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Robin Timsit

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Sorbonne Université

École doctorale de sciences mathématiques de Paris centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Robin Timsit

Homéomorphismes quasiconformes extrémaux et différentielles quadratiques en géométrie CR sphérique

---

dirigée par Elisha Falbel

Soutenue le 17 décembre 2018 devant le jury composé de :

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Résumé

Dans cette thèse, on s'intéresse à l'idée d'homéomorphismes de Teichmüller dans le cadre de la géométrie CR sphérique de dimension 3. On en considère alors deux approches. La première est celle d'homéomorphismes quasiconformes avec des propriétés extrémales. Dans cette direction, on construit explicitement et on prouve l'unicité (à composition avec une rotation autour de l'axe vertical près) d'un minimiseur d'une distortion moyenne entre cylindres du groupe de Heisenberg. On étend par la suite ces résultats aux relevés, par une projection naturelle dans le demi-plan supérieur, de quadrilatères du demi-plan supérieur. Cela permet en particulier de reconstruire l'homéomorphisme quasiconforme connu sous le nom de "radial stretch map" et de retrouver son unicité (à composition avec une rotation autour de l'axe vertical près) comme minimiseur d'une distortion moyenne entre anneaux du groupe de Heisenberg. On montre également un résultat général de relèvement d'homéomorphismes quasiconformes du demi-plan supérieur sur le groupe de Heisenberg.

La seconde approche est celle d'homéomorphismes quasiconformes qui dilatent les trajectoires horizontales d'une différentielle quadratique CR. On parvient à la notion de différentielles quadratiques CR en étudiant une décomposition du complexe de Rumin sur les variétés CR sphériques. On termine par des exemples d'homéomorphismes quasiconformes qui dilatent les trajectoires (horizontales et/ou verticales) de différentielles quadratiques CR. En particulier, on verra que, sous certaines conditions, il existe au plus une famille à deux paramètres d'homéomorphismes quasiconformes qui dilatent les trajectoires horizontales de différentielles quadratiques CR.

Mots-clés

Homéomorphismes quasiconformes extrémaux, différentielles quadratiques, modules de familles de courbes, groupe de Heisenberg, géométrie CR sphérique.

Extremal quasiconformal maps and quadratic differentials in spherical CR geometry

Abstract

In this thesis, we are interested in the idea of Teichmüller homeomorphisms in the setting of 3-dimensional spherical CR geometry. We then consider two approaches of it. The first one is about quasiconformal mappings which have extremal properties. In this direction, we construct explicitly and prove uniqueness (up to composition with a rotation around the vertical axis) of a minimizer of a mean distortion between cylinders in the Heisenberg group. After that, we extend those results to lifts, by a natural projection in the upper half-plane, of quadrilaterals in the upper half-plane. It allows us to reconstruct the quasiconformal map known as the "radial stretch map" and prove its uniqueness (up to composition with a rotation around the vertical axis) as a minimizer of a mean distortion between annuli in the Heisenberg group. We also prove a general result about lifting quasiconformal from the upper half-plane to the Heisenberg group.

The second approach is the one of quasiconformal mappings which dilate horizontal trajectories of CR quadratic differentials. For that, we need to define what are CR quadratic differentials. We manage to do it by considering a decomposition of Rumin complex on spherical CR manifolds. Finally, we give several examples of quasiconformal mappings which dilate (horizontal and/or vertical) trajectories of CR quadratic differentials. In particular, we will see that, under certain conditions, there is at most a two-parameters family of quasiconformal mappings which dilate horizontal trajectories of quadratic differentials.

Keywords

Extremal quasiconformal mappings, quadratic differentials, modulus of curve families, Heisenberg group, spherical CR geometry.

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Introduction

(English version follows).

La théorie de Teichmüller est née de la volonté de répondre au *problème des modules* de Riemann : combien de paramètres (modules) interviennent dans la déformation d'une surface de Riemann ? Étant donnée une surface compacte, connexe, orientable, Σ, l'*espace des modules* de Σ est le quotient de l'ensemble des surfaces de Riemann homéomorphes à Σ par la relation d'équivalence de biholomorphie. Il s'est alors avéré qu'étudier un espace qui "se souvient" de l'homéomorphisme est plus simple. Cet espace est l'espace de Teichmüller de Σ et c'est le revêtement universel de l'espace des modules de Σ.

Lorsque Σ est de genre supérieur ou égal à deux, son espace de Teichmüller peut être identifié à une boule dans un espace vectoriel complexe de dimension (complexe) $3g-3$ où g est le genre de Σ ; identification qui se fait grâce à des homéomorphismes particuliers appelés *homéomorphismes de Teichmüller*. La construction d'un homéomorphisme de Teichmüller entre deux surfaces de Riemann X et Y peut être vue en deux étapes. D'une part, c'est un homéomorphisme qui envoie deux feuilletages singuliers orthogonaux, dits horizontal et vertical, de X (objets entièrement portés par la notion de *différentielle quadratique holomorphe*) sur deux feuilletages singuliers orthogonaux de Y. D'autre part, cet homéomorphisme a un comportement particulier sur ces feuilletages : il dilate le feuilletage horizontal d'un facteur K et le feuilletage vertical d'un facteur $\frac{1}{K}$. Ces homéomorphismes sont également les *homéomorphismes quasiconformes* de distortion minimale dans leur classe d'homotopie.

La géométrie CR est née, quant à elle, de l'envie de comprendre les invariants mis en jeu pour distinguer deux hypersurfaces (réelles) de $\mathbb{C}^{n+1}$. Une structure CR sur une variété lisse est la donnée d'un sous-fibré involutif du complexifié du fibré tangent, qui soit transverse à son conjugué. La structure complexe sur $\mathbb{C}^{n+1}$ induit

alors naturellement une structure CR dite *standard* sur toute hypersurface.

Une variété CR localement CR équivalente à la sphère $\mathbf{S}^{2n+1} \subset \mathbb{C}^{n+1}$ munie de sa structure CR standard est une variété *CR sphérique*. Une autre manière de voir une variété CR sphérique est en tant que variété munie d'une structure (G, X) pour $X = \mathbf{S}^{2n+1}$ et $G = \mathrm{PU}(n, 1)$. La recherche de structures CR sphériques sur une variété donnée est un domaine de recherche actuel et actif.

Ce n'est pas directement ce type de problème qui va nous occuper ici. On s'intéresse plutôt à la question de déformation de structures CR sphériques. L'idée est donc plus de travailler sur un espace de type Teichmüller sur ces variétés, qui a été défini par Wang [56] en utilisant la théorie des applications quasiconformes sur le groupe de Heisenberg développée par Korányi et Reimann [35, 36]. Le but de cette thèse est alors d'étudier la notion d'homéomorphismes de Teichmüller dans ce cadre, que l'on restreindra aux variétés de dimension trois exclusivement.

Quelles sont les applications quasiconformes entre deux variétés CR sphériques qui minimisent la *distortion* dans leur classe d'homotopie ? Cette question est difficile et le travail présenté ici s'inscrit dans ce domaine de recherche actuel. Avant de pouvoir répondre à cette question très générale, obtenir un théorème de type Grötzsch pour des domaines du groupe de Heisenberg semble être une étape nécessaire, mais en l'absence d'un théorème de représentation conforme, chaque domaine doit être traité au cas par cas.

Plutôt que de chercher des applications quasiconformes qui minimisent la distortion maximale, on peut en chercher qui minimisent une distortion "moyenne". C'est le chemin emprunté par Balogh, Fässler et Platis [7]. Ils ont trouvé explicitement un minimiseur pour la distortion moyenne entre deux anneaux du groupe de Heisenberg. Dans leur travail, ils expliquent comment vérifier qu'une application quasiconforme donnée est bel et bien un minimiseur pour la distortion moyenne. Néanmoins, trouver un candidat minimiseur est délicat. Comment peut-on construire un candidat minimiseur de distortion moyenne ? Le début du chapitre 2 concerne la construction explicite d'un tel candidat entre deux cylindres du groupe de Heisenberg. Cette question donne également lieu à un théorème plus général de relèvement des applications quasiconformes dans la lignée d'un résultat de Capogna et Tang [16].

Le second article de Balogh, Fässler et Platis [8] sur le sujet porte sur un résultat d'unicité du minimiseur trouvé entre deux anneaux du groupe de Heisenberg. Une telle unicité pourrait s'avérer utile pour la compréhension de l'espace de Teichmüller puisqu'elle pourrait permettre de choisir un "bon" représentant dans chaque classe

d'homotopie. Dans le chapitre 2 on construit et on montre un résultat d'unicité d'un minimiseur de distortion moyenne entre deux cylindres du groupe de Heisenberg. Quelle raison géométrique se cache derrière ces deux résultats d'unicité ? La fin du chapitre 2 explique en quoi ces deux exemples font en réalité partie d'une même classe d'exemples en extrayant les propriétés géométriques communes entre un anneau et un cylindre.

Comment définir un homéomorphisme de Teichmüller entre deux variétés CR sphériques ? On apporte notre pierre à cet édifice en définissant une notion de *différentielle quadratique CR* au chapitre 3. On utilise ensuite cette notion pour retravailler les exemples du chapitre 2 avec en filigrane l'idée de dilater les *trajectoires horizontales* d'une différentielle quadratique CR. Notamment, dans les exemples des anneaux et cylindres du groupe de Heisenberg, on trouve toutes les applications quasiconformes qui dilatent les trajectoires horizontales pour des différentielles quadratiques CR fixées. Ces exemples offrent un contraste étonnant avec la théorie classique dans laquelle il existe une profusion d'homéomorphismes quasiconformes qui dilatent les trajectoires horizontales d'une différentielle quadratique holomorphe.

Présentation des résultats

Un premier point de vue sur les homéomorphismes de Teichmüller est qu'ils sont ceux qui minimisent la distortion maximale dans leur classe d'homotopie. Ce point de vue est amené par l'étude des homéomorphismes quasiconformes extrémaux. Les exemples les plus simples (et qui par ailleurs fournissent le modèle local d'un homéomorphisme de Teichmüller) de tels homéomorphismes proviennent du théorème de Grötzsch. Dans sa version la plus élémentaire, ce théorème stipule que l'homéomorphisme quasiconforme qui minimise la distortion maximale sur l'ensemble des homéomorphismes entre les deux rectangles $R_{a,b} = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq a,\ 0 \leq \Im(z) \leq b\}$ et $R_{a',b'} = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq a',\ 0 \leq \Im(z) \leq b'\}$ qui sont quasiconformes à l'intérieur de $R_{a,b}$ et qui envoient homéomorphiquement $\{0\} \times [0,b]$ sur $\{0\} \times [0,b']$ et $\{a\} \times [0,b]$ sur $\{a'\} \times [0,b']$est l'homéomorphisme $(x+iy) \longmapsto \frac{a'}{a}x + i\frac{b'}{b}y$. On voudrait donc étudier la possibilité d'un tel résultat dans le cadre des homéomorphismes quasiconformes sur le groupe de Heisenberg.

Le groupe de Heisenberg $\mathcal{H}$ est $\mathbb{C} \times \mathbb{R}$ muni de la loi de groupe :

$$(z,t) * (z',t') = (z+z', t+t'+2\Im(z\overline{z}'))\,.$$

$\mathcal{H}$ est également muni d'une distance invariante à gauche

$$d_{\mathcal{H}}(p,q) = \|p^{-1} * q\|_{\mathcal{H}} \text{ où } \|(z,t)\|_{\mathcal{H}} = \left(|z|^4 + t^2\right)^{\frac{1}{4}}.$$

Un homéomorphisme $f : \Omega \longrightarrow \Omega'$ entre domaines de $\mathcal{H}$ est dit quasiconforme si

$$H(p,f) := \lim_{r \mapsto 0} \sup \frac{\max\limits_{d_{\mathcal{H}}(p,q)=r} d_{\mathcal{H}}(f(p), f(q))}{\min\limits_{d_{\mathcal{H}}(p,q)=r} d_{\mathcal{H}}(f(p), f(q))}$$

est uniformément bornée sur Ω. On dit que f est K-quasiconforme si $\|H(.,f)\|_{L^\infty} \leq K$. Comme dans le cas complexe, on a des définitions analytiques équivalentes. En particulier, si f est un homéomorphisme quasiconforme préservant l'orientation entre domaines de $\mathcal{H}$, en notant $f = (f_1, f_2)$ avec f_1 la partie complexe de f et f_2 sa partie rélle, alors f_1 et $f_2 + i|f_1|^2$ satisfont presque partout la même équation de type Beltrami

$$\overline{Z}u = \mu Z u$$

où μ est une fonction dans la boule unité de L^∞,

$$Z = \frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t} \text{ et } \overline{Z} = \frac{\partial}{\partial \overline{z}} - iz\frac{\partial}{\partial t}.$$

La fonction de distortion de f est alors définie par

$$K(p,f) := \frac{1 + |\mu(p)|}{1 - |\mu(p)|} = \frac{|Zf_1(p)| + |\overline{Z}f_1(p)|}{|Zf_1(p)| - |\overline{Z}f_1(p)|}$$

pour tout $p \in \Omega$ où cette expression a un sens.

Autour des homéomorphismes quasiconformes extrémaux

Suivant un résultat de Balogh, Fässler et Platis [7,8], on s'intéresse au problème de minimisation suivant : on considère une classe $\mathcal{F}$ d'homéomorphismes quasiconformes entre deux domaines (fixés) Ω et Ω' de $\mathcal{H}$. On cherche un homéomorphisme quasiconforme $f_0 \in \mathcal{F}$ tel que

$$\int_\Omega K(p,f_0)^2 \rho_0^4(p) \mathrm{d}L^3(p) = \min_{f \in \mathcal{F}} \int_\Omega K(p,f)^2 \rho_0^4(p) \mathrm{d}L^3(p)$$

pour une densité ρ_0 qui dépend de la géométrie de Ω et où $\mathrm{d}L^3$ est la mesure de Lebesgue sur $\mathbb{R}^3$. Lorsque l'on trouve un tel $f_0 \in \mathcal{F}$, on dit que f_0 minimise la

distortion moyenne sur $\mathcal{F}$ pour la densité ρ_0. Les domaines considérés dans [7, 8] sont des *anneaux sphériques*

$$A_a := \{p \in \mathcal{H} \mid 1 < \|p\|_{\mathcal{H}} < a\} \text{ et } A_{a^k} := \{p \in \mathcal{H} \mid 1 < \|p\|_{\mathcal{H}} < a^k\}$$

où $a > 1$ et $0 < k < 1$ sont deux réels, la classe d'homéomorphismes quasiconformes est l'ensemble $\mathcal{F}$ des homéomorphismes quasiconformes de A_a dans A_{a^k} qui se prolongent homéomorphiquement aux bords et envoient $\{p \in \mathcal{H} \mid \|p\|_{\mathcal{H}} = r\}$ sur $\{p \in \mathcal{H} \mid \|p\|_{\mathcal{H}} = r^k\}$ pour $r = 1, a$ et la densité est $\rho_0(z,t) = \frac{|z|}{\ln(a)\sqrt{t^2+|z|^4}}$.

Théorème 0.0.1 (Balogh, Fässler, Platis [7,8])**.** *L'homéomorphisme quasiconforme f_k, connu sous le nom de "radial stretch map", défini pour tout $(z,t) \in A_a$ par*

$$f_k(z,t) = \left(\sqrt{k}z\left(\frac{t-i|z|^2}{t-ik|z|^2}\right)^{\frac{1}{2}}\left|t+i|z|^2\right|^{\frac{k-1}{2}}, t\frac{|t+i|z|^2|^k}{|t+ik|z|^2|}\right)$$

minimise la distortion moyenne sur $\mathcal{F}$ pour la densité ρ_0. De plus, f_k est l'unique tel minimiseur, à composition avec une rotation autour de l'axe vertical près.

La difficulté pour produire des homéomorphismes quasiconformes dans ce cadre provient du fait que ceux-ci sont, s'ils sont de classe C^1 (et sinon dans un sens faible), des transformations de contact pour la structure de contact sur $\mathcal{H}$ induite par la forme $\omega = \mathrm{d}t - i\overline{z}\mathrm{d}z + iz\mathrm{d}\overline{z}$. Une certaine rigidité se dégage donc qui est le motif principal de l'unicité. Notons d'ailleurs que les résultats de cette thèse ont été obtenus en ne considérant que des homéomorphismes quasiconformes de classe C^2 et qui préservent l'orientation.

Dans le second chapitre, on montre un résultat similaire. On considère deux cylindres

$$C_{a,b} := \{(z,t) \in \mathcal{H} \mid 0 < t < a,\ |z| < \sqrt{b}\} \text{ et}$$

$$C_{a',b'} := \{(z,t) \in \mathcal{H} \mid 0 < t < a',\ |z| < \sqrt{b'}\}$$

où $a, b, a', b' > 0$ sont des réels vérifiant $\frac{ab'}{a'b} > 1$ et la classe $\mathcal{F}$ des homéomorphismes quasiconformes de $C_{a,b}$ dans $C_{a',b'}$ qui se prolongent homéomorphiquement aux bords et envoient $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b},\ t = 0\}$ sur $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b'},\ t = 0\}$ et $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b},\ t = a\}$ sur $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b},\ t = a'\}$. On montre alors :

Théorème 1 (Theorem 2.2.9)**.** *L'homéomorphisme quasiconforme*

$$\begin{array}{rccc} \widetilde{f}_0 : & C_{a,b} & \longrightarrow & C_{a',b'} \\ & (z,t) & \longmapsto & \left(\frac{\sqrt{b'} z e^{\frac{i}{2b}\left(1-\frac{a'b}{ab'}\right)t}}{\sqrt{\left(1-\frac{ab'}{a'b}\right)|z|^2+\frac{ab'}{a'}}}, \frac{a'}{a}t \right) \end{array}$$

minimise la distortion moyenne sur $\mathcal{F}$ pour la densité $\widetilde{\rho}_0(z,t) = \frac{2|z|}{a}$. De plus, à composition avec une rotation autour de l'axe vertical près, $\widetilde{f}_0$ est l'unique tel minimiseur.

L'homéomorphisme $\widetilde{f}_0$ a été obtenu en relevant tous les minimiseurs d'une distortion moyenne entre les rectangles $R_{a,b}$ et $R_{a',b'}$ par la projection de Korányi $\Pi(z,t) = t + i|z|^2$, ce qui a amené à se demander : quels sont les minimiseurs d'une distortion moyenne entre deux domaines Ω, Ω' du demi plan supérieur $\mathbb{H}$ qui se relèvent par Π en homéomorphismes quasiconformes entre $\Pi^{-1}(\Omega)$ et $\Pi^{-1}(\Omega')$? On répond à cette question en montrant le résultat plus général suivant.

Proposition 2 (Proposition 2.3.9)**.** *Soit $g : \Omega \longrightarrow \Omega'$ un homéomorphisme (de classe C^2) entre domaines simplement connexes de $\mathbb{H}$. Notons $\widetilde{\Omega} = \Pi^{-1}(\Omega)$ et $\widetilde{\Omega}' = \Pi^{-1}(\Omega')$.*

— *Si g est un symplectomorphisme pour la forme d'aire hyperbolique sur $\mathbb{H}$ $\left(c'est\ à\ dire\ que\ g^*\tau = \tau\ où\ \tau = \frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2}\right)$, alors il existe une transformation de contact (pour ω) $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ telle que $\Pi \circ f = g \circ \Pi$. De plus, f est de la forme*

$$(z,t) \longmapsto \left(\frac{z}{|z|}\sqrt{\Im(g(t+i|z|^2))}e^{ih(t+i|z|^2)}, \Re(g(t+i|z|^2)) \right)$$

où $h : \Omega \longrightarrow \mathbb{R}$ satisfait pour tout $w \in \Omega$

$$\frac{\partial h}{\partial w}(w) = \frac{1}{4\Im(w)} - \frac{1}{2\Im(g(w))}\frac{\partial \Re(g)}{\partial w}(w).$$

— *Réciproquement, s'il existe une transformation de contact $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ telle que $\Pi \circ f = g \circ \Pi$, alors g est un symplectomorphisme pour la forme d'aire hyperbolique sur $\mathbb{H}$ et f est de la forme donnée précédemment.*

— *Si $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ est une transformation de contact telle que $\Pi \circ f = g \circ \Pi$, alors f est quasiconforme si et seulement si g est quasiconforme. Dans ce cas, la distortion de f en un point $(z,t) \in \widetilde{\Omega}$ est égale à la distortion de g au*

point $t + i|z|^2 \in \Omega$.

Ce résultat a un intérêt au delà des homéomorphismes quasiconformes qui minimisent une distortion moyenne et donne un nouveau moyen de relever des homéomorphismes quasiconformes du plan vers le groupe de Heisenberg différent de [9, 16] par exemple.

Qu'en est-il de l'unicité obtenue dans les théorèmes précédents ; d'où vient cette unicité ? Cette question est plus ouverte et nécessite d'introduire quelques concepts pour être étudiée. Un quadrilatère (Q, I_1, I_2) est la donnée d'une partie Q de $\mathbb{C}$ homéomorphe à un disque fermé dont on distingue deux parties de son bord I_1 et I_2 qui sont disjointes, connexes, non vides et non réduites à un point. Le prototype d'un quadrilatère est un rectangle $R_{a,b} = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq a,\ 0 \leq \Im(z) \leq b\}$ où $a, b > 0$ dont les parties du bord distinguées sont les segments verticaux. De manière générale, tout quadrilatère est *conformément homéomorphe* à un unique tel rectangle à dilatation près, si bien que le quotient $\frac{b}{a}$ est un invariant conforme de quadrilatère appelé son module. Précisons que deux quadrilatères (Q, I_1, I_2) et (Q', I'_1, I'_2) sont (quasi)conformément homéomorphes s'il existe un homéomorphisme $\phi : Q \longrightarrow Q'$ qui envoie homéomorphiquement I_j sur I'_j pour $j = 1, 2$ et qui soit (quasi)conforme à l'intérieur de Q.

L'absence d'un théorème de représentation conforme sur $\mathcal{H}$ nous force à considérer les quadrilatères comme étant différents les uns des autres. Typiquement, résoudre un problème de minimisation entre cylindres est différent de la résolution du même problème entre deux anneaux sphériques, bien que la projection par Π d'un anneau sphérique (fermé) soit un quadrilatère. Le lien entre ces deux exemples est plus subtil.

Prenons deux quadrilatères (Q, I_1, I_2) et (Q', I'_1, I'_2) dont les intérieurs sont dans $\mathbb{H}$ et notons $\mathcal{G}_{Q,Q'}$ l'ensemble des homéomorphismes quasiconformes entre les quadrilatères (Q, I_1, I_2) et (Q', I'_1, I'_2). Considérons les domaines $\widetilde{\Omega}$ et $\widetilde{\Omega}'$ qui sont les intérieurs dans $\mathcal{H}$ des parties $\Pi^{-1}(Q)$ et $\Pi^{-1}(Q')$ respectivement (la projection Π est définie sur $\mathcal{H}$ tout entier et est donc à valeurs dans $\mathbb{H} \cup \mathbb{R}$). On note $\widetilde{I}_j = \Pi^{-1}(I_j)$, $\widetilde{I}'_j = \Pi^{-1}(I'_j)$ pour $j = 1, 2$ et $\mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$ l'ensemble des homéomorphismes quasiconformes de $\widetilde{\Omega}$ sur $\widetilde{\Omega}'$ qui se prolongent homéomorphiquement aux bords et envoient $\widetilde{I}_j$ sur $\widetilde{I}'_j$ pour $j = 1, 2$. Soient les densités

$$\widetilde{\rho}_Q(z,t) = \frac{2|z|}{a}|\alpha'(t + i|z|^2)| \text{ sur } \widetilde{\Omega} \text{ et } \widetilde{\rho}_{Q'}(z,t) = \frac{2|z|}{a'}|\beta'(t + i|z|^2)| \text{ sur } \widetilde{\Omega}'$$

où $\alpha : Q \longrightarrow R_{a,b}$ et $\beta : Q' \longrightarrow R_{a',b'}$ sont des homéomorphismes conformes de quadrilatères. La condition rendant compte des résultats d'unicité est la "pull-back density condition" (Definition 2.3.3) qui demande que $\widetilde{\rho}_Q$ (resp. $\widetilde{\rho}_{Q'}$) réalise le module de la famille de courbes qui correspond au problème que l'on note $\widetilde{\Gamma}_Q$ (resp. $\widetilde{\Gamma}_{Q'}$). Enfin, on note $\widetilde{\Gamma}$ la famille des courbes legendriennes dans $\widetilde{\Omega}$ qui connectent $\widetilde{I}_1$ et $\widetilde{I}_2$. On obtient le résultat suivant.

Théorème 3 (Theorem 2.3.19)**.** *Si $\widetilde{\Gamma}_Q$ et $\widetilde{\Gamma}_{Q'}$ vérifient la "pull-back density condition", alors on a la dichotomie suivante.*

— *Soit il existe un minimiseur de la distortion moyenne sur $\mathcal{G}_{Q,Q'}$ pour la densité $\rho_Q = \frac{1}{a}|\alpha'|$ qui se relève en un homéomorphisme quasiconforme $f \in \mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$. Dans ce cas, f minimise la distortion moyenne sur $\mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$ pour la densité $\widetilde{\rho}_Q$. De plus, tout tel minimiseur est obtenu ainsi.*

— *Soit aucun des minimiseurs de la distortion moyenne sur $\mathcal{G}_{Q,Q'}$ pour la densité ρ_Q ne se relève en un homéomorphisme quasiconforme dans $\mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$. Dans ce cas, pour tout $f \in \mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$,*

$$M\left(f\left(\widetilde{\Gamma}\right)\right) < \int_{\widetilde{\Omega}} K(.,f)^2 \widetilde{\rho}_Q^4 \mathrm{d}L^3$$

où $M\left(f\left(\widetilde{\Gamma}\right)\right)$ est le module de $f\left(\widetilde{\Gamma}\right)$.

Les exemples des anneaux sphériques et des cylindres sont des cas particuliers de ce résultat. On donne un troisième exemple à la fin du chapitre 2 où, cette fois-ci, la deuxième alternative du théorème est vérifiée.

Autour des différentielles quadratiques CR

Une autre manière de voir un homéomorphisme de Teichmüller est comme un homéomorphisme qui *dilate* les trajectoires de différentielles quadratiques holomorphes. Une variété CR sphérique (de dimension 3), est une variété localement modelée sur le groupe de Heisenberg muni de la structure CR $V = span(Z)$. C'est à dire, c'est une variété munie d'un atlas de cartes à valeurs dans $\mathcal{H}$ et dont les fonctions de transition sont des difféomorphismes CR de $(\mathcal{H}, V)$ (i.e. des difféomorphismes préservant V). On peut alors définir l'espace de Teichmüller d'une variété CR sphérique compacte et le munir d'une distance de la même façon que dans le cas des surfaces de Riemann compactes [56] si bien qu'avoir une notion analogue à celle de différentielle quadratique holomorphe semble nécessaire à la compréhension des (potentiels) homéomorphismes de Teichmüller dans ce cadre.

Le chapitre 3 a pour ambition de proposer une notion de *différentielle quadratique CR* sur une variété CR sphérique. Celle-ci est définie grâce à l'étude de la *décomposition de Garfield-Lee* [22] *du complexe de Rumin* [48] qui donne un fibré en droites complexes $\wedge^{1,0}$ au-dessus d'une variété CR sphérique et un opérateur de type Dolbeault, D'', sur ses sections. Opérateur que l'on parvient à "étendre" en un opérateur $\overline{\mathrm{D}}$ sur les sections du fibré en droites complexes $\wedge^{1,0} \otimes \wedge^{1,0}$.

Définition 4 (Definition 3.3.6)**.** Une différentielle quadratique CR sur une variété CR sphérique est une section q de $\wedge^{1,0} \otimes \wedge^{1,0}$ telle que $\overline{\mathrm{D}}q = 0$.

De manière équivalente, si $(U_i, \varphi_i : U_i \longrightarrow U_i')_{i \in I}$ est un atlas CR sphérique d'une variété M, une différentielle quadratique CR sur M est une collection de fonctions $q_i \in C^\infty(U_i')$ telles que

— pour tout $i, j \in I$, en notant $g_{j,i} = \varphi_j \circ \varphi_i^{-1}$ et $g_{j,i}^1$ la partie complexe de $g_{j,i}$,

$$\forall i, j,\ q_i = (q_j \circ g_{j,i}) \left(Z g_{j,i}^1\right)^2 \text{ sur } \varphi_i(U_i \cap U_j) \text{ et}$$

— pour tout $i \in I$,

$$\overline{Z}^3 q_i = 0$$

Les *trajectoires horizontales et verticales* d'une différentielle quadratique CR se définissent alors aisément. On note que, bien qu'il soit tout à fait possible de définir le fibré en droites $\wedge^{1,0} \otimes \wedge^{1,0}$ au-dessus de toute variété CR strictement pseudoconvexe (de dimension 3), l'opérateur $\overline{\mathrm{D}}$ ne semble, lui, défini que lorsque la variété est CR sphérique.

Une fois armé de ces objets, on retourne aux relevés de quadrilatères du chapitre 2. On reprend les notations précédentes concernant les quadrilatères et leurs relevés. Sur Q (resp. Q'), la différentielle quadratique holomorphe à considérer est $q_Q = \alpha^* \mathrm{d}w^2 = (\alpha'(w))^2 \, \mathrm{d}w^2$ (resp. $q_{Q'} = \beta^* \mathrm{d}w^2 = (\beta'(w))^2 \, \mathrm{d}w^2$). Sur $\widetilde{\Omega}$ (resp. $\widetilde{\Omega}'$), on considère la différentielle quadratique CR

$$q_{\widetilde{\Omega}} = \Pi^* q_Q \text{ (resp. } q_{\widetilde{\Omega}'} = \Pi^* q_{Q'})$$

de sorte que la famille de courbes $\widetilde{\Gamma}_Q$ (resp. $\widetilde{\Gamma}_{Q'}$) est exactement la famille des trajectoires horizontales de $q_{\widetilde{\Omega}}$ (resp. $q_{\widetilde{\Omega}'}$) dans $\widetilde{\Omega}$ (resp. $\widetilde{\Omega}'$). Dans ce cadre précis, on introduit ce que signifie de *dilater les trajectoires horizontales de* $\left(q_{\widetilde{\Omega}}, q_{\widetilde{\Omega}'}\right)$ et on obtient le résultat :

Théorème 5 (Theorem 3.4.6)**.** *Supposons que* $\widetilde{\Gamma}_Q$ *et* $\widetilde{\Gamma}_{Q'}$ *vérifient la "pull-back*

density condition" et soit $f \in \mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$. *Si* f *dilate les trajectoires horizontales de* $\left(q_{\widetilde{\Omega}}, q_{\widetilde{\Omega}'}\right)$ *alors il existe* $g \in \mathcal{G}_{Q,Q'}$ *tel que* $\Pi \circ f = g \circ \Pi$.

Ce théorème offre un contraste intéressant avec le cas classique, en particulier lorsqu'il est combiné avec la Proposition 2. En effet, toujours en supposant la "pull-back density condition", la combinaison de ces deux résultats donne que l'ensemble des homéomorphismes quasiconformes dans $\mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$ qui dilatent les trajectoires horizontales de $\left(q_{\widetilde{\Omega}}, q_{\widetilde{\Omega}'}\right)$ est

— soit vide (cela se produit lorsque Q et Q' n'ont pas la même aire hyperbolique par exemple),
— soit une famille à un paramètre,
— soit une famille à deux paramètres

et on donne des exemples de chacun de ces cas. Le cas classique étant celui des homéomorphismes quasiconformes de quadrilatères qui dilatent les trajectoires horizontales de $(q_Q, q_{Q'})$ et ceux-ci existent en quantité (il y en a au moins autant qu'il y a de difféomorphismes croissants d'un segment dans lui-même fixant les deux points du bord).

Plan de thèse

Cette thèse est composée de trois chapitres. Le premier chapitre expose d'une part le cas classique, à savoir les homéomorphismes de Teichmüller entre surfaces de Riemann afin de servir de motivation pour toute la suite. D'autre part, on pose les bases de la géométrie CR (sphérique) et on introduit les homéomorphismes quasiconformes sur le groupe de Heisenberg, nécessaires à la définition de l'espace de Teichmüller d'une variété CR sphérique.

Le second chapitre s'intéresse aux minimiseurs de distortion moyenne. On commence par introduire les modules de familles de courbes nécessaires pour poser et résoudre le problème de minimisation de distortion moyenne. On résoud ensuite explicitement ce problème de minimisation entre cylindres du groupe de Heisenberg. Enfin, on s'intéresse au problème de minimisation sur les relevés de quadrilatères et on termine par des exemples.

Ce chapitre a donné lieu à l'article [53].

Le dernier chapitre a pour ambition de définir les différentielles quadratiques CR. Pour cela, on commence par introduire le complexe de Rumin ainsi que la décom-

position de Garfield-Lee de celui-ci sur les variétés CR strictement pseudoconvexes. Notamment, on y introduit le fibré $\wedge^{1,0}$ et l'opérateur D'' sur ses sections. On fait ensuite un détour vers un autre complexe de formes différentielles, inspiré du complexe de Rumin et de la décomposition de Garfield-Lee de celui-ci, sur les variétés CR strictement pseudoconvexe qui a le bon goût d'avoir un espace de cohomologie de dimension finie. On calcul exactement cet espace de cohomologie dans le cas des variétés CR strictement pseudoconvexes munies d'une action CR du cercle. Après ce léger détour, on analyse la décomposition de Garfield-Lee du complexe de Rumin sur les variétés CR sphériques, ce qui permet de définir les différentielles quadratiques CR sur ces variétés comme sections d'un fibré dans le noyau d'un opérateur $\overline{\mathrm{D}}$. On explique en quoi l'opérateur $\overline{\mathrm{D}}$ est une extension des opérateurs $\overline{\partial}_b$ sur les fonctions et D'' sur les formes de bidegré $(1,0)$. Enfin, on traite d'exemples d'homéomorphismes quasiconformes qui préservent les trajectoires de différentielles quadratiques CR.

Ce chapitre a donné lieu à l'article [54].

Introduction

Teichmüller theory was born from the desire to answer Riemann *moduli problem*: how many parameters (moduli) intervene in the deformation of a Riemann surface? Let Σ be a connected, compact, oriented surface, the *moduli space* of Σ is the space of all Riemann surfaces homeomorphic to Σ up to biholomorphy. It was found that studying a space which "remembers" of the homeomorphism is simpler. This is the Teichmüller space of Σ and it is the universal cover of the moduli space of Σ.

When Σ has genus greater than two, its Teichmüller space can be identified to a ball in a complex vector space with (complex) dimension $3g-3$ where g is the genus of Σ; the identification is given by particular homeomorphisms known as *Teichmüller homeomorphisms*. The construction of a Teichmüller homeomorphism between two Riemann surfaces X and Y can be done in two steps. First, this is a homeomorphism which sends two orthogonal singular foliations, said horizontal and vertical, of X (objects entirely given by the notion of *holomorphic quadratic differential*) to two orthogonal singular foliations of Y. Then, this homeomorphism behaves in a particular way on those foliations: it dilates the horizontal foliation by a factor K and the vertical foliation by a factor $\frac{1}{K}$. These homeomorphisms also are the *quasiconformal mappings* which minimize the distortion in their homotopy class.

The CR geometry was born from the desire to understand the invariants involved to distinguish two (real) hypersurfaces in $\mathbb{C}^{n+1}$. A CR structure on a smooth manifold is an involutive subbundle of the complexified tangent bundle which is transversal to its complex conjugated. The complex structure on $\mathbb{C}^{n+1}$ naturally induces a CR structure called *standard* on any hypersurface.

A CR manifold locally CR equivalent to the unit sphere $\mathbf{S}^{2n+1} \subset \mathbb{C}^{n+1}$ endowed with its standard CR structure is a *spherical CR* manifold. Another way to see a spherical CR manifold is as a manifold endowed with a (G, X)-manifold for $X = \mathbf{S}^{2n+1}$ and $G = \mathrm{PU}(2,1)$. Finding a spherical CR structure on a given manifold is an active and a present-day research area.

We will mainly be interested in the question of deforming spherical CR structures. The idea is to work on a Teichmüller type space on those manifolds which has been defined by Wang [56] using the theory of quasiconformal mappings in the Heisenberg group developed by Korányi and Reimann [35, 36]. The purpose of this thesis is to study the notion of Teichmüller homeomorphisms in this setting, which we will reduce to 3-dimensional manifolds.

What are the quasiconformal mappings between two spherical CR manifolds which minimize the distortion in their homotopy class? This is a difficult question and the work presented here lies in this present-day research area. Before answering this general question, having a Grötzsch type theorem for domains in the Heisenberg group seems to be a necessary step. However, the lack of a Riemann mapping theorem forces us to treat domains separately from each other.

Instead of looking for quasiconformal maps which minimize the maximal distortion, one can look for such maps which minimize a "mean" distortion. This is the path being taken by Balogh, Fässler and Platis [7]. They found explicitly a minimizer of a mean distortion between annuli of the Heisenberg group. In their work, they explain how one can verify that a quasiconformal map is indeed a minimizer of a mean distortion. Nevertheless, finding a candidate for minimizing a mean distortion is tricky. How can we construct a candidate for minimizing a mean distortion? The beginning of chapter 2 deals with the explicit construction of such a map between two cylinders in the Heisenberg group. This question also leads to a general lifting theorem for quasiconformal maps in line with a result of Capogna and Tang [16].

The second article of Balogh, Fässler and Platis [8] on the subject gives a uniqueness result of the minimizer of the mean distortion found between annuli. Such a uniqueness could be useful to understand the Teichmüller space since it could allow to choose a "good" representative in each homotopy class. In chapter 2, we construct and prove a uniqueness result of a minimizer of a mean distortion between cylinders in the Heisenberg group. What geometrical reason hides behind these two uniqueness result? The end of chapter 2 explains how these two examples are part of the same class of examples by extracting the geometrical properties shared between annuli and cylinders.

How can we define a Teichmüller homeomorphism between two spherical CR manifolds? We make a contribution by defining a notion of *CR quadratic differential* in chapter 3. We then use this notion to work again on examples from chapter 2 with the idea of dilating *horizontal trajectories* of CR quadratic differentials. In

particular, in the examples of annuli and cylinders, we find all quasiconformal maps which dilate horizontal trajectories of well-chosen CR quadratic differentials. Those examples contrast with the classical theory where quasiconformal maps which dilate horizontal trajectories of well-chosen holomorphic quadratic differentials exist in a large number.

Statement of results

A first point of view of Teichmüller homeomorphisms is that they are those which minimize the maximal distortion in their homotopy class. This point of view was brought by the study of extremal quasiconformal maps. The simplest examples (and which are the local model of a Teichmüller map) of such maps come from Grötzsch's theorem. In its most elementary version, this theorem states that the quasiconformal map which minimizes the maximal distortion in the set of homeomorphisms between rectangles $R_{a,b} = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq a,\ 0 \leq \Im(z) \leq b\}$ and $R_{a',b'} = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq a',\ 0 \leq \Im(z) \leq b'\}$ which are quasiconformal inside $R_{a,b}$ and send homeomorphically $\{0\} \times [0,b]$ onto $\{0\} \times [0,b']$ and $\{a\} \times [0,b]$ onto $\{a'\} \times [0,b']$ is the map $(x+iy) \longmapsto \frac{a'}{a}x + i\frac{b'}{b}y$. We wish to study the possibility of such a result in the setting of quasiconformal maps in the Heisenberg group.

The Heisenberg group $\mathcal{H}$ is $\mathbb{C} \times \mathbb{R}$ endowed with the group law:

$$(z,t) * (z',t') = (z+z', t+t'+2\Im(z\overline{z}'))\,.$$

$\mathcal{H}$ is also endowed with a left-invariant metric

$$d_{\mathcal{H}}(p,q) = \|p^{-1} * q\|_{\mathcal{H}} \text{ where } \|(z,t)\|_{\mathcal{H}} = \left(|z|^4 + t^2\right)^{\frac{1}{4}}.$$

A homeomorphism $f : \Omega \longrightarrow \Omega'$ between domains of $\mathcal{H}$ is said quasiconformal if

$$H(p,f) := \lim_{r \mapsto 0} \sup \frac{\max\limits_{d_{\mathcal{H}}(p,q)=r} d_{\mathcal{H}}(f(p),f(q))}{\min\limits_{d_{\mathcal{H}}(p,q)=r} d_{\mathcal{H}}(f(p),f(q))}$$

is uniformly bounded on Ω. We say that f is K-quasiconformal if $\|H(.,f)\|_{L^\infty} \leq K$. As in the complex plane case, we have equivalent analytic definitions. In particular, if f is an orientation-preserving quasiconformal map between domains of $\mathcal{H}$, denoting $f = (f_1, f_2)$ with f_1 the complex part f and f_2 its real part, then f_1 and $f_2 + i|f_1|^2$

satisfy almost everywhere the same Beltrami type equation

$$\overline{Z}u = \mu Zu$$

where μ is a function in the unit ball of L^∞,

$$Z = \frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t} \text{ and } \overline{Z} = \frac{\partial}{\partial \overline{z}} - iz\frac{\partial}{\partial t}.$$

The distortion function of f is then defined by

$$K(p,f) := \frac{1+|\mu(p)|}{1-|\mu(p)|} = \frac{|Zf_1(p)| + |\overline{Z}f_1(p)|}{|Zf_1(p)| - |\overline{Z}f_1(p)|}$$

for every $p \in \Omega$ where the expression makes sense.

Around extremal quasiconformal maps

Following a result by Balogh, Fässler and Platis [7, 8], we are interested in the minimization problem: consider a class $\mathcal{F}$ of quasiconformal maps between two (fixed) domains Ω and Ω' of $\mathcal{H}$. We are looking for a quasiconformal map $f_0 \in \mathcal{F}$ such that

$$\int_\Omega K(p,f_0)^2 \rho_0^4(p) \mathrm{d}L^3(p) = \min_{f\in\mathcal{F}} \int_\Omega K(p,f)^2 \rho_0^4(p)\mathrm{d}L^3(p)$$

for a density ρ_0 depending on the geometry of Ω and where $\mathrm{d}L^3$ is the Lebesgue measure on $\mathbb{R}^3$. When we find such a $f_0 \in \mathcal{F}$, we say that f_0 minimizes the mean distortion on $\mathcal{F}$ for the density ρ_0. The domains considered in [7, 8] are *spherical annuli*

$$A_a := \{p \in \mathcal{H} \mid 1 < \|p\|_{\mathcal{H}} < a\} \text{ and } A_{a^k} := \{p \in \mathcal{H} \mid 1 < \|p\|_{\mathcal{H}} < a^k\}$$

where $a > 1$ and $0 < k < 1$ are real numbers, the class of quasiconformal maps is the set $\mathcal{F}$ of quasiconformal maps from A_a to A_{a^k} which extend homeomorphically to the boundaries and send $\{p \in \mathcal{H} \mid \|p\|_{\mathcal{H}} = r\}$ onto $\{p \in \mathcal{H} \mid \|p\|_{\mathcal{H}} = r^k\}$ for $r = 1, a$ and the density is $\rho_0(z,t) = \frac{|z|}{\ln(a)\sqrt{t^2+|z|^4}}$.

Theorem 0.0.1 (Balogh, Fässler, Platis [7,8]). *The quasiconformal map f_k, known as the "radial stretch map", defined for every $(z,t) \in A_a$ by*

$$f_k(z,t) = \left(\sqrt{k}z\left(\frac{t-i|z|^2}{t-ik|z|^2}\right)^{\frac{1}{2}} \left|t+i|z|^2\right|^{\frac{k-1}{2}}, t\frac{|t+i|z|^2|^k}{|t+ik|z|^2|}\right)$$

minimizes the mean distortion on $\mathcal{F}$ for the density ρ_0. Moreover, f_k is the only such minimizer up to composition with a rotation around the vertical axis.

The difficulty to produce quasiconformal maps in the Heisenberg group comes from the fact that, if they are C^1 (and, if not, in a weak sense), they are contact transformations for contact structure on $\mathcal{H}$ induced by the form $\omega = \mathrm{d}t - i\overline{z}\mathrm{d}z + iz\mathrm{d}\overline{z}$. There is some rigidity around quasiconformal maps which is the principal motive of the uniqueness. Notice that the results presented in this thesis were obtained by considering orientation-preserving, C^2 quasiconformal maps.

In the second chapter, we show a similar result. Consider two cylinders

$$C_{a,b} := \{(z,t) \in \mathcal{H} \mid 0 < t < a,\ |z| < \sqrt{b}\} \text{ and}$$

$$C_{a',b'} := \{(z,t) \in \mathcal{H} \mid 0 < t < a',\ |z| < \sqrt{b'}\}$$

where $a, b, a', b' > 0$ are real numbers satisfying $\frac{ab'}{a'b} > 1$ and the class $\mathcal{F}$ of quasiconformal maps from $C_{a,b}$ to $C_{a',b'}$ which extend homeomorphically to the boundaries and send $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b},\ t = 0\}$ onto $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b'},\ t = 0\}$ and $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b},\ t = a\}$ onto $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b},\ t = a'\}$. We prove:

Theorem 1 (Theorem 2.2.9)**.** The quasiconformal map

$$\begin{array}{rccc} \widetilde{f_0}: & C_{a,b} & \longrightarrow & C_{a',b'} \\ & (z,t) & \longmapsto & \left(\dfrac{\sqrt{b'} z e^{\frac{i}{2b}\left(1-\frac{a'b}{ab'}\right)t}}{\sqrt{\left(1-\frac{ab'}{a'b}\right)|z|^2 + \frac{ab'}{a'}}}, \dfrac{a'}{a} t \right) \end{array}$$

minimizes the mean distortion on $\mathcal{F}$ for the density $\widetilde{\rho}_0(z,t) = \frac{2|z|}{a}$. Moreover, up to composition with a rotation around the vertical axis, $\widetilde{f_0}$ is the only such minimizer.

The map $\widetilde{f_0}$ was constructed by liftind every minimizer of a mean distortion between rectangles $R_{a,b}$ and $R_{a',b'}$ by the Korányi map $\Pi(z,t) = t + i|z|^2$, which led us to ask: what are the minimizers of a mean distortion between domains Ω and Ω' of the upper half-plane $\mathbb{H}$ which lift by Π into quasiconformal maps between $\Pi^{-1}(\Omega)$ and $\Pi^{-1}(\Omega')$? We answer this question by showing the more general following theorem.

Proposition 2 (Proposition 2.3.9)**.** Let $g : \Omega \longrightarrow \Omega'$ be a (C^2) homeomorphism between simply connected domains of $\mathbb{H}$. Denote $\widetilde{\Omega} = \Pi^{-1}(\Omega)$ and $\widetilde{\Omega}' = \Pi^{-1}(\Omega')$.

— If g is a symplectomorphism with respect to the hyperbolic area form on $\mathbb{H}$ $\left(\text{meaning that } g^*\tau = \tau \text{ where } \tau = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2}\right)$, then there is a contact map (for

ω) $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ such that $\Pi \circ f = g \circ \Pi$. Moreover, f has the form

$$(z,t) \longmapsto \left(\frac{z}{|z|} \sqrt{\Im(g(t+i|z|^2))} e^{ih(t+i|z|^2)}, \Re(g(t+i|z|^2)) \right)$$

where $h : \Omega \longrightarrow \mathbb{R}$ satisfies for every $w \in \Omega$

$$\frac{\partial h}{\partial w}(w) = \frac{1}{4\Im(w)} - \frac{1}{2\Im(g(w))} \frac{\partial \Re(g)}{\partial w}(w).$$

— Conversely, if there is a contact map $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ such that $\Pi \circ f = g \circ \Pi$, then g is a symplectomorphism with respect to the hyperbolic area form on $\mathbb{H}$ and f has the form given above.

— If $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ is contact map such that $\Pi \circ f = g \circ \Pi$, then f is quasiconformal if and only if g is quasiconformal. In this case, the distortion of f at a point $(z,t) \in \widetilde{\Omega}$ is equal to the distortion of g at the point $t + i|z|^2 \in \Omega$.

This result has an interest beyond quasiconformal maps which minimize a mean distortion and gives a new way to lift quasiconformal maps from the plane to the Heisenberg group different from e.g. [9, 16].

What about the uniqueness in the previous theorems; where does it come from? This question being more open, it needs to introduce a few concepts to be studied. A quadrilateral (Q, I_1, I_2) is the data of a subset Q of $\mathbb{C}$ homeomorphic to a closed disc where we distinguish two parts of its boundary I_1 and I_2 which are disjoint, connected, non-empty and not reduced to a single point. The prototype of a quadrilateral is a rectangle $R_{a,b} = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq a,\ 0 \leq \Im(z) \leq b\}$ with $a, b > 0$ and the distinguished boundary parts are vertical line segments. In general, every quadrilateral is *conformally homeomorphic* to a unique such rectangle up to dilation, so that $\frac{b}{a}$ is a conformal invariant of the quadrilateral called its modulus. Let us specify that two quadrilaterals (Q, I_1, I_2) and (Q', I_1', I_2') are (quasi)conformally homeomorphic if there is a homeomorphism $\phi : Q \longrightarrow Q'$ which sends homeomorphically I_j on I_j' for $j = 1, 2$ and which is (quasi)conformal inside Q.

The lack of a Riemann mapping theorem on $\mathcal{H}$ forces us to consider quadrilaterals as different from each other. Typically, solving a minimization problem between cylinders is different from solving the same problem between spherical annuli even though the projection by Π of a (closed) spherical annulus is a quadrilateral. The

link between these two examples is more subtle.

Let (Q, I_1, I_2) and (Q', I_1', I_2') be two quadrilaterals whose interiors are both contained in $\mathbb{H}$ and denote $\mathcal{G}_{Q,Q'}$ the class of quasiconformal maps between the quadrilaterals (Q, I_1, I_2) and (Q', I_1', I_2'). Consider the domains $\widetilde{\Omega}$ and $\widetilde{\Omega}'$ which are the interiors in $\mathcal{H}$ of the sets $\Pi^{-1}(Q)$ and $\Pi^{-1}(Q')$ respectively (the projection Π is defined on the entire $\mathcal{H}$ and so is with value in $\mathbb{H} \cup \mathbb{R}$). Denote $\widetilde{I}_j = \Pi^{-1}(I_j)$, $\widetilde{I}_j' = \Pi^{-1}(I_j')$ for $j = 1, 2$ and $\mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$ the class of quasiconformal maps from $\widetilde{\Omega}$ to $\widetilde{\Omega}'$ which extend homeomorphically to the boundaries and send $\widetilde{I}_j$ onto $\widetilde{I}_j'$ for $j = 1, 2$. Let

$$\widetilde{\rho}_Q(z,t) = \frac{2|z|}{a}|\alpha'(t + i|z|^2)| \text{ on } \widetilde{\Omega} \text{ and } \widetilde{\rho}_{Q'}(z,t) = \frac{2|z|}{a'}|\beta'(t + i|z|^2)| \text{ on } \widetilde{\Omega}'$$

where $\alpha : Q \longrightarrow R_{a,b}$ and $\beta : Q' \longrightarrow R_{a',b'}$ are conformal homeomorphisms of quadrilaterals. The condition allowing the uniqueness results is the *pull-back density condition* (Definition 2.3.3) which asks that $\widetilde{\rho}_Q$ (resp. $\widetilde{\rho}_{Q'}$) realizes the modulus of the curve family corresponding to the problem which we denote $\widetilde{\Gamma}_Q$ (resp. $\widetilde{\Gamma}_{Q'}$). Finally, denote $\widetilde{\Gamma}$ the family of all legendrian curves in $\widetilde{\Omega}$ connecting $\widetilde{I}_1$ and $\widetilde{I}_2$. We get the following result.

Theorem 3 (Theorem 2.3.19)**.** If $\widetilde{\Gamma}_Q$ and $\widetilde{\Gamma}_{Q'}$ satisfy the pull-back density condition, then we have the following dichotomy.

— Either there is a minimizer of the mean distortion on $\mathcal{G}_{Q,Q'}$ for the density $\rho_Q = \frac{1}{a}|\alpha'|$ which lifts into a quasiconformal map $f \in \mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$. In this cas, f minimizes the mean distortion on $\mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$ for the density $\widetilde{\rho}_Q$. Moreover, every such minimizer is constructed by this process.

— Either none of the minimizers of the mean distortion on $\mathcal{G}_{Q,Q'}$ for the density ρ_Q lifts to a quasiconformal map in $\mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$. In this case, for every $f \in \mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$,

$$M\left(f\left(\widetilde{\Gamma}\right)\right) < \int_{\widetilde{\Omega}} K(.,f)^2 \widetilde{\rho}_Q^4 \mathrm{d}L^3$$

where $M\left(f\left(\widetilde{\Gamma}\right)\right)$ is modulus of $f\left(\widetilde{\Gamma}\right)$.

The examples of spherical annuli and cylinders are special cases of this result. We give a third example at the end of chapter two where, this time, the second alternative is verified.

Around CR quadratic differentials

Another way to see a Teichmüller map is as a map which *dilate* trajectories of holomorphic quadratic differentials. A (3-dimensional) spherical CR manifold is a manifold which is locally modeled on the Heisenberg group endowed with its CR structure $V = span(Z)$. Meaning that it is a manifold endowed with an atlas of charts with value in $\mathcal{H}$ and whose transition functions are CR diffeomorphisms of $(\mathcal{H}, V)$ (i.e. diffeomorphisms preserving V). One can then define the Teichmüller space of a compact spherical CR manifold and its metric in the same way than in the case of compact Riemann surfaces [56] so that having a notion similar to holomorphic quadratic differentials seems needed for the understanding of (potential) Teichmüller maps in this setting.

The purpose of chapter 3 is to propose a notion of *CR quadratic differential* on a spherical CR manifold. It is defined thanks to the study of the *Garfield-Lee decomposition* [22] *of Rumin complex* [48] which gives a complex line bundle $\wedge^{1,0}$ over a spherical CR manifold and a Dolbeault type operator, D'', on its sections. We are able to "extend" this D'' in an operator $\overline{\mathrm{D}}$ on the sections of the complex line bundle $\wedge^{1,0} \otimes \wedge^{1,0}$.

Definition 4 (Definition 3.3.6)**.** A CR quadratic differential on a spherical CR manifold is a section q of $\wedge^{1,0} \otimes \wedge^{1,0}$ such that $\overline{\mathrm{D}}q = 0$.

Equivalently, if $(U_i, \varphi_i : U_i \longrightarrow U_i')_{i \in I}$ is a spherical CR atlas of a manifold M, a CR quadratic differential on M is a collection of functions $q_i \in C^\infty(U_i')$ such that

— for all $i, j \in I$, denoting $g_{j,i} = \varphi_j \circ \varphi_i^{-1}$ and $g_{j,i}^1$ the complex part of $g_{j,i}$,

$$\forall i, j,\ q_i = (q_j \circ g_{j,i}) \left(Z g_{j,i}^1\right)^2 \text{ on } \varphi_i(U_i \cap U_j) \text{ and}$$

— for every $i \in I$,

$$\overline{Z}^3 q_i = 0$$

The *horizontal and vertical trajectories* of a CR quadratic differential can then easily be defined. We notice that, even though one can perfectly define the line bundle $\wedge^{1,0} \otimes \wedge^{1,0}$ over any (3-dimensional) strictly pseudoconvex CR manifold, the operator $\overline{\mathrm{D}}$ seems defined only when the manifold is spherical.

Once we have those objects, we go back to lifts of quadrilaterals. We take back the previous notations. On Q (resp. Q'), the holomorphic quadratic differential to consider is $q_Q = \alpha^* \mathrm{d}w^2 = (\alpha'(w))^2 \, \mathrm{d}w^2$ (resp. $q_{Q'} = \beta^* \mathrm{d}w^2 = (\beta'(w))^2 \, \mathrm{d}w^2$). On $\widetilde{\Omega}$

(resp. $\widetilde{\Omega}'$), consider the CR quadratic differential

$$q_{\widetilde{\Omega}} = \Pi^* q_Q \text{ (resp. } q_{\widetilde{\Omega}'} = \Pi^* q_{Q'})$$

so that the curve family $\widetilde{\Gamma}_Q$ (resp. $\widetilde{\Gamma}_{Q'}$) is exactly the family of horizontal trajectories for $q_{\widetilde{\Omega}}$ (resp. $q_{\widetilde{\Omega}'}$) lying in $\widetilde{\Omega}$ (resp. $\widetilde{\Omega}'$). In this particular setting, we introduce what it means to *dilate horizontal trajectories of* $\left(q_{\widetilde{\Omega}}, q_{\widetilde{\Omega}'}\right)$ and we get the result:

Theorem 5 (Theorem 3.4.6)**.** Assume that $\widetilde{\Gamma}_Q$ and $\widetilde{\Gamma}_{Q'}$ satisfy the pull-back density condition and let $f \in \mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$. If f dilates the horizontal trajectories of $\left(q_{\widetilde{\Omega}}, q_{\widetilde{\Omega}'}\right)$ then there is $g \in \mathcal{G}_{Q,Q'}$ such that $\Pi \circ f = g \circ \Pi$.

This theorem gives an interesting contrast with the classical case, especially when combined with Proposition 2. Indeed, still assuming the pull-back density condition, the combination of these two results gives that the set of quasiconformal maps in $\mathcal{F}_{\widetilde{\Omega},\widetilde{\Omega}'}$ which dilate the horizontal trajectories of $\left(q_{\widetilde{\Omega}}, q_{\widetilde{\Omega}'}\right)$ is

— either empty (it happens for instance when Q and Q' don't have the same hyperbolic area),
— either a one-parameter family,
— either a two-parameters family

and we give examples of each of these cases. The classical case being the one of quasiconformal maps of quadrilaterals which dilate horizontal trajectories of $(q_Q, q_{Q'})$ and they exist at a large number (at least as many as there are increasing diffeomorphisms from a line segment to itself fixing the two boundary points).

Organisation of the thesis

This thesis is composed by three chapters. The first one deals with the basics. For one part, we explain the classical case, that is Teichmüller maps between Riemann surfaces, so that it motivates what follows. For another part, we recall the basics of (spherical) CR geometry and of the theory of quasiconformal maps in the Heisenberg group which are needed for the definition of the Teichmüller space of a spherical CR manifold.

The second chapter deals with minimizers of mean distortions. We begin with an introduction to moduli of curve families which are necessary to expose and solve the problem of minimizing a mean distortion. Then, we solve explicitly this minimiza-

tion problem for cylinders in the Heisenberg group. Finally, we will be interested in the minimization problem for lifts of quadrilaterals and as an end, we give examples.

This chapter gave rise to the article [53].

The purpose of the last chapter is to define and study CR quadratic differentials. For that, we begin with introducing the Rumin complex and its Garfield-Lee decomposition on strictly pseudoconvex CR manifolds. In particular, we introduce the line bundle $\wedge^{1,0}$ and the operator D'' on its sections. Then, we do a deviation toward another differential complex, inspired by Rumin complex and its Garfield-Lee decomposition, on strictly pseudoconvex CR manifolds which has a finite dimensional cohomology space. We compute this cohomology space explicitly in the case of strictly pseudoconvex CR manifolds endowed with a CR action of the circle. After that short deviation, we analyse the Garfield-Lee decomposition of Rumin complex on spherical CR manifolds which leads to the definition of CR quadratic differentials as section of a line bundle in the kernel of an operator $\overline{\mathrm{D}}$. We explain how $\overline{\mathrm{D}}$ extends the operators $\overline{\partial}_b$ on functions and D'' on $(1,0)$-forms. Finally, we give examples of quasiconformal maps which preserve trajectories of CR quadratic differentials.

This chapter gave rise to the article [54].

Chapitre 1

Préliminaires : Théorie de Teichmüller et géométrie CR (sphérique)

Ce premier chapitre a pour vocation de poser les bases (doubles) du cadre dans lequel ce travail s'inscrit. Il s'agit donc d'une part de rappeler brièvement les grandes lignes de l'élaboration de la théorie de Teichmüller dans le cas des surfaces de Riemann. D'autre part, on introduit la géométrie CR, son penchant sphérique et la théorie des applications quasiconformes sur le groupe de Heisenberg nécessaire à la définition de l'espace de Teichmüller d'une variété CR sphérique.

1.1 Le cas classique

(Références principales : $[2, 28, 30]$)

1.1.1 Homéomorphismes quasiconformes

Commençons par les homéomorphismes quasiconformes qui se trouvent à la base de la théorie de Teichmüller. Ceux-ci peuvent être définis d'un certain nombre de manières ; ces différentes définitions étant utilisées selon les propriétés que l'on souhaite mettre en avant. On se concentrera ici sur trois définitions :

— une première dite *analytique* qui est celle généralement utilisée lorsque l'application considérée est explicite (et ce, d'autant plus lorsque l'homéomorphisme est assez régulier),

— une définition dite *métrique* qui permet d'étendre la notion à des espaces métriques plus généraux et enfin,
— une *géométrique* qui fait évidemment intervenir des phénomènes géométriques, mais donne également facilement que la composition des deux homéomorphismes quasiconformes et l'inverse d'un homéomorphisme quasiconforme sont eux-mêmes quasiconformes.

Définition 1.1.1 (Homéomorphisme quasiconforme : définition analytique)**.** Soient $f : U \longrightarrow V$ un homéomorphisme entre ouverts de $\mathbb{C}$ et $K \geq 1$ un réel. On dit que f est K-quasiconforme si ses dérivées partielles au sens des distributions sont localement L^2 (c'est à dire que f est dans l'espace de Sobolev $W^{1,2} = H^1$) et vérifient

$$\left|\frac{\partial f}{\partial \overline{z}}\right| \leq k \left|\frac{\partial f}{\partial z}\right| \text{ dans } L^2_{loc} \tag{1.1}$$

où $0 \leq k = \frac{K-1}{K+1} < 1$.
Le plus petit K pour lequel f est K-quasiconforme est appelé la distortion de f, notée K_f.

Comme expliqué, dans cette première définition il n'est pas clair que l'inverse et la composée d'homéomorphismes quasiconformes soit quasiconforme car l'espace de Sobolev H^1 n'est pas stable par ces opérations. Néanmoins, certaines propriétés peuvent être prouvées. Premièrement, un homéomorphisme 1-quasiconforme est conforme. C'est une conséquence du lemme de Weyl :

Lemme 1.1.2 (Lemme de Weyl)**.** *Soit $f : U \longrightarrow \mathbb{C}$ une distribution sur un ouvert U de $\mathbb{C}$ telle que $\frac{\partial f}{\partial \overline{z}} = 0$ sur U. Alors f est holomorphe sur U.*

Ensuite, la composition (à gauche ou à droite) d'un homéomorphisme K-quasi–conforme et d'une transformation conforme est K-quasiconforme. Cela découle du fait que la composée (à gauche ou à droite) d'un élément de $H^1 \cap C^0$ par une fonction C^1 reste dans $H^1 \cap C^0$ et que les formules classiques exprimant les dérivées partielles d'une composition restent vraies.

La définition géométrique d'un homéomorphisme quasiconforme que l'on va donner est la définition originale d'Ahlfors. Pour la donner, il faut introduire la notion de *quadrilatère*.

Définition 1.1.3 (Quadrilatère)**.** Un quadrilatère (Q, I_1, I_2) de $\mathbb{C}$ est une partie Q de $\mathbb{C}$ homéomorphe à un disque fermé, avec deux composantes de bord I_1 et I_2 qui

sont disjointes et connexes. De plus, I_1 et I_2 sont des segments non vides et non réduits à un point.

De tels objets ont une forme normale qui est celle d'un rectangle $R_M = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq 1,\ 0 \leq \Im(z) \leq M\}$ avec $M > 0$. C'est à dire que pour tout quadrilatère (Q, I_1, I_2) il existe un unique $M > 0$ et un homéomorphisme $\varphi : Q \longrightarrow R_M$ qui soit holomorphe à l'intérieur de Q et envoie I_1 sur le segment $\{0\} \times [0, M]$ et I_2 sur le segment $\{1\} \times [0, M]$. Ce réel M est alors appelé le *module* de (Q, I_1, I_2), noté $\mathrm{Mod}(Q, I_1, I_2)$. Un homéomorphisme K-quasiconforme entre deux quadrilatères (Q, I_1, I_2) et (Q', I_1', I_2') est un homéomorphisme $f : Q \longrightarrow Q'$, K-quasiconforme à l'intérieur de Q et qui envoie homéomorphiquement I_j sur I_j' pour $j = 1, 2$.

Théorème 1.1.4 (Théorème de Grötzsch)**.** *Soit $f : Q \longrightarrow Q'$ un homéomorphisme K-quasiconforme entre les quadrilatères (Q, I_1, I_2) et (Q', I_1', I_2'). Alors*

$$\frac{1}{K}\mathrm{Mod}(Q, I_1, I_2) \leq \mathrm{Mod}(Q', I_1', I_2') \leq K\mathrm{Mod}(Q, I_1, I_2).$$

De plus, l'égalité (à gauche ou à droite, selon que $\mathrm{Mod}(Q, I_1, I_2) \leq \mathrm{Mod}(Q', I_1', I_2')$ ou $\mathrm{Mod}(Q, I_1, I_2) \geq \mathrm{Mod}(Q', I_1', I_2')$) est réalisée si et seulement si pour tout $z = x + iy \in R_{\mathrm{Mod}(Q,I_1,I_2)}$,

$$\left(\psi \circ f \circ \varphi^{-1}\right)(x + iy) = x + i\frac{\mathrm{Mod}(Q', I_1', I_2')}{\mathrm{Mod}(Q, I_1, I_2)}y$$

avec $\varphi : Q \longrightarrow R_{\mathrm{Mod}(Q,I_1,I_2)}$ et $\psi : Q' \longrightarrow R_{\mathrm{Mod}(Q',I_1',I_2')}$ définies comme précédemment.

La preuve originale de ce théorème est appelée *méthode des aires-longueurs* a elle-même permis d'amener la notion de *module d'une famille de courbes* dont on étudiera son penchant sur le groupe de Heisenberg au chapitre suivant (voir [55] pour cette notion dans le cas du plan complexe).

On ressort du théorème de Grötzsch deux intérêts principaux. Tout d'abord, il donne existence et unicité d'un homéomorphisme quasiconforme qui minimise la distortion dans la classe des homéomorphismes quasiconformes de quadrilatères. C'est à dire que, si (Q, I_1, I_2) et (Q', I_1', I_2') sont deux quadrilatères et si l'on note $\mathcal{F}$ la classe des homéomorphismes quasiconformes entre les quadrilatères (Q, I_1, I_2) et (Q', I_1', I_2'), alors pour tout $f \in \mathcal{F}$,

$$K_f \geq \sup\left(\frac{\mathrm{Mod}(Q', I_1', I_2')}{\mathrm{Mod}(Q, I_1, I_2)}, \frac{\mathrm{Mod}(Q, I_1, I_2)}{\mathrm{Mod}(Q', I_1', I_2')}\right)$$

et il existe un unique $f_0 \in \mathcal{F}$ tel que

$$K_{f_0} = \sup\left(\frac{\mathrm{Mod}(Q', I_1', I_2')}{\mathrm{Mod}(Q, I_1, I_2)}, \frac{\mathrm{Mod}(Q, I_1, I_2)}{\mathrm{Mod}(Q', I_1', I_2')}\right).$$

Le deuxième intérêt est que la première partie peut servir de définition pour un homéomorphisme quasiconforme.

Définition 1.1.5 (Homéomorphisme quasiconforme : définition géométrique)**.** Un homéomorphisme $f : U \longrightarrow V$ entre ouverts de $\mathbb{C}$ est K-quasiconforme s'il préserve l'orientation et si, pour tout quadrilatère (Q, I_1, I_2) de U, on a

$$\frac{1}{K}\mathrm{Mod}(Q, I_1, I_2) \leq \mathrm{Mod}(f(Q), f(I_1), f(I_2)) \leq K\mathrm{Mod}(Q, I_1, I_2).$$

De cette définition, il découle immédiatement :

Proposition 1.1.6. *Soient U, V et W trois ouverts de $\mathbb{C}$. Soient $f : U \longrightarrow V$ un homéomorphisme K_1-quasiconforme et $g : V \longrightarrow W$ un homéomorphisme K_2-quasiconforme. Alors, $g \circ f$ est K_1K_2-quasiconforme et f^{-1} est K_1-quasiconforme.*

L'équivalence entre la définition géométrique et la définition analytique est loin d'être évidente. On peut prouver celle-ci en faisant intervenir la notion de *quasisymétrie* (que l'on ne détaillera pas ici). L'idée ensuite est de prouver que la définition analytique implique la définition géométrique (c'est le théorème de Grötzsch), ensuite la définition géométrique implique la quasisymétrie et enfin, la quasisymétrie implique la définition analytique. La difficulté étant de récupérer la régularité de la définition analytique. On termine avec la définition métrique, que l'on mentionne principalement parce qu'elle s'avèrera être le point de départ de la théorie des applications quasiconformes sur le groupe de Heisenberg :

Définition 1.1.7 (Homéomorphisme quasiconforme : définition métrique)**.** Soit $f : U \longrightarrow V$ un homéomorphisme entre ouverts de $\mathbb{C}$ et notons pour tout $z \in U$,

$$H(z) = \limsup_{r \mapsto 0} \frac{\sup\limits_{|w-z|=r} |f(w) - f(z)|}{\inf\limits_{|w-z|=r} |f(w) - f(z)|}.$$

f est quasiconforme si H est essentiellement bornée supérieurement.

On voit bien comment on peut étendre cette définition à n'importe quel espace métrique en remplaçant les modules par des distances. La difficulté à partir de celle-ci

est qu'elle soit équivalente à une définition géométrique ou analytique. Ce qui donne l'occasion de mentionner Heinonen et Koskela [25] qui ont montré l'équivalence entre définition métrique et quasisymétrie dans le cas des groupes de Carnot et de plus, que l'on peut remplacer, dans le cas de $\mathbb{R}^n$, la limite supérieure par la limite inférieure.

Il semble enfin indispensable de mentionner peut-être le théorème le plus important concernant les homéomorphismes quasiconformes à savoir le théorème de représentation de Riemann mesurable, généralement attribué à Ahlfors et Bers, concernant les solutions de l'équation de Beltrami :

Théorème 1.1.8 (Théorème de représentation de Riemann mesurable)**.** *Soit U un ouvert de $\mathbb{C}$ et $\mu \in L^\infty(U)$ avec $\|\mu\|_\infty < 1$. Alors il existe une application quasiconforme $g : U \longrightarrow \mathbb{C}$ solution de l'équation de Beltrami :*

$$\frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}. \tag{1.2}$$

De plus, si h est une autre solution quasiconforme de 1.2, alors $h \circ g^{-1}$ est holomorphe.

1.1.2 Espace et théorème de Teichmüller

Dans cette partie, on se fixe une surface de Riemann compacte X.

Définition 1.1.9. L'espace de Teichmüller de X, noté $\mathcal{T}(X)$, est le quotient de l'ensemble des couples (Y, f) où Y est une surface de Riemann et $f : X \longrightarrow Y$ est un homéomorphisme quasiconforme (un homéomorphisme entre surfaces de Riemann est quasiconforme s'il l'est lu dans les cartes) par la relation d'équivalence : (Y_1, f_1) et (Y_2, f_2) sont équivalents si et seulement si $f_2 \circ f_1^{-1} : Y_1 \longrightarrow Y_2$ est homotope à un biholomorphisme. On notera les classes d'équivalence $[Y, f]$ et $[X, \mathrm{Id}]$ est appelé le point base de $\mathcal{T}(X)$.

On peut alors mettre une distance sur l'espace de Teichmüller de X :

Définition 1.1.10 (Distance de Teichmüller)**.** Pour $[Y_1, f_1]$ et $[Y_2, f_2]$ deux points de $\mathcal{T}(X)$, on pose

$$d\left([Y_1, f_1], [Y_2, f_2]\right) = \inf_f \ln\left(K_f\right)$$

où l'infinimum est pris sur l'ensemble des homéomorphismes quasiconformes $f : Y_1 \longrightarrow Y_2$ homotopes à $f_2 \circ f_1^{-1}$. $(\mathcal{T}(X), d)$ est alors un espace métrique complet.

Si on est capable de comprendre les homéomorphismes quasiconformes de plus petite distortion dans leur classe d'homotopie (si tant est qu'un tel homéomorphisme existe), on comprendra topologiquement $\mathcal{T}(X)$. Ce qui amène aux homéomorphismes de Teichmüller qui, pour être correctement définis, nécessitent de parler de *différentielles quadratiques holomorphes.*

Définition 1.1.11. Une différentielle quadratique holomorphe (resp. méromorphe) sur une surface de Riemann Y est une section holomorphe (resp. méromorphe) du fibré $\mathrm{K}_Y^{\otimes 2}$, K_Y désignant le fibré canonique de Y. On notera Q_Y l'espace des différentielles quadratiques holomorphes sur Y.

Localement, si z est une coordonnée holomorphe sur un ouvert U d'une surface de Riemann Y, une différentielle quadratique holomorphe (resp. méromorphe) est de la forme

$$q(z)\mathrm{d}z^2$$

où q est une fonction holomorphe (resp. méromorphe). La condition de recollement entre coordonnées holomorphes z_i et z_j est donc

$$q_i(z_i)\left(\frac{\mathrm{d}z_i}{\mathrm{d}z_j}\right)^2 = q_j(z_j).$$

C'est à dire, en notant $g_{j,i}(z_i) = z_j$

$$q_i = (q_j \circ g_{j,i})\left(g'_{j,i}\right)^2.$$

Si Y est une surface de Riemann compacte de genre g, alors une application directe du théorème de Riemann-Roch donne la dimension de Q_Y : c'est 1 si $g = 1$ et $3g - 3$ si $g > 1$. Les zéros d'une différentielle quadratique holomorphe non nulle q sur Y sont appelés ses *singularités.* Comptées avec multiplicité, q n'a aucune singularité si $g = 1$ et $4g-4$ si $g > 1$. Une *coordonnée naturelle* pour une différentielle quadratique holomorphe q est une coordonnée holomorphe z telle que

$$q = \mathrm{d}z^2.$$

De telles coordonnées existent partout en dehors des singularités de q. En effet, prenons un point $w_0 \in Y$ tel que $q(w_0) \neq 0$, une coordonnée holomorphe w quelconque

sur un voisinage ouvert U de w_0 sur lequel q ne s'annule pas et notons

$$q = q(w)\mathrm{d}w^2$$

sur U. Quitte à réduire U, il existe une racine carré holomorphe de $q(w)$ sur U. On pose alors

$$z(w) = \int_{w_0}^{w} \sqrt{q(\zeta)}\mathrm{d}\zeta$$

sur U. On a donc

$$\mathrm{d}z^2 = q(w)\mathrm{d}w^2 = q$$

sur U.
De plus, deux coordonnées naturelles ne diffèrent que par translation et signe. Tout ceci nous permet de définir :

Définition 1.1.12 (Trajectoires d'une différentielle quadratique holomorphe)**.** Soit q une différentielle quadratique holomorphe sur une surface de Riemann Y. Soit $\gamma : I \longrightarrow Y$ une courbe paramétrée. On appelle γ une :

— trajectoire horizontale de q si $q(\gamma'(t)) > 0$ pour tout $t \in I$; ou de manière équivalente si γ est une droite horizontale en coordonnée naturelle pour q

— trajectoire verticale de q si $q(\gamma'(t)) < 0$ pour tout $t \in I$; ou de manière équivalente si γ est une droite verticale en coordonnée naturelle pour q.

On parle de trajectoire critique lorsque γ part d'une singularité de q.

L'idée géométrique derrière la donnée d'une différentielle quadratique holomorphe non nulle est qu'elle donne une "métrique" euclidienne en dehors de ses singularités, mais qui dégénère en ses singularités. On renvoie à [49] pour une étude plus approfondie de ces objets et on mentionne [29] pour un lien profond entre différentielles quadratiques holomorphes et feuilletages mesurés.

Définition 1.1.13 (Homéomorphisme de Teichmüller)**.** Soient Y_1 et Y_2 deux surfaces de Riemann compactes, $f : Y_1 \longrightarrow Y_2$ un homéomorphisme et $K \geq 1$ un réel. f est appelé homéomorphisme de Teichmüller de constante K s'il existe deux différentielles quadratiques holomorphes $q_1 \in Q_{Y_1}$ et $q_2 \in Q_{Y_2}$ telles que

1. f envoie les singularités de q_1 sur les singularités de q_2 ;
2. si ζ est une coordonnée naturelle pour q_2, alors

$$\frac{1}{2}\Big((K+1)(\zeta \circ f) - (K-1)\overline{(\zeta \circ f)}\Big)$$

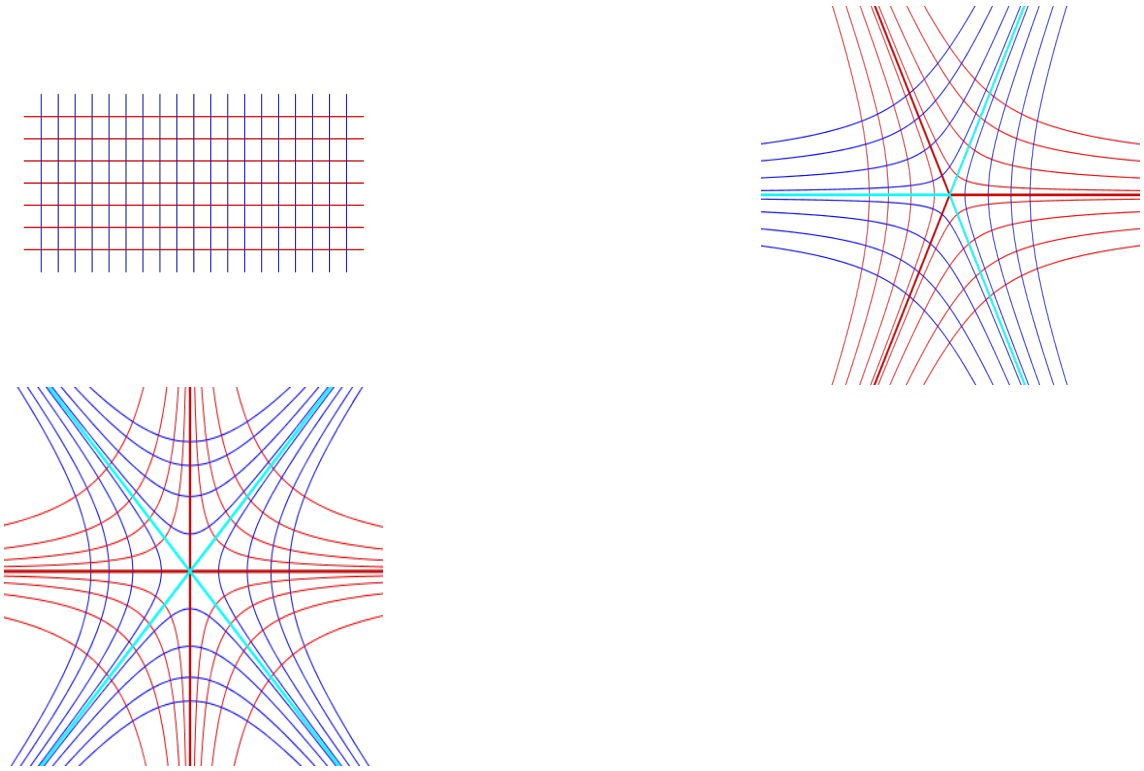

FIGURE 1.1 – Trajectoires des différentielles quadratiques $\mathrm{d}z^2$, $z\mathrm{d}z^2$ et $z^2\mathrm{d}z^2$. Les trajectoires verticales sont en bleu et les trajectoires horizontales en rouge.

est une coordonnée naturelle pour q_1.

Remarque 1.1.14. En reprenant les notations de la définition pour f, K, q_1 et q_2, localement, si $\zeta_i = \xi_i + i\eta_i$ est une coordonnée naturelle pour q_i centrée en x_i (où $x_2 = f(x_1)$ et $q_i(x_i) \neq 0$), alors

$$\zeta_2 \circ f \circ \zeta_1^{-1}(x + iy) = x + i\frac{y}{K}.$$

C'est à dire que f est affine en coordonnées naturelles.

Un homéomorphisme de Teichmüller de constante K envoie donc toute trajectoire horizontale de q_1 sur une trajectoire horizontale de q_2 de même longueur et toute trajectoire verticale de q_2 sur une trajectoire verticale de q_2 de longueur dilatée par $\frac{1}{K}$. C'est ce comportement particulier que l'on va tenter de retrouver pour des homéomorphismes quasiconformes entre variétés CR sphériques.
L'importance principale de ces homéomorphismes réside dans le théorème de Teichmüller, dont le théorème de Grötzsch peut être vu comme une version locale.

Théorème 1.1.15 (Théorème de Teichmüller)**.** *Soit $f : Y_1 \longrightarrow Y_2$ un homéomorphisme de Teichmüller de constante K entre deux surfaces de Riemann compactes de genre strictement supérieur à deux. Soit $g : Y_1 \longrightarrow Y_2$ un homéomorphisme K'-quasiconforme homotope à f.*
Alors $K' \geq K$ avec égalité si et seulement si $f = g$.

1.2 Variétés CR

1.2.1 Généralités

Une manière de voir une structure CR est comme un modèle abstrait d'hypersurface de $\mathbb{C}^{n+1}$. Démarrons tout d'abord par les structures (presque) complexes.

Définition 1.2.1 (Structure presque complexe)**.** Une structure presque complexe sur une variété (réelle) M est une section J du fibré $\mathrm{End}(TM)$ telle que $J^2 = -\mathrm{Id}$.

Remarque 1.2.2.

1. Si une variété admet une structure presque complexe, alors elle est de dimension paire.
2. Soit M une variété complexe et $(U_j, z_j = x_j + iy_j)_{j\in I}$ un atlas holomorphe de M. Alors J, définie par $J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_i}$ et $J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_i}$ est une structure presque complexe sur la variété réelle sous-jacente de M.

3. On étend linéairement J à $\mathbb{C}TM := \mathbb{C}\otimes TM$ et on note $V = \{X - iJX \mid X \in TM\}$ l'espace propre associé à la valeur propre i. Alors $V \cap \overline{V} = \{0\}$ et $\mathbb{C}TM = V \oplus \overline{V}$.
   Réciproquement, soit V un sous-fibré de $\mathbb{C}TM$ tel que $V \cap \overline{V} = \{0\}$ et $\mathbb{C}TM = V \oplus \overline{V}$. Alors, l'endomorphisme J de $\mathbb{C}TM$ dont l'espace propre associé à la valeur propre i est V et l'espace propre associé à la valeur propre $-i$ est $\overline{V}$ définit une structure presque complexe sur M. Ce qui permet de définir de manière équivalente :

Définition 1.2.3 (Structure presque complexe (2))**.** Une structure presque complexe sur une variété M est un sous-fibré V de $\mathbb{C}TM$ tel que

$$V \cap \overline{V} = \{0\} \text{ et } \mathbb{C}TM = V \oplus \overline{V}.$$

Le théorème désormais classique de Newlander et Nirenberg s'énonce :

Théorème 1.2.4 (Newlander-Nirenberg)**.** *Soit V une structure presque complexe sur une variété M. Alors (M, V) définit une structure complexe si et seulement si V est involutive ; c'est à dire que pour toutes sections locale V_1, V_2 de V, le crochet de Lie $[V_1, V_2]$ est toujours dans V.*

Après ce bref rappel, on passe aux hypersurfaces de $\mathbb{C}^{n+1}$. Prenons donc une hypersurface (réelle) M de $\mathbb{C}^{n+1}$ et notons

$$T^{1,0}\mathbb{C}^{n+1} = span\left(\frac{\partial}{\partial z_i}, i = 0, \cdots n\right) \subset \mathbb{C} \otimes T\mathbb{C}^{n+1}$$

la structure (presque) complexe standard de $\mathbb{C}^{n+1}$. On définit alors

$$T^{1,0}M = T^{1,0}\mathbb{C}^{n+1} \cap \mathbb{C}TM \text{ et } T^{0,1}M = \overline{T^{1,0}M} = \overline{T^{1,0}\mathbb{C}^{n+1}} \cap \mathbb{C}TM$$

de sorte que

1. $T^{1,0}M \cap T^{0,1}M = \{0\}$,
2. $\dim_{\mathbb{C}} T^{1,0}M = n$ et
3. $T^{1,0}M$ est involutive car $T^{1,0}\mathbb{C}^{n+1}$ l'est.

Ces trois propriétés ne faisant pas intervenir que M est une hypersurface de $\mathbb{C}^{n+1}$, elles amènent à la définition abstraite de variété CR.

Définition 1.2.5 (Variété CR)**.** Une variété CR (M, V) est la donnée d'une variété (réelle) M de dimension impaire $2n + 1$ et d'un sous-fibré V de $\mathbb{C}TM$ tel que

1. $V \cap \overline{V} = \{0\}$,
2. $\dim_{\mathbb{C}} V = n$ et
3. V est involutif.

Remarque 1.2.6. 1. Les variétés CR telles que définies ici sont parfois appelées *variétés CR de type hypersurface*. Une définition plus générale de variété CR (qui, au passage englobe celle de variété complexe) serait la donnée d'une variété M de dimension $2n + d$ et d'un sous-fibré involutif V de $\mathbb{C}TM$ de rang n tel que $V \cap \overline{V} = \{0\}$. Dans ce cas, d est appelé la *codimension* de la variété CR (M, V). Par conséquent, une variété CR de type hypersurface est une variété CR de codimension 1 tandis qu'une variété complexe est une variété CR de codimension 0.

2. Comme on a vu avant la définition de variété CR, si $M \subset \mathbb{C}^{n+1}$ est une hypersurface, alors $V = T^{1,0}M$ est une structure CR sur M. On appellera cette structure la structure CR *standard* de M.

3. Lorsque $n = 1$ (c'est à dire lorsque M est de dimension 3), la condition d'involutivité est automatiquement vérifiée pour tout sous-fibré de $\mathbb{C}TM$ de rang 1.

4. Comme dans le cas des structures (presque) complexes, on peut donner une définition réelle de variété CR. En effet, prenons (M, V) une variété CR et notons H la distribution (de rang $2n$) telle que $\mathbb{C} \otimes H = V \oplus \overline{V}$. Soit alors J l'endomorphisme de H tel que V est le sous-espace propre associé à la valeur propre i et $\overline{V}$ celui associé à $-i$. Alors tout d'abord, $J^2 = -\mathrm{Id}_H$. De plus, puisque V est involutif, pour tous $X, Y \in H$ il existe $Z \in H$ tel que

$$[X - iJX, Y - iJY] = Z - iJZ.$$

Or,

$$[X - iJX, Y - iJY] = [X, Y] - [JX, JY] - i\left([JX, Y] + [X, JY]\right).$$

Par conséquent,

$$[X, Y] - [JX, JY] \in H \text{ et } J\left([X, Y] - [JX, JY]\right) = [JX, Y] + [X, JY].$$

Réciproquement, soit H une distribution de rang $2n$ et soit J une section de

$\mathrm{End}(H)$ telle que $J^2 = -\mathrm{Id}_H$ et J est intégrable dans le sens que

$$[X,Y] - [JX,JY] \in H \text{ et } J([X,Y] - [JX,JY]) = [JX,Y] + [X,JY].$$

Alors, le sous-espace propre associé à la valeur propre i, noté V est une structure CR sur M. En effet, il est clair que V est de rang (complexe) n et est transverse à son conjugué (qui est le sous-espace propre associé à $-i$). Ce qu'il faut vérifier c'est que V est involutif. Pour cela, on prend $X, Y \in H$ et on veut montrer que $[X - iJX, Y - iJY] \in V$. Or

$$[X - iJX, Y - iJY] = [X,Y] - [JX,JY] - i([JX,Y] + [X,JY])$$

et par hypothèse,

$$[X,Y] - [JX,JY] \in H \text{ et } J([X,Y] - [JX,JY]) = [JX,Y] + [X,JY].$$

Par conséquent, en notant $Z = [X,Y] - [JX,JY] \in H$, on a

$$[X - iJX, Y - iJY] = Z - iJZ \in V.$$

Ce qui permet de définir de manière équivalente :

Définition 1.2.7 (Variété CR (2))**.** Une variété CR (M, H, J) est la donnée d'une variété M de dimension $2n+1$, d'une distribution H de rang $2n$ et d'une section intégrable J de $\mathrm{End}(H)$ telle que $J^2 = -\mathrm{Id}_H$, l'intégrabilité de J étant la vérification pour tous $X, Y \in H$ des propriétés

$$[X,Y] - [JX,JY] \in H \text{ et } J([X,Y] - [JX,JY]) = [JX,Y] + [X,JY].$$

Soit (M, V) une variété CR de dimension $2n+1$. Bien que V soit involutif, il n'y a aucune raison pour que $V \oplus \overline{V}$ le soit. La *forme de Levi* mesure à quel point $V \oplus \overline{V}$ est proche d'être involutif. On la définit comme suit. Soient $(L_1, \cdots L_n)$ un champ de bases au voisinage d'un point x de M et U un champ de vecteur réel (également au voisinage de x) transverse à $V \oplus \overline{V}$. Alors $(L_1, \cdots, L_n, \overline{L}_1, \cdots, \overline{L}_n, U)$ est un champ de bases de $\mathbb{C}TM$ au voisinage de x. On définit une matrice $g(x) = (g_{j,k})_{j,k}$ par : pour tout (j,k),

$$[L_j, \overline{L}_k] = i g_{j,k} U \bmod \{L_1, \cdots L_n, \overline{L}_1, \cdots, \overline{L}_n\}.$$

Cette matrice $g(x)$ est clairement hermitienne. De plus, bien qu'elle soit non-invariante par changement de base, sa *signature* l'est ; la signature de $g(x)$ étant le triplet (p, q, r) où p est le nombre de valeurs propres de $g(x)$ strictement positives, q le nombre de valeurs propres strictement négatives et r le nombre de valeurs propres nulles modulo l'identification de (p, q, r) avec (q, p, r).
En effet, soit $(L'_1, \cdots, L'_n)$ un autre champ de bases de V au voisinage de x et U' un autre champ de vecteur réel transverse à $V \oplus \overline{V}$. On peut alors écrire pour tout $j \in \{1, \cdots, n\}$,

$$L'_j = \sum_{k=1}^{n} A_{j,k} L_k \text{ et } U' = bU \mathrm{mod}\{L_1, \cdots, L_n, \overline{L}_1, \cdots, \overline{L}_n\}$$

où $A = (A_{j,k})_{j,k}$ est inversible et $b \neq 0$. Soit alors la matrice $g'(x) = (g'_{j,k})_{j,k}$ telle que pour tout (j, k),

$$[L'_j, \overline{L}'_k] = ig'_{j,k} \mathrm{mod}\{L'_1, \cdots, L'_n, \overline{L}'_1, \cdots, \overline{L}'_n\}.$$

On a alors pour tout (j, k),

$$\sum_{l,m} A_{j,l} \overline{A}_{k,m} [L_l, \overline{L}_m] = ibg'_{j,k} U \mathrm{mod}\{L_1, \cdots, L_n, \overline{L}_1, \cdots, \overline{L}_n\}.$$

Et donc, pour tout (j, k),

$$\sum_{l,m} A_{j,l} g_{l,m} \overline{A}_{k,m} = bg'_{j,k}.$$

Autrement dit,

$$g'(x) = b^{-1} A g(x) A^*$$

et par conséquent, $g'(x)$ a bien la même signature que $g(x)$ (l'identification de (p, q, r) avec (q, p, r) est faite car b peut très bien être strictement négatif).
Bien que non-invariante par changement de base, on appellera la matrice $g(x)$ la forme de Levi de M au point x. On pourrait donner une définition invariante par changement de base de celle-ci, néanmoins on ne s'intéressera ici qu'à sa signature et par conséquent la définition donnée est suffisante pour la suite.

Définition 1.2.8 (Variété CR strictement pseudoconvexe)**.** Une variété CR (M, V) est dite strictement pseudoconvexe si sa forme Levi en tout point est définie positive ou définie négative, c'est à dire que pour tout x, la matrice $g(x)$ est de signature

$(n, 0, 0)$ ou $(0, n, 0)$.

Comme souvent, lorsque l'on introduit une nouvelle structure, on cherche les applications qui préservent cette structure.

Définition 1.2.9 (Fonction CR)**.** Soit (M, V) une variété CR et $f : M \longrightarrow \mathbb{C}$ une fonction. On dit que f est une fonction CR si pour toute section L de V,

$$\overline{L}f = 0.$$

De manière équivalente, si (M, H, J) est une variété CR et $f : M \longrightarrow \mathbb{C}$ est une fonction, f est CR si

$$\mathrm{d}f \circ J = J_0 \circ \mathrm{d}f$$

où J_0 est la structure complexe standard de $\mathbb{C}$.

Lorsque la variété M est une hypersurface de $\mathbb{C}^{n+1}$, les restrictions à M de fonctions holomorphes sur des ouverts de $\mathbb{C}^{n+1}$ sont naturellement des fonctions CR. La réciproque n'est en générale pas vraie, néanmoins, on a (voir [31, p. 18]) :

Théorème 1.2.10. *Soient M une hypersurface analytique réelle de $\mathbb{C}^{n+1}$, $p \in M$, U un voisinage ouvert de p dans M et $f : U \longrightarrow \mathbb{C}$ une fonction CR analytique réelle. Alors f est la restriction à M d'une fonction holomorphe sur un voisinage ouvert de p dans $\mathbb{C}^{n+1}$*

Pour ce qui est des applications entre variétés CR, on définit :

Définition 1.2.11. Soient (M_1, V_1) et (M_2, V_2) deux variétés CR. Une application $f : M_1 \longrightarrow M_2$ est dite CR si

$$f_* V_1 \subset V_2.$$

De manière équivalente, si (M_1, H_1, J_1) et (M_2, H_2, J_2) sont deux variétés CR, une application $f : M_1 \longrightarrow M_2$ est CR si

$$\mathrm{d}f \circ J_1 = J_2 \circ \mathrm{d}f.$$

Si de plus f est un difféomorphisme, on dit que c'est un difféomorphisme CR.

Deux problèmes naturels se posent en géométrie CR. Le premier est celui du plongement global qui peut être vu comme l'envie d'avoir un analogue CR aux théorèmes de plongement de Whitney et de Kodaira. La question est alors : étant

donnée une variété CR M, existe-t-il un plongement CR de M dans un $\mathbb{C}^N$. Dans les années 70, Boutet de Monvel [11] a montré que toute variété CR strictement pseudoconvexe compacte de dimension supérieure ou égale à 5 peut être CR-plongée dans un $\mathbb{C}^N$. On peut également citer comme exemples de variétés CR-plongeables le cas des 3-variétés CR strictement pseudoconvexes munies d'une action CR du cercle [27, 39]. Mais en général, les variétés CR strictement pseudoconvexes compactes de dimension 3 ne sont pas forcément CR-plongeables [47].
Le second problème est celui du plongement local qui est bien plus spécifique à la géométrie CR. Il est connu que toute variété CR strictement pseudoconvexe de dimension supérieure ou égale à 7 est localement CR-plongeable [3, 38]. Pour ce qui est de la dimension 3, il existe des exemples de variétés strictement pseudoconvexes qui ne sont pas localement CR-plongeables [18, 42].

1.2.2 Le complexe de Cauchy-Riemann tangentiel

Le complexe de Cauchy-Riemann tangentiel, aussi appelé complexe de Kohn-Rossi [34], est un objet important de la géométrie CR. C'est à la fois un analogue CR du complexe de Dolbeault et en même temps un complexe de Dolbeault "induit". On en donne deux constructions (constructions tirées de [10] bien que l'on ait choisi ici de se passer de métriques). La première est une construction pour une hypersurface de $\mathbb{C}^{n+1}$ rendant compte de l'idée d'un complexe de Dolbeault induit, reprenant la terminologie de [10], on l'appellera extrinsèque. La seconde est une construction valable pour toute variété CR, ici encore, on reprendra la terminologie de [10] et on l'appellera intrinsèque.

Commençons donc par l'extrinsèque. Soit M une hypersurface de $\mathbb{C}^{n+1}$ définie par $\rho : \mathbb{C}^{n+1} \longrightarrow \mathbb{R}$ (i.e. ρ est une submersion au-dessus de 0 et $M = \rho^{-1}(0)$). Pour tout (p, q), on note $I^{p,q}$ l'idéal de $\Omega^{p,q}(\mathbb{C}^{n+1})$ engendré par ρ et $\overline{\partial}\rho$ où $\Omega^{p,q}(\mathbb{C}^{n+1})$ est l'espace des formes de bidegré (p, q) sur $\mathbb{C}^{n+1}$ et $\overline{\partial}$ est l'opérateur de Dolbeault. Si bien qu'un élément de $I^{p,q}$ est du type

$$\rho\alpha + \beta \wedge \overline{\partial}\rho \text{ avec } \alpha \in \Omega^{p,q}\left(\mathbb{C}^{n+1}\right) \text{ et } \beta \in \Omega^{p,q-1}\left(\mathbb{C}^{n+1}\right).$$

On définit alors

$$\mathcal{E}^{p,q} = \Omega^{p,q}\left(\mathbb{C}^{n+1}\right) \big/ I^{p,q}$$

et on note $t_M : \Omega^{p,q}(\mathbb{C}^{n+1}) \longrightarrow \mathcal{E}^{p,q}$ la projection canonique.

Définition/Proposition 1.2.12 ($\overline{\partial}_b$ extrinsèque)**.** *Soient $\alpha \in \mathcal{E}^{p,q}$ et $\widetilde{\alpha}$ un relevé de α dans $\Omega^{p,q}(\mathbb{C}^{n+1})$. On pose*

$$\overline{\partial}_b \alpha = t_M\left(\overline{\partial}\widetilde{\alpha}\right).$$

Alors, $\overline{\partial}_b\alpha$ ne dépend pas du choix de $\widetilde{\alpha}$ et ainsi l'opérateur $\overline{\partial}_b : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p,q+1}$ est bien défini. De plus,

$$\overline{\partial}_b^2 = 0.$$

Démonstration. Soient $\widetilde{\alpha}, \widetilde{\beta} \in \Omega^{p,q}(\mathbb{C}^{n+1})$ tels que $t_M(\widetilde{\alpha}) = t_M\left(\widetilde{\beta}\right)$. Alors, par définition,

$$\widetilde{\beta} = \widetilde{\alpha} + \rho\gamma + \delta \wedge \overline{\partial}\rho \text{ avec } \gamma \in \Omega^{p,q}\left(\mathbb{C}^{n+1}\right) \text{ et } \delta \in \Omega^{p,q-1}\left(\mathbb{C}^{n+1}\right).$$

Ainsi,

$$\begin{aligned} \overline{\partial}\widetilde{\beta} &= \overline{\partial}\widetilde{\alpha} + \overline{\partial}\rho \wedge \gamma + \rho \wedge \overline{\partial}\rho + \overline{\partial}\delta \wedge \overline{\partial}\rho \\ &= \overline{\partial}\widetilde{\alpha} + \mu \end{aligned}$$

où $\mu = \rho\overline{\partial}\gamma + \left(\overline{\partial}\delta - \gamma\right) \wedge \overline{\partial}\rho \in I^{p,q+1}$. Donc

$$t_M\left(\overline{\partial}\widetilde{\beta}\right) = t_M\left(\overline{\partial}\widetilde{\alpha}\right)$$

et par conséquent, $\overline{\partial}_b$ est bien défini sur $\mathcal{E}^{p,q}$. Le fait que $\overline{\partial}_b^2 = 0$ découle immédiatement du fait que $\overline{\partial}^2 = 0$. □

$\overline{\partial}_b$ et les $\mathcal{E}^{p,q}$ forment donc un complexe différentiel pour chaque p. Ce sont ces complexes que l'on nomme complexes de Cauchy-Riemann (ici extrinsèques).

Passons maintenant à l'intrinsèque. Soit (M, V) une variété CR de dimension $2n+1$. De manière générale, si E est un fibré vectoriel au-dessus de M, on notera $\Gamma(E)$ l'ensemble de ses sections lisses. Pour tout (p,q), on considère

$$A^{p,q} = \{\alpha \in \wedge^{p+q}\mathbb{C}T^*M \mid \alpha(v_1, \cdots, v_{p+q}) = 0 \text{ si card}\left(\{v_1, \cdots, v_{p+q}\} \cap \overline{V}\right) > q\}$$

et on définit

$$\wedge^{p,q}T^*M = {A^{p,q}}\big/{A^{p+1,q-1}} \text{ et } \mathcal{E}^{p,q} = \Gamma\left(\wedge^{p,q}T^*M\right)$$

où on remarque que $A^{l,m} = \{0\}$ si m est strictement négatif.

Remarque 1.2.13. Il semble intéressant de constater ici une certaine asymétrie qui se détecte entre $\wedge^{1,0}T^*M$ et $\wedge^{0,1}T^*M$. En effet, les trois fibrés mis en jeu dans $\wedge^{1,0}T^*M$ et $\wedge^{0,1}T^*M$ sont

$$A^{1,0} = \{\alpha \in \mathbb{C}T^*M \mid \alpha_{|\overline{V}} = 0\},\ A^{2,-1} = \{0\} \text{ et } A^{0,1} = \mathbb{C}T^*M.$$

Par conséquent, la restriction à $\overline{V}$ d'une 1-forme fournit un isomorphisme naturel entre $\wedge^{0,1}T^*M = {\mathbb{C}T^*M}/{\{\alpha \in \mathbb{C}T^*M \mid \alpha_{|\overline{V}} = 0\}}$ et $\overline{V}^*$ tandis que $\wedge^{1,0}T^*M$ n'est pas isomorphe à V^* puisque $\wedge^{1,0}T^*M = \{\alpha \in \mathbb{C}T^*M \mid \alpha_{|\overline{V}} = 0\}$ est de rang $n+1$ tandis que V^* est de rang n.
C'est cette asymétrie que l'on va contourner au chapitre 3 en considérant la décomposition de Garfield-Lee du complexe de Rumin plutôt que le complexe de Cauchy-Riemann tangentiel pour définir un analogue CR aux formes holomorphes sur une surface de Riemann.

La différentielle extérieure d est bien définie sur les sections de $A^{p,q}$ puisque celles-ci sont des (p,q)-formes sur M.

Lemme 1.2.14. *Pour tout couple (p,q), on a*

$$\mathrm{d}\left(\Gamma\left(A^{p,q}\right)\right) \subset \Gamma\left(A^{p,q+1}\right).$$

Démonstration. Soit α une section de $A^{p,q}$ et soient $v_0, \cdots, v_{p+q}$ des vecteurs tangents à M tels que $\mathrm{card}\left(\{v_0, \cdots, v_{p+q}\} \cap \overline{V}\right) > q+1$. Alors,

$$\begin{aligned}\mathrm{d}\alpha = \sum_{k=0}^{p+q}(-1)^k L_{v_k}\left(\alpha\left(v_0, \cdots, \widehat{v_k}, \cdots, v_{p+q}\right)\right) \\ + \sum_{0\le i<j\le p+q}(-1)^{i+j}\alpha\left([v_i, v_j], v_0, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_{p+q}\right)\end{aligned}$$

où L_v désigne la dérivée de Lie suivant v et l'accent circonflexe signifie l'omission du vecteur dessous. Par définition de α, chaque $\alpha\left(v_0, \cdots, \widehat{v_k}, \cdots, v_{p+q}\right)$ est nul car $\mathrm{card}\left(\{v_0, \cdots, \widehat{v_k}, \cdots, v_{p+q}\} \cap \overline{V}\right) > q$ et, comme $\overline{V}$ est involutif, on a aussi, $\mathrm{card}\left(\{[v_i, v_j], v_0, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_{p+q}\} \cap \overline{V}\right) > q$. Donc $\mathrm{d}\alpha \in \Gamma\left(A^{p,q+1}\right)$. □

Pour tout (p,q), on note $\Pi^{p,q}$ la projection canonique de $\Gamma\left(A^{p,q}\right)$ sur $\mathcal{E}^{p,q}$.

Définition/Proposition 1.2.15 ($\overline{\partial}_b$ intrinsèque)**.** *Il existe $\overline{\partial}_b : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p,q+1}$ tel*

que le diagramme

$$\begin{array}{ccc} \Gamma\left(A^{p,q}\right) & \xrightarrow{\mathrm{d}} & \Gamma\left(A^{p,q+1}\right) \\ \Pi^{p,q}\Big\downarrow & & \Big\downarrow\Pi^{p,q+1} \\ \mathcal{E}^{p,q} & \xrightarrow{\overline{\partial}_b} & \mathcal{E}^{p,q+1} \end{array}$$

commute. De plus,

$$\overline{\partial}_b^2 = 0.$$

Démonstration. Si $\alpha \in \mathcal{E}^{p,q}$, on pose

$$\overline{\partial}_b \alpha = \Pi^{p,q+1}\left(\mathrm{d}\widetilde{\alpha}\right)$$

où $\widetilde{\alpha}$ est une section de $A^{p,q}$ telle que $\Pi^{p,q}\left(\widetilde{\alpha}\right) = \alpha$. $\overline{\partial}_b$ est bien défini car

$$\mathcal{E}^{p,q} = \Gamma\left(A^{p,q}\right)\big/\Gamma\left(A^{p+1,q-1}\right),\ \mathcal{E}^{p,q+1} = \Gamma\left(A^{p,q+1}\right)\big/\Gamma\left(A^{p+1,q}\right) \text{ et}$$

$$\mathrm{d}\left(\Gamma\left(A^{p+1,q-1}\right)\right) \subset \Gamma\left(A^{p+1,q}\right).$$

$\overline{\partial}_b^2 = 0$ découle alors de $\mathrm{d}^2 = 0$. □

Comme pour l'extrinsèque, $\overline{\partial}_b$ et les $\mathcal{E}^{p,q}$ forment un complexe différentiel pour chaque p. Ces complexes sont appelés complexes de Cauchy-Riemann tangentiels (ici, intrinsèques).

Lorsque la variété CR est une hypersurface de $\mathbb{C}^{n+1}$, les complexes de Cauchy-Riemann tangentiels extrinsèques et intrinsèques sont naturellement isomorphes (Theorem 1 de [10, p. 135]). Par conséquent, on parlera juste de complexe de Cauchy-Riemann tangentiel (ou complexe de Kohn-Rossi, ou encore complexe $\overline{\partial}_b$) et on notera $H_b^{p,q}$ ses espaces de cohomologie. Pour le résultat suivant concernant les $H_b^{p,q}$, on renvoie à [21, p. 88].

Proposition 1.2.16. *Soit M une variété CR strictement pseudoconvexe de dimension $2n+1 \geq 5$, alors pour tout p et tout $q \in \{1, \cdots, n-1\}$, $H_b^{p,q}(M)$ est de dimension finie.*

Dans la suite, on va assez vite se limiter aux variétés CR de dimension 3, et parmi celles-ci, une classe toute particulière nous intéressera : les variétés *CR sphériques* [12].

1.3 Géométrie CR sphérique

1.3.1 Premières définitions

Parmi les variétés CR, on a vu que les hypersurfaces de $\mathbb{C}^{n+1}$ jouent un rôle particulier et l'hypersurface de $\mathbb{C}^{n+1}$ la plus basique est la sphère unité S^{2n+1}.

Définition 1.3.1 (Variété CR sphérique)**.** Une variété CR sphérique de dimension $2n+1$ est une variété CR (M, V) de dimension $2n+1$ localement CR-difféomorphe à la sphère unité $S^{2n+1} \subset \mathbb{C}^{n+1}$ munie de sa structure CR standard.

Dans toute la suite de cette partie, on ne considèrera que des variétés CR sphériques de dimension 3.

Pour étudier de telles variétés, il semble pour commencer indispensable de comprendre plus en profondeur S^3 munie de sa structure CR standard. Cela nécessite un léger détour vers la géométrie du plan hyperbolique complexe $\mathbb{H}^2_{\mathbb{C}}$. On ne fera qu'esquisser cette géométrie ici, le but étant d'arriver à la géométrie CR sphérique ce qui nécessite d'introduire proprement le groupe $\mathrm{PU}(2,1)$. Pour une introduction à la géométrie hyperbolique complexe, on renvoie au livre de Goldman [23].

On note $\mathbb{C}^{2,1}$ l'espace vectoriel $\mathbb{C}^3$ muni de la forme hermitienne de signature $(2,1)$, $< .,. >$, définie pour tous vecteurs $u = (u_1, u_2, u_3)$ et $v = (v_1, v_2, v_3)$ de $\mathbb{C}^3$ par

$$< u, v >= u_1\overline{v_1} + u_2\overline{v_2} - u_3\overline{v_3}$$

dont la matrice hermitienne associée est donc

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

On considère alors

$$V_0 = \{z \in \mathbb{C}^{2,1} \backslash \{0\} \mid \ < z, z >= 0\} \text{ et } V_- = \{z \in \mathbb{C}^{2,1} \mid \ < v, v >< 0\}.$$

Soit $\mathbb{P} : \mathbb{C}^{2,1} \backslash \{0\} \longrightarrow \mathbb{CP}^2$ la projection canonique. Dans la carte $\{z_3 \neq 0\} \subset \mathbb{C}^{2,1}$,

$\mathbb{P}$ est donnée pour tout $(z_1, z_2, z_3) \in \{z_3 \neq 0\}$ par

$$\mathbb{P}(z_1, z_2, z_3) = \left(\frac{z_1}{z_3}, \frac{z_2}{z_3}\right).$$

Comme $< .,. >$ est une forme hermitienne, pour tout nombre complexe non nul λ et tout vecteur $z \in \mathbb{C}^{2,1}$, $\lambda z \in V_0$ (resp. $\lambda z \in V_-$) si et seulement si $z \in V_0$ (resp. $z \in V_-$).
Le *modèle projectif* du plan hyperbolique complexe $\mathbb{H}^2_{\mathbb{C}}$ est $\mathbb{P}(V_-)$, son bord est $\mathbb{P}(V_0)$.

En considérant les intersections respectives de V_- et de V_0 avec le plan $\{z_3 = 1\}$ de $\mathbb{C}^{2,1}$, on obtient comme modèle de $\mathbb{H}^2_{\mathbb{C}}$ la boule unité $\mathbb{B}$ de $\mathbb{C}^2$, dont le bord est la sphère unité S^3 de $\mathbb{C}^2$.

Par ailleurs, en prenant pour autre forme hermitienne de signature $(2, 1)$, la forme hermitienne de matrice

$$H_2 = \begin{pmatrix} 0 & 0 & -\frac{i}{2} \\ 0 & 1 & 0 \\ \frac{i}{2} & 0 & 0 \end{pmatrix}$$

on obtient un troisième modèle de $\mathbb{H}^2_{\mathbb{C}}$: le domaine de Siegel

$$\mathbb{S} = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Im(z_1) > |z_2|^2\}$$

dont le bord est

$$\partial\mathbb{S} = \{(z_1, z_2) \in \mathbb{C}^2 \mid \Im(z_1) = |z_2|^2\} \cup \{\infty\}.$$

Notons à ce stade que parfois le domaine de Siegel est défini autrement selon le choix de la forme hermitienne de signature $(2, 1)$. Par exemple, la forme hermitienne de signature $(2, 1)$ dont la matrice est $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ donne comme domaine correspondant à $V_- \cap \{z_3 = 1\}$ le domaine $\{(z_1, z_2) \in \mathbb{C}^2 \mid 2\Re(z_1) + |z_2|^2 < 0\}$ dont le bord est $\{(z_1, z_2) \in \mathbb{C}^2 \mid 2\Re(z_1) + |z_2|^2 = 0\} \cup \{\infty\}$. Le choix de la matrice H_2 a été fait ici afin de faire naturellement apparaître la *projection de Korányi* plus loin.

Le groupe unitaire associé à la forme hermitienne $< .,. >$ est $\mathrm{U}(2, 1) = \{A \in M_3(\mathbb{C}) \mid A^* H_1 A = H_1\}$ où A^* est la transconjuguée de A. $\mathrm{U}(2, 1)$ préserve donc

automatiquement V_-. On obtient de ce fait une action de $\mathrm{U}(2,1)$ sur $\mathbb{B}$ que l'on peut donner explicitement : $A = (a_{i,j})_{1\leq i,j\leq 3} \in \mathrm{U}(2,1)$ agit sur $\mathbb{B}$ par

$$(z_1, z_2) \longmapsto \left(\frac{a_{1,1}z_1 + a_{1,2}z_2 + a_{1,3}}{a_{3,1}z_1 + a_{3,2}z_2 + a_{3,3}}, \frac{a_{2,1}z_1 + a_{2,2}z_2 + a_{2,3}}{a_{3,1}z_1 + a_{3,2}z_2 + a_{3,3}} \right).$$

Réciproquement, tout biholomorphisme de $\mathbb{B}$ dans lui-même est l'action d'un élément de $\mathrm{U}(2,1)$ (voir par exemple [31, p. 41]).

L'action de $\mathrm{U}(2,1)$ sur la boule unité de $\mathbb{C}^2$ n'est pas fidèle puisque pour tout $\lambda \in \mathrm{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\}$, l'action de $\lambda I_3 \in \mathrm{U}(2,1)$ donne l'identité de $\mathbb{B}$. Pour la rendre fidèle, il suffit de quotienter $\mathrm{U}(2,1)$ par $\mathrm{U}(1)$. Ce quotient est le groupe projectif unitaire $\mathrm{PU}(2,1)$. Ainsi donc, $\mathrm{PU}(2,1)$ est le groupe des biholomorphismes de $\mathbb{B}$ dans lui-même.

Comme $\mathrm{U}(2,1)$ préserve aussi V_0, l'action holomorphe de $\mathrm{PU}(2,1)$ sur $\mathbb{B}$ se prolonge en une action CR de $\mathrm{PU}(2,1)$ sur la sphère S^3 (munie de sa structure CR standard). Il s'avère alors que $\mathrm{PU}(2,1)$ est le groupe des difféomorphismes CR de S^3 dans lui-même et qu'il vérifie un théorème de type Liouville (voir par exemple [12, p. 224]).

Théorème 1.3.2 (Théorème de type Liouville)**.** *Soit $f : U \longrightarrow U'$ un difféomorphisme CR entre ouverts de la sphère S^3 (munie de sa structure CR standard). Alors f est la restriction à U de l'action d'un élément de* $\mathrm{PU}(2,1)$.

Le groupe de Heisenberg

Les matrices unipotentes triangulaires supérieures de $\mathrm{SU}(2,1)$ (ici, $\mathrm{SU}(2,1)$ désigne l'ensemble des matrices de déterminant 1 qui préservent la matrice H_2) sont de la forme

$$A_{(z,t)} = \begin{pmatrix} 1 & 2i\overline{z} & t + i|z|^2 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

avec $z \in \mathbb{C}$ et $t \in \mathbb{R}$. On définit alors l'application

$$\begin{array}{rccc} \phi : & \mathbb{C} \times \mathbb{R} & \longrightarrow & \mathrm{SU}(2,1) \\ & (z,t) & \longmapsto & A_{(z,t)} \end{array}$$

Pour faire de ϕ un homomorphisme, il faut munir $\mathbb{C} \times \mathbb{R}$ de la loi de groupe

$$(z,t)(z',t') = (z + z', t + t' + 2\Im(z\overline{z}')).$$

$\mathbb{C} \times \mathbb{R}$ muni de cette loi est le groupe de Heisenberg, que l'on note $\mathcal{H}$. En termes réels, $\mathcal{H}$ est $\mathbb{R}^3$ muni de la loi

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2(x'y - xy')) .$$

Sur $\mathcal{H}$, il vient naturellement deux champs de vecteurs invariants à gauche et linéairement indépendants

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t} \text{ et } Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$$

qui permettent de définir une structure CR sur $\mathcal{H}$: d'abord en termes réels, $\mathbf{P} = span(X, Y)$ pour le champ d'hyperplans et J définie par $JX = Y$ et $JY = -X$ pour la structure presque-complexe sur $\mathbf{P}$ (on a indiqué précédemment que la condition d'intégrabilité est automatique en dimension 3). Puis en termes complexes, $V = span(Z)$ où

$$Z = \frac{1}{2}(X - iY) = \frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t}$$

avec $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$. $(\mathcal{H}, V)$ est de plus strictement pseudoconvexe. En effet, comme $\mathcal{H}$ est de dimension 3, il suffit de vérifier que $Z, \overline{Z}$ et $[Z, \overline{Z}]$ sont linéairement indépendants. Or, un calcul immédiat montre que

$$[Z, \overline{Z}] = -2iT$$

où $T = \frac{\partial}{\partial t}$. Par conséquent, il est clair que $Z, \overline{Z}$ et $[Z, \overline{Z}]$ sont linéairement indépendants.

$(\mathcal{H}, \mathbf{P}, J)$ étant strictement pseudoconvexe, $\mathbf{P}$ est une *structure de contact* sur $\mathcal{H}$. C'est à dire que $\mathbf{P}$ est localement le noyau d'une 1-forme ω telle que $\mathrm{d}\omega$ est non-dégénérée sur $\mathbf{P}$ (ou de manière équivalente, telle que $\omega \wedge \mathrm{d}\omega$ est partout non nulle).

L'homomorphisme ϕ fournit un difféomorphisme CR naturel du groupe de Heisenberg sur le bord (moins le point à l'infini) du domaine de Siegel. Ce difféomorphisme CR est :

$$\begin{array}{rccc} \varphi : & \mathcal{H} & \longrightarrow & \partial\mathbb{S}\backslash\{\infty\} \\ & (z, t) & \longmapsto & (t + i|z|^2, z) \end{array}$$

En effet, comme $\partial\mathbb{S}\backslash\{\infty\}$ est muni de sa structure CR standard, il suffit de montrer

que $(z,t) \longmapsto t + i|z|^2$ et $(z,t) \longmapsto z$ sont des fonctions CR sur $(\mathcal{H}, V)$. Autrement dit, il suffit de remarquer que $\overline{Z}\left((z,t) \longmapsto t + i|z|^2\right) = 0$ et $\overline{Z}\left((z,t) \longmapsto z\right) = 0$.

En rajoutant un point à l'infini, on obtient un difféomorphisme CR entre $(\mathcal{H} \cup \{\infty\}, V)$ et S^3 muni de sa structure CR standard. En effet,

$$\begin{array}{ccc} \mathbb{B} & \longrightarrow & \mathbb{S} \\ (w_1, w_2) & \longmapsto & \left(i\frac{1-w_2}{1+w_2}, \frac{iw_1}{1+w_2}\right) \end{array}$$

est un biholomorphisme qui se prolonge en un difféomorphisme CR ψ de $S^3 \backslash \{(0,-1)\}$ sur $\partial\mathbb{S} \backslash \{\infty\}$ munis de leur structure CR standard respectives. La composée

$$\begin{array}{rccc} \varphi^{-1} \circ \psi : & S^3 \backslash \{(0,-1)\} & \longrightarrow & \mathcal{H} \\ & (w_1, w_2) & \longmapsto & \left(i\frac{w_1}{1+w_2}, \frac{2\Im(w_2)}{|1+w_2|^2}\right) \end{array}$$

est donc un difféomorphisme CR d'inverse

$$(z,t) \longmapsto \left(\frac{-2iz}{1+|z|^2-it}, \frac{1-|z|^2+it}{1+|z|^2-it}\right)$$

et qui se prolonge à $\S^3$. $\mathrm{PU}(2,1)$ s'avère ainsi être également le groupe des automorphismes CR de $(\mathcal{H} \cup \{\infty\}, V)$ et le théorème de type Liouville reste aussi valable sur $\mathcal{H} \cup \{\infty\}$. Il est également connu (voir [23] par exemple) qu'un automorphisme CR de $\mathcal{H} \cup \{\infty\}$ est une composition des quatre types de difféomorphismes CR suivants :

1. les translations à gauche $L_{(z_0,t_0)} : (z,t) \longmapsto (z + z_0, t + t_0 + 2\Im(z_0\overline{z}))$, pour $(z_0, t_0) \in \mathcal{H}$;
2. les rotations autour de l'axe vertical $r_\mu : (z,t) \longmapsto (\mu z, t)$, pour $\mu \in \mathrm{U}(1)$;
3. les dilatations $\delta_\lambda : (z,t) \longmapsto (\lambda z, \lambda^2 t)$, pour $\lambda > 0$; et enfin
4. l'inversion $I : (z,t) \longmapsto \left(\frac{z}{t+i|z|^2}, \frac{t}{t^2+|z|^4}\right)$.

1.3.2 Des variétés CR sphériques à la géométrie CR sphérique

Le théorème de type Liouville est capital. Il permet de munir toute variété CR d'une structure (G, X) avec $G = \mathrm{PU}(2,1)$ et $X = S^3$ ou $\mathcal{H}$.

Définition 1.3.3 (Structure (G, X))**.** Soient X une variété connexe et G un groupe de difféomorphismes de X qui agit transitivement et analytiquement sur X (le caractère analytique de l'action est défini par : si de deux éléments de G coincident

sur un ouvert de X alors ils coincident sur tout X). Une variété M est munie d'une structure (G, X) s'il existe un recouvrement ouvert $(U_i)_{i\in I}$ de M et des plongements $\varphi_i : U_i \longrightarrow X$ dont les fonctions de transitions $\varphi_j \circ \varphi_i^{-1}$ sont des restrictions d'éléments de G.
La donnée d'un tel recouvrement ouvert et de tels plongements est appelé un atlas (G, X) de M.

À une structure (G, X) sur une variété M sont associés deux objets : une application développante et un morphisme d'holonomie. On en rappelle brièvement la construction et on renvoie à [52] pour un traitement plus approfondi. Une application développante est un (G, X)-morphisme du revêtement universel de M, $\widetilde{M}$, dans X. On peut en construire une en prolongeant à $\widetilde{M}$ une carte d'un atlas (G, X) sur $\widetilde{M}$. Pour être plus précis, prenons $(U_i, \varphi_i)_i \in I$ un atlas (G, X) de $\widetilde{M}$ tel que $U_i \cap U_{i+1} \neq \emptyset$. Par définition d'un atlas (G, X), pour tout i, il existe $g_i \in G$ tel que $\varphi_{i-1} = g_i \circ \varphi_i$. Si $p \in U_i$, on pose

$$D(p) = g_1 \circ \cdots \circ g_i(\varphi_i(p)).$$

D est bien définie sur $\widetilde{M}$ car si $p \in U_i \cap U_{i+j}$ alors

$$g_1 \circ \cdots \circ g_{i+j}(\varphi_{i+j}(p)) = g_1 \circ \cdots \circ g_i(\varphi_i(p)).$$

Il est clair que D est un morphisme (G, X), c'est à dire un difféomorphisme local qui, lu dans les cartes, est l'action d'un élément de G.

Par analycité de l'action, deux (G, X)-morphismes D et D' de $\widetilde{M}$ dans X ne diffèrent que par l'action d'un élément de G (i.e. il existe $g \in G$ tel que $D' = g \circ D$). Soit $p \in M$ et notons $\pi_1 = \pi_1(M, p)$ le groupe fondamental de M. Le morphisme d'holonomie associé à une application développante D est l'homomorphisme ρ : $\pi_1 \longrightarrow G$ tel que pour tout $\gamma \in \pi_1$,

$$D \circ \gamma = \rho(\gamma) \circ D.$$

Par ailleurs, si $g \in G$, alors l'holonomie de $g \circ D$ est la conjugaison par g de celle de D. Ainsi, la donnée d'une structure (G, X) sur M est la donnée d'une classe d'équivalence développante-holonomie (D, ρ) pour la relation : (D, ρ) est équivalent à $(gD, g\rho g^{-1})$.

L'existence d'une structure (G, X) sur une variété M se traduit donc comme l'existence d'une représentation du groupe fondamental de M dans G qui soit l'ho-

lonomie d'une structure (G, X). En ce qui concerne les déformations de structure (G, X), le résultat général est le *principe d'Ehresmann-Thurston* qui stipule que si M est compacte et ρ_0 est l'holonomie d'une structure (G, X) sur M, alors toute représentation suffisamment proche de ρ_0 est l'holonomie d'une structure (G, X).

Parmi les structures (G, X), certaines sont naturelles. Par exemple, si Γ est un sous-groupe de G qui agit librement et proprement discontinument sur X, le quotient X/Γ est naturellement muni d'une structure (G, X) et dans ce cas, l'application développante est un revêtement de X. Une structure (G, X) sur une variété pour laquelle l'application développante est un revêtement, est appelée *complète* (certains auteurs appellent une structure complète lorsque la développante est un revêtement sur son image). Lorsque Γ est un sous-groupe de G qui agit librement et proprement discontinument sur un ouvert Ω de X, le quotient Ω/Γ est muni d'une structure (G, X). Une structure (G, X) sur une variété est dite *uniformisable* lorsqu'elle est (G, X)-équivalente à un tel quotient.

Grâce au théorème de type Liouville, on peut donc définir une variété CR sphérique comme une variété munie d'une structure $(\mathrm{PU}(2,1), S^3)$ (ou $(\mathrm{PU}(2,1), \mathcal{H})$). Un atlas $(\mathrm{PU}(2,1), \mathcal{H})$ est appelé un atlas CR sphérique de la variété. La recherche de structure CR sphérique sur une variété est un domaine actif de la recherche actuelle. Parmi les résultats positifs, c'est à dire d'existence de structure CR sphérique, on peut citer par exemple les fibrés en cercles au-dessus de surfaces hyperboliques [19] ou le complémentaire du noeud de huit [17]. Le tore de dimension 3 en revanche est connu pour ne pas admettre de structure CR sphérique [24].

1.4 Homéomorphismes quasiconformes sur le groupe de Heisenberg

On s'intéresse ici à la théorie générale des homéomorphismes quasiconformes sur le groupe de Heisenberg. La première définition est due à Mostow [41]. La théorie générale des homéomorphismes quasiconformes a été ensuite développée (entre autres) par Korányi et Reimann [35] où leurs premières propriétés importantes ont été prouvées sous réserve d'hypothèses de régularité. Pour obtenir des définitions analytiques et géométriques équivalentes que l'on peut trouver dans [36], il a fallu trouver la "bonne" régularité de ces homéomorphismes. Notamment, il a fallu considérer la notion de *P-différentiabilité* [43].

La théorie des homéomorphismes quasiconformes sur le groupe de Heisenberg a

été une théorie pionnière dans l'élaboration de théories analogues sur des espaces métriques plus généraux. Pour de telles théories, on mentionne à nouveau le travail de Heinonen et Koskela [25, 26]. Le premier article sur les groupes de Carnot (dont le groupe de Heisenberg fait partie) et le second sur des espaces métriques plus abstraits.

Dans toute cette partie (et dans toute la suite), on reprendra les notations introduites avant concernant le groupe de Heisenberg, à savoir $X, Y, Z, \overline{Z}, T$, $\mathbf{P}$. De plus, on ne considèrera que des homéomorphismes préservant l'orientation en grande partie car l'espace de Teichmüller d'une variété CR sphérique sera défini à partir d'homéomorphismes quasiconformes préservant l'orientation.

1.4.1 Définition métrique et propriétés des homéomorphismes quasiconformes réguliers

On peut munir le groupe de Heisenberg d'une distance invariante à gauche. Celle-ci est donnée pour tous points $(z,t),(z',t') \in \mathcal{H}$ par

$$d_{\mathcal{H}}((z,t),(z',t')) = \|(z,t)^{-1}(z',t')\|_{\mathcal{H}}$$

où $\|(z,t)\|_{\mathcal{H}} = (t^2 + |z|^4)^{\frac{1}{4}}$. Comme on l'a indiqué, la définition métrique d'un homéomorphisme quasiconforme entre ouverts de $\mathbb{C}$ a pour avantage de pouvoir s'étendre à d'autres contextes, et en particulier à celui du groupe de Heisenberg muni de la distance $d_{\mathcal{H}}$. Soit $f : U \longrightarrow U'$ un homéomorphisme entre domaines de $\mathcal{H}$. On note

$$L_f(p,r) = \sup_{d_h(p,q)=r} d_h\left(f(p), f(q)\right) \text{ et } l_f(p,r) = \inf_{d_h(p,q)=r} d_h\left(f(p), f(q)\right)$$

où $d_{\mathcal{H}}(p, \partial U) > r$ et

$$H_f(p) = \limsup_{r \mapsto 0} \frac{L_f(p,r)}{l_f(p,r)}.$$

Par analogie avec la définition métrique d'un homéomorphisme quasiconforme entre ouverts de $\mathbb{C}$, on définit :

Définition 1.4.1 (Homéomorphisme quasiconforme : définition métrique)**.** Un homéomorphisme $f : U \longrightarrow U'$ entre domaines de $\mathcal{H}$ est K-quasiconforme si

$$\operatorname*{ess\,sup}_{p \in U} |H_f(p)| \leq K.$$

Pour construire des homéomorphismes quasiconformes qui ne sont pas des actions de $\mathrm{PU}(2,1)$, commençons par montrer que les homéomorphismes quasiconformes de classe C^1 sont des transformations de contact.

Théorème 1.4.2. *Soit $f : U \longrightarrow U'$ un homéomorphisme quasiconforme de classe C^1 entre domaines de $\mathcal{H}$. Alors f est une transformation de contact, c'est à dire que $f_*\boldsymbol{P} = \boldsymbol{P}$.*

Démonstration. Soit $p \in U$. On veut montrer que $f_*\mathbf{P}_p = \mathbf{P}_{f(p)}$. Puisque les translations à gauche sont des transformations de contact, on peut supposer que $p = f(p) = 0$.
Soit $V_1 = x_1X + y_1Y \in \mathbf{P}_0$ tel que $f_*V_1 = \widetilde{x}_1X + \widetilde{y}_1Y \in \mathbf{P}_0$. Un tel vecteur V_1 existe car $f_*\mathbf{P}_0)$ est de dimension 2 et intersecte donc $\mathbf{P}_0$. Supposons qu'il existe $V_2 = x_2X + y_2Y \in \mathbf{P}_0$ tel que $f_*V_2 = \widetilde{x}_2X + \widetilde{y}_2Y + cT \notin \mathbf{P}_0$. C'est à dire que $c \neq 0$. f étant de classe C^1, pour tout $\epsilon > 0$ il existe $\delta > 0$ tel que pour $|t| < \delta$ on ait

$$\left| \frac{f(tx_i, ty_i, 0)}{t} - \frac{\partial}{\partial t}\Big|_{t=0} f(tx_i, ty_i, 0) \right|_{eucl} < \epsilon.$$

Ainsi, pour $|t| < \delta$, on a

$$|f(tx_i, ty_i, 0) - tV_i|_{eucl} < \epsilon|t|.$$

En notant $(x_i'(t), y_i'(t), c_i'(t)) = f(tx_i, ty_i, 0)$, on a obtenu les deux inégalités suivantes pour $|t| < \delta$:

$$|(x_1'(t), y_1'(t), c_1'(t)) - (t\widetilde{x}_1, t\widetilde{y}_1, 0)|_{eucl} < \epsilon|t|$$

$$|(x_2'(t), y_2'(t), c_2'(t)) - (t\widetilde{x}_2, t\widetilde{y}_2, tc)|_{eucl} < \epsilon|t|$$

Par conséquent, on a $(x_1'(t) - t\widetilde{x}_1)^2 + (y_1'(t) - t\widetilde{y}_1)^2 + c_1'(t)^2 < \epsilon^2 t^2$ et donc

$$c_1'(t)^2 < \epsilon^2 t^2.$$

De plus, en utilisant l'inégalité triangulaire, on obtient,

$$(|z_1'(t)|^2 + c_1'(t)^2)^{\frac{1}{2}} - |t||\widetilde{z}_1| < \epsilon|t|.$$

Ainsi, $|z_1'(t)|^4 < t^4(\epsilon + |\widetilde{z}_1|)^4$. Par conséquent,

$$\|(z_1'(t), c_1'(t))\|_{\mathcal{H}} \leq (t^4(\epsilon + |\widetilde{z}_1|)^4 + \epsilon^2 t^2)^{\frac{1}{4}}.$$

D'autre part,

$$\|(z_2'(t), c_2'(t))\|_{\mathcal{H}} \geq \|(t\widetilde{z}_2, tc)\|_{\mathcal{H}} - \|(z_2'(t) - t\widetilde{z}_2, c_2'(t) - tc - 2tIm(\widetilde{z}_2\overline{z}_2'(t)))\|_{\mathcal{H}}.$$

Or

$$|c_2'(t) - tc - 2tIm(\widetilde{z}_2\overline{z}_2'(t))| \leq |c_2'(t) - tc| + 2|t||Im(\widetilde{z}_2\overline{z}_2'(t))|.$$

De plus,

$$\begin{aligned} |\widetilde{z}_2||z_2'(t) - t\widetilde{z}_2| &= |\widetilde{z}_2\overline{z}_2'(t) - t|\widetilde{z}_2|^2| \\ &= \left((Re(\widetilde{z}_2\overline{z}_2'(t)) - t|\widetilde{z}_2|^2)^2 + Im(\widetilde{z}_2\overline{z}_2'(t))^2\right)^{\frac{1}{2}} \\ &\geq |Im(\widetilde{z}_2\overline{z}_2'(t))|. \end{aligned}$$

Ainsi, on en déduit que

$$|c_2'(t) - tc - 2tIm(\widetilde{z}_2\overline{z}_2'(t))| \leq \epsilon|t|(1 + 2|t||\widetilde{z}_2|),$$

ce qui implique que

$$\|(z_2'(t), c_2'(t))\|_{\mathcal{H}} \geq (t^4|\widetilde{z}_2|^4 + t^2c^2)^{\frac{1}{4}} - (\epsilon^4t^4 + \epsilon^2t^2(1 + 2|t||\widetilde{z}_2|)^2)^{\frac{1}{4}}.$$

Finalement,

$$\frac{\|(z_2'(t), c_2'(t))\|_{\mathcal{H}}}{\|(z_1'(t), c_1'(t))\|_{\mathcal{H}}} \geq \frac{(t^4|\widetilde{z}_2|^4 + t^2c^2)^{\frac{1}{4}} - (\epsilon^4t^4 + \epsilon^2t^2(1 + 2|t||\widetilde{z}_2|)^2)^{\frac{1}{4}}}{(t^4(\epsilon + |\widetilde{z}_1|)^4 + \epsilon^2t^2)^{\frac{1}{4}}} \underset{t\to 0}{\longrightarrow} \frac{c^{\frac{1}{2}} - \epsilon^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}}.$$

Par conséquent, $\limsup\limits_{t\to 0} \frac{\|(z_2'(t), c_2'(t))\|_{\mathcal{H}}}{\|(z_1'(t), c_1'(t))\|_{\mathcal{H}}} = \infty$, ce qui contredit que f est quasiconforme.

□

On a un résultat réciproque dans une certaine mesure. Soit $f : U \longrightarrow U'$ une transformation de contact de classe C^1. La longueur d'un vecteur $V = aX + bY$ de $\mathbf{P}$ est donnée par $|V|_{\mathcal{H}} = (a^2 + b^2)^{\frac{1}{2}}$.

La version complexe est celle qui sera finalement la plus utilisée par la suite. $\mathbb{C}\mathbf{P} = \mathbb{C} \otimes \mathbf{P}$ est le noyau de la 1-forme $\omega = \mathrm{d}t - i\overline{z}\mathrm{d}z + iz\mathrm{d}\overline{z}$ et a comme base $(Z, \overline{Z})$. La longueur d'un vecteur $V = aZ + b\overline{Z}$ de $\mathbb{C}\mathbf{P}$ est donnée par $|V|_{\mathcal{H}} = (a^2 + b^2)^{\frac{1}{2}}$.

Comme $f_*\mathbf{P} = \mathbf{P}$, on peut considérer pour tout $p \in U$,

$$\lambda_1 = \sup_{V \in \mathbf{P}_p,\ |V|_{\mathcal{H}}=1} |f_*V|_{\mathcal{H}} \text{ et } \lambda_0(p) = \inf_{V \in \mathbf{P}_p,\ |V|_{\mathcal{H}}=1} |f_*V|_{\mathcal{H}}.$$

Le résultat réciproque du théorème précédent s'énonce alors (voir [35, p. 328]) :

Théorème 1.4.3. *Soit $f : U \longrightarrow U'$ une transformation de contact de classe C^2 entre domaines de $\mathcal{H}$. Soit $K \geq 1$ un réel. Si pour tout $p \in U$ on a*

$$\frac{\lambda_1(p)}{\lambda_0(p)} \leq K,$$

alors f est K-quasiconforme.
En particulier, l'action sur un ouvert de $\mathcal{H}$ d'un élément de $\mathrm{PU}(2,1)$ est une transformation conforme (i.e. un homéomorphisme 1-quasiconforme).

Le fait que toute transformation conforme soit la restriction à un ouvert de l'action d'un élément de $\mathrm{PU}(2,1)$, c'est à dire l'analogue du lemme de Weyl dans le cas du groupe de Heisenberg, est aussi vrai. On peut se référer à [35, p. 337] pour le cas C^4 et [14, p.869] pour le cas général.

Une fois ce théorème connu, il devient plus simple de produire des homéomorphismes quasiconformes en prenant un champ de contact, c'est à dire un champ de vecteurs (C^∞) dont le flot en tout temps est une transformation de contact.

Proposition 1.4.4. *Soit $p : \mathcal{H} \longrightarrow \mathbb{R}$ une fonction C^∞. Alors, le champ de vecteurs*

$$V = -\frac{1}{4}(Yp)X + \frac{1}{4}(Xp)Y + pT$$

est un champ de contact sur $\mathcal{H}$. Réciproquement, tout champ de contact est de cette forme avec $p = \omega(V)$, ω étant la forme de contact $\omega = \mathrm{d}t - 2y\mathrm{d}x + 2x\mathrm{d}y$.

Démonstration. C'est une application du fait général suivant (théorème de Liebermann dans la littérature) : soit M une variété et τ une forme de contact sur M. Alors l'ensemble des champs de contact sur M est en bijection avec $C^\infty(M)$ via $V \longmapsto \tau(V)$. Pour obtenir la bijection réciproque de $V \longmapsto \tau(V)$, à une fonction C^∞, H, on associe l'unique champ de vecteur V tel que

$$\tau(V) = H \text{ et } \iota_V \mathrm{d}\tau = \mathrm{d}H(R)\tau - \mathrm{d}H$$

où R est le *champ de Reeb* associé à τ, c'est à dire l'unique champ de vecteurs vérifiant

$$\tau(R) = 1 \text{ et } \iota_R \mathrm{d}\tau = 0.$$

□

Une fois ceci connu, il reste à savoir pour quelles fonctions à valeurs réelles p est-ce que les fonctions λ_0 et λ_1 satisfont la condition du Théorème 1.4.3, ce qui est l'objet du théorème suivant [35, p. 331].

Théorème 1.4.5. *Soit $p : \mathcal{H} \longrightarrow \mathbb{R}$ une fonction de classe C^∞. Notons f_s le flot au temps s du champ de vecteurs*

$$V = -\frac{1}{4}(Yp)X + \frac{1}{4}(Xp)Y + pT.$$

Supposons qu'il existe un réel k tel que

$$|ZZp| \leq k,$$

alors, en tout temps s, f_s est K_s-quasiconforme avec K_s vérifiant

$$\frac{1}{2}\left(K_s + \frac{1}{K_s}\right) \leq e^{\sqrt{2}k|s|}.$$

Cette méthode pour produire des homéomorphismes quasiconformes a notamment été utilisée par Miner [40] pour obtenir que les dilatations sur le groupe de Heisenberg sont quasiconformément conjuguées. Dans ce même article, il est également prouvé que les translations à gauche $L_{(1,0)}$ et $L_{(0,1)}$ ne sont pas quasiconformément conjuguées (ce résultat a été étendu par Kim [32]) et par conséquent les deux cylindres $\mathcal{H}/_{<(1,0)>}$ et $\mathcal{H}/_{<(0,1)>}$ ne sont pas quasiconformément homéomorphes. D'un autre côté, Aebischer et Miner [1] ont montré que les déformations lisses de groupes de Schottky, avec des domaines fondamentaux dont les faces sont des bissecteurs de $\mathbb{H}^2_{\mathbb{C}}$, sont quasiconformément conjuguées (voir aussi [33] pour le résultat analogue pour les déformations lisses des groupes de Schottky avec des domaines fondamentaux dont les faces sont des "packs").

1.4.2 Définitions analytique et géométrique

Afin d'obtenir une définition analytique, il faut introduire une notion de différentiabilité différente de la notion usuelle, à savoir la *P-différentiabilité* définie par Pansu [43].

Définition 1.4.6 (P-différentiabilité)**.** Une application $f : U \longrightarrow U'$ entre ouverts du groupe de Heisenberg est dite P-différentiable en $p \in U$ si

$$\delta_c^{-1} \circ L_{f(p)}^{-1} \circ f \circ L_p \circ \delta_c$$

converge uniformément sur un voisinage de p lorsque $c \mapsto 0$ vers un homomorphisme h de $\mathcal{H}$ dont le morphisme d'algèbre de Lie correspondant h_* préserve $\mathbf{P}$. (Ici, δ_c désigne la dilatation de facteur c et L_p la translation à gauche par p.
On note $D_0 f(p) = h$ et $D_0 f(p)_* = h_*$.

Cette définition est naturelle dans le cadre du groupe de Heisenberg. En effet, si une application $f : U \longrightarrow \mathbb{R}^3$ où U est un ouvert de $\mathbb{R}^3$ est différentiable en $p \in U$, alors pour tout vecteur v de $\mathbb{R}^3$,

$$D_p f(v) = \lim_{t \mapsto 0} \frac{f(p+tv) - f(p)}{t}.$$

L'idée est donc de remplacer les différentes opérations effectuées par leurs analogues sur $\mathcal{H}$. L'addition de deux vecteurs est donc remplacée par la loi de groupe de $\mathcal{H}$, et la multiplication par un scalaire est remplacée par une dilatation. Notons d'ailleurs que la P-différentielle de $f = (f_1, f_2, f_3)$ s'écrit matriciellement dans la base X, Y, T :

$$\begin{pmatrix} Xf_1 & Yf_1 & 0 \\ Xf_2 & Yf_2 & 0 \\ 0 & 0 & Xf_1Yf_2 - Xf_2Yf_1 \end{pmatrix}$$

et en termes complexes, c'est à dire dans la base $Z, \overline{Z}, T$, en notant $f_I = f_1 + if_2$:

$$\begin{pmatrix} Zf_I & \overline{Z}f_I & 0 \\ Z\overline{f}_I & \overline{Zf_I} & 0 \\ 0 & 0 & |Zf_I|^2 - |\overline{Z}f_I|^2 \end{pmatrix}.$$

L'écriture matricielle permet de remarquer que pour tout point $p \in U$,

$$\|D_0 f(p)_*\| := \sup_{V \in \mathbb{C}\mathbf{P} \ | \ |V|_{\mathcal{H}}=1} |D_0 f(p)_*(V)|_{\mathcal{H}} = |Zf_I(p)| + |\overline{Z}f_I(p)| \text{ et}$$

$$J_f(p) := \det(D_0 f(p)_*) = \left(|Zf_I(p)|^2 - |\overline{Z}f_I|^2\right)^2.$$

Pour obtenir la définition analytique d'un homéomorphisme quasiconforme, il faut introduire un espace de Sobolev horizontal. Si $p \geq 1$ et U est un ouvert de $\mathcal{H}$, on définit l'espace $HW^{1,p}_{\text{loc}}(U)$ comme étant l'espace des fonctions $u \in L^p_{\text{loc}}(U)$ telles que $\nabla_0 u$ existe au sens des distributions et $|\nabla_0 u|_{\mathcal{H}} \in L^p_{\text{loc}(U)}$ où ∇_0 est le gradient horizontal :

$$\nabla_0 \phi = (X\phi)X + (Y\phi)Y.$$

Une application $f = (f_1, f_2, f_3) : U \longrightarrow U'$ est dans $HW^{1,p}_{\text{loc}}(U)$ si ses fonctions coordonnées f_1, f_2 et f_3 sont dans $HW^{1,p}_{\text{loc}}(U)$.

La définition analytique d'un homéomorphisme quasiconforme est alors (on renvoie à [15, p. 139] pour cette formulation ne faisant pas intervenir la propriété "ACL") :

Définition 1.4.7 (Homéomorphisme quasiconforme : définition analytique)**.** Soit $f : U \longrightarrow U'$ un homéomorphisme entre domaines de $\mathcal{H}$. On dit que f est K-quasiconforme si $f \in HW^{1,4}_{\text{loc}}(U)$, f est presque partout P-différentiable et sa P-différentielle $D_0 f$ vérifie

$$\|D_0 f(p)_*\|^4 \leq K J_f(p)$$

pour presque tout $p \in U$ où $K \leq 1$ est un réel.

Une seconde définition analytique équivalente fait intervenir un système d'équations ressemblant à l'équation de Beltrami dans le cas complexe.

Définition 1.4.8 (Homéomorphisme quasiconforme : définition analytique (2))**.** Un homéomorphisme $f : U \longrightarrow U'$ entre domaines du groupe de Heisenberg est quasiconforme s'il est dans $HW^{1,4}_{\text{loc}}(U)$, presque partout P-différentiable et satisfait presque partout l'équation

$$\overline{Z} f_I = \mu Z f_I$$

avec $\|\mu\|_\infty < 1$. μ est appelé le coefficient de Beltrami de f. La distortion maximale de f est

$$K_f := \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty}$$

et la distortion de f en un point p où f est P-différentiable est

$$K(p, f) := \frac{1 + |\mu(p)|}{1 - |\mu(p)|}.$$

Remarque 1.4.9. 1. La P-différentiabilité presque partout d'un homéomorphisme quasiconforme f, en particulier le fait que f soit (en un sens faible) une transformation de contact implique que si f_I satisfait presque partout l'équation de Beltrami

$$\overline{Z} f_I = \mu Z f_I$$

alors $f_{II} = f_3 + i|f_I|^2$ satisfait presque partout l'équation de Beltrami pour le même μ.

2. à l'heure actuelle, il n'existe pas de théorème de type représentation mesurable dans ce cadre. Autrement dit, on ne sait pas si pour tout μ dans la boule unité de $L^\infty(\mathcal{H})$ l'équation de Beltrami précédente a une solution quasiconforme.

Une manière de montrer l'équivalence entre la définition analytique et la définition métrique est, comme dans le cas complexe, de montrer que toutes deux sont équivalentes à une définition géométrique. Pour être plus précis, le chemin employé est, comme dans le cas complexe, la définition métrique implique la définition analytique qui implique la définition géométrique qui implique la définition métrique. Dans le cas complexe, la définition géométrique avait nécessité la notion de module d'un quadrilatère. Dans le cas présent, la méthode employée par Korányi et Reimann dans [36] va nécessiter la notion de *capacité d'un anneau.*

Définition 1.4.10 (Capacité d'un anneau)**.** Un anneau A est une partie de $\overline{\mathcal{H}} = \mathcal{H} \cup \{\infty\}$ telle que $\overline{\mathcal{H}} \backslash A$ est la réunion disjointe de deux parties compactes, connexes et non-vides C_0 et C_1. La capacité de A est

$$\mathrm{cap}(A) := \inf_u \int_{\mathcal{H}} |\nabla_0 u|_{\mathcal{H}}^4$$

où u parcourt l'ensemble des fonctions $u \in C^\infty(\overline{\mathcal{H}}, \mathbb{R})$ telles que $u_{|C_0} = 0$ et $u_{|C_1} = 1$.

Par exemple (voir [36, p. 44]),

$$A_{a,b} = \{p \in \mathcal{H} \mid a < \|p\|_{\mathcal{H}} < b\}$$

est un anneau de capacité

$$\mathrm{cap}(A) = \pi^2 \ln \left(\frac{b}{a}\right)^{-3}.$$

Le théorème de type Grötzsch a été obtenu par Pansu (voir [36, p. 46] pour une preuve) et stipule :

Théorème 1.4.11. *Soit $f : U \longrightarrow U'$ un homéomorphisme K-quasiconforme (pour la définition analytique) entre domaines de $\mathcal{H}$. Alors, pour tout anneau $A \in U$,*

$$\mathrm{cap}(A) \leq K^2 \mathrm{cap}(f(A)).$$

Comme dans le cas complexe, ce théorème peut servir comme définition géométrique d'un homéomorphisme quasiconforme.

Définition 1.4.12 (Homéomorphisme quasiconforme : définition géométrique)**.** Un homéomorphisme $f : U \longrightarrow U'$ entre domaines de $\mathcal{H}$ est quasiconforme s'il existe une constante K' telle que

$$\mathrm{cap}(A) \leq K'\mathrm{cap}(f(A))$$

pour tout anneau $A \subset U$.

Le théorème précédent nous donne alors que la définition analytique implique la définition géométrique. On renvoie ensuite à [36, p. 49] pour le fait que la définition géométrique implique la définition métrique et à [36, p. 40] pour le fait que la définition métrique implique la définition analytique.

Remarquons par ailleurs que la définition géométrique donne immédiatement que la composée de deux homéomorphismes quasiconformes est quasiconforme. Le fait que l'inverse d'un homéomorphisme quasiconforme soit quasiconforme est vrai également (voir [36, p. 62]) mais ne découle pas de la définition géométrique. Pour ce qui est des distortions, on résume ce que l'on obtient dans la proposition suivante.

Proposition 1.4.13.

1. *La composition d'un homéomorphisme quasiconforme de distortion maximale K et d'une transformation conforme est un homéomorphisme quasiconforme de distortion maximale K.*
2. *Un homéomorphisme quasiconforme a la même distortion que son inverse.*
3. *La composition d'un homéomorphisme quasiconforme de distortion maximale K_1 par un homéomorphisme quasiconforme de distortion K_2 est un homéomorphisme quasiconforme de distortion maximale K avec $K \leq K_1K_2$.*

Pour finir, on mentionne un résultat général de relèvement des homéomorphismes quasiconformes du plan complexe que l'on peut trouver dans [9, 16] par exemple. Il s'agit de comprendre quels sont les homéomorphismes quasiconformes $f : U \longrightarrow U'$ entre ouverts de $\mathcal{H}$ pour lesquels la partie complexe f_I ne dépend pas de t. En clair, en notant $p : (z,t) \longmapsto z$ la projection de $\mathcal{H} = \mathbb{C} \times \mathbb{R}$ sur $\mathbb{C}$, quels sont les homéomorphismes quasiconformes g entre ouverts de $\mathbb{C}$ pour lesquels il existe un homéomorphisme quasiconforme f entre ouverts de $\mathcal{H}$ tel que $p \circ f = g \circ p$. On va donner une preuve élémentaire dans le cas d'homéomorphismes de classe C^2 car celle-ci donnera l'idée de preuve dans le cas de la projection de Korányi $\Pi : (z,t) \longmapsto t + i|z|^2$ au chapitre 2.

Proposition 1.4.14. *Soit $g : \mathbb{C} \longmapsto \mathbb{C}$ un homéomorphisme de classe C^2.*

— *Si le jacobien de g,* Jac g*, est constant, alors il existe une fonction $f_2 : \mathcal{H} \longmapsto \mathbb{R}$ telle que*

$$\begin{array}{rccc} f: & \mathcal{H} & \longmapsto & \mathcal{H} \\ & (z,t) & \longmapsto & (g(z), f_2(z,t)) \end{array}$$

est une transformation de contact. De plus, pour tout $(z,t) \in \mathcal{H}$, $f_2(z,t) =$ Jac $g(z)t + h(z)$ *où $h : \mathbb{C} \longmapsto \mathbb{R}$ vérifie pour tout $z \in \mathbb{C}$*

$$\frac{\partial h}{\partial z}(z) = -i\overline{z}\text{Jac } g(z) + i\overline{g(z)}\frac{\partial g}{\partial z}(z) - ig(z)\frac{\partial \overline{g}}{\partial z}(z)$$

— *Réciproquement, s'il existe une transformation de contact $f = (f_1, f_2) : \mathcal{H} \longmapsto \mathcal{H}$ tel que $f_1(z,t) = g(z)$ pour tout $(z,t) \in \mathcal{H}$, alors* Jac g *est constant et f est de la forme donnée précédemment.*

— *Si $f : \mathcal{H} \longrightarrow \mathcal{H}$ est une transformation de contact telle que $f_1(z,t) = g(z)$ pour tout $(z,t) \in \mathcal{H}$, alors f est quasiconforme si et seulement si g est quasiconforme. Dans ce cas, la distortion de f en un point $(z,t) \in \mathcal{H}$ est égale à la distortion de g au point $z \in \mathbb{C}$.*

Démonstration. Dans toute la preuve, on notera $\partial_z = \frac{\partial}{\partial z}$, $\partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}}$ et $\partial_t = \frac{\partial}{\partial t}$.

— Pour le premier sens, on doit juste montrer que l'application $f : \mathcal{H} \longmapsto \mathcal{H}$ définie dans le théorème existe et est un homéomorphisme quasiconforme. Par hypothèse, il existe une constante C telle que pour tout $z \in \mathbb{C}$, Jac $g(z) = C$. Considérons $k : \mathbb{C} \longmapsto \mathbb{C}$ définie pour tout $z \in \mathbb{C}$ par

$$k(z) = -iC\overline{z} + i\overline{g(z)}\partial_z g(z) - ig(z)\partial_z \overline{g}(z) \text{ et}$$

$$\beta := k\mathrm{d}z + \overline{k}\mathrm{d}\overline{z}.$$

Un calcul immédiat montre que β est fermée. Le lemme de Poincaré assure alors que β est exacte. Il existe donc une fonction à valeurs réelles (car $\overline{\beta} = \beta$) $h : \mathbb{C} \longmapsto \mathbb{R}$ telle que

$$\partial_z h = k \text{ et } \partial_{\overline{z}} h = \overline{k}.$$

On conclut alors que l'application f définie dans le théorème existe et on vérifie aisément que c'est une transformation de contact.

— Soit $f = (g,k) : \mathcal{H} \longmapsto \mathcal{H}$ une transformation de contact où $k : \mathcal{H} \longmapsto \mathbb{R}$. La condition de contact $f^*\omega = \lambda\omega$ pour une fonction partout non nulle λ, donne le système suivant d'équations aux dérivées partielles que doivent vérifier g

et h :

$$\lambda = \partial_t k$$

$$\partial_z k - i\overline{g}\partial_z g + ig\partial_z \overline{g} = -i\overline{z}\lambda$$

$$\partial_{\overline{z}} k - i\overline{g}\partial_{\overline{z}} g + ig\partial_{\overline{z}} \overline{g} = iz\lambda.$$

En dérivant les deux dernières équations par rapport à t et en remplaçant λ par $\partial_t k$, on trouve

$$\partial^2_{t,z} k = -i\overline{z}\partial^2_{t,t} k \text{ and } \partial^2_{t,\overline{z}} k = iz\partial^2_{t,t} k$$

De plus, en dérivant l'expression de $-iz\partial_t k$ par rapport à $\overline{z}$ et l'expression de $iz\partial_t k$ par rapport à z on obtient

$$\partial^2_{\overline{z},z} k - i\overline{g}\partial^2_{\overline{z},z} g + ig\partial^2_{\overline{z},z}\overline{g} = i|z|^2\partial^2_{t,t} k - i\partial_t k + i\text{Jac } g \text{ et}$$

$$\partial^2_{\overline{z},z} k - i\overline{g}\partial^2_{\overline{z},z} g + ig\partial^2_{\overline{z},z}\overline{g} = i|z|^2\partial^2_{t,t} k + i\partial_t k - i\text{Jac } g.$$

Par conséquent, on obtient

$$\text{Jac } g = \partial_t k.$$

En particulier, $\partial_t k$ est indépendant de t. Ainsi, $\partial^2_{t,t} k = 0$ et on déduit que $\partial_{t,z} k$ et $\partial_{t,\overline{z}} k$ sont aussi identiquement nuls. Cela amène au fait que $\partial_t k$ est constante et donc Jac g également. Finalement, on peut écrire $k(z,t) = \text{Jac } g(z)t + h(z)$ où h est une fonction à valeurs réelles satisfaisant

$$\partial_z h(z) = -i\text{Jac } g(z) + i\overline{g(z)}\partial_z g(z) - ig(z)\partial_z \overline{g}(z).$$

Le troisième point du théorème découle immédiatement de la formule donnée pour la distortion ponctuelle de f.

□

Tout automorphisme CR de $\mathcal{H}$ est l'action d'un élément de $\mathcal{H} \rtimes (U(1) \times \mathbb{R}^+)$ où $((z_0, t_0), \lambda, c)$ agit sur (z,t) par

$$((z_0, t_0), \lambda, c)(z,t) = \left(z_0 + c\lambda z, t_0 + c^2 t - 2\Im\left(c\lambda z \overline{z}_0\right)\right).$$

Autrement dit, un automorphisme CR de $\mathcal{H}$ est la composée d'une translation à gauche, d'une rotation et d'une dilatation. Ce sont donc exactement les relevés par

p de similitudes affines directes de $\mathbb{C}$.

1.5 Espace de Teichmüller d'une variété CR sphérique compacte

Cette partie, plus courte, est consacrée à la définition de l'espace de Teichmüller et de sa métrique comme définis dans [56]. Comme indiqué dans la partie précédente, on supposera que les homéomorphismes considérés ici préservent l'orientation.

Le fait d'avoir un atlas CR sphérique permet de définir aisément un homéomorphisme quasiconforme entre variété CR sphériques. Plus précisément, un homéomorphisme $f : (M, V) \longrightarrow (N, V')$ est quasiconforme s'il l'est lu dans les cartes. Comme les changements de cartes sont des restrictions d'action de $\mathrm{PU}(2, 1)$, la distortion de f en un point $p \in M$ où, lue dans les cartes f est P-différentiable, est indépendante du choix des cartes. On note $K(p, f)$ la distortion de f en p et on définit la distortion maximale de f :

$$K_f = \sup_p K(p, f)$$

où p parcourt l'ensemble des points de M où f est P-différentiable lorsque lue dans les cartes.

On se fixe une variété CR sphérique compacte (M, V).

Définition 1.5.1 (Espace de Teichmüller)**.** L'espace de Teichmüller de (M, V) noté $\mathcal{T}(M, V)$, est le quotient de l'ensemble des couples $((N, W), f)$ où (N, W) est une variété CR sphérique et $f : (M, V) \longrightarrow (N, W)$ est un homéomorphisme quasiconforme par la relation d'équivalence : $((N_1, W_1), f_1)$ et $((N_2, V_2), W_2)$ sont équivalents si et seulement si $f_2 \circ f_1^{-1} : (N_1, W_1) \longrightarrow (N_2, W_2)$ est isotope à un difféomorphisme CR. On note les classes d'équivalence $[(N, W), f]$ et $[(M, V), \mathrm{Id}]$ est appelé le point base de $\mathcal{T}(M, V)$.

Remarque 1.5.2. Les homéomorphismes quasiconformes étant (faiblement) des transformations de contact, on peut voir l'espace de Teichmüller d'une variété CR sphérique comme un espace de déformation de la structure CR sphérique, à structure contact fixée.

Définition 1.5.3 (Pseudo-distance sur l'espace de Teichmüller)**.** La pseudo-distance

entre deux points $[(N_1, W_1), f_1]$ et $[(N_2, W_2), f_2]$ est

$$d\left([(N_1, W_1), f_1], [(N_2, W_2), f_2]\right) = \inf_f \ln\left(K_f\right)$$

où l'infimum est pris sur l'ensemble des homéomorphismes quasiconformes isotopes à $f_2 \circ f_1^{-1}$.

Le fait que d soit bien définie et une pseudo-distance découle immédiatement de la Proposition 1.4.13. Le premier point assure que d est bien définie, le second donne la symétrie de d et le troisième donne l'inégalité triangulaire. On renvoie alors à [56] pour une preuve du théorème suivant.

Théorème 1.5.4. *d est une distance sur $\mathcal{T}(M, V)$ et l'espace métrique $(\mathcal{T}(M, V), d)$ est complet.*

On appelle d la distance de Teichmüller.

Chapter 2

Extremal quasiconformal mappings in the Heisenberg group

Quasiconformal maps with extremal properties between domains of the Heisenberg group are still mysterious. Some progress has been made recently with minimizing a mean distortion. A method relying on the notion of modulus of a curve family has been developed [7] in order to understand when a quasiconformal map minimizes a mean distortion. In particular, the authors gave a condition, easy to check, to verify that a candidate for minimizing a mean distortion is indeed a minimizer. However, finding such a candidate may be quite challenging and this chapter starts with as a goal, a study of how to construct such candidates. Notice that minimizing a mean distortion is different from the classical Grötzsch problem which concerns minimizers for the maximal distortion. Nevertheless, the modulus of curve families method can be used to find lower bounds for the maximal distortion and so, it can also be used for Grötzsch-type problems (see [51] for this type of arguments in CR geometry). For other uses of moduli of curve families, we may quote again [32, 40] who studied quasiconformal conjugacy classes of CR diffeomorphisms.

Another motivation for studying quasiconformal maps which minimize a mean distortion is through their behavior on a family of curves: these maps dilate a family of curves (as we will see several times in the chapter). As we saw, dilating families of curves is one of the defining properties of Teichmüller homeomorphisms so understanding quasiconformal maps which dilate family of curves seems to be a necessary step to the notion of Teichmüller homeomorphisms between spherical CR manifolds.

In this chapter, we are then interested in the construction of quasiconformal

mappings between several domains of the Heisenberg group which minimize a mean distortion. The method employed can be summed up in the following way. We associate a corresponding minimization problem to the one we began with, but the associated problem deals with quasiconformal maps between domains in the upper half plane $\mathbb{H}$. This problem will then be easy to solve and we can try to lift every of its solutions on $\mathcal{H}$. To be precise, we consider the following minimization problem: let $\mathcal{F}$ be a class of quasiconformal maps between two fixed domains Ω and Ω' of $\mathcal{H}$. We wish to find $f_0 \in \mathcal{F}$ such that

$$\int_\Omega K(p, f_0)^2 \rho_0^4(p) \mathrm{d}L^3(p) = \min_{f \in \mathcal{F}} \int_\Omega K(p, f)^2 \rho_0^4(p) \mathrm{d}L^3(p) \tag{2.1}$$

for a density ρ_0 depending on the geometry of Ω and where $\mathrm{d}L^3$ is the Lebesgue measure on $\mathbb{R}^3$. If f_0 satisfies 2.1, we say that f_0 minimizes the mean distortion in $\mathcal{F}$ for the density ρ_0.

We propose here a geometric way to construct such minimizers in specific cases. Our construction relies on the Korányi map

$$\begin{array}{rccc} \Pi : & \mathcal{H} \setminus (\{0\} \times \mathbb{R}) & \longrightarrow & \mathbb{H} \\ & (z, t) & \longmapsto & t + i|z|^2. \end{array}$$

which, as we saw, comes from the identification of $\mathcal{H}$ with the boundary of Siegel domain $\mathbb{S}$. Moreover, $\partial\mathbb{S} \setminus (\{\infty\} \cup \{z_2 = 0\})$ is also diffeomorphic to a trivial circle bundle over the upper half plane $\mathbb{H}$ simply by taking $(z_1, z_2) \longmapsto \left(\frac{z_2}{\sqrt{\Im(z_1)}}, z_1\right)$. Identifying $\partial\mathbb{S} \setminus (\{\infty\} \cup \{z_2 = 0\})$ with $\mathcal{H} \setminus (\{0\} \times \mathbb{R})$, it gives a diffeomorphism

$$\begin{array}{rccc} \Psi^{-1} : & \mathcal{H} \setminus (\{0\} \times \mathbb{R}) & \longrightarrow & S^1 \times \mathbb{H} \\ & (z, t) & \longmapsto & \left(\frac{z}{|z|}, t + i|z|^2\right) \end{array}$$

and the Korányi map Π, is simply the second component of that diffeomorphism. The (topological) identification of $\mathcal{H} \setminus \{z = 0\}$ with a trivial circle bundle over the hyperbolic plane isn't new. For example, Korányi and Reimann in [37] introduced coordinates

$$(a, \alpha, \varphi) \longmapsto (a \cos^{\frac{1}{2}}(\alpha) e^{i\varphi}, a^2 \sin(\alpha))$$

so that (up to rotation and conjugation), (a^2, α) correspond to polar coordinates in $\mathbb{H}$. The first question following that would be: when can a minimizer of a mean distortion between domains of $\mathbb{H}$ be lifted by Π into a quasiconformal map between

domains of $\mathcal{H}$?

Section 2.1 deals with moduli of curve families and explicit the minimization problem considered in $\mathcal{H}$ and its corresponding one in $\mathbb{H}$. In particular, if a minimizer of the corresponding problem in $\mathbb{H}$ can be lifted by Π into a quasiconformal map in $\mathcal{H}$, then this lift is a nice candidate for minimizing the mean distortion considered in $\mathcal{H}$ (see Corollary 2.1.14). Then, in section 2.2 we totally and explicitly solve a minimization problem between cylinders in $\mathcal{H}$. In this case, there is only one solution to the corresponding problem in $\mathbb{H}$ that can be lifted into a quasiconformal map and we compute explicitly this solution, as well as its lift. A small step is then needed to ensure that the lift is indeed a minimizer of the mean distortion we started with. All of this leads to Proposition 2.2.5. After that, Theorem 2.2.6 states that a minimizer of the mean distortion considered is inevitably the lift of a minimizer of the corresponding problem in $\mathbb{H}$. Since there is only one solution of the corresponding problem in $\mathbb{H}$ that can be lifted into a quasiconformal map, uniqueness of the minimizer in $\mathcal{H}$, up to composition with a rotation around the vertical axis, follows.

This uniqueness (already observed in [8] in a similar case) is quite surprising when compared to the abundance of solutions to the corresponding problem in $\mathbb{H}$. This phenomenon made us ask ourself: when does a minimizer of a mean distortion in $\mathcal{H}$ inevitably comes as a lift by Π of a solution of the corresponding problem in $\mathbb{H}$?

Section 2.3 extracts geometric data which made necessary the fact that a minimizer of the mean distortion considered between cylinders is necessarily a lift of a solution to the corresponding problem in $\mathbb{H}$. In particular, the cases of cylinders and spherical annuli from [7,8] both satisfy what we call the *pull-back density condition*. We will see that it is in fact mostly this condition which gives uniqueness. We consider, as domains in $\mathcal{H}$, lifts by Π of quadrilaterals which satisfy the pull-back density condition. (Recall that the lack of a Riemann mapping theorem forces us to consider most of domains in $\mathcal{H}$ as different from each other so that solving a minimization problem between lifts of two quadrilaterals isn't enough to solve the same problem for lifts of all quadrilaterals). Theorem 2.3.19 states that if a solution of the problem between quadrilaterals can be lifted into a minimizer of the mean distortion in $\mathcal{H}$, then every other minimizer is constructed this way. It reduces the problem of finding minimizers to solving an ordinary differential equation with boundary conditions (Proposition 2.3.7). We shall also prove a general lifting theorem for quasiconformal maps between (simply connected) domains of $\mathbb{H}$: Proposition 2.3.9.

We end this chapter with two examples, including the one between spherical annuli for which existence and uniqueness are recovered as a special case of the results from section 2.3.

We assume, in the whole chapter, that every quasiconformal map considered is C^2 and orientation-preserving and every curve is C^1.

2.1 Minimization problem considered in $\mathcal{H}$ and its corresponding one in $\mathbb{H}$

2.1.1 Modulus of a curve family

By analogy with the complex case, in order to understand extremal properties of a quasiconformal map between two domains of the Heisenberg group, we look at its behavior on a well chosen family of rectifiable curves (with respect to $d_{\mathcal{H}}$) that foliates the domain. We restrict the study here to C^1 curves and C^2 orientation-preserving quasiconformal mappings, but most of the results of this section were proved in a general case. First of all, rectifiable curves are *legendrian*.

Definition 2.1.1 (Legendrian curves)**.** A C^1 curve $\gamma :]a, b[\longrightarrow \mathcal{H}$ is called legendrian if its tangents are in the contact distribution $D = ker(\omega)$. This condition is given explicitly by the following. Let $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, $s \in]a, b[$ be a curve in $\mathcal{H}$. Then, γ is *legendrian* if and only if

$$\dot{\gamma}_2(s) = -2\Im(\overline{\gamma}_1(s)\dot{\gamma}_1(s)) \text{ for all } s \in]a, b[.$$

We can then define the modulus of a family of legendrian curves

Definition 2.1.2 (Modulus of a family of legendrian curves)**.** Let Γ be a family of legendrian curves in a domain Ω of $\mathcal{H}$. We denote $\mathrm{adm}\,(\Gamma)$ the set of measurable Borel functions $\rho : \Omega \longrightarrow [0, +\infty]$ such that $\int_\gamma \rho \mathrm{d}l := \int_a^b \rho(\gamma(s))|\dot{\gamma}_1(s)|\mathrm{d}s \geq 1$ for all curves $\gamma \in \Gamma$. We call densities the elements of $\mathrm{adm}(\Gamma)$. The modulus of the family Γ is then defined by

$$M\,(\Gamma) := \inf_{\rho \in \mathrm{adm}(\Gamma)} \int_\Omega \rho(p)^4 \mathrm{d}L^3(p).$$

We say that a density ρ_0 is extremal if it verifies $M\,(\Gamma) = \int_\Omega \rho_0(p)^4 \mathrm{d}L^3(p)$.

When an extremal density exists, it is essentially unique (Proposition 3.4 in [8, p. 143]).

Lemma 2.1.3. *If ρ_1 and ρ_2 are both extremal density for a family Γ of legendrian curves in a domain Ω of $\mathcal{H}$, then $\rho_1 = \rho_2$ a.e. in Ω.*

There is a link between quasiconformality and modulus of a family of legendrian curves. Indeed, we have the following.

Proposition 2.1.4. *Let $f : \Omega \longrightarrow \Omega'$ be a quasiconformal map between domains of $\mathcal{H}$. Then, for every family of legendrian curves Γ in Ω and every $\rho \in \operatorname{adm}(\Gamma)$, one has*

$$M(f(\Gamma)) \quad \leq \quad \int_\Omega K(p,f)^2 \rho(p)^4 \mathrm{d}L^3(p). \tag{2.2}$$

This result is a direct consequence of an essential notion we will need. Fixing a density ρ and a C^1 quasiconformal map $f : \Omega \longrightarrow \Omega'$, one may define a *push-forward* by f of the density ρ (see e.g. [7]).

Definition/Proposition 2.1.5 (Push-forward density)**.** *Let $f = (f_1, f_2) : \Omega \longrightarrow \Omega'$ be a quasiconformal mapping between two domains of $\mathcal{H}$, Γ a family of legendrian curves in Ω and $\rho \in \operatorname{adm}(\Gamma)$. Then,*

$$\rho' = \frac{\rho}{|Zf_1| - |\overline{Z}f_1|} \circ f^{-1} \in \operatorname{adm}\left(f\left(\Gamma\right)\right).$$

Moreover,

$$\int_{\Omega'} \rho'^4 \mathrm{d}L^3 = \int_\Omega K(.,f)^2 \rho^4 \mathrm{d}L^3.$$

Proof. A simple application of the chain rule and the fact that every $\gamma \in \Gamma$ is legendrian lead to the following. For every curve $\gamma = (\gamma_1, \gamma_2) :]a,b[\longrightarrow \Omega,\ \gamma \in \Gamma$, one has

$$(f_1 \dot{\circ} \gamma)(s) = Zf_1(\gamma(s))\dot{\gamma}_1(s) + \overline{Z}f_1(\gamma(s))\dot{\overline{\gamma}}_1(s) \text{ for every } s \in]a,b[.$$

This leads to the important inequality: for every $s \in]a,b[$,

$$|Zf_1(\gamma(s))| - |\overline{Z}f_1(\gamma(s))| \leq \frac{|(f_1 \dot{\circ} \gamma)(s)|}{|\dot{\gamma}_1(s)|} \leq |Zf_1(\gamma(s))| + |\overline{Z}f_1(\gamma(s))|. \tag{2.3}$$

If $\gamma \in \Gamma$, using inequality 2.3 and the fact that $\rho \in \mathrm{adm}(\Gamma)$, we find

$$\int_{f_1\circ\gamma} \rho' \mathrm{d}l = \int_a^b \frac{\rho}{|Zf_1| - |\overline{Z}f_1|} \circ \gamma(s)|(f_1 \dot{\circ} \gamma)(s)|\mathrm{d}s \geq \int_a^b \rho(\gamma(s))|\dot{\gamma}_1(s)|\mathrm{d}s \geq 1.$$

For the second part, this is simply an application of the following change of variable formula for quasiconformal mappings (Theorem 16 in [7, p. 175]). For every non-negative measurable function $u : \Omega' \longrightarrow \mathbb{R}$, we have

$$\int_\Omega (u \circ f)(p)|J(p,f)|\mathrm{d}L^3(p) = \int_{\Omega'} u(q)\mathrm{d}L^3(q)$$

where $J(p,f) = \left(|Zf_1(p)|^2 - |\overline{Z}f_1(p)|^2\right)^2$. Using this formula and the definition of ρ', we have

$$\int_{\Omega'} \rho' \mathrm{d}L^3 = \int_\Omega \rho^4 \frac{\left(|Zf_1|^2 - |\overline{Z}f_1|^2\right)^2}{\left(|Zf_1| - |\overline{Z}f_1|\right)^4} \mathrm{d}L^3 = \int_\Omega K(.,f)^2 \rho^4 \mathrm{d}L^3.$$

□

Just applying the previous proves Proposition 2.1.4.

Remark 2.1.6. In fact, moduli of families of legendrian curves give another geometric definition of a quasiconformal map. Take an annulus A and denote C_0 and C_1 the compact, connected, non-empty components of $\overline{\mathcal{H}} \backslash A$. Then it is known (see e.g. [36, p. 46]) that

$$\mathrm{cap}(A) = M(\Gamma)$$

where Γ is the family of all legendrian curves in A connecting C_0 and C_1.

Now, in order to find a minimizer of a mean distortion in a class of quasiconformal maps, a first place to look would be for maps realizing equality in inequality 2.2 and then, use the following (obvious) proposition.

Proposition 2.1.7. *Let Ω, Ω' be domains of $\mathcal{H}$, $\mathcal{F}$ a subset of the set of all orientation-preserving quasiconformal maps from Ω to Ω', and Γ_0 a family of legendrian curves in Ω with $\rho \in \mathrm{adm}(\Gamma_0)$. Suppose that $f_0 \in \mathcal{F}$ satisfies*

$$M(f_0(\Gamma_0)) = \int_\Omega K(.,f_0)^2 \rho^4 \mathrm{d}L^3$$

and that there is a larger family $\Gamma \supset \Gamma_0$ of legendrian curves in Ω with $\rho \in \mathrm{adm}(\Gamma)$

and verifying

$$M(f_0(\Gamma_0)) \leq M(f(\Gamma))$$

for every $f \in \mathcal{F}$. Then, f_0 minimizes the mean distortion in $\mathcal{F}$ for the density ρ.

Proof. Proposition 2.1.4 ensures that for every $f \in \mathcal{F}$ we have

$$M(f(\Gamma)) \leq \int_\Omega K(.,f)^2 \rho^4 \mathrm{d}L^3.$$

Moreover, for every $f \in \mathcal{F}$,

$$\int_\Omega K(.,f_0)^2 \rho^4 \mathrm{d}L^3 = M(f_0(\Gamma_0)) \leq M(f(\Gamma)).$$

So, for every $f \in \mathcal{F}$,

$$\int_\Omega K(.,f_0)^2 \rho^4 \mathrm{d}L^3 \leq \int_\Omega K(.,f)^2 \rho^4 \mathrm{d}L^3.$$

□

Theorem 1 in [7, p. 153] gives a sufficient condition on a map to ensure that it minimizes the mean distortion in a subset of quasiconformal maps between bounded domains of $\mathcal{H}$ for a nice density.

Theorem 2.1.8. *Let Ω and Ω' be bounded domains of $\mathcal{H}$. Let $\gamma :]a,b[\times\Lambda \longrightarrow \Omega$ be a diffeomorphism that foliates Ω, where $]a,b[\subset \mathbb{R}$ with $a > 0$ and Λ is a domain of $\mathbb{R}^2$,such that $\gamma(\cdot,\lambda)$ is a legendrian curve verifying $|\dot{\gamma}_1(s,\lambda)| \neq 0$ for all $\lambda \in \Lambda$ and $\mathrm{d}L^3(\gamma(s,\lambda)) = |\dot{\gamma}_1(s,\lambda)|^4 \mathrm{d}s\mathrm{d}\mu(\lambda)$ for a measure $\mathrm{d}\mu$ on Λ.*
Then,

$$\rho_0(p) := \frac{1}{(b-a)|\dot{\gamma}_1(\gamma^{-1}(p))|}$$

is an extremal density for the family $\Gamma_0 := \{\gamma(\cdot,\lambda) \mid \lambda \in \Lambda\}$.

Moreover, if $\mathcal{F}$ is a subset of the set of all quasiconformal map from Ω on Ω' and $f_0 \in \mathcal{F}$ is such that:

1. *$\mu_{f_0}(\gamma(s))\frac{\dot{\bar{\gamma}}_1(s)}{\dot{\gamma}_1(s)} < 0$ for all $s \in]a,b[$,*
2. *for all $\lambda \in \Lambda$, $K(\gamma(s,\lambda),f_0)$ does not depend on s, and*
3. *there is $\Gamma \supset \Gamma_0$ such that $\rho_0 \in \mathrm{adm}(\Gamma)$ and $M(f_0(\Gamma_0)) \leq M(f(\Gamma))$ for all $f \in \mathcal{F}$.*

Then f_0 minimizes the mean distortion in $\mathcal{F}$ for the extremal density ρ_0.

2.1.2 The corresponding problem in $\mathbb{H}$

The previous theorem gives a way, once we have a candidate, to check if that candidate minimizes a mean distortion functional. However, finding such a candidate may be quite challenging, one can see for example the complexity of the cartesian expression of the *radial stretch map* from [7, 8] which is, we remind, the only minimizer of a mean distortion, up to composition with a rotation around the vertical axis. Here, we explain how to construct such mappings in specific cases. As said in the introduction, the construction relies on the identification of $\mathcal{H}\backslash\{z=0\}$ with $S^1\times\mathbb{H}$ where S^1 is the unit circle of $\mathbb{C}$ and $\mathbb{H}$ is the Poincaré half plane, identification given by the diffeomorphism Ψ^{-1} defined in the introduction whose inverse is the map

$$\begin{array}{rccc} \Psi: & S^1\times\mathbb{H} & \longrightarrow & \mathcal{H}\backslash\left(\{0\}\times\mathbb{R}\right) \\ & (e^{i\theta},w) & \longmapsto & \left(\sqrt{\Im(w)}e^{i\theta},\Re(w)\right) \end{array}$$

It gives new coordinates on $\mathcal{H}\backslash\left(\{0\}\times\mathbb{R}\right)$ and a simple computation gives the following expression of vector fields Z and $\overline{Z}$

$$\begin{aligned} Z &= 2i\sqrt{\Im(w)}e^{-i\theta}\partial_w - \frac{ie^{-i\theta}}{2\sqrt{\Im(w)}}\partial_\theta \\ \overline{Z} &= -2i\sqrt{\Im(w)}e^{i\theta}\partial_{\overline{w}} + \frac{ie^{i\theta}}{2\sqrt{\Im(w)}}\partial_\theta. \end{aligned}$$

where $\partial_w=\frac{\partial}{\partial w}$, $\partial_{\overline{w}}=\frac{\partial}{\partial\overline{w}}$ and $\partial_\theta=\frac{\partial}{\partial\theta}$.

In the following, we consider $\widetilde{\Omega}$ and $\widetilde{\Omega}'$ domains in $\mathcal{H}\backslash(\{0\}\times\mathbb{R})$ such that $\Psi^{-1}\left(\widetilde{\Omega}\right)=S^1\times\Omega$ and $\Psi^{-1}\left(\widetilde{\Omega}'\right)=S^1\times\Omega'$ with Ω, Ω' domains of $\mathbb{H}$. We will look at lifts by Π of curves in the half-plane.

Lemma 2.1.9. *Let $\gamma:]a,b[\longrightarrow\mathbb{H}$ be a C^1 curve. Then, the only legendrian curves on $\mathcal{H}\backslash(\{0\}\times\mathbb{R})$, $\widetilde{\gamma}=(\gamma_1,\gamma_2)$ such that $\Pi(\widetilde{\gamma})=\gamma$ are the curves $\left(\sqrt{\Im(\gamma)}e^{i\tau},\Re(\gamma)\right)$ where $\dot{\tau}=-\frac{\Re(\dot{\gamma})}{2\Im(\gamma)}$.*

Proof. Saying that $\gamma_2+i|\gamma_1|^2=\gamma$ gives $\gamma_2=\Re(\gamma)$ and $|\gamma_1|=\sqrt{\Im(\gamma)}$. So, we only have to check that $\left(\sqrt{\Im(\gamma)}e^{i\tau},\Re(\gamma)\right)$ is legendrian if and only if $\dot{\tau}=-\frac{\Re(\dot{\gamma})}{2\Im(\gamma)}$, which is a simple application of the definition of a legendrian curve.

□

Remark 2.1.10. Notice that lifting a curve gives a one-parameter family of curves in $\mathcal{H}$, which are obtained by rotating one lift around the vertical axis. Moreover, for a family of curves in $\mathbb{H}$, Γ, we define

$$\widetilde{\Gamma} = \{\widetilde{\gamma} \mid \Pi(\widetilde{\gamma}) \in \Gamma\}$$

and call $\widetilde{\Gamma}$ the *lifted family* of Γ.

Before going further, let's recall how the modulus of a curve family is defined in $\mathbb{C}$. Let Γ be a family of curves $\gamma :]a, b[\longrightarrow \Omega$ in a domain Ω of $\mathbb{C}$. We denote again $\mathrm{adm}(\Gamma)$ the set of measurable Borel functions $\rho : \Omega \longrightarrow [0, \infty]$ such that $\int_\gamma \rho \mathrm{d}l = \int_a^b \rho(\gamma(s))|\dot{\gamma}(s)|\mathrm{d}s \geq 1$. The modulus of the family Γ is

$$M(\Gamma) = \inf_{\rho \in \mathrm{adm}(\Gamma)} \int_\Omega \rho^2 \mathrm{d}L^2$$

where $\mathrm{d}L^2$ is the Lebesgue measure of $\mathbb{R}^2$. With that in mind, we can define the pull-back by Π of a density.

Definition/Proposition 2.1.11 (Pull-back density)**.** *Let Ω be a domain in $\mathbb{H}$ and $\widetilde{\Omega} = \Psi(S^1 \times \Omega)$. Let Γ be a curve family in Ω and denote $\widetilde{\Gamma}$ its lifted family in $\widetilde{\Omega}$. If $\rho \in \mathrm{adm}(\Gamma)$, define $\widetilde{\rho}$ for every $(z, t) \in \widetilde{\Omega}$ by*

$$\widetilde{\rho}(z, t) := |Z\Pi(z, t)|\rho(\Pi(z, t)) = 2|z|\rho(t + i|z|^2).$$

Then $\widetilde{\rho} \in \mathrm{adm}(\widetilde{\Gamma})$. We call the density $\widetilde{\rho}$ the pull-back by Π of ρ.

Proof. Let $\widetilde{\gamma} = (\gamma_1, \gamma_2) \in \widetilde{\Gamma}$. By definition, there is $\gamma \in \Gamma$ such that $\gamma_2 + i|\gamma_1|^2 = \gamma$. Since $\widetilde{\gamma}$ is legendrian, $\dot{\gamma} = 2i\overline{\gamma}_1\dot{\gamma}_1$. Using the definition of $\widetilde{\rho}$, we find

$$\int_{\widetilde{\gamma}} \widetilde{\rho} \mathrm{d}l = \int_a^b 2|\gamma_1(s)|\rho(\gamma(s))|\dot{\gamma}_1(s)|\mathrm{d}s = \int_a^b \rho(\gamma(s))|\dot{\gamma}(s)|\mathrm{d}s \geq 1.$$

□

Notation 2.1.12. — *Let Γ be a curve family in a domain Ω of $\mathbb{H}$ and $\rho \in \mathrm{adm}\,(\Gamma)$. Then, $\Pi^*\rho$ refers to the pull-back by Π of ρ. Moreover, one can define the same notion of push-forward density in $\mathbb{C}$ by setting $g_*\rho = \frac{\rho}{|\partial_w g| - |\partial_{\overline{w}} g|} \circ g^{-1}$ for any quasiconformal map $g : \Omega \longrightarrow \Omega' \subset \mathbb{C}$. In that case, the push-*

forward density satisfies

$$\int_{\Omega'} (g_*\rho)^2 \mathrm{d}L^2 = \int_{\Omega} \rho^2 K(.,g)\mathrm{d}L^2$$

where $K(.,g)$ is the quasiconformal distortion function of g.

— *Let $\widetilde{\Gamma}$ be a family of legendrian curves in a domain $\widetilde{\Omega}$ of $\mathcal{H}$, $\widetilde{\rho} \in \mathrm{adm}\left(\widetilde{\Gamma}\right)$ and $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}' \subset \mathcal{H}$ a quasiconformal map. Then, the push-forward by f of $\widetilde{\rho}$ is denoted $f_*\widetilde{\rho}$.*

The following proposition and corollary explain the link between some minimization problems in the Heisenberg group and corresponding problems in the half plane.

Proposition 2.1.13. *Let Ω and Ω' be domains of $\mathbb{H}$. Denote $\widetilde{\Omega} := \Psi(S^1 \times \Omega)$ and $\widetilde{\Omega}' := \Psi(S^1 \times \Omega')$. Let Γ be a curve family in Ω and $\widetilde{\Gamma}$ its lifted family in $\widetilde{\Omega}$. If $\rho \in \mathrm{adm}(\Gamma)$ and $g : \Omega \longrightarrow \Omega'$ is a quasiconformal map such that there is a quasiconformal map $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ with $\Pi \circ f = g \circ \Pi$, then*

$$\Pi^*(g_*\rho) = f_*(\Pi^*\rho)$$

Before going through the proof, we give a corollary of this.

Corollary 2.1.14. *Let $g : \Omega \longrightarrow \Omega'$ and $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ be as in the previous proposition. Let Γ be a curve family in Ω with continuous extremal density ρ_0 and $\Gamma' := g(\Gamma)$ with continuous extremal density ρ_0'. Suppose the following*

1. *$M(\widetilde{\Gamma}) = \int_{\widetilde{\Omega}} (\Pi^*\rho_0)^4 \mathrm{d}L^3$ where $\widetilde{\Gamma}$ is the lifted family of Γ, and*
2. *$M(\widetilde{\Gamma}') = \int_{\widetilde{\Omega}'} (\Pi^*\rho_0')^4 \mathrm{d}L^3$ where $\widetilde{\Gamma}'$ is the lifted family of Γ'.*

Then,

$$M(f(\widetilde{\Gamma})) = \int_{\widetilde{\Omega}} (\Pi^*\rho_0)^4 K(.,f)^2 \mathrm{d}L^3$$

if and only if

$$M(g(\Gamma)) = \int_{\Omega} K(.,g)\rho_0^2 \mathrm{d}L^2.$$

Proof of Corollary 2.1.14. Notice that $\Gamma' = g(\Gamma) = g(\Pi(\widetilde{\Gamma})) = \Pi(f(\widetilde{\Gamma}))$ and $\widetilde{\Gamma}' =$

$f(\widetilde{\Gamma})$:

$$\begin{aligned}
\widetilde{\delta} \in f(\widetilde{\Gamma}) &\iff f^{-1}(\widetilde{\delta}) \in \widetilde{\Gamma} \\
&\iff \Pi \circ f^{-1}(\widetilde{\delta}) \in \Gamma \\
&\iff g^{-1} \circ \Pi(\widetilde{\delta}) \in \Gamma \\
&\iff \Pi(\widetilde{\delta}) \in \Gamma' \\
&\iff \widetilde{\delta} \in \widetilde{\Gamma}'.
\end{aligned}$$

Now, assume first that $M(\Gamma') = \int_\Omega K(.,g)\rho_0^2 \mathrm{d}L^2$. Then, $g_*\rho_0$ and ρ_0' are both extremal for Γ'. By a result similar to Lemma 2.1.3 for the complex plane, and continuity of $g_*\rho_0$ and ρ_0', $g_*\rho_0 = \rho_0'$ in Ω'. So

$$f_*\Pi^*\rho_0 = \Pi^* g_*\rho_0 = \Pi^*\rho_0'$$

in $\widetilde{\Omega}'$. Thus, we have

$$M(f(\widetilde{\Gamma})) = M(\widetilde{\Gamma}') = \int_{\widetilde{\Omega}'} (f_*\Pi^*\rho_0)^4 \mathrm{d}L^3 = \int_{\widetilde{\Omega}} K(.,f)^2 (\Pi^*\rho_0)^4 \mathrm{d}L^3.$$

Conversely, assume that $M(f(\widetilde{\Gamma})) = \int_{\widetilde{\Omega}'} (f_*\Pi^*\rho_0)^4 \mathrm{d}L^3$. Then, we have

$$\int_{\widetilde{\Omega}'} (\Pi^*\rho_0')^4 \mathrm{d}L^3 = M(\widetilde{\Gamma}') = \int_{\widetilde{\Omega}'} (\Pi^* g_*\rho_0)^4 \mathrm{d}L^3.$$

By Lemma 2.1.3, and continuity of densities, we deduce that $\Pi^* g_*\rho_0 = \Pi^*\rho_0'$ in $\widetilde{\Omega}'$. Using the formula given for pull-back densities, one easily finds that $g_*\rho_0 = \rho_0'$. Leading to

$$M(\Gamma') = M(g(\Gamma)) = \int_\Omega K(.,g)\rho_0^2 \mathrm{d}L^2.$$

□

Proof of Proposition 2.1.13. By definition, $g_*\rho = \frac{\rho}{|\partial_w g| - |\partial_{\overline{w}} g|} \circ g^{-1}$ and so

$$\Pi^*(g_*\rho) = |Z\Pi| \left(\frac{\rho}{|\partial_w g| - |\partial_{\overline{w}} g|} \circ g^{-1} \circ \Pi \right).$$

Moreover, since f is contact, one as (see [35, p. 335])

$$Z(\Pi \circ f) = Z(f_2 + i|f_1|) = 2i\overline{f}_1 Z f_1 \text{ and } \overline{Z}(\Pi \circ f) = \overline{Z}(f_2 + i|f_1|) = 2i\overline{f}_1 \overline{Z} f_1.$$

Thus, since Π is a CR-function, using the chain rule we have

$$\begin{aligned}
|Zf_1| - |\overline{Z}f_1| &= \frac{1}{2|f_1|}\left(|Z(\Pi\circ f)| - |\overline{Z}(\Pi\circ f)|\right)\\
&= \frac{1}{2|f_1|}\left(|Z(g\circ \Pi)| - |\overline{Z}(g\circ \Pi)|\right)\\
&= \frac{|Z\Pi|}{2|f_1|}\left((|\partial_w g| - |\partial_{\overline{w}} g|)\circ \Pi\right)
\end{aligned}$$

Now, computing $f_*(\Pi^*\rho)$, we find

$$\begin{aligned}
f_*(\Pi^*\rho) &= \frac{|Z\Pi|(\rho\circ\Pi)}{|Zf_1| - |\overline{Z}f_1|}\circ f^{-1}\\
&= 2|f_1\circ f^{-1}|\left(\frac{\rho}{|\partial_w g| - |\partial_{\overline{w}} g|}\circ \Pi\circ f^{-1}\right)\\
&= |Z\Pi|\left(\frac{\rho}{|\partial_w g| - |\partial_{\overline{w}} g|}\circ g^{-1}\circ \Pi\right)
\end{aligned}$$

□

Lemma 2.1.15. *Let $g : \Omega \longrightarrow \Omega'$ be a map between domains of $\mathbb{H}$. Denote $\widetilde{\Omega} = \Psi(S^1\times\Omega)$ and $\widetilde{\Omega}' = \Psi(S^1\times\Omega')$. Assume that there is a contact map $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ such that $\Pi\circ f = g\circ\Pi$. Then, for every (z,t) in $\widetilde{\Omega}$,*

$$K((z,t), f) = K(\Pi(z,t), g).$$

Meaning that, f is K-quasiconformal if and only if g is K-quasiconformal.

Proof. Just using

$$Z(\Pi\circ f) = 2i\overline{f}_1 Zf_1 \text{ and } \overline{Z}(\Pi\circ f) = 2i\overline{f}_1\overline{Z}f_1$$

together with the fact that Π is CR, leads to

$$\frac{|Zf_1| - |\overline{Z}f_1|}{|Zf_1| + |\overline{Z}f_1|} = \frac{|\partial_w g| - |\partial_{\overline{w}} g|}{|\partial_w g| + |\partial_{\overline{w}} g|}\circ\Pi$$

which is exactly the statement of the lemma. □

Thus, if we are looking for a quasiconformal map on the Heisenberg group that minimizes a mean distortion for a nice density, a first step would be to look for

solutions of the corresponding problem in $\mathbb{H}$ that can be lifted by Π into contact transformations. Which is what we will do in the next section for cylinders.

2.2 Construction and uniqueness of an extremal quasiconformal map between cylinders

2.2.1 Construction of the map

In this section, we construct explicitly a minimizer of a mean distortion functional in a class of quasiconformal maps between cylinders, our objective is to prove Proposition 2.2.5.

As said, we are looking for a quasiconformal map between cylinders defined as a lift by Π of a quasiconformal map between projections of cylinders. For $r, R > 0$, we denote $C_{r,R}$ the cylinder

$$C_{r,R} = \{(z,t) \in \mathcal{H} \mid 0 < t < r \ \& \ |z| < \sqrt{R}\}.$$

We are here interested in finding a quasiconformal map $f : C_{a,b} \longrightarrow C_{a',b'}$ with $\frac{ab'}{a'b} > 1$ that minimizes a mean distortion functional within the set $\mathcal{F}$ of all orientation-preserving quasiconformal mappings from $C_{a,b}$ to $C_{a',b'}$ that extend homeomorphically to the boundary and map

- the disc $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b} \ \& \ t = 0\}$ to the disc $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b'} \ \& \ t = 0\}$,
- the disc $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b} \ \& \ t = a\}$ to the disc $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b'} \ \& \ t = a'\}$,
- and the cylinder $\{(z,t) \in \mathcal{H} \mid |z| = \sqrt{b} \ \& \ 0 \leq t \leq a\}$ to the cylinder $\{(z,t) \in \mathcal{H} \mid |z| = \sqrt{b} \ \& \ 0 \leq t \leq a\}$.

Projections by Π of $\Pi(C_{a,b} \backslash \{z = 0\})$ and $\Pi(C_{a',b'} \backslash \{z = 0\})$ are respectively the rectangles

$$R_{a,b} = \{w \in \mathbb{C} \mid 0 < \Re(w) < a, \ 0 < \Im(w) < b\}$$

and

$$R_{a',b'} = \{w \in \mathbb{C} \mid 0 < \Re(w) < a', \ 0 < \Im(w) < b'\}.$$

So that cylinders are analogs (up to the projection Π and a rescale of constants by a dilation) to quadrilaterals on their normal form in $\mathbb{C}$. Since extremal quasiconformal maps between quadrilaterals are the local model for Teichmüller homeomorphisms,

it seems appropriate to consider minimization problems for quasiconformal maps between cylinders.

On the cylinder, $C_{a,b}$, there is a natural foliation by legendrian curves given by

$$\widetilde{\Gamma}_0 = \{\widetilde{\gamma}_z(s) = \left(ze^{-\frac{is}{2|z|^2}}, s\right) \mid 0 < |z| < \sqrt{b} \ \& \ 0 < s < a\}$$

which are the legendrian lifts by Π of horizontal lines

$$\gamma_y(s) = s + iy \text{ for } 0 < y < b$$

in $R_{a,b}$ (see Figure 1).

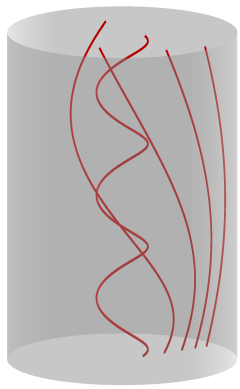

Figure 2.1 – Cylinder foliated by curves in $\widetilde{\Gamma}_0$ (foliation given by rotations around the vertical axis of drawn curves and the vertical axis itself).

To state the minimization problem we are dealing with, we need to find the modulus and extremal density of $\widetilde{\Gamma}_0$.

Lemma 2.2.1. *The curve family $\widetilde{\Gamma}_0$ has modulus*

$$M(\widetilde{\Gamma}_0) = \frac{16\pi b^3}{3a^3}$$

and its extremal density is $\widetilde{\rho}_0(z,t) = \frac{2|z|}{a}$.

Proof. If $\widetilde{\rho} \in \mathrm{adm}(\widetilde{\Gamma}_0)$, by definition we have

$$\int_{\widetilde{\gamma}_z} \widetilde{\rho} \mathrm{d}l = \int_0^a \widetilde{\rho}(\widetilde{\gamma}_z(s)) \frac{\mathrm{d}s}{2|z|} \geq 1$$

for all $0 < |z| < \sqrt{b}$. So,

$$\int_0^a \widetilde{\rho}(\widetilde{\gamma}_z(s))\mathrm{d}s \geq 2|z| \tag{2.4}$$

for all $0 < |z| < \sqrt{b}$. But,

$$\begin{aligned}
\int_{C_{a,b}} \widetilde{\rho}^4 \mathrm{d}L^3 &\underset{\text{cylindrical coordinates}}{=} \int_0^{2\pi}\int_0^{\sqrt{b}}\int_0^a \widetilde{\rho}^4(r,\theta,t) r\mathrm{d}t\mathrm{d}r\mathrm{d}\theta \\
&\underset{\text{substitution with Jacobian 1}}{=} \int_0^{2\pi}\int_0^{\sqrt{b}}\int_0^a \widetilde{\rho}^4\left(r,\theta-\frac{s}{2r^2},s\right) r\mathrm{d}s\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{b}}\int_0^a \widetilde{\rho}^4\left(\widetilde{\gamma}_{re^{i\theta}}(s)\right) r\mathrm{d}s\mathrm{d}r\mathrm{d}\theta.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_0^a \widetilde{\rho}^4(\widetilde{\gamma}_{re^{i\theta}}(s))\mathrm{d}s &\underset{\text{Hölder inequality}}{\geq} \frac{1}{a^3}\left(\int_0^a \widetilde{\rho}(\widetilde{\gamma}_{re^{i\theta}}(s))\mathrm{d}s\right)^4 \\
&\underset{\text{inequality 2.4}}{\geq} \frac{16r^4}{a^3}
\end{aligned}$$

for all $0 < r < \sqrt{b}$ and $0 < \theta < 2\pi$. Thus, we get

$$\int_{C_{a,b}} \widetilde{\rho}^4 \mathrm{d}L^3 \geq \frac{32\pi}{a^3}\int_0^{\sqrt{b}} r^5\mathrm{d}r = \frac{16\pi b^3}{3a^3}$$

for all $\widetilde{\rho} \in \mathrm{adm}(\widetilde{\Gamma}_0)$. Consequently,

$$M(\widetilde{\Gamma}_0) \geq \frac{16\pi b^3}{3a^3}.$$

Moreover, set

$$\widetilde{\rho}_0(z,t) = \frac{2|z|}{a}.$$

Then $\widetilde{\rho}_0 \in \mathrm{adm}(\widetilde{\Gamma}_0)$ and

$$\int_{C_{a,b}} \widetilde{\rho}_0^4 \mathrm{d}L^3 = \frac{16\pi b^3}{3a^3}.$$

□

Let us set $\Gamma_0 = \{\gamma_y \mid 0 < y < b\}$ where $\gamma_y :]0,a[\longrightarrow R_{a,b}$ is defined by $\gamma_y(s) = s + iy$. It is known since the work of Grötzsch that Γ_0 has modulus $\frac{b}{a}$ and extremal

density $\rho_0(w) = \frac{1}{a}$. Let $\mathcal{G}$ be the set of all quasiconformal mappings from $R_{a,b}$ to $R_{a',b'}$ that map homeomorphically the vertical and horizontal boundary components of $R_{a,b}$ on the corresponding ones of $R_{a',b'}$. Here, and in the following, rectangles are considered as subsets of $\mathbb{C}$ and so, boundaries are considered in $\mathbb{C}$. The following is well known (a short proof is given in the appendix A).

Lemma 2.2.2. *Any minimizer of the mean distortion in $\mathcal{G}$ for the density ρ_0 is a map $f_\varphi(x+iy) = \frac{a'}{a}x + i\varphi(y)$ where $\varphi : [0,b] \longrightarrow [0,b']$ is a diffeomorphism such that $\varphi(0) = 0$, $\varphi(b) = b'$ and $\dot{\varphi} \geq \frac{a'}{a}$.*

Notice also that $\widetilde{\rho_0} = \Pi^*\rho_0$ and

$$M(\widetilde{f_\varphi(\Gamma_0)}) = \int_{C_{a',b'}} (\Pi^*\rho_0')^4 \mathrm{d}L^3$$

where $\rho_0'(w) = \frac{1}{a'}$ is extremal for $f_\varphi(\Gamma_0) = \Gamma_0'$ and Γ_0' is the family of horizontal lines in $R_{a',b'}$ (which is true for every φ defined as in the previous lemma). Moreover, every f_φ defined as in the previous lemma satisfies

$$M(f_\varphi(\Gamma_0)) = \int_{R_{a,b}} K(., f_\varphi)\rho_0^2 \mathrm{d}L^2.$$

Then, according to Corollary 2.1.14, if we find a quasiconformal map $\widetilde{f} \in \mathcal{F}$ that maps the vertical axis homeomorphically on the vertical axis and such that $\Pi \circ \widetilde{f} = f_\varphi \circ \Pi$ for a function φ defined as previously, then $\widetilde{f}$ will satisfy

$$M(\widetilde{f}(\widetilde{\Gamma}_0)) = \int_{C_{a,b}} K(., \widetilde{f})^2 \widetilde{\rho}_0^4 \mathrm{d}L^3.$$

Proposition 2.2.3. *There is only one function $\varphi : [0,b] \longrightarrow [0,b']$ with $\varphi(0) = 0$, $\varphi(b) = b'$ and $\dot{\varphi} \geq \frac{a'}{a}$ such that f_φ can be lifted into a quasiconformal map $\widetilde{f} : C_{a,b} \longrightarrow C_{a',b'}$. That function is defined for every $x \in [0,b]$ by*

$$\varphi(x) = \frac{b'x}{\left(1 - \frac{ab'}{a'b}\right)x + \frac{ab'}{a'}}$$

and the lifts are compositions with rotations around the vertical axis of

$$\begin{array}{rccc} \widetilde{f}_0 : & C_{a,b} & \longrightarrow & C_{a',b'} \\ & (z,t) & \longmapsto & \left(\frac{\sqrt{b'} z e^{\frac{i}{2b}\left(1 - \frac{a'b}{ab'}\right)t}}{\sqrt{\left(1 - \frac{ab'}{a'b}\right)|z|^2 + \frac{ab'}{a'}}}, \frac{a'}{a}t \right). \end{array}$$

Proof. To prove this, it will be more convenient to compute in usual cylindrical coordinates on $\mathcal{H} = \mathbb{R}^3$. Meaning, $(r, \theta, t) \longmapsto (re^{i\theta}, t)$. In those coordinates, the contact form writes as

$$\omega = \mathrm{d}t + 2r^2 \mathrm{d}\theta.$$

So, we are looking for four functions R, Θ, T, φ such that

$$T + iR^2(r, \theta, t) = \frac{a'}{a}t + i\varphi(r^2) \text{ and } \mathrm{d}T + 2R^2\mathrm{d}\Theta = \lambda(\mathrm{d}t + 2r^2\mathrm{d}\theta)$$

for a nowhere vanishing function λ. Moreover, for all r, t, $\Theta(r, ., t)$ is 2π-periodic modulo 2π. Since $T + iR^2(r, \theta, t) = \frac{a'}{a}t + i\varphi(r^2)$, we get

$$T(r, \theta, t) = T(t) = \frac{a'}{a}t \text{ and } R^2(r, \theta, t) = R^2(r) = \varphi(r^2).$$

The idea is to use the system of PDEs that the functions R, Θ and T must verify in order to find an ordinary differential equation that φ must verify. In the following, we denote by an index r (resp. θ, resp. t) the partial derivative of a function according to r (resp. θ, resp. t).

Since $T_r = T_\theta = 0$, $T_t(t) = \frac{a'}{a}$, and $\mathrm{d}T + 2R^2\mathrm{d}\Theta = \lambda(\mathrm{d}t + 2r^2\mathrm{d}\theta)$, we get that

$$\Theta_r(r, \theta, t) = 0 \text{ and } \Theta_\theta(r, \theta, t) = \frac{a'r^2}{a\varphi(r^2)} + 2r^2\Theta_t(r, \theta, t).$$

Moreover, from $\Theta_r = 0$, we deduce that

$$\Theta_{r,\theta} = \Theta_{r,t} = 0.$$

So, differentiating $\Theta_\theta(r, \theta, t) = \frac{a'r^2}{a\varphi(r^2)} + 2r^2\Theta_t(r, \theta, t)$ with respect to r gives

$$2\Theta_t(r, \theta, t) = \frac{a'}{a}\frac{r^2\dot{\varphi}(r^2) - \varphi(r^2)}{\varphi^2(r^2)}$$

and we deduce

$$\Theta_\theta(r, \theta, t) = \Theta_\theta(r) = \frac{a'r^4\dot{\varphi}(r^2)}{a\varphi^2(r^2)}.$$

From the fact that $\Theta_{r,\theta} = 0$, putting $x = r^2$, φ must verify the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2\dot{\varphi}(x)}{\varphi^2(x)}\right) = 0$$

whose solutions are the functions

$$\varphi(x) = \frac{x}{Cx + D} \text{ with } C, D \in \mathbb{R}.$$

For such functions, we have $\Theta_\theta(r, \theta, t) = \frac{a'}{a} D$. So, for (R, Θ, T) to be a homeomorphism, we must have $D = \frac{a}{a'}$. Moreover, we want that $\varphi(b) = b'$, so $C = \frac{1}{b'}\left(1 - \frac{ab'}{a'b}\right)$. Consequently

$$\varphi(x) = \frac{b'x}{\left(1 - \frac{ab'}{a'b}\right) x + \frac{ab'}{a'}}$$

for all $x \in [0, b]$ (one may check that $\left(1 - \frac{ab'}{a'b}\right) x + \frac{ab'}{a'} > 0$ if $x \in [0, b]$). Moreover, it is easy to see that

$$\dot{\varphi} \geq \frac{a'}{a}.$$

Replacing φ by its value, we find

$$\begin{aligned}
\Theta_\theta(r, \theta, t) &= 1 \\
\Theta_r(r, \theta, t) &= 0 \\
\Theta_t(r, \theta, t) &= \frac{1}{2b} - \frac{a'}{2ab'}.
\end{aligned}$$

And so

$$\begin{aligned}
\Theta(r, \theta, t) &= \theta + \frac{1}{2}\left(\frac{1}{b} - \frac{a'}{ab'}\right) t + \alpha \text{ with } \alpha \in \mathbb{R} \\
R(r, \theta, t) &= \frac{\sqrt{b'} r}{\sqrt{\left(1 - \frac{ab'}{a'b}\right) r^2 + \frac{ab'}{a'}}} \\
T(r, \theta, t) &= \frac{a'}{a} t
\end{aligned}$$

Which gives in usual coordinates

$$\widetilde{f}_\alpha(z, t) = \left(\frac{\sqrt{b'} e^{i\alpha} z e^{\frac{i}{2b}\left(1 - \frac{a'b}{ab'}\right)t}}{\sqrt{\left(1 - \frac{ab'}{a'b}\right) |z|^2 + \frac{ab'}{a'}}}, \frac{a'}{a} t \right).$$

□

The mappings $\widetilde{f}_\alpha$ are quasiconformal with distortion function

$$K((z,t),\widetilde{f}_\alpha) = \frac{1}{\left(1+\left(\frac{a'}{ab'}-\frac{1}{b}\right)|z|^2\right)^2}$$

and maximal distortion

$$K_{\widetilde{f}_\alpha} = \left(\frac{ab'}{a'b}\right)^2.$$

Moreover, they map the vertical axis homeomorphically to the vertical axis, so according to what we said before the proposition, they satisfy

$$M(\widetilde{f}_\alpha(\widetilde{\Gamma}_0)) = \int_{C_{a,b}} K(.,\widetilde{f}_\alpha)^4\widetilde{\rho}_0^4 \mathrm{d}L^3.$$

It still remains to show that they minimize the mean distortion in $\mathcal{F}$ for the density $\widetilde{\rho}_0$. For that, we will use Proposition 2.1.7 and so, we need to find a larger family of legendrian curves $\widetilde{\Gamma} \supset \widetilde{\Gamma}_0$ for which $\widetilde{\rho}_0 \in \mathrm{adm}(\widetilde{\Gamma})$ and $M(\widetilde{f}_\alpha(\widetilde{\Gamma}_0)) \leq M(f(\widetilde{\Gamma}))$ for every $f \in \mathcal{F}$.

Let $\widetilde{\Gamma}$ be the set of all legendrian curves in $C_{a,b}$ joining the discs $\overline{C_{a,b}} \cap \{t=0\}$ and $\overline{C_{a,b}} \cap \{t=a\}$.

Lemma 2.2.4. *Let $\widetilde{\Gamma}$ be the set of all legendrian curves joining the two boundary discs of $C_{a,b}$ and take an element γ of $\widetilde{\Gamma}$. Then,*

$$\int_\gamma \widetilde{\rho}_0 \mathrm{d}l \geq 1.$$

Moreover, we get equality if and only if $\gamma \in \widetilde{\Gamma}_0$.

Proof. If $\gamma \in \widetilde{\Gamma}$, take a parametrization of γ between 0 and a,

$$\begin{aligned}
\int_\gamma \rho_0 \mathrm{d}l &= \frac{1}{a}\int_0^a 2|\gamma_1(s)||\dot{\gamma}_1(s)|\mathrm{d}s \\
&= \frac{1}{a}\int_0^a \left|\frac{\mathrm{d}}{\mathrm{d}s}(\gamma_2 + i|\gamma_1|^2)(s)\right| \mathrm{d}s \\
&\geq \frac{1}{a}\int_0^a |\dot{\gamma}_2(s)|\mathrm{d}s \\
&= 1.
\end{aligned}$$

Equality happens if and only if $|\gamma_1|^2$ is constant. In that case, one may check that $\gamma(s) = \left(ze^{-i\frac{\zeta(s)}{2|z|^2}}, \zeta(s)\right)$. Meaning, $\gamma \in \widetilde{\Gamma}_0$.

□

Moreover, by definition of $\mathcal{F}$, for every $f \in \mathcal{F}$, $f(\widetilde{\Gamma})$ is the family of all legendrian curves in $C_{a',b'}$ joining the two discs $\overline{C_{a',b'}} \cap \{t = 0\}$ and $\overline{C_{a',b'}} \cap \{t = a'\}$. So, for every $f \in \mathcal{F}$, $\widetilde{f}_\alpha(\widetilde{\Gamma}_0) \subset f(\widetilde{\Gamma})$ and thus we have $M(\widetilde{f}_\alpha(\widetilde{\Gamma}_0)) \leq M(f(\widetilde{\Gamma}))$ for every $f \in \mathcal{F}$. Applying Proposition 2.1.7 is enough to prove the following.

Proposition 2.2.5. *Every composition by a rotation around the t-axis of the map*

$$\begin{array}{rccc} \widetilde{f}_0 : & C_{a,b} & \longrightarrow & C_{a',b'} \\ & (z,t) & \longmapsto & \left(\frac{\sqrt{b'} z e^{\frac{i}{2b}\left(1-\frac{a'b}{ab'}\right)t}}{\sqrt{\left(1-\frac{ab'}{a'b}\right)|z|^2+\frac{ab'}{a'}}}, \frac{a'}{a}t \right) \end{array}$$

is a minimizer of the mean distortion in $\mathcal{F}$ for the density $\widetilde{\rho}_0$.

2.2.2 Uniqueness up to rotations of the map

Here, we are dealing with finding every minimizer of the mean distortion in $\mathcal{F}$ for the density $\widetilde{\rho}_0$. We will show that such minimizers must be constructed as we did in the previous section. Thus, the section is dedicated to the proof of the following.

Theorem 2.2.6. *Suppose that $f \in \mathcal{F}$ minimizes the mean distortion in $\mathcal{F}$ for the density $\widetilde{\rho}_0$. Then, there is $\alpha \in \mathbb{R}$ such that $f = \widetilde{f}_\alpha$.*

The proof is decomposed in two big steps. The first is a reformulation of the beginning of [8] in the setting of cylinders. The idea is to show that a minimizer of the mean distortion in $\mathcal{F}$ for the density $\widetilde{\rho}_0$ *dilates curves in* $\widetilde{\Gamma}_0$. The second step is to prove that if f is a minimizer of the mean distortion considered, then $\Pi \circ f$ does not depend on $\arg(z)$, so $\Pi \circ f$ induces a quasiconformal map $g : R_{a,b} \longrightarrow R_{a',b'}$ which will be in fact a minimizer of the mean distortion in $\mathcal{G}$ for the density ρ_0.

Let $\widetilde{\Gamma}'_0$ be the curve family

$$\widetilde{\Gamma}'_0 = \{\delta_z(s) = \left(ze^{-i\frac{s}{2|z|^2}}, s\right) \mid 0 < |z| < \sqrt{b'}\}.$$

Notice that $f \in \mathcal{F}$ minimizes the mean distortion in $\mathcal{F}$ for the density $\widetilde{\rho}_0$ if and only if $M(\widetilde{\Gamma}'_0) = \int_{C_{a,b}} K(.,f)^2 \widetilde{\rho}_0^4 \mathrm{d}L^3$. Indeed, since $\widetilde{f}_\alpha$ minimizes the mean distortion in $\mathcal{F}$ for the density $\widetilde{\rho}_0$, any other minimizer f satisfies

$$\int_{C_{a,b}} K(.,f)^2 \widetilde{\rho}_0^4 \mathrm{d}L^3 = \int_{C_{a,b}} K(.,\widetilde{f}_\alpha)^2 \widetilde{\rho}_0^4 \mathrm{d}L^3.$$

Moreover, we have

$$M(\widetilde{f}_\alpha(\widetilde{\Gamma}_0)) = M(\widetilde{\Gamma}'_0) = \int_{C_{a,b}} K(.,\widetilde{f}_\alpha)^2 \widetilde{\rho}_0^4 \mathrm{d}L^3$$

So, $f \in \mathcal{F}$ satisfies $\int_{C_{a,b}} K(.,f)^2\widetilde{\rho}_0^4\mathrm{d}L^3 = \int_{C_{a,b}} K(.,\widetilde{f}_\alpha)^2\widetilde{\rho}_0^4\mathrm{d}L^3$ if and only if $M(\widetilde{\Gamma}'_0) = \int_{C_{a,b}} K(.,f)^2\widetilde{\rho}_0^4\mathrm{d}L^3$. Fix f a minimizer of the mean distortion in $\mathcal{F}$ for the density $\widetilde{\rho}_0$. We begin with proving the following.

Proposition 2.2.7. *$f(\widetilde{\Gamma}_0) = \widetilde{\Gamma}'_0$, meaning that for every $0 < |z| < \sqrt{b}$ there is a homeomorphism from $]0,a[$ to $]0,a'[$ ζ_z such that for every $s \in]0,a[$,*

$$f(\gamma_z(s)) = \left(z' e^{-i\frac{\zeta_z(s)}{2|z'|^2}}, \zeta_z(s)\right).$$

Moreover f maps the vertical axis on the vertical axis.

Proof. The fact that f minimizes the mean distortion in $\mathcal{F}$ for the extremal density $\widetilde{\rho}_0$ ensures that $f_*\widetilde{\rho}_0$ is an extremal density of the family $\widetilde{\Gamma}'_0$. But,

$$\widetilde{\rho}'_0(z,t) = \frac{2|z|}{a'}$$

is also an extremal density of the family $\widetilde{\Gamma}'_0$. Then, by Lemma 2.1.3, since $f_*\widetilde{\rho}_0$ and $\widetilde{\rho}'_0$ are continuous,

$$f_*\widetilde{\rho}_0 = \widetilde{\rho}'_0.$$

Let $\delta \in \widetilde{\Gamma}'_0$, according to the previous lemma, we get,

$$1 = \int_\delta \widetilde{\rho}'_0 \mathrm{d}l = \int_\delta f_*\widetilde{\rho}_0 \mathrm{d}l = \int_0^{a'} f_*\widetilde{\rho}_0(\delta(s))|\dot{\delta}_1(s)|\mathrm{d}s.$$

Moreover, since $f^{-1}(\delta) \in \Gamma$, we have, using the fact that $|Z(f^{-1})_1| + |\overline{Z}(f^{-1})_1| = \frac{1}{|Zf_1|-|\overline{Z}f_1|} \circ f^{-1}$ and inequality 2.3,

$$\begin{aligned}
1 &\leq \int_{f^{-1}(\delta)} \widetilde{\rho}_0 \mathrm{d}l \\
&= \int_0^{a'} \widetilde{\rho}_0(f^{-1}(\delta(s)))|((f^{-1})_1^{\cdot} \circ \delta(s))|\mathrm{d}s \\
&\leq \int_0^{a'} \widetilde{\rho}_0(f^{-1}(\delta(s)))|\dot{\delta}_1(s)| \left(|Z(f^{-1})_1(\delta(s))| + |\overline{Z}(f^{-1})_1(\delta(s))|\right) \mathrm{d}s
\end{aligned}$$

$$\begin{aligned} &= \int_0^{a'} f_*\widetilde{\rho}_0(\delta(s))|\dot{\delta}_1(s)|\mathrm{d}s \\ &= 1 \end{aligned}$$

Using Lemma 2.2.4, it means that $f^{-1}(\delta) \in \widetilde{\Gamma}_0$. So,

$$C_{a',b'}\backslash\{z=0\} \subset f(C_{a,b}\backslash\{z=0\}).$$

Since f is a homeomorphism,

$$f(C_{a,b} \cap \{z=0\}) \subset C_{a',b'} \cap \{z=0\}.$$

Moreover, using that $f \in \mathcal{F}$, it implies $f(0,0) = (0,0)$ and $f(0,a) = (0,a')$. So, f maps the vertical axis to itself and $f(\widetilde{\Gamma}_0) = \widetilde{\Gamma}'_0$ because $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}'_0$ are foliations of $C_{a,b}\backslash\{z=0\}$ and $C_{a',b'}\backslash\{z=0\}$ respectively.

□

Now we know that $f\left(ze^{-i\frac{s}{2|z|^2}}, s\right) = \left(z'e^{-i\frac{\zeta_z(s)}{2|z'|^2}}, \zeta_z(s)\right)$, we want to find the functions ζ_z.

Proposition 2.2.8. *For every $0 < |z| < \sqrt{b}$, there is $0 < |z'| < \sqrt{b'}$ such that, for every $s \in]0,a[$, we have*

$$f\left(ze^{-i\frac{s}{2|z|^2}}, s\right) = \left(z'e^{-i\frac{a's}{2a|z'|^2}}, \frac{a'}{a}s\right).$$

We say that f dilates curves in $\widetilde{\Gamma}_0$.

Before giving a proof, we need the following result: Proposition 2.12 in [8, p. 133]. If f is a map as in Theorem 2.5, then for every curve $\gamma \in \widetilde{\Gamma}_0$,

$$|(f_1 \dot{\circ} \gamma)| = \left(|Zf_1(\gamma)| - |\overline{Z}f_1(\gamma)|\right)|\dot{\gamma}_1| \tag{2.5}$$

This property is called the *minimal stretching property.*

Proof. In order to prove this, we consider two vector fields on $\mathcal{H}\backslash\{z=0\}$

$$W := -\frac{iz}{2|z|^2}Z \text{ and } \overline{W} := \frac{i\overline{z}}{2|z|^2}\overline{Z}$$

Then, $W_{\gamma(s)} = \dot{\gamma}_1(s) Z_{\gamma(s)}$ and $\overline{W}_{\gamma(s)} = \dot{\overline{\gamma}}_1(s)\overline{Z}_{\gamma(s)}$ for every $\gamma \in \Gamma_0$. Thus,

$$|Wf_1(\gamma) + \overline{W}f_1(\gamma)| = |\dot{\gamma}_1 Zf_1(\gamma) + \dot{\overline{\gamma}}_1\overline{Z}f_1(\gamma)| = |(f_1 \dot{\circ} \gamma)|.$$

So, using 2.5, we have

$$|Wf_1(\gamma) + \overline{W}f_1(\gamma)| = \big(|Zf_1(\gamma)| - |\overline{Z}f_1(\gamma)|\big)\,|\dot{\gamma}_1|.$$

Since $|Zf_1(\gamma)| - |\overline{Z}f_1(\gamma)| = \frac{1}{|\dot{\gamma}_1|}\big(|Wf_1(\gamma)| - |\overline{W}f_1(\gamma)|\big)$, we get that

$$|Wf_1 + \overline{W}f_1| = |Wf_1| - |Wf_1|.$$

From this and the fact that f is a contact transform, we also deduce

$$|W(\Pi \circ f) + \overline{W}(\Pi \circ f)| = |W(\Pi \circ f)| - |\overline{W}(\Pi \circ f)| = 2|f_1|\left(|Wf_1| - |\overline{W}f_1|\right).$$

Finally, by definition of W and $\overline{W}$, we find

$$|W(\Pi \circ f)| - |\overline{W}(\Pi \circ f)| \;=\; \frac{|f_1|}{|z|}\left(|Zf_1| - |\overline{Z}f_1|\right) \tag{2.6}$$

Now, we know that

$$f_*\widetilde{\rho}_0 \circ f = \frac{\widetilde{\rho}_0}{|Zf_1| - |\overline{Z}f_1|} \underset{by 2.6}{=} \frac{2|f_1|}{a}\frac{1}{|W(\Pi \circ f)| - |\overline{W}(\Pi \circ f)|}$$

and, by continuity and Lemma 2.1.3,

$$f_*\widetilde{\rho}_0 \circ f = \widetilde{\rho}_0' \circ f = \frac{2|f_1|}{a'}.$$

Thus,

$$|W(\Pi \circ f)| - |\overline{W}(\Pi \circ f)| = \frac{a'}{a}$$

Moreover, Proposition 2.2.7 leads to $(\Pi \circ f \circ \gamma_z)(s) = \zeta_z(s) + i|z'|^2$ for every curve $\gamma_z \in \widetilde{\Gamma}_0$. So, $\frac{\mathrm{d}}{\mathrm{d}s}\zeta_z(s) = \frac{\mathrm{d}}{\mathrm{d}s}(\Pi \circ f \circ \gamma_z)(s)$. But,

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}s}(\Pi \circ f \circ \gamma_z)\right| &= |Z(\Pi \circ f)(\gamma_z)\dot{\gamma}_{z,1} + \overline{Z}(\Pi \circ f)(\gamma_z)\dot{\overline{\gamma}}_{z,1}| \\
&= |W(\Pi \circ f)(\gamma_z) + \overline{W}(\Pi \circ f)(\gamma_z)| \\
&= |W(\Pi \circ f)(\gamma_z)| - |\overline{W}(\Pi \circ f)(\gamma_z)|
\end{aligned}$$

$$= \frac{a'}{a}$$

Thus, $\zeta_z(s) = \frac{a'}{a}s$ for every $0 < |z| < \sqrt{b}$. $\square$

In particular, we proved that $f_2(z,t) = \frac{a'}{a}t$. Now, we are in position to prove the announced theorem. We need to show that $f_2 + i|f_1|^2$ does not depend on $\arg(z)$ and that the induced quasiconformal map between $R_{a,b}$ and $R_{a',b'}$ minimizes the mean distortion in $\mathcal{G}$ for the density ρ_0.

Proof of Theorem 2.2.6. As in the proof of Proposition 2.2.3, it is more convenient to write in cylindrical coordinates. In those coordinates, the curves γ_z are the curves $s \mapsto \left(r, \theta - \frac{s}{2r^2}, s\right)$ for $0 < r < \sqrt{b}$ and $\theta \in \mathbb{R}$ and write the map f as (R, Θ, T) (meaning that $(f_1, f_2) = (Re^{i\Theta}, T)$). Since f maps $\widetilde{\Gamma}_0$ to $\widetilde{\Gamma}'_0$, then

$$\frac{\mathrm{d}}{\mathrm{d}s} R\left(r, \theta - \frac{s}{2r^2}, s\right) = 0.$$

Thus,

$$R_\theta\left(r, \theta - \frac{s}{2r^2}, s\right) = 2r^2 R_t\left(r, \theta - \frac{s}{2r^2}, s\right).$$

As it is true for every r, θ, we have

$$R_\theta(r,\theta,t) = 2r^2 R_t(r,\theta,t) \text{ for every } (r,\theta,t) \in]0, \sqrt{b}[\times\mathbb{R}\times]0, a[.$$

Differentiating with respect to r, we also find,

$$R_{r,\theta}(r,\theta,t) = 4rR_t(r,\theta,t) + 2r^2 R_{r,t}(r,\theta,t)$$

for every $(r,\theta,t) \in]0, \sqrt{b}[\times\mathbb{R}\times]0, a[$. Since (R, Θ, T) is a contact map with $T_r(r,\theta,t) = 0$, then

$$\Theta_r(r,\theta,t) = 0.$$

Moreover, there is a nowhere vanishing function λ such that for every $(r,\theta,t) \in]0, \sqrt{b}[\times\mathbb{R}\times]0, a[$, we have

$$\frac{a'}{a} + 2R^2(r,\theta,t)\Theta_t(r,\theta,t) = \lambda(r,\theta,t) \text{ and } 2R^2(r,\theta,t)\Theta_\theta(r,\theta,t) = 2r^2\lambda(r,\theta,t).$$

Leading to, for every $(r,\theta,t) \in]0,\sqrt{b}[\times\mathbb{R}\times]0,a[$,

$$\Theta_\theta(r,\theta,t) \quad = \quad 2r^2\Theta_t(r,\theta,t) + \frac{a'r^2}{aR^2(r,\theta,t)}. \tag{2.7}$$

Now, since $\Theta_r = 0$, differentiating the previous equation with respect to r, we get for every $(r,\theta,t) \in]0,\sqrt{b}[\times\mathbb{R}\times]0,a[$,

$$0 = 4r\Theta_t(r,\theta,t) + \frac{a'}{a}\left(\frac{2r}{R^2(r,\theta,t)} - \frac{2r^2R_r(r,\theta,t)}{R^3(r,\theta,t)}\right).$$

Then, for every $(r,\theta,t) \in]0,\sqrt{b}[\times\mathbb{R}\times]0,a[$

$$2\Theta_t(r,\theta,t) = \frac{a'}{a}\left(\frac{rR_r(r,\theta,t)}{R^3(r,\theta,t)} - \frac{1}{R^2(r,\theta,t)}\right).$$

Replacing in 2.7: for every $(r,\theta,t) \in]0,\sqrt{b}[\times\mathbb{R}\times]0,a[$

$$\Theta_\theta(r,\theta,t) = \frac{a'r^3R_r(r,\theta,t)}{aR^3(r,\theta,t)}.$$

Differentiating the expression of Θ_t with respect to θ, replacing $R_\theta(r,\theta,t)$ by $2r^2R_t(r,\theta,t)$ and $R_{r,\theta}(r,\theta,t)$ by $4rR_t(r,\theta,t) + 2r^2R_{r,t}(r,\theta,t)$, we find that for every $(r,\theta,t) \in]0,\sqrt{b}[\times\mathbb{R}\times]0,a[$,

$$\Theta_{\theta,t}(r,\theta,t) = \frac{a'r^2}{a}\left(\frac{4R_t(r,\theta,t) + rR_{t,r}(r,\theta,t)}{R^3(r,\theta,t)} - \frac{3rR_r(r,\theta,t)R_t(r,\theta,t)}{R^4(r,\theta,t)}\right).$$

Differentiating the expression of Θ_θ with respect to t, we have for every $(r,\theta,t) \in]0,\sqrt{b}[\times\mathbb{R}\times]0,a[$,

$$\Theta_{t,\theta}(r,\theta,t) = \frac{a'r^2}{a}\left(\frac{rR_{t,r}(r,\theta,t)}{R^3(r,\theta,t)} - \frac{3rR_r(r,\theta,t)R_t(r,\theta,t)}{R^4(r,\theta,t)}\right).$$

Since we assumed all our maps to be C^2, by use of Schwarz theorem about commutativity of partial derivatives, we conclude that for every $(r,\theta,t) \in]0,\sqrt{b}[\times\mathbb{R}\times]0,a[$

$$\frac{4R_t(r\theta,t)}{R^3(r,\theta,t)} = 0.$$

Leading to

$$R_t(r,\theta,t) = R_\theta(r,\theta,t) = 0,$$

for every $(r,\theta,t) \in]0,\sqrt{b}[\times\mathbb{R}\times]0,a[$. Then, f must be constructed as a lift of a quasiconformal map g between rectangles. Now, it remains to show that g is a minimizer of the mean distortion in $\mathcal{G}$ for the density ρ_0.

First, notice that

$$M(f(\widetilde{\Gamma}_0)) = \int_{C_{a,b}} K(.,f)^2 \widetilde{\rho}_0^4 \mathrm{d}L^3 \text{ and } \widetilde{\rho}_0 = \Pi^* \rho_0.$$

Moreover, if $\Gamma_0' = \Pi(f(\widetilde{\Gamma}_0))$, then by Proposition 2.2.7, Γ_0' is the family of straight horizontal lines in the rectangle $R_{a',b'}$ which has as extremal density $\rho_0' = \frac{1}{a'}$ and its lifted family is by definition $\widetilde{\Gamma}_0'$ which has as extremal density $\Pi^*\rho_0'$. We can then apply Corollary 2.1.14 to state that

$$M(g(\Gamma_0)) = \int_{R_{a,b}} K(.,g)\rho_0^2 \mathrm{d}L^2.$$

Finally, let Γ be the set of all curves in $R_{a,b}$ connecting the two vertical boundary edges. Then, $\rho_0 \in \mathrm{adm}(\Gamma)$ and for every $h \in \mathcal{G}$, since $g(\Gamma_0) \subset h(\Gamma)$, $M(g(\Gamma_0)) \leq M(h(\Gamma))$. So, by a result similar to Proposition 2.1.7 for quasiconformal maps in the complex plane, g minimizes the mean distortion in $\mathcal{G}$ for the density ρ_0. Thus, by Lemma 2.2.2 and Proposition 2.2.3, f is one of the $\widetilde{f}_\alpha$.

□

To sum up, in section 2.1 we proved

Theorem 2.2.9. *The map*

$$\begin{array}{rccc} \widetilde{f}_0 : & C_{a,b} & \longrightarrow & C_{a',b'} \\ & (z,t) & \longmapsto & \left(\frac{\sqrt{b'} z e^{\frac{i}{2b}\left(1-\frac{a'b}{ab'}\right)t}}{\sqrt{\left(1-\frac{ab'}{a'b}\right)|z|^2+\frac{ab'}{a'}}}, \frac{a'}{a}t \right) \end{array}$$

minimizes the mean distortion in $\mathcal{F}$ for the density $\widetilde{\rho}_0$ and is the unique such minimizer up to composition with a rotation around the vertical axis.

2.3 Generalized construction

2.3.1 Notations and the "pull-back density condition"

In this section, we want to determine conditions in order to generalize the construction we made before to domains in $\mathcal{H}$ that are not conformally equivalent to

cylinders but whose projections on the half plane are (interior of) quadrilaterals. So, let us take two bounded domains of $\mathbb{H}$, $\Omega_{a,b}$ and $\Omega_{a',b'}$ with two biholomorphic maps $\phi : R_{a,b} \longrightarrow \Omega_{a,b}$ and $\psi : R_{a',b'} \longrightarrow \Omega_{a',b'}$ that extend homeomorphically to boundaries (boundaries taken in $\mathbb{C}$).

Notation 2.3.1. — *We denote Γ_0 (resp. Γ'_0) the family of horizontal lines in $R_{a,b}$ (resp. $R_{a',b'}$), then $\Gamma_\phi := \phi(\Gamma_0)$ and $\Gamma_\psi := \psi(\Gamma'_0)$.*

— *We denote again $\rho_0 = \frac{1}{a}$ (resp. $\rho'_0 = \frac{1}{a'}$) the extremal density of Γ_0 (resp. Γ'_0). Set, $\rho_\phi = \phi_*\rho_0 \in \mathrm{adm}(\Gamma_\phi)$ and $\rho_\psi = \psi_*\rho'_0 \in \mathrm{adm}(\Gamma_\psi)$ for the push-forward densities. Since ϕ and ψ are holomorphic, ρ_ϕ (resp. ρ_ψ) is extremal for Γ_ϕ (resp. Γ_ψ).*

— *Consider $R_{a,b}$ and $R_{a',b'}$ as $]0,a[\times]0,b[$ and $]0,a'[\times]0,b'[$ and still write $\phi :]0,a[\times]0,b[\longrightarrow \Omega_{a,b}$ and $\psi :]0,a'[\times]0,b'[\longrightarrow \Omega_{a',b'}$ for the holomorphic maps. Then, for every $s \in [0,a]$, $\partial\Omega_s := \phi(\{s\} \times [0,b])$ and for every $s \in [0,a']$, $\partial\Omega'_s := \psi(\{s\} \times [0,b'])$.*

— *We denote $\mathcal{G}_{\phi,\psi}$ the set of quasiconformal maps from $\Omega_{a,b}$ to $\Omega_{a',b'}$ that extend homeomorphically to boundaries and map $\partial\Omega_0$ to $\partial\Omega'_0$ and $\partial\Omega_a$ to $\partial\Omega'_{a'}$. Going back to quadrilaterals, $\mathcal{G}_{\phi,\psi}$ is the set of quasiconformal maps between the quadrilaterals $\left(\overline{\Omega_{a,b}}, \partial\Omega_0, \partial\Omega_a\right)$ and $\left(\overline{\Omega_{a',b'}}, \partial\Omega'_0, \partial\Omega_{a'}\right)$.*

Recall from Lemma 2.2.2 that every minimizer of the mean distortion in $\mathcal{G}$ for the density ρ_0 is of the form

$$f_\varphi(x+iy) = \frac{a'}{a}x + i\varphi(y)$$

for a function $\varphi : [0,b] \longrightarrow [0,b']$ with $\varphi(0) = 0$, $\varphi(b) = b'$ and $\dot{\varphi} \geq \frac{a'}{a}$. Now, since the modulus of a curve family is a conformal invariant, every minimizer of the mean distortion in $\mathcal{G}_{\phi,\psi}$ for the density ρ_ϕ is one of the

$$g_\varphi = \psi \circ f_\varphi \circ \phi^{-1}.$$

In order to state the results of the section, we need to introduce more notations.

Notation 2.3.2. — *$\widetilde{\Omega}_{a,b} := \overline{\overset{\circ}{\Psi(S^1 \times \Omega_{a,b})}}$ and $\widetilde{\Omega}_{a',b'} := \overline{\overset{\circ}{\Psi(S^1 \times \Omega_{a',b'})}}$. These are the right domains to considers since, as in cylinders, we don't want potential pieces of the vertical axis to be on the topological boundaries.*

— *$\widetilde{\Gamma}_\phi$ (resp. $\widetilde{\Gamma}_\psi$) is the lifted family of Γ_ϕ (resp. Γ_ψ).*

— *Extending Π to a function from $\mathcal{H}$ to $\mathbb{H} \cup \{\Im(w) = 0\}$, we write for every $s \in [0,a]$, $\partial\widetilde{\Omega}_s = \Pi^{-1}(\partial\Omega_s)$ and for every $s \in [0,a']$, $\partial\widetilde{\Omega}'_s = \Pi^{-1}(\partial\Omega'_s)$.*

— *The set of quasiconformal map from $\widetilde{\Omega}_{a,b}$ to $\widetilde{\Omega}_{a',b'}$ that extend homeomorphically to boundaries and map $\partial\widetilde{\Omega}_0$ to $\partial\widetilde{\Omega}'_0$ and $\partial\widetilde{\Omega}_a$ to $\partial\widetilde{\Omega}'_{a'}$ is denoted $\mathcal{F}_{\phi,\psi}$. It is then clear that $\mathcal{F}_{id,id}$ is the set $\mathcal{F}$ from section 2.*

Notice that it could happen that $\partial\widetilde{\Omega}_s$ lies inside $\widetilde{\Omega}_{a,b}$ for $s \in \{0, a\}$ and not in the topological boundary of $\widetilde{\Omega}_{a,b}$.

Definition 2.3.3 (Pull-back density condition)**.** We say that $\widetilde{\Gamma}_\phi$ (resp. $\widetilde{\Gamma}_\psi$) satisfies the pull-back density condition if $\Pi^*\rho_\phi$ (resp. $\Pi^*\rho_\psi$) is extremal for $\widetilde{\Gamma}_\phi$ (resp. $\widetilde{\Gamma}_\psi$).

In the whole section, we will assume that $\widetilde{\Gamma}_\phi$ and $\widetilde{\Gamma}_\phi$ satisfy the pull-back density condition. This is an important condition for both existence and uniqueness results of this section. We will need to get a better understanding of it, but to begin with, let us give an exemple of a holomorphic map ϕ for which $\Pi^*\rho_\phi$ isn't extremal for the family $\widetilde{\Gamma}_\phi$.

Example 2.3.4. *The example is quite simple. We know that in a rectangle $R_{a,b}$, horizontal lines satisfy the above property. But vertical ones don't. Indeed, let Δ be the family of curves*

$$\delta_x(s) = x + is, \text{ for } 0 < x < a \text{ and } 0 < s < b.$$

It is well known that the modulus of the family Δ is $\frac{a}{b}$ with extremal density $\sigma = \frac{1}{b}$. Now, let $\widetilde{\Delta}$ be the lifted family of Δ. One may verify that $\widetilde{\Delta}$ is the family of curves

$$\widetilde{\delta}_{(z,t)}(s) = (sz, t), \text{ for } |z| = 1,\ 0 < t < a \text{ and } 0 < s < \sqrt{b}.$$

With a calculus quite similar to the one done in the proof of Lemma 2.2.1, we find the modulus of $\widetilde{\Delta}$ to be

$$M(\widetilde{\Delta}) = \frac{16\pi a}{27b}$$

with extremal density

$$\widetilde{\sigma}(z,t) = \frac{2}{3b^{\frac{1}{3}}|z|^{\frac{1}{3}}}$$

which is not the pull-back by Π of σ. Now, we may send the rectangle $R_{b,a}$ to the rectangle $R_{a,b}$ by the composition of a rotation of angle $\frac{\pi}{2}$ and a translation by a which is a holomorphic map sending horizontal lines to vertical ones.

The following gives an interpretation of the pull-back density condition in our setting (h' refers to the complex derivative of any holomorphic function h).

Proposition 2.3.5. *If* $\frac{|\phi'|}{\Im(\phi)}$ *is a function of the imaginary part only, then*

$$M(\widetilde{\Gamma}_\phi) = \int_{\widetilde{\Omega}_{a,b}} (\Pi^*\rho_\phi)^4 \mathrm{d}L^3.$$

Conversely, if $M(\widetilde{\Gamma}_\phi) = \int_{\widetilde{\Omega}_{a,b}} (\Pi^*\rho_\phi)^4 \mathrm{d}L^3$*, then* $\frac{|\phi'|}{\Im(\phi)}$ *is a function of the imaginary part only.*

Proof. Let $\gamma_{x,\alpha} \in \widetilde{\Gamma}_\phi$. By definition,

$$\gamma_{x,\alpha}(s) = \left(\sqrt{\Im(\phi(s,x))}e^{i(\alpha+\tau(s,x))}, \Re(\phi(s,x))\right)$$

with $\tau(s,x) = -\int_0^s \frac{\Re(\phi_t(t,x))}{2\Im(\phi(t,x))}\mathrm{d}t$ for every s, x (here, the index t refers to the partial derivative with respect to the first variable). Then, computing, we have for every s, x

$$|\dot{\gamma}_{x,\alpha,1}(s)| = \frac{|\phi'(s,x)|}{2\sqrt{\Im(\phi(s,x))}}$$

where $\gamma_{x,\alpha,1}$ is the first coordinate of the curve $\gamma_{x,\alpha}$. Thus, if $\rho \in \mathrm{adm}(\widetilde{\Gamma}_\phi)$, then, for every x we have

$$\int_0^a \rho(\gamma_{x,\alpha}(s)) \frac{|\phi'(s,x)|}{2\sqrt{\Im(\phi(s,x))}}\mathrm{d}s \geq 1.$$

But, by substitution, we have the following

$$\int_{\widetilde{\Omega}_{a,b}} \rho^4 \mathrm{d}L^3 = \frac{1}{2}\int_0^{2\pi}\int_0^b\int_0^a \rho^4(\gamma_{x,\alpha}(s))|\phi'(s,x)|^2 \mathrm{d}s\mathrm{d}x\mathrm{d}\alpha. \tag{2.8}$$

Moreover, using Hölder inequality, we have for every x, α,

$$\left(\int_0^a \rho(\gamma_{x,\alpha}(s)) \frac{|\phi'(s,x)|}{2\sqrt{\Im(\phi(s,x))}}\mathrm{d}s\right)^4 \leq \left(\int_0^a \rho^4(\gamma_{x,\alpha}(s))|\phi'(s,x)|^2\mathrm{d}s\right) \left(\int_0^a \left(\frac{|\phi'(s,x)|^{\frac{1}{2}}}{2\sqrt{\Im(\phi(s,x))}}\right)^{\frac{4}{3}} \mathrm{d}s\right)^3.$$

So that

$$1 \leq \int_0^a \rho^4(\gamma_{x,\alpha}(s))|\phi'(s,x)|^2\mathrm{d}s \left(\int_0^a \left(\frac{|\phi'(s,x)|^{\frac{1}{2}}}{2\sqrt{\Im(\phi(s,x))}}\right)^{\frac{4}{3}} \mathrm{d}s\right)^3. \tag{2.9}$$

Now, assume first that $\frac{|\phi'|}{\Im(\phi)}$ is a function of x only, then by assumption, there is a real valued function h such that

$$h(x) = \frac{(\Im(\phi(s,x)))^2}{|\phi'(s,x)|^2} \text{ for every } s, x.$$

Thus, using 2.9, we have for every x,

$$\int_0^a \rho^4(\gamma_{x,\alpha}(s))|\phi'(s,x)|^2 \mathrm{d}s \geq \frac{16h(x)}{a^3}.$$

Using 2.8, it leads to,

$$M(\widetilde{\Gamma}_\phi) \geq \frac{16\pi}{a^3} \int_0^b h(x)\mathrm{d}x.$$

Now, by definition,

$$\Pi^* \rho_\phi = \frac{|Z\Pi|}{a|\phi'(\phi^{-1} \circ \Pi)|}.$$

So, again by substitution,

$$\int_{\widetilde{\Omega}_{a,b}} (\Pi^* \rho_\phi)^4 \, \mathrm{d}L^3 = \frac{1}{2} \int_0^{2\pi} \int_0^b \int_0^a \frac{16\,(\Im(\phi(s,x)))^2}{a^4|\phi'(s,x)|^2} \mathrm{d}s\mathrm{d}x\mathrm{d}\alpha = \frac{16\pi}{a^3} \int_0^b h(x)\mathrm{d}x.$$

Thus,

$$M(\widetilde{\Gamma}_\phi) = \int_{\widetilde{\Omega}_{a,b}} (\Pi^* \rho_\phi)^4 \mathrm{d}L^3.$$

Conversely, assume that $M(\widetilde{\Gamma}_\phi) = \int_{\widetilde{\Omega}_{a,b}} (\Pi^* \rho_\phi)^4 \mathrm{d}L^3$. Then, using 2.8 and 2.9, we find for every $\rho \in \mathrm{adm}(\widetilde{\Gamma}_\phi)$,

$$\int_{\widetilde{\Omega}_{a,b}} \rho^4 \mathrm{d}L^3 \geq 16\pi \int_0^b \frac{\mathrm{d}x}{\left(\int_0^a \frac{|\phi'(s,x)|^{\frac{2}{3}}}{(\Im(\phi(s,x)))^{\frac{2}{3}}} \mathrm{d}s\right)^3}.$$

Now, let $\rho \in \mathrm{adm}(\widetilde{\Gamma}_\phi)$ be defined for every x, s, α by

$$\rho(\gamma_{x,\alpha}(s)) := \left(\int_0^a \frac{|\phi'(s,x)|^{\frac{2}{3}}}{2(\Im(\phi(s,x)))^{\frac{2}{3}}} \mathrm{d}s\right)^{-1} \frac{|\phi'(s,x)|^{-\frac{1}{3}}}{(\Im(\phi(s,x)))^{\frac{1}{6}}}$$

Then,

$$\int_{\widetilde{\Omega}_{a,b}} \rho^4 \mathrm{d}L^3 = 16\pi \int_0^b \frac{\mathrm{d}x}{\left(\int_0^a \frac{|\phi'(s,x)|^{\frac{2}{3}}}{(\Im(\phi(s,x)))^{\frac{2}{3}}} \mathrm{d}s\right)^3}.$$

Since, $\Pi^*\rho_\phi$ is extremal, we have $\Pi^*\rho_\phi = \rho$. This leads to

$$\frac{|\phi'(s,x)|^{\frac{2}{3}}}{(\Im(\phi(s,x)))^{\frac{2}{3}}} = \frac{1}{a}\int_0^a \frac{|\phi'(s,x)|^{\frac{2}{3}}}{(\Im(\phi(s,x)))^{\frac{2}{3}}}\mathrm{d}s$$

for every s, x. So, $\frac{|\phi'|}{\Im(\phi)}$ does not depend on s. □

To end (for now, we will come back to it later at the end of chapter 3 when CR quadratic differentials will be defined) the discussion on the pull-back density condition, one can show that, by the previous proposition, it is equivalent to the condition given on the foliation in Theorem 2.1.8.

In the following, we will answer (in this setting) the questions asked in the introduction. First, we will give conditions on the function φ so that g_φ lifts to a quasiconformal map between $\widetilde{\Omega}_{a,b}\backslash\{z=0\}$ and $\widetilde{\Omega}_{a',b'}\backslash\{z=0\}$. To be more precise, we will prove that g_φ can be lifted to a quasiconformal map between $\widetilde{\Omega}_{a,b}\backslash\{z=0\}$ and $\widetilde{\Omega}_{a',b'}\backslash\{z=0\}$ if and only if φ satisfies an ordinary differential equation. Moreover, we will see that when such a φ can be found and if lifts of g_φ can be extend to maps in $\mathcal{F}_{\phi,\psi}$ then those lifts minimize the mean distortion in $\mathcal{F}_{\phi,\psi}$ for the density $\Pi^*\rho_\phi$. After that, we will explain that if we can find a minimizer of the mean distortion functional by that process, then we actually found all of them. We will finish the section and the chapter by giving two non-trivial examples.

2.3.2 Conditions for existence of a lift

As stated, we wish here to find conditions on φ so that g_φ can be lifted by Π into a quasiconformal map between $\widetilde{\Omega}_{a,b}\backslash\{z=0\}$ and $\widetilde{\Omega}_{a',b'}\backslash\{z=0\}$. To do so, we make a change of coordinates in $\mathcal{H}$ more adapted to the problem. New coordinates are then given by the following two maps:

$$\begin{array}{rccc} \Psi_\phi: &]0,a[\times]0,b[\times\mathbb{R} & \longrightarrow & \widetilde{\Omega}_{a,b}\backslash\{z=0\} \\ & (s,x,\theta) & \longrightarrow & \left(\sqrt{\Im(\phi(s,x))}e^{i\theta}, \Re(\phi(s,x))\right) \end{array}$$

$$\begin{array}{rccc} \Psi_\psi: &]0,a'[\times]0,b'[\times\mathbb{R} & \longrightarrow & \widetilde{\Omega}_{a',b'}\backslash\{z=0\} \\ & (s,x,\theta) & \longrightarrow & \left(\sqrt{\Im(\psi(s,x))}e^{i\theta}, \Re(\psi(s,x))\right) \end{array}$$

So that

$$\Pi\circ\Psi_\phi(s,x,\theta) = \phi(s,x) \text{ and } \Pi\circ\Psi_\psi(s,x,\theta) = \psi(s,x)$$

where it makes sense. Moreover, a map $(S, X, \Theta) :]0, a[\times]0, b[\times\mathbb{R} \longrightarrow]0, a'[\times]0, b'[\times\mathbb{R}$ induces a well-defined map $(g_1, g_2) : \widetilde{\Omega}_{a,b}\backslash\{z = 0\} \longrightarrow \widetilde{\Omega}_{a',b'}\backslash\{z = 0\}$ if and only if S, X are 2π-periodic (in the third variable) and Θ is 2π-periodic modulo 2π (also in the third variable). Then, one may verify that a map (S, X, Θ) defines a contact map from $\widetilde{\Omega}_{a,b}\backslash\{z = 0\}$ to $\widetilde{\Omega}_{a',b'}\backslash\{z = 0\}$ if and only if there is a nowhere vanishing function λ such that,

$$\Re(\psi_s(S, X))\mathrm{d}S - \Im(\psi_s(S, X))\mathrm{d}X + 2\Im(\psi(S, X))\mathrm{d}\Theta = \lambda\left(\Re(\phi_s)\mathrm{d}s - \Im(\phi_s)\mathrm{d}x + 2\Im(\phi)\mathrm{d}\theta\right). \quad (2.10)$$

Here again, we denote the partial derivative with respect to s (resp. x , resp. θ) by an index s (resp. x, resp. θ). Finally, a map (S, X, Θ) defines a lift of a g_φ if the diagram

$$\begin{tikzcd}
]0, a[\times]0, b[\times\mathbb{R} \arrow[r, "(S,X,\Theta)"] \arrow[d, "\Pi\circ\Psi_\phi"'] &]0, a'[\times]0, b'[\times\mathbb{R} \arrow[d, "\Pi\circ\Psi_\psi"] \\
\Omega_{a,b} \arrow[r, "g_\varphi"'] & \Omega_{a',b'}
\end{tikzcd}$$

is commutative.

Lemma 2.3.6. *Let $(S, X, \Theta) :]0, a[\times]0, b[\times\mathbb{R} \longrightarrow]0, a'[\times]0, b'[\times\mathbb{R}$. The previous diagram is commutative if and only if, we have*

$$S(s, x, \theta) = S(s) = \frac{a'}{a}s \text{ and } X(s, x, \theta) = X(x) = \varphi(x)$$

for every $(s, x, \theta) \in]0, a[\times]0, b[\times\mathbb{R}$. In other words, $(s, x) \longmapsto (S(s), X(x))$ agrees with f_φ.

Proof. The diagram is commutative if and only if,

$$(g_\varphi \circ \Pi \circ \Psi_\phi)(s, x, \theta) = (\Pi \circ \Psi_\psi)(S(s, x, \theta), X(s, x, \theta), \Theta(s, x, \theta))$$

for every $(s, x, \theta) \in]0, a[\times]0, b[\times\mathbb{R}$. Leading to,

$$(S, X)(s, x, \theta) = (\psi^{-1} \circ g_\varphi \circ \phi)(s, x) = f_\varphi(s, x) = \left(\frac{a'}{a}s, \varphi(x)\right)$$

for every $(s, x, \theta) \in]0, a[\times]0, b[\times\mathbb{R}$. □

To sum up, a map $(S, X, \Theta) :]0, a[\times]0, b[\times\mathbb{R} \longrightarrow]0, a'[\times]0, b'[\times\mathbb{R}$ induces a contact lift of g_φ if and only if, $(s, x) \longmapsto (S(s), X(x))$ agrees with f_φ, Θ is 2π-periodic

modulo 2π in the third variable, and the functions S, X and Θ satisfy the contact condition 2.10. Now, we are able to state conditions for the existence of a contact lift, then quasiconformality of the lift comes from Lemma 2.1.15.

Proposition 2.3.7. *Suppose that (S, X, Θ) is a contact transform from $]0, a[\times]0, b[\times\mathbb{R}$ to $]0, a'[\times]0, b'[\times\mathbb{R}$ such that*

$$g_\varphi \circ \Pi \circ \Psi_\phi = \Pi \circ \Psi_\psi \circ (S, X, \Theta).$$

Then, first, for every $(s, x, \theta) \in]0, a[\times]0, b[\times\mathbb{R}$,

$$S(s, x, \theta) = \frac{a'}{a} s \text{ and } X(s, x, \theta) = \varphi(x).$$

Moreover, φ, ψ and ϕ satisfy for every $(s, x, \theta) \in]0, a[\times]0, b[\times\mathbb{R}$

$$\frac{a'}{a}\dot{\varphi}(x)\frac{\left|\psi'\left(\frac{a'}{a}s, \varphi(x)\right)\right|^2 \left(\Im(\phi(s, x))\right)^2}{|\phi'(s, x)|^2 \left(\Im\left(\psi\left(\frac{a'}{a}s, \varphi(x)\right)\right)\right)^2} = 1. \tag{2.11}$$

Conversely, if those conditions are satisfied, then $(S, X, \Theta) : (s, x, \theta) \longmapsto (\frac{a'}{a}s, \varphi(x), \theta + h(s, x))$ with

$$2h_x(s, x) = \frac{\Im\left(\psi_s\left(\frac{a'}{a}s, \varphi(x)\right)\right)}{\Im\left(\psi\left(\frac{a'}{a}s, \varphi(x)\right)\right)}\dot{\varphi} - \frac{\Im(\phi_s(s, x))}{\Im(\phi(s, x))} \tag{2.12}$$

$$2h_s(s, x) = \frac{\Re(\phi_s(s, x))}{\Im(\phi(s, x))} - \frac{a'}{a}\frac{\Re\left(\psi_s\left(\frac{a'}{a}s, \varphi(x)\right)\right)}{\Im\left(\psi\left(\frac{a'}{a}s, \varphi(x)\right)\right)} \tag{2.13}$$

is a contact transform satisfying $g_\varphi \circ \Pi \circ \Psi_\phi = \Pi \circ \Psi_\psi \circ (S, X, \Theta)$.

Proof. The proof is similar to the one of Proposition 2.2.3, we take information from the map (S, X, Θ) to be contact in order to find the three partial derivatives of Θ. According to the previous lemma, we know that $S(s, x, \theta) = \frac{a'}{a}s$ and $X(s, x, \theta) = \varphi(x)$ for every $(s, x, \theta) \in]0, a[\times]0, b[\times\mathbb{R}$. Now, the contact condition gives the following PDEs :

$$\frac{a'}{a}\Re(\psi_s(S, X)) + 2\Im(\psi(S, X))\Theta_s = \lambda\Re(\phi_s) \tag{2.14}$$

$$\Im(\psi_s(S, X))\dot{\varphi} - 2\Im(\psi(S, X))\Theta_x = \lambda\Im(\phi_s) \tag{2.15}$$

$$\Im(\psi(S,X))\Theta_\theta = \lambda\Im(\phi) \tag{2.16}$$

for a nowhere vanishing function λ. From 2.16, since $\Im(\phi) > 0$, we find

$$\lambda = \frac{\Im(\psi(S,X))}{\Im(\phi)}\Theta_\theta.$$

Replacing in 2.14 and 2.15 and dividing by $\Im(\psi(S,X)) > 0$, we deduce

$$\frac{a'}{a}\frac{\Re(\psi_s(S,X))}{\Im(\psi(S,X))} + 2\Theta_s = \frac{\Re(\phi_s)}{\Im(\phi)}\Theta_\theta \tag{2.17}$$

$$\frac{\Im(\psi_s(S,X))}{\Im(\psi(S,X))}\dot{\varphi} - 2\Theta_x = \frac{\Im(\phi_s)}{\Im(\phi)}\Theta_\theta. \tag{2.18}$$

Differentiating 2.17 with respect to x, 2.18 with respect to s and using Cauchy-Riemann equations for ψ and ϕ, we find

$$-\frac{a'}{a}\dot{\varphi}\left(\frac{\Im(\psi_{s,s}(S,X))}{\Im(\psi(S,X))} + \left(\frac{\Re(\psi_s(S,X))}{\Im(\psi(S,X))}\right)^2\right) + 2\Theta_{x,s} = \frac{\Re(\phi_s)}{\Im(\phi)}\Theta_{x,\theta} - \left(\frac{\Im(\phi_{s,s})}{\Im(\phi)} + \left(\frac{\Re(\phi_s)}{\Im(\phi)}\right)^2\right)\Theta_\theta \tag{2.19}$$

$$\frac{a'}{a}\dot{\varphi}\left(\frac{\Im(\psi_{s,s}(S,X))}{\Im(\psi(S,X))} - \left(\frac{\Im(\psi_s(S,X))}{\Im(\psi(S,X))}\right)^2\right) - 2\Theta_{s,x} = \frac{\Im(\phi_s)}{\Im(\phi)}\Theta_{s,\theta} + \left(\frac{\Im(\phi_{s,s})}{\Im(\phi)} - \left(\frac{\Im(\phi_s)}{\Im(\phi)}\right)^2\right)\Theta_\theta \tag{2.20}$$

Thus, replacing the value of $2\Theta_{s,x}$ from 2.19 in 2.20, and using $|h'|^2 = (\Re(h_s))^2 + (\Im(h_s))^2$ for any holomorphic function, we find

$$-\frac{a'}{a}\dot{\varphi}\frac{|\psi'(S,X)|^2}{(\Im(\psi(S,X)))^2} - \frac{\Re(\phi_s)}{\Im(\phi)}\Theta_{x,\theta} + \frac{|\phi'|^2}{(\Im(\phi))^2}\Theta_\theta = \frac{\Im(\phi_s)}{\Im(\phi)}\Theta_{s,\theta} \tag{2.21}$$

Now, differentiating 2.17 and 2.18 both with respect to θ, we also have

$$\Theta_{s,\theta} = \frac{\Re(\phi_s)}{2\Im(\phi)}\Theta_{\theta,\theta} \text{ and } \Theta_{x,\theta} = -\frac{\Im(\phi_s)}{2\Im(\phi)}\Theta_{\theta,\theta}.$$

So, replacing those in 2.21, we finally have

$$\Theta_\theta = \frac{a'}{a}\dot{\varphi}(x)\frac{\left|\psi'\left(\frac{a'}{a}s,\varphi(x)\right)\right|^2\left(\Im(\phi(s,x))\right)^2}{|\phi'(s,x)|^2\left(\Im\left(\psi\left(\frac{a'}{a}s,\varphi(x)\right)\right)\right)^2}.$$

The term on the right side does not depend on θ. So

$$0 = \Theta_{\theta,\theta} = \Theta_{s,\theta} = \Theta_{x,\theta}.$$

Thus, Θ_θ is constant. So, for Θ to follow the periodicity condition, Θ_θ must be everywhere equal to 1. Which ends the first part of the proof.

For the second part, it is a simple verification that a map (S, X, Θ) defined by $(s, x, \theta) \longmapsto (\frac{a'}{a}s, \varphi(x), \theta + h(s, x))$ with

$$2h_x(s,x) = \frac{\Im\left(\psi_s\left(\frac{a'}{a}s,\varphi(x)\right)\right)}{\Im\left(\psi\left(\frac{a'}{a}s,\varphi(x)\right)\right)}\dot{\varphi}(x) - \frac{\Im(\phi_s(s,x))}{\Im(\phi(s,x))},$$

$$2h_s(s,x) = \frac{\Re(\phi_s(s,x))}{\Im(\phi(s,x))} - \frac{a'}{a}\frac{\Re\left(\psi_s\left(\frac{a'}{a}s,\varphi(x)\right)\right)}{\Im\left(\psi\left(\frac{a'}{a}s,\varphi(x)\right)\right)}$$

and φ satisfying 2.11 is a contact transform satisfying $g_\varphi \circ \Pi \circ \Psi_\phi = \Pi \circ \Psi_\psi \circ (S, X, \Theta)$. Moreover, such a function h exists: using the fact that φ satisfies 2.11, one may verify that

$$\frac{\partial}{\partial s}\left(\frac{\Im\left(\psi_s\left(\frac{a'}{a}s,\varphi(x)\right)\right)}{\Im\left(\psi\left(\frac{a'}{a}s,\varphi(x)\right)\right)}\dot{\varphi}(x) - \frac{\Im(\phi_s(s,x))}{\Im(\phi(s,x))}\right) = \frac{\partial}{\partial x}\left(\frac{\Re(\phi_s(s,x))}{\Im(\phi(s,x))} - \frac{a'}{a}\frac{\Re\left(\psi_s\left(\frac{a'}{a}s,\varphi(x)\right)\right)}{\Im\left(\psi\left(\frac{a'}{a}s,\varphi(x)\right)\right)}\right)$$

and so, the system of equations for h has a solution by Poincaré lemma. □

Finally, let us take a map $(s, x, \theta) \longmapsto (\frac{a'}{a}s, \varphi(x), \theta + h(s, x))$ with φ satisfying 2.11, $\varphi(0) = 0$, $\varphi(b) = b'$, $\dot{\varphi} \geq \frac{a'}{a}$ and h satisfying 2.12 and 2.13. Then, it induces a quasiconformal lift of g_φ, $g : \widetilde{\Omega}_{a,b}\backslash\{z = 0\} \longrightarrow \widetilde{\Omega}_{a',b'}\backslash\{z = 0\}$. Assume moreover that g can be extended continuously to a map in $\mathcal{F}_{\phi,\psi}$. Then, Corollary 2.1.14 ensures that

$$M(g(\widetilde{\Gamma}_\phi)) = \int_{\widetilde{\Omega}_{a,b}} K(.,g)^2(\Pi^*\rho_\phi)^4 \mathrm{d}L^3.$$

Now, let $\widetilde{\Gamma}$ be the family of all legendrian curves in $\widetilde{\Omega}_{a,b}$ connecting $\partial\widetilde{\Omega}_0$ and $\partial\widetilde{\Omega}_a$.

Then, $\Pi^*\rho_\phi \in \mathrm{adm}(\widetilde{\Gamma})$ (see Lemma 2.3.14). Moreover, by definition of $\mathcal{F}_{\phi,\psi}$, for every $f \in \mathcal{F}_{\phi,\psi}$, $g(\widetilde{\Gamma}_\phi) \subset f(\widetilde{\Gamma})$ so that

$$M(g(\widetilde{\Gamma}_\phi)) \leq M(f(\widetilde{\Gamma}))$$

for every $f \in \mathcal{F}_{\phi,\psi}$. Thus, using Proposition 2.1.7, g minimizes the mean distortion in $\mathcal{F}_{\phi,\psi}$ for the density $\Pi^*\rho_\phi$. To sum up, we proved:

Theorem 2.3.8. *Let $g_\varphi : \Omega_{a,b} \longrightarrow \Omega_{a',b'}$ be a minimizer of the mean distortion in $\mathcal{G}_{\phi,\psi}$ for the density ρ_ϕ with φ satisfying 2.11. Then, there is a quasiconformal lift of g_φ, $g : \widetilde{\Omega}_{a,b}\backslash\{z=0\} \longrightarrow \widetilde{\Omega}_{a',b'}\backslash\{z=0\}$. If g can be extended continuously to a map in $\mathcal{F}_{\phi,\psi}$, then g minimizes the mean distortion in $\mathcal{F}_{\phi,\psi}$ for the density $\Pi^*\rho_\phi$.*

To end this section, we give a general theorem for lifting quasiconformal mappings from $\mathbb{H}$. It produces a new way to lift quasiconformal mappings from the plane to the Heisenberg group different from [9, 16]. The formulation of the theorem (to be precise the link between the formula $r^4\frac{\mathrm{Jac}\ g}{(\Im(g))^2} = 1$ and the preservation of the hyperbolic area form) is heavily inspired by a work in progress by Cano and Platis (see [13]). In their work, the authors create a setting in which the lifting theorem fits. The proof they give is therefore more conceptual than the one we give here (in the appendix A) which relies on the same kind of computations than the proof of Proposition 2.3.7.

Proposition 2.3.9. *Let $g : \Omega \longrightarrow \Omega'$ be a C^2 homeomorphism between simply connected domains of $\mathbb{H}$. Let $\widetilde{\Omega} = \Pi^{-1}(\Omega)$ and $\widetilde{\Omega}' = \Pi^{-1}(\Omega')$.*

— *If g is a symplectomorphism with respect to the hyperbolic area form of $\mathbb{H}$ (i.e. for every $w \in \Omega$, $\frac{\mathrm{Jac}\ g(w)}{(\Im(g(w)))^2} = \frac{1}{(\Im(w))^2}$ where $\mathrm{Jac}\ g$ is the Jacobian determinant of g), then there is a contact map $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ such that $\Pi \circ f = g \circ \Pi$. Moreover, f is of the form:*

$$(z,t) \longmapsto \Big(z(\mathrm{Jac}\ g(t+i|z|^2))^{\frac{1}{4}} e^{ih(t+i|z|^2)}, \Re(g(t+i|z|^2))\Big)$$

where $h : \Omega \longrightarrow \mathbb{R}$ is given by the following equations, for every $w \in \Omega$,

$$\frac{\partial h}{\partial w}(w) = \frac{1}{4\Im(w)} - \frac{1}{2\Im(g(w))}\frac{\partial \Re(g)}{\partial w}(w)$$

$$\frac{\partial h}{\partial \overline{w}}(w) = \frac{1}{4\Im(w)} - \frac{1}{2\Im(g(w))}\frac{\partial \Re(g)}{\partial \overline{w}}(w)$$

.

— *Conversely, if g can be lifted by Π into a contact map between $\widetilde{\Omega}$ and $\widetilde{\Omega}'$, then g is a symplectomorphism with respect to the hyperbolic area form of $\mathbb{H}$ and f has the form given previously.*

— *If $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ is contact map such that $\Pi \circ f = g \circ \Pi$, then f is quasiconformal if and only if g is quasiconformal. In this case, the distortion of f at a point $(z,t) \in \widetilde{\Omega}$ is equal to the distortion of g at the point $t + i|z|^2 \in \Omega$.*

Remark 2.3.10. Proposition 2.3.7 is a version of Proposition 2.3.9 in the case

$$g = g_\varphi = \psi \circ f_\varphi \circ \phi^{-1}.$$

In particular, using holomorphicity of ϕ and ψ, and Jac $f_\varphi = \frac{a'}{a}\dot{\varphi}$, condition 2.11 is equivalent to

$$\frac{\text{Jac } (g_\varphi)(\phi(s,x))}{(\Im(g_\varphi(\phi(s,x))))^2} = \frac{a'}{a}\dot{\varphi}(x)\frac{|\psi'(f_\varphi(s,x))|^2}{|\phi'(s,x)|^2(\Im(\psi(f_\varphi(s,x))))^2} = \frac{1}{(\Im(\phi(s,x)))^2}.$$

So that condition 2.11 is equivalent to g_φ being symplectic with respect to the hyperbolic area form.

2.3.3 Uniqueness of the construction

In this section, we will prove that when there is a minimizer of the mean distortion in $\mathcal{G}_{\phi,\psi}$ for the density ρ_ϕ that can be lifted into a quasiconformal map in $\mathcal{F}_{\phi,\psi}$, then every minimizer of the mean distortion in $\mathcal{F}_{\phi,\psi}$ for the density $\Pi^*\rho_\phi$ is constructed by that process. The section is dedicated to the proof of the following theorem, which may be understood as a converse of Proposition 2.1.13 in the case of domains Ω and Ω' being (interior of) quadrilaterals.

Theorem 2.3.11. *Let $f : \widetilde{\Omega}_{a,b} \longrightarrow \widetilde{\Omega}_{a',b'}$ be a quasiconformal map in $\mathcal{F}_{\phi,\psi}$ such that $f_*\Pi^*\rho_\phi = \Pi^*\rho_\psi$. Suppose that $\widetilde{\Gamma}_\phi$ and $\widetilde{\Gamma}_\psi$ satisfy the pull-back density condition. Then, f is a lift of a minimizer of the mean distortion in $\mathcal{G}_{\phi,\psi}$ for the density ρ_ϕ. Namely, there is a quasiconformal map $g \in \mathcal{G}_{\phi,\psi}$ such that $\Pi \circ f = g \circ \Pi$ and $g_*\rho_\phi = \rho_\psi$.*

Remark 2.3.12. It could be tempting to phrase the previous theorem using modulus of a curve family as: if $\widetilde{\Gamma}_\phi$ and $\widetilde{\Gamma}_\psi$ satisfy the pull-back density condition, then every

quasiconformal map $f \in \mathcal{F}_{\phi,\psi}$ such that

$$M(f(\widetilde{\Gamma}_\phi)) = \int_{\widetilde{\Omega}_{a,b}} K(.,f)^2 (\Pi^* \rho_\phi)^4 \mathrm{d}L^3$$

is a lift of a g_φ.
But this is not what Theorem 2.3.11 states. In fact, it concerns maps $f \in \mathcal{F}_{\phi,\psi}$ satisfying

$$M(\widetilde{\Gamma}_\psi) = \int_{\widetilde{\Omega}_{a,b}} K(.,f)^2 (\Pi^* \rho_\phi)^4 \mathrm{d}L^3.$$

Nevertheless, we can still phrase Theorem 2.3.11 using moduli of curve families. Only, not with $f(\widetilde{\Gamma}_\phi)$.

Let $\widetilde{\Gamma}$ be the family of all legendrian curves in $\widetilde{\Omega}_{a,b}$ connecting $\partial\widetilde{\Omega}_0$ and $\partial\widetilde{\Omega}_a$. Then, $\Pi^*\rho_\phi \in \mathrm{adm}(\widetilde{\Gamma})$ (see Lemma 2.3.14) and so it is extremal for $\widetilde{\Gamma}$. In the same way, $\Pi^*\rho_\psi \in \mathrm{adm}(\widetilde{\Gamma}')$ is also extremal where $\widetilde{\Gamma}'$ is the family of all legendrian curves in $\widetilde{\Omega}_{a',b'}$ connecting $\partial\widetilde{\Omega}'_0$ and $\partial\widetilde{\Omega}'_{a'}$. Now, by definition of $\mathcal{F}_{\phi,\psi}$, for every $f \in \mathcal{F}_{\phi,\psi}$, $f(\widetilde{\Gamma}) = \widetilde{\Gamma}'$. So,

$$f_*\Pi^*\rho_\phi = \Pi^*\rho_\psi \iff M(f(\widetilde{\Gamma})) = \int_{\widetilde{\Omega}_{a,b}} K(.,f)^2 (\Pi^* \rho_\phi)^4 \mathrm{d}L^3.$$

Thus, Theorem 2.3.11 is equivalent to the following theorem.

Theorem 2.3.13. *Let $\widetilde{\Gamma}$ be the family of all legendrian curves in $\widetilde{\Omega}_{a,b}$ connecting $\partial\widetilde{\Omega}_0$ and $\partial\widetilde{\Omega}_a$ and $\widetilde{\Gamma}'$ be the family of all legendrian curves in $\widetilde{\Omega}_{a',b'}$ connecting $\partial\widetilde{\Omega}'_0$ and $\partial\widetilde{\Omega}'_{a'}$. Assume that $\widetilde{\Gamma}$ and $\widetilde{\Gamma}'$ satisfy the pull-back density condition. Let $f \in \mathcal{F}_{\phi,\psi}$ be a quasiconformal map satisfying*

$$M(f(\widetilde{\Gamma})) = \int_{\widetilde{\Omega}_{a,b}} K(.,f)^2 (\Pi^* \rho_\phi)^4 \mathrm{d}L^3.$$

Then, f is a lift of one of the g_φ.

The proof of Theorem 2.3.11 follows the same steps as the one of Theorem 2.2.6. We first prove that a quasiconformal map $f = (f_1, f_2)$ as in the previous theorem *dilates curves from $\widetilde{\Gamma}_\phi$ to $\widetilde{\Gamma}_\psi$* then, f has the property: $(f_2 + i|f_1|^2)(z,t)$ does not depend on $\arg(z)$. Let us set a quasiconformal map $f : \widetilde{\Omega}_{a,b} \longrightarrow \widetilde{\Omega}_{a',b'}$ with the hypothesis of the theorem. The first objective is to show that a map $f \in \mathcal{F}_{\phi,\psi}$ such

that $f_*\Pi^*\rho_\phi = \Pi^*\rho_\psi$ must be defined by a map (S, X, Θ) that sends a curve

$$(s, x, \alpha + \tau(s, x)) \text{ with } \tau_s(s, x) = -\frac{\Re(\phi_s(s, x))}{2\Im(\phi(s, x))}$$

on a curve

$$\left(\frac{a'}{a}s, x', \alpha' + \upsilon\left(\frac{a'}{a}s, x'\right)\right) \text{ with } \upsilon_s(s, x) = -\frac{\Re(\psi_s(s, x))}{2\Im(\psi(s, x))}.$$

For that, we will follow essentially what we did in section 2.2. First, a curve $\widetilde{\gamma}(t) = (s(t), x(t), \theta(t))$ in $]0, a[\times]0, b[\times\mathbb{R}$ (resp. in $]0, a'[\times]0, b'[\times\mathbb{R}$) is said to be legendrian if $\Psi_\phi(\widetilde{\gamma}(t))$ is legendrian in $\widetilde{\Omega}_{a,b}$ (resp. $\Psi_\psi(\widetilde{\gamma}(t))$ is legendrian in $\widetilde{\Omega}_{a',b'}$).

Lemma 2.3.14. *Let $\widetilde{\Gamma}$ be the family of all legendrian curves*

$$\widetilde{\gamma}(t) = (s(t), x(t), \theta(t)) \text{ such that } s(0) = 0 \text{ and } s(a) = a'.$$

Then, for every $\widetilde{\gamma} \in \widetilde{\Gamma}$,

$$\int_{\Psi_\phi(\widetilde{\gamma})} \Pi^*\rho_\phi \mathrm{d}l \geq 1.$$

Moreover, equality is achieved if and only if $\Psi_\phi(\widetilde{\gamma}) \in \widetilde{\Gamma}_\phi$.

Proof. Let $\widetilde{\gamma}$ be a curve in $\widetilde{\Gamma}$ and $\gamma = \Psi_\phi(\widetilde{\gamma})$. Then, we have the following

$$\begin{aligned}
\int_\gamma \Pi^*\rho_\phi \mathrm{d}l &= \int_0^a \frac{2|\gamma_1(t)||\dot{\gamma}_1(t)|}{a|\phi'(\phi^{-1}(\Pi\circ\gamma)(t))|}\mathrm{d}t \\
&= \frac{1}{a}\int_0^a \frac{(\Pi \dot{\circ} \gamma)(t)}{|\phi'(\phi^{-1}(\Pi\circ\gamma)(t))|}\mathrm{d}t \\
&= \frac{1}{a}\int_0^a |\dot{s}(t) + i\dot{x}(t)|\mathrm{d}t \\
&\geq \frac{1}{a}\int_0^a |\dot{s}(t)|\mathrm{d}t \\
&= 1
\end{aligned}$$

We have equality here if and only if $\dot{x} = 0$. One may then verify that a curve $(s(t), x_0, \theta(t))$ is legendrian if and only if its image by Ψ_ϕ is an element of $\widetilde{\Gamma}_\phi$.

□

Now, we may prove the following the same way that we proved Proposition 2.2.7.

Proposition 2.3.15. *In our setting we have*

$$f(\widetilde{\Gamma}_\phi) = \widetilde{\Gamma}_\psi.$$

Moreover, f maps (potential pieces of) the vertical axis to itself.

Thus, according to this proposition, we can restrict our attention to $\widetilde{\Omega}_{a,b}\backslash\{z = 0\}$ and $\widetilde{\Omega}_{a',b'}\backslash\{z = 0\}$, meaning that f induces a well defined map $(S, X, \Theta) :]0, a[\times]0, b[\times\mathbb{R} \longrightarrow]0, a'[\times]0, b'[\times\mathbb{R}$ that sends a curve

$$\widetilde{\gamma}_{(x,\alpha)} = (s, x, \alpha + \tau(s, x)) \text{ with } \dot{\tau}(s, x) = -\frac{\Re(\phi_s(s, x))}{2\Im(\phi(s, x))}$$

on a curve

$$\widetilde{\delta}_{(x',\alpha')}(s) = (\zeta_{(x,\alpha)}(s), x', \alpha' + \upsilon(\zeta_{(x,\alpha)}(s), x')) \text{ with } \dot{\upsilon}(s, x) = -\frac{\Re(\psi_s(s, x))}{2\Im(\psi(s, x))}.$$

It remains to show that

$$\zeta_{(x,\alpha)}(s) = \frac{a'}{a}s.$$

Proposition 2.3.16. *The map (S, X, θ) sends a curve $\widetilde{\gamma}_{(x,\alpha)}(s) = (s, x, \alpha + \tau(s, x))$ with $\dot{\tau}(s, x) = -\frac{\Re(\phi_s(s,x))}{2\Im(\phi(s,x))}$ on a curve $\widetilde{\delta}_{(x',\alpha')}(s) = (\frac{a'}{a}s, x', \alpha' + \upsilon(\frac{a'}{a}s, x'))$ with $\dot{\upsilon}(s, x) = -\frac{\Re(\psi_s(s,x))}{2\Im(\psi(s,x))}$.*

We say that f dilates the curves from $\widetilde{\Gamma}_\phi$ to $\widetilde{\Gamma}_\psi$

Proof. Again, the proof is very similar to the one of Proposition 2.2.8. We consider the following two complex vector fields

$$U := -\frac{iz\phi'(\phi^{-1}(t + i|z|^2))}{2|z|^2}Z \text{ and } \overline{U} := \frac{i\overline{z}\overline{\phi}'(\phi^{-1}(t + i|z|^2))}{2|z|^2}\overline{Z}.$$

Then, using the same method as in Proposition 2.2.8, one may check that

$$|U(\Pi \circ f)| - |\overline{U}(\Pi \circ f)| = \frac{|f_1||\phi'(\phi^{-1}(t + i|z|^2))|}{|z|}\left(|Zf_1| - |\overline{Z}f_1|\right).$$

Now, since

$$f_*\widetilde{\rho}_\phi \circ f = \widetilde{\rho}_\psi \circ f \text{ with } \widetilde{\rho}_\phi = \Pi^*\rho_\phi = \Pi^*\phi_*\rho_0 \text{ and } \widetilde{\rho}_\psi = \Pi^*\rho_\psi = \Pi^*\psi_*\rho_0'$$

we have

$$f_*\widetilde{\rho}_\phi \circ f = \frac{2|f_1|}{a\left(|U(\Pi\circ f)| - |\overline{U}(\Pi\circ f)|\right)} = \frac{2|f_1|}{a'|\psi'(\psi^{-1}\circ\Pi\circ f)|}.$$

Leading to

$$|U(\Pi\circ f)| - |\overline{U}(\Pi\circ f)| = \frac{a'}{a}|\psi'(\psi^{-1}\circ\Pi\circ f)|.$$

Now, let $\widetilde{\gamma}_{(x,\alpha)}(s) = (s, x, \alpha + \tau(s,x))$ with $\dot{\tau}(s,x) = -\frac{\Re(\phi_s(s,x))}{2\Im(\phi(s,x))}$ and $\gamma_{(x,\alpha)}(s) = \Psi_\phi(\widetilde{\gamma}_{(x,\alpha)}(s))$. Then we have,

$$\Pi\circ f\circ\gamma_{(x,\alpha)}(s) = \psi(\zeta_{(x,\alpha)}(s), x').$$

Thus,

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\left((\Pi\circ f\circ\gamma_{(x,\alpha)})\right)(s)\right| = |\psi'(\zeta_{(x,\alpha)}(s), x')||\dot{\zeta}_{(x,\alpha)}(s)|.$$

But we also have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}s}\left((\Pi\circ f\circ\gamma_{(x,\alpha)})\right)(s)\right| &= |U(\Pi\circ f)(\gamma_{(x,\alpha)}(s))| - |\overline{U}(\Pi\circ f)(\gamma_{(x,\alpha)}(s))| \\ &= \frac{a'}{a}|\psi'(\psi^{-1}\circ\Pi\circ f\circ\gamma_{(x,\alpha)})(s)| \\ &= \frac{a'}{a}|\psi'(\zeta_{(x,\alpha)}(s), x')|.\end{aligned}$$

So, $\zeta_{(x,\alpha)}(s) = \frac{a'}{a}s$ which ends the proof.

□

Finally, a rather long but elementary calculus (done in the appendix A) shows the following.

Proposition 2.3.17. *The map (S, X, Θ) satisfies $X_\theta = 0$.*

Remark 2.3.18. One notices from the proof that, for the previous to be true, having a map satisfying the hypothesis of the theorem isn't necessary. In fact, the property of the map which is used is that it dilates curves from $\widetilde{\Gamma}_\phi$ to $\widetilde{\Gamma}_\psi$. We shall come back later in chapter 3 to this proposition where we will be able to express it in terms of trajectorties of CR quadratic differentials.

Combining Propositions 2.3.16 and 2.3.17 together with Corollary 2.1.14 is enough to prove Theorem 2.3.11. Before giving examples, we sum up the results obtained in section 2.3. In the following theorem, we denote by $\widetilde{\Gamma}$ the family of all legendrian curves in $\widetilde{\Omega}_{a,b}$ connecting $\partial\widetilde{\Omega}_0$ and $\partial\widetilde{\Omega}_a$.

Theorem 2.3.19. *If $\Pi^*\rho_\phi$ is extremal for $\widetilde{\Gamma}_\phi$ and $\Pi^*\rho_\psi$ is extremal for $\widetilde{\Gamma}_\psi$. Then, we have the dichotomy:*

1. *If there is a minimizer g of the mean distortion in $\mathcal{G}_{\phi,\psi}$ for the density ρ_ϕ that can be lifted into a quasiconformal map $f \in \mathcal{F}_{\phi,\psi}$. Then, f minimizes the mean distortion in $\mathcal{F}_{\phi,\psi}$ for the density $\Pi^*\rho_\phi$. Moreover, every minimizer of the mean distortion in $\mathcal{F}_{\phi,\psi}$ for the density $\Pi^*\rho_\phi$ is a lift of a minimizer of the mean distortion in $\mathcal{G}_{\phi,\psi}$ for the density ρ_ϕ.*

2. *If none of the minimizer of the mean distortion in $\mathcal{G}_{\phi,\psi}$ for the density ρ_ϕ can be lifted into a quasiconformal map in $\mathcal{F}_{\phi,\psi}$, then, for every $f \in \mathcal{F}_{\phi,\psi}$,*

$$M(f(\widetilde{\Gamma})) < \int_{\widetilde{\Omega}_{a,b}} K(.,f)^2(\Pi^*\rho_\phi)^4 \mathrm{d}L^3.$$

Proof. We only have one thing left to prove here: in the first case, every minimizer of the mean distortion in $\mathcal{F}_{\phi,\psi}$ for the density $\Pi^*\rho_\phi$ is a lift of a minimizer of the mean distortion in $\mathcal{G}_{\phi,\psi}$ for the density ρ_ϕ. We will show that in that case, a minimizer f must satisfy $f_*\Pi^*\rho_\phi = \Pi^*\rho_\psi$. Then, the result will follow by Theorem 2.3.11. Pick $f \in \mathcal{F}_{\phi,\psi}$ a minimizer of the mean distortion and denote f_0 a minimizer of the mean distortion coming from the hypothesis. Then, it is clear that $f_0(\widetilde{\Gamma}_\phi) = \widetilde{\Gamma}_\psi$. Moreover, Corollary 2.1.14 ensures that

$$M(f_0(\widetilde{\Gamma}_\phi)) = \int_{\widetilde{\Omega}_{a,b}} K(.,f_0)^2(\Pi^*\rho_\phi)^4 \mathrm{d}L^3.$$

So,

$$M(\widetilde{\Gamma}_\psi) = \int_{\widetilde{\Omega}_{a,b}} K(.,f_0)^2(\Pi^*\rho_\phi)^4 \mathrm{d}L^3 = \int_{\widetilde{\Omega}_{a,b}} K(.,f)^2(\Pi^*\rho_\phi)^4 \mathrm{d}L^3$$

which means that $f_*\Pi^*\rho_\phi$ is extremal for $\widetilde{\Gamma}_\psi$. Since $\Pi^*\rho_\psi$ and $f_*\Pi^*\rho_\phi$ are continuous, they are equal by Lemma 2.1.3.

□

We wish now to give two examples of the construction. The first one is between spherical annuli on the Heisenberg group and comes from [7], [8] where extremality and uniqueness were proved. Here, it is constructed using the holomorphic map $z \longmapsto e^z$. Applying Proposition 2.3.7 and Theorem 2.3.11, we are able to reconstruct the map and prove its uniqueness. The second example uses the translation $z \longmapsto z + i$.

Example 2.3.20. *Let us consider two half-annuli in* $\mathbb{H}$*:*

$$A_a := \{w \in \mathbb{H} \mid 1 < |w| < a^2\} \text{ and } A_{a^k} := \{w \in \mathbb{H} \mid 1 < |w| < a^{2k}\}$$

for $k < 1$ *and* $a > 1$*. Then,*

$$A_a = \phi(]0, 2\ln(a)[\times]0, \pi[) \text{ and } A_{a^k} = \psi(]0, 2k\ln(a)[\times]0, \pi[)$$

where $\phi(s,x) = \psi(s,x) = e^{s+ix}$*. Then,*

$$\frac{|\phi'(s,x)|^2}{\Im(\phi(s,x))^2} = \frac{1}{\sin^2(x)}$$

is a function of x *only. Moreover, we denote*

$$\widetilde{A}_a = \overline{\overset{\circ}{\Pi^{-1}(A_a)}} = \{(z,t) \in \mathcal{H} \mid 1 < \|(z,t)\|_{\mathcal{H}} < a\}$$

and

$$\widetilde{A}_{a^k} = \overline{\overset{\circ}{\Pi^{-1}(A_{a^k})}} = \{(z,t) \in \mathcal{H} \mid 1 < \|(z,t)\|_{\mathcal{H}} < a^k\}$$

the spherical annuli in $\mathcal{H}$*. The set* $\mathcal{F}_{\phi,\psi}$ *is here the set of quasiconformal maps* $f : \widetilde{A}_a \longrightarrow \widetilde{A}_{a^k}$ *that extend homeomorphically on* $\{(z,t) \in \mathcal{H} \mid 1 \leq \|(z,t)\|_{\mathcal{H}} \leq a\}$*, sending* $\{\|(z,t)\|_{\mathcal{H}} = 1\}$ *on* $\{\|(z,t)\|_{\mathcal{H}} = 1\}$ *and* $\{\|(z,t)\|_{\mathcal{H}} = a\}$ *on* $\{\|(z,t)\|_{\mathcal{H}} = a^k\}$*. Finally, the family of curves considered here is the family of radial curves*

$$\gamma_{(x,\alpha)}(s) = \left(\sqrt{e^s \sin x} e^{i(\alpha - \frac{\cot x}{2} s)}, e^s \cos x\right)$$

which has modulus $\pi^2 \ln(a)^{-3}$ *with extremal density*

$$\rho_\phi(z,t) = \frac{|z|}{\ln(a)\sqrt{t^2 + |z|^4}}$$

for $\widetilde{A}_a$ *and* $\pi^2 \ln(a^k)^{-3}$ *with extremal density*

$$\rho_\psi(z,t) = \frac{|z|}{\ln(a^k)\sqrt{t^2 + |z|^4}}$$

for $\widetilde{A}_{a^k}$ *(see Figure 2).*

According to Proposition 2.3.7, if a lift (S, X, Θ) *of* g_φ *exists, it must verify the*

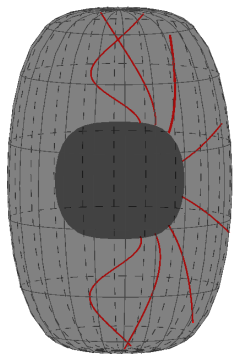

Figure 2.2 – Spherical annulus foliated by radial curves (foliation given by rotations around the vertical axis of drawn curves and the two pieces of the vertical axis itself).

following :

$$S(s,x,\theta) = ks,\ \ X(s,x,\theta) = \varphi(x)$$

$$\dot{\varphi}(x)\frac{\sin^2(x)}{\sin^2(\varphi(x))} = k^{-1}.$$

Thus, solving the ordinary differential equation, we find for every x,

$$\varphi(x) = \cot^{-1}(k^{-1}\cot(x) + D) \text{ where } D \in \mathbb{R}.$$

Moreover, $\dot{\varphi}(x) \geq k$ for every x. For $\varphi(x) = \cot^{-1}(k^{-1}\cot(x) + D)$, this is equivalent to

$$k + 2D\cot(x) + kD^2 \leq 1 \text{ for every } x.$$

Which is possible if and only if $D = 0$. So,

$$\varphi(x) = \cot^{-1}\left(k^{-1}\cot(x)\right)$$

for every $x \in]0,\pi[$. In particular, notice that φ extends continuously in a homeomorphism from $[0,\pi]$ to $[0,\pi]$. By Proposition 2.3.7 again, we know that we can find the function Θ to make (S,X,Θ) define a quasiconformal map between spherical annuli (minus pieces of the vertical axis):

$$\Theta(s,x,\theta) = \theta + h(s,x)$$

where h verifies

$$2h_s(s,x) = 0 \text{ and } 2h_x(s,x) = \dot{\varphi}(x) - 1.$$

Thus, we find

$$h(s,x) = \frac{\varphi(x) - x}{2} + \theta_0 \text{ with } \theta_0 \in \mathbb{R}.$$

Using Ψ_ϕ one is invited to check that in usual coordinates, it gives (up to composition with a rotation around the t-axis) the map

$$\begin{array}{rccc} f: & \widetilde{A}_a \backslash \{z=0\} & \longrightarrow & \widetilde{A}_{a^k} \backslash \{z=0\} \\ & (z,t) & \longmapsto & \left(\sqrt{k} z \left(\frac{t-i|z|^2}{t-ik|z|^2}\right)^{\frac{1}{2}} |t+i|z|^2|^{\frac{k-1}{2}}, t\frac{|t+i|z|^2|^k}{|t+ik|z|^2|}\right) \end{array}$$

Then, f extends in a homeomorphism in $\mathcal{F}_{\phi,\psi}$. This is the map studied in [7] *called the radial stretch map. By Theorem 2.3.8, f minimizes the mean distortion in $\mathcal{F}_\phi$ for the density $\widetilde{\rho}_\phi$. Finally, the first part of Theorem 2.3.19 ensures that, up to composition with vertical rotations, f is the only minimizer of the mean distortion in $\mathcal{F}_{\phi,\psi}$ for the density ρ_ϕ.*

The radial stretch map appeared in various contexts for instance in [40], [5].

Example 2.3.21. *Let us consider a subset of a cylinder*

$$D_{r,R} := \{(z,t) \in \mathcal{H} \mid 0 < t < r,\ 1 < |z|^2 < R+1\}.$$

We are interested in the same minimization problem as in section 2.2 but this time between $D_{a,b}$ and $D_{a',b'}$ with $\frac{a(b'+1)}{a'(b+1)} > 1$. Meaning we consider the foliation of $D_{a,b}$ given by the subset of $\widetilde{\Gamma}_0$ of curves that lie in $D_{a,b}$.

Those cylinders are simply lifts up by Π of rectangles $\phi(R_{a,b})$ and $\phi(R_{a',b'})$ for $\phi(w) = w + i$. According to Proposition 2.3.9, a lift of one of the g_φ exists only if $\phi(R_{a,b})$ and $\phi(R_{a',b'})$ have the same (finite) hyperbolic area. Moreover, equation 2.11 writes as

$$\frac{a'}{a}\dot{\varphi}(x)\frac{(x+1)^2}{(\varphi(x)+1)^2} = 1$$

whose solutions are functions defined for x by

$$\varphi(x) = \frac{a'}{a}\frac{x+1}{Cx+C+1} - 1 \text{ with } C \in \mathbb{R}.$$

From $\varphi(0) = 0$, we deduce that

$$C = \frac{a'}{a} - 1.$$

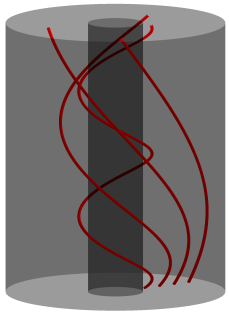

Figure 2.3 – $D_{a,b}$ foliated by a subset of $\widetilde{\Gamma}_0$ (foliation given by rotations around the vertical axis of drawn curves).

So, for every $x \in [0, b]$,

$$\varphi(x) = \frac{a}{a'} \frac{x}{\left(1 - \frac{a}{a'}\right) x + 1}.$$

Then, assume that $\phi(R_{a,b})$ *and* $\phi(R_{a',b'})$ *have the same hyperbolic area, meaning that,*

$$\frac{ab}{b+1} = \frac{a'b'}{b'+1}.$$

Using $\frac{a}{a'} = \frac{b'(b+1)}{b(b'+1)}$, *we have then*

$$\varphi(b) = \frac{a}{a'} \frac{b}{\left(1 - \frac{a}{a'}\right) b + 1} = \frac{b'(b+1)}{b(b'+1) - b'(b+1) + b' + 1} = b'.$$

So, g_φ *can be lifted to a contact map between* $D_{a,b}$ *and* $D_{a',b'}$. *Moreover, for every* $s \in [0, a]$, $x \in [1, b+1]$,

$$g_\varphi(s, x) = \frac{a'}{a} s + i \frac{a'}{a} \frac{x}{\left(\frac{a'}{a} - 1\right)(x - 1) + \frac{a'}{a}} = \frac{a'}{a} s + i \frac{a'}{a} \frac{x}{\left(\frac{a'}{a} - 1\right) x + 1}.$$

Replacing $\frac{a'}{a}$ *by* $\frac{b(b'+1)}{b'(b+1)}$ *in the imaginary part of* g_φ, *we find for every* $s \in [0, a]$, $x \in [1, b]$,

$$g_\varphi(s, x) = \frac{a'}{a} s + i \frac{(b'+1)x}{\left(1 - \frac{b'}{b}\right) x + \frac{b'(b+1)}{b}} = \frac{a'}{a} s + i \frac{(b'+1)x}{\left(1 - \frac{a(b'+1)}{a'(b+1)}\right) x + \frac{a(b'+1)}{a'}}$$

which is exactly the restriction to $[1, b+1]$ *of the function found in section 2.2 from the rectangle* $[0, a] \times [0, b+1]$ *to the rectangle* $[0, a'] \times [0, b'+1]$*. So* g_φ *can be lifted into a quasiconformal map between* $D_{a,b}$ *and* $D_{a,b}$ *and a lift is the restriction to* $D_{a,b}$ *of one of the* $\widetilde{f}_\alpha : C_{a,b+1} \longrightarrow C_{a',b'+1}$ *from Proposition 2.2.3. Finally, Theorem 2.3.8 ensures that lifts of* g_φ *minimize the mean distortion in* $\mathcal{F}_{\phi,\phi}$ *for the density* $\Pi^*\rho_\phi$*. So, Theorem 2.3.19 gives the dichotomy:*

— *If* $\phi(R_{a,b})$ *and* $\phi(R_{a',b'})$ *have the same hyperbolic area, then restrictions of* $\widetilde{f}_\alpha : C_{a,b+1} \longrightarrow C_{a',b'+1}$ *to* $D_{a,b}$ *minimize the mean distortion in* $\mathcal{F}_{\phi,\phi}$ *for the density* $\Pi^*\rho_\phi$ *and they are the only such minimizers.*

— *If* $\phi(R_{a,b})$ *and* $\phi(R_{a',b'})$ *don't have the same hyperbolic area, then for every quasiconformal map* $f \in \mathcal{F}_{\phi,\phi}$*,*

$$M(f(\widetilde{\Gamma})) < \int_{D_{a,b}} K(.,f)^2 (\Pi^*\rho_\phi)^4 \mathrm{d}L^3$$

where $\widetilde{\Gamma}$ *is the family of legendrian curves in* $D_{a,b}$ *connecting the two boundary annuli.*

Chapter 3

CR quadratic differentials on spherical CR manifolds

The construction of a Teichmüller map in the case of Riemann surface is carried out by holomorphic quadratic differentials. A similar notion in the case of spherical CR geometry would be a nice way to get close to the study of extremal quasiconformal maps between spherical CR manifolds. Before getting further, we remind that we restrict ourself to 3-dimensional (spherical) CR manifolds.

Recall that a holomorphic quadratic differential on a Riemann surface X is a holomorphic section of the line bundle $K_X^{\otimes 2}$ where K_X is the canonical bundle (that is, the bundle whose holomorphic sections are precisely the holomorphic forms on X). The two first properties we shall insist on are: $K_X^{\otimes 2}$ is a <u>line</u> bundle and we consider <u>holomorphic</u> sections of it. Then, once we have a holomorphic quadratic differential on X, we can define its trajectories which are particular curves on X. The idea behind the notion of a CR quadratic differential on a spherical CR manifold M is to get back these three properties. Namely, we want to consider a line bundle over M, a natural operator on its sections and the zero set of this operator will give CR quadratic differentials on M. Then, we will be able to define trajectories of a CR quadratic differential.

Before getting to quadratic differentials, one shall already try to define analogues of holomorphic forms on (spherical) CR manifolds. The tangential Cauchy-Riemann complex on a CR manifold M is unpleasant for such a notion because of the asymmetry noticed in the remark 1.2.13: $\wedge^{1,0}T^*M$ is a (complex) rank 2 bundle. In order to get more symmetries, we will consider Rumin complex introduced in [48] and the decomposition of it on strictly pseudoconvex CR manifolds given in [22] (we call it

the Garfield-Lee decomposition of Rumin complex). Those notions give naturally an operator on sections of a line bundle over 3-dimensional strictly pseudoconvex manifolds, from which we deduce a natural analogue of holomorphic forms on such manifolds.

Mixing the Rumin complex with its Garfield-Lee decomposition, we first define a differential complex on 3-dimensional strictly pseudoconvex CR manifolds with a finite dimensional cohomology space (on compact manifolds) which gives a CR invariant. We compute explicitly this cohomology space on strictly pseudoconvex CR 3-manifolds endowed with a CR action of the circle. This complex is also different from Rumin complex since it is not an acyclic resolution of the constant sheaf.

After that, we go back to spherical CR manifolds where we study more precisely the Garfield-Lee decomposition of Rumin complex in this case. This study allows us to naturally define CR quadratic differentials on spherical CR manifolds. The notion of trajectories of a CR quadratic differential follows easily.

Of course, since the Garfield-Lee decomposition of Rumin complex on any strictly pseudoconvex CR 3-manifold M gives rise to a line bundle $\wedge^{1,0}$ over M, one can define a notion of quadratic differential on M simply by taking section of $\wedge^{1,0} \otimes \wedge^{1,0}$. However, defining a natural Dolbeault type operator (that is defining the "CR" part of "CR quadratic differential") in this setting seems to be more difficult (if even it is possible) since $\wedge^{1,0}$ is not a CR line bundle (a CR bundle is a complex bundle with CR transition maps). The difficulty in defining CR quadratic differentials on a spherical CR manifolds is then to find a natural Dolbeault type operator on sections of $\wedge^{1,0} \otimes \wedge^{1,0}$ and the way we do it here seems to be specific to the spherical CR geometry setting.

Finally, we work again with the examples given in chapter 2 of quasiconformal maps between domains of the Heisenberg group. Namely, we express these examples in terms of trajectories of CR quadratic differentials. A translation of Proposition 2.3.17 with trajectories of quadratic differentials (Theorem 3.4.6) and the lifting theorem (Proposition 2.3.9) allow us to compute explicitly, in these examples, all quasiconformal maps which dilate trajectories of several quadratic differentials. To be more accurate, Theorem 3.4.6 and the lifting theorem ensure that, for two well-chosen CR quadratic differentials on two also well-chosen domains of $\mathcal{H}$, up to composition with a rotation around the vertical axis, there is

— no quasiconformal map which dilate horizontal trajectories of a couple of given CR quadratic differentials or

— a unique quasiconformal map which dilates horizontal trajectories of a couple of given CR quadratic differentials or
— a 1-parameter family of quasiconformal maps which dilate horizontal trajectories of a couple of given CR quadratic differentials

and we provide examples for these 3 situations.

The chapter is organized as follows. Rumin complex and its Garfield-Lee decomposition are defined in section 3.1. Section 3.2 is devoted to the introduction of another differential complex, heavily inspired by Rumin complex and its Garfield-Lee decomposition. Contrary to the $\overline{\partial}_b$ cohomology, the complex has a finite dimensional cohomology space (if the manifold is compact) and is a CR invariant. It is also different from Rumin complex since it is not an acyclic resolution of the constant sheaf. In section 3.3, we study the Garfield-Lee decomposition of Rumin complex on spherical CR manifolds. This study allows us to define CR quadratic differentials. Section 3.4 is an interpretation of the results obtained in [7, 8] and in chapter 2 in terms of trajectories of CR quadratic differentials. In particular, we find explicitly all quasiconformal maps between cylinders or spherical annuli which dilate horizontal trajectories of a couple of given CR quadratic differentials.

3.1 Rumin complex and its Garfield-Lee decomposition

3.1.1 Rumin complex on contact manifolds

Let M be an orientable 3-manifold with a contact structure $\mathbf{P}$, that is, $\mathbf{P} \subset TM$ is a rank 2 distribution defined locally as the kernel of a 1-form ω such that $\omega \wedge \mathrm{d}\omega$ is nowhere zero; and so, $(\mathrm{d}\omega)_{|\mathbf{P}}$ is nondegenerate. On such manifolds, Rumin [48] defined a differential complex as follows.

Let $\Omega^\bullet(M)$ be the algebra of differential forms on M. If U is a (sufficiently small) open subset of M, let ω_U be a 1-form on U with kernel $\mathbf{P}$. Then, two such forms differ only by multiplication with a nowhere vanishing real valued function so that the differential ideal generated by all these ω_U, denoted $I^\bullet$, is well defined. To be precise, for every k, with the convention that $\Omega^k = \{0\}$ if $k < 0$,

$$I^k = \{\gamma \in \Omega^k(M) \mid \forall \omega_U \text{ with } \mathbf{P} = \ker \omega_U, \gamma = \omega_U \wedge \alpha + \mathrm{d}\omega_U \wedge \beta \\ \text{with } \alpha \in \Omega^{k-1}(U), \beta \in \Omega^{k-2}(U)\}.$$

Consider also $F^\bullet$ the annihilator of $I^\bullet$. That is, for every k,

$$F^k = \{\alpha \in \Omega^k(M) \mid \forall \omega_U \text{ with } \mathbf{P} = \ker \omega_U, \alpha \wedge \omega_U = \alpha \wedge \mathrm{d}\omega_U = 0\}.$$

Moreover, let $E^k := {\Omega^k(M)}/{I^k}$, then it is easy to see that

$$E^0 = C^\infty(M),\ E^1 = {\Omega^1(M)}/{< \omega_U >},\ E^k = \{0\} \text{ if } k \geq 2,$$

$$F^2 = \{\alpha \in \Omega^2(M) \mid \omega_U \wedge \alpha = 0, \forall \omega_U \text{ with } \mathbf{P} = \ker \omega_U\},\ F^3 = \Omega^3(M) \text{ and}$$

$$F^k = \{0\} \text{ if } k \leq 1.$$

We have then two natural differential operators

$$\mathrm{d}_{\mathbf{P}} = pr_{E_1} \circ \mathrm{d} : E^0 \longrightarrow E^1 \text{ and } \mathrm{d}_{\mathbf{P}} = \mathrm{d}_{|F^2} : F^2 \longrightarrow F^3$$

where pr_{E_1} is the canonical projection from $\Omega^1(M)$ to E^1. In order to get a differential complex, one needs to define an operator $\mathrm{D} : E^1 \longrightarrow F^2$ such that $d_{\mathbf{P}} \circ \mathrm{D} : E^1 \longrightarrow F^3$ and $\mathrm{D} \circ d_{\mathbf{P}} : E^0 \longrightarrow F^2$ vanish. Rumin in [48] defines $D : E^1 \longrightarrow F^2$ in the following way.

Lemma 3.1.1. *If $\alpha \in E^1$, there is a unique representative $\gamma \in \Omega^1(M)$ of α such that $\mathrm{d}\gamma \in F^2$, then we set,*

$$\mathrm{D}\alpha = \mathrm{d}\gamma.$$

Proof. If $\beta \in \Omega^1$ represents α, locally, there is a unique function $f_U \in C^\infty(U)$ such that

$$\omega_U \wedge \mathrm{d}(\beta + f_U \omega_U) = 0.$$

Indeed, if $f_U \in C^\infty(U)$, then

$$\omega_U \wedge \mathrm{d}(\beta + f_U\omega_U) = \omega_U \wedge (\mathrm{d}\beta + \mathrm{d}f_U \wedge \omega_U + f_U \mathrm{d}\omega_U) = \omega_U \wedge (\mathrm{d}\beta + f_U \mathrm{d}\omega_U).$$

Thus,

$$\omega_U \wedge \mathrm{d}(\beta + f_U\omega_U) = 0 \iff (\mathrm{d}\beta + f_U \mathrm{d}\omega_U)_{|\mathbf{P}} = 0.$$

Since $(\mathrm{d}\omega_U)_{|\mathbf{P}}$ is nondegenerate, there is a unique such function.
Therefore, there is a unique $\widetilde{\alpha}_U \in \Omega^1(U)$ representing α on U and such that $\mathrm{d}\widetilde{\alpha}_U \in F^2(U)$. Uniqueness ensures that these $\widetilde{\alpha}_U$ define a global 1-form $\alpha \in \Omega^1(M)$ representing α and such that $\mathrm{d}\widetilde{\alpha} \in F^2(M)$. □

D being defined, we just have to verify that $\mathrm{d}_{\mathbf{P}} \circ \mathrm{D}$ and $\mathrm{D} \circ \mathrm{d}_{\mathbf{P}}$ vanish.
If $\alpha \in E^1$, let $\gamma \in \Omega^1$ be the unique representative of α such that $\mathrm{d}\gamma \in F^2$. Then,

$$(\mathrm{d}_{\mathbf{P}} \circ \mathrm{D})\alpha = (\mathrm{d} \circ \mathrm{d})\gamma = 0.$$

If $f \in C^\infty(M)$, by definition of $\mathrm{d}_{\mathbf{P}}$, $\mathrm{d}f$ is a representative of $\mathrm{d}_{\mathbf{P}} f$ and $\mathrm{d}(\mathrm{d}f) = 0 \in F^2$.
By definition of D,

$$(\mathrm{D} \circ \mathrm{d}_{\mathbf{P}})f = \mathrm{d}(\mathrm{d}f) = 0.$$

Definition 3.1.2. Rumin's complex is the differential complex

$$0 \longrightarrow \mathbb{R} \longrightarrow E^0 \xrightarrow{\mathrm{d}_{\mathbf{P}}} E^1 \xrightarrow{\mathrm{D}} F^2 \xrightarrow{\mathrm{d}_{\mathbf{P}}} F^3 \longrightarrow 0.$$

We write $H^k_R(M)$ the cohomology spaces corresponding to this complex.

It is proved in [48] that this complex is locally exact and so it is an acyclic resolution of the constant sheaf. Its cohomology spaces are then isomorphic to the corresponding ones in de Rham cohomology. In the following, we will denote $H^\bullet_{dR}(M)$ the de Rham cohomology.

Remark 3.1.3. The spaces E^k and F^k can be interpreted as spaces of sections of vector bundles. Let A be the subbundle of T^*M of forms vanishing on the contact distribution $\mathbf{P}$. Then, E^1 is the space of sections of the bundle

$$\wedge^1 = {}^{T^*M}\!/_{A} \simeq \mathbf{P}^*.$$

Let $\wedge^2$ be the subbundle of $\wedge^2 T^*M$ of forms vanishing on $\mathbf{P}^2$, then F^2 is the space of sections of $\wedge^2$.

3.1.2 Garfield-Lee decomposition of Rumin complex on strictly pseudoconvex CR manifolds

In [22], the authors give a decomposition of the Rumin complex on a strictly pseudoconvex CR manifold.

Let (M, V) be a 3-dimensional strictly pseudoconvex CR manifold. Then, $\mathbb{C}\mathbf{P} = \mathbb{C} \otimes \mathbf{P} = V \oplus \overline{V}$ is a (complex) contact structure on M and we can consider Rumin complex on M. In order to decompose the Rumin complex, a first step is to decompose the spaces $\mathbb{C}E^k = \mathbb{C} \otimes E^k$ and $\mathbb{C}F^k = \mathbb{C} \otimes F^k$. Let p, q be non-negative integers, then define

$$E^{p,q} = \{\alpha \in \mathbb{C}E^{p+q} \mid \exists \gamma \in \mathbb{C}\Omega^{p+q} \text{ s.t. } pr_{\mathbb{C}E^1}(\gamma) = \alpha \text{ and } \gamma_{|V \oplus \overline{V}} \text{ is of type } (p, q)\}$$

and

$$F^{p,q} = \{\gamma \in \mathbb{C}F^{p+q} \mid (\iota_X \gamma)_{|V \oplus \overline{V}} \text{ is of type } (p-1, q)\ \forall X \notin V \oplus \overline{V}\}$$

with the standard convention that a (p, q)-form with $p < 0$ or $q < 0$ is 0. The only non-trivial spaces are

$$E^{0,0} = C^\infty(M, \mathbb{C}),\ E^{1,0},\ E^{0,1},\ F^{2,0},\ F^{1,1} \text{ and } F^{2,1} = \Omega^3(M, \mathbb{C}).$$

It is also clear that $\mathbb{C}E^k = \oplus_{p+q=k} E^{p,q}$ and $\mathbb{C}F^k = \oplus_{p+q=k} F^{p,q}$ and, for $p+q \neq 1$, $\mathrm{d}_{\mathbf{P}}$ induces (to simplify the notations, we still write $\mathrm{d}_{\mathbf{P}}$ and D for $\mathrm{d}_{\mathbf{P}} \otimes \mathrm{id}_{\mathbb{C}}$ and $\mathrm{D} \otimes \mathrm{id}_{\mathbb{C}}$ on complex forms)

$$\mathrm{d}_{\mathbf{P}} : E^{p,q} \longrightarrow E^{p+1,q} \oplus E^{p,q+1} \text{ and } \mathrm{d}_{\mathbf{P}} : F^{p,q} \longrightarrow F^{p+1,q} \oplus F^{p,q+1}$$

which allows us to decompose $\mathrm{d}_{\mathbf{P}}$ in two operators $\mathrm{d}' : R^{p,q} \longrightarrow R^{p+1,q}$ and $\mathrm{d}'' : R^{p,q} \longrightarrow R^{p,q+1}$ where $R^{p,q} = E^{p,q}$ if $p + q \leq 1$ and $F^{p,q}$ if not.
When $p + q = 1$, things are not as easy. In general,

$$\mathrm{D} : E^{p,q} \longrightarrow F^{p+1,q} \oplus F^{p,q+1} \oplus F^{p+2,q-1},$$

which gives a decomposition in three operators

$$\mathrm{D}' : E^{p,q} \longrightarrow F^{p+1,q},\ \mathrm{D}'' : E^{p,q} \longrightarrow F^{p,q+1} \text{ and } \mathrm{D}^+ : E^{p,q} \longrightarrow F^{p+2,q-1}.$$

To be precise,

$$\mathrm{D} : E^{1,0} \longrightarrow F^{2,0} \oplus F^{1,1}$$

so that D^+ vanishes on $E^{1,0}$ and

$$\mathrm{D} : E^{0,1} \longrightarrow F^{1,1} \oplus F^{2,0}$$

which means that D'' vanishes on $E^{0,1}$. Relations $\mathrm{d}_{\mathbf{P}} \circ \mathrm{D} = \mathrm{D} \circ \mathrm{d}_{\mathbf{P}} = 0$ and the decomposition according to the bidegree give the following:

$$\mathrm{d}''\mathrm{D}'' = \mathrm{D}''\mathrm{d}'' = 0,$$

$$\mathrm{d}'\mathrm{D}'' + \mathrm{d}''\mathrm{D}' = \mathrm{D}'\mathrm{d}'' + \mathrm{D}''\mathrm{d}' = 0,$$

$$\mathrm{d}'\mathrm{D}^{+} = \mathrm{D}^{+}\mathrm{d}' = 0,$$

$$\mathrm{d}'\mathrm{D}' + \mathrm{d}''\mathrm{D}^{+} = \mathrm{D}'\mathrm{d}' + \mathrm{D}^{+}\mathrm{d}'' = 0$$

Remark 3.1.4. We don't obtain a double complex since $\mathrm{d}'\mathrm{D}'$ and $\mathrm{D}'\mathrm{d}'$ don't vanish.

Definition 3.1.5. The Garfield-Lee decomposition of Rumin complex are the complexes given by the operators d'' and D''. That is, the three complexes

$$0 \longrightarrow \mathrm{CR}\ (M) \longrightarrow E^{0,0} \xrightarrow{\mathrm{d}''} E^{0,1} \xrightarrow{\mathrm{D}''} 0$$

$$0 \longrightarrow \ker\left(\mathrm{D}'' : E^{1,0} \longrightarrow F^{1,1}\right) \longrightarrow E^{1,0} \xrightarrow{\mathrm{D}''} F^{1,1} \xrightarrow{\mathrm{d}''} 0$$

$$0 \longrightarrow \ker\left(\mathrm{d}'' : F^{2,0} \longrightarrow F^{2,1}\right) \longrightarrow F^{2,0} \xrightarrow{\mathrm{d}''} F^{2,1} \longrightarrow 0.$$

Their cohomology spaces are denoted $H^{p,q}_{G-L}(M)$.

It is clear that the $H^{p,q}_{G-L}$ are CR invariants of the manifold. Moreover, Theorem 1 in [22] states that the $H^{p,q}_{G-L}$ are isomorphic to their corresponding spaces in the $\overline{\partial}_b$-complex.

Remark 3.1.6. As for Rumin complex, we can see spaces $E^{p,q}$ and $F^{p,q}$ as spaces of sections of complex line bundles. Let $B^{1,0}$ (resp. $B^{0,1}$) be the subbundle of $\mathbb{C}T^*M$ of forms vanishing on $\overline{V}$ (resp. the subbundle of $\mathbb{C}T^*M$ of forms vanishing on V). Then, $E^{1,0}$ is the space of sections of

$$\wedge^{1,0} = {B^{1,0}}/{A} \simeq V^*$$

and $E^{0,1}$ is the space of sections of

$$\wedge^{0,1} = {B^{0,1}}/{A} \simeq \overline{V}^*.$$

Let $\wedge^{1,1}$ (resp. $\wedge^{2,0}$) be the subbundle of $\wedge^2$ of forms with vanishing interior product with every vector in V (resp. the subbundle of $\wedge^2$ of forms with vanishing interior product with every vector in $\overline{V}$). Then $F^{1,1}$ is the space of sections of $\wedge^{1,1}$ and $F^{2,0}$

is the space of sections of $\wedge^{2,0}$.

Local description

We provide now a local description for forms and for the operators $\mathrm{d}_{\mathbf{P}}, \mathrm{d}', \mathrm{d}'', \mathrm{D}$, D' and D'' (refer to [4] for details).
Let Z be a nowhere vanishing vector field of V in an open set $U \subset M$. We can assume that $L_\omega(Z, Z) = 1$ where L_ω is the Levi form defined for all $V_1, V_2 \in V$ by $L_\omega(V_1, V_2) = i\mathrm{d}\omega\left(V_1, \overline{V}_2\right)$. Let R be the Reeb vector field of ω, that is, the only vector field such that

$$\iota_R \mathrm{d}\omega = 0 \text{ and } \omega(R) = 1.$$

Then, $(Z, \overline{Z}, R)$ is a frame of $\mathbb{C}TM$ in U and we denote $(\zeta, \overline{\zeta}, \omega)$ its dual coframe. By definition of R and Z, we have

$$\mathrm{d}\omega = i\zeta \wedge \overline{\zeta}.$$

Locally, we can write a form α according to its bidegree in the following way

$$(0,0) : \alpha = f,$$

$$(1,0) : \alpha = [f\zeta]_\omega,$$

$$(0,1) : \alpha = [f\overline{\zeta}]_\omega$$

$$(1,1) : \alpha = f\overline{\zeta} \wedge \omega$$

$$(2,0) : \alpha = f\zeta \wedge \omega$$

$$(2,1) : \alpha = f\zeta \wedge \overline{\zeta} \wedge \omega$$

where $f \in C^\infty(U, \mathbb{C})$. Let

$$a = i\zeta([Z, \overline{Z}]),\ b = \zeta([Z, R]) \text{ and } c = \overline{\zeta}([\overline{Z}, R]).$$

Classical differential calculus on manifolds give

$$\mathrm{d}\zeta = ia\zeta \wedge \overline{\zeta} - b\zeta \wedge \omega - c\overline{\zeta} \wedge \omega \text{ and } \mathrm{d}\overline{\zeta} = i\overline{a}\zeta \wedge \overline{\zeta} - \overline{c}\zeta \wedge \omega - \overline{b\zeta} \wedge \omega.$$

Thus, if $f \in C^\infty(U, \mathbb{C})$, we have

$$\mathrm{d}_{\mathbf{P}}(f\zeta \wedge \omega) = (iaf - \overline{Z}f)\zeta \wedge \overline{\zeta} \wedge \omega,$$

$$\mathrm{d}_{\mathbf{P}}(f\overline{\zeta} \wedge \omega) = (i\overline{a}f + Zf)\zeta \wedge \overline{\zeta} \wedge \omega \text{ and}$$

$$\mathrm{d}_{\mathbf{P}} f = [Zf\zeta + \overline{Z}f\overline{\zeta}]_\omega.$$

For $E^{1,0}$, let $f\zeta + h\omega$ with $f, h \in C^\infty(U, \mathbb{C})$ be any representative of $[f\zeta]_\omega$. Then we have

$$\mathrm{d}(f\zeta + h\omega) = (ih + iaf - \overline{Z}f)\zeta \wedge \overline{\zeta} + (Zh - Rf - bf)\zeta \wedge \omega + (\overline{Z}h - cf)\overline{\zeta} \wedge \omega$$

and so,

$$\mathrm{d}(f\zeta + h\omega) \in \mathbb{C}F^2 \iff h = -af - i\overline{Z}f.$$

Therefore, we find

$$\mathrm{D}[f\zeta]_\omega = -(Z(af) + iZ\overline{Z}f + Rf + bf)\zeta \wedge \omega - (\overline{Z}(af) + i\overline{Z}^2 f + cf)\overline{\zeta} \wedge \omega.$$

We can show the same way that, for $f \in C^\infty(U, \mathbb{C})$, we have

$$\mathrm{D}[f\overline{\zeta}]_\omega = (iZ^2 f - Z(\overline{a}f) - \overline{c}f)\zeta \wedge \omega + (i\overline{Z}Zf - \overline{Z}(\overline{a}f) - \overline{b}f - Rf)\overline{\zeta} \wedge \omega.$$

Operators $\mathrm{d}', \mathrm{d}'', \mathrm{D}', \mathrm{D}''$ and D^+, are then

$$\mathrm{d}' f = [(Zf)\zeta]_\omega,\ \mathrm{d}'' f = [(\overline{Z}f)\overline{\zeta}]_\omega,$$

$$\mathrm{D}'[f\zeta]_\omega = -(Z(af) + iZ\overline{Z}f + Rf + bf)\zeta \wedge \omega,$$

$$\mathrm{D}''[f\overline{\zeta}]_\omega = -\left(\overline{Z}(af) + i\overline{Z}^2 f + cf\right)\overline{\zeta} \wedge \omega,$$

$$\mathrm{D}'[f\overline{\zeta}]_\omega = \left(i\overline{Z}Zf - \overline{Z}(\overline{a}f) - \overline{b}f - Rf\right)\overline{\zeta} \wedge \omega,$$

$$\mathrm{D}^+[f\overline{\zeta}]_\omega = \left(iZ^2 f - Z(\overline{a}f) - \overline{c}f\right)\zeta \wedge \omega,$$

$$\mathrm{d}''(f\zeta \wedge \omega) = \left(iaf - \overline{Z}f\right)\zeta \wedge \overline{\zeta} \wedge \omega \text{ and } \mathrm{d}'(f\overline{\zeta} \wedge \omega) = (i\overline{a}f + Zf)\zeta \wedge \overline{\zeta} \wedge \omega$$

and the others vanish.

3.1.3 An example : Heisenberg manifolds

In this section, we compute explicitly the Garfield-Lee cohomology space $H^{1,0}_{G-L}$ of the Heisenberg manifold.

If L is a (cocompact) lattice on $\mathcal{H}$, then $\mathcal{H}/L$ has a natural CR structure induced by $V = span(Z)$ where

$$Z = \frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t}.$$

Moreover,

$$\Omega = \frac{1}{2}\left(dt - i\overline{z}dz + izd\overline{z}\right)$$

is an invariant contact form on $\mathcal{H}$ and $\ker\Omega = V \oplus \overline{V}$. Ω and V define a strictly pseudoconvex structure on $\mathcal{H}/L$ so that Rumin complex and its Garfield-Lee decomposition are defined on it. The Reeb vector field for Ω is $R = 2\frac{\partial}{\partial t}$ and we can verify that $L_\Omega(Z, Z) = 1$. Moreover, the dual coframe of $(Z, \overline{Z}, R)$ is $(\mathrm{d}z, \mathrm{d}\overline{z}, \Omega)$ and easy computations show

$$i[Z, \overline{Z}] = R,\ [Z, R] = [\overline{Z}, R] = 0$$

implying that the functions a, b and c from the previous section vanish. Consequently, for $f \in C^\infty(U, \mathbb{C})$, where U is an open subset of $\mathcal{H}$, we have

$$\mathrm{D}'[f\mathrm{d}z]_\Omega = \left(Rf + iZ\overline{Z}f\right)\Omega \wedge \mathrm{d}z,\ \mathrm{D}''[f\mathrm{d}z]_\Omega = i\overline{Z}^2 f\Omega \wedge \mathrm{d}z,$$

$$\mathrm{D}'[f\mathrm{d}\overline{z}]_\Omega = \left(Rf - i\overline{Z}Zf\right)\Omega \wedge \mathrm{d}\overline{z} \text{ and } \mathrm{D}^+[f\mathrm{d}\overline{z}]_\Omega = -iZ^2 f\Omega \wedge \mathrm{d}z.$$

Thus, a form $\alpha \in \ker \mathrm{D}'' := \ker\left(\mathrm{D}'' : E^{1,0}\left(\mathcal{H}/L\right) \longrightarrow F^{1,1}\left(\mathcal{H}/L\right)\right)$ lifts to a L-invariant function $f \in C^\infty(\mathcal{H})$ such that $\overline{Z}^2 f = 0$. Moreover $H^{1,0}_{G-L}\left(\mathcal{H}/L\right) = \ker \mathrm{D}''$. To sum up, $H^{1,0}_{G-L}\left(\mathcal{H}/L\right)$ is isomorphic to the space

$$K^L(\mathcal{H}) := \{f \in C^\infty(\mathcal{H}), L - \text{invariant} \mid \overline{Z}^2 f = 0\}.$$

Let us fix for the example, $L =< (1, 0), (i, 0) >$.

Proposition 3.1.7. *$H^{1,0}_{G-L}\left(\mathcal{H}/L\right)$ is infinite dimensional.*

Proof. Up to a change of coordinates corresponding to see $\mathcal{H}$ as the group of upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with the multiplication law, we can see $\mathcal{H}$ as $\mathbb{R}^3$ with the group law

$$(p,q,s)(p',q',s') = (p+p', q+q', s+s'+pq')$$

and $L = \mathbb{Z}^3$. Let $M = {}^{\mathcal{H}}\!/_{L}$, we have a well known description of $L^2(M)$ (see [20]): every L-invariant function on $\mathcal{H}$ is of the form

$$g(p,q) + \sum_{n\in\mathbb{Z}^*} e^{2i\pi ns} \sum_{a\in\mathcal{A}_n} \sum_{k\in\mathbb{Z}} f_n(p+k+a) e^{2i\pi n(k+a)q}$$

where $f_n \in L^2(\mathbb{R})$ and g is a $P(L)$-invariant function ($P : (p,q,s) \longmapsto q+ip$). Then, let $F \in K^L(\mathcal{H})$, we have

$$F(p,q,s) = g(p,q) + \sum_{n\in\mathbb{Z}^*} e^{2i\pi ns} \sum_{a\in\mathcal{A}_n} \sum_{k\in\mathbb{Z}} f_n(p+k+a) e^{2i\pi n(k+a)q}$$

with $g \in C^\infty(\mathbb{C},\mathbb{C})$, $P(L)$-invariant and $f_n \in C^\infty(\mathbb{R})$ and notice that, in (p,q,s) coordinates, we have

$$Z = \frac{1}{2}\left(\frac{\partial}{\partial q} - i\frac{\partial}{\partial p} + p\frac{\partial}{\partial s}\right),\ \overline{Z} = \frac{1}{2}\left(\frac{\partial}{\partial q} + i\frac{\partial}{\partial p} + p\frac{\partial}{\partial s}\right) \text{ and } T = \frac{1}{4}\frac{\partial}{\partial s}.$$

So, $\overline{Z}^2 F = 0$ if and only if

$$\frac{\partial^2}{\partial \overline{w}^2} g = 0$$

where $\frac{\partial}{\partial \overline{w}} = \frac{1}{2}\left(\frac{\partial}{\partial q} + i\frac{\partial}{\partial p}\right)$ and for every $n \in \mathbb{Z}^*$ and every $x \in \mathbb{R}$,

$$f_n''(x) + 4\pi n x f_n'(x) + 2\pi n(1+2\pi n x^2) f_n(x) = 0.$$

In particular, $(x \mapsto 1, x \mapsto e^{-\pi x^2}, x \mapsto e^{-2\pi x^2}, \cdots)$ defines an infinite linearly independant family of $K^L(\mathcal{H})$. $\square$

We will come back later to $\ker(\mathrm{D}'' : E^{1,0} \longrightarrow F^{1,1})$ which seems natural as an analogue of holomorphic forms. But the fact that the space is infinite dimensional motivates the introduction of another differential complex.

3.2 Another differential complex, definitions and computations

3.2.1 Definition of the complex

Let us introduce a new differential complex on strictly pseudoconvex 3-dimensional CR manifolds heavily inspired by Rumin complex and its Garfield-Lee decomposition. Let (M, V) be a compact strictly pseudoconvex 3-dimensional CR manifold. We consider again spaces $E^{p,q}$, $F^{p,q}$ and operators $\mathrm{d}_{\mathbf{P}}, \mathrm{d}', \mathrm{d}'', \mathrm{D}, \mathrm{D}', \mathrm{D}''$ and D^+. The complex is:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{CR}(M) \xrightarrow{\mathrm{d}_{\mathbf{P}}} E^{1,0} \xrightarrow{\mathrm{D}} F^2 \xrightarrow{\mathrm{d}_{\mathbf{P}}} F^3 \longrightarrow 0.$$

where $\mathrm{CR}(M)$ is the space of CR functions on M. In the case of compact Riemann surfaces, the space of holomorphic forms is the space of d-closed $(1,0)$-forms. Since there are no nonconstant holomorphic functions on compact Riemann surfaces, there are no d-exact $(1,0)$-forms except 0. Meaning that, the space of holomorphic forms on a compact Riemann surface identifies with the space

$$\{\text{closed } (1,0)\text{-form}\} \big/ \{\text{exact } (1,0)\text{-form}\}.$$

Thus, we will be interested in the space:

$$H^{1,0}(M) = \ker\left(\mathrm{D}_{|E^{1,0}} : E^{1,0} \longrightarrow F^2\right) \Big/ \mathrm{Im}\left(\mathrm{d}_{\mathbf{P}_{|\mathrm{CR}(M)}} : \mathrm{CR}(M) \longrightarrow E^{1,0}\right).$$

Since the space $H^{1,0}(M)$ takes into account only $(1,0)$-forms, we can notice that

$$H^{1,0}(M) = E^{1,0} \cap \ker\left(\mathrm{D} : E^1 \longrightarrow F^2\right) \Big/ E^{1,0} \cap \mathrm{Im}\left(\mathrm{d}_{\mathbf{P}} : C^\infty(M) \longrightarrow E^1\right). \tag{3.1}$$

Remark 3.2.1. Local exactness of the complex seems to be a tricky question. The local exactness for the part

$$\mathrm{CR}(M) \xrightarrow{\mathrm{d}_{\mathbf{P}}} E^{1,0} \xrightarrow{\mathrm{D}} F^2$$

follows from local exactness of Rumin complex. Indeed, let $\gamma \in E^{1,0}$ be a D-closed

form. Then, using local exactness of Rumin complex, locally,

$$\gamma = \mathrm{d}_{\mathbf{P}} f$$

for a C^∞ function f. But, since $\gamma \in E^{1,0}$, $\mathrm{d}'' f = 0$, meaning that f is actually a CR function. Thus, locally, there is a CR function f such that

$$\gamma = \mathrm{d}_{\mathbf{P}} f.$$

For the part

$$F^2 \xrightarrow{\mathrm{d}_{\mathbf{P}}} F^3 \longrightarrow 0,$$

this is just local exactness of Rumin complex.
A difficulty arises for the part

$$E^{1,0} \xrightarrow{\mathrm{D}} F^2 \xrightarrow{\mathrm{d}_{\mathbf{P}}} F^3.$$

Here, local exactness of Rumin complex says that, for every $\mathrm{d}_{\mathbf{P}}$-closed form $\gamma \in F^2$, there is, locally, a form $\alpha \in E^1$ such that

$$\gamma = \mathrm{d}_{\mathbf{P}} \alpha.$$

But α has no reason to be a $(1,0)$-form.
For the Heisenberg group, the question of local exactness is linked with solving linear PDEs involving Lewy's operator

$$Z = \frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t}.$$

Let U be an open subset of $\mathcal{H}$,

$$\gamma = \frac{1}{2i}\left(f \mathrm{d}z \wedge \omega + g \mathrm{d}\overline{z} \wedge \omega\right)$$

a 2-form in F^2, where $\omega = \mathrm{d}t - i\overline{z}\mathrm{d}z + iz\mathrm{d}\overline{z}$ and $f, g \in C^\infty$ (the $\frac{1}{2i}$ is here to remove constants later), and

$$\alpha = [u\mathrm{d}z]_\omega \in E^{1,0}.$$

Then, we saw (in a slightly different way) that

$$\mathrm{D}\alpha = \frac{1}{2i}\Big((Z\overline{Z}u - 2iTu)\mathrm{d}z \wedge \omega + \overline{Z}^2 u \mathrm{d}\overline{z} \wedge \omega\Big)$$

with $T = \frac{\partial}{\partial t}$. γ is $\mathrm{d}_{\mathbf{P}}$-closed if and only if $\overline{Z}f = Zg$. To sum up, the question of local exactness of the part

$$E^{1,0} \xrightarrow{\mathrm{D}} F^2 \xrightarrow{\mathrm{d}_{\mathbf{P}}} F^3$$

reformulates as: let f, g be C^∞ functions satisfying $\overline{Z}f = Zg$. Does the system

$$\begin{cases} Z\overline{Z}u - 2iTu = f \\ \overline{Z}^2 u = g \end{cases}$$

have a C^∞ solution?
Local exactness of the part

$$\mathrm{CR}(M) \xrightarrow{\mathrm{d}_{\mathbf{P}}} E^{1,0} \xrightarrow{\mathrm{D}} F^2$$

says that the equation

$$Zu = f$$

has (locally) a CR solution if and only if

$$Z\overline{Z}f - 2iTf = \overline{Z}^2 f = 0.$$

However, there is something sure:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{CR} \longrightarrow E^{1,0} \longrightarrow F^2 \longrightarrow F^3 \longrightarrow 0$$

is not an acyclic resolution of the constant sheaf since, as we will see in Proposition 3.2.7, there is a strictly pseudoconvex CR manifold M such that

$$\dim_{\mathbb{C}} \left(H^{1,0}(M)\right) \neq \dim_{\mathbb{C}} \left(H^1_{dR}(M, \mathbb{C})\right).$$

The advantage of the space $H^{1,0}(M)$ is that it is finite dimensional.

Proposition 3.2.2. *Let (M, V) be a strictly pseudoconvex 3-dimensional CR manifold, then*

$$\dim_{\mathbb{C}} \left(H^{1,0}(M)\right) \leq \dim_{\mathbb{C}}(H^1_{dR}(M, \mathbb{C})).$$

Proof. The restriction to $\ker(\mathrm{D} : E^1 \longrightarrow F^2) \cap E^{1,0}$ of the canonical projection from $\ker(\mathrm{D} : E^1 \longrightarrow F^2)$ to $H^1_R(M, \mathbb{C})$ has kernel $\mathrm{Im}(\mathrm{d}_{\mathbf{P}}) \cap E^{1,0}$ so that it induces an injective morphism from $H^{1,0}$ to $H^1_R(M, \mathbb{C})$. Since $H^1_R(M, \mathbb{C})$ is isomorphic to $H^1_{dR}(M, \mathbb{C})$ we get the result.

□

It is easy to see that Rumin's operators $\mathrm{d}_{\mathbf{P}}$ and D are natural (that is, for every contact transform $f : (M, \mathbf{P}_M) \longrightarrow (N, \mathbf{P}_N)$, $f^* \circ \mathrm{D} = \mathrm{D} \circ f^*$ and $f^* \circ \mathrm{d}_{\mathbf{P}} = \mathrm{d}_{\mathbf{P}} \circ f^*$). Moreover, a CR diffeomorphism preserves the bidegree. Therefore, $H^{1,0}$ is a CR invariant :

Proposition 3.2.3. *Let (M, V) and (N, W) be strictly pseudoconvex 3-dimensional CR manifolds and $f : M \longrightarrow N$ be a CR diffeomorphism. Then f^* induces an isomorphism from $H^{1,0}(N)$ on $H^{1,0}(M)$.*

3.2.2 Circle bundles over surfaces

An important class of 3-dimensional manifolds is formed by circle bundles over surfaces. Let (M, V) be a strictly pseudoconvex 3-dimensional CR compact manifold endowed with a free action of U(1). The quotient $\Sigma = {}^{M}\!/_{\mathrm{U}(1)}$ is a compact surface of genus g and we denote $\Pi : M \longrightarrow \Sigma$ the projection. Since Π is a submersion, it induces an injective morphism $\Pi^* : H^1_{dR}(\Sigma, \mathbb{C}) \longrightarrow H^1_{dR}(M, \mathbb{C})$ and, since $pr_{E_1} : \Omega^\bullet(M) \longrightarrow E^1$ induces an isomorphism from $H^1_{dR}(M)$ to $H^1_R(M)$, $\overline{\Pi}^* = pr_{E_1} \circ \Pi^* : \Omega^1(\Sigma) \longrightarrow E^1$ also induces an injective morphism $\phi : H^1_{dR}(\Sigma) \longrightarrow H^1_R(M)$.

Lemma 3.2.4. *Let N be a circle bundle over a compact surface S of genus g and let e be the Euler class of the bundle. Then*

$$\begin{cases} \dim(H^1_{dR}(N)) = 2g, \ \textit{if } e \neq 0 \\ \dim(H^1_{dR}(N)) = 2g+1, \ \textit{if } e = 0 \end{cases}.$$

Proof. Gysin long exact sequence reads here as:

$$0 \to H^1_{dR}(S) \to H^1_{dR}(N) \to H^0_{dR}(S) \xrightarrow{e_\wedge} H^2_{dR}(S) \to H^2_{dR}(N) \to H^1_{dR}(S) \to 0$$

where $e_\wedge$ is the exterior product by e. If $e \neq 0$, then $e_\wedge$ is an isomorphism. So, we have the following exact sequence:

$$0 \longrightarrow H^1_{dR}(S) \longrightarrow H^1_{dR}(N) \longrightarrow H^0_{dR}(S) \longrightarrow H^2_{dR}(S) \longrightarrow 0.$$

If $e = 0$, then we have:

$$0 \longrightarrow H^1_{dR}(S) \longrightarrow H^1_{dR}(N) \longrightarrow H^0_{dR}(S) \longrightarrow 0.$$

Since we know $H^k_{dR}(S)$, the result follows easily.

□

This lemma has as important consequence that $\Pi^* : H^1_{dR}(\Sigma, \mathbb{C}) \longrightarrow H^1_{dR}(M, \mathbb{C})$ is in fact an isomorphism as well as $\phi : H^1_{dR}(\Sigma) \longrightarrow H^1_R(M)$ when the circle bundle has non-vanishing Euler class. Moreover, we saw that we can see $H^{1,0}(M)$ as a subset of $H^1_R(M, \mathbb{C})$. Consequently, in the case of a circle bundle with non vanishing Euler class, $H^{1,0}(M)$ is isomorphic to a subspace of $H^1_{dR}(\Sigma, \mathbb{C})$.

We now compute explicitly $H^{1,0}$ for compact CR circle bundles. Let R be the global vector field induced by the U(1)-action on the CR manifold (M, V). We say that the U(1)-action is transversal if for each $x \in M$

$$\mathbb{C}T_xM = \mathbb{C}R_x \oplus V_x \oplus \overline{V}_x$$

and the action is said CR if

$$[R, \Gamma(V)] \subset \Gamma(V)$$

where $\Gamma(V)$ refers to the space of sections of V. A CR circle bundle is a strictly pseudoconvex CR manifold endowed with a free, transversal, CR action of U(1). Let (M, V) be a CR circle bundle, R the vector field induced by the U(1) action and denote ω its dual form (that is, $\omega(R) = 1$ and $\omega_{|V \oplus \overline{V}} = 0$). Then, ω is a (global) contact form on M and R is its associated Reeb vector field. For the computation of $H^{1,0}(M)$ we have to recall a few facts about pseudohermitian manifolds.

A pseudohermitian structure on a manifold N is a pair (ω, J) where ω is a contact form on M and $(\mathbf{P} = \ker(\omega), J)$ is a CR structure on N. From the discussion above, a CR circle bundle admits a natural pseudohermitian structure. One can define a natural connection on a pseudohermitian manifold: the Tanaka-Webser connection (see [50, 57]).

Definition/Proposition 3.2.5. *Let (N, ω, J) be a pseudohermitian manifold. Then, there is a unique connection $\nabla : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$ such that*

1. *for each $X \in \Gamma(TM)$, $\nabla_X (\Gamma(\boldsymbol{P})) \subset \Gamma(\boldsymbol{P})$,*
2. *$\nabla R = 0$, $\nabla J = 0$, $\nabla \mathrm{d}\omega = 0$,*
3. *its torsion T satisfies*

$$\forall X, Y \in \Gamma(\boldsymbol{P}),\ T(X, Y) = \mathrm{d}\omega(X, Y)R \text{ and}$$

$$\forall X \in \Gamma(TM),\ T(R, JX) + JT(R, X) = 0.$$

$T(R, .)$ is called the pseudohermitian torsion of the manifold.

Webster [57, p. 33] proved that the pseudohermitian torsion vanishes if and only if the Reeb vector field induces a 1-parameter family of CR automorphisms. In particular, a CR circle bundle has a vanishing pseudohermitian torsion. The Webster metric g on a pseudohermitian manifold is defined by

$$g(X, Y) = \mathrm{d}\omega(X, JY),\ g(X, R) = g(R, X) = 0,\ g(R, R) = 1$$

for every $X, Y \in \Gamma(\mathbf{P})$. This metric canonically extends to differential forms. Once a metric is given, we can identify E^1 with the orthogonal of I^1 and then, define the adjoint operators of D and $\mathrm{d}_{\mathbf{P}}$ denoted D^* and $\delta_{\mathbf{P}}$ respectively. Following Rumin [48, p. 290], define the laplacian

$$\Delta_{\mathbf{P}} = \mathrm{D}^*\mathrm{D} + (\mathrm{d}_{\mathbf{P}}\delta_{\mathbf{P}})^2 \text{ on } E^1.$$

Then, Rumin proved a Hodge decomposition. If (N, ω, J) is a compact pseudohermitian manifold, then we have an orthogonal decomposition

$$E^1 = \ker\ \Delta_{\mathbf{P}} \oplus \mathrm{Im}\ \Delta_{\mathbf{P}}.$$

Moreover,

$$\ker\ \mathrm{D} = \ker\ \Delta_{\mathbf{P}} \oplus \mathrm{Im}\ \mathrm{d}_{\mathbf{P}}. \tag{3.2}$$

He also showed (see e.g. [48, p. 312]) that $\Delta_{\mathbf{P}}$ preserves the bidegree when the manifold has vanishing pseudohermitian torsion which leads to:

Proposition 3.2.6. *Let M be a compact CR circle bundle. Then $H^{1,0}$ is isomorphic to $E^{1,0} \cap \ker\ \Delta_{\boldsymbol{P}}$.*

Proof. Let $\Psi : E^1 \longrightarrow \ker\ \Delta_{\mathbf{P}}$ be the orthogonal projection. First, we show that $\Psi\left(E^{1,0}\right) \subset E^{1,0}$. Take $\alpha = \beta + \Delta_{\mathbf{P}}\gamma \in E^{1,0}$ with $\Delta_{\mathbf{P}}\beta = 0$ and decompose $\beta = \beta^{1,0} + \beta^{0,1}$, $\gamma = \gamma^{1,0} + \gamma^{0,1}$ with $\beta^{p,q}, \gamma^{p,q} \in E^{p,q}$. Then, since $\alpha \in E^{1,0}$, $\Delta_{\mathbf{P}}\beta = 0$ and $\Delta_{\mathbf{P}}$ preserves the bidegree,

$$\beta^{0,1} = -\Delta_{\mathbf{P}}\gamma^{0,1} \text{ and } \Delta_{\mathbf{P}}\beta^{0,1} = 0.$$

Therefore, $\beta^{0,1} \in \ker\ \Delta_{\mathbf{P}} \cap \operatorname{Im}\ \Delta_{\mathbf{P}} = \{0\}$.

Now, consider $\Psi_{|E^{1,0}\cap \ker\ \mathrm{D}}$. Then, using the decomposition 3.2, its kernel is exactly $E^{1,0} \cap \operatorname{Im}\ \mathrm{d}_{\mathbf{P}}$. Thus $\Psi_{|E^{1,0}\cap \ker\ \mathrm{D}}$ induces an injective linear map

$$\overline{\Psi} : H^{1,0} \longrightarrow E^{1,0} \cap \ker\ \Delta_{\mathbf{P}}.$$

The surjectivity of $\overline{\Psi}$ follows from $\ker\ \Delta_{\mathbf{P}} \subset \ker\ \mathrm{D}$. □

The fact that $\Delta_{\mathbf{P}}$ preserves the bidegree also induces a decomposition of H^1_R for compact pseudohermitian manifolds with vanishing pseudohermitian torsion. Namely, if (N, ω, J) is a compact pseudohermitian manifold with vanishing pseudohermitian torsion, then

$$H^1_R(N) \simeq \ker \Delta_{\mathbf{P}} = E^{1,0} \cap \ker \Delta_{\mathbf{P}} \oplus E^{0,1} \cap \ker \Delta_{\mathbf{P}}.$$

In particular, $H^1_R(N)$ has even dimension. If M is a compact CR circle bundle, using Lemma 3.2.4, we get then,

$$\dim \left(H^1_R(M)\right) = \dim \left(H^1_{dR}(M)\right) = 2g$$

where g is the genus of $M/_{\mathrm{U}(1)}$. In conclusion, we have:

Proposition 3.2.7. *Let M be a compact CR circle bundle. Then*

$$\dim \left(H^{1,0}(M)\right) = g.$$

Even more explicitly, $\Sigma = M/_{\mathrm{U}(1)}$ inherits of a Riemann surface structure for which the projection $\Pi : M \longrightarrow \Sigma$ is a CR map and Π^* induces a natural isomorphism from $K(\Sigma)$ to $H^{1,0}$ where $K(\Sigma)$ is the space of holomorphic forms on Σ.

3.3 CR quadratic differentials on spherical CR manifolds

3.3.1 Differential forms on spherical CR manifolds

Seeing spherical CR manifolds as manifolds endowed with a spherical CR atlas will allow us to define $(1,0)$ and $(0,1)$ forms with cocycle relations.

We denote $T = \frac{\partial}{\partial t}$ on $\mathcal{H}$ and $\omega = dt - i\overline{z}dz + izd\overline{z}$. We saw that

$$i[Z, \overline{Z}] = 2T$$

and other commutators vanish. Moreover, if U is an open subset of $\mathcal{H}$, a function $f : U \longmapsto \mathbb{C}$ is CR if and only if $\overline{Z}f = 0$. Let $g : U \longmapsto \mathcal{H}$ be a map. We denote $g = (g_1, g_2)$ with $g_1 : U \longmapsto \mathbb{C}$ and $g_2 : U \longmapsto \mathbb{R}$. g is CR if and only if g_1 and $g_{II} = g_2 + i|g_1|^2$ are CR functions. Then, we can verify the following formulas for a CR diffeomorphism $g = (g_1, g_2)$ of $\mathcal{H}$:

$$\overline{Z}g_2 = -ig_1\overline{Zg_1},\ Zg_2 = i\overline{g}_1 Zg_1 \text{ and}$$

$$|Zg_1|^2 = Tg_2 - i\overline{g}_1 Tg_1 + ig_1\overline{Tg_1}.$$

Moreover, if $f \in C^\infty(\mathcal{H}, \mathbb{C})$ and $g = (g_1, g_2)$ is a CR diffeomorphism, then

$$\overline{Z}(f \circ g) = \overline{Z}f(g).\overline{Zg_1},\ \overline{Z}(|Zg_1|^2) = -2iTg_1\overline{Zg_1} \text{ and}$$

$$T(f \circ g) = Zf(g)Tg_1 + \overline{Z}f(g)\overline{Tg_1} + Tf(g)|Zg_1|^2.$$

Let (M, V) be a spherical CR manifold, (U, φ) a chart of M and denote $U' = \varphi(U)$. Then, since φ is a CR function, a form α in U is of type $(1, 0)$ if and only if $\varphi_*\alpha$ is a $(1, 0)$ form on U'. That is, if and only if there is a function $f \in C^\infty(U')$ such that

$$\alpha = \varphi^*[f\mathrm{d}z]_\omega.$$

Let (V, ψ) be another chart of M with $U \cap V \neq \emptyset$ and $V' = \psi(V)$. Then, $\beta \in E^{1,0}(V)$ if and only if there is a function $h \in C^\infty(V')$ such that

$$\beta = \psi^*[h\mathrm{d}z]_\omega.$$

Therefore, $\alpha = \beta$ on $U \cap V$ if and only if

$$[f\mathrm{d}z]_\omega = \left(\psi \circ \varphi^{-1}\right)^* [h\mathrm{d}z]_\omega.$$

Denote $g = \psi \circ \varphi^{-1}$ the transition map (which is a CR diffeomorphism between domains of $\mathcal{H}$), then

$$g^*[h\mathrm{d}z]_\omega = [(h \circ g)Zg_1\mathrm{d}z]_\omega.$$

So, $\alpha = \beta$ on $U \cap V$ if and only if

$$f = (h \circ g) Z g_1 \text{ on } \varphi(U \cap V).$$

The same thing can be done for $(0,1)$-forms. So, we can define:

Definition 3.3.1. A $(1,0)$-form on a spherical CR manifold M is a collection of functions $f_i : U_i' \longrightarrow \mathbb{C}$, where $(U_i, \varphi_i : U_i \longrightarrow U_i')_i$ is a spherical CR atlas of M, such that

$$(f_j \circ g_{j,i})\, Z g_{j,i}^1 = f_i \text{ on } \varphi_i(U_i \cap U_j), \text{ where } g_{j,i} = \left(g_{j,i}^1, g_{j,i}^2\right) = \varphi_j \circ \varphi_i^{-1}.$$

A $(0,1)$-form on a spherical CR manifold M is a collection of functions $f_i : U_i' \longrightarrow \mathbb{C}$, where $(U_i, \varphi_i : U_i \longrightarrow U_i')_i$ is a spherical CR atlas of M, such that

$$(f_j \circ g_{j,i})\, \overline{Z g_{j,i}^1} = f_i \text{ on } \varphi_i(U_i \cap U_j), \text{ where } g_{j,i} = \varphi_j \circ \varphi_i^{-1}.$$

Moreover, we saw that the operators D' and D'' on $(1,0)$-forms on $\mathcal{H}$ are:

$$\mathrm{D}'[f \mathrm{d}z]_\omega = \frac{1}{2i}\left(Z\overline{Z}f - 2iTf\right) \mathrm{d}z \wedge \omega \text{ and } \mathrm{D}''[f \mathrm{d}z]_\omega = \frac{1}{2i}\overline{Z}^2 f \mathrm{d}\overline{z} \wedge \omega$$

and D' and D^+ on $(0,1)$-forms on $\mathcal{H}$ are:

$$\mathrm{D}'[f \mathrm{d}\overline{z}]_\omega = \frac{1}{2i}\left(\overline{Z}Zf + 2iTf\right) \omega \wedge \mathrm{d}\overline{z} \text{ and } \mathrm{D}^+[f \mathrm{d}\overline{z}]_\omega = \frac{1}{2i} Z^2 f \omega \wedge \mathrm{d}z.$$

Since the operators $\mathrm{D}', \mathrm{D}''$ and D^+ commute with CR maps, we can define:

Definition 3.3.2. Let M be a spherical CR manifold.

1. Let $\alpha = (f_i)_i$ be a $(1,0)$-form on M. Then α is said:
   — D'-closed if for every i,
   $$Z\overline{Z}f_i = 2iTf_i,$$
   — D''-closed if for every i,
   $$\overline{Z}^2 f_i = 0.$$
2. Let $\alpha = (f_i)_i$ be a $(0,1)$-form on M. Then α is said:
   — D'-closed if for every i,
   $$i\overline{Z}Zf_i = 2Tf_i,$$
   — D^+-closed if for every i,
   $$Z^2 f_i = 0.$$

3. A CR form on M is a D''-closed $(1,0)$-form.

3.3.2 CR quadratic differentials

Before getting to spherical CR manifolds, we begin with quadratic differentials on any open subset U of $\mathcal{H}$. In this section, we will use bundles $\wedge^{p,q}$, $B^{p,q}$ and A defined in remarks 3.1.3 and 3.1.6. A quadratic differential on a subset U of $\mathcal{H}$ is a section of the complex line bundle $\wedge^{1,0} \otimes \wedge^{1,0}$ over U. Since $\wedge^{1,0} \otimes \wedge^{1,0}$ is naturally isomorphic to

$$B^{1,0} \otimes B^{1,0} \big/ B^{1,0} \otimes A + A \otimes B^{1,0},$$

a quadratic differential is the class modulo the contact form ω of a form $q\mathrm{d}z^2$ with $q \in C^\infty(U)$. We write such classes $[q\mathrm{d}z^2]_\omega$. Then, we define an operator

$$\begin{array}{rccc} \overline{\mathrm{D}}: & C^\infty\left(U, (\wedge^{1,0})^2\right) & \longrightarrow & C^\infty\left(U, \wedge^{1,1} \otimes A\right) \\ & [q\mathrm{d}z^2]_\omega & \longmapsto & \overline{Z}^3 q\mathrm{d}\overline{z} \wedge \omega \otimes \omega. \end{array}$$

A quadratic differential on a spherical CR manifold is a section of the line bundle $(\wedge^{1,0})^2$ over the manifold. As for $(1,0)$-forms, it can be understood with cocycle relations.

Let q be a quadratic differential on a spherical CR manifold M and $(U_i, \varphi_i : U_i \longrightarrow U_i')$ a spherical CR atlas of M. Then, for every i, there is a unique function $q_i \in C^\infty(U_i')$ such that

$$q = \varphi_i^*[q_i\mathrm{d}z^2]_\omega.$$

If $U_i \cap U_j \neq \emptyset$, we have

$$[q_i\mathrm{d}z^2]_\omega = g_{j,i}^*[q_j\mathrm{d}z^2]_\omega \text{ on } \varphi_i(U_i \cap U_j) \text{ where } g_{j,i} = \varphi_j \circ \varphi_i^{-1}.$$

That is,

$$[q_i\mathrm{d}z^2]_\omega = [(q_j \circ g_{j,i})(Zg_{j,i}^1)^2\mathrm{d}z^2]_\omega \text{ on } \varphi_i(U_i \cap U_j) \text{ where } g_{j,i} = \varphi_j \circ \varphi_i^{-1}.$$

Therefore we define:

Definition 3.3.3 (Quadratic differential)**.** A quadratic differential q on a spherical CR manifold M is a section of the line bundle $(\wedge^{1,0})^2$. If $(U_i, \varphi_i : U_i \longrightarrow U_i')$ is a spherical CR atlas of M, q can be seen as a collection of functions $q_i \in C^\infty(U_i')$ such

that, for every i, j,

$$q_i = (q_j \circ g_{j,i}) \left(Zg^1_{j,i}\right)^2 \text{ on } \varphi_i(U_i \cap U_j) \text{ where } g_{j,i} = \varphi_j \circ \varphi_i^{-1}.$$

Following that, we need to extend $\overline{\mathrm{D}}$ to global sections of $(\wedge^{1,0})^2$.

Definition/Proposition 3.3.4. *Let q be a quadratic differential on a spherical CR manifold M and $(U_i, \varphi_i : U_i \longrightarrow U_i')$ a spherical CR atlas of M. For every i, we set*

$$\overline{\mathrm{D}}q = \varphi_i^* \overline{\mathrm{D}}[q_i \mathrm{d}z^2]_\omega \text{ where } q = \varphi_i^*[q_i \mathrm{d}z^2]_\omega \text{ on } U_i.$$

Then, $\overline{\mathrm{D}}$ defines an operator from quadratic differentials on M to sections of the line bundle $\wedge^{1,1} \otimes A$ over M.

Proof. In order to prove the proposition, we just have to verify that for every i, j with $U_i \cap U_j \neq \emptyset$,

$$\overline{\mathrm{D}}[q_i \mathrm{d}z^2]_\omega = g_{j,i}^* \left(\overline{\mathrm{D}}[q_j \mathrm{d}z^2]_\omega\right)$$

where $g_{j,i} = \varphi_j \circ \varphi_i^{-1}$. For that, it is enough to prove that we have,

$$g_{j,i}^* \left(\overline{\mathrm{D}}[q_j \mathrm{d}z^2]_\omega\right) = \overline{\mathrm{D}} \left(g_{j,i}^*[q_j \mathrm{d}z^2]_\omega\right).$$

Lemma 3.3.5. *Let U, V be open subsets of $\mathcal{H}$, $q \in C^\infty(U)$ and $g : U \longrightarrow V$ a CR diffeomorphism. Then*

$$g^* \left(\overline{\mathrm{D}}[q \mathrm{d}z^2]_\omega\right) = \overline{\mathrm{D}} \left(g^*[q \mathrm{d}z^2]_\omega\right).$$

Proof. First, we compute

$$\begin{aligned} g^* \left(\overline{\mathrm{D}}[q \mathrm{d}z^2]_\omega\right) &= g^* \left(\overline{Z}^3 q \mathrm{d}\overline{z} \wedge \omega \otimes \omega\right) \\ &= \overline{Z}^3 q(g) \overline{Zg_1} |Zg_1|^4 \mathrm{d}\overline{z} \wedge \omega \otimes \omega. \end{aligned}$$

Then, compute

$$\begin{aligned} \overline{\mathrm{D}} \left(g^*[q \mathrm{d}z^2]_\omega\right) &= \overline{\mathrm{D}}[(q \circ g) (Zg_1)^2 \mathrm{d}z^2]_\omega \\ &= \overline{Z}^3 \left((q \circ g) (Zg_1)^2\right) \mathrm{d}\overline{z} \wedge \omega \otimes \omega. \end{aligned}$$

However, using formulas given in section 3.3.1, we have:

$$\overline{Z} \left((f \circ g) (Zg_1)^2\right) = \overline{Z} f(g) Zg_1 |Zg_1|^2 + 4i(f \circ g) Zg_1 Tg_1.$$

Thus,

$$\begin{aligned}\overline{Z}^2\left((f\circ g)\left(Zg_1\right)^2\right) &= \overline{Z}^2 f(g)|Zg_1|^4 - 2i\overline{Z}f(g)Tg_1|Zg_1|^2\\ &\quad +2i\overline{Z}f(g)Tg_1|Zg_1|^2 + 4i\overline{Z}f(g)Tg_1|Zg_1|^2\\ &\quad -8(f\circ g)\left(Tg_1\right)^2\\ &= \overline{Z}^2 f(g)|Zg_1|^4 + 4i\overline{Z}f(g)Tg_1|Zg_1|^2 - 8(f\circ g)\left(Tg_1\right)^2\end{aligned}$$

and finally,

$$\begin{aligned}\overline{Z}^3\left((f\circ g)\left(Zg_1\right)^2\right) &= \overline{Z}^3 f(g)\overline{Zg_1}|Zg_1|^4 - 4i\overline{Z}^2 f(g)|Zg_1|^2 Tg_1\overline{Zg_1}\\ &\quad +4i\overline{Z}^2 f(g)|Zg_1|^2 Tg_1\overline{Zg_1} + 8\overline{Z}f(g)\left(Tg_1\right)^2\overline{Zg_1}\\ &\quad -8\overline{Z}f(g)\left(Tg_1\right)^2\overline{Zg_1}\\ &= \overline{Z}^3 f(g)\overline{Zg_1}|Zg_1|^4.\end{aligned}$$

Which proves the lemma. □

Now, the result follows from the lemma and the fact that

$$g_{j,i}^*[q_j \mathrm{d}z^2]_\omega = [q_i \mathrm{d}z^2]_\omega$$

since q is a quadratic differential on M. □

Thus, we can define:

Definition 3.3.6 (CR quadratic diffenrential)**.** A CR quadratic differential on a spherical CR manifold M is a quadratic differential q on M satisfying $\overline{\mathrm{D}}q = 0$.
If $(U_i, \varphi_i : U_i \longrightarrow U_i')$ is a spherical CR atlas of M, a CR quadratic differential on M is a collection of functions $q_i \in C^\infty(U_i')$ satisfying:

— the cocycle relation

$$\forall i,j,\ q_i = (q_j \circ g_{j,i})\left(Zg_{j,i}^1\right)^2 \text{ on } \varphi_i(U_i\cap U_j) \text{ where } g_{j,i} = \varphi_j\circ\varphi_i^{-1}$$

— and the *CR condition*

$$\forall i,\ \overline{Z}^3 q_i = 0.$$

Remark 3.3.7. 1. If $\alpha = (f_i)_i$ is a CR form on a spherical CR manifold M, then $\alpha^2 := (f_i^2)_i$ is a CR quadratic differential on M.

2. If M est uniformizable, then we can write $M = {}^{\Omega}\!/_{\Gamma}$ for an open subset Ω of $\mathcal{H}$ and Γ a discrete subgroup of $\mathrm{PU}(2,1)$ acting freely and properly discontinuously on Ω. The space of CR quadratic differentials on M identifies to the space of Γ-invariant CR quadratic differentials on Ω. This space will be denoted $\mathcal{Q}^{\Gamma}(\Omega)$. In other words, the space of CR quadratic differentials on M identifies to

$$\mathcal{Q}^{\Gamma}(\Omega) = \{q \in C^{\infty}(\Omega, \mathbb{C}) \mid \forall \gamma = (\gamma_1, \gamma_2) \in \Gamma, (q \circ \gamma)\,(Z\gamma_1)^2 = q \text{ and } \overline{Z}^3 q = 0\}.$$

Let us explain how the operator $\overline{\mathrm{D}}$ is a natural extension of d'' (on functions) and D'' (on $(1,0)$-forms) in the case of a spherical CR manifold M. For $k \in \mathbb{N}$, define k-th order differential on a spherical CR manifold as a section of the line bundle $(\wedge^{1,0})^k$ over M (when $k = 0$, a k-th order differential is a function on M). On an open subset U of $\mathcal{H}$ a k-th order differential is the class modulo ω of a form $\alpha \mathrm{d}z^k$ with $\alpha \in C^{\infty}(U)$, we write it $[\alpha \mathrm{d}z^k]_{\omega}$. If $(U_i, \varphi_i : U_i \longrightarrow U_i')$ is a spherical CR atlas of M, then a k-th order can be seen as a collection of functions $\alpha_i \in C^{\infty}(U_i')$ satisfying

$$\alpha_i = (\alpha_j \circ g_{j,i})\left(Zg_{j,i}^1\right)^k \text{ on } \varphi_i(U_i \cap U_j) \text{ where } g_{j,i} = \varphi_j \circ \varphi_i^{-1}.$$

On an open subset U of $\mathcal{H}$, define

$$\begin{array}{rccc} \overline{\mathrm{D}}_k : & C^{\infty}(U, (\wedge^{1,0})^k) & \longrightarrow & C^{\infty}(U, \wedge^{1,1} \otimes A^{k-1}) \\ & [\alpha \mathrm{d}z^k]_{\omega} & \longmapsto & \overline{Z}^{k+1} \alpha \mathrm{d}\overline{z} \wedge \omega \otimes \omega^{k-1} \end{array}$$

For $k = 0$, A^{-1} is the dual bundle of A which is naturally isomorphic to ${}^{\mathbb{C}TM}\!/_{\mathbf{P}}$. Then, we have a bundle isomorphism

$$\Psi : \wedge^{1,1} \otimes {}^{\mathbb{C}TM}\!/_{\mathbf{P}} \longrightarrow \wedge^{0,1}$$

defined by

$$\Psi(\alpha \otimes [X]) = [\iota_X \alpha].$$

So that $\overline{\mathrm{D}}_0 : C^{\infty}(U, (\wedge^{1,0})^0) \longrightarrow C^{\infty}(U, \wedge^{1,1} \otimes A^{-1})$ is identified with $\mathrm{d}'' : C^{\infty}(U) \longrightarrow C^{\infty}(U, \wedge^{0,1})$. Then, with a proof similar to the one of Definition/Proposition 3.3.4,

we can define:

Definition 3.3.8. Let α be a k-th order differential on a spherical CR manifold M and $(U_i, \varphi_i : U_i \longrightarrow U_i')$ a spherical CR atlas of M. For every i, we set

$$\overline{\mathrm{D}}_k\alpha = \varphi_i^*\overline{\mathrm{D}}_k[\alpha_i \mathrm{d}z^k]_\omega \text{ where } \alpha = \varphi_i^*[\alpha_i \mathrm{d}z^k]_\omega \text{ on } U_i.$$

Then, $\overline{\mathrm{D}}_k$ defines an operator from k-th order differentials on M to sections of the line bundle $\wedge^{1,1} \otimes A^{k-1}$ over M.

Thus, on a spherical CR manifold, the operator d'' on functions is $\overline{\mathrm{D}}_0$, the operator D'' on $(1,0)$-forms is $\overline{\mathrm{D}}_1$ and the operator $\overline{\mathrm{D}}$ on quadratic differentials is $\overline{\mathrm{D}}_2$. In that sense, $\overline{\mathrm{D}}$ is a natural extension of d'' and D''.

Now, we wish to give examples which illustrate the fact that the CR condition $\overline{Z}^3 q_i = 0$ can be restrictive.

3.3.3 CR quadratic differentials with symmetries

Heisenberg manifolds

Let L be a lattice in $\mathcal{H}$. We denote $\Delta = P(L)$ where $P(z,t) = z$. Then, Δ is a lattice in $\mathbb{C}$ and P induces a circle bundle $\overline{P} : \mathcal{H}/L \longrightarrow \mathbb{C}/\Delta$. We denote $\mathcal{Q}^\Delta(\mathbb{C})$ the space of Δ-invariant holomorphic quadratic differentials on $\mathbb{C}$. The pull-back by P of an element of $\mathcal{Q}^\Delta(\mathbb{C})$ defines a L-invariant CR quadratic differential on $\mathcal{H}$. We can say a little more than this. Let $\widetilde{\Delta} = P^{-1}(\Delta)$, meaning that

$$\widetilde{\Delta} = (\{0\} \times \mathbb{R}) \cup L$$

and consider

$$\mathcal{Q}^{\widetilde{\Delta}}(\mathcal{H}) = \{q \in C^\infty(\mathcal{H}, \mathbb{C}) \mid \forall \delta \in \widetilde{\Delta},\ \forall p \in \mathcal{H},\ q(\delta p) = q(p) \text{ and } \overline{Z}^3 q = 0\}$$

that is, the space of $\widetilde{\Delta}$-invariant CR quadratic differentials on $\mathcal{H}$.

Proposition 3.3.9. *$\mathcal{Q}^{\widetilde{\Delta}}(\mathcal{H})$ is isomorphic to $\mathcal{Q}^\Delta(\mathbb{C})$.*

Proof. First, it is clear that the pull-back by P of an element of $\mathcal{Q}^\Delta(\mathbb{C})$ is an element of $\mathcal{Q}^{\widetilde{\Delta}}(\mathcal{H})$. Let q be an element of $\mathcal{Q}^{\widetilde{\Delta}}(\mathcal{H})$. Since q is $\widetilde{\Delta}$-periodic, we have $q(z, t+t_0) = q(z, t_0)$ for every z, t, t_0 and so

$$\frac{\partial}{\partial t} q = 0.$$

Meaning that q actually defines a quadratic differential on $\mathbb{C}/\Delta$. We just have to show that it is a holomorphic one.
Since $\overline{Z}^3 q = 0$, we have

$$\frac{\partial^3 q}{\partial \overline{z}^3} = 0.$$

Thus, $\frac{\partial^2 q}{\partial \overline{z}^2}$ is a Δ-periodic holomorphic function. So $\frac{\partial^2 q}{\partial \overline{z}^2}$ is a constant function. Denote

$$\frac{\partial^2 q}{\partial \overline{z}^2} = \lambda.$$

Integrating, we get

$$\frac{\partial}{\partial \overline{z}} q(z) = \lambda \overline{z} + h(z)$$

where h is holomorphic. Since $\frac{\partial^2 q}{\partial z \partial \overline{z}} = h'$ is Δ-periodic, h' is constant. So we have

$$\frac{\partial}{\partial \overline{z}} q(z) = \lambda \overline{z} + \mu z + \nu$$

with $\mu, \nu \in \mathbb{C}$. Take a complex number $w \in \Delta \backslash \{0\}$. since $\frac{\partial}{\partial \overline{z}} q$ is Δ-periodic, we have

$$h(z + w) = h(z) - \lambda \overline{w}.$$

From it, we deduce that

$$\mu = -\lambda \frac{\overline{w}}{w}.$$

This is true for every $w \in \Delta \backslash \{0\}$. In particular, if we take two $\mathbb{R}$ linearly independant complex numbers w_1 and w_2 in Δ, we have

$$\lambda \frac{\overline{w}_1}{w_1} = \lambda \frac{\overline{w}_2}{w_2}$$

which is possible only if $\lambda = 0$. So, $\frac{\partial}{\partial \overline{z}} q$ is constant and doing the same reasoning we deduce that $\frac{\partial}{\partial \overline{z}} q$ vanishes. Therefore, q is a Δ-invariant holomorphic function on $\mathbb{C}$. $\square$

$\mathbb{C}$-fuchsian complex hyperbolic groups

An important class of spherical CR manifolds is given by representations of surface groups in $\mathrm{PU}(2,1)$ (refer to [23, 44] for more about the subject). Let $\mathbb{H}^2_{\mathbb{C}}$ be the complex hyperbolic plane and let us fix a compact surface Σ of genus $g \geq 2$ and

denote π_1 its fundamental group. We are interessed in the space

$$\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/_{\mathrm{SU}(2,1)}$$

where $\mathrm{SU}(2,1)$ acts by conjugaison. A representation $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ is called complex hyperbolic quasifuchsian if ρ is discrete, faithful, convex cocompact and totally loxodromic. The complex hyperbolic quasifuchsian space $Q_{\mathbb{C}}(\Sigma)$ is the subset of $\mathrm{Hom}(\pi_1, \mathrm{SU}(2,1))/_{\mathrm{SU}(2,1)}$ formed by such representations.

In $\mathbb{H}^2_{\mathbb{C}}$, there are two types of totally geodesic submanifolds: complex lines and Lagrangian planes. In the unit ball model, it corresponds (up to isometry) to the following two ways of embedding a disc in $\mathbb{H}^2_{\mathbb{C}}$. The first one is given by $u \longmapsto (u, 0)$ and the second one by $u \longmapsto (\Re(u), \Im(u))$. Thus, it gives two types of complex hyperbolic quasifuchsian representations.

— A homomorphism $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ is said $\mathbb{R}$-Fuchsian if $\rho(\pi_1)$ preserves a Lagrangian plane. In that case, ρ is conjugated to a homomorphism $\rho' : \pi_1 \longrightarrow \mathrm{SO}(2,1) \longrightarrow \mathrm{SU}(2,1)$.

— A homomorphism $\rho : \pi_1 \longrightarrow \mathrm{SU}(2,1)$ is said $\mathbb{C}$-Fuchsian if $\rho(\pi_1)$ preserves a complex line. In that case, ρ is conjugated to a homomorphism $\rho' : \pi_1 \longrightarrow \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1,1)) \longrightarrow \mathrm{SU}(2,1)$.

A $\mathbb{C}$-Fuchsian representation induces then a complex hyperbolic structure on a disc bundle over Σ. Indeed, as said, a complex line in $\mathbb{H}^2_{\mathbb{C}}$ is, up to isometry, $\mathbb{D} \times \{0\}$. So, if $\rho(\pi_1)$ preserves $\mathbb{D} \times \{0\}$, it acts on it by isometries. In other words, the projection $(z, w) \longmapsto z$ induces a disc bundle

$$\mathbb{H}^2_{\mathbb{C}}/_{\rho(\pi_1)} \longrightarrow \mathbb{D}/_{\overline{\rho}(\pi_1)}$$

where $\overline{\rho}$ is the representation associated to ρ, with value in $\mathrm{SU}(1,1)$.

To get from complex hyperbolic geometry to spherical CR geometry, we just have to get to the boundary of $\mathbb{H}^2_{\mathbb{C}}$. For that, we extend the projection $(z, w) \longmapsto z$ on $\partial\mathbb{H}^2_{\mathbb{C}} = S^3$ and we remove from it the preimage of the unit circle. In other words, we consider the projection

$$\begin{array}{ccc} S^3 \setminus (S^1 \times \{0\}) & \longrightarrow & \mathbb{D} \\ (z, w) & \longmapsto & z. \end{array}$$

Since $\mathrm{SU}(2,1)$ acts by CR diffeomorphisms on S^3 and since ρ preserves $S^1 \times \{0\}$,

we obtain a spherical CR structure on the circle bundle over Σ:

$$S^3 \backslash \left(S^1 \times \{0\}\right) \big/ \rho(\pi_1) \longrightarrow \mathbb{D} \big/ \overline{\rho}(\pi_1).$$

In the models of Heisenberg group and upper half-plane, one may check that $(z, w) \longmapsto z$ gives the Korányi map

$$\begin{array}{rccc} \Pi : & \mathcal{H}^* & \longrightarrow & \mathbb{H} \\ & (z,t) & \longmapsto & t + i|z|^2 \end{array}$$

where $\mathcal{H}^* = \mathcal{H} \backslash \{z = 0\}$. Thus, denoting $\Lambda = \rho(\pi_1)$ and $\Gamma = \overline{\rho}(\pi_1) < \mathrm{SL}(2, \mathbb{R})$, Π induces a circle bundle:

$$\overline{\Pi} : \mathcal{H}^* \big/ \Lambda \longrightarrow \mathbb{H} \big/ \Gamma.$$

If we see the $\mathbb{C}$-fuchsian group Λ as a group of CR diffeomorphisms of $\mathcal{H} \cup \{\infty\}$, we can describe the associated Fuchsian group Γ as the group of all isometries γ of $\mathbb{H}$ for which there is an element $\delta \in \Lambda$ such that $\Pi \circ \delta = \gamma \circ \Pi$. We can also begin with a Fuchsian group Γ in order to define a $\mathbb{C}$-Fuchsian group by choosing for each element γ of Γ a unique lift by Π and doing it so that we obtain a group. In practice, if we have a system of generators $(\gamma_1, \cdots, \gamma_{2g})$ of Γ, we juste have to choose, for each γ_i, one (and only one) lift by Π denoted $\widetilde{\gamma}_i$. Then, the group generated by the $\widetilde{\gamma}_i$ is $\mathbb{C}$-Fuchsian. A Fuchsian group Γ can also be used in order to define the *lifted group of* Γ, $\widetilde{\Gamma} = \{\widetilde{\gamma},\ \text{CR diffeomorphism of } \mathcal{H} \cup \{\infty\} \mid \Pi \circ \widetilde{\gamma} = \gamma \circ \Pi,\ \text{with } \gamma \in \Gamma\}$.

For the following, let Λ be a $\mathbb{C}$-Fuchsian group, Γ its associated Fuchsian group and $\widetilde{\Gamma}$ the lifted group of Γ. Let q be a holomorphic quadratic differential on $\mathbb{H} \big/ \Gamma$. Up to universal cover, we can see q as a holomorphic function $q : \mathbb{H} \longrightarrow \mathbb{C}$ such that $(q \circ \gamma)(\gamma')^2 = q$ for every $\gamma \in \Gamma$, we denote the space of these functions $\mathcal{Q}^{\Gamma}(\mathbb{H})$. We define the pull-back by Π of q by:

$$\begin{array}{rccc} Q : & \mathcal{H}^* & \longrightarrow & \mathbb{C} \\ & (z,t) & \longmapsto & -4\overline{z}^2 q(t + i|z|^2). \end{array}$$

(The "$-4\overline{z}^2$" corresponds just to $(Z\Pi)^2$.)

It is clear that $\overline{Z}^3 Q = 0$. Moreover, if $\widetilde{\gamma} = (\gamma_1, \gamma_2) \in \widetilde{\Gamma}$, then $(Q \circ \widetilde{\gamma})(Z\gamma_1)^2 = Q$. Indeed, let $\widetilde{\gamma} = (\gamma_1, \gamma_2) \in \widetilde{\Gamma}$ and denote γ the element of Γ such that $\Pi \circ \widetilde{\gamma} = \gamma \circ \Pi$. Simple computation shows that $2i\overline{\gamma}_1 Z\gamma_1 = Z(\Pi \circ \widetilde{\gamma}) = 2i\overline{z}(\gamma' \circ \Pi)$. So,

$$(Q \circ \widetilde{\gamma})(Z\gamma_1)^2 = (q \circ \Pi \circ \widetilde{\gamma})(2i\overline{\gamma}_1 Z\gamma_1)^2 = (q \circ \gamma \circ \Pi)(2i\overline{z}(\gamma' \circ \Pi))^2 = -4\overline{z}^2 (q \circ \Pi).$$

Then, denote $\mathcal{Q}^{\Lambda}(\mathcal{H}^*)$ the space of Λ-invariant CR quadratic differentials on $\mathcal{H}^*$ and $\mathcal{Q}^{\widetilde{\Gamma}}(\mathcal{H}^*)$ the space of $\widetilde{\Gamma}$-invariant CR quadratic differentials on $\mathcal{H}^*$. Since Λ is a subgroup of $\widetilde{\Gamma}$, $\mathcal{Q}^{\widetilde{\Gamma}}(\mathcal{H}^*) \subset \mathcal{Q}^{\Lambda}(\mathcal{H}^*)$. The pull-back by Π defines a natural injection from $\mathcal{Q}^{\Gamma}(\mathbb{H})$ to $\mathcal{Q}^{\widetilde{\Gamma}}(\mathcal{H}^*) \subset \mathcal{Q}^{\Lambda}(\mathcal{H}^*)$. We will show, as in the case of Heisenberg lattices, that $\mathcal{Q}^{\Gamma}(\mathbb{H})$ and $\mathcal{Q}^{\widetilde{\Gamma}}(\mathcal{H}^*)$ are isomorphic.

Let's begin with holomorphic forms. Denote $K^{\Gamma}(\mathbb{H})$ the space of Γ-invariant holomorphic forms on $\mathbb{H}$ and $K^{\widetilde{\Gamma}}(\mathcal{H}^*)$ the space of $\widetilde{\Gamma}$-invariant CR differentials on $\mathcal{H}^*$. In other words,

$$K^{\Gamma}(\mathbb{H}) := \{\beta : \mathbb{H} \longrightarrow \mathbb{C} \text{ holomorphic } \mid (\beta \circ \gamma)\gamma' = \beta, \forall \gamma \in \Gamma\} \text{ and}$$

$$K^{\widetilde{\Gamma}}(\mathcal{H}^*) := \{\alpha : \mathcal{H}^* \longrightarrow \mathbb{C} \text{ smooth} \mid \overline{Z}^2\alpha = 0 \text{ and } (\alpha \circ \widetilde{\gamma})(Z\gamma_1) = \alpha, \\ \forall \widetilde{\gamma} = (\gamma_1, \gamma_2) \in \widetilde{\Gamma}\}.$$

Proposition 3.3.10. *$K^{\Gamma}(\mathbb{H})$ and $K^{\widetilde{\Gamma}}(\mathcal{H}^*)$ are isomorphic.*

Proof. If $\beta \in K^{\Gamma}(\mathbb{H})$, we define the pull-back by Π of β:

$$\begin{array}{rccc} \Pi^*\beta : & \mathcal{H}^* & \longrightarrow & \mathbb{C} \\ & (z,t) & \longmapsto & 2i\overline{z}\beta(t + i|z|^2). \end{array}$$

One can easily verify that Π^* is an injective morphism from $K^{\Gamma}(\mathbb{H})$ to $K^{\widetilde{\Gamma}}(\mathcal{H}^*)$. We need to show that it is also surjective.
Let $\alpha \in K^{\widetilde{\Gamma}}(\mathcal{H}^*)$. Then, for all $\widetilde{\gamma} = (\gamma_1, \gamma_2) \in \widetilde{\Gamma}$, $(\alpha \circ \widetilde{\gamma})Z\gamma_1 = \alpha$. In particular, if $(z,t) \in \mathcal{H}^*$ and $\widetilde{\gamma}$ is the rotation around the vertical axis with angle $arg(z)$, we get:

$$\alpha(z,t) = \alpha(|z|,t)\frac{\overline{z}}{|z|}.$$

Consequently, there is a smooth function $\beta : \mathbb{H} \longrightarrow \mathbb{C}$ such that

$$\alpha(z,t) = 2i\overline{z}\beta(t + i|z|^2)$$

for all $(z,t) \in \mathcal{H}^*$. We just have to show that $\beta \in K^{\Gamma}(\mathbb{H})$.
First, by hypothesis on α, for every $\gamma \in \Gamma$, we have

$$(\beta \circ \gamma)\gamma' = \beta.$$

Meaning that β defines a smooth $(1,0)$-form on $\mathbb{H}/\Gamma$. We need to show that β is holomorphic. For that, denote $\delta = \overline{\partial}\beta$. Then, δ is a d-exact $(1,1)$-form on $\mathbb{H}/\Gamma$. Since $\mathbb{H}/\Gamma$ is a closed manifold, Stokes theorem ensures that

$$\int_{\mathbb{H}/\Gamma} \delta = 0.$$

But, from $\overline{Z}^2\alpha = 0$, we deduce that

$$\frac{\partial}{\partial \overline{w}}\left((\Im(w))^2 \frac{\partial \beta}{\partial \overline{w}}\right) = 0.$$

Moreover, for every $\gamma \in \Gamma$, we have

$$(\Im(\gamma))^2 \left(\left(\frac{\partial \beta}{\partial \overline{w}}\right) \circ \gamma\right) = (\Im(w))^2 \left|\gamma'\right|^2 \left(\left(\frac{\partial \beta}{\partial \overline{w}}\right) \circ \gamma\right) = ((\Im(w))^2 \frac{\partial \beta}{\partial \overline{w}}.$$

Thus, $(\Im(w))^2 \frac{\partial \beta}{\partial \overline{w}}$ defines a holomorphic function on $\mathbb{H}/\Gamma$, holomorphic function which is necessarily constant. So, there is a complex number c such that $\delta = c\omega$ where ω is the hyperbolic volume form on $\mathbb{H}/\Gamma$. Since the integral of δ on $\mathbb{H}/\Gamma$ vanishes, δ vanishes identically. In other words, β is a holomorphic form on $\mathbb{H}/\Gamma$. $\square$

Proposition 3.3.11. *$\mathcal{Q}^{\Gamma}(\mathbb{H})$ and $\mathcal{Q}^{\widetilde{\Gamma}}(\mathcal{H}^*)$ are isomorphic.*

Proof. The proof is similar to the one of the previous proposition. As said, the pull-back by Π defines an injective morphism from $\mathcal{Q}^{\Gamma}(\mathbb{H})$ to $\mathcal{Q}^{\widetilde{\Gamma}}(\mathcal{H}^*)$.

Take $Q \in \mathcal{Q}^{\widetilde{\Gamma}}(\mathcal{H}^*)$. Using the same reasoning as in the previous proposition, there is a smooth function $q : \mathbb{H} \longrightarrow \mathbb{C}$ such that $Q = (Z\Pi)^2(q \circ \Pi)$ and $(q \circ \gamma)(\gamma')^2 = q$ for every $\gamma \in \Gamma$. We just have to show that q is holomorphic. First, remark that $(\Im(w))^2 \frac{\partial q}{\partial \overline{w}}$ defines a holomorphic form on $\mathbb{H}/\Gamma$. Indeed, since $(q \circ \gamma)(\gamma')^2 = q$ for every $\gamma \in \Gamma$, differentiating with respect to $\overline{w}$, we get

$$\frac{\partial q}{\partial \overline{w}} = \left(\frac{\partial q}{\partial \overline{w}} \circ \gamma\right) \gamma' |\gamma'|^2$$

for all $\gamma \in \Gamma$. Thus,

$$(\Im(\gamma))^2 \left(\frac{\partial q}{\partial \overline{w}} \circ \gamma\right) \gamma' = (\Im(w))^2 \frac{\partial q}{\partial \overline{w}}$$

for all $\gamma \in \Gamma$. In other words, $(\Im(w))^2 \frac{\partial q}{\partial \overline{w}}$ defines a $(1,0)$-form one $\mathbb{H}/\Gamma$. Moreover,

from $\overline{Z}^3 Q = 0$, we deduce, with an easy calculus, that

$$\frac{\partial^3}{\partial \overline{w}^3}\left((\Im(w))^2 q\right) = 0.$$

So, there are three folomorphic functions h, j and k such that

$$q(w) = \frac{\overline{w}^2 h(w) + \overline{w} k(w) + j(w)}{(\Im(w))^2}$$

for all w. We compute then

$$\frac{\partial}{\partial \overline{w}}\left((\Im(w))^2 \frac{\partial q}{\partial \overline{w}}\right) = -\frac{w^2 h + wk + j}{2(\Im(w))^2}$$

But, $\frac{\partial}{\partial \overline{w}}\left((\Im(w))^2 \frac{\partial q}{\partial \overline{w}}\right)$ and $\frac{-1}{2(\Im(w))^2}$ define $(1,1)$-forms on $\mathbb{H}/\Gamma$. So $w^2 h + wk + j$ defines a holomorphic function on $\mathbb{H}/\Gamma$. Thus,

$$\frac{\partial}{\partial \overline{w}}\left((\Im(w))^2 \frac{\partial q}{\partial \overline{w}}\right) = \frac{c}{(\Im(w))^2}.$$

Since $\frac{\partial}{\partial \overline{w}}\left((\Im(w))^2 \frac{\partial q}{\partial \overline{w}}\right)$ defines an exact $(1,1)$-form, Stokes theorem implies that $c = 0$. Consequently, $(\Im(w))^2 \frac{\partial q}{\partial \overline{w}}$ defines a holomorphic form on $\mathbb{H}/\Gamma$.

Assume know that this form isn't 0. Then $\frac{q}{(\Im(w))^2 \frac{\partial q}{\partial \overline{w}}}$ is a $(1,0)$-form on $\mathbb{H}/\Gamma \backslash \mathcal{Z}\left(\frac{\partial q}{\partial \overline{w}}\right)$ where $\mathcal{Z}\left(\frac{\partial q}{\partial \overline{w}}\right)$ is the set of zeros of $\frac{\partial q}{\partial \overline{w}}$. Consequently,

$$\frac{1}{(\Im(w))^2} = \frac{\partial}{\partial \overline{w}}\left(\frac{q}{(\Im(w))^2 \frac{\partial q}{\partial \overline{w}}}\right)$$

is an exact $(1,1)$-form on $\mathbb{H}/\Gamma \backslash \mathcal{Z}\left(\frac{\partial q}{\partial \overline{w}}\right)$. Using Stokes theorem, we have

$$\int_{\mathbb{H}/\Gamma} \omega = \int_{(\mathbb{H}/\Gamma)\backslash \mathcal{Z}\left(\frac{\partial q}{\partial \overline{w}}\right)} \omega = 0$$

which is a contradiction. So $\frac{\partial q}{\partial \overline{w}} = 0$. $\square$

3.4 Quasiconformal mappings preserving trajectories of CR quadratic differentials

As an end, we provide examples of quasiconformal maps between domains of $\mathcal{H}$ minimizing the maximal distortion or a mean distortion in a classes of quasiconformal maps and explicit the CR quadratic differentials involved in those examples. Before getting to the examples however, we give the following well-known theorem we shall use (see e.g. [7, p. 177]).

Theorem 3.4.1. *Let $f : \Omega \longrightarrow \Omega'$ be a quasiconformal map between domains in $\mathcal{H}$ and Γ a family of legendrian curves in Ω. Then*

1. *for all $\widetilde{\rho} \in \mathrm{adm}(f(\Gamma))$,*

$$M(\Gamma) \leq \int_{\Omega}' K(f^{-1}(x), f)^2 \widetilde{\rho}^4(x) \mathrm{d}L^3(x),$$

2. *for all $\rho \in \mathrm{adm}(\Gamma)$,*

$$M(f(\Gamma)) \leq \int_{\Omega} K(p, f)^2 \rho^4(p) \mathrm{d}L^3(p),$$

3. *and so*

$$\frac{1}{K_f^2} M(\Gamma) \leq M(f(\Gamma)) \leq K_f^2 M(\Gamma).$$

First of all, we need to define:

Definition 3.4.2 (Trajectories of a CR quadratic differential)**.** Let M be a spherical CR manifold and q a CR quadratic differential on M. Let $\gamma : I \longrightarrow M$ be a parametrized legendrian curve. γ is called:

— a horizontal trajectory for q if, for every $s \in I$, $q(\gamma'(s)) > 0$

— a vertical trajectory for q if, for every $s \in I$, $q(\gamma'(s)) < 0$.

Definition 3.4.3. Let $f : M \longrightarrow N$ be a quasiconformal map between two spherical CR manifolds. Let q be a CR quadratic differential on M and q' a CR quadratic differential on N. We say that f preserves horizontal (resp. vertical) trajectories of (q, q') if for every horizontal (resp. vertical) trajectory γ for q, $f \circ \gamma$ is a horizontal (resp. vertical) trajectory for q'.

We say that f preserves trajectories of (q, q') if it preserves both horizontal and vertical trajectories of (q, q').

Example 3.4.4. *First, we begin with an example of a quasiconformal map between domains of $\mathcal{H}$ with constant distortion. This example was given in* [7, p. 171] *and is constructed using the projection $P(z,t) = z$. Consider a rectangle $R_{a,b} = \{z \in \mathbb{C} \mid 0 < \Re(z) < a,\ 0 < \Im(z) < b\}$ with $a, b > 0$ foliated by vertical lines*

$$\delta_x(s) = x + is$$

for $x \in]0, a[$ and $s \in]0, b[$. Lifting (by P) such a vertical line to a legendrian curve in $\mathcal{H}$ gives a 1-parameter family of curves

$$\widetilde{\delta}_{x,t}(s) = (x + is, t - \Im((x + is)^2)).$$

In this way, we obtain a bounded domain

$$\Omega = \{(z, t - \Im(z^2)) \in \mathcal{H} \mid z \in R_{a,b},\ 0 < t < c\}$$

where $c > 0$. The curves $\widetilde{\delta}_{x,t}$ are vertical trajectories for the CR quadratic differential

$$q = [\mathrm{d}z^2]_\omega.$$

Horizontal trajectories for q lying in Ω are the curves

$$\widetilde{\delta}_{y,t}(s) = (s + iy, t + 2sy)$$

for $s \in]0, a[$, $y \in]0, b[$ and $t + 4ys \in]0, c[$. Consider another domain

$$\Omega' = \{(z, t - \Im(z^2)) \in \mathcal{H} \mid z \in R_{a',b'},\ 0 < t < c'\}$$

with $a', b', c' > 0$ such that

$$\frac{c'}{c} = \frac{a'b'}{ab}.$$

For $y \geq 0$, denote

$$\partial\Omega_y = \{(z, t - \Im(z^2)) \mid \Im(z) = y\}$$

and consider the class $\mathcal{F}$ of all quasiconformal mappings $f : \Omega \longrightarrow \Omega'$ which extend homeomorphically to the boundary with

$$f(\partial\Omega_0) = \partial\Omega'_0 \text{ and } f(\partial\Omega_b) = \partial\Omega'_{b'}.$$

Then, the map f_0 defined by

$$f_0(x+iy,t) = \left(\frac{a'}{a}x + i\frac{b'}{b}y, \frac{a'b'}{ab}t\right)$$

is a quasiconformal map in $\mathcal{F}$ which minimizes the maximal distortion in $\mathcal{F}$. Moreover, f_0 maps the vertical trajectory $s \longmapsto \widetilde{\delta}_{x,t}(s)$ to the vertical trajectory $s \longmapsto \widetilde{\delta}_{\frac{a'}{a}x,\frac{a'b'}{ab}t}(\frac{b'}{b}s)$ and the horizontal trajectory $s \longmapsto \widetilde{\gamma}_{y,t}(s)$ to the horizontal trajectory $s \longmapsto \widetilde{\gamma}_{\frac{b'}{b}y,\frac{a'b'}{ab}t}(\frac{a'}{a}s)$ which is what is expected of a Teichmüller homeomorphism.

Example 3.4.5. *The second example is the counterpart of the previous example using the Korányi map $\Pi(z,t) = t + i|z|^2$. Consider again a rectangle $R_{a,b} = \{z \in \mathbb{C} \mid 0 < \Re(z) < a,\ 0 < \Im(z) < b\}$ with $a, b > 0$ foliated by vertical lines*

$$\delta_x(s) = x + is$$

for $x \in]0,a[$ and $s \in]0,b[$. Lifting such a vertical line to a legendrian curve in $\mathcal{H}$ gives the 1-parameter family of cylindrical radii:

$$\widetilde{\delta}_{z,t}(s) = (sz,t)$$

with $|z| = 1$, $0 < t < a$ and $0 < s < \sqrt{b}$. For $0 < c < 2\pi$, consider then the domain

$$C = \{(z,t) \in \mathcal{H} \mid t + i|z|^2 \in R_{a,b},\ 0 < \arg(z) < c\}.$$

We specify that arguments are taken in $[0, 2\pi[$. The curves $\widetilde{\delta}_{z,t}$ for $|z| = 1$, $0 < \arg(z) < c$ and $t \in]0,a[$ are the vertical trajectories of the CR quadratic differential

$$q = [-4\overline{z}^2 \mathrm{d}z^2]_\omega$$

on C. Horizontal trajectories for q lying in C are

$$\widetilde{\gamma}_z(s) = \left(ze^{-i\frac{s}{2|z|^2}}, s\right)$$

for $0 < |z| < \sqrt{b}$, $0 < s < a$ and $0 < \arg\left(ze^{-i\frac{s}{2|z|^2}}\right) < c$. Consider another domain

$$C' = \{(z,t) \in \mathcal{H} \mid t + i|z|^2 \in R_{a',b'},\ 0 < \arg(z) < c'\}$$

with $a', b' > 0$, $0 < c' < 2\pi$ *such that*

$$\frac{bc}{a} = \frac{b'c'}{a'}.$$

Denote

$$\partial C_y = \{(z,t) \in \overline{C} \mid |z| = \sqrt{y}\}$$

and consider the class $\mathcal{F}$ *of all quasiconformal maps* $f : C \longrightarrow C'$ *which extend homeomorphically to the boundary with*

$$f(\partial C_0) = \partial C'_0 \text{ and } f(\partial C_b) = \partial C'_{b'}.$$

Then, the map f_0 *defined by*

$$f_0(z,t) = \left(\sqrt{\frac{b'}{b}}|z|e^{i\frac{a'b}{ab'}\arg(z)}, \frac{a'}{a}t\right)$$

is a quasiconformal map in $\mathcal{F}$ *which minimizes the maximal distortion in* $\mathcal{F}$*. Indeed, let* Δ_0 *be the family of curves* $\widetilde{\delta}_{z,t} :]0, \sqrt{b}[\longrightarrow C$ *with* $|z| = 1$, $0 < t < a$*. Then, one can compute*

$$M(\Delta_0) = \frac{8ac}{27b}$$

with extremal density

$$\rho_0(z,t) = \frac{2}{3b^{\frac{1}{3}}|z|^{\frac{1}{3}}}.$$

Moreover, let Δ *be the family of all legendrian curves connecting* ∂C_0 *and* ∂C_b*. Then, since* $\rho_0 \in \mathrm{adm}(\Delta)$*, it is extremal for* Δ*. Thus,*

$$M(\Delta) = \frac{8ac}{27b}.$$

Denote Δ' *the corresponding family in* C'*. Then, by hypothesis on* $\mathcal{F}$*, for every* $f \in \mathcal{F}$ *one has*

$$f(\Delta) = \Delta'$$

and the third point in Theorem 3.4.1 gives for every $f \in \mathcal{F}$

$$\frac{1}{K_f^2}\frac{8ac}{27b} \leq \frac{8a'c'}{27b'} \leq K_f^2\frac{8ac}{27b}.$$

Thus, since $\frac{bc}{a} = \frac{b'c'}{a'}$*, one has, for every* $f \in \mathcal{F}$*,*

$$K_f^2 \geq \max\left(\left(\frac{a'b}{ab'}\right)^2, \left(\frac{ab'}{a'b}\right)^2\right) = K_{f_0}^2.$$

Meaning precisely that f_0 *minimizes the maximal distortion in* $\mathcal{F}$*.*
Moreover, f_0 *maps the vertical trajectory* $s \longmapsto \widetilde{\delta}_{z,t}(s)$ *to the vertical trajectory* $s \longmapsto \widetilde{\delta}_{z',t'}\left(\sqrt{\frac{b'}{b}}s\right)$ *with* $z' = z^{\frac{a'b}{ab'}}$ *and* $t' = \frac{a'}{a}t$ *and the horizontal trajectory* $s \longmapsto \widetilde{\gamma}_z(s)$ *to the horizontal trajectory* $s \longmapsto \widetilde{\gamma}_{z'}\left(\frac{a'}{a}s\right)$ *with* $z' = \sqrt{\frac{b'}{b}}|z|e^{i\frac{a'b}{ab'}\arg(z)}$*.*

The next examples are those from chapter 2.

Consider two (interior of) quadrilaterals Ω and Ω' in $\mathbb{H}$ and write the holomorphic maps $\phi : R_{a,b} \longrightarrow \Omega$ and $\psi : R_{a',b'} \longrightarrow \Omega'$ where $R_{a,b}$ and $R_{a',b'}$ are rectangles as in the previous examples. On these domains, define the holomorphic quadratic differentials

$$q_\phi = \phi_* \mathrm{d}w^2 \text{ on } \Omega \text{ and } q_\psi = \psi_* \mathrm{d}w^2 \text{ on } \Omega'.$$

Horizontal trajectories for q_ϕ (resp. for q_ψ) are

$$s \longmapsto \gamma_y^\phi(s) = \phi(s+iy) \ \big(\text{resp. } s \longmapsto \gamma_y^\psi(s) = \psi(s+iy)\big)$$

for $y \in]0,b[$ and $s \in]0,a[$ (resp. $y \in]0,b'[$ and $]0,a'[$) and denote

$$\Gamma_\phi = \{\gamma_y^\phi \mid y \in]0,b[\} \text{ and } \Gamma_\psi = \{\gamma_y^\psi \mid y \in]0,b'[\}.$$

Then, consider the domains in $\mathcal{H}$, $\widetilde{\Omega} = \Pi^{-1}(\Omega)$ and $\widetilde{\Omega}' = \Pi^{-1}(\Omega')$, where $\Pi(z,t) = t + i|z|^2$, endowed with the CR quadratic differentials

$$\widetilde{q}_\phi = \Pi^* q_\phi \text{ and } \widetilde{q}_\psi = \Pi^* q_\psi.$$

Horizontal trajectories for $\widetilde{q}_\phi$ are

$$s \longmapsto \widetilde{\gamma}_{y,\alpha}^\phi(s) = \left(\sqrt{\Im\left(\phi(s+iy)\right)}\alpha e^{i\tau_y(s)}, \Re\left(\phi(s+iy)\right)\right)$$

for $\tau_y(s) = -\int_0^s \frac{\Re\left(\frac{\partial\phi}{\partial x}(x+iy)\right)}{2\Im(\phi(x+iy))}\mathrm{d}x$, $|\alpha| = 1$, $y \in]0,b[$, $s \in]0,a[$ and horizontal trajectories for $\widetilde{q}_\psi$ are

$$s \longmapsto \widetilde{\gamma}_{y,\alpha}^\psi(s) = \left(\sqrt{\Im\left(\psi(s+iy)\right)}\alpha e^{i\sigma_y(s)}, \Re\left(\psi(s+iy)\right)\right)$$

for $\sigma_y(s) = -\int_0^s \frac{\Re\left(\frac{\partial\psi}{\partial x}(x+iy)\right)}{2\Im(\psi(x+iy))}\mathrm{d}x$, $|\alpha| = 1$, $y \in]0, b'[$, $s \in]0, a'[$. Then, denote

$$\widetilde{\Gamma}_\phi = \{\widetilde{\gamma}^\phi_{y,\alpha} \mid y \in]0, b[,\ |\alpha| = 1\} \text{ and } \widetilde{\Gamma}_\psi = \{\widetilde{\gamma}^\psi_{y,\alpha} \mid y \in]0, b'[,\ |\alpha| = 1\}.$$

Now, let $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ be a quasiconformal map and $\lambda > 0$. We say that f *λ-dilates horizontal trajectories of* $(\widetilde{q}_\phi, \widetilde{q}_\psi)$ if for every $(y, \alpha) \in]0, b[\times S^1$, there is $(y', \alpha') \in]0, b'[\times S^1$ such that

$$\forall s \in]0, a[,\ f\left(\widetilde{\gamma}^\phi_{y,\alpha}(s)\right) = \widetilde{\gamma}^\psi_{y',\alpha'}(\lambda s).$$

Moreover, $\rho_\phi : w \longmapsto \frac{1}{a}\left|(\phi^{-1})'(w)\right|$ (resp. $\rho_\psi : w \longmapsto \frac{1}{a}\left|(\psi^{-1})'(w)\right|$) is an extremal density for Γ_ϕ (resp. for Γ_ψ) and $\Pi^*\rho_\phi : (z, t) \longmapsto 2|z|\rho_\phi(t + i|z|^2)$ (resp. $\Pi^*\rho_\psi : (z, t) \longmapsto 2|z|\rho_\psi(t + i|z|^2)$) is an element of $\mathrm{adm}\left(\widetilde{\Gamma}_\phi\right)$ $\left(\text{resp. of } \mathrm{adm}\left(\widetilde{\Gamma}_\psi\right)\right)$. Recall that $\widetilde{\Gamma}_\phi$ (resp. $\widetilde{\Gamma}_\psi$) satisfy the *pull-back density condition* if $\Pi^*\rho_\phi$ (resp. $\Pi^*\rho_\psi$) is extremal for $\widetilde{\Gamma}_\phi$ (resp. for $\widetilde{\Gamma}_\psi$). Then, we can state Proposition 2.3.17 as:

Theorem 3.4.6. *Assume that $\widetilde{\Gamma}_\phi$ and $\widetilde{\Gamma}_\psi$ satisfy the pull-back density condition. Let $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ be a C^2 quasiconformal map that $\frac{a'}{a}$-dilates horizontal trajectories of $(\widetilde{q}_\phi, \widetilde{q}_\psi)$. Then, there is a C^2 quasiconformal map $g : \Omega \longrightarrow \Omega'$ such that*

$$\Pi \circ f = g \circ \Pi.$$

Remark 3.4.7. The map g of the above theorem $\frac{a'}{a}$-dilates horizontal trajectories of (q_ϕ, q_ψ). Moreover, according to the lifting theorem (Proposition 2.3.9), g is symplectic with respect to the hyperbolic area form on $\mathbb{H}$.

Remark 3.4.8. We can give a more geometric sufficient condition for the pull-back density condition to be true. If vertical trajectories of q_ϕ are geodesic for the hyperbolic metric on Ω, then $\widetilde{\Gamma}_\phi$ satisfy the pull-back density condition. Indeed, vertical trajectories for q_ϕ are the curves

$$s \longmapsto \delta^\phi_x(s) = \phi(x + is)$$

for $x \in]0, a[$ and $s \in]0, b[$. On the one hand, saying that they are geodesics for the hyperbolic metric on Ω implies, by the geodesic equation, that

$$\frac{\partial^2\phi_1}{\partial y^2} = \frac{2}{\phi_2}\frac{\partial\phi_1}{\partial y}\frac{\partial\phi_2}{\partial y} \tag{3.3}$$

$$\frac{\partial^2\phi_2}{\partial y^2} = \frac{1}{\phi_2}\left(\left(\frac{\partial\phi_2}{\partial y}\right)^2 - \left(\frac{\partial\phi_1}{\partial y}\right)^2\right) \tag{3.4}$$

where ϕ_1 is the real part of ϕ, ϕ_2 its imaginary part and y is the imaginary part on $R_{a,b}$. On the other hand, by Proposition 2.3.5 and using Cauchy-Riemann equations to express partial derivatives of ϕ_1 and ϕ_2 only with respect to y, saying that $\widetilde{\Gamma}_\phi$ satisfy the pull-back density condition is equivalent to

$$\frac{\partial\phi_2}{\partial y}\frac{\partial^2\phi_1}{\partial y^2} - \frac{\partial\phi_1}{\partial y}\frac{\partial^2\phi_2}{\partial y^2} = \frac{1}{\phi_2}\frac{\partial\phi_1}{\partial y}\left(\left(\frac{\partial\phi_2}{\partial y}\right)^2 + \left(\frac{\partial\phi_1}{\partial y}\right)^2\right). \tag{3.5}$$

Equations 3.3 and 3.4 clearly imply 3.5 and so the statement is true.

It is not clear if the converse holds.

Remark 3.4.9. We chose not to insist on the previous theorem in chapter 2 because it is a deeper result when expressed with the idea of dilating horizontal trajectories of CR quadratic differentials, especially when it is combined with the lifting theorem. Namely, the combination of Theorem 3.4.6 with the lifting theorem gives the following, taking back the notation $\mathcal{F}_{\phi,\psi}$ from chapter 2. If $\widetilde{\Gamma}_\phi$ and $\widetilde{\Gamma}_\psi$ satisfy the pull-back density condition, then, up to composition with a rotation around the vertical axis, there is

— no quasiconformal map in $\mathcal{F}_{\phi,\psi}$ which dilates horizontal trajectories of $(\widetilde{q}_\phi, \widetilde{q}_\psi)$ or

— exactly one quasiconformal map in $\mathcal{F}_{\phi,\psi}$ which dilates horizontal trajectories of $(\widetilde{q}_\phi, \widetilde{q}_\psi)$ or

— a 1-parameter family of quasiconformal maps in $\mathcal{F}_{\phi,\psi}$ which dilate horizontal trajectories of $(\widetilde{q}_\phi, \widetilde{q}_\psi)$.

This is quite different from the classical case where quasiconformal maps between the quadrilaterals Ω and Ω' which dilate horizontal trajectories of their natural holomorphic quadratic differentials exist at a considerable number.

The next examples are applications of this theorem in order to find all quasiconformal maps in appropriate classes of quasiconformal maps between domains of $\mathcal{H}$ which dilate horizontal trajectories. For cylinders, there is, up to composition with a vertical rotation, a unique quasiconformal map (in an appropriate class of quasiconformal maps) which dilates horizontal trajectories. For cylinders from which a smaller cylinder has been removed, there can be no quasiconformal map (in an appropriate class of quasiconformal maps) which dilates horizontal trajectories. Fi-

nally, for spherical annuli, there is a 2-parameters family of quasiconformal maps (in an appropriate class of quasiconformal maps) which dilate horizontal trajectories. Consequently, the three possibilities listed in the previous remark occur.

Example 3.4.10. *Consider cylinders $C_{a,b}$ and $C_{a',b'}$ where*

$$C_{r,R} = \{(z,t) \in \mathcal{H} \mid |z|^2 < R, t \in]0,r[\},$$

for $a, b, a', b' > 0$ satisfying

$$\frac{ab'}{a'b} > 1$$

and the trajectories of the CR quadratic differential

$$q = [-4\overline{z}^2 \mathrm{d}z^2]_\omega.$$

Vertical trajectories are the cylindrical radii

$$\delta_{z,t}(s) = (sz, t)$$

for $|z| = 1$, $t \in]0, a[$ and $s \in]0, \sqrt{b}[$ and horizontal trajectories are the curves

$$\gamma_z(s) = \left(ze^{-i\frac{s}{2|z|^2}}, s\right)$$

for $0 < |z| < \sqrt{b}$ and $s \in]0, a[$ (see Figure 3.1).

Consider the class $\mathcal{F}$ of quasiconformal maps from $C_{a,b}$ to $C_{a',b'}$ which extend homeomorphically to the boundary and map

- *the disc $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b}$ & $t = 0\}$ to the disc $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b'}$ & $t = 0\}$,*
- *and the disc $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b}$ & $t = a\}$ to the disc $\{(z,t) \in \mathcal{H} \mid |z| \leq \sqrt{b'}$ & $t = a'\}$,*

Let Γ_0 be the family of all horizontal trajectories of q. Then we saw that, the map $\widetilde{f}_0$ defined by

$$\widetilde{f}_0(z,t) = \left(\frac{\sqrt{b'}ze^{\frac{it}{2b}\left(1-\frac{a'b}{ab'}\right)}}{\sqrt{\left(1-\frac{ab'}{a'b}\right)|z|^2 + \frac{ab'}{a'}}}, \frac{a'}{a}t\right)$$

is a quasiconformal map in $\mathcal{F}$ which minimizes the mean distortion in $\mathcal{F}$ for the extremal density of Γ_0 and it is the unique such minimizer up to composition with a rotation around the vertical axis. Moreover, f_0 $\frac{a'}{a}$-dilates horizontal trajectory of q: f sends the horizontal trajectory $s \longmapsto \gamma_z(s)$ to the horizontal trajectory $s \longmapsto \gamma_{z'}\left(\frac{a'}{a}s\right)$

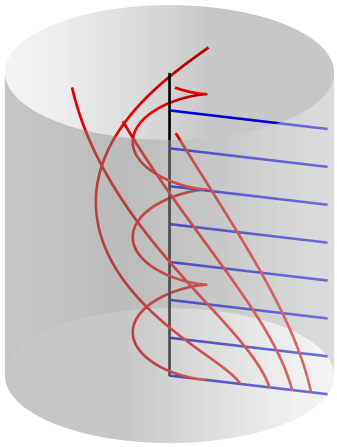

Figure 3.1 – Horizontal and vertical trajectories for the CR quadratic differential $q = [-4\overline{z}^2 \mathrm{d}z^2]_\omega$ lying in a cylinder. Horizontal trajectories are rotations around the vertical axis of the red curves and vertical are rotations around the vertical axis of the blue curves.

with $z' = \frac{\sqrt{b'}z}{\sqrt{\left(1-\frac{ab'}{a'b}\right)|z|^2+\frac{ab'}{a'}}}$. *But it only preserves vertical trajectories of* q*:* f *sends the vertical trajectory* $s \longmapsto \delta_{z,t}(s)$ *to the vertical trajectory* $s \longmapsto \delta_{z',t'}\left(\frac{\sqrt{b'}s}{\sqrt{\left(1-\frac{ab'}{a'b}\right)s^2+\frac{ab'}{a'}}}\right)$ *with* $t' = \frac{a'}{a}t$ *and* $z' = ze^{\frac{it}{2b}\left(1-\frac{a'b}{ab'}\right)}$.

Theorem 3.4.6 says more: up to composition with a vertical rotation, f_0 *is the only* C^2 *quasiconformal map in* $\mathcal{F}$ *which* $\frac{a'}{a}$*-dilates horizontal trajectories of* q.

Indeed, let f *be a quasiconformal map in* $\mathcal{F}$ *which* $\frac{a'}{a}$*-dilates horizontal trajectories of* q*. Then, according to Theorem 3.4.6, there is a quasiconformal map* $g : R_{a,b} \longrightarrow R_{a',b'}$ *such that* $\Pi \circ f = g \circ \Pi$*. Since* f $\frac{a'}{a}$*-dilates horizontal trajectories of* q*, for every* $x + iy \in R_{a,b}$*,*

$$\Re(g(x+iy)) = \frac{a'}{a}x$$

and $\Im(g)$ *depends only on the imaginary part. Then, Proposition 2.3.9 states that* g *must be a symplectomorphism with respect to the hyperbolic area form of the upper half-plane. Thus,* $\varphi = \Im(g)$ *satisfies for every* $x + iy \in R_{a,b}$*,*

$$\frac{a'}{a}\frac{\partial\varphi}{\partial y}(iy)\frac{1}{\varphi(iy)^2} = \frac{1}{y^2}.$$

Solving, we find for every $x+iy \in R_{a,b}$,

$$\varphi(x+iy) = \frac{a'y}{a+a'yC}$$

for a constant C. *Moreover, since* f *sends* $\{(z,t) \in \mathcal{H} \mid |z| = \sqrt{b} \ \& \ 0 \leq t \leq a\}$ *to* $\{(z,t) \in \mathcal{H} \mid |z| = \sqrt{b} \ \& \ 0 \leq t \leq a\}$, *we have*

$$\varphi(ib) = b'$$

and so, $C = b' - \frac{a}{a'b}$. *Thus,* g *is completely determined and so* f *is also determined up to composition with a rotation around the vertical axis (this is a consequence of Proposition 2.3.9). Then,* f *is, up to composition with a rotation around the vertical axis,* $\widetilde{f_0}$.

On the other hand, there is no C^2 *quasiconformal map in* $\mathcal{F}$ *which* $\frac{b'}{b}$*-dilates vertical trajectories of* q. *Indeed, let* f *be such a map and write it in cylindrical coordinates* $r = |z|$, $\theta = \arg(z)$ *and* $t = t$ *as* (R, Θ, T). *Then, the fact that* f *sends vertical trajectories to* $s \longmapsto \delta_{z,t}\left(\sqrt{\frac{b'}{b}}s\right)$ *implies that*

$$\frac{\partial T}{\partial r} = \frac{\partial \Theta}{\partial r} = 0 \text{ and } R = \sqrt{\frac{b'}{b}}r.$$

The contact form ω *in cylindrical coordinates is*

$$\mathrm{d}t + 2r^2\mathrm{d}\theta.$$

Since f *is contact,* R, Θ *and* T *satisfy*

$$\frac{\partial T}{\partial t} + 2\frac{b'}{b}r^2\frac{\partial \Theta}{\partial t} = \frac{1}{2r^2}\frac{\partial T}{\partial \theta} + \frac{b'}{b}\frac{\partial \Theta}{\partial \theta}. \tag{3.6}$$

Then, differentiating it with respect to r *gives*

$$2\frac{b'}{b}r^2\frac{\partial \Theta}{\partial t} = -\frac{1}{2r^2}\frac{\partial T}{\partial \theta}$$

and reporting this expression in 3.6, we obtain

$$\frac{\partial T}{\partial t} + \frac{1}{r^2}\frac{\partial T}{\partial \theta} = \frac{b'}{b}\frac{\partial \Theta}{\partial \theta}$$

leading to

$$\frac{\partial T}{\partial \theta} = 0.$$

This, together with the fact that $\frac{\partial R}{\partial \theta} = 0$ *implies that there is a map* $g : R_{a,b} \longrightarrow R_{a',b'}$ *such that*

$$\Pi \circ f = g \circ \Pi.$$

Since $R = \sqrt{\frac{b'}{b}} r$, *for every* $x + iy \in R_{a,b}$

$$\Im(g(x+iy)) = \frac{b'}{b} y.$$

Again, the lifting theorem implies that g *is a symplectomorphism with respect to the hyperbolic area form of the upper half-plane which leads to*

$$\frac{\partial \Re(g)}{\partial x} = \frac{b'}{b}.$$

Since g *maps* $\{w \in \overline{R_{a,b}} \mid \Re(w) = 0\}$ *on* $\{w \in \overline{R_{a',b'}} \mid \Re(w) = 0\}$, *it must be the dilation with factor* $\frac{b'}{b}$ *which is impossible by the hypothesis* $\frac{ab'}{a'b} > 1$.

Consider two domains

$$D_{a,b} = \{(z,t) \in \mathcal{H} \mid 0 < t < a,\ 1 < |z|^2 < b+1\} \text{ and}$$

$$D_{a',b'} = \{(z,t) \in \mathcal{H} \mid 0 < t < a',\ 1 < |z|^2 < b'+1\}$$

for $a, b, a', b' > 0$ *with*

$$\frac{a(b'+1)}{a'(b+1)} > 1$$

and the trajectories of the CR quadratic differential

$$q = [-4\overline{z}^2 \mathrm{d}z^2]_\omega.$$

Let $\mathcal{F}$ *be the class of quasiconformal maps from* $D_{a,b}$ *to* $D_{a',b'}$ *which extend homeomorphically and map the two boundary annuli to their corresponding ones and the two boundary cylinders to their corresponding ones. Denote* γ_z *the horizontal trajectories of* q. *Then, what was done in Example 2.3.21 can be used to state that a quasiconformal map* $g_0 \in \mathcal{F}$ *which* $\frac{a'}{a}$*-dilates horizontal trajectories of* q *exists if and*

only if

$$\frac{ab}{b+1} = \frac{a'b'}{b'+1}$$

(meaning that the rectangles $\{w \in \mathbb{C} \mid 0 < \Re(w) < a,\ 1 < \Im(w) < b+1\}$ *and* $\{w \in \mathbb{C} \mid 0 < \Re(w) < a',\ 1 < \Im(w) < b'+1\}$ *have the same hyperbolic area). If it is the case, then* g_0 *is, up to composition with a vertical rotation, the restriction to* $D_{a,b}$ *of the map* $\widetilde{f}_0 : C_{a,b+1} \longrightarrow C_{a',b'+1}$ *given previously (with* b *replaced by* $b+1$ *and* b' *by* $b'+1$*).*

Example 3.4.11. *For the last example, consider two spherical annuli* A_a *and* A_{a^k} *for* $a > 1$ *and* $0 < k < 1$*, where*

$$A_r = \{(z,t) \in \mathcal{H} \mid 1 < \|(z,t)\|_{\mathcal{H}} < r\},$$

foliated by radial curves

$$\gamma_{y,\alpha}(s) = \left(\sqrt{e^s \sin(y)}\alpha e^{\frac{is}{2}\cot(y)}, e^s \cos(y)\right)$$

for $|\alpha| = 1$*,* $y \in]0,\pi[$*. These curves are horizontal trajectories for the CR quadratic differential*

$$q = \left[\frac{-4\overline{z}^2}{(t+i|z|^2)^2}\mathrm{d}z^2\right]_\omega$$

and its vertical trajectories are spherical arcs

$$\delta_{x,\alpha}(s) = \left(\sqrt{e^x \sin(s)}\alpha e^{\frac{i}{2}\log(\sin(s))}, e^x \cos(s)\right)$$

for $|\alpha| = 1$*,* $x \in]0, 2\log a[$ *(see Figure 3.2).*

It will be more convenient here to write in logarithmic coordinates introduced in [45]*:*

$$(z,t) = \left(i\cos^{\frac{1}{2}}\psi e^{\frac{\xi+i(\psi-3\eta)}{2}}, -\sin\psi e^{\xi}\right)$$

for $\xi \in \mathbb{R}$*,* $\psi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ *and* $\psi - 3\pi \leq 2\eta < \psi + \pi$*. In those coordinates, up to parametrization, horizontal trajectories of* q *are*

$$\widetilde{\gamma}_{\psi,\eta}(s) = \left(s, \psi, \eta - \frac{\tan\psi}{3}s\right)$$

and its vertical trajectories are

$$\widetilde{\delta}_{\xi,\eta}(s) = (\xi, s, \eta)\,.$$

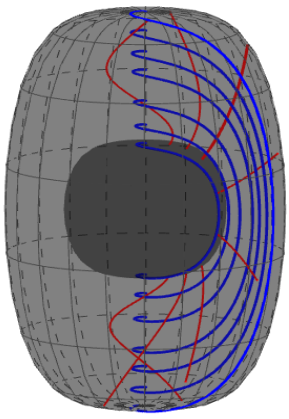

Figure 3.2 – Horizontal and vertical trajectories for the CR quadratic differential $q = \left[\frac{-4\bar{z}^2}{(t+i|z|^2)^2}\mathrm{d}z^2\right]_\omega$ lying in an annulus. Horizontal trajectories are rotations around the vertical axis of the red curves and vertical are rotations around the vertical axis of the blue curves.

Let $\mathcal{F}$ be the class of all quasiconformal maps from A_a to A_{a^k} which extend homeomorphically to the boundary and map the boundary spheres (with respect to $\|.\|_{\mathcal{H}}$) of A_a to the respective boundary spheres of A_{a^k}. Let f_k be the radial stretch map defined in logarithmic coordinates by

$$f_k(\xi, \psi, \eta) = \left(k\xi, \tan^{-1}\left(\frac{\tan\psi}{k}\right), \eta\right).$$

Then, we saw that f_k minimizes the mean distortion in $\mathcal{F}$ for the extremal density of the family of horizontal trajectories of q in A_a and, up to composition with a rotation around the vertical axis, f_k the only such minimizer. Moreover, f_k k-dilates horizontal trajectories of q: f_k maps the horizontal trajectory $s \longmapsto \widetilde{\gamma}_{\psi,\eta}(s)$ to the horizontal trajectory $s \longmapsto \widetilde{\gamma}_{\tan^{-1}\left(\frac{\tan\psi}{k}\right),\eta}(ks)$; and preserves vertical trajectories of q: f_k sends the vertical trajectory $s \longmapsto \widetilde{\delta}_{\xi,\eta}(s)$ to the vertical trajectory $s \longmapsto \widetilde{\delta}_{k\xi,\eta}\left(\tan^{-1}\left(\frac{\tan s}{k}\right)\right)$.

However, contrary to the case of cylinders, composition with rotations around the vertical axis of f_k are not the only quasiconformal maps in $\mathcal{F}$ which k-dilates horizontal trajectories of q. Indeed, for every real number D, the map defined in

logarithmic coordinates by

$$g_D : (\xi, \psi, \eta) \longmapsto \left(k\xi, \tan^{-1}\left(\frac{\tan\psi}{k} + D\right), \eta - \frac{kD\xi}{3} \right)$$

is a quasiconformal map in $\mathcal{F}$ *which sends* $s \longmapsto \widetilde{\gamma}_{\psi,\eta}(s)$ *to* $s \longmapsto \widetilde{\gamma}_{\psi',\eta'}(ks)$*. Up to composition with a vertical rotation, these maps* g_D *are the only* C^2 *quasiconformal maps in* $\mathcal{F}$ *which* k*- dilate horizontal trajectories of* q *(this is a consequence of Theorem 3.4.6).*

On the other hand, as in the case of cylinders, there is no C^2 *quasiconformal map in* $\mathcal{F}$ *which* 1*-dilate vertical trajectories of* q*. The proof of this goes along the same lines as the one for cylinder, except that it is more convenient to use logarithmic coordinates.*

Appendix A

Several proofs left aside in Chapter 2

Proof of Lemma 2.2.2. First, it is a simple verification that maps f_φ minimize the mean distortion in $\mathcal{G}$ for the density ρ_0. Moreover, since those maps satisfy

$$M(\Gamma_0') = \int_{R_{a,b}} K(., f_\varphi)\rho_0^2 \mathrm{d}L^2,$$

every minimizer must also satisfy it. So, with the same argument as in Proposition 2.2.7, if $f \in \mathcal{G}$ minimizes the mean distortion functional, then $f(\Gamma_0) = \Gamma_0'$. So for every s, y,

$$f(s+iy) = \zeta_y(s) + i\varphi(y)$$

for functions ζ_y and φ. Moreover, using a similar argument as the one in Proposition 2.2.8, we find that

$$\zeta_y(s) = \frac{a'}{a}s \text{ for every } s, y.$$

To sum up,

$$f(x+iy) = \frac{a'}{a}x + i\varphi(y) \text{ for every } x+iy \in R_{a,b}.$$

For such a function to be orientation-preserving, $\dot{\varphi} \geq 0$ and, since f sends horizontal boundary components on the corresponding ones, $\varphi(0) = 0$ and $\varphi(b) = b'$. Finally, the minimal stretching property in that case leads to $|\dot{\varphi}| \geq \frac{a'}{a}$. So f is one of the f_φ. □

Proof of Proposition 2.3.9. In this proof, partial derivatives are denoted by an index.

For the first part, let $k : \Omega \longrightarrow \mathbb{C}$ be defined for every $w \in \Omega$ by

$$k(w) = \frac{1}{4\Im(w)} - \frac{(\Re(g))_w(w)}{2\Im(g(w))} \text{ and } \beta = k(w)\mathrm{d}w + \overline{k(w)}\mathrm{d}\overline{w}.$$

Then, using the fact that g is a symplectic map, a calculus shows that β is closed, hence exact by Poincaré lemma. So (since β is a real 1-form) there is a real valued function $h : \Omega \longrightarrow \mathbb{R}$ such that

$$h_w = k \text{ and } h_{\overline{w}} = \overline{k}.$$

Now, it is a simple verification to check that the map f defined in the theorem is a contact map.

For the second part, let $f : \widetilde{\Omega} \longrightarrow \widetilde{\Omega}'$ be a contact lift of g. Here, it will be easier to compute in usual cylindrical coordinates $(r, \theta, t) \longmapsto (re^{i\theta}, t)$. f is then a map (R, Θ, T) with

$$R(r, \theta, t) = \sqrt{\Im(g(t + ir^2))} \text{ and } T(r, \theta, t) = \Re(g(t + ir^2)).$$

The contact form ω is in those coordinates

$$\omega = \mathrm{d}t + 2r^2 \mathrm{d}\theta.$$

So, f is contact if and only if the following three PDEs are satisfied

$$T_r + 2R^2\Theta_r = 0$$

$$T_t + 2R^2\Theta_t = \lambda$$

$$R^2\Theta_\theta = r^2\lambda$$

with λ a nowhere vanishing function. Moreover, for every (r, θ, t),

$$T_r(r, \theta, t) = 2ir\left((\Re(g))_w(t + ir^2) - (\Re(g))_{\overline{w}}(t + ir^2)\right)$$

and

$$T_t(r, \theta, t) = (\Re(g))_w(t + ir^2) + (\Re(g))_{\overline{w}}(t + ir^2).$$

So, we find

$$\Theta_r(r, \theta, t) = \frac{ir}{\Im(g(t + ir^2))}\left((\Re(g))_{\overline{w}}(t + ir^2) - \partial_w(\Re(g))(t + ir^2)\right)$$

and deduce the following two

$$\Theta_{r,\theta} = 0$$

and

$$\Theta_{r,t} = \frac{-ir}{(\Im(g))^2}\left((\Im(g))_w + (\Im(g))_{\overline{w}}\right)\left((\Re(g))_{\overline{w}} - (\Re(g))_w\right) + \frac{ir}{\Im(g)}\left((\Re(g))_{\overline{w},\overline{w}} - (\Re(g))_{w,w}\right).$$

Moreover, we have

$$\Theta_\theta = \frac{r^2}{\Im(g)}\left((\Re(g))_w + (\Re(g))_{\overline{w}}\right) + 2r^2\Theta_t$$

and so, differentiating with respect to r, we find

$$\Theta_t = \frac{r^2 \text{Jac } g}{2(\Im(g))^2} - \frac{(\Re(g))_w + (\Re(g))_{\overline{w}}}{2\Im(g)}.$$

From that, we deduce two things:

$$\Theta_{\theta,t} = 0 \text{ and } \Theta_\theta = r^4\frac{\text{Jac } g}{(\Im(g))^2}.$$

So,

$$\Theta_{\theta,\theta} = 0.$$

Thus, Θ_θ is constant with value 1 (if not so, (R, Θ, T) would not be a homeomorphism). It leads to

$$\frac{\text{Jac } g}{(\Im(g))^2} = \frac{1}{(\Im(w))^2}$$

which ends the proof of the second point.

The third point is simply Lemma 2.1.15.

□

Proof of Proposition 2.3.17. The proof is similar to the one of (the last part of) Theorem 2.2.6. First, by hypothesis, for every $(s, x, \theta) \in]0, a[\times]0, b[\times\mathbb{R}$,

$$S(s, x, \theta) = \frac{a'}{a}s \text{ and } \frac{\mathrm{d}}{\mathrm{d}s}X(s, x, \alpha + \tau(s, x)) = 0.$$

It leads to

$$X_s = \frac{\Re(\phi_s)}{2\Im(\phi)}X_\theta. \tag{A.1}$$

Now, since (S, X, Θ) defines a contact map, there is a nowhere vanishing function λ such that

$$\frac{a'}{a}\Re(\psi_s(S,X)) - \Im(\psi_s(S,X))X_s + 2\Im(\psi(S,X))\Theta_s = \lambda\Re(\phi_s) \tag{A.2}$$

$$\Im(\psi_s(S,X))X_x - 2\Im(\psi(S,X))\Theta_x = \lambda\Im(\phi_s) \tag{A.3}$$

$$-\Im(\psi_s(S,X))X_\theta + 2\Im(\psi(S,X))\Theta_\theta = 2\lambda\Im(\phi). \tag{A.4}$$

From A.4, since $\Im(\phi) > 0$, we find

$$\lambda = \frac{\Im(\psi(S,X))}{\Im(\phi)}\Theta_\theta - \frac{\Im(\psi_s(S,X))}{2\Im(\phi)}X_\theta.$$

Replacing λ by its value in A.2 and A.3, and using A.1, we have the following

$$\Theta_s = \frac{\Re(\phi_s)}{2\Im(\phi)}\Theta_\theta - \frac{a'}{a}\frac{\Re(\psi_s(S,X))}{2\Im(\psi(S,X))} \tag{A.5}$$

$$\Theta_x = \frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_x + \frac{\Im(\phi_s)}{2\Im(\phi)}\frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_\theta - \frac{\Im(\phi_s)}{2\Im(\phi)}\Theta_\theta. \tag{A.6}$$

From now, to follow computations, as a rule, we decide to express all partial derivatives of $\Re(\phi)$, $\Re(\psi)$, $\Im(\phi)$ and $\Im(\psi)$ with respect to s (we do it using Cauchy-Riemann equations). Differentiating A.5 and A.6 both with respect to θ, we find

$$\Theta_{\theta,s} = \frac{\Re(\phi_s)}{2\Im(\phi)}\Theta_{\theta,\theta} - \frac{a'}{a}\left(\frac{\Re(\psi_{s,x}(S,X))}{2\Im(\psi(S,X))} - \frac{\Re(\psi_s(S,X))\Im(\psi_x(S,X))}{2(\Im(\psi(S,X)))^2}\right)X_\theta$$

$$\begin{aligned}\Theta_{\theta,x} = &\left(\frac{\Im(\psi_{s,x}(S,X))}{2\Im(\psi(S,X))} - \frac{\Im(\psi_s(S,X))\Im(\psi_x(S,X))}{2(\Im(\psi(S,X)))^2}\right)X_\theta X_x \\ &+ \frac{\Im(\phi_s)}{2\Im(\phi)}\left(\frac{\Im(\psi_{s,x}(S,X))}{2\Im(\psi(S,X))} - \frac{\Im(\psi_s(S,X))\Im(\psi_x(S,X))}{2(\Im(\psi(S,X)))^2}\right)(X_\theta)^2 \\ &- \frac{\Im(\phi_s)}{2\Im(\phi)}\Theta_{\theta,\theta} + \frac{\Im(\phi_s)}{2\Im(\phi)}\frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_{\theta,\theta} + \frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_{\theta,x}.\end{aligned}$$

Using Cauchy-Riemann equations, we have

$$\Theta_{\theta,s} = \frac{\Re(\phi_s)}{2\Im(\phi)}\Theta_{\theta,\theta} + \frac{a'}{a}\left(\frac{\Im(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} + \frac{(\Re(\psi_s(S,X)))^2}{2(\Im(\psi(S,X)))^2}\right)X_\theta \tag{A.7}$$

$$\begin{aligned}\Theta_{\theta,x} = &\left(\frac{\Re(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} - \frac{\Re(\psi_s(S,X))\Im(\psi_s(S,X))}{2(\Im(\psi(S,X)))^2}\right)X_\theta X_x \\ &+ \frac{\Im(\phi_s)}{2\Im(\phi)}\left(\frac{\Re(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} - \frac{\Re(\psi_s(S,X))\Im(\psi_s(S,X))}{2(\Im(\psi(S,X)))}\right)(X_\theta)^2 \\ &+ \frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_{\theta,x} + \frac{\Im(\phi_s)}{2\Im(\phi)}\frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_{\theta,\theta} - \frac{\Im(\phi_s)}{2\Im(\phi)}\Theta_{\theta,\theta}.\end{aligned} \tag{A.8}$$

Moreover, differentiating A.1 with respect to x gives

$$X_{x,s} = \left(\frac{\Re(\phi_{s,x})}{2\Im(\phi)} - \frac{\Re(\phi_s)\Im(\phi_x)}{2(\Im(\phi))^2}\right)X_\theta + \frac{\Re(\phi_s)}{2\Im(\phi)}X_{x,\theta}$$

and so,

$$X_{x,s} = -\left(\frac{\Im(\phi_{s,s})}{2\Im(\phi)} + \frac{(\Re(\phi_s))^2}{2(\Im(\phi))^2}\right)X_\theta + \frac{\Re(\phi_s)}{2\Im(\phi)}X_{x,\theta}. \tag{A.9}$$

Then, differentiating A.1 with respect to θ:

$$X_{\theta,s} = \frac{\Re(\phi_s)}{2\Im(\phi)}X_{\theta,\theta}. \tag{A.10}$$

Now, we differentiate A.5 with respect to x

$$\begin{aligned}\Theta_{x,s} = &\left(\frac{\Re(\phi_{s,x})}{2\Im(\phi)} - \frac{\Re(\phi_s)\Im(\phi_x)}{2(\Im(\phi))^2}\right)\Theta_\theta + \frac{\Re(\phi_s)}{2\Im(\phi)}\Theta_{x,\theta} \\ &- \frac{a'}{a}\left(\frac{\Re(\psi_{s,x}(S,X))}{2\Im(\psi(S,X))} - \frac{\Re(\psi_s(S,X))\Im(\psi_x(S,X))}{2(\Im(\psi(S,X)))^2}\right)X_x\end{aligned}$$

leading to:

$$\begin{aligned}\Theta_{x,s} = &-\left(\frac{\Im(\phi_{s,s})}{2\Im(\phi)} + \frac{(\Re(\phi_s))^2}{2(\Im(\phi))^2}\right)\Theta_\theta + \frac{\Re(\phi_s)}{2\Im(\phi)}\Theta_{x,\theta} \\ &+ \frac{a'}{a}\left(\frac{\Im(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} + \frac{(\Re(\psi_s(S,X)))^2}{2(\Im(\psi(S,X)))^2}\right)X_x.\end{aligned} \tag{A.11}$$

Differentiate A.6 with respect to s:

$$\begin{aligned}\Theta_{s,x} = \frac{a'}{a}\left(\frac{\Im(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} - \frac{(\Im(\psi_s(S,X)))^2}{2(\Im(\psi(S,X)))^2}\right)X_x \\ + \left(\frac{\Im(\psi_{s,x}(S,X))}{2\Im(\psi(S,X))} - \frac{\Im(\psi_s(S,X))\Im(\psi_x(S,X))}{2(\Im(\psi(S,X)))^2}\right)X_sX_x + \frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_{s,x} \\ + \frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}\left(\frac{\Im(\phi_{s,s})}{2\Im(\phi)} - \frac{(\Im(\phi_s))^2}{2(\Im(\phi))^2}\right)X_\theta \\ + \frac{\Im(\phi_s)}{2\Im(\phi)}\left(\frac{\Im(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} - \frac{(\Im(\psi_s(S,X)))^2}{2(\Im(\psi(S,X)))^2}\right)X_\theta \\ + \frac{\Im(\phi_s)}{2\Im(\phi)}\left(\frac{\Im(\psi_{s,x}(S,X))}{2\Im(\psi(S,X))} - \frac{\Im(\psi_s(S,X))\Im(\psi_x(S,X))}{2(\Im(\psi(S,X)))^2}\right)X_sX_\theta \\ + \frac{\Im(\phi_s)}{2\Im(\phi)}\frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_{s,\theta} - \left(\frac{\Im(\phi_{s,s})}{2\Im(\phi)} - \frac{(\Im(\phi_s))^2}{2(\Im(\phi))^2}\right)\Theta_\theta - \frac{\Im(\phi_s)}{2\Im(\phi)}\Theta_{s,\theta}\end{aligned}$$

and replace X_s by its value from A.1, $X_{s,x}$ by A.9, $X_{s,\theta}$ by A.10 and $\Theta_{s,\theta}$ by A.7:

$$\begin{aligned}\Theta_{s,x} = \frac{a'}{a}\left(\frac{\Im(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} - \frac{(\Im(\psi_s(S,X)))^2}{2(\Im(\psi(S,X)))^2}\right)X_x \\ + \frac{\Re(\phi_s)}{2\Im(\phi)}\left(\frac{\Re(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} - \frac{\Re(\psi_s(S,X))\Im(\psi_s(S,X))}{2(\Im(\psi(S,X)))^2}\right)X_xX_\theta \\ - \frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}\frac{|\phi'|^2}{2(\Im(\phi))^2}X_\theta + \frac{\Re(\phi_s)}{2\Im(\phi)}\frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_{x,\theta} \\ + \frac{a'}{a}\frac{\Im(\phi_s)}{2\Im(\phi)}\left(\frac{\Im(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} - \frac{(\Im(\psi_s(S,X)))^2}{2(\Im(\psi(S,X)))^2}\right)X_\theta \\ + \frac{\Re(\phi_s)\Im(\phi_s)}{4(\Im(\phi))^2}\left(\frac{\Re(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} - \frac{\Re(\psi_s(S,X))\Im(\psi_s(S,X))}{2(\Im(\psi(S,X)))^2}\right)(X_\theta)^2 \\ + \frac{\Re(\phi_s)\Im(\phi_s)}{4(\Im(\phi))^2}\frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_{\theta,\theta} - \left(\frac{\Im(\phi_{s,s})}{2\Im(\phi)} - \frac{(\Im(\phi_s))^2}{2(\Im(\phi))^2}\right)\Theta_\theta \\ - \frac{\Re(\phi_s)\Im(\phi_s)}{4(\Im(\phi))^2}\Theta_{\theta,\theta} - \frac{a'}{a}\frac{\Im(\phi_s)}{2\Im(\phi)}\left(\frac{\Im(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} + \frac{(\Re(\psi_s(S,X)))^2}{2(\Im(\psi(S,X)))^2}\right)X_\theta \quad \text{(A.12)}\end{aligned}$$

where $|h'|^2$ is the Jacobian determinant of h for any holomorphic function. Now, since $\Theta_{s,x} = \Theta_{x,s}$ the previous two are equal. Thus, using the value of $\Theta_{x,\theta}$ found in A.8, we get

$$\Theta_\theta = \frac{a'}{a}\frac{|\psi'(S,X)|^2(\Im(\phi))^2}{|\phi'|^2(\Im(\psi(S,X)))^2}\left(X_x + \frac{\Im(\phi_s)}{2\Im(\phi)}X_\theta\right) + \frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_\theta. \quad \text{(A.13)}$$

Replacing in A.5 we have

$$\Theta_s = \frac{a'}{a}\frac{\Re(\phi_s)}{2\Im(\phi)}\frac{|\psi'(S,X)|^2(\Im(\phi))^2}{|\phi'|^2(\Im(\psi(S,X)))^2}\left(X_x + \frac{\Im(\phi_s)}{2\Im(\phi)}X_\theta\right) + \frac{\Re(\phi_s)}{2\Im(\phi)}\frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_\theta - \frac{a'}{a}\frac{\Re(\psi_s(S,X))}{2\Im(\psi(S,X))}. \quad \text{(A.14)}$$

By Proposition 2.3.5, we can write

$$\frac{|\psi'(S,X)|^2(\Im(\phi))^2}{|\phi'|^2(\Im(\psi(S,X)))^2} = h(.,X)$$

for a real valued function $h :]0,b[\times]0,b'[\longrightarrow \mathbb{R}$. We write h_2 the partial derivative of h with respect to the second variable. Now, differentiate A.13 with respect to s:

$$\begin{aligned}\Theta_{s,\theta} = {} & \frac{a'}{a}h_2(.,X)X_s\left(X_x + \frac{\Im(\phi_s)}{2\Im(\phi)}X_\theta\right) + \frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_{s,\theta} \\ & + \frac{a'}{a}h(.,X)\left(X_{s,x} + \left(\frac{\Im(\phi_{s,s})}{2\Im(\phi)} - \frac{(\Im(\phi_s))^2}{2(\Im(\phi))^2}\right)X_\theta + \frac{\Im(\phi_s)}{2\Im(\phi)}X_{s,\theta}\right) \\ & + \frac{a'}{a}\left(\frac{\Im(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} - \frac{(\Im(\psi_s(S,X)))^2}{2(\Im(\psi(S,X)))^2}\right)X_\theta \\ & + \left(\frac{\Im(\psi_{s,x}(S,X))}{2\Im(\psi(S,X))} - \frac{\Im(\psi_s(S,X))\Im(\psi_x(S,X))}{2(\Im(\psi(S,X)))^2}\right)X_sX_\theta\end{aligned}$$

and use formulas for X_s and $X_{s,\theta}$:

$$\begin{aligned}\Theta_{s,\theta} = {} & \frac{a'}{a}h_2(.,X)X_\theta\left(\frac{\Re(\phi_s)}{2\Im(\phi)}X_x + \frac{\Re(\phi_s)\Im(\phi_s)}{4(\Im(\phi))^2}X_\theta\right) \\ & + \frac{a'}{a}h(.,X)\left(\frac{\Re(\phi_s)}{2\Im(\phi)}X_{x,\theta} - \frac{|\phi'|^2}{2(\Im(\phi))^2}X_\theta + \frac{\Re(\phi_s)\Im(\phi_s)}{4(\Im(\phi))^2}X_{\theta,\theta}\right) \\ & + \frac{\Re(\phi_s)}{2\Im(\phi)}\left(\frac{\Re(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} + \frac{\Re(\psi_s(S,X))\Im(\psi_s(S,X))}{2(\Im(\psi(S,X)))^2}\right)(X_\theta)^2 \\ & + \frac{\Re(\phi_s)}{2\Im(\phi)}\frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_{\theta,\theta} + \frac{a'}{a}\left(\frac{\Im(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} - \frac{(\Im(\psi_s(S,X)))^2}{2(\Im(\psi(S,X)))^2}\right)X_\theta. \quad \text{(A.15)}\end{aligned}$$

Now, differentiating A.14 with respect to θ leads to:

$$\begin{aligned}\Theta_{\theta,s} = {} & \frac{a'}{a}h_2(.,X)X_\theta\left(\frac{\Re(\phi_s)}{2\Im(\phi)}X_x + \frac{\Re(\phi_s)\Im(\phi_s)}{4(\Im(\phi))^2}X_\theta\right) \\ & + \frac{\Re(\phi_s)}{2\Im(\phi)}\left(\frac{\Re(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))} - \frac{\Re(\psi_s(S,X))\Im(\psi_s(S,X))}{2(\Im(\psi(S,X)))^2}\right)(X_\theta)^2\end{aligned}$$

$$+\frac{a'}{a}\frac{\Re(\phi_s)}{2\Im(\phi)}h(.,X)\left(X_{x,\theta}+\frac{\Im(\phi_s)}{2\Im(\phi)}X_{\theta,\theta}\right)+\frac{\Re(\phi_s)}{2\Im(\phi)}\frac{\Im(\psi_s(S,X))}{2\Im(\psi(S,X))}X_{\theta,\theta}$$
$$+\frac{a'}{a}\left(\frac{\Im(\psi_{s,s}(S,X))}{2\Im(\psi(S,X))}+\frac{(\Re(\psi_s(S,X)))^2}{2(\Im(\psi(S,X)))^2}\right)X_\theta. \quad \text{(A.16)}$$

Finally, using the fact that $\Theta_{s,\theta}=\Theta_{\theta,s}$ and the definition of h, we find

$$\frac{|\psi'(S,X)|^2}{(\Im(\psi(S,X)))^2}X_\theta=0$$

which leads to

$$X_\theta=0.$$

□

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