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SORBONNE UNIVERSITÉ  
ÉCOLE DOCTORALE SCIENCES MATHÉMATIQUES PARIS-CENTRE  
LABORATOIRE JACQUES-LOUIS LIONS

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SHAPE OPTIMIZATION AND SPATIAL HETEROGENEITY IN  
REACTION-DIFFUSION EQUATIONS

---

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THÈSE RÉALISÉE SOUS LA DIRECTION DE

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Paresseux! tu l'as dit. Nous l'étions avec  
gloire;  
Ignorants, Dieu le sait! Ce que j'ai fait  
depuis  
A montré clairement si j'avais rien appris.  
Mais quelle douce odeur avait le réfectoire!

---

Musset, *Dupont et Durand*

---

# REMERCIEMENTS

## AUTHOR'S BIBLIOGRAPHY

### ACCEPTED & PUBLISHED ARTICLES

- [MNP19b] *Optimal location of resources maximizing the total population size in logistic models*, with G. Nadin and Y. Privat, Accepted in *Journal de Mathématiques Pures et Appliquées* (2019) ([Hal Link](#)). This paper corresponds to Chapter 2 of this manuscript. This article gave rise to the proceeding [MNP18] for the 89th GAMM Annual Meeting, March 2018, Munich, Germany.
- [Maz19b] *Equilibria and stability issues for diffusive Lotka-Volterra system in a heterogeneous setting for large diffusivities*, Published in *Discrete and continuous dynamical systems, Series B* (2019) ([Hal Link](#), [Journal link](#)). This article is not presented in this manuscript, which we wanted focused on shape optimization problems.
- [Maz19a] *A quantitative inequality for the first eigenvalue of a Schrödinger operator*. Accepted for publication in *Journal of Differential Equations* (2020) ([Hal Link](#)). This article corresponds to Chapter 4 of this manuscript.

### SUBMITTED ARTICLES

- [HMP19] *A semilinear shape optimization problem*, with A. Henrot and Y. Privat. This article corresponds to Chapter 6 of this manuscript.
- [MNP19a] *Optimization of a weighted two-phase eigenvalue*, with Y. Privat and G. Nadin. This article corresponds to Chapter 3 of this manuscript.
- [MRBZ19] *Constrained controllability for bistable reaction-diffusion equations: gene-flow & spatially heterogeneous models*, with D. Ruiz-Balet and E. Zuazua. This article corresponds to Chapter 5 of this manuscript.

## FOREWORD AND STRUCTURE OF THE MANUSCRIPT

This manuscript is organized as follows: in the Introduction, we give the mathematical setting of this thesis, lay down the modelling assumption used throughout the text, present the main problems under consideration before listing our contributions. The rest of the manuscript consists of the articles written (either published or submitted) during my time as a PhD student. Each chapter corresponds to a paper. For most of them, a synthetic presentation has been added in order to underline the main difficulties of the problem and the principal novelties of the methods developed.

- Chapter 2 corresponds to [MNP19b]. It is devoted to maximization of the total population size in logistic-diffusive equations. The main focus is on the bang-bang property for optimizers, and leads to the introduction of new asymptotic methods. We also establish a stationarity property in the one dimensional case.
- Chapter 3 corresponds to [MNP19a]. It is devoted to the study of a spectral optimization problem with respect to the drift and the potential of a linear operator. The emphasis is on (non)-existence, as well as the introduction of a method which is shorter and more elegant than the one usually used to investigate stability of optimal shapes. We introduce new tools to derive stationarity results (i.e for small drifts minimizers are the same as in the case without a drift when the domain is a ball).
- Chapter 4 corresponds to [Maz19a]. It is devoted to the study of a quantitative spectral inequality for a Schrödinger operator in the ball. We introduce new transformations and combine it with new tools developed to handle possible topological changes for competitors.
- Chapter 5 corresponds to [MRBZ19]. It is devoted to the study of a control problem for bistable reaction-diffusion equations. The focus is on existence of control strategy, and the main innovation is the use of perturbative methods to encompass the case of slowly-varying environments.
- Chapter 6 corresponds to [HMP19]. It is devoted to a non-energetic shape optimization problem with a non-linear partial differential equation constraint. It focuses on existence of optimal shapes, as well as the stability of the optimizers for small non-linear perturbations of a shape optimization problem under linear PDE constraint. We introduce a new way to construct perturbations for which optimizers for the unperturbed problem no longer satisfy optimality conditions. This relies on a fine analysis of shape derivatives.

Although the notations used in this manuscript will be properly introduced in due time, we gather here, for the sake of readability, the most common ones used throughout the chapters.

## NOTATIONAL CONVENTIONS

- Sets and subsets:

- (a)  $\mathbb{N}$  is the set of non-negative integers,  $\mathbb{N}^*$  is defined as  $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ ,  $\mathbb{N}_+^*$  (resp.  $\mathbb{N}_-^*$ ) is the set of positive (resp. negative) integers.
- (b)  $\mathbb{R}$  is the set of real numbers,  $\mathbb{R}^*$  is defined as  $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ ,  $\mathbb{R}_+$  (resp.  $\mathbb{R}_-$ ) is the set of non-negative (resp. non-positive) real numbers,  $\mathbb{R}_+^*$  (resp.  $\mathbb{R}_-^*$ ) is the set of positive (resp. negative) real numbers.
- (c) The underlying norm on  $\mathbb{R}^n$  ( $n \in \mathbb{N}^*$ ) is the euclidean norm. The topology used on  $\mathbb{R}^n$  is, unless otherwise specified, the euclidean topology. The topology on any subset  $\Omega \subset \mathbb{R}^n$  is the induced topology.
- (d) For any  $n \in \mathbb{N}^*$ , for any  $x \in \mathbb{R}^n$  and any  $r \geq 0$ ,  $\mathbb{B}(x; r)$  is the euclidean ball of center  $x$  and of radius  $r$ .  $\mathbb{S}(x; r)$  is the euclidean sphere of center  $x$  and of radius  $r$ .
- (e) For any two subsets  $A$  and  $B$  of a set  $E$ , the notation  $A \Delta B$  stands for the symmetric difference of  $A$  and  $B$ .

- Measure theory:

- (a) The underlying measure is the Lebesgue measure. For any  $n \in \mathbb{N}^*$  and any Borel set  $A \subset \mathbb{R}^n$ ,  $|A|$  is the  $n$ -th dimensional Lebesgue measure.
- (b) For any  $n \in \mathbb{N}^*$  and any  $d \in \{0, \dots, n-1\}$ ,  $\mathcal{H}^d$  stands for the  $d$ -dimensional Hausdorff measure. When no confusion is possible, for instance when considering the  $(n-1)$ -dimensional Hausdorff measure of the unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ , this quantity is written in the same way as the Lebesgue measure:  $|\mathbb{S}^{n-1}| := \mathcal{H}^{n-1}(\mathbb{S}^{n-1})$ .

- Integration, derivation and function spaces:

Throughout this list,  $\Omega \subset \mathbb{R}^n$  ( $n \in \mathbb{N}^*$ ) is a Borel set.

- (a) Whenever  $p \in [1; +\infty]$ ,  $L^p(\Omega)$  is the Lebesgue space of order  $p$  associated with  $\Omega$ . The  $L^p$  norm of a function  $u \in L^p(\Omega)$  is

$$\|u\|_{L^p(\Omega)} := \left( \int_{\Omega} |u|^p \right)^{\frac{1}{p}}.$$

- (b) Whenever  $u \in L^1(\Omega)$ , the mean value of  $u$  is defined as

$$\mathop{\int}\limits_{\Omega} u := \frac{1}{|\Omega|} \int_{\Omega} u.$$

- (c) The weak derivative of a function  $u : \Omega \rightarrow \mathbb{R}$  is defined in the sense of distributions.
- (d) Whenever  $k \in \mathbb{N}$  and  $p \in [1; +\infty]$ ,  $W^{k,p}(\Omega)$  is the Sobolev space of order  $(k, p)$ . It consists of functions whose weak derivatives up to order  $k$  lie in  $L^p(\Omega)$ . It is endowed with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \sum_{\ell=0}^k \|\nabla^{\ell} u\|_{L^p(\Omega)}.$$

When  $\Omega$  has a  $\mathcal{C}^{1,1}$  boundary and  $k > 0$ , the set  $W_0^{k,p}(\Omega)$  is the set of functions  $u \in W^{k,p}(\Omega)$  whose trace on  $\partial\Omega$  is 0.

- (e) Assuming that  $\Omega$  is a smooth open set, then, for any  $k \in \mathbb{N}$  and any  $\alpha \in [0; 1]$ , the set  $\mathcal{C}^{k,\alpha}(\Omega)$  is the Hölder space of order  $(k, \alpha)$  and is endowed with the norm

$$\|u\|_{\mathcal{C}^{k,\alpha}(\Omega)} := \sum_{\ell=0}^k \|\nabla^\ell u\|_{L^\infty(\Omega)} + \sup_{x,y \in \Omega, x \neq y} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x - y|^\alpha}.$$

- Variational problems:

When considering variational problems of the form

$$\inf_{x \in X} F(x),$$

we call "solution of the problem" an element  $x^* \in X$  such that  $F(x^*) = \inf_{x \in X} F$ .



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# CHAPTER 1

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## GENERAL INTRODUCTION

La région des mathématiciens est un monde intellectuel, où ce que l'on prend pour des vérités rigoureuses perd absolument cet avantage quand on l'apporte sur notre terre. On en a conclu que c'était à la philosophie expérimentale à rectifier les calculs de la géométrie, et cette conséquence a été avouée, même par les géomètres.

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Diderot,  
*Pensées sur l'interprétation de la Nature*

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## 1.1 PRESENTATION OF THE MODEL AND OF THE MAIN PROBLEMS

### 1.1.1 MAIN OBJECTIVE OF THIS THESIS

A simple way to phrase the main question under scrutiny throughout the works presented here is the following:

*What is the optimal way to spread resources in a domain?*

On a mathematical level, this formulation requires some formalization; however, one can already picture the two main topics of this manuscript: spatial ecology, which will be understood through the lens of reaction-diffusion equations, the qualitative behaviour of which is, by now, well understood, and shape optimisation, a domain where many interesting questions remain, to this day, completely open. We do believe that our works contribute to both these domains, whether it be by shining a new light on the influence of spatial heterogeneity on reaction-diffusion equations or by providing new tools to tackle shape optimisation problems.

Regarding spatial ecology, a strong emphasis was put on **concentration** (and, conversely, **fragmentation**) of resources: is it better to scatter resources across the domain or, on the contrary, to place them all in one spot? We give more details in Section 1.1.3.3.

As for shape optimisation, as will be explained further in Section 1.3, we give new tools to address two ubiquitous problems: one is that of **existence** of optimal shapes, the other is that of the **stability** of these optimisers, in a sense made precise later. The methods we develop in specific cases to tackle these questions might however be applied in a more general setting.

### 1.1.2 STRUCTURE OF THE INTRODUCTION

This Introduction is structured as follows:

We first present the main model for a single-species (which will be the most commonly used throughout the manuscript, and corresponds to Chapters 2, 3, 4), before tackling a controllability problem for bistable reaction-diffusion equations (which corresponds to Chapter 5). We gathered the presentation of all the methods developed in that second part to make the reading easier; furthermore, as some of them are used in different chapters, or seemed worth discussing in a more general setting, an independent presentation looked natural.

Thus, each Section of the first part of the introduction corresponds to one or multiple paragraphs of the second part, according to the correspondance table below:

Problem	Chapter	Model and results	Tools and methods
Total population size	Chapter 2	Paragraph 1.2.1.1	Paragraph 1.3.2
Optimal survival-Spectral optimisation	Chapter 3	Paragraph 1.2.1.2	Paragraphs 1.3.3 and 1.3.4
Quantitative spectral inequality	Chapter 4	Paragraph 1.2.1.3	Paragraphs 1.3.4 and 1.3.4
Controllability for bistable equations	Chapter 5	Paragraph 1.2.2	Paragraph 1.3.6
A non-linear optimisation problem	Chapter 6	Paragraph 1.2.3	Paragraphs 1.3.2 and 1.3.3



### 1.1.3 SINGLE-SPECIES MODELS AND PROPERTIES UNDER INVESTIGATION

We start by introducing the main modelling assumptions underlying most of our work. We emphasize the fact that we will mainly be studying *stationary equations*; in other words, we generally assume that the population has already reached its equilibrium. We also highlight the fact that we work at a *macroscopic scale*: the unknown in our equations is a population density, in contrast to some models where individual behaviours are the relevant unknowns. It is possible to make a link between these two modelling scales, and we give [43, 143] as general references for modelling of biological phenomena. For a historical overview of the development of mathematical biology and ecology, we refer to [11].

The equations governing population dynamics are often called *reaction-diffusion equations* and are ubiquitous since their use in [75, 113]. In the study of these equations, three important phenomena are to be noted:

1. Inherent non-linearities:

This can already be found in Malthus' seminal work, [132]: should too many individuals be present at the same spot, competition between these individuals would have a detrimental effect, and the total population is limited by this effect. We refer to such interactions as *intra-specific interactions*.

2. The interaction of the population with its environment:

Some zones of the environment are *favourable* to the population (for instance, zones where resources can be found), while others are *lethal* or *detrimental*.

3. Spatial dispersal:

Dispersal phenomena will be interpreted in terms of flux and are linked to conservation law; these concepts are at the heart of reaction-diffusion equations.

If we sum it up, we will model the environment with a bounded, smooth domain  $\Omega \subset \mathbb{R}^n$ , in which a population represented by its density  $u$  depending on the spatial variable  $x$  and on the time variable  $t$  evolves, disperses, interacts with itself and the environment. The prototypical reaction-diffusion equation writes

Time evolution = Spatial dispersion + Inter-specific interactions + Interactions with the environment.

The goal of this thesis is to understand the influence of spatial heterogeneity, which we take into account through the "interactions with the environment" term in the above equation.

#### 1.1.3.1 POPULATION DYNAMICS MODEL: THE CASE OF A SINGLE SPECIES

We lay down the main hypothesis in a more precise way.

1. Macroscopic scale, sex and age structure:

As was explained in the previous paragraph, we consider a population density  $u = u(t, x)$ . We work without age-structures (or, equivalently, we assume that individuals reach adulthood as of their birth) or sex-structure.

2. The domain:

The domain is accounted for *via* a smooth, bounded subset  $\Omega \subset \mathbb{R}^n$  (the smoothness assumption will sometimes be dropped, for instance when considering orthotopes and neumann boundary conditions).

3. Inter-specific interactions:

We use a quadratic non-linearity; in other words, we work with the non-linearity

$$f(u) = -u^2.$$

4. Interactions with the environment:

In our works, the interactions with the environment (and, consequently, the influence of spatial heterogeneity) are modeled through resources distributions<sup>1</sup>. For a domain  $\Omega$ , a resources distribution is a function  $m : \Omega \rightarrow \mathbb{R}_+$ . We assume that the growth of the population takes the form  $m \times u$ : if  $m > 0$ , this corresponds to a local growth of the population whereas, if  $m < 0$ , this corresponds to the death of the population. The subset  $\{m \leq 0\}$  will be called *lethal zone*, while the zone  $\{m > 0\}$  will be called *favourable zone*. For obvious modelling reasons, we assume that

$$m \in L^\infty(\Omega).$$

5. (Un)biased dispersal:

The diffusion term in the equation accounts for the movement of individuals in the domain. We will either choose *unbiased diffusion* (i.e the individuals explore each direction with the same probability) or *biased diffusion* (i.e the individuals have some knowledge of their environment, and move accordingly).

The *unbiased flux* is defined as follows: for a real parameter  $\mu > 0$ , we define

$$\mathcal{J}(u) := -\mu \nabla u.$$

The constant  $\mu$  is called *diffusivity* and determines the speed of dispersal. This unbiased model will be used in Chapters 2 and 4.

The *biased flux* is defined as follows: we want to take into account the fact that individuals know where to look for food. Let  $\alpha > 0$  be a parameter quantifying the sensitivity of individuals to resources, and  $m \in L^\infty(\Omega)$  be a resources distribution. The biased flux is

$$\mathcal{J}_0(u) = \nabla u - \alpha u \nabla m,$$

and we refer for instance to [43]. However, the emphasis will be put on a spectral optimization problem (see Chapter 3) for which it is equivalent to work with the following flux, henceforth called biased flux:

$$\mathcal{J}(u) := -(1 + \alpha m) \nabla u.$$

For modelling considerations on biased flux, we refer to [16, 17, 143]. For the equivalence (from the point of view of the spectral optimization problem under consideration) of the two fluxes we refer to Chapter 3, Section 3.1.3 of this manuscript.

6. Boundary conditions: finally, we need to impose boundary conditions; this is a crucial step, both for modelling reasons and for the mathematical analysis of the model. We will choose either Dirichlet boundary conditions (accounting for a lethal boundary) or Neumann boundary conditions (modelling a domain the individuals can not get out of).

In a more mathematical way, these assumptions read as follows: we consider

1. A smooth, bounded domain  $\Omega \subset \mathbb{R}^n$ ,
2. A resources distribution  $m \in L^\infty(\Omega)$  modelling spatial heterogeneity,
3. A diffusivity  $\mu > 0$  or a parameter  $\alpha > 0$  quantifying sensitivity to resources,
4. A boundary operator  $\mathcal{B} : L^2(\partial\Omega) \ni u \mapsto \mathcal{B}u$ . The case

$$\mathcal{B}u = u$$

---

<sup>1</sup>Other interpretations, such as intrinsic birth rates, exist, but we will not focus on them.

corresponds to Dirichlet boundary conditions and will hence be written  $\mathcal{D}$ , the case

$$\mathcal{B}u = \frac{\partial u}{\partial \nu}$$

corresponds to Neumann boundary conditions and will hence be written  $\mathcal{N}$ ,

5. An initial datum  $u_0 \geq 0$ ,  $u_0 \neq 0$ ,  $u_0 \in L^\infty(\Omega)$ .

The equations then write as follows:

• Unbiased model:

$$\begin{cases} \mu \Delta u + u(m - u) = \frac{\partial u}{\partial t} \text{ in } \Omega \times \mathbb{R}_+, \\ u(t = 0, \cdot) = u_0 \text{ in } \Omega, \\ \mathcal{B}u = 0 \text{ on } \partial\Omega, \\ u \geq 0 \text{ in } \Omega. \end{cases} \quad (1.1)$$

This is an evolution equation; our main focus will be on two stationary equations related to it and which, for the quadratic non-linearity under consideration, govern its long-time behaviour: the first one, mostly studied in Chapter 2, is the *logistic-diffusive equation*:

$$\begin{cases} \mu \Delta \theta_{\mathcal{B}} + \theta_{\mathcal{B}}(m - \theta_{\mathcal{B}}) = 0 \text{ in } \Omega, \\ \mathcal{B}\theta_{\mathcal{B}} = 0 \text{ on } \partial\Omega, \\ \theta_{\mathcal{B}} \geq 0 \text{ in } \Omega, \theta_{\mathcal{B}} \neq 0. \end{cases} \quad (1.2)$$

Assuming that this equation has a unique solution, this solution is an equilibrium of (1.1) and one might expect that under some assumptions on  $m$ , any solution of (1.1), for any initial datum  $u_0 \geq 0$ ,  $u_0 \neq 0$ , should satisfy

$$u(t, x) \xrightarrow{t \rightarrow \infty} \theta.$$

To study such existence and uniqueness properties, we introduce the *principal eigenvalue* associated with a resource distribution  $m \in L^\infty(\Omega)$ : for any  $m \in L^\infty(\Omega)$ , the operator

$$\mathcal{L}_m^\mu := \mu \Delta + m$$

along with some boundary conditions  $\mathcal{B}$  has a smallest eigenvalue  $\lambda_1^{\mathcal{B}}(m, \mu)$ , associated with the eigenequation

$$\begin{cases} \mu \Delta \varphi_{m, \mu} + m \varphi_{m, \mu} = \lambda_1^{\mathcal{B}}(m, \mu) \varphi_{m, \mu} \text{ in } \Omega, \\ \mathcal{B}\varphi_{m, \mu} = 0 \text{ on } \partial\Omega. \\ \int_{\Omega} \varphi_{m, \mu}^2 = 1. \end{cases} \quad (1.3)$$

This eigenvalue might also be defined through Rayleigh quotients:

$$\lambda_1^{\mathcal{B}}(m, \mu) := \inf_{u \in X_{\mathcal{B}}(\Omega), u \neq 0} \frac{\mu \int_{\Omega} |\nabla u|^2 - \int_{\Omega} m u^2}{\int_{\Omega} u^2}, \quad (1.4)$$

where the functional space  $X_{\mathcal{B}}$  is either

$$X_{\mathcal{D}} = W_0^{1,2}(\Omega)$$

for Dirichlet boundary conditions, or

$$X_{\mathcal{N}} = W^{1,2}(\Omega)$$

for Neumann boundary conditions.

The linear equation (1.3) can be thought of as the linearization of (1.1) around  $z \equiv 0$ . A first, implicit link between existence and uniqueness for (1.2) and  $\lambda_1^{\mathcal{B}}(m, \mu)$  was observed by Skellam,

[168, Section 3.3, case iii]): for constant resources distributions  $m$ , in the one dimensional case, no non-trivial solutions to (1.2) with Dirichlet boundary conditions can exist when the domain is too small. This observation is turned into a principle by Shigesada and Kawasaki, [166]: roughly speaking, existence of non-trivial solutions to (1.2) amounts to survival of the population, which itself boils down to: the population should increase when the initial datum  $u_0$  is small. This was extended to the multi-dimensional case in [17, 42, 97] for bounded domains and was completely formalized by Berestycki, Hamel and Roques [19]: survival holds for any non-negative initial datum  $u_0 \neq 0$  if and only if existence and uniqueness holds for (1.2) if and only if  $\lambda_1^{\mathcal{B}}(m) < 0$ . Their results are obtained for periodic domains but can easily be adapted here. We sum up the results of [19, 42]: Let  $m \in L^\infty(\Omega)$  and  $\mu > 0$ .

- (a) If  $\lambda_1^{\mathcal{B}}(m, \mu) < 0$ , there exists a unique solution  $\theta_{\mathcal{B}} \neq 0$  of (1.2). Furthermore, any solution  $u$  of (1.1) associated with an initial datum  $u_0 \geq 0$ ,  $u_0 \neq 0$  satisfies

$$u(t, x) \xrightarrow[t \rightarrow \infty]{\mathcal{C}^0(\Omega)} \theta_{\mathcal{B}}.$$

- (b) If  $\lambda_1^{\mathcal{B}}(m, \mu) \geq 0$ , equation (1.2) has no non-zero solution. Furthermore, any solution  $u$  of (1.1) associated with an initial datum  $u_0 \geq 0$ ,  $u_0 \neq 0$  satisfies

$$u(t, x) \xrightarrow[t \rightarrow \infty]{\mathcal{C}^0(\Omega)} 0.$$

**REMARK 1.1.** For Neumann boundary conditions, the Rayleigh quotient formulation (1.4) gives a very simple criterion for survival: using  $u \equiv 1$  as a test function gives

$$\lambda_1^{\mathcal{N}}(m, \mu) \leq - \int_{\Omega} m, \lambda_{\alpha}^{\mathcal{N}}(m) \leq - \int_{\Omega} m,$$

so that requiring  $m \geq 0$ ,  $m \neq 0$  gives existence and uniqueness for (1.2).

**REMARK 1.2 :** *The influence of diffusivity for Dirichlet boundary conditions.* For Dirichlet boundary conditions however, the situation is quite different. As mentioned earlier, it was noted in [168] that, if  $\Omega = (0; L)$ ,  $\mu = 1$  and  $m \equiv 1$ , there exists  $\underline{L} > 0$  such that, for any  $L \leq \underline{L}$ ,

$$\lambda_1^{\mathcal{D}}(m, 1) > 0.$$

In other words, too small an environment is detrimental to the survival of the population. Using the change of variables  $y = \frac{x}{\sqrt{\mu}}$ , this amounts to saying that, in  $\Omega = (0; 1)$ , with  $m \equiv 1$ , there exists  $\bar{\mu} > 0$  such that, for any  $\mu \geq \bar{\mu}$ ,

$$\lambda_1(m, \mu) > 0.$$

In fact, this is valid in any dimension and any resources distribution  $m$ : for any  $\Omega$ , any  $m \in L^\infty(\Omega)$ , there exists a diffusivity threshold for survival. This is in sharp contrast with Neumann boundary conditions.

The question under scrutiny for these two equations are the following:

The first one deals with the total population size, a mathematically challenging problem which we tackle in Chapter 2, corresponding to [MNP19b]. In this Introduction, we present the results obtained for this question in Section 1.2.1.1 and the new method developed in that context in Section 1.3.2.

**Question 1.** Which resources distribution  $m$  yields the largest total population size for (1.2) with Neumann boundary conditions? Can we say something about the geometry of

*optimal resources distributions?*

This question was first raised by Lou in [125]; we recall the results he obtained in Section 1.2.1.1 of this Introduction.

The second one deals with spectral optimisation: given the results of [19, 42] recalled above, it is clear that principal eigenvalues account for the survival capacity associated with a resources distribution. Let us consider the case of Dirichlet boundary conditions, and let us define  $\lambda_1(m, \mu) := \lambda_1^{\mathcal{D}}(m, \mu)$ . Many informations about the minimisation of this eigenvalue in some relevant class are known; we recall some of them in Subsection 1.2.1.2, and refer, for the time being, to [19, 128, 116]. The question of sensitivity of this first eigenvalue is relevant in this context: in other words, knowing a minimiser  $m^*$ , in some admissible class, can we obtain some quantitative information in the form of a quantitative inequality?

**Question 2.** *Can we obtain a quantitative inequality for the first eigenvalue  $\lambda_1(\cdot, \mu)$ ?*

We address this question in Chapter 4, which corresponds to [Maz19a], where we obtain a quantitative inequality in the case of the ball. In this introduction, our results are presented in Section 1.2.1.3 and the methods developed in Paragraphs 1.3.4 and 1.3.4.

- **Biased models:** for biased models, we will only work with Dirichlet boundary conditions and focus on the question of survival of the population. We consider, for any initial datum  $u_0 \geq 0$ ,  $u_0 \neq 0$ , the solution  $u$  of

$$\begin{cases} \nabla \cdot \left( (1 + \alpha m) \nabla u \right) + u(m - u) = \frac{\partial u}{\partial t} \text{ in } \Omega \times \mathbb{R}_+, \\ u(t = 0, \cdot) = u_0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u \geq 0 \text{ in } \Omega. \end{cases} \quad (1.5)$$

along with the associated stationary state

$$\begin{cases} \nabla \cdot \left( (1 + \alpha m) \nabla \psi_{\alpha, m} \right) + \psi_{\alpha, m} (m - \psi_{\alpha, m}) = 0 \text{ in } \Omega, \\ \psi_{\alpha, m} = 0 \text{ on } \partial\Omega, \\ \psi_{\alpha, m} \geq 0 \text{ in } \Omega, \psi_{\alpha, m} \neq 0 \end{cases} \quad (1.6)$$

and the first eigenequation associated with the principal eigenvalue  $\lambda_\alpha(m)$  (as explained, since we only work with Dirichlet boundary conditions, we drop the superscript  $\mathcal{D}$  for this eigenvalue):

$$\begin{cases} \nabla \cdot \left( (1 + \alpha m) \nabla \xi_{\alpha, m} \right) + m \xi_{\alpha, m} = \lambda_\alpha(m) \xi_{\alpha, m} \text{ in } \Omega, \\ \xi_{\alpha, m} = 0 \text{ on } \partial\Omega, \\ \int_\Omega \xi_{\alpha, m}^2 = 1. \end{cases} \quad (1.7)$$

The proofs of [19, 42] are easily adapted to obtain the following result: let  $m \in L^\infty(\Omega)$  and  $\alpha > 0$ .

- (a) If  $\lambda_\alpha(m) < 0$ , there exists a unique non-trivial solution  $\psi_{\alpha, m}$  of (1.6). Furthermore, any solution  $u$  of (1.5) associated with a non-zero initial datum  $u_0 \geq 0$  satisfies

$$u(t, x) \xrightarrow[t \rightarrow \infty]{\mathcal{C}^1(\Omega)} \psi_{\alpha, m}.$$

- (b) If  $\lambda_\alpha(m) \geq 0$ , equation (1.6) has no non-trivial solution. Furthermore, any solution  $u$  of (1.5) associated with a non-zero initial datum  $u_0 \geq 0$  satisfies

$$u(t, x) \xrightarrow[t \rightarrow \infty]{\mathcal{C}^1(\Omega)} 0.$$

These results have the same interpretations as in the case of the biased model: the principal eigenvalue  $\lambda_\alpha(m)$  quantifies the chances of survival of a population, which leads to the following question (see for instance [43, 40, 42] and the references therein):

**Question 3.** *Which resources distribution yields the best chances of survival for biased movement of species?*

In contrast to Question 2, even basic results (e.g existence) were not known when this PhD started. We give an almost complete picture (existence and qualitative properties) of this problem in Chapter 3, which corresponds to [MNP19a]. In this Introduction, our results are presented in Section 1.2.1.2, while the methods are developed in Paragraphs 1.3.3 and 1.3.4.

**BIBLIOGRAPHICAL REMARK.** *We refer to [43, 143] for more details regarding modelling assumptions in reaction-diffusion equations. A vast literature has been devoted to the study of these equations in unbounded domains, via the analysis of travelling waves. We will not tackle this subject and refer to the Thesis' or Habilitations' Introductions [26, 82, 145] for an overview of these works.*

### 1.1.3.2 ADMISSIBLE CLASS

So far, we have not defined the admissible class for resources distributions. For Questions 3,1 and 2, the admissible class will be the same, and we now present it.

We impose  $L^\infty$  and  $L^1$  bounds on resources distributions. This is consistent, from a modelling point of view:  $L^\infty$  bounds model the pointwise environmental limits (i.e it is not possible to store more than a fixed amount of resources in one location), while  $L^1$  bounds stand for the global environmental capacity (i.e it is not possible for a domain to have more than a certain amount of resources).

Let us thus introduce two parameters  $m_0 > 0$  and  $\kappa > 0$ , henceforth fixed for the rest of this Introduction, and define

$$\mathcal{M}(\Omega) := \left\{ m \in L^\infty(\Omega), 0 \leq m \leq \kappa, \frac{1}{|\Omega|} \int_\Omega m = m_0 \right\}. \quad (1.8)$$

Obviously, we require that  $m_0 < \kappa$ , so that this class is non trivial.

**Bang-bang functions** are essential for our upcoming analysis: we recall that a bang-bang function is the characteristic function of a set. In other words, a bang-bang function is of the form

$$m = \kappa \mathbf{1}_E$$

with  $E \subset \Omega$  a measurable subset such that  $|E| = \frac{m_0 |\Omega|}{\kappa}$ . In many situations, the goal is to prove that minimisers of some functionals are bang-bang functions.

**REMARK 1.3.** *We could, as is done in [50], consider sign-changing  $m$  and replace  $0 \leq m \leq \kappa$  with  $-1 \leq m \leq \kappa$ , but this would not change anything to our results. Regarding Question 1, our results rely on functional arguments which are unaffected by the sign of  $m$ , while, for eigenvalue problems, the linearity of the problem would enable us to replace  $m$  with  $m + 1$  without changing the results. Furthermore, as was noted in Remark 1.1, working with non-negative  $m$  guarantees the existence of non-trivial solutions to (1.2).*

The three problems under consideration are then

- maximise the total population size:

$$\sup_{m \in \mathcal{M}(\Omega)} \int_{\Omega} \theta_{m,\mu}, \text{ with } \begin{cases} \mu \Delta \theta_{m,\mu} + \theta_{m,\mu} (m - \theta_{m,\mu}) = 0 \text{ in } \Omega, \\ \frac{\partial \theta_{m,\mu}}{\partial \nu} = 0 \text{ on } \partial \Omega. \end{cases} \quad (P_{\mu})$$

- Derive a quantitative inequality for the unbiased movement:

Establish a quantitative inequality for the first Dirichlet eigenvalue  $\lambda_1(m, \mu)$  of  $\mathcal{L}_m := -\mu \Delta - m$ . (Q)

- Investigate the spectral optimisation problem for the biased movement: derive (non)-existence results and qualitative properties for

$$\inf_{m \in \mathcal{M}(\Omega)} \lambda_{\alpha}(m), \text{ where } \lambda_{\alpha}(m) \text{ is the first Dirichlet eigenvalue of } \mathcal{L}_{\alpha,m} := -\nabla \cdot ((1 + \alpha m) \nabla) - m. \quad (P_{\alpha})$$

### 1.1.3.3 PROPERTIES UNDER INVESTIGATION

Having defined the admissible class, let us present the type of properties we will be investigating for these variational problems, that is, the total population size ( $P_{\mu}$ ) and the biased spectral optimisation problem ( $P_{\alpha}$ ). These problems, being notoriously difficult to solve completely (by which we mean exhibiting the exact minimiser), we will focus on other informations.

**Pointwise and geometric properties** From a qualitative point of view, building on the works of [19, 128, 116], two type of properties are particularly relevant:

1. Pointwise properties:

This amounts to investigating whether or not optimal distributions  $m^*$  for ( $P_{\alpha}$ ) or ( $P_{\mu}$ ) are bang-bang functions: do we have

$$m^* = \kappa \mathbb{1}_E,$$

where  $E$  is a measurable subset of  $\Omega$ ? In other words, is it possible to split the domain into two zones, a favourable one and a lethal one? From a biological point of view, such resources distributions correspond to *patch-models*, the relevance of which is discussed in [19, 42]. In Section 1.3 of this Introduction, we explain the different methods we introduced, throughout this PhD, to give this question an answer in several cases. Let us just keep in mind that this property is usually proved independently of the geometry of the domain and it is often the case that its proof relies on concavity properties of the functional to optimise. For instance, for the spectral optimisation problem ( $P_{\alpha}$ ), the fact that the functional to optimise is energetic makes it easy to prove that, should a minimiser exist, it is bang-bang. For the total population size problem ( $P_{\mu}$ ) however, the functional is no longer energetic, and proving the bang-bang property is therefore very challenging. For the quantitative inequality problem (Q), we will investigate an auxiliary optimisation problem, the solutions of which will be bang-bang functions, and that property will be crucial for the subsequent analysis.

2. Geometric properties: concentration and fragmentation:

Assume now that we already know the solution  $m^*$  to an optimisation problem to be a bang-bang function, for instance

$$m^* = \kappa \mathbb{1}_E.$$

What kind of informations can we have on the set  $E$ ? We must often satisfy ourselves with very broad concentration/fragmentation properties: is the set  $E$  connected? Is it convex? Or

is it, on the other hand, fragmented? This question lies naturally in the field of mathematical biology, since the works [19, 42, 85, 162, 166] where the influence of fragmentation on survival ability is investigated, mainly in the spatially periodic case. We give a rough representation of fragmentation and concentration in Figure 1.1 below:

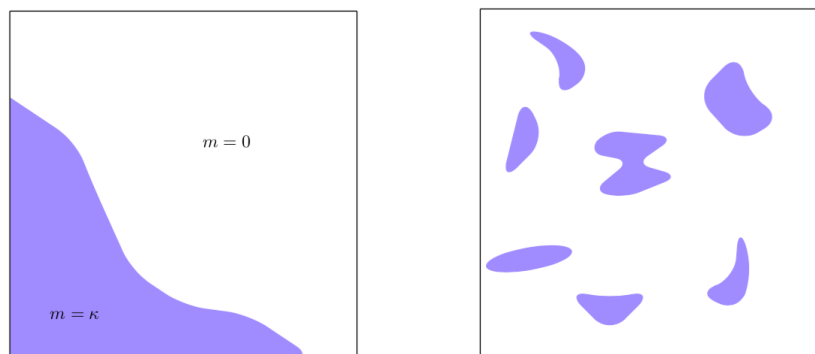


Figure 1.1 –  $\Omega = (0, 1)^2$ . The resources distribution on the left is "more concentrated" than the one on the right.

These informations are hard to obtain, but let us note that they are often derived in "simple" geometries (e.g the ball for Dirichlet boundary conditions, the orthotope for Neumann boundary conditons) using appropriate rearrangements (see Chapters 2, 3 and 4).

Let us give an example: consider, in the one-dimensional case, the first eigenvalue  $\lambda_1^N(m, \mu)$  of  $-\mu \frac{d^2}{dx^2} - m$  in  $\Omega = (0, 1)$  with Neumann boundary conditions. Then it is proved in [19] that, by defining  $m^\# := \kappa \mathbf{1}_{(0, r^*)} \in \mathcal{M}(0, 1)$  for a suitable  $r^*$  we have

$$\forall m \in \mathcal{M}(0, 1), \lambda_1^N(m, \mu) \geq \lambda_1^N(m^\#, \mu). \quad (1.9)$$

Thus, *concentration is always favourable to survival of species*. We represent these two optimisers below:

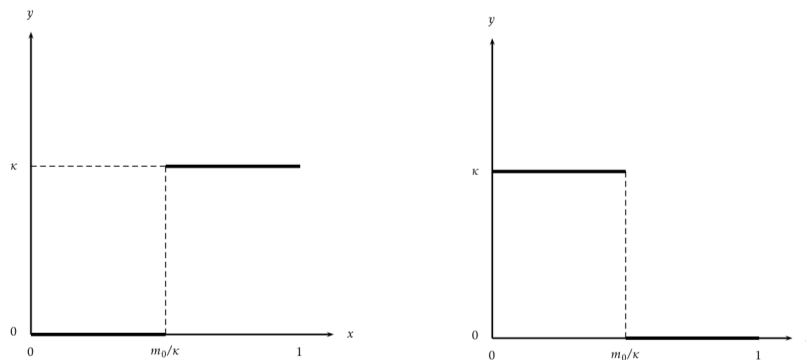


Figure 1.2 –  $\Omega = (0, 1)$ . Two optimal configurations for species survival in the one dimensional case.



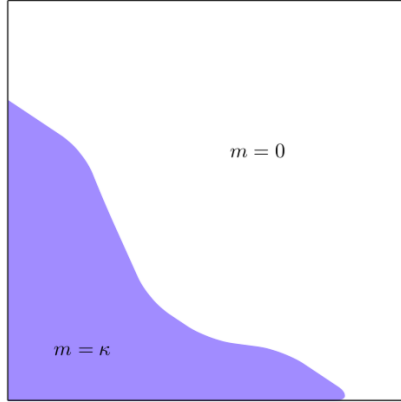


Figure 1.3 –  $\Omega = (0, 1)^2$ . The type of geometry of optimal resources distributions for species survival in an orthotope.

We also highlight the fact that, for the optimisation of the total population size, one of our most striking results of Chapter 2 is that fragmentation is sometimes better for the total population size.

**Minimality of some configurations** Given the difficulty of fully describing the optimisers of variational problems, taking into account the fact that the bang-bang property "should" hold and that variational problems may respect the symmetry of the domain, it is also customary to try and verify the optimality of some configurations.

Although this will mainly be used in Chapters 6, 3 and 4 to check whether or not the ball satisfies optimality conditions, we present the tool used to study these questions in a more general setting. Let us first consider a functional

$$F : \mathcal{M}(\Omega) \ni m \mapsto F(m).$$

Let, for any subset  $E \subset \Omega$  satisfying

$$|E| = \frac{m_0 |\Omega|}{\kappa}$$

define the shape functional  $\mathcal{F}$  as

$$\mathcal{F} : E \mapsto \mathcal{F}(E) =: F(\kappa \mathbb{1}_E).$$

If we want to try the optimality of a particular function  $m^* = \kappa \mathbb{1}_E$ , it is logical to start by considering shape deformations and shape derivatives of the set  $E$ . We only draft the methods used to do so and refer to [4, 83, 93] as well as to Chapters 6, 3 and 4 of this manuscript for a more rigorous presentation. Although these shape deformations (i.e deformations through a vector field on the boundary  $\partial E$ ) do not allow for changes in topology, they are usually instrumental in optimisation problems. For more on the distinction between these shape derivatives and other type of derivatives, we refer to Chapter 4 of this manuscript.

We come back to shape deformations and derivatives: consider a vector field

$$\phi : \partial E \rightarrow \mathbb{R}^n$$

smooth enough so that it can be extended to a compactly supported smooth vector field  $\phi : \Omega \rightarrow \mathbb{R}^n$  so that, for  $t$  small enough,

$$\Phi_t := (Id + t\phi)$$

leaves  $\Omega$  invariant. It is then possible to define

$$\tilde{F}_\phi : t \mapsto F(\kappa \mathbf{1}_{\Phi_t(E)}) = \mathcal{F}(\Phi_t(E)).$$

Under some regularity assumptions, the map  $\tilde{F}_\phi$  is  $\mathcal{C}^2$  at  $t = 0$ , and we may define its derivatives as

$$\tilde{F}'_\phi(0) =: \mathcal{F}'(E)[\phi], \tilde{F}''_\phi(0) =: \mathcal{F}''(E)[\phi, \phi].$$

These derivatives are called *shape derivatives of  $\mathcal{F}$  of first and second order*.

This enables us to write down the optimality conditions for such volume constrained optimisation problems. This volume constraint can be enforced through the use of a Lagrange multiplier.

**DEFINITION 1.1.1** *Let  $\Lambda$  be the Lagrange multiplier associated with the volume constraint*

$$|E| = \frac{m_0}{\kappa}.$$

Let

$$\text{Vol} : E \mapsto |E|,$$

The first and second order optimality condition associated with the problem

$$\inf_{E \subset \Omega, |E| = \frac{m_0}{\kappa}} \mathcal{F}(E)$$

are

$$(\mathcal{F} - \Lambda \text{Vol})'(E)[\phi] = 0, \quad (\mathcal{F} - \Lambda \text{Vol})''(E)[\phi, \phi] > 0 \tag{1.10}$$

for every vector field  $\phi$  such that

$$\phi|_{\partial E} \neq 0, \quad \int_{\partial E} \langle \phi, \nu \rangle = 0.$$

**BIBLIOGRAPHICAL REMARK.** *This approach dates back to Hadamard, [83] We refer to [4, 93] for a rigorous introduction to shape derivatives.*

## 1.2 MAIN CONTRIBUTIONS OF THE THESIS

### 1.2.1 QUALITATIVE RESULTS FOR QUESTIONS 1, 2 AND 3

#### 1.2.1.1 THE INFLUENCE OF RESOURCES DISTRIBUTIONS ON THE TOTAL POPULATION SIZE (CHAPTER 2, QUESTION 1)

Let us first tackle Question 1, that is, optimising the total population size for logistic diffusive models. This corresponds to Chapter 2 of this manuscript and to [MNP19b].

Our methods are developed for Neumann boundary conditions. Indeed, part of our results are carried out in the asymptotic regime  $\mu \rightarrow \infty$ . As explained in Remark 1.2, for Dirichlet boundary conditions, there exists a threshold diffusivity  $\underline{\mu} > 0$  above which the population goes extinct; this does not happen for Neumann boundary conditions.

Let us recall that we are working with unbiased movement, and we are studying the equation

$$\begin{cases} \mu \Delta \theta_{m,\mu} + \theta_{m,\mu}(m - \theta_{m,\mu}) = 0 \text{ in } \Omega, \\ \frac{\partial \theta_{m,\mu}}{\partial \nu} = 0 \text{ on } \partial \Omega, \\ \theta_{m,\mu} \geq 0 \text{ in } \Omega, \theta_{m,\mu} \neq 0 \text{ in } \Omega. \end{cases} \quad (1.11)$$

We introduce the total population size functional

$$\mathcal{F}(\cdot, \mu) : \mathcal{M}(\Omega) \ni m \mapsto \int_{\Omega} \theta_{m,\mu}, \quad (1.12)$$

and the variational problem

$$\boxed{\sup_{m \in \mathcal{M}(\Omega)} \mathcal{F}(m, \mu)}. \quad (P_{\mu})$$

For this problem, we will investigate the bang-bang property alongside geometric properties of maximisers.

**Known results** Let us first review the results available in the literature.

The variational problem  $(P_{\mu})$  was introduced and first studied by Lou in [126, 125]. He shows that the problem has a solution in  $\mathcal{M}(\Omega)$  (and that  $L^1$  and  $L^{\infty}$  are minimal constraints to have existence; if one of them is absent, existence no longer holds). He also establishes the following results:

Let  $\underline{m} \equiv m_0 \in \mathcal{M}(\Omega)$ . Then, for any  $\mu > 0$ ,  $\underline{m}$  is a global minimiser of  $\mathcal{F}(\cdot, \mu)$  in  $\mathcal{M}(\Omega)$ :

$$\forall \mu > 0, \forall m \in \mathcal{M}(\Omega), \mathcal{F}(m, \mu) \geq \mathcal{F}(\underline{m}, \mu).$$

In other words, a homogeneous environment is the worst possible configuration for the total population size. Furthermore, for any fixed  $m \in \mathcal{M}(\Omega)$ , the mapping

$$\mu \in (0; +\infty) \mapsto \mathcal{F}(m, \mu)$$

is continuous, can be continuously extended into a function on  $[0; +\infty]$  by setting  $\mathcal{F}(m, 0) = \mathcal{F}(m, +\infty) := m_0$ , and is minimal at  $\mu = 0$  and  $\mu = +\infty$ .

**REMARK 1.1** *Let us stress that the  $L^{\infty}$  bound is necessary to obtain existence result.*

*Indeed, in [12], the one-dimensional case is studied and some partial informations are obtained: if  $\Omega = (0; 1)$ , if  $\mu = 1$ , then*

$$\sup_{m \in L^{\infty}(\Omega), \int_{\Omega} m = m_0} \mathcal{F}(m, 1) = 3m_0,$$

and this upper bound is never reached.

In [69], a more general functional is considered and several numerical simulations back the natural conjecture that solutions of  $(P_\mu)$  are bang-bang functions.

The bang-bang character of maximisers remained, however, completely open, until a result by Nagahara and Yanagida, [148] and our own article [MNP19b].

In [148], a weak bang-bang property is established, namely: if the set  $\{0 < m < \kappa\}$  has an interior point, then  $m$  is not a solution of  $(P_\mu)$ . The strength of their result lies in the fact that it is valid for any diffusivity, its weakness on the regularity assumption they make which, as exemplified for several functionals, is quite difficult to prove.

No result, however, addressed the problem of the geometry of maximisers of  $\mathcal{F}(\cdot, \mu)$ ; while, as recalled in Subsection 1.1.3.3, concentration is favourable for species survival, it is not clear whether or not it is the case here and, so far, our article [MNP19b] seems to give the most advanced results for the following question:

**Question 4.** *Is fragmentation or concentration of resources favourable for the total population size?*

**Contributions of the thesis for the total population size (Chapter 2, Question 1)** In [MNP19b] (Chapter 2) we prove the following results:

**THEOREM OF THE THESIS 1** [MNP19b] *For the variational problem  $(P_\mu)$ , there holds:*

1. Bang-bang property for large diffusivities (Theorem 2.1.1):

*For any dimension  $n$ , for any smooth enough bounded domain  $\Omega \subset \mathbb{R}^n$ , there exists  $\underline{\mu} = \underline{\mu}(m_0, \kappa, \Omega)$  such that, for any  $\mu \geq \underline{\mu}$ , any solution of  $(P_\mu)$  is a bang-bang function.*

2. Full stationarity for large enough diffusivities in dimension 1 (Theorem 2.1.3):

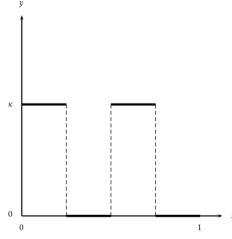
*Let  $m_*^d := \kappa \mathbb{1}_{(1-r^*, r^*)}$  and  $m_*^g := \kappa \mathbb{1}_{(0, r^*)}$  be the unique minimisers of  $\lambda_1^N(\cdot, \mu)$  when  $\Omega = (0; 1)$ . Here, the parameter  $r^*$  is chosen so that  $\kappa r^* = m_0$ . For  $\Omega = (0; 1)$ , there exists  $\bar{\mu} > 0$  such that, for any  $\mu > \bar{\mu}$ ,  $m_*^d$  and  $m_*^g$  are the unique solutions of the variational problem  $(P_\mu)$ .*

3. Concentration in dimension 2 for large diffusivities (Theorem 2.1.2):

*In  $\Omega = [0; 1]^2$ , define, for any  $\mu > 0$ ,  $m_\mu$  as a maximiser of  $\mathcal{F}(\cdot, \mu)$ . Up to a subsequence,  $\{m_\mu\}_{\mu > 0}$  converges, in  $L^1(\Omega)$  and as  $\mu \rightarrow \infty$  to a resources distribution  $m^*$  which is decreasing in every direction. Since this property of being decreasing in every directions holds in particular for optimisers of the species survival we can say that for large diffusivities, the maximisers for the total population size behave as the optimisers for the species survival.*

4. Fragmentation for small diffusivities in dimension 1 (Theorem 2.1.4):

*Let  $\tilde{m} : x \mapsto m_*^g(2x)$  be the double crenel:*



Then there exists  $\tilde{\mu} > 0$  such that

$$\mathcal{F}(\tilde{m}, \tilde{\mu}) > \mathcal{F}(m_*^g, \tilde{\mu}).$$

In particular, concentration of resources can not hold for any diffusivity, and the optimisers for the total population size have a qualitative behaviour different from that of optimisers for species survival.

The proof of the bang-bang property for large diffusivities is the core of [MNP19b], and we refer to Section 1.3.2 for a brief presentation, but highlight the following points:

- The bang-bang property relies on asymptotic expansions of the solutions  $\theta_{m,\mu}$  with respect to the diffusivity  $\mu$  as  $\mu \rightarrow \infty$ . It enables us to bypass regularity assumptions of the type made by Nagahara and Yanagida but, obviously, also has its own limitations in that it requires the diffusivity to be large.
- The stationarity result is proved using fine methods on the switch function, which in this case gives an example of stationarity for a non-energetic functional that does not use rearrangements.

For the time being, we do not know how to give even a formal analysis of the small diffusivity case  $\mu \rightarrow 0$ . However, several numerical simulations seem to indicate that fragmentation is really a relevant term for describing what goes on as  $\mu \rightarrow 0$ . For instance, in [MNP19b], we obtain the following numerical simulations for the optimal resources distributions

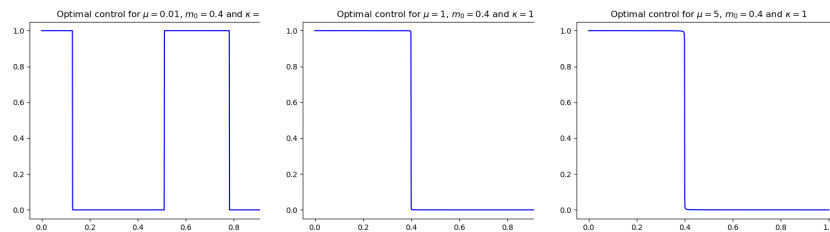


Figure 1.4 –  $m_0 = 0.4$ ,  $\kappa = 1$ . From left to right:  $\mu = 0.01, 1, 5$ . Optimal resources distributions for  $(P_\mu)$ .

We end this Section by commenting on the fact that this is one of the rare case where such a non-energetic, non-linear problem is given an answer, event if it is in an asymptotic regime. The study we carry out in [HMP19] (Chapter 6, Section 1.2.3 of this Introduction) for a prototypical non-linear, non-energetic shape optimisation problem can also be viewed as pertaining to this category of results.

**1.2.1.2 OPTIMAL SURVIVAL OF A POPULATION (CHAPTER 3, QUESTION 3)**

In this Section, devoted to the presentation of our main results regarding spectral optimisation, we will consider two problems. The first one is devoted to spectral optimisation for biased movement of species, while the second one deals with a quantitative inequality for a Schrödinger eigenvalue. Let us start with

$$\boxed{\inf_{m \in \mathcal{M}(\Omega)} \lambda_\alpha(m)} \tag{P_\alpha}$$

where  $\lambda_\alpha(m)$  is the first Dirichlet eigenvalue of

$$-\nabla \cdot ((1 + \alpha m)\nabla) - m.$$

Before presenting our own results, we give some background material on spectral optimisation for the unbiased movement. This corresponds to  $\alpha = 0$  in  $(P_\alpha)$ .

**Background material for  $(P_\alpha)$**

**Unbiased movement of species** Many authors have studied the problem

$$\inf_{m \in \mathcal{M}(\Omega)} \lambda_0(m), \text{ where } \lambda_0(m) \text{ is the first eigenvalue of } -\Delta - m. \tag{1.13}$$

In this case, with Dirichlet boundary conditions, the most striking results obtained in [128, 51] are the following:

1. [128]: for any domain  $\Omega$ , the variational problem (1.13) has a solution. Any solution  $m^*$  of (1.13) is a bang-bang function: there exists  $E \subset \Omega$  such that  $m^* = \kappa \mathbb{1}_E$ . (see [128, Theorem 1.1] for Neumann boundary conditions, but adapting the proof immediately gives the result for Dirichlet boundary conditions)
2. When  $\Omega = \mathbb{B}(0; R)$ , there exists a unique solution of (1.13): let  $r^* > 0$  be such that

$$m^* := \kappa \mathbb{1}_{\mathbb{B}(0; r^*)} \in \mathcal{M}(\Omega).$$

The unique solution of (1.13) is  $m^*$ . Thus, when the domain is ball, we have full concentration.

3. [51]: if  $\Omega$  is a convex domain with an axis of symmetry, if  $m^* = \mathbb{1}_E$  is a solution of (1.13), then  $E$  has the same axis of symmetry. When  $\kappa$  is small enough and  $\Omega$  is simply connected,  $\Omega \setminus E$  is connected. If  $\kappa$  is large enough and if  $\Omega$  is convex, then  $E$  is convex. Concentration thus holds under geometric assumptions on the domain.
4. [51]: there exists  $M_* = M_*(M_0) > 0$  such that, if  $M \geq M_*$ , if  $\Omega = \{M \leq |x| \leq M + 1\}$ , if  $m^* = \kappa \mathbb{1}_E$  is a solution of (1.13),  $E$  is not invariant under the action of rotations. Thus, optimal configurations might not have the same symmetries as the domain.

**REMARK 1.4.** We have written the results of [51] under the form “if  $\kappa$  is large enough” because we defined  $\lambda_0$  as the first eigenvalue of  $-\Delta - m$ . Had it been defined as the first eigenvalue of  $-\mu\Delta - m$ , these results could be rephrased as “if  $\mu$  is small enough”, once again emphasising the role of diffusivity in the geometry of optimal domains.

**Two-phase eigenvalue optimisation** The previous paragraph dealt with optimising an eigenvalue with respect to the potential. Another optimisation problem, which is by now classical, consists

in optimising the first eigenvalue of a differential operator with respect to the drift. Namely, let, for any  $m \in \mathcal{M}(\Omega)$  and  $\varepsilon > 0$ ,  $\tilde{\lambda}_\varepsilon(m)$  be the first eigenvalue of

$$-\nabla \cdot ((1 + \varepsilon m)\nabla)$$

with Dirichlet boundary conditions and consider the optimisation problem

$$\inf_{m \in \mathcal{M}(\Omega)} \tilde{\lambda}_\varepsilon(m). \tag{1.14}$$

Then, following the seminal work by Murat and Tartar [142], which gave the correct way to look at optimality conditions, the following results, some of which are very recent, were established:

1. In [142] it is proved that, if a solution  $m^*$  to (1.14) exists, then it is a bang-bang function and, if  $m^* = \kappa \mathbf{1}_E$  and  $E$  is regular, then  $\Omega$  is a ball  $\Omega = \mathbb{B}(0; R)$  and the set  $E$  is radially symmetric. Their proof relies on the use of Serrin's theorem.
2. In [57], a rearrangement introduced in [7] is used to show that, when  $\Omega$  is a ball, it is sufficient to work with radially symmetric resources distributions, and a proof of existence of a solution is given in that case.
3. In [117], the case of the ball is completely settled for  $\varepsilon$  small enough: Laurain proves that there are two radially symmetric functions  $m_1$  and  $m_2$ , a parameter  $\underline{m}_0$  and  $\underline{\varepsilon}$  such that: if  $m_0 < \underline{m}_0$  and  $\varepsilon \leq \underline{\varepsilon}$ ,  $m_1$  is the only solution of (1.14) while, if  $m_0 > \underline{m}_0$  and  $\varepsilon \leq \underline{\varepsilon}$ ,  $m_2$  is the unique solution of (1.14).
4. In [46, 47, 48], the problem of existence is definitely settled: there, it is proved that (1.14) has a solution if and only if  $\Omega$  is a ball.

In other words, the presence of a drift drastically changes the picture of spectral optimisation problems. As will be explained later, some of our proofs simplify and give a more systematic understanding of the proofs of Laurain [117] for stationarity of optimal configurations when  $\alpha$  (or  $\varepsilon$ ) is small, and shed a new light on the way to build path of admissible resources distributions.

**Biased movement of species** For the biased movement problem  $(P_\alpha)$ , we recall that it was introduced in [17, 59]. The goal of these authors was mainly to understand the effect of adding a drift (i.e going from  $\alpha = 0$  to some small  $\bar{\alpha} > 0$ ) when the resources distribution  $m$  is fixed; they derive monotonicity results for the map

$$\alpha \mapsto \lambda_\alpha^{\mathcal{B}}(m),$$

and insist of the influence of boundary conditions. Let us keep in mind that, for Dirichlet boundary conditions, no monotonicity holds in general: adding a drift when there are resources near the lethal boundary might be detrimental to the population while adding a drift when these resources are far enough from the boundary can be favourable. This is coherent with the intuition one might have of the problem.

A complete study of  $(P_\alpha)$  in the one-dimensional case, for any boundary conditions (whether it be Neumann, Robin or Dirichlet) can be found in [50], where most of the relevant qualitative informations are gathered. For instance, regardless of  $\alpha > 0$ , the solution of the biased problem in  $\Omega = (-1; 1)$  is the same as for the unbiased movement optimisation problem (1.13). Their proof relies on the Sturm-Liouville change of variables which is, unfortunately, unavailable in higher dimensions.

For multi-dimensional domains, the question remained completely open until [MNP19a] (Chapter 3 of this manuscript).

**BIBLIOGRAPHICAL REMARK.** We also mention [10], where the same type of problem is considered for Neumann boundary conditions in the context of reaction-diffusion systems.

We finally mention [84], where a general result for spectral optimisation of eigenvalues with respect to both the drift term and the potential is established; more precisely, let us briefly state a consequence of [84, Theorem 2.1]: let, for any  $m_1, m_2 \in \mathcal{M}(\Omega)$ ,  $A_{m_1, m_2}$  be the operator

$$-\nabla \cdot ((1 + \alpha m_1) \nabla) - m_2$$

and  $\Lambda(m_1, m_2)$  its first Dirichlet eigenvalue. Let  $\Omega^*$  be a ball with the same volume as  $\Omega$ . Then, for any  $m_1, m_2 \in \mathcal{M}(\Omega)$ , there exists two radially symmetric functions  $m_1^*, m_2^* \in \mathcal{M}(\Omega^*)$  such that

$$\Lambda(m_1^*, m_2^*) \leq \Lambda(m_1, m_2).$$

This does not, however, enable us to reach a conclusion, as  $m_1^*$  and  $m_2^*$  might be different.

**Contributions of the thesis for species' survival (Chapter 3, Question 3)** We give an almost complete study of the variational problem  $(P_\alpha)$  in [MNP19a] (Chapter 3 of this manuscript); we sum up the main contributions in the following theorem:

**THEOREM OF THE THESIS 2 [MNP19a]** *Regarding problem  $(P_\alpha)$*

1. Non-existence result (Theorem 3.1.1):

*In any dimension, if  $\Omega$  is not a ball, then  $(P_\alpha)$  does not have a solution.*

2. Stationarity for small  $\alpha$  among radial distributions (Theorem 3.1.2):

*If  $\Omega = \mathbb{B}(0; R)$  if  $n = 2, 3$ , if  $r^* > 0$  is chosen so that*

$$m^* := \kappa \mathbb{1}_{\mathbb{B}(0; r^*)} \in \mathcal{M}(\Omega),$$

*if  $\alpha > 0$  is small enough, then  $m^*$  is a global minimiser among radially symmetric resources distributions. In other words, the solutions (for radially symmetric distributions) is the same for  $\alpha = 0$  and  $\alpha > 0$  small enough.*

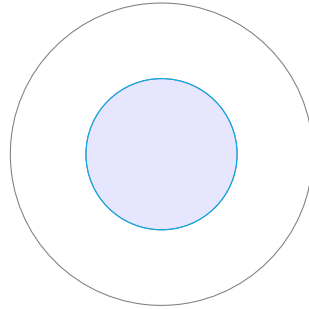


Figure 1.5 – Solution of the problem among radially symmetric distributions.

3. Full stationarity for small  $m_0$  and small  $\alpha$  (Theorem 3.1):

*Let  $\Omega = \mathbb{B}(0; R)$ . There exists  $\bar{m}_0 > 0, \alpha_0 > 0$  such that, if  $m_0 \leq \bar{m}_0$  and  $\alpha \leq \alpha_0$ , then  $m^*$  is the unique minimiser of  $\lambda_\alpha$  in  $\mathcal{M}(\Omega)$ .*

4. Shape minimality (Theorem 3.1.3):



*The resources distribution  $m^* := \kappa \mathbb{1}_{\mathbb{B}(0; r^*)}$  satisfies first and second order optimality conditions in terms of shape derivatives in dimension  $n = 2$  (see Definition 1.1.1).*

For a presentation of the proofs, we refer to Sections 1.3.3 and 1.3.4 of this Introduction. We highlight, for the time being, that the main novelties in our approach are:

- The use of a new method to prove stationarity for small  $\alpha$ ; it relies on a very precise study of the level sets, on a new asymptotic expansion method (across the boundary of the optimal set) and on a fine use of the switch function provided by classical homogenization theory. This requires a careful adaptation of one of the methods introduced in [Maz19a] (Chapter 4).
- The use of a comparison principle method introduced in [Maz19a] (Chapter 4) in a perturbative setting to avoid lengthy computations when discussing optimality conditions in the sense of shape derivatives.

### 1.2.1.3 A QUANTITATIVE INEQUALITY FOR THE SENSITIVITY OF A SCHRÖDINGER EIGENVALUE (CHAPTER 4, QUESTION 2)

We now explain our final contribution to spectral optimisation in this thesis, that is, obtaining a quantitative inequality for a Schrödinger eigenvalue when the underlying domain is a ball. This result has interest in and of itself, but we also believe that [Maz19a] clarifies some points on the main difference between shape derivatives and parametric derivatives, which are two objects used throughout all of this thesis, and which we use in fine combination in this work.

We first briefly recall the main bibliographical references for quantitative inequalities. The general context is the following: consider a shape functional  $\mathcal{F} : \Omega \mapsto \mathcal{F}(\Omega)$ , which is minimised under a volume constraint or a perimeter constraint, and assume that the unique minimiser of  $\mathcal{F}$  is a ball  $\mathbb{B}(0, R)$ . This is the case in the classical isoperimetric inequality or for the Faber-Krahn inequality. Introduce the Fraenkel asymmetry of a domain

$$\mathcal{A}(\Omega) = \inf_{x \in \mathbb{R}^n} \frac{|\Omega \Delta \mathbb{B}(x; R)|}{|\mathbb{B}(x; R)|}.$$

This quantity appears naturally in many quantitative inequalities; the typical quantitative inequality reads

$$\mathcal{F}(\Omega) - \mathcal{F}(\mathbb{B}) \geq C\mathcal{A}(\Omega)^2,$$

and the exponent 2 is usually sharp. This type of inequality was first established in [78] for the classical isoperimetric inequality: there,  $\mathcal{F}(E) = \text{Per}(E)$  is the perimeter, and they consider a volume constraint. In [31], such inequalities are derived for  $\mathcal{F}(E) = \lambda_D(E)$ , where  $\lambda_D(E)$  is the first Dirichlet eigenvalue of the Laplace operator. These were extended to more general eigenvalue problems with Robin boundary conditions in [37]. We give [31] as a general introduction to sharp quantitative inequalities, and refer to the Introduction of Chapter 4 for a more complete list of bibliographical references. We note that a strategy of proof commonly used is first to establish such inequalities for a particular set of domains, namely, normal deformations of the ball  $\mathbb{B}^*$ , (for this step, we refer to the synthetic [64] where a systemic approach is undertaken), and then, through generally very involved steps to prove that every other situation can be handled through this lens.

In [29]<sup>2</sup>, a first result related to quantitative inequalities for interior domains is established; they consider a quantitative inequality for a general Dirichlet energy, where minimisation is carried out with respect to the potential under  $L^p$  constraints and the source term is fixed.

Our approach in [Maz19a] is somehow different, both for the goal and the methods. In it, we work in the case of a ball  $\Omega = \mathbb{B}(0; R)$  and consider the problem we have already encountered of minimising

<sup>2</sup>We thank L. Brasco for pointing out this reference.

the first eigenvalue of a Schrödinger operator:

$$\boxed{\inf_{m \in \mathcal{M}(\Omega)} \lambda(m)} \tag{Q}$$

where  $\lambda(m)$  is the first Dirichlet eigenvalue of the operator

$$-\Delta - m.$$

We then established a qualitative inequality for this problem; this is the main result of Chapter 4.

**THEOREM OF THE THESIS 3** [*Maz19b*] *Let  $\Omega = \mathbb{B}(0, R)$  in dimension  $n = 2, 3$  and let  $m^* := \kappa \mathbb{1}_{\mathbb{B}(0, r^*)} \in \mathcal{M}(\Omega)$  be the unique solution of (Q). There exists  $C > 0$  such that the following quantitative inequality holds:*

$$\forall m \in \mathcal{M}(\Omega), \lambda(m) - \lambda(m^*) \geq C \|m - m^*\|_{L^1(\Omega)}^2.$$

For a mathematical presentation of the proof, we refer to Paragraphs 1.3.4 and 1.3.4 of this Introduction. We underline the two features of this proof which seem relevant to us:

- A comparison principle is used to encompass the case of normal deformations of the optimiser, which greatly simplifies computations and is used when possible competitors can be compared to normal deformations of the ball.
- A technique of expansion of the switch function across the border of the optimiser is used in its full force to handle the case of competitors whose topology might change. This is related to parametric derivatives, that is, derivatives with respects to the coefficients of the equation. This is the technique that would need adapting to extend this inequality to other domains.

These two approaches then need to be combined in order to get access to the full quantitative inequality.

## 1.2.2 CONTROLLABILITY OF SPATIALLY HETEROGENEOUS BISTABLE EQUATIONS

This Section deals with Chapter 5, which corresponds to [MRBZ19].

**REMARK 1.5.** *Throughout this section, the notation  $\theta$  represents a real parameter, in contrast to the previous sections, where it stood for a solution of an equation. The reason we do this is to have a presentation that is coherent with the classical literature devoted to bistable equations.*

### 1.2.2.1 PRESENTATION OF THE MODEL

**Bistable reaction-diffusion equations** One of the equations we have not yet mentioned but which is of great importance for applications is the *bistable reaction-diffusion equation*. We begin with a review of basic background material on bistable equations and their boundary control.

Bistable equations, which model the evolution of the proportion  $p = \frac{n_1}{n_1 + n_2}$  of a subpopulation  $n_1$  in a total population  $N = n_1 + n_2$ , have a wide range of possible interpretations. Each of the population subgroup  $n_1, n_2$  is defined by a trait:  $n_1$  might account for a population of infected mosquitoes while  $n_2$  represents sane mosquitoes,  $n_1$  might be a group of bilingual individuals (speaking both a minority and a majority language), in which case  $n_2$  would for instance be a group of monolingual individuals

(i.e individuals who can only speak the majority language). This equation is characterized by the *Allee effect*: assuming that there is no spatial dependence, there exists a threshold  $\theta \in [0; 1]$  such that, if the initial proportion  $p_0$  is below  $\theta$ , then  $n_1$  will go extinct while, if  $p_0$  is above  $\theta$ ,  $n_2$  will go extinct.

We will first state the equation, before sketching how it might be derived; for further references, we refer to the Introduction of Chapter 5. The usual model goes as follows: consider a *bistable non-linearity* that is, a function  $f : [0; 1] \rightarrow \mathbb{R}$  such that

1.  $f$  is  $\mathcal{C}^\infty$  on  $[0, 1]$ ,
2. There exists  $\theta \in (0; 1)$  such that  $0, \theta$  and  $1$  are the only roots of  $f$  in  $[0, 1]$ ,
3.  $f'(0), f'(1) < 0$  and  $f'(\theta) > 0$ ,

We work under the assumption that

$$\int_0^1 f > 0$$

An example is given by

$$f(\xi) = \xi(\xi - \theta)(1 - \xi),$$

and, in this case,  $\int_0^1 f > 0$  is equivalent to requiring that

$$\theta < \frac{1}{2}.$$

The typical graph of a bistable non-linearity looks like Figure 1.6 below :

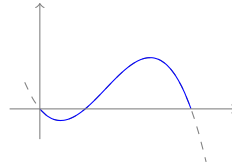


Figure 1.6 – Typical bistable non-linearity.

Such a function accounts for the Allee effect.

The main equation then reads

$$\begin{cases} \frac{\partial p}{\partial t} - \mu \Delta p = f(p) \text{ in } \mathbb{R}_+ \times \Omega, \\ p(t = 0) = p_0, 0 \leq p_0(x) \leq 1, \end{cases} \quad (1.15)$$

along with some boundary conditions that we do not yet specify.

**BIBLIOGRAPHICAL REMARK.** *Since this equation is only studied in Chapter 5, we refer to [143, 43] for more details on the modelling.*

This equation is however not completely satisfactory for several modelling reasons, one of which being that (1.15) is implicitly set in a spatially homogeneous domain  $\Omega$ . Although, in [MRBZ19], we also tackle so-called *gene-flow models* (see Chapter 5), we focus, in this Introduction, on *spatially heterogeneous models* since this is the setting of our main contributions. These models correspond to the case of spatially heterogeneous environments, the spatial heterogeneity being modelled by a

resources distribution  $m : \Omega \rightarrow \mathbb{R}$ . In this case, under certain scaling assumptions detailed in the Introduction of Chapter 5, one can show that the equation on  $p$  becomes

$$\begin{cases} \frac{\partial p}{\partial t} - \mu \Delta p + 2 \langle \frac{\nabla m}{m}, \nabla p \rangle = p(1-p)(p-\theta) =: f(p) \text{ in } \mathbb{R}_+ \times \Omega, \\ p(t=0, \cdot) = p_0(x), 0 \leq p_0 \leq 1. \end{cases} \quad (1.16)$$

We now explain what kind of controllability questions we want to tackle.

**Controllability for bistable equations: background material** For many of the proposed interpretations, for instance for sane and infected mosquitoes (a derivation of the model given in [146] is presented in the Introduction of Chapter 5), control problems naturally arise: is it possible to control the population of infected mosquitoes inside a domain? Such questions are drawing more and more attention from mathematicians working in control theory, and the answers to these questions depend on what we mean by *control*; we might consider distributed control, as is the case in [5] for a non-diffusive model (and the authors obtain a full characterization of optimal controls) or, as will be the case here, boundary controls. This makes sense regarding the biological interpretation of the equation, whether it be in terms of mosquitoes (we might want to act on the boundary of the jungle where the mosquitoes live rather than using an interior control; we refer to [163] and to the references therein) or for bilingualism (where we want to check whether or not an influx of bilingual people will guarantee the survival of the minority language; we refer to [157, 163, 175]). Obviously, given the interpretation of  $p$  as a proportion, natural constraints will be imposed on the control, and this significantly complexifies things.

Let us give more precise definition in the case of spatially homogeneous equations with boundary control, which were first tackled in [157] in the one-dimensional case, and then in [163] in the multi-dimensional case.

A control is a function  $a : \mathbb{R}_+ \times \partial\Omega \rightarrow [0; 1]$  and, for any such control, we consider the *spatially homogeneous control system*

$$\begin{cases} \frac{\partial p}{\partial t} - \mu \Delta p = f(p) \text{ in } \mathbb{R}_+ \times \Omega, \\ p(t=0) = p_0, 0 \leq p_0(x) \leq 1, \\ p(t, \cdot) = a(t, \cdot) \in [0; 1] \text{ on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \quad (1.17)$$

We highlight the constraint

$$a \in [0; 1]. \quad (1.18)$$

We now introduce the targets of the control. Here, *the only targets we consider are*  $z_0 \equiv 0, z_1 \equiv 1$  and  $z_\theta \equiv \theta$ , i.e we want to drive the proportion  $p$  so a spatially homogeneous proportion.

We want to see whether or not it is possible to control any initial datum  $p_0$  to one of these three states in the following sense:

**DEFINITION 1.2.1** *Controllability is defined as follows:*

- *Controllability in finite time: we say that  $p_0$  is controllable to  $\alpha \in \{0, \theta, 1\}$  in finite time if there exists a finite time  $T < \infty$  such that there exists a control  $a$  satisfying the constraints (1.18) and such that the solution  $p = p(t, x)$  of (1.17) satisfies*

$$p(T, \cdot) = z_\alpha \text{ in } \Omega.$$

- *Controllability in infinite time: we say that  $p_0$  is controllable to  $\alpha \in \{0, \theta, 1\}$  in infinite time if there exists a control  $a$  satisfying the constraints (1.18) such that the solution  $p = p(t, x)$  of*

(1.17) satisfies

$$p(t, \cdot) \xrightarrow[t \rightarrow \infty]{\mathcal{C}^0(\Omega)} z_\alpha.$$

The following results are obtained in [157, 163]:

1. In [157] they reach the following conclusions in the one-dimensional case (for  $\Omega = (0; L)$ ): Equation (1.17) is always controllable to 1 in infinite time. Furthermore, there exists a limit diffusivity  $\mu_* = \mu_*(L) > 0$  such that:
  - (a) whenever  $\mu > \mu_*(L)$ , Equation (1.17) is controllable to 0 and 1 in infinite time, and to  $\theta$  in finite (but positive) time.
  - (b) For any  $\mu < \mu_*(L)$ , Equation (1.17) is not controllable to either  $\theta$  or 0.
2. In [163], these results are extended to the multi-dimensional case; the existence of a limit diffusivity for controllability is established and spectral conditions on the domain are given to ensure controllability to  $z_0$  and  $z_\theta$ .

In both these works, the method used is the *staircase method* of [58], see Chapter 5 for a more detailed presentation, but let us keep in mind that the lack of controllability is due to the existence of non-trivial steady-states of the equation.

In a spatially heterogeneous context the natural question is then the following one:

**Question 5.** *Does spatial heterogeneity have an influence on controllability properties? In other words, can the results of [157, 163] be extended in this context, or do spatial variations lead to lack of controllability?*

### 1.2.2.2 CONTRIBUTIONS OF THE THESIS FOR THE CONTROLLABILITY OF BISTABLE EQUATIONS (CHAPTER 5, QUESTION 5)

We summarize the results obtain in [MRBZ19] (Chapter 5 of this manuscript). Let  $m : \Omega \rightarrow \mathbb{R}_+^*$  be a resources distribution. We consider a control  $a$  defined as any function satisfying

$$a : \mathbb{R}_+ \times \partial\Omega \rightarrow [0; 1].$$

For an initial datum

$$p_0 \in [0; 1] \text{ p.p}$$

the spatially heterogeneous control system associated is

$$\begin{cases} \frac{\partial p}{\partial t} - \mu \Delta p + 2(\nabla \ln(m), \nabla p) = f(p) \text{ in } \mathbb{R}_+ \times \Omega, \\ p(t = 0) = p_0, 0 \leq p_0(x) \leq 1, \\ p(t, \cdot) = a(t, \cdot) \in [0; 1] \text{ on } \mathbb{R}_+ \times \partial\Omega. \end{cases} \quad (1.19)$$

Using the notion of controllability similar to that of Definition 1.2.1, we obtain in [MRBZ19] the following result<sup>3</sup>:

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<sup>3</sup>which is stated in the  $n$ -dimensional case with a limit diffusivity while, in the one dimensional case, we focus on the length of the interval, which is more suited to the phase plane analysis of Chapter 5. Note that, while in Chapter 5 we mainly states the results insisting on some spectral and geometric parameters here, to be in agreement with the viewpoint of earlier works presented in the introduction, we adopt the standpoint of diffusivity.

**THEOREM OF THE THESIS 4 :** *[MRBZ19]* For the control system (1.17), there holds:

1. For any type of spatial heterogeneity:

There exists a limit diffusivity  $\mu_*(\Omega) > 0$  below which controllability to 0 does not hold.

2. For slowly varying environments:

Assume that the spatial heterogeneity is slowly varying, i.e, assume that  $m$  writes as

$$m = 1 + \varepsilon \tilde{m}.$$

Then there exists  $\varepsilon_* > 0$  such that, if  $\varepsilon < \varepsilon_*(\Omega, \tilde{m})$ , Equation (1.19) is controllable to 1 in infinite time. If in addition  $\mu > \mu_*(\Omega)$ , then (1.19) is controllable to 0 in infinite time and to  $\theta$  in finite time. This corresponds to Theorem 5.1.2.

3. A prototypical rapidly varying environment:

We consider the one-dimensional case. Fix  $\mu = 1$  and define, for any  $\sigma > 0$  the gaussian

$$m_\sigma(x) := e^{-\frac{x^2}{2\sigma}}.$$

Then, for any  $\sigma > 0$  and  $\alpha \in \{0; 1\}$ , there exists  $L_\sigma^{**}$  such that Equation (1.19) is not controllable to 0,  $\theta$  or 1 on  $(-L_\sigma^{**}, L_\sigma^{**})$ .

Furthermore, let, for any  $\alpha \in \{0; 1\}$ ,  $L_\sigma^*(\alpha)$  be the minimal  $L$  such that controllability to  $\alpha$  fails on  $(-L, L)$ . There holds

$$L_\sigma^*(1) \xrightarrow{\sigma \rightarrow 0} 0, L_\sigma^*(1) \xrightarrow{\sigma \rightarrow \infty} +\infty,$$

and

$$L_\sigma^*(0) \xrightarrow{\sigma \rightarrow 0} 0.$$

This corresponds to Theorem 5.1.4.

In other words, for rapidly varying environments, blocking phenomena appear, which seems to corroborate an observation of [146], where a similar blocking phenomenon was derived for the propagation of travelling waves in a rapidly shifting environment. We underline two points in the proofs of these results:

- The main innovation is the proof for slowly varying environments, which is sketched in Paragraph 1.3.6 of this Introduction, and whose main innovation is the use of perturbative arguments in combination with the staircase method of Coron and Trélat, [58].
- The lack of controllability to 1 is related to the existence of non-trivial solutions  $p \neq z_1$  to

$$\begin{cases} -p'' + 2 \ln(m)'p' = f(p) \text{ in } (-L, L), \\ p(\pm L) = 1. \end{cases}$$

This result seems interesting in and of itself.

### 1.2.3 A PROTOTYPICAL NON-LINEAR, NON-ENERGETIC SHAPE OPTIMISATION PROBLEM IN THE ASYMPTOTIC REGIME

This Section deals with Chapter 6, which corresponds to [HMP19].

Let us mention a non-linear perturbation of a classical shape optimisation problem that we tackle in the asymptotic regime. We sketch the problem and the type of properties investigated; since the main contributions are a new technical outlook on a shape optimisation problem, we postpone a full presentation to Paragraphs 1.3.2 and 1.3.3.

In our articles [MNP19a, MNP19b], the study of optimisation problem heavily relies on perturbative arguments, whether it be when the diffusivity is large or when the drift is small. Here, the point of view is quite different, since it is the domain itself we seek to optimise.

Let us consider, for a small enough parameter  $\rho$ , the unique solution  $u_{\rho,\Omega}$  of

$$\begin{cases} -\Delta u_{\rho,\Omega} + \rho f(u_{\rho,\Omega}) = g & \text{in } \Omega \\ u_{\rho,\Omega} \in W_0^{1,2}(\Omega). \end{cases} \quad (1.20)$$

We consider, for some parameter  $V_0$  and some box  $D$ , the optimisation problem

$$\inf_{\Omega \subset D, |\Omega| \leq V_0} J_\rho(\Omega) \quad \text{where } J_\rho(\Omega) = \frac{1}{2} \int_\Omega |\nabla u_{\rho,\Omega}|^2 - \int_\Omega g u_{\rho,\Omega}. \quad (P_\rho)$$

The non-linearity of the equation, the fact that the criterion to optimise is highly non-linear, make it a difficult problem.

More precisely, we were interested in *existence properties* and in *stability issues*. Namely, we know that, when  $\rho = 0$ , when  $g$  is a non-increasing, radially symmetric function, the unique solution to  $(P_\rho)$  is a centered ball  $\mathbb{B}^*$ , and we wanted to investigate whether or not  $\mathbb{B}^*$  still satisfies optimality conditions when  $\rho > 0$  is small enough. A brief summary of the results obtained in [HMP19] reads

**THEOREM OF THE THESIS 5 : [HMP19]**    1. If  $\inf_D g > 0$  and  $f$  is smooth enough, then there exists  $\underline{\rho} > 0$  such that, for any  $\rho \leq \underline{\rho}$ , the variational problem  $(P_\rho)$  has a solution  $\Omega$ . This result can be extended to non-negative  $g$  by adding monotonicity assumptions on  $f$ .

2. If  $g \equiv 1$  (in which case, for  $\rho = 0$ , the unique solution of  $(P_\rho)$  is a ball  $\mathbb{B}^*$  of volume  $V_0$ ), there exists a smooth non-linearity  $f$  such that, for any  $\rho > 0$  small enough, the centered ball  $\mathbb{B}^*$  does not satisfy second order optimality conditions.

As explained, we postpone the presentation of the methods to Paragraphs 1.3.2 and 1.3.3, but highlight the two main arguments:

- To obtain existence under different monotonicity and sign assumptions on the non-linearity  $f$  and the source term  $g$ , we introduce a relaxed version of the problem which makes it so that we can work in a fixed domain, study the switch function associated with the problem in this domain and pass to the limit to obtain monotonicity of  $J_\rho$ ; it then remains to apply the Buttazzo-DalMaso theorem. This method seems promising to tackle other non-linear, non-energetic problems, at least in some regimes.
- To establish that, if  $g \equiv 1$ , there exists a non-linearity  $f$  such that  $\mathbb{B}^*$  no longer satisfies optimality conditions for  $\rho > 0$  small enough we carry out an asymptotic analysis of the second order shape derivative of the associated lagrangian; to the best of our knowledge, this method is new and looks like it might be adapted to other contexts. For this result, we also adapt some of the methods (e.g comparison principle) introduced in [Maz19a].

## 1.3 SHAPE OPTIMISATION: METHODS DEVELOPED IN THE THESIS

As explained when we laid out the structure of this Introduction, we are now going to present, in this second part, what we think are the main tools and innovations developed throughout the works we carried out. Without dwelling on too many details, we try to give a clear presentation of what we think are the most salient features of the methods we introduced.

### 1.3.1 TYPICAL PROBLEMS

In the same way we presented, in the first part of this Introduction, the typical questions we were led to study, let us introduce the prototypical problems we were faced with and that we managed to overcome in certain situations.

**The existence of optimal shapes** The problem of existence of optimal shapes is ubiquitous and notoriously difficult to solve when the functionals we seek to minimise are not energetic functionals. This question is interesting, both from a mathematical perspective and from an applied perspective; as underlined in Section 1.1.3.3, the existence of optimal shapes may also be a validation of the relevance of *patch models* and are a natural framework to study mathematical biology.

**The typical problem for patch models** Many of the questions we consider in Chapters 2, 3 and 4, whether it be the total population size, the optimal survival of species..., can be recast as a problem under the general form

$$\min_{m \in \mathcal{M}(\Omega)} F(m) \tag{1.21}$$

where we recall that the admissible class is

$$\mathcal{M}(\Omega) := \left\{ m \in L^\infty(\Omega), 0 \leq m \leq \kappa, \int_{\Omega} m = m_0 \right\},$$

and can be linked with a shape optimisation problem: let  $m \in \mathcal{M}(\Omega)$  be a *bang-bang* function, that is,  $m$  rewrites as

$$m = m_E := \kappa \mathbb{1}_E,$$

where  $E \subset \Omega$  is a measurable subset with

$$|E| = \frac{m_0}{\kappa} =: E_0.$$

Define the class of admissible sets

$$\mathcal{O}(\Omega) := \{E \subset \Omega, |E| = E_0\}, \tag{1.22}$$

and consider the problem

$$\min_{E \in \mathcal{O}(\Omega)} \mathcal{F}(E). \tag{1.23}$$

Problem (1.21) is called a *relaxed version of (1.23)*. Then, checking whether or not the solutions of (1.21) are bang-bang functions amounts to answering the following question:

**Question 6.** *Can we prove that (1.21) and (1.23) have the same solutions or, in other words, can we prove that any solution of (1.21) is a bang-bang function?*



Not only is this question relevant for interpretational purposes, but also for the analysis of optimality conditions.

**A more theoretical shape optimisation problem** Another contribution of this thesis (Chapter 6) is the understanding of a non-linear, non-energetic shape optimisation problem presented in Section 1.2.3 of this Introduction. In this article, we consider the following problem: let  $D \subset \mathbb{R}^n$ ,  $V_0 \in (0, |D|)$ ,  $f \in W^{1,\infty}(\mathbb{R})$ ,  $g \in L^2(D)$ ,  $\rho \geq 0$ . For  $\Omega \subset D$  such that  $|\Omega| = V_0$ , consider the solution  $u_{\Omega,\rho}$  of

$$\begin{cases} -\Delta u_{\Omega,\rho} + \rho f(u_{\Omega,\rho}) = g & \text{in } \Omega, \\ u_{\Omega,\rho} = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.24)$$

and the functional

$$J_\rho(\Omega) := \frac{1}{2} \int_{\Omega} |\nabla u_{\Omega,\rho}|^2 - \int_{\Omega} gu.$$

We consider the variational problem

$$\inf_{\Omega \subset D, \Omega \text{ quasi-open}, |\Omega|=V_0} J_\rho(\Omega). \quad (P_\rho)$$

The question is then

**Question 7.** *How can we prove that  $(P_\rho)$  has a solution?*

**Stability of the ball under perturbation** Another interesting question is that of investigating whether or not certain specific configurations satisfy the optimality conditions for the problem under consideration.

For instance, when dealing with problems of the form (1.21), if the domain  $\Omega = \mathbb{B}(0; R)$ , then, when dealing with Dirichlet boundary conditions, a natural candidate to be a minimiser is

$$m^* := \kappa \mathbb{1}_{\mathbb{B}^*}$$

where  $\mathbb{B}^*$  is a centered ball satisfying the required volume constraint. Now, in Chapter 3 (see Section 1.2.1.2 of this Introduction), we consider the following problem: let  $\lambda_\alpha(m)$  be the first Dirichlet eigenvalue of

$$-\nabla \cdot ((1 + \alpha m)\nabla) - m$$

and consider the problem

$$\inf_{m \in \mathcal{M}(\Omega)} \lambda_\alpha(m). \quad (P_\alpha)$$

Let  $\Omega = \mathbb{B}(0, R)$  be a ball. It is known that, for  $\alpha = 0$ ,  $m^* = \kappa \mathbb{1}_{\mathbb{B}^*}$  is the unique solution of  $(P_0)$ . The following question raises the problem of stability of  $m^*$  under perturbation as  $\alpha$  increases from 0:

**Question 8.** *Can we prove that  $\mathbb{B}^*$  remains a minimiser when  $\alpha > 0$  is small enough?*

We also tackled the problem of the stability of the ball as a minimiser under perturbations in the classical shape optimisation problem: namely, for the variational problem  $(P_\rho)$  of Paragraph 1.3.1 (and recalled in the previous paragraph), if the box  $D$  is a ball, if  $g$  is radially symmetric and non-increasing, then it is known that  $\Omega = \mathbb{B}^*$  with  $|\mathbb{B}^*| = V_0$  is the unique solution of  $(P_\rho)$ . It is then natural to investigate the following question:

**Question 9.** *Is it true that, for any non-linearity  $f$  and  $\rho > 0$  small enough,  $\mathbb{B}^*$  satisfies the optimality conditions of Problem  $(P_\rho)$ ? On the contrary, can we build a non-linearity  $f$  such that  $\mathbb{B}^*$  does not satisfy these optimality conditions for  $\rho > 0$  small enough?*

Despite the fact that these problems have different natures ( $(P_\alpha)$  is a parametric optimisation problem, while  $(P_\rho)$  is a shape optimisation one), we gather these two questions in the same paragraph for the reason that we developed the use of the same kind of comparison principle in these two cases. Namely, building on the traditional approach of Lord Rayleigh, we use separation of variables for shape derivatives and derive the desired conclusions in somewhat similar manners in both cases. For Problem  $(P_\rho)$ , we boost this method to build non-linearities for which  $\mathbb{B}^*$  no longer satisfies optimality conditions when  $\rho > 0$ .

**Quantitative stability of the ball** Another natural question when dealing with shape optimisation is that of the stability of minimisers in a quantitative sense. The usual context, as explained in Section 1.2.1.3 (which corresponds to [Maz19a], Chapter 4 of this manuscript), is that of shape optimisation: for an optimisation problem of the form

$$\inf_{E, |E|=V_0} \mathcal{F}(E)$$

is known (for the isoperimetric inequality, for the first eigenvalue of the Laplace operator with a variety of boundary conditions) that, if  $E^*$  is the unique minimiser (usually the ball), then there holds, for some constant  $C > 0$ , the following quantitative inequality:

$$\forall E \subset \mathbb{R}^n, |E| = V_0, \mathcal{F}(E) - \mathcal{F}(E^*) \geq C \inf_{x \in \mathbb{R}^n} |E \Delta (E^* + x)|^2,$$

where we obviously do not specify the type of regularity required. We refer to [30, 64].

A question that arose in the context of Problem  $(Q)$ , that is, the Problem

$$\inf_{m \in \mathcal{M}(\Omega)} \lambda(m), \text{ where } \lambda(m) \text{ is the first Dirichlet eigenvalue of } -\Delta - m$$

was that of establishing such inequalities in this case- we note that quantitative inequalities in this parametric context have not been studied much. In [29], this question is addressed in the case of the natural energy of a differential operator with  $L^p$  constraints on the potential of this operator, and a range of these inequalities is proved in this context. In the case of eigenvalues, however, our contribution is, as far as we know, new.

The question is thus:

**Question 10.** *Can we have a quantitative inequality for  $(Q)$ ?*

We are able to give a positive answer when the domain  $\Omega$  is a ball.

### 1.3.2 METHODS DEVELOPED FOR THE EXISTENCE OF OPTIMAL SHAPE (CHAPTERS 2 AND 6)

We first address Questions 6 and 7.

**An asymptotic expansion method to prove the bang-bang property (Chapter 2)** In Chapter 2, we consider the problem of optimising the total population size in a logistic diffusive model. The problem writes as

$$\sup_{m \in \mathcal{M}(\Omega)} \int_{\Omega} \theta_{m,\mu},$$

where  $\theta_{m,\mu}$  solves (1.11):

$$\begin{cases} \mu \Delta \theta_{m,\mu} + \theta_{m,\mu}(m - \theta_{m,\mu}) = 0 & \text{in } \Omega, \\ \frac{\partial \theta_{m,\mu}}{\partial \nu} = 0 & \text{on } \partial \Omega, \\ \theta_{m,\mu} \geq 0, \\ \theta_{m,\mu} \neq 0 & \text{in } \Omega \end{cases}.$$

Defining

$$\mathcal{F}(\mu, m) := \int_{\Omega} \theta_{m,\mu},$$

this rewrites

$$\sup_{m \in \mathcal{M}(\Omega)} \mathcal{F}(\mu, m).$$

The existence of a maximiser  $m^*$  is a consequence of the direct method in the calculus of variations. As explained in Section 1.2.1.1, an interesting question is that of knowing whether or not any solution  $m^*$  is of bang-bang type.

To give a positive answer to this question when  $\mu$  is large enough, we use the fact that bang-bang functions are extreme points of the convex set  $\mathcal{M}(\Omega)$ . Thus, if  $\mathcal{F}(\cdot, \mu)$  is strictly convex, any maximiser is a bang-bang function. This approach is frequently used for energetic problems (see for instance [116, 128]). Here, the fact that  $\mathcal{F}(\cdot, \mu)$  can not be linked to the energy of the underlying equation complexifies things, and obtaining such convexity properties proves to be challenging.

Thus, our strategy is to prove:

$$\text{For } \mu \text{ large enough, } \mathcal{F}(\cdot, \mu) \text{ is convex.} \tag{1.25}$$

To prove (1.25), we define the admissible perturbations: let  $m \in \mathcal{M}(\Omega)$ . We say that  $h \in L^\infty(\Omega)$  is an *admissible perturbation at  $m$*  if, for every  $t > 0$ ,  $t$  small enough,  $m + th \in \mathcal{M}(\Omega)$ . Let, for any admissible perturbation  $h$  at  $m$ ,  $\dot{\mathcal{F}}(\cdot, \mu)$  (resp  $\ddot{\mathcal{F}}(\cdot, \mu)[h, h]$ ) be the first (resp. second) order derivative at  $t = 0$  of

$$t \mapsto \mathcal{F}(m + th, \mu).$$

Proving (1.25) is equivalent to proving that, for every  $\mu$  large enough, for every  $m \in \mathcal{M}(\Omega)$ , for every admissible perturbation  $h \neq 0$  at  $m$ , there holds

$$\ddot{\mathcal{F}}(m, \mu)[h, h] > 0.$$

We first refine a result of [125] that states that

$$\theta_{m,\mu} \xrightarrow{\mu \rightarrow \infty} m_0$$

and prove that there holds, uniformly in  $m \in \mathcal{M}(\Omega)$ ,

$$\theta_{m,\mu} = m_0 + \frac{\eta_{1,m}}{\mu} + \mathcal{O}\left(\frac{1}{\mu^2}\right),$$

where  $\eta_{1,m}$  solves

$$\begin{cases} \Delta \eta_{1,m} + m_0(m - m_0) = 0, \\ \frac{\partial \eta_{1,m}}{\partial \nu} = 0, \\ \int_{\Omega} \eta_{1,m} = \frac{1}{m_0^2} \int_{\Omega} |\nabla \eta_{1,m}|^2. \end{cases}$$

We may thus write

$$\mathcal{F}(m, \mu) = m_0 + \frac{1}{m_0^2} \int_{\Omega} |\nabla \eta_{1,m}|^2 + \mathcal{O}\left(\frac{1}{\mu^2}\right)$$

and, defining

$$\mathcal{F}_1(m) := \frac{1}{m_0^2} \int_{\Omega} |\nabla \eta_{1,m}|^2$$

it is clear that

$\mathcal{F}_1$  is strictly convex in  $m$ :

for any  $m \in \mathcal{M}(\Omega)$ , for any admissible perturbation  $h \neq 0$  at  $m$ , there holds

$$\ddot{\mathcal{F}}_1(m)[h, h] > 0.$$

As is usual in infinite dimensional optimisation, this second derivative is not coercive enough to guarantee on its own that  $\mathcal{F}(\cdot, \mu)$  is strictly convex for  $\mu$  large enough. To overcome this phenomenon, we fully develop  $\theta_{m,\mu}$  and  $\mathcal{F}(\cdot, \mu)$  with respect to  $\frac{1}{\mu}$ . This is done by introducing a sequence of functions  $\{\eta_{k,m}\}_{k \in \mathbb{N}}$  and a sequence of coefficients  $\{\beta_{k,m}\}_{k \in \mathbb{N}}$  such that

- $\theta_{m,\mu}$  expands as

$$\theta_{m,\mu} = \sum_{k=0}^{\infty} \frac{\eta_{k,m}}{\mu^k},$$

- The sequence  $\{\eta_{k,m}\}_{k \in \mathbb{N}}$  satisfies a hierarchy of equations

$$\begin{cases} \eta_0 = m_0, \\ \forall k \geq 0, \Delta \eta_{k+1,m} + H_{k+1}(\eta_{0,m}, \dots, \eta_{k,m}) = 0, \\ \forall k \in \mathbb{N}, \frac{\partial \eta_{k,m}}{\partial \nu} = 0, \\ \int_{\Omega} \eta_{k+1,m} = \beta_{k+1,m} =: \mathcal{F}_k(m). \end{cases}$$

We then carefully analyse the behaviour of the sequence  $\{\beta_{k,m} = \mathcal{F}_k(m)\}_{k \in \mathbb{N}}$  and show that, for every  $k$ , there exists a constant  $A_k$  such that, for any  $m \in \mathcal{M}(\Omega)$  and any admissible perturbation  $h$  at  $m$  there holds

$$\left| \ddot{\mathcal{F}}_k(m)[h, h] \right| \leq A_k \ddot{\mathcal{F}}_1(m)[h, h],$$

and that the power series  $\sum_{k \in \mathbb{N}} A_k x^k$  has a positive radius of convergence; hence, for  $\mu$  large enough, there holds

$$\ddot{\mathcal{F}}(m, \mu)[h, h] = \sum_{k \in \mathbb{N}} \frac{\ddot{\mathcal{F}}_k(m)[h, h]}{\mu^k} \geq \left( \frac{1}{\mu} - \underbrace{\sum_{k \geq 2} \frac{A_k}{\mu^k}}_{=\mathcal{O}\left(\frac{1}{\mu^2}\right)} \right) \ddot{\mathcal{F}}_1(m)[h, h] > 0.$$

This proves (1.25); hence, all maximisers are bang-bang for  $\mu$  large enough. This method has a considerable advantage: since it does not rely on the construction of explicit perturbations to prove the bang-bang property, it is not necessary to make any regularity assumptions on the maximisers. However, its main drawback is that it is only available for large diffusivities.

**A relaxed formulation for a shape optimisation problem (Chapter 6)** We now present the method developed to tackle the shape optimisation problem  $P_\rho$  (see [HMP19], Chapter 6 of this

manuscript). We briefly recall that it is set as follows: let  $D \subset \mathbb{R}^n$  be a box and define, for a parameter  $V_0$ , the admissible class

$$\mathcal{O}(D) = \{\Omega \subset D, \Omega \text{ quasi-open}, |\Omega| \leq V_0\},$$

a function  $g \in L^2(D)$ , a function  $f \in W^{2,\infty}(\mathbb{R})$  and a parameter  $\rho > 0$ . For any  $\Omega \in \mathcal{O}(D)$ , if  $\rho$  is small enough (uniformly in  $\Omega \in \mathcal{O}(D)$ ), we can define the solution  $u_{\rho,\Omega}$  of

$$\begin{cases} -\Delta u_{\rho,\Omega} + \rho f(u_{\rho,\Omega}) = g \text{ in } \Omega, \\ u_{\rho,\Omega} = 0 \text{ on } \partial\Omega, \end{cases} \quad (1.26)$$

and

$$J_\rho(\Omega) = \frac{1}{2} \int_\Omega |\nabla u_{\rho,\Omega}|^2 - \int_\Omega g u_{\rho,\Omega}$$

The problem is

$$\inf_{\Omega \in \mathcal{O}(D)} J_\rho(\Omega). \quad (P_\rho)$$

Using a result of Buttazzo-Dalmaso, existence for  $(P_\rho)$  is guaranteed if  $J_\rho$  is regular enough (in the sense of  $\gamma$ -convergence, see Chapter 6) and monotonous with respect to domain inclusion:

$$\forall \Omega, \Omega' \in \mathcal{O}(D), \Omega \subset \Omega' \Rightarrow J_\rho(\Omega) \geq J_\rho(\Omega').$$

We now focus on proving this monotonicity property.

We introduce the following relaxation of  $(P_\rho)$ : identifying a domain  $\Omega$  with its characteristic function  $\mathbb{1}_\Omega$  we define a relaxation class as

$$\widehat{\mathcal{O}}_{V_0} = \left\{ a \in L^\infty(D, [0, 1]) \text{ et } \int_D a \leq V_0 \right\}$$

For any  $M > 0, \rho > 0$ , we define the relaxed functional  $\widehat{J}_{M,\rho}$  on  $\widehat{\mathcal{O}}_{V_0}$  as

$$\widehat{J}_{M,\rho}(a) = \frac{1}{2} \int_D |\nabla u_{\rho,M,a}|^2 + \frac{M}{2} \int_\Omega (1-a) u_{\rho,M,a}^2 - \langle g, u_{\rho,M,a} \rangle_{H^{-1}(D), H_0^1(D)},$$

où  $u_{\rho,M,a} \in H_0^1(D)$  is the unique solution of

$$\begin{cases} -\Delta u_{\rho,M,a} + M(1-a)u_{\rho,M,a} + \rho f(u_{\rho,M,a}) = g & \text{in } D, \\ u_{\rho,M,a} = 0 & \text{on } \partial D. \end{cases} \quad (1.27)$$

In this parametric context, an asymptotic study as  $\rho \rightarrow 0$  of the switch function of this relaxed problem enables us to prove that, under certain assumptions on  $f$  and  $g$ , we have

$$\forall a_1, a_2 \in \widehat{\mathcal{O}}_{V_0}, a_1 \leq a_2 \Rightarrow \widehat{J}_{M,\rho}(a_1) \geq \widehat{J}_{M,\rho}(a_2).$$

It is furthermore standard to see that, for any  $\Omega \in \mathcal{O}(D)$ , there holds

$$\widehat{J}_{M,\rho}(\mathbb{1}_\Omega) \xrightarrow{M \rightarrow \infty} J_\rho(\Omega).$$

The desired monotonicity thus holds under certain assumptions on  $f$  and  $g$ , which guarantees existence for  $(P_\rho)$ .

### 1.3.3 METHODS DEVELOPED TO STUDY THE (IN)STABILITY OF THE BALL (CHAPTERS 6 AND 3): SHAPE DERIVATIVES

We present here the methods used to study the stability of the ball through the lens of shape derivatives, see Section 1.1.3.3 for definitions; stability is to be understood as "under some perturbation, the ball still satisfies optimality conditions". We consider two problems. The first one is  $(P_\alpha)$  and writes as

$$\inf_{m \in \mathcal{M}(\Omega)} \lambda_\alpha(m)$$

where  $\lambda_\alpha(m)$  is the first eigenvalue of

$$-\nabla \cdot ((1 + \alpha m)\nabla) - m.$$

This problem is studied in Chapter 3 of this manuscript and presented in Section 1.2.1.2 of this Introduction. We consider the case

$$\Omega = \mathbb{B}(0; R).$$

Let  $\mathbb{B}^* = \mathbb{B}(0, r^*)$  be such that  $m^* = \kappa \mathbb{1}_{\mathbb{B}^*}$  is the unique solution of  $(P_\alpha)$  when  $\alpha = 0$ . We want to check whether or not  $m^*$  still satisfies optimality conditions when  $\alpha > 0$ ,  $\alpha$  small enough. The first step is then to compute the shape derivatives of

$$F_\alpha : E \mapsto \lambda_\alpha(\kappa \mathbb{1}_E)$$

and to study the Lagrangian associated with the volume constraint. First of all, we note that it is always a critical point in the sense of a shape:  $m^*$  satisfies first order optimality conditions. Since  $m^*$  is a critical point for any  $\alpha \geq 0$ , which can be established in a straightforward manner, let us introduce the Lagrange multiplier  $\Lambda_\alpha$ , and the Lagrangian

$$L_\alpha := F_\alpha - \Lambda_\alpha \text{Vol}.$$

The second problem we consider is set in a more classical context; it is Problem  $(P_\rho)$  presented in the previous Section 1.3.2 and in Section 1.2.3 of the Introduction. We briefly recall that we want to solve

$$\inf_{\Omega \subset D, |\Omega| \leq V_0} J_\rho(\Omega), \quad J_\rho(\Omega) = \frac{1}{2} \int_\Omega |\nabla u_{\rho, \Omega}|^2 - \int_\Omega g u, \quad \begin{cases} -\Delta u_{\rho, \Omega} + \rho f(u_{\rho, \Omega}) = g & \text{in } \Omega, \\ u_{\rho, \Omega} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since, if  $g$  is radially symmetric and non-increasing, the centered ball  $\mathbb{B}^*$  such that  $|\mathbb{B}^*| = V_0$  is the unique solution for  $\rho = 0$ , we want to see if for  $\rho > 0$  small enough,  $\mathbb{B}^*$  satisfies optimality conditions. It is once again easy to check that  $\mathbb{B}^*$  is a critical point (in the sense of shape derivatives). Let  $\tilde{\Lambda}_\rho$  be the Lagrange multiplier associated with the volume constraint and consider the Lagrangian

$$T_\rho := J_\rho - \tilde{\Lambda}_\rho \text{Vol}.$$

The second order optimality conditions in the sense of shape derivatives writes as follows in both cases: let

$$\Xi := \left\{ \phi \in \mathcal{C}^1, \int_{\partial\mathbb{B}^*} \langle \phi, \nu \rangle = 0 \right\}.$$

In the case of  $L_\alpha$ , we need to require  $\phi$  to have a compact support in the underlying domain  $\Omega$ , while in the case of  $T_\rho$  we only need to require that it has a compact support.

The second order optimality conditions for Problem  $(P_\alpha)$  (resp. Problem  $(P_\rho)$ ) reads as follows

$$\forall \phi \in \Xi, \langle \phi, \nu \rangle|_{\partial\mathbb{B}^*} \neq 0, L''_\alpha(\mathbb{B}^*)[\phi, \phi] > 0 \quad (\text{resp. } T''_\rho(\mathbb{B}^*)[\phi, \phi] > 0).$$

We are going to prove that, in the two-dimensional case, this positivity holds for  $L''_\alpha$  (Theorem 3.1.3 of Chapter 3) whenever  $\alpha > 0$  is small enough, while, depending on  $f$ , when  $g \equiv 1$ , positivity might fail for  $T''_\rho$ , even if  $\rho$  is small.

In both cases, the fact that we are working with the ball  $\mathbb{B}^*$  in two dimensions allows us to decompose  $\langle \phi, \nu \rangle|_{\partial\mathbb{B}^*}$  as its Fourier series

$$\langle \phi, \nu \rangle|_{\partial\mathbb{B}^*} = \sum_{k \geq 1} a_{k,\phi} e^{2ik\pi\theta}.$$

We then prove that there exist two sequences  $\{\omega_{k,\alpha}\}_{k \in \mathbb{N}}$  et  $\{\varepsilon_{k,\rho}\}_{k \in \mathbb{N}}$  such that

$$L''_\alpha(\mathbb{B}^*)[\phi, \phi] = \sum_{k \geq 1} \omega_{k,\alpha} |a_{k,\phi}|^2$$

and

$$T''_\rho(\mathbb{B}^*)[\phi, \phi] = \sum_{k \geq 1} \varepsilon_{k,\rho} |a_{k,\phi}|^2.$$

These two problems then have different qualitative behaviours:

1. For  $L''_\alpha$ :

We begin with a study of the unperturbed problem, i.e, with the study of the sequence  $\{\omega_{k,0}\}_{k \in \mathbb{N}^*}$ . We prove that, for every  $k \geq 1$ ,  $\omega_{k,0} = y_k(r^*)$  for a solution  $y_k$  to an equation of the form

$$\mathcal{L}_0 y_k = -\frac{k^2}{r^2} y_k \text{ on } (0; R), [y'_k(r^*)] = C \text{ independent of } k, y'_k(0) = y_k(R) = 0.$$

We prove that the differential operator  $\mathcal{L}_0$  has a maximum principle, which gives us monotonicity for the sequence  $\{\omega_{k,0}\}_{k \in \mathbb{N}}$ :

$$\forall k \geq 1, \omega_{k+1,0} > \omega_{k,0}.$$

Finally, studying the case  $\omega_{1,0}$  we prove that  $\omega_{1,0} > 0$  so that

$$\omega_{k,0} > \omega_{1,0} \quad (k \geq 2).$$

We then show that, for  $\alpha > 0$ , the coefficient  $\omega_{k,\alpha}$  is defined, in the same way, through a hierarchy of equations that admits a maximum principle. This allows us to prove that, for every  $k \geq 2$ , there holds

$$\omega_{k,\alpha} - \omega_{1,\alpha} \geq -M\alpha$$

for some uniform (in  $k$ ) constant  $M > 0$ . The proof that  $L''_\alpha$  is positive then follows immediately.

2. For  $T''_\rho$ : regardless of  $g$  and  $f$ , we can prove, using the same comparison methods to get that, for the unperturbed problem ( $\rho = 0$ ) there holds

$$\forall k \in \mathbb{N}, \varepsilon_{k+1,0} > \varepsilon_{k,0} \geq \varepsilon_{1,0} \geq 0,$$

and that there exists  $M > 0$  such that, for  $\rho > 0$  small enough, there holds

$$\forall k \in \mathbb{N}, \varepsilon_{k,\rho} - \varepsilon_{1,\rho} \geq -M\rho.$$

We then need to distinguish two cases:

(a) When  $g$  satisfies the stability condition

$$\frac{1}{\pi R^2} \int_{\mathbb{B}^*} g > g(R),$$

then there holds  $\varepsilon_{1,0} > 0$  and hence, for any  $f$  there exists  $\rho_* = \rho_*(f) > 0$  such that, for any  $\rho < \rho_*$ ,  $T_\rho''$  is positive on  $\Xi$ .

(b) When  $g \equiv 1$ , there holds  $\varepsilon_{1,0} = 0$ . We then look for a Taylor expansion of  $\varepsilon_{1,\rho}$  of the form

$$\varepsilon_{1,\rho} = \rho\epsilon(f) + O_{\rho \rightarrow 0}(\rho^2).$$

Computing  $\epsilon(f)$ , we prove that there exists  $f$  such that  $\epsilon(f) < 0$ , which means that, for any  $\rho > 0$  small enough,  $\varepsilon_{1,\rho} < 0$ . Hence,  $\mathbb{B}^*$  does not satisfy the second order optimality conditions.

Both these methods have the advantage to overcome the usually lengthy computations, which involve Bessel functions, and we do believe that this method can be used in many situations; indeed, it only relies on separation of variables and on a maximum principle for the underlying elliptic operator. Such assumptions are verified in many cases. We do note, however, that this method, however useful, can not be used in a straightforward manner to derive local quantitative inequalities- it only gives a  $L^2$  coercivity, while, in classical shape optimisation, a  $H^{\frac{1}{2}}$ -coercivity is usually to be expected (see [64]).

### 1.3.4 ANOTHER WAY TO INVESTIGATE THE STATIONARITY OF THE BALL (CHAPTERS 3 AND 4): PARAMETRIC FRAMEWORK

We now present a method developed in [Maz19a] and [MNP19a] to investigate, once again, the stability of the ball  $m^* = \kappa \mathbb{1}_{\mathbb{B}^*}$  in a parametric setting, i.e when we are trying to optimise an interior domain. While the previous paragraph was devoted to the study of shape derivatives, this paragraph focuses on the optimality among radial distributions, which allows for changes in the topology.

We work with Problem  $(P_\alpha)$ , that is

$$\inf_{m \in \mathcal{M}(\Omega)} \lambda_\alpha(m), \tag{P_\alpha}$$

where  $\lambda_\alpha(m)$  is the first Dirichlet eigenvalue of

$$-\nabla \cdot ((1 + \alpha m)\nabla) - m$$

and with Problem  $(Q)$ , that is

$$\inf_{m \in \mathcal{M}(\Omega)} \lambda(m), \tag{Q}$$

where  $\lambda(m)$  is the first Dirichlet eigenvalue of

$$-\Delta - m.$$

In both cases,

$$\Omega = \mathbb{B}(0, R), \quad m^* = \kappa \mathbb{1}_{\mathbb{B}^*} \in \mathcal{M}(\Omega),$$

and we are going to prove two different results for these problems, using the same method which can, in a crude form, be linked to a quantitative version of the bathtub principle.

**A quantitative bathtub principle** This method can be described as follows in a simpler context:

---



Assume you are given a  $\mathcal{C}^1$  function  $f : E \rightarrow \mathbb{R}$  (for some domain  $E$ ), whose level sets you assume to be regular  $(n-1)$ -dimensional submanifolds. Consider the following optimisation problem, for some parameter  $\delta > 0$ :

$$\inf_{H \in \mathfrak{H}_\delta} - \int_E fH,$$

where

$$\mathfrak{H}_\delta = \left\{ 0 \leq H \leq 1, \int_E H = \delta \right\}.$$

The bathtub principle states that, given the regularity of  $f$ , the unique minimiser for this problem is

$$H^* := \mathbf{1}_{E^*}$$

with

$$E^* = \{f \geq \mu\}$$

a level set of  $f$ . One might then consider a quantitative inequality for this problem. Since we expect an inequality of the form

$$\int_E fH - \int_E fH^* \geq C \|H - H^*\|_{L^1(E)}^2$$

we introduce the auxiliary class

$$\mathcal{H}_\varepsilon := \{H \in \mathfrak{H}_\delta, \|H - H^*\|_{L^1(E)} = \varepsilon\}.$$

We rewrite, for any  $H \in \mathcal{H}_\varepsilon$ ,  $h := H - H^*$ . Then

$$h \in \mathcal{G}_\varepsilon = \{-1 \leq h \leq 1, h\mathbf{1}_{E^*} \leq 0, h\mathbf{1}_{E \setminus E^*} \geq 0, \int_E h = 0, \int_E |h| = \varepsilon\}.$$

It is then easy to see that, to prove the desired quantitative inequality, it suffices to prove that

$$\lim_{\varepsilon \rightarrow 0} \inf_{h \in \mathcal{G}_\varepsilon} \frac{\int_E -fh}{\varepsilon^2} = C > 0.$$

Finally, using once again the bathtub principle, one sees that solution  $h_\varepsilon^*$  of

$$\inf_{h \in \mathcal{H}_\varepsilon} - \int_E fh$$

is

$$h_\varepsilon^* = \mathbf{1}_{E_\varepsilon^1} - \mathbf{1}_{E_\varepsilon^2},$$

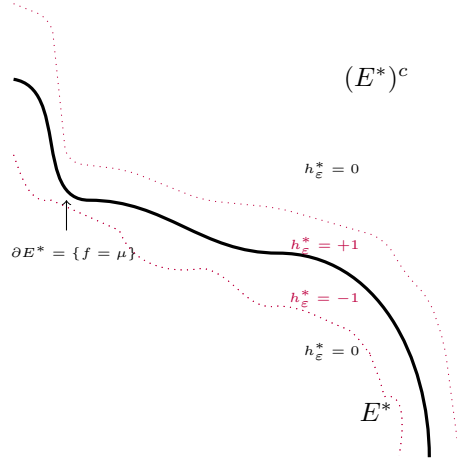
with

$$E_\varepsilon^1 = \{\mu_\varepsilon^1 \geq f \geq \mu\}, E_\varepsilon^2 = \{\mu \geq f \geq \mu_\varepsilon^2\}, |E_\varepsilon^i| = \varepsilon, i = 1, 2.$$

Once again, the regularity assumptions on  $f$  ensure that these solutions are uniquely characterized as these intermediate level sets.

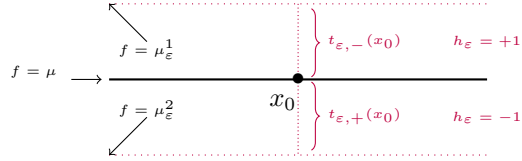
Let us represent the situation in the figure below

---



and the dashed boundaries are level set of  $f$ .

If we zoom around a point  $x_0$  on the boundary  $\partial E^*$ , we get the following situation:



We approximately have

$$t_{\epsilon,+} \approx t_{\epsilon,-} \approx C(x) \|h\|_{L^1(\Omega)},$$

and, if  $\frac{\partial f}{\partial \nu} \neq 0$  on  $\partial E^*$ , we have

$$C(x) \geq C_0 > 0 \text{ on } \partial E^*.$$

If we compute the quantity  $-\int_E h_\epsilon f$  and use the Fubini theorem, we have

$$\begin{aligned} -\int_\Omega h_\epsilon f &\approx \int_{\partial E} \left( \int_0^{t_{\epsilon,+}(x)} f(x - t\nu_{E^*}(x)) dt \right) d\mathcal{H}^{n-1}(x) \\ &\quad - \int_{\partial E} \left( \int_0^{t_{\epsilon,-}(x)} f(x + t\nu_{E^*}(x)) dt \right) d\mathcal{H}^{n-1}(x). \end{aligned}$$

We then write

$$f(x - t\nu_{E^*}(x)) \approx \mu - t \frac{\partial}{\partial \nu} (f)(x), \quad f(x + t\nu_{E^*}(x)) \approx \mu + t \frac{\partial}{\partial \nu} (f)(x),$$

so that

$$\begin{aligned}
 - \int_E h_\varepsilon f &\approx \int_{\partial E^*} \left( \int_0^{t_{\varepsilon,+}(x)} f(x - t\nu_{E^*}(x)) dt \right) d\mathcal{H}^{n-1}(x) \\
 &\quad - \int_{\partial E^*} \left( \int_0^{t_{\varepsilon,-}(x)} f(x + t\nu_{E^*}(x)) dt \right) d\mathcal{H}^{n-1}(x) \\
 &\approx \int_{\partial E^*} f(x) t_{\varepsilon,+}(x) d\mathcal{H}^{n-1}(x) - \int_{\partial E} f(x) t_{\varepsilon,-}(x) d\mathcal{H}^{n-1}(x) \\
 &\quad - \int_{\partial E^*} \frac{\partial}{\partial \nu} (f)(x) (t_{\varepsilon,+}(x)^2 + t_{\varepsilon,-}(x)^2) d\mathcal{H}^{n-1}(x) \\
 &\approx - \int_{\partial E^*} \frac{\partial}{\partial \nu} (f)(x) \left( 2C(x) \|h_\varepsilon\|_{L^1(E)}^2 \right) d\mathcal{H}^{n-1}(x) \\
 &\geq 2C_0 \inf \left| \frac{\partial f}{\partial \nu} \right|_{\partial E^*} \text{Per}(E^*) \|h_\varepsilon\|_{L^1(E)}^2.
 \end{aligned}$$

This is the desired inequality. Note that the key here is the Taylor expansion of the function  $f$  across the boundary of one of its level sets and its regularity. We are going to apply this idea to our two problems.

**Stationarity of the ball among radial distribution (Chapter 3)** We first sketch the proof of the following result:

$$\forall \alpha > 0 \text{ small enough, } \forall m \in \mathcal{M}(\Omega), m \text{ radially symmetric, } \lambda_\alpha(m) \geq \lambda_\alpha(m^*). \quad (1.28)$$

In order to prove (1.28), the naive approach would be the following: argue by contradiction and assume that, for every  $\alpha > 0$ , there exists a radially symmetric  $m_\alpha \in \mathcal{M}(\Omega)$  such that  $\lambda_\alpha(m_\alpha) < \lambda_\alpha(m^*)$ . We can prove that, if this is the case, then  $m_\alpha$  writes as  $m_\alpha = \kappa \mathbb{1}_{E_\alpha}$  and, working a bit more, we can prove that the Hausdorff distance  $d_\alpha := d_H(E_\alpha, \mathbb{B}^*)$  satisfies

$$d_\alpha \xrightarrow{\alpha \rightarrow 0} 0.$$

One might then consider the application

$$f : t \mapsto \lambda_\alpha(m_\alpha + t(m_\alpha - m^*)),$$

compute its derivative and write that there exists  $t_0 \in [0, 1]$  such that

$$\lambda_\alpha(m_\alpha) - \lambda_\alpha(m^*) = f'(t_0) = \int_\Omega (m_\alpha - m^*) (\alpha \kappa |\nabla u_{t_0}|^2 - u_{t_0}^2),$$

where, for every  $\tau \in [0, 1]$ , we define  $u_\tau$  as the eigenfunction associated with  $m_\tau := m_\alpha + \tau(m_\alpha - m^*)$ . At this step, the strategy developed above (i.e expending  $(\alpha \kappa |\nabla u_{t_0}|^2 - u_{t_0}^2)$  across  $\partial \mathbb{B}^*$ ) would work to prove that this last quantity is bounded from below by  $C_0 \|h_\alpha\|_{L^1(\Omega)}^2$ , but this is not a licit computation; indeed,  $(\alpha \kappa |\nabla u_{t_0}|^2 - u_{t_0}^2)$  is not regular enough (it is not even continuous) to apply the method.

To overcome this difficulty, we introduce the following path in [MNP19a]: following [142], introduce, for any  $\tau \in [0, 1]$ ,

$$\Lambda_-(m_\tau) := \frac{1 + \alpha \kappa}{1 + \alpha(\kappa - m_\tau)},$$

define  $\tilde{f}(\tau)$  to be the first eigenvalue of  $-\nabla \cdot (\Lambda_-(m_\tau)\nabla) - m_\tau$ , associated with the eigenfunction  $z_\tau$ . Then we can write, computing once again the derivative of  $\tilde{f}$ ,

$$\lambda_\alpha(m_\alpha) - \lambda_\alpha(m^*) = \tilde{f}(1) - \tilde{f}(0) = \tilde{f}'(t_0) = \int_{\Omega} h_\alpha (\alpha\kappa\Lambda_-(m_\tau)|\nabla z_\tau|^2 - z_\tau^2) =: \int_{\Omega} h_\alpha \Psi_\alpha^\tau.$$

Now, the switch function  $\Psi_\alpha^\tau$  is regular and, from the radially assumption, it is constant on the boundary  $\partial\mathbb{B}^*$ . This is where the radially hypothesis is used. Some work (which is carried out in Chapter 3 of this manuscript) still needs to be done, but the method of expanding  $\Psi_\alpha^\tau$  across  $\partial\mathbb{B}^*$  works in this context and we can conclude to a contradiction. We also note that this method leads to a quicker proof of the stationarity result of [117].

**Quantitative stability of the ball (Chapter 4)** In [Maz19a], we apply this method as a step in the proof of a quantitative inequality for Problem (Q) which, we recall, is defined as

$$\inf_{m \in \mathcal{M}(\Omega)} \lambda(m).$$

Indeed, the first step is to prove the quantitative inequality for radial distributions  $m$ : we want to show that there exists  $C > 0$  such that, for any radially symmetric  $m \in \mathcal{M}(\Omega)$ ,

$$\lambda(m) - \lambda(m^*) \geq C \|m - m^*\|_{L^1(\Omega)}^2.$$

Let us introduce, for a parameter  $\delta > 0$ , the auxiliary class

$$\mathcal{M}_\delta = \{m \in \mathcal{M}(\Omega), m \text{ radially symmetric}, \|m - m^*\|_{L^1(\Omega)} = \delta\}.$$

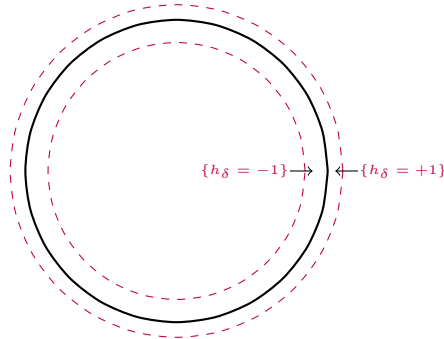
Let  $m_\delta^*$  be the solution of

$$\inf_{m \in \mathcal{M}_\delta(\Omega)} \lambda(m).$$

Obviously it suffices to prove that

$$\lim_{\delta \rightarrow 0} \frac{\lambda(m_\delta^*) - \lambda(m^*)}{\delta} = C_0 > 0.$$

After some careful work, we can prove that the graph of  $h_\delta = m_\delta - m^*$  looks as follows ( $\partial\mathbb{B}^*$  is the black circle):



We introduce, for  $t \in [0, 1]$ ,  $m_t := m^* + th_\delta$  and define  $f(t) := \lambda(m_t)$ , as well as  $u_t$ , which is the associated eigenfunction. We compute the derivative of  $f$  and, combined with the fact that there exists  $t_0 \in [0, 1]$  such that

$$\lambda(m_\delta) - \lambda(m^*) = f'(t_0)$$

we get

$$\lambda(m_\delta) - \lambda(m^*) = - \int_{\Omega} h_\delta u_{t_0}^2.$$

We first notice that  $u_{t_0}$  is constant on  $\partial\mathbb{B}^*$  since it is radial. This is where the radially hypothesis is needed.

Now, using the fact that, for  $\delta > 0$  small enough,  $u_{t_0}$  is close in the  $\mathcal{C}^1$  topology to  $u_0$ , we can apply the very same method to obtain

$$\lambda(m_\delta) - \lambda(m^*) \geq C \|h_\delta\|_{L^1(\Omega)}^2 = C\delta^2.$$

This is the core argument for this step of the proof of the quantitative inequality, and we write down the remaining steps in Chapter 4 of this manuscript.

### 1.3.5 METHODS DEVELOPED FOR THE QUANTITATIVE STABILITY OF THE BALL (CHAPTER 4)

Combining the methods of Paragraph 1.3.4, the comparison methods for shape derivatives of Paragraph 1.3.3 and a new type of surgery, we were able to prove in [Maz19a], a quantitative inequality for the optimization problem

$$\min_{m \in \mathcal{M}(\mathbb{B})} \lambda(m),$$

where  $\lambda(m)$  is the first Dirichlet eigenvalue of  $-\Delta - m$  and  $\Omega = \mathbb{B}(0, R)$  is a centered ball. We recall that, if  $\mathbb{B}^* = \mathbb{B}(0; r^*)$  with  $\kappa|\mathbb{B}^*| = m_0$ , then  $m^* = \kappa\mathbb{1}_{\mathbb{B}^*}$  is the unique solution of this problem.

Our proof may be split in three main steps:

1. Proving it for radially symmetric distributions:

Using the methods of Paragraph 1.3.4, we can prove that there exists  $C > 0$  such that, for any radially symmetric  $m$ ,

$$\lambda(m) - \lambda(m^*) \geq C \|m - m^*\|_{L^1(\Omega)}^2.$$

2. Proving it for normal deformations of  $\mathbb{B}^*$ :

Let  $L : E \mapsto \lambda(\kappa\mathbb{1}_E) - \Lambda\text{Vol}(E)$  be the Lagrangian associated with the volume constraint of the optimisation problem. using the comparison method presented in Paragraph 1.3.3 to study the sign of the second order shape derivative at  $\mathbb{B}^*$ , we can establish that, for any compactly supported vector field  $\phi$  such that  $\int_{\partial\mathbb{B}^*} \langle \phi, \nu \rangle = 0$ , there holds

$$L''(\mathbb{B}(0; r^*))[\phi, \phi] \geq C \|\langle \phi, \nu \rangle\|_{L^2(\partial\mathbb{B}^*)}^2.$$

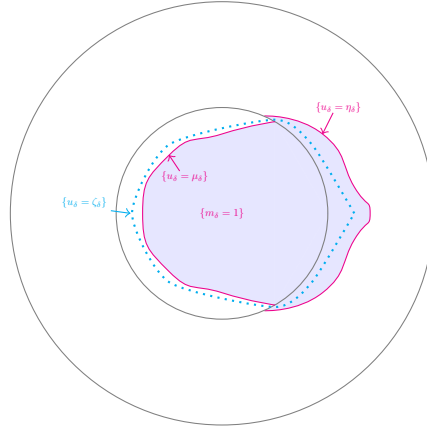
Here, this  $L^2$  coercivity is optimal. Adapting the systematic approach of [64], we derive a local quantitative inequality for normal perturbations of  $\mathbb{B}^*$ .

3. Conclusion

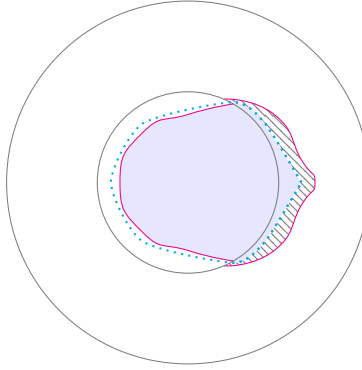
To derive the general inequality, we consider the solution  $\tilde{m}_\delta$  of

$$\inf_{m \in \mathcal{M}(\Omega), \|m - m^*\|_{L^1(\Omega)} = \delta} \lambda(m).$$

We show that there exists  $V_\delta$  such that  $\tilde{m}_\delta = \mathbb{1}_{V_\delta}$ , and that, by defining  $u_\delta$  as the associated eigenfunction, we get the following situation: there exist  $\mu_\delta$  and  $\eta_\delta$  such that we have the following picture



We then study the measure of the striped zone below:



Let us call this volume  $A_\delta$ . We prove that, if  $A_\delta \sim_{\delta \rightarrow 0} C\delta$ , then  $m_\delta$  can be compared to a radially symmetric competitor and that, if  $A_\delta = o_{\delta \rightarrow 0}(\delta)$ , then  $m_\delta$  can be compared to a small normal perturbation of the ball, which, in both cases, was studied in the two previous steps. This concludes the proof.

### 1.3.6 METHODS DEVELOPED FOR THE CONTROLLABILITY OF REACTION-DIFFUSION EQUATIONS (CHAPTER 5)

We conclude this Section with a quick presentation of the method developed in [MRBZ19] to obtain positive controllability results for bistable equations set in a slowly varying environment. The negative results (i.e blocking phenomena leading to lack of controllability) are obtained through involved energy arguments.

We recall that we work in a smooth bounded domain  $\Omega$ , with a bistable non-linearity (see Section 1.2.2.2)  $f$  that has only three roots  $0 < \theta < 1$ . We want, for any initial datum  $p_0$ , to establish that

there exists a control  $a : \partial\Omega \rightarrow [0; 1]$  and a time  $T > 0$  such that the solution  $p$  of

$$\begin{cases} \frac{\partial p}{\partial t} - \mu \Delta p - \varepsilon \langle \nabla N, \nabla p \rangle = f(p) & \text{in } (0, T) \times \Omega, \\ p = a(t, x) & \text{in } (0, T) \times \partial\Omega, \\ 0 \leq p \leq 1, \\ p(t = 0, \cdot) = 0 \leq p_0 \leq 1, \end{cases} \quad (1.29)$$

satisfies  $p(T, \cdot) \equiv \theta, 0$  or  $1$ . We assume that this is possible for any initial datum  $0 \leq p_0 \leq 1$  when  $\varepsilon = 0$ .

The method used in [163, 157] to prove this result for  $\varepsilon = 0$  is the staircase method of [58]. Without getting in too much details, we simply state the following result: if there exists a continuous (in the  $\mathcal{C}^0$  topology) path of steady states linking  $0$  to  $\theta$ , if  $z_0 \equiv 0$  is the unique steady state with homogeneous (zero) Dirichlet boundary conditions, then the this controllability property holds. We first note that if, for  $\varepsilon = 0$ ,  $z_0 \equiv 0$  is the unique steady-state with boundary conditions  $0$ , this uniqueness property still holds for  $\varepsilon > 0$  small enough. In [157], such a path is built in the one-dimensional case; in [163], they prove the existence of such a path in the multi-dimensional case.

In [MRBZ19], we proved the following property, which, however weaker, still guarantees the applicability of the staircase method: for any  $\varepsilon > 0$  small enough, for any  $\delta > 0$  small enough, there exists a sequence of steady-states  $p_{0,\delta,\varepsilon}, p_{1,\delta,\varepsilon}, \dots, p_{N,\delta,\varepsilon}$  of the equation

$$-\mu \Delta p_{i,\delta,\varepsilon} - \varepsilon \langle \nabla N, \nabla p_{i,\delta,\varepsilon} \rangle = f(p_{i,\delta,\varepsilon}), 0 \leq p_{i,\delta,\varepsilon} \leq 1$$

and such that

$$p_{1,\delta,\varepsilon} = 0, p_{N,\delta,\varepsilon} \equiv \theta, \|p_{i+1,\delta,\varepsilon} - p_{i,\delta,\varepsilon}\|_{L^\infty} \leq \delta.$$

To build such a sequence, we first considered a sequence obtained in [163] of steady-states  $p_{0,\delta,0}, p_{1,\delta,0}, \dots, p_{N,\delta,0}$  such that

$$\|p_{i,\delta,0} - p_{i+1,\delta,0}\|_{\mathcal{C}^0} \leq \frac{\delta}{4}.$$

We then tried to perturb each of these  $p_{i,\delta,0}$  to obtain a branch  $\varepsilon \mapsto p_{i,\delta,\varepsilon}$  of steady-states for the equation with drift. The implicit function theorem would enable us to do so, provided the operator

$$L_i : -\Delta - f'(p_{i,\delta,0})$$

with Dirichlet boundary conditions is invertible. This is, in general, not the case. To overcome this difficulty, rather than applying this theorem on  $\Omega$ , we applied it on  $\Omega_\eta := \Omega + \mathbb{B}(0; \eta)$  after extending  $p_{i,\delta,0}$  to  $\Omega_\eta$ . We then choose  $\eta > 0$  small enough, so that the monotonicity of the eigenvalues ensures the invertibility of the operator on  $L_I$  on  $\Omega_\eta$ . We then apply the implicit function theorem on this domain. We represent the situation below:

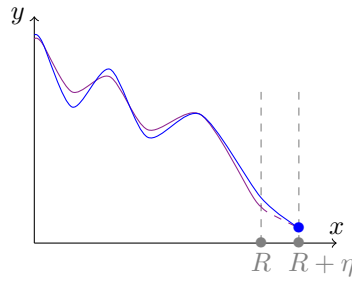


Figure 1.7 – The solution  $p_{i,\delta,0}$  on  $\mathbb{B}(0; R)$  is extended to a solution on  $\mathbb{B}(0; R + \eta)$ , and we apply the implicit function theorem to obtain the blue curve.



## 1.4 OPEN PROBLEMS & RESEARCH PROJECTS

We finally present possible research projects.

### 1.4.1 UNDERSTANDING OSCILLATIONS IN THE SMALL DIFFUSIVITY SETTING

Regarding the optimisation of the total population size addressed in [MNP19b], Chapter 2 and presented in Section 1.2.1.1 of this Introduction, we can say that the situation is rather well understood in the case of large diffusivities. However, even in the one-dimensional case, the case of small diffusivities is rather open: apart from knowing that fragmentation may be better than concentration, no qualitative information is known. This leads us to formulating the following question:

**OPEN QUESTION 1** *Can we give qualitative information about the solutions of  $(P_\mu)$  as  $\mu \rightarrow 0$ ?*

An analysis of the same type than the one carried out for large diffusivities, that is, the derivation of a limit problem, seems delicate to obtain. The idea would be to obtain an asymptotic expansion of the solution  $\theta_{m,\mu}$  of (1.11) under the form

$$\theta_{m,\mu}(x) \approx m(x) + \theta_1 \left( x, \frac{x}{\sqrt{\mu}} \right)$$

but, even on a formal level, the equation on  $\theta_1$  is difficult to study. So far, even a purely formal approach to this problem remains open.

### 1.4.2 AN OPTIMAL CONTROL PROBLEM FOR MONOSTABLE EQUATIONS

Another related issue is that of optimal strategy for parabolic monostable equations. Following [MNP19b], one may consider the control system

$$\begin{cases} \mu \left( \frac{\partial \theta}{\partial t} - \Delta \theta_{m,\mu} \right) - \theta_{m,\mu} (m(t, \cdot) - \theta_{m,\mu}) = 0 \text{ in } \mathbb{R}_+ \times \Omega, \\ \frac{\partial \theta_{m,\mu}}{\partial \nu} = 0 \text{ on } \mathbb{R}_+ \times \partial \Omega, \\ \theta_{m,\mu}(t = 0, \cdot) = \theta_0, \end{cases} \quad (1.30)$$

where the resources distribution  $m$  depends on time and space. Let  $T$  be a time horizon; we assume that

$$m \in \mathcal{M}_1(T; \Omega) := \left\{ m, 0 \leq m(t, x) \leq \kappa, \text{ et } \forall t \in [0; T], \int_{\Omega} m(t, x) dx = m_0 \right\},$$

or that

$$m \in \mathcal{M}_2(T; \Omega) := \left\{ m, 0 \leq m(t, x) \leq \kappa, \text{ et } \int_0^T \int_{\Omega} m(t, x) dx = m_0 \right\},$$

and we consider the two optimisation problems

$$\sup_{m \in \mathcal{M}_1(T; \Omega)} \int_{\Omega} \theta_{m,\mu}(T, x) dx, \quad (1.31)$$

and

$$\sup_{m \in \mathcal{M}_2(T; \Omega)(\Omega)} \int_{\Omega} \theta_{m,\mu}(T, x) dx. \quad (1.32)$$

Then we wish to investigate whether or not we can prove the bang-bang property for these problems (in the sense that optimisers are bang-bang functions). Namely:

**OPEN QUESTION 2** *Are the solutions of (1.31), (1.32) bang-bang functions? What can be said, in terms of concentration and fragmentation, about these maximisers?*

We believe that our methods enable us to provide partial answers to these questions.

We are currently working with E. Zuazua and D. Ruiz-Balet on a turnpike property for (1.31). For an introduction to the turnpike property in optimal control, we refer to [173, 156] and the references therein, but sum it up here as follows: roughly speaking, what we want to prove is that there exists two times  $T_1 < T_2$  that do not depend on  $T$  such that, for  $T$  large enough and  $\|\theta_0\|_{L^\infty}$  small enough:

1. Between 0 and  $T_1$ , the optimiser  $m^*$  is close to an optimiser for the first eigenvalue  $\lambda_1(m, \mu)$ ,
2. between  $T_1$  and  $T_2$ ,  $m^*$  is close to the maximiser for the elliptic optimiser for the total population size,
3. between  $T_1$  and  $T$ ,  $m^*$  goes away from the elliptic maximiser.

We have several promising leads, which lead to investigating the following question:

**OPEN QUESTION 3** *Can we prove this rough turnpike property for (1.31)?*

### 1.4.3 A REARRANGEMENT *à la* ALVINO-TROMBETTI FOR EIGENVALUES WITH DRIFTS AND POTENTIALS

In [MNP19a], Chapter 3, we give a fairly complete study of the variational problem ( $P_\alpha$ ) which consists in optimising the first eigenvalue of

$$\mathcal{L}_m = -\nabla \cdot ((1 + \alpha m)\nabla) - m;$$

we refer to Sections 1.2.1.2 and 1.3.4 of this Introduction. When  $\Omega = \mathbb{B}(0; R)$ , we were able to show that, for  $\alpha > 0$ ,  $m^* = \kappa \mathbb{1}_{\mathbb{B}^*}$  is minimal among radial distributions and among all distributions if  $m_0$  is small enough.

To prove that  $m^*$  is, in general, a global minimiser if  $\alpha > 0$  is small enough, without any smallness assumption on  $m_0$ , the most promising approach seems to be an approach mixing properties of the Schwarz rearrangement and of the Alvino-Trombetti rearrangement.

**OPEN QUESTION 4** *Can we obtain a rearrangement mapping any  $m \in \mathcal{M}(\mathbb{B}(0, R))$  to a radially symmetric function  $m^\# \in M(\mathbb{B}(0, R))$  such that*

$$\lambda_\alpha(m^\#) \leq \lambda_\alpha(m)?$$

Should such a rearrangement exist, our result on the minimality of  $m^*$  among radial distributions would lead to the conclusion.

Finding such a rearrangement seems complicated: the Schwarz rearrangement has a “global” action, while the rearrangement of Alvino and Trombetti has a “local” definition. We do not know yet whether or not it is possible to concile these two approaches.

### 1.4.4 OPEN QUESTIONS IN SPECTRAL OPTIMISATION

We mention two other problems we could study, which are linked to Chapters 3, 4.

1. Disproving a conjecture (Neumann boundary conditions):

This is the following question raised in [116]: consider the spectral optimisation problem

$$\min_{m \in \mathcal{M}(\Omega)} \lambda^{\mathcal{N}}(m),$$

where

- (a)  $\Omega = \mathbb{B}(0; 1)$ ,
- (b)  $\lambda_1^{\mathcal{N}}(m)$  is the first Neumann eigenvalue of

$$\mathcal{L}_m := -\Delta - m.$$

It has been conjectured that the solution of this optimisation problem was a piece of a disk, but we think we might be able to prove that this is not the case. This would be a first step in understanding qualitative properties of optimisers for spectral problem, in a geometry that would not be a square.

2. A stationarity property (Robin boundary conditions):

We might consider another related problem, that of spectral optimisation with Robin boundary conditions, that is, try to study the following variational problem:

$$\min_{m \in \mathcal{M}(\Omega)} \lambda_{\beta}(m) \tag{1.33}$$

where

- (a)  $\Omega = \mathbb{B}(0; 1)$ ,
- (b)  $\lambda_{\beta}(m)$  is the first eigenvalue of the operator

$$\mathcal{L}_m := -\Delta - m$$

with the Robin boundary conditions

$$\frac{\partial u}{\partial \nu} + \beta u = 0.$$

It is known (see [116]) that, as  $\beta \rightarrow \infty$ , the solutions  $m_{\beta}$  of this problem converge to  $m^* = \kappa \mathbb{1}_{\mathbb{B}^*}$ . A natural conjecture backed by the numerical simulations of [116] is the following

$$\exists \beta^* > 0, \quad \forall \beta \geq \beta^*, m_{\beta} = m^*.$$

In the one dimensional case, this is proved in [50].

Adapting some of the tools and methods introduced in [Maz19a], Chapter 4, we think we might be able to give this conjecture a positive answer in the two-dimensional case.

We might sum up these conjectures as follows:

**OPEN QUESTION 5** *Can our methods enable us to derive more qualitative properties on spectral optimisation problems?*

### 1.4.5 EXTENSION OF THE QUANTITATIVE INEQUALITY TO OTHER DOMAINS

A natural extension of [Maz19a], Chapter 4, presented in Sections 1.2.1.3 and 1.3.4 of this Introduction, is the following question:

---

**OPEN QUESTION 6** *Can we prove that, for any domain  $\Omega$ , if  $\mathcal{M}^*$  is the set of maximisers of the first Dirichlet eigenvalue  $\lambda(m)$  of*

$$\mathcal{L} - m := -\Delta - m$$

*we have*

$$\lambda(m) - \lambda(m^*) \geq C \text{dist}_{L^1}(m, \mathcal{M}^*)^2?$$

Here, the main difficulty is the fact that the Schwarz rearrangement, which is a crucial tool to handle topological changes in [Maz19a], is no longer available.

#### 1.4.6 SHAPE OPTIMISATION & OPTIMAL CONTROL

After the work we carried out in [MRBZ19], Chapter 5, presented in Sections 1.2.2 and 1.3.6 of this Introduction, it seems interesting to understand the influence of spatial heterogeneity on controllability properties of equations- one approach may be to understand the following toy problem: let the class of admissible drifts be defined as

$$\mathcal{N}(\Omega) := \left\{ N \in L^\infty(\Omega), 1 \leq N \leq \tilde{\kappa}, \int_{\Omega} N = N_0 \right\}$$

and consider the heat equation with boundary control  $a$ :

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (N \nabla u) = 0 \text{ in } \mathbb{R}_+ \times \Omega, \\ u(t = 0) = 0, \\ u(t, \cdot) = a(t, \cdot) \in [0; 1] \text{ on } \mathbb{R}_+ \times \partial\Omega \end{cases} \quad (1.34)$$

Let, for any  $N \in \mathcal{N}(\Omega)$ ,  $T(N)$  be the minimal control time from 0 to  $\theta$ . Then:

**OPEN QUESTION 7** *What can be said about the optimisers  $N^*$  of*

$$\min_{N \in \mathcal{N}(\Omega)} T(N)?$$



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## CHAPTER 2

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# OPTIMIZATION OF THE TOTAL POPULATION SIZE FOR LOGISTIC DIFFUSIVE MODELS

WITH G. NADIN AND Y. PRIVAT. ACCEPTED FOR PUBLICATION IN *Journal de mathématiques pures et appliquées* (2019).

On peut faire très sérieusement ce qui vous  
amuse, les enfants nous le prouvent tous les  
jours...

---

Bernanos,  
*Dialogue des Carmélites*

## GENERAL PRESENTATION OF THE CHAPTER: MAIN DIFFICULTIES AND METHODS

This Chapter is devoted to the study of the influence of spatial heterogeneity on the total population size. Namely, considering the logistic reaction-diffusion equation

$$\begin{cases} \mu \Delta \theta_{m,\mu}(x) + (m(x) - \theta_{m,\mu}(x))\theta_{m,\mu}(x) = 0 & x \in \Omega, \\ \frac{\partial \theta_{m,\mu}}{\partial \nu} = 0 & x \in \partial\Omega, \end{cases} \quad (2.1)$$

where

$$m \in \mathcal{M}_{m_0,\kappa}(\Omega) := \left\{ 0 \leq m \leq \kappa, \int_{\Omega} m = m_0 \right\},$$

can we characterize or give some properties of the solutions of the variational problem

$$\max_{m \in \mathcal{M}_{m_0,\kappa}(\Omega)} F(m, \mu) := \int_{\Omega} \theta_{m,\mu} \quad (2.2)$$

As explained in the Introduction to this thesis, two qualitative features are important: pointwise properties (the so-called *bang-bang property*) and geometric properties, referred to as *fragmentation and concentration of resources*.

Our four main results, and the methods developed here to obtain them, can be summed up as follows:

- The bang-bang property for large diffusivities:

This is Theorem 2.1.1, which asserts that, for large diffusivities  $\mu$ , any solution  $m^*$  is of bang-bang type, i.e writes  $m^* = \kappa \mathbf{1}_E$  for some subset  $E \subset \Omega$ . The main difficulty in establishing this Theorem is that the convexity of  $m \mapsto F(m, \mu)$  is not clear at all, since the criterion we wish to optimize is non-energetic. To overcome this difficulty, we decompose the solution  $\theta_{m,\mu}$  and  $F(\cdot, \mu)$  in a power series

$$\theta_{m,\mu} = \sum_{k=0}^{\infty} \frac{\eta_{k,m}}{\mu^k}, \quad F(m, \mu) = \sum_{k=0}^{\infty} \frac{F_k(m)}{\mu^k},$$

we establish that  $F_1$  is a strictly convex functional and that the possible loss of convexity due to higher order terms  $F_k$  is controlled by the convexity of  $F_1$ :

$$|\ddot{F}_k| \lesssim \ddot{F}_1,$$

thus establishing the convexity of the functional for large diffusivities. To the best of our knowledge, this method is entirely new.

- A full stationarity result in dimension 1: This is the content of Theorem 2.1.3. Using the optimality conditions we prove that, in dimension 1, in  $\Omega = [0, 1]$ , there exists  $\mu_1 > 0$  such that, for any  $\mu \geq \mu_1$ , the unique solutions of Problem (2.2) are the two following crenels:

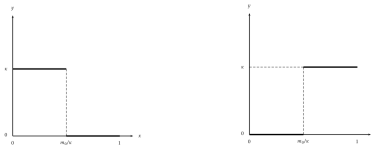


Figure 2.1 –  $\Omega = (0, 1)$ . Plot of the only two maximizers of  $\mathcal{F}_\mu$  over  $\mathcal{M}_{m_0,\kappa}(\Omega)$ .

We also derive concentration properties for large diffusivities in 2 dimensional orthotopes, see Theorem 2.1.2, and provide the full proof of the decreasingness of minimizers of an ergetic variational problem, which we could not locate in the existing literature.

- A more surprising fragmentation result in dimension 1:

This is Theorem 2.1.4, which asserts that, for small enough diffusivities, fragmentation is better (with regards to Problem (2.2)) than concentration (which is known to always be better for optimal survival):

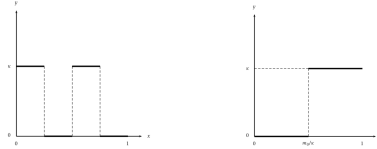


Figure 2.2 –  $\Omega = (0, 1)$ . The left distribution is better than the right for small diffusivities.

This observation is quite striking in the context of optimization in mathematical ecology and provides a strong example of a natural criterion for which fragmenting resources is better than concentrating them. Numerical simulations back this result up, but a full understanding of the phenomenon at play here is not yet available. We plan on coming back to this problem in future works.



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## 2.1 INTRODUCTION

### 2.1.1 MOTIVATIONS AND STATE OF THE ART

In this article, we investigate an optimal control problem arising in population dynamics. Let us consider the population density  $\theta_{m,\mu}$  of a given species evolving in a bounded and connected domain  $\Omega$  in  $\mathbb{R}^n$  with  $n \in \mathbb{N}^*$ , having a  $\mathcal{C}^2$  boundary. In what follows, we will assume that  $\theta_{m,\mu}$  is the positive solution of the steady logistic-diffusive equation (denoted (LDE) in the sequel) which writes

$$\begin{cases} \mu\Delta\theta_{m,\mu}(x) + (m(x) - \theta_{m,\mu}(x))\theta_{m,\mu}(x) = 0 & x \in \Omega, \\ \frac{\partial\theta_{m,\mu}}{\partial\nu} = 0 & x \in \partial\Omega, \end{cases} \quad (\text{LDE})$$

where  $m \in L^\infty(\Omega)$  stands for the resources distribution and  $\mu > 0$  stands for the dispersal ability of the species, also called *diffusion rate*. From a biological point of view, the real number  $m(x)$  is the local intrinsic growth rate of species at location  $x$  of the habitat  $\Omega$  and can be seen as a measure of the resources available at  $x$ .

As will be explained below, we will only consider non-negative resource distributions  $m$ , i.e such that  $m \in L_+^\infty(\Omega) = \{m \in L^\infty(\Omega), m \geq 0 \text{ a.e}\}$ . In view of investigating the influence of spatial heterogeneity on the model, we consider the optimal control problem of maximizing the functional

$$\mathcal{F}_\mu : L_+^\infty(\Omega) \ni m \mapsto \int_\Omega \theta_{m,\mu},$$

where the notation  $f$  denotes the average operator, in other words  $f_\Omega f = \frac{1}{|\Omega|} \int_\Omega f$ . The functional  $F$  stands for the total population size, in order to further our understanding of spatial heterogeneity on population dynamics.

In the framework of population dynamics, the density  $\theta_{m,\mu}$  solving Equation (LDE) can be interpreted as a steady state associated to the following evolution equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \mu\Delta u(t, x) + u(t, x)(m(x) - u(t, x)) & t > 0, x \in \Omega \\ \frac{\partial u}{\partial \nu}(t, x) = 0 & t > 0, x \in \partial\Omega \\ u(0, x) = u^0(x) \geq 0, u^0 \neq 0 & x \in \Omega \end{cases} \quad (\text{LDEE})$$

modeling the spatiotemporal behavior of a population density  $u$  in a domain  $\Omega$  with the spatially heterogeneous resource term  $m$ .

The pioneering works by Fisher [75], Kolmogorov-Petrovski-Piskounov [113] and Skellam [168] on the logistic diffusive equation were mainly concerned with the spatially homogeneous case. Thereafter, many authors investigated the influence of spatial heterogeneity on population dynamics and species survival. In [107], propagation properties in a patch model environment are studied. In [166], a spectral condition for species survival in heterogeneous environments has been derived, while [112] deals with the influence of fragmentation and concentration of resources on population dynamics. These works were followed by [19] dedicated to an optimal design problem, that will be commented in the sequel.

Investigating existence and uniqueness properties of solutions for the two previous equations as well as their regularity properties boils down to the study of spectral properties for the linearized operator

$$\mathcal{L} : \mathcal{D}(\mathcal{L}) \ni f \mapsto \mu\Delta f + mf,$$

where the domain of  $\mathcal{L}$  is  $\mathcal{D}(\mathcal{L}) = \{f \in L^2(\Omega) \mid \Delta f \in L^2(\Omega)\}$  and of its first eigenvalue  $\lambda_1(m, \mu)$ ,

characterized by the Courant-Fischer formula

$$\lambda_1(m, \mu) := \sup_{f \in W^{1,2}(\Omega), \int_{\Omega} f^2 = 1} \left\{ -\mu \int_{\Omega} |\nabla f|^2 + \int_{\Omega} m f^2 \right\}. \quad (2.3)$$

Indeed, the positiveness of  $\lambda(m, \mu)$  is a sufficient condition ensuring the well-posedness of equations (LDEE) and (LDE) ([19]). Then, Equation (LDE) has a unique positive solution  $\theta_{m,\mu} \in W^{1,2}(\Omega)$ . Furthermore, for any  $p \geq 1$ ,  $\theta_{m,\mu}$  belongs to  $W^{2,p}(\Omega)$ , and there holds

$$0 < \inf_{\bar{\Omega}} \theta_{m,\mu} \leq \theta_{m,\mu} \leq \|m\|_{L^\infty(\Omega)}. \quad (2.4)$$

Moreover, the steady state  $\theta_{m,\mu}$  is globally asymptotically stable: for any  $u_0 \in W^{1,2}(\Omega)$  such that  $u_0 \geq 0$  a.e. in  $\Omega$  and  $u_0 \neq 0$ , one has

$$\|u(t, \cdot) - \theta_{m,\mu}\|_{L^\infty(\Omega)} \xrightarrow[t \rightarrow +\infty]{} 0.$$

where  $u$  denotes the unique solution of (LDEE) with initial state  $u_0$  (belonging to  $L^2(0, T; W^{1,2}(\Omega))$  for every  $T > 0$ ).

The importance of  $\lambda_1(m, \mu)$  for stability issues related to population dynamics models was first noted in simple cases by Ludwig, Aronson and Weinberger [130]. Let us mention [70] where the case of diffusive Lotka-Volterra equations is investigated.

To guarantee that  $\lambda_1(m, \mu) > 0$ , it is enough to work with distributions of resources  $m$  satisfying the assumption

$$m \in L_+^\infty(\Omega) \quad \text{where} \quad L_+^\infty(\Omega) = \left\{ m \in L^\infty(\Omega), \int_{\Omega} m > 0 \right\}. \quad (H1)$$

Note that the issue of maximizing this principal eigenvalue was addressed for instance in [103, 104, 116, 129, 85].

In the survey article [127], Lou suggests the following problem: the parameter  $\mu > 0$  being fixed, which weight  $m$  maximizes the total population size among all uniformly bounded elements of  $L^\infty(\Omega)$ ?

In this article, we aim at providing partial answers to this issue, and more generally new results about the influence of the spatial heterogeneity  $m(\cdot)$  on the total population size.

For that purpose, let us introduce the total population size functional, defined for a given  $\mu > 0$  by

$$\mathcal{F}_\mu : L_+^\infty(\Omega) \ni m \mapsto \int_{\Omega} \theta_{m,\mu}, \quad (2.5)$$

where  $\theta_{m,\mu}$  denotes the solution of equation (LDE).

Let us mention several previous works dealing with the maximization of the total population size functional. It is shown in [125] that, among all weights  $m$  such that  $\int_{\Omega} m = m_0$ , there holds  $\mathcal{F}_\mu(m) \geq \mathcal{F}_\mu(m_0) = m_0$ ; this inequality is strict whenever  $m$  is nonconstant. Moreover, it is also shown that the problem of maximizing  $\mathcal{F}_\mu$  over  $L_+^\infty(\Omega)$  has no solution.

**REMARK 2.1** *The fact that  $m \equiv m_0$  is a minimum for  $\mathcal{F}_\mu$  among the resources distributions  $m$  satisfying  $\int_{\Omega} m = m_0$  relies on the following observation: multiplying (LDE) by  $\frac{1}{\theta_{m,\mu}}$  and integrating by parts yields*

$$\mu \int_{\Omega} \frac{|\nabla \theta_{m,\mu}|^2}{\theta_{m,\mu}^2} + \int_{\Omega} (m - \theta_{m,\mu}) = 0. \quad (2.6)$$

and therefore,  $\mathcal{F}_\mu(m) = m_0 + \mu \int_{\Omega} \frac{|\nabla \theta_{m,\mu}|^2}{\theta_{m,\mu}^2} \geq m_0 = \mathcal{F}_\mu(m_0)$  for all  $m \in L_+^\infty(\Omega)$  such that  $\int_{\Omega} m = m_0$ . It follows that the constant function equal to  $m_0$  is a global minimizer of  $\mathcal{F}_\mu$  over  $\{m \in L_+^\infty(\Omega), \int_{\Omega} m =$

$m_0\}$ .

In the recent article [12], it is shown that, when  $\Omega = (0, \ell)$ , one has

$$\forall \mu > 0, \forall m \in L^{\infty}_+(\Omega) \mid m \geq 0 \text{ a.e.}, \quad \int_{\Omega} \theta_{m,\mu} \leq 3 \int_{\Omega} m.$$

This inequality is sharp, although the right-hand side is never reached, and the authors exhibit a sequence  $(m_k, \mu_k)_{k \in \mathbb{N}}$  such that  $\int_{\Omega} \theta_{m_k, \mu_k} / \int_{\Omega} m_k \rightarrow 3$  as  $k \rightarrow +\infty$ , but for such a sequence there holds  $\|m_k\|_{L^{\infty}(\Omega)} \rightarrow +\infty$  and  $\mu_k \rightarrow 0$  as  $k \rightarrow +\infty$ .

In [127], it is proved that, without  $L^1$  or  $L^{\infty}$  bounds on the weight function  $m$ , the maximization problem is ill-posed. It is thus natural to introduce two parameters  $\kappa, m_0 > 0$ , and to restrict our investigation to the class

$$\mathcal{M}_{m_0, \kappa}(\Omega) := \left\{ m \in L^{\infty}(\Omega), 0 \leq m \leq \kappa \text{ a.e.}, \int_{\Omega} m = m_0 \right\}. \quad (2.7)$$

It is notable that in [69], the more general functional  $J_B$  defined by

$$J_B(m) = \int_{\Omega} (\theta_{m,\mu} - Bm^2) \quad \text{for } B \geq 0$$

is introduced. In the case  $B = 0$ , the authors apply the so-called Pontryagin principle, show the Gâteaux-differentiability of  $J_B$  and carry out numerical simulations backing up the conjecture that maximizers of  $J_0$  over  $\mathcal{M}_{m_0, \kappa}(\Omega)$  are of bang-bang type.

However, proving this *bang-bang* property is a challenge. The analysis of optimal conditions is quite complex, because the sensitivity of the functional "total population size" with respect to  $m(\cdot)$  is directly related to the solution of an adjoint state, solving a linearized version of (LDE). Deriving and exploiting the properties of optimal configurations therefore requires a thorough understanding of the  $\theta_{m,\mu}$  behavior as well as the associated state. To do this, we are introducing a new asymptotic method to exploit optimal conditions.

We will investigate two properties of the maximizers of the total population size function  $\mathcal{F}_{\mu}$ .

1. **Pointwise constraints.** The main issue that will be addressed in what follows is the *bang-bang* character of optimal weights  $m^*(\cdot)$ , in other words, whether  $m^*$  is equal to 0 or  $\kappa$  almost everywhere. Noting that  $\mathcal{M}_{m_0, \kappa}(\Omega)$  is a convex set and that bang-bang functions are the extreme points of this convex set, this question rewrites:

*Are the maximizers  $m^*$  extreme points of the set  $\mathcal{M}_{m_0, \kappa}(\Omega)$ ?*

In our main result (Theorem 2.1.1) we provide a **positive answer for large diffusivities**. It is notable that our proof rests upon a well-adapted expansion of the solution  $\theta_{m,\mu}$  of (LDE) with respect to the diffusivity  $\mu$ .

This approach could be considered unusual, since such results are usually obtained by an analysis of the properties of the adjoint state (or switching function). However, since the switching function very implicitly depends on the design variable  $m(\cdot)$ , we did not obtain this result in this way.

2. **Concentration-fragmentation.** It is well known that resource concentration (which means that the distribution of resources  $m$  decreases in all directions, see Definition 2.1.2 for a specific statement) promotes the survival of species [19]. On the contrary, we will say that a resource distribution  $m = \kappa \chi_E$ , where  $E$  is a subset of  $\Omega$ , is *fragmented* when the  $E$  set is disconnected. In the figure 2.3,  $\Omega$  is a square, and the intuitive notion of *concentration-fragmentation* of resource distribution is illustrated.

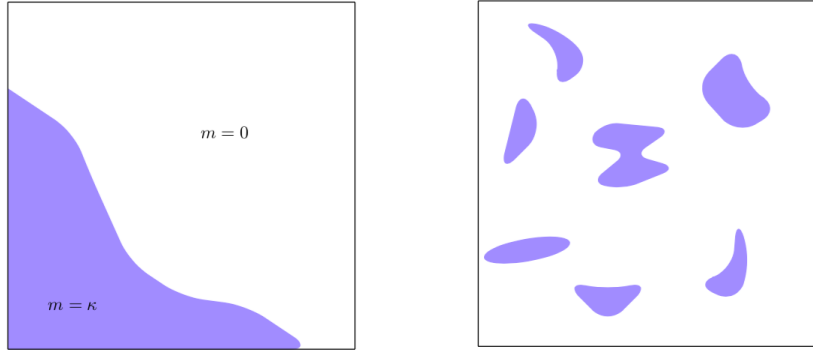


Figure 2.3 –  $\Omega = (0, 1)^2$ . The left distribution is "concentrated" (connected) whereas the right one is fragmented (disconnected).

A natural issue related to qualitative properties of maximizers is thus

*Are maximizers  $m^*$  concentrated? Fragmentated?*

In Theorem 2.1.2, we consider the case of a orthotope shape habitat, and we show that **concentration occurs for large diffusivities**: if  $m_\mu$  maximizes  $\mathcal{F}_\mu$  over  $\mathcal{M}_{m_0, \kappa}(\Omega)$ , the sequence  $\{m_\mu\}_{\mu > 0}$  strongly converges in  $L^1(\Omega)$  to a concentrated distribution as  $\mu \rightarrow \infty$ .

In the one-dimensional case, we also prove that if the diffusivity is large enough, there are only two maximizers, that are plotted on Fig. 2.4 (see Theorem 2.1.3).

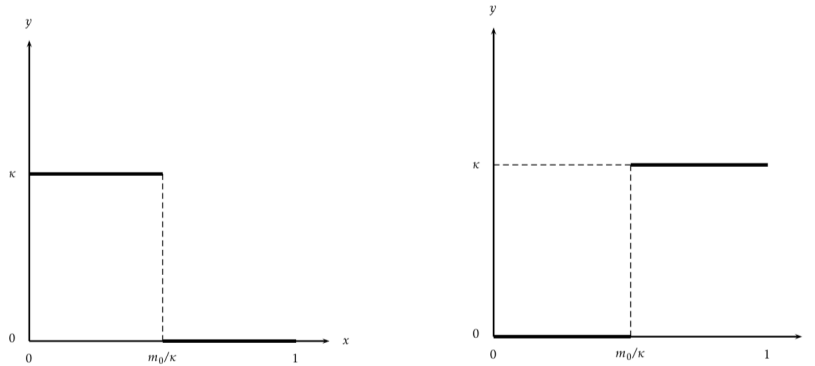


Figure 2.4 –  $\Omega = (0, 1)$ . Plot of the only two maximizers of  $\mathcal{F}_\mu$  over  $\mathcal{M}_{m_0, \kappa}(\Omega)$ .

Finally, in the one-dimensional case, we obtain a surprising result: **fragmentation may be better than concentration for small diffusivities** (see Theorem 2.1.4 and Fig. 2.5 below).

This is surprising because in many problems of optimizing the logistic-diffusive equation, it is expected that the best disposition of resources will be concentrated.

### 2.1.2 MAIN RESULTS

In the whole article, the notation  $\chi_I$  will be used to denote the characteristic function of a measurable subset  $I$  of  $\mathbb{R}^n$ , in other words, the function equal to 1 in  $I$  and 0 elsewhere.

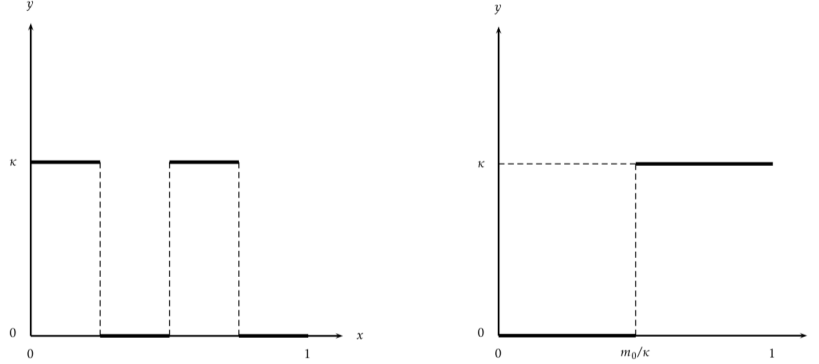


Figure 2.5 –  $\Omega = (0, 1)$ . A double crenel (on the left) is better than a single one (on the right).

For the reasons mentioned in Section 2.1.1, it is biologically relevant to consider the class of admissible weights  $\mathcal{M}_{m_0, \kappa}(\Omega)$  defined by (2.7), where  $\kappa > 0$  and  $m_0 > 0$  denote two positive parameters such that  $m_0 < \kappa$  (so that this set is nontrivial).

We will henceforth consider the following optimal design problem.

**Optimal design problem.** Fix  $n \in \mathbb{N}^*$ ,  $\mu > 0$ ,  $\kappa > 0$ ,  $m_0 \in (0, \kappa)$  and let  $\Omega$  be a bounded connected domain of  $\mathbb{R}^n$  having a  $\mathcal{C}^2$  boundary. We consider the optimization problem

$$\sup_{m \in \mathcal{M}_{\kappa, m_0}(\Omega)} \mathcal{F}_\mu(m). \quad (\mathcal{P}_\mu^n)$$

As will be highlighted in the sequel, the existence of a maximizer follows from a direct argument. We will thus investigate the qualitative properties of maximizers described in the previous section (bang-bang character, concentration/fragmentation phenomena).

For the sake of readability, almost all the proofs are postponed to Section 2.2.

Let us stress that the bang-bang character of maximizer is of practical interest in view of spreading resources in an optimal way. Indeed, in the case where a maximizer  $m^*$  writes  $m^* = \kappa \chi_E$ , the total size of population is maximized by locating all the resources on  $E$ .

### 2.1.2.1 FIRST PROPERTY

We start with a preliminary result related to the saturation of pointwise constraints for Problem  $(\mathcal{P}_\mu^n)$ , valid for all diffusivities  $\mu$ . It is obtained by exploiting the first order optimality conditions for Problem  $(\mathcal{P}_\mu^n)$ , written in terms of an adjoint state.

**PROPOSITION 2.1** Let  $n \in \mathbb{N}^*$ ,  $\mu > 0$ ,  $\kappa > 0$ ,  $m_0 \in (0, \kappa)$ . Let  $m^*$  be a solution of Problem  $(\mathcal{P}_\mu^n)$ . Then, the set  $\{m = \kappa\} \cup \{m = 0\}$  has a positive measure

### 2.1.2.2 THE BANG-BANG PROPERTY HOLDS FOR LARGE DIFFUSIVITIES

For large values of  $\mu$ , we will prove that the variational problem can be recast in terms of a shape optimization problem, as underlined in the next results.

**THEOREM 2.1.1** Let  $n \in \mathbb{N}^*$ ,  $\kappa > 0$ ,  $m_0 \in (0, \kappa)$ . There exists a positive number  $\mu^* = \mu^*(\Omega, \kappa, m_0)$  such that, for every  $\mu \geq \mu^*$ , the functional  $\mathcal{F}_\mu$  is strictly convex. As a consequence, for  $\mu \geq \mu^*$ , any

maximizer of  $\mathcal{F}_\mu$  over  $\mathcal{M}_{m_0, \kappa}(\Omega)$  (or similarly any solution of Problem  $(\mathcal{P}_\mu^n)$ ) is moreover of bang-bang type<sup>1</sup>.

We emphasize that the proof of Theorem 2.1.1 is quite original, since it does not rest upon the exploitation of adjoint state properties, but upon the use of a power series expansions in the diffusivity  $\mu$  of the solution  $\theta_{m, \mu}$  of (LDE), as well as their derivative with respect to the design variable  $m$ . In particular, this expansion is used to prove that, if  $\mu$  is large enough, then the function  $\mathcal{F}_\mu$  is strictly convex. Since the extreme points of  $\mathcal{M}_{m_0, \kappa}(\Omega)$  are bang-bang resources functions, the conclusion readily follows.

This theorem justifies that Problem  $(\mathcal{P}_\mu^n)$  can be recast as a shape optimization problem. Indeed, every maximizer  $m^*$  is of the form  $m^* = \kappa \chi_E$  where  $E$  is a measurable subset such that  $|E| = m_0 |\Omega| / \kappa$ .

**REMARK 2.2** We can rewrite this result in terms of shape optimization, by considering as main unknown the subset  $E$  of  $\Omega$  where resources are located: indeed, under the assumptions of Theorem 2.1.1, there exists a positive number  $\mu^* = \mu^*(\Omega, \kappa, m_0)$  such that, for every  $\mu \geq \mu^*$ , the shape optimization problem

$$\sup_{E \subset \Omega, |E| = m_0 |\Omega| / \kappa} \mathcal{F}_\mu(\kappa \chi_E), \quad (2.8)$$

where the supremum is taken over all measurable subset  $E \subset \Omega$  such that  $|E| = m_0 |\Omega| / \kappa$ , has a solution. We underline the fact that, for such a shape functional, which is “non-energetic” in the sense that the solution of the PDE involved cannot be seen as a minimizer of the same functional, proving the existence of maximizers is usually intricate.

### 2.1.2.3 CONCENTRATION OCCURS FOR LARGE DIFFUSIVITIES

In this section, we state two results suggesting concentration properties for the solutions of Problem  $(\mathcal{P}_\mu^n)$  may hold for large diffusivities.

For that purpose, let us introduce the function space

$$X := W^{1,2}(\Omega) \cap \left\{ u \in W^{1,2}(\Omega), \int_{\Omega} u = 0 \right\} \quad (2.9)$$

and the energy functional

$$\mathcal{E}_m : X \ni u \mapsto \frac{1}{2} \int_{\Omega} |\nabla u|^2 - m_0 \int_{\Omega} m u. \quad (2.10)$$

**THEOREM 2.1.2** [ $\Gamma$ -convergence property]

1. Let  $\Omega$  be a domain with a  $\mathcal{C}^2$  boundary. For any  $\mu > 0$ , let  $m_\mu$  be a solution of Problem  $(\mathcal{P}_\mu^n)$ . Any  $L^1$  closure point of  $\{m_\mu\}_{\mu > 0}$  as  $\mu \rightarrow \infty$  is a solution of the optimal design problem

$$\min_{m \in \mathcal{M}_{m_0, \kappa}(\Omega)} \min_{u \in X} \mathcal{E}_m(u). \quad (2.11)$$

2. In the case of a two 2d orthotope  $\Omega = (0; a_1) \times (0; a_2)$ , any solution of the asymptotic optimization problem (2.11) decreases in every direction.

As will be clear in the proof, this theorem is a  $\Gamma$ -convergence property.

In the one-dimensional case, one can refine this result by showing that, for  $\mu$  large enough, the maximizer is a step function.

**THEOREM 2.1.3** Let us assume that  $n = 1$  and  $\Omega = (0, 1)$ . Let  $\kappa > 0$ ,  $m_0 \in (0, \kappa)$ . There exists  $\hat{\mu} > 0$  such that, for any  $\mu \geq \hat{\mu}$ , any solution  $m$  of Problem  $(\mathcal{P}_\mu^n)$  is equal a.e. to either  $\tilde{m}$  or  $\tilde{m}(1 - \cdot)$ , where  $\tilde{m} = \kappa \chi_{(1-\ell, 1)}$  and  $\ell = m_0 / \kappa$ .

<sup>1</sup>In other words, it is an element of  $\mathcal{M}_{m_0, \kappa}(\Omega)$  equal a.e. to 0 or  $\kappa$  in  $\Omega$ .

### 2.1.2.4 FRAGMENTATION MAY OCCUR FOR SMALL DIFFUSIVITIES

Let us conclude by underlining that the statement of Theorem 2.1.3 cannot be true for all  $\mu > 0$ . Indeed, we provide an example in Section 2.3.1 where a double-crenel growth rate gives a larger total population size than the simple crenel  $\tilde{m}$  of Theorem 2.1.3.

**THEOREM 2.1.4** *The function  $\tilde{m} = \kappa\chi_{(1-\ell,1)}$  (and  $\tilde{m}(1-\cdot) = \kappa\chi_{(0,\ell)}$ ) does not solve Problem  $(\mathcal{P}_\mu^n)$  for small values of  $\mu$ . More precisely, if we extend  $\tilde{m}$  outside of  $(0,1)$  by periodicity, there exists  $\mu > 0$  such that*

$$\mathcal{F}_\mu(\tilde{m}(2\cdot)) > \mathcal{F}_\mu(\tilde{m}).$$

This result is quite unusual. For the optimization of the first eigenvalue  $\lambda_1(m, \mu)$  defined by (2.3) with respect to  $m$  on the interval  $(0;1)$

$$\sup_{m \in \mathcal{M}((0;1))} \lambda_1(m, \mu)$$

we know (see [19]) that the only solutions are  $\tilde{m}$  and  $\tilde{m}(1-\cdot)$ , for any  $\mu$ .

It is notable that the following result is a byproduct of Theorem 2.1.4 above.

**COROLLARY 2.1** *There exists  $\mu > 0$  such that the problems*

$$\sup_{m \in \mathcal{M}((0;1))} \lambda_1(m, \mu)$$

and

$$\sup_{m \in \mathcal{M}((0;1))} \mathcal{F}_\mu(m)$$

do not have the same maximizers.

For further comments on the relationship between the main eigenvalue and the total size of the population, we refer to [Maz19b], where an asymptotic analysis of the main eigenvalue (relative to  $\mu$  as  $\mu \rightarrow +\infty$ ) is performed, and the references therein.

We conclude this section by mentioning the recent work [148], that was reported to us when we wrote this article. They show that, if we assume the optimal distribution of regular resources (more precisely, Riemann integrable), then it is necessarily of *bang-bang* type. Their proof is based on a perturbation argument valid for all  $\mu > 0$ . However, proving such regularity is generally quite difficult. Our proof, although it is not valid for all  $\mu$ , is not based on such a regularity assumption, but these two combined results seem to suggest that all maximizers of this problem are of *bang-bang* type.

## 2.1.3 TOOLS AND NOTATIONS

In this section, we gather some useful tools we will use to prove the main results.

**Rearrangements of functions and principal eigenvalue.** Let us first recall several monotonicity and regularity related to the principal eigenvalue of the operator  $\mathcal{L}$ .

**PROPOSITION 2.2** [70] *Let  $m \in L_+^\infty(\Omega)$  and  $\mu > 0$ .*

- (i) *The mapping  $\mathbb{R}_+^* \ni \mu \mapsto \lambda_1(m, \mu)$  is continuous and non-increasing.*
- (ii) *If  $m \leq m_1$ , then  $\lambda_1(m, \mu) \leq \lambda_1(m_1, \mu)$ , and the equality is true if, and only if  $m = m_1$  a.e. in  $\Omega$ .*

In the proof of Theorem 2.1.3, we will use rearrangement inequalities at length. Let us briefly recall the notion of decreasing rearrangement.



**DEFINITION 2.1.1** For a given function  $b \in L^2(0,1)$ , one defines its monotone decreasing (resp. monotone increasing) rearrangement  $b_{dr}$  (resp.  $b_{br}$ ) on  $(0,1)$  by  $b_{dr}(x) = \sup\{c \in \mathbb{R} \mid x \in \Omega_c^*\}$ , where  $\Omega_c^* = (1 - |\Omega_c|, 1)$  with  $\Omega_c = \{b > c\}$  (resp.  $b_{br}(\cdot) = b_{dr}(1 - \cdot)$ ).

The functions  $b_{dr}$  and  $b_{br}$  enjoy nice properties. In particular, the Polyà-Szego and Hardy-Littlewood inequalities allow to compare integral quantities depending on  $b$ ,  $b_{dr}$ ,  $b_{br}$  and their derivative.

**THEOREM :** [108, 118] Let  $u$  be a non-negative and measurable function.

(i) If  $\psi$  is any measurable function from  $\mathbb{R}_+$  to  $\mathbb{R}$ , then

$$\int_0^1 \psi(u) = \int_0^1 \psi(u_{dr}) = \int_0^1 \psi(u_{br}) \quad (\text{equimeasurability});$$

(ii) If  $u$  belongs to  $W^{1,p}(0,1)$  with  $1 \leq p$ , then

$$\int_0^1 (u')^p \geq \int_0^1 (u'_{br})^p = \int_0^1 (u'_{dr})^p \quad (\text{Pólya inequality});$$

(iii) If  $u, v$  belong to  $L^2(0,1)$ , then

$$\int_0^1 uv \leq \int_0^1 u_{br}v_{br} = \int_0^1 u_{dr}v_{dr} \quad (\text{Hardy-Littlewood inequality});$$

The equality case in the Polyà-Szego inequality is the object of the Brothers-Ziemer theorem (see e.g. [74]).

**Symmetric decreasing functions** In higher dimensions, in order to highlight concentration phenomena, we will use another notion of symmetry, namely monotone symmetric rearrangements that are extensions of monotone rearrangements in one dimension. Here,  $\Omega$  denotes the  $n$ -dimensional orthotope  $\prod_{i=1}^n (a_i, b_i)$ .

**DEFINITION 2.1.2** For a given function  $b \in L^1(\Omega)$ , one defines its symmetric decreasing rearrangement  $b_{sd}$  on  $\Omega$  as follows: first fix the  $n - 1$  variables  $x_2, \dots, x_n$ . Define  $b_{1,sd}$  as the monotone decreasing rearrangement of  $x \mapsto b(x, x_2, \dots, x_n)$ . Then fix  $x_1, x_3, \dots, x_n$  and define  $b_{2,sd}$  as the monotone decreasing rearrangement of  $x \mapsto b_{1,sd}(x_1, x, \dots, x_n)$ . Perform such monotone decreasing rearrangements successively. The resulting function is the symmetric decreasing rearrangement of  $b$ . We define the symmetric increasing rearrangement in a direction  $i$  a similar fashion and write it  $b_{i,id}$ . Note that, in higher dimensions, the definition of decreasing rearrangement strongly depends on the order in which the variables are taken.

Similarly to the one-dimensional case, the Pólya-Szego and Hardy-Littlewood inequalities allow us to compare integral quantities.

**THEOREM :** [19, 20] Let  $u$  be a non-negative and measurable function defined on a box  $\Omega = \prod_{i=1}^n (0; a_i)$ .

(i) If  $\psi$  is any measurable function from  $\mathbb{R}_+$  to  $\mathbb{R}$ , then

$$\int_{\Omega} \psi(u) = \int_{\Omega} \psi(u_{sd}) = \int_{\Omega} \psi(u_{sd}) \quad (\text{equimeasurability});$$

(ii) If  $u$  belongs to  $W^{1,p}(\Omega)$  with  $1 \leq p$ , then, for every  $i \in \mathbb{N}_N$ ,  $[a_i; b_i] = \omega_{1,i} \cup \omega_{2,i} \cup \omega_{3,i}$ , where the map  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N) \mapsto u(x_1, \dots, x_N)$  is decreasing if  $x_i \in \omega_{1,i}$ , increasing if  $x_i \in \omega_{2,i}$  and constant if  $x_i \in \omega_{3,i}$ .

$$\int_{\Omega} |\nabla u|^p \geq \int_{\Omega} |\nabla u_{sd}|^p \quad (\text{Pólya inequality});$$

Furthermore, if, for any  $i \in \mathbb{N}_n$ ,  $\int_{\Omega} |\nabla u|^p = \int_{\Omega} |\nabla u_{i,sd}|^p$  then there exist three measurable subsets  $\omega_{i,1}, \omega_{i,2}$  and  $\omega_{i,3}$  of  $(0; a_i)$  such that

1.  $(0; a_i) = \omega_{i,1} \cup \omega_{i,2} \cup \omega_{i,3}$ ,
2.  $u = u_{i,bd}$  on  $\prod_{k=1}^{i-1} (0; a_k) \times \omega_{i,1} \times \prod_{k=i+1}^n (0; a_k)$ ,
3.  $u = u_{i,id}$  on  $\prod_{k=1}^{i-1} (0; a_k) \times \omega_{i,2} \times \prod_{k=i+1}^n (0; a_k)$ ,
4.  $u = u_{i,bd} = u_{i,id}$  on  $\prod_{k=1}^{i-1} (0; a_k) \times \omega_{i,3} \times \prod_{k=i+1}^n (0; a_k)$ .

(iii) If  $u, v$  belong to  $L^2(\Omega)$ , then

$$\int_{\Omega} uv \leq \int_{\Omega} u_{sd} v_{sd} \quad (\text{Hardy-Littlewood inequality});$$

**Poincaré constants and elliptic regularity results.** We will denote by  $c_{\ell}^{(p)}$  the optimal positive constant such that for every  $p \in [1, +\infty)$ ,  $f \in L^p(\Omega)$  and  $u \in W^{1,p}(\Omega)$  satisfying

$$\Delta u = f \quad \text{in } \mathcal{D}'(\Omega),$$

there holds

$$\|u\|_{W^{2,p}(\Omega)} \leq c_{\ell}^{(p)} (\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}).$$

The optimal constant in the Poincaré-Wirtinger inequality will be denoted by  $C_{PW}^{(p)}(\Omega)$ . This inequality reads: for every  $u \in W^{1,p}(\Omega)$ ,

$$\left\| u - \int_{\Omega} u \right\|_{L^p(\Omega)} \leq C_{PW}^{(p)}(\Omega) \|\nabla u\|_{L^p(\Omega)}. \quad (2.12)$$

We will also use the following regularity results:

**THEOREM** ([169, Theorem 9.1]) *Let  $\Omega$  be a  $\mathcal{C}^2$  domain. There exists a constant  $C_{\Omega} > 0$  such that, if  $f \in L^{\infty}(\Omega)$  and  $u \in W^{1,2}(\Omega)$  solve*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.13)$$

then

$$\|\nabla u\|_{L^1(\Omega)} \leq C_{\Omega} \|f\|_{L^1(\Omega)}. \quad (2.14)$$

**THEOREM** ([53, Theorem 1.1]) *Let  $\Omega$  be a  $\mathcal{C}^2$  domain. There exists a constant  $C_{\Omega} > 0$  such that, if  $f \in L^{\infty}(\Omega)$  and  $u \in W^{1,2}(\Omega)$  solve*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.15)$$

then

$$\|\nabla u\|_{L^{\infty}(\Omega)} \leq C_{\Omega} \|f\|_{L^{\infty}(\Omega)}. \quad (2.16)$$

**REMARK 2.3** *This result is in fact a corollary [53, Theorem 1.1]. In this article it is proved that the  $L^\infty$  norm of the gradient of  $f$  is bounded by the Lorentz norm  $L^{n,1}(\Omega)$  of  $f$ , which is automatically controlled by the  $L^\infty(\Omega)$  norm of  $f$ .*

*Note that Stampacchia's original result deals with Dirichlet boundary conditions. However, the same duality arguments provide the result for Neumann boundary conditions.*

**THEOREM ([121])** *Let  $r \in (1; +\infty)$ . There exists  $C_r > 0$  such that, if  $f \in L^r(\Omega)$  and if  $u \in W^{1,r}(\Omega)$  be a solution of*

$$\begin{cases} -\Delta u = \operatorname{div}(f) & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.17)$$

*then there holds*

$$\|\nabla u\|_{L^r(\Omega)} \leq C_r \|f\|_{L^r(\Omega)}. \quad (2.18)$$

## 2.2 PROOFS OF THE MAIN RESULTS

### 2.2.1 FIRST ORDER OPTIMALITY CONDITIONS FOR PROBLEM $(\mathcal{P}_\mu^n)$

To prove the main results, we first need to state the first order optimality conditions for Problem  $(\mathcal{P}_\mu^n)$ . For that purpose, let us introduce the tangent cone to  $\mathcal{M}_{m_0,\kappa}(\Omega)$  at any point of this set.

**DEFINITION 2.2.1** ([93, chapter 7]) *For every  $m \in \mathcal{M}_{m_0,\kappa}(\Omega)$ , the tangent cone to the set  $\mathcal{M}_{m_0,\kappa}(\Omega)$  at  $m$ , denoted by  $\mathcal{T}_{m,\mathcal{M}_{m_0,\kappa}(\Omega)}$  is the set of functions  $h \in L^\infty(\Omega)$  such that, for any sequence of positive real numbers  $\varepsilon_n$  decreasing to 0, there exists a sequence of functions  $h_n \in L^\infty(\Omega)$  converging to  $h$  as  $n \rightarrow +\infty$ , and  $m + \varepsilon_n h_n \in \mathcal{M}_{m_0,\kappa}(\Omega)$  for every  $n \in \mathbb{N}$ .*

We will show that, for any  $m \in \mathcal{M}_{m_0,\kappa}(\Omega)$  and any admissible perturbation  $h \in \mathcal{T}_{m,\mathcal{M}_{m_0,\kappa}(\Omega)}$ , the functional  $\mathcal{F}_\mu$  is twice Gâteaux-differentiable at  $m$  in direction  $h$ . To do that, we will show that the solution mapping

$$\mathcal{S} : m \in \mathcal{M}_{m_0,\kappa}(\Omega) \mapsto \theta_{m,\mu} \in L^2(\Omega),$$

where  $\theta_{m,\mu}$  denotes the solution of (LDE), is twice Gâteaux-differentiable. In this view, we provide several  $L^2(\Omega)$  estimates of the solution  $\theta_{m,\mu}$ .

**LEMMA 2.1** ([69]) *The mapping  $\mathcal{S}$  is twice Gâteaux-differentiable.*

For the sake of simplicity, we will denote by  $\dot{\theta}_{m,\mu} = d\mathcal{S}(m)[h]$  the Gâteaux-differential of  $\theta_{m,\mu}$  at  $m$  in direction  $h$  and by  $\ddot{\theta}_{m,\mu} = d^2\mathcal{S}(m)[h, h]$  its second order derivative at  $m$  in direction  $h$ .

Elementary computations show that  $\dot{\theta}_{m,\mu}$  solves the PDE

$$\begin{cases} \mu \Delta \dot{\theta}_{m,\mu} + (m - 2\theta_{m,\mu}) \dot{\theta}_{m,\mu} = -h \theta_{m,\mu} & \text{in } \Omega, \\ \frac{\partial \dot{\theta}_{m,\mu}}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.19)$$

whereas  $\ddot{\theta}_{m,\mu}$  solves the PDE

$$\begin{cases} \mu \Delta \ddot{\theta}_{m,\mu} + \ddot{\theta}_{m,\mu} (m - 2\theta_{m,\mu}) = -2 \left( h \dot{\theta}_{m,\mu} - \dot{\theta}_{m,\mu}^2 \right) & \text{in } \Omega, \\ \frac{\partial \ddot{\theta}_{m,\mu}}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.20)$$

It follows that, for all  $\mu > 0$ , the application  $\mathcal{F}_\mu$  is Gâteaux-differentiable with respect to  $m$  in direction  $h$  and its Gâteaux derivative writes

$$d\mathcal{F}_\mu(m)[h] = \int_{\Omega} \dot{\theta}_{m,\mu}.$$

Since the expression of  $d\mathcal{F}_\mu(m)[h]$  above is not workable, we need to introduce the adjoint state  $p_{m,\mu}$  to the equation satisfied by  $\dot{\theta}_{m,\mu}$ , i.e the solution of the equation

$$\begin{cases} \mu\Delta p_{m,\mu} + p_{m,\mu}(m - 2\theta_{m,\mu}) = 1 & \text{in } \Omega, \\ \frac{\partial p_{m,\mu}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.21)$$

Note that  $p_{m,\mu}$  belongs to  $W^{1,2}(\Omega)$  and is unique, according to the Fredholm alternative. In fact, we can prove the following regularity results on  $p_{m,\mu}$ :  $p_{m,\mu} \in L^\infty(\Omega)$ ,  $\|p_{m,\mu}\|_{L^\infty(\Omega)} \leq M$ , where  $M$  is uniform in  $m \in \mathcal{M}_{m_0,\kappa}(\Omega)$  and, for any  $p \in [1; +\infty)$ ,  $p_{m,\mu} \in W^{2,p}(\Omega)$ , so that Sobolev embeddings guarantee that  $p_{m,\mu} \in \mathcal{C}^{1,\alpha}(\Omega)$ .

Now, multiplying the main equation of (2.21) by  $\dot{\theta}_{m,\mu}$  and integrating two times by parts leads to the expression

$$d\mathcal{F}_\mu(m)[h] = - \int_{\Omega} h \theta_{m,\mu} p_{m,\mu}.$$

Now consider a maximizer  $m$ . For every perturbation  $h$  in the cone  $\mathcal{T}_{m,\mathcal{M}_{m_0,\kappa}(\Omega)}$ , there holds  $d\mathcal{F}_\mu(m)[h] \geq 0$ . The analysis of such optimality condition is standard in optimal control theory (see for example [179]) and leads to the following result.

**PROPOSITION 2.3** *Let us define  $\varphi_{m,\mu} = \theta_{m,\mu} p_{m,\mu}$ , where  $\theta_{m,\mu}$  and  $p_{m,\mu}$  solve respectively equations (LDE) and (2.21). There exists  $c \in \mathbb{R}$  such that*

$$\{\varphi_{m,\mu} < c\} = \{m = \kappa\}, \quad \{\varphi_{m,\mu} = c\} = \{0 < m < \kappa\}, \quad \{\varphi_{m,\mu} > c\} = \{m = 0\}.$$

## 2.2.2 PROOF OF PROPOSITION 2.1

An easy but tedious computation shows that the function  $\varphi_{m,\mu}$  introduced in Proposition 2.3 is  $\mathcal{C}^{1,\alpha}(\Omega) \cap W^{1,2}(\Omega)$  function, as a product of two  $\mathcal{C}^{1,\alpha}$  functions and satisfies (in a  $W^{1,2}$  weak sense)

$$\begin{cases} \mu\Delta\varphi_{m,\mu} - 2\mu \left\langle \nabla\varphi_{m,\mu}, \frac{\nabla\theta_{m,\mu}}{\theta_{m,\mu}} \right\rangle + \varphi_{m,\mu} \left( 2\mu \frac{|\nabla\theta_{m,\mu}|^2}{\theta_{m,\mu}^2} + 2m - 3\theta_{m,\mu} \right) = \theta_{m,\mu} & \text{in } \Omega, \\ \frac{\partial\varphi_{m,\mu}}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.22)$$

where  $\langle \cdot, \cdot \rangle$  stands for the usual Euclidean inner product. To prove that  $|\{m = 0\}| + |\{m = \kappa\}| > 0$ , we argue by contradiction, by assuming that  $|\{m = \kappa\}| = |\{m = 0\}| = 0$ . Therefore,  $\varphi_{m,\mu} = c$  a.e. in  $\Omega$  and, according to (2.22), there holds

$$c \left( 2\mu \frac{|\nabla\theta_{m,\mu}|^2}{\theta_{m,\mu}^2} + 2m - 3\theta_{m,\mu} \right) = \theta_{m,\mu}$$

Integrating this identity and using that  $\theta_{m,\mu} > 0$  in  $\Omega$  and  $c \neq 0$ , we get

$$2c \left( \mu \int_{\Omega} \frac{|\nabla\theta_{m,\mu}|^2}{\theta_{m,\mu}^2} + \int_{\Omega} (m - \theta_{m,\mu}) \right) = (c + 1) \int_{\Omega} \theta_{m,\mu}.$$

Equation (2.6) yields that the left-hand side equals 0, so that one has  $c = -1$ . Coming back to the equation satisfied by  $\varphi_{m,\mu}$  leads to

$$m = \theta_{m,\mu} - \mu \frac{|\nabla\theta_{m,\mu}|^2}{\theta_{m,\mu}^2}.$$

The logistic diffusive equation (LDE) is then transformed into

$$\mu\theta_{m,\mu}\Delta\theta_{m,\mu} - \mu|\nabla\theta_{m,\mu}|^2 = 0.$$

Integrating this equation by parts yields  $\int_{\Omega} |\nabla \theta_{m,\mu}|^2 = 0$ . Thus,  $\theta_{m,\mu}$  is constant, and so is  $m$ . In other words,  $m = m_0$ , which, according to (2.6) (see Remark 2.1) is impossible. The expected result follows.

### 2.2.3 PROOF OF THEOREM 2.1.1

The proof of Theorem 2.1.1 is based on a careful asymptotic analysis with respect to the diffusivity variable  $\mu$ .

Let us first explain the outlines of the proof.

Let us fix  $m \in \mathcal{M}_{m_0,\kappa}(\Omega)$  and  $h \in L^\infty(\Omega)$ . In the sequel, the dot or double dot notation  $\dot{f}$  or  $\ddot{f}$  will respectively denote first and second order Gâteaux-differential of  $f$  at  $m$  in direction  $h$ .

According to Lemma 2.1,  $\mathcal{F}_\mu$  is twice Gâteaux-differentiable and its second order Gâteaux-derivative is given by

$$d^2 \mathcal{F}_\mu(m)[h, h] = \int_{\Omega} \ddot{\theta}_{m,\mu},$$

where  $\ddot{\theta}_{m,\mu}$  is the second Gâteaux derivative of  $\mathcal{S}$ , defined as the unique solution of (2.20).

Let  $m_1$  and  $m_2$  be two elements of  $\mathcal{M}_{m_0,\kappa}(\Omega)$  and define

$$\phi_\mu : [0; 1] \ni t \mapsto \mathcal{F}_\mu(tm_2 + (1-t)m_1) - t\mathcal{F}_\mu(m_2) - (1-t)\mathcal{F}_\mu(m_1).$$

One has

$$\frac{d^2 \phi_\mu}{dt^2}(t) = \int_{\Omega} \ddot{\theta}_{(1-t)m_1 + tm_2, \mu}, \quad \text{and} \quad \phi_\mu(0) = \phi_\mu(1) = 0,$$

where  $\ddot{\theta}_{(1-t)m_1 + tm_2, \mu}$  must be interpreted as a bilinear form from  $L^\infty(\Omega)$  to  $W^{1,2}(\Omega)$ , evaluated two times at the same direction  $m_2 - m_1$ . Hence, to get the strict convexity of  $\mathcal{F}_\mu$ , it suffices to show that, whenever  $\mu$  is large enough,

$$\int_{\Omega} \ddot{\theta}_{tm_2 + (1-t)m_1, \mu} > 0$$

as soon as  $m_1 \neq m_2$  (in  $L^\infty(\Omega)$ ) and  $t \in (0, 1)$ , or equivalently that  $d^2 \mathcal{F}_\mu(m)[h, h] > 0$  as soon as  $m \in \mathcal{M}_{m_0,\kappa}(\Omega)$  and  $h \in L^\infty(\Omega)$ . Note that since  $h = m_2 - m_1$ , it is possible to assume without loss of generality that  $\|h\|_{L^\infty(\Omega)} \leq 2\kappa$ .

The proof is based on an asymptotic expansion of  $\theta_{m,\mu}$  into a main term and a reminder one, with respect to the diffusivity  $\mu$ . It is well-known (see e.g. [125, Lemma 2.2]) that one has

$$\theta_{m,\mu} \xrightarrow[\mu \rightarrow \infty]{W^{1,2}(\Omega)} m_0. \quad (2.23)$$

However, since we are working with resources distributions living in  $\mathcal{M}_{m_0,\kappa}(\Omega)$ , such a convergence property does not allow us to exploit it for deriving optimality properties for Problem  $(\mathcal{P}_\mu^n)$ .

For this reason, in what follows, we find a first order term in this asymptotic expansion. To get an insight into the proof's main idea, let us first proceed in a formal way, by looking for a function  $\eta_{1,m}$  such that

$$\theta_{m,\mu} \approx m_0 + \frac{\eta_{1,m}}{\mu}$$

as  $\mu \rightarrow \infty$ . Plugging this formal expansion in (LDE) and identifying at order  $\frac{1}{\mu}$  yields that  $\eta_{1,m}$  satisfies

$$\begin{cases} \Delta \eta_{1,m} + m_0(m - m_0) = 0 & \text{in } \Omega, \\ \frac{\partial \eta_{1,m}}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

To make this equation well-posed, it is convenient to introduce the function  $\hat{\eta}_{1,m}$  defined as the unique

solution to the system

$$\begin{cases} \Delta \hat{\eta}_{1,m} + m_0(m - m_0) = 0 & \text{in } \Omega, \\ \frac{\partial \hat{\eta}_{1,m}}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} \hat{\eta}_{1,m} = 0, \end{cases}$$

and to determine a constant  $\beta_{1,m}$  such that

$$\eta_{1,m} = \hat{\eta}_{1,m} + \beta_{1,m}.$$

In view of identifying the constant  $\beta_{1,m}$ , we integrate equation (LDE) to get

$$\int_{\Omega} \theta_{m,\mu}(m - \theta_{m,\mu}) = 0.$$

which yields, at the order  $\frac{1}{\mu}$ ,

$$\beta_{1,m} = \frac{1}{m_0} \int_{\Omega} \hat{\eta}_{1,m}(m - m_0) = \frac{1}{m_0^2} \int_{\Omega} |\nabla \hat{\eta}_{1,m}|^2.$$

Therefore, one has formally

$$\int_{\Omega} \theta_{m,\mu} \approx m_0 + \frac{1}{\mu} \int_{\Omega} |\nabla \hat{\eta}_{1,m}|^2.$$

As will be proved in Step 1 (paragraph 2.2.3), the mapping  $\mathcal{M}_{m_0,\kappa}(\Omega) \ni m \mapsto \beta_{1,m}$  is convex so that, at the order  $\frac{1}{\mu}$ , the mapping  $\mathcal{M}_{m_0,\kappa}(\Omega) \ni m \mapsto \int_{\Omega} \theta_{m,\mu}$  is convex. We will prove the validity of all the claims above, by taking into account remainder terms in the asymptotic expansion above, to prove that the mapping  $\mathcal{F}_{\mu} : m \mapsto \int_{\Omega} \theta_{m,\mu}$  is itself convex whenever  $\mu$  is large enough.

**REMARK 2.4** *One could also notice that the quantity  $\beta_{1,m}$  arose in the recent paper [89], where the authors determine the large time behavior of a diffusive Lotka-Volterra competitive system between two populations with growth rates  $m_1$  and  $m_2$ . If  $\beta_{1,m_1} > \beta_{1,m_2}$ , then when  $\mu$  is large enough, the solution converges as  $t \rightarrow +\infty$  to the steady state solution of a scalar equation associated with the growth rate  $m_1$ . In other words, the species with growth rate  $m_1$  chases the other one. In the present article, as a byproduct of our results, we maximize the function  $m \mapsto \beta_{1,m}$ . This remark implies that this intermediate result might find other applications of its own.*

Let us now formalize rigorously the reasoning outlined above, by considering an expansion of the form

$$\theta_{m,\mu} = m_0 + \frac{\eta_{1,m}}{\mu} + \frac{\mathcal{R}_{m,\mu}}{\mu^2}.$$

Hence, one has for all  $m \in \mathcal{M}_{m_0,\kappa}(\Omega)$ ,

$$d^2 \mathcal{F}_{\mu}(m)[h, h] = \frac{1}{\mu} \int_{\Omega} \ddot{\eta}_{1,m} + \frac{1}{\mu^2} \int_{\Omega} \ddot{\mathcal{R}}_{m,\mu}$$

We will show that there holds

$$d^2 \mathcal{F}_{\mu}(m)[h, h] \geq \frac{C(h)}{\mu} \left(1 - \frac{\Lambda}{\mu}\right) \tag{2.24}$$

for all  $\mu > 0$ , where  $C(h)$  and  $\Lambda$  denote some positive constants.

The strict convexity of  $\mathcal{F}_{\mu}$  will then follow. Concerning the bang-bang character of maximizers, notice that the admissible set  $\mathcal{M}_{m_0,\kappa}(\Omega)$  is convex, and that its extreme points are exactly the bang-bang functions of  $\mathcal{M}_{m_0,\kappa}(\Omega)$ . Once the strict convexity of  $\mathcal{F}_{\mu}$  showed, we then easily infer that  $\mathcal{F}_{\mu}$  reaches its maxima at extreme points, in other words that any maximizer is bang-bang. Indeed,

assuming by contradiction the existence of a maximizer writing  $tm_1 + (1-t)m_2$  with  $t \in (0, 1)$ ,  $m_1$  and  $m_2$ , two elements of  $\mathcal{M}_{m_0, \kappa}(\Omega)$  such that  $m_1 \neq m_2$  on a positive Lebesgue measure set, one has

$$\mathcal{F}_\mu(tm_1 + (1-t)m_2) < t\mathcal{F}_\mu(m_1) + (1-t)\mathcal{F}_\mu(m_2) < \max\{\mathcal{F}_\mu(m_1), \mathcal{F}_\mu(m_2)\},$$

by convexity of  $\mathcal{F}_\mu$ , whence the contradiction.

The rest of the proof is devoted to the proof of the inequality (2.24). It is divided into the following steps:

**Step 1.** Uniform estimate of  $\int_\Omega \ddot{\eta}_{1,m}$  with respect to  $\mu$ .

**Step 2.** Definition and expansion of the reminder term  $\mathcal{R}_{m,\mu}$ .

**Step 3.** Uniform estimate of  $\mathcal{R}_{m,\mu}$  with respect to  $\mu$ .

**Step 1: minoration of  $\int_\Omega \ddot{\eta}_{1,m}$ .** One computes successively

$$\dot{\beta}_{1,m} = \frac{1}{m_0} \int_\Omega \left( \dot{\eta}_{1,m} m + \dot{\eta}_{1,m} h \right), \quad \ddot{\beta}_{1,m} = \frac{1}{m_0} \int_\Omega \left( 2\dot{\eta}_{1,m} h + \ddot{\eta}_{1,m} h \right) \quad (2.25)$$

where  $\dot{\eta}_{1,m}$  solves the equation

$$\begin{cases} \Delta \dot{\eta}_{1,m} + m_0 h = 0 & \text{in } \Omega \\ \frac{\partial \dot{\eta}_{1,m}}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases} \quad \text{with } \int_\Omega \dot{\eta}_{1,m} = 0. \quad (2.26)$$

Notice moreover that  $\ddot{\eta}_{1,m} = 0$ , since  $\dot{\eta}_{1,m}$  is linear with respect to  $h$ . Moreover, multiplying the equation above by  $\dot{\eta}_{1,m}$  and integrating by parts yields

$$\ddot{\beta}_{1,m} = \frac{2}{m_0^2} \int_\Omega |\nabla \dot{\eta}_{1,m}|^2 > 0 \quad (2.27)$$

whenever  $h \neq 0$ , according to (2.25). Finally, we obtain

$$\int_\Omega \ddot{\eta}_{1,m} = |\Omega| \ddot{\beta}_{1,m} + \int_\Omega \ddot{\eta}_{1,m} = |\Omega| \ddot{\beta}_{1,m} = \frac{2}{m_0^2} \int_\Omega |\nabla \dot{\eta}_{1,m}|^2.$$

It is then notable that  $\int_\Omega \ddot{\eta}_{1,m} \geq 0$ .

**Step 2: expansion of the reminder term  $\mathcal{R}_{m,\mu}$ .** Instead of studying directly the equation (2.20), our strategy consists in providing a well-chosen expansion of  $\ddot{\theta}_{m,\mu}$  of the form

$$\ddot{\theta}_{m,\mu} = \sum_{k=0}^{+\infty} \frac{\zeta_k}{\mu^k}, \text{ where the } \zeta_k \text{ are such that } \sum_{k=2}^{+\infty} \frac{\int_\Omega \zeta_k}{\mu^{k-1}} \leq M \int_\Omega \ddot{\eta}_{1,m}.$$

For that purpose, we will expand formally  $\theta_{m,\mu}$  as

$$\theta_{m,\mu} = \sum_{k=0}^{+\infty} \frac{\eta_{k,m}}{\mu^k}. \quad (2.28)$$

Note that, as underlined previously, since  $\theta_{m,\mu} \xrightarrow{\mu \rightarrow +\infty} m_0$  in  $L^\infty(\Omega)$ , we already know that  $\eta_{0,m} = m_0$ .

Provided that this expansion makes sense and is (two times) differentiable term by term (what will be checked in the sequel) in the sense of Gâteaux, we will get the following expansions

$$\dot{\theta}_{m,\mu} = \sum_{k=0}^{+\infty} \frac{\dot{\eta}_{k,m}}{\mu^k} \quad \text{and} \quad \ddot{\theta}_{m,\mu} = \sum_{k=0}^{+\infty} \frac{\ddot{\eta}_{k,m}}{\mu^k}.$$

Plugging the expression (2.28) of  $\theta_{m,\mu}$  into the logistic diffusive equation (LDE), a formal computation first yields

$$\Delta\eta_{1,m} + m_0(m - m_0) = 0, \quad \frac{\partial\eta_{1,m}}{\partial\nu} = 0 \quad \text{on } \partial\Omega$$

$$\Delta\eta_{2,m} + \eta_{1,m}(m - 2m_0) = 0 \quad \text{in } \Omega, \quad \frac{\partial\eta_{2,m}}{\partial\nu} = 0 \quad \text{on } \partial\Omega$$

and, for any  $k \in \mathbb{N}, k \geq 2$ ,  $\eta_{k,m}$  satisfies the induction relation

$$\Delta\eta_{k+1,m} + (m - 2m_0)\eta_{k,m} - \sum_{\ell=1}^{k-1} \eta_{\ell,m}\eta_{k-\ell,m} = 0 \quad \text{in } \Omega, \quad (2.29)$$

as well as homogeneous Neumann boundary conditions. These relations do not allow to define  $\eta_{k,m}$  in a unique way (it is determined up to a constant). We introduce the following equations to overcome this difficulty: first, we define  $\hat{\eta}_{1,m}$  and  $\hat{\eta}_{2,m}$  as the solutions to

$$\Delta\hat{\eta}_{1,m} + m_0(m - m_0) = 0, \quad \frac{\partial\hat{\eta}_{1,m}}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \quad \int_{\Omega} \hat{\eta}_{1,m} = 0,$$

$$\Delta\hat{\eta}_{2,m} + \eta_{1,m}(m - 2m_0) = 0 \quad \text{in } \Omega, \quad \frac{\partial\hat{\eta}_{2,m}}{\partial\nu} = 0 \quad \text{on } \partial\Omega, \quad \int_{\Omega} \hat{\eta}_{2,m} = 0$$

and, for any  $k \in \mathbb{N}, k \geq 2$ , we define  $\hat{\eta}_{k+1,m}$  as the solution of the PDE

$$\begin{cases} \Delta\hat{\eta}_{k+1,m} + (m - 2m_0)\eta_{k,m} - \sum_{\ell=1}^{k-1} \eta_{\ell,m}\eta_{k-\ell,m} = 0 & \text{in } \Omega \\ \frac{\partial\hat{\eta}_{k+1,m}}{\partial\nu} = 0 & \text{on } \partial\Omega \end{cases} \quad \text{with } \int_{\Omega} \hat{\eta}_{k+1,m} = 0, \quad (2.30)$$

and to define the real number  $\beta_{k,m}$  in such a way that

$$\eta_{k,m} = \hat{\eta}_{k,m} + \beta_{k,m}. \quad (2.31)$$

for every  $k \in \mathbb{N}^*$ . Integrating the main equation of (LDE) yields

$$\int_{\Omega} \theta_{m,\mu}(m - \theta_{m,\mu}) = 0.$$

Plugging the expansion (2.28) and identifying the terms of order  $k$  indicates that we must define  $\beta_{k,m}$  by the induction relation

$$\begin{aligned} \beta_{1,m} &= \frac{1}{m_0^2} \int_{\Omega} |\nabla\hat{\eta}_{1,m}|^2, \\ \beta_{2,m} &= \frac{1}{m_0} \int_{\Omega} m\hat{\eta}_{2,m} - \frac{1}{m_0} \int_{\Omega} \eta_{1,m}^2, \\ \beta_{k+1,m} &= \frac{1}{m_0} \int_{\Omega} m\hat{\eta}_{k+1,m} - \frac{1}{m_0} \sum_{\ell=1}^k \int_{\Omega} \eta_{\ell,m}\eta_{k+1-\ell,m}. \quad (k \geq 2) \end{aligned}$$



This leads to the following cascade system for  $\{\hat{\eta}_{k,m}, \beta_{k,m}, \eta_{k,m}\}_{k \in \mathbb{N}}$ :

$$\left\{ \begin{array}{l} \hat{\eta}_{0,m} = 0, \\ \Delta \hat{\eta}_{1,m} + m_0(m - m_0) = 0 \text{ in } \Omega, \\ \Delta \hat{\eta}_{2,m} + \eta_{1,m}(m - 2m_0) = 0 \text{ in } \Omega, \\ \Delta \hat{\eta}_{k+1,m} + (m - 2m_0)\eta_{k,m} - \sum_{\ell=1}^{k-1} \eta_{\ell,m}\eta_{k-\ell,m} = 0 \text{ in } \Omega, \quad (k \geq 2) \\ \int_{\Omega} \hat{\eta}_{k,m} = 0, \quad (k \geq 0) \\ \frac{\partial \hat{\eta}_{k,m}}{\partial \nu} = 0 \text{ over } \partial\Omega, \quad (k \geq 0) \\ \beta_{0,m} = m_0, \\ \beta_{1,m} = \frac{1}{m_0^2} \int_{\Omega} |\nabla \hat{\eta}_{1,m}|^2, \\ \beta_{2,m} = \frac{1}{m_0} \int_{\Omega} m \hat{\eta}_{2,m} - \frac{1}{m_0} \int_{\Omega} \eta_{1,m}^2, \\ \beta_{k+1,m} = \frac{1}{m_0} \int_{\Omega} m \hat{\eta}_{k+1,m} - \frac{1}{m_0} \sum_{\ell=1}^k \int_{\Omega} \eta_{\ell,m} \eta_{k+1-\ell,m}, \quad (k \geq 2) \\ \eta_{k,m} = \hat{\eta}_{k,m} + \beta_{k,m}. \quad (k \geq 0) \end{array} \right. \quad (2.32)$$

This implies

$$\int_{\Omega} \eta_{k,m} = \beta_{k,m}, \quad \frac{\partial \eta_{k,m}}{\partial \nu} = 0 \text{ over } \partial\Omega. \quad (k \geq 0).$$

Now, the Gâteaux-differentiability of both  $\hat{\eta}_{k,m}$  and  $\beta_{k,m}$  with respect to  $m$  follows from similar arguments as those used to prove Proposition 2.1. Similarly to System (2.32), the system satisfied by the derivatives needs the introduction of two auxiliary sequences  $\{\dot{\hat{\eta}}_{k,m}\}_{k \in \mathbb{N}}$  and  $\{\dot{\beta}_{k,m}\}_{k \in \mathbb{N}}$ . More precisely, we expand  $\dot{\theta}_{m,\mu}$  as

$$\dot{\theta}_{m,\mu} = \sum_{k=0}^{\infty} \frac{\dot{\eta}_{k,m}}{\mu^k}$$

with

$$\dot{\eta}_{k,m} = \dot{\hat{\eta}}_{k,m} + \dot{\beta}_{k,m},$$

and the sequence  $\{\dot{\hat{\eta}}_{k,m}, \dot{\beta}_{k,m}, \dot{\eta}_{k,m}\}_{k \in \mathbb{N}}$  satisfies

$$\left\{ \begin{array}{l} \dot{\eta}_{0,m} = 0, \\ \Delta \dot{\hat{\eta}}_{1,m} + m_0 h = 0 \text{ in } \Omega, \\ \Delta \dot{\hat{\eta}}_{2,m} + \dot{\eta}_{1,m}(m - 2m_0) = -h \eta_{1,m} \text{ in } \Omega, \\ \Delta \dot{\hat{\eta}}_{k+1,m} + (m - 2m_0)\dot{\eta}_{k,m} - 2 \sum_{\ell=1}^{k-1} \dot{\eta}_{\ell,m} \eta_{k-\ell,m} = -h \eta_{k,m} \text{ in } \Omega, \quad (k \geq 2) \\ \int_{\Omega} \dot{\hat{\eta}}_{k,m} = 0, \quad (k \geq 0) \\ \frac{\partial \dot{\hat{\eta}}_{k,m}}{\partial \nu} = 0 \text{ over } \partial\Omega, \quad (k \geq 0) \\ \dot{\beta}_{0,m} = 0, \\ \dot{\beta}_{1,m} = \frac{1}{m_0} \int_{\Omega} (h \hat{\eta}_{1,m} + m \dot{\hat{\eta}}_{1,m}) = \frac{2}{m_0^2} \int_{\Omega} \langle \nabla \hat{\eta}_{1,m}, \nabla \eta_{1,m} \rangle, \\ \dot{\beta}_{2,m} = \frac{1}{m_0} \int_{\Omega} (h \hat{\eta}_{2,m} + m \dot{\hat{\eta}}_{2,m}) - \frac{2}{m_0} \int_{\Omega} \dot{\eta}_{1,m} \eta_{1,m}, \\ \dot{\beta}_{k+1,m} = \frac{1}{m_0} \int_{\Omega} (h \hat{\eta}_{k+1,m} + m \dot{\hat{\eta}}_{k+1,m}) - \frac{2}{m_0} \sum_{\ell=1}^k \int_{\Omega} \dot{\eta}_{\ell,m} \eta_{k+1-\ell,m}, \quad (k \geq 2) \\ \dot{\eta}_{k,m} = \dot{\hat{\eta}}_{k,m} + \dot{\beta}_{k,m}. \quad (k \geq 0) \end{array} \right. \quad (2.33)$$

We note that this implies, for any  $k \in \mathbb{N}$ ,

$$\int_{\Omega} \dot{\eta}_{k,m} = \dot{\beta}_{k,m}, \quad \frac{\partial \dot{\eta}_{k,m}}{\partial \nu} = 0 \text{ on } \partial\Omega. \quad (k \geq 0)$$

Let us also write the system satisfied by the second order differentials. One gets the following hierarchy

for  $\{\ddot{\eta}_{k,m}, \ddot{\beta}_{k,m}, \ddot{\eta}_{k,m}\}_{k \in \mathbb{N}}$ :

$$\left\{ \begin{array}{l} \ddot{\eta}_{0,m} = 0, \\ \Delta \ddot{\eta}_{1,m} = 0 \text{ in } \Omega, \\ \Delta \ddot{\eta}_{2,m} + (m - 2m_0) \ddot{\eta}_{1,m} = -2h \dot{\eta}_{1,m} \text{ in } \Omega, \\ \Delta \ddot{\eta}_{k+1,m} + (m - 2m_0) \ddot{\eta}_{k,m} - 2 \sum_{\ell=1}^{k-1} \ddot{\eta}_{\ell,m} \eta_{k-\ell,m} = 2 \left( \sum_{\ell=1}^{k-1} \dot{\eta}_{\ell,m} \dot{\eta}_{k-\ell,m} - h \dot{\eta}_{k,m} \right) \text{ in } \Omega, \quad (k \geq 2) \\ f_{\Omega} \ddot{\eta}_{k,m} = 0, \quad (k \geq 0) \\ \frac{\partial \ddot{\eta}_{k,m}}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad (k \geq 0) \\ \ddot{\beta}_{0,m} = 0, \\ \ddot{\beta}_{1,m} = \frac{2}{m_0^2} f_{\Omega} |\nabla \dot{\eta}_{1,m}|^2, \\ \ddot{\beta}_{2,m} = \frac{1}{m_0} f_{\Omega} (2h \dot{\eta}_{2,m} + m \ddot{\eta}_{2,m}) - \frac{2}{m_0} f_{\Omega} ((\dot{\eta}_{1,m})^2 + \dot{\eta}_{1,m} \eta_{1,m}), \\ \ddot{\beta}_{k+1,m} = \frac{1}{m_0} f_{\Omega} (2h \dot{\eta}_{k+1,m} + m \ddot{\eta}_{k+1,m}) - \frac{2}{m_0} \sum_{\ell=1}^k f_{\Omega} (\dot{\eta}_{\ell,m} \dot{\eta}_{k+1-\ell,m} + \dot{\eta}_{\ell,m} \eta_{k+1-\ell,m}), \quad (k \geq 2) \\ \ddot{\eta}_{k,m} = \ddot{\eta}_{k,m} + \ddot{\beta}_{k,m}. \quad (k \geq 0) \end{array} \right. \quad (2.34)$$

This gives

$$\int_{\Omega} \ddot{\eta}_{k,m} = \ddot{\beta}_{k,m}, \quad \frac{\partial \ddot{\eta}_{k,m}}{\partial \nu} = 0 \text{ over } \partial\Omega. \quad (k \geq 0).$$

**Step 3: uniform estimates of  $\mathcal{R}_{m,\mu}$ .** This section is devoted to proving an estimate on  $\ddot{\beta}_{k,m}$ , namely

$$\left\{ \begin{array}{l} \forall k \in \mathbb{N}^*, \left| \ddot{\beta}_{k,m} \right| \leq \Lambda(k) \ddot{\beta}_{1,m}, \\ \text{The power series } \sum_{k=1}^{+\infty} \Lambda(k) x^k \text{ has a positive convergence radius.} \end{array} \right. \quad (2.35)$$

This estimate is a key point in our reasoning. Indeed, recall that our goal is to prove that  $\mathcal{F}_{\mu}$  is convex. Assuming that Estimates (2.35) hold true, we expand  $\mathcal{F}_{\mu}$  as follows:

$$\mathcal{F}_{\mu}(m) = \sum_{k=0}^{\infty} \frac{f_{\Omega} \eta_{k,m}}{\mu^k} = \sum_{k=0}^{\infty} \frac{\beta_{k,m}}{\mu^k}.$$

Differentiating this expression twice with respect to  $m$  in direction  $h$  yields

$$\ddot{\mathcal{F}}_{\mu}(m)[h, h] = \sum_{k=1}^{\infty} \frac{\ddot{\beta}_{k,m}}{\mu^k}.$$

Note that the sum starts at  $k = 1$  since  $\beta_{0,m} = m_0$  does not depend on  $m$ .

We can then write

$$\begin{aligned} \mu \ddot{\mathcal{F}}_{\mu}(m)[h, h] &= \ddot{\beta}_{1,m} + \sum_{k=2}^{\infty} \frac{\ddot{\beta}_{k,m}}{\mu^{k-1}} \geq \ddot{\beta}_{1,m} \left( 1 - \sum_{k=2}^{\infty} \frac{\Lambda(k)}{\mu^{k-1}} \right) \\ &= \ddot{\beta}_{1,m} \left( 1 - \frac{1}{\mu} \sum_{k=0}^{\infty} \frac{\Lambda(k+2)}{\mu^k} \right) \end{aligned}$$

Recall that  $\ddot{\beta}_{1,m}$  is positive as soon as  $h$  is not identically equal on 0, according to (2.27). The power series associated with  $\{\Lambda(k+2)\}_{k \in \mathbb{N}}$  also has a positive convergence radius. Then, the right hand side term is positive provided that  $\mu$  be large enough. For the sake of notational clarity, we define  $\delta$  as follows: by the Rellich-Kondrachov embedding theorem, see [34, Theorem 9.16], there exists  $\delta > 0$  such that the continuous embedding  $W^{1,2}(\Omega) \hookrightarrow L^{2+\delta}(\Omega)$  holds. We fix such a  $\delta > 0$ .

Recall that we know from Equation (2.27) that  $\ddot{\beta}_{1,m}$  is proportional to  $\|\nabla\dot{\eta}_{1,m}\|_{L^2(\Omega)}^2$ . From the explicit expression of  $\ddot{\beta}_{k,m}$  in (2.34), one claims that (2.35) follows both from the positivity of  $\ddot{\beta}_{1,m}$  and from the following estimates:

$$\begin{cases} \|\eta_{k,m}\|_{L^\infty(\Omega)}, \|\nabla\eta_{k,m}\|_{L^\infty(\Omega)} \leq \alpha(k), & (I_\alpha^k) \\ \|\nabla\dot{\eta}_{k,m}\|_{L^2(\Omega)} \leq \sigma(k)\|\nabla\dot{\eta}_{1,m}\|_{L^2(\Omega)}, & (I_\sigma^k) \\ \|\dot{\eta}_{k,m}\|_{L^2(\Omega)} \leq \gamma(k)\|\nabla\dot{\eta}_{1,m}\|_{L^2(\Omega)}, & (I_\gamma^k) \\ \|\dot{\eta}_{k,m}\|_{L^{2+\delta}(\Omega)} \leq \tilde{\gamma}(k)\|\nabla\dot{\eta}_{1,m}\|_{L^2(\Omega)}, & (I_{\tilde{\gamma}}^k) \\ \|\nabla\ddot{\eta}_{k,m}\|_{L^1(\Omega)} \leq \delta(k)\|\nabla\dot{\eta}_{1,m}\|_{L^2(\Omega)}^2, & (I_\delta^k) \\ \|\ddot{\eta}_{k,m}\|_{L^1(\Omega)} \leq \varepsilon(k)\|\nabla\dot{\eta}_{1,m}\|_{L^2(\Omega)}^2. & (I_\varepsilon^k) \end{cases}$$

where for all  $k \in \mathbb{N}$ , the numbers  $\alpha(k)$ ,  $\sigma(k)$ ,  $\gamma(k)$ ,  $\tilde{\gamma}(k)$ ,  $\delta(k)$  and  $\varepsilon(k)$  are positive.

In what follows, we will write  $f \lesssim g$  when there exists a constant  $C$  (independent of  $k$ ) such that  $f \leq Cg$ .

The end of the proof is devoted to proving the aforementioned estimates. In what follows, we will mainly deal with the indices  $k \geq 3$ . Indeed, the case  $k = 2$  is much simpler since, according to the cascade systems (2.29)-(2.31)-(2.32)-(2.33)-(2.34), the equations on  $\eta_{k,m}$ ,  $\dot{\eta}_{k,m}$  and  $\ddot{\eta}_{k,m}$  for  $k \geq 3$  involve more terms than the ones on  $\eta_{2,m}$ ,  $\dot{\eta}_{2,m}$  and  $\ddot{\eta}_{2,m}$ .

ESTIMATE ( $I_\alpha^k$ ) This estimate follows from an iterative procedure.

Let us fix  $\alpha(0) = m_0$  and assume that, for some  $k \in \mathbb{N}^*$ , the estimate ( $I_\alpha^k$ ) holds true.

By  $W^{2,p}(\Omega)$  elliptic regularity theorem, there holds

$$\|\eta_{k+1,m}\|_{W^{2,p}(\Omega)} \lesssim \|\eta_{k+1,m}\|_{L^p(\Omega)} + \left\| (m - 2m_0)\eta_{k,m} - \sum_{\ell=1}^{k-1} \eta_{\ell,m}\eta_{k-\ell,m} \right\|_{L^p(\Omega)}.$$

One thus gets from the induction hypothesis

$$\left\| (m - 2m_0)\eta_{k,m} - \sum_{\ell=1}^{k-1} \eta_{\ell,m}\eta_{k-\ell,m} \right\|_{L^p(\Omega)} \lesssim \kappa\alpha(k) + \sum_{\ell=0}^{k-1} \alpha(\ell)\alpha(k-\ell).$$

Moreover, using that  $\|\eta_{k+1,m}\|_{L^p(\Omega)} \leq \|\hat{\eta}_{k+1,m}\|_{L^p(\Omega)} + |\beta_{k+1,m}|$  and the  $L^p$ -Poincaré-Wirtinger inequality (see Section 2.1.3), we get

$$\|\hat{\eta}_{k+1,m}\|_{L^p(\Omega)} \lesssim \|\nabla\hat{\eta}_{k+1,m}\|_{L^p(\Omega)}.$$

We now use the result from [53, Theorem 1.1] recalled in the introduction: it readily yields

$$\|\nabla\eta_{k+1,m}\|_{L^\infty(\Omega)} \lesssim \|\eta_{k,m}\|_{L^\infty(\Omega)} + \left\| \sum_{\ell=0}^k \eta_{\ell,m}\eta_{k-\ell,m} \right\|_{L^\infty(\Omega)} \lesssim \sum_{\ell=0}^k \alpha(\ell)\alpha(k-\ell).$$

The term  $\beta_{k+1,m}$  is controlled similarly, so that

$$|\beta_{k+1,m}| \lesssim \sum_{\ell=0}^k \alpha(\ell)\alpha(k-\ell) + \sum_{\ell=1}^k \alpha(\ell)\alpha(k+1-\ell).$$

Since it is clear that the sequence  $\{\alpha(k)\}_{k \in \mathbb{N}}$  can be assumed to be increasing, we write

$$\begin{aligned} \sum_{\ell=0}^k \alpha(\ell)\alpha(k-\ell) + \sum_{\ell=1}^k \alpha(\ell)\alpha(k+1-\ell) &= \sum_{\ell=0}^{k-1} \alpha(k-\ell)(\alpha(\ell) + \alpha(\ell+1)) + \alpha(0)\alpha(k) \\ &\lesssim \sum_{\ell=0}^{k-1} \alpha(\ell+1)\alpha(k-\ell). \end{aligned}$$

Under this assumption, one has

$$\|\eta_{k+1,m}\|_{L^\infty(\Omega)} \leq |\beta_{k+1,m}| + \|\hat{\eta}_{k+1,m}\|_{L^\infty(\Omega)} \lesssim \sum_{\ell=0}^k \alpha(\ell+1)\alpha(k-\ell).$$

This reasoning guarantees the existence of a constant  $C_1$ , depending only on  $\Omega$ ,  $\kappa$  and  $m_0$ , such that the sequence defined recursively by  $\alpha(0) = m_0$  and

$$\alpha(k+1) = C_1 \sum_{\ell=0}^{k-1} \alpha(\ell+1)\alpha(k-\ell)$$

satisfies the estimate  $(I_\alpha^k)$ .

Setting  $a_k = \alpha(k)/C_1^k$  for all  $k \in \mathbb{N}$ , we know that  $\{a_k\}_{k \in \mathbb{N}}$  is a shifted Catalan sequence (see [161]), and therefore, the power series  $\sum \alpha(k)x^k$  has a positive convergence radius.

ESTIMATES  $(I_\sigma^k)$  AND  $(I_\gamma^k)$ . Obviously, one can assume that  $\sigma(0) = \gamma(0) = 0$ . One again, we work by induction, by assuming these two estimates known at a given  $k \in \mathbb{N}$ . Since  $(I_\sigma^k)$  is an estimate on the  $L^2(\Omega)$ -norm of the gradient of  $\dot{\eta}_{k+1,m}$ , it suffices to deal with  $\dot{\eta}_{k+1,m}$ . According to the Poincaré-Wirtinger inequality, one has  $\int_\Omega |\dot{\eta}_{k+1,m}|^2 \lesssim \int_\Omega |\nabla \dot{\eta}_{k+1,m}|^2$ . Now, using the weak formulation of the equations on  $\dot{\eta}_{k+1,m}$  and  $\dot{\eta}_{1,m}$ , as well as the uniform boundedness of  $\|h\|_{L^\infty(\Omega)}$ , we get

$$\begin{aligned} \int_\Omega |\nabla \dot{\eta}_{k+1,m}|^2 &= \int_\Omega (m - 2m_0)\dot{\eta}_{k,m}\dot{\eta}_{k+1,m} - 2 \sum_{\ell=1}^{k-1} \int_\Omega \eta_{k-\ell,m}\dot{\eta}_{\ell,m}\dot{\eta}_{k+1,m} + \int_\Omega h\eta_{k,m}\dot{\eta}_{k+1,m} \\ &\lesssim \|\dot{\eta}_{k,m}\|_{L^2(\Omega)}\|\dot{\eta}_{k+1,m}\|_{L^2(\Omega)} + \sum_{\ell=1}^k \alpha(k-\ell)\|\dot{\eta}_{k+1,m}\|_{L^2(\Omega)}\|\dot{\eta}_{\ell,m}\|_{L^2(\Omega)} + \\ &\quad \int_\Omega \eta_{k,m}\langle \nabla \dot{\eta}_{1,m}, \nabla \dot{\eta}_{k+1,m} \rangle + \int_\Omega \dot{\eta}_{k+1,m}\langle \nabla \dot{\eta}_{1,m}, \nabla \eta_{k,m} \rangle \\ &\lesssim \|\nabla \dot{\eta}_{k+1,m}\|_{L^2(\Omega)}\|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)} \left( \gamma(k) + \sum_{\ell=1}^k \alpha(k-\ell)\gamma(\ell) + \alpha(k) + \alpha(k) \right) \\ &\lesssim \|\nabla \dot{\eta}_{k+1,m}\|_{L^2(\Omega)}\|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)} \left( \gamma(k) + \sum_{\ell=0}^k \alpha(k-\ell)\alpha(\ell) \right), \end{aligned}$$

where the constants appearing in these inequalities only depend on  $\Omega$ ,  $\kappa$  and  $m_0$ . It follows that there exists a constant  $C_2$  such that, by setting for all  $k \in \mathbb{N}$ ,

$$\sigma(k+1) = C_2 \left( \gamma(k) + \sum_{\ell=0}^k \alpha(k-\ell)\alpha(\ell) \right),$$

the inequality  $(I_\sigma^k)$  is satisfied at rank  $k + 1$ .

Let us now state the estimate  $(I_\gamma^k)$ . By using the Poincaré-Wirtinger inequality, one gets

$$\begin{aligned} \left| \dot{\beta}_{k+1,m} \right| &= \left| \frac{1}{m_0} \int_{\Omega} (h\hat{\eta}_{k+1,m} + m\dot{\eta}_{k+1,m}) - \frac{2}{m_0} \sum_{\ell=1}^k \int_{\Omega} \dot{\eta}_{\ell,m} \eta_{k+1-\ell,m} \right| \\ &\lesssim \int_{\Omega} \langle \nabla \dot{\eta}_{1,m}, \nabla \hat{\eta}_{k+1,m} \rangle + \|\nabla \dot{\eta}_{k+1,m}\|_{L^2(\Omega)} + \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)} \sum_{\ell=1}^k \gamma(\ell) \alpha(k+1-\ell) \\ &\lesssim \alpha(k+1) \|\nabla \eta_{1,m}\|_{L^2(\Omega)} + \sigma(k+1) \|\nabla \eta_{1,m}\|_{L^2(\Omega)} + \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)} \sum_{\ell=1}^k \gamma(\ell) \alpha(k+1-\ell). \end{aligned}$$

Once again, since all the constants appearing in the inequalities depend only on  $\Omega$ ,  $\kappa$  and  $m_0$ , we infer that one can choose  $C_3$  such that, by setting

$$\gamma(k+1) = C_3 \left( \sigma(k+1) + \alpha(k+1) + \sum_{\ell=1}^k \gamma(\ell) \alpha(k+1-\ell) \right),$$

the estimate  $(I_\gamma^k)$  is satisfied. Notice that, by bounding each term  $\alpha(\ell)$ ,  $\ell \leq k$  by  $\alpha(k)$  and by using the explicit formula for  $\sigma(k+1)$ , there exists a constant  $C_4$  depending only on  $\Omega$ ,  $\kappa$  and  $m_0$  such that

$$\gamma(k+1) \leq C_4 \sum_{\ell=0}^k \alpha(k+1-\ell) (\gamma(\ell) + \alpha(\ell)).$$

Under this form, the same arguments as previously guarantee that the associated power series has a positive convergence radius.

ESTIMATE  $(I_\gamma^k)$  This is a simple consequence of the Sobolev embedding  $W^{1,2}(\Omega) \hookrightarrow L^{2+\delta}(\Omega)$ . Let  $C_\delta > 0$  be such that, for any  $u \in W^{1,2}(\Omega)$ ,

$$\|u\|_{L^{2+\delta}(\Omega)} \leq C_\delta \|u\|_{W^{1,2}(\Omega)}. \quad (2.36)$$

Then, Estimates  $(I_\sigma^k)$  and  $(I_\gamma^k)$  rewrite

$$\|\dot{\eta}_{k,m}\|_{W^{1,2}(\Omega)} \leq (\sigma(k) + \gamma(k)) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}.$$

and setting

$$\tilde{\gamma}(k) = C_\delta (\sigma(k) + \gamma(k))$$

concludes the proof of Estimate  $(I_\gamma^k)$ .

ESTIMATES  $(I_\delta^k)$  AND  $(I_\varepsilon^k)$ . For the sake of clarity, let us recall that  $k \in \mathbb{N}$  being fixed, according to Systems (2.33) and (2.34), the functions  $\dot{\eta}_{k\omega,m}$  and  $\ddot{\eta}_{k,m}$  satisfy respectively

$$\Delta \dot{\eta}_{k+1,m} + (m - 2m_0) \dot{\eta}_{k,m} - 2 \sum_{\ell=1}^{k-1} \dot{\eta}_{\ell,m} \eta_{k-\ell,m} = -h \eta_{k,m} \text{ in } \Omega$$

and

$$\Delta \ddot{\eta}_{k+1,m} + (m - 2m_0) \ddot{\eta}_{k,m} - 2 \sum_{\ell=1}^{k-1} \ddot{\eta}_{\ell,m} \eta_{k-\ell,m} = 2 \left( \sum_{\ell=1}^{k-1} \dot{\eta}_{\ell,m} \dot{\eta}_{k-\ell,m} - h \eta_{k,m} \right) \text{ in } \Omega$$

As previously, we first set  $\delta(0) = \varepsilon(0) = 0$  and argue by induction.

To prove these estimates, let us first control  $\|\nabla \ddot{\eta}_{k+1,m}\|_{L^1(\Omega)}$ . To this aim, let us use Estimates  $(I_{\tilde{\gamma}}^k)$ , the Stampacchia regularity Estimate (2.14) and the Lions-Magenes regularity Estimate (2.18). We first use the equation

$$\Delta \dot{\eta}_{1,m} + m_0 h = 0$$

to split the equation on  $\ddot{\eta}_{k+1,m}$  in System (2.34) as follows:

$$h \dot{\eta}_{k,m} = -\frac{1}{m_0} \dot{\eta}_{k,m} \Delta \dot{\eta}_{1,m} = -\frac{1}{m_0} \left( \operatorname{div} (\dot{\eta}_{k,m} \nabla \dot{\eta}_{1,m}) - \langle \dot{\nabla} \eta_{k,m}, \dot{\nabla} \eta_{1,m} \rangle \right).$$

Introduce the function

$$H_k = (m - 2m_0) \ddot{\eta}_{k,m} - 2 \sum_{\ell=1}^{k-1} \ddot{\eta}_{\ell,m} \eta_{k-\ell,m} - 2 \sum_{\ell=1}^{k-1} \dot{\eta}_{\ell,m} \dot{\eta}_{k-\ell,m} + \frac{2}{m_0} \langle \dot{\nabla} \eta_{k,m}, \dot{\nabla} \eta_{1,m} \rangle$$

then  $\ddot{\eta}_{k+1,m}$  solves

$$\Delta \ddot{\eta}_{k+1,m} + H_k = \frac{2}{m_0} \operatorname{div} (\dot{\eta}_{k,m} \nabla \dot{\eta}_{1,m}),$$

along with Neumann boundary conditions, according to (2.35).

By using the induction assumption and the Cauchy-Schwarz inequality, one gets

$$\|H_k\|_{L^1(\Omega)} \lesssim \left( \varepsilon(k) + \sum_{\ell=1}^{k-1} \varepsilon(\ell) \alpha(k-\ell) + \sum_{\ell=1}^{k-1} \gamma(\ell) \gamma(k-\ell) + \gamma(1) \gamma(k) \right) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}^2.$$

Furthermore, let us consider the same number  $\delta > 0$  as the one introduced and used in Estimate  $(I_{\tilde{\gamma}}^k)$ , and define  $r > 1$  such that  $\frac{1}{r} = \frac{1}{2} + \frac{1}{2+\delta}$ , where  $\delta > 0$  is fixed so that (2.36) holds true. By combining Estimate  $(I_{\tilde{\gamma}}^k)$  with the Hölder's inequality, we have

$$\|\dot{\eta}_{k,m} \nabla \dot{\eta}_{1,m}\|_{L^r} \leq \|\dot{\eta}_{k,m}\|_{L^{2+\delta}(\Omega)} \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)} \leq \tilde{\gamma}(k) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}^2. \quad (2.37)$$

Let us introduce  $(\psi_{k+1}, \xi_{k+1})$  as the respective solutions of

$$\begin{cases} \Delta \psi_{k+1} + H_k = 0 & \text{in } \Omega, \\ \frac{\partial \psi_{k+1}}{\partial \nu} = 0 & \text{on } \partial \Omega, \\ \int_{\Omega} \psi_{k+1} = 0, \end{cases} \quad (2.38)$$

and

$$\begin{cases} \Delta \xi_{k+1} = -2 \operatorname{div} (\dot{\eta}_{k,m} \nabla \dot{\eta}_{1,m}) & \text{in } \Omega, \\ \frac{\partial \xi_{k+1}}{\partial \nu} = 0 & \text{on } \partial \Omega, \\ \int_{\Omega} \xi_{k+1} = 0, \end{cases} \quad (2.39)$$

so that  $\ddot{\eta}_{k+1,m} = \psi_{k+1} + \xi_{k+1}$ . Stampacchia's Estimate (2.14) leads to

$$\|\nabla \psi_{k+1}\|_{L^1(\Omega)} \lesssim \|H_k\|_{L^1(\Omega)} \lesssim \left( \varepsilon(k) + \sum_{\ell=1}^{k-1} \varepsilon(\ell) \alpha(k-\ell) + \sum_{\ell=1}^{k-1} \gamma(\ell) \gamma(k-\ell) + \gamma(1) \gamma(k) \right) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}^2,$$

and moreover,

$$\begin{aligned} \|\nabla \xi_{k+1}\|_{L^1(\Omega)} &\lesssim \|\nabla \xi_{k+1}\|_{L^r(\Omega)} \text{ by Hölder's inequality} \\ &\lesssim \|\dot{\eta}_{k,m} \nabla \dot{\eta}_{1,m}\|_{L^r(\Omega)} \text{ by Lions and Magenes Estimate (2.18)} \\ &\lesssim \tilde{\gamma}(k) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}^2 \text{ by Estimate (2.37)}. \end{aligned}$$

We then have

$$\begin{aligned} \|\nabla \ddot{\eta}_{k+1,m}\|_{L^1(\Omega)} &= \|\nabla \psi_{k+1} + \nabla \xi_{k+1}\|_{L^1(\Omega)} \\ &\lesssim \left( \varepsilon(k) + \sum_{\ell=1}^{k-1} \varepsilon(\ell) \alpha(k-\ell) + \sum_{\ell=1}^{k-1} \gamma(\ell) \gamma(k-\ell) + \gamma(1) \gamma(k) + \tilde{\gamma}(k) \right) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}^2. \end{aligned}$$

and we conclude by setting  $\delta(k+1) = \varepsilon(k) + \sum_{\ell=1}^{k-1} \varepsilon(\ell) \alpha(k-\ell) + \sum_{\ell=1}^{k-1} \gamma(\ell) \gamma(k-\ell) + \gamma(1) \gamma(k) + \tilde{\gamma}(k)$ .

Let us now derive  $\varepsilon(k+1)$ . We proceed similarly to the proof of Estimate  $(I_\gamma^k)$ : from the Poincaré-Wirtinger Inequality, there holds

$$\left\| \dot{\eta}_{k+1,m} - \fint_{\Omega} \dot{\eta}_{k+1,m} \right\|_{L^2(\Omega)} \lesssim \|\nabla \dot{\eta}_{k+1,m}\|_{L^2(\Omega)}$$

so that, from Estimate  $(I_\delta^k)$  it suffices to control  $\fint_{\Omega} \ddot{\eta}_{k+1,m}$ . Starting from the expression

$$\fint_{\Omega} \ddot{\eta}_{k+1,m} = \frac{1}{m_0} \fint_{\Omega} (2h\dot{\eta}_{k+1,m} + m\ddot{\eta}_{k+1,m}) - \frac{2}{m_0} \sum_{\ell=1}^k \fint_{\Omega} (\dot{\eta}_{\ell,m} \dot{\eta}_{k+1-\ell,m} + \ddot{\eta}_{\ell,m} \eta_{k+1-\ell,m})$$

stated in (2.34) and using the Cauchy-Schwarz inequality, one gets

$$\begin{aligned} \left| \fint_{\Omega} \ddot{\eta}_{k+1,m} \right| &= \left| \frac{1}{m_0} \fint_{\Omega} (2h\dot{\eta}_{k+1,m} + m\ddot{\eta}_{k+1,m}) - \frac{2}{m_0} \sum_{\ell=1}^k \fint_{\Omega} (\dot{\eta}_{\ell,m} \dot{\eta}_{k+1-\ell,m} + \ddot{\eta}_{\ell,m} \eta_{k+1-\ell,m}) \right| \\ &\lesssim \left| \fint_{\Omega} h\dot{\eta}_{k+1,m} \right| + \|\ddot{\eta}_{k+1,m}\|_{L^2(\Omega)} + \sum_{\ell=1}^k \|\dot{\eta}_{\ell,m}\|_{L^2(\Omega)} \|\dot{\eta}_{k+1-\ell,m}\|_{L^2(\Omega)} \\ &\quad + \sum_{\ell=1}^k \|\ddot{\eta}_{\ell,m}\|_{L^2(\Omega)} \|\eta_{k+1-\ell,m}\|_{L^2(\Omega)} \end{aligned}$$

We then use Equation (2.26) to get

$$\fint_{\Omega} h\dot{\eta}_{k+1,m} = \frac{1}{m_0} \fint_{\Omega} \langle \nabla \dot{\eta}_{1,m}, \nabla \dot{\eta}_{k+1,m} \rangle \leq \frac{1}{m_0} \sigma(1) \sigma(k+1) \|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}^2.$$

Since  $\ddot{\beta}_{1,m}$  is proportional to  $\|\nabla \dot{\eta}_{1,m}\|_{L^2(\Omega)}^2$ , this gives

$$\left| \fint_{\Omega} \ddot{\eta}_{k+1,m} \right| \lesssim \left( \sigma(1) \sigma(k+1) + \delta(k+1) + \sum_{\ell=1}^k (\gamma(\ell) \gamma(k+1-\ell) + \alpha(k+1-\ell) \varepsilon(\ell)) \right) \ddot{\beta}_{1,m}.$$

Setting  $\varepsilon(k+1) = \sigma(1) \sigma(k+1) + \delta(k+1) + \sum_{\ell=1}^k (\gamma(\ell) \gamma(k+1-\ell) + \alpha(k+1-\ell) \varepsilon(\ell))$  concludes the proof.

**Summary.** We have proved here that the functional  $\mathcal{F}_\mu$  has an asymptotic expansion of the form

$$\mathcal{F}_\mu(m) = m_0 + \frac{\beta_{1,m}}{\mu} + R_\mu(m),$$

where  $m \mapsto \beta_{1,m} = \frac{1}{m_0^2} \int_\Omega |\nabla \eta_{1,m}|^2$  is a strictly convex functional, and where  $R_\mu$  satisfies the two following conditions:

1.  $R_\mu = O\left(\frac{1}{\mu^2}\right)$  uniformly in  $\mathcal{M}_{m_0,\kappa}(\Omega)$ ,
2.  $R_\mu$  can be expanded in a power series of  $\frac{1}{\mu}$  as follows:

$$R_\mu(m) = \frac{1}{\mu} \sum_{k=2}^{\infty} \frac{\beta_{k,m}}{\mu^{k-1}}$$

3.  $R_\mu$  is twice Gâteaux-differentiable, and, for any  $m \in \mathcal{M}_{m_0,\kappa}(\Omega)$ , for any admissible variation  $h \in \mathcal{T}_{m,\mathcal{M}_{m_0,\kappa}(\Omega)}$ ,

$$\mu \left| \ddot{R}_\mu[h, h] \right| \lesssim \ddot{\beta}_{1,m}.$$

It immediately follows that the functional  $\mathcal{F}_\mu$  satisfies the following lower bound on its second derivative: for any  $m \in \mathcal{M}_{m_0,\kappa}(\Omega)$ , for any admissible variation  $h \in \mathcal{T}_{m,\mathcal{M}_{m_0,\kappa}(\Omega)}$ ,

$$\left(1 - \frac{1}{\mu}\right) \ddot{\beta}_{1,m} \lesssim \mu \ddot{\mathcal{F}}_\mu(m)[h, h], \quad (2.40)$$

so that it has a positive second derivative, according to (2.27). Hence,  $\mathcal{F}_\mu$  is strictly convex for  $\mu$  large enough.

Since the maximizers of a strictly convex functional defined on a convex set are extreme points, and that the extreme points of  $\mathcal{M}_{m_0,\kappa}(\Omega)$  are bang-bang functions, this ensures that all maximizers of  $\mathcal{F}_\mu$  are bang-bang functions.

## 2.2.4 PROOF OF THEOREM 2.1.2

In what follows, it will be convenient to introduce the functional

$$F_1 : m \mapsto \beta_{1,m} = \frac{1}{m_0^2} \int_\Omega |\nabla \eta_{1,m}|^2 = \int_\Omega \eta_{1,m}$$

where  $\eta_{1,m}$  is defined as a solution to System (2.32). The index in the notation  $F_1$  underlines the fact that  $F_1$  involves the solution  $\eta_{1,m}$ .

According to the proof of Theorem 2.1.1 (Step 1), we already know that  $F_1$  is a convex functional on  $\mathcal{M}_{m_0,\kappa}(\Omega)$ .

### 2.2.4.1 PROOF OF $\Gamma$ -CONVERGENCE PROPERTY FOR GENERAL DOMAINS

To prove this theorem, we proceed into three steps: we first prove weak convergence, then show that maximizers of the functional  $F_1$  are necessarily extreme points of  $\mathcal{M}_{m_0,\kappa}(\Omega)$  and finally recast  $F_1$  using the energy functional  $\mathcal{E}_m$ . Since weak convergence to an extreme point entails strong convergence, this will conclude the proof of the  $\Gamma$ -convergence property.



**Convergence of maximizers.** For  $\mu > 0$ , let  $m_\mu$  be a solution to  $(\mathcal{P}_\mu^n)$ . According to Theorem 2.1.1, there exists  $\mu^* > 0$  such that  $m_\mu = \kappa \chi_{E_\mu}$  for all  $\mu \geq \mu^*$ , where  $E_\mu \subset \Omega$  is such that  $|E_\mu| = m_0 \frac{|\Omega|}{\mu}$ . Since the family  $\{m_\mu\}_{\mu>0}$  is uniformly bounded in  $L^\infty(\Omega)$ , it converges up to a subsequence to some element  $m_\infty \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , weakly star in  $L^\infty(\Omega)$ . Observe that the maximizers of  $\mathcal{F}_\mu$  over  $\mathcal{M}_{m_0, \kappa}(\Omega)$  are the same as the maximizers of  $\mu(\mathcal{F}_\mu - m_0)$ . Recall that, given  $m$  in  $\mathcal{M}_{m_0, \kappa}(\Omega)$ , there holds  $\mu(\mathcal{F}_\mu(m) - m_0) = \int_\Omega \eta_{1,m} + \mathcal{O}\left(\frac{1}{\mu}\right)$  according to the proof of Theorem 2.1.1, where the notation  $\mathcal{O}\left(\frac{1}{\mu}\right)$  stands for a function uniformly bounded in  $L^\infty(\Omega)$ . In other words, we have

$$\mu(\mathcal{F}_\mu - m_0) = F_1 + \mathcal{O}\left(\frac{1}{\mu}\right)$$

with the same notation for  $\mathcal{O}\left(\frac{1}{\mu}\right)$ .

For an arbitrary  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , by passing to the limit in the inequality

$$\mu(\mathcal{F}_\mu(m_\mu) - m_0) \geq \mu(\mathcal{F}_\mu(m) - m_0)$$

one gets that  $m_\infty$  is necessarily a maximizer of the functional  $F_1$  over  $\mathcal{M}_{m_0, \kappa}(\Omega)$ .

We have shown in the proof of Theorem 2.1.1 that  $F_1$  is convex on  $\mathcal{M}_{m_0, \kappa}(\Omega)$  (Step 1). Its maximizers are thus extreme points. It follows that any weak limit of  $\{m_\mu\}_{\mu>0}$  is an extreme point to this set. Thus, the convergence is in fact strong in  $L^1$  ([93, Proposition 2.2.1]).

**“Energetic” expression of  $F_1(m)$ .** Recall that  $F_1$  is given by

$$F_1(m) = \frac{1}{m_0^2} \int_\Omega |\nabla \eta_{1,m}|^2,$$

where  $\eta_{1,m}$  solves

$$\begin{cases} \Delta \eta_{1,m} + m_0(m - m_0) = 0 & \text{in } \Omega, \\ \frac{\partial \eta_{1,m}}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ \int_\Omega \eta_{1,m} = \frac{1}{m_0^2} \int_\Omega |\nabla \eta_{1,m}|^2. \end{cases}$$

The last constraint, which is derived from the integration of Equation (LDE), by passing to the limit as  $\mu \rightarrow +\infty$ , is not so easy to handle. This is why we prefer to deal with  $\hat{\eta}_{1,m}$ , solving the same equation as  $\eta_{1,m}$  completed with the integral condition

$$\int_\Omega \hat{\eta}_{1,m} = 0.$$

Since  $\eta_{1,m}$  and  $\hat{\eta}_{1,m}$  only differ up to an additive constant, we have  $\nabla \eta_{1,m} = \nabla \hat{\eta}_{1,m}$ , so that

$$F_1(m) = \frac{1}{m_0^2} \int_\Omega |\nabla \hat{\eta}_{1,m}|^2 \quad \text{and} \quad \hat{\eta}_{1,m} \in X. \quad (2.41)$$

Regarding then the variational problem

$$\sup_{m \in \mathcal{M}(\Omega)} F_1(m), \quad (PV_1)$$

and standard reasoning on the PDE solved by  $\hat{\eta}_{1,m}$  yields that

$$F_1(m) = -2 \min_{u \in X} \mathcal{E}_m(u),$$

leading to the desired result.

#### 2.2.4.2 PROPERTIES OF MAXIMIZERS OF $F_1$ IN A TWO-DIMENSIONAL ORTHOTOPE

We investigate here the case of the two-dimensional orthotope  $\Omega = (0; a_1) \times (0; a_2)$ . In the last section, we proved that every maximizer  $m$  of  $F_1$  over  $\mathcal{M}_{m_0, \kappa}(\Omega)$  is of the form  $m = \kappa \chi_E$  where  $E$  is a measurable subset of  $\Omega$  such that  $\kappa|E| = m_0|\Omega|$ .

Let  $E^*$  be such a set. We will prove that  $E^*$  is, up to a rotation of  $\Omega$ , decreasing in every direction. It relies on the combination of symmetric decreasing rearrangements properties and optimality conditions for Problem  $(PV_1)$ .

Introduce the notation  $\hat{\eta}_{1, E^*} := \hat{\eta}_{1, \kappa \chi_{E^*}}$ . A similar reasoning to the one used in Proposition 2.3 (see e.g. [179]) yields the existence of a Lagrange multiplier  $c$  such that

$$\{\hat{\eta}_{1, E^*} > c\} = E^*, \quad \{\hat{\eta}_{1, E^*} < c\} = (E^*)^c, \quad \{\hat{\eta}_{1, E^*} = c\} = \partial E^*. \quad (2.42)$$

We already know, thanks to the equality case in the decreasing rearrangement inequality, that any maximizer  $E^*$  is decreasing or increasing in every direction.

To conclude, it remains to prove that  $E^*$  is connected. Let us argue by contradiction, by assuming that  $E^*$  has at least two connected components.

In what follows, if  $E$  denotes a measurable subset of  $\Omega$ , we will use the notation  $\hat{\eta}_{1, E} := \hat{\eta}_{1, \kappa \chi_E}$ . The steps of the proof are illustrated on Figure 2.6 below.

**Step 1:  $E^*$  has at most two components.** It is clear from the equality case in the Pölya-Szegö inequality that  $\hat{\eta}_{1, E^*}$  is decreasing in every direction (i.e, it is either nondecreasing or nonincreasing on every horizontal or vertical line).

Let  $e_1 = (0, 0), e_2 = (a_1, 0), e_3 = (a_1, a_2), e_4 = (0, a_2)$  be the four vertices of the orthotope  $\Omega = (0; a_1) \times (0; a_2)$ . Let  $E_1$  be a connected component of  $E^*$ . Since  $E_1$  is monotonic in both directions  $x$  and  $y$ , thus it necessarily contains at least one vertex. Up to a rotation, one can assume that  $e_1 \in E_1$ . Since  $E_1$  is decreasing in the direction  $y$ , there exists  $\underline{x} \in [0; a_1]$  and a non-increasing function  $f : [0; \underline{x}] \rightarrow [0; a_1]$  such that

$$E_1 = \{(x, t), x \in (0; \underline{x}), t \in [0; f(x)]\}$$

Since  $f$  is decreasing, one has  $E_1 \subseteq [0; \underline{x}] \times [0; f(0)]$ .

Let  $E_2$  be another connected component of  $E^*$ . Since  $E^*$  is monotonic in every direction, the only possibility is that  $E_2$  meet the upper corner  $[\underline{x}; a_1] \times [f(0); a_2]$ , meaning that  $e_3 \in E_2$  and therefore, there exist  $\bar{x} \in [\underline{x}; a_1]$  and a non-decreasing function  $g : [\bar{x}; a_2] \rightarrow [0; a_2]$  such that

$$E_2 = \{(x, t), x \in [\bar{x}; a_1], t \in [a_2 - g(x); a_2]\}$$

**Step 2: geometrical properties of  $E_1$  and  $E_2$ .** We are going to prove that  $g$  or  $f$  is constant and that  $\underline{x} = \bar{x}$ . Let  $b_2$  be the decreasing rearrangement in the direction  $y$ . Let  $E_* := b_2(E^*)$ .

We claim that, by optimality of  $E^*$ , we have

$$F_1(b_2(E^*)) = F_1(E^*) \text{ and } b_2(\hat{\eta}_{1, E^*}) = \hat{\eta}_{1, b_2(E^*)}. \quad (2.43)$$

For the sake of clarity, the proof of (2.43) is postponed to the end of this step.

Since  $b_2(E^*)$  is necessarily a solution of Problem  $(PV_1)$ , it follows, by monotonicity of maximizers, that the mapping  $\tilde{f} : x \in [0; a_1] \mapsto \mathcal{H}^1\left(\{x\} \times [0; a_2] \cap b_2(E^*)\right)$  is also monotonic. However, it is straightforward that  $\tilde{f} = f \chi_{[0; \underline{x}]} + g \chi_{[\bar{x}; a_1]}$ . If  $f$  is nonconstant, it follows that  $\tilde{f}$  is non-increasing. Since  $g$  is non-decreasing and has the same monotonicity as  $\tilde{f}$ , it follows that  $g$  is necessarily constant.

Hence, we get that  $\underline{x} = \bar{x}$  and that  $\inf_{[0;\underline{x}]} f$  is positive. Else,  $\tilde{f}$  would be non-increasing and vanish in  $(\underline{x}; \bar{x})$ . Finally, we also conclude that  $\inf_{[0;\underline{x}]} f \geq g$ . Thus, we can consider the following situation:  $\underline{x} = \bar{x}$ ,  $f \geq \alpha$  and  $f$  is non-increasing and  $g$  is constant, i.e  $g = \alpha$ .

*Proof of (2.43).* Recall that for every  $m \in \mathcal{M}(\Omega)$ ,  $\hat{\eta}_{1,m}$  is the unique minimizer of the energy functional  $\mathcal{E}_m$  over  $X$  where  $\mathcal{E}_m$  and  $X$  are defined by (2.9)-(2.10). For a measurable subset  $E$  of  $\Omega$ , introduce the notations  $F_1(E) := F_1(\kappa\chi_E)$  and  $\mathcal{E}_E := \mathcal{E}_{\kappa\chi_E}$ . Since  $F_1(m) = -\frac{2}{m_0^2}\mathcal{E}_m(\hat{\eta}_{1,m})$  and since  $E^*$  is a maximizer of  $F_1$ , we have

$$F_1(E^*) \geq F_1(b_2(E^*)).$$

Furthermore, one has

$$\begin{aligned} F_1(E^*) &= -\frac{2}{m_0^2}\mathcal{E}_{E^*}(\hat{\eta}_{1,E^*}) \leq -\frac{2}{m_0^2}\mathcal{E}_{b_2(E^*)}(b_2(\hat{\eta}_{1,E^*})) \\ &\leq -\frac{2}{m_0^2}\mathcal{E}_{b_2(E^*)}(\hat{\eta}_{1,b_2(E^*)}) = F_1(b_2(E^*)). \end{aligned}$$

by using successively the Hardy-Littlewood and Pölya-Szegö inequalities.

Thus, all these inequalities are in fact equality, which implies that  $b_2(E^*)$  is also a maximizer of  $F_1$  over  $\mathcal{M}(\Omega)$ . Furthermore, by the equimeasurability property, one has  $b_2(\hat{\eta}_{1,E^*}) \in X$ , so that  $b_2(\hat{\eta}_{1,E^*})$  is a minimizer of  $\mathcal{E}_{b_2(E^*)}$  over  $X$ . The conclusion follows.  $\square$

**Step 3:  $E^*$  has at most one component.** To get a contradiction, let us use the optimality conditions (2.42). This step is illustrated on the bottom of Figure 2.6.

By using the aforementioned properties of maximizers, we get that  $\eta_{1,b_2(E^*)}$  is constant and equal to  $c$  on  $\{\underline{x}\} \times [0; \alpha] \subset b_2(E^*)$ :

$$\hat{\eta}_{1,b_2(E^*)} = c \text{ on } \{\underline{x}\} \times [0; \alpha]. \quad (2.44)$$

Furthermore, since  $b_2(E^*)$  is a maximizer of  $F_1$ , it follows that  $\hat{\eta}_{1,b_2(E^*)}$  is constant on  $\partial b_2(E^*)$ . But one has  $b_2(\hat{\eta}_{1,E^*}) = \hat{\eta}_{1,E^*}$  on  $[0; \underline{x}] \times [0; a_2]$  since  $\hat{\eta}_{1,E^*}$  is decreasing in the vertical direction on this subset. We get that  $\hat{\eta}_{1,b_2(E^*)}$  is equal to  $c$  on  $\partial b_2(E^*)$

However, by the strict maximum principle,  $\hat{\eta}_{1,b_2(E^*)}$  cannot reach its minimum in  $b_2(E^*)$ , which is a contradiction with (2.44). This concludes the proof.

### 2.2.5 PROOF OF THEOREM 2.1.3

As a preliminary remark, we claim that the function  $\theta_{\tilde{m},\mu}$  solving (LDE) with  $m = \tilde{m}$  is positive increasing. Indeed, recall that  $\theta_{\tilde{m},\mu}$  is the unique minimizer of the energy functional

$$\mathcal{E} : W^{1,2}(\Omega, \mathbb{R}_+) \ni u \mapsto \frac{\mu}{2} \int_0^1 u'^2 - \frac{1}{2} \int_0^1 m^* u^2 + \frac{1}{3} \int_0^1 u^3. \quad (2.45)$$

By using the rearrangement inequalities recalled in Section 2.1.3 and the relation  $(\tilde{m})_{br} = \tilde{m}$ , one easily shows that

$$\mathcal{E}(\theta_{\tilde{m},\mu}) \geq \mathcal{E}((\theta_{\tilde{m},\mu})_{br}),$$

and therefore, one has necessarily  $\theta_{\tilde{m},\mu} = (\theta_{\tilde{m},\mu})_{br}$  by uniqueness of the steady-state (see Section 2.1.1). Hence,  $\theta_{\tilde{m},\mu}$  is non-decreasing. Moreover, according to (LDE),  $\theta_{\tilde{m},\mu}$  is convex on  $(0, 1 - \ell)$  and concave on  $(1 - \ell, 1)$  which, combined with the boundary conditions on  $\theta_{\tilde{m},\mu}$ , justifies the positiveness of its derivative. The expected result follows.

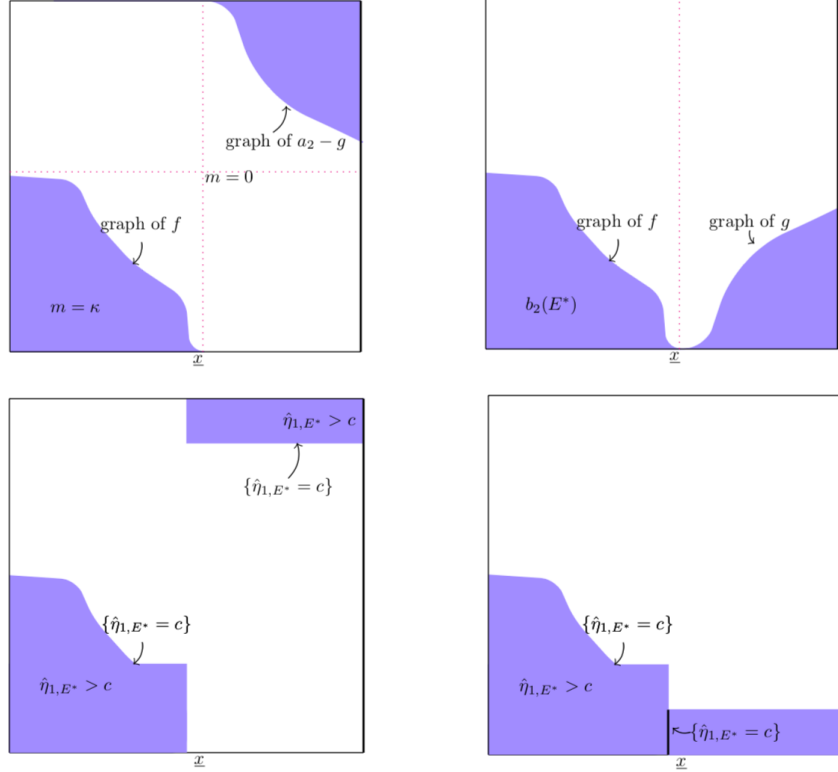


Figure 2.6 – Illustration of the proof and the notations used.

**Step 1: convergence of sequences of maximizers.** As a consequence of Theorem 2.1.2, we get that the functions  $\tilde{m} = \kappa\chi_{(0,\ell)}$  or  $\tilde{m}(1 - \cdot) = \kappa\chi_{(1-\ell,1)}$  are the only closure points of the family  $(m_\mu)_{\mu>0}$  for the  $L^1(0,1)$  topology.

**Step 2: asymptotic behaviour of  $p_{m,\mu}$  and of  $\varphi_{m,\mu}$**  We claim that, as done for the solution  $\theta_{m,\mu}$  of (LDE), the following asymptotic behaviour for the adjoint state  $p_{m,\mu}$

$$p_{m,\mu} = -\frac{1}{m_0} + \mathcal{O}\left(\frac{1}{\mu}\right), \quad \text{in } W^{2,2}(0,1),$$

by using Sobolev embeddings. In particular, this expansion holds in  $\mathcal{C}^1([0,1])$ .

Introduce the function  $z_\mu = \mu(\varphi_{m_\mu,\mu} + 1)$ . Using the convergence results established in the previous steps, in particular that  $(m_\mu)_{\mu>0}$  converges to  $\tilde{m}$  in  $L^1(0,1)$  and that  $\varphi_{m_\mu,\mu} = -1 + \mathcal{O}\left(\frac{1}{\mu}\right)$  uniformly in  $\mathcal{C}^{1,\alpha}([0,1])$ <sup>2</sup> as  $\mu \rightarrow +\infty$ , one infers that  $(z_\mu)_{\mu>0}$  is uniformly bounded in  $\mathcal{C}^{1,\alpha}([0,1])$  and converges, up to a subsequence to  $z_\infty$  in  $\mathcal{C}^1([0,1])$ , where  $z_\infty$  satisfies in particular

$$z_\infty'' + 2(m_0 - \tilde{m}) = 0,$$

with Neumann Boundary conditions in the  $W^{1,2}$  sense.

<sup>2</sup>This is obtained similarly to the proof's technique of theorem 2.1.1, using elliptic estimates and Sobolev embedding for the functions  $\theta_{m,\mu}$  and  $p_{m,\mu}$ .

**Conclusion:**  $m_\mu = \tilde{m}$  or  $\tilde{m}(1 - \cdot)$  whenever  $\mu$  is large enough. According to Theorem 2.1.1 and Proposition 2.3, we know at this step that for  $\mu$  large enough, there exists  $c_\mu \in \mathbb{R}$  such that

$$\{\varphi_{m_\mu, \mu} > c_\mu\} = \{m_\mu = 0\}, \quad \{\varphi_{m_\mu, \mu} < c_\mu\} = \{m_\mu = \kappa\}.$$

We will show that, provided that  $\mu$  be large enough, one has necessarily  $m_\mu = \tilde{m}$  or  $m_\mu = \tilde{m}(1 - \cdot)$ . Since  $\tilde{m} = \kappa$  in  $(0, \ell)$ , it follows that  $z_\infty$  is strictly convex on this interval and since  $z'_\infty(0) = 0$ , one has necessarily  $z'_\infty > 0$  in  $(0, \ell)$ . Similarly, by concavity of  $z_\infty$  in  $(\ell, 1)$ , one has  $z'_\infty > 0$  in this interval.

Furthermore, let us introduce  $d_\mu = \mu(c_\mu + 1)$ . Since  $(z_\mu)_{\mu > 0}$  is bounded in  $C^0((0, 1))$ ,  $(d_\mu)_{\mu > 0}$  converges up to a subsequence to some  $d_\infty$ . By monotonicity of  $z_\infty$  and a compactness argument, there exists a unique  $x_\infty \in [0, \ell]$  such that  $z_\infty(x_\infty) = d_\infty$ . The dominated convergence theorem hence yields

$$|\{z_\infty \leq d_\infty\}| = \kappa\ell, \quad |\{z_\infty \geq d_\infty\}| = \kappa(1 - \ell),$$

and the the aforementioned local convergence results yield

$$\{z_\infty > d_\infty\} \subset \{\tilde{m} = 0\}, \quad \{z_\infty < d_\infty\} \subset \{\tilde{m} = \kappa\}.$$

Hence, the inclusions are equalities (the equality of sets must be understood up to a zero Lebesgue measure set) by using that  $z_\infty$  is increasing.

Moreover, since  $z_\infty$  is increasing, one has  $z_\infty(0) < d_\infty$  and  $z_\infty(1) > d_\infty$ . Since the family  $(z_\mu)_{\mu > 0}$  is uniformly Lipschitz-continuous, there exists  $\varepsilon > 0$  such that for  $\mu$  large enough, there holds

$$z_\mu < d_\mu \text{ in } (0, \varepsilon), \quad z_\mu > d_\mu \text{ in } (1 - \varepsilon, 1), \quad z'_\mu > 0 \text{ in } (\varepsilon, 1 - \varepsilon).$$

This implies the existence of  $x_\mu \in (0, 1)$  such that

$$\{z_\mu < d_\mu\} = [0, x_\mu] \quad \text{and} \quad \{z_\mu > d_\mu\} = (x_\mu, 1],$$

whence the result.

## 2.2.6 PROOF OF THEOREM 2.1.4

Let  $\kappa > 0, m_0 > 0$ , and  $\tilde{m} := \kappa\chi_{[1-\ell, 1]}$  with  $\ell = \frac{m_0}{\kappa}$ , i.e the single crenel distribution.

In order to prove this result, as the function  $\mu > 0 \mapsto \mathcal{F}_\mu(\tilde{m}(2 \cdot))$  has a first local maximizer ([125, Theorem 1.2, Remark 1.4]), we define  $\mu_1$  as its first local maximizer. One gets from a simple change of variables that  $\theta_{\tilde{m}, \mu_1}(2x) = \theta_{\tilde{m}(2 \cdot), \mu_1/4}(x)$  for all  $x \in \Omega$  and thus one has

$$F_{\mu_1}(\tilde{m}) = F_{\frac{\mu_1}{4}}(\tilde{m}(2 \cdot))$$

But our choice of  $\mu_1$  yields that  $\mu \mapsto \mathcal{F}_\mu(\tilde{m}(2 \cdot))$  is increasing on  $(0, \mu_1)$  and thus:

$$F_{\mu_1}(\tilde{m}) = F_{\frac{\mu_1}{4}}(\tilde{m}(2 \cdot)) < F_{\mu_1}(\tilde{m}(2 \cdot)). \quad (2.46)$$

## 2.3 CONCLUSION AND FURTHER COMMENTS

### 2.3.1 ABOUT THE 1D CASE

Let us assume in this section that  $n = 1$  and  $\Omega = (0, 1)$ . We provide hereafter several numerical simulations based on the primal formulation of the optimal design problem  $(\mathcal{P}_\mu^n)$ : on Fig. 2.7, we investigate the general problem  $(\mathcal{P}_\mu^n)$  and we plot the optimal  $m$  determined numerically for several values of  $\mu$ .

These simulations were obtained with an interior point method applied to the optimal control problem  $(\mathcal{P}_\mu^n)$ . We used a Runge-Kutta method of order 4 to discretize the underlying differential equations. The control  $m$  has been also discretized, which has allowed to reduce the optimal control problem to some finite dimensional minimization problem with constraints. We used the code IPOPT (see [176]) combined with AMPL (see [77]) on a standard desktop machine. We considered a regular subdivision of  $(0, 1)$  with  $N$  points, where the order of magnitude of  $N$  is  $1000/\mu$ . The resulting code works out the solution quickly (around 5 to 10 seconds depending on the choice of the parameter  $\mu$ ).

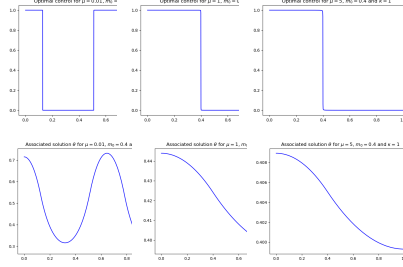


Figure 2.7 –  $m_0 = 0.4$ ,  $\kappa = 1$ . From left to right:  $\mu = 0.01, 1, 5$ . Top: plot of the optimal solution of Problem  $(\mathcal{P}_\mu^n)$  computed with the help of an interior point method. Bottom: plot of the corresponding eigenfunction.

In the cases mentioned above, the algorithm is initialized with several choices of function  $m$ , among which the optimal simple crenel as  $\mu$  is large enough. If  $\mu$  is equal to 1 or 5, the simple crenel is obtained at convergence. Nevertheless, in the case  $\mu = 0.01$ , we obtain a “symmetric” double crenel (in accordance with Theorem 2.1.4) at convergence.

Although we have no guarantee to obtain optimal solutions by using this numerical approach, we checked that a simple crenel is better than a double one in the cases  $\mu = 1, 5$  whereas we observe the contrary in the case  $\mu = 0.01$ .

Notice that we encountered a problem when dealing with too small values of  $\mu$  (for instance  $\mu = 0.001$ ). Indeed, in that case, the stiffness of the discretized system seems to become huge as  $\mu$  takes small positive values and makes the numerical computations hard to converge. Improvements of the numerical method should be found for further numerical investigations.

### 2.3.2 COMMENTS AND OPEN ISSUES

It is also interesting, from a biological point of view, to investigate a more general version of Problem  $(\mathcal{P}_\mu^n)$  for changing-sign weights. In that case, the admissible class of weights is then transformed (for instance) into

$$\widetilde{\mathcal{M}}_{m_0, \kappa}(\Omega) = \left\{ m \in L^\infty(\Omega), m \in [-1; \kappa] \text{ a.e and } \int_{\Omega} m = m_0 \right\},$$

with  $m_0 \in (0, 1)$  (so that  $\lambda_1(m, \mu) > 0$  and Equation (LDE) is well-posed). We claim that the main results of this article can be extended without effort to this new framework and that we will still obtain the bang-bang character of maximizers provided that  $\mu$  be large enough. Such a class has also been considered in the context of principal eigenvalue minimization (see [99, 116]).

Finally, we end this section by providing some open problems for which we did not manage to bring complete answer and that deserve and remain, to our opinion, to be investigated. They are in order:

- (for general domains  $\Omega$ ) we conjecture that maximizers are bang-bang functions for any  $\mu > 0$ .

As outlined in the introduction, this conjecture is supported by Theorem 2.1.1 and the main result of [148].

- (for general domains  $\Omega$ ) use the main results of the present article to determine numerically the maximizer  $m^*$  with the help of an adapted shape optimization algorithm;
- (for  $\Omega = (0; 1)$ ) given that, for  $\mu$  small enough, the optimal configurations for  $\lambda_1(\cdot, \mu)$  and  $\mathcal{F}_\mu$  are not equal, it would be natural and biologically relevant to try to maximize a convex combination of  $\mathcal{F}_\mu$  and  $\lambda_1(\cdot, \mu)$ .
- (for general domains  $\Omega$ ) investigate the asymptotic behavior of maximizer as the parameter  $\mu$  tends to 0? Such a issue appears intricate since it requires a refine study of singular limits for Problem (LDE).

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# APPENDIX

## 2.A CONVERGENCE OF THE SERIES

Let  $\frac{1}{\mu_1^*}$  be the minimum of the convergence radii associated to the power series  $\sum \alpha(k)x^k$ ,  $\sum \sigma(k)x^k$ ,  $\sum \gamma(k)x^k$ ,  $\sum \delta(k)x^k$  and  $\sum \varepsilon(k)x^k$  introduced in the proof of Theorem 2.1.1.

We will show that, whenever  $\mu \geq \mu_1^*$ , the following expansions

$$\sum_{\ell=0}^{+\infty} \frac{\eta_{k,m}}{\mu^k} = \theta_{m,\mu}, \quad \sum_{k=1}^{+\infty} \frac{\dot{\eta}_{k,m}}{\mu^k} = \dot{\theta}_{m,\mu}, \quad \sum_{k=1}^{+\infty} \frac{\ddot{\eta}_{k,m}}{\mu^k} = \ddot{\theta}_{m,\mu}$$

make sense in  $L^2(\Omega)$ . Since the proofs for the series defining  $\dot{\theta}_{m,\mu}$  and  $\ddot{\theta}_{m,\mu}$  are exactly similar to the one for  $\theta_{m,\mu}$ , we only concentrate on the expansion of  $\theta_{m,\mu}$ . By construction, the series  $g_{\infty,\mu} := \sum_{\ell=0}^{+\infty} \frac{\eta_{k,m}}{\mu^k}$  converges in  $W^{1,2}(\Omega)$  to a function  $g_{\infty,\mu}$ . We need to show that  $g_{\infty,\mu} = \theta_{m,\mu}$ .

To this aim, let us set

$$g_{N,\mu} := \sum_{k=0}^N \frac{\eta_{k,m}}{\mu^k}$$

for any  $N \in \mathbb{N}^*$ . Notice that  $g_{N,\mu}$  solves the equation

$$\mu \Delta g_{N,\mu} + g_{N-1,\mu} m - \sum_{k=0}^N \frac{\eta_{k,m}}{\mu^k} g_{N-k,\mu} = 0, \quad \text{in } \Omega \quad (2.47)$$

with Neumann boundary conditions.

In order to pass to the limit  $N \rightarrow \infty$ , one has to determine the limit of  $\tilde{g}_{N,\mu} := \sum_{k=0}^N \frac{\eta_{k,m}}{\mu^k} g_{N-k,\mu}$ . First note that the Cauchy-Schwarz inequality proves the absolute convergence of the sequence  $\{\tilde{g}_{N,\mu}\}_{N \in \mathbb{N}}$  in  $W^{1,2}(\Omega)$  as  $N \rightarrow \infty$ . Let  $H$  denote its limit. Now, let us show that

$$\tilde{g}_{N,\mu} \xrightarrow{N \rightarrow \infty} g_{\infty,\mu}^2 \quad \text{in } L^2(\Omega),$$

whenever  $\mu$  is large enough. Let  $R_1$  be the convergence radius of the power series associated with the sequence  $\{\alpha(k)\}_{k \in \mathbb{N}}$ . This convergence radius is known to be positive. As a consequence, the convergence radius  $R_2$  of the power series associated with the sequence  $\{\alpha(k)^2\}_{k \in \mathbb{N}}$  is also positive and  $R_2 = R_1^2$ .



Let  $\varepsilon > 0$ . Since we are only working with large diffusivities, let us assume that  $\mu \geq 1$  and that  $\mu > (\frac{1}{R_2})^{1/\varepsilon}$ . Noting that, for any  $N \in \mathbb{N}$ , we have

$$\tilde{g}_{N,\mu} - g_{N,\mu}^2 = \sum_{k=0}^N \frac{1}{\mu^k} \eta_{k,m} \left( \sum_{\ell=N-k+1}^N \frac{\eta_{\ell,m}}{\mu^\ell} \right).$$

and using the fact that the sequence  $\{\alpha(k)\}_{k \in \mathbb{N}}$  was built increasing, we get the existence of  $M > 0$  such that

$$\begin{aligned} \frac{1}{|\Omega|} \|\tilde{g}_{N,\mu} - g_{N,\mu}^2\|_{L^2(\Omega)} &= \frac{1}{|\Omega|} \left\| \sum_{k=0}^N \frac{1}{\mu^k} \eta_{k,m} \left( \sum_{\ell=N-k+1}^N \frac{\eta_{\ell,m}}{\mu^\ell} \right) \right\|_{L^2(\Omega)} \\ &\leq \sum_{k=0}^N \frac{\alpha(k)}{\mu^k} \left( \sum_{\ell=N-k+1}^N \frac{\alpha(\ell)}{\mu^\ell} \right) \\ &\leq \alpha(N)^2 \sum_{k=0}^N \frac{1}{\mu^k} \left( \frac{1}{\mu^{N-k+1}} \frac{1 - \frac{1}{\mu^{k-1}}}{1 - \frac{1}{\mu}} \right) && \text{by using that } \alpha(k)\alpha(\ell) \leq \alpha(N)^2 \\ &\leq M \frac{(N+1)\alpha(N)^2}{\mu^{N+1}} \\ &= M \frac{N+1}{(\mu^{1-\varepsilon})^{N+1}} \frac{\alpha(N)^2}{(\mu^\varepsilon)^{N+1}}. \end{aligned}$$

This last quantity converges to zero as  $N \rightarrow \infty$ . Besides, since  $\mu^\varepsilon \geq \frac{1}{R_2}$ , it follows that the sequence  $\left\{ \frac{\alpha(N)^2}{(\mu^\varepsilon)^{N+1}} \right\}_{N \rightarrow \infty}$  is bounded. Assuming moreover that  $\mu^{1-\varepsilon} > 1$ , we get

$$\frac{N+1}{(\mu^{1-\varepsilon})^{N+1}} \xrightarrow{N \rightarrow \infty} 0.$$

We conclude that  $H = g_{\infty,\mu}^2$ . Passing to the limit in Equation (2.47), it follows that

$$\mu \Delta g_{\infty,\mu}^2 + g_{\infty,\mu} (m - g_{\infty,\mu}) = 0 \quad \text{in } \Omega$$

with Neumann boundary conditions.

Finally, we know that  $g_{\infty,\mu} \xrightarrow{\mu \rightarrow +\infty} m_0$  uniformly in  $\mathcal{M}_{m_0,\kappa}(\Omega)$  and moreover, one has  $m_0 > 0$ . It follows that, for  $\mu$  large enough,  $g_{\infty,\mu}$  is positive. The uniqueness of positive solutions of equation (LDE) entails that, for  $\mu$  large enough,  $g_{\infty,\mu} = \theta_{m,\mu}$ . This concludes the proof of the series expansion convergences and thus, the proof of Theorem 2.1.1.

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## CHAPTER 3

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# THE OPTIMAL LOCATION OF RESOURCES PROBLEM FOR BIASED MOVEMENT

WITH G. NADIN AND Y. PRIVAT.

On peut affirmer qu'une traduction s'écarte d'autant plus qu'elle aspire péniblement à la généralité. Car elle cherche alors à imiter jusqu'aux plus fines particularités, évite ce qui est simplement général, et ne peut qu'opposer à chaque propriété une propriété différente. Cela ne doit pourtant pas nous dissuader de traduire.

---

W. Von Humboldt,  
*Introduction à l'Agamemnon*(1816)

## GENERAL PRESENTATION OF THE CHAPTER: MAIN DIFFICULTIES AND METHODS

In this chapter, we study a spectral optimization problem: for any  $\alpha \geq 0$  and any

$$m \in \mathcal{M}(\Omega) := \left\{ 0 \leq m \leq \kappa, \int_{\Omega} m = m_0 \right\},$$

define  $\lambda_{\alpha}(m)$  as the first eigenvalue of the operator

$$\mathcal{L}_{\alpha} := -\nabla \cdot \left( (1 + \alpha m) \nabla \right) - m,$$

and consider the optimization problem

$$\min_{m \in \mathcal{M}(\Omega)} \lambda_{\alpha}(m).$$

The main results of this chapter and the methods developed to obtain them can be summarized as follows:

- Non-existence of minimizers:  
Whenever  $\Omega$  is not a ball, the optimization problem does not have a solution. This is Theorem 3.1.1, and it is proved using classical methods of  $H$ -convergence.
- Stability of the minimizer  $m^*$  in the ball:  
When  $\Omega$  is a ball, we know that the minimizer  $m^*$  for  $\alpha = 0$  is uniquely defined as

$$m^* = \kappa \mathbb{1}_{\mathbb{B}(0, r^*)}.$$

We prove that, when  $\alpha > 0$  is small enough,  $m^*$  remains the unique minimizer among radial resources distributions, using a new technique. This is Theorem 3.1.2. Using this technique and a rearrangement of Alvino-Trombetti, we also prove that, when  $m_0$  is small enough,  $m^*$  remains the unique minimizer in all of  $\mathcal{M}(\Omega)$ . This is Theorem 3.1. Both these results are proved using a new technique of Taylor expansion of a suitably chosen switch function across the boundary of the optimal set.

- Shape stability: We use a comparison method to establish the local shape minimality of  $m^*$ , in the sense that the second order shape derivative is positive. We use a comparison principle to prove that controlling the problem for  $\alpha = 0$  leads to optimality conditions for  $\alpha > 0$  small enough. This is Theorem 3.1.3.

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### 3.1 INTRODUCTION AND MAIN RESULTS

In recent decades, much attention has been paid to extremal problems involving eigenvalues, and in particular to shape optimization problems in which the unknown is the domain where the eigenvalue problem is solved. The study of these last problems is motivated by stability issues of vibrating bodies, wave propagation in composite environments, or also on conductor thermal insulation.

In this article, we are interested in studying a particular extremal eigenvalues problem, involving a drift term. The influence of drift terms on optimal design problems is not so well understood. Such problems naturally arise for instance when looking for optimal shape design for two-phase composite materials. In that case, a possible formulation reads: given  $\Omega$ , a bounded connected open subset of  $\mathbb{R}^n$  and a set of admissible non-negative densities  $\mathcal{M}$  in  $\Omega$ , solve the optimal design problem

$$\inf_{m \in \mathcal{M}} \hat{\lambda}_\alpha(m) \tag{P_\alpha}$$

where  $\hat{\lambda}_\alpha(m)$  denotes the first eigenvalue of the elliptic operator

$$\hat{\mathcal{L}}_\alpha^m : W_0^{1,2}(\Omega) \ni u \mapsto -\nabla \cdot ((1 + \alpha m)\nabla u).$$

Restricting the set of admissible densities to bang-bang ones (in other words to functions taking only two different values) is known to be relevant for the research of structures optimizing the compliance. We refer to Section 3.3 for detailed bibliographical comments.

Mathematically, the main issues regarding Problem ( $\hat{P}_\alpha$ ) concern the existence of optimal densities in  $\mathcal{M}$ , possibly the existence of optimal bang-bang densities (i.e characteristic functions). In this case, it is interesting to try to describe minimizers in a qualitative way.

In what follows, we will consider a refined version of Problem ( $\hat{P}_\alpha$ ), where the operator  $\hat{\mathcal{L}}_\alpha^m$  is replaced by

$$\mathcal{L}_m^\alpha : W_0^{1,2}(\Omega) \ni u \mapsto -\nabla \cdot ((1 + \alpha m)\nabla u) - mu. \tag{3.1}$$

In spite of its intrinsic mathematical interest, the issue of minimizing the first eigenvalue of  $\mathcal{L}_m^\alpha$  with respect to densities  $m$  is motivated by a model of population dynamics.

Before providing a precise mathematical frame of the questions we raise in what follows, let us roughly describe the main results and contributions of this article:

- by adapting the methods developed by Murat and Tartar, [142], and Cox and Lipton, [60], we show that the first eigenvalue of  $\mathcal{L}_m^\alpha$  has no regular minimizer in  $\mathcal{M}$  unless  $\Omega$  is a ball;
- if  $\Omega$  is a ball, denoting by  $m^*$  a minimizer (known to be bang-bang and radial) of  $\mathcal{L}_m^0$  over  $\mathcal{M}$ , we show the following stationarity result:  $m^*$  stil minimizes  $\mathcal{L}_m^\alpha$  over radial distributions of  $\mathcal{M}$  whenever  $\alpha$  is small enough and in small dimension ( $n = 1, 2, 3$ ). Such a result appears unexpectedly difficult to prove. Our approach is based on the use of a well chosen path of quasi-minimizers and on a new type of local argument.
- if  $\Omega$  is a ball, we investigate the local optimality of ball centered distributions among all distributions and prove a quantitative estimate on the second order shape derivative by using a new approach relying on a kind of comparison principle for second order shape derivatives.

Precise statements of these results are given in Section 3.1.2.

### 3.1.1 MATHEMATICAL SET UP

Throughout this article,  $m_0, \kappa$  are fixed positive parameters. Since in our work we want to extend the results of [116], let us define the set of admissible functions

$$\mathcal{M}_{m_0, \kappa}(\Omega) = \left\{ m \in L^\infty(\Omega), 0 \leq m \leq \kappa, \int_{\Omega} m = m_0 \right\},$$

where  $\int_{\Omega} m$  denotes the average value of  $m$  (see Section 3.1.4) and assume that  $m_0 < \kappa$  so that  $\mathcal{M}_{m_0, \kappa}(\Omega)$  is non-empty. Given  $\alpha \geq 0$  and  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , the operator  $\mathcal{L}_m^\alpha$  is symmetric and compact. According to the spectral theorem, it is diagonalizable in  $L^2(\Omega)$ . In what follows, let  $\lambda_\alpha(m)$  be the first positive eigenvalue for this problem. According to the Krein-Rutman theorem,  $\lambda_\alpha(m)$  is simple and its associated  $L^2(\Omega)$ -normalized eigenfunction  $u_{\alpha, m}$  has a constant sign, say  $u_{\alpha, m} \geq 0$ . Let  $R_{\alpha, m}$  be the associated Rayleigh quotient given by

$$R_{\alpha, m} : W_0^{1,2}(\Omega) \ni u \mapsto \frac{\frac{1}{2} \int_{\Omega} (1 + \alpha m) |\nabla u|^2 - \int_{\Omega} m u^2}{\int_{\Omega} u^2}. \quad (3.2)$$

We recall that  $\lambda_\alpha(m)$  can also be defined through the variational formulation

$$\lambda_\alpha(m) := \inf_{u \in W_0^{1,2}(\Omega), u \neq 0} R_{\alpha, m}(u) = R_{\alpha, m}(u_{\alpha, m}). \quad (3.3)$$

and that  $u_{\alpha, m}$  solves

$$\begin{cases} -\nabla \cdot \left( (1 + \alpha m) \nabla u_{\alpha, m} \right) - m u_{\alpha, m} = \lambda_\alpha(m) u_{\alpha, m} & \text{in } \Omega, \\ u_{\alpha, m} = 0 & \text{on } \Omega. \end{cases} \quad (3.4)$$

in a weak  $W_0^{1,2}(\Omega)$  sense. In this article, we address the optimization problem

$$\boxed{\inf_{m \in \mathcal{M}_{m_0, \kappa}(\Omega)} \lambda_\alpha(m)}. \quad (P_\alpha)$$

This problem is a modified version of the standard two-phase problem. It is notable that it is relevant in the framework of population dynamics, when looking for optimal resources configurations in a heterogeneous environment for species survival, see Section 3.1.3.

### 3.1.2 MAIN RESULTS

Before providing the main results of this article, we state a first fundamental property of the investigated model, reducing in some sense the research of general minimizers to the one of *bang-bang* densities. It is notable that, although the set of *bang-bang* densities is known to be dense in the set of all densities for the weak-star topology, such a result is not obvious since it rests upon continuity properties of  $\lambda_\alpha$  for this topology. We overcome this difficulty by exploiting a convexity-like property of  $\lambda_\alpha$ .

**PROPOSITION 3.1 : weak bang-bang property** *Let  $\Omega$  be a bounded connected subset of  $\mathbb{R}^n$  with a Lipschitz boundary and let  $\alpha > 0$  be given. For every  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , there exists a bang-bang function  $\tilde{m} \in \mathcal{M}_{m_0, \kappa}(\Omega)$  such that*

$$\lambda_\alpha(m) \geq \lambda_\alpha(\tilde{m}).$$

*Moreover, if  $m$  is not bang-bang, then we can choose  $\tilde{m}$  so that the previous inequality is strict.*

In other words, given any resources distribution  $m$ , it is always possible to construct a bang-bang

function  $\tilde{m}$  that improves the criterion.

**Non-existence for general domains.** In a series of paper, [46, 47, 48], Casado-Diaz proved that the problem of minimizing the first eigenvalue of the operator  $u \mapsto -\nabla \cdot (1 + \alpha m)\nabla u$  with respect to  $m$  does not have a solution when  $\partial\Omega$  is connected. His proof relies on a study of the regularity for this minimization problem, on homogenization and on a Serrin type argument. The following result is in the same vein, with two differences: it is weaker than his in the sense that it needs to assume higher regularity of the optimal set, but stronger in the sense that we do not make any strong assumption on  $\partial\Omega$ . For further details regarding this literature, we refer to Section 3.3.1.

**THEOREM 3.1.1** *Let  $\Omega$  be a bounded connected subset of  $\mathbb{R}^n$  with a Lipschitz boundary, let  $\alpha > 0$  and  $n \geq 2$ . If the optimization problem  $(P_\alpha)$  has a solution  $\hat{m} \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , then this solution writes  $\hat{m} = \kappa \chi_{\hat{E}}$ , where  $\hat{E}$  is a measurable subset of  $\Omega$ . Moreover, if  $\partial\hat{E}$  is a  $\mathcal{C}^2$  hypersurface and if  $\Omega$  is connected, then  $\Omega$  is a ball.*

The proof of this Theorem relies on methods developed by Murat and Tartar, [142], Cox and Lipton, [60], and on a Theorem of Serrin [165].

**Analysis of optimal configurations in a ball.** According to Theorem 3.1.1, existence of regular solutions fail when  $\Omega$  is not a ball. This suggest to investigate the case  $\Omega = \mathbb{B}(0, R)$ , which is the main goal of what follows.

Let us stress that proving the existence of a minimizer in this setting and characterizing it is a hard thing. Indeed, to underline the difficulty, notice in particular that none of the usual rearrangement techniques (the Schwarz rearrangement or the Alvino-Trombetti one, see Section 3.3.1), that enable to only consider radial densities  $m$ , and thus to get compactness properties, can be applied here.

### The case of radially symmetric distributions

Here, we assume that  $\Omega$  denotes the ball  $\mathbb{B}(0, R)$  with  $R > 0$ . Let

$$m_0^* = \kappa \mathbb{1}_{\mathbb{B}(0, r_0^*)} = \kappa \mathbb{1}_{E_0^*}$$

be the centered distribution known to be the unique minimizer of  $\lambda_0$  in  $\mathcal{M}_{m_0, \kappa}(\Omega)$  (see e.g. [116]).

In what follows, we restrict ourselves to the case of radial resources distributions.

**THEOREM 3.1.2** *Let  $\mathcal{M}_{rad}$  be the subset of radially symmetric distributions of  $\mathcal{M}_{m_0, \kappa}(\Omega)$ . The optimization problem*

$$\inf_{m \in \mathcal{M}_{rad}} \lambda_\alpha(m)$$

*has a solution. Furthermore, when  $n = 1, 2, 3$ , there exists  $\alpha^* > 0$  such that, for any  $\alpha < \alpha^*$ , there holds*

$$\min_{m \in \mathcal{M}_{rad}} \lambda_\alpha(m) = \lambda_\alpha(m_0^*). \tag{3.5}$$

The proof of the existence part of the theorem relies on rearrangement techniques that were first introduced by Alvino and Trombetti in [7] and then refined in [57]. The stationarity result, i.e the fact that  $m^*$  is a minimizer among radially symmetric distributions, was proved in the one-dimensional case in [50]. To extend this result to higher dimensions, we developed an approach involving a homogenized version of the problem under consideration. The small dimensions hypothesis is due to a technical reason, which arises when dealing with elliptic regularity for this equation.

Restricting ourselves to radially symmetric distributions might appear surprising since one could expect this result to be true without restriction, in  $\mathcal{M}_{m_0, \kappa}(\Omega)$ . For instance, a similar result has been shown in the framework of two-phase eigenvalues [57], as a consequence of the Alvino-Trombetti

rearrangement. Unfortunately, regarding Problem  $(P_\alpha)$ , no standard rearrangement technique leads to the expected conclusion, because of the specific form of the involved Rayleigh quotient. A first attempt in the investigation of the ball case is then to consider the case of radially symmetric distributions. It is notable that, even in this case, the proof appears unexpectedly difficult.

Finally, we note that, as a consequence of the methods developed to prove Theorem 3.1.2, when a small amount of resources is available, the centered distribution  $m_0^*$  is optimal.

**COROLLARY 3.1** *Under the same assumptions as in Theorem 3.1.2, there exists  $\underline{m} > 0$ ,  $\underline{\alpha} > 0$  such that, if  $m_0 \leq \underline{m}$  and  $\alpha < \underline{\alpha}$ , then the unique solution of  $(P_\alpha)$  is  $m_0^* = \kappa \mathbb{1}_{E_0^*}$ .*

### Local minimality of the centered distribution among all resources distributions

In what follows, we tackle the issue of the local minimality of  $m_0^*$  in  $\mathcal{M}_{m_0, \kappa}(\Omega)$  with the help of a shape derivative approach. We obtain partial results in dimension  $n = 2$ .

Let  $\Omega$  be a bounded connected domain with a Lipschitz boundary, and consider a *bang-bang* function  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  writing  $m = \kappa \mathbb{1}_E$ , for a measurable subset  $E$  of  $\Omega$  such that  $\kappa|E| = m_0|\Omega|$ . Let us introduce  $\lambda_\alpha(E) := \lambda_\alpha(\mathbb{1}_E)$ , with a slight abuse of notation. Let us assume that  $E$  has a  $\mathcal{C}^2$  boundary. Let  $V : \Omega \rightarrow \mathbb{R}^n$  be a  $W^{3, \infty}$  vector field with compact support, and define for every  $t$  small enough,  $E_t := (\text{Id} + tV)E$ . For  $t$  small enough,  $\phi_t := \text{Id} + tV$  is a smooth diffeomorphism from  $E$  to  $E_t$ , and  $E_t$  is an open connected set with a  $\mathcal{C}^2$  boundary. If  $\mathcal{F} : E \mapsto \mathcal{F}(E)$  denotes a shape functional, the first (resp. second) order shape derivative of  $\mathcal{F}$  at  $E$  in the direction  $V$  is

$$\mathcal{F}'(E)[V] := \left. \frac{d}{dt} \right|_{t=0} \mathcal{F}(E_t) \quad \left( \text{resp.} \quad \left. \frac{d^2}{dt^2} \right|_{t=0} \mathcal{F}(E_t) \right)$$

whenever these quantities exist.

For further details regarding the notion of shape derivative, we refer to [93, Chapter 5].

Since one wants to ensure that  $|E_t| = V_0$ , we impose the condition  $\int_E \nabla \cdot V = 0$  on the vector field  $V$ . We call *admissible at  $E$*  such vector fields and introduce

$$\mathcal{X}(E) := \left\{ V \in W^\infty(\mathbb{R}^n; \mathbb{R}^n), \int_E \nabla \cdot V = 0, \|V\|_{W^{3, \infty}} \leq 1 \right\}. \quad (3.6)$$

A shape  $E \subset \Omega$  with a  $\mathcal{C}^2$  boundary such that  $\kappa|E| = m_0|\Omega|$  is said to be *critical* if

$$\forall V \in \mathcal{X}(E), \quad \lambda'_\alpha(E)[V] = 0. \quad (3.7)$$

or, similarly, if there exist a Lagrange multiplier  $\Lambda_\alpha$  such that  $(\lambda_\alpha - \Lambda_\alpha \text{Vol})'(E)[V] = 0$  for all  $V \in \mathcal{X}(E)$ , where  $\text{Vol} : \Omega \mapsto |\Omega|$  denotes the volume functional. Furthermore, if  $E$  is a local minimizer for Problem  $(P_\alpha)$ , then one has

$$\forall V \in \mathcal{X}(E), \quad (\lambda_\alpha - \Lambda_\alpha \text{Vol})''(E)[V, V] \geq 0. \quad (3.8)$$

**THEOREM 3.1.3** *Let us assume that  $n = 2$ . The ball  $E = \mathbb{B}(0, r_0^*) = \mathbb{B}^*$  satisfies the shape optimality conditions (3.7)-(3.8). Furthermore, if  $\Lambda_\alpha$  is the Lagrange multiplier associated with the volume constraint, there exists two constants  $\bar{\alpha} > 0$  and  $C > 0$  such that, for any  $\alpha \in [0, \bar{\alpha})$  and any vector field  $V \in \mathcal{X}(\mathbb{B}^*)$  normal to  $\partial\mathbb{B}^* = \mathbb{S}^*$  there holds*

$$(\lambda_\alpha - \Lambda_\alpha \text{Vol})''(\mathbb{B}^*)[V, V] \geq C \|V \cdot \nu\|_{L^2(\mathbb{S}^*)}^2.$$

**REMARK 3.1** *The proof requires explicit computation of the shape derivative of the eigenfunction. We note that in [63] such computations are carried out for the two-phase problem and that in [104] such an approach is undertaken to investigate the stability of certain configurations for a weighted Neumann eigenvalue problem.*



The main contribution of this result is to shed light on a monotonicity principle that enables one to lead a careful asymptotic analysis of the second order shape derivative of the functional as  $\alpha \rightarrow 0$ . It is important to note that, although this allows us to deeply analyze the second order optimality conditions, it is expected that the optimal coercivity norm in the right-hand side above is expected to be  $H^{\frac{1}{2}}$  whenever  $\alpha > 0$ , which we do not recover with our method. When  $\alpha = 0$ , we know that the optimal coercivity norm is  $L^2$  (see [Maz19a]).

The rest of this article is dedicated to proofs of the results we have just outlined.

### 3.1.3 A BIOLOGICAL APPLICATION OF THE PROBLEM

Equation (4.2) arises naturally when dealing with simple population dynamics in heterogeneous spaces. For  $\varepsilon \geq 0$ , a parameter of the model, we consider a population density whose flux is given by

$$\mathcal{J}_\varepsilon = -\nabla u + \varepsilon u \nabla m.$$

Since  $\nabla m$  might not make sense if  $m$  is assumed to be only measurable, we temporarily omit this difficulty by assuming it smooth enough so that the expression above makes sense. The term  $u \nabla m$  stands for a bias in the population movement, modeling a tendency of the population to disperse along the gradient of resources and hence move to favorable regions. The parameter  $\varepsilon$  quantifies the influence of the resources distribution on the movement of the species. The complete associated reaction diffusion equation, called “logistic diffusive equation”, reads

$$\frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \varepsilon u \nabla m) + mu - u^2 \quad \text{in } \Omega,$$

completed with suitable boundary conditions. In what follows, we will focus on Dirichlet boundary conditions meaning that the boundary of  $\Omega$  is lethal for the population living inside. Plugging the change of variable  $v = e^{-\varepsilon m} u$  in this equation leads to

$$\frac{\partial v}{\partial t} = \Delta v + \varepsilon \langle \nabla m, \nabla v \rangle + mv - e^{\varepsilon m} v^2 \quad \text{in } \Omega.$$

It is known (see e.g. [17, 16, 143]) that the asymptotic behavior of this equation is driven by the principal eigenvalue of the operator  $\tilde{\mathcal{L}} : u \mapsto -\Delta u - \varepsilon \langle \nabla m, \nabla u \rangle - mu$ . The associated principal eigenfunction  $\psi$  satisfies

$$-\nabla \cdot (e^{\varepsilon m} \nabla \psi) - m e^{\varepsilon m} \psi = \tilde{\lambda}_\varepsilon \psi e^{\varepsilon m} \quad \text{in } \Omega.$$

Following the approach developed in [116], optimal configurations of resources correspond to the ones ensuring the fastest convergence to the steady-states of the PDE above, which comes to minimizing  $\tilde{\lambda}_\varepsilon(m)$  with respect to  $m$ .

By using Proposition 3.1, which enables us to only deal with *bang-bang* densities  $m$ , one shows easily that minimizing  $\tilde{\lambda}_\varepsilon(m)$  over  $\mathcal{M}_{m_0, \kappa}(\Omega)$  is equivalent to minimizing  $\lambda_\varepsilon(m)$  over  $\mathcal{M}_{m_0, \kappa}(\Omega)$ , in other words to Problem  $(P_\alpha)$  with  $\alpha = \varepsilon$ . Theorem 3.1.1 can thus be interpreted as follows in this framework: assuming that the population density moves along the gradient of the resources, it is not possible to lay the resources in an optimal way. Note that the conclusion is completely different in the case  $\alpha = 0$  (see [116]) or in the one-dimensional case (i.e.  $\Omega = (0; 1)$ ) with  $\alpha > 0$  (see [50]), where minimizers exist. In the last case, optimal configurations for three kinds boundary conditions (Dirichlet, Neumann, Robin) have been obtained, by using a new rearrangement technique. Finally, let us mention the related result [84, Theorem 2.1], dealing with Faber-Krahn type inequalities for general operators of the form

$$\mathcal{K} : u \mapsto -\nabla \cdot (A \nabla u) - \langle V, \nabla u \rangle - mu$$

where  $A$  is a positive symmetric matrix. Let us denote the first eigenvalue of  $\mathcal{K}$  by  $E(A, V, m)$ . It is shown, by using new rearrangements, that there exist radially symmetric elements  $A^*, V^*, m^*$  such that

$$0 < \inf A \leq A^* \leq \|A\|_\infty, \quad \|A^{-1}\|_{L^1} = \|(A^*)^{-1}\|_{L^1}, \quad \|V^*\|_{L^\infty} \leq \|V\|_{L^\infty}$$

and  $E(A, V, m) \geq E(A^*, V^*, m^*)$ . We note that applying this result directly to our problem would not allow us to conclude. Indeed, we would get that for every  $\Omega$  of volume  $V_1$  and every  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , if  $\Omega^*$  is the ball of volume  $V_1$ , there exists two radially symmetric functions  $m_1$  and  $m_2$  satisfying  $m_1, m_2 \in \mathcal{M}_{m_0, \kappa}(\Omega)$  such that  $\lambda_\alpha(m) \geq \mu_\alpha(m_1, m_2)$ , where  $\mu_\alpha(m_1, m_2)$  is the first eigenvalue of the operator  $-\nabla \cdot ((1 + \alpha m_1) \nabla) - m_2$ . We note that this result could also be obtained by using the symmetrization techniques of [7].

Finally, let us mention that an optimal control problem involving a similar model but a different cost functional, related to optimal harvesting of a marine resource, has been investigated in the series of articles [32, 33, 54].

### 3.1.4 NOTATIONS AND NOTATIONAL CONVENTIONS, TECHNICAL PROPERTIES OF THE EIGENFUNCTIONS

Let us sum-up the notations used throughout this article.

- $\mathbb{R}_+$  is the set of non-negative real numbers.  $\mathbb{R}_+^*$  is the set of positive real numbers.
- $n$  is a fixed positive integer and  $\Omega$  is a bounded connected domain in  $\mathbb{R}^n$ .
- if  $E$  denotes a subset of  $\Omega$ , the notation  $\chi_E$  stands for the characteristic function of  $E$ , equal to 1 in  $E$  and 0 elsewhere.
- the notation  $\|\cdot\|$  used without subscript refers to the standard Euclidean norm in  $\mathbb{R}^n$ . When referring to the norm of a Banach space  $\mathcal{X}$ , we write it  $\|\cdot\|_{\mathcal{X}}$ .
- The average of every  $f \in L^1(\Omega)$  is denoted by  $f_\Omega := \frac{1}{|\Omega|} \int_\Omega f$ .
- $\nu$  stands for the outward unit normal vector on  $\partial\Omega$ .

## 3.2 PRELIMINARIES

### 3.2.1 SWITCHING FUNCTION

In view of deriving optimality conditions for Problem  $(P_\alpha)$ , we introduce the tangent cone to  $\mathcal{M}_{m_0, \kappa}(\Omega)$  at any point of this set.

**DEFINITION 3.2.1** ([93, chapter 7]) *For every  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , the tangent cone to the set  $\mathcal{M}_{m_0, \kappa}(\Omega)$  at  $m$ , also called the admissible cone to the set  $\mathcal{M}_{m_0, \kappa}(\Omega)$  at  $m$ , denoted by  $\mathcal{T}_m$  is the set of functions  $h \in L^\infty(\Omega)$  such that, for any sequence of positive real numbers  $\varepsilon_n$  decreasing to 0, there exists a sequence of functions  $h_n \in L^\infty(\Omega)$  converging to  $h$  as  $n \rightarrow +\infty$ , and  $m + \varepsilon_n h_n \in \mathcal{M}_{m_0, \kappa}(\Omega)$  for every  $n \in \mathbb{N}$ .*

Notice that, as a consequence of this definition, any  $h \in \mathcal{T}_m$  satisfies  $f_\Omega h = 0$ .

**LEMMA 3.1** *Let  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  and  $h \in \mathcal{T}_m$ . The mapping  $\mathcal{M}_{m_0, \kappa}(\Omega) \ni m \mapsto u_{\alpha, m}$  is twice differentiable at  $m$  in direction  $h$  in a strong  $L^2(\Omega)$  sense and in a weak  $W_0^{1,2}(\Omega)$  sense, and the mapping  $\mathcal{M}_{m_0, \kappa}(\Omega) \ni m \mapsto \lambda_\alpha$  is twice differentiable in a strong  $L^2(\Omega)$  sense.*

The proof of this lemma is technical and is postponed to Appendix 4.B.3.

For  $t$  small enough, let us introduce the mapping  $g_h : t \mapsto \lambda_\alpha(m + th)$ . Hence,  $g_h$  is twice differentiable. The first and second order derivatives of  $\lambda_\alpha$  at  $m$  in direction  $h$ , denoted by  $\dot{\lambda}_\alpha(m)[h]$  and  $\ddot{\lambda}_\alpha(m)[h]$ , are defined by

$$\dot{\lambda}_\alpha(m)[h] := g'_h(0) \quad \text{and} \quad \ddot{\lambda}_\alpha(m)[h] := g''_h(0).$$

**LEMMA 3.2** *Let  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  and  $h \in \mathcal{T}_m$ . The mapping  $m \mapsto \lambda_\alpha(m)$  is differentiable at  $m$  in direction  $h$  in  $L^2$  and its differential reads*

$$\dot{\lambda}_\alpha(m)[h] = \int_\Omega h \psi_{\alpha, m}, \quad \text{with} \quad \psi_{\alpha, m} := \alpha |\nabla u_{\alpha, m}|^2 - u_{\alpha, m}^2. \quad (3.9)$$

The function  $\psi_{\alpha, m}$  is called *switching function*.

*Proof.* According to Lemma 3.1, we can differentiate the variational formulation associated to (4.2) and get that the differential  $\dot{u}_{\alpha, m}[h]$  of  $m \mapsto u_{\alpha, m}$  at  $m$  in direction  $h$  satisfies

$$\begin{cases} -\nabla \cdot (\sigma_\alpha \nabla \dot{u}_{\alpha, m}[h]) - \alpha \nabla \cdot (h \nabla u_{\alpha, m}) = \dot{\lambda}_\alpha(m)[h] u_{\alpha, m} + \lambda_\alpha(m) \dot{u}_{\alpha, m}[h] \\ \quad + m \dot{u}_{\alpha, m}[h] + h u_{\alpha, m} & \text{in } \Omega, \\ \dot{u}_{\alpha, m}[h] = 0 & \text{on } \partial\Omega, \\ \int_\Omega u_{\alpha, m} \dot{u}_{\alpha, m}[h] = 0. \end{cases} \quad (3.10)$$

Multiplying this equation by  $u_{\alpha, m}$ , integrating by parts and using that  $u_{\alpha, m}$  is normalized in  $L^2(\Omega)$  leads to

$$\begin{aligned} \dot{\lambda}_\alpha(m)[h] &= \underbrace{\int_\Omega \sigma_\alpha \langle \nabla \dot{u}_{\alpha, m}[h], \nabla u_{\alpha, m} \rangle - \lambda_\alpha(m) \int_\Omega u_{\alpha, m} \dot{u}_{\alpha, m}[h] - \int_\Omega m u_{\alpha, m} \dot{u}_{\alpha, m}[h]}_{=0 \text{ according to (4.2)}} \\ &\quad + \int_\Omega \alpha h |\nabla u_{\alpha, m}|^2 - \int_\Omega h u_{\alpha, m}^2. \end{aligned}$$

□

### 3.2.2 PROOF OF PROPOSITION 3.1

The proof relies on concavity properties of the functional  $\lambda_\alpha$ . More precisely, let  $m_1, m_2 \in \mathcal{M}_{m_0, \kappa}(\Omega)$ . We will show that the map  $f : [0, 1] \ni t \mapsto \lambda_\alpha((1-t)m_1 + tm_2)$  is strictly concave, i.e. that  $f'' < 0$  on  $[0, 1]$ .

Note that the characterization of the concavity in terms of second order derivatives makes sense, according to Lemma 3.1, since  $\lambda_\alpha$  is twice differentiable. Before showing this concavity property, let us first explain why it implies the conclusion of Proposition 3.1 (the weak bang-bang property). Let  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  assumed to be not bang-bang. The set  $\mathcal{I} = \{0 < m < \kappa\}$  is then of positive Lebesgue measure and  $m$  is therefore not extremal in  $\mathcal{M}_{m_0, \kappa}(\Omega)$ , according to [93, Prop. 7.2.14]. We then infer the existence of  $t \in (0, 1)$  as well as two distinct elements  $m_1$  and  $m_2$  of  $\mathcal{M}_{m_0, \kappa}(\Omega)$  such that  $m = (1-t)m_1 + tm_2$ . Because of the strict concavity of  $\lambda_\alpha$ , the solution of the optimization problem  $\min\{\lambda_\alpha((1-t)m_1 + tm_2)\}$  is either  $m_1$  or  $m_2$ , and moreover,  $m$  cannot solve this problem. Assume that  $m_1$  solves this problem without loss of generality. One thus has  $\lambda_\alpha(m_1) < \lambda_\alpha(m)$ . Since the subset of *bang-bang* functions of  $\mathcal{M}_{m_0, \kappa}(\Omega)$  is dense in  $\mathcal{M}_{m_0, \kappa}(\Omega)$  for the weak-star topology of  $L^\infty(\Omega)$ , there exists a sequence of *bang-bang* functions  $(m^k)_{k \in \mathbb{N}}$  of  $\mathcal{M}_{m_0, \kappa}(\Omega)$  converging weakly-star to  $m_1$  in  $L^\infty(\Omega)$ . Furthermore,  $\lambda_\alpha$  is upper semicontinuous for the weak-star topology of  $L^\infty(\Omega)$ , since it reads as the infimum of continuous linear functionals for this topology. Let  $\varepsilon > 0$ . We

infer the existence of  $k_\varepsilon \in \mathbb{N}$  such that  $\lambda_\alpha(m^{k_\varepsilon}) \leq \lambda_\alpha(m_1) + \varepsilon$ . By choosing  $\varepsilon$  small enough, we infer that  $\lambda_\alpha(m^{k_\varepsilon}) < \lambda_\alpha(m)$ , whence the result.

It now remains to prove that  $f$  is strictly concave. Let  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , and set  $m_1 = m$ ,  $h = m_2 - m_1$ , we observe that  $f''(t) = \ddot{\lambda}_\alpha((1-t)m_1 + tm_2)[h]$  for all  $t \in [0, 1]$ . The differential  $\dot{u}_{\alpha, m}[h]$  of  $m \mapsto u_{\alpha, m}$  at  $m$  in direction  $h$ , denoted  $\dot{u}_{\alpha, m}[h]$ , satisfies (3.10) and the second order Gâteaux-derivatives  $\ddot{u}_{\alpha, m}[h]$  and  $\ddot{\lambda}_\alpha(m)[h]$  solve

$$\begin{cases} -\nabla \cdot (\sigma_\alpha \nabla \ddot{u}_{\alpha, m}[h]) - 2\alpha \nabla \cdot (h \nabla \dot{u}_{\alpha, m}[h]) = & \ddot{\lambda}_\alpha(m)[h] u_{\alpha, m} + 2\dot{\lambda}_\alpha(m)[h] \dot{u}_{\alpha, m}[h] \\ & + \lambda_\alpha(m) \ddot{u}_{\alpha, m}[h] + m \ddot{u}_{\alpha, m}[h] + 2h \dot{u}_{\alpha, m}[h] & \text{in } \Omega, \\ \ddot{u}_{\alpha, m}[h] = 0 & & \text{on } \partial\Omega. \end{cases} \quad (3.11)$$

Multiplying this Equation by  $u_{\alpha, m}$ , using that  $u_{\alpha, m}$  is normalized in  $L^2(\Omega)$  and integrating by parts yields

$$\begin{aligned} \ddot{\lambda}_\alpha(m)[h] &= \underbrace{\int_\Omega \sigma_\alpha \langle \nabla \ddot{u}_{\alpha, m}[h], \nabla u_{\alpha, m} \rangle - \lambda_\alpha(m) \int_\Omega u_{\alpha, m} \ddot{u}_{\alpha, m}[h] - \int_\Omega m u_{\alpha, m} \ddot{u}_{\alpha, m}[h]}_{=0 \text{ according to (4.2)}} \\ &+ 2\alpha \int_\Omega h \langle \nabla \dot{u}_{\alpha, m}[h], \nabla u_{\alpha, m} \rangle - 2 \int_\Omega h u_{\alpha, m} \dot{u}_{\alpha, m}[h] \\ &= 2 \left( - \int_\Omega \sigma_\alpha |\nabla \dot{u}_{\alpha, m}[h]|^2 + \int_\Omega m \dot{u}_{\alpha, m}[h]^2 + \lambda_\alpha(m) \int_\Omega \dot{u}_{\alpha, m}[h]^2 \right) + 2 \underbrace{\dot{\lambda}_\alpha(m)[h] \int_\Omega u_{\alpha, m} \dot{u}_{\alpha, m}[h]}_{=0 \text{ since } \int_\Omega u_{\alpha, m} \dot{u}_{\alpha, m}[h] = 0} \\ &= 2 \int_\Omega \dot{u}_{\alpha, m}[h]^2 (-R_{\alpha, m}[\dot{u}_{\alpha, m}[h]] + \lambda_\alpha(m)) < 0, \end{aligned}$$

where the last inequality comes from the observation that, whenever  $h \neq 0$ , one has  $\dot{u}_{\alpha, m}[h] \neq 0$  and  $\dot{u}_{\alpha, m}[h]$  is in the orthogonal space to the first eigenfunction  $u_{\alpha, m}$  in  $L^2(\Omega)$ . Since the first eigenvalue is simple, the Rayleigh quotient of  $\dot{u}_{\alpha, m}[h]$  is greater than  $\lambda_\alpha(m)$ .

### 3.3 PROOF OF THEOREM 3.1.1

This proof is based on a homogenization argument, inspired from the notions and techniques introduced in [142]. In the next section, we gather the preliminary tools and material involved in what follows.

#### 3.3.1 BACKGROUND MATERIAL ON HOMOGENIZATION AND BIBLIOGRAPHICAL COMMENTS

Let us recall several usual definitions and results in homogenization theory we will need hereafter.

**DEFINITION 3.3.1 : H-convergence** Let  $(m_k)_{k \in \mathbb{N}} \in \mathcal{M}_{m_0, \kappa}(\Omega)^\mathbb{N}$  and for every  $k \in \mathbb{N}$ , define respectively  $\sigma_k$  and  $u_k(f)$  by  $\sigma_k = 1 + \alpha m_k$  and as the unique solution of

$$\begin{cases} -\nabla \cdot (\sigma_k \nabla u_k(f)) = f & \text{in } \Omega \\ u_k(f) = 0 & \text{on } \Omega \end{cases}$$

where  $f \in L^2(\Omega)$  is given. We say that the sequence  $(\sigma_k)_{k \in \mathbb{N}}$  H-converges to  $A : \Omega \rightarrow M_n(\mathbb{R})$  if, for every  $f \in L^2(\Omega)$ , the sequence  $(u_k(f))_{k \in \mathbb{N}}$  converges weakly to  $u_\infty$  in  $W^{1, \infty}(\Omega)$  and the sequence

$(\sigma_k \nabla u_k)_{k \in \mathbb{N}}$  converges weakly to  $A \nabla u_\infty$  in  $L^2(\Omega)$ , where  $u_\infty$  solves

$$\begin{cases} -\nabla \cdot (A \nabla u_\infty) = f & \text{in } \Omega, \\ u_\infty = 0 & \text{on } \Omega \end{cases}$$

In that case, we will write  $\sigma_k \xrightarrow[k \rightarrow \infty]{H} A$ .

**DEFINITION 3.3.2 : arithmetic and geometric means** Let  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  and  $\sigma = 1 + \alpha m$ . We define the arithmetic mean of  $\sigma$  by  $\Lambda_+(m) = \sigma$ , and its harmonic mean by  $\Lambda_-(m) = \frac{1 + \alpha \kappa}{1 + \alpha(\kappa - m)}$ . One has  $\Lambda_-(m) \leq \Lambda_+(m)$ , according to the arithmetic-harmonic inequality, with equality if and only if  $m$  is a bang-bang function.

**PROPOSITION [142, Proposition 10]** Let  $(m_k)_{k \in \mathbb{N}} \in \mathcal{M}_{m_0, \kappa}(\Omega)^\mathbb{N}$  and  $(\sigma_k)_{k \in \mathbb{N}}$  given by  $\sigma_k = 1 + \alpha m_k$ . Up to a subsequence, there exists  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  such that  $(m_k)_{k \in \mathbb{N}} \in \mathcal{M}_{m_0, \kappa}(\Omega)^\mathbb{N}$  converges to  $m$  for the weak-star topology of  $L^\infty$ .

Assume moreover that the sequence  $(\sigma_k)_{k \in \mathbb{N}}$   $H$ -converges to a matrix  $A$ . Then,  $A$  is a symmetric matrix, its spectrum  $\Sigma(A)$  is real, and

$$\Lambda_-(m) \leq \min \Sigma(A) \leq \max \Sigma(A) \leq \Lambda_+(m). \quad (J_1)$$

$$\sum_{j=1}^n \frac{1}{\lambda_j - 1} \leq \frac{1}{\Lambda_-(m) - 1} + \frac{n-1}{\Lambda_+(m) - 1}, \quad (J_2)$$

$$\sum_{j=1}^n \frac{1}{1 + \alpha \kappa - \lambda_j} \leq \frac{1}{1 + \alpha \kappa - \Lambda_-(m)} + \frac{n-1}{1 + \alpha \kappa - \Lambda_+(m)}. \quad (J_3)$$

For a given  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , we introduce

$$M_m^\alpha = \{A : \Omega \rightarrow S_n(\mathbb{R}), A \text{ satisfies } (J_1)-(J_2)-(J_3)\}.$$

For a matrix-valued application  $A \in M_m^\alpha$  for some  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , it is possible to define the principal eigenvalue of  $A$  via Rayleigh quotients as

$$\zeta_\alpha(m, A) := \inf_{u \in W^{1, \infty}(\Omega), \int_\Omega u^2 = 1} \int_\Omega \langle A \nabla u, \nabla u \rangle - \int_\Omega m u^2. \quad (3.12)$$

Note that the dependence of  $\zeta_\alpha$  on the parameter  $\alpha$  is implicitly contained in the condition  $A \in M_m^\alpha$ . We henceforth focus on the following relaxed version of the optimization problem:

$$\inf_{m \in \mathcal{M}_{m_0, \kappa}(\Omega), A \in M_m^\alpha} \zeta_\alpha(m, A). \quad (3.13)$$

for which we have the following result.

**THEOREM [142, Proposition 10]**

1. For every  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  and  $A \in M_m^\alpha$ , there exists a sequence  $(m_k)_{k \in \mathbb{N}} \in \mathcal{M}_{m_0, \kappa}(\Omega)$  such that  $(m_k)_{k \in \mathbb{N}}$  converges to  $m$  for the weak-star topology of  $L^\infty$ , and the sequence  $(\sigma_k)_{k \in \mathbb{N}}$  defined by  $\sigma_k = 1 + \alpha m_k$   $H$ -converges to  $A$ , as  $k \rightarrow +\infty$ .
2. The mapping  $(m, A) \mapsto \lambda_\alpha(m, A)$  is continuous with respect to the  $H$ -convergence (see in particular [149]).

3. The variational problem (3.13) has a solution  $(\bar{m}, \bar{A})$ ; by definition,  $\bar{A} \in M_{\bar{m}}^{\alpha}$ . Furthermore, if  $\bar{u}$  is the associated eigenfunction, then  $\bar{A}\nabla\bar{u} = \Lambda_{-}(\bar{m})\nabla\bar{u}$ .

This theorem allows us to solve Problem (3.13).

**COROLLARY 3.2** [142] *If Problem  $(P_{\alpha})$  has a solution  $\bar{m}$ , then the couple  $(\bar{m}, 1 + \alpha\bar{m})$  solves Problem (3.13).*

*Proof of Corollary 3.2.* Assume that the solution of (3.13) is  $(\bar{m}, \bar{A})$  and that  $\bar{A} \neq 1 + \alpha\bar{m}$ . Then there exists a sequence  $(m_k)_{k \in \mathbb{N}}$  converging weak-star in  $L^{\infty}$  to  $\bar{m}$  and such that the sequence  $(1 + \alpha m_k)_{k \in \mathbb{N}}$   $H$ -converges to  $\bar{A}$ . This means that

$$\lambda_{\alpha}(\bar{m}) = \zeta_{\alpha}(\bar{m}, 1 + \alpha\bar{m}) > \zeta_{\alpha}(\bar{m}, \bar{A}) = \lim_{k \rightarrow \infty} \lambda_{\alpha}(m_k)$$

which immediately yields a contradiction. □

Let us end this section with several bibliographical comments on such problems.

**Bibliographical comments on the two-phase conductors problem.** Problem  $(\hat{P}_{\alpha})$  has drawn a lot of attention in the last decades, since the seminal works by Murat and Tartar, [141, 142]. Roughly speaking, this optimal design problem is, in general, ill-posed and one needs to introduce a relaxed formulation to get existence. We refer to [3, 60, 142, 149].

Let us provide the main lines strategy to investigate existence issues for Problem  $(\hat{P}_{\alpha})$ , according to [141, 142]. If the solution  $(\bar{m}, 1 + \alpha\bar{m})$  to the relaxed problem (3.13) is a solution to the original problem  $(\hat{P}_{\alpha})$ , then there exists a measurable subset  $\bar{E}$  of  $\Omega$  such that  $\bar{m} = \kappa \mathbf{1}_{\bar{E}}$ . If furthermore  $\bar{E}$  is assumed to be smooth enough, then, denoting by  $\bar{u}$  the principal eigenfunction associated with  $(\bar{m}, \lambda_{\alpha}(\bar{m})) = (\bar{m}, \zeta_{\alpha}(\bar{m}, 1 + \alpha\bar{m}))$ , we get that  $\bar{u}$  and  $(1 + \alpha\bar{m})\frac{\partial \bar{u}}{\partial \nu}$  must be constant on  $\partial \bar{E}$ . The function  $1 + \alpha\bar{m}$  being discontinuous across  $\partial E$ , the optimality condition above has to be understood in the following sense: the function  $(1 + \alpha\bar{m})\frac{\partial \bar{u}}{\partial \nu}$ , a priori discontinuous, is in fact continuous across  $\partial E$  and even constant on it. Note that these arguments have been generalized in [60]. These optimality conditions, combined with Serrin's Theorem [165], suggest that Problem  $(\hat{P}_{\alpha})$  could have a solution if, and only if  $\Omega$  is a ball. The best results known to date are the following ones.

**THEOREM** (i) *Let  $\Omega$  be an open set such that  $\partial\Omega$  is  $\mathcal{C}^2$  and connected. Problem  $(\hat{P}_{\alpha})$  has a solution if and only if  $\Omega$  is a ball [48].*

(ii) *If  $\Omega$  is a ball, then Problem  $(\hat{P}_{\alpha})$  has a solution which is moreover radially symmetric [57].*

Regarding the second part of the theorem, the authors used a particular rearrangement coming to replace  $1 + \alpha m$  by its harmonic mean on each level-set of the eigenfunction. Such a rearrangement has been first introduced by Alvino and Trombetti [7]. This drives the author to reduce the class of admissible functions to radially symmetric ones, which allow them to conclude thanks to a compactness argument [6]. These arguments are mimicked to derive the existence part of Theorem 3.1.2.

Finally, let us mention [56, 117], where the optimality of annular configurations in the ball is investigated. A complete picture of the situation is then depicted in the case where  $\alpha$  is small, which is often referred to as the "low contrast regime". We also mention [63], where a shape derivative approach is undertaken to characterize minimizers when  $\Omega$  is a ball.

### 3.3.2 PROOF OF THEOREM 3.1.1

Let us assume the existence of a solution to Problem  $(P_{\alpha})$ , denoted  $\bar{m}$ . According to Proposition 3.1, there exists a measurable subset  $\bar{E}$  of  $\Omega$  such that  $\bar{m} = \kappa \chi_{\bar{E}}$ . Let us introduce  $\bar{\sigma} := 1 + \alpha\bar{m}$  and  $\bar{u}$ , the  $L^2$ -normalized eigenfunction associated to  $\bar{m}$ .

Let us now assume that  $\partial E$  is  $\mathcal{C}^2$ .

**Step 1: derivation of optimality conditions.** What follows is an adaptation of [60]. For this reason, we only recall the main lines. Let us write the optimality condition for the problem

$$\min_{m \in \mathcal{M}_{m_0, \kappa}(\Omega)} \min_{A \in M_m^\alpha} \zeta_\alpha(m, A) = \lambda_\alpha(\bar{m}),$$

where  $\zeta_\alpha$  is given by (3.12). Let  $h$  be an admissible perturbation at  $\bar{m}$ . In [142] it is proved that for every  $\varepsilon > 0$  small enough, there exists a matrix-valued application  $A_\varepsilon \in M_{\bar{m} + \varepsilon h}$  such that

$$A_\varepsilon \nabla \bar{u} = \Lambda_-(\bar{m} + \varepsilon h) \nabla \bar{u} \quad \text{in } \Omega,$$

where  $\Lambda_-$  has been introduced in Definition 3.3.2. Fix  $\varepsilon$  as above. Since  $(\bar{m}, 1 + \alpha \bar{m})$  is a solution of the Problem (3.13), one has

$$\begin{aligned} \int_{\Omega} \langle A_\varepsilon \nabla \bar{u}, \nabla \bar{u} \rangle - \int_{\Omega} (m + \varepsilon h) \bar{u}^2 &\geq \zeta_\alpha(m + \varepsilon h, A_\varepsilon) \\ &\geq \zeta_\alpha(\bar{m}, 1 + \alpha \bar{m}) = \int_{\Omega} \bar{\sigma} |\nabla \bar{u}|^2 - \int_{\Omega} \bar{m} \bar{u}^2. \end{aligned}$$

where one used the Rayleigh quotient definition of  $\zeta_\alpha$  as well as the minimality of  $(\bar{m}, 1 + \alpha \bar{m})$ . Dividing the last inequality by  $\varepsilon$  and passing to the limit yields

$$\int_{\Omega} h \left. \frac{d\Lambda_-(m)}{dm} \right|_{m=\bar{m}} |\nabla \bar{u}|^2 - h \bar{u}^2 \geq 0.$$

Using that  $d\Lambda_-/dm = \alpha \Lambda_-(m)^2 / (1 + \alpha \kappa)$ , and that  $\bar{m}$  is a bang-bang function (so that  $\Lambda_-(\bar{m}) = \bar{\sigma}$ ), we infer that the first order optimality conditions read: there exists  $\mu \in \mathbb{R}$  such that

$$\bar{E} = \{\Psi_\alpha \leq \mu\} \quad \text{where} \quad \Psi_\alpha := \frac{\alpha}{1 + \alpha \kappa} \bar{\sigma}^2 |\nabla \bar{u}|^2 - \bar{u}^2. \quad (3.14)$$

Since the flux  $\bar{\sigma} \frac{\partial \bar{u}}{\partial \nu}$  is continuous across  $\partial \bar{E}$ , one has necessarily  $\Psi_\alpha = \mu$  on  $\partial \bar{E}$ .

Now, let us follow the approach used in [142] and [48] to simplify the writing of the optimality conditions. Notice first that  $\bar{u}$  and  $\bar{\sigma}^2 \left| \frac{\partial \bar{u}}{\partial \nu} \right|^2$  are continuous across  $\partial \bar{E}$ . Let  $\nabla_\tau \bar{u}$  denote the tangential gradient of  $\bar{u}$  on  $\partial \bar{E}$ . For the sake of clarity, the quantities computed on  $\partial \bar{E}$  seen as the boundary of  $\bar{E}$  will be denoted with the subscript *int*, whereas the ones computed on  $\partial \bar{E}$  seen as part of the boundary of  $\bar{E}^c$  will be denoted with the subscript *ext*. According to the optimality conditions (3.14), one has

$$\frac{\alpha}{1 + \alpha \kappa} \bar{\sigma}^2 |\nabla_\tau \bar{u}|^2 + \frac{\alpha}{1 + \alpha \kappa} \bar{\sigma}^2 \left( \frac{\partial \bar{u}}{\partial \nu} \right)^2 - \bar{u}^2 \Big|_{int} \leq \frac{\alpha}{1 + \alpha \kappa} \bar{\sigma}^2 |\nabla_\tau \bar{u}|^2 + \frac{\alpha}{1 + \alpha \kappa} \bar{\sigma}^2 \left( \frac{\partial \bar{u}}{\partial \nu} \right)^2 - \bar{u}^2 \Big|_{ext}$$

on  $\partial \bar{E}$ . By continuity of the flux  $\bar{\sigma} \frac{\partial \bar{u}}{\partial \nu}$ , we infer that  $\alpha \bar{\sigma}^2 |\nabla_\tau \bar{u}|^2 \Big|_{int} \leq \alpha \bar{\sigma}^2 |\nabla_\tau \bar{u}|^2 \Big|_{ext}$  which comes to  $(1 + \alpha \kappa) |\nabla_\tau \bar{u}|^2 \Big|_{int} \leq |\nabla_\tau \bar{u}|^2 \Big|_{ext}$ . Since  $|\nabla_\tau \bar{u}|^2 \Big|_{int} = |\nabla_\tau \bar{u}|^2 \Big|_{ext}$ , we have  $\nabla_\tau \bar{u}|_{\bar{\Sigma}} = 0$ . Therefore,  $\bar{u}$  is constant on  $\partial \bar{E}$  and since  $\Psi_\alpha$  is constant on  $\partial \bar{E}$ , it follows that  $\left| \frac{\partial \bar{u}}{\partial \nu} \right|^2 \Big|_{int}$  is constant as well on  $\partial \bar{E}$ .

To sum-up, the first order necessary conditions drive to the following condition:

$$\text{The functions } \bar{u} \text{ and } |\nabla u| \text{ are constant on } \partial \bar{E}. \quad (3.15)$$

**Step 2: proof that  $\Omega$  is necessarily a ball.** To prove that  $\Omega$  is a ball, we will use Serrin's Theorem, that we recall hereafter.

**THEOREM** [165, Theorem 2] *Let  $\mathcal{E}$  be a connected domain with a  $\mathcal{C}^2$  boundary,  $h$  a  $\mathcal{C}^1(\mathbb{R}; \mathbb{R})$  function and let  $f \in \mathcal{C}^2(\overline{\mathcal{E}})$  be a function satisfying*

$$-\Delta f = h(f), \quad f > 0 \text{ in } \mathcal{E}, \quad f = 0 \text{ on } \partial\mathcal{E}, \quad \frac{\partial f}{\partial \nu} \text{ is constant on } \partial\mathcal{E}.$$

*Then  $\mathcal{E}$  is a ball and  $f$  is radially symmetric.*

According to (3.15), let us introduce  $\bar{\mu} = \bar{u}|_{\partial\overline{E}}$ . One has  $\bar{\mu} > 0$  by using the maximum principle. Let us set  $f = \bar{u} - \bar{\mu}$ ,  $h(z) = (\lambda_\alpha(\bar{m}) + \kappa)z$  and call  $\mathcal{E}$  a given connected component of  $\overline{E}$ . By assumption,  $\overline{E}$  is a  $\mathcal{C}^2$  set, and, according to (3.15), the function  $\partial(\bar{u} - \bar{\mu})/\partial\nu$  is constant on  $\partial\overline{E}$ .

The next result allows us to verify the last assumption of Serrin's theorem.

**LEMMA 3.3** *There holds  $\bar{u} > \bar{\mu}$  in  $\overline{E}$ .*

For the sake of clarity, the proof of this lemma is postponed to the end of this section.

Let us now pick a connected component of  $\overline{E}$  and write it  $\overline{E}_1$ . Applying Serrin's Theorem yields that  $\overline{E}_1$  is a ball centered at some point  $x_0$  and that  $\bar{u}$  is radially symmetric in  $\overline{E}_1$ . Let us introduce  $\mathcal{O} = \cup_{r_1 > 0, \bar{u} \text{ radially symmetric in } \mathbb{B}(x_0, r_1)} \mathbb{B}(x_0, r_1)$  so that  $\mathcal{O}$  is the maximal set in which  $\bar{u}$  is radially symmetric. Let us now show that one has necessarily  $\mathcal{O} = \Omega$ . Since  $\bar{u}$  is radially symmetric in  $\overline{E}_1$ ,  $\mathcal{O}$  is non-empty and there exists  $\tilde{\mu} \geq \bar{\mu}$  such that  $\Psi_\alpha = \tilde{\mu}$  on  $\partial\mathcal{O}$ . We argue by contradiction, assuming that  $\mathcal{O} \neq \Omega$ . It follows that the set  $U_\delta = \{\partial\mathcal{O} + \mathbb{B}(0; \delta)\}$  is contained in  $\Omega$  for  $\delta > 0$  small enough. Let us fix such a  $\delta$ . To get a contradiction we will show that  $\bar{m}$  is in fact radially symmetric in  $U_\delta$ .

If  $\tilde{\mu} = \bar{\mu}$ , then  $\partial\mathcal{O} \subset \partial\overline{E}$  and there exists  $\delta > 0$  such that  $U_\delta \setminus \overline{E} \subset \{\bar{m} = 0\}$  and  $U_\delta \cap \overline{E} \subset \{\bar{m} = \kappa\}$ . In any case,  $\bar{m}$  is radially symmetric in  $U_\delta$  which contradicts the maximality of  $\mathcal{O}$ . If  $\tilde{\mu} < \bar{\mu}$  then, by continuity of  $\Psi_\alpha$  it follows that, for  $\delta > 0$  small enough,  $U_\delta \subset \{\bar{m} = \kappa\}$ , so that  $\bar{m}$  is radially symmetric in  $U_\delta$  and we conclude as before. The conclusion follows. Hence,  $\mathcal{O} = \Omega$ , and  $\Omega$  is a ball.

*Proof of Lemma 3.3.* Let us set  $v = \bar{u} - \bar{\mu}$ , hence  $v$  solves

$$\begin{cases} -\Delta v = (\lambda_\alpha(\bar{m}) + \kappa)v + (\lambda_\alpha(\bar{m}) + \kappa)\bar{\mu} & \text{in } \overline{E}, \\ v = 0 & \text{on } \partial\overline{E}. \end{cases} \quad (3.16)$$

and we are led to show that  $v > 0$  in  $\overline{E}$ . Let  $\lambda^D(\Omega)$  be the first Dirichlet eigenvalue<sup>1</sup> of the Laplace operator in  $E$ . By using the Rayleigh quotient (4.3) we have

$$\lambda_\alpha(\bar{m}) = \min_{u \in W_0^{1,2}(\Omega), u \neq 0} R_{\alpha, m}(u) > \min_{u \in W_0^{1,2}(\Omega), u \neq 0} \frac{\frac{1}{2} \int_\Omega |\nabla u|^2 - \kappa \int_\Omega u^2}{\int_\Omega u^2} = \lambda^D(\Omega) - \kappa,$$

so that  $\lambda_\alpha(\bar{m}) + \kappa > \lambda^D(\Omega) > 0$ . Now, since  $v = 0$  on  $\partial\overline{E}$  and that  $\overline{E}$  is a  $\mathcal{C}^2$  open subset of  $\Omega$ , the extension  $\tilde{v}$  of  $v$  by zero outside  $\overline{E}$  belongs to  $W_0^{1,2}(\Omega)$ . Since  $(\lambda_\alpha(\bar{m}) + \kappa)$  and  $\bar{\mu}$  are non-negative, we get

$$-\Delta \tilde{v} \geq (\lambda_\alpha(\bar{m}) + \kappa)\tilde{v} \text{ in } \overline{E} \quad \text{and} \quad -\Delta \tilde{v} = 0 = (\lambda_\alpha(\bar{m}) + \bar{m})\tilde{v} \text{ in } (\overline{E})^c.$$

We thus have

$$\begin{cases} -\Delta \tilde{v} \geq (\lambda_\alpha(\bar{m}) + \kappa)\tilde{v} & \text{in } \Omega \\ \tilde{v} = 0 & \text{on } \partial\overline{E} \\ \tilde{v} \in W_0^{1,2}(\Omega). \end{cases} \quad (3.18)$$

<sup>1</sup>In other words

$$\lambda^D(\Omega) = \inf_{u \in W_0^{1,2}(\Omega), \int_\Omega u^2 = 1} \int_\Omega |\nabla u|^2 > 0. \quad (3.17)$$



Splitting  $\tilde{v}$  into its positive and negative parts as  $\tilde{v} = \tilde{v}_+ - \tilde{v}_-$  and multiply (3.18) by  $\tilde{v}_-$  we get after an integration by parts

$$-\int_{\Omega} |\nabla \tilde{v}_-|^2 \geq -(\lambda_{\alpha}(\bar{m}) + \kappa) \int_{\Omega} \tilde{v}_-^2.$$

Using that  $\lambda_{\alpha}(\bar{m}) + \kappa > \lambda^D(\Omega) > 0$ , we get

$$\int_{\Omega} |\nabla \tilde{v}_-|^2 \leq (\lambda_{\alpha}(\bar{m}) + \kappa) \int_{\Omega} \tilde{v}_-^2 < \lambda^D(\Omega) \int_{\Omega} \tilde{v}_-^2,$$

which, combined with the Rayleigh quotient formulation of  $\lambda^D(\Omega)$  yields  $\tilde{v}_- = 0$ . Hence  $v$  is nonnegative in  $\bar{E}$ . Using moreover that  $(\lambda_{\alpha}(\bar{m}) + \kappa) \geq 0$  and  $\bar{\mu} \geq 0$  yields that  $-\Delta v \geq 0$  in  $\bar{E}$ . Notice that  $v$  does not vanish identically in  $\bar{E}$ . Indeed,  $u$  would otherwise be constant in  $\bar{E}$  which cannot arise because of (4.2). According to the strong maximum principle, we infer that  $v > 0$  in  $\bar{E}$ .  $\square$

**REMARK 3.2** *Following the arguments by Casado-Diaz in [48], it would be possible to weaken the regularity assumption on  $E$  provided that we assume the stronger hypothesis that  $\partial\Omega$  is simply connected. Indeed, in that case, assuming that  $E$  is only of class  $\mathcal{C}^1$  leads to the same conclusion.*

### 3.4 PROOF OF THEOREM 3.1.2

Throughout this section,  $\Omega$  will denote the ball  $\mathbb{B}(0, R)$ , which will also be denoted  $\mathbb{B}$  for the sake of simplicity. Let  $r_0^* \in (0, R)$  be chosen in such a way that  $m_0^* = \kappa \mathbb{1}_{\mathbb{B}(0, r_0^*)}$  belongs to  $\mathcal{M}_{m_0, \kappa}(\Omega)$ . Let us introduce the notation  $E_0^* = \mathbb{B}(0, r_0^*)$ .

The existence part of the Theorem follows from a straightforward adaptation of [57]. In what follows, we focus on the second part of this theorem, that is, the stationarity of minimizers provided  $\alpha$  is small enough.

#### 3.4.1 STEPS OF THE PROOF FOR THE STATIONARITY

We argue by contradiction, assuming that, for any  $\alpha > 0$ , there exists a radially symmetric distribution  $\tilde{m}_{\alpha}$  such that  $\lambda_{\alpha}(\tilde{m}_{\alpha}) < \lambda_{\alpha}(m_0^*)$ . Consider the resulting sequence  $\{\tilde{m}_{\alpha}\}_{\alpha > 0}$ .

- **Step 1:** we prove that  $\{\tilde{m}_{\alpha}\}_{\alpha \rightarrow 0}$  converges strongly to  $m_0^*$  in  $L^1$ , as  $\alpha \rightarrow 0$ . Regarding the associated eigenfunction, we prove that  $\{u_{\alpha, m_{\alpha}}\}_{\alpha > 0}$  converges strongly to  $u_{0, m_0^*}$  in  $\mathcal{C}^0$  and that  $\alpha \nabla u_{\alpha, m_{\alpha}}$  converges to 0 in  $L^{\infty}(\mathbb{B})$ , as  $\alpha \rightarrow 0$ .
- **Step 2:** by adapting [117, Theorem 3.7], we prove that we can reduce ourselves to considering bang-bang radially symmetric distributions of resources  $\tilde{m}_{\alpha} = \kappa \mathbb{1}_{\tilde{E}}$  such that the Hausdorff distance  $d_H(\tilde{E}, E_0^*)$  is arbitrarily small.
- **Step 3:** this is the main innovation of the proof. Introduce  $h_{\alpha} = \tilde{m}_{\alpha} - m_0^*$ , and consider the path  $\{m_t\}_{t \in [0, 1]}$  from  $m_{\alpha}$  to  $m_0^*$  defined by  $m_t = m_0^* + th_{\alpha}$ . We then consider the mapping

$$f_{\alpha} : t \mapsto \zeta_{\alpha}(m_t, \Lambda_{-}(m_t))$$

where  $\zeta_{\alpha}$  and  $\Lambda_{-}(m_t)$  are respectively given in Def. 3.3.2 and Eq. (3.12). Notice that, since  $m_0^*$  and  $\tilde{m}_{\alpha}$  are bang-bang,  $f_{\alpha}(0) = \lambda_{\alpha}(m_0^*)$  and  $f_{\alpha}(1) = \lambda_{\alpha}(m_{\alpha})$  according to Def. 3.3.2. Let  $u_t$  be a  $L^2$  normalized eigenfunction associated with  $(m_t, \Lambda_{-}(m_t))$ , in other words a solution to the equation

$$\begin{cases} -\nabla \cdot (\Lambda_{-}(m_t) \nabla u_t) = \zeta_{\alpha}(m_t, \Lambda_{-}(m_t)) u_t + m_t u_t & \text{in } \mathbb{B} \\ u_t = 0 & \text{on } \partial\mathbb{B} \\ \int_{\mathbb{B}} u_t^2 = 1. \end{cases} \quad (3.19)$$

According to the proof of the optimality conditions (3.14), one has

$$f'_\alpha(t) = \int_{\mathbb{B}} h_\alpha \left( \frac{\alpha}{1 + \alpha\kappa} \Lambda_-(m_t)^2 |\nabla u_t|^2 - u_t^2 \right).$$

Applying the mean value Theorem yields the existence of  $t_1 \in (0, 1)$  such that  $\lambda_\alpha(\tilde{m}_\alpha) - \lambda_\alpha(m_0^*) = f'(t_1)$ . This enables us to show that, for  $t \in [0, 1]$  and  $\alpha$  small enough, one has

$$f'_\alpha(t) \geq C \int_{\mathbb{B}} |h_\alpha| \operatorname{dist}(\cdot, \mathbb{S}(0, r_0^*))$$

for some  $C > 0$ , giving in turn  $\lambda_\alpha(m_\alpha) - \lambda_\alpha(m^*) \geq C \int_{\mathbb{B}} |h_\alpha| \operatorname{dist}(\cdot, \mathbb{S}(0, r_0^*))$ . (we note that the same quantity is obtained in [117]. Nevertheless, we obtain it in a more straightforward manner which bypasses the exact decomposition of eigenfunctions and eigenvalues used there.)

Let us now provide the details of each step.

### 3.4.2 STEP 1: CONVERGENCE OF QUASI-MINIMIZERS AND OF SEQUENCES OF EIGENFUNCTIONS

We first investigate the convergence of quasi-minimizers.

**LEMMA 3.4** *Let  $\{m_\alpha\}_{\alpha>0}$  be a sequence in  $\mathcal{M}_{m_0, \kappa}(\Omega)$  such that,*

$$\forall \alpha > 0, \quad \lambda_\alpha(m_\alpha) \leq \lambda_\alpha(m_0^*). \quad (3.20)$$

*Then,  $\{m_\alpha\}_{\alpha>0}$  converges strongly to  $m_0^*$  in  $L^1(\Omega)$ .*

*Proof of Lemma 3.4.* The sequence  $(\lambda_\alpha(m_\alpha))_{\alpha>0}$  is bounded from above. Indeed, choosing any test function  $\psi \in W_0^{1,2}(\Omega)$  such that  $\int_\Omega \psi^2 = 1$ , it follows from (4.3) that  $\lambda_\alpha(m_\alpha) \leq (1 + \alpha\kappa) \|\nabla \psi\|_2^2 + \kappa \|\psi\|_2^2$ . Similarly, using once again (4.3), we get that if  $\xi_\alpha$  is the first eigenvalue associated to the operator  $-(1 + \alpha\kappa)\Delta - \kappa$ , then  $\lambda_\alpha(m_\alpha) \geq \xi_\alpha$ . Since  $(\xi_\alpha)_{\alpha>0}$  converges to the first eigenvalue of  $-\Delta - \kappa$  as  $\alpha \rightarrow 0$ ,  $(\xi_\alpha)_{\alpha>0}$  is bounded from below whenever  $\alpha$  is small enough. Combining these facts yields that the sequence  $(\lambda_\alpha(m_\alpha))_{\alpha>0}$  is bounded by some positive constant  $M$  and converges, up to a subfamily, to  $\tilde{\lambda}$ . For any  $\alpha > 0$ , let us denote by  $u_\alpha$  the associated  $L^2$ -normalized eigenfunction associated to  $\lambda_\alpha(m_\alpha)$ . From the weak formulation of equation (4.2) and the normalization condition  $\int_\Omega u_\alpha^2 = 1$ , we infer that

$$\|\nabla u_\alpha\|_2^2 = \int_\Omega |\nabla u_\alpha|^2 \leq \int_\Omega (1 + \alpha m) |\nabla u_\alpha|^2 = \int_\Omega m_\alpha u_\alpha^2 + \lambda_\alpha(m_\alpha) \int_\Omega u_\alpha^2 \leq (M + \kappa).$$

According to the Poincaré inequality and the Rellich-Kondrachov Theorem, the sequence  $(u_\alpha)_{\alpha>0}$  is uniformly bounded in  $W_0^{1,2}(\Omega)$  and converges, up to subfamily, to  $\tilde{u} \in W_0^{1,2}(\Omega)$  weakly in  $W_0^{1,2}(\Omega)$  and strongly in  $L^2(\Omega)$ , and moreover  $\tilde{u}$  is also normalized in  $L^2(\Omega)$ .

Furthermore, since  $L^2$  convergence implies pointwise convergence (up to a subfamily),  $\tilde{u}$  is necessarily nonnegative in  $\Omega$ . Let  $\tilde{m}$  be a closure point of  $(m_\alpha)_{\alpha>0}$  for the weak-star topology of  $L^\infty$ . Passing to the weak limit in the weak formulation of the equation solved by  $u_\alpha$ , namely Eq. (4.2), one gets

$$-\Delta \tilde{u} - \tilde{m} \tilde{u} = \tilde{\lambda} \tilde{u} \quad \text{in } \Omega.$$

Since  $\tilde{u} \geq 0$  and  $\int_{\mathbb{B}(0,R)} \tilde{u}^2 = 1$ , it follows that  $\tilde{u}$  is the principal eigenfunction of  $-\Delta - \tilde{m}$ , so that  $\tilde{\lambda} = \lambda_0(m^*)$ .

Mimicking this reasoning enables us to show in a similar way that, up to a subfamily,  $(\lambda_\alpha(m_0^*))_{\alpha>0}$  converges to  $\lambda_0(m_0^*)$  and  $(u_{\alpha, m_0^*})_{\alpha>0}$  converges to  $u_{0, m_0^*}$  as  $\alpha \rightarrow 0$ . Passing to the limit in the inequality

(3.20) and since  $m_0^*$  is the only minimizer of  $\lambda_0$  in  $\mathcal{M}_{m_0, \kappa}(\Omega)$  according to the Faber-Krahn inequality, we infer that necessarily,  $\tilde{m} = m_0^*$ . Moreover,  $m_0^*$  being an extreme point of  $\mathcal{M}_{m_0, \kappa}(\Omega)$ , the subfamily  $(m_\alpha)_{\alpha > 0}$  converges to  $\tilde{m} = m_0^*$  (see [93, Proposition 2.2.1]), strongly in  $L^1(\Omega)$ .  $\square$

A straightforward adaptation of Lemma 3.4's proof yields that both sets  $\{\lambda_\alpha(m)\}_{m \in \mathcal{M}_{m_0, \kappa}(\Omega)}$  and  $\{\|u_{\alpha, m}\|_{W^{1,2}(\Omega)}\}_{m \in \mathcal{M}_{m_0, \kappa}(\Omega)}$  are uniformly bounded whenever  $\alpha \leq 1$ . Let us hence introduce  $M_0 > 0$  such that

$$\forall \alpha \in [0, 1], \quad \max\{|\lambda_\alpha(m)|, \|u_{\alpha, m}\|_{W_0^{1,2}(\Omega)}\} \leq M_0. \quad (3.21)$$

The next result is the only ingredient of the proof of Theorem 3.1.2 where the low dimension assumption on  $n$  is needed.

**LEMMA 3.5** *Let us assume that  $n = 1, 2, 3$ . There exists  $M_1 > 0$  such that, for every radially symmetric distribution  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$  and every  $\alpha \in [0, 1]$ , there holds*

$$\|u_{\alpha, m}\|_{W^{1, \infty}(\mathbb{B})} \leq M_1.$$

Furthermore, define  $\tilde{\sigma}_{\alpha, m}$ ,  $\tilde{m}$  and  $\varphi_{\alpha, m} : (0, R) \rightarrow \mathbb{R}$  by

$$\forall x \in \mathbb{B}, \quad u_{\alpha, m}(x) = \varphi_{\alpha, m}(|x|), \quad \sigma_{\alpha, m}(x) = \tilde{\sigma}_{\alpha, m}(|x|), \quad m(x) = \tilde{m}(|x|),$$

then  $\tilde{\sigma}_{\alpha, m}(\varphi_{\alpha, m})'$  belongs to  $W^{1, \infty}(0, R)$ .

*Proof of Lemma 3.5.* This proof is inspired by [117, Proof of Theorem 3.3]. It is standard that for every  $\alpha \in [0, 1]$  and every radially symmetric distribution  $m \in \mathcal{M}_{m_0, \kappa}(\Omega)$ , the eigenfunction  $u_{\alpha, m}$  is itself radially symmetric. By rewriting the equation (4.2) on  $u_{\alpha, m}$  in polar coordinates, one sees that  $\varphi_{\alpha, m}$  solves

$$\begin{cases} -\frac{d}{dr} (r^{n-1} \tilde{\sigma}_{\alpha, m} \frac{d}{dr} \varphi_{\alpha, m}) = (\lambda_\alpha(m) \varphi_{\alpha, m} + \tilde{m} \varphi_{\alpha, m}) r^{n-1} & \text{in } (0, R) \\ \varphi_{\alpha, m}(R) = 0. \end{cases} \quad (3.22)$$

By applying the Hardy Inequality<sup>2</sup> on  $f = \varphi_{\alpha, m}$ , we get

$$\begin{aligned} \int_0^R \varphi_{\alpha, m}^2(r) dr &\leq 4 \int_0^R x^2 \varphi_{\alpha, m}'(x)^2 dx \\ &\leq 4R^2 \int_0^R \left(\frac{x}{R^2}\right)^{n-1} \varphi_{\alpha, m}'(x)^2 dx = 4R^{4-2n} \|\nabla u_{\alpha, m}\|_{L^2(\mathbb{B})}^2 \leq M, \end{aligned}$$

since  $n \in \{1, 2, 3\}$ . Hence, there exists  $C > 0$  such that

$$\|\varphi_{\alpha, m}\|_{L^2(0, R)}^2 \leq C. \quad (3.23)$$

We will successively prove that  $\varphi_{\alpha, m}$  is uniformly bounded in  $W_0^{1,2}(0, R)$ , then in  $L^\infty(0, R)$  to infer that  $\varphi_{\alpha, m}'$  is bounded in  $L^\infty(0, R)$ . This proves in particular that  $\sigma_{\alpha, m} \varphi_{\alpha, m}' \in L^\infty(0, R)$ . We will then conclude that  $\sigma_{\alpha, m} \varphi_{\alpha, m}' \in W^{1, \infty}(0, R)$  by using that it is a continuous function whose derivative is uniformly bounded in  $L^\infty$  by the equation on  $\varphi_{\alpha, m}$ .

According to (3.21),  $\|r^{\frac{n-1}{2}} \varphi_{\alpha, m}'\|_{L^2(0, R)} = \|\nabla u_{\alpha, m}\|_{L^2(\mathbb{B})}$  is bounded and therefore,  $r^{n-1} \varphi_{\alpha, m}'(r)$

<sup>2</sup>This inequality reads (see e.g. [150, Lemma 1.3] or [87]): for any non-negative  $f$ ,

$$\int_0^\infty f(x)^2 dx \leq 4 \int_0^\infty x^2 f'(x)^2 dx.$$

converges to 0 as  $r \rightarrow 0$ . Hence, integrating Eq. (3.22) between 0 and  $r > 0$  yields

$$\tilde{\sigma}_{\alpha,m}(r)\varphi_{\alpha,m}'(r) = -\frac{1}{r^{n-1}} \int_0^r t^{n-1} (\lambda_\alpha(m)\varphi_{\alpha,m}(t) + \tilde{m}(t)\varphi_{\alpha,m}(t)) dt.$$

By using the Cauchy-Schwarz inequality and (3.23), we get the existence of  $\tilde{M} > 0$  such that

$$\begin{aligned} \|\varphi_{\alpha,m}'\|_{L^2(0,R)}^2 &\leq \int_0^1 (\tilde{\sigma}_{\alpha,m}\varphi_{\alpha,m}')^2(r) dr \\ &= \int_0^1 \frac{1}{r^{2(n-1)}} \left( \int_0^r t^{n-1} (\lambda_\alpha(m)\varphi_{\alpha,m}(t) + \tilde{m}(t)\varphi_{\alpha,m}(t)) dt \right)^2 dr \\ &\leq \int_0^1 \frac{1}{r^{2(n-1)}} (\lambda_\alpha(m) + \kappa)^2 \|\varphi_{\alpha,m}\|_{L^2(0,R)}^2 \|t^{n-1}\|_{L^2(0;r)}^2 dr \\ &\leq \frac{\tilde{M}}{4n-2} \|\varphi_{\alpha,m}\|_{L^2(0,R)}^2 \leq \tilde{M} \|\varphi_{\alpha,m}\|_{L^2(0,R)}^2 \leq \tilde{M}C, \end{aligned}$$

Hence,  $\varphi_{\alpha,m}$  is uniformly bounded in  $W_0^{1,2}(0,R)$ .

It follows from standard Sobolev embedding's theorems that there exists a constant  $M_2 > 0$ , such that  $\|\varphi_{\alpha,m}\|_{L^\infty(0,R)} \leq M_2$ .

Finally, plugging this estimate in the equality

$$\tilde{\sigma}_{\alpha,m}(r)\varphi_{\alpha,m}'(r) = -\frac{1}{r^{n-1}} \int_0^r t^{n-1} (\lambda_\alpha(m)\varphi_{\alpha,m}(t) + \tilde{m}(t)\varphi_{\alpha,m}(t)) dt$$

and since  $t^{n-1} \leq r^{n-1}$  on  $(0,r)$ , we get that  $\varphi_{\alpha,m}'$  is uniformly bounded in  $L^\infty(0,R)$ .  $\square$

The next lemma is a direct corollary of Lemma 3.4, Lemma 3.5 and the Arzela-Ascoli Theorem.

**LEMMA 3.6** *Let  $(m_\alpha)_{\alpha>0}$  be a sequence of radially symmetric functions of  $\mathcal{M}_{m_0,\kappa}(\Omega)$  such that, for every  $\alpha \in [0,1]$ ,  $\lambda_\alpha(m_\alpha) \leq \lambda_\alpha(m_0^*)$ . Then, up to a subfamily,  $u_{\alpha,m_\alpha}$  converges to  $u_{0,m_0^*}$  for the strong topology of  $\mathcal{C}^0(\bar{\Omega})$  as  $\alpha \rightarrow 0$ .*

### 3.4.3 STEP 2: REDUCTION TO PARTICULAR RESOURCE DISTRIBUTIONS CLOSE TO $m_0^*$

Let us consider a sequence of radially symmetric distributions  $(m_\alpha)_{\alpha>0}$  such that, for every  $\alpha \in [0,1]$ ,  $\lambda_\alpha(m_\alpha) \leq \lambda_\alpha(m_0^*)$ . According to Proposition 3.1, we can assume that each  $m_\alpha$  is a *bang-bang*, in other words that  $m_\alpha = \kappa\chi_{E_\alpha}$  where  $E_\alpha$  is a measurable subset of  $\mathbb{B}(0,R)$ . For every  $\alpha \in [0,1]$ , one introduces  $d_\alpha = d_H(E_\alpha, E_0^*)$ , the Hausdorff distance of  $E_\alpha$  to  $E_0^*$ .

**LEMMA 3.7** *For every  $\varepsilon > 0$  small enough, there exists  $\bar{\alpha} > 0$  such that, for every  $\alpha \in [0,\bar{\alpha}]$ , there exists a measurable subset  $\tilde{E}_\alpha$  of  $\Omega$  such that*

$$\lambda_\alpha(\kappa\chi_{E_\alpha}) \geq \lambda_\alpha(\kappa\chi_{\tilde{E}_\alpha}), \quad |E_\alpha| = |\tilde{E}_\alpha| \quad \text{and} \quad d_H(\tilde{E}_\alpha, E_0^*) \leq \varepsilon.$$

*Proof of Lemma 3.7.* Let  $\alpha \in [0,1]$ . Observe first that  $\lambda_\alpha(m) = \int_{\mathbb{B}} |\nabla u_{\alpha,m}|^2 + \alpha \int_{\mathbb{B}} m |\nabla u_{\alpha,m}|^2 - \int_{\mathbb{B}} m u_{\alpha,m}^2 = \int_{\mathbb{B}} |\nabla u_{\alpha,m}|^2 + \int_{\mathbb{B}} m \psi_{\alpha,m}$ , where  $\psi_{\alpha,m}$  has been introduced in Lemma 3.2. We will first construct  $\tilde{m}_\alpha$  in such a way that

$$\lambda_\alpha(m_\alpha) \geq \int_{\mathbb{B}} |\nabla u_{\alpha,m_\alpha}|^2 + \int_{\Omega} \psi_{\alpha,m_\alpha} \tilde{m}_\alpha \geq \lambda_\alpha(\tilde{m}_\alpha),$$

and, to this aim, we will define  $\tilde{m}_\alpha$  as a suitable level set of  $\psi_{\alpha, m_\alpha}$ . Thus, we will evaluate the Hausdorff distance of these level sets to  $E_0^*$ . The main difficulty here rests upon the lack of regularity of the switching function  $\psi_{\alpha, m_\alpha}$ , which is even not continuous.

According to Lemmas 3.5 and 3.6,  $\psi_{\alpha, m_\alpha}$  converges to  $-u_{0, m_0^*}^2$  for the strong topology of  $L^\infty(\mathbb{B})$ . Recall that  $m_0^* = \kappa \chi_{\mathbb{B}(0, r_0^*)}$  and let  $V_0$  be defined by  $V_0 = |\mathbb{B}(0, r_0^*)|$ . Let us define  $\mu_\alpha^*$  by dichotomy, as the only real number such that

$$|\underline{\omega}_\alpha| \leq V_0 \leq |\bar{\omega}_\alpha|,$$

where  $\underline{\omega}_\alpha = \{\psi_{\alpha, m_\alpha} < \mu_\alpha^*\}$  and  $\bar{\omega}_\alpha = \{\psi_{\alpha, m_\alpha} \leq \mu_\alpha^*\}$ .

Since  $|\{\psi_{0, m_0^*} < -\varphi_{0, m_0^*}^2(r_0^*)\}| = V_0$ , we deduce that  $(\mu_\alpha^*)$  converges to  $-\varphi_{0, m_0^*}^2(r_0^*)$  as  $\alpha \rightarrow 0$ . Since  $\varphi_{0, m_0^*}$  is decreasing, we infer that for any  $\varepsilon > 0$  small enough, there exists  $\bar{\alpha} > 0$  such that: for every  $\alpha \in [0, \bar{\alpha}]$ ,  $\mathbb{B}(0; r_0^* - \varepsilon) \subset \underline{\omega}_\alpha \subset \bar{\omega}_\alpha \subset \mathbb{B}(0; r_0^* + \varepsilon)$ . Therefore, there exists a radially symmetric set  $B_\varepsilon^\alpha$  such that

$$\underline{\omega}_\alpha \subset B_\varepsilon^\alpha \subset \bar{\omega}_\alpha, \quad |B_\varepsilon^\alpha| = V_0, \quad d_H(B_\varepsilon^\alpha, E_0^*) \leq \varepsilon.$$

Since  $E_\alpha$  and  $B_\varepsilon^\alpha$  have the same measure, one has  $|(E_\alpha)^c \cap B_\varepsilon^\alpha| = |E_\alpha \cap (B_\varepsilon^\alpha)^c|$ , we introduce  $\tilde{m}_\alpha = \kappa \chi_{B_\varepsilon^\alpha}$  so that  $\tilde{m}_\alpha$  belongs to  $\mathcal{M}_{m_0, \kappa}(\Omega)$ .

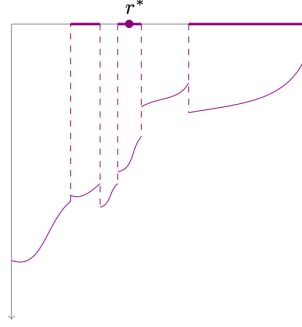


Figure 3.1 – Possible graph of the discontinuous function  $\psi_{\alpha, m_\alpha}$ . The bold intervals on the  $x$  axis correspond to  $\{m_\alpha = 0\}$ .

By construction, one has

$$\begin{aligned} \lambda_\alpha(m_\alpha) &= \int_{\mathbb{B}} (1 + \alpha m_\alpha) |\nabla u_{\alpha, m_\alpha}|^2 - \int_{\mathbb{B}} m_\alpha u_{\alpha, m_\alpha}^2 = \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \int_{\mathbb{B}} \psi_{\alpha, m_\alpha} m_\alpha \\ &= \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \kappa \int_{E_\alpha} \psi_{\alpha, m_\alpha} = \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \kappa \int_{E_\alpha \cap (B_\varepsilon^\alpha)^c} \psi_{\alpha, m_\alpha} + \kappa \int_{E_\alpha \cap B_\varepsilon^\alpha} \psi_{\alpha, m_\alpha} \\ &\geq \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \kappa \mu_\alpha^* |E_\alpha \cap (B_\varepsilon^\alpha)^c| + \kappa \int_{E_\alpha \cap B_\varepsilon^\alpha} \psi_{\alpha, m_\alpha} \\ &= \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \kappa \mu_\alpha^* |(E_\alpha)^c \cap B_\varepsilon^\alpha| + \kappa \int_{E_\alpha \cap B_\varepsilon^\alpha} \psi_{\alpha, m_\alpha} \\ &\geq \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \kappa \int_{(E_\alpha)^c \cap B_\varepsilon^\alpha} \psi_{\alpha, m_\alpha} + \kappa \int_{E_\alpha \cap B_\varepsilon^\alpha} \psi_{\alpha, m_\alpha} = \int_{\mathbb{B}} |\nabla u_{\alpha, m_\alpha}|^2 + \int_{\mathbb{B}} \tilde{m}_\alpha \psi_{\alpha, m_\alpha} \\ &= \int_{\mathbb{B}} \sigma_{\alpha, \tilde{m}_\alpha} |\nabla u_{\alpha, m_\alpha}|^2 - \int_{\mathbb{B}} \tilde{m}_\alpha u_{\alpha, m_\alpha}^2 \geq \lambda_\alpha(\tilde{m}_\alpha), \end{aligned}$$

the last inequality coming from the variational formulation (4.3). The expected conclusion thus follows.  $\square$

From now on we will replace  $m_\alpha$  by  $\kappa\chi_{\tilde{E}_\alpha}$  and still denote this function by  $m_\alpha$  with a slight abuse of notation.

### 3.4.4 STEP 3: CONCLUSION, BY THE MEAN VALUE THEOREM

Recall that, according to Section 3.4.1, for every  $\alpha \in [0, 1]$ , the mapping  $f_\alpha$  is defined by  $f_\alpha(t) := \zeta_\alpha(m_t, \Lambda_-(m_t))$  for all  $t \in [0, 1]$ . We claim that  $f_\alpha$  belongs to  $\mathcal{C}^1$ . This follows from similar arguments to those of the  $L^2$  differentiability of  $m \mapsto \lambda_\alpha(m)$  in Appendix 4.B.3. Following the proof of (3.14), it is also straightforward that for every  $t \in [0, 1]$ , one has

$$f'_\alpha(t) = \int_{\mathbb{B}} \left( \frac{\alpha}{1 + \alpha\kappa} \Lambda_-(m_t)^2 |\nabla u_t|^2 - u_t^2 \right) h_\alpha. \quad (3.24)$$

Finally, since  $m_0^*$  and  $m_\alpha$  are *bang-bang*, it follows from Definition 3.3.2 that  $f_\alpha(0) = \lambda_\alpha(m^*)$  and  $f_\alpha(1) = \lambda_\alpha(m_\alpha)$ .

Since  $m_\alpha$  is assumed to be radially symmetric, so is  $m_t$  for every  $t \in [0, 1]$  thanks to a standard reasoning, and, therefore, so is  $u_t$ . With a slight abuse of notation, we identify  $m_t$ ,  $u_t$  and  $\Lambda_-(m_t)$  with their radial part  $\tilde{m}_t$ ,  $\tilde{u}_t$ ,  $\tilde{\Lambda}_-(m_t)$  defined on  $[0, R]$  by

$$u_t(x) = \tilde{u}_t(|x|), \quad m_t(x) = \tilde{m}_t(|x|), \quad \Lambda_-(m_t)(x) = \tilde{\Lambda}_-(\tilde{m}_t)(|x|).$$

Then the function  $u_t$  (defined on  $[0, R]$ ) solves the equation

$$\begin{cases} -\frac{d}{dr} \left( r^{n-1} \Lambda_-(m_t) \frac{du_t}{dr} \right) = (\zeta_\alpha(m_t, \Lambda_-(m_t)) u_t + m_t u_t) r^{n-1} & r \in [0; R] \\ u_t(R) = 0 \\ \int_0^R r^{n-1} u_t(r)^2 dr = \frac{1}{c_n}, \end{cases} \quad (3.25)$$

where  $c_n = |\mathbb{S}(0, 1)|$ . As a consequence, an immediate adaptation of the proof of Lemma 3.5 yields:

**LEMMA 3.8** *There exists  $M > 0$  such that*

$$\max \left\{ \|u_t\|_{W^{1,\infty}}, \left\| \Lambda_-(m_t) u'_t \right\|_{W^{1,\infty}} \right\} \leq M.$$

Furthermore,  $\Lambda_-(m_t) u'_t$  converges to  $u'_{0,m^*}$  in  $L^\infty(0, R)$  and uniformly with respect to  $t \in [0, 1]$ , as  $\alpha \rightarrow 0$ .

According to the mean value Theorem, there exists  $t_1 = t_1(\alpha) \in [0, 1]$  such that

$$\lambda_\alpha(m_\alpha) - \lambda_\alpha(m^*) = f_\alpha(1) - f_\alpha(0) = f'_\alpha(t_1)$$

and by using Eq. (3.24), one has

$$f'_\alpha(t_1) = \int_{\mathbb{B}} \left( \frac{\alpha}{1 + \alpha\kappa} \Lambda_-(m_{t_1})^2 |\nabla u_{t_1}|^2 - u_{t_1}^2 \right) h_\alpha,$$

where  $h_\alpha = m_\alpha - m_0^*$ . Let us introduce  $I_\alpha^\pm$  as the two subsets of  $[0, R]$  given by  $I_\alpha^\pm = \{h_\alpha = \pm 1\}$ . Let  $\varepsilon > 0$ . According to Lemma 3.7, we have, for  $\alpha$  small enough,  $I_\alpha^+ \subset [r_0^*, r_0^* + \varepsilon]$  and  $I_\alpha^- \subset [r_0^* - \varepsilon, r_0^*]$ . Finally, let us introduce

$$\mathfrak{F}_1 := \frac{\alpha}{1 + \alpha\kappa} \Lambda_-(m_{t_1})^2 |\nabla u_{t_1}|^2 - u_{t_1}^2.$$

According to Lemma 3.8,  $\mathfrak{F}_1$  belongs to  $W^{1,\infty}$  and  $\mathfrak{F}_1 + u_{\alpha, m_0^*}^2$  converges to 0 as  $\alpha \rightarrow 0$ , for the strong topology of  $W^{1,\infty}(0, R)$ . Moreover, there exists  $M > 0$  independent of  $\alpha$  such that for  $\varepsilon > 0$  small

enough,

$$-M \leq 2u_{\alpha, m_0^*} \frac{du_{\alpha, m_0^*}}{dr} \leq -M \quad \text{in } [r_0^* - \varepsilon, r_0^* + \varepsilon]$$

and it follows that

$$\frac{M}{2} \leq \frac{d\mathfrak{F}_1}{dr} \leq 2M \quad \text{in } [r_0^* - \varepsilon, r_0^* + \varepsilon]$$

for  $\alpha$  small enough. Hence, since  $\mathfrak{F}_1$  is Lipschitz continuous and thus absolutely continuous, one has for every  $y \in [0, \varepsilon]$ ,

$$\begin{aligned} \mathfrak{F}_1(r_0^* + y) &= \mathfrak{F}_1(r_0^*) + \int_{r_0^*}^{r_0^* + y} \mathfrak{F}'_1(s) ds \geq \mathfrak{F}_1(r_0^*) + \frac{M}{2}y \\ \text{and } \mathfrak{F}_1(r_0^* - y) &= \mathfrak{F}_1(r_0^*) + \int_{r_0^* - y}^{r_0^*} (-\mathfrak{F}'_1(s)) ds \leq \mathfrak{F}_1(r_0^*) - \frac{M}{2}y. \end{aligned}$$

Since  $h_\alpha \leq 0$  in  $[r_0^* - \varepsilon, r_0^*]$  and  $h_\alpha \geq 0$  in  $[r_0^*, r_0^* + \varepsilon]$ , we have

$$\begin{aligned} h_\alpha(r_0^* + y)\mathfrak{F}_1(r_0^* + y) &\geq h_\alpha(r_0^* + y)\mathfrak{F}_1(r_0^*) + \frac{|h_\alpha|(r_0^* + y)M}{2}y \\ \text{and } h_\alpha(r_0^* - y)\mathfrak{F}_1(r_0^* - y) &\geq h_\alpha(r_0^* - y)\mathfrak{F}_1(r_0^*) + \frac{|h_\alpha|(r_0^* - y)M}{2}y. \end{aligned}$$

for every  $y \in [0, \varepsilon]$ . Hence, using that  $\int_{\mathbb{B}} h_\alpha = 0$ , we infer that

$$\begin{aligned} f'_\alpha(t_1) &= \int_{\mathbb{B}} \left( \frac{\alpha}{1 + \alpha\kappa} \Lambda_-(m_{t_1})^2 |\nabla u_{t_1}|^2 - u_{t_1}^2 \right) h_\alpha = c_n \int_0^R h_\alpha(s) \mathfrak{F}_1(s) s^{n-1} ds \\ &= c_n \left( \int_{r_0^* - \varepsilon}^{r_0^*} h_\alpha \mathfrak{F}_1(s) s^{n-1} ds + \int_{r_0^*}^{r_0^* + \varepsilon} h_\alpha \mathfrak{F}_1(s) s^{n-1} ds \right) \\ &\geq c_n \left( \int_{r_0^* - \varepsilon}^{r_0^*} h_\alpha(s) \mathfrak{F}_1(r_0^*) s^{n-1} ds + \int_{r_0^*}^{r_0^* + \varepsilon} h_\alpha(s) \mathfrak{F}_1(r_0^*) s^{n-1} ds \right) \\ &\quad + \frac{c_n M}{2} \left( \int_{r_0^* - \varepsilon}^{r_0^*} |h_\alpha(s)| |r_0^* - s| s^{n-1} ds + \int_{r_0^*}^{r_0^* + \varepsilon} |h_\alpha(s)| |r_0^* - s| s^{n-1} ds \right) \\ &= \frac{c_n M}{2} \int_{\mathbb{B}} |h_\alpha| \text{dist}(\cdot, \mathbb{S}(0, r_0^*)), \end{aligned}$$

which concludes Step 3. Theorem 3.1.2 is thus proved.

**REMARK 3.3** *Regarding the proof of Theorem 3.1.2, it would have been more natural to consider the path  $t \mapsto (\lambda_\alpha(m_t), m_t)$  rather than  $t \mapsto (\zeta_\alpha(m_t, \Lambda_-(m_t)), m_t)$ . However, we would have been led to consider  $\mathfrak{G}_t = \alpha\kappa |\nabla u_{\alpha, m_t}|^2 - u_{\alpha, m_t}^2$  instead of  $\mathfrak{F}_t$ . Unfortunately, this would have been more intricate because of the regularity of  $\mathfrak{G}_t$ , which is discontinuous and thus, no longer a  $W^{1, \infty}$  function, so that a Lemma analogous to Lemma 3.8 would not be true. Adapting step by step the arguments of [117] would nevertheless be possible although much more technical.*

### 3.5 SKETCH OF THE PROOF OF COROLLARY 3.1

We do not give all details since the proof is then very similar to the ones written previously. We only underline the slight differences in every step.

To prove this result, we consider the following relaxation of our problem, which is reminiscent of the problems considered in [84]. Let us consider, for any pair  $(m_1, m_2) \in \mathcal{M}_{m_0, \kappa}(\Omega)^2$ , the first eigenvalue of the operator  $\mathcal{N} : u \mapsto -\nabla \cdot ((1 + \alpha m_1) \nabla u) - m_2 u$ , and write it  $\eta_\alpha(m_1, m_2)$ . Let  $m^* := \kappa \mathbb{1}_{\mathbb{B}(0, R)}$ . By using the results of [84] or alternatively, applying the rearrangement of Alvino and Trombetti, [7] as it has been done in [57], one proves the existence of a radially symmetric function  $\tilde{m}_1$  such that

$$\eta_\alpha(m_1, m_2) \geq \eta_\alpha(\tilde{m}_1, m^*),$$

so that we are done if we can prove that, for any  $m \in \mathcal{M}(\Omega)$  there holds

$$\eta_\alpha(m, m^*) \geq \eta_\alpha(m^*, m^*). \quad (3.26)$$

We claim that (3.26) holds for any  $m \in \mathcal{M}_{m_0, \kappa}$ , provided that  $m_0$  and  $\alpha$  be small enough. Let us describe the main steps of the proof:

- **Step 1:** mimicking the compactness argument used in [57], one shows that there exists a solution  $m_\alpha$  to the problem

$$\inf_{m \in \mathcal{M}_{m_0, \kappa}(\Omega)} \eta_\alpha(m, m^*),$$

which is radially symmetric and bang-bang. We write it  $m_\alpha = \kappa \mathbb{1}_{E_\alpha}$ .

- **Step 2:** let  $\mu_0$  and  $r_0^*$  be the unique real numbers such that

$$|\{|\nabla u_{0, m^*}|^2 \leq \mu_0\}| = V_0 = |\mathbb{B}(0, r_0^*)|.$$

Introducing  $E_0 = \{|\nabla u_{0, m^*}|^2 \leq \mu_0\}$ , we prove that  $m_\alpha$  converges in  $L^1(\Omega)$  to  $\kappa \mathbb{1}_{E_0}$  as  $\alpha \rightarrow 0$ .

- **Step 3:** we establish that if  $m_0$  is small enough, then  $E_0 = \mathbb{B}(0, r_0^*)$ . This is done by proving that  $u_{0, m^*}$  converges in  $\mathcal{C}^1$  to the first Dirichlet eigenfunction of the ball as  $r_0^* \rightarrow 0$  and by determining the level-sets of this first eigenfunction, as done in [56, Section 2.2].
- **Step 4:** once this limit identified, we mimick the steps of the proof of Theorem 3.1.2 (reduction to a small Hausdorff distance and mean value Theorem for a well-chosen auxiliary function) to conclude that one necessarily has  $m_\alpha = m^*$  for  $\alpha$  small enough.

## 3.6 PROOF OF THEOREM 3.1.3

Throughout this section, we will denote by  $\mathbb{B}^*$  the ball  $\mathbb{B}(0, r_0^*)$ , where  $r_0^*$  is chosen so that  $m_0^* = \kappa \chi_{\mathbb{B}^*}$  belongs to  $\mathcal{M}_{m_0, \kappa}(\Omega)$ .

When it makes sense, we will write  $f|_{int}(y) = \lim_{x \in \mathbb{B}^*, x \rightarrow y} f(x)$ ,  $f|_{ext}(y) := \lim_{x \in (\mathbb{B}^*)^c, x \rightarrow y} f(y)$ , so that  $[f] = f|_{ext} - f|_{int}$  denotes the jump of  $f$  at the boundary  $\mathbb{S}(0, r_0^*)$ .

### 3.6.1 PRELIMINARIES

For  $\varepsilon > 0$ , let us introduce  $\mathbb{B}_\varepsilon^* := (\text{Id} + \varepsilon V) \mathbb{B}^*$  and define  $u_\varepsilon$  as the  $L^2$ -normalized first eigenfunction associated with  $m_\varepsilon = \kappa \chi_{\mathbb{B}_\varepsilon^*}$ .

It is well known (see e.g. [92, 93]) that  $u_\varepsilon$  expands as

$$u_\varepsilon = u_{0, \alpha} + \varepsilon u_{1, \alpha} + \varepsilon^2 \frac{u_{2, \alpha}}{2} + o_{\varepsilon \rightarrow 0}(\varepsilon^2), \quad (3.27)$$

where, in particular,  $u_{0, \alpha} = u_{\alpha, m_0^*}$ , whereas  $\lambda_\alpha(\mathbb{B}_\varepsilon^*)$  expands as

$$\lambda_\alpha(\mathbb{B}_\varepsilon^*) = \lambda_{0, \alpha} + \varepsilon \lambda'_\alpha(\mathbb{B}^*)[V] + \varepsilon^2 \lambda''_\alpha(\mathbb{B}^*)[V] + o_{\varepsilon \rightarrow 0}(\varepsilon^2)$$



$$= \lambda_{0,\alpha} + \varepsilon \lambda_{1,\alpha} + \varepsilon^2 \lambda_{2,\alpha} + o_{\varepsilon \rightarrow 0}(\varepsilon^2). \quad (3.28)$$

By mimicking the proof of Lemma 3.5, one shows the following symmetry result.

**LEMMA 3.9**  $u_{\alpha, m_0^*}$  is a radially symmetric function. Let  $\varphi_{\alpha, m_0^*}$  and  $\tilde{m}$  be such that  $u_{\alpha, m_0^*} = \varphi_{\alpha, m_0^*}(|\cdot|)$  and  $m_0^* = \tilde{m}(|\cdot|)$ , then  $\varphi_{\alpha, m_0^*}$  satisfies the ODE

$$\begin{cases} -\frac{d}{dr} \left( r^{n-1} \tilde{\sigma}_{\alpha, m_0^*} \varphi'_{\alpha, m_0^*} \right) = (\lambda_{\alpha}(m_0^*) + \tilde{m}) \varphi_{\alpha, m_0^*} r^{n-1} & \text{in } (0, R) \\ \varphi_{\alpha, m_0^*}(R) = 0 \end{cases} \quad (3.29)$$

and one has the following jump conditions

$$[\varphi_{\alpha, m_0^*}](r_0^*) = [\sigma_{\alpha, m_0^*} \varphi'_{\alpha, m_0^*}](r_0^*) = 0, \quad [\sigma_{\alpha} \varphi''_{\alpha, m_0^*}](r_0^*) = \kappa \varphi_{\alpha, m_0^*}(r_0^*). \quad (3.30)$$

Furthermore,  $\varphi_{\alpha, m_0^*}$  converges to  $\varphi_{0, m_0^*}$  for the strong topology of  $\mathcal{C}^1$  as  $\alpha \rightarrow 0$ .

### 3.6.2 COMPUTATION OF THE FIRST AND SECOND ORDER SHAPE DERIVATIVES

Hadamard's structure theorem enables us to work with only normal vector fields  $V$  to compute the second order derivative. Since we are working in dimension 2, this means that one can deal with vector fields  $V$  given in polar coordinates by

$$V(r_0^*, \theta) = g(\theta)(\cos \theta, \sin \theta).$$

The proof of the shape differentiability at the first and second order of  $\lambda_{\alpha}$ , based on the method of [139], is exactly similar to [63, Proof of Theorem 2.2], and are thus omitted in [MNP19a]. Nevertheless, for the sake of completeness, we present the proofs and computations related to these shape differentiability properties in Appendix 3.A.1.1 of this Chapter.

**Computation and analysis of the first order shape derivative.** Let us prove that  $\mathbb{B}^*$  is a critical shape in the sense of (3.7).

**LEMMA 3.10** The first order shape derivative of  $\lambda_{\alpha}$  at  $\mathbb{B}^*$  in direction  $V$  reads

$$\lambda_{1,\alpha} = \lambda'_{\alpha}(\mathbb{B}^*)[V] = \int_{\mathbb{S}(0, r_0^*)} V \cdot \nu. \quad (3.31)$$

For all  $V \in \mathcal{X}(\mathbb{B}^*)$  (defined by (3.6)), one has  $\lambda_{1,\alpha} = 0$  meaning that  $\mathbb{B}^*$  satisfies (3.7).

*Proof of Lemma 3.10.* First, elementary computations show that  $u_{1,\alpha}$  solves

$$\begin{cases} -\nabla \cdot (\sigma_{\alpha} \nabla u_{1,\alpha}) = \lambda_{1,\alpha} u_{0,\alpha} + \lambda_{0,\alpha} u_{1,\alpha} + m^* u_{1,\alpha} & \text{in } \mathbb{B}(0, R), \\ \left[ \sigma_{\alpha} \frac{\partial u_{1,\alpha}}{\partial \nu} \right] (r_0^* \cos \theta, r_0^* \sin \theta) = -\kappa g(\theta) u_{0,\alpha}, \\ [u_{1,\alpha}](r_0^* \cos \theta, r_0^* \sin \theta) = -g(\theta) \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] (r_0^* \cos \theta, r_0^* \sin \theta), \end{cases} \quad (3.32)$$

where the jumps denote the jumps of the functions at  $\mathbb{S}(0, r_0^*)$ . The derivation of the main equation of (3.32) is provided in Appendix 4.B.2. To derive the jump on  $u_{1,\alpha}$ , we follow [63] and differentiate the continuity equation  $[u_{\varepsilon}]_{\partial \mathbb{B}_{\varepsilon}^*} = 0$ . Formally plugging (3.27) in this equation yields

$$u_{1,\alpha}|_{int}(r^*, \theta) + g(\theta) \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} = u_{1,\alpha}|_{ext}(r^*, \theta) + g(\theta) \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{ext},$$

and hence

$$[u_{1,\alpha}] = u_{1,\alpha}|_{ext} - u_{1,\alpha}|_{int} = -g(\theta) \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right].$$

Note that the same goes for the normal derivative: we differentiate the continuity equation

$$\left[ (1 + \alpha m_\varepsilon) \frac{\partial u_{\varepsilon,\alpha}}{\partial \nu} \right]_{\partial \mathbb{B}_\varepsilon^*} = 0,$$

yielding

$$\left[ \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial r} \right] = -g(\theta) \left[ \sigma_\alpha \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right].$$

According to the equation  $-\sigma_\alpha \Delta u_{0,\alpha} = \lambda_\alpha(m^*)u_{0,\alpha} + m^*u_{0,\alpha}$  in  $\mathbb{B}^*$ , this rewrites

$$\left[ \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial r} \right] = -\kappa g(\theta) u_{0,\alpha}. \quad (3.33)$$

Now, using  $u_{0,\alpha}$  as a test function in (3.32), we get

$$\begin{aligned} \lambda_{1,\alpha} &= - \int_{\mathbb{B}(0,R)} \sigma_\alpha u_{0,\alpha} \Delta u_{1,\alpha} - \int_{\mathbb{B}(0,R)} m_0^* u_{1,\alpha} u_{0,\alpha} = - \int_{\mathbb{B}(0,R)} \sigma_\alpha u_{0,\alpha} \Delta u_{1,\alpha} + \int_{\mathbb{B}(0,R)} \sigma_\alpha u_{1,\alpha} \Delta u_{0,\alpha} \\ &= \int_{\mathbb{S}(0,r_0^*)} u_{0,\alpha} \left[ \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial \nu} \right] - \int_{\mathbb{S}(0,r_0^*)} \left[ \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} u_{1,\alpha} \right] \\ &= -r_0^* \int_0^{2\pi} \kappa g(\theta) u_{0,\alpha}(r_0^*)^2 d\theta + r_0^* \int_0^{2\pi} g(\theta) \left( \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \right) \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] d\theta \\ &= r_0^* \int_0^{2\pi} g(\theta) \left( -\kappa u_{0,\alpha}(r_0^*)^2 + \left[ \sigma_\alpha \left( \frac{\partial u_{0,\alpha}}{\partial r} \right)^2 \right] \right) d\theta. \end{aligned}$$

by using that  $\int_{\mathbb{B}(0,R)} u_\varepsilon^2 = 1$ , so that  $\int_{\mathbb{B}(0,R)} u_{0,\alpha} u_{1,\alpha} = 0$  by differentiation.

Since  $u_{0,\alpha}$  is radially symmetric according to Lemma 3.9, we introduce the two real numbers

$$\eta_\alpha = -\kappa u_{0,\alpha}(r_0^*)^2 + \left[ \sigma_\alpha \left( \frac{\partial u_{0,\alpha}}{\partial r} \right)^2 \right] \quad \text{and} \quad \lambda_{1,\alpha} = r_0^* \eta_\alpha \int_0^{2\pi} g(\theta) d\theta. \quad (3.34)$$

It is easy to see that  $V$  belongs to  $\mathcal{X}(\mathbb{B}^*)$  if, and only if  $\int_0^{2\pi} g = 0$  so that we finally have  $\lambda_{1,\alpha} = 0$ .  $\square$

**Computation of the Lagrange multiplier.** The existence of a Lagrange multiplier  $\Lambda_\alpha \in \mathbb{R}$  related to the volume constraint is standard, and one has

$$\forall V \in \mathcal{X}(\mathbb{B}^*), \quad (\lambda_\alpha - \Lambda_\alpha \text{Vol})'(\mathbb{B}^*)[V] = 0,$$

Since

$$\text{Vol}'(\mathbb{B}^*)[V] = \int_{\mathbb{S}(0,r_0^*)} V \cdot \nu = r_0^* \int_0^{2\pi} g(\theta) d\theta.$$

(see e.g. [93, chapitre 5]) and since

$$\lambda'_\alpha(\mathbb{B}^*)[V] = r_0^* \eta_\alpha \int_0^{2\pi} g(\theta) d\theta,$$

where  $\eta_\alpha$  is defined by (3.34), the Lagrange multiplier reads

$$\Lambda_\alpha = \eta_\alpha = -\kappa u_{0,\alpha}(r_0^*)^2 + \left[ \sigma_\alpha \left( \frac{\partial u_{0,\alpha}}{\partial r} \right)^2 \right].$$

### Computation of the second order derivative and second order optimality conditions.

**LEMMA 3.11** *For every  $V \in \mathcal{X}(\mathbb{B}^*)$ , one has for the coefficient  $\lambda_{2,\alpha} = \lambda''_\alpha(\mathbb{B}^*)[V, V]$  introduced in (3.28) the expression*

$$\begin{aligned} \lambda_{2,\alpha} &= 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \partial_r u_{1,\alpha}|_{int} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] V \cdot \nu - 2\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu \\ &\quad + \int_{\mathbb{S}(0,r_0^*)} \left( -\frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] - \frac{\kappa}{r_0^*} u_{0,\alpha}^2 \right) (V \cdot \nu)^2 - 2 \int_{\mathbb{S}(0,r_0^*)} \kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2. \end{aligned}$$

*Proof of Lemma 4.6.* In the computations below, we do not need to make the equation satisfied by  $u_{2,\alpha}$  explicit, but we nevertheless will need several times the knowledge of  $[u_{2,\alpha}]$  at  $\mathbb{S}(0, r_0^*)$ . In the same fashion that we obtained the jump conditions on  $u_{1,\alpha}$ . Let us differentiate two times the continuity equation  $[u_\varepsilon]_{\partial \mathbb{B}_\varepsilon^*} = 0$ . We obtain

$$[u_{2,\alpha}]_{\partial \mathbb{B}_\varepsilon^*} = -2g(\theta) \left[ \frac{\partial u_{1,\alpha}}{\partial r} \right] - g(\theta)^2 \left[ \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right]. \quad (3.35)$$

Now, according to Hadamard second variation formula (see [93, Chapitre 5, page 227] for a proof), if  $\Omega$  is a  $\mathcal{C}^2$  domain and  $f$  is two times differentiable at 0 and taking values in  $W^{2,2}(\Omega)$ , then one has

$$\frac{d^2}{dt^2} \Big|_{t=0} \int_{(\text{Id} + tV)\Omega} f(t) = \int_\Omega f''(0) + 2 \int_{\partial\Omega} f'(0) V \cdot \nu + \int_{\partial\Omega} \left( Hf(0) + \frac{\partial f(0)}{\partial \nu} \right) (V \cdot \nu)^2, \quad (3.36)$$

where  $H$  denotes the mean curvature. We apply it to  $f(\varepsilon) = \sigma_{\alpha,\varepsilon} |\nabla u_\varepsilon|^2 - m_\varepsilon u_\varepsilon^2$  on  $\mathbb{B}(0, R)$ , since  $\lambda_\alpha(m_\varepsilon) = \int_{\mathbb{B}(0,R)} f_\varepsilon$ . Let us distinguish between the two subdomains  $\mathbb{B}_\varepsilon^*$  and  $(\mathbb{B}_\varepsilon^*)^c$ . We introduce

$$D_1 = \frac{d^2}{d\varepsilon^2} \Big|_{\varepsilon=0} \int_{\mathbb{B}_\varepsilon^*} (\sigma_{\alpha,\varepsilon} |\nabla u_\varepsilon|^2 - \kappa u_\varepsilon^2) \quad \text{and} \quad D_2 = \frac{d^2}{d\varepsilon^2} \Big|_{\varepsilon=0} \int_{(\mathbb{B}_\varepsilon^*)^c} (\sigma_{\alpha,\varepsilon} |\nabla u_\varepsilon|^2)$$

so that  $\lambda''_\alpha(\mathbb{B}^*)[V, V] = D_1 + D_2$ .

One has

$$\begin{aligned} D_1 &= \int_{\mathbb{B}^*} 2(1 + \alpha\kappa) \nabla u_{2,\alpha} \cdot \nabla u_{1,\alpha} + 2 \int_{\mathbb{B}_\varepsilon^*} (1 + \alpha\kappa) |\nabla u_{1,\alpha}|^2 \\ &\quad - 2\kappa \int_{\mathbb{B}^*} u_{2,\alpha} u_{0,\alpha} - 2\kappa \int_{\mathbb{B}^*} u_{1,\alpha} u_{0,\alpha} \\ &\quad + 4 \int_{\mathbb{S}(0,r_0^*)} (1 + \alpha\kappa) (\nabla u_{1,\alpha}|_{int} \cdot \nabla u_{0,\alpha}|_{int}) V \cdot \nu - 4\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu \\ &\quad + \int_{\mathbb{S}(0,r_0^*)} \left( \frac{1}{r_0^*} (1 + \alpha\kappa) |\nabla u_{0,\alpha}|_{int}^2 - \frac{\kappa}{r_0^*} u_{0,\alpha}^2 + 2(1 + \alpha\kappa) \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \Big|_{int} \right. \\ &\quad \left. - 2\kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} \right) (V \cdot \nu)^2, \end{aligned}$$

and taking into account that the mean curvature has a sign on  $(\mathbb{B}_\varepsilon^*)^c$ , one has

$$\begin{aligned} D_2 &= \int_{(\mathbb{B}^*)^c} 2\nabla u_{2,\alpha} \cdot \nabla u_{1,\alpha} + 2 \int_{(\mathbb{B}^*)^c} |\nabla u_{1,\alpha}|^2 \\ &\quad - 4 \int_{\mathbb{S}(0,r_0^*)} (\nabla u_{1,\alpha}|_{ext} \cdot \nabla u_{0,\alpha}|_{ext}) V \cdot \nu \\ &\quad + \int_{\mathbb{S}(0,r_0^*)} \left( -\frac{1}{r_0^*} |\nabla u_{0,\alpha}|_{ext}^2 - 2 \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{ext} \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \Big|_{ext} \right) (V \cdot \nu)^2. \end{aligned}$$

Summing these two quantities, we get

$$\begin{aligned} \lambda_\alpha''(\mathbb{B}^*)[V, V] &= 2 \int_{\mathbb{B}(0,R)} \sigma_\alpha \nabla u_{0,\alpha} \cdot \nabla u_{2,\alpha} - 2 \int_{\mathbb{B}(0,R)} m_0^* u_{0,\alpha} u_{2,\alpha} \\ &\quad + 2 \int_{\mathbb{B}(0,R)} \sigma_\alpha |\nabla u_{1,\alpha}|^2 - 2 \int_{\mathbb{B}(0,R)} m_0^* u_{1,\alpha}^2 \\ &\quad - 4 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial u_{1,\alpha}}{\partial r} \right] V \cdot \nu - 4\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu \\ &\quad + \int_{\mathbb{S}(0,r_0^*)} \left( -\frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] - \frac{\kappa}{r_0^*} u_{0,\alpha}^2 \right) (V \cdot \nu)^2 \\ &\quad - 2 \int_{\mathbb{S}(0,r_0^*)} \left[ \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right] (V \cdot \nu)^2 - 2\kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2. \end{aligned}$$

To simplify this expression, let us use Eq. (3.30). Introducing

$$D_3 = \int_{\mathbb{B}(0,R)} \sigma_\alpha \nabla u_{0,\alpha} \cdot \nabla u_{2,\alpha} - \int_{\mathbb{B}(0,R)} u_{0,\alpha} u_{2,\alpha} - \lambda_\alpha(\mathbb{B}^*) \int_{\mathbb{B}(0,R)} u_{0,\alpha} u_{2,\alpha},$$

one has

$$D_3 = \int_{\mathbb{S}(0,r_0^*)} [u_{2,\alpha}] \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r}$$

and hence, by using Equation (3.35), one has

$$\begin{aligned} D_3 &= -2 \int_{\mathbb{S}(0,r_0^*)} [u_{2,\alpha}] \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \\ &= 4 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial u_{1,\alpha}}{\partial r} \right] V \cdot \nu + 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right] (V \cdot \nu)^2. \end{aligned}$$

Similarly, let

$$D_4 = \int_{\mathbb{B}(0,R)} \sigma_\alpha |\nabla u_{1,\alpha}|^2 - \int_{\mathbb{B}(0,R)} m_0^* u_{1,\alpha}^2.$$

By using Eq. (3.32) and the fact that  $\lambda_{1,\alpha} = 0$ , one has

$$\begin{aligned} D_4 &= \lambda_\alpha(\mathbb{B}^*) \int_{\mathbb{B}(0,R)} u_{1,\alpha}^2 - \int_{\mathbb{S}(0,r_0^*)} \left[ u_{1,\alpha} \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial r} \right] \\ &= \lambda_\alpha(\mathbb{B}^*) \int_{\mathbb{B}(0,R)} u_{1,\alpha}^2 - \int_{\mathbb{S}(0,r_0^*)} [u_{1,\alpha}] \left( \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial r} \right) \Big|_{ext} - \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} \left[ \sigma_\alpha \frac{\partial u_{1,\alpha}}{\partial r} \right] \end{aligned}$$

$$= \lambda_\alpha(\mathbb{B}^*) \int_{\mathbb{B}(0,R)} u_{1,\alpha}^2 + \int_{\mathbb{S}(0,r_0^*)} \left( \sigma_\alpha \partial_r u_{1,\alpha}|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] + \kappa u_{1,\alpha}|_{int} u_{0,\alpha} \right) V \cdot \nu.$$

Finally, by differentiating the normalization condition  $\int_{\mathbb{B}(0,R)} u_\varepsilon^2 = 1$ , we get

$$\int_{\mathbb{B}(0,R)} u_{0,\alpha} u_{2,\alpha} + \int_{\mathbb{B}(0,R)} u_{1,\alpha}^2 = 0. \quad (3.37)$$

Combining the equalities above, one gets

$$\begin{aligned} \lambda_{2,\alpha} &= 2\lambda_\alpha(\mathbb{B}^*) \left( \int_{\mathbb{B}(0,R)} u_{0,\alpha} u_{2,\alpha} + \int_{\mathbb{B}(0,R)} u_{1,\alpha}^2 \right) + 4 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial u_{1,\alpha}}{\partial r} \right] V \cdot \nu \\ &+ 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right] (V \cdot \nu)^2 + 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \partial_r u_{1,\alpha}|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] V \cdot \nu \\ &+ 2\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu - 4 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \left[ \frac{\partial u_{1,\alpha}}{\partial r} \right] V \cdot \nu \\ &- 4\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu - \int_{\mathbb{S}(0,r_0^*)} \left( \frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] + \frac{\kappa}{r_0^*} u_{0,\alpha}^2 \right) (V \cdot \nu)^2 \\ &- 2 \int_{\mathbb{S}(0,r_0^*)} \left[ \sigma_\alpha \frac{\partial u_{0,\alpha}}{\partial r} \frac{\partial^2 u_{0,\alpha}}{\partial r^2} \right] (V \cdot \nu)^2 - 2 \int_{\mathbb{S}(0,r_0^*)} \kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2 \\ &= 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \partial_r u_{1,\alpha}|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] V \cdot \nu - 2\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu \\ &- \int_{\mathbb{S}(0,r_0^*)} \left( \frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] + \frac{\kappa}{r_0^*} u_{0,\alpha}^2 \right) (V \cdot \nu)^2 - 2 \int_{\mathbb{S}(0,r_0^*)} \kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2 \end{aligned}$$

We have then obtained the desired expression.  $\square$

**Strong stability** The second derivative of the volume is known to be

$$\text{Vol}''(\mathbb{B}^*)[V, V] = \int_{\mathbb{S}(0,r_0^*)} H(V \cdot \nu)^2. \quad (3.38)$$

Hence, introducing  $D_5 = (\lambda_\alpha - \eta_\alpha \text{Vol})''(\mathbb{B}^*)[V, V]$  and taking into account Lemma 4.6, (3.34) and (3.38), we have

$$\begin{aligned} D_5 &= 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \partial_r u_{1,\alpha}|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] V \cdot \nu - 2 \int_{\mathbb{S}(0,r_0^*)} \kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2 \\ &- 2\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{ext} u_{0,\alpha} V \cdot \nu + \int_{\mathbb{S}(0,r_0^*)} \left( -\frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] - \frac{\kappa}{r_0^*} u_{0,\alpha}^2 \right) (V \cdot \nu)^2 \\ &+ \kappa \int_{\mathbb{S}(0,r_0^*)} \frac{1}{r_0^*} u_{0,\alpha}^2 (V \cdot \nu)^2 - \int_{\mathbb{S}(0,r_0^*)} \frac{1}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] (V \cdot \nu)^2 \\ &= 2 \int_{\mathbb{S}(0,r_0^*)} \sigma_\alpha \partial_r u_{1,\alpha}|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] V \cdot \nu - 2 \int_{\mathbb{S}(0,r_0^*)} \kappa u_{0,\alpha} \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2 \\ &- 2\kappa \int_{\mathbb{S}(0,r_0^*)} u_{1,\alpha}|_{int} u_{0,\alpha} V \cdot \nu - \int_{\mathbb{S}(0,r_0^*)} \frac{2}{r_0^*} [\sigma_\alpha |\nabla u_{0,\alpha}|^2] (V \cdot \nu)^2 \end{aligned}$$

We are then led to determine the signature of the quadratic form

$$\begin{aligned}
 \mathcal{F}_\alpha[V, V] &= \frac{1}{2}(\lambda_\alpha(\mathbb{B}^*) - \eta_\alpha \text{Vol})''[V, V] \\
 &= \int_{\mathbb{S}(0, r_0^*)} \sigma_\alpha \partial_r u_{1, \alpha}|_{ext} \left[ \frac{\partial u_{0, \alpha}}{\partial r} \right] V \cdot \nu - \kappa \int_{\mathbb{S}(0, r_0^*)} u_{1, \alpha}|_{int} u_{0, \alpha} V \cdot \nu \\
 &\quad + \int_{\mathbb{S}(0, r_0^*)} \left( -\frac{2}{r^*} [\sigma_\alpha |\nabla u_{0, \alpha}|^2] \right) (V \cdot \nu)^2 - \int_{\mathbb{S}(0, r_0^*)} \kappa u_{0, \alpha} \frac{\partial u_{0, \alpha}}{\partial r} \Big|_{int} (V \cdot \nu)^2.
 \end{aligned} \tag{3.39}$$

### 3.6.3 ANALYSIS OF THE QUADRATIC FORM $\mathcal{F}_\alpha$

**Separation of variables and first simplification.** Each perturbation  $g \in L^2(0, 2\pi)$  such that  $\int g = 0$  expands as

$$g = \sum_{k=1}^{\infty} (\gamma_k \cos(k \cdot) + \beta_k \sin(k \cdot)), \quad \text{with } \gamma_0 = 0.$$

For every  $k \in \mathbb{N}^*$ , let us introduce  $g_k := \cos(k \cdot)$  and  $\tilde{g}_k := \sin(k \cdot)$ . For any  $k \in \mathbb{N}^*$ , let  $u_{1, \alpha}^{(k)}$  be the solution of Eq. (3.32) associated with the perturbation  $g_k$ . It is readily checked that there exists a function  $\psi_{k, \alpha} : [0; R] \rightarrow \mathbb{R}$  such that

$$\forall (r, \theta) \in [0; R] \times [0; 2\pi], \quad u_{1, \alpha}^{(k)}(r, \theta) = g_k(\theta) \psi_{k, \alpha}(r).$$

Furthermore,  $\psi_{k, \alpha}$  solves the ODE

$$\begin{cases} -\sigma_\alpha \psi_{k, \alpha}'' - \frac{\sigma_\alpha}{r} \psi_{k, \alpha}'(r) = \left( \lambda_{0, \alpha} - \frac{k^2}{r^2} \right) \psi_{k, \alpha} + m^* \psi_{k, \alpha} & \text{in } (-R; R), \\ \left[ \sigma_\alpha \psi_{k, \alpha}' \right] (r_0^*) = -\kappa u_{0, \alpha}(r_0^*) \\ \left[ \psi_{k, \alpha} \right] (r_0^*) = - \left[ \frac{\partial u_{0, \alpha}}{\partial r} \right] (r_0^*), \\ \psi_{k, \alpha}(-R) = \psi_{k, \alpha}(R) = 0. \end{cases} \tag{3.40}$$

Regarding  $\tilde{g}_k$ , if we define  $\tilde{u}_{1, \alpha}^{(k)}$  in a similar fashion, it is readily checked that

$$\forall (r, \theta) \in [0; R] \times [0; 2\pi], \quad \tilde{u}_{1, \alpha}^{(k)}(r, \theta) = \tilde{g}_k(\theta) \psi_{k, \alpha}(r).$$

Therefore, any admissible perturbation  $g$  writes

$$g = \sum_{k=1}^{\infty} \{ \gamma_k g_k + \beta_k \tilde{g}_k \} \quad \text{with } \gamma_0 = 0$$

and the solution  $u_{1, \alpha}$  associated with  $g$  writes

$$u_{1, \alpha} = \sum_{k=1}^{\infty} \{ \gamma_k u_{1, \alpha}^{(k)} + \beta_k \tilde{u}_{1, \alpha}^{(k)} \}.$$

Using (3.39) and the orthogonality properties of the family  $\{g_k\}_{k \in \mathbb{N}^*} \cup \{\tilde{g}_k\}_{k \in \mathbb{N}^*}$ , one gets

$$\mathcal{F}_\alpha[V, V] = \frac{r_0^*}{2} \sum_{k=1}^{\infty} \left( \sigma_\alpha \psi_{k, \alpha}'(r_0^*)|_{ext} \left[ \frac{\partial u_{0, \alpha}}{\partial r} \right] - \kappa \psi_{k, \alpha}|_{int} u_{0, \alpha}(r_0^*) \right) (\gamma_k^2 + \beta_k^2)$$

$$\begin{aligned}
 & -\frac{r_0^*}{2} \sum_{k=0}^{\infty} \kappa u_{0,\alpha}(r_0^*) \frac{\partial u_{0,\alpha}}{\partial r}(r_0^*) (\gamma_k^2 + \beta_k^2) - \sum_{k=1}^{\infty} 2 [\sigma_\alpha |\nabla u_{0,\alpha}|^2] (\gamma_k^2 + \beta_k^2) \\
 = & \frac{r_0^* \kappa u_{0,\alpha}(r_0^*)}{2} \sum_{k=1}^{\infty} \left( -\frac{\partial u_{0,\alpha}}{\partial r}(r_0^*) \Big|_{int} - \psi_{k,\alpha}|_{int} \right) (\gamma_k^2 + \beta_k^2) \\
 & + \sum_{k=1}^{\infty} \left( -2 [\sigma_\alpha |\nabla u_{0,\alpha}|^2] + \sigma_\alpha \psi'_{k,\alpha}(r_0^*)|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] \right) (\gamma_k^2 + \beta_k^2) \tag{3.41}
 \end{aligned}$$

Define, for any  $k \in \mathbb{N}$ ,

$$\omega_{k,\alpha} := -\frac{\partial u_{0,\alpha}}{\partial r}(r_0^*) \Big|_{int} - \psi_{k,\alpha}|_{int} \quad \text{and} \quad \zeta_{k,\alpha} := -2 [\sigma_\alpha |\nabla u_{0,\alpha}|^2] + \sigma_\alpha \psi'_{k,\alpha}(r_0^*)|_{ext} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right].$$

Thus,

$$\mathcal{F}_\alpha[V, V] = \sum_{k=1}^{\infty} (\omega_{k,\alpha} + \zeta_{k,\alpha}) (\gamma_k^2 + \beta_k^2).$$

The end of the proof is devoted to proving the local shape minimality of the centered ball, which relies on an asymptotic analysis of the sequences  $\{\omega_{k,\alpha}\}_{k \in \mathbb{N}}$  and  $\{\zeta_{k,\alpha}\}_{k \in \mathbb{N}}$  as  $\alpha$  converges to 0.

**PROPOSITION 3.2** *There exists  $C > 0$  and  $\bar{\alpha} > 0$ , there exists  $M \in \mathbb{R}$  such that for any  $\alpha \leq \bar{\alpha}$  and any  $k \in \mathbb{N}$ , one has*

$$\omega_{k,\alpha} \geq C > 0, \quad \text{and} \quad \zeta_{k,\alpha} \geq -M\alpha. \tag{3.42}$$

The last claim of Theorem 3.1.3 is then an easy consequence of this proposition. The rest of the proof is devoted to the proof of Proposition 3.2, which follows from the combination of the following series of lemmas.

**LEMMA 3.12** *There exists  $\bar{\alpha} > 0$  such that, for every  $\alpha \in [0, \bar{\alpha}]$ ,  $\psi_{1,\alpha}$  is nonnegative on  $(0, R)$ .*

*Proof of Lemma 3.12.* For the sake of notational simplicity, we temporarily drop the dependence on  $\alpha$  and denote  $\psi_{1,\alpha}$  by  $\psi_\alpha$ . The function  $\psi_\alpha$  solves the ODE

$$\begin{cases} -\sigma_\alpha \psi''_\alpha - \frac{\sigma_\alpha}{r} \psi'_\alpha(r) = \left( \lambda_{0,\alpha} - \frac{1}{r^2} \right) \psi_\alpha + m^* \psi_\alpha & \text{in } (0, R), \\ [\sigma_\alpha \psi'_\alpha](r_0^*) = -\kappa u_{0,\alpha}(r_0^*) \\ [\psi_\alpha](r_0^*) = -\left[ \frac{\partial u_{0,\alpha}}{\partial r} \right](r_0^*), \\ \psi_\alpha(R) = 0. \end{cases}$$

Let us introduce  $p_\alpha = \psi_\alpha / u_{0,\alpha}$ . One checks easily that  $p_\alpha$  solves the ODE

$$-\sigma_\alpha p''_\alpha - \frac{\sigma_\alpha}{r} p'_\alpha = \frac{1}{r^2} p_\alpha + 2p'_\alpha \frac{u'_{0,\alpha}}{u_{0,\alpha}} \quad \text{in } (0, R).$$

Furthermore,  $p_\alpha$  satisfies the jump conditions

$$[p_\alpha](r_0^*) = -\frac{[\partial_r u_{0,\alpha}](r_0^*)}{u_{0,\alpha}(r_0^*)} = \frac{-\alpha \kappa \partial_r u_{0,\alpha}|_{int}}{u_{0,\alpha}(r_0^*)} > 0 \quad \text{and} \quad [\sigma_\alpha p'_\alpha](r_0^*) = -\kappa + \frac{\sigma_\alpha \partial_r u_{0,\alpha}}{u_{0,\alpha}(r_0^*)^2} \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right].$$

To show that  $\psi_\alpha$  is nonnegative, we argue by contradiction and consider first the case where a negative minimum is reached at an interior point  $r_- \neq r_0^*$ . Then,  $p_\alpha$  is  $\mathcal{C}^2$  in a neighborhood of  $r_-$  and we have

$$0 \geq -p''_\alpha(r_-) = -\frac{p_\alpha(r_-)}{(r_-)^2} > 0,$$

whence the contradiction.

To exclude the case  $r_- = R$ , let us notice that, according to l'Hospital's rule, one has  $p_\alpha(R) = \psi'_\alpha(R)/u'_{0,\alpha}(R)$ . According to the Hopf lemma applied to  $u_{0,\alpha}$ , this quotient is well-defined. If  $p_\alpha(R) < 0$  then it follows that  $\psi'_\alpha(R) > 0$ . However, one has  $p'_\alpha(r) \sim \psi'_\alpha(r)/(2u_{0,\alpha}(r)) > 0$  as  $r \rightarrow R$ , which contradicts the fact that a minimum is reached at  $R$ .

Let us finally exclude the case where  $r_- = r_0^*$ . Mimicking the elliptic regularity arguments used in the proofs of Lemmas 3.5 and 3.6, we get that  $p_\alpha$  converges to  $p_0$  as  $\alpha \rightarrow 0$  for the strong topologies of  $\mathcal{C}^0([0, r_0^*])$  and  $\mathcal{C}^0([r_0^*, R])$ .

To conclude, it suffices hence to prove that  $p_0$  is positive in a neighborhood of  $r_0^*$ . We once again argue by contradiction and assume that  $p_0$  reaches a negative minimum at  $r_- \in [0, R]$ . Notice that  $r_- \neq r_0^*$  since  $[p_0](r_0^*) = 0$  and  $[p'_0](r_0^*) = -\kappa < 0$ .

If  $r_- \in (0, R)$ , since  $r_- \neq r_0^*$ , we claim that  $p_0$  is  $\mathcal{C}^2$  in a neighborhood of  $r_-$  and, if  $p_0(r_-) < 0$ , the contradiction follows from

$$0 \geq -p''_0(r_-) = -\frac{p_0(r_-)}{(r_-)^2} > 0.$$

For the same reason, a negative minimum cannot be reached at  $r = 0$ .

If  $r_- = R$ , we observe that  $p_0(R) = \psi'_0(R)/u'_{0,0}(R)$ . According to the Hopf lemma applied to  $u_{0,0}$ , this quantity is well-defined. If  $p_0(R) < 0$ , then it follows that  $\psi'_0(R) > 0$ . However,  $p'_0(r) \sim \psi'_0(r)/(2u(r)) > 0$  as  $r \rightarrow R$ , which contradicts the fact that  $R$  is a minimizer.

Therefore  $p_0$  is positive in a neighborhood of  $r_0^*$  and we infer that  $p_\alpha$  is non-negative, so that, in turn,  $\psi_\alpha \geq 0$  in  $[0, R]$ .  $\square$

**LEMMA 3.13** *Let  $\bar{\alpha}$  be defined as in Lemma 3.12. Then, for every  $\alpha \in [0, \bar{\alpha}]$  and every  $k \in \mathbb{N}$ ,*

$$\psi_{k,\alpha} \leq \psi_{1,\alpha}. \quad (3.43)$$

*As a consequence, for any  $\alpha \leq \bar{\alpha}$  and any  $k \in \mathbb{N}$ , there holds  $\omega_{k,\alpha} \geq \omega_{1,\alpha}$ .*

*Proof of Lemma 3.13.* Since  $\omega_{k,\alpha} - \omega_{1,\alpha} = -\psi_{k,\alpha}|_{int}(r_0^*) + \psi_{1,\alpha}|_{int}(r_0^*)$ , the fact that  $\omega_{k,\alpha} \geq \omega_{1,\alpha}$  will follow from (3.43), on which we now focus on. Let us set  $\Psi_k = \psi_{1,\alpha} - \psi_{k,\alpha}$ . From the jump conditions on  $\psi_{1,\alpha}$  and  $\psi_{k,\alpha}$ , one has  $[\Psi_k](r_0) = [\sigma_\alpha \Psi'_k](r_0) = 0$ . The function  $\Psi_k$  satisfies

$$\begin{aligned} -\sigma_\alpha \Psi''_k - \sigma_\alpha \frac{\Psi'_k}{r} &= -\left(\lambda_{0,\alpha} - \frac{k^2}{r^2}\right) \psi_{k,\alpha} - m_0^* \psi_{k,\alpha} + \left(\lambda_{0,\alpha} - \frac{1}{r^2}\right) \psi_{1,\alpha} + m_0^* \psi_{1,\alpha} \\ &> \left(\lambda_{0,\alpha} - \frac{k^2}{r^2}\right) \psi_{k,\alpha} - m_0^* \psi_{k,\alpha} + \left(\lambda_{0,\alpha} - \frac{k^2}{r^2}\right) \psi_{1,\alpha} + m_0^* \psi_{1,\alpha} \\ &> \left(\lambda_{0,\alpha} - \frac{k^2}{r^2}\right) \Psi_k + m_0^* \Psi_k. \end{aligned}$$

since  $\psi_{1,\alpha} \geq 0$ , according to Lemma 3.12. Since  $\Psi_k$  satisfies Dirichlet boundary conditions,  $\Psi_k \geq 0$  in  $(0, R)$ .  $\square$

**LEMMA 3.14** *There exists  $C > 0$  such that, for every  $\alpha \in [0, \bar{\alpha}]$ , where  $\bar{\alpha}$  is introduced on Lemma (3.12), one has  $\omega_{1,\alpha} \geq C$ .*

*Proof of Lemma 5.5.3.* Let us introduce  $\Psi = -\partial u_{0,\alpha}/\partial r - \psi_{k,\alpha}$ . According to (3.29), we have  $[\Psi](r_0^*) = [\Psi'](r_0^*) = 0$ . Furthermore,  $\Psi(R) = -\frac{\partial u_{0,\alpha}}{\partial r}(R) > 0$  according to the Hopf Lemma and  $\Psi(0) = 0$ . Finally, since  $\Psi$  solves the ODE, one has

$$-\frac{1}{r}(\sigma_\alpha \Psi')' = \left(\lambda_{0,\alpha} - \frac{1}{r^2}\right) \Psi + m^* \Psi \quad \text{in } (0, R),$$



it follows that  $\Psi$  is positive in  $(0, R]$ . Furthermore,  $\Psi$  converges to  $\Psi_0$  for the strong topology of  $\mathcal{C}^0([0, R])$  and  $\Psi_0$  solves the ODE

$$\begin{cases} -\frac{1}{r}(\Psi_0)' = (\lambda_{0,0} - \frac{1}{r^2})\Psi_0 + m_0^*\Psi_0 & \text{in } (0, R) \\ \Psi_0(R) = -\frac{\partial u_{0,0}}{\partial r}(R) > 0. \end{cases}$$

Hence there exists  $C > 0$  such that, for every  $\alpha \in [0, \bar{\alpha}]$ , one has  $\Psi(r_0^*) \geq C > 0$ .  $\square$

It remains to prove the second inequality of (4.56). As a consequence of the convergence result stated in Lemma 3.9, one has

$$[\sigma_\alpha |\nabla u_{0,\alpha}|^2] = O(\alpha), \quad \left[ \frac{\partial u_{0,\alpha}}{\partial r} \right] = \alpha \kappa \frac{\partial u_{0,\alpha}}{\partial r} \Big|_{int} < 0. \quad (3.44)$$

It follows that we only need to prove that there exists a constant  $M > 0$  such that, for any  $\alpha \in [0, \bar{\alpha}]$ , and any  $k \in \mathbb{N}^*$ ,

$$M \geq \sigma_\alpha \psi'_{k,\alpha} \Big|_{ext}(r_0^*) \quad (3.45)$$

so that

$$\zeta_{k,\alpha} = O(\alpha) + \sigma_\alpha \psi'_{k,\alpha} \Big|_{ext}(r_0^*) \alpha \kappa \frac{\partial u_{0,\alpha}}{\partial r}(r_0^*) \Big|_{int} \geq O(\alpha) - M \alpha \kappa \left| \frac{\partial u_{0,\alpha}}{\partial r}(r_0^*) \Big|_{int}$$

To show the estimate (3.45), let us distinguish between small and large values of  $k$ . To this aim, we introduce  $N \in \mathbb{N}$  as the smallest integer such that

$$\lambda_{0,\alpha} + m_0^* - \frac{k^2}{r^2} < 0 \text{ in } (0, R) \quad (3.46)$$

for every  $k \geq N$  and  $\alpha \in [0, \bar{\alpha}]$ . The existence of such an integer follows immediately from the convergence of  $(\lambda_{0,\alpha})_{\alpha>0}$  to  $\lambda_0(m_0^*)$  as  $\alpha \rightarrow 0$ .

First, we will prove that, for every  $k \geq N$ ,

$$\psi'_{k,\alpha}(r_0^*) \Big|_{ext} < 0 \quad (3.47)$$

and that there exists  $M > 0$  such that, for every  $k \leq N$ ,

$$|\psi'_{k,\alpha}(r_0^*) \Big|_{ext}| \leq M \quad (3.48)$$

which will lead to (3.45) and thus yield the desired conclusion.

To show (3.47), let us argue by contradiction, assuming that  $\psi'_{k,\alpha}(r_0^*) \Big|_{ext} > 0$ . Since the jump  $[\sigma_\alpha \psi'_{k,\alpha}] = -\kappa u_{0,\alpha}(r_0^*)$  is negative, it follows that

$$(1 + \alpha \kappa) \psi'_{k,\alpha}(r_0^*) \Big|_{int} = \psi'_{k,\alpha}(r_0^*) \Big|_{ext} - [\sigma_\alpha \psi'_{k,\alpha}] > 0.$$

By mimicking the reasonings in the proof of Lemma 3.12,  $\psi_{k,\alpha}$  cannot reach a negative minimum on  $(0, r_0^*)$  since (3.46) holds true. Therefore, since  $\psi_{k,\alpha}(0) = 0$  and  $\psi'_{k,\alpha}(r_0^*) \Big|_{int} > 0$ , one has necessarily  $\psi_{k,\alpha}(r_0^*) \Big|_{int} > 0$ , which in turn gives  $\psi_{k,\alpha}(r_0^*) \Big|_{ext} > 0$  since  $[\psi_{k,\alpha}] = -\alpha \kappa \frac{\partial u_{0,\alpha}}{\partial r} > 0$ .

Furthermore,  $\psi_{k,\alpha}(R) = 0$ . Since  $\psi_{k,\alpha}(r_0^*) \Big|_{ext} > 0$  and  $\psi'_{k,\alpha}(r_0^*) \Big|_{ext} > 0$ , it follows that  $\psi_{k,\alpha}$  reaches a positive maximum at some interior point  $r_1$ , satisfying hence

$$0 \leq -\psi''_{k,\alpha}(r_-) = \left( \lambda_{0,\alpha} + m_0^* - \frac{k^2}{r^2} \right) \psi_{k,\alpha}(r_-) < 0,$$

leading to a contradiction.

Let us now deal with small values of  $k$ , by assuming  $k \leq N$ . We will prove that (3.48) holds true. To this aim, we will compute  $\psi_{k,\alpha}$ . Let  $J_k$  (resp.  $Y_k$ ) be the  $k$ -th Bessel function of the first (resp. the second) kind. One has

$$\psi_{k,\alpha}(r) = \begin{cases} A_{k,\alpha} J_k\left(\sqrt{\frac{\lambda_{0,\alpha} + \kappa}{1 + \alpha\kappa}} \frac{r}{R}\right) & \text{if } r \leq r_0^*, \\ B_{k,\alpha} J_k\left(\sqrt{\lambda_{0,\alpha}} \frac{r}{R}\right) + C_{k,\alpha} Y_k\left(\sqrt{\lambda_{0,\alpha}} \frac{r}{R}\right) & \text{if } r_0^* \leq r \leq R, \end{cases}$$

where  $X_{k,\alpha} = (B_{k,\alpha}, C_{k,\alpha}, A_{k,\alpha})$  solves the linear system

$$\mathcal{A}_{k,\alpha} X_{k,\alpha} = b_\alpha$$

where

$$b_\alpha = \begin{pmatrix} 0 \\ -\kappa u_{0,\alpha}(r_0^*) \\ -[\partial_R u_{0,\alpha}] \end{pmatrix}$$

and

$$\mathcal{A}_{k,\alpha} = \begin{pmatrix} J_k(\sqrt{\lambda_0}) & Y_k(\sqrt{\lambda_0}) & 0 \\ \sqrt{\lambda_{0,\alpha}} J'_k\left(\sqrt{\lambda_{0,\alpha}} + \kappa \frac{r_0^*}{R}\right) & \sqrt{\lambda_{0,\alpha}} Y'_k\left(\sqrt{\lambda_{0,\alpha}} + \kappa \frac{r_0^*}{R}\right) & -\sqrt{\frac{\lambda_{0,\alpha} + \kappa}{1 + \alpha\kappa}} J'_k\left(\sqrt{\frac{\lambda_{0,\alpha} + \kappa}{1 + \alpha\kappa}} \frac{r_0^*}{R}\right) \\ J_k\left(\sqrt{\lambda_{0,\alpha}} \frac{r_0^*}{R}\right) & Y_k\left(\sqrt{\lambda_{0,\alpha}} \frac{r_0^*}{R}\right) & -J_k\left(\sqrt{\frac{\lambda_{0,\alpha} + \kappa}{1 + \alpha\kappa}} \frac{r_0^*}{R}\right) \end{pmatrix}.$$

It is easy to check that

$$\|\mathcal{A}_{k,\alpha} - \mathcal{A}_{k,0}\| \leq M\alpha$$

where  $M$  only depends<sup>3</sup> on  $N$ . Hence it is enough to prove that  $|\psi'_{k,0}(r_0^*)| \leq M$  for some  $M > 0$  depending only on  $N$ , which is straightforward since the set of indices is finite. The expected conclusion follows.

### 3.6.4 CONCLUSION

From Eq. (4.56) and Lemma 5.5.3, there exists  $C > 0$  and  $M > 0$  such that  $\omega_{k,\alpha} \geq C > 0$  and  $\zeta_{k,\alpha} \geq -M\alpha$  for every  $\alpha \in [0, \bar{\alpha}]$  and  $k \in \mathbb{N}$ , from which we infer that

$$\mathcal{F}_\alpha[V, V] \geq (C - M\alpha) \sum_{k=1}^{\infty} (\gamma_k^2 + \beta_k^2) \geq \frac{C}{2} \|V \cdot \nu\|_{L^2}^2.$$

according to Eq. (5.35).

<sup>3</sup>Indeed,  $\{J_k, Y_k\}_{k \leq N}$  are uniformly bounded in  $\mathcal{C}^2([r_0^*/R - \varepsilon, R])$  for every  $\varepsilon > 0$  small enough. Since we consider a finite number of indices  $k$ , there exists  $\delta > 0$  (depending only on  $N$ ) such that

$$\forall k \in \{0, \dots, N\}, \quad \det(\mathcal{A}_{k,\alpha}) \geq \delta > 0.$$

Then, since  $\|X_\alpha - X_0\| \leq M\alpha$ , it follows from the Cramer formula that there exists  $M$  (depending only on  $N$ ) such that  $\|X_{k,\alpha} - X_{k,0}\|_{L^\infty} \leq M\alpha$ .



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# APPENDIX

## 3.A PROOF OF LEMMA 3.1

We prove hereafter that the mapping  $m \mapsto (u_{\alpha, m}, \lambda_\alpha(m))$  is twice differentiable (and even  $\mathcal{C}^\infty$ ) in the  $L^2$  sense, the proof of the differentiability in the weak  $W^{1,2}(\Omega)$  sense being similar. Let  $m^* \in \mathcal{M}_{m_0, \kappa}(\Omega)$ ,  $\sigma_\alpha := 1 + \alpha m^*$ , and  $(u_0, \lambda_0)$  be the eigenpair associated with  $m^*$ . Let  $h \in \mathcal{T}_{m^*}$  (see Def. 3.2.1). Let  $m_h^* := m^* + h$  and  $\sigma_{m^*+h} := 1 + \alpha(m^* + h)$ . Let  $(u_h, \lambda_h)$  be the eigenpair associated with  $m_h^*$ . Let us introduce the mapping  $G$  defined by

$$G : \begin{cases} \mathcal{T}_{m^*} \times W_0^{1,2}(\Omega) \times \mathbb{R} \rightarrow W^{-1,2}(\Omega) \times \mathbb{R}, \\ (h, v, \lambda) \mapsto (-\nabla \cdot (\sigma_{m^*+h} \nabla v)) - \lambda v - m_h^* v, \int_\Omega v^2 - 1). \end{cases}$$

From the definition of the eigenvalue, one has  $G(0, u_0, \lambda_0) = 0$ . Moreover,  $G$  is  $\mathcal{C}^\infty$  in  $\mathcal{T}_{m^*} \cap B \times W_0^{1,2}(\Omega) \times \mathbb{R}$ , where  $B$  is an open ball centered at 0. The differential of  $G$  at  $(0, u_0, \lambda_0)$  reads

$$D_{v, \lambda} G(0, u_0, \lambda_0)[w, \mu] = \left( -\nabla \cdot (\sigma_\alpha \nabla w) - \mu u_0 - \lambda_0 w - m^* w, \int_\Omega 2u_0 w \right).$$

Let us show that this differential is invertible. We will show that, if  $(z, k) \in W^{-1,2}(\Omega) \times \mathbb{R}$ , then there exists a unique pair  $(w, \mu)$  such that  $D_{v, \lambda} G(0, u_0, \lambda_0)[w, \mu] = (z, k)$ . According to the Fredholm alternative, one has necessarily  $\mu = -\langle z, u_0 \rangle$  and for this choice of  $\mu$ , there exists a solution  $w_1$  to the equation

$$-\nabla \cdot (\sigma_\alpha \nabla w) - \mu u_0 - \lambda_0 w - m^* w = z \quad \text{in } \Omega.$$

Moreover, since  $\lambda_0$  is simple, any other solution is of the form  $w = w_1 + t u_0$  with  $t \in \mathbb{R}$ . From the equation  $2 \int_\Omega u_0 w = k$ , we get  $t = k/2 - \int_\Omega w_1 u_0$ . Hence, the pair  $(w, \mu)$  is uniquely determined. According to the implicit function theorem, the mapping  $h \mapsto (u_h, \lambda_h)$  is  $\mathcal{C}^\infty$  in a neighbourhood of  $\vec{0}$ .

### 3.A.1 SHAPE DIFFERENTIABILITY AND COMPUTATION OF THE SHAPE DERIVATIVES

#### 3.A.1.1 PROOF OF THE SHAPE DIFFERENTIABILITY

*Proof of the shape differentiability.* Let  $\mathbb{B}^*$  be the centered ball of radius  $r_0$ ,  $m^* := \kappa \chi_{\mathbb{B}^*}$ ,  $\sigma_\alpha := 1 + \alpha m^*$ ,  $(u_0, \lambda_0)$  be the eigenpair associated with  $m^*$ , and let  $V$  be an admissible vector field at  $\mathbb{B}^*$ .

In particular,  $V$  is a  $W^{1,\infty}$  vector field. Let  $T_V := (Id + V)$  and  $E_V^* := T_V(E_{\mathbb{B}}^*)$ . Let  $u_V$  be the eigenvalue associated with  $\kappa\chi_{\mathbb{B}_V^*}$  and  $\lambda_V$  be the associated eigenvalue. If we introduce

$$J(V) := \det(DV), A(V) := J(V)DT_V^{-1} (DT_V^{-1})^t$$

then the weak formulation of the equation on  $u_V$  is: for any  $v \in W_0^{1,2}(\Omega)$ ,

$$\int_{\Omega} \sigma_{\alpha} \langle A(V) \nabla u_V, \nabla v \rangle = \lambda_V \int_{\Omega} u_V v J(V) + \int_{\Omega} m^* u_V v J(V).$$

We define the map  $F$  in the following way:

$$F : \begin{cases} W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \times W_0^{1,2}(\Omega) \times \mathbb{R} \rightarrow W^{-1,2}(\Omega) \times \mathbb{R}, \\ (V, v, \lambda) \mapsto (-\nabla \cdot (\sigma_{\alpha} A(V) \nabla v) - \lambda v J(V) - m^* v J(V), \int_{\Omega} v^2 J(V) - 1). \end{cases}$$

It is clear from the definition of the eigenvalue that

$$F(0, u_0, \lambda_0) = 0.$$

Furthermore, the same arguments as in [63, Lemma 2.3] show that  $F$  is  $\mathcal{C}^{\infty}$  in  $B \times W_0^{1,2}(\Omega) \times \mathbb{R}$ , where  $B$  is an open ball centered at  $\vec{0}$ .

The differential of  $F$  at  $(0, u_0, \lambda_0)$  is given by

$$D_{v,\lambda} F(0, u_0, \lambda_0)[w, \mu] = \left( -\nabla \cdot (\sigma_{\alpha} \nabla w) - \mu u_0 - \lambda_0 w - m^* w, \int_{\Omega} 2u_0 w \right).$$

To prove that this differential is invertible, it suffices to show that, if  $(z, k) \in W^{-1,2}(\Omega) \times \mathbb{R}$ , then there exists a unique couple  $(w, \mu)$  such that

$$D_{v,\lambda} F(0, u_0, \lambda_0)[w, \mu] = (z, k).$$

By the Fredholm alternative, we know that

$$\mu = -\langle z, u_0 \rangle.$$

There exists a solution  $w_1$  to the equation

$$-\nabla \cdot (\sigma_{\alpha} \nabla w) - \mu u_0 - \lambda_0 w - m^* w = z.$$

We fix such a solution. Any other solution is of the form  $w = w_1 + t u_0$  for a real parameter  $t$ . We look for such a  $t$ . From the equation

$$2 \int_{\Omega} u_0 w = k$$

there comes

$$t = \frac{k}{2} - \int_{\Omega} w_1 u_0.$$

hence the couple  $(w, \mu)$  is uniquely determined. From the implicit function theorem, the map  $V \mapsto (u_V, \lambda_V)$  is  $\mathcal{C}^{\infty}$  in a neighbourhood of  $\vec{0}$ .  $\square$

### 3.A.1.2 COMPUTATION OF THE FIRST ORDER SHAPE DERIVATIVE

*Computation of the first order shape derivative.* Let  $V$  be a smooth vector field defined on  $\mathbb{S}^* = \mathbb{S}(0, r_0^*)(0; r^*)$  and normal to  $\mathbb{S}(0, r_0^*)^*$ . Let, for any  $t > 0$  small enough,  $B_t = T_t(\mathbb{B}^*)$  where

$T_t := Id + tV$ .

Let  $(\lambda_t, u_t)$  be the eigencouple associated with  $m_t := \kappa \chi_{\mathbb{B}_t^*}$ . We first define

$$v_t := u_t \circ T_t : \mathbb{B} \rightarrow \mathbb{R}.$$

The derivative of  $v_t$  with respect to  $t$  will be denoted  $\dot{u}$ . This is the material derivative, while we aim at computing the shape derivative  $u' := u_{1,\alpha}$  defined as

$$u' = \dot{u} + \langle V, \nabla u_0 \rangle.$$

For more on these notions, we refer to [93].

Obviously  $v_t \in W_0^{1,2}(\Omega)$ . The weak formulation on  $u_t$  writes: for any  $\varphi \in W_0^{1,2}(\Omega)$ ,

$$\int_{\mathbb{B}(0;R)} \sigma_t \langle \nabla u_t, \nabla \varphi \rangle = \lambda_t \int_{\mathbb{B}(0;R)} u_t \varphi + \kappa \int_{\mathbb{B}_t^*} u_t \varphi.$$

We do the change of variables

$$x = T_t(y),$$

so that

$$dx = \det(\nabla T_t)(y).$$

Furthermore,

$$\nabla v_t(x) = \nabla T_t(x) \nabla u_t(T_t(x)).$$

and, for any test function  $\varphi$ :

$$\int_{\mathbb{B}(0;R)} \sigma_0 \langle \det(\nabla T_t) \nabla T_t^{-1} (\nabla T_t^{-1})^T \nabla v_t, \nabla \varphi \rangle = \lambda_t \int_{\mathbb{B}(0;R)} v_t \varphi \det(\nabla T_t) + \kappa \int_{\mathbb{B}^*} v_t \varphi \det(\nabla T_t).$$

We follow the notations of [63] and define

$$J(t, x) := \det(\nabla T_t)(x).$$

It is known that

$$\mathcal{J}(x) := \left. \frac{\partial J}{\partial t} \right|_{t=0} (t, x) = \nabla \cdot V.$$

We also introduce

$$A(t, x) := \det(\nabla T_t) \nabla T_t^{-1} (\nabla T_t^{-1})^T,$$

and

$$\mathcal{A}(x) := \left. \frac{\partial A}{\partial t} \right|_{t=0} (t, x) = (\nabla \cdot V) I_n - (\nabla V + (\nabla V)^T).$$

We recall that  $\mathcal{A}$  has the following property: if  $V_1$  and  $V_2$  are two vector fields, there holds

$$\langle \mathcal{A}V_1, V_2 \rangle = \nabla \cdot (\langle V_1, V_2 \rangle V) - \langle \nabla(V \cdot V_1), V_2 \rangle - \langle \nabla(V \cdot V_2), V_1 \rangle. \quad (3.49)$$

The weak formulation on  $v_t$  is thus

$$\int_{\mathbb{B}(0;R)} A(t, x) \langle \sigma_0 \nabla v_t, \nabla \varphi \rangle = \lambda_t \int_{\mathbb{B}(0;R)} J(t, x) v_t \varphi + \kappa \int_{\mathbb{B}^*} J(t, x) v_t \varphi.$$

We differentiate this equation with respect to  $t$  to get the following equation on  $\dot{u}$ :

$$\int_{\mathbb{B}(0;R)} \langle \nabla \varphi, \sigma_0 \nabla \dot{u} + \sigma_0 \mathcal{A} \nabla u_0 \rangle = \dot{\lambda} \int_{\mathbb{B}(0;R)} u_0 \varphi + \lambda_0 \int_{\mathbb{B}(0;R)} \mathcal{J} u_0 \varphi + \lambda_0 \int_{\mathbb{B}(0;R)} \dot{u} \varphi + \kappa \int_{\mathbb{B}^*} \dot{u} \varphi + \kappa \int_{\mathbb{B}^*} \mathcal{J}(x) u_0 \varphi. \quad (3.50)$$

Through Property (4.69) $\mathcal{A}$ , we get

$$\sigma_0 \langle \mathcal{A} \nabla u_0, \nabla \varphi \rangle = \sigma_0 \nabla \cdot (\langle \nabla u_0, \nabla \varphi \rangle V) - \sigma_0 \langle \nabla (\langle V, \nabla u_0 \rangle), \nabla \varphi \rangle - \sigma_0 \langle \nabla (\langle V, \nabla \varphi \rangle), \nabla u_0 \rangle.$$

We deal with these three terms separately: from the divergence Formula

$$\int_{\mathbb{B}(0;R)} \sigma_0 \nabla \cdot (\langle \nabla u_0, \nabla \varphi \rangle V) = - \int_{\mathbb{S}(0, r_0^*)^*} [\langle \sigma_0 \nabla u_0, \nabla \varphi \rangle] \langle V, \nu \rangle.$$

We do not touch the second term.

The third term is dealt with using the weak equation on  $u_0$ :

$$\int_{\mathbb{B}(0;R)} \langle \nabla (\langle V, \nabla \varphi \rangle), \sigma_0 \nabla u_0 \rangle = \lambda_0 \int_{\mathbb{B}(0;R)} \langle V, \nabla \varphi \rangle u_0 + \kappa \int_{\mathbb{B}^*} u_0 \langle V, \nabla \varphi \rangle - \int_{\mathbb{S}^*} [\langle \sigma_0 \nabla u_0, \nabla \varphi \rangle] \langle V, \nu \rangle$$

On a finalement

$$\begin{aligned} \int_{\mathbb{B}(0;R)} \langle \sigma_0 \mathcal{A} \nabla u_0, \nabla \varphi \rangle &= \int_{\mathbb{B}(0;R)} \sigma_0 \nabla \cdot (\langle \nabla u_0, \nabla \varphi \rangle V) - \sigma_0 \langle \nabla (\langle V, \nabla u_0 \rangle), \nabla \varphi \rangle - \sigma_0 \langle \nabla (\langle V, \nabla \varphi \rangle), \nabla u_0 \rangle \\ &= - \int_{\mathbb{B}(0;R)} \langle \sigma_0 \nabla (\langle V, \nabla u_0 \rangle), \nabla \varphi \rangle \\ &\quad - \lambda_0 \int_{\mathbb{B}(0;R)} \langle V, \nabla \varphi \rangle u_0 - \kappa \int_{\mathbb{B}^*} u_0 \langle V, \nabla \varphi \rangle. \end{aligned}$$

The left hand term of 4.70 becomes

$$\begin{aligned} \int_{\mathbb{B}(0;R)} \langle \sigma_0 \nabla \dot{u} + \sigma_0 \mathcal{A} \nabla u_0, \nabla \varphi \rangle &= \int_{\mathbb{B}(0;R)} \langle \sigma_0 \nabla (\dot{u} - V \cdot \nabla u_0), \nabla \varphi \rangle \\ &\quad - \lambda_0 \int_{\mathbb{B}(0;R)} \langle V, \nabla \varphi \rangle u_0 - \kappa \int_{\mathbb{B}^*} u_0 \langle V, \varphi \rangle. \end{aligned}$$

Thus

$$\begin{aligned} &\int_{\mathbb{B}(0;R)} \langle \sigma_0 \nabla (\dot{u} - V \cdot \nabla u_0), \nabla \varphi \rangle - \lambda_0 \int_{\mathbb{B}(0;R)} \langle V, \nabla \varphi \rangle u_0 - \kappa \int_{\mathbb{B}^*} u_0 \langle V, \nabla \varphi \rangle \\ &= \dot{\lambda} \int_{\mathbb{B}(0;R)} u_0 \varphi + \lambda_0 \int_{\mathbb{B}(0;R)} \mathcal{J}(x) u_0 \varphi + \lambda_0 \int_{\mathbb{B}(0;R)} \dot{u} \varphi + \kappa \int_{\mathbb{B}^*} \dot{u} \varphi + \kappa \int_{\mathbb{B}^*} \mathcal{J}(x) u_0 \varphi. \end{aligned}$$

By rearranging the terms, we get

$$\begin{aligned} &\int_{\mathbb{B}(0;R)} \langle \sigma_0 \nabla (\dot{u} - V \cdot \nabla u_0), \nabla \varphi \rangle \\ &= + \dot{\lambda} \int_{\mathbb{B}(0;R)} u_0 \varphi + \lambda_0 \left( \int_{\mathbb{B}(0;R)} \mathcal{J}(x) u_0 \varphi + \int_{\mathbb{B}(0;R)} \langle V, \nabla \varphi \rangle u_0 \right) \end{aligned}$$

$$+\lambda_0 \int_{\mathbb{B}(0;R)} \dot{u}\varphi + \kappa \int_{\mathbb{B}^*} \dot{u}\varphi + \kappa \int_{\mathbb{B}^*} \mathcal{J}(x)u_0\varphi + \kappa \int_{\mathbb{B}^*} u_0 \langle V, \nabla\varphi \rangle.$$

However, since  $\mathcal{J}(x) = \nabla \cdot V(x)$ , we have

$$\mathcal{J}(x)\varphi + \langle V, \nabla\varphi(x) \rangle = \nabla \cdot (\varphi V).$$

Hence

$$\begin{aligned} \int_{\mathbb{B}(0;R)} \mathcal{J}(x)u_0\varphi + \int_{\mathbb{B}(0;R)} \langle V, \nabla\varphi \rangle u_0 &= \int_{\mathbb{B}(0;R)} \nabla \cdot (V\varphi)u_0 \\ &= - \int_{\mathbb{B}(0;R)} \varphi \langle V, \nabla u_0 \rangle, \end{aligned}$$

because  $u_0$  satisfies homogeneous Dirichlet boundary conditions. In the same way

$$\begin{aligned} \kappa \int_{\mathbb{B}^*} \mathcal{J}(x)u_0\varphi + \kappa \int_{\mathbb{B}^*} \langle V, \nabla\varphi \rangle u_0 &= \int_{\mathbb{B}(0;R)} \nabla \cdot (V\varphi)u_0 \\ &= -\kappa \int_{\mathbb{B}^*} \varphi \langle V, \nabla u_0 \rangle + \int_{\partial\mathbb{B}^*} \langle V, \nu \rangle u_0\varphi. \end{aligned}$$

We turn back to the shape derivative; recall that it is defined as

$$u' := \dot{u} - \langle V, \nabla u \rangle.$$

The previous equation rewrites

$$\begin{aligned} \int_{\mathbb{B}(0;R)} \langle \sigma_0 \nabla u', \nabla\varphi \rangle &= \dot{\lambda}_0 \int_{\mathbb{B}(0;R)} u_0\varphi + \lambda_0 \int_{\mathbb{B}(0;R)} u'\varphi + \kappa \int_{\mathbb{B}^*} u'\varphi \\ &\quad + \int_{\partial\mathbb{B}^*} \langle V, \nu \rangle u_0\varphi \end{aligned}$$

Thus there appears that  $u' = u_{1,\alpha}$  solves

$$-\sigma_0 \Delta u_{1,\alpha} = \lambda_{1,\alpha} u_{0,\alpha} + \lambda_{0,\alpha} u_{1,\alpha} + m^* u_{1,\alpha}$$

along with Dirichlet boundary conditions and

$$\left[ \sigma_0 \frac{\partial u_{1,\alpha}}{\partial r} \right] = -\kappa \langle V, \nu \rangle u_{0,\alpha}.$$

Obtaining the jump conditions on  $u_{1,\alpha}, u_{2,\alpha}$  is done by differentiating the continuity condition

$$\frac{d[u_t]}{dt} = 0$$

exactly as in [63]. □





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## CHAPTER 4

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# A QUANTITATIVE INEQUALITY FOR THE FIRST EIGENVALUE OF A SCHRÖDINGER OPERATOR IN THE BALL

Il m'arrive encore de sortir de ma maison,  
sur ma colline, au-dessus de la baie de San  
Francisco, et là, en pleine vue, en pleine  
lumière, je jongle avec trois oranges, tout  
ce que je peux faire aujourd'hui.

---

Gary, *La promesse de l'aube*

**GENERAL PRESENTATION OF THE CHAPTER: MAIN DIFFICULTIES  
AND METHODS**

This Chapter, which corresponds to [Maz19a], is devoted to the following quantitative inequality: let

$$\mathcal{M}(\Omega) := \left\{ V, 0 \leq V \leq 1, \int_{\Omega} V = V_0 \right\}$$

and let

$$\Omega = \mathbb{B}(0; R).$$

It is known that there exists  $r^* > 0$  such that  $V^* = \mathbb{1}_{\mathbb{B}(0; r^*)}$  is the only solution of the minimization problem

$$\min_{V \in \mathcal{M}(\Omega)} \lambda(V)$$

where  $\lambda(V)$  is the first eigenvalue of

$$\mathcal{L}_V := -\Delta - V.$$

We establish that there exists a constant  $C > 0$  such that

$$\lambda(V) - \lambda(V^*) \geq C \|V - V^*\|_{L^1}^2.$$

In order to prove it, we introduce an auxilliary problem, but the key parts are Steps 3 and 5 of the Proof of Theorem 4.1.1, which deal with possible topological changes of competitors.

Roughly speaking, we first prove the inequality for normal deformations of  $\mathbb{B}(0; r^*)$ , using classical techniques of shape derivatives. This involves the use of a comparison principle, which alleviates the computations. We then proceed to study an auxilliary problem for radially symmetric functions. This part is, to the best of our knowledge, quite new, and uses an original method of Taylor expansion across the boundary. Finally, in Step 5 of the proof, we introduce a new transformation to show that handling radial perturbations and normal perturbations of the optimizer are enough to handle every possible competitors. This last step is, once again to the best of our knowledge, new.

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## 4.1 INTRODUCTION

### 4.1.1 STRUCTURE OF THE PAPER

In the first part (Section 4.1.2) of the introduction, we lay out the mathematical setting of this article, the main results and give the relevant definitions. In the second part of the Introduction, we give bibliographical references concerning quantitative inequalities and the biological motivations this study stems from. We then explain, in Subsection 4.2.1, how the proof differs from that of other quantitative inequalities.

The core of this paper is devoted to the proof of Theorem 4.1.1. In the conclusion, we give a conjecture and a final comment on a possible way to obtain an optimal exponent using parametric derivatives.

### 4.1.2 MATHEMATICAL SETTING

The optimization of eigenvalues of elliptic operators defined on domains with a zero-order term (i.e. with a potential) with respect to either the domain (with a fixed potential defined on a bigger domain) or the potential (with a fixed domain) is a classical question in optimization under partial differential equations constraints. The example under scrutiny here is the operator

$$\mathcal{L}_V : u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \mapsto -\Delta u - Vu, \quad (4.1)$$

where  $\Omega$  is a smooth domain. Under the assumption that  $V \in L^\infty(\Omega)$ , this operator is known to have a first, simple eigenvalue, denoted by  $\lambda(V)$ , and associated with an eigenfunction  $u_V$ , which solves

$$\begin{cases} -\Delta u_V - Vu_V = \lambda(V)u_V \text{ in } \Omega, \\ u_V = 0 \text{ on } \partial\Omega. \\ \int_{\Omega} u_V^2 = 1. \end{cases} \quad (4.2)$$

Alternatively, this eigenvalue admits the following variational formulation in terms of Rayleigh quotients:

$$\lambda(V) = \inf_{u \in W_0^{1,2}(\Omega), \int_{\Omega} u^2 = 1} \left\{ \int_{\Omega} |\nabla u|^2 - \int_{\Omega} Vu^2 \right\}. \quad (4.3)$$

We make stronger assumptions on the potential  $V$  and require that it lies in

$$\mathcal{M}(\Omega) := \left\{ V \in L^\infty(\Omega), 0 \leq V \leq 1, \frac{1}{|\Omega|} \int_{\Omega} V = V_0 \right\}, \quad (4.4)$$

where  $V_0 > 0$  is a real parameter such that

$$V_0 < 1$$

so that  $\mathcal{M}(\Omega)$  is non-empty. The optimization problem we focus on in this paper is

$$\inf_{V \in \mathcal{M}(\Omega)} \lambda(V). \quad (4.5)$$

This problem has drawn a lot of attention from the mathematical community over the last decades, and is quite a general one. It is particularly relevant in the context of mathematical biology, see Section 4.1.4.2.

### 4.1.3 MAIN RESULTS

#### 4.1.3.1 NOTATIONS

Here and throughout, the underlying domain is  $\Omega = \mathbb{B}(0; R)$ . The parameter  $r^*$  is chosen so that

$$V^* := \chi_{\mathbb{B}(0; r^*)} = \chi_{E^*} \in \mathcal{M}(\Omega).$$

The constant  $c_n > 0$  is the  $(n - 1)$ -dimensional volume of the unit sphere in dimension  $n$ . For any  $E \subset \mathbb{B}$ , we define  $\lambda(E) := \lambda(\chi_E)$ .

#### 4.1.3.2 QUANTITATIVE INEQUALITY

A classical application of Schwarz' rearrangement shows that  $V^*$  is the unique minimizer of  $\lambda$  in  $\mathcal{M}(\Omega)$ :

$$\forall V \in \mathcal{M}(\Omega), \lambda(V) \geq \lambda(V^*).$$

We refer to [108] for an introduction to the Schwarz rearrangement and to [116] for its use in this context. For the sake of completeness, we also prove this result in Annex 4.A.

The goal of this paper is to establish the following quantitative spectral inequality:

**THEOREM 4.1.1** *Let  $n = 2$ . There exists a constant  $C > 0$  such that, for any  $V \in \mathcal{M}(\Omega)$ , there holds*

$$\lambda(V) - \lambda(V^*) \geq C \|V - V^*\|_{L^1}^2. \quad (4.6)$$

This result has a Corollary which fits in the natural context of estimates using the Fraenkel asymmetry: as we will explain in Subsection 4.1.4.1, this is the natural property to expect in the context of quantitative inequalities. Indeed, if  $V = \chi_E \in \mathcal{M}(\Omega)$ , then

$$\|V - V^*\|_{L^1} = |E^* \Delta E|$$

where  $\Delta$  stands for the symmetric difference, and, if we define the Fraenkel asymmetry of  $E$  as

$$\mathcal{A}(E) := \inf_{\mathbb{B}(x; r), \chi_{\mathbb{B}(x; r)} \in \mathcal{M}(\Omega)} |E \Delta \mathbb{B}(x; r)|$$

then it follows from Theorem 4.1.1 that

$$\lambda(\chi_E) - \lambda(\chi_{E^*}) \geq C \mathcal{A}(E)^2.$$

We thus have a parametric version of a quantitative inequality with what is in fact a sharp exponent.

#### 4.1.3.3 A COMMENT ON PARAMETRIC AND SHAPE DERIVATIVES

The proof of Theorem 4.1.1 relies on parametric and shape derivatives, and the aim of this Section is to give possible links between the two notions and, most notably, to give a situation where this link is no longer possible. This is obviously in sharp contrast with classical shape optimization, since here we are only optimizing with respect to the potential, while it is customary to derive Faber-Krahn type inequalities, i.e to optimize with respect to the domain  $\Omega$  itself, see Subsection 4.1.4.1.

Roughly speaking, there are two ways to tackle spectral optimization problems such as (4.5): the parametric approach and the shape derivative approach. By parametric approach we mean the following:

**DEFINITION 4.1.1** *We define, for any  $V \in \mathcal{M}(\Omega)$ , the tangent cone to  $\mathcal{M}(\Omega)$  at  $V$  as*

$$\mathcal{T}_V := \{h : \Omega \rightarrow [-1; 1], \forall \varepsilon \leq 1, V + \varepsilon h \in \mathcal{M}(\Omega)\}$$

and define, provided it exists, the parametric derivative of  $\lambda$  at  $V$  in the direction  $h \in \mathcal{T}_V$  as

$$\dot{\lambda}(V)[h] := \lim_{t \rightarrow 0} \frac{\lambda(V + th) - \lambda(V)}{t}.$$

In this case, the optimality condition reads

$$\forall h \in \mathcal{T}_V, \dot{\lambda}(V)[h] \geq 0.$$

In the conclusion of this article, we will explain why there holds

$$\forall h \in \mathcal{T}_{V^*}, \dot{\lambda}(V^*)[h] \geq C \|h\|_{L^1}^2.$$

Since, for any  $V \in \mathcal{M}(\Omega)$ ,  $h := V - V^* \in \mathcal{T}_{V^*}$ , what we get is

$$\dot{\lambda}(V^*)[V - V^*] \geq C \|V - V^*\|_{L^1}^2$$

that is, an infinitesimal version of Estimate (4.6), and one might wonder whether such infinitesimal estimates might lead to global qualitative inequality of the type (4.6). This is customary, in the context of shape derivatives. We need to define the notion of shape derivative before going further:

**DEFINITION 4.1.2** *Let  $\mathcal{F} : E \mapsto \mathcal{F}(E) \in \mathbb{R}$  be a shape functional. We define*

$$\mathcal{X}_1(V^*) := \{ \Phi : \mathbb{B}(0; R) \rightarrow \mathbb{R}^2, \|\Phi\|_{W^{1,\infty}} \leq 1, \forall t \in (-1; 1), \chi_{(Id+t\Phi)(\mathbb{B}^*)} \in \mathcal{M}(\Omega). \}$$

as the set of admissible perturbations at  $E^*$ . The shape derivative of first (resp. second) order of a shape function  $\mathcal{F}$  at  $V^*$  in the direction  $\Phi$  is

$$\begin{aligned} \mathcal{F}'(E^*)[\Phi] &= \lim_{t \rightarrow 0} \frac{\mathcal{F}\left((Id + t\Phi)E^*\right) - \mathcal{F}(E^*)}{t} \\ (\text{resp. } \mathcal{F}''(E^*)[\Phi, \Phi] &:= \lim_{t^2 \rightarrow 0} \frac{\mathcal{F}\left((Id + t\Phi)E^*\right) - \mathcal{F}(E^*) - \mathcal{F}'(E^*)(\Phi)}{t^2}.) \end{aligned} \quad (4.7)$$

A customary way to derive quantitative inequality is to show that, at a given shape  $E$ , there holds

$$\mathcal{F}'(E)[\Phi] = 0, \mathcal{F}''(E)[\Phi, \Phi] > 0$$

and to lift the last inequality to a quantitative inequality of the form

$$\mathcal{F}'(E)[\Phi, \Phi] \geq C \|\Phi\|_s^2$$

where  $\|\cdot\|_s$  is a suitable norm; we refer to [64] for more details but for instance one might have  $\|\Phi|_{\partial E}\|_{L^1}^2$  which often turns out to be the suitable exponent for a quantitative inequality. This quantitative inequality for shape deformations is usually not enough, and we refer to Section 4.1.4.1 for more details and bibliographical references.

#### 4.1.3.4 A REMARK ON THE PROOF

The main innovation of this paper is the proof of Theorem 4.1.1 which, although it uses shape derivatives as is customary while proving quantitative inequalities, see Subsection 4.1.4.1, relies heavily on parametric derivatives. This is allowed by the fact that we are working with a potential defined on the interior of the domain.

Furthermore, we also prove that, unlike classical shape optimization, our coercivity norm for the

second order shape derivative is the  $L^2$  norm.

#### 4.1.4 BIBLIOGRAPHICAL REFERENCES

##### 4.1.4.1 QUANTITATIVE SPECTRAL INEQUALITIES

**Spectral deficit for Faber-Krahn type inequalities:** Quantitative spectral inequalities have received a lot of attention for a few decades, and are usually set in a context which is more general than the one introduced here. The main goal of such inequalities were to derive quantitative versions of the Faber-Krahn inequality: for a given parameter  $\beta \in (0; +\infty]$  and a bounded domain  $\Omega \subset \mathbb{R}^n$ , consider the first eigenvalue  $\eta_\beta(\Omega)$  of the Laplacian with Robin boundary conditions:

$$\begin{cases} -\Delta u_{\beta,\Omega} = \eta_\beta(\Omega)u_{\beta,\Omega} \text{ in } \Omega, \\ \frac{\partial u_{\beta,\Omega}}{\partial \nu} + \beta u_{\beta,\Omega} = 0 \text{ on } \partial\Omega, \end{cases} \quad (4.8)$$

with the convention that  $\eta_\infty(\Omega)$  is the first Dirichlet eigenvalue of the Laplace operator. Although it has been known since the independent works of Faber [73] and Krahn [114] that, whenever  $\Omega^*$  is a ball with the same volume as  $\Omega$ , there holds

$$\eta_\infty(\Omega) \geq \eta_\infty(\Omega^*),$$

the question of providing a sharp lower bound for the so-called *spectral deficit*

$$R_\infty(\Omega) = \eta_\infty(\Omega) - \eta_\infty(\Omega^*)$$

remained largely open until Nadriashvili and Hansen [86] and Melas, [136], using Bonnesen type inequalities, obtained a lower bound on the spectral deficit involving quantities related to the geometry of domain  $\Omega$  through the inradius. In a later work, Brasco, De Philippis and Velichkov, [31], the sharp version of the quantitative inequality, namely:

$$R_\infty(\Omega) \geq C\mathcal{A}(\Omega)^2. \quad (4.9)$$

Their proof relies uses as a first step a second order shape derivative argument, a series of reduction to small asymmetry regime and, finally, a quite delicate selection principle. We comment in the next paragraph on the role of second order shape derivative for generic quantitative inequalities and the part it plays in our proof.

For a survey of the history and proofs of quantitative Faber-Krahn inequalities, we refer to the survey [30] and the references therein.

For quantitative forms of spectral inequalities with general Robin boundary conditions, the Bossel-Daners inequality, first derived by Bossel in dimension 2 in [25] and later extended by Daners in all dimensions in [66] reads:

$$R_\beta(\Omega) := \eta_\beta(\Omega) - \eta_\beta(\mathbb{B}) \geq 0, \beta > 0,$$

and a quantitative version of the Inequality was proved by Bucur, Ferone, Nitsch and Trombetti in [37]:

$$R_\beta(\Omega) \geq C\mathcal{A}(\Omega)^2.$$

Their method for  $\beta < \infty$  is different from the case of Dirichlet eigenvalue and relies on a free boundary approach.

**The role of second order shape derivatives:** We only want to mention here the results we draw our inspiration from in Step 4 of the proof, and do not aim at giving out the rigorous mathematical setting of the results mentioned below. We refer to [64] for a thorough presentation of the link between



second order shape derivatives, local shape stability and local quantitative inequalities.

As we said in the previous paragraph, most proofs of quantitative inequalities start with a local quantitative inequality for shape perturbation of the optimum  $E_1$ : namely, if  $\mathcal{F}$  is a regular enough shape functional and  $E_1$  is an admissible set such that

$$\mathcal{F}(E_1)[\Phi] = 0, \mathcal{F}''(E_1)[\Phi, \Phi] > 0 \quad (4.10)$$

for any  $\Phi \in \mathcal{X}_1(E_1)$  then it is proved in [62], [65] that  $E_1$  is a strict local minimizer in a  $\mathcal{C}^{2,\alpha}$  neighbourhood of  $E$  (actually, in these two articles, the authors assume a coercivity of the second order derivative in  $H^{\frac{1}{2}}$  norm on  $\Phi$ ). In [64], Dambrine and Lamboley proved that the same conditions imply a local quantitative inequality under certain technical assumptions. Roughly speaking, their result implies the following result: Condition (4.10) implies that if, for any function  $h \in \mathcal{C}^0(\partial\Omega) \cap W^{2,\infty}(\partial\Omega)$  we define  $E_1^h$  as the domain bounded by

$$\partial E_1^h := \{x + h(x)\nu_{E_1}(x), x \in \partial E_1\}$$

then there exists  $\ell > 0$  such that, for any  $\|h\|_{W^{2,\infty}(\partial\Omega)} \leq \ell$  there holds

$$\mathcal{F}(E_1) + C\|h\|_{L^1}^2 \leq \mathcal{F}(E_1^h)$$

for some  $C > 0$ .

Their result actually holds in stronger norm, but this is the version we wanted to mention here since it is the one we will adapt in our parametric setting. We note however, that the authors, in their problems, prove their inequalities for shape using a  $H^{\frac{1}{2}}$  coercivity norm for the second derivative: they usually have

$$\mathcal{F}(E_1)''[h, h] \geq C\|h\|_{H^{\frac{1}{2}}}^2.$$

In this expression, we have defined  $\mathcal{F}''(E_1)[h, h] := \mathcal{F}(E_1)''[x + h\nu_{E_1}(x), x + h\nu_{E_1}(x)]$ . Here, in Step 4, we will show, using comparison principles, that the optimal coercivity norm is the  $L^2$  norm.

**Difference with our proofs and contribution:** To the best of our knowledge, quantitative inequalities in a parametric setting have not been studied yet, and this article aims at providing a first step in that direction. As we will see while proving Theorem 4.1.1, a shape derivative approach can not be sufficient in of its own for our purposes, as is usually the case, but is needed. To tackle the second order shape derivative, we use a comparison principle. The parametric setting also enables us more freedom while dealing with other competitors and enables us to introduce a new method for dealing with such problems.

#### 4.1.4.2 MATHEMATICAL BIOLOGY

We briefly sketch some of the biological motivations for the problem under scrutiny here. Following the works of Fisher, [75], Kolmogoroff, Petrovsky and Piscounoff [113], a popular model for population dynamics in a bounded domain is the following so-called logistic-diffusive equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + u(m - u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u(t = 0) = u_0 \geq 0, u_0 \neq 0. \end{cases} \quad (4.11)$$

In this equation,  $m \in L^\infty(\Omega)$  accounts for the spatial heterogeneity and can be interpreted in terms of resources distribution: the zones  $\{m \geq 0\}$  are favorable to the growth of the population, while the zones  $\{m \leq 0\}$  are detrimental to this population. The particular structure of the non-linearity  $-u^2$  (which accounts for the Malthusian growth of the population) makes it so that two linear steady

states equations are relevant to our study: the steady-logistic diffusive equation

$$\begin{cases} \Delta\theta + \theta(m - \theta) = 0 \text{ in } \Omega, \\ \theta = 0 \text{ on } \partial\Omega, \\ \theta \geq 0, \end{cases} \quad (4.12)$$

and the first eigenvalue equation of the linearization of (4.11) around the solution  $z \equiv 0$ :

$$\begin{cases} -\Delta\varphi_m - m\varphi_m = \lambda(m)\varphi \text{ in } \Omega, \\ \varphi = 0 \text{ on } \partial\Omega, \end{cases} \quad (4.13)$$

where  $\lambda(m)$  is the first eigenvalue of the operator  $\mathcal{L}_m$  defined in (4.1). More precisely, it is known (see [19, 40, 168]) that

1. Whenever  $\lambda(m) < 0$ , has a unique solution  $\theta_m$ , and any solution  $u = u(t, x)$  of (4.11) with initial datum  $u_0 \geq 0, u_0 \neq 0$  converges in any  $L^p$  to  $\theta_m$  as  $t \rightarrow \infty$ .
2. Whenever  $\lambda(m) \geq 0$ , any solution  $u = u(t, x)$  of (4.11) with initial datum  $u_0 \geq 0, u_0 \neq 0$  converges in any  $L^p$  to 0 as  $t \rightarrow \infty$ .

The eigenvalue which we seek to minimize can thus be interpreted as a measure of the survival ability given by a resources distribution, and later works investigated the problem of minimizing  $\lambda(m)$  with respect to  $m$  under the constraint  $m \in \mathcal{M}(\Omega)$ , where  $\mathcal{M}(\Omega)$  is defined in (4.4). In other words, this is the problem (4.5).

In the case of Neumann boundary conditions, Berestycki, Hamel and Roques introduced the use of a rearrangement (due to Berestycki and Lachand-Robert,[20]) in that context, see [19] and [108] for an introduction to rearrangement, and further geometrical properties of optimizers were derived by Lou and Yanagida, [128], by Kao, Lou and Yanagida [104]. We do not wish to be exhaustive regarding the literature of this domain and refer to [116] where Lamboley, Laurain, Nadin and Privat investigate several properties of solutions of (4.5) under a variety of boundary conditions, and the references therein.

## 4.2 PROOF OF THEOREM 4.1.1

### 4.2.1 BACKGROUND ON (4.5) AND STRUCTURE OF THE PROOF

We recall that we work in  $\Omega = \mathbb{B}(0; R) \subset \mathbb{R}^2$ , that

$$\mathcal{M} := \mathcal{M}(\mathbb{B}(0; R)) = \left\{ 0 \leq V \leq 1, \int_{\Omega} V = V_0 \right\}$$

and that  $r^*$  is chosen so that

$$|\mathbb{B}(0; r^*)| = V_0$$

i.e such that

$$V^* = \chi_{\mathbb{B}(0; r^*)} \in \mathcal{M}.$$

We define

$$\mathbb{S}^* := \partial\mathbb{B}^*.$$

We first recall the following simple consequence of Schwarz' rearrangement:

**LEMMA 4.1**  *$V^*$  is the unique minimizer of  $\lambda$  in  $\mathcal{M}$ : for any  $V \in \mathcal{M}$ ,*

$$\lambda(V^*) \leq \lambda(V),$$

and equality holds if and only if  $V = V^*$ . The associated eigenfunction  $u_*$  is decreasing and radially symmetric.

This result is well-known, but for the sake of completeness we prove it in Annex 4.A. The proof of Theorem 4.1.1 relies on the study of two auxiliary problems: we introduce, for a given  $\delta > 0$ , the new admissible sets

$$\mathcal{M}_\delta := \{V \in \mathcal{M}, \|V - V^*\|_{L^1} = \delta\}, \quad (4.14)$$

$$\tilde{\mathcal{M}}_\delta := \{V \text{ radially symmetric}, V \in \mathcal{M}, \|V - V^*\|_{L^1} = \delta\} \quad (4.15)$$

and study the two variational problems

$$\inf_{V \in \mathcal{M}_\delta} \lambda(V) \quad (4.16)$$

and

$$\inf_{V \in \tilde{\mathcal{M}}_\delta} \lambda(V). \quad (4.17)$$

Obviously, Theorem 4.1.1 is equivalent to the existence of  $C > 0$  such that

$$\forall \delta > 0, \forall V \in \mathcal{M}_\delta, \lambda(V) - \lambda(V^*) \geq C\delta^2. \quad (4.18)$$

**REMARK 4.1** *This is a parametric version of the selection principle of [1], that was developed in [31]. We refer to [30] for a synthetic presentation of this selection principle. We note however that the fact that they use a perimeter constraint enables them to prove that a solution to their auxiliary problem is a normal deformation of the optimal shape. The main difficulty in the analysis of [31] is establishing  $\mathcal{C}^2$  bounds for this normal deformation. Here, working with subsets as shape variables gives, from elliptic regularity, enough regularity to carry out this step when the solution of the auxiliary problem is a normal deformation of  $\mathbb{B}^*$ . However, we conjecture that the solutions of (4.16) and (4.17) are equal and are disconnected (see Step 3 and the Conclusion for a precise conjecture), so that the core difficulty is proving that handling the inequality for normal deformations and for radial distributions is enough to get the inequality for all other sets.*

To prove (A<sub>1</sub>) we follow the steps below:

1. We first show that (4.17) and (4.16) have solutions. The solutions of (4.17) will be denoted by  $\mathcal{V}_\delta$ , the solution of (4.16) will be denoted by  $\mathcal{H}_\delta$ .
2. We prove that it suffices to establish (A<sub>1</sub>) for  $\delta$  small enough.
3. For (4.16) we fully characterize the solutions for  $\delta > 0$  small enough and prove that

$$\forall \delta > 0, \forall V \in \tilde{\mathcal{M}}_\delta, \lambda(V) - \lambda(V^*) \geq C\delta^2. \quad (4.19)$$

In other words, we prove that Theorem 4.1.1 holds for radially symmetric functions.

4. We compute the first and second order shape derivatives of the associated Lagrangian at the ball  $\mathbb{B}^*$  and prove a  $L^2$ -coercivity estimate for the second order derivative. We comment upon the fact that (unlike many shape optimization problems) this is the optimal coercivity norm at the beginning of this Step. We use this information to prove that Theorem 4.1.1 holds for domains that are small normal deformations of  $\mathbb{B}^*$  with bounded mean curvature.
5. We establish a dichotomy for the behaviour of  $\mathcal{V}_\delta$  and prove that (A<sub>1</sub>) holds for any  $V$  by using (4.19) and the inequality for normal deformations of the ball.

### 4.2.2 STEP 1: EXISTENCE OF SOLUTIONS TO (4.16)-(4.17)

We prove the following Lemma:

**LEMMA 4.2** *The optimization problems (4.16) and (4.17) have solutions.*

*Proof of Lemma 4.2.* The proof follows from the following claim:

**CLAIM 4.2.1**  $\mathcal{M}_\delta$  and  $\tilde{\mathcal{M}}_\delta$  are compact for the weak  $L^\infty$  topology.

We postpone the proof to the end of this Proof.

Lemma 4.2 follows from this claim, and we only write the details for (4.17). Let  $\{V_k\}_{k \in \mathbb{N}}$  be a minimizing sequence for  $\lambda$  in  $\mathcal{M}_\delta$ . From Claim 4.2.1, there exists  $V_\infty \in \mathcal{M}_\delta$  such that

$$V_k \rightharpoonup V_\infty.$$

The notation  $\rightharpoonup$  stands for the weak convergence in the weak  $L^\infty$ -\* sense. Let, for any  $k \in \mathbb{N}$ ,  $u_k := u_{V_k}$  and  $\lambda_k := \lambda(V_k)$ .

We first note that the sequence  $\{\lambda_k\}_{k \in \mathbb{N}}$  is bounded. Indeed, let  $\varphi \in W_0^{1,2}(\Omega)$  be such that  $\int_\Omega \varphi^2 = 1$ . From the formulation in terms of Rayleigh quotients and  $V_k \geq 0$  there holds

$$\lambda_k \leq \int_\Omega |\nabla \varphi|^2 - \int_\Omega V_k \varphi^2 \leq \int_\Omega |\nabla \varphi|^2.$$

This gives an upper bound. For a lower bound, let  $\lambda_1(\Omega)$  be the first Dirichlet eigenvalue of  $\Omega$  (equivalently, this is the eigenvalue associated with  $V = 0$ ). From  $V \leq 1$ ,  $\int_\Omega u_k^2 = 1$  and the variational formulation for  $\lambda_1(\Omega)$  there holds

$$\lambda_1(\Omega) - 1 \leq \int_\Omega |\nabla u_k|^2 - \int_\Omega V u_k^2 \leq \lambda_k$$

so that the sequence also admits a lower bound. It is straightforward to see that  $\{u_k\}_{k \in \mathbb{N}}$  is bounded in  $W_0^{1,2}(\Omega)$  so that from the Rellich-Kondrachov Theorem there exists  $u_\infty \in W_0^{1,2}(\Omega)$  such that  $\{u_k\}_{k \in \mathbb{N}}$  converges strongly in  $L^2$  and weakly in  $W_0^{1,2}(\Omega)$  to  $u_\infty$ . Passing to the limit in the weak formulation

$$\forall v \in W_0^{1,2}(\Omega), \int_\Omega \langle \nabla u_k, \nabla v \rangle - \int_\Omega V_k u_k v = \lambda(V_k) \int_\Omega u_k v,$$

in the normalization condition

$$\int_\Omega u_k^2 = 1$$

and in

$$u_k \geq 0$$

readily shows that  $u_\infty$  is a non-trivial eigenfunction of  $\mathcal{L}_{V_\infty}$ . Furthermore, it is non-negative. Since the first eigenfunction is the only eigenfunction with a constant sign, this proves that  $\lambda_\infty = \lambda(V_\infty)$  and that  $u_\infty$  is the eigenfunction associated with  $V_\infty$ . Thus:

$$\lambda(V_\infty) = \inf_{V \in \mathcal{M}_\delta} \lambda(V).$$

It remains to prove Claim 4.2.1:

*Proof of Claim 4.2.1.* We only prove it for  $\mathcal{M}_\delta$ .

Let  $\{V_k\}_{k \in \mathbb{N}} \in \mathcal{M}_\delta^{\mathbb{N}}$ . We define, for any  $k \in \mathbb{N}$

$$h_k := V_k - V^*.$$

Since  $V^* = \chi_{\mathbb{B}^*}$  and  $0 \leq V_k \leq 1$ , the following sign conditions hold on  $h_k$ :

$$h_k \geq 0 \text{ in } (\mathbb{B}^*)^c, h_k \leq 0 \text{ in } \mathbb{B}^*. \quad (4.20)$$

Since  $\int_{\Omega} V_k = \int_{\Omega} V^*$  there holds

$$\int_{\mathbb{B}^*} h_k = - \int_{(\mathbb{B}^*)^c} h_k. \quad (4.21)$$

Finally from

$$\|V_k - V^*\|_{L^1} = \delta$$

there comes

$$\delta = \int_{\Omega} |V_k - V^*| \quad (4.22)$$

$$= \int_{\Omega} |h_k| \quad (4.23)$$

$$= \int_{(\mathbb{B}^*)^c} h_k - \int_{\mathbb{B}^*} h_k \text{ from (4.20)} \quad (4.24)$$

$$= -2 \int_{\mathbb{B}^*} h_k \text{ from (4.21)} \quad (4.25)$$

$$= 2 \int_{(\mathbb{B}^*)^c} h_k. \quad (4.26)$$

We see  $h_k \chi_{(\mathbb{B}^*)^c}$  as an element of  $L^\infty((\mathbb{B}^*)^c)$ . Let  $h_\infty^+$  be a weak- $L^\infty$  closure point of  $\{h_k\}_{k \in \mathbb{N}}$  in  $L^\infty((\mathbb{B}^*)^c)$ . From (4.26) and (4.20) we have

$$1 \geq h_\infty^+ \geq 0, \int_{(\mathbb{B}^*)^c} h_\infty^+ = \frac{\delta}{2}. \quad (4.27)$$

For the same reason, there exists  $h_\infty^- \in L^\infty(\mathbb{B}^*)$  such that

$$-1 \leq h_\infty^- \leq 0, \int_{\mathbb{B}^*} h_\infty^- = -\frac{\delta}{2}$$

and

$$h_k \rightharpoonup h_\infty \text{ weak-* in } L^\infty(\mathbb{B}^*). \quad (4.28)$$

We define

$$h_\infty := h_\infty^- \chi_{(\mathbb{B}^*)^c} + h_\infty^- \chi_{\mathbb{B}^*}$$

and it is clear that

$$h_k \rightharpoonup h_\infty \text{ weak-* in } L^\infty(\Omega).$$

Setting  $V_\infty := V^* + h_\infty$  there holds

$$V_k \rightharpoonup V_\infty$$

and, by (4.27),(4.28),  $V_\infty \in \mathcal{M}_\delta(\Omega)$ . □

□

### 4.2.3 STEP 2: REDUCTION TO SMALL $L^1$ NEIGHBOURHOODS OF $V^*$

We now prove the following Lemma:

**LEMMA 4.3** *To prove Theorem 4.1.1, it suffices to prove (A<sub>1</sub>) for  $\delta$  small enough, in other words it suffices to prove that there exists  $C > 0$  such that*

$$\liminf_{\delta \rightarrow 0} \left( \inf_{V \in \mathcal{M}_\delta} \frac{\lambda(V) - \lambda(V^*)}{\delta^2} \right) \geq C.$$

*Proof of Lemma 4.3.* We define, for any  $V \neq V^*$ ,

$$G(V) := \frac{\lambda(V) - \lambda(V^*)}{\|V - V^*\|_{L^1}^2}$$

and consider a minimizing sequence  $\{V_k\}_{k \in \mathbb{N}}$  for  $G$ . Then we either have, up to subsequence,

$$\forall k \in \mathbb{N}, \|V_k - V^*\|_{L^1} \geq \varepsilon > 0$$

or

$$\|V_k - V^*\|_{L^1} \xrightarrow{k \rightarrow \infty} 0.$$

In the first case, up to a converging subsequence,  $V_k \rightharpoonup V_\infty$  in a weak  $L^\infty$ -\* sense and, by the same arguments as in the Proof of Claim 4.2.1,

$$\|V_\infty - V^*\|_{L^1} \geq \varepsilon > 0.$$

Furthermore, by the same arguments as in the proof of Lemma 4.2,

$$\lambda(V_k) \xrightarrow{k \rightarrow \infty} \lambda(V_\infty)$$

so that

$$G(V_k) \xrightarrow{k \rightarrow \infty} \frac{\lambda(V_\infty) - \lambda(V^*)}{\|V_\infty - V^*\|_{L^1}}$$

and, by Lemma 4.1,  $G(V_\infty) > 0$ . Hence we only need to study the case  $\|V_k - V^*\|_{L^1} \xrightarrow{k \rightarrow \infty} 0$ , as claimed.  $\square$

The same arguments yield the following Lemma:

**LEMMA 4.4** (4.19) *is equivalent to proving that there exists  $C > 0$  such that*

$$\liminf_{\delta \rightarrow 0} \left( \inf_{V \in \mathcal{M}_\delta} \frac{\lambda(V) - \lambda(V^*)}{\delta^2} \right) \geq C.$$

#### 4.2.4 STEP 3: PROOF OF (4.19)

In this Subsection we prove (4.19) or, in other words, we prove that Theorem 4.1.1 holds for radial distributions.

*Proof of (4.19).* We recall that, by Lemma 4.2, there exists a solution to (4.16). Let  $\mathcal{H}_\delta$  be such a minimizer.

We first characterize  $\mathcal{H}_\delta$  for  $\delta$  small enough. Let

$$\mathbb{A}_\delta := \{|x| \leq r^* + r'_\delta\} \setminus \{x, r^* - r_\delta \leq |x| \leq r^* + r'_\delta\} \quad (4.29)$$

be the annular structure such that

$$\chi_{\mathbb{A}_\delta} \in \tilde{\mathcal{M}}_\delta. \quad (4.30)$$

We represent it below

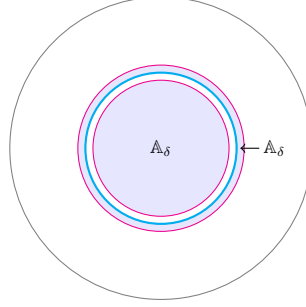


Figure 4.1 – An example of  $\mathbb{A}_\delta$

Since  $r'_\delta$  is defined by the relation

$$|\{r^* \leq |x| \leq r^* + r'_\delta\}| = \frac{\delta}{2}$$

the set  $\mathbb{A}_\delta$  is uniquely defined. We claim the following:

**CLAIM 4.2.2** *There exists  $\bar{\delta} > 0$  such that, for any  $\delta \leq \bar{\delta}$ ,*

$$\mathcal{H}_\delta = \chi_{\mathbb{A}_\delta}.$$

*Proof of Claim 4.2.2.* To prove this claim, we need the optimality conditions associated with (4.16). We first note that, if  $u_\delta$  is the eigenfunction associated with  $\mathcal{H}_\delta$ , there exist two real numbers  $\eta_\delta, \mu_\delta$  such that

1.  $\mathcal{H}_\delta = \chi_{\{u_\delta > \mu_\delta\} \cap \mathbb{B}^*}$  in  $\mathbb{B}^*$ ,
2.  $\mathcal{H}_\delta = \chi_{\{\sup_{\mathbb{S}^*} u_\delta \geq u_\delta > \eta_\delta\} \cap (\mathbb{B}^*)^c}$  in  $(\mathbb{B}^*)^c$ ,
3.  $|\{u_\delta > \mu_\delta\} \cap \mathbb{B}^*| = V_0 - \frac{\delta}{2}$ ,  $|\{\sup_{\mathbb{S}^*} u_\delta \geq u_\delta > \eta_\delta\} \cap (\mathbb{B}^*)^c| = \frac{\delta}{2}$ .

This is readily seen from the Rayleigh quotient formulation (4.3). We only prove 1: let  $\mu_\delta \in \mathbb{R}$  be the only real number such that

$$|\{u_\delta > \mu_\delta\} \cap \mathbb{B}^*| = V_0 - \frac{\delta}{2} = \int_{\mathbb{B}^*} \mathcal{H}_\delta$$

and replace  $\mathcal{H}_\delta$  by

$$\tilde{\mathcal{H}}_\delta := \chi_{\{u_\delta > \mu_\delta\} \cap \mathbb{B}^*} + \mathcal{H}_\delta \chi_{(\mathbb{B}^*)^c}.$$

Since  $u_\delta$  is radially symmetric (because  $\mathcal{H}_\delta$  is radially symmetric),  $\tilde{\mathcal{H}}_\delta$  is radially symmetric. Then, because  $\int_{\mathbb{B}^*} \tilde{\mathcal{H}}_\delta = \int_{\mathbb{B}^*} \mathcal{H}_\delta$ , we have, by the bathtub principle (see [93])

$$\int_{\mathbb{B}^*} u_\delta^2 \mathcal{H}_\delta \leq \int_{\mathbb{B}^*} u_\delta^2 \chi_{\{u_\delta > \mu_\delta\} \cap \mathbb{B}^*},$$

hence

$$\lambda(\mathcal{H}_\delta) \geq \int_{\mathbb{B}} |\nabla u_\delta|^2 - \int_{\mathbb{B}} \tilde{\mathcal{H}}_\delta u_\delta^2 \geq \lambda(\tilde{\mathcal{H}}_\delta).$$

This gives the required property.

We now need to exploit these optimality conditions. First of all, since  $u_\delta$  is radially symmetric, we

can define

$$\zeta_\delta := u_\delta|_{\mathbb{S}^*}.$$

By standard elliptic estimates, we also have

$$u_\delta \xrightarrow{\delta \rightarrow 0} u_* \text{ in } \mathcal{C}^{1,s} \text{ ( } s < 1),$$

where  $u_*$  is the eigenfunction associated with  $V^*$ . Since  $\frac{\partial u_*}{\partial r} < -c$  on  $\{|x| \geq \varepsilon\}$ ,  $u_\delta$  is radially decreasing in  $\{|x| > \varepsilon\}$  for  $\delta > 0$  small enough. It follows that  $\mu_\delta > \zeta_\delta$  for  $\delta$  small enough. For the same reason,  $\zeta_\delta > \eta_\delta$  for  $\delta$  small enough. Hence we have

$$\mu_\delta > \eta_\delta > \zeta_\delta, \mathcal{H}_\delta = \chi_{\{u_\delta > \mu_\delta\}} + \chi_{\{\zeta_\delta \geq u_\delta > \eta_\delta\}}.$$

Finally, once again because  $u_\delta$  is radially decreasing on  $\{|x| > \varepsilon\}$  for  $\delta$  small enough, both level sets  $\{u_\delta > \mu_\delta\}$  and  $\{\zeta_\delta \geq u_\delta > \eta_\delta\}$  are connected, and  $\mathcal{H}_\delta$  is the characteristic function of a centered ball and of an annulus, i.e

$$\mathcal{H}_\delta = \chi_{\{\|x\| \leq r^* - z_\delta\}} + \chi_{\{r^* \leq \|x\| \leq r^* + y_\delta\}}.$$

Since

$$|\mathbb{A}_\delta \Delta \mathbb{B}^*| = \int_{\mathbb{B}} |\mathcal{H}_\delta - V^*|$$

there holds

$$\mathcal{H}_\delta = \chi_{\mathbb{A}_\delta}$$

for  $\delta$  small enough, as claimed. □

We now turn to the proof of (4.19): since  $\mathcal{H}_\delta$  is the minimizer of  $\lambda$  in  $\tilde{\mathcal{M}}_\delta$  we are going to prove that there exists a constant  $C > 0$  such that

$$\lambda(\mathcal{H}_\delta) \geq \lambda(V^*) + C\delta^2 \tag{4.31}$$

for  $\delta \leq \bar{\delta}$ . Because of Lemma 4.3, (4.19) will follow. To prove (4.31), we use parametric derivatives. Let us fix notations:

1. For any  $\delta > 0$  small enough so that  $\mathcal{H}_\delta = \chi_{\mathbb{A}_\delta}$ , we set

$$h_\delta := \mathcal{H}_\delta - V^*.$$

2. For any  $t \in [0; 1]$  we define  $V_{\delta,t}$  as

$$V_{\delta,t} := V^* + th_\delta$$

and  $u_{\delta,t}$  as the eigenfunction associated with  $V_{\delta,t}$ :

$$\begin{cases} -\Delta u_{\delta,t} - V_{\delta,t} u_{\delta,t} = \lambda_t u_{\delta,t} \text{ in } \Omega, \\ u_{\delta,t} = 0 \text{ on } \partial\Omega. \\ \int_{\Omega} u_{\delta,t}^2 = 1. \end{cases} \tag{4.32}$$

3. For any such  $\delta$ ,  $\dot{u}_\delta$  is the first order parametric derivative of  $u$  at  $V_{\delta,t}$  in the direction  $h_\delta$  and  $\lambda_{\delta,t}$  is the first order parametric derivative of  $\lambda$  at  $V_{\delta,t}$  in the direction  $h_\delta$ .

As is proved in Annex 4.B, these objects are well-defined. Differentiating the equation with respect



to  $t$  gives

$$\begin{cases} -\Delta \dot{u}_{\delta,t} = \lambda(V^*)\dot{u}_{\delta,t} + V^*\dot{u}_{\delta,t} + \dot{\lambda}_{\delta,t}u_* + h_{\delta,t}u^*, & \text{in } \Omega, \\ \dot{u}_{\delta,t} = 0 \text{ on } \partial\Omega, \\ \int_{\mathbb{B}} u_* \dot{u}_{\delta,t} = 0, \end{cases} \quad (4.33)$$

and , multiplying the first equation by  $u_{\delta,t}$  and integrating by parts ,

$$\dot{\lambda}_{\delta,t} = - \int_{\mathbb{B}} h_{\delta} u_{\delta,t}^2.$$

We apply the mean value Theorem to  $f : t \mapsto \lambda_{\delta,t}$ . This gives the existence of  $t_1 \in [0; 1]$  such that

$$\lambda(\mathcal{H}_{\delta}) - \lambda(\mathbb{B}^*) = f(1) - f(0) = f'(t_1) = - \int_{\mathbb{B}} h_{\delta} u_{\delta,t_1}^2.$$

Our goal is now to prove that

$$- \int_{\mathbb{B}} h_{\delta} u_{\delta,t_1}^2 \geq C\delta^2 \quad (4.34)$$

for some constant  $C > 0$  whenever  $\delta$  is small enough. We will actually prove the existence of  $\underline{\delta} > 0$  such that, for any  $t \in [0; 1]$  and any  $\delta \leq \underline{\delta}$ , there holds

$$- \int_{\mathbb{B}} h_{\delta} u_{\delta,t}^2 \geq C\delta^2 \quad (4.35)$$

for some  $C > 0$ .

*Proof of Estimate (4.35).* We recall that  $r_{\delta}$  and  $r'_{\delta}$  were defined in (4.29)-(4.30). We can rewrite  $\mathcal{H}_{\delta} = \chi_{\mathbb{A}_{\delta}} \in X_{\delta}$  under the form

$$\int_{r^*-r_{\delta}}^{r^*} t^{n-1} dt + \int_{r^*}^{r^*+r'_{\delta}} t^{n-1} dt = \frac{\delta}{c_n}$$

where  $c_n = \mathcal{H}^{n-1}(\mathbb{S}(0; 1))$  and the condition  $\int_{\Omega} h_{\delta} = 0$  implies

$$\int_{r^*-r_{\delta}}^{r^*} t^{n-1} dt = \int_{r^*}^{r^*+r'_{\delta}} t^{n-1} dt.$$

An explicit computation yields the existence of a constant  $C > 0$  such that

$$r_{\delta}, r'_{\delta} \underset{\delta \rightarrow 0}{\sim} C\delta. \quad (4.36)$$

Let  $I_{\delta}^{\pm} := \{h_{\delta} = \pm 1\}$ . Since  $h_{\delta}$  is radial, for any  $t \in [0; 1]$  the function  $u_{\delta,t}$  is radial.

**First facts regarding  $u_{\delta,t}$**  Identifying  $u_{\delta,t}$  (resp.  $V_{\delta,t}$ ) with the unidimensional function  $\tilde{u}_{\delta,t}$  (resp.  $\tilde{V}_{\delta,t}$ ) such that

$$u_{\delta,t}(x) = \tilde{u}_{\delta,t}(|x|) \text{ (resp. } V_{\delta,t}(x) = \tilde{V}_{\delta,t}(|x|))$$

we have the following equation on  $u_{\delta,t}$ :

$$\begin{cases} -\frac{1}{r^{n-1}} \left( r^{n-1} u'_{\delta,t} \right)' = V_{\delta,t} u_{\delta,t} + \lambda_t u_{\delta,t} \text{ in } [0; R], \\ u'_{\delta,t}(0) = u_{\delta,t}(1) = 0, \\ \int_0^1 x u_{\delta,t}(x)^2 dx = \frac{1}{c_n}. \end{cases} \quad (4.37)$$

Since  $V_{\delta,t}$  is constant in  $(r^* - r_\delta; r^*) \cup (r^*; r^* + r'_\delta)$  and since  $u_{\delta,t}$  is uniformly bounded in  $L^\infty$  by standard elliptic estimates,  $u_{\delta,t}$  is  $\mathcal{C}^2$  in  $(r^* - r_\delta; r^*) \cup (r^*; r^* + r'_\delta)$ . Furthermore, Equation (4.37) readily gives the existence of a constant  $M$  such that, uniformly in  $\delta$  and in  $t \in [0, 1]$ ,

$$\|u_{\delta,t}\|_{W^{2,\infty}} \leq M. \quad (4.38)$$

Finally, it is standard to see that Equation (4.32) gives

$$\|u_{\delta,t} - u_*\|_{\mathcal{C}^1} \xrightarrow{\delta \rightarrow 0} 0 \quad (4.39)$$

uniformly in  $t$ . As a consequence, since

$$u'_*(r^*) < 0$$

there exists  $\underline{\delta}_1 > 0$  such that, for any  $\delta \leq \underline{\delta}_1$  and any  $t \in [0, 1]$ ,

$$u'_{\delta,t}(r^*) \leq -C < 0. \quad (4.40)$$

**End of the Proof** For any  $x \in I_\delta^\pm$  and any  $t \in [0, 1]$ , a Taylor expansion gives

$$u_{\delta,t}^2(x) = u_{\delta,t}^2|_{\mathbb{S}^*} \mp 2u_{\delta,t}|\nabla u_{\delta,t}|_{\mathbb{S}^*} \text{dist}(x; \mathbb{S}^*) + o(\text{dist}(x; \mathbb{S}^*)),$$

and  $o(\text{dist}(x; \mathbb{S}^*))$  is uniform in  $\delta > 0$  small enough and  $t \in [0, 1]$  by Estimate (4.38). This Taylor expansion gives

$$\begin{aligned} \dot{\lambda}_{\delta,t} &= - \int_{\mathbb{B}} h u_{\delta,t}^2 \\ &= - u_{\delta,t}^2|_{\mathbb{S}^*} \int_{\mathbb{B}} h_\delta \left( = 0 \text{ because } \int_{\mathbb{B}} h = 0 \right) \\ &\quad + \int_{I_\delta^-} 2u_{\delta,t}|\nabla u_{\delta,t}|_{\mathbb{S}^*} \text{dist}(x; \mathbb{S}^*) + o\left(\int_{I_\delta^-} \text{dist}(x; \mathbb{S}^*)\right) \\ &\quad + \int_{I_\delta^+} 2u_{\delta,t}|\nabla u_{\delta,t}|_{\mathbb{S}^*} \text{dist}(x; \mathbb{S}^*) + o\left(\int_{I_\delta^+} \text{dist}(x; \mathbb{S}^*)\right) \end{aligned}$$

where the  $o\left(\int_{I_\delta^\pm} \text{dist}(x; \mathbb{S}^*)\right)$  are uniform in  $t \in [0, 1]$  and  $\delta$ . Furthermore,

$$u_{\delta,t}|\nabla u_{\delta,t}|_{\mathbb{S}^*} \geq C > 0$$

for some constant  $C > 0$  independent of  $\delta$  and  $t$  by Estimate (4.40).

Hence

$$\dot{\lambda}_{\delta,t} \underset{\delta \rightarrow 0}{\sim} 2u_{\delta,t}|\nabla u_{\delta,t}|_{\mathbb{S}^*} \left( \int_{I_\delta^+} \text{dist}(x; \mathbb{S}^*) + \int_{I_\delta^-} \text{dist}(x; \mathbb{S}^*) \right).$$

However,

$$\begin{aligned} \int_{I_\delta^+} \text{dist}(x; \mathbb{S}^*) &= \int_{r^*}^{r^*+r'_\delta} t^{n-1}(t - r^*) dt \\ &= (r^*)^n r'_\delta \left( \frac{1}{n+1} \binom{n+1}{n} - \frac{1}{n} \binom{n}{n-1} \right) \\ &\quad + (r^*)^{n-1} (r'_\delta)^2 \left( \frac{1}{n+1} \binom{n+1}{n-1} - \frac{1}{n} \binom{n}{n-2} \right) \end{aligned}$$

$$\begin{aligned}
 &+ o_{\delta \rightarrow 0}(\delta^2) \quad \text{by (4.36)} \\
 &= (r^*)^{n-1} (r'_\delta)^2 + o_{\delta \rightarrow 0}(\delta^2) \\
 &\underset{\delta \rightarrow 0}{\sim} C\delta^2
 \end{aligned}$$

for some  $C > 0$  by (4.36). In the same manner,

$$\int_{I_\delta^-} \text{dist}(x; \mathbb{S}^*) = \int_{r^*-r_\delta}^{r^*} t^{n-1} (r^* - t) dt \underset{\delta \rightarrow 0}{\sim} C'\delta^2$$

and so, combining these estimates gives

$$\dot{\lambda}_{\delta,t} \underset{\delta \rightarrow 0}{\geq} C\delta^2 + o(\delta^2)$$

uniformly in  $\delta$  and  $t$ , which concludes the proof.  $\square$

$\square$

$\square$

## 4.2.5 STEP 4: SHAPE DERIVATIVES AND QUANTITATIVE INEQUALITY FOR GRAPHS

### 4.2.5.1 PRELIMINARIES AND NOTATIONS

In this Subsection, we aim at proving Theorem 4.1.1 for  $V = \chi_E$  and where  $E$  can be obtained as a normal graph over  $\mathbb{B}^*$ .

**Introduction of the Lagrangian and optimality conditions** We introduce the Lagrange multiplier  $\tau$  associated with the volume constraint and define the Lagrangian

$$L_\tau : E \mapsto \lambda(E) - \tau \text{Vol}(E).$$

From classical results in the calculus of variations, we have the following optimality conditions.

**CLAIM 4.2.3** *The necessary optimality conditions for a shape  $E$  to be a local minimizer (however they are not sufficient) are:*

$$\forall \Phi \in \mathcal{X}_1(E), \lambda'(E)[\Phi] = 0, \quad \forall \Phi \in \mathcal{X}_1(E), L''_\tau(E)[\Phi, \Phi] > 0.$$

Here, we use the notion of shape derivatives introduced in Definition 4.1.2. Since we only want a local quantitative inequality for shapes that can be obtained as normal graphs over the ball  $\mathbb{B}^*$ , we introduce some notations.

**Notations** We consider in this parts functions  $g$  belonging to

$$\mathcal{X}_0(\mathbb{B}^*) = \left\{ g \in W^{1,\infty}(\mathbb{B}^*), \|g\|_{L^\infty(\partial\mathbb{B}^*)} \leq 1, \int_{\partial\mathbb{B}^*} g = 0 \right\}.$$

Whenever  $g \in \mathcal{X}_0(\mathbb{B}^*)$ , there exists  $\Phi_g \in \mathcal{X}_1(\mathbb{B}^*)$  such that

$$\langle \Phi_g, \nu \rangle = g \text{ on } \partial\mathbb{B}^*.$$

The set  $\mathcal{X}_0$  corresponds to a linearization of the volume constraints for normal graphs and can be seen as a subset of the set of admissible perturbations  $\mathcal{X}_1(\mathbb{B}^*)$  defined in Definition 4.1.2, in the sense that we restrict admissible perturbations to normal graphs.

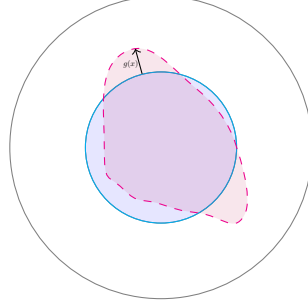


Figure 4.2 – A normal deformation of the ball. The dotted line can be understood as the graph of  $g$ .

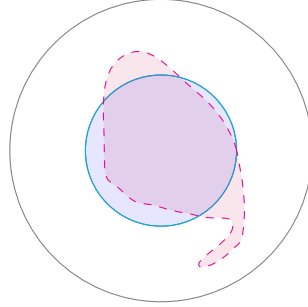


Figure 4.3 – A perturbation of the ball which cannot be seen as the graph of a function

We define, for any  $g \in \mathcal{X}_0(\mathbb{B}^*)$  and any  $t \in [0; T]$  (with  $T$  uniform in  $g$  because of the  $L^\infty$  constraint) the set  $\mathbb{B}_{t,g}$  whose boundary is defined as

$$\partial\mathbb{B}_{t,g} := \{x + tg(x)\nu(x), x \in \partial\mathbb{B}^*\}$$

i.e a slight deformation of  $E^*$ .

We define

$$\lambda_{g,t} := \lambda(\mathbb{B}_{t,g}).$$

Recall that  $\tau$  is the Lagrange multiplier associated with the volume constraint. By defining  $L'_\tau(\mathbb{B}^*)[g] := L'_\tau(\mathbb{B}^*)[\Phi_g]$  and  $L''_\tau(\mathbb{B}^*)[g, g] := L''_\tau(\mathbb{B}^*)[\Phi_g, \Phi_g]$ , and with the same convention for other shape functionals involved, necessary optimality conditions are

$$\forall g \in \mathcal{X}_0(\mathbb{B}^*), L'_\tau(\mathbb{B}^*)[g] = 0, L''_\tau(\mathbb{B}^*)[g, g] > 0. \quad (4.41)$$

We first prove that these optimality conditions hold in the case of a ball and then use them to obtain Theorem 4.1.1 for normal perturbations.

4.2.5.2 STRATEGY OF PROOF AND COMMENT ON THE COERCIVITY NORM

The strategy of proof is the same as the one used in many articles devoted to quantitative spectral inequalities. For example, we refer to [1, 31] for applications of these methods and to the recent [64], which presents a general framework for the study of stability and local quantitative inequalities using second order shape variations.

Although our method of proof is similar, we point out that the main thing to be careful with here is the coercivity norm for the second-order shape derivative. Indeed, let  $J : E \mapsto J(E)$  be a differentiable shape function. In the context of shape spectral optimization, the "typical" coercivity norm at a local minimum  $E^*$  for  $J$  is the  $H^{\frac{1}{2}}$  norm: in [64] a summary of shape functionals known to satisfy

$$J''(E^*)[\Phi, \Phi] \geq C \|\langle \Phi, \nu \rangle\|_{H^{\frac{1}{2}}(\partial E^*)}^2$$

is established and proofs of these coercivity properties are given. In this estimate,  $\Phi$  is an admissible vector field at  $E^*$ .

Here, in the context of parametric shape derivatives, i.e when the shape is a subdomain, it appears (see Subsubsection 4.2.5.5) that the natural coercivity norm is the  $L^2$  norm:

$$L''_{\tau}(E^*)[\Phi, \Phi] \geq C \|\langle \Phi, \nu \rangle\|_{L^2(\partial A^*)}^2$$

and this coercivity norm is optimal. This makes things a bit more complicated when dealing with the terms of the second order derivative that involve the mean curvature. This lack of coercivity might be accounted for by the fact that, while in shape optimization, it is the normal derivative of the shape derivative of the eigenfunction that is involved (see [64]) here, it is just the trace of the shape derivative of the eigenfunction on the boundary of the optimal shape that matters.

Once this  $L^2$  coercivity is established, we will prove that there exists a constant  $\xi, M, C > 0$  such that, for any  $g$  satisfying  $\|g\|_{W^{1,\infty}} \leq \xi$  and such that the mean curvature of  $\mathbb{B}_{g,t}$  is bounded by  $M$  for any  $t \leq T$ , there holds

$$|L''_{\tau}(\mathbb{B}_{t,g})[\Phi_g, \Phi_g] - L''_{\tau}(\mathbb{B}^*)[\Phi_g, \Phi_g]| \leq (C + M)\|g\|_{W^{1,\infty}}\|g\|_{L^2}^2. \quad (4.42)$$

We then apply the Taylor-Lagrange formula to  $f : t \mapsto L_{\tau}(\mathbb{B}_{t,g})$  to get the desired conclusion, see Subsubsection 4.2.5.7.

4.2.5.3 ANALYSIS OF THE FIRST ORDER SHAPE DERIVATIVE AT THE BALL AND COMPUTATION OF THE LAGRANGE MULTIPLIER

The aim of this section is to prove the following Lemma:

**LEMMA 4.5**  $\mathbb{B}^*$  is a critical shape and the Lagrange multiplier associated with the volume constraint is

$$\tau = -u_*^2|_{\partial \mathbb{B}^*}.$$

*Proof of Lemma 4.5.* We recall (see [93]) that

$$Vol'(\mathbb{B}^*)[g] = \int_{\partial \mathbb{B}^*} g (= 0 \text{ if } g \in \mathcal{X}_0(\mathbb{B}^*)).$$

We now compute the first order shape derivative of  $\lambda$ . The shape differentiability of  $\lambda$  follows from an application of the implicit function Theorem of Mignot, Murat and Puel, [139], and is proved in Appendix 4.B.

Let, for any  $g \in \mathcal{X}_0(\mathbb{B}^*)$ ,  $u'_g$  be the shape derivative of the  $u_{\mathbb{B}_{t,g}}$  at  $t = 0$ . We recall that this derivative is defined as follows (see [93, Chapitre 5] for more details): we first define  $v_t := u_{g,t} \circ \Phi_{g,t}$ , we define

$\dot{u}_g$  as the derivative in  $W_0^{1,2}(\Omega)$  of  $v_t$  with respect to  $t$  at  $t = 0$ , and set

$$u'_g := \dot{u}_g - \langle \Phi_g, \nabla u_0 \rangle.$$

We proceed formally to get the equation on  $u'_g$  (for rigorous computations we refer to Appendix 4.B): we first differentiate the main equation

$$-\Delta u_{g,t} = \lambda_{g,t} u_{g,t} + V_{g,t} u_{g,t}$$

with respect to  $t$ , yielding

$$-\Delta u'_g = \lambda'_g u_* + \lambda_* u'_g + (V^*) u'_g.$$

We then differentiate the continuity equations to get the jump conditions: if we define

$$[f](x) := \lim_{y \rightarrow x, y \in (\mathbb{B}^*)^c} f(y) - \lim_{y \rightarrow x, y \in \mathbb{B}^*} f(y),$$

we have

$$[u_{g,t}]|_{\partial \mathbb{B}_{t,g}} = \left[ \frac{\partial u_{g,t}}{\partial \nu} \right] \Big|_{\partial \mathbb{B}_{t,g}} = 0,$$

yielding

$$[u'_g]|_{\partial \mathbb{B}^*} = -g \left[ \frac{\partial u_*}{\partial \nu} \right] \Big|_{\partial \mathbb{B}^*} = 0$$

because  $u_*$  is  $\mathcal{C}^1$ , and

$$\left[ \frac{\partial u'_g}{\partial \nu} \right] \Big|_{\partial \mathbb{B}^*} = -g \left[ \frac{\partial^2 u_*}{\partial \nu^2} \right] \Big|_{\partial \mathbb{B}^*}.$$

However, from the Equation on  $u^*$  we see that

$$\left[ \frac{\partial^2 u_*}{\partial \nu^2} \right] \Big|_{\partial \mathbb{B}^*} = u_*|_{\partial \mathbb{B}^*},$$

so that we finally have the following equation on  $u'_g$ :

$$\begin{cases} -\Delta u'_g = \lambda'_g u_* + \lambda_* u_* + (V^*) u_* & \text{in } \mathbb{B}(0; R), \\ \left[ \frac{\partial u'_g}{\partial \nu} \right] = -g u_*|_{\partial \mathbb{B}^*}. \end{cases} \quad (4.43)$$

The weak formulation of this equation reads: for any  $\varphi \in W_0^{1,2}(\mathbb{B})$ ,

$$\int_{\mathbb{B}} \langle \nabla u'_g, \nabla \varphi \rangle - \int_{\partial \mathbb{B}^*} g u_* \varphi = \lambda'_g \int_{\mathbb{B}} \varphi u_* + \lambda_* \int_{\mathbb{B}} \varphi u_* + \int_{\mathbb{B}} (V^*) \varphi u_*. \quad (4.44)$$

We finally remark that, by differentiating  $\int_{\mathbb{B}} u_{g,t}^2 = 1$ , we get

$$\int_{\mathbb{B}} u_* u'_g = 0.$$

Taking  $u_*$  as a test function in (4.44) and using the normalization condition  $\int_{\mathbb{B}} u_*^2 = 1$  thus gives

$$\lambda'_g = \int_{\mathbb{B}} \langle \nabla u_*, \nabla u'_g \rangle - \lambda_* \int_{\mathbb{B}} u'_g u_* - \int_{\mathbb{B}} (V^*) u_* u'_g - \int_{\partial \mathbb{B}^*} g u_*^2. \quad (4.45)$$

However, since  $u'_g$  does not have a jump at  $\partial\mathbb{B}^*$ , we have

$$\int_{\mathbb{B}} \langle \nabla u_*, \nabla u'_g \rangle - \lambda_* \int_{\mathbb{B}} u'_g u_* - \int_{\mathbb{B}} (V^*) u_* u'_g = 0$$

by using the Equation on  $u_*$ .

In the end, we get

$$\lambda'_g = - \int_{\partial\mathbb{B}^*} g u_*^2.$$

Since  $u_*^2 = \mu_*$  is constant on  $\partial\mathbb{B}^*$  and  $g \in \mathcal{X}_0(\mathbb{B}^*)$ ,

$$\lambda'_g = -\mu_* \int_{\partial\mathbb{B}^*} g = 0.$$

This also enables us to compute the Lagrange multiplier: for a function  $g \in W^{1,\infty}(\partial\mathbb{B}^*)$  which is no longer assumed to satisfy  $\int_{\partial\mathbb{B}^*} g = 0$ , one must have

$$L'_\tau[\mathbb{B}^*](g) = 0.$$

Indeed, we know, from Lemma 4.1, that  $\mathbb{B}^*$  is the unique minimizer of  $\lambda$  under the volume constraint. However, the same computations show that

$$L'_\tau[\mathbb{B}^*](g) = -\mu^* \int_{\partial\mathbb{B}^*} g - \tau \int_{\partial\mathbb{B}^*} g,$$

and the Lagrange multiplier is thus

$$\tau = -\mu^* = -u_*^2|_{\mathbb{B}^*}. \quad (4.46)$$

□

We now compute the second order shape derivative of  $L_\tau$  at any given shape.

#### 4.2.5.4 COMPUTATION OF THE SECOND ORDER SHAPE DERIVATIVE OF $\lambda$

We explained in Subsection 4.2.5.2 that we need to compute the second order derivative at any given shape in order to apply the Taylor-Lagrange formula. Thus, the objectif of this section is the proof of the following Lemma:

**LEMMA 4.6** *The second order derivative of the eigenvalue  $\lambda$  at a shape  $E$  in the direction  $\Phi \in \mathcal{X}_1(E)$  is given by*

$$\lambda''(E)[\Phi, \Phi] = -2 \int_{\partial E} u u' \langle \Phi, \nu \rangle + 2 \int_{\partial E} \frac{\partial u}{\partial \nu} \left( [\nabla^2 u[\Phi, \Phi]] - \left[ \frac{\partial^2 u}{\partial \nu^2} \right] \langle \Phi, \nu \rangle^2 \right) + \int_{\partial E} \left( -H u^2 - 2u \frac{\partial u}{\partial \nu} \right) \langle \Phi, \nu \rangle^2,$$

where  $u'$  is defined by Equation (4.43) and  $H$  is the mean curvature of  $E$ .

*Proof of Lemma 4.6.* To compute  $\lambda''(E)[\Phi, \Phi]$ , we use Hadamard's second variation formula (see [93, Chapitre 5, page 227]): let  $K$  be a  $\mathcal{C}^2$  domain,  $f(t)$  be a shape differentiable function, then

$$\frac{d^2}{dt^2} \Big|_{t=0} \int_{K_{\Phi,t}} f(t) = \int_K f''(0) + 2 \int_{\partial K} f'(0) g + \int_{\partial K} \left( H f(0) + \frac{\partial f(0)}{\partial \nu} \right) g^2. \quad (4.47)$$

Let  $u'$  be the shape derivative of  $u_E$  with respect to  $t$  and  $u''$  the second order shape derivative of  $u_E$  with respect to  $t$ . We successively apply (4.47) to  $E_{\Phi,t}$  and  $f(t) = |\nabla u_t|^2 - u_t^2$  and to  $(E_{\Phi,t})^c$  and

$$f(t) = |\nabla u_t|^2.$$

Since  $\lambda(E_{\Phi,t}) = \int_{E_{\Phi,t}} (|\nabla u_t|^2 - u_t^2) + \int_{E_{\Phi,t}^c} |\nabla u_t|^2$ , this gives

$$\lambda''(E)[\Phi, \Phi] = 2 \int_{\Omega} \langle \nabla u'', \nabla u \rangle - 2 \int_E u'' u \quad (4.48)$$

$$+ 2 \int_{\Omega} |\nabla u'|^2 - 2 \int_E (u')^2 \quad (4.49)$$

$$+ 4 \int_{\partial E} [\langle \nabla u, \nabla u' \rangle] \langle \Phi, \nu \rangle - 4 \int_{\partial E} u' u \langle \Phi, \nu \rangle \quad (4.50)$$

$$+ \int_{\partial E} \left( -H u^2 - 2 \frac{\partial u}{\partial \nu} \left[ \frac{\partial^2 u}{\partial \nu^2} \right] - 2u \frac{\partial u}{\partial \nu} \right) \langle \Phi, \nu \rangle^2. \quad (4.51)$$

Let us simplify this expression: First of all the weak formulation of the equation on  $u'$  gives

$$2 \int_{\Omega} |\nabla u'|^2 - 2 \int_E (u')^2 = \underbrace{2\lambda' \int_{\Omega} u' u}_{=0 \text{ since } \int_{\Omega} u' u = 0} + 2\lambda \int_{\Omega} (u')^2 - 2 \int_{\partial E} \left[ u' \frac{\partial u'}{\partial \nu} \right].$$

We also note that by differentiating  $\int_{\Omega} u_t^2 = 1$  twice with respect to  $t$  we get

$$\int_{\Omega} u'' u + \int_{\Omega} (u')^2 = 0. \quad (4.52)$$

We note one last simplification to handle Line (4.50) in the expression for  $\lambda''$ : we decompose

$$\nabla u = \frac{\partial u}{\partial \nu} \nu + \nabla^{\perp} u, \quad \langle \nabla^{\perp} u, \nu \rangle = 0.$$

We adopt the same decomposition for  $u'$  and notice that, since  $u'$  does not have a jump at  $\partial E$ ,

$$[\nabla^{\perp} u] = \vec{0}.$$

The notation  $\vec{0}$  stands for the zero vector in  $\mathbb{R}^n$ . The same holds true for  $u$ , and so, since  $\frac{\partial u}{\partial \nu}$  has no jump at  $\partial E$ , we get

$$[\langle \nabla u, \nabla u' \rangle] = \frac{\partial u}{\partial \nu} \left[ \frac{\partial u'}{\partial \nu} \right].$$

Finally, using the weak formulation of the equation on  $u$  (Equation (4.2)) we get

$$2 \int_{\Omega} \langle \nabla u'', \nabla u \rangle - 2 \int_E u'' u = 2\lambda \int_{\Omega} u'' u - \int_{\partial E} [u''] \frac{\partial u}{\partial \nu}.$$

We then only need to compute the jump  $[u'']$  at  $\partial E$ . However, invoking the  $W^{2,2}$  regularity of the material derivative  $\ddot{u}$ , we get for the shape derivative  $u''$ :

$$[u''] = -2 [\nabla u'[\Phi]] - [\nabla u[D\Phi(\Phi)]] - [\nabla^2 u[[\Phi, \Phi]]].$$

We now use the fact that

$$[\nabla u] = [\nabla^{\perp} u] = [\nabla^{\perp} u'] = 0 \text{ on } \partial E$$

to rewrite

$$[u''] = -2 \left[ \frac{\partial u'}{\partial \nu} \right] \langle \Phi, \nu \rangle - [\nabla^2 u[[\Phi, \Phi]]].$$



If we gather these expressions we get

$$\begin{aligned}
 \lambda''(E)[\Phi, \Phi] &= 2\lambda \int_{\Omega} u''u + 2\lambda \int_{\Omega} (u')^2 && (= 0 \text{ because}) \\
 &+ 4 \int_{\partial E} \frac{\partial u}{\partial \nu} \left[ \frac{\partial u'}{\partial \nu} \right] \langle \Phi, \nu \rangle + 2 \int_{\partial E} \frac{\partial u}{\partial \nu} [\nabla^2 u][\Phi, \Phi] \\
 &+ 2 \int_{\partial E} u'u \\
 &- 4 \int_{\partial E} \frac{\partial u}{\partial \nu} \left[ \frac{\partial u'}{\partial \nu} \right] \langle \Phi, \nu \rangle - 4 \int_{\partial E} u'u \langle \Phi, \nu \rangle \\
 &+ \int_{\partial E} \left( -Hu^2 - 2 \frac{\partial u}{\partial \nu} \left[ \frac{\partial^2 u}{\partial \nu^2} \right] - 2u \frac{\partial u}{\partial \nu} \right) \langle \Phi, \nu \rangle^2 \\
 &= -2 \int_{\partial E} uu' \langle \Phi, \nu \rangle + 2 \int_{\partial E} \frac{\partial u}{\partial \nu} \left( [\nabla^2 u][\Phi, \Phi] - \left[ \frac{\partial^2 u}{\partial \nu^2} \right] \langle \Phi, \nu \rangle^2 \right) + \int_{\partial E} \left( -Hu^2 - 2u \frac{\partial u}{\partial \nu} \right) \langle \Phi, \nu \rangle^2
 \end{aligned}$$

□

#### 4.2.5.5 ANALYSIS OF THE SECOND ORDER SHAPE DERIVATIVE AT THE BALL

The aim of this paragraph is to prove the following Lemma:

**PROPOSITION 4.1** *There exists a constant  $C > 0$  such that*

$$\forall g \in \mathcal{X}_0(\mathbb{B}^*), L''_{\tau}[B^*](g, g) \geq C \|g\|_{L^2(\partial \mathbb{B}^*)}^2.$$

We note that the proof of this Lemma relies on a monotonicity principle, which guarantees the weak  $L^2$  coercivity. In fact, this is to be the optimal coercivity, in sharp contrast with shape optimization with respect to the boundary of the whole domain  $\partial \Omega$ , where the optimal coercivity usually occurs in the  $H^{\frac{1}{2}}$  norm, as noted in Subsection 4.2.5.2.

*Proof of Proposition 4.1.* We proceed in several steps. We identify  $g$  with the normal vector field  $\Phi_g$  that can be constructed from  $g$ , and, to alleviate notations, write  $\Phi = \Phi_g$ .

1. Computation of  $\lambda''$  We use Lemma 4.6 and first note that, since the vector field  $\Phi$  associated with  $g$  is normal,

$$[\nabla^2 u][\Phi, \Phi] = \left[ \frac{\partial^2 u}{\partial \nu^2} \right] \langle \Phi, \nu \rangle^2.$$

In the case of a ball,  $H = \frac{1}{r^*}$ . For notational simplicity, we stick to the notation

$$H^* = \frac{1}{r^*}.$$

The second derivative of  $\lambda$  becomes

$$\lambda''(\mathbb{B}^*)[\Phi, \Phi] = \int_{\partial \mathbb{B}^*} \left( -H^* u_*^2 - 2u_* \frac{\partial u_*}{\partial \nu} \right) \langle \Phi, \nu \rangle^2 - 2 \int_{\partial \mathbb{B}^*} u_* u' \langle \Phi, \nu \rangle.$$

Taking into account the value of the Lagrange multiplier  $\tau$  associated with the volume constraint, see Equation (4.46), and

$$Vol''(\mathbb{B}^*)[\Phi, \Phi] = \int_{\partial \mathbb{B}^*} H^* \langle \Phi, \nu \rangle^2$$

we get

$$L''_{\tau}(\mathbb{B}^*)[\Phi, \Phi] = 2 \int_{\partial\mathbb{B}^*} -u_* \frac{\partial u_*}{\partial \nu} \langle \Phi, \nu \rangle^2 - u_* u' \langle \Phi, \nu \rangle. \quad (4.53)$$

2. Separation of variables and first simplifications We identify  $g$  with a function  $g : [0; 2\pi] \rightarrow \mathbb{R}$ . We write the decomposition of  $g$  as a Fourier series:

$$g = \sum_{k \in \mathbb{N}} \alpha_k \cos(k \cdot) + \beta_k \sin(k \cdot).$$

Since  $g \in \mathcal{X}_0(\mathbb{B}^*)$  we have  $\alpha_0 = 0$  and thus

$$g = \sum_{k=1}^{\infty} \alpha_k \cos(k \cdot) + \beta_k \sin(k \cdot). \quad (4.54)$$

We define, for any  $k \in \mathbb{N}^*$ ,  $u'_k$  (resp.  $w'_k$ ) as the shape derivative of  $u$  with respect to the perturbation  $g = \cos(k \cdot)$  (resp.  $g = \sin(k \cdot)$ ). Since  $\mathbb{B}^*$  is a critical shape from Lemma 4.5,  $u'_k$  satisfies

$$\begin{cases} -\Delta u'_k = \lambda_* u'_k + V^* u'_k, \\ [u'_k] = -u_* \cos(k \cdot) \text{ on } \partial\mathbb{B}^*, \\ u'_k = 0 \text{ on } \partial\Omega. \end{cases}$$

Since  $u_*$  is constant on partial  $\mathbb{B}^*$ , we can write, in polar coordinates

$$u'_k(r, \theta) = \psi_k(r) \cos(k\theta)$$

where  $\psi_k$  satisfies the following equation (and we identify  $V^*$  with the one dimensional function  $\tilde{V}^*$  such that  $(V^*)(x) = \tilde{V}^*(|x|) = \chi_{|x| \leq r^*}$ ):

$$\begin{cases} -\frac{1}{r}(r\psi'_k)' = \left(\lambda_* + V^* - \frac{k^2}{r^2}\right) \psi_k \\ [\psi'_k](r^*) = -u_*(r^*) \\ \psi_k(R) = 0. \end{cases} \quad (4.55)$$

In the same way, we have

$$w'_k(r, \theta) = \psi_k(r) \sin(k\theta).$$

Whenever  $g$  admits the Fourier decomposition (4.54), the linearity (with respect to  $g$ ) of the equation on  $u'_g$  gives

$$u'_g = \sum_{k=1}^{\infty} \alpha_k u'_k + \beta_k w'_k.$$

Plugging this in the expression of  $L''_{\tau}$ , see Equation (4.53), and using the orthogonality properties of  $\{\cos(k \cdot), \sin(k \cdot)\}_{k \geq 1}$  finally yields

$$L''_{\tau}(\mathbb{B}^*)[g, g] = L''_{\tau}(\mathbb{B}^*)[\Phi, \Phi] = \sum_{k=1}^{\infty} \{\alpha_k^2 + \beta_k^2\} u_*|_{\partial\mathbb{B}^*} \left( -\frac{\partial u_*}{\partial \nu} \Big|_{\partial\mathbb{B}^*} - \psi_k(r^*) \right).$$

We define the relevant sequence  $\{\omega_k\}_{k \in \mathbb{N}^*}$ :

$$\forall k \in \mathbb{N}^*, \omega_k := -\frac{\partial u_*}{\partial \nu} \Big|_{\partial\mathbb{B}^*} - \psi_k(r^*), \quad (4.56)$$

so that

$$L''_\tau(\mathbb{B}^*)[g, g] = u_*|_{\partial\mathbb{B}^*} \sum_{k=1}^{\infty} \omega_k \{ \alpha_k^2 + \beta_k^2 \}. \quad (4.57)$$

Our goal is now the following Lemma:

**LEMMA 4.7** *There exists  $C > 0$  such that*

$$\forall k \in \mathbb{N}^*, \omega_k \geq C > 0.$$

In order to prove this Lemma, we use a comparison principle for one-dimensional differential equations.

3. Proof of Lemma 4.7: monotonicity principle We will first prove that

$$\forall k \in \mathbb{N}^*, \omega_k \geq \omega_1. \quad (4.58)$$

We first note that

$$\psi_1 > 0 \text{ in } (0; R). \quad (4.59)$$

Before we prove, let us see how (4.59) implies (4.58). Let, for any  $k \geq 2$ ,  $z_k$  be the function defined as

$$z_k := \psi_k - \psi_1.$$

Since  $\psi_k$  can be expressed as  $\psi_k = A_k J_k(\frac{r}{R})$  for  $r \leq r^*$  where  $J_k$  is the  $k$ -th Bessel function of first kind, we have  $z_k(0) = z_k(R) = 0$ . Furthermore,  $z_k$  satisfies

$$\begin{aligned} -\frac{1}{r}(rz'_k)' &= \left( \lambda_* + V^* - \frac{k^2}{r^2} \right) \psi_k - \left( \lambda_* + V^* - \frac{1}{r^2} \right) \psi_1 \\ &\leq \left( \lambda_* + V^* - \frac{k^2}{r^2} \right) \psi_k - \left( \lambda_* + V^* - \frac{k^2}{r^2} \right) \psi_1 && \text{because } \psi_1 > 0 \text{ by (4.59)} \\ &\leq \left( \lambda_* + V^* - \frac{k^2}{r^2} \right) z_k. \end{aligned} \quad (4.60)$$

We also have

$$[z'_k](r^*) = 0.$$

However, this equation and this no-jump condition imply

$$z_k \leq 0. \quad (4.61)$$

For  $k$  large enough, this simply follows by a contradiction argument: if  $\lambda_* + V^* - \frac{k^2}{r^2} < 0$  in  $(0; R)$  then, if  $z_k$  reached a positive maximum at some interior point  $\bar{r}$ , we should have

$$0 \leq -\frac{1}{\bar{r}}(\bar{r}z''_k(\bar{r})) \leq \left( \lambda_* + V^* - \frac{k^2}{\bar{r}^2} \right) z_k(\bar{r}) < 0,$$

yielding a contradiction.

A proof of (4.61) that is valid for all values of  $k$  reads as follows: identifying  $u_*$  with its one-dimensional counterpart (i.e with the function  $\tilde{u}_* : [0; R] \rightarrow \mathbb{R}$  such that  $u_*(x) = \tilde{u}_*(|x|)$ ) we define

$$p_k := \frac{z_k}{u_*}.$$

We notice that  $p_k(0) = 0$  and that

$$[p'_k](r^*) = 0.$$

Furthermore, by straightforward computation,  $p_k$  satisfies

$$-\frac{1}{r}(rp'_k)' = \frac{-1}{u_*}(rz'_k)' + \frac{z_k}{u_*^2} \frac{1}{r}(ru'_*)' + 2\frac{u'_*}{u_*} p'_k.$$

By (4.60) and by non-negativity of  $u_*$  we get

$$-\frac{1}{r}(rp'_k)' \leq -\frac{k^2}{r^2} p_k.$$

From this it is straightforward to see by a contradiction argument that  $p_k$  can not reach a positive maximum at an interior point. It remains to exclude the case  $p_k(R) > 0$ .

We argue, once again, by contradiction, and assume that  $p_k(R) > 0$ . Since by l'Hospital rules we have

$$p_k(r) \underset{r \rightarrow R}{\sim} \frac{z'_k(r)}{u'_*(r)}$$

we must have  $z'_k(r) \leq 0$ . However once again by l'Hospital's rule,

$$p'_k(r) = \frac{u_* z'_k}{u_*^2} - \frac{u'_* z_k}{u_*^2} \underset{r \rightarrow 0}{\sim} \frac{1}{2} z'_k 2u_* < 0.$$

Hence  $p_k$  is locally decreasing at  $R$ , yielding a contradiction. Thus  $p_k \leq 0$  and in turn  $\psi_k - \psi_1 = z_k \leq 0$ , completing the proof of (4.58).

The proof of (4.59) follows from the same arguments: we define

$$\Psi_1 := \frac{\psi_1}{u_*}$$

and observe that

$$-\frac{1}{r}(r\Psi'_1)' = -\frac{1}{r^2}\Psi_1 + 2\frac{u'_*}{u_*}\Psi'_1, [\Psi'_1](r^*) = -u_*(r^*).$$

We once again argue by contradiction and assume that  $\Psi_1$  reaches a negative minimum. From the jump condition at  $r^*$ , if this maximum is reached at an interior point, it cannot be at  $r = r^*$  and the contradiction follows from the Equation. We exclude the case of a negative minimum at  $R$  through the same reasons as for  $p_k$ .

It follows that  $\psi_k \leq \psi_1$  so that

$$\omega_k - \omega_1 = \psi_1(r^*) - \psi_k(r^*) \geq 0.$$

To conclude the proof of Lemma 4.7, it remains to prove that

$$\omega_1 > 0. \tag{4.62}$$

*Proof of (4.62).* We define

$$\Psi := u'_* + \psi_1.$$

We note that

$$-\frac{1}{r}(r(u'_*)')' = \left( \lambda_* + V^* - \frac{1}{r^2} \right) u'_*, \quad [(u'_*)'](r^*) = [u''](r^*) = u_*(r^*).$$

By Hopf's Lemma,  $u'_*(R) < 0$  and, since  $u_*$  is  $\mathcal{C}^2$  in  $\mathbb{B}^*$ ,  $u'_*(0) = 0$ . We get the following equation on  $\Psi$ :

$$-\frac{1}{r}(r\Psi')' = \left(\lambda_* + V^* - \frac{1}{r^2}\right)\Psi, \quad [\Psi'](r^*) = 0, \quad \Psi(0) = 0, \quad \Psi(R) < 0.$$

Defining

$$\Theta := \frac{\Psi}{u_*}$$

we get

$$-\frac{1}{r}(r\Theta')' = -\frac{1}{r^2}\Theta + 2\frac{u'_*}{u_*}\Theta'$$

and  $\Theta$  can thus not reach a positive maximum at an interior point. Since it is negative at  $r = R$  we get  $\Theta \leq 0$  in  $[0; R]$ . Furthermore, it is not identically zero since  $\Theta(R) \neq 0$ , and the strong maximum principle implies  $\Theta < 0$  in  $(0; R)$ . This gives

$$\Theta(r^*) < 0$$

or, equivalently

$$\omega_1 = -\Theta(r^*) > 0$$

and this concludes the proof of Lemma 4.7.  $\square$

4. Conclusion of the proof To prove Proposition 4.1, we simply write

$$\begin{aligned} L''_{\tau}(\mathbb{B}^*)[g, g] &= \sum_{k=1}^{\infty} \omega_k \{ \alpha_k^2 + \beta_k^2 \} && \text{by (4.57)} \\ &\geq C \sum_{k=1}^{\infty} \{ \alpha_k^2 + \beta_k^2 \} && \text{by Lemma 4.7} \\ &= C \|g\|_{L^2}^2 \\ &= C \|\langle \Phi, \nu \rangle\|_{L^2}^2. \end{aligned}$$

The proof of the Proposition is now complete.  $\square$

#### 4.2.5.6 TAYLOR-LAGRANGE FORMULA AND CONTROL OF THE REMAINDER

We now state the main estimate which will enable us to apply the Taylor-Lagrange formula.

**PROPOSITION 4.2** *Let  $M > 0$  and  $\eta > 0$ . There exists  $s \in (0; 1)$ ,  $\varepsilon = \varepsilon(\eta) > 0$  such that for any  $\Phi \in \mathcal{X}_1(\mathbb{B}^*)$  satisfying*

$$\|\Phi\|_{\mathcal{C}^{1,1}} \leq M, \quad \|\Phi\|_{\mathcal{C}^{1,s}} \leq \varepsilon$$

there holds

$$|L''_{\tau}(\mathbb{B}_{\Phi})[\Phi, \Phi] - L''_{\tau}(\mathbb{B}^*)[\Phi, \Phi]| \leq \eta \|\langle \Phi, \nu \rangle\|_{L^2}^2. \quad (4.63)$$

As mentioned earlier, this will prove a quantitative inequality for sets of bounded curvature that are in a  $\mathcal{C}^{1,s}$  neighbourhood of  $\mathbb{B}^*$ . Note that working in the  $\mathcal{C}^{1,s}$  norm rather than in a  $\mathcal{C}^{2,s}$  norm will be enough, since elliptic regularity estimates will prove sufficient for our proofs.

The proof of this proposition is technical but not unexpected in this context. We postpone the proof to Appendix 4.C.

4.2.5.7 CONCLUSION OF THE PROOF OF STEP 4

Recall that we have defined

$$f(t) := \lambda(\mathbb{B}_t\Phi).$$

Then:

$$\begin{aligned} \lambda(\mathbb{B}_\Phi) - \lambda(\mathbb{B}^*) &= f(1) - f(0) \\ &= f'(0) && (= 0 \text{ because } \mathbb{B}^* \text{ is critical}) \\ &+ \int_0^1 (1-t)f''(t)dt \\ &= \frac{1}{2}f''(0) + \int_0^1 (1-t)(f''(t) - f''(0))dt \\ &\geq C\|\langle \Phi, \nu \rangle\|_{L^2}^2 - \eta\|\langle \Phi, \nu \rangle\|_{L^2}^2 \\ &\geq \frac{C}{2}\|\langle \Phi, \nu \rangle\|_{L^2}^2 \\ &\geq C'\|\langle \Phi, \nu \rangle\|_{L^1}^2 && \text{by the Cauchy-Schwarz Inequality} \\ &= C'\delta^2 && \text{because } |\mathbb{B}_\Phi \Delta \mathbb{B}^*| = \delta. \end{aligned}$$

whenever  $\|\nabla\Phi\|_{L^\infty}$  is small enough. This concludes the proof of Step 4.

4.2.5.8 A REMARK ON THE COERCIVITY NORM

The  $L^2$  coercivity established in Proposition 4.1 is not only sufficient, but also optimal. Indeed, since  $\psi_k$  is non-negative in  $(0; R)$ , we immediately have the bound

$$0 \leq \omega_k \leq -\frac{\partial u_*}{\partial \nu}.$$

In other words, the coercivity norm for the second derivative is the  $L^2$  (rather than the  $H^{\frac{1}{2}}$ ) norm of the perturbation. This is due to the fact that here, in the context of parametric optimization, it is the value of the shape derivative  $u'$  rather than the value of its normal derivative that is involved in the second order shape derivative. We note that this is in sharp contrast with classical shape optimization, where the optimization is carried out with respect to the whole domain  $\Omega$ , and where the coercivity norm is the  $H^{\frac{1}{2}}$  norm, see [64] and the references therein.

4.2.6 STEP 5: CONCLUSION OF THE PROOF OF THEOREM 4.1.1

We now conclude the proof of Theorem 4.1.1: let  $\mathcal{V}_\delta$  be a solution of the variational problem (4.17) In the same way we derived the optimality conditions for the radial version of the optimization problem, that is, for the variational problem (4.16), it is easy to see that  $V_\delta$  is equal to 0 or 1 almost everywhere and that, furthermore, if  $u_\delta$  is the associated eigenfunction, that there exists two real numbers  $\mu_\delta$  and  $\eta_\delta$  such that

$$\{\mathcal{V}_\delta = 1\} = \left( \{u_\delta \geq \mu_\delta\} \cap \mathbb{B}^* \right) \cup \left( \{u_\delta \geq \eta_\delta\} \cap (\mathbb{B}^*)^c \right).$$

**REMARK 4.2** *We actually expect that*

$$\mathcal{V}_\delta = \mathcal{H}_\delta$$

where  $\mathcal{H}_\delta$  was defined in Step 3, at least for  $\delta > 0$  small enough, in which case Step 4 would prove irrelevant. We were however not able to prove this fact. Put otherwise, we expect the solution to (4.17) to be a radially symmetric set, given the symmetries properties involved.

We introduce one last parameter: let  $\zeta_\delta$  be the unique real number such that

$$|\{u_\delta > \zeta_\delta\}| = V_0. \quad (4.64)$$

In the two figures below, we represent the two most extreme cases we might face (note that we always represent sets that are symmetric with respect to the  $x$ -axis; this is allowed by Steiner's rearrangement but this property will not be used in what follows)

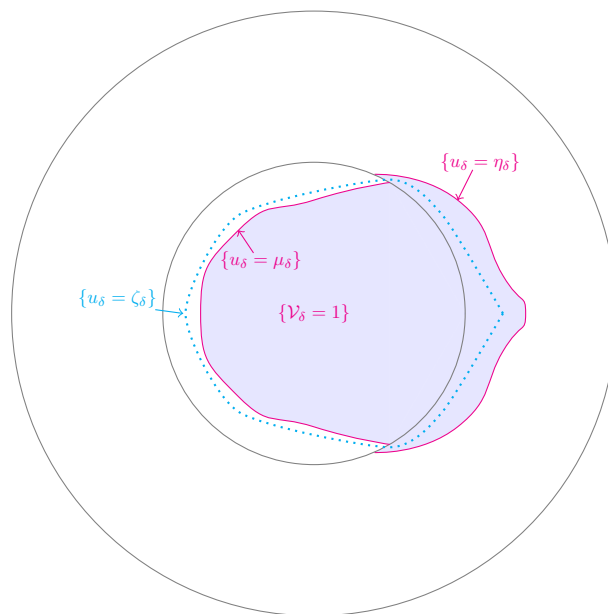


Figure 4.4 – Here, the set  $\mathcal{V}_\delta$  is connected, and we might compare it with a normal deformation of  $\mathbb{B}^*$ .

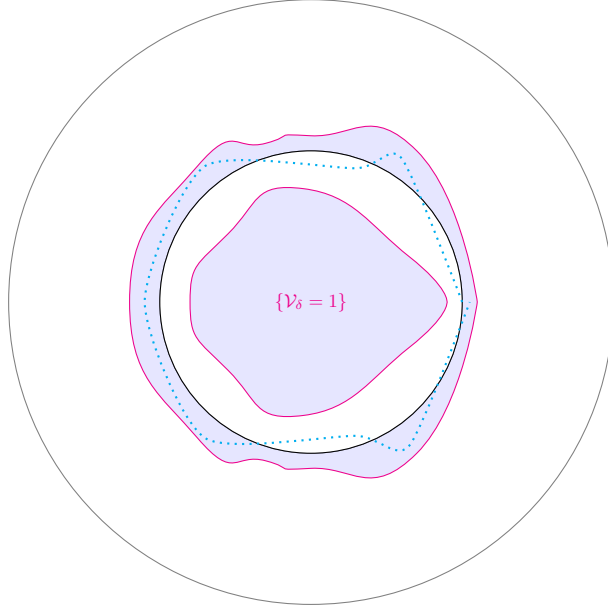


Figure 4.5 – Here, the set  $\mathcal{V}_\delta$  is disconnected, and we might compare it with a radial distribution.

To formalize this, we introduce the quantity

$$f(\delta) := |\{u_\delta \geq \zeta_\delta\} \Delta (\{u_\delta \geq \eta_\delta\} \cap (\mathbb{B}^*)^c)|.$$

Since

$$|\{u_\delta \geq \eta_\delta\} \cap (\mathbb{B}^*)^c| = \frac{\delta}{2}$$

because  $\mathcal{V}_\delta \in \mathcal{M}_\delta$ ,

$$f(\delta) \leq \frac{\delta}{2}.$$

We now distinguish two cases:

1. First case: comparison with a radial distribution The first case is defined by

$$\frac{f(\delta)}{\delta} \xrightarrow{\delta \rightarrow 0} \ell > 0.$$

In that case,  $f(\delta) \underset{\delta \rightarrow 0}{\sim} \ell\delta$ . We now apply the bathub principle: let  $E_\delta^1$  be the solution of

$$\inf_{E \subset \{u_\delta > \zeta_\delta\}, |E| = V_0 - f(\delta)} - \int_{\{u_\delta \geq \zeta_\delta\} \cap E} u_\delta^2.$$

If  $\zeta_{\delta,1}$  is defined through

$$|\{u_\delta > \zeta_{\delta,1}\}| = V_0 - f(\delta)$$

then  $\zeta_{\delta,1} > \zeta_\delta$  and consequently

$$E_\delta^1 = \{u_\delta > \zeta_{\delta,1}\}.$$



In the same way, we define  $E_\delta^2$  as the solution of

$$E \subset \left( \{u_\delta > \zeta_\delta\} \right)^c, |E|=f(\delta) - \int_{\{u_\delta \geq \zeta_\delta\}^c \cap E} u_\delta^2$$

and, if we define  $\zeta_{\delta,2}$  through the equation

$$\left| \{\zeta_\delta > u_\delta > \zeta_{\delta,2}\} \right| = f(\delta)$$

then

$$E_\delta^2 = \{\zeta_\delta > u_\delta > \zeta_{\delta,2}\}.$$

We replace  $V_\delta$  by

$$\mathcal{W}_\delta := \chi_{E_\delta^1} + \chi_{E_\delta^2}.$$

From the bathtub principle,

$$- \int_{\Omega} \mathcal{V}_\delta u_\delta^2 \geq - \int_{\Omega} \mathcal{W}_\delta u_\delta^2.$$

However,  $\mathcal{W}_\delta$  might not satisfy

$$\int_{\Omega} |\mathcal{W}_\delta - \chi_{\mathbb{B}^*}| = \delta.$$

We represent  $E_\delta^i$ ,  $i = 1, 2$ , below:

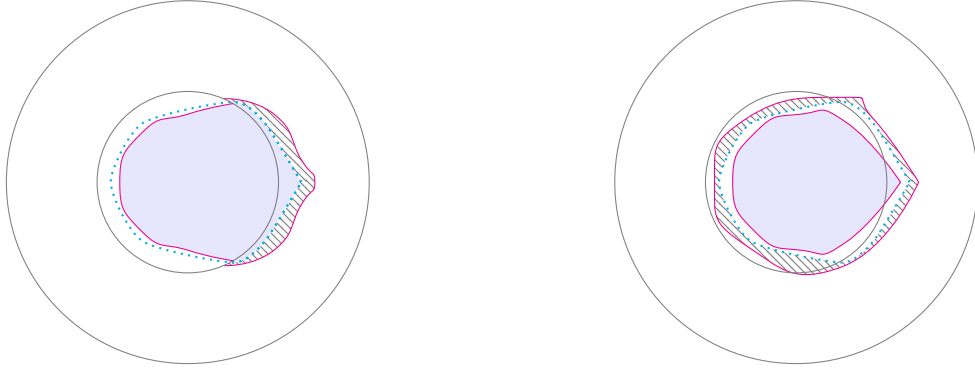
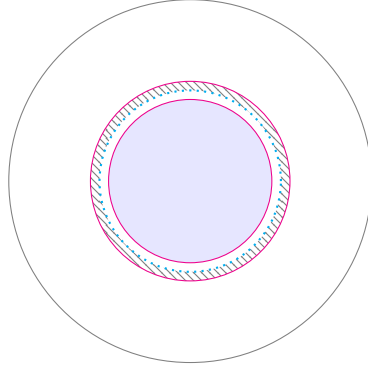


Figure 4.6 – An illustration of the process

Finally, following the notations of step 3, we recall that  $\mathbb{A}_{f(\delta)}$  is defined as

$$\mathbb{A}_{f(\delta)} := \{|x| \leq r^* - r_{f(\delta)}\} \cup \{r^* \leq |x| \leq r^* + r'_{f(\delta)}\}, \chi_{\mathbb{A}_{f(\delta)}} \in \mathcal{M}_{f(\delta)}.$$

Our competitor is  $\mathbb{A}_{f(\delta)}$ :



Let  $u_\delta^*$  be the Schwarz rearrangement of  $u_\delta$ . By equimeasurability of the Schwarz rearrangement, we have

$$\int_{\mathbb{A}_{f(\delta)}} (u_\delta^*)^2 = \int_{\Omega} \mathcal{W}_\delta u_\delta^2.$$

By the Polya-Szego Inequality (see [108]),

$$\int_{\Omega} |\nabla u_\delta^*|^2 \leq \int_{\Omega} |\nabla u_\delta|^2.$$

Finally, we have established the chain of inequalities

$$\begin{aligned} \lambda(\mathcal{V}_\delta) &\geq \int_{\Omega} |\nabla u_\delta|^2 - \int_{\Omega} \mathcal{W}_\delta u_\delta^2 \\ &\geq \int_{\Omega} |\nabla u_\delta^*|^2 - \int_{\Omega} \chi_{\mathbb{A}_{f(\delta)}} (u_\delta^*)^2 \\ &\geq \lambda(\mathbb{A}_{f(\delta)}) \end{aligned} \quad \text{by the Rayleigh quotient formulation (4.3).}$$

Now, by (4.19),

$$\lambda(\mathbb{A}_{f(\delta)}) \geq \lambda_* + Cf(\delta)^2$$

and thus, since

$$f(\delta) \underset{\delta \rightarrow 0}{\sim} \ell\delta$$

we have

$$\lambda(\mathcal{V}_\delta) - \lambda_* \geq C'\ell^2\delta^2$$

which concludes the proof.

2. Second case: comparison with a normal deformation The second case is defined by

$$\frac{f(\delta)}{\delta} \underset{\delta \rightarrow 0}{\rightarrow} 0. \quad (4.65)$$

In this case, we use Step 4 of the proof, i.e the quantitative inequality for normal deformations of the ball.

Let us replace  $\mathcal{V}_\delta$  with

$$\mathcal{W}_\delta := \chi_{\{u_\delta > \zeta_\delta\}}.$$

Recall that  $\zeta_\delta$  was defined in such a way that  $\mathcal{W}_\delta \in \mathcal{M}(\Omega)$ . By the bathtub principle,

$$\lambda(\mathcal{V}_\delta) \geq \lambda(\mathcal{W}_\delta).$$

Furthermore, Condition (4.65) implies

$$|\{u_\delta > \zeta_\delta\} \Delta \mathbb{B}^*| = \delta + o_{\delta \rightarrow 0}(\delta).$$

Indeed,

$$\frac{\delta}{2} = |\{u_\delta > \eta_\delta\} \cap (\mathbb{B}^*)^c| = |\{u_\delta > \zeta_\delta\} \cap (\mathbb{B}^*)^c| + f(\delta).$$

Finally, standard elliptic estimates imply (in dimension 2 and 3) that

$$u_\delta \xrightarrow[\delta \rightarrow 0]{\mathcal{C}^{1,s}(\overline{\Omega})} u_* \quad (4.66)$$

and, since  $\frac{\partial u_*}{\partial \nu} \Big|_{\partial \mathbb{B}^*} \neq 0$  and  $\zeta_\delta \xrightarrow[\delta \rightarrow 0]{} u_* \Big|_{\partial \mathbb{B}^*}$  it follows that  $\partial \{u_\delta > \zeta_\delta\}$  is a  $\mathcal{C}^1$  hypersurface by the implicit function Theorem.

It remains to prove that  $\partial \{u_\delta > \zeta_\delta\}$  is a graph above  $\partial \mathbb{B}^*$ . We start by noticing that (4.66) implies

$$d_H(\{u_\delta > \zeta_\delta\}, \mathbb{B}^*) \xrightarrow[\delta \rightarrow 0]{} 0, \quad (4.67)$$

where  $d_H$  is the Hausdorff distance. We then argue by contradiction and assume that, for every  $\delta > 0$  there exists  $x_\delta \in \partial \mathbb{B}^*$ ,  $t_1 \neq t_2 \in \mathbb{R}$  such that

$$x_\delta + t_k \nu(x_\delta) \in \partial \{u_\delta > \zeta_\delta\}, \quad k = 1, 2.$$

It follows that

$$u_\delta(x_\delta + t_1 \nu(x_\delta)) = u_\delta(x_\delta + t_2 \nu(x_\delta))$$

and by the intermediate value Theorem and (4.67), there exists  $t_\delta \in \mathbb{R}$  such that

$$\langle \nabla u_\delta(x_\delta + t_\delta \nu(x_\delta)), \nu(x_\delta) \rangle = 0, \quad t_\delta \xrightarrow[\delta \rightarrow 0]{} 0.$$

By passing to the limit in this equation up to a subsequence, there exists a point  $x_* \in \partial \mathbb{B}^*$  such that

$$\langle \nabla u_*(x_*), \nu(x_*) \rangle = 0.$$

This is a contradiction since

$$\frac{\partial u_*}{\partial \nu} \Big|_{\partial \mathbb{B}^*} \neq 0.$$

We can then say that  $\partial \{u_\delta > \zeta_\delta\}$  is the graph of a function  $\varphi_\delta$  over  $\partial \mathbb{B}^*$ . Besides, the Convergence result (4.66) implies that

$$\varphi_\delta \xrightarrow[\delta \rightarrow 0]{\mathcal{C}^1(\partial \mathbb{B}^*)} 0.$$

Finally, since the set  $\{u_\delta \geq \mu_\delta\}$  converges in the  $\mathcal{C}^{1,s}$  topology to  $\mathbb{B}^*$ , there exists a uniform radius  $r > 0$  such that, for any  $x \in \partial \{u_\delta \geq \mu_\delta\}$ , there exists  $y_x$  satisfying

$$x \in \mathbb{B}(y_x, r), \quad \mathbb{B}(y_x, r) \subset \{u_\delta \geq \mu_\delta\}.$$

Since  $V_\delta$  is constant in  $\mathbb{B}(y_x, r)$ , we can apply elliptic regularity results to get a uniform  $\mathcal{C}^2$  norm on  $u_\delta$  in  $\{u_\delta \geq \mu_\delta\}$ : there exists  $M > 0$  such that, for any  $x \in \{u_\delta \geq \mu_\delta\}$ ,  $\text{dist}(x, \partial \{u_\delta, \mu_\delta\}) \leq r$ ,  $|\nabla^2 u(x)| \leq M$ . Hence, the curvature of  $\{u_\delta \geq \mu_\delta\}$  is uniformly bounded by some constant  $M$ .

To prove that the curvature of  $\{u_\delta > \zeta_\delta\}$  is uniformly bounded as well, we note the following

fact: for any  $x \in \partial\{u_\delta > \zeta_\delta\}$ , let  $x_\delta$  be its orthogonal projection on  $\{u_\delta \geq \mu_\delta\}$ . Then,

$$\mu_\delta - \zeta_\delta \underset{\delta \rightarrow 0}{\sim} |x_\delta - x| \frac{\partial u_\delta}{\partial \nu}(x_\delta)$$

and so the map  $\partial\{u_\delta \geq \zeta_\delta\} \ni (x, y) \mapsto \frac{|x - x_\delta|}{|y - y_\delta|}$  converges uniformly to 1.  $\{u_\delta > \zeta_\delta\}$  can thus be described, asymptotically, as  $\{u_\delta \geq \mu_\delta\} + \mathbb{B}(0; t_\delta)$ , so that it also has a uniformly bounded curvature.

**REMARK 4.3** *We could have worked directly with  $\{u_\delta \geq \mu_\delta\} + \mathbb{B}(0; t_\delta)$ , by choosing a suitable  $t_\delta$  but in this context, it seemed more relevant to work with level sets.*

We can hence apply Step 4:

$$\begin{aligned} \lambda(\mathcal{V}_\delta) &> \lambda(\mathcal{W}_\delta) \\ &\geq \lambda_* + C |\{u_\delta > \zeta_\delta\} \Delta \mathbb{B}^*|^2 \text{ by Step 4} \\ &\geq \lambda_* + C(\delta + \underset{\delta \rightarrow 0}{o}(\delta))^2 \\ &\geq \lambda_* + C'\delta^2. \end{aligned}$$

The proof of Theorem 4.1.1 is now complete. □

## 4.3 CONCLUDING REMARKS AND CONJECTURE

### 4.3.1 EXTENSION TO OTHER DOMAINS

We do believe that this quantitative inequality is valid not only in the ball but for more general domains. Let, for any domain  $\Omega$ ,  $V_\Omega$  be a solution of (4.5). Let  $u_\Omega$  be the associated eigenfunction. By the bathtub principle, it is easy to see that there exists  $\mu_\Omega \in \mathbb{R}$  such that

$$V_\Omega = \chi_{\{u_\Omega \geq \mu_\Omega\}} = \chi_{E_\Omega}.$$

We give the following conjecture:

**CONJECTURE 4.1** *Assume that*

1. *The minimizer is regular in the sense that  $\frac{\partial u_\Omega}{\partial \nu} \leq -C < 0$  on  $\partial E_\Omega$ ,*
2.  *$E_\Omega$  is a non-degenerate shape minimizer: for any admissible variation  $\Phi \in \mathcal{X}_1(E_\Omega)$ , if  $L_\tau$  is the associated lagrangian, there holds*

$$L''_\tau(E_\Omega)[\Phi, \Phi] > 0.$$

*Then there exists a parameter  $\eta > 0$  such that, for any  $V \in \mathcal{M}(\Omega)$ ,*

$$\|V - V_\Omega\|_{L^1(\Omega)} \leq \eta \Rightarrow \lambda(V) - \lambda(V^*) \geq C\|V - V^*\|_{L^1(\Omega)}^2.$$

Here the main difficulty lies not only in the quantitative inequality for normal perturbations of the domain (Step 4 of the proof of Theorem 4.1.1) but also in the quantitative inequality for possibly disconnected competitors (Step 3 of the proof). Indeed, since the parametric derivatives  $\dot{u}_\Omega$  are no

longer constant on the boundary of the set  $E_\Omega$ , the approach used in Step 3 might fail. We note, however, that the infinitesimal quantitative bound

$$\dot{\lambda}(V_\Omega)[h] \geq C\|h\|_{L^1(\Omega)}^2$$

still holds. To see why, we notice that

$$\dot{\lambda}(V_\Omega)[h] = - \int_{\Omega} hu_\Omega^2$$

and consider, for a parameter  $\delta$ , the solution  $h_\delta$  of

$$\min_{h \text{ admissible at } V_\Omega, \|h\|_{L^1(\Omega)}=\delta} \dot{\lambda}(V_\Omega)[h].$$

By the bathtub principle,  $h_\delta$  can be written for any  $\delta$  as a level set of  $u_\delta$  and, for  $\delta$  small enough, one can prove that  $h_\delta$  writes as follows:

$$h_\delta = \chi_{E_\delta^+} - \chi_{E_\delta^-},$$

where  $E_\delta^+ \subset E_\Omega^c$  and  $E_\delta^- \subset E_\Omega$  and can be described as follows: if  $\nu$  is the unit normal vector to  $E_\Omega$ ,

$$E_\delta^\pm := \{x \pm t\nu(x), t \in (0; t_\pm^\delta(x))\}.$$

We can then prove that

$$\frac{t_\pm^\delta}{\delta} \xrightarrow{\delta \rightarrow 0} f_\pm > 0$$

uniformly in  $x \in \partial E$ . It remains to apply the methods of Step 3 of the Proof of Theorem 4.1.1 and to do a Taylor expansion of  $-u_\Omega^2$  at  $\partial E_\Omega$  to get

$$\dot{\lambda}(V_\Omega)[h_\delta] \geq C\delta^2$$

for some constant  $C$  that depends on  $\inf \frac{\partial u_\Omega}{\partial \nu}$ . Thus, the infinitesimal inequality seems valid. However, it seems complicated to go further using only this information, since the parametric derivatives  $\dot{u}$  are no longer constant on  $\partial E_\Omega$ .

### 4.3.2 OTHER CONSTRAINTS

It would be relevant to consider perimeter constraints instead of volume constraints, but we expect the behaviour of the sequences of solutions to the auxiliary problems to be quite different. We nonetheless believe that the free boundary techniques used in [31] might apply directly to get regularity.

### 4.3.3 A CONJECTURE

Our conjecture is that the solution of the optimization problem (4.17) is, for any  $\delta > 0$  small enough, equal to  $\mathcal{H}_\delta$ . This would make Step 4 of the proof irrelevant. Moreover, this problem seems interesting in of itself. Je sais comment faire pour une des deux parties: pour tout  $x$ ,  $h_-(x) = h_+(x)$  en termes de longueur. Donc ça passe. Il faut également noter que  $h_- > h_+$

---

# APPENDIX

## 4.A PROOF OF LEMMA 4.1

We briefly recall that the Schwarz rearrangement of a function  $u \in W_0^{1,2}(\mathbb{B})$ ,  $u \geq 0$  is defined as the only radially symmetric non-increasing function  $u^*$  such that, for any  $t \in \mathbb{R}$ ,

$$|\{u \geq t\}| = |\{u^* \geq t\}|.$$

*Proof of Lemma 4.1.* We use the Polya-Szego Inequality for the Schwarz rearrangement: for any  $u \in W_0^{1,2}(\mathbb{B})$ ,  $u \geq 0$ ,

$$\int_{\mathbb{B}} |\nabla u^*|^2 \leq \int_{\mathbb{B}} |\nabla u|^2.$$

We also use the Hardy-Littlewood Inequality: for any  $u, v \in L^2(\Omega)$ ,

$$\int_{\mathbb{B}} u^* v^* \geq \int_{\mathbb{B}} uv$$

and the equimeasurability of the rearrangement:

$$\int_{\mathbb{B}} u^2 = \int_{\mathbb{B}} (u^*)^2.$$

We refer to [108] for proofs. Using the Rayleigh quotient formulation (4.3), for any  $V \in \mathcal{M}(\mathbb{B})$ ,

$$\begin{aligned} \lambda(V) &= \frac{\int_{\mathbb{B}} |\nabla u_V|^2 - \int_{\mathbb{B}} V u_V^2}{\int_{\mathbb{B}} u_V^2} \\ &\geq \frac{\int_{\mathbb{B}} |\nabla u_V^*|^2 - \int_{\mathbb{B}} V^* (u_V^*)^2}{\int_{\mathbb{B}} (u_V^*)^2} \\ &\geq \lambda(V^*). \end{aligned}$$

This also proves that  $u_{V^*} = u_{V^*}^*$ . Since the eigenvalue is simple, the eigenfunction is radially symmetric. The fact that it is decreasing follows from the Equation satisfied by  $u_{V^*}$  in polar coordinates.  $\square$

## 4.B PROOF OF THE SHAPE DIFFERENTIABILITY OF $\lambda$

### 4.B.1 PROOF OF THE SHAPE DIFFERENTIABILITY

*Proof of the shape differentiability.* Let  $E$  be a regular subdomain of  $\mathbb{B}$ ,  $(u_0, \lambda_0)$  be the eigenpair associated with  $V := \chi_E$ , and let  $\Phi$  be an admissible vector field at  $E$ . Let  $T_\Phi := (Id + \Phi)$  and  $E_\Phi^* := T_\Phi(E)$ . Let  $u_\Phi$  be the eigenvalue associated with  $V_\Phi := \chi_{E_\Phi^*}$  and  $\lambda_\Phi$  be the associated eigenvalue. If we introduce

$$J_\Omega(\Phi) := \det(\nabla T_\Phi), A_\Phi := J_\Omega(\Phi)DT_\Phi^{-1} (DT_\Phi^{-1})^t$$

then the weak formulation of the equation on  $u_\Phi$  is: for any  $v \in W_0^{1,2}(\Omega)$ ,

$$\int_{\mathbb{B}} \langle A_\Phi \nabla u_\Phi, \nabla v \rangle = \lambda_\Phi \int_{\mathbb{B}} u_\Phi v J_\Omega(\Phi) + \int_{\mathbb{B}} V_\Phi u_\Phi v J_\Omega(\Phi).$$

We define the map  $F$  in the following way:

$$F : \begin{cases} W^{1,\infty}(\mathbb{R}^n, \mathbb{R}^n) \times W_0^{1,2}(\mathbb{B}) \times \mathbb{R} \rightarrow W^{-1,2}(\mathbb{B}) \times \mathbb{R}, \\ (\Phi, v, \lambda) \mapsto (-\nabla \cdot (-\nabla \cdot (A_\Phi \nabla v) - \lambda v J_\Omega(\Phi) - V_\Phi v J_\Omega(\Phi), \int_{\mathbb{B}} v^2 J_\Omega(\Phi) - 1). \end{cases}$$

It is clear from the definition of the eigenvalue that

$$F(0, u_0, \lambda_0) = 0.$$

Furthermore, the same arguments as in [63, Lemma 2.3] show that  $F$  is  $\mathcal{C}^\infty$  in  $B \times W_0^{1,2}(\Omega) \times \mathbb{R}$ , where  $B$  is an open ball centered at  $\bar{0}$ .

The differential of  $F$  at  $(0, u_0, \lambda_0)$  is given by

$$D_{v,\lambda} F(0, u_0, \lambda_0)[w, \mu] = \left( -\Delta w - \mu u_0 - \lambda_0 w - V w, \int_{\mathbb{B}} 2u_0 w \right).$$

To prove that this differential is invertible, it suffices to show that, if  $(z, k) \in W^{-1,2}(\Omega) \times \mathbb{R}$ , then there exists a unique couple  $(w, \mu)$  such that

$$D_{v,\lambda} F(0, u_0, \lambda_0)[w, \mu] = (z, k).$$

By the Fredholm alternative, we know that we must have

$$\mu = -\langle z, u_0 \rangle.$$

There exists a solution  $w_1$  to the equation

$$-\Delta w - \mu u_0 - \lambda_0 w - m^* w = z.$$

We fix such a solution. Any other solution is of the form  $w = w_1 + t u_0$  for a real parameter  $t$ . We look for such a  $t$ . From the equation

$$2 \int_{\Omega} u_0 w = k$$

there comes

$$t = \frac{k}{2} - \int_{\Omega} w_1 u_0.$$

hence the couple  $(w, \mu)$  is uniquely determined. From the implicit function theorem, the map  $\Phi \mapsto (u_\Phi, \lambda_\Phi)$  is  $\mathcal{C}^\infty$  in a neighbourhood of  $\vec{0}$ .  $\square$

### 4.B.2 COMPUTATION OF THE FIRST ORDER SHAPE DERIVATIVE

*Proof of Lemma 4.6.* Let  $\Phi$  be a smooth vector field at  $E$  and  $E_t = T_t(E)$  where  $T_t := Id + t\Phi$ . We define  $J_t := J_\Omega(t\Phi)$  and  $A_t := A_{t\Phi}$ . The other notations are the same as in the previous paragraph. Let  $(\lambda_t, u_t)$  be the eigencouple associated with  $V_t := \chi_{E_t}$ . We first define

$$v_t := u_t \circ T_t : \mathbb{B} \rightarrow \mathbb{R}.$$

The derivative of  $v_t$  with respect to  $t$  will be denoted  $\dot{u}$ . This is the material derivative, while we aim at computing the shape derivative  $u'$  defined as

$$u' = \dot{u} + \langle \Phi, \nabla u_0 \rangle.$$

For more on these notions, we refer to [93].

Obviously  $v_t \in W_0^{1,2}(\Omega)$ . The weak formulation on  $u_t$  writes: for any  $\varphi \in W_0^{1,2}(\Omega)$ ,

$$\int_{\mathbb{B}} \langle \nabla u_t, \nabla \varphi \rangle = \lambda_t \int_{\mathbb{B}} u_t \varphi + \int_{E_t} u_t \varphi.$$

We do the change of variables

$$x = T_t(y),$$

so that, for any test function  $\varphi$ ,

$$\int_{\mathbb{B}} \langle A_t \nabla v_t, \nabla \varphi \rangle = \lambda_t \int_{\mathbb{B}} v_t \varphi J_t + \int_E v_t \varphi J_t. \quad (4.68)$$

It is known that

$$\mathcal{J}(x) := \left. \frac{\partial J_t}{\partial t} \right|_{t=0} (t, x) = \nabla \cdot \Phi,$$

and that

$$\mathcal{A}(x) := \left. \frac{\partial A_t}{\partial t} \right|_{t=0} (t, x) = (\nabla \cdot \Phi) I_n - (\nabla \Phi + (\nabla \Phi)^T).$$

We recall that  $\mathcal{A}$  has the following property: if  $\Phi_1$  and  $\Phi_2$  are two vector fields, there holds

$$\langle \mathcal{A}\Phi_1, \Phi_2 \rangle = \nabla \cdot (\langle \Phi_1, \Phi_2 \rangle \Phi) - \langle \nabla(\Phi \cdot \Phi_1), \Phi_2 \rangle - \langle \nabla(\Phi \cdot \Phi_2), \Phi_1 \rangle. \quad (4.69)$$

We differentiate Equation (4.68) with respect to  $t$  to get the following equation on  $\dot{u}$ :

$$\int_{\mathbb{B}} \langle \nabla \varphi, \nabla \dot{u} + \mathcal{A} \nabla u_0 \rangle = \dot{\lambda} \int_{\mathbb{B}} u_0 \varphi + \lambda_0 \int_{\mathbb{B}} \mathcal{J} u_0 \varphi + \lambda_0 \int_{\mathbb{B}} \dot{u} \varphi + \int_E \dot{u} \varphi + \int_E \mathcal{J}(x) u_0 \varphi. \quad (4.70)$$

Through Property (4.69) we get

$$\langle \mathcal{A} \nabla u_0, \nabla \varphi \rangle = \nabla \cdot (\langle \nabla u_0, \nabla \varphi \rangle \Phi) - \langle \nabla(\langle \Phi, \nabla u_0 \rangle), \nabla \varphi \rangle - \langle \nabla(\langle \Phi, \nabla \varphi \rangle), \nabla u_0 \rangle.$$

We deal with these three terms separately: from the divergence Formula

$$\int_{\mathbb{B}} \nabla \cdot (\langle \nabla u_0, \nabla \varphi \rangle \Phi) = - \int_{\partial E} [(\nabla u_0, \nabla \varphi)] \langle \Phi, \nu \rangle.$$



We do not touch the second term.

The third term is dealt with using the weak equation on  $u_0$ :

$$\int_{\mathbb{B}} \langle \nabla(\langle \Phi, \nabla \varphi \rangle), \nabla u_0 \rangle = \lambda_0 \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 + \int_E u_0 \langle \Phi, \nabla \varphi \rangle - \int_{\partial E} [\langle \nabla u_0, \nabla \varphi \rangle] \langle \Phi, \nu \rangle.$$

Hence

$$\begin{aligned} \int_{\mathbb{B}} \langle \mathcal{A} \nabla u_0, \nabla \varphi \rangle &= \int_{\mathbb{B}} \nabla \cdot (\langle \nabla u_0, \nabla \varphi \rangle \Phi) - \langle \nabla(\langle \Phi, \nabla u_0 \rangle), \nabla \varphi \rangle - \langle \nabla(\langle \Phi, \nabla \varphi \rangle), \nabla u_0 \rangle \\ &= - \int_{\mathbb{B}} \langle \nabla(\langle \Phi, \nabla u_0 \rangle), \nabla \varphi \rangle \\ &\quad - \lambda_0 \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 - \int_E u_0 \langle \Phi, \nabla \varphi \rangle. \end{aligned}$$

The left hand term of (4.70) becomes

$$\begin{aligned} \int_{\mathbb{B}} \langle \nabla \dot{u} + \mathcal{A} \nabla u_0, \nabla \varphi \rangle &= \int_{\mathbb{B}} \langle \nabla(\dot{u} - \Phi \cdot \nabla u_0), \nabla \varphi \rangle \\ &\quad - \lambda_0 \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 - \int_E u_0 \langle \Phi, \nabla \varphi \rangle. \end{aligned}$$

Thus

$$\begin{aligned} &\int_{\mathbb{B}} \langle \nabla(\dot{u} - \Phi \cdot \nabla u_0), \nabla \varphi \rangle - \lambda_0 \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 - \int_E u_0 \langle \Phi, \nabla \varphi \rangle \\ &= \dot{\lambda} \int_{\mathbb{B}} u_0 \varphi + \lambda_0 \int_{\mathbb{B}} \mathcal{J}(x) u_0 \varphi + \lambda_0 \int_{\mathbb{B}} \dot{u} \varphi + \int_E \dot{u} \varphi + \int_E \mathcal{J}(x) u_0 \varphi. \end{aligned}$$

By rearranging the terms, we get

$$\begin{aligned} &\int_{\mathbb{B}} \langle \nabla(\dot{u} - \Phi \cdot \nabla u_0), \nabla \varphi \rangle \\ &= + \dot{\lambda} \int_{\mathbb{B}} u_0 \varphi + \lambda_0 \left( \int_{\mathbb{B}} \mathcal{J}(x) u_0 \varphi + \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 \right) \\ &\quad + \lambda_0 \int_{\mathbb{B}} \dot{u} \varphi + \int_E \dot{u} \varphi + \int_E \mathcal{J}(x) u_0 \varphi + \int_E u_0 \langle \Phi, \nabla \varphi \rangle. \end{aligned}$$

However, since  $\mathcal{J}(x) = \nabla \cdot \Phi(x)$ , we have

$$\mathcal{J}(x) \varphi + \langle \Phi, \nabla \varphi(x) \rangle = \nabla \cdot (\varphi \Phi).$$

Hence

$$\begin{aligned} \int_{\mathbb{B}} \mathcal{J}(x) u_0 \varphi + \int_{\mathbb{B}} \langle \Phi, \nabla \varphi \rangle u_0 &= \int_{\mathbb{B}} \nabla \cdot (\Phi \varphi) u_0 \\ &= - \int_{\mathbb{B}} \varphi \langle \Phi, \nabla u_0 \rangle, \end{aligned}$$

because  $u_0$  satisfies homogeneous Dirichlet boundary conditions. In the same way

$$\begin{aligned} \int_E \mathcal{J}(x)u_0\varphi + \int_E \langle \Phi, \nabla\varphi \rangle u_0 &= \int_{\mathbb{B}} \nabla \cdot (\Phi\varphi)u_0 \\ &= - \int_E \varphi \langle \Phi, \nabla u_0 \rangle + \int_{\partial E} \langle \Phi, \nu \rangle u_0 \varphi. \end{aligned}$$

We turn back to the shape derivative; recall that it is defined as

$$u' := \dot{u} - \langle \Phi, \nabla u \rangle.$$

The previous equation rewrites

$$\begin{aligned} \int_{\mathbb{B}} \langle \nabla u', \nabla \varphi \rangle &= \lambda_0 \int_{\mathbb{B}} u_0 \varphi + \lambda_0 \int_{\mathbb{B}} u' \varphi + \int_E u' \varphi \\ &\quad + \int_{\partial E} \langle \Phi, \nu \rangle u_0 \varphi \end{aligned}$$

Thus there appears that  $u'$  solves

$$-\Delta u' = \lambda' u_0 + \lambda_0 u_1 + V u'$$

along with Dirichlet boundary conditions and

$$\left[ \frac{\partial u'}{\partial r} \right] = -\langle \Phi, \nu \rangle u_0.$$

Obtaining the jump condition on  $u''$  is done in the same way as in [63].  $\square$

### 4.B.3 GÂTEAUX-DIFFERENTIABILITY OF THE EIGENVALUE

The parametric differentiability is also proved using the implicit function theorem applied to the following map:

$$G : \begin{cases} L^\infty(\Omega) \times W_0^{1,2}(\Omega) \times \mathbb{R} \rightarrow W^{-1,2}(\Omega) \times \mathbb{R}, \\ (h, v, \lambda) \mapsto (-\Delta v - \lambda v - (V + h)v, \int_\Omega v^2 - 1). \end{cases}$$

The invertibility of the differential follows from the same arguments as the ones used to prove the invertibility of  $DF$  in the previous section.

## 4.C PROOF OF PROPOSITION 4.2

*Proof of Proposition 4.2.* We can not apply in a straightforward manner the methods of [64], which are well-suited for the proof of a convergence in the  $H^{\frac{1}{2}}$  topology. Some minor adjustments are in order.

Let us define  $T_\Phi := (Id + \Phi)$  and, for any function  $f : \Omega \rightarrow \mathbb{R}$ ,

$$\hat{f} := f \circ T_\Phi.$$

We define the surface Jacobian

$$J_\Sigma(\Phi) := \det(\nabla T_\Phi) \left| ({}^t \nabla T_\Phi^{-1}) \nu \right|,$$

the volume Jacobian

$$J_\Omega(\Phi) := \det(\nabla\Phi)$$

and, finally

$$A_\Phi := J_\Omega(\Phi)(Id + \nabla\Phi)^{-1}(Id + {}^t\nabla\Phi)^{-1}.$$

It is known (see [64, Lemma 4.8]) that

$$\|J_{\Omega/\Sigma}(\Phi) - 1\|_{L^\infty} \leq C\|\Phi\|_{W^{1,\infty}}, \|A_\Phi - 1\|_{L^\infty} \leq C\|\Phi\|_{W^{1,\infty}}. \quad (4.71)$$

We define  $u_0$  as the eigenfunction associated with  $\mathbb{B}^*$  and  $u'_0$  the shape derivative of  $u_0$  in the direction  $\Phi$ .

Finally let  $u'_\Phi$  be the shape derivative in the direction  $\Phi$  and  $\hat{u}'_\Phi := u'_\Phi \circ T_\Phi$ . Let  $H_\Phi$  be the mean curvature of  $\mathbb{B}_\Phi$ . Using the change of variable  $y = T_\Phi(x)$ , the fact that  $\Phi$  is normal to  $\mathbb{B}^*$  and the value of the Lagrange multiplier  $\tau$  given by (4.46), we get

$$\begin{aligned} L''_\tau(\mathbb{B}_\Phi)[\Phi, \Phi] &= -2 \int_{\partial\mathbb{B}^*} J_\Sigma(\Phi) \hat{u}_\Phi \hat{u}'_\Phi \langle \Phi, \nu \rangle + \int_{\partial\mathbb{B}^*} J_\Sigma(\Phi) \left( -\hat{H} \hat{u}_\Phi^2 - 2\hat{u}_\Phi \frac{\partial \hat{u}_\Phi}{\partial \nu} \right) \langle \Phi, \nu \rangle^2 \\ &\quad - \tau \int_{\partial\mathbb{B}^*} \hat{H} \langle \Phi, \nu \rangle^2 \\ &= -2 \int_{\partial\mathbb{B}^*} J_\Sigma(\Phi) \hat{u}_\Phi \hat{u}'_\Phi \langle \Phi, \nu \rangle + \int_{\partial\mathbb{B}^*} J_\Sigma(\Phi) \left( \hat{H} (\hat{u}_\Phi^2 - u_0^2) - 2\hat{u}_\Phi \frac{\partial \hat{u}_\Phi}{\partial \nu} \right) \langle \Phi, \nu \rangle^2. \end{aligned}$$

Hence we have

$$\begin{aligned} L''_\tau(\mathbb{B}_\Phi)[\Phi, \Phi] - L_\tau(\mathbb{B}^*)[\Phi, \Phi] &= -2 \int_{\partial\mathbb{B}^*} (J_\Sigma(\Phi) \hat{u}_\Phi \hat{u}'_\Phi - u_0 u'_0) \langle \Phi, \nu \rangle \\ &\quad + \int_{\partial\mathbb{B}^*} \left( \hat{H} (J_\Sigma(\Phi) \hat{u}_\Phi^2 - u_0^2) \right) \langle \Phi, \nu \rangle^2 \\ &\quad + \int_{\partial\mathbb{B}^*} \left( 2u_0 \frac{\partial u_0}{\partial \nu} - 2J_\Sigma(\Phi) \hat{u}_\Phi \frac{\partial \hat{u}_\Phi}{\partial \nu} \right) \langle \Phi, \nu \rangle^2. \end{aligned} \quad (4.72)$$

We will prove the Proposition using the following estimates

**CLAIM 4.C.1** *For any  $\eta > 0$  there exists  $\varepsilon > 0$  such that, for any  $\Phi$  satisfying*

$$\|\Phi\|_{\mathcal{C}^1} \leq \varepsilon$$

*there holds*

1.

$$\|\hat{u}_\Phi - u_*\|_{\mathcal{C}^1(\Omega)} \leq \eta, \quad (4.73)$$

2.

$$\|\hat{u}'_\Phi - u'_0\|_{W_0^{1,2}} \leq \eta \|\langle \Phi, \nu \rangle\|_{L^2(\Sigma)}. \quad (4.74)$$

*Proof of Claim 4.C.1.* Estimate (6.34) follows from a simple contradiction argument and by using the fact that, if a sequence  $\{\Phi_k\}_{k \in \mathbb{N}}$  converges in the  $\mathcal{C}^1$  norm to 0, then  $u_{\Phi_k}$  converges, in every  $\mathcal{C}^{1,s}(\Omega)$  ( $s < 1$ ) to  $u_0$ . To prove (4.74), we first prove that there exists a constant  $M$  such that

$$\|\hat{u}'_\Phi\|_{W_0^{1,2}} \leq M \|\langle \Phi, \nu \rangle\|_{L^2(\Sigma)}. \quad (4.75)$$

By the change of variable  $y := T_\Phi(x)$ , we see that  $\hat{u}'_\Phi$  satisfies

$$-\nabla \cdot \left( A_\Phi \nabla \hat{u}'_\Phi \right) = J_\Omega(\Phi) (\lambda_\Phi \hat{u}'_\Phi + (V^*) \hat{u}'_\Phi + \lambda'_\Phi \hat{u}_\Phi), [A_\Phi \partial_\nu \hat{u}'_\Phi] = -J_\Sigma \langle \Phi, \nu \rangle \hat{u}_\Phi \quad (4.76)$$

with homogeneous Dirichlet boundary conditions. The orthogonality conditions gives

$$\int_\Omega J_\Omega(\Phi) \hat{u}_\Phi \hat{u}'_\Phi = 0$$

and we will use a Spectral Gap Estimate (4.81) combined with a bootstrap argument.

**Spectral gap estimate** For any  $V \in \mathcal{M}(\mathbb{B})$ ,  $\lambda(V)$  was defined as the first eigenvalue of the operator  $\mathcal{L}_V$  defined in (4.1). We recalled in the Introduction that this eigenvalue is simple. Let, for any  $V \in \mathcal{M}(\Omega)$ ,  $\lambda_2(V) > \lambda(V)$  and  $u_{2,V}$  be the second eigenvalue and an associated eigenfunction (we choose a  $L^2$  normalization). We claim there exists  $\omega > 0$  such that, for any  $V \in \mathcal{M}(\Omega)$ ,

$$\omega \leq \lambda_2(V) - \lambda(V). \quad (4.77)$$

To prove this, we use a direct argument. Let  $S(V) := \lambda_2(V) - \lambda(V)$  be the spectral gap associated with  $V$ . We consider a minimizing sequence  $\{V_k\}_{k \in \mathbb{N}} \in \mathcal{M}(\Omega)$  (the radially assumption is not necessary here) which, up to a subsequence, converges weakly in  $L^\infty - *$  to some  $V_\infty \in \mathcal{M}(\Omega)$ . It is standard to see that

$$\lambda(V_k) \xrightarrow[k \rightarrow \infty]{} \lambda(V_\infty), u_{V_k} \xrightarrow[k \rightarrow \infty]{} u_{V_\infty} \text{ strongly in } L^2(\mathbb{B}), \text{ weakly in } W_0^{1,2}(\mathbb{B}).$$

The only part which is not completely classical is to prove that

$$\lambda_2(V_k) \xrightarrow[k \rightarrow \infty]{} \lambda_2(V_\infty). \quad (4.78)$$

However, for any  $k \in \mathbb{N}$ ,  $\lambda_2(V_k)$  is defined as

$$\lambda_2(V_k) = \min_{u \in W_0^{1,2}(\mathbb{B}), \int_{\mathbb{B}} u^2 = 1, u \in \langle u_{V_k} \rangle^\perp} R_{V_k}[u], \quad (4.79)$$

where  $\langle u \rangle^\perp$  is the subspace of functions that are  $L^2$ -orthogonal to  $u$ , and  $u_{2,V_k}$  is defined as a minimizer for this problem (there a possibly multiple eigenfunctions). In the same way we proved that  $\lambda(V)$  is uniformly bounded in  $V$ , one proves that  $\lambda_2(V)$  is uniformly bounded in  $V$ . Let  $\lambda_{2,\infty}$  be such that

$$\lambda_2(V_k) \xrightarrow[k \rightarrow \infty]{} \lambda_{2,\infty}.$$

Standard elliptic estimates prove that there exists a function  $u_{2,\infty} \in W_0^{1,2}(\Omega)$  such that  $u_{2,V_k} \xrightarrow[k \rightarrow \infty]{} u_{2,\infty}$  strongly in  $L^2(\mathbb{B})$  and weakly in  $W_0^{1,2}(\mathbb{B})$ . Passing to the limit in

$$\int_{\mathbb{B}} u_{2,V_k} u_{V_k} = 0$$

gives

$$\int_{\mathbb{B}} u_{2,\infty} u_{V_\infty} = 0. \quad (4.80)$$

Passing to the limit in the weak formulation of the equation on  $u_{2,V_k}$  proves that  $u_{2,\infty}$  is an eigenfunction of  $\mathcal{L}_{V_\infty}$  associated with  $\lambda_{2,\infty}$ . It follows from the orthogonality relation (4.80) that

$$\lambda_{2,\infty} \geq \lambda_2(V_\infty).$$

Hence

$$\liminf_{k \rightarrow \infty} S(V_k) \geq \lambda_2(V_\infty) - \lambda(V_\infty) \geq \omega_1 > 0$$

because  $\lambda(V_\infty)$  is a simple eigenvalue.

As a consequence of the spectral gap estimate (4.77), we get the following estimate:

$$\forall V \in \mathcal{M}(\Omega), \forall u \in \langle u_V \rangle^\perp, \omega \int_\Omega u^2 \leq \int_\Omega |\nabla u|^2 - \int_\Omega V u^2 - \lambda(V) \int_{\mathbb{B}} u^2. \quad (4.81)$$

Indeed, let  $V \in \mathcal{M}(\Omega)$  and  $u \in \langle u_V \rangle^\perp, u \neq 0$ . Then, by the Rayleigh quotient formulation on  $\lambda_2(V)$ , see Equation (4.79),

$$\int_{\mathbb{B}} |\nabla u|^2 - \int_{\mathbb{B}} V u^2 \geq \lambda_2(V) \int_{\mathbb{B}} u^2 \geq \omega \int_{\mathbb{B}} u^2 + \lambda(V) \int_{\mathbb{B}} u^2 \text{ by (4.77),}$$

which is exactly the desired conclusion.

**Proof of (4.75)** First of all, multiplying (4.76) by  $\hat{u}'_\Phi$  and integrating by parts gives

$$\int_\Omega A_\Phi |\nabla \hat{u}'_\Phi|^2 - \int_\Omega V^* J_\Omega(\Phi) (\hat{u}'_\Phi)^2 = \int_{\partial \mathbb{B}^*} J_\Sigma \hat{u}_\Phi \hat{u}'_\Phi \langle V, \nu \rangle.$$

By the Spectral gap estimate, using the fact that eigenfunctions are uniformly bounded and by continuity of the trace operator we get the existence of a constant  $M$  such that

$$\int_\Omega (\hat{u}'_\Phi)^2 \leq M \|\langle \Phi, \nu \rangle\|_{L^2(\partial \mathbb{B}^*)} \|\hat{u}'_\Phi\|_{W_0^{1,2}(\Omega)}. \quad (4.82)$$

We rewrite

$$\|\hat{u}'_\Phi\|_{W_0^{1,2}(\Omega)} = \|\hat{u}'_\Phi\|_{L^2(\Omega)} + \|\nabla \hat{u}'_\Phi\|_{L^2(\Omega)}.$$

By the shape differentiability of  $E \mapsto (\lambda(E), u_E)$ , there exists  $C$  such that

$$\|\nabla \hat{u}'_\Phi\|_{L^2(\Omega)} \leq C$$

for any  $\Phi$  such that  $\|\Phi\|_{W^{1,\infty}} \leq 1$ .

We then let  $X := \|\hat{u}'_\Phi\|_{L^2(\Omega)}$ . The estimate (4.82) rewrites

$$X^2 \leq M \|\langle \Phi, \nu \rangle\|_{L^2(\partial \Omega)} X + MC \|\langle \Phi, \nu \rangle\|_{L^2(\partial \Omega)},$$

from where it follows that there exists  $M > 0$  such that

$$\|\hat{u}'_\Phi\|_{L^2(\Omega)} \leq M \sqrt{\|\langle \Phi, \nu \rangle\|_{L^2(\partial \Omega)}}.$$

We now multiply (4.75) by  $\hat{u}'_\Phi$  and integrate by part, giving, for some constant  $M$ ,

$$\int_\Omega |\nabla \hat{u}'_\Phi|^2 \leq C \|\langle \Phi, \nu \rangle\|_{L^2(\partial \Omega)} + \|\langle \Phi, \nu \rangle\|_{L^2(\partial \Omega)} \|\nabla \hat{u}'_\Phi\|_{L^2(\Omega)} + \|\langle \Phi, \nu \rangle\|_{L^2(\partial \Omega)}^{\frac{3}{2}}$$

which in turn yields, using the same arguments,

$$\|\nabla \hat{u}'_\Phi\|_{L^2(\Omega)} \leq \sqrt{\|\langle \Phi, \nu \rangle\|_{L^2(\partial \Omega)}}.$$

We use this in (4.82), giving

$$\|\hat{u}'_\Phi\|_{L^2(\Omega)}^2 \leq M \left( \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \|\hat{u}'_\Phi\|_{L^2(\Omega)} + \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}^{\frac{3}{2}} \right).$$

This yields

$$\|\hat{u}'_\Phi\|_{L^2(\Omega)} \leq M \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}$$

and, finally, from the weak formulation of the equation,

$$\|\nabla \hat{u}'_\Phi\|_{L^2(\Omega)} \leq M \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}.$$

**Proof of (4.74)** We now turn to the proof of the continuity estimate (4.74), for which we will apply the same kind of bootstrap arguments, combined with a version of the splitting method, see [64, Lemma 4.10].

Let us define  $H_\Phi$  as the solution of

$$\begin{cases} -\Delta H_\Phi = V_\Phi H_\Phi, \\ \left[ \frac{\partial H_\Phi}{\partial \nu} \right] = u_\Phi \langle \Phi, \nu_\Phi \rangle \text{ on } \partial E_t, \\ H_\Phi = 0 \text{ on } \partial\Omega. \end{cases} \quad (4.83)$$

Then it appears that

$$\lambda'_\Phi = - \int_{\partial E_t} u_\Phi^2 \langle \Phi, \nu_\Phi \rangle = \lambda_\Phi \int_{\Omega} H_\Phi u_\Phi.$$

We can prove using the same bootstrap arguments used to prove (4.75) that

$$\|\hat{H}_\Phi\|_{W_0^{1,2}(\Omega)} \leq C \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}.$$

Indeed, multiplying (4.84) by  $H_\Phi$ , doing a change of variables and integrating by parts gives

$$\int_{\Omega} A_t |\nabla \hat{H}_\Phi|^2 - \int_{\Omega} J_t \hat{H}_\Phi^2 \leq \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \|\hat{H}_\Phi\|_{W_0^{1,2}(\Omega)}$$

and, by the variational formulation of the eigenvalue,

$$\int_{\Omega} \hat{H}_\Phi^2 \leq \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \|\hat{H}_\Phi\|_{W_0^{1,2}(\Omega)}.$$

We then use the same bootstrap argument: we first prove that this implies  $\|\hat{H}_\Phi\|_{L^2(\Omega)} \leq M \sqrt{\|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)}}$  and plug this estimate in the weak formulation of the equation. The conclusion follows.

We turn back to (4.74).

Let  $\pi_\Phi$  be the orthogonal projection on  $\langle u_\Phi \rangle^\perp$ . We decompose  $u'_\Phi$  as

$$u'_\Phi = -\pi_\Phi H_\Phi + \xi_\Phi$$

where  $\xi_\Phi$  solves

$$\begin{cases} -\Delta \xi_\Phi = \lambda_\Phi \xi_\Phi + V_\Phi \xi_\Phi - \lambda_\Phi \pi_\Phi H_\Phi, \\ \xi_\Phi = 0 \text{ on } \partial\Omega, \\ \int_{\Omega} \xi_\Phi u_\Phi = 0. \end{cases} \quad (4.84)$$

Thanks to the Fredholm alternative, such a  $\xi_\Phi$  exists and is uniquely defined.

We now prove that

$$\left\| \hat{H}_\Phi - H_0 \right\|_{W_0^{1,2}(\Omega)} \leq M \eta \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \quad (4.85)$$

for  $\|\Phi\|_{\mathcal{C}^1}$  small enough. To that end, we define

$$\mathfrak{H}_\Phi := \hat{H}_\Phi - H_0.$$

Direct computation shows that

$$-\Delta \mathfrak{H}_\Phi = (V^*)\mathfrak{H}_\Phi + (V^*)\hat{H}_\Phi(J_t - 1) + \nabla \cdot \left( (A_t - Id)\nabla \hat{H}_\Phi \right)$$

along with Dirichlet boundary conditions and

$$\left[ \frac{\partial \mathfrak{H}_\Phi}{\partial \nu} \right] = (u_0 - J_\Sigma \hat{u}_\Phi)\langle \Phi, \nu \rangle + \left[ \langle (Id - A_t)\nabla \hat{H}_\Phi, \nu \rangle \right].$$

We proceed in the same fashion: we first multiply the equation on  $\mathfrak{H}_\Phi$  by  $\mathfrak{H}_\Phi$ , integrate by parts and use the variational formulation of the eigenvalue to get

$$\begin{aligned} \|\mathfrak{H}_\Phi\|_{L^2(\Omega)}^2 &\leq \|J_t - 1\|_{L^\infty} \|\hat{H}_\Phi\|_{L^2(\Omega)} + \|A_t - Id\|_{L^\infty(\Omega)} \|\nabla \hat{H}_\Phi\|_{L^2(\Omega)} \|\nabla \mathfrak{H}_\Phi\|_{L^2(\Omega)} \\ &\quad + \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \|\mathfrak{H}_\Phi\|_{W_0^{1,2}(\Omega)} \|u_0 - J_\Sigma \hat{u}_\Phi\|_{L^\infty(\partial\Omega)} \end{aligned}$$

up to a multiplicative constant. This first gives, using (6.34),

$$\|\mathfrak{H}_\Phi\|_{L^2(\Omega)} \leq M \sqrt{\|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} (\|\nabla \Phi\|_{L^\infty} + \eta)}.$$

We then apply the same bootstrap method to get the desired conclusion.

Finally, we need to show the following estimate, which will conclude the proof:

$$\|\hat{\xi}_\Phi - \xi_0\|_{W_0^{1,2}(\Omega)} \leq \eta \|\langle \Phi, \nu \rangle\|_{L^2(\partial\Omega)} \tag{4.86}$$

for  $\|\Phi\|_{\mathcal{C}^1}$  small enough. However, this follows from the same arguments as in [64, Lemma 4.10, Paragraph 3 of the proof] and from the same bootstraps arguments.  $\square$

Finally, going back to (4.72), it suffices to use the continuity of the trace to control the terms involving  $u'_\Phi$  and Estimates (6.34)-(4.74) to conclude the proof of Proposition 4.2.  $\square$

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# CHAPTER 5

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## CONTROL OF A BISTABLE REACTION-DIFFUSION EQUATION IN A HETEROGENEOUS ENVIRONMENT

WITH D. RUIZ-BALUET AND E. ZUAZUA

Les petites choses n'ont l'air de rien, mais  
elles donnent la paix.

---

Bernanos,  
*Journal d'un curé de campagne*

Ils vont trop vite, ils se casseront le cou.

---

Robespierre, cité par Michelet,  
*Histoire de la Révolution Française*



## GENERAL PRESENTATION OF THE CHAPTER: MAIN DIFFICULTIES AND METHODS

In this Chapter, which corresponds to [MRBZ19], we mainly focus on the controllability of bistable reaction-diffusion equations in spatially heterogeneous environments. Our two main results, Theorems 5.1.2 and 5.1.4 may be summed up as follows:

- For slowly varying environments: This corresponds to Theorem 5.1.2. We prove, using a new technique coupled with a careful analysis of the staircase method of [58], that the controllability properties of the equation are the same as that of the spatially homogeneous equation. This new technique uses perturbation methods but on a possibly varying domain.
- For rapidly varying environments: This corresponds to Theorem 5.1.4. We study a particular example of rapidly varying environment to prove that the behaviour of the equation might be drastically different. Namely, a sharp transition might lead to a lack of controllability to any of the three steady-states of the bistable reaction-diffusion equation, in sharp contrast with the spatially homogeneous case. Our analysis, in that case, relies on a fine study of the phase portrait, and is completed by several numerical simulations.

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## 5.1 INTRODUCTION

### 5.1.1 SETTING AND MAIN RESULTS

**Motivations** Reaction-diffusion equations have drawn a lot of attention from the mathematical community over the last decades, but most usually in spatially homogeneous setting, while the literature devoted to spatially heterogeneous domains only started developing recently. This growing interest led to many interesting questions regarding the possible effects of spatial heterogeneity on, for instance, the dynamics of the equation, or on optimization and control problems: how do these heterogeneities impact the dynamics or the criteria under consideration? Can the results obtained in the homogeneous case be obtained in the heterogeneous one, which is more relevant for applications? In this article, we study some of these questions and the influence of spatial heterogeneity from the angle of control theory. Some of our proofs and results are, however, of independent interest for reaction-diffusion equations.

We investigate a boundary control problem arising naturally from population dynamics models and which has several interpretations. For instance, one might consider the following situation: given a population of mosquitoes, a proportion of which is carrying a disease, is it possible, acting only on the proportion of sick mosquitoes on the boundary, to drive this population to a state where only sane mosquitoes remain? Such questions have drawn the attention of the mathematical community in the past years, see for instance [5]. Another example might be that of linguistic dynamics: considering a population of individuals, a part of which is monolingual (speaking only the dominant language), the other part of which is bilingual (speaking the dominant and a minority language), is it possible, acting only on the proportion of bilingual speakers on the boundary of the domain, to drive the population to a state where there remains a non-zero proportion of bilingual speakers, thus ensuring the survival of the minority language? Such models are proposed, for instance, in [175]. In both cases, the influence of the spatial heterogeneity has still not been investigated, and the aim of this work is to provide some informations on such matters.

We give, in Section 5.1.2, more bibliographical references related to modelling issues and the mathematical analysis of the equations studied here.

**The equation and the control system** We now present the main equations that will be studied here. We refer to Section 5.1.2 for more informations on modelling.

In this article, we consider a boundary control problem for bistable reaction-diffusion equations. Such bistable equations are well-suited to describe the evolution of a proportion of a population and are characterized by the so-called *Allee effect*: there exists a threshold for the proportion of the population under scrutiny such that, in the absence of spatial diffusion, above this threshold, this subgroup will invade the whole domain (and drive the other subgroup to extinction) while, under this threshold, this subgroup of the population will go extinct. This Allee effect is, on a mathematical level, taken into account via a *bistable non-linearity*, that is, a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

1.  $f$  is  $\mathcal{C}^\infty$  on  $[0, 1]$ ,
2. There exists  $\theta \in (0; 1)$  such that  $0, \theta$  and  $1$  are the only three roots of  $f$  in  $[0, 1]$ , This parameter  $\theta$  accounts for the Allee effect mentioned above.
3.  $f'(0), f'(1) < 0$  and  $f'(\theta) > 0$ ,
4. Without loss of generality, we assume that  $\int_0^1 f > 0$ .

We give an example of such a bistable non-linearity in Figure 5.1 below:

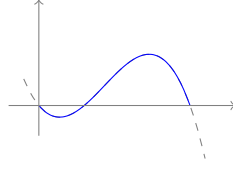


Figure 5.1 – Graph of a typical bistable non-linearity.

The typical example of such a non-linearity is

$$f(\xi) = \xi(\xi - \theta)(1 - \xi),$$

and in this case requiring that  $\int_0^1 f > 0$  is equivalent to asking that  $\theta$  satisfies  $\theta < \frac{1}{2}$ .

Models with spatial diffusion were studied from the angle of control theory in [163, 157], see Section 5.1.2. Here, we want to study more precise version of this equation and take into account two phenomenons of great relevance for applications, see Section 5.1.2: *gene-flow models* and *spatially heterogeneous models*. To write these models in a synthetic way, we will consider, in general, a function  $N = N(x, p)$ . As will be explained later, gene-flow models correspond to  $N = N(p)$  and spatially heterogeneous models correspond to  $N = N(x)$ .

With a bistable non-linearity  $f$  and such a function  $N$ , in a domain  $\Omega \subset \mathbb{R}^d$ , the equation we consider writes, in its most general form

$$\frac{\partial p}{\partial t} - \Delta p - 2\langle \nabla (\ln(N(x, p))), \nabla p \rangle = f(p). \quad (5.1)$$

Here, once again,  $p$  stands for a proportion of the total population (for instance, the proportion of infected mosquitoes or of monolingual speakers). Of particular relevance are the spatially homogeneous steady-states of this equation:  $p \equiv 0$ ,  $p \equiv \theta$  and  $p \equiv 1$ . *Our objective in this article is to investigate whether or not it is possible to control any initial datum to these spatially heterogeneous steady-states.*

Let us formalize this control problem. Given an initial datum  $p_0 \in L^2(\Omega)$  such that

$$0 \leq p_0 \leq 1$$

we consider the control system

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p - 2\langle \nabla \ln(N), \nabla p \rangle = f(p) & \text{in } (0, T) \times \Omega, \\ p = u(t, x) & \text{on } (0, T) \times \partial\Omega, \\ p(t = 0, \cdot) = 0 \leq p_0 \leq 1, \end{cases} \quad (5.2)$$

where, for every  $t \geq 0, x \in \partial\Omega$ ,

$$u(t, x) \in [0, 1] \quad (5.3)$$

is the control function. Our goal is the following:

*Given any initial datum  $0 \leq p_0 \leq 1$ ,  
is it possible to drive  $p_0$  to 0,  $\theta$ , or 1 in  
(in)finite time with a control  $u$  satisfying (5.3)?*

In other words, can we drive any initial datum to one of the spatially homogeneous steady-states of the equation? If one thinks about infected mosquitoes, driving any initial population to 0 is relevant

for controlling the disease while, if one thinks about mono or bilingual speakers, driving the initial datum to the intermediate steady-state  $\theta$  ensures the survival of the minority language.

Let us denote the steady-states as follows

$$\forall a \in \{0, \theta, 1\}, z_a \equiv a.$$

By controllability, we mean the following: let  $a \in \{0, \theta, 1\}$ , then

- Controllability in finite time: we say that  $p_0$  is controllable to  $a$  in finite time if there exists a finite time  $T < \infty$  such that there exists a control  $u$  satisfying the constraints (5.3) and such that the solution  $p = p(t, x)$  of (5.2) satisfies

$$p(T, \cdot) = z_a \text{ in } \Omega.$$

- Controllability in infinite time: we say that  $p_0$  is controllable to  $z_a$  in infinite time if there exists a control  $u$  satisfying the constraints (5.3) such that the solution  $p = p(t, x)$  of (5.2) satisfies

$$p(t, \cdot) \xrightarrow[t \rightarrow \infty]{\mathcal{C}^0(\Omega)} z_a.$$

**REMARK 5.1** Note that, in the definition of controllability in finite time, we do not ask that the controllability time be small; it might actually be large because of the constraint  $0 \leq u \leq 1$ , and the question of the minimal controllability time for this problem is, as far as the authors know, still open.

**DEFINITION 5.1.1** We say that (5.2) is controllable to  $z_a$  in (in)finite time if it is controllable to  $z_a$  in (in)finite time for any initial datum  $0 \leq p_0 \leq 1$ .

Here, for modelling reasons (which we present in the next paragraph), we only consider two cases for the flux  $N = N(x, p)$ :

- The gene-flow model: In this case, the function  $N = N(x, p)$  assumes the form

$$N(x, p) = N(p). \tag{H_1}$$

This model is referred to as the gene-flow models and appears in many situations (we refer to Section 5.1.2 and mention that this corresponds to a limit case of a system of coupled reaction-diffusion equations). In this case, the environment is spatially homogeneous, and we prove that the controllability results established in [163, 157] still hold under the same assumptions.

- The spatially heterogeneous model: In this case,  $N = N(x, p)$  is of the form

$$N = N(x). \tag{H_2}$$

This case corresponds to a spatially heterogeneous environment: when  $p$  is a proportion of a total population, this term accounts for the spatial variations of the total population, see [146] and Section 5.1.2. We mention that this corresponds to another limit in a system of coupled reaction-diffusion equations. Regarding this spatially heterogeneous model, we will focus on two situations: a slowly varying environment, in a sense made precise in the statement of Theorem 5.1.2, and a rapidly varying environment, see Theorem 5.1.4. In the first case, we prove that controllability still holds while, in the second case, we give an example of  $N$  that proves that it is in general hopeless to try and control the equation in a rapidly varying environment under the constraints (5.3).

## 5.1.2 MOTIVATIONS AND KNOWN RESULTS

### 5.1.2.1 MODELLING CONSIDERATIONS

In this paragraph, we lay out the biological motivations for our work.

Reaction-diffusion equations such as (5.2) have been used since the seminal works [75, 113] to give mathematical models of population dynamics. The bistable non-linearity accounting for the Allee affect is omnipresent in mathematical biology and we refer, for instance, to [9, 15, 14] for some of its uses in population dynamics. We also point to [43, 143, 152] for modelling issues, or to [101, 175], where a game theory approach is undertaken. Here,  $p$  stands for the frequency of some trait, or of the proportion of a type of a population, and the drift term accounts for either the spatial heterogeneity of the environment, or for the gene-flow phenomenon.

We note that gene-flow models have been used in the modelling of evolutionary processes of differentiation, see [79, 135]. We point, for further references regarding the adaptative point of view on gene-flow, to [23], as well as [49]. A mathematical study of the impact of gene-flow models on adaptative dynamics is carried out in [140], while a traveling-wave point of view is studied in [146].

In [146, Section 6], a possible derivation of the equations under study in our article is carried out. We can briefly sketch their arguments as follows: let us consider a population with size  $N = n_1 + n_2$  where, for  $i = 1, 2$ ,  $n_i$  is the number of individuals with trait  $i$  (e.g infected or sane mosquitoes). Let us define the proportion  $p = \frac{n_1}{N}$ . We assume the population evolves in a spatially heterogeneous environment  $\Omega$ , and that the heterogeneity is modelled by a resources distribution  $m : \Omega \rightarrow \mathbb{R}$ . We introduce the death rates associated to each group  $d_1 > d_2$ , the fertility rates  $F_1 < F_2$ . The following system is proposed in [170] to model this situation:

$$\begin{cases} \frac{\partial n_1}{\partial t} - \Delta n_1 = F_1 n_1 (1 - \frac{N}{m}) - \delta d_2 n_1, \\ \frac{\partial n_2}{\partial t} - \Delta n_2 = F_2 n_2 (1 - p)(1 - \frac{N}{m}) - d_2 n_2, \end{cases} \quad (5.4)$$

for some  $\delta > 0$ .

One then shows that  $p$  solves

$$\frac{\partial p}{\partial t} - \mu \Delta p - 2 \langle \nabla \ln(N), \nabla p \rangle = p(1-p)(F_1(1-N)(p-1) + d_1(1-\delta)).$$

The authors of [146] distinguish two limits

- Homogeneous environment and large birth rate:

Assuming that  $F_2 \gg 1$  and that  $m$  is constant, it is possible to show that there exists  $h = h(p)$  such that, when  $F_2 \rightarrow \infty$ ,  $p$  solves

$$\frac{\partial p}{\partial t} - \mu \Delta p + 2 |\nabla p|^2 \frac{h'(p)}{h(p)} = p(1-p)(p - \theta)$$

for some  $\theta \in (0; 1)$  which is the gene-flow model studied in this article.

- Heterogeneous environment and large birth rate:

We apply the same reasoning, with  $F_2 \gg 1$  and the additional assumption that  $|\frac{\Delta m}{m}| \ll 1$ . We then obtain

$$\frac{\partial p}{\partial t} - \mu \Delta p + 2 \left\langle \frac{\nabla m}{m}, \nabla p \right\rangle = p(1-p)(p - \theta),$$

for some  $\theta \in (0; 1)$ , which is the spatially heterogeneous model under consideration here.

As was explained earlier in this Section, one can think of the unknown  $p$  as the infection frequency in a population of mosquitoes, as is the case in [5, 146] or as the proportion of mono or bilingual speakers as proposed in [175]. This last interpretation was one of the motivations of [157, 172]. Thus, we may think of wanting to drive  $p_0$  to  $\theta$  as wanting to reach an equilibrium regarding the languages spoken inside a community, for instance to preserve the existence of this minority language.

The main contribution of this article is understanding how spatial heterogeneity might affect this controllability. We insist upon the fact that such questions pertain to a growing field, see [19, 116, MNP19a, 166] for an optimization approach to spatial heterogeneity for monostable case. In the case of bistable equations, a possible reference from the mathematical point of view is [146]. We refer to [166] for a more biology oriented presentation of such topics in mathematical biology.

Namely, we will prove that, provided the environment is not rapidly varying, the controllability properties still hold, while giving examples where sharp changes keep us from controlling the equation. Intuitively, this result makes sense: if there is a sharp transition in the environment in the center of the domain, it is hopeless to control what is happening inside the domain only using the boundary. To give an example of such quick transitions, we will investigate the case where the spatial heterogeneity is a gaussian, and give a qualitative analysis of the controllability properties when the variance is either small or large.

### 5.1.2.2 KNOWN RESULTS REGARDING THE CONSTRAINED CONTROLLABILITY OF BISTABLE EQUATIONS

The influence of spatial heterogeneity on population dynamics and its interplay with optimization problems has drawn a lot of attention in the past years. Regarding the controllability properties of these equations, the available literature is scarce.

In [157], the controllability to 0,  $\theta$  or 1 of the equation

$$\frac{\partial p}{\partial t} - \Delta p = f(p)$$

with a constraints on the boundary control is carried out using a phase portrait analysis. In their case, the domain is  $\Omega = [-L, L]$ . Namely, they prove, using comparison principles that, regardless of  $L$ , the static strategy  $u = 1$  allows you to control to  $z_1 \equiv 1$  in infinite time. They prove that there exists a threshold  $L^*$  such that control to 0 is possible if and only if  $L < L^*$ , in which case the static strategy  $u \equiv 0$  works. This threshold is established by proving that there exists  $L^*$  such that, for any  $L \geq L^*$  there exists a non trivial solution to the equation with homogeneous Dirichlet boundary conditions; this solution acts as a barrier and prevents controllability. Finally, they prove, using a precise analysis of the phase portrait of the equation and the staircase method of [58] that the equation is controllable to  $z_\theta \equiv \theta$  in finite time if and only if  $L < L^*$ .

These results were extended, for the same equation, to the multi-dimensional case in [163].

In [172], the equation

$$\frac{\partial p}{\partial t} - \Delta p = p(p - \theta(t))(1 - p)$$

is considered, but this time, it is the Allee parameter  $\theta = \theta(t)$  that is the control parameter and the target is a travelling wave solution.

In [5], an optimal control problem for the equation without diffusion

$$\frac{\partial p}{\partial t} = f(p) + u(t),$$

and with an interior control  $u$  (rather than a boundary one) is considered. We underline that, in their study,  $u$  only depends on the time, and not on the space variable.

Finally, we mention [146], in which the existence of traveling-waves for the gene-flow model ( $H_1$ )

is established and (non)-existence and properties of traveling-waves solutions for the  $(H_2)$  model are studied. The authors prove that, under certain assumptions on the heterogeneity  $N$  (for instance, a high exponential growth on a large enough interval of  $\mathbb{R}$ ), the invasion of the front is blocked. This result seems loosely to the lack of controllability in a rapidly varying environment, see Theorem 5.1.4.

### 5.1.3 STATEMENT OF THE MAIN CONTROLLABILITY RESULTS

We recall that we work with Equation (5.2)

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p - 2\langle \nabla \ln(N), \nabla p \rangle = f(p) & \text{in } \mathbb{R}_+ \times \Omega, \\ p = u(t, x) & \text{on } (0, T) \times \partial\Omega, \\ p(t = 0, \cdot) = p_0 & \text{in } \Omega, \end{cases}$$

where  $u \in [0, 1]$  and that we want to control any initial datum to 0,  $\theta$  or 1.

#### 5.1.3.1 A BRIEF REMARK ON THE STATEMENT OF THE THEOREMS

We are going to present controllability and non-controllability results for the gene-flow models and the spatially heterogeneous ones. Regarding obstructions to controllability, the main obstacles are the existence of non-trivial steady-states, namely solutions to

$$-\Delta\varphi - 2\left\langle \frac{\nabla N}{N}, \nabla\varphi \right\rangle = f(\varphi) \text{ in } \Omega$$

associated with the boundary conditions  $\varphi = 0$  or  $\varphi = 1$ . However, given that the existence of non-trivial solutions for the Dirichlet boundary conditions  $\varphi = 0$  is obtained through a sub and super solution methods, the natural quantity appearing is the inradius of the domain, i.e

$$\rho_\Omega = \sup\{r > 0, \exists x \in \Omega, \mathbb{B}(x, r) \subset \Omega\},$$

while non-existence of non-trivial solutions is usually done through the study of the first Laplace-Dirichlet eigenvalue

$$\lambda_1^D(\Omega) := \inf_{u \in W_0^{1,2}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2},$$

which explains why both quantities  $\rho_\Omega$  and  $\lambda_1^D(\Omega)$  appear in the statements. Using Hayman-type inequalities, see [30], we could rewrite  $\lambda_1^D(\Omega)$  in terms of the inradius when the set  $\Omega$  is convex. Indeed, it is proved in [30, Proposition 7.75] that, when  $\Omega$  is a convex set with  $\rho_\Omega < \infty$  then

$$\frac{1}{c\rho_\Omega^2} \leq \lambda_1^D(\Omega) \leq \frac{C}{\rho_\Omega^2},$$

so that the theorems can be recast in terms of inradius only in the case of convex domains.

#### 5.1.3.2 GENE-FLOW MODELS

For the gene flow model  $(H_1)$ , i.e when  $N$  assumes the form

$$N = N(u),$$



the main equation of (5.2) reads

$$\frac{\partial p}{\partial t} - \Delta p - 2 \frac{N'}{N}(p) |\nabla p|^2 = f(p).$$

Then the controllability properties of the equation are the same as in [163]:

**THEOREM 5.1.1** *Let, for any  $\Omega \subset \mathbb{R}^d$ ,  $\rho_\Omega$  be its inradius:*

$$\rho_\Omega = \sup\{r > 0, \exists x \in \Omega, \mathbb{B}(x, r) \subset \Omega\}. \quad (5.5)$$

When  $N$  satisfies  $(H_1)$ , there exists  $\rho^* = \rho_*(f)$  such that, for any smooth bounded domain  $\Omega$ ,

1. Lack of controllability for large inradii: If  $\rho_\Omega > \rho^*$ , then (5.2) is not controllable to 0 in (in)finite time in the sense of Definition 5.1.1: there exist initial data  $0 \leq p_0 \leq 1$  such that, for any control  $u$  satisfying the constraints (5.3), the solution  $p$  of (5.2) does not converge to 0 as  $t \rightarrow \infty$ .
2. Controllability for large Dirichlet eigenvalue If  $\lambda_1^D(\Omega) > \|f\|_{L^\infty}$ , then (5.2) is controllable to 0, 1 in infinite time for any initial datum  $0 \leq p_0 \leq 1$ , and to  $\theta$  in finite time for any initial datum  $0 \leq p_0 \leq 1$ .

Hence the situation is exactly the same as in [163, 157], and we give, in Figure 5.2 and 5.3 schematic representations of domains where controllability might hold or not.

**REMARK 5.2** *The result makes sense: even if the domain has a large measure, if it is also very thin, it makes sense that a boundary control should work while if it has a big bulge, it is intuitive that a lack of boundary controllability should occur:*

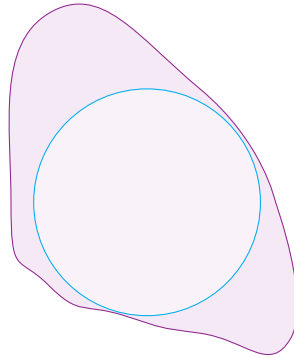


Figure 5.2 – A domain with a large inradius, for which constrained boundary control does not enable us to control the population to an intermediate trait.

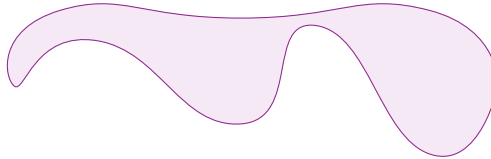


Figure 5.3 – A domain with a large eigenvalue, for which constrained boundary control enables us to control the population to an intermediate trait.

5.1.3.3 SPATIALLY HETEROGENEOUS MODELS

In this case, we work under assumption  $H_2$ , i.e with  $N = N(x)$  in  $\Omega$ .

As explained in the introduction, we need to distinguish between two cases: that of a slowly varying environment and that of sharp changes in the environment.

**Slowly varying environment** In the first part of this paragraph, we consider the case of a slowly varying total population size: we consider, for a homogeneous steady state  $z_a \equiv a$ ,  $a \in \{0, \theta, 1\}$ , a function  $n \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R})$  and a parameter  $\varepsilon > 0$  the control system

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p - \varepsilon \langle \nabla n, \nabla p \rangle = f(p) & \text{in } \mathbb{R}_+ \times \Omega, \\ p = u(t, x) & \text{on } \partial\Omega, \\ 0 \leq u \leq 1, \\ p(t = 0, \cdot) = p_0, 0 \leq p_0 \leq 1, \end{cases} \quad (5.6)$$

which models an environment with small spatial changes in the total population size; this amounts to requiring that

$$\left| \frac{\nabla N}{N} \right| \ll 1,$$

where  $N$  satisfies  $(H_2)$ . Indeed, we can then formally write

$$N \approx N_0 + \frac{\varepsilon}{2} n(x),$$

where  $N_0$  is a constant<sup>1</sup>.

**REMARK 5.3** For simplicity, we assume that  $n$  is defined on  $\mathbb{R}^d$  rather than on  $\Omega$ . Since we already assumed that  $N$  was  $\mathcal{C}^1$ , this amounts to requiring that  $n$  can be extended in a  $\mathcal{C}^1$  function outside of  $\Omega$ , which once again would follow from regularity assumptions on  $\Omega$ .

**THEOREM 5.1.2** Let, for any  $\Omega \subset \mathbb{R}^d$ ,  $\rho_\Omega$  be its inradius (defined in Equation (5.5)).

Let  $n \in \mathcal{C}^1(\mathbb{R}^d)$ .

1. *Lack of controllability for large inradii:* There exists  $\rho^* = \rho_*(n, f) > 0$  such that if  $\rho_\Omega > \rho^*$ , then (5.6) is not controllable to 0 in (in)finite time in the sense of Definition 5.1.1: there exist initial data  $p_0$  such that, for any control  $u$  satisfying the constraints (5.3), the solution  $p$  of (5.6) does not converge to 0 as  $t \rightarrow \infty$ .
2. *Controllability for large Dirichlet eigenvalue and small spatial variations:* If  $\lambda_1^D(\Omega) > \|f\|_{L^\infty}$ , there exists  $\varepsilon_* = \varepsilon_*(n, f, \Omega)$  such that, when  $\varepsilon \leq \varepsilon_*$ , the Equation (5.6) is controllable to 0 and 1 in infinite time  $f$  and to  $\theta$  in finite time in the sense of Definition 5.1.1.

To prove this theorem, we have to introduce perturbative arguments to the staircase method of [58], which we believe sheds a new light on this method as well as on the influence of spatial heterogeneity on reaction-diffusion equations.

**The case of radial drifts** The previous result, however general, is proved using a very implicit method that does not enable us to give explicit bounds on the perturbation  $\varepsilon$ . In the case where the total population size  $n : \Omega \rightarrow \mathbb{R}_+^*$  can be extended into a radial function  $n : \mathbb{R}^d \rightarrow \mathbb{R}_+^*$  we can give en

<sup>1</sup>We can assume, without loss of generality, that  $N_0 = 1$ . Indeed, the equation (5.2) is invariant under the scaling  $N \mapsto \lambda N$  where  $\lambda \in \mathbb{R}_+^*$ .

explicit bound on the decay rate of  $N$  to ensure the controllability of

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p - \langle \frac{\nabla N}{N}(x), \nabla p \rangle = f(p) & \text{in } \Omega \times (0, T), \\ p = u(t, x) & \text{on } \partial\Omega \times (0, T), \\ 0 \leq p, u(t, x) \leq 1, \\ p(t = 0, \cdot) = \varphi_0, 0 \leq \varphi_0 \leq 1, \end{cases} \quad (5.7)$$

In other words, when the total population size is the restriction to the domain  $\Omega$  of a radial function, we can obtain controllability results.

**THEOREM 5.1.3** *Let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^d$ . Let  $N \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}_+^*)$ ,  $\inf N > 0$  and  $N$  be radially symmetric. Let*

$$\lambda_1^D(\Omega, N) := \inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_{\Omega} N^2 |\nabla u|^2}{\int_{\Omega} N^2 u^2}$$

be the weighted eigenvalue associated with  $N$ .

If

$$\|f'\|_{L^\infty} \leq \lambda_1^D(\Omega, N) \quad (5.8)$$

and if

$$N'(r) \geq -\frac{d-1}{2r}N(r), \quad (A_1)$$

then the equation (5.7) is controllable to  $z_0$  in infinite time and to  $\theta$  in finite time, for any initial datum  $0 \leq p_0 \leq 1$ .

This Theorem is proved using energy methods and adapting the proofs of [163].

### Lack of controllability for rapidly varying total population size: blocking phenomena

As mentioned, the lack of controllability occurs when barriers appear. For instance, if a non-trivial solution to

$$\begin{cases} -\Delta\varphi - 2\langle \frac{\nabla N}{N}, \nabla\varphi \rangle = f(\varphi) & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

exists, then it must reach its maximum above  $\theta$  and thus, from the maximum principle, it is not possible to drive an initial datum  $p_0 \geq \varphi_0$  to 0 with constrained controls. This kind of counter-examples appear when the drift is absent, see [163, 157]. They are usually constructed by means of sub and super solutions of the equation. What is more surprising however is that adding a drift actually leads to the existence of non-trivial solutions to

$$\begin{cases} -\Delta\varphi_1 - 2\langle \frac{\nabla N}{N}, \nabla\varphi_1 \rangle = f(\varphi_1) & \text{in } \Omega, \\ \varphi = 1 & \text{on } \partial\Omega, \end{cases}$$

which never happens when no drift is present, meaning that driving the population from an initial datum  $p_0 \leq \varphi_1$  to  $z_1$  is impossible. Here, we need to carry out a precise analysis of the equation: Equation (5.11) has a variational formulation but since  $z_1 \equiv 1$  is always a global minimizer of the natural energy associated with Equation (5.11), using an energy argument is not possible.

In this paragraph we give an explicit example of some  $N$  in the one-dimensional case such that the equation is not controllable to either 0,  $\theta$  or 1. Let, for any  $\sigma > 0$ , the gaussian of variance  $\sqrt{\sigma}$  be defined as

$$N_\sigma(x) := \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}},$$

so that the control problem (5.2) becomes, in the one dimensional case

$$\left\{ \begin{array}{l} \frac{\partial \varphi}{\partial t} - \frac{\partial \varphi}{\partial x^2} + \frac{2x}{\sigma} \frac{\partial \varphi}{\partial x} = f(\varphi) \text{ in } \Omega, \\ \varphi(-L) = u(t, -L), \varphi(L) = u(t, L), \\ 0 \leq \varphi, u(t, x) \leq 1, \\ \varphi(t = 0, \cdot) = \varphi_0, 0 \leq \varphi_0 \leq 1, \\ \varphi(t = T, \cdot) = z_a, T \in ]0; +\infty], \end{array} \right. \quad (5.9)$$

Introduce the following barrier equations (i.e, if there exists a non-trivial solution to these equations, controllability might fail) on some interval  $[-L, L]$ :

$$\left\{ \begin{array}{l} -\frac{\partial^2 \varphi}{\partial x^2} + \frac{2x}{\sigma} \frac{\partial \varphi}{\partial x} = f(\varphi) \text{ in } [-L, L], \\ \varphi(\pm L) = 0, \\ 0 \leq \varphi \leq 1, \end{array} \right. \quad (5.10)$$

and

$$\left\{ \begin{array}{l} -\frac{\partial^2 \varphi}{\partial x^2} + \frac{2x}{\sigma} \frac{\partial \varphi}{\partial x} = f(\varphi) \text{ in } [-L, L], \\ \varphi(\pm L) = 1, \\ 0 \leq \varphi \leq 1, \end{array} \right. \quad (5.11)$$

**THEOREM 5.1.4** *1. Existence of critical lengths  $L_\sigma$ : For any  $\sigma > 0$ , there exists  $L_\sigma(0) > 0$  (resp.  $L_\sigma(1)$ ) such that the Equation (5.10) (resp. Equation (5.11)) has a non-trivial solution in  $\Omega = [-L_\sigma(0); L_\sigma(0)]$  (resp.  $[-L_\sigma(1); L_\sigma(1)]$ ). As a consequence, for  $a = 0, 1$ , Equation (5.9) is not controllable in infinite time to a or  $\theta$  on  $[-L_\sigma(a); L_\sigma(a)]$ . For any  $L \geq L_\sigma^*(1)$ , Equation (5.11) has a non-trivial solution on  $[-L, L]$ .*

*2. Asymptotic analysis of  $L_\sigma$ : Define  $L_\sigma(a)^*$  as the minimal length of the interval such that controllability fails on  $[-L_\sigma(a)^*; L_\sigma(a)^*]$ , then*

$$L_\sigma(1)^* \xrightarrow{\sigma \rightarrow \infty} +\infty, L_\sigma(1)^* \xrightarrow{\sigma \rightarrow 0} 0, L_\sigma^*(0) \xrightarrow{\sigma \rightarrow 0} 0.$$

*In other words, the sharper the transition, the smaller the interval where lack of controllability occurs.*

*3. Double-blocking phenomenon: There exists  $L_\sigma^{**}$  such that both Equations (5.10) and (5.11) have a non-trivial solution on  $[-L_\sigma^{**}, L_\sigma^{**}]$ . Equation (5.9) is not controllable to either 0,  $\theta$  or 1 on  $[-L_\sigma^{**}, L_\sigma^{**}]$ .*

We illustrate the existence of non-trivial solutions in Figures 5.4, 5.5, 5.6 and 5.7. To prove this Theorem, we will study the energy

$$\mathcal{E} : (u, v) \mapsto \frac{1}{2}v^2 + F(u).$$

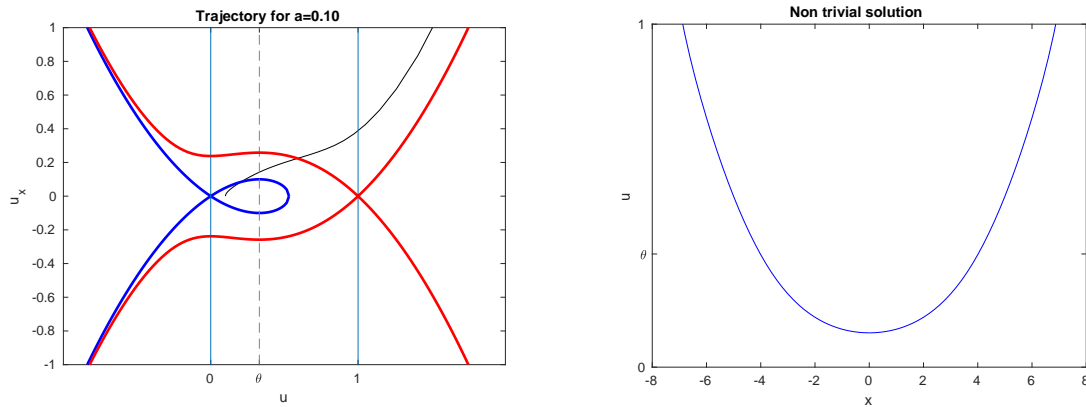


Figure 5.4 –  $\sigma = 40$  and  $f(s) = s(1 - s)(s - \theta)$ ,  $\theta = 0.33$ . Phase portrait (Left): the trajectory corresponding to the nontrivial solution is in black, the energy set  $\{\mathcal{E} = F(1)\}$  in red, the energy set  $\{\mathcal{E} = F(0)\}$  in blue. Nontrivial solution of (5.11) (Right).

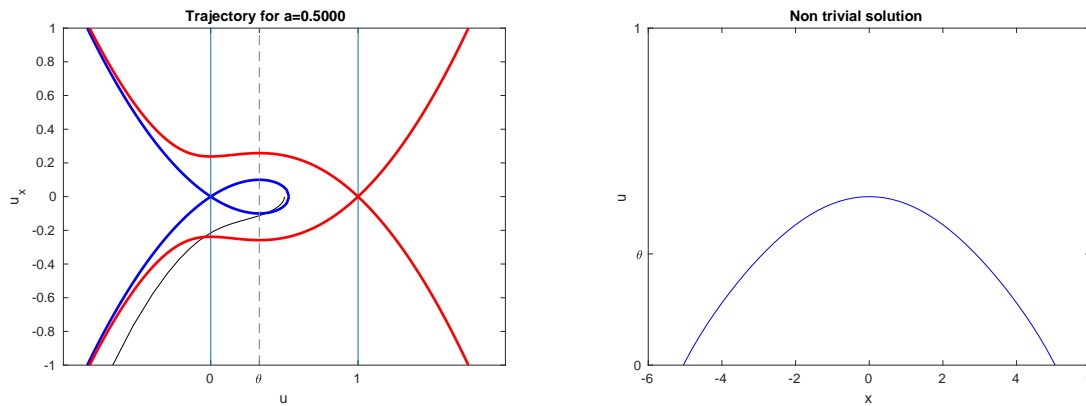


Figure 5.5 – Same class of parameters  $\sigma, \theta, f$ . Phase portrait (Left): the trajectory corresponding to the nontrivial solution is in black, the energy set  $\{\mathcal{E} = F(1)\}$  in red, the energy set  $\{\mathcal{E} = F(0)\}$  in blue. Nontrivial solution of (5.10) (Right).

We also observe this "double-blocking" phenomenon (i.e the existence of non-trivial solutions to (5.11) and (5.10) in the same interval) numerically, when trying to control an initial datum to  $\theta$ :

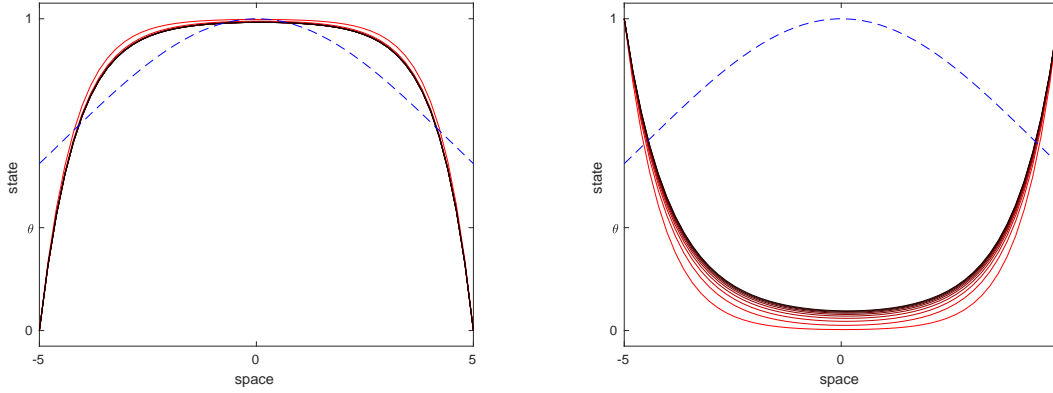


Figure 5.6 –  $N(x) = e^{-\frac{x^2}{\sigma}}$ ,  $\sigma = 40$ ,  $L = 5$ , (Left) initial datum  $u_0 = 1$ , (Right) initial datum  $u_0 = 0$ .

There can also be controllability from 0 to  $\theta$ , but not from 1 to  $\theta$ , as shown, numerically, below:

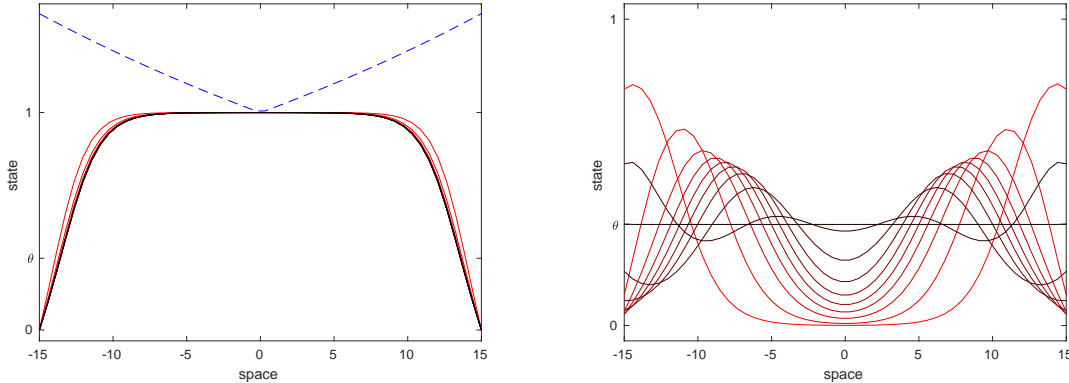


Figure 5.7 –  $N(x) = e^{\frac{|x|}{\sigma}}$ ,  $\sigma = 40$ ,  $T = 150$ ,  $L = 15$ . (Left) initial datum  $u_0 = 1$ , (Right) initial datum  $u_0 = 0$ .

**REMARK 5.4** As noted, these sharp changes in the total population size have been known, since [146], to provoke blocking phenomena for the traveling-waves solutions of the bistable equation, and our results seem to lead to the same kind of interpretation: when a sudden change occurs in  $N$ , it is hopeless for a population coming from the boundary to settle everywhere in the domain. We prove this result using a careful analysis of the phase portrait for the non-autonomous system to establish existence of non-trivial solutions to the steady-state equations with homogeneous Dirichlet boundary conditions equal to either 0 or 1.

We note that our proofs could be extended to the multi-dimensional case, when considering a multi-dimensional gaussian distribution.

## 5.2 PROOF OF THEOREM 5.1.1: GENE-FLOW MODELS

*Proof of Theorem 5.1.1.* The proof consists in a simple transformation of the equation, already used in [146, Proof of Theorem 1], which will turn the equation into the classical bistable reaction diffusion equation already considered in [163]. We consider the equation

$$\frac{\partial p}{\partial t} - \Delta p - 2\frac{N'}{N}(p)|\nabla p|^2 = f(p), \quad (5.12)$$

and introduce the anti-derivative of  $N$  as

$$\mathcal{N} : x \mapsto \int_0^x N^2(\xi) d\xi.$$

We first note that multiplying  $N$  by any factor  $\lambda$  leaves the equation (5.12) invariant. We thus fix

$$\int_0^1 N^2(\xi) d\xi = 1.$$

Multiplying (5.12) by  $N^2$  we get

$$N^2(p) \frac{\partial p}{\partial t} - N^2(p) \Delta p - 2N'(p)|\nabla p|^2 = (\mathcal{N}(p))_t - \nabla \cdot (N^2(p) \nabla p) = (\mathcal{N}(p))_t - \Delta(\mathcal{N}(p)).$$

Hence, as  $\mathcal{N}$  is a diffeomorphism the function  $\tilde{p} := \mathcal{N}(p)$  satisfies

$$\frac{\partial \tilde{p}}{\partial t} - \Delta \tilde{p} = \tilde{f}(\mathcal{N}^{-1}(\tilde{p})) N^2(\mathcal{N}^{-1}(\tilde{p})) =: \tilde{f}(\tilde{p}).$$

However, it is easy to see that,  $f$  being bistable, so is  $\tilde{f}$ . Furthermore,  $\mathcal{N}$  is a  $\mathcal{C}^1$  diffeomorphism of  $[0, 1]$ , and it is easy to see that  $p$  is controllable to  $0, \theta$  or  $1$  if and only if  $\tilde{p}$  is controllable to  $0, \theta$  or  $1$ , and we are thus reduced to the statement of [163, Theorem 1.2], from which the conclusion follows.  $\square$

## 5.3 PROOF OF THEOREM 5.1.2: SLOWLY VARYING TOTAL POPULATION SIZE

### 5.3.1 LACK OF CONTROLLABILITY TO 0 FOR LARGE INRADIUS

We prove here the first point of Theorem 5.1.2. Recall that we want to prove that, if the inradius  $\rho_\Omega$  is bigger than a threshold  $\rho^*$  depending only on  $f$ , then equation (5.6) is not controllable to  $0$  in (in)finite time.

Following [157], we claim that this lack of controllability occurs when the equation

$$\begin{cases} -\Delta \eta - \varepsilon \langle \nabla n, \nabla \eta \rangle = f(\eta) & \text{in } \Omega, \\ \eta = 0 & \text{on } \partial\Omega, \\ 0 \leq \eta \leq 1 \end{cases} \quad (5.13)$$

has a non-trivial solution, i.e a solution such that  $\eta \neq 0$ . Indeed, we have the following Claim:

**CLAIM 5.3.1** *If there exists a non-trivial solution  $\eta \neq 0$  to (5.13), then (5.6) is not controllable to  $0$  in infinite time.*

*Proof of Claim 5.3.1.* This is an easy consequence of the maximum principle. Indeed, let  $\eta$  be a non-trivial solution of (5.13) and let  $p_0$  be any initial datum satisfying

$$\eta \leq p_0 \leq 1.$$

Let  $u : \mathbb{R}_+ \times \partial\Omega \rightarrow [0, 1]$  be a boundary control. Let  $p^u$  be the solution of

$$\begin{cases} \frac{\partial p^u}{\partial t} - \Delta p^u - \varepsilon \langle \nabla n, \nabla p^u \rangle = f(p^u) & \text{in } \Omega, \\ p^u = u & \text{on } \mathbb{R}_+^* \times \partial\Omega, \\ p^u(t = 0, \cdot) = p_0, \end{cases} \quad (5.14)$$

From the parabolic maximum principle [158, Theorem 12], we have for every  $t \in \mathbb{R}_+$ ,

$$\eta(t, \cdot) \leq p^u(t, \cdot),$$

so that  $p^u$  cannot converge to 0 as  $t \rightarrow \infty$ . This concludes the proof.  $\square$

It thus remains to establish the following Lemma:

**LEMMA 5.1** *There exists  $\rho^* = \rho^*(n, f)$  such that, for any  $\Omega$  satisfying*

$$\rho_\Omega > \rho^*$$

*there exists a non-trivial solution  $\eta \neq 0$  to equation (5.13).*

Since the proof of this Lemma is a straightforward adaptation of [163, Proposition 3.1], we postpone it to Appendix 5.A.

### 5.3.2 CONTROLLABILITY TO 0 AND 1

We now prove the second part of Theorem 5.1.2, which we rewrite as the following claim:

**CLAIM 5.3.2** 1. Controllability to 0: *There exists  $\rho_* = \rho_*(n, f)$  such that, for any  $\Omega$ , if  $\rho_\Omega \leq \rho_*$ , Equation (5.2) is controllable to 0 in infinite time.*

2. Controllability to 1: *There exists  $\bar{\varepsilon} > 0$  such that, for any  $\varepsilon \leq \bar{\varepsilon}$ , Equation (5.2) is controllable to 1 in infinite time.*

*Proof of Claim 5.3.2.* 1. Controllability to 0:

The key part is the following thing:

There exists  $\rho_* > 0$  such that, if  $\rho_\Omega < \rho_*$ , then  $y \equiv 0$  is the only solution to

$$\begin{cases} -\Delta y - \varepsilon \langle \nabla n, \nabla y \rangle = f(y), & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.15)$$

Indeed, assuming that the uniqueness result (5.15) holds, consider the static control  $u \equiv 0$  and the solution of

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p - \varepsilon \langle \nabla n, \nabla p \rangle = f(p), & \text{in } \mathbb{R}_+ \times \Omega, \\ p = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ p(t = 0, \cdot) = p^0 & \text{in } \Omega. \end{cases}$$



From standard parabolic regularity and the Arzela-Ascoli theorem,  $p$  converges uniformly in  $\Omega$  to a solution  $\bar{p}$  of

$$\begin{cases} -\Delta \bar{p} - \varepsilon \langle \nabla n, \nabla \bar{p} \rangle = f(\bar{p}), & \text{in } \mathbb{R}_+ \times \Omega \\ \bar{p} = 0 & \text{on } (0, T) \times \partial\Omega. \end{cases}$$

However, by the uniqueness result (5.15), we have  $\bar{p} = 0$ , whence

$$p(t, \cdot) \xrightarrow[t \rightarrow \infty]{\mathcal{C}^0(\bar{\Omega})} 0,$$

which means that the static strategy drives  $p_0$  to 0.

Finally, we claim that (5.15) follows from spectral arguments: first of all, uniqueness holds for

$$\begin{cases} -\Delta y - \varepsilon \langle \nabla n, \nabla y \rangle = f(y), & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases}$$

if the first eigenvalue  $\lambda(\varepsilon, n, \Omega)$  of the operator

$$\mathcal{L}_{\varepsilon, n} = -\nabla \cdot (e^{\varepsilon n} \nabla u)$$

with Dirichlet boundary conditions satisfies

$$\lambda_1(\varepsilon, n, \Omega) > \|f'\|_{L^\infty} e^{\varepsilon \|n\|_{L^\infty}},$$

as is standard from classical theory for non-linear elliptic PDE, see [21].

We now notice that,  $n$  being positive, the Rayleigh quotient formulation for the eigenvalue

$$\lambda(\varepsilon, n, \Omega) = \inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_{\Omega} e^{\varepsilon n} |\nabla u|^2}{\int_{\Omega} u^2}$$

yields that

$$\lambda(\varepsilon, n, \Omega) \geq \lambda_1^D(\Omega)$$

where  $\lambda_1^D(\Omega)$  is the first eigenvalue of the Laplace operator with Dirichlet boundary conditions. Thus we are reduced to checking that

$$\lambda_1^D(\Omega) > \|f'\|_{L^\infty} e^{\varepsilon \|n\|_{L^\infty}},$$

as claimed. If the condition  $\lambda_1^D(\Omega) > \|f'\|_{L^\infty}$ , taking the limit as  $\varepsilon \rightarrow 0$  yields the desired result.

2. Controllability to 1 Using the same arguments, we claim that controllability to 1 can be achieved through the static control  $u \equiv 1$  provided the only solution to

$$\begin{cases} -\Delta \bar{p} - \varepsilon \langle \nabla n, \nabla \bar{p} \rangle = f(\bar{p}), & \text{in } \Omega, \\ \bar{p} = 1 & \text{on } \partial\Omega, \\ 0 \leq \bar{p} \leq 1 \end{cases} \quad (5.16)$$

is  $\bar{p} \equiv 1$ .

We already know (see [163, 157]) that uniqueness holds for  $\varepsilon = 0$ . Now this implies that uniqueness holds for  $\varepsilon$  small enough. Indeed, argue by contradiction and assume that, for every  $\varepsilon > 0$  there exists a non-trivial solution  $\bar{p}_\varepsilon$  to (5.16). Since  $\bar{p}_\varepsilon \neq 1$ ,  $p$  reaches a minimum at some  $\bar{x}_\varepsilon \in \Omega$ , and so

$$f(\bar{p}_\varepsilon(\bar{x}_\varepsilon)) < 0$$

which means that

$$\bar{p}_\varepsilon(\bar{x}_\varepsilon) < \theta.$$

Standard elliptic estimates entail that, as  $\varepsilon \rightarrow 0$ ,  $p_\varepsilon$  converges in  $W^{1,2}(\Omega)$  and in  $\mathcal{C}^0(\bar{\Omega})$  to  $\bar{p}$  satisfying

$$\begin{cases} -\Delta \bar{p} = f(\bar{p}) & \text{in } \Omega, \\ \bar{p} = 1 & \text{on } \partial\Omega, \\ 0 \leq \bar{p} \leq 1 \end{cases} \quad (5.17)$$

and such that there exists a point  $\bar{x}$  satisfying

$$\bar{p}(\bar{x}) < \theta$$

which is a contradiction since we now uniqueness holds for (5.16). This concludes the proof.  $\square$

### 5.3.3 PROOF OF THE CONTROLLABILITY TO $\theta$ FOR SMALL INRADIUSSES

#### 5.3.3.1 STRUCTURE OF THE PROOF: THE STAIRCASE METHOD

We recall that we want to control the semilinear heat equation

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p - \varepsilon \langle \nabla n, \nabla p \rangle = f(p) & \text{in } \Omega, \\ p = u(t) & \text{on } \partial\Omega, \\ p(t=0, \cdot) = y_0 \end{cases} \quad (5.18)$$

to  $z_\theta \equiv \theta$ .

We give the following local exact controllability result from [157, Lemma 1] or [154, Lemma 2.1], which is the starting point of the method:

**PROPOSITION 5.1** [Local exact controllability] *Let  $T > 0$ . There exists  $\delta_1 > 0$  such that for all steady state  $y_f$  of (5.18), for all  $0 \leq y_d \leq 1$  satisfying*

$$\|y_d - y_f\|_{\mathcal{C}^0} \leq \delta_1$$

*then (5.18) is controllable from  $y_d$  to  $z_\theta$  in finite time  $T < \infty$  through a control  $u$ . Furthermore, letting  $\bar{u} = y_f|_{\partial\Omega}$ , the control function  $u = u(t)$  satisfies*

$$\|u(t) - \bar{u}\|_{\mathcal{C}^0(\partial\Omega)} \leq C(T)\delta_1 \quad (5.19)$$

*for some constant  $C(T) > 0$ .*

We now assume that  $\rho_\Omega \leq \rho^*$ , that is, thanks to Claim 5.3.2, we assume that we have uniqueness for the equation

$$\begin{cases} -\Delta y - \varepsilon \langle \nabla n, \nabla y \rangle = f(y) & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega \end{cases}$$

The way to prove controllability is to proceed along two different steps:

- Step 1: Starting from any initial condition  $0 \leq p_0 \leq 1$ , we first set the static control

$$u(t, x) = 0.$$

Since  $n$  is  $\mathcal{C}^1$ , standard parabolic estimates and the Arzela-Ascoli theorem ensures that the

solution  $p^u$  of (5.6) converges uniformly, as  $t \rightarrow \infty$  to a solution  $\bar{\eta}$  of

$$\begin{cases} -\Delta\bar{\eta} - \varepsilon\langle \nabla n, \nabla\bar{\eta} \rangle = f(\bar{\eta}) & \text{in } \Omega, \\ \bar{\eta} = 0 & \text{on } \partial\Omega, \\ 0 \leq \bar{\eta} \leq 1 \end{cases} \quad (5.20)$$

However, from Claim 5.3.2,  $\rho_\Omega \leq \rho_*(n, f)$  implies that  $z_0 \equiv 0$  is the unique solution of this equation. Thus, this static control guarantees that, for every  $\delta_1 > 0$ , there exists  $T_1 > 0$  such that, for any  $t \geq T_1$

$$\|p^u(t, \cdot)\|_{L^\infty} \leq \delta.$$

- Step 2: We prove that there exists a steady state  $p_0$  of (5.18) such that

$$0 < \inf_{x \in \Omega} p_0(x) \leq \|p_0\|_{L^\infty} \leq \frac{\delta}{2}$$

and, applying Proposition 5.1, we drive  $p^u(T_1, \cdot)$  to  $p_0$  in finite time.

- Step 3: If we can drive  $p_0$  to  $\theta$ , then we are done. Thus, we are, in this setting, reduced to the controllability of initial datum in a small neighbourhood of 0 to  $\theta$ . This is what we are going to prove, using the staircase method.

**The staircase method** The key idea to do that is the same as in [157], that is, we want to use the staircase method of Coron and Trélat, see [58] for the one-dimensional case (which uses quasi-static deformations) and [154] for a full derivation. We briefly recall the most important features of this method and the way we wish to apply it to our problem.

Assume that there exists a  $\mathcal{C}^0$ -continuous path of steady-states of (5.18)  $\Gamma = \{p_s\}_{s \in [0,1]}$  such that  $p_0 = y_0$  and  $p_1 = y_1$ .

Then (5.18) is controllable from  $y_0$  to  $y_1$  in finite time. Indeed, as is usually done, we consider a subdivision

$$0 = s_{i_1} < \dots < s_{i_N} = 1$$

of  $[0, 1]$  such that

$$\forall j \in \{0, \dots, N_1\}, \|p_{s_{i_j}} - p_{s_{i_{j+1}}}\|_{\mathcal{C}^0(\Omega)} \leq \delta_1$$

where  $\delta_1$  is the controllability parameter given by the local exact controllability result. We then control each  $p_{s_{i_j}}$  to  $p_{s_{i_{j+1}}}$  in finite time using Proposition 5.1.

This result does not necessarily yield constrained controls, but, thanks to estimate (5.19) we can enforce these constraints, by choosing a control parameter  $\delta_1$  small enough.

Thus, the tricky part seems to be finding a continuous path of steady-states for the perturbed system with slowly varying total population size (5.6). However, it suffices to have a finite numbers of steady-states that are close enough to each other, starting at  $y_0$  and ending at  $y_1$ . We represent the situation in Figure 5.8 below:

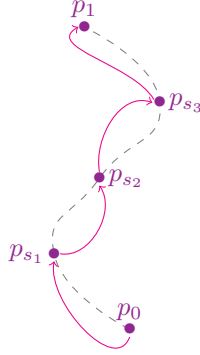


Figure 5.8 – The dashed curve is the path of steady states (for instance in  $W^{1,2}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ ), and the points are the close enough steady states. We represent the exact control in finite time  $T$  with the pink arrows.

### 5.3.3.2 PERTURBATION OF A PATH OF STEADY-STATES

We are going to perturb the path of steady-states using the implicit function Theorem in order to get a sequence of close enough steady-states, so that the previous staircase strategy still applies.

**REMARK 5.5** Here, if we were to try and prove, for  $\varepsilon$  small enough, the existence of a continuous path of steady states, the idea would be to start from a path  $(p_{s,0})_{s \in [0,1]}$  for  $\varepsilon = 0$  (which we know exists from [163, 157]) and to try and perturb it into a path for  $\varepsilon > 0$  small enough, thus giving us a path  $\{p_{\varepsilon,s}\}_{s \in [0,1], \varepsilon > 0}$ . However, doing it for the whole path requires some kind of implicit function theorem or, at least, some bifurcation argument. Namely, to construct the path, we would need to ensure that either

$$\mathcal{L}^{s,\varepsilon} := -\nabla \cdot (e^{\varepsilon n} \nabla) - e^{\varepsilon n} f'(p^{0,s})$$

has no zero eigenvalue for  $\varepsilon = 0$  or that it has a non-zero crossing number (namely, a non zero number of eigenvalues enter or leave  $\mathbb{R}_+^*$  as  $\varepsilon$  increases from  $-\delta$  to  $\delta$ ). In the first case, the implicit function theorem would apply; in the second case, Bifurcation Theory (see [111, Theorem II.7.3]) would ensure the existence of a branch  $p_{\varepsilon,s}$  for  $\varepsilon$  small enough. These conditions seem too hard to check for a general path of continuous of steady states.

Hence, we focus on perturbing a finite number of points close enough on the path since, as we noted, this is enough to ensure exact controllability.

We will strongly rely on the properties of the path of steady-states built in [163, 157].

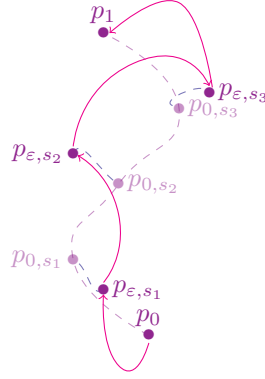


Figure 5.9 – In dark purple, the perturbed steady states, linked to the unperturbed steady states. We do not know whether or not a continuous path of steady states linking these new states exists; however, such points enable us to do exact controllability again and to apply the stair case method.

Henceforth, our goal is the following proposition:

**PROPOSITION 5.2** *Let  $\delta > 0$ . There exists  $N > 0$  and  $\bar{\varepsilon} > 0$  such that, for any  $\varepsilon \leq \bar{\varepsilon}$ , there exists a sequence  $\{p_{\varepsilon, i}\}_{i=1, \dots, N}$  satisfying:*

- For every  $i = 1, \dots, N$ ,  $p_{\varepsilon, i}$  is a steady-state of (5.6):

$$-\Delta p_{\varepsilon, i} - \varepsilon \langle \nabla n, \nabla p_{\varepsilon, i} \rangle = f(p_{\varepsilon, i}),$$

- $p_{\varepsilon, N} = z_\theta \equiv \theta$ ,  $0 < \inf p_{\varepsilon, 1} \leq \|p_{\varepsilon, 1}\|_{L^\infty} \leq \delta$

- For every  $i = 1, \dots, N$ ,

$$\frac{\delta}{2} \leq p_{\varepsilon, i} \leq \|p_{\varepsilon, i}\|_{L^\infty} \leq 1 - \frac{\delta}{2},$$

- For every  $i = 1, \dots, N - 1$ ,

$$\|p_{\varepsilon, i+1} - p_{\varepsilon, i}\|_{L^\infty} \leq \delta.$$

As explained, this Proposition gives us the desired conclusion:

**CLAIM 5.3.3** *Proposition 5.2 implies the controllability to  $\theta$  for any initial datum  $p_0$  in Equation (5.2).*

Before we prove Proposition 5.2, we recall how the paths of steady-states are constructed when  $\varepsilon = 0$ .

**Known constructions of a path of steady-states** For the multi-dimensional case, it has been shown in [163] that one can construct a path of steady-states linking  $z_0 \equiv 0$  to  $z_\theta \equiv \theta$  in the following way: let  $\Omega$  be the domain where the equation is set and let  $R_\Omega > 0$  be such that

$$\Omega \subseteq \mathbb{B}(0; R_\Omega).$$

The path of steady state is defined as follows: first of all, if uniqueness holds for

$$\begin{cases} -\Delta y = f(y) \text{ in } \mathbb{B}(0; R_\Omega), \\ y = 0. \end{cases}$$

then, for  $\eta > 0$  small enough, there exists a unique solution to

$$\begin{cases} -\Delta y_\eta = f(y_\eta) & \text{in } \mathbb{B}(0; R_\Omega) \\ y_\eta = \eta & \text{on } \partial\mathbb{B}(0; R_\Omega). \end{cases}$$

Define, for any  $s \in [0, 1]$ , let  $p^{0,s}$  be the unique solution to the problem

$$\begin{cases} -\Delta p^{0,s} = f(p^{0,s}) & \text{in } \mathbb{B}(0; R), \\ p^{0,s}(0) = s\theta + (1-s)y_\eta(0), \\ p^{0,s} \text{ is radial.} \end{cases} \quad (5.21)$$

Using the polar coordinates, the authors prove that the equation above has a unique solution, and that this solution is admissible, i.e that we even have, for any  $0 < s_0 < 1$ ,

$$0 < \inf_{s \in [s_0; 1], x \in \mathbb{B}(0; R)} p^{0,s}(x) \leq \sup_{s \in [0, 1], x \in \mathbb{B}(0; R)} p^{0,s}(x) < 1.$$

This is done using energy type methods and gives a path on  $\mathbb{B}(0; R_\Omega)$ . To construct the path on  $\Omega$ , it suffices to set

$$\tilde{p}^{0,s} := p^{0,s}|_\Omega.$$

Furthermore, by elliptic regularity or by studying the equation in polar coordinates, we see that, for every  $s \in [0, 1]$ ,

$$p^{0,s} \in \mathcal{C}^{2,\alpha}(\mathbb{B}(0; R_\Omega))$$

for any  $0 < \alpha < 1$ . Instead of perturbing the functions  $\tilde{p}^{0,s} \in \mathcal{C}^{2,\alpha}(\Omega)$ , we will perturb the functions  $p^{0,s} \in \mathcal{C}^2(\mathbb{B}(0; R_\Omega))$ .

Henceforth, the parameter  $R_\Omega > 0$  is fixed and, for any  $s \in [0, 1]$ ,  $p^{0,s}$  is the unique solution to (5.21)

*Proof of Proposition 5.2.* Let  $\delta > 0$ . Let  $\{s_i\}_{i=1, \dots, N}$  be a sequence of points such that

$$0 < p^{0,s_0} \leq \|p^{0,s_0}\|_{L^\infty} \leq \frac{\delta}{2}, \quad (5.22)$$

and

$$\forall i \in \{0, \dots, N-1\}, \|p^{0,s_i} - p^{0,s_{i+1}}\|_{L^\infty} \leq \frac{\delta}{4}. \quad (5.23)$$

We define, for any  $i = 1, \dots, N$ ,

$$p_{0,i} = p^{0,s_i}.$$

Fix a parameter  $\alpha \in (0; 1)$ . We define a one-parameter family of mappings as follows: for any  $i = 1, \dots, N$ , let

$$\mathcal{F}_i : \begin{cases} \mathcal{C}^{2,\alpha}(\mathbb{B}(0; R_\Omega)) \times [-1; 1] & \rightarrow \mathcal{C}^{0,\alpha}(\mathbb{B}(0; R_\Omega)) \times \mathcal{C}^0(\partial\mathbb{B}(0; R_\Omega)), \\ (u, \varepsilon) & \mapsto (-\nabla \cdot (e^{\varepsilon n} \nabla u) - f(u)e^{\varepsilon n}, u|_{\partial\mathbb{B}(0; R_\Omega)} - p_{0,i}^i|_{\partial\mathbb{B}(0; R_\Omega)}). \end{cases}$$

We note that

$$\forall i \in \{0, \dots, N\}, \mathcal{F}_i(p_{0,i}, 0) = 0.$$

We wish to apply the implicit function theorem, which is permitted provided the operator

$$\mathcal{L}_i : u \mapsto -\Delta u - f'(p_{0,i})u$$

with Dirichlet boundary conditions is invertible. If this is the case we know that there exists a

continuous path  $p_{\varepsilon,i}$  starting from  $p_{0,i}$  such that

$$\mathcal{F}_i(p_{\varepsilon}^i, \varepsilon) = 0$$

Denoting, for any differential operator  $\mathcal{A}$  its spectrum by  $\Sigma(\mathcal{A})$ , this invertibility property amounts, thanks to elliptic regularity (see [80]) to requiring that

$$\forall i \in \{1, \dots, N\}, 0 \notin \Sigma(\mathcal{L}_i). \quad (5.24)$$

If the condition (5.24) is satisfied, then  $p_{0,i}$  perturbs into  $p_{\varepsilon}^i$  and we can define

$$\tilde{p}_{\varepsilon}^i := p_{\varepsilon}^i|_{\Omega}$$

as a suitable sequence of steady states in  $\Omega$ . Since we are working with a finite number of points, taking  $\varepsilon$  small enough guarantees

$$\forall i = 1, \dots, N, \|p_{\varepsilon,i} - p_{0,i}\|_{L^\infty} \leq \frac{\delta}{4}$$

and we would then have, for any  $i = 1, \dots, N_1$ ,

$$\begin{aligned} \|p_{\varepsilon,i+1} - p_{\varepsilon,i}\|_{L^\infty} &\leq \|p_{\varepsilon,i+1} - p_{0,i+1}\|_{L^\infty} + \|p_{0,i} - p_{\varepsilon,i}\|_{L^\infty} + \|p_{0,i+1} - p_{0,i}\|_{L^\infty} \\ &\leq \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{2} = \delta, \end{aligned}$$

which is what we require of the sequence.

Let us define the set of resonant points (i.e the points where the implicit function Theorem does not apply) as

$$\Gamma := \{j \in \{1, \dots, N\}, 0 \in \Sigma(\mathcal{L}_j)\}.$$

We note that  $0 \notin \Gamma$  because the first eigenvalue of

$$\mathcal{L}_0 = -\Delta - f'(0)$$

is positive: indeed, since  $f'(0) < 0$ , this first eigenvalue is bounded from below by the first Dirichlet eigenvalue of the ball  $\mathbb{B}(0; R)$ . Hence  $0 \notin \Gamma$ . We proceed as follows:

1. Whenever  $i \notin \Gamma$ , we can apply the implicit function Theorem to obtain the existence of a continuous path  $p_{\varepsilon,i}$  starting from  $p_{0,i}$  such that

$$p_{\varepsilon,i}|_{\partial\mathbb{B}(0;R_\Omega)} = p_{0,i}|_{\partial\mathbb{B}(0;R_\Omega)}, \mathcal{F}(p_{\varepsilon,i}, \varepsilon) = 0,$$

so that, taking  $\varepsilon$  small enough, we can ensure that, for any  $i \notin \Gamma$ ,

$$\|p_{\varepsilon,i} - p_{0,i}\|_{L^\infty} \leq \frac{\delta}{4}.$$

2. Whenever  $i \in \Gamma$ , we apply the implicit function theorem on a larger domain  $\mathbb{B}(0; R_\Omega + \tilde{\delta})$ ,  $\tilde{\delta} > 0$ .

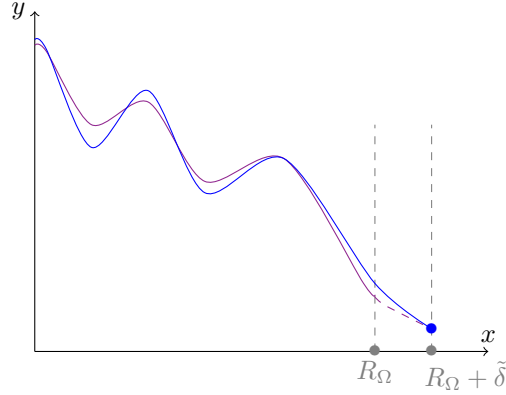


Figure 5.10 – The initial solution  $p_{0,i}$  on  $\mathbb{B}(0; R_\Omega)$  is continued into a solution on  $\mathbb{B}(0; R_\Omega + \tilde{\delta})$ , and we apply the implicit function theorem on this domain to obtain the blue curve.

Let, for any  $i \in \Gamma$ ,  $\lambda_i(k, R_\Omega)$  be the  $k$ -th eigenvalue of  $\mathcal{L}_i$  with Dirichlet boundary conditions on  $\mathbb{B}(0; R_\Omega)$ .

Let, for any  $i \in \Gamma$ ,

$$k_i := \sup \{k, \lambda_i(k, R) = 0\}.$$

Obviously, there exists  $M > 0$  such that  $k_i \leq M$  uniformly in  $i$ , since  $\lambda_i(k, R_\Omega) \rightarrow \infty$  as  $k \rightarrow \infty$ . We then invoke the monotonicity of the eigenvalues with respect to the domain. Let, for any  $\tilde{\delta}$ ,  $p_{0,i}^{\tilde{\delta}}$  be the extension of  $p_{0,i}$  to  $\mathbb{B}(0; R_\Omega + \tilde{\delta})$ ; this is possible given that  $p_{0,i}$  is given by the radial equation (5.21).

Let  $\tilde{\mathcal{L}}_i : u \mapsto -\Delta u - f'(p_{0,i}^{\tilde{\delta}})u$  and  $\tilde{\lambda}_i(\cdot, R_\Omega + \tilde{\delta})$  be its eigenvalues. By the min-max principle of Courant (see [92]) we have, for any  $k \in \mathbb{N}$  and any  $\tilde{\delta} > 0$ ,

$$\tilde{\lambda}_i(k, R_\Omega + \tilde{\delta}) < \lambda_i(k, R_\Omega).$$

Hence, for every  $i \in \Gamma$ , there exists  $\tilde{\delta}_i > 0$  small enough so that, for any  $0 < \tilde{\delta} < \tilde{\delta}_i$ ,

$$0 \notin \Sigma(\tilde{\mathcal{L}}_i).$$

We then choose  $\tilde{\delta} = \min_{i \in \Gamma} \tilde{\delta}_i$  and apply the implicit function theorem on  $\mathbb{B}(0; R_\Omega + \frac{\tilde{\delta}}{2})$ . This gives the existence of  $\tilde{\varepsilon} > 0$  such that, for any  $\varepsilon < \tilde{\varepsilon}$  and any  $i \in \Gamma$ , there exists a solution  $p_{\varepsilon,i}^{\tilde{\delta}}$  of

$$\left\{ \begin{array}{l} -\Delta p_{\varepsilon,i}^{\tilde{\delta}} - \varepsilon \langle \nabla n, \nabla p_{\varepsilon,i}^{\tilde{\delta}} \rangle = f(p_{\varepsilon,i}^{\tilde{\delta}}) \text{ in } \mathbb{B}(0; R_\Omega + \frac{\tilde{\delta}}{2}), \\ p_{\varepsilon,i}^{\tilde{\delta}} = p_{0,i} \big|_{\mathbb{S}(0; R_\Omega + \frac{\tilde{\delta}}{2})}, \\ p_{\varepsilon,i} \xrightarrow[\varepsilon \rightarrow 0]{\mathcal{C}^0(\mathbb{B}(0; R_\Omega + \frac{\tilde{\delta}}{2}))} p_{0,i} \end{array} \right. \quad (5.25)$$

Furthermore,

$$p_{0,i}^{\tilde{\delta}} \big|_{\mathbb{S}(0; R_\Omega + \delta)} \xrightarrow[\delta \rightarrow 0]{} p_{0,i} \big|_{\partial \mathbb{B}(0; R_\Omega)}.$$



Thus, by choosing  $\tilde{\delta}$  small enough, we can guarantee that, by defining

$$\tilde{p}_{\varepsilon,i} := p_{\varepsilon,i}|_{\mathbb{B}(0;R_\Omega)}$$

we have for every  $\varepsilon$  small enough

$$\|\tilde{p}_{\varepsilon,i} - p_{0,i}\|_{L^\infty} \leq \frac{\delta}{4}.$$

We note that  $\tilde{p}_{\varepsilon,i}$  does not satisfy, on  $\partial\mathbb{B}(0;R_\Omega)$  the same boundary condition as  $p_{0,i}$ , but this would be too strong a requirement.

This concludes the proof.  $\square$

## 5.4 PROOF OF THEOREM 5.1.3

*Proof of Theorem 5.1.3.* Proceeding along the same lines as in Theorem 5.1.2, we prove that for any drift  $N \in \mathcal{C}^\infty(\Omega; \mathbb{R}^d)$  (regardless of whether or not it is the restriction of a radial drift  $N$  to the domain  $\Omega$ ), we prove that, when condition (5.8) then  $z_0 \equiv 0$  is the only solution to

$$\begin{cases} -\Delta p - 2\langle \frac{\nabla N}{N}, \nabla p \rangle = f(p) & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega, \\ 0 \leq p \leq 1, \end{cases} \quad (5.26)$$

and note that the main equation is equivalent to

$$-\nabla \cdot (N^2 \nabla u) = f(p)N^2.$$

Indeed, assuming there exists a non-trivial solution to (5.26) then from the mean value theorem, we can write

$$f(p) = f'(y)p$$

for some function  $y$  and, multiplying the equation by  $p$  and integrating by parts gives, using the Rayleigh quotient formulation of  $\lambda_1^D(\Omega)$ :

$$\lambda_1^D(\Omega) \int_{\Omega} p^2 \leq \int_{\Omega} |\nabla p|^2 \leq \int_{\Omega} N^2 |\nabla p|^2 \leq \|f'\|_{L^\infty} \|N^2\|_{L^\infty} \int_{\Omega} p^2,$$

which is contradiction unless  $p = z_0$ .

Once we have uniqueness for (5.26) we follow, for any initial datum  $p_0$ , the staircase procedure explained in the proof of Theorem 5.1.2: we first set the static control  $u = 0$ , we drive the solution to a  $\mathcal{C}^0$  neighbourhood of  $z_0$ , then to a steady-state solution of (5.7) in this neighborhood. Thus, we only need to prove the existence of a path of steady states linking  $z_0$  to  $z_\theta$ . In order to prove that such a path of steady states exists under assumption  $(A_1)$ , we use an energy method.

Let  $R > 0$  be such that  $\Omega \subset \mathbb{B}(0; R)$ . As in [163], we define, for any  $s \in [0, 1]$ ,  $p_s$  as the unique solution of

$$\begin{cases} -\Delta p_s - 2\langle \frac{\nabla N}{N}, \nabla p_s \rangle = f(p_s), & \text{in } \mathbb{B}(0; R_\Omega) \\ p_s \text{ is radial in } \mathbb{B}(0; R), \\ p_s(0) = s\theta. \end{cases} \quad (5.27)$$

We notice that the first equation in (5.27) rewrites as

$$-\nabla \cdot (N^2 \nabla p_s) = f(p_s)N.$$

Since  $N$  is radially symmetric, this amounts to solving, in radial coordinates

$$\begin{cases} -\frac{1}{r^{d-1}} (r^{d-1} N^2 p_s')' = f(p_s) N^2 \text{ in } [0; R], \\ p_s(0) = s\theta, p_s'(0) = 0. \end{cases} \quad (5.28)$$

We prove the existence and uniqueness of solutions to (5.28) below but underline that the core difficulty here is ensuring that

$$0 \leq p_s \leq 1.$$

**CLAIM 5.4.1** *For any  $s \in [0, 1]$ , there exists a unique solution to (5.28).*

**CLAIM 5.4.2** *Under Assumption  $A_1$  the path is admissible: we have, for any  $s \in [0, 1]$ ,*

$$0 \leq p_s \leq 1. \quad (5.29)$$

Furthermore, the path  $\{p_s\}_{s \in [0, 1]}$  is continuous in the  $\mathcal{C}^0$  topology.

*Proof of Claim 5.4.2.* 1. Admissibility of the path under Assumption  $A_1$ : We now prove Estimate (5.29), which proves that the path of steady states is admissible (with respect to the constraints).

To do so, we introduce the energy functional

$$\mathcal{E}_1 : x \mapsto \frac{1}{2} (p_s'(x))^2 + F(u(x)).$$

Differentiating  $\mathcal{E}_1$  with respect to  $x$ , we get

$$\begin{aligned} \mathcal{E}_1'(x) &= (p_s''(x) + f(p_s)) p_s'(x) \\ &= \left( -\frac{d-1}{r} - 2 \frac{N'(r)}{N(r)} \right) (p_s'(r))^2 && \text{from Equation (5.28)} \\ &\leq 0 && \text{from Hypothesis } A_1. \end{aligned}$$

In particular, we have, for any  $s \neq 0$ ,  $p_s \neq 0$  in  $(0; R)$ : arguing by contradiction if, for  $\underline{x} \in (0; R)$  we had  $p_s(\underline{x}) = 0$  then

$$\mathcal{E}_1(\underline{x}) = \frac{1}{2} (p_s'(\underline{x}))^2 \geq 0.$$

However,  $\mathcal{E}_1(0) = F(s\theta) < 0$ , so that a contradiction follows. For the same reason,  $p_s \neq 1$  in  $[0; R]$ , for otherwise, if  $p_s(\bar{x}) = 1$  at some  $\bar{x} \in [0, 1]$  we would have

$$\mathcal{E}_1(\bar{x}) \geq F(1) > 0,$$

which is once again a contradiction. It follows that, for any  $s \in (0; 1]$ ,

$$0 \leq p_s \leq 1,$$

as claimed. This concludes the proof of the admissibility of the path.

2. Continuity of the path: We want to prove the  $\mathcal{C}^0$  continuity of the path. Let  $s \in [0, 1]$  and let  $\{s_k\}_{k \in \mathbb{N}} \in [0, 1]^{\mathbb{N}}$  be a sequence such that

$$s_k \xrightarrow[k \rightarrow \infty]{} s.$$

Let  $p_k := p_{s_k}$ . Our goal is to show that

$$p_k \xrightarrow[k \rightarrow \infty]{\mathcal{C}^0(\mathbb{B}(0; R))} p_s. \quad (5.30)$$

We will use elliptic regularity to ensure that. We first derive a  $W^{1,\infty}$  estimate from the one-dimensional equation and use it to derive a  $\mathcal{C}^{2,\alpha}$  estimate for the equation set in  $\mathbb{B}(0; R)$ .

By the admissibility of the path we have, for every  $k \in \mathbb{N}$ ,

$$0 \leq p_k \leq 1.$$

Passing into radial coordinates and integrating Equation (5.28) between 0 and  $x$  gives

$$-p'_k(x) = \frac{1}{N^2(x)x^{d-1}} \int_0^x f(p_k(t)) N^2(t) t^{d-1} dt. \quad (5.31)$$

From Equation (5.31) we see that  $\{p_k\}_{k \in \mathbb{N}}$  is uniformly bounded in  $W^{1,\infty}((0; 1))$ . We now consider the Equation in  $\mathbb{B}(0; R)$ , i.e we work with (5.27). Since  $\{p_k\}_{k \in \mathbb{N}}$  is uniformly bounded in any  $\mathcal{C}^{0,\alpha}(\mathbb{B}(0; R))$  by the first step and since  $N \in \mathcal{C}^\infty(\mathbb{B}(0; R))$ , it follows from Hölder elliptic regularity (see [80] that there exists  $M \in \mathbb{R}$  such that, for every  $k \in \mathbb{N}$ ,

$$\|p_k\|_{\mathcal{C}^{2,\alpha}(\mathbb{B}(0; R))} \leq M$$

hence  $\{p_k\}_{k \in \mathbb{N}}$  converges in  $\mathcal{C}^1(\mathbb{B}(0; R))$ , up to a subsequence, to  $p_\infty$ . Passing to the limit in the weak formulation of the equation, we see that  $p_\infty$  satisfies

$$-\nabla \cdot (N^2 \nabla p_\infty) = f(p_\infty) N^2.$$

Passing to the limit in

$$\forall k \in \mathbb{N}, p_k(0) = s_k$$

we get  $p_\infty(0) = s$  and, finally, since for every  $k \in \mathbb{N}$ ,  $p_k$  is radial, i.e

$$\forall k \in \mathbb{N}, \forall i, j \in \{1, \dots, d\}, x_j \frac{\partial p_k}{\partial x_i} - x_i \frac{\partial p_k}{\partial x_j} = 0,$$

we can pass to the limit in this identity to obtain that  $p_\infty$  is radial. In particular,

$$p_\infty = p_s$$

and so the continuity of the path holds. □

To conclude the proof of Theorem 5.1.3, it suffices to apply the staircase method. □

## 5.5 PROOF OF THEOREM 5.1.4: BLOCKING PHENOMENON

*Proof of Theorem 5.1.4.* The proof that there exists  $L_\sigma(0) > 0$  such that, for any  $L \geq L_\sigma$  a non-trivial solution to (5.10) exists is exactly the same as for the case without a drift a relies on an energy argument, as is done in the proof of Lemma 5.1.

Such energy arguments fail however when trying to prove that (5.11) has a solution, since the natural energy of the equation (5.11) satisfies, when  $N$  is the gaussian,

$$\mathcal{E}[L, u] = \int_{-L}^L e^{-\frac{x^2}{\sigma}} (u')^2 - \int_{-L}^L e^{-\frac{x^2}{\sigma}} F(u) \geq \mathcal{E}[L, z_1] \text{ with } z_1 \equiv 1,$$

because  $F(u) \geq F(1)$ .

To prove the existence of a non-trivial solution, we give a fine study of the phase portrait which will

ensure that  $L_\sigma(1)$  is well-defined and that  $L_\sigma(1) > 0$ .

Let  $\alpha \in (0; \theta)$ . Let  $u_{\alpha, \sigma}$  be the solution, in  $\mathbb{R}$ , of

$$\begin{cases} -u''_{\alpha, \sigma} + 2\frac{x}{\sigma}u'_{\alpha, \sigma} = f(u_{\alpha, \sigma}), \\ u_{\alpha, \sigma}(0) = \alpha, \\ u'_{\alpha, \sigma}(0) = 0. \end{cases} \quad (5.32)$$

We first note that  $u_{\alpha, \sigma}$  is an even function.

Our goal is to prove that

$$\exists \alpha_\sigma \in [0; \theta] \text{ such that } \left( \exists x_{\alpha_\sigma} > 0, \text{ and } u_{\alpha_\sigma}(x_{\alpha_\sigma}) > 1, \forall t \in (0; x_{\alpha_\sigma}), 1 > u_{\alpha_\sigma}(t) > 0. \right). \quad (5.33)$$

If this holds, choosing  $\alpha_\sigma$  and defining  $L_\sigma(1) := x_{\alpha_\sigma}$  automatically yields the desired conclusion.

Since we use in this proof energy arguments, let us define, for any  $x \in \mathbb{R}_+$ ,

$$\mathcal{E}_\alpha(x) := \frac{1}{2}(u'_{\alpha, \sigma}(x))^2 + F(u_{\alpha, \sigma}(x))$$

and carry out some elementary computations.

First of all, we notice that

$$\begin{aligned} \frac{d}{dx} \mathcal{E}_\alpha(x) &= \frac{2x}{\sigma} u'_{\alpha, \sigma}(x)^2 \\ &= \frac{4x}{\sigma} (\mathcal{E}_\alpha(x) - F(u_{\alpha, \sigma}(x))) \\ &\geq \frac{4x}{\sigma} (\mathcal{E}_\alpha(x) - F(1)). \end{aligned}$$

*Proof of (5.33).* We start with the following result:

**CLAIM 5.5.1** *For any  $\sigma, \alpha > 0$ , there exists  $x_{\alpha, \sigma, \theta} > 0$  such that*

$$u_{\alpha, \sigma}(x_{\alpha, \sigma, \theta}) = \theta, 0 < u_{\alpha, \sigma} < \theta \text{ on } (0; x_{\alpha, \sigma, \theta}), u'_{\alpha, \sigma}(x_{\alpha, \sigma, \theta}) > 0, u'_{\alpha, \sigma} > 0 \text{ on } (0; x_{\alpha, \sigma, \theta}).$$

*Proof of Claim 5.5.1.* Since  $u_{\alpha, \sigma}$  is continuous and since  $u_{\alpha, \sigma}(0) = \alpha < \theta$ , there exists  $\delta > 0$  such that

$$u_{\alpha, \sigma}([0; \delta]) \subset [0; \theta].$$

Let  $x_{\alpha, \sigma, \theta}$  be defined as

$$x_{\alpha, \sigma, \theta} := \sup \{ \delta > 0, u_{\alpha, \sigma}([0; \delta]) \subset [0; \theta] \} > 0.$$

Note that we might have  $x_{\alpha, \sigma, \theta} = +\infty$ , but we will exclude this case: we prove that  $u_{\alpha, \sigma}$  is increasing on  $[0; x_{\alpha, \sigma, \theta}]$  and, we show at the same time, that  $x_{\alpha, \sigma, \theta} < +\infty$ .

As a consequence,  $u_{\alpha, \sigma}(x_{\alpha, \sigma, \theta}) = \theta$  and  $0 < u_{\alpha, \sigma} < \theta$  on  $[0; x_{\alpha, \sigma, \theta}]$ .

$u_{\alpha, \sigma}$  is increasing on  $[0; x_{\alpha, \sigma, \theta}]$ : On  $[0; x_{\alpha, \sigma, \theta})$ , we have  $f(u_{\alpha, \sigma}) < 0$ , whence

$$u''_{\alpha, \sigma}(x) > \frac{2x}{\sigma} u'_{\alpha, \sigma}(x). \quad (5.34)$$

From the Grönwall inequality, it follows that, for every  $x \in (0; x_{\alpha, \sigma, \theta})$ ,  $u'_{\alpha, \sigma}(x) > 0$ . Thus,  $u_{\alpha, \sigma} \geq \alpha$  on  $(0; x_{\alpha, \sigma, \theta})$  so

$$u_{\alpha, \sigma}(x) \xrightarrow{x \rightarrow x_{\alpha, \sigma, \theta}} \theta.$$

$x_{\alpha,\sigma,\theta} < \infty$ : Let  $\bar{x} \in (0; x_{\alpha,\sigma,\theta})$ . For any  $x \in (\bar{x}; x_{\alpha,\sigma,\theta})$  the Grönwall inequality applied to Equation (5.34) gives

$$u'_{\alpha,\sigma}(x) \geq e^{\frac{2}{\sigma}(x^2 - \bar{x}^2)} u'_{\alpha,\sigma}(\bar{x}) > u'_{\alpha,\sigma}(\bar{x}) > 0.$$

It follows that  $x_{\alpha,\sigma,\theta} < \infty$  and so  $u_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) = \theta$ ,  $u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) > 0$  and  $u_{\alpha,\sigma}(x) > 0$  on  $(0; x_{\alpha,\sigma,\theta}]$ .  $\square$

**CLAIM 5.5.2** *There holds*

$$x_{\alpha,\sigma,\theta} \xrightarrow{\alpha \rightarrow 0, \alpha > 0} +\infty.$$

*Proof of Claim 5.5.2.* This is a consequence of the Grönwall inequality applied to

$$\xi(x) := \frac{1}{2}(u_{\alpha,\sigma}^2 + u_{\alpha,\sigma}'^2).$$

First of all note that we know from Claim 5.5.1 that  $u_{\alpha,\sigma}, u'_{\alpha,\sigma} > 0$  on  $[0; x_{\alpha,\sigma,\theta}]$ . Let  $L > 0$  be defined as

$$L := \sup_{x \in [0,1]} \frac{-f(t)}{t} > 0.$$

Differentiating  $\xi$  gives

$$\begin{aligned} \frac{d\xi}{dx} &= u'_{\alpha,\sigma}(x) (u_{\alpha,\sigma}(x) + u''_{\alpha,\sigma}(x)) \\ &= u'_{\alpha,\sigma}(x) \left( u_{\alpha,\sigma}(x) - f(u_{\alpha,\sigma}) + 2\frac{x}{\sigma} u'_{\alpha,\sigma}(x) \right) \\ &\leq u'_{\alpha,\sigma}(x) \left( u_{\alpha,\sigma}(x) + Lu_{\alpha,\sigma}(x) + 2\frac{x}{\sigma} u'_{\alpha,\sigma}(x) \right) \\ &\leq u'_{\alpha,\sigma}(x) u_{\alpha,\sigma}(x) (L+1) + 2\frac{x}{\sigma} u_{\alpha,\sigma}'^2(x) \\ &\leq \frac{L+1}{2} (u_{\alpha,\sigma}'^2(x) + u_{\alpha,\sigma}(x)^2) + 2\frac{x}{\sigma} (u_{\alpha,\sigma}'^2(x) + u_{\alpha,\sigma}(x)^2) \\ &\leq \xi(x) \left( L+1 + 4\frac{x}{\sigma} \right). \end{aligned}$$

Since  $\xi(0) = \frac{1}{2}\alpha^2$  we conclude from Grönwall's lemma that

$$\xi(x) \leq \frac{\alpha^2}{2} e^{(L+1)x + 2\frac{x^2}{\sigma}}.$$

As a consequence, at  $x_{\alpha,\sigma,\theta}$ , we must have

$$\theta \leq \sqrt{\xi(x_{\alpha,\sigma,\theta})} \leq \frac{\alpha}{\sqrt{2}} e^{\frac{L+1}{2}x_{\alpha,\sigma,\theta} + \frac{x_{\alpha,\sigma,\theta}^2}{\sigma}}.$$

Thus, we have

$$x_{\alpha,\sigma,\theta} \xrightarrow{\alpha \rightarrow 0} +\infty,$$

as claimed.  $\square$

**CLAIM 5.5.3** *Let  $x_{\alpha,\sigma,\theta}$  be defined as*

$$x_{\alpha,\sigma,\theta} := \inf \{ x, x \text{ satisfies the conclusion of Claim 5.5.1} \}.$$

*For any  $\sigma > 0$ , there exists  $\alpha \in (0; \theta)$  such that*

$$u'_{\alpha}(x) \xrightarrow{x \rightarrow \infty} +\infty, u'_{\alpha,\sigma}(x) > 0 \text{ on } [x_{\alpha,\sigma,\theta}; +\infty).$$

*Proof of Claim 5.5.3.* We first notice that

$$\frac{d}{dx}u'_{\alpha,\sigma} = -f(u_{\alpha,\sigma}) + \frac{2x}{\sigma}u'_{\alpha,\sigma}(x) =: g(x) \quad (5.35)$$

The non-linearity changes sign at  $x_{\alpha,\sigma,\theta}$ .

We note that  $g(x_{\alpha,\sigma,\theta}) = u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) > 0$ . We want to ensure that the right-hand side of (5.35) enjoys some monotonicity property.

To guarantee this, we first note that, on  $[0; x_{\alpha,\sigma,\theta}]$ ,

$$\frac{d}{dx}\mathcal{E}_\alpha(x) = \frac{2x}{\sigma}u'_{\alpha,\sigma}(x)^2 \geq 0$$

and so

$$\mathcal{E}_\alpha(x) \geq \mathcal{E}_\alpha(0) = F(\alpha).$$

Thus,

$$u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) \geq \sqrt{2(F(\alpha) - F(\theta))}.$$

We hence assume that  $\alpha \leq \frac{\theta}{2}$ , which implies  $F(\alpha) \geq F(\frac{\theta}{2})$ . This gives a uniform lower bound of the form

$$u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) \geq c_0 > 0.$$

Now, regarding the monotonicity of the right-hand side of (5.35), we note that

$$\begin{aligned} g'(x) &= -u'_{\alpha,\sigma}f'(u_{\alpha,\sigma}) + \frac{2}{\sigma}u'_{\alpha,\sigma} + \frac{2x}{\sigma} \left( \frac{2x}{\sigma}u'_{\alpha,\sigma} - f(u_{\alpha,\sigma}) \right) \\ &= u'_{\alpha,\sigma} \left( -f'(u_{\alpha,\sigma}) + \frac{2}{\sigma} + 4\frac{x^2}{\sigma^2} - \frac{f(u_{\alpha,\sigma})}{u'_{\alpha,\sigma}} \right) \\ &= u'_{\alpha,\sigma}G(x, u_{\alpha,\sigma}, u'_{\alpha,\sigma}) \end{aligned}$$

with

$$G(x, u, v) := -f'(u) + \frac{2}{\sigma} + 4\frac{x^2}{\sigma^2} - \frac{f(u)}{v}.$$

We now want to ensure the following condition:

$$\forall v \geq c_0, \forall x \geq x_{\alpha,\sigma,\theta}, G(x, u, v) \geq 0. \quad (5.36)$$

Extending if need be  $f$  into a  $W^{1,\infty}$  function outside of  $[0, 1]$ , we see that this condition is guaranteed if, for any  $x \geq x_{\alpha,\sigma,\theta}$  we have

$$\|f'\|_{L^\infty} + \frac{\|f\|_{L^\infty}}{c_0} \leq \frac{2}{\sigma} + 4\frac{x^2}{\sigma^2}. \quad (5.37)$$

This in turn implies a condition on  $x_{\alpha,\sigma,\theta}$ , and we need to guarantee that  $x_{\alpha,\sigma,\theta}$  can be chosen arbitrarily large as  $\alpha \rightarrow 0$ . However this is a consequence of Claim 5.5.2.

As a consequence, coming back to (5.35), we see that, since  $g$  is locally positive because  $g(x_{\alpha,\sigma,\theta}) = u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) > c_0$  we can define

$$A_1 := \sup\{A \in \mathbb{R}_+^*, u'_{\alpha,\sigma} \geq u'_{\alpha,\sigma}(x_{\alpha,\sigma,\theta}) \text{ in } [x_{\alpha,\sigma,\theta}; x_{\alpha,\sigma,\theta} + A]\} > 0$$

and we now show that

$$A_1 = +\infty.$$

We first note that  $g$  is non-decreasing on  $[x_{\alpha,\sigma,\theta}; x_{\alpha,\sigma,\theta} + A]$ , by (5.36) and since  $g' = u'_{\alpha,\sigma}G(x, u_{\alpha,\sigma}, u'_{\alpha,\sigma})$ .

That  $A_1 = \infty$  is now an easy consequence of this fact: indeed, we have

$$\frac{d}{dx}(u'_{\alpha,\sigma}) = g(x) \geq g(x_{\alpha,\sigma,\theta}) > 0$$

and so we have

$$u'_{\alpha,\sigma}(x) \xrightarrow{x \rightarrow \infty} +\infty.$$

□

As a consequence of these two claims, there exists  $x_{\alpha,\sigma,1}$  such that

$$u_{\alpha}(x_{\alpha,\sigma,1}) = 1, u_{\alpha,\sigma} > 0 \text{ on } [0; x_{\alpha,\sigma,1}].$$

This concludes the proof of (5.33).

□

This concludes the proof of the existence part of Theorem 5.1.4 by setting  $L_{\sigma}^{(1)} = x_{\alpha,\sigma,1}$ . Let us set

$$L_{\sigma}^*(1) := \inf\{L > 0, (5.11) \text{ has a non-trivial solution in } [-L, L]\} > 0.$$

We now prove that  $L_{\sigma}^*(1) \xrightarrow{\sigma \rightarrow \infty} \infty/0$ . The analysis when  $\sigma \rightarrow \infty$  is quite easy, while the case  $\sigma \rightarrow 0$  is harder to tackle.

**CLAIM 5.5.4** *It holds*

$$L_{\sigma}^*(1) \xrightarrow{\sigma \rightarrow \infty} +\infty.$$

*Proof of Claim 5.5.4.* Let  $u_1$  be a non-trivial solution of (5.11). We can assume that  $u_1$  is a radially symmetric solution, i.e  $u_1(x) = u_1(-x)$ . Assume this is not the case, and set  $x_1$  such that

$$u_1(x_1) = \min u_1 < \theta$$

by the maximum principle. We can assume without loss of generality that  $x_1 > 0$ . Define

$$\varphi_1 : [0; L] \ni x \mapsto u_1(x_1)\mathbf{1}_{[0;x_1]} + u_1(x)$$

and extend it by parity to  $[-L, L]$ . Then  $\varphi_1$  is a radially symmetric supersolution of the equation, and  $z_0 \equiv 0$  is a subsolution of the same equation. The constructive iterative procedure of the construction of sub and super solutions gives the existence of a radially symmetric solution to the equation.

Thus we assume that  $u_1$  is even.

Let  $\alpha := u_1(0)$ , we then have  $u_1 = u_{\alpha,\sigma}$ , where  $u_{\alpha,\sigma}$  was constructed in the first part of the proof of the Theorem.

We start by noticing that integrating Equation (5.32) we have, for every  $x \in [0; L]$ ,

$$\begin{aligned} e^{-\frac{x^2}{\sigma}} u'_{\alpha,\sigma}(x) &= \int_0^x (-f(u_{\alpha,\sigma}))(t) e^{-\frac{t^2}{\sigma}} dt \\ &\leq \int_0^x \|f\|_{L^\infty} dt \end{aligned}$$

Thus, for any  $x > 0$ , we have

$$u'_{\alpha,\sigma}(x) \leq e^{\frac{x^2}{\sigma}} x \|f\|_{L^\infty}.$$

Integrating this inequality between 0 and  $L$ , we get

$$1 \leq \frac{e^{\frac{L^2}{\sigma}} L^2 \|f\|_{L^\infty}}{2}.$$

As a consequence,  $L_\sigma^*(1)$  can not stay bounded as  $\sigma \rightarrow \infty$ , which concludes the proof.  $\square$

We now pass to the proof of the following Claim:

**CLAIM 5.5.5** *There holds*

$$L_\sigma^*(1) \xrightarrow{\sigma \rightarrow 0} 0.$$

*Proof of Claim 5.5.5.* We argue by contradiction. Assume that there exists a sequence  $\{\sigma_k\}_{k \in \mathbb{N}}$  such that

$$L_{\sigma_k}^*(1) \not\xrightarrow{k \rightarrow \infty} 0, \sigma_k \xrightarrow{k \rightarrow \infty} 0$$

Let

$$\underline{L} := \liminf_{k \rightarrow \infty} L_{\sigma_k}^*(0) > 0.$$

Let  $\alpha > 0$  be fixed. From Claim 5.5.1 we know that, for every  $\sigma > 0$ , there exists  $x_{\alpha, \sigma, \theta} > 0$  such that

$$u_{\alpha, \sigma}(x_{\alpha, \sigma, \theta}) = \theta, u_{\alpha, \sigma} \text{ is increasing on } [0; x_{\alpha, \sigma, \theta}].$$

Let

$$u_k := u_{\alpha, \sigma_k}, \quad x_k := x_{\alpha, \sigma_k, \theta}.$$

We reach a contradiction by distinguishing two cases:

1. 0 is an accumulation point of  $\{x_k\}$ : Up to a subsequence, we can assume that

$$x_k \xrightarrow{k \rightarrow \infty} 0.$$

From the mean value theorem, there exists  $\{y_k\}_{k \in \mathbb{N}}$  such that

$$y_k \xrightarrow{k \rightarrow \infty} 0, u'_k(y_k) = \frac{\theta - \alpha}{x_k} \xrightarrow{k \rightarrow \infty} +\infty.$$

This implies that  $u'_k \rightarrow +\infty$  "uniformly on  $[y_k; y_k + \varepsilon]$  for every  $\varepsilon$  small enough" as made precise in the following statement:

$$\forall \varepsilon > 0, \forall M \in \mathbb{R}_+^*, \exists k_M \in \mathbb{N}, \forall k \geq k', u'_k \geq M \text{ on } [y_k; y_k + \varepsilon].$$

This is once again an application of the Grönwall Lemma: we have

$$\frac{d}{dx} u'_k \geq -\|f\|_{L^\infty} + \frac{2x}{\sigma} u'_k.$$

This implies

$$\forall t \geq 0, u'_k(y_k + t) \geq (u'_k(y_k) - \|f\|_{L^\infty} t) e^{\frac{(y_k + t)^2 - y_k^2}{\sigma}},$$

giving the desired conclusion.

Fixing  $\varepsilon = \frac{L}{2}$  and using  $u'_k(y_k) \xrightarrow{k \rightarrow \infty} +\infty$  gives the desired conclusion.

It immediately follows that, for  $k$  large, enough, there exists  $x_{k,1}$  such that  $|x_{k,1} - y_k| \leq \frac{L}{2}$ ,  $u_k(x_{k,1}) = 1$  and  $u_k > 0$  on  $(0; x_{k,1})$ , which is obviously a contradiction.



2. 0 is not an accumulation point of  $\{x_k\}$ : Assuming 0 is not an accumulation point of  $\{x_k\}_{k \in \mathbb{N}}$ , a contradiction ensues in the following manner: we now that there thus exists a point  $\underline{x} > 0$  such that

$$y \leq \liminf_{k \rightarrow \infty} x_k, \lim_{k \rightarrow \infty} u_k(y) \leq \theta - \delta$$

for some  $\delta > 0$ . Then we note that, by integration of (5.32) we get

$$u'_k(y) = e^{\frac{y^2}{\sigma}} \int_0^y e^{-\frac{t^2}{\sigma}} (-f(u_k(t))) dt. \quad (5.38)$$

Since  $y \in [0; x_k]$  for every  $k$  large enough, we have  $\alpha \leq u_k \leq \theta - \delta$  for every  $t \in [0; y]$ , so that

$$\exists \delta' > 0, f(u_k) \leq -\delta' \text{ on } [0; y].$$

Plugging this in the integral formulation (5.38) gives the lower bound

$$u'_k(y) \geq \delta e^{\frac{y^2}{\sigma}} \int_0^y e^{-\frac{t^2}{\sigma}} dt.$$

We estimate the right hand side using Laplace's method:

$$\int_0^y e^{-\frac{t^2}{\sigma}} dt \underset{\sigma \rightarrow 0}{\sim} C\sqrt{\sigma}$$

for some  $C > 0$ , which immediately gives

$$u'_k(y) \underset{k \rightarrow \infty}{\rightarrow} +\infty.$$

We conclude as in the first case. □

We use the same type of arguments to analyse the behaviour of  $L_\sigma^*(0)$  as  $\sigma \rightarrow 0$ . Obviously, when  $\sigma \rightarrow \infty$ ,  $L_\sigma^*(0)$  goes to  $L^*$ , the threshold for existence of a non-trivial solution to (5.10) already studied in [157]. We now prove that adding a gaussian drift yields the existence of a non-trivial drift with a small variance  $\sigma$  leads to the existence of non-trivial solutions around 0, even when the length of the interval is quite small.

**CLAIM 5.5.6** *There holds*

$$L_\sigma^*(0) \underset{\sigma \rightarrow 0}{\rightarrow} 0.$$

*Proof of Claim 5.5.6.* Here we only need to prove

$$\forall L > 0, \exists \sigma_L > 0, \forall \sigma \leq \sigma_L, (5.10) \text{ has a non-trivial solution,}$$

which is stronger than what we require.

To do so, we use an energy argument similar to that of the proof of Theorem 5.1.2: introduce, for a given  $L > 0$ , the energy functional of Equation (5.10):

$$\mathcal{E}^\sigma : W_0^{1,2}((-L, L]) \ni u \mapsto \frac{1}{2} \int_L^L |\nabla u|^2 e^{-\frac{x^2}{\sigma}} - \int_L^L F(u) e^{-\frac{x^2}{\sigma}}.$$

We now consider a smooth function  $\varphi \in \mathcal{C}^\infty((-L; L))$  with compact support and with

$$0 \leq \varphi \leq \varphi(0) = 1, \varphi \equiv 1 \text{ on } \left(-\frac{L}{2}; \frac{L}{2}\right).$$

We apply the Laplace method to

$$\mathcal{E}^\sigma[\varphi].$$

We first note that we immediately have, since  $\varphi$  is a fixed test function with a zero derivative on a neighborhood of 0,

$$\int_L^L |\nabla u|^2 e^{-\frac{x^2}{\sigma}} = o_{\sigma \rightarrow 0}(\sqrt{\sigma}).$$

On the other hand, the right hand side satisfies, thanks to the Laplace method,

$$\int_L^L F(u) e^{-\frac{x^2}{\sigma}} \underset{\sigma \rightarrow 0}{\sim} CF(1)\sqrt{\sigma}$$

where  $C > 0$  is a positive constant. Thus, for  $\sigma$  small enough, we have

$$\mathcal{E}^\sigma[\varphi] < 0,$$

hence the existence of a non-trivial solution for (5.10). □

□

□

## 5.6 CONCLUSION

### 5.6.1 OBTAINING THE RESULTS FOR GENERAL COUPLED SYSTEMS

As explained in Section 5.1.2 of the Introduction, the equations considered in this article correspond to some scaling limits for more general coupled systems of reaction-diffusion equations, and it seems interesting to investigate whether or not the results we obtained in this article might be generalized to encompass the case of such general systems. As was explained in Section 5.1.2, these models can be used to control populations of infected mosquitoes and arise in evolutionary dynamics. Obtaining a finer understanding of the real underlying dynamics rather than the simplified version under scrutiny here seems, however, challenging. Indeed, although controllability results for linear systems of equations exist (see for instance [123]), the non-linear case has not yet been completely studied.

However, given that, as explained in the Introduction, gene-flow models and spatially heterogeneous models are limits in a certain scaling of such systems, it would be interesting to see whether or not our perturbation arguments, that were introduced to pass from the spatially homogeneous model to the slowly varying one, could work to pass from this scaling limit to the whole system in a certain regime.

### 5.6.2 OPEN PROBLEM: THE MINIMAL CONTROLLABILITY TIME AND SPATIAL HETEROGENEITY

Let us now list a few questions which, to the best of our knowledge, are still open and seem worth investigating.

- The qualitative properties of time optimal controls:

As suggested in [157] one might try to optimize the control with respect to the controllability time. Indeed, it is known that, under constraints on the control, parabolic equations have a minimal controllability time, see for instance [177, 154].

For constrained controllability it is known that there exists a minimal controllability time to control, for instance, from 0 to  $\theta$  (see [157]). We may try to optimize the control strategies so as

to minimize the controllability time. In our case, that is, the spatially heterogeneous case, are these controls of bang-bang type? Another qualitative question that is relevant in this context is that of symmetry: in the one dimensional case, when working on an interval  $[-L, L]$ , are time-optimal controls symmetric? In the multi-dimensional case, when the domain  $\Omega$  is a ball, is it possible to prove radial symmetry of time optimal controls?

- The influence of spatial heterogeneity on controllability time:

Adding a drift (which corresponds to the spatially heterogeneous model) modifies the controllability time. As we have seen, such heterogeneities might lead to a lack of controllability. However, it is also suggested in the numerical experiments shown below that adding a drift might be beneficial for the controllability time. It might be interesting to consider the following question: given  $L^\infty$  and  $L^1$  bounds on the spatial heterogeneity  $N$ , which is the drift yielding the minimal controllability time? In other terms: how can we design the domain so as to minimize the controllability time? In the simulation below, we thus considered the following optimization problem: letting, for any drift  $N$ ,  $T(N)$  be the minimal controllability time from 0 to  $\theta$  of the spatially heterogeneous equation (5.2) (with  $T(N) \in (0; +\infty]$ ), solve

$$\inf_{-M \leq N \leq M, \int_{-L}^L N = 0} T(N).$$

We obtain the following graph with  $M = 250$  and  $L = 2.5$ :

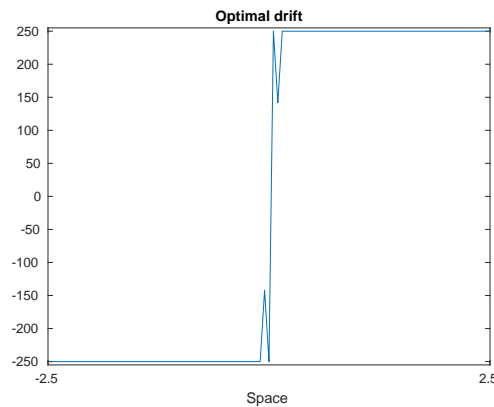


Figure 5.11 – Time optimal spatial heterogeneity.

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# APPENDIX

## 5.A PROOF OF LEMMA 5.1

*Proof of Lemma 5.1.* Let us first remark that (5.13) has a variational structure. Indeed,  $u$  is a solution of

$$-\Delta u + \varepsilon \langle \nabla n, \nabla u \rangle = f(u), u \in W_0^{1,2}(\Omega)$$

if and only if

$$-\nabla \cdot (e^{\varepsilon n} \nabla u) = f(u) e^{\varepsilon n}, u \in W_0^{1,2}(\Omega). \quad (5.39)$$

Following the arguments of [21, Remark II.2], we introduce the energy functional associated with (5.39): let

$$\mathcal{E}_1 : W_0^{1,2}(\Omega) \ni u \mapsto \frac{1}{2} \int_{\Omega} e^{\varepsilon n} |\nabla u|^2 - \int_{\Omega} e^{\varepsilon n} F(u),$$

From standard arguments in the theory of sub and super solutions [21], if there exists  $v \in W_0^{1,2}(\Omega)$  such that

$$\mathcal{E}_1(v) < 0 \quad (5.40)$$

then there exists a non-trivial solution to (5.13). We now prove that there exists  $v \in W_0^{1,2}(\Omega)$  such that (5.40) holds, by adapting the construction of [163]: let  $\mathbb{B}(\bar{x}; \rho_{\Omega})$  be one of the ball of maximum radius inscribed in  $\Omega$ . Up to a translation, we assume that  $\bar{x} = 0$ .

Let  $\delta > 0$ . We define  $v_{\delta}$  as follows

$$v_{\delta} : \begin{cases} x \in \mathbb{B}(0; \rho_{\Omega} - \delta) \mapsto 1, \\ x \in \mathbb{B}(0; \rho_{\Omega}) \setminus \mathbb{B}(0; \rho_{\Omega} - \delta) \mapsto \frac{\rho_{\Omega}^2 - \|x\|^2}{\rho_{\Omega}^2 - (\rho_{\Omega} - \delta)^2}, \\ x \in \Omega \setminus \mathbb{B}(0; \rho_{\Omega}) \mapsto 0. \end{cases}$$

An explicit computation yields

$$\int_{\Omega} |\nabla v|^2 \sim_{\delta \rightarrow 0} C \rho_{\Omega}^d$$

for some constant  $C > 0$ , and

$$\int_{\Omega} F(v) = F(1) \rho_{\Omega}^d + \mathcal{O}_{\delta \rightarrow 0}(\rho_{\Omega}^{d-1}).$$

Hence, since  $n$  is bounded, the conclusion: as  $\rho_{\Omega} \rightarrow \infty$  ad  $\delta \rightarrow 0$  the energy of  $v_1$  is negative.

□

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# CHAPTER 6

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## AN EXISTENCE THEOREM FOR A NON-LINEAR SHAPE OPTIMIZATION PROBLEM

WITH A. HENROT AND Y. PRIVAT.

On ne peut pas ruiner un homme qui ne  
possède rien.

---

Balzac,  
*Gobseck*

## GENERAL PRESENTATION OF THE CHAPTER: MAIN DIFFICULTIES AND METHODS

In this Chapter, which corresponds to [HMP19], we investigate a shape optimization problem: let  $D$  be a box, and let for any  $\Omega \subset D$  regular enough  $u_\Omega$  be a solution of the semi-linear equation

$$\begin{cases} -\Delta u_\Omega + \rho f(u_\Omega) = g \text{ in } \Omega, \\ u_\Omega = 0 \text{ on } \partial\Omega. \end{cases}$$

The function  $g : D \rightarrow \mathbb{R}$  is fixed,  $f$  is regular,  $\rho > 0$  is a small parameter; we consider the shape optimization problem

$$\inf_{\Omega \subset D, |\Omega|=V_0} \frac{1}{2} \int_{\Omega} |\nabla u_\Omega|^2 - \int_{\Omega} gu.$$

The main results can be summed up as follows:

- Existence properties:

The main problem here is that the functional is not energetic. Using a porous medium approximation and a fine analysis of the witch function associated with the problem, we prove that, under various monotonicity assumptions on the non-linearity  $f$ , or under sign assumptions on  $g$ , the optimization problem has a solution. These results can be found in Theorem 6.2.1.

- (In)stability of minimizers:

We then analyse a particular case, that of a radially symmetric, non-increasing  $g$ . It is known that  $\Omega^* = \mathbb{B}(0; r^*)$  is the unique minimizer for  $\rho = 0$ . We then show, using comparison techniques, that  $\Omega^*$  satisfies second order optimality conditions under certain assumptions on  $g$  for  $\rho$  small enough while proving that, for  $g \equiv 1$ , we can construct a non-linearity  $f$  such that  $\Omega^*$  is not a minimizer for any  $\rho > 0$  small enough. The technique to prove this instability is new, and uses a fine understanding of the second order shape derivatives. More specifically, we use the (in)stability of  $\Omega^*$  as a minimizer for  $\rho = 0$  to obtain our results. These results can be found in Theorem 6.2.2.

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## 6.1 INTRODUCTION

### 6.1.1 MOTIVATIONS AND STATE OF THE ART

Existence and characterization of domains minimizing or maximizing a given shape functional under constraint is a long story. Such issues have been much studied over the last decades (see e.g. [36, 68, 92, 110, 93]). Recent progress has been made in understanding such issues for problems involving for instance spectral functionals (see e.g. [95]).

The issue of minimizing the Dirichlet energy (in the linear case) with respect to the domain is a basic and academical shape optimization problem under PDE constraint, which is by now well understood. This problem reads:

*Let  $d \in \mathbb{N}^*$  and  $D$  be a smooth compact set of  $\mathbb{R}^d$ . Given  $g \in H^{-1}(D)$  and  $m \leq |D|$ , minimize the Dirichlet energy*

$$J(\Omega) = \frac{1}{2} \int_{\Omega} |\nabla u_{\Omega}|^2 - \langle g, u_{\Omega} \rangle_{H^{-1}(\Omega), W_0^{1,2}(\Omega)},$$

*where  $u_{\Omega}$  is the unique solution of the Dirichlet problem<sup>1</sup> on  $\Omega$  associated to  $g$ , among all open bounded sets  $\Omega \subset D$  of Lebesgue measure  $|\Omega| \leq m$ .*

As such, this problem is not well-posed and it has been shown (see e.g. [88] or [93, Chap. 4] for a survey of results about this problem) that optimal sets only exist within the class

$$\mathcal{O}_m = \{\Omega \in \mathcal{A}(D), |\Omega| \leq m\}, \tag{6.1}$$

where  $\mathcal{A}(D)$  denotes the class of quasi-open sets<sup>2</sup> of  $D$ .

This article is motivated by the observation that, in general, the techniques used to prove existence, regularity and even characterization of optimal shapes for this problem rely on the fact that the functional is "energetic", in other words that the PDE constraint can be handled by noting that the full shape optimization problem rewrites

$$\min_{\substack{\Omega \in \mathcal{A}(D) \\ |\Omega| \leq m}} \min_{u \in W_0^{1,2}(D)} \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \langle g, u \rangle_{H^{-1}(\Omega), W_0^{1,2}(\Omega)} \right\}.$$

In this article, we introduce and investigate a prototypal problem close to the standard "Dirichlet energy shape minimization", involving a nonlinear differential operator. The questions we wish to study here concern existence of optimal shapes and stability issues for "non energetic" models. We note that the literature regarding existence and qualitative properties for non-energetic, non-linear optimization problems is scarce. We nevertheless mention [MNP19b], where existence results are established in certain asymptotic regimes for a shape optimization problem arising in population dynamics.

Since our aim is to investigate the optimization problems in the broadest classes of measurable domains, a volume constraint, known to lead to potential difficulties due to lack of compactness for standard topologies, is considered.

<sup>1</sup>in other words

$$u_{\Omega} = \operatorname{argmin}_{u \in W_0^{1,2}(D)} \left\{ \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \langle g, u \rangle_{H^{-1}(\Omega), W_0^{1,2}(\Omega)} \right\}.$$

<sup>2</sup>Recall that  $\Omega \subset D$  is said quasi-open whenever there exists a non-increasing sequence  $(\omega_n)_{n \in \mathbb{N}}$  such that

$$\forall n \in \mathbb{N}, \Omega \cup \omega_n \text{ is open} \quad \text{and} \quad \lim_{n \rightarrow +\infty} \operatorname{cap}(\omega_n) = 0$$

In the close version of the Dirichlet problem we will deal with, the linear PDE solved by  $u_\Omega$  is changed into a nonlinear one but the functional to minimize remains the same. Since, in such a case, the problem is not "energetic" anymore (in the sense made precise above), the PDE constraint cannot be incorporated into the shape functional. This calls for new tools to be developed in order to overcome this difficulty. Among others, we are interested in the following issues:

- **Existence:** is the resulting shape optimization problem well-posed?
- **Stability of optimal sets:** given a minimizer  $\Omega_0^*$  for the Dirichlet energy in the linear case, is  $\Omega_0^*$  still a minimizer when considering a "small enough" non-linear perturbation of the problem?

This article is organized as follows: the main results, related to the existence of optimal shapes for Problem (6.3) and the criticality/stability of the ball are gathered in Section 6.2. Section 6.3 is dedicated to the proofs of the existence results whereas Section 6.4 is dedicated to the proofs of the stability results.

### 6.1.2 THE SHAPE OPTIMIZATION PROBLEM

In what follows, we consider a modified version of the problem described above, where the involved PDE constraint is now nonlinear.

Let  $d \in \mathbb{N}^*$ ,  $D$  a compact set (i.e the closure of a bounded open set) of  $\mathbb{R}^d$ ,  $g \in L^2(D)$  and  $f \in W^{1,\infty}(\mathbb{R})$ . For a small enough positive parameter  $\rho$ , let  $u_\Omega \in W_0^{1,2}(\Omega)$  be the unique solution of the problem

$$\begin{cases} -\Delta u_{\rho,\Omega} + \rho f(u_{\rho,\Omega}) = g & \text{in } \Omega \\ u_{\rho,\Omega} \in W_0^{1,2}(\Omega). \end{cases} \quad (6.2)$$

For  $m \leq |D|$ , solve the problem:

$$\inf_{\Omega \in \mathcal{O}_m} J_\rho(\Omega) \quad \text{where } J_\rho(\Omega) = \frac{1}{2} \int_\Omega |\nabla u_{\rho,\Omega}|^2 - \int_\Omega g u_{\rho,\Omega}, \quad (6.3)$$

where  $\mathcal{O}_m$  is defined in (6.1).

In this problem, the smallness assumption on the parameter  $\rho$  guarantees the well-posedness of the PDE problem (6.2) for generic choices of nonlinearities  $f$ .

**LEMMA 6.1** *There exists  $\underline{\rho} > 0$  such that, for any  $\Omega \in \mathcal{O}_m$ , for any  $\rho \in [0, \underline{\rho}]$ , Equation (6.2), understood through its variational formulation, has a unique solution in  $W_0^{1,2}(\Omega)$ .*

This follows from a simple fixed-point argument: let  $\lambda_1(\Omega)$  be the first eigenvalue of the Dirichlet Laplacian on  $\Omega$ . We note that the operator

$$\begin{aligned} T : W_0^{1,2}(\Omega) &\longrightarrow W_0^{1,2}(\Omega) \\ u &\longmapsto w_\Omega, \end{aligned}$$

where  $w_\Omega$  is the unique solution of

$$\begin{cases} -\Delta w - g = -\rho f(u) & \text{in } \Omega \\ w \in W_0^{1,2}(\Omega), \end{cases}$$

is Lipschitz with Lipschitz constant  $C_T(\Omega)$  such that  $C_T(\Omega) \leq \rho \frac{1}{\lambda_1(\Omega)} \|f\|_{W^{1,\infty}}$ . By the monotonicity of  $\lambda_1$  with respect to domain inclusion (see [92]), we have, for every  $\Omega \in \mathcal{O}_m$ ,  $\lambda_1(D) \leq \lambda_1(\Omega)$ , so that  $C_T(\Omega) \leq \frac{\rho \|f\|_{W^{1,\infty}}}{\lambda_1(D)}$ .

## 6.2 MAIN RESULTS OF THE PAPER

### 6.2.1 EXISTENCE RESULTS

We state hereafter a partial existence result inherited from the linear case. Indeed, we will exploit a monotonicity property of the shape functional  $J_\rho$  together with its lower-semi continuity for the  $\gamma$ -convergence to apply the classical theorem by Buttazzo-DalMaso (see Subsection 6.3.1). Our approach takes advantage of the analysis of a relaxed formulation of Problem (6.3). To introduce it, let us first consider a given box  $D \subset \mathbb{R}^n$  (i.e a smooth, compact subset of  $\mathbb{R}^n$ ) such that  $|D| > V_0$ .

In the minimization problem (6.3), let us identify a shape  $\Omega$  with its characteristic function  $\mathbf{1}_\Omega$ . This leads to introducing the “relaxation” set

$$\widehat{\mathcal{O}}_m = \left\{ a \in L^\infty(D, [0, 1]) \text{ such that } \int_D a \leq m \right\}$$

For a given positive relaxation parameter  $M$ , we define the (relaxed) functional  $\hat{J}_{M,\rho}$  by

$$\hat{J}_{M,\rho}(a) = \frac{1}{2} \int_D |\nabla u_{M,\rho,a}|^2 + \frac{M}{2} \int_\Omega (1-a) u_{M,\rho,a}^2 - \int_\Omega g u_{\rho,\Omega}, \quad (6.4)$$

for every  $a \in \widehat{\mathcal{O}}_m$ , where  $u_{M,\rho,a} \in W_0^{1,2}(D)$  denotes the unique solution of the non-linear problem

$$\begin{cases} -\Delta u_{M,\rho,a} + M(1-a)u_{M,\rho,a} + \rho f(u_{M,\rho,a}) = g & \text{in } D \\ u_{M,\rho,a} \in W_0^{1,2}(D). \end{cases} \quad (6.5)$$

Our existence result involves a careful asymptotic analysis of  $u_{M,\rho,a}$  as  $\rho \rightarrow 0$  to derive a monotonicity property.

Standard elliptic estimates entail that, for every  $M > 0$  and  $a \in \widehat{\mathcal{O}}_m$ , one has  $u_{M,\rho,a} \in \mathcal{C}^0(\overline{\Omega})$ .

**REMARK 6.1** *Such an approximation of  $u_{\rho,\Omega}$  is rather standard in the framework of fictitious domains. The introduction of the term  $M(1-a)$  in the PDE has an interpretation in terms of porous materials (see e.g. [72]) and it may be expected that  $u_{M,\rho,a}$  converges in some sense to  $u_{\rho,\Omega}$  as  $M \rightarrow +\infty$  and whenever  $a = \mathbf{1}_\Omega$ . This will be confirmed in the analysis to follow.*

Roughly speaking, the existence result stated in what follows requires the right-hand side of equation (6.2) to have a constant sign. To write the hypothesis down, we need a few notations related to the relaxed problem (6.5), which is the purpose of the next lemma.

**LEMMA 6.2** *Let  $m \in [0, |D|]$ ,  $a \in \widehat{\mathcal{O}}_m$  and  $g \in L^2(D)$  be nonnegative. There exists a positive constant  $N_{m,g}$  such that*

$$\forall a \in \widehat{\mathcal{O}}_m, \forall M > 0, \forall \rho \in [0, \underline{\rho}], \quad \|u_{M,\rho,a}\|_\infty \leq N_{m,g}, \quad (6.6)$$

where  $\underline{\rho}$  is defined in Lemma 6.1,  $u_{M,\rho,a}$  denotes the unique solution to (6.5). In what follows,  $N_{m,g}$  will denote the optimal constant in the inequality above, namely

$$N_{m,g} = \sup\{\|u_{M,\rho,a}\|_{L^\infty(\Omega)}, a \in \widehat{\mathcal{O}}_m, M > 0, \rho \in [0, \underline{\rho}]\}.$$

This follows from standard arguments postponed to Section 6.A.

We now state the main results of this section. Let us introduce the assumptions we will consider hereafter:

**(H<sub>1</sub>)** There exist two positive numbers  $g_0, g_1$  such that  $g_0 < g_1$  and  $g_0 \leq g(\cdot) \leq g_1$  a.e. in  $D$ .

**(H<sub>2</sub>)** One has  $f \in W^{1,\infty}(\mathbb{R}) \cap D^2$ , where  $D^2$  is the set of twice differentiable functions (with second derivatives not necessarily continuous). Moreover,  $f(0) \leq 0$  and there exists  $\delta > 0$  such that the mapping  $x \mapsto xf(x)$  is non-decreasing on  $[0, N_{m,g} + \delta]$  where  $N_{m,g}$  is given by Lemma 6.2.

**THEOREM 6.2.1** *Let us assume that one of the following assumptions holds true:*

- $g$  or  $-g$  satisfies the assumption **(H<sub>1</sub>)**;
- $g$  is non-negative and the function  $f$  satisfies the assumption **(H<sub>2</sub>)** or  $g$  is non-positive and the function  $-f$  satisfies the assumption **(H<sub>2</sub>)**;

*Then, there exists a positive constant  $\rho_0 = \rho_0(D, f(0), \|f\|_{W^{1,\infty}})$  such that the shape optimization problem (6.3) has a solution  $\Omega^*$  for every  $\rho \in (0, \rho_0)$ .*

**REMARK 6.2** *The proof of Theorem 6.2.1 rests upon a monotonicity property of the relaxed functional  $\hat{J}_{M,\rho}$  given by (6.4). This is the first ingredient that subsequently allows the well-known existence result of Buttazzo and Dal-Maso to be applied.*

*It is natural to wonder whether or not it would be possible to obtain this result in a more direct way, for instance by using shape derivatives. We claim that such an approach would require to consider domains  $\Omega$  having a smooth boundary so that the shape derivative (in the sense of Hadamard) of  $J_\rho$  at  $\Omega$  in direction  $V$ , where  $V$  denotes an adequate vector field, makes sense. This relaxed version enables us to encompass less regular domains.*

## 6.2.2 STABILITY RESULTS

In what follows, we will work in  $\mathbb{R}^2$ .  $\mathbb{B}^*$  denotes the centered ball with radius  $R > 0$  such that  $\mathbb{B}^* \in \mathcal{O}_m$  and we introduce  $\mathbb{S}^* = \partial\mathbb{B}^*$ . The notation  $\nu$  stands for the outward unit vector on  $\mathbb{S}^*$ , in other words  $\nu(x) = x/|x|$  for all  $x \in \mathbb{S}^*$ .

In this section, we will discuss the local optimality of the ball for small nonlinearities. We will in particular highlight that the local optimality of the ball can be either preserved or lost depending on the choice of the right-hand side  $g$ . Indeed, if  $\rho = 0$  and if  $g$  is radially symmetric and non-increasing, the Schwarz rearrangement<sup>3</sup> ensures that, for any  $\Omega \in \mathcal{O}_m$ ,  $J_0(\Omega) \geq J_0(\mathbb{B}^*)$ . Without such assumptions, not much is known about the qualitative properties of the optimizers.

According to the considerations above, we will assume in the whole section that

**(H<sub>3</sub>)**  $g$  is a non-increasing, radially symmetric and non-negative function in  $L^2(\mathbb{B}^*)$  and  $f$  is a given  $\mathcal{C}^2$  non-linearity.

Notice that the analysis to follow can be generalized to sign-changing  $g$ . Here, this assumption allows us to avoid distinguishing between the cases where the signs of normal derivatives on  $\mathbb{S}^*$  are positive or negative. For the sake of simplicity, for every  $\rho \geq 0$ , we will call  $u_\rho$  the solution of the PDE

$$\begin{cases} -\Delta u_\rho + \rho f(u_\rho) = g & \text{in } \mathbb{B}^* \\ u_\rho \in W_0^{1,2}(\mathbb{B}^*) & \text{on } \partial\mathbb{B}^* = \mathbb{S}^*. \end{cases} \quad (6.7)$$

Proving a full stationarity result<sup>4</sup> is too intricate to tackle, since we do not know the minimizers topology. Hereafter, we investigate the local stability of the ball  $\mathbb{B}^*$ : we will prove that the ball is always a critical point, and show that we obtain different stability results, related to the non-negativity of the second shape derivative of the Lagrangian, depending on  $f$  and  $g$ .

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<sup>3</sup>see e.g. [108] for an introduction to the Schwarz rearrangement.

<sup>4</sup>in other words, proving that for any  $\rho \leq \rho^*$   $\mathbb{B}^*$  is the unique minimizer of  $J_\rho$  in  $\mathcal{O}_m$

To compute the first and second order shape derivatives, it is convenient to consider vector fields  $V \in W^{3,\infty}(\mathbb{R}^2)$  and to introduce, for a given admissible vector field  $V$  (i.e such that, for  $t$  small enough,  $(\text{Id} + tV)\mathbb{B}^* \in \mathcal{O}_m$ ), the mapping

$$f_V : t \mapsto J_\rho((\text{Id} + tV)\mathbb{B}^*).$$

The first (resp. second) order shape derivative of  $J_\rho$  in the direction  $V$  is defined as

$$J'_\rho(\mathbb{B}^*)[V] := f'_V(0), \text{ (resp. } J''_\rho(\mathbb{B}^*)[V, V] := f''_V(0)).$$

To enforce the volume constraint  $|\Omega| = m$ , we work with the unconstrained functional

$$\mathcal{L}_{\Lambda_\rho} : \Omega \mapsto J_\rho(\Omega) - \Lambda_\rho (\text{Vol}(\Omega) - m),$$

where  $\text{Vol}$  denotes the Lebesgue measure in  $\mathbb{R}^2$  and  $\Lambda_\rho$  denotes a Lagrange multiplier associated with the volume constraint. Recall that, for every domain  $\Omega$  with a  $\mathcal{C}^2$  boundary and every vector field in  $W^{3,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ , we have

$$\text{Vol}'(\Omega)[V] = \int_{\partial\Omega} V \cdot \nu \quad \text{and} \quad \text{Vol}''(\Omega)[V, V] = \int_{\partial\Omega} H(V \cdot \nu)^2,$$

where  $H$  stands for the mean curvature of  $\partial\Omega$ . The local first and second order optimality conditions for Problem (6.3) read as follow:

$$\left. \begin{array}{l} \mathcal{L}'_{\Lambda_\rho}(\Omega)[V] = 0 \\ \mathcal{L}''_{\Lambda_\rho}(\Omega)[V, V] \geq 0 \end{array} \right\} \text{ for every } V \in W^{3,\infty}(\mathbb{R}^2, \mathbb{R}^2) \text{ such that } \int_{\mathbb{S}^*} V \cdot \nu = 0.$$

For further informations about shape derivatives, we refer for instance to [93, Chapitre 5]. Let us state the main result of this section. In what follows,  $\rho$  is chosen small enough so that Equation (6.2) has a unique solution.

**THEOREM 6.2.2** *Let  $f$  and  $g$  satisfying the assumption  $(\mathbf{H}_3)$ . Let  $V \in W^{3,\infty}(\mathbb{R}^2, \mathbb{R}^2)$  denote a vector field such that  $\int_{\mathbb{S}^*} V \cdot \nu = 0$ .*

1. *(Shape criticality)  $\mathbb{B}^*$  is a critical shape, in other words  $J'_\rho(\mathbb{B}^*)[V] = 0$ .*
2. *(Shape stability) Assume that*

$$\pi R^2 g(R) \leq \int_{\mathbb{B}^*} g \quad \text{and} \quad 0 < \int_{\mathbb{B}^*} g, \tag{6.8}$$

where  $R$  denotes the radius of the ball  $\mathbb{B}^*$ . Let  $\Lambda_\rho$  be the Lagrange multiplier associated with the volume constraint. There exists  $\bar{\rho} > 0$  and  $C > 0$  such that, for any  $\rho \leq \bar{\rho}$ ,

$$(J_\rho - \Lambda_\rho \text{Vol})''(\mathbb{B}^*)[V, V] \geq C \|V \cdot \nu\|_{L^2(\Omega)}^2. \tag{6.9}$$

3. *(Shape instability) Assume that  $g$  is the constant function equal to 1 and that  $f$  is a  $\mathcal{C}^1$  non-negative function such that  $f' < -1$  on  $[0, 2\|u_0\|_{L^\infty})$ , where  $u_0$  is the solution of (6.2) with  $\rho = 0$  and  $\Omega = \mathbb{B}^*$ . Then, the second order optimality conditions are not fulfilled on  $\mathbb{B}^*$ : there exists  $\bar{\rho} > 0$  and  $\hat{V} \in W^{3,\infty}(\mathbb{R}^2, \mathbb{R}^2)$  such that  $\int_{\mathbb{S}^*} \hat{V} \cdot \nu = 0$  and, for any  $\rho \leq \bar{\rho}$ ,*

$$(J_\rho - \Lambda_\rho \text{Vol})''(\mathbb{B}^*)[\hat{V}, \hat{V}] < 0.$$

**REMARK 6.3** *Let us comment on the strategy of proof. It is known that estimates of the kind (6.9)*

can lead to local quantitative inequalities [64]. We first establish (6.9) in the case  $\rho = 0$ , and then extend it to small parameters  $\rho$  with the help of a perturbation argument. Assumptions of the type (6.8) are fairly well-known, and amount to requiring that  $\mathbb{B}^*$  is a stable shape minimizer [65, 94]. Finally, the instability result rests upon the following observation: if  $g = 1$  and if  $V_1$  is the vector field given by  $V(r \cos(\theta), r \sin(\theta)) = \cos(\theta)(r \cos(\theta), r \sin(\theta))$ , then one has

$$(J_0'' - \Lambda_0 \text{Vol})''(\mathbb{B}^*)[V, V] = 0$$

while higher order modes are stable [65, 94]. It therefore seems natural to consider such perturbations when dealing with for small parameters  $\rho$ .

It should also be noted that our proof uses a comparison principle, which shortens many otherwise lengthy computations.

### 6.3 PROOF OF THEOREM 6.2.1

#### 6.3.1 GENERAL OUTLINE OF THE PROOF

The proof of Theorem 6.2.1 rests upon an adaptation of the standard existence result by Buttazzo-DalMaso (see either the original article [38] or [93, Thm 4.7.6] for a proof), based on the notion of  $\gamma$ -convergence, that we recall below.

**DEFINITION 6.3.1** For any quasi-open set  $\Omega \in \mathcal{O}_m$ , let  $R_\Omega$  be the resolvent of the Laplace operator on  $\Omega$ . We say that a sequence of quasi-open sets  $(\Omega_k)_{k \in \mathbb{N}}$  in  $\mathcal{O}_m$   $\gamma$ -converges to  $\Omega \in \mathcal{O}_m$  if, for any  $\ell \in H^{-1}(D)$ ,  $(R_{\Omega_k}(\ell))_{k \in \mathbb{N}}$  converges in  $W_0^{1,2}(D)$  to  $R_\Omega(\ell)$ .

The aforementioned existence theorem reads as follows.

theorem[Buttazzo-DalMaso] Let  $J : \mathcal{O}_m \rightarrow \mathbb{R}$  be a shape functional satisfying the two following assumptions:

1. (monotonicity) For every  $\Omega_1, \Omega_2 \in \mathcal{O}_m$ ,  $\Omega_1 \subseteq \Omega_2 \Rightarrow J(\Omega_2) \leq J(\Omega_1)$ .
2. ( $\gamma$ -continuity)  $J$  is lower semi-continuous for the  $\gamma$ -convergence.

Then the shape optimization problem

$$\inf_{\Omega \in \mathcal{O}_m} J(\Omega)$$

has a solution. theorem

As is customary when using this result, the lower semi-continuity for the  $\gamma$ -convergence is valid regardless of any sign assumptions on  $g$  or of any additional hypothesis on  $f$ . This is the content of the next result, whose proof is standard and thus, postponed to Appendix 6.B.

**PROPOSITION 6.1** Let  $f \in W^{1,\infty}(\mathbb{R})$  and  $\rho \geq 0$ . The functional  $J_\rho$  is lower semi-continuous for the  $\gamma$ -convergence.

It remains hence to investigate the monotonicity of  $J_\rho$ . Our approach uses a relaxed version of  $J_\rho$ , namely the functional  $\hat{J}_{M,\rho}$  defined by (6.4). More precisely, we will prove under suitable assumptions that

$$\forall M \geq 0, \forall a_1, a_2 \in \widehat{\mathcal{O}}_m, a_1 \leq a_2 \implies \hat{J}_{M,\rho}(a_1) \geq \hat{J}_{M,\rho}(a_2). \quad (6.10)$$

The following result, whose proof is postponed to Appendix 6.C for the sake of clarity, allows us to make the link between  $\hat{J}_{M,\rho}$  and  $J_\rho$ .

**LEMMA 6.3** Let  $\Omega \in \mathcal{O}_m$ . There exists  $(M_n)_{n \in \mathbb{N}}$  such that

$$M_n \rightarrow +\infty \quad \text{and} \quad \lim_{n \rightarrow +\infty} \hat{J}_{M_n,\rho}(\mathbf{1}_\Omega) = J_\rho(\Omega).$$

Setting then  $a_1 = \mathbb{1}_{\Omega_1}$ ,  $a_2 = \mathbb{1}_{\Omega_2}$ ,  $M = M_n$ , where  $M_n$  is chosen as in the statement of Lemma 6.3, and passing to the limit in (6.10) as  $n \rightarrow \infty$  gives the monotonicity of  $J_\rho$ .

In the next sections, we will concentrate on showing the monotonicity property (6.10). To this aim, we will carefully analyze the so-called “switching function” (representing the gradient of the functional  $\hat{J}_{M,\rho}$ ) as the parameter  $M$  is large enough.

### 6.3.2 STRUCTURE OF THE SWITCHING FUNCTION

It is notable that, in this section, we will not make any assumption on  $g$  or  $f$ . Let  $M > 0$ . Considering the following relaxed version of Problem (6.3)

$$\inf_{a \in \hat{\mathcal{O}}_m} \hat{J}_{M,\rho}(a), \quad (6.11)$$

it is convenient to introduce the set of *admissible perturbations* in view of deriving first order optimality conditions.

**DEFINITION 6.3.2 : tangent cone, see e.g. [55]** Let  $a^* \in \mathcal{M}$  and  $\mathcal{T}_{a^*}$  be the tangent cone to the set  $\hat{\mathcal{O}}_m$  at  $a^*$ . The cone  $\mathcal{T}_{a^*}$  is the set of functions  $h \in L^\infty(D)$  such that, for any sequence of positive real numbers  $\varepsilon_n$  decreasing to 0, there exists a sequence of functions  $h_n \in L^\infty(D)$  converging to  $h$  as  $n \rightarrow +\infty$ , and  $a^* + \varepsilon_n h_n \in \mathcal{O}_m$  for every  $n \in \mathbb{N}$ .

In what follows, for any  $a \in \hat{\mathcal{O}}_m$ , any element  $h$  of the tangent cone  $\mathcal{T}_a$  will be called an *admissible direction*.

**LEMMA 6.4 : Differential of  $\hat{J}_{M,\rho}$**  Let  $a \in \hat{\mathcal{O}}_m$  and  $h \in \mathcal{T}_a$ . Let  $v_{M,\rho,a}$  be the unique solution of

$$\begin{cases} -\Delta v_{M,\rho,a} + M(1-a)v_{M,\rho,a} + \rho f'(u_{M,\rho,a})v_{M,\rho,a} = \rho f(u_{M,\rho,a}) & \text{in } D \\ v_{M,\rho,a} \in W_0^{1,2}(D). \end{cases} \quad (6.12)$$

Then,  $\hat{J}_{M,\rho}$  is differentiable in the sense of Fréchet at  $a$  in the direction  $h$  and its differential reads  $\langle d\hat{J}_{M,\rho}(a), h \rangle = \int_D h \Psi_a$ , where  $\Psi_a$  is the so-called “switching function” defined by

$$\Psi_a = -M \left( v_{M,\rho,a} + \frac{u_{M,\rho,a}}{2} \right) u_{M,\rho,a}.$$

*Proof of Lemma 6.4.* The Fréchet-differentiability of  $\hat{J}_{M,\rho}$  and of the mapping  $\mathcal{O}_m \ni a \mapsto u_{M,\rho,a} \in W_0^{1,2}(D)$  at  $m^*$  is standard (see e.g. [93, Chap. 8]). Let us consider an admissible perturbation  $h$  of  $a$  and let  $\dot{u}_{M,\rho,a}$  be the differential of  $u_{M,\rho,a}$  at  $a$  in direction  $h$ . One has

$$\begin{aligned} \langle d\hat{J}_{M,\rho}(a), h \rangle &= \int_D \nabla u_{M,\rho,a} \cdot \nabla \dot{u}_{M,\rho,a} + M \int_D (1-a)u_{M,\rho,a} \dot{u}_{M,\rho,a} - \frac{M}{2} \int_D h u_{M,\rho,a}^2 \\ &\quad - \langle g, \dot{u}_{M,\rho,a} \rangle_{H^{-1}(\Omega), W_0^{1,2}(\Omega)}, \end{aligned}$$

where  $\dot{u}_{M,\rho,a}$  solves the system

$$\begin{cases} -\Delta \dot{u}_{M,\rho,a} + M(1-a)\dot{u}_{M,\rho,a} + \rho f'(u_{M,\rho,a})\dot{u}_{M,\rho,a} = M h u_{M,\rho,a} & \text{in } D \\ \dot{u}_{M,\rho,a} \in W_0^{1,2}(D). \end{cases} \quad (6.13)$$

Let us multiply the main equation of (6.5) by  $\dot{u}_{M,\rho,a}$  and then integrate by parts. We get

$$\int_D \nabla u_{M,\rho,a} \cdot \nabla \dot{u}_{M,\rho,a} + M \int_D (1-a)u_{M,\rho,a} \dot{u}_{M,\rho,a} + \rho \int_D f(u_{M,\rho,a}) \dot{u}_{M,\rho,a} = \langle g, \dot{u}_{M,\rho,a} \rangle_{H^{-1}(\Omega), W_0^{1,2}(\Omega)}$$

and therefore,

$$\langle d\hat{J}_{M,\rho}(a), h \rangle = -\frac{M}{2} \int_D h u_{M,\rho,a^*}^2 - \rho \int_D f(u_{M,\rho,a^*}) \dot{u}_{M,\rho,a^*}$$

Let us multiply the main equation of (6.13) by  $v_{M,\rho,a}$  and then integrate by parts. We get

$$\int_D \nabla v_{M,\rho,a} \cdot \nabla \dot{u}_{M,\rho,a} + M \int_D (1-a) v_{M,\rho,a} \dot{u}_{M,\rho,a} + \rho \int_D f'(u_{M,\rho,a}) \dot{u}_{M,\rho,a} v_{M,\rho,a} = M \int_D h u_{M,\rho,a} v_{M,\rho,a}.$$

Similarly, multiplying the main equation of (6.12) by  $\dot{u}_{M,\rho,a}$  and then integrating by parts yields

$$\int_D \nabla v_{M,\rho,a} \cdot \nabla \dot{u}_{M,\rho,a} + M \int_D (1-a) v_{M,\rho,a} \dot{u}_{M,\rho,a} + \rho \int_D f'(u_{M,\rho,a}) \dot{u}_{M,\rho,a} v_{M,\rho,a} = \rho \int_D f(u_{M,\rho,a}) \dot{u}_{M,\rho,a}.$$

Combining the two relations above leads to

$$\rho \int_D f(u_{M,\rho,a}) \dot{u}_{M,\rho,a} = M \int_D h u_{M,\rho,a} v_{M,\rho,a}.$$

Plugging this relation into the expression of  $\langle d\hat{J}_{M,\rho}(a), h \rangle$  above yields the expected conclusion.  $\square$

### 6.3.3 PROOF THAT (6.10) HOLDS TRUE WHENEVER $\rho$ IS SMALL ENOUGH

Let us consider each set of assumptions separately.

Existence under the first assumption:  $g$  or  $-g$  satisfies the assumption  $(\mathbf{H}_1)$ .

According to the discussion carried out in Section 6.3.1, proving Theorem 6.2.1 boils down to proving monotonicity properties for the functional  $\hat{J}_{M,\rho}$  whenever  $\rho$  is small enough, which is the purpose of the next result.

**LEMMA 6.5** *Let  $a_1$  and  $a_2$  be two elements of  $\hat{\mathcal{O}}_m$  such that  $a_1 \leq a_2$  a.e. in  $D$ . If  $g$  or  $-g$  satisfies the assumption  $(\mathbf{H}_1)$ , then there exists  $\rho_1 = \rho_1(D, g_0, g_1, \|f\|_{W^{1,\infty}}) > 0$  such that*

$$\rho \in (0, \rho_1) \Rightarrow \hat{J}_{M,\rho}(a_1) \geq \hat{J}_{M,\rho}(a_2).$$

*Proof of Lemma 6.5.* Assume without loss of generality that  $g_0 > 0$ , the case  $g_1 < 0$  being easily inferred by modifying all the signs in the proof below. Then, one has

$$-\Delta u_{M,\rho,a} + M(1-a)u_{M,\rho,a} = g - \rho f(u_{M,\rho,a}) \geq 0 \quad \text{in } D,$$

whenever  $\rho \in (0, g_0/\|f\|_\infty)$ , and therefore, one has  $u_{M,\rho,a} \geq 0$  by the comparison principle.

Similarly, notice that

$$-\Delta u_{M,\rho,a} \leq g_1 + \rho \|f\|_\infty \quad \text{in } D,$$

which implies that  $u_{M,\rho,a} \leq (g_1 + \rho \|f\|_\infty) w_D$  where  $w_D$  is the torsion function of  $D$ . By the classical Talenti's estimate of the torsion function [171], we have  $\|w_D\|_\infty \leq \frac{1}{2d} \left( \frac{|D|}{\omega_d} \right)^{2/d}$  (where  $\omega_d$  is the volume of the unit ball). Thus

$$\|u_{M,\rho,a}\|_\infty \leq (g_1 + \rho \|f\|_\infty) \frac{1}{2d} \left( \frac{|D|}{\omega_d} \right)^{2/d} := C(g_0, \rho, \|f\|_\infty, D).$$



Setting  $U_{M,\rho,a} = \frac{1}{2}u_{M,\rho,a} + v_{M,\rho,a}$ , elementary computations show that  $U_{M,\rho,a}$  solves the problem

$$\begin{cases} -\Delta U_{M,\rho,a} + (M(1-a) + \rho f'(u_{M,\rho,a}))U_{M,\rho,a} = \frac{\rho}{2}(f(u_{M,\rho,a}) + u_{M,\rho,a}f'(u_{M,\rho,a})) + \frac{g}{2} & \text{in } D, \\ U_{M,\rho,a} = 0 & \text{on } \partial D. \end{cases} \quad (6.14)$$

**LEMMA 6.6** *Let us choose  $\rho_1$  in such a way that*

$$\rho_1(\|f\|_\infty + C(g_0, \frac{1}{2}, \|f\|_\infty, D)\|f'\|_\infty) < g_0, \quad \text{and} \quad \rho_1\|f'\|_\infty \leq \frac{\lambda_1(D)}{2}, \quad (6.15)$$

where  $\lambda_1(D)$  denotes the first eigenvalue of the Dirichlet-Laplacian operator on  $D$ . For every  $\rho \in [0, \rho_1)$ ,  $U_{M,\rho,a}$  is non-negative in  $D$ .

*Proof of Lemma 6.6.* The result follows immediately from the generalized maximum principle which claims that if a function  $v$  satisfies

$$-\Delta v + a(\cdot)v \geq 0 \quad \text{with } a(\cdot) > -\lambda_1(D) \quad (6.16)$$

and  $v = 0$  on  $\partial D$ , then  $v \geq 0$  a.e. in  $D$ . Here we have chosen  $\rho_1$  in such a way that

$$M(1-a) + \rho f'(u_{M,\rho,a}) \geq -\lambda_1(D)$$

and the right-hand side of (6.14) is non-negative which yields the result.  $\square$

Coming back to the proof of Lemma 6.5, consider  $h = a_2 - a_1$ . According to the mean value theorem, there exists  $\varepsilon \in (0, 1)$  such that

$$\hat{J}_{M,\rho}(a_2) - \hat{J}_{M,\rho}(a_1) = \langle d\hat{J}_{M,\rho}(a_1 + \varepsilon h), h \rangle = -M \int_D h u_{M,a_1 + \varepsilon h} U_{M,a_1 + \varepsilon h} \leq 0,$$

according to the combination of the analysis above with Lemma 6.4. The expected conclusion follows.  $\square$

**Existence under the second assumption:**  $g$  is non-negative and the function  $f$  satisfies the assumption  $(\mathbf{H}_2)$  or  $g$  is non-positive and the function  $-f$  satisfies the assumption  $(\mathbf{H}_2)$ .

The main difference with the previous case is that  $g$  might possibly be zero. Deriving the conclusion is therefore trickier and relies on a careful asymptotic analysis of the solution  $u_{M,\rho,a}$  as  $\rho \rightarrow 0$ .

**PROPOSITION 6.2** *There exists  $C = C(D, \|f\|_{L^\infty(\Omega)}) > 0$  such that, for any  $M \in \mathbb{R}_+$ , any  $a \in \hat{\mathcal{O}}_m$  such that  $a \geq 0$  a.e. in  $D$ , there holds*

$$\|u_{M,\rho,a} - u_{M,0,a}\|_{L^\infty(D)} \leq C\rho. \quad (6.17)$$

*Proof.* Let us set  $z_\rho = u_{M,\rho,a} - u_{M,0,a}$  for any  $\rho > 0$ . A direct computation yields that  $z_\rho$  satisfies

$$-\Delta z_\rho + M(1-a)z_\rho = -\rho f(u_{M,\rho,a}).$$

By comparison with the torsion function  $w_D$  of  $D$ , this implies

$$\|z_\rho\|_\infty \leq \rho \|f\|_\infty \|w_D\|_\infty$$

and the result follows, with a constant  $C$  explicit by Talenti's Theorem like in the proof of Lemma 6.5.  $\square$

Let us consider the switching function  $\Psi = -MU_{M,\rho,a}u_{M,\rho,a}$  where  $u_{M,\rho,a}$  and  $U_{M,\rho,a}$  respectively solve (6.5) and (6.14), and we will prove that both  $u_{M,\rho,a}$  and  $U_{M,\rho,a}$  are non-negative, so that one can conclude similarly to the previous case.

**LEMMA 6.7** *The functions  $u_{M,\rho,a}$  and  $U_{M,\rho,a}$  non-negative whenever  $\rho$  is small enough.*

*Proof.* Let us choose  $\rho$  such that  $\rho\|f\|_\infty < \lambda_1(D)$ . Since  $u_{M,\rho,a}$  satisfies

$$-\Delta u_{M,\rho,a} + M(1-a)u_{M,\rho,a} + \rho f(u_{M,\rho,a}) \geq 0$$

the non-negativity of  $u_{M,\rho,a}$  is a consequence of the generalized maximum principle (6.16). Indeed, for  $\rho$  small enough, we have

$$M(1-a) + \rho f(u_{M,\rho,a}) > -\lambda_1(D).$$

Since  $U_{M,\rho,a}$  satisfies (6.14), the proof follows the same lines assuming the  $\rho\|f'\|_\infty < \lambda_1(D)$  and using the assumption **(H<sub>2</sub>)** to get non-negativity of the right-hand side. By mimicking the reasoning done at the end of the first case, one gets that (6.10) is true if  $\rho$  is small enough.  $\square$

Thus, in both cases, the monotonicity of the functional is established, so that the theorem of Buttazzo and Dal Maso applies.

## 6.4 PROOF OF THEOREM 6.2.2

Note first that the functional  $J_\rho$  is shape differentiable, which follows from standard arguments, see e.g. [93, Chapitre 5].

Our proof of Theorem 6.2.2 is divided into two steps: after proving the criticality of  $\mathbb{B}^*$  for  $\rho$  small enough, we compute the second order shape derivative of the Lagrangian associated with the problem at the ball. Next, we establish that, under Assumption (6.8), there exists a positive constant  $C_0$  such that, for any admissible  $V$ , one has

$$(J_0 - \Lambda_0 \text{Vol})''(\mathbb{B}^*)[V, V] \geq C_0\|V \cdot \nu\|_{L^2(\Omega)}^2. \quad (6.18)$$

Finally, we prove that, for any radially symmetric, non-increasing non-negative  $g$ , there exists  $M \in \mathbb{R}$  such that, for any admissible  $V$ , one has

$$(J_\rho - \Lambda_\rho \text{Vol})''(\mathbb{B}^*)[V, V] \geq (J_0 - \Lambda_0 \text{Vol})''(\mathbb{B}^*)[V, V] - M\rho\|V \cdot \nu\|_{L^2(\Omega)}^2. \quad (6.19)$$

Local shape minimality of  $\mathbb{B}^*$  for  $\rho$  small enough can then be inferred in a straightforward way.

If  $V$  is an admissible vector field, we will denote by  $u'_{\rho,V}$  and  $u''_{\rho,V}$  the first and second order (eulerian) shape derivatives of  $u_\rho$  at  $\mathbb{B}^*$  with respect to  $V$ .

### 6.4.1 PRELIMINARY MATERIAL

**LEMMA 6.8** *Under the assumptions of Theorem 6.2.2, i.e when  $g$  is radially symmetric and non-increasing function, for  $\rho$  small enough, the function  $u_\rho$  is radially symmetric nonincreasing. We write it  $u_\rho = \varphi_\rho(|\cdot|)$ . Furthermore, if  $\rho = 0$ , one has*

$$-\frac{\partial u_0}{\partial \nu} \geq \frac{R}{2}g(R).$$

*Proof of Lemma 6.8.* The fact that  $u_\rho$  is a radially symmetric nonincreasing function follows from a simple application of the Schwarz rearrangement. Integrating the equation on the ball  $\mathbb{B}^*$  yields

$$-\int_{\mathbb{B}^*} \Delta u_0 = -\int_{\partial \mathbb{B}^*} \frac{\partial u_0}{\partial \nu} = -2\pi \frac{\partial u_0}{\partial \nu}$$

on the one-hand, while using the fact that  $g$  is decreasing:

$$-\int_{\mathbb{B}^*} \Delta u_0 = \int_{\mathbb{B}^*} g \geq 2\pi g(R) \int_0^R r dr = \pi R g(R)$$

□

By differentiating the main equation and the boundary conditions (see e.g. [93, Chapitre 5]), we get that the functions  $u'_{\rho,V}$  and  $u''_{\rho,V}$  satisfy

$$\begin{cases} -\Delta u'_{\rho,V} + \rho f'(u_\rho) u'_{\rho,V} = 0 & \text{in } \mathbb{B}^* \\ u'_{\rho,V} = -\frac{\partial u_\rho}{\partial \nu} V \cdot \nu & \text{on } \partial \mathbb{B}^* \end{cases} \quad (6.20)$$

and

$$\begin{cases} -\Delta u''_{\rho,V} + \rho f'(u_\rho) u''_{\rho,V} + \rho f''(u_\rho) (u'_{\rho,V})^2 = 0 & \text{in } \mathbb{B}^* \\ u''_{\rho,V} = -2\frac{\partial u'_{\rho,V}}{\partial \nu} V \cdot \nu - (V \cdot \nu)^2 \frac{\partial^2 u_\rho}{\partial \nu^2} & \text{on } \partial \mathbb{B}^*. \end{cases} \quad (6.21)$$

#### 6.4.2 PROOF OF THE SHAPE CRITICALITY OF THE BALL

Proving the shape criticality of the ball boils down to showing the existence of a Lagrange multiplier  $\Lambda_\rho \in \mathbb{R}$  such that for every admissible vector field  $V \in W^{3,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ , one has

$$(J_\rho - \Lambda_\rho \text{Vol})'(\mathbb{B}^*)[V] = 0 \quad (6.22)$$

Standard computations (see e.g. [93, chapitre 5]) yield

$$\begin{aligned} J'_\rho(\mathbb{B}^*)[V] &= \int_{\mathbb{B}^*} \langle \nabla u_\rho, \nabla u'_{\rho,V} \rangle - \int_{\mathbb{B}^*} g u'_{\rho,V} + \int_{\mathbb{S}^*} \frac{1}{2} |\nabla u_\rho|^2 V \cdot \nu \\ &= \int_{\mathbb{S}^*} u'_{\rho,V} \frac{\partial u_\rho}{\partial \nu} - \rho \int_{\mathbb{B}^*} u'_{\rho,V} f(u_\rho) + \int_{\mathbb{S}^*} \frac{1}{2} |\nabla u_\rho|^2 V \cdot \nu \\ &= - \int_{\mathbb{S}^*} \left( \frac{\partial u_\rho}{\partial \nu} \right)^2 V \cdot \nu + \int_{\mathbb{S}^*} \frac{1}{2} |\nabla u_\rho|^2 V \cdot \nu - \rho \int_{\mathbb{B}^*} u'_{\rho,V} f(u_\rho) \\ &= -\frac{1}{2} \int_{\mathbb{S}^*} |\nabla u_\rho|^2 V \cdot \nu - \rho \int_{\mathbb{B}^*} u'_{\rho,V} f(u_\rho). \end{aligned}$$

We introduce the adjoint state  $p_\rho$  as the unique solution of

$$\begin{cases} -\Delta p_\rho + \rho p_\rho f'(u_\rho) + \rho f(u_\rho) = 0 & \text{in } \mathbb{B}^* \\ p_\rho = 0 & \text{on } \mathbb{S}^*. \end{cases} \quad (6.23)$$

Since  $u_\rho$  is radially symmetric, so is  $p_\rho$ . Multiplying the main equation of (6.23) by  $u'_{\rho,V}$  and integrating by parts yields

$$-\rho \int_{\mathbb{B}^*} u'_{\rho,V} f(u_\rho) = \int_{\mathbb{S}^*} \frac{\partial p_\rho}{\partial \nu} \frac{\partial u_\rho}{\partial \nu} V \cdot \nu,$$

and finally

$$J'_\rho(\mathbb{B}^*)[V] = \int_{\mathbb{S}^*} \left( \frac{\partial p_\rho}{\partial \nu} \frac{\partial u_\rho}{\partial \nu} - \frac{1}{2} \left( \frac{\partial u_\rho}{\partial \nu} \right)^2 \right) V \cdot \nu.$$

Observe that  $\frac{\partial p_\rho}{\partial \nu}$  and  $\frac{\partial u_\rho}{\partial \nu}$  are constant on  $\mathbb{S}^*$  since  $u_\rho$  and  $p_\rho$  are radially symmetric. Introduce the real number  $\Lambda_\rho$  given by

$$\Lambda_\rho = \left. \frac{\partial p_\rho}{\partial \nu} \frac{\partial u_\rho}{\partial \nu} - \frac{1}{2} \left( \frac{\partial u_\rho}{\partial \nu} \right)^2 \right|_{\mathbb{S}^*}, \quad (6.24)$$

we get that (6.22) is satisfied, whence the result.

In what follows, we will exploit the fact that the adjoint state is radially symmetric. In the following definition, we sum-up the notations we will use in what follows.

**DEFINITION 6.4.1** *Recall that  $\varphi_\rho$  (defined in Lemma 6.8) is such that*

$$u_\rho(x) = \varphi_\rho(|x|), \quad \forall x \in \mathbb{B}^*.$$

*Since  $p_\rho$  is also radially symmetric, introduce  $\phi_\rho$  such that*

$$p_\rho(x) = \phi_\rho(|x|), \quad \forall x \in \mathbb{B}^*.$$

### 6.4.3 SECOND ORDER OPTIMALITY CONDITIONS

Let us focus on the second and third points of Theorem 6.2.2, especially on (6.9). Since  $\mathbb{B}^*$  is a critical shape, it is enough to work with normal vector fields, in other words vector fields  $V$  such that  $V = (V \cdot \nu)\nu$  on  $\mathbb{S}^*$ . Consider such a vector field  $V$ . For the sake of notational simplicity, let us set  $J''_\rho = J''_\rho(\mathbb{B}^*)[V, V]$ ,  $\mathcal{L}''_{\Lambda_\rho} = (J_\rho - \Lambda_\rho \text{Vol})''(\mathbb{B}^*)[V, V]$ ,  $u = u_\rho$ ,  $u' = u'_{\rho, V}$  and  $u'' = u''_{\rho, V}$ .

#### 6.4.3.1 COMPUTATION OF THE SECOND ORDER DERIVATIVE

To compute the second order derivative, we use the Hadamard second order formula [93, Chap. 5, p. 227] for normal vector fields, namely

$$\left. \frac{d^2}{dt^2} \right|_{t=0} \int_{(\text{Id} + tV)\mathbb{B}^*} f(t) = \int_{\mathbb{B}^*} f''(0) + 2 \int_{\mathbb{S}^*} f'(0) V \cdot \nu + \int_{\mathbb{S}^*} \left( \frac{1}{R} f(0) + \frac{\partial f(0)}{\partial \nu} \right) (V \cdot \nu)^2,$$

applied to  $f(t) = \frac{1}{2} |\nabla u_t|^2 - g u_t$ , where  $u_t$  denotes the solution of (6.2) on  $(\text{Id} + tV)\mathbb{B}^*$ .

The Hadamard formula along with the weak formulation of Equations (6.20)-(6.21) yields

$$\begin{aligned} J''_\rho &= \int_{\mathbb{B}^*} \langle \nabla u, \nabla u'' \rangle - \int_{\mathbb{B}^*} g u'' + \int_{\mathbb{B}^*} |\nabla u'|^2 + 2 \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu - 2 \int_{\mathbb{S}^*} g u' V \cdot \nu \\ &+ \int_{\mathbb{S}^*} \left( \frac{1}{2R} |\nabla u|^2 + \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial \nu^2} - g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 \\ &= -\rho \int_{\mathbb{B}^*} f(u) u'' - \rho \int_{\mathbb{B}^*} (u')^2 f'(u) + \int_{\mathbb{S}^*} u'' \frac{\partial u}{\partial \nu} + \int_{\mathbb{S}^*} u' \frac{\partial u'}{\partial \nu} \\ &+ 2 \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu - 2 \int_{\mathbb{S}^*} g u' V \cdot \nu + \int_{\mathbb{S}^*} \left( \frac{1}{2R} |\nabla u|^2 + \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial \nu^2} - g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 \\ &= -\rho \int_{\mathbb{B}^*} f(u) u'' - \rho \int_{\mathbb{B}^*} (u')^2 f'(u) + \int_{\mathbb{S}^*} \left( -2 \frac{\partial u'}{\partial \nu} V \cdot \nu - \frac{\partial^2 u}{\partial \nu^2} (V \cdot \nu)^2 \right) \frac{\partial u}{\partial \nu} \\ &- \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu + 2 \int_{\mathbb{S}^*} \left( \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} - g u' \right) V \cdot \nu + \int_{\mathbb{S}^*} \left( \frac{1}{2R} |\nabla u|^2 + \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial \nu^2} - g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 \\ &= -\rho \int_{\mathbb{B}^*} f(u) u'' - \rho \int_{\mathbb{B}^*} (u')^2 f'(u) - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial \nu^2} (V \cdot \nu)^2 - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} (V \cdot \nu) \end{aligned}$$

$$\begin{aligned}
 & + 2 \int_{\mathbb{S}^*} g \frac{\partial u}{\partial \nu} (V \cdot \nu)^2 + \int_{\mathbb{S}^*} \left( \frac{1}{2R} |\nabla u|^2 + \frac{\partial u}{\partial \nu} \frac{\partial^2 u}{\partial \nu^2} - g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 \\
 & = -\rho \int_{\mathbb{B}^*} f(u) u'' - \rho \int_{\mathbb{B}^*} (u')^2 f'(u) + \int_{\mathbb{S}^*} \left( \frac{1}{2R} \left( \frac{\partial u}{\partial \nu} \right)^2 + g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu.
 \end{aligned}$$

As such, the two first terms of the sum in the expression above are not tractable. Let us rewrite them. Multiplying the main equation of (6.23) by  $u''$  and integrating two times by parts yields

$$-\rho \int_{\mathbb{B}^*} f(u) u'' = \int_{\mathbb{S}^*} u'' \frac{\partial p_\rho}{\partial \nu} - \rho \int_{\mathbb{B}^*} (u')^2 p_\rho f''(u).$$

To handle the last term of the right-hand side, let us introduce the function  $\lambda_\rho$  defined as the solution of

$$\begin{cases} -\Delta \lambda_\rho + \rho \lambda_\rho f'(u) + \rho u' p_\rho f''(u) = 0 & \text{in } \mathbb{B}^* \\ \lambda_\rho = 0 & \text{on } \mathbb{S}^*. \end{cases} \quad (6.25)$$

Multiplying this equation by  $u'$  and integrating by parts gives

$$-\rho \int_{\mathbb{B}^*} f''(u) (u')^2 = \int_{\mathbb{S}^*} u' \frac{\partial \lambda_\rho}{\partial \nu}.$$

To handle the term  $-\rho \int_{\mathbb{B}^*} (u')^2 f'(u)$  of  $J''_\rho$ , we introduce the function  $\eta_\rho$ , defined as the only solution to

$$\begin{cases} -\Delta \eta_\rho + \rho \eta_\rho f'(u) + \rho u' f'(u) = 0 & \text{in } \mathbb{B}^* \\ \eta_\rho = 0 & \text{on } \mathbb{S}^*. \end{cases} \quad (6.26)$$

Multiplying this equation by  $u'$  and integrating by parts gives

$$-\rho \int_{\mathbb{B}^*} (u')^2 f'(u) = \int_{\mathbb{S}^*} u' \frac{\partial \eta_\rho}{\partial \nu} = - \int_{\mathbb{S}^*} V \cdot \nu \frac{\partial \eta_\rho}{\partial \nu} \frac{\partial u}{\partial \nu}.$$

Gathering these terms, we have

$$\begin{aligned}
 J''_\rho & = \int_{\mathbb{S}^*} u'' \frac{\partial p_\rho}{\partial \nu} + \int_{\mathbb{S}^*} u' \frac{\partial \lambda_\rho}{\partial \nu} - \int_{\mathbb{S}^*} \frac{\partial \eta_\rho}{\partial \nu} \frac{\partial u}{\partial \nu} V \cdot \nu - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu \\
 & + \int_{\mathbb{S}^*} \left( \frac{1}{2R} \left( \frac{\partial u}{\partial \nu} \right)^2 + g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2.
 \end{aligned}$$

Using that

$$\Lambda_\rho = \frac{\partial p_\rho}{\partial \nu} \frac{\partial u_\rho}{\partial \nu} - \frac{1}{2} \left( \frac{\partial u_\rho}{\partial \nu} \right)^2 \Big|_{\mathbb{S}^*} \quad \text{and} \quad \text{Vol}''(\mathbb{B}^*) = \int_{\mathbb{S}^*} \frac{1}{R} (V \cdot \nu)^2,$$

one computes

$$\begin{aligned}
 \mathcal{L}''_{\Lambda_\rho} & = \int_{\mathbb{S}^*} u'' \frac{\partial p_\rho}{\partial \nu} + \int_{\mathbb{S}^*} u' \frac{\partial \lambda_\rho}{\partial \nu} - \int_{\mathbb{S}^*} \frac{\partial \eta_\rho}{\partial \nu} \frac{\partial u}{\partial \nu} V \cdot \nu - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu \\
 & + \int_{\mathbb{S}^*} \left( -\frac{\Lambda_\rho}{R} + \frac{1}{2R} \left( \frac{\partial u}{\partial \nu} \right)^2 + g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2
 \end{aligned} \quad (6.27)$$

## 6.4.3.2 EXPANSION IN FOURIER SERIES

In this section, we recast the expression of  $\mathcal{L}''_{\Lambda_\rho}$  in a more tractable form, by using the method introduced by Lord Rayleigh: since we are dealing with vector fields normal to  $\mathbb{S}^*$ , we expand  $V \cdot \nu$  as a Fourier series. This leads to introduce the sequences of Fourier coefficients  $(\alpha_k)_{k \in \mathbb{N}^*}$  and  $(\beta_k)_{k \in \mathbb{N}^*}$  defined by:

$$V \cdot \nu = \sum_{k \in \mathbb{N}^*} \left( \alpha_k \cos(k \cdot) + \beta_k \sin(k \cdot) \right),$$

the equality above being understood in a  $L^2(\mathbb{S}^*)$  sense.

Let  $v_{k,\rho}$  (resp.  $w_{k,\rho}$ ) denote the function  $u'$  associated to the perturbation choice  $V_k$  given by  $V_k = V_k^c := \cos(k \cdot) \nu$  (resp.  $V_k = V_k^s := \sin(k \cdot) \nu$ ), in other words,  $v_{k,\rho} = u'_{\rho, V_k^c}$  (resp.  $w_{k,\rho} = u'_{\rho, V_k^s}$ ). Then, one shows easily (by uniqueness of the solutions of the considered PDEs) that for every  $k \in \mathbb{N}$ , there holds

$$v_{k,\rho}(r, \theta) = \psi_{k,\rho}(r) \cos(k\theta) \quad (\text{resp. } w_{k,\rho}(r, \theta) = \psi_{k,\rho}(r) \sin(k\theta)),$$

where  $(r, \theta)$  denote the polar coordinates in  $\mathbb{R}^2$ , where  $\psi_{k,\rho}$  solves

$$\begin{cases} -\frac{1}{r}(r\psi'_{k,\rho})' = -\left(\frac{k^2}{r^2} + \rho f'(u)\right) \psi_{k,\rho} & \text{in } (0, R) \\ \psi_{k,\rho}(R) = -\varphi'_\rho(R). \end{cases} \quad (6.28)$$

By linearity, we infer that

$$u' = \sum_{k \in \mathbb{N}^*} \alpha_k v_{k,\rho} + \beta_k w_{k,\rho}.$$

For every  $k \in \mathbb{N}^*$ , let us introduce  $\eta_{k,\rho}$  as the solution of (6.26) associated with  $v_{k,\rho}$ . One shows that  $\eta_{k,\rho}$  satisfies

$$\begin{cases} -\Delta \eta_{k,\rho} + \rho f'(u) \eta_{k,\rho} + \rho f'(u) v_{k,\rho} = 0 & \text{in } \mathbb{B}^* \\ \eta_{k,\rho} = 0 & \text{on } \mathbb{S}^*. \end{cases} \quad (6.29)$$

Similarly, one shows easily that

$$\eta_{k,\rho}(r, \theta) = \xi_{k,\rho}(r) \cos(k\theta),$$

where  $\xi_{k,\rho}$  satisfies

$$\begin{cases} -\frac{1}{r}(r\xi'_{k,\rho})' = -\left(\frac{k^2}{r^2} + \rho f'(u)\right) \xi_{k,\rho} - \rho \psi_{k,\rho} & \text{in } (0, R) \\ \xi_{k,\rho}(R) = 0. \end{cases} \quad (6.30)$$

Notice that one has  $\xi_{k,\rho} = 0$  whenever  $\rho = 0$ , which can be derived obviously from (6.26).

Since  $u_\rho$  is radially symmetric we denote by  $r \mapsto \varphi_\rho(r)$  this radial function.

Finally, we introduce a last set of equations related to  $\lambda_\rho$ . Let us define  $\zeta_{k,\rho}$  as the solution of

$$\begin{cases} -(r\zeta'_{k,\rho})' = -\frac{k^2}{r^2} \zeta_{k,\rho} - r\rho \zeta_{k,\rho} f'(u) - \rho r \psi_{k,\rho} \phi_\rho f''(u) & \text{in } (0, R) \\ \zeta_{k,\rho}(R) = 0. \end{cases} \quad (6.31)$$

and verify that  $\lambda_\rho = \zeta_{k,\rho}(r) \cos(k\theta)$  whenever  $V = V_k$ .

**PROPOSITION 6.3** *The quadratic form  $\mathcal{L}''_{\Lambda_\rho}$  expands as*

$$\mathcal{L}''_{\Lambda_\rho} = \sum_{k=1}^{\infty} \omega_{k,\rho} (\alpha_k^2 + \beta_k^2), \quad (6.32)$$

where, for any  $k \in \mathbb{N}^*$ ,

$$\begin{aligned} \omega_{k,\rho} = \pi R \left( -2\psi'_{k,\rho}(R)\phi'_\rho(R) - \varphi'_\rho(R)\zeta'_{k,\rho}(R) - \varphi''_\rho(R)\phi'_\rho(R) \right. \\ \left. - \xi'_{k,\rho}(R)\varphi'_\rho(R) - \frac{\Lambda_\rho}{R} + \frac{1}{2R}(\varphi'_\rho)^2 + g(R)\varphi'_\rho(R) - \varphi'_\rho(R)\psi'_{k,\rho}(R) \right), \end{aligned} \quad (6.33)$$

the functions  $\psi_{k,\rho}$ ,  $\xi_{k,\rho}$ ,  $\zeta_{k,\rho}$  being respectively defined by (6.28), (6.30), (6.31), and  $\Lambda_\rho$  is given by (6.24).

*Proof of Proposition 6.3.* Let us first deal with the particular case  $V \cdot \nu = \cos(k \cdot)$ . According to (6.20), (6.21) and (6.27), one has

$$\begin{aligned} \mathcal{L}''_{\Lambda_\rho} &= \int_{\mathbb{S}^*} u'' \frac{\partial p_\rho}{\partial \nu} + \int_{\mathbb{S}^*} u' \frac{\partial \lambda_\rho}{\partial \nu} - \int_{\mathbb{S}^*} \frac{\partial \eta_\rho}{\partial \nu} \frac{\partial u}{\partial \nu} V \cdot \nu - \int_{\mathbb{S}^*} \frac{\partial u}{\partial \nu} \frac{\partial u'}{\partial \nu} V \cdot \nu \\ &\quad + \int_{\mathbb{S}^*} \left( -\frac{\Lambda_\rho}{R} + \frac{1}{2R} \left( \frac{\partial u}{\partial \nu} \right)^2 + g \frac{\partial u}{\partial \nu} \right) (V \cdot \nu)^2 \\ &= R \int_0^{2\pi} (-2 \cos(k\theta)^2 \psi'_{k,\rho}(R) - \cos(k\theta)^2 \varphi''_\rho(R)) \phi'_\rho(R) d\theta - R \int_0^{2\pi} \cos(k\theta)^2 \varphi'_\rho(R) \zeta'_{k,\rho}(R) d\theta \\ &\quad - R \int_0^{2\pi} \cos(k\theta)^2 \xi'_{k,\rho}(R) \varphi'_\rho(R) d\theta + R \int_0^{2\pi} \cos(k\theta)^2 \left( -\frac{\Lambda_\rho}{R} + \frac{1}{2R} (\varphi'_\rho)^2 + g(R) \varphi'_\rho(R) \right) d\theta \\ &\quad - R \int_0^{2\pi} \cos(k\theta)^2 \varphi'_\rho(R) \psi'_{k,\rho}(R) d\theta \end{aligned}$$

and therefore

$$\begin{aligned} \frac{\mathcal{L}''_{\Lambda_\rho}}{\pi R} &= -2\psi'_{k,\rho}(R)\phi'_\rho(R) - \varphi'_\rho(R)\zeta'_{k,\rho}(R) - \varphi''_\rho(R)\phi'_\rho(R) - \xi'_{k,\rho}(R)\varphi'_\rho(R) \\ &\quad - \frac{\Lambda_\rho}{R} + \frac{1}{2R}(\varphi'_\rho)^2 + g(R)\varphi'_\rho(R) - \varphi'_\rho(R)\psi'_{k,\rho}(R) \end{aligned}$$

We have then obtained the expected expression for this particular choice of vector field  $V$ . Similar computations enable us to recover the formula when dealing with the vector field  $V$  given by  $V \cdot \nu = \sin(k \cdot)$ . Finally, for general  $V$ , one has to expand the square  $(V \cdot \nu)^2$ , and the computation follows exactly the same lines as before. Note that all the crossed terms of the sum (i.e. the term that do not write as squares of real numbers) vanish, by using the  $L^2(\mathbb{S})$  orthogonality properties of the families  $(\cos(k \cdot), \sin(k \cdot))_{k \in \mathbb{N}^*}$ .  $\square$

#### 6.4.3.3 COMPARISON PRINCIPLE ON THE FAMILY $\{\omega_{k,\rho}\}_{k \in \mathbb{N}^*}$

The next result allows us to recast the ball stability issue in terms of the sign of  $\omega_{1,\rho}$ .

**PROPOSITION 6.4** *There exists  $M > 0$  such that, for any  $\rho$  small enough,*

$$\forall k \in \mathbb{N}^*, \quad \omega_{k,\rho} - \omega_{1,\rho} \geq -M\rho \quad \text{and} \quad |\omega_{1,\rho} - \omega_{1,0}| \leq M\rho.$$

*Proof of Proposition 6.4.* Fix  $k \in \mathbb{N}$  and introduce  $\tilde{\omega}_{k,\rho} = \omega_{k,\rho}/(\pi R)$ . Using (6.33), one computes

$$\begin{aligned} \tilde{\omega}_{k,\rho} - \tilde{\omega}_{1,\rho} &= (-\varphi'_\rho(R) - 2\phi'_\rho(R)) (\psi'_{k,\rho}(R) - \psi'_{1,\rho}(R)) \\ &\quad - \varphi'_\rho(R) (\xi'_{k,\rho}(R) - \xi'_{1,\rho}(R) + \zeta'_{k,\rho}(R) - \zeta'_{1,\rho}(R)). \end{aligned}$$

We need to control each term of the expression above, which is the goal of the next results, whose proofs are postponed at the end of this section.

**LEMMA 6.9** *There exists  $M > 0$  and  $\bar{\rho} > 0$  such that for  $\rho \in [0, \bar{\rho}]$ , one has*

$$\max \{ \|\varphi'_\rho - \varphi'_0\|_{L^\infty(0,R)}, \|\phi'_\rho\|_{L^\infty(0,R)}, \|\xi'_{k,\rho}\|_{L^\infty} \} \leq M\rho \quad \text{and} \quad \|\zeta'_{k,\rho}\|_{L^\infty} \leq M\rho^2.$$

According to Lemma 6.8, one has in particular  $\varphi'_0(R) < 0$ . We thus infer from Lemma 6.9 the existence of  $\delta > 0$  such that

$$\min\{-\varphi'_\rho(R) - 2\phi'_\rho(R), -\varphi'_\rho(R)\} \geq \delta > 0.$$

for  $\rho$  small enough. Furthermore, Lemma 6.9 also yields easily the estimate

$$|\zeta'_{k,\rho}(R) - \zeta'_{1,\rho}(R)| \leq M\rho^2$$

Hence, we are done by applying the following result.

**LEMMA 6.10** *There exists  $\hat{M} > 0$  and  $\bar{\rho} > 0$  such that for  $\rho \in [0, \bar{\rho}]$ , one has*

$$\psi'_{k,\rho}(R) - \psi'_{1,\rho}(R) \geq 0 \quad \text{and} \quad |\xi'_{k,\rho}(R) - \xi'_{1,\rho}(R)| \leq \hat{M}\rho. \quad (6.34)$$

Indeed, the results above lead to

$$\omega_{k,\rho} - \omega_{1,\rho} \geq \delta(\psi'_{k,\rho}(R) - \psi'_{1,\rho}(R) + \xi'_{k,\rho}(R) - \xi'_{1,\rho}(R)) \geq 0$$

for every  $k \geq 1$  and  $\rho$  small enough.

Finally, the proof of the second inequality follows the same lines and are left to the reader.  $\square$

*Proof of Lemma 6.9.* These convergence rates are simple consequences of elliptic regularity theory. Since the reasonings for each terms are similar, we only focus on the estimate of  $\|\phi'_\rho\|_{L^\infty(\Omega)}$ . Recall that  $p_\rho$  solves the equation (6.23). Multiplying this equation by  $p_\rho$ , integrating by parts and using the Poincaré inequality yield the existence of  $C > 0$  such that

$$(1 - \rho C \|f'\|_{L^\infty(\mathbb{B}^*)}) \|\nabla p_\rho\|_{L^2(\mathbb{B}^*)}^2 \leq \rho \|f\|_{L^\infty(\mathbb{B}^*)} \|p_\rho\|_{L^2(\mathbb{B}^*)},$$

so that  $\|p_\rho\|_{W_0^{1,2}(\mathbb{B}^*)}$  is uniformly bounded for  $\rho$  small enough. Hence, the elliptic regularity theory yields that  $p_\rho$  is in fact uniformly bounded in  $W^{2,2}(\mathbb{B}^*)$ , and there exists  $\hat{M} > 0$  such that  $\|p_\rho\|_{W_0^{2,2}(\mathbb{B}^*)} \leq \hat{M}\rho$  and, since  $\mathbb{B}^* \subset \mathbb{R}^2$ , we get

$$\|p_\rho\|_{L^\infty(\mathbb{B}^*)} \leq M\rho.$$

Since  $\Delta p_\rho = \rho p_\rho f'(u_\rho) + \rho f(u_\rho)$  and the right-hand side belongs to  $L_0^p(\mathbb{B}^*)$  for all  $p \geq 1$ , the elliptic regularity theory yields the existence of  $C > 0$  such that

$$\|p_\rho\|_{W_0^{2,p}(\Omega)} \leq C(\rho \|p_\rho\|_{L^\infty} \|f'\|_{L^\infty} + \rho \|f\|_{L^\infty}) \leq M\rho$$

and using the embedding  $W^{2,p} \hookrightarrow \mathcal{C}^{1,\alpha}$  for  $p$  large enough, one finally gets

$$\|\nabla p_\rho\|_{L^\infty(\mathbb{B}^*)} \leq M\rho.$$

$\square$

*Proof of Lemma 6.10.* The two estimates are proved using the maximum principle. Let us first prove that, for any  $k$  and any  $\rho$  small enough,  $\psi_{k,\rho}$  is non-negative on  $(0, R)$ . Since, for  $\rho$  small enough,  $-\varphi'_\rho(R)$  is positive, and therefore  $\psi_{k,\rho}(R) > 0$ . Since  $v_k$  belongs to  $W_0^{1,2}$ , one has necessarily  $\psi_{k,\rho}(0) =$



0. Furthermore, according to (6.28), by considering  $\rho > 0$  small enough so that

$$-\frac{1}{r^2} + \rho \|f'\|_{L^\infty} \leq -\frac{1}{2r^2}$$

it follows that

$$-\frac{1}{r}(r\psi'_{k,\rho})' = c_{k,\rho}(r)\psi_{k,\rho} \quad \text{with} \quad c_{k,\rho} = -\frac{k^2}{r^2} - \rho f'(u_0) < 0.$$

Let us argue by contradiction, assuming that  $\psi_{k,\rho}$  reaches a negative minimum at a point  $r_1$ . Because of the boundary condition,  $r_1$  is necessarily an interior point of  $(0, R)$ . Then, from the equation,

$$0 \geq -\psi''_{k,\rho}(r_1) = c_{k,\rho}(r_1)\psi_{k,\rho}(r_1) > 0,$$

which is a contradiction. Thus there exists  $\bar{\rho} > 0$  small enough such that, for any  $\rho \leq \bar{\rho}$  and every  $k \in \mathbb{N}^*$ ,  $\psi_{k,\rho}$  is non-negative on  $(0, R)$ .

Now, introduce  $z_k = \psi_{k,\rho} - \psi_{1,\rho}$  for every  $k \geq 1$  and notice that it satisfies

$$-\frac{1}{r}(rz'_k)' = \frac{1}{r^2}\psi_{1,\rho} - \frac{k^2}{r^2}\psi_{k,\rho} - \rho f'(u_0)z_k.$$

Since  $\psi_{k,\rho}$  is non-negative, it implies

$$-\frac{1}{r}(rz'_k)' \leq \left(-\frac{k^2}{r^2} - \rho f'(u_0)\right)z_k, \quad \text{and} \quad z_k(R) = z_k(0) = 0.$$

Up to decreasing  $\bar{\rho}$ , one may assume that for  $\rho \leq \bar{\rho}$ ,  $-\frac{k^2}{r^2} - \rho f'(u_0) < 0$  in  $(0, R)$ . If  $z_k$  reached a positive maximum, it would be at an interior point  $r_1$ , but we would have

$$0 \leq -z''_k(r_1) < \left(-\frac{k^2}{r^2} - \rho f'(u_0)\right)z_k(r_1) < 0.$$

Hence, one has necessarily  $z_k \leq 0$  in  $(0, R)$  and  $z_k$  reaches a maximum at  $R$ , which means in particular that  $z'_k(R) = \psi'_{k,\rho}(R) - \psi'_{1,\rho}(R) \geq 0$ .  $\square$

#### 6.4.4 SHAPE (IN)STABILITY OF $\mathbb{B}^*$

##### 6.4.4.1 UNDER ASSUMPTION (6.8)

Stability under Assumption (6.8) is well known (see [65]) in the case where  $\rho = 0$ . Hereafter, we recall the proof, showing by the same a stability result for  $\rho > 0$ .

**LEMMA 6.11** *Under assumption (6.8), one has  $\omega_{1,0} > 0$ .*

This Lemma concludes the proof of the second part of Theorem 6.2.2. Indeed, according to Propositions 6.3 and 6.4, there holds

$$\begin{aligned} \mathcal{L}''_{\Lambda_\rho}(\mathbb{B}^*)[V, V] &\geq \left(\frac{\omega_{1,0}}{2} + O(\rho)\right) \sum_{k=1}^{\infty} (\alpha_k^2 + \beta_k^2) \\ &= \left(\frac{\omega_{1,0}}{2} + O(\rho)\right) \|V \cdot \nu\|_{L^2}^2 \end{aligned}$$

for  $\rho$  small enough.

*Proof of Lemma 6.11.* To compute  $\omega_{1,0}$ , recall that, for  $\rho = 0$ , the function  $\psi_{1,0}$  solves

$$-\frac{1}{r}(r\psi'_{1,0})' = -\frac{1}{r^2}\psi_{1,0} \quad \text{and} \quad \psi_{1,0}(R) = -\varphi'_0(R),$$

and therefore,  $\psi_{1,0}(r) = -\frac{r}{R}\varphi'_0(R)$  for all  $r \in [0, R]$ , so that

$$\begin{aligned} \frac{\omega_{1,0}}{\pi R} &= -\frac{\Lambda_0}{R} + \frac{1}{2R}(\varphi'_0(R))^2 + g(R)\varphi'_0(R) - \varphi'_0(R)\psi'_{1,0}(R) \\ &= \frac{1}{R}(\varphi'_0(R))^2 + g(R)\varphi'_0(R) + \frac{1}{R}(\varphi'_0(R))^2 \\ &= \frac{2}{R}(\varphi'_0(R))^2 + g(R)\varphi'_0(R) \\ &= -\varphi'_0(R) \left( -\frac{2}{R}\varphi'_0(R) - g(R) \right) \end{aligned}$$

where the expression of  $\Lambda_0$  is given by (6.24). Since  $-R\varphi'_0(R) = \int_0^R tg(t)dt = \frac{1}{2\pi} \int_{\mathbb{B}^*} g$ , and  $\varphi'_0(R) < 0$ , we infer that the sign of  $\omega_{1,0}$  is the sign of

$$-\frac{2}{R}\varphi'_0(R) - g(R) = \frac{1}{\pi R^2} \int_{\mathbb{B}^*} g - g(R),$$

and the positivity of this last quantity is exactly Assumption (6.8). The conclusion follows.

#### 6.4.4.2 AN EXAMPLE OF INSTABILITY

In this part, we will assume that  $g$  is the constant function equal to 1, i.e.  $g = 1$ . Even if the ball  $\mathbb{B}^*$  is known to be a minimizer in the case  $\rho = 0$ , it is a degenerate one in the sense that  $\omega_{1,0} = 0$  coming from the invariance by translations of the problem. In what follows, we exploit this fact and will construct a suitable nonlinearity  $f$  such that  $\mathbb{B}^*$  is not a local minimizer for  $\rho$  small enough, in other words such that  $\omega_{1,\rho} < 0$ .

We assume without loss of generality that  $R = 1$  for the sake of simplicity.

**LEMMA 6.12** *There holds*

$$\omega_{1,\rho} = \frac{\rho}{4}(w_1 + w'_1)(1) + O(\rho^2)$$

where  $w_1$  solves

$$\begin{cases} -(rw'_1)' = -\frac{1}{r}w_1 - \frac{r^2}{2}f'(\varphi_0) - \frac{r^2}{2} & \text{in } (0, 1) \\ w_1(1) = -\int_0^1 tf(\varphi_0) dt. \end{cases} \quad (6.35)$$

*Proof of Lemma 6.12.* The techniques to derive estimates follow exactly the same lines as in Lemma 6.9. First, we claim that

$$\varphi_\rho = \varphi_0 + \rho\varphi_1 + O(\rho^2) \quad \text{in } \mathcal{C}^1, \quad (6.36)$$

where  $\varphi_1$  satisfies

$$\begin{cases} -\frac{1}{r}(r\varphi'_1)' = -f(\varphi_0) & \text{in } (0, 1) \\ \varphi_1(1) = 0. \end{cases} \quad (6.37)$$

Indeed, considering the function  $\delta = \varphi_\rho - \varphi_0 - \rho\varphi_1$ , one shows easily that it satisfies

$$\begin{cases} -\frac{1}{r}(r\delta')' = \rho(f(\varphi_0) - f(\varphi_\rho)) & \text{in } (0, 1) \\ \delta(1) = 0. \end{cases}$$

Therefore, by mimicking the reasonings done in the proof of Lemma 6.9, involving the elliptic regularity theory, and the fact that  $\|\varphi_\rho - \varphi_0\|_{W^{1,\infty}} = O(\rho)$ , we infer that  $\|\delta\|_{\mathcal{C}^1} = O(\rho^2)$ , whence the result.

Using that  $\varphi_\rho$  satisfies  $-\frac{1}{r}(r\varphi'_\rho)' + \rho f(\varphi_\rho) = g$  and integrating this equation yields

$$-\varphi'_\rho(1) = \frac{1}{2} - \rho \int_0^1 tf(\varphi_\rho) dt = \frac{1}{2} - \rho \int_0^1 tf(\varphi_0(t)) dt + O(\rho^2). \quad (6.38)$$

The Equation on  $\phi_\rho$  reads

$$\begin{cases} -(r\phi'_\rho)' = r \left( -\rho\phi_\rho f'(\varphi_\rho) - \rho f(\varphi_\rho) \right) & \text{in } (0, 1) \\ \phi_\rho(0) = 0. \end{cases}$$

and according to Lemma 6.9, there holds  $\|\phi_\rho\|_{L^\infty} = O(\rho)$ . We thus infer that

$$-\phi'_\rho(1) = -\rho \int_0^1 tf(\varphi_0) dt + O(\rho^2). \quad (6.39)$$

From (6.38) and (6.39), we infer that

$$\begin{aligned} \Lambda_\rho &= \frac{1}{2}((\varphi'_\rho(1))^2 - \phi'_\rho(1)\varphi'_\rho(1)) = \frac{1}{2}\varphi'_0(1)^2 - \rho\varphi'_0(1) \int_0^1 tf(\varphi_0) dt + \rho\varphi'_0(1) \int_0^1 tf(\varphi_0) dt + O(\rho^2) \\ &= \frac{1}{2}\varphi'_0(1)^2 + O(\rho^2). \end{aligned} \quad (6.40)$$

Regarding  $\psi_{1,\rho}$  and using that it satisfies (6.28), we get

$$\psi_{1,\rho}(1) = -\varphi'_\rho(1) = \frac{1}{2} - \rho \int_0^1 tf(\varphi_0) dt.$$

We then infer that  $\|\psi_{1,\rho} + r\varphi'_{0,\rho}(1)\|_{\mathcal{C}^1} = O(\rho)$ . Plugging this estimate in (6.28) allows us to show that

$$\psi_{1,\rho}(r) = -\varphi'_0(1)r + \rho y_1(r) + O(\rho^2) \quad \text{in } \mathcal{C}^1(0, 1), \quad (6.41)$$

where  $y_1$  solves

$$\begin{cases} -(ry'_1)' = -\frac{1}{r}y_1 + r^2\varphi'_0(1)f'(\varphi_0) & \text{in } (0, 1) \\ y_1(1) = -\int_0^1 tf(\varphi_0) dt. \end{cases} \quad (6.42)$$

Regarding  $\xi_{1,\rho}$  and using that it satisfies (6.30), we easily get that  $\|\xi_{1,\rho}\|_{W^{1,\infty}} = O(\rho)$ , according to Lemma 6.9. This allows us to write

$$\xi_{1,\rho} = \rho z_1 + O(\rho^2) \quad \text{in } \mathcal{C}^1(0, 1) \quad (6.43)$$

where  $z_1$  satisfies

$$\begin{cases} -(rz'_1)' = -\frac{1}{r}z_1 + r^2\varphi'_0(1) & \text{in } (0, 1) \\ z_1(1) = 0. \end{cases} \quad (6.44)$$

Let us now expand  $\omega_{1,\rho}$  with respect to the parameter  $\rho$ . Recall that

$$\begin{aligned} \omega_{1,\rho} &= \frac{1}{2} \left( -2\psi'_{1,\rho}(1)\phi'_{1,\rho}(1) - \varphi''_\rho(1)\phi'_{1,\rho}(1) - \varphi'_\rho(R)\zeta'_{1,\rho}(R) \right. \\ &\quad \left. - \xi'_{1,\rho}(1)\varphi'_\rho(1) + \Lambda_\rho + \frac{1}{2}(\varphi'_\rho)^2 + \varphi'_\rho(1) - \varphi'_\rho(1)\psi'_{1,\rho}(1) \right). \end{aligned}$$

Regarding the term  $\varphi'_{0,\rho}(R)\zeta'_{1,\rho}(R)$ , we know from Lemma 6.9 that  $\|\zeta'_{1,\rho}(R)\|_{L^\infty} = O(\rho^2)$ .

Using this estimate and plugging the expansions (6.36)-(6.40)-(6.41)-(6.43) in the expression above yields successively

$$\begin{aligned}
 -2\psi'_{1,\rho}(1)\phi'_\rho(1) &= 2\rho\varphi'_0(1) \int_0^1 tf(\varphi_0) dt + O(\rho^2) = -\rho \int_0^1 tf(\varphi_0) dt + O(\rho^2). \\
 -\varphi''_\rho(1)\phi'_\rho(1) &= -\varphi''_0(1)\phi'_\rho(1) + O(\rho^2) = \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt + O(\rho^2). \\
 -\xi'_{1,\rho}(1)\varphi'_\rho(1) &= -\varphi'_0(1)\xi'_{1,\rho}(1) + O(\rho^2) = \frac{\rho}{2}z'_1(1) \\
 \Lambda_\rho + \frac{1}{2}(\varphi'_\rho)^2 &= \varphi'_0(1)^2 - \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt + O(\rho^2) = \frac{1}{4} - \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt + O(\rho^2) \\
 \varphi'_\rho(1) &= -\frac{1}{2} + \rho \int_0^1 tf(\varphi_0) dt + O(\rho^2) \\
 -\varphi'_\rho(1)\psi'_{1,\rho}(1) &= \varphi'_0(1)^2 - \rho\varphi'_0(1)y'_1(1) + \rho\varphi'_0(1) \int_0^1 tf(\varphi_0) dt + O(\rho^2) \\
 &= \frac{1}{4} + \frac{\rho}{2}y'_1(1) - \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt,
 \end{aligned}$$

by using that  $\|\phi_\rho\|_{W^{1,\infty}} = O(\rho)$  and  $\|\xi_{1,\rho}\|_{W^{1,\infty}} = O(\rho)$ . This gives

$$\begin{aligned}
 \omega_{1,\rho} &= -\rho \int_0^1 tf(\varphi_0) dt + \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt + \frac{\rho}{2}z'_1(1) \\
 &\quad + \frac{1}{4} - \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt - \frac{1}{2} + \rho \int_0^1 tf(\varphi_0) dt + \frac{1}{4} + \frac{\rho}{2}y'_1(1) - \frac{\rho}{2} \int_0^1 tf(\varphi_0) dt + O(\rho^2).
 \end{aligned}$$

As expected, the zero order terms cancel each other out and we get

$$\omega_{1,\rho} = -\frac{\rho}{2} \int_0^1 tf(\varphi_0) dt + \frac{\rho}{2}z'_1(1) + \frac{\rho}{2}y'_1(1) + O(\rho^2),$$

which concludes the proof by setting  $w_1 = y_1 + z_1$ . □

**Construction of the non-linearity.** Recall that we are looking for a non-linearity  $f$  such that  $\omega_{1,\rho} < 0$ , in other words such that  $(w_1 + w'_1)(1) < 0$  according to Lemma 6.12. To this aim, let us consider the function  $w_1$  solving (6.35). Let us consider a non-negative function  $f$  such that

$$f'(\cdot) < -1 \quad \text{on } [0, \|\varphi_0\|_{L^\infty}]. \tag{6.45}$$

It follows that

$$w_1(1) = - \int_0^1 tf(\varphi_0) dt < 0.$$

Besides,

$$-(rw'_1)' = -\frac{1}{r}w_1 - \frac{r^2}{2}(f'(\varphi_0) + 1) \geq -\frac{1}{r}w_1$$

by using (6.45). Thus  $w_1$  cannot reach a local negative minimum in  $(0, 1)$ . Moreover, by using that  $w_1$  is regular ( $w_1$  is the sum of two functions at least  $\mathcal{C}^1$  according to the proof of Lemma 6.12) and

integrating the equation above yields

$$-rw_1'(r) + \frac{1}{2} \int_0^r s^2 (f(\varphi_0(s)) + 1) ds = - \int_0^r \frac{w_1(s)}{s} ds$$

for  $r > 0$ . The left-hand side is well-defined and it follows that so is the right-hand side, which implies that necessarily  $w_1(0) = 0$  (else, we would immediately reach a contradiction).

Since  $w_1$  cannot reach a local minimum on  $(0, 1)$  and since  $0 = w_1(0) > w_1(1)$ , we get that  $w_1$  is decreasing on  $(1 - \delta, 1)$  for some  $\delta > 0$ , ensuring that  $w_1'(1) < 0$ . The conclusion follows.  $\square$

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# APPENDIX

## 6.A PROOF OF LEMMA 6.2

Recall that we want to establish a uniform (with respect to  $a$  and  $M$ )  $L^\infty$  bound on the solutions of

$$\begin{cases} -\Delta u_{M,\rho,a} + M(1-a)u_{M,\rho,a} + \rho f(u_{M,\rho,a}) = g, & \text{in } D, \\ u_{M,\rho,a} \in W_0^{1,2}(\Omega). \end{cases} \quad (6.46)$$

Here, it is assumed that  $g$  is non-negative.

Define  $\phi_g$  as the solution of

$$\begin{cases} -\Delta \phi_g + \rho f(\phi_g) = g, & \text{in } D, \\ \phi_g \in W_0^{1,2}(\Omega). \end{cases}$$

Standard  $L^p$  estimates show that  $\phi_g$  is continuous and that

$$\|\phi_g\|_{L^\infty(\Omega)} < +\infty.$$

Define  $z := \phi_g - u_{M,\rho,a} \in W_0^{1,2}(\Omega)$ . We can write

$$-\Delta z + \rho \frac{f(\phi_g) - f(u_{M,\rho,a})}{\phi_g - u_{M,\rho,a}} z = M(1-a)u_{M,\rho,a} \geq 0$$

The generalized maximum principle, and the fact that  $f$  is Lipschitz entails that  $z$  reaches its minimum on the boundary  $\partial D$ , so that  $z$  is non-negative. Thus

$$0 \leq u_{M,\rho,a} \leq \phi_g \leq \|\phi_g\|_{L^\infty(\Omega)} < +\infty$$

and we conclude by noting that the quantity in the right-hand side is uniformly bounded with respect to  $\rho \in [0, \underline{\rho}]$ .

## 6.B PROOF OF PROPOSITION 6.1

We recall that we want to establish that if  $(\Omega_k)_{k \in \mathbb{N}} \in \mathcal{O}_m^{\mathbb{N}}$   $\gamma$ -converges to  $\Omega$ , then

$$J_\rho(\Omega) \leq \liminf_{k \rightarrow \infty} J_\rho(\Omega_k).$$

Fix such a sequence  $(\Omega_k)_{k \in \mathbb{N}}$  that  $\gamma$ -converges to  $\Omega$ . For the sake of clarity, we drop the subscript  $\rho$ ,  $f$  and  $g$  and define, for every  $k \in \mathbb{N}$ ,  $u_k \in W_0^{1,2}(D)$  the unique solution to

$$\begin{cases} -\Delta u_k + \rho f(u_k) = g \text{ in } \Omega_k, \\ u_k \in W_0^{1,2}(\Omega_k), \\ u_k \text{ is extended by continuity as a function in } W_0^{1,2}(D). \end{cases}$$

First note that, for any  $k \in \mathbb{N}$ , multiplying the equation by  $u_k$  and integrating by parts immediately yields

$$\begin{aligned} \lambda_1(D) \int_D u_k^2 &= \lambda_1(D) \int_{\Omega_k} u_k^2 \leq \lambda_1(\Omega_k) \int_{\Omega_k} u_k^2 \leq \int_{\Omega_k} |\nabla u_k|^2 \\ &\leq \|g\|_{L^2(\Omega_k)} \|u\|_{L^2(\Omega_k)} + \rho \|f\|_{L^\infty(\mathbb{R})} |\Omega_k|^{\frac{1}{2}} \|u_k\|_{L^2(\Omega_k)}. \end{aligned}$$

The sequence  $(u_k)_{k \in \mathbb{N}}$  is thus uniformly bounded in  $W_0^{1,2}(D)$ . By the Rellich-Kondrachov Theorem,  $(u_k)_{k \in \mathbb{N}}$  converges (up to a subsequence, strongly in  $L^2(D)$  and weakly in  $W_0^{1,2}(D)$ ) to a function  $u \in W_0^{1,2}(D)$ .

The dominated convergence theorem then yields that the sequence  $(f(u_k))_{k \in \mathbb{N}}$  converges strongly in  $W_0^{-1,2}(D)$ , to  $f(u)$ . Thus, the sequence  $(g - f(u_k))_{k \in \mathbb{N}}$  converges strongly in  $W_0^{-1,2}(D)$  to  $g - f(u)$ . Since by assumption  $(\Omega_k)_{k \in \mathbb{N}}$   $\gamma$ -converges to  $\Omega$  and since the right hand term converges strongly to  $g - \rho f(u)$  in  $W_0^{-1,2}(D)$ , it follows that  $(u_k)_{k \in \mathbb{N}}$  converges strongly in  $W_0^{1,2}(D)$  to  $u$  and that  $u$  solves

$$\begin{cases} -\Delta u + \rho f(u) = g \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

This strong convergence immediately implies that

$$J(\Omega) \leq \liminf_{k \rightarrow \infty} J(\Omega_k),$$

thus concluding the proof of Proposition 6.1.

## 6.C PROOF OF LEMMA 6.3

Let us first prove that  $(u_{M,\rho,a})_{M \geq 0}$  is uniformly bounded in  $W_0^{1,2}(D)$  with respect to  $M$  and  $\rho$ . To this aim, let us multiply (6.2) by  $u_{M,\rho,a}$  and integrate by parts. One gets

$$\begin{aligned} \int_D |\nabla u_{M,\rho,a}|^2 &\leq \int_D |\nabla u_{M,\rho,a}|^2 + M(1-a)u_{M,\rho,a}^2 \\ &\leq \|g\|_{H^{-1}(D)} \|u_{M,\rho,a}\|_{L^2(D)} + \rho(f(0) + \|f\|_{W^{1,\infty}}) \|u_{M,\rho,a}\|_{L^2(D)}. \end{aligned}$$

By using the Poincaré inequality, we infer an uniform estimate of  $u_{M,\rho,a}$  in  $W_0^{1,2}(D)$ . According to the Rellich-Kondrachov Theorem, there exists  $u^* \in W_0^{1,2}(D)$  such that, up to a subfamily,  $(u_{M,\rho,a})_{M \geq 0}$  converges to  $u^*$  weakly in  $H^1(D)$  and strongly in  $L^2(D)$ . As a consequence, up to a subsequence,  $(f(u_{M,\rho,a}))_{M \geq 0}$  converges to  $f(u^*)$  in  $L^2(D)$  by using that  $f$  is Lipschitz and  $(\langle g, u_{M,\rho,a} \rangle_{H^{-1}, H_0^1})_{M \geq 0}$  converges to  $\langle g, u^* \rangle_{H^{-1}, H_0^1}$ . By rewriting (6.5) under variational form with  $u = u_{M,\rho,a}$ , and passing to the limit as  $M \rightarrow +\infty$  after having adequately extracted subsequences, we infer that  $u^*$  is the unique solution of (6.5). By using the previous convergence results and the fact that

$$\hat{J}_{M,\rho}(a) = -\frac{\rho}{2} \int_D u_{M,\rho,a} f(u_{M,\rho,a}) - \frac{1}{2} \langle g, u_{M,\rho,a} \rangle_{H^{-1}(D), W_0^{1,2}(D)}$$

we have

$$\hat{J}_{M,\rho}(a) \rightarrow -\frac{\rho}{2} \int_D u^* f(u^*) - \frac{1}{2} \langle g, u^* \rangle_{H^{-1}(D), W_0^{1,2}(D)} \quad \text{as } M \rightarrow +\infty.$$

Finally, if  $a = \mathbf{1}_\Omega$ , by multiplying (6.5) by  $u_{M,\rho,a}$  and integrating by parts, one gets

$$\int_D |\nabla u_{M,\rho,a}|^2 + M \int_{D \setminus \Omega} u_{M,\rho,a}^2 = \int_D (g - \rho f(u_{M,\rho,a})) u_{M,\rho,a},$$

and since the right-hand side is uniformly bounded with respect to  $M$ , we infer that  $\sqrt{M}u_{M,\rho,a}$  is bounded in  $L^2(D \setminus \Omega)$  so that  $u^* \in W_0^{1,2}(\Omega)$ .

The conclusion follows.





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## Résumé

Cette thèse est dédiée à l'étude de problèmes d'optimisation de forme et de contrôle qui apparaissent naturellement en écologie spatiale. Considérant une population dont la densité dépend d'un terme de ressource à travers l'équation aux dérivées partielles de Fisher-KPP hétérogène en espace, on cherche à déterminer une répartition de ressources garantissant sa survie ou optimisant la taille de la population. Dans cette perspective, plusieurs approches reposant sur l'introduction et l'analyse de problèmes d'optimisation de forme et de contrôle mettant en jeu la solution de cette EDP et/ou une quantité spectrale dépendant du terme de ressource sont envisagés. L'analyse de ces problèmes nécessite

- le développement de méthodes asymptotiques pour étudier l'existence et certaines propriétés qualitatives (concentration et fragmentation des ressources) de formes optimales, ou encore la stabilité de certaines configurations de ressources ;
- l'établissement d'une inégalité spectrale quantitative pour un opérateur de Schrödinger dans la boule ;
- l'introduction d'une méthode perturbative pour étudier la contrôlabilité des équations de réaction-diffusion en milieu hétérogène.

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**Mots clés:** Équations de réaction-diffusion, Optimisation spectrale, Optimisation de formes, Contrôle des EDP, Inégalités quantitatives, Analyse qualitative.

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## Summary

This thesis is devoted to the study of shape optimisation and control problems stemming from the study of spatial ecology. Assuming we are working with a population whose density depends on a spatially heterogeneous Fisher-KPP equation involving a resources distribution, we wish to investigate which of these resources distributions optimises the survival, or total population size of the population. In order to study such questions, we introduce and analyse several shape optimisation and control problems involving the solutions of reaction-diffusion PDEs and/or spectral quantities that depend on the resources distribution.

The analysis of these problems leads to

- developing asymptotic methods to study the existence and some qualitative properties (e.g concentration, fragmentation) of optimal shapes, as well as their stability;
- proving a quantitative spectral inequality for a Schrödinger operator in the ball;
- introducing a perturbative method to study controllability properties of spatially heterogeneous reaction-diffusion equations.

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**Keywords:** Reaction-diffusion equations, Spectral optimization, Shape optimization, PDE Control, Quantitative inequalities, Qualitative analysis.