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de l'Information et de la Communication*
Spécialité : *Mathématiques et leurs interactions*

Par

Angelo ROSELLO

Limites d'échelles pour des modèles cinétiques stochastiques

Thèse présentée et soutenue à Rennes, le 16 juillet 2020
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“ A regular wind-up toy world this is, I think.
Once a day the wind-up bird has to come and wind the springs of this world. ”

Haruki Murakami, The Wind-Up Bird Chronicle

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Introduction

Les *équations cinétiques* sont une classe d'équations aux dérivées partielles issues de la physique statistique, utilisées entre autre en théorie cinétique des gaz, en physique des plasmas ou encore en théorie du transfert radiatif. À une échelle intermédiaire entre *microscopique* (l'échelle des particules) et *macroscopique* (l'échelle de la chambre renfermant le gaz) que d'aucun pourrait qualifier de *mésoscopique*, on cherche à décrire non pas la dynamique individuelle de chacune des particules, mais plutôt le devenir statistique de l'ensemble, au travers d'une équation évolutive portant sur la fonction de distribution probabiliste des particules. En 1872, le physicien autrichien Ludwig Boltzmann généralise les travaux de l'écossais James Clerk Maxwell portant sur l'équilibre thermique des gaz, en proposant un modèle régissant un système non-homogène et en évolution : désignant par $f(t, x, v)$ la densité de probabilité qu'une particule, au temps t , se situe à la position x et admette la vitesse v , l'*équation de Boltzmann* s'écrit sous la forme générale

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = C[f],$$

où F désigne le champ des forces extérieures appliquées au gaz et $C[f]$ est un opérateur rendant compte de la collision entre deux particules. Dans le cadre de la physique des plasmas, où les collisions binaires sont bien souvent négligeables, on est ramené à l'*équation de Vlasov* qui consiste en un transport libre :

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = 0,$$

où $F = E + B \times v$ rend compte des forces issues du champ électromagnétique extérieur et induit par les particules.

Pour de tels modèles intermédiaires, à l'interface entre microscopique et macroscopique, la question des limites d'échelles se pose naturellement : comment et en quel sens s'effectue la transition entre les différents niveaux de description ? D'une part, on peut s'intéresser à l'obtention d'une description cinétique continue à partir du système de particules discret sous-jacent, dans la limite où le nombre de particules tend vers l'infini. D'autre part, à grandes échelles de temps et d'espace, on peut examiner le passage d'une équation cinétique, portant sur une distribution statistique, à une équation fluide décrivant l'évolution de quantités macroscopiques.

Du mésoscopique au macroscopique

Commençons par aborder la transition vers les équations fluides. Il est à noter que l'espace des phases d'une équation cinétique est relativement grand, si bien que, dans la pratique, les descriptions cinétiques nécessitent des calculs extrêmement coûteux. Pour décrire le comportement d'un fluide observé à l'échelle macroscopique, on a donc tout intérêt à remplacer ce modèle cinétique par un modèle *hydrodynamique*, mettant en jeu les quantités intégrées

$$\rho(t, x) = \int f(t, x, v) dv, \quad u(t, x) = \frac{\int v f(t, x, v) dv}{\rho(t, x)}$$

qui correspondent à la densité et à la vitesse moyenne du fluide en un point. Pour étudier cette transition d'un régime à l'autre, on effectue un changement d'échelle de temps et d'espace en posant

$$f^\varepsilon(t, x, v) = f(t/\tau(\varepsilon), x/\varepsilon, v)$$

où $\tau(\varepsilon) \rightarrow 0$ lorsque $\varepsilon \rightarrow 0$. Si f est régie par l'équation de Boltzmann, la quantité f^ε vérifie ainsi

$$\tau(\varepsilon)\partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon + F \cdot \nabla_v f^\varepsilon = C[f^\varepsilon]$$

et l'on cherche à déterminer l'équation limite satisfaite par ρ^ε (et/ou u^ε) lorsque $\varepsilon \rightarrow 0$: on parle alors de limite hydrodynamique. De nombreux travaux se sont penchés sur cette asymptotique : ainsi, selon le régime étudié – et notamment en fonction du ratio $\tau(\varepsilon)/\varepsilon$ considéré – l'équation de Boltzmann conduira par exemple aux équations fluides d'Euler, Stokes ou Navier-Stokes, compressible ou incompressible. Nombre de ces limites hydrodynamiques sont par exemple présentées dans [Vil01]. À titre d'illustration, considérons ici l'équation de Boltzmann sans force extérieure, avec un opérateur de collision linéaire

$$C[f] = L[f] := \rho M - f,$$

où $M(v) = (2\pi)^{-d/2} \exp(-|v|^2/2)$ désigne la distribution Maxwellienne des vitesses. La renormalisation temporelle $\tau(\varepsilon) = \varepsilon^2$ conduit à l'équation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} L[f^\varepsilon]. \quad (0.1)$$

Déterminons heuristiquement la limite hydrodynamique associée. En appliquant la méthode du développement de Hilbert, c'est à dire en posant l'Ansatz

$$f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + O(\varepsilon^3) \quad (0.2)$$

et en identifiant les termes de même ordre en ε dans (0.1), on obtient le système d'équations

$$L[f_0] = 0, \quad (0.3)$$

$$L[f_1] = v \cdot \nabla_x f_0, \quad (0.4)$$

$$L[f_2] = \partial_t f_0 + v \cdot \nabla_x f_1. \quad (0.5)$$

On montre facilement que l'équation $L[f] = g$ d'inconnue f admet des solutions si et seulement si $\int g(x, v) dv = 0$, auquel cas ces solutions sont de la forme $f(x, v) = -g(x, v) + q(x)M(v)$.

L'équation (0.3) exprime le fait que f^ε converge vers une distribution f_0 qui présente un équilibre Maxwellien des vitesses, c'est à dire de la forme

$$f_0(t, x, v) = \rho(t, x)M(v).$$

Il en résulte immédiatement que la condition $\int v \cdot \nabla_x f_0 dv = 0$ est satisfaite, si bien qu'une solution f_1 de (0.4) est donnée par

$$f_1(t, x, v) = -v \cdot \nabla_x f_0(t, x, v) + q(t, x)M(v).$$

Enfin, en intégrant l'équation (0.5) en vitesse, on obtient

$$\int (\partial_t f_0 + v \cdot \nabla_x f_1) dv = 0,$$

ce qui, à l'aide des expressions précédentes, se réécrit

$$\partial_t \rho - \nabla_x \cdot (K \nabla_x \rho) = 0, \quad (0.6)$$

où la matrice définie positive K est donnée par

$$K = \int (v \otimes v) M(v) dv.$$

Cette analyse suggère ainsi que, dans la limite $\varepsilon \rightarrow 0$, la quantité $f^\varepsilon(t, x, v)$ solution de l'équation cinétique (0.1) peut être approchée par $\rho(t, x)M(v)$, où ρ est solution de l'équation diffusive (0.6). En ce sens, on parle parfois d'*approximation-diffusion*. Au passage, notons que l'essentiel du raisonnement que l'on vient de développer repose sur notre capacité à résoudre rigoureusement le problème inverse $L[f] = g$ associé à l'opérateur L . Une condition nécessaire (et suffisante ici) pour garantir que $g \in \text{Im}(L)$ s'exprime comme l'annulation de l'intégrale $\int g dv$, et on sait alors définir (explicitement ici) un antécédent f en fonction de g . Nous reviendrons sur le caractère central de ce genre de problème inverse un peu plus tard.

Du microscopique au mésoscopique

Évoquons à present le passage d'un modèle discret, microscopique, à une EDP cinétique. À cette fin, considérons une équation de Vlasov non-linéaire, où le champ de force F dépend (non-localement) de la configuration du système f :

$$\partial_t f + v \cdot \nabla_x f + F[f](x) \cdot \nabla_v f = 0. \quad (0.7)$$

On peut penser au cas hamiltonien où la force résulte d'un potentiel d'interaction $V(x)$, soit

$$F[f](x) = -(\nabla V * \rho)(x) = - \int \int \nabla V(x - x') f(x', v') dx' dv'.$$

La dynamique d'un tel système peut être décrite naturellement d'un point de vue microscopique en considérant le système de particules en interaction associé à (0.7). Pour $i \in \{1, \dots, N\}$, notant $X_t^{i,N}, V_t^{i,N}$ la position et vitesse de la i -ème particule au temps t , la dynamique newtonienne conduit au système différentiel ordinaire

$$\begin{cases} \frac{d}{dt} X_t^{i,N} = V_t^{i,N}, \\ \frac{d}{dt} V_t^{i,N} = -\frac{1}{N} \sum_{j=1}^N \nabla V(X_t^{i,N} - X_t^{j,N}). \end{cases}$$

En introduisant la mesure empirique

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{i,N}, V_t^{i,N})} \in \mathcal{P}(\mathbb{R}^{2d}),$$

l'équation régissant la vitesse peut se mettre sous la forme

$$\frac{d}{dt} V_t^{i,N} = F[\mu_t^N](X_t^{i,N}), \text{ avec } F[\mu](x) = - \int \int \nabla V(x - x') \mu(dx' dv').$$

Ceci vient mettre en évidence le fait que l'interaction entre les particules est de type *champ moyen*. Les particules jouent un rôle symétrique : chacune d'entre elle interagit avec la "moyenne" de toutes les autres, au travers de la mesure empirique. Cette dernière, par construction, satisfait l'équation (0.7) au sens des distributions : pour tout $N \geq 1$ fixé,

$$\partial_t \mu^N + v \cdot \nabla_x \mu^N + F[\mu^N] \cdot \nabla_v \mu^N = 0.$$

Par souci de simplicité, on se permet dans la suite de maintenir un voile sur le sens précis attribué aux différentes notions de convergence. Lorsque $N \rightarrow \infty$, pour peu que la distribution initiale des particules μ_0^N converge vers une distribution continue $\mu_0(dx dv) = f_0(x, v) dx dv$, on montre typiquement la convergence de la mesure empirique μ_t^N vers une distribution de densité $f(t, x, v)$, solution de l'équation de Vlasov (0.7). On parle alors de *limite de champ moyen*. Quant à la trajectoire individuelle des particules, on déduit aisément que pour tout $i \geq 1$ fixé, pour peu qu'initialement

$$(X_0^{i,N}, V_0^{i,N}) \xrightarrow{N \rightarrow \infty} z := (x, v),$$

pour tout $t \geq 0$, on a la convergence

$$(X_t^{i,N}, V_t^{i,N}) \xrightarrow{N \rightarrow \infty} Z_t^f(z)$$

où $Z_t^f(z) = (X_t^f(z), V_t^f(z))$ désigne le flot des caractéristiques

$$\begin{cases} \frac{d}{dt} X_t^f(z) = V_t^f(z), \\ \frac{d}{dt} V_t^f(z) = F[f_t](X_t^f(z)), \end{cases} \text{ avec } \begin{cases} X_0^f(z) = x, \\ V_0^f(z) = v. \end{cases} \quad (0.8)$$

On parle ici de caractéristiques non-linéaires, au sens où les coefficients de l'équation différentielle (0.8) dépendent de la solution $f_t = \lim_{N \rightarrow \infty} \mu_t^N$ de (0.7), qui est elle-même donnée par le transport de la donnée initiale par le flot :

$$f_t = (Z_t^f)^* f_0, \quad (0.9)$$

c'est à dire, pour tout $\phi \in C_b(\mathbb{R}^{2d})$,

$$\int \phi(z) f_t(z) dz = \int \phi(Z_t^f(z)) f_0(z) dz.$$

En ce sens, à z fixé, l'équation (0.8) vérifiée par $Z_t^f(z)$ dépend des trajectoires de toutes les particules $\{Z_t^f(z'), z' \in \mathbb{R}^{2d}\}$, et l'égalité (0.9) s'interprète comme un point fixe. Ces résultats de convergence de champ moyen peuvent être aisément établis dans le cas élémentaire où le champ de force ∇V est globalement Lipschitzien.

Dans le même esprit, on peut considérer des équations de Vlasov avec un terme diffusif :

$$\partial_t f + v \cdot \nabla_x f + F[f](x) \cdot \nabla_v f = \frac{\sigma^2}{2} \Delta_v f. \quad (0.10)$$

Il est alors pertinent d'interpréter (0.10) comme une équation de Fokker-Planck non-linéaire, en introduisant le processus stochastique $Z_t^f = (X_t^f, V_t^f)$ régi par

$$\begin{cases} dX_t^f = V_t^f dt, \\ dV_t^f = F[f_t](X_t^f) dt + \sigma dB_t \end{cases} \quad \text{avec } Z_0^f = \xi_0 \sim f_0 \quad (0.11)$$

où $\xi_0 \in \mathbb{R}^{2d}$ est une variable aléatoire de loi f_0 et $(B_t)_{t \geq 0}$ est un mouvement Brownien dans \mathbb{R}^d indépendant de ξ_0 . Dans ce cadre, le point fixe (0.9) est naturellement remplacé par

$$f_t = \mathcal{L}(Z_t^f),$$

où $\mathcal{L}(Z_t^f)$ désigne la loi de la variable aléatoire Z_t^f . Une diffusion "non-linéaire" $(Z_t)_{t \geq 0}$ satisfaisant (0.11) avec $f_t = \mathcal{L}(Z_t)$ s'appelle un processus de McKean-Vlasov. Considérons maintenant le système de particules stochastique $Z^{i,N} = (X^{i,N}, V^{i,N})$ donné par

$$\begin{cases} dX_t^{i,N} = V_t^{i,N} dt, \\ dV_t^{i,N} = F[\mu_t^N](X_t^{i,N}) dt + \sigma dB_t^i, \end{cases} \quad \text{avec } Z_0^{i,N} = \xi_0^i \sim f_0, \quad (0.12)$$

où $(\xi_0^i)_{i \geq 1}$ est une famille de variables aléatoires i.i.d de loi f_0 et les $(B^i)_{i \geq 1}$ sont des mouvements Browniens mutuellement indépendants, et indépendants de $(\xi_0^i)_{i \geq 1}$. On peut à nouveau établir un résultat de champ moyen : lorsque $\mu_0^N(dx dv) \rightarrow f_0(x, v) dx dv$, la mesure empirique (aléatoire cette fois) μ_t^N converge vers une mesure à densité f_t satisfaisant (0.10). Concernant les trajectoires des particules, pour tout $i \geq 1$ fixé, on a la convergence

$$Z_t^{i,N} = (X_t^{i,N}, V_t^{i,N}) \xrightarrow{N \rightarrow \infty} Z_t^i,$$

où $(Z_t^i)_{t \geq 0}$ est le processus de McKean-Vlasov satisfaisant (0.11) avec la condition initiale $Z_0^i = \xi_0^i$. Plus précisément, ce contexte est le cadre typique d'apparition du phénomène de *propagation du chaos*. S'agissant d'un système de particules en interaction, il est clair que deux particules $Z_t^{i,N}$, $Z_t^{j,N}$ pour $i \neq j$ sont corrélées dès lors que $t > 0$. Cependant, dans la limite où le nombre de particules est grand, les corrélations fini-dimensionnelles disparaissent : si l'on observe un nombre fixé de particules, tout se passe comme si elles évoluaient de façon indépendante, en suivant chacune la dynamique dictée par le processus McKean-Vlasov (0.11). Ce résultat peut s'exprimer comme suit : pour tout $r \geq 1$ et $\phi_1, \dots, \phi_r \in C_b(\mathbb{R}^{2d})$,

$$\mathbb{E} \left[\prod_{i=1}^r \phi_i(Z_t^{i,N}) \right] \xrightarrow{N \rightarrow \infty} \prod_{i=1}^r \mathbb{E} \left[\phi_i(Z_t^i) \right] = \prod_{i=1}^r \left(\int \phi_i(z) f_t(z) dz \right),$$

où $f_t = \mathcal{L}(Z_t^i)$ satisfait l'équation (0.10). Encore une fois, ces résultats sont classiquement établis dans le cadre où ∇V est globalement Lipschitzien. On pourra par exemple se référer à [Szn91].

Les processus de McKean-Vlasov constituent ainsi une première inclusion naturelle d'aléa au sein d'un modèle cinétique, afin de faire apparaître un terme additionnel diffusif. Il est cependant à noter que, conformément à l'intuition donnée par la loi des grands nombres, l'aléa introduit dans le système de particules (0.12) disparaît à la limite par moyennisation, si bien que l'EDP (0.10) reste déterministe. L'introduction d'un bruit de nature aléatoire au niveau de l'équation cinétique elle-même est pourtant riche de sens, tant mathématiquement que du point de vue de la modélisation, afin de rendre compte d'incertitudes diverses.

Cette thèse se propose donc d'étudier quelques transitions d'échelles pour des modèles cinétiques bruités, où l'objet limite est encore de nature aléatoire, régi par une *équation aux dérivées partielles stochastique* (EDPS). Dans le chapitre 1, on s'intéresse au passage du mésoscopique au macroscopique, en considérant un système d'équations cinétique (modélisant le comportement d'un "spray" dans un fluide ambiant) perturbé par un processus de Markov mélangeant. Après un changement d'échelle approprié, on montre la convergence de la densité $\rho = \int f dv$ vers une limite hydrodynamique, qui s'exprime comme une équation de conservation stochastique. Dans un second temps, on étudie des équations cinétiques stochastiques correspondant à des modèles biologiques de mouvement collectif. Cette étude se divise en deux travaux qui se consacrent à deux modèles distincts. Dans le chapitre 2, on s'intéresse à la limite de champ moyen du système de particules stochastique associé à des perturbations aléatoires du modèle classique de Cucker-Smale. Dans le chapitre 3, on étudie un modèle plus fin, rendant compte d'une interaction locale entre les individus, pour lequel on démontre l'existence de solutions martingales.

Contenu du chapitre 1 : Limite hydrodynamique pour un modèle cinétique de spray perturbé par un bruit markovien mélangeant

Dans ce premier chapitre, on considère un modèle-jouet visant à décrire le comportement d'un "spray" de molécules immergées dans un fluide ambiant : on peut penser à des molécules de parfum vaporisées dans l'air, ou encore à des particules de polluant dans un cours d'eau. On désigne par $f \equiv f(t, x, v)$ la densité cinétique associée aux particules composant le spray, dont l'évolution est donnée par l'équation de Vlasov (sous forme conservative)

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F(t, x, v)f) = 0,$$

où le terme de force

$$F(t, x, v) = (u(t, x) - v) + m(t, \varepsilon x)$$

exprime l'alignement de la vitesse d'une particule avec la vitesse $u(t, x)$ du fluide ambiant. Cet alignement est perturbé au cours du temps par un terme de force aléatoire, modélisé par un processus de Markov centré à valeurs fonctionnelles $m = (m(t, x))_{t \geq 0}$. On suppose de plus que l'influence du spray sur le fluide ambiant est très faible, si bien que l'équation régissant ce dernier est donnée par

$$\partial_t u - \Delta_x u = \varepsilon^2 \int (v - u) f dv.$$

On s'intéresse alors au changement d'échelle

$$f^\varepsilon(t, x, v) = f(t/\varepsilon^2, x/\varepsilon, v), \quad u^\varepsilon(t, x) = \varepsilon^{-1} u(t/\varepsilon^2, x/\varepsilon), \quad m^\varepsilon(t, x) = m(t/\varepsilon^2, x),$$

ce qui nous amène au système d'équations

$$\begin{cases} \partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon + \frac{1}{\varepsilon^2} \nabla_v \cdot [(\varepsilon u^\varepsilon - v + m^\varepsilon) f^\varepsilon] = 0, \\ \partial_t u^\varepsilon - \Delta_x u^\varepsilon = \int (v - \varepsilon u^\varepsilon) f^\varepsilon dv. \end{cases} \quad (0.13)$$

La première étape est bien-sûr de montrer l'existence et l'unicité d'une solution de (0.13) à ε fixé. À cette fin, on cherche à résoudre l'équation de transport sur f^ε par la méthode des caractéristiques,

ce qui nécessite que la fonction u^ε correspondante soit Lipschitzienne. On doit dans ce cas se restreindre à la dimension 1 ($x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ et $v \in \mathbb{R}$) et imposer la condition suivante sur les données initiales, pour $\varepsilon > 0$ fixé : $f_0^\varepsilon \geq 0$ avec $\int_x \int_v f_0^\varepsilon(x, v) dx dv = 1$, et

$$\int_x \int_v (1 + |v|^4) |f_0^\varepsilon|^2 dx dv + \int_x \int_v (1 + |v|) |\nabla_{x,v} f_0^\varepsilon| dx dv < \infty, \quad u_0^\varepsilon \in H_x^\eta \text{ pour } \eta > 3/2. \quad (0.14)$$

En ce qui concerne le processus de Markov $m = (m_t)_{t \geq 0}$ dirigeant l'équation (0.13), pour des raisons similaires, il est raisonnable de considérer qu'il est à valeurs Lipschitziennes. Plus précisément, son espace d'état E , muni d'une norme $\|\cdot\|_E$, est supposé complet, séparable et borné :

$$m_t \in E \subset W_x^{2,\infty}, \quad \|\cdot\|_{W_x^{2,\infty}} \lesssim \|\cdot\|_E, \quad \sup_{n \in E} \|n\|_E < \infty.$$

On suppose que le processus $m = (m_t)_{t \geq 0}$ possède une mesure stationnaire $\nu \in \mathcal{P}(E)$ centrée :

$$\forall h \in C(\mathbb{T}), \quad \int \langle h, n \rangle d\nu(n) = 0.$$

L'hypothèse de mélange s'exprime de la manière suivante : pour tout $n \in E$, il existe un couplage càdlàg $(m_t^*(n), \tilde{m}_t^*)_{t \geq 0}$ dont les distributions marginales sont celles du processus m avec les lois initiales respectives δ_n et ν , et tel que

$$\forall t \geq 0, \quad \mathbb{E}[\|m_t^*(n) - \tilde{m}_t^*\|_E] \leq \gamma_{mix}(t). \quad (0.15)$$

La fonction γ_{mix} , qui caractérise la vitesse de mélange, est supposée intégrable en temps. On peut typiquement penser à une chaîne de Markov à temps continu $(m_t)_{t \geq 0}$ irréductible et récurrente positive, à valeurs dans un espace d'état dénombrable E constitué de fonctions régulières. La propriété de mélange (0.15) sera par exemple satisfaite si l'espace d'état E est fini, ou encore si les sauts de $(m_t)_{t \geq 0}$ sont tirés indépendamment selon la loi ν .

On démontre le résultat de convergence suivant pour la quantité intégrée $\rho^\varepsilon = \int f^\varepsilon dv$.

Théorème 1. *Supposons que les données initiales $f_0^\varepsilon, u_0^\varepsilon$ satisfont (0.14), la convergence*

$$\rho_0^\varepsilon \rightarrow \rho_0 \text{ dans } L_x^1, \quad u_0^\varepsilon \rightarrow u_0 \text{ dans } H_x^\alpha \text{ pour un } \alpha \in [1/2, 3/2],$$

ainsi que la borne uniforme

$$\sup_{\varepsilon > 0} \int_x \int_v |v|^3 f_0^\varepsilon dx dv < \infty.$$

Alors, pour tout $\gamma > 1/2$ et $\beta < \alpha$, il existe une sous-suite $(\rho^{\varepsilon_k}, u^{\varepsilon_k})_{k \geq 1}$ telle que

$$(\rho^{\varepsilon_k}, u^{\varepsilon_k}) \rightarrow (\rho, u) \text{ en loi dans } C([0, T]; H_x^{-\gamma} \times H_x^\beta)$$

où (ρ, u) est une solution faible (au sens probabiliste, c'est à dire du problème de martingale) de

$$\begin{cases} d\rho + \partial_x[\rho(u - a)]dt + \partial_x[\rho Q^{1/2} \circ dW_t] = 0, \\ \partial_t u - \partial_x^2 u = 0. \end{cases} \quad (0.16)$$

Dans (0.16), $W = (W_t)_{t \geq 0}$ désigne un processus de Wiener cylindrique dans L_x^2 . Le coefficient $a \equiv a(x)$ et l'opérateur de trace Q sur L_x^2 sont déterminés explicitement à partir de la loi du processus de Markov $m = (m_t)_{t \geq 0}$.

Ainsi, dans la limite $\varepsilon \rightarrow 0$, le processus aléatoire complexe $(\rho^\varepsilon)_{t \geq 0}$, dirigé par un processus de Markov relativement général, peut être approché par un processus de diffusion $(\rho_t)_{t \geq 0}$, dont le drift et la covariance sont entièrement déterminés par la loi de $(m_t)_{t \geq 0}$, ou plus précisément du processus stationnaire $(\tilde{m}_t)_{t \in \mathbb{R}}$ associé (de loi initiale $\tilde{m}_0 \sim \nu$). Dans cet esprit, on pourrait avancer que le Théorème 1 constitue un résultat d'"approximation-diffusion" en un sens probabiliste.

L'essentiel de la preuve de ce théorème est basé sur la méthode de la fonction test perturbée introduite par [GPV77] et développée dans [FGPS07]. Ce travail s'inscrit dans la continuité des résultats développés notamment dans [DV12, DDMV16, DV17] où la méthode de la fonction test

perturbée est exploitée pour établir des limites hydrodynamiques stochastiques. Une des spécificités du modèle considéré ici est la forme "transport libre", c'est à dire l'absence de terme de collision dans (0.13). En conséquence, le processus limite satisfait une équation de conservation stochastique (sous forme Stratonovich) plutôt qu'une EDPS diffusive. Par ailleurs, en contraste avec les travaux précédents, une attention particulière est portée à la construction rigoureuse des correcteurs impliqués dans la méthode de la fonction test perturbée, en exploitant l'hypothèse de mélange du processus de Markov m . Afin de souligner ce dernier point, donnons maintenant les grandes lignes de la preuve du Théorème 1.

À partir de l'équation (0.13), on détermine aisément le générateur infinitésimal \mathcal{L}^ε du processus de Markov $(f_t^\varepsilon, u_t^\varepsilon, m_t^\varepsilon)_{t \geq 0}$. On le décompose selon les différents ordres en ε en posant

$$\forall \phi \equiv \phi(f, u, n), \quad \mathcal{L}^\varepsilon \phi = \frac{1}{\varepsilon^2} \mathcal{L}_\# \phi + \frac{1}{\varepsilon} \mathcal{L}_\flat \phi + \mathcal{L}_0 \phi + \varepsilon \mathcal{L}_1 \phi. \quad (0.17)$$

Puisque la loi d'un processus de Markov est essentiellement déterminée par son générateur infinitésimal, afin de dégager l'équation limite satisfaite par la loi de $(\rho^\varepsilon, u^\varepsilon)$, il convient d'étudier $\mathcal{L}^\varepsilon \phi(f, u, n)$ pour une fonction test de la forme

$$\phi(f, u, n) = \phi(f, u) = \phi(\rho, u).$$

Typiquement, il est suffisant de considérer, pour $\xi, \zeta \in C^\infty(\mathbb{T}_x)$,

$$\phi(f, u) = \langle \rho, \xi \rangle + \langle u, \zeta \rangle \quad \text{et} \quad \phi(f, u) = |\langle \rho, \xi \rangle|^2 + |\langle u, \zeta \rangle|^2.$$

La méthode de la fonction test perturbée apparaît alors en quelque sorte comme une version duale de la méthode de Hilbert (0.2) : au lieu de chercher un développement de la solution f_t^ε en puissance de ε , on vient perturber la fonction test ϕ en

$$\phi^\varepsilon(f, u, n) = \phi(\rho, u) + \varepsilon \phi_1(f, u, n) + \varepsilon^2 \phi_2(f, u, n).$$

Lorsque $\varepsilon \rightarrow 0$, on a d'une part $\phi^\varepsilon = \phi + O(\varepsilon)$, et d'autre part on cherche à avoir

$$\mathcal{L}^\varepsilon \phi^\varepsilon(f, u, n) = \mathcal{L} \phi(\rho, u) + O(\varepsilon) \quad (0.18)$$

pour des correcteurs ϕ_1 et ϕ_2 bien choisis. L'opérateur \mathcal{L} s'interprète alors comme le générateur infinitésimal du processus $(\rho_t, u_t)_{t \geq 0}$ limite. Identifiant les ordres en ε , on cherche à résoudre le système d'équations

$$\mathcal{L}_\# \phi = 0, \quad (0.19)$$

$$\mathcal{L}_\# \phi_1 = -\mathcal{L}_\flat \phi, \quad (0.20)$$

$$\mathcal{L}_\# \phi_2 = \mathcal{L} \phi - \mathcal{L}_\flat \phi_1 - \mathcal{L}_0 \phi. \quad (0.21)$$

Comme dans le cadre de la méthode de Hilbert présentée précédemment dans un contexte déterministe, il apparaît clairement que l'identification du générateur limite $\mathcal{L} \phi$ et la construction des correcteurs ϕ_1 et ϕ_2 sont intimement liées à la résolution de problèmes inverses de la forme

$$\mathcal{L}_\# \psi = \theta, \quad (0.22)$$

d'inconnue ψ . Il importe donc d'étudier l'opérateur $\mathcal{L}_\#$, qui est naturellement le terme dominant dans le développement (0.17) de \mathcal{L}^ε . À cette fin, on introduit un processus de Markov auxiliaire $(g_t, m_t)_{t \geq 0}$ dont le générateur infinitésimal est précisément donné par $\mathcal{L}_\#$. Cela revient essentiellement à ne conserver que les termes d'ordre ε^{-2} dans (0.13), de sorte que l'évolution d'une trajectoire $(g_t(f, n), m_t(n))_{t \geq 0}$ issue de (f, n) est donnée par

$$\begin{cases} \partial_t g + \partial_v [(-v + m)g] = 0, \\ g_0 = f, \quad m_0 = n. \end{cases}$$

La résolution de l'équation des caractéristiques conduit à l'expression explicite

$$g_t(f, n)(x, v) = e^t f\left(x, e^t[v - w_t(n)(x)]\right), \quad \text{avec} \quad w_t(n) = \int_0^t e^{-s} m_{t-s}(n) ds.$$

On détermine alors une famille de mesure stationnaires $\{\mu_\rho, \rho \in L_x^1\}$ pour le processus auxiliaire : la densité intégrée

$$\rho(g_t(f, n)) := \int g_t(f, n) dv = \int f dv = \rho$$

étant un invariant de l'équation, il est naturel que cette famille soit indexée par ρ . On définit (partiellement) la mesure $\mu_\rho \in \mathcal{P}(\mathbb{T}_x \times \mathbb{R}_v \times E)$ par

$$\forall \theta \equiv \theta(f, n), \quad \langle \langle \theta, \mu_\rho \rangle \rangle := \int \theta(f, n) d\mu_\rho(f, n) = \mathbb{E} \left[\theta(\rho \otimes \delta_{\tilde{w}}, \tilde{m}_0) \right] \quad (0.23)$$

quand cette expression fait sens. Dans (0.23), $(\tilde{m}_t)_{t \in \mathbb{R}}$ désigne le processus stationnaire associé à m , et la mesure $\rho \otimes \delta_{\tilde{w}}$ est définie par

$$\forall \xi \in C_b(\mathbb{T}_x \times \mathbb{R}_v), \quad \langle \xi, \rho \otimes \delta_{\tilde{w}} \rangle = \int_{\mathbb{T}_x} \xi(x, \tilde{w}(x)) \rho(x) dx,$$

où la vitesse

$$\tilde{w} = \int_0^\infty e^{-s} \tilde{m}_{-s} ds$$

correspond à la limite en loi de $w_t(n)$ en temps long. En exploitant l'hypothèse de mélange (0.15) faite sur le processus $(m_t)_{t \geq 0}$, on établit le résultat suivant : notant $(Q_t)_{t \geq 0}$ le semi-groupe du processus $(g_t, m_t)_{t \geq 0}$, on a en temps long, pour un large panel de fonctions test $\theta \equiv \theta(f, n)$,

$$Q_t \theta(f, n) \xrightarrow[t \rightarrow \infty]{} \langle \langle \theta, \mu_\rho \rangle \rangle \quad \text{où } \rho = \int f dv$$

avec la vitesse de convergence intégrable

$$\begin{aligned} \left| Q_t \theta(f, n) - \langle \langle \theta, \mu_\rho \rangle \rangle \right| &\leq C \left(\int \int (1 + |v|^2) f dx dv \right) r(t), \\ \left| D_f \left[Q_t \theta(f, n) - \langle \langle \theta, \mu_\rho \rangle \rangle \right] (x, v) \right| &\leq C(1 + |v|^2) r(t), \\ \left| \nabla_{x,v} D_f \left[Q_t \theta(f, n) - \langle \langle \theta, \mu_\rho \rangle \rangle \right] (x, v) \right| &\leq C(1 + |v|^2) r(t), \end{aligned}$$

où $r \equiv r(t) \in L^1(\mathbb{R}_+)$. Revenant au problème inverse (0.22), une condition nécessaire à l'existence de solution est

$$\forall \rho, \quad \langle \langle \theta, \mu_\rho \rangle \rangle = 0.$$

Lorsque cette condition est satisfaite, pour peu que la fonction test θ soit convenable, le résultat de convergence précédent garantit que

$$\psi(f, n) = - \int_0^\infty Q_t \theta(f, n) dt.$$

est bien défini et dans le domaine de l'opérateur $\mathcal{L}_\#$. On montre qu'il s'agit alors d'une solution de (0.22) : c'est ainsi l'examen précis du comportement en temps long du processus auxiliaire $(g_t, m_t)_{t \geq 0}$, reposant sur l'hypothèse de mélange (0.15), qui nous permet de construire rigoureusement les correcteurs ϕ_1 et ϕ_2 .

Puisque $\phi(f, u) = \phi(\rho, u)$ et que ρ est un invariant du processus auxiliaire, l'équation (0.19) est satisfaite. On montre que $\langle \langle \mathcal{L}_b \phi, \mu_\rho \rangle \rangle = 0$, si bien que l'on peut définir (explicitement) un premier correcteur $\phi_1 \equiv \phi_1(f, u, n)$ solution de (0.20). Par suite, la condition nécessaire d'existence d'une solution ϕ_2 à (0.21) impose la forme suivante pour le générateur limite :

$$\mathcal{L} \phi(\rho, u) = \langle \langle \mathcal{L}_b \phi_1(u), \mu_\rho \rangle \rangle + \langle \langle \mathcal{L}_0 \phi(u), \mu_\rho \rangle \rangle.$$

Le calcul montre que la première contribution peut se réécrire

$$\langle \langle \mathcal{L}_b \phi_1(u), \mu_\rho \rangle \rangle = \left\langle \mathcal{A} \rho, D_f \phi(\rho, u) \right\rangle + \frac{1}{2} \text{Tr} \left(\partial_x \left[\rho Q^{1/2} \right]^* D_f^2 \phi(\rho, u) \partial_x \left[\rho Q^{1/2} \right] \right),$$

où \mathcal{A} est un opérateur différentiel d'ordre 2, ce qui correspond à l'EPDS limite (0.16) écrite sous forme Itô. Quant à la seconde contribution, il s'agit simplement de

$$\langle \langle \mathcal{L}_0 \phi(u), \mu_\rho \rangle \rangle = \langle \partial_x^2 u, D_u \phi(\rho, u) \rangle,$$

ce qui correspond à l'équation de la chaleur déterministe. Par suite, on peut définir rigoureusement le second correcteur ϕ_2 solution de (0.21).

Pour tout $\varepsilon > 0$ fixé, le problème de martingale associé au générateur \mathcal{L}^ε du processus $(f_t^\varepsilon, u_t^\varepsilon, m_t^\varepsilon)_{t \geq 0}$ peut s'exprimer de la manière suivante : pour tout $s_1 < \dots < s_n = s < t$

$$\mathbb{E} \left[\left(\phi^\varepsilon(f_t^\varepsilon, u_t^\varepsilon, m_t^\varepsilon) - \phi^\varepsilon(f_s^\varepsilon, u_s^\varepsilon, m_s^\varepsilon) - \int_s^t \mathcal{L}^\varepsilon \phi^\varepsilon(f_\sigma^\varepsilon, u_\sigma^\varepsilon, m_\sigma^\varepsilon) d\sigma \right) H^\varepsilon \right] = 0,$$

où $H^\varepsilon = h(\rho_{s_1}^\varepsilon, u_{s_1}^\varepsilon, \dots, \rho_{s_n}^\varepsilon, u_{s_n}^\varepsilon)$ avec $h : (H_x^{-\gamma} \times H_x^\beta)^n \rightarrow \mathbb{R}$ continu et borné. Par construction de la perturbation ϕ^ε , on déduit de l'égalité (0.18) que

$$\mathbb{E} \left[\left(\phi(\rho_t^\varepsilon, u_t^\varepsilon) - \phi(\rho_s^\varepsilon, u_s^\varepsilon) - \int_s^t \mathcal{L} \phi(\rho_\sigma^\varepsilon, u_\sigma^\varepsilon) d\sigma \right) H^\varepsilon \right] = O(\varepsilon).$$

Après avoir montré la tension des lois de $(\rho^\varepsilon, u^\varepsilon)$ dans l'espace $C([0, T]; H_x^{-\gamma} \times H_x^\beta)$ (pour $\gamma > 1/2$ et $\beta < \alpha$) on utilise le théorème de représentation de Skorokhod pour passer à la limite dans l'égalité précédente et obtenir

$$\mathbb{E} \left[\left(\phi(\rho_t, u_t) - \phi(\rho_s, u_s) - \int_s^t \mathcal{L} \phi(\rho_\sigma, u_\sigma) d\sigma \right) H \right] = 0,$$

où $H = h(\rho_{s_1}, u_{s_1}, \dots, \rho_{s_n}, u_{s_n})$. On en déduit que le processus

$$N_\phi(t) = \phi(\rho_t, u_t) - \phi(\rho_0, u_0) - \int_0^t \mathcal{L} \phi(\rho_s, u_s) ds, \quad t \geq 0$$

définit une martingale continue adaptée à la filtration $\mathcal{F}_t = \sigma((\rho_s, u_s) \in H_x^{-\gamma} \times H_x^\beta, s \in [0, t])$. Reprenant ces arguments avec la fonction test $|\phi|^2$, on identifie également la variation quadratique de $N_\phi(t)$. On peut alors classiquement utiliser le théorème de représentation de martingale [DPZ14], Theorem 8.2, pour déduire que la loi de $(\rho_t)_{t \geq 0}$ correspond à une solution faible de (0.16).

Le résultat d'existence faible du Théorème 1 est complété l'unicité trajectorielle suivante.

Théorème 2. Soient $\rho_0 \in L_x^1$ et $u_0 \in W_x^{4, \infty}$. On suppose que le coefficient $a \equiv a(x)$ et le noyau $k \equiv k(x, y)$ de l'opérateur de trace Q impliqués dans (0.16) satisfont

$$a \in W_x^{4, \infty}, \quad \sup_{y \in \mathbb{T}} \|k(\cdot, y)\|_{W_x^{10+\delta, \infty}} < \infty \text{ pour un } \delta > 0.$$

Alors, pour $\gamma > 1/2$, une solution forte $\rho \in C([0, T]; H_x^{-\gamma})$ (p.s) de l'EPDS (0.16) telle que

$$\|\rho\|_{L_x^\infty L_t^\infty L_x^1} \leq 1$$

est trajectoriellement unique.

En diagonalisant l'opérateur Hilbert-Schmidt $Q^{1/2}$ dans L_x^2 , on peut introduire des fonctions $(\phi_k)_{k \geq 0}$ et des mouvements Browniens indépendants $(\beta_k)_{k \geq 0}$ de sorte que

$$Q^{1/2} \circ dW_t = \sum_k \phi_k \circ d\beta_t^k.$$

L'équation de conservation stochastique (0.16) peut donc se réécrire, sous forme Itô,

$$\begin{cases} d\rho + \left(\partial_x((a-u)\rho) - \frac{1}{2} \sum_k \partial_x(\phi_k \partial_x(\phi_k \rho)) \right) dt = \sum_k \partial_x(\phi_k \rho) d\beta_t^k, \\ \rho(0) = \rho_0 \in L_x^1. \end{cases}$$

L'unicité trajectorielle pour cette équation est déduite de l'existence d'une solution forte suffisamment régulière à l'équation duale, qui prend naturellement la forme d'une EDPS rétrograde :

$$\begin{cases} d\psi + \left((u - a)\partial_x \psi + \frac{1}{2} \sum_k \phi_k \partial_x (\phi_k \partial_x \psi) - \sum_k \phi_k \partial_x Z^k \right) dt = \sum_k Z^k d\beta_t^k, \\ \psi(T) = \psi_T, \end{cases}$$

d'inconnue (ψ, Z) . Pour montrer l'existence d'une solution à cette dernière équation, on commence par la perturber avec le terme additionnel $\frac{\varepsilon}{2} \partial_x^2 \psi$. Dans ce cadre super-parabolique, avec l'hypothèse de régularité sur les coefficients, [DM10] Theorem 2.3 garantit l'existence d'une solution $(\psi^\varepsilon, Z^\varepsilon)$ satisfaisant

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\psi_t^\varepsilon\|_{H_x^4}^2 \right] + \mathbb{E} \left[\int_0^T \|\psi_t^\varepsilon\|_{H_x^5}^2 dt \right] + \sum_k \mathbb{E} \left[\int_0^T \|Z_t^{\varepsilon, k}\|_{H_x^4}^2 dt \right] < \infty.$$

On établit alors une estimation uniforme de $(\psi^\varepsilon, Z^\varepsilon)$, avant d'envoyer ε vers 0 pour conclure.

Contenu du chapitre 2 : Limites de champ moyen pour des modèles de Cucker-Smale stochastiques

Dans ce deuxième chapitre, on s'intéresse à des modèles cinétiques issus de la biologie, destinés à décrire le mouvement collectif d'un groupe d'individus, et plus précisément l'émergence d'un alignement des trajectoires, d'un consensus adopté par le groupe malgré l'absence de meneur. Ce phénomène, que l'on observe invariablement chez différentes populations (nuées d'étourneaux, essaims d'abeilles, bancs de sardines, certaines colonies de bactéries...) est communément dénommé *flocking* (ou *swarming*) dans la littérature. Un des premiers modèles mathématiques destinés à décrire ce genre d'organisation spontanée est celui de Cucker-Smale, introduit dans les papiers fondateurs [CS07a, CS07b]. Il s'agit d'un modèle particulière : désignant par $X_t^{i, N}, V_t^{i, N} \in \mathbb{R}^d$ la position et la vitesse d'un agent, l'évolution du système à N agents est donnée par

$$\begin{cases} \frac{d}{dt} X_t^{i, N} = V_t^{i, N}, \\ \frac{d}{dt} V_t^{i, N} = \frac{1}{N} \sum_{j=1}^N \psi(X^{i, N} - X^{j, N}) (V^{j, N} - V^{i, N}). \end{cases} \quad (0.24)$$

Chaque individu cherche ainsi à aligner son vecteur vitesse avec ceux de ses voisins. Cet alignement est pondéré par une fonction $\psi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ qui décroît à mesure que les individus sont éloignés, typiquement de la forme

$$\psi(x - y) = \frac{\lambda}{(1 + |x - y|^2)^\gamma}, \quad \lambda, \gamma > 0.$$

Afin de modéliser des incertitudes de différentes natures, il est naturel de vouloir introduire du bruit dans ce système d'équations. On peut penser à la perturbation stochastique générale suivante pour le terme de force :

$$\begin{aligned} dV_t^{i, N} = & \left[\frac{1}{N} \sum_{j=1}^N \psi(X^{i, N} - X^{j, N}) (V^{j, N} - V^{i, N}) \right] dt + \sigma(X_t^{i, N}, V_t^{i, N}) \circ dB_t^i \\ & + F(X_t^{i, N}, V_t^{i, N}) \circ dW_t + \left[\frac{1}{N} \sum_{j=1}^N \tilde{\psi}(X^{i, N} - X^{j, N}) (V^{j, N} - V^{i, N}) \right] \circ d\beta_t. \end{aligned}$$

Le terme de diffusion $\sigma \circ dB_t^i$, dirigé par des mouvements Browniens indépendants $(B^i)_{i \geq 1}$, est un bruit individuel qui rend compte du degré de liberté de chaque individu. Les termes dirigés par les mouvements Browniens W et β sont des bruits communs : $F \circ dW_t$ rend compte d'un aléa environnemental affectant simultanément l'ensemble des individus ; le dernier terme rend compte d'un aléa dans l'interaction, obtenu en perturbant la fonction de poids ψ en $\psi + \tilde{\psi} \circ d\beta_t$.

La limite de champ moyen pour un tel modèle avec bruit individuel a été étudié dans [BCC11] pour $\sigma(x, v) = \sqrt{D}Id$, et dans [CS19] pour $\sigma(x, v) = \mathcal{R}(v)$, fonction de troncation de la vitesse.

Pour ce qui est du bruit commun, on trouve dans la littérature des résultats de convergence en champ moyen pour des coefficients de diffusion linéaires : un bruit environnemental de type $F(x, v) = D(v - v_c)$ pour $v_c \in \mathbb{R}^d$ constant est étudié dans [AH10] ; une perturbation de l'interaction avec $\tilde{\psi} = \sqrt{2\sigma}$ constant est étudié dans [CS17] et [HJR19]. On souhaite ici étendre ces résultats à des coefficients de diffusion plus généraux.

En notant $Z_t^{i,N} = (X_t^{i,N}, V_t^{i,N}) \in \mathbb{R}^{2d}$ et en introduisant la mesure empirique

$$\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{Z_t^{i,N}}, \quad (0.25)$$

le modèle de Cucker-Smale perturbé peut s'exprimer comme un système de particules stochastique à champ moyen, de la forme générale

$$dZ_t^{i,N} = B[\mu_t^N](Z_t^{i,N})dt + C[\mu_t^N](Z_t^{i,N}) \circ d\beta_t + \sigma(Z_t^{i,N}) \circ dB_t^i, \quad (0.26)$$

où $B[\mu]$, $C[\mu]$ désignent les opérateurs à noyaux

$$B[\mu](z) = \int_{\mathbb{R}^{2d}} b(z, z')d\mu(z'), \quad C[\mu](z) = \int_{\mathbb{R}^{2d}} c(z, z')d\mu(z').$$

Notons que l'on a choisi d'introduire des bruits sous la forme Stratonovich, car il s'agit de la forme d'intégration stochastique ayant le plus de sens physique. Le système (0.26) peut se réécrire sous forme Itô avec un terme de drift additionnel

$$dZ_t^{i,N} = \left(B[\mu_t^N](Z_t^{i,N}) + S[\mu_t^N](Z_t^{i,N}) \right) dt + C[\mu_t^N](Z_t^{i,N})d\beta_t + \sigma(Z_t^{i,N})dB_t^i,$$

avec $S[\mu]$ sous la forme

$$S[\mu](z) = \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} s(z, z', z'')d\mu(z')d\mu(z''),$$

où le coefficient $s \equiv s(z, z', z'')$ s'exprime en fonction de c , $\nabla_z c$, σ et $\nabla_z \sigma$. On s'intéresse à la limite de champ moyen associée à (0.26), c'est à dire au comportement de la mesure empirique μ^N dans la limite $N \rightarrow \infty$. Ce comportement est bien connu et établi dans le cadre où les coefficients b, c, σ, s sont globalement Lipschitziens : on pourra se référer par exemple à [Szn91] pour le cas de bruits individuels, et à [CF14] pour le bruit commun. Cependant, il est clair que les modèles de Cucker-Smale perturbés qui font l'objet de notre étude dépassent ce simple cadre, les coefficients impliqués dans (0.24) étant seulement localement Lipschitziens.

La convergence de la mesure empirique est étudiée dans les espaces de Wasserstein : étant donné un espace de Banach séparable $(E, \|\cdot\|)$, on définit, pour $p \geq 1$,

$$\mathcal{P}_p(E) = \left\{ \mu \in \mathcal{P}(E), \int_E \|x\|^p d\mu(x) < \infty \right\}$$

muni de la distance

$$W_p[\mu, \nu] = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{E \times E} \|x^1 - x^2\|^p d\pi(x^1, x^2) \right)^{1/p}$$

où $\Pi(\mu, \nu)$ désigne l'ensemble des mesures de probabilités sur $E \times E$, de marginales μ et ν . Dans un premier temps, sous des hypothèses très générales, on démontre le résultat de champ moyen faible suivant.

Théorème 3. *Notons $\mathcal{C} = C([0, T]; \mathbb{R}^{2d})$ l'espace des trajectoires. On suppose les coefficients b, c, σ, s localement Lipschitziens et sous-linéaires :*

$$|b(z, z')| + |c(z, z')| + |\sigma(z)| + |s(z, z', z'')| \lesssim 1 + |z| + |z'| + |z''|.$$

On suppose que les données initiales satisfont, pour un $\delta > 0$,

$$\mu_0^N \rightarrow \mu_0 \text{ dans } \mathcal{P}_2(\mathbb{R}^{2d}) \text{ et } \sup_N \int |z|^{2+\delta} d\mu_0^N(z) < \infty.$$

Alors il existe une sous-suite $(\mu^{N_k})_{k \geq 1}$ telle que

$$\mu^{N_k} \rightarrow \mu \text{ en loi dans } \mathcal{P}_2(\mathcal{C})$$

où μ est une solution faible (au sens probabiliste) de l'EPDS

$$d\mu_t + \nabla \cdot (B[\mu_t]\mu_t)dt + \nabla \cdot (C[\mu_t]\mu_t) \circ d\beta_t + \frac{1}{2} \nabla \cdot (\text{Tr}(\nabla \sigma \sigma^T)\mu_t)dt = \frac{1}{2} \nabla \cdot (\nabla \cdot (\sigma \sigma^T \mu_t))dt. \quad (0.27)$$

Ainsi, l'introduction d'un bruit environnemental sous forme Stratonovich $C[\mu^N](Z_t^{i,N}) \circ d\beta_t$ au niveau des particules conduit naturellement au terme de conservation $\nabla \cdot (C[\mu]\mu) \circ d\beta_t$ au niveau de l'équation limite (0.27). Le bruit individuel est quant à lui moyennisé pour conduire au terme diffusif $\frac{1}{2} \nabla \cdot \nabla \cdot (\sigma \sigma^T \mu)$ dans le membre de droite.

Donnons à présent les grandes lignes de la démonstration du Théorème 3. Voyant la mesure empirique μ^N comme une variable aléatoire à valeurs dans l'espace des mesures de probabilité $\mathcal{P}(\mathcal{C})$ portant sur les trajectoires, il s'agit essentiellement d'un résultat de compacité en loi dans l'espace de Wasserstein $\mathcal{P}_2(\mathcal{C})$. Étant donnée une variable aléatoire à valeurs mesures $\mu : \Omega \rightarrow \mathcal{P}(E)$, on définit son intensité $I(\mu) \in \mathcal{P}(E)$ comme la mesure déterministe donnée par

$$\forall f \in C_b(E), \quad \langle f, I(\mu) \rangle = \mathbb{E}[\langle f, \mu \rangle].$$

Il est classiquement établi que la tension des lois d'une famille de variables aléatoires $(\mu^N)_N$ à valeurs dans $\mathcal{P}(E)$ se réduit à la simple tension des mesures déterministes $(I(\mu^N))_N$. On peut par exemple se référer à [Szn91], p178. On montre que ce résultat s'étend aux espaces de Wasserstein de la façon suivante : étant donnée une famille de variables aléatoires $(\mu^N)_N$ à valeurs dans $\mathcal{P}_2(E)$, il y a équivalence entre la relative compacité des intensités $(I(\mu^N))_N$ dans l'espace $\mathcal{P}_2(E)$, et l'existence d'un espace de probabilité $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ muni de variables aléatoires $(\bar{\mu}^{N_k})_k$ et $\bar{\mu}$ telles que

$$\begin{cases} \forall k, \bar{\mu}^{N_k} \sim \mu^{N_k} \text{ en loi,} \\ W_2[\bar{\mu}^{N_k}, \bar{\mu}] \xrightarrow[k \rightarrow \infty]{} 0 \text{ } \bar{\mathbb{P}}\text{-presque sûrement et dans } L^2(\bar{\Omega}). \end{cases}$$

Dans le cas d'une mesure empirique (0.25), la mesure intensité s'obtient simplement comme la moyenne des lois des trajectoires des particules :

$$I(\mu^N) = \frac{1}{N} \sum_{i=1}^N \mathbb{P}[Z^{i,N} \in \cdot].$$

La relative compacité de $(I(\mu^N))_N$ dans $\mathcal{P}_2(\mathcal{C})$ découle donc d'estimations uniformes du système de particule (0.26), déduites de la sous-linéarité des coefficients. Comme développé dans le chapitre 1, on peut alors passer à la limite dans le problème de martingale associé à (0.26), la convergence dans l'espace de Wasserstein étant suffisante pour assurer le passage à la limite dans les termes non-linéaires. On utilise enfin le théorème de représentation de martingale pour déduire l'existence d'une solution faible à l'EDPS limite (0.27).

Dans un second temps, sous des hypothèses plus restrictives, on souhaite établir un résultat de convergence forte. On se limite pour cela au cas d'un bruit commun, si bien que l'équation limite attendue est l'équation de conservation stochastique

$$d\mu_t + \nabla \cdot (B[\mu_t]\mu_t)dt + \nabla \cdot (C[\mu_t]\mu_t) \circ d\beta_t = 0. \quad (0.28)$$

Comme mentionné précédemment dans le cadre d'une équation de Vlasov déterministe, une solution de (0.28) est naturellement recherchée sous la forme d'un transport de la donnée initiale

$$\mu = (Z^\mu)^* \mu_0 \quad (0.29)$$

par les caractéristiques stochastiques non-linéaires

$$dZ_t^\mu(z) = B[\mu_t](Z_t^\mu(z))dt + C[\mu_t](Z_t^\mu(z)) \circ d\beta_t, \text{ avec } Z_0^\mu(z) = z. \quad (0.30)$$

On peut alors énoncer le résultat suivant.

Théorème 4. *On suppose les coefficients b, s sous-linéaires, le coefficient c borné*

$$|b(z, z')| + |s(z, z', z'')| \lesssim 1 + |z| + |z'| + |z''|, \quad |c(z, z')| \lesssim 1.$$

ainsi que les estimations Lipschitz

$$\begin{aligned} |b(z_1, z'_1) - b(z_2, z'_2)| &\lesssim L_b(z) \left(|z_1 - z_2| + |z'_1 - z'_2| \right), \\ |s(z_1, z'_1, z''_1) - s(z_2, z'_2, z''_2)| &\lesssim L_s(z) \left(|z_1 - z_2| + |z'_1 - z'_2| + |z''_1 - z''_2| \right), \\ |c(z_1, z'_1) - c(z_2, z'_2)| &\lesssim L_c(z) \left(|z_1 - z_2| + |z'_1 - z'_2| \right), \end{aligned}$$

avec, pour un $\theta \in [0, 1)$,

$$\begin{aligned} L_b(z) &= 1 + |z_1|^{2\theta} + |z'_1|^{2\theta} + |z'_2|^{2\theta} + |z''_2|^{2\theta}, \\ L_s(z) &= 1 + |z_1|^{2\theta} + |z'_1|^{2\theta} + |z''_1|^{2\theta} + |z'_2|^{2\theta} + |z''_2|^{2\theta} + |z''_2|^{2\theta}, \\ L_c(z) &= 1 + |z_1|^\theta + |z'_1|^\theta + |z'_2|^\theta + |z''_2|^\theta. \end{aligned}$$

On suppose les mesures initiales μ_0^N à support dans un compact $K \subset \mathbb{R}^{2d}$ et $\mu_0^N \rightarrow \mu_0$ dans $\mathcal{P}_p(\mathbb{R}^{2d})$ pour un $p \geq 2$. Alors on a la convergence

$$\mu^N \rightarrow \mu \text{ dans } L^p(\Omega; \mathcal{P}_p(\mathcal{C}))$$

où μ est l'unique solution de (0.28) de la forme transport $\mu = (Z^\mu)^ \mu_0$.*

Les hypothèses faites ici sur les coefficients permettent notamment de considérer un modèle de Cucker-Smale stochastique avec une perturbation de la forme

$$dV_t^{i,N} = \frac{1}{N} \sum_{j=1}^N \psi(X_t^{i,N} - X_t^{j,N}) (V_t^{j,N} - V_t^{i,N}) dt + \frac{1}{N} \sum_{j=1}^N \tilde{\psi}(X_t^{i,N} - X_t^{j,N}) \mathcal{R}(V_t^{j,N} - V_t^{i,N}) \circ d\beta_t$$

où $\mathcal{R} \in C_c^\infty(\mathbb{R}^d)$ est une fonction de troncature de la vitesse.

Décrivons les éléments clés de la preuve du Théorème 4. On note d'abord que pour tout $N \geq 1$, la mesure empirique μ^N est de la forme (0.29) et les caractéristiques $Z_t^{\mu^N}$ associées correspondent précisément au système à N particules. Dès lors, estimer la distance de Wasserstein entre deux mesures empiriques revient à comparer les trajectoires des systèmes de particules associés : si l'on introduit un couplage optimal $\pi \in \mathcal{P}(\mathbb{R}^{2d} \times \mathbb{R}^{2d})$ de sorte que

$$W_p^p[\mu_0^{N_1}, \mu_0^{N_2}] = \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} |z - z'|^p d\pi(z, z'),$$

il s'en suit que

$$W_p^p[\mu_t^{N_1}, \mu_t^{N_2}] \leq \int_{\mathbb{R}^{2d} \times \mathbb{R}^{2d}} \left| Z_t^{\mu^{N_1}}(z) - Z_t^{\mu^{N_2}}(z') \right|^p d\pi(z, z').$$

Les coefficients de (0.30) étant seulement localement Lipschitziens, il est naturel d'introduire un temps d'arrêt afin de contraindre les trajectoires de toutes les particules à rester dans un ensemble borné. Considérant des mesures initiales μ_0^N à support dans un compact $K \subset \mathbb{R}^{2d}$, on déduit ainsi le contrôle suivant : pour tout $R > 0$ fixé,

$$\mathbb{E} \left[\sup_{t \leq \tau_R} W_p^p[\mu_t^{N_1}, \mu_t^{N_2}] \right] \leq C_R W_p^p[\mu_0^{N_1}, \mu_0^{N_2}] \quad (0.31)$$

où le temps d'arrêt τ_R est donné par

$$\tau_R = \inf \left\{ t \geq 0, \sup_{z \in K} \left[|Z_t^{\mu^{N_1}}(z)| + |Z_t^{\mu^{N_2}}(z)| \right] \geq R \right\} \wedge T.$$

Par suite, afin d'obtenir une convergence globale en temps pour μ^N , on est amené à estimer $\sup_{z \in K} |Z_t^\mu(z)|$, c'est à dire la taille du support de la mesure $\mu_t = (Z_t^\mu)^* \mu_0$ lorsque μ_0 est à support dans K . Il est à noter que cette difficulté est spécifique au cadre aléatoire. Dans le cas

déterministe, les caractéristiques d'une équation de conservation à coefficients sous-linéaires issues d'un compact restent contraintes à un compact en temps borné. Dans le cas stochastique, les trajectoires Browniennes ont une portée arbitrairement grande avec probabilité positive, si bien que l'on peut avoir $\mathbb{P}[\tau_R < T] > 0$ pour tout $R > 0$ fixé.

Sous l'hypothèse où le coefficient de diffusion $C[\mu]$ est borné, avec une méthode similaire à celle développée dans [BCC11], on établit l'existence de moments exponentiels pour les caractéristiques : pour tout $\alpha_0 \in (0, 1]$, il existe $\alpha_T > 0$ tel que

$$\forall t \in [0, T], \quad \mathbb{E} \left[\exp \left(\alpha_T \left(\left| Z_t^\mu(z) \right|^2 + \int |z'|^2 d\mu_t(z') \right) \right) \right] \lesssim \exp \left(\alpha_0 \left(|z|^2 + \int |z'|^2 d\mu_0(z') \right) \right).$$

Lorsque la mesure initiale μ_0 est support dans un compact $K \subset \mathbb{R}^{2d}$, on exploite ces moments ainsi que l'hypothèse sur le caractère Lipschitzien des coefficients afin d'obtenir une estimation de Kolmogorov, pour tout $p > 1$,

$$\forall z, z' \in K, \quad \mathbb{E} \left[\sup_{t \in [0, T]} \left| Z_t^\mu(z) - Z_t^\mu(z') \right|^p \right] \lesssim |z - z'|^p,$$

d'où l'on déduit finalement l'estimation désirée

$$\mathbb{E} \left[\sup_{t \in [0, T]} \sup_{z \in K} \left| Z_t^\mu(z) \right|^p \right] \lesssim 1.$$

Revenant à (0.31), on peut alors établir la convergence des mesures empiriques μ^N dans $L^p(\Omega; \mathcal{P}_p(\mathcal{C}))$, et passer à la limite dans le point fixe $\mu^N = (Z^{\mu^N})^* \mu_0$ pour obtenir $\mu = (Z^\mu)^* \mu_0$.

Sous les mêmes hypothèses, on démontre par ailleurs un résultat de propagation du chaos conditionnel semblable à celui obtenu dans [CF14] dans le cas de coefficients globalement Lipschitziens.

Théorème 5. *On suppose que les coefficients b, c, s satisfont les hypothèses du Théorème 4.*

Soit $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d})$ à support dans un compact $K \subset \mathbb{R}^{2d}$ et $(\mathcal{F}_t^\beta)_{t \geq 0}$ la filtration canonique de β .

Introduisant $(\xi_0^i)_{i \geq 1}$ une suite de variables aléatoires dans \mathbb{R}^{2d} , i.i.d, \mathcal{F}_0^β -mesurables, de loi μ_0 , on considère $Z^{i, N}$ la solution du système de particule (0.26) avec la condition initiale $Z_0^{i, N} = \xi_0^i$. Alors, pour tout $p \geq 2$, on a la convergence

$$\mu^N \rightarrow \mu \text{ dans } L^p(\Omega; \mathcal{P}_p(\mathcal{C}))$$

où μ est l'unique solution de (0.28) de la forme transport $\mu = (Z^\mu)^* \mu_0$. De plus, pour tout $r \geq 1$ et $\phi_1, \dots, \phi_r \in C_b(\mathcal{C})$,

$$\mathbb{E} \left[\prod_{i=1}^r \phi_i(Z^{i, N}) \Big| \mathcal{F}_T^\beta \right] \rightarrow \prod_{i=1}^r \langle \phi_i, \mu \rangle \text{ dans } L^1(\Omega).$$

Enfinement, soit Z^i la solution de

$$dZ_t^i = B[\mu_t](Z_t^i) dt + C[\mu_t](Z_t^i) \circ d\beta_t, \quad \text{avec } Z_0^i = \xi_0^i.$$

Alors μ est une version de la loi conditionnelle $\mathcal{L}(X^i | \mathcal{F}_T^\beta)$ et on a pour tout $i \geq 1$ la convergence

$$Z^{i, N} \rightarrow Z^i \text{ dans } L^p(\Omega; \mathcal{C}).$$

Comme dans les cadres classiques considérés dans [Szn91] ou [CF14], ce résultat de propagation du chaos repose essentiellement sur la symétrie du système de particule (0.26) : pour $\phi_1, \phi_2 \in C_b(\mathcal{C})$ avec $|\phi_1|, |\phi_2| \leq M$, on peut écrire

$$\mathbb{E} \left[\mathbb{E} \left[\phi_1(Z^{1, N}) \phi_2(Z^{2, N}) \Big| \mathcal{F}_T^\beta \right] - \langle \phi_1, \mu \rangle \langle \phi_2, \mu \rangle \right] \leq \mathbb{E}[A^N] + \mathbb{E}[B^N]$$

avec

$$\begin{aligned} A^N &= \left| \mathbb{E} \left[\phi_1(Z^{1, N}) \phi_2(Z^{2, N}) \Big| \mathcal{F}_T^\beta \right] - \mathbb{E} \left[\langle \phi_1, \mu^N \rangle \langle \phi_2, \mu^N \rangle \Big| \mathcal{F}_T^\beta \right] \right|, \\ B^N &= \left| \mathbb{E} \left[\langle \phi_1, \mu^N \rangle \langle \phi_2, \mu^N \rangle \Big| \mathcal{F}_T^\beta \right] - \mathbb{E} \left[\langle \phi_1, \mu \rangle \langle \phi_2, \mu \rangle \Big| \mathcal{F}_T^\beta \right] \right|. \end{aligned}$$

Par symétrie,

$$\begin{aligned} A^N &= \left| \frac{1}{N^2 - N} \sum_{i \neq j} \mathbb{E} \left[\phi_1(Z^{i,N}) \phi_2(Z^{j,N}) \middle| \mathcal{F}_T^\beta \right] - \frac{1}{N^2} \sum_{i,j} \mathbb{E} \left[\phi_1(Z^{i,N}) \phi_2(Z^{j,N}) \middle| \mathcal{F}_T^\beta \right] \right| \\ &\leq \left(\frac{1}{N^2 - N} - \frac{1}{N^2} \right) (N^2 - N) M^2 + \frac{1}{N} M^2 = 2 \frac{M^2}{N} \rightarrow 0. \end{aligned}$$

D'autre part, $\mathbb{E}[B^N]$ tend vers 0 par des arguments similaires à ceux développés dans la preuve du Théorème 4.

Contenu du chapitre 3 : Existence de solutions martingales pour des modèles stochastiques de mouvement collectif avec alignement local

On poursuit maintenant l'étude des modèles cinétiques de mouvement collectif en s'intéressant à un modèle plus précis, rendant compte d'un alignement local des vitesses. Dans [MT11], les auteurs mettent en évidence les limites du modèle de Cucker-Smale (0.24) en terme de cohérence, lorsqu'on le confronte à des distributions très hétérogènes d'individus (deux groupes évoluant à une très grande distance l'un de l'autre par exemple). Afin de remédier à ces incohérences, le modèle particulière suivant, dit de Motsch-Tadmor, est proposé :

$$\begin{cases} \frac{d}{dt} X_t^{i,N} = V_t^{i,N}, \\ \frac{d}{dt} V_t^{i,N} = \frac{1}{\sum_{j=1}^N \phi(X^{i,N} - X^{j,N})} \sum_{j=1}^N \phi(X^{i,N} - X^{j,N}) (V^{j,N} - V^{i,N}). \end{cases}$$

La fonction de poids $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^+$ est typiquement à support compact, et l'interaction entre deux individus est cette fois renormalisée par la somme totale des interactions, traduisant un alignement local des vecteurs vitesse. Comme suggéré dans [KMT13], il peut être judicieux de considérer un modèle hybride, en laissant l'alignement de type Motsch-Tadmor (MT) gouverner l'interaction locale, et le terme du modèle de Cucker-Smale (CS) diriger l'interaction à grande distance. Ceci conduit naturellement à l'équation de conservation suivante, régissant la distribution $f(t, x, v)$ des individus :

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \left((L^{CS}[f] + L^{MT}[f]) f \right) = 0, \quad (0.32)$$

où les termes de forces sont donnés par

$$\begin{aligned} L^{CS}[f](x, v) &= \int \int \psi(x - x') (v' - v) f(x', v') dx' dv', \\ L^{MT}[f](x, v) &= \frac{\int \int \phi(x - x') (v' - v) f(x', v') dx' dv'}{\int \int \phi(x - x') f(x', v') dx' dv'}. \end{aligned}$$

Dans [KMT13], le modèle de Motsch-Tadmor est également considéré dans le cas limite où la fonction de poids ϕ converge vers la distribution de Dirac en 0, conduisant ainsi au terme d'alignement fortement local (SLA : *Strongly Local Alignment*)

$$L^{SLA}[f](x, v) = \frac{\int (v' - v) f(x, v') dv'}{\int f(x, v') dv'} = \frac{j(x)}{\rho(x)} - v$$

mettant en jeu les quantités intégrées

$$\rho(x) = \int f(x, v') dv', \quad j(x) = \int v' f(x, v') dv'.$$

On considère alors l'équation de conservation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \left((L^{CS}[f] + L^{SLA}[f]) f \right) = 0. \quad (0.33)$$

L'existence de solutions aux équations (0.32) et (0.33) est démontrée dans [KMT13]. Par ailleurs, dans [KMT12], les auteurs établissent la convergence des solutions de (0.32) vers celles de (0.33) lorsque ϕ converge convenablement vers la distribution de Dirac δ_0 . Comme mis en place dans le chapitre précédent dans le cadre du modèle de Cucker-Smale, on cherche ici à étendre ces résultats déterministes au cas où les équations (0.32), (0.33) sont perturbées par un bruit dirigé par un mouvement Brownien. On s'intéresse ainsi à l'équation de conservation stochastique

$$df + (v \cdot \nabla_x f) dt + \nabla_v \cdot \left((L^{CS}[f] + L[f])f \right) dt + \nabla_v \cdot \left(K[f]f \right) \circ d\beta_t = 0, \quad (0.34)$$

avec $L = L^{MT}$ ou $L = L^{SLA}$ et le coefficient de diffusion

$$K[f](x, v) = F(x) + \int \int \tilde{\psi}(x - x')(v' - v)f(x', v')dx'dv'.$$

Ceci correspond à l'ajout d'un bruit environnemental $F(x) \circ d\beta_t$, ainsi qu'une perturbation de la fonction de poids ψ de l'alignement de Cucker-Smale en $\psi + \tilde{\psi} \circ d\beta_t$. On démontre alors le résultat suivant.

Théorème 6. *On fait les hypothèses suivantes :*

$$\begin{aligned} |\partial_x^\alpha \psi(x)| + |\partial_x^\alpha \tilde{\psi}(x)| &\lesssim 1 \text{ pour tout } 0 \leq |\alpha| \leq 4, \\ \phi &\in C^\infty(\mathbb{R}^d) \text{ et } \exists 0 < r_1 < r_2 < \infty, \inf \phi(B(0, r_1)) > 0, \text{ et } \text{Supp}(\varphi) \subset B(0, r_2), \\ F &\in C^\infty(\mathbb{R}^d) \text{ et } |F(x)| \lesssim 1 + |x|. \end{aligned} \quad (0.35)$$

Soit $f_0 : \mathbb{R}^{2d} \rightarrow \mathbb{R}^+$ avec $\int_{\mathbb{R}^{2d}} f_0(z) dz = 1$ satisfaisant, pour un $\delta > 1$ et $\theta \in (0, 1)$,

$$\int_{\mathbb{R}^{2d}} |f_0(z)|^p dz + \int_{\mathbb{R}^{2d}} (|x|^\delta + |v|^k) f_0(z) dz < \infty, \quad p = 1 + \frac{1}{\theta}, \quad k > \frac{\max(d+2, 4)}{1-\theta}.$$

Alors il existe une solution faible f (au sens probabiliste) de (0.34) avec $L = L^{MT}$.

La preuve de ce résultat s'effectue en deux temps. On commence par introduire un paramètre de régularisation $R > 0$, et on considère l'équation (0.34), sous forme Itô, avec des coefficients régularisés :

$$\begin{aligned} df^R + (v \cdot \nabla_x f^R) dt + \nabla_v \cdot \left((L_R^{CS}[f^R] + L_R^{MT}[f^R] + S_R[f^R])f^R \right) dt \\ + \nabla_v \cdot \left(K_R[f^R]f^R \right) d\beta_t = 0, \end{aligned} \quad (0.36)$$

de sorte que qu'on ait, pour $R > 0$ fixé, les estimations Lipschitz

$$\begin{aligned} \left| L_R^{CS}[\mu](z) - L_R^{CS}[\nu](z') \right| + \left| L_R^{MT}[\mu](z) - L_R^{MT}[\nu](z') \right| &\leq C_R \left(|z - z'| + W_1[\mu, \nu] \right), \\ \left| K_R[\mu](z) - K_R[\nu](z') \right| + \left| S_R[\mu](z) - S_R[\nu](z') \right| &\leq C_R \left(|z - z'| + W_1[\mu, \nu] \right). \end{aligned}$$

On peut alors écrire le système de particules stochastique $Z^{i,N} = (X^{i,N}, V^{i,N})$ associé

$$\begin{cases} dX_t^{i,N} = V_t^{i,N}, \\ dV_t^{i,N} = \left(L_R^{CS}[\mu_t^N] + L_R^{MT}[\mu_t^N] + S_R[\mu_t^N] \right) (Z_t^{i,N}) dt + K_R[\mu_t^N](Z_t^{i,N}) d\beta_t. \end{cases}$$

Les coefficients de cette équation étant globalement Lipschitziens (c'est le cadre considéré dans [CF14]), par des arguments similaires à ceux exposés dans le chapitre 2, on établit la limite de champ moyen. Pour tout $p \geq 1$, sous l'hypothèse où μ_0^N converge vers une mesure à densité $f_0(z) dz$ dans $\mathcal{P}_p(\mathbb{R}^{2d})$, on a

$$\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{Z^{i,N}} \rightarrow f^R(z) dz \text{ dans } L^p(\Omega; \mathcal{P}_p(\mathcal{C})),$$

où f^R désigne l'unique solution de (0.36) donnée par le transport $f^R = (Z^R)^* f_0$ de la donnée initiale par les caractéristiques stochastiques non-linéaires associées :

$$\begin{cases} dX_t^R(z) = V_t^R(z) dt, \\ dV_t^R(z) = \left(L_R^{CS}[f_t^R] + L_R^{MT}[f_t^R] + S_R[f_t^R] \right) (Z_t^R(z)) dt + K_R[f_t^R](Z_t^R(z)) d\beta_t. \end{cases}$$

Les coefficients étant réguliers, la théorie des flots stochastiques développée par [Kun84] garantit que $z \mapsto Z_t^R(z)$ définit un difféomorphisme. En inversant ce flot, on déduit ainsi l'expression explicite

$$f^R(t, z) = \det \left[D_z \left((Z_t^R)^{-1}(z) \right) \right] f_0 \left((Z_t^R)^{-1}(z) \right)$$

si bien que pour une condition initiale f_0 régulière, f^R définit presque sûrement une solution forte (au sens des EDP) de l'équation (0.36). Dans un second temps, on établit les estimations suivantes sur f^R , uniformément en $R > 0$:

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \|f^R(t)\|_{L^p}^p \right] &\lesssim \|f_0\|_{L^p}^p, \\ \mathbb{E} \left[\sup_{t \in [0, T]} \int_{\mathbb{R}^{2d}} (|x|^\delta + |v|^k) f^R(t, z) dz \right] &\lesssim 1 + \int_{\mathbb{R}^{2d}} (|x|^\delta + |v|^k) f_0(z) dz, \end{aligned}$$

et, pour un certain $q > 1$,

$$\forall t, s \in [0, T], \quad \mathbb{E} \left[\|f^R(t) - f^R(s)\|_{H^{-2}}^{2q} \right] \lesssim |t - s|^q.$$

De ces estimations, on déduit la tension de la famille de variables aléatoires $(f^R)_{R>0}$ dans l'espace

$$C([0, T]; H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d})), \quad \sigma > 0,$$

où $H_{W^{-1}}^{-\sigma}$ désigne un espace de Sobolev à poids. Cette convergence faible en espace n'est pas suffisante pour passer à la limite dans l'équation non-linéaire (0.36). Afin d'obtenir une convergence plus forte, on exploite un lemme de moyenne stochastique (similaire à celui énoncé dans [DDMV15]) en généralisant au cadre aléatoire la méthode L^2 déterministe exposée dans [BD99]. On obtient ainsi la régularité

$$\mathbb{E} \left[\|\rho_\varphi^R\|_{L_t^2 H_x^\eta}^2 \right] \lesssim 1, \quad (\text{avec } \eta = 1/6)$$

pour les quantités

$$\rho_\varphi^R(t, x) = \int_{\mathbb{R}^d} \varphi(v) f^R(t, x, v) dv, \quad \varphi \in C_c^\infty(\mathbb{R}^d).$$

Il en résulte, pour tout $\varphi \in C_c^\infty(\mathbb{R}^d)$, la tension de la famille de variables aléatoires $(\rho_\varphi^R)_{R>0}$ dans $L^2([0, T]; L^2(\mathbb{R}^d))$. Il s'agit enfin de passer à la limite $R \rightarrow \infty$ dans le problème de martingale associé à l'équation (0.36) :

$$M_\Psi^R(t) = \langle \Psi, f^R(t) \rangle - \langle \Psi, f_0 \rangle - \int_0^t \left\langle \mathcal{L}_R[f^R(s)] \Psi, f^R(s) \right\rangle ds, \quad t \geq 0,$$

où l'opérateur $\mathcal{L}_R[f]$ est défini par

$$\mathcal{L}_R[f] = v \cdot \nabla_x \Psi + \left(L_R^{CS}[f] + L_R^{MT}[f] + S_R[f] \right) \cdot \nabla_v \Psi + \frac{1}{2} \sum_{1 \leq i, j \leq d} K_R[f]^i K_R[f]^j \partial_{v_i v_j}^2 \Psi.$$

Le terme linéaire $v \cdot \nabla_x \Psi$ ne pose pas de difficulté. Les non-linéarités issues de $L_R^{CS}[f]$ et $S_R[f]$ sont de même nature que celles rencontrées dans le chapitre 2. Le terme le plus délicat est bien-sûr celui impliquant la non-linéarité de Motsch-Tadmor $L^{MT}[f]$, qui met en jeu un ratio de deux quantités intégrées. Un examen soigneux du terme en question, avec la convergence dans $L^2([0, T]; L^2(\mathbb{R}^d))$ des ρ_φ^R , permet de conclure. De nouveau, on utilise le théorème de représentation de martingale pour déduire l'existence d'une solution faible.

En s'appuyant sur les mêmes arguments, on démontre également la convergence des solutions dans le régime $\phi \rightarrow \delta_0$, telle qu'elle est établie dans [KMT12] dans le cas déterministe. Ce résultat s'énonce de la manière suivante.

Théorème 7. *On suppose les hypothèses du Théorème 6 satisfaites.*

Soit $\phi_1 \in C_c^\infty(\mathbb{R}^d)$ satisfaisant (0.35). Pour $r > 0$, considérons la fonction $\phi_r(x) = r^d \phi_1(x/r)$. Alors les solutions faibles f^r de (0.34) avec $L = L^{MT}$ et $\phi = \phi_r$ construites dans le Théorème 6 satisfont, pour une sous suite $r_n \rightarrow \infty$,

$$f^{r_n} \rightarrow f \text{ en loi dans } C([0, T]; H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d})) \text{ pour tout } \sigma > 0,$$

où f définit une solution faible de (0.34) avec $L = L^{SLA}$. L'espace de Sobolev $H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d})$ est pondéré par $W^{-1}(z) = (1 + |z|)^{-1}$.

1

Diffusion-approximation for a kinetic spray-like system with markovian forcing

1.1 Introduction

1.1.1 Overview of the kinetic model

We consider a toy model for spray-like behavior, describing the motion of some particles immersed in an ambient fluid, given by the following conservation equation

$$\partial_t f + \varepsilon v \cdot \nabla_x f + \nabla_v \cdot (F(t, x, v) f) = 0. \quad (1.1)$$

The forcing term

$$F(t, x, v) = (u_t(x) - v) + m_t(x)$$

expresses how the speed of a particle tends to align with the local speed $u(x)$ of the ambient fluid. It is perturbed by some random forcing naturally modeled by a space-dependent, centered Markov process $(m_t(x))_{t \geq 0}$. Assuming that the particles have very little impact on the behavior of the ambient fluid, $u \equiv u_t(x)$ evolves according to

$$\partial_t u - \Delta u = \varepsilon^2 \int_v (v - u) f dv. \quad (1.2)$$

To simplify the resolution of the system of coupled equations (1.1) and (1.2), we omit the Burgers transport term $u \cdot \nabla_x u$ in equation (1.2). This term plays no major role in the arguments developed in the present work and could hence be added without too much difficulty. In order for the system (1.1)-(1.2) to admit a solution $(f_t(x, v), u_t(x))$, we need, in fact, to restrict ourselves to the one-dimensional case :

$$x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad v \in \mathbb{R}.$$

One can refer to section 1.8 below (and in particular Remark 1.8.2) for details regarding the well-posedness of this system. The rescaled variables

$$f_t^\varepsilon(x, v) := f_{\varepsilon^{-2}t}(x, v), \quad u_t^\varepsilon(x) := \varepsilon^{-1} u_{\varepsilon^{-2}t}(x), \quad m_t^\varepsilon(x) := m_{\varepsilon^{-2}t}(x),$$

then satisfy

$$\begin{cases} \partial_t f^\varepsilon + \varepsilon^{-1} v \partial_x f^\varepsilon + \varepsilon^{-2} \partial_v \left[(\varepsilon u^\varepsilon(x) - v + m^\varepsilon(x)) f^\varepsilon \right] = 0, \\ \partial_t u^\varepsilon - \partial_x^2 u^\varepsilon = \int (v - \varepsilon u^\varepsilon) f^\varepsilon dv. \end{cases} \quad (1.3)$$

Introducing the integrated quantity

$$\rho^\varepsilon(x) = \int f^\varepsilon(x, v) dv, \quad (1.4)$$

we are interested in the behavior of $(\rho^\varepsilon, u^\varepsilon)$ as ε goes to zero. In essence, we wish to describe how the random forcing m added on a "mesoscopic level" translates on a "macroscopic level" in the

scaling limit. More precisely, we shall determine the stochastic partial differential equation satisfied by the limiting law of ρ^ε as ε goes to 0.

This work is naturally in line with the various stochastic diffusion-approximation results developed in [DDMV16, DV12, DV17]. In the case considered here, it is to be noted that due to the absence of collision term in the first equation of (1.3), the limiting SPDE satisfied by ρ^ε will be of order one : it is in fact a stochastic conservation equation when written in Stratonovich form. In particular, the question of uniqueness for this limiting SPDE is not trivial, and is dealt with using a completely different approach from the one used in [DDMV16, DV12, DV17]. The study of the scaling limit relies essentially on the perturbed test function method, first introduced in [GPV77] and developed further in [FGPS07]. In contrast with previous works on the matter, particular attention will be paid to the rigorous construction of the correctors involved in the perturbed test function method. This will be achieved by handling infinitesimal generators and their domains in an overall more cautious way, and by exploiting the mixing assumptions made on the driving Markov process to properly solve Poisson equations associated with these generators.

1.1.2 Main results

Throughout our study, we will be led to work with the following spaces :

$$\begin{aligned} L_{x,v}^p &:= L^p(\mathbb{T} \times \mathbb{R}), & L_x^p &:= L^p(\mathbb{T}), \\ W_x^{s,p} &:= W^{s,p}(\mathbb{T}), & H_x^\sigma &:= W^{2,\sigma}(\mathbb{T}). \end{aligned}$$

We will denote by $\mathcal{P}(E)$ the set of probability measures on a metric space E . When it is well defined, we will denote

$$\langle f, g \rangle = \int f(x, v)g(x, v)dx dv \quad \text{or} \quad \int f(x)g(x)dx.$$

depending on the context. Finally, we shall sometimes use the handy notation $A(z) \lesssim B(z)$ to signify that there exists a constant $C > 0$ such that $A(z) \leq CB(z)$ for all z .

Let us now state our diffusion-approximation result.

Theorem 1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space equipped with a Markov process $m = (m_t)_{t \geq 0}$ satisfying Assumptions 1 through 4, described in section 1.2.2 below. Let $T > 0$. Assume that the initial data $(f_0^\varepsilon, u_0^\varepsilon)_{\varepsilon > 0}$ satisfy*

$$\begin{aligned} f_0^\varepsilon &\geq 0 \quad \text{and} \quad \int \int f_0^\varepsilon(x, v)dx dv = 1, \\ \rho_0^\varepsilon &\rightarrow \rho_0 \quad \text{in } L_x^1, \quad u_0^\varepsilon \rightarrow u_0 \quad \text{in } H_x^\alpha \quad \text{for some } \alpha \in (1/2, 3/2] \end{aligned}$$

and, for all fixed $\varepsilon > 0$,

$$\int_x \int_v (1 + |v|^4) |f_0^\varepsilon|^2 dx dv + \int_x \int_v (1 + |v|) |\nabla_{x,v} f_0^\varepsilon| dx dv < \infty, \quad u_0^\varepsilon \in H_x^\eta \quad \text{for } \eta(\varepsilon) > 3/2. \quad (1.5)$$

Assume the following uniform bound :

$$\sup_{\varepsilon > 0} \int_x \int_v |v|^3 f_0^\varepsilon dx dv < \infty. \quad (1.6)$$

Let $(f^\varepsilon, u^\varepsilon)$ be the unique solution of (1.3) with initial data $(f_0^\varepsilon, u_0^\varepsilon)$ (in the sense of (1.33), given in Proposition 1.3.1 below). Then for all $\gamma > 1/2$ and $\beta < \alpha$, there exists a subsequence $(\rho^{\varepsilon_k}, u^{\varepsilon_k})_{k \geq 1}$ such that

$$(\rho^{\varepsilon_k}, u^{\varepsilon_k}) \rightarrow (\rho, u) \quad \text{in law in } C([0, T]; H_x^{-\gamma} \times H_x^\beta)$$

where u satisfies the heat equation

$$\partial_t u - \partial_x^2 u = 0, \quad \text{with } u(0) = u_0,$$

and ρ is a martingale solution (in the sense of [DPZ14], Chapter 8) of

$$d\rho = \partial_x \left[(a - u)\rho \right] dt + \partial_x \left[\rho Q^{1/2} \circ dW_t \right], \quad \text{with } \rho(0) = \rho_0, \quad (1.7)$$

satisfying $\sup_{t \in [0, T]} \|\rho_t\|_{L_x^1} \leq 1$ a.s. In (1.7), $(W_t)_{t \geq 0}$ denotes a cylindrical Wiener process on L_x^2 . The coefficient $a \equiv a(x)$ and the trace operator Q on L_x^2 are completely determined by the law of the stationary process $(\tilde{m}_t)_{t \geq 0}$ associated with the random driving term m (see section 1.2.2 below for a definition of \tilde{m}) :

$$a(x) = \frac{1}{2} \int_0^\infty \mathbb{E} \left[\partial_x \tilde{m}_0(x) \tilde{m}_t(x) - \tilde{m}_0(x) \partial_x \tilde{m}_t(x) \right] dt + \int_0^\infty e^{-t} \mathbb{E} \left[\tilde{m}_0(x) \partial_x \tilde{m}_t(x) \right] dt, \quad (1.8)$$

$$Qf(x) = \int_y k(x, y) f(y) dy, \quad \text{with } k(x, y) = \int_{\mathbb{R}} \mathbb{E} \left[\tilde{m}_0(x) \tilde{m}_t(y) \right] dt. \quad (1.9)$$

Note that assumption (1.5) is only necessary to ensure that the system (1.3) can indeed be uniquely solved for fixed $\varepsilon > 0$, as will be stated later on, in Proposition 1.3.1. The result of Theorem 1 holds when the underlying probability space Ω is equipped with the probability \mathbb{P}_μ – so that $\mathbb{P}_\mu[m_0 \in A] = \mu(A)$ – for any initial law $\mu \in \mathcal{P}(E)$. For instance, one may consider the case $\mu = \delta_n$ where the process $(m_t)_{t \geq 0}$ starts at some fixed $n \in E$; or one may consider the case where $\mu = \nu$ is the invariant measure of m , so that $(m_t)_{t \geq 0} := (\tilde{m}_t)_{t \geq 0}$ is a stationary process (see again section 1.2.2 below).

The weak existence result established in Theorem 1 is completed by the following strong uniqueness result.

Theorem 2. *Let $\rho_0 \in L_x^1$ and $u_0 \in W_x^{4, \infty}$. Assume furthermore that the coefficient a and the kernel k defined in (1.8), (1.9) satisfy*

$$a \in W_x^{4, \infty}, \quad \sup_{y \in \mathbb{T}} \|k(\cdot, y)\|_{W_x^{10+\delta, \infty}} < \infty \quad \text{for some } \delta > 0. \quad (1.10)$$

Then a solution $\rho \in C([0, T]; H_x^{-\gamma})$ of (1.7) satisfying the bound

$$\sup_{t \in [0, T]} \|\rho_t\|_{L_x^1} \leq 1 \quad \text{a.s.}$$

is path-wise unique.

The regularity properties (1.10) will be satisfied under some adequate assumptions on the driving process $(m_t)_{t \geq 0}$ (see the end of section 1.2.2).

In the limit where ε goes to 0, the rather complex random process $(\rho_t^\varepsilon)_{t \geq 0}$ driven by a general Markov process $m = (m_t)_{t \geq 0}$ can therefore be approximated by a much simpler diffusion $(\rho_t)_{t \geq 0}$ whose drift and covariance are explicitly determined by the law of the associated stationary process $\tilde{m} = (\tilde{m}_t)_{t \geq 0}$. Let us give a quick overview of the proof of Theorems 1 and 2.

In section 1.2 we start by introducing a rigorous framework for solving Poisson equations associated with generators of Markov processes, which will be of great use in section 1.5 in particular. The tools and results presented here are all very classical, however the authors could not find a reference really suited for the specific setup adopted in the present work. We then describe the assumptions made on the driving process m and their consequences in section 1.2.2. In section 1.3, we establish some estimates regarding the solutions of (1.3) and rigorously determine the generator of the process $(f^\varepsilon, u^\varepsilon, m^\varepsilon)$. The essential work is developed in section 1.5 where we use the perturbed test function method to determine the limiting equation (1.7) and pave the way towards the convergence of the martingale problem. Section 1.6 concludes the proof by proving the convergence announced in Theorem 1. Finally, the path-wise uniqueness of (1.7) stated in Theorem 2 is looked upon in section 1.7.

1.2 Generators and Poisson equations

1.2.1 Generalities on generators

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$, let us consider a càdlàg, time-homogeneous Markov process $X = (X_t)_{t \geq 0}$ taking values in some separable metric space $\mathcal{X} \equiv (\mathcal{X}, d_{\mathcal{X}})$. Throughout this paper, we will naturally use the handy notation $(X_t(x))_{t \geq 0}$ to denote $(X_t)_{t \geq 0}$ under the probability \mathbb{P}_{δ_x} (that is $X_0(x) = x$ a.s.). All the processes we shall encounter in this work satisfy the following assumptions :

— X is stochastically continuous :

$$\forall x \in \mathcal{X}, \forall t \geq 0, \forall \varepsilon > 0, \mathbb{P}\left[d_{\mathcal{X}}(X_s(x), X_t(x)) > \varepsilon\right] \xrightarrow{s \rightarrow t} 0. \quad (1.11)$$

— X is locally bounded (in space and time) :

$$\forall r > 0, \forall T > 0, \exists R > 0, \forall x \in B(0, r), \mathbb{P}\left[\forall t \in [0, T], X_t(x) \in B(0, R)\right] = 1. \quad (1.12)$$

Let us denote by $(Q_t)_{t \geq 0}$ the semigroup associated to $(X_t)_{t \geq 0}$. The local boundedness assumption (1.12) allows to define it over locally bounded test functions : let us introduce the spaces

$$\mathcal{B}^{loc}(\mathcal{X}) = \{\psi : \mathcal{X} \rightarrow \mathbb{R} \text{ measurable, } \psi \text{ maps bounded sets onto bounded sets}\} \quad (1.13)$$

$$C_b^{loc}(\mathcal{X}) = \{\psi : \mathcal{X} \rightarrow \mathbb{R} \text{ continuous, } \psi \text{ maps bounded sets onto bounded sets}\} \quad (1.14)$$

and define the semigroup $Q_t : \mathcal{B}^{loc}(\mathcal{X}) \rightarrow \mathcal{B}^{loc}(\mathcal{X})$ as

$$\forall \psi \in \mathcal{B}^{loc}(\mathcal{X}), Q_t \psi(x) = \mathbb{E}[\psi(X_t(x))].$$

Note that, using assumption (1.11), the map $t \mapsto Q_t \psi(x)$ is continuous whenever $\psi \in C_b^{loc}(\mathcal{X})$. The Markov property

$$\forall \psi \in \mathcal{B}^{loc}(\mathcal{X}), \mathbb{E}[\psi(X_{t+s}) | \mathcal{F}_s] = Q_t \psi(X_s) \text{ a.s}$$

yields in particular the semigroup property $Q_{t+s} = Q_t Q_s$ on $\mathcal{B}^{loc}(\mathcal{X})$.

Definition 1.2.1 (Infinitesimal generator). *The infinitesimal generator of X is the operator*

$$A : D(A) \subset \mathcal{B}^{loc}(\mathcal{X}) \rightarrow C_b^{loc}(\mathcal{X})$$

defined as follows : for all $\psi \in \mathcal{B}^{loc}(\mathcal{X})$ and $\theta \in C_b^{loc}(\mathcal{X})$,

$$\psi \in D(A) \text{ and } A\psi = \theta \text{ if and only if } \frac{Q_t \psi - \psi}{t} \xrightarrow[t \rightarrow 0^+]{b.p.c} \theta \in C_b^{loc}(\mathcal{X})$$

where *b.p.c* denotes the (locally) bounded pointwise convergence :

$$\theta_t \xrightarrow{b.p.c} \theta \iff \begin{cases} \forall x \in \mathcal{X}, \theta_t(x) \rightarrow \theta(x) \\ \forall B \subset \mathcal{X} \text{ bounded, } \sup_{t > 0} \sup_{x \in B} |\theta_t(x)| < \infty. \end{cases}$$

We then have the following classical result.

Proposition 1.2.1 (Kolmogorov's equation). *For all $\psi \in D(A)$, for all $t \geq 0$,*

$$\frac{Q_{t+s}\psi - Q_t\psi}{s} \xrightarrow[s \rightarrow 0]{b.p.c} Q_t A\psi.$$

In particular, for all $x \in \mathcal{X}$,

$$\frac{d}{dt} Q_t \psi(x) = Q_t A\psi(x).$$

Remark 1.2.1. *Provided that $Q_t A\psi \in C_b^{loc}(\mathcal{X})$, this states that $Q_t \psi \in D(A)$ and $AQ_t \psi = Q_t A\psi$.*

Proof. For $s > 0$, we may write

$$\frac{Q_{t+s}\psi - Q_t\psi}{s}(x) = Q_t \left[\frac{Q_s\psi - \psi}{s} \right](x) = \mathbb{E} \left[\frac{Q_s\psi(X_t(x)) - \psi(X_t(x))}{s} \right].$$

Making use of (1.11) and (1.12), we may use dominated convergence as s goes to 0 to conclude for the right derivative. As for the left derivative, we simply note that

$$\frac{Q_t\psi - Q_{t-s}\psi}{s} = Q_{t-s} \left[\frac{Q_s\psi - \psi}{s} - A\psi \right] + Q_{t-s} A\psi$$

and conclude in the same way, keeping in mind that $A\psi \in C_b^{loc}(\mathcal{X})$ so that $u \mapsto Q_u A\psi(x)$ is continuous. . \square

The following property will be of great interest, and could in fact be an alternative way of defining the generator A .

Proposition 1.2.2 (Martingale problem). *For all $\psi \in D(A)$, for all $x \in \mathcal{X}$,*

$$M_\psi(t) = \psi(X_t(x)) - \int_0^t A\psi(X_s(x))ds, \quad t \geq 0$$

defines an $(\mathcal{F}_t)_{t \geq 0}$ -martingale.

Proof. Let us write $X_t := X_t(x)$ for simplicity. We need to check that, for all h bounded, \mathcal{F}_s -measurable,

$$\mathbb{E}\left[\left(\psi(X_t) - \psi(X_s)\right)h\right] = \mathbb{E}\left[\left(\int_s^t A\psi(X_u)du\right)h\right].$$

Using the Markov property (twice) and Proposition 1.2.1, the right hand side can be written as

$$\begin{aligned} \mathbb{E}\left[\mathbb{E}\left[\int_s^t A\psi(X_u)du \middle| \mathcal{F}_s\right]h\right] &= \mathbb{E}\left[\left(\int_s^t \mathbb{E}A\psi(X_u) \middle| \mathcal{F}_s\right)du\right)h\right] = \mathbb{E}\left[\left(\int_s^t Q_{u-s}A\psi(X_s)du\right)h\right] \\ &= \mathbb{E}\left[\left(\int_0^{t-s} \frac{d}{du}Q_u\psi(X_s)du\right)h\right] = \mathbb{E}\left[\left(Q_{t-s}\psi(X_s) - \psi(X_s)\right)h\right] \\ &= \mathbb{E}\left[\left(\psi(X_t) - \psi(X_s)\right)h\right]. \end{aligned}$$

Again, we have used the fact $u \mapsto Q_uA\psi(x)$ is continuous since $A\psi$ is continuous. \square

Let us now give a natural context in which one can interchange generator and integral.

Proposition 1.2.3 (Generator under the integral sign). *Let $(t, x) \in \mathbb{R}^+ \times \mathcal{X} \mapsto \psi_t(x) \in \mathbb{R}$ be a measurable map, such that $\psi_t \in D(A)$ for all $t \geq 0$. Assume that, for all $B \subset \mathcal{X}$ bounded,*

$$\int_0^\infty \sup_{x \in B} |\psi_t(x)|dt < \infty \quad (1.15)$$

$$\int_0^\infty \sup_{x \in B} |A\psi_t(x)|dt < \infty \quad (1.16)$$

Then $\psi = \int_0^\infty \psi_t dt \in D(A)$ and $A\psi = \int_0^\infty A\psi_t dt$.

Proof. Let us not get into the questions of measurability. First, ψ is well defined, in $\mathcal{B}^{loc}(X)$, and we have $Q_s\psi(x) = \int_0^\infty Q_s\psi_t(x)dt$. The interversion is justified by Fubini's theorem since, using (1.15),

$$\mathbb{E}\left[\int_0^\infty |\psi_t|(X_s(x))dt\right] \leq \int_0^\infty \sup_{y \in B} |\psi_t(y)|dt < \infty$$

where $B \subset \mathcal{X}$ is a bounded set such that $\forall \sigma \in [0, s], X_\sigma(x) \in B$, \mathbb{P} a.s. It follows that

$$\frac{Q_s\psi(x) - \psi(x)}{s} = \int_0^\infty \frac{Q_s\psi_t(x) - \psi_t(x)}{s} dt.$$

For $x \in B(0, r)$ for some $r > 0$, for $s \leq 1$, we have the bound

$$\left|\frac{Q_s\psi_t(x) - \psi_t(x)}{s}\right| \leq \sup_{\sigma \in [0, s]} \left|\frac{d}{d\sigma}Q_\sigma\psi_t(x)\right| = \sup_{\sigma \in [0, s]} \left|Q_\sigma A\psi_t(x)\right| \leq \sup_{y \in B} \left|A\psi_t(y)\right|$$

where $B \subset \mathcal{X}$ is bounded and such that $\forall x \in B(0, r), \forall \sigma \in [0, 1], X_\sigma(x) \in B$, \mathbb{P} -a.s. Dominated convergence then gives the b.p.c of $\int_0^\infty s^{-1}(Q_s\psi_t(x) - \psi_t(x))dt$ to $\int_0^\infty A\psi_t(x)dt$. Assumption (1.16) and the fact that $A\psi_t \in C_b^{loc}(\mathcal{X})$ for all $t \geq 0$ guarantee that $\int_0^\infty A\psi_t dt$ is in $C_b^{loc}(\mathcal{X})$, so that we can indeed state $\psi \in D(A)$. \square

Proposition 1.2.4 (Resolution of a Poisson equation $A\psi = \theta$). *Let $\theta \in D(A) \cap C_b^{loc}(\mathcal{X})$. Assume that, for all $B \subset \mathcal{X}$ bounded,*

$$\int_0^\infty \sup_{x \in B} |Q_t \theta(x)| dt < \infty \quad (1.17)$$

$$\int_0^\infty \sup_{x \in B} |Q_t A\theta(x)| dt < \infty \quad (1.18)$$

Then $\psi = - \int_0^\infty Q_t \theta dt \in D(A)$ and $A\psi = \theta$.

Proof. Since $Q_t A\theta$ is not necessarily continuous, we cannot exactly state that $\psi_t = Q_t \theta \in D(A)$. Let us hence be a little more careful : using the same arguments as the previous proof, we can show that

$$\frac{Q_s \psi(x) - \psi(x)}{s} \xrightarrow[s \rightarrow 0]{b.p.c.} - \int_0^\infty Q_t A\theta(x) dt = \lim_{T \rightarrow \infty} - \int_0^T \frac{d}{dt} Q_t \theta(x) dt = \theta(x) - \lim_{T \rightarrow \infty} Q_T \theta(x).$$

Hence, $\lim_{T \rightarrow \infty} Q_T \theta(x)$ exists and assumption (1.17) implies $\lim_{T \rightarrow \infty} Q_T \theta(x) = 0$. We have shown

$$\frac{Q_s \psi - \psi}{s} \xrightarrow[s \rightarrow 0]{b.p.c.} \theta$$

and θ is in $C_b^{loc}(\mathcal{X})$ by assumption, so that we have indeed $\psi \in D(A)$. \square

Remark 1.2.2. *Assuming (1.17) only, one can still define the $\mathcal{B}^{loc}(\mathcal{X})$ function $\psi = - \int_0^\infty Q_t \theta dt$. If for some reason we know in advance that $\psi \in D(A)$, then we can conclude immediately that $A\psi = \theta$ without assumption (1.18). Indeed,*

$$Q_s \psi(x) = - \int_0^\infty Q_{t+s} \theta(x) dt = - \int_s^\infty Q_t \theta(x) dt$$

which gives, since $t \mapsto Q_t \theta(x)$ is continuous,

$$\left. \frac{d}{ds^+} Q_s \psi(x) \right|_{s=0} = Q_0 \theta(x) = \theta(x)$$

and using Kolmogorov's equation, we know that

$$\left. \frac{d}{ds^+} Q_s \psi(x) \right|_{s=0} = A\psi(x).$$

1.2.2 The random driving term

We now describe the assumptions made on the process $m = (m_t)_{t \geq 0}$ driving equation (1.3). Let us define its canonical filtration

$$\mathcal{F}_t = \sigma(m_s, 0 \leq s \leq t), \quad t \geq 0.$$

We assume that $m = (m_t)_{t \geq 0}$ is a càdlàg, stochastically continuous Markov process taking values in some separable, complete normed space $(E, \|\cdot\|_E)$, satisfying the continuous embedding

$$m_t \in E \subset W_x^{2,\infty}, \quad \text{with } \|\cdot\|_{W_x^{2,\infty}} \lesssim \|\cdot\|_E. \quad (1.19)$$

This enables us to consider linear test functions of the form

$$\psi_h : n \in E \mapsto \langle h, n \rangle, \quad h \in (W_x^{2,\infty})'.$$

Assumption 1 (Bounded state space). *For some $C^* > 0$,*

$$\|n\|_E \leq C^* \quad \text{for all } n \in E. \quad (1.20)$$

We denote by $(P_t)_{t \geq 0}$ the semigroup associated to m and by $(M, D(M))$ its infinitesimal generator, in the sense of the b.p.c convergence introduced in section 1.2.1.

Assumption 2 (Stationary measure). *There exists some measure $\nu \in \mathcal{P}(E)$ satisfying :*

$$\forall t \geq 0, \forall \psi \in \mathcal{B}(E), \quad \int_E P_t \psi(n) d\nu(n) = \int_E \psi(n) d\nu(n) \quad (\text{Stationary})$$

$$\forall h \in (W_x^{2,\infty})', \quad \int \psi_h(n) d\nu(n) = 0. \quad (\text{Centered})$$

We shall denote by $(\tilde{m}_t)_{t \geq 0}$ the process with initial law ν .

Note that it is possible to build a càdlàg stationary process $(\hat{m}_t)_{t \in \mathbb{R}}$ indexed by $t \in \mathbb{R}$: one may indeed use the Kolmogorov extension theorem with the finite dimensional distributions

$$\mathbb{P} \left[\hat{m}_s \in A_0, \hat{m}_{s+t_1} \in A_1, \dots, \hat{m}_{s+t_n} \in A_n \right] = \mathbb{P} \left[\tilde{m}_0 \in A_0, \tilde{m}_{t_1} \in A_1, \dots, \tilde{m}_{t_n} \in A_n \right]$$

for $s \in \mathbb{R}$ and $t_1, \dots, t_n \geq 0$. This extended stationary process will still be denoted by $(\tilde{m}_t)_{t \in \mathbb{R}}$.

Furthermore, we require that m satisfies the following mixing property.

Assumption 3 (Mixing). *There exists a non-increasing, integrable function $\gamma_{mix} \in L^1(\mathbb{R}^+)$ such that, for any initial state $n \in E$, there exists a càdlàg coupling $(m_t^*(n), \tilde{m}_t^*)_{t \geq 0}$, whose marginal laws are respectively those of $(m_t(n))_{t \geq 0}$ and $(\tilde{m}_t)_{t \geq 0}$, satisfying*

$$\mathbb{E} \|m_t^*(n) - \tilde{m}_t^*\|_E \leq \gamma_{mix}(t), \quad t \geq 0. \quad (1.21)$$

An immediate consequence is the following : for any Lipschitz-continuous function $\psi : E \rightarrow \mathbb{R}$ such that $\int_E \psi(n) d\nu(n) = 0$,

$$\left| P_t \psi(n) \right| = \left| \mathbb{E} [\psi(m_t^*(n)) - \psi(\tilde{m}_t^*)] \right| \leq \|\psi\|_{Lip} \gamma_{mix}(t) \in L_t^1(\mathbb{R}^+). \quad (1.22)$$

As a result, the coefficient $a(x)$ and the kernel $k(x, y)$ introduced in (1.8), (1.9) are well-defined : for $k(x, y)$ we may for instance write (using the stationarity of \tilde{m})

$$\int_{\mathbb{R}} \left| \mathbb{E} [\tilde{m}_0(x) \tilde{m}_t(y)] \right| dt = \int_0^\infty \left| \mathbb{E} [\tilde{m}_0(x) \tilde{m}_t(y)] \right| dt + \int_0^\infty \left| \mathbb{E} [\tilde{m}_0(y) \tilde{m}_t(x)] \right| dt$$

and note that, using the Markov property,

$$\int_0^\infty \left| \mathbb{E} [\tilde{m}_0(x) \tilde{m}_t(y)] \right| dt = \int_0^\infty \left| \int_E n(x) \mathbb{E} [m_t(n)(y)] d\nu(n) \right| dt \leq C^* \int_0^\infty \gamma_{mix}(t) dt < \infty.$$

As discussed in section 1.2.1, the property (1.22) allows to solve a number of Poisson equations. We may indeed define, for $h \in (W_x^{2,\infty})'$,

$$M^{-1} \psi_h(n) = - \int_0^\infty P_t \psi_h(n) dt = - \int_0^\infty \mathbb{E} \langle h, m_t(n) \rangle dt.$$

It is then natural to introduce the function $n \in E \mapsto M^{-1} I(n)$ defined by

$$\forall h \in (W_x^{2,\infty})', \quad \langle M^{-1} I(n), h \rangle = M^{-1} \psi_h(n).$$

In other words,

$$M^{-1} I(n) = - \int_0^\infty \mathbb{E} [m_t(n)] dt \in W_x^{2,\infty}. \quad (1.23)$$

Note that the function $M^{-1} I$ is bounded : since $\mathbb{E} [\tilde{m}_t] = \int_E n d\nu(n) = 0$ in $W_x^{2,\infty}$, we have

$$\begin{aligned} \|M^{-1} I(n)\|_{W_x^{2,\infty}} &\leq \int_0^\infty \|\mathbb{E} [m_t(n)]\|_{W_x^{2,\infty}} dt = \int_0^\infty \|\mathbb{E} [m_t^*(n) - \tilde{m}_t^*]\|_{W_x^{2,\infty}} dt \\ &\leq \int_0^\infty \mathbb{E} \|m_t^*(n) - \tilde{m}_t^*\|_E dt \leq \int_0^\infty \gamma_{mix}(t) dt < \infty. \end{aligned}$$

Assumption 4 (Generator). For all $h \in (W_x^{2,\infty})'$, $M^{-1}\psi_h \in D(M)$ and $M^{-1}I$ satisfies

$$\|M^{-1}I(n) - M^{-1}I(n')\|_{W_x^{2,\infty}} \lesssim \|n - n'\|_E. \quad (1.24)$$

Furthermore, for all $h \in L_x^1$,

$$|M^{-1}\psi_h|^2 \in D(M) \quad \text{and} \quad \left| M|M^{-1}\psi_h|^2(n) \right| \lesssim \|h\|_{L_x^1}^2. \quad (1.25)$$

Note that, as stated in Remark 1.2.2, the assumption $M^{-1}\psi_h \in D(M)$ guarantees that

$$M \langle M^{-1}I(n), h \rangle = \langle n, h \rangle.$$

It is typically obtained under the assumptions of Proposition 1.2.4. Combining (1.24) with the mixing property (1.21) essentially allows to define $M^{-2}I$, which shall prove necessary in the calculations developed later on.

Let us conclude by discussing the additional assumption (1.10) made in Theorem 2. This requires of course some more regularity for the stationary process \tilde{m} . A natural setup would be the following : assume that the stationary measure ν is reversible (that is $(\tilde{m}_{-t})_{t \geq 0} = (\tilde{m}_t)_{t \geq 0}$ in law) and supported in $W_x^{10+\delta,\infty}$, with

$$\mathbb{E} \|\tilde{m}_t\|_{W_x^{10+\delta,\infty}}^2 = \int_E \|n\|_{W_x^{10+\delta,\infty}}^2 d\nu(n) < \infty.$$

Making use of the time-reversibility (1.8) and (1.9) become

$$a(x) = \int_0^\infty e^{-t} \mathbb{E} [\tilde{m}_0(x) \partial_x \tilde{m}_t(x)] dt, \quad k(x, y) = 2 \int_0^\infty \mathbb{E} [\tilde{m}_0(x) \tilde{m}_t(y)] dt \quad (1.26)$$

and it follows that

$$\|a\|_{W_x^{4,\infty}} \lesssim \left(\int_0^\infty e^{-t} dt \right) \left(\int_E \|n\|_{W_x^{5,\infty}}^2 d\nu(n) \right) < \infty,$$

and

$$\begin{aligned} \|k(\cdot, y)\|_{W_x^{10+\delta,\infty}} &\leq 2 \int_0^\infty \left\| \int_E n \mathbb{E} [m_t(n)(y)] d\nu(n) \right\|_{W_x^{10+\delta,\infty}} dt \\ &\leq 2 \left(\int_0^\infty \gamma_{mix}(t) dt \right) \left(\int_E \|n\|_{W_x^{10+\delta,\infty}} d\nu(n) \right) < \infty. \end{aligned}$$

Alternatively, one could drop the reversibility assumption on ν and assume that the whole process $(m_t)_{t \geq 0}$ takes values in $E \subset W_x^{10+\delta,\infty}$ with $\|\cdot\|_{W_x^{10+\delta,\infty}} \lesssim \|\cdot\|_E$.

1.2.3 An elementary example

Let us give a simple example of a Markov process $m = (m_t)_{t \geq 0}$ satisfying the assumptions detailed above, in the form of a jump process. Consider a discrete-time Markov chain $(M_i)_{i \in \mathbb{N}}$ whose (bounded) countable state space E consists of sufficiently smooth functions :

$$E = \{n^j, j \in \mathbb{N}^*\} \subset C^k(\mathbb{T}_x), \quad \|\cdot\|_E := \|\cdot\|_{C_x^k},$$

for some k large enough, say $k > 10$. Let us denote by $P = (P(n, n'))_{n, n' \in E}$ its transition matrix. Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where an i.i.d sequence $(U_i)_{i \in \mathbb{N}^*}$ of random variables uniformly distributed in $[0, 1)$ is given. The trajectory of $(M_i(n))$ for any initial data $n \in E$ can then classically be simulated in the following manner : $M_0(n) = n$ and, for all $i \in \mathbb{N}$,

$$M_{i+1}(n) = T(M_i(n); U_{i+1}),$$

where the jump function is given by

$$T(M; U) = n^j \quad \text{when} \quad \sum_{\ell=1}^{j-1} P(M, n^\ell) \leq U < \sum_{\ell=1}^j P(M, n^\ell).$$

We may define the time-continuous, constant-rate jump process associated with the discrete Markov chain (M_i) : considering a Poisson process $(N(t))_{t \geq 0}$ of rate 1 with $N(0) = 0$, independent of (U_i) , we simply define

$$m_t(n) = M_{N(t)}(n), \quad t \geq 0.$$

Let us assume that the transition matrix $P = (P(n, n'))$ is irreducible, aperiodic and positive recurrent. As a result, let us introduce the unique invariant law $\nu \in \mathcal{P}(E)$ of the chain $(N_i)_{i \in \mathbb{N}}$ (which we may assumed to be centered, in accordance with Assumption 2).

Let us prove that the mixing property given in Assumption 3 is satisfied. To this intent, we introduce a coupling similar to the one developed in the classical proof of the ergodic theorem for Markov chains. Let us consider a random variable $\tilde{n} \in E$ of distribution ν and an i.i.d sequence $(\tilde{U}_i)_{i \in \mathbb{N}^*}$ of uniform random variables, mutually independent and independent of $(U_i)_{i \in \mathbb{N}^*}$. One can then introduce the Markov chain $(\tilde{M}_i(\tilde{n}))_{i \in \mathbb{N}^*}$ defined by $\tilde{M}_0(\tilde{n}) = \tilde{n}$ and, for all $i \in \mathbb{N}$,

$$\tilde{M}_{i+1}(\tilde{n}) = T(\tilde{M}_i(\tilde{n}); \tilde{U}_{i+1}).$$

As a result, $(\tilde{M}_i(\tilde{n}))$ defines a stationary Markov chain of transition matrix P , independent of $(M_i(n))$. For fixed $n \in E$, we then define the stopping time

$$\tau = \inf \left\{ i \geq 0, M_i(n) = \tilde{M}_i(\tilde{n}) \right\}$$

and the process

$$M_i^*(n) = \begin{cases} M_i(n) & \text{if } i \leq \tau, \\ \tilde{M}_i(\tilde{n}) & \text{if } i > \tau. \end{cases}$$

One can easily see that $(M_i^*(n))$ defines a Markov chain of transition matrix P , so that

$$\left(M_i^*(n), i \geq 0 \right) \sim \left(M_i(n), i \geq 0 \right) \text{ in law.}$$

Moreover, $(C_i)_{i \in \mathbb{N}} = (M_i(n), \tilde{M}_i(\tilde{n}))_{i \in \mathbb{N}}$ defines a Markov chain on $E \times E$, of transition matrix $P \otimes P$. The assumptions made on P guarantee that $P \otimes P$ is irreducible, and since $\nu \otimes \nu$ is an invariant probability, it is also positive recurrent. As a result, since τ is the first hitting time of the diagonal $D = \{(n, n), n \in E\}$ by (C_i) , we deduce that $\mathbb{E}[\tau] < \infty$. Now, defining the continuous-time processes

$$m_t^*(n) = M_{N(t)}^*(n), \quad \tilde{m}_t^* = \tilde{M}_{N(t)}(\tilde{n}), \quad t \geq 0,$$

the coupling $(m_t^*(n), \tilde{m}_t^*)_{t \geq 0}$ has the expected marginal laws and

$$\mathbb{P} \left[m_t^*(n) \neq \tilde{m}_t^* \right] = \mathbb{P} \left[N(t) < \tau \right] = \mathbb{E} \left[e^{-t} \sum_{k=0}^{\tau-1} \frac{t^k}{k!} \right].$$

This tends to zero as t goes to infinity by dominated convergence. Regarding the integrability in time, simple calculations give

$$\int_0^\infty \mathbb{P} \left[m_t^*(n) \neq \tilde{m}_t^* \right] dt = \sum_{k=0}^\infty (k+1) \mathbb{P}(\tau > k) = \frac{1}{2} \mathbb{E}[\tau(\tau+1)].$$

The mixing Assumption 3 is therefore satisfied as soon as the hitting time τ admits a second moment. This holds for instance if the state space E is finite : it is indeed a well-known fact that hitting times of Markov chains with finite state space have exponential tails (see e.g [Roc15], Lemma 3.25). Another possible setup is the one where the transition matrix is given by $P(n, n') = \nu(n')$, that is when M_{i+1} is drawn on E according to the law ν , independently of M_i .

In the general case, the generator of $(m_t)_{t \geq 0}$ is given by

$$M\psi(n) = \sum_{n' \in E} P(n, n')\psi(n') - \psi(n)$$

and we derive that $M^{-1}I = (M^{-1}I(n))_{n \in E}$ satisfies the linear system

$$\forall n \in E, \quad \sum_{n' \in E} P(n, n')M^{-1}I(n') - M^{-1}I(n) = n.$$

When the state space E is finite, it is clear that $M^{-1}I$ satisfies the Assumption 4. In the case where $P(n, n') = \nu(n')$, since ν is centered, we get simply

$$M^{-1}I(n) = -n.$$

1.3 Solutions of the kinetic system

For $k \geq 0$, let us introduce the Banach space $(G^k, \|\cdot\|_{G^k})$ defined by

$$G^k := \left\{ f \in L^1_{x,v}, \quad \|f\|_{G^k} := \int \int (1 + |v|^k) |f(x, v)| dx dv < \infty \right\}. \quad (1.27)$$

When $f \in G_2$, we shall denote

$$\rho(f)(x) = \int f(x) dv, \quad J(f)(x) = \int v f(x) dv, \quad K(f) = \int v^2 f(x) dv.$$

Since solutions f^ε of (1.3) are of course expected to be probability densities, we also introduce the complete normed space $(G_0^k, \|\cdot\|_{G^k})$ defined by

$$G_0^k := G^k \cap \left\{ f \geq 0, \quad \int \int f(x, v) dx dv = 1 \right\}. \quad (1.28)$$

1.3.1 Path-wise weak solutions

Let us start by stating the well-posedness of the PDE system (1.3). Taking $\varepsilon = 1$ for simplicity, it may be written as

$$\begin{cases} \partial_t f + v \partial_x f + \partial_v [(m + u - v) f] = 0, \\ \partial_t u - \partial_x^2 u = J(f) - \rho u. \end{cases} \quad (1.29)$$

We consider a fixed $\omega \in \Omega$, so that a trajectory $m \equiv m(\omega)$ belongs to $D([0, T]; E)$, space of càdlàg functions taking values in the separable complete space E .

Whenever $w \in D([0, T]; E)$ is given, the solution $f[w]$ to the linear conservation equation

$$\begin{cases} \partial_t f + v \partial_x f + \partial_v [(w - v) f] = 0, \\ f_0 \in L^1_{x,v}, \end{cases}$$

is naturally expressed as

$$f_t[w](x, v) = e^t f_0 \circ \Phi_0^t(x, v), \quad (1.30)$$

where $\Phi_s^t(x, v) = (X_s^t(x, v), V_s^t(x, v))$ is the flow associated to the characteristics :

$$\begin{cases} \frac{d}{ds} X_s^t = V_s^t, & X_t^t(x, v) = x, \\ \frac{d}{ds} V_s^t = w_s(X_s^t) - V_s^t, & V_t^t(x, v) = v, \end{cases} \quad (1.31)$$

which is clearly well defined since w is globally Lipschitz-continuous in the space variable. Note that the fact that w is only càdlàg in time causes no issue since this differential problem should be thought of in integral form. It is in fact easy to check that, for $x \in D([0, T]; \mathbb{R})$,

$$\forall t \in [0, T], \quad y_t = \int_0^t x_s ds \quad \text{if and only if} \quad \forall t \in [0, T], \quad \frac{dy}{dt^+} = x_t, \quad \frac{dy}{dt^-} = x_{t^-}.$$

In the following, we will sometimes write "full" derivatives $\frac{d}{dt}$ for simplicity, but rigorously speaking one should rather consider left and right derivatives. We may now state our well-posedness result.

Proposition 1.3.1. *Assume that*

$$\int_x \int_v (1 + |v|^4) |f_0|^2 dx dv + \int_x \int_v (1 + |v|) |\nabla_{x,v} f_0| dx dv < \infty, \quad u_0 \in H_x^\eta \text{ for } \eta \in (3/2, 2). \quad (1.32)$$

Then, for all $\beta \in (3/2, \eta)$, there exists a unique couple (f, u) with $u \in C([0, T]; H_x^\beta)$ solution of the system (1.29) in the sense that

$$\begin{cases} f_t = f_t[m + u] \text{ as defined in (1.30),} \\ u_t = S(t)u_0 + \int_0^t S(t-s) [J(f_s) - \rho_s u_s] ds, \end{cases} \quad (1.33)$$

where $S(t) = e^{t\partial_x^2}$ denotes the semigroup associated to the heat equation. For any $k \geq 0$, assuming $f_0 \in G_0^k$, we have additionally $f \in C([0, T]; G_0^k)$.

The proof of Proposition 1.3.1 is based on a rather classical fixed point argument. Since it is not a central aspect of the present work, it is postponed to section 1.8.

Remark 1.3.1. *The uniqueness of the solutions via an explicit iterative construction method classically guarantees that $\omega \mapsto (f_t(\omega), u_t(\omega)) \in G_0^k \times H_x^\beta$ is $\mathcal{F}_t = \sigma(m_s, 0 \leq s \leq t)$ -measurable. It is of course to be expected that $(f_t, u_t, m_t)_{t \geq 0}$ defines a Markov process. We will not however attempt to rigorously prove it, since it is in fact not necessary for the purpose of this work.*

Let us simply justify here that the unique solution (f, u) in the sense of (1.33) satisfies some weak formulation of (1.29). In particular, we derive some estimates that will be useful to rigorously determine the generator of the process $(f_t, u_t, m_t)_{t \geq 0}$ later on.

In order to properly express the weak formulation of the equation satisfied by $f_t \in G^k$, we now introduce the space \mathcal{C}_k which should be thought of as the natural dual of G^k "along" equation (1.29).

Definition 1.3.1. *For $k \geq 0$ and $\xi \in C^1(\mathbb{T} \times \mathbb{R})$ let us introduce the norm*

$$N_k(\xi) = \left\| \frac{\xi}{1 + |v|^k} \right\|_{L_{x,v}^\infty} + \left\| \frac{(1 + |v|)\nabla_{x,v}\xi}{1 + |v|^k} \right\|_{L_{x,v}^\infty} \quad (1.34)$$

and the associated normed space, given by the closure

$$\mathcal{C}_k = \overline{\{\xi \in C^1(\mathbb{T}^d \times \mathbb{R}^d), N_k(\xi) < \infty\}}.$$

Proposition 1.3.2. *Let $k \geq 0$ and assume that $f_0 \in G^k$. Then for all $\xi \in \mathcal{C}_k$,*

$$\langle f_t, \xi \rangle - \langle f_0, \xi \rangle = \int_0^t \langle f_s, v\partial_x \xi + (u_s + m_s - v)\partial_v \xi \rangle ds$$

and the following estimate holds : for all $t, s \in [0, T]$,

$$\left| \frac{\langle f_{t+s}, \xi \rangle - \langle f_t, \xi \rangle}{s} \right| \leq C(k, C^*, \|u\|_{L_{t,x}^\infty}, T) N_k(\xi) \|f_0\|_{G^k}. \quad (1.35)$$

Proof. Let us denote $\Phi_t := \Phi_t^0$ for simplicity. Since

$$\frac{\langle f_{t+s}, \xi \rangle - \langle f_t, \xi \rangle}{s} = \int \int f_0(x, v) \frac{\xi \circ \Phi_{t+s}(x, v) - \xi \circ \Phi_t(x, v)}{s} dx dv,$$

we want to dominate the integrand uniformly for $t, t + s \in [0, T]$. For all $t \geq 0$,

$$\frac{\xi \circ \Phi_{t+s}(x, v) - \xi \circ \Phi_t(x, v)}{s} = \frac{1}{s} \int_t^{t+s} \frac{d}{d\sigma} \xi(\Phi_\sigma(x, v)) d\sigma.$$

with

$$\frac{d}{dt} \xi(\Phi_t(x, v)) = V_t(x, v) \partial_x \xi \circ \Phi_t(x, v) + ((u_t + m_t)(X_t(x, v)) - V_t(x, v)) \partial_v \xi \circ \Phi_t(x, v)$$

which, by design, can be bounded by

$$(1 + C^* + \|u\|_{L_{t,x}^\infty}) N_k(\xi) (1 + |V_t(x, v)|^k).$$

From the sub-linearity of the equation of characteristics (1.31), we easily derive

$$\forall t \in [0, T], |V_t(x, v)|^k \leq C(C^*, \|u\|_{L_{t,x}^\infty}, T) (1 + |v|^k).$$

Combining these bounds, we have dominated

$$\left| f(x, v) \frac{\xi \circ \Phi_{t+s}(x, v) - \xi \circ \Phi_t(x, v)}{s} \right|$$

uniformly for $t, t + s \in [0, T]$ by an expression of the form

$$C(k, C^*, \|u\|_{L_{t,x}^\infty}, T) |f_0(x, v)| (1 + |v|^k).$$

This is exactly the expected estimate, and the dominated convergence theorem leads to

$$\frac{d}{dt} \langle f_t, \xi \rangle = \int_{x,v} f(x, v) \frac{d}{dt} \xi \circ \Phi_t(x, v) dx dv = \langle f_t, v\partial_x \xi + (u_t + m_t - v)\partial_v \xi \rangle.$$

□

Similarly, u is a weak solution of the corresponding equation :

Proposition 1.3.3. *For all $\xi \in C^2(\mathbb{T})$,*

$$\langle u_t, \xi \rangle - \langle u_0, \xi \rangle = \int_0^t \langle u_s, \partial_x^2 \xi \rangle + \langle J(f_s) - \rho_s u_s, \xi \rangle ds.$$

We deduce the estimate : for all $t, s \in [0, T]$,

$$\left| \frac{\langle u_{t+s}, \xi \rangle - \langle u_t, \xi \rangle}{s} \right| \leq C(\|u\|_{L_{t,x}^\infty}, \|f_0\|_{G_1}, T) \left(\|\xi\|_\infty + \|\partial_x^2 \xi\|_\infty \right). \quad (1.36)$$

Proof. The weak formulation is classically derived from the mild form (1.33) (cf. [Bal77] for example). The estimate is easily deduced. \square

1.3.2 Estimates uniform in ε

We now establish some uniform bounds for the solution $(f^\varepsilon, u^\varepsilon)$ of (1.3).

Proposition 1.3.4. *For all $\varepsilon \in (0, 1)$, we have (almost surely)*

$$\forall t \in [0, T], \quad \|f_t^\varepsilon\|_{L_{x,v}^1} = \|f_0^\varepsilon\|_{L_{x,v}^1} = 1$$

and

$$\begin{aligned} \sup_{t \in [0, T]} \|f_t^\varepsilon\|_{G_1} &\leq C(T, C^*, \|u_0^\varepsilon\|_{L_x^\infty}, \|f_0^\varepsilon\|_{G_1}), \\ \sup_{t \in [0, T]} \|u_t^\varepsilon\|_{L_x^\infty} &\leq C(T, C^*, \|u_0^\varepsilon\|_{L_x^\infty}, \|f_0^\varepsilon\|_{G_1}). \end{aligned}$$

Démonstration. The conservation of the L^1 norm of f^ε is clear from the transport form (1.33). Assuming $f_0^\varepsilon \geq 0$, $f_0^\varepsilon \in C_c^\infty$ (we conclude in the general case by density), we get

$$\frac{d}{dt} \int_x \int_v |v| f_t^\varepsilon(x, v) dx dv = -\varepsilon^{-2} \int_x \int_v |v| \partial_v \left[(m_t^\varepsilon + \varepsilon u_t^\varepsilon - v) f_t^\varepsilon \right] dx dv$$

hence denoting $\bar{J}(f) = \int_x \int_v |v| f dx dv$, (so that $\|f\|_{G_1} = \|f\|_{L_{x,v}^1} + \bar{J}(f)$) we get

$$\frac{d}{dt} \bar{J}(f_t^\varepsilon) \leq \varepsilon^{-2} (\|m_t^\varepsilon\|_{L_x^\infty} + \varepsilon \|u_t^\varepsilon\|_{L_x^\infty}) - \varepsilon^{-2} \bar{J}(f_t^\varepsilon)$$

and Grönwall's lemma gives

$$\bar{J}(f_t^\varepsilon) \leq e^{-\varepsilon^{-2}t} \bar{J}(f_0^\varepsilon) + \int_0^t \varepsilon^{-2} e^{-\varepsilon^{-2}(t-s)} (\|m_s^\varepsilon\|_{L_x^\infty} + \varepsilon \|u_s^\varepsilon\|_{L_x^\infty}) ds.$$

We may note that $\int_0^t \varepsilon^{-2} e^{-\varepsilon^{-2}s} ds = 1 - e^{-\varepsilon^{-2}t} \leq 1$ so that we get

$$\bar{J}(f_t^\varepsilon) \leq \bar{J}(f_0^\varepsilon) + (C^* + \varepsilon \sup_{s \in [0, t]} \|u_s^\varepsilon\|_{L_x^\infty}). \quad (1.37)$$

On the other hand, using the mild form (1.33) for u^ε , usual heat-semigroup estimates, and making use of the embedding $H_x^{1/2+\delta} \subset L_x^\infty$ for any $\delta > 0$, we have

$$\begin{aligned} \|u_t^\varepsilon\|_{L_x^\infty} &\leq \|u_0^\varepsilon\|_{L_x^\infty} + C \int_0^t |t-s|^{-1/2-\delta} \|J(f_s^\varepsilon) - \varepsilon \rho_s^\varepsilon u_s^\varepsilon\|_{L_x^1} ds \\ &\leq \|u_0^\varepsilon\|_{L_x^\infty} + C \int_0^t |t-s|^{-1/2-\delta} \|J(f_s^\varepsilon)\|_{L_x^1} + C\varepsilon \int_0^t |t-s|^{-1/2-\delta} \|u_s^\varepsilon\|_{L_x^\infty} ds \\ &\leq \|u_0^\varepsilon\|_{L_x^\infty} + C(\|J(f_0^\varepsilon)\|_{L_x^1} + C^*) + C\varepsilon \int_0^t |t-s|^{-1/2-\delta} \sup_{\sigma \in [0, s]} \|u_\sigma^\varepsilon\|_{L_x^\infty} ds, \end{aligned}$$

using (1.37). We finally get an estimate of the form

$$\sup_{s \in [0, t]} \|u_s^\varepsilon\|_{L_x^\infty} \leq C(\|u_0^\varepsilon\|_{L_x^\infty}, \|f_0^\varepsilon\|_{G_1}, C^*) \left(1 + \varepsilon \int_0^t |t-s|^{-1/2-\delta} \sup_{\sigma \in [0, s]} \|u_\sigma^\varepsilon\|_{L_x^\infty} ds \right)$$

and deduce the desired estimates using a singular Grönwall inequality. \square

The higher v -moments for f are obtained by induction using the same arguments. For the sake of brevity, we omit the proof.

Proposition 1.3.5 (Higher moments for f). *For all $k \geq 1$, we have (almost surely)*

$$\sup_{t \in [0, T]} \|f_t^\varepsilon\|_{G^k} \leq C(k, T, C^*, \|u_0^\varepsilon\|_{L_x^\infty}, \|f_0^\varepsilon\|_{G^k}).$$

Let us finally state some compactness estimate for u^ε .

Proposition 1.3.6 (Compactness for u). *For $\alpha \in (1/2, 3/2)$, we have (almost surely)*

$$\begin{aligned} \|u_t^\varepsilon\|_{H_x^\alpha} &\leq C(\alpha, T, C^*, \|u_0^\varepsilon\|_{H_x^\alpha}, \|f_0^\varepsilon\|_{G_1}) \\ \forall \beta < \alpha \quad \|u_t^\varepsilon - u_s^\varepsilon\|_{H_x^\beta} &\leq C(\alpha, \beta, T, C^*, \|u_0^\varepsilon\|_{H_x^\alpha}, \|f_0^\varepsilon\|_{G_1}) \int_s^t \theta(\sigma) d\sigma, \end{aligned}$$

where $\theta \in L^1([0, T])$ is independent of ε .

Démonstration. Consider the mild form

$$u_t^\varepsilon = S(t)u_0^\varepsilon + \int_0^t S(t-\sigma)g_\sigma^\varepsilon d\sigma, \quad \text{where } g_\sigma^\varepsilon = J(f_\sigma^\varepsilon) - \varepsilon \rho_\sigma^\varepsilon u_\sigma^\varepsilon. \quad (1.38)$$

The embedding $L_x^1 \subset H_x^{-1/2-\delta}$ for all $\delta > 0$, and a classical regularity estimate for the heat kernel $S(t)$ lead to

$$\|S(t)w\|_{H_x^\alpha} \lesssim (1 + t^{-(\alpha+1/2+\delta)/2}) \|w\|_{L_x^1}, \quad (1.39)$$

from which we derive

$$\|u_t^\varepsilon\|_{H_x^\alpha} \lesssim \|u_0^\varepsilon\|_{H_x^\alpha} + \int_0^t (1 + |t-\sigma|^{-(\alpha+1/2+\delta)/2}) \|g_\sigma^\varepsilon\|_{L_x^1} d\sigma.$$

Using Proposition 1.3.4, we have $\|g_t^\varepsilon\|_{L_x^{1,v}} \leq C(T, C^*, \|u_0^\varepsilon\|_{L^\infty}, \|f_0^\varepsilon\|_{G_1})$, which leads to the first estimate, since $(\alpha + 1/2 + \delta)/2 < 1$ for δ small enough.

For the second estimate, we first write, for all $\xi \in H_x^{-\beta}$,

$$\begin{aligned} \left| \langle S(t)u_0^\varepsilon - S(s)u_0^\varepsilon, \xi \rangle \right| &= \left| \int_s^t \langle S(\sigma)u_0^\varepsilon, \partial_x^2 \xi \rangle d\sigma \right| \leq \|\xi\|_{H_x^{-\beta}} \int_s^t \|S(\sigma)u_0^\varepsilon\|_{H_x^{2+\beta}} d\sigma \\ &\leq C\|\xi\|_{H_x^{-\beta}} \|u_0^\varepsilon\|_{H_x^\alpha} \int_s^t (1 + \sigma^{-(2+\beta-\alpha)/2}) d\sigma, \end{aligned}$$

which gives $\|S(t)u_0^\varepsilon - S(s)u_0^\varepsilon\|_{H_x^\beta} \leq \|u_0^\varepsilon\|_{H_x^\alpha} \int_s^t \theta_1(\sigma) d\sigma$ with $\theta_1 \in L^1$ since $(2 + \beta - \alpha)/2 < 1$. Finally,

$$\left\| \int_s^t S(t-\sigma)g_\sigma^\varepsilon d\sigma \right\|_{H_x^\beta} \leq \int_s^t \|S(t-\sigma)g_\sigma^\varepsilon\|_{H_x^\beta} d\sigma$$

and we use again (1.39) with β instead of α to conclude from the mild form (1.38). \square

1.3.3 Generator of the process

As previously mentioned, $(f_t^\varepsilon, u^\varepsilon)_{t \geq 0}$ is not a Markov process in itself, however the process $(f_t^\varepsilon, u^\varepsilon, m_t^\varepsilon)_{t \geq 0}$ is. In view of (1.3), its generator is naturally expected to be

$$\mathcal{L}^\varepsilon \psi = \frac{1}{\varepsilon^2} \mathcal{L}_\# \psi + \frac{1}{\varepsilon} \mathcal{L}_\flat \psi + \mathcal{L}_0 \psi + \varepsilon \mathcal{L}_1 \psi, \quad (1.40)$$

where

$$\begin{cases} \mathcal{L}_\# \psi(f, u, n) = -\left\langle \partial_v [(n-v)f], D_f \psi(f, u, n) \right\rangle + M\psi(f, u, n), \\ \mathcal{L}_\flat \psi(f, u, n) = -\left\langle v \partial_x f + u \partial_v f, D_f \psi(f, u, n) \right\rangle, \\ \mathcal{L}_0 \psi(f, u, n) = \left\langle \partial_x^2 u + J(f), D_u \psi(f, u, n) \right\rangle, \\ \mathcal{L}_1 \psi(f, u, n) = -\left\langle \rho u, D_u \psi(f, u, n) \right\rangle. \end{cases} \quad (1.41)$$

Let us quickly get into the details of the notation "D" used here. For $\psi : G^k \rightarrow \mathbb{R}$, let $\text{Diff}_f \psi$ denote the usual differential in $f \in G^k$ of ψ , that is

$$\forall f \in G^k, \quad \text{Diff}_f \psi(f) \in (G^k)' = L^1\left(\mathbb{T}^d \times \mathbb{R}^d, (1 + |v|^k) dx dv\right)' \simeq L_{x,v}^\infty.$$

There exists $h(f) \in L_{x,v}^\infty$ such that

$$\forall g \in G^k, \quad \text{Diff}_f \psi(f)(g) = \left\langle g, h(f) \right\rangle_{G^k, (G^k)'} = \left\langle g, (1 + |v|^k) h(f) \right\rangle.$$

The differential $\text{Diff}_f \psi(f)$ is naturally identified with the $L_{x,v}^\infty$ function $h(f)$. For convenience, we shall rather denote by $D_f \psi(f)$ the function $(1 + |v|^k) h(f)$, so that

$$\forall g \in G^k, \quad \text{Diff}_f \psi(f)(g) = \left\langle g, D_f \psi(f) \right\rangle.$$

To avoid cluttering notation, we do not indicate the dependence of $D_f \psi(f)$ on the choice of the parameter $k \geq 1$. In section 1.5, where the perturbed test-function method is developed, the value $k = 3$ is fixed for good. The natural bound on $D_f \psi$ is consequently

$$\left\| \frac{D_f \psi(f)}{1 + |v|^k} \right\|_{L_{x,v}^\infty} = \|\text{Diff}_f \psi(f)\|_{G^{k'}} < \infty.$$

Similarly, for $\psi : H_x^\beta \rightarrow \mathbb{R}$, $D_u \psi$ is defined so that

$$\forall w \in H_x^\beta, \quad \text{Diff}_u \psi(u)(w) = \left\langle w, D_u \psi(u) \right\rangle.$$

We will not properly identify the complete domain of the infinitesimal generator \mathcal{L}^ε : for the purpose of this paper, it will in fact be sufficient to show that \mathcal{L}^ε satisfies the martingale problem associated with $(f^\varepsilon, u^\varepsilon, m^\varepsilon)$ for a large enough class of test functions.

Definition 1.3.2 (Good test function). *Let $k \geq 1$ and $\alpha \in (1/2, 3/2)$ be given. Recalling the space G_0^k introduced (1.28), define the state space for the process $(f_t^\varepsilon, u_t^\varepsilon, m_t^\varepsilon)_{t \geq 0}$:*

$$\mathcal{X}_k^\alpha = G_0^k \times H_x^\alpha \times E. \quad (1.42)$$

A function $\psi : (f, u, n) \in G^k \times H_x^\alpha \times E \rightarrow \mathbb{R}$ is said to be a "good test function" on \mathcal{X}_k^α when :

1. $\psi \in \mathcal{B}^{loc}(\mathcal{X}_k^\alpha)$
2. For all $(f, u) \in G_0^k \times H_x^\alpha$, $\psi(f, u, \cdot) \in D(M)$ and $M\psi \in C_b^{loc}(\mathcal{X}_k^\alpha)$
3. For all $(u, n) \in H_x^\alpha \times E$, $\psi(\cdot, u, n)$ is G^k -differentiable and

$$D_f \psi : \begin{array}{ccc} \mathcal{X}_k^\alpha & \longrightarrow & (\mathcal{C}_k, N_k) \\ (f, u, n) & \longmapsto & D_f \psi(f, u, n) \end{array} \quad \text{is } C_b^{loc}(\mathcal{X}_k^\alpha; \mathcal{C}_k),$$

where the normed space (\mathcal{C}_k, N_k) has been introduced in Definition (1.3.1)

4. For all $(f, n) \in G_0^k \times E$, $\psi(f, \cdot, n)$ is H_x^α -differentiable and

$$D_u \psi : \begin{array}{ccc} \mathcal{X}_k^\alpha & \longrightarrow & C^2(\mathbb{T}) \\ (f, u, n) & \longmapsto & D_u \psi(f, u, n) \end{array} \quad \text{is } C_b^{loc}(\mathcal{X}_k^\alpha; C^2(\mathbb{T})).$$

Note that the local boundedness required in point 3. can be expressed as

$$\left\{ \begin{array}{l} |D_f \psi(f, u, n)| \leq C \left(\|f\|_{G^k}, \|u\|_{H_x^\alpha} \right) (1 + |v|^k), \\ |v \partial_x D_f \psi(f, u, n)| \leq C \left(\|f\|_{G^k}, \|u\|_{H_x^\alpha} \right) (1 + |v|^k), \\ |(1 + |v|) \partial_v D \psi(f, n)| \leq C \left(\|f\|_{G^k}, \|u\|_{H_x^\alpha} \right) (1 + |v|^k), \end{array} \right.$$

which is a natural condition to ensure that $\mathcal{L}^\varepsilon \psi$ is well-defined and $C_b^{loc}(\mathcal{X}_k^\alpha)$. This suggests that, when $f \in G_0^k$, typical good test functions involve moments of order at most $k - 1$ for f . Later on, we will for instance consider the test function

$$\psi(f, n) = \left\langle J_{k-1}(f), (\partial_x M^{-1} I(n)) \xi \right\rangle, \quad \xi \in W_x^{1, \infty},$$

and linear combinations of such functions. Introducing the canonical filtration of m^ε

$$\mathcal{F}_t^\varepsilon = \sigma\left(m_s^\varepsilon, 0 \leq s \leq t\right) = \mathcal{F}_{\varepsilon^{-2}t}, \quad t \geq 0,$$

the process $t \mapsto (f_t^\varepsilon, u_t^\varepsilon) \in G_0^k \times H_x^\alpha$ is naturally $(\mathcal{F}_t^\varepsilon)$ -adapted, as mentioned in Remark 1.3.1. We can now state the following result :

Proposition 1.3.7. *Let $k \geq 1$, and $\alpha \in (1/2, 3/2)$ be given. Let $\psi \equiv \psi(f, u, n)$ be a good test function on \mathcal{X}_k^α . Then ψ satisfies the martingale problem for \mathcal{L}^ε in the sense that*

$$\forall (f, u, n) \in \mathcal{X}_k^\alpha, \quad \mathcal{L}^\varepsilon \psi(f, u, n) \text{ is well-defined, } \mathcal{L}^\varepsilon \psi \in C_b^{loc}(\mathcal{X}_k^\alpha)$$

and for any $(f_0^\varepsilon, u_0^\varepsilon, n) \in \mathcal{X}_k^\alpha$ (with $f_0^\varepsilon, u_0^\varepsilon$ satisfying the assumption (1.32)),

$$M_\psi^\varepsilon(t) = \psi(f_t^\varepsilon, u_t^\varepsilon, m_t^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon \psi(f_s^\varepsilon, u_s^\varepsilon, m_s^\varepsilon) ds, \quad t \geq 0$$

defines a càdlàg, L^2 martingale with respect to the filtration $(\mathcal{F}_t^\varepsilon)_{t \geq 0}$ under $\mathbb{P}_{(f_0^\varepsilon, u_0^\varepsilon, n)}$.

Moreover, if $|\psi|^2$ is also a good test function, the predictable quadratic variation is given by :

$$\left[M_\psi^\varepsilon\right](t) = \frac{1}{\varepsilon^2} \int_0^t \left(M|\psi|^2 - 2\psi M\psi\right)(f_s^\varepsilon, u_s^\varepsilon, m_s^\varepsilon, s) ds.$$

Remark 1.3.2. *The predictable quadratic variation $\left[M\right](t)$ is the unique càdlàg, predictable, increasing process such that $|M(t)|^2 - \left[M\right](t)$ defines a martingale. When $M(t)$ is moreover continuous, this coincides with the standard quadratic variation (see [JS03], p.38).*

Proof. The assumptions on ψ clearly guarantee that $\mathcal{L}^\varepsilon \psi$ is well defined and $C_b^{loc}(\mathcal{X}_k^\alpha)$. To simplify, we shall consider the case $\varepsilon = 1$ and write (f_t, u_t, m_t) instead of $(f_t^\varepsilon, u_t^\varepsilon, m_t^\varepsilon)$. The estimates established in the previous subsections show that, (f_0, u_0) being fixed,

$$\forall T > 0, \quad \forall t \in [0, T], \quad \|f_t\|_{G^k} + \|u_t\|_{H_x^\alpha} \leq C(T), \quad \|m_t\|_E \leq C^*,$$

almost surely under the probability $\mathbb{P}_{(f_0^\varepsilon, u_0^\varepsilon, n)}$. It follows that $M_\psi^\varepsilon(t)$ is well defined and in $L^\infty(\Omega)$ for $t \geq 0$. We now need to prove that, for $s < t$ and h bounded and $\mathcal{F}_s^\varepsilon$ -measurable,

$$\mathbb{E}\left[\left(\psi(f_t, u_t, m_t) - \psi(f_s, u_s, m_s) - \int_s^t \mathcal{L}^\varepsilon \psi(f_\sigma, u_\sigma, m_\sigma) d\sigma\right)h\right] = 0. \quad (1.43)$$

To this purpose, let us formulate two natural lemmas :

Lemma 1.3.1. *We have*

$$\begin{aligned} \frac{d}{dt} \psi(f_t, u, n) &= \left\langle D_f \psi(f_t, u, n), \partial_t f_t \right\rangle \\ &:= \left\langle f_t, v \partial_x D_f \psi(f_t, u, n) + (u_t + m_t - v) \partial_v D_f \psi(f_t, u, n) \right\rangle. \end{aligned}$$

Proof of the lemma. One can simply write

$$\frac{\psi(f_{t+s}, u, n) - \psi(f_t, u, n)}{s} = \left\langle D_f \psi(f_t, u, n), \frac{f_{t+s} - f_t}{s} \right\rangle + R_s$$

with

$$R_s = \int_0^1 \left\langle D_f \psi(f_t + \theta(f_{t+s} - f_t), u, n) - D_f \psi(f_t, u, n), \frac{f_{t+s} - f_t}{s} \right\rangle d\theta.$$

Since $D_f \psi(f_t, u, n) \in \mathcal{C}_k$, Proposition 1.3.2 guarantees that

$$\left\langle D_f \psi(f_t, u, n), \frac{f_{t+s} - f_t}{s} \right\rangle \xrightarrow{s \rightarrow 0} \left\langle D_f \psi(f_t, u, n), \partial_t f_t \right\rangle$$

and that

$$|R_s| \lesssim \sup_{\theta \in [0, 1]} N_k \left(D_f \psi(f_t + \theta(f_{t+s} - f_t), u, n) - D_f \psi(f_t, u, n) \right)$$

which tends to 0 as $s \rightarrow 0$, since $f_{t+s} \rightarrow f_t$ in G_0^k and $D_f \psi$ is continuous. \square

This second lemma is proved in a similar fashion.

Lemma 1.3.2. *We have*

$$\begin{aligned} \frac{d}{dt}\psi(f, u_t, n) &= \left\langle D_u\psi(f, u_t, n), \partial_t u_t \right\rangle \\ &:= \left\langle u_s, \partial_x^2 D_u\psi(f, u_t, n) \right\rangle + \left\langle J(f_t) - \rho_t u_t, D_u\psi(f, u_t, n) \right\rangle. \end{aligned}$$

Introducing a subdivision $t_0 = s < t_1 < \dots < t_N = t$ of $[s, t]$ of step $\delta = \max_i |t_{i+1} - t_i|$, we may split (1.43) into 3 terms :

$$\begin{aligned} A &= \mathbb{E} \left[\left(\sum_i \psi(f_{t_i}, u_{t_i}, m_{t_{i+1}}) - \psi(f_{t_i}, u_{t_i}, m_{t_i}) - \int_{t_i}^{t_{i+1}} M\psi(f_\sigma, u_\sigma, m_\sigma) d\sigma \right) h \right], \\ B &= \mathbb{E} \left[\left(\sum_i \psi(f_{t_{i+1}}, u_{t_{i+1}}, m_{t_{i+1}}) - \psi(f_{t_i}, u_{t_{i+1}}, m_{t_{i+1}}) - \int_{t_i}^{t_{i+1}} \left\langle D_f\psi(f_\sigma, u_\sigma, m_\sigma), \partial_\sigma f_\sigma \right\rangle d\sigma \right) h \right], \\ C &= \mathbb{E} \left[\left(\sum_i \psi(f_{t_i}, u_{t_{i+1}}, m_{t_{i+1}}) - \psi(f_{t_i}, u_{t_i}, m_{t_{i+1}}) - \int_{t_i}^{t_{i+1}} \left\langle D_u\psi(f_\sigma, u_\sigma, m_\sigma), \partial_\sigma u_\sigma \right\rangle d\sigma \right) h \right]. \end{aligned}$$

Let us treat the first term A : conditioning with respect to $\mathcal{F}_{t_i}^\varepsilon$ and using the Markov property,

$$\mathbb{E} \left[\left(\psi(f_{t_i}, u_{t_i}, m_{t_{i+1}}) - \psi(f_{t_i}, u_{t_i}, m_{t_i}) \right) h \right] = \mathbb{E} \left[\left(P_{t_{i+1}-t_i}[\psi(f_{t_i}, u_{t_i}, \cdot)](m_{t_i}) - \psi(f_{t_i}, u_{t_i}, m_{t_i}) \right) h \right].$$

We note that, for all $t \geq 0$ and $(f, u, n) \in \mathcal{X}_k^\alpha$,

$$P_t[\psi(f, u, \cdot)](n) - \psi(f, u, n) = \int_0^t P_\sigma M\psi(f, u, n) d\sigma = \mathbb{E}_n \int_0^t M\psi(f, u, m_\sigma) d\sigma,$$

where we have used Kolmogorov's forward equation from Proposition 1.2.1 and the interversion $\mathbb{E} f = f \mathbb{E}$ is justified since $M\psi$ is locally bounded. We are hence led to

$$\begin{aligned} \mathbb{E} \left[\left(\psi(f_{t_i}, u_{t_i}, m_{t_{i+1}}) - \psi(f_{t_i}, u_{t_i}, m_{t_i}) \right) h \right] &= \mathbb{E} \left[\left(\mathbb{E}_{m_{t_i}} \int_0^{t_{i+1}-t_i} M\psi(f_{t_i}, u_{t_i}, m_\sigma) d\sigma \right) h \right] \\ &= \mathbb{E} \left[\left(\int_0^{t_{i+1}-t_i} M\psi(f_{t_i}, u_{t_i}, m_{t_i+\sigma}) d\sigma \right) h \right] \\ &= \mathbb{E} \left[\left(\int_{t_i}^{t_{i+1}} M\psi(f_{t_i}, u_{t_i}, m_\sigma) d\sigma \right) h \right] \end{aligned}$$

where we have once again used the Markov property. It follows that A can be rewritten as

$$A = \mathbb{E} \left[\left(\int_s^t a_\delta(\sigma) d\sigma \right) h \right]$$

with

$$a_\delta(\sigma) = \sum_i \mathbb{1}_{[t_i, t_{i+1}]}(\sigma) \left(M\psi(f_{t_i}, u_{t_i}, m_\sigma) - M\psi(f_\sigma, u_\sigma, m_\sigma) \right).$$

For fixed $\sigma \in [s, t]$, since $t \mapsto (f_t, u_t) \in G_0^k \times H_x^\alpha$ and $M\psi$ are continuous, it is clear that, as $\delta \rightarrow 0$, we have $a_\delta(\sigma) \rightarrow 0$ almost surely (hence in probability). Moreover, $M\psi$ being locally bounded, we get $a_\delta \in L^\infty([s, t] \times \Omega)$ with a bound uniform in δ . In particular, $\mathbb{E}[a_\delta(\sigma)h]$ tends to 0 and is uniformly integrable in $\sigma \in [s, t]$, so that we may conclude $A \xrightarrow{\delta \rightarrow 0} 0$.

Using the two lemmas introduced earlier, one can write in a similar fashion

$$B = \mathbb{E} \left[\left(\int_s^t b_\delta(\sigma) d\sigma \right) h \right], \quad C = \mathbb{E} \left[\left(\int_s^t c_\delta(\sigma) d\sigma \right) h \right],$$

with

$$\begin{aligned} b_\delta(\sigma) &= \sum_i \mathbb{1}_{[t_i, t_{i+1}]}(\sigma) \left\langle D_f\psi(f_\sigma, u_{t_{i+1}}, m_{t_{i+1}}) - D_f\psi(f_\sigma, u_\sigma, m_\sigma), \partial_\sigma f_\sigma \right\rangle \\ c_\delta(\sigma) &= \sum_i \mathbb{1}_{[t_i, t_{i+1}]}(\sigma) \left\langle D_u\psi(f_{t_i}, u_\sigma, m_{t_{i+1}}) - D_u\psi(f_\sigma, u_\sigma, m_\sigma), \partial_\sigma u_\sigma \right\rangle \end{aligned}$$

and conclude in the same way using the estimates from Propositions 1.3.2 and 1.3.3 and the fact that $(m_t)_{t \geq 0}$ is stochastically continuous and $D_f \psi, D_u \psi$ are $C_b^{loc}(\mathcal{X}_k^\alpha)$.

If $|\psi|^2$ is also a good test function, then $\mathcal{L}^\varepsilon |\psi|^2$ is locally bounded, and $M_{|\psi|^2}^\varepsilon(t)$ defines a martingale. Using these facts, (we may refer to the proof of Theorem B.3 in [DV17] for details) one can show that the predictable quadratic variation of M_ψ^ε is given by

$$d[M_\psi^\varepsilon](t) = \left(\mathcal{L}^\varepsilon |\psi|^2 - 2\psi \mathcal{L}^\varepsilon \psi \right) (f_t^\varepsilon, u_t^\varepsilon, m_t^\varepsilon) dt$$

which, in our case, boils down to $\varepsilon^{-2}(M|\psi|^2 - 2\psi M\psi)$ since $\mathcal{D}|\psi|^2 = 2\psi \mathcal{D}\psi$ for any first order differential operator \mathcal{D} . \square

1.4 The auxiliary process

In section 1.5, we will be led to solve Poisson equations relative to the partial generator $\mathcal{L}_\#$ defined in (1.41). To this purpose, it is natural to study the auxiliary Markov process $(g_t, m_t)_{t \geq 0}$ whose generator is expected to be $\mathcal{L}_\#$. We hence define $g_t \equiv g_t(f, n)$ as the solution of

$$\begin{cases} \partial_t g_t + \partial_v [(m_t(n) - v)g_t] = 0, \\ g_0(f, n) = f, \quad m_0(n) = n, \end{cases} \quad (1.44)$$

and let $(Q_t)_{t \geq 0}$ be the corresponding semigroup

$$\forall \psi \in \mathcal{B}^{loc}(G_0^k \times E), \quad Q_t \psi(f, n) = \mathbb{E}[\psi(g_t(f, n), m_t(n))].$$

The characteristics associated with the conservation equation (1.44) solve

$$\dot{V}_t(v) = m_t(x) - V_t(v),$$

leading to the explicit expression

$$V_t(v) = e^{-t} \left(v + \int_0^t e^u m_u(x) du \right).$$

Definition 1.4.1. *The solution of (1.44) is defined by*

$$g_t(f, n)(x, v) = e^t f \left(x, e^t \left[v - w_t(n)(x) \right] \right), \quad t \geq 0 \quad (1.45)$$

where

$$w_t(n) = \int_0^t e^{-(t-s)} m_s(n) ds = \int_0^t e^{-s} m_{t-s}(n) ds \in W_x^{2, \infty}. \quad (1.46)$$

It is easy to verify that, for all $f \in G_0^k$, we have $g(f, n) \in C([0, T]; G_0^k)$ with a bound

$$\|g_t(f, n)\|_{G^k} \leq C(T, C^*, \|f\|_{G^k}).$$

Additionally, $g_t(f, n)$ defines a weak solution of (1.44) just as in Proposition 1.3.2.

1.4.1 Generator of the auxiliary process

With the explicit expression (1.45), one could easily prove that the auxiliary process $(g_t, m_t)_{t \geq 0}$ rigorously defines a (locally bounded) Markov process on the state space $G_0^k \times E$. In order to solve Poisson equations later, it is of interest this time to truly characterize the generator in terms of b.p.c convergence.

Proposition 1.4.1. *Let $k \geq 1$ be given, and let $\psi \equiv \psi(f, n)$ be a good test function on $G_0^k \times E$. Then $\psi \in D(\mathcal{L}_\#)$ in the sense that $\mathcal{L}_\# \psi \in C_b^{loc}(G_0^k \times E)$ and*

$$\frac{Q_t \psi(f, n) - \psi(f, n)}{t} \xrightarrow[t \rightarrow 0^+]{b.p.c} \mathcal{L}_\# \psi(f, n).$$

Proof. The properties satisfied by the good test function ψ clearly guarantee that $\mathcal{L}_\# \psi$ is well defined and $C_b^{loc}(G_0^k \times E)$. Let us denote $(g_t(f, n), m_t(n))$ by (g_t, m_t) for simplicity. We split the ratio into two terms

$$\frac{Q_t \psi(f, n) - \psi(f, n)}{t} = \mathbb{E} \left[\frac{\psi(f, m_t) - \psi(f, n)}{t} \right] + \mathbb{E} \left[\frac{\psi(g_t, m_t) - \psi(f, m_t)}{t} \right].$$

The first term is simply $t^{-1}(P_t \psi(f, n) - \psi(f, n))$ which converges to $M\psi(f, n)$ and can be bounded as $t \rightarrow 0$ by

$$\sup_{t \in [0, 1]} \left| \frac{d}{ds} P_s \psi(f, n) \right| = \sup_{t \in [0, 1]} \left| P_t M \psi(f, n) \right|$$

which is locally bounded in (f, n) . For the second term, we write

$$\frac{\psi(g_t, m_t) - \psi(f, m_t)}{t} = \left\langle D_f \psi(f, n), \frac{g_t - f}{t} \right\rangle + R_t \quad (1.47)$$

with

$$R_t = \int_0^1 \left\langle D_f \psi(f + \theta(g_t - f), m_t) - D_f \psi(f, n), \frac{g_t - f}{t} \right\rangle d\theta.$$

As in the proof of Lemma 1.3.1, we see that (1.47) converges to $\left(D_f \psi(f, n), -\partial_v[(n - v)f] \right)$ and is uniformly locally bounded in (f, n) as $t \rightarrow 0$. We conclude by dominated convergence. \square

1.4.2 Limiting measures and mixing properties

We will now determine to some extent the limiting law of $(g_t, m_t)(f, n)$ as t goes to infinity, which naturally corresponds to an invariant measure for the semi-group $(Q_t)_{t \geq 0}$. More precisely, we shall describe how the mixing assumption (1.21) on the driving process $(m_t)_{t \geq 0}$ impacts the speed of convergence of Q_t to this invariant measure, along a certain type of test functions. From the explicit form (1.45) for example, we see that

$$\rho(g_t(f, n)) = \rho(f),$$

for all $t \geq 0$. The density ρ being invariant along the trajectory $(g_t, m_t)(f, n)$, the limiting law (which we define in (1.49)) will therefore be naturally parametrized by ρ .

Definition 1.4.2. For $\rho : \mathbb{T} \rightarrow \mathbb{R}_+$ with $\int \rho(x) dx = 1$ and $w : \mathbb{T} \rightarrow \mathbb{R}$, we define the probability measure $\rho \otimes \delta_w$ on $\mathbb{T} \times \mathbb{R}$ as

$$\forall \xi \in C_b(\mathbb{T} \times \mathbb{R}), \quad \langle \xi, \rho \otimes \delta_w \rangle = \int_x \xi(x, w(x)) \rho(x) dx,$$

Keeping in mind expression (1.45), we easily have

$$e^t f(x, e^t[v - w(x)]) \xrightarrow{t \rightarrow \infty} \rho(x) \otimes \delta_{w(x)}$$

in the distributions sense. Moreover, we will see that $w_t(n) = \int_0^t e^{-s} m_{t-s}(n) ds \xrightarrow{t \rightarrow \infty} \tilde{w}$ in law, where

$$\tilde{w} = \int_{-\infty}^0 e^s \tilde{m}_s ds. \quad (1.48)$$

We therefore expect the limiting measures to have the following form.

Definition 1.4.3 (Invariant measures). For $\rho : \mathbb{T} \rightarrow \mathbb{R}_+$ with $\int \rho(x) dx = 1$, we (partially) define the probability measure μ_ρ on $\mathcal{P}(\mathbb{T} \times \mathbb{R}) \times E$ by

$$\forall \psi \equiv \psi(f, n), \quad \langle \psi, \mu_\rho \rangle := \int \psi(f, n) d\mu_\rho(f, n) = \mathbb{E} \left[\psi(\rho \otimes \delta_{\tilde{w}}, \tilde{m}_0) \right]. \quad (1.49)$$

when this expression makes sense.

Remaining on a formal level, one may compute the first moments

$$\begin{aligned}\rho(\rho \otimes \delta_{\tilde{w}}) &:= \int \rho \otimes \delta_{\tilde{w}} dv = \rho, \\ J(\rho \otimes \delta_{\tilde{w}}) &:= \int v \rho \otimes \delta_{\tilde{w}} dv = \tilde{w}\rho, \\ K(\rho \otimes \delta_{\tilde{w}}) &:= \int v^2 \rho \otimes \delta_{\tilde{w}} dv = \tilde{w}^2\rho.\end{aligned}$$

We are now ready to state the following result.

Proposition 1.4.2 (Mixing speed for the auxiliary process). *Let us work with $f \in G_0^k$ for $k \geq 3$. Let ψ be a function of the form*

$$\psi(f, n) = \langle \xi_1(n)\rho, \xi_2(n) \rangle \quad \text{or} \quad \langle J(f), \xi_1(n) \rangle \quad \text{or} \quad \langle K(f), \xi \rangle \quad (1.50)$$

with $\xi_1, \xi_2 : E \rightarrow W_x^{1,\infty}$ Lipschitz-continuous functions :

$$\|\xi_i(n) - \xi_i(n')\|_{W_x^{1,\infty}} \lesssim \|n - n'\|_E$$

or $\xi \in W_x^{1,\infty}$. Then for all $(f, n) \in G_0^k \times E$, letting $\rho = \rho(f)$, we have

$$Q_t\psi(f, n) \xrightarrow[t \rightarrow \infty]{} \langle \langle \psi, \mu_\rho \rangle \rangle.$$

More precisely, for all $B \subset G_0^k \times E$ bounded,

$$\begin{aligned}\sup_{(f,n) \in B} \left| Q_t\psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle \right| &\in L^1(\mathbb{R}^+), \\ \sup_{(f,n) \in B} N_k \left(D_f \left[Q_t\psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle \right] \right) &\in L^1(\mathbb{R}^+).\end{aligned}$$

Remark 1.4.1. — Note that in particular, when $\langle \langle \psi, \mu_\rho \rangle \rangle = 0$, we have

$$\int_0^\infty \sup_{(f,n) \in B} \left| Q_t\psi(f, n) \right| dt < \infty, \quad \int_0^\infty \sup_{(f,n) \in B} N_k(D_f Q_t\psi(f, n)) dt < \infty.$$

This allows to define $\psi_1(f, n) = \int_0^\infty Q_t\psi(f, n) dt$ which then satisfies

$$\psi_1 \in \mathcal{B}^{loc}(G_0^k \times E) \quad \text{and} \quad D_f\psi_1 \in C_b^{loc}(G_0^k \times E; \mathcal{C}_k^1)$$

as in Definition 1.3.2 of a good test function.

- Later on, we will typically consider $\xi_i(n) = n, M^{-1}I(n), \partial_x M^{-1}I(n)$ which are indeed Lipschitz-continuous functions according to the assumptions made on the driving process.
- The proof of this result will in fact emphasize that it holds for any test function ψ such that $\psi(g_t(f, n), m_t(n))$ depends "at most quadratically on n " with the general rule of thumb :

$$\begin{array}{ccc} \xi(n) \text{ (with } \xi \text{ Lipschitz)} & \longleftrightarrow & n, \\ \rho & \longleftrightarrow & 1, \\ J(f) & \longleftrightarrow & n, \\ K(f) & \longleftrightarrow & n^2. \end{array}$$

For instance, the test function $\psi(f, n) = \langle \rho, \xi(n) \rangle \langle J(f), \xi \rangle$ can be considered.

Proof. Recalling (1.45), simple computations lead to

$$\begin{aligned}\rho(g_t(f, n)) &= \rho(f) = \rho \\ J(g_t(f, n)) &= e^{-t}J(f) + w_t(n)\rho(f) \\ K(g_t(f, n)) &= e^{-2t}K(f) + 2e^{-t}w_t(n)J(f) + w_t(n)^2\rho(f).\end{aligned}$$

Let us introduce the coupling $(m_t^*(n), \tilde{m}_t^*)_{t \geq 0}$ given by the mixing Assumption 3, and define

$$w_t^*(n) := \int_0^t e^{-s} m_{t-s}^*(n) ds, \quad \tilde{w}_t^* := \int_{-\infty}^0 e^s \tilde{m}_{t+s}^* ds.$$

Recalling (1.46) and (1.48), one can observe that, since $(\tilde{m}_t)_{t \in \mathbb{R}}$ is stationary,

$$(w_t(n), m_t(n)) \sim (w_t^*(n), m_t^*(n)) \text{ in law}, \quad (\tilde{w}, \tilde{m}_0) \sim (\tilde{w}_t^*, \tilde{m}_t^*) \text{ in law.}$$

The mixing assumption gives

$$\begin{aligned} \mathbb{E} \|w_t^*(n) - \tilde{w}_t^*\|_E &\leq \int_0^t e^{-s} \mathbb{E} \|m_{t-s}^*(n) - \tilde{m}_{t-s}^*\|_E ds + \int_t^\infty e^{-s} \mathbb{E} \|\tilde{m}_{t-s}^*\|_E ds \\ &\lesssim \int_0^t e^{-s} \gamma_{\text{mix}}(t-s) ds + e^{-t}. \end{aligned}$$

Noting that

$$\begin{aligned} \int_0^t e^{-s} \gamma_{\text{mix}}(t-s) ds &\leq \int_0^\infty e^{-s} \gamma_{\text{mix}}(t-s) ds \xrightarrow{t \rightarrow \infty} 0, \\ \int_{t=0}^\infty \int_{s=0}^t e^{-s} \gamma_{\text{mix}}(t-s) ds dt &= \int_{s=0}^\infty e^{-s} \int_{t=s}^\infty \gamma_{\text{mix}}(t-s) dt ds = \int_0^\infty \gamma_{\text{mix}}(t) dt < \infty, \end{aligned}$$

we have shown that

$$t \mapsto \mathbb{E} \|w_t^*(n) - \tilde{w}_t^*\|_E \in L^1(\mathbb{R}^+)$$

and tends to 0 as $t \rightarrow \infty$. Let us first consider $\psi(f, n) = \langle \xi_1(n) \rho, \xi_2(n) \rangle$. In this case,

$$\begin{aligned} Q_t \psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle &= \mathbb{E} \langle \xi_1(m_t^*(n)) \rho, \xi_2(m_t^*(n)) \rangle - \mathbb{E} \langle \xi_1(\tilde{m}_t^*) \rho, \xi_2(\tilde{m}_t^*) \rangle, \\ D_f [Q_t \psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle] &= \mathbb{E} [\xi_1(m_t^*(n)) \xi_2(m_t^*(n)) - \xi_1(\tilde{m}_t^*) \xi_2(\tilde{m}_t^*)]. \end{aligned}$$

The Lipschitz assumption on ξ_i and the inequality

$$|\langle h, k \rangle - \langle \tilde{h}, \tilde{k} \rangle| \leq \|h\| \|k - \tilde{k}\| + \|k\| \|h - \tilde{h}\| \quad (1.51)$$

lead to

$$|Q_t \psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle| \lesssim \|f\|_{G_0} \gamma_{\text{mix}}(t), \quad N_k \left(D_f [Q_t \psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle] \right) \lesssim \gamma_{\text{mix}}(t).$$

Let us now consider $\psi(f, n) = \langle J(f), \xi_1(n) \rangle$. In this case,

$$\begin{aligned} Q_t \psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle &= e^{-t} \mathbb{E} \langle J(f), \xi_1(m_t(n)) \rangle + \left[\mathbb{E} \langle w_t^*(n) \rho, \xi_1(m_t^*(n)) \rangle - \mathbb{E} \langle \tilde{w}_t^*(n) \rho, \xi_1(\tilde{m}_t^*) \rangle \right] \\ D_f [Q_t \psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle] &= e^{-t} \mathbb{E} [v \xi_1(m_t(n))] + \mathbb{E} [w_t^*(n) \xi_1(m_t^*(n)) - \tilde{w}_t^*(n) \xi_1(\tilde{m}_t^*)]. \end{aligned}$$

The first terms can be bounded by some Ce^{-t} , and we can use (1.51) again to bound the second terms. Finally, let us consider $\psi(f, n) = \langle K(f), \xi \rangle$. In this case,

$$\begin{aligned} Q_t \psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle &= e^{-2t} \mathbb{E} \langle K(f), \xi \rangle + 2e^{-t} \mathbb{E} \langle w_t(n) J(f), \xi \rangle \\ &\quad + \left[\mathbb{E} \langle w_t^*(n)^2 \rho, \xi \rangle - \mathbb{E} \langle (\tilde{w}_t^*)^2 \rho, \xi \rangle \right], \\ D_f [Q_t \psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle] &= e^{-2t} v^2 \xi + 2e^{-t} \mathbb{E} [w_t(n)] v \xi + \mathbb{E} [w_t^*(n)^2 - (\tilde{w}_t^*)^2] \xi. \end{aligned}$$

We conclude in the same way. \square

Let us complete this section by pointing out some identities regarding the laws of \tilde{w} , \tilde{m}_0 , and $M^{-1}I(\tilde{m}_0)$ which will be useful for calculations later on.

Proposition 1.4.3. *We have the following identities :*

$$\mathbb{E}\left[\tilde{w}(x)\tilde{w}(y)\right] = \frac{1}{2} \int_{\mathbb{R}} e^{-|t|} \mathbb{E}\left[\tilde{m}_0(x)\tilde{m}_t(y)\right] dt, \quad (1.52)$$

$$\mathbb{E}\left[\tilde{w}(x)M^{-1}I(\tilde{m}_0)(y)\right] = - \int_{t=0}^{\infty} (1 - e^{-t}) \mathbb{E}\left[\tilde{m}_0(x)\tilde{m}_t(y)\right] dt \quad (1.53)$$

$$= \mathbb{E}\left[\tilde{m}_0(x)M^{-1}I(\tilde{m}_0)(y) + \tilde{w}(x)\tilde{m}_0(y)\right]. \quad (1.54)$$

Proof. Recalling Definition 1.48 and using the fact that \tilde{m} is stationary, the quantity $\mathbb{E}\left[\tilde{w}(x)\tilde{w}(y)\right]$ can be written

$$\int_{s=-\infty}^0 \int_{t=-\infty}^0 e^{t+s} \mathbb{E}[\tilde{m}_s(x)\tilde{m}_t(y)] dt ds = \int_{s=-\infty}^0 \int_{t=-\infty}^0 e^{t+s} \mathbb{E}[\tilde{m}_0(x)\tilde{m}_{t-s}(y)] dt ds.$$

We set $r = t - s$ and use Fubini's Theorem to obtain the expression

$$\int_{r=-\infty}^0 \left(\int_{s=-\infty}^0 e^{2s} ds \right) e^r \mathbb{E}[\tilde{m}_0(x)\tilde{m}_r(y)] dr + \int_{r=0}^{\infty} \left(\int_{s=-\infty}^{-r} e^{2s} ds \right) e^r \mathbb{E}[\tilde{m}_0(x)\tilde{m}_r(y)] dr,$$

which gives (1.52). As for the second relation (1.53), we have

$$\mathbb{E}\left[\tilde{w}(x)M^{-1}I(\tilde{m}_0)(y)\right] = \int_{-\infty}^0 e^s \mathbb{E}\left[\tilde{m}_s(x)M^{-1}I(\tilde{m}_0)(y)\right] ds. \quad (1.55)$$

For $s \leq 0$, recalling definition (1.23) and using the Markov property,

$$\begin{aligned} \mathbb{E}\left[\tilde{m}_s(x)M^{-1}I(\tilde{m}_0)(y)\right] &= \int_E \mathbb{E}\left[m_s(n)(x)M^{-1}I(n)\right] d\nu(n) \\ &= - \int_E \int_0^{\infty} \mathbb{E}\left[m_s(n)(x)\mathbb{E}[m_t(n)(y)|\mathcal{F}_0]\right] dt d\nu(n) \\ &= - \int_E \int_0^{\infty} \mathbb{E}\left[m_s(n)(x)m_t(n)(y)\right] dt d\nu(n) \\ &= - \int_0^{\infty} \mathbb{E}\left[\tilde{m}_s(x)\tilde{m}_t(y)\right] dt. \end{aligned} \quad (1.56)$$

It follows that (1.55) is equal to

$$- \int_{s=-\infty}^0 \int_{t=0}^{\infty} e^s \mathbb{E}\left[\tilde{m}_s(x)\tilde{m}_t(y)\right] ds dt = - \int_{s=-\infty}^0 \int_{t=0}^{\infty} e^s \mathbb{E}\left[\tilde{m}_0(x)\tilde{m}_{t-s}(y)\right] ds dt.$$

Setting again $r = t - s$ and using Fubini's Theorem leads to the expression

$$- \int_{r=0}^{\infty} (1 - e^{-r}) \mathbb{E}\left[\tilde{m}_0(x)\tilde{m}_r(y)\right] dr$$

for (1.55). Using (1.56), we also recognize this to be (1.54) □

1.5 The perturbed test function method

From this point on, we shall work in the state space $(f, u, n) \in \mathcal{X}_k^\alpha = G_0^k \times H_x^\alpha \times E$, for $k = 3$ and $\alpha \in (1/2, 3/2)$. The aim of this section is to identify a "limiting generator" \mathcal{L} satisfying, for a large class of test functions ϕ ,

$$\mathcal{L}^\varepsilon \phi^\varepsilon(f, u, n) = \mathcal{L}\phi(\rho, u) + O(\varepsilon) \quad (1.57)$$

where ϕ^ε is a perturbed version of ϕ of the form

$$\phi^\varepsilon(f, u, n) = \phi(f, u) + \varepsilon\phi_1(f, u, n) + \varepsilon^2\phi_2(f, u, n)$$

and the correctors ϕ_1, ϕ_2 are good test functions to be identified. The $O(\varepsilon)$ is to be understood as some $R^\varepsilon(f, u, n)$ such that $\sup_B |R^\varepsilon(f, u, n)| = O(\varepsilon)$ as $\varepsilon \rightarrow 0$ for all bounded $B \subset \mathcal{X}_k^\alpha$. Let us start with a good test function $\phi \equiv \phi(f, u) \in \mathcal{B}^{loc}(\mathcal{X}_k^\alpha)$ of the form

$$\phi(f, u) = \Phi(\langle f, \xi \rangle; \langle u, \zeta \rangle) = \Phi(\langle \rho, \xi \rangle; \langle u, \zeta \rangle) \quad (1.58)$$

with $\Phi \in C^3(\mathbb{R}^2)$, and $\xi \equiv \xi(x), \zeta \equiv \zeta(x) \in C^2(\mathbb{T})$. Since, for fixed initial data f , the quantity $\rho(x)$ is conserved along the auxiliary equation (1.44), in the following we shall often write $\Phi, \partial_i \Phi$ (etc...) instead of specifying $\Phi(\langle \rho, \xi \rangle; \langle u, \zeta \rangle), \partial_i \Phi(\langle \rho, \xi \rangle; \langle u, \zeta \rangle)$ (etc...). Identifying the different orders in ε in (1.57), we get the set of equations

$$\begin{cases} \varepsilon^{-2} : \mathcal{L}_\# \phi(f, u, n) = 0, \\ \varepsilon^{-1} : \mathcal{L}_\# \phi_1(f, u, n) + \mathcal{L}_b \phi(f, u, n) = 0, \\ \varepsilon^0 : \mathcal{L}_\# \phi_2(f, u, n) + \mathcal{L}_b \phi_1(f, u, n) + \mathcal{L}_0 \phi(f, u, n) = \mathcal{L} \phi(\rho, u), \end{cases}$$

as well as

$$\begin{cases} \varepsilon^1 : \mathcal{L}_1 \phi(f, u, n) + \mathcal{L}_0 \phi_1(f, u, n) + \mathcal{L}_b \phi_2(f, u, n) = O(1), \\ \varepsilon^2 : \mathcal{L}_1 \phi_1(f, u, n) + \mathcal{L}_0 \phi_2(f, u, n) = O(\varepsilon^{-1}), \\ \varepsilon^3 : \mathcal{L}_1 \phi_2(f, u, n) = O(\varepsilon^{-2}). \end{cases} \quad (1.59)$$

1.5.1 Order ε^{-2}

Given the form (1.58) for ϕ , we have $\phi(f, u) = \phi(\rho(f), u)$, which is invariant along the trajectories of the auxiliary process $(g_t, m_t)_{t \geq 0}$. It follows that, indeed $\mathcal{L}_\# \phi(f, u, n) = 0$. Furthermore, one may observe that

$$\mathcal{L}_b \phi(f, u) = \left\langle J(f), \partial_x D_f \phi(f, u) \right\rangle = \left\langle J(f), \partial_x \xi \right\rangle \partial_1 \Phi$$

is a good test function satisfying the assumptions of Proposition 1.4.2.

1.5.2 Order ε^{-1} : first corrector

We wish to solve the Poisson equation in ϕ_1 :

$$\mathcal{L}_\# \phi_1(f, u, n) = -\mathcal{L}_b \phi(f, u).$$

The solution is expected to be given by

$$\phi_1(f, u, n) = \int_0^\infty Q_t[\mathcal{L}_b \phi](f, u, n) dt.$$

Proposition 1.4.2 guarantees that this corrector is indeed well-defined and locally bounded as long as $\langle \langle \mathcal{L}_b \phi, \mu_\rho \rangle \rangle = 0$, and indeed, since \tilde{m} is centered,

$$\langle \langle \mathcal{L}_b \phi, \mu_\rho \rangle \rangle = \mathbb{E} \left\langle \rho \tilde{w}, \partial_x D_f \phi(\rho, u) \right\rangle = 0.$$

The first corrector ϕ_1 can in fact be calculated explicitly :

$$\phi_1(f, u, n) = \int_0^\infty \mathbb{E} \left\langle f, w_t(n) \partial_x D_f \phi(f, u) \right\rangle dt = \left\langle f, \int_0^\infty \mathbb{E}[w_t(n)] dt \partial_x D_f \phi(f, u) \right\rangle,$$

with

$$\begin{aligned} \int_0^\infty \mathbb{E}[w_t(n)] dt &= \int_{t=0}^\infty \int_{s=0}^t e^{-(t-s)} \mathbb{E}[m_s(n)] ds dt = \int_{s=0}^\infty e^s \mathbb{E}[m_s(n)] \int_{t=s}^\infty e^{-t} dt ds \\ &= \int_{s=0}^\infty \mathbb{E}[m_s(n)] ds = -M^{-1} I(n). \end{aligned}$$

We hence get the expression

$$\phi_1(f, u, n) = \left\langle \partial_x [M^{-1} I(n) \rho - J(f)], D_f \phi(f, u) \right\rangle, \quad (1.60)$$

that is, with (1.58),

$$\phi_1(f, u, n) = \left\langle J(f) - M^{-1}I(n)\rho, \partial_x \xi \right\rangle \partial_1 \Phi,$$

which is clearly a good test function, thus in the domain of generator $\mathcal{L}_\#$. Making use of Remark 1.2.2, we have hence rigorously proven the following.

Proposition 1.5.1. *The first corrector given by (1.60) is a good test function which satisfies*

$$\mathcal{L}_\# \phi_1(f, u, n) = -\mathcal{L}_b \phi(f, u).$$

A simple calculation then leads to

$$\begin{aligned} \left\langle D_f \phi_1(f, u, n), h \right\rangle &= \left\langle \partial_x [M^{-1}I(n)\rho(h) - J(h)], D_f \phi(\rho, u) \right\rangle \\ &\quad + D_f^2 \phi(\rho, u) \left(\rho(h), \partial_x [M^{-1}I(n)\rho - J(f)] \right). \end{aligned} \quad (1.61)$$

1.5.3 Order ε^0 : Limiting generator and second corrector

We now wish to solve the Poisson equation in ϕ_2 :

$$\mathcal{L}_\# \phi_2(f, u, n) = -\mathcal{L}_b \phi_1(f, u, n) - \mathcal{L}_0 \phi(f, u, n) + \mathcal{L} \phi(\rho, u).$$

Again, the solution is expected to be given by

$$\phi_2(f, u, n) = \int_0^\infty Q_t \left[\mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi - \mathcal{L} \phi(\rho, u) \right] (f, u, n) dt.$$

which should converge whenever

$$\mathcal{L} \phi(\rho, u) = \langle \langle \mathcal{L}_b \phi_1, \mu_\rho \rangle \rangle + \langle \langle \mathcal{L}_0 \phi, \mu_\rho \rangle \rangle. \quad (1.62)$$

This last condition determines the form of the limiting generator $\mathcal{L} \phi$.

Limiting generator

We first compute $\mathcal{L}_b \phi_1(f, u, n)$: letting $h = -(v \partial_x f + u \partial_v f)$ in (1.61), noting that

$$\rho(h) = -\partial_x J(f), \quad J(h) = -\partial_x K(f) + u \rho$$

leads to the expression

$$\begin{aligned} \mathcal{L}_b \phi_1(f, u, n) &= \left\langle \partial_x \left[-M^{-1}I(n)\partial_x J(f) + \partial_x K(f) - u \rho \right], D_f \phi(\rho, u) \right\rangle \\ &\quad + D_f^2 \phi(\rho, u) \left(-\partial_x J(f), \partial_x [M^{-1}I(n)\rho - J(f)] \right). \end{aligned} \quad (1.63)$$

According to (1.62), the first contribution \mathcal{L}_ρ to the limiting generator is hence given by

$$\begin{aligned} \mathcal{L}_\rho \phi(\rho, u) := \langle \langle \mathcal{L}_b \phi_1, \mu_\rho \rangle \rangle &= \mathbb{E} \left\langle \partial_x \left[\partial_x [(\tilde{w})^2 \rho] - M^{-1}I(\tilde{m}_0) \partial_x [\tilde{w} \rho] - u \rho \right], D_f \phi(\rho, u) \right\rangle \\ &\quad + \mathbb{E} D_f^2 \phi(\rho, u) \left(\partial_x [\tilde{w} \rho], \partial_x \left[(\tilde{w} - M^{-1}I(\tilde{m}_0)) \rho \right] \right). \end{aligned} \quad (1.64)$$

Similarly, recalling that

$$\mathcal{L}_0 \phi(f, u) = \left\langle \partial_x^2 u + J(f), D_u \phi(f, u) \right\rangle \quad (1.65)$$

the second contribution \mathcal{L}_u to the limiting generator is given by

$$\mathcal{L}_u \phi(\rho, u) := \langle \langle \mathcal{L}_0 \phi, \mu_\rho \rangle \rangle = \left\langle \partial_x^2 u, D_u \phi(\rho, u) \right\rangle. \quad (1.66)$$

Defining the operator \mathcal{L} as

$$\mathcal{L} \phi(\rho, u) = \mathcal{L}_\rho \phi(\rho, u) + \mathcal{L}_u \phi(\rho, u), \quad (1.67)$$

let us now describe the corresponding process.

Limiting SPDE

The limiting process corresponding to the generator \mathcal{L} , will be denoted by $(\rho_t, u_t)_{t \geq 0}$ for simplicity in this subsection only. Since it appears to be a diffusion process, we are naturally led to determine the associated SPDE.

Firstly, note that this limiting process $(\rho_t, u_t)_{t \geq 0}$ is decoupled : the contribution $\mathcal{L}_u \phi$ given by (1.66) corresponds to the simple deterministic PDE

$$\partial_t u = \partial_x^2 u.$$

Let us hence focus on the contribution $\mathcal{L}_\rho \phi$ given by (1.64) which describes the evolution of $(\rho_t)_{t \geq 0}$. The second order differential in ϕ is associated to the diffusive part of the SPDE : we must identify the corresponding covariance operator. Let us assume that $D_f^2 \phi(\rho, u)$ is associated to some symmetric kernel Ψ . Then,

$$\begin{aligned} & \mathbb{E} \left[D_f^2 \phi(\rho, u) \left(\partial_x [\tilde{w} \rho], \partial_x \left[\left(\tilde{w} - M^{-1} I(\tilde{m}_0) \right) \rho \right] \right) \right] \\ &= \mathbb{E} \int_x \int_y \Psi(x, y) \partial_x [\tilde{w} \rho](x) \partial_y \left[\left(\tilde{w} - M^{-1} I(\tilde{m}_0) \right) \rho \right](y) dx dy \\ &= \mathbb{E} \int_x \int_y \Psi(x, y) \partial_x \left[\partial_y \left[\tilde{w}(x) \left(\tilde{w} - M^{-1} I(\tilde{m}_0) \right) (y) \rho(x) \rho(y) \right] \right] dx dy. \end{aligned} \quad (1.68)$$

We are thus naturally led to study the kernel $\mathbb{E} \left[\tilde{w}(x) \left(\tilde{w}(y) - M^{-1} I(\tilde{m}_0)(y) \right) \right]$. Note that since Ψ is symmetric in (1.68), we may as well consider the symmetrized version

$$\mathbb{E} \left[\tilde{w}(x) \tilde{w}(y) \right] - \frac{1}{2} \left(\mathbb{E} \left[\tilde{w}(x) M^{-1} I(\tilde{m}_0)(y) \right] + \mathbb{E} \left[\tilde{w}(y) M^{-1} I(\tilde{m}_0)(x) \right] \right).$$

Making use of the expressions obtained in Proposition 1.4.3 as well as the stationary character of $(\tilde{m}_t)_{t \in \mathbb{R}}$, we easily derive

$$\mathbb{E} \left[\tilde{w}(x) \tilde{w}(y) \right] - \frac{1}{2} \left(\mathbb{E} \left[\tilde{w}(x) M^{-1} I(\tilde{m}_0)(y) \right] + \mathbb{E} \left[\tilde{w}(y) M^{-1} I(\tilde{m}_0)(x) \right] \right) = \frac{1}{2} k(x, y) \quad (1.69)$$

where $k(x, y)$ is the kernel defined in (1.9). Let us denote by Q the linear operator on L_x^2 associated to this kernel :

$$\forall f \in L_x^2, \quad Qf(x) = \int_y k(x, y) f(y) dy. \quad (1.70)$$

Proposition 1.5.2. *The operator Q is self-adjoint, compact and non-negative :*

$$\forall f \in L^2(\mathbb{T}; \mathbb{R}) \quad (Qf, f) \geq 0.$$

We refer to [DV12], Lemma 1 for a proof. As a result, we can introduce the square root $Q^{1/2}$ of the operator Q , and denote by q the associated kernel :

$$Q^{1/2} f(x) = \int_y q(x, y) f(y) dy, \quad k(x, y) = \int_z q(x, z) q(z, y) dz.$$

We may now define the operator on L_x^2

$$\partial_x \left[\rho Q^{1/2} \right] (f)(x) := \int_y \partial_x \left[\rho(x) q(x, y) f(y) \right] dy, \quad (1.71)$$

whose adjoint is given by

$$\partial_x \left[\rho Q^{1/2} \right]^* (g)(y) = \int_x \partial_x \left[\rho(x) q(x, y) \right] g(x) dx.$$

A straight-forward calculation then leads to

$$\partial_x \left[\rho Q^{1/2} \right]^* D^2 \phi(\rho, u) \partial_x \left[\rho Q^{1/2} \right] (f)(t) = \int_z N(t, z) f(z) dz$$

where

$$N(t, z) = \int_x \int_y \partial_x [\rho(x)q(x, t)] \Psi(x, y) \partial_y [\rho(y)q(y, z)] dx dy.$$

We can now get back to (1.68) and derive

$$\begin{aligned} \mathbb{E} \left[D_f^2 \phi(\rho, u) \left(\partial_x [\tilde{w}\rho], \partial_x \left[\left(\tilde{w} - M^{-1}I(\tilde{m}_0) \right) \rho \right] \right) \right] &= \frac{1}{2} \int_x \int_y \Psi(x, y) \partial_x \left[\partial_y \left[\rho(x)\rho(y)k(x, y) \right] \right] dx dy \\ &= \frac{1}{2} \int_x \int_y \int_z \Psi(x, y) \partial_x \left[\partial_y \left[\rho(x)\rho(y)q(x, z)q(z, y) \right] \right] dx dy dz = \frac{1}{2} \int_z N(z, z) dz \\ &= \frac{1}{2} \text{Tr} \left(\partial_x \left[\rho Q^{1/2} \right]^* D_f^2 \phi(\rho) \partial_x \left[\rho Q^{1/2} \right] \right). \end{aligned}$$

Consequently, the contribution \mathcal{L}_ρ to the limiting generator \mathcal{L} may finally be rewritten as

$$\mathcal{L}_\rho \phi(\rho, u) = \left\langle \mathcal{A}^I \rho, D_f \phi(\rho, u) \right\rangle + \frac{1}{2} \text{Tr} \left(\partial_x \left[\rho Q^{1/2} \right]^* D_f^2 \phi(\rho, u) \partial_x \left[\rho Q^{1/2} \right] \right), \quad (1.72)$$

where \mathcal{A}^I denotes the second order differential operator

$$\mathcal{A}^I \rho = \mathbb{E} \left[\partial_x \left(\partial_x [\tilde{w}^2 \rho] - M^{-1}I(\tilde{m}_0) \partial_x [\tilde{w}\rho] - \rho u \right) \right]. \quad (1.73)$$

This naturally corresponds to the following SPDE, given in Itô form

$$d\rho_t = \mathcal{A}^I \rho_t dt + \partial_x \left[\rho_t Q^{1/2} dW_t \right]. \quad (1.74)$$

Finally, let us determine the Stratonovich form associated with (1.74). Recalling (1.69), the Itô operator \mathcal{A}^I may be rewritten as

$$\mathcal{A}^I \rho = \frac{1}{2} \partial_x^2 \left[k(x, x) \rho \right] + \partial_x \left[\mathbb{E} \left(\partial_x [M^{-1}I(\tilde{m}_0)] \tilde{w} \right) \rho \right] - \partial_x [u \rho].$$

The correction from the Itô to the Stratonovich form is given by the formula

$$\partial_x \left[\rho Q^{1/2} \circ dW_t \right] = \partial_x \left[\rho Q^{1/2} dW_t \right] + \mathcal{A}^{I \rightarrow S} \rho$$

where

$$\mathcal{A}^{I \rightarrow S} \rho := \frac{1}{2} \partial_x \left[\int_y q(x, y) \partial_x \left(\rho(x)q(x, y) \right) dy \right] = \frac{1}{2} \partial_x^2 \left[k(x, x) \rho \right] - \frac{1}{4} \partial_x \left[\left(\partial_x k(x, x) \right) \rho \right]$$

since $k(x, x) = \int_y q(x, y)^2 dy$. Hence, (1.74) can be expressed in Stratonovich form as

$$d\rho_t = \mathcal{A}^S \rho_t dt + \partial_x \left[\rho_t Q^{1/2} \circ dW_t \right]$$

where the differential operator \mathcal{A}^S is of order one :

$$\mathcal{A}^S \rho = \mathcal{A}^I \rho - \mathcal{A}^{I \rightarrow S} \rho = \partial_x \left[\left(\mathbb{E} \partial_x [M^{-1}I(\tilde{m}_0)] \tilde{w} + \frac{1}{4} \partial_x k(x, x) - u \right) \rho \right].$$

Using once again expression (1.69) and the identities from Proposition 1.4.3, straight-forward calculations give

$$\mathcal{A}^S \rho = \partial_x \left[(a - u) \rho \right]$$

where $a \equiv a(x)$ is given by (1.8).

Second corrector

Using the form (1.58) for ϕ , recalling (1.63) and (1.65), calculations lead to the expressions

$$\begin{aligned} \mathcal{L}_b \phi_1(f, u, n) &= - \left\langle J(f), \partial_x (M^{-1}I(n) \partial_x \xi) \right\rangle \partial_1 \Phi + \left\langle K(f), \partial_x^2 \xi \right\rangle \partial_1 \Phi + \left\langle \rho, u \partial_x \xi \right\rangle \partial_1 \Phi \\ &\quad - \left\langle J(f), \partial_x \xi \right\rangle \left\langle \rho, M^{-1}I(n) \partial_x \xi \right\rangle \partial_1^2 \Phi + \left\langle J(f), \partial_x \xi \right\rangle^2 \partial_1^2 \Phi \end{aligned} \quad (1.75)$$

and

$$\mathcal{L}_0 \phi(f, u) = \left\langle u, \partial_x^2 \zeta \right\rangle \partial_2 \Phi + \left\langle J(f), \zeta \right\rangle \partial_2 \Phi. \quad (1.76)$$

We now prove the following result.

Proposition 1.5.3. *The second corrector given by*

$$\phi_2(f, u, n) = \int_0^\infty Q_t \left[\mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi - \mathcal{L} \phi \right] (f, u, n) dt$$

defines a good test function which satisfies

$$\mathcal{L}_\# \phi_2(f, u, n) = -\mathcal{L}_b \phi_1(f, u, n) - \mathcal{L}_0 \phi(f, u, n) + \mathcal{L} \phi(\rho, u), \quad (1.77)$$

where \mathcal{L} is the limiting generator given by (1.67).

Proof. Firstly, note that $\theta := \mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi - \mathcal{L} \phi \in C_b^{loc}(\mathcal{X}_k^\alpha)$ (recall that $k = 3$) has the form of a good test function. In particular, it is in the domain of $\mathcal{L}_\#$. For $B \subset \mathcal{X}_k^\alpha$ bounded, we shall prove the following points :

1. **First point** : $\int_0^\infty \sup_{(f,u,n) \in B} \left| Q_t \left[\mathcal{L}_b \phi_1 \mathcal{L}_0 \phi - \mathcal{L} \phi \right] (f, u, n) \right| dt < \infty$,
2. **Second point** : $\int_0^\infty \sup_{(f,u,n) \in B} \left| Q_t \mathcal{L}_\# \left[\mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi - \mathcal{L} \phi \right] (f, u, n) \right| dt < \infty$,
3. **Third point** : $\int_0^\infty \sup_{(f,u,n) \in B} N_k \left(D_f Q_t \left[\mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi - \mathcal{L} \phi \right] (f, u, n) \right) dt < \infty$,
4. **Fourth point** : $\int_0^\infty \sup_{(f,u,n) \in B} \| D_u Q_t \left[\mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi - \mathcal{L} \phi \right] (f, u, n) \|_{C^2(\mathbb{T})} dt < \infty$,

Point 1 shows that ϕ_2 is well defined and in $\mathcal{B}^{loc}(\mathcal{X}_k^\alpha)$. Once point 2 is obtained, we can use Corollary 1.2.4 to conclude immediately that $\phi_2 \in D(\mathcal{L}_\#)$ and satisfies (1.77). It then remains to prove that ϕ_2 is a good test function.

Points 3 and 4 guarantee that ϕ_2 meets the last two requirements of Definition 1.3.2. Moreover, using the same decomposition as in the proof of Proposition 1.4.1, we may write

$$\frac{P_t \phi_2(f, u, n) - \phi_2(f, u, n)}{t} = \frac{Q_t \phi_2(f, u, n) - \phi_2(f, u, n)}{t} - \mathbb{E} \left[\frac{\phi_2(g_t, u, m_t) - \phi_2(f, u, m_t)}{t} \right].$$

Considering the b.p.c convergence of each term of the right hand side as t goes to zero, we obtain $\phi_2(f, u, \cdot) \in D(M)$ and

$$M \phi_2(f, u, n) = \mathcal{L}_\# \phi_2(f, u, n) - \left\langle \partial_v [(v - n)f], D_f \phi_2(f, u, n) \right\rangle$$

which is in $C_b^{loc}(\mathcal{X}_k^\alpha)$, concluding the proof. Let us now review the different points.

First point It is clear from the explicit expressions (1.75) and (1.76) that $\mathcal{L}_b \phi_1$ and $\mathcal{L}_0 \phi$ are linear combinations of functions satisfying the assumptions of Proposition 1.4.2, i.e essentially "quadratic in n ". By definition, $\mathcal{L} \phi(f, u) = \langle \langle \mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi, \mu_\rho \rangle \rangle$, so that

$$Q_t \left[\mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi - \mathcal{L} \phi \right] = Q_t \left[\mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi \right] - \langle \langle \mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi, \mu_\rho \rangle \rangle.$$

One can easily check that the estimates obtained from Proposition 1.4.2 are also locally bounded in $u \in L_x^\infty \subset H_x^\alpha$ (since $\alpha > 1/2$).

Second point Let us compute explicitly $\mathcal{L}_\# \left[\mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi - \mathcal{L} \phi \right]$, recalling that

$$\mathcal{L}_\# \psi(f, n) = \left\langle h, D_f \psi(f, n) \right\rangle + M \psi(f, n)$$

where $h = \partial_v [(v - n)f]$. Since $\mathcal{L} \phi \equiv \mathcal{L} \phi(\rho, u)$, we have $\mathcal{L}_\# [\mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi - \mathcal{L} \phi] = \mathcal{L}_\# \mathcal{L}_b \phi_1 + \mathcal{L}_\# \mathcal{L}_0 \phi$. Using Proposition 1.4.2 once again, we simply need to show that the test functions involved have the right form and are centered with respect to the invariant measure μ_ρ of the auxiliary process. Calculations give, for $\chi \equiv \chi(x)$,

$$\begin{aligned} \langle h, \chi \rangle &= \langle \rho(h), \chi \rangle = 0, \\ \langle J(h), \chi \rangle &= \langle n\rho - J(f), \chi \rangle, \\ \langle K(h), \chi \rangle &= 2 \left\langle nJ(f) - K(f), \chi \right\rangle. \end{aligned}$$

Let us review the different terms in (1.75) and express the corresponding parts in $\mathcal{L}_\# \mathcal{L}_\flat \phi_1$.

The first term is $\psi_1(f, n) = -\langle J(f), \partial_x(M^{-1}I(n)\partial_x\xi) \rangle \partial_1\Phi$. We get

$$\begin{aligned} M\psi_1(f, n) &= -\langle J(f), \partial_x(n.\partial_x\xi) \rangle \partial_1\Phi, \\ \langle h, D\psi_1(f, n) \rangle &= \langle J(f) - n\rho, \partial_x(M^{-1}I(n).\partial_x\xi) \rangle \partial_1\Phi, \end{aligned}$$

so that $\mathcal{L}_\#\psi_1$ has the form required in Proposition 1.4.2. Calculations give the following expression for $\langle\langle \mathcal{L}_\#\psi_1, \mu_\rho \rangle\rangle$:

$$\mathbb{E}\left[-\langle \tilde{w}\rho, \partial_x(\tilde{m}_0.\partial_x\xi) \rangle + \langle \tilde{w}\rho, \partial_x(M^{-1}I(\tilde{m}_0).\partial_x\xi) \rangle - \langle \tilde{m}_0\rho, \partial_x(M^{-1}I(\tilde{m}_0).\partial_x\xi) \rangle\right] \partial_1\Phi.$$

Using the identity (1.54) from Proposition 1.4.3, this is indeed 0.

The next term is $\psi_2(f, n) = \langle K(f), \partial_x^2\xi \rangle \partial_1\Phi$. We get

$$\begin{aligned} M\psi_2(f, n) &= 0, \\ \langle h, D\psi_2(f, n) \rangle &= 2\langle nJ(f) - K(f), \partial_x^2\xi \rangle \partial_1\Phi, \end{aligned}$$

so that $\mathcal{L}_\#\psi_2$ satisfies the assumption of Proposition 1.4.2. Here,

$$\langle\langle \mathcal{L}_\#\psi_2, \mu_\rho \rangle\rangle = 2\mathbb{E}\langle\langle \tilde{m}_0\tilde{w} - \tilde{w}^2 \rangle\rangle \rho, \partial_x^2\xi \rangle \partial_1\Phi.$$

Using the identity (1.52) from Proposition 1.4.3 as well as the stationarity of \tilde{m} , we see that $\mathbb{E}(\tilde{m}_0\tilde{w}) = \mathbb{E}(\tilde{w}^2)$, so that this is indeed 0.

The next term $\langle \rho, u\partial_x\xi \rangle \partial_1\Phi$ is in fact constant along the trajectories of $(g_t, m_t)_{t \geq 0}$, giving no contribution. The following term is $\psi_3(f, n) = -\langle J(f), \partial_x\xi \rangle \langle \rho, M^{-1}I(n)\partial_x\xi \rangle \partial_1^2\Phi$. We get

$$\begin{aligned} M\psi_3(f, n) &= -\langle J(f), \partial_x\xi \rangle \langle \rho, n\partial_x\xi \rangle \partial_1^2\Phi, \\ \langle h, D\psi_3(f, n) \rangle &= -\langle n\rho - J(f), \partial_x\xi \rangle \langle \rho, M^{-1}I(n).\partial_x\xi \rangle \partial_1^2\Phi, \end{aligned}$$

so that $\mathcal{L}_\#\psi_3$ satisfies the assumption of Proposition 1.4.2. Here,

$$\begin{aligned} \langle\langle \mathcal{L}_\#\psi_3, \mu_\rho \rangle\rangle &= \mathbb{E}\left[-\langle \tilde{w}\rho, \partial_x\xi \rangle \langle \rho, \tilde{m}_0.\partial_x\xi \rangle - \langle \tilde{m}_0\rho, \partial_x\xi \rangle \langle \rho, M^{-1}I(\tilde{m}_0).\partial_x\xi \rangle \right. \\ &\quad \left. + \langle \tilde{w}\rho, \partial_x\xi \rangle \langle \rho, M^{-1}I(\tilde{m}_0).\partial_x\xi \rangle\right] \partial_1^2\Phi. \end{aligned}$$

Again, using the identity (1.54) from Proposition 1.4.3, this is indeed 0. Finally, the last term is

$\psi_4(f, n) = \langle J(f), \partial_x\xi \rangle^2 \partial_1^2\Phi$. We get

$$\begin{aligned} M\psi_4(f, n) &= 0, \\ \langle h, D\psi_4(f, n) \rangle &= 2\langle n\rho - J(f), \partial_x\xi \rangle \langle J(f), \partial_x\xi \rangle \partial_1^2\Phi, \end{aligned}$$

so that $\mathcal{L}_\#\psi_4$ satisfies the assumption of Proposition 1.4.2. Here,

$$\langle\langle \mathcal{L}_\#\psi_4, \mu_\rho \rangle\rangle = 2\mathbb{E}\left[\langle \tilde{m}_0\rho, \partial_x\xi \rangle \langle \tilde{w}\rho, \partial_x\xi \rangle - \langle \tilde{w}\rho, \partial_x\xi \rangle \langle \tilde{w}\rho, \partial_x\xi \rangle\right] \partial_1^2\Phi. \quad (1.78)$$

Using the identity (1.52) as well as the stationarity of \tilde{m} , we see that

$$\frac{1}{2}\mathbb{E}[\tilde{m}_0(x)\tilde{w}(y) + \tilde{m}_0(y)\tilde{w}(x)] = \mathbb{E}[\tilde{w}(x)\tilde{w}(y)]$$

hence, making use of the symmetry of (1.78), it is indeed 0. It remains to study $\mathcal{L}_\# \mathcal{L}_0\phi(f, u)$. The first term $\langle u, \partial_x^2\zeta \rangle \partial_2\Phi$ is constant. As for the second term $\langle J(f), \zeta \rangle \partial_2\Phi$, we get

$$\mathcal{L}_\#\left[\langle J(f), \zeta \rangle \partial_2\Phi\right] = \langle J(h), \zeta \rangle \partial_2\Phi = \langle n\rho - J(f), \zeta \rangle \partial_2\Phi$$

which satisfies the assumption of Proposition 1.4.2 and

$$\langle\langle \langle n\rho - J(f), \zeta \rangle \partial_2\Phi, \mu_\rho \rangle\rangle = \mathbb{E}\langle\langle \tilde{m}_0\rho - \tilde{w}\rho, \zeta \rangle\rangle \partial_2\Phi = 0.$$

This concludes the proof of the second point.

Third point Once again, we simply use Proposition 1.4.2 (the part regarding the mixing speed of $D_f Q_t \psi$) since, as previously mentioned,

$$Q_t [\mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi - \mathcal{L} \phi] = \left(Q_t [\mathcal{L}_b \phi_1] - \langle \langle \mathcal{L}_b \phi_1, \mu_\rho \rangle \rangle \right) + \left(Q_t [\mathcal{L}_0 \phi] - \langle \langle \mathcal{L}_0 \phi, \mu_\rho \rangle \rangle \right)$$

and the different terms in (1.75), (1.76) are good test functions satisfying the requested assumptions.

Fourth point Reviewing $Q_t [\mathcal{L}_b \phi_1 + \mathcal{L}_0 \phi - \mathcal{L} \phi]$, we see that it is a sum of terms of the form

$$\left(Q_t \psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle \right) \partial_j^i \Phi((f, \xi); (u, \zeta)).$$

with $i, j = 1$ or 2 and ψ of the form required in Proposition 1.4.2 (the terms involving u in (1.75) and (1.76) are constant along the trajectories of $(g_t, m_t)_{t \geq 0}$ and are hence canceled out). The differential with respect to u of such a term is hence simply

$$\left(Q_t \psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle \right) \partial_2 \partial_j^i \Phi((f, \xi); (u, \zeta)) \zeta.$$

Since we assumed $\Phi \in C^3$, its $\|\cdot\|_{C^2(\mathbb{T})}$ norm over $B \subset \mathcal{X}_k^\alpha$ can be bounded by

$$C \sup_{(f, u, n) \in B} \left| Q_t \psi(f, n) - \langle \langle \psi, \mu_\rho \rangle \rangle \right| \|\zeta\|_{C^2(\mathbb{T})}$$

which is integrable over \mathbb{R}^+ according to Proposition 1.4.2. \square

1.5.4 Consequence

Note that since ϕ, ϕ_1, ϕ_2 are good test functions, the equations with a positive order (1.59) are automatically satisfied since $\mathcal{L}_\star \phi, \mathcal{L}_\star \phi_i$ are locally bounded for $i = 1, 2$, and $\star = 0, 1, b$. Let us summarize the results we have obtained : for all ϕ good test-function of the form

$$\phi(f) = \Phi((\rho, \xi); (u, \zeta))$$

with $\Phi \in C^3(\mathbb{R}^2)$, $\xi \equiv \xi(x)$, $\zeta \equiv \zeta(x) \in C^2(\mathbb{T})$, there exist correctors ϕ_1, ϕ_2 which are good test functions, such that the perturbed test function $\phi^\varepsilon = \phi + \varepsilon \phi_1 + \varepsilon^2 \phi_2$ satisfies

$$\mathcal{L}^\varepsilon \phi^\varepsilon(f, u, n) = \mathcal{L} \phi(\rho, u) + R^\varepsilon(f, u, n)$$

where \mathcal{L} is the limiting generator identified in (1.67), and $R^\varepsilon/\varepsilon$ is locally bounded. As a result, we know from Proposition 1.3.7 that

$$M_\phi^\varepsilon(t) = \phi^\varepsilon(f_t^\varepsilon, u_t^\varepsilon, m_t^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon \phi^\varepsilon(f_s^\varepsilon, u_s^\varepsilon, m_s^\varepsilon) ds, \quad t \geq 0$$

defines a $(\mathcal{F}_t^\varepsilon)_{t \geq 0}$ -martingale, so that

$$N_\phi^\varepsilon(t) = \phi(\rho_t^\varepsilon, u_t^\varepsilon) - \int_0^t \mathcal{L} \phi(\rho_s^\varepsilon, u_s^\varepsilon) ds, \quad t \geq 0$$

is "almost" a martingale, up to some $O(\varepsilon)$. This is an essential step towards our diffusion-approximation result.

1.6 Diffusion-approximation

1.6.1 Tightness

We now prove the following result.

Proposition 1.6.1. *Let $\alpha \in (1/2, 3/2]$ and assume that*

$$\sup_{\varepsilon} \|u_0^\varepsilon\|_{H_x^\alpha} + \sup_{\varepsilon} \|f_0^\varepsilon\|_{G_3} < \infty.$$

Then the family of random variables $(\rho^\varepsilon, u^\varepsilon)_{\varepsilon>0}$ is tight in the space

$$C([0, T]; H_x^{-\gamma}) \times C([0, T]; H_x^\beta), \quad \gamma > 1/2, \quad \beta < \alpha.$$

Démonstration. The estimates from Proposition 1.3.6 guarantee that $(u^\varepsilon)_\varepsilon$ lies in a (deterministic) compact subset of $C([0, T]; H_x^\beta)$. We only need to prove the tightness of $(\rho^\varepsilon)_\varepsilon$ in $C([0, T]; H_x^{-\gamma})$. To this intent, we will once again use the perturbed test function method, but this time only up to the first order. Firstly, note that since the injection $L_x^1 \subset H_x^{-\gamma}$ is compact for $\gamma > 1/2$, it is sufficient to prove

$$\limsup_{\varepsilon} \mathbb{P}(\|\rho^\varepsilon\|_{L_t^\infty L_x^1} > M) \xrightarrow{M \rightarrow \infty} 0, \quad (1.79)$$

$$\forall r > 0, \quad \limsup_{\varepsilon} \mathbb{P}(w_\gamma(\rho^\varepsilon; \delta) > r) \xrightarrow{\delta \rightarrow 0} 0, \quad (1.80)$$

where $w_\gamma(\rho; \delta)$ denote the modulus of continuity

$$w_\gamma(\rho; \delta) = \sup_{|t-s| \leq \delta} \|\rho_t - \rho_s\|_{H_x^{-\gamma}}.$$

One may refer to [Bil99], Chapter 2, for details on this tightness criterion. The first point (1.79) is obvious since $\|\rho_t^\varepsilon\|_{L_x^1} = \|\rho_0^\varepsilon\|_{L_x^1} = 1$. Note that for any $\eta > \gamma$ an interpolation inequality gives, for some $\theta \in (0, 1)$,

$$w_\gamma(\rho; \delta) \leq \left(2 \sup_{t \in [0, T]} \|\rho_t\|_{L_x^1}\right)^\theta w_\eta(\rho; \delta)^{1-\theta}$$

so that it is in fact sufficient to establish (1.80) for γ large enough. Let us define the diagonal operator

$$J = (Id - \Delta_x)^{-1/2}$$

that is, considering the $L^2(\mathbb{T})$ Hilbert-base $\{e_j(x) = (2\pi)^{-1/2} e^{ijx}, j \in \mathbb{Z}\}$,

$$J\xi = \sum_j (1 + |j|^2)^{-1/2} \langle \xi, e_j \rangle e_j,$$

so that

$$\|J^\gamma \rho\|_{L^2}^2 = \sum_j |\langle \rho, J^\gamma e_j \rangle|^2 = \sum_j (1 + |j|^2)^{-\gamma} |\langle \rho, e_j \rangle|^2 = \|\rho\|_{H_x^{-\gamma}}^2.$$

For fixed $j \in \mathbb{Z}$, we consider the good test function

$$\phi_j(f) = \langle \rho, J^\gamma e_j \rangle. \quad (1.81)$$

which is indeed of the form (1.58) used in the perturbed test function method, with

$$\|J^\gamma e_j\|_{C^2} = (2\pi)^{-1/2} |j|^2 (1 + |j|^2)^{-\gamma/2}.$$

The associated first corrector

$$\phi_j^1(f, n) = \left\langle J(f) - M^{-1}I(n)\rho, \partial_x J^\gamma e_j \right\rangle \quad (1.82)$$

is designed so that $\mathcal{L}_\# \phi_j^1(f, n) = -\mathcal{L}_\flat \phi_j(f)$. Defining the first order perturbation

$$\phi_j^\varepsilon(f, n) = \phi_j(\rho) + \varepsilon \phi_j^1(f, n)$$

it results that

$$\mathcal{L}^\varepsilon \phi_j^\varepsilon(f, u, n) = \mathcal{L}_\flat \phi_j^1(f, u, n) = O(1) \quad (1.83)$$

in the sense that it can be bounded by $C\|J^\gamma e_j\|_{C^2}$ when (f, n) lies in a bounded $B \subset \mathcal{X}_k^\alpha$. Let us choose γ large enough so that

$$\sum_j \|J^\gamma e_j\|_{C^2}^2 < \infty,$$

that is in fact, $\gamma > 5/2$. We now define

$$\theta^\varepsilon(t) = J^{-\gamma} \sum_j \phi_j^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) e_j \in H_x^{-\gamma}$$

and notice that, using the form (1.82) and Proposition 1.3.4, for all $t \in [0, T]$,

$$\begin{aligned} \|\rho_t^\varepsilon - \theta^\varepsilon(t)\|_{H_x^{-\gamma}} &= \|J^\gamma \rho_t^\varepsilon - J^\gamma \theta^\varepsilon(t)\|_{L^2} = \sum_j \left| \phi_j(\rho_t^\varepsilon) - \phi_j^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) \right|^2 = \varepsilon \sum_j \left| \phi_j^1(f_t^\varepsilon, m_t^\varepsilon) \right|^2 \\ &\leq \varepsilon C(T, C^*, \|u_0^\varepsilon\|_{L_x^\infty}, \|f_0^\varepsilon\|_{G^1}) \sum_j \|J^\gamma e_j\|_{C^1}^2 \lesssim \varepsilon. \end{aligned}$$

We deduce that (1.80) can be derived from the tightness of $(\theta^\varepsilon)_\varepsilon$ in $D([0, T]; H_x^{-\gamma})$: indeed, ρ^ε being continuous,

$$w_\gamma(\rho^\varepsilon; \delta) \lesssim w'_\gamma(\rho^\varepsilon; \delta) \lesssim w'_\gamma(\theta^\varepsilon; \delta) + \varepsilon$$

where w'_γ denotes the modulus of continuity in $D([0, T]; H_x^{-\gamma})$ (see again [Bil99], Chapter 3, for details). We apply Aldou's criterion to prove the tightness of $(\theta^\varepsilon)_\varepsilon$ and conclude : let $\delta > 0$ and τ_1, τ_2 be two $(\mathcal{F}_t^\varepsilon)_{t \geq 0}$ -stopping time satisfying

$$\tau_1 \leq \tau_2 \leq \tau_1 + \delta, \quad \tau_2 \leq T. \quad (1.84)$$

Introducing the square integrable martingale

$$M_j^\varepsilon(t) = \phi_j^\varepsilon(f_t^\varepsilon, m_t^\varepsilon) - \int_0^t \mathcal{L}^\varepsilon \phi_j^\varepsilon(f_s^\varepsilon, u_s^\varepsilon, m_s^\varepsilon) ds$$

we have

$$\begin{aligned} \mathbb{E} \|\theta^\varepsilon(\tau_2) - \theta^\varepsilon(\tau_1)\|_{H_x^{-\gamma}}^2 &= \sum_j \left| \phi_j^\varepsilon(f_{\tau_2}^\varepsilon, m_{\tau_2}^\varepsilon) - \phi_j^\varepsilon(f_{\tau_1}^\varepsilon, m_{\tau_1}^\varepsilon) \right|^2 \\ &= \sum_j \mathbb{E} \left| M_j^\varepsilon(\tau_2) - M_j^\varepsilon(\tau_1) - \int_{\tau_1}^{\tau_2} \mathcal{L}^\varepsilon \phi_j^\varepsilon(f_s^\varepsilon, u_s^\varepsilon, m_s^\varepsilon) ds \right|^2. \end{aligned}$$

On one hand, using Doob's optional sampling theorem, we have

$$\mathbb{E} \left| M_j^\varepsilon(\tau_2) - M_j^\varepsilon(\tau_1) \right|^2 = \mathbb{E} \left(|M_j^\varepsilon(\tau_2)|^2 - |M_j^\varepsilon(\tau_1)|^2 \right) = \mathbb{E} \left([M_j^\varepsilon](\tau_2) - [M_j^\varepsilon](\tau_1) \right)$$

where $[M_j^\varepsilon](t)$ denotes the predictable quadratic variation of $M_j^\varepsilon(t)$. We know from Proposition 1.3.7 that it is given by

$$[M_j^\varepsilon](t) = \frac{1}{\varepsilon^2} \int_0^t \left(M |\phi_j^\varepsilon|^2 - 2\phi_j^\varepsilon M \phi_j^\varepsilon \right) (f_s^\varepsilon, m_s^\varepsilon) ds = \int_0^t \left(M |\phi_j^1|^2 - 2\phi_j^1 M \phi_j^1 \right) (f_s^\varepsilon, m_s^\varepsilon) ds$$

after some straight-forward calculation. Using expression (1.82) and the quadratic inequality (1.25), we deduce that the integrand can be bounded over $[0, T]$ by $C\|J^\gamma e_j\|_{C^1}^2$, so that

$$\sum_j \mathbb{E} \left| M_j^\varepsilon(\tau_2) - M_j^\varepsilon(\tau_1) \right|^2 \lesssim \delta. \quad (1.85)$$

On the other hand, making use of (1.83),

$$\left| \int_s^t \mathcal{L}^\varepsilon \phi_j^\varepsilon(f_s^\varepsilon, u_s^\varepsilon, m_s^\varepsilon) ds \right|^2 \lesssim \int_s^t |\mathcal{L}^\varepsilon \phi_j^\varepsilon(f_s^\varepsilon, u_s^\varepsilon, m_s^\varepsilon)|^2 ds \lesssim \|J^\gamma e_j\|_{C^2}^2 |t - s|$$

so that the choice of γ guarantees that

$$\sum_j \mathbb{E} \left| \int_{\tau_1}^{\tau_2} \mathcal{L}^\varepsilon \phi_j^\varepsilon(f_s^\varepsilon, u_s^\varepsilon, m_s^\varepsilon) ds \right|^2 \lesssim \delta. \quad (1.86)$$

Combining (1.85) and (1.86) and using Markov's inequality, we conclude that

$$\forall r > 0, \limsup_\varepsilon \sup_{\tau_1, \tau_2} \mathbb{P} \left(\left| \theta^\varepsilon(\tau_2) - \theta^\varepsilon(\tau_1) \right| > r \right) \xrightarrow{\delta \rightarrow 0} 0,$$

where the supremum is taken over all stopping times τ_1, τ_2 satisfying (1.84). Aldou's criterion is met, which concludes the proof. \square

In the following, let us fix $\gamma > 1/2$ and $\beta < \alpha$ and a subsequence $(\rho^{\varepsilon_j}, u^{\varepsilon_j})_{j \geq 1}$ which converges in law to some (ρ, u) . Using Skorokhod's representation theorem, we may introduce some probability space (Ω^*, \mathbb{P}^*) hosting some random variables $(\rho_*^{\varepsilon_j}, u_*^{\varepsilon_j})_{j \geq 1}$, (ρ_*, u_*) satisfying

$$(\rho_*^{\varepsilon_j}, u_*^{\varepsilon_j}) \sim (\rho^{\varepsilon_j}, u^{\varepsilon_j}), \quad (\rho_*, u_*) \sim (\rho, u) \quad \text{in law in } C([0, T]; H_x^{-\gamma}) \times C([0, T]; H_x^\beta)$$

and

$$(\rho_*^{\varepsilon_j}, u_*^{\varepsilon_j}) \xrightarrow{j \rightarrow \infty} (\rho_*, u_*) \text{ a.s in } C([0, T]; H_x^{-\gamma}) \times C([0, T]; H_x^\beta).$$

Additionally, note that passing to the limit in the equality $\|\rho_*^\varepsilon(t)\|_{L_x^1} = 1$ leads to the inequality

$$\sup_{[0, T]} \|\rho_*(t)\|_{L_x^1} \leq 1 \quad \text{a.s}$$

or equivalently, regarding the limiting law,

$$\mathbb{P} \left[\sup_{t \in [0, T]} \|\rho_t\|_{L_x^1} \leq 1 \right] = 1. \quad (1.87)$$

1.6.2 Convergence of the martingale problem

Let us consider a good test-function ϕ of the form (1.58). The final result obtained in section 1.5 can be summarized as follows : for any $s_1 < s_2 < \dots < s_n = s < t$ and any $\theta : (H_x^{-\gamma} \times H_x^\beta)^n \rightarrow \mathbb{R}$ continuous and bounded,

$$\mathbb{E} \left[\left(\phi(\rho_t^\varepsilon, u_t^\varepsilon) - \phi(\rho_s^\varepsilon, u_s^\varepsilon) - \int_s^t \mathcal{L} \phi(\rho_\sigma^\varepsilon, u_\sigma^\varepsilon) d\sigma \right) \theta(\rho_{s_1}^\varepsilon, u_{s_1}^\varepsilon, \dots, \rho_{s_n}^\varepsilon, u_{s_n}^\varepsilon) \right] = O(\varepsilon). \quad (1.88)$$

Regarding the adaptedness of the process, note that since $\omega \mapsto (\rho_t^\varepsilon, u_t^\varepsilon)(\omega) \in L_x^1 \times H_x^\alpha$ is $\mathcal{F}_t^\varepsilon$ -measurable, and since the injection $L_x^1 \times H_x^\alpha \subset H_x^{-\gamma} \times H_x^\beta$ is continuous, the random variable $\omega \mapsto (\rho_t^\varepsilon, u_t^\varepsilon)(\omega) \in H_x^{-\gamma} \times H_x^\beta$ is also $\mathcal{F}_t^\varepsilon$ -measurable. The martingale property (1.88) only depends on the law of $(\rho^\varepsilon, u^\varepsilon) \in C([0, T]; H_x^{-\gamma} \times H_x^\beta)$, so that we may in fact write, on the new probability space (Ω^*, \mathbb{P}^*) ,

$$\mathbb{E}^* \left[\left(\phi(\rho_*^{\varepsilon_j}(t), u_*^{\varepsilon_j}(t)) - \phi(\rho_*^{\varepsilon_j}(s), u_*^{\varepsilon_j}(s)) - \int_s^t \mathcal{L} \phi(\rho_*^{\varepsilon_j}(\sigma), u_*^{\varepsilon_j}(\sigma)) d\sigma \right) \theta_* \right] = O(\varepsilon),$$

where $\theta_* = \theta(\rho_*^{\varepsilon_j}(s_1), \dots, u_*^{\varepsilon_j}(s_n))$. From the expressions (1.64), (1.66) and (1.67), we easily deduce that $\mathcal{L} \phi$ is continuous and locally bounded on $H_x^{-\gamma} \times H_x^\beta$ as soon as the functions ξ and ζ involved in (1.58) have enough regularity (namely $\xi \in H_x^{\gamma+2}$, $\zeta \in H_x^{2-\beta}$). We may then apply the dominated convergence theorem on the probability space Ω^* to send ε_j to zero. Using once again the invariance of the law, we obtain

$$\mathbb{E} \left[\left(\phi(\rho_t, u_t) - \phi(\rho_s, u_s) - \int_s^t \mathcal{L} \phi(\rho_\sigma, u_\sigma) d\sigma \right) \theta(\rho_{s_1}, u_{s_1}, \dots, \rho_{s_n}, u_{s_n}) \right] = 0.$$

This equality holding true for any $s_1 < \dots < s_n = s < t$ and θ , we deduce that

$$N_\phi(t) = \phi(\rho_t, u_t) - \phi(\rho_0, u_0) - \int_0^t \mathcal{L} \phi(\rho_s, u_s) ds, \quad t \geq 0$$

defines a continuous martingale, starting at 0, with respect to the natural filtration

$$\mathcal{F}_t = \sigma\left((\rho_s, u_s) \in H_x^{-\gamma} \times H_x^\beta, s \in [0, t]\right), \quad t \geq 0.$$

This reasoning also holds for the good test function $|\phi|^2$ which is still of the form (1.58), so that $N_{|\phi|^2}(t)$ defines a continuous martingale as well. As mentioned in the proof of Proposition 1.3.7 (see again [DV17], Theorem B.3) we derive that the quadratic variation of $N_\phi(t)$ is given by

$$\left[N_\phi\right](t) = \int_0^t \left(\mathcal{L}|\phi|^2 - 2\phi\mathcal{L}\phi\right)(\rho_s, u_s) ds.$$

Let us consider $\phi(\rho, u) = (\rho, \xi) + (u, \zeta)$. Simple calculations from the expression (1.72) give

$$\begin{aligned} \mathcal{L}\phi(\rho, u) &= (\mathcal{A}^I \rho, \xi) + (\partial_x^2 u, \zeta), \\ \left(\mathcal{L}|\phi|^2 - 2\phi\mathcal{L}\phi\right)(\rho, u) &= \int_x \int_y k(x, y) \rho(x) \partial_x \xi(x) \rho(y) \partial_y \xi(y) dx dy. \end{aligned}$$

where the operator \mathcal{A}^I and the kernel $k(x, y)$ are defined in (1.73) and (1.9). Let us introduce the $H_x^{-\gamma-2} \times H_x^{\beta-2}$ -valued process

$$N(t) = \left(\rho_t - \rho_0 - \int_0^t \mathcal{A}^I \rho_s ds, u_t - u_0 - \int_0^t \partial_x^2 u_s ds\right), \quad t \geq 0.$$

Then for all $(\xi, \zeta) \in H_x^{\gamma+2} \times H_x^{2-\beta}$,

$$\left\langle N(t), (\xi, \zeta) \right\rangle, \quad t \geq 0$$

defines a continuous real-valued $(\mathcal{F}_t)_{t \geq 0}$ -martingale of quadratic variation

$$\left[\left\langle N, (\xi, \zeta) \right\rangle\right](t) = \int_0^t \int_x \int_y k(x, y) \rho_t(x) \partial_x \xi(x) \rho_t(y) \partial_y \xi(y) dx dy. \quad (1.89)$$

Using the polarization formula $\langle N, h_1 \rangle \langle N, h_2 \rangle = \frac{1}{4} \left(\langle N, h_1 + h_2 \rangle - \langle N, h_1 - h_2 \rangle \right)$, we derive from (1.89) that

$$\left\langle N(t), (\xi_1, \zeta_1) \right\rangle \left\langle N(t), (\xi_2, \zeta_2) \right\rangle - \left\langle V(t) \xi_1, \xi_2 \right\rangle, \quad t \geq 0$$

is a continuous real-valued $(\mathcal{F}_t)_{t \geq 0}$ -martingale, where the operator $V(t)$ is given by

$$\begin{aligned} \left\langle V(t) \xi_1, \xi_2 \right\rangle &= \int_0^t \int_x \int_y k(x, y) \rho_t(x) \partial_x \xi_1(x) \rho_t(y) \partial_y \xi_2(y) dx dy \\ &= \int_z \left(\int_x \partial_x (\rho_t(x) q(x, z)) \xi_1(x) dx \right) \left(\int_x \partial_x (\rho_t(y) q(y, z)) \xi_2(y) dy \right) dz \\ &= \int_z \left(\partial_x [\rho_t Q^{1/2}]^* \xi_1(z) \right) \left(\partial_x [\rho_t Q^{1/2}]^* \xi_2(z) \right) dz. \end{aligned}$$

We have used the notation introduced in (1.71). That is in fact

$$V(t) = \partial_x [\rho_t Q^{1/2}] \partial_x [\rho_t Q^{1/2}]^*.$$

We may therefore apply the martingale representation theorem [DPZ14, Theorem 8.2] : there exists some filtered probability space $(\widehat{\Omega}, (\widehat{\mathcal{F}}_t)_{t \geq 0})$ equipped with a Wiener process \widehat{W} and a process $(\widehat{\rho}, \widehat{u})$ defined on $\widehat{\Omega}$ whose law is that of (ρ, u) , such that the process

$$\widehat{N}(t) = \left(\widehat{\rho}_t - \rho_0 - \int_0^t \mathcal{A}^I \widehat{\rho}_s ds, \widehat{u}_t - u_0 - \int_0^t \partial_x^2 \widehat{u}_s ds\right), \quad t \geq 0.$$

can be represented as the Wiener integral

$$\widehat{N}(t) = \left(\int_0^t \partial_x \left[\widehat{\rho}_t Q^{1/2} d\widehat{W}_t \right]; 0 \right).$$

This shows that the limiting process $(\rho_t, u_t)_{t \geq 0}$ is a weak solution of the expected SPDE (1.7), which concludes the proof of Theorem 1.

1.7 Uniqueness for the limiting SPDE

We now prove the path-wise uniqueness for the limiting SPDE stated in Theorem 2. For convenience, let us drop the hats on the quantities $\hat{\rho}, \hat{u}$ introduced earlier and consider the limiting process $(\rho, u) \in C([0, T]; H_x^{-\gamma} \times H_x^\beta)$, where ρ satisfies the SPDE of (1.7) given on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0})$ equipped with a Wiener process $W = (W_t)_{t \geq 0}$. Of course, $u = (u_t)_{t \in [0, T]}$ is completely determined as the solution the deterministic heat equation with initial data u_0 . Let us denote

$$b_t(x) = a(x) - u_t(x),$$

with a given by (1.8). Diagonalizing the compact operator $Q^{1/2}$ involved in (1.7), we may introduce functions $(\phi_k)_{k \geq 0}$ such that

$$Q^{1/2} \circ dW_t = \sum_k \phi_k \circ d\beta_t^k, \quad (\beta^k) \text{ independent Brownian motions.}$$

As a consequence, equation (1.7) can be re-written as

$$\begin{cases} d\rho_t + \partial_x(b_t \rho_t) dt = \partial_x(\rho_t \circ \sum_k \phi_k d\beta_t^k), \\ \rho(0) = \rho_0 \in L_x^1, \end{cases}$$

or, equivalently, in Itô form,

$$\begin{cases} d\rho_t + \left(\partial_x(b_t \rho_t) - \frac{1}{2} \sum_k \partial_x(\phi_k \partial_x(\phi_k \rho_t)) \right) dt = \sum_k \partial_x(\phi_k \rho_t) d\beta_t^k \\ \rho(0) = \rho_0 \in L_x^1, \end{cases} \quad (1.90)$$

Recall that, as mentioned in (1.87), the solution ρ constructed as a weak limit of $(\rho^\varepsilon)_\varepsilon$ in Theorem 1 satisfies the additional bound

$$\sup_{t \in [0, T]} \|\rho_t\|_{L_x^1} \leq 1, \quad a.s. \quad (1.91)$$

Under the additional regularity assumptions of Theorem 2, it is clear that $b \in L^\infty([0, T]; W_x^{4, \infty})$. Regarding the regularity of ϕ_k , one may note that, for $\sigma > 4 + 1/2$,

$$\sum_k \|\phi_k\|_{W_x^{4, \infty}}^2 \leq \sum_k \|\phi_k\|_{H_x^\sigma}^2 = \|Q^{1/2}\|_{\mathcal{L}_2(L^2; H_x^\sigma)}^2 = \text{Tr}_{L^2}(J^{-2\sigma} Q)$$

where J is the positive regularizing diagonal operator introduced in (1.6.1). Applying Corollary C.2 and Proposition C.3 from appendix C of [DPZ14], we deduce, for any $\theta > 1$,

$$\text{Tr}_{L_x^2}(J^{-2\sigma} Q) = \text{Tr}_{L_x^2}(J^\theta J^{-\theta} J^{-2\sigma} Q) \leq \text{Tr}_{L_x^2}(J^\theta) \|J^{-(\theta+2\sigma)} Q\|_{\mathcal{L}(L_x^2)}.$$

On one hand, $\text{Tr}_{L_x^2}(J^\theta) = \sum_k (1 + |k|^2)^{-\theta/2} < \infty$ since $\theta > 1$. On the other hand, for $f \in L_x^2$, we have

$$J^{-(\theta+2\sigma)} Q f(x) = \int_y \left(Id - \partial_x^2 \right)^{(\theta+2\sigma)/2} k(x, y) f(y) dy$$

so that the bound $\|J^{-(\theta+2\sigma)} Q\|_{\mathcal{L}(L_x^2)} < \infty$ is ensured as soon as

$$\sup_{y \in \mathbb{T}} \|k(\cdot, y)\|_{W_x^{\theta+2\sigma, \infty}} < \infty.$$

The number $\theta + 2\sigma$ can be chosen as close to $1 + 2 \times (4 + 1/2) = 10$ as desired. Hence, the regularity assumption (1.10) of Theorem 2 leads to

$$\sum_k \|\phi_k\|_{W_x^{4, \infty}}^2 < \infty.$$

It is now clear that the proof of Theorem 2 boils down to that of the following result.

Proposition 1.7.1. *Assume that $\rho_0 \in L_x^1$ and*

$$b \in L^\infty([0, T]; W_x^{4, \infty}), \quad \sum_k \|\phi_k\|_{W_x^{4, \infty}}^2 < \infty. \quad (1.92)$$

Then a solution $\rho \in C([0, T]; H_x^{-\gamma})$ a.s of (1.90) satisfying (1.91) is path-wise unique.

We are naturally led to consider the dual BSPDE (Backward SPDE) associated with (1.90), of unknown (ψ, Z) :

$$\begin{cases} d\psi_t + \left(-b_t \partial_x \psi_t + \frac{1}{2} \sum_k \phi_k \partial_x (\phi_k \partial_x \psi_t) - \sum_k \phi_k \partial_x Z_t^k \right) dt = \sum_k Z_t^k d\beta_t^k \\ \psi(T) = \psi_T. \end{cases} \quad (1.93)$$

For $\sigma > 0$, let us define the spaces (where "prog." stands for "progressively measurable")

$$\begin{aligned} \mathbb{H}_\psi^\sigma &= \left\{ \psi : \Omega \times [0, T] \rightarrow H_x^\sigma \text{ prog.}, \|\psi\|_{\mathbb{H}^\sigma}^2 := \mathbb{E} \int_0^T \|\psi_t\|_{H_x^\sigma}^2 dt < \infty \right\}, \\ \mathbb{H}_Z^\sigma &= \left\{ Z = (Z^k)_k : \Omega \times [0, T] \rightarrow H_x^\sigma \text{ prog.}, \|Z\|_{\mathbb{H}^\sigma}^2 := \sum_k \mathbb{E} \int_0^T \|Z_t^k\|_{H_x^\sigma}^2 dt < \infty \right\}. \end{aligned}$$

The existence theory for (1.93) is linked to the uniqueness for (1.90) in the following way.

Lemma 1.7.1. *Assume that for some $s > 1/2$, for all \mathcal{F}_T -measurable $\psi_T : \Omega \rightarrow C^\infty(\mathbb{T}_x)$, (1.93) has a solution $(\psi, Z) \in \mathbb{H}_\psi^{s+2} \times \mathbb{H}_Z^{s+1}$. Then the conclusion of Proposition 1.7.1 holds.*

Proof of the lemma. Considering two solutions ρ^1, ρ^2 of (1.90), let us define $\rho = \rho^1 - \rho^2$, so that $\rho(0) = 0$. We start by regularizing (1.90) using an appropriate kernel $k_\delta \in C^\infty(\mathbb{T}_x)$, $\delta > 0$ (one may for instance consider the Féjer kernel). Denoting $\rho_\delta = k_\delta * \rho$ and $\partial_\delta = k_\delta * \partial_x$, we have

$$\begin{cases} d\rho_t^\delta + \left(\partial_\delta(b_t \rho_t) - \frac{1}{2} \sum_k \partial_\delta(\phi_k \partial_x(\phi_k \rho_t)) \right) dt = \sum_k \partial_\delta(\phi_k \rho_t) d\beta_t^k \\ \rho^\delta(0) = k_\delta * \rho_0 := \rho_0^\delta \end{cases}$$

in the "strong sense" that $(\rho_t^\delta)_{t \geq 0}$ is an L_x^2 -valued semi-martingale. Itô's formula is now licit (one may verify that the assumptions of [DPZ14] Theorem 4.17 are satisfied) and gives

$$\begin{aligned} d\langle \rho_t^\delta, \psi_t \rangle &= \left[-\langle \partial_\delta(b_t \rho_t), \psi_t \rangle - \langle \rho_t^\delta, b \partial_x \psi_t \rangle \right] dt \\ &\quad + \left[\frac{1}{2} \sum_k \langle \partial_\delta(\phi_k \partial_x \phi_k \rho_t), \psi_t \rangle - \langle \rho_t^\delta, \phi_k \partial_x(\phi_k \partial_x \psi_t) \rangle \right] dt \\ &\quad + \left[\sum_k \langle \rho_t^\delta, \phi_k \partial_x Z_t^k \rangle + \langle \partial_\delta(\phi_k \rho_t), Z_t^k \rangle \right] dt \\ &\quad + \sum_k \left[\langle \partial_\delta(\phi_k \rho_t), \psi_t \rangle + \langle \rho_t^\delta, Z_t^k \rangle \right] d\beta_t^k. \end{aligned}$$

The assumptions easily guarantee that the last term defines a square-integrable martingale. We may hence write

$$\begin{aligned} \mathbb{E} \left[\langle \rho_T, k_\delta * \psi_T \rangle \right] &= -\mathbb{E} \int_0^T \left\langle \rho_t, b_t \partial_x(k_\delta * \psi_t) - k_\delta * [b_t \partial_x \psi_t] \right\rangle dt \\ &\quad - \frac{1}{2} \mathbb{E} \sum_k \int_0^T \left\langle \rho_t, \phi_k \partial_x(\phi_k \partial_x(k_\delta * \psi_t)) - k_\delta * [\phi_k \partial_x(\phi_k \partial_x \psi_t)] \right\rangle dt \quad (1.94) \\ &\quad - \mathbb{E} \sum_k \int_0^T \left\langle \rho_t, k_\delta * [\phi_k \partial_x Z_t^k] - \phi_k \partial_x(k_\delta * Z_t^k) \right\rangle dt. \quad (1.95) \end{aligned}$$

The left-hand side easily goes to $\mathbb{E}[\langle \rho_T, \psi_T \rangle]$, while the right-hand side vanishes as $\delta \rightarrow 0$. For instance, using (1.91), one may control the second term (1.94) with

$$\sum_k \mathbb{E} \int_0^T \|\phi_k \partial_x(\phi_k \partial_x(k_\delta * \psi_t)) - k_\delta * [\phi_k \partial_x(\phi_k \partial_x \psi_t)]\|_{L_x^\infty} dt$$

The assumptions guarantee that $\psi_t \in H_x^{s+2} \subset W_x^{2,\infty}$ a.s, so that the integrand tends to 0 for all fixed k, t and almost every ω . With the uniform bound

$$\|\phi_k \partial_x (\phi_k \partial_x (k_\delta * \psi_t)) - k_\delta * [\phi_k \partial_x (\phi_k \partial_x \psi_t)]\|_{L_x^\infty} \leq C \|\phi_k\|_{W_x^{1,\infty}}^2 \|\psi_t\|_{W_x^{2,\infty}} \leq C \|\phi_k\|_{W_x^{1,\infty}}^2 \|\psi_t\|_{H_x^{s+2}}$$

and the regularity assumption (1.92), we may apply dominated convergence to conclude. Note that the bound for the third term (1.95) is

$$\|k_\delta * [\phi_k \partial_x Z_t^k] - \phi_k \partial_x (k_\delta * Z_t^k)\|_{L_x^\infty} \leq \|\phi_k\|_{L^\infty} \|Z_t^k\|_{W_x^{1,\infty}} \leq \|\phi_k\|_{L^\infty}^2 + \|Z_t^k\|_{H_x^{s+1}}^2.$$

Since $\|\rho_T\|_{L_x^1} \leq 1$ a.s, the equality $\mathbb{E}[\langle \rho_T, \psi_T \rangle] = 0$ for all \mathcal{F}_T -measurable $\psi_T : \Omega \rightarrow C^\infty(\mathbb{T})$ leads to $\rho_T(x) = 0$ for almost every $(x, \omega) \in \mathbb{T} \times \Omega$. This holds for all $T > 0$ and since $\rho \in C([0, T]; H_x^{-\gamma})$ almost surely, we conclude $\rho = 0$ a.s. \square

It now remains to prove the existence of sufficiently regular solutions to the BSPDE (1.93). Let us consider the equation perturbed with an additional $\frac{\varepsilon}{2} \partial_x^2 \psi$:

$$\begin{cases} d\psi_t + \left((-b_t + \frac{1}{2} \sum_k \phi_k \partial_x \phi_k) \partial_x \psi_t - \sum_k \phi_k \partial_x Z_t^k \right) dt + \frac{1}{2} \left(\sum_k \phi_k^2 + \varepsilon \right) \partial_x^2 \psi_t dt = \sum_k Z_t^k d\beta_t^k \\ \psi(T) = \psi_T. \end{cases} \quad (1.96)$$

In this super-parabolic setup, in view of the assumptions (1.92) on the coefficients, assuming furthermore that

$$\psi_T \in L^2\left((\Omega, \mathcal{F}_T); H_x^4\right),$$

Theorem 2.3 of [DM10] guarantees the existence of a solution $(\psi^\varepsilon, Z^\varepsilon)$ of (1.96) such that

$$(\psi^\varepsilon, Z^\varepsilon) \in \mathbb{H}^5 \times \mathbb{H}^4 \quad \text{and} \quad \mathbb{E}\left[\sup_{t \in [0, T]} \|\psi_t^\varepsilon\|_{H_x^4}^2\right] < \infty. \quad (1.97)$$

Of course, the corresponding estimates in these spaces may depend on $\varepsilon > 0$.

Proposition 1.7.2. *For all $0 \leq m \leq 4$, the solution $(\psi^\varepsilon, Z^\varepsilon) \equiv (\psi, Z)$ of (1.96) satisfies the estimate*

$$\begin{aligned} & \frac{1}{2} \|\partial_x^m \psi_T\|_{L_x^2}^2 + \frac{1}{2} \int_t^T \left(\varepsilon \|\partial_x^{m+1} \psi_s\|_{L_x^2}^2 + \sum_k \|\partial_x^m Z_s^k + m(\partial_x^m \psi_s)(\partial_x \phi_k) - \partial_x(\partial_x^m \psi_s \phi_k)\|_{L_x^2}^2 \right) ds \\ & \leq \frac{1}{2} \|\partial_x^m \psi_T\|_{L_x^2}^2 + C \int_t^T \left(\|\psi_s\|_{H_x^m}^2 + \sum_k \|Z_s^k\|_{H_x^{m-1}}^2 \right) ds - \sum_k \int_t^T \langle \partial_x^m \psi_s, \partial_x^m Z_s^k \rangle d\beta_s^k, \end{aligned} \quad (1.98)$$

with the convention $\|Z\|_{H_x^{m-1}} = 0$ when $m = 0$, where the constant C depends only on the quantities involved in (1.92).

Remark 1.7.1. *Note that (1.97) guarantees that, for $0 \leq m \leq 4$,*

$$M_t = \sum_k \int_0^t \langle \partial_x^m \psi_s, \partial_x^m Z_s^k \rangle d\beta_s^k, \quad t \geq 0$$

defines a real-valued L^1 martingale. Indeed,

$$V_T := \sum_k \int_0^T \left| \langle \partial_x^m \psi_t, \partial_x^m Z_t^k \rangle \right|^2 dt \leq \sup_{t \in [0, T]} \|\psi_t\|_{H_x^m}^2 \sum_k \int_0^T \|Z_t^k\|_{H_x^m}^2 dt < \infty \quad a.s$$

ensures that M_t is a local martingale. Burkholder-Davis-Gundy's inequality [BDG72] then gives

$$\mathbb{E}\left[\sup_{t \in [0, T]} |M_t|\right] \lesssim \mathbb{E}\left[V_T^{1/2}\right] \lesssim \mathbb{E}\left[\sup_{t \in [0, T]} \|\psi_t\|_{H_x^m}^2\right] + \sum_k \mathbb{E} \int_0^T \|Z_t^k\|_{H_x^m}^2 dt < \infty.$$

Let us now give a proof of Proposition 1.7.2.

Proof. We apply Itô's formula to $\|\partial_x^m \psi_t\|_{L_x^2}^2$. To make it rigorous (so that the assumptions of [DPZ14] Theorem 4.17 are again satisfied), one may start by regularizing the coefficients b and $(\phi_k)_k$ involved in (1.96), so that [DM10] provides the regularity $(\psi, Z) \in \mathbb{H}^6 \times \mathbb{H}^5$. We obtain

$$\frac{1}{2}d\|\partial_x^m \psi_t\|_{L_x^2}^2 = \left(A_t^1 + A_t^2 + A_t^3 + A_t^4 + A_t^5 \right) dt + \left\langle \partial_x^m \psi_t, \partial_x^m Z_t dW_t \right\rangle,$$

with

$$\begin{aligned} A_t^1 &:= -\frac{\varepsilon}{2} \left\langle \partial_x^m \psi_t, \partial_x^{m+2} \psi_t \right\rangle, & A_t^2 &:= \left\langle \partial_x^m \psi_t, \partial_x^m (b_t \partial_x \psi_t) \right\rangle, & A_t^3 &:= \sum_k \left\langle \partial_x^m \psi_t, \partial_x^m (\phi_k \partial_x Z_t^k) \right\rangle, \\ A_t^4 &:= \frac{1}{2} \sum_k \|\partial_x^m Z_t^k\|_{L_x^2}^2, & A_t^5 &:= -\frac{1}{2} \sum_k \left\langle \partial_x^m \psi_t, \partial_x^m (\phi_k \partial_x (\phi_k \partial_x \psi_t)) \right\rangle dt. \end{aligned}$$

Let us review the different terms. The first term A_t^1 appears untouched in (1.98). We may expand the second term into

$$\begin{aligned} A_t^2 &= \left\langle \partial_x^m \psi_t, b_t \partial_x^{m+1} \psi_t \right\rangle + \sum_{l=0}^{m-1} \binom{m}{l} \left\langle \partial_x^m \psi_t, \partial_x^{m-l} b_t \partial_x^{l+1} \psi_t \right\rangle \\ &= -\frac{1}{2} \left\langle |\partial_x^m \psi_t|^2, \partial_x b_t \right\rangle + \sum_{l=0}^{m-1} \binom{m}{l} \left\langle \partial_x^m \psi_t, \partial_x^{m-l} b_t \partial_x^{l+1} \psi_t \right\rangle, \end{aligned}$$

which can be bounded by $C\|b\|_{L_t^\infty W_x^{m,\infty}} \|\psi_t\|_{H_x^m}^2$. Similarly, the third term can be expanded into

$$\begin{aligned} A_t^3 &= \left[\sum_k \left\langle \partial_x^m \psi_t, \phi_k \partial_x^{m+1} Z_t^k \right\rangle + m \left\langle \partial_x^m \psi_t, \partial_x \phi_k \partial_x^m Z_t^k \right\rangle \right] \\ &\quad + \left[\sum_k \sum_{l=0}^{m-2} \binom{m}{l} \left\langle \partial_x^m \psi_t \partial_x^{m-l} \phi_k, \partial_x^{l+1} Z_t^k \right\rangle \right] \quad := B_t^1 + B_t^2. \end{aligned}$$

The term B_t^2 can be bounded by

$$C \sum_k \|\psi_t\|_{H_x^m} \|\phi_k\|_{W_x^{m,\infty}} \|Z_t^k\|_{H_x^{m-1}} \lesssim \|\psi_t\|_{H_x^m}^2 \sum_k \|\phi_k\|_{W_x^{m,\infty}}^2 + \sum_k \|Z_t^k\|_{H_x^{m-1}}^2.$$

Using a polarizing identity, the term B_t^1 may be rewritten as

$$\begin{aligned} B_t^1 &= -\sum_k \left\langle \partial_x (\partial_x^m \psi_t \phi_k) - m \partial_x^m \psi_t \partial_x \phi_k, \partial_x^m Z_t^k \right\rangle \\ &= -\frac{1}{2} \sum_k \|\partial_x^m Z_t^k\|_{L_x^2}^2 - \frac{1}{2} \sum_k \|\partial_x (\partial_x^m \psi_t \phi_k) - m \partial_x^m \psi_t \partial_x \phi_k\|_{L_x^2}^2 \\ &\quad + \frac{1}{2} \sum_k \|\partial_x^m Z_t^k + m \partial_x^m \psi_t \partial_x \phi_k - \partial_x (\partial_x^m \psi_t \phi_k)\|_{L_x^2}^2 \\ &=: C_t^1 + C_t^2 + C_t^3. \end{aligned}$$

The term C_t^1 cancels out A_t^4 . The term C_t^3 appears untouched in (1.98). We are hence left to study

$$D_t := A_t^5 + C_t^2 = -\frac{1}{2} \sum_k \left(\partial_x^m \psi_t, \partial_x^m (\phi_k \partial_x (\phi_k \partial_x \psi_t)) \right) - \frac{1}{2} \sum_k \|\partial_x (\partial_x^m \psi_t \phi_k) - m \partial_x^m \psi_t \partial_x \phi_k\|_{L_x^2}^2.$$

By expanding the derivatives and integrating by parts as done previously, we get the expression

$$-\frac{1}{2} \sum_k \left[\left(|\partial_x^m \psi_t|^2, \partial_x (\phi_k \partial_x \phi_k) \right) - \left(|\partial_x^{m+1} \psi_t|^2, |\phi_k|^2 \right) + \sum_{l=0}^{m-2} \binom{m}{l} \left(\partial_x^m \psi_t, \partial_x^{m-l} \phi_k \partial_x^{l+1} (\phi_k \partial_x \psi_t) \right) \right]$$

for the term A_t^5 , and

$$C_t^2 = -\frac{1}{2} \sum_k \left[\left(|\partial_x^m \psi_t|^2, (m-1)^2 |\partial_x \phi_k|^2 + (m-1) \partial_x (\phi_k \partial_x \phi_k) \right) + \left(|\partial_x^{m+1} \psi_t|^2, |\phi_k|^2 \right) \right].$$

We see that the unwanted terms involving $\partial_x^{m+1} \psi_t$ therefore cancel out in D_t , and the remaining terms can once again be bounded by $C\|\psi_t\|_{H_x^m}^2 \sum_k \|\phi_k\|_{W_x^{m,\infty}}^2$. \square

From Proposition 1.7.2, we derive the following uniform bound.

Corollary 1.7.1. *The solution $(\psi^\varepsilon, Z^\varepsilon)$ of (1.96) satisfies the bound*

$$\|\psi^\varepsilon\|_{\mathbb{H}^4}^2 + \|Z^\varepsilon\|_{\mathbb{H}^3}^2 \lesssim 1.$$

The constant involved in \lesssim depends only on T and the quantities of (1.92).

Proof. Let us simply denote $(\psi, Z) = (\psi^\varepsilon, Z^\varepsilon)$. As mentioned in Remark 1.7.1, we may take the expectation in (1.98), which gives, in particular, for $0 \leq m \leq 4$, the estimate

$$\mathbb{E}\|\psi_t\|_{H_x^m}^2 \leq \mathbb{E}\|\psi_T\|_{H_x^m}^2 + C\left(\int_t^T \mathbb{E}\|\psi_s\|_{H_x^m}^2 ds + \sum_k \int_t^T \mathbb{E}\|Z_s^k\|_{H_x^{m-1}}^2 ds\right). \quad (1.99)$$

Writing

$$\|\partial_x^m Z_s^k\|_{L_x^2} \leq \|\partial_x^m Z_s^k + m(\partial_x^m \psi_s)(\partial_x \phi_k) - \partial_x(\partial_x^m \psi_s \phi_k)\|_{L_x^2} + \|m(\partial_x^m \psi_s)(\partial_x \phi_k) - \partial_x(\partial_x^m \psi_s \phi_k)\|_{L_x^2},$$

and using (1.98) again, we also obtain

$$\sum_k \int_t^T \mathbb{E}\|Z_s^k\|_{H_x^m}^2 ds \leq \mathbb{E}\|\psi_T\|_{H_x^m}^2 + C\left(\int_t^T \mathbb{E}\|\psi_s\|_{H_x^{m+1}}^2 ds + \sum_k \int_t^T \mathbb{E}\|Z_s^k\|_{H_x^{m-1}}^2 ds\right). \quad (1.100)$$

Recalling the convention $\|Z\|_{H_x^{-1}} = 0$, in Proposition 1.7.2, inequality (1.100) yields, for $0 \leq m \leq 3$,

$$\sum_k \int_t^T \mathbb{E}\|Z_s^k\|_{H_x^m}^2 ds \lesssim \mathbb{E}\|\psi_T\|_{H_x^3}^2 + \int_t^T \mathbb{E}\|\psi_s\|_{H_x^4}^2 ds. \quad (1.101)$$

Coming back to (1.99), we get, for $0 \leq m \leq 4$,

$$\mathbb{E}\|\psi_t\|_{H_x^4}^2 \lesssim \mathbb{E}\|\psi_T\|_{H_x^4}^2 + \int_t^T \mathbb{E}\|\psi_s\|_{H_x^4}^2 ds$$

and Grönwall's Lemma provides the desired estimate for ψ . The estimate for Z is obtained by taking $t = 0$ in (1.101). \square

Now, let us consider $\varepsilon_1, \varepsilon_2 > 0$ and denote $(\psi, Z) = (\psi^{\varepsilon_1} - \psi^{\varepsilon_2}, Z^{\varepsilon_1} - Z^{\varepsilon_2})$. In the same fashion as Proposition 1.7.2, one can prove the following estimate, for $0 \leq m \leq 3$:

$$\begin{aligned} & \frac{1}{2}\|\partial_x^m \psi_t\|_{L_x^2}^2 + \frac{1}{2}\int_t^T \sum_k \|\partial_x^m Z_s^k + m(\partial_x^m \psi_s)(\partial_x \phi_k) - \partial_x(\partial_x^m \psi_s \phi_k)\|_{L_x^2}^2 ds \\ & \leq C|\varepsilon_2 - \varepsilon_1| \int_0^T \left(\|\psi_s^{\varepsilon_1}\|_{H_x^{m+1}}^2 + \|\psi_s^{\varepsilon_2}\|_{H_x^{m+1}}^2\right) ds + C \int_t^T \left(\|\psi_s\|_{H_x^m}^2 + \sum_k \|Z_s^k\|_{H_x^{m-1}}^2\right) ds \\ & \quad - \sum_k \int_t^T \left(\partial_x^m \psi_s, \partial_x^m Z_s^k\right) d\beta_s^k. \end{aligned}$$

Relying on the bound from Corollary 1.7.1 and applying similar calculations as in its proof, we deduce

$$\|\psi^{\varepsilon_1} - \psi^{\varepsilon_2}\|_{\mathbb{H}^3}^2 + \|Z^{\varepsilon_1} - Z^{\varepsilon_2}\|_{\mathbb{H}^2}^2 \lesssim |\varepsilon_1 - \varepsilon_2|.$$

As a consequence, $(\psi^\varepsilon, Z^\varepsilon)$ converges to some (ψ, Z) in the Banach space $\mathbb{H}^3 \times \mathbb{H}^2$ as ε goes to zero. It is now easy to send ε to zero in (1.96) and verify that $(\psi, Z) \in \mathbb{H}^3 \times \mathbb{H}^3$ is indeed a solution of (1.93). Lemma 1.7.1 may be applied, which concludes the proof of Proposition 1.7.1, and therefore that of Theorem 2.

1.8 Appendix : Well-posedness of the path-wise kinetic system

For the sake of completeness, we wish to prove the well-posedness of the deterministic PDE system

$$\begin{cases} \partial_t f + v \partial_x f + \partial_v \left[(m + u - v) f \right] = 0, \\ \partial_t u - \partial_x^2 u = J(f) - \rho u. \end{cases} \quad (1.102)$$

Here, we consider a fixed $\omega \in \Omega$, so that $m \equiv m(\omega)$ belongs to $D([0, T]; E)$, space of càdlàg functions taking values in the separable complete space E .

1.8.1 L^1 and L^2 estimates for a conservation equation with Lipschitz coefficients

Recall that, whenever $w \in D([0, T]; E)$ is given, the solution $f[w]$ to the linear conservation equation

$$\begin{cases} \partial_t f + v \partial_x f + \partial_v [(w - v)f] = 0, \\ f_0 \in L^1_{x,v}, \end{cases}$$

is naturally expressed as

$$f_t[w](x, v) = e^t f_0 \circ \Phi_0^t(x, v),$$

where $\Phi_s^t(x, v) = (X_s^t(x, v), V_s^t(x, v))$ is the flow associated to the characteristics :

$$\begin{cases} \frac{d}{ds} X_s^t = V_s^t, & X_t^t(x, v) = x \\ \frac{d}{ds} V_s^t = w_s(X_s^t) - V_s^t, & V_t^t(x, v) = v \end{cases} \quad (1.103)$$

Grönwall's lemma immediately gives the following result.

Lemma 1.8.1. *Let $w, \tilde{w} \in D([0, T]; E)$ and let $\Phi[w], \Phi[\tilde{w}]$ denote the respective flows. Then,*

$$\forall s, t \in [0, T], \quad \left\| \Phi_s^t[w] - \Phi_s^t[\tilde{w}] \right\|_{L^\infty_{x,v}} \lesssim \|w - \tilde{w}\|_{L^\infty_{t,x}}.$$

The constant involved in \lesssim depends on $T, \|\partial_x w\|_{L^\infty_{t,x}}, \|\partial_x \tilde{w}\|_{L^\infty_{t,x}}$ only.

Let us now fix some $w \in D([0, T]; E)$ and give some estimates regarding the v -moments of $f_t[w]$. Recall the notation, for $f \in G^1$,

$$\rho(f)(x) = \int f(x, v) dv, \quad J(f)(x) = \int v f(x, v) dv.$$

Lemma 1.8.2 (L^1 moments estimates). *Let $f_0 \in L^1_{x,v}$. Then for all $t \in [0, T]$,*

$$\|f_t[w]\|_{L^1_{x,v}} = \|f_0\|_{L^1_{x,v}}.$$

Assume furthermore that $f_0 \in G^k$, $k \geq 1$. Then for all $t \in [0, T]$,

$$\|f_t[w]\|_{G^k} \lesssim \|f_0\|_{G^k}.$$

The constant involved in \lesssim depends on T, k and $\|w\|_{L^\infty_{t,x}}$ only.

The estimate for the first moment is precisely :

$$\|J(f_t[w])\|_{L^1_{x,v}} \leq \left(\|J(f_0)\|_{L^1_x} + t \sup_{s \in [0, t]} \|w_s\|_{L^\infty_x} \|f_0\|_{L^1_{x,v}} \right) e^{-t}$$

Proof. Without loss of generality, we may assume $f_0 \geq 0$. When $w \in D([0, 1]; E)$ is smooth, it is clear that the flow $\Phi[\tilde{w}]$ is a globally-defined C^∞ -diffeomorphism, whose Jacobian determinant is given by $J\Phi_0^t[\tilde{w}] = \det(D\Phi_0^t[\tilde{w}]) = e^t$. The conservation of the L^1 -norm is then obtained by a change of variable using the form (1.30). The result is generalized to $w \in D([0, 1]; E)$ by density using Lemma 1.8.1.

For simplicity, we only prove the estimate in G^1 : the estimates on higher moments can be obtained by induction using the same process. Assuming $f_0 \in C_c^\infty(\mathbb{T} \times \mathbb{R})$, the following calculations are justified :

$$\begin{aligned} \frac{d}{dt} \|J(f_t)\|_{L^1_x} &= \frac{d}{dt} \int_x \int_v |v| f_t(x, v) dx dv = - \int_x \int_v |v| \left(v \partial_x f + \partial_v [(w_t(x) - v) f_t(x, v)] \right) dx dv \\ &= - \int_v \text{sign}(v) w_t(x) f_t(x, v) dv + \int_v |v| f_t(x, v) dv, \end{aligned}$$

so that

$$\frac{d}{dt} \|J(f_t)\|_{L^1_x} \leq \|w_t\|_{L^\infty_x} \|f_0\|_{L^1_x} - \|J(f_t)\|_{L^1_x} \leq \sup_{s \in [0, t]} \|w_s\|_{L^\infty_x} \|f_0\|_{L^1_x} - \|J(f_t)\|_{L^1_x},$$

that is, integrating over $[0, t]$,

$$\|J(f_t)\|_{L_x^1} \leq \left(\|J(f_0)\|_{L_x^1} + t \sup_{s \in [0, t]} \|w_s\|_{L_x^\infty} \|f_0\|_{L_x^1} \right) - \int_0^t \|J(f_s)\|_{L_x^1} ds.$$

Grönwall's lemma gives the expected result. Again, the inequality holds for a general initial data $f_0 \geq 0$ by a density argument, using Fatou's Lemma. \square

Corollary 1.8.1. *Let $f_0 \in L_{x,v}^2$. Then for all $t \in [0, T]$,*

$$\|f_t[w]\|_{L_{x,v}^2} = e^{t/2} \|f_0\|_{L_{x,v}^2}.$$

Assume furthermore that $|f_0|^2 \in G^k$, $k \geq 1$. Then for all $t \in [0, T]$,

$$\|f_t[w]^2\|_{G^k} \lesssim \|f_0^2\|_{G^k}.$$

The constant involved in \lesssim depends on T , k and $\|w\|_{L_{t,x}^\infty}$ only.

Proof. Simply note that $h_t := e^{-t}(f_t[w])^2 = e^t f_0^2 \circ \Phi_t^0$ and apply Lemma 1.8.2 to h . \square

Corollary 1.8.2 (L^2 moments estimates). *Let $f_0 \in L_{x,v}^2$ and $f_0^2 \in G_4$. Then, for all $t \in [0, T]$,*

$$\begin{aligned} \|\rho(f_t[w])\|_{L_x^2} &\lesssim \|f_0^2\|_{G_2}^{1/2}, \\ \|J(f_t[w])\|_{L_x^2} &\lesssim \|f_0^2\|_{G_4}^{1/2}. \end{aligned}$$

The constants involved in \lesssim depends on T and $\|w\|_{L_{t,x}^\infty}$ only.

Proof. For instance, for the second estimate, Hölder's inequality gives

$$\left(\int_v |v| f dv \right)^2 = \left(\int_v (1 + |v|) |v| f \times \frac{1}{1 + |v|} dv \right)^2 \leq \left(\int_v (1 + |v|)^2 |v|^2 f^2 dv \right) \left(\int_v \frac{1}{(1 + |v|)^2} dv \right)$$

from which we deduce $\int_x J(f)^2 dx \leq C \|f^2\|_{G_4}$, and we use the estimate from Corollary 1.8.1. \square

We will now establish some kind of stability estimate for f as a function of w . Let us start with a lemma regarding the gradient.

Lemma 1.8.3 (L^1 gradient estimates). *Assume that $\|\nabla_{x,v} f_0\|_{L_{x,v}^1} < \infty$. Then,*

$$\|\nabla_{x,v} f_t[w]\|_{L_{x,v}^1} \leq C(T, \|\partial_x w\|_{L_{t,x}^\infty}) \|\nabla_{x,v} f_0\|_{L_{x,v}^1}.$$

Assume furthermore that $\|v \nabla_{x,v} f_0\|_{L_{x,v}^1} < \infty$. Then,

$$\|v \nabla_{x,v} f_t[w]\|_{L_{x,v}^1} \leq C(T, \|w\|_{L_{t,x}^\infty}, \|\partial_x w\|_{L_{t,x}^\infty}, \|\nabla_{x,v} f_0\|_{L_{x,v}^1}, \|v \nabla_{x,v} f_0\|_{L_{x,v}^1}).$$

Proof. Let us assume that $f_0 \in C_c^\infty$, so that we are dealing with smooth, compact-supported, strong solutions. The general case is once again deduced by density arguments. From

$$\partial_t f + v \partial_x f + (w - v) \partial_v f - f = 0$$

we derive that $g_1 = \partial_x f$, $g_2 = \partial_v f$ satisfy

$$\partial_t g_1 + v \partial_x g_1 + \partial_x w g_2 + (w - v) \partial_v g_1 - g_1 = 0, \quad (1.104)$$

$$\partial_t g_2 + g_1 + v \partial_x g_2 + (w - v) \partial_v g_2 - 2g_2 = 0 \quad (1.105)$$

Multiplying (1.104) by $\text{sign}(g_1)$ and integrating over time (to be rigorous, one should replace $\text{sign}(x)$ by $s_\eta(x) = x/(|x| + \eta)$ and carefully let η go to zero), we are easily led to

$$\frac{d}{dt} \int_x \int_v |g_1| dx dv \leq \|\partial_x w\|_{L_{t,x}^\infty} \int_x \int_v |g_2| dx dv.$$

Similarly, multiplying (1.105) by $\text{sign}(g_2)$ we get

$$\frac{d}{dt} \int_x \int_v |g_2| dx dv \leq \int_x \int_v |g_1| dx dv + \int_x \int_v |g_s| dx dv.$$

Summing these and applying Grönwall's inequality gives the expected estimate. Let us now establish the estimate on $\|v\nabla_{x,v}f\|_{L^1_{x,v}}$. From (1.104), (1.105), we derive that $h_1 := v\partial_x f$ and $h_2 := v\partial_v f$ satisfy

$$\begin{aligned}\partial_t h_1 + v\partial_x h_1 + (\partial_x w)h_2 + v(w-v)\partial_v g_1 - h_1 &= 0, \\ \partial_t h_2 + h_1 + v\partial_x h_2 + v(w-v)\partial_v g_2 - 2h_2 &= 0\end{aligned}$$

Now since $(w-v)\partial_v h_i = (w-v)g_i + v(w-v)\partial_v g_i$, we can rewrite

$$v(w-v)\partial_v g_i = (w-v)\partial_v h_i + h_i - wg_i$$

and get

$$\begin{aligned}\partial_t h_1 + v\partial_x h_1 + (\partial_x w)h_2 + (w-v)\partial_v h_1 - wg_1 &= 0, \\ \partial_t h_2 + h_1 + v\partial_x h_2 + (w-v)\partial_v h_2 - h_2 - wg_2 &= 0.\end{aligned}$$

We then use the same approach to obtain

$$\begin{aligned}\frac{d}{dt}\|h_1(t)\|_{L^1_{x,v}} &\leq \|\partial_x w\|_{L^\infty_{t,x}}\|h_2(t)\|_{L^1_{x,v}} - \|h_1(t)\|_{L^1_{x,v}} + \|w\|_{L^\infty_{t,x}}\|g_1(t)\|_{L^1_{x,v}}, \\ \frac{d}{dt}\|h_2(t)\|_{L^1_{x,v}} &\leq \|h_1(t)\|_{L^1_{x,v}} + \|w\|_{L^\infty_{t,x}}\|g_2(t)\|_{L^1_{x,v}}\end{aligned}$$

and conclude by applying the previous estimates and Grönwall's inequality. \square

Corollary 1.8.3 (L^1 stability). *Let $w, \tilde{w} \in D([0, T]; E)$. Assume*

$$\|(1 + |v|)\nabla_{x,v}f_0\|_{L^1_{x,v}} < +\infty.$$

Then the following estimates hold :

$$\begin{aligned}\|\rho_t[w] - \rho_t[\tilde{w}]\|_{L^1_x} &\leq \|f_t[w] - f_t[\tilde{w}]\|_{L^1_{x,v}} \lesssim \|w - \tilde{w}\|_{L^\infty_{t,x}}, \\ \|J(f_t[w]) - J(f_t[\tilde{w}])\|_{L^1_x} &\lesssim \|w - \tilde{w}\|_{L^\infty_{t,x}}.\end{aligned}$$

The constants involved in \lesssim depend on T , $\|(1 + |v|)\nabla_{x,v}f_0\|_{L^1_{x,v}}$, $\|w\|_{L^\infty_{t,x}}$, $\|\tilde{w}\|_{L^\infty_{t,x}}$, $\|\partial_x w\|_{L^\infty_{t,x}}$, and $\|\partial_x \tilde{w}\|_{L^\infty_{t,x}}$ only.

Proof. Let us once again assume that $f_0 \in C_c^\infty$. Let us denote $f_t = f_t[w]$, $\tilde{f}_t = f_t[\tilde{w}]$, $F = f - \tilde{f}$ and $W = w - \tilde{w}$. These quantities satisfy

$$\partial_t F + v\partial_x F + \partial_v \left[(w-v)F + W\tilde{f} \right] = 0$$

which can be rewritten as

$$\partial_t F + v\partial_x F + (w-v)\partial_v F - F + W\partial_v \tilde{f} = 0.$$

Multiplying by $\text{sign}(F)$, we get

$$\partial_t |F| + v\partial_x |F| + (w-v)\partial_v |F| - |F| + \text{sign}(F) W\partial_v \tilde{f} = 0$$

and we deduce

$$\frac{d}{dt}\|F(t)\|_{L^1_{x,v}} \leq \|W\|_{L^\infty_{t,x}}\|\nabla_{x,v}\tilde{f}\|_{L^1_{x,v}}.$$

Using the L^1 gradient estimates on \tilde{f} from Lemma 1.8.3, we get the first inequality. Now let $G = vF$. Multiplying (1.8.1) by v , we get

$$\partial_t G + v\partial_x G + v(w-v)\partial_v F - G + vW\partial_v \tilde{f} = 0.$$

Since $(w-v)\partial_v G = (w-v)F + v(w-v)\partial_v F$ this can be rewritten as

$$\partial_t G + v\partial_x G + (w-v)\partial_v G - wF + vW\partial_v \tilde{f} = 0.$$

Multiplying by $\text{sign}(G)$, we get

$$\partial_t |G| + v \partial_x |G| + (w - v) \partial_v |G| - \text{sign}(G) w F + \text{sign}(G) W v \partial_v \tilde{f} = 0$$

and we deduce

$$\frac{d}{dt} \|G(t)\|_{L^1_{x,v}} \leq -\|G(t)\|_{L^1_{x,v}} + \|w\|_{L^\infty_{t,x}} \|F(t)\|_{L^1_{x,v}} + \|W\|_{L^\infty_{t,x}} \|v \nabla_{x,v} \tilde{f}(t)\|_{L^1_{x,v}}.$$

Using the L^1 gradient estimates on \tilde{f} and the inequality we just showed, Grönwall's lemma gives the expected estimate. \square

1.8.2 Existence and uniqueness

We are now ready to prove the following result.

Proposition 1.8.1. *Assume that*

$$\int_x \int_v (1 + |v|^4) |f_0|^2 dx dv + \int_x \int_v (1 + |v|) |\nabla_{x,v} f_0| dx dv < \infty, \quad u_0 \in H_x^\eta \text{ for } \eta \in (3/2, 2).$$

Then, for all $\beta \in (3/2, \eta)$, there exists a unique couple (f, u) with $u \in C([0, T]; H_x^\beta)$ solution of the system (1.29) in the sense that

$$\begin{cases} f_t = f_t[m + u] \text{ as defined in (1.8.1),} \\ u_t = S(t)u_0 + \int_0^t S(t-s) [J(f_s) - \rho_s u_s] ds, \end{cases}$$

where $S(t) = e^{t\partial_x^2}$ denotes the semigroup associated to the heat equation. For any $k \geq 0$, assuming $f_0 \in G^k$, we have additionally $f \in C([0, T]; G^k)$.

Remark 1.8.1. *The "numerical constants" used in the following proof may all depend on the quantity $\sup_{t \in [0, T]} \|m_t(\omega)\|_{W_x^{1, \infty}}$, which is in any case assumed to be bounded in Assumption 1.*

Let us define the recursive sequences

$$\begin{aligned} f^n &\in C([0, T]; L^1_{x,v}), \quad n \geq 0 \\ u^n &\in C([0, T]; H_x^\eta), \quad n \geq 0 \end{aligned}$$

as follows : $(f_t^0, u_t^0) = (f_0, u_0)$ and

$$f_t^{n+1} = f_t[m + u^n] \text{ as defined in (1.8.1),} \quad (1.106)$$

$$u_t^{n+1} = S(t)u_0 + \int_0^t S(t-s) [J(f_s^{n+1}) - \rho_s^{n+1} u_s^n] ds. \quad (1.107)$$

Claim 1 : these sequences are well-defined

Since $\eta > 3/2$, and we work in dimension 1, the embedding $H_x^\eta \subset W_x^{1, \infty}$ guarantees that $f_t[m + u^n]$ can be properly defined as long as u^n takes values in H_x^η . We have

$$\|u_t^{n+1}\|_{H_x^\eta} \leq \|u_0\|_{H_x^\eta} + \int_0^t \left\| S(t-s) [J(f_s^{n+1}) - \rho_s^{n+1} u_s^n] \right\|_{H_x^\eta} ds$$

with the classical estimate

$$\forall w \in L^2, \quad \|S(t)w\|_{H_x^\eta} \lesssim (1 + t^{-\eta/2}) \|w\|_{L_x^2}.$$

The L^2 estimates for $f_t^{n+1} = f_t[m + u^n]$ from Lemma 1.8.2 give

$$\|J(f_s^{n+1}) - \rho_s^{n+1} u_s^n\|_{L_x^2} \leq C \left(\sup_{t \in [0, T]} \|u_t^n\|_{L_x^\infty} \right) \left(1 + \sup_{t \in [0, T]} \|u_t^n\|_{L_x^\infty} \right)$$

from which we deduce

$$\|u_t^{n+1}\|_{H_x^\eta} \leq \|u_0\|_{H_x^\eta} + C \left(\sup_{t \in [0, T]} \|u_t^n\|_{L_x^\infty} \right) \int_0^t (1 + s^{-\eta/2}) ds. \quad (1.108)$$

Since $\eta < 2$, it follows that, for all $n \geq 0$, $u^n \in C([0, T]; H_x^\eta)$ and f^n is well defined.

Remark 1.8.2. *This calculation explains why we have to restrict ourselves to the one-dimensional case $x \in \mathbb{T}$.*

Claim 2 : $(u^n)_n$ is bounded in $C([0, T]; L_x^\infty)$

We have

$$\|u_t^{n+1}\|_{L_x^\infty} \leq \|u_0\|_{L_x^\infty} + \int_0^t \left\| S(t-s) \left[J(f_s^{n+1}) - \rho_s^{n+1} u_s^n \right] \right\|_{L_x^\infty} ds$$

and since $H_x^{1/2+\delta} \subset L_x^\infty$ for $\delta > 0$, we get $L_x^1 \subset H_x^{-1/2-\delta}$ and the estimate

$$\forall w \in L_x^1, \quad \|S(t)w\|_{L_x^\infty} \leq \|S(t)w\|_{H_x^{1/2+\delta}} \lesssim (1+t^{-1/2-\delta})\|w\|_{H_x^{-1/2-\delta}} \lesssim (1+t^{-1/2-\delta})\|w\|_{L_x^1}.$$

The L^1 estimates for $f_t^{n+1} = f_t[m + u^n]$ from Lemma 1.8.2 give

$$\|J(f_s^{n+1}) - \rho_s^{n+1} u_s^n\|_{L_x^1} \lesssim 1 + \sup_{\sigma \in [0, s]} \|u_\sigma^n\|_{L_x^\infty} + \|u_s^n\|_{L_x^\infty} \lesssim 1 + \sup_{\sigma \in [0, s]} \|u_\sigma^n\|_{L_x^\infty}.$$

It follows that (forgetting the $\delta > 0$ for simplicity)

$$\|u_t^{n+1}\|_{L_x^\infty} \leq \|u_0\|_{L_x^\infty} + C \int_0^t \left(1 + (t-s)^{-1/2}\right) \left(1 + \sup_{\sigma \in [0, s]} \|u_\sigma^n\|_{L_x^\infty}\right) ds.$$

Introducing, for $\mu > 0$, the norm

$$\forall w \in C([0, T], L_x^\infty), \quad \|w\|_\mu = \sup_{t \in [0, T]} e^{-\mu t} \|w_t\|_{L_x^\infty}, \quad (1.109)$$

is easy to check that $\|u^{n+1}\|_\mu \leq a_\mu \|u^n\|_\mu + b$ with $a_\mu \in [0, 1)$ for μ chosen large enough. It follows that $\sup_n \|u^n\|_\mu < \infty$, hence the result claimed.

Claim 3 : $(u^n)_n$ is bounded in $C([0, T]; H_x^\eta)$

Since $\sup_{t \in [0, T]} \|u_t^n\|_{L_x^\infty} \lesssim 1$, this follows directly from (1.108).

Claim 4 : $(u^n)_n$ is compact in $C([0, T]; H_x^\beta)$

Noting that, with $g^n = J(f^n) - \rho^n u^{n-1}$,

$$\forall \xi \in C_c^\infty(\mathbb{T}), \quad \langle u_t^n, \xi \rangle - \langle u_0^n, \xi \rangle = \int_0^t \langle u_s^n, \partial_x^2 \xi \rangle + \langle g_s^n, \xi \rangle ds,$$

we are led to

$$\begin{aligned} \left| \langle u_{t_2}^n - u_{t_1}^n, \xi \rangle \right| &\leq \int_{t_1}^{t_2} |\langle u_s^n, \partial_x^2 \xi \rangle| + |\langle g_s^n, \xi \rangle| ds \\ &\leq |t_2 - t_1| \left(\|u^n\|_{C([0, T]; H_x^\eta)} \|\xi\|_{H_x^{2-\eta}} + \|g^n\|_{C([0, T]; L_{x,v}^2)} \|\xi\|_{L_x^2} \right) \\ &\lesssim |t_2 - t_1| \|\xi\|_{H_x^{2-\eta}}, \end{aligned}$$

that is in fact $u^n \in C^{0,1}([0, T]; H_x^{\eta-2})$ with a Lipschitz constant uniform in n . Ascoli's theorem therefore guarantees the compactness of $(u^n)_n \in C([0, T]; H_x^{\eta-2})$. Recalling the bound from Claim 3, we get the compactness of $(u^n)_n \in C([0, T]; H_x^\beta)$ by interpolation.

Conclusion

We simply note that the mapping $u_n \mapsto u_{n+1}$ is a contraction on $C([0, T]; L_x^\infty)$:

$$\begin{aligned} \left\| u_t^{n+1} - u_t^n \right\|_{L_x^\infty} &= \left\| \int_0^t S(t-s) \left[J(f_s^{n+1}) - J(f_s^n) - (\rho_s^{n+1} u_s^n - \rho_s^n u_s^{n-1}) \right] ds \right\|_{L_x^\infty} \\ &\lesssim \int_0^t \left(1 + (t-s)^{-1/2}\right) \left(\|J(f_s^{n+1}) - J(f_s^n)\|_{L_x^1} + \|\rho_s^{n+1} u_s^{n+1} - \rho_s^n u_s^{n-1}\|_{L_x^1} \right) ds. \end{aligned} \quad (1.110)$$

Since $f^{n+1} = f[m + u^n]$ and $f^n = f[m + u^{n-1}]$, using Corollary (1.8.3) and the fact that $(u^n)_n$ is bounded in $C([0, T]; W_x^{1, \infty})$, we get for all $s \in [0, T]$

$$\|J(f_s^{n+1}) - J(f_s^n)\|_{L_x^1} \lesssim \sup_{\sigma \in [0, s]} \|u_\sigma^n - u_\sigma^{n-1}\|_{L_x^\infty},$$

and

$$\begin{aligned} \|\rho_s^{n+1} u_s^n - \rho_s^n u_s^{n-1}\|_{L_x^1} &\leq \|\rho_s^{n+1} (u_s^n - u_s^{n-1})\|_{L_x^1} + \|(\rho_s^{n+1} - \rho_s^n) u_s^{n-1}\|_{L_x^1} \\ &\lesssim \sup_{\sigma \in [0, s]} \|u_\sigma^n - u_\sigma^{n-1}\|_{L_x^\infty}. \end{aligned}$$

Therefore,

$$\|u_t^{n+1} - u_t^n\|_{L_x^\infty} \lesssim \int_0^t \left(1 + (t-s)^{-1/2}\right) \sup_{\sigma \in [0, s]} \|u_\sigma^n - u_\sigma^{n-1}\|_{L_x^\infty} ds.$$

Using once again the norm $\|\cdot\|_\mu$ defined in (1.109), we are easily led to

$$\|u^{n+1} - u^n\|_\mu \leq k_\mu \|u^n - u^{n-1}\|_\mu$$

with $k_\mu \in [0, 1)$ for μ large enough. As a result, u^n converges to some u in the Banach space $C([0, T]; L_x^\infty)$. Since $(u^n)_n$ is also compact in $C([0, T]; H_x^\beta)$ we deduce that $u^n \rightarrow u$ in $C([0, T]; H_x^\beta)$. We may now introduce $f = f[m + u]$. Using the same estimates as before (only replacing u^{n+1} by u in (1.110)), we easily derive, as n goes to infinity,

$$u_t = S(t)u_0 + \int_0^t S(t-s) [J(f_s) - \rho_s u_s] ds.$$

Hence, (f, u) is indeed a solution to (1.102). If (\tilde{f}, \tilde{u}) is another solution with $\tilde{u} \in C([0, T]; H_x^\beta)$ and $\tilde{f} = f[m + \tilde{u}]$, the same estimates lead to

$$\|u - \tilde{u}\|_\mu \leq k_\mu \|u - \tilde{u}\|_\mu,$$

so that $\tilde{u} = u$ and $\tilde{f} = f$. This concludes the proof of the well-posedness of (1.102).

Finally, let us assume that $f_0 \in G^k$ and prove that $f \in C([0, T]; G^k)$. It is clearly enough to show that $\|f_t - f_0\|_{G^k} \rightarrow 0$ as t goes to 0. Given the form (1.8.1) and Lemma 1.8.1, this result is easily obtained by dominated convergence when $f_0 \in C_c(\mathbb{R}^{2d})$. It may then be generalized by density to any $f_0 \in G^k$, since, given $\tilde{f}_0 \in C_c(\mathbb{R}^{2d})$

$$\forall t \in [0, 1], \quad \|f_t - \tilde{f}_t\|_{G^k} = \int (1 + |V_0^t(z)|^k) |f_0(z) - \tilde{f}_0(z)| dz \lesssim \|f_0 - \tilde{f}_0\|_{G^k}.$$

We have used the estimate $|V_0^t(x, v)|^k \lesssim 1 + |v|^k$ easily derived from the sub-linearity of (1.103), with $\|m + u\|_{L_{t,x}^\infty} < \infty$.

2

Weak and strong mean-field limits for stochastic Cucker-Smale particle systems

2.1 Introduction

2.1.1 Overview of the model.

Flocking, or swarming, is a phenomenon consistently observed in nature where individuals from a population (birds, fish, insects, bacterias...) tend to naturally align their trajectories without the need of a leadership. One of the most commonly studied model which intends to describe this kind of behavior is the Cucker-Smale model, introduced in [CS07a] and [CS07b].

In this model, each individual interacts with the group in a mean-field-like manner : denoting by $X^{i,N}, V^{i,N} \in \mathbb{R}^d$ the position and velocity of the i -th individual, the behavior of the system can be written as

$$\begin{cases} \frac{d}{dt} X_t^{i,N} &= V_t^{i,N} \\ \frac{d}{dt} V_t^{i,N} &= \frac{1}{N} \sum_{j=1}^N \psi(X_t^{i,N} - X_t^{j,N})(V_t^{j,N} - V_t^{i,N}) \end{cases} \quad (2.1)$$

where the weight function $\psi : \mathbb{R} \rightarrow \mathbb{R}^+$ is even and bounded, typically of the form

$$\psi(x - y) = \frac{\lambda}{(1 + |x - y|^2)^\gamma}, \quad \lambda, \gamma > 0.$$

In order to take into account unpredictable phenomena of different natures, it is rather natural to perturb this deterministic model with some noise. In [CDP18], where the flocking phenomenon (alignment of speeds, distance between the individuals bounded over time) is studied in a variety of different stochastic Cucker-Smale models, three different kinds of perturbations are identified.

The first one considers the degree of freedom of each individual by adding some independent noise, dragged by a brownian motion B^i , to each of them :

$$dV_t^{i,N} = \frac{1}{N} \sum_{j=1}^N \psi(X_t^{i,N} - X_t^{j,N})(V_t^{j,N} - V_t^{i,N})dt + \sigma(X_t^{i,N}, V_t^{i,N}) \circ dB_t^i. \quad (2.2)$$

This setting typically appears in the propagation of chaos framework. The flocking behavior for (2.2) has been studied in [HLL09]. The mean-field limit as N goes to infinity is considered in [BCC11], in the case of a constant diffusion coefficient $\sigma(x, v) = \sqrt{D}Id$, and more recently in [CS19] for $\sigma(x, v) = \mathcal{R}(v)$ a "truncation function" of the speed. Note that, when presenting new models, we insist on introducing noise in Stratonovich form, since it is the most physically relevant form.

Another kind of perturbation might emerge from the environment in which the individuals evolve. In this case, we add some common noise dragged by a Wiener process $dW = \sum_k \sigma_k dW^k$:

$$dV_t^{i,N} = \frac{1}{N} \sum_{j=1}^N \psi(X_t^{i,N} - X_t^{j,N})(V_t^{j,N} - V_t^{i,N})dt + \sum_k \sigma_k(X_t^{i,N}, V_t^{i,N}) \circ dW_t^k. \quad (2.3)$$

A version of (2.3), with a diffusion coefficient of the form $\sigma(x, v) = D(v - v_e)$ for some constant $v_e \in \mathbb{R}^d$, is studied in [AH10].

Lastly, one may consider that the weight function ψ modeling the interaction between individuals is perturbed into $\tilde{\psi} = \psi + d\xi$, where ξ is some space-dependent Wiener process given by $d\xi = \sum_k \phi_k d\beta^k$, leading to

$$\begin{aligned} dV_t^{i,N} &= \frac{1}{N} \sum_{j=1}^N \psi(X_t^{i,N} - X_t^{j,N})(V_t^{j,N} - V_t^{i,N}) dt \\ &\quad + \frac{1}{N} \sum_{j=1}^N \sum_k \phi_k(X_t^{i,N} - X_t^{j,N})(V_t^{j,N} - V_t^{i,N}) \circ d\beta_t^k. \end{aligned} \quad (2.4)$$

The mean-field limit and flocking for (2.4) is looked upon in [CS17] and more recently in [HJR19] in the particular case where the perturbation ξ does not depend on x : $d\xi = \sqrt{2\sigma} d\beta_t$.

In this paper, we focus on the mean-field limit of these particle systems. Namely, we intend to extend the results mentioned above by studying the behavior of the empirical measure

$$\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{(X^{i,N}, V^{i,N})}$$

as N goes to infinity, for general stochastic Cucker-Smale model of the form (2.2), (2.3) or (2.4) (or combinations of these models). Let us keep the notions of convergence a little vague for a moment, in order to give a quick overview of the results to come : we will show for instance that, for (2.4), under the assumptions

$$\sum_k \|\phi_k\|_\infty^2 < \infty, \quad \sum_k \|\phi_k\|_{\text{lip}}^2 < \infty, \quad (2.5)$$

where $\|\phi\|_{\text{lip}} = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|}$, provided that $\mu_0^N \rightarrow \mu_0$, the (random) empirical measure μ_t^N converges in law, up to a subsequence, to some μ_t which is a weak solution of the expected limiting stochastic PDE

$$d\mu_t + v \cdot \nabla_x \mu_t dt + \nabla_v \cdot (F[\mu_t] \mu_t) dt + \sum_k \nabla_v \cdot (F_k[\mu_t] \mu_t) \circ d\beta_t^k = 0, \quad (2.6)$$

with

$$F[\mu](x, v) = \int \psi(x - y)(w - v) d\mu(y, w), \quad F_k[\mu](x, v) = \int \phi_k(x - y)(w - v) d\mu(y, w).$$

It is of some interest to note here that the noise added in Stratonovich form in (2.4) directly translates into the expected conservative form (2.6) for the limiting equation, which emphasizes the physical relevance of Stratonovich's integration over Itô's.

Regarding the flocking phenomenon, the method developed in [CS17] could in fact be easily extended to the model (2.6). Given a solution $\mu = (\mu_t)_{t \geq 0}$ of (2.6), the average velocity $\bar{v}_t = \int v d\mu_t$ is conserved over time. Assuming that

$$\psi_m := \min_x \psi(x) > 0, \quad \sum_k \|\phi_k\|_\infty^2 < \infty$$

and denoting

$$E_t = \int_{\mathbb{R}^{2d}} |v - \bar{v}_t|^2 d\mu_t(x, v)$$

calculations easily lead to

$$\frac{d}{dt} \mathbb{E}[E_t] \leq -2 \left(\psi_m - 4 \sum_k \|\phi_k\|_\infty^2 \right) \mathbb{E}[E_t].$$

Therefore, under the condition $\psi_m > 4 \sum_k \|\phi_k\|_\infty^2$, the model (2.6) exhibits a flocking behavior in the sense that $\mathbb{E}[E_t] \rightarrow 0$ exponentially fast as t goes to infinity.

Under the same assumptions (2.5), the "strong" mean-field convergence $\mu_t^N \rightarrow \mu_t$ (see Theorem 4 below for details) is obtained for the whole sequence if we consider "truncated velocities" in the perturbative term, that is a model given by

$$\begin{aligned} dV_t^{i,N} = & \frac{1}{N} \sum_{j=1}^N \psi(X_t^{i,N} - X_t^{j,N})(V_t^{j,N} - V_t^{i,N})dt \\ & + \frac{1}{N} \sum_{j=1}^N \sum_k \phi_k(X_t^{i,N} - X_t^{j,N})\mathcal{R}(V_t^{j,N} - V_t^{i,N}) \circ d\beta_t^k \end{aligned} \quad (2.7)$$

where $\mathcal{R} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is smooth and compactly-supported, similarly to the case considered in [CS19]. We are in fact allowed slightly more general truncation functions, as will be detailed later on in section 2.3.2.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space, and let $\beta, (B^i)_{i \geq 1}$ be independent, respectively \mathbb{R} -valued and \mathbb{R}^d -valued (\mathcal{F}_t) -brownian motions on Ω , starting from 0. Throughout the rest of this paper, we extend our study to a stochastic interacting particle system in \mathbb{R}^d of the general mean-field form

$$\begin{aligned} dX_t^{i,N} = & B[\mu_t^N](X_t^{i,N})dt + C[\mu_t^N](X_t^{i,N}) \circ d\beta_t + \sigma(X_t^{i,N}) \circ dB_t^i, \\ i \in & \{1, \dots, N\}, \end{aligned} \quad (2.8)$$

where

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}}, \quad B[\mu](x) = \int b(x, y)d\mu(y), \quad C[\mu](x) = \int c(x, y)d\mu(y) \quad (2.9)$$

for some coefficients $b, c : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$. The particles in (2.8) are subject to two noises of different nature : some individual noise dragged by B_t^i and some common noise dragged by β_t . The case $c(x, y) = c(x)$ corresponds to a noisy environment (as in (2.3)) whereas the case $c(x, y) = c(x - y)$ corresponds to a noisy interaction (as in (2.4)).

For simplicity purposes, from this point on we choose to only consider a "one-dimensional" common noise $c(x, y) \circ d\beta_t$. It may of course be replaced with a more general $\sum_{k=1}^{\infty} c_k(x, y) \circ d\beta_t^k$. The results presented in this paper will still hold, provided essentially that the assumptions made here on c are satisfied by all c_k , with constants which are square-summable over k , as suggested in (2.5).

In view of usual stochastic mean-field results, it is natural to expect that the limiting equation for the empirical measure μ_t^N associated to (2.8) as N goes to infinity is given by

$$d\mu_t + \nabla \cdot (B[\mu_t]\mu_t)dt + \nabla \cdot (C[\mu_t]\mu_t) \circ d\beta_t + \frac{1}{2} \nabla \cdot (\text{Tr}(\nabla \sigma \sigma^T)\mu_t)dt = \frac{1}{2} \nabla \cdot (\nabla \cdot (\sigma \sigma^T \mu_t))dt \quad (2.10)$$

where we have used the slight abuse of notation :

$$\left(\text{Tr}(\nabla \sigma \sigma^T) \right)_i = \text{Tr}((\nabla \sigma_i) \sigma^T) = \sum_{k,l=1}^d (\partial_k \sigma_{i,l}) \sigma_{k,l}. \quad (2.11)$$

Due to the driving noise β_t which is common to all particles, (2.10) is an SPDE, so that the limiting measure $(\mu_t)_{t \geq 0}$ is still a stochastic process. The individual noises σdB_t^i are expected to average into the elliptic operator $\frac{1}{2} \nabla \cdot (\nabla \cdot (\sigma \sigma^T \cdot))$. The first order operator $\frac{1}{2} \nabla \cdot (\text{Tr}(\nabla \sigma \sigma^T) \cdot)$ only results from the correction from Stratonovich to Itô integration. In the particular case $\sigma(x) \equiv \sigma Id$, we are simply left with

$$d\mu_t + \nabla \cdot (B[\mu_t]\mu_t)dt + \nabla \cdot (C[\mu_t]\mu_t) \circ d\beta_t = \frac{\sigma^2}{2} (\Delta \mu_t)dt.$$

The mean-field limit of the particle system (2.8) is well known and established when the coefficients b, c and σ are globally Lipschitz-continuous (see e.g [CF14]). In this article, we want

to consider Cucker-Smale perturbations of the form (2.2), (2.3) and (2.4). This corresponds to $Z_t^{i,N} = (X_t^{i,N}, V_t^{i,N})$ satisfying (2.8) in \mathbb{R}^{2d} with coefficients of the form

$$b((x, v); (y, w)) = \begin{pmatrix} v \\ \psi(x - y)(w - v) \end{pmatrix} \quad c((x, v); (y, w)) = \begin{pmatrix} 0 \\ \phi(x - y)(w - v) \end{pmatrix} \quad (2.12)$$

which, when ψ and ϕ are globally Lipschitz-continuous, are only locally Lipschitz-continuous. This leads to additional difficulties compared to the "globally Lipschitz" case. A classical way to deal with such difficulties is to introduce suitable stopping times. In the case considered here, the problem is more difficult since the non-linear terms in equation (2.10) depend on the trajectories of all the particles. This requires to stop every particle at once, leading us to essentially derive estimates on

$$\sup_{i \in \{1, \dots, N\}} \sup_{t \in [0, T]} |X_t^{i,N}| \quad (2.13)$$

as made clear in Proposition 2.3.2 and developed in section 2.3.4. The bound (2.13) is the crucial tool in [CS17] for instance, where it is derived from a stochastic Gronwall inequality that relies on the simple linear form of the noise. In our case where the noise is more complex, this bound can be obtained through the use of exponential moments for the particles, using a method similar to the one suggested in [BCC11]. This is dealt with in more details in section 2.3.3.

Under the assumption that the coefficients b , c and σ are only locally Lipschitz-continuous and sublinear, we prove the convergence in law (up to a subsequence) of the empirical measure associated to (2.8) to a weak solution of the limiting SPDE (2.10). In a more restrictive setting, considering only common noise, requiring boundedness for c and additional assumptions regarding the growth of local Lipschitz norms of the coefficients, this weak convergence is turned into a strong $L^p(\Omega)$ convergence and the propagation of chaos is established. Precise assumptions and results are stated in section 2.1.2 below.

Note that (2.8) and (2.10) have only been given in the (heuristic) Stratonovich form. In section 2.1.3, we shall determine the corresponding Itô forms and derive a proper definition for solutions of (2.8) and particularly (2.10) (see Definition 2.1.2).

2.1.2 Main results.

In the rest of this paper, $\mathcal{P}(E)$ shall denote the set of probability measures on some space E . The results presented here along with their proofs involve some considerations regarding Wasserstein spaces $\mathcal{P}_p(E)$.

Definition 2.1.1. *Given $(E, \|\cdot\|)$ a separable Banach space and $p \geq 1$, the p th-Wasserstein space*

$$\mathcal{P}_p(E) = \left\{ \mu \in \mathcal{P}(E), \int_{x \in E} \|x\|^p d\mu(x) < \infty \right\}$$

is equipped with the distance

$$W_p[\mu, \nu] = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{x^1, x^2 \in E} \|x^1 - x^2\|^p d\pi(x^1, x^2) \right)^{1/p},$$

where

$$\Pi(\mu, \nu) = \left\{ \pi \in \mathcal{P}(E^2), \int_{x^2 \in E} \pi(\cdot, dx^2) = \mu \text{ and } \int_{x^1 \in E} \pi(dx^1, \cdot) = \nu \right\}.$$

In the rest of this paper, we shall sometimes use the notation $A(z) \lesssim B(z)$ to signify that there exists a constant $C > 0$ independent of the variable z considered such that $A(z) \leq CB(z)$ for all z . Defining the Stratonovich corrective terms (see section 2.1.3)

$$s_1(x, y, z) = \frac{1}{2} \nabla_x c(x, y) c(x, z) + \nabla_y c(x, y) c(y, z), \quad (2.14)$$

$$S_2(x) = \text{Tr}((\nabla \sigma_i) \sigma^T) = \sum_{k, l=1}^d (\partial_k \sigma_{i,l}) \sigma_{k,l}, \quad (2.15)$$

we shall first make the following assumptions on the coefficients of (2.8) :

Assumption 1 (Sublinearity).

$$\begin{aligned} |b(x, y)| &\lesssim 1 + |x| + |y|, & |c(x, y)| &\lesssim 1 + |x| + |y|, & |\sigma(x)| &\lesssim 1 + |x| \\ |s_1(x, y, z)| &\lesssim 1 + |x| + |y| + |z|, & |S_2(x)| &\lesssim 1 + |x|. \end{aligned}$$

Assumption 2 (Locally Lipschitz).

$$b, c, \sigma, \nabla c, \nabla \sigma \text{ are locally Lipschitz-continuous.}$$

In this rather general setup, the local Lipschitz-continuity alone is not enough to ensure "standard" estimates of the form

$$\mathbb{E} \left[W_2^2[\mu_t^N, \mu_t^M] \right] \leq C W_2^2[\mu_0^N, \mu_0^M]$$

and we are not able to establish the "strong" convergence of the particle system. Instead, we rely on compactness arguments to prove the following weak mean-field limit result.

Theorem 3. *Let $T > 0$ and $\mathcal{C} := C([0, T]; \mathbb{R}^d)$ equipped with $\|x\|_\infty = \sup_{t \in [0, T]} |x_t|$. Suppose that Assumptions 1 and 2 are satisfied. Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ such that*

$$\int |x|^{2+\delta} d\mu_0(x) < \infty \text{ for some } \delta > 0.$$

Let $(X_t^{i,N})_{t \geq 0}^{i=1, \dots, N}$ be a solution of (2.8) and $\mu^N \in \mathcal{P}(\mathcal{C})$ the associated empirical measure. Provided that

$$\mu_0^N \rightarrow \mu_0 \text{ in } \mathcal{P}_2(\mathbb{R}^d) \text{ and } \sup_N \int |x|^{2+\delta} d\mu_0^N(x) < \infty,$$

there exists a subsequence $(\mu^{N'})_{N'}$ such that

$$\mu^{N'} \rightarrow \mu \text{ in law, in } \mathcal{P}_2(\mathcal{C})$$

and μ is a martingale solution of (2.10) (in the sense of [DPZ14], Chapter 8) : there exists some other probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ equipped with a brownian motion $\tilde{\beta}$ such that μ satisfies the assumptions of Definition 2.1.2 below on $\tilde{\Omega}$.

This weak convergence can be strengthened into a strong convergence for compactly supported initial measures, under some more restrictive assumptions on the coefficients.

First, we shall only consider the case of common noise (adding individual noises would require additional work, see Remark 2.3.4).

Assumption 3 (Common noise only).

$$\sigma = 0.$$

In this case, the limiting SPDE (2.10) becomes a stochastic conservation equation :

$$d\mu_t + \nabla \cdot (B[\mu_t]\mu_t)dt + \nabla \cdot (C[\mu_t]\mu_t) \circ d\beta_t = 0. \quad (2.16)$$

Non-linear stochastic conservation equations resembling (2.16) (with local non-linearities) have been studied for instance in [GS15], [LPS14]. A solution of (2.16) is naturally expected to be "of the transport form" $\mu = (X^\mu)^* \mu_0$, i.e μ is given by the push-forward measure of the initial data by the (non-linear) stochastic characteristics

$$\begin{cases} dX_t^\mu(x) = B[\mu_t](X_t^\mu(x))dt + C[\mu_t](X_t^\mu(x)) \circ d\beta_t, \\ X_0^\mu(x) = x \in \mathbb{R}^d. \end{cases}$$

A precise statement on measures of the transport form is made in Definition 2.3.1. Let us make some additional assumptions on the coefficients :

Assumption 4 (Sublinear drift, bounded diffusion coefficient).

$$\begin{aligned} |b(x, y)| &\lesssim 1 + |x| + |y|, \\ |s_1(x, y, z)| &\lesssim 1 + |x| + |y| + |z|, \\ |c(x, y)| &\lesssim 1, \end{aligned}$$

Assumption 5 (Growth of the local Lipschitz constants).

$$\begin{aligned} |b(x, y) - b(x', y')| &\lesssim L_b(x, y, x', y') \left(|x - x'| + |y - y'| \right), \\ |s_1(x, y, z) - s_1(x', y', z')| &\lesssim L_s(x, y, z, x', y', z') \left(|x - x'| + |y - y'| + |z - z'| \right), \\ |c(x, y) - c(x', y')| &\lesssim L_c(x, y, x', y') \left(|x - x'| + |y - y'| \right). \end{aligned}$$

where, for some $\theta \in (0, 1)$

$$\begin{aligned} L_b(x, y, x', y') &= 1 + |x|^{2\theta} + |y|^{2\theta} + |x'|^{2\theta} + |y'|^{2\theta}, \\ L_s(x, y, z, x', y', z') &= 1 + |x|^{2\theta} + |y|^{2\theta} + |z|^{2\theta} + |x'|^{2\theta} + |y'|^{2\theta} + |z'|^{2\theta}, \\ L_c(x, y, x', y') &= 1 + |x|^\theta + |y|^\theta + |x'|^\theta + |y'|^\theta. \end{aligned}$$

Theorem 4. Let $T > 0$, $p \geq 2$ and $\mathcal{C} := C([0, T]; \mathbb{R}^d)$.

Suppose that Assumptions 3, 4 and 5 are satisfied. With the same notations as before, provided that μ_0^N is uniformly supported in some compact set $K \subset \mathbb{R}^d$ and $\mu_0^N \rightarrow \mu_0$ in $\mathcal{P}_p(\mathbb{R}^d)$, we have the convergence

$$\mu^N \rightarrow \mu \text{ in } L^p(\Omega; \mathcal{P}_p(\mathcal{C})),$$

where $\mu = (\mu_t)_{t \in [0, T]}$ is the unique solution of the transport form of (2.16), in the sense of Definitions 2.1.2 and 2.3.1 below.

Finally, let us complete this last statement by presenting a result of (conditional) propagation of chaos similar to the one formulated in [CF14].

Theorem 5. Let $T > 0$, $p \geq 2$ and $\mathcal{C} := C([0, T]; \mathbb{R}^d)$.

Suppose that Assumptions 3, 4 and 5 are satisfied, and let $(\mathcal{F}_t^\beta)_{t \in [0, T]}$ denote the canonical filtration associated with β . Given $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ supported in some compact set $K \subset \mathbb{R}^d$, let us introduce

$$(\xi_0^i)_{i \geq 1} \text{ i.i.d. } \mathcal{F}_0^\beta\text{-measurable, } \mathbb{R}^d\text{-valued random variables with law } \mu_0.$$

Let $(X_t^{i, N})_{t \geq 0}^{i=1, \dots, N}$ be the solution of (2.8) with the initial conditions $X_0^{i, N} = \xi_0^i$, and let $\mu^N \in \mathcal{P}(\mathcal{C})$ be the associated empirical measure. Then we have the convergence

$$\mu^N \rightarrow \mu \text{ in } L^p(\Omega; \mathcal{P}_p(\mathcal{C})),$$

where $\mu = (\mu_t)_{t \in [0, T]}$ is the unique solution of the transport form of (2.16), in the sense of Definitions 2.1.2 and 2.3.1 below. Additionally, for all $r \geq 1$ and $\phi_1, \dots, \phi_r \in C_b(\mathbb{C})$ we have

$$\mathbb{E} \left[\phi_1(X^{1, N}) \dots \phi_r(X^{r, N}) | \mathcal{F}_T^\beta \right] \rightarrow \prod_{i=1}^r \langle \phi_i, \mu \rangle \text{ in } L^1(\Omega).$$

Finally, for all $i \geq 1$, let X^i be the solution of

$$\begin{cases} dX_t^i = B[\mu_t](X_t^i) dt + C[\mu_t](X_t^i) \circ d\beta_t, \\ X_0^i = \xi_0^i. \end{cases}$$

Then the limiting measure $\mu \in \mathcal{P}(\mathcal{C})$ is a version of the conditional law $\mathcal{L}(X^i | \mathcal{F}_T^\beta)$ and we have the convergence

$$X^{i, N} \rightarrow X^i \text{ in } L^p(\Omega; \mathcal{C}).$$

2.1.3 Itô form.

Let us now determine the proper Itô form expressions of (2.8) and (2.10). Itô's formula gives

$$\begin{aligned} d[C[\mu_t^N](X_t^{i, N})] &= \frac{1}{N} \sum_j d[c(X_t^{i, N}, X_t^{j, N})] \\ &= \left(\frac{1}{N} \sum_j \nabla_x c(X_t^{i, N}, X_t^{j, N}) C[\mu_t^N](X_t^{i, N}) + \nabla_y c(X_t^{i, N}, X_t^{j, N}) C[\mu_t^N](X_t^{j, N}) \right) d\beta_t \\ &\quad + dV_t^{i, j} + dM_t^{i, j} \end{aligned}$$

where $V^{i,j}$ is a process with bounded variation and

$$dM_t^{i,j} = \frac{1}{N} \sum_j \left(\nabla_x c(X_t^{i,N}, X_t^{j,N}) \sigma(X_t^{i,N}) dB_t^i + \nabla_y c(X_t^{i,N}, X_t^{j,N}) \sigma(X_t^{j,N}) dB_t^j \right).$$

It follows that the correction from Stratonovich to Itô is given by

$$C[\mu_t^N](X_t^{i,N}) \circ d\beta_t = C[\mu_t^N](X_t^{i,N}) d\beta_t + S_1[\mu_t^N](X_t^{i,N}) dt$$

with

$$S_1[\mu](x) = \int \int s_1(x, y, z) d\mu(y) d\mu(z) \quad (2.17)$$

where

$$s_1(x, y, z) = \frac{1}{2} \nabla_x c(x, y) c(x, z) + \nabla_y c(x, y) c(y, z)$$

as defined in (2.14). Similarly, the correction for the individual noise is given by

$$\sigma(X_t^{i,N}) \circ dB_t^i = \sigma(X_t^{i,N}) dB_t^i + S_2(X_t^{i,N}) dt$$

with

$$S_2(x) = \frac{1}{2} \text{Tr}(\nabla \sigma(x) \sigma^T(x))$$

as defined in (2.15). We may now rewrite the particle system (2.8) as

$$dX_t^{i,N} = \left(B[\mu_t^N](X_t^{i,N}) + S[\mu_t^N](X_t^{i,N}) \right) dt + C[\mu_t^N](X_t^{i,N}) d\beta_t + \sigma(X_t^{i,N}) dB_t^i, \quad (2.18)$$

where

$$S[\mu](x) = S_1[\mu](x) + S_2(x) \text{ is defined in (2.17) and (2.15) .}$$

As for the SPDE (2.10), it is to be understood in the following weak sense : for any $\psi \in C_c^2(\mathbb{R}^d)$,

$$d\langle \psi, \mu_t \rangle = \langle (B[\mu_t] + \frac{1}{2} \text{Tr}(\nabla \sigma \sigma^T)) \cdot \nabla \psi, \mu_t \rangle dt + \langle C[\mu_t] \cdot \nabla \psi, \mu_t \rangle \circ d\beta_t + \frac{1}{2} \langle \text{Tr}(\sigma(\nabla^2 \psi) \sigma^T), \mu_t \rangle dt.$$

Let us determine the correction corresponding to the Stratonovich term. We have

$$d \left[\langle C[\mu_t] \cdot \nabla \psi, \mu_t \rangle \right] = \langle C[\mu_t] \cdot \nabla \psi, d\mu_t \rangle + \langle d \left[\langle C[\mu_t] \cdot \nabla \psi \right], \mu_t \rangle.$$

On one hand,

$$\begin{aligned} \langle C[\mu_t] \cdot \nabla \psi, d\mu_t \rangle &= \langle C[\mu_t] \cdot \nabla (C[\mu_t] \cdot \nabla \psi), \mu_t \rangle d\beta_t + dV_t^{(1)} \\ &= \left(\langle (\nabla C[\mu_t] C[\mu_t]) \cdot \nabla \psi, \mu_t \rangle + \langle \nabla^2 \psi \cdot C[\mu_t]^{\otimes 2}, \mu_t \rangle \right) d\beta_t + dV_t^{(1)} \end{aligned}$$

where $V^{(1)}$ is a process with bounded variation. On the other hand,

$$C[\mu_t](x) \cdot \nabla \psi(x) = \int \phi(y) d\mu_t(y) = \langle \phi, \mu_t \rangle \text{ with } \phi(y) = c(x, y) \cdot \nabla \psi(x)$$

so that

$$\begin{aligned} d \left[\langle C[\mu_t] \cdot \nabla \psi \right] (x) &= d\langle \phi, \mu_t \rangle = \langle C[\mu_t] \cdot \nabla \phi, \mu_t \rangle d\beta_t + dV_t^{(2)}(x) \\ &= \left(\left(\int \nabla_y c(x, y) C[\mu_t](y) d\mu_t(y) \right) \cdot \nabla \psi(x) \right) d\beta_t + dV_t^{(2)}(x) \end{aligned}$$

where $V^{(2)}(x)$ is a process with bounded variation. Combining both expressions, we are led to

$$d\left[\langle C[\mu_t] \cdot \nabla \psi, \mu_t \rangle\right] = \langle 2S_1[\mu_t] \cdot \nabla \psi + \nabla^2 \psi \cdot (C[\mu_t]^{\otimes 2}), \mu_t \rangle d\beta_t + dU_t$$

where $U_t = \int_0^t \left(dV_s^{(1)} + \langle dV_s^{(2)}, \mu_s \rangle \right)$ is a process with bounded variation. The correction is therefore given by

$$\langle C[\mu_t] \cdot \nabla \psi, \mu_t \rangle \circ d\beta_t = \langle C[\mu_t] \cdot \nabla \psi, \mu_t \rangle d\beta_t + \left(\langle S_1[\mu_t] \cdot \nabla \psi, \mu_t \rangle + \frac{1}{2} \langle \nabla^2 \psi \cdot (C[\mu_t]^{\otimes 2}), \mu_t \rangle \right) dt.$$

Consequently, the Itô form corresponding to the SPDE (2.10) is exactly

$$d\mu_t + \nabla \cdot \left((B[\mu_t] + S[\mu_t])\mu_t \right) dt + \nabla \cdot (C[\mu_t]\mu_t) d\beta_t = \frac{1}{2} \nabla \cdot \nabla \cdot \left((\sigma\sigma^T + C[\mu_t]C[\mu_t]^T)\mu_t \right) dt \quad (2.19)$$

with $S[\mu_t]$ as in (2.18). This allows us to precisely define the notion of solution for (2.10).

Definition 2.1.2. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space equipped with an (\mathcal{F}_t) -brownian motion β . Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$.

A measure-valued process $\mu = (\mu_t)_{t \in [0, T]} : \Omega \rightarrow \mathcal{P}(\mathbb{R}^d)^{[0, T]}$ is said to be a solution of the SPDE (2.10) (or equivalently (2.19)) with initial value μ_0 when for all $\psi \in C_c^2(\mathbb{R}^d)$, the process $(\langle \psi, \mu_t \rangle)_{t \in [0, T]}$ is adapted with a continuous version and satisfies

$$\langle \psi, \mu_t \rangle = \langle \psi, \mu_0 \rangle + \int_0^t \langle (B[\mu_s] + S[\mu_s]) \cdot \nabla \psi + \mathcal{A}[\mu_s]\psi, \mu_s \rangle ds + \int_0^t \langle C[\mu_s] \cdot \nabla \psi, \mu_s \rangle d\beta_s, \quad (2.20)$$

where $S[\mu]$ is defined in (2.18) and the second order operator $\mathcal{A}[\mu]$ is given by

$$\mathcal{A}[\mu]\psi = \frac{1}{2} \sum_{i,j} \left(\sum_k \sigma_{i,k} \sigma_{j,k} + C_i[\mu] C_j[\mu] \right) \partial_{i,j}^2 \psi. \quad (2.21)$$

Remark 2.1.1. Comparing the particle system (2.18) and the SPDE (2.19) expressed in Itô form, we see that the correction from Stratonovich to Itô integration adds some "virtual" interaction kernel $S[\mu]$ to the system. On the SPDE (2.19), it additionally results in the operator $\mathcal{A}[\mu]$ which is of order 2 and consequently is not "visible" on the particle system (2.18).

2.2 Weak mean-field convergence

2.2.1 Properties of the coefficients.

In the entirety of section 2.2, we shall assume that Assumptions 1 and 2 are satisfied.

Assumption 2 guarantees that the coefficients of the SDE system expressed in Itô form (2.18) are locally Lipschitz-continuous, which classically provides the local existence and uniqueness of solutions. The sublinearity Assumption 1 immediately results in

$$|B[\mu](x)|, |C[\mu](x)|, |S[\mu](x)| \lesssim \left(1 + |x| + \int |y| d\mu(y) \right). \quad (2.22)$$

Of course, Assumptions 1 and 2 are satisfied in the classical "globally-Lipschitz" setup when

$$|\nabla b|, |\nabla c|, |\nabla \sigma| \lesssim 1, \quad \nabla c \text{ and } \nabla \sigma \text{ locally Lipschitz-continuous.}$$

Most importantly, we are indeed allowed to consider coefficients with the Cucker-Smale form : let b and c be given by (2.12). Assuming that ψ, ϕ are bounded and locally Lipschitz-continuous, b and c are clearly sublinear. Moreover, a simple calculation gives, with $z_i = (x_i, v_i)$,

$$s_1(z_1, z_2, z_3) = \left(\begin{array}{c} 0 \\ -\phi(x_1 - x_2)\phi(x_1 - x_3)(v_3 - v_1) + \phi(x_1 - x_2)\phi(x_2 - x_3)(v_3 - v_2) \end{array} \right)$$

which is sublinear as well.

2.2.2 Estimates for the particle system.

Firstly, Assumption 1 naturally guarantees some moment estimates for the solutions of (2.8).

Proposition 2.2.1 (Moment estimates, global existence).

Let $T > 0$, $q \geq 2$ and $\mu_0^N = \frac{1}{N} \sum_i \delta_{X_0^{i,N}}$ be such that $\int |x|^q d\mu_0^N(x) < \infty$. Then the SDE system (2.8) (or equivalently (2.18)) has a unique solution defined on $[0, T]$, which satisfies,

$$\mathbb{E} \left[\sup_{t \in [0, T]} \int |x|^q d\mu_t^N(x) \right] \lesssim 1 + \int |x|^q d\mu_0^N(x) \quad (2.23)$$

and for all $i \in \{1, \dots, N\}$,

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{i,N}|^q \right] \lesssim 1 + \left(|X_0^{i,N}|^q + \int |x|^q d\mu_0^N(x) \right). \quad (2.24)$$

The constants involved in \lesssim depend on T and q only.

Proof. The assumptions guarantee that the coefficients of the SDE (2.18) are locally Lipschitz-continuous, which provides the local existence and uniqueness of the solution. To simplify the notation, we shall consider that all stochastic integrals are well defined : for a more rigorous framework, one should consider the solution of the truncated equations with a suitable stopping time; classically, estimate (2.24) (uniform on the truncation) then ensures that the solution is globally defined. Using (2.22), one can write

$$\begin{aligned} |X_t^{i,N}|^q &\lesssim |X_0^{i,N}|^q + \int_0^t (|B[\mu_s^N](X_s^{i,N})|^q + |S[\mu_s^N](X_s^{i,N})|^q) ds + |M_t^i|^q \\ &\lesssim 1 + |X_0^{i,N}|^q + \int_0^t (|X_s^{i,N}|^q + \int |x|^q d\mu_s^N) ds + |M_t^i|^q \end{aligned} \quad (2.25)$$

where $M_t^i = \int_0^t C[\mu_s^N](X_s^{i,N}) d\beta_s + \int_0^t \sigma(X_s^{i,N}) dB_s^i$. Taking the mean over i , and letting

$$\overline{|X_t|^q} = \int |x|^q d\mu_t^N$$

we are led to

$$\overline{|X_t|^q} \lesssim 1 + \overline{|X_0|^q} + \int_0^t \overline{|X_s|^q} ds + \frac{1}{N} \sum_i |M_t^i|^q$$

and therefore,

$$\sup_{\sigma \in [0, t]} \overline{|X_\sigma|^q} \lesssim 1 + \overline{|X_0|^q} + \int_0^t \sup_{\sigma \in [0, s]} \overline{|X_\sigma|^q} ds + \frac{1}{N} \sum_i \sup_{\sigma \in [0, t]} |M_\sigma^i|^q. \quad (2.26)$$

Burkholder-Davis-Gundy's inequality from [BDG72] states that $\mathbb{E} \left[\sup_{\sigma \in [0, t]} |M_\sigma^i|^q \right] \lesssim \mathbb{E} \left(\left[M^i \right]_t^{q/2} \right)$.

Using (2.22),

$$\left[M^i \right]_t = \int_0^t |C[\mu_s^N](X_s^{i,N})|^2 + |\sigma(X_s^{i,N})|^2 ds \lesssim 1 + \int_0^t |X_s^{i,N}|^2 + \overline{|X_s|^2} ds$$

hence $\left[M^i \right]_t^{q/2} \lesssim 1 + \int_0^t |X_s^{i,N}|^q + \overline{|X_s|^q} ds$. Coming back to (2.26),

$$\mathbb{E} \left[\sup_{\sigma \in [0, t]} \overline{|X_\sigma|^q} \right] \lesssim 1 + \overline{|X_0|^q} + \int_0^t \mathbb{E} \left[\sup_{\sigma \in [0, s]} \overline{|X_\sigma|^q} \right] ds \quad (2.27)$$

and we use Grönwall's Lemma to get the first estimate of Proposition 2.2.1. We can now get back to (2.25) to get

$$\sup_{\sigma \in [0, t]} |X_t^{i,N}|^q \lesssim 1 + |X_0^{i,N}|^q + \int_0^t \sup_{\sigma \in [0, s]} |X_s^{i,N}|^q ds + \sup_{\sigma \in [0, T]} \overline{|X_\sigma|^q} + \sup_{\sigma \in [0, t]} |M_t^i|^q.$$

Using the previously established estimate and Burkholder-Davis-Gundy's inequality once again,

$$\mathbb{E} \left[\sup_{\sigma \in [0, t]} |X_t^{i, N}|^q \right] \lesssim \left(1 + |X_0^{i, N}|^q + \overline{|X_0|^q} \right) + \int_0^t \mathbb{E} \left[\sup_{\sigma \in [0, s]} |X_s^{i, N}|^q \right] ds$$

and we may apply Grönwall's Lemma to obtain the second estimate of Proposition 2.2.1. \square

Remark 2.2.1. Given some $\psi \in C^2(\mathbb{R}^d)$ with $|\nabla\psi|, |\nabla^2\psi| \lesssim 1$, Itô's formula gives

$$\begin{aligned} d\psi(X_t^{i, N}) &= \nabla\psi(X_t^{i, N}) \cdot (B[\mu_t^N] + S[\mu_t^N])(X_t^{i, N})dt + \nabla\psi(X_t^{i, N}) \cdot C[\mu_t^N](X_t^{i, N})d\beta_t \\ &\quad + \nabla\psi(X_t^{i, N}) \cdot \sigma(X_t^{i, N})dB_t^i + \mathcal{A}[\mu_t^N](X_t^{i, N})dt, \end{aligned}$$

hence taking the mean in $i \in \{1, \dots, N\}$ we are led to

$$\begin{aligned} \langle \psi, \mu_t^N \rangle &= \langle \psi, \mu_0^N \rangle + \int_0^t \langle (B[\mu_s^N] + S[\mu_s^N]) \cdot \nabla\psi + \mathcal{A}[\mu_s^N]\psi, \mu_s^N \rangle ds + \int_0^t \langle C[\mu_s^N] \cdot \nabla\psi, \mu_s^N \rangle d\beta_s \\ &\quad + \frac{1}{N} \sum_i \int_0^t \nabla\psi(X_s^{i, N}) \cdot (\sigma(X_s^{i, N})dB_s^i). \end{aligned} \quad (2.28)$$

Given the bounds on $\mathbb{E}[\int |x|^2 d\mu_t^N(x)]$, it is easy to see that the stochastic integrals involved are continuous martingales. Aside from the last term, which is expected to vanish as N goes to infinity, this is exactly the SPDE (2.19).

Let us now establish some estimates regarding the regularity of solutions of (2.8).

Proposition 2.2.2 (Kolmogorov continuity for the particle system).

Let $T > 0$, $q \geq 2$ and $\mu_0^N = \frac{1}{N} \sum_i \delta_{X_0^{i, N}}$ be such that $\int |x|^q d\mu_0^N(x) < \infty$.

The following estimate holds uniformly for $t, s \in [0, T]$

$$\frac{1}{N} \sum_i \mathbb{E}|X_t^{i, N} - X_s^{i, N}|^q \lesssim \left(1 + \int |x|^q d\mu_0^N \right) |t - s|^{q/2}.$$

The constant involved in \lesssim depends on T and q only.

Proof. Again, one can write

$$\begin{aligned} |X_t^{i, N} - X_s^{i, N}|^q &\lesssim \left| \int_s^t (B[\mu_\sigma^N] + S[\mu_\sigma^N])(X_\sigma^{i, N})d\sigma \right|^q + |M_t^i - M_s^i|^q \\ &\lesssim |t - s|^{q-1} \int_s^t (|B[\mu_\sigma^N](X_\sigma^{i, N})|^q + |S[\mu_\sigma^N](X_\sigma^{i, N})|^q) d\sigma + |M_t^i - M_s^i|^q, \end{aligned}$$

hence using the estimates from Proposition 2.2.1,

$$\frac{1}{N} \sum_i \mathbb{E}|X_t^{i, N} - X_s^{i, N}|^q \lesssim |t - s|^q \left(1 + \int |x|^q d\mu_0^N \right) + \frac{1}{N} \sum_i \mathbb{E}|M_t^i - M_s^i|^q.$$

Burkholder-Davis-Gundy's inequality gives

$$\begin{aligned} \mathbb{E}|M_t^i - M_s^i|^q &\lesssim \mathbb{E} \left| [M^i]_t - [M^i]_s \right|^{q/2} \\ &\lesssim \mathbb{E} \left| \int_s^t 1 + \overline{|X_\sigma^N|^2} + |X_\sigma^{i, N}|^2 d\sigma \right|^{q/2} \lesssim |t - s|^{q/2-1} \int_s^t 1 + \mathbb{E}[\overline{|X_\sigma^N|^q} + |X_\sigma^{i, N}|^q] d\sigma \end{aligned}$$

since $q/2 \geq 1$, hence

$$\frac{1}{N} \sum_i \mathbb{E}|M_t^i - M_s^i|^q \lesssim |t - s|^{q/2} \left(1 + \int |x|^q d\mu_0^N \right)$$

which concludes the proof \square

2.2.3 Tightness of measure-valued random variables.

In this subsection, we state general results regarding the tightness of random measures, which we shall later apply in our special case.

Let $(E, \|\cdot\|)$ be a separable Banach space. The space $\mathcal{P}(E)$ of probability measures on E is equipped with the topology of the weak convergence. More precisely, we shall consider that $\mathcal{P}(E)$ is equipped with the Lévy-Prokhorov metric, which also makes it a polish space (one may refer to [Bil99] for details regarding this topology and tightness in general).

Definition 2.2.1. For a random measure $\mu : \Omega \rightarrow \mathcal{P}(E)$, we define the intensity $I(\mu)$ of μ by

$$\forall f : E \rightarrow \mathbb{R} \text{ measurable, bounded, } \langle f, I(\mu) \rangle = \mathbb{E}[\langle f, \mu \rangle],$$

that is $I(\mu) \in \mathcal{P}(E)$ is a deterministic probability measure on E .

The following result (mentioned e.g in [Szn91] p178) establishes a link between the relative compactness in law of μ and the tightness of its intensity measure $I(\mu)$.

Proposition 2.2.3.

For a sequence $(\mu^N)_{N \geq 1}$ of random measures on E , the two following statements are equivalent.

- i) The sequence of $\mathcal{P}(E)$ -valued random variables $(\mu^N)_N$ is tight.
- ii) The sequence $(I(\mu^N))_N$ of measures on E is tight.

Proof. Firstly i) clearly implies ii) since

$$[\mu^N \rightarrow \mu \text{ in } \mathcal{P}(E) \text{ weakly on } \Omega] \text{ implies } [I(\mu^N) \rightarrow I(\mu) \text{ weakly on } E].$$

Let us assume ii) : we introduce a sequence $(C_m)_{m \geq 1}$ of compacts of E such that

$$\forall m \geq 1, \sup_N I(\mu^N)(C_m^c) \leq 4^{-m}.$$

For a given $\varepsilon > 0$, let us define

$$K_\varepsilon = \{ \mu \in \mathcal{P}(E), \forall m \geq 1, \mu(C_m^c) \leq \varepsilon^{-1} 2^{-m} \}. \quad (2.29)$$

Prokhorov's theorem on tightness states that K_ε is a compact of $\mathcal{P}(E)$ equipped with the Lévy-Prokhorov metric. Now, for all $N \geq 1$, using simply Markov's inequality,

$$\mathbb{P}[\mu^N \notin K_\varepsilon] \leq \sum_m \mathbb{P}[\mu^N(C_m^c) > \varepsilon^{-1} 2^{-m}] \leq \varepsilon \sum_m I(\mu^N)(C_m^c) 2^m \leq \varepsilon.$$

□

Let us extend this reasoning to the Wasserstein space $\mathcal{P}_2(E)$ recalled in Definition 2.1.1. Firstly, the following convergence criteria is well known.

Proposition 2.2.4.

For a sequence of measure $(\mu^N)_N \in \mathcal{P}_2(E)^{\mathbb{N}}$, the following statements are equivalent :

- i) $\mu^N \rightarrow \mu$ in $\mathcal{P}_2(E)$
- ii) $\mu^N \rightarrow \mu$ in $\mathcal{P}(E)$ and $\limsup_N \int_{\|x\| > R} \|x\|^2 d\mu^N(x) \xrightarrow{R \rightarrow \infty} 0$
- iii) $\langle \phi, \mu^N \rangle \rightarrow \langle \phi, \mu \rangle$ for all ϕ continuous such that $\phi(x) \leq C(1 + |x|^2)$.

One could refer to [Vil08], section 6 (Theorem 6.9) for a proof. This immediately results in

Corollary 2.2.1 (Compact subsets of $\mathcal{P}_2(E)$).

A subset $A \subset \mathcal{P}_2(E)$ is relatively compact if and only if

- The family of measures $(\mu)_{\mu \in A}$ is tight
- $\sup_{\mu \in A} \left[\int_{\|x\| > R} \|x\|^2 d\mu(x) \right] \rightarrow 0$ as $R \rightarrow \infty$.

We can now state the following criteria for the relative compactness in law in $\mathcal{P}_2(E)$, which we conveniently express through the means of a Skorokhod representation theorem.

Proposition 2.2.5. *Let $\mu^N : \Omega \rightarrow \mathcal{P}_2(E)$, $N \geq 1$, be a sequence of random measures on E . The following statements are equivalent :*

- i) $\{\mu^N, N \geq 1\}$ is relatively compact in $\mathcal{P}_2(E)$
- ii) The family of measures $(I(\mu^N))_N$ is tight and $\sup_N \int_{\|x\|>R} \|x\|^2 dI(\mu^N)(x) \xrightarrow{R \rightarrow \infty} 0$
- iii) Out of any subsequence of $(\mu^N)_N$, one can extract a subsequence $(\mu^{N'})_{N'}$ satisfying the following : there exists some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables $\tilde{\mu}^{N'}, \tilde{\mu} : \tilde{\Omega} \rightarrow \mathcal{P}_2(E)$ such that

$$\begin{aligned} \forall N', \tilde{\mu}^{N'} &\sim \mu^{N'} \text{ in law} \\ W_2[\tilde{\mu}^{N'}, \tilde{\mu}] &\xrightarrow{N' \rightarrow \infty} 0 \tilde{\mathbb{P}}\text{-a.s and in } L^2(\tilde{\Omega}). \end{aligned}$$

Remark 2.2.2. *Note that the statement expressed in point iii) is in fact slightly stronger than the tightness of the $\mathcal{P}_2(E)$ -valued random variables $(\mu^N)_N$ since, up to a change of probability space, we are able to obtain a convergence in $L^2(\tilde{\Omega}; \mathcal{P}_2(E))$.*

Proof of Proposition 2.2.5. The equivalence between i) and ii) is exactly stated in Corollary 2.2.1. Assuming ii), we can introduce $(C_m)_{m \geq 1}$ compacts of E and $(R_m)_m$ with $R_m \rightarrow \infty$ such that for all $m \geq 1$,

$$\sup_N I(\mu^N)(C_m^c) \leq 4^{-m}, \quad \sup_N \int_{\|x\|>R_m} \|x\|^2 dI(\mu^N)(x) \leq 4^{-m}.$$

For a given $\varepsilon > 0$, let us define

$$K_\varepsilon = \left\{ \mu \in \mathcal{P}_2(E), \forall m \geq 1, \mu(C_m^c) \leq \varepsilon^{-1} 2^{-m}, \int_{\|x\|>R_m} \|x\|^2 d\mu(x) \leq \varepsilon^{-1} 2^{-m} \right\}$$

which is a relatively compact subset of $\mathcal{P}_2(E)$ by Corollary 2.2.1. Then, using Markov's inequality

$$\begin{aligned} \mathbb{P}[\mu^N \notin K_\varepsilon] &\leq \sum_{m \geq 1} \mathbb{P}[\mu^N(C_m^c) > \varepsilon^{-1} 2^{-m}] + \mathbb{P}\left[\int_{\|x\|>R_m} \|x\|^2 d\mu^N(x) > \varepsilon^{-1} 2^{-m}\right] \\ &\leq \varepsilon \sum_{m \geq 1} 2^m I(\mu^N)(C_m^c) + 2^m \int_{\|x\|>R_m} \|x\|^2 dI(\mu^N)(x) \leq 2\varepsilon \end{aligned}$$

which proves the tightness of the $\mathcal{P}_2(E)$ -valued random variables $(\mu^N)_N$. Let us introduce a subsequence $(\mu^{N'})_{N'}$ which converges in law to some $\mu \in \mathcal{P}_2(E)$. Applying Skorokhod's representation theorem on the polish space $\mathcal{P}_2(E)$ we get, on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}})$, $\tilde{\mu}^{N'} \rightarrow \tilde{\mu}$ a.s in $\mathcal{P}_2(E)$.

To conclude regarding the convergence in $L^2(\tilde{\Omega}; \mathcal{P}_2(E))$, it suffices to show that $W_2^2[\tilde{\mu}^{N'}, \tilde{\mu}]$ is uniformly integrable in N' . To this purpose, one can simply write

$$W_2^2[\tilde{\mu}^{N'}, \tilde{\mu}] \lesssim \int \|x\|^2 d\tilde{\mu}^{N'} + \int \|x\|^2 d\tilde{\mu} \quad (2.30)$$

and note that for all $R, M > 0$,

$$\begin{aligned} \tilde{\mathbb{E}}\left[\left(\int \|x\|^2 d\tilde{\mu}^{N'}\right) \mathbf{1}_{\left\{\int \|x\|^2 d\tilde{\mu}^{N'} > R\right\}}\right] &\leq \tilde{\mathbb{E}}\left[\int_{\|x\|^2 > M} \|x\|^2 d\tilde{\mu}^{N'}\right] + M \tilde{\mathbb{P}}\left[\int \|x\|^2 d\tilde{\mu}^{N'} > R\right] \\ &\leq \sup_N \int_{\|x\|^2 > M} \|x\|^2 dI(\mu^N) + \frac{M}{R} \sup_N \int \|x\|^2 dI(\mu^N). \end{aligned}$$

This shows that for all $M > 0$,

$$\limsup_{R \rightarrow \infty} \sup_N \tilde{\mathbb{E}}\left[\int \|x\|^2 d\tilde{\mu}^{N'} \mathbf{1}_{\left\{\int \|x\|^2 d\tilde{\mu}^{N'} > R\right\}}\right] \leq \sup_N \int_{\|x\|^2 > M} \|x\|^2 dI(\mu^N) \rightarrow 0 \text{ as } M \rightarrow \infty,$$

hence the first term in (2.30) is uniformly integrable. As for the second one, a use of Fatou's lemma gives $\tilde{\mathbb{E}} \left[\int \|x\|^2 d\tilde{\mu} \right] \leq \sup_N \int \|x\|^2 dI(\mu^N) < \infty$. We have proved *iii*).

Finally, let us show that *iii*) implies *i*). Let us introduce $\tilde{\pi}(\omega)$ an optimal plan between $\tilde{\mu}^{N'}(\omega)$ and $\tilde{\mu}(\omega)$, that is

$$W_2^2[\tilde{\mu}^{N'}, \tilde{\mu}](\omega) = \int_{x^1, x^2 \in E} \|x^1 - x^2\|^2 d\tilde{\pi}(\omega)(x^1, x^2).$$

Note that such a coupling exists, and can indeed be selected to be measurable, see [Vil08], Theorem 4.1 and Corollary 5.22. Then $\tilde{\pi} \in \Pi[\tilde{\mu}^{N'}, \tilde{\mu}]$ (for every $\omega \in \tilde{\Omega}$) and it is clear from the definition that

$$I(\tilde{\pi}) \in \Pi[I(\tilde{\mu}^{N'}), I(\tilde{\mu})] = \Pi[I(\mu^{N'}), I(\mu)].$$

Therefore it follows that

$$\begin{aligned} W_2^2[I(\mu^{N'}), I(\mu)] &\leq \int_{x^1, x^2 \in E} \|x^1 - x^2\|^2 dI(\tilde{\pi})(x^1, x^2) \\ &= \tilde{\mathbb{E}} \left[\int_{x^1, x^2 \in E} \|x^1 - x^2\|^2 d\tilde{\pi}(x^1, x^2) \right] = \mathbb{E} \left[W_2^2[\tilde{\mu}^{N'}, \tilde{\mu}] \right] \rightarrow 0 \text{ as } N' \rightarrow \infty. \end{aligned}$$

□

2.2.4 Proof of the weak convergence.

We will now prove the result stated in Theorem 3. Consider $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ satisfying

$$\int |x|^{2+\delta} d\mu_0(x) < \infty \text{ for some } \delta > 0$$

and a sequence of empirical measures $(\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_0^{i,N}})_N$ such that

$$\mu_0^N \rightarrow \mu_0 \text{ in } \mathcal{P}_2(\mathbb{R}^d), \quad \sup_N \int |x|^{2+\delta} d\mu_0^N(x) < \infty.$$

Let $(X_t^{i,N})_{t \in [0, T]}$, $i \in \{1, \dots, N\}$ be the solution of (2.18) with initial data $X_0^{i,N} = x_0^{i,N}$. We shall look at these processes as random variables taking values in the (separable Banach) space of continuous functions :

$$\begin{aligned} X^{i,N} : \Omega &\longrightarrow \mathcal{C} = C([0, T]; \mathbb{R}^d) \\ \omega &\longmapsto (X^{i,N}(\omega))_{t \in [0, T]} \end{aligned} \quad (2.31)$$

where \mathcal{C} is naturally equipped with the norm $\|x\|_\infty = \sup_{t \in [0, T]} |x_t|$.

The associated empirical measure $\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X^{i,N}}$ is hence seen as a random element of $\mathcal{P}(\mathcal{C})$. Its intensity measure is given by

$$I(\mu^N)(A) = \frac{1}{N} \sum_{i=1}^N \mathbb{P}(X^{i,N} \in A), \quad A \in \mathcal{B}(\mathcal{C}). \quad (2.32)$$

Let us verify the assumptions of Proposition 2.2.5 to establish the compactness in law in $\mathcal{P}_2(\mathcal{C})$:

1. Firstly, $\sup_N \int_{x \in \mathcal{C}} |x_0|^2 dI(\mu^N) = \sup_N \int_{x \in \mathbb{R}^d} |x|^2 d\mu_0^N < \infty$ and Proposition 2.2.2 gives

$$\begin{aligned} \sup_N \int_{x \in \mathcal{C}} |x_t - x_s|^{2+\delta} dI(\mu^N) &= \sup_N \frac{1}{N} \sum_i \mathbb{E} |X_t^{i,N} - X_s^{i,N}|^{2+\delta} \\ &\lesssim \left(1 + \sup_N \int |x|^{2+\delta} d\mu_0^N \right) |t - s|^{1+\delta/2}. \end{aligned}$$

Classically, using Kolmogorov's continuity criterium, for any $\alpha \in]0, 1/2[$, defining the compact subset of \mathcal{C}

$$K_R = \left\{ x \in \mathcal{C}, |x_0| \leq R, \sup_{t, s \in [0, T]} |x_t - x_s| \leq R|t - s|^\alpha \right\},$$

we get $\sup_N I(\mu^N)(K_R^c) \xrightarrow{R \rightarrow \infty} 0$ the sequence $(I(\mu^N))_N$ is tight.

2. We have

$$\int_{\|x\|_\infty \geq R} \|x\|_\infty^2 dI(\mu^N)(x) = \mathbb{E} \left[\int_{\|x\|_\infty \geq R} \|x\|_\infty^2 d\mu^N(x) \right] \leq \frac{1}{R^{2\delta}} \mathbb{E} \left[\int_x \|x\|_\infty^{2+\delta} d\mu^N(x) \right]$$

and taking the mean over i in Proposition 2.2.1,

$$\mathbb{E} \left[\int_x \|x\|_\infty^{2+\delta} d\mu^N(x) \right] = \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^{i, N}|^{2+\delta} \right] \lesssim 1 + \int |x|^{2+\delta} d\mu_0^N(x)$$

$$\text{hence } \limsup_N \int_{\|x\|_\infty \geq R} \|x\|_\infty^2 dI(\mu^N)(x) \xrightarrow{R \rightarrow \infty} 0.$$

We can therefore introduce a subsequence and some probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$ with random variables $\tilde{\mu}^{N'}, \tilde{\mu} : \tilde{\Omega} \rightarrow \mathcal{P}_2(\mathcal{C})$ such that

$$\begin{aligned} \forall N', \quad \tilde{\mu}^{N'} &\sim \mu^{N'} \text{ in law} \\ W_2[\tilde{\mu}^{N'}, \tilde{\mu}] &\xrightarrow{N' \rightarrow \infty} 0 \text{ a.s (and in } L^2(\tilde{\Omega})). \end{aligned}$$

Now, considering the law of the process $(\mu_t^N)_{t \geq 0}$ only, equation (2.28) can be translated as : for all $\psi \in C^2(\mathbb{R}^d)$ with $|\nabla \psi|, |\nabla^2 \psi| \lesssim 1$,

$$\begin{aligned} M_\psi^N(t) &= \langle \psi, \mu_t^N \rangle - \langle \psi, \mu_0^N \rangle - \int_0^t \langle L[\mu_s^N] \psi, \mu_s^N \rangle ds, \quad t \geq 0, \\ \text{where } L[\mu] \psi &= \left(B[\mu] + S[\mu] \right) \cdot \nabla \psi + \mathcal{A}[\mu] \psi, \end{aligned} \quad (2.33)$$

is a continuous L^2 martingale on Ω with respect to the canonical filtration of $(\mu_t^N)_t$, whose quadratic variation is given by

$$\begin{aligned} [M_\psi^N](t) &= \int_0^t \left| \langle C[\mu_s^N] \cdot \nabla \psi, \mu_s^N \rangle \right|^2 ds + \frac{1}{N^2} \sum_i \int_0^t \left| \sigma^T(X_s^{i, N}) \nabla \psi(X_s^{i, N}) \right|^2 ds \\ &= \int_0^t \left| \langle C[\mu_s^N] \cdot \nabla \psi, \mu_s^N \rangle \right|^2 ds + \frac{1}{N} \int_0^t \langle |\sigma^T \nabla \psi|^2, \mu_s^N \rangle ds. \end{aligned}$$

Equivalently, this can be expressed as

$$\mathbb{E} \left[\left(M_\psi^N(t) - M_\psi^N(s) \right) h(\mu_{t_1}^N, \dots, \mu_{t_m}^N) \right] = 0 \quad (2.34)$$

$$\mathbb{E} \left[\left| M_\psi^N(t) - M_\psi^N(s) \right|^2 h(\mu_{t_1}^N, \dots, \mu_{t_m}^N) \right] = \mathbb{E} \left[\left([M_\psi^N](t) - [M_\psi^N](s) \right) h(\mu_{t_1}^N, \dots, \mu_{t_m}^N) \right] \quad (2.35)$$

for all $0 \leq t_1, \dots, t_m \leq s \leq t$ and $h : \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ continuous bounded. Since $\tilde{\mu}^{N'} \sim \mu^{N'}$ in law in $\mathcal{P}_2(\mathcal{C})$, it is clear that the processes $(\mu_t^{N'})_{t \in [0, T]}$ and $(\tilde{\mu}_t^{N'})_{t \in [0, T]}$ have the same law. It follows that (2.34) and (2.35) also hold on the probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$ for

$$\tilde{M}_\psi^{N'}(t) = \langle \psi, \tilde{\mu}_t^{N'} \rangle - \langle \psi, \mu_0^{N'} \rangle - \int_0^t \langle L[\tilde{\mu}_s^{N'}] \psi, \tilde{\mu}_s^{N'} \rangle ds, \quad t \geq 0$$

making it a continuous L^2 martingale on $\tilde{\Omega}$, with respect to the canonical filtration of $(\tilde{\mu}_t^{N'})_t$, with quadratic variation $[\tilde{M}_\psi^{N'}](t) = \int_0^t \left| \langle C[\tilde{\mu}_s^{N'}] \cdot \nabla \psi, \tilde{\mu}_s^{N'} \rangle \right|^2 ds + \frac{1}{N'} \int_0^t \langle |\sigma^T \nabla \psi|^2, \tilde{\mu}_s^{N'} \rangle ds$.

We can now establish the following result.

Proposition 2.2.6. *For all $\psi \in C^2(\mathbb{R}^d)$ such that $|\nabla \psi|, |\nabla^2 \psi| \lesssim 1$,*

$$\tilde{M}_\psi(t) = \langle \psi, \tilde{\mu}_t \rangle - \langle \psi, \mu_0 \rangle - \int_0^t \langle L[\tilde{\mu}_s] \psi, \tilde{\mu}_s \rangle ds, \quad t \geq 0$$

is a continuous L^2 martingale on $\tilde{\Omega}$ with respect to the canonical filtration of $(\tilde{\mu}_t)_t$, whose quadratic variation is given by

$$[\tilde{M}_\psi](t) = \int_0^t \left| \langle C[\tilde{\mu}_s] \cdot \nabla \psi, \mu_s \rangle \right|^2 ds =: V_\psi(t).$$

Proof. Let us work on $\tilde{\Omega}$, but drop the tildas on μ, \mathbb{P} and the primes on N for clarity.

Given $0 \leq s \leq t \leq T$ and $0 \leq t_1 \leq \dots \leq t_m \leq s$ and h continuous bounded, we wish to send $N \rightarrow \infty$ in (2.34) and (2.35). It is enough to verify the following points :

1. $M_\psi^N(t) \rightarrow M_\psi(t)$ in probability
2. $|M_\psi^N(t)|^2$ is uniformly integrable in N
3. $[M_\psi^N](t) \rightarrow V_\psi(t)$ in probability
4. $[M_\psi^N](t)$ is uniformly integrable in N .

Given that $W_2[\mu^N, \mu] \rightarrow 0$ almost surely, using the immediate inequality

$$W_2^2[\mu \otimes \mu', \nu \otimes \nu'] \leq W_2^2[\mu, \nu] + W_2^2[\mu', \nu'],$$

we also derive that $W_2[(\mu^N)^{\otimes 2}, \mu^{\otimes 2}], W_2[(\mu^N)^{\otimes 3}, \mu^{\otimes 3}] \rightarrow 0$. Recalling (2.33), let us review the different terms involved in $M_\psi^N(t)$.

Since $|\psi(x)| \lesssim 1 + |x|$, we deduce from Proposition 2.2.4 that $\langle \psi, \mu_t^N \rangle \rightarrow \langle \psi, \mu \rangle$ a.s. Moreover, $|\langle \psi, \mu_t^N \rangle| \lesssim 1 + \int \|x\|_\infty d\mu^N(x)$.

The term $\int_0^t \langle B[\mu_s^N] \cdot \nabla \psi, \mu_s^N \rangle ds$ can be written as

$$\int_0^t \int_{(\mathbb{R}^d)^2} b(x, y) \cdot \nabla \psi(x) d(\mu_s^N \otimes \mu_s^N)(x, y) ds = \int_{\mathcal{C}^2} \left(\int_0^t b(x_s, y_s) \cdot \nabla \psi(x_s) ds \right) d(\mu^N \otimes \mu^N)(x, y)$$

which converges almost surely to the expected term since the functional is indeed sub-quadratic ($|b(x, y)| \lesssim |x| + |y|$ and $|\nabla \psi| \lesssim 1$). The term involving $S[\mu_t^N]$ is treated in the same way.

Recalling the form (2.21), the term $\int_0^t \langle \mathcal{A}[\mu_s^N] \psi, \mu_s^N \rangle ds$ can be written similarly as

$$\sum_{i,j} \int_0^t \int_x \int_y \int_z \left[\sum_k \sigma_{i,k}(x) \sigma_{j,k}(x) + c_i(x, y) c_j(x, z) \right] \partial_{i,j}^2 \psi(x) d(\mu_s^N \otimes \mu_s^N \otimes \mu_s^N)(x, y, z) ds.$$

again, this converges almost surely since

$$\left| \sigma_{i,k}(x) \sigma_{j,k}(x) \right| \lesssim 1 + |x|^2, \quad \left| c_i(x, y) c_j(x, z) \right| \lesssim 1 + |x|^2 + |y|^2 + |z|^2, \quad |\nabla^2 \psi| \lesssim 1.$$

Point 1. is hence proven. Let us skip Point 2. for now and consider $[M_\psi^N](t)$. Using the same arguments as before, $\left| \langle C[\mu_s^N] \cdot \nabla \psi, \mu_s^N \rangle \right|^2 \rightarrow \left| \langle C[\mu_s] \cdot \nabla \psi, \mu_s \rangle \right|^2$ almost surely for fixed $s \in [0, t]$. The bound

$$\left| \langle C[\mu_s^N] \cdot \nabla \psi, \mu_s^N \rangle \right|^2 \lesssim 1 + \int \|x\|_\infty^2 d\mu^N(x), \quad \text{with} \quad \sup_N \mathbb{E} \left[\int \|x\|_\infty^{2+\delta} d\mu^N(x) \right] < \infty$$

guarantees the uniform integrability in (t, ω) , so that

$$\int_0^t \left| \langle C[\mu_s^N] \cdot \nabla \psi, \mu_s^N \rangle \right|^2 \rightarrow \int_0^t \left| \langle C[\mu_s] \cdot \nabla \psi, \mu_s \rangle \right|^2 = V_\psi(t) \text{ in } L^1(\tilde{\Omega}),$$

in particular in probability. Additionally, $\left| \langle \sigma^T \nabla \psi|^2, \mu_s^N \rangle \right| \lesssim 1 + \int \|x\|_\infty^2 d\mu^N(x)$ which is bounded in $L^1(\tilde{\Omega})$ uniformly in N , so that

$$\frac{1}{N} \int_0^t \langle |\sigma^T \nabla \psi|^2, \mu_s^N \rangle \rightarrow 0 \text{ in } L^1(\tilde{\Omega}),$$

in particular in probability, which proves point 3.

We have in fact just seen that $[M_\psi^N](t)^{1+\delta/2} \lesssim 1 + \int \|x\|_\infty^{2+\delta} d\mu^N(x)$ which is bounded in $L^1(\tilde{\Omega})$ uniformly in N , hence giving point 4. Finally, Burkholder-Davis-Gundy's inequality gives

$$\mathbb{E} \left| M_\psi^N(t) \right|^{2+\delta} \lesssim \mathbb{E} \left[[M_\psi^N](t) \right]^{1+\delta/2}$$

and we derive point 2. from point 4. □

From $(\tilde{\mu}_t)_{t \geq 0}$ satisfying this martingale problem stated in Proposition 2.2.6, we classically construct a weak solution using a martingale representation theorem in some Hilbert space.

We start by noting that $\mathcal{P}_2(\mathbb{R}^d)$ is continuously embedded in the Sobolev space $H^{-\gamma} = (H^\gamma)'$ (where $H^\gamma = W^{\gamma,2}(\mathbb{R}^d)$) as soon as $\gamma > 1 + d/2$. Indeed, for $\psi \in H^\gamma$,

$$\left| \langle \psi, \mu \rangle - \langle \psi, \nu \rangle \right| \leq \|\nabla \psi\|_\infty W_2[\mu, \nu] \leq C \|\psi\|_{H^\gamma} W_2[\mu, \nu]$$

where we have used the continuous Sobolev embedding $H^\gamma \subset C_b^1$ for $\gamma > 1 + d/2$. We may consider the $H^{-\gamma-2}$ -valued process

$$\widetilde{M}(t) = \tilde{\mu}_t - \mu_0 - \int_0^t L[\tilde{\mu}_s]^* \tilde{\mu}_s ds, \quad t \in [0, T]$$

which satisfies, for all $\psi \in H^{\gamma+2}$ (a Sobolev embedding gives $|\nabla \psi|, |\nabla^2 \psi| \lesssim 1$),

$$\langle \widetilde{M}(t), \psi \rangle = \widetilde{M}_\psi(t), \quad t \in [0, T],$$

which is a continuous L^2 martingale with respect to the filtration

$$\widetilde{\mathcal{F}}_t = \sigma(\tilde{\mu}_s \in \mathcal{P}_2(\mathbb{R}^d), s \in [0, t]), \quad t \in [0, T],$$

with quadratic variation $V_\psi(t)$. Using a polarisation formula, we deduce more precisely that for $\psi_1, \psi_2 \in H^{\gamma+2}$,

$$\langle \widetilde{M}(t), \psi_1 \rangle \langle \widetilde{M}(t), \psi_2 \rangle - \langle V(t) \psi_1, \psi_2 \rangle, \quad t \in [0, T] \quad (2.36)$$

is a continuous $(\widetilde{\mathcal{F}}_t)_t$ -martingale, where $\langle V(t) \psi_1, \psi_2 \rangle = \int_0^t \langle C[\tilde{\mu}_s] \cdot \nabla \psi_1, \tilde{\mu}_s \rangle \langle C[\tilde{\mu}_s] \cdot \nabla \psi_2, \tilde{\mu}_s \rangle ds$. The martingale representation theorem from [DPZ14] p222 (Theorem 9.2) then holds, giving another probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ with a filtration $(\widetilde{\mathcal{F}}_t)_{t \in [0, T]}$ and a $(\widetilde{\mathcal{F}}_t \times \widetilde{\mathcal{F}}_t)$ -brownian motion $(W_t)_{t \in [0, T]}$ on $(\widetilde{\Omega} \times \widetilde{\Omega}, \widetilde{\mathbb{P}} \otimes \widetilde{\mathbb{P}})$ such that

$$\widetilde{M}(t)(\tilde{\omega}, \hat{\omega}) := \widetilde{M}(t)(\tilde{\omega}) = - \int_0^t \nabla \cdot (C[\tilde{\mu}_s(\tilde{\omega})] \tilde{\mu}_s(\tilde{\omega})) dW_s(\tilde{\omega}, \hat{\omega}). \quad (2.37)$$

It follows that $(\tilde{\omega}, \hat{\omega}) \mapsto (\tilde{\mu}_t(\tilde{\omega}))_{t \in [0, T]}$ is a solution of (2.10) on $\widetilde{\Omega} \times \widetilde{\Omega}$ according to Definition 2.1.2 (whose law is of course the same as that of $\tilde{\omega} \mapsto (\tilde{\mu}_t(\tilde{\omega}))_{t \in [0, T]}$).

2.3 Strong mean-field convergence

In this section, for simplicity, we restrict ourselves to the setting of a common noise, according to Assumption 3.

In this case, the limiting SPDE (2.10) becomes a stochastic conservation equation (given by (2.16)) and solutions μ_t are naturally expected to be obtained as the push-forward measures of μ_0 through the flow of the associated (non-linear) stochastic characteristics.

2.3.1 Stochastic characteristics.

Let us suppose that Assumptions 1 and 2 hold.

Definition 2.3.1.

Given some random $\mu \in \mathcal{P}_2(\mathcal{C})$ such that $\mathbb{E} \left[\int \|x\|_\infty^2 d\mu(x) \right] < \infty$, the characteristics X^μ are defined as the solution of

$$\begin{cases} dX_t^\mu(x) = \left(B[\mu_t] + S[\mu_t] \right) (X_t^\mu(x)) dt + C[\mu_t](X_t^\mu(x)) d\beta_t, & t \in [0, T], \\ X_0^\mu(x) = x \in \mathbb{R}^d. \end{cases} \quad (2.38)$$

A random measure $\mu : \Omega \rightarrow \mathcal{P}_2(\mathcal{C})$ is said to be "of the transport form" if it satisfies the fixed-point like identity

$$\mu = (X^\mu)^* \mu_0 \quad \text{a.s.} \quad (2.39)$$

where $X^\mu = (X_t^\mu(x))_{t \in [0, T], x \in \mathbb{R}^d}$ is the flow of characteristics associated to (2.38) and the measure $(X_t^\mu)^* \mu_0 \in \mathcal{P}(\mathcal{C})$ is defined by : for all $m \geq 1$, $t_1, \dots, t_m \in [0, T]$, $\psi \in C_b((\mathbb{R}^d)^m)$,

$$\int_{\mathcal{C}} \psi(x_{t_1}, \dots, x_{t_m}) d((X^\mu)^* \mu_0)(x) = \int_{\mathbb{R}^d} \psi(X_{t_1}^\mu(x), \dots, X_{t_m}^\mu(x)) d\mu_0(x).$$

Remark 2.3.1. Using Assumptions 1 and 2, given $\mu \in \mathcal{P}_2(\mathcal{C})$ satisfying $\mathbb{E} \left[\int \|x\|_\infty^2 d\mu(x) \right] < \infty$, one may easily establish that, for any fixed $x \in \mathbb{R}^d$, $\mathbb{E} \left[\sup_{t \in [0, T]} |X_t^\mu(x)|^2 \right] < \infty$, so that (2.38) admits a unique global solution.

Moreover, it is to be noted that the flow $x \mapsto X^\mu(x) \in \mathcal{C}$ is almost-surely continuous, so that the push-forward measure $(X^\mu)^* \mu_0$ is indeed well defined. This can easily be seen in the case where the kernels b, c, s_1 are globally Lipschitz-continuous, since we can derive some Kolmogorov estimate of the form

$$\mathbb{E} \left[\sup_{t \in [0, T]} |X_t^\mu(x) - X_t^\mu(x')|^p \right] \lesssim |x - x'|^p.$$

The result follows in the locally Lipschitz-continuous case using a classic stopping-time method (found e.g in [FZ03]).

Remark 2.3.2. For some fixed $N \geq 1$, let $(X^{i, N})_{i=1, \dots, N}$ be the solution of the particle system (2.8) with initial data $(x_0^{i, N})_{i=1, \dots, N}$ and let $\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X^{i, N}}$ the associated empirical measure. Then one can see that $X_t^{i, N} = X_t^{\mu^N}(x_0^{i, N})$ so that $\mu^N = (X^{\mu^N})^* \mu_0^N$ is of the transport form.

Measures of the transport form are, by design, solutions of the conservation equation (2.16)

Proposition 2.3.1. Let $\mu_0 \in \mathcal{P}_2(\mathcal{C})$ and $\mu = (X^\mu)^* \mu_0$ be of the transport form. Then $\mu = (\mu_t)_{t \in [0, T]}$ satisfies the SPDE (2.16) in the sense of Definition 2.1.2.

Proof. Firstly, using the same reasoning as in the proof of Proposition 2.2.1, one can easily show

$$\mathbb{E} \int_{\mathcal{C}} \|x\|_\infty^2 d\mu(x) = \mathbb{E} \int_{\mathbb{R}^d} \sup_{t \in [0, T]} |X_t^\mu(x)|^2 d\mu_0(x) \lesssim 1 + \int_{\mathbb{R}^d} |x|^2 d\mu_0(x), \quad (2.40)$$

so that the characteristics (2.38) are globally well-defined. For any $\psi \in C_c^2(\mathbb{R}^d)$, since $\mu \in \mathcal{P}(\mathcal{C})$, the process $(\langle \psi, \mu_t \rangle)_{t \in [0, T]}$ is automatically (adapted and) almost surely continuous. Itô's formula then results in

$$\begin{aligned} \psi(X_t^\mu(x)) &= \psi(x) + \int_0^t \nabla \psi(X_s^\mu(x)) \cdot (B[\mu_s] + S[\mu_s] + \mathcal{A}[\mu_s])(X_s^\mu(x)) ds \\ &\quad + \int_0^t \nabla \psi(X_s^\mu(x)) \cdot C[\mu_t](X_s^\mu(x)) d\beta_s. \end{aligned}$$

Note that, using the sublinearity Assumption 1 and the estimate (2.40), the stochastic integral involved here easily defines a square-integrable martingale. Integrating with respect to $d\mu_0(x)$ using a stochastic Fubini theorem gives exactly (2.20), so that Definition 2.1.2 is satisfied. \square

We now formulate an estimate which locally compares two solutions of the transport form.

Proposition 2.3.2 (\mathcal{P}_p -comparison estimate for compactly-supported initial data).

Let $\mu_0, \tilde{\mu}_0 \in \mathcal{P}(\mathbb{R}^d)$ be supported in some compact set $K \subset \mathbb{R}^d$, and let $\mu, \tilde{\mu} : \Omega \rightarrow \mathcal{P}_2(\mathcal{C})$ be of the transport form. For all $p > 1$, $R > 0$, defining the stopping time

$$\tau_R = \inf \left\{ t \geq 0, \sup_{x \in K} (|X_t^\mu(x)| + |X_t^{\tilde{\mu}}(x)|) \geq R \right\} \wedge T$$

there exists some constant $C_{p, R, T} > 0$ such that

$$\mathbb{E} \left[\sup_{t \in [0, \tau_R]} W_p^p[\mu_t, \tilde{\mu}_t] \right] \leq C_{p, R, T} W_p^p[\mu_0, \tilde{\mu}_0]. \quad (2.41)$$

Remark 2.3.3. Given some sequence $(x_k)_{k \geq 1}$ dense in K , the continuity of $x \mapsto X_0^\mu(x) \in \mathcal{C}$ gives

$$\{\tau_R \geq t\} = \left\{ \sup_{x \in K} \sup_{[0, t]} \left(|X_s^\mu(x)| + |X_s^{\tilde{\mu}}(x)| \right) \leq R \right\} = \bigcap_{k \geq 1} \{\tau_R(x_k) \geq t\}$$

where $\tau_R(x) = \inf \left\{ t \geq 0, |X_t^\mu(x)| + |X_t^{\tilde{\mu}}(x)| \geq R \right\} \wedge T$ is a stopping time, so that τ_R is indeed a stopping time.

Remark 2.3.4. The proof of this comparison estimate relies strongly on measures of the transport form (2.39) which are natural solutions of the stochastic conservation equation (2.16). Whenever $\sigma \neq 0$, solutions of SPDE (2.10) no longer exhibit a natural "transport form". Moreover, for fixed $N \geq 1$, the empirical measure μ^N cannot be written as the solution of some SPDE (see (2.28)). The case of a particle system with independent noise therefore requires additional work.

Proof of Proposition 2.3.2. For the sake of making calculations clearer, we only treat the case $p = 2$. Let us forget about $S[\mu]$ since it plays the same role as $B[\mu]$. Let us introduce a local Lipschitz constant $c_R > 0$ so that for all $|x|, |y|, |x'|, |y'| \leq R$,

$$\begin{aligned} |b(x, y) - b(x', y')| &\leq c_R (|x - x'| + |y - y'|) \\ |c(x, y) - c(x', y')| &\leq c_R (|x - x'| + |y - y'|) \\ |c(x, y)| &\leq c_R. \end{aligned}$$

This easily results in the following for all $|x|, |x'| \leq R$, ν, ν' with support in $B(0, R)$,

$$\begin{cases} |B[\nu](x) - B[\nu'](x')| \leq c_R (|x - x'| + W_1[\nu, \nu']) \leq c_R (|x - x'| + W_2[\nu, \nu']) \\ |C[\nu](x) - C[\nu'](x')| \leq c_R (|x - x'| + W_1[\nu, \nu']) \leq c_R (|x - x'| + W_2[\nu, \nu']) \\ |C[\nu](x)| \leq c_R. \end{cases} \quad (2.42)$$

Using Theorem 4.1 from [Vil08], we may introduce an optimal plan $\pi \in \Pi(\mu_0, \tilde{\mu}_0)$ so that

$$W_2^2[\mu_0, \tilde{\mu}_0] = \int_{K^2} |x - y|^2 d\pi(x, y).$$

Since $\mu, \tilde{\mu}$ are of the transport form, denoting $X_t = X_t^\mu$ and $\tilde{X}_t = \tilde{X}_t^{\tilde{\mu}}$, introducing the mapping $T : (x, y) \in (\mathbb{R}^d)^2 \mapsto (X_t(x), \tilde{X}_t(y)) \in (\mathbb{R}^d)^2$, one can easily see that $T^*\pi \in \Pi(\mu_t, \tilde{\mu}_t)$. It follows that

$$W_2^2[\mu_t, \tilde{\mu}_t] \leq J_t := \int_{K^2} |X_t(x) - \tilde{X}_t(y)|^2 d\pi(x, y). \quad (2.43)$$

We now apply Itô's formula to $\eta_t(x, y) = X_t(x) - \tilde{X}_t(y)$ to get

$$\begin{aligned} d|\eta_t(x, y)|^2 &= \left(2\eta_t(x, y) \cdot \left[B[\mu_t](X_t(x)) - B[\tilde{\mu}_t](\tilde{X}_t(y)) \right] + \left| C[\mu_t](X_t(x)) - C[\tilde{\mu}_t](\tilde{X}_t(y)) \right|^2 \right) dt \\ &\quad + 2\eta_t(x, y) \cdot \left(C[\mu_t](X_t(x)) - C[\tilde{\mu}_t](\tilde{X}_t(y)) \right) d\beta_t. \end{aligned}$$

Applying (2.42), we deduce, for some $C_R > 0$,

$$\begin{aligned} d|\eta_{t \wedge \tau_R}(x, y)|^2 &\leq C_R \left(|\eta_{t \wedge \tau_R}(x, y)|^2 + W_2^2[\mu_{t \wedge \tau_R}, \tilde{\mu}_{t \wedge \tau_R}] \right) dt + dM_{t \wedge \tau_R}(x, y) \\ &\leq C_R \left(|\eta_{t \wedge \tau_R}(x, y)|^2 + J_{t \wedge \tau_R} \right) dt + dM_{t \wedge \tau_R}(x, y) \end{aligned} \quad (2.44)$$

with $M_t(x, y) = \int_0^t 2\eta_s(x, y) \cdot \left(C[\mu_s](X_s(x)) - C[\tilde{\mu}_s](\tilde{X}_s(y)) \right) d\beta_s$. We may integrate this expression with respect to $d\pi(x, y)$ using a stochastic Fubini theorem to get

$$dJ_{t \wedge \tau_R} \leq C_R J_{t \wedge \tau_R} dt + dM_{t \wedge \tau_R} \quad (2.45)$$

with $M_t = \int_0^t \left(\int_{K^2} 2\eta_s(x, y) \cdot \left(C[\mu_s](X_s(x)) - C[\tilde{\mu}_s](\tilde{X}_s(y)) \right) d\pi(x, y) \right) dt$. Taking the expectation in (2.45) and applying Grönwall's lemma leads to

$$\forall t \in [0, T], \quad \mathbb{E}[J_{t \wedge \tau_R}] \leq C_{R,T} W_2^2[\mu_0, \tilde{\mu}_0].$$

Coming back to (2.45) one may now write, using Burkholder-Davis-Gundy's inequality

$$\mathbb{E} \left[\sup_{[0, T]} J_{t \wedge \tau_R} \right] \leq C_R \int_0^T \mathbb{E}[J_{t \wedge \tau_R}] dt + \mathbb{E} \left[\sup_{[0, T]} M_{t \wedge \tau_R} \right] \leq C_{R,T} W_2^2[\mu_0, \tilde{\mu}_0] + C \mathbb{E} \left([M]_{T \wedge \tau_R}^{1/2} \right).$$

With

$$\begin{aligned} [M]_{T \wedge \tau_R} &= 4 \int_0^T \left| \int_{K^2} \eta_{t \wedge \tau_R}(x, y) \cdot \left(C[\mu_{t \wedge \tau_R}](X_{t \wedge \tau_R}(x)) - C[\tilde{\mu}_{t \wedge \tau_R}](\tilde{X}_{t \wedge \tau_R}(y)) \right) d\pi(x, y) \right|^2 dt \\ &\leq 4C_R \int_0^T \left(\int_{K^2} |\eta_{t \wedge \tau_R}(x, y)| \left(\eta_{t \wedge \tau_R}(x, y) + W_2[\mu_{t \wedge \tau_R}, \tilde{\mu}_{t \wedge \tau_R}] \right) d\pi(x, y) \right)^2 dt \\ &\leq C_R \int_0^T (J_{t \wedge \tau_R})^2 dt \leq C_R \left(\sup_{[0, T]} J_{t \wedge \tau_R} \right) \int_0^T J_{t \wedge \tau_R} dt. \end{aligned}$$

Hölder's inequality classically gives

$$\mathbb{E} \left[\sup_{[0, T]} J_{t \wedge \tau_R} \right] \leq C_{R,T} W_2^2[\mu_0, \tilde{\mu}_0] + C_{R,T} \mathbb{E} \left[\sup_{[0, T]} J_{t \wedge \tau_R} \right]^{1/2} W_2[\mu_0, \tilde{\mu}_0]$$

from which we easily derive

$$\mathbb{E} \left[\sup_{t \in [0, \tau_R]} W_2^2[\mu_t, \tilde{\mu}_t] \right] \leq \mathbb{E} \left[\sup_{[0, T]} J_{t \wedge \tau_R} \right] \leq C_{R,T} W_2^2[\mu_0, \tilde{\mu}_0]. \quad (2.46)$$

□

Remark 2.3.5. Seeing $\mu = (X)^* \mu_0$ and $\tilde{\mu} = (\tilde{X})^* \tilde{\mu}_0$ as random elements of $\mathcal{P}_p(\mathcal{C})$, we can in fact be a little more precise. With $\pi \in \Pi(\mu_0, \tilde{\mu}_0)$ an optimal plan between μ_0 and $\tilde{\mu}_0$, we have

$$W_p^p[\mu, \tilde{\mu}] \leq J_T^* := \int_{K^2} \sup_{t \in [0, T]} |X_t(x) - \tilde{X}_t(y)|^p d\pi(x, y) \quad (2.47)$$

and one could easily adapt the proof (apply Burkholder-Davis-Gundy's inequality in (2.44) before integrating) to get the estimate

$$\mathbb{E} [J_{\tau_R}^*] \leq C_{p,R,T} W_p^p[\mu_0, \tilde{\mu}_0]. \quad (2.48)$$

The result from Proposition 2.3.2 makes it clear that, given a compactly-supported measure μ_0 , one should naturally require some estimates regarding the growth of the support of μ_t , that is, estimates on $\sup_{x \in K} |X_t^\mu(x)|$. In [CS17] and [HJR19] for instance, where the diffusion coefficient $c(x, y)$ is linear, precise almost-sure estimates for the support of μ_t are achieved using some stochastic Grönwall inequality.

The assumptions from Theorem 4 provide another setting (where, in particular, the diffusion coefficient is bounded) in which we are able to obtain a bound on the moments

$$\mathbb{E} \left[\sup_{x \in K} \sup_{[0, T]} |X_t^\mu(x)|^p \right], \quad p \geq 1.$$

2.3.2 Properties of the coefficients.

From this point on, we suppose that Assumptions 4 and 5 are satisfied.

Note that, as mentioned in the introduction, these assumptions allow us to consider stochastic Cucker-Smale models with "truncated velocities" in the interaction perturbation, given by (2.7). Indeed, this corresponds to coefficients of the form

$$b((x, v); (y, w)) = \begin{pmatrix} v \\ \psi(x-y)(w-v) \end{pmatrix} \quad c((x, v); (y, w)) = \begin{pmatrix} 0 \\ \phi(x-y)\mathcal{R}(w-v) \end{pmatrix}. \quad (2.49)$$

Provided that the weight functions ψ, ϕ and the truncation function \mathcal{R} satisfy, for some $\theta \in [0, 1]$,

$$\begin{aligned} |\psi(x)|, |\phi(x)|, |\mathcal{R}(v)| &\lesssim 1, \\ |\psi(x) - \psi(y)| &\lesssim (1 + |x|^\theta + |y|^\theta)|x - y|, \\ |\phi(x) - \phi(y)| &\lesssim (1 + |x|^\theta + |y|^\theta)|x - y|, \\ |\mathcal{R}(v) - \mathcal{R}(w)| &\lesssim (1 + |v|^\theta + |w|^\theta)|v - w|, \\ |\nabla\mathcal{R}(v) - \nabla\mathcal{R}(w)| &\lesssim (1 + |v|^{2\theta} + |w|^{2\theta})|v - w|, \end{aligned}$$

one can check that all the required assumptions are satisfied, with, denoting $z_i = (x_i, v_i)$,

$$s_1(z_1, z_2, z_3) = \begin{pmatrix} 0 \\ -\phi(x_1 - x_2)\phi(x_1 - x_3)\nabla\mathcal{R}(v_2 - v_1)\mathcal{R}(v_3 - v_1) \\ +\phi(x_1 - x_2)\phi(x_2 - x_3)\nabla\mathcal{R}(v_2 - v_1)\mathcal{R}(v_3 - v_2) \end{pmatrix}.$$

From Assumptions 4 and 5, we easily derive

$$|B[\mu](x)| \lesssim 1 + |x| + \int |y| d\mu(y) \quad (2.50)$$

$$|B[\mu](x) - B[\mu](x')| \lesssim \left(1 + |x|^{2\theta} + |x'|^{2\theta} + \int |y|^{2\theta} d\mu(y)\right) |x - x'| \quad (2.51)$$

and similar estimates for $S[\mu]$, as well as

$$|C[\mu](x)| \lesssim 1 \quad (2.52)$$

$$|C[\mu](x) - C[\mu](x')| \lesssim \left(1 + |x|^\theta + |x'|^\theta + \int |y|^\theta d\mu(y)\right) |x - x'|. \quad (2.53)$$

2.3.3 Estimates for the stochastic characteristics.

In this context, let us start by establishing some exponential moments for the stochastic characteristics.

Lemma 2.3.1 (Exponential moments).

Let $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ and $\mu : \Omega \rightarrow \mathcal{P}_2(\mathcal{C})$ be of the transport form.

Then, for all $T > 0$, $\alpha_0 \in (0, 1]$, there exists $\alpha_T > 0$ and $C_T > 0$ such that for $t \in [0, T]$, $x \in \mathbb{R}^d$,

$$\mathbb{E} \left[\exp \left(\alpha_T \left(|X_t^\mu(x)|^2 + \int |y|^2 d\mu_t(y) \right) \right) \right] \leq C_T \exp \left(\alpha_0 \left(|x|^2 + \int |y|^2 d\mu_0(y) \right) \right). \quad (2.54)$$

Proof. This method is inspired from the one developed in [BCC11], Lemma 3.5. Again, let us forget $S[\mu]$ since it satisfies the same estimates as $B[\mu]$. Itô's formula gives

$$d \left[|X_t^\mu(x)|^2 \right] = (2X_t^\mu(x) \cdot B[\mu_t](X_t^\mu(x)) + |C[\mu_t](X_t^\mu(x))|^2) dt + 2X_t^\mu(x) \cdot C[\mu_t](X_t^\mu(x)) d\beta_t.$$

Integrating with respect to $d\mu_0(x)$ leads to

$$d \left[\int |y|^2 d\mu_t(y) \right] = \left[\int \left(2y \cdot B[\mu_t](y) + |C[\mu_t](y)|^2 \right) d\mu_t(y) \right] dt + \left[\int \left(2y \cdot C[\mu_t](y) \right) d\mu_t(y) \right] d\beta_t.$$

Hence, letting $Y_t = |X_t^\mu(x)|^2 + \int |y|^2 d\mu_t$ and summing these two identities, we get

$$dY_t = a_t dt + \sigma_t d\beta_t$$

with, using (2.50) and (2.52), $|a_t|, |\sigma_t|^2 \lesssim (1 + Y_t)$. Let $\alpha(t)$ be a deterministic, positive smooth function to be fixed later on. Letting $Z_t = \exp\left(\alpha(t)Y_t\right)$, it follows that, for some $C > 0$,

$$\begin{aligned} dZ_t &= Z_t \left(\alpha'(t)Y_t + \alpha(t)a_t + \frac{\alpha(t)^2}{2}\sigma_t^2 \right) dt + Z_t \alpha(t)\sigma_t d\beta_t \\ &\leq Z_t \left(\alpha'(t)Y_t + C(1 + Y_t)(\alpha(t) + \alpha(t)^2) \right) dt + Z_t \alpha(t)\sigma_t d\beta_t \\ &= Z_t Y_t \left(\alpha'(t) + C\alpha(t) + C\alpha^2(t) \right) dt + CZ_t \left(\alpha(t) + \alpha(t)^2 \right) dt + Z_t \alpha(t)\sigma_t d\beta_t. \end{aligned}$$

Choosing $\alpha(t)$ so that $\alpha' + C\alpha + C\alpha^2 \leq 0$, that is for instance $\alpha(t) = \alpha_0 e^{-2Ct}$, we are led to

$$dZ_t \leq C(\alpha_0 + \alpha_0^2) Z_t dt + Z_t \alpha(t)\sigma_t d\beta_t, \quad (2.55)$$

hence taking the expectation (again, one may use a stopping time to be more rigorous) and applying Grönwall's lemma gives

$$\mathbb{E} \left[Z_t \right] \leq \exp(C(\alpha_0 + \alpha_0^2)T) \exp \left(\alpha_0 \left(|x|^2 + \int |y|^2 d\mu_0(y) \right) \right).$$

This is the expected result with $\alpha_T = \inf_{t \in [0, T]} \alpha(t) = \alpha_0 e^{-2CT}$ and $C_T = \exp(C(\alpha_0 + \alpha_0^2)T)$. \square

We can now establish some bounds regarding the support of μ_t .

Proposition 2.3.3 (Kolmogorov continuity estimates for the stochastic characteristics).

Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ be supported in some compact set $K \subset \mathbb{R}^d$, and $\mu : \Omega \rightarrow \mathcal{P}_2(\mathcal{C})$ be of the transport form. For all $T > 0$, $p > 1$, there exists a constant $C_{K, T, p}$ such that

$$\forall x, x' \in K, \quad \mathbb{E} \left[\sup_{t \in [0, T]} |X_t^\mu(x) - X_t^\mu(x')|^p \right] \leq C_{K, T, p} |x - x'|^p.$$

Using Kolmogorov's continuity theorem, one can then bound all the moments of the α -Hölder constant

$$N_\alpha(X^\mu) = \sup_{x, x' \in K} \sup_{t \in [0, T]} \frac{|X_t^\mu(x) - X_t^\mu(x')|}{|x - x'|^\alpha}$$

for all $\alpha \in (0, 1)$. The set K being compact, an immediate consequence is the following.

Corollary 2.3.1. Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ be supported in some compact set $K \subset \mathbb{R}^d$, and $\mu : \Omega \rightarrow \mathcal{P}_2(\mathcal{C})$ be of the transport form. For all $T > 0$, $p > 1$, there exists a constant $C_{K, T, p}$ such that

$$\mathbb{E} \left[\sup_{x \in K} \sup_{[0, T]} |X_t^\mu(x)|^p \right] \leq C_{K, T, p}.$$

Proof of Proposition 2.3.3. Let us once again forget $S[\mu]$. Letting $\eta_t = X_t^\mu(x) - X_t^\mu(x')$, Itô's formula gives

$$d|\eta_t|^{2p} = 2p|\eta_t|^{2p-2}\eta_t \cdot d\eta_t + 2p(p-1) \sum_{i, j=1}^d |\eta_t|^{2p-4} \eta_t^i \eta_t^j d[\eta^i, \eta^j]_t + p \sum_{i=1}^d |\eta_t|^{2p-2} d[\eta^i]_t$$

with

$$\begin{aligned} d\eta_t &= (B[\mu_t](X_t^\mu(x)) - B[\mu_t](X_t^\mu(x'))) dt + (C[\mu_t](X_t^\mu(x)) - C[\mu_t](X_t^\mu(x'))) d\beta_t \\ d[\eta^i, \eta^j]_t &= (C^i[\mu_t](X_t^\mu(x)) - C^i[\mu_t](X_t^\mu(x'))) (C^j[\mu_t](X_t^\mu(x)) - C^j[\mu_t](X_t^\mu(x'))) dt. \end{aligned}$$

Using the Lipschitz estimates from (2.51) and (2.53), we derive that, for some $C_* \geq 1$,

$$d|\eta_t|^{2p} \leq \lambda_t |\eta_t|^{2p} dt + dM_t, \quad \text{with} \quad (2.56)$$

$$\lambda_t \equiv \lambda_t(x, x') := C_* \left(1 + |X_t^\mu(x)|^{2\theta} + |X_t^\mu(x')|^{2\theta} + \int |y|^{2\theta} d\mu_t \right), \quad (2.57)$$

$$M_t = \int_0^t 2p|\eta_s|^{2p-2} \eta_s \cdot (C[\mu_s](X_s^\mu(x)) - C[\mu_s](X_s^\mu(x'))) d\beta_s. \quad (2.58)$$

Let us define

$$\Lambda_t \equiv \Lambda_t(x, x') := \gamma \int_0^t \lambda_s(x, x') ds, \quad (2.59)$$

where the constant $\gamma \geq 1$ is to be fixed later on. One can now write

$$\begin{aligned} \mathbb{E} \left[\sup_{[0, T]} |\eta|^p \right] &\leq \mathbb{E} \left[\exp \left(\frac{\Lambda_T}{2} \right) \sup_{[0, T]} \left(\exp \left(-\frac{\Lambda_t}{2} \right) |\eta_t|^p \right) \right] \\ &\leq \mathbb{E} \left[\exp(\Lambda_T) \right]^{1/2} \mathbb{E} \left[\sup_{[0, T]} (\exp(-\Lambda_t) |\eta_t|^{2p}) \right]^{1/2}. \end{aligned} \quad (2.60)$$

Let us fix $\alpha_0 = 1$ and introduce $\alpha_T > 0$ such that the estimate from Proposition 2.3.1 holds. Then,

$$\begin{aligned} \mathbb{E} \left[\exp(\Lambda_T) \right] &\leq \int_0^T \mathbb{E} \left[\exp(\gamma \lambda_t) \right] dt \\ &= \int_0^T \mathbb{E} \left[\exp \left(\gamma C_* \left(1 + |X_t^\mu(x)|^{2\theta} + |X_t^\mu(x')|^{2\theta} + \int |y|^{2\theta} d\mu_t \right) \right) \right] dt \\ &\leq C_\# \int_0^T \mathbb{E} \left[\exp \left(\frac{\alpha_T}{2} \left(|X_t^\mu(x)|^2 + |X_t^\mu(x')|^2 + \int |y|^2 d\mu_t(y) \right) \right) \right] dt, \end{aligned} \quad (2.61)$$

where the constant $C_\# = C_\#(T, \gamma) > 0$ is chosen large enough so that (recall that $\theta \in [0, 1)$)

$$\forall u \in \mathbb{R}^+, \quad \exp(\gamma C_* (1 + u^{2\theta})) \leq C_\# \exp\left(\frac{\alpha_T}{2} u^2\right).$$

We may now use Hölder's inequality in (2.61) and apply the estimate from Proposition 2.3.1 to conclude

$$\mathbb{E} \left[\exp(\Lambda_T) \right] \leq C_{K, T}. \quad (2.62)$$

Combining (2.60) and (2.62), it only remains to prove that

$$\mathbb{E} \left[\sup_{[0, T]} (\exp(-\Lambda_t) |\eta_t|^{2p}) \right] \leq C_{K, T} |x - x'|^{2p}. \quad (2.63)$$

By design, we derive from (2.56) with Itô's formula that

$$\exp(-\Lambda_t) |\eta_t|^{2p} \leq |x - x'|^{2p} + \int_0^t \exp(-\Lambda_s) dM_s$$

so that, denoting N_t the martingale term, with Burkholder-Davis-Gundy's inequality,

$$\mathbb{E} \left[\sup_{[0, T]} (\exp(-\Lambda_t) |\eta_t|^{2p}) \right] \lesssim |x - x'|^{2p} + \mathbb{E} \left([N]_T^{1/2} \right). \quad (2.64)$$

This quadratic variation is given by

$$\begin{aligned} [N]_T &= 4p^2 \int_0^T \exp(-2\Lambda_t) \left| |\eta_s|^{2p-2} \eta_s \cdot (C[\mu_s](X_s^\mu(x)) - C[\mu_s](X_s^\mu(x'))) \right|^2 dt \\ &\lesssim \int_0^T \exp(-2\Lambda_t) \lambda_t |\eta_t|^{4p} dt \lesssim \sup_{[0, T]} (\exp(-\Lambda_t) |\eta_t|^{2p}) \int_0^T \exp(-\Lambda_t) \lambda_t |\eta_t|^{2p} dt, \end{aligned}$$

so that (2.64) leads to

$$\begin{aligned} &\mathbb{E} \left[\sup_{[0, T]} (\exp(-\Lambda_t) |\eta_t|^{2p}) \right] \\ &\lesssim |x - x'|^{2p} + \mathbb{E} \left[\sup_{[0, T]} (\exp(-\Lambda_t) |\eta_t|^{2p}) \right]^{1/2} \left(\int_0^T \mathbb{E} \left[\exp(-\Lambda_t) \lambda_t |\eta_t|^{2p} \right] dt \right)^{1/2}. \end{aligned}$$

The estimate (2.63) will therefore hold if we can establish

$$\forall t \in [0, T], \quad \mathbb{E} \left[\exp(-\Lambda_t) \lambda_t |\eta_t|^{2p} \right] \leq C_K |x - x'|^{2p}. \quad (2.65)$$

The integration by part formula gives

$$\begin{aligned} d \left[\exp(-\Lambda_t) \lambda_t |\eta_t|^{2p} \right] &= \exp(-\Lambda_t) \left(|\eta_t|^{2p} d\lambda_t + \lambda_t d(|\eta_t|^{2p}) + d[\lambda, |\eta|^{2p}]_t - \Lambda'_t \lambda_t |\eta_t|^{2p} dt \right) \\ &= \exp(-\Lambda_t) \left(|\eta_t|^{2p} d\lambda_t + \lambda_t d(|\eta_t|^{2p}) + d[\lambda, |\eta|^{2p}]_t - \gamma \lambda_t^2 |\eta_t|^{2p} dt \right). \end{aligned} \quad (2.66)$$

Given that $\mathbb{E} \left[\exp(-\Lambda_0) \lambda_0 |\eta_0|^{2p} \right] \leq C_K |x - x'|^{2p}$, it is enough to prove that the drift terms in (2.66) are all negative for γ chosen large enough.

Recalling (2.56), $\lambda_t d(|\eta_t|^{2p}) \lesssim \lambda_t^2 |\eta_t|^{2p} dt + \lambda_t dM_t$.

Moreover, using the sublinearity Assumption 5, we easily get from the expression (2.57)

$$d\lambda_t \lesssim \lambda_t + dm_t(x) + dm_t(x') + \int dm_t(y) d\mu_0(y), \quad (2.67)$$

$$\text{where } dm_t(y) = 2\theta |X_t^\mu(y)|^{2\theta-2} X_t^\mu(y) \cdot C[\mu_t](X_t^\mu(y)) d\beta_t, \quad (2.68)$$

so that $|\eta_t|^{2p} d\lambda_t \lesssim \lambda_t |\eta_t|^{2p} dt + |\eta_t|^{2p} (dm_t(x) + dm_t(x') + \int dm_t(y) d\mu_0(y))$.

We conclude by noting that

$$d[\lambda, |\eta|^{2p}]_t = d \left[m(x) + m(x') + \int m(y) d\mu_0(y), M \right]_t \lesssim \lambda_t |\eta_t|^{2p} dt$$

since for all $y \in K$, $d \left[m(y), M \right]_t$ can be written

$$\begin{aligned} & (2\theta |X_t^\mu(y)|^{2\theta-2} X_t^\mu(y) \cdot C[\mu_t](X_t^\mu(y))) (2p |\eta_t|^{2p-2} \eta_t \cdot (C[\mu_t](X_t^\mu(x)) - C[\mu_t](X_t^\mu(x')))) dt \\ & \lesssim |X_t^\mu(y)|^{2\theta-1} \left(1 + |X_t^\mu(x)|^\theta + |X_t^\mu(x')|^\theta + \int |z|^\theta d\mu_t(z) \right) |\eta_t|^{2p} dt \\ & \lesssim \left(1 + |X_t^\mu(x)|^{2\theta+\theta-1} + |X_t^\mu(x')|^{2\theta+\theta-1} + |X_t^\mu(y)|^{2\theta+\theta-1} + \int |z|^{2\theta+\theta-1} d\mu_t(z) \right) |\eta_t|^{2p} dt \\ & \lesssim \left(1 + |X_t^\mu(x)|^{2\theta} + |X_t^\mu(x')|^{2\theta} + |X_t^\mu(y)|^{2\theta} + \int |z|^{2\theta} d\mu_t(z) \right) |\eta_t|^{2p} dt. \end{aligned}$$

Note that $\lambda_t \geq 1$ so that $\lambda_t \leq \lambda_t^2$. The proof is complete. \square

2.3.4 Proof of the strong convergence.

Now that the support estimate from Corollary 2.3.1 is acquired, we can prove Theorem 4.

Let us fix some $T > 0$ and $p \geq 2$. Let $K \subset \mathbb{R}^d$ be a compact set, $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ be a probability measure with support in K , and $(\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_0^{i,N}})_N$ be a sequence of empirical measures with support in K such that

$$W_p[\mu_0^N, \mu_0] \xrightarrow{N \rightarrow \infty} 0.$$

Letting $(X_t^{i,N})_{t \in [0, T]}$ be the solution of (2.8) with initial data $X_0^{i,N} = x_0^{i,N}$, let us introduce $\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{X^{i,N}}$ the empirical measure associated with the particle system, which is naturally of the transport form : $\mu^N = (X^{\mu^N})^* \mu_0^N$.

Consequently, for $N, M \geq 1$, Proposition 2.3.2 gives, for all $R > 0$,

$$\mathbb{E} \left[\sup_{t \in [0, \tau_R^{N,M}]} W_p^p[\mu_t^N, \mu_t^M] \right] \leq C_{p,R,T} W_p^p[\mu_0^N, \mu_0^M]$$

where the stopping time is given by

$$\tau_R^{N,M} = \inf \left\{ t \geq 0, \sup_{x \in K} \left(|X_t^{\mu^N}(x)| + |X_t^{\mu^M}(x)| \right) \geq R \right\} \wedge T. \quad (2.69)$$

Let us in fact be more precise and use the inequality (2.48) : introducing an optimal plan $\pi^{N,M} \in \Pi(\mu_0^N, \mu_0^M)$ so that

$$W_p^p[\mu_0^N, \mu_0^M] = \int_{K^2} |x - y|^p d\pi^{N,M}(x, y),$$

we have

$$W_p^p[\mu^N, \mu^M] \leq J_T^* := \int_{K^2} \sup_{t \in [0, T]} \left| X_t^{\mu^N}(x) - X_t^{\mu^M}(y) \right|^p d\pi^{N,M}(x, y), \quad (2.70)$$

with the inequality

$$\mathbb{E} \left[J_{\tau_R^*}^{*N, M} \right] \leq C_{p, R, T} W_p^p[\mu_0^N, \mu_0^M]. \quad (2.71)$$

It then follows that

$$\begin{aligned} \mathbb{E} [W_p^p[\mu^N, \mu^M]] &\leq \mathbb{E} [J_T^*] \leq \mathbb{E} \left[J_T^* \mathbb{1}_{\{\tau_R^{N, M} = T\}} \right] + \mathbb{E} \left[J_T^* \mathbb{1}_{\{\tau_R^{N, M} < T\}} \right] \\ &\leq \mathbb{E} \left[J_{\tau_R^*}^{*N, M} \right] + C_p \mathbb{E} \left[\mathbb{1}_{\{\tau_R^{N, M} < T\}} \left(\int_{x \in C} \|x\|_\infty^p d\mu^N(x) + \int_{x \in C} \|x\|_\infty^p d\mu^M(x) \right) \right] \\ &\leq C_{p, R, T} W_p^p[\mu_0^N, \mu_0^M] + C_p \mathbb{P} \left(\tau_R^{N, M} < T \right)^{1/2} \left(\sup_N \mathbb{E} \left[\int_{x \in C} \|x\|_\infty^{2p} d\mu^N(x) \right] \right)^{1/2}. \end{aligned} \quad (2.72)$$

Therefore, using the bound from Proposition 2.2.1,

$$\mathbb{E} [W_p^p[\mu^N, \mu^M]] \leq C_{p, R, T} W_p^p[\mu_0^N, \mu_0^M] + C_{p, K} \mathbb{P} \left(\tau_R^{N, M} < T \right)^{1/2}.$$

Now using Corollary 2.3.1, Markov's inequality leads to

$$\mathbb{P} \left(\tau_R^{N, M} < T \right) \leq \mathbb{P} \left(\sup_{t \in [0, T]} \sup_{x \in K} \left(|X_t^{\mu^N}(x)| + |X_t^{\mu^M}(x)| \right) \geq R \right) \leq C_{p, K} R^{-2}.$$

We conclude that

$$\limsup_{N, M \rightarrow \infty} \mathbb{E} [W_p^p[\mu^N, \mu^M]] \leq C_{p, K} R^{-1} \xrightarrow{R \rightarrow 0} 0.$$

This shows that $(\mu^N)_{N \geq 1}$ is a Cauchy sequence, hence converges in $L^p(\Omega; \mathcal{P}_p(\mathcal{C}))$ to some μ .

Let us now prove that the limiting measure μ is also of the transport form and therefore satisfies the SPDE (2.10) with initial data μ_0 (according to Proposition 2.3.1).

Denoting $\nu = (X^\mu)^* \mu_0 \in \mathcal{P}_p(\mathcal{C})$, we may slightly adapt the proof of Proposition 2.3.2 to obtain an estimate of $W_p^p[\mu^N, \nu]$. Indeed, introducing an optimal plan $\pi^N \in \Pi(\mu_0^N, \mu_0)$ so that

$$W_p^p[\mu_0^N, \mu_0] = \int_{K^2} |x - y|^p d\pi^N(x, y),$$

we have this time

$$W_p^p[\mu^N, \nu] \leq I_T^* := \int_{K^2} \sup_{t \in [0, T]} \left| X_t^{\mu^N}(x) - X_t^\mu(y) \right|^p d\pi^N(x, y) \quad (2.73)$$

and we are naturally led to study $\eta_t(x, y) = X_t^{\mu^N}(x) - X_t^\mu(y)$. Setting $p = 2$ for simplicity, calculations give, as in (2.44),

$$\begin{aligned} d|\eta_{t \wedge \tau_R^N}(x, y)|^2 &\leq C_R \left(|\eta_{t \wedge \tau_R^N}(x, y)|^2 + W_2^2[\mu_{t \wedge \tau_R^N}^N, \mu_{t \wedge \tau_R^N}] \right) dt + dM_{t \wedge \tau_R^N}(x, y) \\ &\leq C_R \left(|\eta_{t \wedge \tau_R^N}(x, y)|^2 + W_2^2[\mu^N, \mu] \right) dt + dM_{t \wedge \tau_R^N}(x, y) \end{aligned}$$

where the stopping time is defined as

$$\tau_R^N = \inf \left\{ t \geq 0, \sup_{x \in K} \left(|X_t^{\mu^N}(x)| + |X_t^\mu(x)| \right) \geq R \right\} \wedge T.$$

We can now carry on as in the proof of Proposition 2.3.2 to obtain the estimate

$$\mathbb{E} \left[I_{\tau_R^N}^* \right] \leq C_{p,R,T} \left(W_p^p[\mu_0^N, \mu_0] + \mathbb{E} \left[W_p^p[\mu^N, \mu] \right] \right) \leq C'_{p,R,T} \mathbb{E} \left[W_p^p[\mu^N, \mu] \right]. \quad (2.74)$$

From (2.73) and (2.74), using the same method as in (2.72), since μ_0^N, μ_0 are supported in $K \subset \mathbb{R}^d$, we are led to

$$\mathbb{E} \left[W_p^p[\mu^N, \nu] \right] \leq C_{p,R,T} \mathbb{E} \left[W_p^p[\mu^N, \mu] \right] + C_{p,K} \mathbb{P} \left(\tau_R^N < T \right)^{1/2}.$$

Sending $N, R \rightarrow \infty$ in the same fashion as before, we conclude that $\mathbb{E} \left[W_p^p[\mu, \nu] \right] = 0$, that is $\mu = \nu$ almost surely : therefore μ is of the transport form.

Lastly, the uniqueness of a solution of the transport form for μ_0 supported in $K \subset \mathbb{R}^d$ is again obtained in the same way : given $\mu = (X^\mu)^* \mu_0$ and $\tilde{\mu} = (X^{\tilde{\mu}})^* \mu_0$ two solutions of the transport form, we may apply Proposition 2.3.2 (or more precisely equation (2.48)) and the method used in (2.72) to obtain

$$\mathbb{E} \left[W_p^p[\mu, \tilde{\mu}] \right] \leq C_{p,K} \mathbb{P} \left(\tau_R < T \right)^{1/2}$$

with

$$\tau_R = \inf \left\{ t \geq 0, \sup_{x \in K} \left(|X_t^\mu(x)| + |X_t^{\tilde{\mu}}(x)| \right) \geq R \right\} \wedge T.$$

We may then send $R \rightarrow \infty$ to get $\mu = \tilde{\mu}$ almost surely. This concludes the proof of Theorem 4.

2.3.5 Propagation of chaos.

We now prove Theorem 5. To this purpose, we naturally extend the definition of the stochastic characteristics introduced in (2.38) in the following way.

Given some random measure $\mu : \Omega \rightarrow \mathcal{P}_2(\mathcal{C})$ with $\mathbb{E} \left[\int \|x\|_\infty^2 d\mu(x) \right] < \infty$ and an \mathcal{F}_0 -measurable random variable $\xi_0 : \Omega \rightarrow \mathbb{R}^d$, we shall denote by $X^\mu(\xi_0) = (X_t^\mu(\xi_0))_{t \in [0, T]}$ the solution of

$$\begin{cases} dX_t^\mu(\xi_0) = \left(B[\mu_t] + S[\mu_t] \right) (X_t^\mu(\xi_0)) dt + C[\mu_t] (X_t^\mu(\xi_0)) d\beta_t, & t \in [0, T], \\ X_0^\mu(\xi_0) = \xi_0. \end{cases} \quad (2.75)$$

Let us start by establishing a link between the law of $X^\mu(\xi_0)$ given by (2.75) and the flow of characteristics defined in (2.38).

Proposition 2.3.4.

Let $\mu : \Omega \rightarrow \mathcal{P}_2(\mathcal{C})$ such that $\mathbb{E} \int \|x\|_\infty^2 d\mu(x) < \infty$.

Let $\xi_0 : \Omega \rightarrow \mathbb{R}^d$ be an \mathcal{F}_0 -measurable random variable with law μ_0 supported in some $K \subset \mathbb{R}^d$.

Then, letting $X^\mu(\xi_0)$ be defined by (2.75), for all $\phi \in C_b(\mathcal{C})$,

$$\mathbb{E} \left[\phi(X^\mu(\xi_0)) \middle| \mathcal{F}_T^\beta \right] = \int_{\mathbb{R}^d} \phi(X^\mu(x)) d\mu_0(x) \quad a.s.,$$

where $(X^\mu(x))_{x \in \mathbb{R}^d}$ is defined in (2.38).

Proof. Let us first consider the case where ξ_0 takes a finite number of values : let us introduce a partition $(A_k)_{k \in \{1, \dots, n\}} \in (\mathcal{F}_0)^n$ of Ω and $(x_k)_{k \in \{1, \dots, n\}} \in (\mathbb{R}^d)^n$ such that

$$\xi_0 = \sum_{k=1}^n x_k \mathbb{1}_{A_k}. \quad (2.76)$$

One can easily prove that, in this case,

$$X_t^\mu(\xi_0) = \sum_{k=1}^n X_t^\mu(x_k) \mathbb{1}_{A_k}, \quad t \in [0, T], \quad a.s.$$

by checking that the left and right hand side both satisfy problem (2.75), for which path-wise uniqueness is established. Given $\phi \in C_b(\mathcal{C})$, it follows that

$$\begin{aligned} \mathbb{E} \left[\phi(X^\mu(\xi_0)) \middle| \mathcal{F}_T^\beta \right] &= \sum_{k=1}^n \mathbb{E} \left[\phi(X^\mu(x_k)) \mathbb{1}_{A_k} \middle| \mathcal{F}_T^\beta \right] \\ &= \sum_{k=1}^n \phi(X^\mu(x_k)) \mu_0(\{x_k\}) = \int_{\mathbb{R}^d} \phi(X^\mu(x)) d\mu_0(x), \end{aligned}$$

where we have used the facts that $X^\mu(x) \in \mathcal{C}$ is \mathcal{F}_T^β -measurable and \mathcal{F}_0 is independent of \mathcal{F}_T^β . Given a general random variable ξ_0 of law μ_0 , let us introduce a sequence $(\xi_0^k)_{k \geq 1}$ of random variables of the form (2.76) such that $\xi_0^k \rightarrow \xi_0$ a.s. Denoting by μ_0^k the law of ξ_0^k , since the almost sure convergence implies the convergence of the laws, we deduce that $\mu_0^k \rightarrow \mu_0$ weakly. For all $k \geq 1$,

$$\mathbb{E} \left[\phi(X^\mu(\xi_0^k)) \middle| \mathcal{F}_T^\beta \right] = \int_{\mathbb{R}^d} \phi(X^\mu(x)) d\mu_0^k(x) \text{ a.s.}$$

Recalling that the mapping $x \mapsto X^\mu(x) \in \mathcal{C}$ is almost-surely continuous (see Remark 2.3.1), the result is deduced by taking the limit in $L^1(\Omega)$ as k goes to infinity, using the dominated convergence theorem, \square

With definition (2.75) in mind, introducing the unique solution μ of (2.16) of the transport form (given by Theorem 4) the variables considered in Theorem 5 may be rewritten as

$$X^{i,N} = X^{\mu^N}(\xi_0^i), \quad X^i = X^\mu(\xi_0^i), \quad (2.77)$$

where the empirical measure is of the transport form $\mu^N = (X^{\mu^N})^* \mu_0^N$ with the random initial measure $\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{\xi_0^i}$. Note that the strong law of large numbers gives, for any $\psi \in C_b(\mathbb{R}^d)$,

$$\langle \psi, \mu_0^N \rangle = \frac{1}{N} \sum_{i=1}^N \psi(\xi_0^i) \xrightarrow{N \rightarrow \infty} \int \psi(x) d\mu_0(x) \text{ a.s.}$$

Since μ_0^N and μ_0 are supported in the compact set $K \subset \mathbb{R}^d$, we easily deduce that

$$\mu_0^N \rightarrow \mu_0 \text{ in } L^p(\Omega; \mathcal{P}_p(\mathbb{R}^d)).$$

From here, we may easily proceed as in the proof of Proposition 2.3.2 to obtain

$$\mathbb{E} \left[\sup_{t \in [0, \tau_R^N]} W_p^p[\mu_t^N, \mu_t] \right] \leq C_{p,R,T} \mathbb{E} [W_p^p[\mu_0^N, \mu_0]] \xrightarrow{N \rightarrow \infty} 0,$$

where the stopping time is defined as

$$\tau_R^N = \inf \left\{ t \geq 0, \sup_{x \in K} (|X_t^{\mu^N}(x)| + |X_t^\mu(x)|) \geq R \right\} \wedge T.$$

The arguments developed in section 2.3.4 then provide the first convergence in Theorem 5 :

$$\mathbb{E} [W_p^p[\mu^N, \mu]] \rightarrow 0.$$

The mean-field limit being established, the propagation of chaos announced in (5) will be deduced by exploiting the following symmetry property : using (a slightly tweaked version of) Proposition 2.3.4, one can write, for all $r \geq 1$ and $\phi \in C_b(\mathcal{C}^r)$,

$$\begin{aligned} \mathbb{E} \left[\phi(X^{1,N}, \dots, X^{r,N}) \middle| \mathcal{F}_T^\beta \right] &= \mathbb{E} \left[\phi(X^{\mu^N}(\xi_0^1), \dots, X^{\mu^N}(\xi_0^r)) \middle| \mathcal{F}_T^\beta \right] \\ &= \int_{(\mathbb{R}^d)^r} \phi(X^{\mu^N}(x_1), \dots, X^{\mu^N}(x_r)) d\mu_0^{\otimes r}(x_1, \dots, x_r) \text{ a.s.} \end{aligned}$$

since, by independence, $\mathcal{L}(\xi_0^1, \dots, \xi_0^r) = \mu_0^{\otimes r}$. In particular, this shows that

$$\forall \sigma \in \mathcal{S}_r, \quad \mathbb{E} \left[\phi(X^{\sigma(1),N}, \dots, X^{\sigma(r),N}) \middle| \mathcal{F}_T^\beta \right] = \mathbb{E} \left[\phi(X^{1,N}, \dots, X^{r,N}) \middle| \mathcal{F}_T^\beta \right] \text{ a.s.} \quad (2.78)$$

The convergence (5) can now be proved in the same way as in [CF14], Theorem 24. We detail the proof here for the sake of completeness : for simplicity, let us consider the case $r = 2$. For $\phi_1, \phi_2 \in C_b(\mathcal{C})$ with $|\phi_1|, |\phi_2| \leq M$, let us write

$$\mathbb{E} \left| \mathbb{E} \left[\phi_1(X^{1,N}) \phi_2(X^{2,N}) \middle| \mathcal{F}_T^\beta \right] - \langle \phi_1, \mu \rangle \langle \phi_2, \mu \rangle \right| \leq \mathbb{E}[A^N] + \mathbb{E}[B^N]$$

where

$$\begin{aligned} A^N &= \left| \mathbb{E} \left[\phi_1(X^{1,N}) \phi_2(X^{2,N}) \middle| \mathcal{F}_T^\beta \right] - \mathbb{E} \left[\langle \phi_1, \mu^N \rangle \langle \phi_2, \mu^N \rangle \middle| \mathcal{F}_T^\beta \right] \right|, \\ B^N &= \left| \mathbb{E} \left[\langle \phi_1, \mu^N \rangle \langle \phi_2, \mu^N \rangle \middle| \mathcal{F}_T^\beta \right] - \mathbb{E} \left[\langle \phi_1, \mu \rangle \langle \phi_2, \mu \rangle \middle| \mathcal{F}_T^\beta \right] \right|. \end{aligned}$$

On one hand, using the symmetry property (2.78), we may rewrite A^N as

$$\begin{aligned} A^N &= \left| \frac{1}{N^2 - N} \sum_{i \neq j} \mathbb{E} \left[\phi_1(X^{i,N}) \phi_2(X^{j,N}) \middle| \mathcal{F}_T^\beta \right] - \frac{1}{N^2} \sum_{i,j} \mathbb{E} \left[\phi_1(X^{i,N}) \phi_2(X^{j,N}) \middle| \mathcal{F}_T^\beta \right] \right| \\ &\leq \left(\frac{1}{N^2 - N} - \frac{1}{N^2} \right) (N^2 - N) M^2 + \frac{1}{N} M^2 = 2 \frac{M^2}{N} \rightarrow 0. \end{aligned}$$

On the other hand,

$$\mathbb{E}[B^N] \leq \mathbb{E} \left| \langle \phi_1, \mu^N \rangle \langle \phi_2, \mu^N \rangle - \langle \phi_1, \mu \rangle \langle \phi_2, \mu \rangle \right| \rightarrow 0$$

using the dominated convergence theorem, given that $\mu^N \rightarrow \mu$ in $\mathcal{P}_p(\mathcal{C})$ (hence in particular $\langle \phi, \mu^N \rangle \rightarrow \langle \phi, \mu \rangle$) in probability. This proves (5).

Furthermore, for all $\phi \in C_b(\mathcal{C})$, since $\mu = (X^\mu)^* \mu_0$ is of the transport form, Proposition 2.3.4 gives

$$\mathbb{E} \left[\phi(X^i) \middle| \mathcal{F}_T^\beta \right] = \mathbb{E} \left[\phi(X^\mu(\xi_0^i)) \middle| \mathcal{F}_T^\beta \right] = \int_{\mathbb{R}^d} \phi(X^\mu(x)) d\mu_0(x) = \int_{\mathcal{C}} \phi(x) d\mu(x) \text{ a.s.}$$

so that $\mu \in \mathcal{P}(\mathcal{C})$ is indeed a version of the conditional law $\mathcal{L}(X^i | \mathcal{F}_T^\beta)$.

Lastly, one may again use (a slightly tweaked version of) Proposition 2.3.4 to write

$$\mathbb{E} \|X^{i,N} - X^i\|_\infty^p = \mathbb{E} \|X^{\mu^N}(\xi_0^i) - X^\mu(\xi_0^i)\|_\infty^p = \mathbb{E} \int_K \sup_{t \in [0, T]} \left| X_t^{\mu^N}(x) - X_t^\mu(x) \right|^p d\mu_0(x).$$

Once again, the arguments developed in section 2.3.4 (see (2.73) and below) give the required convergence : we obtain $X^{i,N} \rightarrow X^i$ in $L^p(\Omega; \mathcal{C})$, which concludes the proof of Theorem 5.

3

Existence of martingale solutions for stochastic flocking models with local alignment

3.1 Introduction, main results

3.1.1 Collective motion with local alignment

The emergence of a consensus or ordered motion amongst a population of interacting agents has been drawing a fair amount of attention among the scientific community in recent years. This phenomenon, consistently observed in nature, from schooling fish to swarming bacteria, is usually referred to as flocking. One of the earliest and most commonly studied mathematical models describing this kind of behavior is the celebrated Cucker-Smale model, introduced in [CS07a, CS07b]. In this model, agents interact in a mean-field manner : for $1 \leq i \leq N$, denoting by $X_t^{i,N}, V_t^{i,N} \in \mathbb{R}^d$ the position and velocity of the i -th individual, the evolution of the system is given by

$$\begin{cases} \frac{d}{dt} X_t^{i,N} = V_t^{i,N}, \\ \frac{d}{dt} V_t^{i,N} = \frac{1}{N} \sum_{j=1}^N \psi(X_t^{i,N} - X_t^{j,N})(V_t^{j,N} - V_t^{i,N}), \end{cases}$$

where the weight function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^+$ is even, typically of the form

$$\psi(x - y) = \frac{\lambda}{(1 + |x - y|^2)^\gamma}, \quad \lambda, \gamma > 0.$$

Equivalently, one may consider the conservation equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (L^{CS}[f]f) = 0 \tag{3.1}$$

where the Cucker-Smale alignment term $L^{CS}[f]$ is given by the convolution

$$L^{CS}[f](x, v) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(x - y)(w - v)f(y, w)dydw. \tag{3.2}$$

Equation (3.1) is naturally associated to the particle system, since it is satisfied by the empirical measure

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{i,N}, V_t^{i,N})}$$

in the sense of distributions. In [MT11], Motsch and Tadmor brought to light several drawbacks regarding the physical relevance of the Cucker-Smale model when confronted to strongly non-homogeneous distributions of agents, due to the normalizing constant $\frac{1}{N}$ in the alignment force, which involves the whole group of individuals. To remedy these issues, they proposed a new model where the influence between two agents is normalized by the total influence :

$$\begin{cases} \frac{d}{dt} X_t^{i,N} = V_t^{i,N}, \\ \frac{d}{dt} V_t^{i,N} = \frac{1}{\sum_{j=1}^N \phi(X_t^{i,N} - X_t^{j,N})} \sum_{j=1}^N \phi(X_t^{i,N} - X_t^{j,N})(V_t^{j,N} - V_t^{i,N}). \end{cases}$$

Considering some weight function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^+$ with compact support, we may naturally consider a hybrid model, letting the Cucker-Smale forcing dictate the long-range interaction and the Motsch-Tadmor term dictate the short-range interaction :

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot ((L^{CS}[f] + L^{MT}[f])f) = 0 \quad (3.3)$$

where the $L^{MT}[f]$ is given by

$$L^{MT}[f](x, v) = \frac{\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x-y)(w-v)f(y, w)dydw}{\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x-y)f(y, w)dydw}. \quad (3.4)$$

Note that the Motsch-Tadmor forcing term can also be written

$$L^{MT}[f](x, v) = u[f](x) - v,$$

expressing the alignment of the speed with the local average velocity $u[f]$, defined as

$$u[f](x) = \frac{\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x-y)w f(y, w)dydw}{\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x-y)f(y, w)dydw}.$$

As suggested in [KMT13], we may also consider (3.3) in the singular limit where the weight function ϕ governing the short-range interaction converges to the Dirac function δ_0 , leading to

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot ((L^{CS}[f] + L^{SLA}[f])f) = 0. \quad (3.5)$$

In (3.5), the Strong Local Alignment term $L^{SLA}[f]$ is given by

$$L^{SLA}[f](x, v) = u_0[f](x) - v, \quad (3.6)$$

where the local velocity $u_0[f]$ is given by

$$u_0[f](x) = \frac{\int_{\mathbb{R}^d} w f(x, w)dw}{\int_{\mathbb{R}^d} f(x, w)dw}.$$

The existence of solutions to the kinetic equations (3.3) and (3.5) has been established in [KMT13]. Moreover, in [KMT12], the authors rigorously explore the limit $\phi \rightarrow \delta_0$: considering some $\phi_1 \in C_c(\mathbb{R}^d)$ and weight functions of the form $\phi^r(x) = r^{-d}\phi_1(x/r)$, solutions $(f^r)_{r \geq 0}$ of (3.3) converge (up to some subsequence) to a solution f of (3.5).

In order to take into account random phenomena emerging from the environment, or unpredictable interactions between the agents, it is rather natural to perturb the deterministic equations (3.3) and (3.5) with some noise, driven by a Wiener process $dW(z) = \sum_k K_k[f](z)d\beta_t^k$, leading to the stochastic conservation equation

$$df_t + \left[v \cdot \nabla_x f_t + \nabla_v \cdot ((L[f_t]f_t) \right] dt + \sum_k \nabla_v \cdot (K_k[f_t]f_t) \circ d\beta_t^k = 0$$

with $L[f] = L^{CS}[f] + L^{MT}[f]$ or $L[f] = L^{CS}[f] + L^{SLA}[f]$. Here, for simplicity purposes, we choose to only consider a "one-dimensional" noise driven by a real valued Brownian motion $\beta = (\beta_t)_{t \geq 0}$, leading to equations

$$df_t + \left[v \cdot \nabla_x f_t + \nabla_v \cdot ((L^{CS}[f_t] + L^{MT}[f_t])f_t) \right] dt + \nabla_v \cdot (K[f_t]f_t) \circ d\beta_t = 0 \quad (3.7)$$

and

$$df_t + \left[v \cdot \nabla_x f_t + \nabla_v \cdot ((L^{CS}[f_t] + L^{SLA}[f_t])f_t) \right] dt + \nabla_v \cdot (K[f_t]f_t) \circ d\beta_t = 0. \quad (3.8)$$

The methods developed in the present paper shall not rely on this particular form, so that the results may be easily generalized to SPDEs with multiple Brownian motions. Note that these

stochastic conservation equations are written in Stratonovitch form, since it is the most physically relevant form. As in the deterministic case, equation (3.7) is naturally associated to the stochastic particle system

$$\begin{cases} dX_t^{i,N} = V_t^{i,N} dt, \\ dV_t^{i,N} = L[\mu_t^N](X_t^{i,N}, V_t^{i,N}) dt + K[\mu_t^N](X_t^{i,N}, V_t^{i,N}) \circ d\beta_t \end{cases} \quad (3.9)$$

where $\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{(X_t^{i,N}, V_t^{i,N})}$ and $L[\mu] = L^{CS}[\mu] + L^{MT}[\mu]$.

The mean-field convergence of (3.9) to the corresponding limiting SPDE has been studied in the litterature in the case of the Cucker-Smale interaction, that is with $L[\mu] = L^{CS}[\mu]$. The diffusion coefficient $K[\mu](x, v) = D(v - v_c)$ for some constant $v_c \in \mathbb{R}^d$ is considered in [AH10]; the coefficient $K[\mu](x, v) = \sqrt{2\sigma}(v - \bar{v})$, where $\bar{v} = \int w d\mu(y, w)$, is looked upon in [CS17] and [HJR19]; some more general (non linear) diffusion coefficients are considered in [Ros19].

As for the Motsch-Tadmor model, that is for $L[\mu] = L^{MT}[\mu]$, the flocking phenomenon for the particle system (3.9) (alignment of speeds, distance between the individuals bounded over time) is studied in [LX15] in the case of a multiplicative noise $K[\mu](x, v) = Dv$. However, even in the deterministic case, due to the singular ratio involved in the non-linear term $L^{MT}[\mu]$, the mean-field limit of the Motsch-Tadmor particle system is a very delicate question (as suggested in [MT11] and [MT14]) to which the authors could not find a proper answer in the litterature. It should also be noted that the strong local alignment term $L^{SLA}[f]$ given by (3.6) is ill-defined when f is a general measure, so that the particle system associated with equation (3.8) cannot in fact be written.

In the present work, we shall consider a diffusion coefficient of the form

$$K[f](x, v) = F(x) + \int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{\psi}(x - y)(w - v)f(y, w) dy dw \quad (3.10)$$

which corresponds to some random environmental forcing $F(x) \circ d\beta_t$, as well as a random perturbation $\psi \leftarrow \psi + \tilde{\psi} \circ d\beta_t$ of the weight function involved in the Cucker-Smale alignment term (3.2) (as considered in [CS17] and [Ros19] in the case of the Cucker-Smale model only). Note that the choice $F = 0$ and $\tilde{\psi} = \sqrt{2\sigma}$ leads in particular to the simpler coefficient $K[\mu](x, v) = \sqrt{2\sigma}(v - \bar{v})$. The arguments developed in the present work also easily apply to the case $K[\mu](x, v) = Dv$ looked upon in [LX15].

In this paper, we extend the work developed in [KMT13] and [KMT12] in the deterministic case to establish the existence of martingale solutions (see Definition 3.1.1 below) for the stochastic conservation equations (3.7) and (3.8). To this intent, we start by regularizing the coefficients : in section 3.2, we prove the existence of a unique solution of equation (3.7) with regularized coefficients, which is naturally constructed as the mean-field limit of the corresponding stochastic particle system. Then, in section 3.3, we prove the tightness of these approximate solutions with respect to the regularizing parameter, and rigorously pass to the limit in the martingale problem associated with (3.7).

3.1.2 Assumptions and main results

The weight functions $\psi, \tilde{\psi} : \mathbb{R}^d \rightarrow \mathbb{R}^+$ involved in (3.2) and (3.10) are assumed to satisfy

$$|\psi(x)| + |\tilde{\psi}(x)| \lesssim 1, \quad |\partial_x^\alpha \psi(x)| + |\partial_x^\alpha \tilde{\psi}(x)| \lesssim 1 \text{ for } 1 \leq |\alpha| \leq 4. \quad (3.11)$$

The weight function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^+$ involved in (3.4) is assumed to be smooth and compactly supported around zero : $\phi \in C^\infty(\mathbb{R}^d)$ and for some $0 < r_1 < r_2 < \infty$,

$$\inf_{x \in B(0, r_1)} \phi(x) > 0, \quad \text{Supp}(\phi) \subset B(0, r_2). \quad (3.12)$$

The forcing F involved in (3.10) is assumed to be smooth and sublinear :

$$|F(x)| \lesssim 1 + |x|.$$

Simple calculations show that the proper Itô form associated with SPDE (3.7) is

$$df_t + \nabla_v \cdot \left(\left(L^{CS}[f_t] + L^{MT}[f_t] + S[f_t] \right) f_t \right) dt + \nabla_v \cdot (K[f_t] f_t) d\beta_t = \frac{1}{2} \nabla_v^2 (K[f_t] K[f_t]^T f_t) dt$$

where we have used the notation

$$\nabla_v^2(K[f]K[f]^T f) = \sum_{1 \leq i, j \leq d} \partial_{v_i v_j}^2 [K[f]^i K[f]^j f]$$

and the additional drift forcing term $S[f]$ is given by

$$S[f](z) = \frac{1}{2} \int_{\mathbb{R}^{2d}} \tilde{\psi}(x-y) \left(K[f](y, w) - K[f](x, v) \right) f(y, w) dy dw.$$

This motivates the following definition.

Definition 3.1.1. Let $T > 0$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}, \beta)$ be a filtered probability space equipped with an (\mathcal{F}_t) -brownian motion β . Let $f_0 : \mathbb{R}^{2d} \rightarrow \mathbb{R}_+$ with $\int_{\mathbb{R}^{2d}} f_0(z) dz = 1$.

A process $f \in L^\infty(\Omega; L^\infty([0, T]; L^1(\mathbb{R}^{2d})))$ with

$$\left[\int_{\mathbb{R}^{2d}} |f(\omega)(t, z)| dz \leq 1, \quad dt\text{-a.e.} \right], \quad \mathbb{P}\text{-a.s.} \quad (3.13)$$

satisfying the estimate

$$\mathbb{E} \left[\int_0^T \int_{\mathbb{R}^{2d}} (1 + |v|^2) |f(t, z)| dz dt \right] < \infty \quad (3.14)$$

is said to be a solution of (3.7) on (Ω, β) (with initial value f_0) when, for any test function $\Psi \in C_c^\infty(\mathbb{R}^{2d})$, the process $(\langle f(t), \Psi \rangle)_{0 \leq t \leq T}$ is adapted with a continuous version and satisfies

$$\langle f(t), \Psi \rangle = \langle f_0, \Psi \rangle + \int_0^t \langle \mathcal{L}[f(s)] \Psi, f(s) \rangle ds + \int_0^t \langle K[f(s)] \cdot \nabla_v \Psi, f(s) \rangle d\beta_s, \quad t \in [0, T]$$

where $\mathcal{L}[f]$ denotes the second order operator

$$\mathcal{L}[f] \Psi = v \cdot \nabla_x \Psi + \left(L^{CS}[f] + L^{MT}[f] + S[f] \right) \cdot \nabla_v \Psi + \frac{1}{2} \sum_{1 \leq i, j \leq d} K[f]^i K[f]^j \partial_{v_i v_j}^2 \Psi. \quad (3.15)$$

If there exists some probability tuple $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{0 \leq t \leq T}, \bar{\mathbb{P}}, \bar{\beta})$ and a solution \bar{f} of (3.7) on $(\bar{\Omega}, \bar{\beta})$, we say that equation (3.7) has a martingale solution (in the sense of [DPZ14], Chapter 8).

Estimates (3.13) and (3.14) are quite natural since we expect solutions of (3.7) to be densities. One can easily check that these estimates guarantee that the process $(\langle f(t), \Psi \rangle)_{t \geq 0}$ is well defined, the stochastic integral being a square integrable martingale. Solutions of equation (3.8) are defined similarly. We now state our main results.

Theorem 6 (Stochastic Motsch-Tadmor flocking).

Let $f_0 : \mathbb{R}^{2d} \rightarrow \mathbb{R}_+$ with $\int_{\mathbb{R}^{2d}} f_0(z) dz = 1$ such that, for some $\delta > 1$ and $\theta \in (0, 1)$,

$$\int_{\mathbb{R}^{2d}} |f_0(z)|^p dz + \int_{\mathbb{R}^{2d}} (|x|^\delta + |v|^k) f_0(z) dz < \infty, \quad p = 1 + \frac{1}{\theta}, \quad k > \frac{\max(d+2, 4)}{1-\theta}. \quad (3.16)$$

Then there exists a martingale solution f of equation (3.7) with initial data f_0 .

We also prove the existence of a martingale solution of (3.8), which can be constructed as a weak limit of solutions of (3.7) as the function ϕ involved in the Motsch-Tadmor alignment term (3.4) properly approaches the Dirac function δ_0 , similarly to the result established in [KMT12].

Theorem 7 (Stochastic flocking with strong local alignment).

Let $\phi_1 \in C_c^\infty(\mathbb{R}^d)$ satisfy (3.12). Let us consider the sequence of functions $(\phi_r)_{r>0}$ given by

$$\forall x \in \mathbb{R}^d, \quad \phi_r(x) = r^d \phi_1(x/r).$$

Assuming (3.16), the martingale solution f^r of equation (3.7) with $\phi = \phi_r$ constructed in Theorem 6 satisfies, along some subsequence $r_n \rightarrow 0$,

$$f^{r_n} \rightarrow f \text{ in law, in } C([0, T]; H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d})) \text{ for all } \sigma > 0,$$

where f defines a martingale solution of (3.8). The weighted Sobolev space $H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d})$, with $W^{-1}(z) = (1 + |z|)^{-1}$ is introduced in (3.78) below.

Note that, although f only converges weakly in space, the use of a stochastic averaging lemma (developed in Proposition 3.3.4 below) will guarantee that integrated quantities of the form

$$\rho_\varphi = \int_{\mathbb{R}^d} \varphi(v) f dv$$

converge in the strong space $L^2([0, T]; L^2(\mathbb{R}^d))$. This allows to properly pass to the limit in the non-linear equation (3.7).

3.2 Regularized equation

In this section, we prove existence and uniqueness of a solution of equation (3.7) with regularized coefficients. This solution will be naturally obtained as the mean-field limit of the corresponding particle system.

For $R > 0$ let us introduce smooth, compactly supported truncation functions

$$\chi_R \in C_c^\infty(\mathbb{R}^d; \mathbb{R}), \quad \theta_R \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$$

satisfying

$$\begin{cases} \chi_R(x) = 1 & \text{if } |x| \leq R, & |\chi_R(x)| \leq 1, \\ \theta_R(v) = v & \text{if } |v| \leq R, & |\theta_R(v)| \leq |v|, \\ |\nabla_v \cdot \theta_R(v)| \lesssim 1 & \text{uniformly in } R > 0. \end{cases} \quad (3.17)$$

We may then introduce the following regularized coefficients :

$$\left\{ \begin{array}{l} L_R^{CS}[\mu](z) = \int \chi_R(x-y) \psi(x-y) \theta_R(w-v) d\mu(y, w), \\ u_R[\mu](x) = \frac{\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x-y) \theta_R(w) d\mu(y, w)}{R^{-1} + \int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x-y) d\mu(y, w)}, \\ L_R^{MT}[\mu](z) = u_R[\mu](x) - v, \\ K_R[\mu](z) = \chi_R(x) F(x) + \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_R(x-y) \tilde{\psi}(x-y) \theta_R(w-v) d\mu(y, w), \\ S_R[\mu](z) = \frac{1}{2} \int_{\mathbb{R}^{2d}} \tilde{\psi}(x-y) \left(K_R[\mu](y, w) - K_R[\mu](x, v) \right) d\mu(y, w). \end{array} \right. \quad (3.18)$$

Simple calculations show that, for fixed R , these regularized coefficients are globally Lipschitz continuous in the following sense : for all $z, z' \in \mathbb{R}^{2d}$, $\mu, \nu \in \mathcal{P}(\mathbb{R}^{2d})$,

$$\left| L_R^{CS}[\mu](z) - L_R^{CS}[\nu](z') \right| + \left| L_R^{MT}[\mu](z) - L_R^{MT}[\nu](z') \right| \lesssim |z - z'| + W_1[\mu, \nu], \quad (3.19)$$

$$\left| K_R[\mu](z) - K_R[\nu](z') \right| + \left| S_R[\mu](z) - S_R[\nu](z') \right| \lesssim |z - z'| + W_1[\mu, \nu]$$

where the constants involved in \lesssim in (3.19) depend on R , and W_1 denotes the Wasserstein distance. The assumptions made also guarantee the uniform sub-linearity of some coefficients : for all $z \in \mathbb{R}^{2d}$, $\mu \in \mathcal{P}(\mathbb{R}^{2d})$,

$$\left| L_R^{CS}[\mu](z) \right| + \left| K_R[\mu](z) \right| + \left| S_R[\mu](z) \right| \lesssim 1 + |z| + \int |z'| d\mu(z') \quad (3.20)$$

where the constant involved in \lesssim in (3.20) does not depend on R .

3.2.1 Mean-field limit of the associated particle system

We may now consider the associated mean-field particle system on $Z^{i,N} = (X^{i,N}, V^{i,N}) \in \mathbb{R}^{2d}$, $i \in \{1, \dots, N\}$:

$$\begin{cases} dX_t^{i,N} = V_t^{i,N} dt \\ dV_t^{i,N} = \left(L_R^{CS}[\mu_t^N] + L_R^{MT}[\mu_t^N] + S_R[\mu_t^N] \right) (Z_t^{i,N}) dt + K_R[\mu_t^N](Z_t^{i,N}) d\beta_t, \end{cases} \quad (3.21)$$

where μ_t^N denotes the empirical measure

$$\mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{Z_t^{i,N}}.$$

From the sub-linearity of the coefficients, we easily deduce the following result.

Proposition 3.2.1. *For any $T > 0$ and $z_0^{i,N} \in \mathbb{R}^{2d}$, $i \in \{1, \dots, N\}$, the SDE system (3.21) with initial condition $Z_0^{i,N} = z_0^{i,N}$ has a unique global solution $(Z_t^{i,N})_{t \in [0, T]}^{i \in \{1, \dots, N\}}$ which satisfies the following estimates : for any $p \geq 1$,*

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \int_{\mathbb{R}^{2d}} |z|^p d\mu_t^N(z) \right] &\lesssim 1 + \int_{\mathbb{R}^{2d}} |z|^p d\mu_0^N(z), \\ \mathbb{E} \left[\sup_{t \in [0, T]} |Z_t^{i,N}|^p \right] &\lesssim 1 + |z_0^{i,N}|^p + \int_{\mathbb{R}^{2d}} |z|^p d\mu_0^N(z). \end{aligned}$$

The constants involved in \lesssim depend on R, p and T only.

Proof. The coefficients of (3.21) being locally Lipschitz-continuous, the local existence and uniqueness of solutions is guaranteed. The estimates of Proposition 3.2.1 should first be established with the stopping time

$$\tau_M = \inf \left\{ t \geq 0, \max_{1 \leq i \leq N} |Z_t^{i,N}| \geq M \right\} \wedge T,$$

which should then be sent to T , as M goes to infinity. For the sake of simplicity, we omit this stopping time in the following. Given the sub-linearity of the coefficients, Itô's formula gives

$$|Z_t^{i,N}|^p = |z_0^{i,N}|^p + \int_0^t A_s^i ds + \int_0^t B_s^i d\beta_s \quad (3.22)$$

with

$$A_t^i \lesssim 1 + |Z_t^{i,N}|^p + \int |z|^p d\mu_t^N, \quad B_t^i = p |Z_t^{i,N}|^{p-2} Z_t^{i,N} \cdot K_R[\mu_t^N](Z_t^{i,N}).$$

Averaging (3.22) over i , we are led to

$$\int |z|^p d\mu_t^N = \int |z|^p d\mu_0^N + \int_0^t \left(\frac{1}{N} \sum_{i=1}^N A_s^i \right) ds + \int_0^t \left(\frac{1}{N} \sum_{i=1}^N B_s^i \right) d\beta_s \quad (3.23)$$

from which we easily deduce, for all $0 \leq r \leq T$

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, r]} \int |z|^p d\mu_t^N \right] &\lesssim 1 + \mathbb{E} \left[\int |z|^p d\mu_0^N \right] + \int_0^r \mathbb{E} \left[\sup_{t \in [0, s]} \int |z|^p d\mu_t^N \right] ds \\ &\quad + \mathbb{E} \left[\sup_{t \in [0, r]} \int_0^t \left(\frac{1}{N} \sum_{i=1}^N B_s^i \right) d\beta_s \right]. \end{aligned}$$

Burkholder-Davis-Gundy's inequality (from [BDG72]) gives

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, r]} \int_0^t \left(\frac{1}{N} \sum_{i=1}^N B_s^i \right) d\beta_s \right] &\lesssim \mathbb{E} \left[\left(\int_0^r \left| \frac{1}{N} \sum_{i=1}^N B_s^i \right|^2 ds \right)^{1/2} \right] \lesssim \mathbb{E} \left[\left(\int_0^r \left| 1 + \int |z|^p d\mu_s^N \right|^2 ds \right)^{1/2} \right] \\ &\lesssim 1 + \mathbb{E} \left[\left(\sup_{t \in [0, r]} \int |z|^p d\mu_t^N \right)^{1/2} \left(\int_0^r \int |z|^p d\mu_s^N ds \right)^{1/2} \right] \\ &\lesssim 1 + \mathbb{E} \left[\sup_{t \in [0, r]} \int |z|^p d\mu_t^N \right]^{1/2} \int_0^r \mathbb{E} \left[\sup_{t \in [0, s]} \int |z|^p d\mu_t^N \right]^{1/2} ds \end{aligned}$$

so that we may come back to (3.23) and get

$$\mathbb{E} \left[\sup_{t \in [0, r]} \int |z|^p d\mu_t^N \right] \lesssim 1 + \mathbb{E} \left[\int |z|^p d\mu_0^N \right] + \int_0^r \mathbb{E} \left[\sup_{t \in [0, s]} \int |z|^p d\mu_t^N \right] ds.$$

We may now apply Gronwall's lemma to derive the first estimate. Coming back to (3.22), a similar reasoning leads to the second estimate. \square

Let us now consider the space of trajectories

$$\mathcal{C} = C([0, T]; \mathbb{R}^{2d}), \quad \|z\|_\infty = \sup_{t \in [0, T]} |z_t|$$

and view the empirical measure as a (random) probability over \mathcal{C} :

$$\mu^N = \frac{1}{N} \sum_{i=1}^N \delta_{Z^{i, N}} \in \mathcal{P}(\mathcal{C}).$$

More precisely, for $p \geq 1$, we introduce the Wasserstein space

$$\mathcal{P}_p(\mathcal{C}) = \left\{ \mu \in \mathcal{P}(\mathcal{C}), \int_{z \in \mathcal{C}} \|z\|_\infty^p d\mu < \infty \right\}$$

equipped with the usual distance

$$W_p[\mu, \nu] = \inf_{\pi \in \Pi(\mu, \nu)} \left(\int_{(z_1, z_2) \in \mathcal{C}^2} \|z_1 - z_2\|_\infty^p d\pi(z_1, z_2) \right)^{1/p}$$

where

$$\Pi(\mu, \nu) = \left\{ \pi \in \mathcal{P}(\mathcal{C}^2), \int_{z_2 \in \mathcal{C}} \pi(dz_1, dz_2) = \mu(dz_1), \int_{z_1 \in \mathcal{C}} \pi(dz_1, dz_2) = \nu(dz_2) \right\}.$$

We may now state the following mean-field limit result.

Proposition 3.2.2. *Let $p > 1$. As $N \rightarrow \infty$, provided that $\mu_0^N \rightarrow \mu_0$ in $\mathcal{P}_p(\mathbb{R}^{2d})$, we have $\mu^N \rightarrow \mu$ in $L^p(\Omega; \mathcal{P}_p(\mathcal{C}))$ where μ solves the regularized SPDE*

$$\begin{aligned} d\mu_t + v \cdot \nabla_x \mu_t dt + \nabla_v \cdot \left(\left(L_R^{CS}[\mu_t] + L_R^{MT}[\mu_t] + S_R[\mu_t] \right) \mu_t \right) dt + \nabla_v \cdot (K_R[\mu_t] \mu_t) d\beta_t \\ = \frac{1}{2} \nabla_v \cdot \left(\nabla_v \cdot (K_R[\mu_t] K_R[\mu_t]^T \mu_t) \right) dt \end{aligned} \quad (3.24)$$

in the following sense : denoting the operator

$$\mathcal{L}_R[\mu] \Psi = v \cdot \nabla_x \Psi + \left(L_R^{CS}[\mu] + L_R^{MT}[\mu] + S_R[\mu] \right) \cdot \nabla_v \Psi + \frac{1}{2} \sum_{1 \leq i, j \leq d} K_R[\mu]^i K_R[\mu]^j \partial_{v_i v_j}^2 \Psi \quad (3.25)$$

we have, for any test function $\Psi \in C_c^\infty(\mathbb{R}^{2d})$,

$$\langle \mu_t, \Psi \rangle = \langle \mu_0, \Psi \rangle + \int_0^t \langle \mathcal{L}_R[\mu_s] \Psi, \mu_s \rangle ds + \int_0^t \langle K_R[\mu_s] \cdot \nabla_v \Psi, \mu_s \rangle d\beta_s, \quad t \in [0, T].$$

More precisely, μ is the unique element of $L^2(\Omega; \mathcal{P}_p(\mathcal{C}))$ given by the push-forward measure of the initial data by the non-linear characteristics :

$$\mu = (Z^\mu)^* \mu_0 \text{ in } \mathcal{P}(\mathcal{C}), \quad a.s \quad (3.26)$$

where $Z^\mu : z \in \mathbb{R}^{2d} \mapsto (X_t^\mu(z), V_t^\mu(z))_{t \in [0, T]} \in \mathcal{C}$ is the flow associated with the SDE

$$\begin{cases} dX_t^\mu(z) = V_t^\mu(z) dt, \\ dV_t^\mu(z) = \left(L_R^{CS}[\mu_t] + L_R^{MT}[\mu_t] + S_R[\mu_t] \right) (Z_t^\mu(z)) dt + K_R[\mu_t] (Z_t^\mu(z)) d\beta_t, \\ Z_0^\mu(z) = z. \end{cases} \quad (3.27)$$

Démonstration. For simplicity, let us consider the case $p = 2$. We start by noticing that, for fixed $N \geq 1$, the empirical measure μ^N naturally satisfies the fixed-point-like equation (3.26). For all $N, M \geq 1$, introducing an optimal plan $\pi \in \Pi(\mu_0^N, \mu_0^M)$ (one may refer to [Vil08] for details) such that

$$W_2^2[\mu_0^N, \mu_0^M] = \int_{(z_1, z_2) \in (\mathbb{R}^{2d})^2} |z_1 - z_2|^2 d\pi(z_1, z_2),$$

it follows that

$$W_2^2[\mu^N, \mu^M] \leq \int_{(z_1, z_2) \in (\mathbb{R}^{2d})^2} \|Z^{\mu^N}(z_1) - Z^{\mu^M}(z_2)\|_\infty^2 d\pi(z_1, z_2) =: J_T^{N,M} \quad (3.28)$$

and we may simply re-write

$$J_T^{N,M} = \sum_{1 \leq i \leq N} \sum_{1 \leq j \leq M} \sup_{t \in [0, T]} \left| Z_t^{i,N} - Z_t^{j,M} \right|^2 \pi(\{z_0^{i,N}, z_0^{j,M}\}) \quad (3.29)$$

where $Z^{i,N}$ is the solution of (3.21). Itô's formula easily leads to

$$d \left| Z_t^{i,N} - Z_t^{j,M} \right|^2 = \left(\zeta_t^1 + \zeta_t^2 + \zeta_t^3 + \zeta_t^4 \right) dt + \zeta_t^5 d\beta_t$$

where

$$\begin{aligned} \zeta_t^1 &= 2 \left(Z_t^{i,N} - Z_t^{j,M} \right) \cdot \left(L_R^{CS}[\mu_t^N](Z_t^{i,N}) - L_R^{CS}[\mu_t^M](Z_t^{j,M}) \right), \\ \zeta_t^2 &= 2 \left(Z_t^{i,N} - Z_t^{j,M} \right) \cdot \left(L_R^{MT}[\mu_t^N](Z_t^{i,N}) - L_R^{MT}[\mu_t^M](Z_t^{j,M}) \right), \\ \zeta_t^3 &= 2 \left(Z_t^{i,N} - Z_t^{j,M} \right) \cdot \left(S_R[\mu_t^N](Z_t^{i,N}) - S_R[\mu_t^M](Z_t^{j,M}) \right), \\ \zeta_t^4 &= \left| K_R[\mu_t^N](Z_t^{i,N}) - K_R[\mu_t^M](Z_t^{j,M}) \right|^2, \\ \zeta_t^5 &= 2 \left(Z_t^{i,N} - Z_t^{j,M} \right) \cdot \left(K_R[\mu_t^N](Z_t^{i,N}) - K_R[\mu_t^M](Z_t^{j,M}) \right). \end{aligned}$$

Using the Lipschitz estimate (3.19), we deduce (for fixed $R > 0$)

$$\begin{aligned} d \left| Z_t^{i,N} - Z_t^{j,M} \right|^2 &\lesssim \left(\left| Z_t^{i,N} - Z_t^{j,M} \right|^2 + W_1[\mu_t^N, \mu_t^M]^2 \right) dt + \zeta_t^5 d\beta_t \\ &\lesssim \left(\left| Z_t^{i,N} - Z_t^{j,M} \right|^2 + J_t^{N,M} \right) dt + \zeta_t^5 d\beta_t. \end{aligned} \quad (3.30)$$

Taking the expectation in (3.30) and applying Gronwall's lemma leads to

$$\mathbb{E} \left| Z_t^{i,N} - Z_t^{j,M} \right|^2 \lesssim \left(|z_0^{i,N} - z_0^{j,M}|^2 + \mathbb{E}[J_t^{N,M}] \right).$$

Coming back to (3.30), we may write

$$\sup_{\sigma \in [0, t]} \left| Z_\sigma^{i,N} - Z_\sigma^{j,M} \right|^2 \lesssim |z_0^{i,N} - z_0^{j,M}|^2 + \int_0^t \left(\left| Z_s^{i,N} - Z_s^{j,M} \right|^2 + J_s^{N,M} \right) ds + \sup_{\sigma \in [0, t]} \int_0^\sigma \zeta_s^5 d\beta_s$$

and therefore

$$\mathbb{E} \left[\sup_{\sigma \in [0, t]} \left| Z_\sigma^{i,N} - Z_\sigma^{j,M} \right|^2 \right] \lesssim |z_0^{i,N} - z_0^{j,M}|^2 + \int_0^t \mathbb{E}[J_s^{N,M}] ds + \mathbb{E} \left[\sup_{\sigma \in [0, t]} \int_0^\sigma \zeta_s^5 d\beta_s \right]. \quad (3.31)$$

Burkholder-Davis-Gundy's inequality gives

$$\begin{aligned} \mathbb{E} \left[\sup_{\sigma \in [0, t]} \int_0^\sigma \zeta_s^5 d\beta_s \right] &\lesssim \mathbb{E} \left[\left(\int_0^t |\zeta_s^5|^2 ds \right)^{1/2} \right] \\ &\lesssim \mathbb{E} \left[\sup_{s \in [0, t]} \left| Z_s^{i,N} - Z_s^{j,M} \right| \left(\int_0^t \left| K_R[\mu_s^N](Z_s^{i,N}) - K_R[\mu_s^M](Z_s^{j,M}) \right|^2 ds \right)^{1/2} \right] \\ &\lesssim \mathbb{E} \left[\sup_{s \in [0, t]} \left| Z_s^{i,N} - Z_s^{j,M} \right|^2 \right]^{1/2} \left(\int_0^t \mathbb{E} \left| K_R[\mu_s^N](Z_s^{i,N}) - K_R[\mu_s^M](Z_s^{j,M}) \right|^2 ds \right)^{1/2}. \end{aligned}$$

Making use of (3.19) again, we get

$$\mathbb{E} \left[\sup_{\sigma \in [0, t]} \int_0^\sigma \zeta_s^5 d\beta_s \right] \lesssim \mathbb{E} \left[\sup_{s \in [0, t]} \left| Z_s^{i,N} - Z_s^{j,M} \right|^2 \right]^{1/2} \left(\int_0^t \mathbb{E} \left| Z_s^{i,N} - Z_s^{j,M} \right|^2 + \mathbb{E}[J_s^{N,M}] ds \right)^{1/2}$$

and we may come back (3.31) to obtain

$$\mathbb{E} \left[\sup_{\sigma \in [0, t]} \left| Z_{\sigma}^{i, N} - Z_{\sigma}^{j, M} \right|^2 \right] \lesssim |z_0^{i, N} - z_0^{j, M}|^2 + \int_0^t \left(\mathbb{E} \left| Z_s^{i, N} - Z_s^{j, M} \right|^2 + E[J_s^{N, M}] \right) ds.$$

Summing over i and j as in (3.29) finally leads to

$$\mathbb{E} \left[J_t^{N, M} \right] \lesssim W_2^2[\mu_0^N, \mu_0^M] + \int_0^t \mathbb{E} [J_s^{N, M}] ds.$$

Gronwall's lemma hence gives $\mathbb{E} [J_T^{N, M}] \lesssim W_2^2[\mu_0^N, \mu_0^M]$ so that, coming back to (3.28), we get

$$\mathbb{E} \left[W_2^2[\mu^N, \mu^M] \right] \lesssim W_2^2[\mu_0^N, \mu_0^M] \rightarrow 0$$

as N, M are sent to infinity. We have shown that μ^N converges to some μ in the complete space $L^2(\Omega; \mathcal{P}_2(\mathcal{C}))$. Let us now prove that μ satisfies the fixed-point identity (3.26). First, since $\mathbb{E} \left[\int_{z \in \mathcal{C}} \|z\|_{\infty}^2 d\mu \right] < \infty$, one could easily deduce from the sub-linearity of the coefficients that

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left| Z_t^{\mu}(z) \right|^2 \right] < \infty,$$

thereby guaranteeing that the solution $Z_t^{\mu}(z)$ of (3.27) is unique and global. Moreover, it is a well known fact (see e.g [Kun84]) that the flow $Z^{\mu} : z \in \mathbb{R}^{2d} \mapsto (Z_t^{\mu}(z))_{t \in [0, T]}$ is almost-surely continuous, so that the push-forward measure involved in (3.26) is indeed well-defined. This could be seen in this case by establishing a Kolmogorov estimate

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left| Z_t^{\mu}(z) - Z_t^{\mu}(z') \right|^2 \right] \lesssim |z - z'|^2, \quad \forall z, z' \in \mathbb{R}^{2d}.$$

Let us introduce the measure $\nu = (Z^{\mu})^* \mu_0$. Introducing an optimal plan $\pi \in \Pi(\mu_0^N, \mu_0)$ so that

$$W_2^2[\mu_0^N, \mu_0] = \int_{(z_1, z_2) \in (\mathbb{R}^{2d})^2} |z_1 - z_2|^2 d\pi(z_1, z_2)$$

we have this time

$$W_2^2[\mu^N, \nu] \leq \int_{(z_1, z_2) \in (\mathbb{R}^{2d})^2} \sup_{t \in [0, T]} \left| Z_t^{\mu^N}(z_1) - Z_t^{\mu}(z_2) \right|^2 d\pi(z_1, z_2) =: J_T^N.$$

As in (3.30), Itô's formula gives an expression of the form

$$\begin{aligned} d \left| Z_t^{\mu^N}(z_1) - Z_t^{\mu}(z_2) \right|^2 &\lesssim \left(\left| Z_t^{\mu^N}(z_1) - Z_t^{\mu}(z_2) \right|^2 + W_1^2[\mu_t^N, \mu_t] \right) dt + \zeta_t d\beta_t \\ &\lesssim \left(\left| Z_t^{\mu^N}(z_1) - Z_t^{\mu}(z_2) \right|^2 + W_2^2[\mu^N, \mu] \right) dt + \zeta_t d\beta_t. \end{aligned}$$

Proceeding as in the first part of the proof, we eventually obtain

$$\mathbb{E} \left[W_2^2[\mu^N, \nu] \right] \lesssim \mathbb{E} \left[W_2^2[\mu^N, \mu] \right].$$

Letting N go to infinity, we conclude that $\mu = \nu$ a.s, that is exactly (3.26).

We may once again use the same arguments to prove that the fixed-point-like equation (3.26) has a unique solution : considering μ and ν such that $\mu = (Z^{\mu})^* \mu_0$ a.s and $\nu = (Z^{\nu})^* \nu_0$ a.s, we are led to

$$\mathbb{E} \left[W_2^2[\mu, \nu] \right] \lesssim W_2^2[\mu_0, \nu_0]$$

so that $\mu_0 = \nu_0$ implies $\mu = \nu$ a.s.

Finally, let us notice that any μ satisfying (3.26) defines a solution of (3.24). Indeed, for any test function $\Psi \in C_c^{\infty}(\mathbb{R}^{2d})$, Itô's formula gives exactly

$$\Psi(Z_t^{\mu}(z)) = \Psi(z) + \int_0^t \mathcal{L}_R[\mu_s] \Psi(Z_s^{\mu}(z)) ds + \int_0^t \nabla \Psi(Z_s^{\mu}(z)) \cdot K_R[\mu_s](Z_s^{\mu}(z)) d\beta_s.$$

where $\mathcal{L}_R[\mu] \Psi$ is given by (3.25). Since $\langle \Psi, \mu_t \rangle = \int_{z \in \mathbb{R}^{2d}} \Psi(Z_t^{\mu}(z)) d\mu_0(z)$, integrating with respect to $d\mu_0(z)$ using a stochastic Fubini theorem gives the expected result. \square

3.2.2 Flow of characteristics, regular solutions

For some fixed $R > 0$, let us consider the unique solution μ of (3.24) constructed in Proposition 3.2.2. Although this measure, as well as the associated characteristics $(Z_t^\mu)_{t \geq 0}$, depend on $R > 0$, we shall hide this dependence in the following expressions to avoid cluttering notation. One may note from expressions (3.18) that the coefficients

$$L_R^{CS}[\mu_t](z), \quad L_R^{MT}[\mu_t](z), \quad S_R[\mu_t](z), \quad K_R[\mu_t](z)$$

involved in the SDE (3.27) have the regularity $C^4(\mathbb{R}^{2d})$ in the z variable. More precisely, assumption (3.11) guarantees that, for fixed $R > 0$, for $1 \leq |\alpha| \leq 4$,

$$\left| \partial_z^\alpha L_R^{CS}[\mu_t] \right| + \left| \partial_z^\alpha L_R^{MT}[\mu_t] \right| + \left| \partial_z^\alpha S_R[\mu_t] \right| + \left| \partial_z^\alpha K_R[\mu_t] \right| \lesssim 1$$

uniformly in $t \in [0, T]$ and $\omega \in \Omega$. In particular, the first, second and third order z -derivatives of the coefficients are globally Lipschitz-continuous. As a result, Theorem 4.4 of [Kun84], Chapter II, yields

$$\left[\forall t \in [0, T], \quad Z_t^\mu : z \in \mathbb{R}^{2d} \mapsto Z_t^\mu(z) \in \mathbb{R}^{2d} \text{ is a } C^3\text{-diffeomorphism} \right], \quad \mathbb{P} - a.s. \quad (3.32)$$

More precisely, for $0 \leq s \leq t$, denoting by $Z_{s,t}^\mu(z)$ the solution of the SDE

$$\begin{cases} X_{s,t}^\mu(z) = x + \int_{r=s}^t V_{s,r}^\mu(z) dr, \\ V_{s,t}^\mu(z) = v + \int_{r=s}^t \left(L_R^{CS}[\mu_r] + L_R^{MT}[\mu_r] + S_R[\mu_r] \right) (Z_{s,r}^\mu(z)) dr + \int_{r=s}^t K_R[\mu_r](Z_{s,r}^\mu(z)) d\beta_r, \end{cases}$$

the inverse map $(Z_{s,t}^\mu)^{-1}(z)$ satisfies the corresponding backward SDE

$$\begin{cases} (X_{s,t}^\mu)^{-1}(z) = x - \int_{r=s}^t (V_{r,t}^\mu)^{-1}(z) dr, \\ (V_{s,t}^\mu)^{-1}(z) = v - \int_{r=s}^t \left(L_R^{CS}[\mu_r] + L_R^{MT}[\mu_r] + S_R[\mu_r] - \tilde{S}_R[\mu_r] \right) ((Z_{r,t}^\mu)^{-1}(z)) dr \\ \quad - \int_{r=s}^t K_R[\mu_r]((Z_{r,t}^\mu)^{-1}(z)) \widehat{d\beta}_r, \end{cases}$$

where $\tilde{S}_R[\mu] = \nabla_v K_R[\mu] K_R[\mu]$ and $\int_s^t \cdot \widehat{d\beta}_t$ denotes the backward Stratonovich integral (see again [Kun84] Theorem 7.3 for this result and p.194 for the definition of the backward integral). When $s = 0$, we simply denote

$$Z_t^\mu(z) = Z_{0,t}^\mu(z), \quad (Z_t^\mu)^{-1}(z) = (Z_{0,t}^\mu)^{-1}(z).$$

In the particular case where the initial measure μ_0 admits a density $f_0 \in L^1(\mathbb{R}^{2d})$ with respect to the Lebesgue measure on \mathbb{R}^{2d} , for any test function $\Psi \in C_b(\mathbb{R}^{2d})$ we may write

$$\int_{z \in \mathbb{R}^{2d}} \Psi(z) d\mu_t(z) = \int_{z \in \mathbb{R}^{2d}} \Psi(Z_t^\mu(z)) f_0(z) dz = \int_{z \in \mathbb{R}^{2d}} \Psi(z) |J_t(z)^{-1}| f_0((Z_t^\mu)^{-1}(z)) dz$$

where $J_t(z)^{-1}$ denotes the jacobian determinant

$$J_t(z)^{-1} = \det \left[D_z((Z_t^\mu)^{-1})(z) \right]. \quad (3.33)$$

Considering a countable separating family of such test functions Ψ , from $\mu = (Z^\mu)^* f_0$ in $\mathcal{P}(\mathcal{C})$, we deduce that

$$\left[\forall t \in [0, T], \quad d\mu_t(z) = f(t, z) dz \right], \quad \mathbb{P} - a.s.,$$

where

$$f(t, z) = J_t(z)^{-1} f_0((Z_t^\mu)^{-1}(z)). \quad (3.34)$$

Note that we may drop the absolute value in (3.34) since $J_t(z)^{-1} \geq 0$ a.s. Let us now give some estimates regarding the forward and backward characteristics.

Proposition 3.2.3.

For all $p \geq 1$, let $f_0 \in L^1(\mathbb{R}^{2d})$ such that $\int |z'|^p f_0(z) dz' < \infty$. Let $\mu \in L^2(\Omega; \mathcal{P}_2(\mathcal{C}))$ such that $\mu = (Z^\mu)^* f_0$ as in (3.26). Then for all $z, z_1, z_2 \in \mathbb{R}^{2d}$, $t, t_1, t_2 \in [0, T]$ and $1 \leq k \leq 3$,

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Z_t^\mu(z)|^p \right] \lesssim 1 + |z|^p + \int |z'|^p f_0(z) dz', \quad (3.35)$$

$$\mathbb{E} \left[|D_z^k(Z_t^\mu)(z)|^p \right] \lesssim 1 + \int |z'|^p f_0(z) dz',$$

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Z_t^\mu(z_1) - Z_t^\mu(z_2)|^p \right] \lesssim |z_1 - z_2|^p, \quad (3.36)$$

$$\mathbb{E} \left[|(Z_t^\mu)^{-1}(z)|^p \right] \lesssim |z|^p + \int |z'|^p f_0(z) dz',$$

$$\mathbb{E} \left[\left| (Z_{t_1}^\mu)^{-1}(z_1) - (Z_{t_2}^\mu)^{-1}(z_2) \right|^p \right] \lesssim |z_1 - z_2|^p + \left(1 + |z_1|^p + \int |z'|^p f_0(z) dz' \right) |t_1 - t_2|^{p/2}, \quad (3.37)$$

$$\mathbb{E} \left[|D_z^k((Z_t^\mu)^{-1})(z)|^p \right] \lesssim 1 + \int |z'|^p f_0(z) dz', \quad (3.38)$$

$$\mathbb{E} \left[\left| D_z^k((Z_{t_1}^\mu)^{-1})(z) - D_z^k((Z_{t_2}^\mu)^{-1})(z') \right|^p \right] \lesssim |z - z'|^p + |t_1 - t_2|^{p/2}. \quad (3.39)$$

The constants involved in \lesssim depend on p, k, T, R only.

These estimates are deduced in a classical manner from the sub-linearity and the global Lipschitz continuity of the coefficients of the equations satisfied by $Z_t^\mu(z)$ and $(Z_t^\mu)^{-1}(z)$. Applying Kolmogorov's lemma to (3.37) and (3.39), we deduce that, for all $0 \leq k \leq 3$,

$$(t, z) \in [0, T] \times \mathbb{R}^{2d} \mapsto D^k((Z_t^\mu)^{-1})(z) \quad (3.40)$$

is continuous \mathbb{P} -a.s. Additionally, we may deduce from (3.35) and (3.36) the following estimate : given a compact set $K \subset \mathbb{R}^{2d}$ and $z_0 \in K$, for any $p \geq 1$ and $\alpha \in (0, 1)$,

$$\mathbb{E} \left[\sup_{z \in K} \sup_{t \in [0, T]} |Z_t^\mu(z)|^p \right] \lesssim \mathbb{E} \left[\sup_{t \in [0, T]} |Z_t^\mu(z_0)|^p \right] + \mathbb{E} \left[\|Z^\mu\|_{C_z^\alpha L_t^\infty}^p \right] \text{diam}(K)^{\alpha p} \leq C_{K, T, p} < \infty. \quad (3.41)$$

In (3.41), $\text{diam}(K) = \sup \{|z_1 - z_2|, z_1, z_2 \in K\}$ and $\|Z^\mu\|_{C_z^\alpha L_t^\infty}$ denotes the α -Hölder semi-norm

$$\|Z^\mu\|_{C_z^\alpha L_t^\infty} = \sup_{z_1 \neq z_2 \in \mathbb{R}^{2d}} \sup_{t \in [0, T]} \frac{|Z_t^\mu(z_1) - Z_t^\mu(z_2)|}{|z_1 - z_2|^\alpha}$$

which satisfies indeed $\mathbb{E} \left[\|Z^\mu\|_{C_z^\alpha L_t^\infty}^p \right] \lesssim 1$ by applying Kolmogorov's lemma to (3.36). We may now establish the following result.

Proposition 3.2.4 (Regular solution). *Let $f_0 \in C_c^2(\mathbb{R}^{2d})$. Let $\mu \in L^2(\Omega; \mathcal{P}_2(\mathcal{C}))$ such that $\mu = (Z^\mu)^* f_0$ as in (3.26), and let $f(t, z)$ be defined as in (3.34), so that μ may be represented as $d\mu_t(z) = f(t, z) dz$. Then f is a regular solution of (3.24) in the following sense :*

— \mathbb{P} -almost surely, $f(t, \cdot) \in C^2(\mathbb{R}^{2d})$ for all $t \in [0, T]$ and the maps

$$(t, z) \in [0, T] \times \mathbb{R}^{2d} \mapsto \partial_z^\alpha f(t, z), \quad 0 \leq |\alpha| \leq 2$$

are continuous.

— Denoting the operator

$$(\mathcal{L}_R[f])^* g = -v \cdot \nabla_x g - \nabla_v \cdot \left((L_R^{CS}[f] + L_R^{MT}[f] + S_R[f])g \right) + \frac{1}{2} \nabla_v \cdot \nabla_v \left(K_R[f] K_R[f]^T g \right), \quad (3.42)$$

we have for all $z \in \mathbb{R}^{2d}$, $t \in [0, T]$, \mathbb{P} -almost surely,

$$f(t, z) = f_0(z) + \int_0^t (\mathcal{L}_R[f(s)])^* f(s, z) ds - \int_0^t \nabla_v \cdot \left(K_R[f(s)] f(s, z) \right) d\beta_s. \quad (3.43)$$

Démonstration. As a consequence of (3.40), it is clear from expression (3.34) that the first condition is met. Furthermore, $f(t, \cdot)$ is almost surely compactly supported, uniformly in $t \in [0, T]$, with

$$\forall t \in [0, T], \text{ Supp}(f(t, \cdot)) \subset \left\{ z \in \mathbb{R}^{2d}, |z| \leq \sup_{z' \in K} \sup_{t \in [0, T]} |Z_t^\mu(z')| \right\}.$$

Since f is a solution of (3.24), for any $\Psi \in C_c^\infty(\mathbb{R}^{2d})$, \mathbb{P} -almost surely,

$$\langle f(t), \Psi \rangle = \langle f_0, \Psi \rangle + \int_0^t \left\langle (\mathcal{L}_R[f(s)])^* f(s), \Psi \right\rangle ds - \int_0^t \langle \nabla_v \cdot (K_R[f(s)]f(s)), \Psi \rangle d\beta_s.$$

We can then interchange the integrals

$$\int_0^t \langle \cdot, \Psi \rangle ds = \left\langle \int_0^t \cdot, \Psi \right\rangle ds, \quad \int_0^t \langle \cdot, \Psi \rangle d\beta_s = \left\langle \int_0^t \cdot, \Psi \right\rangle d\beta_s.$$

Since all functions are compactly supported (for fixed $\omega \in \Omega$), the integrals with respect to ds cause no issue. As for the stochastic integral, we may use a stochastic Fubini theorem, as long as

$$\int_0^t \int_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\left| \nabla_v \cdot (K_R[f(s)]f(s)) \Psi(z) \right|^2 \right] dz ds < \infty.$$

From expressions (3.18), we deduce (for fixed $R > 0$)

$$|K_R[f(t)](z)| + |\nabla_v \cdot (K_R[f(t)])(z)| \lesssim 1,$$

From expression (3.34), (and $|\det[A]| \lesssim |A|^{2d}$) it is clear that

$$|f(t, z)|^2 + |\nabla_z f(t, z)|^2 \lesssim \left(1 + \max_{k=1,2} \left| D_z^k ((Z_t^\mu)^{-1})(z) \right|^q \right) \left(\|f_0\|_{L^\infty}^2 + \|\nabla_z f_0\|_{L^\infty}^2 \right)$$

for some $q = q(d) \geq 1$. We may hence write

$$\mathbb{E} \left| \nabla_v \cdot (K_R[f(s)]f(s))(z) \right|^2 \lesssim 1 + \mathbb{E} \left[\max_{k=1,2} \left| D_z^k ((Z_t^\mu)^{-1})(z) \right|^q \right] \lesssim 1 \quad (3.44)$$

thanks to the bound (3.38). It follows that

$$\int_0^t \int_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\left| \nabla_v \cdot (K_R[f(s)]f(s)) \Psi(z) \right|^2 \right] dz ds \lesssim \int_{z \in \mathbb{R}^{2d}} |\Psi(z)|^2 dz < \infty.$$

Consequently, \mathbb{P} -a.s., (3.43) holds when integrated against any $\Psi \in C_c^\infty(\mathbb{R}^{2d})$. We deduce that (3.43) holds for almost every $z \in \mathbb{R}^{2d}$. Since both sides of (3.43) are continuous with respect to z (thanks to (3.44) again), we conclude that the equality holds for every $z \in \mathbb{R}^{2d}$. \square

3.3 Weak convergence of approximate solutions

From now on, let us fix some initial data f_0 satisfying (at least) for some $\delta > 1$,

$$f_0 \geq 0, \quad \int_{z \in \mathbb{R}^{2d}} f_0(z) dz = 1, \quad \int_{z \in \mathbb{R}^{2d}} |z|^\delta f_0(z) dz < \infty.$$

For any $R > 0$, considering the particle system (3.21), with initial data satisfying

$$\mu_0^N(dz) \rightarrow f_0(z) dz \text{ in } \mathcal{P}_\delta(\mathbb{R}^{2d}),$$

we may introduce the solution μ_R of (3.24) constructed in Proposition 3.2.2. As previously discussed, we naturally identify μ^R with its density $f^R = (f^R(t, z))_{t \in [0, T], z \in \mathbb{R}^{2d}}$ defined by (3.34).

3.3.1 Uniform estimates

In this section, we shall establish some estimates on f^R uniformly on the regularization parameter R .

Proposition 3.3.1. *Let $p \geq 1$ and $f_0 \in L^p(\mathbb{R}^{2d})$. Then $f^R \in L^\infty([0, T]; L^p(\mathbb{R}^{2d}))$ a.s, with the estimate*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|f^R(t)\|_{L^p}^p \right] \lesssim \|f_0\|_{L^p}^p.$$

The constant involved in \lesssim depends on p and T only.

Démonstration. Let us start by considering $f_0 \in C_c^2(\mathbb{R}^{2d})$ supported in some compact $K \subset \mathbb{R}^{2d}$. Then f^R is a regular solution of (3.24) in the sense of Proposition 3.2.4. For any $z \in \mathbb{R}^{2d}$, applying Itô's formula to $|f^R(t, z)|^p$ hence gives

$$\begin{aligned} |f^R(t, z)|^p &= f_0(z)^p - p \int_0^t |f^R(s, z)|^{p-1} \left[v \cdot \nabla_x f^R(s, z) + \nabla_v \cdot \left(B_R[f^R(s)] f^R(s, z) \right) \right] ds \\ &\quad + \frac{p}{2} \int_0^t |f^R(s, z)|^{p-1} \nabla_v \cdot \left(K_R[f^R(s)] K_R[f^R(s)]^T f^R(s, z) \right) ds \\ &\quad + \frac{p(p-1)}{2} \int_0^t |f^R(s, z)|^{p-2} \left| \nabla_v \cdot \left(K_R[f^R(s)] f^R(s, z) \right) \right|^2 ds \\ &\quad - p \int_0^t |f^R(s, z)|^{p-1} \nabla_v \cdot \left(K_R[f^R(s)] f^R(s, z) \right) d\beta_s, \end{aligned} \quad (3.45)$$

where $B_R[f]$ denotes the drift coefficient $L_R^{CS}[f] + L_R^{MT}[f] + S_R[f]$. We can now integrate (3.45) with respect to dz and interchange the integrals :

$$\int_{z \in \mathbb{R}^{2d}} \int_0^t \cdot ds dz = \int_0^t \int_{z \in \mathbb{R}^{2d}} \cdot dz ds, \quad \int_{z \in \mathbb{R}^{2d}} \int_0^t \cdot d\beta_s dz = \int_0^t \int_{z \in \mathbb{R}^{2d}} \cdot dz d\beta_s,$$

Since all the integrands in (3.45) are compactly supported in z uniformly for $t \in [0, T]$ (for fixed $\omega \in \Omega$), the integrals with respect to ds cause no issue. As for the stochastic integral, we may use a stochastic Fubini theorem if we can justify that

$$\mathcal{E} := \mathbb{E} \left[\int_0^T \int_{z \in \mathbb{R}^{2d}} \left| f^R(s, z)^{p-1} \nabla_v \cdot \left(K_R[f^R(s)] f^R(s, z) \right) \right|^2 dz ds \right] < \infty. \quad (3.46)$$

As in the proof of Proposition 3.2.4, one can see that (with $\mu(dz) = f^R(z) dz$)

$$\left| f^R(s, z)^{p-1} \nabla_v \cdot \left(K_R[f^R(s)] f^R(s, z) \right) \right|^2 \lesssim 1 + \max_{k=1,2} \left| D_z^k((Z_t^\mu)^{-1})(z) \right|^q$$

for some $q = q(d, p) \geq 1$. Denoting $S = \sup_{t \in [0, T]} \sup_{z' \in K} |Z_t^\mu(z')|$, it follows that

$$\int_{z \in \mathbb{R}^{2d}} \left| f^R(s, z)^{p-1} \nabla_v \cdot \left(K_R[f^R(s)] f^R(s, z) \right) \right|^2 dz \lesssim \int_{|z| \leq S} \max_{k=1,2} \left| D_z^k((Z_t^\mu)^{-1})(z) \right|^q dz$$

and, fixing some $m > 2d$,

$$\begin{aligned} \int_{|z| \leq S} \max_{k=1,2} \left| D_z^k((Z_t^\mu)^{-1})(z) \right|^q dz &= \int_{|z| \leq S} (1 + |z|)^{m/2} \frac{\max_{k=1,2} \left| D_z^k((Z_t^\mu)^{-1})(z) \right|^q}{(1 + |z|)^{m/2}} dz \\ &\lesssim \int_{|z| \leq S} (1 + |z|)^m dz + \int_{z \in \mathbb{R}^{2d}} \max_{k=1,2} \left| D_z^k((Z_t^\mu)^{-1})(z) \right|^{2q} \frac{dz}{(1 + |z|)^m} \\ &\lesssim (1 + S^{m+1}) + \int_{z \in \mathbb{R}^{2d}} \max_{k=1,2} \left| D_z^k((Z_t^\mu)^{-1})(z) \right|^{2q} \frac{dz}{(1 + |z|)^m}. \end{aligned}$$

Using (3.41) and (3.38), we deduce

$$\mathcal{E} \lesssim 1 + \mathbb{E} \left[S^{m+1} \right] + \int_{z \in \mathbb{R}^{2d}} \mathbb{E} \left[\max_{k=1,2} \left| D_z^k((Z_t^\mu)^{-1})(z) \right|^{2q} \right] \frac{dz}{(1 + |z|)^m} < \infty$$

We may hence integrate (3.45) with respect to z , which leads to

$$\int_{\mathbb{R}^{2d}} |f^R(t, z)|^p dz = \int_{\mathbb{R}^{2d}} f_0(z)^p dz + I_1(t) + I_2(t) + I_3(t) + I_4(t), \quad (3.47)$$

where

$$\begin{aligned} I_1(t) &= -p \int_0^t \int_{\mathbb{R}^{2d}} |f^R(s, z)|^{p-1} \left[v \cdot \nabla_x f^R(s, z) + \nabla_v \cdot \left(B_R[f^R(s)] f^R(s, z) \right) \right] dz ds, \\ I_2(t) &= \frac{p}{2} \int_0^t \int_{\mathbb{R}^{2d}} |f^R(s, z)|^{p-1} \nabla_v^2 \left(K_R[f^R(s)] K_R[f^R(s)]^T f^R(s, z) \right) dz ds, \\ I_3(t) &= \frac{p(p-1)}{2} \int_0^t \int_{\mathbb{R}^{2d}} |f^R(s, z)|^{p-2} \left| \nabla_v \cdot \left(K_R[f^R(s)] f^R(s, z) \right) \right|^2 dz ds, \\ I_4(t) &= -p \int_0^t \int_{\mathbb{R}^{2d}} |f^R(s, z)|^{p-1} \nabla_v \cdot \left(K_R[f^R(s)] f^R(s, z) \right) dz d\beta_s. \end{aligned}$$

Simple calculations lead to the classical identity

$$\int |f^R|^{p-1} \nabla_v \cdot (B_R f^R) dz = \frac{p-1}{p} \int |f^R|^p (\nabla_v \cdot B_R) dz \quad (3.48)$$

so that

$$\begin{aligned} I_1(t) &= \frac{p-1}{p} \int_0^t \int_{\mathbb{R}^{2d}} |f^R(s, z)|^p (\nabla_v \cdot B_R[f(s)](z)) dz ds \\ &\leq \left\| \nabla_v \cdot B_R[f^R] \right\|_{L_{t,z}^\infty([0, T] \times \mathbb{R}^{2d})} \int_0^t \int_{\mathbb{R}^{2d}} |f^R(s, z)|^p dz ds. \end{aligned}$$

From expressions (3.18), we derive

$$\begin{aligned} \nabla_v \cdot L_R^{CS}[f](z) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_R(x-y) \psi(x-y) \nabla_v \cdot \left(\theta_R(w-v) \right) f(y, w) dy dw, \\ \nabla_v \cdot L_R^{MT}(z) &= -d, \\ \nabla_v \cdot K_R[f](z) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_R(x-y) \tilde{\psi}(x-y) \nabla_v \cdot \left(\theta_R(w-v) \right) f(y, w) dy dw, \\ \nabla_v \cdot S_R[f](z) &= -\frac{1}{2} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{\psi}(x-y) f(y, w) dy dw \right) \nabla_v \cdot K_R[f](z) \end{aligned}$$

Assumptions (3.11), (3.17) guarantee that these terms are bounded uniformly in $t \in [0, T]$, $z \in \mathbb{R}^{2d}$, $\omega \in \Omega$, $R > 0$, so that

$$I_1(t) \lesssim \int_0^t \int_{\mathbb{R}^{2d}} |f^R(s, z)|^p dz ds.$$

Similarly, we have the following identity :

$$\begin{aligned} &\int |f^R|^{p-1} \nabla_v^2 (K_R K_R^T f^R) dz + (p-1) \int |f^R|^{p-2} |\nabla_v \cdot (K_R f^R)|^2 dz \\ &= (p-1) \int |f^R|^{p-2} \left[|\nabla_v \cdot (K_R f^R)|^2 - \nabla_v f^R \cdot \nabla_v \cdot (K_R K_R^T f^R) \right] dz \\ &= (p-1) \int |f^R|^p |\nabla_v \cdot K_R|^2 \end{aligned}$$

so that, again,

$$I_2(t) + I_3(t) = \frac{p(p-1)}{2} \int_0^t \int_{\mathbb{R}^{2d}} |f^R(s, z)|^p \left| \nabla_v \cdot K_R[f^R(s)] \right|^2 dz ds \lesssim \int_0^t \int_{\mathbb{R}^{2d}} |f^R(s, z)|^p dz ds.$$

Since (3.46) guarantees that $I_4(t)$ defines a (square integrable) martingale, we may take the expectation in (3.47) and apply Gronwall's lemma to derive

$$\mathbb{E} \left[\int_{\mathbb{R}^{2d}} |f^R(t, z)|^p dz \right] \lesssim \|f_0\|_{L^p}^p.$$

It follows that

$$\mathbb{E} \left[\sup_{\sigma \in [0, t]} \int_{\mathbb{R}^{2d}} |f^R(\sigma, z)|^p dz \right] \lesssim \int_0^t \mathbb{E} \left[\int_{\mathbb{R}^{2d}} |f^R(s, z)|^p dz \right] ds + \mathbb{E} \left[\sup_{\sigma \in [0, t]} I_4(\sigma) \right]$$

and Burkholder-Davis-Gundy's inequality yields (making use of (3.48) again)

$$\begin{aligned} \mathbb{E} \left[\sup_{\sigma \in [0, t]} I_4(\sigma) \right] &\lesssim \mathbb{E} \left[\left(\int_0^t \left| \int_{\mathbb{R}^{2d}} |f^R|^{p-1} \nabla_v \cdot (K_R[f^R] f^R) dz \right|^2 d\sigma \right)^{1/2} \right] \\ &\lesssim \mathbb{E} \left[\left(\int_0^t \left| \int_{\mathbb{R}^{2d}} |f^R(\sigma, z)|^p dz \right|^2 d\sigma \right)^{1/2} \right] \\ &\leq \frac{1}{2} \mathbb{E} \left[\sup_{\sigma \in [0, t]} \int_{\mathbb{R}^{2d}} |f^R(\sigma, z)|^p dz \right] + C \int_0^t \mathbb{E} \left[\int_{\mathbb{R}^{2d}} |f^R(\sigma, z)|^p dz \right] d\sigma \end{aligned}$$

which gives the expected result. We now extend the estimate to any $f_0 \in L^p(\mathbb{R}^{2d})$ satisfying $\int |z|^\delta f_0(z) dz < \infty$ by considering a sequence of densities $f_0^k \in C_c^2(\mathbb{R}^{2d})$ such that, as k goes to infinity,

$$\begin{aligned} f_0^k &\rightarrow f_0 \text{ a.s and in } L^p(\mathbb{R}^{2d}), \\ \sup_{k \geq 1} \int_{z \in \mathbb{R}^{2d}} |z|^\delta f_k(z) dz &< \infty. \end{aligned}$$

It is easy to see that these assumptions imply in particular

$$f_0^k \rightarrow f_0 \text{ in } \mathcal{P}_r(\mathbb{R}^{2d})$$

for any $1 < r < \delta$. Denoting by f^k the solution of (3.24) with initial data f_0^k constructed in Proposition 3.2.2, we may deduce (as in the proof of Proposition 3.2.2),

$$\mathbb{E} \left[\sup_{t \in [0, T]} W_r^r[f_t^k, f_t] \right] \leq \mathbb{E} \left[W_r^r[f^k, f] \right] \lesssim W_r^r[f_0^k, f_0] \rightarrow 0.$$

Up to a subsequence, we may hence assume that

$$\sup_{t \in [0, T]} W_r[f_t^k, f_t] \rightarrow 0, \quad \mathbb{P} - a.s. \quad (3.49)$$

From the estimates

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|f^k(t)\|_{L^p}^p \right] \lesssim \|f_0^k\|_{L^p}^p, \quad k \geq 1,$$

we derive that $(f^k)_{k \geq 1}$ is bounded in $L_\omega^p L_t^\infty L_z^p$ and therefore, up to a subsequence

$$f^k \rightharpoonup g \text{ weak } * \text{ in } L^p \left(\Omega; L^\infty([0, T]; L^p(\mathbb{R}^{2d})) \right) \quad (3.50)$$

where g satisfies the bound

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|g(t)\|_{L^p}^p \right] \lesssim \limsup_k \|f_0^k\|_{L^p}^p = \|f_0\|_{L^p}^p. \quad (3.51)$$

Let us consider $\Psi \in C_c([0, T] \times \mathbb{R}^{2d})$, $\xi \in L^\infty(\Omega)$ and introduce $\Phi(\omega, t, z) = \xi(\omega)\Psi(t, z)$. From (3.49) we deduce

$$\left[\forall t \in [0, T], \int_z f^k(t, z) \Psi(t, z) dz \rightarrow \int_z f(t, z) \Psi(t, z) dz \right], \quad \mathbb{P} - a.s.$$

and the bound $\left| \int_z f^k(t, z) \Psi(t, z) dz \right| \leq \|\Psi\|_{L_{t,z}^\infty}$ guarantees

$$\int_{t=0}^T \int_z f^k(t, z) \Psi(t, z) dz dt \rightarrow \int_{t=0}^T \int_z f(t, z) \Psi(t, z) dz dt, \quad \mathbb{P} - a.s.$$

Finally, the bound $\left| \int_{t=0}^T \int_z \xi(\omega) f^k(\omega, t, z) \Psi(t, z) dz dt \right| \leq T \|\xi\|_{L_\omega^\infty} \|\Psi\|_{L_{t,z}^\infty}$ guarantees

$$\mathbb{E} \left[\xi(\omega) \int_{t=0}^T \int_z f^k(\omega, t, z) \Psi(t, z) dz dt \right] \rightarrow \mathbb{E} \left[\xi(\omega) \int_{t=0}^T \int_z f(\omega, t, z) \Psi(t, z) dz dt \right]$$

so that, according to (3.50),

$$\mathbb{E} \left[\xi \int_{t=0}^T \int_z g(t, z) \Psi(t, z) dz dt \right] = \mathbb{E} \left[\xi \int_{t=0}^T \int_z f(t, z) \Psi(t, z) dz dt \right].$$

We easily derive that $f = g$ in $L_\omega^p L_t^\infty L_z^p$ and the bound (3.51) concludes the proof. \square

Proposition 3.3.2. *For all $f : \mathbb{R}^{2d} \rightarrow \mathbb{R}^+$ and $p \geq 1$,*

$$\int_{z \in \mathbb{R}^{2d}} |u_R[f](x)|^p f(z) dz \lesssim \int_{z \in \mathbb{R}^{2d}} |v|^p f(z) dz.$$

The constant involved in \lesssim depends on ϕ only.

Démonstration. It is clear from the expression of $u_R[f]$ (3.18) and Jensen's inequality that

$$\begin{aligned} \int_{z \in \mathbb{R}^{2d}} |u_R[f](x)|^p f(z) dz &\leq \int_x \int_v \frac{\int_y \int_w |w|^p \phi(x-y) f(y, w) dy dw}{\int_y \int_w \phi(x-y) f(y, w) dy dw} f(x, v) dx dv \\ &= \int_y \int_w \left(\int_x \frac{\rho(x)}{\tilde{\rho}(x)} \phi(x-y) dx \right) |w|^p f(y, w) dx dy dw \end{aligned}$$

where

$$\rho(x) = \int_{v \in \mathbb{R}^d} f(x, v), \quad \tilde{\rho}(x) = (\phi * \rho)(x) = \int_{y \in \mathbb{R}^d} \int_{w \in \mathbb{R}^d} \phi(x-y) f(y, w) dy dw.$$

The desired estimate hence follows from the inequality

$$\forall y \in \mathbb{R}^d, \quad \int_{x \in \mathbb{R}^d} \frac{\rho(x)}{\tilde{\rho}(x)} \phi(x-y) dx \leq C(\phi) \quad (3.52)$$

where, with assumption (3.12) in mind, $C(\phi)$ is some constant proportional to

$$C(\phi) \propto \frac{\sup_{B(0, r_2)} \phi}{\inf_{B(0, r_1)} \phi} (R/r)^d. \quad (3.53)$$

The proof of (3.52) is given in [KMT13], Lemma 5.2. \square

Proposition 3.3.3. *Let $k \geq 2$, $1 < \delta \leq 2$ and f_0 be a density satisfying $\int_z (|x|^\delta + |v|^k) f_0(z) dz < \infty$. Then,*

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} \int_{\mathbb{R}^{2d}} |v|^k f^R(t, z) dz \right] &\lesssim 1 + \int_{\mathbb{R}^{2d}} |v|^k f_0(z) dz. \\ \mathbb{E} \left[\sup_{t \in [0, T]} \int_{\mathbb{R}^{2d}} |x|^\delta f^R(t, z) dz \right] &\lesssim 1 + \int_{\mathbb{R}^{2d}} (|x|^\delta + |v|^2) f_0(z) dz. \end{aligned}$$

The constants involved in \lesssim depends on k, δ, T and ϕ only.

Démonstration. The first estimate should first be established with the stopping time

$$\tau_M = \inf \left\{ t \geq 0, \quad \int_{z \in \mathbb{R}^{2d}} |v|^k f_t(z) dz \geq M \right\} \wedge T,$$

which should then be sent to T , as M goes to infinity. For the sake of simplicity, we omit this stopping time in the following. Let us denote $f(t, z) = f^R(t, z)$ and the associated characteristics $Z_t(z) = Z_t^{\mu^R}(z)$ (with $\mu^R(dz) = f^R(z) dz$), satisfying (3.27). We have

$$\int_{z \in \mathbb{R}^{2d}} |z|^k f(t, z) dz = \int_{z \in \mathbb{R}^{2d}} \left| Z_t(z) \right|^k f_0(z) dz$$

and Itô's formula gives, for fixed $z \in \mathbb{R}^{2d}$,

$$\begin{aligned} d\left[|V_t(z)|^k\right] &= k|V_t(z)|^{k-2}V_t(z) \cdot \left(L_R^{CS}[f_t] + L_R^{MT}[f_t] + S_R[f_t]\right)(Z_t(z))dt \\ &\quad + k|V_t(z)|^{k-2}V_t(z) \cdot K_R[f_t](Z_t(z))d\beta_t \\ &\quad + k(k/2 - 1)|V_t|^{k-4}\left|V_t(z) \cdot K_R[f_t](Z_t(z))\right|^2 dt + k|V_t(z)|^{k-2}\left|K_R[f_t](Z_t(z))\right|^2 dt. \end{aligned}$$

From the uniform sublinearity (3.20), we derive

$$|V_t(z)|^k = |v|^k + \int_0^t \left(a_s^1(z) + a_s^2(z)\right)ds + \int_0^t a_s^3(z)d\beta_s, \quad (3.54)$$

where

$$\begin{aligned} |a_t^1(z)| + |a_t^3(z)| &\lesssim \left(1 + |V_t(z)|^k + \int_{z'} |v'|^k f_t(z')dz'\right), \\ a_t^2(z) &= k|V_t(z)|^{k-2}V_t(z) \cdot \left(u_R[f_t](X_t(z)) - V_t(z)\right). \end{aligned}$$

We may then integrate (3.54) with respect to $f_0(z)dz$ using a stochastic Fubini theorem, which leads to

$$\int_{z \in \mathbb{R}^{2d}} |v|^k f(t, z)dz = \int_{z \in \mathbb{R}^{2d}} |v|^k f_0(z)dz + \int_0^t \left(A_s^1 + A_s^2\right)ds + \int_0^t A_s^3 d\beta_s, \quad (3.55)$$

where

$$\begin{aligned} |A_t^1| + |A_t^3| &\lesssim \left(1 + \int_z |v|^k f_t(z)dz\right), \\ A_t^2 &= k \int_{z \in \mathbb{R}^{2d}} |v|^{k-2}v \cdot \left(u_R[f_t](x) - v\right) f_t(z)dz. \end{aligned} \quad (3.56)$$

To deal with A_t^2 , one may write

$$v \cdot (u - v) = \frac{1}{4} \left(|v + (u - v)|^2 - |v - (u - v)|^2 \right) \leq \frac{1}{4} |u|^2$$

so that

$$\begin{aligned} \int_z |v|^{k-2}v \cdot (u(x) - v) f(z)dz &\leq \frac{1}{4} \int_z |v|^{k-2} |u(x)|^2 f(z)dz \\ &\leq \frac{1}{4} \int_z |v|^k f(z)dz + \frac{1}{4} \int_z |u(x)|^k f(z)dz. \end{aligned}$$

From Proposition 3.3.2, we hence deduce

$$|A_t^2| \lesssim \int |v|^k f_t(z)dz. \quad (3.57)$$

From SDE (3.55) with the sublinear terms (3.56) and (3.57), using Gronwall's lemma and Burkholder-Davis-Gundy's inequality, we classically obtain the first estimate. Moreover, from

$$|X_t(z)|^\delta = \left| x + \int_0^t V_s(z)ds \right|^\delta \lesssim 1 + |x|^\delta + \int_0^t |V_s(z)|^2 ds$$

since $\delta \leq 2$, we derive the second estimate, which concludes the proof. \square

3.3.2 Stochastic averaging lemma

Proposition 3.3.4. *Let us assume that the initial data f_0 satisfies, for some $\theta \in (0, 1)$,*

$$\int_{z \in \mathbb{R}^{2d}} |f_0(z)|^p dz + \int_{z \in \mathbb{R}^{2d}} (1 + |v|^k) f_0(z) dz < \infty \quad \text{with } p = 1 + \frac{1}{\theta}, \quad k > \frac{4}{1 - \theta}. \quad (3.58)$$

For all, $\varphi \in C_c^\infty(\mathbb{R}^d)$, letting $\eta = 1/6$, the averaged quantity

$$\rho_\varphi^R(t, x) = \int_{\mathbb{R}^d} \varphi(v) f^R(t, x, v) dx$$

lies in $L^2([0, T]; H^\eta(\mathbb{R}^d))$ almost surely, with the estimate

$$\mathbb{E} \left[\|\rho_\varphi^R\|_{L_t^2 H_x^\eta}^2 \right] \lesssim 1. \quad (3.59)$$

The constant involved in \lesssim in (3.59) depends on f_0 , φ , T and ϕ only.

Démonstration. The proof of this result is based on a classical L^2 averaging lemma (see e.g [BD99] for the deterministic case), which we adapt here to the stochastic case, in a similar fashion to [DDMV15], Lemma 4.3. Let us first consider some initial data $f_0 \in C_c^2(\mathbb{R}^{2d})$ so that $f = f^R$ is a regular solution of (3.24) in the sense of Proposition 3.2.4, which may be written as follows : \mathbb{P} -a.s, for all $z \in \mathbb{R}^{2d}$,

$$\begin{aligned} df(t, z) + v \cdot \nabla_x f(t, z) dt = & \left(\sum_{1 \leq i \leq d} \partial_{v_i} G^i(t, z) + \sum_{1 \leq i, j \leq d} \partial_{v_i v_j}^2 G^{ij}(t, z) \right) dt \\ & + \left(\sum_{1 \leq i \leq d} \partial_{v_i} H^i(t, z) \right) d\beta_t \end{aligned} \quad (3.60)$$

where we have introduced the coefficients

$$\begin{cases} G^i(t, z) = \left(L_R^{CS}[f(t)] + L_R^{MT}[f(t)] + S_R[f(t)] \right) (z)^i f(t, z), \\ G^{ij}(t, z) = K_R[f(t)](z)^i K_R[f(t)](z)^j f(t, z), \\ H^i(t, z) = K_R[f(t)]^i f(t, z). \end{cases} \quad (3.61)$$

Let us fix $\xi \in \mathbb{R}^d$. For simplicity, let us drop the summation signs in (3.60) and integrate it with respect to $e^{-i\xi \cdot x} dx$. This is possible (for every $v \in \mathbb{R}^d$) thanks to the bound (3.46) with $p = 1$ established previously. Denoting the x -Fourier transform

$$\hat{f}(t, \xi, v) = \int_{x \in \mathbb{R}^d} e^{-i\xi \cdot x} f(t, x, v) dx,$$

we are led to

$$d\hat{f}(t, \xi, v) + iv \cdot \xi \hat{f}(t, \xi, v) dt = + \left(\widehat{\partial_{v_i} G^i} + \widehat{\partial_{v_i v_j}^2 G^{ij}} \right) (t, \xi, v) dt. + \widehat{\partial_{v_i} H^i} (t, \xi, v) d\beta_t$$

Therefore, introducing some $\lambda = \lambda(\xi) > 0$, we get

$$\begin{aligned} d\hat{f}(t, \xi, v) + (\lambda + iv \cdot \xi) \hat{f}(t, \xi, v) dt = & \lambda \hat{f}(t, \xi, v) dt + \left(\widehat{\partial_{v_i} G^i} + \widehat{\partial_{v_i v_j}^2 G^{ij}} \right) (t, \xi, v) dt \\ & + \widehat{\partial_{v_i} H^i} (t, \xi, v) d\beta_t, \end{aligned}$$

from which we deduce the expression

$$\begin{aligned} \hat{f}(t, \xi, v) = & e^{-(\lambda + iv \cdot \xi)t} \hat{f}_0(\xi, v) + \lambda \int_0^t e^{-(\lambda + iv \cdot \xi)(t-s)} \hat{f}(s, \xi, v) ds \\ & + \int_0^t e^{-(\lambda + iv \cdot \xi)(t-s)} \left(\widehat{\partial_{v_i} G^i} + \widehat{\partial_{v_i v_j}^2 G^{ij}} \right) (s, \xi, v) ds \\ & + \int_0^t e^{-(\lambda + iv \cdot \xi)(t-s)} \widehat{\partial_{v_i} H^i} (s, \xi, v) d\beta_s. \end{aligned} \quad (3.62)$$

We now integrate (3.62) with respect to $\varphi(v) dv$. This is possible since one could show that

$$\begin{aligned} & \mathbb{E} \int_0^t \int_{v \in \mathbb{R}^d} \left| \varphi(v) e^{-(\lambda + iv \cdot \xi)(t-s)} \widehat{\partial_{v_i} H^i} (s, \xi, v) \right|^2 dv ds \\ & \lesssim \mathbb{E} \int_0^T \int_{v \in \mathbb{R}^d} (1 + |v|^2) \left| \int_{x \in \mathbb{R}^d} \left| \nabla_v \cdot (K_R[f(s)] f(s)) \right| dx \right|^2 dv ds < \infty \end{aligned}$$

for fixed $R > 0$, by a method similar to the one employed to establish (3.46) in the proof Proposition 3.3.1. Introducing the (x, v) -Fourier transform

$$(\mathcal{F}f)(t, \xi, \zeta) = \int_{v \in \mathbb{R}^d} \int_{x \in \mathbb{R}^d} e^{-i(\xi \cdot x + \zeta \cdot v)} f(t, x, v) dx dv,$$

equation (3.62) leads to

$$\begin{aligned} \widehat{\rho}_\varphi(t, \xi) &= e^{-\lambda t} \mathcal{F}(f_0 \varphi)(\xi, \xi t) + \lambda \int_0^t e^{-\lambda(t-s)} \mathcal{F}(f \varphi)(s, \xi, \xi(t-s)) ds \\ &\quad + \int_0^t e^{-\lambda(t-s)} \left(\mathcal{F}((\partial_{v_i} G^i) \varphi) + \mathcal{F}((\partial_{v_i v_j}^2 G^{ij}) \varphi) \right) (s, \xi, \xi(t-s)) ds \\ &\quad + \int_0^t e^{-\lambda(t-s)} \mathcal{F}((\partial_{v_i} H^i) \varphi)(s, \xi, \xi(t-s)) d\beta_s. \end{aligned}$$

Note that since

$$(\partial_{v_i} G^i) \varphi = \partial_{v_i} (G^i \varphi) - G^i \partial_{v_i} \varphi, \quad (3.63)$$

$$(\partial_{v_i v_j}^2 G^{ij}) \varphi = \partial_{v_i v_j}^2 (G^{ij} \varphi) - \partial_{v_i} (G^{ij} \partial_j \varphi) - \partial_{v_j} (G^{ij} \partial_i \varphi) + G^{ij} \partial_{v_i v_j}^2 \varphi \quad (3.64)$$

forgetting the summation signs again, we may as well only consider terms of the form

$$\begin{aligned} \widehat{\rho}_\varphi(t, \xi) &= e^{-\lambda t} \mathcal{F}(f_0 \Psi)(\xi, \xi t) + \lambda \int_0^t e^{-\lambda(t-s)} \mathcal{F}(f \Psi)(s, \xi, \xi(t-s)) ds \\ &\quad + \int_0^t e^{-\lambda(t-s)} \mathcal{F}(\partial_v^\alpha (G^\beta \Psi))(s, \xi, \xi(t-s)) ds \\ &\quad + \int_0^t e^{-\lambda(t-s)} \mathcal{F}(\partial_v^\gamma (H^i \Psi))(s, \xi, \xi(t-s)) d\beta_s \end{aligned} \quad (3.65)$$

where $\Psi \equiv \Psi(v) \in C_c^\infty(\mathbb{R}^d)$ and α, β and γ are multi-indexes of order $0 \leq |\alpha| \leq 2$, $1 \leq |\beta| \leq 2$ and $0 \leq |\gamma| \leq 1$. From (3.65) Itô's isometry gives

$$\mathbb{E} \int_0^T |\widehat{\rho}_\varphi(t, \xi)|^2 dt \lesssim J_1(\xi) + J_2(\xi) + J_3(\xi) + J_4(\xi) \quad (3.66)$$

where

$$\begin{aligned} J_1(\xi) &= \int_0^T e^{-2\lambda t} |\mathcal{F}(f_0 \Psi)(\xi, \xi t)|^2 dt, \\ J_2(\xi) &= \lambda^2 \mathbb{E} \int_0^T \left| \int_0^t e^{-\lambda(t-s)} \mathcal{F}(f \Psi)(s, \xi, \xi(t-s)) ds \right|^2 dt, \\ J_3(\xi) &= \mathbb{E} \int_0^T \left| \int_0^t e^{-\lambda(t-s)} \mathcal{F}(\partial_v^\alpha (G^\beta \Psi))(s, \xi, \xi(t-s)) ds \right|^2 dt, \\ J_4(\xi) &= \mathbb{E} \int_0^T \int_0^t e^{-2\lambda(t-s)} \left| \mathcal{F}(\partial_v^\gamma (H^i \Psi))(s, \xi, \xi(t-s)) \right|^2 ds dt. \end{aligned}$$

Using a trace lemma (see e.g [Tri83], Theorem 2.7.2) we get, for some $s > (d-1)/2$,

$$\begin{aligned} J_1(\xi) &\leq \int_{\mathbb{R}} \left| \mathcal{F}(f_0 \Psi)(\xi, \xi t) \right|^2 dt = |\xi|^{-1} \int_{\mathbb{R}} \left| \mathcal{F}(f_0 \Psi)(\xi, \frac{\xi}{|\xi|} t) \right|^2 dt \\ &\lesssim |\xi|^{-1} \int_{\mathbb{R}^d} \left| (Id - \Delta_w)^s \mathcal{F}(f_0 \Psi)(\xi, w) \right|^2 dw. \end{aligned}$$

It then follows by Plancherel's identity, since Ψ is compactly supported, that

$$J_1(\xi) \lesssim |\xi|^{-1} \int_{\mathbb{R}^d} (1 + |v|^2)^s |\Psi(v)|^2 |\hat{f}_0(\xi, v)|^2 dv \lesssim |\xi|^{-1} \int_{\mathbb{R}^d} |\hat{f}_0(\xi, v)|^2 dv. \quad (3.67)$$

For the second term $J_2(\xi)$, Jensen's inequality gives

$$\begin{aligned} J_2(\xi) &\leq \lambda \mathbb{E} \int_0^T \int_0^t e^{-\lambda(t-s)} \left| \mathcal{F}(f\Psi)(s, \xi, \xi(t-s)) \right|^2 ds dt \\ &\leq \lambda \mathbb{E} \int_0^T \int_s^T \left| \mathcal{F}(f\Psi)(s, \xi, \xi(t-s)) \right|^2 dt ds \leq \lambda \mathbb{E} \int_0^T \left(\int_{\mathbb{R}} \left| \mathcal{F}(f\Psi)(s, \xi, \xi t) \right|^2 dt \right) ds \end{aligned}$$

and the same manipulation leads to

$$J_2(\xi) \lesssim \lambda |\xi|^{-1} \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\hat{f}(s, \xi, v)|^2 dv ds. \quad (3.68)$$

Similarly, for the third term $J_3(\xi)$, Jensen's inequality gives

$$\begin{aligned} J_3(\xi) &\leq \lambda^{-1} \mathbb{E} \int_0^T \int_0^t e^{-\lambda(t-s)} \left| \mathcal{F}(\partial_v^\alpha (G^\beta \Psi))(s, \xi, \xi(t-s)) \right|^2 ds dt \\ &\leq \lambda^{-1} \mathbb{E} \int_0^T \int_0^t e^{-\lambda(t-s)} |\xi|^4 (t-s)^4 \left| \mathcal{F}(G^\beta \Psi)(s, \xi, \xi(t-s)) \right|^2 ds dt \\ &\lesssim \lambda^{-5} |\xi|^4 \mathbb{E} \int_0^T \left(\int_{\mathbb{R}} \left| \mathcal{F}(G^\beta \Psi)(s, \xi, \xi t) \right|^2 dt \right) ds \end{aligned}$$

where we have used $e^{-\lambda(t-s)}(t-s)^4 \lesssim \lambda^{-4}$. The same manipulation hence leads to

$$J_3(\xi) \lesssim \lambda^{-5} |\xi|^3 \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\widehat{G}^\beta(s, \xi, v)|^2 dv ds. \quad (3.69)$$

Finally, for the fourth term $J_4(\xi)$, we get

$$\begin{aligned} J_4(\xi) &\leq \mathbb{E} \int_0^T \int_0^t e^{-2\lambda(t-s)} |\xi|^2 (t-s)^2 \left| \mathcal{F}(H^i \Psi)(s, \xi, \xi(t-s)) \right|^2 ds dt \\ &\lesssim \lambda^{-2} |\xi|^2 \mathbb{E} \int_0^T \left(\int_{\mathbb{R}} \left| \mathcal{F}(H^i \Psi)(s, \xi, \xi(t-s)) \right|^2 dt \right) ds \end{aligned}$$

and the same manipulation leads to

$$J_4(\xi) \lesssim \lambda^{-2} |\xi| \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\widehat{H}^i(s, \xi, v)|^2 dv ds. \quad (3.70)$$

Now fixing some $\eta > 0$, let us consider

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^d} (1 + |\xi|^{2\eta}) |\widehat{\rho}_\varphi(t, \xi)|^2 d\xi dt = I_1 + I_2$$

where

$$I_1 = \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\widehat{\rho}_\varphi(t, \xi)|^2 d\xi dt + \mathbb{E} \int_0^T \int_{|\xi| \leq 1} |\xi|^{2\eta} |\widehat{\rho}_\varphi(t, \xi)|^2 d\xi dt, \quad (3.71)$$

$$I_2 = \mathbb{E} \int_0^T \int_{|\xi| > 1} |\xi|^{2\eta} |\widehat{\rho}_\varphi(t, \xi)|^2 d\xi dt. \quad (3.72)$$

On one hand, making use of Plancherel's identity, and since φ is compactly supported,

$$\begin{aligned} I_1 &\leq 2 \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\widehat{\rho}_\varphi(t, \xi)|^2 d\xi dt \lesssim \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\rho_\varphi(t, x)|^2 dx dt \\ &\lesssim \mathbb{E} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(t, x, v)|^2 dv dx dt \lesssim 1 \end{aligned}$$

thanks to Proposition 3.3.1, and the initial bound (3.58). On the other hand, from the bounds (3.67), (3.68), (3.69) and (3.70) we deduce (using Plancherel's identity once again)

$$\begin{aligned} I_2 &\lesssim \sup_{|\xi| > 1} \left[|\xi|^{2\eta-1} + \lambda(\xi) |\xi|^{2\eta-1} + \lambda(\xi)^{-5} |\xi|^{2\eta+3} + \lambda(\xi)^{-2} |\xi|^{2\eta+1} \right] \\ &\quad \times \left[\int_{z \in \mathbb{R}^{2d}} |f_0|^2 dz + \mathbb{E} \int_0^T \int_{z \in \mathbb{R}^{2d}} \left(|f|^2 + |G^\beta|^2 + |H^i|^2 \right) dz dt \right]. \quad (3.73) \end{aligned}$$

Expressions (3.61) and the sublinearity estimate (3.20) immediately give

$$|G^i(t, z)|^2 \lesssim \left(1 + |v|^2 + \int |v'|^2 f(t, z') dz'\right) |f(t, z)|^2, \quad (3.74)$$

$$|G^{ij}(t, z)|^2 \lesssim \left(1 + |v|^4 + \int |v'|^4 f(t, z') dz'\right) |f(t, z)|^2, \quad (3.75)$$

$$|H^i(t, z)|^2 \lesssim \left(1 + |v|^2 + \int |v'|^2 f(t, z') dz'\right) |f(t, z)|^2, \quad (3.76)$$

from which we easily deduce (using the same method as in the proof of Proposition 3.3.3 for the term involving $u_R[f(t)]$), for $\theta \in (0, 1)$,

$$\begin{aligned} \int_{z \in \mathbb{R}^{2d}} |G^i|^2 dz &\lesssim \int_{z \in \mathbb{R}^{2d}} (1 + |v|^{\frac{2}{1-\theta}}) f(t) dz + \int_{z \in \mathbb{R}^{2d}} |f(t)|^{1+\frac{1}{\theta}} dz, \\ \int_{z \in \mathbb{R}^{2d}} |G^{ij}|^2 dz &\lesssim \int_{z \in \mathbb{R}^{2d}} (1 + |v|^{\frac{4}{1-\theta}}) f(t) dz + \int_{z \in \mathbb{R}^{2d}} |f(t)|^{1+\frac{1}{\theta}} dz, \\ \int_{z \in \mathbb{R}^{2d}} |H^i|^2 dz &\lesssim \int_{z \in \mathbb{R}^{2d}} (1 + |v|^{\frac{2}{1-\theta}}) f(t) dz + \int_{z \in \mathbb{R}^{2d}} |f(t)|^{1+\frac{1}{\theta}} dz, \end{aligned}$$

so that, thanks again to Proposition 3.3.1, Proposition 3.3.3 and the initial bound (3.58), (3.73) yields

$$I_2 \lesssim \sup_{|\xi| > 1} \left[|\xi|^{2\eta-1} + \lambda(\xi) |\xi|^{2\eta-1} + \lambda(\xi)^{-5} |\xi|^{2\eta+3} + \lambda(\xi)^{-2} |\xi|^{2\eta+1} \right].$$

Considering $\lambda(\xi) = |\xi|^r$, this supremum is bounded under the requirements

$$2\eta - 1 \leq 0, \quad 2\eta - 1 + r \leq 0, \quad 2\eta + 3 - 5r \leq 0, \quad 2\eta + 1 - 2r \leq 0,$$

which can be met as soon as $\eta \leq 1/6$. For such η , we have shown that

$$\mathbb{E} \left[\|\rho_\varphi\|_{L_t^2 H_x^\eta}^2 \right] \lesssim \mathbb{E} \int_0^T \int_{\mathbb{R}^d} (1 + |\xi|^{2\eta}) |\widehat{\rho}_\varphi(t, \xi)|^2 d\xi dt \lesssim 1.$$

which concludes the proof in the case of a regular initial data $f_0 \in C_c^2(\mathbb{R}^{2d})$. We may extend this last inequality to general initial data similarly to the proof of Proposition 3.3.1. \square

3.3.3 Tightness

Given an increasing weight function, say

$$W(x, v) = 1 + |x| + |v|,$$

let us introduce the weighted Sobolev space

$$H_W^2(\mathbb{R}^{2d}) = \left\{ \Psi \in D'(\mathbb{R}^{2d}), \|\Psi\|_{H_W^2}^2 := \max_{|\beta| \leq 2} \int |\partial_z^\beta \Psi|^2(z) W(z) dz < \infty \right\} \quad (3.77)$$

and the dual space

$$H_{W^{-1}}^{-2}(\mathbb{R}^{2d}) = \left(H_{W^{-1}}^2(\mathbb{R}^{2d}) \right)' \text{ with } \|h\|_{H_{W^{-1}}^{-2}} = \sup \left\{ \langle h, \Psi \rangle, \|\Psi\|_{H_W^2} = 1 \right\}. \quad (3.78)$$

We may also define the intermediate Sobolev spaces $H_W^\sigma(\mathbb{R}^{2d})$, $H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d})$ for non-integer $0 < \sigma < 2$. Note that $\|\Psi\|_{H^2} \leq \|\Psi\|_{H_W^2}$ so that $\|h\|_{H_{W^{-1}}^{-2}} \leq \|h\|_{H^{-2}}$.

Proposition 3.3.5. *Let us assume that the initial data f_0 satisfies, for some $\delta > 1$ and $\theta \in (0, 1)$,*

$$\int_{z \in \mathbb{R}^{2d}} |f_0(z)|^p dz + \int_{z \in \mathbb{R}^{2d}} (|x|^\delta + |v|^k) f_0(z) dz < \infty \text{ with } p = 1 + \frac{1}{\theta}, \quad k > \frac{4}{1-\theta}. \quad (3.79)$$

Then for all $\sigma > 0$, the family of random variables $(f^R)_{R>0}$ is tight in $C([0, T]; H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d}))$.

Démonstration. Without loss of generality, let $\sigma \in (0, 2)$. For some $\alpha \in (0, 1/2)$ and $M > 0$, let us introduce the set

$$K_M = \left\{ f \in C([0, T]; H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d})) \mid \|f\|_{L_t^\infty L_x^2} \leq M, \|f\|_{C_t^\alpha H_{W^{-1}}^{-2}} \leq M \right\},$$

where $\|\cdot\|_{C_t^\alpha H_{W^{-1}}^{-2}}$ denotes the α -Hölder semi-norm

$$\|f\|_{C_t^\alpha H_{W^{-1}}^{-2}} := \sup_{t \neq s} \frac{\|f(t) - f(s)\|_{H_{W^{-1}}^{-2}}}{|t - s|^\alpha} \leq \|f\|_{C_t^\alpha H_{W^{-1}}^{-2}}.$$

Since the embedding $H_W^\sigma(\mathbb{R}^{2d}) \subset L^2(\mathbb{R}^{2d})$ is compact, the dual embedding $L^2(\mathbb{R}^{2d}) \subset H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d})$ is compact. Additionally, for $f \in K_M$, an interpolation inequality (in weighted Sobolev spaces) yields, for some $\tau \in (0, 1)$,

$$\|f\|_{C_t^\alpha H_{W^{-1}}^{-\sigma}} \lesssim \|f\|_{L_t^\infty L_{W^{-1}}^2}^\tau \|f\|_{C_t^\alpha H_{W^{-1}}^{-2}}^{1-\tau} \lesssim M^2.$$

Consequently, Arzelà-Ascoli's theorem guarantees that K_M is a relatively compact subset of the separable, complete space $C([0, T]; H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d}))$. Markov's inequality gives

$$\mathbb{P}\left[f^R \notin K_M\right] \leq M^{-2} \mathbb{E}\left[\sup_{t \in [0, T]} \|f^R(t)\|_{L^2}^2\right] + M^{-\gamma} \mathbb{E}\left[\|f^R\|_{C_t^\alpha H_{W^{-1}}^{-2}}^\gamma\right].$$

for $\gamma > 0$. The first term is bounded uniformly in R thanks to Proposition 3.3.1. The bound on the second term results directly from Kolmogorov's continuity theorem and the following lemma.

Lemma 3.3.1. *For some $q > 1$, for all $t, s \in [0, T]$,*

$$\mathbb{E}\left[\|f^R(t) - f^R(s)\|_{H_{W^{-1}}^{-2}}^{2q}\right] \lesssim |t - s|^q.$$

The constant involved in \lesssim depends on q, f_0 and T only.

We now prove this lemma : let $0 \leq s \leq t \leq T$. Let us simply denote $f = f^R$ and note that

$$\mathbb{E}\left[\|f(t) - f(s)\|_{H_{W^{-1}}^{-2}}^{2q}\right] = \mathbb{E}\left[\left|\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |\xi|^4 + |\zeta|^4)^{-1} \left|\mathcal{F}f(t, \xi, \zeta) - \mathcal{F}f(s, \xi, \zeta)\right|^2 d\xi d\zeta\right|^q\right].$$

As in the proof of Proposition (3.3.4), we may apply the (x, v) -Fourier transform $\mathcal{F}f(t, \xi, \zeta)$ to equation (3.60) to get (forgetting the summation signs)

$$\mathcal{F}f(t) - \mathcal{F}f(s) = \int_s^t \left(-i\xi \cdot \mathcal{F}(vf) + i\zeta^i \mathcal{F}(G^i) - \zeta^i \zeta^j \mathcal{F}(G^{ij}) \right) d\sigma + i \int_s^t \zeta^i \mathcal{F}(H^i) d\beta_\sigma.$$

Itô's formula results in

$$\begin{aligned} \left|\mathcal{F}f(t) - \mathcal{F}f(s)\right|^2 &= 2 \int_s^t \overline{\mathcal{F}f(\sigma) - \mathcal{F}f(s)} \left(-i\xi \cdot \mathcal{F}(vf) + i\zeta^i \mathcal{F}(G^i) - \zeta^i \zeta^j \mathcal{F}(G^{ij}) \right) d\sigma \\ &\quad + \int_s^t |\zeta^i|^2 |\mathcal{F}(H^i)|^2 d\sigma + 2i \int_s^t \overline{\mathcal{F}f(\sigma) - \mathcal{F}f(s)} \zeta^i \mathcal{F}(H^i) d\beta_\sigma \end{aligned}$$

so that, integrating against $(1 + |\xi|^4 + |\zeta|^4)^{-1} d\xi d\zeta$, we get

$$\|f(t) - f(s)\|_{H_{W^{-1}}^{-2}}^2 \lesssim \int_s^t \|f(\sigma) - f(s)\|_{H_{W^{-1}}^{-2}}^2 d\sigma + D_t + M_t \quad (3.80)$$

where

$$\begin{aligned} D_t &= \int_s^t \int_{\mathbb{R}^{2d}} \left(|\mathcal{F}(vf)|^2 + |\mathcal{F}(G^i)|^2 + |\mathcal{F}(G^{ij})|^2 + |\mathcal{F}(H^i)|^2 \right) d\xi d\zeta d\sigma, \\ M_t &= 2i \int_s^t \left(\int_{\mathbb{R}^{2d}} (1 + |\xi|^4 + |\zeta|^4)^{-1} \overline{\mathcal{F}f(\sigma) - \mathcal{F}f(s)} \zeta^i \mathcal{F}(H^i) d\xi d\zeta \right) d\beta_\sigma. \end{aligned}$$

From (3.80) we derive

$$\mathbb{E}\|f(t) - f(s)\|_{H^{-2}}^{2q} \lesssim \int_s^t \mathbb{E}\|f(\sigma) - f(s)\|_{H^{-2}}^{2q} d\sigma + \mathbb{E}[|D_t|^q] + \mathbb{E}[|M_t|^q]. \quad (3.81)$$

Plancherel's identity gives

$$\begin{aligned} |D_t|^q &= \left| \int_s^t \left(\|vf\|_{L^2}^2 + \|G^i\|_{L^2}^2 + \|G^{ij}\|_{L^2}^2 + \|H^i\|_{L^2}^2 \right) d\sigma \right|^q \\ &\lesssim |t-s|^{q-1} \int_s^t \left(\|vf\|_{L^2}^{2q} + \|G^i\|_{L^2}^{2q} + \|G^{ij}\|_{L^2}^{2q} + \|H^i\|_{L^2}^{2q} \right) d\sigma \end{aligned}$$

so that

$$\mathbb{E}[|D_t|^q] \lesssim |t-s|^q \mathbb{E} \left[\sup_{t \in [0, T]} \left(\|vf(t)\|_{L^2}^{2q} + \|G^i(t)\|_{L^2}^{2q} + \|G^{ij}(t)\|_{L^2}^{2q} + \|H^i(t)\|_{L^2}^{2q} \right) \right]$$

Considering (3.74), (3.75), (3.76), we see that we essentially need a bound (uniform in R) on

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left| \int_{\mathbb{R}^{2d}} (1 + |v|^4) |f(t, z)|^2 dz \right|^q \right]. \quad (3.82)$$

This is possible since, for any $m > 2d$,

$$\begin{aligned} \left| \int_{\mathbb{R}^{2d}} (1 + |v|^4) |f(t, z)|^2 dz \right|^q &\lesssim \int_{\mathbb{R}^{2d}} (1 + |v|^{4q})(1 + |z|)^{m(q-1)} |f|^{2q} dz \\ &\lesssim \int_{\mathbb{R}^{2d}} (1 + |v|^{4q+m(q-1)} + |x|^{m(q-1)}) |f|^{2q} dz \end{aligned}$$

so that, for $\tau \in (0, 1)$,

$$\begin{aligned} \left| \int_{\mathbb{R}^{2d}} (1 + |v|^4) |f(t, z)|^2 dz \right|^q &\lesssim \int_{\mathbb{R}^{2d}} (1 + |x|^{\frac{m(q-1)}{1-\tau}} + |v|^{\frac{4q+m(q-1)}{1-\tau}}) f dz + \int_{\mathbb{R}^{2d}} |f|^{1+\frac{2q-1}{\tau}} dz \\ &\lesssim \int_{\mathbb{R}^{2d}} (1 + |x|^\delta + |v|^k) f dz + \int_{\mathbb{R}^{2d}} |f|^p dz \end{aligned}$$

whenever, recalling (3.79), for some $\gamma > 0$,

$$\frac{2q-1}{\tau} = \frac{1}{\theta}, \quad \frac{m(q-1)}{1-\tau} \leq \delta, \quad \frac{4q+m(q-1)}{1-\tau} \leq k := \frac{4+\gamma}{1-\theta}.$$

These requirements can be met for some $q = q(\gamma) > 1$ close enough to 1 (and τ close to θ). As for the martingale term, Burkholder-Davis-Gundy's inequality gives

$$\begin{aligned} \mathbb{E}[|M_t|^q] &\lesssim \mathbb{E} \left[\left| \int_s^t \left(\int_{\mathbb{R}^{2d}} (1 + |\xi|^4 + |\zeta|^4)^{-1} |\mathcal{F}f(\sigma) - \mathcal{F}f(s)| |\zeta| |\mathcal{F}(H^i)| d\xi d\zeta \right)^2 d\sigma \right|^{q/2} \right] \\ &\lesssim \mathbb{E} \left[\left| \int_s^t \|f(\sigma) - f(s)\|_{H^{-2}}^2 \|H^i\|_{L^2}^2 d\sigma \right|^{q/2} \right] \\ &\lesssim \mathbb{E} \left[\left| \sup_{t \in [0, T]} \|H^i(t)\|_{L^2}^2 \right|^{q/2} \left| \int_s^t \|f(\sigma) - f(s)\|_{H^{-2}}^2 d\sigma \right|^{q/2} \right] \\ &\lesssim \mathbb{E} \left[\sup_{t \in [0, T]} \|H^i(t)\|_{L^2}^{2q} \right] + \int_s^t \mathbb{E}\|f(\sigma) - f(s)\|_{H^{-2}}^{2q} d\sigma. \end{aligned} \quad (3.83)$$

Considering (3.76), the first term in (3.83) is again controlled by the bound on (3.82). We may then come back to (3.81) and use Grönwall's lemma to conclude. \square

Proposition 3.3.6. *Let us assume that the initial data f_0 satisfies (3.79).*

Then, for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, the family of random variables $(\rho_\varphi^R)_{R>0}$ is tight in $L^2([0, T]; L^2(\mathbb{R}^d))$.

Démonstration. Let us introduce the weight function

$$V(x) = 1 + |x|$$

and the associated weighted spaces $H_V^2(\mathbb{R}^d)$ and $H_{V^{-1}}^{-2}(\mathbb{R}^d)$ as in (3.77) and (3.78). One could prove the following lemma as previously.

Lemma 3.3.2. *For some $q > 1$, for all $t, s \in [0, T]$,*

$$\mathbb{E} \left[\|\rho_\varphi^R(t) - \rho_\varphi^R(s)\|_{H^{-2}}^{2q} \right] \lesssim |t - s|^q.$$

The constant involved in \lesssim depends on q, f_0, φ and T only.

We may now fix some $M > 0$ and naturally introduce the set

$$K_M = \left\{ \rho_\varphi := \int \varphi(v) f dv \mid f \in \mathcal{F}_M \right\},$$

where \mathcal{F}_M denotes the set of functions $f \equiv f(t, x, v)$ satisfying, for $\eta = 1/6$ and some $\alpha \in (0, 1/2)$,

$$\sup_{t \in [0, T]} \|f(t)\|_{L_z^p}^p \leq M, \quad (3.84)$$

$$\sup_{t \in [0, T]} \int (1 + |x|^\delta + |v|^k) f(t) dz \leq M, \quad (3.85)$$

$$\|\rho_\varphi\|_{L_t^2 H_x^\eta}^2 \leq M, \quad (3.86)$$

$$\|\rho_\varphi\|_{C_t^\alpha H^{-2}} \leq M.$$

Markov's inequality, gives, for some $\gamma > 0$,

$$\begin{aligned} \mathbb{P} \left[\rho_\varphi^R \notin K_M \right] &\leq M^{-1} \mathbb{E} \left[\sup_{t \in [0, T]} \|f^R(t)\|_{L_z^p}^p \right] + M^{-1} \mathbb{E} \left[\sup_{t \in [0, T]} \int (1 + |x|^\delta + |v|^k) f^R(t) dz \right] \\ &\quad + M^{-1} \mathbb{E} \left[\|\rho_\varphi^R\|_{L_t^2 H_x^\eta}^2 \right] + M^{-\gamma} \mathbb{E} \left[\|\rho_\varphi^R\|_{C_t^\alpha H^{-2}}^\gamma \right] \end{aligned}$$

which tends to zero uniformly in $R > 0$ as M goes to infinity, thanks to Proposition 3.3.1, 3.3.3, 3.3.4 and Lemma 3.3.2. It only remains to prove that K_M is a relatively compact subset of $L^2([0, T]; L^2(\mathbb{R}^d))$. Let us introduce a sequence $(\rho_\varphi^n)_{n \geq 1}$ in K_M .

First, let us show that $(\rho_\varphi^n)_n$ is compact locally in space, that is in $L^2([0, T]; L^2(B(0, r)))$ for any $r > 0$. Since $\|\rho_\varphi^n\|_{L_t^\infty L_x^1} \leq \|\varphi\|_{L^\infty}$, we deduce from (3.84) that

$$\|\rho_\varphi^n\|_{L_t^\infty L_x^2} \lesssim M.$$

Similarly to the proof of Proposition 3.3.5, we may then use Arzelà-Ascoli's theorem to deduce that $(\rho_\varphi^n)_n$ converges in $C([0, T]; H_{V^{-1}}^{-2}(\mathbb{R}^d))$ up to some subsequence (which we omit for clarity). An interpolation inequality (in weighted Sobolev spaces : $H_{V^{-1}}^\eta \subset L_{V^{-1}}^2 \subset H_{V^{-1}}^{-2}$) yields

$$\|\rho\|_{L^2(B(0, r))} \lesssim \|\rho\|_{L_{V^{-1}}^2} \lesssim \|\rho\|_{H_{V^{-1}}^{-2}}^\tau \|\rho\|_{H_{V^{-1}}^\eta}^{1-\tau} \lesssim \|\rho\|_{H_{V^{-1}}^{-2}}^\tau \|\rho\|_{H^\eta}^{1-\tau}$$

where $\tau = \frac{\eta}{\eta+2} \in (0, 1)$. It follows that

$$\begin{aligned} \int_0^T \|\rho_\varphi^n(t) - \rho_\varphi^m(t)\|_{L^2(B(0, r))}^2 dt &\leq \int_0^T \|\rho_\varphi^n(t) - \rho_\varphi^m(t)\|_{H_{V^{-1}}^{-2}}^{2\tau} \times \|\rho_\varphi^n(t) - \rho_\varphi^m(t)\|_{H^\eta}^{2(1-\tau)} dt \\ &\leq \left(\int_0^T \|\rho_\varphi^n(t) - \rho_\varphi^m(t)\|_{H_{V^{-1}}^{-2}}^2 dt \right)^\tau \left(\int_0^T \|\rho_\varphi^n(t) - \rho_\varphi^m(t)\|_{H^\eta}^2 dt \right)^{1-\tau} \\ &\leq T^\tau \|\rho_\varphi^n - \rho_\varphi^m\|_{C([0, T]; H_{V^{-1}}^{-2})}^{2\lambda} \times \|\rho_\varphi^n - \rho_\varphi^m\|_{L_t^2 H^\eta}^{2(1-\tau)}. \end{aligned}$$

Thanks to (3.86), we deduce that

$$\|\rho_\varphi^n - \rho_\varphi^m\|_{L^2([0, T]; L^2(B(0, r)))} \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

To derive compactness globally in space, that is in $L^2([0, T]; L^2(\mathbb{R}^d))$, it is enough to establish a uniform integrability estimate of the form

$$\sup_{n \geq 1} \left[\int_0^T \int_{|x| \geq r} |\rho_\varphi^n(t, x)|^2 dx dt \right] \rightarrow 0 \text{ as } r \rightarrow \infty.$$

To this intent, since φ is compactly supported, we may simply write

$$\begin{aligned} \int_{|x| \geq r} |\rho_\varphi^n(x)|^2 dx &\lesssim \int_{\mathbb{R}^d} \int_{|x| \geq r} |f^n(x, v)|^2 dx dv \leq r^{-\gamma} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 + |x|^\gamma) |f^n(x, v)|^2 dx dv \\ &\lesssim r^{-\gamma} \left(\int_{\mathbb{R}^{2d}} (1 + |x|^\delta) f^n(z) dz + \int_{\mathbb{R}^{2d}} |f^n(z)|^p dz \right) \end{aligned}$$

where $\gamma = \delta(1 - \theta) > 0$, according to (3.79), and then use the bounds (3.84) and (3.85) to conclude. \square

3.3.4 Convergence of the martingale problem

Let us introduce a sequence $R_n \rightarrow \infty$ and a countable subset \mathcal{D} of $C_c^\infty(\mathbb{R}^d)$, which we assume to contain the truncation functions

$$\{\theta_{R_n}, n \geq 1\} \subset \mathcal{D} \subset C_c^\infty(\mathbb{R}^d). \quad (3.87)$$

Recall that the function $\theta_R(v)$ has been introduced in (3.17). Since \mathcal{D} is countable, it follows from Proposition 3.3.5, Proposition 3.3.6 and Tykhonov's theorem that the family of random variables $(f^{R_n}, (\rho_\varphi^{R_n})_{\varphi \in \mathcal{D}})_{n \geq 1}$ is tight in the space

$$C([0, T]; H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d})) \times \left(L^2([0, T]; L^2(\mathbb{R}^d)) \right)^{\mathcal{D}}$$

for $\sigma > 0$. Using Skorokhod's representation theorem, up to a subsequence of $(R_n)_n$ which we omit for simplicity, we may introduce random variables $\bar{f}^n, \bar{\rho}_\varphi^n, \bar{f}, \bar{\rho}_\varphi$ defined on some other probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ such that, for all $n \geq 1$,

$$(\bar{f}^n, (\bar{\rho}_\varphi^n)_{\varphi \in \mathcal{D}}) \sim (f^{R_n}, (\rho_\varphi^{R_n})_{\varphi \in \mathcal{D}}) \text{ in law, in } C([0, T]; H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d})) \times \left(L^2([0, T]; L^2(\mathbb{R}^d)) \right)^{\mathcal{D}},$$

and the following convergences hold $\bar{\mathbb{P}}$ -almost surely :

$$\begin{aligned} \bar{f}^n &\rightarrow \bar{f} \text{ in } C([0, T]; H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d})) \text{ a.s.}, \\ \forall \varphi \in \mathcal{D}, \bar{\rho}_\varphi^n &\rightarrow \bar{\rho}_\varphi \text{ in } L^2([0, T]; L^2(\mathbb{R}^d)) \text{ a.s.} \end{aligned}$$

More precisely, since Proposition 3.3.1 and Proposition 3.3.4 provide the bounds

$$\bar{\mathbb{E}} \left[\|\bar{f}^n\|_{L_t^\infty L_x^2}^2 \right] = \mathbb{E} \left[\|f^n\|_{L_t^\infty L_x^2}^2 \right] \lesssim 1, \quad \bar{\mathbb{E}} \left[\|\bar{\rho}_\varphi^n\|_{L_t^2 H_x^\eta}^2 \right] = \mathbb{E} \left[\|\rho_\varphi^n\|_{L_t^2 H_x^\eta}^2 \right] \lesssim 1 \quad (3.88)$$

uniformly in $n \geq 1$, we derive in particular that the families of random variables $(\bar{f}^n)_n$ and $(\bar{\rho}_\varphi^n)_n$ are uniformly integrable, and therefore

$$\bar{f}^n \rightarrow \bar{f} \text{ a.s and in } L^1(\bar{\Omega}; C([0, T]; H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d}))). \quad (3.89)$$

$$\forall \varphi \in \mathcal{D}, \bar{\rho}_\varphi^n \rightarrow \bar{\rho}_\varphi \text{ a.s and in } L^1(\bar{\Omega}; L^2([0, T]; L^2(\mathbb{R}^d))). \quad (3.90)$$

Consequently, up to a subsequence, we may also assume that

$$\forall \varphi \in \mathcal{D}, \left[\bar{\rho}_\varphi^n(t, x) \rightarrow \bar{\rho}_\varphi(t, x), dt \otimes dx\text{-a.e.} \right], \bar{\mathbb{P}}\text{-a.s.} \quad (3.91)$$

Remark 3.3.1. For all $\varphi \in \mathcal{D}$, from the equality $\rho_\varphi^n = \int \varphi(v) f^n dv$, \mathbb{P} -a.s, it is clear that $\bar{\rho}_\varphi^n = \int_{\mathbb{R}^d} \varphi(v) \bar{f}^n dv$, $\bar{\mathbb{P}}$ -a.s. The convergence (3.89) then guarantees that, for all $\varphi \in \mathcal{D}$, $\bar{\rho}_\varphi$ is indeed given by

$$\bar{\rho}_\varphi = \int_{\mathbb{R}^d} \varphi(v) \bar{f} dv, \quad \bar{\mathbb{P}}\text{-a.s.}$$

Let us now introduce the averaged quantities

$$\rho = \int_{\mathbb{R}^d} f dv, \quad j = \int_{\mathbb{R}^d} v f dv.$$

By requiring some greater moments for the initial data, we may extend the convergence (3.90) to ρ and j .

Lemma 3.3.3. *Assume that the initial data f_0 satisfies, for some $\theta \in (0, 1)$,*

$$\int_{z \in \mathbb{R}^{2d}} |f_0(z)|^p dz + \int_{z \in \mathbb{R}^{2d}} (1 + |v|^k) f_0(z) dz < \infty \quad \text{with } p = 1 + \frac{1}{\theta}, \quad k > \frac{d+2}{1-\theta}. \quad (3.92)$$

Then the following convergences hold in $L^1(\bar{\Omega}; L^2([0, T]; L^2(\mathbb{R}^d)))$:

$$\begin{aligned} \bar{\rho}^n &\rightarrow \bar{\rho}, & \bar{j}^n &\rightarrow \bar{j}, & \bar{\rho}_{\theta R_n}^n &\rightarrow \bar{j}, \\ \phi * \bar{\rho}^n &\rightarrow \phi * \bar{\rho}, & \phi * \bar{j}^n &\rightarrow \phi * \bar{j}, & \phi * \bar{\rho}_{\theta R_n}^n &\rightarrow \phi * \bar{j} \end{aligned}$$

Consequently, up to a subsequence, we may also assume that these convergences hold $dt \otimes dx$ almost everywhere, $\bar{\mathbb{P}}$ almost surely.

Démonstration. Let us, for instance, prove the convergence of \bar{j}^n . For fixed $N \geq 1$, denoting $\varphi = \theta_{R_N}$,

$$\|\bar{j}^n - \bar{j}\|_{L_t^2 L_x^2} \leq \|\bar{j}^n - \bar{\rho}_\varphi^n\|_{L_t^2 L_x^2} + \|\bar{\rho}_\varphi^n - \bar{\rho}\|_{L_t^2 L_x^2} + \|\bar{\rho}_\varphi - \bar{j}\|_{L_t^2 L_x^2} \quad (3.93)$$

and we note that, for any $\gamma > 0$ and $m > d$,

$$\begin{aligned} |\bar{j}^n - \bar{\rho}_\varphi^n|^2 &= \left| \int |v - \theta_{R_N}(v)| \bar{f}^n dv \right|^2 \lesssim \left| \int_{|v| \geq R_N} |v| \bar{f}^n dv \right|^2 \leq R_N^{-2\gamma} \left| \int_{\mathbb{R}^d} |v|^{1+\gamma} \bar{f}^n dv \right|^2 \\ &\lesssim R_N^{-2\gamma} \int_{\mathbb{R}^d} (1 + |v|)^{2+2\gamma+m} |\bar{f}^n|^2 dv \\ &\lesssim R_N^{-2\gamma} \left(\int_{\mathbb{R}^d} (1 + |v|)^{\frac{2+2\gamma+m}{1-\theta}} \bar{f}^n dv + \int_{\mathbb{R}^d} |\bar{f}^n|^{1+\frac{1}{\theta}} dv \right) \\ &\lesssim R_N^{-2\gamma} \left(\int_{\mathbb{R}^d} (1 + |v|)^k \bar{f}^n dv + \int_{\mathbb{R}^d} |\bar{f}^n|^p dv \right) \end{aligned}$$

for (γ, m) close enough to $(0, d)$. As a result,

$$\sup_{n \geq 1} \bar{\mathbb{E}} \left[\|\bar{j}^n - \bar{\rho}_\varphi^n\|_{L_t^2 L_x^2}^2 \right] \lesssim R_N^{-2\gamma} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Since, from (3.89), we classically derive

$$\bar{\mathbb{E}} \left[\int_0^T \int_{\mathbb{R}^{2d}} \left((1 + |v|)^k \bar{f} + |\bar{f}|^p \right) dz dt \right] \leq \sup_{n \geq 1} \bar{\mathbb{E}} \left[\int_0^T \int_{\mathbb{R}^{2d}} \left((1 + |v|)^k \bar{f}^n + |\bar{f}^n|^p \right) dz dt \right] \lesssim 1$$

we deduce similarly that $\bar{\mathbb{E}} \left[\|\bar{j} - \bar{\rho}_\varphi\|_{L_t^2 L_x^2}^2 \right] \rightarrow 0$ as N goes to infinity. We may hence come back to (3.93) and conclude. The convergence of the convoluted functions is easily deduced. \square

For fixed $n \geq 1$, f^{R_n} defines a (strong) solution of (3.24) on $(\Omega, \mathcal{F}, \mathbb{P})$. In particular, it satisfies the associated martingale problem : recalling the operator $\mathcal{L}_R[f]$ defined in (3.25), for all $\Psi \in C_c^\infty(\mathbb{R}^{2d})$, the process

$$M_\Psi^n(t) = \langle \Psi, f^{R_n}(t) \rangle - \langle \Psi, f_0 \rangle - \int_0^t \left\langle \mathcal{L}_{R_n}[f^{R_n}(s)] \Psi, f^{R_n}(s) \right\rangle ds, \quad t \in [0, T] \quad (3.94)$$

defines a continuous, real valued L^2 martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the filtration

$$\mathcal{F}_t^n = \sigma \left(f^{R_n}(s) \in H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d}), \quad s \in [0, t] \right), \quad t \in [0, T].$$

Its quadratic variation is given by

$$\left[M_{\Psi}^n \right](t) = V_{\Psi}^n(t) := \int_0^t \left| \left\langle K_{R_n}[f^{R_n}(s)] \cdot \nabla_v \Psi, f^{R_n}(s) \right\rangle \right|^2 ds. \quad (3.95)$$

We are now ready to state the following result.

Proposition 3.3.7. *Let us introduce, on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, the filtration*

$$\bar{\mathcal{F}}_t = \sigma\left(\bar{f}(s) \in H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d}), s \in [0, t]\right), \quad t \in [0, T].$$

Recalling the operator $\mathcal{L}[f]$ defined in (3.15), for all test functions Ψ of the form

$$\Psi(x, v) = \Psi_1(x)\Psi_2(v), \quad \Psi_1, \Psi_2 \in C_c^\infty(\mathbb{R}^d) \text{ with } \nabla_v \Psi_2 \in \mathcal{D}, \quad (3.96)$$

the process

$$\bar{M}_{\Psi}(t) = \langle \Psi, \bar{f}(t) \rangle - \langle \Psi, f_0 \rangle - \int_0^t \left\langle \mathcal{L}[\bar{f}(s)]\Psi, \bar{f}(s) \right\rangle ds, \quad t \in [0, T]$$

defines a continuous, real-valued L^2 martingale with respect to $(\bar{\mathcal{F}}_t)_{t \geq 0}$, with quadratic variation

$$\left[\bar{M}_{\Psi} \right](t) = \bar{V}_{\Psi}(t) := \int_0^t \left| \left\langle K[\bar{f}(s)] \cdot \nabla_v \Psi, \bar{f}(s) \right\rangle \right|^2 ds.$$

Remark 3.3.2. *The assumption $\nabla_v \Psi_2 \in \mathcal{D}$ in (3.96) is only technical : for any given countable family $F = (\Psi_2)_{\Psi_2 \in F}$ of test functions in $C_c^\infty(\mathbb{R}^d)$, on can initially choose the countable subset \mathcal{D} such that $\{\nabla_v \Psi_2, \Psi_2 \in F\} \subset \mathcal{D}$, so that the conclusion of Proposition 3.3.7 holds true for all $\Psi(x, v) = \Psi_1(x)\Psi_2(v)$ with $\Psi_1 \in C_c^\infty(\mathbb{R}^d)$ and $\Psi_2 \in F$.*

Proof of Proposition 3.3.7. The martingale problem set on Ω may be expressed as

$$\begin{aligned} \mathbb{E} \left[\left(M_{\Psi}^n(t) - M_{\Psi}^n(s) \right) H^n \right] &= 0, \\ \mathbb{E} \left[\left| M_{\Psi}^n(t) - M_{\Psi}^n(s) \right|^2 H^n \right] &= \mathbb{E} \left[\left(V_{\Psi}^n(t) - V_{\Psi}^n(s) \right) H^n \right] \end{aligned}$$

for all $H^n = h\left(f^{R_n}(t_i), 1 \leq i \leq m\right)$, where $0 \leq t_1, \dots, t_m \leq s \leq t$, and $h : (H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d}))^m \rightarrow \mathbb{R}$ is continuous and bounded, Since the laws of f^n and \bar{f}^n coincide, it follows that, on $\bar{\Omega}$,

$$\bar{\mathbb{E}} \left[\left(\bar{M}_{\Psi}^n(t) - \bar{M}_{\Psi}^n(s) \right) \bar{H}^n \right] = 0, \quad (3.97)$$

$$\bar{\mathbb{E}} \left[\left| \bar{M}_{\Psi}^n(t) - \bar{M}_{\Psi}^n(s) \right|^2 \bar{H}^n \right] = \bar{\mathbb{E}} \left[\left(\bar{V}_{\Psi}^n(t) - \bar{V}_{\Psi}^n(s) \right) \bar{H}^n \right] \quad (3.98)$$

where $\bar{H}^n = h\left(\bar{f}^n(t_i), 1 \leq i \leq m\right)$ and \bar{M}_{Ψ}^n and $\bar{V}_{\Psi}^n(t)$ are naturally defined on $\bar{\Omega}$, as \bar{M}_{Ψ} and $\bar{V}_{\Psi}(t)$. We may decompose these into

$$\bar{M}_{\Psi}^n(t) = \langle \Psi, \bar{f}^n(t) \rangle - \langle \Psi, f_0 \rangle - \int_0^t \left(\left\langle v \cdot \nabla_x \Psi, \bar{f}^n(s) \right\rangle + \sum_{i=1}^4 \mathcal{M}_i^n(\bar{f}^n(s)) \right) ds, \quad (3.99)$$

$$\bar{V}_{\Psi}^n(t) = \int_0^t \left| \mathcal{M}^5(\bar{f}^n(s)) \right|^2 ds,$$

where, recalling expressions (3.18),

$$\begin{aligned}
 \mathcal{M}_n^1(f) &= \int_x \int_v L_{R_n}^{CS}[f] \cdot \nabla_v \Psi f dx dv = \int_x \int_v (\Phi_n^1 * f) \cdot \nabla_v \Psi f dx dv, \\
 \mathcal{M}_n^2(f) &= \int_x \int_v S_{R_n}[f] \cdot \nabla_v \Psi f dx dv \\
 &= \frac{1}{2} \int_x \int_v \left(\tilde{\psi} * (\Phi_n^2 + \Phi_n^3 * f) - \|\tilde{\psi}\|_{L^1} (\Phi_n^2 + \Phi_n^3 * f) \right) \cdot \nabla_v \Psi f dx dv, \\
 \mathcal{M}_n^3(f) &= \frac{1}{2} \sum_{1 \leq i, j \leq d} \int_x \int_v K_{R_n}^i[f] K_{R_n}^j[f] \partial_{v_i v_j}^2 \Psi f dx dv \\
 &= \frac{1}{2} \sum_{1 \leq i, j \leq d} \int_x \int_v \left(\Phi_n^2 + \Phi_n^3 * f \right)^i \left(\Phi_n^2 + \Phi_n^3 * f \right)^j \partial_{v_i v_j}^2 \Psi f dx dv, \\
 \mathcal{M}_n^4(f) &= \int_x \int_v L_{R_n}^{MT}[f] \cdot \nabla_v \Psi f dx dv = \int_x \int_v (u_R[f] - v) \cdot \nabla_v \Psi f dx dv \\
 &= \int_x \frac{\phi * \rho_{\theta_{R_n}}}{R^{-1} + \phi * \rho} \cdot \rho_{\nabla \Psi_2} \Psi_1 dx - \int_x \int_v v \cdot \nabla_v \Psi f dx dv \\
 \mathcal{M}_n^5(f) &= \int_x \int_v K_{R_n}[f] \cdot \nabla_v \Psi f dx dv = \int_x \int_v (\Phi_n^2 + \Phi_n^3 * f) \cdot \nabla_v \Psi f dx dv
 \end{aligned}$$

with

$$\begin{aligned}
 \Phi_n^1(x, v) &= \chi_{R_n}(x) \psi(x) \theta_{R_n}(-v), \\
 \Phi_n^2(x) &= \chi_{R_n}(x) F(x), \\
 \Phi_n^3(x, v) &= \chi_{R_n}(x) \tilde{\psi}(x) \theta_{R_n}(-v),
 \end{aligned}$$

We wish to send n to infinity in (3.97) and (3.98). Thanks to the convergence (3.89), the first three linear terms in (3.99) cause no issue; let us hence focus on the remaining terms. Let us consider the term involving $\mathcal{M}_n^1(f)$: defining the natural limiting term

$$\mathcal{M}^1(f) = \int_x \int_v (\Phi^1 * f) \cdot \nabla_v \Psi f dx dv$$

with $\Phi^1(x, v) = \psi(x)(-v)$, we have

$$\left| \int_s^t \mathcal{M}_n^1(\bar{f}^n(\sigma)) d\sigma - \int_s^t \mathcal{M}^1(\bar{f}(\sigma)) d\sigma \right| \leq \mathcal{J}_n^1 + \mathcal{J}_n^2 \quad (3.100)$$

where

$$\begin{aligned}
 \mathcal{J}_n^1 &= \int_s^t \int_x \int_v \left| (\Phi_n^1 * \bar{f}^n) - (\Phi^1 * \bar{f}) \right| |\nabla_v \Psi| |\bar{f}| dx dv d\sigma, \\
 \mathcal{J}_n^2 &= \left| \int_s^t \int_x \int_v (\Phi_n^1 * \bar{f}^n) \cdot \nabla_v \Psi (\bar{f}^n - \bar{f}) dx dv d\sigma \right|.
 \end{aligned}$$

First, for \mathcal{J}_n^1 we have, thanks to (3.88),

$$\begin{aligned}
 \mathbb{E}[\mathcal{J}^1] &\lesssim \mathbb{E} \left[\int_s^t \int_{z \in \text{Supp}(\Psi)} \left| (\Phi_n^1 * \bar{f}^n) - (\Phi^1 * \bar{f}) \right|^2 dz d\sigma \right]^{1/2} \mathbb{E} \left[\int_s^t \int_z |\bar{f}|^2 dz d\sigma \right]^{1/2} \\
 &\lesssim \mathbb{E} \left[\int_s^t \int_{z \in \text{Supp}(\Psi)} \left| (\Phi_n^1 * \bar{f}^n) - (\Phi^1 * \bar{f}) \right|^2 dz d\sigma \right]^{1/2}.
 \end{aligned} \quad (3.101)$$

For all $n \geq m \geq 1$, for fixed $(x, v) \in \text{Supp}(\Psi)$, we may write

$$\begin{aligned}
 \mathbb{E} \left[\left| \Phi_n^1 * \bar{f}^n - \Phi^1 * \bar{f} \right|^2 \right] &\lesssim \mathbb{E} \left[\left| \Phi_m^1 * \bar{f}^n - \Phi_m^1 * \bar{f} \right|^2 \right] + \mathbb{E} \left[\left| (\Phi_n^1 - \Phi_m^1) * \bar{f}^n \right|^2 \right] \\
 &\quad + \mathbb{E} \left[\left| (\Phi^1 - \Phi_m^1) * \bar{f} \right|^2 \right]
 \end{aligned}$$

which converges to 0 as n goes to infinity thanks to the convergence (3.89), and the bounds

$$\begin{aligned} \mathbb{E} \left[\left| (\Phi_n^1 - \Phi_m^1) * \bar{f}^n \right|^2 \right] &\lesssim \mathbb{E} \left[\left| \int \int_{|v-w| \geq R_m} |v-w| f^n(y, w) dy dw \right|^2 \right] \\ &\lesssim R_m^{-2} \left(|v|^4 + \mathbb{E} \left[\int |w|^4 f^n dy dw \right] \right) \lesssim R_m^{-2} (1 + |v|^4) \end{aligned}$$

and, for some $\gamma > 0$, recalling that $(x, v) \in \text{Supp}(\Psi)$,

$$\mathbb{E} \left[\left| \Phi_m^1 * \bar{f}^n \right|^{2+\gamma}(x, v) \right] \lesssim 1 + \mathbb{E} \left[\int |w|^{2+\gamma} \bar{f}^n dy fw \right] \lesssim 1$$

for all $n \geq m$. Note that this last bound also guarantees the uniform integrability in (ω, σ, x, v) of the integrand in (3.101), so that $\mathbb{E} \left[\mathcal{J}_n^1 \right] \rightarrow 0$. Additionally, for $\gamma > 0$ small enough, the bound

$$\begin{aligned} \mathbb{E} \left[\int_s^t \int_{z \in \text{Supp}(\Psi)} \left| \Phi_n^1 * \bar{f}^n \right|^{2+\gamma} |\bar{f}|^{2+\gamma} dz d\sigma \right] \\ \lesssim \mathbb{E} \left[\int_s^t \int_{z \in \text{Supp}(\Psi)} \left(1 + \int |w|^{2+\gamma} \bar{f}^n dy dw \right) |\bar{f}|^{2+\gamma} dz d\sigma \right] \\ \lesssim \mathbb{E} \left[\int_s^t \int_{\mathbb{R}^{2d}} |v|^k \bar{f}^n + |\bar{f}|^p dz d\sigma \right] \lesssim 1 \end{aligned}$$

guarantees that $\mathbb{E} \left[\left| \mathcal{J}_n^1 \right|^{2+\gamma} \right] \lesssim 1$. Similarly, for \mathcal{J}_n^2 , we write, for all $n \geq m \geq 1$,

$$\begin{aligned} \mathcal{J}_n^2 &\leq \left| \int_s^t \int_x \int_v (\Phi_m^1 * \bar{f}^m) \cdot \nabla_v \Psi(\bar{f}^n - \bar{f}) dx dv d\sigma \right| \\ &\quad + \int_s^t \int_x \int_v \left| (\Phi_m^1 * \bar{f}^m) - (\Phi_n^1 * \bar{f}^n) \right| |\nabla_v \Psi| (|\bar{f}^n| + |\bar{f}|) dx dv d\sigma \end{aligned}$$

so that

$$\begin{aligned} \mathbb{E} \left[\mathcal{J}_n^2 \right] &\lesssim \mathbb{E} \left[\left| \int_s^t \int_x \int_v \left((\Phi_m^1 * \bar{f}^m) \cdot \nabla_v \Psi \right) (\bar{f}^n - \bar{f}) dx dv d\sigma \right| \right] \\ &\quad + \mathbb{E} \left[\int_s^t \int_{z \in \text{Supp}(\Psi)} \left| (\Phi_m^1 * \bar{f}^m) - (\Phi_n^1 * \bar{f}^n) \right|^2 dz d\sigma \right]. \end{aligned}$$

Since, for fixed $m \geq 1$, $(\Phi_m^1 * \bar{f}^m) \cdot \nabla_v \Psi \in C_c^\infty(\mathbb{R}^{2d})$, we conclude in a similar fashion that $\mathbb{E} \left[\mathcal{J}_n^2 \right] \rightarrow 0$ and $\mathbb{E} \left[\left| \mathcal{J}_n^2 \right|^{2+\gamma} \right] \lesssim 1$. Coming back to (3.100), we have shown that

$$\begin{aligned} \int_s^t \mathcal{M}_n^1(\bar{f}^n(\sigma)) d\sigma &\rightarrow \int_s^t \mathcal{M}^1(\bar{f}(\sigma)) d\sigma \text{ in probability,} \\ \mathbb{E} \left[\left| \int_s^t \mathcal{M}_n^1(\bar{f}^n(\sigma)) d\sigma \right|^{2+\gamma} \right] &\lesssim 1, \end{aligned}$$

which is sufficient to pass to the limit in the corresponding term of (3.97) and the left-hand side of (3.98).

The terms involving $\mathcal{M}_n^i(f)$ for $i = 2, 3, 5$ can be treated with similar arguments. Let us now handle the more delicate term, involving $\mathcal{M}_n^4(f)$: we wish to prove that

$$\int_s^t \int_x \frac{\phi * \bar{\rho}_{R_n}^n}{R_n^{-1} + \phi * \bar{\rho}^n} \cdot \bar{\rho}_{\nabla \Psi_2}^n \Psi_1 dx d\sigma = \int_s^t \int_x u_R[\bar{f}^n] \cdot \bar{\rho}_{\nabla \Psi_2}^n \Psi_1 dx d\sigma \quad (3.102)$$

converges to the expected limiting term. Let us introduce

$$\bar{J}^n := u_{R_n}[\bar{f}^n] \cdot \bar{\rho}_{\nabla \Psi_2}^n = u_{R_n}[\bar{f}^n] \cdot \int_v \nabla_v \Psi_2(v) \bar{f}^n dv. \quad (3.103)$$

For some $q > 2$ and $\tau \in (0, 1)$, we have

$$\begin{aligned} \int |\bar{J}^n|^q dx &\lesssim \int_x \int_v |u_{R_n}[\bar{f}^n]|^q |\bar{f}^n|^q dx dv \lesssim \int_x \int_v |u_{R_n}[\bar{f}^n]|^{\frac{q}{1-\tau}} \bar{f}^n dx dv + \int_x \int_v |\bar{f}^n|^{1+\frac{q-1}{\tau}} dx dv \\ &\lesssim \int_x \int_v (1 + |u_{R_n}[\bar{f}^n]|^k) \bar{f}^n dx dv + \int_x \int_v |\bar{f}^n|^p dx dv \end{aligned}$$

as soon as $\tau = \theta(q-1)$ and $\frac{q}{1-\tau} \leq \frac{4}{1-\theta}$, which can be met for q close enough to 2. Hence, thanks to Proposition 3.3.2, we derive

$$\bar{\mathbb{E}} \left[\sup_{t \in [0, T]} \int_x |\bar{J}^n(t, x)|^q dx \right] \lesssim \mathbb{E} \left[\sup_{t \in [0, T]} \left(\int_z (1 + |v|)^k f^n(t, z) dz + \int_z |f^n(t, z)|^p dz \right) \right] \lesssim 1. \quad (3.104)$$

As a consequence, up to some subsequence which we omit for simplicity,

$$\bar{J}^n \rightharpoonup \bar{J} \quad \text{weak * in } L^q(\bar{\Omega}; L^\infty([0, T]; L^q(\mathbb{R}^d))). \quad (3.105)$$

It is clear that this weak convergence is enough for the term (3.102) to pass to the limit in (3.97). Therefore, it only remains to identify the limit as

$$\bar{J} = \bar{u} \cdot \bar{\rho}_{\nabla \Psi_2}, \quad \text{where } \bar{u}(x, t) := u[\bar{f}](x, t) = \begin{cases} \frac{(\phi * \bar{j})(t, x)}{(\phi * \bar{\rho})(t, x)} & \text{if } (\phi * \bar{\rho})(t, x) \neq 0 \\ 0 & \text{if } (\phi * \bar{\rho})(t, x) = 0. \end{cases} \quad (3.106)$$

First, considering the set, for $r > 0$, $\omega \in \bar{\Omega}$,

$$A_r(\omega) = \{(t, x) \in [0, T] \times B(0, r) \mid (\phi * \bar{\rho})(\omega, t, x) = 0\},$$

we have

$$\int_{A_r} |\bar{J}^n| dx dt \lesssim \left(\int_{(t, x) \in A_r} \int_{v \in \text{Supp}(\Psi_2)} |u_{R_n}[\bar{f}^n]|^2 \bar{f}^n dx dt \right)^{1/2} \left(\int_{A_r} \bar{\rho}^n dx dt \right)^{1/2} \quad (3.107)$$

so that, with Proposition 3.3.2,

$$\bar{\mathbb{E}} \left[\int_{A_r(\omega)} |\bar{J}^n| dx dt \right] \lesssim \mathbb{E} \left[\int_0^T \int_{\mathbb{R}^{2d}} |v|^2 f^{R_n} dz dt \right]^{1/2} \bar{\mathbb{E}} \left[\int_{A_r(\omega)} \bar{\rho}^n dx dt \right]^{1/2} \lesssim \bar{\mathbb{E}} \left[\int_{A_r(\omega)} \bar{\rho}^n dx dt \right]^{1/2}.$$

From (3.105) and Lemma 3.3.3, we deduce

$$\bar{\mathbb{E}} \left[\int_0^T \int_{A_r(\omega)} |\bar{J}| dx dt \right] \lesssim \bar{\mathbb{E}} \left[\int_0^T \int_{\{x \in B(0, r) \mid \phi * \bar{\rho} = 0\}} \bar{\rho} dx dt \right]^{1/2} = 0.$$

Indeed, it is easy to see that, when $\rho \in C(\mathbb{R}^d)$,

$$\int_{\{x \in B(0, r) \mid \phi * \bar{\rho} = 0\}} \rho(x) dx = 0$$

and the same equality is deduced for any $\rho \in L^2(\mathbb{R}^d)$ by density. Hence, $\bar{\mathbb{P}}$ almost surely, $\bar{J} = 0$ a.e on A_r . Since this holds for any $r > 0$, we deduce that, $\bar{\mathbb{P}}$ almost surely, $\bar{J} = 0$ holds whenever $\phi * \bar{\rho} = 0$, so that we only have to check equality (3.106) whenever $(\phi * \bar{\rho})(t, x) \neq 0$.

Recalling (3.91) and Lemma 3.3.3, we have

$$\bar{J}^n = \frac{\phi * \bar{\rho}_{\theta R_n}^n}{R_n^{-1} + \phi * \bar{\rho}^n} \cdot \bar{\rho}_{\nabla \Psi_2}^n \rightarrow \frac{\phi * \bar{j}}{\phi * \bar{\rho}} \cdot \bar{\rho}_{\nabla \Psi_2} = \bar{u} \cdot \rho_{\nabla \Psi_2}$$

almost everywhere on the set $\{(t, x) \in [0, T] \times \mathbb{R}^d \mid \phi * \bar{\rho} \neq 0\}$, $\bar{\mathbb{P}}$ almost surely. It follows that equality (3.106) holds.

Finally, it remains to pass (3.102) to the limit in the quadratic equality (3.98). To this intent, we simply notice that

$$\left| \int_s^t \int_x \frac{\phi * \bar{\rho}_{\theta R_n}^n}{R_n^{-1} + \phi * \bar{\rho}^n} \cdot \bar{\rho}_{\nabla \Psi_2}^n \Psi_1 dx d\sigma \right|^2 = \int_{\sigma_1} \int_{\sigma_2} \int_x \int_y \bar{J}^n(\sigma_1, x) \bar{J}^n(\sigma_2, y) \Psi_1(x) \Psi_2(y) dx dy d\sigma_1 d\sigma_2$$

where $\bar{J}^n(\sigma, x)$ is defined as in (3.103). We have the uniform bound

$$\mathbb{E}\left[\left[\sup_{\sigma_1 \in [0, T]} \sup_{\sigma_2 \in [0, T]} \int_x \int_y \left|\bar{J}^n(\sigma_1, x) \bar{J}^n(\sigma_2, y)\right|^q dx dy\right]^{1/2}\right] = \mathbb{E}\left[\left[\sup_{\sigma \in [0, T]} \int_x \left|\bar{J}^n(\sigma, x)\right|^q dx\right]^{1/2}\right] \lesssim 1$$

with $q/2 > 1$ so that, up to some subsequence

$$\bar{J}^n(\sigma_1, x) \bar{J}^n(\sigma_2, y) \rightharpoonup \bar{J}(\sigma_1, \sigma_2, x, y) \text{ weak } * \text{ in } L^{q/2}(\bar{\Omega}; L^\infty([0, T]^2; L^q((\mathbb{R}^d)^2))).$$

This time, thanks to (3.104)

$$\begin{aligned} \mathbb{E}\left[\int_{(x, \sigma_1) \in A_r} \int_{(y, \sigma_2) \in B(0, r) \times [0, T]} \left|\bar{J}^n(\sigma_1, x) \bar{J}^n(\sigma_2, y)\right| dx d\sigma_1 dy d\sigma_2\right] \\ \lesssim \mathbb{E}\left[\left(\int_{A_r} \left|\bar{J}^n\right| dx dt\right)^2\right]^{1/2} \mathbb{E}\left[\int_0^T \int_{B(0, r)} \left|\bar{J}^n\right|^2 dx dt\right]^{1/2} \lesssim \mathbb{E}\left[\left(\int_{A_r} \left|\bar{J}^n\right| dx dt\right)^2\right]^{1/2}. \end{aligned}$$

Using (3.107), it follows that

$$\begin{aligned} \mathbb{E}\left[\left(\int_{A_r} \left|\bar{J}^n\right| dx dt\right)^2\right] &\lesssim \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^{2d}} |u_{R_n}[f^{R_n}]|^4 |f^{R_n}|^2 dz dt\right]^{1/2} \mathbb{E}\left[\int_{A_r} |\bar{\rho}^n|^2 dx dt\right]^{1/2} \\ &\lesssim \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^{2d}} |v|^k f^{R_n} + |f^{R_n}|^p dz dt\right]^{1/2} \mathbb{E}\left[\int_{A_r} |\bar{\rho}^n|^2 dx dt\right]^{1/2} \\ &\lesssim \mathbb{E}\left[\int_{A_r} |\bar{\rho}^n|^2 dx dt\right]^{1/2} \end{aligned}$$

Using again Lemma 3.3.3, we deduce that, $\bar{\mathbb{P}}$ almost surely, $\bar{J} = 0$ almost everywhere on

$$\{(\sigma_1, \sigma_2, x, y) \in [0, T]^2 \times (\mathbb{R}^d)^2 \mid (\phi * \bar{\rho})(\sigma_1, x) = 0 \text{ or } (\phi * \bar{\rho})(\sigma_2, y) = 0\}.$$

The limit is then determined to be $\bar{J}(\sigma_1, \sigma_2, x, y) = \left(\bar{u}(x) \cdot \rho_{\nabla \Psi_2}(\sigma_1, x)\right) \left(\bar{u}(y) \cdot \rho_{\nabla \Psi_2}(\sigma_2, y)\right)$ on the complementary set using pointwise convergence, as done previously. \square

From the martingale problem of Proposition 3.3.7, we classically construct a martingale solution of (3.7), in the sense of Definition 3.1.1, using a martingale representation theorem. First, note that estimates (3.13) and (3.14) are easily derived from the convergence (3.89) using Fatou's Lemma. Introducing the process

$$\bar{M}(t) = \bar{f}(t) - f_0 - \int_0^t (\mathcal{L}[\bar{f}(s)])^* \bar{f}(s) ds, \quad t \in [0, T] \quad (3.108)$$

we see that, for all test function $\Psi \in C_c^\infty(\mathbb{R}^{2d})$ of the form (3.96),

$$\langle \bar{M}(t), \Psi \rangle = \bar{M}_\Psi(t), \quad t \in [0, T]$$

which is a continuous L^2 martingale with respect to the filtration $(\bar{\mathcal{F}}_t)_{t \in [0, T]}$, with quadratic variation \bar{V}_Ψ . With Remark 3.3.2 in mind, by density, we may carefully extend this statement to any test function Ψ in some separable Hilbert space \mathcal{H} . One may convince oneself for instance that the weighted Sobolev space

$$\mathcal{H} = \left\{ \Psi \in D'(\mathbb{R}^{2d}), \max_{|\beta| \leq 2} \int_{\mathbb{R}^{2d}} e^{|z|} |\partial_z^\beta \Psi(z)|^2 dz \right\}$$

is suitable. Using a polarization formula, we deduce that for $\Psi_1, \Psi_2 \in \mathcal{H}$,

$$\langle \bar{M}(t), \Psi_1 \rangle \langle \bar{M}(t), \Psi_2 \rangle - \langle \bar{V}(t) \Psi_1, \Psi_2 \rangle, \quad t \in [0, T]$$

defines a continuous $(\bar{\mathcal{F}}_t)_{t \in [0, T]}$ -martingale, where the operator $\bar{V}(t)$ is defined through

$$\langle \bar{V}(t) \Psi_1, \Psi_2 \rangle = \int_0^t \langle K[\bar{f}(s)] \cdot \nabla_v \Psi_1, \bar{f}(s) \rangle \langle K[\bar{f}(s)] \cdot \nabla_v \Psi_2, \bar{f}(s) \rangle ds.$$

The martingale representation theorem from [DPZ14] (p222, Theorem 9.2) holds for the \mathcal{H}' -valued process (3.108), giving another probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ equipped with a filtration $(\hat{\mathcal{F}}_t)_{t \in [0, T]}$, and a $(\bar{\mathcal{F}}_t \times \hat{\mathcal{F}}_t)$ -brownian motion $(W_t)_{t \in [0, T]}$ on $\bar{\Omega} \times \hat{\Omega}$ such that

$$\bar{M}(t)(\bar{\omega}, \hat{\omega}) := \bar{M}(t)(\bar{\omega}) = - \int_0^t \nabla_v \cdot \left(K[\bar{f}(s)]\bar{f}(s) \right) dW_s(\bar{\omega}, \hat{\omega}).$$

It follows that $(\bar{\omega}, \hat{\omega}) \in \bar{\Omega} \times \hat{\Omega} \mapsto (\bar{f}(t)(\bar{\omega}))_{t \in [0, T]}$ defines a solution of (3.7) on $\bar{\Omega} \times \hat{\Omega}$. This concludes the proof of Theorem 6.

3.3.5 Strong local alignment

The proof of Theorem 7 can be established using the same arguments. Indeed, the only role played by the weight function ϕ in the estimates of sections 3.3.1 and 3.3.2 is through the constant $C(\phi)$ in Proposition 3.3.2, given by (3.53). Introducing ϕ_r of the form $\phi_r(x) = r^d \phi_1(x/r)$ we notice that

$$C(\phi_r) \propto \frac{r^d \sup_{B(0, r_2)} \phi_1}{\inf_{B(0, r_1)} \phi_1} \left(\frac{r_2}{r_1 \times r} \right)^d = \frac{\sup_{B(0, r_2)} \phi_1}{\inf_{B(0, r_1)} \phi_1} \left(\frac{r_2}{r_1} \right)^d \propto C(\phi_1)$$

uniformly in $r > 0$. As a result, introducing f^r a weak solution of (3.7) with $\phi = \phi^r$ constructed (in law) as previously, the following estimates hold uniformly in $r > 0$: for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\mathbb{E} \left[\sup_{t \in [0, T]} \int_{\mathbb{R}^{2d}} |f^r(t, z)|^p + (1 + |x|^2 + |v|^k) f^r(t, z) dz \right] + \mathbb{E} \left[\|\rho_\varphi^r\|_{L_t^2 H_x^q}^2 \right] \lesssim 1$$

and for some $q > 1$,

$$E \left[\|f^r(t) - f^r(s)\|_{H^{-2}}^{2q} \right] + \mathbb{E} \left[\|\rho^r(t) - \rho^r(s)\|_{H^{-2}}^{2q} \right] \leq |t - s|^q.$$

From here, we may use the same arguments as previously to establish the tightness of $(f^r)_{r>0}$ (resp. $(\rho_\varphi^r)_{r>0}$) in $C([0, T]; H_{W^{-1}}^{-\sigma}(\mathbb{R}^{2d}))$ for $\sigma > 0$ (resp. in $L^2([0, T]; L^2(\mathbb{R}^d))$) and then pass to the limit in the martingale problem satisfied by f^{r_n} as n goes to infinity.

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Titre : Limites d'échelles pour des modèles cinétiques stochastiques

Mots clés : Équations aux dérivées partielles stochastiques, Limites hydrodynamiques, Systèmes de particules stochastiques, Limites de champ moyen

Résumé : Cette thèse se propose d'étudier quelques transitions d'échelles pour des modèles cinétiques bruités où l'objet limite est encore de nature aléatoire, régi par une équation aux dérivées partielles stochastique. Dans le premier chapitre, on s'intéresse au passage d'une description mésoscopique à une description macroscopique en considérant un système d'équations – modélisant le comportement d'un « spray » dans un fluide ambiant – perturbé par un processus de Markov mélangeant. Après un changement d'échelle approprié, en s'appuyant sur la méthode de la fonction test perturbée, on montre la convergence de la densité de particules vers une limite hydrodynamique qui s'exprime comme une équation de conservation stochastique dirigée par un processus de Wiener.

Dans un second temps, on étudie des équations cinétiques stochastiques correspondant à des modèles biologiques de mouvement collectif. Cette étude se divise en deux travaux consacrés à deux modèles distincts. Dans le chapitre 2, on s'intéresse à la limite de champ moyen du système de particules stochastique associé à des perturbations aléatoires du modèle classique de Cucker-Smale. Dans le chapitre 3, on étudie un modèle plus fin, rendant compte d'une interaction locale entre les individus, pour lequel on établit l'existence de solutions martingales.

Title : Scaling limits for stochastic kinetic models

Keywords : Stochastic partial differential equations, Hydrodynamic limits, Stochastic particle systems, Mean-field limits

Abstract : This thesis aims at providing an understanding of certain scaling limits for kinetic models perturbed with some random noise, where the limiting object remains of stochastic nature, governed by a stochastic partial differential equation. In the first chapter, the transition from a mesoscopic to a macroscopic description is studied through a kinetic system of equations – corresponding to the behavior of a “spray” of particles embedded in an ambient fluid – perturbed by a mixing Markov process. Under a suitable scaling, relying on the perturbed test function method, we establish the convergence of the density of particles to a hydrodynamic limit which can be expressed as the solution of a stochastic conservation equation driven by a Wiener process.

Next, we focus on stochastic kinetic equations derived from biological models of collective motion. This study is split into two different works, devoted to distinct models. In chapter 2, we first examine the mean-field limits of a few different particle systems which correspond to random perturbations of the classical Cucker-Smale model. Then, in chapter 3, we establish the existence of martingale solutions for some more advanced model, which allows local interactions between individuals.