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Market Impact in Systematic Trading and Option Pricing

Emilio Said

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Market Impact in Systematic Trading and Option Pricing

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THÈSE

présentée pour obtenir

LE GRADE DE DOCTEUR EN MATHÉMATIQUES APPLIQUÉES
DE L'UNIVERSITÉ PARIS-SACLAY

par

Emilio SAID

Market Impact in Systematic Trading and Option Pricing

Soutenue le 23 Juin 2020 devant un jury composé de

Aurélien ALFONSI	<i>Ecole des Ponts ParisTech</i>	rapporteur
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À la gloire de mes parents...

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Résumé

L'objectif principal de cette thèse est de comprendre les divers aspects du market impact. Elle se compose de quatre chapitres dans lesquelles le market impact est étudié dans différents contextes et à différentes échelles.

Le premier chapitre présente une étude empirique du market impact des ordres limites sur les marchés actions européens. Cette étude a été menée sur deux ans de données – de Janvier 2016 à Décembre 2017 – regroupant toutes les transactions effectuées par BNP Paribas sur les marchés européens ainsi que les événements du carnet d'ordres associés avec pour précision la microseconde. Nous avons mis en place un algorithme de traitement systématique de ces bases de données permettant la reconstruction des métaordres selon une méthodologie définie au préalable. Nous nous sommes particulièrement intéressés aux ordres limites agressifs et passifs et avons étudié l'impact de ces métaordres sur le processus de formation du prix.

Dans le deuxième chapitre, nous avons étendu la méthodologie présentée pour les marchés actions aux marchés options. Après avoir identifié les principales différences que présentaient les deux types de métaordres, actions et options, nous avons mené une étude empirique sur deux ans de données – de Juillet 2016 à Juin 2018 – provenant d'une activité de market making haute fréquence sur l'un des principaux marchés asiatiques où BNP Paribas intervient en tant que teneur de marché principal. Cette étude empirique a mis en évidence que notre définition d'un *métaordre options* nous permet de retrouver la totalité des résultats mis en évidence sur les marchés actions.

Le troisième chapitre s'intéresse au market impact dans le contexte de l'évaluation des produits dérivés. Ce chapitre tente d'apporter une composante microstructure à l'évaluation des options notamment en proposant une théorie des perturbations du market impact au cours du processus de *re-hedging*. Notre approche prédit l'existence de métaordres particuliers, appelés *métaordres de re-hedge* composés d'ordres de taille décroissante et présentant un impact linéaire en fonction du volume. D'autre part, en s'appuyant sur une hypothèse de *fair pricing* – validé empiriquement dans les deux premiers chapitres –, nous mettons en évidence que la relaxation de ces *métaordres de re-hedge* est variable et dépend notamment des caractéristiques du carnet d'ordres et du portefeuille du trader au début de l'exécution.

Nous explorons dans le quatrième chapitre un modèle assez simple pour la relaxation des métaordres. La relaxation des métaordres est traitée dans cette partie en tant que processus informationnel qui se transmet au marché. Ainsi, partant du point de départ qu'à la fin de l'exécution d'un métaordre l'information portée par celui-ci est maximale, nous proposons une interprétation du phénomène de relaxation comme étant le résultat de la dégradation de cette information au détriment du bruit extérieur du marché.

Abstract

The main objective of this thesis is to understand the various aspects of market impact. It consists of four chapters in which the market impact is studied in different contexts and at different scales.

The first chapter presents an empirical study of the market impact of limit orders on European equity markets. This study was carried out on two years of data - from January 2016 to December 2017 - gathering all the transactions carried out by BNP Paribas on the European markets as well as the events of the associated order book at the microsecond scale. We have implemented an algorithm for the systematic processing of these databases allowing the reconstruction of metaorders according to a methodology defined beforehand. We were particularly interested in aggressive and passive limit orders and studied the impact of these metaorders on the price formation process.

In the second chapter, we have extended the methodology presented for the equity markets to the options markets. After identifying the main differences between the two types of metaorders, stocks and options, we conducted an empirical study over two years of data - from July 2016 to June 2018 - from a high frequency market making activity on one of the main Asian markets where BNP Paribas acts as the main market maker. This empirical study has shown that our definition of an *options metaorder* allows us to retrieve all the results highlighted on the equity markets.

The third chapter focuses on market impact in the context of option pricing. This chapter is based on previous works of option pricing with market impact and attempts to bring a microstructure component to pricing option theory by proposing a perturbation theory of market impact during the *re-hedging* process. Our framework predicts the existence of a certain kind of metaorders, the *hedging metaorders*, showing linear impact with the size of the metaorder. On the other hand, by leaning on an assumption of *fair pricing* - validated empirically in the first two chapters -, we highlight that the relaxation of these *re-hedging metaorders* is variable and can vary with the characteristics of the limit order book and trader's portfolio at the beginning of the execution.

We explore in the fourth chapter a fairly simple model for the relaxation of metaorders. The relaxation of metaorders is treated in this part as an informational process which is transmitted to the market. Thus, starting from the starting point that at the end of the execution of a metaorder the information carried by this one is maximum, we propose an interpretation of the relaxation phenomenon as being the result of the degradation of this information at the expense of outside market noise.

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Introduction

The guiding principle of this thesis is to study various aspects of market impact in different markets and at different time scales. We aim at simultaneously proposing useful algorithms to reconstruct metaorders executed on the market, building estimation procedures to measure market impact and proposing connections with other fields such as option pricing and information theory in finance. Let us begin with presenting and motivating the different questions on which we want to shed some light in this thesis.

Motivations

The market impact of metaorders is crucial in describing and controlling the behaviour of modern financial markets. Being able to quantify this impact is clearly a question of great relevance when studying the price formation process, and it has also become a major practical issue for optimal trading. The significant growth in electronic trading in the recent years has created a need among regulators, academics and practitioners to develop better understanding of the mechanisms of market impact. Are the properties of market impact universal? Or alternatively, are they subject to the market, the trading frequency or market participants' practices? We contribute to this debate by considering the following questions:

Question 1. *What are the main stylized facts of market impact in equity market?*

Addressing Question 1, we highlight the main properties of market impact across all European venues. After giving a definition of an algorithmic metaorder to reconstruct the large orders - intentional or not - executed by BNP Paribas, we empirically demonstrate that the goal of the participants as well as the nature of the orders used - aggressive or passive - do not affect the properties of market impact. The study of the characteristics of the identified metaorders makes it possible to retrieve many results obtained from broker data. Thus it appears that an algorithmic metaorder, namely a series of orders executed by the same actor on the same product on the financial market, behaves like a metaorder and therefore can be considered as such. This pushes us to wonder whether our algorithmic metaorder approach can be adapted to options market. In addition to this, market impact in options market have never been deeply studied. Consequently, we naturally consider the following question:

Question 2. *How could we extend our algorithmic metaorder approach to options market?*

When answering this question, we propose an algorithmic definition for options metaorders inspired from our previous approach. In this framework, we provide a complete methodology to study and measure market impact on options market. The main difference with stock markets is that options traders buy and sell more *sensitivities* than the products themselves in order to reduce the expositions of their portfolio i.e. they think and plan not in terms of options (e.g. calls, puts), but rather in terms of sensitivities of their portfolio. Thus their actions on the market aim to generate local deformations of the implied volatility surface and are based on the parametrization of their implied volatility model. We conduct an empirical study of the high-frequency options market and confirm the relevance of our approach by showing a pretty good agreement with some observations already highlighted in the stock markets. As we are interested in exploring these results to option hedging and pricing, we consider the following questions:

Question 3. *How to build an option pricing theory with market impact compatible with some previous empirical observations and non-arbitrage?*

As an answer to Question 3, we propose a perturbation theory of option hedging with market impact enabling us to establish a pricing equation leading to exact replication for European options. Thanks to the simplicity of the model, computations are explicit. However, it has some limitations. First, it restrains the theory to small child orders. Even if the sum of small child orders n_k can give birth to a large order $N = \sum_k n_k$, we have to consider first orders to derive the pricing equation. Besides, although our approach is compatible with some of our empirical results such as market impact dynamics, fair pricing and price reversion of metaorders, it leads us to a linear relation between the peak impact and the metaorder size of the metaorder, which is not really realistic according to our answers to Questions 1 and 2. The reason for this lies in the fact that we neglect the feedback of the limit order book during the hedging process since we assume that the market depth λ remains constant during the execution of the hedging metaorder: it is only updated when the hedging process is over. Consequently, it is impossible to build a realistic option pricing theory with market impact in agreement with empirical observations without considering a precise feedback mechanism between the trader and the limit order book. However our model sheds light on the fact that the ratio between the permanent impact and the permanent impact takes its values in $[1/2, 1]$ and can vary with the spot, the gamma and the market depth. Besides it leads to the prediction of the existence of *hedging metaorders* that are metaorders with decreasing small child orders showing linear market impact. Such conclusions are to our knowledge the firsts of their kind in the litterature. This arises a central question: Can we observe *hedging metaorders* in the market? This also brings us to reopen an old controversy: the existence or not of permanent impact. Hence, the final question of this thesis is the following:

Question 4. *How can we explain the existence or not of permanent market impact?*

Outline

Each chapter of this thesis addresses one of the questions above. This work can be separated into two main parts: one empirical and the other theoretical. In Chapters I and II, we give empirical answers to Questions 1 and 2 by analysing unique datasets labelled by BNP Paribas market participants identities, with microsecond granularity covering the trades and orders of the European securities in Chapter I and Kospi Index options in Chapter II. In Chapters III and IV, we deal with Questions 3 and 4 by proposing some models taking into account empirical observations of Chapters I and II.

In Chapter I, we answer Question 1 by studying the trades and orders of the European securities executed by BNP Paribas, on all European venues from January 2016 to December 2017. First, we develop a methodology to identify and reconstruct the metaorders executed by BNP Paribas and then we analyse the market impact of these large orders. Second, we evaluate the trading practices of BNP Paribas traders by assessing their share in traded amounts, and whether they are mainly liquidity providers or liquidity consumers. We show that their trades are mainly aggressive and passive limit orders. Furthermore, we provide an empirical study of the market impact for each type of order. Finally, we shed light on the similarities between the two previous studies.

In Chapter II, we tackle Question 2 by investigating the same type of data as that of Chapter I, from July 2016 to June 2018 with a special focus on high-frequency options traders. We start with looking how to define a metaorder on options market and give an algorithmic definition based on the parametrization of the implied volatility surface. To conduct our empirical study of the options market we use a proprietary parametric implied volatility model fitting the market with accuracy. Such an empirical study is to our knowledge the first of its kind in the literature. In contrast to several earlier market impact studies, ours is the only of its kind focused specifically on the high-frequency options market. In particular, we show that our study of the market impact of options market, in agreement with our definition of options metaorder, reproduces all the stylized facts observed in equity market.

Answer to Question 3 lies in Chapter III. Using a perturbation type approach, we are able to establish the modified spot dynamics caused by market impact, the pricing equation and the replication of European options. By considering local linear market impact, we show that arbitrage opportunities between the different execution strategies vanish when the trading frequency goes to infinity. Furthermore, we establish that there is a strong connection between linear market impact and exponential linear market impact as they represent the two extreme cases when one considers the execution of metaorders with local linear market impact. This model enables us to bridge some gaps between market impact empirical observations and Black-Scholes type option pricing theory. Indeed, the option pricing derived can be seen as a generalization of the Black-Scholes model since we retrieve the Black-Scholes equation by cancelling market impact in our approach.

We address Question 4 in Chapter IV. For this, we restrict ourselves to the modelling of the price reversion of the metaorders. We propose an approach based on information diffusion. More precisely, we study how the information of a metaorder competes with market noise. Under mild assumptions, we study the conditions of existence or not of permanent market impact. Thanks to these results, we are able to connect the value of the ratio between the permanent impact and the peak impact with the rate of convergence of the interaction – as an informational process – between the metaorder and the market. Furthermore, it is the convergence of this interaction which leads to market equilibrium.

1 Part I: Empirical Analysis of Equity Traders Market Impact

In Part I, we conduct empirical analyses in order to answer Question 1. We analyse the metaorders executed by BNP Paribas in the European stock market from January 2016 to December 2017 with a specific focus on the market impact of *limit orders*. As a matter of fact, market orders are generally not used by institutional investors because of the lack of control they imply. On the contrary, limit orders, whether they are *aggressive* - crossing the spread - or *passive*, form the vast majority of orders that are actually sent to the market during the execution of a large trade. As such, they should be the main subject of interest in a study of market impact.

1.1 Chapter I - Market Impact: A Systematic Study of Limit Orders

In Chapter I, we answer Question 1: What are the main stylized facts of market impact in equity market? We focus on the behaviour of traders on an intraday scale by studying their market impact. To do so, we have access to a unique proprietary database consisting of appropriately selected limit orders executed on the European equity market between January 2016 and December 2017. Our study relies on an algorithmic-based identification and reconstruction of metaorders from the database of all orders.

1.1.1 Algorithmic definition of a metaorder

Loosely speaking a **metaorder** is a large trading order that is split into small pieces and executed incrementally. In order to perform rigorous statistical analyses, a more specific and precise definition of a metaorder is required, and given in Definition 1 below:

Definition 1. *A metaorder is a series of orders sequentially executed during the same day and having those same attributes:*

- *agent* i.e. a participant on the market (an algorithm, a trader...);
- *product id* i.e. a financial instrument (a share, an option...);
- *direction* (buy or sell);

1. Part I: Empirical Analysis of Equity Traders Market Impact

AGENT	PRODUCT_ID	DIRECTION	TIMESTAMP	CURRENCY	EXEC_PRICE	EXEC_QTY	PRICE_TYPE	ORDER_TYPE
agent01	product01	Sell	01-APR-16 16:52:00.536612	EUR	87,5	127	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:03:48.203844	EUR	38,475	393	Aggressive	Limit
agent01	product01	Sell	01-APR-16 13:33:09.566350	EUR	87,4	64	Aggressive	Limit
agent01	product01	Sell	01-APR-16 16:52:00.536641	EUR	87,5	52	Aggressive	Limit
agent04	product20	Sell	01-APR-16 09:46:08.135532	EUR	4,128	570	Aggressive	Limit
agent11	product13	Sell	01-APR-16 15:10:32.279258	EUR	15,2	459	Aggressive	Limit
agent11	product02	Sell	01-APR-16 13:39:30.559504	EUR	4,263	2000	Aggressive	Limit
agent05	product07	Sell	01-APR-16 16:00:29.447697	EUR	58,5	173	Aggressive	Limit
agent05	product11	Buy	01-APR-16 16:34:49.039137	EUR	39,08	374	Aggressive	Limit
agent11	product09	Buy	01-APR-16 13:03:01.846048	EUR	17,005	500	Aggressive	Limit
agent11	product04	Sell	01-APR-16 15:14:18.662973	EUR	19,23	746	Aggressive	Limit
agent11	product02	Buy	01-APR-16 16:12:15.194178	EUR	4,194	942	Aggressive	Limit
agent05	product07	Sell	01-APR-16 14:23:20.168576	EUR	58,32	8	Aggressive	Limit
agent11	product04	Buy	01-APR-16 09:44:28.440733	EUR	19,535	500	Aggressive	Limit
agent01	product01	Sell	01-APR-16 13:33:09.566321	EUR	87,4	100	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:07:52.669195	EUR	38,45	404	Aggressive	Limit

Figure .1 – Sample of the history of orders before the identification process of the algorithm. Each color corresponds to a different metaorder according to the methodology introduced in Definition 1.

AGENT	PRODUCT_ID	DIRECTION	TIMESTAMP	CURRENCY	EXEC_PRICE	EXEC_QTY	PRICE_TYPE	ORDER_TYPE
agent01	product01	Sell	01-APR-16 13:33:09.566321	EUR	87,4	100	Aggressive	Limit
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agent04	product20	Sell	01-APR-16 09:46:08.135532	EUR	4,128	570	Aggressive	Limit
agent05	product07	Sell	01-APR-16 14:23:20.168576	EUR	58,32	8	Aggressive	Limit
agent05	product07	Sell	01-APR-16 16:00:29.447697	EUR	58,5	173	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:03:48.203844	EUR	38,475	393	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:07:52.669195	EUR	38,45	404	Aggressive	Limit
agent05	product11	Buy	01-APR-16 16:34:49.039137	EUR	39,08	374	Aggressive	Limit
agent11	product09	Buy	01-APR-16 13:03:01.846048	EUR	17,005	500	Aggressive	Limit
agent11	product04	Buy	01-APR-16 09:44:28.440733	EUR	19,535	500	Aggressive	Limit
agent11	product04	Sell	01-APR-16 15:14:18.662973	EUR	19,23	746	Aggressive	Limit
agent11	product13	Sell	01-APR-16 15:10:32.279258	EUR	15,2	459	Aggressive	Limit
agent11	product02	Sell	01-APR-16 13:39:30.559504	EUR	4,263	2000	Aggressive	Limit
agent11	product02	Buy	01-APR-16 16:12:15.194178	EUR	4,194	942	Aggressive	Limit

Figure .2 – Sample of the history of orders after the identification process of the algorithm. Each color corresponds to a different metaorder according to the methodology introduced in Definition 1.

AGENT	PRODUCT_ID	DIRECTION	TIMESTAMP	CURRENCY	EXEC_PRICE	EXEC_QTY	PRICE_TYPE	ORDER_TYPE
agent01	product01	Sell	01-APR-16 13:33:09.566350	EUR	87,4	164	Aggressive	Limit
agent01	product01	Sell	01-APR-16 16:52:00.536641	EUR	87,5	179	Aggressive	Limit
agent04	product20	Sell	01-APR-16 09:46:08.135532	EUR	4,128	570	Aggressive	Limit
agent05	product07	Sell	01-APR-16 14:23:20.168576	EUR	58,32	8	Aggressive	Limit
agent05	product07	Sell	01-APR-16 16:00:29.447697	EUR	58,5	173	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:03:48.203844	EUR	38,475	393	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:07:52.669195	EUR	38,45	404	Aggressive	Limit
agent05	product11	Buy	01-APR-16 16:34:49.039137	EUR	39,08	374	Aggressive	Limit
agent11	product09	Buy	01-APR-16 13:03:01.846048	EUR	17,005	500	Aggressive	Limit
agent11	product04	Buy	01-APR-16 09:44:28.440733	EUR	19,535	500	Aggressive	Limit
agent11	product04	Sell	01-APR-16 15:14:18.662973	EUR	19,23	746	Aggressive	Limit
agent11	product13	Sell	01-APR-16 15:10:32.279258	EUR	15,2	459	Aggressive	Limit
agent11	product02	Sell	01-APR-16 13:39:30.559504	EUR	4,263	2000	Aggressive	Limit
agent11	product02	Buy	01-APR-16 16:12:15.194178	EUR	4,194	942	Aggressive	Limit

Figure .3 – *Sample of the history of the orders executed corresponding to the previous figure after merging the orders executed during the same second. The yellow metaorder which was previously a metaorder of length $N = 4$ is now a metaorder of length $N = 2$.*

1.1.2 Market Impact definitions

Notation	Definition
ω	a metaorder
$S(\omega)$	financial instrument of the metaorder ω
$t_0(\omega)$	start time of the metaorder ω
$T(\omega)$	duration of the metaorder ω
$N(\omega)$	length of the metaorder ω
$Q(\omega)$	size of the metaorder ω
$V(\omega)$	volume traded on the day $d(\omega)$ on the instrument $S(\omega)$ during $[t_0(\omega), t_0(\omega) + T(\omega)]$
$\epsilon(\omega)$	sign of the metaorder ω
$P(\omega)$	price of $S(\omega)$
Ω	set of all the metaorders identified by the algorithm
$\Omega_{n^*} \subset \Omega$	subset of the metaorders with $N \geq n^*$

Table .1 – *Notations and definitions*

The **market impact curve** of a metaorder ω measures the magnitude of the relative price variation between the starting time of the metaorder t_0 and the current time $t > t_0$. Let $\mathcal{I}_t(\omega)$ be a proxy for the realized price variation between time t_0 and time $t_0 + t$. In line with many authors [Almgren et al., 2005] [Bershova and Rakhlin, 2013] [Bacry et al., 2015], we use the *return proxy* defined by

$$\mathcal{I}_t = \frac{P_t - P_{t_0}}{P_{t_0}}, \quad (1)$$

where P_t represents either the execution price of the financial instrument S during the *execution part* of the metaorder, or the mid-price during its *relaxation part* starting when the metaorder has been fully executed. Hence the market impact proxy used in this study is $\epsilon \times \mathcal{I}_t$.

1.1.3 Aggressive metaorders

Firstly, we study the metaorders generated by aggressive limit orders, that is, limit orders that actually cross the spread in order to trigger an immediate transaction. Such orders are sometimes rather loosely considered as market orders in the modelling literature on limit order books, but it is clear that they behave differently, as their execution price is always equal to that of the best available limit and can never trigger transactions at higher (buy) or lower (sell) price.

Data

- Study period : **1st Jan 2016 - 31st Dec 2017**
- Markets : **European Equity Markets**
- Order types : **Aggressive Limit Orders**
- Filters : **metaorders** $\omega \in \Omega$
- Number of metaorders : **1 561 505**

Distribution of length N

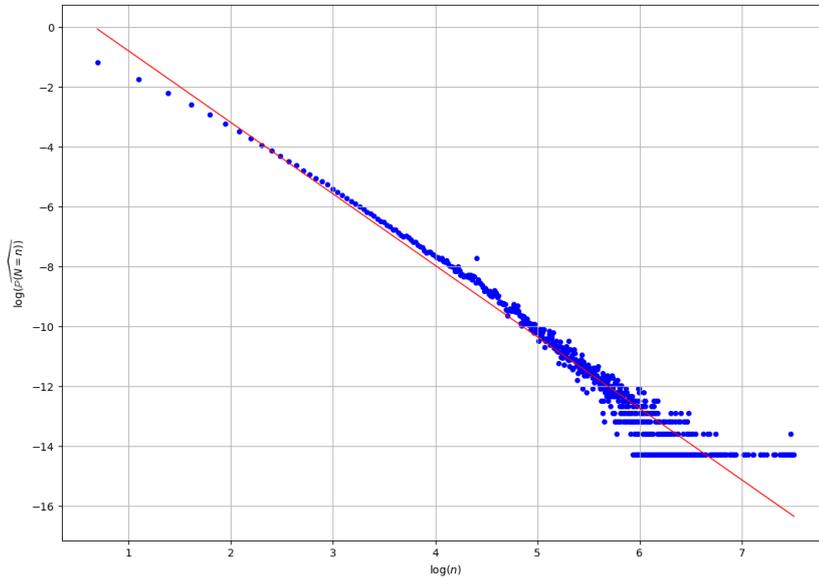


Figure .4 - Fitting Length distribution of the aggressive metaorders

The apparently linear relation suggests that N follows a discrete Pareto distribution. $\widehat{\mathbb{P}(N = n)}$ stands for the natural frequency estimator of the probability $\mathbb{P}(N = n)$. As regards the parameter β introduced in [Farmer et al., 2013] - see Section I.A.5 - one obtains the estimate $\beta \approx 1.4$. Therefore, N is distributed as a power law, in agreement with [Vaglica et al., 2008] who reconstructed metaorders on the Spanish stock exchange using data with brokerage codes and found that N is distributed as power law for large N with $\beta \approx 1.7$.

Result 1. *In the case of aggressive metaorders, N follows a discrete Pareto distribution with parameter $1 + \beta$ such as $\beta \approx 1.4$.*

Market Impact dynamics

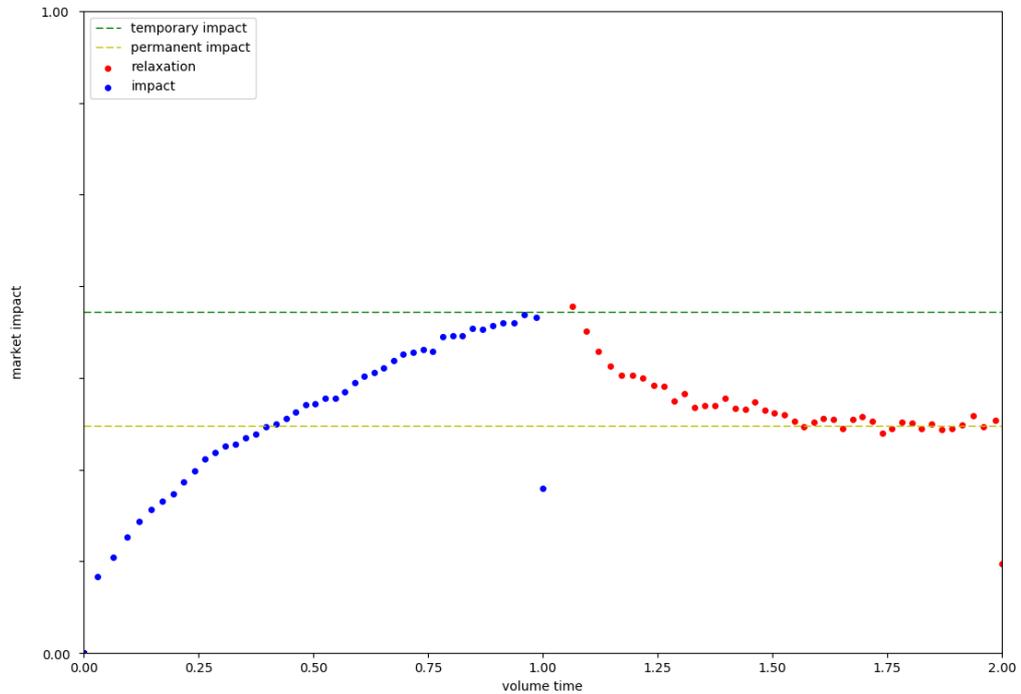


Figure .5 – Market impact dynamics with relaxation in the case of the aggressive metaorders (set : Ω , 1 561 505 metaorders, temporary impact : 0.53, permanent impact : 0.35)

Result 2. *In the case of aggressive orders, the concave shape of the temporary impact and the convex relaxation curve are in line with the empirical results in [Bacry et al., 2015] and [Bershova and Rakhlin, 2013]. Also, and more interestingly, the market impact and relaxation curves confirm the theoretical findings of [Farmer et al., 2013] that the impact should be concave and increasing, and that the final impact after the execution is performed should relax to about two-thirds of the peak impact.*

1.1.4 Execution metaorders

Secondly, we analyse metaorders executed *via* passive limit orders, and compare the resulting market impact curves with those previously obtained. Note that the scales on the graphs concerning the aggressive and passive metaorders are exactly the same, thereby making comparisons possible.

Data

- Study period : **1st Jan 2016 - 31st Dec 2017**
- Markets : **European Equity Markets**
- Order types : **All Orders**¹
- Filters : **metaorders** $\omega \in \Omega$
- Number of metaorders : **74 552**

Distribution of length N

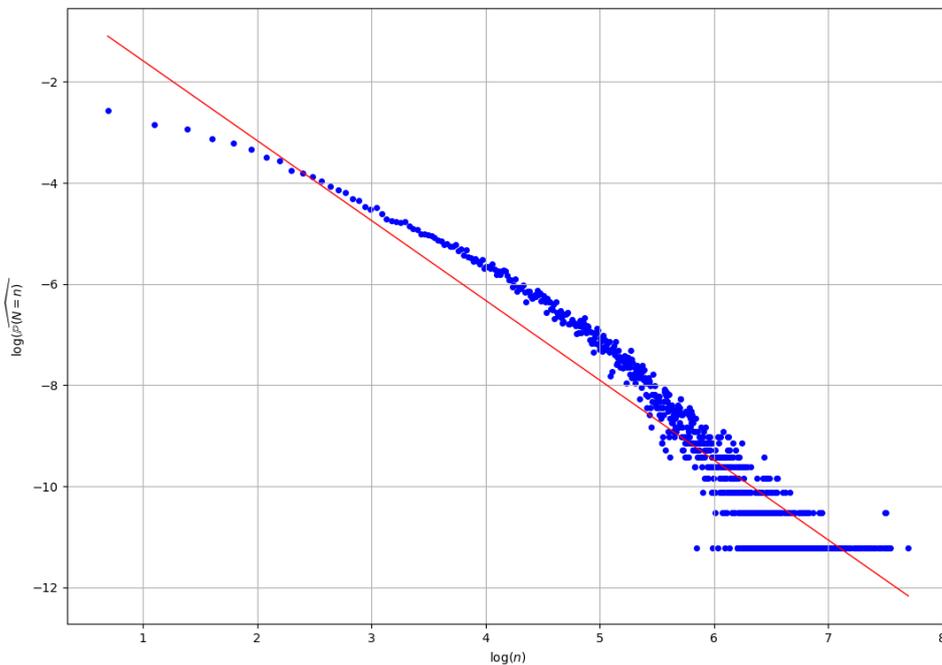


Figure .6 – Fitting Length distribution of the aggressive metaorders

Again, a linear relation in log-log scale seems rather clear (Figure .6), although it is noisier, with $\beta \approx 1.8$ in this case. This tends to confirm the relevance of the discrete Pareto distribution independently of the nature of the orders.

Result 3. *In the case of execution metaorders, N follows a discrete Pareto distribution with parameter $1 + \beta$ such as $\beta \approx 1.8$.*

¹ ~ 65% passive limit orders, ~ 30% aggressive limit orders and 5% others

Market Impact dynamics

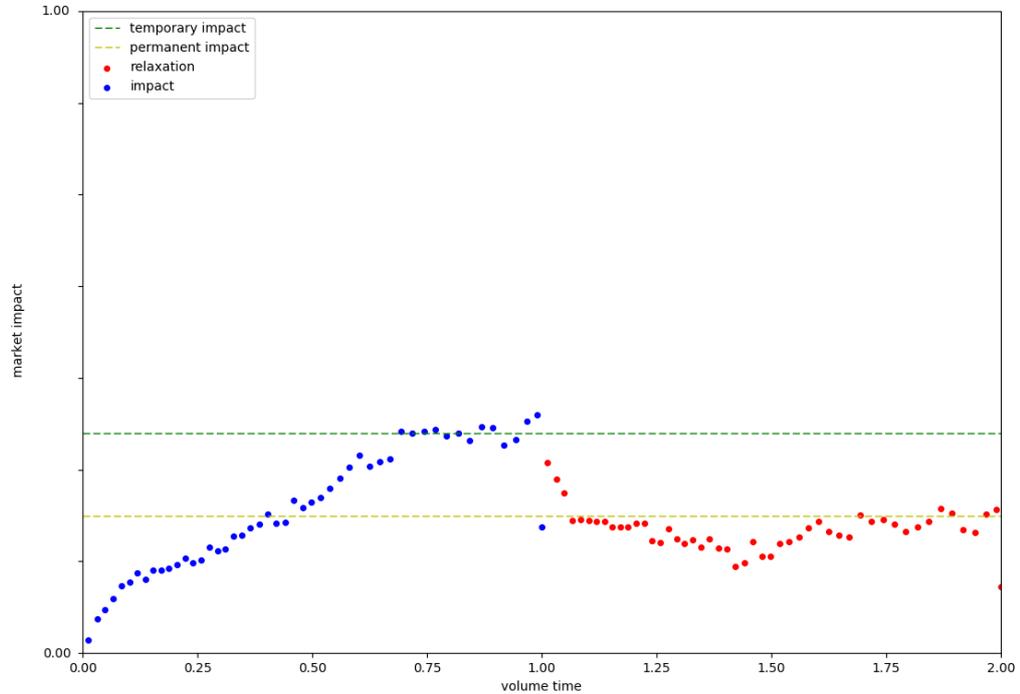


Figure .7 – Market impact dynamics with relaxation in the case of the execution metaorders (set : Ω , 74 552 metaorders, temporary impact : 0.34, permanent impact : 0.22)

Result 4. *In the case of execution metaorders, one observes a positive, increasing and concave market impact curve. The impacts appears to be smaller than in the aggressive case (Figure .5) although participation rates are higher and metaorders last longer. Moreover, the impact is insensitive to the size of the metaorders. These two features are a clear indication that it is beneficial to use passive limit orders. An execution metaorder has the main constraint of executing on the market a known and predefined size in advance, whatever the chosen strategy is. The choice of an execution strategy less brutal than one relying solely on aggressive metaorders allows to limit and control its impact.*

1.1.5 Fair Pricing

Finally, the fair pricing condition for metaorders is examined. In the interest of homogeneity, we consider normalized prices, so that for every stock and at every time, the value of the

price is 1 whenever the execution of a metaorder begins.

First of all, the VWAP of a metaorder ω is defined by

$$\mathcal{P}_{VWAP}(\omega) = \frac{\sum_{i=0}^{N-1} Q_i(\omega) P_{t_i(\omega)}(\omega)}{Q(\omega)},$$

where $t_0(\omega), \dots, t_{N-1}(\omega)$ represent the transaction times of the metaorder ω and $Q(\omega) = \sum_{i=0}^{N-1} Q_i(\omega)$. Hence, we want to compare $1 + \mathcal{R}_{VWAP} = \frac{\mathcal{P}_{VWAP}}{P_{t_0}}$ with $1 + \mathcal{R}_{t_0+2T} = \frac{P_{t_0+2T}}{P_{t_0}}$.

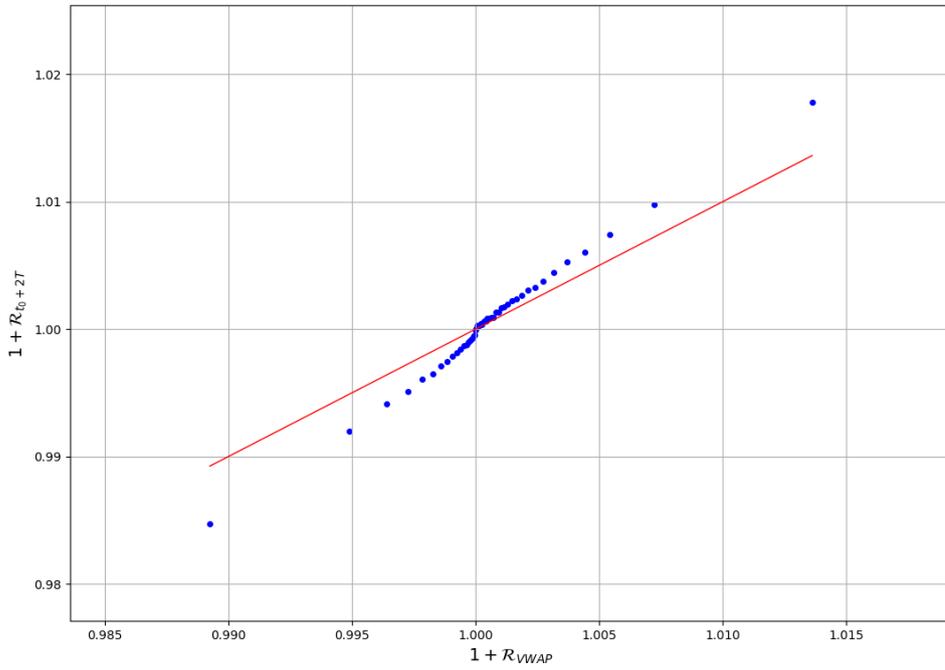


Figure .8 – Fair pricing condition in the case of the aggressive metaorders

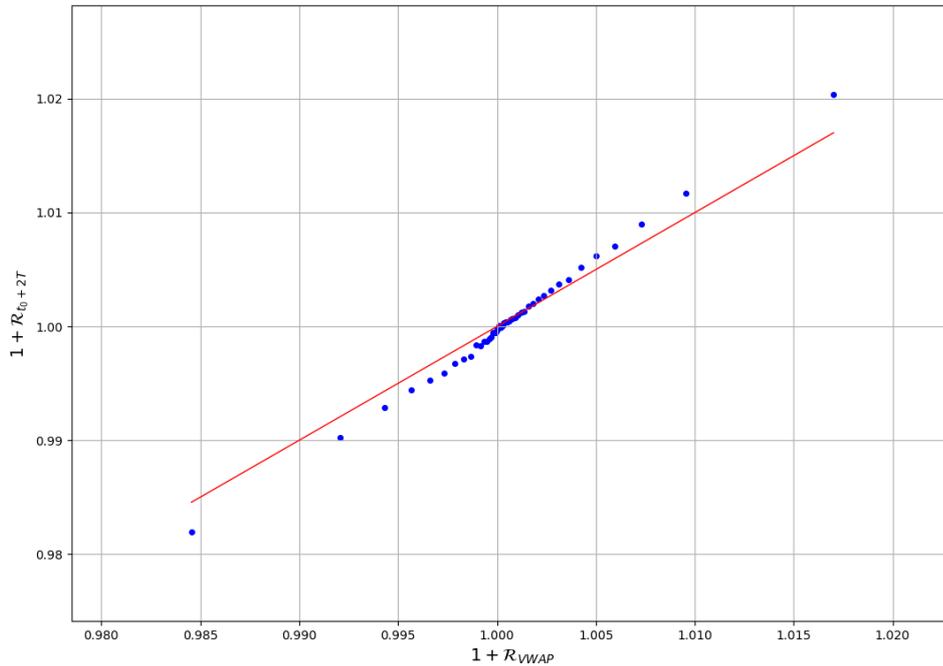


Figure .9 – Fair pricing condition in the case of the execution metaorders

Result 5. *It appears from Figures .8 and .9 that the fair pricing condition can reasonably be assumed to hold. It also appears that, the greater the absolute price variations, the more one moves away from the perfect fair pricing condition. This last point was predictable: high variations are generally associated to longer and larger metaorders that are therefore more affected by the diffusive nature of the prices.*

One of our remarkable findings in this chapter is that the properties of market impact in equity market seem to be the same no matter the nature of the orders used to execute a metaorder. Nevertheless, we have seen that incorporating passive limit orders in an execution strategy can reduce its impact cost. In what follows, we aim at analysing these limit orders – rarely studied in literature – in options market: are they present similar properties than those observed in equity market? We address this issue in Chapter II.

2 Part II: Empirical Analysis of High-Frequency Options Traders Market Impact

2.1 Chapter II - Market Impact: A Systematic Study of the High Frequency Options Market

In this chapter, we answer Question 2: How could we extent our algorithmic metaorder approach to options market? We present some evidence concerning the impact of aggressive orders on the volatility formation process in the high-frequency options market. For this, we conduct a study on KOSPI 200 index options from July 2016 to June 2018. Over the analysed period, we use both trade and limit orders book data to describe the stylized facts of market impact in options market. We summarise in the following our main findings.

2.1.1 Algorithmic definition of an options metaorder

In order to conduct market impact studies on options market, Definition 1 must be adapted to fit the options market. Options are a bit more complex than equities. Traders buy and sell volatility and deal directly with the implied volatility surface, and therefore, with their implicit volatility parameters. As such, an options metaorder can naturally be defined as a sequence of transactions that generate some specific deformations of the volatility surface.

Definition 2. *An options metaorder with respect to an implied volatility parameter θ is a series of orders sequentially executed during the same day and having those same attributes:*

- *agent* i.e. a participant on the market (an algorithm, a trader...);
- *underlying product id* i.e. the underlying financial instrument;
- *direction* regarding the sign of $\mathcal{S}^\theta := Q \times \frac{\partial \mathcal{C}}{\partial \theta}$ where Q is the algebraic quantity (positive for a buy order and negative for a sell order) and \mathcal{C} the price of the option traded;

Note that in Definition 2, the *product id* condition introduced in Definition 1 is dropped. As a matter of fact, trading an option with a given strike K and maturity T also affects those with nearby strikes and maturities, so that trades on options with different strikes and maturity can very well belong to the same metaorder.

This approach is in line with what is presented in [Said et al., 2018], and leads to a systematic study of market impact studies.

In what follows we will use the term θ -metaorder to refer to an options metaorder with respect to the implied volatility parameter θ as defined in Definition 2 and will focus on the two following parameters: *at the money forward* volatility and skew.

We summarize in Table .4 the differences between equity metaorders and options metaorders.

	Equity Metaorder	Options Metaorder
Object of study	stock market	options market
Quantity of interest	stock price P	implied volatility parameters θ
Effective size	quantity Q	sensitivity \mathcal{S}^θ
Effective variation	$P_t - P_{t_0}$	$\theta_t - \theta_{t_0}$
Effective Sign ϵ	$\mathbb{1}_{\{Q>0\}} - \mathbb{1}_{\{Q<0\}}$	$\mathbb{1}_{\{\mathcal{S}^\theta>0\}} - \mathbb{1}_{\{\mathcal{S}^\theta<0\}}$
Market impact proxy	$\epsilon \times (P_t - P_0)$	$\epsilon \times (\theta_t - \theta_0)$

Table .2 – Comparison of the quantities of interest between equity and options metaorders

2.1.2 Market Impact definitions

Notation	Definition
ω	A metaorder
$O(\omega)$	Option of the metaorder ω
$d(\omega)$	Execution day of the metaorder ω
$t_0(\omega)$	Start time of the metaorder ω
$T(\omega)$	Duration of the metaorder ω
$N(\omega)$	Length of the metaorder ω
$\mathcal{V}^\theta(\omega)$	θ -Sensitivity of the metaorder ω
$V^\theta(\omega)$	θ -Sensitivity traded the day $d(\omega)$ on all the options of the universe
$\epsilon^\theta(\omega)$	Sign of $\mathcal{V}^\theta(\omega)$
$\mathcal{O}(\omega)$	Price of $O(\omega)$
Ω^θ	Set of all the θ -metaorders identified by the algorithm
$\Omega_{n^*}^\theta \subset \Omega$	Subset of the θ -metaorders with $N \geq n^*$

Table .3 – Notations and definitions

The **market impact curve** of a metaorder ω quantifies the magnitude of the relative θ -variation between the starting time of the metaorder t_0 and the current time $t > t_0$, θ being a parameter of the implied volatility model. Let $\mathcal{I}_t(\omega)$ be a proxy for the realized θ -parameter variation between time t_0 and time $t_0 + t$. We use the *variation proxy* defined by

$$\mathcal{I}_t = \theta_t - \theta_{t_0}, \quad (2)$$

Hence the market impact proxy used in this study is $\epsilon \times \mathcal{I}_t$.

2.1.3 At the money forward volatility metaorders

Data

- Study period : **1st July 2016 – 30th June 2018**
- Order types : **Aggressive Limit Orders**

- Parameter : $\theta \equiv$ **at the money forward volatility**
- Filters : **metaorders** $\omega \in \Omega^\theta$
- Number of orders : **1,026,197**
- Number of metaorders : **149,441**

Market Impact dynamics

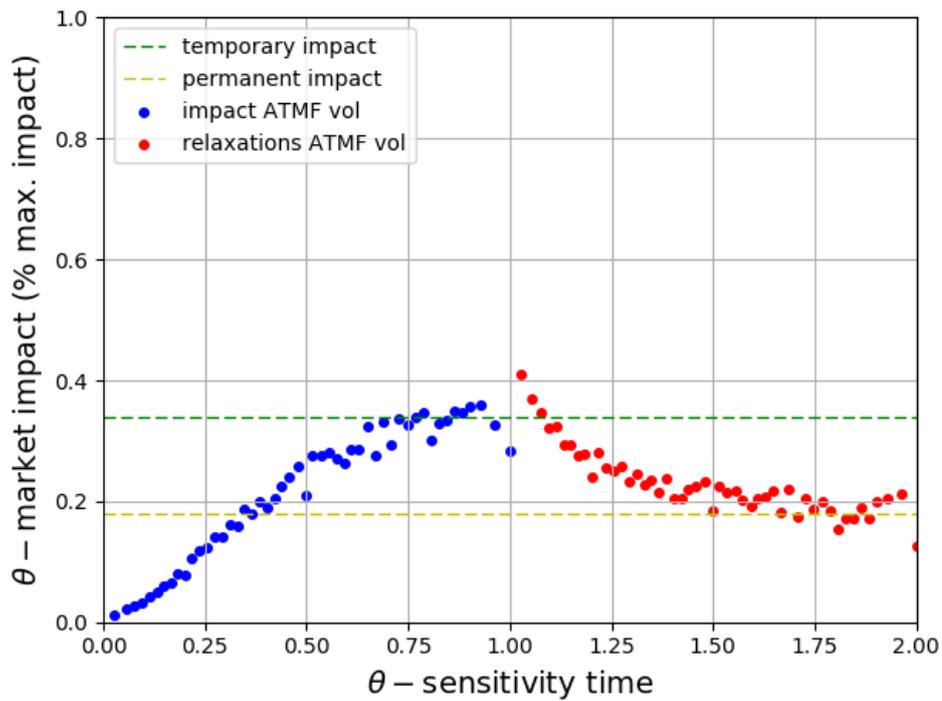


Figure .10 – Market impact dynamics in the case of the *at the money forward volatility* metaorders ($\theta \equiv$ ATMF volatility, set : Ω , 1,026,197 orders, 149,441 metaorders, temporary impact: 0.34, permanent impact: 0.17)

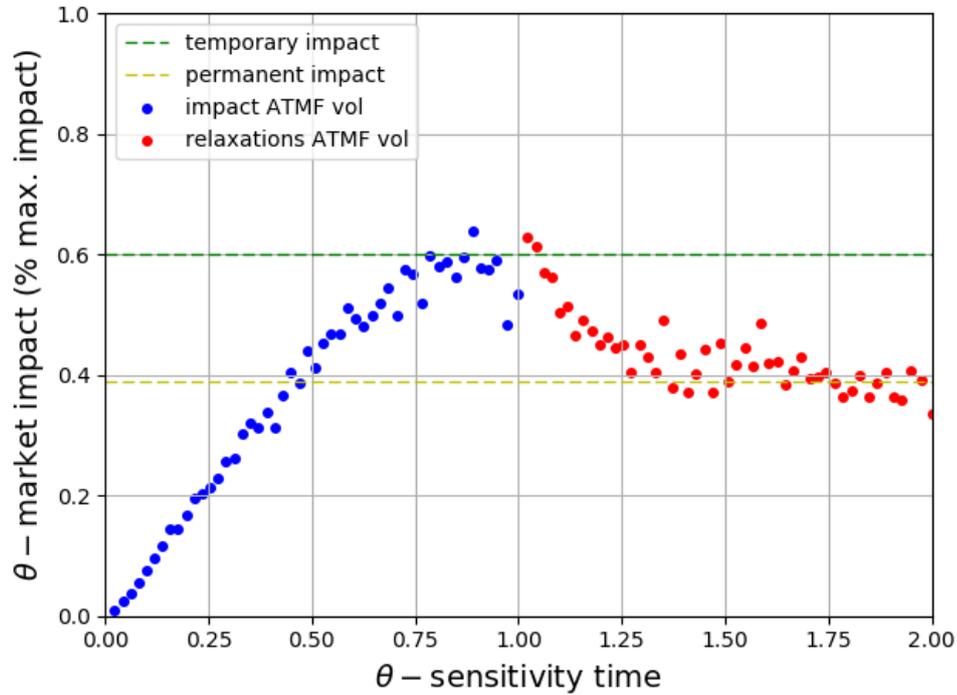


Figure .11 – Market impact dynamics in the case of the *at the money forward volatility* metaorders ($\theta \equiv$ ATMF volatility, set : Ω_{10} , 215,274 orders, 17,286 metaorders, temporary impact: 0.60, permanent impact: 0.39)

Result 6. *The analysis of the at the money forward volatility metaorders clearly yields an increasing, concave market impact curve. Also note that on the two figures .10 and .11, the larger the metaorders, the higher the impacts: 0.34 and 0.60 for the temporary market impact. Besides, it appears quite clearly in Figure .11 that the permanent impact is around 2/3 of the peak impact.*

2.1.4 At the money forward skew metaorders

Data

- Study period : **1st July 2016 – 30th June 2018**
- Order types : **Aggressive Limit Orders**
- Parameter : $\theta \equiv$ **at the money forward skew**
- Filters : **metaorders** $\omega \in \Omega^\theta$
- Number of orders : **1,304,714**
- Number of metaorders : **174,091**

Market Impact dynamics

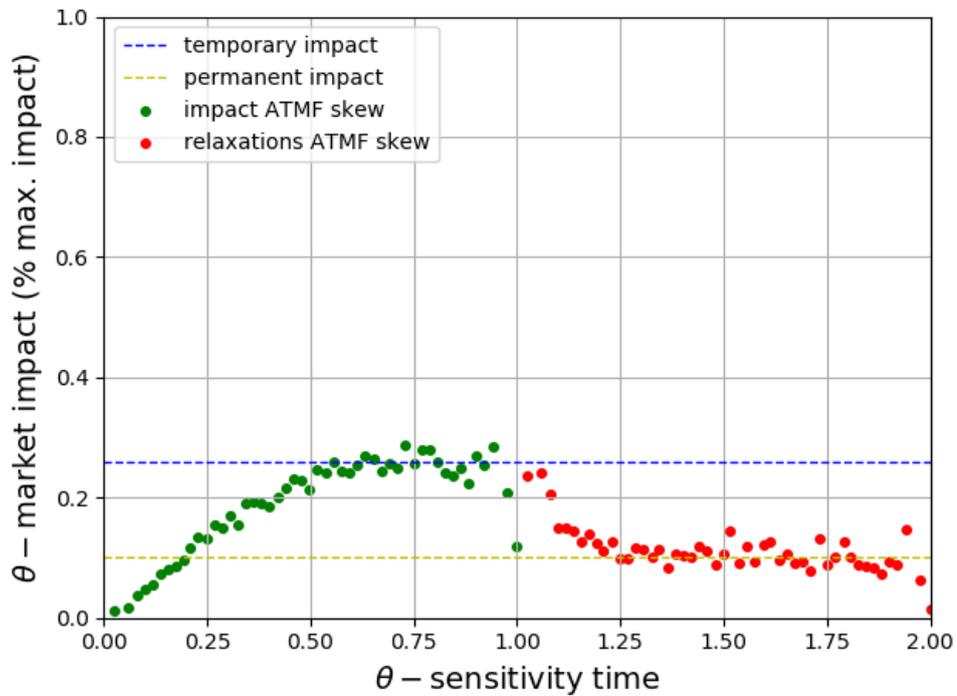


Figure .12 – Market impact dynamics in the case of the *at the money forward skew* metaorders ($\theta \equiv$ ATMF skew, set : Ω , 1,304,714 orders, 174,091 metaorders, temporary impact: 0.26, permanent impact: 0.10)

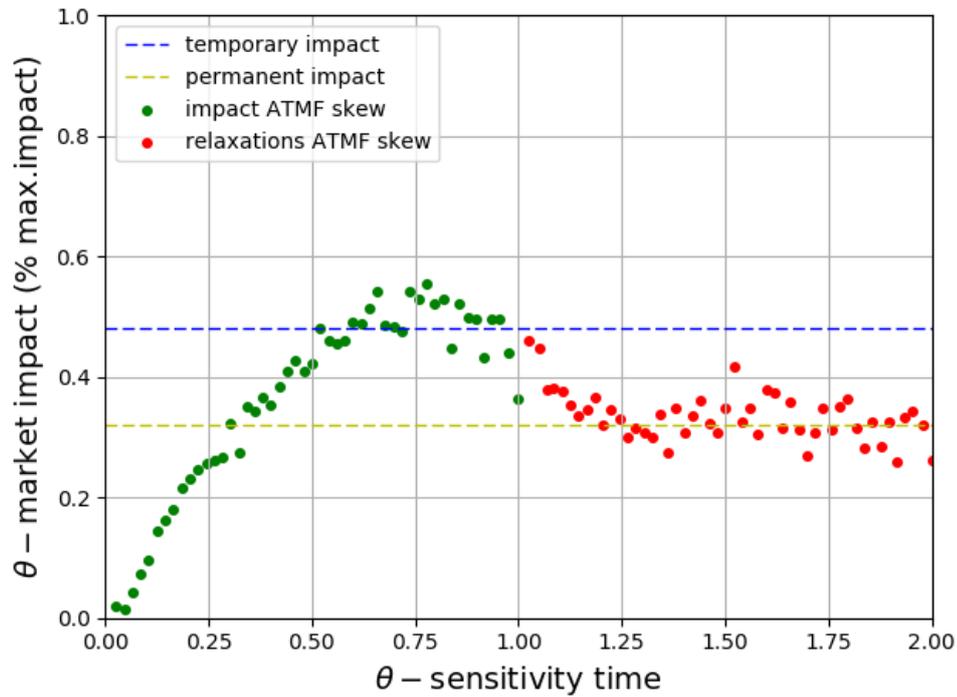


Figure .13 – Market impact dynamics in the case of the *at the money forward skew* metaorders ($\theta \equiv$ ATMF skew, set : Ω_{10} , 405,918 orders, 30,932 metaorders, temporary impact: 0.48, permanent impact: 0.32)

Result 7. *The concave shape of the temporary impact and the convex relaxation curve concerning the at the money forward skew metaorders are in line with the empirical results observed on the at the money forward volatility metaorders. Also, and more interestingly, the market impact and relaxation curves confirm the theoretical findings of [Farmer et al., 2013] that the impact should be concave and increasing, and that the final impact after the execution is performed should relax to about two-thirds of the peak impact as underlined in Figure .13.*

Square-Root laws

The results presented in this section are certainly the most important of the chapter. They confirm the consistency of the *Square-Root Law* already observed in the equity market [Almgren et al., 2005], [Bershova and Rakhlin, 2013], [Gomes and Waelbroeck, 2015], [Mastromatteo et al., 2014], [Moro et al., 2009] and [Toth et al., 2011] but also in the bitcoin market [Donier and Bonart, 2015].

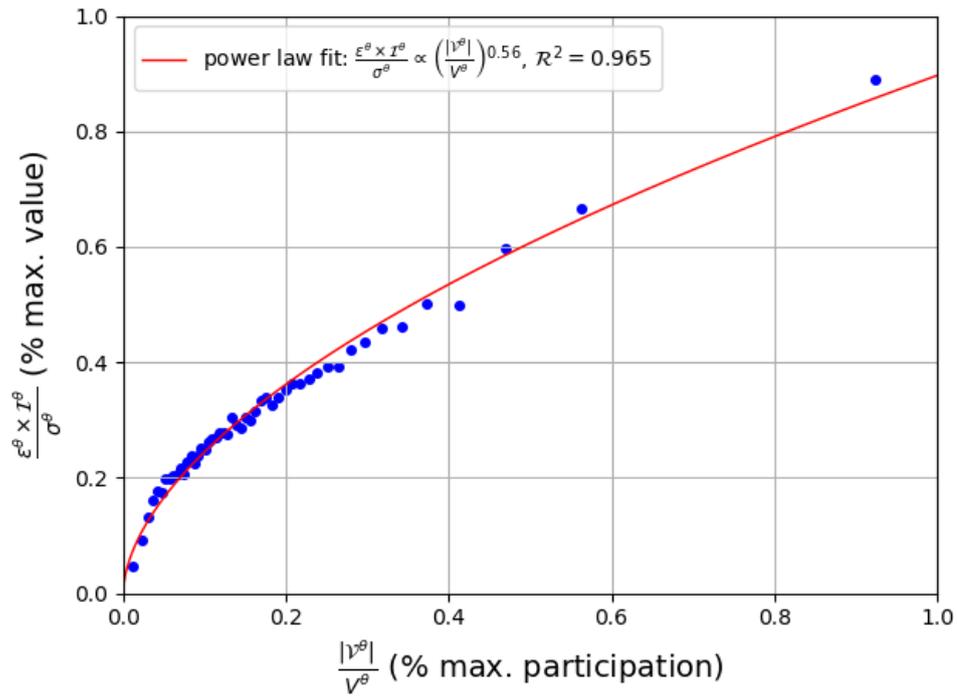


Figure .14 – The square-root law in the case of the *at the money forward volatility* metaorders, power law fit: $y \propto x^{0.56}$, $R^2 = 0.965$.

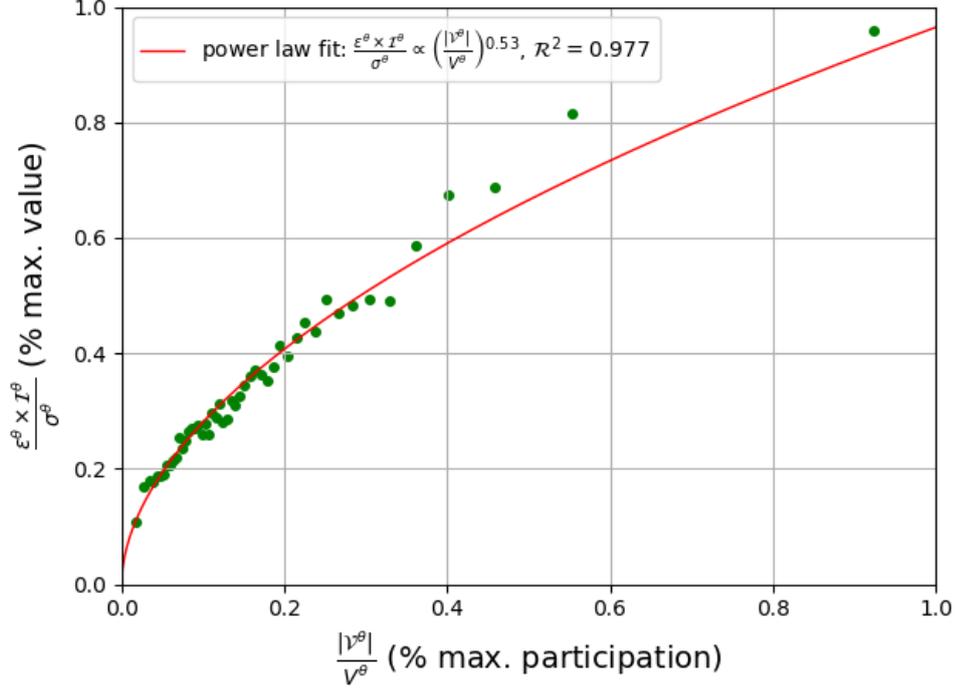


Figure .15 – The square-root law in the case of the *at the money forward skew* metaorders, power law fit: $y \propto x^{0.53}$, $\mathcal{R}^2 = 0.977$.

Result 8. *In both cases, the analysis shows that our options metaorders, present a market impact following a theoretical curve of the form $\sigma \sqrt{\mathcal{R}}$ with σ a volatility factor and \mathcal{R} a participation rate factor. Those findings support the idea for a universal underlying mechanism in market microstructure.*

Fair Pricing

We examine the fair pricing condition by considering the portfolio generated during a metaorder. Let us define the portfolio value of a metaorder ω as the quantity defined by

$$\mathcal{P}(\omega) = \sum_{i=0}^{N(\omega)-1} |Q_i(\omega)| \mathcal{O}_{t_i(\omega)}(\omega)$$

where $t_0(\omega), \dots, t_{N(\omega)-1}(\omega)$ represent the instants, $|Q_0(\omega)|, \dots, |Q_{N(\omega)-1}(\omega)|$ the quantity (positive) and $\mathcal{O}_0(\omega), \dots, \mathcal{O}_{N(\omega)-1}(\omega)$ the prices of the transactions of the metaorder ω . Hence we want to compare $\frac{\mathcal{P} - \mathcal{P}_{t_0}}{\mathcal{P}_{t_0}}$ with $\frac{\mathcal{P}_{t_0+2T} - \mathcal{P}_{t_0}}{\mathcal{P}_{t_0}}$ (Fig. .16 and .17) where \mathcal{P}_{t_0} and \mathcal{P}_{t_0+2T} are respectively the prices of the same portfolio at t_0 and $t_0 + 2T$. The red line represents the perfect fair pricing condition as it corresponds to $\frac{\mathcal{P} - \mathcal{P}_{t_0}}{\mathcal{P}_{t_0}} = \frac{\mathcal{P}_{t_0+2T} - \mathcal{P}_{t_0}}{\mathcal{P}_{t_0}}$.

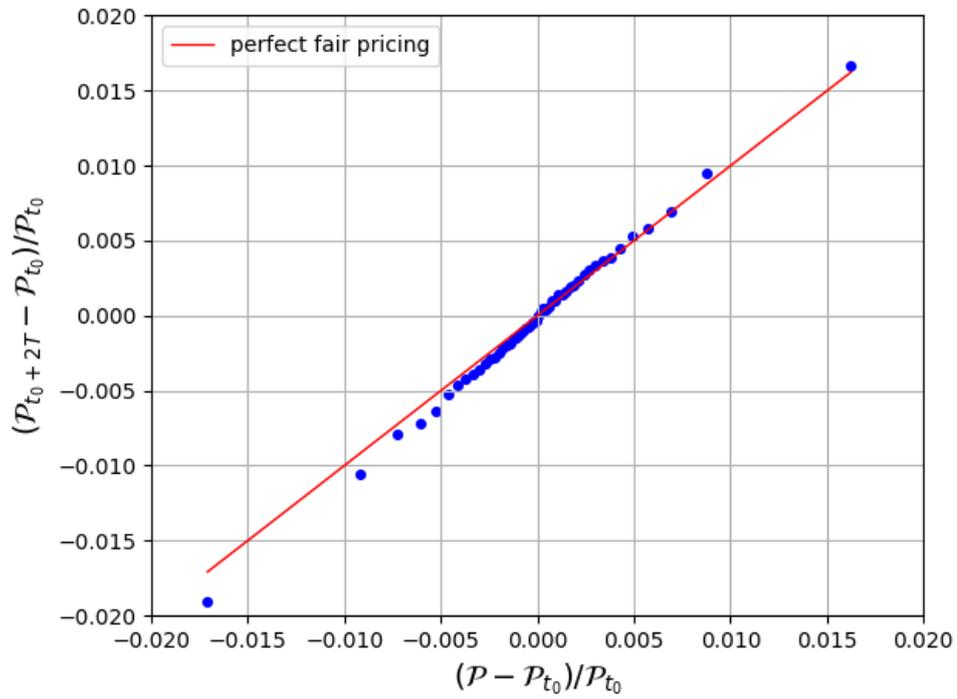


Figure .16 – Fair pricing of the portfolio value in the case of the *at the money forward volatility* metaorders

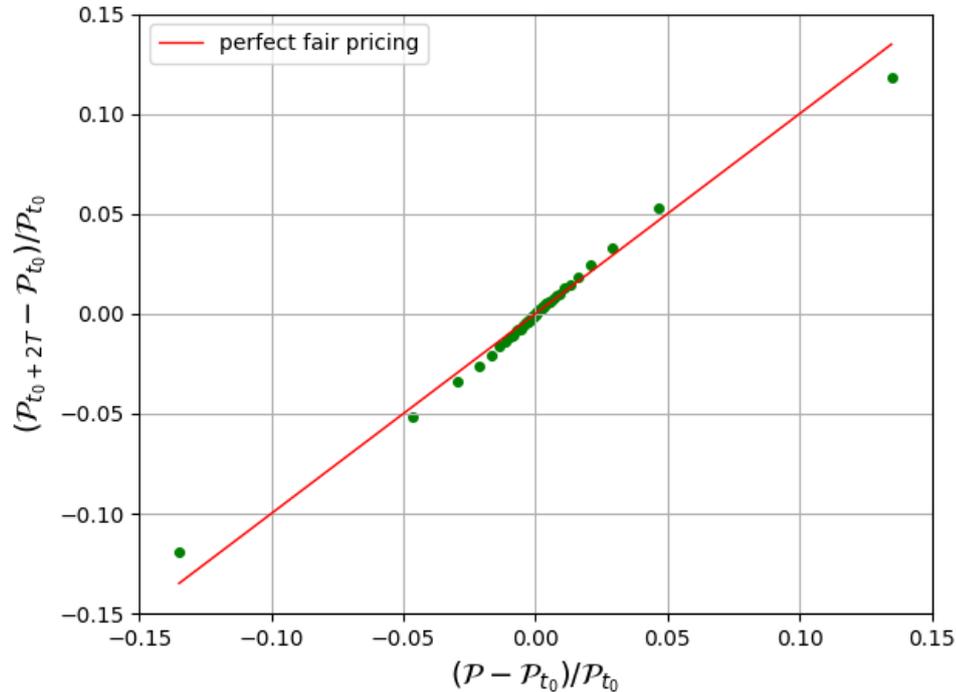


Figure .17 – Fair pricing of the portfolio value in the case of the *at the money forward skew* metaorders

Result 9. *The fair pricing seems to be also hold on the options market. This confirms the fair pricing hypothesis introduced in [Farmer et al., 2013] as a universal mechanism concerning the metaorders and their interaction with the price formation process.*

In this chapter a new algorithmic definition of an *options metaorder* has been proposed. The statistical results based on this definition show a pretty good agreement with some observations already highlighted in the stock markets: *Square-Root Law*, *Fair Pricing* and *Market Impact Dynamics*. In both cases, the analysis shows that the temporary impact is increasing and concave, with a convex decreasing relaxation phase. More precisely, the price reversion after the completion of a trade yields a permanent impact such that its ratio to the maximum impact observed at the last fill is roughly two-third, already highlighted empirically on equity markets.

3 Part III: Option Pricing and Market Impact

3.1 Chapter III - How Option Hedging Shapes Market Impact

Chapter III addresses the following question: How to build an option pricing theory with market impact compatible with some previous empirical observations and non-arbitrage? We

present a perturbation theory of market impact in which we consider only *local linear market impact*. We study the execution process of hedging derivatives and show how these *hedging metaorders* can explain some stylized facts observed in the empirical market impact literature. As we are interested in the hedging process we will establish that the arbitrage opportunities that exist in discrete time execution strategies vanish when the trading frequency goes to infinity letting us to derive a pricing equation.

3.1.1 Perturbation theory with local linear market impact

Let us consider that we have sold an European style option whose value is $u(t, s)$ with a fixed maturity T . Greeks are given as usual by

$$\begin{aligned}\Delta(t, s) &= \partial_s u(t, s), \\ \Gamma(t, s) &= \partial_{ss} u(t, s) = \partial_s \Delta(t, s), \\ \Theta(t, s) &= \partial_t u(t, s).\end{aligned}$$

We consider linear market impact in the framework of our perturbation theory. Therefore in what follows we will take:

$$S_* \rightarrow S_* + \lambda S_*^{1+\zeta} N_* \quad (3)$$

i.e. the impact of an order to buy N_* stocks at the price S_* is $\lambda S_*^{1+\zeta} N_*$ when the size N_* of the order is sufficiently small (*linear market impact*) and $\zeta \in \{0, 1\}^2$. We set the parameter ϕ defined by

$$\phi \equiv \phi(t, S) := \lambda S^{1+\zeta} \partial_{ss} u(t, S) = \lambda S^{1+\zeta} \Gamma(t, S). \quad (4)$$

We assume an initial spot moves from S to $S + dS$, dS supposed to be small and $S > 0$. By following an iterative hedging strategy one has to adjust the hedge by ΓdS stocks after the initial spot move $S \rightarrow S + dS$, which then again impacts the spot price by $dS_1 = \lambda(S + dS_0)^{1+\zeta} \Gamma dS_0$ according to the linear market impact rule presented above with $dS_0 = dS$. This spot move is then followed by a second hedge adjustment of ΓdS_1 , which in turn impacts the spot price by $dS_2 = \lambda(S + dS_0 + dS_1)^{1+\zeta} \Gamma dS_1$ and so on and so forth. Hence one has

$$\begin{cases} dS_0 = dS \\ \forall n \in \mathbb{N}, dS_{n+1} = \lambda \left(S + \sum_{k=0}^n dS_k \right)^{1+\zeta} \Gamma dS_n = \phi \left(1 + \frac{1}{S} \sum_{k=0}^n dS_k \right)^{1+\zeta} dS_n \end{cases} \quad (5)$$

with the assumption that $\Gamma \equiv \Gamma(t, S)$ remain constant during the hedging procedure described just above.

In the context of our perturbation theory of market impact, we need to study the convergence properties of the market impact series $\sum_{n \in \mathbb{N}} dS_n$. Let us introduce the following definitions:

Definition 3. A sequence $(dS_n)_{n \in \mathbb{N}}$ defined by (5) is said to be a *market impact scenario* starting from $dS \in \mathbb{R}$.

² $\zeta = 0$ corresponds to [Abergel and Loeper, 2017] and $\zeta = 1$ to [Loeper, 2018]

Definition 4. A market impact scenario $(dS_n)_{n \in \mathbb{N}}$ is said to be *admissible from a trading perspective* if there exists $R > 0$ such that for any $dS \in (-R, R)$, $\sum_{n \in \mathbb{N}} dS_n$ converges.

The following result gives an equivalent criterion on the parameter ϕ for a market impact scenario to be admissible:

Result 10. The ϕ -market impact scenario $(dS_n)_{n \in \mathbb{N}}$ is admissible from a trading perspective if, and only if, $\phi \in (-\infty, 1)$.

3.1.2 Execution strategies with local linear market impact

We study metaorders execution in our framework and show that when the metaorder is enough fragmented, there is no execution strategy better than another.

Let us consider the market impact rule (3) when $\zeta = 0$

$$S_* \rightarrow S_* + \lambda S_* N_*. \quad (6)$$

Assume an agent wants to execute incrementally an order of size N with $\mathcal{K} \in \mathbb{N}^*$ child orders of size $n_1, n_2, \dots, n_{\mathcal{K}}$ satisfying

$$\sum_{k=1}^{\mathcal{K}} n_k = N.$$

Without loss of generality we will suppose that $N \in \mathbb{R}_+^*$ and $n_1, \dots, n_{\mathcal{K}} \in \mathbb{R}_+^*$ - i.e. a buy order, the same holds for a sell order - such that

$$\lim_{\mathcal{K} \rightarrow +\infty} \sup_{1 \leq k \leq \mathcal{K}} |n_k| = 0. \quad (7)$$

The condition (7) is needed to ensure that (6) can be applied to $n_1, \dots, n_{\mathcal{K}}$ for \mathcal{K} large enough. Applying this when $\mathcal{K} = 2$ leads to

$$S \xrightarrow{n_1} S + \lambda S n_1 \xrightarrow{n_2} S + \lambda S n_1 + \lambda(S + \lambda S n_1) n_2,$$

which can be written

$$S \xrightarrow{n_1} S(1 + \lambda n_1) \xrightarrow{n_2} S(1 + \lambda(n_1 + n_2) + \lambda^2 n_1 n_2).$$

Let us denote $\xrightarrow{n_1, \dots, n_{\mathcal{K}}}$ the contraction of $\xrightarrow{n_1} \dots \xrightarrow{n_{\mathcal{K}}}$. By a straightforward induction we have for all $\mathcal{K} \in \mathbb{N}^*$,

$$S \xrightarrow{n_1, \dots, n_{\mathcal{K}}} S \left(1 + \sum_{k=1}^{\mathcal{K}} \lambda^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq \mathcal{K}} n_{i_1} n_{i_2} \dots n_{i_k} \right).$$

Result 11. When the trading frequency \mathcal{K} goes to infinity, the arbitrage opportunities that exist when \mathcal{K} is finite vanish. Besides, when $\mathcal{K} < +\infty$ is given, the worst execution strategy is the equally-sized strategy.

3.1.3 The pricing equation

We consider the framework of covered options, hence we start from a delta-hedged portfolio. Let us assume that the stock price S moves initially by dS such that

$$dS = S(\nu dt + \sigma dW_t), \quad (8)$$

where $(W_t)_{t \geq 0}$ is a Brownian motion. By following the hedging strategy presented above we move the spot from S to $S + \tilde{dS}$ where \tilde{dS} represents the cumulative market impact at the end of the re-hedging procedure

$$\tilde{dS} := \sum_{n=0}^{+\infty} dS_n, \quad (9)$$

which gives

$$\tilde{dS} = \frac{dS}{1-\phi} + \frac{1}{S} \frac{(1+\zeta)\phi}{1-\phi} \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n dS_k \right) dS_n + \frac{1}{S^2} \frac{\zeta\phi}{1-\phi} \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n dS_k \right)^2 dS_n. \quad (10)$$

This gives at the leading order

$$\tilde{dS} = \frac{dS}{1-\phi} + \tilde{\nu} S dt + o(dt). \quad (11)$$

The value V of the hedging portfolio containing $\Delta(t, S)$ stocks at t evolves as

$$dV = \Delta \tilde{dS} + \mathcal{R}$$

as S moves to $S + \tilde{dS}$ with

$$\begin{aligned} \mathcal{R} &= N \times (\text{Final price of the stocks bought} - \text{Average execution price}) \\ &= \frac{1}{2} \lambda S N^2 \end{aligned}$$

at the leading order, with $N = \Gamma \tilde{dS}$ the number of stocks bought during the re-hedging procedure. This gives

$$dV = \Delta \tilde{dS} + \frac{1}{2} \lambda S^{1+\zeta} (\Gamma \tilde{dS})^2. \quad (12)$$

Besides assuming that the option is sold at its fair price, we have $dV = du$ with at the leading order, for S moving to $S + \tilde{dS}$,

$$du = \partial_t u dt + \partial_s u \tilde{dS} + \frac{1}{2} \partial_{ss} u (\tilde{dS})^2 + o(dt). \quad (13)$$

Therefore we obtain the pricing equation

$$\begin{aligned} \partial_t u + \frac{1}{2} \sigma^2 s^2 \partial_{ss} u \frac{1}{1-\phi} &= 0, \\ \phi &= \lambda s^{1+\zeta} \partial_{ss} u. \end{aligned} \quad (14)$$

Result 12. *Under mild assumptions, every European style contingent claim can be perfectly replicated via a δ -hedging strategy given by the unique, smooth away from the maturity T , solution to the pricing equation (14).*

3.1.4 The hedging metaorders

In Chapter I we define a metaorder as a large trading order split into small pieces and executed incrementally the same day by the same *agent* on the same *stock* and all having the same *direction* (buy or sell). This motivates the following definition:

Definition 5. Let $(dS_n)_{n \in \mathbb{N}}$ a regular³ ϕ -market impact scenario. The market impact scenario $(dS_n)_{n \in \mathbb{N}}$ is said to be an **hedging metaorder** if, and only if, $\phi \in (0, 1)$.

Result 13. Under Fair Pricing, the ratio between permanent impact I and temporary impact \mathcal{I} satisfies

$$\frac{I}{\mathcal{I}} = 1 - \frac{\phi}{2}.$$

Therefore

$$\frac{1}{2} \leq \frac{I}{\mathcal{I}} \leq 1.$$

Now we want to focus on the price reversion of metaorders and study the relaxation of the metaorders as a part of an informational process.

4 Part IV: Information and Market Impact

4.1 Chapter IV - The Structure of Information in Market Impact

We formulate a theory which underlines the role of information on the price reversion process of metaorders and show that the relaxation of a metaorder results from interactions between the information conveyed by metaorders and the market. As a consequence the *permanent impact* of metaorders appears to be the result of this informational process.

4.1.1 Empirical observations

Empirical study	Permanent impact / Temporary impact
[Moro et al., 2009]	0.5 ~ 0.7 (single day metaorders)
[Bershova and Rakhlin, 2013]	~ 2/3 (single day metaorders)
[Gomes and Waelbroeck, 2015]	~ 2/3 (informed) - ~ 0 (uninformed) after 10 days
[Said et al., 2018]	~ 2/3 (single day metaorders)
[Bucci et al., 2019b]	~ 2/3 at the end of the same day - ~ 1/3 after 50 days

Table .4 - Decay of the impact obtained in some empirical studies

The traditional view in finance is that market impact is just a reflection of information and postulates that the functional form of market impact is the expression of how informed the agents are who trade with a given volume. As information is difficult to define and measure

³This notion will be defined in Chapter III

the metaorder size and duration have been used as explanatory variables for the temporary market impact. If it seems reasonable to assume that the characteristics of metaorders can determine the shape of the temporary impact, they cannot explain the permanent impact: Once executed, the information reflected in the metaorder is subject to market noise. So the permanent market impact must be the result of this interference. In the following we provide a model for the information content of metaorders.

4.1.2 Model description

As market impact is just a reflection of information we will postulate that the decay of the impact is just the response of the market to the information conveyed through metaorders trade execution. The existence of a permanent market impact – in contrast of a temporary impact – means that once the relaxation of the metaorder is over there is only a fraction of this information absorbed by the market. We will denote the fraction of the metaorder information absorbed by the market by \mathcal{R} . Note that if the market is efficient as expressed by [Fama, 1970], which means that security prices at any time fully reflect all available information, then \mathcal{R} must be equal to the ratio between the temporary impact and the permanent market impact. We will make the assumption that the dynamics of the process \mathcal{R} is given by

$$\mathcal{R}_0 = 1 \quad \mathbb{P} - a.s. \quad (15)$$

and the process \mathcal{R} after the $(n + 1)$ -th event is updated such as

$$\mathcal{R}_{n+1} = \underbrace{\frac{1}{n+2}\mathcal{R}_0}_{\text{lost of information}} + \underbrace{\left(1 - \frac{1}{n+2}\right)\mathcal{R}_n\Delta_{n+1}}_{\text{perturbated multiplicative noise}} \quad \mathbb{P} - a.s., \quad (16)$$

for any $n \in \mathbb{N}$ and where for all $n \geq 1$, Δ_n is a random variable valued in $[0, 1]$ representing the effect of the n -th noise on the process \mathcal{R} such that $\mathbb{P}(\Delta_n = 0) < 1$.

As we are interested in computing statistical averages let us introduce Ω the set of the price trajectories of a single stock after the full execution of a metaorder. Hence in what follows we will consider $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}}, \mathbb{P})$ a filtered probability space with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and for all $n \in \mathbb{N}^*$, $\mathcal{F}_n = \sigma(\Delta_k, 1 \leq k \leq n)$. We set for all $n \in \mathbb{N}$,

$$r_n := \mathbb{E}[\mathcal{R}_n] \quad (17)$$

and

$$\delta_n := \mathbb{E}[\Delta_n]. \quad (18)$$

Hence the system (1)-(2) gives

$$\begin{cases} r_0 = 1, \\ \forall n \in \mathbb{N}, r_{n+1} = \frac{1 + (n+1)r_n\delta_{n+1}}{n+2}. \end{cases} \quad (19)$$

In what follows the sequence $(r_n)_{n \in \mathbb{N}}$ will be the quantity of interest.

Result 14. *There exists a critical rate of convergence giving by $1 - \delta_n \sim_{+\infty} \frac{l}{n}$ for some $l > 0$, which authorizes the establishment of permanent market impact in a such way that it is equal to $\frac{1}{l+1}$ of the temporary impact. This can also be seen as the fraction of the metaorder information absorbed by the market at equilibrium. Thus which is determinant in the establishment of permanent impact is the speed of the informational interaction between the metaorder and the market. On one hand, if the information conveyed by the metaorder to the market is rapidly kept – in the sense that $(\delta_n)_{n \in \mathbb{N}^*}$ converges quickly to 1 –, then there is no relaxation. On the other hand, if the information is slowly kept, it vanishes and there is no permanent impact. Between those two extreme situations there is a critical regime leading to permanent market impact.*

Part I

Empirical Analysis of Equity Traders Market Impact

Market Impact: A Systematic Study of Limit Orders

Abstract

This paper is devoted to the important yet little explored subject of the market impact of *limit orders*. Our analysis is based on a proprietary database of metaorders - large orders that are split into smaller pieces before being sent to the market. We first address the case of *aggressive* limit orders and then, that of *passive* limit orders. In both cases, we provide empirical evidence of a power law behaviour for the temporary market impact. The relaxation of the price following the end of the metaorder is also studied, and the long-term impact is shown to stabilize at a level of approximately two-thirds of the maximum impact. Finally, a *fair pricing* condition during the life cycle of the metaorders is empirically validated.

Keywords: *Market microstructure, statistical finance, market impact, fair pricing, automated trading, limit orders.*

1 Introduction

It is a commonly acknowledged fact that market prices move during the execution of a trade - they go up for a large buy order and decrease for a large sell order. This is, loosely stated, the phenomenon known as *market impact*.

The market impact of large trading orders that are split into pieces - better known as **metaorders** - and executed incrementally through a sequence of orders of smaller sizes is crucial in describing and controlling the behaviour of modern financial markets. Being able to quantify this impact is clearly a question of great relevance when studying the price formation process, and it has also become a major practical issue for optimal trading. Indeed, in order to know whether a trade will be profitable, it is essential to monitor transaction costs, which are directly linked to market impact. Measuring and modelling market impact has therefore become a central question of interest, both for researchers and practitioners, in the

field of market microstructure.

Given the importance of the subject, there exists but a few research articles pertaining to the empirical estimation of market impact, mostly due to the scarcity of data. In fact, trades and quotes, or even order book databases are not sufficient to perform the analysis: what is required is a clear identification of metaorders. Metaorders have started being recorded as such in a systematic way only recently, and mostly in proprietary databases that are not readily accessible to academic researchers in the field of market microstructure. The analyses presented in [Almgren et al., 2005] [Moro et al., 2009] [Gatheral, 2010] [Toth et al., 2011] [Bershova and Rakhlin, 2013] [Bacry et al., 2015] [Zarinelli et al., 2015] [Gomes and Waelbroeck, 2015] essentially cover all that is published about the market impact of large orders.

Although difficult to measure in practice, market impact has been studied from a theoretical point of view. In an economic theory perspective, the information held by investors, which governs their decisions, should have some predictive power over future prices. This point was thoroughly investigated by [Farmer et al., 2013], a paper which we find enlightening and use as reference for the theoretical measurements of market impact. In a related study, [Bershova and Rakhlin, 2013] uses their own proprietary database to perform an empirical analysis of a set of large institutional orders, and validates some predictions of the [Farmer et al., 2013] model. Such a comparison will also be performed in the present work.

Our paper is a contribution to this strand of research, with a specific focus on the market impact of *limit orders*. As a matter of fact, market orders are generally not used by institutional investors because of the lack of control they imply. On the contrary, limit orders, whether they are *aggressive* - crossing the spread - or *passive*, form the vast majority of orders that are actually sent to the market during the execution of a large trade. As such, they should be the main subject of interest in a study of market impact. To the best of our knowledge, ours is the first academic study of market impact with an emphasis on limit orders. The statistical results presented in this paper are obtained from a proprietary database consisting of appropriately selected limit orders executed on the European equity market between January 2016 and December 2017. The originality of our approach lies in the fact that the study relies on an algorithmic-based identification and reconstruction of metaorders from the database of all orders.

The paper is organized as follows: Section 2 is a short review of the literature on market impact. Section 3 introduces our main definitions of metaorders and market impact measures. Section 4 presents our empirical results: They confirm that limit orders behave in agreement with some *stylized* facts already established in the literature, and shed a new light on the influence of passive orders in an execution strategy. Section 5 considers the so-called *Square Root Law* regarding to our metaorders. Section 6 deals with the fair pricing condition concerning our metaorders. Section 7 is a discussion of our results and their implications, including some comparisons with the literature. Appendix I.A recalls for reference the framework and main results of the theoretical, agent-based market impact model developed in [Farmer et al., 2013].

2 A short review of the market impact literature

The strategic reasons underlying the incremental execution of metaorders were originally analyzed by [Kyle, 1985], where a model considering an insider trader with monopolistic information about future prices is introduced. It is shown that the optimal strategy for such a trader consists of breaking its metaorder into pieces and execute them incrementally at a uniform rate, gradually incorporating its information into the price. In Kyle's theory the total impact is a linear function of size, and the price increases linearly with time as the auctions take place. The prediction of linearity also appeared in the work of [Huberman and Stanzl, 2004]. They showed that, in the case of constant liquidity providing, and in order to prevent arbitrage, the permanent impact must be linear, i.e. incremental impact per share remains constant during the metaorder life. It is however the case that real data contradict these predictions: metaorders do not exhibit linear market impact. In fact, most empirical studies consistently highlight a concave impact, in sharp contrast with the theoretical linear shape. The first relevant empirical study of market impact is [Almgren et al., 2005], which directly measures the market impact of large metaorders in the US equity market and provides empirical evidence of a concave temporary impact. Later studies [Moro et al., 2009] [Toth et al., 2011] [Bacry et al., 2015] also find a concave market impact, in rough agreement with a square root formula. Following these experimental findings, some theoretical efforts have been made to reconcile Kyle's model with a concave dependence on size, essentially by adding the hypothesis that larger metaorders contain less information per share than smaller ones. [Farmer et al., 2013] presents a model enriching Kyle's approach with this concave dependence on size. Whereas Kyle considers a single, monopolistic informed trader, [Farmer et al., 2013] introduces several competitive traders receiving a common information signal and then choosing independently the size of the order they submit to an algorithmic execution service. This set-up is close in spirit to the real-life organization of the major players in the market, which operate by setting up an internal market with a stakeholder recovering all the orders before executing them on the external market.

To be even more specific, market impact can be studied from two different perspectives. The first one, introduced in the previous paragraph, addresses the effect of a metaorder being executed on the price formation process. This effect is commonly termed the temporary market impact. It is clearly an important explanatory variable of the price discovery and is studied as such in several papers [Kyle, 1985] [Hautsch and Huang, 2012] [Farmer et al., 2004]. Temporary market impact is obviously the main source of trading costs, and models based on empirical measurements can be used in optimal trading schemes [Almgren et al., 2005] [Gatheral, 2010] [Lehalle and Dang, 2010] [Almgren and Chriss, 2001] [Gatheral and Schied, 2013], or used by an investment firm in order to understand its trading costs [Bershova and Rakhlin, 2013] [Brokmann et al., 2015] [Mastromatteo et al., 2014]. One common conclusion to the studies is that the temporary market impact is mainly characterized by three components. The first, obvious one is the size of the metaorder, suitably rescaled by a quantity reflecting the traded volume of the security under scrutiny. The daily participation or trading rates capture most of the dynamics of this component. Note that some empirical studies such as

[Brokmann et al., 2015] prefer to consider as a scaling factor the participation rate during the metaorder life. This approach presents the advantage to include a duration effect not captured by the daily participation rate, and it will be used in the present study. One must also allow for some dependency on the price uncertainty while the metaorder is executed, and the volatility or the bid-ask spread are typical measures of this uncertainty. Last but not least, it seems necessary to capture the information leakage generated by the metaorder, the number of orders executed during the metaorder or its duration being good proxies.

There is a second, more controversial type of market impact, pertaining to the persistence of a shift in the price after the metaorder is fully executed. This effect is commonly called the *permanent market impact*. Research papers dealing with permanent impact can be separated in two categories. The first one considers the permanent impact as the consequence of a mechanical process. The second ones consider the permanent impact as the trace of new information in the price. Among those who share the mechanical vision of the permanent market impact, there are two approaches. The first picture of [Farmer et al., 2013] and [Bershova and Rakhlin, 2013] says that the permanent impact is important and roughly equals to 2/3 of the temporary impact. This is derived from a fair-pricing hypothesis. In the second picture [Bouchaud, 2010], there is actually no such thing as a permanent impact, the slow decay of the market impact being the result of the long memory of the order flow. These two approaches are incompatible. In the present paper, the studies performed on the price relaxation seem to advocate in favour of a permanent impact at the approximate two-thirds level, in agreement with the existing empirical literature as well as the fair pricing condition of [Farmer et al., 2013].

3 Definitions, Algorithm and Market Impact measures

3.1 Basic Definitions

Some basic concepts, and the algorithmic definition of a metaorder, are introduced here.

Definition 6. *A limit order is an order that sets the maximum or minimum price at which an agent is willing to buy or sell a given quantity of a particular stock.*

Definition 7. *An aggressive limit order is one that instantaneously removes liquidity from the order book by triggering a transaction. An aggressive order crosses the Bid-Ask spread. In other words an aggressive buy order will be placed on the ask, and an aggressive sell order will be placed on the bid.*

A limit order that is not aggressive is termed **passive**. Passive orders sit in the order book until they are executed or cancelled.

Loosely speaking a **metaorder** is a large trading order that is split into small pieces and executed incrementally. In order to perform rigorous statistical analyses, a more specific and precise definition of a metaorder is required, and given in Definition 8 below:

3. Definitions, Algorithm and Market Impact measures

Definition 8. A *metaorder* is a series of orders sequentially executed during the same day and having those same attributes:

- *agent* i.e. a participant on the market (an algorithm, a trader...);
- *product id* i.e. a financial instrument (a share, an option...);
- *direction* (buy or sell);

The advantage of adopting such a definition is that it is no longer necessary to work directly on raw metaorder data. Indeed, a series of orders executed by the same actor on the same product on the financial market will behave like a metaorder and therefore can be considered as such (Figure I.1).

AGENT	PRODUCT_ID	DIRECTION	TIMESTAMP	CURRENCY	EXEC_PRICE	EXEC_QTY	PRICE_TYPE	ORDER_TYPE
agent01	product01	Sell	01-APR-16 16:52:00.536612	EUR	87,5	127	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:03:48.203844	EUR	38,475	393	Aggressive	Limit
agent01	product01	Sell	01-APR-16 13:33:09.566350	EUR	87,4	64	Aggressive	Limit
agent01	product01	Sell	01-APR-16 16:52:00.536641	EUR	87,5	52	Aggressive	Limit
agent04	product20	Sell	01-APR-16 09:46:08.135532	EUR	4,128	570	Aggressive	Limit
agent11	product13	Sell	01-APR-16 15:10:32.279258	EUR	15,2	459	Aggressive	Limit
agent11	product02	Sell	01-APR-16 13:39:30.559504	EUR	4,263	2000	Aggressive	Limit
agent05	product07	Sell	01-APR-16 16:00:29.447697	EUR	58,5	173	Aggressive	Limit
agent05	product11	Buy	01-APR-16 16:34:49.039137	EUR	39,08	374	Aggressive	Limit
agent11	product09	Buy	01-APR-16 13:03:01.846048	EUR	17,005	500	Aggressive	Limit
agent11	product04	Sell	01-APR-16 15:14:18.662973	EUR	19,23	746	Aggressive	Limit
agent11	product02	Buy	01-APR-16 16:12:15.194178	EUR	4,194	942	Aggressive	Limit
agent05	product07	Sell	01-APR-16 14:23:20.168576	EUR	58,32	8	Aggressive	Limit
agent11	product04	Buy	01-APR-16 09:44:28.440733	EUR	19,535	500	Aggressive	Limit
agent01	product01	Sell	01-APR-16 13:33:09.566321	EUR	87,4	100	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:07:52.669195	EUR	38,45	404	Aggressive	Limit

Figure I.1 – Sample of the history of orders before the identification process of the algorithm. Each color corresponds to a different metaorder according to the methodology introduced in Definition 8.

AGENT	PRODUCT_ID	DIRECTION	TIMESTAMP	CURRENCY	EXEC_PRICE	EXEC_QTY	PRICE_TYPE	ORDER_TYPE
agent01	product01	Sell	01-APR-16 13:33:09.566321	EUR	87,4	100	Aggressive	Limit
agent01	product01	Sell	01-APR-16 13:33:09.566350	EUR	87,4	64	Aggressive	Limit
agent01	product01	Sell	01-APR-16 16:52:00.536612	EUR	87,5	127	Aggressive	Limit
agent01	product01	Sell	01-APR-16 16:52:00.536641	EUR	87,5	52	Aggressive	Limit
agent04	product20	Sell	01-APR-16 09:46:08.135532	EUR	4,128	570	Aggressive	Limit
agent05	product07	Sell	01-APR-16 14:23:20.168576	EUR	58,32	8	Aggressive	Limit
agent05	product07	Sell	01-APR-16 16:00:29.447697	EUR	58,5	173	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:03:48.203844	EUR	38,475	393	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:07:52.669195	EUR	38,45	404	Aggressive	Limit
agent05	product11	Buy	01-APR-16 16:34:49.039137	EUR	39,08	374	Aggressive	Limit
agent11	product09	Buy	01-APR-16 13:03:01.846048	EUR	17,005	500	Aggressive	Limit
agent11	product04	Buy	01-APR-16 09:44:28.440733	EUR	19,535	500	Aggressive	Limit
agent11	product04	Sell	01-APR-16 15:14:18.662973	EUR	19,23	746	Aggressive	Limit
agent11	product13	Sell	01-APR-16 15:10:32.279258	EUR	15,2	459	Aggressive	Limit
agent11	product02	Sell	01-APR-16 13:39:30.559504	EUR	4,263	2000	Aggressive	Limit
agent11	product02	Buy	01-APR-16 16:12:15.194178	EUR	4,194	942	Aggressive	Limit

Figure I.2 – Sample of the history of orders after the identification process of the algorithm. Each color corresponds to a different metaorder according to the methodology introduced in Definition 8.

I. Market Impact: A Systematic Study of Limit Orders

Let us mention that the current study is not restricted to metaorders executed on a single market, due to the fact that an instrument can be simultaneously traded on several markets. Also note that orders executed during the same second are aggregated in order to avoid time stamping issues: the quantities are summed up, the local VWAP is set as the execution price, and the `TIMESTAMP` of the last order is retained for all orders during the same second. One can see below (see Figure I.3) how those simplifications reduce the complexity of Figure I.2:

AGENT	PRODUCT_ID	DIRECTION	TIMESTAMP	CURRENCY	EXEC_PRICE	EXEC_QTY	PRICE_TYPE	ORDER_TYPE
agent01	product01	Sell	01-APR-16 13:33:09.566350	EUR	87,4	164	Aggressive	Limit
agent01	product01	Sell	01-APR-16 16:52:00.536641	EUR	87,5	179	Aggressive	Limit
agent04	product20	Sell	01-APR-16 09:46:08.135532	EUR	4,128	570	Aggressive	Limit
agent05	product07	Sell	01-APR-16 14:23:20.168576	EUR	58,32	8	Aggressive	Limit
agent05	product07	Sell	01-APR-16 16:00:29.447697	EUR	58,5	173	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:03:48.203844	EUR	38,475	393	Aggressive	Limit
agent05	product11	Buy	01-APR-16 12:07:52.669195	EUR	38,45	404	Aggressive	Limit
agent05	product11	Buy	01-APR-16 16:34:49.039137	EUR	39,08	374	Aggressive	Limit
agent11	product09	Buy	01-APR-16 13:03:01.846048	EUR	17,005	500	Aggressive	Limit
agent11	product04	Buy	01-APR-16 09:44:28.440733	EUR	19,535	500	Aggressive	Limit
agent11	product04	Sell	01-APR-16 15:14:18.662973	EUR	19,23	746	Aggressive	Limit
agent11	product13	Sell	01-APR-16 15:10:32.279258	EUR	15,2	459	Aggressive	Limit
agent11	product02	Sell	01-APR-16 13:39:30.559504	EUR	4,263	2000	Aggressive	Limit
agent11	product02	Buy	01-APR-16 16:12:15.194178	EUR	4,194	942	Aggressive	Limit

Figure I.3 – *Sample of the history of the orders executed corresponding to the previous figure after merging the orders. The yellow metaorder which was previously a metaorder of length $N = 4$ is now a metaorder of length $N = 2$.*

3.2 Algorithmic procedure

To carry out the study of market impact, the algorithm is trained and run on a proprietary database. Figure I.1 in the previous section represents the database in its initial state, as the input of the algorithm. Figure I.2 shows an intermediate state of the data extracted from the initial data, during the metaorder reconstruction phase. Figure I.3 displays the data in their final state, directly exploitable for statistical studies. Note that the recovery and cleaning of market data are done simultaneously during the metaorder reconstruction.

3.3 Market Impact definitions

The same framework as that in [Bacry et al., 2015] is adopted. Let Ω be the set of metaorders under scrutiny, that is, metaorders that are fully executed during a single market session, and pick $\omega \in \Omega$ executed on an instrument S and during a given day d . Its execution starts at some time $t_0(\omega)$ and ends the same day at time $t_0(\omega) + T(\omega)$. Thus $T(\omega)$ represents the duration of the metaorder. Denote by $Q(\omega)$ and $N(\omega)$ respectively the number of shares and the number of orders that have been executed during the life cycle of the metaorder ω . Hence $Q(\omega)$ is the size and $N(\omega)$ the length of ω . Let $V(\omega)$ be the volume traded the same day d on the instrument S (summed over all European trading venues) between time t_0 and $t_0 + T$, i.e. during the life cycle of ω . The sign of ω will be denoted by $\epsilon(\omega)$ with $\epsilon = 1$ for a buy order and $\epsilon = -1$ for a sell order. Clearly, all the quantities introduced in this section depend on ω .

For the sake of simplicity, we chose to omit this dependence whenever there is no ambiguity and will often write T, N, Q, V, ϵ instead of $T(\omega), N(\omega), Q(\omega), V(\omega), \epsilon(\omega)$.

The **market impact curve** of a metaorder ω measures the magnitude of the relative price variation between the starting time of the metaorder t_0 and the current time $t > t_0$. Let $\mathcal{I}_t(\omega)$ be a proxy for the realized price variation between time t_0 and time $t_0 + t$. In line with many authors [Almgren et al., 2005] [Bershova and Rakhlin, 2013] [Bacry et al., 2015], we use the *return proxy* defined by

$$\mathcal{I}_t = \frac{P_t - P_{t_0}}{P_{t_0}}, \quad (1)$$

where P_t represents either the execution price of the financial instrument S during the *execution part* of the metaorder, or the mid-price during its *relaxation part* starting when the metaorder has been fully executed. This estimation relies on the assumption that the *exogenous market moves* W_t will cancel out once averaged, i.e. as a random variable, W_t should have finite variance and basically satisfy $\mathbb{E}(\epsilon(\omega)W_t(\omega)) = 0$. One can thus write

$$\epsilon(\omega)\mathcal{I}_t(\omega) = \eta_t(\omega) + \epsilon(\omega)W_t(\omega), \quad (2)$$

where $\eta_t(\omega)$ represents the market impact curve and $W_t(\omega)$, the exogenous variation of the price corresponding to the relative price move that would have occurred if the metaorder had not been sent to the market.

4 Empirical study

4.1 Notations

Notation	Definition
ω	a metaorder
$S(\omega)$	financial instrument of the metaorder ω
$t_0(\omega)$	start time of the metaorder ω
$T(\omega)$	duration of the metaorder ω
$N(\omega)$	length of the metaorder ω
$Q(\omega)$	size of the metaorder ω
$V(\omega)$	volume traded on the day $d(\omega)$ on the instrument $S(\omega)$ during $[t_0(\omega), t_0(\omega) + T(\omega)]$
$\epsilon(\omega)$	sign of the metaorder ω
$P(\omega)$	price of $S(\omega)$
Ω	set of all the metaorders identified by the algorithm
$\Omega_{n^*} \subset \Omega$	subset of the metaorders with $N \geq n^*$

Table I.1 – *Notations and definitions*

Remark 1. *As we only consider metaorders that have at least two executed transactions, $\Omega = \Omega_2$.*

4.2 Aggressive Limit Orders

The subject of interest of this section is the metaorders generated by aggressive limit orders, that is, limit orders that actually cross the spread in order to trigger an immediate transaction. Such orders are sometimes rather loosely considered as market orders in the modelling literature on limit order books, but it is clear that they behave differently, as their execution price is always equal to that of the best available limit and can never trigger transactions at higher (buy) or lower (sell) price.

4.2.1 Data

- Study period : **1st Jan 2016 - 31st Dec 2017**
- Markets : **European Equity Markets**
- Order types : **Aggressive Limit Orders**
- Filters : **metaorders** $\omega \in \Omega$
- Number of metaorders : **1 561 505**

Duration distribution

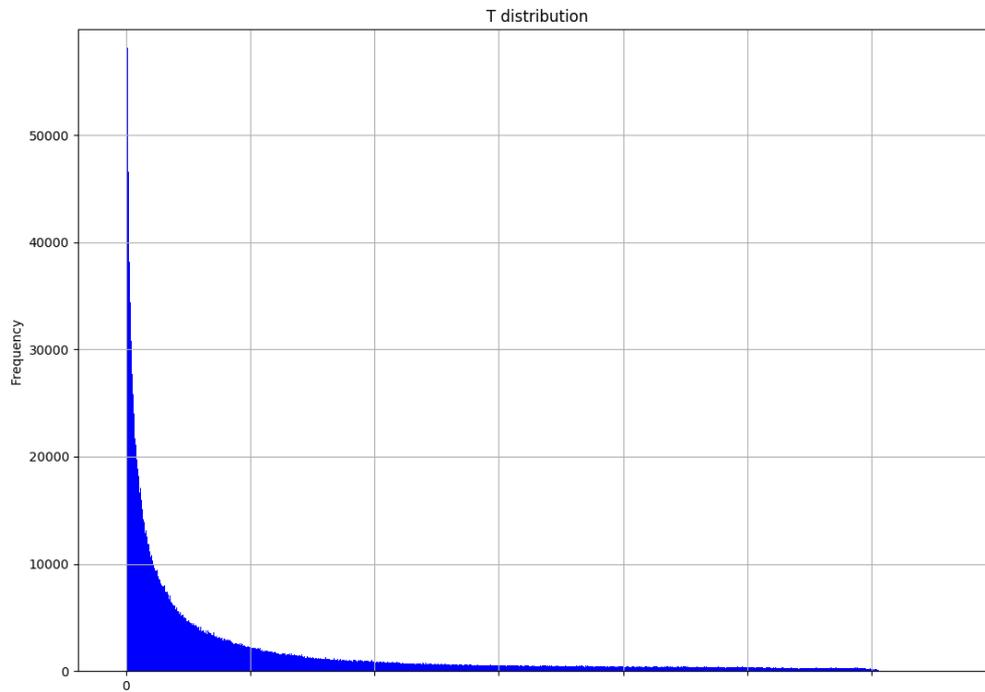


Figure I.4 – Duration distribution of the aggressive metaorders

One can observe that, in agreement with the intuition, metaorders with shorter durations are more frequent (Figure I.4).

Length distribution

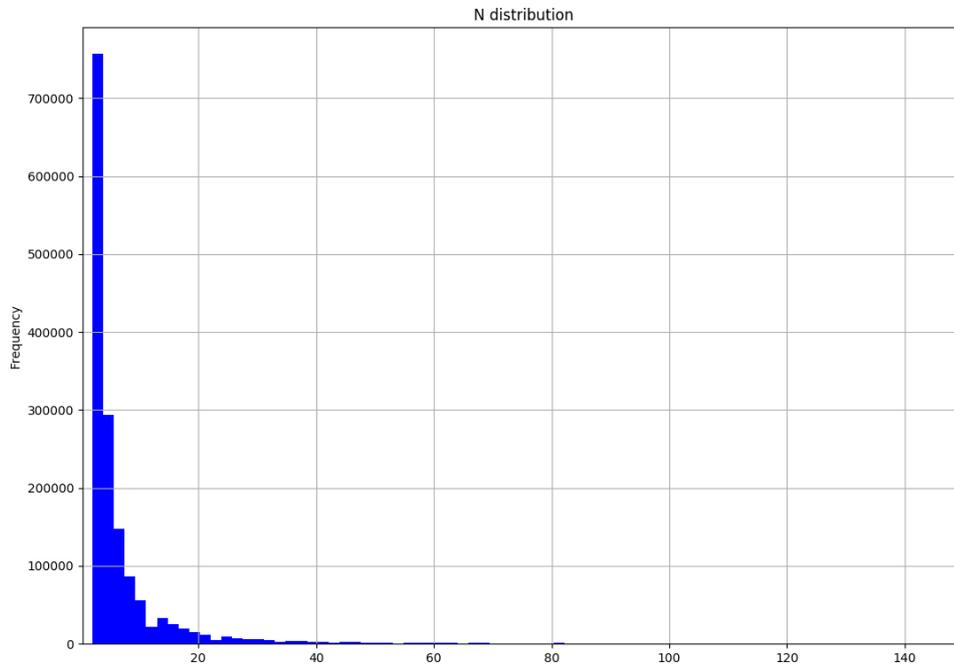


Figure I.5 – Length distribution of the aggressive metaorders

mean 8
median 4

As already observed for the duration, shorter metaorders (in length) are more represented. This observation is obviously not a surprise, since the quantities N and T are expected to be highly positively correlated. A log-log scale (Figure I.6) gives a more precise idea of the distribution of N . The apparently linear relation suggests that N follows a discrete Pareto distribution.

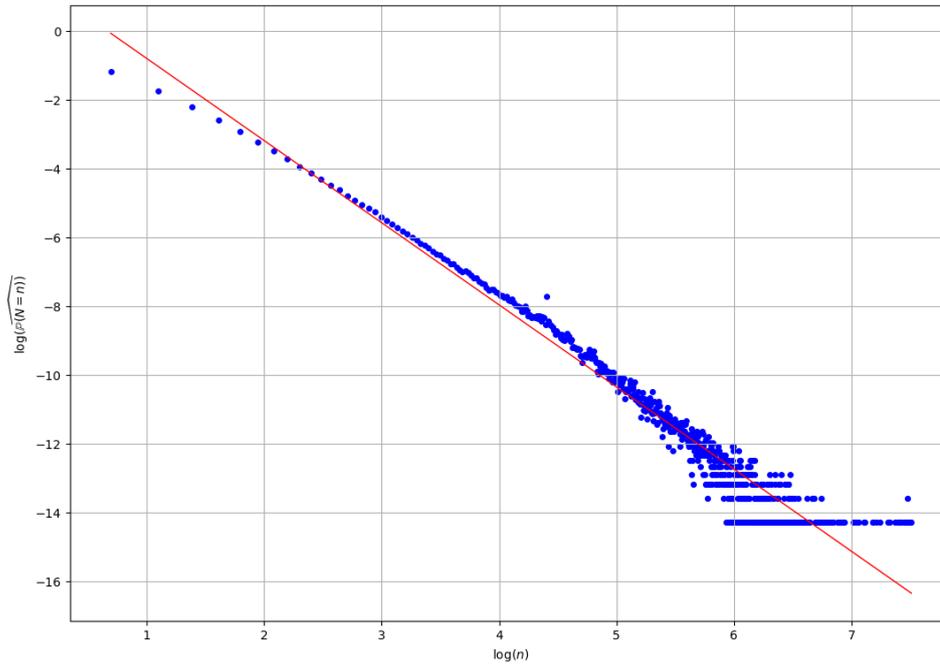


Figure I.6 - Fitting Length distribution of the aggressive metaorders

$\widehat{\mathbb{P}}(N=n)$ stands for the natural frequency estimator of the probability $\mathbb{P}(N=n)$. As regards the parameter β introduced in [Farmer et al., 2013] - see Section I.A.5 - one obtains the estimate $\beta \approx 1.4$. Therefore, N is distributed as a power law, in agreement with [Vaglica et al., 2008] who reconstructed metaorders on the Spanish stock exchange using data with brokerage codes and found that N is distributed as power law for large N with $\beta \approx 1.7$.

Participation rate distribution

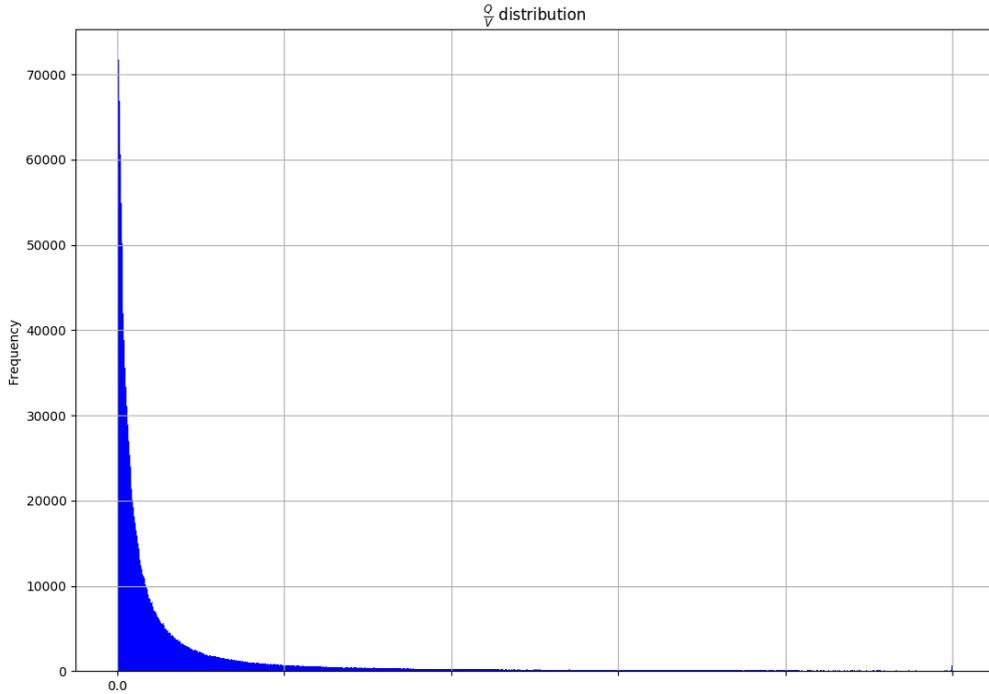


Figure I.7 - $\frac{Q}{V}$ distribution of the aggressive metaorders

Similar to the observations for duration and length (see Figures I.4 and I.5), metaorders with smaller participation rates are more represented. Again, and as expected, the quantities N and $\frac{Q}{V}$ are highly positively correlated (Figure I.7).

4.2.2 Market Impact curves

The main results of Section 4.2 are now given, namely, the *market impact curves* for aggressive metaorders. In order to plot the market impact dynamics, a *bucketing method* is used: Consider for example that one wants to plot y as a function of x , x, y being two arrays of data. One starts by ordering the couple of values (x_i, y_i) according to the values of x and then divides the **sorted (by x) distribution** $(x, y)_{sorted}$ into N_{bucket} . This procedure yields N_{bucket} subsets of the distribution $(x, y)_{sorted}$, $(x_i, y_i)_{i \in I_1}$, $(x_i, y_i)_{i \in I_2}$, ..., $(x_i, y_i)_{i \in I_{N_{bucket}}}$, and for each bucket

I_k the *mean values* (\bar{x}_k, \bar{y}_k) is computed. The last step of this bucketing method is to plot the points $(\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2), \dots, (\bar{x}_{N_{bucket}}, \bar{y}_{N_{bucket}})$.

Market Impact Dynamics

To study the dynamics of the market impact, one plots $(\epsilon(\omega) \mathcal{I}_t(\omega))_{\omega \in \Omega, t_0(\omega) \leq t \leq t_0(\omega) + 2T(\omega)}$. The first sub-interval $t_0(\omega) \leq t \leq t_0(\omega) + T(\omega)$ corresponds to the execution of the metaorder, whereas the second $t_0(\omega) + T(\omega) \leq t \leq t_0(\omega) + 2T(\omega)$ corresponds to the relaxation. The study of relaxation presents a degree of arbitrariness, since a choice has to be made as to the elapsed time after the metaorder is completed. For the sake of homogeneity, the relaxation is measured over the same duration as the execution. This choice seems to be a good compromise to cope with two antagonistic requirements, one being to minimize this elapsed time because of the diffusive nature of prices, the other being to maximize it so as to make sure that the relaxation is achieved. Further studies performed on our database actually show very little dependence of the permanent impact level on this time parameter, a result which we found quite comforting, and in line with some previous results in the literature, see e.g. [Gomes and Waelbroeck, 2015].

In order to perform an extensive statistical analysis involving metaorders of varying lengths in physical and volume time, a rescaling in time is necessary, see e.g. [Bacry et al., 2015]. With this convention, all orders are executed on the time interval $[0, 1]$ and price relaxation occurs in the time interval $[1, 2]$. For each metaorder ω , one considers $[0, 1]$ instead of $[t_0(\omega), t_0(\omega) + T(\omega)]$ $\left([0, 1] = \frac{[t_0(\omega), t_0(\omega) + T(\omega)] - t_0(\omega)}{T(\omega)} \right)$ for the *execution part* of ω and $[1, 2]$ instead of $[t_0(\omega) + T(\omega), t_0(\omega) + 2T(\omega)]$ for the *relaxation part* of ω , and then averages using the *bucketing method* previously described on the time-rescaled volume quantities.

The time variable $t \in [0, 1]$ in Figure I.8 is actually the *volume time*, i.e., the ratio between the part of the metaorder already executed at the time of the observation and the total size of the metaorder - of course, at the end of the execution part this quantity is always equal to 1.

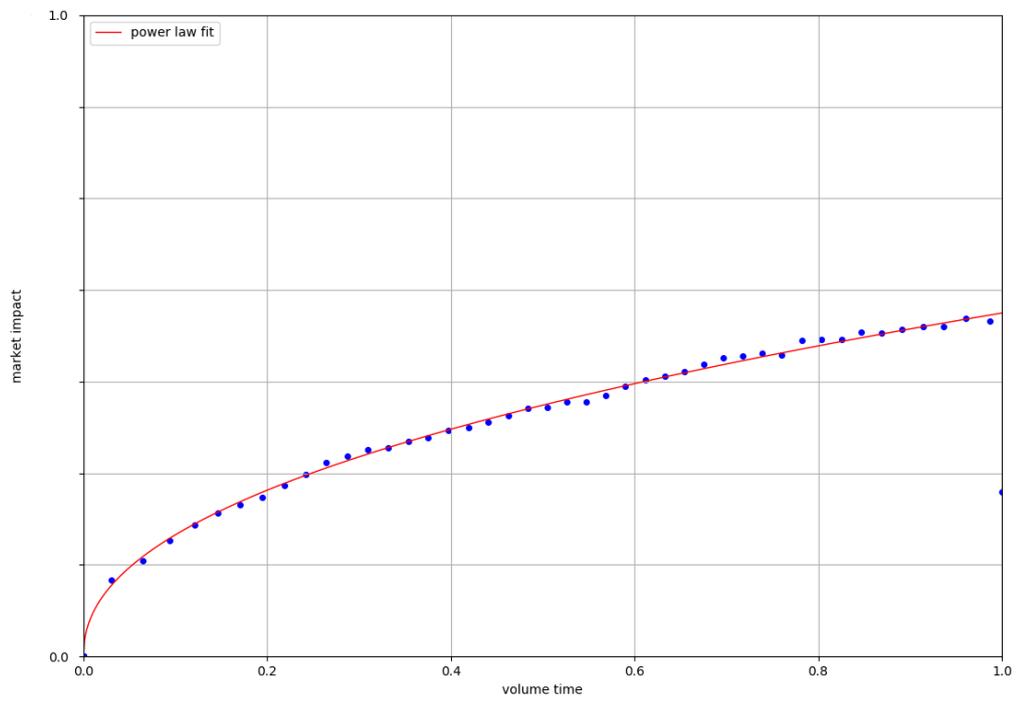


Figure I.8 – Market impact dynamics during the *execution part* in the case of the aggressive metaorders (set: Ω , 1 561 505 metaorders, power law fit: $y = 0.54 \times x^{0.45}$)

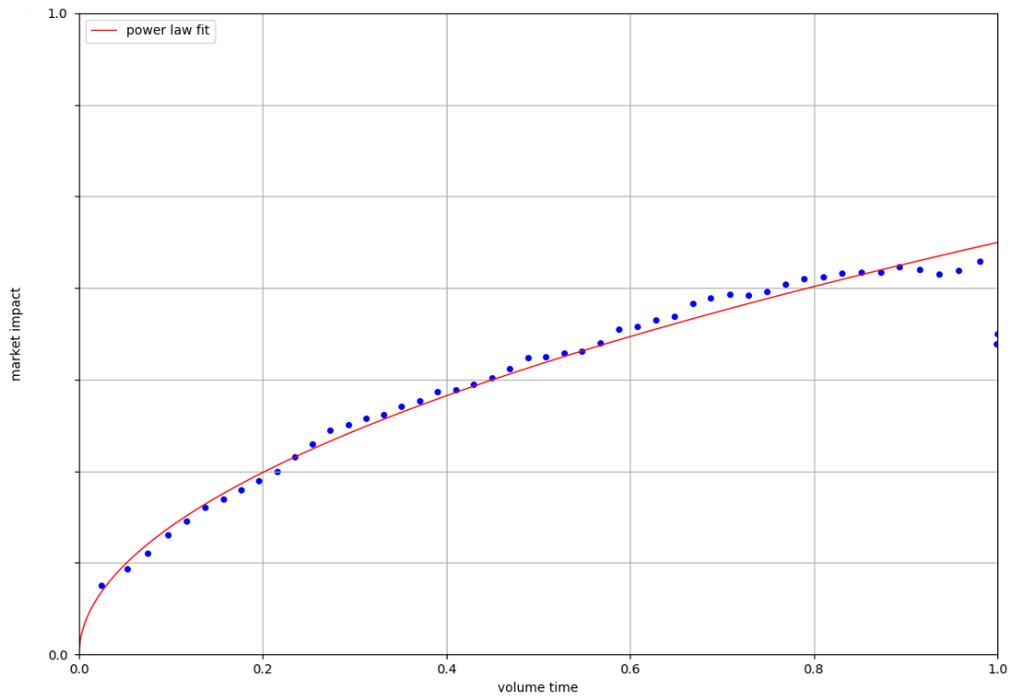


Figure I.9 - Market impact dynamics during the *execution part* in the case of the aggressive metaorders (set: Ω_{10} , 275 969 metaorders, power law fit: $y = 0.63 \times x^{0.51}$)

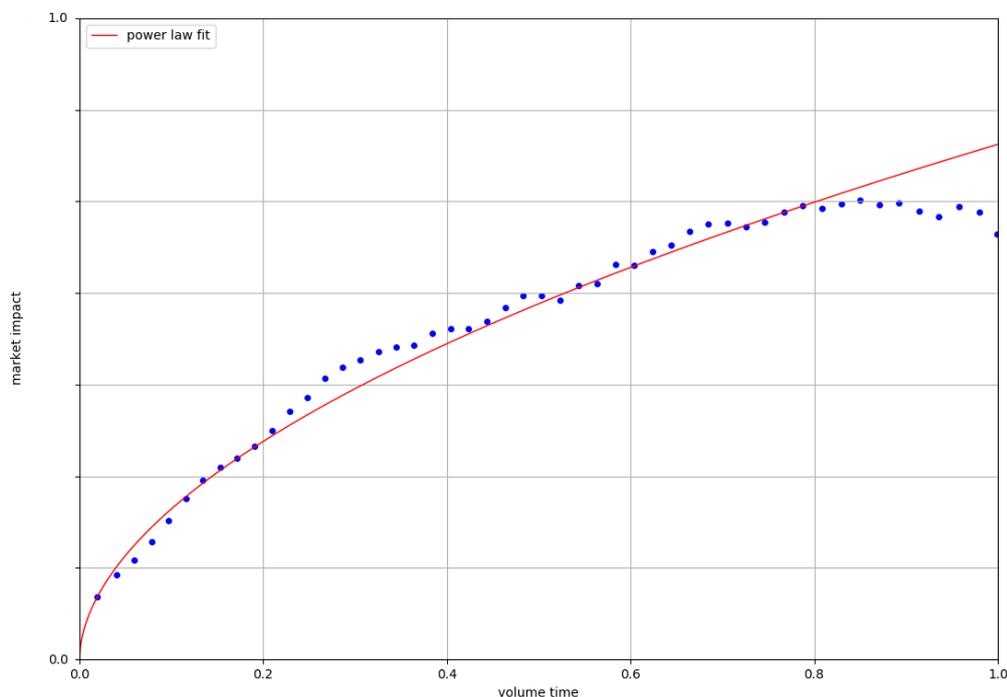


Figure I.10 – Market impact dynamics during the *execution part* in the case of the aggressive metaorders (set: Ω_{30} , 65 683 metaorders, power law fit: $y = 0.80 \times x^{0.53}$)

The analysis clearly yields an increasing, concave market impact curve. The decay observed in the last points of the curve is an artifact that can be explained by the larger number of metaorders of smaller lengths and with lower impact, as shown in Figure I.5. Also note that on the three figures I.8, I.9 and I.10, the larger the metaorders, the higher the impacts: 0.53, 0.60 and then 0.71 for the temporary market impact. However, the last curve I.10 indicates that, for large metaorders, the impact reaches a plateau, as if the market has adjusted to the information it received in such a way that the execution of the metaorder no longer affects it.

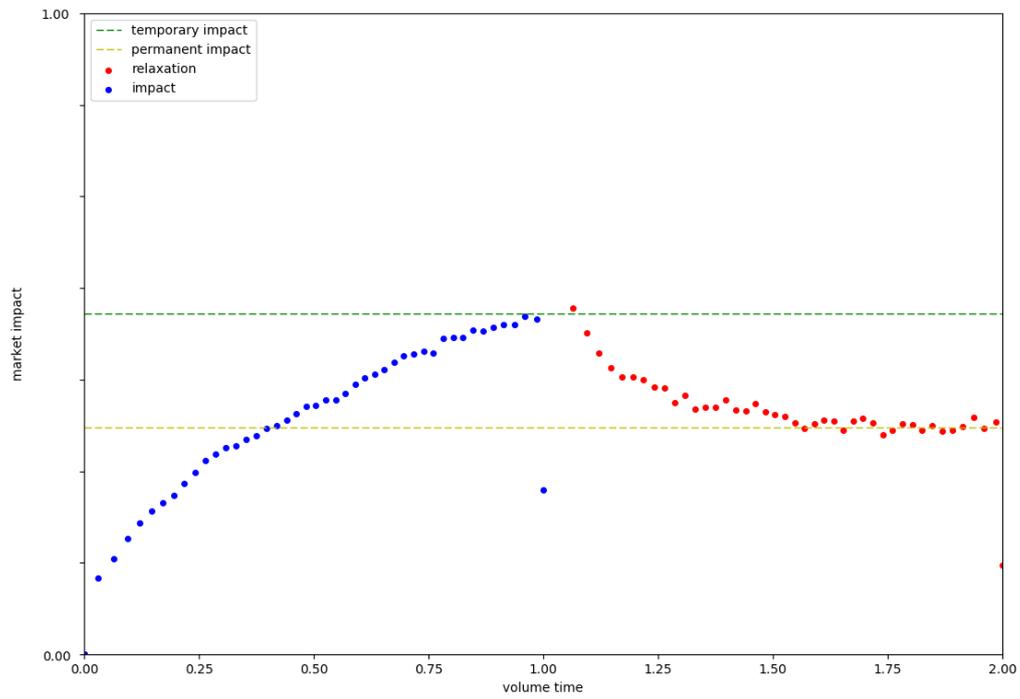


Figure I.11 – Market impact dynamics with relaxation in the case of the aggressive metaorders (set: Ω , 1 561 505 metaorders, temporary impact: 0.53, permanent impact: 0.35)

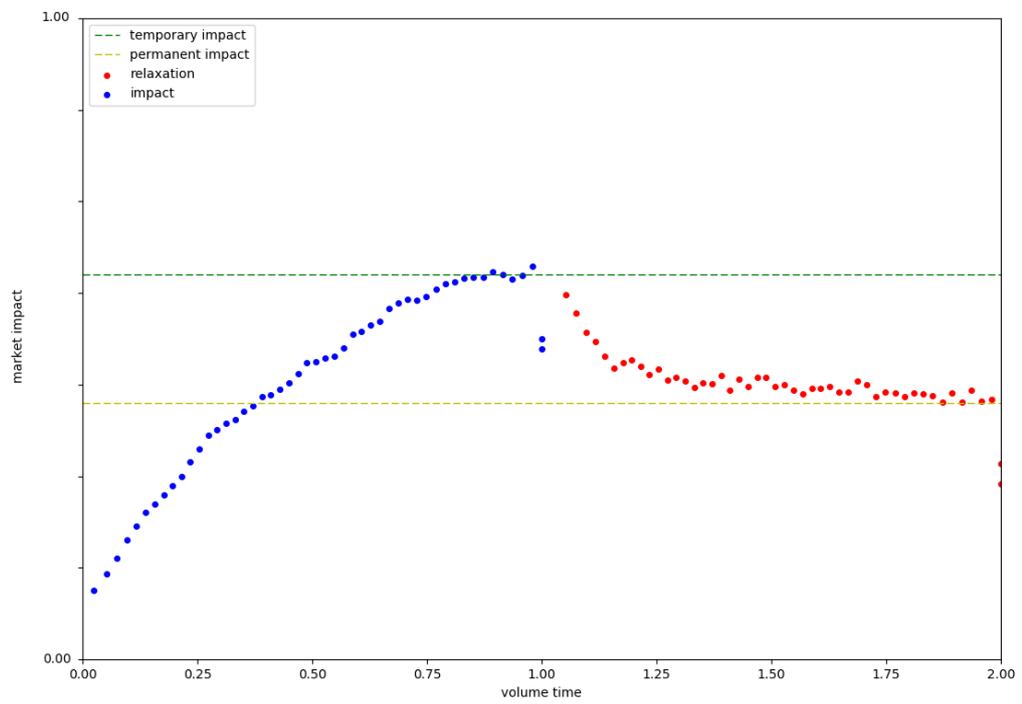


Figure I.12 – Market impact dynamics with relaxation in the case of the aggressive metaorders (set: Ω_{10} , 275 969 metaorders, temporary impact: 0.60, permanent impact: 0.40)

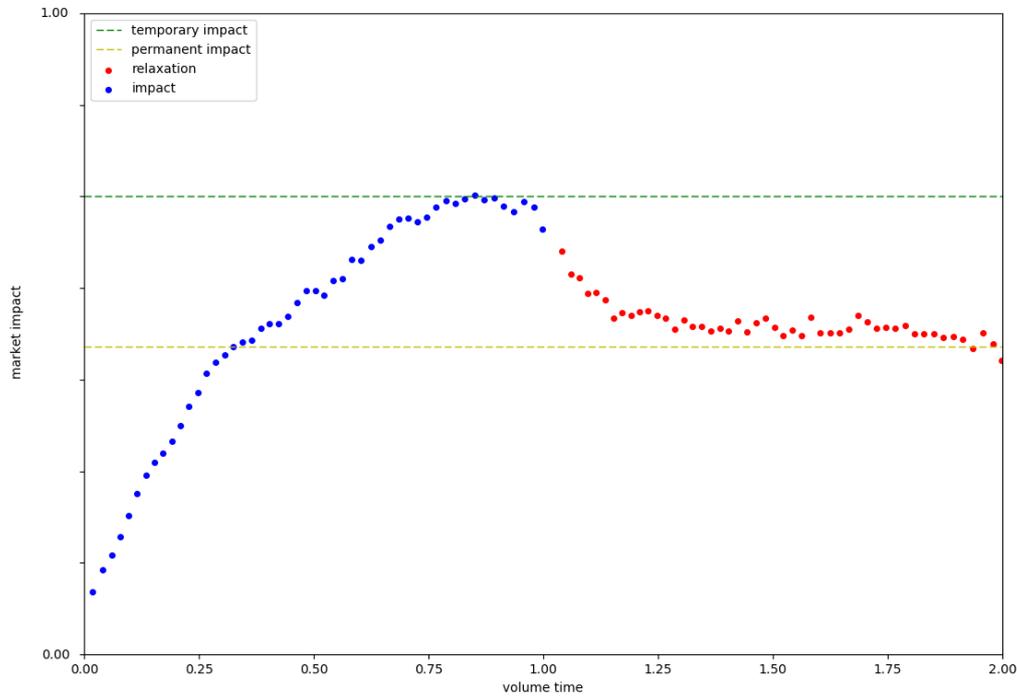


Figure I.13 – Market impact dynamics with relaxation in the case of the aggressive metaorders (set: Ω_{30} , 65 683 metaorders, temporary impact: 0.71, permanent impact: 0.48)

The blue points correspond to execution prices and the red points correspond to mid-prices observed at identical times starting from the end of the metaorder. The isolated points observed in Figures I.11 and I.12 are due to the fact that we have considered all metaorders, so metaorders with a small length, especially those with length $N = 2$ are over-represented in **volume time** = 1.0, therefore inducing a bias towards the end of the curve. One can make this artifact vanish when considering only metaorders with larger sizes, see e.g. I.13.

Figures I.11 I.12 and I.13 clearly exhibit the concave shape of market impact during the execution part, followed by a convex and decreasing relaxation. Simply by eyeballing Figure I.11, one can safely assume that relaxation is complete and stable at a level around 0.35. However, on Figures I.12 and I.13, relaxation does not seem to be quite complete. This behaviour for larger metaorders is quite probably due to the fact that it is not always possible to reach the final time $t_0 + 2T$ during the intraday observation period. Hence, the relaxation of larger metaorders may be hindered by the closing of the market.

A conclusion to this section is that the concave shape of the temporary impact and the convex relaxation curve are in line with the empirical results in [Bacry et al., 2015] and

[Bershova and Rakhlin, 2013]. Also, and more interestingly, the market impact and relaxation curves confirm the theoretical findings of [Farmer et al., 2013] that the impact should be concave and increasing, and that the final impact after the execution is performed should relax to about two-thirds of the peak impact.

4.3 Execution strategies

The purpose of this section is to analyse metaorders executed *via* passive limit orders, and compare the resulting market impact curves with those previously obtained. Note that the scales on the graphs concerning the aggressive and passive metaorders are exactly the same, thereby making comparisons possible.

At this stage, some comments are in order: by nature, passive limit orders are not always executed and therefore, the notion of market impact for such orders has to be taken with a grain of salt. When an agent places a new passive limit order, especially within the bulk of the order book, it is not clear whether this should indicate that the market is moving in any direction at all, and an impact should not always be expected. In fact, a strategy purely based on limit orders can be even considered to have only negative market impact, since the order is executed only after the price has moved in favour of the agent. And of course, an execution strategy relying only on limit orders will often fail to achieve its target, thereby facing an implementation shortfall that will have to be dealt at some point in the future.

In order to cope with this inherent difficulty and also to present results that are consistent with the task at hand, namely the study of market impact during the execution of a metaorder, the analyses in this section are performed on a database of *execution strategies*. Metaorders in this database are executed using (on average) $\sim 65\%$ of passive limit orders and $\sim 30\%$ of aggressive limit orders, the remaining 5% consisting of orders of various types (market, market on close...).

4.3.1 Data

- Study period : **1st Jan 2016 - 31st Dec 2017**
- Markets : **European Equity Markets**
- Order types : **All Orders**¹
- Filters : **metaorders** $\omega \in \Omega$
- Number of metaorders : **74 552**

¹~ 65% passive limit orders, ~ 30% aggressive limit orders and 5% others

Duration distribution

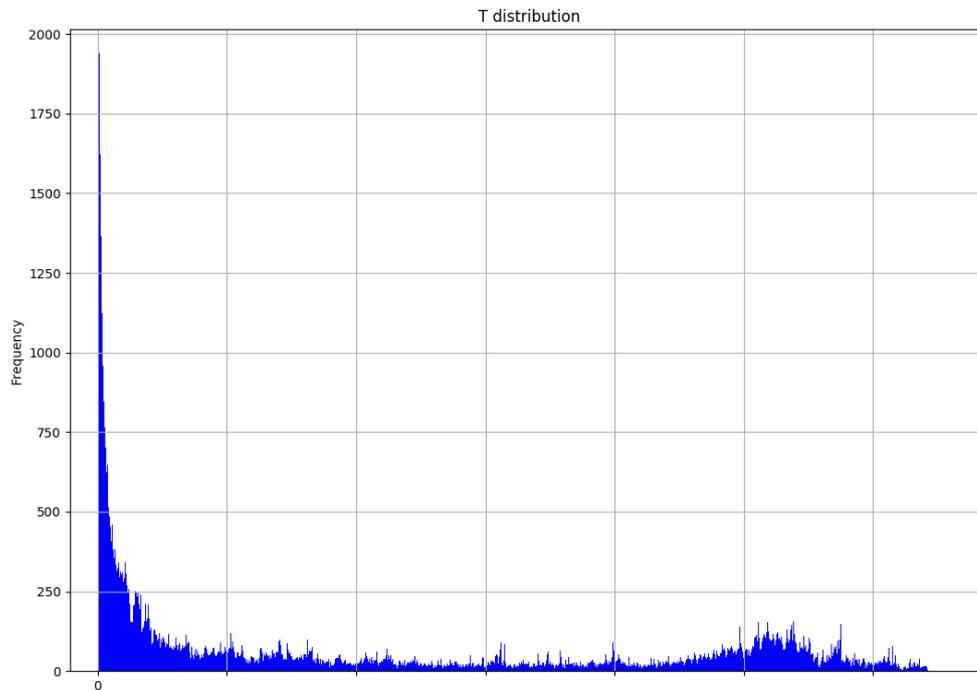


Figure I.14 – Duration distribution in seconds of the execution metaorders

As expected, and previously shown for the aggressive metaorders, see Figure I.4, one observes again (Figure I.14) that, the shorter the metaorders are, the more they are represented. Different from the case of the aggressive metaorders, the distribution is seen to have a fatter tail, an indication of the "intelligence" of the execution strategies.

Length distribution

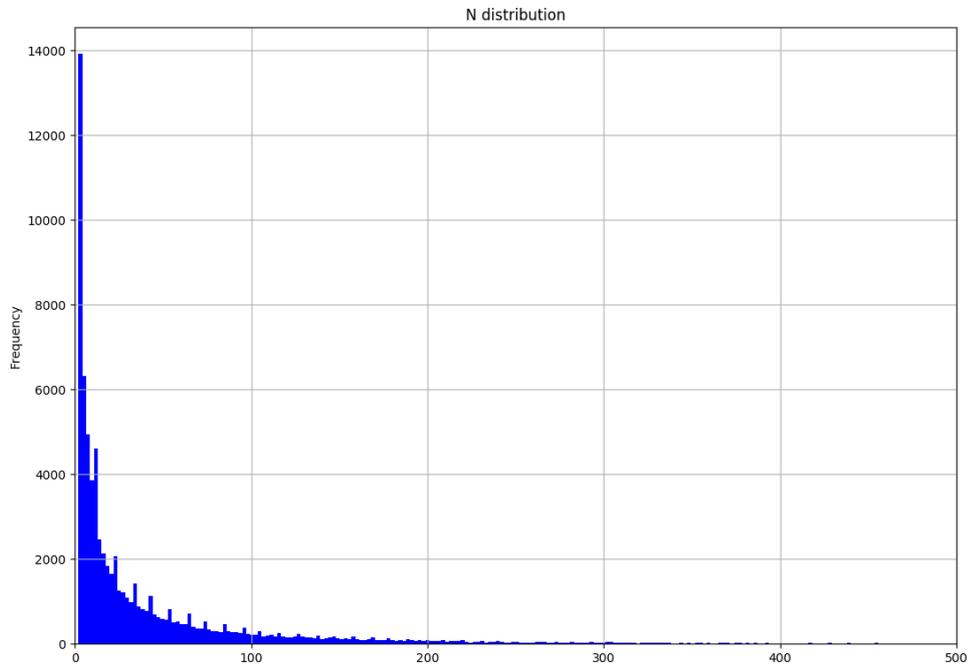


Figure I.15 – Length distribution of the execution metaorders

mean 50
median 17

The shape of the distribution in Figure I.15 is very similar to that obtained for the aggressive metaorders in Figure I.5.

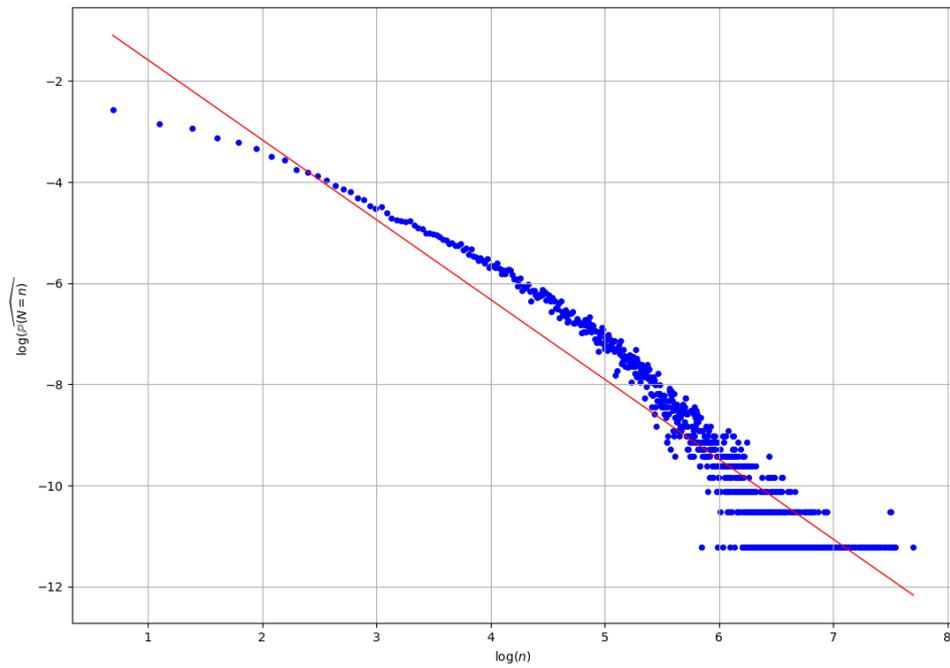


Figure I.16 – Length distribution of the execution metaorders with a log-log scale

Again, a linear relation in log-log scale seems rather clear (Figure I.16), although it is noisier, with $\beta \approx 1.8$ in this case. This tends to confirm the relevance of the discrete Pareto distribution independently of the nature of the orders.

Participation rate distribution

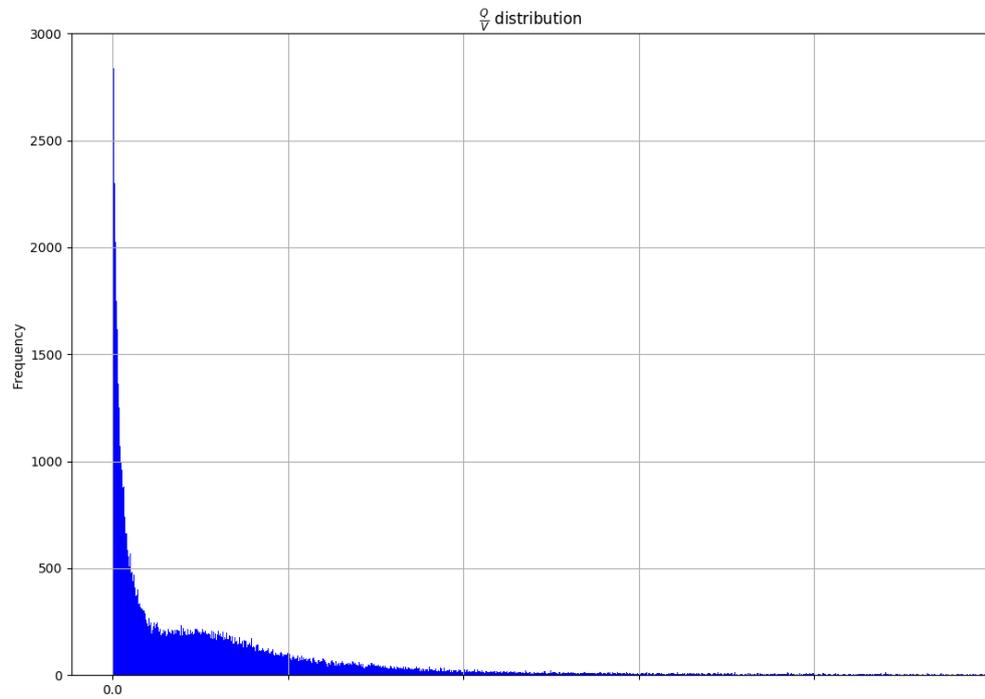


Figure I.17 - $\frac{Q}{V}$ distribution of the execution metaorders

Once again the quantities N , T and $\frac{Q}{V}$ are highly positively correlated (Figure I.17). Besides, and not surprisingly, the execution metaorders have larger participation rates than those observed in the aggressive case (Figure I.7).

4.3.2 Market Impact Dynamics

The market impact study proceeds along the same lines as those presented in 4.2.2. Hence, to plot $(\epsilon(\omega)\mathcal{I}_t(\omega))_{\omega \in \Omega, t_0(\omega) \leq t \leq t_0(\omega) + T(\omega)}$, one has to consider its rescaled time version $(\epsilon(\omega)\mathcal{I}_s(\omega))_{\omega \in \Omega, 0 \leq s \leq 1}$, i.e. for each metaorder ω we consider $[0, 1]$ instead of $[t_0(\omega), t_0(\omega) + T(\omega)]$ such as $[0, 1] = \frac{[t_0(\omega), t_0(\omega) + T(\omega)] - t_0(\omega)}{T(\omega)}$.

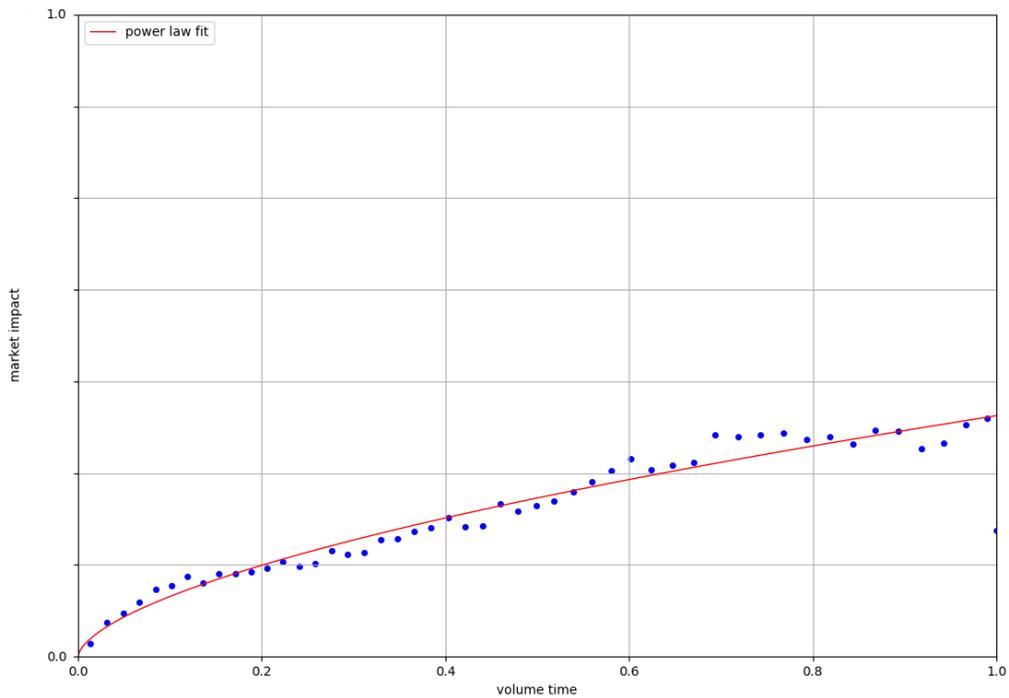


Figure I.18 – Market impact dynamics during the *execution part* in the case of the execution metaorders (set: Ω , 74 552 metaorders, fit: $y = 0.37 \times x^{0.60}$)

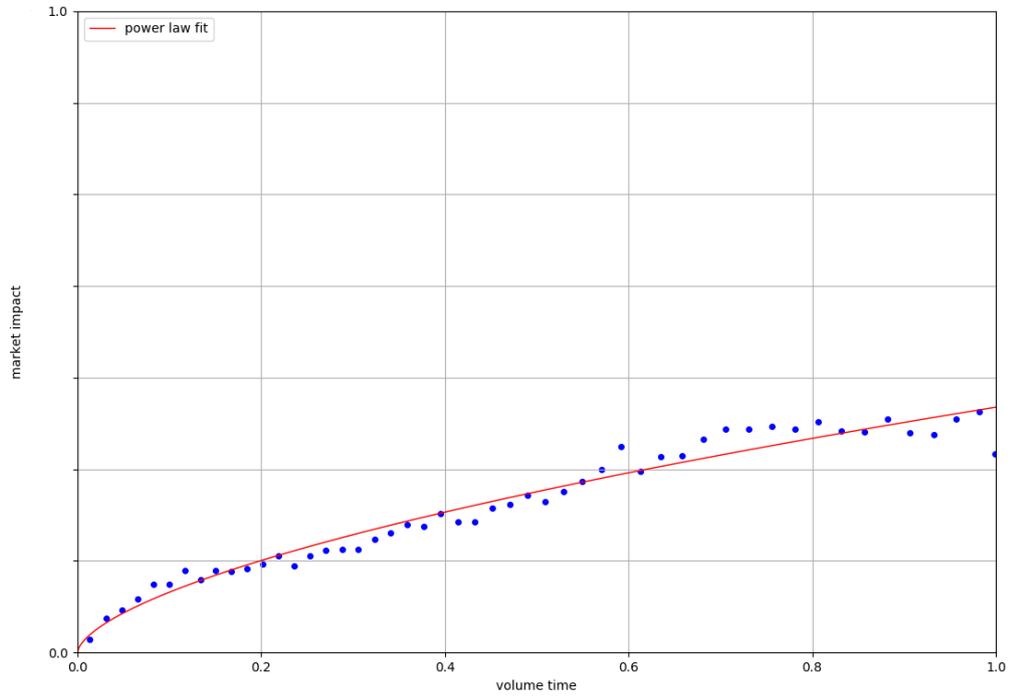


Figure I.19 – Market impact dynamics during the *execution part* in the case of the execution metaorders (set: Ω_{10} , 47 243 metaorders, power law fit: $y = 0.37 \times x^{0.61}$)

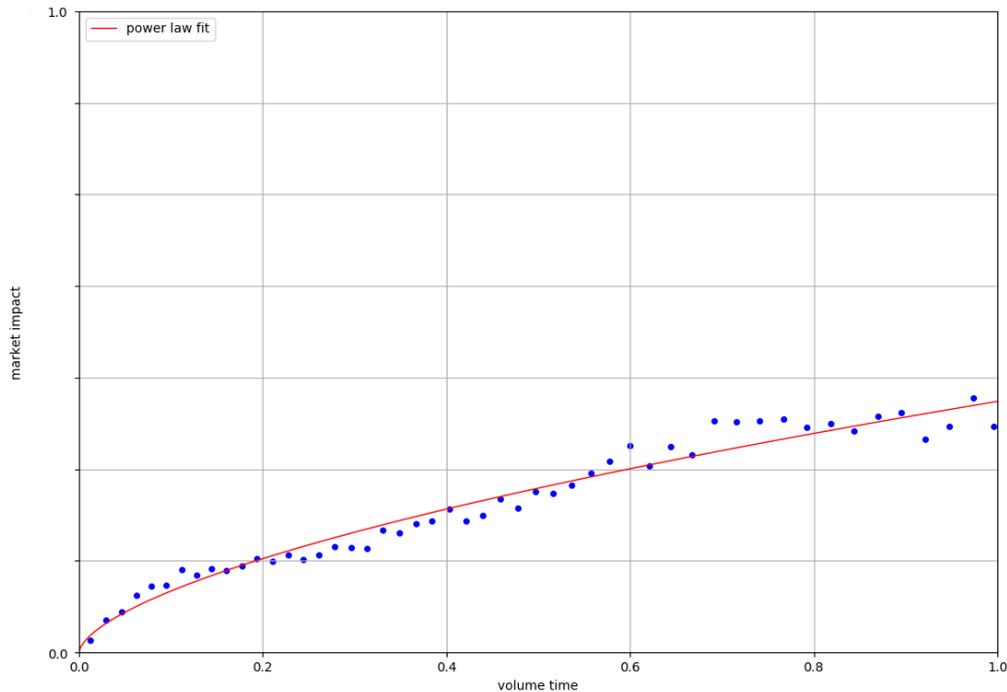


Figure I.20 – Market impact dynamics during the *execution part* in the case of the execution metaorders (set: Ω_{30} , 27 710 metaorders, power law fit: $y = 0.40 \times x^{0.62}$)

One observes a positive, increasing and concave market impact curve. The impacts appear to be smaller than in the aggressive case (Figure I.8) although participation rates are higher and metaorders last longer. Moreover, the three curves Figures I.18 I.19 and I.20 show that the impact is insensitive to the size of the metaorders. These two features are a clear indication that it is beneficial to use passive limit orders. An execution metaorder has the main constraint of executing on the market a known and predefined size in advance, whatever the chosen strategy is. The choice of an execution strategy less brutal than one relying solely on aggressive metaorders allows to limit and control its impact.

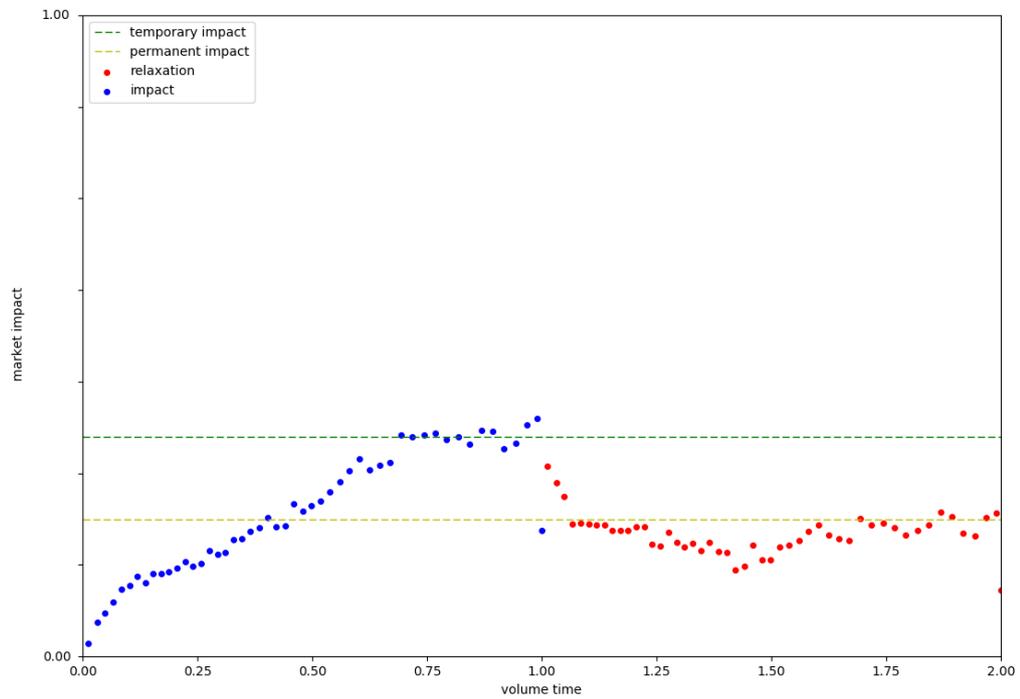


Figure I.21 – Market impact dynamics with relaxation in the case of the execution metaorders (set: Ω , 74 552 metaorders, temporary impact: 0.34, permanent impact: 0.22)

At first glance, see Figures I.21 I.22 and I.23, the dynamic seems fairly similar to that obtained in the aggressive case, with a slightly rougher curve due to a much smaller data set for the execution metaorders: There is an increasing, concave temporary impact curve and a decreasing, convex relaxation curve that finishes at about two-thirds of the peak impact. However, the monotonicity and convexity of the relaxation curve are less pronounced. Two possible explanations come to mind. The first is that the relaxation is faster because the temporary impacts are less important and therefore, a return to equilibrium is easier. In this hypothesis we can consider the end of the relaxation curve as an other metaorder starting in the market. The second hypothesis is that this is a noisy artifact due to a smaller data set.

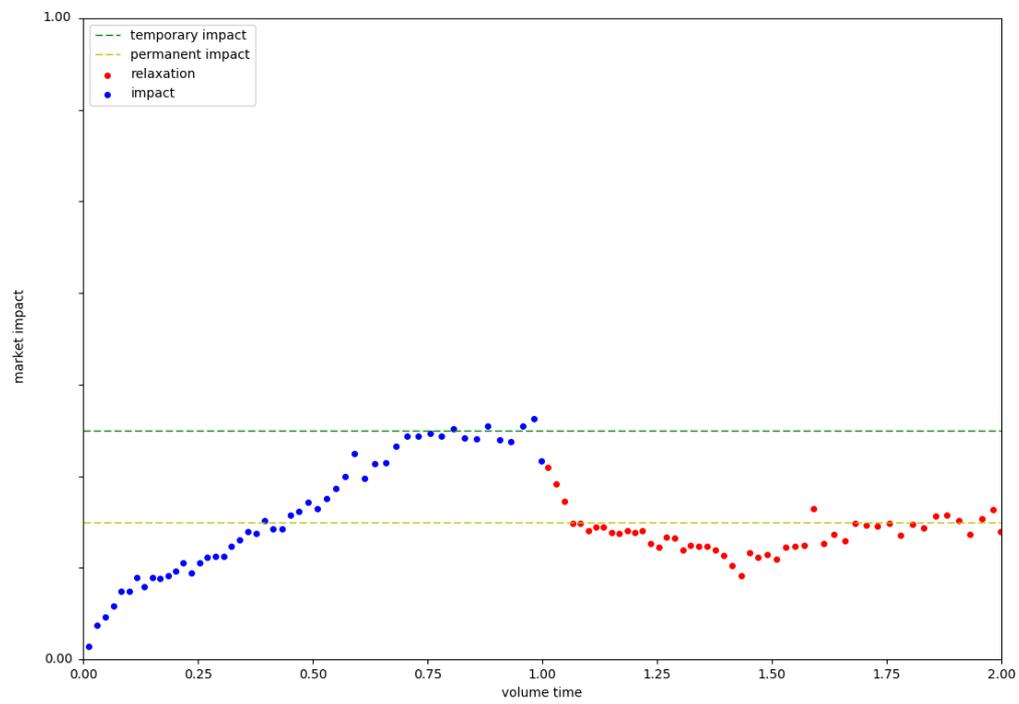


Figure I.22 – Market impact dynamics with relaxation in the case of the execution metaorders (set: Ω_{10} , 47 423 metaorders, temporary impact: 0.36, permanent impact: 0.23)

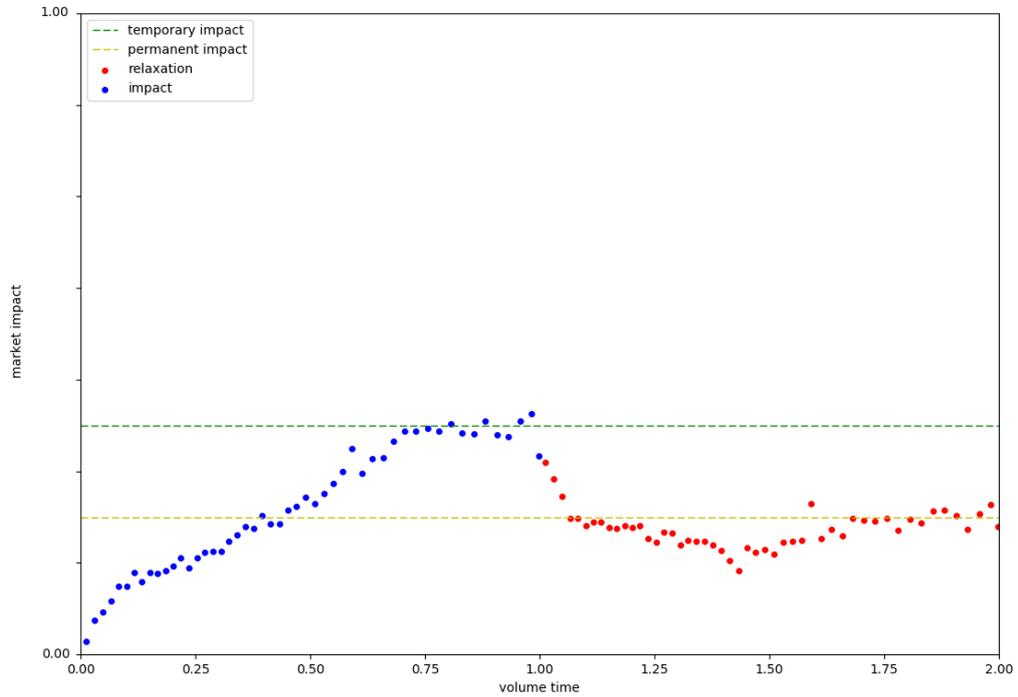


Figure I.23 – Market impact dynamics with relaxation in the case of the execution metaorders (set: Ω_{30} , 27 710 metaorders, temporary impact: 0.37, permanent impact: 0.24)

5 Square-Root Law

In this section we are interested in what is commonly called the *Square-Root Law*. The *Square-Root Law* is the fact that the impact curve should not depend on the duration of the metaorder. Indeed, almost all studies now agree on the fact that the impact is more or less close to be proportional to the square root of the volume executed. However, the so-called *Square-Root Law* states much more than that. It basically claims that the market impact does not depend on the metaorder duration. This last point remains a controversial matter: does the market impact depends on the metaorder duration or not.

At first glance, it does not seem clear that the market impact of a metaorder should depend solely on its size. In the industry, several models are based, at the first order, on a theoretical curve of the form $\sigma\sqrt{\frac{Q}{V}}$ with σ a volatility factor and $\frac{Q}{V}$ a participation rate factor **over the metaorder time scale**, see e.g. [Brokmann et al., 2015]. Considering such a local participation rate already introduces a duration effect. Nevertheless, a scatter plot of the impacts in

terms of the local participation rates (cf. Figures I.24 and I.25)) does not highlight any additional dependency on the duration, apart from that already built in the participation rate. What one can see is that, for the same participation rate, the dispersion of market impact is higher for longer metaorders. This is simply due to the diffusive nature of the prices that creates a noisier impact.

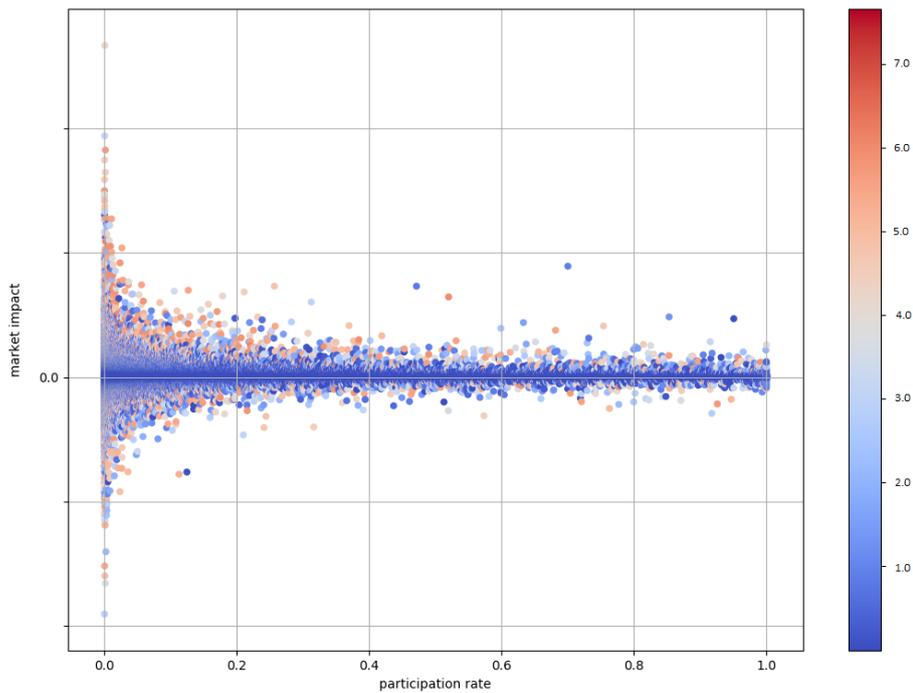


Figure I.24 – Impact against participation rate, the duration of the order is increasing from dark blue to dark red in the case of the *aggressive* metaorders.

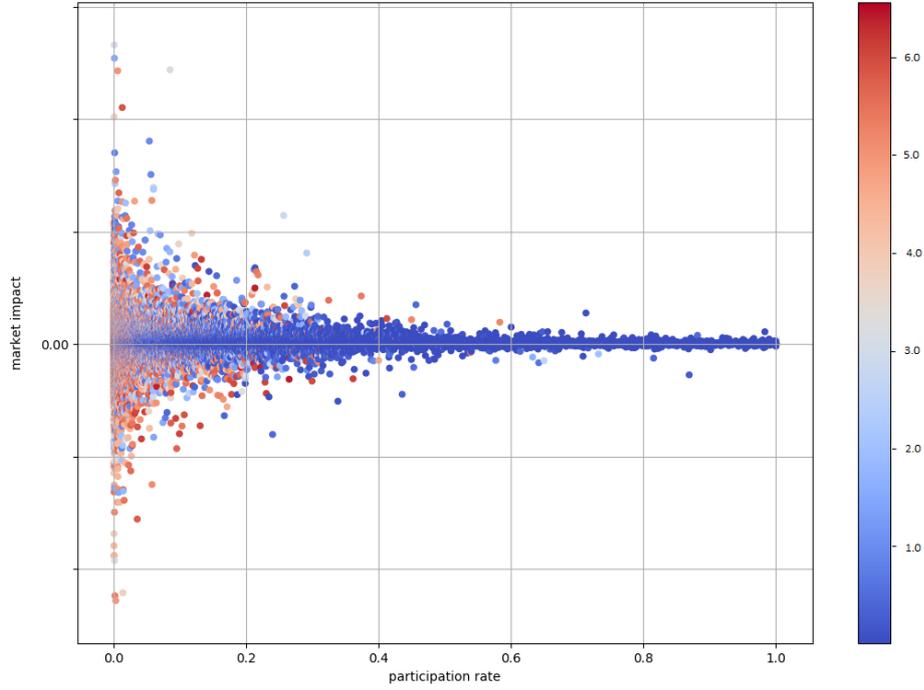


Figure I.25 – Impact against participation rate, the duration of the order is increasing from dark blue to dark red in the case of the *execution* metaorders.

6 Fair Pricing

In this final, short section, the fair pricing condition for metaorders is examined. In the interest of homogeneity, we consider normalized prices, so that for every stock and at every time, the value of the price is 1 whenever the execution of a metaorder begins.

First of all, the VWAP of a metaorder ω is defined by

$$\mathcal{P}_{VWAP}(\omega) = \frac{\sum_{i=0}^{N-1} Q_i(\omega) P_{t_i(\omega)}(\omega)}{Q(\omega)},$$

where $t_0(\omega), \dots, t_{N-1}(\omega)$ represent the transaction times of the metaorder ω and $Q(\omega) = \sum_{i=0}^{N-1} Q_i(\omega)$. Hence, we want to compare $1 + \mathcal{R}_{VWAP} = \frac{\mathcal{P}_{VWAP}}{P_{t_0}}$ with $1 + \mathcal{R}_{t_0+2T} = \frac{P_{t_0+2T}}{P_{t_0}}$.

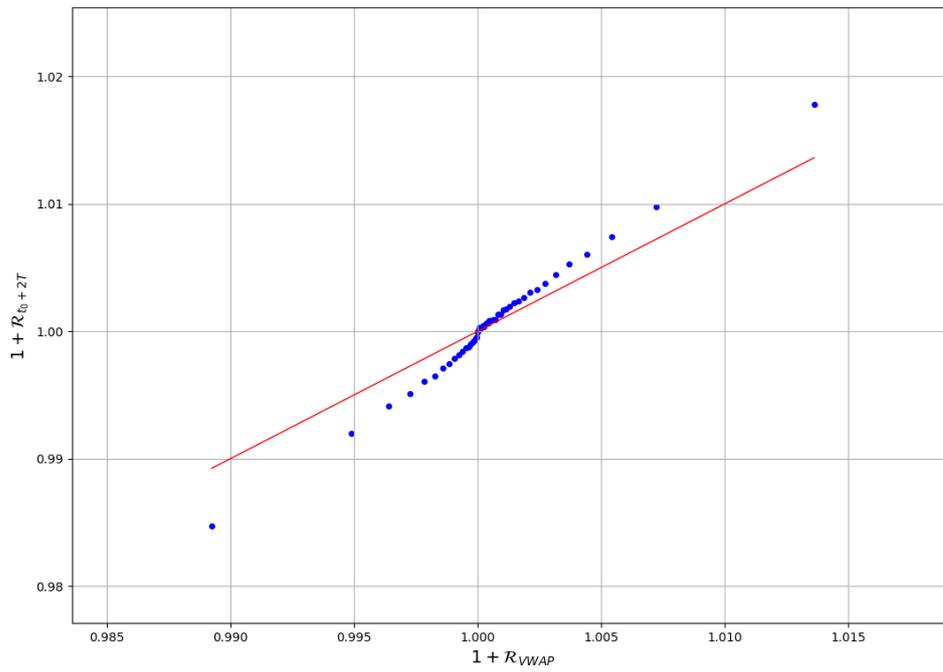


Figure I.26 - Fair pricing condition in the case of the aggressive metaorders

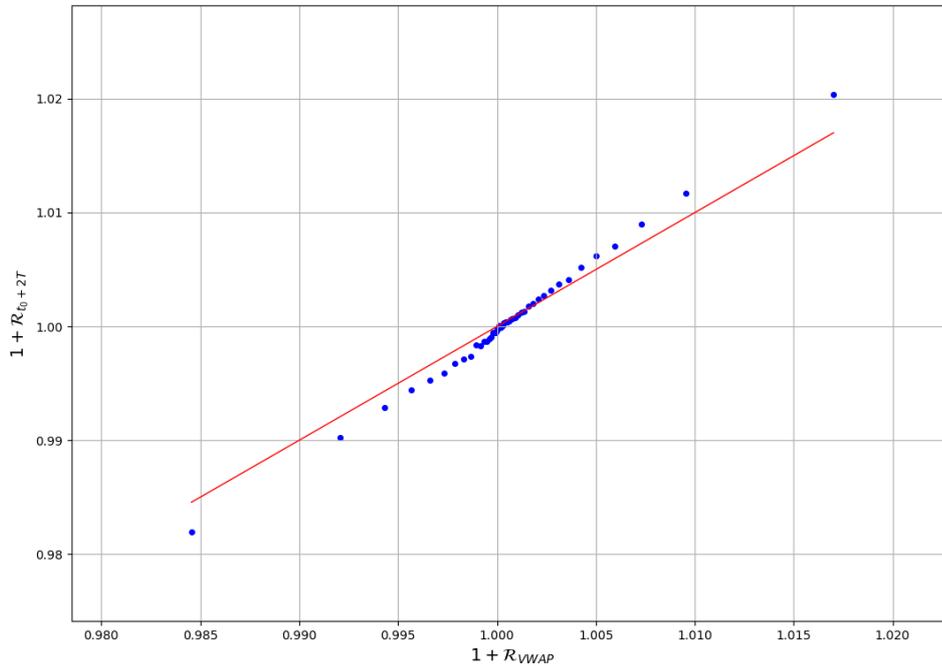


Figure I.27 – Fair pricing condition in the case of the execution metaorders

The red lines in Figures I.26 and I.27 represent the perfect fair pricing condition, corresponding to the identity $1 + \mathcal{R}_{VWAP} = 1 + \mathcal{R}_{t_0+2T}$.

We conclude from Figures I.26 and I.27 that the fair pricing condition can reasonably be assumed to hold. It also appears that, the greater the absolute price variations, the more one moves away from the perfect fair pricing condition. This last point was predictable: high variations are generally associated to longer and larger metaorders that are therefore more affected by the diffusive nature of the prices. Note that, in the model of [Farmer et al., 2013], the discrete Pareto distribution of N of parameter $\beta + 1 = 2.5$ associated with martingale and fair pricing conditions (see Appendix I.A) leads to a relaxation at $2/3$ of the peak impact, a level that is experimentally verified in our study as well as in the already cited papers [Bershova and Rakhlin, 2013] and [Zarinelli et al., 2015].

7 Conclusion

The work presented here is an empirical study of a large set of metaorders executed using limit orders in the European equity markets. A new algorithmic definition of a metaorder has been proposed. The statistical results show a good agreement with some predictions of the

market impact model in [Farmer et al., 2013] in the case of limit orders.

Our study contains two distinct subgroups of orders: a set of aggressive limit orders, and a database of execution strategies predominantly composed of passive limit orders. In both cases, the analysis shows that the temporary market impact is increasing and concave, and it also confirms that the length distribution of metaorders follows a Pareto distribution of parameter $\beta + 1$ with $\beta \approx 1.4$ in the case of the aggressive metaorders and $\beta \approx 1.8$ in the case of the execution metaorders.

As for the relaxation phase, a convex, decreasing functional form is obtained, as mentioned in [Bacry et al., 2015] and other related studies. More precisely, the price reversion after the completion of a trade yields a permanent impact such that its ratio to the maximum impact observed at the last fill is roughly 2/3 as predicted in the article of [Farmer et al., 2013] and highlighted empirically [Bershova and Rakhlin, 2013].

Finally, we have shown that the square-root law and also the fair pricing condition seem to be empirically verified: the VWAP of a meatorder is equal to the final price of the security after the relaxation phase is over.

I.A The Market Impact Model of [Farmer et al., 2013]

The main results of [Farmer et al., 2013] are recalled. The central goal of the model is to understand the way order splitting affects market impact.

I.A.1 Model description

- A filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is given.
- At $t = 0$, before the opening of the market, the \mathcal{K} long-term traders have a common information signal α and each trader $k = 1, \dots, \mathcal{K}$ formulate an order and submit it to the algorithmic trading firm that bundles them together into a metaorder which will be executed **in lots of equal size**.
- There are N auctions following each other $t = 1, \dots, N$, N representing the number of orders necessary to execute fully the metaorder is bounded, $N \leq M$.
- At $t = N + 1$, corresponding to the relaxation, the last instant in the game, i.e. the instant after the metaorder is fully executed, is announced with the final price $\tilde{X}_N = G(X_0, \alpha, \eta_1, \dots, \eta_N)$, $N > 0$, where X_0 is the initial price and G a function whose form is not important. **We will use the tilde notation to refer to a quantity depending on (η_t)** , where (η_t) is a zero mean i.i.d random process modeling market noise.

I.A.2 Notations

- As regards statistical averages, \tilde{S}_t denotes a specific realization of transaction prices, whereas S_t stands for an average price over the signal η_i . Likewise, \tilde{X}_N is a specific realization of the final price at the end of the metaorder whereas X_N is an average price over the signal η_i ;
- The final price averaging over η_t is denoted by X_N i.e $X_N = \mathbb{E}[\tilde{X}_N]$;
- $X_{t-1} = S_{t-1} - R_{t-1}^-$, $t = 2, \dots, M$ and $X_M = S_M$ thus $R_M^- = 0$, M corresponding to the end of the market session;
- $S_t = X_0 + \sum_{i=0}^{t-1} R_i^+$, $t = 1, \dots, M$, where $R_t^+ = S_{t+1} - S_t$ and $R_t^- = S_t - X_t$ are the corresponding incremental average impacts;
- $\mathcal{I}_t = S_t - X_0$ is the **average immediate impact** at t and $I_N = X_N - X_0$ denotes the **average permanent impact**;
- Let $m \in \{0, 1\}$ an indicator variable where $m = 1$ if the metaorder is present and $m = 0$ if it is absent
- We will consider $p_t = \mathbb{P}(t \leq N < t + 1 | m = 1)$ at each period t and \mathcal{P}_t the probability that the metaorder will continue given that it is still active at timestep t , i.e.

$$\mathcal{P}_t = \frac{\sum_{i \geq t+1} p_i}{\sum_{i \geq t} p_i} = \mathbb{P}(N \geq t+1 | N \geq t, m=1).$$

The possible price paths for a buy are summarized in the graph below:

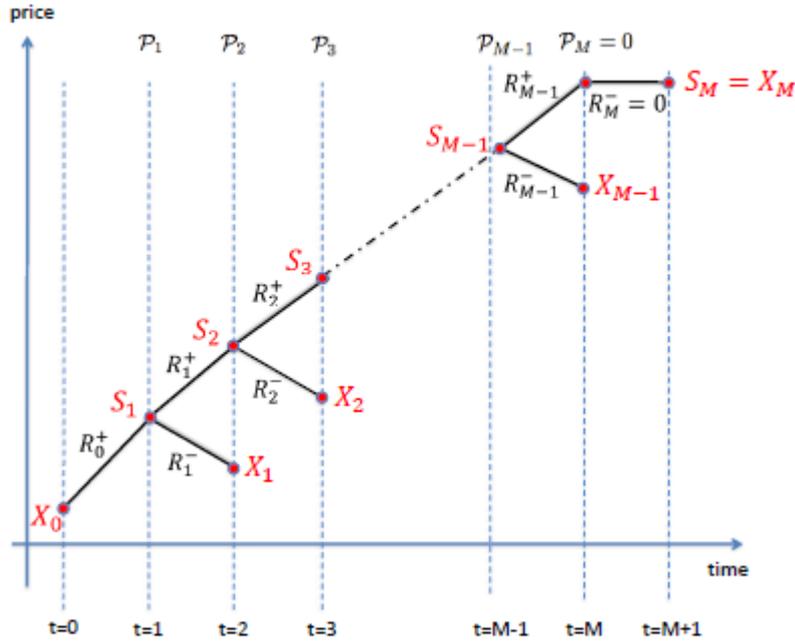


Figure I.28 – *Tree of possible price paths for a buy metaorder for different sizes N .*
(extracted from [Farmer et al., 2013])

In Figure I.28, the metaorder is supposed to be present and only expected price paths, averaged over the day trader’s information (which is why the notation does not include tildes), are shown. The price is initially X_0 , after the first lot is executed it is $S_1 = X_0 + R_0^+$. If $N = 1$ it is finished and the price reverts to $X_1 = S_1 - R_1^-$, but if $N > 1$ another lot is executed and it rises to $S_2 = S_1 + R_1^+$. This proceeds similarly until the execution of the metaorder is completed. At any given point the probability that the metaorder has size $N > t$, i.e. that the order continues, is \mathcal{P}_t . A typical price path (rather than the expected price paths shown here) subject to a large day trader’s noisy information signal would look more like a random walk with a time-varying drift caused by the impact of the metaorder.

I.A.3 Martingale condition

In this section, (\tilde{S}_t) is supposed to be a martingale: $\mathbb{E}[\tilde{S}_{t+1} | \mathcal{F}_t] = \tilde{S}_t$. Therefore, the following properties hold true:

- $\mathbb{E}[\tilde{S}_{t+1}|\mathcal{F}_t] = \mathbb{P}(m = 1|\mathcal{F}_t)\mathbb{E}[\tilde{S}_{t+1}|\mathcal{F}_t, m = 1] + \mathbb{P}(m = 0|\mathcal{F}_t)\mathbb{E}[\tilde{S}_{t+1}|\mathcal{F}_t, m = 0]$, and:

$$\mathbb{P}(m = 1|\mathcal{F}_t) = \mu'_t$$

considering that μ'_t is the market maker's best estimate of the probability that the metaorder is present;

- For the price in the next period, it is necessary to average over three possibilities :

$$\begin{aligned} \mathbb{E}[\tilde{S}_{t+1}|\mathcal{F}_t, m = 1] &= \mathbb{P}(N = t|\mathcal{F}_t, m = 1)\mathbb{E}[\tilde{S}_{t+1}|\mathcal{F}_t, m = 1, N = t] \\ &\quad + \mathbb{P}(N > t|\mathcal{F}_t, m = 1)\mathbb{E}[\tilde{S}_{t+1}|\mathcal{F}_t, m = 1, N > t] \end{aligned}$$

with:

- $\mathbb{E}[\tilde{S}_{t+1}|\mathcal{F}_t, m = 1, N > t] = \tilde{S}_t + R_t^+$
- $\mathbb{E}[\tilde{S}_{t+1}|\mathcal{F}_t, m = 1, N = t] = \tilde{S}_t - R_t^-$
- $\mathbb{E}[\tilde{S}_{t+1}|\mathcal{F}_t, m = 0] = \tilde{S}_t$
- $\mathbb{P}(N > t|\mathcal{F}_t, m = 1) = \mathbb{P}(N \geq t + 1|N \geq t, m = 1) = \mathcal{P}_t$.

thus:

$$\begin{aligned} \tilde{S}_t &= \mu'_t(\mathcal{P}_t(\tilde{S}_t + R_t^+) + (1 - \mathcal{P}_t)(\tilde{S}_t - R_t^-)) + (1 - \mu'_t)\tilde{S}_t \\ \mathcal{P}_t R_t^+ - (1 - \mathcal{P}_t)R_t^- &= 0 \end{aligned}$$

Since (\tilde{S}_t) is a martingale, there holds:

Proposition 1.

$$\frac{R_t^+}{R_t^-} = \frac{1 - \mathcal{P}_t}{\mathcal{P}_t}, t \geq 2.$$

I.A.4 Fair pricing condition

The martingale condition derived in the previous section only sets the value of the ratio R_t^+/R_t^- at each auction t . Another condition is required to derive the values of R_t^+ and of R_t^- and therefore, to obtain the expression for the immediate and the permanent impact.

- **The fair pricing condition** states that for any N ,

$$\pi_N = \frac{1}{N} \sum_{t=1}^N S_t - X_N = 0.$$

This implies that we have

$$I_N = \frac{1}{N} \sum_{t=1}^N \mathcal{I}_t.$$

Assuming that the martingale condition holds for $t = 1, \dots, M$ and the fair pricing condition holds for $t = 2, \dots, M - 1$ leads to a system of $2M - 2$ homogeneous equations with $2M - 1$ unknowns, so we choose R_1^+ as an undetermined constant.

Proposition 2. *The system of martingale conditions and fair pricing conditions has the solution*

$$R_t^+ = \frac{1}{t} \frac{1 - \mathcal{P}_t}{\mathcal{P}_t} \frac{1}{\mathcal{P}_1 \mathcal{P}_2 \dots \mathcal{P}_{t-1}} R_1^+, t \geq 2$$

and

$$R_t^- = \frac{1}{t} \frac{1}{\mathcal{P}_1 \mathcal{P}_2 \dots \mathcal{P}_{t-1}} R_1^+, t \geq 2.$$

Corollary 1. *The immediate impact is*

$$\mathcal{I}_t = S_t - X_0 = R_0^+ + R_1^+ \left(1 + \sum_{k=2}^{t-1} \frac{1}{k} \frac{1 - \mathcal{P}_k}{\mathcal{P}_k} \frac{1}{\mathcal{P}_1 \dots \mathcal{P}_{k-1}} \right), t \geq 2.$$

Since $X_N - X_0 = X_N - S_N + S_N - X_0 = \mathcal{I}_N - R_N^-$, we have

Corollary 2. *The permanent impact is*

$$I_N = R_0^+ + R_1^+ \left(1 + \sum_{k=2}^{N-1} \frac{1}{k} \frac{1 - \mathcal{P}_k}{\mathcal{P}_k} \frac{1}{\mathcal{P}_1 \dots \mathcal{P}_{k-1}} - \frac{1}{N} \frac{1}{\mathcal{P}_1 \dots \mathcal{P}_{N-1}} \right), N \geq 2.$$

I.A.5 Dependence on the metaorder size distribution

According to [Farmer et al., 2013] which cite a host of other relevant studies, there is a considerable evidence that - in the large size limit and for most major equity markets - the metaorder size V is distributed as $\mathbb{P}(V > v) \sim v^{-\beta}$, with $\beta \approx 1.5$.

- $\mathbb{P}(V > v - 1) - \mathbb{P}(V > v) \sim v^{-\beta} \left(\left(1 - \frac{1}{v} \right)^{-\beta} - 1 \right) \approx \frac{\beta}{v^{\beta+1}}$: an exact zeta distribution for all $n \geq 1$ is considered,

$$p_n = \mathbb{P}(N = n | m = 1) = \frac{1}{\zeta(\beta + 1)} \frac{1}{n^{\beta+1}}$$

so that

$$\mathcal{P}_t = \frac{\zeta(1 + \beta, t + 1)}{\zeta(1 + \beta, t)} \approx \frac{\int_{t+1}^{+\infty} \frac{dx}{x^{\beta+1}}}{\int_t^{+\infty} \frac{dx}{x^{\beta+1}}} = \left(1 + \frac{1}{t} \right)^{-\beta} \sim 1 - \frac{\beta}{t};$$

- As a consequence,

$$R_t^+ = \frac{1}{t^{2+\beta}} \frac{\zeta(1 + \beta)}{\zeta(1 + \beta, t) \zeta(1 + \beta, t + 1)} R_1^+ \sim \frac{1}{t^{2-\beta}};$$

- Thus, the immediate impact \mathcal{I}_t behaves asymptotically for large t as

$$\mathcal{I}_t \sim \begin{cases} t^{\beta-1} & \text{for } \beta \neq 1 \\ \log(t+1) & \text{for } \beta = 1 \end{cases};$$

- Recalling the fact that the fair pricing gives $I_N = \frac{1}{N} \sum_{t=1}^N \mathcal{I}_t$, there holds

$$I_N \sim \frac{1}{N} \int_0^N x^{\beta-1} dx = \frac{1}{\beta} N^{\beta-1}.$$

Finally

Proposition 3.

$$\frac{I_N}{\mathcal{I}_N} = \frac{1}{\beta}.$$

For further use, one can observe that for a value of $\beta = 1.5$, we have $\frac{I_N}{\mathcal{I}_N} = \frac{2}{3}$.

Part II

Empirical Analysis of High-Frequency Options Traders Market Impact

Market Impact: A Systematic Study of the High Frequency Options Market

Abstract

This paper deals with a fundamental subject that has seldom been addressed in recent years, that of market impact in the options market. Our analysis is based on a proprietary database of metaorders - large orders that are split into smaller pieces before being sent to the market - on one of the main Asian markets. In line with our previous work on the equity market [Said et al., 2018], we propose an algorithmic approach to identify metaorders, based on some implied volatility parameters, the *at the money forward volatility* and *at the money forward skew*. In both cases, we obtain results similar to the now well understood equity market: *Square-Root Law*, *Fair Pricing Condition* and *Market Impact Dynamics*.

Keywords: *Market microstructure, market impact, statistical finance, fair pricing, automated trading, limit orders, options market, implied volatility, high frequency.*

1 Introduction

In recent years market impact has become a topic of interest for most market participants. The advent of algorithmic trading has significantly increased the traded volumes and the number of transactions. The whole point of electronic markets is to directly match participants that are willing to sell an asset with participants that are willing to buy it. This is mainly done via two types of orders: market orders and limit orders. Market orders are sent by participants that are willing to either buy or sell the asset immediately. Limit orders, however, do not share this urgency: these orders show the interest of the participant to buy or sell the asset at a pre-assigned price. Market orders are generally not used by institutional investors because of the lack of control they imply. On the contrary, limit orders, whether they are *aggressive* - crossing the spread - or *passive*, form the vast majority of orders actually sent to the market during the execution of an algorithmic trading strategy, for example a market making strategy or an optimal execution strategy. Most strategies referred to as algorithmic trading fall into the cost-reduction category. The basic idea is to break down a large order -

a **metaorder** – into small orders and send them to the market over time. The reasons for the incremental execution of metaorders are originally due to what we have called the *liquidity paradox*: The volume of buy or sell limit orders typically available in the order book at a given instant of time is quite small and represents only the order of 1% of the traded daily volume, i.e. $10^{-4} - 10^{-5}$ of the market cap for stocks [Bouchaud et al., 2009]. Hence the fact that the outstanding liquidity is so small has an immediate consequence: trades must be fragmented.

Recently, market impact questions related to options market have appeared in option pricing issues. However, the constraints of non-arbitrage often impose some limitations such as linear market impact [Bouchard et al., 2017] [Loeper, 2018]. In these papers the authors examine how linear market impact on the spot market can affect option prices. The point of view adopted here is slightly different and seeks to highlight how dealing with options can affect directly volatility and therefore option prices.

Many studies have been conducted to understand the influences of metaorders on the price formation process. Most of them are concerned with the equity markets, see e.g. [Almgren et al., 2005], [Moro et al., 2009], [Toth et al., 2011], [Bershova and Rakhlin, 2013], [Mastromatteo et al., 2014], [Bacry et al., 2015], [Gomes and Waelbroeck, 2015], [Brokmann et al., 2015] and [Said et al., 2018]. All these studies have shown a common behavior during the execution of a metaorder, namely, a concave and temporary impact followed by a convex and decreasing relaxation. More recently, [Donier and Bonart, 2015] have observed similar effects in the bitcoin market, and a short note [Tóth et al., 2016] indicated that the *Square root law* seems to be hold for the options market.

This paper is intended as an analysis of market impact in the options market. To the best of our knowledge, this is the first detailed, in-depth academic study of this phenomenon.

The paper is organized as follow: Section 2 recalls our algorithmic definition of an *equity metaorder*, introduces that of an *options metaorder* and presents some market impact measures. Section 3 introduces the data set of the study and presents our approach for *options metaorders*. Sections 4, 5 and 6 present our findings and empirical results: They confirm that the market impact laws observed in the equity market also hold true in the options market. Section 7 is a discussion of our results and their implications.

2 Definitions, Algorithm and Market Impact measures

2.1 Basic Definitions

Some basic concepts, and the algorithmic definition of an options metaorder, are introduced here.

Definition 9. *A limit order is an order that sets the maximum or minimum price at which an agent is willing to buy or sell a given quantity of a particular stock.*

Definition 10. An *aggressive limit order* is one that instantaneously removes liquidity from the order book by triggering a transaction. An aggressive order crosses the Bid–Ask spread. In other words an aggressive buy order will be placed on the ask, and an aggressive sell order will be placed on the bid.

A limit order that is not aggressive is termed **passive**. Passive orders sit in the order book until they are executed or cancelled.

Loosely speaking, a **metaorder** is a large trading order that is split into small pieces and executed incrementally. In order to perform rigorous statistical analyses, a more specific and precise definition of a metaorder is required, and given in Definition 11 below:

Definition 11. A *metaorder* is a series of orders sequentially executed during the same day and having those same attributes:

- *agent* i.e. a participant on the market (an algorithm, a trader...);
- *product id* i.e. a financial instrument (a share, an option...);
- *direction* (buy or sell);

Clearly, Definition 11 must be adapted to fit the options market.

Options are a bit more complex than equities. Traders buy and sell volatility and deal directly with the implied volatility surface, and therefore, with their implicit volatility parameters. As such, an options metaorder can naturally be defined as a sequence of transactions that generate some specific deformations of the volatility surface.

Definition 12. An *options metaorder* with respect to an implied volatility parameter θ is a series of orders sequentially executed during the same day and having those same attributes:

- *agent* i.e. a participant on the market (an algorithm, a trader...);
- *underlying product id* i.e. the underlying financial instrument;
- *direction* regarding the sign of $\mathcal{S}^\theta := Q \times \frac{\partial \mathcal{C}}{\partial \theta}$ where Q is the algebraic quantity (positive for a buy order and negative for a sell order) and \mathcal{C} the price of the option traded;

Note that in Definition 12, the *product id* condition introduced in Definition 11 is dropped. As a matter of fact, trading an option with a given strike K and maturity T also affects those with nearby strikes and maturities, so that trades on options with different strikes and maturity can very well belong to the same metaorder.

This approach is in line with what is presented in [Said et al., 2018], and leads to a systematic study of options market impact.

In what follows we will use the term θ –metaorder to refer to an options metaorder with respect to the implied volatility parameter θ as defined in Definition 12.

2.2 Market Impact definitions

The framework is similar to that introduced in [Said et al., 2018]. Let Ω^θ be the set of θ -metaorders under scrutiny, that is, θ -metaorders that are fully executed during a single market session, and pick $\omega \in \Omega^\theta$ executed on (possibly) several options with the same underlying and during a given day d . Its execution starts at some time $t_0(\omega)$ and ends the same day at time $t_0(\omega) + T(\omega)$. Thus $T(\omega)$ represents the duration of the metaorder. Denote by $N(\omega)$ the number of orders that have been executed during the life cycle of the metaorder ω : $N(\omega)$ is the length of ω . Let $t_0(\omega), t_1(\omega), \dots, t_{N(\omega)-1}(\omega)$ be the transaction times of the metaorder ω , we define $\mathcal{V}^\theta(\omega) := \sum_{i=0}^{N(\omega)-1} \mathcal{S}_{t_i(\omega)}^\theta$ as the sensitivity of the metaorder ω regarding to the parameter θ . Let $V^\theta := \sum_{t \in \mathcal{T}_{market}} |\mathcal{S}_t^\theta|$ be the sensitivity traded the same day d on all the options of the universe, which means all the options traded by the algorithms summed over all market transactions \mathcal{T}_{market} occurred in the day d . Hence V^θ can be viewed as the *absolute sensitivity* traded by the market regarding to the parameter θ . Note that this quantity depends only on the universe the day d . Therefore all the θ -metaorders executed the same day will share the same *absolute sensitivity*. Hence we define $\frac{|\mathcal{V}^\theta(\omega)|}{V^\theta}$ as the θ -daily participation rate. The sign of ω will be noted $\epsilon(\omega)$ (i.e. $\epsilon = 1$ for a positive sensitivity options metaorder and $\epsilon = -1$ for a sensitive negative one), the sign of $\epsilon(\omega)$ is also the sign of $\mathcal{S}_{t_0(\omega)}^\theta, \mathcal{S}_{t_1(\omega)}^\theta, \dots, \mathcal{S}_{t_{N(\omega)-1}(\omega)}^\theta$ which is invariant during the life of ω . Clearly, most of the quantities introduced in this section depend on ω . For the sake of simplicity, we chose to omit this dependence whenever there is no ambiguity and will often write $T, N, \mathcal{V}^\theta, \epsilon$ instead of $T(\omega), N(\omega), \mathcal{V}^\theta(\omega), \epsilon(\omega)$.

The **market impact curve** of a metaorder ω quantifies the magnitude of the relative θ -variation between the starting time of the metaorder t_0 and the current time $t > t_0$, θ being a parameter of the implied volatility model. Let $\mathcal{I}_t(\omega)$ be a proxy for the realized θ -parameter variation between time t_0 and time $t_0 + t$. We use the *variation proxy* defined by

$$\mathcal{I}_t = \theta_t - \theta_{t_0}, \quad (1)$$

This estimation relies on the assumption that the *exogenous market moves* W_t will cancel out once averaged, i.e. as a random variable, W_t should have finite variance and basically satisfy $\mathbb{E}(\epsilon(\omega)W_t(\omega)) = 0$.

One can thus write

$$\epsilon(\omega)\mathcal{I}_t(\omega) = \eta_t(\omega) + \epsilon(\omega)W_t(\omega), \quad (2)$$

where $\eta_t(\omega)$ represents the market impact curve and $W_t(\omega)$, the exogenous variation corresponding to the relative move that would have occurred if the metaorder had not been sent to the market.

	Equity Metaorder	Options Metaorder
Object of study	stock market	options market
Quantity of interest	stock price P	implied volatility parameters θ
Effective size	quantity Q	sensitivity \mathcal{S}^θ
Effective variation	$P_t - P_{t_0}$	$\theta_t - \theta_{t_0}$
Effective Sign ϵ	$\mathbb{1}_{\{Q>0\}} - \mathbb{1}_{\{Q<0\}}$	$\mathbb{1}_{\{\mathcal{S}^\theta>0\}} - \mathbb{1}_{\{\mathcal{S}^\theta<0\}}$
Market impact proxy	$\epsilon \times (P_t - P_0)$	$\epsilon \times (\theta_t - \theta_0)$

Table II.1 – Comparison of the quantities of interest between equity and options metaorders

2.3 Implied volatility model

In the previous section we have introduced the following proxy

$$\epsilon \times (\theta_t - \theta_{t_0}) \quad (3)$$

to measure the market impact of θ which is a parameter of the proprietary implied volatility model used to conduct this study. Although the approach used is model dependent, we think that for the two parameters presented here (*at the money forward* volatility and skew) the results must be the same for any implied volatility model fitting the market. This is also one of the reasons why we limited ourselves to study the market impact on the *at the money forward* volatility and skew in our article as every reasonable implied volatility model must have at least those two variables. We have respectively sketched on Figures II.1 and II.2 the expected market impact variations of the *at the money forward* volatility and skew. For sake of clarity let us recall that the terminology *at the money forward* refers to quantities function of the strike evaluated at the forward.

3 Data

3.1 Data description

The data set we use for this analysis contains trade orders executed by the BNP Paribas options trading desk for the 2-year period from June 2016 through June 2018 on the KOSPI 200 options. The KOSPI 200 Index is a capitalization-weighted index of 200 Korean stocks which make up 93% of the total market value of the Korea Stock Exchange. In order to perform rigorous statistical analyses we need to be able to calibrate the parameters of the implied volatility model as often as possible. This is necessary to observe the variations of the parameters at a frequency similar to that of the transactions. To make this possible with sufficient accuracy, we have only considered executions trading mostly short maturities options.

Because of the high frequency in the execution of the orders, **we only consider options metaorders with at least 5 completed transactions**. This underlines the fact that we want to keep only metaorders that reasonably act as liquidity takers and could impact significantly

the market. Indeed, while **the equity metaorders studied in [Said et al., 2018] had an average time life of several hours, the options metaorders presented here last a few tenths of seconds.**

3.2 Data representation

For sake of confidentiality numerical values of market impact, daily participation rates and durations are given in percentage of reference values – related to the market and BNP Paribas options desk – that cannot be disclosed. Nevertheless we will give when it is possible the value of the ratio of these reference values in order to make feasible comparisons between the *at the money forward* volatility and skew metaorders. For instance the reference value of impact is the same in Figures II.9, II.10 and II.11 for the volatility metaorders, so it is possible to make comparisons between these graphs. The same applies for skew metaorders in Figures II.12, II.13 and II.14.

In what follows we will use the following convention: All graphs relating to volatility (resp. skew) metaorders will be drawn in blue (resp. green).

3.3 Options Market Impact - The Liquidity Taker Mode (Aggressive Orders)

Let us now focus on the market impact generated by a series of aggressive executions on the options market. More specifically, we consider the two kinds of metaorders: namely the *at the money forward volatility* and the *at the money forward skew* metaorders. We recall that aggressive limit orders are limit orders that cross the spread in order to trigger an immediate transaction.

It is important to understand the fundamental difference between these two types of metaorders. For example, if someone wants to buy *at the money forward volatility* (Fig. II.1), the simplest strategy is to buy options with strikes close to the money forward. Whereas if one wants to buy *at the money forward skew*, a simple way to do so consists in buying out of the money options while at the same time, selling in the money options (Fig. II.2).

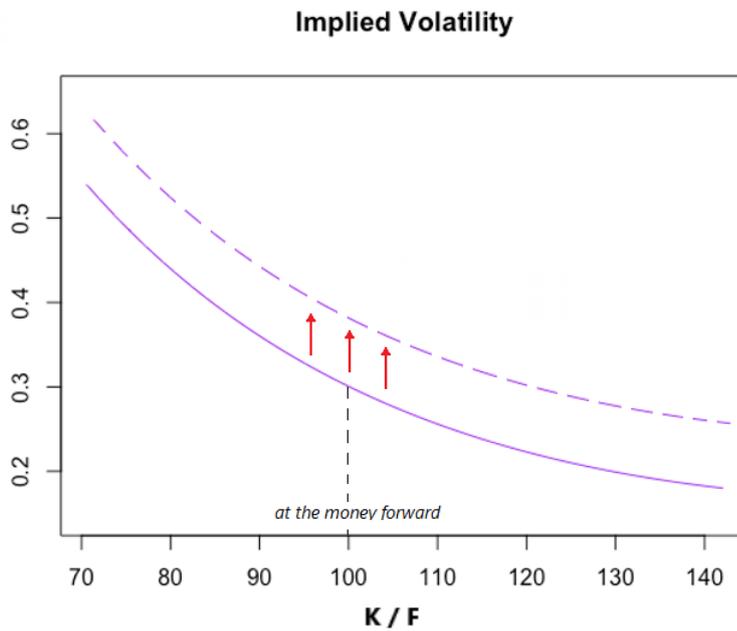


Figure II.1 - Market impact on the implied volatility surface for a given maturity T under the effect of buying options with strikes near the money forward. One can notice how this could increase the *at the money forward volatility* parameter as emphasized by the red arrows.

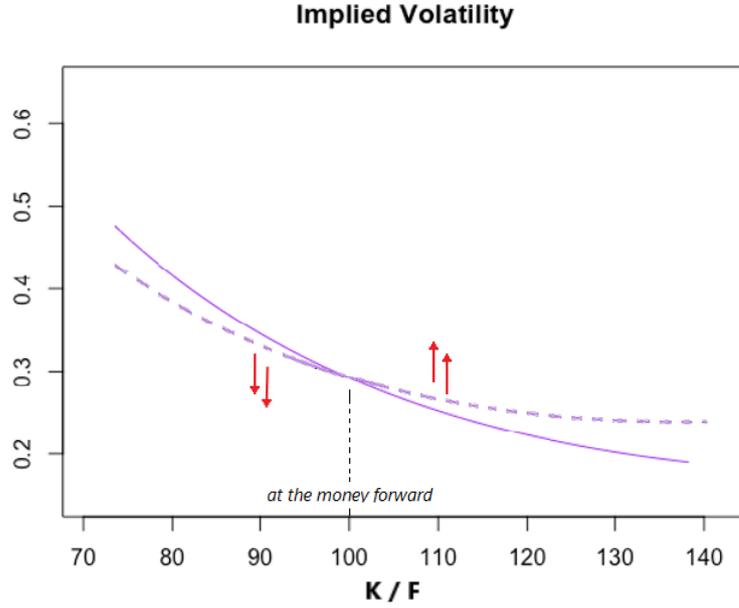


Figure II.2 – Market impact on the implied volatility surface for a given maturity T under the effect of buying options with strikes at the right of the money forward and sell options with strikes at the left of the money forward at the same time. One can see how this could increase the *at the money forward skew* parameter as emphasized by the red arrows.

A simple way to check that the previous intuitions presented in Figures II.1 and II.2 are not totally irrelevant is to study for the two kinds of metaorders the relation of the θ -market impact curve as a function of the (unbiased) standard deviation (weighted by the θ -sensitivity) of the strikes of the child orders pertaining for each metaorder denoted by $\sigma_{K/F}$ (Fig. II.3 and II.4). Hence for each θ -metaorder ω the expression of $\sigma_{K/F}^\theta(\omega)$ is given by

$$\sigma_{K/F}^\theta(\omega) = \sqrt{\frac{\sum_{i=0}^{N(\omega)-1} \mathcal{S}_{t_i(\omega)}^\theta (K_i / F_{t_i(\omega)} - \mu_{K/F}^\theta(\omega))^2}{\mathcal{V}^\theta(\omega) - 1}}, \quad (4)$$

where

$$\mu_{K/F}^\theta(\omega) = \frac{\sum_{i=0}^{N(\omega)-1} \mathcal{S}_{t_i(\omega)} K_i / F_{t_i(\omega)}}{\mathcal{V}^\theta(\omega)} \quad (5)$$

is the weighted mean with K_i the strikes of the corresponding child orders and F_{t_i} the forward prices at the instants t_i . Figures II.3 and II.4 are obtained by following the methodology presented in [Said et al., 2018] and briefly recalled in Sec. 4.1.

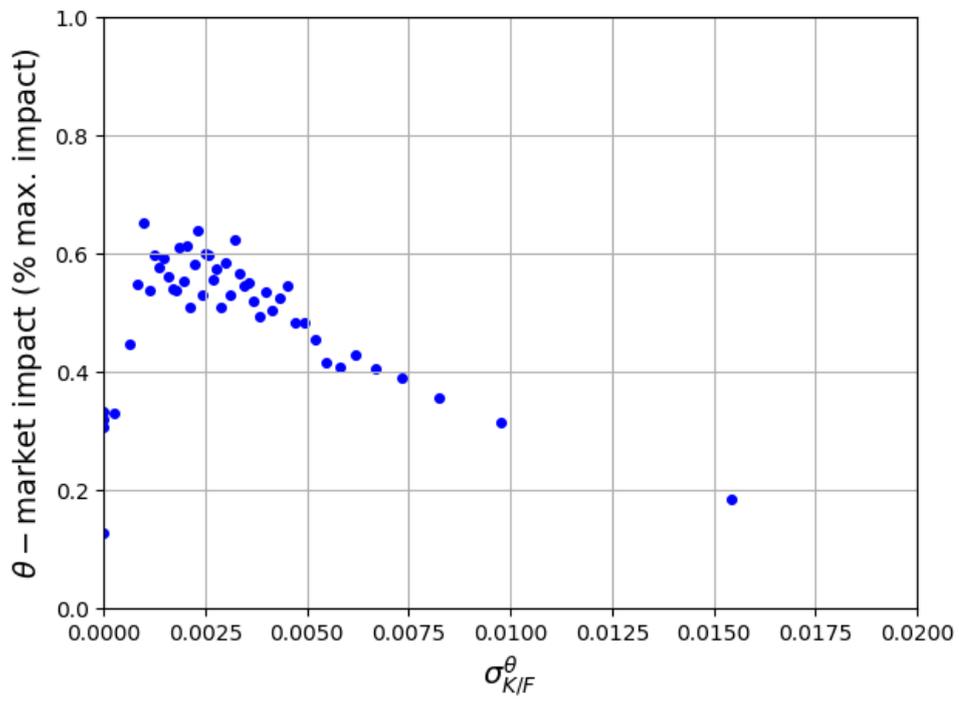


Figure II.3 – Market impact as a function of $\sigma_{K/F}$ in the case of the *at the money forward volatility* metaorders ($\theta \equiv$ ATM volatility)

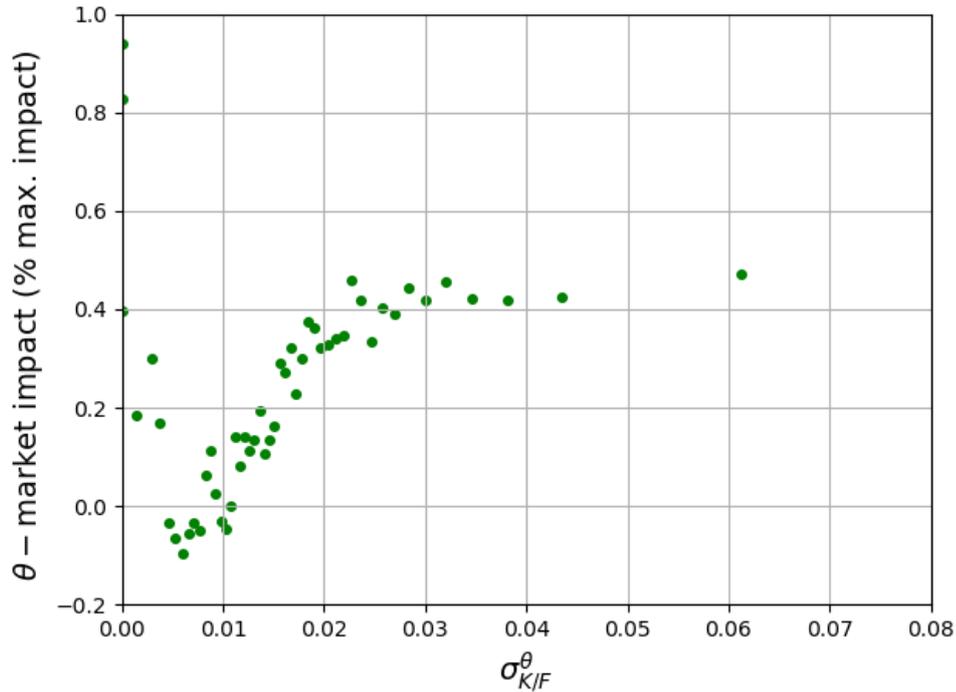


Figure II.4 – Market impact as a function of $\sigma_{K/F}$ in the case of the *at the money forward skew* metaorders ($\theta \equiv$ ATMF skew)

A careful scrutiny of Figures II.3 and II.4 shows that in the case of the *at the money forward volatility metaorders*, the less diversified the strikes are, the more important the market impact. It is the opposite in the case of the *at the money forward skew metaorders*: the metaorders with the higher values of $\sigma_{K/F}$ match those which present the most important market impacts. Besides as expected, one can note that the average values of $\sigma_{K/F}^\theta$ are higher for the *at the money skew metaorders*. Those observations are in line with the intuitive predictions presented in Figures II.1 and II.2. The first points appearing on the Figures II.1 and II.2 actually correspond to the shortest metaorders in length and duration, so they are just noisy points.

3.3.1 The ATMF volatility metaorders

In this section we focus on metaorders impacting the *at the money forward volatility* parameter of the implied volatility model.

Data

- Study period : **1st July 2016 – 30th June 2018**
- Order types : **Aggressive Limit Orders**
- Parameter : $\theta \equiv$ **at the money forward volatility**

- Filters : **metaorders** $\omega \in \Omega^\theta$
- Number of orders : **1,026,197**
- Number of metaorders : **149,441**

Duration distribution

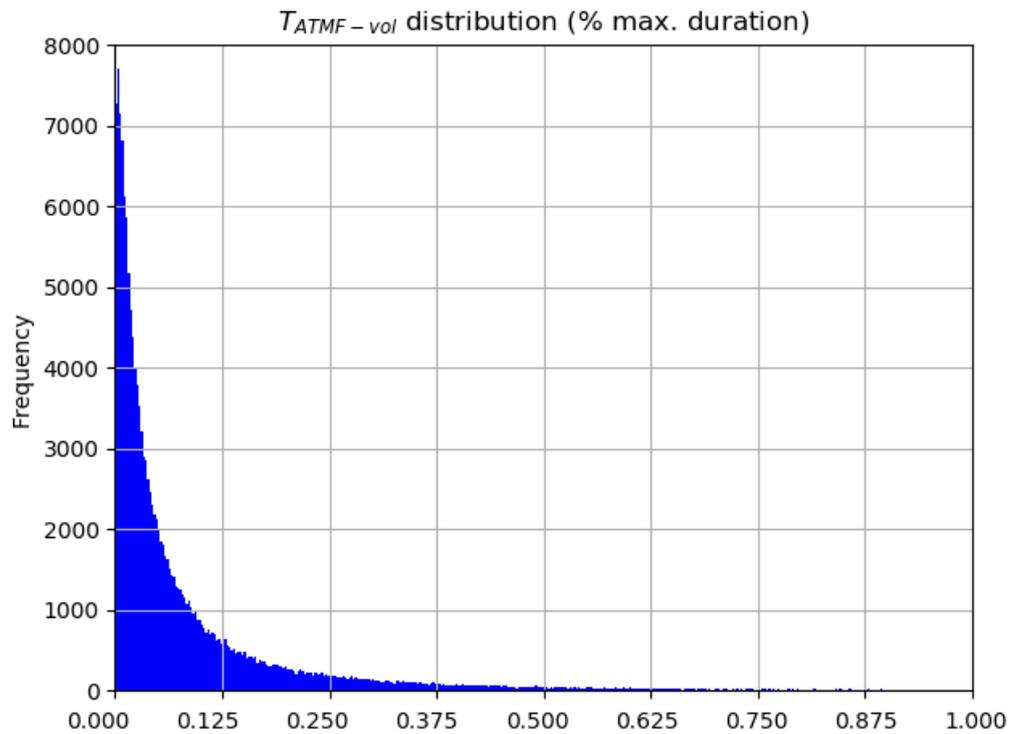


Figure II.5 - Duration distribution of the *at the money forward volatility* metaorders

One can observe that, in agreement with the intuition, metaorders with shorter durations are more frequent (Fig. II.5). This histogram is quite similar to the one of presented for the equity aggressive metaorders in [Said et al., 2018].

Daily participation rate distribution

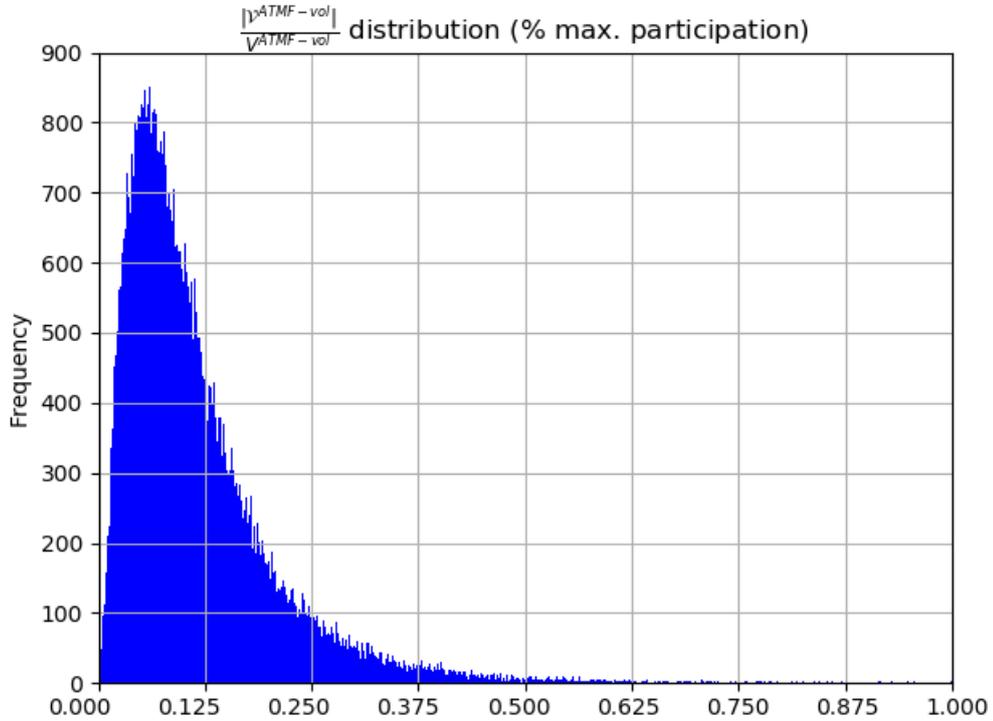


Figure II.6 – Daily participation rate distribution of the *at the money forward volatility* metaorders

One can note in Figure II.6 that metaorders with very high participation rates are evenly represented.

3.3.2 The ATMF skew metaorders

In this section we focus on metaorders whose *at the money forward skew* parameter is the parameter of interest in the implied volatility model.

Data

- Study period : **1st July 2016 – 30th June 2018**
- Order types : **Aggressive Limit Orders**
- Parameter : $\theta \equiv$ **at the money forward skew**
- Filters : **metaorders** $\omega \in \Omega^\theta$

- Number of orders : **1,304,714**
- Number of metaorders : **174,091**

Duration distribution

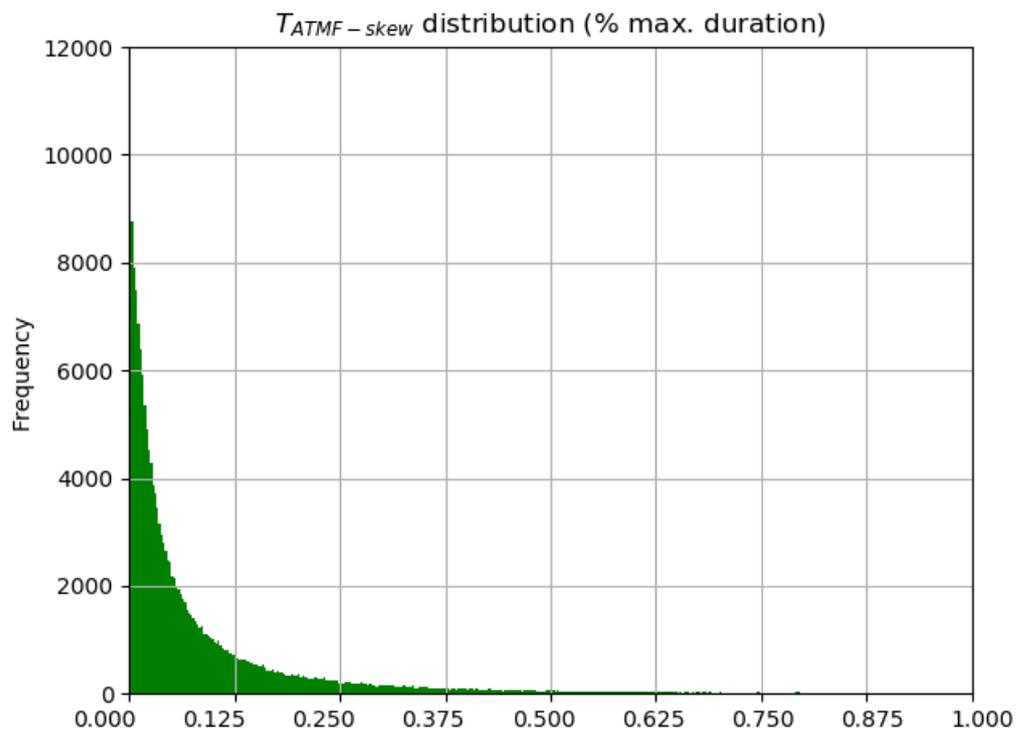


Figure II.7 - Duration distribution of the *at the money forward skew* metaorders

The distribution of the durations of the *at the money forward skew* metaorders (Fig. II.7) is quite similar to the distribution observed in Figure II.5 for the *at the money forward volatility* metaorders. One more time we notice that metaorders with shorter durations are more frequent.

Daily participation rate distribution

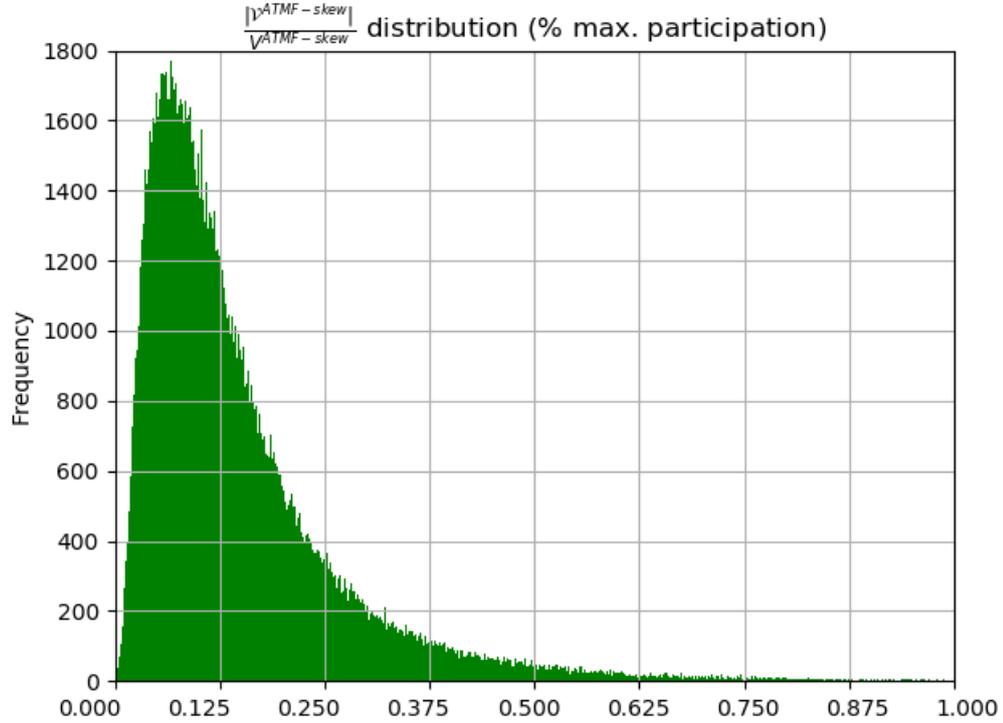


Figure II.8 – Daily participation rate distribution of the *at the money forward skew* metaorders

One can note in Figure II.8 that metaorders with very high participation rates are evenly represented. As already observed in Sec. 3.3.1 there is a maximum frequency for metaorders around a certain value closed to the maximum observed in Figure II.6.

3.3.3 Comparisons of Volatility and Skew metaorders distributions

In Figures II.5, II.6, II.7 and II.8 durations and daily participation rates are disclosed as a percentage of reference maximum values relative to the market and BNP Paribas options desk. We will denote $\langle T \rangle_{ref}^{ATMF-vol}$ and $\langle T \rangle_{ref}^{ATMF-skew}$ these reference durations and

$\left\langle \frac{|V|}{V} \right\rangle_{ref}^{ATMF-vol}$ and $\left\langle \frac{|V|}{V} \right\rangle_{ref}^{ATMF-skew}$ these reference daily participation rates. As

$$\langle T \rangle_{ref}^{ATMF-vol} \approx \langle T \rangle_{ref}^{ATMF-skew}$$

and

$$\left\langle \frac{|V|}{V} \right\rangle_{ref}^{ATMF-vol} \approx \left\langle \frac{|V|}{V} \right\rangle_{ref}^{ATMF-skew},$$

we can say that the duration and participation distributions are quite similar for the two types of metaorders. Also the two kinds of metaorders present closed number of orders and metaorders during the same period as disclosed in Sec. 3.3.1 and Sec. 3.3.2, so we can conclude that *at the money forward* volatility and skew metaorders must behave in the same manner. Hence the differences in terms of market impact between the two types of metaorders could be only explained by the way the market reacts to each type of metaorder.

3.4 Notations

Notation	Definition
ω	A metaorder
$O(\omega)$	Option of the metaorder ω
$d(\omega)$	Execution day of the metaorder ω
$t_0(\omega)$	Start time of the metaorder ω
$T(\omega)$	Duration of the metaorder ω
$N(\omega)$	Length of the metaorder ω
$\mathcal{V}^\theta(\omega)$	θ -Sensitivity of the metaorder ω
$V^\theta(\omega)$	θ -Sensitivity traded the day $d(\omega)$ on all the options of the universe
$\epsilon^\theta(\omega)$	Sign of $\mathcal{V}^\theta(\omega)$
$\mathcal{O}(\omega)$	Price of $O(\omega)$
Ω^θ	Set of all the θ -metaorders identified by the algorithm
$\Omega_{n^*}^\theta \subset \Omega$	Subset of the θ -metaorders with $N \geq n^*$

Table II.2 – *Notations and definitions*

Remark 2. *As we only consider metaorders that have at least 5 executed transactions, $\Omega^\theta = \Omega_5^\theta$.*

4 Market Impact Dynamics

4.1 Market impact curves

The main results related to market impact dynamics are now given, namely, the *market impact curves* for the *at the money forward volatility* and the *at the money forward skew* metaorders. In order to plot the market impact dynamics, a similar *bucketing* method as the one presented in [Said et al., 2018] is used: Let x, y being two arrays of data and consider for example that one wants to plot y as a function x . First one starts by ordering the couple of values (x_i, y_i) according to the values of x and then divides the **sorted (by x) distribution** $(x, y)_{sorted}$ into N_{bucket} equally-sized buckets. This procedure yields N_{bucket} subsets of the distribution $(x, y)_{sorted}$, $(x_i, y_i)_{i \in I_1}, (x_i, y_i)_{i \in I_2}, \dots, (x_i, y_i)_{i \in I_{N_{bucket}}}$, and for each bucket I_k the *means values* (\bar{x}_k, \bar{y}_k) is computed. The last step of this bucketing method is to plot the points $(\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2), \dots, (\bar{x}_{N_{bucket}}, \bar{y}_{N_{bucket}})$.

To study the dynamics of the market impact, one plots $(\epsilon(\omega), \mathcal{F}_t(\omega))_{\omega \in \Omega, t_0(\omega) \leq t \leq t_0(\omega) + 2T(\omega)}$. The first sub-interval $t_0(\omega) \leq t \leq t_0(\omega) + T(\omega)$ corresponds to the execution of the metaorder, whereas the second $t_0(\omega) + T(\omega) \leq t \leq t_0(\omega) + 2T(\omega)$ corresponds to the relaxation. The study of relaxation presents a degree of arbitrariness, since a choice has to be made as to the elapsed time after the metaorder is completed. For the sake of homogeneity, the relaxation is measured over the same duration as the execution. This choice seems to be a good compromise to cope with two antagonistic requirements, one being to minimize this elapsed time because of the diffusive nature of prices - which affects also the parameters -, the other being to maximize it so as to make sure that the relaxation is achieved.

In order to perform an extensive statistical analysis involving metaorders of varying lengths in physical and θ -sensitivity time, a rescaling in time is necessary, see e.g. [Bacry et al., 2015] and [Said et al., 2018]. With this convention, all orders are executed on the time interval $[0, 1]$ and parameter relaxation occurs in the time interval $[1, 2]$. For each metaorder ω , one considers $[0, 1]$ instead of $[t_0(\omega), t_0(\omega) + T(\omega)]$ $\left([0, 1] = \frac{[t_0(\omega), t_0(\omega) + T(\omega)] - t_0(\omega)}{T(\omega)} \right)$ for the *execution part* of ω and $[1, 2]$ instead of $[t_0(\omega) + T(\omega), t_0(\omega) + 2T(\omega)]$ for the *relaxation part* of ω , and then averages using the *bucketing method* previously described on the time-rescaled θ -sensitivity quantities. In fact the θ -sensitivity time here plays the same role as the volume time in [Said et al., 2018]. Indeed, traders in the options markets buy and sell more *sensitivities* than the products themselves in order to reduce the expositions of their portfolio i.e. they think and plan not in terms of options (e.g. calls, puts), but rather in terms of sensitivities of their portfolio.

The time variable $t \in [0, 1]$ in the Figures of Sec. 4 is actually the θ -sensitivity time, i.e., the ratio between the θ -sensitivity of the metaorder already executed at the time of the observation and the total θ -sensitivity of the metaorder - of course, at the end of the execution part this quantity is always equal to 1.

The market impacts disclosed in Figures II.9, II.10 and II.11 in Sec. 4.2 (resp. Figures II.12, II.13 and II.14 in Sec. 4.3) are given in percentage of the same reference value making possible comparisons between them.

4.2 The ATMF volatility market impact dynamics

The blue points correspond to θ -values during execution, θ being the *at the money forward volatility* parameter, and the red points correspond to θ -values observed at identical times starting from the end of the metaorder.

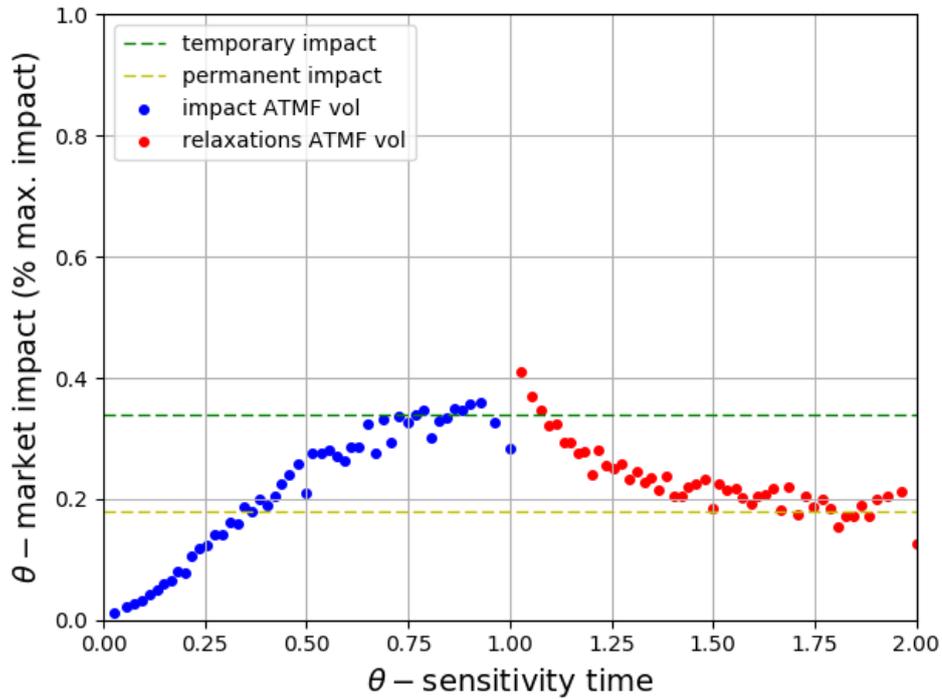


Figure II.9 – Market impact dynamics in the case of the *at the money forward volatility* metaorders ($\theta \equiv$ ATMF volatility, set: Ω , 1,026,197 orders, 149,441 metaorders, temporary impact: 0.34, permanent impact: 0.17)

The analysis clearly yields an increasing, concave market impact curve. However, one can observe that the curve has a linear behavior at the beginning and becomes more concave towards the end. This is explained in particular by the fact that the duration of the metaorders is quite short. The decay observed in the last points (in $t = 1.0$ and $t = 2.0$) of the curve is an artifact, already discussed in [Said et al., 2018], inducing a bias towards the end of the curve. It can be explained by the larger number of metaorders of smaller lengths and with lower impact. Also note that on the three figures II.9, II.10 and II.11, the larger the metaorders, the higher the impacts: 0.34, 0.60 and then 0.86 for the temporary market impact.

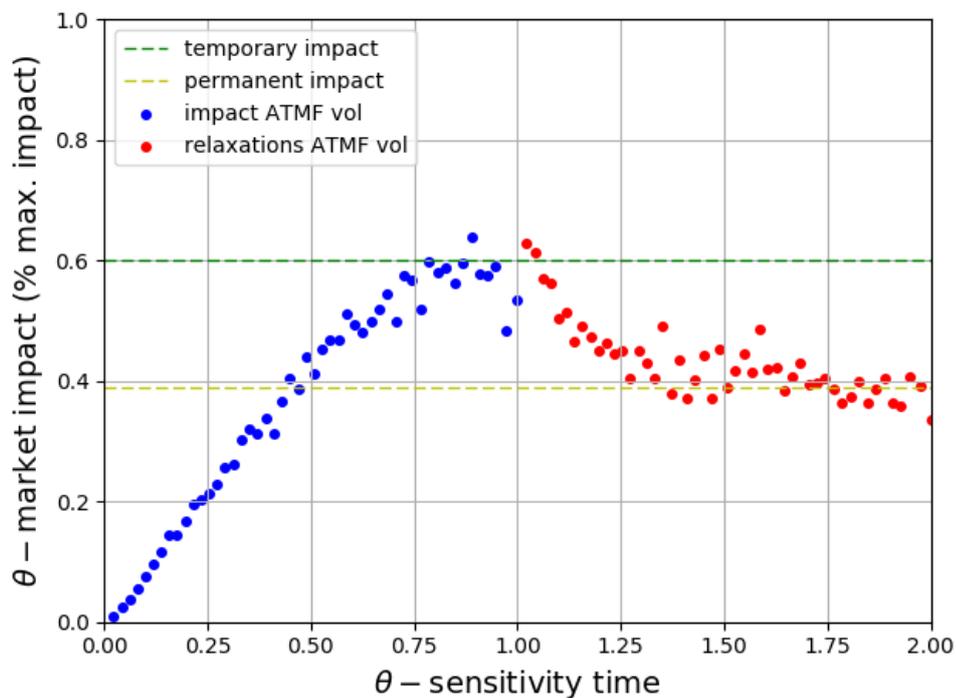


Figure II.10 - Market impact dynamics in the case of the *at the money forward volatility* metaorders ($\theta \equiv$ ATMF volatility, set: Ω_{10} , 215,274 orders, 17,286 metaorders, temporary impact: 0.60, permanent impact: 0.39)

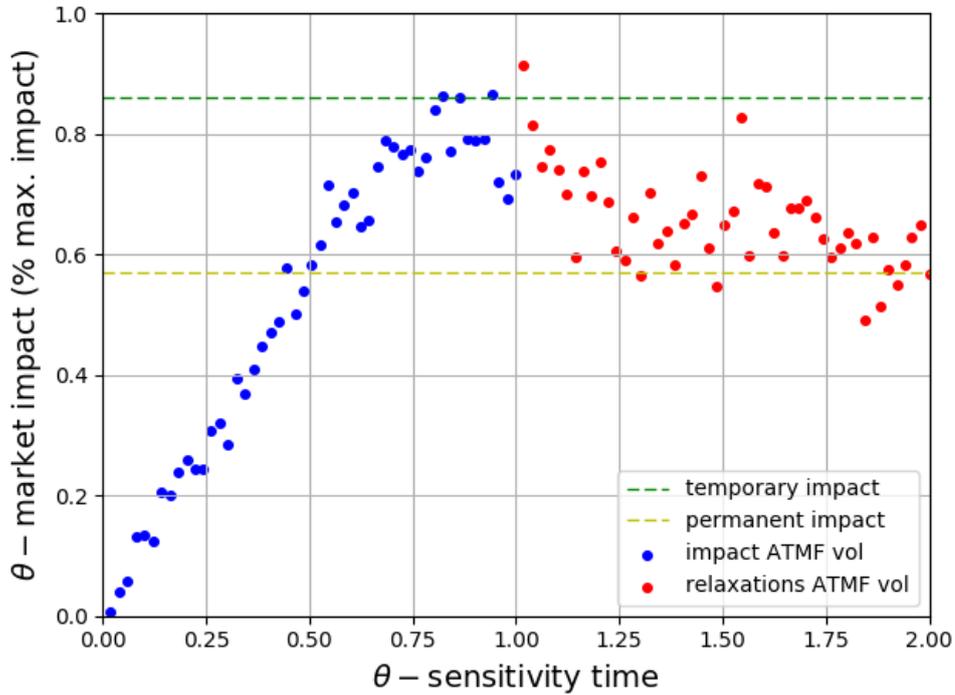


Figure II.11 – Market impact dynamics in the case of the *at the money forward volatility* metaorders ($\theta \equiv$ ATMF volatility, set: Ω_{15} , 54,203 orders, 2,958 metaorders, temporary impact: 0.86, permanent impact: 0.57)

Figures II.9, II.10 and II.11 clearly exhibit the concave shape of market impact during the execution part, followed by a convex and decreasing relaxation. Simply by eyeballing Figures II.9, II.10 and II.11, one can safely assume that relaxation is complete and stable at a level around respectively 0.17, 0.39 and 0.57. However, on Figure II.11, relaxation does not seem to be quite smooth. This behaviour for those larger metaorders is essentially due to the fact that the set Ω_{15} contains much less metaorders.

A conclusion to this section is that the concave shape of the temporary impact and the convex relaxation curve concerning the *at the money forward volatility* metaorders are in line with the empirical results observed on the equity markets and highlighted in [Bacry et al., 2015], [Bershova and Rakhlin, 2013] and [Said et al., 2018]. Also, and more interestingly, the market impact and relaxation curves confirm the theoretical findings of [Farmer et al., 2013] that the impact should be concave and increasing, and that the final impact after the execution is performed should relax to about two-thirds of the peak impact. Indeed while on the Figure II.9 the ratio between the *permanent market impact* and the *temporary market impact* seems to be much closer to 1/2, one can observe in Figures II.10 and II.11 – which correspond to larger metaorders and therefore more significant –, how this ratio gets close to 2/3.

4.3 The ATMF skew market impact dynamics

The main results related to skew market impact dynamics are now given, namely, the *market impact curves* for the *at the money forward skew* metaorders. To this purpose we use the same *bucketting method* introduced in Sec. 4.1.

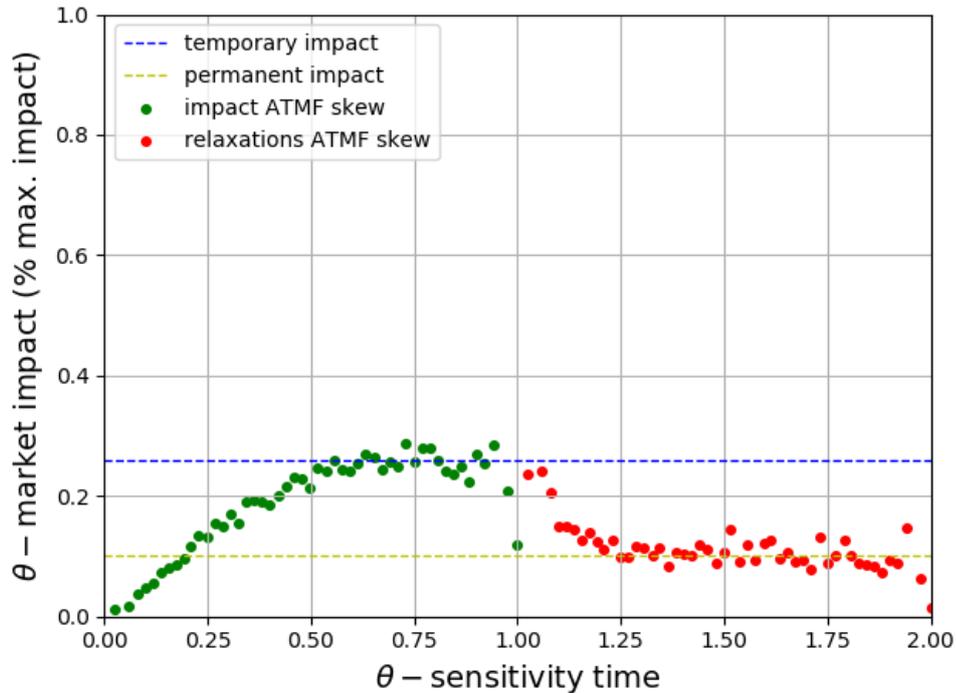


Figure II.12 – Market impact dynamics in the case of the *at the money forward skew* metaorders ($\theta \equiv$ ATMF skew, set: Ω , 1,304,714 orders, 174,091 metaorders, temporary impact: 0.26, permanent impact: 0.10)

The green points correspond to θ -values during execution, θ being the *at the money forward skew* parameter, and the red points correspond to θ -values observed at identical times starting from the end of the metaorder.

The results show an increasing, concave market impact curve, with a linear behavior at the beginning. The curve becomes more concave towards the end. As mentioned in Sec. 4.2, this is due to the fact that the durations of the metaorders are quite short. The decay observed in the last points (in $t = 1.0$ and $t = 2.0$) is the same effect observed in the *at the money forward volatility* metaorders. Also note that on the three figures II.12, II.13 and II.14, the larger the metaorders, the higher the impacts: 0.26, 0.48 and then 0.77 for the temporary market impact.

Figures II.12, II.13 and II.14 clearly exhibit the concave shape of market impact during the execution part, followed by a convex and decreasing relaxation. Simply by eyeballing Figures

II.12, II.13 and II.14, one can safely assume that relaxation is complete and stable at a level around respectively 0.10, 0.32 and 0.51. However, on Figure II.14, relaxation does not seem to be quite smooth. This behaviour for those larger metaorders is essentially due to the fact that the set Ω_{15} contains much less metaorders.

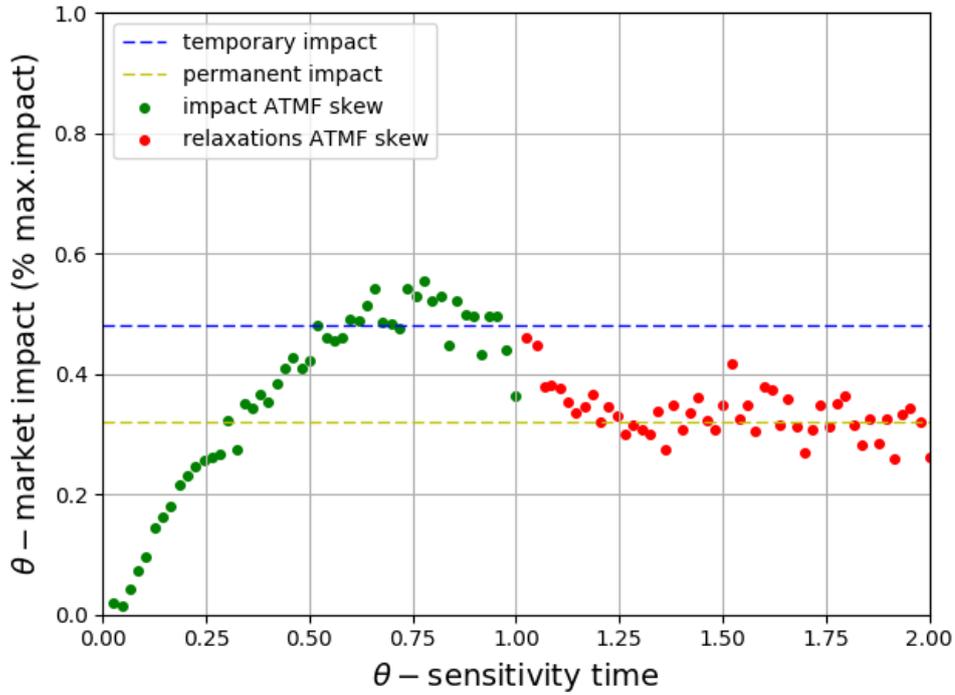


Figure II.13 – Market impact dynamics in the case of the *at the money forward skew* metaorders ($\theta \equiv$ ATMF skew, set: Ω_{10} , 405,918 orders, 30,932 metaorders, temporary impact: 0.48, permanent impact: 0.32)

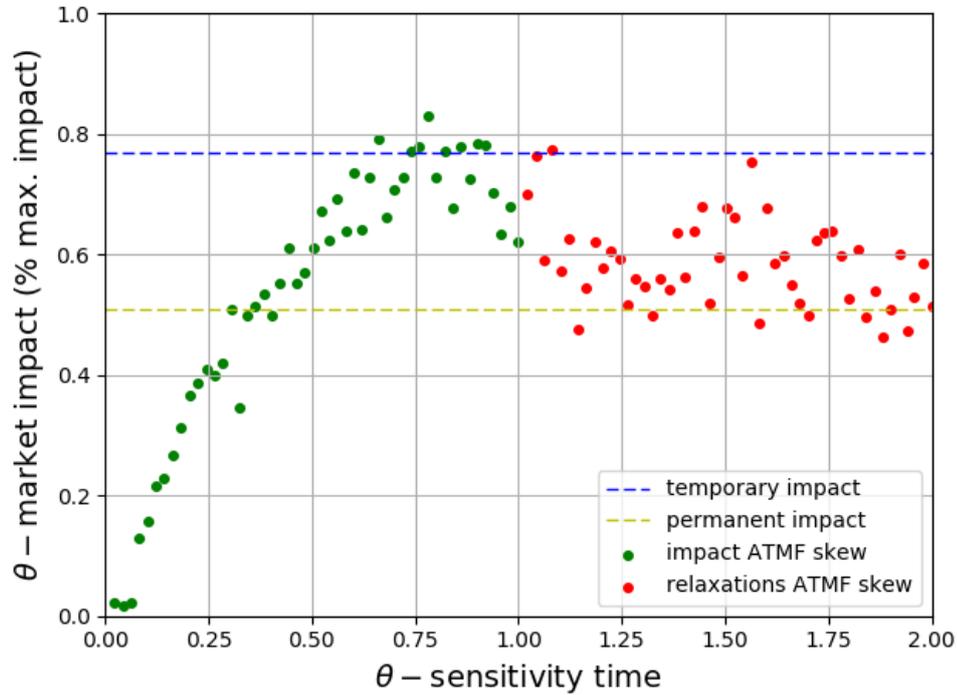


Figure II.14 – Market impact dynamics in the case of the *at the money forward volatility* metaorders ($\theta \equiv$ ATMF skew, set: Ω_{15} , 136,410 orders, 7,260 metaorders, temporary impact: 0.77, permanent impact: 0.51)

A conclusion to this section is that the concave shape of the temporary impact and the convex relaxation curve concerning the *at the money forward skew* metaorders are in line with the empirical results observed on the *at the money forward volatility* metaorders. Also, and more interestingly, the market impact and relaxation curves confirm the theoretical findings of [Farmer et al., 2013] that the impact should be concave and increasing, and that the final impact after the execution is performed should relax to about two-thirds of the peak impact. Indeed while on the Figure II.12 the ratio between the *permanent market impact* and the *temporary market impact* seems to be much closer to 0.4, one can observe in Figures II.10 and II.11 – which correspond to larger metaorders and therefore more significant –, how this ratio gets close to $2/3$.

5 Square-Root Law

5.1 Volatility and Skew metaorders square-root laws

The results presented in this section are certainly the most important of the article. They confirm the consistency of the *Square-Root Law* already observed in the equity market [Almgren et al., 2005], [Bershova and Rakhlin, 2013], [Gomes and Waelbroeck, 2015],

[Mastromatteo et al., 2014], [Moro et al., 2009] and [Toth et al., 2011], the bitcoin market [Donier and Bonart, 2015] and more recently in a short note [Tóth et al., 2016] the authors highlighted that the *Square-Root Law* also holds for option markets according to their definition of the *implied volatility* metaorders which is more global. The method presented here which relies on the definition 12 tends to be more local focusing on the local deformations of the implied volatility surface through the variations of the parameters of the model which are in fact the projections of those local deformations.

The *Square-Root Law* is the fact that the impact curve should not depend on the duration of the metaorder. Indeed, almost all studies now agree on the fact that the impact is more or less close to be proportional to the square root of the volume executed. Considering options market, which plays the role of an executed volume is in fact the θ -sensitivity executed regarding to a θ -metaorder. However, the so-called *Square-Root Law* states much more than that. It basically claims that the market impact does not depend on the metaorder duration.

In what follows one plots the θ -market impact normalized by a volatility factor $\epsilon^\theta \times \frac{\mathcal{I}^\theta}{\sigma^\theta}$ as a function of the θ -daily participation rate $\frac{|V^\theta|}{V^\theta}$ for both the *at the money forward volatility* (Fig. II.15) and the *at the money forward skew* (Fig. II.16) metaorders, σ^θ being the daily standard deviation of the parameter θ and $\mathcal{I}_t^\theta = \theta_t - \theta_{t_0}$ as defined in Equation (1). One observes that a power-law fit gives exponents for the θ -daily participation rate close to 0.5 (≈ 0.56 for the *at the money forward volatility* metaorders and ≈ 0.53 for the *at the money forward skew* metaorders).

In both cases, the analysis shows that our options metaorders, present a market impact following a theoretical curve of the form $\sigma\sqrt{\mathcal{R}}$ with σ a volatility factor and \mathcal{R} a participation rate factor. Those findings support the idea for a universal underlying mechanism in market microstructure.

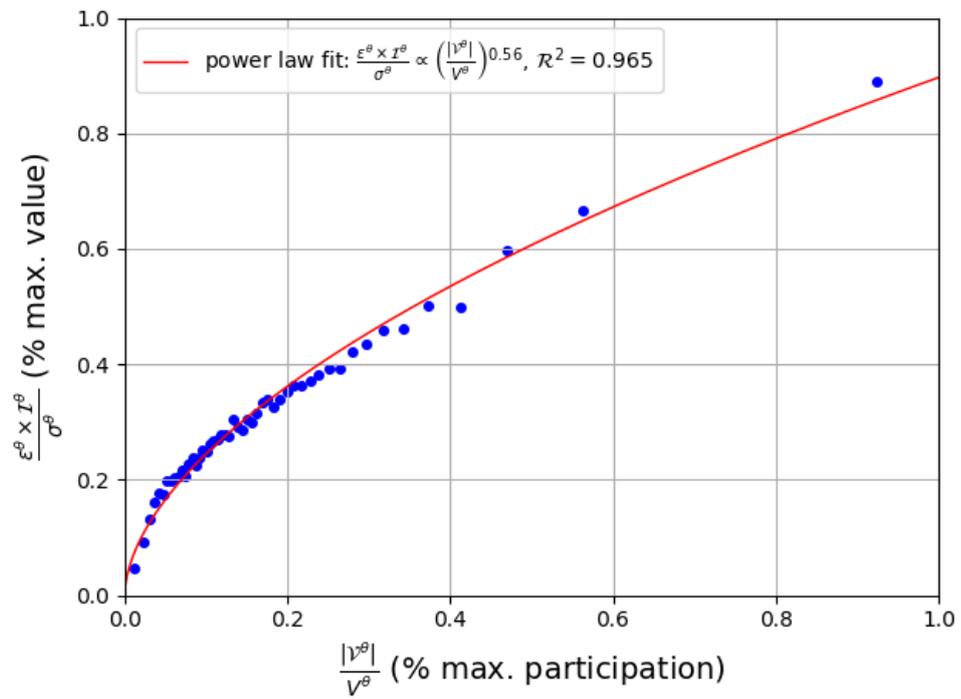


Figure II.15 - The square-root law in the case of the *at the money forward volatility* metaorders, power law fit: $y \propto x^{0.56}$, $R^2 = 0.965$.

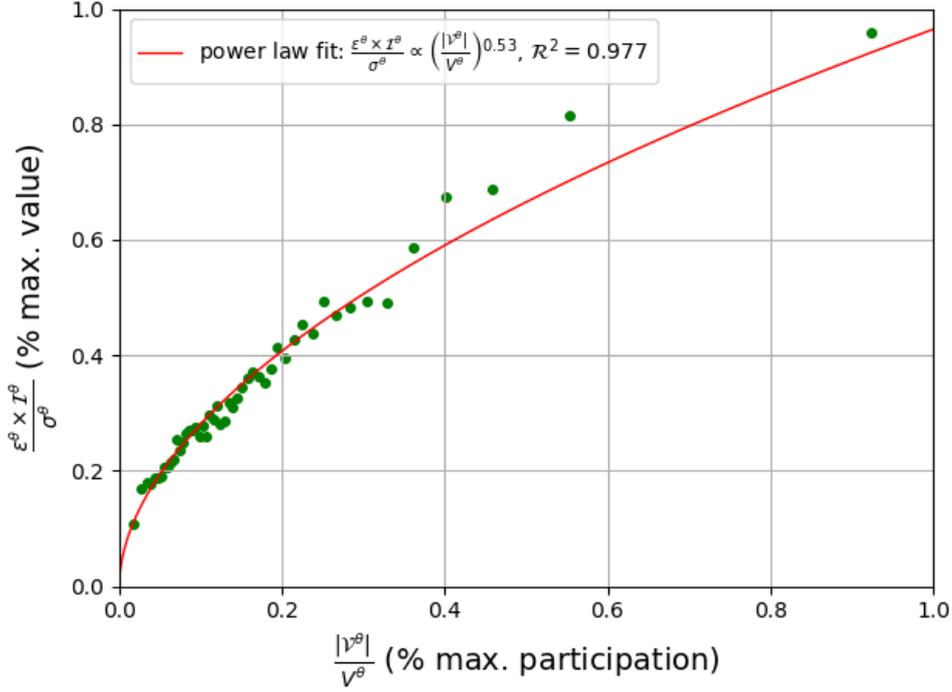


Figure II.16 – The square-root law in the case of the *at the money forward skew* metaorders, power law fit: $y \propto x^{0.53}$, $\mathcal{R}^2 = 0.977$.

5.2 Comparisons of Volatility and Skew metaorders square-root laws

As already done in Sec. 3.3.3 we are going now to make some comparisons between the square-root laws presented in Sec. 5.1. In Figures II.15 and II.16 normalized market impacts and daily participation rates are disclosed in percentage of reference maximum values.

We will denote respectively $\left\langle \frac{\epsilon \times \mathcal{I}}{\sigma} \right\rangle_{ref}^{ATMF-vol}$, $\left\langle \frac{\epsilon \times \mathcal{I}}{\sigma} \right\rangle_{ref}^{ATMF-skew}$, $\left\langle \frac{|V|}{V} \right\rangle_{ref}^{ATMF-vol}$ and $\left\langle \frac{|V|}{V} \right\rangle_{ref}^{ATMF-skew}$ these reference values. The following relations

$$\left\langle \frac{\epsilon \times \mathcal{I}}{\sigma} \right\rangle_{ref}^{ATMF-vol} \approx 2 \times \left\langle \frac{\epsilon \times \mathcal{I}}{\sigma} \right\rangle_{ref}^{ATMF-skew}$$

and

$$\left\langle \frac{|V|}{V} \right\rangle_{ref}^{ATMF-vol} \approx 0.7 \times \left\langle \frac{|V|}{V} \right\rangle_{ref}^{ATMF-skew},$$

show in comparison to skew metaorders, that lower participation rates is needed to obtain larger normalized impacts in volatility metaorders. Hence although volatility and skew

metaorders present similarities as underlined in Sec. 3.3.3 the market reacts much more easily to metaorders of volatility. This can be explained by the fact that *trading volatility* is more common than *trading skew* in options market.

6 Fair Pricing

In this section we deal with the fair pricing condition of our options metaorders. First of all let us define the \mathcal{S} -WAP (sensitivity weighted average parameter) of a metaorder ω as the quantity defined by

$$\theta_{\mathcal{S}\text{-WAP}}(\omega) = \frac{\sum_{i=0}^{N(\omega)-1} \mathcal{S}_i^\theta(\omega) \theta_{t_i(\omega)}(\omega)}{\mathcal{V}^\theta(\omega)}$$

where $t_0(\omega), \dots, t_{N(\omega)-1}(\omega)$ represent the times of the transactions of the metaorder ω and $\mathcal{V}^\theta(\omega) = \sum_{i=0}^{N(\omega)-1} \mathcal{S}_i^\theta(\omega)$. Hence we want to compare $\theta_{\mathcal{S}\text{-WAP}} - \theta_{t_0}$ with $\theta_{t_0+2T} - \theta_{t_0}$ (Fig. II.17 and II.18). The red line represents the perfect fair pricing condition as it corresponds to $\theta_{\mathcal{S}\text{-WAP}} - \theta_{t_0} = \theta_{t_0+2T} - \theta_{t_0}$.

It appears from Figures II.17 and II.18 that the fair pricing condition can reasonably be assumed to hold. This is in line with what has already been observed in the equity market and mentioned in [Said et al., 2018]. One observes also that the greater the absolute θ -variations, the more one moves away from the perfect fair pricing condition. In agreement with the intuition high variations are generally associated to longer and larger metaorders that are therefore more affected by the diffusive nature of the prices.

One can note that the fair pricing condition is more effective for the *at the money forward volatility* metaorders. This can be explained by the fact that *trading volatility* is more common than *trading skew* on the options market as already mentioned previously.

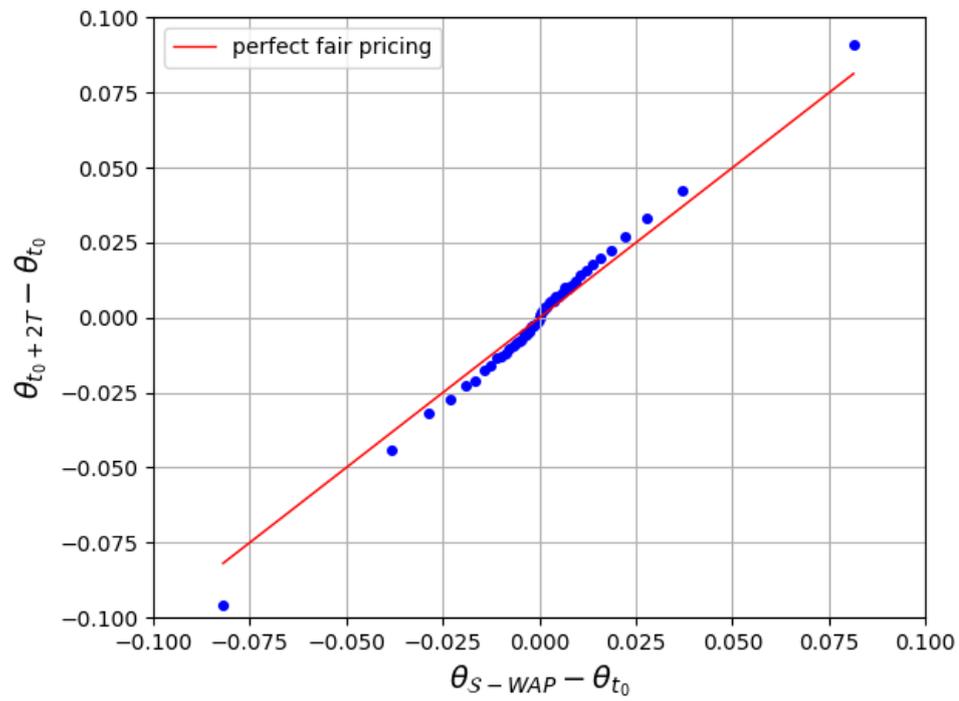


Figure II.17 – Fair pricing of the *at the money forward volatility* parameter in the case of the *at the money forward volatility* metaorders

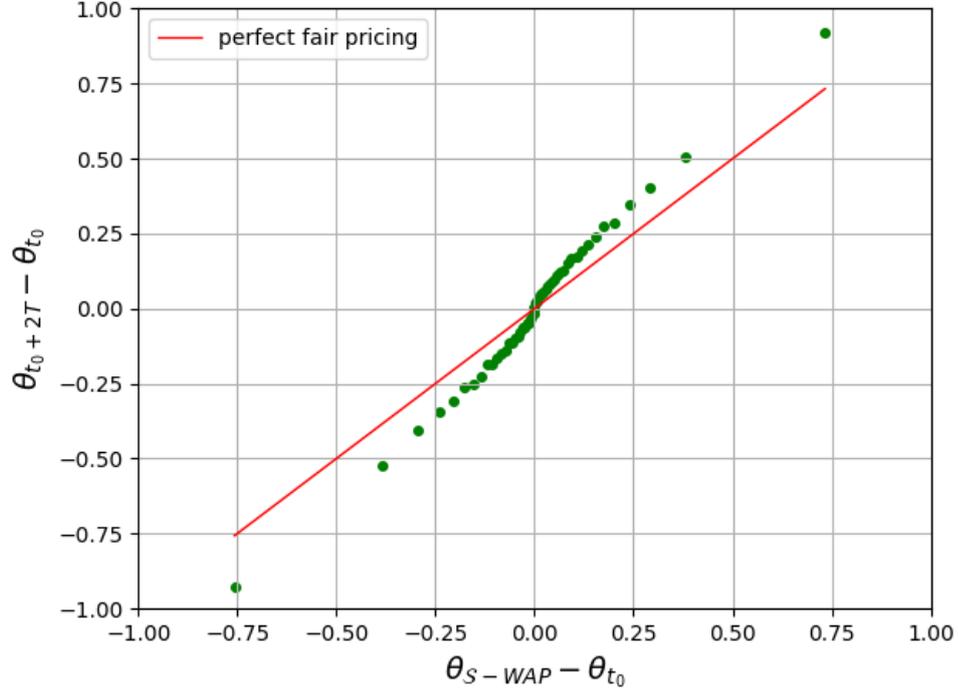


Figure II.18 – Fair pricing of the *at the money forward skew* parameter in the case of the *at the money forward skew* metaorders

For now, we have considered the fair pricing condition by studying the variations of the parameters of the metaorders in question. However, it is more relevant to examine the fair pricing condition by considering the portfolio generated during a metaorder. Let us define the portfolio value of a metaorder ω as the quantity defined by

$$\mathcal{P}(\omega) = \sum_{i=0}^{N(\omega)-1} |Q_i(\omega)| \mathcal{O}_{t_i(\omega)}(\omega)$$

where $t_0(\omega), \dots, t_{N(\omega)-1}(\omega)$ represent the instants, $|Q_0(\omega)|, \dots, |Q_{N(\omega)-1}(\omega)|$ the quantity (positive) and $\mathcal{O}_0(\omega), \dots, \mathcal{O}_{N(\omega)-1}(\omega)$ the prices of the transactions of the metaorder ω . Hence we want to compare $\frac{\mathcal{P} - \mathcal{P}_{t_0}}{\mathcal{P}_{t_0}}$ with $\frac{\mathcal{P}_{t_0+2T} - \mathcal{P}_{t_0}}{\mathcal{P}_{t_0}}$ (Fig. II.19 and II.20) where \mathcal{P}_{t_0} and \mathcal{P}_{t_0+2T} are respectively the prices of the same portfolio at t_0 and $t_0 + 2T$. The red line represents the perfect fair pricing condition as it corresponds to $\frac{\mathcal{P} - \mathcal{P}_{t_0}}{\mathcal{P}_{t_0}} = \frac{\mathcal{P}_{t_0+2T} - \mathcal{P}_{t_0}}{\mathcal{P}_{t_0}}$.

To conclude this section, we observe that the fair pricing seems to be also hold on the options market. This confirms the fair pricing hypothesis introduced in [Farmer et al., 2013] as a universal mechanism concerning the metaorders and their interaction with the price formation process.

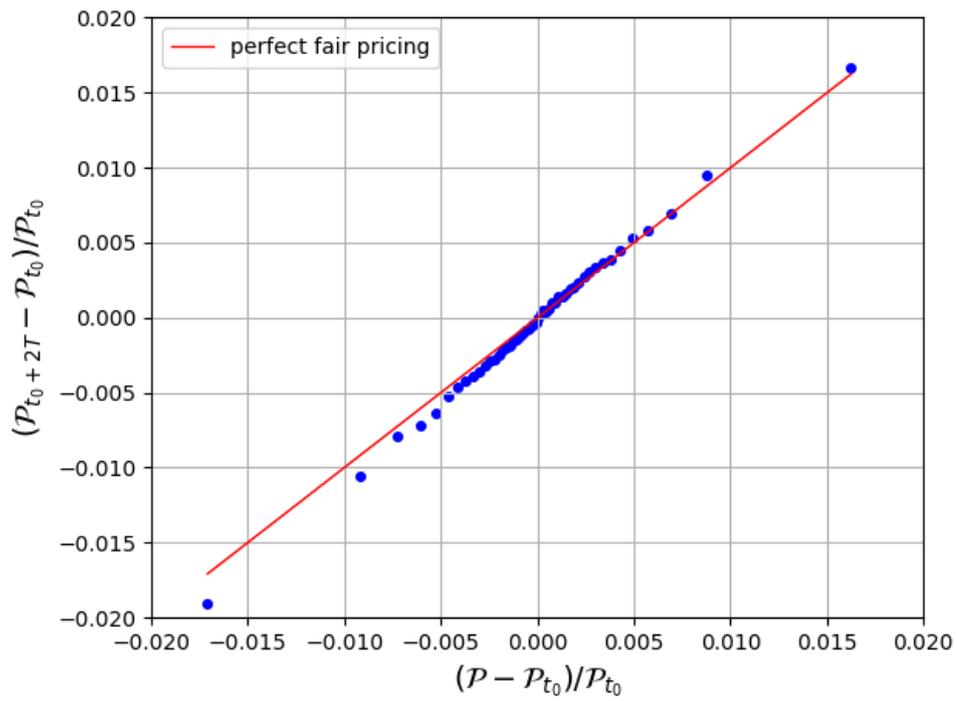


Figure II.19 – Fair pricing of the portfolio value in the case of the *at the money forward volatility* metaorders

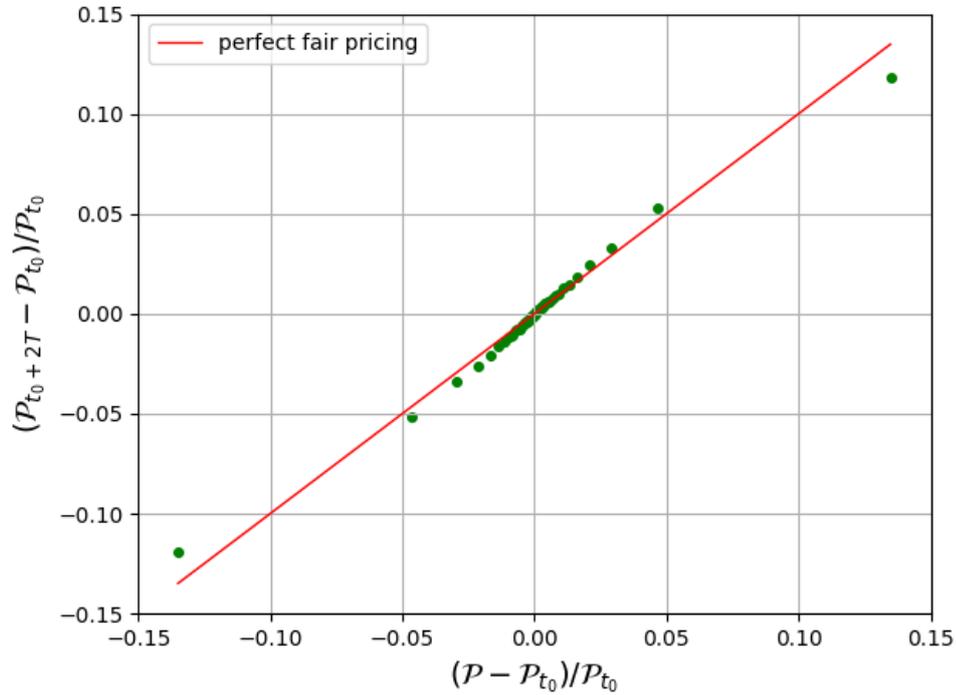


Figure II.20 – Fair pricing of the portfolio value in the case of the *at the money forward skew* metaorders

7 Conclusion

This work is an empirical study of a large set of metaorders in one of the main Asian index options market. A new algorithmic definition of an *options metaorder* has been proposed. Our study contains two distinct groups of metaorders using aggressive limit orders: a set of *at the money forward volatility* metaorders, and a database of *at the money forward skew* metaorders. The statistical results based on this definition show a pretty good agreement with some observations already highlighted in the stock markets: *Square-Root Law*, *Fair Pricing* and *Market Impact Dynamics*. In both cases, the analysis shows that the temporary impact is increasing and concave, with a convex decreasing relaxation phase. More precisely, the price reversion after the completion of a trade yields a permanent impact such that its ratio to the maximum impact observed at the last fill is roughly two-third, as predicted in the paper of [Farmer et al., 2013] and already highlighted empirically on equity markets.

Part III

Option Pricing and Market Impact

How Option Hedging Shapes Market Impact

Abstract

We present a perturbation theory of market impact in which we consider only *local linear market impact*. We study the execution process of hedging derivatives and show how these *hedging metaorders* can explain some stylized facts observed in the empirical market impact literature. As we are interested in the hedging process we will establish that the arbitrage opportunities that exist in discrete time execution strategies vanish when the trading frequency goes to infinity letting us to derive a pricing equation. Furthermore our approach retrieves several results already established in the option pricing literature such that the spot dynamics modified by market impact and predicts the existence of a new kind of metaorders called *hedging metaorders* showing linear impact. We also study the relaxation of our *hedging metaorders* based on the *fair pricing* hypothesis and establish a relation between the *immediate impact* and the *permanent impact* in agreement with recent empirical studies dealing with the subject.

Keywords: *Market microstructure, option pricing, market impact, metaorders execution, option hedging, metaorders relaxation, fair pricing.*

1 Introduction

This paper presents a perturbation theory of market impact. We will consider the framework of covered options. To illustrate our perturbative approach, let us consider an option's hedger who has to deal with a feedback mechanism between the underlying price dynamics and the option's delta-hedging, better known as *market impact*. Let us consider a market impact rule obeying:

$$S_* \rightarrow S_* + f(S_*, N_*), \quad (1)$$

which means that the impact on the stock price of an order to buy N_* stocks at a price S_* is $f(S_*, N_*)$. From the point of view of the option's hedger, if the spot moves from S to $S + dS_0$, he will buy ΓdS_0 stocks, but doing this, due to market impact effects (1) the spot price will move to $S + dS_0 + dS_1$ where dS_1 will be given by $dS_1 = f(S + dS_0, \Gamma dS_0)$. But again

to adjust the hedging, he has to buy ΓdS_1 stocks which in turns impacts the spot price by $dS_2 = f(S + dS_0 + dS_1, \Gamma dS_1)$ and so on and so forth. This perturbative approach taking to the limit $n \rightarrow +\infty$, where n represents the number of transactions of the re-hedging procedure, will move the spot:

$$S \rightarrow S + \sum_{n=0}^{+\infty} dS_n. \quad (2)$$

From a realistic trading perspective, a re-hedging procedure leading to the divergence of the *market impact series* $\sum_{n \geq 0} dS_n$ cannot be acceptable, therefore we need to study some convergence properties of the market impact series derived from the market impact rule (1) described above. The convergence of the market impact series $\sum_{n \geq 0} dS_n$ has several main physical consequences that seem necessary. First, it implies that the market impact has only a limited effect on the stock price. Secondly, the incremental impacts dS_n vanish where $n \rightarrow +\infty$. Those properties have already been observed empirically in the U.S. stock market [Bucci et al., 2019a] and in the European equity market [Said et al., 2018], highlighting the fact that at the end of large metaorders the incremental impacts vanish letting the total market impact reaching a plateau. In our framework, as we will only consider a perturbative approach, the initial exogenous spot move dS will be supposed small enough. In that case, the following re-hedging trades will be considered also small and we will show how this implies to consider at this scale *local linear market impact* rule. Indeed it is well established that the response to individual small orders are linear and it is only the aggregation of those small orders executed consecutively, better known as *metaorders*, which lead to concave market impact in agreement to a *square-root law* in equity market [Moro et al., 2009] [Bacry et al., 2015] [Toth et al., 2011] but also in options market [Tóth et al., 2016] [Said et al., 2019]. The multi-timescale property of the market impact has also been recently addressed in [Benzaquen and Bouchaud, 2018]. Linear market impact has often been associated to non-arbitrage since [Kyle, 1985] and [Huberman and Stanzl, 2004]. The absence of arbitrage being key in every option pricing theory, our approach predicts the existence of a certain type of metaorders, namely *hedging metaorders*, showing total impact that are linear function of size. This can be explained by the very special structure of the *hedging metaorders* among other metaorders that are metaorders with decreasing child orders. The existence of *hedging metaorders* shows that it is possible to conciliate execution, market impact and option pricing.

The rest of the paper is organized as follows. Section 2 presents our order book assumptions and the way it leads in our perturbative approach to *local linear market impact*. Section 3 gives the main results presented in the paper. Section 4 introduces the main definitions of our perturbation theory and gives the convergence results about the market impact series. Section 5 presents some results about the execution of metaorders in our framework. Section 6 presents the derivation of the pricing equation based on the results of the previous sections. Section 7 studies the shape and the relaxation of our *hedging metaorders*. Section 8 recalls and discusses our main results and makes some connections with related works.

2 Order book modeling and local linear market impact

Let us consider an order book parametrized by a mid price \bar{S} and a supply intensity $\eta(t, s)$ such as the units of stocks available at the instant t on the limit order book between S and $S + dS$ is equal to $\eta(t, S)dS$. Of course the order book approach presented here is a trivial simplification of what it is really observed on real markets. More accurate descriptions are given in [Biais et al., 1995] and [Toth et al., 2012]. For a more detailed order book modeling, one can also see [Cont et al., 2010] and [Abergel and Jedidi, 2013].

In our order book perspective, the execution of an amount A (expressed in currency) will consume the order book up to $\bar{S} + \varepsilon(A)$, where $\varepsilon(A)$ is defined such as

$$A = \int_{\bar{S}}^{\bar{S} + \varepsilon(A)} s \eta(t, s) ds,$$

whereas the number $N(A)$ of stocks purchased to execute the amount A must satisfy

$$N(A) = \int_{\bar{S}}^{\bar{S} + \varepsilon(A)} \eta(t, s) ds.$$

In agreement with the intuition, we expect that a small order has only a small impact on the spot price, hence we will assume that $\varepsilon(0) = 0$ and ε is continuous in 0. As we will consider only small orders, we will be interested in the behaviour of ε around 0 at the leading order in A . The function ε as defined above allows us to capture the liquid character or not of an underlying. Indeed, to illustrate this, let us consider two particular cases: the one of a super liquid stock with $A \gg \varepsilon(A)$ and the second of a very poor liquid equity satisfying $A \sim \varepsilon(A)$ which reads that A and $\varepsilon(A)$ are the same order of magnitude. Those considerations motivate to define the *market depth* $L(t, \bar{S})$ at \bar{S} by

$$L(t, \bar{S}) := \lim_{A \rightarrow 0} \frac{A}{\varepsilon(A)}.$$

We have

$$\frac{A}{\varepsilon(A)} = \frac{1}{\varepsilon(A)} \int_{\bar{S}}^{\bar{S} + \varepsilon(A)} s \eta(t, s) ds \xrightarrow{A \rightarrow 0} \eta(t, \bar{S}) \bar{S},$$

and

$$\frac{N(A)}{\varepsilon(A)} = \frac{1}{\varepsilon(A)} \int_{\bar{S}}^{\bar{S} + \varepsilon(A)} \eta(t, s) ds \xrightarrow{A \rightarrow 0} \eta(t, \bar{S}).$$

This gives that

$$\varepsilon(A) = \frac{\varepsilon(A)}{N(A) \bar{S}} \times N(A) \bar{S} \sim_{A \rightarrow 0} \frac{1}{\eta(t, \bar{S}) \bar{S}} \times N(A) \bar{S}.$$

By setting

$$\lambda(t, \bar{S}) := \frac{1}{L(t, \bar{S})} = \frac{1}{\eta(t, \bar{S}) \bar{S}}, \quad (3)$$

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we have $\varepsilon(A) \sim_{A \rightarrow 0} \lambda(t, \bar{S})N(A)\bar{S}$ corresponding to *linear market impact* for A small enough. Therefore we have established the following *local linear market impact* rule:

$$S_* \rightarrow S_* + \lambda(t, S_*)S_*N_*, \quad (4)$$

which is valid when the order size N_* is small enough. When $\lambda(t, \bar{S}) \equiv \lambda \in \mathbb{R}_+$ we retrieve the market impact rule given in [Abergel and Loeper, 2017] where the impact is given by

$$S_*(e^{\lambda N_*} - 1) \sim_{N_* \rightarrow 0} \lambda S_* N_*. \quad (5)$$

The case $\lambda(t, \bar{S}) \equiv \lambda \bar{S}$ with $\lambda \in \mathbb{R}_+$ corresponds to the market impact rule presented in [Loeper, 2018]:

$$S_* \rightarrow S_* + \lambda S_*^2 N_*. \quad (6)$$

In what follows we will consider indifferently (5) or (6) by introducing

$$S_* \rightarrow S_* + \lambda S_*^{1+\zeta} N_*, \quad (7)$$

where $\zeta \in \{0, 1\}$. One must notice that depending of the choice of ζ the dimension of the parameter λ can vary.

3 Context and main results

Let us suppose the market impact rule (7) holds, and consider for instance an agent who is short of an European style option. Taking in to account market impact, if the spot moves from S to $S + dS$ the agent is going to try to react to the exogenous market move dS by adjusting his hedge and purchasing N stocks. This will result in a final state

$$S + dS + (S + dS)^{1+\zeta} \lambda N,$$

and as the trader wants to be hedged at the end of the day, N must satisfy the following equation

$$\Gamma(t, S + dS)(dS + (S + dS)^{1+\zeta} \lambda N) = N$$

leading to

$$N = \frac{\Gamma(t, S + dS)dS}{1 - (S + dS)^{1+\zeta} \lambda \Gamma(t, S + dS)}.$$

Let us assume that $x \mapsto \frac{\Gamma(t, x)}{1 - \lambda x^{1+\zeta} \Gamma(t, x)}$ has a Taylor series expansion in the neighbourhood of any S , hence N can be read

$$\begin{aligned} N &= \frac{\Gamma(t, S + dS)}{1 - (S + dS)^{1+\zeta} \lambda \Gamma(t, S + dS)} \\ &= \frac{\Gamma(t, S)dS}{1 - \lambda S^{1+\zeta} \Gamma(t, S)} + c_2(t, S)(dS)^2 + c_3(t, S)(dS)^3 + \dots \end{aligned} \quad (8)$$

giving $N \approx \frac{\Gamma(t, S)dS}{1 - \lambda S^{1+\zeta}\Gamma(t, S)}$ at the leading order in dS . So everything happens as if the market impact has adjusted the dynamics of the spot by

$$d\tilde{S} = \frac{dS}{1 - \lambda S^{1+\zeta}\Gamma(t, S)} \quad (9)$$

at the first order. This approach was developed in [Loeper, 2018] by considering (9) as an *ansatz* to derive his pricing equation. Modified spot dynamics generated by market impact has been also discussed in [Bouchard et al., 2016]. As dS and $d\tilde{S}$ need to have the same sign, we have necessarily

$$\sup_{(t, S) \in [0, T] \times \mathbb{R}_+} \lambda S^{1+\zeta}\Gamma(t, S) < 1 \quad (10)$$

with $T > 0$ fixed. The condition (10) and its variants are admitted in several papers on the topic (see [Liu and Yong, 2005] [Abergel and Loeper, 2017] and [Loeper, 2018] for instance). However this important question has always been left aside and often assumed in order to derive a pricing equation with exact replication of European style options. From a mathematical point of view it is always possible to replace the pricing equation $\mathcal{P} = 0$ by $\max(\mathcal{P}, \lambda S^{1+\zeta}\Gamma(t, S) - 1) = 0$ as suggested in [Loeper, 2018] but without ensuring replication of European options.

At this stage, there is also an other point that needs to be discussed. Most papers on option pricing and hedging with market impact deal with linear impact as done in [Bouchard et al., 2016] [Bouchard et al., 2017] and [Loeper, 2018] for exemple. However, although this approach is acceptable for *small* trades is clearly not realistic for *large* orders in terms of market microstructure. Besides when it comes trades with sufficiently large size, it is not realistic to state that a large order can be executed in a single trade because of the *liquidity paradox* which states that only a small fraction of the daily volume executed sits on the limit order book. Hence it becomes necessary to propose a pricing theory taking into accounts those market microstructure considerations such as the liquidity paradox, market impact and execution of metaorders.

The main objective of this paper is to show however that the option pricing approaches developed mainly in [Liu and Yong, 2005] [Abergel and Loeper, 2017] and [Loeper, 2018] are compatible with the market impact foundations mainly based on the study of metaorders. The first step done in this direction has been presented in [Abergel and Loeper, 2017]. In their paper the authors integrate in their option pricing model a relaxation factor to illustrate the results obtained of the *permanent impact* in the metaorders. Our paper is a contribution of this strand of research as our main goal is to provide some realistic connections between *option pricing theory* and *market impact empirical findings*. To this end our approach is mainly inspired by the original definition of a metaorder which is nothing less than a large order split into several small orders to be executed incrementally. Besides it has been shown in two companion papers ([Said et al., 2018] and [Said et al., 2019]) that metaorders can obey to an *algorithmic definition*. This implies that they are not necessarily driven by *execution strategies* but they are more often simply *opportunistic* or *hedging* trades. So as the vast majority of

meatorders executed appear to be hedging trades they deserve special attention. We will introduce a hedging procedure – as presented in Section 1 based on a local linear market impact rule (4) – composed of successive small orders decreasing in size and each of them impacting the price by dS_n giving birth to the concept of *hedging metaorders*. Under a Gamma constant approximation we will show that the market impact series $\sum_{n \geq 0} dS_n$ of these *hedging metaorders* is convergent if and only if the condition (10) holds. This result gives a physical meaning to the condition (10) as it linked the convergence of the market impact series with the possibility to derive a pricing equation. We will also show that the sum of the market impact series $\sum_{n \geq 0} dS_n$ can be expressed as in (8). Hence what was considered as an *ansatz* (9) in [Loeper, 2018] is now simply a consequence of the convergence of the market impact series.

4 Market Impact and Hedging: A perturbation theory of market impact

In this section we give the main results based of our perturbation theory of market impact needed to derive the pricing equation. For ease of notations, we will not consider any interest rates or dividends in the rest of the paper.

Let us consider that we have sold an European style option whose value is $u(t, s)$ with a fixed maturity T . Greeks are given as usual by

$$\begin{aligned}\Delta(t, s) &= \partial_s u(t, s), \\ \Gamma(t, s) &= \partial_{ss} u(t, s) = \partial_s \Delta(t, s), \\ \Theta(t, s) &= \partial_t u(t, s).\end{aligned}$$

We have already established in Section 2 that this leads to consider linear market impact in the framework of our perturbation theory. Therefore in what follows we will take:

$$S_* \rightarrow S_* + \lambda S_*^{1+\zeta} N_* \tag{11}$$

i.e. the impact of an order to buy N_* stocks at the price S_* is $\lambda S_*^{1+\zeta} N_*$ when the size N_* of the order is sufficiently small (*linear market impact*). We set the parameter ϕ defined by

$$\phi \equiv \phi(t, S) := \lambda S^{1+\zeta} \partial_{ss} u(t, S) = \lambda S^{1+\zeta} \Gamma(t, S). \tag{12}$$

We assume an initial spot move from S to $S + dS$, dS supposed to be small and $S > 0$. By following an iterative hedging strategy one has to adjust the hedge by ΓdS stocks after the initial spot move $S \rightarrow S + dS$, which then again impacts the spot price by $dS_1 = \lambda(S + dS_0)^{1+\zeta} \Gamma dS_0$ according to the linear market impact rule presented above with $dS_0 = dS$. This spot move is then followed by a second hedge adjustment of ΓdS_1 , which in turn impacts the spot price by $dS_2 = \lambda(S + dS_0 + dS_1)^{1+\zeta} \Gamma dS_1$ and so on and so forth. Hence one has

$$\begin{cases} dS_0 = dS \\ \forall n \in \mathbb{N}, dS_{n+1} = \lambda \left(S + \sum_{k=0}^n dS_k \right)^{1+\zeta} \Gamma dS_n = \phi \left(1 + \frac{1}{S} \sum_{k=0}^n dS_k \right)^{1+\zeta} dS_n \end{cases}$$

with the assumption that $\Gamma \equiv \Gamma(t, S)$ remain constant during the hedging procedure described just above. This Gamma approximation appears also in [Almgren and Li, 2016] and considerably simplifies the problem by eliminating the dependence of the variable state S_t in the expression of Γ . Hence the approximation that Γ is constant during the hedging procedure allows us to exhibit the essential features of the local hedging without losing ourselves in complexities. Furthermore from the numerical point of view, this hypothesis is clearly justified in the context of the perturbative approach proposed here. In what follows we will show that for dS small enough, the total market impact $\sum_{n=0}^{+\infty} dS_n$ will be also small enough. In the context of our approach as the market impact is only considered as a perturbation we have that

$$\Gamma\left(t, S + \sum_{n=0}^{+\infty} dS_n\right) \approx \Gamma(t, S) + \partial_S \Gamma(t, S) \sum_{n=0}^{+\infty} dS_n \approx \Gamma(t, S).$$

Besides considering the fact that the price is bounded below by 0 it is more relevant to write

$$\begin{cases} dS_0 = dS \vee (-S) \\ \forall n \in \mathbb{N}, dS_{n+1} = \phi\left(1 + \frac{1}{S} \sum_{k=0}^n dS_k\right)^{1+\zeta} dS_n \vee \left(-\sum_{k=0}^n dS_k - S\right) \end{cases} \quad (13)$$

with for all $a, b \in \mathbb{R}$, $\max(a, b) = a \vee b$. For all $n \in \mathbb{N}$, dS_n being a function of dS , we will denote x the exogenous spot move dS and u_n the n -th impact dS_n . Hence we introduce for ease of notations the sequence of real valued functions $(u_n)_{n \in \mathbb{N}}$ defined by

$$\begin{cases} u_0(x) = x \vee (-S) \\ \forall n \in \mathbb{N}, u_{n+1}(x) = \phi\left(1 + \frac{s_n(x)}{S}\right)^{1+\zeta} u_n(x) \vee (-s_n(x) - S) \end{cases} \quad (14)$$

for any $x \in \mathbb{R}$ with $s_n := \sum_{k=0}^n u_k$, $n \in \mathbb{N}$ the partial sums of the series $\sum_{n \in \mathbb{N}} u_n$. As s_n represents the cumulative market impact after n successive re-hedging trades, from a realistic trading perspective this quantity cannot exploded. Hence we will add the following constraint:

Assumption 1. *For all $x \in \mathbb{R}$, the sequence $(s_n(x))_{n \in \mathbb{N}}$ is bounded.*

In the context of our perturbation theory of market impact, it is necessary to study the convergence properties of the market impact series derived from the local linear market impact relation (4) given previously. To this end, we introduce the following definition:

Definition 13. *A sequence $(u_n)_{n \in \mathbb{N}}$ defined by (14) is said to be a **market impact scenario** starting from $x \in \mathbb{R}$.*

A *market impact scenario* can be seen as an extension of the definition given for an hedging metaorder (see Definition 20 introduced in Section 7). We have this first elementary result:

Proposition 4. *Let $(u_n)_{n \in \mathbb{N}}$ an ϕ -market impact scenario. Then $(s_n)_{n \in \mathbb{N}}$ is uniformly bounded below by $-S$.*

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Proof. Let $(u_n)_{n \in \mathbb{N}}$ an ϕ -market impact scenario starting from $x \in \mathbb{R}$. We proceed by induction.

- $s_0(x) = u_0(x) = x \vee (-S) \geq -S$.
- Let $n \in \mathbb{N}$ and suppose that $u_n(x) \geq -S$. We have $u_{n+1}(x) \geq -s_n(x) - S$ which gives $s_{n+1}(x) \geq -S$.

Hence we have shown that for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, $s_n(x) \geq -S$. □

Among the market impact scenarios, we must distinguish between those that are acceptable from those that are not. The following definition is here to give an admissibility criterion:

Definition 14. *A market impact scenario $(u_n)_{n \in \mathbb{N}}$ is said to be **admissible from a trading perspective** if there exists $R > 0$ such that for any $x \in (-R, R)$, $\sum_{n \in \mathbb{N}} u_n(x)$ converges.*

The following result gives an equivalent criterion on the parameter ϕ for a market impact scenario to be admissible:

Theorem 1. *The ϕ -market impact scenario $(u_n)_{n \in \mathbb{N}}$ is admissible from a trading perspective if, and only if, $\phi \in (-\infty, 1)$.*

Proof. see Appendix III.A.1. □

The previous result establishes a straightforward connection between the values the parameter ϕ can take and the convergence of the market impact series. Actually the proof of Theorem 1 given in Appendix III.A.1 states much more than that and leads to the following result:

Theorem 2. *Let $(u_n)_{n \in \mathbb{N}}$ an ϕ -market impact scenario. The following statements are equivalent:*

- (i) *The market impact scenario $(u_n)_{n \in \mathbb{N}}$ is admissible from a trading perspective.*
- (ii) *$\phi \in (-\infty, 1)$.*
- (iii) *There exists $R > 0$ such that for any $x \in (-R, R)$, $\sum_{n \geq 0} u_n(x)$ converges.*
- (iv) *There exists $R > 0$ such that for any $x \in (-R, R)$, $\sum_{n \geq 0} u_n(x)$ converges absolutely.*

Proof. This result is an immediate corollary of the proofs given for the Propositions 1, 2, 3 and 4. □

From now on, we consider only market impact scenarios that are admissible from a trading perspective which is, according to Theorem 1, the same to consider values of ϕ in $(-\infty, 1)$. We give now an expression of the market impact series for an admissible market impact scenario, in particular one can notice that we retrieve the first order term of the Taylor expansion 8 introduced previously in Sec. 3.

Proposition 5. *Let $(u_n)_{n \in \mathbb{N}}$ an ϕ -market impact scenario admissible from a trading perspective starting from $x \in \mathbb{R}$ and*

$$N(x) := \inf \left\{ n \in \mathbb{N} \mid s_n(x) = -S \right\} \quad (15)$$

with the convention $\inf \emptyset = +\infty$. Then there exists $R > 0$ such that for all $x \in (-R, R)$, $N(x) \in \mathbb{N}^ \cup \{+\infty\}$ and*

$$\sum_{n=0}^{N(x)} u_n(x) = \frac{x}{1-\phi} + \frac{1}{S} \frac{(1+\zeta)\phi}{1-\phi} \sum_{n=0}^{N(x)} s_n(x) u_n(x) + \frac{1}{S^2} \frac{\zeta\phi}{1-\phi} \sum_{n=0}^{N(x)} s_n^2(x) u_n(x). \quad (16)$$

Proof. see Appendix III.A.2. □

Among the admissible market impact scenarios, three cases can occur. Firstly, the trivial scenario where nothing strictly happens corresponding to the case the price reaches 0 after the first exogenous spot move. Namely, this corresponds to $u_0(x) = -S$. In this case there is nothing much to say. The second is to reach 0 after a finite number of re-hedging trades. In that case it is the market impact that brings the price stock to 0. And finally the third one and more realistic one implying an infinite number of re-hedging trades and moving the stock price to a non-zero different level. The definition given just below introduces the three possibilities:

Definition 15. *Let $(u_n)_{n \in \mathbb{N}}$ an ϕ -market impact scenario admissible from a trading perspective starting from $x \in \mathbb{R}$ and $N(x)$ as defined in (15). The market impact scenario $(u_n)_{n \in \mathbb{N}}$ is said to be:*

- **null** when $N(x) = 0$.
- **chaotic** when $N(x) \in \mathbb{N}^*$.
- **regular** when $N(x) = +\infty$.

In the rest of the section, we will only consider chaotic or regular market impact scenarios. Especially we will show (see Proposition 6) that for $\phi \in (-1, 1)$ – when the initial spot move is small enough, which is clearly the case of interest in our perturbative approach –, it is always possible to consider only regular market impact scenarios. Particularly, it will be the case for our *hedging metaorders* discussed later (see Section 7).

Proposition 6. *Let $(u_n)_{n \in \mathbb{N}}$ an ϕ -market impact scenario admissible from a trading perspective. For any $\phi \in (-1, 1)$, there exists $R > 0$, such that for all $x \in (-R, R)$, the ϕ -market impact scenario $(u_n)_{n \in \mathbb{N}}$ is regular.*

Proof. see Appendix III.A.3. □

The following proposition shows that in the case of regular market impact scenarios with $\phi \in (-1, 1)$ the market impact series is nothing less than a power series. In particular we obtain a closed form for the second order term of the market impact series. We recall that the first linear order term has been already given for a chaotic or regular market impact scenario in Proposition 5.

Proposition 7. *Let $\phi \in (-1, 1)$ and $(u_n)_{n \in \mathbb{N}}$ an ϕ -market impact scenario admissible from a trading perspective. There exists $R > 0$, such that for all $x \in (-R, R)$, the ϕ -market impact scenario $(u_n)_{n \in \mathbb{N}}$ is regular. Hence the series function $\sum_{n \geq 0} u_n$ can be expressed as a power series on $(-R, R)$, is of class \mathcal{C}^∞ on $(-R, R)$ and the first two terms of the decomposition of $\sum_{n \geq 0} u_n$ as a power series are such that*

$$\sum_{n=0}^{+\infty} u_n(x) = \frac{1}{1-\phi} x + \frac{1}{S} \frac{(1+\zeta)\phi}{(1-\phi)^3(1+\phi)} x^2 + o(x^2) \text{ as } x \rightarrow 0.$$

Proof. see Appendix III.A.4. □

5 Market Impact and Metaorders Execution

The aim of this section is to provide general results on metaorders execution in our framework. We will show that when the metaorder is enough fragmented, there is no execution strategy better than another.

5.1 $\lambda(t, S) \equiv \lambda$

Let us consider the market impact rule (11) when $\zeta = 0$

$$S_* \rightarrow S_* + \lambda S_* N_*. \quad (17)$$

Assume an agent wants to execute incrementally an order of size N with $\mathcal{K} \in \mathbb{N}^*$ child orders of size $n_1, n_2, \dots, n_{\mathcal{K}}$ satisfying

$$\sum_{k=1}^{\mathcal{K}} n_k = N.$$

Without loss of generality we will suppose that $N \in \mathbb{R}_+^*$ and $n_1, \dots, n_{\mathcal{K}} \in \mathbb{R}_+^*$ - i.e. a buy order, the same holds for a sell order - such that

$$\lim_{\mathcal{K} \rightarrow +\infty} \sup_{1 \leq k \leq \mathcal{K}} |n_k| = 0. \quad (18)$$

Note that the condition (18) is needed to ensure that (17) can be applied to $n_1, \dots, n_{\mathcal{K}}$ for \mathcal{K} large enough. Applying this when $\mathcal{K} = 2$ leads to

$$S \xrightarrow{n_1} S + \lambda S n_1 \xrightarrow{n_2} S + \lambda S n_1 + \lambda(S + \lambda S n_1) n_2,$$

which can be written

$$S \xrightarrow{n_1} S(1 + \lambda n_1) \xrightarrow{n_2} S(1 + \lambda(n_1 + n_2) + \lambda^2 n_1 n_2).$$

Let us denote $\xrightarrow{n_1, \dots, n_{\mathcal{K}}}$ the contraction of $\xrightarrow{n_1} \dots \xrightarrow{n_{\mathcal{K}}}$. By a straightforward induction we have for all $\mathcal{K} \in \mathbb{N}^*$,

$$S \xrightarrow{n_1, \dots, n_{\mathcal{K}}} S \left(1 + \sum_{k=1}^{\mathcal{K}} \lambda^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq \mathcal{K}} n_{i_1} n_{i_2} \dots n_{i_k} \right).$$

Definition 16. Let $N \in \mathbb{R}_+^*$ and $\mathcal{K} \in \mathbb{N}^*$. A finite sequence $(n_1, \dots, n_{\mathcal{K}}) \in (\mathbb{R}_+^*)^{\mathcal{K}}$ such that $\sum_{k=1}^{\mathcal{K}} n_k = N$ is said to be an (N, \mathcal{K}) -execution strategy. An (N, \mathcal{K}) -execution strategy is considered *admissible* if it satisfies (18).

Definition 17. Let $(n_1, \dots, n_{\mathcal{K}})$ an (N, \mathcal{K}) -execution strategy. Let us denote by

$$S_{N,k}(n_1, \dots, n_{\mathcal{K}}) := S \left(1 + \sum_{i=1}^k \lambda^i \sum_{1 \leq j_1 < j_2 < \dots < j_i \leq k} n_{j_1} n_{j_2} \dots n_{j_i} \right)$$

the k -th price of this (N, \mathcal{K}) -execution strategy for $k \in \llbracket 0, \mathcal{K} \rrbracket$. Then let us set its *market impact* by

$$I_{N,\mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) := S_{N,\mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) - S$$

and its *average execution price* by

$$\bar{S}_{N,\mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) := \frac{1}{N} \sum_{k=1}^{\mathcal{K}} n_k S_{N,k-1}(n_1, \dots, n_{\mathcal{K}}).$$

Proposition 8. Let $N \in \mathbb{R}_+^*$ and $\mathcal{K} \in \mathbb{N}^*$. The market impact of an (N, \mathcal{K}) -execution strategy depends only on the sizes of the \mathcal{K} child orders. The order in which the child orders are executed does not affect the final value of $I_{N,\mathcal{K}}(n_1, \dots, n_{\mathcal{K}})$.

Proof. Let $(n_1, \dots, n_{\mathcal{K}})$ an (N, \mathcal{K}) -execution strategy and $\sigma \in \mathcal{S}_{\mathcal{K}}$ the set of the permutations of $\llbracket 1, \mathcal{K} \rrbracket$. The result is a straightforward consequence of the fact that for any permutation $\sigma \in \mathcal{S}_{\mathcal{K}}$, $I_{N,\mathcal{K}}(n_{\sigma(1)}, \dots, n_{\sigma(\mathcal{K})}) = I_{N,\mathcal{K}}(n_1, \dots, n_{\mathcal{K}})$. \square

Proposition 8 shows that for a given (N, \mathcal{K}) -execution strategy any permutation of $(n_1, \dots, n_{\mathcal{K}})$ leads to the same market impact. The point underlined by Proposition 8 is in fact a well known property of market impact which appears in the litterature through the *square root formula* or other related formulas. The square root formula expresses the final market impact of a metaorder as a function of its size but it does not tell much about the dynamics itself of the metaorder.

The following result (Theorem 3) shows that among all the (N, \mathcal{K}) -execution strategies the one which impacts the most the market is the most predictable. Indeed given a metaorder size N and \mathcal{K} child orders, the first execution strategy that comes in mind is to split the metaorder in lots of equal size. But adoptiong such a strategy considerably increases the probability to be detected by the market makers especially when \mathcal{K} becomes large enough. Hence it seems normal that the most expensive strategy in terms of market impact and average execution price is associated to this execution strategy.

Theorem 3. Let $N \in \mathbb{R}_+^*$ and $\mathcal{K} \in \mathbb{N}^*$. The market impact and the average execution price of an (N, \mathcal{K}) -execution strategy are bounded and reach their upper bound if, and only if the strategy is equally-sized i.e. $n_1 = \dots = n_{\mathcal{K}} = \frac{N}{\mathcal{K}}$. Besides the following inequalities hold

$$S + \lambda SN \leq S_{N,\mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) \leq S e^{\lambda N} \quad (19)$$

and

$$S + \frac{1}{2} \lambda S \left(N - \sup_{1 \leq k \leq \mathcal{K}} n_k \right) \leq \bar{S}_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) \leq S \frac{e^{\lambda N} - 1}{\lambda N}. \quad (20)$$

Proof. see Appendix III.A.5. □

The lower bound of (19) corresponds to the linear market impact model presented in [Loeper, 2018] and the upper bound to the one presented in [Abergel and Loeper, 2017]. One can notice that these two bounds do not depend of the execution strategy chosen – they do not vary with the sizes and the number \mathcal{K} of child orders – as they are only functions of the final size N of the metaorder. This has the advantage of avoiding arbitrage opportunities, as well as not being sensitive to the hedging frequency. One can notice that $S_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) \approx S + \lambda NS$ when $N \rightarrow 0$ in agreement with (17). Hence the lower bound of (19) can be reached when N is small enough. More interesting, regarding metaorders with a size N much larger, we will show that the upper bound of (19) corresponds to the limit case when the number of child orders \mathcal{K} goes to infinity. This result seems pretty obvious from a market making perspective. Indeed when $\mathcal{K} \rightarrow +\infty$ the probability that the market makers detect the metaorder is going to converge towards 1 and it is expected that the market impact attains its peak value.

Theorem 4. *Let $(n_1, \dots, n_{\mathcal{K}})$ an admissible (N, \mathcal{K}) -execution strategy i.e. such that*

$$\lim_{\mathcal{K} \rightarrow +\infty} \sup_{1 \leq k \leq \mathcal{K}} |n_k| = 0.$$

Then we have that

$$\lim_{\mathcal{K} \rightarrow +\infty} S_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) = S e^{\lambda N} \quad (21)$$

and

$$\lim_{\mathcal{K} \rightarrow +\infty} \bar{S}_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) = S \frac{e^{\lambda N} - 1}{\lambda N}. \quad (22)$$

Hence when \mathcal{K} goes to infinity the market impact attains its peak value. Furthermore the market impact and the average execution price do not depend longer of the execution strategy chosen. This implies that arbitrage opportunities vanish when the trading frequency tends to infinity.

Proof. see Appendix III.A.6. □

Remark 3. *Theorem 4 is actually true for $n_1, \dots, n_{\mathcal{K}} \in \mathbb{R}$ not necessarily positive by weakening (18) and assuming that for some $\varepsilon > 0$, $\sup_{1 \leq k \leq \mathcal{K}} |n_k| = O\left(\left(\frac{1}{\mathcal{K}}\right)^{1/2+\varepsilon}\right)$.*

Remark 3 states in fact that when $\mathcal{K} \rightarrow +\infty$ there does not exist any trading strategy that could affect the average execution price or the final market impact. In other words it is not possible to manipulate the price formation process by executing *round trip trades* as defined and studied in [Gatheral, 2010] and [Alfonsi and Schied, 2010] for instance. Theorem 4 motivates to extent our market impact and average execution price definitions for continuous execution strategies by:

Definition 18. Let $N \in \mathbb{R}$. A function $F : [0, 1] \rightarrow \mathbb{R}$ of bounded variation and continuous such that $F(0) = 0$ and $F(1) = N$ is said to be an $(N, +\infty)$ -execution strategy. Any $(0, +\infty)$ -execution strategy is said to be a *round trip trade*.

Definition 19. Let $N \in \mathbb{R}$. The *market impact* of an $(N, +\infty)$ -execution strategy is given by

$$I_{N,+\infty} := Se^{\lambda \int_0^1 dF(s)} - S = Se^{\lambda N} - S \quad (23)$$

and the *average execution price* by

$$\bar{S}_{N,+\infty} := \frac{\int_0^1 Se^{\lambda F(s)} dF(s)}{\int_0^1 \lambda dF(s)} = S \frac{e^{\lambda N} - 1}{\lambda N}. \quad (24)$$

One can see in (23)-(24) how the choice of the continuous execution strategy F does not affect the market impact and the average execution price in agreement with Theorem 4.

5.2 $\lambda(t, S) \equiv \lambda S$

Let us consider now the market impact rule (11) when $\zeta = 1$

$$S_* \rightarrow S_* + \lambda S_*^2 N_*. \quad (25)$$

Let us show that under the condition (18) all the results can be derived from the previous ones by considering the following approximation

$$(1+x)^2 \approx 1+2x,$$

when x is small enough. When $\mathcal{K} = 2$ we have

$$S \xrightarrow{n_1} S + \lambda S^2 n_1 \xrightarrow{n_2} S + \lambda S^2 n_1 + \underbrace{\lambda S^2 (1 + \lambda S n_1)^2 n_2}_{\approx \lambda S^2 (1 + 2\lambda S n_1) n_2}$$

leading approximately to

$$S \xrightarrow{n_1} S + \lambda S^2 n_1 \xrightarrow{n_2} S \left(1 + \frac{1}{2} (2\lambda S) (n_1 + n_2) + \frac{1}{2} (2\lambda S)^2 n_1 n_2 \right).$$

A straightforward induction gives for all $\mathcal{K} \in \mathbb{N}^*$,

$$S \xrightarrow{n_1, \dots, n_{\mathcal{K}}} S \left(1 + \frac{1}{2} \sum_{k=1}^{\mathcal{K}} (2\lambda S)^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq \mathcal{K}} n_{i_1} n_{i_2} \dots n_{i_k} \right).$$

Hence all the results derived in the previous section still hold. Particularly, Theorems 3 and 4 can now be written

Theorem 5. Let $N \in \mathbb{R}_+^*$ and $\mathcal{K} \in \mathbb{N}^*$. The market impact and the average execution price of an (N, \mathcal{K}) -execution strategy are bounded and reach their upper bound if, and only if the strategy is equally-sized i.e. $n_1 = \dots = n_{\mathcal{K}} = \frac{N}{\mathcal{K}}$. Besides the following inequalities hold

$$S + \lambda S^2 N \leq S_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) \leq \frac{S}{2} (1 + e^{2\lambda SN}) \quad (26)$$

and

$$S + \frac{1}{2} \lambda S^2 \left(N - \sup_{1 \leq k \leq \mathcal{K}} n_k \right) \leq \bar{S}_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) \leq \frac{S}{2} \left(1 + \frac{e^{2\lambda SN} - 1}{2\lambda SN} \right). \quad (27)$$

Theorem 6. Let $(n_1, \dots, n_{\mathcal{K}})$ an admissible (N, \mathcal{K}) -execution strategy i.e. such that

$$\lim_{\mathcal{K} \rightarrow +\infty} \sup_{1 \leq k \leq \mathcal{K}} |n_k| = 0.$$

Then we have that

$$\lim_{\mathcal{K} \rightarrow +\infty} S_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) = \frac{S}{2} (1 + e^{2\lambda SN}) \quad (28)$$

and

$$\lim_{\mathcal{K} \rightarrow +\infty} \bar{S}_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) = \frac{S}{2} \left(1 + \frac{e^{2\lambda SN} - 1}{2\lambda SN} \right). \quad (29)$$

6 Market Impact and Option Pricing

6.1 Pricing equation

We consider the framework of covered options, hence we start from a delta-hedged portfolio. Let us assume that the stock price S moves initially by dS such that $(dS_n)_{n \in \mathbb{N}}$ is a **regular** ϕ -market impact scenario given by (13) with

$$dS = S(vdt + \sigma dW_t), \quad (30)$$

where $(W_t)_{t \geq 0}$ is a Brownian motion. By following the hedging strategy presented above we move the spot from S to $S + \tilde{d}S$ where $\tilde{d}S$ represents the cumulative market impact at the end of the re-hedging procedure

$$\tilde{d}S := \sum_{n=0}^{+\infty} dS_n, \quad (31)$$

hence from (16) we have

$$\tilde{d}S = \frac{dS}{1 - \phi} + \frac{1}{S} \frac{(1 + \zeta)\phi}{1 - \phi} \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n dS_k \right) dS_n + \frac{1}{S^2} \frac{\zeta\phi}{1 - \phi} \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n dS_k \right)^2 dS_n, \quad (32)$$

which can be rewritten

$$\tilde{d}S = \frac{dS}{1 - \phi} + \frac{\mu}{S} (dS)^2 + o((dS)^2),$$

where μ is only a function of the parameter ϕ . Plugging equation (30) in the previous equation leads to

$$\tilde{dS} = \frac{dS}{1-\phi} + \mu\sigma^2 S dt + o(dt),$$

as $(dS)^2 = \sigma^2 S^2 dt + o(dt)$. This gives at the leading order

$$\tilde{dS} = \frac{dS}{1-\phi} + \tilde{\nu} S dt + o(dt). \quad (33)$$

The value V of the hedging portfolio containing $\Delta(t, S)$ stocks at t evolves as

$$dV = \Delta \tilde{dS} + \mathcal{R}$$

as S moves to $S + \tilde{dS}$ with

$\mathcal{R} = N \times (\text{Final price of the stocks bought} - \text{Average execution price})$

$$\begin{aligned} &= N \times \left(S + \sum_{n=0}^{+\infty} dS_n - \frac{\sum_{n=0}^{+\infty} \Gamma dS_n \left(S + \sum_{k=0}^n dS_k \right)}{\sum_{n=0}^{+\infty} \Gamma dS_n} \right) \\ &= N \times \left(\sum_{n=0}^{+\infty} dS_n - \frac{\sum_{n=0}^{+\infty} \Gamma dS_n \left(\sum_{k=0}^n dS_k \right)}{\sum_{n=0}^{+\infty} \Gamma dS_n} \right), \end{aligned}$$

$N = \Gamma \tilde{dS}$ being the number of stocks bought during the re-hedging procedure. As $dS \rightarrow 0$, we have that $\sup_{n \in \mathbb{N}} |dS_n| \rightarrow 0$. Besides the proof given of Theorem 1 shows that for n large enough there exists $r \in (0, 1)$ such that $|dS_{n+1}| < r |dS_n|$, hence we can apply Remark 3 and use the results stated in Theorem 4. When $\lambda(t, S) \equiv \lambda$ we have that

$$\text{Final price of the stocks bought} = S + S(e^{\lambda N} - 1)$$

and

$$\text{Average execution price} = S + S \left(\frac{e^{\lambda N} - 1}{\lambda N} - 1 \right),$$

giving at the leading order

$$\mathcal{R} = \frac{1}{2} \lambda S N^2.$$

The same can be done when $\lambda(t, S) \equiv \lambda S$ and gives

$$\text{Final price of the stocks bought} = S + \frac{S}{2} (e^{2\lambda S N} - 1)$$

and

$$\text{Average execution price} = S + \frac{S}{2} \left(\frac{e^{2\lambda SN} - 1}{2\lambda SN} - 1 \right),$$

implying also at the leading order

$$\mathcal{R} = \frac{1}{2} \lambda S^2 N^2.$$

Finally one can write in any case

$$\mathcal{R} = \frac{1}{2} \lambda N^2 S^{1+\zeta}$$

with $N = \Gamma \tilde{dS}$. This gives

$$dV = \Delta \tilde{dS} + \frac{1}{2} \lambda S^{1+\zeta} (\Gamma \tilde{dS})^2. \quad (34)$$

Besides assuming that the option is sold at its fair price, we have $dV = du$ with at the leading order, for S moving to $S + \tilde{dS}$,

$$du = \partial_t u dt + \partial_s u \tilde{dS} + \frac{1}{2} \partial_{ss} u (\tilde{dS})^2 + o(dt). \quad (35)$$

Therefore from equations (34) and (35) we get

$$\partial_t u dt + \frac{1}{2} (\tilde{dS})^2 [\partial_{ss} u - \lambda S^{1+\zeta} (\partial_{ss} u)^2] = o(dt). \quad (36)$$

If we plugg the spot dynamics (30) in the expression of \tilde{dS} (33), we obtain at the leading order

$$(\tilde{dS})^2 = \frac{\sigma^2 S^2}{(1-\phi)^2} dt + o(dt). \quad (37)$$

Finally by reinjecting (37) in (36), we obtain the pricing equation

$$\begin{aligned} \partial_t u + \frac{1}{2} \sigma^2 s^2 \partial_{ss} u \frac{1}{1-\phi} &= 0, \\ \phi &= \lambda s^{1+\zeta} \partial_{ss} u. \end{aligned} \quad (38)$$

6.2 Perfect replication

Any contingent claim can be perfectly replicated at zero cost, as long as one can exhibit a solution to the pricing equation (38). One possibility is to study the parabolicity of the given pricing equation. Furthermore, we have established that only market impact scenarios admissible from a trading perspective have to be considered which implies

$$\sup_{(t,s) \in [0,T] \times \mathbb{R}_+} \lambda s^{1+\zeta} \partial_{ss} u < 1, \quad (39)$$

according to Theorem 1, T being the maturity of the option. In fact we have the following result

Theorem 7. *There holds the two following propositions:*

(i) *The non-linear backward partial differential operator*

$$\partial_t \cdot + \frac{1}{2} \sigma^2 s^2 \frac{\partial_{ss} \cdot}{1 - \lambda s^{1+\zeta} \partial_{ss} \cdot}$$

is parabolic.

(ii) *Every European style contingent claim with payoff Φ satisfying the terminal constraint*

$$\sup_{s \in \mathbb{R}_+} \lambda s^{1+\zeta} \partial_{ss} \Phi < 1$$

can be perfectly replicated via a δ -hedging strategy given by the unique, smooth away from the maturity T , solution to equation (38).

Proof. The parabolic nature of the operator is determined by the monotonicity of the function

$$F: p \in (-\infty, 1) \mapsto \frac{p}{1-p}.$$

F is globally increasing, therefore the pricing equation is globally well-posed. Besides, given that the payoff satisfies the terminal constraint, classical results on the maximum principle for the second derivative of the solution ensure that the same constraint is satisfied globally for $t \leq T$ which constitutes in fact the constraint (39). Hence, the perfect replication is possible. Results on maximum principle for the second derivative can be found in [Wang, 1992a] and [Wang, 1992b] for instance. A proof of this result in a more general setting is also given in [Abergel and Loeper, 2017]. \square

6.3 SDE formulation

In this section, we provide an alternative description of our theory based on a system of a differential equations. This has the advantage of providing a more tractable framework from the practitioner's point of view but also leaves aside several questions that cannot be raised in a stochastic formulation.

6.3.1 The system of SDE

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ a filtered probability space supporting an $(\mathcal{F}_t)_{t \geq 0}$ standard Brownian motion $(W_t)_{t \geq 0}$. Considering the result established in Theorem 1, we will assume the following uniform condition:

Assumption 2. *There exists a constant $\varepsilon > 0$, such that for all $t \geq 0$, \mathbb{P} -a.s.,*

$$1 - \lambda(t, S_t) S_t \Gamma_t \geq \varepsilon. \tag{40}$$

III. How Option Hedging Shapes Market Impact

Let us consider the following system of stochastic differential equations:

$$d\delta_t = a_t dt + \Gamma_t dS_t, \quad (41)$$

$$\frac{dS_t}{S_t} = \sigma(t, S_t) dW_t + \left(v(t, S_t) + \frac{\sigma^2(t, S_t) S_t \Gamma_t (\lambda(t, S_t) + \partial_x \lambda(t, S_t))}{1 - \lambda(t, S_t) S_t \Gamma_t} \right) dt + \lambda(t, S_t) d\delta_t, \quad (42)$$

$$dV_t = \delta_t dS_t + \frac{1}{2} \lambda(t, S_t) S_t d\langle \delta, \delta \rangle_t. \quad (43)$$

where δ , S and V are three processes starting respectively from δ_0 , S_0 and V_0 at $t = 0$. We set

$$\alpha_t \equiv \alpha(t, S_t, \Gamma_t) := \frac{\bar{\sigma}(t, S_t) \Gamma_t}{1 - \tilde{\lambda}(t, S_t) \Gamma_t},$$

and

$$\beta_t \equiv \beta(t, S_t, \Gamma_t) := \frac{a_t}{1 - \tilde{\lambda}(t, S_t) \Gamma_t} + \frac{\Gamma_t}{1 - \tilde{\lambda}(t, S_t) \Gamma_t} \left(\tilde{v}(t, S_t) + \frac{\bar{\sigma}^2(t, S_t) \Gamma_t \partial_x \tilde{\lambda}(t, S_t)}{1 - \tilde{\lambda}(t, S_t) \Gamma_t} \right),$$

by setting also

$$\bar{\sigma}(t, x) := x\sigma(t, x),$$

$$\tilde{v}(t, x) := xv(t, x),$$

and

$$\tilde{\lambda}(t, x) := x\lambda(t, x)$$

for all $t \in [0, T]$, $x \in \mathbb{R}_+$. Hence the system of stochastic differential equations (41)-(42)-(43) can be rewritten

$$d\delta_t = \alpha_t dW_t + \beta_t dt, \quad (44)$$

$$dS_t = \bar{\sigma}(t, S_t) dW_t + (\tilde{v}(t, S_t) + \alpha_t (\bar{\sigma} \partial_x \tilde{\lambda})(t, S_t)) dt + \tilde{\lambda}(t, S_t) d\delta_t, \quad (45)$$

$$dV_t = \delta_t dS_t + \frac{1}{2} \tilde{\lambda}(t, S_t) \alpha_t^2 dt. \quad (46)$$

Theorem 8. *Suppose the following assumptions hold:*

- $(\alpha_t)_{t \geq 0}$ and $(\Gamma_t)_{t \geq 0}$ are uniformly bounded adapted processes
- $(\Gamma_t)_{t \geq 0}$ satisfies uniformly the condition (40)
- there exists a constant $K > 0$, such that for all $(t, x) \in [0, T] \times \mathbb{R}$,

$$|x\sigma(t, x)| + |xv(t, x)| \leq K$$

- there exists a constant $K > 0$, such that for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$|x\sigma(t, x) - y\sigma(t, y)| + |xv(t, x) - yv(t, y)| \leq K|x - y|$$

- $(t, x) \in [0, T] \times \mathbb{R} \mapsto \lambda(t, x)$ is of class $\mathcal{C}^{1,2}$ and all its partial derivatives are bounded.
- $(t, x) \in [0, T] \times \mathbb{R} \mapsto x\lambda(t, x)$ has all its partial derivatives bounded.

Then there exists a unique strong solution to the system of stochastic differential equations (41)-(42)-(43) starting from $(\delta_0, S_0, V_0) \in \mathbb{R}^3$ at $t = 0$.

Proof. By plugging (44) in (45), we have

$$dS_t = \underbrace{(\bar{\sigma}(t, S_t) + \alpha_t \tilde{\lambda}(t, S_t))}_{:=a(t, S_t, \Gamma_t)} dW_t + \underbrace{(\tilde{v}(t, S_t) + \alpha_t (\bar{\sigma} \partial_x \tilde{\lambda})(t, S_t) + \beta_t \tilde{\lambda}(t, S_t))}_{:=b(t, S_t, \Gamma_t)} dt \quad (47)$$

with S starting from $S_0 \in \mathbb{R}$. $a(t, S_t, \Gamma_t)$ and $b(t, S_t, \Gamma_t)$ can be expressed as a sum of terms of the form

$$h(t, S_t, \Gamma_t) := b_t f(\tilde{\lambda}(t, S_t) \Gamma_t) g(t, S_t)$$

where $(b_t)_{t \geq 0}$ is a uniformly bounded adapted process, f bounded Lipschitz-continuous on $(-\infty, 1 - \varepsilon)$ and g bounded Lipschitz-continuous in the space variable on $[0, T] \times \mathbb{R}$. For all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} h(t, x, \Gamma_t) - h(t, y, \Gamma_t) &= b_t (f(\tilde{\lambda}(t, x) \Gamma_t) g(t, x) - f(\tilde{\lambda}(t, y) \Gamma_t) g(t, y)) \\ &= b_t (f(\tilde{\lambda}(t, x) \Gamma_t) - f(\tilde{\lambda}(t, y) \Gamma_t)) g(t, x) \\ &\quad + b_t f(\tilde{\lambda}(t, y) \Gamma_t) (g(t, x) - g(t, y)). \end{aligned}$$

Therefore there exists a constant $C_1 > 0$ such that for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$|h(t, x, \Gamma_t) - h(t, y, \Gamma_t)| \leq C_1 |x - y| \quad \mathbb{P} - a.s.$$

giving that there exists a constant $C_2 > 0$ such that for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$,

$$|a(t, x, \Gamma_t) - a(t, y, \Gamma_t)| + |b(t, x, \Gamma_t) - b(t, y, \Gamma_t)| \leq C_2 |x - y| \quad \mathbb{P} - a.s..$$

This implies that the stochastic differential equation (47) with the initial condition $S_0 \in \mathbb{R}$ admits a unique strong solution. Therefore we deduce that the system of stochastic differential equations (44)-(45)-(46) admits a unique strong solution so does the system (41)-(42)-(43). \square

In what follows we will show that the system of stochastic differential equations (44)-(45)-(46) can be derived from a discrete rebalancing trading strategy taking to the limit $n \rightarrow +\infty$, where n represents the number of auctions of the trading strategy. To this end, let us consider for all $n \in \mathbb{N}^*$, the following discrete time rebalancing time grid $(t_i^n)_{i \in [0, n]}$ such that

$$0 = t_0^n < t_1^n < \dots < t_{n-1}^n < t_n^n = T.$$

with $T > 0$ a fixed maturity. We set for all $n \in \mathbb{N}^*$,

$$\Delta_n := \sup_{1 \leq i \leq n} |t_i^n - t_{i-1}^n|.$$

III. How Option Hedging Shapes Market Impact

Let us set $(\delta^n, S^n, V^n)_{n \in \mathbb{N}^*}$ the sequence of processes defined on $[0, T]$ such that for all $t \in [0, T]$,

$$\delta_t^n = \sum_{i=0}^{n-1} \delta_{t_i^n} \mathbb{1}_{\{t_i^n \leq t < t_{i+1}^n\}} + \delta_T \mathbb{1}_{\{t=T\}}, \quad (48)$$

$$\begin{aligned} S_t^n := S_0 &+ \sum_{k=1}^n \mathbb{1}_{\{t_{k-1}^n \leq t \leq T\}} \left(\int_{t_{k-1}^n}^{t \wedge t_k^n} \tilde{\sigma}(s, S_s^n) dW_s + \int_{t_{k-1}^n}^{t \wedge t_k^n} \tilde{\nu}(s, S_s^n) ds \right) \\ &+ \sum_{k=1}^n \mathbb{1}_{\{t_k^n \leq t \leq T\}} (\delta_{t_k^n} - \delta_{t_{k-1}^n}) \tilde{\lambda}(t_k^n, S_{t_k^-}^n). \end{aligned} \quad (49)$$

$$\begin{aligned} V_t^n := V_0 &+ \sum_{k=1}^n \mathbb{1}_{\{t_{k-1}^n \leq t \leq T\}} \delta_{t_{k-1}^n} (S_{t \wedge t_k^n}^n - S_{t_{k-1}^n}^n) \\ &+ \sum_{k=1}^n \mathbb{1}_{\{t_k^n \leq t \leq T\}} \left[\frac{1}{2} (\delta_{t_k^n} - \delta_{t_{k-1}^n})^2 \tilde{\lambda}(t_k^n, S_{t_k^-}^n) + \delta_{t_{k-1}^n} (\delta_{t_k^n} - \delta_{t_{k-1}^n}) \tilde{\lambda}(t_k^n, S_{t_k^-}^n) \right]. \end{aligned} \quad (50)$$

Theorem 9. *Suppose the assumptions of Theorem 8 hold. Let $X := (\delta, S, V)$ the unique strong solution of the system of stochastic differential equations (44)-(45)-(46) on $[0, T]$ starting from $(\delta_0, S_0, V_0) \in \mathbb{R}^3$ at $t = 0$ and $(X^n)_{n \in \mathbb{N}^*} := (\delta^n, S^n, V^n)_{n \in \mathbb{N}^*}$ defined as in (48)-(49)-(50). Assume that $\Delta_n = O\left(\frac{1}{n}\right)$. Then there exists a constant $C > 0$ such that for all $n \in \mathbb{N}^*$,*

$$\sup_{t \in [0, T]} \mathbb{E} [\|X_t - X_t^n\|_2^2] \leq C \Delta_n.$$

In particular, we have that

$$X^n \xrightarrow[n \rightarrow +\infty]{\mathbb{L}^2, \mathbb{P}, \mathcal{L}} X.$$

Furthermore if $\sum_{n=1}^{+\infty} \Delta_n < +\infty$, we have also that

$$X^n \xrightarrow[n \rightarrow +\infty]{\mathbb{P}\text{-a.s.}, \mathbb{L}^2, \mathbb{P}, \mathcal{L}} X.$$

Proof. The first convergence result is a straightforward application of Propositions 5, 6 and 7.

Assume now that $\sum_{n=1}^{+\infty} \Delta_n < +\infty$. Let $t \in [0, T]$, we have for all $n \in \mathbb{N}^*$,

$$\mathbb{E} [\|X_t - X_t^n\|_2^2] \leq C \Delta_n.$$

Hence

$$\sum_{n=1}^{+\infty} \mathbb{E} [\|X_t - X_t^n\|_2^2] = \mathbb{E} \left[\sum_{n=1}^{+\infty} \|X_t - X_t^n\|_2^2 \right] < +\infty,$$

which implies that

$$\sum_{n=1}^{+\infty} \|X_t - X_t^n\|_2^2 < +\infty \quad \mathbb{P}\text{-a.s.}$$

and

$$\lim_{n \rightarrow +\infty} \|X_t - X_t^n\|_2 = 0 \quad \mathbb{P}\text{-a.s.}$$

□

Corollary 3. *Suppose the assumptions of Proposition 8 hold. Let $X := (\delta, S, V)$ the unique strong solution of the system of stochastic differential equations (44)-(45)-(46) on $[0, T]$ starting from $(\delta_0, S_0, V_0) \in \mathbb{R}^3$ at $t = 0$ and $(X^n)_{n \in \mathbb{N}^*} := (\delta^n, S^n, V^n)_{n \in \mathbb{N}^*}$ defined as in (48)-(49)-(50). Assume that one of the two following conditions hold:*

(i) *There exists $\varepsilon > 0$ such that $\Delta_n = O\left(\frac{1}{n^{1+\varepsilon}}\right)$.*

(ii) *$(\Delta_n)_{n \in \mathbb{N}^*}$ is a non-increasing sequence such that $\sum_{n=1}^{+\infty} \Delta_n < +\infty$.*

Then we have

$$X^n \xrightarrow[n \rightarrow +\infty]{\mathbb{P}\text{-a.s., } \mathbb{L}^2, \mathbb{P}, \mathcal{L}} X.$$

Proof. If we suppose (i), using the results on Riemann series and Theorem 9 leads to the conclusion. Let us suppose now (ii) and let us show that this implies that $\Delta_n = o\left(\frac{1}{n}\right)$ when $n \rightarrow +\infty$. Let $n \in \mathbb{N}^*$, we have

$$\sum_{k=1}^{2n} \Delta_k - \sum_{k=1}^n \Delta_k = \sum_{k=n+1}^{2n} \Delta_k \geq n\Delta_{2n},$$

hence $\lim_{n \rightarrow +\infty} n\Delta_{2n} = 0$. Besides we have also

$$\begin{aligned} (2n+1)\Delta_{2n+1} &\leq (2n+1)\Delta_{2n} \\ &\leq 2n\Delta_{2n} + \Delta_{2n}. \end{aligned}$$

Therefore we have

$$\lim_{n \rightarrow +\infty} 2n\Delta_{2n} = \lim_{n \rightarrow +\infty} (2n+1)\Delta_{2n+1} = 0,$$

wich implies that

$$\lim_{n \rightarrow +\infty} n\Delta_n = 0.$$

The conclusion is now a straightforward application of Theorem 9. □

6.3.2 The pricing equation

In option theory pricing the establishment of the valuation partial differential equation is often connected to the *replication problem* which consists in finding a self-financed strategy and an initial wealth V_0 such that $\Phi(S_T) = V_T$, \mathbb{P} -a.s. where Φ is a terminal *payoff*. Let (δ, S, V) a strong solution to the system of stochastic differential equations (44)-(45)-(46), which can take the form given in (41)-(42)-(43), and $u \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ a smooth function such that $u(T, \cdot) = \Phi$.

III. How Option Hedging Shapes Market Impact

The absence of arbitrage opportunities implies that for all $t \in [0, T]$, \mathbb{P} -a.s., $u(t, S_t) = V_t$. We consider the strategy given by $\delta(t, S_t) = \partial_s u(t, S_t)$, hence we have $\Gamma_t = \partial_{ss} u(t, S_t)$ \mathbb{P} -a.s. by the unicity of the Ito decomposition. Applying Ito's lemma to u gives

$$du(t, S_t) = \partial_t u(t, S_t)dt + \partial_s u(t, S_t)dS_t + \frac{1}{2}\partial_{ss} u(t, S_t)d\langle S, S \rangle_t, \quad (51)$$

besides we have also

$$dV_t = \partial_s u(t, S_t)dS_t + \frac{1}{2}\lambda(t, S_t)\Gamma_t^2 d\langle S, S \rangle_t. \quad (52)$$

By combining (41) and (42), we obtain

$$dS_t = \frac{\sigma(t, S_t)S_t}{1 - \lambda(t, S_t)S_t\Gamma_t}dW_t + \frac{\mu_t S_t + a_t \lambda(t, S_t)}{1 - \lambda(t, S_t)S_t\Gamma_t}dt$$

with

$$\mu_t := \nu(t, S_t) + \frac{\sigma^2(t, S_t)\Gamma_t(\lambda(t, S_t) + S_t\partial_x \lambda(t, S_t))}{1 - \lambda(t, S_t)S_t\Gamma_t}. \quad (53)$$

Hence by plugging

$$d\langle S, S \rangle_t = \frac{\sigma^2(t, S_t)S_t^2}{(1 - \lambda(t, S_t)S_t\Gamma_t)^2},$$

in the equality between Equations (51) and (52)

$$\partial_t u(t, S_t) + \frac{1}{2}\sigma^2(t, S_t)S_t^2\Gamma_t \frac{1}{1 - \lambda(t, S_t)S_t\Gamma_t} = 0 \quad \mathbb{P} - a.s..$$

Therefore $u \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ is a solution of

$$\partial_t u + \frac{1}{2}\sigma^2 s^2 \partial_{ss} u \frac{1}{1 - \phi} = 0, \quad (54)$$

$$\phi(t, s) = \lambda(t, s)s\partial_{ss} u(t, s), \quad (55)$$

$$1 - \phi \geq \varepsilon, \quad (56)$$

$$u(T, \cdot) = \Phi. \quad (57)$$

7 The Market Impact of Hedging Metaorders

In [Said et al., 2018] the authors define a metaorder as a large trading order split into small pieces and executed incrementally the same day by the same *agent* on the same *stock* and all having the same *direction* (buy or sell). This definition and Proposition 6 motivate the following:

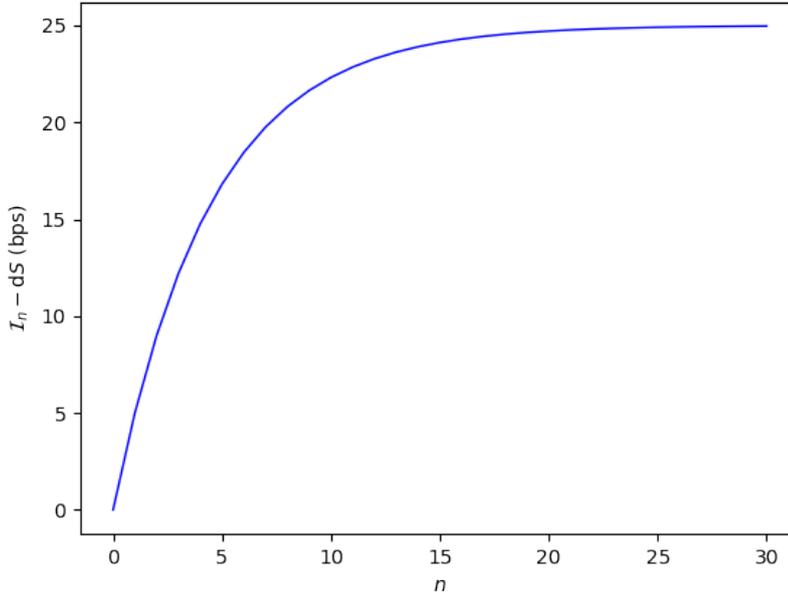
Definition 20. *Let $(u_n)_{n \in \mathbb{N}}$ a regular ϕ -market impact scenario. The market impact scenario $(u_n)_{n \in \mathbb{N}}$ is said to be a **metaorder** if, and only if, $\phi \in (0, 1)$. We will also refer to these metaorders as **hedging metaorders**.*

7.1 Dynamics of Hedging Metaorders

Let us consider an *hedging metaorder* $(dS_n)_{n \in \mathbb{N}}$. The proofs given to establish Theorem 1 show that there exists $r \in (0, 1)$ such that for all $n \in \mathbb{N}$, $|dS_{n+1}| \leq r \times |dS_n|$. Without loss of generality we will consider in the rest of the section that $(dS_n)_{n \in \mathbb{N}}$ is a buy metaorder i.e. $dS_n > 0$ for any $n \in \mathbb{N}$. Let us set for all $n \in \mathbb{N}$,

$$\mathcal{I}_n := \sum_{k=0}^n dS_k \tag{58}$$

the temporary market impact after n trades. As the sequence $(dS_n)_{n \in \mathbb{N}}$ is strictly non-increasing, the plot $(n, \mathcal{I}_n)_{n \in \mathbb{N}}$ has a concave non-decreasing shape reaching a plateau as $n \rightarrow +\infty$ since $\lim_{n \rightarrow +\infty} dS_n = 0$. In order to illustrate those ideas let us plot the dynamics of an hedging metaorder in the case that the $dS_{n+1} = r \times dS_n$ for any $n \in \mathbb{N}$ with $r = 0.8$ and $dS_1 = 5$ bps.



7.2 Immediate Impact and Size of the Hedging Metaorder

Let us consider an *hedging metaorder* $(dS_n)_{n \in \mathbb{N}}$ and recall that the number of shares executed during the *hedging metaorder* is given by

$$N = \Gamma \sum_{n=0}^{+\infty} dS_n.$$

This leads to the following linear relation

$$\mathcal{I} = \Gamma^{-1} N,$$

7.3 Relaxation of Hedging Metaorders

Let us consider an *hedging metaorder* $(dS_n)_{n \in \mathbb{N}}$. By Proposition 6 we can consider that $N(dS) = +\infty$. We have already established that at the leading order

$$\mathcal{R} = \sum_{n=0}^{+\infty} \Gamma dS_n \times \left(\sum_{n=0}^{+\infty} dS_n - \frac{\sum_{n=0}^{+\infty} \Gamma dS_n \left(\sum_{k=0}^n dS_k \right)}{\sum_{n=0}^{+\infty} \Gamma dS_n} \right) = \frac{1}{2} \lambda \left(\sum_{n=0}^{+\infty} \Gamma dS_n \right)^2 S^{1+\zeta},$$

leading to

$$\frac{\sum_{n=0}^{+\infty} \Gamma dS_n \left(\sum_{k=0}^n dS_k \right)}{\sum_{n=0}^{+\infty} \Gamma dS_n} = \left(1 - \frac{\phi}{2} \right) \sum_{n=0}^{+\infty} dS_n.$$

It has been shown in several empirical works that the market impact of metaorders possesses two distinct phases: A *temporary impact* as a consequence of the execution of the order followed by a relaxation phenomenon leading to a *permanent impact*. The relaxation phenomenon has been studied from an empirical point of view in stocks market [Bershova and Rakhlin, 2013] [Bacry et al., 2015] [Said et al., 2018] and with a theoretical insight in [Bouchaud et al., 2004] [Gatheral et al., 2011] and [Farmer et al., 2013] for instance. Recently the same has been done empirically concerning options market [Said et al., 2019]. Let us consider $(dS_n)_{n \in \mathbb{N}}$ an hedging metaorder starting from $dS \neq 0$, hence $N(dS) = +\infty$. In the terminology of metaorders let us denote by

$$\mathcal{I} := \sum_{n=0}^{+\infty} dS_n \tag{59}$$

as the *temporary impact* of the metaorder $(dS_n)_{n \in \mathbb{N}}$. We will denote by I the *permanent impact* of the metaorder $(dS_n)_{n \in \mathbb{N}}$. According to the *Fair Pricing* condition – predicted theoretically in [Farmer et al., 2013] and validated empirically in equity [Said et al., 2018] and options [Said et al., 2019] market – the *permanent impact* I satisfies the following identity:

$$S + I = \frac{\sum_{n=0}^{+\infty} \Gamma dS_n \times (S + \mathcal{I}_n)}{\sum_{n=0}^{+\infty} \Gamma dS_n} \tag{60}$$

where for all $n \in \mathbb{N}$, $\mathcal{I}_n := \sum_{k=0}^n dS_k$ as defined in (58), giving that

$$I = \frac{\sum_{n=0}^{+\infty} \Gamma dS_n \times \mathcal{I}_n}{\sum_{n=0}^{+\infty} \Gamma dS_n}. \tag{61}$$

The fair pricing identity (60), introduced in [Farmer et al., 2013], is in fact a *non-arbitrage* condition as it states that the final price – after the price reversion of the metaorder – reaches a level such that it equals to the volume-weighted average price of the metaorder. The fair pricing identity (61) reads that the *permanent impact* is equal to the volume-weighted average *temporary impacts* of the metaorder. Hence we have shown that:

Theorem 10.

$$\frac{I}{\mathcal{I}} = 1 - \frac{\phi}{2}, \quad (62)$$

giving the level of the relaxation phenomenon of the metaorder. Thus

$$\frac{1}{2} \leq \frac{I}{\mathcal{I}} \leq 1. \quad (63)$$

Theorem 10 connects directly the intensity ϕ of the *hedging metaorder* with the intensity of the price reversion after the end of the metaorder. Particularly, it states that the more the intensity is the more the relaxation will be. According to recent empirical and theoretical works [Bershova and Rakhlin, 2013] [Farmer et al., 2013] [Said et al., 2018] [Bucci et al., 2019b], we have that $\langle 1 - \frac{1}{2}\phi \rangle_{metaorders} = 2/3$ for intraday metaorders where $\langle \dots \rangle_{metaorders}$ stand for a mean value over all the metaorders.

8 Conclusion

In this paper we have presented a perturbation theory of market impact connecting option pricing theory with market microstructure empirical findings. From the option pricing point of view our model appears to be an extension of the model presented in [Loeper, 2018] in which we study the hedging process. Furthermore we have introduced what we have called the *hedging metaorders* to establish a connection between option hedging and market microstructure metaorders. Particularly, we have shown in our framework that our *hedging metaorders* obey to linear market impact in the size of the metaorder and possess a relaxation factor in $[1/2, 1]$ directly connected to their intensity characterized by the parameter ϕ . An interesting finding in our framework is the prediction of the existence of *hedging metaorders* that are metaorders with decreasing child orders showing linear market impact function of size. Some questions can be raised from our findings: Although most empirical studies consistently highlight a concave impact, is it possible to design specific execution strategies with linear impact? Does exist a correlation between concave impact and the fact to be detected by market makers for a metaorder?

III.A Proofs

III.A.1 Proof of Theorem 1

The proof of Theorem 1 is a consequence of the following propositions (see Propositions 1, 2, 3 and 4). It gives an equivalent criterion on the parameter ϕ for a market impact scenario to be admissible. We will establish that a market impact scenario is admissible from a trading perspective if, and only if, $\phi \in (-\infty, 1)$.

Proposition 1. *Let $(u_n)_{n \in \mathbb{N}}$ an ϕ -market impact scenario. If $\phi \geq 1$, then $(u_n)_{n \in \mathbb{N}}$ is not admissible from a trading perspective.*

Proof. Let $R > 0$ and $x \in (0, R)$. Hence for all $n \in \mathbb{N}$, $u_n(x) > 0$ which implies that for all $n \in \mathbb{N}$, $s_n(x) > 0$. Thus for all $n \in \mathbb{N}$,

$$-s_n(x) - S \leq 0$$

and

$$\phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x) \geq 0,$$

which gives for all $n \in \mathbb{N}$,

$$u_{n+1}(x) = \phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x) \geq \phi u_n(x).$$

In that case the series $\sum_{n \geq 0} u_n(x)$ diverges and $\sum_{n=0}^{+\infty} u_n(x) = +\infty$. \square

Proposition 2. *Let $(u_n)_{n \in \mathbb{N}}$ an ϕ -market impact scenario. If $\phi \in [0, 1)$, then $(u_n)_{n \in \mathbb{N}}$ is admissible from a trading perspective.*

Proof. When $\phi = 0$, for all $n \in \mathbb{N}$, $u_n = 0$ and $(u_n)_{n \in \mathbb{N}}$ is obviously admissible from a trading perspective. We consider now the case $\phi \in (0, 1)$. Let $r \in (\phi, 1 \wedge 4\phi)$, $R = (1 - r)S \left(\sqrt{\frac{r}{\phi}} - 1 \right)$

and $x \in [-R, R]$. Set $\mathcal{A}(x) = \left\{ n \in \mathbb{N} \mid s_n(x) = -S \right\}$.

- If $\mathcal{A}(x) \neq \emptyset$, $\mathcal{A}(x)$ has a least element $n_0(x)$ and for all $n > n_0(x)$, $u_n(x) = 0$. This gives the absolute convergence of the series $\sum_{n \geq 0} u_n(x)$.
- Let us suppose $\mathcal{A}(x) = \emptyset$, hence for all $n \in \mathbb{N}$, $\phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x) \geq -s_n(x) - S$ and for all $n \in \mathbb{N}$, $u_{n+1}(x) = \phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x)$. Let us show by induction that for all $n \in \mathbb{N}$, $|u_{n+1}(x)| \leq r|u_n(x)|$.

- If $n = 0$, $|u_1(x)| = \phi \left| 1 + \frac{x}{S} \right|^{1+\zeta} |u_0(x)|$ and $|x| \leq R < S$. Thus

$$|u_1(x)| \leq \phi \left(1 + \frac{|x|}{S} \right)^2 |u_0(x)|$$

with $\frac{|x|}{S} \leq \frac{|x|}{(1-r)S} \leq \sqrt{\frac{r}{\phi}} - 1$, which gives $|u_1(x)| \leq r|u_0(x)|$.

- Let $n \geq 1$ and suppose that for all $k \in \llbracket 0, n-1 \rrbracket$, $|u_{k+1}(x)| \leq r|u_k(x)|$. Hence for all $k \in \llbracket 0, n \rrbracket$, $|u_k(x)| \leq r^k|x|$ and $\frac{|s_n(x)|}{S} \leq \frac{1}{S} \sum_{k=0}^n r^k|x| \leq \frac{|x|}{(1-r)S} \leq \sqrt{\frac{r}{\phi}} - 1 < 1$ which implies

$$|u_{n+1}(x)| = \phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} |u_n(x)| \leq \phi \left(1 + \frac{|s_n(x)|}{S} \right)^2 |u_n(x)| \leq r|u_n(x)|.$$

Thus for all $n \in \mathbb{N}$, $|u_{n+1}(x)| \leq r|u_n(x)|$ leading to the absolute convergence of the series $\sum_{n \geq 0} u_n(x)$.

Finally for all $x \in [-R, R]$, $\sum_{n \geq 0} u_n(x)$ converges absolutely. \square

Proposition 3. *Let $(u_n)_{n \in \mathbb{N}}$ an ϕ -market impact scenario. If $\phi \in (-1, 0)$, then $(u_n)_{n \in \mathbb{N}}$ is admissible from a trading perspective.*

Proof. Let $r \in (|\phi|, 1 \wedge 4|\phi|)$, $R = S \left(\sqrt{\frac{r}{|\phi|}} - 1 \right)$ and $x \in [-R, R]$. $\sum_{n \geq 0} u_n(x)$ is an alternating series and for all $n \in \mathbb{N}$, $|s_n(x)| \leq |x| \leq R < S$. Set $\mathcal{A}(x) = \{n \in \mathbb{N} \mid s_n(x) = -S\}$.

- If $\mathcal{A}(x) \neq \emptyset$, $\mathcal{A}(x)$ has a least element $n_0(x)$ and for all $n > n_0(x)$, $u_n(x) = 0$. This gives the absolute convergence of the series $\sum_{n \geq 0} u_n(x)$.
- Let us suppose $\mathcal{A}(x) = \emptyset$, hence for all $n \in \mathbb{N}$, $\phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x) \geq -s_n(x) - S$ and for all $n \in \mathbb{N}$, $u_{n+1}(x) = \phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x)$. We have for all $n \in \mathbb{N}$,

$$|u_{n+1}(x)| \leq |\phi| \left(1 + \frac{s_n(x)}{S} \right)^2 |u_n(x)|$$

and

$$|\phi| \left(1 + \frac{s_n(x)}{S} \right)^2 \leq |\phi| \left(1 + \frac{|x|}{S} \right)^2 \leq r.$$

Thus for all $n \in \mathbb{N}$, $|u_{n+1}(x)| \leq r|u_n(x)|$ which gives the absolute convergence of the series $\sum_{n \geq 0} u_n(x)$.

Finally for all $x \in [-R, R]$, $\sum_{n \geq 0} u_n(x)$ converges absolutely. \square

Proposition 4. *Let $(u_n)_{n \in \mathbb{N}}$ an ϕ -market impact scenario. If $\phi \in (-\infty, -1]$, then $(u_n)_{n \in \mathbb{N}}$ is admissible from a trading perspective.*

Proof. Let $x \in \mathbb{R}$, the case $x = 0$ being trivial let us consider now $x \neq 0$. Without loss of generality let us assume $x > 0$, the proof would be similar for $x < 0$. As $\phi < 0$, $\sum_{n \geq 0} u_n(x)$ is an alternating series. Set

$$\mathcal{A}(x) = \left\{ n \in \mathbb{N} \mid s_n(x) = -S \right\}.$$

- If $\mathcal{A}(x) \neq \emptyset$ it has a least element denoted by $n_0(x)$. Hence for all $n \geq n_0(x)$, $u_n(x) = 0$ and the series $\sum_{n \geq 0} u_n(x)$ converges absolutely.
- Let us suppose $\mathcal{A}(x) = \emptyset$. Hence for all $n \in \mathbb{N}$, $\phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x) \geq -s_n(x) - S$ and for all $n \in \mathbb{N}$, $u_{n+1}(x) = \phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x)$. Let

$$\mathcal{B}(x) = \left\{ n \in \mathbb{N} \mid s_n(x) < -S + \frac{S}{|\phi|^{1/(1+\zeta)}} \right\}.$$

Let us assume $\mathcal{B}(x) = \emptyset$, hence for all $n \in \mathbb{N}$, $s_n(x) \geq -S + \frac{S}{|\phi|^{1/(1+\zeta)}}$. As $x \neq 0$, for all $n \in \mathbb{N}$, $u_n(x) \neq 0$ and $\frac{|u_{n+1}(x)|}{|u_n(x)|} = |\phi| \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta}$ implying that $|u_{n+1}(x)| \geq |u_n(x)|$. Hence $(s_{2n}(x))_{n \in \mathbb{N}}$ is a non-decreasing real valued sequence and $(s_{2n+1}(x))_{n \in \mathbb{N}}$ is non-increasing. The sequence $(s_n(x))_{n \in \mathbb{N}}$ being bounded by Assumption 1, $(s_{2n}(x))_{n \in \mathbb{N}}$ and $(s_{2n+1}(x))_{n \in \mathbb{N}}$ are convergent. Let $l(x) = \lim_{n \rightarrow +\infty} s_{2n+1}(x)$ and $l'(x) = \lim_{n \rightarrow +\infty} s_{2n}(x)$. As $l(x) < 0$ and $l'(x) > 0$, we have $l(x) < l'(x)$. Besides the sequence $(|u_n(x)|)_{n \in \mathbb{N}}$ is convergent and $\lim_{n \rightarrow +\infty} |u_n(x)| = l'(x) - l(x) > 0$, hence $\lim_{n \rightarrow +\infty} \frac{|u_{2n+1}(x)|}{|u_{2n}(x)|} = 1$. We have also for all $n \in \mathbb{N}$,

$$\frac{|u_{2n+1}(x)|}{|u_{2n}(x)|} = |\phi| \left(1 + \frac{s_{2n}(x)}{S} \right)^{1+\zeta}$$

which gives

$$\lim_{n \rightarrow +\infty} \frac{|u_{2n+1}(x)|}{|u_{2n}(x)|} = |\phi| \left(1 + \frac{l'(x)}{S} \right)^{1+\zeta}.$$

This leads to $l'(x) = -S + \frac{S}{|\phi|^{1/(1+\zeta)}} \leq l(x)$ which implies a contradiction. Thus $\mathcal{B}(x)$ is a non empty subset of \mathbb{N} and has a least element $n_1(x)$. Without loss of generality let us suppose that $n_1(x)$ is odd hence we can write $n_1(x) = 2k(x) + 1$ with $k(x) \in \mathbb{N}$. Set

$$\mathcal{C}(x) = \left\{ n \in \mathbb{N} \mid s_n(x) \vee s_{n+1}(x) < -S + \frac{S}{|\phi|^{1/(1+\zeta)}} \right\}.$$

If $\mathcal{C}(x) = \emptyset$ then $(s_{2n}(x))_{n \geq k(x)}$ and $(s_{2n+1}(x))_{n \geq k(x)}$ are two sequences non-increasing bounded from below. Thus $(s_{2n}(x))_{n \geq k(x)}$ and $(s_{2n+1}(x))_{n \geq k(x)}$ are convergent. Let

denote by $l(x)$ and $l'(x)$ their respective limit such that $l(x) = \lim_{n \rightarrow +\infty} s_{2n+1}(x)$ and $l'(x) = \lim_{n \rightarrow +\infty} s_{2n}(x)$. As $l(x) < -S + \frac{S}{|\phi|^{1/(1+\zeta)}}$ and $-S + \frac{S}{|\phi|^{1/(1+\zeta)}} \leq l'(x)$ we have $l(x) < l'(x)$. As previously this leads to $l(x) = l'(x) = -S + \frac{S}{|\phi|^{1/(1+\zeta)}}$ which gives a contradiction. Thus $\mathcal{C}(x)$ is a non empty subset of \mathbb{N} and has a least element $n_2(x) \geq n_1(x)$ since $\mathcal{C}(x) \subset \mathcal{B}(x)$. For all $n > n_2(x)$, $\frac{|u_{n+1}(x)|}{|u_n(x)|} = |\phi| \left(1 + \frac{s_{n_2(x)}(x)}{S}\right)^{1+\zeta} < 1$.

Thus the series $\sum_{n \geq 0} u_n(x)$ is absolutely convergent for any $x \in \mathbb{R}$. \square

III.A.2 Proof of Proposition 5

Proof. Let $R' > 0$ such that $(-R', R') \subset (-S, +\infty)$. Besides by Theorem 2 there exists also $R'' > 0$ such that for all $x \in (-R'', R'')$, $\sum_{n \geq 0} u_n(x)$ converges absolutely. Let $R = R' \wedge R''$, for all $x \in (-R, R)$, $N(x) \geq 1$.

- When $N(x) < +\infty$, for all $n \geq N(x)$, $s_n(x) = -S$ and for all $n \geq N(x) + 1$, $u_n(x) = 0$. Hence for all $n \in \mathbb{N}$, $u_{n+1}(x) = \phi \left(1 + \frac{s_n(x)}{S}\right)^{1+\zeta} u_n(x)$.
- When $N(x) = +\infty$, for all $n \in \mathbb{N}$, $\phi \left(1 + \frac{s_n(x)}{S}\right)^{1+\zeta} u_n(x) \geq -s_n(x) - S$ and for all $n \in \mathbb{N}$, $u_{n+1}(x) = \phi \left(1 + \frac{s_n(x)}{S}\right)^{1+\zeta} u_n(x)$.

In any case for all $n \in \mathbb{N}$, $u_{n+1}(x) = \phi \left(1 + \frac{(1+\zeta)s_n(x)}{S} + \frac{\zeta s_n^2(x)}{S^2}\right) u_n(x)$. This implies

$$\begin{aligned} \sum_{n=1}^{N(x)} u_n(x) &= \phi \sum_{n=0}^{N(x)} u_n(x) + \frac{(1+\zeta)\phi}{S} \sum_{n=0}^{N(x)} s_n(x) u_n(x) + \frac{\zeta\phi}{S^2} \sum_{n=0}^{N(x)} s_n^2(x) u_n(x) \\ \sum_{n=0}^{N(x)} u_n(x) &= x + \phi \sum_{n=0}^{N(x)} u_n(x) + \frac{(1+\zeta)\phi}{S} \sum_{n=0}^{N(x)} s_n(x) u_n(x) + \frac{\zeta\phi}{S^2} \sum_{n=0}^{N(x)} s_n^2(x) u_n(x) \\ (1-\phi) \sum_{n=0}^{N(x)} u_n(x) &= x + \frac{(1+\zeta)\phi}{S} \sum_{n=0}^{N(x)} s_n(x) u_n(x) + \frac{\zeta\phi}{S^2} \sum_{n=0}^{N(x)} s_n^2(x) u_n(x) \end{aligned}$$

which finally gives

$$\sum_{n=0}^{N(x)} u_n(x) = \frac{x}{1-\phi} + \frac{1}{S} \frac{(1+\zeta)\phi}{1-\phi} \sum_{n=0}^{N(x)} s_n(x) u_n(x) + \frac{1}{S^2} \frac{\zeta\phi}{1-\phi} \sum_{n=0}^{N(x)} s_n^2(x) u_n(x).$$

\square

III.A.3 Proof of Proposition 6

Proof. • $\phi = 0$, in that case for all $x \in \mathbb{R}$, $N(x) = +\infty$.

- $|\phi| \in (0, 1)$, let $r \in (|\phi|, 1 \wedge \frac{9}{4}|\phi|)$, $R = (1-r)S \left(\sqrt{\frac{r}{|\phi|}} - 1 \right)$ and $x \in [-R, R]$. Let us show by induction that for all $n \in \mathbb{N}$, $|u_{n+1}(x)| \leq r|u_n(x)|$.

- If $n = 0$, $|u_1(x)| = |\phi| \left| 1 + \frac{x}{S} \right|^{1+\zeta} |u_0(x)|$ and $|x| \leq R < S$. Thus

$$|u_1(x)| \leq |\phi| \left(1 + \frac{|x|}{S} \right)^2 |u_0(x)|$$

with $\frac{|x|}{S} \leq \frac{|x|}{(1-r)S} \leq \sqrt{\frac{r}{|\phi|}} - 1$, which gives $|u_1(x)| \leq r|u_0(x)|$.

- Let $n \geq 1$ and suppose that for all $k \in \llbracket 0, n-1 \rrbracket$, $|u_{k+1}(x)| \leq r|u_k(x)|$. Hence for all $k \in \llbracket 0, n \rrbracket$, $|u_k(x)| \leq r^k|x|$ and

$$|s_n(x)| \leq \sum_{k=0}^n r^k|x| \leq \frac{|x|}{(1-r)} \leq S \left(\sqrt{\frac{r}{|\phi|}} - 1 \right) < S.$$

Besides

$$-s_n(x) - S \leq -S + \frac{|x|}{1-r}$$

and

$$\begin{aligned} \left| \phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x) \right| &\leq |\phi| \left(1 + \frac{s_n(x)}{S} \right)^2 |u_n(x)| \\ &\leq |\phi| \left(1 + \frac{|s_n(x)|}{S} \right)^2 |u_n(x)| \\ &\leq r|u_n(x)| \\ &\leq r^{n+1}|x| \\ &\leq \frac{|x|}{1-r}. \end{aligned}$$

Hence

$$-s_n(x) - S \leq -S + \frac{|x|}{1-r} \leq -\frac{|x|}{1-r} \leq \phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x)$$

and

$$u_{n+1}(x) = \phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x)$$

which implies

$$|u_{n+1}(x)| = |\phi| \left(1 + \frac{s_n(x)}{S} \right)^2 |u_n(x)| \leq |\phi| \left(1 + \frac{|s_n(x)|}{S} \right)^2 |u_n(x)| \leq r|u_n(x)|.$$

Thus for all $x \in [-R, R]$, $n \in \mathbb{N}$, $|u_{n+1}(x)| \leq r|u_n(x)|$ and $|s_n(x)| \leq S \left(\sqrt{\frac{r}{|\phi|}} - 1 \right)$ which gives the absolute convergence of the series $\sum_{n \geq 0} u_n(x)$ with

$$\left| \sum_{n=0}^{+\infty} u_n(x) \right| \leq S \left(\sqrt{\frac{r}{|\phi|}} - 1 \right) < S.$$

Therefore for all $x \in [-R, R]$, $N(x) = +\infty$. □

III.A.4 Proof of Proposition 7

Proof. • By Theorem 2 and Proposition 6 there exists $R > 0$ such that for all $x \in (-R, R)$, $N(x) = +\infty$ and $\sum_{n \geq 0} u_n(x)$ converges absolutely. Let $x \in (-R, R)$, by Proposition 5 we have

$$\underbrace{\sum_{n=0}^{+\infty} u_n(x)}_{:=u(x)} = \frac{x}{1-\phi} + \frac{1}{S} \frac{(1+\zeta)\phi}{1-\phi} \sum_{n=0}^{+\infty} s_n(x) u_n(x) + \frac{1}{S^2} \frac{\zeta\phi}{1-\phi} \sum_{n=0}^{+\infty} s_n^2(x) u_n(x)$$

and for all $n \in \mathbb{N}$,

$$\begin{cases} u_0(x) = x \\ \forall n \in \mathbb{N}, u_{n+1}(x) = \phi \left(1 + \frac{s_n(x)}{S} \right)^{1+\zeta} u_n(x). \end{cases}$$

Hence by induction for all $n \in \mathbb{N}$, $x \in (-R, R)$, $u_n(x) = P_n(x)$ with $P_n \in \mathbb{R}[X]$ and the sequence of polynomials $(P_n)_{n \in \mathbb{N}}$ satisfying the following properties:

$$\begin{aligned} & - \begin{cases} P_0 = X \\ \forall n \in \mathbb{N}, P_{n+1} = \phi \left(1 + \frac{1}{S} \sum_{k=0}^n P_k \right)^{1+\zeta} P_n \end{cases} \\ & - \forall n \in \mathbb{N}, P_n(0) = 0. \end{aligned}$$

The series $\sum_{n \geq 0} u_n(x)$ being absolutely convergent is also unconditionally convergent which implies that $\sum_{n \geq 0} u_n$ has a power series expansion on $(-R, R)$.

- For all $n \in \mathbb{N}$, $P_n(0) = 0$ hence there exists $Q_n \in \mathbb{R}[X]$ such that $P_n = XQ_n$. The sequence of polynomials $(Q_n)_{n \in \mathbb{N}}$ satisfy the following recurrence relation

$$\begin{cases} Q_0 = 1 \\ \forall n \in \mathbb{N}, Q_{n+1} = \phi \left(1 + \frac{1}{S} \sum_{k=0}^n P_k \right)^{1+\zeta} Q_n \end{cases}$$

which implies for all $n \in \mathbb{N}$, $Q_n(0) = \phi^n$. Besides

$$u(x) = \frac{x}{1-\phi} + \frac{1}{S} \frac{(1+\zeta)\phi}{1-\phi} \underbrace{\sum_{n=0}^{+\infty} \left(\sum_{k=0}^n P_k(x) \right) P_n(x)}_{:=v(x)} + \frac{1}{S^2} \frac{\zeta\phi}{1-\phi} \underbrace{\sum_{n=0}^{+\infty} \left(\sum_{k=0}^n P_k(x) \right)^2 P_n(x)}_{:=w(x)}.$$

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Let us show that $v(x) = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n Q_k(0) \right) Q_n(0) x^2 + o(x^2)$ and $w(x) = o(x^2)$ as $x \rightarrow 0$.

Let $x \in \mathbb{R}$ such that $|x| < R \wedge (1-r)S \left(\sqrt{\frac{r}{|\phi|}} - 1 \right)$ with $r \in (|\phi|, 1 \wedge \frac{9}{4}|\phi|)$. As shown in Proposition 6 for all $n \in \mathbb{N}$, $|P_n(x)| \leq r^n |x|$ which gives for all $n \in \mathbb{N}$, $|Q_n(x)| \leq r^n$. Let $x \neq 0$, we have $\frac{v(x)}{x^2} = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n Q_k(x) \right) Q_n(x)$ and for all $n \in \mathbb{N}$,

$$\begin{aligned} \left| \left(\sum_{k=0}^n Q_k(x) \right) Q_n(x) \right| &\leq \left(\sum_{k=0}^n |Q_k(x)| \right) |Q_n(x)| \\ &\leq \left(\sum_{k=0}^n r^k \right) r^n \\ &\leq \frac{1-r^{n+1}}{1-r} r^n \end{aligned}$$

and $\sum_{n=0}^{+\infty} (1-r^{n+1})r^n < +\infty$. This leads to $\lim_{x \rightarrow 0} \frac{v(x)}{x^2} = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n Q_k(0) \right) Q_n(0)$. Similarly we

have $\lim_{x \rightarrow 0} \frac{w(x)}{x^2} = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n Q_k(0) \right)^2 P_n(0) = 0$. We have also

$$\begin{aligned} \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n Q_k(0) \right) Q_n(0) &= \sum_{n=0}^{+\infty} \left(\sum_{k=0}^n \phi^k \right) \phi^n \\ &= \sum_{n=0}^{+\infty} \frac{1-\phi^{n+1}}{1-\phi} \phi^n \\ &= \frac{1}{1-\phi} \sum_{n=0}^{+\infty} [\phi^n - \phi^{2n+1}] \\ &= \frac{1}{1-\phi} \left(\frac{1}{1-\phi} - \frac{\phi}{1-\phi^2} \right) \\ &= \frac{1}{1-\phi} \left(\frac{1}{1-\phi} - \frac{\phi}{(1-\phi)(1+\phi)} \right) \\ &= \frac{1}{(1+\phi)(1-\phi)^2} \end{aligned}$$

leading to

$$u(x) = \frac{1}{1-\phi} x + \frac{1}{S} \frac{(1+\zeta)\phi}{(1-\phi)^3(1+\phi)} x^2 + o(x^2) \text{ as } x \rightarrow 0.$$

□

III.A.5 Proof of Theorem 3

Proof. The final price of an (N, \mathcal{K}) -execution strategy is given by

$$S_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) = S \left(1 + \sum_{k=1}^{\mathcal{K}} \lambda^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq \mathcal{K}} n_{i_1} n_{i_2} \dots n_{i_k} \right).$$

Using Maclaurin's inequalities (see e.g. [Hardy et al., 1952]) we have

$$S_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) \leq S \left(1 + \sum_{k=1}^{\mathcal{K}} \frac{(\lambda N)^k}{\mathcal{K}^k} \binom{\mathcal{K}}{k} \right)$$

with equality exactly if and only if all the n_k are equal. Furthermore we have

$$S \left(1 + \lambda \sum_{k=1}^{\mathcal{K}} n_k \right) \leq S_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) \leq S \left(1 + \sum_{k=1}^{\mathcal{K}} \frac{(\lambda N)^k}{k!} \right),$$

leading straightforwardly to (19). To derive (20) let us study the maximum of the following function

$$f : \mathbb{R}^{\mathcal{K}} \longrightarrow \mathbb{R} \\ (x_1, \dots, x_{\mathcal{K}}) \longmapsto \sum_{k=1}^{\mathcal{K}} x_k \prod_{i=1}^{k-1} (1 + x_i)$$

on

$$\Lambda^* := \left\{ (x_1, \dots, x_{\mathcal{K}}) \in (\mathbb{R}_+^*)^{\mathcal{K}} \mid \sum_{k=1}^{\mathcal{K}} x_k = \lambda N \right\}.$$

by setting for all $k \in \llbracket 1, \mathcal{K} \rrbracket$, $x_k := \lambda n_k$. Let us consider the compact set

$$\Lambda := \left\{ (x_1, \dots, x_{\mathcal{K}}) \in (\mathbb{R}_+)^{\mathcal{K}} \mid \sum_{k=1}^{\mathcal{K}} x_k = \lambda N \right\}.$$

As the function f is continuous $f|_{\Lambda}$ is bounded and reaches its upper bound, so there exists $a \in \Lambda$ such that $f(a) = \sup_{x \in \Lambda} f(x)$. Let us suppose that there exists $i \in \llbracket 1, \mathcal{K} \rrbracket$ such that $a_i = 0$.

Without loss of generality by rearranging the terms a_k we can take $i = \mathcal{K}$. We set

$$k^* := \max \{ 1 \leq i \leq \mathcal{K} \mid a_i > 0 \} < \mathcal{K},$$

well-defined as $\sum_{k=1}^{\mathcal{K}} a_k = 1$. Let $0 < \varepsilon < a_{k^*}$, we have that

$$f(\underbrace{a_1, \dots, a_{k^*} - \varepsilon}_{k^* \text{ first terms}}, \varepsilon, 0, \dots, 0) - f(\underbrace{a_1, \dots, a_{k^*}}_{k^* \text{ first terms}}, 0, \dots, 0) = \varepsilon (a_{k^*} - \varepsilon) \prod_{i=1}^{k^*-1} (1 + a_i) > 0$$

giving a contradiction. Hence for all $i \in \llbracket 1, \mathcal{K} \rrbracket$, $a_i > 0$ and $\sup_{x \in \Lambda} f(x) = \sup_{x \in \Lambda^*} f(x) = f(a)$. Let us set

$$g : \mathbb{R}^{\mathcal{K}} \longrightarrow \mathbb{R} \\ (x_1, \dots, x_{\mathcal{K}}) \longmapsto \sum_{k=1}^{\mathcal{K}} x_k - \lambda N$$

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hence $\{x \in (\mathbb{R}_+^*)^{\mathcal{K}} \mid g(x) = 0\} = \Lambda^*$. f and g are of class \mathcal{C}^1 , so there exists a Lagrange multiplier $\beta \in \mathbb{R}$, such that for all $k \in \llbracket 1, \mathcal{K} \rrbracket$,

$$\frac{\partial f}{\partial x_k}(a) = \beta \frac{\partial g}{\partial x_k}(a)$$

which reads

$$\prod_{\substack{1 \leq i \leq \mathcal{K} \\ i \neq k}} (1 + a_i) = \frac{\prod_{1 \leq i \leq \mathcal{K}} (1 + a_i)}{1 + a_k} = \beta.$$

Thus we have that $a_1 = \dots = a_{\mathcal{K}} = \frac{\lambda N}{\mathcal{K}}$. Noticing that

$$1 + \sum_{k=1}^{\mathcal{K}} \lambda^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq \mathcal{K}} n_{i_1} n_{i_2} \dots n_{i_k} = \prod_{k=1}^{\mathcal{K}} (1 + \lambda n_k),$$

we get that

$$\begin{aligned} \bar{S}_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) &\leq \bar{S}_{N, \mathcal{K}}\left(\frac{N}{\mathcal{K}}, \dots, \frac{N}{\mathcal{K}}\right) \\ &\leq \frac{1}{N} \sum_{k=1}^{\mathcal{K}} \frac{N}{\mathcal{K}} S \prod_{i=1}^{k-1} \left(1 + \frac{\lambda N}{\mathcal{K}}\right) \\ &\leq \frac{S}{\mathcal{K}} \sum_{k=0}^{\mathcal{K}-1} \left(1 + \frac{\lambda N}{\mathcal{K}}\right)^k \\ &\leq \frac{S}{\mathcal{K}} \frac{\left(1 + \frac{\lambda N}{\mathcal{K}}\right)^{\mathcal{K}} - 1}{\frac{\lambda N}{\mathcal{K}}} \\ &\leq S \frac{e^{\lambda N} - 1}{\lambda N}. \end{aligned}$$

Besides we have also

$$\begin{aligned} \bar{S}_{N, \mathcal{K}}(n_1, \dots, n_{\mathcal{K}}) &\geq \frac{1}{N} \sum_{k=1}^{\mathcal{K}} n_k S \left(1 + \lambda \sum_{l=1}^{k-1} n_l\right) \\ &\geq S + \frac{\lambda S}{N} \sum_{1 \leq l < k \leq \mathcal{K}} n_k n_l \\ &\geq S + \frac{1}{2N} \lambda S \sum_{\substack{1 \leq l, k \leq \mathcal{K} \\ l \neq k}} n_k n_l \\ &\geq S + \frac{1}{2N} \lambda S \left(\sum_{1 \leq k, l \leq \mathcal{K}} n_k n_l - \sum_{k=1}^{\mathcal{K}} n_k^2 \right) \\ &\geq S + \frac{1}{2} \lambda S \left(N - \frac{1}{N} \sum_{k=1}^{\mathcal{K}} n_k^2 \right) \\ &\geq S + \frac{1}{2} \lambda S \left(N - \sup_{1 \leq k \leq \mathcal{K}} n_k \right) \end{aligned}$$

since the following inequality holds

$$\frac{N}{\mathcal{K}} \leq \frac{1}{N} \sum_{k=1}^{\mathcal{K}} n_k^2 \leq \sup_{1 \leq k \leq \mathcal{K}} n_k.$$

□

III.A.6 Proof of Theorem 4

Proof. Let us notice that

$$1 + \sum_{k=1}^{\mathcal{K}} \lambda^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq \mathcal{K}} n_{i_1} n_{i_2} \dots n_{i_k} = \prod_{k=1}^{\mathcal{K}} (1 + \lambda n_k).$$

We have

$$\begin{aligned} 0 \leq \lambda N - \ln \left(\prod_{k=1}^{\mathcal{K}} (1 + \lambda n_k) \right) &= \sum_{k=1}^{\mathcal{K}} (\lambda n_k - \ln(1 + \lambda n_k)) \\ &\leq \frac{\lambda^2}{2} \sum_{k=1}^{\mathcal{K}} n_k^2 \\ &\leq \frac{\lambda^2}{2} N \sup_{1 \leq k \leq \mathcal{K}} n_k \xrightarrow{\mathcal{K} \rightarrow +\infty} 0. \end{aligned}$$

Hence

$$\lim_{\mathcal{K} \rightarrow +\infty} \left(1 + \sum_{k=1}^{\mathcal{K}} \lambda^k \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq \mathcal{K}} n_{i_1} n_{i_2} \dots n_{i_k} \right) = e^{\gamma \lambda N}.$$

Besides we also have that

$$\begin{aligned} \left| \frac{e^{\lambda N} - 1}{\lambda} - \sum_{k=1}^{\mathcal{K}} n_k \prod_{i=1}^{k-1} (1 + \lambda n_i) \right| &= \left| \int_0^N e^{\lambda n} dn - \sum_{k=1}^{\mathcal{K}} n_k \prod_{i=1}^{k-1} (1 + \lambda n_i) \right| \\ &\leq \left| \int_0^N e^{\lambda n} dn - \sum_{k=1}^{\mathcal{K}} n_k e^{\lambda \sum_{i=1}^{k-1} n_i} \right| \\ &\quad + \left| \sum_{k=1}^{\mathcal{K}} n_k e^{\lambda \sum_{i=1}^{k-1} n_i} - \sum_{k=1}^{\mathcal{K}} n_k e^{\sum_{i=1}^{k-1} \ln(1 + \lambda n_i)} \right| \\ &\leq \left| \int_0^N e^{\lambda n} dn - \sum_{k=1}^{\mathcal{K}} n_k e^{\lambda \sum_{i=1}^{k-1} n_i} \right| \\ &\quad + \sum_{k=1}^{\mathcal{K}} n_k \left(e^{\lambda \sum_{i=1}^{k-1} n_i} - e^{\sum_{i=1}^{k-1} \ln(1 + \lambda n_i)} \right). \end{aligned}$$

In addition we have that,

$$\begin{aligned}
 0 &\leq \sum_{k=1}^{\mathcal{K}} n_k \left(e^{\lambda \sum_{i=1}^{k-1} n_i} - e^{\sum_{i=1}^{k-1} \ln(1+\lambda n_i)} \right) \\
 &\leq e^{\lambda N} \sum_{k=1}^{\mathcal{K}} n_k \left(\lambda \sum_{i=1}^{k-1} n_i - \sum_{i=1}^{k-1} \ln(1+\lambda n_i) \right) \\
 &\leq e^{\lambda N} \sum_{k=1}^{\mathcal{K}} n_k \frac{\lambda^2}{2} N \sup_{1 \leq k \leq \mathcal{K}} n_k \\
 &\leq e^{\lambda N} \frac{(\lambda N)^2}{2} \sup_{1 \leq k \leq \mathcal{K}} n_k
 \end{aligned}$$

where we have used that for all $(a, b) \in [0, \lambda N]^2$, $|e^a - e^b| \leq e^{\lambda N} |a - b|$. As

$$\lim_{\mathcal{K} \rightarrow +\infty} \sup_{1 \leq k \leq \mathcal{K}} |n_k| = 0,$$

we get that

$$\lim_{\mathcal{K} \rightarrow +\infty} \sum_{k=1}^{\mathcal{K}} n_k \prod_{i=1}^{k-1} (1 + \lambda n_i) = \int_0^N e^{\lambda n} dn.$$

Consequently,

$$\lim_{\mathcal{K} \rightarrow +\infty} \sum_{k=1}^{\mathcal{K}} n_k \prod_{i=1}^{k-1} (1 + \lambda n_i) = \frac{e^{\lambda N} - 1}{\lambda}.$$

□

III.A.7 Proof of Theorem 9

Proposition 5. *Suppose the assumptions of Theorem 8 hold. Let $(\delta^n)_{n \in \mathbb{N}}$ a sequence of processes defined on $[0, T]$ such that for all $t \in [0, T]$,*

$$\delta_t^n := \sum_{i=0}^{n-1} \delta_{t_i^n} \mathbb{1}_{\{t_i^n \leq t < t_{i+1}^n\}} + \delta_T \mathbb{1}_{\{t=T\}}. \quad (64)$$

Assume that $\lim_{n \rightarrow +\infty} \Delta_n = 0$. Then there exists a constant $C > 0$ such that for all $n \in \mathbb{N}^*$,

$$\sup_{t \in [0, T]} \mathbb{E}[|\delta_t - \delta_t^n|^2] \leq C \Delta_n.$$

Proof. Let $t \in [0, T]$, if $t = T$, then we have $\delta_t^n = \delta_t$. Let us take now $t \in [t_{i-1}^n, t_i^n)$ with $i \in [1, n]$. For all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned}
 \delta_t - \delta_t^n &= \delta_t - \delta_{t_{i-1}^n} \\
 &= \int_{t_{i-1}^n}^t \alpha_s dW_s + \int_{t_{i-1}^n}^t \beta_s ds.
 \end{aligned}$$

This gives by Ito's lemma

$$\begin{aligned} d|\delta_t - \delta_{t_{i-1}^n}|^2 &= 2(\delta_t - \delta_{t_{i-1}^n})(\alpha_t dW_t + \beta_t dt) + \alpha_t^2 dt \\ &= 2(\delta_t - \delta_{t_{i-1}^n})\alpha_t dW_t + [2(\delta_t - \delta_{t_{i-1}^n})\beta_t + \alpha_t^2]dt \end{aligned}$$

and for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned} \mathbb{E}[|\delta_t - \delta_{t_{i-1}^n}|^2] &= 2\mathbb{E}\left[\int_{t_{i-1}^n}^t (\delta_s - \delta_{t_{i-1}^n})\beta_s ds\right] + \mathbb{E}\left[\int_{t_{i-1}^n}^t \alpha_s^2 ds\right] \\ &\leq \mathbb{E}\left[\int_{t_{i-1}^n}^t |\delta_s - \delta_{t_{i-1}^n}|^2 ds\right] + \mathbb{E}\left[\int_{t_{i-1}^n}^t \beta_s^2 ds\right] + \mathbb{E}\left[\int_{t_{i-1}^n}^t \alpha_s^2 ds\right] \\ &\leq \int_{t_{i-1}^n}^t \mathbb{E}[|\delta_s - \delta_{t_{i-1}^n}|^2] ds + \int_{t_{i-1}^n}^t (\mathbb{E}[\beta_s^2] + \mathbb{E}[\alpha_s^2]) ds. \end{aligned}$$

Therefore there exists a constant $C_1 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\delta_t - \delta_{t_{i-1}^n}|^2] \leq C_1 \int_{t_{i-1}^n}^t \mathbb{E}[|\delta_s - \delta_{t_{i-1}^n}|^2] ds + C_1 \Delta_n$$

which implies by Gronwall's lemma, for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\delta_t - \delta_{t_{i-1}^n}|^2] \leq C_1 \Delta_n \exp(C_1(t - t_{i-1}^n)) \leq \underbrace{C_1 \Delta_n \exp(C_1 \Delta_n)}_{=O(\Delta_n)},$$

hence there exists a constant $C_2 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\delta_t - \delta_{t_{i-1}^n}|^2] \leq C_2 \Delta_n.$$

Let $(\tau_k^{i,n})_{k \in \mathbb{N}}$ a non-decreasing sequence such that $\lim_{k \rightarrow +\infty} \tau_k^{i,n} = t_i^n$, by Fatou's lemma we have

$$\mathbb{E}\left[\liminf_{k \rightarrow +\infty} |\delta_{\tau_k^{i,n}} - \delta_{t_{i-1}^n}|^2\right] \leq \liminf_{k \rightarrow +\infty} \mathbb{E}[|\delta_{\tau_k^{i,n}} - \delta_{t_{i-1}^n}|^2] \leq C_2 \Delta_n,$$

and

$$\mathbb{E}\left[\liminf_{k \rightarrow +\infty} |\delta_{\tau_k^{i,n}} - \delta_{t_{i-1}^n}|^2\right] = \mathbb{E}\left[\lim_{k \rightarrow +\infty} |\delta_{\tau_k^{i,n}} - \delta_{t_{i-1}^n}|^2\right] = \mathbb{E}[|\delta_{t_i^n} - \delta_{t_{i-1}^n}|^2]$$

since δ is a continuous process. Therefore we have for all $n \in \mathbb{N}^*$, $i \in \llbracket 1, n \rrbracket$, $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\delta_t - \delta_t^n|^2] \leq C_2 \Delta_n,$$

and

$$\sup_{t \in [0, T]} \mathbb{E}[|\delta_t - \delta_t^n|^2] = \max_{1 \leq i \leq n} \sup_{t \in [t_{i-1}^n, t_i^n]} \mathbb{E}[|\delta_t - \delta_t^n|^2].$$

This gives that for all $n \in \mathbb{N}^*$,

$$\sup_{t \in [0, T]} \mathbb{E}[|\delta_t - \delta_t^n|^2] \leq C_2 \Delta_n.$$

□

Proposition 6. *Suppose the assumptions of Theorem 8 hold. Let $(S^n)_{n \in \mathbb{N}}$ a sequence of processes defined on $[0, T]$ such that for all $t \in [0, T]$,*

$$\begin{aligned} S_t^n := & S_0 + \sum_{k=1}^n \mathbb{1}_{\{t_{k-1}^n \leq t \leq T\}} \left(\int_{t_{k-1}^n}^{t \wedge t_k^n} \bar{\sigma}(s, S_s^n) dW_s + \int_{t_{k-1}^n}^{t \wedge t_k^n} \tilde{v}(s, S_s^n) ds \right) \\ & + \sum_{k=1}^n \mathbb{1}_{\{t_k^n \leq t \leq T\}} (\delta_{t_k^n} - \delta_{t_{k-1}^n}) \tilde{\lambda}(t_k^n, S_{t_k^n}^n). \end{aligned} \quad (65)$$

Assume that $\Delta_n = O\left(\frac{1}{n}\right)$. Then there exists a constant $C > 0$ such that for all $n \in \mathbb{N}^*$,

$$\sup_{t \in [0, T]} \mathbb{E}[|S_t - S_t^n|^2] \leq C \Delta_n.$$

Proof. Let $n \in \mathbb{N}^*$, $\Delta S^n := S - S^n$ and $t \in [t_{i-1}^n, t_i^n)$ for $i \in \llbracket 1, n \rrbracket$. We have for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned} dS_t &= \bar{\sigma}(t, S_t) dW_t + (\tilde{v}(t, S_t) + \alpha_t (\bar{\sigma} \partial_x \tilde{\lambda})(t, S_t)) dt + \tilde{\lambda}(t, S_t) d\delta_t, \\ dS_t^n &= \bar{\sigma}(t, S_t^n) dW_t + \tilde{v}(t, S_t^n) dt, \end{aligned}$$

which gives

$$\begin{aligned} d\Delta S_t^n &= dS_t - dS_t^n \\ &= (\bar{\sigma}(t, S_t) - \bar{\sigma}(t, S_t^n) + \alpha_t \tilde{\lambda}(t, S_t)) dW_t \\ &\quad + (\tilde{v}(t, S_t) - \tilde{v}(t, S_t^n) + \alpha_t (\bar{\sigma} \partial_x \tilde{\lambda})(t, S_t) + \beta_t \tilde{\lambda}(t, S_t)) dt. \end{aligned}$$

Hence by Ito's lemma,

$$d|\Delta S_t^n|^2 = 2\Delta S_t^n d\Delta S_t^n + d\langle \Delta S^n, \Delta S^n \rangle_t.$$

Therefore for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned}
 \mathbb{E}[|\Delta S_t^n|^2] &= \mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] \\
 &+ 2\mathbb{E}\left[\int_{t_{i-1}}^t \Delta S_s^n (\tilde{v}(s, S_s) - \tilde{v}(s, S_s^n) + \alpha_s(\tilde{\sigma}\partial_x\tilde{\lambda})(s, S_s) + \beta_s\tilde{\lambda}(s, S_s)) ds\right] \\
 &+ \mathbb{E}\left[\int_{t_{i-1}}^t (\tilde{\sigma}(s, S_s) - \tilde{\sigma}(s, S_s^n) + \alpha_s\tilde{\lambda}(s, S_s))^2 ds\right] \\
 &= \mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] \\
 &+ 2\mathbb{E}\left[\int_{t_{i-1}}^t \Delta S_s^n (\tilde{v}(s, S_s) - \tilde{v}(s, S_s^n)) ds\right] \\
 &+ 2\mathbb{E}\left[\int_{t_{i-1}}^t \Delta S_s^n (\alpha_s(\tilde{\sigma}\partial_x\tilde{\lambda})(s, S_s) + \beta_s\tilde{\lambda}(s, S_s)) ds\right] \\
 &+ \mathbb{E}\left[\int_{t_{i-1}}^t (\tilde{\sigma}(s, S_s) - \tilde{\sigma}(s, S_s^n) + \alpha_s\tilde{\lambda}(s, S_s))^2 ds\right] \\
 &\leq \mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] \\
 &+ 2\mathbb{E}\left[\int_{t_{i-1}}^t |\Delta S_s^n| |\tilde{v}(s, S_s) - \tilde{v}(s, S_s^n)| ds\right] \\
 &+ \mathbb{E}\left[\int_{t_{i-1}}^t (|\Delta S_s^n|^2 + |\alpha_s(\tilde{\sigma}\partial_x\tilde{\lambda})(s, S_s) + \beta_s\tilde{\lambda}(s, S_s)|^2) ds\right] \\
 &+ 2\mathbb{E}\left[\int_{t_{i-1}}^t (|\tilde{\sigma}(s, S_s) - \tilde{\sigma}(s, S_s^n)|^2 + |\alpha_s\tilde{\lambda}(s, S_s)|^2) ds\right],
 \end{aligned}$$

hence there exists a constant $C_1 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\Delta S_t^n|^2] \leq \mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] + C_1\Delta_n + C_1 \int_{t_{i-1}}^t \mathbb{E}[|\Delta S_s^n|^2] ds. \quad (66)$$

Applying Gronwall's lemma we obtain for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned}
 \mathbb{E}[|\Delta S_t^n|^2] &\leq \left(\mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] + C_1\Delta_n\right) \exp(C_1(t - t_{i-1}^n)) \\
 &\leq \left(\mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] + C_1\Delta_n\right) \exp(C_1\Delta_n). \quad (67)
 \end{aligned}$$

Plugging (67) in (66) leads to, for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned}
 \mathbb{E}[|\Delta S_t^n|^2] &\leq \mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] + C_1\Delta_n + C_1\Delta_n \left(\mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] + C_1\Delta_n\right) \exp(C_1\Delta_n) \\
 &\leq \mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] \left(1 + \underbrace{C_1\Delta_n \exp(C_1\Delta_n)}_{=O(\Delta_n)}\right) + \underbrace{C_1\Delta_n + C_1^2\Delta_n^2 \exp(C_1\Delta_n)}_{=O(\Delta_n)}.
 \end{aligned}$$

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This implies that there exists a constant $C_2 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n]$,

$$\mathbb{E}[|\Delta S_t^n|^2] \leq \mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2](1 + C_2\Delta_n) + C_2\Delta_n. \quad (68)$$

We have

$$d((\delta_t - \delta_{t_{i-1}^n})\tilde{\lambda}(t, S_t^n)) = (\delta_t - \delta_{t_{i-1}^n})d(\tilde{\lambda}(t, S_t^n)) + \tilde{\lambda}(t, S_t^n)d(\delta_t - \delta_{t_{i-1}^n}) + d\langle \delta - \delta_{t_{i-1}^n}, \tilde{\lambda}(\cdot, S^n) \rangle_t$$

and

$$d\tilde{\lambda}(t, S_t^n) = (\bar{\sigma}\partial_x\tilde{\lambda})(t, S_t^n)dW_t + \left(\partial_t\tilde{\lambda} + \tilde{\nu}\partial_x\tilde{\lambda} + \frac{1}{2}\bar{\sigma}^2\partial_{xx}\tilde{\lambda} \right)(t, S_t^n)dt.$$

This gives that

$$\begin{aligned} d((\delta_t - \delta_{t_{i-1}^n})\tilde{\lambda}(t, S_t^n)) &= ((\delta_t - \delta_{t_{i-1}^n})(\bar{\sigma}\partial_x\tilde{\lambda})(t, S_t^n) + \alpha_t\tilde{\lambda}(t, S_t^n))dW_t \\ &\quad + (\delta_t - \delta_{t_{i-1}^n}) \left(\partial_t\tilde{\lambda} + \tilde{\nu}\partial_x\tilde{\lambda} + \frac{1}{2}\bar{\sigma}^2\partial_{xx}\tilde{\lambda} \right)(t, S_t^n)dt \\ &\quad + (\beta_t\tilde{\lambda}(t, S_t^n) + \alpha_t(\bar{\sigma}\partial_x\tilde{\lambda})(t, S_t^n))dt, \end{aligned}$$

which implies that for all $t \in [t_{i-1}^n, t_i^n]$,

$$(\delta_t - \delta_{t_{i-1}^n})\tilde{\lambda}(t, S_t^n) = \mathcal{R}_t^{i,n},$$

by setting for all $t \in [t_{i-1}^n, t_i^n]$,

$$\begin{aligned} \mathcal{R}_t^{i,n} &:= \int_{t_{i-1}^n}^t ((\delta_s - \delta_{t_{i-1}^n})(\bar{\sigma}\partial_x\tilde{\lambda})(s, S_s^n) + \alpha_s\tilde{\lambda}(s, S_s^n))dW_s \\ &\quad + \int_{t_{i-1}^n}^t (\delta_s - \delta_{t_{i-1}^n}) \left(\partial_t\tilde{\lambda} + \tilde{\nu}\partial_x\tilde{\lambda} + \frac{1}{2}\bar{\sigma}^2\partial_{xx}\tilde{\lambda} \right)(s, S_s^n)ds \\ &\quad + \int_{t_{i-1}^n}^t (\beta_s\tilde{\lambda}(s, S_s^n) + \alpha_s(\bar{\sigma}\partial_x\tilde{\lambda})(s, S_s^n))ds. \end{aligned}$$

We set for all $t \in [t_{i-1}^n, t_i^n]$, $\tilde{S}_t^n := S_t^n + (\delta_t - \delta_{t_{i-1}^n})\tilde{\lambda}(t, S_t^n)$. Hence

$$\lim_{t \rightarrow (t_i^n)^-} \tilde{S}_t^n = S_{t_i^n}^n + (\delta_{t_i^n} - \delta_{t_{i-1}^n})\tilde{\lambda}(t_i^n, S_{t_i^n}^n) = S_{t_i^n}^n.$$

Let us consider now for all $t \in [t_{i-1}^n, t_i^n]$, $\Delta\tilde{S}_t^n := S_t - \tilde{S}_t^n = \Delta S_t^n - \mathcal{R}_t^{i,n}$. We have

$$\begin{aligned} d\Delta\tilde{S}_t^n &= d\Delta S_t^n - d\mathcal{R}_t^{i,n} \\ &= (\bar{\sigma}(t, S_t) - \bar{\sigma}(t, S_t^n) + \alpha_t\tilde{\lambda}(t, S_t))dW_t \\ &\quad + (\tilde{\nu}(t, S_t) - \tilde{\nu}(t, S_t^n) + \alpha_t(\bar{\sigma}\partial_x\tilde{\lambda})(t, S_t) + \beta_t\tilde{\lambda}(t, S_t))dt \\ &\quad - ((\delta_t - \delta_{t_{i-1}^n})(\bar{\sigma}\partial_x\tilde{\lambda})(t, S_t^n) + \alpha_t\tilde{\lambda}(t, S_t^n))dW_t \\ &\quad - (\delta_t - \delta_{t_{i-1}^n}) \left(\partial_t\tilde{\lambda} + \tilde{\nu}\partial_x\tilde{\lambda} + \frac{1}{2}\bar{\sigma}^2\partial_{xx}\tilde{\lambda} \right)(t, S_t^n)dt \\ &\quad - (\beta_t\tilde{\lambda}(t, S_t^n) + \alpha_t(\bar{\sigma}\partial_x\tilde{\lambda})(t, S_t^n))dt \end{aligned}$$

This gives by Ito's lemma,

$$d|\Delta\tilde{S}_t^n|^2 = 2\Delta\tilde{S}_t^n d\Delta\tilde{S}_t^n + d\langle\Delta\tilde{S}^n, \Delta\tilde{S}^n\rangle_t$$

which implies that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned} \mathbb{E}[|\Delta\tilde{S}_t^n|^2] &= \mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] \\ &+ 2\mathbb{E}\left[\int_{t_{i-1}}^t \Delta\tilde{S}_s^n (\tilde{v}(s, S_s) - \tilde{v}(s, S_s^n)) ds\right] \\ &+ 2\mathbb{E}\left[\int_{t_{i-1}}^t \Delta\tilde{S}_s^n \alpha_s ((\tilde{\sigma}\partial_x\tilde{\lambda})(s, S_s) - (\tilde{\sigma}\partial_x\tilde{\lambda})(s, S_s^n)) ds\right] \\ &+ 2\mathbb{E}\left[\int_{t_{i-1}}^t \Delta\tilde{S}_s^n \beta_s (\tilde{\lambda}(s, S_s) - \tilde{\lambda}(s, S_s^n)) ds\right] \\ &- 2\mathbb{E}\left[\int_{t_{i-1}}^t \Delta\tilde{S}_s^n (\delta_s - \delta_{t_{i-1}}^n) \left(\partial_t\tilde{\lambda} + \tilde{v}\partial_x\tilde{\lambda} + \frac{1}{2}\tilde{\sigma}^2\partial_{xx}\tilde{\lambda}\right)(s, S_s^n) ds\right] \\ &+ \mathbb{E}\left[\int_{t_{i-1}}^t \left(\tilde{\sigma}(s, S_s) - \tilde{\sigma}(s, S_s^n) + \alpha_s(\tilde{\lambda}(s, S_s) - \tilde{\lambda}(s, S_s^n)) - (\delta_s - \delta_{t_{i-1}}^n)(\tilde{\sigma}\partial_x\tilde{\lambda})(s, S_s^n)\right)^2 ds\right] \end{aligned}$$

leading to

$$\begin{aligned} \mathbb{E}[|\Delta\tilde{S}_t^n|^2] &\leq \mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] \\ &\leq \mathbb{E}\left[\int_{t_{i-1}}^t |\Delta\tilde{S}_s^n| |\tilde{v}(s, S_s) - \tilde{v}(s, S_s^n)| ds\right] \\ &+ 2\mathbb{E}\left[\int_{t_{i-1}}^t |\Delta\tilde{S}_s^n| |\alpha_s| |(\tilde{\sigma}\partial_x\tilde{\lambda})(s, S_s) - (\tilde{\sigma}\partial_x\tilde{\lambda})(s, S_s^n)| ds\right] \\ &+ 2\mathbb{E}\left[\int_{t_{i-1}}^t |\Delta\tilde{S}_s^n| |\beta_s| |\tilde{\lambda}(s, S_s) - \tilde{\lambda}(s, S_s^n)| ds\right] \\ &+ \mathbb{E}\left[\int_{t_{i-1}}^t \left(|\Delta\tilde{S}_s^n|^2 + |\delta_s - \delta_{t_{i-1}}^n|^2 \left|\left(\partial_t\tilde{\lambda} + \tilde{v}\partial_x\tilde{\lambda} + \frac{1}{2}\tilde{\sigma}^2\partial_{xx}\tilde{\lambda}\right)(s, S_s^n)\right|^2\right) ds\right] \\ &+ 3\mathbb{E}\left[\int_{t_{i-1}}^t |\tilde{\sigma}(s, S_s) - \tilde{\sigma}(s, S_s^n)|^2 ds\right] \\ &+ 3\mathbb{E}\left[\int_{t_{i-1}}^t |\alpha_s|^2 |\tilde{\lambda}(s, S_s) - \tilde{\lambda}(s, S_s^n)|^2 ds\right] \\ &+ 3\mathbb{E}\left[\int_{t_{i-1}}^t |\delta_s - \delta_{t_{i-1}}^n|^2 |(\tilde{\sigma}\partial_x\tilde{\lambda})(s, S_s^n)|^2 ds\right] \end{aligned}$$

for any $t \in [t_{i-1}^n, t_i^n)$. Hence there exists a constant $C_3 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\Delta\tilde{S}_t^n|^2] \leq \mathbb{E}[|\Delta S_{t_{i-1}}^n|^2] + C_3 \int_{t_{i-1}}^t \left(\mathbb{E}[|\delta_s - \delta_{t_{i-1}}^n|^2] + \mathbb{E}[|\Delta S_s^n|^2] + \mathbb{E}[|\Delta\tilde{S}_s^n|^2]\right) ds. \quad (69)$$

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Besides by Proposition 5 there exists a constant $C_4 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\delta_t - \delta_{t_{i-1}^n}|^2] \leq C_4 \Delta_n, \quad (70)$$

therefore by plugging (68) and (70) in the inequality (69) we obtain that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned} \mathbb{E}[|\Delta \tilde{S}_t^n|^2] &\leq \mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2] + C_3 C_4 \Delta_n^2 + C_3 \Delta_n \left(\mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2] (1 + C_2 \Delta_n) + C_2 \Delta_n \right) + C_3 \int_{t_{i-1}^n}^t \mathbb{E}[|\Delta \tilde{S}_s^n|^2] ds \\ &\leq \mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2] \left(1 + \underbrace{C_3 \Delta_n + C_2 C_3 \Delta_n^2}_{=O(\Delta_n)} \right) + \underbrace{(C_2 + C_4) C_3 \Delta_n^2}_{=O(\Delta_n^2)} + C_3 \int_{t_{i-1}^n}^t \mathbb{E}[|\Delta \tilde{S}_s^n|^2] ds. \end{aligned}$$

Hence there exists a constant $C_5 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\Delta \tilde{S}_t^n|^2] \leq \mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2] (1 + C_5 \Delta_n) + C_5 \Delta_n^2 + C_5 \int_{t_{i-1}^n}^t \mathbb{E}[|\Delta \tilde{S}_s^n|^2] ds, \quad (71)$$

which implies by Gronwall's lemma that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned} \mathbb{E}[|\Delta \tilde{S}_t^n|^2] &\leq \exp(C_5(t - t_{i-1}^n)) \left(\mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2] (1 + C_5 \Delta_n) + C_5 \Delta_n^2 \right) \\ &\leq \exp(C_5 \Delta_n) \left(\mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2] (1 + C_5 \Delta_n) + C_5 \Delta_n^2 \right). \end{aligned} \quad (72)$$

By plugging the inequality (72) in (71), we get

$$\begin{aligned} \mathbb{E}[|\Delta \tilde{S}_t^n|^2] &\leq \mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2] (1 + C_5 \Delta_n) + C_5 \Delta_n^2 + C_5 \Delta_n \exp(C_5 \Delta_n) \left(\mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2] (1 + C_5 \Delta_n) + C_5 \Delta_n^2 \right) \\ &\leq \mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2] \left(1 + \underbrace{C_5 \Delta_n + C_5 \Delta_n \exp(C_5 \Delta_n) + C_5^2 \Delta_n^2 \exp(C_5 \Delta_n)}_{=O(\Delta_n)} \right) \\ &\quad + \underbrace{C_5 \Delta_n^2 + C_5^2 \Delta_n^3 \exp(C_5 \Delta_n)}_{=O(\Delta_n^2)}. \end{aligned}$$

Therefore there exists a constant $C_6 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\Delta \tilde{S}_t^n|^2] \leq \mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2] (1 + C_6 \Delta_n) + C_6 \Delta_n^2.$$

Let $(\tau_k^{i,n})_{k \in \mathbb{N}}$ a non-decreasing sequence such that $\lim_{k \rightarrow +\infty} \tau_k^{i,n} = t_i^n$, by Fatou's lemma we have

$$\mathbb{E} \left[\liminf_{k \rightarrow +\infty} |\Delta \tilde{S}_{\tau_k^{i,n}}^n|^2 \right] \leq \liminf_{k \rightarrow +\infty} \mathbb{E}[|\Delta \tilde{S}_{\tau_k^{i,n}}^n|^2] \leq \mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2] (1 + C_6 \Delta_n) + C_6 \Delta_n^2,$$

and

$$\mathbb{E} \left[\liminf_{k \rightarrow +\infty} |\Delta \tilde{S}_{\tau_k^{i,n}}^n|^2 \right] = \mathbb{E} \left[\lim_{k \rightarrow +\infty} |\Delta \tilde{S}_{\tau_k^{i,n}}^n|^2 \right] = \mathbb{E}[|\Delta S_{t_i^n}^n|^2]$$

since $\lim_{t \rightarrow (t_i^n)^-} \Delta \tilde{S}_t^n = \Delta S_{t_i^n}^n$. Therefore we have for all $n \in \mathbb{N}^*$, $i \in \llbracket 1, n \rrbracket$,

$$\begin{aligned} \mathbb{E}[|\Delta S_{t_i^n}^n|^2] &\leq \mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2](1 + C_6 \Delta_n) + C_6 \Delta_n^2 \\ &\leq \mathbb{E}[|\Delta S_{t_{i-2}^n}^n|^2](1 + C_6 \Delta_n)^2 + C_6 \Delta_n^2(1 + C_6 \Delta_n) + C_6 \Delta_n^2 \\ &\dots \\ &\leq \mathbb{E}[|\Delta S_{t_0^n}^n|^2](1 + C_6 \Delta_n)^i + C_6 \Delta_n^2 \sum_{k=0}^{i-1} (1 + C_6 \Delta_n)^k. \end{aligned}$$

Considering the fact that $\Delta S_{t_0^n}^n = 0$, we have for all $n \in \mathbb{N}^*$, $i \in \llbracket 1, n \rrbracket$,

$$\begin{aligned} \mathbb{E}[|\Delta S_{t_i^n}^n|^2] &\leq C_6 \Delta_n^2 \frac{(1 + C_6 \Delta_n)^i - 1}{C_6 \Delta_n} \\ &\leq \Delta_n (1 + C_6 \Delta_n)^n \\ &\leq \Delta_n \exp(n \ln(1 + C_6 \Delta_n)) \\ &\leq \underbrace{\Delta_n \exp(C_6 n \Delta_n)}_{O(\Delta_n)}. \end{aligned}$$

Hence there exists a constant $C_7 > 0$ such that for all $n \in \mathbb{N}^*$, $i \in \llbracket 0, n \rrbracket$,

$$\mathbb{E}[|\Delta S_{t_i^n}^n|^2] \leq C_7 \Delta_n$$

and for all $n \in \mathbb{N}^*$, $i \in \llbracket 0, n \rrbracket$, $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned} \mathbb{E}[|\Delta S_t^n|^2] &\leq \mathbb{E}[|\Delta S_{t_{i-1}^n}^n|^2](1 + C_2 \Delta_n) + C_2 \Delta_n^2 \\ &\leq \underbrace{C_7 \Delta_n (1 + C_2 \Delta_n) + C_2 \Delta_n^2}_{=O(\Delta_n)} \end{aligned}$$

which implies that there exists a constant $C_8 > 0$ such that for all $n \in \mathbb{N}^*$, $i \in \llbracket 1, n \rrbracket$,

$$\sup_{t \in [t_{i-1}^n, t_i^n]} \mathbb{E}[|\Delta S_t^n|^2] \leq C_8 \Delta_n.$$

This gives that for all $n \in \mathbb{N}^*$,

$$\sup_{t \in [0, T]} \mathbb{E}[|\Delta S_t^n|^2] = \max_{1 \leq i \leq n} \sup_{t \in [t_{i-1}^n, t_i^n]} \mathbb{E}[|\Delta S_t^n|^2] \leq C_8 \Delta_n.$$

□

Proposition 7. *Suppose the assumptions of Theorem 8 hold. Let $(V^n)_{n \in \mathbb{N}}$ a sequence of processes on $[0, T]$ such that for all $t \in [0, T]$,*

$$\begin{aligned} V_t^n &:= V_0 + \sum_{k=1}^n \mathbb{1}_{\{t_{k-1}^n \leq t \leq T\}} \delta_{t_{k-1}^n} (S_{t \wedge t_k^n}^n - S_{t_{k-1}^n}^n) \\ &\quad + \sum_{k=1}^n \mathbb{1}_{\{t_k^n \leq t \leq T\}} \left[\frac{1}{2} (\delta_{t_k^n} - \delta_{t_{k-1}^n})^2 \tilde{\lambda}(t_k^n, S_{t_k^n}^n) + \delta_{t_{k-1}^n} (\delta_{t_k^n} - \delta_{t_{k-1}^n}) \tilde{\lambda}(t_k^n, S_{t_{k-1}^n}^n) \right]. \end{aligned} \quad (73)$$

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Assume that $\Delta_n = O\left(\frac{1}{n}\right)$. Then there exists a constant $C > 0$ such that for all $n \in \mathbb{N}^*$,

$$\sup_{t \in [0, T]} \mathbb{E}[|V_t - V_t^n|^2] \leq C\Delta_n.$$

Proof. Let $n \in \mathbb{N}^*$, $\Delta V^n := V - V^n$ and $t \in [t_{i-1}^n, t_i^n]$ for $i \in \llbracket 1, n \rrbracket$. We have for all $t \in [t_{i-1}^n, t_i^n]$,

$$\begin{aligned} dV_t &= \delta_t dS_t + \frac{1}{2} \tilde{\lambda}(t, S_t) \alpha_t^2 dt \\ dV_t^n &= \delta_{t_{i-1}^n} dS_t^n, \end{aligned}$$

which leads to

$$\begin{aligned} d\Delta V_t^n &= dV_t - dV_t^n \\ &= (\delta_t \tilde{\sigma}(t, S_t) - \delta_{t_{i-1}^n} \tilde{\sigma}(t, S_t^n) + \delta_t \alpha_t \tilde{\lambda}(t, S_t)) dW_t \\ &\quad + \left(\delta_t \tilde{v}(t, S_t) - \delta_{t_{i-1}^n} \tilde{v}(t, S_t^n) + \frac{1}{2} \alpha_t^2 \tilde{\lambda}(t, S_t) + \delta_t \beta_t \tilde{\lambda}(t, S_t) + \delta_t \alpha_t (\tilde{\sigma} \partial_x \tilde{\lambda})(t, S_t) \right) dt. \end{aligned}$$

By Ito's lemma we have

$$d|\Delta V_t^n|^2 = 2\Delta V_t^n d\Delta V_t^n + d\langle \Delta V^n, \Delta V^n \rangle_t,$$

which gives for all $t \in [t_{i-1}^n, t_i^n]$,

$$\begin{aligned} \mathbb{E}[|\Delta V_t^n|^2] &= \mathbb{E}[|\Delta V_{t_{i-1}^n}^n|^2] \\ &\quad + 2\mathbb{E}\left[\int_{t_{i-1}^n}^t \Delta V_s^n (\delta_s \tilde{v}(s, S_s) - \delta_{t_{i-1}^n} \tilde{v}(s, S_s^n)) ds\right] \\ &\quad + 2\mathbb{E}\left[\int_{t_{i-1}^n}^t \Delta V_s^n \left(\frac{1}{2} \alpha_s^2 \tilde{\lambda}(s, S_s) + \delta_s \beta_s \tilde{\lambda}(s, S_s) + \delta_s \alpha_s (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s)\right) ds\right] \\ &\quad + \mathbb{E}\left[\int_{t_{i-1}^n}^t (\delta_s \tilde{\sigma}(s, S_s) - \delta_{t_{i-1}^n} \tilde{\sigma}(s, S_s^n) + \delta_s \alpha_s \tilde{\lambda}(s, S_s))^2 ds\right] \\ &\leq \mathbb{E}[|\Delta V_{t_{i-1}^n}^n|^2] \\ &\quad + \mathbb{E}\left[\int_{t_{i-1}^n}^t (|\Delta V_s^n|^2 + |\delta_s \tilde{v}(s, S_s) - \delta_{t_{i-1}^n} \tilde{v}(s, S_s^n)|^2) ds\right] \\ &\quad + \mathbb{E}\left[\int_{t_{i-1}^n}^t \left(|\Delta V_s^n|^2 + \left|\frac{1}{2} \alpha_s^2 \tilde{\lambda}(s, S_s) + \delta_s \beta_s \tilde{\lambda}(s, S_s) + \delta_s \alpha_s (\tilde{\sigma} \partial_x \tilde{\lambda})(t, S_s)\right|^2\right) ds\right] \\ &\quad + 2\mathbb{E}\left[\int_{t_{i-1}^n}^t (|\delta_s \tilde{\sigma}(s, S_s) - \delta_{t_{i-1}^n} \tilde{\sigma}(s, S_s^n)|^2 + |\delta_s \alpha_s f(s, S_s)|^2) ds\right]. \end{aligned}$$

Besides for all function $h : (t, x) \in \mathbb{R}^2 \mapsto \mathbb{R}$ bounded and Lipschitz in the space variable we have for all $t \in [t_{i-1}^n, t_i^n]$,

$$\begin{aligned} \mathbb{E}[|\delta_s h(s, S_s) - \delta_{t_{i-1}^n} h(s, S_s^n)|^2] &\leq 2\mathbb{E}[|h(s, S_s^n)|^2 |\delta_s - \delta_{t_{i-1}^n}|^2] + 2\mathbb{E}[|\delta_{t_{i-1}^n}|^2 |h(s, S_s) - h(s, S_s^n)|^2] \\ &\leq 2\mathbb{E}[|h(s, S_s^n)|^2 |\delta_s - \delta_{t_{i-1}^n}|^2] + 2\sqrt{\mathbb{E}[|\delta_{t_{i-1}^n}|^4]} \sqrt{\mathbb{E}[|h(s, S_s) - h(s, S_s^n)|^4]} \end{aligned}$$

which implies that there exists a constant $C_h > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\delta_s h(s, S_s) - \delta_{t_{i-1}^n} h(s, S_s^n)|^2] \leq C_h (\mathbb{E}[|S_s - S_s^n|^2] + \mathbb{E}[|\delta_s - \delta_s^n|^2]). \quad (74)$$

Hence by (74) there exists a constant $C_1 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned} \mathbb{E}[|\Delta V_t^n|^2] &\leq \mathbb{E}[|\Delta V_{t_{i-1}^n}^n|^2] \\ &\quad + C_1 \int_{t_{i-1}^n}^t (1 + \mathbb{E}[|\Delta V_s^n|^2] + \mathbb{E}[|S_s - S_s^n|^2] + \mathbb{E}[|\delta_s - \delta_s^n|^2]) ds. \end{aligned} \quad (75)$$

By plugging the Propositions 5 and 6 in the inequality (75), we obtain that there exists a constant $C_2 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\Delta V_t^n|^2] \leq \mathbb{E}[|\Delta V_{t_{i-1}^n}^n|^2] + C_2 \Delta_n + C_2 \int_{t_{i-1}^n}^t \mathbb{E}[|\Delta V_s^n|^2] ds. \quad (76)$$

By applying Gronwall's lemma, similarly as the inequality (66) leads to the inequality (68), the inequality (76) implies also that there exists a constant $C_3 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\Delta V_t^n|^2] \leq \mathbb{E}[|\Delta V_{t_{i-1}^n}^n|^2] (1 + C_3 \Delta_n) + C_3 \Delta_n. \quad (77)$$

We have for all $t \in [t_{i-1}^n, t_i^n)$,

$$d(|\delta_t - \delta_{t_{i-1}^n}|^2 \tilde{\lambda}(t, S_t^n)) = |\delta_t - \delta_{t_{i-1}^n}|^2 d(\tilde{\lambda}(t, S_t^n)) + \tilde{\lambda}(t, S_t^n) d(|\delta_t - \delta_{t_{i-1}^n}|^2) + d(|\delta_t - \delta_{t_{i-1}^n}|^2, \tilde{\lambda}(\cdot, S^n))_t$$

with

$$d\tilde{\lambda}(t, S_t^n) = (\bar{\sigma} \partial_x \tilde{\lambda})(t, S_t^n) dW_t + \left(\partial_t \tilde{\lambda} + \tilde{\nu} \partial_x \tilde{\lambda} + \frac{1}{2} \bar{\sigma}^2 \partial_{xx} \tilde{\lambda} \right)(t, S_t^n) dt,$$

and

$$d|\delta_t - \delta_{t_{i-1}^n}|^2 = 2(\delta_t - \delta_{t_{i-1}^n}) \alpha_t dW_t + (2(\delta_t - \delta_{t_{i-1}^n}) \beta_t + \alpha_t^2) dt.$$

Therefore for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned} d(|\delta_t - \delta_{t_{i-1}^n}|^2 \tilde{\lambda}(t, S_t^n)) &= |\delta_t - \delta_{t_{i-1}^n}|^2 (\bar{\sigma} \partial_x \tilde{\lambda})(t, S_t^n) dW_t \\ &\quad + 2(\delta_t - \delta_{t_{i-1}^n}) \alpha_t \tilde{\lambda}(t, S_t^n) dW_t \\ &\quad + |\delta_t - \delta_{t_{i-1}^n}|^2 \left(\partial_t \tilde{\lambda} + \tilde{\nu} \partial_x \tilde{\lambda} + \frac{1}{2} \bar{\sigma}^2 \partial_{xx} \tilde{\lambda} \right)(t, S_t^n) dt \\ &\quad + (2(\delta_t - \delta_{t_{i-1}^n}) \beta_t + \alpha_t^2) \tilde{\lambda}(t, S_t^n) dt \\ &\quad + 2(\delta_t - \delta_{t_{i-1}^n}) \alpha_t (\bar{\sigma} \partial_x \tilde{\lambda})(t, S_t^n) dt, \end{aligned}$$

which implies that for all $t \in [t_{i-1}^n, t_i^n)$,

$$|\delta_t - \delta_{t_{i-1}^n}|^2 \tilde{\lambda}(t, S_t^n) = \mathcal{I}_t^{i,n},$$

by setting for all $t \in [t_{i-1}^n, t_i^n]$,

$$\begin{aligned}
 \mathcal{S}_t^{i,n} &:= \int_{t_{i-1}^n}^t |\delta_s - \delta_{t_{i-1}^n}|^2 (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s^n) dW_s \\
 &+ \int_{t_{i-1}^n}^t 2(\delta_s - \delta_{t_{i-1}^n}) \alpha_s \tilde{\lambda}(s, S_s^n) dW_s \\
 &+ \int_{t_{i-1}^n}^t |\delta_s - \delta_{t_{i-1}^n}|^2 \left(\partial_t \tilde{\lambda} + \tilde{\nu} \partial_x \tilde{\lambda} + \frac{1}{2} \tilde{\sigma}^2 \partial_{xx} \tilde{\lambda} \right) (s, S_s^n) ds \\
 &+ \int_{t_{i-1}^n}^t (2(\delta_s - \delta_{t_{i-1}^n}) \beta_s + \alpha_s^2) \tilde{\lambda}(s, S_s^n) ds \\
 &+ \int_{t_{i-1}^n}^t 2(\delta_s - \delta_{t_{i-1}^n}) \alpha_s (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s^n) ds.
 \end{aligned}$$

We set for all $t \in [t_{i-1}^n, t_i^n]$, $\tilde{V}_t^n := V_t^n + \delta_{t_{i-1}^n} (\delta_t - \delta_{t_{i-1}^n}) \tilde{\lambda}(t, S_t^n) + \frac{1}{2} (\delta_t - \delta_{t_{i-1}^n})^2 \tilde{\lambda}(t, S_t^n)$. Hence

$$\lim_{t \rightarrow (t_i^n)^-} \tilde{V}_t^n = V_{t_i^n}^n + \delta_{t_{i-1}^n} (\delta_{t_i^n} - \delta_{t_{i-1}^n}) \tilde{\lambda}(t_i^n, S_{t_i^n}^n) + \frac{1}{2} (\delta_{t_i^n} - \delta_{t_{i-1}^n})^2 \tilde{\lambda}(t_i^n, S_{t_i^n}^n) = V_{t_i^n}^n.$$

Let us consider now for all $t \in [t_{i-1}^n, t_i^n]$, $\Delta \tilde{V}_t^n := V_t - \tilde{V}_t^n = \Delta V_t^n - \delta_{t_{i-1}^n} \mathcal{R}_t^{i,n} - \frac{1}{2} \mathcal{S}_t^{i,n}$, where $\mathcal{R}_t^{i,n}$ is defined, as previously, such that for all $t \in [t_{i-1}^n, t_i^n]$,

$$\begin{aligned}
 \mathcal{R}_t^{i,n} &:= \int_{t_{i-1}^n}^t ((\delta_s - \delta_{t_{i-1}^n}) (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s^n) + \alpha_s \tilde{\lambda}(s, S_s^n)) dW_s \\
 &+ \int_{t_{i-1}^n}^t (\delta_s - \delta_{t_{i-1}^n}) \left(\partial_t \tilde{\lambda} + \tilde{\nu} \partial_x \tilde{\lambda} + \frac{1}{2} \tilde{\sigma}^2 \partial_{xx} \tilde{\lambda} \right) (s, S_s^n) ds \\
 &+ \int_{t_{i-1}^n}^t (\beta_s \tilde{\lambda}(s, S_s^n) + \alpha_s (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s^n)) ds.
 \end{aligned}$$

We have by Ito's lemma

$$d|\Delta \tilde{V}_t^n|^2 = 2\Delta \tilde{V}_t^n d\Delta \tilde{V}_t^n + d\langle \Delta \tilde{V}^n, \Delta \tilde{V}^n \rangle_t,$$

hence for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned}
 d|\Delta \tilde{V}_t^n|^2 &= 2\Delta \tilde{V}_t^n (\delta_t \tilde{\sigma}(t, S_t) - \delta_{t_{i-1}^n} \tilde{\sigma}(t, S_t^n) + \delta_t \alpha_t \tilde{\lambda}(t, S_t)) dW_t \\
 &\quad + 2\Delta \tilde{V}_t^n (\delta_t \tilde{v}(t, S_t) - \delta_{t_{i-1}^n} \tilde{v}(t, S_t^n) + \frac{1}{2} \alpha_t^2 \tilde{\lambda}(t, S_t) + \delta_t \beta_t \tilde{\lambda}(t, S_t) + \delta_t \alpha_t (\tilde{\sigma} \partial_x \tilde{\lambda})(t, S_t)) dt \\
 &\quad - 2\Delta \tilde{V}_t^n \delta_{t_{i-1}^n} ((\delta_t - \delta_{t_{i-1}^n}) (\tilde{\sigma} \partial_x \tilde{\lambda})(t, S_t^n) + \alpha_t \tilde{\lambda}(t, S_t^n)) dW_t \\
 &\quad - 2\Delta \tilde{V}_t^n \delta_{t_{i-1}^n} (\delta_t - \delta_{t_{i-1}^n}) \left(\partial_t \tilde{\lambda} + \tilde{v} \partial_x \tilde{\lambda} + \frac{1}{2} \tilde{\sigma}^2 \partial_{xx} \tilde{\lambda} \right) (t, S_t^n) dt \\
 &\quad - 2\Delta \tilde{V}_t^n \delta_{t_{i-1}^n} (\beta_t \tilde{\lambda}(t, S_t^n) + \alpha_t (\tilde{\sigma} \partial_x \tilde{\lambda})(t, S_t^n)) dt \\
 &\quad - 2\Delta \tilde{V}_t^n \frac{1}{2} |\delta_t - \delta_{t_{i-1}^n}|^2 (\tilde{\sigma} \partial_x \tilde{\lambda})(t, S_t^n) dW_t \\
 &\quad - 2\Delta \tilde{V}_t^n (\delta_t - \delta_{t_{i-1}^n}) \alpha_t \tilde{\lambda}(t, S_t^n) dW_t \\
 &\quad - 2\Delta \tilde{V}_t^n \frac{1}{2} |\delta_t - \delta_{t_{i-1}^n}|^2 \left(\partial_t \tilde{\lambda} + \tilde{v} \partial_x \tilde{\lambda} + \frac{1}{2} \tilde{\sigma}^2 \partial_{xx} \tilde{\lambda} \right) (t, S_t^n) dt \\
 &\quad - 2\Delta \tilde{V}_t^n \frac{1}{2} (2(\delta_t - \delta_{t_{i-1}^n}) \beta_t + \alpha_t^2) \tilde{\lambda}(t, S_t^n) dt \\
 &\quad - 2\Delta \tilde{V}_t^n (\delta_t - \delta_{t_{i-1}^n}) \alpha_t (\tilde{\sigma} \partial_x \tilde{\lambda})(t, S_t^n) dt \\
 &\quad + |(\delta_t \tilde{\sigma}(t, S_t) - \delta_{t_{i-1}^n} \tilde{\sigma}(t, S_t^n) + \delta_t \alpha_t \tilde{\lambda}(t, S_t)) \\
 &\quad \quad - \delta_{t_{i-1}^n} ((\delta_t - \delta_{t_{i-1}^n}) (\tilde{\sigma} \partial_x \tilde{\lambda})(t, S_t^n) + \alpha_t \tilde{\lambda}(t, S_t^n)) \\
 &\quad \quad - \frac{1}{2} |\delta_t - \delta_{t_{i-1}^n}|^2 (\tilde{\sigma} \partial_x \tilde{\lambda})(t, S_t^n) \\
 &\quad \quad - (\delta_t - \delta_{t_{i-1}^n}) \alpha_t \tilde{\lambda}(t, S_t^n)|^2 dt.
 \end{aligned}$$

Therefore, for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned}
 \mathbb{E}[|\Delta \tilde{V}_t^n|^2] &= \mathbb{E}[|\Delta V_{t_{i-1}}^n|^2] \\
 &+ 2\mathbb{E}\left[\int_{t_{i-1}^n}^t \Delta \tilde{V}_s^n (\delta_s \tilde{v}(s, S_s) - \delta_{t_{i-1}}^n \tilde{v}(s, S_s^n)) ds\right] \\
 &+ 2\mathbb{E}\left[\int_{t_{i-1}^n}^t \Delta \tilde{V}_s^n \frac{1}{2} \alpha_s^2 (\tilde{\lambda}(s, S_s) - \tilde{\lambda}(s, S_s^n)) ds\right] \\
 &+ 2\mathbb{E}\left[\int_{t_{i-1}^n}^t \Delta \tilde{V}_s^n \beta_s (\delta_s \tilde{\lambda}(s, S_s) - \delta_{t_{i-1}}^n \tilde{\lambda}(s, S_s^n)) ds\right] \\
 &+ 2\mathbb{E}\left[\int_{t_{i-1}^n}^t \Delta \tilde{V}_s^n \alpha_s (\delta_s (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s) - \delta_{t_{i-1}}^n (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s^n)) ds\right] \\
 &- 2\mathbb{E}\left[\int_{t_{i-1}^n}^t \Delta \tilde{V}_s^n \delta_{t_{i-1}}^n (\delta_s - \delta_{t_{i-1}}^n) \left(\partial_t \tilde{\lambda} + \tilde{v} \partial_x \tilde{\lambda} + \frac{1}{2} \tilde{\sigma}^2 \partial_{xx} \tilde{\lambda}\right)(s, S_s^n) ds\right] \\
 &- 2\mathbb{E}\left[\int_{t_{i-1}^n}^t \Delta \tilde{V}_s^n \frac{1}{2} |\delta_s - \delta_{t_{i-1}}^n|^2 \left(\partial_t \tilde{\lambda} + \tilde{v} \partial_x \tilde{\lambda} + \frac{1}{2} \tilde{\sigma}^2 \partial_{xx} \tilde{\lambda}\right)(s, S_s^n) ds\right] \\
 &- 2\mathbb{E}\left[\int_{t_{i-1}^n}^t \Delta \tilde{V}_s^n (\delta_s - \delta_{t_{i-1}}^n) (\beta_s \tilde{\lambda}(s, S_s^n) + \alpha_s (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s^n)) ds\right] \\
 &+ \mathbb{E}\left[\int_{t_{i-1}^n}^t |\gamma_s^{i,n}|^2 ds\right],
 \end{aligned}$$

with $\gamma^{i,n}$ defined such as for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned}
 \gamma_t^{i,n} &:= \delta_t \tilde{\sigma}(t, S_t) - \delta_{t_{i-1}}^n \tilde{\sigma}(t, S_t^n) \\
 &+ \alpha_t (\delta_t \tilde{\lambda}(t, S_t) - \delta_{t_{i-1}}^n \tilde{\lambda}(t, S_t^n)) \\
 &- (\delta_t - \delta_{t_{i-1}}^n) (\delta_{t_{i-1}}^n (\tilde{\sigma} \partial_x \tilde{\lambda})(t, S_t^n) + \frac{1}{2} (\delta_t - \delta_{t_{i-1}}^n) (\tilde{\sigma} \partial_x \tilde{\lambda})(t, S_t^n) + \alpha_t \tilde{\lambda}(t, S_t^n)).
 \end{aligned}$$

Hence, this gives that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned}
 \mathbb{E}[|\Delta \tilde{V}_t^n|^2] &\leq \mathbb{E}[|\Delta V_{t_{i-1}}^n|^2] \\
 &+ \mathbb{E} \left[\int_{t_{i-1}^n}^t (|\Delta \tilde{V}_s^n|^2 + |\delta_s \tilde{v}(s, S_s) - \delta_{t_{i-1}^n} \tilde{v}(s, S_s^n)|^2) ds \right] \\
 &+ \mathbb{E} \left[\int_{t_{i-1}^n}^t (|\Delta \tilde{V}_s^n|^2 + \left| \frac{1}{2} \alpha_s^2 \right|^2 |\tilde{\lambda}(s, S_s) - \tilde{\lambda}(s, S_s^n)|^2) ds \right] \\
 &+ \mathbb{E} \left[\int_{t_{i-1}^n}^t (|\Delta \tilde{V}_s^n|^2 + |\beta_s|^2 |\delta_s \tilde{\lambda}(s, S_s) - \delta_{t_{i-1}^n} \tilde{\lambda}(s, S_s^n)|^2) ds \right] \\
 &+ \mathbb{E} \left[\int_{t_{i-1}^n}^t (|\Delta \tilde{V}_s^n|^2 + |\alpha_s|^2 |\delta_s (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s) - \delta_{t_{i-1}^n} (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s^n)|^2) ds \right] \\
 &+ \mathbb{E} \left[\int_{t_{i-1}^n}^t \left(|\Delta \tilde{V}_s^n|^2 + \left| \left(\partial_t \tilde{\lambda} + \tilde{v} \partial_x \tilde{\lambda} + \frac{1}{2} \tilde{\sigma}^2 \partial_{xx} \tilde{\lambda} \right) (s, S_s^n) \right|^2 |\delta_{t_{i-1}^n} (\delta_s - \delta_{t_{i-1}^n})|^2 \right) ds \right] \\
 &+ \mathbb{E} \left[\int_{t_{i-1}^n}^t \left(|\Delta \tilde{V}_s^n|^2 + \left| \frac{1}{2} \left(\partial_t \tilde{\lambda} + \tilde{v} \partial_x \tilde{\lambda} + \frac{1}{2} \tilde{\sigma}^2 \partial_{xx} \tilde{\lambda} \right) (s, S_s^n) \right|^2 |\delta_s - \delta_{t_{i-1}^n}|^4 \right) ds \right] \\
 &+ \mathbb{E} \left[\int_{t_{i-1}^n}^t \left(|\Delta \tilde{V}_s^n|^2 + |(\beta_s \tilde{\lambda}(s, S_s^n) + \alpha_s (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s^n))|^2 |\delta_s - \delta_{t_{i-1}^n}|^2 \right) ds \right] \\
 &+ 5\mathbb{E} \left[\int_{t_{i-1}^n}^t |\delta_s \tilde{\sigma}(s, S_s) - \delta_{t_{i-1}^n} \tilde{\sigma}(s, S_s^n)|^2 ds \right] \\
 &+ 5\mathbb{E} \left[\int_{t_{i-1}^n}^t |\alpha_s|^2 |\delta_s \tilde{\lambda}(s, S_s) - \delta_{t_{i-1}^n} \tilde{\lambda}(s, S_s^n)|^2 ds \right] \\
 &+ 5\mathbb{E} \left[\int_{t_{i-1}^n}^t |(\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s^n)|^2 |\delta_{t_{i-1}^n} (\delta_s - \delta_{t_{i-1}^n})|^2 ds \right] \\
 &+ 5\mathbb{E} \left[\int_{t_{i-1}^n}^t \left| \frac{1}{2} (\tilde{\sigma} \partial_x \tilde{\lambda})(s, S_s^n) \right|^2 |\delta_s - \delta_{t_{i-1}^n}|^4 ds \right] \\
 &+ 5\mathbb{E} \left[\int_{t_{i-1}^n}^t |\alpha_s \tilde{\lambda}(s, S_s^n)|^2 |\delta_s - \delta_{t_{i-1}^n}|^2 ds \right].
 \end{aligned}$$

which implies that there exists a constant $C_4 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned}
 \mathbb{E}[|\Delta \tilde{V}_t^n|^2] &\leq \mathbb{E}[|\Delta V_{t_{i-1}}^n|^2] \\
 &\quad + C_4 \int_{t_{i-1}}^t (\mathbb{E}[|\Delta \tilde{V}_s^n|^2] + \mathbb{E}[|S_s - S_s^n|^2] + \mathbb{E}[|\delta_s - \delta_s^n|^2]) ds \\
 &\quad + C_4 \int_{t_{i-1}}^t (\mathbb{E}[|\delta_{t_{i-1}}^n (\delta_s - \delta_{t_{i-1}}^n)|^2] + \mathbb{E}[|\delta_s - \delta_{t_{i-1}}^n|^4]) ds \\
 &\leq \mathbb{E}[|\Delta V_{t_{i-1}}^n|^2] \\
 &\quad + C_4 \int_{t_{i-1}}^t (\mathbb{E}[|\Delta \tilde{V}_s^n|^2] + \mathbb{E}[|S_s - S_s^n|^2] + \mathbb{E}[|\delta_s - \delta_s^n|^2]) ds \\
 &\quad + C_4 \int_{t_{i-1}}^t \left(\sqrt{\mathbb{E}[|\delta_{t_{i-1}}^n|^4]} \sqrt{\mathbb{E}[|\delta_s - \delta_{t_{i-1}}^n|^4]} + \mathbb{E}[|\delta_s - \delta_{t_{i-1}}^n|^4] \right) ds. \tag{78}
 \end{aligned}$$

Besides by the generalized Minkowski inequality and the Burkholder-Davis-Gundy inequality (see for instance Section 7.8.1 in [Pagès, 2018]) we have that there exists $K > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\|\delta_t - \delta_{t_{i-1}}^n\|_4 \leq K\sqrt{\Delta_n},$$

hence

$$\sqrt{\mathbb{E}[|\delta_s - \delta_{t_{i-1}}^n|^4]} = \|\delta_t - \delta_{t_{i-1}}^n\|_4^2 \leq K^2 \Delta_n. \tag{79}$$

By plugging (74), (79), Propositions 5 and 6 in the inequality (78), we obtain that there exists a constant $C_5 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\Delta \tilde{V}_t^n|^2] \leq \mathbb{E}[|\Delta V_{t_{i-1}}^n|^2] + C_5 \Delta_n^2 + C_5 \int_{t_{i-1}}^t \mathbb{E}[|\Delta \tilde{V}_s^n|^2] ds. \tag{80}$$

Hence by Gronwall's lemma we have that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned}
 \mathbb{E}[|\Delta \tilde{V}_t^n|^2] &\leq \exp(C_5(t - t_{i-1}^n)) \left(\mathbb{E}[|\Delta V_{t_{i-1}}^n|^2] + C_5 \Delta_n^2 \right) \\
 &\leq \exp(C_5 \Delta_n) \left(\mathbb{E}[|\Delta V_{t_{i-1}}^n|^2] + C_5 \Delta_n^2 \right), \tag{81}
 \end{aligned}$$

wich leads by plugging the inequality (81) in (80) to

$$\begin{aligned}
 \mathbb{E}[|\Delta \tilde{V}_t^n|^2] &\leq \mathbb{E}[|\Delta V_{t_{i-1}}^n|^2] + C_5 \Delta_n^2 + C_5 \Delta_n \exp(C_5 \Delta_n) \left(\mathbb{E}[|\Delta V_{t_{i-1}}^n|^2] + C_5 \Delta_n^2 \right) \\
 &\leq \mathbb{E}[|\Delta V_{t_{i-1}}^n|^2] \left(1 + \underbrace{C_5 \Delta_n \exp(C_5 \Delta_n)}_{=O(\Delta_n)} \right) + \underbrace{C_5 \Delta_n^2 + C_5^2 \Delta_n^3 \exp(C_5 \Delta_n)}_{=O(\Delta_n^2)}
 \end{aligned}$$

for any $t \in [t_{i-1}^n, t_i^n)$. Therefore there exists a constant $C_6 > 0$ such that for all $t \in [t_{i-1}^n, t_i^n)$,

$$\mathbb{E}[|\Delta \tilde{V}_t^n|^2] \leq \mathbb{E}[|\Delta V_{t_{i-1}}^n|^2] (1 + C_6 \Delta_n) + C_6 \Delta_n^2. \tag{82}$$

Let $(\tau_k^{i,n})_{k \in \mathbb{N}}$ a non-decreasing sequence such that $\lim_{k \rightarrow +\infty} \tau_k^{i,n} = t_i^n$, by Fatou's lemma we have

$$\mathbb{E} \left[\liminf_{k \rightarrow +\infty} |\Delta \tilde{V}_{\tau_k^{i,n}}^n|^2 \right] \leq \liminf_{k \rightarrow +\infty} \mathbb{E} [|\Delta \tilde{V}_{\tau_k^{i,n}}^n|^2] \leq \mathbb{E} [|\Delta V_{t_{i-1}}^n|^2] (1 + C_6 \Delta_n) + C_6 \Delta_n^2,$$

and

$$\mathbb{E} \left[\liminf_{k \rightarrow +\infty} |\Delta \tilde{V}_{\tau_k^{i,n}}^n|^2 \right] = \mathbb{E} \left[\lim_{k \rightarrow +\infty} |\Delta \tilde{V}_{\tau_k^{i,n}}^n|^2 \right] = \mathbb{E} [|\Delta V_{t_i}^n|^2]$$

since $\lim_{t \rightarrow (t_i^n)^-} \Delta \tilde{V}_t^n = \Delta V_{t_i}^n$. Therefore we have for all $n \in \mathbb{N}^*$, $i \in \llbracket 1, n \rrbracket$,

$$\begin{aligned} \mathbb{E} [|\Delta V_{t_i}^n|^2] &\leq \mathbb{E} [|\Delta V_{t_{i-1}}^n|^2] (1 + C_6 \Delta_n) + C_6 \Delta_n^2 \\ &\leq \mathbb{E} [|\Delta V_{t_{i-2}}^n|^2] (1 + C_6 \Delta_n)^2 + C_6 \Delta_n^2 (1 + C_6 \Delta_n) + C_6 \Delta_n^2 \\ &\dots \\ &\leq \mathbb{E} [|\Delta V_{t_0}^n|^2] (1 + C_6 \Delta_n)^i + C_6 \Delta_n^2 \sum_{k=0}^{i-1} (1 + C_6 \Delta_n)^k. \end{aligned}$$

Considering the fact that $\Delta V_{t_0}^n = 0$, we have for all $n \in \mathbb{N}^*$, $i \in \llbracket 1, n \rrbracket$,

$$\begin{aligned} \mathbb{E} [|\Delta V_{t_i}^n|^2] &\leq C_6 \Delta_n^2 \frac{(1 + C_6 \Delta_n)^i - 1}{C_6 \Delta_n} \\ &\leq \Delta_n (1 + C_6 \Delta_n)^n \\ &\leq \Delta_n \exp(n \ln(1 + C_6 \Delta_n)) \\ &\leq \underbrace{\Delta_n \exp(C_6 n \Delta_n)}_{O(\Delta_n)}. \end{aligned}$$

Hence there exists a constant $C_7 > 0$ such that for all $n \in \mathbb{N}^*$, $i \in \llbracket 0, n \rrbracket$,

$$\mathbb{E} [|\Delta V_{t_i}^n|^2] \leq C_7 \Delta_n$$

and for all $n \in \mathbb{N}^*$, $i \in \llbracket 0, n \rrbracket$, $t \in [t_{i-1}^n, t_i^n)$,

$$\begin{aligned} \mathbb{E} [|\Delta V_t^n|^2] &\leq \mathbb{E} [|\Delta V_{t_{i-1}}^n|^2] (1 + C_3 \Delta_n) + C_3 \Delta_n \\ &\leq \underbrace{C_7 \Delta_n (1 + C_3 \Delta_n) + C_3 \Delta_n}_{=O(\Delta_n)} \end{aligned}$$

which implies that there exists a constant $C_8 > 0$ such that for all $n \in \mathbb{N}^*$, $i \in \llbracket 1, n \rrbracket$,

$$\sup_{t \in [t_{i-1}^n, t_i^n]} \mathbb{E} [|\Delta V_t^n|^2] \leq C_8 \Delta_n.$$

This gives that for all $n \in \mathbb{N}^*$,

$$\sup_{t \in (0, T]} \mathbb{E} [|\Delta V_t^n|^2] = \max_{1 \leq i \leq n} \sup_{t \in [t_{i-1}^n, t_i^n]} \mathbb{E} [|\Delta V_t^n|^2] \leq C_8 \Delta_n.$$

□

Part IV

Information and Market Impact

The Structure of Information in Market Impact

Abstract

We present a model that explores the role of information on market impact and are particularly interested in the relaxation of metaorders. We formulate a theory which underlines the role of information on the price reversion process of metaorders and show that the relaxation of a metaorder results from interactions between the information conveyed by metaorders and the market. As a consequence the *permanent impact* of metaorders appears to be the result of this informational process.

Keywords: *Market microstructure, market efficiency, metaorders relaxation, metaorders information, permanent market impact.*

1 Introduction

It is commonly accepted that although market information can come through many channels, it has been shown since [Grossman and Stiglitz, 1980] [Grossman et al., 1989] that market impact is the main mechanism whereby information is conveyed to the market through trade execution. In recent years many studies on different markets have been conducted to understand the influence of metaorders on the price formation process: [Almgren et al., 2005] [Moro et al., 2009] [Toth et al., 2011] [Bershova and Rakhlin, 2013] [Bacry et al., 2015] [Gomes and Waelbroeck, 2015] [Said et al., 2018] for stock market, [Tóth et al., 2016] [Said et al., 2019] for options market and [Donier and Bonart, 2015] for the bitcoin market. All of these studies agree that market impact is a two-phase process. A first characterized by a *temporary* market impact – concave and increasing –, followed by a relaxation – convex and decreasing –, and giving rise to what is called the *permanent* market impact.

This second type of market impact is more controversial and research papers dealing with permanent market impact can be separated in two categories. On one hand the permanent market impact can be seen as the consequence of a pure mechanical process. On the other hand the permanent market impact is considered to be a trace of new information in the

price. In the pure mechanical vision prices move because of the activity of all the market participants. So if the buy pressure takes advantage on the sell pressure the price go up, and if selling pressure is stronger than buying pressure the price go down. This is the econophysicist point of view which main goal is to determine the behavior of these two forces and how they generate impact on prices' dynamic. The second school of thought is the economist point of view: The informational vision says prices move because new information is made available to investors who update their expectations. As a consequence the market participants change their offer and demand which gives birth to a global new equilibrium resulting in new prices levels. In this picture, as emphasized in [Hasbrouck, 2007], orders do not impact prices and it is more accurate to say that orders forecast prices. Among those whose share the meachanical vision of the permanent impact there are also two pictures. On one side, there is the framework proposed by [Bouchaud, 2010] where there is no such thing as permanent impact but only the long memory of the sign of the metaorder flow. On the other side, the picture of [Farmer et al., 2013] states that permanent impact can be important and roughly equals to 2/3 of the peak impact. This is the fair pricing hypothesis. A range of papers have analyzed all sorts of metaorder databases reaching conclusions in favour of one position or the other as summarized in Table IV.1. In the present paper we are not going to make a stand for one or the other vision as we embrace the thoughts expressed in [Bacry et al., 2015] that market impact must be a mechanical - informational dual process.

If it is widely recognized that temporary impact has a concave dependence on size [Almgren et al., 2005] [Engle et al., 2012] [Bacry et al., 2015], the functional form of permanent impact is harder to measure and remains an open question. Especially differences arise in the price reversion following the end of a metaorder. The existing empirical literature of decay metaorders market impact is limited [Moro et al., 2009] [Bershova and Rakhlin, 2013] [Gomes and Waelbroeck, 2015] [Said et al., 2018] [Bucci et al., 2019b] due to the difficulty of obtaining data.

Empirical study	Permanent impact / Temporary impact
[Moro et al., 2009]	0.5 ~ 0.7 (single day metaorders)
[Bershova and Rakhlin, 2013]	~ 2/3 (single day metaorders)
[Gomes and Waelbroeck, 2015]	~ 2/3 (informed) - ~ 0 (uninformed) after 10 days
[Said et al., 2018]	~ 2/3 (single day metaorders)
[Bucci et al., 2019b]	~ 2/3 at the end of the same day - ~ 1/3 after 50 days

Table IV.1 - *Decay of the impact obtained in some empirical studies*

The traditional view in finance is that market impact is just a reflection of information and postulates that the functional form of market impact is the expression of how informed the agents are who trade with a given volume. As information is difficult to define and measure the metaorder size and duration have been used as explanatory variables for the temporary market impact. If it seems reasonable to assume that the characteristics of metaorders can determine the shape of the temporary impact, they cannot explain the permanent impact: Once executed, the information reflected in the metaorder is subject to market noise. So the

permanent market impact must be the result of this interference. By market noise we mean any event after the last fill not related to the metaorder. The main motivation of this paper is to understand the mechanisms that shape the permanent impact. To this end we propose a toy model where we investigate the role played by market noise on the information revealed by the metaorders. Our main motivation is theoretical: Understanding and modelling the mechanisms of market impact is a question of interest in market microstructure as market impact reflects the shape of excess demand.

2 Model Description

2.1 Framework

As market impact is just a reflection of information we will postulate that the decay of the impact is just the response of the market to the information conveyed through metaorders trade execution. The existence of a permanent market impact – in contrast of a temporary impact – means that once the relaxation of the metaorder is over there is only a fraction of this information absorbed by the market. We will denote the fraction of the metaorder information absorbed by the market by \mathcal{R} . Note that if the market is efficient as expressed by [Fama, 1970], which means that security prices at any time fully reflect all available information, then \mathcal{R} must be equal to the ratio between the temporary impact and the permanent market impact. The value of the process \mathcal{R} is updated every time a new information is made available to the market participants. Regarding the metaorder this new information can be referred as market noise. Hence the information carried by the metaorder competes with other information. Let t_0 be the time corresponding to the end of the metaorder and t_n the arrival time of the n -th information. Hence the start value of \mathcal{R} is

$$\mathcal{R}_0 = 1 \quad \mathbb{P} - a.s.. \quad (1)$$

Besides we will make the assumption that the process \mathcal{R} after the $(n+1)$ -th event is given by

$$\mathcal{R}_{n+1} = \underbrace{\frac{1}{n+2}\mathcal{R}_0}_{\text{lost of information}} + \underbrace{\left(1 - \frac{1}{n+2}\right)\mathcal{R}_n\Delta_{n+1}}_{\text{perturbated multiplicative noise}} \quad \mathbb{P} - a.s., \quad (2)$$

for any $n \in \mathbb{N}$ and where for all $n \geq 1$, Δ_n is a random variable valued in $[0, 1]$ representing the effect of the n -th noise on the process \mathcal{R} such that $\mathbb{P}(\Delta_n = 0) < 1$. This ensures that for any $n \in \mathbb{N}^*$, $\mathbb{E}[\Delta_n] > 0$. We will also consider that the perturbations $(\Delta_n)_{n \in \mathbb{N}^*}$ are independant. The dynamics of \mathcal{R} described in Equation (2) can be seen as a slightly more complicated version of a process with multiplicative noise as we have for n large enough $\mathcal{R}_{n+1} \approx \mathcal{R}_n\Delta_{n+1}$. Besides as we want to take into account that the full information revealed at the end of the metaorder competes with the perturbations Δ_n the term \mathcal{R}_0 appears with a decay prefactor in the first term of the right hand side of (2). Note that for any $n \in \mathbb{N}$, \mathcal{R}_{n+1} can be affected positively or negatively by the noise Δ_{n+1} . For example one can imagine cross-impact effects [Benzaquen et al., 2017].

2.2 Averaging

As we are interested in computing statistical averages let us introduce Ω the set of the price trajectories of a single stock after the full execution of a metaorder. Hence in what follows we will consider $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in \mathbb{N}}, \mathbb{P})$ a filtered probability space with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and for all $n \in \mathbb{N}^*$, $\mathcal{F}_n = \sigma(\Delta_k, 1 \leq k \leq n)$. We set for all $n \in \mathbb{N}$,

$$r_n := \mathbb{E}[\mathcal{R}_n] \quad (3)$$

and

$$\delta_n := \mathbb{E}[\Delta_n]. \quad (4)$$

Hence the system (1)-(2) gives

$$\begin{cases} r_0 = 1, \\ \forall n \in \mathbb{N}, r_{n+1} = \frac{1 + (n+1)r_n\delta_{n+1}}{n+2}. \end{cases} \quad (5)$$

In what follows the sequence $(r_n)_{n \in \mathbb{N}}$ will be the quantity of interest.

3 Market equilibrium, permanent impact and metaorder information

One of the main goal of this paper is to shed some light on the relaxation of the metaorders which can be seen as the result of a competition between the information conveyed by the metaorders and market noise. This continues until market equilibrium is reached. Hence we have the following definitions:

Definition 21. (i) *Market equilibrium is reached if, and only if the sequence $(r_n)_{n \in \mathbb{N}}$ as defined in (5) converges.*

(ii) *Permanent market impact exists when market equilibrium is reached and $\lim_{n \rightarrow +\infty} r_n > 0$.*

Definition 21 precises the conditions of the existence of permanent market impact. As a consequence when $\lim_{n \rightarrow +\infty} r_n = 0$ we will say that there is no permanent market impact. The following result gives the asymptotic structure of the market in case a state equilibrium is not reached. More precisely we have

Proposition 9. *The set \mathcal{L} of the limit points of the sequence $(r_n)_{n \in \mathbb{N}}$ is given by*

$$\mathcal{L} = \left[\liminf_{n \rightarrow +\infty} r_n, \limsup_{n \rightarrow +\infty} r_n \right].$$

Proof. \mathcal{L} being the set of the limit points of a bounded sequence is compact. Besides we have that

$$0 \leq \liminf_{n \rightarrow +\infty} r_n \leq \limsup_{n \rightarrow +\infty} r_n \leq 1.$$

Let us show that \mathcal{L} is an interval. Let $\alpha, \beta \in \mathcal{L}$ such that $\alpha < \beta$. Let $\gamma \in (\alpha, \beta)$, $\varepsilon > 0$ and $N \in \mathbb{N}$ with ε such that $\varepsilon < \min(\frac{\gamma-\alpha}{2}, \frac{\beta-\gamma}{2})$. One has for all $n \in \mathbb{N}$,

$$\begin{aligned} r_{n+1} - r_n &= \frac{1}{n+2} + \frac{\delta_{n+1}(n+1)r_n}{n+2} - r_n \\ &= \frac{1}{n+2} + r_n \left(\frac{\delta_{n+1}(n+1)}{n+2} - 1 \right) \\ &\leq \frac{1}{n+2}. \end{aligned}$$

Hence there exists $n_0 \geq N$ such that for all $n \geq n_0$, $r_{n+1} - r_n < \varepsilon$, $n_1 \geq n_0$ such that $r_{n_1} \in (\alpha - \varepsilon, \alpha + \varepsilon)$ and $n_2 > n_1$ such that $r_{n_2} \in (\beta - \varepsilon, \beta + \varepsilon)$. Let

$$A := \{n \geq n_1 \mid r_n > \gamma + \varepsilon\}.$$

Since $n_2 \in A$, A is a non empty subset of \mathbb{N} and thus has a least element m . Besides $m > n_1$ since $r_{n_1} \leq \gamma + \varepsilon$. By definition $r_{m-1} \leq \gamma + \varepsilon$ and $r_{m-1} = r_m - (r_m - r_{m-1}) > \gamma + \varepsilon - \varepsilon = \gamma$ with $m \geq N$. Hence $\gamma \in \mathcal{L}$ and $[\alpha, \beta] \subset \mathcal{L}$. Thus \mathcal{L} is a compact interval given by

$$\mathcal{L} = \left[\liminf_{n \rightarrow +\infty} r_n, \limsup_{n \rightarrow +\infty} r_n \right].$$

□

To fully understand the implications of Proposition 9 one needs to remind that in situations where the market is efficient a good proxy for \mathcal{R}_n must be given by

$$\mathcal{R}_n = \frac{P_{t_n} - P_*}{P_{t_0} - P_*}, \quad (6)$$

where P_* and P_{t_n} are respectively the stock price at the beginning of the metaorder and at the arrival time of the n -th information. Thus P_{t_0} is the price at the end of the execution. Besides one has also

$$r_n = \frac{\mathbb{E}[P_{t_n}] - P_*}{P_{t_0} - P_*}. \quad (7)$$

So the previous result states that in situations where market equilibrium cannot be reached the average price is confined and fluctuates between a lower and an upper limit. This means that the competition between the information conveyed by the metaorder and market noise never ends. As we believe that this last situation does not correspond to any realistic pattern we will focus only on the cases leading to market equilibrium.

As we are interested in the establishment of market equilibrium we need first of all to see how the terms $(\delta_k)_{1 \leq k \leq n}$ shape r_n .

Theorem 11. *There exists a couple of two positive sequences $(r_n^+)_{n \in \mathbb{N}}$ and $(r_n^-)_{n \in \mathbb{N}}$ starting with $r_0^+ = r_0^- = 1$ such that $(r_n^+)_{n \in \mathbb{N}}$ is non-decreasing, $(r_n^-)_{n \in \mathbb{N}}$ is non-increasing with for every $n \in \mathbb{N}$,*

$$r_n^- \leq 1 \leq r_n^+, \quad (8)$$

and

$$r_n = r_n^+ \times r_n^- \quad (9)$$

Furthermore we have for any $n \in \mathbb{N}$,

$$r_n = \frac{1}{n+1} \sum_{k=0}^n \prod_{i=0}^{k-1} \delta_{n-i}. \quad (10)$$

Hence a representation of (r^-, r^+) is given by

$$r_n^- = \prod_{k=1}^n \delta_k, \quad (11)$$

$$r_n^+ = \frac{1}{n+1} \sum_{k=0}^n \prod_{l=1}^k \delta_l^{-1}. \quad (12)$$

for every $n \in \mathbb{N}$.

Proof. We have for all $n \in \mathbb{N}$, \mathcal{R}_n is \mathcal{F}_n -measurable and

$$\mathcal{R}_{n+1} = \frac{1 + (n+1)\mathcal{R}_n\Delta_{n+1}}{n+2},$$

which gives that for all $n \in \mathbb{N}$,

$$\mathbb{E}[\mathcal{R}_{n+1} | \mathcal{F}_n] = \frac{1 + (n+1)\mathcal{R}_n\delta_{n+1}}{n+2} \geq \mathcal{R}_n\delta_{n+1},$$

leading that for all $n \in \mathbb{N}$,

$$\mathbb{E}\left[\frac{\mathcal{R}_{n+1}}{\delta_1 \dots \delta_{n+1}} \middle| \mathcal{F}_n\right] \geq \frac{\mathcal{R}_n}{\delta_1 \dots \delta_n}.$$

Hence $\left(\frac{\mathcal{R}_n}{\prod_{1 \leq k \leq n} \delta_k}\right)_{n \in \mathbb{N}}$ is an $(\mathcal{F}_n)_{n \in \mathbb{N}}$ -submartingale. Then by Doob's decomposition there exists a martingale $(M_n)_{n \in \mathbb{N}}$ and an integrable non-decreasing predictable process $(A_n)_{n \in \mathbb{N}}$ relative to $(\mathcal{F}_n)_{n \in \mathbb{N}}$ starting with $A_0 = 0$ such that for all $n \in \mathbb{N}$,

$$\frac{\mathcal{R}_n}{\prod_{1 \leq k \leq n} \delta_k} = M_n + A_n.$$

This leads that for all $n \in \mathbb{N}$,

$$\begin{aligned} r_n &= \left(\prod_{1 \leq k \leq n} \delta_k\right) (\mathbb{E}[M_n] + \mathbb{E}[A_n]) \\ &= \left(\prod_{1 \leq k \leq n} \delta_k\right) (\mathbb{E}[M_0] + \mathbb{E}[A_n]) \\ &= (1 + \mathbb{E}[A_n]) \left(\prod_{1 \leq k \leq n} \delta_k\right) \\ &= r_n^+ \times r_n^- \end{aligned}$$

by setting for all $n \in \mathbb{N}$,

$$r_n^+ := 1 + \mathbb{E}[A_n],$$

and

$$r_n^- := \prod_{1 \leq k \leq n} \delta_k.$$

Besides by a straightforward induction we have for any $n \in \mathbb{N}$,

$$r_n = \frac{1}{n+1} \sum_{k=0}^n \prod_{i=0}^{k-1} \delta_{n-i} = \frac{1 + \delta_n + \delta_n \delta_{n-1} + \dots + \delta_n \dots \delta_1}{n+1}.$$

Thus for any $n \in \mathbb{N}$,

$$r_n^+ = 1 + \mathbb{E}[A_n] = \frac{1}{n+1} \sum_{k=0}^n \prod_{l=1}^k \delta_l^{-1}.$$

□

The previous result shows that the asymptotic behaviour of the sequence $(r_n)_{n \in \mathbb{N}}$ depends of the properties of $(\delta_n)_{n \in \mathbb{N}^*}$. More precisely if we consider r^- and r^+ as defined in (11)-(12) we can see that the asymptotic behaviour of $(r_n)_{n \in \mathbb{N}}$ is only the product of the limit of the sequence $(r_n^-)_{n \in \mathbb{N}}$ and the Cesàro limit of the sequence $\left(\frac{1}{r_n^-}\right)_{n \in \mathbb{N}}$. The sequence $(r_n^-)_{n \in \mathbb{N}}$ being always convergent in $[0, 1]$, $(r_n^+)_{n \in \mathbb{N}}$ always converges in $[1, +\infty]$. When $(r_n^-)_{n \in \mathbb{N}}$ converges to a positive real number, $(r_n^+)_{n \in \mathbb{N}}$ converges to the inverse positive number and $(r_n)_{n \in \mathbb{N}}$ converges to 1, which means that there is no relaxation. Hence the interesting case is when $\lim_{n \rightarrow +\infty} r_n^- = 0$ and $\lim_{n \rightarrow +\infty} r_n^+ = +\infty$. In fact

Proposition 10. *When permanent market impact exists, $(\delta_n)_{n \in \mathbb{N}^*}$ is convergent and $\lim_{n \rightarrow +\infty} \delta_n = 1$.*

Proof. For every $n \in \mathbb{N}$, $r_n > 0$ and

$$\delta_{n+1} = \frac{(n+2)r_{n+1}}{(n+1)r_n} - \frac{1}{(n+1)r_n}.$$

Thus when $(r_n)_{n \in \mathbb{N}}$ converges to a real positive number, $(\delta_n)_{n \in \mathbb{N}^*}$ converges to 1. □

Hence the existence of permanent market impact is only possible when information conveyed to the market by metaorders is persistent and not asymptotically annihilated by market noise. In the terminology of [Gomes and Waelbroeck, 2015] we will say that

Definition 22. *Metaorders are informed when $(\delta_n)_{n \in \mathbb{N}^*}$ converges to 1. Otherwise we will say that they are uninformed.*

The following proposition underlines the fact that there is no chance to observe permanent market impact when metaorders are uninformed:

Proposition 11. (i) *If $\limsup_{n \rightarrow +\infty} \delta_n < 1$, then $(r_n)_{n \in \mathbb{N}}$ converges and $\lim_{n \rightarrow +\infty} r_n = 0$.*

(ii) If $\liminf_{n \rightarrow +\infty} \delta_n < 1$ and $(r_n)_{n \in \mathbb{N}}$ converges, then $\lim_{n \rightarrow +\infty} r_n = 0$.

Proof. (i) There exists $\alpha \in (0, 1)$, $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\delta_n < \alpha$. Besides for any $n \geq n_0$,

$$\begin{aligned} r_n &= \frac{1 + \delta_n + \delta_n \delta_{n-1} + \dots + \delta_n \cdots \delta_{n_0+2} + \delta_n \cdots \delta_{n_0+1} (n_0 + 1) r_{n_0}}{n + 1} \\ &\leq \frac{1 + \alpha + \alpha^2 + \dots + \alpha^{n-n_0-1} + \alpha^{n-n_0} (n_0 + 1) r_{n_0}}{n + 1} \\ &\leq \frac{1}{n + 1} \sum_{k=0}^{+\infty} \alpha^k + \alpha^n \frac{n_0 + 1}{n + 1} \frac{r_{n_0}}{\alpha^{n_0}} \\ &\leq \frac{1}{n + 1} \frac{1}{1 - \alpha} + \alpha^n \frac{n_0 + 1}{n + 1} \frac{r_{n_0}}{\alpha^{n_0}}. \end{aligned}$$

Let $\varepsilon > 0$, hence there exists $n_1 \geq n_0$, such that for all $n \geq n_1$, $r_n < \varepsilon$ which gives that the sequence $(r_n)_{n \in \mathbb{N}}$ tends to 0.

(ii) By (5) one has that any limit point of the sequence $(\delta_{n+1} r_n)_{n \in \mathbb{N}}$ is a limit point of $(r_n)_{n \in \mathbb{N}}$. It follows from Proposition 9 that

$$\liminf_{n \rightarrow +\infty} r_n \leq \liminf_{n \rightarrow +\infty} (\delta_{n+1} r_n) \leq \limsup_{n \rightarrow +\infty} (\delta_{n+1} r_n) \leq \limsup_{n \rightarrow +\infty} r_n.$$

Besides for all $n \in \mathbb{N}$, $\delta_{n+1} r_n \leq r_n$ which leads to

$$\liminf_{n \rightarrow +\infty} (\delta_{n+1} r_n) \leq \liminf_{n \rightarrow +\infty} r_n.$$

Therefore

$$\liminf_{n \rightarrow +\infty} (\delta_{n+1} r_n) = \liminf_{n \rightarrow +\infty} r_n.$$

Hence

$$\lim_{n \rightarrow +\infty} r_n = \liminf_{n \rightarrow +\infty} r_n = \left(\liminf_{n \rightarrow +\infty} \delta_n \right) \left(\lim_{n \rightarrow +\infty} r_n \right),$$

which implies that $\lim_{n \rightarrow +\infty} r_n = 0$. □

From now, we will consider informed metaorders. We have already seen that if $(r_n^-)_{n \in \mathbb{N}}$ converges to a positive real number, then $(r_n^+)_{n \in \mathbb{N}}$ converges to the inverse positive number and there is no relaxation. For instance, this is the case when $1 - \delta_n = O\left(\frac{1}{n^\alpha}\right)$ for some $\alpha > 1$. Thus it seems that the convergence speed of the sequence $(\delta_n)_{n \in \mathbb{N}^*}$ plays a major role in the establishment or not of permanent market impact. Indeed, the rate of convergence of $(\delta_n)_{n \in \mathbb{N}^*}$ characterizes the speed at which the market deals with the information conveyed by metaorders as illustrated below:

Theorem 12. *Let us suppose that $(\delta_n)_{n \in \mathbb{N}^*}$ converges to 1 such that*

$$\lim_{n \rightarrow +\infty} n(1 - \delta_n) = l \in \mathbb{R}_+ \cup \{+\infty\}.$$

Then the sequence $(r_n)_{n \in \mathbb{N}}$ converges and $\lim_{n \rightarrow +\infty} r_n = \frac{1}{l + 1}$.

Proof. Let set for all $n \in \mathbb{N}^*$, $\varepsilon_n := 1 - \delta_n$.

- Let $l \in \mathbb{R}_+$ such that $\lim_{n \rightarrow +\infty} n\varepsilon_n = l$. Let $\varepsilon > 0$. We set for any $n \in \mathbb{N}$, $w_n := n\varepsilon_n$. There exists $N \in \mathbb{N}^*$ such that

- $\forall n > N$, $|w_n - l| \leq \varepsilon$,
- $\forall p, q > N$, $\left| \frac{1}{p^2} + \dots + \frac{1}{q^2} \right| \leq \varepsilon$,
- $\forall n > N$, $\frac{l + \varepsilon}{n} < \frac{1}{2}$.

Let $n > N$. On one hand

$$\begin{aligned}
 r_n &= \frac{1}{n+1} \sum_{k=0}^n \prod_{i=0}^{k-1} (1 - \varepsilon_{n-i}) \\
 &= \frac{1}{n+1} \sum_{k=0}^{n-N} \prod_{i=0}^{k-1} \exp\left(\ln\left(1 - \frac{w_{n-i}}{n-i}\right)\right) + \frac{1}{n+1} \sum_{k=n-N+1}^n \prod_{i=0}^{k-1} (1 - \varepsilon_{n-i}) \\
 &\leq \frac{1}{n+1} \sum_{k=0}^{n-N} \exp\left(\sum_{i=0}^{k-1} \ln\left(1 - \frac{w_{n-i}}{n-i}\right)\right) + \frac{N}{n+1} \\
 &\leq \frac{1}{n+1} \sum_{k=0}^{n-N} \exp\left(\sum_{i=0}^{k-1} \ln\left(1 - \frac{l - \varepsilon}{n-i}\right)\right) + \frac{N}{n+1} \\
 &\leq \frac{1}{n+1} \sum_{k=0}^{n-N} \exp\left(- (l - \varepsilon) \sum_{i=0}^{k-1} \frac{1}{n-i}\right) + \frac{N}{n+1} \\
 &\leq \frac{1}{n+1} \sum_{k=0}^{n-N} \exp\left(- (l - \varepsilon) \left(\frac{1}{n-1} + \dots + \frac{1}{n-k}\right)\right) + \frac{N}{n+1} \\
 &\leq \frac{1}{n+1} \sum_{k=0}^{n-N} \exp\left((l - \varepsilon) \ln\left(1 - \frac{k}{n}\right)\right) + \frac{N}{n+1} \\
 &\leq \frac{1}{n+1} \sum_{k=0}^{n-N} \left(1 - \frac{k}{n}\right)^{l - \varepsilon} + \frac{N}{n+1},
 \end{aligned}$$

which gives that for all $\varepsilon > 0$, $\limsup_{n \rightarrow +\infty} r_n \leq \frac{1}{l + 1 - \varepsilon}$. This gives that

$$\limsup_{n \rightarrow +\infty} r_n \leq \frac{1}{l + 1}. \tag{13}$$

On the other hand

$$\begin{aligned}
 r_n &= \frac{1}{n+1} \sum_{k=0}^{n-N} \prod_{i=0}^{k-1} \exp\left(\ln\left(1 - \frac{w_{n-i}}{n-i}\right)\right) + \frac{1}{n+1} \sum_{k=n-N+1}^n \prod_{i=0}^{k-1} (1 - \varepsilon_{n-i}) \\
 &\geq \frac{1}{n+1} \sum_{k=0}^{n-N} \exp\left(\sum_{i=0}^{k-1} \ln\left(1 - \frac{l+\varepsilon}{n-i}\right)\right) + \frac{1}{n+1} \sum_{k=n-N+1}^n \prod_{i=0}^{k-1} (1 - \varepsilon_{n-i}) \\
 &\geq \frac{1}{n+1} \sum_{k=0}^{n-N} \exp\left(- (l+\varepsilon) \sum_{i=0}^{k-1} \frac{1}{n-i} - (l+\varepsilon)^2 \sum_{i=0}^{k-1} \frac{1}{(n-i)^2}\right) \\
 &\quad + \frac{1}{n+1} \sum_{k=n-N+1}^n \prod_{i=0}^{k-1} (1 - \varepsilon_{n-i}) \\
 &\geq \frac{1}{n+1} \sum_{k=0}^{n-N} \exp\left(- (l-\varepsilon) \left(\frac{1}{n-1} + \dots + \frac{1}{n-k}\right) - (l-\varepsilon)^2 \left(\frac{1}{(n-1)^2} + \dots + \frac{1}{(n-k)^2}\right)\right) \\
 &\quad + \frac{1}{n+1} \sum_{k=n-N+1}^n \prod_{i=0}^{k-1} (1 - \varepsilon_{n-i}) \\
 &\geq e^{-(l+\varepsilon)^2 \varepsilon} \frac{1}{n+1} \sum_{k=0}^{n-N} \exp\left((l+\varepsilon) \ln\left(1 - \frac{k}{n-1}\right)\right) + \frac{1}{n+1} \sum_{k=n-N+1}^n \prod_{i=0}^{k-1} (1 - \varepsilon_{n-i}) \\
 &\geq e^{-(l+\varepsilon)^2 \varepsilon} \frac{1}{n+1} \sum_{k=0}^{n-N} \left(1 - \frac{k}{n-1}\right)^{l+\varepsilon} + \frac{1}{n+1} \sum_{k=n-N+1}^n \prod_{i=0}^{k-1} (1 - \varepsilon_{n-i})
 \end{aligned}$$

where we have used that for all $x \in [0, 1/2)$, $\ln(1-x) \geq -x - x^2$. Hence for all $\varepsilon > 0$, $\liminf_{n \rightarrow +\infty} r_n \geq \frac{e^{-(l+\varepsilon)^2 \varepsilon}}{l+1+\varepsilon}$ which gives that

$$\liminf_{n \rightarrow +\infty} r_n \geq \frac{1}{l+1}. \quad (14)$$

From (13) and (14) one has

$$\limsup_{n \rightarrow +\infty} r_n = \liminf_{n \rightarrow +\infty} r_n = \frac{1}{l+1},$$

which gives the convergence of the sequence $(r_n)_{n \in \mathbb{N}}$ to the limit $\frac{1}{l+1}$.

- Suppose $l = +\infty$. Let $l' \geq 0$. Let $(\varepsilon'_n)_{n \in \mathbb{N}^*}$ and $(r'_n)_{n \in \mathbb{N}}$ two sequences defined such that for any $n \geq 1$,

$$\varepsilon'_n = \min\left(\varepsilon_n, \frac{l'}{n}\right)$$

and

$$\begin{cases} r'_0 = 1, \\ \forall n \in \mathbb{N}, r'_{n+1} = \frac{1 + (n+1)r'_n(1 - \varepsilon'_{n+1})}{n+2} \end{cases}$$

Recalling that $(r_n)_{n \in \mathbb{N}}$ is given by

$$\begin{cases} r_0 = 1, \\ \forall n \in \mathbb{N}, r_{n+1} = \frac{1 + (n+1)r_n(1 - \varepsilon_{n+1})}{n+2}, \end{cases}$$

we get that for all $n \in \mathbb{N}$, $r'_n \geq r_n$, leading to $\limsup_{n \rightarrow +\infty} r'_n \geq \limsup_{n \rightarrow +\infty} r_n$. Furthermore as $\lim_{n \rightarrow +\infty} n\varepsilon'_n = l'$, we have that $\lim_{n \rightarrow +\infty} r'_n = \frac{1}{l'+1}$. Thus for all $l' \geq 0$, $\frac{1}{l'+1} \geq \limsup_{n \rightarrow +\infty} r_n$. This leads to $\limsup_{n \rightarrow +\infty} r_n = 0$.

□

Hence the previous result establishes that there exists a critical rate of convergence giving by $1 - \delta_n \sim_{+\infty} \frac{l}{n}$ for some $l > 0$, which authorizes the establishment of permanent market impact in a such way that it is equal to $\frac{1}{l+1}$ of the temporary impact. This can also be seen as the fraction of the metaorder information absorbed by the market at equilibrium. Thus which is determinant in the establishment of permanent impact is the speed of the informational interaction between the metaorder and the market. On one hand, if the information conveyed by the metaorder to the market is rapidly kept - in the sense that $(\delta_n)_{n \in \mathbb{N}^*}$ converges quickly to 1 -, then there is no relaxation. On the other hand, if the information is slowly kept, it vanishes and there is no permanent impact. Between those two extreme situations there is a critical regime leading to permanent market impact.

4 Conclusion and connections with related works

We have shown in the previous section that the absence of permanent impact - the picture of [Bouchaud, 2010] - arises in two possible situations: First, when the metaorder is uninformed i.e. the information conveyed to the market through its trades is totally lost - corresponding to the case when $(\delta_n)_{n \in \mathbb{N}^*}$ does not converge to 1 -, and secondly, when the digestion of the metorder information by the market is too slow - corresponding to the case when $(\delta_n)_{n \in \mathbb{N}^*}$ converges to 1 with $\lim_{n \rightarrow +\infty} n(1 - \delta_n) = +\infty$. This underlines the fact that contrary to what has been suggested in [Gomes and Waelbroeck, 2015], the presence or not of permanent impact for a metaorder has nothing to do with the fact that the metaorder is *informed* or not. As underlined by [Bouchaud, 2010] since trading on modern electronic markets is anonymous, there cannot be any obvious difference between *informed* trades and *uninformed* trades if the strategies used for their execution are similar. We rather believe that it is the way how the market deals with the information conveyed by the metaorder that shapes the establishment of permanent market impact. The results of recent empirical studies on the price reversion of metaorders presented in Table IV.1 argue in favor of the existence of permanent market impact according to the predictions of [Farmer et al., 2013]. The theory of [Farmer et al., 2013] states that the ratio of the permanent impact to the immediate impact is $\frac{1}{\beta}$ where $1 + \beta$

is the parameter of the Pareto distribution of the metaorder size N . There is now considerable evidence that N is distributed as a power law with $\beta \approx 1.5$ [Vaglica et al., 2008] [Bershova and Rakhlin, 2013] [Said et al., 2018]. This gives, in our framework, that the market digests the metaorder information such that $\delta_n \approx 1 - \frac{\beta - 1}{n}$ when n becomes large enough.

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Titre: Impact de Marché en Trading algorithmique et Pricing d'Options

Mots clés: Mathématiques Financières, Impact de Marché, Trading Algorithmique, Évaluation d'Options

Résumé: L'objectif principal de cette thèse est de comprendre les divers aspects du market impact. Elle se compose de quatre chapitres dans lesquelles le market impact est étudié dans différents contextes et à différentes échelles. Le premier chapitre présente une étude empirique du market impact des ordres limites sur les marchés actions européens. Dans le deuxième chapitre, nous avons étendu la méthodologie présentée pour les marchés actions aux marchés options. Cette étude empirique a mis en évidence que notre définition d'un métaordre options nous permet de retrouver la totalité des résultats mis en évidence sur les marchés actions. Le troisième chapitre s'intéresse au market impact dans le contexte de l'évaluation des pro-

duits dérivés. Ce chapitre tente d'apporter une composante microstructure à l'évaluation des options notamment en proposant une théorie des perturbations du market impact au cours du processus de re-hedging. Nous explorons dans le quatrième chapitre un modèle assez simple pour la relaxation des métaordres. La relaxation des métaordres est traitée dans cette partie en tant que processus informationnel qui se transmet au marché. Ainsi, partant du point de départ qu'à la fin de l'exécution d'un métaordre l'information portée par celui-ci est maximale, nous proposons une interprétation du phénomène de relaxation comme étant le résultat de la dégradation de cette information au détriment du bruit extérieur du marché.

Title: Market Impact in Systematic Trading and Option Pricing

Keywords: Financial Mathematics, Market Impact, Systematic Trading, Option Pricing

Abstract: The main objective of this thesis is to understand the various aspects of market impact. It consists of four chapters in which the market impact is studied in different contexts and at different scales. The first chapter presents an empirical study of the market impact of limit orders on European equity markets. In the second chapter, we have extended the methodology presented for the equity markets to the options markets. This empirical study has shown that our definition of an options metaorder allows us to retrieve all the results highlighted on the equity markets. The third chapter focuses on market impact in the context of option pricing. This chapter is based on previ-

ous works of option pricing with market impact and attempts to bring a microstructure component to pricing option theory by proposing a perturbation theory of market impact during the re-hedging process. We explore in the fourth chapter a fairly simple model for the relaxation of metaorders. The relaxation of metaorders is treated in this part as an informational process which is transmitted to the market. Thus, starting from the starting point that at the end of the execution of a metaorder the information carried by this one is maximum, we propose an interpretation of the relaxation phenomenon as being the result of the degradation of this information at the expense of outside market noise.