Limites d'ensembles quasiminimaux et existence d'ensembles minimaux sous contraintes topologiques

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Thèse de doctorat

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Chapter 1

Introduction (version française)

1.1 Des films de savon au problème de Plateau

Le problème de Plateau consiste à minimiser l'aire d'une surface s'appuyant sur un bord. Ce paragraphe est destiné à expliquer cet énoncé et ses origines. Joseph Plateau était un physicien et mathématicien belge du dix-neuvième siècle. Il est renommé pour ses travaux précurseurs sur la persistence rétinienne et son appareil d'animation, le phénakistiscope. Il a aussi étudié la tension superficielle et les films de savon. En plongeant un contour rigide dans une solution savonneuse, on forme un film de savon bordé par le contour. Le contour joue le rôle de *bord* (ou *frontière*) de la surface, tandis que la surface *s'appuie* sur le contour. Dans le cas le plus simple, le bord est un cercle et le film de savon est le disque correspondant. Si le bord est le squelette d'un polyhèdre, la surface est un suprenant système de faces que Plateau appelait *système laminaire*. Plateau a observé que ces systèmes étaient réguliers et

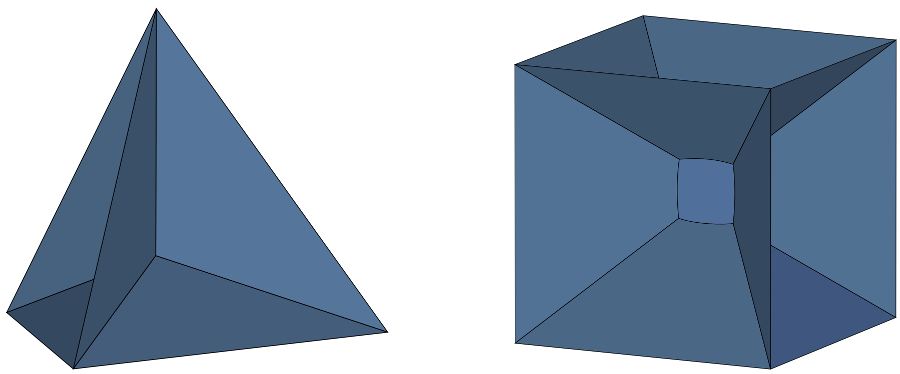
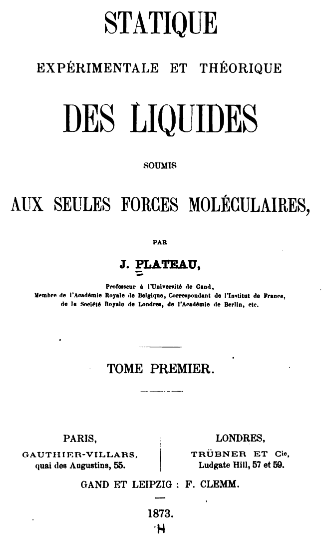


Figure 1.1: Film de savon s'appuyant sur le squelette d'un tétrahèdre (gauche) et le squelette d'un cube (droite).

symétriques et il a entrepris de comprendre leur disposition géométrique.

Comment s'expliquent les formes singulières des films de savon? Les molécules de savon forment une interface entre l'air et l'eau, donnant de l'élasticité et de la stabilité à la surface. Celle-ci est alors portée à minimiser son aire autant que possible pour atteindre une position d'équilibre (cette propriété s'appelle la *tension superficielle*). Dans notre situation «autant que possible» signifie «tout en s'appuyant sur le bord». Les mathématiciens ont isolé cette propriété. Ils définissent un problème de Plateau comme la donnée des «surfaces s'appuyant sur un bord donné» et de leur «aire». Une solution est une surface s'appuyant sur le bord avec une aire minimale pour ces définitions. Les problèmes de Plateau simplifient la réalité physique des films de savon mais ils l'a généralisent aussi car ils peuvent se formuler dans des espaces plus généraux que notre espace euclidien à 3 dimensions. Toutefois, ces concepts se sont révélés difficiles à formaliser.



STATIQUE

EXPÉRIMENTALE ET THÉORIQUE

DES LIQUIDES

SOUMIS

AUX SEULES FORCES MOLÉCULAIRES,

PAR

J. PLATEAU,

Professeur à l'Université de Gand,
Membre de l'Académie Royale de Belgique, Correspondant de l'Institut de France,
de la Société Royale de Londres, de l'Académie de Berlin, etc.

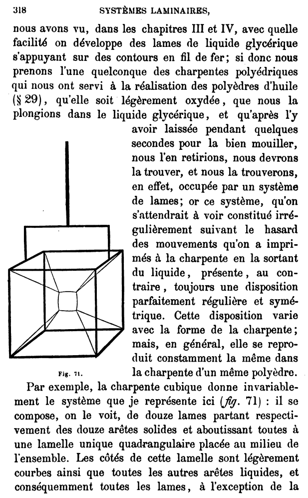
TOME PREMIER.

PARIS,
GAUTHIER-VILLARS,
quai des Augustins, 55.

LONDRES,
TRÜBNER ET Cie,
Ludgate Hill, 57 et 59.

GAND ET LEIPZIG : F. CLEMM.

1873.



318 SYSTÈMES LAMINAIRES,

nous avons vu, dans les chapitres III et IV, avec quelle facilité on développe des lames de liquide glycérique s'appuyant sur des contours en fil de fer; si donc nous prenons l'une quelconque des charpentes polyédriques qui nous ont servi à la réalisation des polyèdres d'huile (§ 29), qu'elle soit légèrement oxydée, que nous la plongions dans le liquide glycérique, et qu'après l'y avoir laissée pendant quelques secondes pour la bien mouiller, nous l'en retirions, nous devrons la trouver, et nous la trouverons, en effet, occupée par un système de lames; or ce système, qu'on s'attendrait à voir constitué irrégulièrement suivant le hasard des mouvements qu'on a imprimés à la charpente en la sortant du liquide, présente, au contraire, toujours une disposition parfaitement régulière et symétrique. Cette disposition varie avec la forme de la charpente; mais, en général, elle se reproduit constamment la même dans la charpente d'un même polyèdre.

Fig. 71.

Par exemple, la charpente cubique donne invariablement le système que je représente ici (*fig.* 71) : il se compose, on le voit, de douze lames partant respectivement des douze arêtes solides et aboutissant toutes à une lamelle unique quadrangulaire placée au milieu de l'ensemble. Les côtés de cette lamelle sont légèrement courbes ainsi que toutes les autres arêtes liquides, et conséquemment toutes les lames, à l'exception de la

Figure 1.2: Dans son livre *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires* [Pl], Plateau présente ses expériences et développe des lois décrivant les systèmes laminaires.

1.2 Approches du problème de Plateau

Qu'attendons-nous d'une formulation optimale du problème de Plateau?

1. Elle devrait inclure une définition des «surfaces s'appuyant sur un bord donné» (aussi appelées *compétiteurs*) et de leur «aire».

2. Elle devrait se prêter à la méthode directe du calcul des variations (trouver une solution comme limite d'une suite minimisante). Il est en général difficile d'avoir à la fois un principe de compacité sur la classe des compétiteurs et un principe de semicontinuité inférieure sur l'aire.

3. Elle devrait rester proche des motivations de Plateau: décrire les films de savon. Cela veut dire que l'on veut travailler avec le genre de frontière que l'on peut construire avec un fil de fer et obtenir les films de savon correspondant comme solutions.

On présente quelques grandes approches de problème (le lecteur peut aussi être intéressé par [D5]). La frontière sera notée Γ.

L'approche classique minimise des paramétrisations dont le domaine est un disque. C'est la formulation de Radò et Douglas dans les années 30 (voir [Rado] et [Doug]). Regardons d'un peu plus près. Soit D^2 et S^1, le disque unité et le cercle unité du plan euclidien respectivement. On fixe une courbe de Jordan rectifiable Γ dans $\mathbf{R}^n$ (une image homéomorphe de S^1 dont la longueur est finie). Une surface s'appuyant sur Γ est défine comme une application continue $f\colon D^2 \to \mathbf{R}^n$ telle que f envoie S^1 homéomorphiquement sur Γ. L'aire correspondante est définie comme la *variation totale* de f (Radò et Douglas l'appellent *area-integral* ou *area functional*). Si f est une application C^1, la variation totale peut être calculée directement par la formule

$$\int_D Jf \,\mathrm{d}x$$

où Jf est le Jacobien f,

$$Jf = \begin{vmatrix} |\partial_1 f|^2 & \partial_1 f \cdot \partial_2 f \\ \partial_1 f \cdot \partial_2 f & |\partial_2 f|^2 \end{vmatrix}^{\frac{1}{2}}.$$

La variation totale diffère de l'aire de l'image à cause des éventuelles multiplicités mais elle a l'avantage d'être semicontinue inférieurement. Radò et Douglas cherchent une immersion lisse f qui minimise la variation totale. Ainsi, f paramétrerait localement une surface minimale au sens classique. Il reste difficile de faire converger une suite minimisante vers une immersion lisse. Le problème doit particulièrement sa complexité à la diversité des paramétrisations d'une surface donnée. Même une suite d'entre elles pourrait ne pas converger du tout. Comme les courbes qui ont les paramétrisations par la longueur d'arc, les surfaces de dimension 2 ont des paramétrisations privilégiées: les paramétrisations conformes[1]. De plus, une paramétrisation conforme f décrit une surface minimale si et seulement si f est harmonique. Cela amène Radò et Douglas à spécifier le problème comme la recherche d'une application harmonique et conforme qui minimise

[1]Une paramétrisation f est conforme si $|\partial_1 f| = |\partial_2 f|$ et $\partial_1 f \cdot \partial_f = 0$.

la variation totale. Radò travaille avec une suite de compétiteurs "approximativement conformes" minimisant la variation totale. Douglas minimise une fonctionnelle plus simple pour laquelle la variation première ne peut s'annuler qu'aux applications conformes. Le principale défaut de cette approche est qu'elle ne se généralise pas à trois variables ou plus car elle dépend trop de la théorie des applications conformes. De plus, on préfererait des solutions qui décrivent les singularités observées par Plateau. Une solution de Radò et Douglas paramétre localement une surface minimale mais son image complète n'est pas d'aire minimale. Prenons par exemple un noeud Γ dont la solution a des auto-intersections. Les morceaux qui se traversent sont indépendemment minimaux mais la façon dont ils s'intersectent n'est pas prise en compte par la variation totale. Ceci explique que des plans transversent se trouvent dans les solutions de Radò et Douglas alors qu'ils n'existent pas dans les films de savon. On pointe enfin que cette approche ne compare que des paramétrisations définis sur le disque, fixant ainsi le type topologique des compétiteurs.

Dans les années soixante sont apparues deux approches qui incluent toutes les dimensions et codimensions et dont les compétiteurs ont un type topologique variable: Federer-Fleming et Reifenberg. Federer et Fleming travaillent avec les *courants intégraux* et minimisent leur *masse* (une aire calculée avec multiplicité). Ils ont développée la *convergence plate* pour laquelle les courants intégraux disposent d'un principe de compacité et la masse est semicontinue inférieurement. Ils déduisent l'existence de courants rectifiables minimiseurs de masse dont la frontière est donnée. Cependant, leurs solutions sont trop régulières loin de la frontière pour décrire les films de savon. Au lieu de minimiser la masse, on pourrait minimiser la taille (l'aire du support). Un tel point de vue est très proche de celui de Reifenberg. Reifenberg travaille avec des *ensembles* de l'espace euclidien qui s'appuient sur la frontière au sens de la topologique algébrique et minimise leur *mesure de Hausdorff* (sphérique). Un ensemble E de dimension d s'appuie sur une frontière Γ si E contient Γ et annule les $(d-1)$-cycles de Γ (ou un sous-groupe d'entre eux).

Definition (Compétiteurs de Reifenberg)**.** Fixons Γ un compact de $\mathbf{R}^n$ et L un sous-groupe du groupe d'homologie $H_{d-1}(\Gamma)$. Un compétiteur de Reifenberg est un compact $E \subset \mathbf{R}^n$ tel que E contient Γ et le morphisme induit par inclusion,

$$H_{d-1}(\Gamma) \longrightarrow H_{d-1}(E),$$

est nul sur L.

On rappelle la définiton de la mesure de Hausdorff sphérique d-dimensionnelle.

Definition (Mesure de Hausdorff sphérique)**.**

$$S^d(E) := \lim_{\delta \to 0^+} \inf \left\{ \sum_k \operatorname{diam}(B_k)^d \;\middle|\; E = \cup_{k \in \mathbf{N}} B_k, \operatorname{diam}(B_k) \leq \delta \right\}.$$

où (B_k) est une suite de boules.

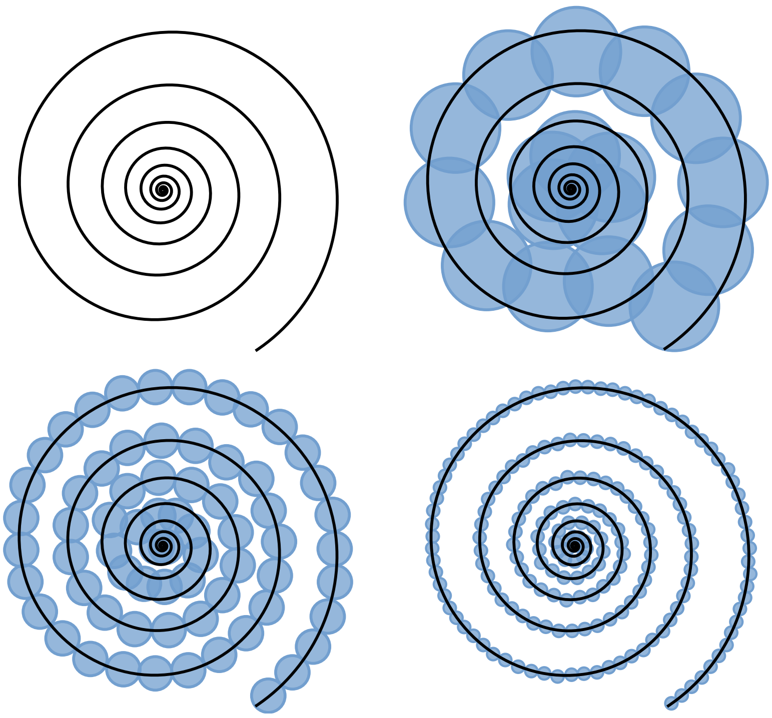


Figure 1.3: Calcul de la mesure S^1 d'une spirale.

L'exemple le plus classique est le cas où Γ est composé de deux cercles $C_1 \cup C_2$ de même rayon et centrés sur un même axe de symétrie. Les films de savon s'appuyant sur Γ sont les disques parallèles, la caténoide et la caténoide avec un disque (deux morceaux de caténoides qui rejoignent un disque intermédiaire avec un angle de 120 degrés). Ces deux derniers n'existent que si la distance entre les cercles est suffisamment petite comparée à leurs rayons. On peut relier ce phénomère au fait que l'aire de ces films est alors plus petite que l'aire des deux disques. On aborde le problème du point de vue de Reifenberg en supposant que les cercles sont proches pour ignorer les disques parallèles. On va voir que le choix de L discrimine la caténoide et la caténoide avec un disque. Si L est tout le groupe $H_1(\Gamma)$, alors la caténoide n'est pas un compétiteur et le minimiseur est la caténoide avec un disque. Si L est le sous-groupe engendré par $\gamma_2 - \gamma_1$, où γ_i est le cycle correspondant à

C_i, alors le minimiseur de Reifenberg est la caténoide. On conclut que pour interpréter un film de savon donné comme un minimiseur de Reifenberg, on doit spécifier le groupe à annuler.

Reifenberg a prouvé l'existence de solutions à son problème en 1960 pour la théorie homologique de Čech et des groupes de coefficient abéliens compacts ([Rei]). La propriété de continuité de la théorie de Čech implique qu'une limite de Hausdorff de compétiteurs est un compétiteur. Ainsi, une solution peut être cherchée comme une limite de Hausdoff d'une suite minimisante.

Theorem (Continuité de l'homologie de Čech)**.** *Soit (E_k) une suite décroissante de compacts. Alors pour tous $d \in \mathbf{N}$,*

$$H_d(E_\infty) = \lim_k H_d(E_k),$$

où $E_\infty = \bigcap_k E_k$, $\lim_k H_d(E_k)$ est la limite inverse par rapport aux morphismes induits par les inclusions $E_{k+1} \subset E_k$ et le signe $=$ est le morphisme induit pur les inclusions $E_\infty \subset E_k$.

En revanche, l'aire n'est pas semicontinue inférieurement par rapport aux limites de Hausdorff. Par exemple, on peut imaginer une suite minimisante qui présente des tentacules de plus en plus denses si bien que le limite est trop grosse. Reifenberg a travaillé avec des groupe de coefficient compacts pour disposer de l'axiome d'excision et ainsi couper les tentacules et reboucher les trous. Sa construction aboutit à suite minimisante alternative pour laquelle l'aire est semicontinue inférieurement. Nakauchi ([Na]) a énoncé et résolu une variante à frontière libre en 1984 (pour des groupes de coefficient compacts également). On signifie par là que l'intersection $E \cap \Gamma$ varie parmi les compétiteurs E.

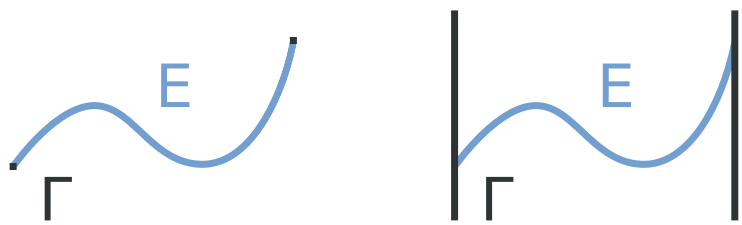


Figure 1.4: Deux types de problème de Plateau. Les compétiteurs ont un bord fixe à gauche et un bord libre à droite.

Definition (Compétiteurs de Nakauchi)**.** Fixons Γ un compact de $\mathbf{R}^n$ et L un sous-groupe de $H_{d-1}(\Gamma)$. Les compétiteurs de Nakauchi sont les compacts $E \subset \mathbf{R}^n$ tels que pour tous $v \in L$, il existe $u \in H_{d-1}(E \cap \Gamma)$ tel que $i_*(u) = v$

et $i'_*(u) = 0$ où i_* et i'_* sont les morphismes induits par inclusions:

$$\begin{tikzcd}
& H_{d-1}(\Gamma) \\
H_{d-1}(E \cap \Gamma) \arrow[ur, "i_*"] \arrow[dr, "i'_*"] & \\
& H_{d-1}(E).
\end{tikzcd}$$

Nakauchi minimise $H^d(E \setminus \Gamma)$ mais il serait également intéressant de minimiser $H^d(E)$ pour prendre en compte le bord libre. En 2015, Fang ([Fn]) a donné une nouvelle preuve des problèmes de Reifenberg et Nakauchi pour tous les groupes de coefficient et minimisant $H^d(E \setminus \Gamma)$. Son idée est de tirer parti de la semicontinuité inférieure de l'aire sur les ensembles quasiminimaux (on les présente dans le prochain paragraphe). Grâce à une construction de Feuvrier ([Feuv]), il obtient une suite minimisante alternative composée d'ensembles quasiminimaux. Fang remplace aussi les mesures de Hausdorff par des fonctionnelles plus générales appelées *intégrants elliptiques*.

Inspiré par les *restricted sets* d'Almgren ([Alm]), David a introduit les *compétiteurs glissant* dans [DS] et [D6]. Il compare l'aire d'une surface avec son image sous l'action de déformations continues. Une *déformation glissante* d'un ensemble E est une application lipschitz $\phi\colon E \to \mathbf{R}^n$ telle qu'il existe une application continue $\Phi\colon [0,1] \times E \to \mathbf{R}^n$ satisfaisant $\Phi(0,\cdot) = \mathrm{id}$, $\Phi(1,\cdot) = \phi$, $\Phi(t,\cdot) = \mathrm{id}$ en dehors d'un compact et $\Phi(t, E \cap \Gamma) \subset \Gamma$. On peut penser à un film de savon se mouvant le long d'un tube par exemple.

Definition (Compétiteurs glissants)**.** Fixons Γ un compact de $\mathbf{R}^n$ et E_0 un compact H^d fini de $\mathbf{R}^n$. Les compétiteurs glissants induits par E_0 sont les images de E_0 par les déformations glissantes.

L'existence de solutions est encore inconnue! On peut se demander pourquoi la déformation initiale ϕ est lipschitz tandis que l'homotopie Φ est simplement continue. L'hypothèse lipschitz est un héritage d'Almgren. Elle se révèle utile car les fonctions lipschitz ont une actions sur les courants, les ensembles H^d mesurables, H^d finis et H^d rectifiables. L'hypothèse de continuité sur Φ sert à imposer une contrainte topologique de la façon la plus générale possible. On peut observer que les solutions de Reifenberg et du problème glissant satisfont une propriété intermédiaire: leur aire est minimale sous l'actions des déformations glissantes. Ceci motive la notion d'*ensembles minimaux glissants*. Ce sont des ensembles fermés et H^d localement finis $E \subset \mathbf{R}^n$ tels que pour toute déformation glissante ϕ de E,

$$H^d(E \cap W) \leq H^d(\phi(E \cap W)).$$

où H^d est la mesure de Hausdorff de dimension d et W est l'ensemble

$$W = \{ x \in \mathbf{R}^n \mid \phi(x) \neq x \} .$$

Cette propriété fait écho à l'élasticité et la stabilité des films de savon sous l'actions de (petites) déformations. Malheureusement, ce n'est qu'en petite dimension et loin du bord que nous connaissons de bons résultats de régularité à leur propos (voir le théorème de Jean Taylor, [Ta], [D4]). Les *ensembles quasiminimaux* satisfont une généralisation de cette définition où la mesure H^d est remplacée par des fonctionnelles plus générales et l'ensemble W est localisé dans des petites boules. On connait des résulats faibles de régularité sur les ensembles quasiminimaux en toutes dimensions (voir [DS], [D6], [D2] et [D3]). David a étudié les propriétés de passage à la limite des suites quasiminimisantes ([D1]). Il a prouvé en particulier qu'une limite de Hausdorff locale d'ensembles quasiminimaux est encore un ensemble quasiminimal ([D6], Theorem 10.8).

Dans [DLGM] et [DPDRG1], De Lellis, De Philippis, De Rosa, Ghiraldin et Maggi ont introduit un nouveau type de méthode directe. Comme Reifenberg, ils travaillent avec des ensembles et des mesures de Hausdorff de l'espace euclidien mais ils remplacent les limites de Hausdorff par des limites faibles. Présentons leur travail avec plus de détails. Les auteurs fixent un fermé $\Gamma \subset \mathbf{R}^n$ (la frontière) et se placent dans $\mathbf{R}^n \setminus \Gamma$ (loin de la frontière). Ils considèrent une classe de sous-ensembles relativement fermés $E \subset \mathbf{R}^n \setminus \Gamma$ (les compétiteurs). À une suite minimisante de compétiteurs (E_k) pour H^d, ils associent la suite de mesures de Radon $(H^d \llcorner E_k)$. Ils font alors converger la suite faiblement vers une mesure de Radon μ dans l'espace ambiant $\mathbf{R}^n \setminus \Gamma$. Le point essentiel de cette méthode est que, si la classe des compétiteurs est suffisamment complète, on a

$$\mu = H^d \llcorner E_\infty$$

où E_∞ est le support de μ dans $\mathbf{R}^n \setminus \Gamma$. En ce sens, l'ensemble E_∞ est une limite faible de la suite minimisante (E_k). Cette stratégie donne en particulier la semicontinuité inférieure de l'aire:

$$H^d(E_\infty) \leq \lim_k H^d(E_k).$$

La démonstration repose sur la construction de bons compétiteurs. Les auteurs passent par une information intermédiaire: ils montrent que E_∞ est un ensemble H^d rectifiable grâce au théorème de rectifiabilité de Preiss. Pour résoudre le problème de Plateau, il reste à voir si E_∞ est un compétiteur. Les auteurs montrent que c'est le cas pour le problème de Harrison-Pugh ([HP]).

Definition (Compétiteurs de Harrison et Pugh)**.** Fixons Γ un compact de $\mathbf{R}^n$ et $\mathcal{C}$ un ensemble de plongements lisses $\gamma \colon S^{n-d} \to \mathbf{R}^n \setminus \Gamma$ qui est stable par homotopie dans $\mathbf{R}^n \setminus \Gamma$. Les compétiteurs sont les ensembles relativement fermés $E \subset \mathbf{R}^n \setminus \Gamma$ tels que pour tout $\gamma \in \mathcal{C}$, $\gamma \cap E \neq \emptyset$.

Ils montrent que c'est aussi le cas pour le problème de Reifenberg (avec un groupe de coefficient compact) dans [DPDRG3]. Dans le cas général, ce n'est pas forcément vrai mais les auteurs obtiennent tout de même une information sur la structure de E_∞: l'ensemble E_∞ est Almgren-minimal dans $\mathbf{R}^n \setminus \Gamma$. Dans les articles [DLDRG3] et [DPDRG3], les auteurs remplacent les mesures de Hausdorff par des Lagrangiens anisotropes plus généraux. Le théorème de rectifiabilité de Preiss est à son tour remplacé par une extension du théorème de rectifiabilité d'Allard pour les varifolds ([DPDRG2]).

Dans cette thèse, on généralise aux suites quasiminimisantes, la limite faible de suites minimisantes introduite par De Lellis, De Philippis, De Rosa, Ghiraldin et Maggi. On montre qu'une limite faible d'ensembles quasiminimaux est quasiminimal. Ce résultat est analogue au théorème de passage à la limite de David pour la convergence de Hausdorff locale. Notre démonstration est inspirée par celle de David tout en étant plus simple. De plus, elle n'a pas recours pas au théorème de rectifiabilité de Preiss, ni à la théorie des varifolds. Elle n'est pas non plus limitée à l'espace ambiant $\mathbf{R}^n \setminus \Gamma$. Ceci signifie que l'on peut prendre en compte la partie du compétiteur qui se trouve sur la frontière. Bien que l'on minimise la mesure de Hausdorff, nos techniques pourraient s'adapter aux intégrants elliptiques. On déduit une méthode directe pour prouver l'existence de solutions à divers problèmes de Plateau, même quand on minimise les compétiteurs sur la frontière. On l'applique ensuite à deux variantes du problème de Reifenberg (minimisant ou non la frontière libre) pour tous les groupes de coefficient. D'autre part, on propose une structure pour construire des projections de Federer-Fleming ainsi qu'une nouvelle estimation sur le choix des centres de projection.

1.3 Principaux résultats

Notre espace ambiant est un ouvert X de $\mathbf{R}^n$. On fixe un entier $1 \leq d \leq n$. L'expression *un ensemble fermé* $S \subset X$ signifie que S est relativement fermé dans X. Pour $x \in X$ et $s \in [0, \infty]$, on définit

$$r_s(x) = \min \left\{ \tfrac{s}{1+s} \mathrm{d}(x, X^c), s \right\}.$$

On exprime l'échelle d'une boule $B(x, r)$ dans X comme le paramètre $s \in [0, \infty]$ tel que $r = r_s$. L'interval $[0, 1]$ est noté I. Étant donné un ensemble $E \subset X$ et une fonction $F \colon I \times E \to X$, la notation F_t signifie $F(t, \cdot)$. Étant donnés deux ensembles $A, B \subset \mathbf{R}^n$, la notation $A \subset\subset B$ signifie qu'il existe un compact $K \subset \mathbf{R}^n$ tel que $A \subset K \subset B$. Pour une boule U de centre x et de rayon r pour $h \geq 0$, le symbole hU désigne la boule de centre x et de rayon hr. La frontière est un ensemble fermé $\Gamma \subset X$ qui a de la structure et de la régularité (on en discutera à la fin de l'introduction).

Definition (Déformation glissante le long d'une frontière)**.** Soit E un fermé H^d localement fini de X. Une *déformation glissante* de E dans un ouvert $U \subset X$ est une application lipschitz $f\colon E \to X$ telle qu'il existe une homotopie continue $F\colon I \times E \to X$ réalisant les conditions suivantes:

$$\begin{aligned}
&F_0 = \mathrm{id}\\
&F_1 = f\\
&\forall t \in I,\ F_t(E \cap \Gamma) \subset \Gamma\\
&\forall t \in I,\ F_t(E \cap U) \subset U\\
&\forall t \in I,\ F_t = \mathrm{id} \text{ in } E \setminus K,
\end{aligned}$$

où K est un sous-ensemble compact de $E \cap K$. Alternativement, le dernier axiome peut s'écrire

$$\{\, x \in E \mid \exists\, t \in I,\ F_t(x) \neq x \,\} \subset\subset E \cap U.$$

On définit les déformations glissantes globales dans U comme les déformations glissantes de X dans U (l'ensemble E est remplacé par X dans la définition précédente).

Les principaux objets de cette thèse sont les *ensembles minimaux* et *quasiminimaux*. Les ensembles quasiminimaux sont des surfaces dont les déformations glissantes ne peuvent pas réduire l'aire en dessous d'un pourcentage fixé κ^{-1} (et modulo une erreur de petite taille). La contrainte topologique empêchant l'écrasement de l'aire peut venir de U car $F_t(E \cap U) \subset U$ et $F_t = \mathrm{id}$ dans $X \setminus U$ ou de le frontière car $F_t(E \cap \Gamma) \subset \Gamma$

Definition (Ensembles quasiminimaux)**.** Soit E un fermé H^d localement fini de X. Soit $\mathcal{P} = (\kappa, h, s)$ un triplet de paramètres composé de $\kappa \geq 1$, $h \geq 0$ et d'une échelle$s \in]0, \infty]$. On dit que E est *$\mathcal{P}$-quasiminimal* dans X si pour tout $x \in E$, pour tout $0 < r \leq r_s(x)$ et pour toute déformation glissante f de E dans $U = B(x, r)$, on a

$$H^d(W_f) \leq \kappa H^d(f(W_f)) + hH^d(E \cap hU),$$

où

$$W_f = \{\, x \in E \mid f(x) \neq x \,\}.$$

On dit que E est minimal dans le cas $\mathcal{P} = (1, 0, \infty)$.

Ces notions trouvent leur inspiration dans les travaux d'Almgren. Elles ont été introduites par David et Semmes dans [DS] (sans frontière) puis généralisées par David dans [D6] (avec une frontière). Le facteur κ permet d'inclure les graphes lipschitziens parmi les ensembles quasiminimaux. De son côté, $hH^d(E \cap hU)$ est un un terme de plus petit ordre qui élargit la classe des fonctionnelles minimisées. En pratique, on suppose que h est assez petit en fonction de n et Γ. On peut aussi considérer un terme de la form $h\mathrm{diam}(U)^d$ mais il pose problème pour notre théorème de passage à la limite. Les premières propriétés des ensembles quasiminimaux sont l'invariance bilipschitz (Remarque 3.1.2), l'Ahlfors régularité (Proposition 5.1.1) et la rectifiabilité (Corollaire 5.1.1).

Le principal résultat de la thèse est qu'une limite faible d'ensembles quasiminimaux est un ensemble quasiminimal. On omet les hypothèses sur la frontière dans les deux prochains énoncés. Les résultats complets sont le Théorème 5.2.1 et le Corollaire 6.1.1.

Theorem (Théorème de passage à la limite). *On fixe un triplet de paramètres* (κ, h, s) *et un paramètre additionnel* $\kappa_0 \geq 1$. *On suppose que* h *est assez petit (en fonction de* n *et* Γ*). Soit* (E_i) *une suite de fermés* H^d *localement finis de* X *satisfaisant les conditions suivantes:*

1. *la suite des mesures de Radon* $(H^d \llcorner E_i)_i$ *a une limite faible* μ *dans* X*;*

2. *pour tout* $x \in \mathrm{spt}(\mu)$*, pour tout* $0 < r \leq r_s(x)$*, il existe une suite* $(\varepsilon_i) \to 0$ *telle que pour toute déformation glissante globale* f *dans* $U = B(x,r)$*,*

$$H^d(E_i \cap W_f) \leq \kappa H^d(f(E_i \cap W_f)) + hH^d(E_i \cap hU) + \varepsilon_i$$

   et

$$H^d(E_i \cap U) \leq \kappa_0 H^d(f(E_i \cap U)) + \varepsilon_i.$$

Alors $E = \mathrm{spt}(\mu)$ *est* $(\kappa, \kappa_0 h, s)$*-quasiminimal et on a*

$$H^d \llcorner E \leq \mu \leq \kappa_0 H^d \llcorner E.$$

Ce théorème fait suite aux travaux de David ainsi que de De Lellis, De Philippis, De Rosa, Ghiraldin et Maggi. David a démontré qu'une limite de Hausdorff locale d'ensembles quasiminimaux est quasiminimal dans [D6] (Theorem 10.8). Les limites de Hausdorff sont utilisées pour résoudre des problèmes de Plateau depuis Reifenberg. D'autre part, l'équipe italienne a développé une notion de limite faible pour les suites minimisantes de compétiteurs. Notre théorème étend cette notion aux suites quasiminimisantes et montre qu'une limite faible d'ensembles quasiminimaux est quasiminimal. On déduit une nouvelle démonstration de la méthode directe de l'équipe italienne (voir [DLGM], théorème 7 et [DPDRG1], théorème 1.8) mais elle ne repose ni sur le théorème de rectifiabilité de Preiss, ni sur la théorie des varifolds. Notre version permet de minimiser les compétiteurs sur la frontière.

Corollary (Méthode directe). *Soit* $\mathcal{C}$ *une classe de fermés de* X *telle que*

$$m = \inf \{ H^d(E) \mid E \in \mathcal{C} \} < \infty$$

et on suppose que pour tout $E \in \mathcal{C}$, pour toute déformation glissante f de E dans X,

$$m \leq H^d(f(E)).$$

Soit (E_k) une suite minimisante pour H^d dans $\mathcal{C}$. À une sous-suite près, il existe un ensemble coral[2] *minimal E_∞ dans X tel que*

$$H^d \llcorner E_k \rightharpoonup H^d \llcorner E_\infty.$$

où la flèche $\rightharpoonup$ dénote la convergence faible des mesures de Radon dans X. En particulier, $H^d(E_\infty) \leq m$.

En application, on définit les compétiteurs de Reifenberg à bord libre et on résout deux formulation du problème (avec ou sans le bord libre). Dans les travaux de Reifenberg, la limite de Hausdorff d'une suite minimisante est un compétiteur mais l'aire n'est pas semicontinue inférieurement. Reifenberg a travaillé avec un groupe de coefficient compact pour construire une suite minimisante alternative. Avec les limites faibles, la semicontinuité inférieure découle du théorème précédent mais il nous faut encore montrer que la limite est un compétiteur. On le démontre pour tout groupe de coefficient en construisant des recouvrements appropriés.

Definition (Compétiteurs de Reifenberg)**.** Fixons un fermé Γ de $\mathbf{R}^n$ et un sous-groupe L de $H_{d-1}(\Gamma)$. Un compétiteur de Reifenberg est un compact $E \subset \mathbf{R}^n$ tel que le morphisme induit par inclusion,

$$H_{d-1}(\Gamma) \longrightarrow H_{d-1}(E \cup \Gamma),$$

est nul sur L.

On verra que cette formulation est essentiellement équivalente à celle de Nakauchi.

Lemma. *Soit $(E_k) \subset \mathbf{R}^n$ une suite de compétiteurs de Reifenberg. Soit E un compact de $\mathbf{R}^n$. On suppose que*

1. *il existe un compact $C \subset \mathbf{R}^n$ tel que pour tout k, $E_k \subset C$;*
2. *pour tout ouvert $V \subset \mathbf{R}^n$ contenant $E \cup \Gamma$,*

$$\lim_k H^d(E_k \setminus V) = 0.$$

Alors E est un compétiteur de Reifenberg.

[2]Un ensemble $E \subset X$ est coral dans X si E est le support de $H^d \llcorner E$ dans X. De façon équivalente, E est fermé dans X et pour tout $x \in E$ et pour tout $r > 0$, $H^d(E \cap B(x,r)) > 0$.

Theorem (Reifenberg en minimisant le bord libre). *on suppose que*

$$m = \inf \{ H^d(E) \mid E \textit{ compétiteur de Reifenberg} \} < \infty$$

et qu'il existe un compact $C \subset \mathbf{R}^n$ tel que

$$m = \inf \{ H^d(E) \mid E \textit{ compétiteur de Reifenberg, } E \subset C \} .$$

Alors il existe un compétiteur de Reifenberg $E \subset C$ tel que $H^d(E) = m$.

Le prochain théorème est similaire au théorème 1.3 de [Fn] (qui se fonde sur la construction de Feuvrier) et au théorème 3.4 de [DPDRG3] (qui se fonde sur les limites faibles de suites minimimisantes). Cependant, nous n'avons pas encore traité les intégrants elliptiques.

Theorem (Reifenberg sans minimiser le bord libre). *On suppose que*

$$m = \inf \{ H^d(E \setminus \Gamma) \mid E \textit{ compétiteur de Reifenberg} \} < \infty$$

et qu'il existe un compact $C \subset \mathbf{R}^n$ tel que

$$m = \inf \{ H^d(E \setminus \Gamma) \mid E \textit{ compétiteur de Reifenberg, } E \subset C \} .$$

Alors il existe un compétiteur de Reifenberg $E \subset C$ tel que $H^d(E \setminus \Gamma) = m$.

Remark. Si Γ est compact et $H^d(\Gamma) < \infty$, cela revient à minimiser $H^d(E)$ parmi les compétiteurs de Reifenberg contenant Γ.

1.4 Outils du théorème de passage à la limite

1.4.1 Une classe de déformations

Le lemme suivant est inspiré des techniques de David dans [D6]. Il contient la principale partie technique de la preuve. La constante de lipschitz d'une application f est notée $\|f\|_L$.

Lemma. *Soit f une déformation glissante globale dans un ouvert $U \subset X$. Soit W un ouvert de U et soit $E \subset W$ un ensemble H^d mesurable, H^d fini et H^d rectifiable. Pour tout $\varepsilon > 0$, il existe une déformation glissante gloable g dans U et un ouvert $V \subset W$ telle que $g - f$ a un support compact inclu dans W, $|g - f| \leq \varepsilon$, $\|g - f\|_L \leq C\|f\|_L$ (where $C \geq 1$ depends on n, Γ) and*

$$\begin{aligned} &H^d(E \setminus V) \leq \varepsilon \\ &H^d(g(V)) \leq H^d(f(E)) + \varepsilon. \end{aligned}$$

En résumé, on construit une déformation glissante g qui «écrase» un presque-voisinage de $E \cap W$ sur $f(E \cap W)$.

1.4.2 La projection de Federer-Fleming

La projection de Federer-Fleming pour les ensembles a été introduite par David et Semmes dans [DS] en suivant les idées de la projection pour les courants. Il s'agit d'envoyer un ensemble E de dimension d dans le d-squelette d'un réseau de cube tout en contrôlant la mesure de l'image. Pour commencer, on choisit un centre de projection non adhérent à E dans chaque cube. On réalise alors une projection radiale dans chaque cube pour envoyer E sur les faces de dimensions $(n-1)$. Comme les centres de projection ne sont pas adhérents à E, chaque projection radiale est lipschitz sur E et la mesure de E est multipliée par la constante de lipschitz. Malheureusement, on ne contrôle pas cette constante a priori. David et Semmes sont parvenus à choisir des centres de projection tels que la mesure de E est multipliée par une constante qui ne dépend que de n. On itère le procédé dans chaque face de dimension $(n-1)$ pour projeter E dans les faces de dimension $(n-2)$. On s'arrête en général quand E est envoyé dans les faces de dimension d car on ne peut plus s'assurer qu'il existe des centres de projections non adhérents à l'image de E. On présente le lemme de David et Semmes sur le choix des centres de projection:

Lemma ([DS], Lemma 3.22)**.** *Soit Q un cube $\mathbf{R}^n$, soit E un sous-ensemble borélien de Q. Alors*

$$\frac{1}{|Q|}\int_{\frac{1}{2}Q}^{*} H^d(\phi_x(E))\,\mathrm{d}x \leq CH^d(E),$$

où ϕ_x est la projection radiale de $Q \setminus x$ sur ∂Q, C est une constante qui ne dépend que de n.

On développe un lemme analogue pour une autre jauge qui a l'avantage d'annuler la partie purement non rectifiable de E. Commençons par rappeler quelques notations. La grassmanienne $G(d,n)$ est l'ensemble de tous les d-plans vectoriels de $\mathbf{R}^n$. Un plan vectoriel V peut être représenté par sa projection orthogonale p_V. Ainsi, la norme d'opérateur par rapport à la norme euclidenne induit une distance naturelle sur $G(d,n)$. Cet espace est aussi muni d'une mesure canonique invariante $\mathrm{d}V$. On renvoit à l'appendice B (et alternativement, [Mat], Section 3). On définit la jauge ζ^d sur les boréliens de $\mathbf{R}^n$ par

$$\zeta^d(E) := \int_{G(d,n)} H^d(p_V(E))\,\mathrm{d}V.$$

On appelle *cellule* une face de cube de dimension quelconque. Pour une cellule A, on pose la restriction de cette jauge à A:

$$\zeta^d \llcorner A(E) := \int_{G(\mathrm{aff}(A),d)} H^d(p_V(E \cap A))\,\mathrm{d}V,$$

où $\mathrm{aff}(A)$ est l'enveloppe affine de A et $G(\mathrm{aff}(A), d)$ est l'ensemble des d-plans vectoriels de $\mathrm{aff}(A)$ centrés en un point quelconque de $\mathrm{aff}(A)$. On rappelle aussi la notion d'intégrale supérieure. Pour un ensemble $S \subset \mathbf{R}^n$ muni d'une mesure μ et pour une fonction $f\colon S \to [0, +\infty]$, l'*intégrale supérieure* de f est définie par

$$\int_S^* f \,\mathrm{d}\mu := \inf_\psi \int_S \psi \,\mathrm{d}\mu$$

pour tout fonction μ-mesurable $\psi\colon S \to [0, +\infty]$ telle que $f \leq \psi$. Il n'est pas sûr qu'on ait réellement besoin de l'intégrale supérieure mais on ne s'en soucie pas.

Lemma. *Soit Q un cube de $\mathbf{R}^n$, soit E un sous-ensemble borélien de Q. Alors*

$$\frac{1}{|Q|} \int_{\frac{1}{2}Q}^* \sup_A \zeta^d \llcorner A(\phi_x(E)) \,\mathrm{d}x \leq C\zeta^d(E),$$

où ϕ_x est la projection radiale de $Q \setminus x$ sur ∂Q, le sup *est indexé par toutes les cellules $A \subset \partial Q$ et C est une constante qui ne dépend que de n.*

La gauge $\zeta^d \llcorner A$ se simplifie en $H^d \llcorner A$ quand A est une cellule de dimension d. À la dernière étape de la projection, une fois que l'ensemble E a été envoyé dans le d-squelette, on obtient une estimation de la forme

$$H^d(\phi(E)) \leq C \int_{G(d,n)} H^d(p_V(E)) \,\mathrm{d}V.$$

En particulier, la partie purement non rectifiable de E est annulée par le côté droit. Feuvrier avait déjà montré qu'il était possible de choisir les centres de projections pour supprimer la partie non rectifiable avec une méthode différente ([Feuv], Lemma 4.3.16).

Considérons un ensemble quasiminimal E. On peut tester la propriété de quasiminimalité contre des projections de Federer-Fleming dans des grilles de cubes. Cela mène à des estimations de densité (see Corollary 5.1.1): pour tout $x \in E^*$ (le support de $H^d \llcorner E$) et pour tous les petits rayons $r > 0$,

$$C^{-1} r^d \leq H^d(E \cap B(x, r)) \leq C \int_{G(d,n)} H^d(p_V(E \cap B(x, 16\sqrt{n}r))) \,\mathrm{d}V.$$

On obtient en même temps l'Ahlfors régularité d-dimensionnelle de E^*,

$$C^{-1} r^d \leq H^d(E \cap B(x, r)) \leq C r^d,$$

et la rectifiabilité H^d de E. On utilisera de telles estimations pour montrer que l'ensemble E_∞ du théorème de passage à la limite est H^d rectifiable. La projection de Federer-Fleming est donc un outil important de la démonstration. Si on veut que les projections de Federer-Fleming soient des déformations glissantes, il faut néanmoins qu'elles préservent la frontière.

1.4.3 Une frontière régulière

L'ensemble Γ qui joue le rôle de frontière est d'abord un fermé de X. On est susceptible d'utiliser trois hypothèses supplémentaires.

1. Premièrement, on peut vouloir que Γ soit un *retract lipschitz local* de X. Ceci signifie qu'il existe un ouvert $O \subset X$ contenant Γ et une application lipschitz $p\colon O \to \Gamma$ telle que $p = \mathrm{id}$ sur Γ.

2. Deuxième, on peut vouloir que pour tout $\varepsilon > 0$, il existe un ouvert $O \subset X$ et une application $(1+\varepsilon)$-lipschitz $p\colon O \cup \Gamma \to \Gamma$ tels que $H^d(\Gamma \setminus O) = 0$ et $p = \mathrm{id}$ sur Γ. L'intérêt des retractions $(1+\varepsilon)$-lipschitz est qu'elles n'agrandissent pas trop les mesures de Hausdorff. La condition précédente autorise des coins non convexes pourvu qu'ils soient H^d négligeables.

3. Troisièmement, on voudrait pouvoir construire des projections de Federer-Fleming qui préservent la frontière.

Au chapitre 4, on propose une structure générale pour construire des projections de Federer-Fleming. Ainsi, on peut définir une large de classe de frontières compatibles avec les projections de Federer-Fleming. On considère un ensemble K de faces de cube de diamètres variables; c'est la liste des faces dans lesquelles on effectuer une projection radiale. Si la relation d'inclusion $\subset$ dans K est compatible avec la topologie en un certain sens, ces projections radiales peuvent être recollées et composées pour former une projection de Federer-Fleming. La structure de K est similaire à un CW-complexe mais le bord d'une face n'est pas forcément recouvert par d'autres faces. Sur la figure 1.5 par exemple, les arêtes externes ne font pas partie de K.

Definition. Une *cellule* est une face de cube dans $\mathbf{R}^n$. L'*intérieur* d'une cellule est l'intérieur relatif à son enveloppe affine. Le *support* d'un ensemble K de cellules est défini par

$$|K| = \bigcup \{\, A \in K \,\}.$$

Un *complexe* K est un ensemble de cellules tels que

1. les intérieurs des cellules $\{\, \mathrm{int}(A) \mid A \in K \,\}$ sont mutuellement disjoints;

2. tout $x \in |K|$ admet un voisinage relatif dans $|K|$ qui rencontre un nombre fini de cellules $A \in K$;

3. pour tout cellule $A \in K$, l'ensemble

$$V_A := \bigcup \{\, \mathrm{int}(B) \mid B \in K \text{ contains } A \,\}$$

est un voisinage relatif de $\mathrm{int}(A)$ dans $|K|$.

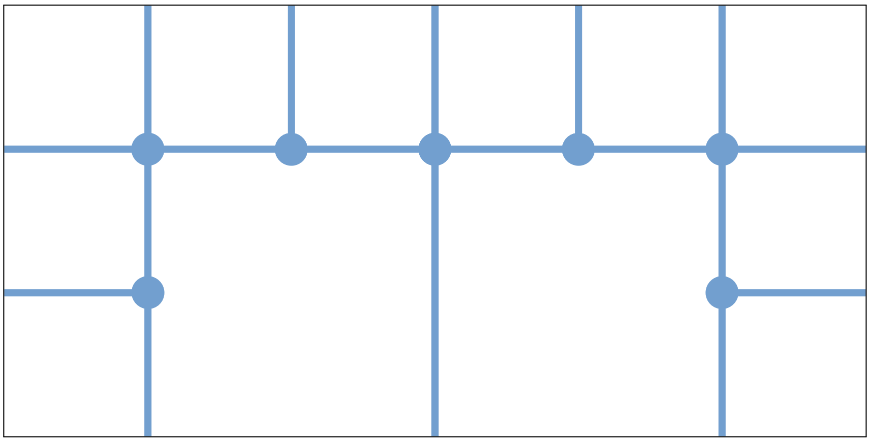


Figure 1.5: Dans cet example, on fait une projection radiale dans chaque carré (blanc), puis dans chaque arête interne (bleu). On regroupe ces faces dans un ensemble K.

On présente une méthode simple pour construire de tels complexes en recollant des complexes élémentaires. Cette technique est analogue à une limite directe en algèbre. Ainsi, on construit des complexes qui remplissent un ouvert de façon similaire à une décomposition de Whitney. Enfin, on définit nos frontières comme des images bilipschitz d'unions de cellules de tels complexes.

Chapter 2

Introduction

2.1 From soap films to the Plateau problem

The Plateau problem consists in minimizing the area of a surface spanning a boundary. This paragraph is devoted to explaining this statement and its origins. Joseph Plateau was a Belgian physicist and mathematician of the nineteenth century. He is renowned for his pioneering works on persistence of vision and his animation device, the phenakistiscope. He also studied surface tension and *soap films*. By dipping a rigid frame in a soapy solution, one gives form to a soap film bordered by the frame. The frame acts as the *boundary* of the soap film whereas the soap film *spans* the frame. In the simplest case, the boundary is a circle and the soap film is the correspnoding disk. If the boundary is the skeleton of a polyhedron, the surface is a suprising system of faces that Plateau called *laminar system*. Plateau observed that these

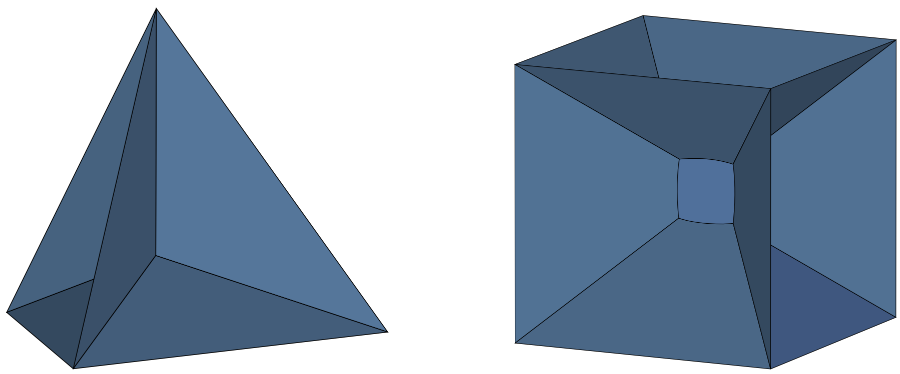
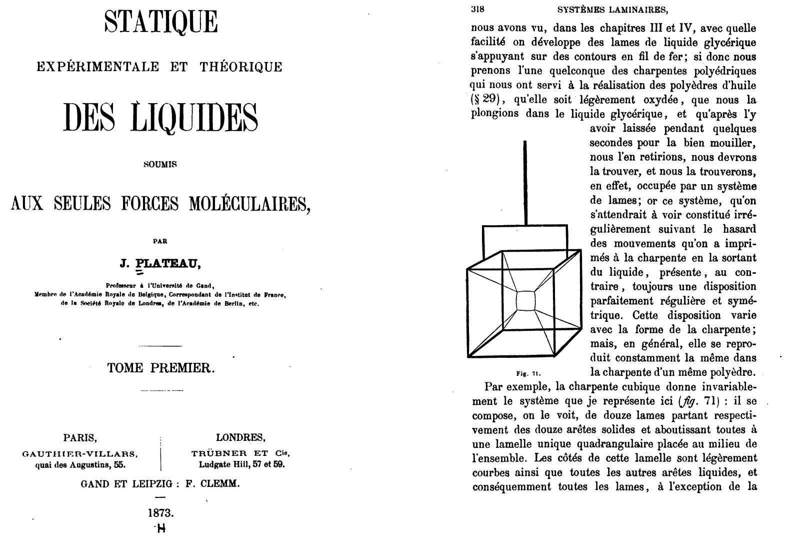


Figure 2.1: Soap films spanning the skeleton of a tetrahedron (left) and the skeleton of a cube (right).

systems were regular and symmetric and he undertook to understand their geometrical configuration.

How do we explain the singular shapes of soap films? The molecules of soap interface the air and the water, giving elasticity and stability to the surface. This one tends to minimize its area as much as possible to reach an equilibrium position. In our situation, "as much as possible" means "while spanning the boundary". Mathematicians have isolated this property. They define a Plateau problem as the statement of the "surfaces spanning a given boundary" and of their "area". As solution is a surface spanning the boundary with a minimal area in the sense of theses definition. Plateau problem simplify the physical reality of soap films but they also generalize it because they can be formulated in more general spaces than our 3-dimensional Euclidian space. However, these concepts proved intricate to formalize.


STATIQUE

EXPÉRIMENTALE ET THÉORIQUE

DES LIQUIDES

SOUMIS

AUX SEULES FORCES MOLÉCULAIRES,

PAR

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1873.

318 SYSTÈMES LAMINAIRES,

nous avons vu, dans les chapitres III et IV, avec quelle facilité on développe des lames de liquide glycérique s'appuyant sur des contours en fil de fer; si donc nous prenons l'une quelconque des charpentes polyédriques qui nous ont servi à la réalisation des polyèdres d'huile (§ 29), qu'elle soit légèrement oxydée, que nous la plongions dans le liquide glycérique, et qu'après l'y avoir laissée pendant quelques secondes pour la bien mouiller, nous l'en retirions, nous devrons la trouver, et nous la trouverons, en effet, occupée par un système de lames; or ce système, qu'on s'attendrait à voir constitué irrégulièrement suivant le hasard des mouvements qu'on a imprimés à la charpente en la sortant du liquide, présente, au contraire, toujours une disposition parfaitement régulière et symétrique. Cette disposition varie avec la forme de la charpente; mais, en général, elle se reproduit constamment la même dans la charpente d'un même polyèdre.

Fig. 71.

Par exemple, la charpente cubique donne invariablement le système que je représente ici (*fig.* 71) : il se compose, on le voit, de douze lames partant respectivement des douze arêtes solides et aboutissant toutes à une lamelle unique quadrangulaire placée au milieu de l'ensemble. Les côtés de cette lamelle sont légèrement courbes ainsi que toutes les autres arêtes liquides, et conséquemment toutes les lames, à l'exception de la

Figure 2.2: In his book *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires* [Pl], Plateau presents his experiments and develops laws describing laminar systems.

2.2 Approaches of the Plateau problem

What do we expect from an optimal formulation of the Plateau problem?

1. It should include a definition of "surfaces spanning a given boundary" (also called *competitors*) and their "area".

2. It should lend itself to the direct method of the calculus of variation (find a solution as a limit of a minimizing sequence). It is in general dif-

ficult to have both a compactness principle on the class of competitors and a lower semicontinuity principle on the area.

3. It should stay close to Plateau's orginal motivations: describing soap films. This means that we want to work with the kind of boundaries one can build with a wire and obtain the corresponding soap films as solutions.

We present a few approaches below (the reader can also be interested by [D5]). The boundary will be denoted by Γ.

The classical approaches minimizes over parametrizations whose base space is a disk. This is the formulation of Radò and Douglas in the 1930s (see [Rado] and [Doug]). Let us say a bit more. Let the closed unit disk and the unit circle of the Euclidean plane be denoted by D^2 and S^1 respectively. Fix a rectifiable Jordan curve Γ in $\mathbf{R}^3$ (a homeomorphic image of S^1 whose length is finite). A surface spanning Γ is defined as a continuous map $f\colon D^2 \to \mathbf{R}^3$ such that f sends S^1 homemorphically onto Γ. The corresponding area is defined as the *total variation* of f (Radò and Douglas called it the *area-integral* or *area functional*). If f is a C^1 map, the total variation can be computed directly by the formula

$$\int_D Jf \, \mathrm{d}x$$

where Jf is the Jacobian of f,

$$Jf = \begin{vmatrix} |\partial_1 f|^2 & \partial_1 f \cdot \partial_2 f \\ \partial_1 f \cdot \partial_2 f & |\partial_2 f|^2 \end{vmatrix}^{\frac{1}{2}}.$$

The total variation differs from the area of the image because of eventual multiplicities but it has the advantage to be lower semicontinuous. Radò and Douglas are looking for a smooth immersion f which minimizes the total variation. Thus, f would locally parametrize a minimal surface in the classical sense. It is yet difficult to let converge a minimizing sequence of parametrizations to a smooth immersion. The problem owes particularly its complexity to the diversity of parametrizations of a given surface. Even a sequence of them may not converge at all. As curves which have the parametrization by arc length, 2-dimensional surfaces have privileged parametrizations: the conformal parametrizations[1]. Moreover, a conformal parametrization f describes a minimal surface if and only if f is harmonic. This leads Radò and Douglas to specify the problem as the research of a harmonic and conformal map minimizing the total variation. Radò works with a sequence of "approximately conformal" competitors minimizing the total variation. Douglas minimizes a simpler functional for which the first variation can only

[1] A map f is conformal if $|\partial_1 f| = |\partial_2 f|$ and $\partial_1 f \cdot \partial_2 f = 0$.

vanish at conformal maps. The main drawback of this approach is that it cannot be generalized to three or more variables because it depends too much on the theory of conformal mappings. In addition, we would prefer solutions which describe the singularies observed by Plateau. A solution of Radò and Douglas locally parametrizes a minimal surface but its full image has not a minimal area. Let us take for example a knot Γ whose solution has self-intersections. The pieces that meet are independently minimal but the way they intersect is not taken into account by the total variation. This explains why transverse planes can be found in the solutions of Radó and Douglas whereas they do not exists in soap films. We also point out that this approach only compares parametrizations defined on the disk, fixing thus the topological type of the competitors.

In the sixties appeared two approaches which include all dimensions and codimensions and whose competitors have a varying topological type: Federer-Fleming and Reifenberg. Federer and Fleming work with *integral currents* and minimize their *mass* (an area computed with multiplicity). They have developped the *flat convergence* for which integral currents enjoy a compactness principle and the mass is lower semicontinuous. They deduce the existence of mass minimizing rectifiable currents whose boundary is given. However, their solutions are too smooth away from the boundary to describe the singularities of soap films. Instead of minimizing the mass, one could minimize the size (the area of the support). Such point of view is very close to the one of Reifenberg. Reifenberg works with *sets* of the Euclidean space which span a boundary in the sense of algebraic topology and minimizes their (spherical) *Hausdorff measure*. A d-dimensional set E spans a boundary Γ if E contains Γ and cancel the $(d-1)$-cycles of Γ (or a subgroup of them).

Definition (Reifenberg competitors)**.** Fix Γ a compact subset of $\mathbf{R}^n$ and let L be a subgroup of the homology group $H_{d-1}(\Gamma)$. A Reifenberg competitor is a compact subset $E \subset \mathbf{R}^n$ such that E contains Γ and the morphism induced by inclusion,

$$H_{d-1}(\Gamma) \longrightarrow H_{d-1}(E),$$

is zero on L.

We recall the definition of the spherical d-dimensional Hausdorff measure.

Definition (Spherical Hausdorff measure)**.**

$$S^d(E) := \lim_{\delta \to 0^+} \inf \left\{ \sum_k \mathrm{diam}(B_k)^d \;\middle|\; E = \cup_{k \in \mathbf{N}} B_k, \mathrm{diam}(B_k) \leq \delta \right\}.$$

where (B_k) is a sequence of balls.

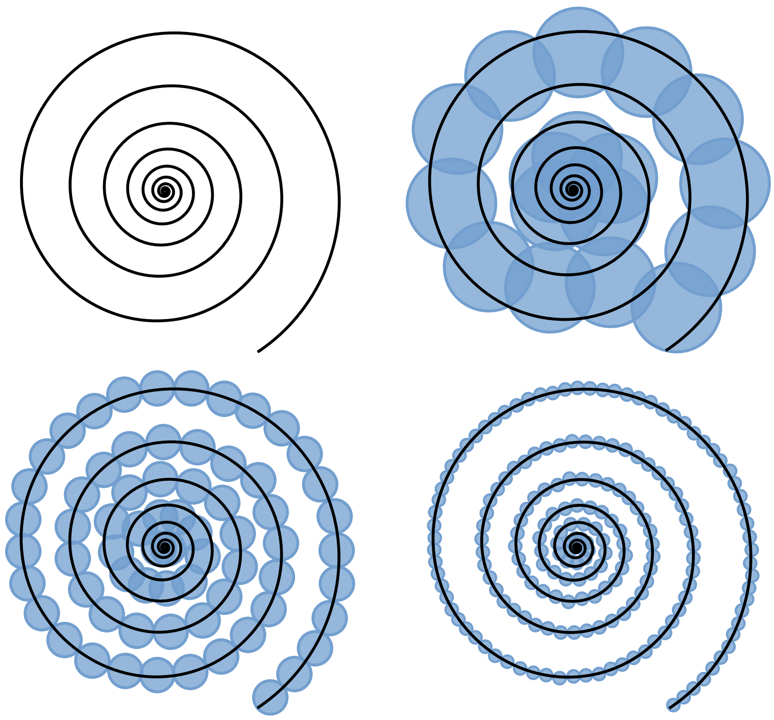

Figure 2.3: Computation of the S^1 measure of a spiral.

The most classical example is the case where Γ is composed of two disks $C_1 \cup C_2$ of same radius and centred on a same axis of symmetry. The soap films spanning Γ are the two parallel disks, the catenoid and the catenoid with a disk (this surface is composed of two pieces of catenoid which meet an intermediate disk with an angle of 120 degrees). The two formers only exist if the distance between the circles is small enough compared to their radius. One can relate this phenomenon to the fact that the area of these films area is less than the area of the two disks. We adress the point of view of Reifenberg, assuming that the circles are close to ignore the parallel disks. We are going to see that the choice of L discriminates the catenoid and the catenoid with disk. If L is the whole group $H_1(\Gamma; \mathbf{Z})$, then the catenoid is not a competitor and the minimizer is the catenoid with a disk. If L is the subgroup of $H_1(\Gamma; \mathbf{Z})$ generated by $\gamma_2 - \gamma_1$, where γ_i is a cycle corresponding to C_i, then the Reifenberg minimizer is the catenoid. We conclude that to interpret a given soap film as a Reifenberg minimizer, one has to specify the group to be canceled.

Reifenberg proved the existence of a solution to his problem in 1960 for the Čech homology theory and compact Abelian coefficient groups ([Rei]). The continuity property of the Čech theory implies that a Hausdorff limit of competitors is a competitor. Thus, a solution can be searched as a Hausdorff limit of a minimizing sequence.

Theorem (Continuity of the Čech Homology)**.** *Let (E_k) be a decreasing sequence of compact sets. Then for every $d \in \mathbf{N}$,*

$$H_d(E_\infty) = \lim_k H_d(E_k),$$

where $E_\infty = \bigcap_k E_k$, $\lim_k H_d(E_k)$ is the inverse limit with respect to the morphisms induced by the inclusions $E_{k+1} \subset E_k$ and the $=$ sign is the morphism induced by the inclusions $E_\infty \subset E_k$.

However, the area is not lower semicontinuous with respect to Hausdorff limits. For instance, one can imagine a minimizing sequence which has more and more dense tentacles so that the limit set is too large. Reifenberg worked with compact coefficient groups to have the Excision Axiom and thus to be able to cut out the tentacles and patch the holes. His construction leads to an alternative minimizing sequence for which the area is lower semicontinuous. Nakauchi ([Na]) stated and solved a free boundary variant in 1984 (for compact Abelian coefficient groups as well). We mean by this that the intersection $E \cap \Gamma$ varies among the competitors E.

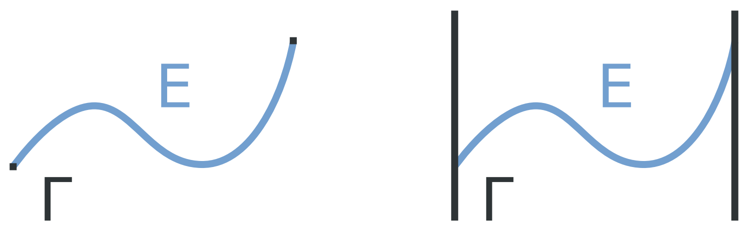

Figure 2.4: Two types of Plateau problem. The competitors have a fixed boundary on the left and a free boundary on the right.

Definition (Nakauchi competitors)**.** Fix a compact subset Γ of $\mathbf{R}^n$ and a subgroup L of $H_{d-1}(\Gamma)$. The Nakauchi competitors are the compacts sets $E \subset \mathbf{R}^n$ such that for all $v \in L$, there exists $u \in H_{d-1}(E \cap \Gamma)$ such that $i_*(u) = v$ and $i'_*(u) = 0$ where i_* and i'_* are the morphisms induced by inclusions:

$$\begin{tikzcd} & H_{d-1}(\Gamma) \\ H_{d-1}(E \cap \Gamma) \arrow[ur, "i_*"] \arrow[dr, "i'_*"'] & \\ & H_{d-1}(E). \end{tikzcd}$$

Nakauchi minimizes $H^d(E \setminus \Gamma)$ but it would be also interesting to minimize $H^d(E)$ to take into account the free boundary. In 2015, Fang ([Fn]) gave a new proof of the Reifenberg and Nakauchi problems for all coefficient groups and minimizing $H^d(E \setminus \Gamma)$. His idea is to take advantage of the lower semicontinuity of the area on quasiminimal sets (we present them in the next paragraph). Thanks to a construction of Feuvrier ([Feuv]), he obtains an alternative minimizing sequence composed of quasiminimal sets. Fang also replaces the Hausdorff measures with more general functionals, called *elliptic integrands*.

Inspired by *restricted sets* of Almgren ([Alm]), David introduced the *sliding competitors* in [DS] and [D6]. He compares the area of surface with its image under continuous deformations. A *sliding deformation* of a set E is a Lipschitz map $\phi\colon E \to \mathbf{R}^n$ such that there exists a continuous map $\Phi\colon [0,1] \times E \to \mathbf{R}^n$ satisfying $\Phi(0,\cdot) = \mathrm{id}$, $\Phi(1,\cdot) = \phi$, $\Phi(t,\cdot) = \mathrm{id}$ outside a compact set and $\Phi(t, E \cap \Gamma) \subset \Gamma$. One can think of a soap film moving along a tube for example.

Definition (Sliding competitors)**.** Fix Γ a compact subset of $\mathbf{R}^n$ and E_0 a compact, H^d finite subset of $\mathbf{R}^n$. The sliding competitors induced by E_0 are the images of E_0 under sliding deformations.

The existence of solutions is still unknown! One may wonder why the initial deformation ϕ is Lipschitz whereas the homotopy Φ is merely continuous. The Lipschitz assumption is a legacy of the restricted sets of Almgren. It turns useful because Lipschitz maps have an action on currents, H^d measurable sets, H^d finite sets and H^d rectifiable sets. The continuity assumption on Φ is used to impose a topological constraint in the most general way. One can observe that the solutions of the Reifenberg problem or of the sliding problem satisfy an intermediate property: their area is minimal under sliding deformations. This motivates the notion of *sliding minimal sets*. They are a closed and H^d locally finite sets $E \subset \mathbf{R}^n$ such that for every sliding deformation ϕ of E,

$$H^d(E \cap W) \leq H^d(\phi(E \cap W)).$$

where H^d is the d-dimensional Hausdorff measure and W is the set

$$W = \{\, x \in \mathbf{R}^n \mid \phi(x) \neq x \,\}.$$

This property echoes the elasticity and stability of soap films under (small) deformations. Unfortunately, it is only in small dimensions and away from the boundary that we know good regularity results about them (see the Theorem of Jean Taylor, [Ta], [D4]). *Quasiminimal sets* satisfy a generalization of this definition where the measure H^d is replaced more general functionals and where the set W is localised in small balls. We know weak results of regularity on quasiminimal sets in all dimensions (we refer to [DS], [D6], [D2] and

[D3]). David studied the limiting properties of quasiminimizing sequences ([D1]). He proved especially that a local Hausdorff limit of quasiminimals sets is again a quasiminimal set ([D6], Theorem 10.8).

In [DLGM] and [DPDRG1], De Lellis, De Philippis, De Rosa, Ghiraldin and Maggi introduced a new type of direct method to solve Plateau problems. As Reifenberg, they work with sets and Hausdorff measures of the Euclidean space but they replace local Hausdorff limits by weak limits. Let us present their work in more details. The authors fix a closed set $\Gamma \subset \mathbf{R}^n$ (the boundary) and place themselves in $\mathbf{R}^n \setminus \Gamma$ (away from the boundary). They consider a class of relatively closed subsets of $\mathbf{R}^n \setminus \Gamma$ (the competitors). To a minimizing sequence (E_k) for H^d, they associate the sequence of Radon measures $(H^d \llcorner E_k)$. Then they let converge the sequence weakly to a Radon measure μ in the ambiant space $\mathbf{R}^n \setminus \Gamma$. The main point of the method is that, if the class of competitors is complete enough, one have

$$\mu = H^d \llcorner E_\infty$$

where E_∞ is the support of μ in $\mathbf{R}^n \setminus \Gamma$. In that sense, the set E_∞ is a weak limit of the minimizing sequence (E_k). This strategy yields in particular the lower semicontinuity of the area:

$$H^d(E_\infty) \leq \lim_k H^d(E_k).$$

The proof is based on the construction of good competitors. The authors go through an intermediate information: they show that E_∞ is H^d rectifiable thanks to the Preiss's rectifiability theorem. To solve the Plateau problem, it is left to see if E_∞ is a competitor. The authors show that this is the case for the problem of Harrison-Pugh ([HP]).

Definition (Harrison and Pugh competitors)**.** Fix Γ a compact subset of $\mathbf{R}^n$ and $\mathcal{C}$ a set of smooth embeddings $\gamma\colon S^{n-d} \to \mathbf{R}^n \setminus \Gamma$ which is stable under homotopies in $\mathbf{R}^n \setminus \Gamma$. The corresponding competitors are the relatively closed sets $E \subset \mathbf{R}^n \setminus \Gamma$ such that for every $\gamma \in \mathcal{C}$, $\gamma \cap E \neq \emptyset$.

They also show that this is the case for the problem of Reifenberg (with a compact coefficient group) in [DPDRG3]. In the general case, this is not necessarily true but the authors still obtain a piece of information about the structure of E_∞: the set E_∞ is Almgren minimal in $\mathbf{R}^n \setminus \Gamma$. In the articles [DLDRG3] and [DPDRG3], they authors replace the Hausdorff measure H^d by more general anisotropic Lagrangians. The Preiss's rectifiability theorem is in turn replaced by an extension of Allard's rectifiability theorem on varifolds ([DPDRG2]).

In this thesis, we generalize to quasiminimizing sequences, the weak limit of minimizing sequences introduced by De Lellis, De Philippis, De Rosa, Ghiraldin and Maggi. We show that a weak limit of quasiminimal sets is quasiminimal. This result is analogous to the limiting theorem of David for the local Hausdorff convergence. Our proof is inspired by David's one while being simpler. In addition, it does not resort to the Preiss's rectifiability theorem, neither the theory of varifolds. It is neither limited to the ambiant space $\mathbf{R}^n \setminus \Gamma$. This means that we are able to take into account the part of the competitor that lies on the boundary. Although we minimize the Hausdorff measure, our techniques could be adapted to elliptic integrands. We deduce a direct method to prove existence of solutions to various Plateau problem, even one minimizes the competitors on the boundary. We apply this method to two variants of the Reifenberg problem (minimizing or not the free boundary) for all coefficient groups. Furthermore, we propose a structure to build Federer-Fleming projections as well as a new estimate on the choice of the projection centers.

2.3 Main results

Our ambiant space is an open set X of $\mathbf{R}^n$. We fix an integer $1 \leq d \leq n$. The term *a closed set* $S \subset X$ means that S is relatively closed in X. For $x \in X$ and $s \in [0, \infty]$, we define

$$r_s(x) = \min\left\{\, \tfrac{s}{1+s}\mathrm{d}(x, X^c), s \,\right\}.$$

We express the scale of a ball $B(x,r)$ in X as the parameter $s \in [0,\infty]$ such that $r = r_s$. The interval $[0,1]$ is denoted by the capital letter I. Given a set $E \subset X$ and a function $F \colon I \times E \to X$, the notation F_t means $F(t, \cdot)$. Given two sets $A, B \subset \mathbf{R}^n$, the notation $A \subset\subset B$ means that there exists a compact set $K \subset \mathbf{R}^n$ such that $A \subset K \subset B$. For a ball U of center x and radius r, for $h \geq 0$, the symbol hU denotes the ball of center x and radius hr. The boundary is a closed subset $\Gamma \subset X$ which has some structure and regularity (we will discuss about it at the end of the introduction).

Definition (Sliding deformation along a boundary)**.** Let E be a closed, H^d locally finite subset of X. A *sliding deformation* of E in an open set $U \subset X$ is a Lipschitz map $f \colon E \to X$ such that there exists a continuous homotopy $F \colon I \times E \to X$ satisfying the following conditions:

$$\begin{aligned}
&F_0 = \mathrm{id}\\
&F_1 = f\\
&\forall t \in I,\ F_t(E \cap \Gamma) \subset \Gamma\\
&\forall t \in I,\ F_t(E \cap U) \subset U\\
&\forall t \in I,\ F_t = \mathrm{id} \text{ in } E \setminus K,
\end{aligned}$$

where K is some compact subset of $E \cap U$. Alternatively, the last axiom can be stated as

$$\{\, x \in E \mid \exists\, t \in I,\ F_t(x) \neq x \,\} \subset\subset E \cap U.$$

We define the global sliding deformations in U as the sliding deformations of X in U (the set E is replaced by X in the previous definition).

The main objects of the thesis are *minimal* and *quasiminimal sets*. Quasiminimal sets are surfaces for which sliding deformations cannot decrease the area below a fixed percentage κ^{-1} (and modulo an error of small size). The topological constraint preventing the collapse might come from U as $F_t(E \cap U) \subset U$ and $F_t = \mathrm{id}$ in $X \setminus U$ or from the boundary as $F_t(E \cap \Gamma) \subset \Gamma$.

Definition (Quasiminimal sets)**.** Let E be a closed, H^d locally finite subset of X. Let $\mathcal{P} = (\kappa, h, s)$ be a triple of parameters composed of $\kappa \geq 1$, $h \geq 0$ and a scale $s \in]0, \infty]$. We say that E is $\mathcal{P}$*-quasiminimal* in X if for all $x \in E$, for all $0 < r \leq r_s(x)$ and for all sliding deformation f of E in $U = B(x, r)$, we have

$$H^d(W_f) \leq \kappa H^d(f(W_f)) + hH^d(E \cap hU),$$

where $W_f = \{\, x \in E \mid f(x) \neq x \,\}$. We say that E is a minimal set in the case $\mathcal{P} = (1, 0, \infty)$.

These notions find their inspirations in the works of Almgren. They have been introduced by David and Semmes in [DS] (without boundary), then generalised by David in [D6] (with a boundary). The factor κ includes Lipschitz graphs among quasiminimal sets. For its part, $hH^d(E \cap hU)$ is a lower order term which broadens the class of functionals that are to be minimized. In practice, h is assumed small enough depending only on n and Γ. On can also consider a term of the form $h\mathrm{diam}(U)^d$ but it poses a problem to our limiting theorem. The first properties of quasiminimal sets are the bilipschitz invariance (Remark 3.1.2), the Ahlfors regularity (Proposition 5.1.1) and the rectifiability (Corollary 5.1.1).

The main result of this thesis is that a weak limit of quasiminimal sets is a quasiminimal set. We omit the assumptions on the boundary in the two next statements. The complete results are Theorem 5.2.1 and Corollary 6.1.1.

Theorem (Limiting theorem)**.** *Fix a triple of parameters (κ, h, s) and an additionnal parameter $\kappa_0 \geq 1$. Assume that h is small enough (depending on n and Γ). Let (E_i) be a sequence of closed, H^d locally finite subsets of X satisfying the following conditions:*

1. *the sequence of Radon measures $(H^d \llcorner E_i)$ has a weak limit μ in X;*
2. *for all $x \in \mathrm{spt}(\mu)$, for all $0 < r \leq r_s(x)$, there exists a sequence $(\varepsilon_i) \to 0$ such that for all global sliding deformation f in $U = B(x, r)$,*

$$H^d(E_i \cap W_f) \leq \kappa H^d(f(E_i \cap W_f)) + hH^d(E_i \cap hU) + \varepsilon_i$$

and

$$H^d(E_i \cap U) \leq \kappa_0 H^d(f(E_i \cap U)) + \varepsilon_i.$$

Then, $E = \mathrm{spt}(\mu)$ is $(\kappa, \kappa_0 h, s)$-quasiminimal in X and we have

$$H^d \llcorner E \leq \mu \leq \kappa_0 H^d \llcorner E.$$

This theorem follows the works of David and De Lellis, De Philippis, De Rosa, Ghiraldin and Maggi. David proved that a local Hausdorff limit of quasiminimal sets is quasiminimal in [D6] (Theorem 10.8). Hausdorff limits are used to solve Plateau problems since Reifenberg. On the other hand, the Italian team developped a notion of weak limit for minimizing sequences of competitors. Our theorem extends this notion to quasiminizing sequence and shows that a weak limit of quasiminimal sets is quasiminimal. We deduce a new proof of the direct method of the Italian team (see [DLGM], Theorem 7 and [DPDRG1], Theorem 1.8) but it does not rely on the Preiss's rectifiability theorem, neither the theory of varifolds. Our version allows to minimize the competitors on the boundary.

Corollary (Direct method)**.** *Let $\mathcal{C}$ be a class of closed subsets of X such that*

$$m = \inf \{ H^d(E) \mid E \in \mathcal{C} \} < \infty$$

and assume that for all $E \in \mathcal{C}$, for all sliding deformations f of E in X,

$$m \leq H^d(f(E)).$$

Let (E_k) be a minimizing sequence for H^d in $\mathcal{C}$. Up to a subsequence, there exists a coral[2] minimal set E_∞ in X such that

$$H^d \llcorner E_k \rightharpoonup H^d \llcorner E_\infty.$$

where the arrow $\rightharpoonup$ denotes the weak convergence of Radon measures in X. In particular, $H^d(E_\infty) \leq m$.

As an application, we define Reifenberg competitors with free boundary and solve two formulations of the problem (minimizing or not the free boundary). In the works of Reifenberg, the Hausdorff limit of a minimizing sequence is a competitor but the area is not lower semicontinuous. Reifenberg worked with a compact coefficient group to build an alternative minimizing sequence. With weak limits, the lower semicontinuity follows from the previous theorem but we have yet to show that the limit is a competitor. We prove it for any coefficient group by building suitable coverings.

Definition (Reifenberg competitors)**.** Fix a closed subset Γ of $\mathbf{R}^n$ and a subgroup L of $H_{d-1}(\Gamma)$. A Reifenberg competitor is a compact subset $E \subset \mathbf{R}^n$ such that the morphism induced by inclusion,

$$H_{d-1}(\Gamma) \longrightarrow H_{d-1}(E \cup \Gamma),$$

is zero on L.

[2] A set $E \subset X$ is coral in X if E is the support of $H^d \llcorner E$ in X. Equivalently, E is closed in X and for all $x \in E$ and for all $r > 0$, $H^d(E \cap B(x,r)) > 0$.

We will see that this formulation is essentially equivalent to the formulation of Nakauchi.

Lemma. *Let $(E_k) \subset \mathbf{R}^n$ be a sequence of Reifenberg competitors. Let E be a compact subset of $\mathbf{R}^n$. We assume that*

1. *there exists a compact set $C \subset \mathbf{R}^n$ such that for all k, $E_k \subset C$;*
2. *for all open sets V containing $E \cup \Gamma$,*

$$\lim_k H^d(E_k \setminus V) = 0.$$

Then E is a Reifenberg competitor.

Theorem (Reifenberg - minimizing the free boundary)**.** *We assume that*

$$m = \inf \{ H^d(E) \mid E \text{ Reifenberg competitor} \} < \infty$$

and that there exists a compact set $C \subset \mathbf{R}^n$ such that

$$m = \inf \{ H^d(E) \mid E \text{ Reifenberg competitor}, \; E \subset C \} .$$

Then there exists a Reifenberg competitor $E \subset C$ such that $H^d(E) = m$.

The next theorem is similar to Theorem 1.3 of [Fn] (which is based on Feuvrier's construction) and Theorem 3.4 of [DPDRG3] (which is based on weak limits of minimizing sequences). However, we have not yet dealt with elliptic integrands.

Theorem (Reifenberg - without minimizing the free boundary)**.** *We assume that*

$$m = \inf \{ H^d(E \setminus \Gamma) \mid E \text{ Reifenberg competitor} \} < \infty$$

and that there exists a compact set $C \subset \mathbf{R}^n$ such that

$$m = \inf \{ H^d(E \setminus \Gamma) \mid E \text{ Reifenberg competitor}, \; E \subset C \} .$$

Then there exists a Reifenberg competitor $E \subset C$ such that $H^d(E \setminus \Gamma) = m$.

Remark. If Γ is compact and $H^d(\Gamma) < \infty$, this amounts to minimizing $H^d(E)$ among Reifenberg competitors containing Γ.

2.4 Tools of the limiting theorem

2.4.1 A class of deformations

The next lemma is inspired by the techniques of David in [D6]. It contains the main technical part of the proof. The Lipschitz constant of a map $f \colon \mathbf{R}^n \to \mathbf{R}^n$ is denoted by $\|f\|_L$.

Lemma. *Let f be a global sliding deformation in an open set $U \subset X$. Let W be an open subset of U and let $E \subset W$ be a H^d measurable, H^d finite and H^d rectifiable set. For all $\varepsilon > 0$, there exists a global sliding deformation g in U and an open set $V \subset W$ such that $g - f$ has a compact support included in W, $|g - f| \leq \varepsilon$, $\|g - f\|_L \leq C\|f\|_L$ (where $C \geq 1$ depends on n, Γ) and*

$$\begin{aligned} &H^d(E \setminus V) \leq \varepsilon \\ &H^d(g(V)) \leq H^d(f(E)) + \varepsilon. \end{aligned}$$

In summary, we build a sliding deformation g which "smashes" an almost-neighborhood of $E \cap W$ onto $f(E \cap W)$.

2.4.2 The Federer-Fleming projection

The Federer-Fleming projection for sets was introduced by David and Semmes in [DS] following the ideas of the projection for currents. It consists in sending a d-dimensional set E in the d-skeleton of a lattice of cubes which controling the measure of the image. To start with, we choose a projection center away from the closure of E in each cube. We then carry out a radial projection in each cube to send E in the $(n-1)$-dimensional faces. Since the projection centers are not in the closure of E, each radial projection is Lipschitz on E and the measure of E is multiplied by the Lipschitz constant. Unfortunalty, we do not control this constant a priori. David and Semmes achieved to choose projection centers such that the measure of the E is multiplied by a constant that depends only on n. We iterate the process in each $(n-1)$-dimensional face to project E in the $(n-2)$-dimensional faces, etc. We stop in general when E is sent in the d-skeleton because we cannot anymore make sure that there exists a projection center away from the closure of the image of E. We present the Lemma of David and Semmes on the choice of projection centers:

Lemma ([DS], Lemma 3.22)**.** *Let Q be a cube of $\mathbf{R}^n$, let E be a Borel subset of Q. Then*

$$\frac{1}{|Q|} \int_{\frac{1}{2}Q}^{*} H^d(\phi_x(E)) \, \mathrm{d}x \leq C H^d(E),$$

where ϕ_x is the radial projection from $Q \setminus x$ onto ∂Q, C is a constant that depends only on n.

We develop a similar lemma for another gauge which has the advantage to cancel the purely nonrectifiable part. Let us start by recalling some notation. The Grassmannian $G(d,n)$ is the set of all d-linear planes in $\mathbf{R}^n$. A linear plane can be represented by its orthogonal projection p_V. Thus, the operator norm with respect to the Euclidean norm induces a natural metric in $G(d,n)$. This space is also equipped of a canonic invariant measure $\mathrm{d}V$. We refer to

Appendix B (and alternatively, [Mat], Section 3). We define the gauge ζ^d on Borel subsets of $\mathbf{R}^n$ by

$$\zeta^d(E) := \int_{G(d,n)} H^d(p_V(E))\,\mathrm{d}V.$$

We call *cell* a face of cube of any dimension. For a cell A, we restrict this gauge to A

$$\zeta^d \llcorner A(E) := \int_{G(\mathrm{aff}(A),d)} H^d(p_V(E \cap A))\,\mathrm{d}V,$$

where $\mathrm{aff}(A)$ is the affine span of A and $G(\mathrm{aff}(A), d)$ is the set of all d-linear planes of $\mathrm{aff}(A)$ centered at an arbitrary point of $\mathrm{aff}(A)$. We also recall the notion of upper-integral. For a set $S \subset \mathbf{R}^n$ equipped with a measure μ and for a function $f\colon S \to [0, +\infty]$, the *upper integral* of f is defined by.

$$\int_S^* f\,\mathrm{d}\mu = \inf_\psi \int_S \psi\,\mathrm{d}\mu$$

where ψ run through the μ-measurable functions $S \to [0, +\infty]$ such that $f \leq \psi$. It is not sure we really need upper-integral but we do not bother.

Lemma. *Let Q be a cube of $\mathbf{R}^n$, let E be a Borel subset of Q. Then*

$$\frac{1}{|Q|} \int_{\frac{1}{2}Q}^* \sup_A \zeta^d \llcorner A(\phi_x(E))\,\mathrm{d}x \leq C\zeta^d(E),$$

where ϕ_x is the radial projection from $Q \setminus x$ onto ∂Q, the sup *is indexed by all cells $A \subset \partial Q$ and C is a constant that depends only on n.*

The gauge $\zeta^d \llcorner A$ simplifies to $H^d \llcorner A$ when A is a d-dimensional cell. At the last step of the projection, once the set E has been sent in the d-dimensional skeleton, we obtain an estimate of the form

$$H^d(\phi(E)) \leq C \int_{G(d,n)} H^d(p_V(E))\,\mathrm{d}V.$$

In particular, the purely non-rectifiable part of E is canceled by the right hand side. Feuvrier had already proved that it was possible to choose the projection centers to nullify the non rectifiable part with a different method ([Feuv], Lemma 4.3.16).

Let us consider a quasiminimal set E. We can test the quasiminimality of E against Federer-Fleming projections in grids on cubes. This leads to density estimates (see Corollary 5.1.1): for $x \in E^*$ (the support of $H^d \llcorner E$) and for all small radii $r > 0$,

$$C^{-1} r^d \leq H^d(E \cap B(x,r)) \leq C \int_{G(d,n)} H^d(p_V(E \cap B(x, 16\sqrt{n}r)))\,\mathrm{d}V.$$

We obtain at the same time the d-dimensional Ahlfors regularity of E^*,

$$C^{-1}r^d \leq H^d(E \cap B(x,r)) \leq Cr^d,$$

and the H^d rectifiability of E. We will use such estimate to prove that the set E_∞ of the limiting theorem is H^d rectifiable. The Federer-Fleming projection is therefore an important tool of the proof. If one wants the Federer-Fleming projections to be sliding deformations, they however have to preserve the boundary.

2.4.3 A regular boundary

The set Γ which plays the role of boundary is first of all a closed set of X. We might use three additional assumptions.

1. First, we might want Γ to be a *Lipschitz neighborhood retract* of X. This means that there exists an open set $O \subset X$ containing Γ and there exists a Lipschitz map $p\colon O \to \Gamma$ such that $p = \mathrm{id}$ on Γ.

2. Secondly, we might want that for all $\varepsilon > 0$, there exists an open set $O \subset X$ such that $H^d(\Gamma \setminus O) = 0$ and a $(1+\varepsilon)$-Lipschitz map $p\colon O \cup \Gamma \to \Gamma$ such that $p = \mathrm{id}$ on Γ. The point of $(1+\varepsilon)$-Lipschitz retractions is that they do not enlarge too much the Hausdorff measures. The previous condition allows non convex corners as long as they are H^d negligible.

3. Thirdly, we would like to build Federer-Fleming projections which preserve the boundary.

In chapter 4, we propose a structure to build Federer-Fleming projections. Thus, we are able to define a large class of boundaries compatible with Federer-Fleming projections. We consider a set K of faces of cubes of varying diameters; this is the list of faces in which we are going to perform a radial projection. If the inclusion relation $\subset$ in K is compatible with the topology in some sense, these radial projections can be pasted and composed to form a Federer-Fleming projection. The structure of K is similar to a CW-complex but the boundary of a face may not be covered by other faces. On Figure 2.5 for example, the external edges are not part of K.

Definition. A *cell* is a face of cube in $\mathbf{R}^n$. The *interior* of a cell is the interior relative to its affine span. The *support* of a set K of cells is defined by

$$|K| = \bigcup \{ A \in K \}.$$

A *complex* K is a set of cells such that

1. the cells interior $\{ \mathrm{int}(A) \mid A \in K \}$ are mutually disjoint;

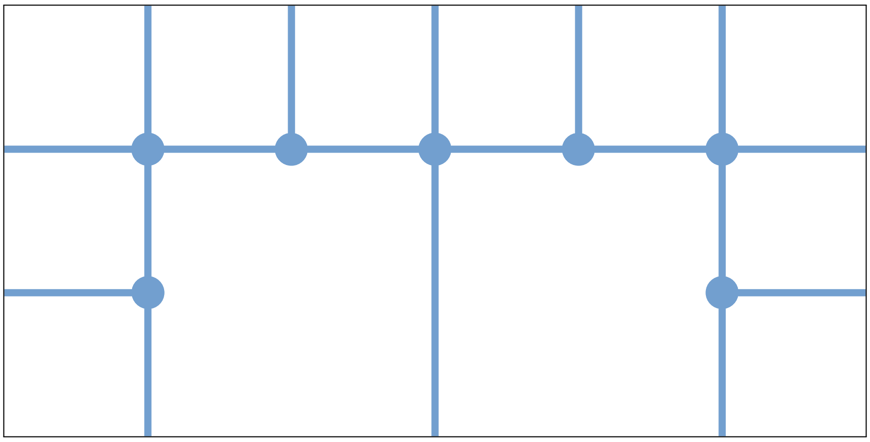

Figure 2.5: In this example, we make a radial projection in each square (white), then in each internal edge (blue). We gather these cells in a set K.

2. every $x \in |K|$ admits a relative neighborhood in $|K|$ which meets a finite number of cells $A \in K$;

3. for every cell $A \in K$, the set

$$V_A := \bigcup \{\, \mathrm{int}(B) \mid B \in K \text{ contains } A \,\}$$

is a relative neighborhood of $\mathrm{int}(A)$ in $|K|$.

We present a simple method to build such complexes by pasting elementary complexes. This technique is analogous to a direct limit in algebra. Thus, we build complexes which fill an open set in a similar way as Whitney decomposition. Finally, we define our boundaries as bilipschitz images of unions of cells of varying dimensions of such complexes.

Chapter 3

Sliding deformations

3.1 Definitions

Our ambiant space is an open set X of $\mathbf{R}^n$. We fix an integer $1 \leq d \leq n$. The term *a closed set* $S \subset X$ means that S is relatively closed in X. For $x \in X$ and $s \in [0, \infty]$, we define

$$r_s(x) = \min \{ \tfrac{s}{1+s} \mathrm{d}(x, X^c), s \} . \tag{3.1}$$

We express the scale of a ball $B(x, r)$ in X as the parameter $s \in [0, \infty]$ such that $r = r_s$. The interval $[0, 1]$ is denoted by the capital letter I. Given a set $E \subset X$ and a function $F \colon I \times E \to X$, the notation F_t means $F(t, \cdot)$. Given two sets $A, B \subset \mathbf{R}^n$, the notation $A \subset\subset B$ means that there exists a compact set $K \subset \mathbf{R}^n$ such that $A \subset K \subset B$. For a ball U of center x and radius r, for $h \geq 0$, the symbol hU denotes the ball of center x and radius hr. We fix a closed subset Γ of X (the boundary).

Definition 3.1.1 (Sliding deformation along a boundary)**.** Let E be a closed, H^d locally finite subset of X. A *sliding deformation* of E in an open set $U \subset X$ is a Lipschitz map $f \colon E \to X$ such that there exists a continuous homotopy $F \colon I \times E \to X$ satisfying the following conditions:

$$F_0 = \mathrm{id} \tag{3.2a}$$
$$F_1 = f \tag{3.2b}$$
$$\forall t \in I,\ F_t(E \cap \Gamma) \subset \Gamma \tag{3.2c}$$
$$\forall t \in I,\ F_t(E \cap U) \subset U \tag{3.2d}$$
$$\forall t \in I,\ F_t = \mathrm{id} \text{ in } E \setminus K, \tag{3.2e}$$

where K is some compact subset of $E \cap U$. Alternatively, the last axiom can be stated as

$$\{ x \in E \mid \exists t \in I,\ F_t(x) \neq x \} \subset\subset E \cap U. \tag{3.3}$$

Definition 3.1.2 (Quasiminimal sets)**.** Let E be a closed, H^d locally finite subset of X. Let $\mathcal{P} = (\kappa, h, s)$ be a triple of parameters composed of $\kappa \geq 1$, $h \geq 0$ and a scale $s \in]0, \infty]$. We say that E is $\mathcal{P}$*-quasiminimal* in X if for all $x \in E$, for all $0 < r \leq r_s(x)$ and for all sliding deformation f of E in $U = B(x, r)$, we have

$$H^d(W_f) \leq \kappa H^d(f(W_f)) + hH^d(E \cap hU), \tag{3.4}$$

where $W_f = \{\, x \in E \mid f(x) \neq x \,\}$. We say that E is a minimal set in the case $\mathcal{P} = (1, 0, \infty)$.

Remark 3.1.1 (Replacing W_f by a larger open set). We keep the previous notations and we consider an open set W such that $W_f \subset W$. Then, (3.4) implies

$$H^d(E \cap W) \leq (\kappa + 1)H^d(f(E \cap W)) + hH^d(E \cap hU). \tag{3.5}$$

Taking $W = U$ and $h < 1$, we obtain the simplier inequality

$$H^d(E \cap U) \leq \tfrac{(\kappa+1)}{1-h} H^d(f(E \cap U)). \tag{3.6}$$

Remark 3.1.2 (Action of bilipschitz maps). Let $T \colon x \mapsto Tx$ be bilipschitz map from X onto an open set $T(X)$ of $\mathbf{R}^n$. Let α be the Lipschitz constant of T, β be the the Lipschitz constant of T^{-1}, let $\gamma = \alpha\beta \geq 1$. Let E be (κ, s, h)-quasiminimal set along Γ in X. We show that the set $T(E)$ is quasiminimal along $T(\Gamma)$ in $T(X)$ with respect to deformed parameters that depends only on κ, s, h, α and β.

First of all, we want to check that for all $x \in X$,

$$\mathrm{d}(Tx, T(X)^c) \leq \alpha \mathrm{d}(x, X^c). \tag{3.7}$$

For $0 \leq r < \mathrm{d}(Tx, T(X)^c)$, we observe that

$$T(X \cap B(x, \alpha^{-1}r)) \subset \overline{B}(Tx, r) \subset T(X) \tag{3.8}$$

so

$$X \cap B(x, \alpha^{-1}r) \subset T^{-1}(\overline{B}(Tx, r)) \subset X \tag{3.9}$$

and thus

$$X \cap B(x, \alpha^{-1}r) = T^{-1}(\overline{B}(Tx, r)) \cap B(x, \alpha^{-1}r). \tag{3.10}$$

This set is both relatively open and relatively closed in $B(x, \alpha^{-1}r)$ so by connectedness, $B(x, \alpha^{-1}r) \subset X$. This means that $r \leq \alpha \mathrm{d}(x, X^c)$. We let r converges to its supremum value to obtain $\mathrm{d}(Tx, T(X)^c) \leq \alpha \mathrm{d}(x, X^c)$.

Now, we fix $x \in E$ and $0 < r \leq r_t(Tx)$ for some $t > 0$ to be determined. Let g be a sliding deformation of $T(E)$ along $T(\Gamma)$ in $V = B(Tx, r)$. We consider the conjugate map $f = T^{-1} \circ g \circ T$. It is easy to check that $T(W_f) = W_g$ and that f is a sliding deformation of E along Γ in $U = B(x, \beta r)$.

In order to apply the quasiminimality of E against f in U, we need that $\beta r \leq r_s(x)$. Thus, we determine a rough $t > 0$ such that $\beta r_t(Tx) \leq r_s(x)$. As $\mathrm{d}(Tx, T(X)^c) \leq \alpha \mathrm{d}(x, X^c)$, we have

$$\begin{aligned} r_t(Tx) &= \min\left\{ \tfrac{t}{1+t}\mathrm{d}(Tx, T(X)^c), t \right\} &(3.11)\\ &\leq t \min\left\{ \mathrm{d}(Tx, T(X)^c), 1 \right\} &(3.12)\\ &\leq \max\left\{ \alpha, 1 \right\} t \min\left\{ \mathrm{d}(x, X^c), 1 \right\} &(3.13) \end{aligned}$$

and on the other hand,

$$\begin{aligned} r_s(x) &= \min\left\{ \tfrac{s}{1+s}\mathrm{d}(x, X^c), s \right\} &(3.14)\\ &\geq \tfrac{s}{1+s} \min\left\{ \mathrm{d}(x, X^c), 1 \right\}. &(3.15) \end{aligned}$$

We conclude that it suffices to take

$$t = \frac{1}{\beta \max\left\{ \alpha, 1 \right\}} \frac{s}{1+s}. \tag{3.16}$$

Assuming $r \leq r_t(y)$, we have finally

$$\begin{aligned} H^d(T(E) \cap W_g) &= H^d(T(E \cap W_f)) &(3.17)\\ &\leq \alpha^d H^d(E \cap W_f) &(3.18)\\ &\leq \alpha^d \kappa H^d(f(E \cap W_f)) + \alpha^d h H^d(E \cap hU) &(3.19)\\ &\leq \gamma^d \kappa H^d(g(T(E) \cap W_g)) \\ &\qquad + \gamma^d h H^d(T(E) \cap \gamma h V). &(3.20) \end{aligned}$$

This shows that $T(E)$ is $(\gamma^d \kappa, \gamma^d h, t)$-quasiminimal in $T(X)$ along $T(\Gamma)$. Note that if T is an affine similitude of $\mathbf{R}^n$, $\gamma = 1$ so T just changes the scale.

Remark 3.1.3 (Global sliding deformations). We test the quasiminimality against sliding deformations defined only on E. Alternatively, we could work with deformations defined on the ambiant space X. We define the *global sliding deformation in U* as the sliding deformations of X in U (the set E is replaced by X in Definition 3.1.1). We feel that sliding deformations defined only on E are more natural than global sliding deformations. On the other hand, global sliding deformations are handier for our limiting theorem (Theorem 5.2.1). In Section 3.3, we prove that if the boundary is a Lipschitz neighborhood retract in X (Definition 3.1.3), these two notions induce the same quasiminimal sets.

We cannot build much sliding deformations without assuming that the boundary is at least a Lipschitz neighborhood retract of X.

Definition 3.1.3. A *Lipschitz neighborhood retract* of X is a closed subset $\Gamma \subset X$ for which there exists an open set $O \subset X$ containing Γ and a Lipschitz map $r \colon O \to \Gamma$ such that $r = \mathrm{id}$ on Γ.

We sometimes need that the retraction r has a Lipschitz constant sufficiently close to 1. Thus, r does not enlarge too much the Hausdorff measures. However, we also want to allow boundaries with nonconvex corners. We find a compromise by asking that the boundary admits $(1+\varepsilon)$-Lipschitz retractions outside a null set.

Definition 3.1.4. A closed set $\Gamma \subset X$ is said H^d *regular* if for all $\varepsilon > 0$, there exists an open set $O \subset X$ and a $(1+\varepsilon)$-Lipschitz map $r \colon O \cup \Gamma \to \Gamma$ such that $H^d(\Gamma \setminus O) = 0$ and $r = \mathrm{id}$ on Γ.

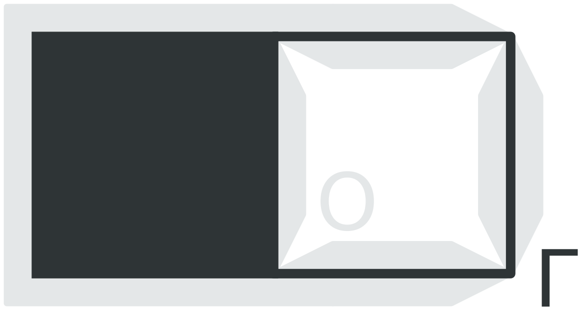

Figure 3.1: A closed set Γ in black. An open set O containing almost all Γ in gray. One can builds a retraction from $O \cup \Gamma$ onto Γ with a good Lipschitz constant because O avoids the nonconvex corners.

We will often need to localise a retraction of the boundary in a given open set.

Lemma 3.1.1. *Let Γ be a Lipschitz neighborhood retract of X. For all open sets $U \subset X$ and for all $\varepsilon > 0$, there exists a Lipschitz map $p \colon \mathbf{R}^n \to \mathbf{R}^n$ and an open subset $O \subset X$ such that $\Gamma \cap U \subset O \subset U$ and*

$$|p - \mathrm{id}| \leq \varepsilon \tag{3.21a}$$

$$p(O) \subset \Gamma \tag{3.21b}$$

$$p = \mathrm{id} \textit{ in } \Gamma \cup (\mathbf{R}^n \setminus U). \tag{3.21c}$$

Moreoever, we can build p such that its Lipschitz constant depends only on n and Γ (but not U and ε).

Proof. Let r be a Lipschitz retraction from an open set O_0 containing Γ onto Γ. Let $\varepsilon > 0$, we define the open set

$$O = \{\, x \in U \cap O_0 \mid |r(x) - x| < \varepsilon \min\{\, \mathrm{d}(x, U^c), 1 \,\} \,\}. \tag{3.22}$$

In particular, $\Gamma \cap U \subset O \subset U$ and $r(O) \subset \Gamma$. Consider the partially defined map

$$p = \begin{cases} r & \text{in } O \\ \mathrm{id} & \text{in } X \setminus U. \end{cases} \tag{3.23}$$

Notice that $\Gamma \subset O \cup (X \setminus U)$ so $p = \mathrm{id}$ on Γ. It is straightforward that $|p - \mathrm{id}| \leq \varepsilon$ because $|r - \mathrm{id}| \leq \varepsilon$ in O. Next, we estimate the Lipschitz constant of $p - \mathrm{id}$. Let $\|r\|$ be the Lipschitz constant of r. For $x, y \in O$,

$$\begin{aligned} |(p - \mathrm{id})(x) - (p - \mathrm{id})(y)| &\leq |r(x) - r(y)| + |x - y| && (3.24) \\ &\leq (1 + \|r\|)|x - y|. && (3.25) \end{aligned}$$

For $x \in O$ and for $y \in X \setminus U$,

$$\begin{aligned} |(p - \mathrm{id})(x) - (p - \mathrm{id})(y)| &\leq |r(x) - x| && (3.26) \\ &\leq \varepsilon \mathrm{d}(x, X \setminus U) && (3.27) \\ &\leq \varepsilon |x - y|. && (3.28) \end{aligned}$$

We assume $\varepsilon \leq 1$ so that $p - \mathrm{id}$ is $(1 + \|r\|)$-Lipschitz on its domain. Finally we apply Lemma A.2.1 in Appendix A to $p - \mathrm{id}$. Thus, p extends as a Lipschitz map $p \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $(p - \mathrm{id})$ is $C(1 + \|r\|)$-Lipschitz (where C depends only on n) and $|p - \mathrm{id}| \leq \varepsilon$. □

3.2 Stability of sliding deformations

The next lemma says that a slight modification of a sliding deformation is still a sliding deformation.

Lemma 3.2.1. *Let Γ be a Lipschitz neighborhood retract of X. Let E be a closed subset of X. Let f be a sliding deformation of E in an open subset $U \subset X$. Let W be an open set such that $W \subset\subset E \cap U$. Then there exists $\delta > 0$ such that all Lipschitz maps $g \colon E \to \mathbf{R}^n$ satisfying*

$$\begin{aligned} &|g - f| \leq \delta, && (3.29a) \\ &g(E \cap \Gamma) \subset \Gamma, && (3.29b) \\ &g = \mathrm{id} \text{ in } E \setminus W, && (3.29c) \end{aligned}$$

are sliding deformations of E in U.

Proof. Let F be a sliding homotopy associated to f. We define

$$W_0 = W \cup \{\, x \in E \mid \exists t \in I,\ F_t(x) \neq x \,\} \tag{3.30}$$

and we underline that $W_0 \subset\subset E \cap U$. Thus, there exists an open set U_0 such that

$$F(I \times \overline{W_0}) \subset U_0 \subset\subset U. \tag{3.31}$$

We fix $\delta_0 > 0$ such that $\mathrm{d}(U_0, X \setminus U) \geq \delta_0$. We apply Lemma 3.1.1 to obtain a Lipschitz function $p \colon X \to \mathbf{R}^n$ and an open set $O \subset X$ containing Γ such that

$$|p - \mathrm{id}| \leq \delta_0 \tag{3.32a}$$
$$p(O) \subset \Gamma \tag{3.32b}$$
$$p = \mathrm{id} \text{ in } \Gamma. \tag{3.32c}$$

The map p will not be used in step 1 but the open set O will be needed.

Step 1. Let $\delta > 0$ (to be precised momentarily) and let g be as the statement. We introduce the homotopy $G_t = F_t + t(g - f)$. It is clear that $G_0 = \mathrm{id}$, $G_1 = g$ and for all $t \in I$, $G_t = \mathrm{id}$ in $E \setminus W_0$. The condition $|g - f| \leq \delta$ also implies that for all $t \in I$, $|G_t - F_t| \leq \delta$. We are going to see that if δ is sufficiently small, then for all $t \in I$,

$$G_t(E \cap \Gamma) \subset O, \tag{3.33}$$
$$G_t(E \cap U_0) \subset U_0. \tag{3.34}$$

Since $I \times \overline{W_0}$ is compact and $F(I \times \overline{W_0}) \subset U_0$, we can take $\delta > 0$ small enough so that for all $t \in I$, for all $x \in \overline{W_0}$,

$$\mathrm{d}(F_t(x), \mathbf{R}^n \setminus U_0) > \delta. \tag{3.35}$$

Thus, for all $t \in I$, $x \in W_0$,

$$G_t(x) \in \overline{B}(F_t(x), \delta) \subset U_0. \tag{3.36}$$

We deduce that for all $t \in I$, $G_t(E \cap U_0) \subset U_0$ as $G_t = \mathrm{id}$ in $E \cap U_0 \setminus W_0$. Similarly, $I \times (\Gamma \cap \overline{W_0})$ is compact and $F(I \times (\Gamma \cap \overline{W_0})) \subset O$ because $F(I \times (\Gamma \cap E)) \subset \Gamma \subset O$. We take $\delta > 0$ small enough so for all $t \in I$, for all $x \in \Gamma \cap \overline{W_0}$,

$$\mathrm{d}(F_t(x), \mathbf{R}^n \setminus O) > \delta \tag{3.37}$$

and we deduce that for all $t \in I$, $G_t(E \cap \Gamma) \subset O$.

Step 2. We would like to retract $G_t(\Gamma)$ onto Γ so we define

$$H_t = \begin{cases} \mathrm{id} & \text{in } (0 \times E) \cup (I \times (E \setminus W_0)) \\ p \circ G_t & \text{in } I \times (E \cap \Gamma) \\ g & \text{in } 1 \times E. \end{cases} \tag{3.38}$$

This map is continuous as a pasting of continuous maps in closed domains (relative to $I \times E$). As $G_t(E \cap \Gamma) \subset O$, we have $H_t(E \cap \Gamma) \subset \Gamma$. Since $|p - \mathrm{id}| \leq \delta_0$, H also satisfies the inequality $|H_t - G_t| \leq \delta_0$. We apply the Tietze Extension Theorem (Lemma A.1.1, Appendix A) in the working space $I \times E$ to $H - G$. Thus, we extend H into a continuous function $H \colon E \to \mathbf{R}^n$ such that $|H_t - G_t| \leq \delta_0$ in E. Combining $|H_t - G_t| \leq \delta_0$ and $G_t(E \cap U_0) \subset U_0$, we deduce that $H_t(E \cap U_0) \subset U$ by definition of δ_0. Moreover, $H_t = \mathrm{id}$ on $E \setminus U_0$ so we have in fact $H_t(U) \subset U$. We conclude that H is a sliding homotopy. $\square$

3.3 Global sliding deformations

We recall that a global sliding deformation in an open set $U \subset X$ is a sliding deformation of X in U (the set E is replaced by X in Definition 3.1.1). Our goal is to show that global sliding deformations induce the same quasiminimal sets. First, we present a necessary and sufficient condition for a sliding deformations on E to extend as a global deformation.

Lemma 3.3.1 (Sliding Deformation Extension)**.** *Let Γ be a Lipschitz neighborhood retract of X. Let E be a closed subset of X. Let f be a sliding deformation of E in an open set $U \subset X$. Then f extends as a global sliding deformation in U if and only if there exists a constant $C \geq 1$ such that for all $x \in E$,*

$$\mathrm{d}(f(x), \Gamma) \leq C\mathrm{d}(x, \Gamma). \tag{3.39}$$

Proof. Let us justify that the condition is necessary. Assume that there exists a global sliding deformation g in U which coincides with f on E. For all $x \in E$, for all $y \in \Gamma$, we have $g(y) \in \Gamma$ so

$$\begin{aligned} \mathrm{d}(f(x), \Gamma) &\leq |f(x) - g(y)| && (3.40)\\ &\leq |g(x) - g(y)| && (3.41)\\ &\leq \|g\||x - y|, && (3.42) \end{aligned}$$

where $\|g\|$ is the Lipschitz constant of g. Since $y \in \Gamma$ is arbitrary, it follows that $\mathrm{d}(f(x), \Gamma) \leq \|g\|\mathrm{d}(x, \Gamma)$. From now on, we assume (3.39) and we build an extension of f.

This paragraph is devoted to introducing a few objects and notation. Let F be a sliding homotopy associated to f. Our extension of F risks overstepping U so we are going to work in a smaller open set U_0 which is relatively compact in U. Let $K \subset E \cap U$ be a compact set such that for all t, $F_t = \mathrm{id}$ in $E \setminus K$. As $F(I \times K)$ is a compact subset of U, there exists an open set $U_0 \subset\subset U$ such that $F(I \times K) \subset U_0$. In particular $K \subset U_0$ because $F_0 = \mathrm{id}$. Let W be an open set such that $K \subset W \subset\subset U_0$. The point of such set W is that there exists a constant $M > 0$ such that for all $x \in E \cap K$,

$$|f(x) - x| \leq M\mathrm{d}(x, X \setminus W). \tag{3.43}$$

Of course, the inequality still holds for all $x \in E$ since $f = \mathrm{id}$ outside K. This inequality will allow to extend f in a Lipschitz way by $f = \mathrm{id}$ in $X \setminus W$. Finally, we want a Lipschitz retraction onto Γ. We apply Lemma 3.1.1 to obtain a Lipschitz map $p \colon X \to \mathbf{R}^n$ and an open subset $O \subset X$ containing Γ such that

$$\begin{aligned} &|p - \mathrm{id}| \leq \varepsilon && (3.44a)\\ &p(O) \subset \Gamma && (3.44b)\\ &p = \mathrm{id} \text{ in } \Gamma, && (3.44c) \end{aligned}$$

where ε is a small positive constant that we will specify later.

Step 1. The first part of the proof consists in building a continuous function $G\colon I \times X \to \mathbf{R}^n$ which is an extension of F and such that

$$\begin{align}
&G_0 = \mathrm{id} \tag{3.45a}\\
&G_1 \text{ is Lipschitz} \tag{3.45b}\\
&\forall t \in [0,1],\ G_t(\Gamma) \subset O \tag{3.45c}\\
&\forall t \in [0,1],\ G_t(U_0) \subset U_0 \tag{3.45d}\\
&\forall t \in [0,1],\ G_t = \mathrm{id} \text{ in } X \setminus W. \tag{3.45e}
\end{align}$$

The partially defined function

$$\begin{cases} F_t & \text{in } I \times E \\ \mathrm{id} & \text{in } (0 \times X) \cup (I \times (X \setminus W)) \end{cases} \tag{3.46}$$

is continuous because it is obtained by pasting continuous functions in closed domains. We apply the Tietze Theorem to obtain a continuous extension $G\colon I \times X \to \mathbf{R}^n$. In order to obtain the conditions (3.45), we will reparametrize G_t. By compactness, the inclusion $I \times K \subset G^{-1}(U_0)$ implies the existence of an open set $V \subset X$ such that $K \subset V$ and

$$I \times V \subset G^{-1}(U_0). \tag{3.47}$$

We apply the same argument in $I \times \Gamma$ where the inclusion $I \times (\Gamma \cap K) \subset G^{-1}(\Gamma) \subset G^{-1}(O)$ implies the existence of a relative open set $V_\Gamma \subset \Gamma$ such that $\Gamma \cap K \subset V_\Gamma$ and

$$I \times V_\Gamma \subset G^{-1}(O). \tag{3.48}$$

Let $\varphi\colon X \to [0,1]$ be a continuous function such that $\varphi = 1$ in K and $\varphi = 0$ in $X \setminus V$ and $\Gamma \setminus V_\Gamma$. We define

$$G'_t(x) = G_{t\varphi(x)}(x). \tag{3.49}$$

Hence G' is a continuous function which satisfies

$$\begin{align}
&G'_0 = \mathrm{id} \tag{3.50a}\\
&\forall t \in [0,1],\ G'_t(V_\Gamma) \subset O \tag{3.50b}\\
&\forall t \in [0,1],\ G'_t(V) \subset U_0 \tag{3.50c}\\
&\forall t \in [0,1],\ G'_t = \mathrm{id} \text{ in } (X \setminus W) \cup (X \setminus V) \cup (\Gamma \setminus V_\Gamma). \tag{3.50d}
\end{align}$$

In addition, G' coincides with F on $I \times E$. Combining $G'_t(V) \subset U_0$ and $G'_t = \mathrm{id}$ in $X \setminus V$, one deduces that $G'_t(U_0) \subset U_0$. Similarly, $G'_t(\Gamma) \subset O$. Nex, we replace G'_1 with a Lipschitz approximation. As $W \subset\subset U_0$, $G(I \times \overline{W})$ is a compact subset of U_0 and there exists $\delta > 0$ such that for all $t \in I$, for all $x \in \overline{W}$,

$$\overline{B}(G_t(x), \delta) \subset U_0. \tag{3.51}$$

The set $G(I \times (\Gamma \cap \overline{W}))$ is also a compact subset of O so we can assume that for all $t \in I$, for all $x \in \Gamma \cap \overline{W}$,

$$\overline{B}(G_t(x), \delta) \subset O. \tag{3.52}$$

We are going to replace G'_1 with a Lipschitz function $g \colon X \to \mathbf{R}^n$ such that $g = G'_1$ in $E \cup (X \setminus W)$ and $|g - G'_1| \leq \delta$ in X. We start by checking that G'_1 is Lipschitz in $E \cup (X \setminus W)$. Indeed $G_1 = f$ in E, $G_1 = \mathrm{id}$ in $X \setminus W$ and for $x \in E$ and $y \in X \setminus W$, (3.43) yields

$$\begin{aligned}
|G_1(x) - G_1(y)| &= |f(x) - y| && (3.53)\\
&\leq |f(x) - x| + |x - y| && (3.54)\\
&\leq M\mathrm{d}(x, X \setminus W) + |x - y| && (3.55)\\
&\leq (M+1)|x - y|. && (3.56)
\end{aligned}$$

We can apply Lemma A.2.1 to $G'_1 - \mathrm{id}$ as it is continuous with compact support. Hence we obtain a Lipschitz function $v \colon X \to \mathbf{R}^n$ such that $v = G'_1 - \mathrm{id}$ in $E \cup (X \setminus W)$ and $|G'_1 - \mathrm{id} - v| < \delta$. Then we define $g = v + \mathrm{id}$ and we replace G' with

$$G''_t = G'_t + t(g - G'_1). \tag{3.57}$$

Combining (3.51), (3.52) and the facts that $|g - G'_1| < \delta$ and $g = \mathrm{id}$ in $X \setminus W$, one can see that $G''_t(\Gamma) \subset O$ and $G''_t(U_0) \subset U_0$. We conclude that G'' solves step 1. It will be denoted G in the next step.

Step 2. We would like to retract $G_t(\Gamma)$ onto Γ so we define

$$H_t = \begin{cases} \mathrm{id} & \text{in } 0 \times X \\ p \circ G_t & \text{in } I \times \Gamma \\ G_t & \text{in } I \times (E \cup (X \setminus W)). \end{cases} \tag{3.58}$$

This function is continuous as a pasting of continuous functions in closed domains. As $G_t(\Gamma) \subset O$, we have $H_t(\Gamma) \subset \Gamma$. It satisfies the inequality $|H_t - G_t| \leq \varepsilon$ because $|p - \mathrm{id}| \leq \varepsilon$. Let us check that H_1 is Lipschitz on its domain. The Lipschitz constants of p, $p - \mathrm{id}$ and g are denoted by $\|p\|$, $\|p - \mathrm{id}\|$ and $\|g\|$ respectively. Note that for all $x \in \mathbf{R}^n$, for all $y \in \Gamma$,

$$\begin{aligned}
|p(x) - x| &= |(p - \mathrm{id})(x) - (p - \mathrm{id})(y)| && (3.59)\\
&\leq \|p - \mathrm{id}\| |x - y| && (3.60)
\end{aligned}$$

whence for all $x \in X$,

$$|x - p(x)| \leq \|p - \mathrm{id}\| \mathrm{d}(x, \Gamma). \tag{3.61}$$

Using the Lemma assumption (3.39), we deduce that for $x \in E$ and $y \in \Gamma$, we have

$$|H_1(x) - H_1(y)| = |f(x) - pg(y)| \tag{3.62}$$
$$\leq |f(x) - pf(x)| + |pg(x) - pg(y)| \tag{3.63}$$
$$\leq \|p - \mathrm{id}\| \mathrm{d}(f(x), \Gamma) + \|p\| \|g\| |x - y| \tag{3.64}$$
$$\leq \|p - \mathrm{id}\| C \mathrm{d}(x, \Gamma) + \|p\| \|g\| |x - y| \tag{3.65}$$
$$\leq \|p - \mathrm{id}\| C |x - y| + \|p\| \|g\| |x - y|. \tag{3.66}$$

We also have for $x \in X \setminus W$ and $y \in \Gamma$,

$$|H_1(x) - H_1(y)| = |x - pg(y)| \tag{3.67}$$
$$\leq |x - p(x)| + |pg(x) - pg(y)| \tag{3.68}$$
$$\leq \|p - \mathrm{id}\| \mathrm{d}(x, \Gamma) + \|p\| \|g\| |x - y| \tag{3.69}$$
$$\leq \|p - \mathrm{id}\| |x - y| + \|p\| \|g\| |x - y|. \tag{3.70}$$

We apply the extension Lemma A.2.1 to extend H_1 as a Lipschitz function $H_1 \colon X \to \mathbf{R}^n$ such that $|H_1 - g| \leq \varepsilon$. Then, we use the Tietze Extension Lemma (Lemma A.1.1, Appendix A) to extend H as a continuous function $H \colon I \times X \to \mathbf{R}^n$ such that $|H_t - G_t| \leq \varepsilon$. We assume that ε is small enough so that for all $x \in U_0$, $\overline{B}(x, \varepsilon) \subset U$. Thus, the conditions $G_t(U_0) \subset U_0$ and $|H_t - G_t| \leq \varepsilon$ ensure that $H_t(U_0) \subset U$. Moreover $H_t = \mathrm{id}$ in $X \setminus U_0$, so we have in fact $H_t(U) \subset U$. We conclude that H solves the lemma. ☐

Combining the previous lemmas, we prove that every sliding deformation can be replaced by an equivalent global sliding deformation.

Lemma 3.3.2 (Sliding Deformation Alternative)**.** *Let Γ be a Lipschitz neighborhood retract of X. Let E be a closed subset of X which is H^d locally finite in X. Let f be a sliding deformation of E in an open subset $U \subset X$. Then for all $\varepsilon > 0$, there exists a global sliding deformation g in U such that $|g - f| \leq \varepsilon$, $E \cap W_g \subset\subset W_f$ and*

$$H^d(g(W_f) \setminus f(W_f)) \leq \varepsilon \tag{3.71}$$

where

$$W_g = \{\, x \in X \mid g(x) \neq x \,\}, \tag{3.72}$$
$$W_f = \{\, x \in E \mid f(x) \neq x \,\}. \tag{3.73}$$

Proof. Given Lemmas 3.3.1 and 3.2.1, it suffices to build a Lipschitz function $g \colon E \to \mathbf{R}^n$ which satisfies the following conditions: $|g - f| \leq \varepsilon$, $g(E \cap \Gamma) \subset \Gamma$, $W_g \subset\subset W_f$, there exists $C \geq 1$ such that for all $x \in E$,

$$\mathrm{d}(g(x), \Gamma) \leq C \mathrm{d}(x, \Gamma) \tag{3.74}$$

and finally,

$$H^d(g(W_f) \setminus f(W_f)) \leq \varepsilon_0. \tag{3.75}$$

We fix $\varepsilon_0 > 0$. The construction will brings into play an intermediate variable $\varepsilon > 0$. First, we want to build a Lipschitz function $p\colon X \to \mathbf{R}^n$ (whose Lipschitz constant depends only on Γ) such that $|p - \mathrm{id}| \leq \varepsilon_0$, $p = \mathrm{id}$ on Γ and such that there exists an open set O with $\Gamma \subset O \subset X$ and $p(O) \subset \Gamma$. Moreover, we want that

$$H^d(W_f \cap W_p) \leq \varepsilon_0 \tag{3.76}$$

and

$$H^d(W_f \cap f^{-1}(W_p)) \leq \varepsilon_0, \tag{3.77}$$

where $W_p = \{\, x \in X \mid p(x) \neq x \,\}$. Let us proceed to build p. Since $\overline{W_f}$ is a compact subset of E, we have $H^d(W_f) < \infty$ so we can find an open set V such that $\Gamma \subset V \subset X$ and

$$H^d(W_f \cap V \setminus \Gamma) \leq \varepsilon_0 \tag{3.78}$$

and

$$H^d(W_f \cap f^{-1}(V \setminus \Gamma)) \leq \varepsilon_0. \tag{3.79}$$

Then we apply Lemma 3.1.1 in the open set V: there exists a Lipschitz function $p\colon X \to \mathbf{R}^n$ (whose Lipschitz constant depends only on Γ) and an open set O such that $\Gamma \cap V \subset O \subset V$ and

$$|p - \mathrm{id}| \leq \varepsilon_0 \tag{3.80a}$$

$$p(O) \subset \Gamma \tag{3.80b}$$

$$p = \mathrm{id} \text{ in } \Gamma \cup (X \setminus V). \tag{3.80c}$$

As $W_p \subset V \setminus \Gamma$, we deduce

$$H^d(W_f \cap W_p) \leq \varepsilon_0 \tag{3.81}$$

and

$$H^d(W_f \cap f^{-1}(W_p)) \leq \varepsilon_0. \tag{3.82}$$

Next, we truncate f in view of obtaining the property $W_g \subset\subset W_f$. We introduce the set

$$W_\varepsilon = \{\, x \in E \mid \mathrm{d}(x, E \setminus W_f) \geq \varepsilon \,\} \tag{3.83}$$

and since W_f is a relative an open subset of E, we can assume ε small enough so that

$$H^d(W_f \setminus W_{2\varepsilon}) \leq \varepsilon_0. \tag{3.84}$$

Let f' be partially defined by

$$f' = \begin{cases} f & \text{in } W_{2\varepsilon} \\ \mathrm{id} & \text{in } E \setminus W_\varepsilon. \end{cases} \tag{3.85}$$

We are going to estimate $|f - f'|$ and the Lipschitz constant of $f - f'$. The Lipschitz constant of $f - \mathrm{id}$ plays a special role in these estimates and is denoted by L. We start by proving that for $x \in E \setminus W_\varepsilon$,

$$|f(x) - x| \leq L\varepsilon \tag{3.86}$$

Indeed, for $x \in E \setminus W_\varepsilon$, there exists $y \in E \setminus W_f$ such that $|x - y| \leq \varepsilon$ whence

$$\begin{aligned} |f(x) - x| &= |(f - \mathrm{id})(x) - (f - \mathrm{id})(y)| && (3.87) \\ &\leq L|x - y| && (3.88) \\ &\leq L\varepsilon. && (3.89) \end{aligned}$$

Now, it is straighforward that $|f' - f| \leq L\varepsilon$ on the domain of f'. We are going to see that $f' - f$ is L-Lipschitz on the domain of f'. It is clear that $f - f'$ is L-Lipschitz on $W_{2\varepsilon}$ and $E \setminus W_\varepsilon$ respectively. For $x \in W_{2\varepsilon}$ and $y \in X \setminus W_\varepsilon$, we have $|x - y| \geq \varepsilon$ so

$$\begin{aligned} \big|(f' - f)(x) - (f' - f)(y)\big| &= |f(y) - y| && (3.90) \\ &\leq L\varepsilon && (3.91) \\ &\leq L|x - y|. && (3.92) \end{aligned}$$

We apply Lemma A.2.1 to $f' - f$ so as to extend f' as a Lipschitz map $f' \colon E \to \mathbf{R}^n$ such that $|f' - f| \leq L\varepsilon$ with a Lipschitz depending only on n and f. Before moving to the next paragraph, we require $f'(E) \subset X$. As $f(E) \subset X$ and $\overline{W_f}$ is a compact subset of E, we can assume ε small enough so that for all $x \in \overline{W_f}$,

$$\mathrm{d}(f(x), \mathbf{R}^n \setminus X) > L\varepsilon. \tag{3.93}$$

This implies $f'(E) \subset X$ as $f' = f$ in $E \setminus W_f$ and $|f' - f| \leq L\varepsilon$.

We finally define the map g on E by

$$g = p \circ f' + \mathrm{id} - p. \tag{3.94}$$

We have $|p - \mathrm{id}| \leq \varepsilon_0$ and $|f' - f| \leq L\varepsilon$ so $|g - f| \leq 2\varepsilon_0 + L\varepsilon$. We assume ε small enough so that $|g - f| \leq 3\varepsilon_0$. Observe that $g = \mathrm{id}$ in $E \setminus W_\varepsilon$ so $W_g \subset\subset W_f$. Next, we prove that there exists a constant $C \geq 1$ such that for all $x \in E$,

$$\mathrm{d}(g(x), \Gamma) \leq C\mathrm{d}(x, \Gamma). \tag{3.95}$$

This inequality is clearly true for $x \in E \setminus W_f$ so we focus on W_f. As Γ is relatively closed in X and $\overline{W_f}$ is compact subset of X, $\Gamma \cap \overline{W_f}$ is compact.

Its image $f(\Gamma \cap \overline{W_f})$ is a compact subset of $\Gamma \subset O$ so there exists $\delta > 0$ such that for all $y \in f(\Gamma \cap \overline{W_f})$,

$$\mathrm{d}(y, X \setminus O) \geq \delta. \tag{3.96}$$

We introduce

$$O_\delta = \{\, y \in X \mid \mathrm{d}(y, X \setminus O) \geq \delta \,\} \tag{3.97}$$

so the set $f^{-1}(O_\delta)$ is a relative open set of E containing $\Gamma \cap \overline{W_f}$. By compactness, we can assume ε small enough so that

$$\{\, x \in \overline{W_f} \mid \mathrm{d}(x, \Gamma) \leq \varepsilon \,\} \subset f^{-1}(O_\delta). \tag{3.98}$$

Then for $x \in \overline{W_f}$ such that $\mathrm{d}(x, \Gamma) \leq \varepsilon$, we have $f(x) \in O_\delta$ and thus $f'(x) \in O$ assuming ε small enough so that $|f' - f| \leq \delta$. We are also going to need the fact that for $x \in E$,

$$|p(x) - x| \leq \|p - \mathrm{id}\| \mathrm{d}(x, \Gamma), \tag{3.99}$$

where $\|p - \mathrm{id}\|$ is the Lipschitz constant of $p - \mathrm{id}$. Indeed for all $x \in E$ and all $y \in \Gamma$,

$$|p(x) - x| = |(p - \mathrm{id})(x) - (p - \mathrm{id})(y)| \tag{3.100}$$

$$\leq \|p - \mathrm{id}\| |x - y| \tag{3.101}$$

and since y is arbitrary in Γ, $|p(x) - x| \leq \|p - \mathrm{id}\| \mathrm{d}(x, \Gamma)$. We are ready to prove (3.95). For $x \in W_f$, we have either $\mathrm{d}(x, \Gamma) \leq \varepsilon$, either $\mathrm{d}(x, \Gamma) \geq \varepsilon$. In the first case, $f'(x) \in O$ so $p \circ f(x) \in \Gamma$ and then

$$\mathrm{d}(g(x), \Gamma) \leq |g(x) - p \circ f(x)| \tag{3.102}$$

$$\leq |p(x) - x| \tag{3.103}$$

$$\leq \|p - \mathrm{id}\| \mathrm{d}(x, \Gamma). \tag{3.104}$$

In the second case,

$$\mathrm{d}(g(x), \Gamma) \leq \sup \{\, \mathrm{d}(g(u), \Gamma) \mid u \in \overline{W_f} \,\} \tag{3.105}$$

$$\leq \sup \{\, \mathrm{d}(g(u), \Gamma) \mid u \in \overline{W_f} \,\} \, \varepsilon^{-1} \mathrm{d}(x, \Gamma). \tag{3.106}$$

In both cases, we have $\mathrm{d}(g(x), \Gamma) \leq C \mathrm{d}(x, \Gamma)$, where $C \geq 1$ is a constant that does not depends on x. To finish the proof, we show that

$$H^d(g(W_f) \setminus f(W_f)) \leq 3\|g\|\varepsilon_0 \tag{3.107}$$

where $\|g\|$ is the Lipschitz constant of g (it depends only on n, f and Γ). Observe that $g = f$ on $W_{2\varepsilon} \setminus (W_p \cup f^{-1}(W_p))$. Moreover by (3.84),

$$H^d(W_f \setminus W_{2\varepsilon)} \leq \varepsilon_0 \tag{3.108}$$

and by (3.76), (3.77),

$$H^d(W_f \cap (W_p \cup f^{-1}(W_p))) \leq 2\varepsilon_0. \tag{3.109}$$

The result follows. $\square$

Chapter 4

Rigid and Lipschitz boundaries

4.1 Complexes

The Federer-Fleming projection for sets was introduced by David and Semmes in [DS] following the ideas of the Federer-Fleming projection for currents. Given a lattice of cubes, the Federer-Fleming projection of a d-dimensional set E is a technique to send E into the d-dimensional skeleton of the lattice. In each cube, we choose a center of projection which is not in the closure of E. We perfom a radial projection in each cube to send E into the $(n-1)$-dimensional faces. Since the centers of projection are not in the closure of E, the radial projections are Lipschitz on E. The measure of the image is therefore multiplied by the Lipschitz constant to the power $(n-1)$. However, we may not have a good control over this Lipschitz constant. David and Semmes proved that in average among centers of projection, the measure of the image is multiplied by a constant that depends only on n. Then one iterates the process in each $(n-1)$-dimensional face to project once again E into the $(n-2)$-dimensional faces, etc. One usually stops when E is sent into the d-skeleton because it is no longer possible to ensure that there exists a center of projection away from E.

The goal of this section is to axiomatize the structure which supports Federer-Fleming. As a consequence, we will be able to define a large class of boundaries preserved by Federer-Fleming projections. We consider a set K made of the faces of varying dimensions in which we intend to perform radial projections. In order to glue and compose radial projections, the inclusion relation $\subset$ between the faces of K should be compatible with the topology in some sense. We adopt a formalism close the one of CW-complex , except that the boundary of a cell may not be covered by other subcells. On Figure 4.1 for example, the external edges are not part of the set K.

We define a *cell* as a face of cubes of $\mathbf{R}^n$. We could take a more general definition but it is easier to build complexes that way. The *dimension* of a cell A is the dimension of its affine span. Its *interior* $\mathrm{int}(A)$ and *boundary*

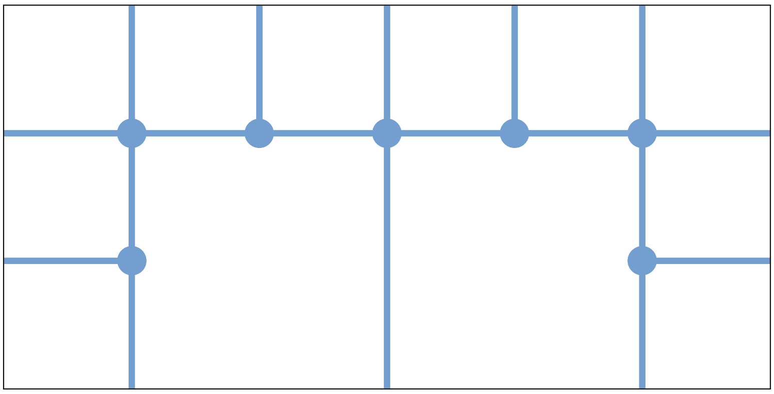

Figure 4.1: A set of cells composed of 2-cells (white squares), 1-cells (blue edges) and 0-cells (blue vertices). The associated Federer-Fleming projection consists in making a radial projection in each white square and then in each blue edge. We take the convention that a radial projection in a 0-cell is the identity map.

∂A are the interior and boundary relative to its affine span. Given a set of cells K, we define the *support* of K by

$$|K| := \bigcup \{ A \mid A \in K \} . \tag{4.1}$$

For an integer $i = 0, \dots, n$, the subset of K composed of the i-dimensional cells is denoted by K^i. Let us motivate our next axioms. Say K is a set of cells in $\mathbf{R}^n$ and we want to glue radial projections in its n-cells, then in its $(n-1)$-cells, etc. It is natural to ask that the cells have disjoint interiors because we want the gluing of our radial projections to be well-defined. It is more problematic to ensure that the gluing is continuous. We will have to glue a family of continuous maps (radial projections)

$$\phi_A \colon A \to A \tag{4.2}$$

indexed by $A \in K^d$ (for some $d \in \{ 0, \dots, n \}$) and a continuous map (the identity map)

$$\phi \colon |K| \setminus \bigcup \{ \operatorname{int}(A) \mid A \in K, \ \dim A \geq d \} \to |K|. \tag{4.3}$$

We make use of the classical argument which says that a pasting of continuous maps on two closed domains is a continuous map. This also works for a locally finite family of closed domains[1]. We require the cells to constitute a

[1]In a topological space X, a family of sets (A_i) is locally finite provided that for every $x \in X$, there exists a neighborhood U of x such that $\{ i \mid A_i \cap U \neq \emptyset \}$ is finite. As a consequence, if the sets A_i are closed, their union $\bigcup A_i$ is closed in X.

locally finite family in $|K|$ so that we can glue continuously the maps (ϕ_A). Next, we require the set

$$|K| \setminus \bigcup \{ \operatorname{int}(A) \mid A \in K,\ \dim A \geq d \} \tag{4.4}$$

to be closed. The corresponding axiom is more precise: we require that for every $A \in K$, the set $\bigcup \{ \operatorname{int}(B) \mid B \in K,\ A \subset B \}$ is a neighborhood of $\operatorname{int}(A)$.

Definition 4.1.1 (Complex)**.** A *complex* K is a set of cells such that

1. the cells interior $\{ \operatorname{int}(A) \mid A \in K \}$ are mutually disjoint;
2. every $x \in |K|$ admits a relative neighborhood in $|K|$ which meets a finite number of cells $A \in K$;
3. for every cell $A \in K$, the set

$$V_A := \bigcup \{ \operatorname{int}(B) \mid B \in K \text{ contains } A \} \tag{4.5}$$

is a relative neighborhood of $\operatorname{int}(A)$ in $|K|$.

These are expected properties of CW-complexes but in our case, the boundary of a cell may not be covered by other cells. We present a few general properties of a complex K. By the local finiteness axiom, K is at most countable. For each $A \in K$, the set V_A is relatively open in $|K|$. Indeed for each cell $B \in K$ containing A, V_A is also a neighborhood of $\operatorname{int}(B)$. Finally, we are going to see that a nonempty intersection of the form $\operatorname{int}(A) \cap B$ is meaningful.

Lemma 4.1.1. *Let K be a complex.*

- *For $A, B \in K$, $\operatorname{int}(A) \cap B \neq \emptyset$ implies $A \subset B$.*
- *For $A, B \in K$ such that $\dim A > \dim B$, we have $\operatorname{int}(A) \cap B = \emptyset$.*
- *For $A, B \in K$ of the same dimension, $\operatorname{int}(A) \cap B \neq \emptyset$ implies $A = B$.*

Proof. Let us prove the first point. The set V_A is a relative open set of $|K|$ which meets $B = \overline{\operatorname{int}(B)}$ so it also meets $\operatorname{int}(B)$. As the cells of K have disjoint interiors, we deduce that B contains A. The second point is now obvious. Next, we assume that A and B have the same dimension and we prove that $A = B$. According to the inclusion $A \subset B$ and a dimension argument, the affines spaces $\operatorname{aff}(A)$ and $\operatorname{aff}(B)$ are equals. As $\operatorname{int}(A)$ is relatively open in $\operatorname{aff}(B)$, we deduce $\operatorname{int}(A) \subset \operatorname{int}(B)$. Then $A = B$ because the cells have disjoint interiors. $\square$

We introduce the natural subspace of a complex.

Definition 4.1.2 (Subcomplex)**.** Let K be a complex. A *subcomplex* of K is a subset $L \subset K$ such that for all $A \in L$,

$$\{ B \in K \mid B \text{ contains } A \} \subset L. \tag{4.6}$$

The *rigid open set* induced by L is the set

$$U(L) := \bigcup \{ \operatorname{int} A \mid A \in L \}. \tag{4.7}$$

If L is a subcomplex of K, then L is a also a complex. Moreover, $U(L)$ is relatively is relatively open in K by the second axiom of Definition 4.1.1.

We are going to present a way of building complexes by pasting pieces of grids. The result is called a n-complex. As an example, we will see that the Whitney decomposition of an open set is a n-complex. The model case is the canonical n-complex.

Definition 4.1.3 (Canonical n-complex)**.** The *canonical n-complex* of $\mathbf{R}^n$ is

$$E_n = \left\{ \prod_{i=1}^{n} [0, \alpha_i] \;\middle|\; \alpha \in \{ -1, 0, 1 \}^n \right\}. \tag{4.8}$$

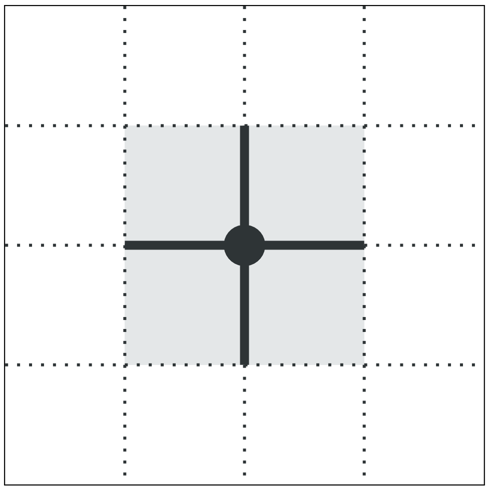

Figure 4.2: The canonical grid of the plane is represented in dotted lines. The complex E_2 is made of the four gray squares, the four black edges et the black vertice.

We are going to show that E_n is a complex. Moreover, we will see that there exists a constant $\kappa \geq 1$ (depending only on n) such that for all $A \in E_n$, the set

$$V_A(\kappa) := \left\{ x \in |K| \;\middle|\; \mathrm{d}(x, A) < \kappa^{-1} \mathrm{d}(x, \partial A) \right\} \tag{4.9}$$

is contained in V_A. The set $V_A(\kappa)$ is a neighborhood of $\operatorname{int}(A)$ and the parameter κ quantifies how wide it is.

Proof. Consider $\alpha \in \{-1, 0, 1\}^n$ and the corresponding cell $A = \prod[0, \alpha_i]$. The interior of A is

$$\operatorname{int}(A) = \prod_{i=1}^{n} (0, \alpha_i), \tag{4.10}$$

where $(0, \alpha_i) =]0, \alpha_i[$ if $\alpha_i \neq 0$ and $(0, \alpha_i) = \{0\}$ otherwise. Observe that different index α, β induce cells $\prod[0, \alpha]$, $\prod[0, \beta]$ which have disjoint interiors. Thus, the cells of E_n have disjoint interiors. Next, we show that there exists $\kappa \geq 1$ (depending only on n) such that for all $A \in E_n$,

$$\left\{ x \in \mathbf{R}^n \;\middle|\; \mathrm{d}(x, A) < \kappa^{-1} \mathrm{d}(x, \partial A) \right\} \subset V_A. \tag{4.11}$$

Without loss of generality, we work with $A = [0, 1]^d \times \{0\}^{n-d}$. Thus,

$$V_A =]0, 1[^d \times] - 1, 1[^{n-d}. \tag{4.12}$$

We proceed by contraposition. Fix $x \in \mathbf{R}^n \setminus V_A$. We have either $x_i \notin]0, 1[$ for some $i = 1, \ldots, d$ or $x_i \notin] - 1, 1[$ for some $i = d + 1, \ldots, n$. In the first case, the distance $\mathrm{d}(x, A)$ is attained on ∂A so $\mathrm{d}(x, A) = \mathrm{d}(x, \partial A)$. In the second case we have $\mathrm{d}(x, A) \geq 1$. As the triangular inequality yields

$$\begin{aligned} \mathrm{d}(x, \partial A) &\leq \mathrm{d}(x, A) + \operatorname{diam}(A) && (4.13) \\ &\leq \mathrm{d}(x, A) + \sqrt{d}, && (4.14) \end{aligned}$$

we deduce

$$\mathrm{d}(x, \partial A) \leq (1 + \sqrt{d}) \mathrm{d}(x, A). \tag{4.15}$$

In both cases, one has $\mathrm{d}(x, \partial A) \leq \kappa \mathrm{d}(x, A)$, where κ depends only on n. Finally, it is clear that E_n is locally finite because it has 3^n cells. □

Definition 4.1.4 (Subordinate complex)**.** We say that a complex L is *subordinate* to a complex K when for every $A \in L$, there exists $B \in K$ (necessary unique) such that

$$\operatorname{int}(A) \subset \operatorname{int}(B). \tag{4.16}$$

This relation is denoted by $L \preceq K$.

We call *n-charts* the image of a subcomplex of E_n by a similarity of $\mathbf{R}^n$ (translations, isometries, homothetys and their compositions).

Definition 4.1.5 (n-complex)**.** A *n-complex* K is a complex of $\mathbf{R}^n$ such that for every $A \in K$, there exists a n-chart $L \preceq K$ containing A.

Lemma 4.1.2. *Let K be a n-complex.*

1. *There exists a constant $\kappa \geq 1$ (depending only on n) such that for all $A \in K$, the set*

$$V_A(\kappa) := \{ x \in \mathbf{R}^n \mid \mathrm{d}(x, \partial A) < \kappa^{-1} \mathrm{d}(x, A) \} \tag{4.17}$$

is contained in V_A;

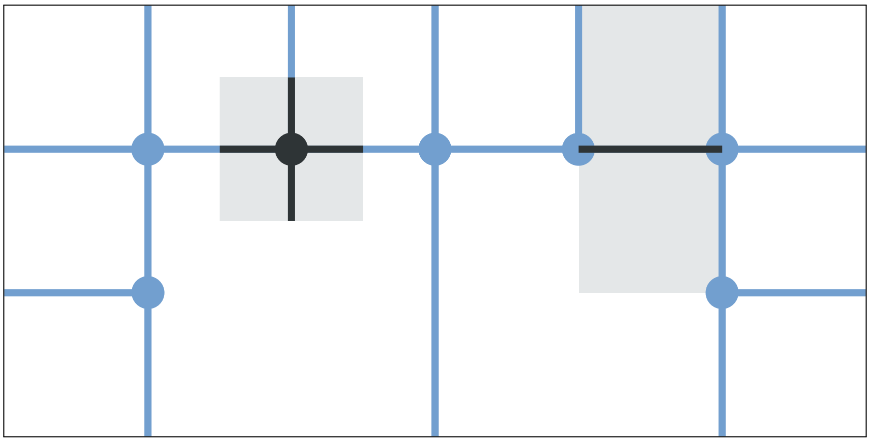

Figure 4.3: A n-complex (white squares, blue edges, blue vertices) and two subordinate n-charts (gray/black).

2. *The rigid open sets of K are open in $\mathbf{R}^n$;*

3. *Every cell $A \in K$ is included in at most 3^n cells $B \in K$.*

Proof. We make a few observations before justifying each properties. Let $A \in K$ and let $F \preceq K$ be a n-chart containing A. As F inherit the properties of E_n, we have

$$V_A(\kappa) \subset \bigcup \{ \operatorname{int}(B) \mid B \in F,\ A \subset B \}, \tag{4.18}$$

where κ is the constant that we found when we studied E_n. For every $B \in F$, there exists a unique $\pi(B) \in K$ such that $\operatorname{int}(B) \subset \operatorname{int}(\pi(B))$. Let us show that

$$\{ \pi(B) \mid B \in F,\ A \subset B \} = \{ C \in K,\ A \subset C \}. \tag{4.19}$$

For $B \in F$ containing A, the inclusion $\operatorname{int}(B) \subset \operatorname{int}(\pi(B))$ implies the inclusion of the closures $B \subset \pi(B)$. Hence $\pi(B)$ contains A as well. Reciprocally, consider $C \in K$ containing A. The set $V_A(\kappa)$ contains $\operatorname{int}(A)$ so it meets C. As $V_A(\kappa)$ is open and C is the closure of $\operatorname{int}(C)$, $V_A(\kappa)$ also meets $\operatorname{int}(C)$. By (4.18), we deduce that there exists $B \in F$ containing A such that $\operatorname{int}(B) \cap \operatorname{int}(C) \neq \emptyset$. Finally $C = \pi(B)$ as the cells of K have disjoint interiors.

Now, the lemma is easy to prove. Combining (4.18) and (4.19), we have

$$V_A(\kappa) \subset \{ \operatorname{int}(C) \mid C \in K,\ A \subset C \}. \tag{4.20}$$

We deduce that V_A is a neighborhood of $\operatorname{int}(A)$ in $\mathbf{R}^n$ (as opposed to being only a relative neighborhood in $|K|$). As a consequence, for every subcomplex L of K and every $A \in L$, $U(L)$ is a neighborhood of $\operatorname{int}(A)$ in $\mathbf{R}^n$. This explains why $U(L)$ is an open set of $\mathbf{R}^n$. Also, it is clear from (4.19) and the definition of E_n that for every cell A, there exists at most 3^n cells C containing A. $\square$

The natural way of building n-complexes consists in pasting a family of n-charts (K_α). This principle is close to the notion of *direct limit* in algebra (see the remark below Definition 4.1.6). We paste the family by "identifying" the cells A,B for which there exists another cell C such that $\mathrm{int}(A), \mathrm{int}(B) \subset \mathrm{int}(C)$. We need a few assumptions to ensure that each equivalent class has a maximal cell and that such cells form a n-complex.

Definition 4.1.6. A *system of n-charts* is a family of n-charts (K_α) such that

1. for all $A, B \in \bigcup K_\alpha$
$$\mathrm{int}(A) \cap \mathrm{int}(B) \neq \emptyset \implies \mathrm{int}(A) \subset \mathrm{int}(B) \text{ or } \mathrm{int}(B) \subset \mathrm{int}(A) \quad (4.21)$$
2. for every $x \in \bigcup |K_\alpha|$, there exists a relative neighborhood V of x in $\bigcup |K_\alpha|$ and a finite finite set $S \subset \bigcup K_\alpha$ such that whenever a cell $A \in \bigcup K_\alpha$ meets V, there exists $B \in S$ such that $\mathrm{int}(A) \subset \mathrm{int}(B)$.

We define a *maximal cell of the system* as a cell $A \in \bigcup K_\alpha$ such that for all $B \in \bigcup K_\alpha$,
$$\mathrm{int}(A) \cap \mathrm{int}(B) \neq \emptyset \implies \mathrm{int}(B) \subset \mathrm{int}(A). \quad (4.22)$$
The collection of all maximal cells is called the *limit of the system* K_∞. It is a n-complex such that $|K_\infty| = \bigcup |K_\alpha|$ and whose local charts are given by the family (K_α).

We prove our claims on K_∞ below the remark.

Remark 4.1.1. The goal of this remark is to precise the analogy between a system of n-charts and the notion of *direct system* in algebra. Let the index set of (K_α) be denoted by M. Equip M with the following quasi-order: $\alpha \leq \beta$ if $K_\alpha \preceq K_\beta$. When $\alpha \leq \beta$, there exists a natural function
$$\pi_\alpha^\beta \colon K_\alpha \to K_\beta \quad (4.23)$$
which associate to each $A \in K_\alpha$, the unique $B \in K_\beta$ such that $\mathrm{int}(A) \subset \mathrm{int}(B)$. It is clear that
$$\pi_\alpha^\alpha = \text{identity} \quad (4.24)$$
and for $\alpha \leq \beta \leq \gamma$ in M,
$$\pi_\beta^\gamma \pi_\alpha^\beta = \pi_\alpha^\gamma. \quad (4.25)$$
Given α and β, the first axiom of our definition allows to paste K_α and K_β. The result is a n-complex K such that $K_\alpha, K_\beta \preceq K$ and which can be added to the system. In this way, M becomes a directed set. Then we say that two

cells $A \in K_\alpha$ and $B \in K_\beta$ are equivalent if there exists some $\gamma \geq \alpha, \beta$ such that $\pi_\alpha^\gamma(A) = \pi_\beta^\gamma(B)$. The *direct limit* of such system in algebra would be the set of all equivalent classes. Our definition took a few shortcuts because the abstract equivalent classes are replaced by a collection of maximal cells. We could not avoid a long axiom to say that these maximal cells are locally finite.

Proof. It is straightforward from the definition that the cells of K_∞ have disjoint interiors. Moreover, the second axiom implies that K_∞ is locally finite in $\bigcup |K_\alpha|$. Let us prove that for every $A \in \bigcup K_\alpha$, there exists $\hat{A} \in K_\infty$ such that

$$\operatorname{int}(A) \subset \operatorname{int}(\hat{A}). \tag{4.26}$$

It will follow immediatly that $|K_\infty| = \bigcup |K_\alpha|$, that the charts K_α are subordinate to K_∞ and that they induce a n-complex structure. We search a biggest element in the set

$$X = \{\, B \in \bigcup K_\alpha \mid \operatorname{int}(A) \subset \operatorname{int}(B) \,\} \tag{4.27}$$

and then we will prove that it belongs to K_∞. Using the second axiom for any point $x \in \operatorname{int}(A)$, we obtain a finite subset $S \subset \bigcup K_\alpha$ such that for all $B \in X$, there exists $C \in S$ such that $\operatorname{int}(B) \subset \operatorname{int}(C)$. In particular, $B \subset C$ and $C \in X$. We deduce that the biggest element of X should be searched for in the finite set $X \cap S$. Moreover, the first axiom implies that X is totally ordered. It follows that X has a biggest element $\hat{A}$. Let us prove that $\hat{A} \in K_\infty$. For all $B \in \bigcup K_\alpha$ such that $\operatorname{int}(B) \cap \operatorname{int}(\hat{A}) \neq \emptyset$, the first axiom says that we have either $\operatorname{int}(B) \subset \operatorname{int}(\hat{A})$ or $\operatorname{int}(\hat{A}) \subset \operatorname{int}(B)$. In the second case, $B \in X$ so we have in fact $\hat{A} = B$ by maximality of $\hat{A}$. We conclude that $\hat{A} \in K_\infty$. □

Example 4.1.1. Remember

$$E_n = \left\{ \prod_{i=1}^{n} [0, \alpha_i] \;\middle|\; \alpha \in \{\, -1, 0, 1 \,\}^n \right\}. \tag{4.28}$$

The family of translations of E_n by $\mathbf{Z}^n$,

$$\{\, p + E_n \mid p \in \mathbf{Z}^n \,\}, \tag{4.29}$$

is a system of n-charts. Its limit is the canonic lattice of $\mathbf{R}^n$. We could have defined this lattice directly as the set of all cells of the form

$$p + \prod_{i=1}^{n} [0, \alpha_i], \tag{4.30}$$

where $p \in \mathbf{Z}^n$ and $\alpha \in \{\, -1, 0, 1 \,\}^n$.

Example 4.1.2. Let X an open set of $\mathbf{R}^n$. We build a n-complex describing X and which is analogous to a Whitney decomposition. For $k \in \mathbf{N}$ and $p \in 2^{-k}\mathbf{Z}^n$, we introduce the dyadic chart of center p and sidelength 2^{-k}:

$$E_n(p,k) = p + 2^{-k}E_n, \tag{4.31}$$

i.e.

$$E_n(p,k) = \left\{ \prod_{i=1}^{n} [p_i, p_i + 2^{-k}\alpha_i] \;\middle|\; \alpha \in \{ -1, 0, 1 \}^n \right\}. \tag{4.32}$$

Then consider the family

$$\left\{ E_n(p,k) \;\middle|\; k \in \mathbf{N},\ p \in 2^{-k}\mathbf{Z}^n,\ |E_n(p,k)| \subset X \right\}. \tag{4.33}$$

It satisfies the first axiom of definition 4.1.6 as a general property of dyadic cells. Let us focus on the second axiom. Instead of working with the Euclidean norm, we work with the maximum norm $|\cdot|_\infty$, with its corresponding distance d_∞ and its open (cubic) balls U. For $x \in X$, define $r_x = \min\{ 1, \mathrm{d}_\infty(x, X^c) \}$. Let $k \in \mathbf{N}$ be such that $2^{-k+1} \leq r_x \leq 2^{-k+2}$. There exists $p \in 2^{-k-1}\mathbf{Z}^n$ such that $|x-p|_\infty < 2^{-k-1}$. Then, the triangular inequality gives

$$U(x, \tfrac{1}{8}r_x) \subset U(p, 2^{-k}) \subset \overline{U}(p, 2^{-k}) \subset U(x, r_x). \tag{4.34}$$

Or equivalently,

$$U(x, \tfrac{1}{8}r_x) \subset U(E_n(p,k)) \subset |E_n(p,k)| \subset U(x, r_x). \tag{4.35}$$

By the right hand side inclusion and the definition of r_x, $E_n(p,k)$ belongs to the system. Let A be a cell of the system which meets $U(E_n(p,k))$. Then $\mathrm{int}(A)$ also meets $U(E_n(p,k))$ because $U(E_n(p,k))$ is an open set and A is the closure of $\mathrm{int}(A)$. Thus, $\mathrm{int}(A)$ meets the interior of a cell $B \in E_n(p,k)$. Either the sidelength of A is $\geq 2^{-k}$, either it is $\leq 2^{-k}$ and $\mathrm{int}(A) \subset \mathrm{int}(B)$. In both cases, $\mathrm{int}(A)$ is contained in $\mathrm{int}(B)$, where B is a cell of the system of sidelength $\geq 2^{-k}$. By local finitness of dyadic lattices in $\mathbf{R}^n$, there exists a finite number of dyadic cells of sidelength $\geq 2^{-k}$ which meet $U(x, r_x)$. We deduce the second axiom of definition 4.1.6.

In Definition 4.1.2, we have seen the notion of rigid open set. We define a rigid closed set as an union of cells of varying dimensions.

Definition 4.1.7. Let K be a complex. A rigid open set of K is a set of the form

$$U(L) = \bigcup \{ \mathrm{int}(A) \mid A \in L \}, \tag{4.36}$$

where L is a subcomplex of K. A rigid closed set of K is a set of the form

$$|M| = \bigcup \{ A \in M \} \tag{4.37}$$

where $M \subset K$.

It is natural to ask whether the rigid closed sets are the complements of rigid open sets. Consider a subset $M \subset K$. The set

$$L = \{ A \in K \mid \nexists B \in M,\ A \subset B \} \tag{4.38}$$

is a subcomplex of K and we clearly have $|K \setminus L| = |M|$. We are led to wonder whether the subcomplexes L of K satisfies $|K \setminus L| = |K| \setminus U(L)$ in general. First, Lemma 4.1.1 shows that

$$|K \setminus L| \subset |K| \setminus U(L). \tag{4.39}$$

This inclusion may not be an equality. Indeed, the right-hand-side contains the "boundary" $|K| \setminus U(K)$ which may not be covered by cells of $K \setminus L$. In the case where $|K| = U(K)$, one sees that we always have $|K \setminus L| = |K| \setminus U(L)$ so the two points of view are equivalent. Now, we introduce the notion of a Whitney complex. It is a n-complex which decompose an open set and where one can "work locally in big charts". Here, "working locally in big charts" reminds Lebesgue's number Lemma where one can works locally in big balls subordinated to an open set of a cover. This definition is motivated by Example 4.1.2. In the introduction of Chapter 3, the reader can find the definition of the gauge r_s and its interpretation.

Definition 4.1.8 (Whitney complex)**.** A Whitney complex of $\mathbf{R}^n$ is a n-complex K such that

1. $|K| = U(K)$;
2. there exists $s > 0$ such that for all $x \in U(K)$, there exists an image F of E_n by a similarity such that $F \preceq K$ and

$$B(x, r_s(x)) \subset |F|. \tag{4.40}$$

Finally, we come to the notion of a Lipschitz subset of an open set X. They will play the role of boundaries in the next chapters.

Definition 4.1.9 (Lipschitz subset)**.** Let X be an open set of $\mathbf{R}^n$. A Lipschitz subset of X is a closed subset $\Gamma \subset X$ for which there exists a Whitney complex K of $\mathbf{R}^n$, a bijective and bilipschitz map $T\colon |K| \to X$ and a subset $M \subset K$ such that $\Gamma = T(|M|)$.

By the previous discussion, there also exists a subcomplex L of K such that $\Gamma = T(|K| \setminus U(L))$.

4.2 Existence of retractions

The next proposition justifies that a Lipschitz subset of an open set X is a also Lipschitz neighborhood retract of X. We build a retraction by following the scheme of the Federer-Fleming projection.

Proposition 4.2.1 (Scheme of Federer-Fleming projection)**.** *Let K be a n-complex, let L be a subcomplex of K. Then there exists a Lipschitz function $\phi\colon |K| \to |K|$ satisfying the following properties:*

1. *for all $A \in K$, $\phi(A) \subset A$;*
2. *$\phi = \mathrm{id}$ in $|K| \setminus U(L)$;*
3. *there exists a relative open set $O \subset |K|$ containing $|K| \setminus U(L)$ such that*

$$\phi(O) \subset |K| \setminus U(L). \tag{4.41}$$

Proof. We build by induction a family of locally Lipschitz functions

$$(\phi_m)\colon |K| \to |K| \tag{4.42}$$

indexed by a decreasing integer $m = n+1, \dots, 0$. We define

$$U_m(L) = \bigcup \{\, \mathrm{int}(A) \mid A \in L,\ \dim A \geq m \,\}. \tag{4.43}$$

and require that

1. for all $A \in K$, $\phi_m(A) \subset A$,
2. $\phi_m = \mathrm{id}$ in $|K| \setminus U_m(L)$,
3. there exists a relative open set $O \subset |K|$ containing $|K| \setminus U_m(L)$ such that
$$\phi(O) \subset |K| \setminus U_m(L). \tag{4.44}$$

The induction starts with $\phi_{n+1} = \mathrm{id}$. Assume that ϕ_{m+1} is well-defined for some $m \leq n$. We will post-compose ϕ_{m+1} with a function ψ_m made of radial projections in the cells of L of dimension m. First we define

$$\psi_m = \mathrm{id} \text{ in } |K| \setminus U_m(L). \tag{4.45}$$

We want to extend ψ_m as a Lipschitz function on $|K| \setminus U_{m+1}(L)$ by defining ψ_m on each cell $A \in L^m$ as a radial projection. For $A \in L^m$, let x_A be the center of A and let $\delta_A > 0$ be such that

$$A \cap \overline{B}(x_A, \delta_A) \subset \mathrm{int}(A). \tag{4.46}$$

Define ψ_m in $A \setminus B(x_A, \delta_A)$ to be the radial projection centered in x_A onto ∂A (if A is a 0-cell, $A \setminus B(x_A, \delta_A)$ is empty and we are not doing anything). In particular ψ_m is Lipschitz: for all $x, y \in A \setminus B(x_A, \delta_A)$,

$$|\psi_m(x) - \psi_m(y)| \leq C \mathrm{diam}(A) \delta_A^{-1} |x - y|. \tag{4.47}$$

We extend ψ_m as a $C\mathrm{diam}(A)\delta_A^{-1}$-Lipschitz function $\psi_m \colon A \to A$. Remember from Lemma 4.1.1 that $|L^m|$ is disjoint from $U_{m+1}(L)$. It follows that

$$|K| \setminus U_{m+1}(L) = |L^m| \cup (|K| \setminus U_m(L)) \tag{4.48}$$

and ψ_m is now a well-defined function on $|K| \setminus U_{m+1}(L)$. Let us prove that it is Lipschitz. Consider $A, B \in L^m$ such that $A \neq B$ and let $x \in A$ and $y \in B$. According to Lemma 4.1.1, we have $V_A \cap B = \emptyset$. And by Lemma 4.1.2, there exists a constant $\kappa \geq 1$ such that $V_A(\kappa) \subset V_A$. We deduce that $y \notin V_A(\kappa)$, i.e. $\mathrm{d}(y, \partial A) \leq \kappa \mathrm{d}(y, A)$. Let $z \in \partial A$ be such that $|y - z| = \mathrm{d}(y, \partial A)$, in particular

$$|y - z| \leq \kappa |x - y|. \tag{4.49}$$

Similarly, one can find $z' \in \partial B$ such that

$$\left|x - z'\right| \leq \kappa |x - y|. \tag{4.50}$$

Using the triangular inequality, we see that

$$|x - z|, \left|y - z'\right|, \left|z - z'\right| \leq C\kappa |x - y|. \tag{4.51}$$

Thus

$$\begin{aligned} |\psi_m(x) - \psi_m(y)| &\leq |\psi_m(x) - \psi_m(z)| + \left|\psi_m(z) - \psi_m(z')\right| \\ &\quad + \left|\psi_m(z') - \psi_m(y)\right| && (4.52) \\ &\leq |\psi_m(x) - \psi_m(z)| + \left|z - z'\right| + \left|\psi_m(z') - \psi_m(y)\right| && (4.53) \\ &\leq C\kappa(\mathrm{diam}(A)\delta_A^{-1} + \mathrm{diam}(B)\delta_B^{-1} + 1)|x - y|. && (4.54) \end{aligned}$$

Next consider $A \in L^m$, let $x \in A$ and $y \in |K| \setminus U_m(L)$. We have $V_A(\kappa) \subset V_A \subset U_m(L)$ so $\mathrm{d}(y, \partial A) \leq \kappa \mathrm{d}(y, A)$. Let $z \in \partial A$ be such that $|y - z| = \mathrm{d}(y, \partial A)$, in particular

$$|y - z| \leq \kappa |x - y|. \tag{4.55}$$

Using the triangular inequality, we see that $|x - z| \leq C\kappa |x - y|$. It follows that

$$\begin{aligned} |\psi_m(x) - \psi_m(y)| &\leq |\psi_m(x) - \psi_m(z)| + |\psi_m(z) - \psi_m(y)| && (4.56) \\ &\leq |\psi_m(x) - \psi_m(z)| + |z - y| && (4.57) \\ &\leq C\kappa(\mathrm{diam}(A)\delta_A^{-1} + 1)|x - y|. && (4.58) \end{aligned}$$

Note that we can always choose δ_A in such way that $\mathrm{diam}(A)\delta_A^{-1} \leq C$. We didn't make this simplification earlier because we aim to use this proof where we have no control over the ratio $\mathrm{diam}(A)\delta_A^{-1}$. This concludes the proof that ψ_m is well-defined and C-Lipschitz on $|K| \setminus U_{m+1}(L)$.

In this paragraph, we extend ψ_m over $|K|$ in such way that for each $A \in K$, $\psi_m(A) \subset A$. Let us start with the extension process and we will check afterward that ψ_m preserves every face. If $m = n$, we have

$$|K| \setminus U_{m+1}(L) = |K| \tag{4.59}$$

so ψ_m is already defined over $|K|$. Assume that $m < n$. For $A \in L^{m+1}$, Lemma 4.1.1 says that

$$A \setminus U_{m+1}(L) = \partial A \tag{4.60}$$

so ψ_m is defined and C-Lipschitz on ∂A and we can extend it as a C-Lipschitz function $\psi_m \colon A \to A$. Then, ψ_m is well-defined on $|K| \setminus U_{m+2}(L)$ and we can show that it is Lipschitz using a similar proof as in the previous paragraph. The main difference is that we no longer have $\psi_m = \mathrm{id}$ on the boundaries but it is enough to know that ϕ is Lipschitz on the boundaries. We continue the process until ψ_m is defined on $|K|$. Let us prove that ψ_m preserves every face $A \in K$. There are two cases to distinguish. If $A \subset |K| \setminus U_m(L)$, then $\psi_m = \mathrm{id}$ on A by (4.45). If $A \cap U_m(L) \neq \emptyset$, there exists $B \in L$ such that $\dim(B) \geq m$ and $A \cap \mathrm{int}(B) \neq \emptyset$. By the first point of Lemma 4.1.1, $B \subset A$ so $\mathrm{diam}(A) \geq m$ and, by the properties of subcomplex, $A \in L$. We conclude that ψ_m preserves A by construction.

We finally define $\phi_m = \psi_m \circ \phi_{m+1}$. By construction, ϕ_m is C-Lipschitz and satisfies the first and second requirement of the induction. Let us check that it satisfies the third requirement. Let O be a relative open set of $|K|$ containing $|K| \setminus U_{m+1}(L)$ and such that

$$\phi_{m+1}(O) \subset |K| \setminus U_{m+1}(L) \tag{4.61}$$

We will solve the induction with the set

$$O' = O \setminus \phi_{m+1}^{-1}\left(\bigcup_{A \in L^m} A \cap \overline{B}(x_A, \delta_A)\right). \tag{4.62}$$

The set O' contains $|K| \setminus U_m(L)$ because $|K| \setminus U_m(L) \subset |K| \setminus U_{m+1}(L) \subset O$ and $\phi_{m+1} = \mathrm{id}$ on $|K| \setminus U_m(L)$ and for $A \in L^m$,

$$A \cap \overline{B}(x_A, \delta_A) \subset \mathrm{int}(A) \subset U_m(L). \tag{4.63}$$

Let us justify that O' is open. The family $(A \cap \overline{B}(x_A, \delta_A))_A$ is a locally finite family of closed sets in $|K|$ so its union is relatively closed and O' is relatively open. Finally,

$$\phi_m(O') \subset \psi_m \circ \phi_{m+1}(O') \tag{4.64}$$

$$\subset \psi_m(|K| \setminus (U_{m+1}(L) \cup \bigcup_{A \in L^m} A \setminus \overline{B}(x_A, \delta_A)) \tag{4.65}$$

$$\subset |K| \setminus U_m(L). \tag{4.66}$$

□

Corollary 4.2.1. *Let X be an open set of $\mathbf{R}^n$. A Lipschitz subset of X is also a Lipschitz neighborhood retract.*

Finally, we precise the notion of "convex corner" for a set of cells M. We are going to see that if the nonconvex corners of M are H^d negligible, then $|M|$ is H^d regular. We expect the H^d regularity to be preserved by diffeomorphism but we don't formalize it.

Definition 4.2.1. Let M be a set of cells. A cell $A \in M$ is called *regular* if the set $\bigcup\{\, B \in M \mid A \subset B \,\}$ is convex. The *regular support* of M is the set

$$\operatorname{reg}(M) = \bigcup\{\, \operatorname{int}(A) \mid A \in M,\ \text{regular} \,\}\,. \tag{4.67}$$

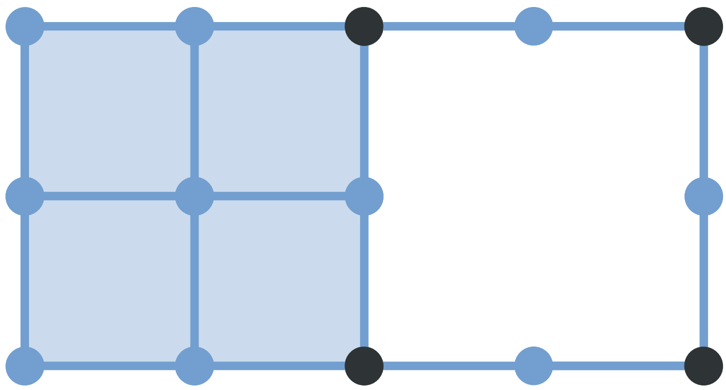

Figure 4.4: A set of cells (light blue squares, blue edges, black/blue vertices). The nonregular cells are black.

We are going to see that for all $\varepsilon > 0$, we can construct a $(1+\varepsilon)$-retraction onto $\operatorname{reg}(M)$.

Proposition 4.2.2 (Almost retraction on the regular part)**.** *Let K be a complex and M be a subset of K. For all $\varepsilon > 0$, there exists a relative open set $O \subset |K|$ containing* $\operatorname{reg}(M)$ *and a $(1+\varepsilon)$-Lipschitz map $p\colon O \cup |M| \to |M|$ such that $p = \operatorname{id}$ on $|M|$.*

Proof. Consider a regular cell $A \in M$. The set $[A] = \bigcup\{\, B \in M \mid A \subset B \,\}$ is convex and also compact by local finitness of K. Let the orthogonal projection onto $[A]$ be denoted by p_A. We want to define a relative open set $O_A \subset |K|$ which contains $\operatorname{int}(A)$ and such that all $x \in O_A$, the projection $p_A(x)$ is the unique distance minimizer of x in M. We introduce

$$O_A := \{\, x \in |K| \mid |x - p_A(x)| < \varepsilon \mathrm{d}(x, |K| \setminus V_A) \,\}\,. \tag{4.68}$$

This set is relatively open in $|K|$. Moreover, it contains $\operatorname{int}(A)$ because $p_A = \operatorname{id}$ on $\operatorname{int}(A)$ and V_A is a relative neighborhood of $\operatorname{int}(A)$. Now we justify that for all $x \in O_A$, $p_A(x)$ is the unique distance minimizer of x in M. Let $y \in |M|$ be distinct from $p_A(x)$. We distinguish two cases. If $y \in V_A$, consider a cell $B \in M$ such that $y \in B$. As $B \cap V_A \neq \emptyset$, we deduce $A \subset B$ by Lemma 4.1.1. We conclude that $y \in [A]$ and then $|x - p_A(x)| < |x - y|$ because x has a unique distance minimizer in $[A]$. If $y \notin V_A$, the definition of O_A shows that (assuming $\varepsilon \leq 1$)

$$|x - p_A(x)| < \varepsilon |x - y| \leq |x - y|. \tag{4.69}$$

This finishes the proof that $p_A(x)$ is the unique distance minimizer in $|M|$.

Now, we define O as the reunion of all the sets O_A, where A is a regular cell of $|M|$. For $x \in O \cup |M|$, we define $p(x)$ as the unique distance minimizer of x in M. Let us show that p has a good Lipschitz constant. We consider $x, y \in O \cup |M|$ and we distinguish three cases. First, we assume $x, y \in |M|$. We have $p(x) = x$ and $p(y) = y$ so $|p(x) - p(y)| = |x - y|$. Secondly, we assume $x \in O_A$, where A is a regular cell of M and $y \in |M|$. If $y \in V_A$, we have $y \in [A]$ (see the first paragraph) so $p(y) = y = p_A(y)$ and

$$|p(x) - p(y)| = |p_A(x) - p_A(y)| \leq |x - y|. \tag{4.70}$$

If $y \notin V_A$, we have $|x - p_A(x)| < \varepsilon |x - y|$ (see the first paragraph) so

$$\begin{aligned} |p(x) - p(y)| &= |p_A(x) - y| && (4.71)\\ &\leq |p_A(x) - x| + |x - y| && (4.72)\\ &\leq (1 + \varepsilon)|x - y|. && (4.73) \end{aligned}$$

Thirdly, we assume $x \in O_A$ and $y \in O_B$, where A, B are regular cells of M. If $p(y) \in V_A$, we have $p(y) \in [A]$ (first paragraph) so $p(y) = p_A(y)$ and

$$|p(x) - p(y)| = |p_A(x) - p_A(y)| \leq |x - y|. \tag{4.74}$$

If $p(x) \in V_B$, we reason similarly. If $p(y) \notin V_A$ and $p(x) \notin V_B$, we have

$$|x - p(x)| < \varepsilon |x - p(y)| \leq \varepsilon(|x - y| + |y - p(y)|) \tag{4.75a}$$

$$|y - p(y)| < \varepsilon |y - p(x)| \leq \varepsilon(|x - y| + |x - p(x)|) \tag{4.75b}$$

whence

$$|x - p(x)| + |y - p(y)| \leq \frac{2\varepsilon}{1 - \varepsilon}|x - y|. \tag{4.76}$$

We conclude that

$$\begin{aligned} |p(x) - p(y)| &\leq |p(x) - x| + |x - y| + |y - p(y)| && (4.77)\\ &\leq (1 + \frac{2\varepsilon}{1 - \varepsilon})|x - y|. && (4.78) \end{aligned}$$

□

4.3 The Federer-Fleming projection

We state the Federer-Fleming projection in the language of complexes and we present a new estimate (see (4.81) below).

Proposition 4.3.1 (Federer-Fleming projection)**.** *Let K be n-complex. Let $E \subset |K|$ be a Borel set such that $H^{d+1}(|K| \cap \overline{E}) = 0$. Then there exists a locally Lipschitz function $\phi \colon |K| \to |K|$ satisfying the following properties:*

1. *for all $A \in K$, $\phi(A) \subset A$;*
2. *$\phi = \mathrm{id}$ in $|K| \setminus \bigcup \{ \mathrm{int}(A) \mid \dim A > d \}$;*
3. *there exists a relative open set $O \subset |K|$ such that $E \subset O$ and*
$$\phi(O) \subset |K| \setminus \bigcup \{ \mathrm{int}(A) \mid \dim A > d \}; \tag{4.79}$$
4. *for all $A \in K$,*
$$H^d(\phi(\mathrm{int}(A) \cap E)) \leq C H^d(\mathrm{int}(A) \cap E); \tag{4.80}$$
5. *for all $A \in K^d$,*
$$H^d(A \cap \phi(E)) \leq C \int_{G(d,n)} H^d(p_V(V_A \cap E) \, \mathrm{d}V, \tag{4.81}$$

where $C \geq 1$ is a constant that depends only on n. In addition, let us assume that there exists a constant $M \geq 1$ such that

1. *for all $A, B \in K$ such that $A \subset B$, $\mathrm{diam}(B) \leq M \mathrm{diam}(A)$;*
2. *for all bounded subset $S \subset E$, for all radius $0 < r \leq \mathrm{diam}(S)$, the set $E \cap S$ can be covered by at most $M r^{-d} \mathrm{diam}(S)^d$ balls of radius $r > 0$.*

Then, we can build such ϕ with a Lipschitz constant that depends only on n and M.

Before proving Proposition 4.3.1, we introduce the two preliminary lemmas which are the origins of the estimates (4.80) and (4.81). David and Semmes showed in [DS] (Lemma 3.22) that, in average among centers of radial projections, the measure of the image is not much larger that the original set. We state the Lemma without proof below.

Lemma 4.3.1 ([DS], Lemma 3.22)**.** *Let Q be a cube of $\mathbf{R}^n$, let E be a Borel subset of Q. Then*

$$\mathrm{diam}(Q)^{-n} \int_{\frac{1}{2}Q} H^d(\phi_x(E)) \, \mathrm{d}x \leq C H^d(E), \tag{4.82}$$

where ϕ_x is the radial projection from $Q \setminus x$ onto ∂Q and C is a constant that depends only on n.

We develop a similar lemma for another jauge which has the advantage to cancel the purely nonrectifiable part. We recall that the Grassmannian $G(d,n)$ is the set of all d-linear planes in $\mathbf{R}^n$. A linear plane can be represented by its orthogonal projection p_V. Thus, the operator norm with respect to the Euclidean norm induces a natural metric on $G(d,n)$. This space is also equipped with a canonic invariant measure $\mathrm{d}V$. We refer to Appendix B (and possibly, [Mat], Section 3). Following the notation of Federer ([Fe], 2.10.5), we define the gauge ζ^d on Borel subsets of $\mathbf{R}^n$ by

$$\zeta^d(E) := \int_{G(d,n)} H^d(p_V(E))\,\mathrm{d}V. \tag{4.83}$$

For a cell A, we define the restriction of this gauge to A

$$\zeta^d \llcorner A(E) := \int_{G(\mathrm{aff}(A),d)} H^d(p_V(A \cap E))\,\mathrm{d}V, \tag{4.84}$$

where $\mathrm{aff}(A)$ is the affine span of A and $G(\mathrm{aff}(A),d)$ is the set of all d-linear planes of $\mathrm{aff}(A)$ centered at an arbitrary point. Finally, we recall the notion of upper-integral. Given a set $S \subset \mathbf{R}^n$ equipped with a measure μ and given any function $f\colon S \to [0,+\infty]$, the *upper-integral* of f is defined by

$$\int_S^* f\,\mathrm{d}\mu = \inf_{\psi} \int_S \psi\,\mathrm{d}\mu \tag{4.85}$$

where ψ run through the μ-measurable functions $S \to [0,+\infty]$ such that $f \leq \psi$. It is not certain we really need an upper-integral but we do not want to check.

Lemma 4.3.2. *Let Q be a cube of $\mathbf{R}^n$, let E be a Borel subset of Q. Then*

$$\mathrm{diam}(Q)^{-n} \int_{\frac{1}{2}Q}^* \sup_A \zeta^d \llcorner A(\phi_x(E))\,\mathrm{d}x \leq C\zeta^d(E), \tag{4.86}$$

where ϕ_x is the radial projection from $Q \setminus x$ onto ∂Q, the sup *is indexed on all cells $A \subset \partial Q$ and C is a constant that depends only on n.*

Proof. The principle is that for all cell $A \subset \partial Q$, for all $x \in \frac{1}{2}Q \setminus E$,

$$\mathrm{diam}(Q)^{-d} \sup_A \zeta^d \llcorner A(\phi_x(E)) \leq C\gamma_{n-d,n}(\{\, W \mid (x+W) \cap E \neq \emptyset \,\}) \tag{4.87}$$

and, by a Fubini argument,

$$\mathrm{diam}(Q)^{-n} \int_Q \gamma_{n-d,n}(\{\, W \mid (x+W) \cap E \neq \emptyset \,\})\,\mathrm{d}x \leq C\mathrm{diam}(Q)^{-d}\zeta^d(E). \tag{4.88}$$

To start with, we work with $Q = [-1,1]^n$ and $A = [-1,1]^{n-1} \times \{1\}$. We will explain in step 3 how to deal with general cells $A \subset \partial Q$.

Step 1. We prove inequality (4.87) for $x = 0$ (this is not a suspicious loss of generality). Let ϕ be the radial projection from $Q \setminus 0$ onto ∂Q. We want to show

$$\zeta^d \llcorner A(\phi(E)) \leq C\gamma_{n-d,n}(\{ W \mid W \cap E \neq \emptyset \}). \tag{4.89}$$

Let L_0 be the line generated by the vector $(0, \cdots, 0, 1)$. In particular L_0 is orthogonal to $\mathrm{aff}(A)$. We apply the disintegration formula (see Proposition B.2.1),

$$\begin{aligned}
&\gamma_{n-d,n}(\{ W \mid W \cap E \neq \emptyset \}) \\
&= \int_{G(1,n)} \gamma_{d,L^\perp}(\{ V \mid (L + V^\perp) \cap E \neq \emptyset \}) \, \mathrm{d}L && (4.90)\\
&\geq \int_{B(L_0,\alpha)} \gamma_{d,L^\perp}(\{ V \mid (L + V^\perp) \cap E \neq \emptyset \}) \, \mathrm{d}L, && (4.91)
\end{aligned}$$

where $V \in G(d, L^\perp)$ and $V^\perp$ is the orthogonal complement of V in $L^\perp$, $\alpha \in (0,1)$ is a constant close to 1 that we will specify later and $B(L_0, \alpha)$ is the ball of center L_0 and radius α in $G(1,n)$. For $L \in G(1,n)$ such that $\mathrm{d}(L, L_0) \leq \alpha$, we show that

$$\begin{aligned}
&\gamma_{d,L^\perp}(\{ V \mid (L + V^\perp) \cap E \neq \emptyset \}) \\
&\qquad \geq C(\alpha)^{-1} \gamma_{d,L_0^\perp}(\{ V \mid (L + V^\perp) \cap E \neq \emptyset \}), \quad (4.92)
\end{aligned}$$

where $C(\alpha)$ is a constant that depends on n and α. Let us define

$$u \colon L_0^\perp \to L^\perp \tag{4.93}$$

to be the orthogonal projection onto $L^\perp$. Since $\mathrm{d}(L, L_0) \leq \alpha < 1$, (B.1) in Appendix B says that u is an isomorphism with $\|u\| \, \|u^{-1}\| \leq C(\alpha)$. According to Lemma B.1.3, u induces a $C(\alpha)$-bi-Lipschitz and one-to-one correspondence from $G(L_0^\perp, d)$ onto $G(L^\perp, d)$. Finally, notice that for all $V \in G(L_0^\perp, d)$,

$$L + u(V^\perp) = L + V^\perp \tag{4.94}$$

because, for $x \in L_0^\perp$, $u(x) - x \in L$. By the action of Lipschitz functions on Hausdorff measures, we conclude that

$$\begin{aligned}
&\int_{G(L_0,\alpha)} \gamma_{d,L^\perp}(\{ V \mid (L + V^\perp) \cap E \neq \emptyset \}) \, \mathrm{d}L \\
&\qquad \geq C(\alpha)^{-1} \int_{G(L_0,\alpha)} \gamma_{d,L_0^\perp}(\{ V \mid (L + V^\perp) \cap E \neq \emptyset \}). \quad (4.95)
\end{aligned}$$

The right hand side allows an application of Fubini,

$$\int_{B(L_0,\alpha)} \gamma_{d,L_0^\perp}(\{ V \mid (L+V^\perp)\cap E \neq \emptyset \})\,\mathrm{d}L \\ = \int_{G(d,L_0^\perp)} \gamma_{1,n}(\{ L \in B(L_0,\alpha) \mid (L+V^\perp)\cap E \neq \emptyset \})\,\mathrm{d}V. \quad (4.96)$$

We apply Lemma B.2.2 to see that for some universal $\alpha \in]0,1[$ (close enough to 1), for all $V \in G(L_0^\perp, d)$,

$$\gamma_{1,n}(\{ L \in B(L_0,\alpha) \mid (L+V^\perp)\cap E \neq \emptyset \}) \\ \geq C^{-1} H^{n-1}(\{ x \in 2A \mid (L(x)+V^\perp)\cap E \neq \emptyset \}), \quad (4.97)$$

where $L(x)$ is the linear line generated by x. Moreover, it clear from the definition of ϕ that

$$\{ x \in A \mid (L(x)+V^\perp)\cap E \neq \emptyset \} \\ = \{ x \in 2A \mid (x+V^\perp)\cap \phi(E) \neq \emptyset \}. \quad (4.98)$$

Using the decomposition of the Lebesgue measure in $\mathrm{aff}(A)$,

$$\mathrm{vol}_{n-1} = (\mathrm{vol}_d \llcorner V) \times (\mathrm{vol}_{n-1-d} \llcorner V^\perp), \quad (4.99)$$

we finally obtain

$$H^{n-1}(\{ x \in 2A \mid (x+V^\perp)\cap \phi(E) \neq \emptyset \}) \geq C^{-1} H^d(p_V(A \cap \phi(E))). \quad (4.100)$$

Step 2. By duality of $G(d,n)$,

$$\int_Q \gamma_{n,n-d}(\{ W \mid (x+W)\cap E \neq \emptyset \})\,\mathrm{d}x \\ = \int_Q \gamma_{n,d}(\{ V \mid (x+V^\perp)\cap E \neq \emptyset \})\,\mathrm{d}x. \quad (4.101)$$

We apply Fubini,

$$\int_Q \gamma_{n,d}(\{ V \mid (x+V^\perp)\cap E \neq \emptyset \})\,\mathrm{d}x \\ = \int_{G(d,n)} \mathrm{vol}_n(\{ x \in Q \mid (x+V^\perp)\cap E \neq \emptyset \})\,\mathrm{d}V. \quad (4.102)$$

Using the decomposition of the Lebesgue measure in $\mathbf{R}^n$,

$$\mathrm{vol}_n = (\mathrm{vol}_d \llcorner V) \times (\mathrm{vol}_{n-d} \llcorner V^\perp), \quad (4.103)$$

we obtain

$$\mathrm{vol}_n(\{\, x \in Q \mid (x + V^{\perp}) \cap E \neq \emptyset \,\}) \leq CH^d(p_V(E)). \tag{4.104}$$

Step 3. In this step, we explain how to deal with arbitrary cells. Let A be any cell included in ∂Q. A convexity argument justifies that there exists a $(n-1)$-face R of Q containing A. In the previous part, we have proved that

$$\mathrm{diam}(Q)^{-n} \int_{\frac{1}{2}Q}^{*} \zeta^d \llcorner R(\phi_x(E)) \, \mathrm{d}x \leq C\zeta^d(E). \tag{4.105}$$

To extend this inequality to A, we show that $\zeta^d \llcorner A \leq \zeta^d \llcorner R$. We apply (4.105) with E being replaced by a Borel set $F \subset R$ and the inequality simplifies to

$$\zeta^d \llcorner R(F) \leq C\zeta^d(F). \tag{4.106}$$

It follows that for all affine hyperplane V, for all bounded Borel subset $F \subset V$,

$$\zeta^d \llcorner V(F) \leq C\zeta^d(F) \tag{4.107}$$

because we can artificially include F in a $(n-1)$-face of a cube. We deduce by induction that for all affine planes V, W such that $V \subset W$, for all bounded Borel subset $F \subset V$,

$$\zeta^d \llcorner V(F) \leq C\zeta^d \llcorner W(F) \tag{4.108}$$

It is now clear that $\zeta^d \llcorner A \leq \zeta^d \llcorner R$ on Borel subsets because $\mathrm{aff}(A) \subset \mathrm{aff}(R)$. $\square$

We are prepared to prove Proposition 4.3.1.

Proof of Proposition 4.3.1. In the following proof, the letter C plays the role of a constant ≥ 1 that depends on n. Its value can increase from one line to another (but a finite number of times). We postpone the case where E is semiregular to the end of the proof. The principle of the proof is to build by induction a family of locally Lipschitz functions

$$(\phi_m)\colon |K| \to |K| \tag{4.109}$$

indexed by a decreasing integer $m = n+1, \ldots, d+1$. We define

$$U_m(K) = \bigcup \{\, \mathrm{int}(A) \mid A \in K,\ \dim A \geq m \,\}. \tag{4.110}$$

and require that

1. for all $A \in K$, $\phi_m(A) \subset A$;

2. $\phi_m = \mathrm{id}$ in $|K| \setminus U_m(K)$;

3. there exists a relative open set $O \subset |K|$ containing $|K| \setminus U_m(K)$ such that
$$\phi(O) \subset |K| \setminus U_m(K); \tag{4.111}$$

4. for all $A \in K$,
$$H^d(\phi_m(\text{int}(A) \cap E)) \leq CH^d(\text{int}(A) \cap E); \tag{4.112}$$

5. for all $A \in K$,
$$\zeta^d \llcorner A(\text{int}(A) \cap \phi_m(E)) \leq C\zeta^d(V_A \cap E). \tag{4.113}$$

The proof follows the same scheme as Proposition 4.2.1 (where the subcomplex L is the set of cells $A \in K$ of dimension $\geq d+1$) but we choose the center of projection wisely. The induction starts with $\phi_{n+1} = \text{id}$. Assume that ϕ_{m+1} is well-defined for some $m \leq n$. We are going to post-compose ϕ_{m+1} with a function ψ_m made of radial projections in the cells of K of dimension m. First we define
$$\psi_m = \text{id in } |K| \setminus U_m(K). \tag{4.114}$$
Then we define ψ_m on each cell $A \in K^m$ as a radial projection. Fix $A \in K^m$. For $x \in \text{int}(A)$, let ϕ_x be the radial projection onto ∂A centered at x. We want a center of projection $x_A \in \frac{1}{2}A$ such that

1. $x_A \notin \overline{\phi_{m+1}(E)}$;

2. for all $B \in K$ containing A,
$$H^d(\phi_{x_A}(A_B)) \leq CH^d(A_B), \tag{4.115}$$
where $A_B = \text{int}(A) \cap \phi_{m+1}(\text{int}(B) \cap E)$;

3. for all $B \in K$ contained in ∂A,
$$\zeta^d \llcorner B(\phi_{x_A}(A_*)) \leq C\zeta^d \llcorner A(A_*), \tag{4.116}$$
where $A_* = \text{int}(A) \cap \phi_{m+1}(E)$.

We are going to show that such centers x_A exists. First, we justify that
$$H^{d+1}(\text{int}(A) \cap \overline{\phi_{m+1}(E)}) = 0. \tag{4.117}$$
This means that the first requirement is satisfied for H^m-almost every $x_A \in \text{int}(A)$. For $y \in \text{int}(A) \cap \phi_{m+1}(E)$, there exists $x \in E$ such that $y = \phi_{m+1}(x)$ and there exists $B \in K$ such that $x \in B$. According to the induction

assumptions, $\phi_{m+1}(B) \subset B$ so $\operatorname{int}(A) \cap B \neq \emptyset$ and then $A \subset B$ by Lemma 4.1.1. Thus

$$\operatorname{int}(A) \cap \phi_{m+1}(E) \subset \bigcup_{A \subset B} \phi_{m+1}(B \cap E). \tag{4.118}$$

There exists only a finite number of $B \in K$ containing A (by local finitness or by Lemma 4.1.2) so the union $\bigcup_{A \subset B} \phi_{m+1}(B \cap \overline{E})$ is compact. Moreover for each $B \in K$ containing A,

$$H^{d+1}(\phi_{m+1}(B \cap \overline{E})) = 0 \tag{4.119}$$

because ϕ_{m+1} is Lipschitz on B. We deduce (4.117). Now let's deal with with the second requirement on x_A. Let $B \in K$ containing A. We use Lemma 4.3.1 in combination with the Markov inequality to estimate that for $C' > 0$,

$$\operatorname{diam}(A)^{-m} H^m(\{ x \in \operatorname{int}(A) \mid H^d(\phi_x(A_B)) \geq C' H^d(A_B) \}) \leq \frac{C}{C'} \tag{4.120}$$

If C' is big enough (depending on n), there is a big set of $x \in \frac{1}{2}A$ such that $H^d(\phi_x(A_B)) \leq C' H^d(A_B)$. As there are at most 3^n cells B containing A by Lemma 4.1.2, we can also require that this condition is true for every $B \in K$ containing A (the constant C' increases a finite number of times). We obtain the third requirement in the same fashion using Lemma 4.3.2. Now that x_A is chosen, we let $\delta_A > 0$ be such that

$$A \cap \overline{B}(x_A, \delta_A) \subset \operatorname{int}(A) \setminus \phi_{m+1}(E) \tag{4.121}$$

and we define ψ_m in $A \setminus B(x_A, \delta_A)$ to be the radial projection centered in x_A onto ∂A. In particular ψ_m is $C\operatorname{diam}(A)\delta_A^{-1}$ Lipschitz and we extend ψ_m as a $C\operatorname{diam}(A)\delta_A^{-1}$-Lipschitz function $\psi_m \colon A \to A$. The construction of ψ_m continues just as in the proof of Lemma 4.2.1 and ϕ_m is finally defined as $\phi_m = \psi_m \circ \phi_{m+1}$.

We only show the fourth and the fifth requirement of the induction because the other requirements are practically proved in 4.2.1. Let us start with the fourth. We fix $B \in K$ and we prove that

$$H^d(\phi_m(\operatorname{int}(B) \cap E)) \leq C H^d(\operatorname{int}(B) \cap E)). \tag{4.122}$$

We recall that

$$\phi_{m+1}(E) \subset |K| \setminus \bigcup \{ \operatorname{int}(A) \mid \dim A \geq m+1 \}. \tag{4.123}$$

For $y \in \phi_m(\operatorname{int}(B) \cap E)$, there exists $x \in \operatorname{int}(B) \cap E$ such that $y = \phi_m(x)$. Either $\phi_{m+1}(x) \in \bigcup \{ \operatorname{int}(A) \mid \dim A \geq m \}$ and then $y = \phi_{m+1}(x)$; either $\phi_{m+1}(x) \in \operatorname{int}(A)$ for some cell $A \in K^m$. In the latter case, notice that

$\mathrm{int}(A) \cap B \neq \emptyset$ because $\phi_{m+1}(x) \in \mathrm{int}(A) \cap B$. It follows that $A \subset B$ by Lemma 7. We thus have in the second case,

$$\phi_{m+1}(x) \in A_B := \mathrm{int}(A) \cap \phi_{m+1}(\mathrm{int}(B) \cap E)), \tag{4.124}$$

where $A \in K^m$ is such that $A \subset B$. In conclusion,

$$\begin{aligned} & H^d(\phi_m(\mathrm{int}(B) \cap E)) \\ & \leq H^d(\phi_{m+1}(\mathrm{int}(B) \cap E)) + \sum_A H^d(\psi_m(\phi_{m+1}(A_B))) && (4.125) \\ & \leq H^d(\phi_{m+1}(\mathrm{int}(B) \cap E)) + \sum_A H^d(\phi_{x_A}(A_B)) && (4.126) \\ & \leq CH^d(\phi_{m+1}(\mathrm{int}(B) \cap E)) + C \sum_A H^d(A_B) && (4.127) \\ & \leq CH^d(\phi_{m+1}(\mathrm{int}(B) \cap E)) && (4.128) \\ & \leq CH^d(\mathrm{int}(B) \cap E), && (4.129) \end{aligned}$$

where $\sum_A$ is indexed by the cells $A \in K^m$ such that $A \subset B$. We have used the second requirement on x_A, the additivity of the measure H^d and the induction assumption 4. Now we check the fifth requirement. We fix $B \in K$ and we prove that

$$\zeta^d \llcorner B(\mathrm{int}(B) \cap \phi_m(E)) \leq C\zeta^d(V_B \cap E). \tag{4.130}$$

The reasoning is similar except that B is not an initial cell but a final cell. For $y \in \mathrm{int}(B) \cap \phi_m(E)$, there exists $x \in E$ such that $y = \phi_m(x)$. Either $\phi_{m+1}(x) \in \bigcup \{ \mathrm{int}(A) \mid \dim A \geq m \}$ and $y = \phi_{m+1}(x)$; either $\phi_{m+1}(x) \in \mathrm{int}(A)$ for some cell $A \in K^m$. In the latter case, notice that $\mathrm{int}(B) \cap \partial A \neq \emptyset$ because $y = \psi_m \circ \phi_{m+1}(x) \in \mathrm{int}(B) \cap \partial A$. It follows that $B \subset A$ by Lemma 4.1.1 but we can say more. Since $\mathrm{int}(A) \cap \partial A = \emptyset$, we necessarily have $A \neq B$ which implies that A and B have disjoint interiors. We deduce that $\mathrm{int}(B) \subset \partial A$ and, by taking the closure, $B \subset \partial A$. Thus, for the second case,

$$\phi_{m+1}(x) \in A_* := \mathrm{int}(A) \cap \phi_{m+1}(E), \tag{4.131}$$

where $A \in K^m$ is such that $B \subset \partial A$. In sum,

$$\begin{aligned} & \zeta^d \llcorner B(\mathrm{int}(B) \cap \phi_m(E)) \\ & \leq \zeta^d \llcorner B(\mathrm{int}(B) \cap \phi_{m+1}(E)) + \sum_A \zeta^d \llcorner B(\psi_m(A_*)) && (4.132) \\ & \leq \zeta^d \llcorner B(\mathrm{int}(B) \cap \phi_{m+1}(E)) + \sum_A \zeta^d \llcorner B(\phi_{x_A}(A_*)) && (4.133) \\ & \leq \zeta^d \llcorner B(\mathrm{int}(B) \cap \phi_{m+1}(E)) + \sum_A \zeta^d \llcorner A(A_*) && (4.134) \\ & \leq \zeta^d(E \cap V_B), && (4.135) \end{aligned}$$

where $\sum_A$ is indexed by the cells $A \in K^m$ such that $B \subset \partial A$. Here we have used the third requirement on x_A, we have bounded the sum because there are at most 3^n cells containing B (ζ^d is not additive!) and we have used the induction assumption 5.

In the last paragraph, we assume that there exists a constant $M \geq 1$ such that

1. for all $A, B \in K$ such that $A \subset B$, $\mathrm{diam}(A) \leq M\mathrm{diam}(A)$;
2. for all bounded subset $S \subset E$, for all radius $0 < r \leq \mathrm{diam}(S)$, the set S can be covered by at most $Mr^{-d}\mathrm{diam}(S)^d$ balls of radius $r > 0$.

The letter C plays now the role of constant ≥ 1 that depends on on n and M. The principle is that, given a C-Lipschitz ϕ_{m+1}, we want to build a C-Lipschitz ψ_m. According to the proof of Lemma 4.2.1, it suffices that for all $A \in K^m$,

$$\mathrm{diam}(A)\delta_A^{-1} \leq C. \tag{4.136}$$

Thus, we prove the existence of $x_A \in \frac{1}{2}$ such that

1. $\mathrm{diam}(A)\mathrm{d}(x, \phi_{m+1}(E))^{-1} \leq C$;
2. for all $B \in K$ containing A,
$$H^d(\phi_{x_A}(A_B)) \leq CH^d(A_B), \tag{4.137}$$
where $A_B = \mathrm{int}(A) \cap \phi_{m+1}(\mathrm{int}(B) \cap E)$;
3. for all $B \in K$ contained in ∂A,
$$\zeta^d \llcorner B(\phi_{x_A}(A_*)) \leq C\zeta^d \llcorner A(A_*), \tag{4.138}$$
where $A_* = \mathrm{int}(A) \cap \phi_{m+1}(E)$.

Remember that to obtain the second requirement on x_A, we have chosen C big enough (depending on n) so that for $B \in K$ containing A,

$$\mathrm{diam}(A)^{-m}H^m(\{\, x \in \mathrm{int}(A) \mid H^d(\phi_x(A_B)) \geq CH^d(A_B) \,\}) \tag{4.139}$$

is sufficiently small (depending on n). We have obtained the third requirement on x_A similarly. Now, we also want C big enough (depending on n, M) so that

$$\mathrm{diam}(A)^{-m}H^m(\{\, x \in \mathrm{int}(A) \mid \mathrm{d}(x, \phi_{m+1}(E)) \leq C^{-1}\mathrm{diam}(A) \,\}) \tag{4.140}$$

is sufficiently small (depending on n). Thus, the points $x \in \mathrm{int}(A)$ that do not satisfy all our criteria will have a small H^m-measure compare to $H^m(A)$. Fix $0 < \delta \leq 1$. We recall that

$$\mathrm{int}(A) \cap \phi_{m+1}(E) \subset \bigcup_B \phi_{m+1}(B \cap E), \tag{4.141}$$

where $\bigcup_B$ is indexed by the cells $B \in K$ containing A. For such B, we can cover $E \cap B$ by at most $C\delta^{-d}$ balls of radius $\delta \mathrm{diam}(A)$. Since ϕ_{m+1} is C-Lipschitz and since, by Lemma 4.1.2, there are at most 3^n cells $B \in K$ containing A, the set $\mathrm{int}(A) \cap \phi_{m+1}(E)$ is covered by at most $C\delta^{-d}$ balls of radius $C\delta \mathrm{diam}(A)$. We deduce that

$$H^m(\{ x \in \mathrm{int}(A) \mid \mathrm{d}(x, \phi_{m+1}(E)) < C\delta \mathrm{diam}(A) \}) \\ \leq C\delta^{m-d} \mathrm{diam}(A)^m. \quad (4.142)$$

As $m > d$ and δ is arbitrary small, we can find C big enough so that

$$\mathrm{diam}(A)^{-m} H^m(\{ x \in \mathrm{int}(A) \mid \mathrm{d}(x, \phi_{m+1}(E)) \leq C^{-1} \mathrm{diam}(A) \}) \quad (4.143)$$

is sufficiently small. $\square$

Remark 4.3.1. Let us assume that there exists $s > 0$ and $M \geq 1$ such that for all $x \in E$, for all $0 < r \leq \mathrm{diam}(E)$,

$$M^{-1} r^d \leq H^d(E \cap B(x,r)) \leq M r^d, \quad (4.144)$$

then we show that for all bounded subset $S \subset E$, for all radius $0 < r \leq \mathrm{diam}(S)$, the set S can be covered by at most $(2^d M)^2 r^{-d} \mathrm{diam}(S)^d$ balls of radius r. Let us note that the upper inequality of (4.144) holds for all $r > 0$. Indeed, $E \subset B(x, \mathrm{diam}(E))$ so for $r \geq \mathrm{diam}(E)$,

$$\begin{aligned} H^d(E \cap B(x,r)) &= H^d(E \cap B(x, \mathrm{diam}(E))) && (4.145) \\ &\leq M \mathrm{diam}(E)^d && (4.146) \\ &\leq M r^d. && (4.147) \end{aligned}$$

Let S be a (nonempty) bounded subset of E and let $0 < r \leq \mathrm{diam}(S)$. Let (x_i) be a maximal sequence of points in S such that $|x_i - x_j| \geq r$. By maximality, S is covered by the balls $B(x_i, r)$. Then, we estimate the cardinal N of such family. The balls $E \cap B(x_i, \frac{1}{2}r)$ are disjoints and included in $E \cap B(x_S, 2\mathrm{diam}(S))$, where x_S is any fixed point of S. It follows that

$$\sum H^d(E \cap B(x_i, \tfrac{1}{2}r)) \leq H^d(E \cap B(x_S, 2\mathrm{diam}(S))). \quad (4.148)$$

We apply the Ahlfors regularity of E in this inequality and obtain

$$(2^d M)^{-1} N r^d \leq 2^d M \mathrm{diam}(S)^d. \quad (4.149)$$

Chapter 5

Properties of quasiminimal sets

5.1 Ahlfors regularity and rectifiability

The following proposition is a adaptation of ([DS], Proposition 4.1) or ([D6], Propositions 4.1 and 4.74) to our formalism of boundaries (Definition 4.1.9).

Proposition 5.1.1 (Ahlfors regularity). *Fix a Lipschitz subset Γ of X. Fix a triple of parameters $\mathcal{P} = (\kappa, h, s)$ assuming h small enough depending on n, Γ. Let E be a $\mathcal{P}$-quasiminimal set in X. There exists $C > 1$ (depending on n, κ, Γ) and $t > 0$ (depending on n, s, Γ) such that for all $x \in E^*$, for all $0 < r \leq r_t(x)$,*

$$C^{-1} r^d \leq H^d(E \cap B(x,r)) \leq C r^d. \tag{5.1}$$

Proof. Let K be a Whitney complex of $\mathbf{R}^n$, let $T\colon |K| \to X$ be a bijective and bilipschitz map and let L be a subcomplex of K such that $\Gamma = T(|K| \setminus U(L))$. Remember that by Remark 3.1.2, the image $T^{-1}(E)$ is quasiminimal along $T^{-1}(\Gamma)$ in $|K|$ (with respect to deformed parameters). As property (5.1) is preserved by bilipschitz maps, it suffices to prove it for $T(E)$. Thus, we can assume $T = \mathrm{id}$ without loss of generality. In this case, h and C will depend on n and κ only.

The proof consists in building Federer-Fleming projections of E in small finite n-complexes $M \preceq K$. The condition $M \preceq K$ ensures that a Federer-Fleming projection of $U(M) \cap E$ in M induces a global sliding deformation in every open ball containing $|M|$. Let us justify this claim. Let ϕ be a Federer-Fleming projection of $U(M) \cap E$ in M (as in Proposition 4.3.1). We know that ϕ is locally Lipschitz in $|M|$ but since M is finite, ϕ is Lipschitz in $|M|$. We justify that ϕ can be extended as a Lipschitz function $\phi\colon \mathbf{R}^n \to \mathbf{R}^n$ by $\phi = \mathrm{id}$ in $\mathbf{R}^n \setminus |M|$. By Lemma 4.1.2, $U(M)$ is an open set of $\mathbf{R}^n$ included in $|M|$ so $\partial |M| \subset |M| \setminus U(M)$. For $x \in \mathbf{R}^n \setminus |M|$ and $y \in |M|$, the segment $[x,y]$ meets $\partial |M|$ at a point z and then $\phi(z) = z$ because $z \in |M| \setminus U(M)$.

As a consequence, we have

$$|\phi(y) - \phi(x)| \leq |\phi(y) - \phi(z)| + |\phi(z) - \phi(x)| \tag{5.2}$$
$$\leq \|\phi\||y - z| + |x - z| \tag{5.3}$$
$$\leq (\|\phi\| + 1)|x - y|, \tag{5.4}$$

where $\|\phi\|$ is the Lipschitz constant of ϕ in $|M|$. Finally, we show that the homotopy

$$\phi_t = (1 - t)\mathrm{id} + t\phi \tag{5.5}$$

preserves Γ. This amounts to proving that for all $x \in \Gamma$, $[x, \phi(x)] \subset \Gamma$. For $x \in \Gamma$, either $x \notin U(M)$ and $\phi(x) = x$ or there exists $A \in M$ such that $x \in \mathrm{int}(A)$. We focus on the case $x \in \mathrm{int}(A)$. By the properties of Federer-Fleming projections, $\phi(x) \in A$ and by convexity of cells, $[x, \phi(x)] \subset A$. As $M \preceq K$, there exists $B \in K$ such that $\mathrm{int}(A) \subset \mathrm{int}(B)$. Since $x \in \mathrm{int}(A) \cap \Gamma$, we have $x \in \mathrm{int}(B) \setminus U(L)$ so $B \notin L$. We deduce that $B \subset X \setminus U(L)$ by the first point of Lemma 4.1.1 and the definition of a subcomplex. In summary, $A \subset B \subset \Gamma$ so $[x, \phi(x)] \subset \Gamma$.

Step 1. There exists $C \geq 1$ *(depending on* n, κ*) and* $t > 0$ *(depending on* n, s, Γ*) such that for all* $x \in E$ *and for all* $0 < r \leq r_t(x)$,

$$H^d(E \cap B(x, r)) \leq Cr^d. \tag{5.6}$$

We are going to reduce the problem. According to Definition 4.1.8, there exists $t_0 > 0$ such that for all $x \in X$, there exists an image F of E_n by a similarity such that $F \preceq K$ and

$$B(x, r_{t_0}(x)) \subset |F|. \tag{5.7}$$

In addition, we assume $t_0 \leq \frac{1}{5}$. Let $x \in E$ and let $F \preceq K$ be a n-chart similar to E_n such that $B(x, r_{t_0}(x)) \subset |F|$. To simplify the notations, we work in a new coordinate system so as to assume that $F = E_n$. For $0 < r \leq r_{t_0}(x)$, we are going to approximate $B(x, r)$ by a chart of the form

$$E_n(p, k) = p + 2^{-k} E_n, \tag{5.8}$$

where $k \in \mathbf{N}$ and $p \in \bigcup_i 2^{-i}\mathbf{Z}^n$. Let the maximal norm (in the new coordinate system) be denoted by $|\cdot|_\infty$ and the associated open balls by $U(\cdot, r)$. As $r_{t_0} \leq t_0 \leq \frac{1}{5}$, we have $r < \frac{1}{4}$ so there exists $k \in \mathbf{N}$ such that $2^{-k-3} \leq r < 2^{-k-2}$. There also exists $p \in 2^{-k-1}\mathbf{Z}^n$ such that $|x - p|_\infty \leq 2^{-k-2}$. According to the triangular inequality with respect to the norm $|\cdot|_\infty$,

$$U(x, r) \subset U(p, 2^{-k-1}) \subset \overline{U}(p, 2^{-k}) \subset U(x, 16r) \tag{5.9}$$

whence

$$B(x, r) \subset U(p, 2^{-k-1}) \subset \overline{U}(p, 2^{-k}) \subset B(x, 16\sqrt{n}r) \tag{5.10}$$

We are going to work in $E_n(p,k)$ and prove that there exists $C \geq 1$ (depending on n, κ) such that

$$H^d(E \cap U(p, 2^{-k-1})) \leq C(2^{-k})^d. \tag{5.11}$$

The proof will require that $E_n(p,k) \preceq K$. This point holds as soon as $16\sqrt{n}r \leq r_{t_0}(x)$ because $B(x, r_{t_0}(x)) \subset |E_n|$ and $E_n \preceq K$. Moreover, we will need that $16\sqrt{n}r \leq r_s$ to be able to apply the quasiminimality in the ball $U = B(x, 16\sqrt{n}r)$. Thus, we restrict the previous construction to the radii $r \leq r_t(x)$ where $0 < t \leq t_0$ is chosen such that

$$16\sqrt{n}r_t(x) \leq \min\{\, r_{t_0}(x), r_s(x)\,\}. \tag{5.12}$$

This choice depends only on n, s and t_0. For example, we have

$$r_t(x) = \min\{\, \tfrac{t}{1+t}\mathrm{d}(x, X^c), t\,\} \tag{5.13}$$

$$\leq t \min\{\, \mathrm{d}(x, X^c), 1\,\} \tag{5.14}$$

and

$$r_s(x) = \min\{\, \tfrac{s}{1+s}\mathrm{d}(x, X^c), s\,\} \tag{5.15}$$

$$\geq \tfrac{s}{1+s} \min\{\, \mathrm{d}(x, X^c), 1\,\}. \tag{5.16}$$

so we can take $t \leq \frac{1}{16\sqrt{n}}\frac{s}{1+s}$, etc. Finally, we change the coordinate system once again to assume $E_n(p,k) = E_n$.

The letter C plays the role of a constant ≥ 1 that depends on n, κ. Its value can increase from one line to another (but a finite number of times). We are going to prove that

$$H^d(E \cap \tfrac{1}{2}]{-1}, 1[) \leq C. \tag{5.17}$$

We fix a parameter $0 < \mu < 1$ which is close enough to 1 (this will be precised later). We aim to apply the Federer-Fleming projection in a sequence of complexes $(M_k)_{k \in \mathbf{N}}$ whose supports is of the form

$$|M_k| = (1 - \mu^k)[-1, 1]^n. \tag{5.18}$$

We also want the complexes M_k to be composed of dyadic cells so that $M_k \preceq E_n$. Thus, we define a sequence $(q(k))_{k \in \mathbf{N}}$ of nonnegative integers such that $(1 - \sum_{i \geq k} 2^{-q(i)})[-1, 1]^n$ will be the support of M_k and, in some sense,

$$\sum_{i \geq k} 2^{-q(i)} = \frac{1}{2} - \sum_{i < k} 2^{-q(i)} \sim \mu^k. \tag{5.19}$$

We define $(q(k))_k$ by induction. Assuming that $q(0), \cdots, q(k-1)$ have been built and $1 - \sum_{i<k} 2^{-q(i)} > 0$, we define $q(k)$ as the smallest nonnegative integer such that

$$\frac{\frac{1}{2} - \sum_{i \leq k} 2^{-q(i)}}{\frac{1}{2} - \sum_{i < k} 2^{-q(i)}} \geq \mu. \tag{5.20}$$

We present the main properties of this sequence, some of which will only be useful in step 2. It is straightforward that $q(k) \geq 2$ otherwise the numerator of (5.20) is nonpositive. We rewrite (5.20) as $q(k)$ being the smallest nonnegative integer such that

$$\frac{2^{-q(k)}}{\frac{1}{2} - \sum_{i<k} 2^{-q(i)}} \leq 1 - \mu. \tag{5.21}$$

We can deduce that $(q(k))$ is nondecreasing. Indeed, by definition of $q(k+1)$ we have,

$$\frac{2^{-q(k+1)}}{\frac{1}{2} - \sum_{i<k} 2^{-q(i)}} \leq \frac{2^{-q(k+1)}}{\frac{1}{2} - \sum_{i<k+1} 2^{-q(i)}} \leq 1 - \mu. \tag{5.22}$$

so $q(k+1) \geq q(k)$. More generally, the minimality of $q(k)$ with respect to (5.21) is equivalent to

$$\frac{1}{2}(1-\mu) < \frac{2^{-q(k)}}{\frac{1}{2} - \sum_{i<k} 2^{-q(i)}} \leq (1 - \mu). \tag{5.23}$$

An induction on (5.20) shows that for all $k \geq 0$,

$$\frac{1}{2} - \sum_{i<k} 2^{-q(i)} \geq \frac{1}{2}\mu^k. \tag{5.24}$$

We combine (5.23) and (5.24) to obtain

$$2^{-q(k)} \geq \frac{1}{4}(1-\mu)\mu^k, \tag{5.25}$$

Finally, we justify that $\sum_i 2^{-q(i)} = \frac{1}{2}$. It is clear that $\sum_i 2^{-q(i)} \leq \frac{1}{2}$ so $\lim\limits_{i\to\infty} 2^{-q(i)} = 0$ and then (5.23) implies that $\frac{1}{2} - \sum_i 2^{-q(i)} = 0$.

Now, we are ready to build the complexes $(M_k)_k$. For $k \geq 0$, let S_k be the system of dyadic charts of sidelength $2^{-q(k)}$ in the cube $(\frac{1}{2} + \sum_{i<k} 2^{-q(i)})[-1,1]^n$ (see Definition 4.1.6). We define M_k as the limit of $\bigcup_{i=0}^{k} S_i$. The set M_k is a finite n-complex subordinated to E_n and

$$|M_k| = (\tfrac{1}{2} + \sum_{i<k} 2^{-q(i)})[-1,1]^n, \tag{5.26a}$$

$$U(M_k) = (\tfrac{1}{2} + \sum_{i<k} 2^{-q(i)})]-1,1[^n. \tag{5.26b}$$

Moreover, we have $M_k \subset M_{k+1}$. Indeed, for $A \in M_k$ and for $B \in \bigcup S_{k+1}$,

$$\operatorname{int}(A) \cap \operatorname{int}(B) \neq \emptyset \implies \operatorname{int}(B) \subset \operatorname{int}(A) \tag{5.27}$$

because A and B are dyadic cells and the sidelength of B is less than or equal to the sidelength of A.

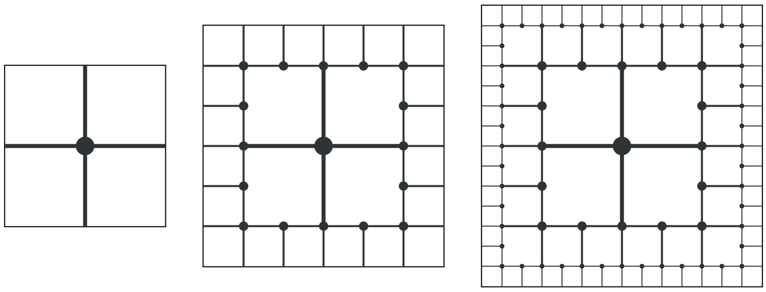

Figure 5.1: From left to right, examples of M_0, M_1, M_2.

We define $U_k = U(M_k)$ and $U_\infty = \bigcup_k U_k =]-1,1[^n$. Let ϕ be a Federer-Fleming projection of $E \cap U_{k+1}$ in M_{k+1}. We apply the quasiminimality of E with respect to ϕ in $U = B(x, 16\sqrt{n}r)$. We assume h small enough (depending on n) such that $hH^d(E \cap hU) \leq \frac{1}{2} H^d(E \cap B(x,r))$. We have then

$$H^d(E \cap U_{k+1})) \leq CH^d(\phi(E \cap U_{k+1})) + \tfrac{1}{2} H^d(E \cap U_{k+1}) \tag{5.28}$$

so

$$H^d(E \cap U_{k+1})) \leq CH^d(\phi(E \cap U_{k+1})). \tag{5.29}$$

We decompose $E \cap U_{k+1}$ in two parts: $E \cap U_k$ and $E \cap U_{k+1} \setminus U_k$. First, we claim $H^d(\phi(E \cap U_k)) \leq H^d(M_{k+1})^d$. Indeed, by the properties of the Federer-Fleming projections,

$$\phi(E \cap U_k) \subset |M_{k+1}| \setminus \bigcup \{\, \mathrm{int}(A) \mid A \in M_{k+1},\ \dim(A) > d \,\}. \tag{5.30}$$

As $M_k \subset M_{k+1}$, ϕ preserves the cells of M_k so $\phi(U_k) \subset |M_k|$. We conclude that

$$\begin{aligned}
\phi(E \cap U_k) &\subset |M_k| \setminus \bigcup \{\, \mathrm{int}(A) \mid A \in M_{k+1},\ \dim(A) > d \,\} && (5.31)\\
&\subset U_{k+1} \setminus \bigcup \{\, \mathrm{int}(A) \mid A \in M_{k+1},\ \dim(A) > d \,\} && (5.32)\\
&\subset \bigcup \{\, \mathrm{int}(A) \mid A \in M_{k+1},\ \dim(A) \leq d \,\}. && (5.33)
\end{aligned}$$

Next, we claim that $H^d(\phi(E \cap U_{k+1} \setminus U_k)) \leq CH^d(E \cap U_{k+1} \setminus U_k)$. This is deduced from the observation that $U_{k+1} \setminus U_k = \bigcup \{\, \mathrm{int}(A) \mid A \in M_{k+1} \setminus M_k \,\}$ and the fact that for all $A \in M_{k+1}$,

$$H^d(\phi(E \cap \mathrm{int}(A))) \leq CH^d(E \cap \mathrm{int}(A)). \tag{5.34}$$

In conclusion,

$$H^d(E \cap U_{k+1}) \leq CH^d(M_{k+1}^d) + CH^d(E \cap U_{k+1} \setminus U_k). \tag{5.35}$$

We rewrite this inequality as

$$H^d(E \cap U_k) - \lambda H^d(E \cap U_{k+1}) \leq H^d(M^d_{k+1}), \tag{5.36}$$

where $\lambda = C^{-1}(C-1)$. We multiply both sides of the inequation by λ^k:

$$\lambda^k H^d(E \cap U_k) - \lambda^{k+1} H^d(E \cap U_{k+1}) \leq \lambda^k H^d(M^d_{k+1}). \tag{5.37}$$

We sum this inequality over $k \geq 0$ and since $0 < \lambda < 1$ and $H^d(E \cap U_\infty) < \infty$, the telescopic side simplifies to

$$H^d(E \cap U_0) \leq \sum_k \lambda^k H^d(M^d_{k+1}). \tag{5.38}$$

It is left to bound $\sum_k \lambda^k H^d(M^d_{k+1}) \leq C$. We choose μ so that $\mu^{n-d} = \lambda$. As the the cells of M_{k+1} have a sidelength $\sim \mu^k$, we will see that this choice implies

$$\lambda^k (H^d(M^d_{k+1}) - H^d(M^d_k)) \leq C|U_{k+1} \setminus U_k| \tag{5.39}$$

where $|\cdot|$ denotes the Lebesgue measure. Summing (5.39) over $k \geq 0$ gives then

$$\begin{aligned}(1-\lambda)\sum_k \lambda^k H^d(M^d_{k+1}) &\leq C \sum_k |U_{k+1} \setminus U_k| &(5.40)\\ &\leq C. &(5.41)\end{aligned}$$

Now, we justify (5.39). By the choice $\mu^{n-d} = \lambda$, the fact that $\mu^k \leq C 2^{-q(k+1)}$ (see (5.25)) and the fact that the cells of M_{k+1} are of sidelength $2^{-q(k+1)}$, we have

$$\begin{aligned}\lambda^k (H^d(M^d_{k+1}) - H^d(M^d_k)) &\leq \lambda^k \sum_{A \in M^d_{k+1} \setminus M^d_k} \operatorname{diam}(A)^d &(5.42)\\ &\leq C \sum_{A \in M^d_{k+1} \setminus M^d_k} \operatorname{diam}(A)^n &(5.43)\\ &\leq C|U_{k+1} \setminus U_k|. &(5.44)\end{aligned}$$

Step 2. There exists $C \geq 1$ (depending on n, κ) and $t > 0$ (depending on n, s, Γ) such that for all $x \in E^*$ and for all $0 < r \leq r_t(x)$,

$$H^d(E \cap B(x,r)) \geq C^{-1} r^d. \tag{5.45}$$

We reduce the problem just as in step 1. The letter C plays the role of a constant ≥ 1 that depends on n, κ. Its value can increase from one line to another (but a finite number of times). We are going to prove by contradiction that

$$H^d(E \cap \,]-1,1[^n) \geq C^{-1}. \tag{5.46}$$

We fix $0 < \mu < 1$ close enough to 1 (this will be precised later). We aim to apply the Federer-Fleming projection in a sequence of complexes $(M_k)_{k \in \mathbf{N}}$ whose supports is of the form

$$|M_k| = (\tfrac{1}{2} + \mu^k)[-1,1]^n. \tag{5.47}$$

We also want the complexes M_k to be composed of dyadic cells so that $M_k \preceq E_n$. We define the same sequence $(q(k))_{k \in \mathbf{N}}$ as in step 1 such that

$$\sum_{i \geq k} 2^{-q(i)} = \frac{1}{2} - \sum_{i<k} 2^{-q(i)} \sim \mu^k. \tag{5.48}$$

For each k, let M_k be the set of dyadic cells of sidelength $2^{-q(k)}$ subdivising the cube $(1 - \sum_{i<k} 2^{-q(i)})[-1,1]^n$. The set M_k is a finite n-complex subordinated to E_n and

$$|M_k| = (1 - \sum_{i<k} 2^{-q(i)})[-1,1]^n, \tag{5.49a}$$

$$U(M_k) = (1 - \sum_{i<k} 2^{-q(i)})]-1,1[^n. \tag{5.49b}$$

Moreover, we have $M_{k+1} \preceq M_k$. Indeed, $U(M_{k+1}) \subset U(M_k)$ and the dyadic cells composing M_{k+1} have a sidelength which is less than or equal to those of M_k.

We define $U_k = U(M_k)$ and $U_\infty = \bigcap_k U_k = \frac{1}{2}[-1,1]^n$. Let ϕ be a Federer-Fleming projection of $E \cap U_k$ in M_k. By the properties of the Federer-Fleming projection, there exists $C_0 \geq 1$ (depending on n) such that for all $A \in M_k^d$,

$$H^d(\phi(E \cap U_k) \cap A) \leq C_0 H^d(E \cap U_k). \tag{5.50}$$

As the cells of M_k have sidelength $2^{-q(k)}$, the area $H^d(\frac{1}{2}A)$ is of the form $C2^{-dq(k)}$. We deduce that there exists $C_1 > 0$ (depending only on n) such that

$$H^d(E \cap U_k) \leq C_1^{-1} 2^{-dq(k)} \implies H^d(\phi(E \cap U_k) \cap \tfrac{1}{2}A) < H^d(\tfrac{1}{2}A). \tag{5.51}$$

In this case, the set $\phi(E \cap U_k) \cap A$ does not include $\frac{1}{2}A$. It can be postcomposed with a radial projection centered in $\frac{1}{2}A$ and sent to ∂A. If $H^d(E \cap U_k) \leq C_1^{-1} 2^{-dq(k)}$, we can thus assume that for all $A \in M_k^d$, $H^d(\phi(E \cap U_k) \cap A) = 0$. That being done, we apply the quasiminimality of E with respect to ϕ in $U = B(x, 16\sqrt{n}r)$. We assume h small enough (depending on n) such that $hH^d(E \cap hU) \leq \frac{1}{2} H^d(E \cap B(x,r)$. We have then

$$H^d(E \cap U_k)) \leq CH^d(\phi(E \cap U_k)) + \tfrac{1}{2} H^d(E \cap U_k) \tag{5.52}$$

so

$$H^d(E \cap U_k)) \leq CH^d(\phi(E \cap U_k)). \tag{5.53}$$

We decompose $E \cap U_k$ in two parts. The points of $E \cap U_{k+1}$ are sent into the d-dimensional skeleton of M_k (as in step 1) and then radially projected in the $(d-1)$-skeleton so their image is H^d-negligible. On the other hand, $U_k \setminus U_{k+1} = \bigcup \{ \operatorname{int}(A) \mid A \in M_k \setminus M_{k+1} \}$ so $H^d(\phi(E \cap U_k \setminus U_{k+1})) \leq CH^d(E \cap U_{k+1} \setminus U_k)$. In sum,

$$H^d(E \cap U_k) \leq CH^d(E \cap U_k \setminus U_{k+1}). \tag{5.54}$$

We rewrite this inequality as

$$H^d(E \cap U_{k+1}) - \lambda H^d(E \cap U_k) \leq 0. \tag{5.55}$$

where $\lambda = C^{-1}(C-1)$. Multiplying this inequality by λ^{-k}, we obtain a telescopic term:

$$\lambda^{-(k+1)} H^d(E \cap U_{k+1}) - \lambda^k H^d(E \cap U_k) \leq 0. \tag{5.56}$$

We choose μ such that $\mu^d = \lambda$. Next, we show that there exists a constant $C_3 \geq 1$ (depending on n, κ) such that if $H^d(E \cap U_0) \leq C_3^{-1}$, then (5.56) holds for all $k \in \mathbf{N}$. The idea is to observe that if (5.56) holds for $i = 0, \ldots, k-1$, we can sum this telescopic inequality and obtain

$$\lambda^{-k} H^d(E \cap U_k) - H^d(E \cap U_0) \leq 0. \tag{5.57}$$

so

$$H^d(E \cap U_k) \leq \lambda^k H^d(E \cap U_0). \tag{5.58}$$

According to the choice $\mu^d = \lambda$ and (5.23), there exists $C_2 \geq 1$ (depending on n, κ) such that for all $k \in \mathbf{N}$, $2^{-dq(k)} \geq C_2^{-1} \lambda^k$. We conclude that if $H^d(E \cap U_0) \leq (C_1 C_2)^{-1}$, then

$$H^d(E \cap U_k) \leq C_1^{-1} 2^{-dq(k)} \tag{5.59}$$

and the process can be iterated. However, taking the limit $k \to \infty$ in (5.57) yields a contradiction because $0 < \lambda < 1$, $H^d(E \cap U_\infty) > 0$ (remember that $x \in E^*$) and $H^d(E \cap U_0) < \infty$. $\square$

We use our new estimate on the Federer-Fleming projection to improve the Ahlfors regularity. As a consequence, we obtain the rectifiability of quasiminimal sets thanks to the Besicovitch-Federer Theorem ([Mat], Theorem 18.1).

Corollary 5.1.1. *Fix a Lipschitz subset Γ of X. Fix a triple of parameters $\mathcal{P} = (\kappa, h, s)$ assuming h small enough (depending on n, Γ). Let E be a $\mathcal{P}$-quasiminimal set in X, then E is H^d rectifiable.*

Proof. Let K be a Whitney complex of $\mathbf{R}^n$, let $T\colon |K| \to X$ be a bijective and bilipschitz map and let L be a subcomplex of K such that $\Gamma = T(|K| \setminus U(L))$. Remember that by Remark 3.1.2, the image $T^{-1}(E)$ is quasiminimal along $T^{-1}(\Gamma)$ in $|K|$ (with respect to deformed parameters). As the H^d rectifiability is preserved by bilipschitz maps, it suffices to prove the property for $T(E)$. Thus, we can assume that $T = \mathrm{id}$ without loss of generality. In this case, we prove a stronger property than the H^d rectifiability. *There exists $C \geq 1$ (depending on n, κ) and $t > 0$ (depending on n, s, Γ) such that for all $x \in E^*$, for all $0 < r \leq r_t(x)$,*

$$H^d(E \cap B(x,r)) \leq C \int_{G(d,n)} H^d(p_V(E \cap B(x, 16\sqrt{n}r)))\,\mathrm{d}V. \tag{5.60}$$

This implies the H^d rectifiability of E because the right hand side integral cancels the purely nonrectifiable part.

We reduce the problem as in Proposition 5.1.1. The letter C plays the role of a constant ≥ 1 that depends on n, κ. Its value can increase from one line to another (but a finite number of times). We are going to prove that

$$H^d(E \cap \tfrac{1}{2}]{-1}, 1[^n) \leq C \int_{G(d,n)} H^d(p_V(E \cap]{-1}, 1[^n))\,\mathrm{d}V. \tag{5.61}$$

Since E is Alfhors-regular (Proposition 5.1.1), we can assume that

$$H^d(E \cap]{-1}, 1[^n) \leq C H^d(E \cap \tfrac{1}{2}]{-1}, 1[^n) \tag{5.62}$$

We fix $q \in \mathbf{N}^*$. For $0 \leq k \leq 2^q$, let M_k be the set of dyadic cells of sidelength 2^{-q} subdivising the cube $(1 - k2^{-q})[-1,1]^n$ (except the boundary). The set M_k is a finite n-complex subordinated to E_n and

$$|M_k| = (1 - k2^{-q})[-1,1]^n, \tag{5.63a}$$

$$U(M_k) = (1 - k2^{-q})]{-1}, 1[^n. \tag{5.63b}$$

Moreover, it is clear that $M_{k+1} \subset M_k$. We define $U_k = U(M_k)$. Let ϕ be a Federer-Fleming projection of $E \cap U_k$ in M_k. We apply the quasiminimality of E with respect to ϕ in $U = B(x, 16\sqrt{n}r)$. We assume h small enough (depending on n) such that $hH^d(E \cap hU) \leq \frac{1}{2} H^d(E \cap B(x,r))$. We have then

$$H^d(E \cap U_k) \leq CH^d(\phi(E \cap U_k)) + \tfrac{1}{2} H^d(E \cap U_k) \tag{5.64}$$

so

$$H^d(E \cap U_k) \leq CH^d(\phi(E \cap U_k)). \tag{5.65}$$

We decompose $E \cap U_k$ in two parts: $E \cap U_{k+1}$ and $E \cap U_k \setminus U_{k+1}$. First, we have (as in step 1 of Proposition 5.1.1) ,

$$\phi(E \cap U_{k+1}) \subset \bigcup \{\, A \in M_k \mid \dim(A) \leq d \,\}. \tag{5.66}$$

We recall that for $A \in M_k^d$,

$$H^d(\phi(E) \cap A) \leq C \int_{G(d,n)} H^d(p_V(E \cap U_0))\, \mathrm{d}V \tag{5.67}$$

and since M_k^d contains at most $C2^q$ cells,

$$H^d(\phi(E \cap U_{k+1})) \leq C2^{qd} \int_{G(d,n)} H^d(p_V(E \cap U_0))\, \mathrm{d}V. \tag{5.68}$$

Next, we have $H^d(\phi(E \cap U_k \setminus U_{k+1})) \leq CH^d(E \cap U_{k+1} \setminus U_k)$ because $U_k \setminus U_{k+1} = \bigcup \{ \operatorname{int}(A) \mid A \in M_k \setminus M_{k+1} \}$. In sum,

$$\begin{aligned} H^d(E \cap U_k) \leq C2^q \int_{G(d,n)} & H^d(p_V(E \cap U_0))\, \mathrm{d}V \\ & + CH^d(E \cap U_k \setminus U_{k+1}). \end{aligned} \tag{5.69}$$

We use a Chebychev argument to find an index $0 \leq k \leq 2^q$ (depending on n, κ) such that $\frac{1}{2}]-1,1[^n \subset U_k$ and $H^d(E \cap U_{k+1} \setminus U_k) \leq \frac{1}{2} H^d(E \cap \frac{1}{2}]-1,1[^n)$. We start by observing that the sets $E \cap U_{k+1} \setminus U_k$ are disjoint so

$$\sum_k H^d(E \cap U_{k+1} \setminus U_k) \leq H^d(E \cap U_0). \tag{5.70}$$

By (5.63), there are at least 2^{q-1} index k such that $\frac{1}{2}]-1,1[^n \subset U_k$ so there exists an index k such that $\frac{1}{2}]-1,1[^n \subset U_k$ and

$$2^{q-1} H^d(E \cap U_{k+1} \setminus U_k) \leq H^d(E \cap U_0). \tag{5.71}$$

By Ahlfors regularity (5.62), we deduce that

$$H^d(E \cap U_{k+1} \setminus U_k) \leq C2^{-q} H^d(E \cap \tfrac{1}{2}]-1,1[^n). \tag{5.72}$$

so we can choose q big enough (depending on n, κ) such that

$$H^d(E \cap U_{k+1} \setminus U_k) \leq \tfrac{1}{2} H^d(E \cap \tfrac{1}{2}]-1,1[^n), \tag{5.73}$$

Now, (5.69) becomes

$$H^d(E \cap \tfrac{1}{2}]-1,1[^n) \leq C \int_{G(d,n)} H^d(p_V(E \cap U_0))\, \mathrm{d}V. \tag{5.74}$$

The constant 2^q has been absorbed in C because q depends now on n, κ. $\square$

5.2 Weak limits of quasiminimal sets

In this section, we prove that a weak limit of quasiminimal sets (of uniform parameters) is also a quasiminimal set (of the same parameters). Our working space is an open set X of $\mathbf{R}^n$.

Theorem 5.2.1 (Limiting Theorem)**.** *Fix a Lipschitz subset Γ of X which is H^d regular. Fix a triple of parameters (κ, h, s) and an additionnal parameter $\kappa_0 \geq 1$. Assume that h is small enough (depending on n and Γ). Let (E_i) be a sequence of closed, H^d locally finite subsets of X satisfying the following conditions:*

1. *the sequence of Radon measures $(H^d \llcorner E_i)$ has a weak limit μ in X;*
2. *for all $x \in \operatorname{spt}(\mu)$, for all $0 < r \leq r_s(x)$, there exists a sequence $(\varepsilon_i) \to 0$ such that for all global sliding deformation f in $U = B(x, r)$,*
$$H^d(E_i \cap W_f) \leq \kappa H^d(f(E_i \cap W_f)) + h H^d(E_i \cap hU) + \varepsilon_i \tag{5.75}$$
and
$$H^d(E_i \cap U) \leq \kappa_0 H^d(f(E_i \cap U)) + \varepsilon_i. \tag{5.76}$$

Then, $E = \operatorname{spt}(\mu)$ is $(\kappa, \kappa_0 h, s)$-quasiminimal in X and we have
$$H^d \llcorner E \leq \mu \leq \kappa_0 H^d \llcorner E. \tag{5.77}$$

Theorem 5.2.1 is proved by constructing relevant sliding deformations. We distinguish three intermediate result:

1. *The limit measure μ is d-dimensional Ahlfors regular and H^d rectifiable.* We adapt the techniques used to prove that quasiminimal sets are Ahlfors regular and rectifiable. The rectifiability plays an essential role in the two next steps. Here, an error term of the form $h\operatorname{diam}(U)^d$ would pose a problem to the lower Ahlfors regularity and the rectifiability.

2. *For all sliding deformation f of $E = \operatorname{spt}(\mu)$ in local balls U, we have*
$$\mu(W_f) \leq \kappa H^d(f(W_f)) + h\mu(hU). \tag{5.78}$$
This part follows from a technical lemma which is inspired by the techniques of David in [D6]. The lemma was not conceptualised in [D6] and allows significant simplifications. A similar argument shows that $\mu \leq \kappa_0 H^d \llcorner E$.

3. *We have $\mu \geq H^d \llcorner E$.* We make use of an argument introduced by Fang in [Fn]. It bypasses the concentration Lemma of Dal Maso, Morel and Solimini when the limit is already known to be rectifiable.

Although we minimize the Hausdorff measure, our techniques could be adapted to elliptic integrands.

5.2.1 Technical lemmas

Given a map $f\colon \mathbf{R}^n \to \mathbf{R}^n$, the symbol $\|f\|_L$ means the Lipschitz constant of f.

Lemma 5.2.1. *Let W be an open set of $\mathbf{R}^n$ and let $E \subset W$ be a H^d measurable, H^d finite and H^d rectifiable set. Let $f\colon \mathbf{R}^n \to \mathbf{R}^n$ be a Lipschitz map. For all $\varepsilon > 0$, there exists a Lipschitz map $g\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $g - f$ has a compact support included in W, $|g - f| \leq \varepsilon$, $\|g - f\|_L \leq \|f\|_L$ and such that there is an open set $V \subset W$ satisfying*

$$H^d(E \setminus V) \leq \varepsilon \tag{5.79a}$$

$$H^d(g(V)) \leq H^d(f(E)) + \varepsilon. \tag{5.79b}$$

Roughly speaking, g smashes an almost neighborhood V of E onto $f(E)$.

Proof. Let us get rid of the case $\|f\|_L = 0$, that is f constant. In this case, we take $g = f$. If $E \neq \emptyset$, then $f(W) = f(E)$ so we can choose $V = W$. If $E = \emptyset$, we choose $V = \emptyset$. From now on, we assume $\|f\|_L > 0$. To start with, we justify that the (outer) measures

$$\mu\colon A \to H^d(E \cap f^{-1}(A)) \tag{5.80}$$

and

$$\lambda\colon A \to H^d(f(E) \cap A) \tag{5.81}$$

are Radon measures in $\mathbf{R}^n$. We adopt the formalism of Mattila ([Mat], Definition 1.15). In particular, we recall that an (outer) measure in $\mathbf{R}^n$ is a Radon measure if and only if it is Borel regular and H^d locally finite ([Mat], Corollary 1.11). We recall that the measure H^d is Borel regular ([Mat], Corollary 4.5). We also recall that the image of an H^d measurable set by a Lipschitz map is H^d measurable; this follows from the approximation theorem ([Mat], Theorem 1.10) applied to H^d. As H^d is Borel regular and $f(E)$ is H^d measurable and $H^d(f(E)) \leq L^d H^d(E) < \infty$, we conclude that λ is a Radon measure ([Mat], Theorem 1.9). Let us pass to μ. The measure μ is finite because $H^d(E) < \infty$. As of the Borel regularity, it suffices to show that for all $A \subset \mathbf{R}^n$,

$$\mu(A) = \inf\left\{\, H^d(C) \mid C \text{is a Borel set containing } A \,\right\}. \tag{5.82}$$

Let U be an open set of $\mathbf{R}^n$ containing $f^{-1}(A)$. Then the set

$$B = \{\, y \in \mathbf{R}^n \mid f^{-1}(y) \subset U \,\} \tag{5.83}$$

$$= \mathbf{R}^n \setminus f(\mathbf{R}^n \setminus U) \tag{5.84}$$

is a Borel set (because $\mathbf{R}^n \setminus U$ is σ-compact for example), it contains A and

$$\mu(B) = H^d(E \cap f^{-1}(B)) \leq H^d(E \cap U). \tag{5.85}$$

To conclude, we observe that by regularity of the Radon measure $H^d \llcorner E$,

$$\mu(A) = \inf \{ H^d(E \cap U) \mid f^{-1}(A) \subset U \text{ is open} \}. \tag{5.86}$$

We are going to differentiate μ with respect to λ. For $t > 0$, we introduce Y_t: the set of points $y \in f(E)$ such that

$$\liminf_{r \to 0} \frac{H^d(E \cap f^{-1}(B(y,r)))}{H^d(f(E) \cap B(y,r))} \leq t. \tag{5.87}$$

Using [Mat] (Remark 2.10 and Lemma 2.13), we draw two properties of Y_t. First, Y_t is H^d measurable. Secondly, if we define $E_1 = E \cap f^{-1}(Y_t)$ and $E_2 = E \setminus f^{-1}(Y_t)$, then for all $A \subset \mathbf{R}^n$,

$$H^d(E_1 \cap f^{-1}(A)) \leq t H^d(A \cap Y_t) \tag{5.88}$$

and

$$H^d(E_2 \cap f^{-1}(A)) \geq t H^d(A \cap Y_t). \tag{5.89}$$

Finally, we fix a constant $\varepsilon_0 > 0$ for all the proof. We will deal independently with E_1 and E_2 in step 1 and step 2 respectively. However, the parameter t is arbitrary in step 1 whereas it is chosen in step 2.

Step 0. We build two open sets $W_1, W_2 \subset\subset W$ such that $\overline{W_1} \cap \overline{W_2} = \emptyset$ and

$$H^d(E_i \setminus W_i) \leq \tfrac{1}{3}\varepsilon_0. \tag{5.90}$$

According to the approximation theorem ([Mat], Theorem 1.10), there exists two compact sets $K_1, K_2 \subset W$ and two open sets $O_1, O_2 \subset W$ such that $K_i \subset E_i \subset O_i$ and

$$H^d \llcorner E(O_i \setminus K_i) \leq \tfrac{1}{3}\varepsilon_0. \tag{5.91}$$

As K_1 and K_2 are disjoint compact sets such that $K_i \subset O_i$, there exists two open sets $W_i \subset\subset O_i$ ($i = 1, 2$) such that $K_i \subset W_i$ and $\overline{W_1} \cap \overline{W_2} = \emptyset$. As $E_i \subset O_i$,

$$H^d(E_i \setminus W_i) \leq H^d(E \cap O_i \setminus K_i) \leq \tfrac{1}{3}\varepsilon_0. \tag{5.92}$$

Step 1. We build a Lipschitz map $g_1 \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $g_1 - f$ has a compact support included in W_1, $|g_1 - f| \leq \varepsilon_0$, $\|g_1 - f\|_L \leq \|f\|_L$ and such that there is an open set $V_1 \subset W_1$ satisfying

$$H^d(E_1 \setminus V_1) \leq \varepsilon_0, \tag{5.93a}$$

$$H^d(g_1(V_1)) \leq H^d(f(E)) + \varepsilon_0. \tag{5.93b}$$

We are going to use classical properties of rectifiable sets (see [Mat], 15.17, 15.19 and 16.2). *From now on, the letter C plays the role of a constant ≥ 1 that depends on n. Its value can increase from one line to another (but a* *finite number of times). For $x \in \mathbf{R}^n$, for V a plane passing through x and for $0 < \varepsilon \leq 1$, we introduce the open cone*

$$C(x, V, \varepsilon) := \{ \, z \in \mathbf{R}^n \mid \mathrm{d}(z, V) < \varepsilon |z - x| \, \} \, . \tag{5.94}$$

We take the convention that the H^d measure of a d-dimensional disk of radius $r \geq 0$ equals $(2r)^d$.

The set Y_t is H^d measurable, H^d finite and H^d rectifiable so for H^d-ae. $y \in Y_t$,

$$\lim_{r \to 0} (2r)^{-d} H^d(Y_t \cap B(y, r)) = 1. \tag{5.95}$$

Let us fix $0 < a < 1$ (to be specified later). The previous property implies that for H^d-ae. $y \in Y_t$,

$$\lim_{r \to 0} (2r)^{-d} H^d(Y_t \cap B(y, r) \setminus aB(y, r)) = 1 - a^d \tag{5.96}$$

whence

$$\limsup_{r \to 0} r^{-d} H^d(Y_t \cap B(y, r) \setminus aB(y, r)) \leq C(1 - a). \tag{5.97}$$

Moreoever, for H^d-ae. $y \in Y_t$, there exists a unique d-plane V_y such that for all $0 < \varepsilon \leq 1$,

$$\lim_{r \to 0} r^{-d} H^d(Y_t \cap B(y, r) \setminus C(x, V, \varepsilon)) = 0. \tag{5.98}$$

Let $0 < \varepsilon \leq 1$. We apply the Vitali covering theorem ([Mat], Theorem 2.8) see to obtain a *finite* sequence of open balls $(B_j)_j$ (of center $y_j \in Y_t$ and radius $0 < r_j \leq 1$) such that the closure $(\overline{B_j})$ are disjoint, $H^d(Y_t \setminus \bigcup_j \overline{B_j}) \leq \varepsilon$ and

$$H^d(Y_t \cap B_j) \geq (1 + \varepsilon)^{-1} (2r_j)^d \tag{5.99a}$$

$$H^d(Y_t \cap B_j \setminus aB_j) \leq C(1 - a) r_j^d \tag{5.99b}$$

$$H^d(Y_t \cap B_j \setminus C_j) \leq \varepsilon r_j^d, \tag{5.99c}$$

where $C_j = C(y_j, V_{y_j}, \varepsilon)$. In addition, we require $H^d(Y_t \cap \partial B_j) = 0$ and we obtain the "almost-coverering" condition on the open balls,

$$H^d(Y_t \setminus \bigcup_j B_j) \leq \varepsilon. \tag{5.99d}$$

In each ball B_j, we define a projection onto V_{y_j}, the tangent plane to Y_t at y_j. Let π_j be partially defined by

$$\pi_j(x) = \begin{cases} y' & \text{in } aB_j \cap C_j \\ y & \text{in } \mathbf{R}^n \setminus B_j, \end{cases} \tag{5.100}$$

where y' is the orthogonal projection of y onto V_{y_j}. By definition of C_j, we have $|\pi_j - \mathrm{id}| \leq \varepsilon r_j$ in $aB_j \cap C_j$. Next, we estimate the Lipschitz constant of $\pi_j - \mathrm{id}$. The map $\pi_j - \mathrm{id}$ is 1-Lipschitz in $aB_j \cap C_j$ by properties of orthogonal projection. As

$$\mathrm{d}(aB_j, \mathbf{R}^n \setminus B_j) \geq (1-a)r_j, \tag{5.101}$$

we have for $x \in aB_j \cap C_j$ and for $y \in \mathbf{R}^n \setminus B_j$,

$$\begin{aligned} |(\pi_j - \mathrm{id})(x) - (\pi_j - \mathrm{id})(y)| &\leq |(\pi_j - \mathrm{id})(x)| && (5.102)\\ &\leq \varepsilon r_j && (5.103)\\ &\leq \frac{\varepsilon}{1-a}|x-y|. && (5.104) \end{aligned}$$

We choose $a = 1 - \sqrt{\varepsilon}$ so that

$$|(\pi_j - \mathrm{id})(x) - (\pi_j - \mathrm{id})(y)| \leq C\sqrt{\varepsilon} \tag{5.105}$$

If ε is small enough, then $\pi_j - \mathrm{id}$ is this 1-Lipschitz on its domain. We apply the Kirzbraun theorem to $\pi_j - \mathrm{id}$ and extend π_j as a Lipschitz map $\pi_j \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $|\pi_j - \mathrm{id}| \leq \varepsilon r_j$ and $\pi_j - \mathrm{id}$ is 1-Lipschitz. It is left to paste together the functions π_j into a function π:

$$\pi = \begin{cases} \pi_j & \text{in } B_j \\ \mathrm{id} & \text{in } \mathbf{R}^n \setminus \bigcup_j B_j. \end{cases} \tag{5.106}$$

It is clear that $|\pi - \mathrm{id}| \leq \varepsilon$. The map $\pi - \mathrm{id}$ is also locally 1-Lipschitz. Indeed, $\pi = \mathrm{id}$ in the open set $\mathbf{R}^n \setminus \bigcup_j \overline{B_j}$ and for each index j, $\pi = \pi_j$ in the open set $\mathbf{R}^n \setminus \bigcup_{i \neq j} \overline{B_i}$. These these open sets cover $\mathbf{R}^n$ because the closed balls $(\overline{B_i})_i$ are disjoints. We deduce that π is globally 1-Lipschitz by convexity of $\mathbf{R}^n$. Eventually, we define

$$V_1' = \bigcup_j aB_j \cap C_j \tag{5.107}$$

and we estimate $H^d(Y_t \setminus V_1')$ and $H^d(\pi(V_1'))$. According to (5.99) and the choice $a = 1 - \sqrt{\varepsilon}$, we have

$$\begin{aligned} H^d(Y_t \setminus V_1') \leq H^d(Y_t \setminus \bigcup_j B_j) + \sum_j H^d(Y_t \cap (B_j \setminus aB_j)) \\ + \sum_j H^d(Y_t \cap (B_j \setminus C_j)) \end{aligned} \tag{5.108}$$

$$\leq \sum_j C(1-a) r_j^d + \varepsilon r_j^d + \varepsilon \tag{5.109}$$

$$\leq C\sqrt{\varepsilon} \sum_j r_j^d + \varepsilon \tag{5.110}$$

$$\leq C\sqrt{\varepsilon} \sum_j H^d(Y_t \cap B_j) + \varepsilon \tag{5.111}$$

$$\leq C\sqrt{\varepsilon} H^d(f(E)) + \varepsilon \tag{5.112}$$

and

$$H^d(\pi(V_1')) \leq \sum_j H^d(\pi(aB_j \cap C_j)) \tag{5.113}$$

$$\leq \sum_j (2r_j)^d \tag{5.114}$$

$$\leq (1+\varepsilon) \sum_j H^d(Y_t \cap B_j) \tag{5.115}$$

$$\leq (1+\varepsilon) H^d(f(E)) \tag{5.116}$$

Now, we want to set $g_1 = \pi \circ f$ but we also need that $g_1 = f$ in $\mathbf{R}^n \setminus W_1$. By definition of W_1, $H^d(E_1 \setminus W_1) \leq \frac{1}{3}\varepsilon_0$ so there exists open sets $W_1'' \subset\subset W_1' \subset\subset W_1$ such that

$$H^d(E_1 \setminus W_1'') \leq \tfrac{2}{3}\varepsilon_0. \tag{5.117}$$

Now, we define

$$V_1 = W_1'' \cap f^{-1}(V_1') \tag{5.118}$$

and

$$g_1 = \begin{cases} \pi \circ f & \text{in } V_1 \\ f & \text{in } \mathbf{R}^n \setminus W_1'. \end{cases} \tag{5.119}$$

It is straightforward that $|g_1 - f| \leq \varepsilon$ because $|\pi - \mathrm{id}| \leq \varepsilon$. In particular, $|g_1 - f| \leq \varepsilon_0$ for ε small enough. Let us estimate the Lipschitz constant of $g_1 - f$. For $x, y \in V_1$, we have

$$|(g_1 - f)(x) - (g_1 - f)(y)| \leq |(\pi - \mathrm{id})f(x) - (\pi - \mathrm{id})f(y)| \tag{5.120}$$

$$\leq |f(x) - f(y)| \tag{5.121}$$

$$\leq \|f\|_L |x - y| \tag{5.122}$$

because $(\pi - \mathrm{id})$ is 1-Lipschitz. For $x \in V_1$ and for $y \in \mathbf{R}^n \setminus W_1'$,

$$\begin{aligned} |(g_1 - f)(x) - (g_1 - f)(y)| &\leq |(\pi - \mathrm{id}) f(x)| && (5.123)\\ &\leq \varepsilon && (5.124)\\ &\leq \varepsilon \mathrm{d}(W_1'', \mathbf{R}^n \setminus W_1')^{-1} |x - y|. && (5.125) \end{aligned}$$

We can take ε small enough so that $g_1 - f$ is $\|f\|_L$-Lipschitz on its domain. We apply Lemma A.2.1 in Appendix A to $g_1 - f$ and obtain a Lipschitz map $g_1 \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\|g_1 - f\|_L \leq \|f\|_L$. Next, we estimate $H^d(E_1 \setminus V_1)$ and $H^d(g_1(V_1))$. By (5.88) and (5.112), we have

$$\begin{aligned} H^d(E_1 \setminus V_1) &\leq H^d(E_1 \setminus W_1'') + H^d(E_1 \setminus f^{-1}(V_1')) && (5.126)\\ &\leq \tfrac{2}{3}\varepsilon_0 + t H^d(Y_t \setminus V_1') && (5.127)\\ &\leq \tfrac{2}{3}\varepsilon_0 + t(C\sqrt{\varepsilon} H^d(f(E)) + \varepsilon). && (5.128) \end{aligned}$$

By the definition of g_1 and (5.116), we have

$$H^d(g_1(V_1)) \leq H^d(\pi(V_1')) \leq (1 + \varepsilon) H^d(f(E)). \qquad (5.129)$$

We take one last time ε small enough so that

$$\begin{aligned} H^d(E_1 \setminus V_1) &\leq \varepsilon_0, && (5.130)\\ H^d(g_1(V_1)) &\leq H^d(f(E)) + \varepsilon_0. && (5.131) \end{aligned}$$

Step 2. We build a Lipschitz map $g_2 \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $g_2 - f$ has a compact support included in W_2, $|g_2 - f| \leq \varepsilon$, $\|g_2 - f\|_L \leq \|f\|_L$ and such that there is an open set $V_2 \subset W_2$ satisfying

$$\begin{aligned} H^d(E_2 \setminus V_2) &\leq \varepsilon_0, && (5.132a)\\ H^d(g_2(V_2)) &\leq \varepsilon_0. && (5.132b) \end{aligned}$$

Let $0 < \varepsilon \leq 1$, let $0 < a < 1$ (to be chosen later). We recall that for H^d-ae. $x \in E_2$,

$$\begin{aligned} &\lim_{r \to 0} (2r)^{-d} H^d(E \cap B(x,r)) = 1 && (5.133)\\ &\limsup_{r \to 0} r^{-d} H^d(E \cap a\overline{B}(x,r) \setminus B(x,r)) \leq C(1-a) r^d && (5.134)\\ &\lim_{r \to 0} r^{-d} H^d(E \cap \overline{B}(x,r) \setminus C(x, V_x, \varepsilon) = 0. && (5.135) \end{aligned}$$

Moreover, by the properties of Lipschitz functions and rectifiable sets, for H^d-ae. $x \in E_2$, there exists a (unique) affine map $T_x f \colon V_x \to \mathbf{R}^n$ and a radius $r > 0$ such that for all $y \in V_x \cap B(x,r)$,

$$|f(y) - T_x f(y)| \leq \varepsilon |y - x|. \qquad (5.136)$$

By definition of W_2, $H^d(E_2 \setminus W_2) \le \frac{1}{3}\varepsilon_0$ so there exists a compact set $K \subset E_2 \cap W_2$ and a radius $0 < r_0 \le 1$ such that

$$H^d(E_2 \setminus K) \le \tfrac{2}{3}\varepsilon_0 \tag{5.137}$$

and for all $x \in K$,

$$\overline{B}(x, r_0) \subset W_2 \tag{5.138a}$$
$$\forall\, 0 < r \le r_0,\ (2r)^{-d} H^d(E \cap B(x,r)) \ge \tfrac{1}{2} \tag{5.138b}$$
$$\forall\, 0 < r \le r_0,\ H^d(E \cap \overline{B}(x,r) \setminus aB(x,r)) \le C(1-a)r^d \tag{5.138c}$$
$$\forall\, 0 < r \le r_0,\ H^d(E \cap \overline{B}(x,r) \setminus C(x, V_x, \varepsilon) \le \varepsilon r^d \tag{5.138d}$$
$$\forall y \in V_x \cap B(x, r_0),\ |f(y) - T_x f(y)| \le \varepsilon |y - x|. \tag{5.138e}$$

Let (D_j) be a sequence of closed balls (of center $y_j \in f(K)$ and radius $0 < \rho_j \le \varepsilon r_0$) such that

$$f(K) \subset \bigcup_j D_j \tag{5.139}$$

and

$$\sum_j \operatorname{diam}(D_j)^d \le 2^d H^d(f(K)) + \varepsilon. \tag{5.140}$$

The compact set K is covered by open balls of center $x \in K$ and radius $r > 0$ such that for some j, $x \in f^{-1}(D_j)$ and $r = \varepsilon^{-1}\rho_j$. We extract a *finite* covering of K by open balls (B_k) of center $x_k \in K$ and radius $r_k > 0$ such that for all k, there exists j such that $x_k \in f^{-1}(D_j)$ and $r_k = \varepsilon^{-1}\rho_j$. Let us note that $|f(x_k) - y_j| \le \varepsilon r_k$. In any metric space, a *finite* family of balls $(B_k)_k$ admits a a subfamily of disjoint balls $(B_l)_l$ such that $\bigcup_k B_k \subset \bigcup_l 3B_l$. Thus, we can also require that the reduced balls $(\frac{1}{3}B_k)$ are disjoint while $K \subset \bigcup_k B_k$.

Since $\rho_j \le \varepsilon r_0$, we have $r_k \le r_0 \le 1$. In particular, $\overline{B_k} \subset W_2$ by (5.138a). Moreover, (5.138b) and the fact that the balls $(\frac{1}{3}B_k)$ are disjoint yields

$$\sum_k r_k^d \le C H^d(E). \tag{5.141}$$

In each ball B_k, we are going to replace f by an orthogonal projection onto $\operatorname{Im} T_{x_k} f$. For each k, we define p_k as the orthogonal projection onto $\operatorname{Im} T_{x_k}$. We can write p_k as

$$p_k = f(x_k) + \vec{p}_k(\cdot - f(x_k)), \tag{5.142}$$

where $\vec{p}_k$ is the linear orthogonal projection onto the direction of $\operatorname{Im} T_{x_k} f$. We consider a maximal sequence $(\vec{\pi}_l)_l$ of linear orthogonal projection of rank $\le d$ and of mutual distances $> \varepsilon$. The index set of $(\vec{\pi}_l)_l$ contains at most $C(\varepsilon)$ elements, where $C(\varepsilon)$ depends on n and ε. For each k, we define $j(k)$ to be an index such that $x_k \in E \cap f^{-1}(D_{j(k)})$ and $r_k = \varepsilon^{-1}\rho_{j(k)}$. We also define $l(k)$ to be an index such that $\|\vec{p}_k - \vec{\pi}_{l(k)}\| \leq \varepsilon$. Finally, we let π_k be the orthogonal projection onto the affine plane $y_{j(k)} + \operatorname{Im} \vec{\pi}_{l(k)}$, that is

$$\pi_k = y_{j(k)} + \vec{\pi}_{l(k)}(\cdot - y_{j(k)}). \tag{5.143}$$

Without loss of generality we assume that the domain of definition of (B_k) is totally ordered and that the radius (r_k) are decreasing. We define

$$g_2 = \begin{cases} \pi_k \circ f & \text{in } aB_k \cap C_k \setminus \bigcup_{i<k} B_i \\ f & \text{in } \mathbf{R}^n \setminus \bigcup_k B_k, \end{cases} \tag{5.144}$$

where $C_k = C(x_k, V_{x_k}, \varepsilon)$. We start by estimating $|g_2 - f|$. For all $x \in aB_k \cap C_k \setminus \bigcup_{i<k} B_i$, we have

$$\begin{aligned} |g_2(x) - f(x)| &= |\pi_k f(x) - f(x)| && (5.145) \\ &\leq |\pi_k f(x) - p_k f(x)| + |p_k f(x) - f(x)|. && (5.146) \end{aligned}$$

By construction

$$\begin{aligned} |\pi_k f(x) - p_k f(x)| &\leq \big|f(x_k) - y_{j(k)}\big| + \big|\vec{\pi}_{l(k)}(f(x_k) - y_{j(k)})\big| \\ &\quad + \big|(\vec{p}_k - \vec{\pi_{l(k)}})(f(x) - f(x_k))\big| && (5.147) \\ &\leq 2\big|f(x_k) - y_{j(k)}\big| + \big\|\vec{p}_k - \vec{\pi}_{l(k)}\big\| |f(x) - f(x_k)| && (5.148) \\ &\leq 2\varepsilon r_k + \|f\|_L \varepsilon r_k. && (5.149) \end{aligned}$$

On the other hand, by the properties of orthogonal projections, the definition of C_k and (5.138e),.

$$\begin{aligned} |p_k f(x) - f(x)| &= \mathrm{d}(f(x), \operatorname{Im} T_{x_k} f) && (5.150) \\ &\leq \big|f(x) - T_{x_k}(x' - x_k)\big| && (5.151) \\ &\leq \big|f(x) - f(x')\big| + \big|f(x') - T_{x_k}(x' - x_k)\big| && (5.152) \\ &\leq \|f\|_L \varepsilon r_k + \varepsilon r_k, && (5.153) \end{aligned}$$

where x' is the orthogonal projection of x onto V_{x_k}. In conclusion, $|g_2 - f| \leq C(\|f\|_L + 1)\varepsilon r_k$ in $aB_k \cap C_k \setminus \bigcup_{i<k} B_i$. Next, we estimate the Lipschitz constant of $g_2 - f$. In each set $aB_k \cap C_k \setminus \bigcup_{i<k} B_i$, the map $g_2 - f = (\pi_k - \mathrm{id}) f$ is $\|f\|_L$-Lipschitz because $\pi_k - \mathrm{id}$ is 1-Lipschitz. As

$$\mathrm{d}(aB_k, \mathbf{R}^n \setminus B_k) \geq (1 - a) r_k, \tag{5.154}$$

we have for $x \in aB_k \cap C_k \setminus \bigcup_{i<k} B_i$ and $y \in \mathbf{R}^n \setminus \bigcup_k B_k$,

$$\begin{aligned} |(g_2 - f)(x) - (g_2 - f)(y)| &\leq |g_2(x) - f(x)| && (5.155) \\ &\leq C(\|f\|_L + 1)\varepsilon r_k && (5.156) \\ &\leq C(\|f\|_L + 1)\frac{\varepsilon}{1 - a}|x - y|. && (5.157) \end{aligned}$$

For $x \in aB_k \cap C_k \setminus \bigcup_{i<k} B_i$ and $y \in aB_l \cap C_l \setminus \bigcup_{i<l} B_i$ where $k < l$, we have similarly

$$
\begin{aligned}
|(g_2 - f)(x) - (g_2 - f)(y)| &\le |g_2(x) - f(x)| + |g_2(y) - f(y)| && (5.158)\\
&\le C(\|f\|_L + 1)\varepsilon r_k + C(\|f\|_L + 1)\varepsilon r_l && (5.159)\\
&\le C(\|f\|_L + 1)\varepsilon r_k && (5.160)\\
&\le C(\|f\|_L + 1)\frac{\varepsilon}{1-a}|x - y|. && (5.161)
\end{aligned}
$$

Taking $a = 1 - \sqrt{\varepsilon}$, we conclude that $g_2 - f$ is $C(\|f\|_L + 1)\sqrt{\varepsilon}$-Lipschitz. We assume ε so that we can apply Lemma A.2.1 Appendix A to $g_2 - f$ and extend g_2 as a Lipchitz map $\mathbf{R}^n \to \mathbf{R}^n$ such that $\|g_2 - f\|_L \le \|f\|_L$ and $|g_2 - f| \le \varepsilon_0$.

Now, we define

$$V_2 = \bigcup_k B_k \setminus A, \tag{5.162}$$

where A is the compact set $A = \bigcup_k \overline{B_k} \setminus (aB_k \cap C_k)$ and we estimate $H^d(g_2(V_2))$ and $H^d(E_2 \setminus V_2)$. By (5.138), (5.141) and the choice $a = 1 - \sqrt{\varepsilon}$,

$$
\begin{aligned}
H^d(E_2 \setminus V_2) &\le H^d(E_2 \setminus K) + H^d(K \setminus V) && (5.163)\\
&\le \tfrac{2}{3}\varepsilon_0 + H^d(K \cap A) && (5.164)\\
&\le \tfrac{2}{3}\varepsilon_0 + \sum_k H^d(E \cap \overline{B_k} \setminus aB_k) \\
&\qquad + \sum_k H^d(E \cap \overline{B_k} \setminus C_k) && (5.165)\\
&\le \tfrac{2}{3}\varepsilon_0 + C(1-a)\sum_k r_k^d + \varepsilon \sum_k r_k^d && (5.166)\\
&\le \tfrac{2}{3}\varepsilon_0 + C\sqrt{\varepsilon}\sum_k r_k^d && (5.167)\\
&\le \tfrac{2}{3}\varepsilon_0 + C\sqrt{\varepsilon} H^d(E) && (5.168)
\end{aligned}
$$

By the definition of g_2,

$$
\begin{aligned}
g_2(V_2) &\subset \bigcup_k g_2(B_k \setminus A) && (5.169)\\
&\subset \bigcup_k g_2(aB_k \cap C_k \setminus \bigcup_{i<k} B_i) && (5.170)\\
&\subset \bigcup_k (y_{j(k)} + \operatorname{Im} \vec{\pi}_{l(k)}) \cap B(f(x_k), C(\|f\|_L + 1)r_k) && (5.171)\\
&\subset \bigcup_k (y_{j(k)} + \operatorname{Im} \vec{\pi}_{l(k)}) \cap B(y_{j(k)}, C(\|f\|_L + 1)r_k) && (5.172)
\end{aligned}
$$

Then by the definition of (D_j) and (5.89),

$$H^d(g_2(V_2)) \leq C(\varepsilon)(\|f\|_L + 1)^d \sum_j \operatorname{diam}(D_j)^d \tag{5.173}$$
$$\leq C(\varepsilon)(\|f\|_L + 1)^d H^d(f(K)) \tag{5.174}$$
$$\leq C(\varepsilon)(\|f\|_L + 1)^d t^{-1} H^d(E). \tag{5.175}$$

We fix ε small enough so that $H^d(E_2 \setminus V) \leq \varepsilon_0$. Then we fix t big enough such that $H^d(g_2(V_2)) \leq \varepsilon_0$.

Step 3. We define $V = V_1 \cup V_2$ and g the piecewise function

$$g(z) = \begin{cases} g_1(z) & \text{in } W_1 \\ g_2(z) & \text{in } W_2 \\ f(z) & \text{in } \mathbf{R}^n \setminus (W_1 \cup W_2). \end{cases} \tag{5.176}$$

We have $|g - f| \leq \varepsilon_0$ and $\|g - f\|_L \leq \|f\|_L$ ($g - f$ is locally $\|f\|_L$-Lipschitz because $g_i - f$ has a compact support included in W_i and then globally $\|f\|_L$-Lipschitz by convexity of $\mathbf{R}^n$). Finally, we estimate

$$H^d(g(V)) \leq H^d(g_1(V_1)) + H^d(g_2(V_2)) \tag{5.177}$$
$$\leq H^d(f(E))) + 2\varepsilon_0 \tag{5.178}$$

and

$$H^d(E \setminus V) \leq H^d(E_1 \setminus V_1) + H^d(E_2 \setminus V_2) \tag{5.179}$$
$$\leq 2\varepsilon_0. \tag{5.180}$$

□

Lemma 5.2.2 (Adaptation of Lemma 5.2.1 for sliding deformations)**.** *Fix Γ a Lipschitz neighborhood retract of X which is H^d regular. Let f be a global sliding deformation in an open set $U \subset X$. Let W be an open subset of U and let $E \subset W$ be a H^d measurable, H^d finite and H^d rectifiable set. For all $\varepsilon > 0$, there exists a global sliding deformation g in U and an open set $V \subset W$ such that $g - f$ has a compact support included in W, $|g - f| \leq \varepsilon$, $\|g - f\|_L \leq C\|f\|_L$ (where $C \geq 1$ depends on n, Γ) and*

$$H^d(E \setminus V) \leq \varepsilon \tag{5.181a}$$
$$H^d(g(V)) \leq H^d(f(E)) + \varepsilon. \tag{5.181b}$$

Proof. This proof is an adaptation of the proof of Lemma 5.2.1 but we want build a map which preserves the boundary. *In the following proof, the letter C plays the role of a constant ≥ 1 that depends on n and Γ. Its value can* *increase from one line to another (but a finite number of times). For a radius $\delta > 0$, the δ-neighborhood of Γ is denoted by $\Gamma(\delta)$, i.e.*

$$\Gamma(\delta) := \{ x \in X \mid \mathrm{d}(x, \Gamma) < \delta \}. \tag{5.182}$$

Without loss of generality, we assume $W \subset\subset U$. We extend f as a Lipschitz map $f\colon \mathbf{R}^n \to \mathbf{R}^n$ with a Lipschitz constant multiplied by C. We can get rid of the case $\|f\|_L = 0$ as in Lemma 5.2.1 and assume $\|f\|_L > 0$. We fix $\varepsilon_0 > 0$. For $t > 0$, we define Y_t to be the set of points $y \in f(E)$ such that

$$\liminf_{r \to 0} \frac{H^d(E \cap f^{-1}(B(y,r)))}{H^d(f(E) \cap B(y,r))} \leq t. \tag{5.183}$$

As in Lemma 5.2.1, this set is H^d measurable. We define $E_1 = E \cap f^{-1}(Y_t)$ and $E_2 = E \setminus f^{-1}(Y_t)$, and thus for all $A \subset \mathbf{R}^n$,

$$H^d(E_1 \cap f^{-1}(A)) \leq t H^d(A \cap Y_t) \tag{5.184}$$

and

$$H^d(E_2 \cap f^{-1}(A)) \geq t H^d(A \cap Y_t). \tag{5.185}$$

The parameter t is arbitrary in step 1 whereas it is chosen in step 2. Our constructions will rely on intermediate variables $0 < \delta, \varepsilon \leq 1$. These variables will be as small as we want but the choice of ε will be subordinated to the choice of δ.

We introduce an "almost-retraction" p_1 on the boundary with a good Lipschitz constant and a retraction p_2 on the boundary with a bad Lipschitz constant. The boundary Γ is H^d regular so there exists an open set $O_1 \subset X$ and a $(1+\delta)$-Lipschitz map $p_1\colon \Gamma \cup O_1 \to \Gamma$ such that $H^d(\Gamma \setminus O_1) = 0$ and $p_1 = \mathrm{id}$ on Γ. The boundary Γ is also a Lipschitz neighborhood retract so there exists an open set $O_2 \subset X$ containing Γ and a C-Lipschitz map $p_2\colon O_2 \to \Gamma$ such that $p_2 = \mathrm{id}$ on Γ. We underline that the Lipschitz constant of p_2 depends only on n and Γ. We restrict each set O_i so as to assume that $O_i \subset \Gamma(\frac{\delta}{2})$ and $|p_i - \mathrm{id}| \leq \frac{1}{2}\delta^2 \leq \frac{1}{2}\delta$ (for $i = 1, 2$). We extend p_i on $X \setminus \Gamma(\delta)$ by $p_i = \mathrm{id}$. Let us check that this extension is still Lipschitz with the same Lipschitz constant: $(1+\delta)$ for p_1 and C for p_2. For $x \in O_i$ and $y \in X \setminus \Gamma(\delta)$, we have $|x - y| \geq \frac{\delta}{2}$ so

$$\begin{aligned}
|p_i(x) - p_i(y)| &\leq |p_i(x) - y| && (5.186)\\
&\leq |p_i(x) - x| + |x - y| && (5.187)\\
&\leq \tfrac{1}{2}\delta^2 + |x - y| && (5.188)\\
&\leq (1+\delta)|x - y|. && (5.189)
\end{aligned}$$

Now, we use Lemma A.2.1 to extend p_2 as a C-Lipschitz map $p_2\colon X \to \mathbf{R}^n$ such that $|p_2 - \mathrm{id}| \leq \frac{\delta}{2}$. However, we have to be careful with p_1 because we do not want to lose its good Lipschitz constant. We use the Kirzbraun theorem to extend p_1 as a $(1+\delta)$-Lipschitz map $p_1\colon X \to \mathbf{R}^n$. At the same time, we would like that $|p_1 - \mathrm{id}| \leq \frac{\delta}{2}$. This is true on $X \setminus \Gamma(\delta)$ because $p_1 = \mathrm{id}$ on $X \setminus \Gamma(\delta)$. For $x \in \Gamma(\delta)$, there exists $y \in \Gamma$ such that $|x-y| \leq \delta$ so

$$\begin{aligned} |p_1(x) - x| &\leq |p_1(x) - p_1(y)| + |x-y| && (5.190)\\ &\leq (1+\delta)|x-y| + |x-y| && (5.191)\\ &\leq (1+\delta)\delta + \delta. && (5.192)\end{aligned}$$

To simplify, we assume $|p_1 - \mathrm{id}| \leq \frac{\delta}{2}$.

Step 0. We build two open sets $W_1, W_2 \subset\subset W$ such that $\overline{W_1} \cap \overline{W_2} = \emptyset$ and

$$H^d(E_i \setminus W_i) < \tfrac{1}{3}\varepsilon_0. \tag{5.193}$$

We refer to the corresponding step of Lemma 5.2.1.

Step 1. We build a Lipschitz map $h_1\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $h_1 - f$ has a compact support included in W_1, $|h_1 - f| \leq \varepsilon_0$, $\|h_1 - f\|_L \leq \|f\|_L$, $h_1(\Gamma) \subset \Gamma$ and such that there is an open set $V_1 \subset W_1$ satisfying

$$H^d(E_1 \setminus V_1) \leq \varepsilon_0, \tag{5.194a}$$
$$H^d(h_1(V_1)) \leq H^d(f(E)) + \varepsilon_0. \tag{5.194b}$$

By definition of W_1, $H^d(E_1 \setminus W_1) < \frac{1}{3}\varepsilon_0$ so there exists open sets $W_1'' \subset\subset W_1' \subset\subset W_1$ such that

$$H^d(E_1 \setminus W_1'') \leq \tfrac{1}{3}\varepsilon_0. \tag{5.195}$$

We appy step 1 of Lemma 5.2.1 to find a Lipschitz map $g\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $g - f$ has a compact support included in W_1', $|g - f| \leq \varepsilon$, $\|g - f\|_L \leq \|f\|_L$, and such that such that there is an open set $V \subset W_1'$ satisfying

$$H^d(E_1 \setminus V) \leq \tfrac{1}{3}\varepsilon_0, \tag{5.196a}$$
$$H^d(g(V)) \leq H^d(f(E)) + \tfrac{1}{2}\varepsilon_0. \tag{5.196b}$$

We want to ensure that $g(X) \subset X$. As $\overline{W_1'}$ is a compact subset of X and $f(X) \subset X$, we can assume that ε is small enough so that for all $x \in \overline{W_1'}$,

$$\mathrm{d}(f(x), \mathbf{R}^n \setminus X) > \varepsilon. \tag{5.197}$$

We deduce that $g(X) \subset X$ because $g = f$ in $X \setminus W_1'$ and $|g - f| \leq \varepsilon$. In addition, we would like that

$$g(V) \subset (X \setminus \Gamma(\delta)) \cup O_1 \tag{5.198}$$

to be able to retract $g(V \cap \Gamma)$ onto the boundary without enlarging too much its measure. We consider the open set

$$V'_{\delta,\varepsilon} := \{\, y \in X \mid \mathrm{d}(y, \Gamma) > \delta + \varepsilon \text{ or } \mathrm{d}(y, X \setminus O_1)) > \varepsilon \,\}. \tag{5.199}$$

When $\varepsilon \to 0$, the sets $V_{\delta,\varepsilon}$ are increasing to

$$V'_\delta := \{\, y \in X \mid \mathrm{d}(y,\Gamma) > \delta \text{ or } y \in O_1 \,\} \tag{5.200}$$

and when $\delta \to 0$, the sets V_δ are increasing to

$$\begin{aligned} V'_0 &:= \{\, y \in X \mid y \notin \Gamma \text{ or } y \in O_1 \,\} && (5.201)\\ &:= X \setminus (\Gamma \setminus O_1) && (5.202) \end{aligned}$$

because Γ is a closed subset of X. Let us recall that $H^d(\Gamma \setminus O_1) = 0$. As $H^d(f(E)) < \infty$, we can choose $\delta > 0$ small enough so that

$$H^d(f(E) \setminus V'_\delta) < \tfrac{1}{3} t^{-1} \varepsilon_0 \tag{5.203}$$

and then $\varepsilon > 0$ small enough so that

$$H^d(f(E) \setminus V'_{\delta,\varepsilon}) \leq \tfrac{1}{3} t^{-1} \varepsilon_0. \tag{5.204}$$

Now, we replace the set V by

$$V_1 = V \cap W''_1 \cap f^{-1}(V'_{\delta,\varepsilon}). \tag{5.205}$$

In particular, $g(V_1) \subset (X \setminus \Gamma(\delta)) \cup O_1$ because $g(V_1) \subset X$ and $|f - g| \leq \varepsilon$. By (5.184) and (5.204), we estimate

$$\begin{aligned} H^d(E_1 \setminus V_1) &\leq H^d(E_1 \setminus V) + H^d(E_1 \setminus W''_1) + H^d(E_1 \setminus f^{-1}(V_{\delta,\varepsilon})) && (5.206)\\ &\leq \varepsilon_0. && (5.207) \end{aligned}$$

Finally, we compose g with $p_2 \circ p_1$. More precisely, we consider the partially defined map

$$h_1 = \begin{cases} p_2 \circ p_1 \circ g & \text{in } V_1 \cup \Gamma \\ g \ (= f) & \text{in } \mathbf{R}^n \setminus W'_1. \end{cases} \tag{5.208}$$

The map h_1 is well-defined because $V_1 \subset W''_1 \subset W'_1$ and $p_i = \mathrm{id}$ on Γ ($i = 1, 2$). We start by estimating $|h_1 - f|$. As $|p_2 p_1 - \mathrm{id}| \leq \delta$, we have $|h_1 - g| \leq \delta$. Moreover, $|g - f| \leq \varepsilon$ so $|h_1 - f| \leq \delta + \varepsilon$. We can assume δ and ε small enough so that $|h_1 - f| \leq \varepsilon_0$. Next, we show that $\|h_1 - f\|_L \leq C\|f\|_L$. It suffices to have $\|h_1 - g\|_L \leq C\|f\|_L$. The first ingredients are the facts that $|p_2 p_1 - \mathrm{id}| \leq \delta$ and, for δ small enough, $\mathrm{d}(W''_1, \mathbf{R}^n \setminus W'_1) \geq \delta \|f\|_L^{-1}$. As a consequence, for $x \in V_1$ and $y \in \mathbf{R}^n \setminus W'_1$,

$$\begin{aligned} |(h_1 - g)(x) - (h_1 - g)(y)| &\leq |(h_1 - g)(x)| + |(h_1 - g)(y)| && (5.209)\\ &\leq 2\delta && (5.210)\\ &\leq 2\|f\|_L |x - y|. && (5.211) \end{aligned}$$

The second ingredients are the facts that p_2p_1 is C-Lipschitz and $p_2p_1 = \mathrm{id}$ on Γ. It implies that, for all $x \in X$,

$$|p_2p_1(x) - x| \leq \|p_2p_1 - \mathrm{id}\|_L \mathrm{d}(x, \Gamma) \tag{5.212}$$
$$\leq C\mathrm{d}(x, \Gamma). \tag{5.213}$$

As a consequence, for $x \in \Gamma$ and $y \in \mathbf{R}^n \setminus W_1'$,

$$|(h_1 - g)(x) - (h_1 - g)(y)| \leq |h_1(x) - g(x)| \tag{5.214}$$
$$\leq C\mathrm{d}(g(x), \Gamma) \tag{5.215}$$
$$\leq C|g(x) - f(x)| \tag{5.216}$$
$$\leq C|(g - f)(x) - (g - f)(y)| \tag{5.217}$$
$$\leq C\|f\|_L |x - y|. \tag{5.218}$$

We extend h_1 as a Lipschitz map $h_1 \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $\|h_1 - f\|_L \leq C\|f\|_L$ and $|h_1 - f| \leq \varepsilon_0$. Next, we prove that $h_1(\Gamma) \subset \Gamma$. In view of the definition of p_2 and O_2, it suffices that $g(\Gamma) \subset p_1^{-1}(O_2)$. As Γ is relatively closed in X and $\overline{W_1'}$ is a compact subset of X, the intersection $\Gamma \cap \overline{W_1'}$ is compact. Its image $f(\Gamma \cap \overline{W_1'})$ is a compact subset of $\Gamma \subset p_1^{-1}(O_2)$. We take ε is small enough so that for all $x \in \Gamma \cap \overline{W_1'}$,

$$\mathrm{d}(f(x), \mathbf{R}^n \setminus p_1^{-1}(O_2)) > \varepsilon. \tag{5.219}$$

We deduce that $g(\Gamma) \subset p_1^{-1}(O_2)$ as $g = f$ in $\mathbf{R}^n \setminus W_1'$ and $|g - f| \leq \varepsilon$. Finally, we estimate $H^d(h_1(V_1))$. We recall the key argument:

$$g_1(V_1) \subset (X \setminus \Gamma(\delta)) \cup O_1. \tag{5.220}$$

This implies that $p_1g_1(V_1) \subset (X \setminus \Gamma(\delta)) \cup \Gamma$ and then $p_2 = \mathrm{id}$ on $p_1g_1(V_1)$. We deduce that p_2p_1 is $(1 + \delta)$-Lipschitz on $g_1(V_1)$. Hence

$$H^d(h_1(V_1)) \leq (1 + \delta)^d H^d(g(V_1)) \tag{5.221}$$
$$\leq (1 + \delta)^d (H^d(f(E \cap W)) + \tfrac{1}{2}\varepsilon_0) \tag{5.222}$$

We choose $\delta > 0$ small enough so that $H^d(h_1(V_1)) \leq H^d(f(E \cap W)) + \varepsilon_0$.

Step 2. We build a Lipschitz map $h_2 \colon \mathbf{R}^n \to \mathbf{R}^n$ such that $h_2 - f$ has a compact support included in W_2, $|h_2 - f| \leq \varepsilon$, $\|h_2 - f\|_L \leq C\|f\|_L$ and such that there is an open set $V_2 \subset W_2$ satisfying

$$H^d(E_2 \setminus V_2) \leq \varepsilon_0, \tag{5.223a}$$
$$H^d(h_2(V_2)) \leq \varepsilon_0. \tag{5.223b}$$

By definition of W_2, $H^d(E_2 \setminus W_2) < \frac{1}{3}\varepsilon_0$ so there exists open sets $W_2'' \subset\subset W_2' \subset\subset W_2$ such that

$$H^d(E_2 \setminus W_2'') \leq \tfrac{1}{2}\varepsilon_0. \tag{5.224}$$

We appy step 2 of Lemma 5.2.1 to find a Lipschitz map $g\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $g - f$ has a compact support included in W_2', $|g - f| \leq \varepsilon$, $\|g - f\|_L \leq \|f\|_L$, and such that such that there is an open set $V \subset W_2'$ satisfying

$$H^d(E_2 \setminus V) \leq \tfrac{1}{2}\varepsilon_0, \tag{5.225a}$$
$$H^d(g(V)) \leq C^{-d}\varepsilon_0. \tag{5.225b}$$

We replace the open set V by

$$V_2 = V \cap W_2''. \tag{5.226}$$

We still have

$$\begin{aligned} H^d(E_2 \setminus V_2) &\leq H^d(E_2 \setminus V) + H^d(E_2 \setminus W_2'') &(5.227)\\ &\leq \varepsilon_0. &(5.228) \end{aligned}$$

Next, we compose g with p_2. We consider the partially defined map

$$h_2 = \begin{cases} p_2 \circ g & \text{in } V_2 \cup \Gamma \\ g\ (= f) & \text{in } \mathbf{R}^n \setminus W_2'. \end{cases} \tag{5.229}$$

We proceed as before to extend h_2 as a Lipschitz map $h_2\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $|h_2 - f| \leq \varepsilon_0$ and $\|h_2 - f\|_L \leq C\|f\|_L$. We can also assume ε small enough so that $h_2(\Gamma) \subset \Gamma$. Since p_2 is C-Lipschitz, we have finally

$$H^d(h_2(V_2)) \leq C^d H^d(g(V_2)) \leq \varepsilon_0. \tag{5.230}$$

Step 3. We define $V = V_1 \cup V_2$ and h the piecewise function

$$h(z) = \begin{cases} h_1(z) & \text{in } W_1 \\ h_2(z) & \text{in } W_2 \\ f(z) & \text{in } \mathbf{R}^n \setminus (W_1 \cup W_2). \end{cases} \tag{5.231}$$

We can conclude as in step 3 of Lemma 5.2.1 but in addition $h(\Gamma) \subset \Gamma$. As $W \subset\subset U$ and $h = f$ in $X \setminus W$, Lemma 3.2.1 says that h is a global sliding deformation, provided that ε_0 is small enough. □

5.2.2 Proof of the limiting theorem

Proof. In this proof, the letter C is an unspecified constant ≥ 1 that depends on n, κ, Γ. Its value can increase from one line to another (but a finite number of times). Similarly, the letter t is an unspecified constant > 0 that depends on n, s, Γ and whose value can decrease from one line to another.
Let E be the support of μ in X. Let K be a Whitney complex of $\mathbf{R}^n$, let $T\colon |K| \to X$ be a bijective and bilipschitz map and let L be a subcomplex of K such that $\Gamma = T(|K| \setminus U(L))$.

Step 1. We show that for all $x \in E$, for all $0 < r \leq r_t(x)$ and for i big enough,

$$C^{-1}r^d \leq H^d(E_i \cap B(x,r)) \leq Cr^d. \tag{5.232}$$

Before proving step 1, we draw its main consequences. It yields that for $x \in E$ and for $0 < r \leq r_t(x)$,

$$C^{-1}r^d \leq \mu(B(x,r)) \leq Cr^d. \tag{5.233}$$

According to the density theorems on Radon measures (see [Mat], Theorem 6.9), we have

$$C^{-1}H^d \llcorner E \leq \mu \leq CH^d \llcorner E. \tag{5.234}$$

so E is Ahlfors regular.

Now, we proceed to the proof of step 1. It relies on two ingredients:

i) the sets E_i satisfy (5.75) in local balls centered in E;

ii) for all $x \in E$, for all $r > 0$, $\liminf_i H^d(E_i \cap B(x,r)) > 0$.

As T is a bilipschitz map, it suffices to prove (5.232) for the sets $T^{-1}(E_i)$ with respect to $T^{-1}(E)$. Morever, the sets $T^{-1}(E_i)$ satisfy the same ingredients with respect to $T^{-1}(E)$ (the parameters of (5.75) are deformed as in Remark 3.1.2). Thus, we assume $T = \mathrm{id}$ directly.

Let $x \in E$ and $0 < r \leq r_t(x)$. We build sliding deformations in the open ball $U = B(x, 16\sqrt{n}r)$ to get that

$$H^d(E_i \cap B(x,r)) \leq Cr^d \tag{5.235}$$

$$H^d(E_i \cap B(x, 16\sqrt{n}r)) \geq C^{-1}r^d. \tag{5.236}$$

We can reduce the problem by assuming

$$B(x,r) \subset \tfrac{1}{2}]-1,1[^n \subset [-1,1]^n \subset B(x, 16\sqrt{n}r), \tag{5.237}$$

$E_n \preceq K$, $16\sqrt{n}r \leq r_s(x)$ and by proving that

$$H^d(E_i \cap \tfrac{1}{2}]-1,1[^n) \leq Cr^d, \tag{5.238}$$

$$H^d(E_i \cap]-1,1[) \geq C^{-1}r^d. \tag{5.239}$$

Let (ε_i) be the sequence associated to U. As $\liminf_i H^d(E_i \cap hU) > 0$, we deduce that for i big enough, $\varepsilon_i \leq hH^d(E_i \cap hU)$. Then, we can proceed as in the proofs of the reduced problems of Proposition 5.1.1.

Step 2. We show that E is H^d-rectifiable. The proof relies on on two ingredients:

i) the sets E_i satisfy (5.75) in local balls centered in E;

ii) the set E is Ahlfors regular of dimension d.

As T is a bilipschitz map, it suffices to prove that $T^{-1}(E)$ is H^d rectifiable. Morever, the sets $T^{-1}(E_i)$ satisfy the same ingredients with respect to $T^{-1}(E)$. Thus, we assume $T = \mathrm{id}$ directly.

Let $x \in E$ and $0 < r \leq r_t(x)$. We build sliding deformations in the open ball $U = B(x, 16\sqrt{n}r)$ to get that

$$H^d(E \cap B(x,r)) \leq C \int_{G(d,n)} H^d(p_V(E \cap B(x, 16\sqrt{n}r))\, \mathrm{d}V. \tag{5.240}$$

We can reduce the problem by assuming

$$B(x,r) \subset \tfrac{1}{2}]-1,1[^n \subset [-1,1]^n \subset B(x, 16\sqrt{n}r), \tag{5.241}$$

$E_n \preceq K$, $16\sqrt{n}r \leq r_s(x)$ and by proving that

$$H^d(E \cap \tfrac{1}{2}]-1,1[^n) \leq C \int_{G(d,n)} H^d(p_V(E \cap]-1,1[^n))\, \mathrm{d}V. \tag{5.242}$$

Let (ε_i) be the sequence associated to U. We have $\liminf_i H^d(E_i \cap hU) > 0$ so for i big enough, $\varepsilon_i \leq hH^d(E_i \cap hU)$. We want to proceed as in the proof of the reduced problem of Corollary 5.1.1 but there is a difference. We are not going to make Federer-Fleming projection of the sets E_i but of the set E instead.

We fix $q \in \mathbf{N}^*$. For $0 \leq k \leq 2^q$, let M_k be the set of dyadic cells of sidelength 2^{-q} subdivising the cube $(1-k2^{-q})[-1,1]^n$ (except the boundary). The set M_k is a finite n-complex subordinated to E_n and

$$|M_k| = (1 - k2^{-q})[-1,1]^n, \tag{5.243a}$$

$$U(M_k) = (1 - k2^{-q})]-1,1[^n. \tag{5.243b}$$

Moreover, it is clear that $M_{k+1} \subset M_k$. We define $U_k = U(M_k)$. Let ϕ be a Federer-Fleming projection of $E \cap U_k$ in M_k (not $E_i \cap U_k$!). We apply the quasiminimality of E_i with respect to ϕ in $U = B(x, 16\sqrt{n}r)$. We assume h small enough (depending only on n) such that $hH^d(E_i \cap hU) \leq \frac{1}{4}H^d(E_i \cap B(x,r))$. We also recall that i is big enough so that $\varepsilon_i \leq hH^d(E_i \cap hU)$. We have then

$$H^d(E_i \cap U_k) \leq CH^d(\phi(E_i \cap U_k)) + \tfrac{1}{2}H^d(E_i \cap U_k) \tag{5.244}$$

so

$$H^d(E_i \cap U_k) \leq CH^d(\phi(E_i \cap U_k)). \tag{5.245}$$

We prove that there exists an open set $O \subset U_{k+1}$ which contains $E \cap U_{k+1}$ and such that

$$H^d(\phi(O)) \leq C2^q \int_{G(d,n)} H^d(p_V(E \cap U_0))\, \mathrm{d}V + \varepsilon \tag{5.246}$$

where $\varepsilon > 0$ is an error term which is small compared to $H^d(E_i \cap U_k)$. Later, we will decompose $E_i \cap U_k$ in two parts: $E_i \cap O$ and $E_i \cap U_k \setminus O$. By the properties of the Federer-Fleming projection (Proposition 4.3.1), there exists an open set $O' \subset U_k$ containing $E \cap U_k$ and such that

$$\phi(O') \subset |M_k| \setminus \bigcup \{ \operatorname{int}(A) \mid A \in M_k,\ \dim(A) > d \}. \tag{5.247}$$

As $M_{k+1} \subset M_k$, ϕ preserves the cells of M_{k+1} so $\phi(U_{k+1}) \subset |M_{k+1}|$. We deduce that

$$\begin{aligned} \phi(O' \cap U_{k+1}) &\subset |M_{k+1}| \setminus \bigcup \{ \operatorname{int}(A) \mid A \in M_k,\ \dim(A) > d \} && (5.248)\\ &\subset U_k \setminus \bigcup \{ \operatorname{int}(A) \mid A \in M_k,\ \dim(A) > d \} && (5.249)\\ &\subset S_k && (5.250) \end{aligned}$$

where

$$S_k = \bigcup \{ \operatorname{int}(A) \mid A \in M_k,\ \dim(A) \leq d \}. \tag{5.251}$$

We recall that for $A \in M_k^d$,

$$H^d(\phi(E) \cap A) \leq C \int_{G(d,n)} H^d(p_V(E \cap U_0))\, \mathrm{d}V. \tag{5.252}$$

Since $\phi(E \cap U_{k+1}) \subset S$ and M_k^d contains at most $C2^q$ cells, we deduce that

$$H^d(\phi(E \cap U_{k+1})) \leq C2^q \int_{G(d,n)} H^d(p_V(E \cap U_0))\, \mathrm{d}V. \tag{5.253}$$

The set O that we intend to build is a small neighborhood of $E \cap U_{k+1}$ for which this estimate still holds with an error term ε. First, we observe that $H^d \llcorner S$ is a Radon measure because M is finite. According to the regularity properties of Radon measure, there exists an open set O'' containing $\phi(E \cap U_{k+1})$ such that

$$H^d(S \cap O'') \leq H^d(\phi(E \cap U_{k+1})) + \varepsilon. \tag{5.254}$$

Since $\phi(O' \cap U_{k+1}) \subset S$, the set $O = O' \cap U_{k+1} \cap \phi^{-1}(O'')$ is a solution: it is an open subset of U_{k+1} which contains $E \cap U_{k+1}$ and which satisfies (5.246). In conclusion,

$$\begin{aligned} H^d(E_i \cap U_k) &\leq CH^d(\phi(E_i \cap U_k)) && (5.255)\\ &\leq CH^d(\phi(O)) + CH^d(E_i \cap U_k \setminus O)) && (5.256)\\ &\leq C2^q \int_{G(d,n)} H^d(p_V(E \cap U_0))\, \mathrm{d}V + C\varepsilon \\ &\qquad + CH^d(\phi(E_i \cap U_k \setminus O)). && (5.257) \end{aligned}$$

As $\liminf_i H^d(E_i \cap B(x,r)) > 0$, we can take ε small enough (independently from i) so that $C\varepsilon < \frac{1}{2} \liminf_i H^d(E_i \cap B(x,r))$ and then assume i big enough so that

$$C\varepsilon \leq \tfrac{1}{2} H^d(E_i \cap U_k). \tag{5.258}$$

Thus,

$$H^d(E_i \cap U_k) \leq C2^q \int_{G(d,n)} H^d(p_V(E \cap U_0))\,\mathrm{d}V + \\ CH^d(\phi(E_i \cap U_k \setminus O)). \tag{5.259}$$

Moreover, E is Ahlfors regular of dimension d so we can assume that ϕ is C-Lipschitz (see the second part of Proposition 4.3.1 as well as Remark 4.3.1 applied to $E \cap U_k$). Thus,

$$H^d(E_i \cap U_k) \leq C2^q \int_{G(d,n)} H^d(p_V(E \cap U_0))\,\mathrm{d}V \\ + CH^d(E_i \cap U_k \setminus O). \tag{5.260}$$

We pass to the limit $i \to \infty$,

$$H^d(E \cap U_k) \leq C2^q \int_{G(d,n)} H^d(p_V(E \cap U_0))\,\mathrm{d}V \\ + CH^d(E \cap |M_k| \setminus O). \tag{5.261}$$

The open set O contains $E \cap U_{k+1}$ so

$$H^d(E \cap U_k) \leq C2^q \int_{G(d,n)} H^d(p_V(E \cap U_0))\,\mathrm{d}V \\ + CH^d(E \cap |M_k| \setminus U_{k+1})). \tag{5.262}$$

This is (almost) the same inequality as (5.69) in the proof of Corollary 5.1.1. We can conclude in the same way by a Chebychev argument.

Step 3. We show that for all $x \in E$, for all $0 < r \leq r_s(x)$, for all sliding deformation f of E in $U = B(x,r)$,

$$\mu(W_f) \leq \kappa H^d(f(E \cap W_f)) + h\mu(hU). \tag{5.263}$$

This step relies mainly on Lemma 5.2.2. Let us fix $x \in E$ and $0 < r \leq r_s(x)$. Let f be a sliding deformation of E in $U = B(x,r)$. Let $\varepsilon > 0$. Let K be a compact subset of $E \cap W_f$ such that $\mu(W_f \setminus K) \leq \varepsilon$. There exists $\delta > 0$ such that $|f - \mathrm{id}| > \delta$ on K. According to Lemma 3.3.2, there exists a global sliding deformation f_1 in U such that $|f_1 - f| \leq \frac{\delta}{2}$, $E \cap W_{f_1} \subset\subset W_f$ and

$$H^d(f_1(W_f) \setminus f(W_f)) \leq \varepsilon. \tag{5.264}$$

In particular, $K \subset W_{f_1}$ because $|f - f_1| < |f - \mathrm{id}|$ on K. We deduce that

$$H^d(E \cap W_f \setminus W_{f_1}) \leq \varepsilon. \tag{5.265}$$

We are going to apply Lemma 5.2.2. There exists a global sliding deformation g in U (whose Lipschitz constant does not depend on ε) and an open set $V \subset W_{f_1}$ such that $W_g \subset W_{f_1}$, $|g - f_1| \leq \frac{\delta}{2}$ and

$$H^d(E \cap W_{f_1} \setminus V) \leq \varepsilon \tag{5.266a}$$

$$H^d(g(V)) \leq H^d(f_1(E \cap W_{f_1})) + \varepsilon. \tag{5.266b}$$

Let us draw a few consequences. It is straightforward that $E \cap \overline{W_g} \subset W_f$. Moreover, $K \subset W_g$ because $|g - f| < |f - \mathrm{id}|$ on K. The conditions (5.265), (5.266a) imply

$$\mu(W_f \setminus V) \leq \mu(W_f \setminus W_{f_1}) + \mu(W_{f_1} \setminus V) \tag{5.267}$$

$$\leq C\varepsilon. \tag{5.268}$$

and the conditions (5.264), (5.266b) imply

$$H^d(g(V)) \leq H^d(f_1(E \cap W_{f_1})) + \varepsilon \tag{5.269}$$

$$\leq H^d(f_1(W_f)) + \varepsilon \tag{5.270}$$

$$\leq H^d(f(W_f)) + 2\varepsilon. \tag{5.271}$$

Now, we apply the quasiminimality of E_i with respect to g in U,

$$H^d(E_i \cap W_g) \leq \kappa H^d(g(E_i \cap W_g)) + hH^d(E_i \cap hU) + \varepsilon_i. \tag{5.272}$$

By construction,

$$H^d(g(E_i \cap W_g)) \leq H^d(g(V)) + H^d(g(E_i \cap W_g \setminus V)) \tag{5.273}$$

$$\leq H^d(f(W_f)) + \|g\| H^d(E_i \cap W_g \setminus V) + 2\varepsilon \tag{5.274}$$

where $\|g\|$ is the Lipschitz constant of g. Passing to the limit in i, we have

$$\limsup_i H^d(g(E_i \cap W_g)) \leq H^d(f(W_f)) + \|g\| \mu(\overline{W_g} \setminus V) + 2\varepsilon \tag{5.275}$$

$$\leq H^d(f(W_f)) + \|g\| \mu(W_f \setminus V) + 2\varepsilon \tag{5.276}$$

$$\leq H^d(f(W_f)) + C\|g\|\varepsilon + 2\varepsilon. \tag{5.277}$$

On the other hand, we have

$$\liminf_i \mu(E_i \cap W_g) \geq \mu(W_g) \geq \mu(K). \tag{5.278}$$

We conclude that

$$\mu(W_f) \leq \mu(K) + \varepsilon \tag{5.279}$$

$$\leq \kappa H^d(f(W_f)) + h\mu(h\overline{U}) + \kappa(C\|g\|\varepsilon + 2\varepsilon). \tag{5.280}$$

The Lipschitz constant of g does not depend on ε so we can make $\varepsilon \to 0$ to obtain

$$\mu(W_f) \leq \kappa H^d(f(W_f)) + h\mu(h\overline{U}). \tag{5.281}$$

To replace $\overline{U}$ by U in the lower order term, we can apply the previous reasoning in a slightly smaller ball U' where f is still a sliding deformation.

Step 4. We show that

$$\mu \leq \kappa_0 H^d \llcorner E \tag{5.282}$$

This is quite the same reasoning. We fix $x \in E$, $0 < r \leq r_s(x)$. We consider $f = \mathrm{id}$ and $U = B(x,r)$. Let $\varepsilon > 0$. According to Lemma 5.2.2, there exists a global sliding deformation g in U (whose Lipschitz constant does not depend on ε) and an open set $V \subset U$ such that

$$H^d(E \cap U \setminus V) \leq \varepsilon \tag{5.283a}$$

$$H^d(g(V)) \leq H^d(E \cap U) + \varepsilon. \tag{5.283b}$$

Now, we apply the quasiminimality of E_i with respect to g in U,

$$H^d(E_i \cap U) \leq \kappa_0 H^d(g(E_i \cap U)) + \varepsilon_i. \tag{5.284}$$

By construction,

$$H^d(g(E_i \cap U)) \leq H^d(g(V)) + H^d(g(E_i \cap U \setminus V)) \tag{5.285}$$

$$\leq H^d(E \cap U) + \|g\| H^d(E_i \cap U \setminus V) + \varepsilon \tag{5.286}$$

where $\|g\|$ is the Lipschitz constant of g. Passing to the limit in i, we have

$$\limsup_i H^d(g(E_i \cap U)) \leq H^d(E \cap U) + \|g\| \mu(\overline{U} \setminus V) + \varepsilon \tag{5.287}$$

$$\leq H^d(E \cap U) + \|g\| \mu(\partial U) + C\|g\|\varepsilon + \varepsilon. \tag{5.288}$$

On the other hand, we have

$$\liminf_i \mu(E_i \cap U) \geq \mu(U). \tag{5.289}$$

We conclude that

$$\mu(U) \leq \kappa_0 H^d(E \cap U) + \kappa_0(\|g\| \mu(\partial U) + C\|g\|\varepsilon + \varepsilon). \tag{5.290}$$

The Lipschitz constant of g does not depend on ε so we can make $\varepsilon \to 0$ to obtain

$$\mu(U) \leq \kappa_0 H^d(E \cap U) + \kappa_0 \|g\| \mu(\partial U). \tag{5.291}$$

To get rid of $\mu(\partial U)$, we can apply the previous reasoning in slightly smaller balls U' where f is still a sliding deformation and such that $\mu(\partial U') = 0$.

Step 5. We show that

$$H^d \llcorner E \leq \mu. \tag{5.292}$$

We take the convention that the H^d measure of a d-dimensional disk of radius $r \geq 0$ equals $(2r)^d$. According to the density Theorem ([Mat], Theorem 6.9), it suffices to prove that for H^d-almost all $x \in E$,

$$\liminf_{r} (2r)^{-d} \mu(B(x,r)) \geq 1. \tag{5.293}$$

The proof relies on an argument introduced by Fang ([Fn]). It is an alternative to the concentration Lemma of Dal Maso, Morel and Solimini and relies on the fact that E is H^d-rectifiable. First of all, we define the "good" points $x \in X$ we want to work with. As μ is a rectifiable measure, for μ-ae. $x \in X$, there exists a d-plane V passing through x and a positive constant $\theta > 0$ such that

$$r^{-d} \mu_{x,r} \rightharpoonup \theta H^d \llcorner V, \tag{5.294}$$

where $\mu_{x,r} \colon A \mapsto \mu(x + r(A - x))$ and the arrow $\rightharpoonup$ denotes the weak convergence of Radon measures as $r \to 0$. Moreover, for μ-ae. $x \in \Gamma$,

$$\lim_{r \to 0} r^{-d} \mu(B(x,r) \setminus \Gamma) = 0. \tag{5.295}$$

This can be justified by applying ([Mat], Theorem 6.2) to the set $E \setminus \Gamma$. Fix $x \in \Gamma$ satisfying (5.294) and (5.295). We prove that for all $0 < \varepsilon \leq 1$, there exists $r > 0$ such that

$$V \cap B(x,r) \subset \{\, y \in X \mid \mathrm{d}(y,\Gamma) \leq \varepsilon |y - x| \,\}. \tag{5.296}$$

If the statement is not true, there exists a sequence $y_k \in V$ such that $y_k \to x$ and $\mathrm{d}(y_k, \Gamma) > \varepsilon |y_k - x|$. As $x \in \Gamma$ and $y_k \notin \Gamma$, we necessarily have $y_k \neq x$. Let $r_k = |y_k - x|$ and $\hat{y_k} = x + r_k^{-1}(y_k - x)$. The point $\hat{y_k}$ belongs to $V \cap S(x,1)$ where $S(x,1)$ is the unit sphere centred at x. By compactness, we can assume that $\hat{y_k} \to \hat{z} \in V \cap S(x,1)$. Observe that by definition,

$$y_k = x + r_k(\hat{y_k} - x). \tag{5.297}$$

We consider the sequence (z_k) defined by

$$z_k = x + r_k(\hat{z} - x) \tag{5.298}$$

which has the advantage that, by the weak convergence (5.294),

$$\lim_k r_k^{-d} \mu(B(z_k, \tfrac{1}{2}\varepsilon r_k)) = \theta H^d(V \cap B(z, \tfrac{1}{2}\varepsilon)) > 0. \tag{5.299}$$

On the other hand, we have $|y_k - z_k| \leq r_k |\hat{y_k} - \hat{z}|$ so, whenever k is big enough such that $|\hat{y_k} - \hat{z}| \leq \frac{1}{2}\varepsilon$,

$$\mathrm{d}(z_k, \Gamma) \geq \mathrm{d}(y_k, \Gamma) - |z_k - y_k| \geq \tfrac{1}{2}\varepsilon r_k. \tag{5.300}$$

This means that the open ball $B(z_k, \frac{1}{2}\varepsilon r_k)$ is disjoint from Γ. Then, by (5.295),

$$\limsup_k r_k^{-d} \mu(B(z_k, \tfrac{1}{2}\varepsilon r_k)) \leq \limsup_k r_k^{-d} \mu(B(x, 2r_k) \setminus \Gamma) = 0. \tag{5.301}$$

This is a contradiction!

Now, we fix x which satisfies the weak convergence (5.294) and which satisfies either $x \notin \Gamma$, either $x \in \Gamma$ and (5.295). Our goal is to prove that $\theta \geq 1$. We proceed by contradiction and assume that $\theta < 1$. This assumptions will allow us build special sliding deformations in small balls centered at x. To simplify the notations, we assume that $x = 0$ and that

$$V = \{\, x \in \mathbf{R}^n \mid x_i = 0 \text{ for } i = d+1, \dots, n \,\}. \tag{5.302}$$

In particular, we identify V to $\mathbf{R}^d$. For $r > 0$, let $Q_r = [-r, r]^n$. We fix $0 < a \leq \frac{1}{3}$ (close to 0). We introduce the disk

$$D_r = (1 - 2a)[-r, r]^d, \tag{5.303}$$

we define layers surrounding this disk,

$$R_r = (1 - 2a)[-r, r]^d \times a[-r, r]^{n-d} \tag{5.304}$$

$$R'_r = (1 - a)[-r, r]^d \times 2a[-r, r]^{n-d} \tag{5.305}$$

$$R''_r = [-r, r]^d \times 3a[-r, r]^{n-d}. \tag{5.306}$$

and we define an area of Q_r away from V,

$$S_r = Q_r \setminus (V \times a]{-r}, r[^{n-d}). \tag{5.307}$$

We are going to project $E_i \cap R_r$ onto ∂D_r. The layers R_r, R'_r, R''_r will help us to make a transition between this projection and the identity outside Q_r. The set S_r is a transition area where we do not control well the image of E_i. To solve this problem, we are going to make a Federer-Fleming projection of $E_i \cap S_r$ into a $(d-1)$-skeleton so as to assume that $H^d(E_i \cap S_r) = 0$. This strategy is analogous to the proof of Theorem 1.3 in [DPDRG1] but we have to deal with the boundary.

We introduce a neighborhood of S_r,

$$O_r =]{-2r}, 2r[^n \setminus (V \times \tfrac{a}{2}[-r, r]^{n-d}). \tag{5.308}$$

The main property of O_r is that we can assume that E_i has a very small measure H^d in O_r. Let us precise this claim. Observe that by the weak convergence $H^d \llcorner E_i \rightharpoonup \mu$ in X, we have

$$\limsup_i H^d(E_i \cap O_r) \leq \mu(\overline{O_r}) \tag{5.309}$$

and by the weak convergence $r^{-d}\mu_{0,r} \rightharpoonup \theta H^d \llcorner V$,

$$\lim_r r^{-d}\mu(\overline{O_r}) = H^d \llcorner V(]-2,2[^n \setminus (V \times \tfrac{a}{2}[-1,1]^{n-d}) \tag{5.310}$$
$$= 0. \tag{5.311}$$

Now we fix $\varepsilon > 0$ (to be chosen later). For $r > 0$ small enough, $\mu(\overline{O_r}) < \varepsilon r^d$ and then for i big enough (depending on r),

$$H^d(E_i \cap \overline{O_r}) < \varepsilon r^d. \tag{5.312}$$

Then, we build a global sliding deformation ϕ in O_r such that

$$H^d(\phi(E_i \cap O_r)) \leq C H^d(E \cap O_r) \tag{5.313}$$
$$H^d(\phi(E_i \cap O_r) \cap S_r) = 0. \tag{5.314}$$

The idea consists in applying a Federer-Fleming projection to the set $E_i \cap S_r$. However, we work with its image $T^{-1}(E_i \cap S_r)$ because the rigid boundary is easier to deal with. Let $x' = T^{-1}(0)$, there exists a cell $A \in K$ such that $x' \in \mathrm{int}(A)$. By definition of a n-complex, there exists a n-chart (the image of a subcomplex of E_n by a similitude) which contains A and which is subordinated to K. We deduce that there exists a Cartesian frame $(O, (e_1, \dots, e_n))$ and a length $\ell > 0$ such that

$$A = O + \ell \sum_{i=1}^{m} [0,1] e_i, \text{ where } m = \dim(A), \tag{5.315}$$

and for all cells of the form

$$D = O + \ell \sum_{i=1}^{n} [0,\alpha_i] e_i, \text{ where } \alpha \in \{\, -1, 0, 1 \,\}^n, \tag{5.316}$$

if D contains A (that is, $\alpha_i = 1$ for $i = 1, \dots, m$), there exists $B \in K$ such that $\mathrm{int}(D) \subset \mathrm{int}(B)$. Fix $q \in \mathbf{N}^*$ and let M_r be the complex composed of all the cells of the form

$$O + \sum_{i=1}^{n} [p_i, p_i + (2^{-q} r)\alpha_i] e_i, \tag{5.317}$$

where $p \in (2^{-q} r)\mathbf{Z}^n$ and $\alpha \in \{\, -1, 0, 1 \,\}^n$. This is a uniform grid of sidelength $2^{-q} r$ as in Example 4.1.1 of Chapter 4. Then, we consider the subcomplex

$$N_r = \{\, D \in M_r \mid D \cap T^{-1}(S_r) \neq \emptyset \,\}. \tag{5.318}$$

It is clear that $T^{-1}(S_r) \subset \mathrm{int}(N_r)$. Moreover, we are going to see if q is big enough (depending on n, κ, Γ, a),

$$|N_r| \subset T^{-1}(O_r). \tag{5.319}$$

By definition of a cell $D \in N_r$, the image $T(D)$ meets S_r. As O_r contains a $(C^{-1}ar)$-neighborhood of S_r and $\operatorname{diam}(D) \leq C2^{-q}r$, we deduce that if q is big enough (depending on n, κ, Γ, a but not r, i), $T(D) \subset O_r$. Next, we justify that $N_r \preceq K$ for r small enough. This is an important property to ensure that a Federer-Fleming projection in N_r preserves the cells of K. First, observe that the set

$$O + \ell \sum_{i \leq m}]0,1[e_i + \ell \sum_{i > m}]-1,1[e_i \tag{5.320}$$

is a neighborhood of $\operatorname{in}(A)$ and in particular of $x' = T^{-1}(0)$. We assume $r > 0$ so that $T^{-1}(2Q_r)$ is included in this neighborhood. Let D be a cell of N_r. It can be written

$$D = O + \sum_{i=1}^{n} [p_i, p_i + (2^{-q}r)\alpha_i]e_i, \tag{5.321}$$

where $p \in (2^{-q}r)\mathbf{Z}^n$ and $\alpha \in \{-1,0,1\}^n$. Since $|D| \subset T^{-1}(O_r) \subset T^{-1}(2Q_r)$, we have in particular

$$D \subset O + \ell \sum_{i \leq m}]0,1[e_i + \ell \sum_{i > m}]-1,1[e_i. \tag{5.322}$$

By projection on the coordinate axis, it follows that for $i = 1, \ldots, m$,

$$[p_i, p_i + (2^{-q}r)\alpha_i] \subset \ell]0,1[\tag{5.323}$$

and for $i = m+1, \ldots, n$,

$$[p_i, p_i + (2^{-q}r)\alpha_i] \subset \ell]-1,1[. \tag{5.324}$$

As $p_i \in (2^{-q}r)\mathbf{Z}$ and $\alpha_i \in \{-1,0,1\}$, we deduce that for $i = m+1, \ldots, n$, there exists $\beta_i \in \{-1,0,1\}$ such that

$$(p_i, p_i + (2^{-q}r)\alpha_i) \subset \ell(0, \beta_i) \tag{5.325}$$

where (x,y) means $]x,y[$ if $x \neq y$ and $\{x\}$ if $x = y$. In summary, $\operatorname{int}(D) \subset \operatorname{int}(D')$ where

$$D' = O + \ell \sum_{i \leq m} [0,1]e_i + \ell \sum_{i > m} [0,\beta_i]e_i. \tag{5.326}$$

By definition of the Cartesian frame $(O,(e_1,\ldots,e_n))$ and the sidelength ℓ, there exists $B \in K$ such that $\operatorname{int}(D') \subset \operatorname{int}(B)$. In particular, $\operatorname{int}(D) \subset \operatorname{int}(B)$. This proves that $N_r \preceq K$.

Let ϕ' be a Federer-Fleming projection of $T^{-1}(E_i) \cap \operatorname{int}(N_r)$ in N_r. Thus, $\phi' \colon |N_r| \to |N_r|$ is a Lipschitz map such that

1. for all $D \in N_r$, $\phi'(D) \subset D$;

2. $\phi' = \mathrm{id}$ in $|N_r| \setminus \mathrm{int}(N_r)$;

3. for all $D \in N_r$,

$$H^d(\phi'(T^{-1}(E_i) \cap \mathrm{int}(D))) \leq CH^d(T^{-1}(E_i) \cap \mathrm{int}(D)). \tag{5.327}$$

By (5.327), we have

$$H^d(\phi'(T^{-1}(E_i) \cap \mathrm{int}(N_r))) \leq CH^d(T^{-1}(E_i) \cap \mathrm{int}(N_r)). \tag{5.328}$$

By construction, $|N_r| \subset T^{-1}(O_r)$ so the previous ineqality gives

$$\begin{aligned} H^d(\phi'(T^{-1}(E_i) \cap \mathrm{int}(N_r))) &\leq CH^d(T^{-1}(E_i) \cap T^{-1}(O_r)) && (5.329)\\ &\leq CH^d(E_i \cap O_r). && (5.330) \end{aligned}$$

We extend ϕ' over $\mathbf{R}^n$ by $\phi' = \mathrm{id}$ in $\mathbf{R}^n \setminus \mathrm{int}(N_r)$ (this extension is still Lipschitz, as we have check at the beginning of the proof of Lemma 5.1.1). It follows that

$$\begin{aligned} &H^d(\phi'(T^{-1}(E_i) \cap T^{-1}(O_r)))\\ &\leq H^d(\phi'(T^{-1}(E_i) \cap T^{-1}(O_r) \cap \mathrm{int}(N_r))) && (5.331)\\ &\quad + H^d(\phi'(T^{-1}(E_i) \cap T^{-1}(O_r) \setminus \mathrm{int}(N_r)))\\ &\leq H^d(\phi'(T^{-1}(E_i) \cap \mathrm{int}(N_r))) + H^d(T^{-1}(E_i) \cap T^{-1}(O_r)) && (5.332)\\ &\leq CH^d(E_i \cap O_r). && (5.333) \end{aligned}$$

As $\phi'(T^{-1}(E_i)) \cap \mathrm{int}(N_r) \subset \phi'(T^{-1}(E_i) \cap \mathrm{int}(N_r))$, we have also

$$H^d(\phi'(T^{-1}(E_i)) \cap \mathrm{int}(N_r)) \leq CH^d(E_i \cap O_r). \tag{5.334}$$

We recall that $H^d(E_i \cap O_r) \leq C\varepsilon r^d$. The cells of N_r have a sidelength of $2^{-q}r$ so if $\varepsilon > 0$ is chosen small enough (depending on n, κ, Γ, a but not r, i), we can make additional projections in the d-dimensional cells $D \in N_r^d$ and obtain that

$$H^d(\phi'(T^{-1}(E_i)) \cap \mathrm{int}(N_r)) = 0; \tag{5.335}$$

in particular

$$H^d(\phi'(T^{-1}(E_i)) \cap T^{-1}(S_r)) = 0. \tag{5.336}$$

As $N_r \preceq K$, ϕ' preserves each cell of K and, by convexity, the homotopy $t \mapsto t\phi' + (1-t)\mathrm{id}$ also preserves the cells of K. We conclude that ϕ' is a global sliding deformation along $T^{-1}(\Gamma)$ in $T^{-1}(O_r)$. Thus, $\phi = T \circ \phi' \circ T^{-1}$ is a global sliding deformation along Γ in O_r. By (5.333) and (5.336),

$$\begin{aligned} &H^d(\phi(E_i \cap O_r)) \leq CH^d(E_i \cap O_r) && (5.337)\\ &H^d(\phi(E_i) \cap S_r) = 0. && (5.338) \end{aligned}$$

In the next part, we work with the set $\phi(E_i)$ instead of E_i.

We are going to squash the points of $\phi(E_i) \cap R_r$ to ∂D_r. Let π be the orthogonal projection from $\mathbf{R}^n$ onto D_r. We want to postcompose π with a radial projection onto ∂D_r. We prove that there exists a center of projection $c_i \in \mathrm{int}(D_r) \setminus \pi(\phi(E_i) \cap R_r)$ by comparing the measure H^d of $\phi(E_i) \cap R_r$ to the measure of D_r. As $\phi = \mathrm{id}$ outside O_r and by (5.337), we have

$$\begin{aligned} H^d(\phi(E_i) \cap R_r) &\leq H^d(\phi(E_i \cap O_r) \cap R_r) + H^d(\phi(E_i \setminus O_r) \cap R_r) && (5.339)\\ &\leq H^d(\phi(E_i \cap O_r)) + H^d(E_i \cap R_r) && (5.340)\\ &\leq CH^d(E_i \cap O_r) + H^d(E_i \cap R_r). && (5.341)\end{aligned}$$

Moreover,

$$\begin{aligned} \limsup_r \limsup_i r^{-d} H^d(E_i \cap R_r) &\leq \lim_r r^{-d}\mu(R_r) && (5.342)\\ &\leq \theta H^d([-1,1]^d) && (5.343)\end{aligned}$$

and

$$\lim_r \limsup_i r^{-d} H^d(E_i \cap O_r) = 0 \qquad (5.344)$$

We deduce that if r is big enough and i big enough (depending on r),

$$H^d(\phi(E_i) \cap R_r) < \tfrac{\theta+1}{2} H^d([-r,r]^d). \qquad (5.345)$$

Finally, we assume a small enough (depending on θ) such that $\frac{\theta+1}{2} < (1 - 2a)^d$. Thus,

$$\begin{aligned} H^d(\phi(E_i) \cap R_r) &< \tfrac{\theta+1}{2} H^d([-r,r]^d) && (5.346)\\ &< (1-2a)^d H^d([-r,r]^d) && (5.347)\\ &< H^d(D_r). && (5.348)\end{aligned}$$

The map π is 1-Lipschitz so we have also $H^d(\pi(\phi(E_i) \cap R_r)) < H^d(D_r)$. We conclude that there exists a point $c_i \in \mathrm{int}(D_r)$ such that $c_i \notin \pi(\phi(E_i) \cap R_r)$. As $\phi(E_i) \cap R_r$ is compact, there also exists a small radius $r_i > 0$ such that

$$B(c_i, r_i) \subset \mathrm{int}(D_r) \setminus \pi(\phi(E_i) \cap R_r). \qquad (5.349)$$

Let ψ be the radial projection from $D_r \setminus B(c_i, r_i)$ onto ∂D_r centered at c_i. We extend ψ as a Lipschitz function $\psi \colon D_r \to D_r$. Let $\varphi \colon \mathbf{R}^n \to [0,1]$ be a $C(ar)^{-1}$-Lipschitz function such that $\varphi = 1$ on R_r and $\varphi = 0$ on $\mathbf{R}^n \setminus R'_r$. Then we define

$$f = \varphi(\psi \circ \pi) + (1 - \varphi)\mathrm{id}. \qquad (5.350)$$

Observe that $f = \psi \circ \pi$ in R_r and $f = \mathrm{id}$ in $\mathbf{R}^n \setminus R'_r$. We have

$$f(R'_r) \subset R'_r \qquad (5.351)$$

because $\psi \circ \pi$ takes its values in $D_r \subset R'_r$ and because R'_r is convex. This also implies that $f(Q_r) \subset Q_r$. The map f is Lipschitz on $\mathbf{R}^n$ because

$$f - \mathrm{id} = \varphi \cdot (\psi \circ \pi - \mathrm{id}), \tag{5.352}$$

φ is Lipschitz with compact support and $(\psi \circ \pi - \mathrm{id})$ is Lipschitz. We do not have a good control over this Lipschitz constant but we can show that f is C-Lipschitz on $Q_r \setminus (R_r \cup S_r)$. On this set, the values of π belongs to ∂D_r so

$$f - \mathrm{id} = \varphi \cdot (\psi \circ \pi - \mathrm{id}) \tag{5.353}$$

$$= \varphi \cdot (\pi - \mathrm{id}). \tag{5.354}$$

On the one hand, φ is $C(ar)^{-1}$-Lipschitz and $|\varphi| \leq 1$. On the other hand, $(\pi - \mathrm{id})$ is 2-Lipschitz and $|\pi - \mathrm{id}| \leq Car$ on $Q_r \setminus (R_r \cup S_r)$.

Next, we want to postcompose f with a retraction onto the boundary to obtain a sliding deformation. If $0 \notin \Gamma$, then for $r > 0$ small enough, Q_r is disjoint from Γ and there is nothing to do. We assume that $0 \in \Gamma$ and we define our retraction. There exists an open set $O \subset X$ containing Γ and a C-Lipschitz map $r\colon O \to \Gamma$ such that $r = \mathrm{id}$ on Γ. We are going to see that if r is small enough, then $|r - \mathrm{id}| \leq Car$ in R'_r. By (5.296), we can assume that $r > 0$ is small enough so that

$$[-r, r]^d \subset \{\, y \in X \mid \mathrm{d}(y, \Gamma) \leq ar \,\}. \tag{5.355}$$

Let $y \in R'_r$ and let u be the orthogonal projection of y onto V. Thus, $u \in [-r, r]^d$ and $|y - u| \leq Car$. By (5.355), $\mathrm{d}(u, \Gamma) \leq ar$ so there exists $v \in \Gamma$ such that $|y - v| \leq Car$. Then,

$$|(r - \mathrm{id})(y)| = |(r - \mathrm{id})(y) - (r - \mathrm{id})(v)| \tag{5.356}$$

$$\leq C|y - v| \tag{5.357}$$

$$\leq Car. \tag{5.358}$$

We restrict r to $\Gamma \cup R'_r$ and then extend this restriction as a C-Lipschitz map $r\colon X \to \mathbf{R}^n$ such that $|r - \mathrm{id}| \leq Car$. We still have $r(R'_r) \subset \Gamma$ and $r = \mathrm{id}$ on Γ. Let $\varphi'\colon X \to [0, 1]$ be a $C(ar)^{-1}$-Lipschitz function such that $\varphi' = 1$ on R'_r and $\varphi = 0$ on $X \setminus R''_r$. We finally define

$$p = \varphi' r + (1 - \varphi')\mathrm{id}. \tag{5.359}$$

Observe that $p = r$ in R'_r and $p = \mathrm{id}$ in $X \setminus R''_r$ and $p = \mathrm{id}$ on Γ because $r = \mathrm{id}$ on Γ. The map p is C-Lipschitz on X because

$$p - \mathrm{id} = \varphi'(r - \mathrm{id}) \tag{5.360}$$

and φ' is $C(ar)^{-1}$-Lipschitz, $|\varphi'| \leq 1$, $r - \mathrm{id}$ is C-Lipschitz, $|r - \mathrm{id}| \leq Car$. As $|p - \mathrm{id}| \leq Car$, we can assume a small enough so that

$$p(Q_r) \subset \mathrm{int}(2Q_r). \tag{5.361}$$

Now, we consider $g = p \circ f$. It is clear that $g(Q_r) \subset \mathrm{int}(2Q_r)$, $g = \mathrm{id}$ in $\mathbf{R}^n \setminus \mathrm{int}(Q_r)$. We have $g(\Gamma) \subset \Gamma$ because $f(R'_r) \subset R'_r$ and $p(R'_r) \subset \Gamma$ and $f = \mathrm{id}$ outside R'_r. Finally, we justify that the map g is a global sliding deformation in $\mathrm{int}(2Q_r)$. We introduce

$$G = \begin{cases} \mathrm{id} & \text{in } (\{\,0\,\} \times \mathbf{R}^n) \cup (I \times (\mathbf{R}^n \setminus \mathrm{int}(Q_r))) \\ p \circ g_t & \text{in } I \times \Gamma \\ g & \text{in } \{\,1\,\} \times \mathbf{R}^n, \end{cases} \tag{5.362}$$

where $g_t = (1-t)\mathrm{id} + tg$. We see as previously that $G_t(\Gamma) \subset \Gamma$. The map G is continuous as a pasting of continuous maps in closed domains of X. According to the Tietze extension theorem, it can be extend as a continuous map $G\colon I \times \mathbf{R}^n \to \mathbf{R}^n$ such that $G_t(Q_r) \subset \mathrm{int}(2Q_r)$. Thus, g is a sliding deformation in $\mathrm{int}(2Q_r)$.

Both g and ϕ are sliding deformations in $\mathrm{int}(2Q_r)$. We apply the quasiminimality of E_i with respect to $g\phi$ in $\mathrm{int}(2Q_r)$. We assume h small enough (depending only on n) such that $hH^d(E_i \cap hB(0, 2\sqrt{n}r)) \leq \frac{1}{2}H^d(E_i \cap \mathrm{int}(Q_r))$. As $g\phi = \mathrm{id}$ outside $\mathrm{int}(Q_r) \cup O_r$, the quasiminimality gives

$$H^d(E_i \cap (\mathrm{int}(Q_r) \cup O_r)) \leq CH^d(g\phi[E_i \cap (\mathrm{int}(Q_r) \cup O_r)]) + \varepsilon_i. \tag{5.363}$$

Since $g = p \circ f$ and p is C-Lipschitz, we have

$$H^d(g\phi[E_i \cap (\mathrm{int}(Q_r) \cup O_r)]) \leq CH^d(f\phi[E_i \cap (\mathrm{int}(Q_r) \cup O_r)]). \tag{5.364}$$

By the fact that $f = \mathrm{id}$ outside Q_r, that $\phi = \mathrm{id}$ outside O_r and by (5.337), we estimate

$$\begin{aligned}
&H^d(f\phi[E_i \cap (\mathrm{int}(Q_r) \cup O_r)]) \\
&\leq H^d(f(\phi(E_i) \cap Q_r)) + H^d(f(\phi[E_i \cap (\mathrm{int}(Q_r) \cup O_r)] \setminus Q_r)) && (5.365)\\
&\leq H^d(f(\phi(E_i) \cap Q_r)) + H^d(\phi[E_i \cap (\mathrm{int}(Q_r) \cup O_r)] \setminus Q_r) \\
&\leq H^d(f(\phi(E_i) \cap Q_r)) + H^d(\phi(E_i \cap O_r)) && (5.366)\\
&\leq H^d(f(\phi(E_i) \cap Q_r)) + CH^d(E_i \cap O_r). && (5.367)
\end{aligned}$$

We decompose the set $\phi(E_i) \cap Q_r$ in three parts by taking its intersection with R_r, S_r and $Q_r \setminus (R_r \cup S_r)$. First, $f(\phi(E_i) \cap R_r) \subset \partial D_r$ so

$$H^d(f(\phi(E_i) \cap R_r)) = 0. \tag{5.368}$$

Next, by (5.338),

$$H^d(f(\phi(E_i) \cap S_r)) = 0. \tag{5.369}$$

Finally, f is C-Lipschitz on $Q_r \setminus (R_r \cup S_r)$ so

$$\begin{aligned}
&H^d(f(\phi(E_i) \cap Q_r \setminus (R_r \cup S_r))) \\
&\leq CH^d(\phi(E_i) \cap Q_r \setminus (R_r \cup S_r)) && (5.370)\\
&\leq CH^d(E_i \cap O_r) + CH^d(E_i \cap Q_r \setminus (R_r \cup S_r)). && (5.371)
\end{aligned}$$

In sum,

$$H^d(E_i \cap \operatorname{int}(Q_r)) \leq CH^d(E_i \cap Q_r \setminus (R_r \cup S_r)) \\ + CH^d(E_i \cap O_r) + \varepsilon_i. \quad (5.372)$$

Then we pass to the limit $i \to +\infty$ and obtain

$$\mu(\operatorname{int}(Q_r)) \leq C\mu(Q_r \setminus (R_r \cup S_r)) + C\mu(\overline{O_r}). \quad (5.373)$$

We multiply both sides by r^{-d} and we pass to the limit $r \to 0$. In particular, $\limsup_r r^{-d}\mu(\overline{O_r}) = 0$ so

$$H^d([-1,1]^d) \leq CH^d([-1,1]^d \setminus (1-2a)[-1,1]^d). \quad (5.374)$$

This is true for all a small enough so we can make $a \to 0$ and obtain

$$H^d([-1,1]^d) = 0. \quad (5.375)$$

Contradiction! □

Chapter 6

Direct method

6.1 Scheme

We use Theorem 5.2.1 to derive a scheme of direct method. This is the same strategy as [DLGM], [DPDRG1] but we can minimize the competitors on the boundary. Our working space is an open set X of $\mathbf{R}^n$.

Corollary 6.1.1 (Direct method)**.** *Fix a Lipschitz subset Γ of X which is H^d regular. Let $\mathcal{C}$ be a class of closed subsets of X such that*

$$m = \inf \{ H^d(E) \mid E \in \mathcal{C} \} < \infty \tag{6.1}$$

and assume that for all $E \in \mathcal{C}$, for all sliding deformations f of E in X,

$$m \leq H^d(f(E)). \tag{6.2}$$

Let (E_k) be a minimizing sequence for H^d in $\mathcal{C}$. Up to a subsequence, there exists a coral[1] *minimal set E_∞ in X such that*

$$H^d \llcorner E_k \rightharpoonup H^d \llcorner E_\infty. \tag{6.3}$$

where the arrow $\rightharpoonup$ denotes the weak convergence of Radon measures in X. In particular, $H^d(E_\infty) \leq m$.

Remark 6.1.1. The limit E_∞ may not belong to $\mathcal{C}$.

Proof. Let (E_k) be a minimizing sequence. Then $(H^d \llcorner E_k)$ is a bounded sequence of Radon measure in X and, up to a subsequence, there exists a Radon measure μ in X such that

$$H^d \llcorner E_k \rightharpoonup \mu. \tag{6.4}$$

[1] A set $E \subset X$ is coral in X if E is the support of $H^d \llcorner E$ in X. Equivalently, E is closed in X and for all $x \in E$ and for all $r > 0$, $H^d(E \cap B(x,r)) > 0$.

For all k, for all global sliding deformation f in X,

$$H^d(E_k) \leq m + o(1) \tag{6.5}$$
$$\leq H^d(f(E_k)) + o(1) \tag{6.6}$$
$$\leq H^d(f(E_k \cap W_f)) + H^d(E_k \setminus W_f) + o(1) \tag{6.7}$$

whence

$$H^d(E_k \cap W_f) \leq H^d(f(E_k \cap W_f)) + o(1). \tag{6.8}$$

According to Theorem 5.2.1, we have $\mu = H^d \llcorner E$, where E is the support of μ. Moreover, for all sliding deformation f in X,

$$H^d(E \cap W_f) \leq H^d(f(E \cap W_f)). \tag{6.9}$$

$\square$

6.2 Application to the Reifenberg problem

6.2.1 Reifenberg competitors

A Reifenberg competitor is a set which cancels the holes of the boundary in the language of algebraic topology. The goal of this section is to define Reifenberg competitors and to prove a nice limiting theorem: the class of Reifenberg competitors is closed under weak limits. *Given a topological space X and an integer k , $H_k(X;G)$ is the k^{th} Čech homology group of X over an Abelian coefficient group G. We abbreviate this notation as $H_k(X)$ since the coefficient group is not significant for us. Throughout this section, we fix a closed set Γ of $\mathbf{R}^n$ and a subgroup L of $H_{d-1}(\Gamma)$.*

Definition 6.2.1 (Reifenberg competitor)**.** A Reifenberg competitor is a compact subset $E \subset \mathbf{R}^n$ such that the morphism induced by inclusion,

$$H_{d-1}(\Gamma) \longrightarrow H_{d-1}(E \cup \Gamma),$$

is zero on L.

This definition is different from the original definition of Reifenberg because the competitor E may not contain Γ. In the remainder of this paragraph, we compare this definition to the definition of Nakauchi ([Na]). Let E be a compact subset of $\mathbf{R}^n$ and consider the following commutative diagram induced by the inclusions:

$$\begin{tikzcd} & H_{d-1}(\Gamma) \arrow[dr, "j_*"] & \\ H_{d-1}(E \cap \Gamma) \arrow[ur, "i_*"] \arrow[dr, "i'_*"'] & & H_{d-1}(E \cup \Gamma). \\ & H_{d-1}(E) \arrow[ur, "j'_*"'] & \end{tikzcd}$$

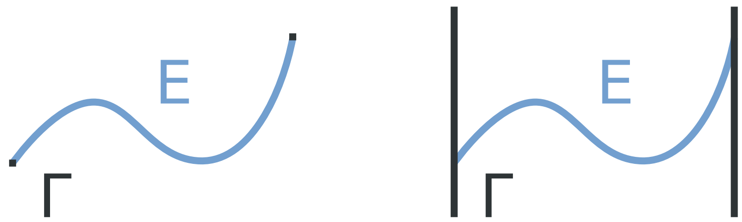

Figure 6.1: Two types of Plateau problem. The competitors have a fixed boundary on the left and a free boundary on the right.

The set E is a Nakauchi competitor provided that for all $v \in L$, there exists $u \in H_{d-1}(E \cap \Gamma)$ such that $i_*(u) = v$ and $i'_*(u) = 0$. Assuming that the Mayer Vietoris sequence holds for the sets Γ, E in $E \cup \Gamma$, the following sequence is exact:

$$H_{d-1}(E \cap \Gamma) \xrightarrow{(i_*, i'_*)} H_{d-1}(\Gamma) \otimes H_{d-1}(E) \xrightarrow{j_* - j'_*} H_{d-1}(E \cup \Gamma).$$

Observe that E satisfies Definition 6.2.1 if and only if all elements of the form $(v, 0) \in L \otimes H_{d-1}(E)$ are in the kernel of $j_* - j'_*$. And E is a Nakauchi competitor if and only if all elements of the form $(v, 0) \in L \otimes H_{d-1}(E)$ are in the image of (i_*, i'_*). Thus, the Mayer Vietoris sequence implies that Definition 6.2.1 is equivalent to the definition of Nakauchi. In that sense, we consider these definitions to be essentially equivalent. We favor Definition 6.2.1 because we are able to prove that it is stable under weak limits (see Lemma 6.2.3).

6.2.2 Operations on the competitors

We present three operations that preserve the Reifenberg competitors: supsets, continuous image by sliding deformations and weak limits.

Lemma 6.2.1. *Let E be a Reifenberg competitor. Let F be a compact subset of $\mathbf{R}^n$ containing E. Then F is a Reifenberg competitor.*

Proof. This follows from the following commutative diagram

$$\begin{tikzcd} H_{d-1}(\Gamma) \arrow[r] \arrow[dr] & H_{d-1}(E \cup \Gamma) \arrow[d] \\ & H_{d-1}(F \cup \Gamma) \end{tikzcd}$$

where the arrows are the morphisms induced by inclusion. □

Lemma 6.2.2. *Let E be a Reifenberg competitor. Let $f\colon E \cup \Gamma \to \mathbf{R}^n$ be a continuous map such that there exists a continuous map $F\colon I \times \Gamma \to \Gamma$ satisfying $F_0 = \mathrm{id}$ and $F_1 = f$. Then $f(E)$ is a Reifenberg competitor.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} H_{d-1}(\Gamma) & \longrightarrow & H_{d-1}(E \cup \Gamma) \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\ H_{d-1}(\Gamma) & \longrightarrow & H_{d-1}(f(E) \cup \Gamma) \end{array}$$

where the unlabeled arrows are the morphisms induced by inclusion. As $f\colon \Gamma \to \Gamma$ is homotopic to id, $f_* = \mathrm{id}$ on $H_{d-1}(\Gamma)$. □

The lemma assumed f to be defined on $E \cup \Gamma$ but the image $f(E)$ depends only on the values of f on E. In the two following remarks, we are going to see that it is generally enough for f to be defined on E. In particular, the second remark applies to sliding deformations.

Remark 6.2.1. Let $f\colon E \to \mathbf{R}^n$ be a continuous map such that $f = \mathrm{id}$ on $E \cap \Gamma$. As E and Γ are closed sets of $\mathbf{R}^n$, the gluing

$$g = \begin{cases} f & \text{in } E \\ \mathrm{id} & \text{in } \Gamma \end{cases} \tag{6.10}$$

is continuous. Then $G_t = (1-t)\mathrm{id} + tg$ is a continuous homotopy from id to g and $G_t = \mathrm{id}$ on Γ. We deduce that $f(E)$ is a Reifenberg competitor.

Remark 6.2.2. Let $f\colon E \to \mathbf{R}^n$ be a continuous map such that there exists a continuous map $F\colon I \times (E \cap \Gamma) \to \Gamma$ satisfying $F_0 = \mathrm{id}$ and $F_1 = f$. Let us assume that Γ is a neighborhood retract i.e. there exists an open set $O \subset \mathbf{R}^n$ and a continuous map $r\colon O \to \Gamma$ such that $r = \mathrm{id}$ on Γ. According to the Homotopy Extension Lemma, F extends as a continuous map $F\colon I \times \Gamma \to \Gamma$. Moreover, the gluing

$$g = \begin{cases} f & \text{in } E \\ F_1 & \text{in } \Gamma \end{cases} \tag{6.11}$$

is continous because E and Γ are closed sets of $\mathbf{R}^n$. We deduce that $f(E)$ is a Reifenberg competitor.

We finally present our lemma about weak limits of Reifenberg competitors.

Lemma 6.2.3. *Let $(E_k) \subset \mathbf{R}^n$ be a sequence of Reifenberg competitors. Let E be a compact subset of $\mathbf{R}^n$. We assume that*

1. *there exists a compact set $C \subset \mathbf{R}^n$ such that for all k, $E_k \subset C$;*

2. *for all open sets V containing $E \cup \Gamma$,*

$$\lim_k H^d(E_k \setminus V) = 0. \tag{6.12}$$

Then E is a Reifenberg competitor.

The proof requires a preliminary lemma about the general position of spheres. For $x \in \mathbf{R}^n$ and $r > 0$, let $S(x, r)$ denote the Euclidean sphere of center x and radius r of $\mathbf{R}^n$. Given an integer k, a *k-sphere* is an Euclidean sphere of positive radius relative to a $(k+1)$-affine plane. We extend this definition to $k < 0$, by calling *k-sphere* the empty set.

Lemma 6.2.4. *Let S^k be a k-sphere in $\mathbf{R}^n$ and let x be a point in $\mathbf{R}^n$. Then for all $r > 0$ (except for at most one value), $S^k \cap S(x, r)$ is a subset of a $(k-1)$-sphere.*

Proof. We assume $k \geq 1$. The proof is based on the observation that the intersection of a sphere with a k-affine plane is either empty, a point, or a $(k-1)$-sphere. In all cases, this intersection is part of a $(k-1)$-sphere. Let P_0 be the $(k+1)$-affine plane associated to S^k, let $x_0 \in P_0$ be the center of S^k and $r_0 > 0$ be its radius. For $r > 0$, a point $y \in S^k \cap S(x, r)$ is characterized by the system

$$y \in P_0 \tag{6.13a}$$
$$|y - x| = r \tag{6.13b}$$
$$|y - x_0| = r_0 \tag{6.13c}$$

or equivalently

$$y \in P_0 \tag{6.14a}$$
$$|y - x| = r \tag{6.14b}$$
$$|y - x|^2 - |y - x_0|^2 = r^2 - r_0^2. \tag{6.14c}$$

Assume $x = x_0$. If $r \neq r_0$ (this removes one value of r), equation (6.14c) has no solutions. Then, $S^k \cap S(x, r)$ is empty and it is part of a $(k-1)$-sphere. Assume $x \neq x_0$. Equation (6.14c) defines an hyperplane and, if $|x - x_0|^2 \neq r^2 - r_0^2$ (this removes at most one value of r), this hyperplane does not contain x_0. Then, the intersection of the two planes (6.14a) and (6.14c) is included in a k-affine plane. The intersection of this plane with the sphere (6.14b) is part of a $(k-1)$-sphere as seen in introduction. □

Proof of Lemma 6.2.3. Observe that the sequence $(E_k \cup E)_k$ also satisfies the Lemma assumptions. So without loss of generality, we assume that for all k, $E \subset E_k$. We define a *general covering* as an open family $\gamma = (\gamma_j)_{j \in V_\gamma}$ of $\mathbf{R}^n$ satisfying the following properties:

1. there exists k such that $E_k \cup \Gamma \subset \bigcup_{j \in V_\gamma} \gamma_j$;
2. for every subset $S \subset V_\gamma$ of cardinal $d + 1$,

$$\bigcap_S \gamma_j \neq \emptyset \implies (E \cup \Gamma) \cap \bigcap_S \gamma_j \neq \emptyset. \tag{6.15}$$

The main goal of the proof is to show that for any open covering $\alpha = (\alpha_i)_i$ of $E \cup \Gamma$, there exists a general covering $\gamma = (\gamma_j)_{j \in V_\gamma}$ such that $((E \cup \Gamma) \cap \gamma_j)_j$ is a refinement of α. Let us explain how to conclude from there. A general covering γ induces simplicial complexes:

$$K(\Gamma) = \{ S \subset V_\gamma \text{ finite} \mid \Gamma \cap \bigcap_S \gamma_j \neq \emptyset \}, \tag{6.16}$$

$$K(E \cup \Gamma) = \{ S \subset V_\gamma \text{ finite} \mid (E \cup \Gamma) \cap \bigcap_S \gamma_j \neq \emptyset \}, \tag{6.17}$$

$$K(E_k \cup \Gamma) = \{ S \subset V_\gamma \text{ finite} \mid (E_k \cup \Gamma) \cap \bigcap_S \gamma_j \neq \emptyset \}. \tag{6.18}$$

The inclusions $K(\Gamma) \subset K(E \cup \Gamma) \subset K(E_k \cup \Gamma)$ induce morphisms i_* and j_*:

$$H_{d-1}(K(\Gamma)) \xrightarrow{i} H_{d-1}(K(E \cup \Gamma)) \xrightarrow{j} H_{d-1}(K(E_k \cup \Gamma)).$$

As E_k is a Reifenberg set, we have $j_* \circ i_* = 0$ on L. However, the second axiom of general coverings says that the simplicial complexes $K(E \cup \Gamma)$ and $K(E_k \cup \Gamma)$ have the same d-simplexes. Hence the d-chains of $K(E \cup \Gamma)$ and $K(E_k \cup \Gamma)$ are identical and they induce the same boundaries. We deduce that j_* is injective and then, $i_* = 0$ on L. Since every open covering α of $E \cup \Gamma$ is refined by such general covering γ, we conclude that the morphism induced by inclusion $H_{d-1}(\Gamma) \to H_{d-1}(E \cup \Gamma)$ is nul on L. *Step 1.* We fix a relative open covering $\alpha = (\alpha_i)_i$ of $E \cup \Gamma$ and we build a locally finite open sequence $\beta = (\beta_j)_{j \in \mathbf{N}}$ in $\mathbf{R}^n$ such that

1. β cover $E \cup \Gamma$ and $((E \cup \Gamma) \cap \beta_j)_j$ is a refinement of α;
2. for every finite subset $S \subset \mathbf{N}$, the intersection of boundaries $\bigcap_S \partial \beta_i$ is included in a finite union of $(n-m)$-spheres, where m is the cardinal of S;
3. for every finite subset $S \subset \mathbf{N}$,

$$\bigcap_S \beta_j \neq \emptyset \implies (E \cup \Gamma) \cap \bigcap_S \beta_j \neq \emptyset. \tag{6.19}$$

We work with the closed set $F := E \cup \Gamma$. For all $x \in F$, there exists i such that $x \in \alpha_i$ so there exists an open ball B centred at x such that $F \cap 2B \subset \alpha_i$. We extract a sequence of open ball $(B_j)_{j \in \mathbf{N}}$ covering F such that $(2B_j)_j$ is locally finite in $\mathbf{R}^n$ and $(F \cap 2B_j)_j$ is a refinement of α. Next, we build by induction an open sequence $(\beta_j)_{j \in \mathbf{N}}$ such that for all j,

1. $F \cap \overline{B_j} \subset \beta_j$ and there exists i such that $F \cap \beta_j \subset \alpha_i$.
2. for every subset $S \subset \{ 1, \dots, j \}$, the intersection of boundaries $\bigcap_S \partial \beta_i$ is included in a finite union of $(n-m)$-spheres, where m is the cardinal of S;

3. for every subset $S \subset \{1, \dots, j\}$,

$$\bigcap_S \beta_i \neq \emptyset \implies F \cap \bigcap_S \beta_i \neq \emptyset. \tag{6.20}$$

Assume that $\beta_0, \dots, \beta_{j-1}$ has been built and let us built β_j. For all $x \in F \cap \overline{B_j}$, there exists an open ball B centered at x such that

1. $B \subset 2B_j$;
2. for all finite subset $S \subset \{1, \dots, j-1\}$, the intersection of boundaries $\partial B \cap \bigcap_S \partial \beta_i$ is included in a finite union of $(n-m-1)$-spheres, where m is the cardinal of S;
3. for all finite subset $S \subset \{1, \dots, j-1\}$,

$$(F \cap \overline{B_j}) \subset \mathbf{R}^n \setminus \bigcap_S \overline{\beta_j} \implies \overline{B} \subset \mathbf{R}^n \setminus \bigcap_S \overline{\beta_j} \tag{6.21}$$

   or, équivalently

$$\overline{B} \cap \bigcap_S \overline{\beta_j} \neq \emptyset \implies (F \cap \overline{B_j}) \cap \bigcap_S \overline{\beta_j} \neq \emptyset. \tag{6.22}$$

Extract a finite covering of $F \cap \overline{B_j}$ by such balls B and denote β_j their union. Then β_j solves the next step of the induction.

Step 2. We complete the family β with an open set β_∞ to obtain a covering of one of the E_k. We take care not to introduce new d-simplexes on $E \cup \Gamma$. We want to reduce the problem to the case where for some k, $E_k \setminus \bigcup_j \beta_j$ is a $(d-1)$-dimensional grid. Using a Federer-Fleming projection, we are going to project E_k in a $(d-1)$-dimensional grid away from $E \cup \Gamma$. Let $\ell > 0$ and consider a complex K describing a uniform grid of sidelength ℓ in $\mathbf{R}^n$ (as in Example 4.1.1 of Chapter 4). In particular, $\mathbf{R}^n = |K| = U(K)$ and the cells of K have a diameter $\leq \sqrt{n}\ell$. We select the cells in which we want to perform the Federer-Fleming projection. Let B_0 be an open ball such that for all k, $E_k \subset \overline{B_0}$. Let L be the subcomplex of K defined by

$$L = \{ A \in K \mid \exists x \in A,\ x \in \overline{2B_0} \text{ and } \mathrm{d}(x, E \cup \Gamma) \geq 2\sqrt{n}\ell \}. \tag{6.23}$$

Consider $x \in \overline{2B_0}$ such that $\mathrm{d}(x, E \cup \Gamma) \geq 2\sqrt{n}\ell$. As $\mathbf{R}^n = U(K)$, there exists a cell $A \in K$ such that $x \in \mathrm{int}(A)$ and, in particular, $A \in L$. We deduce that

$$\{ x \in \overline{2B_0} \mid \mathrm{d}(x, E \cup \Gamma) \geq 2\sqrt{n}\ell \} \subset U(L) \tag{6.24}$$

As $E \cup \Gamma$ is a closed set included in $\bigcup_j \beta_j$, the function $x \mapsto \mathrm{d}(x, E \cup \Gamma)$ is positive on $\mathbf{R}^n \setminus \bigcup_j \beta_j$. Moreover, $\overline{2B_0} \setminus \bigcup_j \beta_j$ is compact so the function $x \mapsto \mathrm{d}(x, E \cup \Gamma)$ has a positive minimum on $\overline{2B_0} \setminus \bigcup_j \beta_j$. This minimum does not depend on ℓ so we can assume ℓ small enough so that for all $x \in \overline{2B_0} \setminus \bigcup_j \beta_j$, $\mathrm{d}(x, E \cup \Gamma) > 4\sqrt{n}\ell$. By contraposition,

$$\{ x \in \overline{2B_0} \mid \mathrm{d}(x, E \cup \Gamma) \leq 4\sqrt{n}\ell \} \subset \bigcup_j \beta_j. \tag{6.25}$$

Next, we introduce the Federer-Fleming projection of $E_k \cap |L|$ in L. First, we justify that $H^d(E_k \cap |L|) < \infty$. In fact, we are going to have much better. By local finitness of K, $|L|$ is a closed subset of $\mathbf{R}^n$. Since the cells of K have a diameter $\leq \sqrt{n}\ell$, the definition of L implies that the cells of L cannot meet $E \cup \Gamma$. Thus, the set $V = \mathbf{R}^n \setminus |L|$ is open and contains $E \cup \Gamma$. According to the Lemma assumptions,

$$\lim_k H^d(E_k \cap |L|) = 0. \tag{6.26}$$

Now, we apply Lemma 4.3.1 of Chapter 4 and we obtain a continuous map $\phi\colon |L| \to |L|$ such that

1. for all $A \in L$, $\phi(A) \subset A$;
2. $\phi(E_k \cap |L|) \subset |L| \setminus \bigcup \{ \mathrm{int}(A) \mid A \in L,\ \dim(A) > d \}$;
3. for all $A \in L^d$,

$$H^d(\phi(E_k \cap |L|) \cap A) \leq C H^d(E_k \cap |L|) \tag{6.27}$$

where C is a positive constant that depends only on n. When k is big enough (depending on ℓ), $H^d(E_k \cap |L|)$ becomes sufficiently small so that one can perform additional projections in the d-dimensional cells of L. Thus, the second axiom becomes

$$\phi(E_k \cap |L|) \subset |L| \setminus \bigcup \{ \mathrm{int}(A) \mid A \in L,\ \dim(A) \geq d \}; \tag{6.28}$$

In particular,

$$\phi(E_k \cap |L|) \cap U(L) \subset \bigcup \{ \mathrm{int}(A) \mid A \in L,\ \dim(A) \leq d-1 \}. \tag{6.29}$$

The sets $E \cup \Gamma$ and $|L|$ are disjoint and closed so we can extend ϕ continuously on $E \cup \Gamma$ by $\phi = \mathrm{id}$. Observe that $|\phi - \mathrm{id}| \leq \sqrt{n}\ell$ because ϕ preserves the cells of L. We can extend ϕ continuously on $\mathbf{R}^n$ in such that way that $|\phi - \mathrm{id}| \leq \sqrt{n}\ell$. Let us show that

$$\phi(E_k) \subset |L^{d-1}| \cup \bigcup_{j \in \mathbf{N}} \beta_j. \tag{6.30}$$

Remember that $E_k \subset \overline{B_0}$. We assume ℓ small enough so that $\sqrt{n}\ell \leq 1$ whence $\phi(E_k) \subset \overline{2B_0}$. For $x \in E_k$, we distinguish two cases. If $\mathrm{d}(x, E \cup \Gamma) \leq$

$3\sqrt{n}\ell$, then $\mathrm{d}(\phi(x), E \cup \Gamma) \leq 4\sqrt{n}\ell$ so $\phi(x) \in \bigcup_j \beta_j$ by (6.25). If $\mathrm{d}(x, E \cup \Gamma) \geq 3\sqrt{n}\ell$, then we have both $\mathrm{d}(x, E \cup \Gamma) \geq 2\sqrt{n}\ell$ and $\mathrm{d}(\phi(x), E \cup \Gamma) \geq 2\sqrt{n}\ell$ so (6.25) shows that $x \in U(L)$ and $\phi(x) \in U(L)$. By (6.29), we have thus $\phi(x) \in \left|L^{d-1}\right|$.

Now, we are all set to introduce

$$\beta_\infty = \mathbf{R}^n \setminus (E \cup \Gamma \cup \bigcup_{|S|=d} \bigcap_S \overline{\beta_j}). \tag{6.31}$$

First, we justify that β_∞ is open. It suffices to show that the family

$$\left(\bigcap_S \overline{\beta_j}\right)_{|S|=d} \tag{6.32}$$

is locally finite in $\mathbf{R}^n$. In step 1, we have built the family $(\beta_j)_{j \in \mathbf{N}}$ such that it is locally finite: for all $x \in \mathbf{R}^n$, there exists an open set U containing x such that the set

$$S_0 = \{\, j \in \mathbf{N} \mid U \cap \beta_j \neq \emptyset \,\} \tag{6.33}$$

is finite. Let S be a subset of $\mathbf{N}$ with cardinal d such that U meets $\bigcap_S \overline{\beta_j}$. Then for all $j \in S$, $U \cap \overline{\beta_j} \neq \emptyset$ and thus $U \cap \beta_j \neq \emptyset$ because U is open. This means that $S \subset S_0$. We deduce that there exists only a finite number of subsets $S \subset \mathbf{N}$ of cardinal d such that U meets $\bigcap_S \overline{\beta_j} \neq \emptyset$. We conclude that β_∞ is open. Observe that β_∞ is disjoint from $E \cup \Gamma$ and that for all $S \subset \mathbf{N}$ of cardinal d,

$$\beta_\infty \cap \bigcap_S \overline{\beta_j} = \emptyset. \tag{6.34}$$

In other words, for all $S \subset \mathbf{N}$ of cardinal $d+1$, the condition $\bigcap_S \beta_j \neq \emptyset$ implies $S \subset \mathbf{N}$. This means that the family $(\beta_j)_{j \in \mathbf{N} \cup \{\,\infty\,\}}$ does not induce additional d-simplexes. Finally, we would like

$$\phi(E_k) \subset \beta_\infty \cup \bigcup_{j \in \mathbf{N}} \beta_j. \tag{6.35}$$

This is where $(d-1)$-dimensional grid helps us a lot. According to (6.30), the condition (6.35) holds if $|L^{d-1}| \setminus \beta_\infty \subset \bigcup_j \beta_j$, that is, for all $S \subset \mathbf{N}$ of cardinal d,

$$|L^{d-1}| \cap \bigcap_S \overline{\beta_j} \subset \bigcup_j \beta_j. \tag{6.36}$$

We are going to see that a suitable translation of K allows to assume that $|K^{d-1}|$ is disjoint from the intersection of boundaries $\bigcap_S \partial\beta_j$. Fix $S \subset \mathbf{N}$ of cardinal d. As $\bigcap_S \partial\beta_j$ is included in a finite union of $(n-d)$-spheres, we deduce that for all $(d-1)$-linear plane P,

$$H^n(\bigcap_S \partial\beta_j + P) = 0. \tag{6.37}$$

In particular,

$$H^n(\bigcap_S \partial\beta_j + (-|K^{d-1}|\,)) = 0. \tag{6.38}$$

This means that for almost every $x \in \mathbf{R}^n$, $x+|K^{d-1}|$ is disjoint from $\bigcap_S \partial\beta_j$. There are only a countable number of subsets $S \subset \mathbf{N}$ of cardinal d so we can find x such that this is true for all of them. To simplify the notation, we assume that $x = 0$ and that (6.36 holds).

We are about to finish the proof. We define the domain $V_\gamma = \mathbf{N} \cup \{\,\infty\,\}$ and for $j \in V_\gamma$, we define the open set $\gamma_j = \phi^{-1}(\beta_j)$. Remember that $\phi = \mathrm{id}$ on $E \cup \Gamma$ so for all $j \in \mathbf{N} \cup \{\,\infty\,\}$,

$$(E \cup \Gamma) \cap \gamma_j = (E \cup \Gamma) \cap \beta_j. \tag{6.39}$$

The family γ covers $E_k \cup \Gamma$ because $(\beta_j)_{j \in \mathbf{N} \cup \{\,\infty\,\}}$ covers $E \cup \Gamma$ and $\phi(E_k)$. The family $((E \cup \Gamma) \cap \gamma_j)_{j \in V_\gamma}$ is a refinement of α because $(E \cup \Gamma) \cap \gamma_\infty = \emptyset$ and because for all $j \in \mathbf{N}$, γ_j coincides with β_j on $E \cup \Gamma$. Finally, for all $S \subset V_\gamma$ of cardinal $d+1$, the condition

$$\bigcap_S \gamma_j \neq \emptyset \tag{6.40}$$

implies $\bigcap_S \beta_j \neq \emptyset$ and then by (6.34), $S \subset \mathbf{N}$. By construction of $(\beta_j)_{j \in \mathbf{N}}$, we have $(E \cup \Gamma) \cap \bigcap_S \beta_j \neq \emptyset$ or equivalently,

$$(E \cup \Gamma) \cap \bigcap_S \gamma_j \neq \emptyset \tag{6.41}$$

since γ_j coincides with β_j on $E \cup \Gamma$. □

6.2.3 Existence of Plateau solutions

We solve two formulations of the Reifenberg Plateau problem. In the first one, we work in $X = \mathbf{R}^n$ and minimize $H^d(E)$ among Reifenberg competitors E. In the second one, we work in $X = \mathbf{R}^n \setminus \Gamma$ (that is, away from the boundary) and minimize $H^d(E \setminus \Gamma)$ among Reifenberg competitors E. In this second case, we do not require regularity on the boundary.

Theorem 6.2.1 (Reifenberg - minimizing the free boundary)**.** *Fix a Lipschitz subset Γ of $\mathbf{R}^n$ which is H^d regular and fix a subgroup L of $H_{d-1}(\Gamma)$. We assume that*

$$m = \inf\{\, H^d(E) \mid E \text{ Reifenberg competitor}\,\} < \infty \tag{6.42}$$

and that there exists a compact set $C \subset \mathbf{R}^n$ such that

$$m = \inf\{\, H^d(E) \mid E \text{ Reifenberg competitor},\ E \subset C\,\}. \tag{6.43}$$

Then there exists a Reifenberg competitor $E \subset C$ such that $H^d(E) = m$.

Proof. We work in $X = \mathbf{R}^n$ and we consider the class

$$\mathcal{C} = \{\, E \mid E \text{ is a Reifenberg competitor} \,\}\,. \tag{6.44}$$

By Lemma 6.2.2, the class $\mathcal{C}$ is preserved by sliding deformations in $\mathbf{R}^n$ so it satisfies the requirement of Corollary 6.1.1. Let (E_k) be a minimizing sequence of $\mathcal{C}$ such that for all k, $E_k \subset C$. According to Corollary 6.1.1, there exists a coral set E_∞ of $\mathbf{R}^n$ such that

$$H^d \llcorner E_k \rightharpoonup H^d \llcorner E_\infty. \tag{6.45}$$

We prove that E_∞ is a Reifenberg competitor. First, we show that E_∞ is a compact subset of C. Observe that $\mathbf{R}^n \setminus C$ is an open set and that by lower semicontinuity,

$$H^d(E_\infty \setminus C) \leq \liminf_k H^d(E_k \setminus C) = 0. \tag{6.46}$$

This proves that the support of $H^d \llcorner E_\infty$ is included in C. As E is coral, E is a subset of C and therefore compact. Next, we appy Lemma 6.2.3 to the set E_∞. For all open set V containing $E_\infty \cup \Gamma$,

$$\limsup_k H^d(E_k \setminus V) = \limsup_k H^d(E_k \cap C \setminus V) \tag{6.47}$$

$$\leq H^d(E_\infty \cap C \setminus V) \tag{6.48}$$

$$\leq 0. \tag{6.49}$$

We conclude that E_∞ is a Reifenberg competitor. Finally, we show that $H^d(E_\infty) = m$. As E_∞ is a Reifenberg competitor, we have of course $H^d(E_\infty) \geq m$. The fact that $H^d(E_\infty) \leq m$ was already observed in Corollary 6.1.1. □

The next theorem is similar to Theorem 1.3 of [Fn] (which is based on Feuvrier's construction) and Theorem 3.4 of [DPDRG3] (which is based on weak limits of minimizing sequences). However, we have not dealt with elliptic integrands yet.

Theorem 6.2.2 (Reifenberg - without minimizing the free boundary)**.** *Fix a closed set* Γ *of* $\mathbf{R}^n$ *and a subgroup* L *of* $H_{d-1}(\Gamma)$*. We assume that*

$$m = \inf \{\, H^d(E \setminus \Gamma) \mid E \textit{ Reifenberg competitor} \,\} < \infty \tag{6.50}$$

and that there exists a compact set $C \subset \mathbf{R}^n$ *such that*

$$m = \inf \{\, H^d(E \setminus \Gamma) \mid E \textit{ Reifenberg competitor},\ E \subset C \,\}\,. \tag{6.51}$$

Then there exists a Reifenberg competitor $E \subset C$ *such that* $H^d(E \setminus \Gamma) = m$.

Remark 6.2.3. If Γ is compact and $H^d(\Gamma) < \infty$, this amounts to minimizing $H^d(E)$ among Reifenberg competitors containing Γ.

Proof. We work in $X = \mathbf{R}^n \setminus \Gamma$ (away from the boundary) and we consider the class

$$\mathcal{C} = \{\, E \setminus \Gamma \mid E \text{ is a Reifenberg competitor} \,\}. \tag{6.52}$$

By Lemma 6.2.2, the class $\mathcal{C}$ is preserved by sliding deformations in X so it satisfies the requirement of Corollary 6.1.1. Let (E_k) be a sequence of Reifenberg competitor such that $(E_k \setminus \Gamma)$ is a minimizing sequence of $\mathcal{C}$ and for all k, $E_k \subset C$. According to Corollary 6.1.1, there exists a coral set S_∞ of X such that

$$H^d \llcorner (E_k \setminus \Gamma) \rightharpoonup H^d \llcorner S_\infty \text{ in } X. \tag{6.53}$$

We prove that there exists a Reifenberg competitor $E_\infty \subset C$ such that $S_\infty = E_\infty \setminus \Gamma$. First, we justify that $S_\infty \subset C$. Observe that $X \setminus C$ is an open set of X and that by lower semicontinuity,

$$H^d(S_\infty \setminus C) \leq \liminf_k H^d((E_k \setminus \Gamma) \setminus C) = 0. \tag{6.54}$$

As a consequence, the support of $H^d \llcorner S_\infty$ in X is included in C. As S_∞ is coral in X, S_∞ is a subset of C. Now, let

$$\begin{aligned} E_\infty &= (S_\infty \cup \Gamma) \cap C && (6.55)\\ &= S_\infty \cup (\Gamma \cap C). && (6.56) \end{aligned}$$

The set S_∞ is closed in X so $S_\infty \cup \Gamma$ is closed in $\mathbf{R}^n$ and E_∞ is compact. We appy Lemma 6.2.3 to the set E_∞. For all open set V containing $E_\infty \cup \Gamma$, $C \setminus V$ is a compact subset of X so

$$\begin{aligned} \limsup_k H^d(E_k \setminus V) &= \limsup_k H^d(E_k \cap C \setminus V) && (6.57)\\ &= \limsup_k H^d((E_k \setminus \Gamma) \cap C \setminus V) && (6.58)\\ &\leq H^d(E_\infty \cap C \setminus V) && (6.59)\\ &\leq 0. && (6.60) \end{aligned}$$

In conclusion, E_∞ is a Reifenberg competitor and $S_\infty = E_\infty \setminus \Gamma \in \mathcal{C}$. Finally, we show that $H^d(S_\infty) = m$. As $S_\infty \in \mathcal{C}$, we have of course $H^d(S_\infty) \geq m$. The fact that $H^d(S) \leq m$ has already been observed in Corollary 6.1.1. $\square$

Appendices

Appendix A

Continuous and Lipschitz extensions

A.1 Continuous extensions

Lemma A.1.1 (Tietze extension)**.** *Let X be a metric space and A be a closed subset of X. Any continuous function $f\colon A \to \mathbf{R}^n$ has a continous extension $g\colon X \to \mathbf{R}^n$.*

Remark A.1.1. Note that we can post-compose g with the orthogonal projection onto the closed convex hull of $f(A)$. Thus, we obtain another continuous extension whose image is included in the convex hull of $f(A)$. For example, if $|f| \leq M$, we can assume $|g| \leq M$ as well.

A.2 Lipschitz extensions

The McShane-Whitney formula is a simple technique to build Lipschitz extensions.

Lemma A.2.1 (McShane-Whitney extension)**.** *Let X be a metric space and $A \subset X$. Any Lipschitz function $f\colon A \to \mathbf{R}^n$ has a Lipschitz extension $g\colon X \to \mathbf{R}^n$.*

Proof. We cover the case $n = 1$ because it suffices to extend each coordinate functions independantly. Let L be the Lipschitz constant of f. Then the McShane-Whitney extension of f is given by the formula

$$g(x) = \inf_{y \in A} \{ f(y) + L|y - x| \} . \tag{A.1}$$

One can check that g is real-valued, coincides with f in A and is L-Lipschitz. $\square$

Remark A.2.1. If $\|f\|$ is the Lipschitz constant of f, the McShane-Whitney extension is $C\|f\|$-Lipschitz, where C is a posivite constant that depends only on n. In the case $X = \mathbf{R}^m$, the Kirzbraun theorem gives an extension g with the same Lipschitz constant as f. As before, we can post-compose g with the orthogonal projection onto the closed convex hull of $f(A)$. We obtain another Lipschitz extension whose image is included in the closed convex hull of $f(A)$ (and this operation does not change the Lipschitz constant). For example, if $|f| \leq M$, we can assume $|g| \leq M$ as well without changing the Lipschitz constant of g.

We also want to approximate continuous functions by Lipschitz functions.

Lemma A.2.2. *Let X be a metric space. Let $f\colon X \to \mathbf{R}^n$ be a bounded and uniformly continuous function. Then for all $\varepsilon > 0$, there exists a Lipschitz function $g\colon X \to \mathbf{R}^n$ such that $|g - f| \leq \varepsilon$.*

Proof. We cover the case $n = 1$ because it suffices to approximate each coordinate functions independently. We denote $M = \sup|f|$. Let us fix $\varepsilon > 0$; there exists $\delta > 0$ such that for all $x, y \in X$ with $|x - y| \leq \delta$, $|f(x) - f(y)| \leq \varepsilon$. Define

$$g(x) = \inf_{y \in X} \{\, f(y) + 2M\delta^{-1}|x - y| \,\}\,. \tag{A.2}$$

One can check that g is real-valued, $g \leq f$ and g is $2M\delta^{-1}$-Lipschitz. Next, we check that $f \leq g + \varepsilon$. For $x, y \in X$, either $|x - y| \geq \delta$ and then

$$f(x) \leq f(y) + 2M \tag{A.3}$$
$$\leq f(y) + 2M\delta^{-1}|x - y| \tag{A.4}$$

or $|x - y| \leq \delta$ and then

$$f(x) \leq f(y) + \varepsilon \tag{A.5}$$
$$\leq f(y) + 2M\delta^{-1}|x - y| + \varepsilon. \tag{A.6}$$

In both cases, $f(x) \leq f(y) + 2M\delta^{-1}|x - y| + \varepsilon$ and since $y \in X$ is arbitrary, $f(x) \leq g(x) + \varepsilon$. □

Corollary A.2.1. *Let X be a metric space. Let $f\colon X \to \mathbf{R}^n$ be a bounded uniformily continuous function which is Lipschitz on some subset $A \subset X$. Then, for all $\varepsilon > 0$, there exists a Lipschitz function $g\colon X \to \mathbf{R}^n$ such that $|g - f| < \varepsilon$ in X and $g = f$ in A.*

Proof. According to Lemma A.2.2, there exists a Lipschitz function $g\colon X \to \mathbf{R}^n$ such that $|g - f| \leq \frac{\varepsilon}{2}$. The function $u = f - g$ is Lipschitz on A and satisfies $|u| \leq \frac{\varepsilon}{2}$, so Lemma A.2.1 say that it admits a Lipschitz extension $v\colon X \to \mathbf{R}^n$ with $|v| \leq \frac{\varepsilon}{2}$. We conclude that $g + v$ is a solution to our problem. □

Appendix B

Grassmannian space

Let E be an Euclidean vector space and d be a nonnegative integer. The Grassmannian $G(d, E)$ is the set of all d-linear planes of E. In the case $E = \mathbf{R}^n$, $G(d, E)$ is simply denoted by $G(d, n)$. The operator norm with respect to the Euclidean norm of a linear map $u \colon E \to E$ is denoted by $\|u\|_E$ or $\|u\|$ when there is no ambiguity.

B.1 Metric structure

Each linear plane $V \in G(d, n)$ is uniquely identified by the orthogonal projection p_V onto V. This correspondance induces a metric on $G(d, n)$:

$$\mathrm{d}(V, W) = \|p_V - p_W\|.$$

Remark B.1.1. The action of $O(n)$ on $G(d, n)$, $(g, V) \mapsto g(V)$, is distance-preserving because $p_{g(V)} = g p_V g^{-1}$.

Remark B.1.2. The application $G(d, n) \to G(n - dn)$, $V \to V^\perp$ is an isometry. It establishes a duality between $G(d, n)$ and $G(n - d, n)$.

It is helpful to compute the norm $\|p_V - p_W\|$ on specific subspaces.

Lemma B.1.1. *For $V, W \in G(d, n)$,*

$$\begin{aligned} \mathrm{d}(V, W) &= \max \left\{ \|p_V - p_W\|_V, \|p_V - p_W\|_{V^\perp} \right\} \\ &= \max \left\{ \|p_V - p_W\|_V, \|p_V - p_W\|_W \right\}. \end{aligned}$$

Proof. For $x \in \mathbf{R}^n$, we denote $x_V = p_V(x)$ and $x_{V^\perp} = p_{V^\perp}(x)$. Note that $(p_V - p_W)(x_V) \in W^\perp$ and $(p_V - p_W)(x_{V^\perp}) \in W$ so

$$|p_V(x) - p_W(x)|^2 = |p_V(x_V) - p_W(x_V)|^2 + |p_V(x_{V^\perp}) - p_W(x_{V^\perp})|^2.$$

We deduce that

$$\|p_V - p_W\| \leq \max \left\{ \|p_V - p_W\|_V, \|p_V - p_W\|_{V^\perp} \right\}.$$

Since the maps $(p_V - p_W)\colon V^\perp \to W$ and $(p_V - p_W)\colon W \to V^\perp$ are adjoints of one another, we have $\|p_V - p_W\|_{V^\perp} = \|p_V - p_W\|_W$. Finally, it is clear that

$$\max(\|p_V - p_W\|_V, \|p_V - p_W\|_W) \le \|p_V - p_W\|.$$

□

The next lemma describes the local structure of $G(d, n)$.

Lemma B.1.2. *1. For all $V, W \in G(d, n)$, $\mathrm{d}(V, W) \le 1$ and there is equality if and only if $V \cap W^\perp \neq 0$ or $V^\perp \cap W \neq 0$.*

2. Let $V, W \in G(d, n)$ be such that $\mathrm{d}(V, W) < 1$, then for all $x \in W$,

$$|x| \le \frac{1}{\sqrt{1 - \mathrm{d}(V, W)^2}} |p_V(x)| \tag{B.1}$$

Thus p_V induces an isomorphism from W to V and $W = \{ x + \varphi(x) \mid x \in V \}$, where $\varphi\colon V \to V^\perp$, $x \to p_V^{-1}(x) - x$.

3. Let $V, W \in G(d, n)$ and assume there exists a linear application $\varphi\colon V \to V^\perp$ such that $W = \{ x + \varphi(x) \mid x \in V \}$, then

$$\mathrm{d}(V, W) = \frac{\|\varphi\|}{\sqrt{1 + \|\varphi\|^2}}. \tag{B.2}$$

Proof. **1)** For $x \in \mathbf{R}^n$, the orthogonal projection $p_V(x)$ satisfies the equation $p_V(x) \cdot (x - p_V(x)) = 0$ which is equivalent to $\left|p_V(x) - \frac{x}{2}\right| = \frac{|x|}{2}$ (we have a similar property for $p_W(x)$). According to the triangular inequality,

$$\begin{aligned} |p_V(x) - p_W(x)| &\le \left|p_V(x) - \frac{x}{2}\right| + \left|p_W(x) - \frac{x}{2}\right| \\ &\le \frac{|x|}{2} + \frac{|x|}{2} \\ &\le |x|. \end{aligned}$$

so $\|p_V - p_W\| \le 1$. Assume that $|p_V(x) - p_W(x)| = |x|$ for some $x \in \mathbf{R}^n \setminus 0$. The previous triangular inequality becomes an egality:

$$|p_V(x) - p_W(x)| = \left|p_V(x) - \frac{x}{2}\right| + \left|p_W(x) - \frac{x}{2}\right|.$$

We deduce that $p_V(x)$ and $p_W(x)$ are antipodals on the sphere $S(\frac{x}{2}, \frac{|x|}{2})$, whence

$$p_V(x) - \frac{x}{2} = -\left(p_W(x) - \frac{x}{2}\right)$$

or equivalently

$$p_V(x) + p_W(x) = x.$$

As $x \neq 0$, we have either $p_V(x) \neq 0$ or $p_{V^\perp}(x) \neq 0$. In the first case,

$$p_V(x) = p_{W^\perp}(x) \in V \cap W^\perp \setminus 0$$

and in the second case

$$p_{V^\perp}(x) = p_W(x) \in V^\perp \cap W \setminus 0.$$

The converse is straighforward.

2) For all $x \in W$,

$$\begin{aligned} |x|^2 &= |p_V(x)|^2 + |x - p_V(x)|^2 \\ &= |p_V(x)|^2 + |p_W(x) - p_V(x)|^2 \\ &\leq |p_V(x)|^2 + \mathrm{d}(V,W)^2 |x|^2 \end{aligned}$$

so

$$|x| \leq \frac{1}{\sqrt{1 - \mathrm{d}(V,W)^2}} |p_V(x)|.$$

3) First, we show that

$$W^\perp = \left\{ x - \varphi^*(x) \;\middle|\; x \in V^\perp \right\},$$

where $\varphi^* \colon V^\perp \to V$ is the adjoint of φ. It is easy to check that

$$\{ x - \varphi^*(x) \mid x \in V^\perp \} \subset W^\perp$$

using the fact that $W = \{ x + \varphi(x) \mid x \in V \}$. For all $x \in V$ and for all $y \in V^\perp$,

$$\begin{aligned} (x + \varphi(x)) \cdot (y - \varphi^*(y)) &= \varphi(x) \cdot y - x \cdot \varphi^*(y) \\ &= 0 \end{aligned}$$

By a dimension argument, this inclusion is in fact an equality.

Next we prove that $\mathrm{d}(V,W) \leq \frac{\|\varphi\|}{\sqrt{1+\|\varphi\|^2}}$. According to Lemma B.1.1,

$$\mathrm{d}(V,W) = \max \{ \|\mathrm{id} - p_W\|_V, \|\mathrm{id} - p_{W^\perp}\|_{V^\perp} \}.$$

If we show that

$$\|\mathrm{id} - p_W\|_V \leq \frac{\|\varphi\|}{\sqrt{1 + \|\varphi\|^2}}, \tag{B.3}$$

the same proof will yield by duality,

$$\|\mathrm{id} - p_{W^\perp}\|_{V^\perp} \leq \frac{\|-\varphi^*\|}{\sqrt{1 + \|-\varphi^*\|^2}} = \frac{\|\varphi\|}{\sqrt{1 + \|\varphi\|^2}}.$$

Therefore, we only prove (B.3). Let us fix $x \in V \setminus 0$. As $|x - p_W(x)| = \mathrm{d}(x, W)$, we have for all $t \in \mathbf{R}$, $|x - p_W(x)| \leq |x - t(x + \varphi(x))|$. The right hand side attains its minimum for

$$t = \frac{x \cdot (x + \varphi(x))}{|x + \varphi(x)|^2} = \frac{|x|^2}{|x + \varphi(x)|^2}$$

and one can compute

$$\begin{aligned} |x - t(x + \varphi(x)|^2 &= (1-t)^2|x|^2 + t^2|\varphi(x)|^2 \\ &= \frac{|x|^2|\varphi(x)|^2}{|x|^2 + |\varphi(x)|^2} \\ &\leq \frac{\|\varphi\|^2}{1 + \|\varphi\|^2}|x|^2. \end{aligned}$$

We conclude that $\mathrm{d}(V, W) \leq \frac{\|\varphi\|}{\sqrt{1+\|\varphi\|^2}}$. It is left to prove the reverse inequality. For all $x \in V$,

$$\begin{aligned} |\varphi(x)| &= |(x + \varphi(x)) - x| \\ &= |p_W(x + \varphi(x)) - p_V(x + \varphi(x))| \\ &\leq \mathrm{d}(V, W)|x + \varphi(x)|. \end{aligned}$$

By $(B.1)$, we have

$$\begin{aligned} |x + \varphi(x)| &\leq \frac{1}{\sqrt{1 - \mathrm{d}(V, W)^2}}|p_V(x + \varphi(x))| \\ &\leq \frac{1}{\sqrt{1 - \mathrm{d}(V, W)^2}}|x|. \end{aligned}$$

We deduce that $\|\varphi\| \leq \frac{\mathrm{d}(V,W)}{\sqrt{1-\mathrm{d}(V,W)^2}}$ or equivalently,

$$\frac{\|\varphi\|}{\sqrt{1 + \|\varphi\|^2}} \leq \mathrm{d}(V, W).$$

□

Finally, we estimate the Lipschitz constant of a linear isomorphism acting on $G(d, n)$.

Lemma B.1.3. *Let $u\colon \mathbf{R}^n \to \mathbf{R}^n$ be a linear isomorphism. Then for all $V, W \in G(d, n)$,*

$$\mathrm{d}(u(V), u(W)) \leq \|u\| \, \|u^{-1}\| \, \mathrm{d}(V, W).$$

Proof. According to Lemma B.1.1,

$$\mathrm{d}(u(V), u(W)) = \max\left\{ \left\|\mathrm{id} - p_{u(W)}\right\|_{u(V)}, \left\|\mathrm{id} - p_{u(V)}\right\|_{u(W)} \right\}.$$

By symmetry, we only need to show that

$$\left\|\mathrm{id} - p_{u(W)}\right\|_{u(V)} \leq \|u\|\|u^{-1}\|\mathrm{d}(V, W).$$

Fix $y \in u(V)$. By the properties of orthogonal projection,

$$\left|(\mathrm{id} - p_{u(W)})(y)\right| = \mathrm{d}(y, u(W)).$$

Taking an element $x \in V$ such that $y = u(x)$, we compute

$$\begin{aligned}
\mathrm{d}(y, u(W)) &= \mathrm{d}(u(x), u(W)) \\
&\leq |u(x) - u(p_W(x))| \\
&\leq \|u\| \, |x - p_W(x)| \\
&\leq \|u\| \, |p_V(x) - p_W(x)| \\
&\leq \|u\| \, \mathrm{d}(V, W)|x| \\
&\leq \|u\| \, \|u^{-1}\| \, \mathrm{d}(V, W)|y|.
\end{aligned}$$

□

B.2 Invariant measure

Let E be an Euclidean vector space and d be a nonnegative integer. The measure $\gamma_{d,E}$ is the unique Radon measure on $G(d, E)$ whose total mass is 1 and which is invariant under the action of $O(n)$ (see [Mat], for existence and unicity). In the case $E = \mathbf{R}^n$, it is simply denoted by $\gamma_{d,n}$. We admit that $\gamma_{d,n}$ coincides, up to a multiplicative constant, to the Hausdorff measure of dimension $d(n - d)$.

Lemma B.2.1 (Disintegration formula)**.** *Let p, q, n be non-negative integers with $p + q \leq n$. For all Borel set $A \subset G(n, p + q)$,*

$$\gamma_{p+q,n}(A) = \int_{G(p,n)} \gamma_{q,V^\perp}(\{\, W \mid V + W \in A \,\}) \, \mathrm{d}V.$$

We omit $W \in G(q, V^\perp)$ for ease of notation.

Proof. We introduce some notation so as to interpret the right-hand side as a pushforward measure. Let us define the space

$$X = \{\, (V, W) \in G(p, n) \times G(q, n) \mid V \perp W \,\}.$$

This space is closed in $G(p,n) \times G(q,n)$ because it can be written as

$$\bigcap_{x,y \in \mathbf{R}^n} \{ (V,W) \in G(p,n) \times G(q,n) \mid p_V(x) \cdot p_W(y) = 0 \} .$$

and for each $x \in \mathbf{R}^n$, the evaluation map $V \mapsto p_V(x)$ is $|x|$-Lipschitz (and so is $W \mapsto p_W(y)$). We equip X with the Radon measure $\gamma_{p\perp q,n}$ defined by

$$\gamma_{p\perp q,n}(A) = \int_{G(p,n)} \gamma_{V^\perp,q}(\{ W \mid (V,W) \in A \}) \, \mathrm{d}V$$

for all Borel set $A \subset X$. Finally we define the map $f \colon X \to G(p+q,n)$, $(V,W) \to V+W$. It is Lipschitz because $V \perp W$ implies $p_{V+W} = p_V + p_W$.

Now, the lemma reduces to showing that

$$\gamma_{p+q,n} = f_\# \gamma_{n,p\perp q}.$$

According to ([Mat], Theorem 1.18) $f_\# \gamma_{n,p\perp q}$ is a Radon measure on $G(p+q,n)$. It is invariant by linear isometries and its total mass is 1. By unicity of uniformly distributed measures on $G(p+q,n)$, it coincides with $\gamma_{p+q,n}$. $\square$

Lemma B.2.2. *Let H be an affine hyperplane in $\mathbf{R}^{n+1}$ which does not pass through 0. Then for all bounded subset $A \subset H$,*

$$H^n(A) \leq C \left(\frac{r^2}{r_0} \right)^n \gamma_{n+1,1}(\{ L \in B(L_0,\alpha) \mid L \cap A \neq \emptyset \}),$$

where L_0 is the linear line orthogonal to H, $r = \sup_{x \in A} |x|$, $r_0 = \mathrm{d}(0,H)$, $\alpha = \sqrt{1 - \left(\frac{r_0}{r}\right)^2}$ and C is a constant that depends only on n.

Proof. Let x_0 be the orthogonal projection of 0 onto H, in particular L_0 is the line generated by x_0. For $L \in B(L_0,1)$, define x_L to be the intersection point of L and H. This point exists because there is a linear application $\varphi_L \colon L_0 \to L_0^\perp$ such that

$$L = \{ x + \varphi_L(x) \mid x \in L_0 \}$$

and $x_L = x_0 + \varphi_L(x_0)$.

Step 1. We show that for $L \in B(L_0,1)$,

$$\mathrm{d}(L,L_0) = \sqrt{1 - \left(\frac{x_0}{x_L} \right)^2}.$$

According to (B.2), we have

$$\mathrm{d}(L,L_0) = \frac{\|\varphi_L\|}{\sqrt{1 + \|\varphi_L\|^2}}.$$

Since φ_L is defined on a line, it is easy to compute $\|\varphi\|$:

$$\|\varphi_L\| = \frac{|\varphi_L(x_0)|}{|x_0|} = \frac{\sqrt{|x_L|^2 - |x_0|^2}}{|x_0|}$$

and the result follows.

Step 2. We want to evaluate the Lipschitz constant of $f\colon L \mapsto x_L$ in a neighborhood of L_0. For $L_1, L_2 \in B(L_0, 1)$, we show that

$$|x_{L_1} - x_{L_2}| \le \frac{|x_{L_1}||x_{L_2}|}{r_0} \mathrm{d}(L_1, L_2).$$

We start by computing $p_{L_i}(x_0)$ for $i = 1, 2$. The function $t \mapsto |tx_{L_i} - x_0|$ attains its minimum at

$$t = \frac{x_{L_i} \cdot x_0}{|x_{L_i}|^2} = \frac{|x_0|^2}{|x_{L_i}|^2}$$

As L_i is a linear line generated by x_{L_i}, we deduce that $p_{L_i}(x_0) = \frac{|x_0|^2}{|x_{L_i}|^2} x_{L_i}$. Thus

$$|p_{L_1}(x_0) - p_{L_2}(x_0)| = |x_0|^2 \left| \frac{x_{L_1}}{|x_{L_1}|^2} - \frac{x_{L_2}}{|x_{L_2}|^2} \right|.$$

One can check that for all $u, v \in \mathbf{R}^n \setminus 0$, $\left| \frac{u}{|u|^2} - \frac{v}{|v|^2} \right| = \frac{|u-v|}{|u||v|}$ so

$$|p_{L_1}(x_0) - p_{L_2}(x_0)| = \frac{|x_0|^2}{|x_{L_1}||x_{L_2}|} |x_{L_1} - x_{L_2}|.$$

By definition of $\mathrm{d}(L_1, L_2)$,

$$|p_{L_1}(x_0) - p_{L_2}(x_0)| \le \mathrm{d}(L_1, L_2)|x_0|$$

and this concludes step 2.

Step 3. Conclusion. For $x \in A$, let $L(x)$ be the line generated by x. Since $L(x)$ and L_0 are not orthogonal, we have $\mathrm{d}(L(x), L_0) < 1$. More precisely, we can bound $\mathrm{d}(L(x), L_0)$ according to step 1:

$$\mathrm{d}(L(x), L_0) \le \sqrt{1 - \left(\frac{r_0}{r}\right)^2} = \alpha$$

hence

$$A \subset f(\{\, L \in B(L_0, \alpha) \mid f(L) \in A \,\}).$$

Step 2 shows that f is $\frac{r^2}{r_0}$-Lipschitz on $\{\, L \in B(L_0, \alpha) \mid f(L) \in A \,\}$. According to the properties of Hausdorff measures, we conclude that

$$H^n(A) \le C \left(\frac{r^2}{r_0}\right)^n \gamma_{n+1,1}(\{\, L \in B(L_0, \alpha) \mid f(L) \in A \,\}).$$

$\square$

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Titre: Limites d'ensembles quasiminimaux et existence d'ensembles minimaux sous contraintes topologiques

Mots clés: problème de plateau, surfaces minimales, théorie géométrique de la mesure, calcul des variations

Résumé: Au dix-neuvième siècle, Joseph Plateau a décrit la disposition géométrique des films de savons. Leur forme s'explique par leur tendance à minimiser leur aire pour atteindre une position d'équilibre. Les mathématiciens ont abstrait le concept de « surface d'aire minimale s'appuyant sur un bord » et ont nommé le problème de minimisation correspondant, « problème de Plateau ». Il fait l'objet de différentes formulations qui correspondent à autant de façons de définir la classe des « surfaces bordées par une frontière fixée » et « l'aire » à minimiser.

Dans cette thèse, on généralise aux suites quasiminimisantes, la limite faible de suites minimisantes introduite par De Lellis, De Philippis, De Rosa, Ghiraldin et Maggi. On montre qu'une limite faible d'ensembles quasiminimaux est quasiminimal. Ce résultat est analogue au théorème de passage à la limite de David pour la convergence de Hausdorff locale. Notre démonstration est inspirée par celle de David tout en étant plus simple. On déduit une méthode directe pour prouver l'existence de solutions à divers problèmes de Plateau, même avec une frontière libre. On l'applique ensuite à deux variantes du problème de Reifenberg (frontière libre ou fixe) pour tous les groupes de coefficient. D'autre part, on propose une structure pour construire des projections de Federer-Fleming ainsi qu'une nouvelle estimation sur le choix des centres de projection.

Title: Limits of quasiminimal sets and existence of minimal sets under topological constraints

Keywords: plateau problem, minimal surfaces, geometric measure theory, calculus of variations

Abstract: In the nineteenth century, Joseph Plateau described the geometrical disposition of soap films. Their shape is explained by their tendency to minimize their area to a reach an equilibrium. Mathematicians have abstracted the concept of "surface with minimal area spanning a boundary" and have named the corresponding minimization problem, "Plateau problem". It has different formulations corresponding to as many ways of defining the class of "surfaces spanning a given boundary" and the "area" to minimize.

In this thesis, we generalize to quasiminimizing sequences, the weak limit of minimizing sequences introduced by De Lellis, De Philippis, De Rosa, Ghiraldin and Maggi. We show that a weak limit of quasiminimal sets is quasiminimal. This result is analogous to the limiting theorem of David for the local Hausdorff convergence. Our proof is inspired by David's one while being simpler. We deduce a direct method to prove existence of solutions to various Plateau problem, even with a free boundary. We apply it then to two variants of the Reifenberg problem (fixed or free boundary) for all coefficient groups. Furthermore, we propose a structure to build Federer-Fleming projections as well as a new estimate on the choice of projection centers.

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