



HAL
open science

The comparison between two mortar spectral element methods for Darcy's equations

Kangzheng Xing

► **To cite this version:**

Kangzheng Xing. The comparison between two mortar spectral element methods for Darcy's equations. Spectral Theory [math.SP]. Université Pierre et Marie Curie - Paris VI, 2015. English. NNT : 2015PA066740 . tel-02901567

HAL Id: tel-02901567

<https://theses.hal.science/tel-02901567>

Submitted on 17 Jul 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

UNIVERSITÉ PIERRE ET MARIE CURIE

Ecole doctorale de sciences mathématiques de Paris-Centre (ED 386)

Comparaison de deux méthodes d'éléments spectraux avec joints pour les équations de Darcy

Par **Kangzheng XING**

Thèse de doctorat de **Mathématiques Appliquées**

Dirigée par **Christine BERNARDI**

Pour obtenir le grade de

DOCTEUR de l'UNIVERSITÉ PIERRE ET MARIE CURIE - PARIS 6

Présentée et soutenue publiquement le 10 Novembre 2015

Devant le jury composé de :

Rapporteur :	Madame Florence Hubert
Rapporteur :	Monsieur Azaïez Mejdî
Président de jury :	Monsieur Yvon Maday
Examineur :	Madame Francesca RAPETTI
Examineur :	Monsieur Adel Blouza
Examineur :	Monsieur Faker Ben Belgacem

Cette thèse est dédiée aux membres de ma famille,
sans lesquels je n'aurais jamais vu le jour.

Remerciements

Le thèse est un projet qui exige de l'endurance, de l'engagement et de la patience. A travers les quatre dernières années que je suis un doctorant à UPMC, je rencontrais des gens qui m'aidé avec la charge émotionnelle et scientifique qui vient avec un travail de ce genre. À la fin de ce travail, je tiens à exprimer ma gratitude et à adresser quelques mots à tous ceux qui, de près ou de loin, ont été présents et ont contribué à sa réalisation.

Je tiens tout d'abord à exprimer ma profonde reconnaissance envers Christine BERNARDI. Ces années de thèse sous sa direction furent pour moi un vrai enrichissement. Je le remercie d'avoir accepté d'encadrer cette thèse. J'ai pu bénéficier de son intuition mathématique. Je voudrais le remercier pour tous ses encouragements constants, sa disponibilité et sa profonde gentillesse qui m'ont permis d'avancer sereinement dans mon travail. Je n'oublierai jamais son soutien et sa disponibilité dans les moments de doute.

Juger un travail de thèse n'est pas une chose aisée. Pour cela, je suis extrêmement honorée que Madame Florence Hubert et Monsieur Mejdî Azaiez aient accepté de consacrer du temps à lire ce mémoire et à rédiger un rapport. Ils ont également contribué par leurs remarques et suggestions à améliorer la qualité de ce mémoire, et je les en remercie. Je leur suis également très reconnaissante d'avoir pris la peine de se déplacer pour assister à ma soutenance.

Je tiens à remercier très vivement M. Yvon Maday qui me fait l'honneur de présider mon jury de thèse. Je souhaiterais remercier le Professeur Faker Ben Belgacem qui a eu la gentillesse de me conseiller, pour le temps qu'il m'a accordé à plusieurs reprises au cours de cette thèse ainsi que pour l'intérêt qu'il a porté à mes travaux.

J'adresse une pensée particulière à Khashayar Dadras et Christian David pour son sourire, sa disponibilité, sa gentillesse et son amitié. Je le remercie également pour l'impression de ce manuscrit malgré son état de santé. Je remercie Antoine Le Hyaric pour les nombreuses fois où il m'a aidé avec toutes sortes de problèmes (logiciels scientifiques ou autres) ainsi que pour son amitié et sa gentillesse à mon égard.

Je voudrais remercier mes collègues dans bureau 1516-316 qui m'ont accompagné pendant ces années de ma thèse à Jussieu, j'ai connu des nouveaux collègues dans bureau 1516-303 qui je remercie pour les nombreuses fois où il m'a aidé. Je voudrais remercier pour ces années passées au laboratoire Jacques-Louis Lions tous ses membres pour la bonne humeur, l'atmosphère à la fois studieuse.

Je souhaiterais remercier mes rapporteurs pour le temps qu'ils ont accordé à la lecture de cette thèse et à l'élaboration de leur rapport.

Je remercie mes amis chinois de Paris pour l'ambiance amicale qu'ils ont créée, je pense à Ma Yue, Zhu Jiamin, Guo Hao et bien d'autres.

Je remercie particulièrement Yang Xingdi de m'avoir beaucoup aidé et accompagné. Merci à lui pour les moments agréables que nous avons passés ensemble à Paris. Merci pour son soutien et pour tout ce qu'elle a fait pour moi.

Mes derniers remerciements iront évidemment à mon famille.

Contents

1	Introduction	9
2	Darcy's Equations	17
2.1	Introduction	17
2.2	Sobolev spaces	18
2.3	Variational formulations	21
2.3.1	Velocity in $H(\operatorname{div}, \Omega)$	22
2.3.2	Velocity in $L^2(\Omega)$	24
2.4	Well-posedness of the problem	25
2.4.1	Case of velocity in $H(\operatorname{div}, \Omega)$	25
2.4.2	Case of velocity in $L^2(\Omega)$	28
2.5	Regularity properties	31
3	Spectral Discretization of Darcy's Equations	35

3.1	Orthogonal polynomials	35
3.2	Quadrature Formula	37
3.3	Polynomial approximation	39
3.3.1	Polynomial approximation on the interval	39
3.3.2	Polynomial approximation on the square or cube	40
3.4	Polynomial interpolation	41
3.4.1	Polynomial interpolation on the interval	42
3.4.2	Polynomial interpolation on the square or cube	43
3.5	Spectral discretization of the Darcy's problem	44
3.5.1	First discretization	44
3.5.2	Second discretization	50
3.6	Numerical results	59
3.6.1	The first discretization	59
3.6.2	The second discretization	59
4	Mortar Spectral Discretization of Darcy's Equations(I)	67
4.1	Introduction	67
4.2	Description of the mortar spectral element discretization	69

4.2.1	Approximation of divergence-free functions.	76
4.2.2	Error estimate	80
4.3	Numerical implementation	89
5	Mortar Spectral Discretization of Darcy's Equations(II)	93
5.1	Introduction	93
5.2	Analysis of the mortar SE discretization	94
5.3	Error estimates	101
5.4	Numerical implementation	111
5.5	Comparison between the two methods	112
6	Conclusion and Perspectives	115
7	Appendix : A Spectral Method for 4-th Differential Equations	117
7.1	Introduction	117
7.2	Chebyshev polynomials	119
7.3	Spectral discretization scheme	120
7.3.1	Linear equation	120
7.3.2	Nonlinear equation	123
7.4	Numerical results	124

7.5	Preliminaries	128
7.6	Convergence of the schemes	129
7.6.1	Linear equation	129
7.6.2	Nonlinear equation	130
7.7	Conclusion	133

Chapter 1

Introduction

Motivation

In this thesis, we mainly talk about the problem of the flow of a viscous incompressible fluid moving in a rigid porous inhomogeneous medium. A good example of this problem is that water moves in the subsurface.

In the subsurface, rock is deposited in layers. Fluid flow within and between the rock layers is governed by the permeability of the rocks. However, to account for permeability, it must be measured in both the vertical and horizontal directions. For example, shale typically has permeabilities that are much lower vertically than horizontally (assuming flat lying shale beds). This means that it is difficult for fluid to flow up and down through a shale bed but much easier for it to flow from side to side. Ultimately, if the pressure difference between a hydraulically fractured zone and a fresh water aquifer is not great, the distance between the zones is relatively large, and there are rocks with low vertical permeabilities in between the deeper and the shallower zones, flow between the zones is unlikely to occur. The exception to this is where there is a separate flow pathway such as an open bore hole or a series of faults or joints that intersect both the fractured zone and the fresh water aquifer. Under either of these circumstances, the pressure difference and distance will be the

determining factors as to whether fluid can migrate from the lower to the upper zone. The principle that governs how fluid moves in the subsurface is called Darcy's law. Darcy's law is an equation that defines the ability of a fluid to flow through a porous media such as rock. The law was formulated by Henry Darcy based on the results of experiments [33] on the flow of water through beds of sand. It also forms the scientific basis of fluid permeability used in the earth sciences, particularly in hydrogeology.

Although Darcy's law (an expression of conservation of momentum) was determined experimentally by Darcy, it has since been derived from the Navier-Stokes equations via homogenization. It is analogous to Fourier's law in the field of heat conduction, Ohm's law in the field of electrical networks, or Fick's law in diffusion theory.

One application of Darcy's law is to water flow through an aquifer; Darcy's law along with the equation of conservation of mass are equivalent to the groundwater flow equation, one of the basic relationships of hydrogeology. Darcy's law is also used to describe oil, water, and gas flows through petroleum reservoirs.

Darcy's law at constant elevation is a simple proportional relationship between the instantaneous discharge rate through a porous medium, the viscosity of the fluid and the pressure drop over a given distance. Darcy found that his data could be described by

$$U = \frac{-kA}{\mu} \frac{P_b - P_a}{L}, \quad (1.0.1)$$

where, U (units of volume per time, e.g., m^3/s) is equal to the product of the intrinsic permeability of the medium, $k(m^2)$, the cross-sectional area to flow, A (units of area, e.g., m^2), $(P_b - P_a)$ (Pascals) is the total pressure drop, and L , the length over which the pressure drop.

The negative sign is needed because fluid flows from high pressure to low pressure. Note: the elevation head must be taken into account if the inlet and outlet are at different elevations. If the change in pressure is negative (where $P_a > P_b$), then the flow will be in the positive ' x ' direction. Dividing both sides of the equation by the

area and using more general notation leads

$$u = \frac{-k}{\mu} \nabla P, \quad (1.0.2)$$

where, u is the flux (discharge per unit area, with units of length per time, m/s) and ∇P is the pressure gradient vector (Pa/m).

For very short time scales, a time derivative of flux may be added to Darcy's law, which results in valid solutions at very small times (in heat transfer, this is called the modified form of Fourier's law),

$$\tau \frac{\partial u}{\partial t} + u = \frac{-k}{\mu} \nabla h, \quad (1.0.3)$$

where τ is a very small time constant which causes this equation to reduce to the normal form of Darcy's law at "normal" times.

At last, in fact the water is the incompressible fluid, so the equation will also satisfy the incompressible condition:

$$\nabla \cdot (u) = 0.$$

A long this thesis we will focus on the following model : *find a velocity \mathbf{u} and pressure p solutions of :*

$$\left\{ \begin{array}{ll} \alpha \mathbf{u} + \mathbf{grad} p = \alpha \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = k & \text{on } \Gamma_1, \\ p = p_0 & \text{on } \Gamma_2 \end{array} \right. \quad (1.0.4)$$

Spectral, spectral element and mortar element methods

Spectral methods are a class of techniques used in applied mathematics and scientific computing to numerically solve certain differential equations, often involving the

use of the Fast Fourier Transform. The idea is to write the solution of the differential equation as a sum of certain "basis functions" (for example, as a Fourier series which is a sum of sinusoids) and then to choose the coefficients in the sum in order to satisfy the differential equation as well as possible.

Spectral methods have been used extensively during the last decades for the numerical solution of partial differential equations (PDE) due to their bigger accuracy when compared to Finite Differences (FD) and Finite Elements (FE) methods. The rate of convergence of spectral approximations depends only on the smoothness of the solution, yielding the ability to achieve high precision with a small number of data. This fact is known in literature as "spectral accuracy".

Spectral element techniques are high order methods which allow for either obtaining very accurate results or reducing the number of degrees of freedom for a fixed standard accuracy. As firstly explained by [51, 41], they rely on high degree piecewise polynomial approximation, and on the use of tensorized bases of polynomials.

For these reasons, the basic geometries for these methods are tensorized, i.e. they are square or cubes. We refer to [38, 43, 21] for a general presentation and analysis of the spectral methods in these geometries, and also to [36, 29, 55] for extensions to triangles or tetrahedra. The discrete spaces for spectral methods are simply polynomial spaces, and the discretization parameter is the maximal degree of these polynomials.

The mortar element method, introduced in [24], is a domain decomposition technique which allows for working on general partitions of the domain without conformity restrictions. It is particularly important when combined with spectral-type discretizations, since handling complex geometries from very simple subdomains can be performed with this method in a very efficient way. It can also be used to couple different kinds of variational discretizations on the subdomains, such as finite elements or spectral methods. So it leads to discrete problems which are most often non-conforming in the Hodge sense, which means that the discrete space is not con-

tained in the variational one. It was first analysed in the case of the 2D Laplace equation [24] which admits a natural variational formulation in the usual Sobolev space $H^1(\Omega)$ of functions with square-integrable first-order derivatives. We also refer to [9] for the first 3D results. It was extended [14] to the bilaplacian equation where the variational space is the standard space $H^2(\Omega)$ of functions with square-integrable first-order and second-order derivatives and also to the Stokes problem which is of saddle-point type; however, it still involves the usual Sobolev spaces. We also quote [13] for an application of the mortar technique to weighted Sobolev spaces, in order to handle discontinuous boundary conditions for the Navier-Stokes equations.

Objectives of the thesis

In this thesis, we mainly talk about the analysis of the mortar spectral element discretization of the problem (5.1.1 where Ω be a bounded, connected, open set in \mathbb{R}^d , $d = 2$ or 3 , with a Lipschitz continuous boundary $\partial\Omega$, and let \mathbf{n} denote the unit outward normal vector to Ω on $\partial\Omega$, Γ_1 and Γ_2 be a partition of $\partial\Omega$ without overlap. α is constant or piecewise continuous coefficients. The data are now the function \mathbf{f} and the boundary conditions k and p_0 . The unknowns are the velocity \mathbf{u} and the pressure (or hydraulic head, according to the model) p . we are interested in the case where this function is not globally continuous but only piecewise smooth and also such that the ratio of its maximal value to its minimal value is large. This models, for instance, the flow of a viscous incompressible fluid in a rigid porous inhomogeneous medium. A very interesting and unusual feature of Darcy's equations is that they admit two equivalent variational formulations, whether the space of velocities is more regular than the space of pressures or not, see [6, 2, 19]. The drag coefficient α will be considered as piecewise constant or piecewise smooth functions. The variational spaces can be chosen as $(u; p)$ in $H(\text{div}; \Omega) \times L^2(\Omega)$ or in $L^2(\Omega) \times H^1(\Omega)$. So two mortar discrete problems based on Galerkin numerical integration (GNI) will be constructed. Then we obtain the well-posedness and regularity properties of the problems. Next, it is necessary to prove a priori error estimates of spectral type: the order of convergence

only depends on the regularity of the solution, more precisely on its local regularity in each subdomain. For the implementation of the mortar technique, it mainly relies on an appropriate treatment of the matching conditions on the interfaces.

In this thesis, we will also develop a spectral method for fourth-order differential equations in one dimension. A Legendre Petrov-Galerkin method for linear fourth-order differential equations and a Legendre Petrov-Galerkin and Chebyshev Collocation method for the nonlinear Kuramoto-Sivashinsky equation will be presented. Also we will prove the optimal rate of convergence in L^2 norm of the method, and numerical experiments will be given which demonstrate the efficiency of proposed schemes.

Thesis outline

The thesis is composed by 2 parts. The first part is dedicated to the main subject of my work and concerns the numerical analysis of the mortar method applied to the Darcy problem. During the first part of my thesis I worked on the analysis of a Legendre Petrov-Galerkin method for linear fourth-order differential equations in one dimension and a Legendre Petrov-Galerkin and Chebyshev collocation method for the nonlinear Kuramoto-Sivashinsky equation have been developed. The second part contains the description of this contribution. This investigation will be given in the end of the manuscript (see Appendix).

After a short introduction given in chapter one, we will consider in chapter 2 the Darcy's equations with general boundary conditions and piecewise continuous coefficients in a bounded domain. This problem can be formulated in two different spaces. Then we prove the well-posedness of the equivalent variational problem of the Darcy's equation. We also give the regularity properties.

In chapter 3 we present the discretization of the steady Darcy problem. We propose spectral discretizations of this problem based on the Galerkin with Numerical Integration (G-NI) variants. The numerical analysis of the discrete problem is performed. We also present the two-dimensional numerical experiments.

In chapter 4 we consider Darcy's equations with general boundary conditions and

piecewise continuous coefficients in a bounded domain. This problem can be formulated in spaces of square-integrable functions with square-integrable divergence. We propose a spectral element discretization of this problem which relies on the mortar domain decomposition technique. The Numerical analysis of the discrete problem is performed. We also present the two-dimensional numerical experiments. They turn out to be in good coherency with the theoretical results.

In chapter 5 we also consider Darcy's equations with general boundary conditions and piecewise continuous coefficients in a bounded domain. A spectral element discretization of this problem relies on the mortar domain decomposition technique is proposed. We present the numerical analysis and two-dimensional numerical experiments, which turn out to be in good coherency. This chapter ends with the comparison of the two spectral element methods will be presented in Chapter 6.

Chapter 2

Darcy's Equations

2.1 Introduction

We first talk about the analysis of the mortar spectral element discretization of the problem introduced by Darcy [33] as follows

$$\left\{ \begin{array}{ll} \alpha \mathbf{u} + \mathbf{grad} p = \alpha \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = k & \text{on } \Gamma_1, \\ p = p_0 & \text{on } \Gamma_2 \end{array} \right. \quad (2.1.1)$$

where Ω is a bounded, connected, open set in \mathbb{R}^d , d equals to 2 or 3, with a Lipschitz continuous boundary $\partial\Omega$, and \mathbf{n} denotes the unit outward normal vector to Ω on $\partial\Omega$, Γ_1 and Γ_2 , with a positive measure, being a partition of $\partial\Omega$ without overlap. The data are now the equation \mathbf{f} and the boundary conditions k and p_0 . The unknowns are the velocity \mathbf{u} and the pressure (or hydraulic head, according to the model) p . The coefficients α are constant or piecewise continuous. We are interested in the case where this function is not globally continuous but only piecewise smooth and also such

that the ratio of its maximal value to its minimal value is large. This models, for instance, the flow of a viscous incompressible fluid in a rigid porous inhomogeneous medium.

2.2 Sobolev spaces

In this section, we recall the main notions and results, concerning the classical Sobolev spaces, which will be used in later sections, also some properties. Although they are stated without proof, these results are complete, rigorous and fairly general.

We define $\mathfrak{D}(\Omega)$ to be the linear space of infinitely differentiable functions, with compact support on Ω . Then we set

$$\mathfrak{D}(\bar{\Omega}) = \{\phi|_{\Omega}; \phi \in \mathfrak{D}(\mathbb{R}^d)\}$$

or equivalently, if \mathcal{O} denotes any open subset of \mathbb{R}^N such that $\bar{\Omega} \subset \mathcal{O}$,

Now, let $\mathfrak{D}'(\Omega)$ denote the dual space of $\mathfrak{D}(\Omega)$, often called the space of distributions on Ω . We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $\mathfrak{D}'(\Omega)$ and $\mathfrak{D}(\Omega)$ and we remark that when f is locally integrable function, then f can be identified with a distribution by

$$\langle f, \phi \rangle = \int_{\Omega} f(x)\phi(x)dx, \quad \forall \phi \in \mathfrak{D}(\Omega)$$

In other words, $\langle \cdot, \cdot \rangle$ is an extension of the scalar product of $L^2(\Omega)$.

For $\mathbf{v} = (v_1, v_2, \dots, v_N)$, we define the divergence operator by:

$$\operatorname{div} \mathbf{v} = \sum_{i=1}^N (\partial v_i / \partial x_i).$$

Note the identity:

$$\operatorname{div}(\mathbf{grad} v) = \Delta v.$$

So far, we have been mainly interested in subspaces of $H^1(\Omega)^N$; but subsequently, it will be worthwhile to use functions with less regularity. Bearing this in mind, we introduce the following spaces:

$$H(\operatorname{div}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^N; \operatorname{div} \mathbf{v} \in L^2(\Omega)\},$$

which is clearly a Hilbert space for the norm:

$$\|\mathbf{v}\|_{H(\operatorname{div}; \Omega)} = (\|\mathbf{v}\|_{0, \Omega}^2 + \|\operatorname{div} \mathbf{v}\|_{0, \Omega}^2)^{1/2}. \quad (2.2.1)$$

Next we are coming to the trace theorems. The trace on the boundary Γ of a function $v \in H^s(\Omega)$ is, in a sense to make precise, the value of v restricted to Γ . If we denote by $\mathcal{C}^0(\bar{\Omega})$ the space of continuous functions on $\bar{\Omega}$, the precise result reads as follows, see ([40], Chapter I):

Theorem 2.1 *Let Ω be a bounded open set of \mathbb{R}^N with Lipschitz-continuous boundary Γ and let $s > 1/2$.*

1. *There exists a unique linear continuous map $\gamma_0 : H^s(\Omega) \rightarrow H^{s-1/2}(\Gamma)$ such that $\gamma_0 v = v|_{\Gamma}$ for each $v \in H^s(\Omega)$.*
2. *There exists a linear continuous map $\mathfrak{R}_0 : H^{s-1/2}(\Gamma) \rightarrow H^s(\Omega)$ such that $\gamma_0 \mathfrak{R}_0 \varphi = \varphi$ for each $\varphi \in H^{s-1/2}(\Gamma)$.*

The next theorem concerns the normal component of boundary values of functions of $H(\operatorname{div}; \Omega)$.

Theorem 2.2 *The mapping $\gamma_n : \mathbf{v} \rightarrow \mathbf{v} \cdot \mathbf{n}|_{\Gamma}$ defined on $\mathfrak{D}(\bar{\Omega})^N$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_n , from $H(\operatorname{div}; \Omega)$ to $H^{-1/2}(\Gamma)$.*

By extension, $\gamma_n \mathbf{v}$ is called the normal component of \mathbf{v} on Γ and is denoted simply by $\mathbf{v} \cdot \mathbf{n}$.

From the above two theorem, we derive the following Green's formula:

$$(\mathbf{v}, \mathbf{grad} \phi) + (\operatorname{div} \mathbf{v}, \phi) = \langle \mathbf{v} \cdot \mathbf{n}, \phi \rangle_{\Gamma}, \quad \forall \mathbf{v} \in \mathbf{H}(\operatorname{div}; \Omega), \forall \phi \in H^1(\Omega). \quad (2.2.2)$$

As a consequence, we can now extend Green's formula for the Laplace operator to a wider range of functions.

Corollary 2.1 *Let $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$. Then $\partial u / \partial n \in H^{-1/2}(\Gamma)$ and*

$$(\mathbf{grad} u, \mathbf{grad} v) = -(\Delta u, v) + \langle \partial u / \partial n, v \rangle_{\Gamma}, \quad \forall v \in H^1(\Omega). \quad (2.2.3)$$

Two useful applications of Green's formula will also be given here.

Lemma 2.1 *Let Ω be a bounded open subset of \mathbb{R}^N with a Lipschitz-continuous boundary Γ .*

1. *For u and v in $H^1(\Omega)$ and for $1 \leq i \leq N$, we have*

$$\int_{\Omega} u (\partial v / \partial x_i) dx = - \int_{\Omega} (\partial u / \partial x_i) v dx + \int_{\Gamma} \gamma_0 u \gamma_0 v n_i ds \quad (2.2.4)$$

2. *If in addition $u \in H^2(\Omega)$ we have*

$$\sum_{i=1}^N \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx = - \sum_{i=1}^N \int_{\Omega} \frac{\partial^2 u}{\partial x_i^2} v + \sum_{i=1}^N \int_{\Gamma} \left(\frac{\partial u}{\partial x_i} v \right) n_i ds \quad (2.2.5)$$

Adopting the usual notations

$$\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}, \quad \mathbf{grad} u = (\partial u / \partial x_1, \dots, \partial u / \partial x_N)$$

(2.2.5) becomes:

$$(\mathbf{grad}u, \mathbf{grad}v) = -(\Delta u, v) + \int_{\Gamma} (\partial u / \partial n) \gamma_0 v ds. \quad (2.2.6)$$

When Σ is an open (non-empty) Lipschitz continuous subset of the boundary Γ , we introduce the space

$$H_{\Sigma}^1(\Omega) := \{v \in H^1(\Omega) | \gamma_{\Sigma} v = 0\}.$$

An important result, which will find widespread application in the sequel, is the so-called Poincaré inequality, see ([25], Chapter 1, Lemma 2.6):

Theorem 2.3 *Assume that Ω is a bounded connected open set of \mathbb{R}^N . Then there exists a constant $C_{\Omega}(\Sigma) > 0$ such that*

$$\int_{\Omega} v^2(\mathbf{x}) d\mathbf{x} \leq C_{\Omega}(\Sigma) \int_{\Omega} |\nabla v(\mathbf{x})|^2 d\mathbf{x} \quad (2.2.7)$$

for each $v \in H_{\Sigma}^1(\Omega)$.

We next recall Sobolev embedding theorem ([3], Chapter 4).

Theorem 2.4 *Assume that Ω is a (bounded or unbounded) open set of \mathbb{R}^d with a Lipschitz continuous boundary, and that $1 \leq p < \infty$. Then the following continuous embedding hold:*

1. *If $0 \leq sp < d$, then $W^{s,p}(\Omega) \subset L^{p^*}(\Omega)$ for $p^* = dp/(d - sp)$;*
2. *If $sp = d$, then $W^{s,p}(\Omega) \subset L^q(\Omega)$ for any q such that $p \leq q < \infty$;*
3. *If $sp > d$, then $W^{s,p}(\Omega) \subset C^0(\bar{\Omega})$.*

2.3 Variational formulations

A very interesting feature of Darcy's equations is that they admit two equivalent variational formulations, whether the space of velocities is more regular than the space

of pressures or not, see [6, 2, 19]. The drag coefficient α will be considered as piecewise constants or piecewise smooth functions. The variational spaces can be chosen as $(u; p)$ in $H(\text{div}; \Omega) \times L^2(\Omega)$ or in $L^2(\Omega) \times H^1(\Omega)$. So two variational formulations will be constructed.

Throughout the thesis, we make the following assumptions on the function α : there exists a finite number of domains $\Omega_l, 1 \leq l \leq L$, such that :

- they form a partition of Ω without overlapping:

$$\bar{\Omega} = \sum_{l=1}^L \bar{\Omega}_l, \quad \Omega_l \cap \Omega_{l'} = \emptyset, \quad 1 \leq l \neq l' \leq L, \quad (2.3.1)$$

- the restriction of α to each $\bar{\Omega}_l, 1 \leq l \leq L$, is bounded and positive, i.e. there exists constants α_l^{max} and α_l^{min} such that

$$\alpha_l^{max} = \sup_{\mathbf{x} \in \bar{\Omega}_l} \alpha(\mathbf{x}) < +\infty, \quad \alpha_l^{min} = \inf_{\mathbf{x} \in \bar{\Omega}_l} \alpha(\mathbf{x}) > 0. \quad (2.3.2)$$

We set

$$\alpha^{max} = \max_{1 \leq l \leq L} \alpha_l^{max}, \quad \alpha^{min} = \min_{1 \leq l \leq L} \alpha_l^{min}. \quad (2.3.3)$$

2.3.1 Velocity in $H(\text{div}, \Omega)$

In this subsection, for each domain \mathcal{O} in \mathbb{R}^d with a Lipschitz-continuous boundary, we use the full scale of Sobolev spaces $H^s(\mathcal{O})$ and $H_0^s(\mathcal{O}), s \geq 0$, their trace spaces on $\partial\mathcal{O}$ and their dual spaces. We denote by $\mathcal{C}^\infty(\bar{\mathcal{O}})$ the space of restrictions to \mathcal{O} of indefinitely differentiable functions of \mathbb{R}^d and by $\mathcal{D}(\mathcal{O})$ its subspace of functions with a compact support in \mathcal{O} .

Let Γ be any part of $\partial\Omega$ with positive measure. We also recall that $H_{00}^{\frac{1}{2}}(\Gamma)$ is defined as the space of functions in $H^{\frac{1}{2}}(\Gamma)$ such that their extension by zero belongs to $H^{\frac{1}{2}}(\partial\Omega)$.

We have the following formula

$$\forall q \in H_{00}^{\frac{1}{2}}(\Gamma), \quad \langle (\mathbf{v} \cdot \mathbf{n}), q(\mathbf{x}) \rangle = \int_{\Omega} (\operatorname{div} \mathbf{v}) \bar{q}(\mathbf{x}) dx + \int_{\Omega} \mathbf{v} \cdot (\mathbf{grad} \bar{q})(\mathbf{x}) dx$$

where \bar{q} is any lifting in $H^1(\Omega)$ of the extension by zero of q to $\partial\Omega$. So the normal trace on Γ of a function \mathbf{v} in $H(\operatorname{div}, \Omega)$ makes sense in the dual space $H_{00}^{\frac{1}{2}}(\Gamma)'$ of $H_{00}^{\frac{1}{2}}(\Gamma)$. In fact, the integral in the left-hand side of the above equality represents a duality pairing.

We next define

$$H_0(\operatorname{div}, \Omega) = \{\mathbf{v} \in H(\operatorname{div}, \Omega); \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\} \quad (2.3.4)$$

Then $\mathcal{D}(\Omega)^d$ is dense in $H_0(\operatorname{div}, \Omega)$, and both $H_0(\operatorname{div}, \Omega)$ and $H(\operatorname{div}, \Omega)$ are Hilbert spaces for the scalar product associated with the norm defined in (2.2.1).

We now introduce the variational spaces

$$\begin{aligned} X(\Omega) &= H(\operatorname{div}, \Omega), \\ X_0(\Omega) &= \{\mathbf{v} \in X(\Omega); \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma_1\}. \end{aligned} \quad (2.3.5)$$

Both of them are equipped with the norm

$$\|\mathbf{v}\|_{X(\Omega)} = \|\mathbf{v}\|_{H(\operatorname{div}, \Omega)} = (\|\mathbf{v}\|_{L^2(\Omega)^d}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega)}^2)^{\frac{1}{2}}. \quad (2.3.6)$$

The variational problem we consider reads: Find (\mathbf{u}, p) in $X(\Omega) \times L^2(\Omega)$ such that $\mathbf{u} \cdot \mathbf{n} = k$ on Γ_1 and that

$$\begin{aligned} \forall \mathbf{v} \in X_0(\Omega), \quad a_\alpha(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \mathcal{L}(\mathbf{v}), \\ \forall q \in L^2(\Omega), \quad b(\mathbf{u}, q) &= 0, \end{aligned} \quad (2.3.7)$$

where the bilinear forms and linear form are defined by

$$\begin{aligned} a_\alpha(\mathbf{u}, \mathbf{v}) &= \sum_{l=1}^L \alpha_l \int_{\Omega_l} \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) d\mathbf{x}, \\ b(\mathbf{v}, q) &= - \int_{\Omega} (\operatorname{div} \mathbf{v})(\mathbf{x}) \cdot q(\mathbf{x}) d\mathbf{x}. \\ \mathcal{L}(\mathbf{v}) &= \sum_{l=1}^L \alpha_l \int_{\Omega_l} \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) d\mathbf{x} - \int_{\Gamma_2} p_0(\mathbf{s}) (\mathbf{v} \cdot \mathbf{n})(\mathbf{s}) d\mathbf{s} \end{aligned}$$

We make the following assumption on the data

$$\mathbf{f} \in X_0(\Omega)', \quad k \in (H_{00}^{\frac{1}{2}}(\Gamma_1))', \quad p_0 \in H_{00}^{\frac{1}{2}}(\Gamma_2), \quad (2.3.8)$$

2.3.2 Velocity in $L^2(\Omega)$

The variational problem that we consider now reads : Find (\mathbf{u}, p) in $L^2(\Omega)^d \times H^1(\Omega)$ such that $p = p_0$ on Γ_2 and that

$$\begin{aligned} \forall \mathbf{v} \in L^2(\Omega)^d, \quad a_\alpha(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \mathcal{L}(\mathbf{v}), \\ \forall q \in H_{\Gamma_2}^1(\Omega), \quad b(\mathbf{u}, q) &= \int_{\Gamma_1} k(\tau) q(\tau) d\tau, \end{aligned} \quad (2.3.9)$$

where the bilinear forms and linear form are defined by

$$a_\alpha(\mathbf{u}, \mathbf{v}) = \sum_{l=1}^L \alpha_l \int_{\Omega_l} \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) d\mathbf{x}, \quad (2.3.10)$$

$$b(\mathbf{v}, q) = \int_{\Omega} \mathbf{v}(\mathbf{x}) \cdot (\mathbf{grad} q)(\mathbf{x}) d\mathbf{x}. \quad (2.3.11)$$

and

$$\mathcal{L}(\mathbf{v}) = \int_{\Omega} \alpha \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) d\mathbf{x} \quad (2.3.12)$$

$$H_{\Gamma_2}^1(\Omega) = \{q \in H^1(\Omega), q = 0 \text{ on } \Gamma_2\}. \quad (2.3.13)$$

2.4 Well-posedness of the problem

2.4.1 Case of velocity in $H(\operatorname{div}, \Omega)$

Proposition 2.1 *Assume that the partition of $\partial\Omega$ into Γ_1 and Γ_2 is sufficiently smooth for $\mathcal{D}(\Omega \cup \Gamma_2)$ to be dense in $X_0(\Omega)$. If the data (\mathbf{f}, k, p_0) satisfy assumption (2.3.8), any smooth enough pair of functions (\mathbf{u}, p) in $X(\Omega) \times L^2(\Omega)$ is a solution of problem (2.3.7) if and only if it is a solution of problem (2.1.1).*

Proof: We refer to [15]. If $Y(\Omega) = \mathfrak{D}(\bar{\Omega})(\bar{\Omega}) \cap X_0(\Omega)$, then $Y(\Omega)$ is dense in $X_0(\Omega)$. When multiplying the first line in (2.1.1) by a function \mathbf{v} in $Y(\Omega)$, integrating by parts, we obtain that the first equation of (2.3.7) is satisfied for all \mathbf{v} in $Y(\Omega)$, hence for all \mathbf{v} in $X_0(\Omega)$ by density. Conversely, letting \mathbf{v} run through $\mathcal{D}(\Omega \cup \Gamma_2) \cap X_0(\Omega)$ gives the first line of (2.1.1) in the distribution sense and letting \mathbf{v} run through $Y(\Omega)$ leads to the fourth line. Due to the density of $\mathcal{D}(\Omega)$ in $L^2(\Omega)$, the second line in (2.1.1), when taken in the distribution sense, is fully equivalent to the second equation in (2.3.7). To prove the well-posedness of problem (2.3.7), we first construct a lifting of the boundary conditions. Before that we introduce the α -dependent norms

$$\|\mathbf{v}\|_{\alpha, X(\Omega)} = \left(\sum_{l=1}^L \alpha_l (\|\mathbf{v}\|_{L^2(\Omega_l)^d}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega_l)}^2) \right)^{\frac{1}{2}} \quad (2.4.1)$$

$$\|q\|_{\alpha^*, L^2(\Omega)} = \left(\sum_{l=1}^L \frac{1}{\alpha_l} \|q\|_{L^2(\Omega_l)}^2 \right)^{\frac{1}{2}} \quad (2.4.2)$$

Lemma 2.2 *There exists a divergence-free function \mathbf{u}_b in $X(\Omega)$ which satisfies*

$$\mathbf{u}_b \cdot \mathbf{n} = k \quad \text{on } \Gamma_1, \quad (2.4.3)$$

and

$$\|\mathbf{u}_b\|_{\alpha, X(\Omega)} \leq C \sqrt{\alpha_{\max}} \|k\|_{(H_{00}^{\frac{1}{2}}(\Gamma_1))'}. \quad (2.4.4)$$

where C is dependent with α .

Proof: We now denote by $I(\Omega)$ the space

$$I(\Omega) = \{\mu \in H^1(\Omega); \mu = 0 \text{ on } \Gamma_2\},$$

and we consider the problem :

Find λ in $I(\Omega)$ such that ,

$$\forall \mu \in I(\Omega), \int_{\Omega} (\mathbf{grad} \lambda) \cdot (\mathbf{grad} \mu)(\mathbf{x}) d\mathbf{x} = \int_{\Gamma_1} (k \mu)(\mathbf{x}) d\tau \quad (2.4.5)$$

This problem has a unique solution. Moreover the function $\mathbf{u}_b = \mathbf{grad} \lambda$ is divergence-free on Ω and satisfies $\mathbf{u}_b \cdot \mathbf{n} = k$ on Γ_1 , and

$$\|\mathbf{u}_b\|_{\alpha, X(\Omega)} \leq C \sqrt{\alpha_{\max}} \|k\|_{(H_{00}^{\frac{1}{2}}(\Gamma_1))'}. \quad (2.4.6)$$

To go further, we set: $\mathbf{u}_0 = \mathbf{u} - \mathbf{u}_b$, and consider the following problem: Find (\mathbf{u}_0, p) in $X_0(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} \forall \mathbf{v} \in X_0(\Omega), \quad a_{\alpha}(\mathbf{u}_0, \mathbf{v}) + b(\mathbf{v}, p) &= -a_{\alpha}(\mathbf{u}_b, \mathbf{v}) + \mathcal{L}(\mathbf{v}), \\ \forall q \in L^2(\Omega), \quad b(\mathbf{u}_0, q) &= 0. \end{aligned} \quad (2.4.7)$$

Then we can see that the kernel

$$V(\Omega) = \{\mathbf{v} \in X_0(\Omega); \forall q \in L^2(\Omega), b(\mathbf{v}, q) = 0\} \quad (2.4.8)$$

coincides with the space of functions in $X_0(\Omega)$ which are divergence-free on Ω . So it is easy to have the following Lemma about $a_{\alpha}(\cdot, \cdot)$:

Lemma 2.3 *The following ellipticity property holds*

$$\forall \mathbf{v} \in V(\Omega), a_\alpha(\mathbf{v}, \mathbf{v}) = \|\mathbf{v}\|_{\alpha, X(\Omega)}^2. \quad (2.4.9)$$

Proof: $\forall \mathbf{v} \in V(\Omega), a_\alpha(\mathbf{v}, \mathbf{v}) = \sum_{l=1}^L \alpha_l \int_{\Omega_l} \mathbf{v} \cdot \mathbf{v} d\mathbf{x} = \|\mathbf{v}\|_{\alpha, X(\Omega)}^2.$

Lemma 2.4 *There exists a constant $\beta > 0$ such that the following inf-sup condition holds*

$$\forall q \in L^2(\Omega), \sup_{\mathbf{v} \in X_0(\Omega)} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\alpha, X(\Omega)}} \geq \beta \sqrt{\frac{\alpha_{\min}}{\alpha_{\max}}} \|q\|_{\alpha^*, L^2(\Omega)}. \quad (2.4.10)$$

Proof: We first define the space

$$W = \{\mathbf{x} | \mathbf{x} \in H^1(\Omega), \mathbf{x} = 0 \text{ on } \Gamma_2\},$$

then for all $q \in L^2(\Omega)$, there exists $Y \in W$, such that

$$\begin{cases} -\Delta Y = q, & \text{in } \Omega \\ \mathbf{grad} Y \cdot \mathbf{n} = 0, & \text{on } \Gamma_1 \\ Y = 0 & \text{on } \Gamma_2 \end{cases} \quad (2.4.11)$$

Letting $\mathbf{v} = \mathbf{grad} Y$, then $\mathbf{v} \in X_0(\Omega)$ and $-\text{div } \mathbf{v} = q$ in Ω . Obviously,

$$b(\mathbf{v}, q) = - \int_{\Omega} \text{div } \mathbf{v} q d\mathbf{x} = \|q\|_{L^2(\Omega)}^2.$$

Thanks to Green's formula and Poincaré-Friedrichs inequality

$$\|\mathbf{grad} Y\|_{L^2(\Omega)}^2 = \int_{\Omega} q Y d\mathbf{x} \leq \|q\|_{L^2(\Omega)} \|Y\|_{H^1(\Omega)} \leq c \|q\|_{L^2(\Omega)} \|\mathbf{grad} Y\|_{L^2(\Omega)}.$$

It follows with

$$\|\mathbf{v}\|_{L^2(\Omega)} = \|\mathbf{grad} Y\|_{L^2(\Omega)} \leq c \|q\|_{L^2(\Omega)}$$

So

$$\|\mathbf{v}\|_{\alpha, X(\Omega)} = \left(\sum_{l=1}^L \alpha_l (\|\mathbf{v}\|_{L^2(\Omega_l)^d}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega_l)}^2) \right)^{\frac{1}{2}} \leq C \sqrt{\alpha_{\max}} \|q\|_{L^2(\Omega)}$$

Then we obtain the result

$$\forall q \in L^2(\Omega), \quad \sup_{\mathbf{v} \in X_0(\Omega)} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\alpha, X(\Omega)}} \geq c \sqrt{\frac{\alpha_{\min}}{\alpha_{\max}}} \|q\|_{\alpha^*, L^2(\Omega)}.$$

It follows from Lemma 2.3 and Lemma 2.4 that problem (2.4.7) has a unique solution (\mathbf{u}_0, p) in $X_0(\Omega) \times L^2(\Omega)$ and that this solution satisfies

$$\|\mathbf{u}_0\|_{\alpha, X(\Omega)} + c \sqrt{\frac{\alpha_{\min}}{\alpha_{\max}}} \|p\|_{\alpha^*, L^2(\Omega)} \leq C \frac{1}{\sqrt{\alpha_{\min}}} (\|\mathbf{u}_b\|_{\alpha, X(\Omega)} + \|\mathbf{f}\|_{L^2(\Omega)} + \|p_0\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}) \quad (2.4.12)$$

We now obtain the main result of this section.

Theorem 2.5 *For any data satisfying (2.3.8), problem (2.3.7) has a unique solution (\mathbf{u}, p) in $X(\Omega) \times L^2(\Omega)$. Moreover this solution satisfies*

$$\|\mathbf{u}\|_{\alpha, X(\Omega)} + c \sqrt{\frac{\alpha_{\min}}{\alpha_{\max}}} \|p\|_{L^2(\Omega)} \leq c \frac{1}{\sqrt{\alpha_{\min}}} (\sqrt{\alpha_{\max}} \|k\|_{(H_{00}^{\frac{1}{2}}(\Gamma_1))'} + \|\mathbf{f}\|_{L^2(\Omega)} + \|p_0\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}). \quad (2.4.13)$$

2.4.2 Case of velocity in $L^2(\Omega)$

Proposition 2.2 *Assume that the partition of $\partial\Omega$ into Γ_1 and Γ_2 is sufficiently smooth for $\mathcal{D}(\Omega \cup \Gamma_1)$ to be dense in $H_{\Gamma_2}^1(\Omega)$. Any smooth enough pair of functions (\mathbf{u}, p) is a solution of problem (2.3.9) if and only if it is a solution of problem (2.1.1) in the distribution sense.*

Proof: Due to the density of $\mathcal{D}(\Omega)^d$ in $L^2(\Omega)^d$, the first line in (2.1.1), when taken in the distribution sense, is fully equivalent to the first equation in (2.3.9). On the other hand, when multiplying the second line in (2.1.1) by a function q in $\mathcal{D}(\Omega \cup \Gamma_1)$,

integrating by parts and using the third line, we obtain that the second equation of (2.3.9) is satisfied for all q in $\mathcal{C}^\infty(\Omega) \cap H_{\Gamma_2}^1(\Omega)$, hence for all q in $H_{\Gamma_2}^1(\Omega)$ by density according to [15]. Conversely, letting q run through $\mathcal{D}(\Omega \cup \Gamma_1)$ gives the second line of (2.1.1) in the distribution sense and letting q run through $\mathcal{C}^\infty(\Omega) \cap H_{\Gamma_2}^1(\Omega)$ leads to the third line.

We now introduce the α -dependent norms

$$\|\mathbf{v}\|_\alpha = \left(\sum_{l=1}^L \alpha_l \|\mathbf{v}\|_{L^2(\Omega)^d}^2 \right)^{\frac{1}{2}}, \quad \|q\|_{\alpha_*} = \left(\sum_{l=1}^L \frac{1}{\alpha_l} |q|_{H^1(\Omega_l)}^2 \right)^{\frac{1}{2}} \quad (2.4.14)$$

To prove the well-posedness of problem (2.3.9), we first cite the following Property ([48], Chap. I, § 11).

Property 2.1 *For all $p_0 \in H_{00}^{\frac{1}{2}}(\Gamma_2)$, the extension \tilde{p}_0 by zero from Γ_2 to $\partial\Omega$, such that $\tilde{p}_0|_{\Gamma_2} = p_0$ belongs to $H^{\frac{1}{2}}(\partial\Omega)$ and satisfies $\|\tilde{p}_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C \|p_0\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}$.*

Then we construct a lifting of the boundary conditions of \tilde{p}_0 . We give the following lemma.

Lemma 2.5 *There exists a linear continuous map $\mathfrak{R}_0 : H^{s-1/2}(\Gamma) \rightarrow H^s(\Omega)$ such that $\mathfrak{R}_0 \tilde{p}_0 = \bar{P}$, and*

$$\inf_{\gamma_0(\bar{P})=\tilde{p}_0} \|\bar{P}\|_{H^1(\Omega)} = \|\tilde{p}_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C \|p_0\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)} \quad (2.4.15)$$

To go further, we set: $P_1 = p - \bar{P}$, and consider the following problem: Find (\mathbf{u}, P_1) in $L^2(\Omega)^d \times H_{\Gamma_2}^1(\Omega)$ such that

$$\begin{aligned} \forall \mathbf{v} \in L^2(\Omega), \quad a_\alpha(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, P_1) &= -b(\mathbf{v}, \bar{P}) + \mathcal{L}(\mathbf{v}), \\ \forall q \in H_{\Gamma_2}^1(\Omega), \quad b(\mathbf{u}, q) &= \langle k, q \rangle_{\Gamma_1}. \end{aligned} \quad (2.4.16)$$

Indeed, the form $a_\alpha(\cdot, \cdot)$ and $b(\cdot, \cdot)$ satisfy the continuity properties and we have the ellipticity property

$$\forall \mathbf{v} \in L^2(\Omega)^d, a_\alpha(\mathbf{v}, \mathbf{v}) = \|\mathbf{v}\|_\alpha^2. \quad (2.4.17)$$

We also have the inf-sup condition on the form $b(\cdot, \cdot)$ as follows:

Lemma 2.6

$$\forall q \in H_{\Gamma_2}^1(\Omega), \sup_{\mathbf{v} \in L^2(\Omega)^d} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_\alpha} \geq \|q\|_{\alpha^*} \quad (2.4.18)$$

Proof: $\forall q \in H_{\Gamma_2}^1(\Omega)$, letting $\mathbf{v} = \alpha^{-1} \nabla q$, then $\mathbf{v} \in L^2(\Omega)^d$. $b(\mathbf{v}, q) = \int_\Omega \mathbf{v} \cdot \nabla q = \|q\|_{\alpha^*}^2$. So $\frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_\alpha} \geq \frac{\|q\|_{\alpha^*}^2}{\|q\|_{\alpha^*}} \geq \|q\|_{\alpha^*}$

It follows from (2.4.17) and (2.4.18) that problem (2.4.16) has a unique solution (\mathbf{u}, P_1) in $L^2(\Omega)^d \times H_{\Gamma_2}^1(\Omega)$ and that this solution satisfies

$$\|\mathbf{u}\|_\alpha + \|P_1\|_{\alpha^*} \leq C(\|\mathbf{f}\|_\alpha + \|\bar{P}\|_{\alpha^*}) \quad (2.4.19)$$

We are now state the main result of this section.

Theorem 2.6 For any data (f, k, p_0) in $L^2(\Omega)^d \times H^{-\frac{1}{2}}(\Gamma_1) \times H_{00}^{\frac{1}{2}}(\Gamma_2)$, problem (2.3.9) has a unique solution (\mathbf{u}, p) in $L^2(\Omega)^d \times H^1(\Omega)$. Moreover this solution satisfies

$$\|\mathbf{u}\|_\alpha + \|p\|_{\alpha^*} \leq C(\|\mathbf{f}\|_\alpha + \|p_0\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)} + \sqrt{\alpha_{max}} \|k\|_{(H_{00}^{\frac{1}{2}}(\Gamma_1))'}). \quad (2.4.20)$$

Proof: We establish successively the existence and uniqueness of the solution.

1) It follows from the Lax-Milgram lemma, combined with Bramble-Hilbert inequality, that there exists a unique μ in $H_{\Gamma_2}^1(\Omega)$ such that

$$\forall \varphi \in H_{\Gamma_2}^1(\Omega), \int_\Omega (\mathbf{grad} \mu) \cdot (\mathbf{grad} \varphi) dx = \langle k, \varphi \rangle_{\Gamma_1}. \quad (2.4.21)$$

Thus, the function $\mathbf{u}_b = \mathbf{grad} \mu$ satisfies

$$\|\mathbf{u}_b\|_\alpha \leq c\sqrt{\alpha_{\max}} \|k\|_{(H_{00}^{\frac{1}{2}}(\Gamma_1))'}. \quad (2.4.22)$$

On the other hand, it follows from the standard results on saddle-point problems ([40], Chap,I, Cor.4.1) that the problems

Find (\mathbf{u}_0, P_1) in $L^2(\Omega)^d \times H_{\Gamma_2}^1(\Omega)$ such that

$$\begin{aligned} \forall \mathbf{v} \in L^2(\Omega), \quad a_\alpha(\mathbf{u}_0, \mathbf{v}) + b(\mathbf{v}, P_1) &= -b(\mathbf{v}, \bar{P}) - a_\alpha(\mathbf{u}_b, \mathbf{v}) + \mathcal{L}(\mathbf{v}), \\ \forall q \in H_{\Gamma_2}^1(\Omega), \quad b(\mathbf{u}_0, q) &= 0. \end{aligned} \quad (2.4.23)$$

has a unique solution (\mathbf{u}_0, P_1) which moreover satisfies

$$\|\mathbf{u}_0\|_\alpha + \|P_1\|_{\alpha^*} \leq C(\|\mathbf{u}_b\|_\alpha + \|\mathbf{f}\|_\alpha + \|\bar{P}\|_{\alpha^*}) \quad (2.4.24)$$

Then, the pair (\mathbf{u}, p) with $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_b$, $p = P_1 + \bar{P}$, is a solution of problem (??)-(2.3.9), and estimate (2.4.20) follows from (2.4.22) and (2.4.24). The pair $(\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_b, p = P_1 + \bar{P})$ is a solution of problem (??)-(2.3.9), and estimate (2.4.20) is a consequence of (2.4.15) and (2.4.19). On the other hand, let (\mathbf{u}_1, p_1) and (\mathbf{u}_2, p_2) be two solutions of problem (2.3.9). Then the difference $(\mathbf{u}_1 - \mathbf{u}_2, p_1 - p_2)$ is a solution of problem (2.4.16) with data $\mathbf{u}_b, \mathbf{f}, P_1$ equal to zero, also p_0 equal to zero. Thus, it follows from (2.4.20) that it is zero. So the solution of problem (2.3.9) is unique.

2.5 Regularity properties

Let \mathcal{O} be a subdomain of Ω such that α is a constant in a neighbourhood of $\bar{\mathcal{O}}$ in Ω . We first set

$$H(\mathit{curl}, \mathit{div}; \mathcal{O}) = \mathbf{H}(\mathit{curl}, \mathcal{O}) \cap \mathbf{H}(\mathit{div}, \mathcal{O}).$$

Then we introduce a space

$$W_0 = \{\mathbf{u} \in H(\mathbf{curl}, \operatorname{div}; \mathcal{O}) \mid \mathbf{u} \cdot \mathbf{n} \in L^2(\partial\mathcal{O})\}.$$

Due to ([5], Remark 2.16) and [30], when \mathcal{O} is a general Lipschitz domain, the space W_0 is imbedded in $H^{\frac{1}{2}}(\mathcal{O})$. When the domain \mathcal{O} is convex, due to ([5], Theorem 2.17), then the space W_0 is continuously imbedded in $H^1(\mathcal{O})$, in fact, $\mathbf{u} \cdot \mathbf{n}$ has even higher regularity, $\mathbf{u} \cdot \mathbf{n} \in H^{\frac{1}{2}}(\partial\mathcal{O})$. In the subdomain \mathcal{O} , taking the **curl** of the first equation in (2.1.1), yields

$$\mathbf{curl}(\alpha\mathbf{u}) = \mathbf{curl}(\alpha\mathbf{f}) \quad \text{in } \mathcal{O}.$$

This leads to the following local regularity result.

Proposition 2.3 *Let \mathcal{O} be a subdomain of Ω such that α is a constant in a neighbourhood of $\bar{\mathcal{O}}$ in Ω . Then the mapping: $(\mathbf{f}, k, p_0) \mapsto (\mathbf{u}, p)$, where (\mathbf{u}, p) is the solution of problem (2.1.1), is continuous from $H^s(\Omega)^d \times H^{s-\frac{1}{2}}(\Omega) \times H^{s+\frac{1}{2}}(\Omega)$ into $H^s(\mathcal{O})^d \times H^{s+1}(\mathcal{O})$, (i) for $0 \leq s \leq \frac{1}{2}$ in the general case, (ii) for $0 \leq s \leq 1$ when $\bar{\mathcal{O}}$ is convex.*

It follows from [31] that these results still hold with \mathcal{O} replaced by $\mathcal{O} \cap \Omega_l$

- i) in dimension $d = 2$, when the Ω_l are polygons and $\bar{\mathcal{O}}$ does not contain any vertex of the Ω_l ,
- ii) in dimension $d = 3$, when the Ω_l are polyhedras and $\bar{\mathcal{O}}$ neither contains a vertex of the Ω_l nor intersects an edge of the Ω_l .

However when $\bar{\mathcal{O}}$ either contains a vertex of the Ω_l or intersects an edge of the Ω_l and when we consider the global regularity properties, the results will be weaker. More

properties can be proved only in dimension $d = 2$ and thus we only consider this case in what follows.

We first consider the local regularity properties. We introduce here also a space

$$W = \{\mathbf{u} \in H(\text{curl}, \text{div}; \Omega) \mid \mathbf{u} \cdot \mathbf{n} \in L^2(\partial\Omega)\}.$$

We know from [30] when Ω is a polygon, a function \mathbf{u} in W can be written as

$$\mathbf{u} = \mathbf{u}_r + \mathbf{grad}S, \tag{2.5.1}$$

where \mathbf{u}_r belongs to $H^1(\Omega)^2$ and S is a linear combination of the singularities of the Laplace equation provided with Neumann boundary conditions. We recall that each singularity in the neighbourhood of a corner of the polygon with aperture ω has the form

$$r^{\frac{\pi}{\omega}}(\varphi(\theta) + (\log r)^p \psi(\theta)),$$

where r is the distance to the corner, θ the corresponding angular variable, p is equal to 0 except when $\frac{\pi}{\omega}$ is an integer where it is equal to 1. More generally, any such function \mathbf{u} which has the further property

$$\text{div } \mathbf{u} \in H^s(\Omega), \quad \mathbf{curl } \mathbf{u} \in H^s(\Omega)^3, \tag{2.5.2}$$

admits the expansion (2.5.1) with \mathbf{u}_r in $H^{s+1}(\Omega)^2$ for all $s, 0 < s < \frac{2\pi}{\omega} - 1$.

Taking the **curl** of the first equation in (2.1.1), which yields

$$\mathbf{curl}(\alpha \mathbf{u}) = \mathbf{curl}(\alpha \mathbf{f}) \quad \text{in } \Omega.$$

So when Ω is a polygon, \mathbf{u} admits the expansion (2.5.1), and if $\mathbf{curl } \mathbf{f}$ belongs to $H^s(\Omega)^2$, $0 < s < \frac{2\pi}{\omega} - 1$, where ω denotes the largest angle of Ω , the regular part \mathbf{u}_r in this expansion belongs to $H^{s+1}(\Omega)$.

We next give the regularity property of p . Taking **div** of the first equation in

(2.1.1), we consider the following mixed boundary conditions problem:

$$\begin{cases} \operatorname{div} \left(\frac{1}{\alpha} \mathbf{grad} p \right) = \operatorname{div} \mathbf{f} & \text{in } \Omega, \\ \frac{1}{\alpha} \partial_{\mathbf{n}} p = \mathbf{f} \cdot \mathbf{n} & \text{on } \Gamma, \\ p = p_0 & \text{on } \Gamma_2 \end{cases} \quad (2.5.3)$$

We know from [34] that there exists a real number s_0 , $0 < s_0 < \frac{1}{2}$, only depending on the geometry of Ω , α_{\min} and α_{\max} , such that when $\operatorname{div} \mathbf{f} \in H^{s-1}(\Omega)$ and $p_0 \in H_{\diamond}^{s+\frac{1}{2}}(\Gamma_2)$, the solution p of the equation (2.5.3) belongs to $H^{s+1}(\Omega)$ for all $s \leq s_0$, where $H_{\diamond}^{s+\frac{1}{2}}(\Gamma_2)$ stands for the space of functions such that their restrictions to any edge e contained in $\bar{\Gamma}_2$ belongs to $H^{s+\frac{1}{2}}(e)$. Moreover, when Ω is a polygon, according to ([42], Chapter 5) and [34], p will belong to $H^{s+1}(\Omega)$, $s \leq \min\{s_0, \frac{2\pi}{\omega} - 1\}$.

We then consider the local regularity properties when $\bar{\mathcal{O}}$ contains either vertex of the Ω_l or intersects an edge of the Ω_l . we are now at the last conclusion of this section.

Suppose $\mathbf{f}|_{\Omega_l}, k, p_0$ belongs to $H^{s_l}(\Omega_l)^2 \times H^{s_l-\frac{1}{2}}(\Omega_l) \times H^{s_l+\frac{1}{2}}(\Omega_l)$ for any real positive number s_l : *i*) When $\bar{\Omega}_l$ contains no corner of Ω , the regularity of (\mathbf{u}, p) on Ω_l only depends on the data \mathbf{f} , and (\mathbf{u}, p) belongs to $H^{s_l}(\Omega_l)^2 \times H^{s_l+1}(\Omega_l)$.

ii) When $\bar{\Omega}_l$ contains a corner of Ω with ω , we derive from (2.5.1) that $\mathbf{u}|_{\Omega_l} = \mathbf{u}_r + \mathbf{grad} S$, $p|_{\Omega_l} = p_r + S$. So the first equation of (5.1.1) can be rewritten as

$$\frac{1}{\alpha} \mathbf{grad} p' = \mathbf{f} - \mathbf{u}_r \text{ where } p' = p_r + (1 + \alpha) S. \quad (2.5.4)$$

Taking the **curl** of the first equation of (5.1.1) and taking *div* of the equation (2.5.4), we can get that (\mathbf{u}_r, p_r) belongs to $H^s(\Omega_l)^2 \times H^{s+1}(\Omega_l)$ for $s \leq s_l$ and $s < \frac{2\pi}{\omega}$.

Chapter 3

Spectral Discretization of Darcy's Equations

3.1 Orthogonal polynomials

We firstly work on the interval $\Lambda = [-1, 1]$, since all results on Ω are deduced from their analogues on Λ thanks to tensorization arguments. Over Λ , all the approximation properties rely on a proper choice of an orthogonal basis of $L^2(\Lambda)$: the Legendre polynomials.

The family of Legendre polynomials is a family $(L_k)_{k \geq 0}$ of polynomials in one variable, which are orthogonal to each other in $L^2(\Lambda)$. For any integer $k \geq 0$, the polynomial L_k has degree k . If $L_k(x)$ is normalized so that $L_k(1) = 1$, then for any k :

$$L_k(x) = \frac{1}{2^k} \sum_{(l=0)}^{[k/2]} (-1)^l \binom{k}{l} \binom{2k-2l}{k} x^{k-2l}, \quad (3.1.1)$$

where $[k/2]$ denotes the integral part of $k/2$.

It can be noted that, for all k , the k zeros of L_k are distinct real numbers in Λ and

also that the polynomial L_k is even if k is even and odd if k is odd.

The norm of each L_k is given by the formula

$$\int_{-1}^1 L_k^2(x) dx = \frac{1}{k + \frac{1}{2}}. \quad (3.1.2)$$

We recall from ([1], Chapter 22) some very important properties of these polynomials, firstly the differential equation

$$((1 - x^2)L_k')' + k(k + 1)L_k = 0, k \geq 0, \quad (3.1.3)$$

secondly the integral equation

$$(2k + 1)L_k = L_{k+1}' - L_{k-1}', k \geq 1, \quad (3.1.4)$$

thirdly the induction formula, which allows for computing them:

$$L_{k+1}(x) = \frac{2k + 1}{k + 1}xL_k(x) - \frac{k}{k + 1}L_{k-1}(x), \quad (3.1.5)$$

where $L_0(x) = 1$ and $L_1(x) = x$.

We consider now discrete Legendre series. Since explicit formulas for the quadrature nodes are not known, such points have to be computed numerically as zeros of approximate polynomials. The quadrature weights can be expressed in closed form in terms of the nodes, as indicated in the following formulas: ([44], Chapter 2)

- Legendre- Gauss (LG) : for $j = 1, \dots, N$

$$x_j \text{ are zeros of } L_{N+1}; \quad (3.1.6)$$

$$\omega_j = \frac{2}{(1 - x_j^2)[L_{N+1}'(x_j)]^2}. \quad (3.1.7)$$

- Legendre -Gauss-Lobatto (LGL) : for $j = 0, \dots, N$

$$x_j \text{ are zeros of } (1 - x^2)L'_N; \quad (3.1.8)$$

$$\omega_j = \frac{2}{N(N+1)} \frac{1}{[L'_N(x_j)]^2}. \quad (3.1.9)$$

3.2 Quadrature Formula

As we know that, the most natural quadrature formula in the polynomial context is the Gauss Lobatto formula associated with the Lebesgue measure, since it allows for easily handling boundary conditions and leads to diagonal mass matrices. We firstly describe it, next we study the corresponding interpolation operators.

The Gauss Lobatto formula

Let us set $x_0 = -1$ and $x_N = 1$. There exists a unique set of $N - 1$ nodes x_j , $1 \leq j \leq N - 1$ and a unique set of $N + 1$ weights ρ_j , $0 \leq j \leq N$, such that the following exactness property holds

$$\forall \phi \in \mathbb{P}_{2N-1}(\Lambda), \quad \int_{-1}^1 \phi(x) dx = \sum_{j=0}^N \phi(x_j) \rho_j. \quad (3.2.1)$$

The zeros x_j , $1 \leq j \leq N - 1$, are the zeros of L'_N and the weights ρ_j are positive, given by

$$\rho_j = \frac{2}{N(N+1)L_N^2(x_j)}, \quad 0 \leq j \leq N. \quad (3.2.2)$$

We conclude with a further useful property of the quadrature formula.

Lemma 3.1 *There exists a constant c such that*

$$\forall \phi_N \in \mathbb{P}_N(\Lambda), \quad \|\phi_N\|_{L^2(\Lambda)}^2 \leq \sum_{j=0}^N \phi_N^2(x_j) \rho_j \leq 3 \|\phi_N\|_{L^2(\Lambda)}^2. \quad (3.2.3)$$

Proof: We write the expansion of any polynomial ϕ_N in $L^2(\Lambda)$:

$$\phi_N = \sum_{n=0}^N \alpha_n L_n. \quad (3.2.4)$$

We deduce from the exactness property (3.2.1) that

$$\begin{aligned} \|\phi_N\|_{L^2(\Lambda)}^2 &= \sum_{n=0}^N \alpha_n^2 \|L_n\|_{L^2(\Lambda)}^2, \\ \sum_{j=0}^N \phi_N^2(x_j) \rho_j &= \sum_{n=0}^{N-1} \alpha_n^2 \|L_n\|_{L^2(\Lambda)}^2 + \alpha_N^2 \sum_{j=0}^N L_N^2(x_j) \rho_j, \end{aligned}$$

so that it suffices to prove the results with $\phi_N = L_N$. Noting that the degree of the polynomial $L_N^2 + N^{-2}(1-x^2)L_N'^2$ is less than $2N-1$ (this comes by computing the coefficient of x^{2N}), we derive that

$$\int_{-1}^1 (L_N^2(x) + N^{-2}(1-x^2)L_N'^2(x)) dx = \sum_{j=0}^N (L_N^2(x_j) + N^{-2}(1-x_j^2)L_N'^2(x_j)) \rho_j, \quad (3.2.5)$$

or equivalently, since the x_j are the zeros of $(1-x^2)L_N'$,

$$\sum_{j=0}^N L_N^2(x_j) \rho_j = \int_{-1}^1 L_N^2(x) dx + N^{-2} \int_{-1}^1 L_N'^2(x) (1-x^2) dx. \quad (3.2.6)$$

The second integral is easily computed, leading to

$$\sum_{j=0}^N L_N^2(x_j) \rho_j = \left(2 + \frac{1}{N}\right) \|L_N\|_{L^2(\Lambda)}^2. \quad (3.2.7)$$

Since $N \geq 1$, this gives the desired result.

3.3 Polynomial approximation

3.3.1 Polynomial approximation on the interval

Let now N be a fixed positive integer ≥ 2 . In what follow, c stands for a positive constant independent of N . We introduce the orthogonal projection operator π_N from $L^2(\Lambda)$ onto $\mathbb{P}_N(\Lambda)$. Then the projection operator can be expressed in the following way:

$$\forall \phi \in L^2(\Lambda), \quad \pi_N \phi = \sum_{n=0}^N \phi_n L_n, \quad (3.3.1)$$

where

$$\phi_n = \frac{1}{\|L_n\|_{L^2(\Lambda)}^2} \int_{-1}^1 \phi(x) L_n(x) dx. \quad (3.3.2)$$

Lemma 3.2 *For any real numbers $s \geq 0$, the following estimate holds for any ϕ in $H^s(\Lambda)$:*

$$\|\phi - \pi_N \phi\|_{L^2(\Lambda)} \leq cN^{-s} \|\phi\|_{H^s(\Lambda)}. \quad (3.3.3)$$

For the proof of this lemma, we propose ([23], Proposition 1).

We now consider the orthogonal projection operator $\pi_N^{1,0}$ from $H_0^1(\Lambda)$ onto $\mathbb{P}_N^0(\Lambda)$, for the scalar product associated with the norm $|\cdot|_{H^1(\Lambda)}$. Its properties are derived from the formula $(\pi_N^{1,0})' = \pi_{N-1}$ for the bounded in $H^1(\Lambda)$, plus a duality argument for the bounded in $L^2(\Lambda)$.

Proposition 3.1 ([23], Proposition 2) *For any real number $s \geq 1$, the following estimate holds for any ϕ in $H^s(\Lambda) \cap H_0^1(\Lambda)$:*

$$|\phi - \pi_N^{1,0} \phi|_{H^1(\Lambda)} + N \|\phi - \pi_N^{1,0} \phi\|_{L^2(\Lambda)} \leq cN^{1-s} \|\phi\|_{H^s(\Lambda)}. \quad (3.3.4)$$

Finally, with any function ϕ in $H^1(\Lambda)$, hence continuous, we associate the function

ϕ_0 defined by

$$\phi_0(x) = \phi(x) - \phi(-1)\frac{1-x}{2} - \phi(1)\frac{1+x}{2}, \quad (3.3.5)$$

and since it belongs to $H_0^1(\Lambda)$, we define the polynomial $\pi_N^1\phi$ by

$$(\pi_N^1\phi)(x) = (\pi_N^{1,0}\phi_0)(x) + \phi(-1)\frac{1-x}{2} + \phi(1)\frac{1+x}{2}. \quad (3.3.6)$$

This operator does not correspond to an orthogonal projection, but it preserves the values in ± 1 and its properties are easily derived from the formula $\phi - \pi_N^1\phi = \phi_0 - \pi_N^{1,0}\phi_0$.

Proposition 3.2 (*[23], Proposition 3*) *For any real number $s \geq 1$, the following estimate holds for any ϕ in $H^s(\Lambda) \cap H_0^1(\Lambda)$:*

$$|\phi - \pi_N^1\phi|_{H^1(\Lambda)} + N\|\phi - \pi_N^1\phi\|_{L^2(\Lambda)} \leq cN^{1-s}\|\phi\|_{H^s(\Lambda)}. \quad (3.3.7)$$

3.3.2 Polynomial approximation on the square or cube

For each $n \geq 0$, let $\mathbb{P}_n(\Omega)$ be the space of polynomials worth degree $\leq n$ with respect to each variable, and \mathbb{P}_n^0 be its subspace made of all polynomials which vanish on the boundary $\partial\Omega$.

As previously, we fix an integer $N \geq 2$. The best fit of a given function is realized via projection operators onto $\mathbb{P}_N(\Omega)$ or $\mathbb{P}_N^0(\Omega)$, and analyzing their properties relies on tensorization arguments. For instance, if Π_N denotes the orthogonal projection operator from $L^2(\Omega)$ onto $\mathbb{P}_N(\Omega)$, it coincides with $\pi_N^{(x)} \circ \pi_N^{(y)}$ in dimension $d = 2$, with $\pi_N^{(x)} \circ \pi_N^{(y)} \circ \pi_N^{(z)}$ in dimension $d = 3$, where the exponent after an operator denotes the direction in which it is applied. So the approximation properties of the operator Π_N are derived from the triangular inequality, in dimension $d = 2$ for instance:

$$\|v - \Pi_N v\|_{L^2(\Omega)} \leq \|v - \pi_N^{(x)} v\|_{L^2(\Omega)} + \|\pi_N^{(x)}(v - \pi_N^{(y)} v)\|_{L^2(\Omega)}. \quad (3.3.8)$$

Proposition 3.3 ([23], Proposition 4) *For any real number $s \geq 0$, the following estimate holds for any v in $H^s(\Omega)$:*

$$\|v - \Pi_N v\|_{L^2(\Omega)} \leq cN^{-s} \|v\|_{H^s(\Omega)}. \quad (3.3.9)$$

Let now $\Pi_N^{1,0}$ stand for the orthogonal projection operator from $H_0^1(\Omega)$ onto $\mathbb{P}_N^0(\Omega)$. Using a further duality argument, we derive the next proposition.

Proposition 3.4 ([23], Proposition 5) *For any real number $s \geq 1$, the following estimate holds for any v in $H^s(\Omega) \cap H_0^1(\Omega)$:*

$$\|v - \Pi_N^{1,0} v\|_{H^1(\Omega)} + N \|v - \Pi_N^{1,0} v\|_{L^2(\Omega)} \leq cN^{1-s} \|v\|_{H^s(\Omega)}. \quad (3.3.10)$$

Finally, the approximation properties of the orthogonal projection operator Π_N^1 from $H^1(\Omega)$ onto $\mathbb{P}_N(\Omega)$ are derived from the same triangular inequality .

Proposition 3.5 ([23], Proposition 6) *For any real number $s \geq 1$, the following estimate holds for any v in $H^s(\Omega)$:*

$$\|v - \Pi_N^1 v\|_{H^1(\Omega)} + N \|v - \Pi_N^1 v\|_{L^2(\Omega)} \leq cN^{1-s} \|v\|_{H^s(\Omega)}. \quad (3.3.11)$$

3.4 Polynomial interpolation

In this section, we will introduce the interpolation operator on Gauss-Lobatto points.

3.4.1 Polynomial interpolation on the interval

Our aim is to derive the approximation properties of the Lagrange interpolation operator i_N at the nodes $x_j, 0 \leq j \leq N$, with values in $\mathbb{P}_N(\Lambda)$, which means that

$$(i_N\phi)(x_j) = \phi(x_j), \quad 0 \leq j \leq N. \quad (3.4.1)$$

Lemma 3.3 ([23], Lemma 3) *There exists a positive constant c such that, for any function ϕ in $H_0^1(\Lambda)$,*

$$\|i_N\phi\|_{L^2(\Lambda)} \leq c(\|\phi\|_{L^2(\Lambda)} + N^{-1}\|\phi\|_{H^1(\Lambda)}). \quad (3.4.2)$$

Applying the above inequality to the function $\phi - \pi_N^1\phi$ obviously gives an estimate for the interpolation error of functions which belongs to $H^1(\Lambda)$. However, we have rather state a slightly more general result which allows for interpolating less smooth functions.

Proposition 3.6 ([23], Proposition 7) *For any real number $s > \frac{1}{2}$, the following estimate holds for any ϕ in $H^s(\Lambda)$:*

$$\|\phi - i_N\phi\|_{L^2(\Lambda)} \leq cN^{-s}\|\phi\|_{H^s(\Lambda)}. \quad (3.4.3)$$

Proposition 3.7 ([21], Theorem 13.4) *For any real number r and s satisfying $s > \max\{\frac{1}{2}, r\}$ and $0 \leq r \leq 1$, there exists a positive constant c depending only on s such that, for any function ϕ in $H^s(\Lambda)$, the following estimate holds:*

$$\|\phi - i_N\phi\|_{H^r(\Lambda)} \leq cN^{r-s}\|\phi\|_{H^s(\Lambda)}. \quad (3.4.4)$$

3.4.2 Polynomial interpolation on the square or cube

By tensorization, we associate with the nodes x_j the grid

$$\Xi_N = \begin{cases} \{(x_i, x_j), 0 \leq i, j \leq N\} & \text{in dimension } d=2, \\ \{(x_i, x_j, x_k), 0 \leq i, j, k \leq N\} & \text{in dimension } d=3, \end{cases} \quad (3.4.5)$$

and also the discrete product: for all continuous functions u and v on $\bar{\Omega}$,

$$((u, v))_N = \begin{cases} \sum_{i=0}^N \sum_{j=0}^N u(x_i, x_j) v(x_i, x_j) \rho_i \rho_j & \text{in dimension } d=2, \\ \sum_{i=0}^N \sum_{j=0}^N \sum_{k=0}^N u(x_i, x_j, x_k) v(x_i, x_j, x_k) \rho_i \rho_j \rho_k & \text{in dimension } d=3, \end{cases} \quad (3.4.6)$$

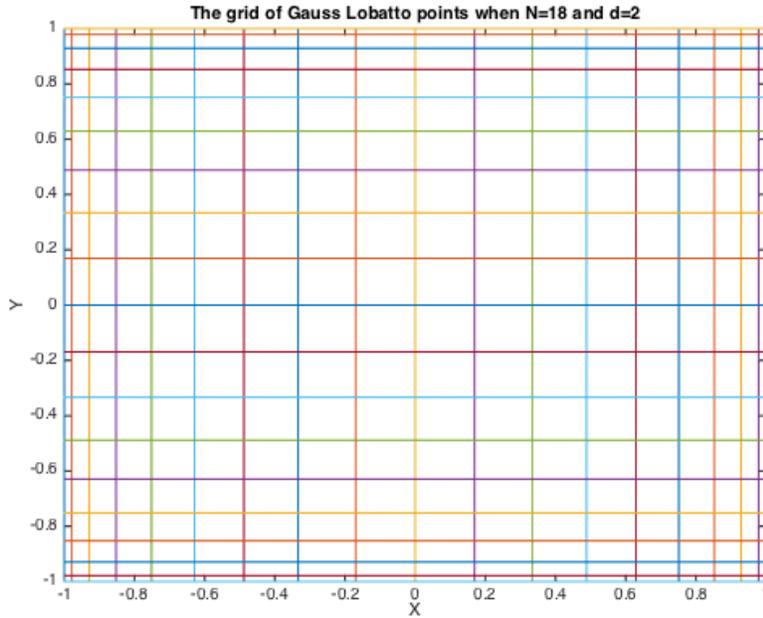


Figure 3-1: The Grid of Gauss Lobatto points when $N=18$ and $d=2$

We denote by \mathcal{I}_N the Lagrange interpolation operator on the grid Ξ_N , with values in $\mathbb{P}_N(\Omega)$. The operator coincides with $i_N^{(x)} \circ i_N^{(y)}$ in dimension $d = 2$, and with $i_N^{(x)} \circ i_N^{(y)} \circ i_N^{(z)}$ in dimension $d = 3$. So, proving its approximation properties is now obvious.

Proposition 3.8 For any real number $s > \frac{d}{2}$, the following estimate holds for any v in $H^s(\Omega)$:

$$\|v - \mathcal{I}_N v\|_{L^2(\Omega)} \leq cN^{-s} \|v\|_{H^s(\Omega)}. \quad (3.4.7)$$

For any real number r and s satisfying $s > \max\{\frac{d}{2}, r\}$ and $0 \leq r \leq 1$, there exists a positive constant c depending only on s such that, for any function ϕ in $H^s(\Omega)$, the following estimate holds:

$$\|v - \mathcal{I}_N v\|_{H^r(\Omega)} \leq cN^{r-s} \|v\|_{H^s(\Omega)}. \quad (3.4.8)$$

3.5 Spectral discretization of the Darcy's problem

According to the variational forms in Chapter 2, we here discuss the discrete problems.

3.5.1 First discretization

According to the variational formulation, we introduce the discrete spaces as follows:

$$\mathbb{D}_N = \mathbb{P}_N(\Omega)^d, \quad (3.5.1)$$

also we introduce another space:

$$\mathbb{D}_N^0 = \{\mathbf{u}_N \in \mathbb{D}_N, \mathbf{u}_N \cdot \mathbf{n} = 0 \text{ on } \Gamma_1\}. \quad (3.5.2)$$

For the space of pressure, there are different choices. Usually we consider the spurious modes on the pressure, namely the set

$$Z_n = \{q_N \in \mathbb{P}_N(\Omega), \forall \mathbf{v}_N \in \mathbb{D}_N^0, b_N(\mathbf{v}_N, q_N) = 0\} \quad (3.5.3)$$

which is analyzed in [8] and [25]. The space of pressure \mathbb{M}_N will be taken as the orthogonal space of Z_n in \mathbb{P}_N .

But here we choose another pressure space as $\mathbb{M}_N = \mathbb{P}_{N-2}(\Omega)$ which has not been discussed yet in the literature as far as the author's knowledge. Assuming that the functions f, k, p_0 has continuous restrictions to $\bar{\Omega}$ and $\bar{\Gamma}_1, \bar{\Gamma}_2$ respectively. Then the discrete problem built from the variational formulation reads: Find (\mathbf{u}_N, p_N) in $\mathbb{D}_N \times \mathbb{M}_N$ such that

$$\mathbf{u}_N \cdot \mathbf{n} = \mathcal{I}_N^{\Gamma_1} k \quad \text{on } \Gamma_1, \quad (3.5.4)$$

and

$$\begin{aligned} \forall \mathbf{v}_N \in \mathbb{D}_N^0, \quad a_\alpha^N(\mathbf{u}_N, \mathbf{v}_N) + b_N(\mathbf{v}_N, p_N) &= \mathcal{L}_N(\mathbf{v}_N), \\ \forall q_N \in \mathbb{M}_N, \quad b_N(\mathbf{u}_N, q_N) &= 0. \end{aligned} \quad (3.5.5)$$

where the bilinear forms $a_\alpha^N(\cdot, \cdot)$ and $b_N(\cdot, \cdot)$ are defined by

$$\begin{aligned} a_\alpha^N(\mathbf{u}_N, \mathbf{v}_N) &= \alpha((\mathbf{u}_N, \mathbf{v}_N))_N, \\ b_N(\mathbf{v}_N, q_N) &= -((div \mathbf{v}_N, q_N))_N, \\ \mathcal{L}_N(\mathbf{v}_N) &= ((\alpha \mathbf{f}, \mathbf{v}_N))_N - ((p_0, \mathbf{v}_N \cdot \mathbf{n}))_N^{\Gamma_2}. \end{aligned} \quad (3.5.6)$$

Next we introduce the α - dependent norms

$$\begin{aligned} \|\mathbf{v}_N\|_\alpha &= (\alpha(\|\mathbf{v}_N\|^2 + \|div \mathbf{v}_N\|^2))^{\frac{1}{2}}, \\ \|q_N\|_{\alpha^*} &= (\alpha^{-1}\|q_N\|^2)^{\frac{1}{2}}. \end{aligned} \quad (3.5.7)$$

Lemma 3.4 *There exists a divergence-free function \mathbf{u}_N^b in \mathbb{D}_N which satisfies*

$$\mathbf{u}_N^b \cdot \mathbf{n} = \mathcal{I}_N^{\Gamma_1} k \quad \text{on } \Gamma_1, \quad (3.5.8)$$

and

$$\|\mathbf{u}_N^b\|_\alpha \leq C \sqrt{\alpha_{\max}} \|\mathcal{I}_N^{\Gamma_1} k\|_{(H_{00}^{\frac{1}{2}}(\Gamma_1))'}. \quad (3.5.9)$$

where C is dependent with α .

To go further, we set: $\mathbf{u}_N^0 = \mathbf{u} - \mathbf{u}_N^b$, and consider the following problem: Find (\mathbf{u}_N^0, p) in $\mathbb{D}_N^0 \times \mathbb{M}_N$ such that

$$\begin{aligned} \forall \mathbf{v}_N \in \mathbb{D}_N^0, \quad a_\alpha^N(\mathbf{u}_N^0, \mathbf{v}) + b(\mathbf{v}, p) &= -a_\alpha^N(\mathbf{u}_N^b, \mathbf{v}) + \mathcal{L}_N(\mathbf{v}_N), \\ \forall q \in \mathbb{M}_N, \quad b_N(\mathbf{u}_N^0, q) &= 0. \end{aligned} \quad (3.5.10)$$

Then we can see that the kernel

$$V_N = \{\mathbf{v}_N \in \mathbb{D}_N^0; \forall q_N \in \mathbb{M}_N, b_N(\mathbf{v}_N, q_N) = 0\} \quad (3.5.11)$$

coincides with the space of functions in \mathbb{D}_N^0 which are divergence-free on Ω . So it is easy to have the following Lemma about $a_\alpha^N(\cdot, \cdot)$:

Lemma 3.5 *There exists a constant c independent of the discretization, such that the following ellipticity property holds*

$$\forall \mathbf{v}_N \in V_N, a_\alpha^N(\mathbf{v}_N, \mathbf{v}_N) \geq c \|\mathbf{v}_N\|_\alpha^2. \quad (3.5.12)$$

Next, we come to the inf-sup condition.

Lemma 3.6 *There exists a constant $\beta > 0$ such that the following inf-sup condition holds*

$$\forall q_N \in \mathbb{M}_N, \quad \sup_{\mathbf{v}_N \in \mathbb{D}_N^0} \frac{b_N(\mathbf{v}_N, q_N)}{\|\mathbf{v}_N\|_\alpha} \geq \beta \|q_N\|_{\alpha^*}. \quad (3.5.13)$$

Proof: The proof is given only in the case $d = 2$, since the corresponding proof in the case $d = 3$ is completely similar. Any function q_N in \mathbb{M}_N has the expansion

$$q_N(x, y) = \sum_{m=0}^{N-2} \sum_{n=0}^{N-2} \alpha_{mn} L_m(x) L_n(y). \quad (3.5.14)$$

We can derive that

$$\|q_N\|_{L^2(\Omega)}^2 = \sum_{m=0}^{N-2} \sum_{n=0}^{N-2} \alpha_{mn}^2 \frac{1}{(m + \frac{1}{2})(n + \frac{1}{2})} \quad (3.5.15)$$

The idea is now to choose $\mathbf{v}_N = (v1_N, v2_N)$, with

$$v1_N(x, y) = \sum_{m=0}^{N-2} \sum_{n=0}^m \alpha_{mn} \frac{L_{m+1}(x) - L_{m-1}(x)}{2m + 1} L_n(y) \quad (3.5.16)$$

and

$$v2_N(x, y) = \sum_{m=0}^{N-2} \sum_{n=m+1}^{N-2} \alpha_{mn} L_m(x) \frac{L_{m+1}(y) - L_{m-1}(y)}{2m + 1}. \quad (3.5.17)$$

In fact, the polynomial \mathbf{v}_N belongs to \mathbb{D}_N^0 , and which satisfies $\operatorname{div} \mathbf{v}_N = -q_N$. Next, we note that

$$\frac{\partial v1_N(x, y)}{\partial x} = \sum_{m=0}^{N-2} \sum_{n=0}^m \alpha_{mn} L_m(x) L_n(y) \quad (3.5.18)$$

which allows for bounding $\|\frac{\partial v1_N(x, y)}{\partial x}\|_{L^2(\Omega)}$ by comparing with (3.5.15). Using a similar argument for $\frac{\partial v2_N(x, y)}{\partial y}$, we derive that

$$\left\| \frac{\partial v1_N(x, y)}{\partial x} \right\|_{L^2(\Omega)} + \left\| \frac{\partial v2_N(x, y)}{\partial y} \right\|_{L^2(\Omega)} \leq c \|q_N\|_{L^2(\Omega)}, \quad (3.5.19)$$

and also, thanks to the Poincaré-Friedrichs inequality applied with respect either to x or y ,

$$\|v1_N(x, y)\|_{L^2(\Omega)} + \|v2_N(x, y)\|_{L^2(\Omega)} \leq c \|q_N\|_{L^2(\Omega)}, \quad (3.5.20)$$

then the inf-sup condition is obviously.

It follows from Lemma 3.5.12 and Lemma 3.6 that problem (3.5.10) has a unique solution (\mathbf{u}_N^0, p) in $\mathbb{D}_N^0 \times \mathbb{M}_N$ and that this solution satisfies

$$\|\mathbf{u}_0\|_\alpha + \|p\|_{\alpha^*} \leq C(\|\mathbf{u}_b\|_\alpha + \|\mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^d} + \|\mathcal{I}_N^{\Gamma_2} p_0\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}) \quad (3.5.21)$$

We now state the main result of this section.

Theorem 3.1 *For any data (\mathbf{f}, k, p_0) such that each \mathbf{f} , and k, p_0 are continuous on $\bar{\Omega}_k$ and on Γ_1, Γ_2 , problem (3.5.4) – (3.5.5) has a unique solution (\mathbf{u}_N, p_N) in $\mathbb{D}_N \times \mathbb{M}_N$. Moreover this solution satisfies*

$$\|\mathbf{u}_N\|_\alpha + \|p_N\|_{\alpha^*} \leq c(\|\mathcal{I}_N^{\Gamma_1} k\|_{(H_{00}^{\frac{1}{2}})'}(\Gamma_1)} + \|\mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^d} + \|\mathcal{I}_N^{\Gamma_2} p_0\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}). \quad (3.5.22)$$

Proof: The pair $(\mathbf{u} = \mathbf{u}_N^0 + \mathbf{u}_N^b, p)$ is a solution of problem (3.5.4)-(3.5.5), and estimate (3.5.22) is a consequence of (3.5.21) and (3.5.62). On the other hand, let (\mathbf{u}_1, p_1) and (\mathbf{u}_2, p_2) be two solutions of problem (3.5.4)-(3.5.5). Then the difference $(\mathbf{u}_1 - \mathbf{u}_2, p_1 - p_2)$ is a solution of problem (3.5.10) with data $\mathbf{u}_b, \mathbf{f}, p_0$ equal to zero. Thus, it follows from (3.5.22) that it is zero. So the solution of problem (3.5.4)-(3.5.5) is unique.

• **Error estimate :** here we introduce a new norm as follows:

$$\|\mathbf{u}_N\|_{\alpha, L^2(\Omega)} = (\alpha \|\mathbf{u}_N\|^2)^{\frac{1}{2}}.$$

We prove an error estimates, first for the velocity, second for the pressure. Let u be the solution of equation (2.1.1), u_N the solution of equation (3.5.4)-(3.5.5) and $\boldsymbol{\omega}_N$ any function in the kernel $V_N(\Omega)$. Multiply the first line of (2.1.1) by $\boldsymbol{\omega}_N$ gives

$$\alpha \int_{\Omega} \mathbf{u}_0 \cdot \boldsymbol{\omega}_N d\mathbf{x} + b(\boldsymbol{\omega}_N, p) = \alpha \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\omega}_N d\mathbf{x} - \alpha \int_{\Omega} \mathbf{u}_b \cdot \boldsymbol{\omega}_N d\mathbf{x}. \quad (3.5.23)$$

Using the definition of $V_N(\Omega)$ thus implies, for any q_N in $\mathbb{M}_N(\Omega)$,

$$\alpha \int_{\Omega} \mathbf{u}_0 \cdot \boldsymbol{\omega}_N d\mathbf{x} + b(\boldsymbol{\omega}_N, p - q_N) = \alpha \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\omega}_N d\mathbf{x} - \alpha \int_{\Omega} \mathbf{u}_b \cdot \boldsymbol{\omega}_N d\mathbf{x}. \quad (3.5.24)$$

In (3.5.24), letting $\boldsymbol{\omega}_N = \mathbf{u}_N^0 - \mathbf{v}_N$ and subtracting the first line of (3.5.10) leads to

$$\begin{aligned}
 \|\mathbf{u}_N^0 - \mathbf{v}_N\|_{\alpha, L^2(\Omega)}^2 &\leq \alpha \int_{\Omega} (\mathbf{u}_0 - \mathbf{v}_N) \cdot (\mathbf{u}_N^0 - \mathbf{v}_N) d\mathbf{x} + \alpha \int_{\Omega} \mathbf{v}_N \cdot (\mathbf{u}_N^0 - \mathbf{v}_N) d\mathbf{x} \\
 &\quad - \alpha ((\mathbf{v}_N, \mathbf{u}_N^0 - \mathbf{v}_N))_N^k + b(\mathbf{u}_N^0 - \mathbf{v}_N, p - q_N) + \alpha ((\mathbf{f}, \mathbf{u}_N^0 - \mathbf{v}_N))_N^k \\
 &\quad - \alpha \int_{\Omega} \mathbf{f} \cdot (\mathbf{u}_N^0 - \mathbf{v}_N) d\mathbf{x} + \alpha \int_{\Omega} \mathbf{u}^b \cdot (\mathbf{u}_N^0 - \mathbf{v}_N) d\mathbf{x} - \\
 &\quad \alpha ((\mathbf{u}_N^b, \mathbf{u}_N^0 - \mathbf{v}_N))_N^k + \int_{\Gamma_2} p_0 \cdot [(\mathbf{u}_N^0 - \mathbf{v}_N) \cdot \mathbf{n}] ds - ((p_0, (\mathbf{u}_N^0 - \mathbf{v}_N) \cdot \mathbf{n}))_N^{\Gamma_2}.
 \end{aligned} \tag{3.5.25}$$

Using a triangle inequality yields

$$\begin{aligned}
 \|\mathbf{u} - \mathbf{u}_N\|_{\alpha, L^2(\Omega)} &\leq \|\mathbf{u}_0 - \mathbf{v}_N\|_{\alpha, L^2(\Omega)} + \|\mathbf{u}_N^0 - \mathbf{v}_N\|_{\alpha, L^2(\Omega)} + \|\mathbf{u}_b - \mathbf{u}_N^b\|_{\alpha, L^2(\Omega)} \\
 &\leq C \left\{ \inf_{\mathbf{v}_N \in V_N} \|\mathbf{u}_0 - \mathbf{v}_N\|_{\alpha, L^2(\Omega)} + \sup_{\boldsymbol{\omega}_N \in \mathbb{D}_N} \left\{ \frac{\alpha \int_{\Omega} \mathbf{v}_N \cdot \boldsymbol{\omega}_N d\mathbf{x} - \alpha ((\mathbf{v}_N, \boldsymbol{\omega}_N))_N^k}{\|\boldsymbol{\omega}_N\|_{\alpha, L^2(\Omega)}} \right. \right. \\
 &\quad \left. \left. + \frac{\alpha ((\mathbf{f}, \boldsymbol{\omega}_N))_N^k - \alpha \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\omega}_N d\mathbf{x}}{\|\boldsymbol{\omega}_N\|_{\alpha, L^2(\Omega)}} + \frac{\alpha \int_{\Omega} \mathbf{u}_b \cdot \boldsymbol{\omega}_N d\mathbf{x} - \alpha ((\mathbf{u}_N^b, \boldsymbol{\omega}_N))_N^k}{\|\boldsymbol{\omega}_N\|_{\alpha, L^2(\Omega)}} \right. \right. \\
 &\quad \left. \left. + \frac{\int_{\Gamma_2} p_0 \cdot (\boldsymbol{\omega}_N \cdot \mathbf{n}) ds - ((p_0, (\boldsymbol{\omega}_N \cdot \mathbf{n}))_N^{\Gamma_2})}{\|\boldsymbol{\omega}_N\|_{\alpha, L^2(\Omega)}} + \frac{b(\boldsymbol{\omega}_N, p - q_N)}{\|\boldsymbol{\omega}_N\|_{\alpha, L^2(\Omega)}} + \|\mathbf{u}_b - \mathbf{u}_N^b\|_{\alpha, L^2(\Omega)} \right\}.
 \end{aligned} \tag{3.5.26}$$

All the items above can be found in ([25], Chapter V), we will not prove here and we will give the first theorem about the velocity as follows.

Theorem 3.2 *Assume that the solution (\mathbf{u}, p) of the problem (2.1.1) in $H^m(\Omega)^d \times H^m(\Omega)$ for $m \geq d/2$; the function \mathbf{f} belongs to $H^m(\Omega)^d$ for $m \geq d/2$; the function k on Γ_1 in $H^s(\Gamma_1)$ for $s > \frac{d-1}{2}$, and p_0 on Γ_2 in $H^t(\Omega)$ for $t > \frac{d-1}{2}$, we have the following estimate:*

$$\begin{aligned}
 \|\mathbf{u} - \mathbf{u}_N\|_{\alpha, L^2(\Omega)} &\leq c(N^{-m} (\|\mathbf{u}\|_{H^m(\Omega)^d} + \|p\|_{H^m(\Omega)}) \\
 &\quad + N^{-r} \|\mathbf{f}\|_{H^r(\Omega)^d} + N^{-s} \|k\|_{H^s(\Gamma_1)} + N^{-t} \|p_0\|_{H^t(\Gamma_2)}).
 \end{aligned} \tag{3.5.27}$$

Due to the inf – sup condition, for any q_N in $\mathbb{M}_N(\Omega)$, we derive that

$$\beta_D \|p_N - q_N\|_{\alpha^*} \leq \sup_{\mathbf{v}_N \in \mathbb{D}_N^0(\Omega)} \frac{-\int_{\Omega} (p_N - q_N) (\operatorname{div} \mathbf{v}_N) d\mathbf{x}}{\|\mathbf{v}_N\|_{\alpha}}. \tag{3.5.28}$$

In order to evaluate $-\int_{\Omega}(p_N - q_N)(\operatorname{div} \mathbf{v}_N)d\mathbf{x}$, we first use the discrete problem (3.5.5):

$$\begin{aligned} -\int_{\Omega}(p_N - q_N)(\operatorname{div} \mathbf{v}_N)d\mathbf{x} &= \alpha((\mathbf{f}, \mathbf{v}_N))_N^k - ((p_0, \mathbf{v}_N \cdot \mathbf{n}))_N^{\Gamma_2} \\ &\quad - \alpha((\mathbf{u}_N, \mathbf{v}_N))_N^k + \int_{\Omega} q_N(\operatorname{div} \mathbf{v}_N)d\mathbf{x}. \end{aligned} \quad (3.5.29)$$

Next, we apply equation (2.1.1) to the function \mathbf{v}_N , integrate by parts and add it to the previous line, which yields

$$\begin{aligned} -\int_{\Omega}(p_N - q_N)(\operatorname{div} \mathbf{v}_N)d\mathbf{x} &= \alpha((\mathbf{f}, \mathbf{v}_N))_N^k - \alpha \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_N d\mathbf{x} \\ &\quad + \alpha \int_{\Omega} \mathbf{u} \cdot \mathbf{v}_N d\mathbf{x} - \alpha((\mathbf{u}, \mathbf{v}_N))_N^k \\ + \int_{\Gamma_2} p_0 \cdot (\mathbf{v} \cdot \mathbf{n}) ds - ((p_0, \mathbf{v}_N \cdot \mathbf{n}))_N^{\Gamma_2} &+ \int_{\Omega} (q_N - p) \operatorname{div} \mathbf{v}_N d\mathbf{x} \end{aligned} \quad (3.5.30)$$

Now, it is easy to derive the next theorem.

Theorem 3.3 *Assume that the solution (\mathbf{u}, p) of the problem 2.1.1 in $H^m(\Omega)^d \times H^m(\Omega)$ for $m \geq d/2$; the function \mathbf{f} belongs to $H^m(\Omega)^d$ for $m \geq d/2$; the function k on Γ_1 in $H^s(\Gamma_1)$ for $s > \frac{d-1}{2}$, and p_0 on Γ_2 in $H^t(\Omega)$ for $t > \frac{d-1}{2}$, we have the following estimate:*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_N\|_{\alpha, L^2(\Omega)} + \|p - p_N\|_{\alpha^*} &\leq c(N^{-m}(\|\mathbf{u}\|_{H^m(\Omega)^d} + \|p\|_{H^m(\Omega)}) \\ &\quad + N^{-r}\|\mathbf{f}\|_{H^r(\Omega)^d} + N^{-s}\|k\|_{H^s(\Gamma_1)} + N^{-t}\|p_0\|_{H^t(\Gamma_2)}). \end{aligned} \quad (3.5.31)$$

3.5.2 Second discretization

• **Well-posedness of the solution** : According to the variational formulation, we introduce the discrete spaces as follows:

$$\mathbb{X}_N = \mathbb{P}_N(\Omega)^d, \quad \mathbb{M}_N = \mathbb{P}_N(\Omega), \quad (3.5.32)$$

also we introduce another space:

$$\mathbb{M}_N^1 = \{q_N \in \mathbb{M}_N, q_N = 0 \text{ on } \Gamma_2\}. \quad (3.5.33)$$

Assuming that the functions f , k , p_0 has continuous restrictions to all $\bar{\Omega}_k$, $1 \leq k \leq K$ and $\bar{\Gamma}_1, \bar{\Gamma}_2$ respectively. Then the discrete problem built from the variational formulation reads: Find (\mathbf{u}_N, p_N) in $\mathbb{X}_N \times \mathbb{M}_N$ such that

$$p_N = p_0 \text{ on } \Gamma_2, \quad (3.5.34)$$

and

$$\begin{aligned} \forall \mathbf{v}_N \in \mathbb{X}_N, \quad a_\alpha^N(\mathbf{u}_N, \mathbf{v}_N) + b_N(\mathbf{v}_N, p_N) &= \mathcal{L}_N(\mathbf{v}_N), \\ \forall q_N \in \mathbb{M}_N^1, \quad b_N(\mathbf{u}_N, q_N) &= ((k, q_N))_N^{\Gamma_1}. \end{aligned} \quad (3.5.35)$$

where the bilinear forms $a_\alpha^N(\cdot, \cdot)$ and $b_N(\cdot, \cdot)$ are defined by

$$\begin{aligned} a_\alpha^N(\mathbf{u}_N, \mathbf{v}_N) &= \alpha((\mathbf{u}_N, \mathbf{v}_N))_N, \\ b_N(\mathbf{v}_N, q_N) &= ((\mathbf{v}_N, \mathbf{grad} q_N))_N, \\ \mathcal{L}_N(\mathbf{v}_N) &= ((\alpha \mathbf{f}, \mathbf{v}_N))_N. \end{aligned} \quad (3.5.36)$$

We introduce here related norm and semi-norm as follows:

$$\|\mathbf{u}_N\|_\alpha = (\alpha \|\mathbf{u}_N\|^2)^{\frac{1}{2}}, \quad \|q_N\|_{\alpha^*} = (\alpha^{-1} |q_N|^2)^{\frac{1}{2}}.$$

We construct a lifting of the boundary condition of p_0 , we give the following lemma according to the reference ([25], Th. III.3.1) or ([7], Lemma 4.1).

Lemma 3.7 *If p_0 is continuous on Γ_2 , then there exists a function p_N^b in \mathbb{M}_N and a constant c independent of N , such that*

$$p_N^b = \mathcal{I}_N^{\Gamma_2} p_0 \text{ on } \Gamma_2, \quad (3.5.37)$$

and

$$\|p_N^b\|_{\alpha^*} \leq c \sqrt{\frac{1}{\alpha_{\min}}} \|\mathcal{I}_N^{\Gamma_2} p_0\|_{H^{\frac{1}{2}}(\Gamma_2)}. \quad (3.5.38)$$

To go further, we set $p_N^0 = p_N - p_N^b$, and consider the following problem: Find (\mathbf{u}_N, p_N^0) in $\mathbb{X}_N \times \mathbb{M}_N^1$ such that

$$\begin{aligned} \forall \mathbf{v}_N \in \mathbb{X}_N, \quad a_\alpha^N(\mathbf{u}_N, \mathbf{v}_N) + b_N(\mathbf{v}_N, p_N^0) &= L_N(\mathbf{v}_N) - b_N(\mathbf{v}_N, p_N^b), \\ \forall q_N \in \mathbb{M}_N^1, \quad b_N(\mathbf{u}_N, q_N) &= ((k, q_N))_N^{\Gamma_1}. \end{aligned} \quad (3.5.39)$$

Lemma 3.8 *The form $a_\alpha^N(\cdot, \cdot)$ satisfies the following continuity and ellipticity properties*

$$\forall \mathbf{u}_N \in \mathbb{X}_N, \forall \mathbf{v}_N \in \mathbb{X}_N, a_\alpha^N(\mathbf{u}_N, \mathbf{v}_N) \leq c \|\mathbf{u}_N\|_\alpha \|\mathbf{v}_N\|_\alpha, \quad (3.5.40)$$

$$\forall \mathbf{u}_N \in \mathbb{X}_N, a_\alpha^N(\mathbf{u}_N, \mathbf{u}_N) \geq \|\mathbf{u}_N\|_\alpha^2. \quad (3.5.41)$$

Proof: We use the Cauchy-Schwarz inequality to (3.4.6), we can derive

$$((\mathbf{u}, \mathbf{v}))_N \leq ((\mathbf{u}, \mathbf{u}))_N^{\frac{1}{2}} ((\mathbf{v}, \mathbf{v}))_N^{\frac{1}{2}}. \quad (3.5.42)$$

For all \mathbf{u}_N in \mathbb{X}_N , because α_k is piecewise constant, we have

$$a_\alpha^N(\mathbf{u}_N, \mathbf{u}_N) = \alpha \sum_{i=0}^N \sum_{j=0}^N \mathbf{u}_N^2(x_i, x_j) \rho_i \rho_j, \quad (3.5.43)$$

then we can easily derive:

$$\forall \mathbf{u}_N \in \mathbb{X}_N, \|\mathbf{u}_N\|_\alpha^2 \leq a_\alpha^N(\mathbf{u}_N, \mathbf{u}_N) \leq c \|\mathbf{u}_N\|_\alpha^2. \quad (3.5.44)$$

We next come to the inf-sup condition:

Lemma 3.9 *The bilinear form $b_N(\cdot, \cdot)$ satisfies the inf-sup condition*

$$\forall q_N \in \mathbb{M}_N^1, \quad \sup_{\mathbf{v}_N \in \mathbb{X}_N} \frac{b_N(\mathbf{v}_N, q_N)}{\|\mathbf{v}_N\|_\alpha} \geq C \|q_N\|_{\alpha^*}. \quad (3.5.45)$$

Proof: We note that for any q_N in \mathbb{M}_N^1 , we can define the function \mathbf{v}_N by

$$\mathbf{v}_N = \alpha^{-1} \mathbf{grad} q_N, \quad (3.5.46)$$

It is obvious that $\mathbf{v}_N \in \mathbb{X}_N$. Then we have the above inf-sup condition. From (3.5.40-3.5.41) and the inf-sup condition (3.5.45), the saddle-point problem (3.5.39) has a unique solution (\mathbf{u}_N, p_N^0) which satisfies

$$\|\mathbf{u}_N\|_\alpha + \|p_N^0\|_{\alpha^*} \leq C (\|p_N^b\|_{\alpha^*} + \|\mathcal{I}_N^{\Gamma_1} k\|_{H^{-\frac{1}{2}}(\Gamma_1)} + \|\mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^d}) \quad (3.5.47)$$

Now we are in the position of the main result of this section:

Theorem 3.4 *For any data (\mathbf{f}, k, p_0) such that each \mathbf{f} , and k, p_0 are continuous on $\bar{\Omega}_k$ and on Γ_1, Γ_2 respectively, problem (3.5.34) – (3.5.35) has a unique solution (\mathbf{u}_N, p_N) in $\mathbb{X}_N \times \mathbb{M}_N$. Moreover, there exists a constant c independent of N such that this solution satisfies*

$$\|\mathbf{u}_N\|_\alpha + \|p_N\|_{\alpha^*} \leq C (\|\mathcal{I}_N^{\Gamma_2} p_0\|_{H^{\frac{1}{2}}(\Gamma_2)} + \|\mathcal{I}_N^{\Gamma_1} k\|_{H^{-\frac{1}{2}}(\Gamma_1)} + \|\mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^d}) \quad (3.5.48)$$

Proof: We establish successively the existence and uniqueness of the solution.

1) It follows from the Lax-milgram lemma, combined with Bramble-Hilbert inequality, that there exists a unique φ_N in \mathbb{X}_N such that

$$\forall \phi_N \in \mathbb{M}_N^1, \quad ((\mathbf{grad} \varphi_N, \mathbf{grad} \phi_N))_N = ((k, \phi_N))_N^{\Gamma_1}. \quad (3.5.49)$$

Thus, the function $\mathbf{u}_N^b = \mathbf{grad} \varphi_N$, satisfies

$$\|\mathbf{u}_N^b\|_\alpha \leq c\sqrt{\alpha_{\max}} \|\mathcal{I}_N^{\Gamma_1} k\|_{H^{-\frac{1}{2}}(\Gamma_1)}. \quad (3.5.50)$$

On the other hand, it follows from the standard results on saddle-point problems ([40], Chap.I, Cor.4.1), that the problem : Find (\mathbf{u}_N^0, p_N^0) in $\mathbb{X}_N \times \mathbb{M}_N^1$ such that

$$\begin{aligned} \forall \mathbf{v}_N \in \mathbb{X}_N, \quad a_\alpha^N(\mathbf{u}_N^0, \mathbf{v}_N) + b_N(\mathbf{v}_N, p_N^0) &= L_N(\mathbf{v}_N) - b_N(\mathbf{v}_N, p_N^b) - a_\alpha^N(\mathbf{u}_N^b, \mathbf{v}_N), \\ \forall q_N \in \mathbb{M}_N^1, \quad b_N(\mathbf{u}_N^0, q_N) &= 0. \end{aligned} \quad (3.5.51)$$

has a unique solution (\mathbf{u}_N^0, p_N^0) which satisfies

$$\|\mathbf{u}_N^0\|_\alpha + \|p_N^0\|_{\alpha^*} \leq C (\|p_N^b\|_{\alpha^*} + \|\mathbf{u}_N^b\|_\alpha + \|\mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^d}) \quad (3.5.52)$$

Then, the pair (\mathbf{u}_N, p_N) , with $\mathbf{u}_N = \mathbf{u}_N^0 + \mathbf{u}_N^b$, $p_N = p_N^0 + p_N^b$, is a solution of problem (3.5.34) - (3.5.35), and estimate (3.5.48) follows from (3.5.50) and (3.5.52).

2) The pair $(\mathbf{u}_N, p_N = p_N^0 + p_N^b)$ is a solution of problem (3.5.34) - (3.5.35), and estimate (3.5.48) is a consequence of (3.5.38) and (3.5.47). On the other hand, let $(\mathbf{u}_{N1}, p_{N1})$ and $(\mathbf{u}_{N2}, p_{N2})$ be two solutions of problem (3.5.34) - (3.5.35). Then the difference $(\mathbf{u}_{N1} - \mathbf{u}_{N2}, p_{N1} - p_{N2})$ is a solution of problem (3.5.39) with data $\mathbf{u}_N^b, \mathbf{f}, p_0$ equal to zero. Thus, it follows from (3.5.48) that it is zero. So the solution of problem (3.5.34) - (3.5.35) is unique.

To conclude, we introduce the discrete kernel

$$V_N = \{\mathbf{v}_N \in \mathbb{X}_N; \forall q_N \in \mathbb{M}_N^1, b_N(\mathbf{v}_N, q_N) = 0\}. \quad (3.5.53)$$

As usual, it plays a key role in the numerical analysis of problem (3.5.34) - (3.5.35).

• Error estimate

This section is devoted to the error estimates, first for the velocity, second for the pressure. We intend to prove here an estimate between the solution (\mathbf{u}, p) of problem (2.3.9) and the solution (\mathbf{u}_N, p_N) of problem (3.5.34)-(3.5.35).

Let $\boldsymbol{\omega}_N$ be any function in the kernel V_N . Multiplying the first line of (2.1.1) by $\boldsymbol{\omega}_N$ gives

$$a_\alpha(\mathbf{u}_0, \boldsymbol{\omega}_N) + b(\boldsymbol{\omega}_N, P_1) = \alpha \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\omega}_N d\mathbf{x} - a_\alpha(\mathbf{u}_b, \boldsymbol{\omega}_N) - b(\boldsymbol{\omega}_N, \bar{P}). \quad (3.5.54)$$

which implies, due to the definition of V_N , that for any q_N in $\mathbb{M}_N^1(\Omega)$,

$$a_\alpha(\mathbf{u}_0, \boldsymbol{\omega}_N) + b(\boldsymbol{\omega}_N, P_1 - q_N) = \alpha \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\omega}_N d\mathbf{x} - a_\alpha(\mathbf{u}_b, \boldsymbol{\omega}_N) - b(\boldsymbol{\omega}_N, \bar{P}). \quad (3.5.55)$$

Next, we deduce from the ellipticity property, that we have for any \mathbf{v}_N in V_N

$$\|\mathbf{u}_N^0 - \mathbf{v}_N\|_\alpha^2 \leq a_\alpha^N(\mathbf{u}_N^0 - \mathbf{v}_N, \mathbf{u}_N^0 - \mathbf{v}_N). \quad (3.5.56)$$

Adding (3.5.55) with $\boldsymbol{\omega}_N = \mathbf{u}_N^0 - \mathbf{v}_N$ and subtracting the first line of (3.5.35) leads to

$$\begin{aligned} \|\mathbf{u}_N^0 - \mathbf{v}_N\|_\alpha^2 &\leq a_\alpha^N(\mathbf{u}_N^0 - \mathbf{v}_N, \mathbf{u}_N^0 - \mathbf{v}_N) \leq a_\alpha(\mathbf{u}_0 - \mathbf{v}_N, \mathbf{u}_N^0 - \mathbf{v}_N) + a_\alpha(\mathbf{v}_N, \mathbf{u}_N^0 - \mathbf{v}_N) \\ &\quad - a_\alpha^N(\mathbf{v}_N, \mathbf{u}_N^0 - \mathbf{v}_N) + b(\mathbf{u}_N^0 - \mathbf{v}_N, P_1 - q_N) + ((\alpha \mathbf{f}, \mathbf{u}_N^0 - \mathbf{v}_N))_N \\ &\quad - \alpha \int_{\Omega} \mathbf{f} \cdot (\mathbf{u}_N^0 - \mathbf{v}_N) d\mathbf{x} + a_\alpha(\mathbf{u}_b, \mathbf{u}_N^0 - \mathbf{v}_N) \\ &\quad - a_\alpha^N(\mathbf{u}_b, \mathbf{u}_N^0 - \mathbf{v}_N) + b(\mathbf{u}_N^0 - \mathbf{v}_N, \bar{P}) - b_N(\mathbf{u}_N^0 - \mathbf{v}_N, p_N^b). \end{aligned} \quad (3.5.57)$$

It is obvious that

$$\forall \mathbf{v}_N \in \mathbb{X}_N, \forall q_N \in \mathbb{M}_N^1, \quad b_N(\mathbf{v}_N, q_N) \leq \|\mathbf{v}_N\|_\alpha \|q_N\|_{\alpha^*}. \quad (3.5.58)$$

So, combining the above inequality and triangle inequality, we derive that the error $\|\mathbf{u} - \mathbf{u}_N\|_\alpha$ is bounded, up to a multiplicative constant, by the sum of seven terms:

- the approximation error in \mathbb{X}_N

$$\inf_{\mathbf{v}_N \in V_N} \|\mathbf{u}_0 - \mathbf{v}_N\|_\alpha, \quad (3.5.59)$$

- the error approximation in \mathbb{M}_N^1

$$\inf_{q_N \in \mathbb{M}_N^1} \|P_1 - q_N\|_{\alpha^*}, \quad (3.5.60)$$

- three terms issued from numerical integration

$$\sup_{\boldsymbol{\omega}_N \in \mathbb{X}_N} \frac{a_\alpha(\mathbf{v}_N, \boldsymbol{\omega}_N) - a_\alpha^N(\mathbf{v}_N, \boldsymbol{\omega}_N)}{\|\boldsymbol{\omega}_N\|_\alpha}, \quad (3.5.61)$$

$$\sup_{\boldsymbol{\omega}_N \in \mathbb{X}_N} \frac{a_\alpha(\mathbf{u}_b, \boldsymbol{\omega}_N) - a_\alpha^N(\mathbf{u}_b, \boldsymbol{\omega}_N)}{\|\boldsymbol{\omega}_N\|_\alpha}, \quad (3.5.62)$$

$$\sup_{\boldsymbol{\omega}_N \in \mathbb{X}_N} \frac{((\alpha \mathbf{f}, \boldsymbol{\omega}_N))_\alpha - \alpha \int_\Omega \mathbf{f} \cdot \boldsymbol{\omega}_N d\mathbf{x}}{\|\boldsymbol{\omega}_N\|_\alpha}, \quad (3.5.63)$$

$$\sup_{\boldsymbol{\omega}_N \in \mathbb{X}_N} \frac{b(\boldsymbol{\omega}_N, \bar{P}) - b_N(\boldsymbol{\omega}_N, p_N^b)}{\|\boldsymbol{\omega}_N\|_\alpha}. \quad (3.5.64)$$

and

$$\|\mathbf{u}_b - \mathbf{u}_N^b\|_\alpha \quad (3.5.65)$$

We refer ([32], Section 2.6.2), the following estimate could be found there.

For $r \geq 2$,

$$\sup_{\boldsymbol{\omega}_N \in \mathbb{X}_N} \frac{((\alpha \mathbf{f}, \boldsymbol{\omega}_N))_\alpha - \alpha \int_\Omega \mathbf{f} \cdot \boldsymbol{\omega}_N d\mathbf{x}}{\|\boldsymbol{\omega}_N\|_\alpha} \leq cN^{-r} \|\mathbf{f}\|_{H^r(\Omega)^d}. \quad (3.5.66)$$

For $s \geq 1$,

$$\sup_{\boldsymbol{\omega}_N \in \mathbb{X}_N} \frac{a_\alpha(\mathbf{u}_b, \boldsymbol{\omega}_N) - a_\alpha^N(\mathbf{u}_b, \boldsymbol{\omega}_N)}{\|\boldsymbol{\omega}_N\|_\alpha} \leq cN^{-s} \|\mathbf{u}_b\|_{H^s(\Omega)^d}. \quad (3.5.67)$$

Lemma 3.10 ([32], Lemma 2.6.2) *For all $m \geq d-2$, and the function \mathbf{u} in $H^m(\Omega)^d$,*

we have:

$$\inf_{\mathbf{v}_N \in V_N} \|\mathbf{u}_0 - \mathbf{v}_N\|_\alpha \leq cN^{-m} \|\mathbf{u}_0\|_{H^m(\Omega)^d}. \quad (3.5.68)$$

Then,

$$\inf_{q_N \in \mathbb{M}_N^1} \|P_1 - q_N\|_{\alpha^*} \leq cN^{-m} \|P_1\|_{H^{m+1}(\Omega)}. \quad (3.5.69)$$

So, according to the above estimate, we have the first approximation error for the velocity.

Theorem 3.5 *Assume that the solution (\mathbf{u}, p) of the problem (2.1.1) in $H^m(\Omega)^d \times H^{m+1}(\Omega)$ for $m \geq d - 2$; the function \mathbf{f} belongs to $H^r(\Omega)^d$ for $r \geq 2$; the function k on Γ_1 in $H^s(\Gamma_1)$ for $s > \frac{d-1}{2}$, and p_0 on Γ_2 in $H^t(\Omega)$ for $t > d - 1$, we have the following estimate:*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_N\|_\alpha &\leq c(N^{-m} (\|\mathbf{u}\|_{H^m(\Omega)^d} + \|p\|_{H^{m+1}(\Omega)}) \\ &+ N^{-r} \|\mathbf{f}\|_{H^r(\Omega)^d} + N^{-s} \|k\|_{H^s(\Gamma_1)} + N^{-t} \|p_0\|_{H^t(\Gamma_2)}). \end{aligned} \quad (3.5.70)$$

Then we come to the error estimate for the pressure. From the inf – sup condition (3.5.45), we derive that, for any q_N in \mathbb{M}_N^1 ,

$$\beta \|p_N^0 - q_N^0\|_{\alpha^*} \leq \sup_{\mathbf{v}_N \in \mathbb{X}_N} \frac{b_N(\mathbf{v}_N, p_N^0 - q_N^0)}{\|\mathbf{v}_N\|_\alpha}. \quad (3.5.71)$$

We first use the discrete problem (3.5.35)

$$b_N(\mathbf{v}_N, p_N^0 - q_N^0) = ((\alpha \mathbf{f}, \mathbf{v}_N))_N - a_\alpha^N(\mathbf{u}_N - \mathbf{v}_N) - b_N(\mathbf{v}_N, p_N^b) - b_N(\mathbf{v}_N, q_N^0). \quad (3.5.72)$$

Next, we apply equation (2.3.9) to the function \mathbf{v}_N , and adding it to the previous

line. This yields

$$\begin{aligned}
 b_N(\mathbf{v}_N, p_N^0 - q_N^0) &= \alpha \int_{\Omega} (\mathbf{u} - \mathbf{u}_N) \cdot \mathbf{v}_N d\mathbf{x} + \alpha \int_{\Omega} \mathbf{u}_N \cdot \mathbf{v}_N d\mathbf{x} - a_{\alpha}^N(\mathbf{u}_N - \mathbf{v}_N) \\
 &\quad + \int_{\Omega} \mathbf{v}_N \cdot \mathbf{grad} (P_1 - q_N^0) d\mathbf{x} + \int_{\Omega} \mathbf{v}_N \cdot \mathbf{grad} (\bar{P} - p_N^b) d\mathbf{x} \\
 &\quad + ((\alpha \mathbf{f}, \mathbf{v}_N))_N \alpha \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_N d\mathbf{x}.
 \end{aligned} \tag{3.5.73}$$

According to the triangle inequality, we have

$$\|p - q_N\|_{\alpha^*} = \|P_1 - q_N^0\|_{\alpha^*} + \|p_N^0 - q_N^0\|_{\alpha^*} + \|\bar{P} - p_N^b\|_{\alpha^*} \tag{3.5.74}$$

So using the same arguments as in the estimation of terms issued from numerical integration together with the above triangle inequality yields

$$\begin{aligned}
 \|p - q_N\|_{\alpha^*} &\leq c(\|\mathbf{u} - \mathbf{u}_N\|_{\alpha} + \sup_{\mathbf{v}_N \in \mathbb{X}_N} \frac{\alpha \int_{\Omega} \mathbf{u}_N \cdot \mathbf{v}_N d\mathbf{x} - a_{\alpha}^N(\mathbf{u}_N, \mathbf{v}_N)}{\|\mathbf{v}_N\|_{\alpha}} \\
 &\quad + \|P_1 - q_N^0\|_{\alpha^*} + \sup_{\mathbf{v}_N \in \mathbb{X}_N} \frac{((\alpha \mathbf{f}, \mathbf{v}_N))_N - \alpha \int_{\Omega} \mathbf{f}_N \cdot \mathbf{v}_N d\mathbf{x}}{\|\mathbf{v}_N\|_{\alpha}} \\
 &\quad + \sup_{\mathbf{v}_N \in \mathbb{X}_N} \frac{b(\mathbf{v}_N, \bar{P}) - b_N(\mathbf{v}_N, q_N^0)}{\|\mathbf{v}_N\|_{\alpha}} + \|\bar{P} - p_N^b\|_{\alpha^*}).
 \end{aligned} \tag{3.5.75}$$

Then we give the second theorem about the error estimate of the pressure:

Theorem 3.6 *Assume that the solution (\mathbf{u}, p) of the problem (2.1.1) in $H^m(\Omega)^d \times H^{m+1}(\Omega)$ for $m \geq d - 2$; the function \mathbf{f} belongs to $H^r(\Omega)^d$ for $r \geq 2$; the function k on Γ_1 in $H^s(\Gamma_1)$ for $s > \frac{d-1}{2}$, and p_0 on Γ_2 in $H^t(\Omega)$ for $t > d - 1$, we have the following estimate:*

$$\begin{aligned}
 \|\mathbf{u} - \mathbf{u}_N\|_{\alpha} + \|p - p_N\|_{\alpha^*} &\leq c(N^{-m}(\|\mathbf{u}\|_{H^m(\Omega)^d} + \|p\|_{H^{m+1}(\Omega)}) \\
 &\quad + N^{-r}\|\mathbf{f}\|_{H^r(\Omega)^d} + N^{-s}\|k\|_{H^s(\Gamma_1)} + N^{-t}\|p_0\|_{H^t(\Gamma_2)}).
 \end{aligned} \tag{3.5.76}$$

3.6 Numerical results

In this section, we will talk about the numerical computation for the schemes proposed in the above sections. Because α is piecewise constant in our work, for simplicity, we suppose here a constant in the whole domain, and we set $\alpha = 1$.

For the details of the computation, including the linear system, Uzawa algorithm and the numerical implementation, we refer to ([25], V.4) and ([32], Chapter 2). Here we only give some numerical results to support our estimate convergence. For the simplicity of the computation, we set here $\Gamma_1 = \partial\Omega$.

3.6.1 The first discretization

N	5	9	13	17	19
$\frac{\ \mathbf{u}-\mathbf{u}_N\ _\alpha}{\ \mathbf{u}\ _\alpha}$	1.97e-2	9.42e-5	9.42e-9	4.68e-13	4.8e-15
$\frac{\ p-p_N\ _{\alpha^*}}{\ p\ _{\alpha^*}}$	4.93e-2	1.99e-5	1.86e-9	1.38e-13	3.4e-15

Table 2: The relative error for the second case

3.6.2 The second discretization

The first example is as follows with analytical velocity and the pressure is also with regular property:

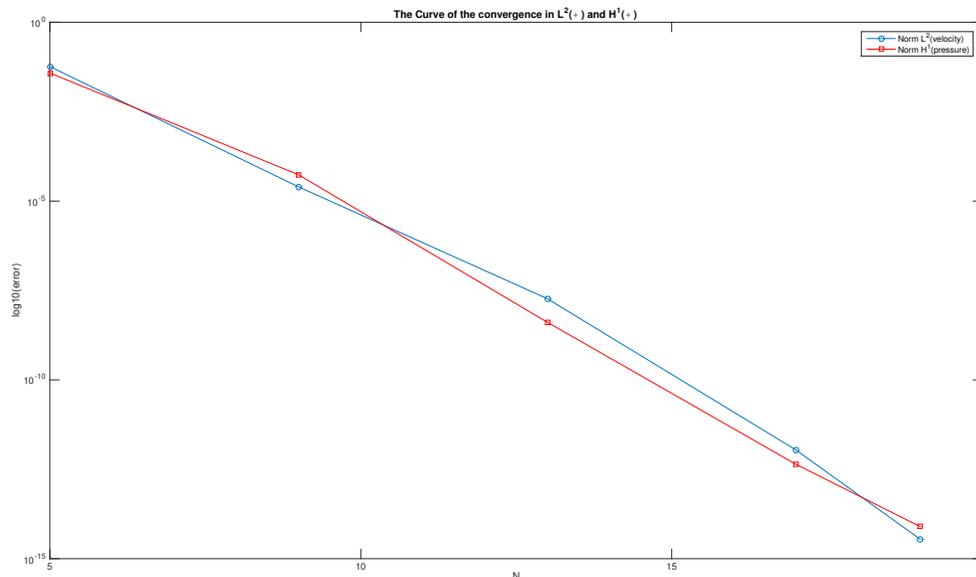
$$\mathbf{u}(x, y) = \begin{pmatrix} \sin(\pi x) \cos(\pi y) \\ -\cos(\pi x) \sin(\pi y) \end{pmatrix}, \quad p(x, y) = \sin(\pi(x + y)). \quad (3.6.1)$$

The following table indicate the spectral approximation property, which means the the rate of convergence is exponential.

N	5	9	13	17	19
$\frac{\ \mathbf{u}-\mathbf{u}_N\ _{\alpha, L^2(\Omega)}}{\ \mathbf{u}\ _{\alpha, L^2(\Omega)}}$	3.81e-2	5.42e-5	4.08e-9	4.33e-13	7.83e-15
$\frac{\ p-p_N\ _{\alpha^*}}{\ p\ _{\alpha^*}}$	5.73e-2	2.49e-5	1.86e-8	1.09e-12	3.47e-15

Table: The relative error for the first case

Where (\mathbf{u}_N, p_N) is the velocity and pressure respectively computed on the Legendre-Gauss-Labatto points. We can know that the error with order 10^{-15} . The following figure represents the logarithm of error, in norm $L^2(\Omega)^2$ and $H^1(\Omega)$. It shows that the error decays fast when N increase. In addition, the error decays expontionaly when N increase. These results confirm the numerical analysis obtained in the above section.


 Figure 3-2: The curve of convergence in $L^2(\Omega)$ and $H^1(\Omega)$

The second example is given as non-homogeneous:

$$\mathbf{u}(x, y) = \begin{pmatrix} \cos(\pi x) \sin(\pi y) \\ -\sin(\pi x) \cos(\pi y) \end{pmatrix}, \quad p(x, y) = \sin(\pi(x + y)). \quad (3.6.2)$$

The boundary conditions are:

$$k(x, y) = \begin{cases} -\sin(\pi y), & \text{if } -1 \leq y \leq 1 \text{ and } x = 1 \text{ and } x = -1 \\ \sin(\pi x), & \text{if } -1 \leq x \leq 1 \text{ and } y = 1 \text{ and } y = -1 \end{cases} \quad (3.6.3)$$

The next table give the numerical results for the solution given above.

N	9	12	15	18	21
$\frac{\ \mathbf{u}-\mathbf{u}_N\ _{\alpha, L^2(\Omega)}}{\ \mathbf{u}\ _{\alpha, L^2(\Omega)}}$	e-2	3e-6	4e-9	4.3e-12	2.8e-14
$\frac{\ p-p_N\ _{\alpha^*}}{\ p\ _{\alpha^*}}$	2.73e-4	5.49e-7	6e-10	3e-13	1.4e-14

Table : The relative error for the first case

We can know that the error with order 10^{-14} . The following figure represents the logarithm of error, in norm $L^2(\Omega)^2$ and $H^1(\Omega)$. It shows that the error decays fast when N increase. In addition, the error decays exponentially when N increase. We

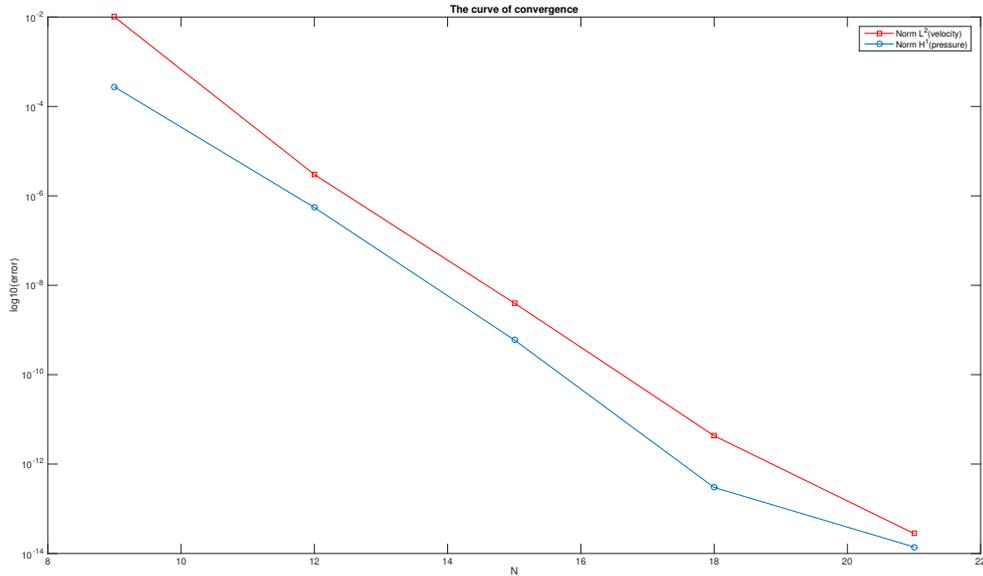


Figure 3-3: The curve of convergence in $L^2(\Omega)$ and $H^1(\Omega)$

are also interested in the case where the solution of the problem is not so regular, so

the third example as follows:

$$\mathbf{u}(x, y) = \begin{pmatrix} -\frac{5}{2}y(x^2 + y^2)^{\frac{1}{4}} \\ \frac{5}{2}x(x^2 + y^2)^{\frac{1}{4}} \end{pmatrix}, \quad p(x, y) = \sin(\pi(x + y)). \quad (3.6.4)$$

The boundary conditions are:

$$k(x, y) = \begin{cases} -\frac{5}{2}y(1 + y^2)^{\frac{1}{4}}, & \text{if } -1 \leq y \leq 1 \text{ and } x = 1 \text{ and } x = -1 \\ \frac{5}{2}x(x^2 + 1)^{\frac{1}{4}}, & \text{if } -1 \leq x \leq 1 \text{ and } y = 1 \text{ and } y = -1 \end{cases} \quad (3.6.5)$$

The results are given in the following table:

N	5	9	13	17	19
$\frac{\ \mathbf{u}-\mathbf{u}_N\ _{\alpha, L^2(\Omega)}}{\ \mathbf{u}\ _{\alpha, L^2(\Omega)}}$	e-3	7e-6	4e-9	6.3e-14	2.8e-15
$\frac{\ p-p_N\ _{\alpha^*}}{\ p\ _{\alpha^*}}$	2.4e-3	5.7e-5	e-8	3e-12	8.4e-15

Table : The relative error for the first case

The next figure represents the logarithm of error, in norm $L^2(\Omega)^2$ and $H^1(\Omega)$. It shows that the error decays fast when N increase. In addition, the error decays exponentially when N increase.

In the end, we present the following problems, the knowns datas are:

$$\mathbf{f}(x, y) = \begin{pmatrix} -y^2 \\ (1 - x^2) \end{pmatrix}, \quad \text{in } \Omega, \quad k(x, y) = 0 \text{ on } \partial\Omega \quad (3.6.6)$$

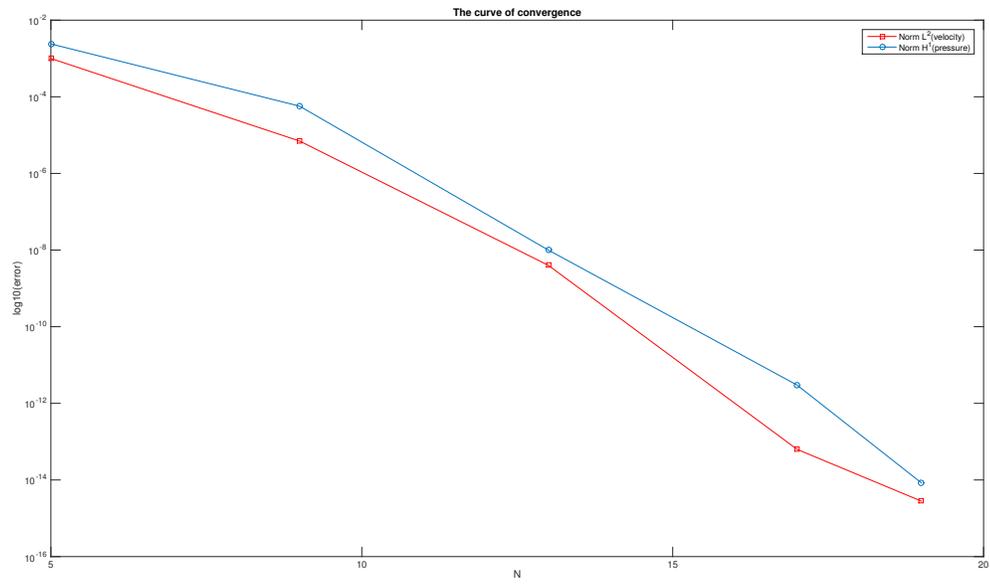


Figure 3-4: The curve of convergence in $L^2(\Omega)$ and $H^1(\Omega)$

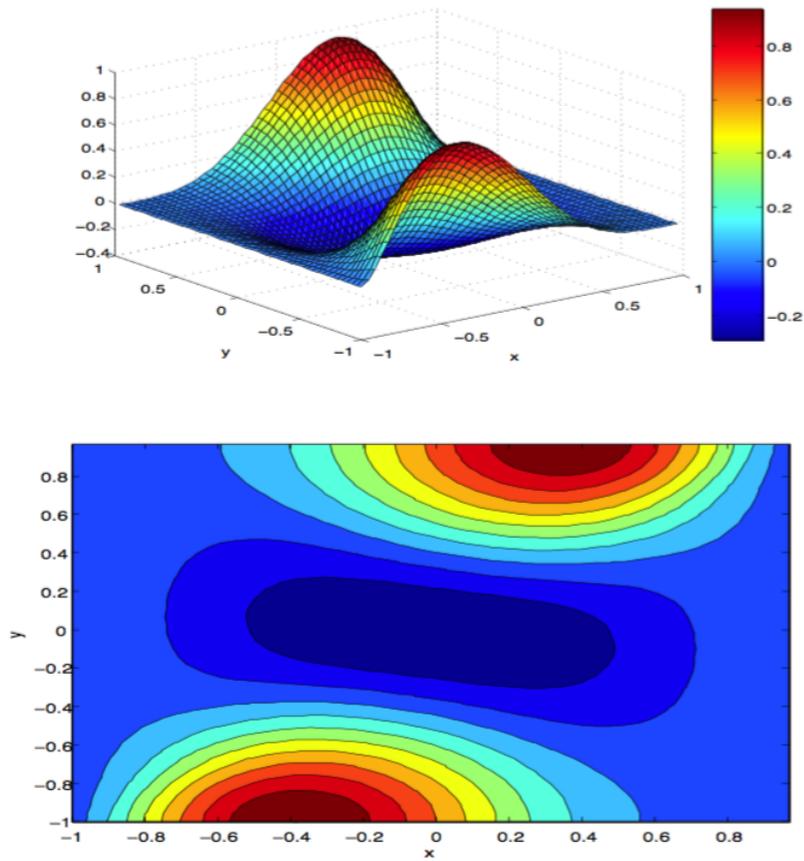


Figure 3-5: The curve and isovalue of the first component of velocity obtained by $N=48$

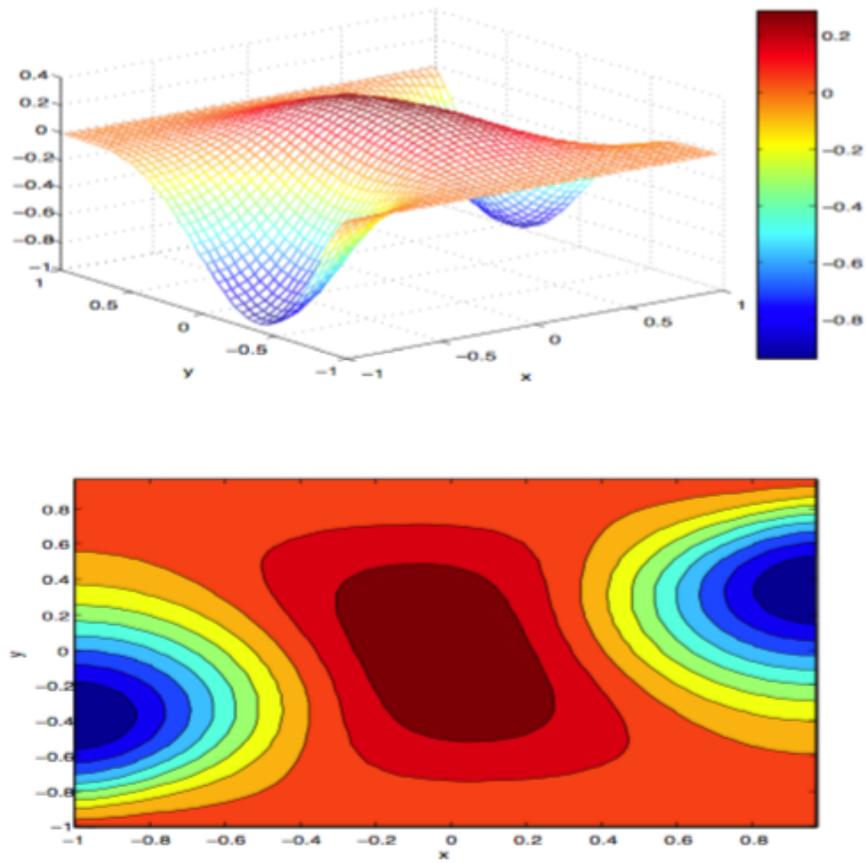


Figure 3-6: The curve and isovalue of the second component of velocity obtained by $N=48$

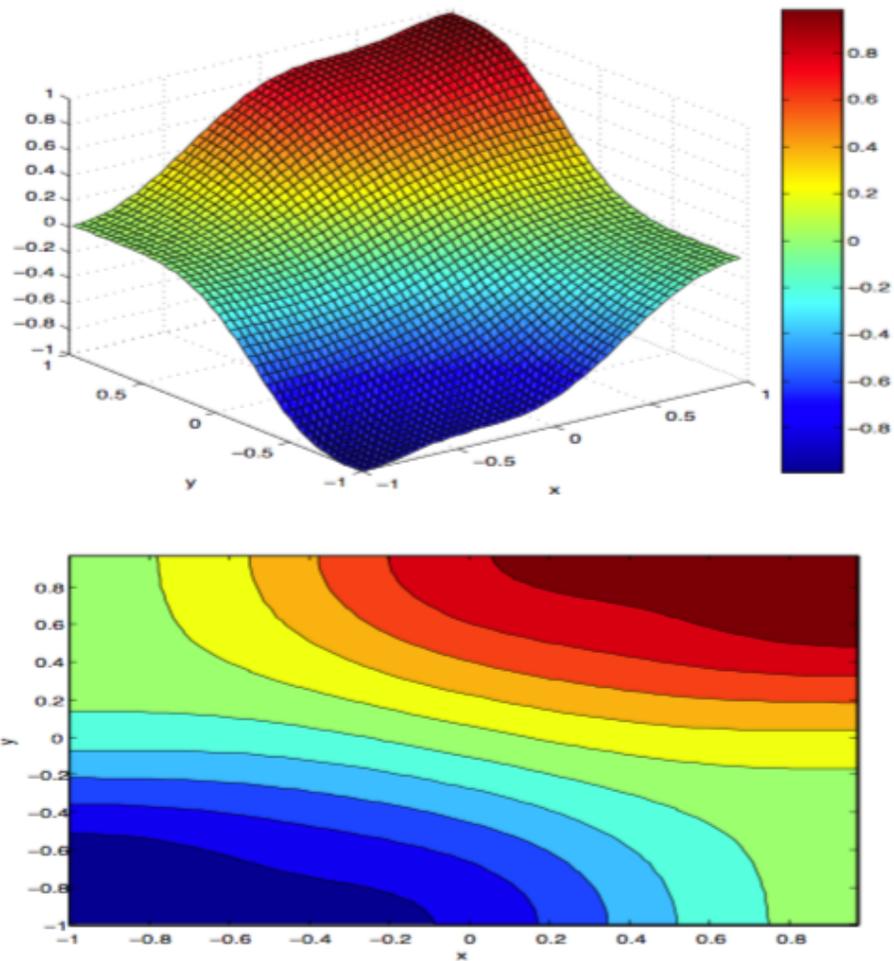


Figure 3-7: The curve and isovalue of the pressure obtained by $N=48$

Chapter 4

Mortar Spectral Discretization of Darcy's Equations(I)

4.1 Introduction

Let Ω be a bounded, connected, open set in \mathbb{R}^d , $d = 2$ or 3 , with a Lipschitz continuous boundary $\partial\Omega$, and let \mathbf{n} denote the unit outward normal vector to Ω on $\partial\Omega$, Γ_1 and Γ_2 be a partition of $\partial\Omega$ without overlap. We will talk about the Darcy's equations as follows

$$\left\{ \begin{array}{ll} \alpha \mathbf{u} + \mathbf{grad} p = \alpha \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = k & \text{on } \Gamma_1, \\ p = p_0 & \text{on } \Gamma_2 \end{array} \right. \quad (4.1.1)$$

The data are now the function \mathbf{f} and the boundary conditions k and p_0 . The unknowns are the velocity \mathbf{u} and the pressure (or hydraulic head, according to the model) p . We assume that the function α is piecewise constant: There exists a partition of the

domain Ω without overlap into connected subdomains $\Omega_l, 1 \leq l \leq L$, such that α is constant equal to α_l on Ω_l . We introduce the notation

$$\alpha_{min} = \min_{1 \leq l \leq L} \alpha_l \quad \text{and} \quad \alpha_{max} = \max_{1 \leq l \leq L} \alpha_l,$$

and, without restriction, we assume that α_{min} is positive.

The mortar element method, due to Bernardi, Maday and Petera [24], is a domain decomposition technique which allows for working on general partitions of the domain, without conformity restrictions. It is widely used to discrete second-order elliptic equations when set in standard Sobolev spaces, we refer to [16] and [6]. In [6], Azaïez, Belgacem and Bernardi extend this method to problems formulated in spaces of square-integrable functions with square-integrable divergence, where Darcy's equations with homogeneous Dirichlet boundary is discussed.

We firstly consider the key situation where the function α is piecewise constant. We refer to ([58], Subsec. 1.5) for a first application of this method to discontinuous coefficient in the finite element framework. For spectral element framework, we refer to [16] for the second-order elliptic equations and [32] for Darcy's equations formulated in standard Sobolev spaces. For the boundary conditions, we introduce a subspace of $H(\text{div}, \Omega)$ with trace $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ_1 . Thanks to the recent paper [15] by Bernard, we can derive its density results and then the well-posedness of the problem.

The approximation properties of divergence-free functions in the mortar discrete space are needed for the numerical analysis of the discrete problem. According to this, we can prove a spectral type error estimate: the order of convergence only depends on the regularity of the solution, more precisely on its local regularity in each subdomain.

Finally, an appropriate treatment of the matching conditions on the interface is

the main point of the implementation of the mortar technique, we refer to [6]. We also refer to [10] to see another way of treating these conditions. Two dimensional numerical experiments are presented. They are in good coherency with the numerical analysis.

4.2 Description of the mortar spectral element discretization

Throughout this section, we work with a piecewise constant function α . We now assume that Ω admits a disjoint decomposition into a finite number of (open) rectangles in dimension $d = 2$, rectangular parallelepipeds in dimension $d = 3$, denoted by Ω_k , with edges all parallel to the coordinate axes,

$$\bar{\Omega} = \bigcup_{k=1}^K \bar{\Omega}_k \quad \text{and} \quad \Omega_k \cap \Omega_{k'} = \emptyset, 1 \leq k \neq k' \leq K. \quad (4.2.1)$$

We make the further assumption that the intersection of each $\partial\Omega_k$ with $\partial\Omega$, if not empty, is a corner, a whole edge or a whole face of Ω_k . For $1 \leq k \leq K$, we denote by $\Gamma_{k,l}, 1 \leq l \leq L(k)$, the (open) edges in dimension $d = 2$, faces in dimension $d = 3$, of Ω_k which are not contained in $\partial\Omega$. We denote by \mathbf{n}_k the unit outward normal vector to Ω_k on $\partial\Omega_k$. The decomposition is said to be conforming when the intersection of two different $\bar{\Omega}_k$, if not empty, is a corner or a whole edge or a whole face of both of them. For simplicity, we denote by α_k the constant value of α on each $\Omega_k, 1 \leq k \leq K$.

The skeleton \mathcal{S} of the decomposition, equal to $\bigcup_{k=1}^K \partial\Omega_k \setminus \partial\Omega$, admits a decomposition without overlapping into mortars

$$\bar{\mathcal{S}} = \bigcup_{m=1}^M \bar{\gamma}_m \quad \text{and} \quad \gamma_m \cap \gamma_{m'} = \emptyset, 1 \leq m \neq m' \leq M. \quad (4.2.2)$$

where each $\gamma_m = \Gamma_{k(m),l(m)}$ is a whole edge in dimension $d = 2$, face in dimension $d = 3$,

of a subdomain Ω_k , denoted by $\Omega_{k(m)}$. Note that the choice of this decomposition is not unique, however it is decided a priori for all the discretizations we work with.

In order to describe the discrete problem, we introduce the discretization parameter δ , which is here a K -tuple of positive integers $N_k, 1 \leq k \leq K$, with each $N_k \geq 2$. Indeed, the local discrete space on each Ω_k is the space $\mathbb{P}_{N_k}(\Omega_k)$ of restrictions to Ω_k of polynomials with degree $\leq N_k$ with respect to each variable. In all that follows, c stands for a generic constant which may vary from line to line but is always independent of δ .

We first introduce the discrete spaces. From the trace theory and (2.3.5) we know that the space $X(\Omega)$ coincides with the space of functions \mathbf{v} such that their restrictions to each $\Omega_k, 1 \leq k \leq K$, belong to $H(\text{div}, \Omega_k)$ and their normal traces are continuous through the skeleton \mathcal{S} . For $1 \leq m \leq M$, we define the mortar function φ on each γ_m by

$$\varphi|_{\gamma_m} = \mathbf{v}_\delta|_{\Omega_{k(m)}} \cdot \mathbf{n}_{k(m)}.$$

Then we define the corresponding discrete space of velocities $\mathbb{D}_\delta(\Omega)$ of functions \mathbf{v}_δ such that

1. their restrictions $\mathbf{v}_\delta|_{\Omega_k}$ to each $\Omega_k, 1 \leq k \leq K$, belong to $\mathbb{P}_{N_k}(\Omega_k)^d$,
2. such the following matching condition holds on each edge $\Gamma_{k,l}, 1 \leq k \leq K, 1 \leq l \leq L(k)$, which is not a mortar:

$$\forall \Psi \in \mathbb{P}_{N_k-2}(\Gamma_{k,l}), \int_{\Gamma_{k,l}} (\mathbf{v}_\delta|_{\Omega} \cdot \mathbf{n}_k + \varphi)(\mathbf{t}) \Psi(\mathbf{t}) d\mathbf{t} = 0. \quad (4.2.3)$$

We also define the following space

$$\mathbb{D}_\delta^0(\Omega) = \{\mathbf{v}_\delta \in \mathbb{D}_\delta(\Omega) \mid \mathbf{v}_\delta \cdot \mathbf{n} = 0 \text{ on } \Gamma_1\} \quad (4.2.4)$$

According to ([8], Lemma 4.1) and ([24], Thm 24.1), the space \mathbb{M}_δ of the discrete

pressures is introduced by:

$$\mathbb{M}_\delta(\Omega) = \{q_\delta \in L^2(\Omega); q_\delta|_{\Omega_k} \in \mathbb{P}_{N_k-2}(\Omega_k), 1 \leq k \leq K\}. \quad (4.2.5)$$

Assuming that the functions f, k and p_0 have continuous restrictions to all $\bar{\Omega}_k, 1 \leq k \leq K$ and $\bar{\Gamma}_1, \bar{\Gamma}_2$ respectively. Then the discrete problem built from the variational formulation (2.3.7) by the Galerkin method with numerical integration(GNI) reads : Find $(\mathbf{u}_\delta, p_\delta)$ in $\mathbb{D}_\delta(\Omega) \times \mathbb{M}_\delta(\Omega)$ such that

$$\mathbf{u}_\delta \cdot \mathbf{n} = \mathcal{I}_\delta^{\Gamma_1} k \quad \text{on } \Gamma_1, \quad (4.2.6)$$

and

$$\begin{aligned} \forall \mathbf{v}_\delta \in \mathbb{D}_\delta^0(\Omega), \quad a_\alpha^\delta(\mathbf{u}_\delta, \mathbf{v}_\delta) + b_\delta(\mathbf{v}_\delta, p_\delta) &= \mathcal{L}_\delta(\mathbf{v}_\delta), \\ \forall q_\delta \in \mathbb{M}_\delta(\Omega), \quad b_\delta(\mathbf{u}_\delta, q_\delta) &= 0. \end{aligned} \quad (4.2.7)$$

where the bilinear forms $a_\alpha^\delta(\cdot, \cdot)$ and b_δ are defined by

$$\begin{aligned} a_\alpha^\delta(\mathbf{u}_\delta, \mathbf{v}_\delta) &= \sum_{k=1}^K \alpha_k ((\mathbf{u}_\delta, \mathbf{v}_\delta))_\delta^k, \\ b_\delta(\mathbf{v}_\delta, q_\delta) &= - \sum_{k=1}^K ((\text{div } \mathbf{v}_\delta, q_\delta))_\delta^k, \\ \mathcal{L}_\delta(\mathbf{v}_\delta) &= \sum_{k=1}^K \alpha_k ((\mathbf{f}_\delta, \mathbf{v}_\delta))_\delta^k - ((p_0, \mathbf{v}_\delta \cdot \mathbf{n}))_\delta^{\Gamma_2}. \end{aligned} \quad (4.2.8)$$

Thanks to the exactness property of the quadrature formula, we note that

$$\forall \mathbf{v}_\delta \in \mathbb{D}_\delta^0(\Omega), \quad \forall q_\delta \in \mathbb{M}_\delta(\Omega), \quad b_\delta(\mathbf{v}_\delta, q_\delta) = - \int_\Omega q_\delta (\text{div } \mathbf{v}_\delta) dx. \quad (4.2.9)$$

Since $\mathbb{D}_\delta(\Omega)$ is not contained in $H(\text{div}, \Omega)$ in the general case, we need the "bro-

ken"norm defined by

$$\begin{aligned}\|\mathbf{v}\|_{\alpha,\delta} &= \left(\sum_{k=1}^K \alpha_k \|\mathbf{v}\|_{H(\text{div}, \Omega_k)}^2\right)^{\frac{1}{2}}, \\ \|q\|_{\alpha^*,\delta} &= \left(\sum_{k=1}^K \alpha_k^{-1} \|q\|_{L^2(\Omega_k)}^2\right)^{\frac{1}{2}}.\end{aligned}\tag{4.2.10}$$

We then construct a lifting of the boundary conditions.

Lemma 4.1 *There exists a divergence free function \mathbf{u}_δ^b in $\mathbb{D}_\delta(\Omega)$ which satisfies*

$$\mathbf{u}_\delta^b \cdot \mathbf{n} = \mathcal{I}_\delta^{\Gamma_1} k \quad \text{on } \Gamma_1,\tag{4.2.11}$$

and

$$\|\mathbf{u}_\delta^b\|_{\alpha,\delta} \leq C \sqrt{\alpha_{\max}} \|\mathcal{I}_\delta^{\Gamma_1} k\|_{H^{-\frac{1}{2}}(\Gamma_1)}.\tag{4.2.12}$$

where C is independent of δ and α .

We set $\mathbf{u}_\delta^0 = \mathbf{u}_\delta - \mathbf{u}_\delta^b$. To go further, we consider the following problem: Find $(\mathbf{u}_\delta^0, p_\delta)$ in $\mathbb{D}_\delta^0(\Omega) \times \mathbb{M}_\delta(\Omega)$ such that

$$\begin{aligned}\forall \mathbf{v}_\delta \in \mathbb{D}_\delta^0(\Omega), \quad a_\alpha^\delta(\mathbf{u}_\delta^0, \mathbf{v}_\delta) + b_\delta(\mathbf{v}_\delta, p_\delta) &= -a_\alpha^\delta(\mathbf{u}_\delta^b, \mathbf{v}_\delta) + \mathcal{L}_\delta(\mathbf{v}_\delta), \\ \forall q_\delta \in \mathbb{M}_\delta(\Omega), \quad b_\delta(\mathbf{u}_\delta^0, q_\delta) &= 0.\end{aligned}\tag{4.2.13}$$

We introduce the kernel

$$V_\delta(\Omega) = \{\mathbf{v}_\delta \in \mathbb{D}_\delta^0(\Omega); \forall q_\delta \in \mathbb{M}_\delta, b_\delta(\mathbf{v}_\delta, q_\delta) = 0\}\tag{4.2.14}$$

We can also write it like this:

$$V_\delta(\Omega) = \{\mathbf{v}_\delta \in \mathbb{D}_\delta^0(\Omega); \forall q_\delta \in \mathbb{M}_\delta, b(\mathbf{v}_\delta, q_\delta) = 0\}\tag{4.2.15}$$

For the sake of generality, we prove the ellipticity property on a space that contains all the $V_\delta(\Omega)$.

Lemma 4.2 *There exists a constant c independent of the discretization, such that the following ellipticity property about $a_\alpha^\delta(\cdot, \cdot)$ holds*

$$\forall \mathbf{v}_\delta \in V_\delta(\Omega), \quad a_\alpha^\delta(\mathbf{v}_\delta, \mathbf{v}_\delta) \geq c \|\mathbf{v}_\delta\|_{\alpha, \delta}^2 \quad (4.2.16)$$

We next prove the inf-sup condition: First introduce two spaces associated with $\mathbb{D}_\delta(\Omega)$ and $\mathbb{M}_\delta(\Omega)$:

$$\mathbb{M}_D(\Omega) = \{q \in L_0^2(\Omega); q|_{\Omega_k} \in \mathbb{R}, 1 \leq k \leq K\}; \quad (4.2.17)$$

Clearly this space only depends on the decomposition.

Next, for a positive integer L , we consider the space $\mathbb{D}_{DD}(\Omega)$ of functions \mathbf{v}_δ such that

1. their restrictions $\mathbf{v}_\delta|_{\Omega_k}$ to each $\Omega_k, 1 \leq k \leq K$, belong to $\mathbb{P}_L(\Omega_k)$,
2. $\mathbf{v}_\delta \cdot \mathbf{n}$ vanish on $\partial\Omega$.
3. which are continuous on $\bar{\Omega}$.

Note that the space only depends on the decomposition and on the integer L . Note also that it is contained in all spaces $\mathbb{D}_\delta^0(\Omega)$ such that the N_k are larger than L . We intend to establish that an inf-sup condition links the spaces $\mathbb{D}_D(\Omega)$ and $\mathbb{M}_D(\Omega)$, for an appropriate choice of L . According to the reference ([12], Proposition 3.1), we have the following lemma.

Lemma 4.3 *There exists a constant L only depending on the decomposition, such that for any function q in $\mathbb{M}_D(\Omega)$ satisfying*

$$\forall \mathbf{v} \in \mathbb{D}_D(\Omega), \quad b(\mathbf{v}, q) = 0,$$

then q must be zero on the whole domain $\bar{\Omega}$.

We can then derive the inf-sup condition.

Lemma 4.4 *There exists an integer N_D only depending on the decomposition of Ω such that, if all the $N_k \geq N_D$, the following inf-sup condition holds*

$$\forall q_\delta \in \mathbb{M}_\delta(\Omega), \quad \sup_{\mathbf{v}_\delta \in \mathbb{D}_\delta^0(\Omega)} \frac{-\int_\Omega q_\delta (\operatorname{div} \mathbf{v}_\delta) d\mathbf{x}}{\|\mathbf{v}_\delta\|_{\alpha, \delta}} \geq \beta_D \|q_\delta\|_{\alpha^*, \delta}, \quad (4.2.18)$$

for a positive constant β_D depending only on the decomposition of Ω but not δ .

Proof: Let q_δ be any function in $\mathbb{M}_\delta(\Omega)$, according to the Boland and Nicolaidis technique [28], the idea consists in writing q_δ as

$$q_\delta = \bar{q}_\delta + \tilde{q}_\delta, \quad \text{with } \bar{q}_\delta|_{\Omega_k} = \frac{1}{\operatorname{meas}(\Omega_k)} \int_{\Omega_k} q_\delta(\mathbf{x}) d\mathbf{x}, \quad 1 \leq k \leq K.$$

The decomposition is orthogonal in the sense of

$$\|q_\delta\|_{L^2(\Omega)}^2 = \|\bar{q}_\delta\|_{L^2(\Omega)}^2 + \|\tilde{q}_\delta\|_{L^2(\Omega)}^2. \quad (4.2.19)$$

Since $\tilde{q}_\delta|_{\Omega_k}$ has a null integral on Ω_k , it follows from [8] that there exists a function \mathbf{v}_k in $\mathbb{P}_{N_k} \cap H_0(\operatorname{div}, \Omega_k)$ such that

$$-\operatorname{div} \mathbf{v}_k = \tilde{q}_\delta|_{\Omega_k} \quad \text{and} \quad \|\mathbf{v}_k\|_{H(\operatorname{div}, \Omega_k)} \leq \tilde{c} \|\tilde{q}_\delta|_{\Omega_k}\|_{L^2(\Omega)}. \quad (4.2.20)$$

So, the function $\tilde{\mathbf{v}}_\delta$ defined by $\mathbf{v}_\delta|_{\Omega_k} = \mathbf{v}_k$, $1 \leq k \leq K$, belongs to $\mathbb{D}_\delta^0(\Omega)$ (and even to $H_0(\operatorname{div}, \Omega)$).

Concerning the function \bar{q}_δ , (note that its integral on Ω is equal to the integral of \tilde{q}_δ , hence to zero), we refer to ([12], Proposition 3.1) that there exists at least a function $\bar{\mathbf{v}}_\delta$ in \mathbb{D}_D , with $\bar{\mathbf{v}}_\delta|_{\Omega_k}$ in $\mathbb{P}_{N_D}(\Omega_k)^d$, such that

$$-\operatorname{div} \bar{\mathbf{v}}_\delta = \bar{q}_\delta \quad \text{and} \quad \|\bar{\mathbf{v}}_\delta\|_{H(\operatorname{div}, \Omega)} \leq \bar{c} \|\bar{q}_\delta\|_{L^2(\Omega)}. \quad (4.2.21)$$

This function $\bar{\mathbf{v}}_\delta$ belongs to $\mathbb{D}_\delta^0(\Omega)$ when all the $N_k \geq N_D$.

We finally take: $\mathbf{v}_\delta = \tilde{\mathbf{v}}_\delta + \mu \bar{\mathbf{v}}_\delta$, for a positive parameter μ . Indeed, $\operatorname{div} \tilde{\mathbf{v}}_\delta$ is orthogonal to \bar{q}_δ (this comes from the Stokes formula applied on each Ω_k and $\mathbf{grad} \bar{q}_\delta = 0$),

so that, from the previous two equations, we have

$$-\int_{\Omega} q_{\delta} (\operatorname{div} \mathbf{v}_{\delta}) d\mathbf{x} \geq \|\tilde{q}_{\delta}\|_{L^2(\Omega)}^2 + \mu \|\bar{q}_{\delta}\|_{L^2(\Omega)}^2 - \mu \|\operatorname{div} \bar{\mathbf{v}}_{\delta}\|_{L^2(\Omega)} \|\tilde{q}_{\delta}\|_{L^2(\Omega)}.$$

Then it is readily checked that

$$-\int_{\Omega} q_{\delta} (\operatorname{div} \mathbf{v}_{\delta}) d\mathbf{x} \geq \|\tilde{q}_{\delta}\|_{L^2(\Omega)}^2 + \mu \|\bar{q}_{\delta}\|_{L^2(\Omega)}^2 - \mu \bar{c} \|\bar{q}_{\delta}\|_{L^2(\Omega)} \|\tilde{q}_{\delta}\|_{L^2(\Omega)} \quad (4.2.22)$$

$$\geq \frac{1}{2} \|\tilde{q}_{\delta}\|_{L^2(\Omega)}^2 + \mu \left(1 - \frac{\mu \bar{c}^2}{2}\right) \|\bar{q}_{\delta}\|_{L^2(\Omega)}^2. \quad (4.2.23)$$

Choosing $\mu = \frac{1}{\bar{c}^2}$ and using (4.2.19) gives

$$-\int_{\Omega} q_{\delta} (\operatorname{div} \mathbf{v}_{\delta}) d\mathbf{x} \geq c \|\tilde{q}_{\delta}\|_{L^2(\Omega)}^2.$$

On the other hand, we have

$$\|\mathbf{v}_{\delta}\|_{\alpha, \delta} \leq c (\|\tilde{q}_{\delta}\|_{L^2(\Omega)} + \mu \|\bar{q}_{\delta}\|_{L^2(\Omega)}) \leq c \|q_{\delta}\|_{\alpha^*, \delta}.$$

This ends the proof.

It follows from Lemma 4.2 and Lemma 4.4 that problem (4.2.13) has a unique solution $(\mathbf{u}_{\delta}^0, p_{\delta})$ in $\mathbb{D}_{\delta}^0(\Omega) \times \mathbb{M}_{\delta}(\Omega)$ and that this solution satisfies

$$\|\mathbf{u}_{\delta}^0\|_{\alpha, \delta} + \|p_{\delta}\|_{\alpha^*, \delta} \leq C (\|\mathbf{u}_{\delta}^b\|_{\alpha, \delta} + \|\mathbf{f}\|_{\mathbb{D}_{\delta}^0(\Omega)'} + \|p_0\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}) \quad (4.2.24)$$

We now have the main result of this section.

Proposition 4.1 *For any data (\mathbf{f}, k, p_0) such that each $\mathbf{f}|_{\Omega_k}, 1 \leq k \leq K$, and k, p_0 are continuous on $\bar{\Omega}_k$ and on Γ_1, Γ_2 respectively, problem (4.2.6)–(4.2.7) has a unique solution $(\mathbf{u}_{\delta}, p_{\delta})$ in $\mathbb{D}_{\delta}(\Omega) \times \mathbb{M}_{\delta}(\Omega)$. Moreover, there exists a constant c independent of δ such that this solution satisfies*

$$\|\mathbf{u}_{\delta}\|_{\alpha, \delta} + \|p_{\delta}\|_{\alpha^*, \delta} \leq c (\|\mathcal{I}_{\delta}^{\Gamma_1} k\|_{H^{-\frac{1}{2}}(\Gamma_1)} + \|\mathcal{I}_{\delta} \mathbf{f}\|_{\mathbb{D}_{\delta}^0(\Omega)'} + \|\mathcal{I}_{\delta}^{\Gamma_2} p_0\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}) \quad (4.2.25)$$

Proof: The pair $(\mathbf{u}_\delta = \mathbf{u}_\delta^0 + \mathbf{u}_\delta^b, p_\delta)$ is a solution of problem (4.2.6) and (4.2.7), and estimate (4.2.25) is a consequence of (4.2.24) and (4.2.12). On the other hand, let $(\mathbf{u}_{\delta 1}, p_{\delta 1})$ and $(\mathbf{u}_{\delta 2}, p_{\delta 2})$ be two solutions of problem (4.2.6) and (4.2.7). Then the difference $(\mathbf{u}_{\delta 1} - \mathbf{u}_{\delta 2}, p_{\delta 1} - p_{\delta 2})$ is a solution of problem (4.2.13) with data $\mathbf{u}_\delta^b, \mathbf{f}, p_0$ equal to zero. Thus, it follows from (4.2.25) that it is zero. So the solution of problem (4.2.6) and (4.2.7) is unique.

4.2.1 Approximation of divergence-free functions.

We now intend to estimate the distance of a divergence-free function to the space $V_\delta(\Omega)$ introduced in (4.2.14). Here we consider separately the case of dimension $d = 2$ and $d = 3$. For each $\delta = (N_1, N_2, \dots, N_L)$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_L)$, we define the parameter μ_δ

- equal to the maximum of the ratios $\alpha_l/\alpha_{l'}$ in the case of a conforming decomposition,
- equal to the maximum of the ratios $(\alpha_l N_l)/(\alpha_{l'} N_{l'})$ for all adjacent subdomains Ω_l and $\Omega_{l'}$, $1 \leq l, l' \leq L$, where Ω_l and $\Omega_{l'}$ share a part of an edge in dimension $d = 2$, or a part of a face in dimension $d = 3$.

We also define the quantity μ as the largest ratios of $\alpha_l/\alpha_{l'}$.

We note that the constant μ only depends on the decomposition and on the choice of the mortars but not on the discretization parameter. However μ_δ depends on δ for a nonconforming decomposition.

We introduce here new norms as follows:

$$\|v\|_{\alpha, L^2(\Omega)} = \left(\sum_{k=1}^K \alpha_k \|v|_{\Omega_k}\|_{L^2(\Omega_k)}^2 \right)^{\frac{1}{2}}$$

$$\|v\|_{\alpha, H^1(\Omega)} = \left(\sum_{k=1}^K \alpha_k \|v|_{\Omega_k}\|_{H^1(\Omega_k)}^2 \right)^{\frac{1}{2}}$$

Proposition 4.2 Assume the function \mathbf{u} in V such that each $\mathbf{u}|_{\Omega_k}$, $1 \leq k \leq K$, belongs to $H^{s_k}(\Omega_k)^d$, $s_k \geq \frac{1}{2}$. In the case of dimension $d = 2$, there exists a constant c independent of δ such that

$$\inf_{\mathbf{v}_\delta \in V(\Omega)} \|\mathbf{u} - \mathbf{v}_\delta\|_{\alpha, L^2(\Omega)} \leq c(1 + \mu^{\frac{1}{2}} + \mu_\delta^{\frac{1}{2}}) \left(\sum_{k=1}^K \alpha_k N_k^{-2s_k} \|\mathbf{u}\|_{H^{s_k}(\Omega)^d}^2 \right)^{\frac{1}{2}}. \quad (4.2.26)$$

Proof: Since \mathbf{u} is divergence-free, there exists a stream-function ψ in $H^1(\Omega)$ such that $\mathbf{u} = \mathbf{curl}\psi$, with each $\psi|_{\Omega_l}$ in $H^{s_l+1}(\Omega_l)$. According to the isotropic case in ([24], Appendix B) and [16], there exists a function ψ_δ :

- which is equal to ψ on $\partial\Omega$,
- which preserves the values of ψ in all the corners of the Ω_l ,
- such that each $\psi_\delta|_{\Omega_l}$ belongs to $\mathbb{P}_{N_l}(\Omega_l)$ and satisfies on each $\Gamma_{k,l}$

$$\forall \chi \in \mathbb{P}_{N_l-2}(\Gamma_{k,l}), \int_{\Gamma_{k,l}} (\psi_\delta - \phi) \chi d\tau = 0. \quad (4.2.27)$$

Here, ϕ stands for the mortar function associated with ψ_δ , equal to $\psi_\delta|_{\Omega_{l(m)}}$ on each γ_m .

- and finally which satisfies (refer to ([16], Lemma 3.4))

$$\|\psi - \psi_\delta\|_{\alpha, H^1(\Omega)} \leq c(1 + \mu^{\frac{1}{2}} + \mu_\delta^{\frac{1}{2}}) \left(\sum_{k=1}^K \alpha_k N_k^{-2s_k} \|\psi\|_{H^{s_k+1}(\Omega)^d}^2 \right)^{\frac{1}{2}}. \quad (4.2.28)$$

Taking $\mathbf{v}_\delta = \mathbf{curl}\psi_\delta$, the mortar function associated with \mathbf{v}_δ is defined by, for $1 \leq m \leq M$,

$$\phi_1 = \mathbf{v}_\delta|_{\Omega_{l(m)}} \cdot \mathbf{n}_{l(m)} = \partial_\tau \psi_\delta|_{\Omega_{l(m)}} = \phi' \quad (4.2.29)$$

which gives the results (4.2.26). So we only need to check that it satisfies the matching condition. Let $\boldsymbol{\tau}$ denotes the unit vector directly orthogonal to $\mathbf{n}_{l(m)}$, the mortar function associated with \mathbf{v}_δ is defined by, for $1 \leq m \leq M$,

$$\phi_1 = \mathbf{v}_\delta|_{\Omega_{l(m)}} \cdot \mathbf{n}_{l(m)} = \partial_\tau \psi_\delta|_{\Omega_{l(m)}} = \phi'. \quad (4.2.30)$$

So, we have on each $\Gamma_{k,l}$ which is not a mortar (recall that $\psi_\delta|_{\Omega_k} - \phi$ vanishes at each endpoint of $\Gamma_{k,l}$)

$$\begin{aligned} \forall \chi \in \mathbf{P}_{N_l-2}(\Gamma_{k,l}), \int_{\Gamma_{k,l}} (\mathbf{v}_\delta|_{\Omega_k} \cdot \mathbf{n}_k + \phi_1) \chi d\tau &= - \int_{\Gamma_{k,l}} (\partial_\tau \psi_\delta|_{\Omega_k} - \phi') \chi d\tau \\ &= \int_{\Gamma_{k,l}} (\psi_\delta|_{\Omega_k} - \phi) \chi' d\tau = 0 \end{aligned}$$

which is the desired matching condition.

We now consider the case of dimension $d = 3$, which is much more complex. In order to prove the approximation results, we first introduce the mortar space $\mathbb{C}_\delta(\Omega)$ which is the analogue of $\mathbb{D}_\delta(\Omega)$ for the approximations in $H(\mathit{curl}, \Omega)$. It is the space of functions \mathbf{v}_δ such that:

- their restrictions $\mathbf{v}_\delta|_{\Omega_k}$ to each Ω_k , $1 \leq k \leq K$, belongs to $\mathbb{P}_{N_k}(\Omega_k)^3$,
- the mortar function φ being defined on each γ_m by

$$\varphi|_{\gamma_m} = \mathbf{v}_\delta|_{\Omega_{k(m)}} \times \mathbf{n}_{k(m)},$$

the following matching condition holds on each edge $\Gamma_{k,l}$, $1 \leq k \leq K, 1 \leq l \leq L(k)$, which is not a mortar:

$$\forall \chi \in \mathbf{P}_{N_k-2}(\Gamma_{k,l})^3, \int_{\Gamma_{k,l}} (\mathbf{v}_\delta|_{\Omega_k} \times \mathbf{n}_k + \varphi)(\boldsymbol{\tau}) \cdot \boldsymbol{\chi}(\boldsymbol{\tau}) d\boldsymbol{\tau} = 0. \quad (4.2.31)$$

Here we first consider the case of a conforming decomposition, with some restrictions on the choice of the N_k .

Lemma 4.5 *Assume that the function \mathbf{u} is equal to $\mathit{curl} \boldsymbol{\xi}$ for a function $\boldsymbol{\xi}$ in $H(\mathit{curl}, \Omega)$, and is such that each $\mathbf{u}|_{\Omega_k}, 1 \leq k \leq K$, belongs to $H^{s_k}(\Omega_k)^d, s_k \geq \frac{3}{2}$. In the case of dimension $d = 3$, if the decomposition is conforming and if moreover for each mortar $\gamma_m, 1 \leq m \leq M$, which is a face of both $\Omega_{k(m)}$ and Ω_k , $N_{k(m)} \geq N_k$,*

there exists a constant c independent of δ such that

$$\inf_{\boldsymbol{\xi}_\delta \in \mathbb{C}_\delta(\Omega)} \|\mathbf{u} - \mathbf{curl} \boldsymbol{\xi}_\delta\|_{\alpha, L^2(\Omega)^3} \leq c(1 + \mu^{\frac{1}{2}}) \left(\sum_{k=1}^K \alpha_k N_k^{-2s_k} \|\mathbf{u}\|_{H^{s_k}(\Omega_k)^3}^2 \right)^{\frac{1}{2}} \quad (4.2.32)$$

Proof: It follows from ([11], Them 4.9) that for $1 \leq k \leq K$ and $s_k \geq \frac{3}{2}$, there exists a polynomial $\boldsymbol{\xi}_\delta^k$ in $\mathbb{P}_{N_k}(\Omega)^3$ such that $\boldsymbol{\xi}_\delta^k \times \mathbf{n}$ coincide with $\boldsymbol{\xi}^k \times \mathbf{n}$ and that

$$\alpha_k \|\mathbf{curl} \boldsymbol{\xi} - \mathbf{curl} \boldsymbol{\xi}_\delta^k\|_{L^2(\Omega_k)} \leq c \alpha_k N_k^{-s_k} \|\mathbf{u}\|_{H^{s_k}(\Omega_k)^3}. \quad (4.2.33)$$

Moreover, for $1 \leq l \leq L(k)$, this function $\boldsymbol{\xi}_\delta^k$ satisfies

$$\forall \mathbf{g} \in \mathbb{P}_{N_k-2}(\Gamma_{k,l})^3, \quad \int_{\Gamma_{k,l}} (\boldsymbol{\xi} - \boldsymbol{\xi}_\delta^k) \times \mathbf{n} \cdot \mathbf{g} d\boldsymbol{\tau} = 0. \quad (4.2.34)$$

In the case of a conforming decomposition and thanks to the assumption on the N_k , equation (4.2.34) implies that the function $\boldsymbol{\xi}_\delta$ defined by $\boldsymbol{\xi}_\delta|_{\Omega_k} = \boldsymbol{\xi}_\delta^k$, $1 \leq k \leq K$, belongs to \mathbb{C}_δ , whence the result.

Proposition 4.3 *Assume that the function \mathbf{u} in V such that each $\mathbf{u}|_{\Omega_k}$, $1 \leq k \leq K$, belongs to $H^{s_k}(\Omega_k)^d$, $s_k \geq \frac{3}{2}$. In the case of dimension $d = 3$, if the decomposition is conforming and if moreover for each mortar γ_m , $1 \leq m \leq M$, which is a face of both $\Omega_{k(m)}$ and Ω_k , $N_{k(m)} \geq N_k$, there exists a constant c independent of δ such that*

$$\inf_{\mathbf{v}_\delta \in V_\delta(\Omega)} \|\mathbf{u} - \mathbf{v}_\delta\|_{\alpha, L^2(\Omega)^3} \leq c(1 + \mu^{\frac{1}{2}}) \left(\sum_{k=1}^K \alpha_k N_k^{-2s_k} \|\mathbf{u}\|_{H^{s_k}(\Omega_k)^3}^2 \right)^{\frac{1}{2}} \quad (4.2.35)$$

Proof: It follows from [5] that there exists a vector potential $\boldsymbol{\xi}$ such that $\mathbf{u} = \mathbf{curl} \boldsymbol{\xi}$. Estimate (4.2.32) yields that the function $\mathbf{v}_\delta = \mathbf{curl} \boldsymbol{\xi}_\delta$ satisfies the desired property (4.2.35). Moreover, it can be checked ([11], Chapter 4) that, on each face Γ of Ω_k , $\mathbf{curl} \boldsymbol{\xi}_\delta \cdot \mathbf{n}$ is equal to the orthogonal projection (in $L^2(\Gamma)$) of $\mathbf{u} \cdot \mathbf{n}$ onto $\mathbb{P}_{N_k-1}(\Gamma)$. So the jump of $\mathbf{curl} \boldsymbol{\xi}_\delta \cdot \mathbf{n}$ through Γ is zero (or its trace if Γ is contained in $\partial\Omega$), and the function \mathbf{v}_δ belongs to $V_\delta(\Omega)$.

4.2.2 Error estimate

We prove an error estimates, first for the velocity, second for the pressure. Let u be the solution of equation (4.1.1), u_δ the solution of equation (4.2.6)-(4.2.7) and $\boldsymbol{\omega}_\delta$ any function in the kernel $V_\delta(\Omega)$. Multiply the first line of (4.1.1) by $\boldsymbol{\omega}_\delta$ gives

$$\begin{aligned} \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{u}^0 \cdot \boldsymbol{\omega}_\delta d\mathbf{x} + b(\boldsymbol{\omega}_\delta, p) &= \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k \int_{\Omega_l} \mathbf{u}^b \cdot \boldsymbol{\omega}_\delta d\mathbf{x} \\ &\quad - \int_{\Gamma_2} p \cdot \boldsymbol{\omega}_\delta \cdot \mathbf{n} ds - \int_S p \cdot [\boldsymbol{\omega}_\delta \cdot \mathbf{n}] d\tau \end{aligned}$$

where the notation $[\cdot]$ means the jump across S .

Using the definition of $V_\delta(\Omega)$ thus implies, for any q_δ in $\mathbb{M}_\delta(\Omega)$,

$$\begin{aligned} \sum_{k=1}^K \alpha_k \int_{\Omega_k} u^0 \cdot \boldsymbol{\omega}_\delta d\mathbf{x} + b(\boldsymbol{\omega}_\delta, p - q_\delta) &= \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k \int_{\Omega_l} u^b \cdot \boldsymbol{\omega}_\delta d\mathbf{x} \\ &\quad - \int_{\Gamma_2} p \cdot \boldsymbol{\omega}_\delta \cdot \mathbf{n} ds - \int_S p \cdot [\boldsymbol{\omega}_\delta \cdot \mathbf{n}] d\tau \end{aligned} \tag{4.2.36}$$

Due to (??), we have for any \mathbf{v}_δ in V_δ

$$\|\mathbf{u}_\delta^0 - \mathbf{v}_\delta\|_{\alpha, L^2(\Omega)}^2 \leq \sum_{k=1}^K \alpha_k ((\mathbf{u}_\delta^0 - \mathbf{v}_\delta, \mathbf{u}_\delta^0 - \mathbf{v}_\delta))_\delta^k.$$

In (4.2.36), letting $\boldsymbol{\omega}_\delta = \mathbf{u}_\delta^0 - \mathbf{v}_\delta$ and subtracting the first line of (4.2.7) leads to

$$\begin{aligned} \|\mathbf{u}_\delta^0 - \mathbf{v}_\delta\|_{\alpha, L^2(\Omega)}^2 &\leq \sum_{k=1}^K \alpha_k \int_{\Omega_k} (\mathbf{u}^0 - \mathbf{v}_\delta) \cdot (\mathbf{u}_\delta^0 - \mathbf{v}_\delta) d\mathbf{x} + \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{v}_\delta \cdot (\mathbf{u}_\delta^0 - \mathbf{v}_\delta) d\mathbf{x} \\ &\quad - \sum_{k=1}^K \alpha_k ((\mathbf{v}_\delta, \mathbf{u}_\delta^0 - \mathbf{v}_\delta))_\delta^k + b(\mathbf{u}_\delta^0 - \mathbf{v}_\delta, p - q_\delta) + \int_S [(\mathbf{u}_\delta^0 - \mathbf{v}_\delta) \cdot \mathbf{n}] \cdot p d\tau \\ &\quad + \sum_{k=1}^K \alpha_k ((\mathbf{f}, \mathbf{u}_\delta^0 - \mathbf{v}_\delta))_\delta^k - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot (\mathbf{u}_\delta^0 - \mathbf{v}_\delta) d\mathbf{x} + \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{u}^b \cdot (\mathbf{u}_\delta^0 - \mathbf{v}_\delta) d\mathbf{x} \\ &\quad - \sum_{k=1}^K \alpha_k ((\mathbf{u}_\delta^b, \mathbf{u}_\delta^0 - \mathbf{v}_\delta))_\delta^k \int_{\Gamma_2} p_0 \cdot [(\mathbf{u}_\delta^0 - \mathbf{v}_\delta) \cdot \mathbf{n}] ds - ((p_0, (\mathbf{u}_\delta^0 - \mathbf{v}_\delta) \cdot \mathbf{n}))_\delta^{\Gamma_2}. \end{aligned}$$

Using a triangle inequality yields

$$\begin{aligned}
 & \|\mathbf{u} - \mathbf{u}_\delta\|_{\alpha, L^2(\Omega)} \leq \|\mathbf{u}^0 - \mathbf{v}_\delta\|_{\alpha, L^2(\Omega)} + \|\mathbf{u}_\delta^0 - \mathbf{v}_\delta\|_{\alpha, L^2(\Omega)} + \|\mathbf{u}^b - \mathbf{u}_\delta^b\|_{\alpha, L^2(\Omega)} \\
 & \leq C \left\{ \inf_{\mathbf{v}_\delta \in V_\delta} \|\mathbf{u}^0 - \mathbf{v}_\delta\|_{\alpha, L^2(\Omega)} + \sup_{\boldsymbol{\omega}_\delta \in \mathbb{D}_\delta} \left\{ \frac{\sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{v}_\delta \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k ((\mathbf{v}_\delta, \boldsymbol{\omega}_\delta))_\delta^k}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} \right. \right. \\
 & \quad \left. \left. + \frac{\sum_{k=1}^K \alpha_k ((\mathbf{f}, \boldsymbol{\omega}_\delta))_\delta^k - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\omega}_\delta d\mathbf{x}}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} \right. \right. \\
 & \quad \left. \left. + \frac{\sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{u}^b \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k ((\mathbf{u}_\delta^b, \boldsymbol{\omega}_\delta))_\delta^k}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} + \frac{\int_{\Gamma_2} p_0 \cdot (\boldsymbol{\omega}_\delta \cdot \mathbf{n}) ds - ((p_0, (\boldsymbol{\omega}_\delta \cdot \mathbf{n}))_{\Gamma_2}^2)}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} \right. \right. \\
 & \quad \left. \left. + \frac{b(\boldsymbol{\omega}_\delta, p - q_\delta)}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} + \frac{\int_{\mathbf{S}} [\boldsymbol{\omega}_\delta \cdot \mathbf{n}] (p - q_\delta) d\boldsymbol{\tau}}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} + \|\mathbf{u}^b - \mathbf{u}_\delta^b\|_{\alpha, L^2(\Omega)} \right\}. \right. \\
 & \hspace{15em} (4.2.37)
 \end{aligned}$$

Next, let $\mathbf{\Pi}_{N_{k-1}}$ be defined as the orthogonal projection operator from $L^2(\Omega_k)$ onto $\mathbb{P}_{N_{k-1}}(\Omega_k)$. By adding and subtracting the function $\mathbf{\Pi}_{N_{k-1}}\mathbf{u}^0$, we deduce from the exactness of the quadrature formula that

$$\begin{aligned}
 & \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{v}_\delta \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k ((\mathbf{v}_\delta, \boldsymbol{\omega}_\delta))_\delta^k \\
 & = \sum_{k=1}^K \alpha_k \int_{\Omega_k} (\mathbf{v}_\delta - \mathbf{\Pi}_{N_{k-1}}\mathbf{u}^0) \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k ((\mathbf{v}_\delta - \mathbf{\Pi}_{N_{k-1}}\mathbf{u}^0), \boldsymbol{\omega}_\delta)_\delta^k \\
 & \hspace{15em} (4.2.38)
 \end{aligned}$$

So we have

$$\begin{aligned}
 & \sup_{\boldsymbol{\omega}_\delta \in \mathbb{D}_\delta} \frac{\sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{v}_\delta \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k ((\mathbf{v}_\delta, \boldsymbol{\omega}_\delta))_\delta^k}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} \leq (3^d + 1) \left(\sum_{k=1}^K \alpha_k \|\mathbf{v} - \mathbf{\Pi}_{N_{k-1}}\mathbf{u}^0\|_{L^2(\Omega_k)}^2 \right)^{\frac{1}{2}} \\
 & \leq (3^d + 1) \|\mathbf{u}^0 - \mathbf{v}_\delta\|_{\alpha, L^2(\Omega)} + (3^d + 1) \left(\sum_{k=1}^K \alpha_k \|\mathbf{u}^0 - \mathbf{\Pi}_{N_{k-1}}\mathbf{u}^0\|_{L^2(\Omega_k)}^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

Suppose the solution $\mathbf{u}_{|\Omega_k}^0$ is in $H^{s_k}(\Omega_k)^d$, $s_k \geq 0$, then according to the property of the orthogonal projection operator,

$$\begin{aligned} & \sup_{\boldsymbol{\omega}_\delta \in \mathbb{D}_\delta} \frac{\sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{v}_\delta \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k ((\mathbf{v}_\delta, \boldsymbol{\omega}_\delta)_\delta)^k}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} \\ & \leq 4\|\mathbf{u}^0 - \mathbf{v}_\delta\|_{\alpha, L^2(\Omega)} + c \left(\sum_{k=1}^K \alpha_k N_k^{-2s_k} \|\mathbf{u}_{|\Omega_k}^0\|_{H^{s_k}(\Omega_k)^d}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (4.2.39)$$

Similarly, we also have

$$\begin{aligned} & \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k ((\mathbf{f}, \boldsymbol{\omega}_\delta)_\delta)^k \\ & = \sum_{k=1}^K \alpha_k \int_{\Omega_k} (\mathbf{f} - \mathbf{\Pi}_{N_k-1} \mathbf{f}) \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k ((\mathbf{f} - \mathbf{\Pi}_{N_k-1} \mathbf{f}), \boldsymbol{\omega}_\delta)_\delta^k \\ & \leq \sqrt{\alpha_{\max}} \left(10 \left(\sum_{k=1}^K \|\mathbf{f} - \mathbf{\Pi}_{N_k-1} \mathbf{f}\|_{L^2(\Omega_k)^d}^2 \right)^{\frac{1}{2}} + 9 \|\mathbf{f} - \mathcal{I}_\delta \mathbf{f}\|_{L^2(\Omega_k)^d} \right) \|\boldsymbol{\omega}_\delta\|_{L^2(\Omega_k)^d}. \end{aligned} \quad (4.2.40)$$

Suppose the function $\mathbf{f}_{|\Omega_k}$ is in $H^{\sigma_k}(\Omega_k)^d$, $\sigma_k \geq 0$, then according to the property of the orthogonal projection operator $\mathbf{\Pi}_{N_k-1}$ and interpolation operator \mathcal{I}_δ ,

$$\begin{aligned} & \sup_{\boldsymbol{\omega}_\delta \in \mathbb{D}_\delta} \frac{\sum_{k=1}^K \alpha_k ((\mathbf{f}, \boldsymbol{\omega}_\delta)_\delta)^k - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\omega}_\delta d\mathbf{x}}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} \\ & \leq c \sqrt{\alpha_{\max}} \left(\sum_{k=1}^K N_k^{-2\sigma_k} \|\mathbf{f}_{|\Omega_k}\|_{H^{\sigma_k}(\Omega_k)^d}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (4.2.41)$$

Similarly suppose the function p_0 is in $H^{\theta_k}(\Gamma_2 \cap \partial\Omega_k)$, $\theta_k \geq 0$, then we have

$$\begin{aligned} & \sup_{\boldsymbol{\omega}_\delta \in \mathcal{D}_\delta} \frac{\int_{\Gamma_2} p_0 \cdot (\boldsymbol{\omega}_\delta \cdot \mathbf{n}) ds - ((p_0, (\boldsymbol{\omega}_\delta \cdot \mathbf{n}))_\delta^{\Gamma_2})}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} \\ & \leq c \sqrt{\frac{1}{\alpha_{\min}}} \left(\sum_{k=1}^{K_0} N_k^{-2\theta_k} \|p_0\|_{H^{\theta_k}(\Gamma_2 \cap \partial\Omega_k)}^2 \right)^{\frac{1}{2}} \end{aligned} \quad (4.2.42)$$

We refer to ([21], Thm.7.1) and ([21], Thm.14.2) for the approximation properties of

the operators $\mathbf{\Pi}_{N_k-1}$ and \mathcal{I}_δ , respectively.

According to Lemma 4.1, we also obtain that

$$\begin{aligned} & \sup_{\boldsymbol{\omega}_\delta \in \mathbb{D}_\delta} \frac{\sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{u}^b \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k ((\mathbf{u}_\delta^b, \boldsymbol{\omega}_\delta))_\delta^k}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} \\ & \leq c \sqrt{\alpha_{\max}} \left(\sum_{k=1}^K N_k^{-2s_k} \|\mathbf{u}_{\Omega_k}^b\|_{H^{s_k}(\Omega_k)^d}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (4.2.43)$$

According to ([6], Lemma 6.1 and (6.9)) we know that

$$\begin{aligned} & \int_{\mathbf{S}} [\boldsymbol{\omega}_\delta \cdot \mathbf{n}] (p - q_\delta) d\boldsymbol{\tau} \\ & \leq c \sqrt{\frac{1}{\alpha_{\min}}} (1 + \mu) \left(\sum_{k=1}^K \sum_{l=1}^{L(k)} N_k^2 \|p - q_\delta\|_{L^2(\Gamma_{k,l})}^2 \right)^{\frac{1}{2}} \|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}. \end{aligned} \quad (4.2.44)$$

So we have

$$\sup_{\boldsymbol{\omega}_\delta \in \mathbb{D}_\delta} \frac{\int_{\mathbf{S}} [\boldsymbol{\omega}_\delta \cdot \mathbf{n}] (p - q_\delta) d\boldsymbol{\tau}}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} \leq c \sqrt{\frac{1}{\alpha_{\min}}} (1 + \mu) \left(\sum_{k=1}^K \sum_{l=1}^{L(k)} N_k^2 \|p - q_\delta\|_{L^2(\Gamma_{k,l})}^2 \right)^{\frac{1}{2}}. \quad (4.2.45)$$

the estimation of norm $\|p - q_\delta\|_{L^2(\Gamma_{k,l})}^2$ will be showed later.

Because of the ellipticity property of $a_\alpha^\delta(\cdot, \cdot)$ in V_δ , we have

$$\begin{aligned} b(\boldsymbol{\omega}_\delta, p - q_\delta) &= - \sum_{k=1}^K \int_{\Omega_k} \operatorname{div}(\boldsymbol{\omega}_\delta) (p - q_\delta) d\mathbf{x} \\ &\leq \|\operatorname{div}(\boldsymbol{\omega}_\delta)\|_{\alpha, L^2(\Omega)} \cdot \sum_{k=1}^K \frac{1}{\sqrt{\alpha_{\min}}} \|p - q_\delta\|_{L^2(\Omega_k)} \end{aligned} \quad (4.2.46)$$

So we can get

$$\sup_{\boldsymbol{\omega}_\delta \in \mathbb{D}_\delta} \frac{b(\boldsymbol{\omega}_\delta, p - q_\delta)}{\|\boldsymbol{\omega}_\delta\|_{\alpha, L^2(\Omega)}} \leq c \sum_{k=1}^K \frac{1}{\sqrt{\alpha_{\min}}} \|p - q_\delta\|_{L^2(\Omega_k)}. \quad (4.2.47)$$

Next, we choose q_δ by: $q_\delta|_{\Omega_k} = \mathbf{\Pi}_{N_k-2}^1 p$, where $\mathbf{\Pi}_{N_k-2}^1$ stands for the orthogonal

projection operator from $H^1(\Omega_k)$ onto $\mathbb{P}_{N_k-2}(\Omega_k)$ defined by

$$\forall q \in \mathbb{P}_{N_k-2}(\Omega_k), \int_{\Omega_k} \mathbf{grad}(p - \mathbf{\Pi}_{N_k-2}^1 p) \cdot \mathbf{grad} q dx = 0, \int_{\Omega_k} (p - \mathbf{\Pi}_{N_k-2}^1 p) dx = 0$$

In fact, the following approximation properties of this operator are well-known: if the function $p|_{\Omega_k}$ belongs to $H^{s_k+1}(\Omega_k)$, $s_k \geq 0$,

$$\|p - \mathbf{\Pi}_{N_k-2}^1 p\|_{H^1(\Omega_k)} + N_k \|p - \mathbf{\Pi}_{N_k-2}^1 p\|_{L^2(\Omega_k)} \leq c N_k^{-s_k} \|p\|_{H^{s_k+1}(\Omega_k)}. \quad (4.2.48)$$

While estimating the norm $\|p - \mathbf{\Pi}_{N_k-2}^1 p\|_{L^2(\Gamma_{k,l})}$ results from a duality argument and we refer to ([6], Lemma 6.2) for the following lemma.

Lemma 4.6 *For $1 \leq k \leq K$, there exists a constant c independent of N such that, if the function $p|_{\Omega_k}$ belongs to $H^{s_k+1}(\Omega_k)$, $s_k \geq 0$,*

$$\|p - \mathbf{\Pi}_{N_k-2}^1 p\|_{L^2(\Gamma_{k,l})} \leq c N_k^{-s_k - \frac{1}{2}} \|p\|_{H^{s_k+1}}. \quad (4.2.49)$$

We are in a position to write the first error estimate, due to the above analysis and using the approximation properties stated in Propositions 4.2 and Propositions 4.3.

Theorem 4.1 *Assume the data \mathbf{f} such that each $\mathbf{f}|_{\Omega_k}, 1 \leq k \leq K$, belongs to $H^{\sigma_k}(\Omega_k)^d, \sigma_k > \frac{d}{2}$, the solution (\mathbf{u}, p) of problem (4.1.1) such that each $(\mathbf{u}|_{\Omega_k}, p|_{\Omega_k}), 1 \leq k \leq K$, belongs to $H^{s_k}(\Omega_k)^d \times H^{s_k+1}(\Omega_k), s_k \geq d - \frac{3}{2}$. Then, in the two cases*

1. *in dimension $d = 2$,*
2. *in dimension $d = 3$, if the decomposition is conforming and if moreover for each mortar $\gamma^m, 1 \leq m \leq M$, which is a face of both $\Omega_{k(m)}$ and $\Omega_k, N_{k(m)} \geq N_k$, the following error estimate holds between the velocity \mathbf{u} and the velocity \mathbf{u}_δ of problem (4.2.6) – (4.2.7):*

$$\begin{aligned}
 \|\mathbf{u} - \mathbf{u}_\delta\|_{\alpha, L^2(\Omega)} &\leq c \sum_{k=1}^K \left\{ (1 + \mu + \mu_\delta) \sqrt{\alpha_{\max}} N_k^{-s_k} \|\mathbf{u}|_{\Omega_k}\|_{H^{s_k}(\Omega_k)} \right. \\
 &+ (1 + \mu_\delta) \frac{1}{\sqrt{\alpha_{\min}}} N_k^{-s_k + \frac{1}{2}} \|p|_{\Omega_k}\|_{H^{s_k+1}(\Omega_k)} + \sqrt{\alpha_{\max}} N_k^{-\sigma_k} \|f|_{\Omega_k}\|_{H^{\sigma_k}(\Omega_k)} \left. \right\} \\
 &+ c \sum_{k=1}^{K_0} \frac{1}{\sqrt{\alpha_{\min}}} N_k^{-s_k - \frac{1}{2}} \|p_0\|_{H^{s_k + \frac{1}{2}}(\Gamma_2 \cap \partial\Omega_k)} \\
 &+ c \sum_{k=1}^{K_1} (1 + \mu + \mu_\delta) \sqrt{\alpha_{\max}} N_k^{-s_k} \|k\|_{H^{s_k - \frac{1}{2}}(\Gamma_1 \cap \partial\Omega_k)}.
 \end{aligned} \tag{4.2.50}$$

Remark 4.1 *Estimate (4.2.50) is not fully optimal, however, we know that for a general decomposition (of course including the nonconforming case), when $(1 + \mu + \mu_\delta) \max\{\sqrt{\alpha_{\max}}, \frac{1}{\sqrt{\alpha_{\min}}}\}$ is bounded (in fact, this is most often the case in practical situations), the lack of optimality is only $N_k^{\frac{1}{2}}$. In fact, the estimate results of the two cases when the decomposition is conforming, seems to be improved, but we don't prove the results here.*

A more explicit estimate can be deduced from the previously quoted regularity results in the two-dimension case of a polygon Ω . We refer to ([6], Corollary 6.4) and [25] for the next result.

Corollary 4.1 *In the two-dimensional case of a polygon Ω , we have the same assumptions with Theorem 4.1 about the data f . Then the following error estimate holds between the velocity \mathbf{u} and the velocity \mathbf{u}_δ of problem (4.2.6) – (4.2.7):*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_\delta\|_{\alpha, L^2(\Omega)} &\leq c(1 + \mu + \mu_\delta) \max\{\sqrt{\alpha_{\max}}, \frac{1}{\sqrt{\alpha_{\min}}}\} \sum_{k=1}^K N_k^{\frac{1}{2}} E_k \|f|_{\Omega_k}\|_{H^{\sigma_k}(\Omega_k)} \\ &\quad + c \sum_{k=1}^{K_0} \frac{1}{\sqrt{\alpha_{\min}}} N_k^{-\frac{1}{2}} E_k \|p_0\|_{H^{\sigma_k + \frac{1}{2}}(\Gamma_2 \cap \partial\Omega_k)} \\ &\quad + c(1 + \mu + \mu_\delta) \max\{\sqrt{\alpha_{\max}}, \frac{1}{\sqrt{\alpha_{\min}}}\} \sum_{k=1}^{K_1} E_k \|k\|_{H^{\sigma_k - \frac{1}{2}}(\Gamma_1 \cap \partial\Omega_k)} \end{aligned}$$

where $E_k, 1 \leq k \leq K$ is equal to $N_k^{-\sigma_k}$ if $\bar{\Omega}_k$ contains no corner of Ω , $\sup\{N_k^{-\sigma_k}, N_k^{-\frac{4}{3}}(\log N_k)^{\frac{1}{2}}\}$, if $\bar{\Omega}_k$ contains a nonconvex corner of Ω and $\sup\{N_k^{-\sigma_k}, N_k^{-4}(\log N_k)^{\frac{3}{2}}\}$, if $\bar{\Omega}_k$ contains a no nonconvex corner of Ω .

Next we will estimate the error of the pressure which is easier now.

Theorem 4.2 *If the assumptions of Theorem 5.1 are satisfied and in case (i) and (ii) of this theorem, the following error estimate holds between the pressure p of problem (4.1.1) and the pressure p_δ of (4.2.7):*

$$\begin{aligned} &\|p - p_\delta\|_{\alpha^*, \delta} \\ &\leq c \sum_{k=1}^K \{(1 + \mu + \mu_\delta) \sqrt{\alpha_{\max}} N_k^{-s_k} \|\mathbf{u}|_{\Omega_k}\|_{H^{s_k}(\Omega_k)} \\ &\quad + (1 + \mu + \mu_\delta) \max\{\sqrt{\alpha_{\max}}, \frac{1}{\sqrt{\alpha_{\min}}}\} N_k^{-s_k} \max\{N_k^{\frac{1}{2}}, (\log N_k)^\nu\} \|p|_{\Omega_k}\|_{H^{s_k+1}(\Omega_k)} \\ &\quad + \sqrt{\alpha_{\max}} N_k^{-s_k} \|f|_{\Omega_k}\|_{H^{\sigma_k}(\Omega_k)}\} + c \sum_{k=1}^{K_0} \frac{1}{\sqrt{\alpha_{\min}}} N_k^{-s_k - \frac{1}{2}} \|p_0\|_{H^{s_k + \frac{1}{2}}(\Gamma_2 \cap \partial\Omega_k)} \\ &\quad + c \sum_{k=1}^{K_1} (1 + \mu + \mu_\delta) \sqrt{\alpha_{\max}} N_k^{-s_k} \|k\|_{H^{s_k - \frac{1}{2}}(\Gamma_1 \cap \partial\Omega_k)}. \end{aligned}$$

where ν is equal to zero if the decomposition is conforming, to 1 otherwise.

Proof: Due to the inf – sup condition, for any q_δ in $\mathbb{M}_\delta(\Omega)$, we derive that

$$\beta_D \|p_\delta - q_\delta\|_{\alpha^*, \delta} \leq \sup_{\mathbf{v}_\delta \in \mathbb{D}_\delta^0(\Omega)} \frac{-\int_\Omega (p_\delta - q_\delta)(\operatorname{div} \mathbf{v}_\delta) d\mathbf{x}}{\|\mathbf{v}_\delta\|_{\alpha, \delta}}. \quad (4.2.51)$$

In order to evaluate $-\int_\Omega (p_\delta - q_\delta)(\operatorname{div} \mathbf{v}_\delta) d\mathbf{x}$, we first use the discrete problem (4.2.7):

$$\begin{aligned} -\int_\Omega (p_\delta - q_\delta)(\operatorname{div} \mathbf{v}_\delta) d\mathbf{x} &= \sum_{k=1}^K \alpha_k ((\mathbf{f}, \mathbf{v}_\delta)_\delta^k - ((p_0, \mathbf{v}_\delta \cdot \mathbf{n}))_\delta^{\Gamma_2} \\ &\quad - \sum_{k=1}^K \alpha_k ((\mathbf{u}_\delta, \mathbf{v}_\delta)_\delta^k + \int_\Omega q_\delta (\operatorname{div} \mathbf{v}_\delta) d\mathbf{x}. \end{aligned} \quad (4.2.52)$$

Next, we apply equation (4.1.1) to the function \mathbf{v}_δ , integrate by parts and add it to the previous line, which yields

$$\begin{aligned} -\int_\Omega (p_\delta - q_\delta)(\operatorname{div} \mathbf{v}_\delta) d\mathbf{x} &= \sum_{k=1}^K \alpha_k ((\mathbf{f}, \mathbf{v}_\delta)_\delta^k - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \mathbf{v}_\delta d\mathbf{x} + \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{u} \cdot \mathbf{v}_\delta d\mathbf{x} \\ &\quad - \sum_{k=1}^K \alpha_k ((\mathbf{u}, \mathbf{v}_\delta)_\delta^k + \int_{\Gamma_2} p_0 \cdot (\mathbf{v} \cdot \mathbf{n}) ds - ((p_0, \mathbf{v}_\delta \cdot \mathbf{n}))_\delta^{\Gamma_2} \\ &\quad + \int_\Omega (q_\delta - p) \operatorname{div} \mathbf{v}_\delta d\mathbf{x} + \int_S [\mathbf{v} \cdot \mathbf{n}] \cdot p d\boldsymbol{\tau}. \end{aligned}$$

Using a triangle inequality yields

$$\begin{aligned} \|p - p_\delta\|_{\alpha^*, \delta} &\leq c(\|p - q_\delta\|_{\alpha^*, \delta} + \|\mathbf{u} - \mathbf{u}_\delta\|_{\alpha, L^2(\Omega)} \\ &\quad + \sup_{\boldsymbol{\omega}_\delta \in \mathbb{D}_\delta^0} \left\{ \frac{\sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{u}_\delta \mathbf{v}_\delta d\mathbf{x} - \sum_{k=1}^K \alpha_k ((\mathbf{u}_\delta, \mathbf{v}_\delta)_\delta^k}{\|\mathbf{v}_\delta\|_{\alpha, L^2(\Omega)}} \right. \\ &\quad + \frac{\sum_{k=1}^K \alpha_k ((\mathbf{f}, \mathbf{v}_\delta)_\delta^k - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \mathbf{v}_\delta d\mathbf{x}}{\|\mathbf{v}_\delta\|_{\alpha, L^2(\Omega)}} \\ &\quad + \frac{\int_{\Gamma_2} p_0 \cdot (\mathbf{v}_\delta \cdot \mathbf{n}) ds - ((p_0, (\mathbf{v}_\delta \cdot \mathbf{n}))_\delta^{\Gamma_2}}{\|\mathbf{v}_\delta\|_{\alpha, L^2(\Omega)}} \\ &\quad \left. + \frac{\int_S [\mathbf{v}_\delta \cdot \mathbf{n}] p d\mathbf{x}}{\|\mathbf{v}_\delta\|_{\alpha, \delta}} \right\}. \end{aligned} \quad (4.2.53)$$

All the right-hand side terms have been estimated previously, except the last one which corresponds to the consistency error. We refer to ([16], Lemma 3.3) for the following result:

$$\sup_{\omega_\delta \in \mathbb{D}_\delta^0} \frac{\int_S [\mathbf{v}_\delta \cdot \mathbf{n}] p d\mathbf{x}}{\|\mathbf{v}_\delta\|_{\alpha, \delta}} \leq c(1 + \mu) \left(\sum_{k=1}^K \alpha_k N_k^{-2s_k} (\log N_k)^\nu \|p\|_{s_{k+1}}^2 \right)^{\frac{1}{2}}. \quad (4.2.54)$$

where ν is equal to zero if the decomposition is conforming, to 1 otherwise.

Combining all the results above, yields the desired estimate.

As in Corollary 4.1, we also have the following result:

Corollary 4.2 *In the two-dimensional case of a polygon Ω , if the same assumptions as in Theorem 4.1 hold about the data f . Then the following error estimate holds between the velocity p and the velocity p_δ of problem (4.2.7):*

$$\begin{aligned} & \|p - p_\delta\|_{\alpha^*, \delta} \\ & \leq c(1 + \mu + \mu_\delta) \max\{\sqrt{\alpha_{\max}}, \frac{1}{\sqrt{\alpha_{\min}}}\} \sum_{k=1}^K \max\{N_k^{\frac{1}{2}}, (\log N_k)^\nu\} E_k \|f\|_{\Omega_k} \|_{H^{\sigma_k}(\Omega_k)} \\ & \quad + c \sum_{k=1}^{K_0} \frac{1}{\sqrt{\alpha_{\min}}} N_k^{-\frac{1}{2}} E_k \|p_0\|_{H^{\sigma_k + \frac{1}{2}}(\Gamma_2 \cap \partial\Omega_k)} \\ & \quad + c(1 + \mu + \mu_\delta) \max\{\sqrt{\alpha_{\max}}, \frac{1}{\sqrt{\alpha_{\min}}}\} \sum_{k=1}^{K_1} E_k \|k\|_{H^{\sigma_k - \frac{1}{2}}(\Gamma_1 \cap \partial\Omega_k)} \end{aligned} \quad (4.2.55)$$

where ν is equal to zero if the decomposition is conforming, to 1 otherwise; and $E_k, 1 \leq k \leq K$ are given by:

$$E_k = \begin{cases} N_k^{-\sigma_k}, & \text{if } \bar{\Omega}_k \text{ contains no corner of } \Omega, \\ \sup\{N_k^{-\sigma_k}, N_k^{-4} (\log N_k)^{\frac{3}{2}}\}, & \text{if } \bar{\Omega}_k \text{ contains a no nonconvex corner of } \Omega, \\ \sup\{N_k^{-\sigma_k}, N_k^{-\frac{4}{3}} (\log N_k)^{\frac{1}{2}}\}, & \text{if } \bar{\Omega}_k \text{ contains a nonconvex corner of } \Omega. \end{cases} \quad (4.2.56)$$

4.3 Numerical implementation

Firstly, we briefly explain how to implement the discrete problem in section before. The full set of unknowns of the discrete system is given by

- the vector U of the values if \mathbf{u}_δ at all nodes $(x_i^k, y_i^k), 0 \leq i, j \leq N_k, 1 \leq k \leq K$, in dimension $d = 2$, $(x_i^k, y_i^k, z_p^k), 0 \leq i, j, p \leq N_k, 1 \leq k \leq K$, in dimension $d = 3$,
- the vector P of the values if p_δ at all nodes $(x_i^k, y_i^k), 0 \leq i, j \leq N_k, 1 \leq k \leq K$, in dimension $d = 2$, $(x_i^k, y_i^k, z_p^k), 0 \leq i, j, p \leq N_k, 1 \leq k \leq K$, in dimension $d = 3$.

Let (U, P) denote the vector made of these values. Then the mortar condition can be expressed in the following way: there exists a rectangular matrix Q such that the boundary and matching conditions can be enforced, which leads to a new vector QU . Then problem (4.2.6) – (4.2.7) can now equivalent to the following linear system

$$\begin{pmatrix} Q^T A Q & Q^T B \\ B^T Q & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} Q^T F Q \\ 0 \end{pmatrix}, \quad (4.3.1)$$

where Q^T stands for the transposed matrix of Q . The matrix A is fully diagonal, its diagonal terms are the $\rho_i^{x,k}, \rho_j^{y,k}$ or the $\rho_i^{x,k}, \rho_j^{y,k}, \rho_p^{z,k}$ according to the dimension. The matrix B is only block-diagonal, with K blocks B_k on the diagonal, one for each Ω_k . For the first experiment, we concern the 2D L-shaped domain

$$\Omega = [-1, 3]^2 \setminus [1, 3]^2,$$

partitioned into three subdomains illustrated in Fig.4-1 (upper panel). Γ_1 will be chosen to be $[-1, 1] \times \{3\} \cup \{-1\} \times [-1, 1]$, then $\Gamma_2 = \partial\Omega \setminus \Gamma_1$. We divide the three domains as follows: $\Omega_1 = [-1, 1] \times [1, 3], \Omega_2 = [-1, 1] \times [-1, 1], \Omega_3 = [1, 3] \times [-1, 1]$, which associated with $N_1 = 22, N_2 = 17, N_3 = 17$. We set here $\alpha_1 = 1, \alpha_2 = 10, \alpha_3 = 100$. We use our spectral method to compute an approximation of the analytical

solution (\mathbf{u}, p) given by

$$\mathbf{u}(x, y) = \begin{pmatrix} -\sin(\pi x) \cos(\pi y) \\ \cos(\pi x) \sin(\pi y) \end{pmatrix}, p(x, y) = \sin(\pi(x + y)). \quad (4.3.2)$$

The lower panel of Fig.4-1 depicts, in a semi-logarithmic scale, the curves of the errors $\|\mathbf{u} - \mathbf{u}_\delta\|_{\alpha, L^2(\Omega)}$ and $\|p - p_\delta\|_{\alpha^*, L^2(\Omega)}$ as a function of N . As can be seen from estimates in section 5, the convergence rate here is nearly exponential despite the non-conformity of the discretization.

We then consider the square $\Omega = [-1, 3]^2$, with a nonconforming decomposition into two squares $\Omega_1 = [-1, 1]^2$, and $\Omega_3 = [1, 3] \times [-1, 1]$ and a rectangle $\Omega_2 = [-1, 3] \times [1, 3]$. For an integer $N \geq 8$, we take all the N_k equal to N . We work here with the solutions (\mathbf{u}, p) given by

$$\mathbf{u}(x, y) = \begin{pmatrix} -\sin(\pi x) \cos(\pi y) \\ \cos(\pi x) \sin(\pi y) \end{pmatrix}, p(x, y) = \sin(\pi x) \cos(\pi y). \quad (4.3.3)$$

In Fig.5-2 the curves of the errors $\|\mathbf{u} - \mathbf{u}_\delta\|_{\alpha, L^2(\Omega)}$ and $\|p - p_\delta\|_{\alpha^*, L^2(\Omega)}$, for both cases as a function of N , are plotted. For the smooth solution, a linear/logarithmic scale is used and we observe that the exponential decay of the error is preserved despite the non-conforming domain decomposition.

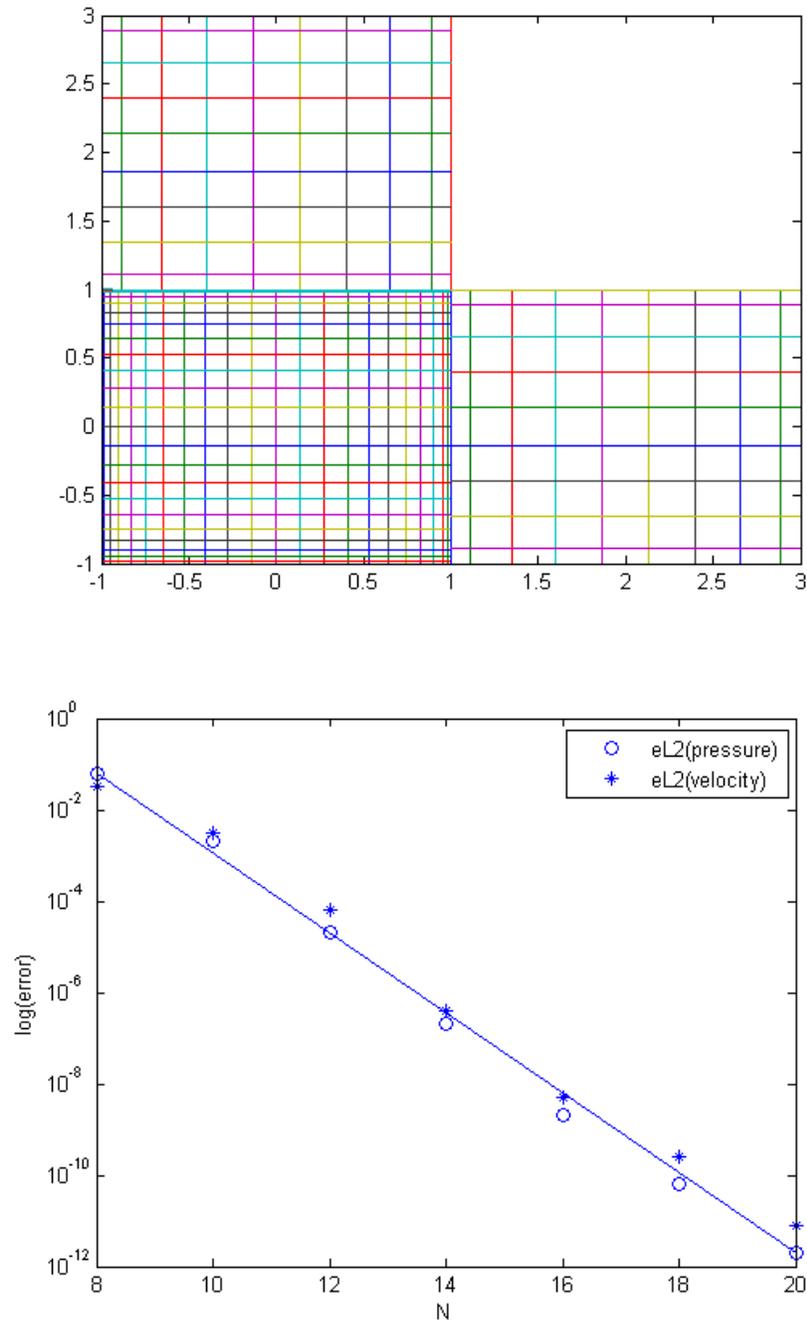


Figure 4-1: The non-matching grids for $N = 22$ and the velocity and pressure error curves

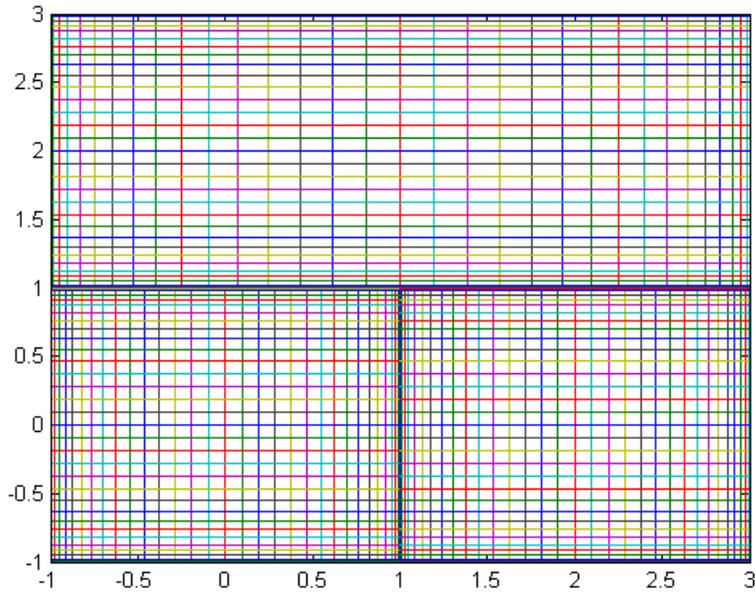


Figure 4-2: The non-matching grids for a non-conforming decomposition with $N = 32$

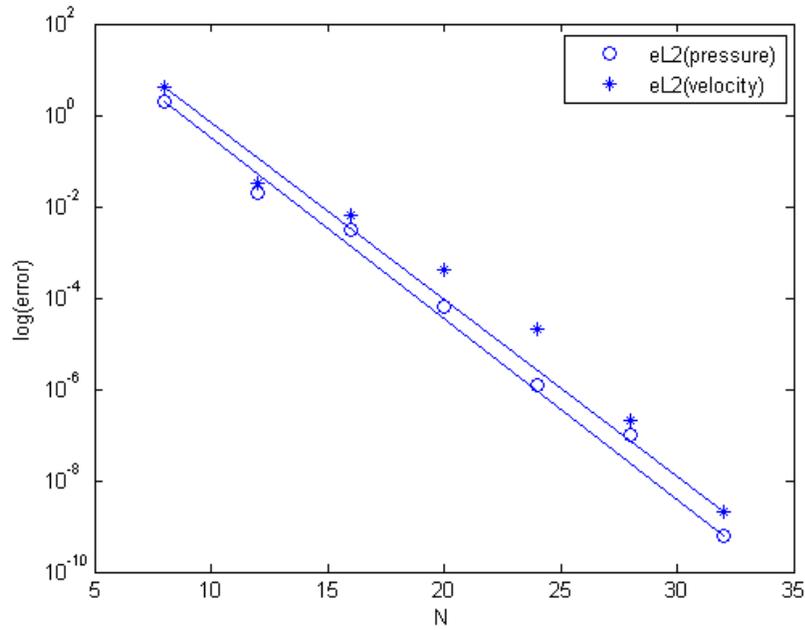


Figure 4-3: The error curves for an analytical solution

Chapter 5

Mortar Spectral Discretization of Darcy's Equations(II)

5.1 Introduction

In this chapter, we mainly talk about the analysis of the mortar spectral element discretization of the problem introduced by Darcy [33] as follows

$$\left\{ \begin{array}{ll} \alpha \mathbf{u} + \mathbf{grad} p = \alpha \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = k & \text{on } \Gamma_1, \\ p = p_0 & \text{on } \Gamma_2 \end{array} \right. \quad (5.1.1)$$

where Ω be a bounded, connected, open set in \mathbb{R}^d , $d = 2$ or 3 , with a Lipschitz continuous boundary $\partial\Omega$, and let \mathbf{n} denote the unit outward normal vector to Ω on $\partial\Omega$, Γ_1 and Γ_2 be a partition of $\partial\Omega$ without overlap. The data are now the function \mathbf{f} and the boundary conditions k and p_0 . The unknowns are the velocity \mathbf{u} and the pressure (or hydraulic head, according to the model) p . we are interested in the case

where this function is not globally continuous but only piecewise smooth and also such that the ratio of its maximal value to its minimal value is large.

This problem is an expansion of [32], where the problem with special boundary condition has been discussed. Here the idea of this paper is different: our problem has the general boundary conditions, including \mathbf{u} and p . So new spaces with special traces have to be introduced.

5.2 Analysis of the mortar SE discretization

Throughout this section, we work with a piecewise constant function α . We now assume that Ω admits a disjoint decomposition into a finite number of (open) rectangles in dimension $d = 2$, rectangular parallelepipeds in dimension $d = 3$, denoted by Ω_k , with edges all parallel to the coordinate axes,

$$\bar{\Omega} = \bigcup_{k=1}^K \bar{\Omega}_k \quad \text{and} \quad \Omega_k \cap \Omega_{k'} = \emptyset, 1 \leq k \neq k' \leq K. \quad (5.2.1)$$

We make the further assumption that the intersection of each $\partial\Omega_k$ with $\partial\Omega$, if not empty, is a corner, a whole edge or a whole face of Ω_k . For $1 \leq k \leq K$, we denote by $\Gamma_{k,l}$, $1 \leq l \leq L(k)$, the (open) edges in dimension $d = 2$, faces in dimension $d = 3$, of Ω_k which are not contained in $\partial\Omega$. We denote by \mathbf{n}_k the unit outward normal vector to Ω_k on $\partial\Omega_k$. The decomposition is said to be conforming means that the intersection of two different $\bar{\Omega}_k$, if not empty, is a corner or a whole edge of both of them.

The skeleton \mathcal{S} of the decomposition, equal to $\bigcup_{k=1}^K \partial\Omega_k \setminus \partial\Omega$, admits a decomposition without overlapping into mortars

$$\bar{\mathcal{S}} = \bigcup_{m=1}^M \bar{\gamma}_m \quad \text{and} \quad \gamma_m \cap \gamma_{m'} = \emptyset, 1 \leq m \neq m' \leq M. \quad (5.2.2)$$

where each $\gamma_m = \Gamma_{k(m),l(m)}$ is a whole edge in dimension $d = 2$, face in dimension $d = 3$, of a subdomain Ω_k , denoted by $\Omega_{k(m)}$. Note that the choice of this decomposition is not unique, however it is decided a priori for all the discretizations we work with.

In order to describe the discrete problem, we introduce the discretization parameter δ , which is here a K -tuple of positive integers $N_k, 1 \leq k \leq K$ with each $N_k \geq 2$. Indeed, the local discrete space on each Ω_k is the space $\mathbb{P}_{N_k}(\Omega_k)$ of restrictions to Ω_k of polynomials with degree $\leq N_k$ with respect to each variable. In all that follows, c stands for a generic constant which may vary from one line to the other but is always independent of δ .

We here introduce the discrete space according to (??)-(2.3.9). For each $k, 1 \leq k \leq K$, the discrete space of velocities \mathbb{X}_δ is defined by

$$\mathbb{X}_\delta = \{\mathbf{v}_\delta \in L^2(\Omega)^d; \mathbf{v}_\delta|_{\Omega_k} \in \mathbb{P}_{N_k}(\Omega_k)^d, 1 \leq k \leq K\}. \quad (5.2.3)$$

We give the mortar function $\Phi(q)$ for each piecewise regular function q : On each $\gamma_m, 1 \leq m \leq M$, the restriction of $\Phi(q)$ to γ_m is equal to the trace of $q|_{\Omega_{k(m)}}$. We then define the discrete space of pressures \mathbb{M}_δ of functions q_δ :

1. which belong to $L^2(\Omega)$,
2. their restriction to each Ω_k belongs to $\mathbb{P}_{N_k}(\Omega_k), 1 \leq k \leq K$,
3. the following matching condition holds on all subdomains $\Omega_k, 1 \leq k \leq K$, and for all $\Gamma_{k,j}$ of Ω_k that are not contained in $\partial\Omega$ and not a mortar,

$$\forall \varphi \in \mathbb{P}_{N_k-2}(\Gamma_{k,j}), \int_{\Gamma_{k,j}} (q_\delta|_{\Omega_k} - \Phi(q_\delta))\varphi d\tau = 0, \quad (5.2.4)$$

where $\mathbb{P}_{N_k-2}(\Gamma_{k,j})$ is the space of polynomials with degree $\leq N_k - 2$ on $\Gamma_{k,j}$.

We also introduce another discrete space \mathbb{M}_δ^1 associated with \mathbb{M}_δ :

$$\mathbb{M}_\delta^1 = \{q_\delta \in \mathbb{M}_\delta \mid q_\delta = 0 \text{ on } \Gamma_2\}. \quad (5.2.5)$$

Assuming that the functions f, k and p_0 has continuous restrictions to all $\bar{\Omega}_k, 1 \leq k \leq K$ and $\bar{\Gamma}_1, \bar{\Gamma}_2$ respectively. Then the discrete problem built from the variational formulation(2.3.9) reads:

Find $(\mathbf{u}_\delta, p_\delta)$ in $\mathbb{X}_\delta(\Omega) \times \mathbb{M}_\delta(\Omega)$ such that

$$p_\delta = \mathcal{I}_\delta^{\Gamma_2} p_0 \text{ on } \Gamma_2, \quad (5.2.6)$$

and

$$\begin{aligned} \forall \mathbf{v}_\delta \in \mathbb{X}_\delta(\Omega), \quad a_\alpha^\delta(\mathbf{u}_\delta, \mathbf{v}_\delta) + b_\delta(\mathbf{v}_\delta, p_\delta) &= L_\delta(\mathbf{v}_\delta), \\ \forall q_\delta \in \mathbb{M}_\delta^1(\Omega), \quad b_\delta(\mathbf{u}_\delta, q_\delta) &= ((k, q_\delta))_\delta^{\Gamma_1}. \end{aligned} \quad (5.2.7)$$

where the bilinear forms $a_\alpha^\delta(\cdot, \cdot)$ and b_δ are defined by

$$\begin{aligned} a_\alpha^\delta(\mathbf{u}_\delta, \mathbf{v}_\delta) &= \sum_{k=1}^K \alpha_k ((\mathbf{u}_\delta, \mathbf{v}_\delta))_\delta^k, \\ b_\delta(\mathbf{v}_\delta, q_\delta) &= \sum_{k=1}^K ((\mathbf{v}_\delta, \mathbf{grad} q_\delta))_\delta^k, \\ \mathcal{L}_\delta(\mathbf{v}_\delta) &= ((\alpha \mathbf{f}, \mathbf{v}_\delta))_\delta. \end{aligned} \quad (5.2.8)$$

Since \mathbb{M}_δ is not contained in $H^1(\Omega)$, we define the ‘‘broken’’ norm and semi-norm

$$\|\mathbf{v}_\delta\|_\alpha = \left(\sum_{k=1}^K \alpha_k \|\mathbf{u}_\delta\|_{L^2(\Omega_k)^d}^2 \right)^{\frac{1}{2}}, \quad \|q_\delta\|_{\alpha^*} = \left(\sum_{k=1}^K \alpha_k^{-1} |q_\delta|_{H^1(\Omega_k)}^2 \right)^{\frac{1}{2}}$$

Let M be the maximal number of points of which are contained in the same edge $\Gamma_{k,j}$. We first introduce the following space \mathbb{M}_D^1 of functions q defined on Ω , such that:

1. which belongs to $L^2(\Omega)$,
2. their restriction to each Ω_k belongs to $H^1(\Omega_k), 1 \leq k \leq K$,
3. vanish on Γ_2 ,
4. the following matching condition holds on all subdomains $\Omega_k, 1 \leq k \leq K$, and for all $\Gamma_{k,j}$ of Ω_k that are not contained in $\partial\Omega$ and not a mortar,

$$\forall \varphi \in \mathbb{P}_{M-2}(\Gamma_{k,j}), \int_{\Gamma_{k,j}} (q|_{\Omega_k} - \Phi)\varphi d\tau = 0, \quad (5.2.9)$$

where $\mathbb{P}_{M-2}(\Gamma_{k,j})$ is the space of polynomials with degree $\leq M - 2$ on $\Gamma_{k,j}$.

Not that the definition of \mathbb{M}_D is similar to that of \mathbb{M}_δ^1 , however \mathbb{M}_D^1 is no longer a discrete space: its dimension is infinite and it only depends on the decomposition of the domain. Not also that \mathbb{M}_D^1 contains all the spaces \mathbb{M}_δ^1 , such that the N_k in δ are larger than M .

Lemma 5.1 *There exists a constant C independent of δ such that*

$$\forall q \in \mathbb{M}_D^1(\Omega), \sum_{k=1}^K \|q\|_{L^2(\Omega_k)}^2 \leq C \sqrt{\alpha_{\max}} \|q\|_{\alpha^*}^2. \quad (5.2.10)$$

and C is only dependent with the decomposition of Ω .

Proof: Firstly, we consider a function q in \mathbb{M}_D^1 such that all the right hand side of (5.2.10) is zero. Thus, q is a constant on each Ω_k and due to the choice of M , according to ([22], Proposition 20), which yield that q is zero on Ω . Next, we observe that

$$\|q\|_{H^1(\cup\Omega)} = c(\|q\|_{L^2(\Omega)}^2 + \sum_{k=1}^K \int_{\Omega_k} (\mathbf{grad} q|_{\Omega_k})^2 dx)^{\frac{1}{2}}. \quad (5.2.11)$$

Also, since the imbedding of each $H^1(\Omega_k)$ in $L^2(\Omega_k)$ is compact, the imbedding of \mathbb{M}_D^1 into $L^2(\Omega)$ is compact. So, the desired result follows from the Peetre-Tartar lemma ([40], Chap.I, Theorem 2.1).

We construct a lifting of the boundary condition of p_0 , we give the following lemma according to the reference ([25], Th. III.3.1) or ([7], Lemma 4.1).

Lemma 5.2 *If p_0 is continuous on Γ_2 , then there exists a function p_δ^b in $\mathbb{M}_\delta(\Omega)$ and a constant c independent of δ , such that*

$$p_\delta^b = \mathcal{I}_\delta^{\Gamma_2} p_0 \quad \text{on } \Gamma_2, \quad (5.2.12)$$

and

$$\|p_\delta^b\|_{\alpha^*} \leq c \sqrt{\frac{1}{\alpha_{\min}}} \|\mathcal{I}_\delta^{\Gamma_2} p_0\|_{H_{00}^{\frac{1}{2}}(\Gamma_2)}. \quad (5.2.13)$$

To go further, we set $p_\delta^0 = p_\delta - p_\delta^b$, and consider the following problem:

Find $(\mathbf{u}_\delta, p_\delta^0)$ in $\mathbb{X}_\delta(\Omega) \times \mathbb{M}_\delta^1(\Omega)$ such that

$$\begin{aligned} \forall \mathbf{v}_\delta \in \mathbb{X}_\delta(\Omega), \quad a_\alpha^\delta(\mathbf{u}_\delta, \mathbf{v}_\delta) + b_\delta(\mathbf{v}_\delta, p_\delta^0) &= L_\delta(\mathbf{v}_\delta) - b_\delta(\mathbf{v}_\delta, p_\delta^b), \\ \forall q_\delta \in \mathbb{M}_\delta^1(\Omega), \quad b_\delta(\mathbf{u}_\delta, q_\delta) &= ((k, q_\delta))_\delta^{\Gamma_1}. \end{aligned} \quad (5.2.14)$$

Lemma 5.3 *The form $a_\alpha^\delta(\cdot, \cdot)$ satisfies the following continuity and ellipticity properties*

$$\forall \mathbf{u}_\delta \in \mathbb{X}_\delta(\Omega), \forall \mathbf{v}_\delta \in \mathbb{X}_\delta(\Omega), a_\alpha^\delta(\mathbf{u}_\delta, \mathbf{v}_\delta) \leq c \|\mathbf{u}_\delta\|_\alpha \|\mathbf{v}_\delta\|_\alpha, \quad (5.2.15)$$

$$\forall \mathbf{u}_\delta \in \mathbb{X}_\delta(\Omega), a_\alpha^\delta(\mathbf{u}_\delta, \mathbf{u}_\delta) \geq \|\mathbf{u}_\delta\|_\alpha^2. \quad (5.2.16)$$

The proof is similar to the reference ([32], Chap.3, Lemma 3.21).

We next come to the inf-sup condition:

Lemma 5.4 *The bilinear form $b_\delta(\cdot, \cdot)$ satisfies the inf-sup condition*

$$\forall q_\delta \in \mathbb{M}_\delta^1(\Omega), \sup_{\mathbf{v}_\delta \in \mathbb{X}_\delta} \frac{b_\delta(\mathbf{v}_\delta, q_\delta)}{\|\mathbf{v}_\delta\|_\alpha} \geq C \|q_\delta\|_{\alpha^*}. \quad (5.2.17)$$

Proof: We note that for any q_δ in $\mathbb{M}_\delta^1(\Omega)$, we can define the function \mathbf{v}_δ by

$$\mathbf{v}_\delta|_{\Omega_k} = \alpha_k^{-1} \mathbf{grad}(q_\delta|_{\Omega_k}), 1 \leq k \leq K, \quad (5.2.18)$$

It is obvious that $\mathbf{v}_\delta \in \mathbb{X}_\delta(\Omega)$. Then we have the above inf-sup condition.

From (5.2.16) and the inf-sup condition (5.2.17), the saddle-point problem (5.2.14) has a unique solution $(\mathbf{u}_\delta, p_\delta^0)$ which satisfies

$$\|\mathbf{u}_\delta\|_\alpha + \|p_\delta^0\|_{\alpha^*} \leq C (\|p_\delta^b\|_{\alpha^*} + \|I_\delta^{\Gamma_1} k\|_{H^{-\frac{1}{2}}(\Gamma_1)} + \|\mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^d}) \quad (5.2.19)$$

Now we are in the position to give the main result of this section:

Theorem 5.1 *For any data (\mathbf{f}, k, p_0) such that each $\mathbf{f}|_{\Omega_k}, 1 \leq k \leq K$, and k, p_0 are continuous on $\bar{\Omega}_k$ and on Γ_1, Γ_2 respectively, problem (5.2.6) – (5.2.7) has a unique solution $(\mathbf{u}_\delta, p_\delta)$ in $\mathbb{X}_\delta(\Omega) \times \mathbb{M}_\delta(\Omega)$. Moreover, there exists a constant c independent of δ such that this solution satisfies*

$$\|\mathbf{u}_\delta\|_\alpha + \|p_\delta\|_{\alpha^*} \leq C (\|\mathcal{I}_\delta^{\Gamma_2} p_0\|_{H^{\frac{1}{2}}(\Gamma_2)} + \|I_\delta^{\Gamma_1} k\|_{H^{-\frac{1}{2}}(\Gamma_1)} + \|\mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^d}) \quad (5.2.20)$$

Proof: We establish successively the existence and uniqueness of the solution.

1) It follows from the Lax-Milgram lemma and (5.2.10), that there exists a unique φ_δ in \mathbb{X}_δ such that

$$\forall \phi_\delta \in \mathbb{M}_\delta^1, \quad ((\mathbf{grad} \varphi_\delta, \mathbf{grad} \phi_\delta))_\delta = ((k, \phi_\delta))_\delta^{\Gamma_1}. \quad (5.2.21)$$

Thus, the function $\mathbf{u}_\delta^b = \mathbf{grad} \varphi_\delta$, satisfies

$$\|\mathbf{u}_\delta^b\|_\alpha \leq c \sqrt{\alpha_{\max}} \|I_\delta^{\Gamma_1} k\|_{H^{\frac{1}{2}}(\Gamma_1)}. \quad (5.2.22)$$

On the other hand , it follows from the standard results on saddle-point problems ([40], Chap.I, Cor.4.1), that the problem

Find $(\mathbf{u}_\delta^0, p_\delta^0)$ in $\mathbb{X}_\delta(\Omega) \times \mathbb{M}_\delta^1(\Omega)$ such that

$$\begin{aligned} \forall \mathbf{v}_\delta \in \mathbb{X}_\delta(\Omega), \quad a_\alpha^\delta(\mathbf{u}_\delta^0, \mathbf{v}_\delta) + b_\delta(\mathbf{v}_\delta, p_\delta^0) &= L_\delta(\mathbf{v}_\delta) - b_\delta(\mathbf{v}_\delta, p_\delta^b) - a_\alpha^\delta(\mathbf{u}_\delta^b, \mathbf{v}_\delta), \\ \forall q_\delta \in \mathbb{M}_\delta^1(\Omega), \quad b_\delta(\mathbf{u}_\delta^0, q_\delta) &= 0. \end{aligned} \quad (5.2.23)$$

has a unique solution $(\mathbf{u}_\delta^0, p_\delta^0)$ which moreover satisfies

$$\|\mathbf{u}_\delta^0\|_\alpha + \|p_\delta^0\|_{\alpha^*} \leq C (\|p_\delta^b\|_{\alpha^*} + \|\mathbf{u}_\delta^b\|_\alpha + \|\mathcal{I}_N \mathbf{f}\|_{L^2(\Omega)^d}) \quad (5.2.24)$$

Then, the pair $(\mathbf{u}_\delta, p_\delta)$, with $\mathbf{u}_\delta = \mathbf{u}_\delta^0 + \mathbf{u}_\delta^b$, $p_\delta = p_\delta^0 + p_\delta^b$, is a solution of problem (5.2.7), and estimate (5.2.20) follows from (5.2.22) and (5.2.24).

2)The pair $(\mathbf{u}_\delta, p_\delta = p_\delta^0 + p_\delta^b)$ is a solution of problem (5.2.6) and (5.2.7), and estimate (5.2.20) is a consequence of (5.2.19) and (5.2.13). On the other hand, let $(\mathbf{u}_{\delta 1}, p_{\delta 1})$ and $(\mathbf{u}_{\delta 2}, p_{\delta 2})$ be two solutions of problem (5.2.6) and (5.2.7). Then the difference $(\mathbf{u}_{\delta 1} - \mathbf{u}_{\delta 2}, p_{\delta 1} - p_{\delta 2})$ is a solution of problem (5.2.14) with data $\mathbf{u}_\delta^b, \mathbf{f}, p_0$ equal to zero. Thus, it follows from (5.2.20) that it is zero. So the solution of problem (5.2.6) and (5.2.7) is unique.

To conclude, we introduce the discrete kernel

$$V_\delta = \{\mathbf{v}_\delta \in \mathbb{X}_\delta; \forall q_\delta \in \mathbb{M}_\delta^1, b_\delta(\mathbf{v}_\delta, q_\delta) = 0\}. \quad (5.2.25)$$

As usual, it plays a key role in the numerical analysis of problem (5.2.7).

5.3 Error estimates

This section is devoted to the error estimates, first for the velocity, second for the pressure. We intend to prove here an estimate between the solution (\mathbf{u}, p) of problem (2.3.9) and the solution $(\mathbf{u}_\delta, p_\delta)$ of problem (5.2.7).

Let $\boldsymbol{\omega}_\delta$ be any function in the kernel V_δ . Multiplying the first line of (5.1.1) by $\boldsymbol{\omega}_\delta$ gives

$$a_\alpha(\mathbf{u}_0, \boldsymbol{\omega}_\delta) + b(\boldsymbol{\omega}_\delta, P_1) = \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - a_\alpha(\mathbf{u}_b, \boldsymbol{\omega}_\delta) - b(\boldsymbol{\omega}_\delta, \bar{P}). \quad (5.3.1)$$

which implies, due to the definition of V_δ , that for any q_δ in $\mathbb{M}_\delta(\Omega)$,

$$a_\alpha(\mathbf{u}_0, \boldsymbol{\omega}_\delta) + b(\boldsymbol{\omega}_\delta, P_1 - q_\delta) = \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - a_\alpha(\mathbf{u}_b, \boldsymbol{\omega}_\delta) - b(\boldsymbol{\omega}_\delta, \bar{P}). \quad (5.3.2)$$

Next, we deduce from the ellipticity property (5.2.16), that we have for any \mathbf{v}_δ in V_δ

$$\|\mathbf{u}_\delta^0 - \mathbf{v}_\delta\|_\alpha^2 \leq a_\alpha^\delta(\mathbf{u}_\delta^0 - \mathbf{v}_\delta, \mathbf{u}_\delta^0 - \mathbf{v}_\delta). \quad (5.3.3)$$

Adding (5.3.2) with $\boldsymbol{\omega}_\delta = \mathbf{u}_\delta^0 - \mathbf{v}_\delta$ and subtracting the first line of (5.2.7) leads to

$$\begin{aligned} \|\mathbf{u}_\delta^0 - \mathbf{v}_\delta\|_\alpha^2 &\leq a_\alpha^\delta(\mathbf{u}_\delta^0 - \mathbf{v}_\delta, \mathbf{u}_\delta^0 - \mathbf{v}_\delta) \leq a_\alpha(\mathbf{u}_0 - \mathbf{v}_\delta, \mathbf{u}_\delta^0 - \mathbf{v}_\delta) \\ &\quad + a_\alpha(\mathbf{v}_\delta, \mathbf{u}_\delta^0 - \mathbf{v}_\delta) - a_\alpha^\delta(\mathbf{v}_\delta, \mathbf{u}_\delta^0 - \mathbf{v}_\delta) + b(\mathbf{u}_\delta^0 - \mathbf{v}_\delta, P_1 - q_\delta) \\ &\quad + ((\alpha \mathbf{f}, \mathbf{u}_\delta^0 - \mathbf{v}_\delta))_\delta - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot (\mathbf{u}_\delta^0 - \mathbf{v}_\delta) d\mathbf{x} + a_\alpha(\mathbf{u}_b, \mathbf{u}_\delta^0 - \mathbf{v}_\delta) \\ &\quad - a_\alpha^\delta(\mathbf{u}_b, \mathbf{u}_\delta^0 - \mathbf{v}_\delta) + b(\boldsymbol{\omega}_\delta, \bar{P}) - b_\delta(\boldsymbol{\omega}_\delta, p_\delta^b). \end{aligned} \quad (5.3.4)$$

It is obvious that

$$\forall \mathbf{v}_\delta \in \mathbb{X}_\delta, \quad \forall q_\delta \in \mathbb{M}_\delta, \quad b_\delta(\mathbf{v}_\delta, q_\delta) \leq \|\mathbf{v}_\delta\|_\alpha \|q_\delta\|_{\alpha^*}. \quad (5.3.5)$$

So, combining the above inequality and triangle inequality, we derive that the error $\|\mathbf{u} - \mathbf{u}_\delta\|_\alpha$ is bounded, up to a multiplicative constant, by the sum of seven terms:

- the approximation error in \mathbb{X}_δ

$$\inf_{\mathbf{v}_\delta \in V_\delta} \|\mathbf{u}_0 - \mathbf{v}_\delta\|_\alpha, \quad (5.3.6)$$

- the error approximation in \mathbb{M}_δ^1

$$\inf_{q_\delta \in \mathbb{M}_\delta^1} \|P_1 - q_\delta\|_{\alpha^*}, \quad (5.3.7)$$

- three terms issued from numerical integration

$$\sup_{\boldsymbol{\omega}_\delta \in \mathbb{X}_\delta} \frac{a_\alpha(\mathbf{v}_\delta, \boldsymbol{\omega}_\delta) - a_\alpha^\delta(\mathbf{v}_\delta, \boldsymbol{\omega}_\delta)}{\|\boldsymbol{\omega}_\delta\|_\alpha}, \quad (5.3.8)$$

$$\sup_{\boldsymbol{\omega}_\delta \in \mathbb{X}_\delta} \frac{a_\alpha(\mathbf{u}_b, \boldsymbol{\omega}_\delta) - a_\alpha^\delta(\mathbf{u}_b, \boldsymbol{\omega}_\delta)}{\|\boldsymbol{\omega}_\delta\|_\alpha}, \quad (5.3.9)$$

$$\frac{((\alpha \mathbf{f}, \boldsymbol{\omega}_\delta))_\alpha - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\omega}_\delta d\mathbf{x}}{\|\boldsymbol{\omega}_\delta\|_\alpha}, \quad (5.3.10)$$

$$\sup_{\boldsymbol{\omega}_\delta \in \mathbb{X}_\delta} \frac{b(\boldsymbol{\omega}_\delta, \bar{P}) - b_\delta(\boldsymbol{\omega}_\delta, p_\delta^b)}{\|\boldsymbol{\omega}_\delta\|_\alpha}. \quad (5.3.11)$$

and

$$\|\mathbf{u}_b - \mathbf{u}_\delta^b\|_\alpha \quad (5.3.12)$$

We first estimate the terms issued from numerical integration, which is easy since they can be evaluated separately on each subdomain Ω_k . Let Π_{N_k-1} denote the orthogonal projection operator from $L^2(\Omega_k)$ onto $\mathbb{P}_{N_k-1}(\Omega_k)$ for each $k, 1 \leq k \leq K$. For any $\boldsymbol{\omega}_\delta$ in \mathbb{X}_δ , since each product of $\Pi_{N_k-1} \mathbf{u}$ by $\boldsymbol{\omega}_\delta$ belongs to $\mathbb{P}_{2N_k-1}(\Omega_k)$, it follows

from the exactness property (??) that

$$a_\alpha(\mathbf{v}_\delta, \boldsymbol{\omega}_\delta) - a_\alpha^\delta(\mathbf{v}_\delta, \boldsymbol{\omega}_\delta) = \sum_{k=1}^K \alpha_k \left(\int_{\Omega_k} (\mathbf{v}_\delta - \Pi_{N_k-1} \mathbf{u}_0) \cdot \boldsymbol{\omega}_\delta d\mathbf{x} - ((\mathbf{v}_\delta - \Pi_{N_k-1} \mathbf{u}_0, \boldsymbol{\omega}_\delta)_{N_k}^k) \right).$$

So, we deduce from the continuity property (5.2.15) that

$$\begin{aligned} \sup_{\boldsymbol{\omega}_\delta \in \mathbb{X}_\delta} \frac{a_\alpha(\mathbf{v}_\delta, \boldsymbol{\omega}_\delta) - a_\alpha^\delta(\mathbf{v}_\delta, \boldsymbol{\omega}_\delta)}{\|\boldsymbol{\omega}_\delta\|_\alpha} &\leq 10 \left(\sum_{k=1}^K \alpha_k \|\mathbf{v}_\delta - \Pi_{N_k-1} \mathbf{u}_0\|_{L^2(\Omega_k)^d}^2 \right)^{\frac{1}{2}} \\ &\leq 10 \|\mathbf{u}_0 - \mathbf{v}_\delta\|_\alpha + 10 \left(\sum_{k=1}^K \alpha_k \|\mathbf{u}_0 - \Pi_{N_k-1} \mathbf{u}_0\|_{L^2(\Omega_k)^d}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Because of the well-known properties of the operator Π_{N_k-1} ([21], Theorem 7.3), we can get the following estimate: if the solution \mathbf{u}_0 satisfies that $\mathbf{u}_0|_{\Omega_k}$ belongs to $H^{s_k+1}(\Omega_k)^d$, $s_k \geq 0$

$$\sup_{\boldsymbol{\omega}_\delta \in \mathbb{X}_\delta} \frac{a_\alpha(\mathbf{v}_\delta, \boldsymbol{\omega}_\delta) - a_\alpha^\delta(\mathbf{v}_\delta, \boldsymbol{\omega}_\delta)}{\|\boldsymbol{\omega}_\delta\|_\alpha} \leq 4 \|\mathbf{u}_0 - \mathbf{v}_\delta\|_\alpha + c \left(\sum_{k=1}^K \alpha_k N_k^{-2s_k} \|\mathbf{u}_0|_{\Omega_k}\|_{H^{s_k}(\Omega_k)^d}^2 \right)^{\frac{1}{2}}. \quad (5.3.13)$$

Similarly, for any $\boldsymbol{\omega}_\delta$ in \mathbb{X}_δ , we have

$$\begin{aligned} ((\alpha \mathbf{f}, \boldsymbol{\omega}_\delta)_\alpha) - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\omega}_\delta d\mathbf{x} &= \sum_{k=1}^K \alpha_k \left(((\mathcal{I}_\delta \mathbf{f} - \Pi_{N_k-1} \mathbf{f}, \boldsymbol{\omega}_\delta)_{N_k}^k) \right. \\ &\quad \left. - \int_{\Omega_k} (\mathbf{f} - \Pi_{N_k-1} \mathbf{f}) \cdot \boldsymbol{\omega}_\delta d\mathbf{x} \right). \end{aligned}$$

So, we also apply the exactness property of the Gauss-Lobatto formula, which yields

$$\begin{aligned} ((\alpha \mathbf{f}, \boldsymbol{\omega}_\delta)_\alpha) - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\omega}_\delta d\mathbf{x} &\leq \sqrt{\alpha_{\max}} \left(10 \left(\sum_{k=1}^K \|\mathbf{f} - \Pi_{N_k-1} \mathbf{f}\|_{L^2(\Omega_k)^d}^2 \right)^{\frac{1}{2}} \right. \\ &\quad \left. + 9 \|\mathbf{f} - \mathcal{I}_\delta \mathbf{f}\|_{L^2(\Omega_k)^d} \|\boldsymbol{\omega}_\delta\|_{L^2(\omega)^d} \right). \end{aligned}$$

According to the approximation properties of the operator \mathcal{I}_δ ([21], Theorem 7.1) and Π_{N_k-1} ([21], Theorem 7.3), we derive that, if the function \mathbf{f} which satisfies that $\mathbf{f}|_{\Omega_k}$

belongs to $H^{\sigma_k}(\Omega_k)^d$, $\sigma_k > 1$.

$$\sup_{\boldsymbol{\omega}_\delta \in \mathbb{X}_\delta} \frac{((\alpha \mathbf{f}, \boldsymbol{\omega}_\delta))_\alpha - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\omega}_\delta d\mathbf{x}}{\|\boldsymbol{\omega}_\delta\|_\alpha} \leq c\sqrt{\alpha_{\max}} \left(\sum_{k=1}^K N_k^{-2\sigma_k} \|\mathbf{f}|_{\Omega_k}\|_{H^{\sigma_k}(\Omega_k)^d}^2 \right)^{\frac{1}{2}}. \quad (5.3.14)$$

By analogy, we estimate the numerical integration term (5.3.9), we have

$$\sup_{\boldsymbol{\omega}_\delta \in \mathbb{X}_\delta} \frac{a_\alpha(\mathbf{u}_b, \boldsymbol{\omega}_\delta) - a_\alpha^\delta(\mathbf{u}_\delta^b, \boldsymbol{\omega}_\delta)}{\|\boldsymbol{\omega}_\delta\|_\alpha} \leq c\sqrt{\alpha_{\max}} \left(\sum_{k=1}^K N_k^{-2s_k} \|u_b|_{\Omega_k}\|_{H^{s_k}(\Omega_k)^d}^2 \right)^{\frac{1}{2}}. \quad (5.3.15)$$

Also, the term (5.3.11) derives that

$$\sup_{\boldsymbol{\omega}_\delta \in \mathbb{X}_\delta} \frac{b(\boldsymbol{\omega}_\delta, \bar{P}) - b_\delta(\boldsymbol{\omega}_\delta, p_\delta^b)}{\|\boldsymbol{\omega}_\delta\|_\alpha} \leq c\sqrt{\alpha_{\max}} \left(\sum_{k=1}^K N_k^{-2s_k} \|\bar{P}|_{\Omega_k}\|_{H^{s_k+1}(\Omega_k)^d}^2 \right)^{\frac{1}{2}}. \quad (5.3.16)$$

Due to (2.4.21) and (5.2.21), combined with ([18], Chap. VI, Lemma 5.1), we can get the estimate results of (5.3.12)

$$\|\mathbf{u}_b - \mathbf{u}_\delta^b\|_\alpha \leq c\sqrt{\alpha_{\max}} \sum_{k=1}^{K_1} N_k^{-s_k} \|k\|_{H^{s_k - \frac{1}{2}}(\Gamma_1 \cap \partial\Omega_k)}. \quad (5.3.17)$$

where K_1 is such that $\Gamma_1 \cap \partial\Omega_k \neq \emptyset$.

We are now in the position to derive the approximation error in \mathbb{X}_δ (5.3.6). We refer to ([32], Chap.3, Lemma. 3.24) for the following lemma.

Lemma 5.5 *There exists a constant c independent of δ such that*

$$\inf_{\mathbf{v}_\delta \in V_\delta} \|\mathbf{u}_0 - \mathbf{v}_\delta\|_\alpha \leq c \left(\inf_{\mathbf{z}_\delta \in \mathbb{X}_\delta} \|\mathbf{u}_0 - \mathbf{z}_\delta\|_\alpha + \sup_{q_\delta \in \mathbb{M}_\delta^1} \frac{\int_S (\mathbf{u}_0 \cdot \mathbf{n}) [q_\delta] d\tau}{\|q_\delta\|_{\alpha^*}} \right). \quad (5.3.18)$$

Proof: Let \mathbf{z}_δ be an arbitrary element of \mathbb{X}_δ . The inf – sup condition (5.2.17) and

([40], Chpt. 1, Lemma 4.1) prove that there exists a unique $\mathbf{t}_\delta \in V_\delta^\perp$ such that

$$b_\delta(\mathbf{t}_\delta, q_\delta) = b_\delta(\mathbf{z}_\delta, q_\delta), \quad \text{and} \quad \|\mathbf{t}_\delta\|_\alpha \leq \frac{1}{\beta} \sup_{q_\delta \in \mathbb{M}_\delta^1} \frac{b_\delta(\mathbf{z}_\delta, q_\delta)}{\|q_\delta\|_{\alpha^*}}. \quad (5.3.19)$$

Thus, if we set $\mathbf{v}_\delta = \mathbf{z}_\delta - \mathbf{t}_\delta$, then by combing the exactness property (??), we have

$$b_\delta(\mathbf{u}_0, q_\delta) = \sum_{k=1}^K \int_{\Omega_k} \mathbf{u}_0 \cdot \mathbf{grad} q_\delta dx = \sum_{k=1}^K \int_{\partial\Omega_k} (\mathbf{u}_0 \cdot \mathbf{n}) q_\delta d\tau = \int_S (\mathbf{u}_0 \cdot \mathbf{n}) [q_\delta] d\tau,$$

therefore

$$\|\mathbf{t}_\delta\|_\alpha \leq c \sup_{q_\delta \in \mathbb{M}_\delta^1} \frac{b_\delta(\mathbf{u}_0 - \mathbf{z}_\delta, q_\delta) - \int_S (\mathbf{u}_0 \cdot \mathbf{n}) [q_\delta] d\tau}{\|q_\delta\|_{\alpha^*}}. \quad (5.3.20)$$

This inequality and triangle inequality implies

$$\begin{aligned} \|\mathbf{u}_0 - \mathbf{v}_\delta\|_\alpha &\leq \|\mathbf{u}_0 - \mathbf{z}_\delta\|_\alpha + \|\mathbf{t}_\delta\|_\alpha \\ &\leq c(\|\mathbf{u}_0 - \mathbf{z}_\delta\|_\alpha + \sup_{q_\delta \in \mathbb{M}_\delta^1} \frac{\int_S (\mathbf{u}_0 \cdot \mathbf{n}) [q_\delta] d\tau}{\|q_\delta\|_{\alpha^*}}) \end{aligned}$$

Since \mathbf{z}_δ is arbitrary element of \mathbb{X}_δ , this implies the lemma.

So the approximation error in \mathbb{X}_δ (5.3.6) can be divided into two parts, the first term is the approximatio error in \mathbb{X}_δ , another term is the consistency error.

We first estimate the approximation error in \mathbb{X}_δ . Taking \mathbf{z}_δ equal to the orthogonal projection operator $\Pi_{N_k-1} \mathbf{u}_0$ on each Ω_k , we can derive the following lemma.

Lemma 5.6 *Assume that the solution (\mathbf{u}, p) of problem (??) – (2.3.9) is such that each $\mathbf{u}|_{\Omega_k}$ belongs to $H^{s_k}(\Omega_k)^d$ for a real number s_k , $s_k \geq 0$. The following estimate holds*

$$\inf_{\mathbf{z}_\delta \in \mathbb{X}_\delta} \|\mathbf{u}_0 - \mathbf{z}_\delta\|_\alpha \leq c \left(\sum_{k=1}^K \alpha_k N_k^{-2s_k} \|\mathbf{u}|_{\Omega_k}\|_{H^{s_k}(\Omega_k)^d}^2 \right)^{\frac{1}{2}}. \quad (5.3.21)$$

Next we come to the consistency error. It involves the quantity μ , defined as the

largest ratio of $\frac{\alpha_k}{\alpha_{k'}}$, it is

$$\mu = \left(\max \frac{\alpha_k}{\alpha_{k'}} \right)^{\frac{1}{2}}. \quad (5.3.22)$$

We refer to ([16], Lemma 3.3) and ([32], Chap 3, Lemma 3.26) for the following lemma.

Lemma 5.7 *For any function \mathbf{u} such that each $\mathbf{u}_{|\Omega_k}$, $1 \leq k \leq K$, belongs to $H^{s_k}(\Omega_k)^d$ for a real number $s_k, s_k \geq \frac{1}{2}$, the following estimate holds*

$$\sup_{q_\delta \in \mathbb{M}_\delta^1} \frac{\int_S (\mathbf{u}_0 \cdot \mathbf{n}) [q_\delta] d\tau}{\|q_\delta\|_{\alpha^*}} \leq c(1 + \mu) \left(\sum_{k=1}^K \alpha_k N_k^{-2s_k} (\log N_k) \|\mathbf{u}_{|\Omega_k}\|_{H^{s_k}(\Omega_k)^d}^2 \right)^{\frac{1}{2}}. \quad (5.3.23)$$

Remark 5.1 *In fact, $(\log N_k)^{\frac{1}{2}}$ in (5.3.23) disappears when all the edges of $\partial\Omega_k$ which are not mortars are contained either in $\partial\Omega$ or in one mortar, however it is negligible in comparison with the $N_k^{-s_k}$ when N_k is large enough.*

Estimating the approximation error in \mathbb{M}_δ^1 is more technical and more complex. We first introduce the quantity which depends on δ ,

$$\mu_\delta = \left(\max \left(\frac{\alpha_k N_k}{\alpha_{k'} N_{k'}} \right) \right)^{\frac{1}{2}}. \quad (5.3.24)$$

The nonconforming case for $d = 3$ is much more complicate, so we here only consider the conforming case of $d = 3$ and the general case of $d = 2$. We refer to ([18], Chap. V, Theorem 5.1) and ([32], Chap.3, Lemma3.27) for the following lemma.

Lemma 5.8 *Assume that the solution (\mathbf{u}, p) of problem (??) – (2.3.9) is such that each $p_{|\Omega_k}$, $1 \leq k \leq K$, belongs to $H^{s_k+1}(\Omega_k)$ for a real number $s_k, s_k > \frac{d-1}{2}$, then in the two cases*

(i) *in dimension $d = 2$,*

(ii) *in dimension $d = 3$, if the decomposition is conforming,*

the following estimate holds

$$\inf_{q_\delta \in \mathbb{M}_\delta^1} \|P_1 - q_\delta\|_{\alpha^*} \leq c(1 + \mu + \mu'_\delta) \left(\sum_{k=1}^K \alpha_k^{-1} N_k^{-2s_k} \|P_1|_{\Omega_k}\|_{H^{s_k+1}(\Omega_k)}^2 \right)^{\frac{1}{2}}. \quad (5.3.25)$$

where

$$\mu'_\delta = \begin{cases} \mu & , \text{ for } d=2 \text{ and } 3, \text{ and a conforming case,} \\ \mu_\delta & \text{ for } d=2, \text{ and a nonconforming case.} \end{cases} \quad (5.3.26)$$

Remark 5.2 When $(1 + \mu + \mu'_\delta)$ is bounded (this is most often the case in practical situations), for a general decomposition of dimension $d = 3$, the lack of optimality is $(\log N_k)N_k^{\frac{3}{2}}$. For further more results, we refer to ([18], Chap.V).

According to the above result, now we arrive at the first theorem in this section.

Theorem 5.2 Assume that the function α is constant on each Ω_k , $1 \leq k \leq K$. If the solution (\mathbf{u}, p) of problem (2.3.9) is such that its restriction to each Ω_k , $1 \leq k \leq K$, belongs to $H^{s_k}(\Omega_k)^d \times H^{s_k+1}(\Omega_k)$, $s_k > \frac{d-1}{2}$, and the function \mathbf{f} is such that its restriction to each Ω_k , $1 \leq k \leq K$, belongs to H^{σ_k} , $\sigma_k > \frac{d}{2}$, in the two cases:

(i) in dimension $d = 2$,

(ii) in dimension $d = 3$, if the decomposition is conforming,

the following error estimate holds between this solution (\mathbf{u}, p) and the solution $(\mathbf{u}_\delta, p_\delta)$ of problem (5.2.7)

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_\delta\|_\alpha &\leq c((1 + \mu + \mu'_\delta) \sum_{k=1}^K N_k^{-s_k} (\sqrt{\alpha_{\max}} (\log N_k) \|\mathbf{u}|_{\Omega_k}\|_{H^{s_k}(\Omega_k)^d} + \sqrt{1/\alpha_{\min}} \\ &\|p|_{\Omega_k}\|_{H^{s_k+1}(\Omega_k)}) + \sqrt{\alpha_{\max}} \sum_{k=1}^K N_k^{-\sigma_k} \|\mathbf{f}|_{\Omega_k}\|_{H^{\sigma_k}(\Omega_k)^d} + \sqrt{\alpha_{\max}} \sum_{k=1}^{K_1} N_k^{-s_k} \|k\|_{H^{s_k-\frac{1}{2}}(\Gamma_1 \cap \partial\Omega_k)}). \end{aligned}$$

where μ'_δ is the same as it in (5.3.26) and the constant c is independent of the parameter δ and the function α .

So it is readily checked that the following improved estimate holds for a conforming decomposition.

Corollary 5.1 *If the decomposition is conforming and if the assumptions of Theorem 4.1 are satisfied, the following error estimate holds between the solution (\mathbf{u}, p) of problem (2.3.9) and the solution $(\mathbf{u}_\delta, p_\delta)$ of problem (5.2.7)*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_\delta\|_\alpha &\leq c((1 + \mu) \sum_{k=1}^K N_k^{-s_k} (\sqrt{\alpha_{\max}} \|\mathbf{u}\|_{H^{s_k}(\Omega_k)^d} + \sqrt{\frac{1}{\alpha_{\min}}} \|p\|_{H^{s_k+1}(\Omega_k)}) \\ &\quad + \sqrt{\alpha_{\max}} \sum_{k=1}^K N_k^{-\sigma_k} \|\mathbf{f}\|_{H^{\sigma_k}(\Omega_k)^d} + \sqrt{\alpha_{\max}} \sum_{k=1}^{K_1} N_k^{-\tau_k} \|k\|_{H^{\tau_k}(\Gamma_1 \cap \partial\Omega_k)}). \end{aligned}$$

where the constant c is independent of the parameter δ and the function α .

Estimating the error on the pressure now is easy.

From the inf – sup condition (5.2.17), we derive that, for any q_δ in M_δ^1 ,

$$\beta \|p_\delta^0 - q_\delta^0\|_{\alpha^*} \leq \sup_{\mathbf{v}_\delta \in \mathbb{X}_\delta} \frac{b_\delta(\mathbf{v}_\delta, p_\delta^0 - q_\delta^0)}{\|\mathbf{v}_\delta\|_\alpha}. \quad (5.3.27)$$

We first use the discrete problem (5.2.7)

$$b_\delta(\mathbf{v}_\delta, p_\delta^0 - q_\delta^0) = ((\alpha \mathbf{f}, \mathbf{v}_\delta))_\delta - a_\alpha^\delta(\mathbf{u}_\delta - \mathbf{v}_\delta) - b_\delta(\mathbf{v}_\delta, p_\delta^b) - b_\delta(\mathbf{v}_\delta, q_\delta^0). \quad (5.3.28)$$

Next, we apply equation (2.3.9) to function \mathbf{v}_δ , and adding it to the previous line.

This yields

$$\begin{aligned} b_\delta(\mathbf{v}_\delta, p_\delta^0 - q_\delta^0) &= \sum_{k=1}^K \alpha_k \int_{\Omega_k} (\mathbf{u} - \mathbf{u}_\delta) \cdot \mathbf{v}_\delta d\mathbf{x} + \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{u}_\delta \cdot \mathbf{v}_\delta d\mathbf{x} \\ &\quad - a_\alpha^\delta(\mathbf{u}_\delta - \mathbf{v}_\delta) + \int_{\Omega} \mathbf{v}_\delta \cdot \mathbf{grad} (P_1 - q_\delta^0) d\mathbf{x} \\ &\quad + \int_{\Omega} \mathbf{v}_\delta \cdot \mathbf{grad} (\bar{P} - p_\delta^b) d\mathbf{x} + ((\alpha \mathbf{f}, \mathbf{v}_\delta))_\delta - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f} \cdot \mathbf{v}_\delta d\mathbf{x}. \end{aligned}$$

According to the triangle inequality, we have

$$\|p - q_\delta\|_{\alpha^*} = \|P_1 - q_\delta^0\|_{\alpha^*} + \|p_\delta^0 - q_\delta^0\|_{\alpha^*} + \|\bar{P} - p_\delta^b\|_{\alpha^*} \quad (5.3.29)$$

So using the same arguments as in the estimation of terms issued from numerical integration together with the above triangle inequality yields

$$\begin{aligned} \|p - q_\delta\|_{\alpha^*} \leq & c(\|\mathbf{u} - \mathbf{u}_\delta\|_\alpha + \sup_{\mathbf{v}_\delta \in \mathbb{X}_\delta} \frac{\sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{u}_\delta \cdot \mathbf{v}_\delta d\mathbf{x} - a_\alpha^\delta(\mathbf{u}_\delta, \mathbf{v}_\delta)}{\|\mathbf{v}_\delta\|_\alpha} \\ & + \|P_1 - q_\delta^0\|_{\alpha^*} + \sup_{\mathbf{v}_\delta \in \mathbb{X}_\delta} \frac{((\alpha \mathbf{f}, \mathbf{v}_\delta))_\delta - \sum_{k=1}^K \alpha_k \int_{\Omega_k} \mathbf{f}_\delta \cdot \mathbf{v}_\delta d\mathbf{x}}{\|\mathbf{v}_\delta\|_\alpha} \\ & + \sup_{\mathbf{v}_\delta \in \mathbb{X}_\delta} \frac{b(\mathbf{v}_\delta, \bar{P}) - b_\delta(\mathbf{v}_\delta, q_\delta^0)}{\|\mathbf{v}_\delta\|_\alpha} + \|\bar{P} - p_\delta^b\|_{\alpha^*}). \end{aligned} \quad (5.3.30)$$

All the terms in the right-hand side have been estimated previously except the term $\|\bar{P} - p_\delta^b\|_{\alpha^*}$. Because p_δ^b is in \mathbb{M}_δ , so the estimate result of this term is simily to the term $\|P_1 - q_\delta^0\|_{\alpha^*}$. We refer to ([18], Chap. VI, Lemma 5.1 and Lemma 5.2) for the following result.

Lemma 5.9 *Suppose p_0 be a function in $H^{\frac{1}{2}}(\Gamma_2)$ such that each $p_0|_{\Gamma_2 \cap \partial\Omega_k}$ belongs to $H^{s_k + \frac{1}{2}}(\Gamma_2 \cap \partial\Omega_k)$, $\theta_k \geq 0$, in the two cases:*

(i) *in dimension $d = 2$,*

(ii) *in dimension $d = 3$, if the decomposition is conforming,*

the following error estimate holds

$$\|\bar{P} - p_\delta^b\|_{\alpha^*} \leq c(1 + \mu + \mu'_\delta) \sqrt{\frac{1}{\alpha_{\min}}} \sum_{k=1}^{K_2} N_k^{-s_k} \|p_0\|_{H^{s_k + \frac{1}{2}}(\Gamma_2 \cap \partial\Omega_k)}. \quad (5.3.31)$$

where μ'_δ is the same as it in (5.3.26) and the constant c is independent of the parameter δ and the function α .

Now we get the second estimate result of this section on the pressure.

Theorem 5.3 *If the assumptions of Theorem 6.3 are satisfied in dimension 2. In dimension 3, it needs that the decomposition be conforming and the the following error estimate holds between the pressure p of problem (??) and the pressure p_δ of problem (5.2.7):*

$$\begin{aligned} \|p - p_\delta\|_{\alpha^*} &\leq c((1 + \mu + \mu'_\delta) \sum_{k=1}^K N_k^{-s_k} (\sqrt{\alpha_{\max}} (\log N_k)) \|\mathbf{u}|_{\Omega_k}\|_{H^{s_k}(\Omega_k)^d} \\ &+ \sqrt{\frac{1}{\alpha_{\min}}} \|p|_{\Omega_k}\|_{H^{s_k+1}(\Omega_k)} + \sqrt{\alpha_{\max}} \sum_{k=1}^K N_k^{-\sigma_k} \|\mathbf{f}|_{\Omega_k}\|_{H^{\sigma_k}(\Omega_k)^d} \\ &+ \sqrt{\alpha_{\max}} \sum_{k=1}^{K_1} N_k^{-s_k} \|k\|_{H^{s_k-\frac{1}{2}}(\Gamma_1 \cap \partial\Omega_k)} + (1 + \mu + \mu'_\delta) \sqrt{\frac{1}{\alpha_{\min}}} \sum_{k=1}^{K_2} N_k^{-s_k} \|p_0\|_{H^{s_k+\frac{1}{2}}(\Gamma_2 \cap \partial\Omega_k)}). \end{aligned}$$

In the two-dimensional case of a polygon Ω , a more explicit estimate can be deduced from the previously quoted regularity results. In the end of this part, we give the following corollary according to ([20]) and ([6]).

Corollary 5.2 *In the two-dimensional case of a polygon Ω , assume that the data (\mathbf{f}, k, p_0) belongs to $H^{\sigma_k}(\Omega_k)^2 \times H^{\sigma_k-\frac{1}{2}}(\Gamma_1 \cap \partial\Omega_k) \times H^{\sigma_k+\frac{1}{2}}(\Gamma_2 \cap \partial\Omega_k)$, $\sigma_k > 1$ Then, the following error estimate holds between the solution pair (\mathbf{u}, p) of problem (??) – (2.3.9) and the solution pair $(\mathbf{u}_\delta, p_\delta)$ of problem (5.2.7):*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_\delta\|_\alpha + \|p - p_\delta\|_{\alpha^*} &\leq c(1 + \mu + \mu'_\delta) \max\{\sqrt{\alpha_{\max}}, \sqrt{\frac{1}{\alpha_{\min}}}\} \left(\sum_{k=1}^K E_k (\log N_k)^\nu \right. \\ &\left. \|\mathbf{f}|_{\Omega_k}\|_{H^{\sigma_k}(\Omega_k)^2} + \sum_{k=1}^{K_1} E_k \|k\|_{H^{\sigma_k-\frac{1}{2}}(\Gamma_1 \cap \partial\Omega_k)} + \sum_{k=1}^{K_2} E_k \|p_0\|_{H^{\sigma_k+\frac{1}{2}}(\Gamma_2 \cap \partial\Omega_k)} \right). \end{aligned}$$

where ν equals to zero in the conforming decomposition and to 1 otherwise, and $E_k, 1 \leq k \leq K$ is equal to $N_k^{-\sigma_k}$ if $\bar{\Omega}_k$ contains no corner of Ω , $\sup\{N_k^{-\sigma_k}, N_k^{-\frac{4}{3}} (\log N_k)^{\frac{1}{2}}\}$, if $\bar{\Omega}_k$ contains a nonconvex corner of Ω and $\sup\{N_k^{-\sigma_k}, N_k^{-4} (\log N_k)^{\frac{3}{2}}\}$, if $\bar{\Omega}_k$ contains a no nonconvex corner of Ω .

5.4 Numerical implementation

Firstly, we briefly explain how to implement the discrete problem in section before. The full set of unknowns of the discrete system is given by

- the vector U of the values if \mathbf{u}_δ at all nodes $(x_i^k, y_i^k), 0 \leq i, j \leq N_k, 1 \leq k \leq K$, in dimension $d = 2$, $(x_i^k, y_i^k, z_p^k), 0 \leq i, j, p \leq N_k, 1 \leq k \leq K$, in dimension $d = 3$,
- the vector P of the values if p_δ at all nodes $(x_i^k, y_i^k), 0 \leq i, j \leq N_k, 1 \leq k \leq K$, in dimension $d = 2$, $(x_i^k, y_i^k, z_p^k), 0 \leq i, j, p \leq N_k, 1 \leq k \leq K$, in dimension $d = 3$.

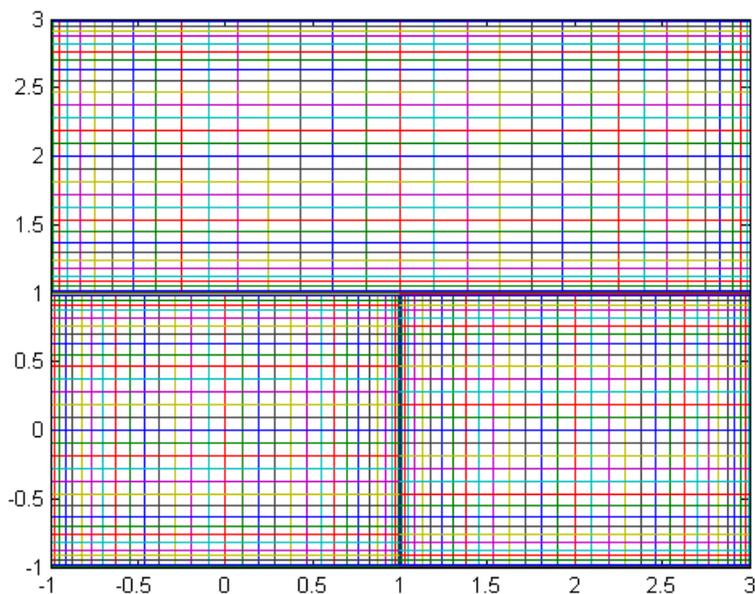
Let (U, P) denote the vector made of these values. Then the mortar condition can be expressed in the following way: there exists a rectangular matrix Q such that the boundary and matching conditions can be enforced, which leads to a new vector QU . Then problem 5.2.6 – 5.2.7 can now equivalent to the following linear system

$$\begin{pmatrix} A & BQ \\ Q^T B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ Q^T G \end{pmatrix}, \quad (5.4.1)$$

where Q^T stands for the transposed matrix of Q . The matrix A is fully diagonal, its diagonal terms are the $\rho_i^{x,k}, \rho_j^{y,k}$ or the $\rho_i^{x,k}, \rho_j^{y,k}, \rho_p^{z,k}$ according to the dimension. The matrix B is only block-diagonal, with K blocks B_k on the diagonal, one for each Ω_k . Then the system can be solved by conjugate gradient algorithm. We then consider the square $\Omega = [-1, 3]^2$, with a nonconforming decomposition into two squares $\Omega_1 = [-1, 1]^2$, and $\Omega_3 = [1, 3] \times [-1, 1]$ and a rectangle $\Omega_2 = [-1, 3] \times [1, 3]$, see Fig.5-3. For an integer $N \geq 8$, we take all the N_k equal to N . The mortars are chosen here as $\{1\} \times [1, 3] \cup [-1, 3] \times \{1\}$.

set here $\alpha_1 = 1, \alpha_2 = 10, \alpha_3 = 100$. Then we work here with the solutions (\mathbf{u}, p) given by

$$\mathbf{u}(x, y) = \begin{pmatrix} -\sin(\pi x) \cos(\pi y) \\ \cos(\pi x) \sin(\pi y) \end{pmatrix}, p(x, y) = \sin(\pi x) \cos(\pi y). \quad (5.4.2)$$

Figure 5-1: Non-matching grids for a non-conforming decomposition with $N = 32$

5.5 Comparison between the two methods

In our work, we mainly talk about the generally decomposition in $d = 2$, including the conforming decomposition and non-conforming decomposition, and the conforming decomposition in the case of $d = 3$. No convergence results are proven in the case of dimension 3 and with a non-conforming decomposition. For the same problem with the same assumptions for \mathbf{f}, k, p_0 , we return to the Theorem 5.1 and Theorem 5.2 in Chapter 5 for the first decomposition with velocity in $H(\text{div}, \Omega)$, and the Theorem 6.2 and Theorem 6.3 in Chapter 6 for the second decomposition with velocity in $L^2(\Omega)^d$. For the first decomposition, with velocity in $H(\text{div}, \Omega)$ we can observe that from Theorem 5.1, the estimate about the velocity $\|\mathbf{u} - \mathbf{u}_\delta\|_{\alpha, L^2(\Omega)}$ is fully optimal. However, from Theorem 5.2, the estimate about the pressure $\|p - p_\delta\|_{\alpha^*, \delta}$ is not fully optimal, even if the decomposition for $d = 2$ and $d = 3$ are both conforming. The lack of the optimality is $\max\{N_k^{-\frac{1}{2}}, (\log N_k)^\nu\}$, where ν is equal to 0 if the decomposition is conforming, to 1 otherwise.

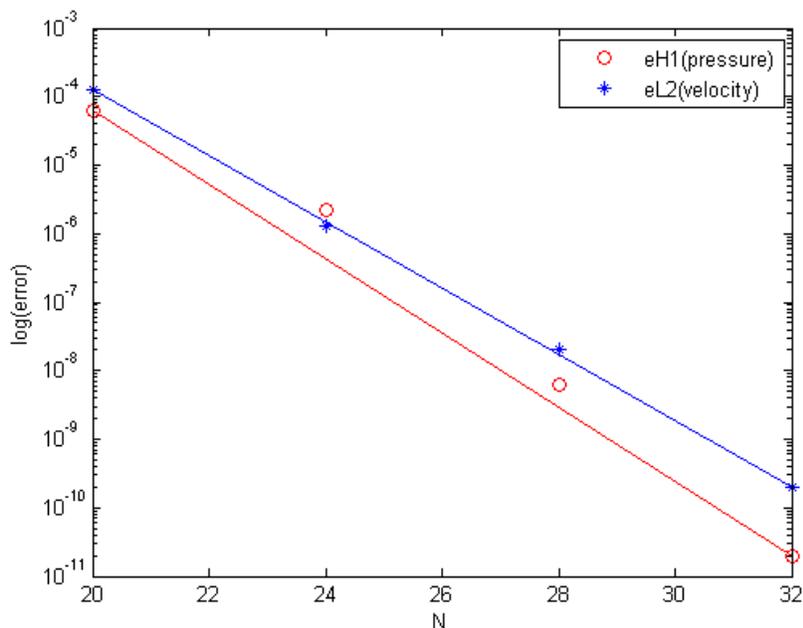


Figure 5-2: The error curves for an analytical solution

For the second decomposition, with velocity in $L^2(\Omega)^d$, we can see that from Theorem 6.3, the estimate about the pressure $\|p - p_\delta\|_{\alpha^*}$ is fully optimal. However from Theorem 6.2, the estimate about the velocity $\|\mathbf{u} - \mathbf{u}_\delta\|_\alpha$ is not fully optimal, for both conforming and non-conforming cases. The lack of the optimality is $\log N_k$.

We can see from the above statements that the first mortar spectral discretization derives a better (optimal) convergence result for the velocity. The second mortar spectral discretization derives a better (optimal) convergence result for the pressure.

The results about the numerical computation

For the two mortar discretizations, we mainly consider Darcy's equations on the same domain as follows. We consider the square $\Omega = [-1, 3]^2$, with a nonconforming decomposition into two squares $\Omega_1 = [-1, 1]^2$, and $\Omega_3 = [1, 3] \times [-1, 1]$ and a rectangle $\Omega_2 = [-1, 3] \times [1, 3]$, see Fig.5-3. For an integer $N \geq 8$, we take all the N_k equal to N . The mortars are chosen here as $\{1\} \times [-1, 1] \cup [-1, 3] \times \{1\}$. The condition of the

numerical experiment are the same then previously.

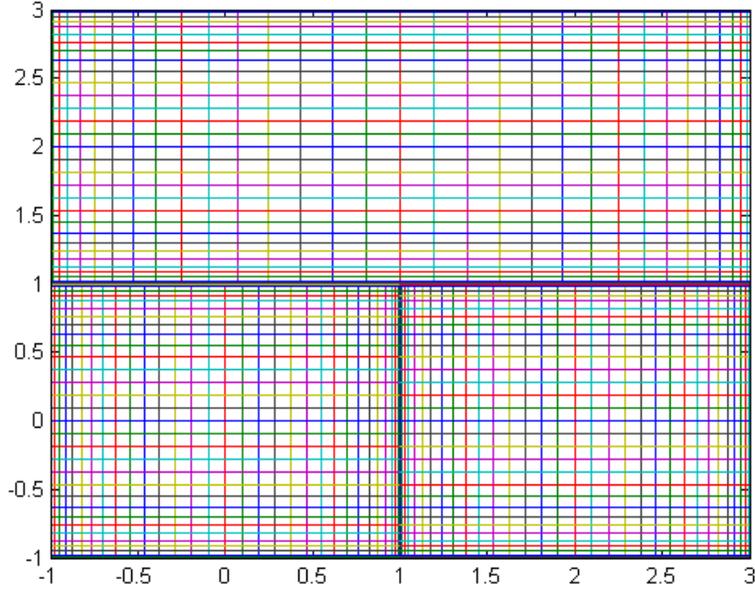


Figure 5-3: Non-matching grids for a non-conforming decomposition with $N = 32$

In the following figure, we can see the comparison of the error curves for an analytical solution, where the left subfigure denotes the first discretization and the right subfigure denotes the second discretization. In Figure 5-4 are plotted, the curves

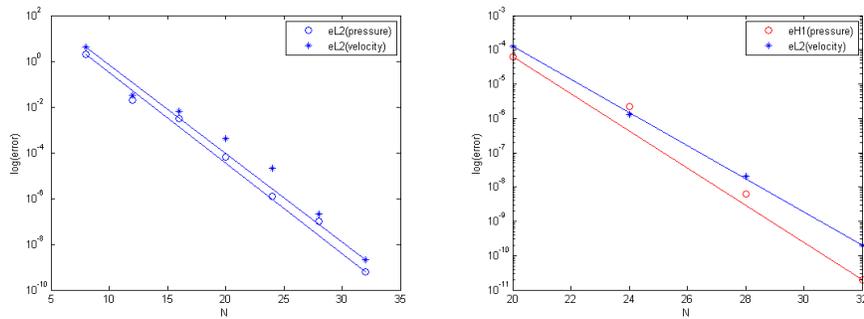


Figure 5-4: Comparison of the error curves for an analytical solution

of the errors $\|\mathbf{u} - \mathbf{u}_\delta\|_\alpha$ and $\|p - p_\delta\|_{\alpha^*}$ for both cases as a function of N . For the smooth solution, a linear or logarithmic scale is used. We observe that the exponential decaying of the error is preserved despite the nonconforming domain decomposition.

Chapter 6

Conclusion and Perspectives

In the whole thesis, we are mainly interested to the spectral approximation and mortar spectral element method for the Darcy's Equations with general boundary conditions.

In Chapter 3, we have presented the spectral discretization for the Darcy's equations. The spectral method has been proven optimal in the sense that the order of convergence is only limited by the regularity of the solution. The chapter 4 and Chapter 5 are the key chapters in our thesis. We propose two different discretization relies on the mortar spectral method. Both the numerical analysis of the discretization problems are performed and numerical experiments are presented, which turn out to be in good coherency with the theoretical results. The last Chapter is an appendix, which is an exception in our thesis, because a spectral scheme have been presented for fourth-order equations. We here have developed a Legendre-Petrov-Galerkin method for linear fourth-order differential equations in one dimension and a Legendre-Petrov-Galerkin and Chebyshev Collocation method for nonlinear fourth-order equations. We propose two different discretization relies on the mortar spectral method. Both the numerical analysis of the discretization problems are performed and numerical experiments are presented, which turn out to be in good coherency with the theoretical

results.

In our later work, we will extend to unsteady Darcy's system, which models the time-dependent flow of an incompressible fluid such as water in a rigid porous medium.

$$\left\{ \begin{array}{ll} \partial_t \mathbf{u}(\mathbf{x}, t) + \alpha \mathbf{u}(\mathbf{x}, t) + \mathbf{grad} p(\mathbf{x}, t) = \alpha \mathbf{f}(\mathbf{x}, t) & \text{in } \Omega \times [0, T], \\ \operatorname{div} \mathbf{u}(\mathbf{x}, t) = 0 & \text{in } \Omega \times [0, T], \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}, 0) & \text{in } \Omega, \\ \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n} = k(\mathbf{x}, t) & \text{on } \Gamma_1 \times [0, T], \\ p(\mathbf{x}, t) = p_0(\mathbf{x}, t) & \text{on } \Gamma_2 \times [0, T] \end{array} \right. \quad (6.0.1)$$

It also admits two equivalent variational formulations as the steady Darcy's system. For time adaptivity, in order to avoid Courant-Friedrichs-Lévy conditions, an implicit scheme such as the backward Euler's scheme could be used. Concerning the space discretization, it can be extended from the previous problems. A priori error estimates will also be proved to justify the optimal convergence properties of the discretizations.

Chapter 7

Appendix : A Spectral Method for 4-th Differential Equations

In this chapter, we develop a Legendre Petrov-Galerkin method for linear fourth-order differential equations in one dimension and a Legendre Petrov-Galerkin and Chebyshev collocation method for the nonlinear Kuramoto-Sivashinsky equation. The numerical experiments are given which demonstrate the efficiency of proposed schemes. Finally, we give the optimal rate of convergence in L^2 -norm of the methods.

7.1 Introduction

There are many applications of Galerkin methods and collocation methods in the literatures for fourth-order differential equations. In [52, 53], Shen proposes a class of spectral-Galerkin methods for the direct solution of second- and fourth-order equations based on Legendre and Chebyshev polynomials. In [27], a Legendre spectral Galerkin method is presented for the solution of the biharmonic Dirichlet problem. Their approach is based on a mixed method which gives rise to a variational formula-

tion for two second order differential equations. A similar approach has been applied in the Legendre spectral collocation solution of the same problem in [26]. In [35], the authors present some efficient spectral algorithms based on the Jacobi-Galerkin methods for fourth-order equations. Theoretical work of the two methods is referred to [17, 21, 39].

However, to the authors' knowledge, there is little previous work on Petrov-Galerkin methods for fourth-order differential equations. Because of the merits of Petrov-Galerkin methods, that is, the test functions do not satisfy all the boundary conditions, they are widely used in computation [37, 45, 49, 50]. This motivated our interest in such methods.

As is pointed out in [41, 53], the direct application of the tau method for fourth-order equations is numerically unstable, because it results by ill conditioned system. Meanwhile, it is difficult to get optimal error estimate in L^2 -norm. Therefore, in this study, we present a Legendre Petrov-Galerkin (LPG) method for fourth-order equations in one dimension. Through the numerical experiments, we can note that the LPG method is an efficient algorithm. Moreover, optimal error estimates in L^2 -norm of the LPG method can be obtained.

In a recent work [54], a linear steady fourth-order equation has been computed by the LPG method and get desirable numerical results. However, the algebraic systems and error estimates are not given. In this work, we give the algebraic scheme of LPG method and prove the optimal rate of convergence in L^2 -norm for linear fourth-order differential equations. Moreover, for nonlinear equations such as the Kuramoto-Sivashinsky(K-S) equation which is an important equation in physical models, a Legendre Petrov-Galerkin and Chebyshev collocation (LPG-CC) method is developed. The scheme of the linear part is formulated in the LPG form while the nonlinear term is treated with the Chebyshev collocation method. The time discretization is a classical leapfrog/Crank-Nicolson scheme. To get optimal error estimate, we apply the approach of [54] in which a norm stronger than standard L^2 -norm is used.

The chapter is organized as follows. In the next section, we propose the LPG scheme and the LPG-CC scheme for linear and nonlinear fourth-order equations, respectively. Some numerical experiments are presented in Section 3. In Section 4, we recall some recent results on some projections and a Chebyshev-Gauss-Lobatto interpolation with a negative weight. In Section 5, optimal error estimates of the two schemes are given. The final section is for concluding remarks.

7.2 Chebyshev polynomials

Classical reference on the Chebyshev polynomials are [41]. The Chebyshev polynomials of the first kind, $T_k(x)$, $k = 0, 1, \dots$, are the eigenfunctions of the singular Sturm-Liouville problem

$$(\sqrt{1-x^2}T_k'(x))' + \frac{k^2}{\sqrt{1-x^2}}T_k(x) = 0.$$

For any k , $T_k(x)$ is even if k is even, and odd if k is odd. If T_k is normalized so that $T_k(1) = 1$, then

$$T_k(x) = \cos k\theta, \quad \theta = \arccos x. \quad (7.2.1)$$

Thus, the Chebyshev polynomials are nothing but cosine functions after a change of independent variable. Due to the trigonometric relation

$$\cos(k+1)\theta + \cos(k-1)\theta = 2\cos\theta\cos(k\theta),$$

we will give the recursion relation of Chebyshev polynomials:

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad (7.2.2)$$

with $T_0(x) = 1$ and $T_1(x) = x$.

Next we will introduce the Chebyshev-Gauss-lobatto points and weights:

$$\begin{cases} x_0 = 1, x_N = -1, \\ \omega_0 = \omega_N = \frac{\Pi}{2N}, \\ x_j = \cos \frac{\pi j}{N}, \omega_j = \frac{\Pi}{N}, 1 \leq j \leq N-1. \end{cases}$$

Let $\{x - i\}_{i=0}^N$ be the Chebyshev-Gauss-Lobatto points, i.e. $x_i = \cos \frac{i\Pi}{N}$, and u be a function on $[-1, 1]$. The Chebyshev interpolation polynomial I_N^C is defined as a polynomial of degree less than or equal to N such that

$$I_N^C u(x_i) = u(x_i), \quad i = 0, 1, \dots, N. \quad (7.2.3)$$

7.3 Spectral discretization scheme

7.3.1 Linear equation

At first, we consider the following 1 dimension linear fourth-order equation

$$\begin{cases} \partial_x^4 U - \beta \partial_x^2 U + \alpha U = g, & x \in (-1, 1), \\ U(\pm 1) = \partial_x U(\pm 1) = 0, \end{cases} \quad (7.3.1)$$

where $\alpha, \beta > 0$.

Let $\mathcal{V}_N = \mathbb{P}_N(I) \cap H_0^2(I) \cap H^3(I)$, $\mathcal{W}_{N-2} = \mathbb{P}_{N-2}(I) \cap H_0^1(I) \cap H^2(I)$, where $I = [-1, 1]$.

A Legendre Petrov-Galerkin method for the problem (7.3.1) is to find $u_N \in \mathcal{V}_N$ such that for any $v \in \mathcal{W}_{N-2}$,

$$\alpha(u_N, v) - \beta(\partial_x^2 u_N, v) - (\partial_x^3 u_N, \partial_x v) = (g, v). \quad (7.3.2)$$

Appropriate base functions of \mathcal{V}_N and \mathcal{W}_{N-2} are chosen to set up the corresponding system of equations as in [52]. For $0 \leq n \leq N - 4$, we let

$$\begin{cases} c_n = 1/\sqrt{2(2n+3)^2(2n+5)}, \\ d_n = 1/(2n+1), \end{cases}$$

and

$$\begin{cases} \phi_n(x) = c_n[L_n(x) - \frac{2(2n+5)}{2n+7}L_{n+2}(x) + \frac{2n+3}{2n+7}L_{n+4}(x)], \\ \psi_n(x) = d_{n+1}[L_n(x) - L_{n+2}(x)], \end{cases}$$

where $L_n(x)$ is the Legendre polynomial of degree n . We note that all the polynomials $L_n(x)$, $0 \leq n \leq N$ are orthogonal, and

$$\|L_n(x)\|_{L^2(I)}^2 = \frac{1}{n + \frac{1}{2}}.$$

Expanding $u_N(x) = \sum_{n=0}^{N-4} \hat{u}_n \phi_n(x)$ and taking $v = \psi_m(x)$ in (7.3.2) lead to

$$\sum_{n=0}^{N-4} [\alpha(\phi_n, \psi_m) - \beta(\partial_x^2 \phi_n, \psi_m) - (\partial_x^3 \phi_n, \partial_x \psi_m)] \hat{u}_n = (g, \psi_m), \quad 0 \leq m \leq N - 4. \quad (7.3.3)$$

Let $a_{nm} = (\phi_n, \psi_m)$, $b_{nm} = (\partial_x^2 \phi_n, \psi_m)$, $p_{nm} = -(\partial_x^3 \phi_n, \partial_x \psi_m)$, then

$$a_{nm} = \begin{cases} 2c_n d_{m+1} (d_n + 2d_{n+3}), & n = m, \\ -2c_n d_{m+1} d_n, & n = m + 2, \\ -2c_n d_{m+1} (2d_{n+3} + d_{n+3} d_{n+4}/d_{n+1}), & n = m - 2, \\ 2c_n d_{m+1} d_{n+3} d_{n+4}/d_{n+1}, & n = m - 4, \\ 0, & \text{otherwise.} \end{cases}$$

$$b_{nm} = c_n(2n+3)((L_{n+3} - L_{n+1}), L_{m+1}) = \begin{cases} 2c_n(2n+3)d_{n+3}, & n = m - 2, \\ -2c_n(2n+3)d_{n+1}, & n = m, \\ 0, & \text{otherwise.} \end{cases}$$

$$p_{nm} = c_n(2n+3)(2n+5)(L'_{n+2}, L_{m+1}) = \begin{cases} 0, & m \geq n + 1, \\ c_n(2n+3)(2n+5)(1 - (-1)^{m+n+3}), & m \leq n. \end{cases}$$

We denote

$$A = \{a_{nm}\}_{0 \leq n, m \leq N-4}, \quad B = \{b_{nm}\}_{0 \leq n, m \leq N-4}, \quad P = \{p_{nm}\}_{0 \leq n, m \leq N-4},$$

$$\hat{g}_m = (g, \psi_m), \quad \hat{g} = [\hat{g}_0, \hat{g}_1, \dots, \hat{g}_{N-4}]^T.$$

The scheme (7.3.3) can be written in a matrix form as follows:

$$(\alpha A - \beta B + P)\hat{u} = \hat{g}, \quad \hat{u} = [\hat{u}_0, \hat{u}_1, \dots, \hat{u}_{N-4}]^T. \quad (7.3.4)$$

Remark 7.1 In Table 1, we list the condition numbers of the fourth-order term of the LPG method and that of the Legendre tau (LT) method which we expand $u_N(x) = \sum_{n=0}^{N-4} \hat{u}_n \phi_n(x)$ [44] and take $v = L_m(x)$, $0 \leq m \leq N - 4$. The command "cond" is used in Matlab to compute the condition numbers.

Table 1 Condition numbers for the fourth-order equation

N	16	32	64	128
LT	3.97e+2	4.72e+3	5.46e+4	6.23e+5
LPG	1.58e+1	4.64e+1	1.34e+2	3.82e+2

We note that the condition numbers of the LPG method are smaller than the LT method from Table 1.

7.3.2 Nonlinear equation

Next, we consider the Kuramoto-Sivashinsky equation

$$\begin{cases} \partial_t U + \partial_x F(U) + \partial_x^2 U + \partial_x^4 U = g(x, t) & (x, t) \in (-1, 1) \times (0, T) \\ U(\pm 1, t) = \partial_x U(\pm 1, t) = 0 & t \in (0, T) \\ U(x, 0) = U_0(x) & x \in (-1, 1). \end{cases} \quad (7.3.5)$$

where $F(z)$ is a smooth function of z , and $T > 0$

The semidiscrete LPG-CC method for the problem is to find $u_N(t) \in \mathcal{V}_N$ such that for any $v \in \mathcal{W}_{N-2}$,

$$\begin{cases} (\partial_t u_N, v) + (\partial_x I_N^C F(u_N), v) + (\partial_x^2 u_N, v) - (\partial_x^3 u_N, \partial_x v) = (I_N^C g(t), v) & t \in (0, T) \\ u_N(0) = I_N^C u_0, \end{cases} \quad (7.3.6)$$

where I_N^C is the Chebyshev interpolation operator at the Chebyshev-Gauss-Lobatto points.

Let τ be the step size in time space and $S_t = \{k\tau : k = 1, 2, \dots, n_t; t = n_t\tau\}$. For simplicity, denote $u_{\hat{t}} = \frac{1}{2\tau}(u(t+\tau) - u(t-\tau))$, $\bar{u}(t) = \frac{1}{2}(u(t+\tau) + u(t-\tau))$. Using leapfrog/Crank-Nicolson scheme in time such that the linear part is treated implicitly and the nonlinear part explicitly, we get the fully discrete LPG-CC approximation to (7.3.5): to find $u_N \in \mathcal{V}_N$ such that for any $v \in \mathcal{W}_{N-2}$,

$$\begin{cases} (u_{N\hat{t}}, v) + (\partial_x I_N^C F(u_N), v) + (\partial_x^2 \bar{u}_N, v) - (\partial_x^3 \bar{u}_N, \partial_x v) = (I_N^C \bar{g}, v), & t \in S_{T-\tau}, \\ (u_N(\tau), v) = (I_N^C [u_0 + \tau \partial_t u(0)], v), \\ u_N(0) = I_N^C u_0. \end{cases} \quad (7.3.7)$$

Note that $\partial_t u(0) = g(0) - \partial_x^2 u_0 - \partial_x^4 u_0 - \partial_x F(u_0)$.

In computation, expanding $u_N(x) = \sum_{n=0}^{N-4} \hat{u}_n \phi_n(x)$ and taking $v = \psi_m(x)$, the scheme of the linear part can be written in a matrix form as in (7.3.4). The nonlinear term

and the right-hand term can be computed by the fast Legendre transform (FLT) [4] between the coefficients of the Legendre series and its values at the CGL points, such as

$$\{\hat{u}_n\} \xrightarrow{FLT} \{u_N(x_j)\} \rightarrow \{F(u_N(x_j))\} \xrightarrow{FLT} \{(\widehat{I_N^C F(u_N)})_n^L\},$$

where $(\widehat{I_N^C F(u_N)})_n^L$ are the Legendre expansion coefficients of $I_N^C F(u_N)$.

7.4 Numerical results

We first consider linear fourth-order differential equations.

Example 7.1 *The 1-D fourth-order equation*

$$\partial_x^4 U - \partial_x^2 U + U = ((k\pi)^4 + (k\pi)^2 + 1) \sin(k\pi x), \quad x \in I,$$

with a smooth exact solution $U(x) = \sin(k\pi x)$.

We compute Example 7.1 by the LPG method compared with the LT method with $k=6$. The maximum error is listed in Table 2.

Table 2 Maximum error for Example 7.1

N	16	32	64	128
LT	4.02	2.37e-5	4.35e-13	4.14e-13
LPG	5.28	1.15e-5	3.71e-13	5.14e-14

From Example 7.1, we note that the Legendre Petrov-Galerkin method is a little more accurate than the Legendre Tau method although the boundary conditions which test functions satisfy of the tau method are less than that of the Petrov-Galerkin method. Furthermore, through the appropriate combination of base functions, the tau method also possess the property of numerical stability.

Example 7.2 *The 1-D fourth-order equation*

$$\partial_x^4 U + 4U = 1, \quad x \in I.$$

with an analytical solution

$$U(x) = 0.25(1 - (\sin(1) \sinh(1) \sin(x) \sinh(x) + \cos(1) \cosh(1) \cos(x) \cosh(x)) / (\cos(2) + \cosh(2))).$$

This example is taken from [56] in order to make a comparison with the Sixth-order method based on non-polynomial spline functions in [56]. We list the maximum error in Table 3.

Table 3 Maximum error for Example 7.2

N	Sixth-order method in [56]	N	LPG
16	9.99e-9	8	2.58e-8
32	1.89e-10	10	7.92e-11
64	2.82e-12	12	2.17e-13

Example 7.2 shows that our method is of the same accuracy as the Sixth-order method in [56] through taking less points.

Example 7.3 *The 1-D fourth-order equation*

$$\partial_x^4 U + xU = -(8 + 7x + x^3)e^x, \quad 0 < x < 1.$$

with an analytical solution $U(x) = x(1 - x)e^x$.

This example is also taken from [56] in order to make a comparison with the Sixth-order method based on non-polynomial spline functions in [56]. The maximum errors of U and its n th-order derivatives ($n = 1, 2, 3, 4$) by the Sixth-order method in [56]

are listed in Table 4. The results by our method are listed in Table 5. Here U_i and i_{Ni} denote the analytical results and numerical results.

Table 4 Maximum errors of U and its n th-order derivatives ($n = 1, 2, 3, 4$) by the Sixth-order method in [56] for Example 7.3

N	$\max U_i - u_{Ni} $	$\max U'_i - u'_{Ni} $	$\max U''_i - u''_{Ni} $	$\max U'''_i - u'''_{Ni} $	$\max U_i^{(iv)} - u_{Ni}^{(iv)} $
8	3.70e-10	4.91e-6	3.42e-4	7.20e-3	1.85e-10
16	7.51e-12	1.99e-7	2.24e-5	1.20e-3	4.16e-12
32	6.99e-14	6.21e-9	1.85e-6	1.78e-4	3.90e-14

Table 5 Maximum errors of U and its n th-order derivatives ($n = 1, 2, 3, 4$) by our method for Example 7.3

N	$\max U_i - u_{Ni} $	$\max U'_i - u'_{Ni} $	$\max U''_i - u''_{Ni} $	$\max U'''_i - u'''_{Ni} $	$\max U_i^{(iv)} - u_{Ni}^{(iv)} $
8	2.80e-8	1.76e-7	1.08e-6	2.36e-5	5.05e-4
12	1.06e-14	1.10e-13	1.12e-12	4.96e-11	2.74e-9
16	5.55e-17	2.22e-16	4.44e-16	8.88e-16	3.11e-15

We note that the maximum errors computed by our method are more accurate than by the Sixth-order method in [56], especially for the errors of n th-order derivatives ($n = 1, 2, 3, 4$).

Finally, we consider a nonlinear evolution equation.

Example 7.4 *The Kuramoto-Sivashisky equation*

$$\partial_t U + U \partial_x U + \partial_x^2 U + \partial_x^4 U = 0, \quad x \in I.$$

with exact solution

$$U(x, t) = C + \frac{15}{19} \sqrt{\frac{11}{19}} [11 \tanh^3(k(x - Ct - x_0)) - 9 \tanh(k(x - Ct - x_0))]$$

We compute Example 7.4 by the scheme (7.3.7) at $T = 1$ with $C = 0.1, k = \frac{1}{2}\sqrt{\frac{11}{19}}, x_0 = -10, I = [-30, 30]$. To see the order of accuracy, we let τ decrease from 10^{-1} to 10^{-4} for large $N = 256$ and let N increase from 64 to 256 for small $\tau = 10^{-3}$. The results are given in Table 6.

Table 6 Error at $T = 1$ for Example 7.4

τ	N	$L^\infty - error$	$L^2 - error$
1e-1	256	7.21e-5	1.07e-4
1e-2		7.21e-7	1.07e-6
1e-3		7.19e-9	1.07e-8
1e-4		7.08e-11	1.07e-10
1e-3	64	5.25e-2	9.37e-2
	128	3.17e-5	1.28e-4
	256	7.19e-9	1.07e-8

In Table 7, L^∞ and L^2 errors are recorded at time $t = 0.1; 0.3; 0.5; 0.7; 1$. We compare the results with those presented in [57] by the mesh-free collocation method based on multiquadric (MQ) with $c(\text{MQ})=2.5$ advancing in time by the Crank-Nicolson scheme. For the nonlinear term, they linearize it.

Table 7 Maximum error of MQ method in [57] and LPG-CC method for Example 7.4 at time $t = 0.1; 0.3; 0.5; 0.7; 1$ with $N = 120; \tau = 0.001; I = [-30, 30]; C = 0.1; x_0 = -10$

	t	0.1	0.3	0.5	0.7	1.0
$L^\infty - error$	MQ method	1.03e-4	1.85e-4	2.89e-4	3.85e-4	5.23e-4
	LPG-CC	2.56e-5	2.58e-5	2.53e-5	2.57e-5	3.52e-5
$L^2 - error$	MQ method	1.73e-5	3.86e-4	7.30e-4	7.13e-4	1.51e-3
	LPG-CC	6.77e-5	8.30e-5	9.40e-5	1.03e-4	1.15e-4

It is noticed that the LPG-CC method shows better accuracy than the MQ method in [57].

7.5 Preliminaries

In this section, we state approximation properties of some projection operators and an interpolation operator, which will be needed in the proof of convergence. Throughout this chapter, C will denote a generic positive constant.

Let $I = (-1, 1)$, and $\omega_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ be a certain weight function in the usual sense. Denote by $(\cdot, \cdot)_{\omega_{\alpha,\beta}}$ and $\|\cdot\|_{\omega_{\alpha,\beta}}$ the inner product and the norm of the weighted space $L^2_{\omega_{\alpha,\beta}}(I)$, respectively. We adopt the standard notation $H^\sigma_{\omega_{\alpha,\beta}}(I)$ for weighted Sobolev space with norm $\|\cdot\|_{\omega_{\alpha,\beta},\sigma}$ and semi-norm $|\cdot|_{\omega_{\alpha,\beta},\sigma}$. The subscript ω will be omitted whenever $\omega_{\alpha,\beta}(x) \equiv 1$. Let $\mathbb{P}_N(I)$ be the space of polynomials of degree at most N on the interval I .

The Jacobi orthogonal projection operator with the weight $\omega_{\alpha,\beta}(x)$ is defined $\mathcal{P}_N^{\alpha,\beta} : L^2_{\omega_{\alpha,\beta}}(I) \rightarrow \mathbb{P}_N(I)$. For simplicity, we set $P_N = \mathcal{P}_N^{0,0}$. Next, we introduce the spectral projection operator $P_N^k : H^k(I) \rightarrow \mathbb{P}_N(I)$ [21, 46, 54] defined by

$$P_N^k u = u(-1) + \int_{-1}^x (P_{N-1}^{k-1} \partial_\xi u) d\xi, \quad k \geq 1,$$

where $P_N^0 = P_N$. The projection operator has the following properties [46]:

$$(\partial_x^l P_N^k u)(1) = \partial_x^l u(1), \quad (\partial_x^l P_N^k u)(-1) = \partial_x^l u(-1), \quad 0 \leq l \leq k-1, \quad (7.5.1)$$

$$(\partial_x^k (P_N^k u - u), v) = 0, \quad \forall v \in \mathbb{P}_{N-k}. \quad (7.5.2)$$

Here we need a stronger result, which is in negative weight.

Lemma 7.1 ([54]) *If $u \in H^r(I)$ then it holds that for all $0 \leq l \leq k \leq r$,*

$$\|\partial_x^l (P_N^k u - u)\|_{\omega_{l-k, l-k}} \leq CN^{l-r} \|\partial_x^r u\|_{\omega_{r-k, r-k}}, \quad (7.5.3)$$

Denoting by I_N^C the Chebyshev-Gauss-Lobatto (CGL) interpolation operator, we have

Lemma 7.2 ([46]) *If $u \in H^r(I)$ and $r \geq 1$, then it holds that for all $0 \leq l \leq 1$,*

$$\|\partial_x^l(I_N^C u - u)\|_{\omega_{l-1,l-1}} \leq CN^{l-r} \|\partial_x^r u\|_{\omega_{r-1,r-1}}. \quad (7.5.4)$$

7.6 Convergence of the schemes

7.6.1 Linear equation

At first, we give an optimal rate of convergence for the scheme (7.3.2). Assume that the theoretical solution $U(x)$ is smooth enough. Let $\eta = U - u^*$ and $e = u_N - u^*$, where $u^* = P_N^3 U$. From (7.3.1) and (7.3.2), we have

$$\alpha(e, v) - \beta(\partial_x^2 e, v) - (\partial_x^3 e, \partial_x v) = \alpha(\eta, v) - \beta(\eta, \partial_x^2 v). \quad (7.6.1)$$

where we have used the fact $(\partial_x^3 \eta, \partial_x v) = 0$ for all $v \in \mathcal{W}_{N-2}$, according to (7.5.2). Because of $e(\pm 1) = \partial_x e(\pm 1) = 0$ by (7.5.1), we can take $v = \omega_{-1,-1} e$. It is easy to derive the following equalities

$$-(\partial_x^3 e, \partial_x v) = 2|\partial_x v(1)|^2 + 2|\partial_x v(-1)|^2 + 4\|\partial_x v\|^2 + \|\partial_x^2 v\|_{\omega_{1,1}}^2.$$

$$-(\partial_x^2 e, v) = \|\partial_x v\|_{\omega_{1,1}}^2 + \|v\|^2.$$

Therefore, we have

$$\begin{aligned} & \alpha\|e\|_{\omega_{-1,-1}}^2 + \beta\|v\|^2 + \beta\|\partial_x v\|_{\omega_{1,1}}^2 + 2|\partial_x v(1)|^2 + 2|\partial_x v(-1)|^2 + 4\|\partial_x v\|^2 + \|\partial_x^2 v\|_{\omega_{1,1}}^2 \\ & \leq \beta\|\eta\|_{\omega_{-1,-1}}\|\partial_x^2 v\|_{\omega_{1,1}} + \alpha\|\eta\|\|v\| \\ & \leq \beta^2\|\eta\|_{\omega_{-1,-1}}^2 + \frac{1}{4}\|\partial_x^2 v\|_{\omega_{1,1}}^2 + \frac{\alpha^2}{\beta}\|\eta\|^2 + \frac{\beta}{4}\|v\|^2, \end{aligned}$$

According to (7.5.3), we have

$$\|e\|_{\omega_{-1,-1}} \leq C^* N^{-r}.$$

Furthermore, from the inverse property of polynomials (see (3.17) of [47]), we have

$$\|\partial_x e\| \leq CN \|e\|_{\omega_{-1,-1}} \leq CN^{1-r},$$

which means the method also admits optimal error estimate in H^1 -norm.

We arrive at the following convergence result via the triangle inequality and Lemma 7.1.

Theorem 7.1 *Assume that $r \geq 3$ and $U(x) \in H^r(I)$, then*

$$\|U - u_N\| \leq \|U - u_N\|_{\omega_{-1,-1}} \leq CN^{-r},$$

and

$$\|U - u_N\| + N^{-1} \|\partial_x(U - u_N)\| \leq CN^{-r}.$$

7.6.2 Nonlinear equation

Next, we consider the Kuramoto-Sivashinsky equation (7.3.5). We give the stability and an optimal rate of convergence for the semidiscrete scheme (7.3.6) of the LPG-CC method.

Suppose that u_N and the term on the right-hand side in (7.3.6) have the error \tilde{u}_N and \tilde{g} , respectively. Then by (7.3.6) we have

$$(\partial_t \tilde{u}_N, v) + (\partial_x I_N^C \tilde{F}, v) + (\tilde{u}_N, \partial_x^2 v) - (\partial_x^3 \tilde{u}_N, \partial_x v) = (\tilde{g}, v).$$

where $\tilde{F} = F(u_N + \tilde{u}_N) - F(u_N)$. Taking $v = \omega_{-1,-1}\tilde{u}_N \in \mathcal{W}_{N-2}$, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\tilde{u}_N\|_{\omega_{-1,-1}}^2 + 2|\partial_x v(1)|^2 + 2|\partial_x v(-1)|^2 + 4\|\partial_x v\|^2 + \|\partial_x^2 v\|_{\omega_{1,1}}^2 \\ \leq & \|I_N^C \tilde{F}\|^2 + \frac{1}{4}\|\partial_x v\|^2 + \|\tilde{u}_N\|_{\omega_{-1,-1}}^2 + \frac{1}{4}\|\partial_x^2 v\|_{\omega_{1,1}}^2 + \|\tilde{g}\|_{-1}^2 + \frac{1}{4}\|\partial_x v\|^2. \end{aligned}$$

Let

$$\begin{aligned} u_M &= \max_{0 \leq s \leq T} \{ \|u_N(s)\|_{L^\infty(I)} + \|\partial_x u_N(s)\|_{L^\infty(I)} \}, \\ C_F(z_1, z_2) &= \max_{|z| \leq |z_1| + |z_2|} |\partial_z F(z)| + (|z_1| + |z_2|) \max_{|z| \leq |z_1| + |z_2|} |\partial_z^2 F(z)|. \end{aligned}$$

Assume

$$\|\tilde{u}_N(s)\|_{L^\infty(I)} \leq C_1, \quad \forall s \in (0, t].$$

By (7.5.4), we have

$$\begin{aligned} \|I_N^C \tilde{F}\| &\leq \|I_N^C \tilde{F}\|_{\omega_{-1,-1}} \leq \|I_N^C \tilde{F} - \tilde{F}\|_{\omega_{-1,-1}} + \|\tilde{F}\|_{\omega_{-1,-1}} \\ &\leq CN^{-1}\|\partial_x \tilde{F}\| + \|\tilde{F}\|_{\omega_{-1,-1}} \\ &\leq CN^{-1} \left\| \int_0^1 \tilde{F}''(u_N(s) + \theta \tilde{u}_N(s)) (\partial_x u_N(s) + \theta \partial_x \tilde{u}_N(s)) \tilde{u}_N(s) + \tilde{F}'(u_N(s) + \theta \tilde{u}_N(s)) \partial_x \tilde{u}_N(s) d\theta \right\|_{\omega_{-1,-1}} \\ &+ \left\| \int_0^1 \tilde{F}'(u_N(s) + \theta \tilde{u}_N(s)) \tilde{u}_N(s) d\theta \right\|_{\omega_{-1,-1}} \\ &\leq C_F(u_M, C_1) (N^{-1} \|\partial_x \tilde{u}_N\|_{\omega_{-1,-1}} + \|\tilde{u}_N\|_{\omega_{-1,-1}}) \\ &\leq CC_F(u_M, C_1) \|\tilde{u}_N\|_{\omega_{-1,-1}} \end{aligned}$$

so,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\tilde{u}_N\|_{\omega_{-1,-1}}^2 + 2|\partial_x v(1)|^2 + 2|\partial_x v(-1)|^2 + 4\|\partial_x v\|^2 + \|\partial_x^2 v\|_{\omega_{1,1}}^2 \\ \leq & C^* \|\tilde{u}_N\|_{\omega_{-1,-1}}^2 + \frac{1}{2}\|\partial_x v\|^2 + \frac{1}{4}\|\partial_x^2 v\|_{\omega_{1,1}}^2 + \|\tilde{g}\|_{-1}^2. \end{aligned}$$

Therefore, integrating the inequality in time leads to

$$E(t) \leq \rho(t) + C^* \int_0^t E(\tau) d\tau,$$

where C^* is a positive constant dependent on $C_F(u_M, C_1)$, and let

$$E(t) = \|\tilde{u}_N(t)\|_{\omega_{-1,-1}}^2 + \int_0^t 2|\partial_x v(1)|^2 + 2|\partial_x v(-1)|^2 + \frac{7}{2}|\partial_x v|^2 + \frac{3}{4}\|\partial_x^2 v\|_{\omega_{1,1}}^2 ds,$$

$$\rho(t) = \|\tilde{u}_N(0)\|_{\omega_{-1,-1}}^2 + \int_0^t \|\tilde{g}\|_{-1}^2 ds.$$

We can get the following theorem. The process of the proof for this theorem is similar to that of the proof in [54].

Theorem 7.2 *If $\rho(T) \leq 2C_1^2 e^{-C^*T}/(N+1)^2$, then*

$$E(t) \leq \rho(t)e^{C^*t}.$$

Now, we consider the convergence of the scheme (7.3.6). Let $u^* = P_N^3 U$, $\eta = U - u^*$ and $e = u_N - u^*$. By (7.3.5), (7.3.6), (7.5.2), we have

$$\begin{aligned} & (\partial_t e, v) + (\partial_x I_N^C \tilde{G}, v) - (\partial_x^3 e, \partial_x v) \\ = & (\partial_t \eta, v) + (I_N^C g - g, v) + (\partial_x (F(U) - I_N^C F(u^*)), v) - (e, \partial_x^2 v) + (\eta, \partial_x^2 v) + (\partial_x^3 \eta, \partial_x v) \\ = & (\partial_t \eta, v) + (I_N^C g - g, v) + (\partial_x I_N^C (F(U) - F(u^*)), v) + (\partial_x (F(U) - I_N^C F(U)), v) \\ & - (e, \partial_x^2 v) + (\eta, \partial_x^2 v) \\ = & I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \end{aligned}$$

where $\tilde{G} = F(u^* + e) - F(u^*)$. Taking $v = \omega_{-1,-1} e$, we can get

$$\begin{aligned} |I_1| &= |(\partial_t \eta, v)| \leq \|\partial_t \eta\|_{-1}^2 + \frac{1}{4}\|\partial_x v\|^2 \leq CN^{-2r}\|\partial_t U\|_{r-1}^2 + \frac{1}{4}\|\partial_x v\|^2, \\ |I_2| &= |(I_N^C g - g, v)| \leq \|I_N^C g - g\|_{-1}\|\partial_x v\| \\ &\leq CN^{-2r}\|g\|_{r-1}^2 + \frac{1}{4}\|\partial_x v\|^2. \end{aligned}$$

Let

$$U_M = \max_{0 \leq s \leq T} \{\|U(s)\|_{L^\infty(I)} + \|\partial_x U(s)\|_{L^\infty(I)}\}.$$

$$\begin{aligned}
|I_3| &= |(I_N^C(F(U) - F(u^*)), \partial_x v)| \leq \|I_N^C(F(U) - F(u^*))\|^2 + \frac{1}{4}\|\partial_x v\|^2 \\
&\leq \|I_N^C(F(U) - F(u^*)) - (F(U) - F(u^*))\|^2 + \|F(U) - F(u^*)\|^2 + \frac{1}{4}\|\partial_x v\|^2 \\
&\leq CC_F(U_M, \|U - u^*\|_{L^\infty(I)})N^{-2r}\|U\|_r^2 + \frac{1}{4}\|\partial_x v\|^2,
\end{aligned}$$

$$\begin{aligned}
|I_4| &= |(F(U) - I_N^C F(U), \partial_x v)| \leq \|F(U) - I_N^C F(U)\|^2 + \frac{1}{4}\|\partial_x v\|^2 \\
&\leq CN^{-2r}\|F(U)\|_r^2 + \frac{1}{4}\|\partial_x v\|^2,
\end{aligned}$$

$$|I_5| = |(e, \partial_x^2 v)| \leq \|e\|_{\omega_{-1,-1}}^2 + \frac{1}{4}\|\partial_x^2 v\|_{\omega_{1,1}}^2,$$

$$|I_6| = |(\eta, \partial_x^2 v)| \leq \|\eta\|_{\omega_{-1,-1}}^2 + \frac{1}{4}\|\partial_x^2 v\|_{\omega_{1,1}}^2 \leq CN^{-2r}\|U\|_r^2 + \frac{1}{4}\|\partial_x^2 v\|_{\omega_{1,1}}^2,$$

For the initial error, we have from (7.5.3) and (7.5.4) that

$$\|e(0)\|_{\omega_{-1,-1}} \leq \|(I_N^C - P_N^3)U_0\|_{\omega_{-1,-1}} \leq CN^{-r}\|U_0\|_r.$$

Just as the proof of stability, we can get the following convergence theorem.

Theorem 7.3 *Assume that $r \geq 3$ and $U \in H^1(0, T; H^r(I) \cap H_0^2(I))$, $F(z) \in C^r(\mathbb{R})$ and $g \in L^2(0, T; H^{r-1}(I))$, then there exists a positive constant C^* such that when N is large enough, we have*

$$\|U(t) - u_N(t)\| \leq \|U(t) - u_N(t)\|_{\omega_{-1,-1}} \leq C^*N^{-r}, \quad t \in (0, T]. \quad (7.6.2)$$

7.7 Conclusion

In this work, we develop the Legendre Petrov-Galerkin method for linear fourth-order equation and the Legendre Petrov-Galerkin and Chebyshev collocation method in space combined with leapfrog/Crank-Nicolson method in time for the Kuramoto-Sivashinsky equation. We present some numerical results which demonstrate the effi-

ciency of proposed schemes, and coincide very well with theoretical analysis. Optimal error estimates in L^2 -norm of the two methods for fourth-order differential equations are also given.

Bibliography

- [1] Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. Number 55. Courier Corporation, 1964.
- [2] Yves Achdou, Christine Bernardi, and Frédéric Coquel. A priori and a posteriori analysis of finite volume discretizations of Darcy’s equations. *Numerische Mathematik*, 96(1):17–42, 2003.
- [3] Robert A Adams and John JF Fournier. *Sobolev spaces*, volume 140. Academic press, 2003.
- [4] Bradley K Alpert and Vladimir Rokhlin. A fast algorithm for the evaluation of legendre expansions. *SIAM Journal on Scientific and Statistical Computing*, 12(1):158–179, 1991.
- [5] Cherif Amrouche, Christine Bernardi, Monique Dauge, and Vivette Girault. Vector potentials in three-dimensional non-smooth domains. *Mathematical Methods in the Applied Sciences*, 21(9):823–864, 1998.
- [6] Mejdı Azaïez, Faker Ben Belgacem, and Christine Bernardi. The mortar spectral element method in domains of operators. part I: The divergence operator and Darcy’s equations. *IMA journal of numerical analysis*, 26(1):131–154, 2006.
- [7] Mejdı Azaïez, Faker Ben Belgacem, Christine Bernardi, and Nejmeddine Chorfi. Spectral discretization of Darcy’s equations with pressure dependent porosity. *Applied Mathematics and Computation*, 217(5):1838–1856, 2010.
- [8] Mejdı Azaïez, Christine Bernardi, and Miloslav Grundmann. Spectral methods applied to porous media. *East- West Journal on Numerical Analysis*, 2:91–105, 1994.
- [9] Faker Ben Belgacem. *Discrétisations 3D non conformes par la méthode de décomposition de domaine des éléments avec joints: Analyse mathématique et mise en œuvre pour le probleme de Poisson*. PhD thesis, L’Université Pierre et Marie Curie, 1993.

- [10] Faker Ben Belgacem. The mortar finite element method with Lagrange multipliers. *Numerische Mathematik*, 84(2):173–197, 1999.
- [11] Faker Ben Belgacem and Christine Bernardi. Spectral element discretization of the Maxwell equations. *Mathematics of Computation of the American Mathematical Society*, 68(228):1497–1520, 1999.
- [12] Faker Ben Belgacem, Cristine Bernardi, Nejmeddine Chorfi, and Yvon Maday. Inf-sup conditions for the mortar spectral element discretization of the Stokes problem. *Numerische Mathematik*, 85(2):257–281, 2000.
- [13] Zakania Belhachmi, Christine Bernardi, and Andreas Karageorghis. Spectral element discretization of the circular driven cavity. part III: The Stokes equations in primitive variables. *Journal of Mathematical Fluid Mechanics*, 5(1):24–69, 2003.
- [14] Zakaria Belhachmi. *Méthodes d'éléments spectraux avec joints pour la résolution de problèmes d'ordre quatre*. PhD thesis, L'Université Pierre et Marie Curie, 1994.
- [15] Jean-Marie Emmanuel Bernard. Density results in Sobolev spaces whose elements vanish on a part of the boundary. *Chinese Annals of Mathematics, Series B*, 32(6):823–846, 2011.
- [16] Christine Bernardi and Nejmeddine Chorfi. Mortar spectral element methods for elliptic equations with discontinuous coefficients. *Mathematical Models and Methods in Applied Sciences*, 12(04):497–524, 2002.
- [17] Christine Bernardi, Giuseppe Coppoletta, and Yvon Maday. Some spectral approximations of two-dimensional fourth-order problems. *Mathematics of Computation*, 59(199):63–76, 1992.
- [18] Christine Bernardi, Monique Dauge, and Yvon Maday. Polynomials in Sobolev spaces and application to the mortar spectral element method. *Preprint*.
- [19] Christine Bernardi, Frédéric Hecht, and Zoubida Mghazli. Mortar finite element discretization for the flow in a nonhomogeneous porous medium. *Computer methods in applied mechanics and engineering*, 196(8):1554–1573, 2007.
- [20] Christine Bernardi and Yvon Maday. Polynomial approximation of some singular functions. *Applicable Analysis*, 42(1-4):1–32, 1991.
- [21] Christine Bernardi and Yvon Maday. Spectral methods. In P. G. Ciarlet and J.-L. Lions, editors, *Handbook of numerical analysis*, volume V. North-Holland, 1997.
- [22] Christine Bernardi and Yvon Maday. Spectral element method and mortar element method in theory and numericals of differential equations. In J. P. Coleman J. F. Blowey and A. W. Craig, editors, *Durham2000*, pages 1–57. Springer, 2001.

- [23] Christine Bernardi and Yvon Maday. Spectral, spectral element and mortar element methods. In *Theory and Numerics of Differential Equations*, pages 1–57. Springer, 2001.
- [24] Christine Bernardi, Yvon Maday, and Anthony T Patera. A new nonconforming approach to domain decomposition: the mortar element method. In H. Brezis and J.-L. Lions, editors, *Collège de France Seminar XI*, pages 13–51. Pitman, 1994.
- [25] Christine Bernardi, Yvon Maday, and Franscesca Rapetti. *Discrétisations variationnelles de problèmes aux limites elliptiques*, volume 45. Springer Science & Business Media, 2004.
- [26] Bernard Bialecki and Andreas Karageorghis. A Legendre spectral collocation method for the biharmonic Dirichlet problem. *ESAIM: Mathematical Modelling and Numerical Analysis*, 34(03):637–662, 2000.
- [27] Bernard Bialecki and Andreas Karageorghis. A Legendre spectral Galerkin method for the biharmonic Dirichlet problem. *SIAM Journal on Scientific Computing*, 22(5):1549–1569, 2001.
- [28] J. M Boland and Roy A Nicolaides. Stability of finite elements under divergence constraints. *SIAM Journal on Numerical Analysis*, 20(4):722–731, 1983.
- [29] Gary Cohen, Patrick Joly, JE Roberts, and N Tordjman. Higher order triangular finite elements with mass lumping for the wave equation. *SIAM Journal on Numerical Analysis*, 38(6):2047–2078, 2001.
- [30] Martin Costabel and Monique Dauge. Un résultat de densité pour les équations de maxwell régularisées dans un domaine lipschitzien. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 327:849–854, 1998.
- [31] Martin Costabel, Monique Dauge, and Serge Nicaise. Singularities of Maxwell interface problems. *ESAIM: Mathematical Modelling and Numerical Analysis*, 33(03):627–649, 1999.
- [32] Mouna Daadaa. *Discrétisation spectrale et par éléments spectraux des équations de Darcy*. PhD thesis, L'Université Pierre et Marie Curie, 2009.
- [33] Henry Darcy. *Les fontaines publiques de la ville de Dijon*. Victor Dalmont, 1856.
- [34] Monique Dauge. Neumann and mixed problems on curvilinear polyhedra. *Integral equations and operator theory*, 15(2):227–261, 1992.
- [35] EH Doha and AH Bhrawy. Efficient spectral-Galerkin algorithms for direct solution of fourth-order differential equations using Jacobi polynomials. *Applied Numerical Mathematics*, 58(8):1224–1244, 2008.

- [36] Moshe Dubiner. Spectral methods on triangles and other domains. *Journal of Scientific Computing*, 6(4):345–390, 1991.
- [37] Elsayed ME Elbarbary. Efficient Chebyshev–Petrov–Galerkin method for solving second-order equations. *Journal of Scientific Computing*, 34(2):113–126, 2008.
- [38] Daniele Funaro. *Polynomial approximation of differential equations*, volume 8. Springer, 1992.
- [39] Daniele Funaro and Wilhelm Heinrichs. Some results about the pseudospectral approximation of one-dimensional fourth-order problems. *Numerische Mathematik*, 58(1):399–418, 1990.
- [40] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms*, volume 5. Springer-Verlag, Berlin, New-York, 1986.
- [41] David Gottlieb and Steven A Orszag. *Numerical analysis of spectral methods*, volume 2. SIAM, 1977.
- [42] Pierre Grisvard. *Boundary Value Problems in Non-Smooth Domains*. Pitman, London, 1985.
- [43] M Yousuff Hussaini and Thomas A Zang. Spectral methods in fluid dynamics. *Annual Review of Fluid Mechanics*, 19(1):339–367, 1987.
- [44] MY Hussaini, A Quarteroni, Th A Zang, and CG Canuto. Spectral methods: Fundamentals in single domains, 2006.
- [45] MS Ismail. Numerical solution of complex modified Korteweg–de Vries equation by Petrov–Galerkin method. *Applied Mathematics and Computation*, 202(2):520–531, 2008.
- [46] Huiyuan Li. *Super spectral viscosity methods for nonlinear conservation laws, Chebyshev collocation methods and their applications*. PhD thesis, Shanghai University, 2001.
- [47] Jian Li, Heping Ma, and Weiwei Sun. Error analysis for solving the Korteweg-de Vries equation by a Legendre pseudo-spectral method. *Numerical Methods for Partial Differential Equations*, 16(6):513–534, 2000.
- [48] Jacques Louis Lions and Enrico Magenes. *Problemes aux limites non homogenes et applications*. Dunod, 1968.
- [49] Heping Ma and Weiwei Sun. A Legendre–Petrov–Galerkin and Chebyshev Collocation Method for third-order differential equations. *SIAM Journal on Numerical Analysis*, 38(5):1425–1438, 2000.

- [50] Heping Ma and Weiwei Sun. Optimal error estimates of the Legendre–Petrov–Galerkin method for the Korteweg–de Vries equation. *SIAM Journal on Numerical Analysis*, 39(4):1380–1394, 2001.
- [51] Steven A Orszag. Comparison of pseudospectral and spectral approximation. *Studies in Applied Mathematics*, 51(3):253, 1972.
- [52] Jie Shen. Efficient spectral-galerkin method I. direct solvers of second-and fourth-order equations using Legendre polynomials. *SIAM Journal on Scientific Computing*, 15(6):1489–1505, 1994.
- [53] Jie Shen. Efficient spectral-galerkin method II. direct solvers of second-and fourth-order equations using Chebyshev polynomials. *SIAM Journal on Scientific Computing*, 16(1):74–87, 1995.
- [54] Ting-Ting Shen, Zhong-Qiang Zhang, and He-Ping Ma. Optimal error estimates of the Legendre tau method for second-order differential equations. *Journal of Scientific Computing*, 42(2):198–215, 2010.
- [55] Spencer Sherwin and George Em Karniadakis. Tetrahedral hp finite elements: Algorithms and flow simulations. *Journal of Computational Physics*, 124(1):14–45, 1996.
- [56] Ikram A Tirmizi and Saadat Ashraf. A class of methods based on non-polynomial spline functions for the solution of a special fourth-order boundary-value problems with engineering applications. *Applied Mathematics and Computation*, 174(2):1169–1180, 2006.
- [57] Marjan Uddin and Sirajul Haq. A mesh-free numerical method for solution of the family of Kuramoto–Sivashinsky equations. *Applied Mathematics and Computation*, 212(2):458–469, 2009.
- [58] Barbara I Wohlmuth. *Discretization Techniques Based on Domain Decomposition*. Springer, 2001.

Résumé

Nous parlons essentiellement dans cette thèse la simulation numérique par la méthode spectrale de l'écoulement stable dans un milieu poreux rigide qui est simulé par les équations de Darcy avec des conditions aux limites générales. La méthode s'avère optimale en ce sens que l'erreur obtenue n'est limitée que par la régularité de la fonction. Un des paramètre dépend de la perméabilité du milieu et, lorsqu'il n'est pas homogène, les variations de ce paramètre peuvent être extrêmement importantes. Pour traiter ce phénomène, nous proposons deux discrétisation différente par éléments spectraux avec joints. Nous donnons des estimations *a priori* de l'erreur et nous confirmons l'étude théorique par des résultats numériques. En outre, nous développons une Lagendre-Petrov-Galerkin méthode pour l'équations différentielles linéaires du 4ème ordre à une dimension, et un Legendre Petrov-Galerkin et Chebyshev collocation méthode pour l'équation non linéaire Kuramoto-Sivashinsky. Nous effectuons l'analyse *a priori* de cette discrétisation et présentons quelques expériences numériques qui confirment les résultats de l'analyse.

Mots-clés: milieu poreux, équations de Darcy, méthode spectrale, éléments spectraux avec joints, estimation d'erreur, équation de Kuramoto-Sivashinsky.

Abstract

We mainly talk about in this thesis the numerical simulation of the steady flow in a rigid porous medium which is simulated by Darcy's equations with a general boundary conditions, by spectral method. The method has been proven optimal in the sense that the order of convergence is only limited by the regularity of the solution. One of the parameters of the system depends on the permeability of the medium and, when this one is not homogeneous, the variations of the parameter could be very high. To handle this phenomenon, we propose two different discretization relies on the mortar spectral method. Both the numerical analysis of the discretization problems are performed and numerical experiments are presented, which turn out to be in good coherency with the theoretical results. In addition, we develop a Legendre Petrov-Galekin method for linear fourth-order differential equations in one dimension and a Legendre Petrov-Galerkin and Chebyshev collocation method for the nonlinear Kuramoto-Sivashinsky equation. The numerical experiments are given which demonstrate the efficient of proposed schemes. Finally, we give the optimal rate of convergence in L^2 -norm of the methods.

Keywords: porous medium, Darcys's equations, spectral method, mortar spectral element method, piecewise continuous coefficients, fourth-order equations, Kuramoto-Sivashinsky equation, Legendre Petrov-Galekin method.

