy of some cosimplicial object X^\bullet , we need to show that the maps $X^n \rightarrow M_n(X^\bullet)$ are fibrant for all $n \geq 0$, where $M_n(X^\bullet)$ is the *matching object* given by

$$M_n(X^\bullet) = \lim_{\substack{\varphi: [n] \rightarrow [i] \\ i \neq n}} X^i.$$

We can write this down more explicitly as

$$\begin{aligned} M_0(X^\bullet) &= \{*\} \\ M_1(X^\bullet) &= X^0 \\ M_2(X^\bullet) &= X^1 \times_{X^0} X^1 \end{aligned}$$

and so on.

It is a purely formal consequence of the simplicial identities that any *simplicial* set is Reedy *cofibrant* in the *injective* model structure; formally dual to this is the fact that any *cosimplicial* set is Reedy *fibrant* in the *projective* model structure. For example, the fact that $X^1 \rightarrow M_1(X^\bullet) = X^0$ is fibrant (i.e. surjective) is due to the fact that it admits a right inverse (namely either of the two face maps $X^0 \rightarrow X^1$), thanks to the (co)simplicial identities (namely $s_0^0 f_1^0 = s_0^0 f_0^0 = \text{id}_{[0]}$).

10.2 Equivalences

(10.2.1) **Lemma.** The left adjoint i^* of the adjunction in (10.1.18) is conservative.

Proof. Let $f: \mathcal{F}^\star \rightarrow \mathcal{G}^\star$ be a morphism in $\text{LCCoh}(X)$ such that i^*f is an isomorphism in $\text{LCoh}^{\text{cart}}(X_\bullet^{\mathcal{U}})$. By definition of the weak equivalences in $\text{Coh}^{\text{cart}}(X_\bullet^{\mathcal{U}})$, and the calculation of i^* in (10.1.19), over each $U_{\alpha_0 \dots \alpha_p} \in X_p^{\mathcal{U}}$, this says that the map

$$i^*f = f \Big|_{U_{\alpha_0 \dots \alpha_p}: \mathcal{F}^\star} \Big|_{U_{\alpha_0 \dots \alpha_p} \rightarrow \mathcal{G}^\star} \Big|_{U_{\alpha_0 \dots \alpha_p}}$$

is a quasi-isomorphism. But this is just saying that every restriction of f to some open subset of X is a quasi-isomorphism, which implies that f is a quasi-isomorphism, i.e. a weak equivalence in $\text{CCoh}(X)$. \square

^[1]We thank Maximilien Péroux for suggesting this reference.

(10.2.2) Lemma. Let $\varphi^\bullet: \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ be a morphism^[1] in $\text{Sh}^{\text{cart}}(X_\bullet^{\mathcal{U}})$. Then φ^\bullet is a weak equivalence if and only if $\varphi^0: \mathcal{F}^0 \rightarrow \mathcal{G}^0$ is a weak equivalence.

Proof. If φ^\bullet is a weak equivalence then, by definition, each φ^p is a weak equivalence, and so it remains only to show ‘if’ part of the claim. Recalling that pulling back along the inclusion of an open subset is the same as restriction to that same open subset, the cartesian condition on \mathcal{F}^\bullet tells us that, for all $U_{\alpha_0 \dots \alpha_p} \in \mathcal{U}$,

$$\mathcal{F}^0 \big|_{U_{\alpha_0 \dots \alpha_p}} \xrightarrow{\sim} \mathcal{F}^p \big|_{U_{\alpha_0 \dots \alpha_p}}$$

as (complexes of) sheaves over $U_{\alpha_0 \dots \alpha_p}$, and similarly for \mathcal{G}^\bullet . Combining this with the commutative square that follows from the definition of what it means to be a morphism of sheaves on the nerve, we get the commutative square

$$\begin{array}{ccc} \mathcal{F}^0 & \xrightarrow{\sim} & \mathcal{F}^p \\ \varphi^0 \downarrow & & \downarrow \varphi^p \\ \mathcal{G}^0 & \xrightarrow{\sim} & \mathcal{G}^p \end{array}$$

from which it follows that, if $\varphi^0: \mathcal{F}^0 \rightarrow \mathcal{G}^0$ is a weak equivalence, then so too is φ^p for all $p \in \mathbb{N}$, and hence also φ^\bullet . \square

(10.2.3) Lemma. The counit

$$i^* \circ \int \implies \text{id}_{\text{LCoh}^{\text{cart}}(X_\bullet^{\mathcal{U}})}$$

of the adjunction in (10.1.18) is a weak equivalence.

Proof. Let $\mathcal{F}^{\bullet, \star} \in \text{Coh}^{\text{cart}}(X_\bullet^{\mathcal{U}})$. First of all, by (10.2.2), we know that it suffices to show that the counit is a weak equivalence in simplicial degree zero. But we can further simplify things: recalling the definition of \int , and using the fact that i^* is simply the pullback along i , it suffices to show that the induced morphism

$$\left(i^* \text{holim}_{p \in \mathbb{N}} (\iota_* \mathcal{F}^{\bullet, \star})^p \right) \big|_{U_\alpha} \longrightarrow \mathcal{F}^{0, \star} \big|_{U_\alpha} \quad (10.2.3.1)$$

in $\text{Coh}^{\text{cart}}(X_\bullet^{\mathcal{U}})$ is a weak equivalence for all $U_\alpha \in \mathcal{U}$. But since the composite $U_\alpha \hookrightarrow \mathcal{U} \rightarrow X$ is exactly $U_\alpha \hookrightarrow X$, we see that pulling back along i and then restricting to U_α is the same as restricting directly to U_α . Finally, just to simplify notation, we write $F^{\bullet, \star}$ to mean the cosimplicial object $(\iota_* \mathcal{F}^{\bullet, \star})^\bullet$, we write d_F to mean the differential $d_{\mathcal{F}^{\bullet, \star}}$ coming from the \star -grading of $\mathcal{F}^{\bullet, \star}$, and we write a subscript α to denote restriction to U_α . All together then, (10.2.3.1) becomes

$$\left(\text{holim}_{p \in \mathbb{N}} F^{p, \star} \right) \big|_{U_\alpha} \longrightarrow \mathcal{F}_\alpha^{0, \star} \quad (10.2.3.2)$$

But now we can proceed almost exactly as in the proof of (10.1.17): we use the total construction, and construct the same commuting triangle but with the horizontal arrow going in the other direction. \square

^[1]Where we omit from our notation the internal grading \star of the complexes, writing e.g. \mathcal{F}^\bullet instead of $\mathcal{F}^{\bullet, \star}$.

(10.2.4) The counit of an adjunction being a weak equivalence (or an isomorphism, in the 1-categorical case) is equivalent to the right adjoint being fully faithful.

(10.2.5) **Lemma.** Let $C : (L \dashv R) : \mathcal{D}$ be an adjunction with L conservative and R fully faithful. Then $(L \dashv R)$ gives an equivalence $C \simeq \mathcal{D}$.

Proof. It suffices to show that R is essentially surjective, so let $c \in C$, and define $d = L(c)$. Then $LR(d) \rightarrow d$ is an equivalence (because R being fully faithful is equivalent to the counit of the adjunction being an equivalence). But $LR(d) \rightarrow d$ is, by definition, $LRL(c) \rightarrow L(c)$, and since this is an equivalence and L is conservative, we see that $RL(c) \rightarrow c$ is an equivalence. That is, $R(d) \simeq c$. \square

(10.2.6) **Corollary.** The adjunction in (10.1.18) gives an equivalence of $(\infty, 1)$ -categories

$$\mathrm{LCoh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) \simeq \mathrm{LCoh}_{\mathcal{U}}(X)$$

and thus an equivalence

$$\mathrm{hocolim}_{\mathcal{U}} \mathrm{LCoh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}}) \simeq \mathrm{hocolim}_{\mathcal{U}} \mathrm{LCoh}_{\mathcal{U}}(X).$$

(10.2.7) **Lemma.** The composite functor

$$\mathrm{hocolim}_{\mathcal{U}} \mathrm{LGreen}(X_{\bullet}^{\mathcal{U}}) \xrightarrow{\textcircled{3}\textcircled{2}\textcircled{1}} \mathrm{hocolim}_{\mathcal{U}} \mathrm{LCoh}_{\mathcal{U}}(X)$$

is essentially surjective.

Proof. Since $\textcircled{3}$ is an equivalence (10.2.6), it suffices to show that

$$\textcircled{2}\textcircled{1}: \mathrm{hocolim}_{\mathcal{U}} \mathrm{LGreen}(X_{\bullet}^{\mathcal{U}}) \rightarrow \mathrm{hocolim}_{\mathcal{U}} \mathrm{LCoh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$$

is essentially surjective. So let $\mathcal{F}^{\bullet, \star} \in \mathrm{Coh}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$.

By definition, for all $U_{\alpha} \in \mathcal{U}$, there exists some complex $\mathcal{G}_{\alpha}^{\star}$ of coherent sheaves on U_{α} such that $\mathcal{F}_{\alpha}^{0, \star} \simeq \mathcal{G}_{\alpha}^{\star}$. We know that we can always locally resolve $\mathcal{G}_{\alpha}^{\star}$ by locally free sheaves, and so, by possibly taking a refinement $\mathcal{V} \supset \mathcal{U}$ (and using α, β, \dots to now label the open sets $V_{\alpha}, V_{\beta}, \dots$ of the refinement), we can obtain some (bounded) complex $\mathcal{H}_{\alpha}^{\star}$ of free sheaves (of finite rank) on V_{α} such that $\mathcal{F}_{\alpha}^{0, \star} \simeq \mathcal{H}_{\alpha}^{\star}$.

But this is simply saying that $\mathcal{F}_{\alpha}^{0, \star}$ is perfect, and so [OTT85, Proposition 1.2.3] (or [Wei16, Proposition 3.20]), tells us that, after possibly taking another refinement of our cover, there exists some holomorphic twisting resolution of $\mathcal{F}^{0, \star}$. Applying the construction of Green's resolution then gives us some $\mathcal{E}^{\bullet, \star} \xrightarrow{\simeq} j^*(\mathcal{F}^{0, \star})^{\bullet}$, where $\mathcal{E}^{\bullet, \star}$ is a complex of locally free sheaves on the Čech nerve of $X_0^{\mathcal{U}} = \coprod_{\alpha} V_{\alpha}$, and j is the map from the Čech nerve of $X_0^{\mathcal{U}}$ to $X_0^{\mathcal{U}}$ itself. Note, however, that the Čech nerve of $X_0^{\mathcal{U}}$ is identical to the Čech nerve of X , and so it suffices to prove

that $j^*(\mathcal{F}^{0,\star})^\bullet \simeq \mathcal{F}^{\bullet,\star}$ as sheaves on X_\bullet^u , since ① and ② are both simply inclusions of full subcategories.

By (10.2.2), it suffices to show that we have a weak equivalence in simplicial degree zero, but $j^*(\mathcal{F}^{0,\star})^0 \xrightarrow{\simeq} \mathcal{F}^{0,\star}$ is simply the identity map. \square

(10.2.8) Lemma. Let C be a partial model category, and $(\mathcal{D}_\lambda)_{\lambda \in \mathcal{P}}$ be a diagram of full subcategories of C indexed by some filtered poset^[1] \mathcal{P} . Assume further that each \mathcal{D}_λ is stable under weak equivalences. Then

$$\mathrm{hocolim}_{\lambda \in \mathcal{P}} \mathrm{L}\mathcal{D}_\lambda \simeq \bigcup_{\lambda \in \mathcal{P}} \mathrm{L}\mathcal{D}_\lambda$$

where $\bigcup_{\lambda \in \mathcal{P}} \mathrm{L}\mathcal{D}_\lambda$ is the full sub- $(\infty, 1)$ -category of LC spanned by the union of the objects of all of the $\mathrm{L}\mathcal{D}_\lambda$.

In particular, the induced map

$$\mathrm{hocolim}_{\lambda \in \mathcal{P}} \mathrm{L}\mathcal{D}_\lambda \rightarrow \mathrm{LC}$$

is fully faithful.

Proof. Since partial model categories present $(\infty, 1)$ -categories, it suffices to prove the corresponding claim for $\mathrm{hocolim} \mathcal{D}_\lambda$ instead of $\mathrm{hocolim} \mathrm{L}\mathcal{D}_\lambda$. Write Y_\bullet^λ to mean the complete Segal space $\mathrm{L}\mathcal{D}_\lambda$, and X_\bullet to mean the complete Segal space LC . But since each \mathcal{D}_λ is a full subcategory of C stable under weak equivalences, each Y_n^λ is a union of connected components of X_n , corresponding to the span of the objects of \mathcal{D}_λ . This means that, in particular, for each $\mathcal{D}_\lambda \hookrightarrow \mathcal{D}_\mu$, the maps $Y_n^\lambda \hookrightarrow Y_n^\mu$ are all closed embeddings, and thus cofibrations. Hence

$$\mathrm{hocolim}_\lambda Y_\bullet^\lambda \cong \mathrm{colim}_\lambda Y_\bullet^\lambda.$$

Similarly, each $Y_n^\lambda \hookrightarrow X_n$ is a closed embedding, and thus a cofibration.

We claim that

$$\mathrm{colim}_\lambda Y_\bullet^\lambda \cong \bigcup_\lambda Y_\bullet^\lambda$$

where $\bigcup_\lambda Y_\bullet^\lambda$ is the subspace of X_\bullet spanned by the connected components of all the Y_\bullet^λ . Note that this is the complete Segal space $\bigcup_\lambda \mathrm{L}\mathcal{D}_\lambda$, and so proving the above claim will finish the proof.

By the universal property of the colimit, we have a commutative diagram

$$\begin{array}{ccc} (Y_\bullet^\lambda)_{\lambda \in \mathcal{P}} & & \\ (f_\lambda)_{\lambda \in \mathcal{P}} \downarrow & \searrow \iota & \\ \mathrm{colim}_\lambda Y_\bullet^\lambda & \xrightarrow{f} & \bigcup_\lambda Y_\bullet^\lambda \end{array}$$

^[1]The poset assumption is not necessarily necessary

and we wish to show that f is an isomorphism. Since ι is simply the inclusion of each Y_\bullet^λ into the union, it is surjective. Thus, given any $y \in \bigcup_\lambda Y_\bullet^\lambda$, there exists some λ such that $y \in Y_\bullet^\lambda$, and then, by commutativity, $f(f_\lambda(y)) = y$, which shows surjectivity. To show injectivity, let $y, z \in \text{colim}_\lambda Y_\bullet^\lambda$ be such that $f(y) = f(z)$. Since \mathcal{P} is filtered, there exists some λ such that $f(y), f(z) \in Y_\bullet^\lambda$. Now $f(y) = f f_\lambda f(y)$ and $f(z) = f f_\lambda f(z)$, whence $f f_\lambda f(y) = f f_\lambda f(z)$. But f is surjective, and so $f f_\lambda(y) = f f_\lambda(z)$, but $f f_\lambda = \iota_\lambda$ is simply the inclusion of Y_\bullet^λ into the union, whence $y = z$. \square

(10.2.9) Theorem. There is an equivalence of $(\infty, 1)$ -categories

$$\text{hocolim}_{\mathcal{U}} \text{LGreen}_{\nabla, 0}(X_\bullet^{\mathcal{U}}) \simeq \text{hocolim}_{\mathcal{U}} \text{LCoh}_{\mathcal{U}}(X).$$

Proof. Note that, in the following, all the \textcircled{i} are the functors at the level of localisations. By (10.2.3) and (10.2.4), $\textcircled{3}$ is fully faithful; since $\textcircled{1}$ and $\textcircled{2}$ are inclusions of full subcategories, they remain fully faithful at the level of localisations, and we can use (10.2.8) to see that they remain fully faithful after taking homotopy colimits; this means that the composite functor $\textcircled{3}\textcircled{2}\textcircled{1}$ is also fully faithful. (10.2.7) tells us that $\textcircled{3}\textcircled{2}\textcircled{1}$ is essentially surjective. All together then, this tells us that the composite functor $\textcircled{3}\textcircled{2}\textcircled{1}$ is an equivalence, and since all of the composite functors are fully faithful, each one must itself also be an equivalence. By (10.1.13), we know that $\textcircled{4}$ is an equivalence. \square

(10.2.10) Lemma. The functor

$$\text{hocolim}_{\mathcal{U}} \text{LCoh}_{\mathcal{U}}(X) \rightarrow \text{LCCoh}(X)$$

induced by the full embedding $\text{Coh}_{\mathcal{U}}(X) \hookrightarrow \text{CCoh}(X)$ is essentially surjective.

Proof. We need to show that, given any \mathcal{K}^\bullet with coherent cohomology, there exists some cover \mathcal{U} such that, on each $U_\alpha \in \mathcal{U}$, there exists some bounded complex of coherent sheaves quasi-isomorphic to $\mathcal{K}^\bullet|_{U_\alpha}$. Following the proof of [KS90, Proposition 1.7.11], we see that it suffices to show that, for any surjective morphism $u: \mathcal{G} \rightarrow \mathcal{H}$ of analytic sheaves with \mathcal{H} coherent, and any point $x \in X$, there exists a neighbourhood U_x of x , a coherent sheaf \mathcal{F} on U_x , and a morphism $t: \mathcal{F} \rightarrow \mathcal{G}$ such that the composite morphism $ut: \mathcal{F} \rightarrow \mathcal{H}$ is surjective.

To see this, let $x \in X$. Since \mathcal{H} is coherent, \mathcal{H}_x is of finite type over the local ring $\mathcal{O}_{X,x}$, and so can be generated by a finite number of sections s_1, \dots, s_r . Since the map $\mathcal{G}_x \rightarrow \mathcal{H}_x$ is surjective, we can lift these sections to sections t_1, \dots, t_r of \mathcal{G}_x defined on some neighbourhood U_x of x . Define the sheaf \mathcal{F} on U_x to be the free sheaf $(\mathcal{O}_{U_x})^r$, and define the morphism $t: \mathcal{F} \rightarrow \mathcal{G}$ by the sections t_1, \dots, t_r . Consider the cokernel \mathcal{C} of $ut: \mathcal{F} \rightarrow \mathcal{H}$, which is coherent, since both \mathcal{F} and \mathcal{H} are. Since \mathcal{C} is coherent, its support is an analytic set, and thus closed. By construction, the map $(ut)_x: \mathcal{F} \rightarrow \mathcal{H}$ is surjective. Since the complement of $\text{supp}(\mathcal{C})$ is open, and not equal to $\{x\}$, there exists a neighbourhood V_x of x that does not intersect with $\text{supp}(\mathcal{C})$. Thus $ut: \mathcal{F} \rightarrow \mathcal{H}$ is surjective on V_x . \square

(10.2.11) Lemma. The functor

$$\mathrm{hocolim}_{\mathcal{U}} \mathrm{LCoh}_{\mathcal{U}}(X) \rightarrow \mathrm{LCCoh}(X)$$

induced by the full embedding $\mathrm{Coh}_{\mathcal{U}}(X) \hookrightarrow \mathrm{CCoh}(X)$ is fully faithful.

Proof. If we identify each $\mathrm{LCoh}_{\mathcal{U}}(X)$ with its ‘weakly-essential’ image in $\mathrm{LCCoh}(X)$ (i.e. the subcategory spanned by objects *weakly equivalent* to those in the image of the inclusion), then we can simply apply (10.2.8). \square

(10.2.12) Definition. Let $\underline{\mathrm{Vect}}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$ be the subcategory of $\mathrm{Vect}^{\mathrm{cart}}(X_{\bullet}^{\mathcal{U}})$ spanned by complexes that actually *are* cartesian complexes of locally free sheaves on the nerve; let $\underline{\mathrm{Green}}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ be the subcategory of $\mathrm{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ spanned by complexes that actually *are* Green complexes.

(10.2.13) Lemma. $\mathrm{hocolim}_{\mathcal{U}} \mathrm{LGreen}_{\nabla,0}(X_{\bullet}^{\mathcal{U}}) \simeq \mathrm{hocolim}_{\mathcal{U}} \underline{\mathrm{LGreen}}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$.

Proof. The homotopy colimit over refinements of all covers is equivalent to the homotopy colimit over some truncation below of refinements over all covers, i.e. we can always assume that our covers are as fine as we wish when computing the homotopy colimit. But, by taking a fine enough cover \mathcal{U} , every object of $\mathrm{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ restricted to each U_{α} is free, and so, as in the proof of (10.2.7), we can invert quasi-isomorphisms whose target is in $\mathrm{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$. This means that, given some morphism in the localisation $\mathrm{LGreen}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$, expressed as a chain of roofs with the left-legs all quasi-isomorphisms, we can invert the quasi-isomorphisms and compose the resulting morphisms to obtain a single morphism in $\mathrm{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ which is equal to that in the localisation with which we started. This means that $\mathrm{LGreen}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$ is equivalent to $\underline{\mathrm{LGreen}}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$, and so their homotopy colimits agree. \square

(10.2.14) Corollary. There is an equivalence of $(\infty, 1)$ -categories

$$\mathrm{hocolim}_{\mathcal{U}} \underline{\mathrm{LGreen}}_{\nabla,0}(X_{\bullet}^{\mathcal{U}}) \simeq \mathrm{LCCoh}(X).$$

Proof. This is a consequence of Sections 10.2.9 to 10.2.11 and (10.2.13). \square

10.3 General properties

(10.3.1) Lemma. Let \mathcal{L} be a holomorphic line bundle, given by transition maps $\{g_{\alpha\beta}\} \in \check{C}^1(\mathbb{C}^{\times})$. Then

$$\mathrm{tr} \int_{|\Delta^{\bullet}|} \hat{\mathrm{at}}_{\mathcal{L}}^{\circ 1} = c_1(\mathcal{L}),$$

where $c_1(\mathcal{L})$ denotes the (first) Chern class of the line bundle given by the connecting homomorphism from the Picard group in the long exact sequence associated to the *exponential sheaf sequence*

$$0 \rightarrow 2\pi i\mathbb{Z} \hookrightarrow \mathcal{O}_X \xrightarrow{\text{exp}} \mathcal{O}_X^\times \rightarrow 0$$

(that is, ‘the’ classical definition, as given in e.g. [Huy05, Definition 2.2.13]).

Proof. We have already calculated, in (4.1.12) and (6.2.2), changing notation from $M_{\alpha\beta}$ to $g_{\alpha\beta}$, that the left-hand side is equal to $g_{\alpha\beta}^{-1}dg_{\alpha\beta}$. That the right-hand side gives the same result can be found in e.g. the proof of the Proposition in [GH94, Chapter 1, §1, *Chern Classes of Line Bundles*, p. 141]. \square

(10.3.2) Lemma. Let $f: Y \rightarrow X$ be a continuous map of complex-analytic manifolds, and let \mathcal{F} be a coherent sheaf on X . Then

$$f^* \left(\text{tr} \int_{|\Delta^\bullet|} \hat{\text{at}}_{\mathcal{F}}^{\circ k} \right) = \text{tr} \int_{|\Delta^\bullet|} \hat{\text{at}}_{f^*\mathcal{F}}^{\circ k}$$

for all $k \in \mathbb{N}$, where f^* denotes the *derived pullback*.

Proof. This is basically the combination of the following facts: the derived pullback is exact; the simplicial Atiyah class is defined by taking a resolution; the derived pullback of a locally free sheaf agrees with the non-derived pullback; the non-simplicial Atiyah class of the non-derived pullback of a locally free sheaf is exactly the pullback of the non-simplicial Atiyah class of the locally free sheaf. We spell out how to join up these facts in slightly more detail below.

Since the derived pullback is exact, we know that, given Green’s resolution $\mathcal{E}^{\bullet,\star}$ of $\mathcal{F}^\bullet = (X \xrightarrow{\mathcal{U}} X)^*\mathcal{F}$ by vector bundles on the nerve, the derived pullback $f^*\mathcal{E}^{\bullet,\star}$ is a resolution for $f^*\mathcal{F}^\bullet$. But the derived pullback on locally free objects agrees with the non-derived pullback (since we are tensoring with something locally free), and so $f^*\mathcal{E}^{\bullet,\star}$ can be calculated by the non-derived pullback. Now we can use the fact^[1] that, for a single vector bundle, the Atiyah class of the pullback is the pullback of the Atiyah class; for a single vector bundle on the nerve, the simplicial Atiyah class is determined entirely by the $\omega_{\alpha_0\alpha_i}$ (which represent the

^[1]We are working with vector bundles, which are locally free, and so, as previously mentioned, the derived pullback is just exactly the non-derived pullback. There is some subtlety however, since we are using the word ‘pullback’ to mean a few different things here. When we talk about pulling back the Atiyah class of E , we mean first applying the pullback of sheaves, to get some class in $H^1(X, f^*\Omega_X^1 \otimes \mathcal{E}nd(f^*E))$ (using the fact that pullbacks commute with $\mathcal{E}nd$ for finite-dimensional locally free sheaves), and then applying the canonical map $f^*\Omega_X^1 \rightarrow \Omega_Y^1$ (which is what we really mean when we talk about pulling back forms). The fact that this construction sends at_E to at_{f^*E} follows from the fact that short exact sequences are distinguished triangles, and so specifying a morphism between the first and last (non-zero) terms of two SESs extends uniquely to a morphism of the SESs (in that we get a unique morphism between the middle terms of the two SESs such that everything commutes).

non-simplicial Atiyah class). Combining all the above, we see that all the forms defining the simplicial Atiyah class of the derived pullback of \mathcal{F} are exactly the pullbacks of the forms defining the simplicial Atiyah class of \mathcal{F} . In particular, then, taking the trace and then fibre integrating (both of which commute with the pullback of forms on X) gives us the required result. \square

(10.3.3) Lemma. If the total simplicial Atiyah classes are additive on every split exact sequence of coherent sheaves then they are additive on every short exact sequence of coherent sheaves.

Proof. Let $0 \rightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \rightarrow 0$ be a short exact sequence of coherent sheaves on X , and $t \in \mathbb{C}$ (which can be thought of as $t \in \Gamma(\mathbb{C}, \mathbb{P}^1)$). Write $p: X \times \mathbb{P}^1 \rightarrow X$ to mean the projection map. Define

$$\mathcal{N} = \text{Ker}\left(p^*\mathcal{G}(1) \oplus p^*\mathcal{H} \xrightarrow{\pi(1)-t \cdot \text{id}} p^*\mathcal{H}(1)\right)$$

where $(\pi(1) - t \cdot \text{id}): (g \otimes y, h) \mapsto \pi(g) \otimes y - h \otimes t$. We claim that this gives a short exact sequence

$$0 \rightarrow p^*\mathcal{F}(1) \rightarrow \mathcal{N} \rightarrow p^*\mathcal{H} \rightarrow 0$$

of sheaves over $X \times \mathbb{P}^1$, where the maps are the ‘obvious’ ones: $p^*\mathcal{F}(1) \rightarrow \mathcal{N}$ is the map $\iota(1): p^*\mathcal{F}(1) \rightarrow p^*\mathcal{G}(1)$ included into $p^*\mathcal{G}(1) \oplus p^*\mathcal{H}$ (which we prove lands in \mathcal{N} below); and $\mathcal{N} \rightarrow p^*\mathcal{H}$ is the projection $p^*\mathcal{G}(1) \oplus p^*\mathcal{H} \rightarrow p^*\mathcal{H}$ restricted to \mathcal{N} .

To prove surjectivity, let $h \in \Gamma(U, p^*\mathcal{H})$. Then $h \otimes t \in \Gamma(U, p^*\mathcal{H}(1))$. But $\pi: \mathcal{G} \rightarrow \mathcal{H}$ is surjective, and thus so too is the induced map $\pi(1): p^*\mathcal{G}(1) \rightarrow p^*\mathcal{H}(1)$. Hence there exists $g \otimes y \in \Gamma(U, p^*\mathcal{G}(1))$ such that $\pi(g \otimes y) = h \otimes t$. So $(g \otimes y, h) \in \mathcal{N}$ maps to h .

To prove injectivity (and that this map is indeed well defined), we use the fact that tensoring with $\mathcal{O}(1)$ is exact, and so, in particular, $\iota(1): p^*\mathcal{F}(1) \rightarrow p^*\mathcal{G}(1)$ is injective. The inclusion into the direct sum $p^*\mathcal{G}(1) \oplus p^*\mathcal{H}$ is injective by the definition of a direct sum, so all that remains to show is that the image of this composite map is contained inside \mathcal{N} . Let $f \otimes x \in \Gamma(U, p^*\mathcal{F}(1))$. Then this maps to $(\iota(f) \otimes x, 0) \in p^*\mathcal{G}(1) \oplus p^*\mathcal{H}$, but this is clearly in the kernel of $\pi(1) - t \cdot \text{id}$ since $\pi \iota(f) = 0$.

To prove exactness, it suffices to show that $\text{Ker}(\mathcal{N} \rightarrow p^*\mathcal{F}) \cong p^*\mathcal{F}(1)$. But

$$\begin{aligned} \text{Ker}(\mathcal{N} \rightarrow p^*\mathcal{F}) &= \{(g \otimes y, h) \in \mathcal{N} \mid h = 0\} \\ &= \{(g \otimes y, h) \in p^*\mathcal{G}(1) \oplus p^*\mathcal{H} \mid h = 0 \text{ and } \pi(g) \otimes y - h \otimes t = 0\} \\ &= \{(g \otimes y, h) \in p^*\mathcal{G}(1) \oplus p^*\mathcal{H} \mid \pi(g) \otimes y = 0\} \\ &= \{(g \otimes y, h) \in p^*\mathcal{G}(1) \oplus p^*\mathcal{H} \mid (g \otimes y) \in \text{Im } \iota(1)\} \\ &\cong p^*\mathcal{F}(1). \end{aligned}$$

Now we claim that the short exact sequence is split for $t = 0$, and has $\mathcal{N} \cong p^*\mathcal{G}$ for $t \neq 0$. Formally, we do this by looking at the pullback of the map $X \times \{t\} \rightarrow X \times \mathbb{P}^1$, but we can think of this as just ‘picking a value for t ’.

For $t = 0$, by definition,

$$\begin{aligned} \mathcal{N} &= \text{Ker}\left(p^*\mathcal{G}(1) \oplus p^*\mathcal{H} \xrightarrow{\pi(1)-\text{id}} p^*\mathcal{H}(1)\right) \\ &= \text{Ker}\left(p^*\mathcal{G}(1) \oplus p^*\mathcal{H} \xrightarrow{\pi(1)} p^*\mathcal{H}(1)\right) \\ &\cong \text{Ker}\left(p^*\mathcal{G}(1) \xrightarrow{\pi(1)} p^*\mathcal{H}(1)\right) \oplus p^*\mathcal{H} \\ &\cong p^*\mathcal{F}(1) \oplus p^*\mathcal{H}. \end{aligned}$$

For $t \neq 0$, define the injective morphism $\varphi: p^*\mathcal{G} \rightarrow \mathcal{N}$ of coherent sheaves by $\varphi(g) = (g \otimes t, \pi(g))$. To see that this is also surjective, let $(g \otimes y, h) \in \mathcal{N}$. If $y = 0$ then we must have $h = 0$, and so $(g \otimes y, h) = (0, 0) = \varphi(0 \otimes 0)$. If $y \neq 0$ then $\pi(g) \otimes y - h \otimes t = 0$, with $y, t \neq 0$, whence $\pi(g) = \frac{y}{t}h$. Then $(g \otimes y, h) = (\frac{t}{y}g \otimes t, \pi(\frac{t}{y}g)) = \varphi(\frac{t}{y}g)$.

As one final ingredient, note that any coherent sheaf on X pulled back to a sheaf on $X \times \mathbb{P}^1$ is flat over \mathbb{P}^1 , and so \mathcal{N} is flat over \mathbb{P}^1 , since both $\mathcal{F}(1)$ and \mathcal{H} are. Thus, for $\tau_t: X \times \{t\} \rightarrow X \times \mathbb{P}^1$ given by a choice of $t \in \mathbb{C}$, the derived pullback $\mathbb{L}\tau_t^*\mathcal{N}$ agrees with the usual pullback $\tau_t^*\mathcal{N}$.

Now we use the \mathbb{P}^1 -homotopy invariance of de Rham cohomology, which is the following statement: the induced map

$$\tau_t^*: H^\bullet(X \times \mathbb{P}^1, \Omega_{X \times \mathbb{P}^1}^\bullet) \rightarrow H^\bullet(X \times \{t\}, \Omega_{X \times \{t\}}^\bullet)$$

does *not* depend on the choice of t . Since $X \times \{t\}$ is (canonically) homotopic to X , we can identify $(p\tau_t)^*$ with the identity on $H^\bullet(X, \Omega_X^\bullet)$. Since the SES splits for $t = 0$, by our hypothesis, flatness, and the fact that Green’s construction is functorial under derived pullback, we know that

$$\begin{aligned} \tau_0^*c(\mathcal{N}) &= c(\mathbb{L}\tau_0^*\mathcal{N}) = c(\tau_0^*\mathcal{N}) = c(\tau_0^*p^*\mathcal{F}(1) \oplus \tau_0^*p^*\mathcal{H}) \\ &= c(\mathcal{F}) \wedge c(\mathcal{H}). \end{aligned}$$

But we also know that $\mathcal{N} \cong p^*\mathcal{G}$ for $t \neq 0$, and so

$$c(\mathcal{G}) = (p\tau_t)^*c(\mathcal{G}) = \tau_t^*c(\mathcal{N}).$$

So, finally, the t -invariance of τ^* tells us that

$$c(\mathcal{G}) = c(\mathcal{F}) \wedge c(\mathcal{H}). \quad \square$$

(10.3.4) Lemma. The total simplicial Atiyah classes are additive on every split exact sequence of coherent sheaves.

Proof. This exactly [Gre80, Lemma 2.6], but the essence of the proof is simple: twisting cochains behave nicely with respect to direct sums, as do all of the constructions giving the simplicial connections generated in degree zero. \square

10.4 The compact case

(10.4.1) We have previously said that the trace of the Atiyah class is ‘basically the same as’ the Chern class, and we can now formalise that in the specific case where X is **compact**, using [Gri09, Theorem 6.5]. Since tDR cohomology is ‘nice enough’ (namely, it satisfies conditions (α) – (δ) in [Gri09, §6.2]), it suffices^[1] to check the following things in order to show that the trace of the Atiyah class really is the same as the Chern class:

- (i) *The construction agrees with ‘the’ classical one for line bundles.* Although [Gre80, Lemma 2.5] tells us that the construction actually agrees with the classical one for *arbitrary* vector bundles, the proof is maybe lacking a bit in explicit details. It turns out that the case of line bundles is easy enough to do by hand anyway, hence our inclusion of (10.3.1).
- (ii) *The construction is functorial under pullbacks.* This is (10.3.2).
- (iii) *The construction gives us the Whitney sum formula for short exact sequences.* We split the proof of this into two steps: firstly, showing in (10.3.3) that it suffices to prove that we have the Whitney sum formula for *split* exact sequences; then showing in (10.3.4) that we do indeed have the Whitney sum formula for split exact sequences.
- (iv) *The construction gives us the Grothendieck-Riemann-Roch formula for closed immersions.* Although this is given as a requirement in [Gri09, Theorem 6.5], it is actually not necessary, thanks to e.g. [Gri12, Proposition 3.1], whose proof relies only on the three properties above, after some classical algebraic geometry shenanigans.

(10.4.2) Again, when X is compact, [Gre80, Lemma 2.7] gives a direct proof of the fact that the (p, p) -term of the fibre integral of the trace of these simplicial Atiyah classes agrees (up to a constant factor) with the Chern classes of Atiyah-Hirzenbruch.

10.5 The whole story: a summary

(10.5.1) Starting with some complex of coherent sheaves $\mathcal{F}^\bullet \in \text{Coh}(X)$, with the category of complexes of coherent sheaves being defined as in (10.1.3), we take Green’s resolution, which produces a complex $\mathcal{E}^{\bullet, \bullet}$ of vector bundles on the nerve (of some possible refined cover) resolving $i^* \mathcal{F}^\bullet$. Then, also by Green’s resolution, we get simplicial connections on each of the $\mathcal{E}^{\bullet, i}$, and (7.7.15) tells us that these are admissible. So, applying the generalised invariant polynomial $P_\bullet = \{\text{tr} \circ \mu_n\}_{n \in \mathbb{N}}$

^[1]As we explain somewhat further in (C.2.3).

(where μ_n is the multiplication map that sends an n -fold tensor product of endomorphisms to the endomorphism given by composition (resp. wedge product)) to the curvatures of each of the simplicial connections, we obtain the simplicial exponential (resp. standard) Atiyah classes of each $\mathcal{E}^{\bullet,i}$. Using the alternating-sum convention, this gives us the simplicial exponential (or standard) Atiyah class of \mathcal{F} . Finally, by fibre integration of the trace of these classes, we obtain closed classes in the Čech-de Rham bicomplex, and thus classes in de Rham (or even tDR) cohomology.

The fact that this construction is independent of the choice of twisting cochain (and thus cover) and of local connections is explained in the proof of [Gre80, Theorem 2.4].

(10.5.2) Of particular importance is (10.2.13), which tells us that the homotopy colimit (over refinements of covers) of the localisation of the category of Green complexes ($\underline{\text{Green}}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$) is equivalent to the homotopy colimit of the localisation of the category of complexes which are locally quasi-isomorphic to Green complexes ($\text{Green}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$). In particular, we only know how to construct admissible simplicial connections for objects of the former, and so it is necessary that our construction of $\mathcal{E}^{\bullet,\star}$ in the above is an object of $\underline{\text{Green}}_{\nabla,0}(X_{\bullet}^{\mathcal{U}})$.

IV. APPENDICES (FACTUAL)

Dupont's fibre integration

A

(A.0.1) Purpose. *As the oft-cited aphorism goes: “the hardest thing in homological algebra is ensuring that you make an even number of sign errors” — from my experience, working with fibre integration of simplicial differential forms is exactly the same. In particular, the choice of convention discussed in (A.1.3) can lead to much confusion. Hopefully, anybody hoping to do any explicit calculations with fibre integration themselves can use this chapter as a reference for all the mistakes I've already made for them.*

A.1 Sign conventions

(A.1.1) We follow the Koszul sign convention for the product of two complexes:

$$a \otimes b = (-1)^{|a||b|} b \otimes a$$

$$(a \otimes b) \wedge (x \otimes y) = (-1)^{|b||x|} (a \wedge x) \otimes (b \wedge y).$$

(A.1.2) Lemma. The simplicial de Rham complex is isomorphic to the total complex of the de Rham bicomplex of simplicial forms grouped by type, i.e.

$$\left(\Omega_{Y_\bullet \times \Delta^\bullet}^{\bullet, \Delta}, d_{Y_\bullet \times \Delta^\bullet} \right) \simeq \left(\text{Tot}^\bullet \Omega_{Y_\bullet \times \Delta^\bullet}^{i, j}, d_{Y_\bullet} + (-1)^i d_{\Delta^\bullet} \right).$$

There is a subtle abuse of notation^[1] in the above. What we should really say is the following: $d_{Y_\bullet} + (-1)^i d_{\Delta^\bullet}$ is a differential on

$$\pi_1^{-1} \Omega_{Y_\bullet}^{\bullet, \Delta} \otimes_{(\pi_1^{-1} \mathcal{O}_{Y_\bullet} \otimes \pi_2^{-1} \mathcal{O}_{\Delta^\bullet})} \pi_2^{-1} \Omega_{\Delta^\bullet}^{\bullet, \Delta}$$

where π_1, π_2 are the projection maps of $Y_\bullet \times \Delta^\bullet$. This differential then extends to a differential on

$$\pi_1^{-1} \Omega_{Y_\bullet}^{\bullet, \Delta} \otimes_{(\pi_1^{-1} \mathcal{O}_{Y_\bullet} \otimes \pi_2^{-1} \mathcal{O}_{\Delta^\bullet})} \pi_2^{-1} \Omega_{\Delta^\bullet}^{\bullet, \Delta} \otimes_{(\pi_1^{-1} \mathcal{O}_{Y_\bullet} \otimes \pi_2^{-1} \mathcal{O}_{\Delta^\bullet})} \mathcal{O}_{Y_\bullet \times \Delta^\bullet}$$

which is exactly $d_{Y_\bullet \times \Delta^\bullet}$.

(A.1.3) The first trap that we can fall into with simplicial differential forms is the choice of writing $X \times \Delta$ or $\Delta \times X$. We opted for the former, since it makes things like currying (see (F.6.3)) seem more natural, and it's also the convention used in [Gre80]. The convention used in [Dup76] is the opposite, and it seems like this was definitely the good choice: the map $\sigma: X \times \Delta \rightarrow \Delta \times X$ that sends (x, t) to (t, x) only induces a morphism of complexes if we multiply by $(-1)^{ij}$, and so when we want to fibre integrate using *our* convention (of writing $X \times \Delta$) we need to make sure to multiply by this sign factor every single time. This is not really very fun, but we have made our bed and so must lie in it.



^[1]What a surprise.

(A.1.4) Lemma/Definition. Let M be an oriented manifold with boundary ∂M , and orientations on each connected component $(\partial M)_\alpha$ of the boundary (*not necessarily* the canonically-induced ones). Define

$$\varepsilon_\alpha = \begin{cases} +1 & \text{if } (\partial M)_\alpha \text{ has the canonically-induced orientation;} \\ -1 & \text{otherwise.} \end{cases}$$

Then, for any $(n-1)$ -form ω on M , we have that

$$\int_M d\omega = \sum_\alpha \varepsilon_\alpha \int_{(\partial M)_\alpha} \omega.$$

(A.1.5) Recalling (3.0.2), we write the coordinates of the (topological) p -simplex as $\{(t_0, \dots, t_p) \in [0, 1]^{p+1} \mid \sum_i t_i = 1\} \subset \mathbb{R}^{p+1}$. For defining the orientation on the p -simplex, however, it is easier to use the other standard convention: we have coordinates $\{(t_1, \dots, t_p) \in [0, 1]^p \mid \sum_i t_i \leq 1\} \subset \mathbb{R}^p$, and can recover the convention that we use by setting $t_0 = 1 - \sum_{i=1}^p t_i$.

We take the t_i to all be positively oriented, and pick an orientation on the p -simplex such that

$$\int_{\Delta^p} dt_1 \wedge \dots \wedge dt_p > 0.$$

Orient the i -th face $(\partial \Delta^p)_i \simeq \Delta^{p-1}$ using the positive orientation on Δ^{p-1} , i.e. one that makes $\int_{\Delta^{p-1}} dt_1 \wedge \dots \wedge dt_{p-1}$ positive. Then $\varepsilon_i = (-1)^i$ in the notation of (A.1.4).

A.2 Calculations

(A.2.1) ([Gre80, p. 36]). Since the integral of a k -form over an ℓ -dimensional manifold is only non-zero when $k = \ell$, we see that the fibre integral of some simplicial differential r -form $\omega = \{\omega_p^{i,j}\}_{p \in \mathbb{N}, i+j=r}$ is determined entirely by the type- $(r-p, p)$ parts on the p -simplices:

$$\int_{\Delta^\bullet} \omega = \int_{\Delta^0} \omega_0^{r,0} + (-1)^{(r-1)} \int_{\Delta^1} \omega_1^{r-1,1} + \dots + (-1)^{(r-p)(p)} \int_{\Delta^p} \omega_p^{r-p,p} + \dots + \int_{\Delta^r} \omega_r^{0,r}$$

where the signs come from (A.1.3).

(A.2.2) Theorem. ([Dup76, Theorem 2.3]). Fibre integration gives us a quasi-isomorphism from the complex of simplicial forms to the total complex of the Čech complex of the de Rham complex, consisting of the morphisms

$$\int_{\Delta^\bullet} : \Omega_{X^u}^r \rightarrow \text{Tot}^r \check{C}^\bullet(\Omega_X^\bullet).$$



Proof. **N.B. This proof uses Dupont's convention of writing $\Delta \times X$, and so fibre integration doesn't have the signs that it does in (A.2.1).** Our goal is to construct the quasi-inverse to fibre integration, and we proceed step-by-step.^[1]

- **Elementary forms.** Let $I = (i_0, \dots, i_k)$ be an $(k + 1)$ -tuple of integers, and define $dt_I = dt_{i_0} \wedge \dots \wedge dt_{i_k}$. We also define the *elementary form* β_I by

$$\beta_I = \sum_{a=0}^k (-1)^a t_{i_a} dt_{i_0} \wedge \dots \wedge \widehat{dt_{i_a}} \wedge \dots \wedge dt_{i_k}.$$

Note that this satisfies the relation $d\beta_I = (k + 1)dt_I$

- **Coface maps.** Given some $p \in \mathbb{N}$ and some $I = (i_0, \dots, i_k)$ with $i_0 < i_1 < \dots < i_k \leq p$, we have the *associated coface map*^[2] $f_{I,p} : [k] \rightarrow [p]$ in the abstract simplex category, given by mapping a to i_a . This induces a map $X_{\bullet}^{\mathcal{U}}(f_{I,p}) : X_p^{\mathcal{U}} \rightarrow X_k^{\mathcal{U}}$.
- **The morphism.** We define $\mathcal{E} : \text{Tot}^r \check{C}^{\bullet}(\Omega_X^{\bullet}) \rightarrow \Omega_{X^{\bullet}}^r$ on $\omega \in \check{C}^k(\Omega_X^{\ell})$ by

$$(\mathcal{E}\omega)_p = \begin{cases} 0 & \text{if } p < k; \\ k! \sum_{\substack{|I|=k+1 \\ i_k \leq p}} \beta_I \wedge X_{\bullet}^{\mathcal{U}}(f_{I,p})^* \omega & \text{otherwise.} \end{cases}$$

- **The image is a simplicial form.** Fix some $p \in \mathbb{N}$ and let $0 \leq i \leq p$. Then we are interested in the restriction of $\mathcal{E}\omega$ to $\{t_i = 0\} \times X_p^{\bullet}$ (the i th face of Δ_p). But, since $t_i = 0$,

$$\beta_I |_{\{t_i = 0\}} = \begin{cases} \beta_I & \text{if } i \notin I; \\ 0 & \text{if } i \in I \end{cases}$$

and so, for $p \geq k$,

$$(\mathcal{E}\omega)_p = k! \sum_{\substack{|I|=k+1 \\ i_k \leq p \\ i \notin I}} \beta_I \wedge X_{\bullet}^{\mathcal{U}}(f_{I,p})^* \omega.$$

Here we are implicitly using the coordinates $\{t_0, \dots, \widehat{t_i}, \dots, t_p\}$ on Δ_{p-1} , but if we relabel them as $\{t_0, \dots, t_{p-1}\}$ then we can write

$$(\mathcal{E}\omega)_p = k! \sum_{\substack{|J|=k+1 \\ j_k \leq p-1}} \beta_J \wedge X_{\bullet}^{\mathcal{U}}(f_{J,p})^* \omega$$

^[1]Although a proof is given for [Dup76, Theorem 2.3], this is for *smooth* forms, and so we reproduce it (in part) here for forms that are *holomorphic* on the X component just to reassure ourselves that things all still work just fine.

^[2]In [Dup76] this is denoted by μ_I , but we use $f_{I,p}$ to reinforce the fact that it depends also on p .

where J' is defined by

$$J' = \{j \mid j \in J, j < i\} \cup \{j+1 \mid j \in J, j \geq i\}.$$

But then $f_{J',p} = f_p^i \circ f_{J,p-1}$, where $f_p^i: [p-1] \rightarrow [p]$ is the i th coface map, and so

$$(\mathcal{E}\omega)_p = k! \sum_{\substack{|J|=k+1 \\ j_k \leq p-1}} \beta_J \wedge (X_{\bullet}^{\mathcal{U}}(f_{J,p-1}) \circ X_{\bullet}^{\mathcal{U}}(f_p^i))^* \omega = X_{\bullet}^{\mathcal{U}}(f_p^i)^* (\mathcal{E}\omega)_{p-1}.$$

– **This is a morphism of complexes.** First, consider

$$d(\mathcal{E}\omega)_p = k! \sum_{\substack{|I|=k+1 \\ i_k \leq p}} [d\beta_I \wedge X_{\bullet}^{\mathcal{U}}(f_{I,p})^* \omega + (-1)^k \beta_I \wedge X_{\bullet}^{\mathcal{U}}(f_{I,p})^* d\omega].$$

The second term in the sum is exactly $(-1)^k (\mathcal{E}d\omega)_p$, and the first term is equal to

$$(k+1)! \sum_{\substack{|I|=k+1 \\ i_k \leq p}} dt_I \wedge X_{\bullet}^{\mathcal{U}}(f_{I,p})^* \omega,$$

using the aforementioned fact that $d\beta_I = (k+1)dt_I$. Then we look at

$$\begin{aligned} (\mathcal{E}\check{\delta}\omega)_p &= (k+1)! \sum_{\substack{|J|=k+2 \\ j_k \leq p}} \beta_J \wedge X_{\bullet}^{\mathcal{U}}(f_{J,p})^* \check{\delta}\omega \\ &= (k+1)! \sum_{\substack{|J|=k+2 \\ j_k \leq p}} \beta_J \wedge X_{\bullet}^{\mathcal{U}}(f_{J,p})^* \sum_{i=0}^{k+1} (-1)^i X_{\bullet}^{\mathcal{U}}(f_{k+1}^i)^* \omega \\ &= (k+1)! \sum_{\substack{|J|=k+2 \\ j_k \leq p}} \sum_{i=0}^{k+1} (-1)^i \beta_J \wedge X_{\bullet}^{\mathcal{U}}(f_{J,p} \circ f_{k+1}^i)^* \omega. \end{aligned}$$

This map $f_{J,p} \circ f_{k+1}^i$ sends $(0, \dots, k)$ to $(j_0, \dots, \widehat{j_i}, \dots, j_{k+1})$, and this gives us a way to group terms. That is, given $L = (\ell_0, \dots, \ell_k)$ with $\ell_0 < \dots < \ell_k \leq p$ (and setting $\ell_{-1} = 0$ and $\ell_{k+1} = p$) we can add any $\ell' \in (\ell_{i-1}, \ell_i)$ and take $J = L \cup \{\ell'\}$ (reordered), and always get the same image. So we sum all the corresponding β_J , i.e.

$$\sum_{\ell' \notin L} (-1)^{\text{the position of } \ell' \text{ in } L} \cdot \beta_{L \cup \{\ell'\}} = \sum_{\ell'=0}^{\ell_1-1} \beta_{\ell' \ell_0 \ell_1 \dots \ell_{k+1}} - \sum_{\ell'=\ell_1+1}^{\ell_2-1} \beta_{\ell_0 \ell' \ell_1 \dots \ell_{k+1}} + \dots$$

$$= \sum_{\substack{\ell' \in \{0, \dots, p\} \\ \ell' \notin L}} \left[t_{\ell'} dt_{\ell_0} \wedge dt_{\ell_1} \wedge \dots \wedge dt_{\ell_{k+1}} - t_{\ell_0} dt_{\ell'} \wedge dt_{\ell_1} \wedge \dots \wedge dt_{\ell_{k+1}} + \dots \right].$$

But all terms except the first are zero (since $\sum dt_{\ell'} = 0$), and the first one sums to exactly dt_L , whence we find exactly the first term of $d(\mathcal{E}\omega)_p$. Since the differential on $\text{Tot}^r \check{C}^\bullet(\Omega_X^\bullet)$ is given by $\check{d} + (-1)^i d$, we are done.

The fact that this morphism really is a quasi-inverse to fibre integration can be proven in exactly the same way as in [Dup76, Theorem 2.3]. \square

Derivations, connections, and curvature

B

(B.0.1) Purpose. *As somebody with a very weak background in differential geometry, there was much to learn about the Chern-Weil approach to characteristic classes, the fundamental of which being the theory of connections and their curvatures. There is nothing original in this chapter, and the interested reader would be much better off with a proper reference on the subject, but, having already made notes for myself, it seemed like I might as well include them here. To understand connections and curvature in the case where we have a hermitian metric, a classic reference is [GH94, Chapter 0, §5], or [Huy05, §§4.2, 4.3]; the whole principal-bundle side of the story (Ehresmann connections and covariant derivatives) is explained beautifully in [Son15]; to understand the links between different notions of connections, take a look at [Gol]. Since this thesis only really uses connections to get characteristic classes from their curvatures, I don't delve too deep into the theory: what is below is really not much more than rather bland statements of definitions and consequent properties.*

(B.0.2) Throughout, unless otherwise stated, let k be a field (almost always assumed to be \mathbb{C}) and A a commutative k -algebra (almost always assumed to be $k[x_1, \dots, x_n]$).

B.1 Kähler differentials and the cotangent bundle

(B.1.1) A *derivation on an A -module M* is a k -linear map $D: A \rightarrow M$ such that

$$D(ab) = D(a)b + aD(b).$$

We write $\text{Der}_k(A, M)$ to mean the collection of all such maps.

As a corollary to this definition, note that $D(\lambda) = 0$ for all $\lambda \in k$.

(B.1.2) The *module of Kähler differentials of A over k* is the data of an A -module $\Omega_{A/k}^1$ and an A -linear map $d: A \rightarrow \Omega_{A/k}^1$ that are *universal*, i.e. such that

$$\text{Hom}_A(\Omega_{A/k}^1, M) \cong \text{Der}_k(A, M)$$

for all A -modules M .

(B.1.3) We can explicitly construct such a module and a map as a free k -module, generate by formal symbols da , with certain relations imposed:

$$\Omega_{A/k}^1 = k \left\langle da \mid d\lambda, d(s+t) - (ds+dt), d(st) - (d(s)t + sd(t)) \right\rangle_{a \in A, \lambda \in k}$$

which, in the case of $A = k[x_1, \dots, x_n]$, reduces to

$$\Omega_{k[x_1, \dots, x_n]/k}^1 = \bigoplus_{i=1}^n k[x_1, \dots, x_n] \langle dx_i \rangle.$$

(B.1.4) We can define the *algebraic de Rham complex* $(\Omega_{A/k}^\bullet, d)$ by setting $\Omega_{A/k}^n = \bigwedge^n \Omega_{A/k}^1$.

(B.1.5) This definition doesn't behave very well with the analytic definition (i.e. the 'classical' de Rham complex), as evidenced by the following example. Let X be some smooth manifold (say, complex analytic), and consider $\mathcal{O}_X(U)$ for some open subset $U \subseteq X$, which is a \mathbb{C} -algebra. Then we can define a *sheaf* $\Omega_{\mathcal{O}_X/\mathbb{C}}^1$ on X by

$$\Omega_{\mathcal{O}_X/\mathbb{C}}^1(U) = \Omega_{\mathcal{O}_X(U)/\mathbb{C}}^1$$

where the latter is the module of Kähler differentials with $d = d_{\text{de Rham}}$.

But for non-polynomial $f \in \mathcal{O}_X(U)$ we have a problem: for example, $d \exp(x)$ is **not** equal to $\exp(x)dx$ in $\Omega_{\mathcal{O}_X(U)/\mathbb{C}}^1$. In general, we have to impose a new relation:

$$df = \sum_{i=1}^n \frac{df}{dx_i} dx_i$$

for $f \in \mathcal{O}_X(U)$. This (we claim) gives us a free $\mathcal{O}_X(U)$ -module generated by the dx_i , and the locally free sheaf $\Omega_{\mathcal{O}_X/\mathbb{C}}^1$ is called the *cotangent bundle* Ω_X^1 . Note that this locally free sheaf is locally isomorphic to the classical cotangent bundle T^*X , and the de Rham differential is given exactly by $d: \mathcal{O}_X \rightarrow \Omega_{\mathcal{O}_X/\mathbb{C}}^1 \twoheadrightarrow \Omega_X^1$, where the second map is the quotient given by enforcing the above relation.

(B.1.6) In the above, we use the fact that if X is smooth then Ω_X^1 (in the classical sense) is locally free, and if we choose a chart on any open $U \subseteq X$ then $\Omega_X^1|_U$ (in the classical sense) is free.

B.2 Connections and curvatures

(B.2.1) Now we really do just assume that $k = \mathbb{C}$, and work in the complex-analytic case, and so d means the de Rham differential^[1] and Ω_X^\bullet can be taken to be the classical cotangent bundle.

^[1]Although we could really work with any derivation, using the comment at the end of (B.1.5) and the fact that the Kähler differential $d: \mathcal{O}_X \rightarrow \Omega_{\mathcal{O}_X/\mathbb{C}}^1$ is universal.

(B.2.2) Let (X, \mathcal{O}_X) be a complex-analytic manifold of dimension n , and E a locally free sheaf of rank r of \mathcal{O}_X -modules on X . A *linear*, or *Koszul*, *connection* ∇ on E is a \mathbb{C} -linear map of sheaves of \mathcal{O}_X -modules

$$\nabla: E \rightarrow E \otimes_{\mathcal{O}_X} \Omega_X^1$$

such that, over any open $U \subseteq X$,

$$\nabla(fs) = f\nabla(s) + s \otimes df.$$

This condition is called the *Leibniz rule*, and generalises to higher degrees (i.e. for maps $\tilde{\nabla}: E \otimes \Omega_X^r \rightarrow E \otimes \Omega_X^{r+1}$) as

$$\nabla(s \otimes \omega) = \nabla(s) \wedge \omega + s \otimes d\omega.$$

Enforcing this ensures that $\tilde{\nabla}(fs \otimes \omega) = \tilde{\nabla}(s \otimes f\omega)$.

(B.2.3) Given a vector bundle E over X , and a continuous map $f: Y \rightarrow X$, we get local frames of f^*E over any point $y \in Y$ by pulling back local frames over $f(y) \in X$. So although it is *not* true that any section of a pullback bundle is just the pullback of a section of the original bundle, we get some linear version of this: any section of the pullback bundle can be written as a linear combination of elements of some local frame, and this local frame can be written as the pullback of a local frame of the original bundle. So, to define the pullback connection $f^*\nabla$ on f^*E , it suffices to define how $f^*\nabla$ acts on f^*s , where $s \in \Gamma(U, E)$ and we assume that E is trivial (since we can always work locally, and then just apply the Leibniz rule).

(B.2.4) If we have two connections ∇_1 and ∇_2 on E , then they differ (thanks to the Leibniz rule) by an \mathcal{O}_X -linear morphism:

$$\nabla_1(fs) - \nabla_2(fs) = f(\nabla_1(s) - \nabla_2(s)).$$

This means that, given one connection ∇ , we can recover all others by simply adding \mathcal{O}_X -linear morphisms $E \rightarrow E \otimes \Omega_X^1$. (This can be formalised using the notion of *affine torsors*.)

(B.2.5) If V and W are vector spaces with W free and of finite rank then $\text{Hom}(V, V \otimes W) \cong \text{Hom}(V, V) \otimes W$. Thus

$$\text{Hom}_{\mathcal{O}_X}(E, E \otimes \Omega_X^1) = \Gamma(X, \mathcal{H}\text{om}(E, E \otimes \Omega_X^1)) \cong \Gamma(X, \mathcal{H}\text{om}(E, E) \otimes \Omega_X^1) \quad (\star)$$

which we call the collection of *differential endomorphism-valued 1-forms*; an object of this looks **locally** like an $(r \times r)$ -matrix of 1-forms.

If U is now such that $E|_U$ is trivial (i.e. such that $E|_U \cong \mathcal{O}_X^r$), then $\Omega_X^1|_U \cong \mathcal{O}_X^n$, and we have a natural connection, called the *trivial connection*, on E given exactly by the differential $d: E \rightarrow \Omega_X^1$. What do we mean by this? Well, let $E|_U \cong \mathcal{O}_X \langle s_1, \dots, s_r \rangle$, so that $s \in \Gamma(U, E)$ can be written as $\sum_{i=1}^r f_i s_i$ for some $f_i \in \mathcal{O}_X$. Then define $\nabla_d(s) = \sum_{i=1}^r s_i \otimes df_i$.

This means that, **locally**, any connection ∇ can be written as $\nabla_d + \bar{\omega}$, where $\bar{\omega} \in \mathcal{H}\text{om}(E, E) \otimes \Omega_X^1$, by (B.2.4) and (\star) .

(B.2.6) By definition, a connection ∇ is **not** \mathcal{O}_X -linear, because

$$\nabla(fs) = f\nabla(s) + s \otimes df \neq f\nabla(s)$$

(but it is \mathbb{C} -linear). By enforcing the Leibniz rule, as described in (B.2.2), ∇ induces a map $E \otimes \Omega_X^1 \rightarrow E \otimes \Omega_X^2$, which we also (confusingly, but unambiguously) call ∇ . Applying ∇ twice then gives us a map $E \rightarrow E \otimes \Omega_X^2$, which we denote by $\kappa(\nabla)$ and call the *curvature of ∇* , and this map is \mathcal{O}_X -linear (i.e. $\kappa(\nabla)(fs) = f\kappa(\nabla)(s)$).

Since any connection can **locally** be written as $\nabla_d + \bar{\omega}$, the curvature can also **locally** be written as $d\bar{\omega} + \bar{\omega} \cdot \bar{\omega}$, where the product \cdot is taken to mean ‘matrix multiplication, where we multiply the elements using the wedge product of forms’.

(B.2.7) A connection is said to be *flat* if its curvature is zero; a bundle is said to be *flat* if it admits a flat connection; a section s of a bundle is said to be *flat* for a given connection ∇ if $\nabla(s) = 0$.

We motivate these definitions in the next section.

B.3 Frobenius integrability and local systems

(B.3.1) Consider a *linear differential system*: the problem of trying to find (smooth) functions $f_1(x_1, \dots, x_n), \dots, f_r(x_1, \dots, x_n)$ such that

$$\frac{df}{dx_j} = \sum_{i=1}^r \mu_{i,j} f_i$$

for given functions $\mu_{i,y}$, where we write $f = (f_1, \dots, f_r)$. To turn this into language similar to that used so far in this chapter, we can think of f as a section of the trivial rank- r bundle on an n -manifold with local coordinates x_1, \dots, x_n .

As an example, with $n = r = 2$, consider the linear differential system given by

$$\begin{aligned} \frac{df_1}{dx_1} &= af_1 + bf_2, & \frac{df_2}{dx_1} &= pf_1 + qf_2, \\ \frac{df_1}{dx_2} &= cf_1 + df_2, & \frac{df_2}{dx_2} &= rf_1 + sf_2. \end{aligned}$$

Using that $df_i = \frac{df_i}{dx_1} dx_1 + \frac{df_i}{dx_2} dx_2$ we can rewrite this as

$$d \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} adx_1 + cdx_2 & bdx_1 + ddx_2 \\ pdx_1 + rdx_2 & qdx_1 + sdx_2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

which is exactly $(d + \bar{\omega})f = 0$, with $-\bar{\omega}$ being the (2×2) -matrix of 1-forms on the right-hand side.

So, if we have some initial-value condition $f(x_0) = v$, then solving our system is exactly the same as finding a section s of the trivial rank- r bundle with fixed

value at a given point (i.e. $s(x_0) = v \in E_{x_0}$, where E_{x_0} is the fibre of E above the point $x_0 \in X$) that is flat for the connection $\nabla = d + \bar{\omega}$.

Further, if $\nabla(s) = 0$ then $\kappa(\nabla)(s) = 0$, whence $\kappa(\nabla)(s)(x_0) = \kappa(\nabla)(v) = 0$, and so the curvature measures the *obstruction* to solving our system: we can only solve for initial conditions $v \in E_{x_0}$ such that $v \in \text{Ker } \kappa(\nabla)$. So being able to solve for all initial conditions is equivalent to the curvature being zero at x_0 .

(B.3.2) Frobenius integrability theorem. Let ∇ be a flat connection on E , $x_0 \in X$, and $v \in E_{x_0}$. Then there exists a local section $s \in \Gamma(U, E)$ such that $s(x_0) = v$ and $\nabla(s) = 0$.

(B.3.3) What we say is really a theorem of *analytic* geometry, and **not algebraic** geometry: we are taking X to have the usual \mathbb{C}^n -induced topology (not Zariski) and \mathcal{O}_X to be the collection of holomorphic forms (not regular algebraic functions). For example, flat sections of $\nabla = d - 1/z$ on $\mathbb{C} \setminus \{0\}$ are logarithms, which exist locally as holomorphic functions, but are not rational.

(B.3.4) Using the Frobenius integrability theorem, we have the following general method of solving linear differential systems (or, equivalently, studying flat connections on a rank- r bundle):

1. take a basis $\{v_1, \dots, v_r\}$ of E_{x_0} ;
2. extend these elements to flat sections $\{s_1, \dots, s_r\}$;
3. trivialise your bundle using these flat sections;

and then, in this basis, we have that ∇ is exactly d (i.e. $\bar{\omega} = 0$), since

$$\nabla(fs_i) = f\nabla(s_i) + s_i \otimes df = s_i \otimes df.$$

In other words, locally, flat connections on a rank- r bundle are canonically modelled by the trivial connection on the trivial rank- r bundle.

(B.3.5) Now assume that E is a flat bundle of rank r . Then we can look at the *sheaf of flat sections* (which is a sheaf because it is the kernel of a sheaf morphism). Since, locally, this sheaf is given by solving $df = 0$, $f = (f_1, \dots, f_r)$ is locally constant. That is, the sheaf of flat sections is a locally-constant sheaf (which is sometimes referred to as a *local system*).

Conversely, let L be a local system. Then, since the differential d is \mathbb{C} -linear, the induced map

$$d: L \otimes_{\mathbb{C}_X} \mathcal{O}_X \rightarrow L \otimes_{\mathbb{C}_X} \Omega_X^1 \cong L \otimes_{\mathbb{C}_X} \mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega_X^1$$

(where we write \mathbb{C}_X to mean the constant sheaf on X of value \mathbb{C}) lets us realise $L \otimes_{\mathbb{C}_X} \mathcal{O}_X$ as a flat bundle with connection $\nabla = d$.

We claim that this correspondence between flat bundles and local systems induces an equivalence of categories.^[1] This is interesting, because flat bundles are ‘analytic objects’, but local systems are purely topological.^{[2][3]} In one dimension, this is demonstrated by the fact that line bundles are in bijection with $H^1(X, \mathcal{O}_X)$, whereas **flat** line bundles are in bijection with $H^1(X, (\mathbb{C} \setminus \{0\})_X)$, which depends only on the topological structure of X .

^[1]This is a special case of the Riemann-Hilbert correspondence

^[2]In fact, local systems are parametrised by representations of the fundamental group in \mathbb{C}^r , or (under nice conditions on X) functors from the fundamental groupoid of X into the category of sets (when we are interested in sheaves of sets).

^[3]As an example, the *orientation sheaf* on a manifold is a local system.

Characteristic classes

C

(C.0.1) Purpose. Basically, I can never remember the which polynomials are which in the fundamental theorem of symmetric polynomials, and I always got confused with which relations held between the Chern classes, exponential Chern classes, Chern characters, etc. This appendix is mostly just a collection of notational conventions, but also serves as a brief introduction to the multiple possible definitions of Chern classes.

(C.0.2) Conventions. Throughout, unless otherwise stated, we work with a rank- r (holomorphic) vector bundle on a complex-analytic manifold X of dimension n .

C.1 Background

(C.1.1) The main idea. Characteristic classes are invariants of principal bundles that tell us how close the bundle is to being trivial, in that they ‘measure’ the difference between the local structure of the bundle and the global (product) structure.

There are different types of characteristic classes depending on the setting in question, and the coefficients of cohomology that we wish to study: a real (oriented) vector bundle $E \rightarrow X$ has *Pontryagin classes* $p_k(E) \in H^{4k}(X; \mathbb{Z})$, *Stiefel-Whitney classes* $w_k(E) \in H^k(X; \mathbb{Z}/2\mathbb{Z})$, and an *Euler class* $e(E) \in H^{r(E)}(X; \mathbb{Z})$. If we have instead a *complex* vector bundle, then we can study its *Chern classes* $c_k \in H^{2k}(X; \mathbb{C})$.

Characteristic classes *tend to* satisfy some variation of the following axioms, where we write γ to mean ‘some sort of characteristic class’.

1. *Rank vanishing.* For sufficiently large k (usually either greater than either $r(E)$ or $\dim X$), the classes are zero.
2. *Naturality/functoriality.* For any continuous $f: Y \rightarrow X$, the classes respect pullbacks along f , in that $f^*(\gamma(E)) = \gamma(f^*(E))$.
3. *Whitney sum/product formula.* The class of a direct sum of bundles is the cup product of the classes of each summand: $\gamma(E \oplus F) = \gamma(E) \smile \gamma(F)$.
4. *Normalisation.* The class of a bundle of rank 1 is forced to be something ‘simple’, such as $1 + e(E_{\mathbb{R}})$.

We are often interested in *stable* characteristic classes, which are those that are invariant under direct summing with a trivial bundle.^[1] All of the aforementioned classes are stable except for the Euler class.

^[1]In a more abstract sense, stability refers to some sort of invariance under the inclusion $\mathcal{B}G(n) \hookrightarrow \mathcal{B}G(n+1)$.

Characteristic classes can often be understood as generalisations of other classical constructions. For example, the Euler class of the tangent bundle of a smooth manifold recovers exactly the Euler characteristic.

Note that we haven't actually said *how* to define any of these characteristic classes yet. In general, there are multiple equivalent ways. We discuss the various constructions of Chern classes in (C.2).

(C.1.2) Topology vs. geometry. As with many objects, there are definitions of characteristic classes in both algebraic topology and algebraic geometry. For Chern classes (the specific type of characteristic classes that we study in this thesis), the topological notion gives classes that live in *cohomology*, be it Deligne, de Rham, Hodge, singular, or whatever really; the geometric notion, however, gives classes that live in the *Chow ring*.

Since we are working in the *analytic* case, and not the algebraic one, it is the topological approach that interests us the most: we want *cohomology-valued* classes.

(C.1.3) References. For the complex-analytic constructions of Chern classes, see [Huy05, §4.4] or [Voi08, §11.2]; for the general topological construction of characteristic classes, see [MS74] or [Hat17, Chapter 3].

C.2 Defining Chern classes

(C.2.1) Chern-Weil. If we have some connection form^[1] ω on our bundle E , then we can define Chern classes in terms of the curvature Ω of the connection: the classes $c_k(E) \in H_{\text{dR}}^{2k}(X; \mathbb{C})$ are defined by demanding that they satisfy the equation

$$\det\left(\frac{it\Omega}{2\pi} + 1\right) = \sum_{k=0}^r c_k(E)t^k \tag{*}$$

where 1 is the identity matrix, and t a formal variable. By using the identity

$$\text{tr}(\ln(A)) = \ln(\det(A))$$

and the Maclaurin expansion for $\ln(1 + A)$, we can expand the left-hand side of (*) in order to get explicit definitions for the Chern classes:

$$\sum_{k=0}^r c_k(E)t^k = 1 + i\frac{\text{tr}(\Omega)}{2\pi}t + \frac{\text{tr}(\Omega^2) - \text{tr}(\Omega)^2}{8\pi^2}t^2 + i\frac{-2\text{tr}(\Omega^3) + 3\text{tr}(\Omega^2)\text{tr}(\Omega) - \text{tr}(\Omega)^3}{48\pi^3}t^3 + \dots$$

^[1]See Appendix B.

(C.2.2) Classifying spaces. Thinking of vector bundles as their associated GL_r -principal bundles, we can define characteristic classes by using the construction of the universal bundle (and, in particular, Chern classes arise in this way by thinking of a holomorphic vector bundle as a $GL_r(\mathbb{C})$ -principal bundle). On an abstract level, characteristic classes in this sense can be thought of as natural transformations from the functor $\text{Bund}_G: \text{Top}^{\text{op}} \rightarrow \text{Set}$ to the functor $H^\bullet: \text{Top}^{\text{op}} \rightarrow \text{Set}$, where the former sends a topological space X to the set of isomorphism classes of G -principal bundles on X , and the latter is some sort of cohomology theory (composed with the forgetful functor into Set , so as to forget any algebraic structure). Writing down what this means, we see that a characteristic class γ consists of, for each G -principal bundle P on X , an element $\gamma(P) \in H^\bullet(X)$, and the naturality condition of a natural transformation then tells us that $\gamma(f^*P) = f^*(\gamma(P))$ for any continuous $f: X \rightarrow Y$.

Apologising in advance for the jargon, here is how we can define characteristic classes using universal bundles. Write $\mathcal{B}G$ to mean the geometric realisation of the classifying groupoid (or delooping) of G , $\mathbb{H} = \text{Top}$, and let A be an arbitrary abelian group. Then any G -principal bundle $P \rightarrow X$ is classified by some map $\kappa_P: X \rightarrow \mathcal{B}G$; any cocycle $\gamma: \mathcal{B}G \rightarrow \mathcal{B}^n A$ gives a cohomology class

$$[\gamma] \in H^n(\mathcal{B}G, A) = \pi_0 \mathbb{H}(\mathcal{B}G, \mathcal{B}^n A);$$

and so we can pull back any $\gamma \in H^k(\mathcal{B}G, A)$ to get $[\gamma(\kappa_P)] \in H^k(X, A)$ by simply precomposing with κ . This means that it suffices to calculate the characteristic classes of $\mathcal{B}G$ in order to calculate the characteristic classes of any G -principal bundle, since we can just then pull them back along the classifying map of the bundle.

For example, it can be shown (by induction on n , and using the Gysin sequence, similar to as in [Hat17, Theorem 3.9]) that

$$H^\bullet(\mathcal{B}U(n); \mathbb{Z}) = \mathbb{Z}[\gamma_1, \dots, \gamma_n]$$

where γ_k is of degree $2k$. This tells us that the k th characteristic class of any $U(n)$ -principal bundle will live in cohomological degrees $2k$, and this agrees with what we find in any other construction of Chern classes. In general, calculating the cohomology of classifying spaces of G -principal bundles, for $G \leq GL_n$, can be done by looking at direct limits of Grassmanians of increasing dimension. The ‘important’ groups G that we study are usually exactly subgroups of GL_n , with $G = U(n)$ recovering Chern classes; $G = O(n)$ recovering Pontryagin and Stiefel-Whitney classes; and $G = SO(n)$ recovering the Euler class.

(C.2.3) Axiomatic. We can define the $c_k \in H^{2k}(X; \mathbb{Z})$ by a formal axiomatisation, followed by some argument showing uniqueness (which we don’t give here). We give three possible sets of axioms below, where F denotes any other (holomorphic) rank- s vector bundle on X , and Y denotes any other complex-analytic manifold of dimension m . For nicer notation, we define the *total Chern class* $c(E)$ of E as the sum of all its Chern classes: $c(E) = \sum_{k=0}^r c_k(E)$.

• **Classical.**

1. *Unitality.* $c_0(E) = 1$;
2. *Naturality.* $c_k(f^*E) = f^*c_k(E)$ for any continuous map $f: Y \rightarrow X$;
3. *Whitney sum.* $c_k(E \oplus F) = \sum_{i=0}^k c_i(E) \smile c_{k-i}(F)$, or, equivalently, $c(E \oplus F) = c(E) \smile c(F)$;
4. *Normalisation.* The total Chern class of the tautological line bundle over $\mathbb{C}P^p$ is $1 - H$, where H is Poincaré dual to $\mathbb{C}P^{p-1} \subset \mathbb{C}P^p$.

• **Grothendieck.** In [Gro58], Grothendieck showed^[1] that the total Chern class of any complex vector bundle (of finite rank) can be defined in terms of the first Chern class of a tautological line bundle.^[2] This means that the list of axioms is slightly smaller, and he also generalised the Whitney sum formula to arbitrary (that is, not necessarily split) short exact sequences.

1. *Naturality.* $c_k(f^*E) = f^*c_k(E)$ for any continuous map $f: Y \rightarrow X$;
2. *Additivity.* $c(E) = c(F) \smile c(F')$ for any short exact sequence $0 \rightarrow F \rightarrow E \rightarrow F' \rightarrow 0$;
3. *Normalisation.* If E is a line bundle then $c(E) = 1 + e(E_{\mathbb{R}})$, where $e(E_{\mathbb{R}})$ is the Euler class of the underlying real vector bundle of E .

In special cases (but, importantly, not in the setting of this thesis), we can use the *splitting principle* along with the principle of scindage to calculate Chern classes easily. See, for example, the *Birkhoff-Grothendieck theorem*.

• **Grivaux.** In [Gri09, Theorem 6.5], there is another axiomatisation given, which concerns the *Grothendieck-Riemann-Roch theorem*:

Let X and Y be smooth quasi-projective manifolds, $f: X \rightarrow Y$ a proper morphism, and \mathcal{F} a coherent algebraic sheaf on X . Then the following identity holds in the Chow ring of Y :

$$f_*(\text{ch}(\mathcal{F})\text{td}(X)) = \sum_{i \geq 0} (-1)^i \text{ch}(R^i f_*(\mathcal{F}))\text{td}(Y)$$

where $\text{td}(X)$ is the Todd class of X , and $\text{ch}(X)$ is the Chern character of X .

What interests us in the context of [Gri09] is the analytic version of this, which, for *compact* complex-analytic manifolds X and Y , and Chern classes

^[1]Using the *Leray-Hirsch theorem*, which describes, under certain hypotheses, an isomorphism between the singular cohomology of a fibre bundle and the tensor product of the singular cohomologies of its fibre and its base.

^[2]This is called the principle of *scindage*, and tells us that, if we know the Chern classes $1, a, a^2, \dots, a^{k-1}$ of the projectivisation $\mathbb{P}(E)$ of E , then we can define the Chern classes of E by writing $-a^k$ as a linear combination of the $1, a, \dots, a^{k-1}$ (which we can do, thanks to Leray-Hirsch), and taking $c_i(E)$ to be the coefficient of a^{k-i} .

in the *Hodge rings* $\bigoplus_i H^i(X, \Omega_X^i)$, is shown in [OTT85]. The axiomatisation is then as follows.

1. *Naturality.* $c_k(f^*E) = f^*c_k(E)$ for any continuous map $f: Y \rightarrow X$;
2. *Additivity.* $c(E) = c(F) \smile c(F')$ for any short exact sequence $0 \rightarrow F \rightarrow E \rightarrow F' \rightarrow 0$;
3. *Normalisation.* If E is a line bundle then $c(L) = 1 + c_1(L)$;
4. *GRR.* The Grothendieck-Riemann-Roch theorem holds for *closed immersions*.

This axiomatisation is the one that we use in Section 10.4.

C.3 Chern ‘things’

(C.3.1) Symmetric polynomials. Consider polynomials in some formal variables x_1, \dots, x_ℓ over some ring R , along with the canonical action (i.e. permuting the variables) of the symmetric group S_ℓ on such polynomials.

- The *symmetric polynomials* $p(x_1, \dots, x_\ell)$ are those polynomials which are invariant under the S_ℓ -action.
- The *elementary symmetric polynomials* $\sigma_k(x_1, \dots, x_\ell)$ are the symmetric polynomials defined to satisfy, for a formal variable t ,

$$\prod_{k=1}^{\ell} (t + x_k) = \sum_{k=1}^{\ell} \sigma_k(x_1, \dots, x_\ell) t^{\ell-k};$$

$$\sigma_k(x_1, \dots, x_\ell) = 0 \text{ for } k > \ell.$$

They can equivalently be defined by

$$\sigma_k(x_1, \dots, x_\ell) = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq \ell} x_{j_1} x_{j_2} \dots x_{j_k} \text{ for } 0 \leq k \leq \ell.$$

- The *power-sum symmetric polynomials* $\tau_k(x_1, \dots, x_\ell)$ are defined by

$$\tau_k(x_1, \dots, x_\ell) = \frac{1}{k!} (x_1^k + \dots + x_\ell^k).$$

- The *fundamental theorem of symmetric polynomials* says that there exist polynomials P_k and Q_k such that

$$\tau_k(\underline{x}) = P_k(\sigma_1(\underline{x}), \dots, \sigma_k(\underline{x}))$$

$$\sigma_k(\underline{x}) = Q_k(\tau_1(\underline{x}), \dots, \tau_k(\underline{x}))$$

where $\underline{x} = (x_1, \dots, x_\ell)$.

(C.3.2) Chern classes and polynomials.

- The *Chern classes* $c_0(E) = 1$ and $c_1(E), \dots, c_r(E)$ are defined by any one of the constructions in (C.2).
- The *total Chern class* $c(E)$ is defined to be the sum $\sum_k c_k(E)$ of the Chern classes.
- The *Chern polynomial* $c_t(E) = \sum_k c_k(E)t^k$ is the formal polynomial corresponding to the total Chern class; its roots are called the *Chern roots*, and denoted $a_i(E)$.
- The *exponential Chern classes* $ch_0(E) = r$ and $ch_1(E), \dots, ch_n(E)$ are defined by the polynomials found in the fundamental theorem of symmetric polynomials evaluated on the Chern classes:

$$ch_k(E) = P_k(c_1(E), \dots, c_r(E)).$$

- The *total exponential Chern class*, or *Chern character*, $ch(E)$ is defined to be the sum $\sum_k ch_k(E)$ of the exponential Chern classes.

(C.3.3) Relations.

- $ch(E) = r + c_1(E) + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + \dots$
- If $E = L_1 \oplus \dots \oplus L_r$ is a direct sum of line bundles, then $a_i(E) = c_1(L_i)$.
- $c_k(E) = \sigma_k(a_1(E), \dots, a_k(E)) = Q_k(ch_1(E), \dots, ch_k(E))$.
- $ch(L) = \exp(c_1(L))$ for any line bundle L , where the right-hand side is defined to equal $\sum_{i=0}^{\infty} (c_1(L)^i)/i!$.

(C.3.4) Atiyah classes. The *Atiyah class(es)* of E are defined in (4.1.2), and we discuss how they are equivalent to the Chern classes in (4.1.3) and Section 10.4.

(C.3.5) K -theory. The total Chern class is an *additive group* morphism from K -theory into the \mathbb{Q} -cohomology^[1] of X ; the total exponential Chern class is a *ring* morphism from K -theory into the \mathbb{Q} -cohomology of X . That is,

$$c(E \oplus F) = c(E) \smile c(F)$$

$$ch(E \oplus F) = ch(E) + ch(F)$$

$$ch(E \otimes F) = ch(E)ch(F)$$

^[1]That is, cohomology with coefficients in \mathbb{Q} .

Relative categories

D

(D.0.1) Purpose. *There are many models for $(\infty, 1)$ -categories, but relative categories work most naturally for what I need in Chapter 10. Rather than place the technical details there, I've put them in this appendix, so that anybody wishing to ignore the ∞ -details can simply avoid them. I skip over quite a few definitions in (D.2.1), assuming that the sort of people who care about the proof of Lemma (D.2.3) are the sort of people who already know far more than I do about such things.*

D.1 Relative categories as presentations of infinity-categories

(D.1.1) Definition. In Chapter 10 we use *relative categories* (and *homotopical categories*) as a way of talking about $(\infty, 1)$ -categories. As we mention there, a relative category is a pair (C, \mathcal{W}) , where C is a category, and \mathcal{W} (whose morphisms we call *weak equivalences*) is a wide subcategory of C . A relative category is said to be a *homotopical category* if its weak equivalences satisfy the *2-out-of-6 property*: if

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

is a sequence of composable morphisms such that the compositions gf and hg are weak equivalences, then f , g , h , and hgf are all weak equivalences too.

In this appendix, we try to always write (C, \mathcal{W}) instead of just C .

(D.1.2) Using the formalism of [Rez00] along with the results of [BK13], we can think of a homotopical category (C, \mathcal{W}) as presenting the $(\infty, 1)$ -category LC , which is the complete Segal space given by taking a Reedy fibrant replacement of the Rezk/simplicial nerve $N(C, \mathcal{W})$. In particular, [BK13, §1.2 (ii)] tells us that any homotopically full relative subcategory of a partial model category is again a partial model category, with model categories being a specific example of partial model categories.

D.2 Bisimplicial sets and model structures

(D.2.1) There is an adjunction

$$\text{bisSet} : (\mathcal{N}_\xi \dashv \mathcal{K}_\xi) : \text{RelCat}$$

between the category of relative categories and the category of bisimplicial sets, described in [BK12, §5.3]. This induces a model structure on RelCat where, in particular, the weak equivalences are exactly those morphisms that become weak

equivalences when we apply N_ξ . But [BK12, Lemma 5.4] tells us that it is equivalent to say that a morphism in RelCat is a weak equivalence if and only if its image under the Rezk nerve $N: \text{RelCat} \rightarrow \text{bisSet}$ is a weak equivalence, where bisSet has the *Rezk complete Segal model structure*. We don't really need many details about these model structures, except for what we use in the proof of (D.2.3), so we give the relevant definitions there.

(D.2.2) Morphisms between relative categories are functors that respect the weak equivalences (that is, they send weak equivalences to weak equivalences). It is *not* true, however, if we have some $F: (C_1, \mathcal{W}_1) \rightarrow (C_2, \mathcal{W}_2)$ such that $F: C_1 \rightarrow C_2$ is an equivalence of plain categories, that $F: \text{LC}_1 \rightarrow \text{LC}_2$ is an equivalence of $(\infty, 1)$ -categories. Consider, for example, the case when $C_1 = C_2$, but \mathcal{W}_1 is given by just isomorphisms, and \mathcal{W}_2 is something strictly larger than this. Then $F = \text{id}_{C_1}$ gives an equivalence $C_1 \simeq C_2$, but $\text{LC}_1 \simeq C_1$ and LC_2 have no reason to be equivalent, let alone equivalent via F . What we can sometimes use, however, is Lemma (D.2.3).

(D.2.3) Lemma. Let (C_1, \mathcal{W}_1) and (C_2, \mathcal{W}_2) be relative categories. Assume that $F: (C_1, \mathcal{W}_1) \rightarrow (C_2, \mathcal{W}_2)$ is a functor of relative categories, inducing an equivalence $C_1 \simeq C_2$, such that the restriction $F: \mathcal{W}_1 \rightarrow \mathcal{W}_2$ is *also* an equivalence of categories. Then F induces an equivalence of $(\infty, 1)$ -categories $\text{LC}_1 \xrightarrow{\sim} \text{LC}_2$.

Proof. Using (D.2.1), we know that we want exactly for

$$N(F): N((C_1, \mathcal{W}_1)) \rightarrow N((C_2, \mathcal{W}_2))$$

to be a weak equivalence. Working simplicial level by simplicial level, let's spell out what exactly $N(F)$ is.

In degree $p = 0$, we have the morphism

$$N(F)_0: (\mathcal{W}_1) \rightarrow (\mathcal{W}_2)$$

which is an equivalence by hypothesis (since $\mathcal{W}_1 \simeq \mathcal{W}_2$, and so their nerves are also equivalent). In degree $p = 1$, we have

$$N(F)_1: (\bullet \xrightarrow{C_1} \bullet, \mathcal{W}_1) \rightarrow (\bullet \xrightarrow{C_2} \bullet, \mathcal{W}_2)$$

where $(\bullet \xrightarrow{C_i} \bullet, \mathcal{W}_i)$ has one 'direction' of 1-simplices being morphisms $x \rightarrow y$ in C_i , and the other 'direction' being commutative squares

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ x' & \longrightarrow & y' \end{array}$$

where \rightsquigarrow denotes a morphism in \mathcal{W}_i . Then $N(F)_1$ is essentially surjective because F is full; it is fully faithful because F induces an equivalence $\mathcal{W}_1 \simeq \mathcal{W}_2$. In general, in degree $p \in \mathbb{N}$, we have

$$N(F)_p : \left(\underbrace{\bullet \xrightarrow{C_1} \bullet \xrightarrow{C_1} \dots \xrightarrow{C_1} \bullet}_{p \text{ arrows}}, \mathcal{W}_1 \right) \longrightarrow \left(\underbrace{\bullet \xrightarrow{C_2} \bullet \xrightarrow{C_2} \dots \xrightarrow{C_2} \bullet}_{p \text{ arrows}}, \mathcal{W}_2 \right)$$

where, on both sides, one ‘direction’ of p -simplices consists of chains of morphisms $x_0 \rightarrow \dots \rightarrow x_p$ in C_i , and the other ‘direction’ of commutative ‘squares’

$$\begin{array}{ccccccc} x_0^0 & \longrightarrow & x_1^0 & \longrightarrow & \dots & \longrightarrow & x_p^0 \\ \downarrow \rightsquigarrow & & \downarrow \rightsquigarrow & & & & \downarrow \rightsquigarrow \\ x_0^1 & \longrightarrow & x_1^1 & \longrightarrow & \dots & \longrightarrow & x_p^1 \\ \downarrow \rightsquigarrow & & \downarrow \rightsquigarrow & & & & \downarrow \rightsquigarrow \\ \vdots & & \vdots & & \ddots & & \vdots \\ \downarrow \rightsquigarrow & & \downarrow \rightsquigarrow & & & & \downarrow \rightsquigarrow \\ x_0^p & \longrightarrow & x_1^p & \longrightarrow & \dots & \longrightarrow & x_p^p. \end{array}$$

But, again, the fact that F induces equivalences $C_1 \simeq C_2$ and $\mathcal{W}_1 \simeq \mathcal{W}_2$ tells us that $N(F)_p$ is indeed an equivalence. \square

Coherent sheaves and perfect complexes

(E.0.1) Purpose. *When working with complexes of coherent algebraic sheaves, there is a well-known equivalence between complexes of actually coherent sheaves, and complexes of sheaves which have coherent internal cohomology. It is not clear, however, that the same is true in the analytic case, so I thought it would be nice to explain this in some detail here.*

(E.0.2) Definition. Let (X, \mathcal{O}_X) be a ringed space, and M^\bullet a cochain complex of \mathcal{O}_X -modules. We say that M^\bullet is *perfect* if it is locally quasi-isomorphic to a bounded complex of locally free sheaves (i.e. free \mathcal{O}_X -modules of finite type). That is, M^\bullet is perfect if, for every point $x \in X$, there exists some open neighbourhood U of x , and some bounded complex L_U^\bullet of locally free sheaves on U , such that $M^\bullet|_U \simeq L_U^\bullet$.

We write $\text{Pf}(X)$ to denote the triangulated category of perfect complexes of \mathcal{O}_X -modules, as a full subcategory of the derived category $D(\text{Sh}(X))$ of sheaves of \mathcal{O}_X -modules.

(E.0.3) Definition. Let (X, \mathcal{O}_X) be a ringed space, and \mathcal{F} a sheaf of \mathcal{O}_X -modules on X . Then \mathcal{F} is *of finite type* if every $x \in X$ has an open neighbourhood U_x such that there exists a surjective morphism

$$\mathcal{O}_X^n|_{U_x} \twoheadrightarrow \mathcal{F}|_{U_x}$$

for some $n \in \mathbb{N}$.

We say that \mathcal{F} is *quasi-coherent* if it is locally the cokernel of free modules, i.e. if there exists some cover $\mathcal{U} = \{U_\alpha\}$ of X such that, for all α , there exist $m_\alpha, n_\alpha \in \mathbb{N} \cup \{\infty\}$ such that

$$\mathcal{O}_X^{m_\alpha}|_{U_\alpha} \rightarrow \mathcal{O}_X^{n_\alpha}|_{U_\alpha} \rightarrow \mathcal{F}|_{U_\alpha} \rightarrow 0$$

is exact.

Finally, \mathcal{F} is said to be *coherent* if

1. it is of finite type; and
2. for every open subset $U \subset X$, every $n \in \mathbb{N}$, and every morphism

$$\varphi: \mathcal{O}_X^n|_U \rightarrow \mathcal{F}|_U$$

of $(\mathcal{O}_X|_U)$ -modules, the kernel $\ker \varphi$ is of finite type.

Note that coherent sheaves are, in particular, quasi-coherent.

E.1 Coherent algebraic sheaves

(E.1.1) For an arbitrary ringed space (X, \mathcal{O}_X) , the category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves of \mathcal{O}_X -modules on X is a full abelian subcategory of the category $\mathrm{Sh}(X)$ of sheaves of \mathcal{O}_X -modules on X , and its objects are those that correspond locally to modules over a ring.

If X is further assumed to be Noetherian, then the category $\mathrm{Coh}(X)$ of coherent sheaves of \mathcal{O}_X -modules on X is a full abelian subcategory of $\mathrm{QCoh}(X)$, and its objects are those that correspond locally to *finitely generated* modules.

By definition, we see that the full subcategory $\mathrm{Pf}(X)$ of $D(\mathrm{Sh}(X))$ is contained inside $D_{\mathrm{coh}}(\mathrm{Sh}(X))$, which is the full subcategory of $D(\mathrm{Sh}(X))$ spanned by objects whose internal (i.e. cochain) cohomology is coherent in each degree.^[1]

(E.1.2) Proposition. Let X be a Noetherian scheme. Then the canonical fully faithful functor

$$D^b(\mathrm{Coh}(X)) \hookrightarrow D(\mathrm{Sh}(X))$$

identifies $D^b(\mathrm{Coh}(X))$ with the full subcategory $D_{\mathrm{coh}}^b(\mathrm{Sh}(X))$ of $D(\mathrm{Sh}(X))$.

Proof. [Ill71b, Corollaire 2.2.2.1]. □

(E.1.3) Proposition. Let X be a smooth scheme. Then there is a canonical equivalence of triangulated categories

$$\mathrm{Pf}(X) \xrightarrow{\sim} D^b(\mathrm{Coh}(X)).$$

Proof. This follows from [Ill71a, Exemples 5.11] combined with (E.1.2). □

(E.1.4) Morally, then, when we want to work with coherent *algebraic* sheaves, if X is nice enough, then working with perfect complexes, complexes of coherent sheaves, or complexes of sheaves with coherent cohomology are all equivalent.

(E.1.5) A nice summary of a lot of the fundamental results of [Ill71a; Ill71b] (SGA 6) concerning perfect complexes can be found in [TT90, §2].

E.2 Coherent analytic sheaves

(E.2.1) For coherent *analytic* sheaves, the story is more subtle: as far as the author is aware, the question of whether or not $D^b(\mathrm{Coh}(X))$ and $D_{\mathrm{coh}}^b(\mathrm{Sh}(X))$ are equivalent is still open, except in low dimensions, where it is known to be true (see [Yu13, §2.2.2]). In particular cases (see [TV08a, Théorème 1.1], for example), we

^[1]Such objects, if bounded, are exactly the *pseudo-coherent* objects, as justified by [Ill71a, Corollaire 3.5].

can sometimes “algebraize” our analytic manifold and then use the nice properties of coherent algebraic sheaves, but this isn’t always viable.

Finally, if we wish to work with quasi-coherent analytic sheaves, then, following [Wei16, Remark 6.4], we need to replace the definition with that of *Fréchet* quasi-coherent sheaves (but this does not really concern us here).

(E.2.2) The definition of ‘(complexes of) coherent analytic sheaves’ that we use in Chapter 10 is such that we have an equivalence with perfect complexes almost immediately by definition (after potentially refining the cover). This fact is used in the proof of (10.2.7).

We then show, in (10.2.10) and (10.2.11) that complexes of coherent sheaves and complexes of sheaves with coherent cohomology are ‘equivalent’, with the technical detail that we take a homotopy colimit over refinements of covers.

To be precise, we do *not* prove that $D^b(\mathrm{Coh}(X))$ and $D_{\mathrm{coh}}^b(\mathrm{Sh}(X))$ are equivalent, but we prove that the $(\infty, 1)$ -categorical version of the latter is equivalent to something that, *locally*, looks like the former.

V. APPENDICES (VAGUE)

Simplicial sheafy things

F

(F.0.1) Purpose. In (4.3.3) I gave some warnings about the definition of sheaves on a simplicial space, and they all largely stem from the fact that I couldn't find any reference for these objects in the literature apart from [Gre80; TT86], where these objects are called simplicial sheaves — a name which I found slightly misleading. I think that these objects are interesting in their own right, and so have attempted to formalise what I can about them, and to introduce them as characters in stories that are already well known. We also briefly discuss the 'simplicial gluing condition' found, for example, in the definition of simplicial forms.

(F.0.2) Although our notation differs, most of what follows (as well as any missing proofs) can be found in [Toë02]. In particular, we assume some level of familiarity with the idea of localisation of categories and the classical categorical approach to sheaves.

(F.0.3) When we talk about 'stacks' we mean as coefficients for cohomology, i.e. simply ∞ -sheaves, without any geometric or algebraic structure. When we talk about 'the category of topological spaces' we always mean 'some category of sufficiently-nice topological spaces'.

F.1 Sheaves

(F.1.1) First of all, let's think about sheaves in a purely categorical way. Given a topological space X , define

$$\mathrm{PrSh}(X) = [\mathrm{Op}(X)^{\mathrm{op}}, \mathrm{Set}]$$

to be the category of *presheaves* on X . Define a set of morphisms

$$W = \{\varphi: A \rightarrow B \text{ in } \mathrm{PrSh}(X) \mid \varphi_x: A_x \rightarrow B_x \text{ is a bijection}\}$$

(where $A_x = \varinjlim_{U \ni x} A(U)$ is the stalk of A at x) and call such morphisms *local isomorphisms*. We can then take the (GZ) localisation $W^{-1}\mathrm{PrSh}(X)$ and note that the sheafification functor $a_{\mathrm{Sh}}: \mathrm{PrSh}(X) \rightarrow \mathrm{Sh}(X)$ factors through $W^{-1}\mathrm{PrSh}(X)$. We denote the right adjoint to a_{Sh} by j_{Sh} .

(F.1.2) Lemma. The induced functor $W^{-1}\mathrm{PrSh}(X) \rightarrow \mathrm{Sh}(X)$ gives rise to an equivalence of categories. That is, the localisation functor $L: \mathrm{PrSh}(X) \rightarrow W^{-1}\mathrm{PrSh}(X)$ has a fully-faithful right adjoint whose essential image is exactly the subcategory of sheaves, realised as *W-local objects*.

F.2 Stacks

(F.2.1) We want to define stacks as ∞ -sheaves somehow, and we do this by taking a simplicial model of the latter. Let

$$\text{sPrSh} = \text{sPrSh}(\text{Top}) = [\text{Top}^{\text{op}}, \text{sSet}]$$

be the category of *topological simplicial presheaves*. That is, an object $A \in \text{sPrSh}$ is the data of a simplicial set $A(X)$ for every space $X \in \text{Top}$, along with a morphism $f^*: A(Y) \rightarrow A(X)$ for every morphism $f: X \rightarrow Y$ in Top , functorial^[1] in f .

(F.2.2) There is an equivalence of categories

$$[\text{Top}^{\text{op}}, [\Delta^{\text{op}}, \text{Set}]] \simeq [\Delta^{\text{op}}, [\text{Top}^{\text{op}}, \text{Set}]]$$

which justifies our definition of simplicial presheaves as presheaves of simplicial sets, not as simplicial objects in the category of presheaves.

(F.2.3) We again define a set of morphisms to be our *local equivalences*

$$W = \{\varphi: A \rightarrow B \text{ in } \text{sPrSh} \mid \varphi_x: A_x \xrightarrow{\sim} B_x \text{ for every } X \in \text{Top} \text{ and } x \in X\}$$

where we recall that a morphism of simplicial sets is an *equivalence* if it induces an isomorphism on all homotopy groups for all choices of basepoint. We also define the set of *global equivalences*

$$W^{\text{Pr}} = \{\varphi: A \rightarrow B \text{ in } \text{sPrSh} \mid \varphi_X: A(X) \xrightarrow{\sim} B(X) \text{ for every } X \in \text{Top}\}.$$

It follows from the definitions that $W^{\text{Pr}} \subseteq W$.

(F.2.4) A *stack*, or ∞ -*sheaf*, is defined to be an object of $\text{St} = W^{-1}\text{sPrSh}$, and a *prestack* an object of $\text{PrSt} = (W^{\text{Pr}})^{-1}\text{sPrSh}$ (which justifies the choice of notation W^{Pr}). There is a *stackification functor* $a_{\text{St}}: \text{PrSt} \rightarrow \text{St}$ due to the fact that the (opposite) category of open neighbourhoods of a point is filtered, and a filtered colimit of equivalences is an equivalence.

(F.2.5) **Lemma.** The stackification functor $a_{\text{St}}: \text{PrSt} \rightarrow \text{St}$ has a right-adjoint j_{St} which is fully-faithful, and a local equivalence between two pre-stacks in the image of j_{St} gives a global equivalence between the two.^[2] That is, there is a full subcategory $\text{sPrSh}^{\text{desc}}$ of sPrSh consisting of W -local simplicial presheaves such that St is equivalent to $(W^{\text{Pr}})^{-1}\text{sPrSh}^{\text{desc}}$. The adjunction morphism $A \rightarrow j_{\text{St}}a_{\text{St}}(A)$ in PrSt is the *associated stack construction*.

^[1] $(g \circ f)^* = f^* \circ g^*$.

^[2]This is the ‘local-to-global’ principle for simplicial presheaves.

F.3 Sheaves and stacks

(F.3.1) Lemma. There is an adjunction $\text{St} : (\pi_0 \dashv j_0) : \text{Sh}$ where $\pi_0 : \text{St} \rightarrow \text{Sh}$ gives the sheaf of connected components and $j_0 : \text{Sh} \rightarrow \text{St}$ sends a sheaf of sets to the corresponding presheaf of discrete simplicial sets. Further, j_0 is fully faithful and its essential image^[1] consists of 0-truncated stacks.^[2] Note also that the composition $j_0 \circ a \circ \mathcal{Y}$ gives us a full embedding of spaces into stacks as *representable stacks*.

F.4 1-stacks-in-groupoids and stacks

(F.4.1) A *fibred category* is a presheaf of groupoids ‘up to isomorphism’: for inclusions $i : U \hookrightarrow V$ and $j : V \hookrightarrow W$ of open subsets, the morphism $\tau_{ij} : (ji)^* \rightarrow i^*j^*$ is only required to be an isomorphism, not necessarily the identity.

We content ourselves in saying that a *1-stack in groupoids* \mathcal{X} is a fibred category $\mathcal{X} \rightarrow \text{Grpd}$ of groupoids that satisfies some extra descent conditions (whatever this might mean, formally). The morphisms between two 1-stacks in groupoids are just commutative diagrams of functors, and we say that two such morphisms are homotopic if there exists a natural isomorphism between them that is compatible with the projection to Grpd . We write 1-St to mean the category of 1-stacks and $\text{Ho}(1\text{-St})$ to mean the homotopy category (relative to the above definition of homotopies).

(F.4.2) Lemma. We can replace a fibred category of groupoids by an actual presheaf of groupoids in a functorial way (the so-called *strictification of fibred categories*). This is done by the 2-Yoneda lemma: for any 1-stack \mathcal{X} we have the presheaf of groupoids^[3] $\mathcal{Y}(\mathcal{X})$ given by

$$\mathcal{Y}(\mathcal{X})(X) := \underline{\text{Hom}}_{\text{Top}}(\underline{X}, \mathcal{X})$$

where \underline{X} is the fibred category represented by X .

(F.4.3) Lemma. Writing $\mathbb{B}G$ to mean the classifying *simplicial set* of a groupoid G , we have the composition $\mathbb{B}\mathcal{Y} : 1\text{-St} \rightarrow \text{sPrSh} \rightarrow \text{St}$. This induces a functor on the level of homotopy categories^[4]

$$\mathbb{B}\mathcal{Y} : \text{Ho}(1\text{-St}) \rightarrow \text{St}$$

that is fully faithful and has essential image being exactly the *1-truncated stacks*.^[5]

^[1]That is, the “weakly essential image”, consisting of objects weakly equivalent to those in the image.

^[2]These are the stacks A such that $\pi_n(A_x; y) = 0$ for all $X \in \text{Top}$, $x \in X$, $y \in A_x$, and $n \geq 1$.

^[3]We are implicitly using the equivalence between Top and Grpd .

^[4]We defined St as a localisation along weak equivalences, so taking the homotopy category changes nothing.

^[5]These are the stacks A such that $\pi_n(A_x; y) = 0$ for all $X \in \text{Top}$, $x \in X$, $y \in A_x$, and $n \geq 2$.

(F.4.4) The functors so far can be summarised in the following diagram:

$$\begin{array}{ccc}
 \text{Top} & & \\
 \downarrow \mathcal{Y} & & \\
 \text{PrSh} & \xrightleftharpoons[a_{\text{Sh}}]{j_{\text{Sh}}} & \text{Sh} \\
 & \searrow \pi_0 \uparrow j_0 & \\
 \text{PrSt} & \xrightleftharpoons[a_{\text{St}}]{j_{\text{St}}} & \text{St} \\
 & \uparrow \mathbb{B}\mathcal{Y} & \\
 & \text{Ho}(1\text{-St}) &
 \end{array}$$

F.5 Sheaves on simplicial spaces

(F.5.1) The composition $(j_0 \circ a_{\text{Sh}} \circ \mathcal{Y}): \text{Top} \rightarrow \text{St}$ lets us view any topological space as a stack. Similarly, any simplicial space X_\bullet can be viewed as a stack $|X_\bullet|$ in one of two (equivalent) ways:

1. think of $X_\bullet \in [\Delta^{\text{op}}, \text{Top}]$ as a simplicial presheaf on Top via Yoneda

$$|X_\bullet| = \text{Hom}_{\text{Top}}(-, X_\bullet) \in \text{sPrSh}$$

and then take the associated stack under localisation $\text{sPrSh} \rightarrow W^{-1}\text{sPrSh} = \text{St}$;
or

2. think of X_\bullet as a simplicial diagram in St , acting via

$$X_\bullet: [n] \mapsto (j_0 \circ a_{\text{Sh}} \circ \mathcal{Y})X_n$$

and then take the homotopy colimit $|X_\bullet| = \text{hocolim } X_\bullet$.

We also have the assignment $X \mapsto \text{St}(X)$, where $\text{St}(X)$ is the ∞ -category of stacks (modelled by simplicial sheaves, as with St) on X . This gives us a stack of categories on Top . Further, given some stack $Y \in \text{St}$, we can define $\text{St}(Y) = \text{holim}_{X \rightarrow Y} \text{St}(X)$ to obtain a stack of categories on all of St . Living inside this category is $\text{Sh}(Y) \hookrightarrow \text{St}(Y)$, the category of *discrete sheaves* (i.e. zero-truncated stacks).

Now we can (finally) talk about the “simplicial sheaves” of [Gre80] (although we stay in the topological, rather than complex-analytic, setting for the moment).

(F.5.2) A “*simplicial sheaf*” on a simplicial space Y_\bullet , as defined in [Gre80], is a collection of sheaves \mathcal{F}^p on Y_p along with *functorial*^[1] morphisms $\mathcal{F}^\bullet(\alpha): (Y_\bullet \alpha)^* \mathcal{F}^p \rightarrow \mathcal{F}^q$ for all $\alpha: [p] \rightarrow [q]$ in Δ .

[1]That is, $\mathcal{F}(\beta \circ \alpha) = \mathcal{F}(\beta) \circ \mathcal{F}(\alpha)$.

(F.5.3) We let $Y = |X_{\bullet}^{\mathcal{U}}| \in \text{St}$ and think of $\text{Sh}(X_{\bullet}^{\mathcal{U}})$ and $\text{St}(X_{\bullet}^{\mathcal{U}})$ as cosimplicial diagrams of categories. By definition, then, we have functors

$$\text{laxlim Sh}(X_{\bullet}^{\mathcal{U}}) \rightarrow \text{lim Sh}(X_{\bullet}^{\mathcal{U}}) \rightarrow \text{holim Sh}(X_{\bullet}^{\mathcal{U}}) \rightarrow \text{Sh}(Y)$$

(where a *lax limit* is a limit where the cones only commute ‘up to higher morphisms’), and similarly for St . We can think of a “simplicial sheaf” as some sort of lax limit object of $\text{Sh}(X_{\bullet}^{\mathcal{U}})$, since the cones don’t necessarily commute, but there does exist a morphism in one direction. That is, they would be true limit objects if the maps $\mathcal{F}^{\bullet}(\alpha)$ were required to be *isomorphisms* (or even some sort of weak equivalences), as explained in (F.5.5). This means that we can think of “simplicial sheaves” as very ‘coarse’ models for discrete sheaves on Y .

(F.5.4) In (4.3.2), we say that a sheaf \mathcal{F}^{\bullet} on a simplicial space Y_{\bullet} is *strongly cartesian* if the maps

$$(Y_{\bullet}\alpha)^* \mathcal{F}^p \xrightarrow{\mathcal{F}^{\bullet}\alpha} \mathcal{F}^q$$

are all *isomorphisms*, for any $\alpha: [p] \rightarrow [q]$; and we say that a complex $\mathcal{F}^{\bullet, \star}$ of sheaves on Y_{\bullet} is *cartesian* if the maps $\mathcal{F}^{\bullet, \star}\alpha$ are *quasi-isomorphisms*.

Since the $\mathcal{F}^{\bullet}\alpha$ are identity maps whenever $\mathcal{F}^{\bullet} = (X_{\bullet}^{\mathcal{U}} \rightarrow X)^*F$ is the pullback (to the nerve) of a (global) bundle, we can think of the *strongly cartesian* condition as being a mild generalisation of this. The (weakly) cartesian condition, for complexes, is then the ‘up to homotopy’ version of this: it tells us that our complex will glue, up to homotopy, to give global objects. This is exactly one of the properties guaranteed by Green in his construction of a resolution of coherent sheaves — see (9.1.4).

Thinking back to (F.5.3), we see that the cartesian condition corresponds to the sheaf being a *true* limit object of $\text{Sh}(X_{\bullet}^{\mathcal{U}})$ instead of simply a *lax* one.

(F.5.5) The construction of sheaves on a simplicial space as lax limit objects (or of cartesian sheaves as limit objects) can be formalised using [Ber12, Definition 3.1], setting $F_{\alpha, \beta}^{\theta} = (X_{\bullet}^{\mathcal{U}}\theta)^*$ and $u_{\alpha, \beta}^{\theta} = \mathcal{F}^{\bullet}(\theta)$.

(F.5.6) We can move into the world of complex geometry (or, indeed, a bunch of other worlds) by replacing simplicial sets in all the above with any other model category (e.g. that of complexes of \mathcal{O}_X -modules), since there is still some sort of equivalence between Y and $|X_{\bullet}|$ when Y is a (maybe paracompact) complex manifold and X_{\bullet} is the nerve of some (Stein, maybe locally-finite) open cover.

(F.5.7) As briefly mentioned in (G.5), we can think of sheaves on simplicial spaces as cosimplicial diagrams: let \mathcal{F}^{\bullet} be some sheaf on $X_{\bullet}^{\mathcal{U}}$, and consider the diagram

$$\Pi_{\alpha} \mathcal{F}(U_{\alpha}) \rightrightarrows \Pi_{\alpha, \beta} \mathcal{F}(U_{\alpha\beta}) \rightrightarrows \Pi_{\alpha, \beta, \gamma} \mathcal{F}(U_{\alpha\beta\gamma}) \rightrightarrows \dots$$

If we are in a nice enough setting (here the keywords are those like ‘softness’), then we can apply \mathfrak{L} , as defined in (G.5.1), to get a cosimplicial diagram of dg-categories which is somehow equivalent to the diagram above, but much simpler to work with.

F.6 The simplicial gluing condition

(F.6.1) For what follows, we don’t really give many references, mostly because we don’t really explain anything in much detail anyway: this section is really just a soliloquy on the subject of simplices, full of vague allusions to the truth. All of the definitions and theorems below can be found, for example, on the nLab, or your other favourite reference for such things.

(F.6.2) We now think a bit about the condition for a form to be simplicial, as stated in (4.3.6). As mentioned there, it is a rather natural condition when we consider how the *fat geometric realisation* $\|Y_\bullet\|$ of a simplicial space Y_\bullet can be defined as a quotient:

$$\|Y_\bullet\| = \left(\coprod_p Y_p \times \Delta_{\text{Top}}^p \right) / \sim_+$$

where \sim_+ is the equivalence relation given by

$$\begin{aligned} ((y, f_*t) \in Y_q \times \Delta_{\text{Top}}^q) \sim_+ ((f^*y, t) \in Y_p \times \Delta_{\text{Top}}^p) \\ \iff \\ f: [p] \rightarrow [q] \text{ is a composition of coface maps.} \end{aligned}$$

(Note that it can also be defined as the coend

$$\|Y_\bullet\| = \int^{[p] \in \Delta_+} Y_p \times \Delta_{\text{Top}}^p$$

where $\Delta_+ \subset \Delta$ is the subcategory of the simplex category that has only coface (i.e. strictly increasing) maps).

An important theorem about the fat^[1] geometric realisation is that *it computes the homotopy colimit*:

$$\|Y_\bullet\| \simeq \text{hocolim}_{[p] \in \Delta} Y_p.$$

[1]There is a similar story about the *ordinary* geometric realisation, which is defined via the same coend, but over $[p] \in \Delta$ instead of $[p] \in \Delta_+$, but we don’t care too much about it here, because for *this* to compute the homotopy colimit, we need to place extra conditions on Y_\bullet .

A very hand-wavy reason why this is true is because the Δ_{Top}^p are all *contractible*, and thus somehow trivial in the context of topological spaces. This can be formalised by the idea of a *framing in a model category*, but this is something that the especially-interested reader can find more information about for themselves.

Now let's look at the algebraic analogue/dual of this story. Take some simplicial space Y_\bullet and consider calculating its homology with coefficients in some field k . This can be expressed as a homotopy limit in the category of complexes of k -modules:

$$H^\star(Y_\bullet, k) = \text{holim}(\text{Hom}^\star(Y_\bullet, k)).$$

If we wish to remember the algebra structure present, however, then we might instead wish to compute this homotopy limit in the category of commutative dg- k -algebras. If so, then we can use the de Rham complex $\Omega_{\Delta_{\text{Top}}}^\star$ to help us: the homotopy limit can be described as a *subobject* of $\text{Hom}(Y_\bullet, k) \otimes_k \Omega_{\Delta_{\text{Top}}}^\star$. Here, the de Rham complex of any topological simplex is simply equivalent to k itself, and so the simplices again play the role of something trivial, and once more we can formalise this with the idea of framings.

So, the (topological) simplices play an equivalent/dual role in both sides of the story:

Algebraic	Topological
$\Omega_{\Delta_\bullet}^\star \simeq k[0]$	$\Delta^\bullet \simeq \{*\}$
$\text{holim } Y_\bullet \hookrightarrow \text{Hom}^\star(Y_\bullet, k) \otimes \Omega_{\Delta_\bullet}^\star$	$Y_\bullet \times \Delta^\bullet \twoheadrightarrow \text{hocolim } Y_\bullet$

In both cases, we have some simplicial-cosimplicial object that takes values in either commutative dg- k -algebras or topological spaces.

(F.6.3) The tensor-hom adjunction in the category of sets is sometimes referred to (especially by computer scientists and programmers) as *currying*. This adjunction the statement that maps (of sets) $A \times B \rightarrow C$ are in bijections with maps $A \rightarrow \{\text{maps } B \rightarrow C\}$. Given some $f: A \times B \rightarrow C$, we define the *curried function* $g_a: B \rightarrow C$ by $g_a(b) = f(a, b)$. With this in mind, a simplicial differential form, which is form (satisfying certain properties) on $X_\bullet \times \Delta^\bullet$, and thus in particular a map from $X_\bullet \times \Delta^\bullet$ to the cotangent bundle, can be thought of as a map from X_\bullet to a ‘simplicially-valued’ cotangent bundle. There are some subtleties here: in particular, we forget that forms are really *sections*, and not just arbitrary maps, but the vague idea still holds a modicum of merit.

Twisting cochains and twisted complexes

G

(G.0.1) Purpose. A ‘fun’ exercise is the following: pick $a \in \{\text{twisting, twisted}\}$ and $b \in \{\text{cochain, complex}\}$, and look up the definition of $(a\ b)$ in your favourite book, website, or other mathematical reference source. Compare your results with the person next to you, and see by how much they differ.

This appendix hopes to serve as a venue for a ‘family reunion’ — a chance for all the different definitions that have spawned from one another to meet up and talk about themselves, as well as seeing how they might still resemble one another. In particular, I could not find any (even mildly) rigorous explanation as to how the twisted complexes of [BK91] have anything to do with the twisting cochains of [TT78], and it turns out that my initial idea of how they might be equivalent was wrong, so the details can be found in (G.6.4). There are also some proofs (such as (G.2.16)) that seem to be omitted from the literature^[1] that I’ve tried to not skim over here. There is a clear focus on some specific points of view and not others, but hopefully this chapter can still serve as an introductory reading guide to the various facets of the subject, at least.

G.1 Historical overview

(G.1.1) Twisting cochains and twisted complexes arise naturally in many different settings. They seem to become a very useful tool when studying complex-analytic manifolds, since they can be used as a substitute for the global resolutions of coherent sheaves that we have in the algebraic setting. Their history is not just a story about complex algebraic geometry, but it is this aspect upon which we really place emphasis in this short exposition.

In particular, we don’t really mention the ‘first’ reference to twisted cochains: [BJ59]; we also don’t follow what happened to the subject when it branched off into differential homological algebra (namely [Moo70]), even though this also predates all of the material that we *do* cover. For a lovely summary of the subject from a differential and lie-algebraic viewpoint, we refer the reader to [Sta09]. Our focus is really split into two parts: firstly (and mainly), the development of twisting cochains by Toledo and Tong, and O’Brian, in [TT76; TT78; OTT81] using Čech cohomology and other such explicit methods; secondly (in both order and importance (to this thesis)), the development of twisted complexes (from [BK91]) and the application of the language of dg-categories in [BHW15; Wei16; Wei19].

(G.1.2) The purpose of this whole appendix is purely to *motivate* the definition of twisting cochains, and so we give a handful of different definitions. Note that the

^[1]Often because they are not technically difficult, but still ...

definition that we finally use (in Chapter 8) is subtly different from any described here.

(G.1.3) We try to be consistent in the use of ‘twisted’ and ‘twisting’, using the former to describe complexes and the latter to describe cochains. There seems to be, however, no such consistent usage in the literature, so this attempt is almost entirely pointless.

(G.1.4) To quote (8.1.2), a good idea to keep in mind is that ‘a twisting cochain is like a complex of coherent sheaves, but not quite as rigidly glued’. That is, rather than having transition maps on overlaps, we have maps that only satisfy the cocycle condition ‘up to homotopy’, and these homotopies only commute ‘up to higher homotopies’, and so on. As we explain in (G.2.12), a complex of coherent sheaves is ‘like’ a truncated twisting cochain, with all homotopies being zero in degrees greater than one.

The other nice thing about twisting cochains is that they let us formalise what we have been doing throughout most of this thesis: thinking of all data above some $U_{\alpha_0 \dots \alpha_p}$ as lying above U_{α_0} , so that we can compare everything on an equal footing.

G.2 Holomorphic twisting cochains from local resolutions

(G.2.1) In [TT78, §§1, 2], they start with arbitrary graded \mathbb{C} -vector spaces and obtain vector bundles by tensoring with \mathcal{O}_X . We, however, start straight away with \mathcal{O}_{U_α} -modules. Apart from that, and a few other subtle differences, though, this section is mostly just a summary of the above paper.

(G.2.2) Let X be some paracompact complex-analytic manifold with a ‘sufficiently nice’^[1] open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. Let $V = \{V_\alpha^\bullet\}$ be a collection of bounded-graded \mathcal{O}_{U_α} -modules. That is,

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q$$

with $V_\alpha^q = 0$ for all but finitely many $q \in \mathbb{N}$. (This is basically asking for a bounded complex of \mathcal{O}_{U_α} -modules over each U_α , but without any differential.)

^[1]In particular, we assume \mathcal{U} to be Stein, locally compact, and, if we are working with some locally free sheaf \mathcal{F} of rank r , such that $\mathcal{F}|_{U_\alpha} \cong \mathcal{O}_X^r$ (i.e. a trivialising cover).

(G.2.3) We define a Čech-style complex^[1] $\hat{C}^\bullet(\mathcal{U}, V^\bullet)$ by letting an element of $\hat{C}^p(\mathcal{U}, V^q)$ be a Čech p -cochain c whose value $c_{\alpha_0 \dots \alpha_p}$ on the p -simplex $U_{\alpha_0 \dots \alpha_p}$ lies in $V_{\alpha_0}^q \mid \alpha_0 \dots \alpha_p$. The Čech differential on this complex is actually a *deleted* Čech differential (written $\hat{\delta}$ instead of the usual $\check{\delta}$) in that it is given by the usual alternating sum of the cochain on $\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_p$ (where the hat denotes omission) *but* starting from $i = 1$, *not* $i = 0$. That is,

$$\begin{aligned} \hat{\delta}: \hat{C}^p(\mathcal{U}, V^q) &\rightarrow \hat{C}^{p+1}(\mathcal{U}, V^q) \\ (\hat{\delta}c)_{\alpha_0 \dots \alpha_{p+1}} &= \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_{p+1}} \end{aligned}$$



This is a reasonably natural definition because we have defined our cochains to take values in $V_{\alpha_0}^q$, and something of the form $c_{\alpha_1 \dots \alpha_{p+1}}$ does *not* live here — it lives in $V_{\alpha_1}^q$. (We will soon see *why* we have this seemingly unnatural construction that depends so heavily on the first coordinate of any given p -simplex.)

Note that, since V is just a collection of graded modules, there is not (yet) a differential $d: V^q \rightarrow V^{q+1}$, and so we don't (yet) have a bicomplex.

(G.2.4) We don't assume that our Čech cochains are skew-symmetric (i.e. that exchanging two indices changes the sign), but this is just a matter of convention. For more details, see (6.2.6).

(G.2.5) The complex $\hat{C}^\bullet(\mathcal{U}, V^\bullet)$ has a natural multiplication structure given by

$$(c^p \cdot d^q)_{\alpha_0 \dots \alpha_{p+q}} = (-1)^q c_{\alpha_0 \dots \alpha_p}^p d_{\alpha_p \dots \alpha_{p+q}}^q$$

which plays nicely with all the differentials involved.^[2] In fact, it often also has an algebra structure, but this isn't too important for us here. See [TT78, §1] for details. We often omit the \cdot and just write cd .

(G.2.6) For each $s \in \mathbb{N}$, we define the collection $\text{End}^q(V)$ of *degree- q endomorphisms* to be given over each $U_{\alpha_0 \dots \alpha_p}$ by

$$\text{End}^q(V) \mid U_{\alpha_0 \dots \alpha_p} = \bigoplus_{i \in \mathbb{Z}} \text{Hom}(V_{\alpha_p}^i \mid U_{\alpha_0 \dots \alpha_p}, V_{\alpha_0}^{i+q} \mid U_{\alpha_0 \dots \alpha_p}).$$

Note that the morphisms are of degree q from $V_{\alpha_p}^\bullet$ to $V_{\alpha_0}^\bullet$, and *not* from $V_{\alpha_0}^\bullet$ to $V_{\alpha_0}^\bullet$. The motivation for this is explained further in (G.2.8).

^[1]Recall (3.0.3). In particular, we might switch from writing V^\bullet to writing V^* whenever we have multiple gradings floating about.

^[2]At the moment there is only the deleted-Čech differential, but later on we consider the case where V has differentials as well.

Again, since V is just a collection of graded modules and thus has no differential, we need not worry about whether the morphisms here are true morphisms of complexes or not (that is, whether or not they commute with the differentials), because V doesn't have the structure of a complex.^[1]



(G.2.7) We get another Čech-style complex $\hat{C}^\bullet(\mathcal{U}, \text{End}^\bullet(V))$ by simply replacing V with $\text{End}(V)$ in (G.2.3), and modifying the deleted-Čech differential $\hat{\delta}$ so that it also omits the *last* term:

$$(\hat{\delta}c)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=1}^p c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}.$$

Here we delete both the first and the last term because we want all of the $c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}$ to live in the same space, i.e. be maps from $V_{\alpha_{p+1}} \big|_{U_{\alpha_0 \dots \alpha_{p+1}}} \rightarrow V_{\alpha_0} \big|_{U_{\alpha_0 \dots \alpha_{p+1}}}$.

Note that we can still define multiplication as in (G.2.5), but where the sign respects the bidegree:

$$(c^{p,q} \cdot d^{\bar{p},\bar{q}})_{\alpha_0 \dots \alpha_{p+\bar{p}}} = (-1)^{q\bar{p}} c_{\alpha_0 \dots \alpha_p}^{p,q} d_{\alpha_p \dots \alpha_{p+\bar{p}}}^{\bar{p},\bar{q}}$$

Again, if V had differentials then this would give us the structure of a bicomplex.

(G.2.8) A holomorphic vector bundle E on X can be described exactly by its transition maps $\{g_{\alpha\beta} \in \text{GL}(n, \mathbb{C})\}_{\alpha,\beta}$, thought of as describing the change in trivialisation from over U_β to over U_α , which satisfy two conditions: the *cocycle condition* ($g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$) and the *invertibility condition* ($g_{\alpha\alpha} = \text{id}$). Already, then, the fact that the transition maps give us a map from (the trivialisation of) $E \big|_{U_\beta}$ to (the trivialisation of) $E \big|_{U_\alpha}$ (over the intersection $U_{\alpha\beta}$) half justifies the choice in (G.2.6) to make elements of $\text{End}(V)$ be maps from V_{α_p} to V_{α_0} (both over the intersection $U_{\alpha_0 \dots \alpha_p}$).

So let's try to define something in $\hat{C}^\bullet(\mathcal{U}, \text{End}^\bullet(V))$ that looks like it could describe a vector bundle.

To begin with, forget the grading on V , and consider a true Čech cochain

$$\mathfrak{a} = \{\mathfrak{a}_{\alpha\beta}\}_{\alpha,\beta \in I} \in \check{C}^1(\mathcal{U}, \text{End}(V))$$

satisfying the cocycle condition ($\mathfrak{a}_{\alpha\beta} \cdot \mathfrak{a}_{\beta\gamma} = \mathfrak{a}_{\alpha\gamma}$) and the invertibility condition ($\mathfrak{a}_{\alpha\alpha} = \text{id}$). But, here, the cocycle condition is equivalent to

$$\hat{\delta}\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a} = 0,$$

^[1]Although when V *does* have differentials, as in Green's construction, we will see that we *do* want these morphisms to be true morphisms of complexes, i.e. morphisms in the homotopy category of the dg-category.

because^[1] $(\hat{\delta}\mathfrak{a})_{\alpha\beta\gamma} = -\mathfrak{a}_{\alpha\gamma}$, and $(\mathfrak{a} \cdot \mathfrak{a})_{\alpha\beta\gamma} = \mathfrak{a}_{\alpha\beta} \cdot \mathfrak{a}_{\beta\gamma}$.

Since we forgot the grading on V , we might as well think of \mathfrak{a} as being of degree zero, and so the idea for a general definition seems readily available: we let \mathfrak{a} be of arbitrary degree and ask that $\hat{\delta}\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a} = 0$. This is exactly how we can come across the definition of a *twisting cochain*.^[2]

(G.2.9) A (*holomorphic*) *twisting cochain over V* is a formal sum^[3]

$$\mathfrak{a} = \sum_{k \in \mathbb{N}} \mathfrak{a}^{k, 1-k}$$

where $\mathfrak{a}^{k, 1-k} \in \hat{\mathcal{C}}^k(\mathcal{U}, \text{End}^{1-k}(V))$ such that

$$\begin{aligned} \hat{\delta}\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a} &= 0 \\ \mathfrak{a}_{\alpha\alpha}^{1,0} &= \text{id}. \end{aligned}$$

In particular then, it *could* be the case that all but finitely-many $\mathfrak{a}^{k, 1-k}$ are zero, but *not* $\mathfrak{a}^{1,0}$, since we need at least $\mathfrak{a}_{\alpha\alpha}^{1,0}$ to be the identity.^[4]

Note that multiplication is not simply done component-wise, but instead by taking all possible combinations. That is, for two formal sums \mathfrak{a} and \mathfrak{b} ,

$$(\mathfrak{a} \cdot \mathfrak{b})^{p,s} = \sum_{\substack{q+q'=p \\ t+t'=s}} \mathfrak{a}^{q,t} \cdot \mathfrak{b}^{q',t'}.$$

Once more, if V had differentials, then \mathfrak{a} would be an element of total degree 1. We return to this fact in (G.2.16)

(G.2.10) At the moment we say nothing about the *existence* of twisting cochains for a given V , but the hope is that, under reasonably mild conditions, one will always exist (and this is indeed the case — see (G.2.15)).

(G.2.11) If we write out explicitly what $\hat{\delta}\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a} = 0$ means at each Čech level k then we can gain some insight into what twisting cochains really look like.

$k = 0 \rightsquigarrow \mathfrak{a}_{\alpha}^{0,1} \mathfrak{a}_{\alpha}^{0,1} = 0$, which says exactly that $\mathfrak{a}_{\alpha}^{0,1}$ gives us a *differential* on V_{α}^{\bullet} , letting us (finally) think of it as a complex.

^[1]Not forgetting (G.2.7): the deleted-Čech differential for $\hat{\mathcal{C}}^{\bullet}(\mathcal{U}, \text{End}^{\bullet}(V))$ omits both the first and last terms of the sum.

^[2]This equation looks (exactly) like the Maurer-Cartan equation, which we discuss later.

^[3]We are rather lax in differentiating between $+$ and \oplus at times, but this hopefully won't cause any problems.

^[4]Unless some V_{α}^{\bullet} are zero, of course.

$k = 1 \rightsquigarrow \mathfrak{a}_\alpha^{0,1} \mathfrak{a}_{\alpha\beta}^{1,0} = \mathfrak{a}_{\alpha\beta}^{1,0} \mathfrak{a}_\beta^{0,1}$, which tells us that, over $U_{\alpha\beta}$, we have a *chain map of complexes*

$$\mathfrak{a}_{\alpha\beta}^{1,0}: (V_\beta^\bullet | U_{\alpha\beta}, \mathfrak{a}_\beta^{0,1}) \rightarrow (V_\alpha^\bullet | U_{\alpha\beta}, \mathfrak{a}_\alpha^{0,1}).$$

$k = 2 \rightsquigarrow -\mathfrak{a}_{\alpha\gamma}^{1,0} + \mathfrak{a}_{\alpha\beta}^{1,0} \mathfrak{a}_{\beta\gamma}^{1,0} = \mathfrak{a}_\alpha^{0,1} \mathfrak{a}_{\alpha\beta\gamma}^{2,-1} + \mathfrak{a}_{\alpha\beta\gamma}^{2,-1} \mathfrak{a}_\gamma^{0,1}$, which says that $\mathfrak{a}_{\alpha\gamma}^{1,0}$ and $\mathfrak{a}_{\alpha\beta}^{1,0} \mathfrak{a}_{\beta\gamma}^{1,0}$ are *chain homotopic* via the homotopy $\mathfrak{a}_{\alpha\beta\gamma}^{2,-1}$. Taking the degenerate simplices $\alpha\beta\alpha$ and $\beta\alpha\beta$, this tells us, in particular, that $\mathfrak{a}_{\alpha\beta}^{1,0}$ and $\mathfrak{a}_{\beta\alpha}^{1,0}$ are chain homotopic inverses, i.e. that the chain map $\mathfrak{a}_{\alpha\beta}^{1,0}$ is a *quasi-isomorphism*.

$k \geq 3 \rightsquigarrow$ some sort of ‘higher-order homotopic gluings’, whatever this might mean, formally (which we explain in (8.3.4)).

(G.2.12) We started with some collection of graded \mathcal{O}_{U_α} -modules over each U_α , with absolutely no compatibility conditions on intersections. If we can find a twisting cochain, however, then this lets us (i) define differentials that are compatible with the grading, and (ii) define quasi-isomorphisms between the (now) complexes (as well as (iii) define higher-order compatibility conditions).

To formalise this, a twisting cochain \mathfrak{a} on V lets us glue the homology of each complex to get a *complex $H^\bullet(\mathfrak{a})$ of coherent^[1] sheaves* in the following way:

- let $H_\alpha^\bullet(\mathfrak{a}) = H^\bullet(V_\alpha^\bullet)$ be the complex of \mathcal{O}_{U_α} -modules given by taking the homology of the complex $(V_\alpha^\bullet, \mathfrak{a}_\alpha^{0,1})$;
- over each intersection $U_{\alpha\beta}$ we have an isomorphism

$$H(\mathfrak{a}_{\alpha\beta}^{1,0}): H_\beta^\bullet(\mathfrak{a}) | U_{\alpha\beta} \xrightarrow{\sim} H_\alpha^\bullet(\mathfrak{a}) | U_{\alpha\beta}$$

induced by the quasi-isomorphism $\mathfrak{a}_{\alpha\beta}^{1,0}$;

- over each intersection $U_{\alpha\beta\gamma}$ we have that $H(\mathfrak{a}_{\alpha\gamma}^{1,0}) = H(\mathfrak{a}_{\alpha\beta}^{1,0})H(\mathfrak{a}_{\beta\gamma}^{1,0})$;
- some higher-order stuff, which we don’t actually need, since we already have *strict* compositional equality of the $H(\mathfrak{a}_{\alpha\beta}^{1,0})$.

Since $H^\bullet(\mathfrak{a})$ forgets all the higher homotopy data (i.e. it uses only $\mathfrak{a}^{0,1} + \mathfrak{a}^{1,0}$), it is a bit too ‘blunt’ for the purposes of this thesis, but we mention it here anyway for completeness.

(G.2.13) If the pair (V, \mathfrak{a}) is such that $H^i(\mathfrak{a}) = 0$ for $i > 0$ then we can think of it as being a *local* resolution of the coherent sheaf $\mathcal{F} = H^0(\mathfrak{a})$ by locally free sheaves (as well as being a (*trivial*) *global* resolution by coherent sheaves).

^[1]The fact that the gluing gives us something coherent is almost immediate by definition, because the $(V_\alpha^\bullet, \mathfrak{a}_\alpha^{0,1})$ give us something *perfect*.

(G.2.14) Example. In (8.2.11) we describe an explicit twisting cochain, and so we can ask what exactly $H^\bullet(\mathfrak{a})$ is. Well, we know what the homology of each ξ_α^\bullet is, by definition, because they are local resolutions: the homology is zero, except for in degree zero, where it is isomorphic to $\mathcal{F} \upharpoonright U_\alpha$. That is, the complex $H^\bullet(\mathfrak{a})$ of coherent sheaves is exactly the trivial resolution of \mathcal{F} , since

$$H^i(\mathfrak{a}) \cong \begin{cases} \mathcal{F} & i = 0; \\ 0 & \text{otherwise.} \end{cases}$$

This is exactly what we should expect: we picked our V to be homologically trivial (i.e. a collection of (local) *resolutions*), so, by definition, $H^\bullet(\mathfrak{a})$ will be trivial. Why bother with this example then? The reason is twofold:

1. this gives us an idea of how it should be possible to construct a twisting cochain when starting with a collection of complexes of vector bundles (formalised in (G.2.15)), rather than starting with a collection of graded vector bundles and then using the degree-zero part of a twisting cochain to define differentials; and
2. this specific twisting cochain is the basis of the example studied in [Gre80], so it's nice to at least mention it.

(G.2.15) Lemma. Let \mathcal{F} be a coherent sheaf on X , with the same hypotheses on X as in (G.2.2). If we have local resolutions ξ_α^\bullet of $\mathcal{F} \upharpoonright U_\alpha$ by locally free sheaves over each U_α then we can construct a twisting cochain (V, \mathfrak{a}) , where $V = \{\xi_\alpha\}$, such that $\mathfrak{a}_\alpha^{0,1} = d_{\xi_\alpha^\bullet}$.

Proof. This is shown in [OTT81, pp. 229–231]. There is also [TT76, Lemma 8.13], which relies on a different treatment of twisting cochains (which we discuss in (G.3)), but is ‘translated’ in [TT78, Proposition 2.4]. \square

(G.2.16) It is routine to show that, for any $\mathfrak{a} \in \text{Tot}^1 \hat{C}^\bullet(\mathcal{U}, \text{End}^\bullet(V))$, the map

$$\begin{aligned} D_\mathfrak{a}: \text{Tot}^r \hat{C}^\bullet(\mathcal{U}, V^\bullet) &\rightarrow \text{Tot}^{r+1} \hat{C}^\bullet(\mathcal{U}, V^\bullet) \\ c &\mapsto \hat{\delta}c + \mathfrak{a} \cdot c \end{aligned}$$

defines a differential (i.e. squares to zero) if and only if \mathfrak{a} is a twisting cochain.

In the hope that this thesis has at least some use, we give the proof, since it doesn't seem to be written down anywhere in the literature, even though it follows immediately from the properties of $\hat{C}^\bullet(\mathcal{U}, \text{End}^\bullet(V))$ and $\hat{C}^\bullet(\mathcal{U}, V^\bullet)$.

Proof. Recall ([TT78, Equation (1.8)]) that $\hat{\delta}$ is a derivation with respect to all products. Then consider the square of the map:

$$D_\mathfrak{a}^2(c) = \hat{\delta}(\hat{\delta}c + \mathfrak{a} \cdot c) + \mathfrak{a} \cdot (\hat{\delta}c + \mathfrak{a} \cdot c)$$

$$\begin{aligned}
 &= \hat{\delta}(\mathfrak{a} \cdot c) + \mathfrak{a} \cdot (\hat{\delta}c) + \mathfrak{a} \cdot \mathfrak{a} \cdot c \\
 &= (\hat{\delta}\mathfrak{a}) \cdot c + (-1)^{|\mathfrak{a}|} \mathfrak{a} \cdot (\hat{\delta}c) + \mathfrak{a} \cdot (\hat{\delta}c) + (\mathfrak{a} \cdot \mathfrak{a}) \cdot c \\
 &= (\hat{\delta}\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a}) \cdot c - \mathfrak{a} \cdot (\hat{\delta}c) + \mathfrak{a} \cdot (\hat{\delta}c) \\
 &= (\hat{\delta}\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a}) \cdot c.
 \end{aligned}$$

This is equal to zero for all c if and only if $(\hat{\delta}\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a}) = 0$, but this is exactly the condition for \mathfrak{a} to be a twisting cochain. \square

In some sense, writing $D_{\mathfrak{a}} = \hat{\delta} + \mathfrak{a}$, we can think of this map as a first-order (since \mathfrak{a} is of degree one) perturbation of the Čech differential, and this is the point of view that we emphasise in (G.3). We can first, however, motivate this definition by giving the two choice examples from [TT78], and so we do so.

(G.2.17) Returning to the example in (G.2.8), where V is just a vector bundle (i.e. ungraded) and $\mathfrak{a}^{1,0}$ is given by the transition maps, we have $\mathfrak{a} = \mathfrak{a}^{0,1} + \mathfrak{a}^{1,0}$, where $\mathfrak{a}_{\alpha}^{0,1} = \text{id}_{V_{\alpha}}$. Then

$$(D_{\mathfrak{a}}c)_{\alpha_0 \dots \alpha_{p+1}} = \mathfrak{a}_{\alpha_0 \alpha_1}^{1,0} c_{\alpha_1 \dots \alpha_{p+1}} + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_{p+1}}$$

for $c \in \hat{C}^p(\mathcal{U}, V)$. Recall that $c_{\alpha_0 \dots \alpha_p} \in V_{\alpha_0} \mid U_{\alpha_0 \dots \alpha_p}$, and so we had to modify the Čech differential by deleting the first term, because $c_{\alpha_1 \dots \alpha_p}$ lives in a fundamentally different space than any $c_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_{p+1}}$ for $i \neq 0$. But here we have found a way of fixing this problem: we can use the ‘quasi-isomorphism’^[1] $\mathfrak{a}_{\alpha_0 \alpha_1}^{1,0}$ to think of $c_{\alpha_1 \dots \alpha_p}$ as living in $V_{\alpha_0} \mid U_{\alpha_0 \dots \alpha_p}$ instead of $V_{\alpha_1} \mid U_{\alpha_0 \dots \alpha_p}$. This means that $D_{\mathfrak{a}}$ can be thought of as really being the *true* Čech differential (and this can be made formal by a spectral sequence argument, as in [TT78, Theorem 2.9]).

(G.2.18) Now consider a slightly richer example, where V is a *complex* of vector bundles, and $\mathfrak{a} = \mathfrak{a}^{0,1} + \mathfrak{a}^{1,0}$ is given by the data of the differentials $d_{V_{\alpha}} = \mathfrak{a}_{\alpha}^{0,1}$ and the transition maps $\mathfrak{a}_{\alpha\beta}^{1,0}$. Then

$$\begin{aligned}
 (D_{\mathfrak{a}}c)_{\alpha_0 \dots \alpha_{p+1}} &= \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_{p+1}} + (\mathfrak{a}^{0,1} \cdot c + \mathfrak{a}^{1,0} \cdot c)_{\alpha_0 \dots \alpha_{p+1}} \\
 &= \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \dots \widehat{\alpha}_i \dots \alpha_{p+1}} + (-1)^p \mathfrak{a}_{\alpha_0}^{0,1} c_{\alpha_0 \dots \alpha_{p+1}} + \mathfrak{a}_{\alpha_0 \alpha_1}^{1,0} c_{\alpha_1 \dots \alpha_{p+1}}
 \end{aligned}$$

^[1]We use scare quotes because there is no structure of a complex on V , and so here ‘quasi-isomorphism’ just means ‘invertible, with inverse $\mathfrak{a}_{\alpha_1 \alpha_0}^{1,0}$ ’.

for $c \in \hat{C}^p(\mathcal{U}, V^s)$. The first and last term combine, as in (G.2.17), to give us the usual Čech differential; the middle term is exactly $(-1)^p d_V$ (recalling the sign from (G.2.5)). That is, we recover exactly the usual total differential on the usual Čech bicomplex: $\check{\delta} + (-1)^p d_V$.

G.3 Holomorphic twisting cochains from total differentials

(G.3.1) Twisting cochains were defined in [TT76] for more general bigraded algebras, and the idea was to *start* with some total differential and try to modify it by some degree-1 perturbation, and ask for control over the ‘curvature’. This is the construction that we now explain.

(G.3.2) Let $A = \bigoplus_{i,j \in \mathbb{Z}} A^{i,j}$ be a bigraded \mathbb{C} -algebra, and $M = \bigoplus_{i,j \in \mathbb{Z}} M^{i,j}$ a bigraded A -module (hence also a bigraded \mathbb{C} -module), such that both $A^{i,j}$ and $M^{i,j}$ are zero for all $i < 0$, and non-zero for only finitely-many j . Assume also that M is a *faithful*^[1] A -module: if $am = 0$ for all m then $a = 0$.

Now suppose that we have two \mathbb{C} -linear maps

$$\begin{aligned} D &= D^{1,0} + D^{0,1}: A^{i,j} \rightarrow A^{i+1,j+1} \\ \nabla &= \nabla^{1,0} + \nabla^{0,1}: M^{i,j} \rightarrow M^{i+1,j+1} \end{aligned}$$

(both of bidegree $(1,1)$, and both decomposed into their $(1,0)$ and $(0,1)$ parts, respectively) that satisfy the following conditions:

(i) they are both derivations with respect to D , i.e.

$$\begin{aligned} D(a \cdot b) &= (Da) \cdot b + (-1)^{|a|} a \cdot (Db) \\ \nabla(a \cdot m) &= (Da) \cdot m + (-1)^{|a|} a \cdot (\nabla m) \end{aligned}$$

where $|a| = i + j$ for $a \in A^{i,j}$;

(ii) both $D^{0,1}$ and $\nabla^{0,1}$ are differentials, i.e. $(D^{0,1})^2 = (\nabla^{0,1})^2 = 0$;

(iii) there exists some $\kappa \in A^{2,0}$ such that $(\nabla^{1,0})^2 = \kappa$, i.e. $(\nabla^{1,0})^2(m) = \kappa \cdot m$ for all $m \in M$;

(iv) the $(1,0)$ and $(0,1)$ parts anticommute, i.e.

$$\begin{aligned} D^{1,0}D^{0,1} &= -D^{0,1}D^{1,0} \\ \nabla^{1,0}\nabla^{0,1} &= -\nabla^{0,1}\nabla^{1,0}. \end{aligned}$$

^[1]This is equivalent to asking that we can identify A with a subalgebra of $\text{End}_{\mathbb{C}}(M)$.

(G.3.3) There are a few examples of such data in [TT76], and so we won't really give any examples in this section, instead referring the interested reader to this source. We point out, however, that this seemingly long list of hypotheses is not grossly difficult to satisfy.

The choice of notation is meant to make us think of D as something that we had hoped was a total differential, but isn't quite so, and ∇ as some sort of connection on a bundle, with curvature κ . We will further justify this analogy as we go.

(G.3.4) With the above assumptions, we have the following relations:

- (i) $\nabla^2 m = \kappa m$ for all $m \in M$;
- (ii) $D^2 a = \kappa a - a\kappa$ for all $a \in A$;
- (iii) $D\kappa = 0$.

Proof. See [TT76, §8] (although all three are reasonably quick consequences of our assumptions). \square

To reinforce the analogy of (G.3.3), note that $\nabla^2 m = \kappa m$ looks like the curvature of a connection, and $D\kappa = 0$ looks like the (second) Bianchi identity for a connection on a principal bundle. The fact that $D^2 a = \kappa a - a\kappa$, however, tells us that D is in general *not* a differential, but only a differential 'up to homotopy' somehow.^[1]

(G.3.5) We know that neither D nor ∇ is a differential, but we are interested^[2] in modifying ∇ in order to get a differential on M . What do we mean by 'modifying'? Well, let's consider a map of the form

$$\nabla_{\mathfrak{a}}: m \mapsto \nabla m + \mathfrak{a} \cdot m$$

for some $\mathfrak{a} = \bigoplus_{i \geq 1} \mathfrak{a}^{i+1, -i} \in A$, where $\mathfrak{a}^{m, n} \in A^{m, n}$, and justify this choice afterwards.

First of all, in order for the degrees to all match up (since ∇ is of bidegree $(1, 1)$), we need $\mathfrak{a} \in A$ to be of total degree 1, and so we can write $\mathfrak{a} = \sum_{k \in \mathbb{N}} \mathfrak{a}^{k, 1-k}$. Now, if we want this to give a differential, then we need for its square to be zero, and a quick calculation tells us that

$$\begin{aligned} \nabla_{\mathfrak{a}} m &= \nabla^2 m + \mathfrak{a} \cdot \nabla m + \nabla(\mathfrak{a}m) + (\mathfrak{a} \cdot \mathfrak{a})m \\ &= \kappa m + (D\mathfrak{a}) \cdot m + (\mathfrak{a} \cdot \mathfrak{a})m \\ &= (D\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a} + \kappa) \cdot m. \end{aligned}$$

So, since M is faithful, we are interested in $\mathfrak{a} \in A$ such that $D\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a} + \kappa = 0$.

^[1]Inspired by the definition that morphisms of complexes f and g are *homotopic* if $f - g = dp - pd$.

^[2]Motivated by the examples in [TT76].

(G.3.6) Now to explain why we consider maps of the form $\nabla_{\mathfrak{a}}$ in the first place. The primary justification in the context of [TT76] is that such maps occur naturally in various situations, and so it's nice to abstractify it. The justification for us, here, is (once more) largely based on (as of yet) unfounded analogy: this looks like the local definition of a connection, in that it is a connection ∇ (of curvature κ) plus some operator of differential degree 0 and total ('algebraic') degree 1. In this light, we are looking exactly for the \mathfrak{a} such that $\nabla_{\mathfrak{a}}$ is *flat* (or, equivalently, such that the 'curvature' of \mathfrak{a} is $-\kappa$).

(G.3.7) A *twisting cochain* for (M, ∇) is an element $\mathfrak{a} = \bigoplus_{i \geq 1} \mathfrak{a}^{i+1, -i} \in A$ such that

$$D\mathfrak{a} + \mathfrak{a} \cdot \mathfrak{a} = -\kappa.$$

The *associated twisted complex* $(M, \nabla_{\mathfrak{a}})$ is the complex given by M with the differential $\nabla_{\mathfrak{a}}: m \mapsto \nabla m + \mathfrak{a} \cdot m$.

Note that $\mathfrak{a} = 0$ is a twisting cochain with curvature $\kappa = 0$, and with this choice we recover exactly our original bicomplex as the associated twisted complex.

(G.3.8) If A is $D^{(0,1)}$ -acyclic then twisting cochains can always be inductively constructed: see [TT76, Lemma 8.13].

G.4 Enrichment of perfect complexes

(G.4.1) Triangulated categories can be understood as some sort of zero truncation of particularly nice dg-categories (as we explain in more detail in (G.4.2)), and so we can ask whether or not we can enrich a specific triangulated category with some dg structure. In particular, we can ask whether or not the (derived) category of perfect complexes on a (sufficiently nice) scheme admits such an enrichment.

Formally, given some triangulated category \mathcal{T} , and a pair (\mathcal{P}, ϵ) , where \mathcal{P} is a stable^[1] dg-category, and $\epsilon: \text{Ho}\mathcal{P} \rightarrow \mathcal{T}$ is a functor of triangulated categories, we say that (\mathcal{P}, ϵ) is a *dg-enhancement* of \mathcal{T} if ϵ is an equivalence. For a more complete introduction, we recommend [CS19].

(G.4.2) We can explain the relationship between triangulated categories and other higher-categorical structures in a bit more detail. In particular, triangulated categories are the H^0 of stable dg-categories, which are semi-strictifications of A_{∞} -categories, which are models for linear stable $(\infty, 1)$ -categories. Then the *triangulated* structure on a triangulated category is essentially the structure that $\text{Ho}(C)$ inherits, for any stable $(\infty, 1)$ -category C , from the *stability* of C . This is summarised in Figure (G.4.2.1).

^[1]Also known as an *enhanced triangulated*, or *pre-triangulated*, dg-category.

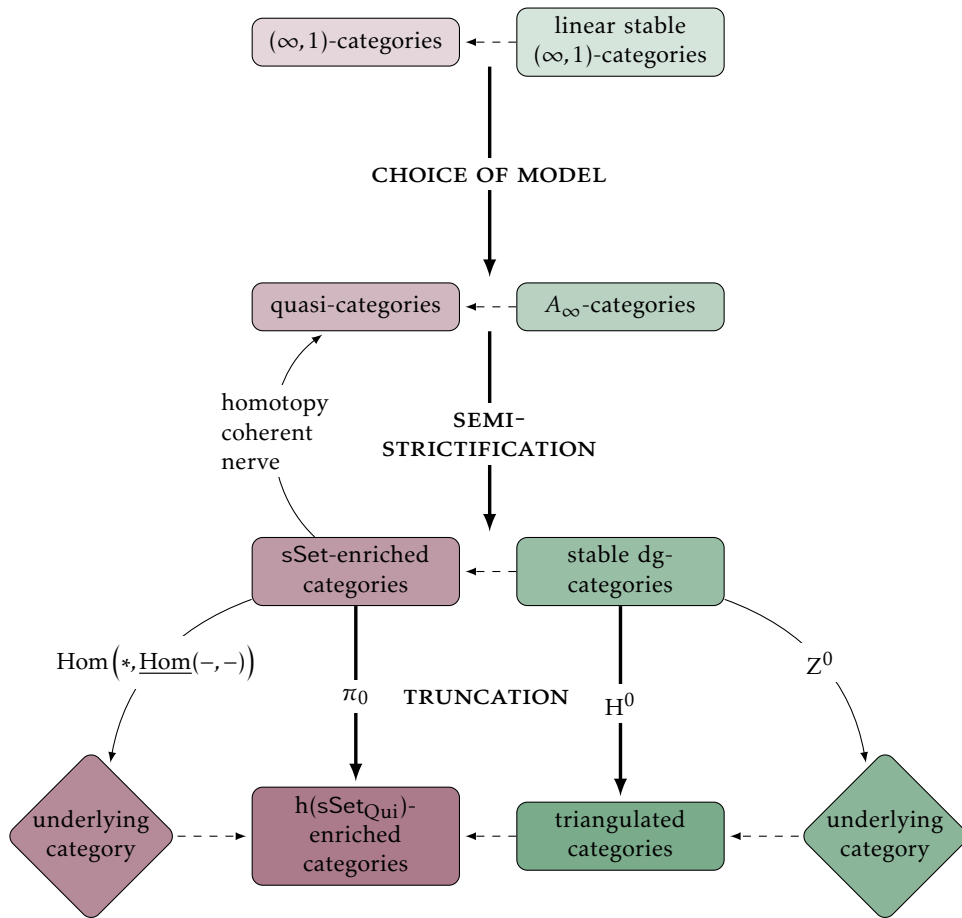


Figure (G.4.2.1). Relationships between various structures on higher categories, as explained in (G.4.2).

(G.4.3) As a familiar example of some of the ‘lower’ categorical structures depicted in Figure (G.4.2.1), consider the dg-category $\text{Ch}_{\text{dg}}(\mathcal{A})$ of chain complexes in an abelian category \mathcal{A} (that is, the category where the morphisms do *not necessarily* commute with the differentials). This is a stable dg-category. The ‘homotopy category’ $K(\mathcal{A})$ is then the triangulated category given by $H^0(\text{Ch}_{\text{dg}}(\mathcal{A}))$; and the ‘naive’ chain-complex category $\text{Ch}(\mathcal{A})$, whose morphisms are those that *must* commute with the differentials, is given by $Z^0(\text{Ch}_{\text{dg}}(\mathcal{A}))$. Then the ‘derived category’ $D(\mathcal{A})$ can be obtained either by inverting weak equivalences in $K(\mathcal{A})$, or by taking the (1-categorical) homotopy category of $\text{Ch}(\mathcal{A})$. So, since $\text{Ho} = H^0$ for any dg-category of chain complexes, we see that $\text{Ch}_{\text{dg}}(\mathcal{A})$ is a dg-enrichment of $K(\mathcal{A})$.

To avoid making this extended digression any longer, we suggest [Coh16] as a nice reference.

(G.4.4) This idea of stabilisation is one of the key motivations for the definition

of twisted complexes in dg-categories. We discuss this in (G.6), but just note here that this point of view, combined with this idea of enrichment of perfect complexes, lets us say that coherent sheaves really are ‘the’ stabilisation of vector bundles, and so topological (i.e. those of OTT) twisting cochains really *do* look like specific examples of the dg-category ones (of Bondal and Kapranov) which we will later see.

(G.4.5) We now summarise some of the work of Z. Wei on perfect twisting cochains.

One of the main results of [Wei16] is Theorem 3.32, which tells us that, given some conditions on a ringed space (X, \mathcal{O}_X) , there is an equivalence of categories

$$\mathrm{Ho} \mathrm{Tw}_{\mathrm{perf}}(X, \mathcal{O}_X, \mathcal{U}) \simeq \mathrm{D}_{\mathrm{perf}}(X),$$

where $\mathrm{Tw}_{\mathrm{perf}}(X, \mathcal{O}_X, \mathcal{U})$ is the category of twisted perfect complexes on $(X, \mathcal{O}_X, \mathcal{U})$, and $\mathrm{D}_{\mathrm{perf}}(X)$ is the triangulated subcategory of $\mathrm{D}(X)$ consisting of perfect complexes, where $\mathrm{D}(X) = \mathrm{D}(\mathrm{Mod}_{\mathcal{O}_X})$ is the derived category of \mathcal{O}_X -modules. In other words, $\mathrm{Tw}_{\mathrm{perf}}(X, \mathcal{O}_X, \mathcal{U})$ is, under certain conditions on X and \mathcal{O}_X , a dg-enhancement of $\mathrm{D}_{\mathrm{perf}}(\mathrm{Mod}_{\mathcal{O}_X})$.

The main theorem of [Wei18] is that there is a quasi-equivalence

$$\mathfrak{L}(X, \mathcal{O}_X) \simeq \mathrm{Tw}_{\mathrm{perf}}(X, \mathcal{O}_X, \mathcal{U}),$$

where $\mathfrak{L}(X, \mathcal{O}_X)$ denotes the dg-category of bounded complexes of *global sections* of finitely-generated locally free sheaves on the ringed space (X, \mathcal{O}_X) . This combines [Wei16, Theorem 3.32] with a more classical result (stated in [Wei18, Theorem 1.1]) which says that, whenever X is compact and \mathcal{O}_X *soft*, there is an equivalence

$$\mathrm{D}_{\mathrm{perf}}(\mathrm{Mod}_{\mathcal{O}_X}) \simeq \mathrm{D}_{\mathrm{perf}}(\mathrm{Mod}_{\mathcal{O}_X(X)}).$$

All this fits together in the following commutative (again, given certain conditions on X , \mathcal{O}_X , and \mathcal{U}) diagram:

$$\begin{array}{ccc} \mathrm{Tw}_{\mathrm{perf}}(X, \mathcal{O}_X, \mathcal{U}) & \xrightarrow{\sim} & \mathfrak{L}(X, \mathcal{O}_X) \\ \mathrm{dg}\text{-enrichment} \downarrow & & \downarrow \mathrm{dg}\text{-enrichment} \\ \mathrm{D}_{\mathrm{perf}}(\mathrm{Mod}_{\mathcal{O}_X}) & \xrightarrow{\sim} & \mathrm{D}_{\mathrm{perf}}(\mathrm{Mod}_{\mathcal{O}_X(X)}) \end{array}$$

This top equivalence ([Wei18, Theorem 3.1]) has an important interpretation in terms of homotopy limits, but we save a discussion of this for (G.5).

Finally, in [Wei19], the generalisation of twisted perfect complexes to the case of *simplicial objects in a ringed site* is studied, and it is shown (see, e.g., [Wei19, Lemma 4.2]) that the construction is somehow functorial.

G.5 Homotopy limits

(G.5.1) With the notation of (G.4.5), we can now talk a bit about descent and homotopy limits. By considering locally free sheaves on each simplicial level of the Čech nerve of (X, \mathcal{U}) , we get the following cosimplicial diagram of dg-categories:

$$\Pi_\alpha \mathfrak{L}(U_\alpha) \rightrightarrows \Pi_{\alpha,\beta} \mathfrak{L}(U_{\alpha\beta}) \rightrightarrows \Pi_{\alpha,\beta,\gamma} \mathfrak{L}(U_{\alpha\beta\gamma}) \rightrightarrows \dots$$

The so-called *descent data* of $\mathfrak{L}(\mathcal{U})$ is given by the homotopy limit of this diagram; [BHW15] shows that this homotopy limit is equivalent to $\mathrm{Tw}_{\mathrm{perf}}(X, \mathcal{O}_X, \mathcal{U})$, i.e. twisting cochains calculate descent data. Combining this with the results described in (G.4.5), we see that twisting cochains actually let us forget all about twisting cochains, in that we obtain a quasi-equivalence

$$\mathfrak{L}(X) \simeq \mathrm{holim}_\alpha \mathfrak{L}(U_\alpha).$$

The simplicial generalisation of twisting cochains in [Wei19] also extends to homotopy limits: [Wei19, Proposition 4.3] tells us that the dg-category of twisted perfect complexes on a simplicial ringed space gives an explicit construction of the homotopy limit of $\mathrm{Perf}(\mathcal{U})$.

G.6 Twisted complexes in dg-categories

(G.6.1) The canonical reference for *twisted complexes* (in the setting of dg-categories) is [BK91], which we will now explore. For how the following fits into the A_∞ - and ∞ -categorical stories, we recommend [Fao15, §4].

Before explaining any motivation, we first repeat [BK91, Definition 1].

(G.6.2) **Definition.** Let \mathcal{A} be a dg-category. Then a *twisted complex* \mathfrak{C} over \mathcal{A} is a collection

$$\mathfrak{C} = \left(E_i \in \mathcal{A}, (q_{ij} : E_i \rightarrow E_j)_{j \in \mathbb{Z}} \right)_{i \in \mathbb{Z}}$$

such that

- only finitely many of the E_i are non-zero;
- the q_{ij} are morphisms of degree $i - j + 1$;
- $dq_{ij} + \sum_s q_{sj}q_{is} = 0$.

(G.6.3) One of the motivations behind Definition (G.6.2) is the following question: “given a dg-category \mathcal{A} , what is the ‘smallest’ dg-category \mathcal{A}' , into which \mathcal{A} embeds, such that we can define shifts and functorial cones in \mathcal{A}' ?” Or, thinking about (G.4.2) and (G.4.4), and taking the point of view of a stable-homotopy theorist, we can pose the following (basically equivalent) question: “an arbitrary dg-category \mathcal{A} has no (pre-)triangulated structure a priori, and so lacks stability ‘upstairs’, so what is the ‘smallest’ way of ‘enriching’ our dg-category to get some dg-category $\widehat{\mathcal{A}}$ which has a stable structure?”

It turns out that, for both questions, $\widehat{\mathcal{A}}$ is given exactly by the category of twisted complexes over \mathcal{A} . The embedding $\mathcal{A} \hookrightarrow \widehat{\mathcal{A}}$ lets us ‘pull back’ the shift and cones from $\widehat{\mathcal{A}}$ to ones in \mathcal{A} . Even nicer, it turns out that the homotopy category of \mathcal{A} is triangulated with this shift functor and these cones.

Further, if \mathcal{A} is pre-triangulated, then this embedding $\mathcal{A} \hookrightarrow \widehat{\mathcal{A}}$ is a quasi-equivalence^[1] of dg-categories. This means that, if \mathcal{A} does already have some pre-triangulated structure, then we don’t really change anything about it.

(G.6.4) Lemma. It is *not* the case that the twisting cochains of e.g. [OTT81] are equivalent to the twisted complexes of [BK91]. In fact, the former is a specific case of the latter. We know, heuristically, that this should be the case, because twisting cochains give us a stabilisation of the category of vector bundles (as discussed in (G.4.4)), which is exactly what twisted complexes are meant to do, but it is interesting to note that twisting cochains are a *strict* example of twisted complexes, as we will now explain.

Recalling (G.2), we start with the data of a graded holomorphic vector bundle V_α on every U_α , and write $V = \{V_\alpha\}_\alpha$. Let $B = \check{C}^\bullet(\mathcal{U}, \mathcal{O}_X)$ and $E_0 = \check{C}^\bullet(\mathcal{U}, V)$. Now, over each U_α we can consider the graded \mathcal{O}_{U_α} -module V_α as a complex V_α^\star with trivial differential, and this gives E_0 the structure of a B -dg-module.

If we then define $E_i = 0$ for $i \neq 0$, we can ask what it means for $\{E_i\}_{i \in \mathbb{Z}}$ to have the structure of a twisted complex: we need to provide a degree-1 B -linear endomorphism $a = q_{00}$ of E_0 such that $da + aa = 0$. But the dg-algebra $\text{End}_B(E_0)$ is exactly^[2] $\check{C}^\bullet(\mathcal{U}, \text{End}(V)^\star)$. Decomposing a into $a^{k, 1-k} \in \check{C}^k(\mathcal{U}, \text{End}^{1-k}(V))$ we recover exactly the Maurer-Cartan condition for twisting cochains. Thus any holomorphic twisting cochain gives us a twisted complex in the dg-category of B -dg-modules.

It is *not* the case, however, that by picking the ‘right’ dg-category \mathcal{A} , we recover the definition of holomorphic twisting cochains from that of twisted complexes. In particular, twisting cochains are twisted complexes with only *one* non-zero object. Further, this object E_0 is not an arbitrary B -module: it comes from $\check{C}^\bullet(\mathcal{U}, V)$, where V is graded, and is thus somehow (via maybe the Serre-Swann theorem) *projective* or *free*.

^[1]That is, it induces an equivalence on the level of homotopy categories.

^[2][OTT81, §1] shows that we get a module structure on $\check{C}^\bullet(\mathcal{U}, V^\star)$ over the algebra $\check{C}^\bullet(\mathcal{U}, \text{End}^\star(V))$

G.7 Dold-Kan and twisting cochains

(G.7.1) As mentioned previously, Green’s resolution (9.1.2) lets us pretty much forget about twisting cochains. It is still interesting, however, to wonder whether or not there is some *functorial* construction from twisting cochains to vector bundles on the nerve, since then we could see if the construction of characteristic classes mirrors that for twisting cochains, as given in [Bre+07] or [Sha10, §2.2]. In general, it would be nice to be able to use the myriad of nice properties of the category of twisting cochains, and the constructions that exist therein, so let’s consider how we might be able to do so.



We repeat the warning: since this is a ‘vague’ appendix, it is not our intention here to give a rigorous explanation, but instead just furnish the reader with the appropriate vocabulary to be able to actually understand what we allude to here by asking people who actually know things (instead of the present author).

The *Dold-Kan correspondence* is an equivalence of categories between simplicial objects (in some abelian category) and connective chain complexes (in the same category). There are many generalisations of the correspondence, but the one that interests us is that for sheaves, which says^[1] that

$$\mathrm{Sh}(X, \mathrm{s}\mathcal{A}) \simeq \mathrm{Sh}(X, \mathrm{Ch}_+(\mathcal{A})),$$

where \mathcal{A} is some abelian category, and $\mathrm{s}\mathcal{A} = [\Delta^{\mathrm{op}}, \mathcal{A}]$ is the category of simplicial objects in \mathcal{A} . Although this, as stated, is an equivalence of 1-categories, it reflects weak equivalences, and induces an equivalence when we take ‘suitable’ localisations of $\mathrm{s}\mathcal{A}$ and $\mathrm{Ch}_+(\mathcal{A})$.

If we just consider sheaves as things that satisfy some equaliser condition^[2], then sheaves of complexes are equivalent to complexes of sheaves, and, similarly, sheaves of simplicial objects are exactly simplicial objects in the category of sheaves. That is, the above equivalence gives us the equivalence

$$\mathrm{sSh}(X, \mathcal{A}) \simeq \mathrm{Ch}_-(\mathrm{Sh}(X, \mathcal{A})).$$

which gives us

$$\mathrm{Ch}_+(\mathrm{sSh}(X, \mathcal{A})) \simeq \mathrm{Ch}_{+,-}(\mathrm{Sh}(X, \mathcal{A})),$$

where we write $\mathrm{Ch}_{+,-}$ to mean bicomplexes concentrated in one quadrant.

When we take $\mathcal{A} = \mathrm{Mod}_{\mathcal{O}_X}$ to be the category of \mathcal{O}_X -modules, twisting cochains can be viewed as elements of the category on the right-hand side, and the result of Green’s resolution can *almost* be viewed as an element of the category on the

^[1]See, for example, item (3) in [Bro73, p. 426].

^[2]As opposed to the finer notion of *homotopy sheaves*, where we replace the equaliser with a homotopy limit.

left-hand side: we just need to figure out the relation between $s\text{Sh}(X, \text{Mod}_{O_X})$ and $\text{Vect}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$. But in (F.5.4) we explain how objects $\text{Vect}^{\text{cart}}(X_{\bullet}^{\mathcal{U}})$ are limit objects of $\text{Sh}(X_{\bullet}^{\mathcal{U}})$, and, although this is in the purely topological setting, we can see how one might recover the desired result: Green’s resolution is somehow a coarse, *non-functorial*^[1] version of Dold-Kan.

To formalise this, [Dyc17] and [Cal18] seem appropriate.

G.8 Alternative viewpoints

(G.8.1) This section just lists all of the twisting cochain constructions that the author has come across during the writing of this thesis. Much of the content here is hurried and unmotivated, aimed at readers already somewhat familiar with the various contexts, and is really meant to serve as nothing more than a ‘minimal completion’ of this survey of the subject. Most things here are explained in more detail on, of course, as always, the nLab.

(G.8.2) Maurer-Cartan. Since Maurer-Cartan elements classify deformations of (finite-dimensional associative unital) algebras, we could better understand (G.3) by thinking of twisting cochains as deformations of the deleted-Čech complex.

A small side note: it is known that Maurer-Cartan elements of $A \otimes \mathfrak{g}$ are equivalent to morphisms of dg-algebras from the Chevalley-Eilenberg complex of \mathfrak{g} to A ; with this in mind, one could ask what twisting cochains look like when we work with endomorphism-valued forms, and if this bears any relevance to the theory developed elsewhere in this thesis.

(G.8.3) Graded mixed complexes. A *graded mixed complex of dg- k -modules* is, as defined towards the end of [Cal+17, §1.1], a family $\{E(p)\}_{p \in \mathbb{Z}}$ of dg- k -modules together with a family $\{\epsilon_p\}_{p \in \mathbb{Z}}$ of morphisms

$$\epsilon_p : E(p) \rightarrow E(p+1)[1]$$

such that $\epsilon_{p+1} \circ \epsilon_p = 0$. It can be shown that a twisted complex (in the sense of Bondal and Kaprinov) is a specific example of such a thing.

(G.8.4) A_{∞} -categories. All of the following is from [Kel01]; another nice reference is [Fao15].

Let \mathcal{A} be an A_{∞} -algebra (with strict identities). We write $C_{\infty}\mathcal{A}$ to mean the category of A_{∞} -modules over \mathcal{A} , and $\mathcal{D}_{\infty}\mathcal{A}$ to mean the homotopy category of $C_{\infty}\mathcal{A}$. In the non- A_{∞} case we can define a category $\text{tria}\mathcal{A}$ as follows: up to isomorphism, the objects of $\text{tria}\mathcal{A}$ are bounded complexes of finitely-generated free \mathcal{A} -modules, and the morphisms “are” homotopy classes of morphisms of complexes. We wish

^[1]Recall that Green’s construction isn’t functorial: (8.3.12). This means that we can think of it as a ‘coarse *object-level* Dold-Kan’.



to formalise the definition of $\text{tria } \mathcal{A}$ in the case where \mathcal{A} is an A_∞ -algebra, starting with the vague definition as the “ \mathcal{A}_∞ -version of the triangulated subcategory of $\mathcal{D}\mathcal{A}$ generated by the free rank-1 \mathcal{A} module”.

It can be shown that the Yoneda functor \mathfrak{Y} factors through an A_∞ -category $\text{tw } \mathcal{A}$, called the *category of twisting cochains*, as

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathfrak{Y}_1} & \text{tw } \mathcal{A} \\ & \searrow \mathfrak{Y} & \downarrow \mathfrak{Y}_2 \\ & & C_\infty \mathcal{A} \end{array}$$

Further, \mathfrak{Y}_1 is (strictly) fully faithful, and \mathfrak{Y}_2 induces an equivalence

$$H^0 \text{tw } \mathcal{A} \xrightarrow{\sim} \text{tria } \mathcal{A}.$$

This lets us formalise what the A_∞ -version of $\text{tw } \mathcal{A}$ really is, and an explicit description is given in [Kel01, §7.6].

To paraphrase [Wei19, Remark 1.5]: to people familiar with A_∞ -categories, the definition of twisted complexes might look similar to the construction of A_∞ -functors; this is explained in [Tsy19, §16] and [AØ19, §4].

(G.8.5) The bar-cobar adjunction. If we have a dg-coalgebra (C, d_C) with comultiplication Δ , and a dg-algebra (A, d_A) with multiplication μ , then we can define a *twisting cochain* to be a morphism $\tau: C \rightarrow A[1]$ such that

$$d_A \circ \tau + \tau \circ d_C + \mu \circ (\tau \otimes \tau) \circ \Delta = 0.$$

Note that this is Maurer-Cartan, since the last term is the product $\tau \star \tau$ in the *convolution algebra*.

We will not say any more on this approach, instead referring the interested reader to, say, the nLab.

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