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THÈSE

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préparée au sein des **laboratoires, LAMA, Université Savoie Mont Blanc
et Laboratoire KALMA, Université Libanaise**
dans les **Écoles Doctorales MSTII, Grenoble et EDST, Beyrouth**

Stabilité et contrôlabilité de quelques systèmes localement couplés

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Stabilité et contrôlabilité de quelques systèmes localement couplés

Résumé

Cette thèse est consacrée à l'étude de la stabilité et de la contrôlabilité de quelques systèmes localement couplés. D'abord, nous avons étudié la stabilisation d'un système de deux équations d'ondes couplées par les termes des vitesses avec un seul amortissement localisé et sous des conditions géométriques appropriées. Pour le cas où les ondes se propagent à la même vitesse, nous avons établi un taux de décroissance exponentielle de l'énergie. Cependant, dans le cas physique naturel où les ondes ne se propagent pas à la même vitesse, nous avons montré que notre système n'est pas uniformément stable et nous avons établi le taux de décroissance polynomial optimal de l'énergie.

Après, nous avons traité la contrôlabilité exacte d'un système des équations d'ondes localement couplées. L'outil principal est le résultat de A. Haraux dans [31] par lequel l'inégalité d'observabilité est équivalente à la stabilité exponentielle. Plus précisément, nous avons fourni une analyse complète de la stabilité exponentielle du système dans deux espaces d'Hilbert différents et sous des conditions géométriques convenables. Ensuite, en utilisant la méthode HUM, nous avons prouvé que le système est exactement contrôlable. Nous avons aussi effectué des études numériques pour valider nos résultats théoriques obtenus.

Enfin, nous avons analysé la stabilité d'un système de Bresse avec un amortissement local de type Kelvin-Voigt avec des conditions aux bords Dirichlet ou Dirichlet-Neumann-Neumann. Dans le cas de trois amortissements locaux, nous avons établi un taux de décroissance exponentielle ou polynomiale de l'énergie. Cependant, lorsque les ondes ne sont soumises qu'à un ou deux amortissements et que, dans les conditions aux bords sont de type Dirichlet-Neumann-Neumann, nous avons démontré que le système n'est pas uniformément stable. Dans le cas d'un seul amortissement local, nous avons établi un taux de décroissance polynomiale de l'énergie.

Dans cette thèse, la méthode de domaine fréquentielle et la technique des multiplicateurs ont été utilisées.

Mots-clés

C_0 -semigroupe, Stabilité exponentielle, Stabilité polynomiale, Méthode fréquentielle, Méthode des multiplicateurs, Inégalité d'observabilité, Contrôlabilité exacte, Méthode HUM, Système de Bresse, Différences finies.

Stability and controllability of some locally coupled systems

Abstract

This thesis is devoted to study the stabilization and exact controllability of some locally coupled systems. First, we studied the stabilization of a system of two wave equations coupled by velocities with only one localized damping and under appropriate geometric conditions. For the case involving waves that propagate at the same speed, we established the exponential energy decay rate. However, the natural physical case also entails waves that do not propagate with equal speed; in such a case, we showed that our system is not uniformly stable and we established an optimal polynomial energy decay rate.

Second, we have investigated the exact controllability of locally coupled wave equations. The main tool is a result of A. Haraux in [31] by which the observability inequality is equivalent to the exponential stability of the system. More precisely, we provided a complete exponential stability analysis of the system in two different Hilbert spaces and under appropriate geometric conditions. Then, using the HUM method, we proved that the system is exactly controllable. Later, we performed numerical experiments to validate our obtained theoretical results.

Last, we analyzed the stability of a Bresse system with local Kelvin-Voigt damping with fully Dirichlet or Dirichlet-Neumann-Neumann boundary conditions. In the case of three local damping, according to their properties (smoothness), we established an exponential or a polynomial energy decay rate. However, when the waves are only subjected to one or two damping and under Dirichlet-Neumann-Neumann boundary conditions, we demonstrated that the Bresse system is not uniformly stable. In the case of one local damping, we established a polynomial energy decay rate.

In this thesis, the frequency domain approach and the multiplier technique were used.

Keywords

C_0 -Semigroup, Exponential stability, Polynomial stability, Frequency domain method, Multiplier method, Observability inequality, Exact Controllability, HUM method, Bresse system, Finite difference discretization.

Avant-propos

La théorie du contrôle et de la stabilisation d'un système physique gouverné par des équations mathématiques, en particulier par des EDP, peut être décrit comme étant le processus qui consiste à influencer le comportement asymptotique du système pour atteindre un but désiré, principalement par l'utilisation d'un contrôle qui modifie son état final. Cette théorie est appliquée dans un large éventail de disciplines scientifiques et techniques comme la réduction du bruit, la vibration de structures, les vagues et les tremblements de terre sismiques, la régulation des systèmes biologiques comme le système cardiovasculaire humain, la conception des systèmes robotiques, le contrôle laser mécanique quantique, les systèmes moléculaires, etc.

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Chapitre 1

Introduction

The theory of stability and controllability of mathematical systems involved from engineering and physical problems (wave equation, beam equation, Schrödinger equation, plates equation, etc) has recently received the attention of many authors.

This thesis treats the stability and exact controllability of some locally coupled systems with different types of internal damping. In practice, it is often not possible to control all the components of the state, either because cost's reasons or technological limitations. Mathematically, this means that some equations of the coupled system are not directly stabilized. Which creates mathematical difficulties, that requires to answer new questions, especially the transmission of the informations (the effect of the control) from the damped equation to the undamped one through the coupling.

For more details about the indirect stabilization or controllability studies of some coupled systems, we refer you to see [2, 8, 6, 18, 22, 23, 68, 65, 67, 57, 37, 54, 16].

Description of the Thesis :

This thesis is devoted to study the stabilization and exact controllability of some locally coupled systems and it contains three chapters.

First, in chapter 2, we study the stability of a system of two wave equations coupled by velocities with only one localized damping and under appropriate geometric conditions. First, we establish the strong stability without geometric conditions. We then study the energy decay rate of our system by distinguishing two cases. The first one is when the waves propagate at same speed. In this case, under appropriate geometric conditions named by Piecewise multiplier geometric condition (PMGC in short), we establish an exponential energy decay rate for usual initial data. Next, in the general case, when the waves are not assumed to propagate at the same speed, we prove the non uniform (exponential) stability and under the same geometric conditions, we establish a polynomial energy decay rate of type $\frac{1}{t}$ for smooth initial data. Finally, in one space dimension, using the real part of the asymptotic expansion of the eigenvalues of the system, we show that the obtained polynomial decay is optimal.

Next, chapter 3 is devoted to the study of the exact controllability of locally coupled wave

equations. First, we show the exponential decay rate of the system when the coupling region is a subset of the damping one and satisfying the geometric control condition (GCC in short). Next, we show that our system is exactly controllable by using a result due to A. Haraux in [31]. Later, we show the exponential decay rate of the system in the weak energy space provided that the damping region satisfies the PMGC condition while the coupling region is a subset of the damping one and satisfying the GCC condition. Finally, we perform a numerical approximation of the problem by a finite difference discretization and we validate our theoretical results.

Finally, in chapter 4, we analyze the stability of a Bresse system with local Kelvin-Voigt damping and non smooth coefficient at interface with either fully Dirichlet or Dirichlet-Neumann-Neumann boundary conditions. First, we establish the well posedness of our system. Next, we prove the strong stability of the system in the lack of the compactness of the resolvent of the generator. Then, we move to study the energy decay rate in the case of local or global distributed dampings. In the case when the dampings are globally distributed, we establish an analytic stability. While in the case of three local dampings, we analyse an exponential decay and a polynomial energy decay rate of type $1/t$ depending on the regularity of the coefficient functions. Last but not least, under Dirichlet-Neumann-Neumann boundary conditions, we prove the lack of uniform stability of the system in the absence of at least one damping. Finally, in the case of one local damping, we prove a polynomial decay rate of type $1/\sqrt{t}$.

Description of the chapter :

This chapter is divided into two parts. The first part contains the introduction of the thesis in English and the second one in French. First section is devoted to the tools used in this thesis. In second section, we present the main results of chapter 2 concerning the stability results of a coupled wave equations under geometric conditions. Section 3 contains the main results of chapter 3 on the Controllability of system of coupled wave equations. Finally, in section 4, we exhibit the results of chapter 4 on the Bresse system with local Kelvin-Voigt damping and non smooth coefficient at interface.

1.1 Introduction in English

1.1.1 Principal methods used

As the analysis in this thesis is based on the semigroup theory, in this section, we exhibit and talk about many recent results on the strong, exponential and polynomial stability of a C_0 -semigroup that will be used to prove our main results in the next chapters. Next, we present some results about observability and exact controllability. Finally we recall some geometric conditions needed in our work. For more details we refer to [19, 24, 25, 29, 36, 49, 55, 58, 59, 60, 62].

Semigroups, Existence and uniqueness of solution

Let $(X, \|\cdot\|_X)$ be a Banach space over \mathbb{C} and \mathcal{H} be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and the induced norm $\|\cdot\|_{\mathcal{H}}$.

In this subsection, we start by the definition of semigroups since the vast majority of the evolution equations can be reduced to the form

$$\begin{cases} U_t(x, t) &= \mathcal{A}U(x, t), \quad t > 0, \\ U(0) &= U_0 \in \mathcal{H}, \end{cases} \quad (1.1.1)$$

where \mathcal{A} is the infinitesimal generator of a C_0 -semigroup $(S(t))_{t \geq 0}$ in a Hilbert space \mathcal{H} .

Definition 1.1.1. *A family $(S(t))_{t \geq 0}$ of bounded linear operators on X is a strongly continuous semigroup (in a short, a C_0 -semigroup) if*

- $S(0) = I$.
- $S(t + s) = S(t)S(s) \quad \forall s, t \geq 0$.
- $\lim_{t \rightarrow 0} \|S(t) - I\|_X = 0$.

□

Definition 1.1.2. *The linear operator \mathcal{A} defined by*

$$\mathcal{A}x = \lim_{t \rightarrow 0} \frac{S(t)x - x}{t}, \quad \forall x \in D(\mathcal{A}),$$

where

$$D(\mathcal{A}) = \left\{ x \in X; \lim_{t \rightarrow 0} \frac{S(t)x - x}{t} \text{ exists} \right\}$$

is the infinitesimal generator of the semigroup $(S(t))_{t \geq 0}$.

□

Some properties of semigroup and its generator operator \mathcal{A} are given in the following theorems :

Theorem 1.1.3. *(Pazy [59]) Let $(S(t))_{t \geq 0}$ be a C_0 -semigroup on a Hilbert space \mathcal{H} . Then there exist two constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{\omega t}, \quad \forall t \geq 0.$$

□

If $\omega = 0$, the semigroup $(S(t))_{t \geq 0}$ is called uniformly bounded. Moreover, if $M = 1$, then it is called a C_0 -semigroup of contractions.

Theorem 1.1.4. *If \mathcal{A} generates a C_0 -semigroup on \mathcal{H} . Then*

- $\overline{D(\mathcal{A})} = \mathcal{H}$.
- \mathcal{A} is closed.

□

Definition 1.1.5. *An unbounded linear operator $(\mathcal{A}, D(\mathcal{A}))$ on X is said to be dissipative if*

$$\|(\lambda I - \mathcal{A})x\|_X \geq \lambda \|x\|_X \quad \forall x \in D(\mathcal{A}) \text{ and } \forall \lambda > 0.$$

□

Proposition 1.1.6. *Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded linear operator on \mathcal{H} , then*

$$\mathcal{A} \text{ is dissipative if and only if } \Re \langle Ax, x \rangle \leq 0, \quad \forall x \in D(\mathcal{A}).$$

□

Definition 1.1.7. *An unbounded linear operator $(\mathcal{A}, D(\mathcal{A}))$ on X is said to be maximal dissipative (m -dissipative) if*

- \mathcal{A} is a dissipative operator.
- $\exists \lambda_0$ such that $R(\lambda_0 I - \mathcal{A}) = X$, i.e. $\forall x \in X, \exists u \in D(\mathcal{A})$ such that $\lambda_0 u - \mathcal{A}u = x$.

□

For the existence of solution of problem (1.1.1), we typically use the following Hille-Yosida and Lumer-Phillips theorems from [59] :

Theorem 1.1.8. (*Hille-Yosida*) *An unbounded linear operator $(\mathcal{A}, D(\mathcal{A}))$ on X generates a C_0 -semigroup of contractions $(S(t))_{t \geq 0}$ if and only if*

- \mathcal{A} is closed and $\overline{D(\mathcal{A})} = X$.
- The resolvent set $\rho(\mathcal{A})$ contains $(0, \infty)$ and $\forall \lambda > 0$,

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

□

Theorem 1.1.9. (*Lumer-Phillips*) *Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded linear operator on X with dense domain $D(\mathcal{A})$ in X . \mathcal{A} is the infinitesimal generator of a C_0 -semigroup of contractions $(S(t))_{t \geq 0}$ if and only if it is a m -dissipative operator.*

□

Corollary 1.1.10. *Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded linear operator on \mathcal{H} . \mathcal{A} is the infinitesimal generator of a C_0 -semigroup of contractions $(S(t))_{t \geq 0}$ if and only if it is a m -dissipative operator.*

□

Consequently, the existence of solution is justified by the following corollary which follows from Lumer-Phillips theorem.

Corollary 1.1.11. *Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded linear operator on \mathcal{H} . Assume that \mathcal{A} is the infinitesimal generator of a C_0 -semigroup of contractions $(S(t))_{t \geq 0}$.*

- *For $U_0 \in D(\mathcal{A})$, the problem (1.1.1) has a unique strong solution*

$$U(t) = S(t)U_0 \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H}).$$

- *For $U_0 \in \mathcal{H}$, the problem (1.1.1) has a unique weak solution*

$$U(t) \in C([0, +\infty), \mathcal{H}).$$

□

Stability of semigroups

In order to show the strong stability of a C_0 -semigroup, we apply the next theorem due to Arendt and Batty in [20].

Theorem 1.1.12. *Assume that \mathcal{A} is the infinitesimal generator of a strongly continuous semigroup of contractions $(S(t))_{t \geq 0}$ on X . If \mathcal{A} has no pure imaginary eigenvalues and if $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable, then $(S(t))_{t \geq 0}$ is strongly stable.*

□

Now, when the C_0 -semigroup is strongly stable, we look for a necessary and sufficient conditions for which a semigroup is exponentially stable. We recall here only the following frequency domain approach method obtained by Huang [36] and Prüss [60].

Theorem 1.1.13. *(Huang [36] and Prüss [60]) Let $(S(t))_{t \geq 0}$ be a C_0 -semigroup of contractions on \mathcal{H} and \mathcal{A} be its infinitesimal generator. Then, $(S(t))_{t \geq 0}$ is exponentially stable if and only if*

- $i\mathbb{R} \subseteq \rho(\mathcal{A})$,
- $\limsup_{\beta \in \mathbb{R}, |\beta| \rightarrow \infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

□

Since some studied systems in this thesis do not achieve the exponential stability, therefore we look for a polynomial one. In general, polynomial stability results are obtained using different methods like : multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them (see [40, 46, 47]). In this thesis, we recall only the frequency domain approach obtained by Borichev- Tomilov in [25, Theorem 2.4].

Theorem 1.1.14. *(Borichev-Tomilov [25]) Let $(S(t))_{t \geq 0}$ be a bounded C_0 -semigroup of contractions on \mathcal{H} generated by \mathcal{A} . If $i\mathbb{R} \subset \rho(\mathcal{A})$, then for a fixed $l > 0$, the following conditions are equivalent :*

- $\limsup_{\beta \rightarrow +\infty} \frac{1}{|\beta|^l} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty$.
- $\|S(t)U_0\|_{\mathcal{H}} \leq \frac{c}{t^{l-1}} \|U_0\|_{D(\mathcal{A})}, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}), \text{ for some } c > 0$.

□

Finally, in order to study the optimality of the obtained decay rate, we refer to a Theorem 3.4.1 in [56].

Theorem 1.1.15. (Wehbe, Najdi 2016) *Let \mathcal{A} be the infinitesimal generator of a C_0 -semigroup of contractions $(S(t))_{t \geq 0}$. Let $(\lambda_{k,n})$ the eigenvalues of \mathcal{A} and $(e_{k,n})$ eigenvectors. Assume that there exist $\mu_{k,n} \rightarrow +\infty$, $\alpha_k > 0$, $\beta_k > 0$ such that $\Re(\lambda_{k,n}) \sim -\frac{\beta_k}{\mu_{k,n}^{\alpha_k}}$, $|\Im(\lambda_{k,n})| \sim \mu_{k,n}$, $i\mathbb{R} \subset \rho(\mathcal{A})$ and for any $u_0 \in D(\mathcal{A})$, there exists constant $M > 0$ such that*

$$\|S(t)u_0\|_{\mathcal{H}} \leq \frac{M}{t^{\frac{1}{l_k}}} \|u_0\|_{D(\mathcal{A})}, \quad l_k = \max_{1 \leq k \leq k_0} (\alpha_k), \quad \forall t > 0. \quad (1.1.2)$$

Then the decay rate (1.1.2) is optimal. □

Observability and exact controllability

In this part, we present briefly the duality between the notion of observability and controllability, which lies at the basis of the Hilbert uniqueness method (HUM) of J .L. Lions [46].

First, we consider the following system :

$$\begin{cases} U_t(x, t) = AU(x, t) + Bv(x), & \text{on } \Omega \times (0, +\infty), \\ U(x, 0) = U_0(x), \end{cases} \quad (1.1.3)$$

where $\Omega \subset \mathbb{R}^d (d \in \mathbb{N}^*)$, U is a scalar or vector-valued function, A is a set of partial differential operators, linear or non linear (at least for the time being), v denotes the control and B maps the "space of controls" into the "state space". The partial differential equation (1.1.3) should include boundary conditions. We do not make them explicit here. They are supposed to be contained in the abstract formulation (1.1.3).

The control v can be either applied inside the domain Ω (in this case v is said to be internal control), or on the boundary Γ of Ω or on part of it (in this case v is said to be a boundary control). If v is applied at points of Ω , v is said to be pointwise control.

It will be assumed that, given v (in a suitable space), problem (1.1.3) uniquely defines a solution. This solution is a function (scalar or vector-valued) of $x \in \Omega$, $t > 0$ and of U_0 and v .

Now, we can introduce the notion of controllability, either exact or approximate. Let $T > 0$ be given and let U_T (the target function) be a given element of the state space. We want to "drive the System" from initial state U_0 at $t = 0$ to final state U_T at $t = T$, that is, we want to find a suitable control v such that

$$U(x, T) = U_T(x), \quad x \in \Omega.$$

If this is possible for any target function U_T in the state place, one can say that the System is controllable (or exactly controllable). For more details for controllability see [46].

In [46], J. L. Lions introduce the Hilbert Uniqueness Method (HUM) to solve controllability problems for linear partial differential equations. This method is closely related to duality between controllability and observability.

We now present a result of A. Haraux in [31] to get the observability inequality. If A is an unbounded, self adjoint, positive and coercive linear operator on a Hilbert space \mathcal{H} and B a bounded linear operator on \mathcal{H} such that $B = B^* \geq 0$, he established a logical equivalence between the exponential decay of solutions of the second order evolution equation $U_{tt} + AU + BU_t = 0$, uniformly on bounded subsets of $D(A^{1/2}) \times \mathcal{H}$ and a "B^{1/2}-controllability" property of the system governed by the undamped equation $\varphi_{tt} + A\varphi = 0$ on some time interval (see Proposition 1 and 2 in [31]).

Remark 1.1.16. *The result of A. Haraux in [31] still valid for a first order evolution equation.*

Geometric conditions

This part is devoted to recall some geometric conditions since we shall use them along our work.

We begin by reviewing the Geometric Control Conditions GCC introduced by Rauch and Taylor in [61] for manifolds without boundaries and by Bardos, Lebeau and Rauch in [21] for domains with boundaries.

Definition 1.1.17. *We say that a subset ω of Ω satisfies the **GCC** if every ray of the geometrical optics starting at any point $x \in \Omega$ at $t = 0$ enters the region ω in finite time T .* □

Next, we recall the Piecewise Multipliers Geometric Condition introduced by K. Liu in [48].

Definition 1.1.18. *We say that ω satisfies the Piecewise Multipliers Geometric Condition (PMGC in short) if there exist $\Omega_j \subset \Omega$ having Lipschitz boundary $\Gamma_j = \partial\Omega_j$ and $x_j \in \mathbb{R}^N$, $j = 1, \dots, J$ such that $\Omega_j \cap \Omega_i = \emptyset$ for $j \neq i$ and ω contains a neighborhood in Ω of the set $\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)$ where $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ and ν_j is the outward unit normal vector to Γ_j .* □

Remark 1.1.19. *The PMGC is the generalization of the Multipliers Geometric Condition (MGC in short) introduced by Lions in [46], saying that ω contains a neighborhood in Ω of the set $\{x \in \Gamma : (x - x_0) \cdot \nu(x) > 0\}$, for some $x_0 \in \mathbb{R}^N$, where ν is the outward unit normal vector to $\Gamma = \partial\Omega$. However, the PMGC is much more restrictive than the GCC.* □

After this general introduction, we move now to detail more the results of this thesis. We divided into three parts.

1.1.2 Chapter 2 : Local Indirect Stabilization of two coupled wave equations under geometric conditions

Let Ω be a non empty open bounded domain of \mathbb{R}^N with boundary Γ of class C^2 . In chapter 2, we consider the following coupled wave equation :

$$u_{tt} - a\Delta u + c(x)u_t + b(x)y_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \quad (1.1.4)$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \quad (1.1.5)$$

$$u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+, \quad (1.1.6)$$

with the following initial data :

$$u(x, 0) = u_0, \quad y(x, 0) = y_0, \quad u_t(x, 0) = u_1 \text{ and } y_t(x, 0) = y, \quad x \in \Omega,$$

where $a > 0$ constant and $b(x) \in C^0(\overline{\Omega}; \mathbb{R})$ is a non-zero function. The damping term $c(x) \in C^0(\overline{\Omega}, \mathbb{R}_+)$ is only applied at the first equation and the second equation is indirectly damped through the coupling between the two equations. This type of indirect control was introduced by D.L. Russel [63] and since this time, it attracted the attention of many authors.

Preceding results :

In [37], B. Kapitov studied the stability of system (1.1.4)-(1.1.6) in the case when the support of b coincide with the support of c . When the waves propagate at the same speed (i.e. $a = 1$), he established an exponential decay of the energy. While when the waves propagate at different speeds, no decay rate was discussed.

In [12], F. Alabau-Boussouira et al. considered the energy decay of the following system :

$$u_{tt} - a\Delta u + \rho(x, u_t) + b(x)y_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (1.1.7)$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (1.1.8)$$

$$u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*, \quad (1.1.9)$$

where $a > 0$ constant, $b \in C^0(\overline{\Omega}, \mathbb{R})$ and $\rho(x, u_t)$ is a non linear damping. Using an approach based on multiplier techniques, weighted nonlinear inequalities and the optimal-weight convexity method (developed in [7]), the authors established an explicit energy decay formula in terms of the behavior of the nonlinear feedback close to the origin in the case that the three following conditions are satisfied : the waves propagate at the same speed ($a = 1$) and the coupling coefficient $b(x)$ is small positive ($0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ where b^* is a constant depending on Ω and on the control region) and both the coupling and the damping regions satisfying an appropriated geometric conditions named Piecewise Multipliers Geometric Conditions (PMGC, in short). But the contrary case, when the waves are not assumed to propagate with equal speed (a is not necessarily equal to 1) and/or b is not assumed to be small and positive has been left as open question even in the linear case i.e. $\rho(x, u_t) = c(x)u_t$. This open question will be our target in chapter 2 in the linear case.

Principal results of the chapter.

The main novelty in this chapter is that the waves are not necessarily propagating at the same speed and the coupling coefficient is not assumed to be positive and small.

First, we begin to study the existence, uniqueness and regularity of the solution of our system using the semigroup approach. Let (u, u_t, y, y_t) be a regular solution of the system (1.1.4)-(1.1.6), its associated total energy is defined by

$$E(t) = \frac{1}{2} \int_{\Omega} (|u_t|^2 + a|\nabla u|^2 + |y_t|^2 + |\nabla y|^2) dx. \quad (1.1.10)$$

A direct computation gives

$$\frac{d}{dt}E(t) = - \int_{\Omega} c(x)|u_t|^2 dx \leq 0. \quad (1.1.11)$$

Consequently, system (1.1.4)-(1.1.6) is dissipative in the sense that its energy is non-increasing with respect to the variable time t . Next, we define the energy space $\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2$ equipped, for all $U = (u, v, y, z)$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in \mathcal{H}$, by the scalar product :

$$(U, \tilde{U})_{\mathcal{H}} = a \int_{\Omega} (\nabla u \cdot \nabla \tilde{u}) dx + \int_{\Omega} v \tilde{v} dx + \int_{\Omega} (\nabla y \cdot \nabla \tilde{y}) dx + \int_{\Omega} z \tilde{z} dx.$$

Setting $U = (u, u_t, y, y_t)$, system (1.1.4)-(1.1.6) may be rewritten as :

$$U_t = \mathcal{A}U, \text{ in } (0, +\infty), U(0) = (u_0, u_1, y_0, y_1),$$

where the unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is given by :

$$D(\mathcal{A}) = \left((H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \right)^2 \quad (1.1.12)$$

and

$$\mathcal{A}U = (v, a\Delta u - bz - cv, z, \Delta y + bv), \quad \forall U = (u, v, y, z) \in D(\mathcal{A}). \quad (1.1.13)$$

The operator \mathcal{A} is m-dissipative in \mathcal{H} and generates a C_0 -semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$. So, system (1.1.4)-(1.1.6) is well posed in \mathcal{H} .

Then, we move to study the asymptotic behavior of $E(t)$. For this aim, we assume that there exists a non empty open $\omega_{c_+} \subset \Omega$ satisfying the following condition

$$\{x \in \Omega : c(x) > 0\} \supset \bar{\omega}_{c_+}. \quad (\text{LH1})$$

On the other hand, as $b(x)$ is not identically null and continuous, then there exists a non empty open sets $\omega_{b_+} \cup \omega_{b_-} \subset \Omega$ such that

$$\{x \in \Omega : b(x) > 0\} \supset \bar{\omega}_{b_+} \quad \text{and} \quad \{x \in \Omega : b(x) < 0\} \supset \bar{\omega}_{b_-}. \quad (\text{LH2})$$

We first prove that our system is strongly stable without geometric condition. This is given by the following theorem :

Theorem 1.1.20. (*Strong Stability*) Assume that $a > 0$, condition (LH1) holds and that $\omega = \omega_{c_+} \cap \omega_{b_+} \neq \emptyset$ (or $\omega_{c_+} \cap \omega_{b_-} \neq \emptyset$). Then the semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$ is strongly stable on the energy space \mathcal{H} , i.e. for any $U_0 \in \mathcal{H}$, we have

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0. \quad (1.1.14)$$

□

Now, we study the energy decay rate by using a frequency domain approach combined with piecewise multiplier technique in two cases : the first one when the waves are assumed to propagate at the same speed (i.e. $a = 1$) and the second case when $a \neq 1$.

The first main result is the following one :

Theorem 1.1.21. (*Exponential decay rate*) Let $a = 1$. Assume that condition (LH1) holds. Assume also that the nonempty open set $\omega = \omega_{c_+} \cap \omega_{b_+}$ (or $\omega = \omega_{c_+} \cap \omega_{b_-}$) satisfies the geometric conditions PMGC and that $b, c \in W^{1,\infty}(\Omega)$. Then there exist positive constants $M \geq 1$, $\theta > 0$ such that for all initial data $(u_0, u_1, y_0, y_1) \in \mathcal{H}$ the energy of the system (1.1.4)-(1.1.6) satisfies the following decay rate :

$$E(t) \leq Me^{-\theta t}E(0), \quad \forall t > 0. \quad (1.1.15)$$

□

Remark 1.1.22. Note that in the previous theorem we have no restriction on the upper bound and the sign of the function b . This theorem is then a generalization in the linear case of the result of [12] where the coupling coefficient considered have to satisfy $0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ where b^* is a constant depending on Ω and on the control region. Nevertheless, the problem still be open in the nonlinear case. □

The condition of equal speed propagation is a necessary and sufficient condition for the exponential stability of our system. In fact, in the case $a \neq 1$, we construct a sequence (U_n) of elements in $D(\mathcal{A})$ and a real sequence (μ_n) such that $\|U_n\| = 1$ and $\|(i\mu_n I - \mathcal{A})U_n\|_{\mathcal{H}} \rightarrow 0$. Hence, the resolvent of \mathcal{A} is not uniformly bounded on the imaginary axis. Following a result of Huang [36] and Prüss [60] we conclude that the semigroup $(e^{t\mathcal{A}})_{t \geq 0}$ is not uniformly stable in \mathcal{H} . So it is natural to look for a polynomial decay of the energy. Consequently, our second main result when the wave propagate at different speeds ($a \neq 1$) can be stated as follows :

Theorem 1.1.23. (*Polynomial decay rate*) Let $a \neq 1$. Assume that condition (LH1) holds. Assume also that the nonempty open set $\omega = \omega_{c_+} \cap \omega_{b_+}$ (or $\omega = \omega_{c_+} \cap \omega_{b_-}$) satisfies the geometric conditions PMGC and that $b, c \in W^{1,\infty}(\Omega)$. Then there exists a positive constant $C > 0$ such that for all initial data $(u_0, u_1, y_0, y_1) \in D(\mathcal{A})$ the energy of the system (1.1.4)-(1.1.6) satisfies the following polynomial decay rate :

$$E(t) \leq C \frac{1}{t} \|U(0)\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \quad (1.1.16)$$

□

Finally, in one space dimension (i.e. $N = 1$), $a \neq 1$ and b is a constant, we prove that there exist $n_0 \in \mathbb{N}$ sufficiently large and a sequence λ_n of simple eigenvalues of the operator \mathcal{A} satisfying the following asymptotic behavior

$$\lambda_n = in\pi - \frac{ib^2}{2(a-1)n\pi} - \frac{cb^2}{2(a-1)^2n^2\pi^2} + O\left(\frac{1}{n^3}\right), \quad \forall |n| \geq n_0. \quad (1.1.17)$$

It follows that the obtained polynomial decay rate is optimal (see Theorem 3.4.1 in [56]).

1.1.3 Chapter 3 : Exact controllability and stabilization of locally coupled wave equations

The aim of this chapter is to investigate the exact controllability of the following system :

$$u_{tt} - a\Delta u + b(x)y_t = c(x)v(t) \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (1.1.18)$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (1.1.19)$$

$$u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*, \quad (1.1.20)$$

with the following initial data

$$u(x, 0) = u_0, \quad y(x, 0) = y_0, \quad u_t(x, 0) = u_1 \text{ and } y_t(x, 0) = y_1, \quad x \in \Omega, \quad (1.1.21)$$

under appropriate geometric conditions. Here, $a > 0$ constant, $b \in C^0(\overline{\Omega}, \mathbb{R})$, $c \in C^0(\overline{\Omega}, \mathbb{R}^+)$ and v is an appropriate control. The idea is to use a result of A. Haraux in [31] for which the observability of the homogeneous system associated to (1.1.18)-(1.1.20) is equivalent to the exponential stability of system (1.1.4)-(1.1.6) in an appropriate Hilbert space. So, we provide a complete analysis for the exponential stability of system (1.1.4)-(1.1.6) in different Hilbert spaces.

Preceding results :

In chapter 2, we studied the stabilization of system (1.1.4)-(1.1.6) in two cases. In the first one, when the waves are assumed propagating at the same speed (i.e. $a = 1$), under the assumption that the coupling region and the damping region have a non empty intersection and satisfying the PMGC condition. In this case, we established an exponential decay rate for weak initial data. On the contrary (i.e. $a \neq 1$) we first proved the lack of the exponential stability of the system. However, under the same geometric condition, an optimal energy decay rate of type $\frac{1}{t}$ was established for smooth initial data.

Our aim in this chapter is to prove the exponential stability of system (1.1.4)-(1.1.6) in two different Hilbert spaces by using geometric conditions more general than that used in chapter 2. And consequently, by using Proposition 2 of A. Haraux [31], we obtain the observability of the homogeneous system associated to (1.1.18)-(1.1.20).

Principal results of the chapter.

First, we need to study the asymptotic behavior of $E(t)$ associated to (1.1.4)-(1.1.6) and given by equation (1.1.10). For this aim, we suppose that there exists a non empty open $\omega_{c_+} \subset \Omega$ satisfying the following condition

$$\{x \in \Omega : c(x) > 0\} \supset \bar{\omega}_{c_+}. \quad (\text{LH1})$$

On the other hand, as $b(x)$ is not identically null and continuous, then there exists a non empty open $\omega_b \subset \Omega$ such that

$$\{x \in \Omega : b(x) \neq 0\} \supset \bar{\omega}_b. \quad (\text{LH2})$$

If $\omega = \omega_{c_+} \cap \omega_b \neq \emptyset$ and condition (LH1) holds, then system (1.1.4)-(1.1.6) is strongly stable using Theorem 1.1.20, i.e.

$$\lim_{t \rightarrow +\infty} \|e^{tA}(u_0, u_1, y_0, y_1)\|_{\mathcal{H}} = 0 \quad \forall (u_0, u_1, y_0, y_1) \in \mathcal{H}.$$

Then, when the waves propagate at the same speed (i.e., $a = 1$), under the condition that the coupling region includes in the damping region and satisfying the called Geometric Control Condition (GCC in Short), we establish the exponential stability of system (1.1.4)-(1.1.6) given by the following theorem

Theorem 1.1.24. *(Exponential decay rate) Let $a = 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that $\omega_b \subset \omega_{c_+}$ satisfies the geometric control conditions GCC and that $b, c \in W^{1,\infty}(\Omega)$. Then there exist positive constants $M \geq 1$, $\theta > 0$ such that for all initial data $(u_0, u_1, y_0, y_1) \in \mathcal{H}$ the energy of the system (1.1.4)-(1.1.6) satisfies the following decay rate :*

$$E(t) \leq M e^{-\theta t} E(0), \quad \forall t > 0. \quad (1.1.22)$$

□

Remark 1.1.25. *The geometric situations covered by Theorem 1.1.24 are richer than that considered in Chapter 2 and [12]. Indeed, in the previous references, the authors consider the PMGC geometric conditions that are more restrictive than GCC. On the other hand, unlike the results in [12], we have no restriction in Theorem 1.1.24 on the upper bound and the sign of the coupling function coefficient b . This theorem is then a generalization in the linear case of the result of [12] where the coupling coefficient considered have to satisfy $0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ where b^* is a constant depending on Ω and on the control region.*

□

Consequently, using Proposition 2 of A. Haraux in [31], an observability inequality of the solution of the homogeneous system associated to (1.1.18)-(1.1.20) in the space $(H_0^1(\Omega) \times L^2(\Omega))^2$ is established. This leads, by the HUM method introduced by Lions in [46], to the exact controllability of system (1.1.18)-(1.1.20) in the space $(H^{-1}(\Omega) \times L^2(\Omega))^2$.

Furthermore, on the contrary when the waves propagate at different speeds, (i.e., $a \neq 1$),

we establish the exponential stability of system (1.1.4)-(1.1.6) in the weak energy space. For this, we introduce the following weak energy space

$$D = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega),$$

equipped with the scalar product

$$(U, \tilde{U}) = \int_{\Omega} (a \nabla u \cdot \nabla \tilde{u} + v \tilde{v} + y \tilde{y} + (-\Delta)^{-1/2} z (-\Delta)^{-1/2} \tilde{z}) dx,$$

for all $U = (u, v, y, z) \in D$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in D$.

Next, we define the unbounded linear operator $\mathcal{A}_d : D(\mathcal{A}_d) \subset D \rightarrow D$ by

$$\mathcal{A}_d U = (v, a \Delta u - bz - cv, z, \Delta y + bv),$$

$$D(\mathcal{A}_d) = ((H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)), \quad \forall U = (u, v, y, z) \in D(\mathcal{A}_d).$$

Its total mixed energy is defined by

$$E_m(t) = \frac{1}{2} (a \|\nabla u\|_{L^2(\Omega)}^2 + \|u_t\|_{L^2(\Omega)}^2 + \|y_t\|_{H^{-1}(\Omega)}^2 + \|y\|_{L^2(\Omega)}^2).$$

Then, we move to study the asymptotic behavior of $E_m(t)$. For this aim, we need to assume that ω_{c_+} satisfies the geometric conditions PMGC, then there exist $\varepsilon > 0$, subsets $\Omega_j \subset \Omega$, $j = 1, \dots, J$, with Lipschitz boundary $\Gamma_j = \partial\Omega_j$ and points $x_j \in \mathbb{R}^N$ such that $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$ and $\omega_c^+ \supset \mathcal{N}_\varepsilon(\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)) \cap \Omega$ with $\mathcal{N}_\varepsilon(\mathcal{O}) = \{x \in \mathbb{R}^N : d(x, \mathcal{O}) < \varepsilon\}$ where $\mathcal{O} \subset \mathbb{R}^N$, $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ where ν_j is the outward unit normal vector to Γ_j and that ω_b satisfies the GCC condition and

$$\omega_b \subset (\Omega \setminus \cup_{j=1}^J \Omega_j). \quad (\text{LH3})$$

Now, our second main result when the waves propagate at different speed (i.e. $a \neq 1$) can be stated as follows :

Theorem 1.1.26. *(Exponential decay rate) Let $a \neq 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that ω_{c_+} satisfies the geometric conditions PMGC, ω_b satisfies GCC condition and (LH3) and $b, c \in L^\infty(\Omega)$. Then there exist positive constants $M \geq 1$, $\theta > 0$ such that for all initial data $(u_0, u_1, y_0, y_1) \in D$ the energy of system (1.1.4)-(1.1.6) satisfies the following decay rate :*

$$E_m(t) \leq M e^{-\theta t} E_m(0), \quad \forall t > 0. \quad (1.1.23)$$

□

Consequently, using Proposition 2 of A. Haraux in [31], an observability inequality of the solution of the homogeneous system associated to (1.1.18)-(1.1.20) is established. This leads, by the HUM method, to the exact controllability of system (1.1.18)-(1.1.20) in the space $L^2(\Omega) \times H^{-1}(\Omega) \times H^{-1}(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))'$, where the duality is according to $L^2(\Omega)$.

Finally, we perform numerical tests in the 1-D case to insure the theoretical results obtained here and in chapter 2. In fact, the numerical results show a better behavior than the one expected by the theoretical results.

1.1.4 Chapter 4 : Stability of a Bresse system with local Kelvin-Voigt damping and non-smooth coefficient at interface

This chapter is devoted to study the stability of an elastic Bresse system with local Kelvin-Voigt damping and non-smooth coefficient at interface under fully Dirichlet or Dirichlet-Neumann-Neumann boundary conditions. The system defined on $(0, L) \times (0, +\infty)$ is governed by the following partial differential equations :

$$\begin{cases} \rho_1 \varphi_{tt} - [k_1 (\varphi_x + \psi + lw) + D_1 (\varphi_{xt} + \psi_t + lw_t)]_x - lk_3 (w_x - l\varphi) - lD_3 (w_{xt} - l\varphi_t) = 0, \\ \rho_2 \psi_{tt} - [k_2 \psi_x + D_2 \psi_{xt}]_x + k_1 (\varphi_x + \psi + lw) + D_1 (\varphi_{xt} + \psi_t + lw_t) = 0, \\ \rho_1 w_{tt} - [k_3 (w_x - l\varphi) + D_3 (w_{xt} - l\varphi_t)]_x + lk_1 (\varphi_x + \psi + lw) + lD_1 (\varphi_{xt} + \psi_t + lw_t) = 0, \end{cases} \quad (1.1.24)$$

The coefficients $\rho_1, \rho_2, k_1, k_2, k_3$ and l are positive constants. D_1, D_2 and D_3 are positive functions over $(0, L)$.

Preceding results :

Kelvin-Voigt material is a viscoelastic structure having properties of both elasticity and viscosity. The Kelvin-Voigt damping can be globally or locally distributed. But the case we are interested in is when it is localized on an arbitrary subinterval of the domain. The regularity and stability properties of a solution depend on the properties of the damping coefficients. Indeed, the system is more effectively controled by the local Kelvin-Voigt damping when the coefficient changes more smoothly near the interface.

Recently, X. Tian and Q. Zhang in [66] considered the following Timoshenko system defined on $(0, L) \times (0, +\infty)$ with fully Dirichlet boundary conditions :

$$\begin{cases} \rho_1 \varphi_{tt} - [k_1 (\varphi_x + \psi)_x + D_1 (\varphi_{xt} - \psi_t)]_x = 0, \\ \rho_2 \psi_{tt} - (k_2 \psi_x + D_2 \psi_{xt})_x + k_1 (\varphi_x + \psi)_x + D_1 (\varphi_{xt} - \psi_t) = 0. \end{cases} \quad (1.1.25)$$

They studied this system with locally or globally distributed Kelvin-Voigt damping when coefficient functions $D_1, D_2 \in C([0, L])$. First, when the damping is globally distributed, they showed that the Timoshenko system (1.1.25) under fully Dirichlet boundary conditions is analytic stable. Next, when the damping are locally distributed near the boundary, they analyzed the exponential or polynomial stability according to the properties of coefficient functions D_1, D_2 . Unlike the results of [66], in this chapter, we studied the Bresse system (1.1.24) subjected to either the fully Dirichlet or Dirichlet-Neumann-Neumann boundary conditions and in the case of Kelvin-Voigt dampings localized on any arbitrary subinterval of the domain.

Principal results of the chapter :

First, we begin to study the well-posedness of our system using the semigroup approach. Let $(\varphi, \varphi_t, \psi, \psi_t, w, w_t)$ be a regular solution of system (1.1.24), its total energy is defined

by

$$E(t) = \frac{1}{2} \left\{ \int_0^L (\rho_1 |\varphi_t|^2 + \rho_2 |\psi_t|^2 + \rho_1 |w_t|^2 + k_1 |\varphi_x + \psi + lw|^2) dx + \int_0^L (k_2 |\psi_x|^2 + k_3 |w_x - l\varphi|^2) dx \right\}. \quad (1.1.26)$$

Hence a straightforward computations gives

$$E'(t) = - \int_0^L (D_1 |\varphi_{xt} + \psi_t + lw_t|^2 + D_2 |\psi_{xt}|^2 + D_3 |w_{xt} - l\varphi_t|^2) dx \leq 0. \quad (1.1.27)$$

Thus, the system (1.1.24) is dissipative in the sense that its energy is non-increasing with respect to the variable time t . Next, we define the following energy spaces :

$$\mathcal{H}_1 = (H_0^1 \times L^2)^3 \quad \text{and} \quad \mathcal{H}_2 = H_0^1 \times L^2 \times (H_*^1 \times L_*^2)^2,$$

where

$$L_*^2 = \{f \in L^2(0, L) : \int_0^L f(x)dx = 0\} \quad \text{and} \quad H_*^1 = \{f \in H^1(0, L) : \int_0^L f(x)dx = 0\}.$$

We define the unbounded linear operators \mathcal{A}_j in $\mathcal{H}_j, j = 1, 2$ by

$$D(\mathcal{A}_1) = \left\{ U \in \mathcal{H}_1 \mid v^2, v^4, v^6 \in H_0^1(0, L), \right. \\ \left. [k_1 (v_x^1 + v^3 + lv^5) + D_1 (v_x^2 + v^4 + lv^6)]_x \in L^2(0, L), \right. \\ \left. [k_2 v_x^3 + D_2 v_x^4]_x \in L^2(0, L), [k_3 (v_x^5 - lv^1) + D_3 (v_x^6 - lv^2)]_x \in L^2(0, L) \right\},$$

$$D(\mathcal{A}_2) = \left\{ U \in \mathcal{H}_2 \mid v^2 \in H_0^1(0, L), v^4, v^6 \in H_*^1(0, L), v_x^3|_{(0,L)} = v_x^5|_{(0,L)} = 0, \right. \\ \left. [k_1 (v_x^1 + v^3 + lv^5) + D_1 (v_x^2 + v^4 + lv^6)]_x \in L^2(0, L), \right. \\ \left. [k_2 v_x^3 + D_2 v_x^4]_x \in L_*^2(0, L), [k_3 (v_x^5 - lv^1) + D_3 (v_x^6 - lv^2)]_x \in L_*^2(0, L) \right\},$$

and

$$\mathcal{A}_j \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^5 \\ v^6 \end{pmatrix} = \begin{pmatrix} v^2 \\ \rho_1^{-1} ([k_1 (v_x^1 + v^3 + lv^5) + D_1 (v_x^2 + v^4 + lv^6)]_x + lk_3 (v_x^5 - lv^1) + lD_3 (v_x^6 - lv^2)) \\ v^4 \\ \rho_2^{-1} ((k_2 v_x^3 + D_2 v_x^4)_x - k_1 (v_x^1 + v^3 + lv^5) - D_1 (v_x^2 + v^4 + lv^6)) \\ v^6 \\ \rho_1^{-1} ([k_3 (v_x^5 - lv^1) + D_3 (v_x^6 - lv^2)]_x - lk_1 (v_x^1 + v^3 + lv^5) - lD_1 (v_x^2 + v^4 + lv^6)) \end{pmatrix}$$

for all $U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(\mathcal{A}_j)$. Setting $U = (\varphi, \varphi_t, \psi, \psi_t, w, w_t)^T$, system (1.1.24) may be rewritten as :

$$U_t = \mathcal{A}_j U, j = 1, 2 \quad U = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1)^T.$$

Then, we prove that the operator \mathcal{A}_j is m-dissipative in the energy space \mathcal{H}_j . Therefore, thanks to Lumer-Phillips Theorem, we deduce that \mathcal{A}_j generates a C_0 -semigroup of contractions $e^{t\mathcal{A}_j}$ in \mathcal{H}_j , and then the problem is well posed in \mathcal{H}_j . Later, using a general criteria of Arendt and Batty [20], we show that the C_0 -semigroup $e^{t\mathcal{A}_j}$ is strongly stable in the absence of the compactness of the resolvent of \mathcal{A}_j and in the presence of at least one local Kelvin-Voigt damping.

Now, we need to study the energy decay rate by using a frequency domain approach combined with piecewise multiplier technique in several cases depending on the regularity of the coefficient functions D_1, D_2, D_3 , the localization of their supports and their numbers.

In the case when the three dampings are globally distributed, we prove an analytic stability. Then, in the case when the positive continuous functions $D_i, i = 1, 2, 3$ satisfy the following condition :

$$\exists d_0 > 0 \text{ such that } D_i \geq d_0 > 0 \text{ for every } x \in (\alpha, \beta), 0 < \alpha < \beta < L, \quad (1.1.28)$$

we establish the uniform stability of the C_0 -semigroup $e^{t\mathcal{A}_j}$ given by the following theorem :

Theorem 1.1.27. (*Exponential decay rate*) Assume that (1.1.28) is satisfied and D_1, D_2 and $D_3 \in W^{1,\infty}(0, L)$. The C_0 -semigroup $e^{t\mathcal{A}_j}$ is exponentially stable, i.e., there exist constants $M \geq 1$ and $\epsilon > 0$ independent of U_0 such that

$$\|e^{t\mathcal{A}_j} U_0\|_{\mathcal{H}_j} \leq M e^{-\epsilon t} \|U_0\|_{\mathcal{H}_j}, \quad t \geq 0, j = 1, 2.$$

□

While in the case when the positive functions $D_i \in L^\infty(0, L), i = 1, 2, 3$ satisfy the following condition :

$$\omega = \text{supp}D_1 \cap \text{supp}D_2 \cap \text{supp}D_3 = (\alpha, \beta) \subset (0, L) \text{ such that } \text{mes}(\omega) > 0, \quad (1.1.29)$$

we establish a polynomial decay given by the following theorem :

Theorem 1.1.28. (*Polynomial decay rate*) Assume that condition (1.1.29) is satisfied. Assume also that D_1, D_2 and $D_3 \in L^\infty(0, L)$. Then, there exists a positive constant $c > 0$ such that for all $U_0 \in D(\mathcal{A}_j), j = 1, 2$, the energy of the system satisfies the following decay rate :

$$E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A}_j)}^2. \quad (1.1.30)$$

□

Moreover, we prove that the system (1.1.24) with Dirichlet-Neumann-Neumann boundary condition and under the following

$$D_1 = 0 \quad \text{and} \quad D_2 = D_3 = 1 \quad \text{on} \quad (0, L), \quad (1.1.31)$$

is not exponentially stable. Then, we prove that this system is also not exponentially stable under the following hypothesis

$$D_1 = D_3 = 0 \quad \text{and} \quad D_2 = 1 \quad \text{on} \quad (0, L). \quad (1.1.32)$$

In fact, to prove the non uniform stability, we construct a sequence (V_n) of elements in $D(\mathcal{A}_2)$ and a real sequence (λ_n) such that $\|V_n\| \rightarrow +\infty$ and $\|(i\lambda_n I - \mathcal{A}_2)U_n\|_{\mathcal{H}_2}$ is bounded as $n \rightarrow +\infty$. Hence, the resolvent of \mathcal{A}_2 is not uniformly bounded on the imaginary axis. Following Huang [36] and Pruss [60] we conclude that the semigroup $e^{t\mathcal{A}_2}$ is not uniformly stable in \mathcal{H}_2 .

Finally, in the case of one local damping, i.e. we assume that there exists $d_2 > 0$ such that

$$D_1 = D_3 = 0 \quad \text{in} \quad (0, L) \quad \text{and} \quad D_2 \geq d_2 > 0 \quad \text{in} \quad (\alpha, \beta) \subset (0, L), \quad (1.1.33)$$

we prove the following theorem :

Theorem 1.1.29. *(polynomial decay rate) Assume that condition (1.1.33) is satisfied. Assume also that $D_2 \in L^\infty(0, L)$. Then, there exists a positive constant $c > 0$ such that for all $U_0 \in D(\mathcal{A}_j)$, $j = 1, 2$ the energy of system (1.1.24) satisfies the following decay rate :*

$$E(t) \leq \frac{c}{\sqrt{t}} \|U_0\|_{D(\mathcal{A}_j)}^2. \quad (1.1.34)$$

□

1.2 Introduction in French

1.2.1 Principales méthodes utilisées

Le fait que l'analyse de cette thèse est basée sur la théorie de semigroupe, dans cette partie, nous exposons et discutons de nombreux résultats récents sur la stabilité forte, exponentielle et polynomiale d'un C_0 -semigroupe qui servira à prouver nos résultats dans les chapitres suivants. Ensuite, nous présentons quelques résultats sur l'observabilité et la contrôlabilité exacte. Enfin, nous rappelons certaines conditions géométriques nécessaires à notre travail. Pour plus de détails nous nous référons à [19, 24, 25, 29, 36, 49, 55, 58, 59, 60, 62].

Semigroupes, Existence et Unicité de la solution

Soit $(X, \|\cdot\|_X)$ un espace de Banach sur \mathbb{C} et \mathcal{H} un espace de Hilbert muni du produit scalaire $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ et la norme induite $\|\cdot\|_{\mathcal{H}}$.

La majorité des équations d'évolution peuvent être réduites sous la forme

$$\begin{cases} U_t(x, t) = \mathcal{A}U(x, t), & t > 0, \\ U(0) = U_0 \in \mathcal{H}, \end{cases} \quad (1.2.1)$$

où \mathcal{A} est l'infinitesimal générateur d'un C_0 -semigroupe $(S(t))_{t \geq 0}$ dans un espace d'Hilbert \mathcal{H} . Pour cela, nous commençons par la définition des semigroupes.

Définition 1.2.1. Une famille $(S(t))_{t \geq 0}$ d'opérateurs linéaires bornés sur X est un semigroupe fortement continu (ou C_0 -semigroupe) si elle vérifie les conditions suivantes :

- $S(0) = I$.
- $S(t+s) = S(t)S(s) \quad \forall s, t \geq 0$.
- $\lim_{t \rightarrow 0} \|S(t) - I\|_X = 0$.

□

Définition 1.2.2. Soit $(S(t))_{t \geq 0}$ un C_0 -semigroupe. Alors, l'opérateur linéaire \mathcal{A} défini par

$$\mathcal{A}x = \lim_{t \rightarrow 0} \frac{S(t)x - x}{t}, \quad \forall x \in D(\mathcal{A}),$$

où

$$D(\mathcal{A}) = \left\{ x \in X; \lim_{t \rightarrow 0} \frac{S(t)x - x}{t} \text{ existe} \right\}$$

est le générateur infinitésimal du C_0 -semigroupe $(S(t))_{t \geq 0}$.

□

Certaines propriétés du semigroupe et de son générateur \mathcal{A} sont données par les théorèmes suivants :

Théorème 1.2.1. (Pazy [59]) Soit $(S(t))_{t \geq 0}$ un C_0 -semigroupe sur \mathcal{H} . Alors, il existe deux constantes $\omega \geq 0$ et $M \geq 1$ tel que

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{\omega t}, \quad \forall t \geq 0.$$

□

Si $\omega = 0$, le semigroupe $(S(t))_{t \geq 0}$ est dit uniformément borné. De plus, si $M = 1$, alors il s'appelle un C_0 -semigroupe de contractions.

Théorème 1.2.2. Si \mathcal{A} engendre un C_0 -semigroupe sur \mathcal{H} , alors

- $\overline{D(\mathcal{A})} = \mathcal{H}$.
- \mathcal{A} est fermé.

□

Définition 1.2.3. Un opérateur linéaire non borné $(\mathcal{A}, D(\mathcal{A}))$ sur X est dit dissipatif si

$$\|(\lambda I - \mathcal{A})x\|_X \geq \lambda \|x\|_X \quad \forall x \in D(\mathcal{A}) \text{ et } \forall \lambda > 0.$$

□

Proposition 1.2.4. Soit $(\mathcal{A}, D(\mathcal{A}))$ un opérateur linéaire non borné sur \mathcal{H} , alors

$$\mathcal{A} \text{ est dissipatif si et seulement si } \Re \langle Ax, x \rangle \leq 0, \quad \forall x \in D(\mathcal{A}).$$

□

Définition 1.2.5. *Un opérateur linéaire non borné $(\mathcal{A}, D(\mathcal{A}))$ sur X est dit maximal dissipatif (m -dissipatif) si*

- \mathcal{A} est un opérateur dissipatif.
- $\exists \lambda_0$ tel que $R(\lambda_0 I - \mathcal{A}) = X$, i.e. $\forall x \in X, \exists u \in D(\mathcal{A})$ tel que $\lambda_0 u - \mathcal{A}u = x$.

□

Pour l'existence de la solution du problème (1.2.1), nous utilisons typiquement les théorèmes de Hille-Yosida et de Lumer-Phillips suivants de [59] :

Théorème 1.2.3. (*Hille-Yosida*) *Un opérateur linéaire non borné $(\mathcal{A}, D(\mathcal{A}))$ sur X engendre un C_0 -semigroupe de contractions $(S(t))_{t \geq 0}$ si et seulement si*

- \mathcal{A} est fermé et $\overline{D(\mathcal{A})} = X$.
- L'ensemble résolvant $\rho(\mathcal{A})$ contient $(0, \infty)$ et $\forall \lambda > 0$,

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{\lambda}.$$

□

Théorème 1.2.4. (*Lumer-Phillips*) *Soit $(\mathcal{A}, D(\mathcal{A}))$ un opérateur linéaire non borné sur X avec un domaine $D(\mathcal{A})$ dense dans X . \mathcal{A} est le générateur infinitésimal d'un C_0 -semigroupe de contractions $(S(t))_{t \geq 0}$ si et seulement s'il est un opérateur m -dissipatif.*

□

Corollaire 1.2.6. *Soit $(\mathcal{A}, D(\mathcal{A}))$ un opérateur linéaire non borné sur \mathcal{H} . \mathcal{A} est l'infinitésimal générateur d'un C_0 -semigroupe de contractions $(S(t))_{t \geq 0}$ si et seulement s'il est un opérateur m -dissipatif.*

□

Par conséquent, l'existence et l'unicité de la solution sont justifiées par le corollaire suivant qui découle de théorème de Lumer-Phillips.

Corollaire 1.2.7. *Soit $(\mathcal{A}, D(\mathcal{A}))$ un opérateur linéaire non borné sur \mathcal{H} . Supposons que \mathcal{A} est l'infinitésimal générateur d'un C_0 -semigroupe de contractions $(S(t))_{t \geq 0}$. Alors :*

- Pour $U_0 \in D(\mathcal{A})$, le problème (1.2.1) admet une solution unique forte

$$U(t) = S(t)U_0 \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H}).$$

- Pour $U_0 \in \mathcal{H}$, le problème (1.2.1) admet une solution unique faible

$$U(t) \in C([0, +\infty), \mathcal{H}).$$

□

Stabilité des semigroupes

Afin de démontrer la stabilité forte d'un C_0 -semigroupe, nous appliquons le théorème suivant dû à Arendt et Batty dans [20].

Théorème 1.2.5. *Supposons que \mathcal{A} est le générateur infinitésimal d'un C_0 -semigroupe de contractions $(S(t))_{t \geq 0}$ fortement continu sur X . Si \mathcal{A} n'a pas de valeurs propres imaginaires pures et si $\sigma(\mathcal{A}) \cap i\mathbb{R}$ est dénombrable, alors $(S(t))_{t \geq 0}$ est fortement stable.*

□

Maintenant, quand le C_0 -semigroupe est fortement stable, nous recherchons des conditions nécessaires et suffisantes pour lesquelles il est exponentiellement stable. Nous rappelons ici uniquement la méthode fréquentielle suivante obtenue par Huang [36] et Prüss [60].

Théorème 1.2.6. (Huang [36] and Prüss [60]) Soit $(S(t))_{t \geq 0}$ un C_0 -semigroupe de contractions sur \mathcal{H} et \mathcal{A} son générateur infinitésimal. Alors, $(S(t))_{t \geq 0}$ est exponentiellement stable si et seulement si

- $i\mathbb{R} \subseteq \rho(\mathcal{A})$,
- $\limsup_{\beta \in \mathbb{R}, |\beta| \rightarrow \infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$. □

Puisque certains systèmes étudiés dans cette thèse n'atteignent pas la stabilité exponentielle, nous cherchons donc un taux polynomial. En général, les résultats de la stabilité polynomiale sont obtenus à l'aide de différentes méthodes telles que : méthode des multiplicateurs, méthode fréquentielle, base de Riesz, analyse de Fourier ou une combinaison de ces méthodes (Voir [40, 46, 47]). Dans cette thèse, nous utilisons la méthode fréquentielle obtenue par Borichev-Tomilov dans [25, Theorem 2.4].

Théorème 1.2.7. (Borichev-Tomilov [25]) Soit $(S(t))_{t \geq 0}$ un C_0 -semigroupe de contractions borné sur \mathcal{H} engendré par \mathcal{A} . Si $i\mathbb{R} \subset \rho(\mathcal{A})$, alors pour $l > 0$ fixé, les conditions suivantes sont équivalentes :

- $\limsup_{\beta \in \mathbb{R}, |\beta| \rightarrow \infty} \frac{1}{|\beta|^l} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty$.
- $\|S(t)U_0\|_{\mathcal{H}} \leq \frac{c}{t^{l-1}} \|U_0\|_{D(\mathcal{A})}, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}),$ pour certains $c > 0$. □

Enfin, pour étudier l'optimalité du taux de décroissance obtenu, nous nous référons au Théorème 3.4.1 de [56].

Théorème 1.2.8. (Wehbe, Najdi 2016) Soit \mathcal{A} le générateur infinitésimal d'un C_0 -semigroupe de contractions $(S(t))_{t \geq 0}$. Soient $(\lambda_{k,n})$ les valeurs propres de \mathcal{A} et $(e_{k,n})$ les vecteurs propres associés. Supposons qu'il existe $\mu_{k,n} \rightarrow +\infty$, $\alpha_k > 0$, $\beta_k > 0$ tel que $\Re(\lambda_{k,n}) \sim -\frac{\beta_k}{\alpha_k}$, $|\Im(\lambda_{k,n})| \sim \mu_{k,n}$, $i\mathbb{R} \subset \rho(\mathcal{A})$ et pour tout $u_0 \in D(\mathcal{A})$, il existe une constant $M > 0$ tel que

$$\|S(t)u_0\|_{\mathcal{H}} \leq \frac{M}{t^{l_k}} \|u_0\|_{D(\mathcal{A})}, \quad l_k = \max_{1 \leq k \leq k_0} (\alpha_k), \quad \forall t > 0. \quad (1.2.2)$$

Alors le taux de décroissance (1.2.2) est optimal. □

Observabilité et contrôlabilité exacte

Dans cette partie, nous présentons brièvement la dualité entre les notions d'observabilité et de contrôlabilité, qui est à la base de la Méthode d'Unicité de Hilbert (HUM)

établie par J. L. Lions dans [46].

Tout d'abord, nous considérons le système suivant :

$$\begin{cases} U_t(x, t) = AU(x, t) + Bv(x), & \text{sur } \Omega \times (0, +\infty), \\ U(x, 0) = U_0(x), \end{cases} \quad (1.2.3)$$

où $\Omega \subset \mathbb{R}^d (d \in \mathbb{N}^*)$, U est un scalaire ou une fonction à valeur vectorielle, A est un ensemble d'équations aux dérivées partielles, linéaires ou non linéaires (au moins pour le moment), v dénote le contrôle et B est la fonction de "l'espace des contrôles" dans "l'espace d'état". Le système (1.2.3) doit inclure les conditions aux bords. Nous ne les faisons pas explicitement ici. Elles sont supposées être contenues dans la formulation abstraite (1.2.3).

Le contrôle v peut être appliqué à l'intérieur du domaine Ω (dans ce cas, on dit que v est un contrôle interne), ou sur la frontière Γ de Ω ou sur une partie de la frontière (dans ce cas, v est considéré comme un contrôle frontière). Si v est appliqué en un point de Ω , v est dit contrôle ponctuel.

On supposera que, pour v (dans un espace approprié), le problème (1.2.3) définit de manière unique une solution. Cette solution est une fonction (valeur scalaire ou vectorielle) de $x \in \Omega$, $t > 0$ et de U_0 et v .

Maintenant, nous pouvons introduire la notion de contrôlabilité, exacte ou approximative. Soit $T > 0$ et U_T (la cible) un élément donné de l'espace d'état. Nous voulons "conduire le système" de l'état initial U_0 à $t = 0$ à l'état final U_T à $t = T$, c'est-à-dire que nous voulons trouver un contrôle approprié v tel que

$$U(x, T) = U_T(x), \quad x \in \Omega.$$

Si cela est possible pour n'importe quelle cible U_T à la place de l'état, on peut dire que le système est contrôlable (ou exactement contrôlable). Pour plus de détails sur la contrôlabilité, voir [46].

Nous présentons maintenant un résultat de A. Haraux dans [31] pour obtenir l'inégalité d'observabilité. Si A est un opérateur linéaire non borné, auto-adjoint, positif et coercive sur un espace de Hilbert \mathcal{H} et B un opérateur linéaire sur \mathcal{H} tel que $B = B^* \geq 0$, il a établi une équivalence logique entre la décroissance exponentielle des solutions de l'équation d'évolution du deuxième ordre $U_{tt} + AU + BU_t = 0$, uniformément sur des sous-ensembles de $D(A^{1/2}) \times \mathcal{H}$ et une propriété " $B^{1/2}$ - contrôlabilité" du système non contrôlable $\varphi_{tt} + A\varphi = 0$ sur un intervalle de temps (voir les propositions 1 et 2 dans [31]).

Remarque 1. *Le résultat de A. Haraux dans [31] reste valable pour une équation d'évolution du premier ordre.*

Conditions géométriques

Cette section est consacrée à rappeler certaines conditions géométriques nécessaires tout au long de notre travail.

Nous commençons par les conditions du contrôle géométrique GCC introduites par Rauch et Taylor dans [61] pour des variétés sans frontière et par Bardos, Lebeau et Rauch dans [21] pour les domaines avec frontière.

Définition 1.2.8. *On dit qu'un sous ensemble ω de Ω satisfait le **GCC** si chaque rayon de l'optique géométrique partant d'un point quelconque $x \in \Omega$ à $t = 0$ doit rencontrer la région ω en temps fini T . \square*

Ensuite, nous rappelons la condition de multiplicateurs géométriques par morceaux introduite par K. Liu dans [48].

Définition 1.2.9. *On dit que ω satisfait la condition de multiplicateurs géométriques par morceaux (PMGC) s'ils existent $\Omega_j \subset \Omega$ ayant une frontière Lipschitzienne $\Gamma_j = \partial\Omega_j$ et $x_j \in \mathbb{R}^N$, $j = 1, \dots, J$ tel que $\Omega_j \cap \Omega_i = \emptyset$ pour $j \neq i$ et ω contient un voisinage dans Ω de l'ensemble $\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)$ où $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ et ν_j est le vecteur normal unitaire extérieure à Γ_j . \square*

Remarque 2. *La condition PMGC est la généralisation de la condition géométrique des multiplicateurs (MGC) introduite par Lions dans [46], en disant que ω contient un voisinage dans Ω de l'ensemble $\{x \in \Gamma : (x - x_0) \cdot \nu(x) > 0\}$, pour certains $x_0 \in \mathbb{R}^N$, où ν est le vecteur unitaire normal extérieur à $\Gamma = \partial\Omega$. Cependant, la condition PMGC est beaucoup plus restrictive que celle de GCC. \square*

Après cette introduction générale, nous allons maintenant détailler les résultats de cette thèse dans les trois sections suivantes.

1.2.2 Chapitre 2 : Stabilité indirecte locale de deux équations d'ondes couplées sous des conditions géométriques

Soit $\Omega \subset \mathbb{R}^N$ un ouvert borné non vide avec une frontière Γ de class C^2 .

Dans chapitre 2, nous sommes intéressés à étudier la stabilité du système suivant de deux équations d'ondes couplées :

$$u_{tt} - a\Delta u + c(x)u_t + b(x)y_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \quad (1.2.4)$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \quad (1.2.5)$$

$$u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+, \quad (1.2.6)$$

avec les données initiales suivantes :

$$u(x, 0) = u_0, \quad y(x, 0) = y_0, \quad u_t(x, 0) = u_1 \text{ and } y_t(x, 0) = y, \quad x \in \Omega,$$

où $a > 0$ est une constante et $b(x) \in C^0(\overline{\Omega}; \mathbb{R})$ est une fonction non identiquement nulle. Le terme d'amortissement $c(x) \in C^0(\overline{\Omega}, \mathbb{R}_+)$ est appliqué seulement à la première équation tandis que la deuxième est amortie par l'intermédiaire du couplage et par suite par un contrôle indirect. La notion du mécanisme de dissipation indirecte a été introduite par D.L. Russel [63], et depuis lors, elle a suscité l'intérêt de nombreux auteurs ces dernières années.

Résultats antérieurs

Dans [37], B. Kapitov a considéré la stabilité du système (1.2.4)-(1.2.6). Il a établi un taux de décroissance exponentiel de l'énergie dans le cas où les trois conditions suivantes sont satisfaites : les ondes se propagent à la même vitesse ($a = 1$), le support de b coïncide avec celui de c et le domaine, où l'amortissement est effectif, satisfait certaines conditions géométriques. En revanche, dans le cas général, quand les vitesses sont différentes le taux de décroissance de l'énergie n'est pas étudié.

Dans [12], F. Alabau-Boussouira et al. ont étudié la stabilité du système suivant :

$$u_{tt} - a\Delta u + \rho(x, u_t) + b(x)y_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (1.2.7)$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (1.2.8)$$

$$u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*, \quad (1.2.9)$$

où $a > 0$ constante, $b(x) \in C^0(\overline{\Omega}, \mathbb{R})$ et $\rho(x, u_t)$ est un amortissement non linéaire. En utilisant une approche basée sur les techniques de multiplicateurs, les inégalités non linéaires avec poids et la méthode de convexité avec poids optimal (développée dans [7]), les auteurs ont établi une formule explicite qui permet d'obtenir un taux de décroissance de l'énergie en fonction du comportement au voisinage de zéro du terme de dissipation non linéaire. Leurs résultats sont obtenus dans le cas où les trois conditions suivantes sont satisfaites : les ondes se propagent à la même vitesse ($a = 1$), le coefficient de couplage b est supposé être positif et petit ($0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ où b^* est une constante qui ne dépend que de Ω et de la zone du contrôle) et les zones de couplage et de contrôle satisfont toutes deux les conditions géométriques PMGC. Alors la stabilisation du système (1.2.7)-(1.2.9) dans le cas où les ondes ne se propagent pas forcément à la même vitesse ($a \neq 1$) et/ou le coefficient de couplage $b(x)$ n'est pas supposé être positif et petit, a été laissée comme problème ouvert même dans le cas où le terme d'amortissement ρ est linéaire par rapport à la deuxième variable *i.e.* $\rho(x, u_t) = c(x)u_t$. Dans le chapitre 2, notre objectif est de répondre à ces questions ouvertes dans le cas linéaire grâce à une analyse complète de la stabilité du système (1.2.4)-(1.2.6) sous des conditions géométriques appropriées.

Principaux résultats obtenus

La principale nouveauté dans ce chapitre est que les ondes ne se propagent pas nécessairement à la même vitesse et que le coefficient de couplage n'est supposé ni positif ni petit.

D'abord, nous commençons à étudier l'existence, l'unicité et la régularité de la solution de notre système en utilisant la théorie de semigroupe. Soit $U = (u, u_t, y, y_t)$ une solution régulière du système (1.2.4)-(1.2.6), son énergie associée est définie par :

$$E(t) = \frac{1}{2} \int_{\Omega} (|u_t|^2 + a|\nabla u|^2 + |y_t|^2 + |\nabla y|^2) dx. \quad (1.2.10)$$

Par un calcul direct, on obtient :

$$\frac{d}{dt} E(t) = - \int_{\Omega} c(x)|u_t|^2 dx \leq 0. \quad (1.2.11)$$

Maintenant, nous définissons l'espace de l'énergie $\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2$ muni, pour tout $U = (u, v, y, z)$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in \mathcal{H}$ du produit scalaire :

$$(U, \tilde{U})_{\mathcal{H}} = a \int_{\Omega} (\nabla u \cdot \nabla \tilde{u}) dx + \int_{\Omega} v \tilde{v} dx + \int_{\Omega} (\nabla y \cdot \nabla \tilde{y}) dx + \int_{\Omega} z \tilde{z} dx.$$

Ensuite, nous définissons l'opérateur linéaire non borné $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ par :

$$D(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega))^2, \quad \mathcal{A}U = (v, a\Delta u - bz - cv, z, \Delta y + bv).$$

Puisque $c(x) \geq 0$, l'opérateur \mathcal{A} est m-dissipatif et engendre un C_0 semigroupe de contractions $e^{t\mathcal{A}}$ sur l'espace d'énergie \mathcal{H} . Comme le système (1.2.4)-(1.2.6) est équivalent à

$$U_t = \mathcal{A}U \text{ dans } \mathcal{H}, \quad t > 0, \quad U(0) = U_0 \quad (1.2.12)$$

avec $U = (u, u_t, y, y_t)$, nous déduisons son caractère bien posé.

Comme souligné précédemment nous allons étudier la stabilité et le taux de décroissance de l'énergie de notre système. Pour cela, nous supposons qu'il existe un ouvert non vide $\omega_{c_+} \subset \Omega$ tel que :

$$\{x \in \Omega : c(x) > 0\} \supset \bar{\omega}_{c_+}. \quad (\text{LH1})$$

D'autre part, comme $b(x)$ est non identiquement nulle et continue, alors il existe un ouvert non vide $\omega_{b_+} \cup \omega_{b_-} \subset \Omega$ tel que

$$\{x \in \Omega : b(x) > 0\} \supset \bar{\omega}_{b_+} \quad \text{ou} \quad \{x \in \Omega : b(x) < 0\} \supset \bar{\omega}_{b_-}. \quad (\text{LH2})$$

Nous montrons d'abord que notre système est fortement stable.

Théorème 1.2.9. (*Stabilité forte*) *Supposons que $a > 0$, l'hypothèse de localisation (LH1) est satisfaite et que $\omega = \omega_{c_+} \cap \omega_{b_+} \neq \emptyset$ (ou $\omega = \omega_{c_+} \cap \omega_{b_-} \neq \emptyset$). Alors pour toute donnée initiale $(u_0, u_1, y_0, y_1) \in \mathcal{H}$ l'énergie du système (1.2.4)-(1.2.6) $E(t) = \frac{1}{2} \|(u, u_t, y, y_t)\|_{\mathcal{H}}^2$, décroît asymptotiquement vers 0.* \square

Ensuite, nous nous intéressons à l'étude du taux de décroissance de l'énergie en utilisant la méthode fréquentielle combinée avec la technique de multiplicateur et en distinguant deux cas : le premier quand les ondes sont supposées propagées à la même vitesse (i.e. $a = 1$) et le deuxième quand $a \neq 1$.

Théorème 1.2.10. (*Taux de décroissance exponentiel*) *Soit $a = 1$. Supposons que l'hypothèse de localisation (LH1) est satisfaite. Supposons de plus que l'ouvert non vide $\omega = \omega_{c_+} \cap \omega_{b_+}$ (ou $\omega = \omega_{c_+} \cap \omega_{b_-}$) satisfait les conditions géométriques PMGC et que $b, c \in W^{1,\infty}(\Omega)$, alors il existe des constantes $\theta > 0$ et $M \geq 1$ telles que, pour toute donnée initiale $U_0 = (u_0, u_1, y_0, y_1) \in \mathcal{H}$, l'énergie $E(t)$ du système (1.2.4)-(1.2.6), vérifie l'estimation suivante :*

$$E(t) \leq Me^{-\theta t} E(0), \quad \forall t > 0. \quad (1.2.13)$$

\square

Remarque 3. *Notons que dans le théorème précédent nous n'avons pas de restriction sur la borne supérieure et le signe de la fonction b . Ce théorème est une généralisation dans le cas linéaire du résultat de [12] où le coefficient de couplage considéré doit satisfaire $0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ où b^* est une constante qui ne dépend que de Ω et de la zone de contrôle. Néanmoins, le problème reste ouvert dans le cas non linéaire.* \square

La condition d'égalité des vitesses de propagation est une condition nécessaire et suffisante pour la stabilité exponentielle de notre système. En effet, dans le cas $a \neq 1$, nous construisons une suite (U_n) d'éléments de $D(\mathcal{A})$ et une suite réelle (μ_n) telles que $\|U_n\|_{\mathcal{H}} = 1$ et $\|(i\mu_n I - \mathcal{A})U_n\|_{\mathcal{H}} \rightarrow 0$. Ainsi, la résolvante de \mathcal{A} n'est pas uniformément bornée sur l'axe imaginaire. D'après un résultat de Huang [36] et Prüss [60], nous concluons que le semigroupe $e^{t\mathcal{A}}$ n'est pas uniformément stable sur \mathcal{H} . Il est donc naturel de penser à un taux de décroissance polynomial de l'énergie du système; nous établissons, alors, l'estimation suivante :

Théorème 1.2.11. (*Taux de décroissance polynomial*) *Soit $a \neq 1$. Supposons que l'hypothèse de localisation (LH1) est satisfaite. Supposons de plus que l'ouvert non vide $\omega = \omega_{c_+} \cap \omega_{b_+}$ (ou $\omega = \omega_{c_+} \cap \omega_{b_-}$) satisfait les conditions géométriques PMGC et que $b, c \in W^{1,\infty}(\Omega)$, alors il existe une constante $C > 0$ telle que, pour toute donnée initiale $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$, l'énergie $E(t)$ du système (1.2.4)-(1.2.6), vérifie l'estimation suivante :*

$$E(t) \leq \frac{C}{t} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \quad (1.2.14)$$

\square

Notons que dans le cas $a \neq 1$ les ondes ne se propagent pas à la même vitesse. Ce théorème généralise, dans le cas linéaire, les résultats de [12] et [37].

Finalement, dans le cas monodimensionnel d'espace (i.e. $N = 1$), $a \neq 1$ et b est une constante, nous montrons qu'il existe $n_0 \in \mathbb{N}$ suffisamment grand et une suite λ_n de valeurs propres simples associées à l'opérateur \mathcal{A} satisfaisant le développement asymptotique suivant :

$$\lambda_n = in\pi - \frac{ib^2}{2(a-1)n\pi} - \frac{cb^2}{2(a-1)^2n^2\pi^2} + O\left(\frac{1}{n^3}\right), \quad \forall |n| \geq n_0. \quad (1.2.15)$$

En utilisant le Théorème 3.4.1 de [56] et l'équation (1.2.15), nous déduisons que le taux polynomial obtenu (1.2.14) est optimal.

1.2.3 Chapitre 3 : Contrôlabilité exacte et stabilité des équations d'ondes localement couplées

L'objectif de ce chapitre est d'étudier la contrôlabilité exacte du système suivant :

$$u_{tt} - a\Delta u + b(x)y_t = c(x)v(t) \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (1.2.16)$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (1.2.17)$$

$$u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*, \quad (1.2.18)$$

avec les données initiales suivantes :

$$u(x, 0) = u_0, \quad y(x, 0) = y_0, \quad u_t(x, 0) = u_1 \text{ and } y_t(x, 0) = y_1, \quad x \in \Omega, \quad (1.2.19)$$

où $a > 0$ constante, $b \in C^0(\overline{\Omega}, \mathbb{R})$, $c \in C^0(\overline{\Omega}, \mathbb{R}^+)$ et v est un contrôle approprié.

L'idée est d'utiliser un résultat de A. Haraux [31] par lequel l'inégalité d'observabilité du système homogène associé à (1.2.16)-(1.2.18) est équivalente à la stabilité exponentielle du système (1.2.4)-(1.2.6) dans un espace d'Hilbert approprié. Alors, nous fournissons une analyse complète de la stabilité exponentielle du système (1.2.4)-(1.2.6) dans des différents espaces d'Hilbert.

Résultats antérieurs

Dans le chapitre 2, nous avons étudié la stabilité du système (1.2.4)-(1.2.6) dans deux cas. Dans le premier cas où les ondes se propagent à la même vitesse (i.e. $a = 1$), en supposant que la région de couplage et la région d'amortissement ont une intersection non vide et satisfaisant la condition PMGC, nous avons établi un taux de décroissance exponentiel de l'énergie pour des données initiales faibles. Dans le cas où $a \neq 1$, nous avons d'abord prouvé que notre système n'est pas uniformément stable. Cependant, sous les mêmes conditions géométriques, un taux de décroissance polynomial optimal de type $\frac{1}{t}$ a été établi pour des données initiales régulières.

Notre but dans ce chapitre est de prouver la stabilité exponentielle du système (1.2.4)-(1.2.6) dans deux espaces de Hilbert différents en utilisant des conditions géométriques plus générales que celles utilisées dans le chapitre 2. Et par conséquent, en utilisant la Proposition 2 de A. Haraux [31], on obtient l'observabilité du système homogène associé à (1.2.16)-(1.2.18).

Principaux résultats obtenus

D'abord, il faut étudier le comportement asymptotique de l'énergie $E(t)$ donnée par (1.2.10) associée à (1.2.4)-(1.2.6). Pour ce but, on suppose qu'il existe un ouvert non vide $\omega_{c_+} \subset \Omega$ satisfaisant la condition suivante :

$$\{x \in \Omega : c(x) > 0\} \supset \overline{\omega_{c_+}}. \quad (\text{LH1})$$

D'autre part, comme $b(x)$ est identiquement non nulle et continue, alors il existe un ouvert non vide $\omega_b \subset \Omega$ tel que

$$\{x \in \Omega : b(x) \neq 0\} \supset \overline{\omega_b}. \quad (\text{LH2})$$

Si $\omega = \omega_{c_+} \cap \omega_b \neq \emptyset$ et l'hypothèse de localisation (LH1) est satisfaite, alors le système (1.2.4)-(1.2.6) est fortement stable en utilisant le Théorème 1.2.5, i.e.

$$\lim_{t \rightarrow +\infty} \|e^{tA}(u_0, u_1, y_0, y_1)\|_{\mathcal{H}} = 0 \quad \forall (u_0, u_1, y_0, y_1) \in \mathcal{H}.$$

Ensuite, quand les ondes se propagent à la même vitesse, sous la condition que la région de couplage est incluse dans la région d'amortissement (i.e. $\omega_b \subset \omega_{c_+}$) et satisfaisant la condition du contrôle géométrique GCC, nous établissons un taux de décroissance exponentielle du système (1.2.4)-(1.2.6) :

Théorème 1.2.12. (*Taux de décroissance exponentiel*) Soit $a = 1$. supposons que les conditions (LH1) et (LH2) sont satisfaites. Supposons aussi que $\omega_b \subset \omega_{c_+}$ satisfait la condition du contrôle géométrique GCC et que $b, c \in W^{1,\infty}(\Omega)$. Alors, il existe des constantes positives $M \geq 1$, $\theta > 0$ tel que pour toute donnée initiale $(u_0, u_1, y_0, y_1) \in \mathcal{H}$ l'énergie du système (1.2.4)-(1.2.6) vérifie l'estimation suivante :

$$E(t) \leq M e^{-\theta t} E(0), \quad \forall t > 0. \quad (1.2.20)$$

□

Remarque 4. Les situations géométriques gouvernées par le Théorème 1.2.12 sont plus riches que celles considérées dans le Chapitre 2 et [12]. En effet, dans les références précédentes, les auteurs considèrent les conditions géométriques de PMGC qui sont plus restrictives que GCC. D'autre part, contrairement aux résultats de [12], nous n'avons aucune restriction dans le Théorème 1.2.12 sur la borne supérieure et le signe du coefficient de la fonction de couplage b . Ce théorème est alors une généralisation dans le cas linéaire du résultat de [12] où le coefficient de couplage considéré doit satisfaire $0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ où b^* est une constante dépendante de Ω et de la région du contrôle. □

En utilisant la Proposition 2 de A. Haraux dans [31], une inégalité d'observabilité de la solution du système homogène associé à (1.2.16)-(1.2.18) dans l'espace $(H_0^1(\Omega) \times L^2(\Omega))^2$ est établie. Cela mène, par la méthode HUM introduite par J. L. Lions dans [46], à la contrôlabilité exacte du système (1.2.4)-(1.2.6) dans l'espace $(H^{-1}(\Omega) \times L^2(\Omega))^2$.

De plus, lorsque les ondes se propagent avec des vitesses différentes, (i.e., $a \neq 1$), nous établissons un taux de décroissance exponentielle de l'énergie du système (1.2.4)-(1.2.6) dans l'espace d'énergie faible. Pour cela, on introduit l'espace d'énergie faible suivant

$$D = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega),$$

muni de produit scalaire

$$(U, \tilde{U}) = \int_{\Omega} (a \nabla u \cdot \nabla \tilde{u} + v \tilde{v} + y \tilde{y} + (-\Delta)^{-1/2} z (-\Delta)^{-1/2} \tilde{z}) dx,$$

pour tout $U = (u, v, y, z) \in D$ et $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in D$.

Ensuite, nous définissons l'opérateur linéaire non borné $\mathcal{A}_d : D(\mathcal{A}_d) \subset D \rightarrow D$ par

$$\mathcal{A}_d U = (v, a \Delta u - bz - cv, z, \Delta y + bv),$$

$$D(\mathcal{A}_d) = ((H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)), \quad \forall U = (u, v, y, z) \in D(\mathcal{A}_d).$$

Son énergie mixte totale est définie par

$$E_m(t) = \frac{1}{2} (a \|\nabla u\|_{L^2(\Omega)}^2 + \|u_t\|_{L^2(\Omega)}^2 + \|y_t\|_{H^{-1}(\Omega)}^2 + \|y\|_{L^2(\Omega)}^2).$$

Ensuite, nous passons à l'étude du comportement asymptotique de $E_m(t)$. Pour ce but, nous devons supposer que ω_{c_+} satisfait le condition géométrique PMGC, alors il existe

$\varepsilon > 0$, sous ensemble $\Omega_j \subset \Omega$, $j = 1, \dots, J$, avec frontière Lipchtizienne, $\Gamma_j = \partial\Omega_j$ et points $x_j \in \mathbb{R}^N$ tel que $\Omega_i \cap \Omega_j = \emptyset$ si $i \neq j$ et $\omega_c^+ \supset \mathcal{N}_\varepsilon(\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)) \cap \Omega$ avec $\mathcal{N}_\varepsilon(\mathcal{O}) = \{x \in \mathbb{R}^N : d(x, \mathcal{O}) < \varepsilon\}$ où $\mathcal{O} \subset \mathbb{R}^N$, $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ où ν_j est le vecteur normal extérieur à Γ_j et que ω_b satisfait le GCC et que

$$\omega_b \subset (\Omega \setminus \cup_{j=1}^J \Omega_j). \quad (\text{LH3})$$

Maintenant, notre deuxième résultat principal lorsque les ondes se propagent avec des vitesses différentes (i.e. $a \neq 1$) est le suivant :

Théorème 1.2.13. (*Taux de décroissance exponentiel*) Soit $a \neq 1$. Supposons que les conditions (LH1) et (LH2) sont satisfaites. Supposons de plus que ω_{c^+} satisfait la condition géométrique PMGC, ω_b satisfait la condition GCC et (LH3) et $b, c \in L^\infty(\Omega)$. Alors il existe des constantes positives $M \geq 1$, $\theta > 0$ tel que pour toute donnée initiale $(u_0, u_1, y_0, y_1) \in D$ l'énergie du système (1.2.4)-(1.2.6) vérifie l'estimation suivante :

$$E_m(t) \leq M e^{-\theta t} E_m(0), \quad \forall t > 0. \quad (1.2.21)$$

□

Par conséquent, en utilisant la Proposition 2 de A. Haraux [31], une inégalité d'observabilité de la solution du système homogène associé à (1.2.16)-(1.2.18) est établie. Cela conduit, par la méthode HUM, à la contrôlabilité exacte du système (1.2.16)-(1.2.18) dans l'espace d'énergie $L^2(\Omega) \times H^{-1}(\Omega) \times H^{-1}(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))'$, où la dualité est par rapport à $L^2(\Omega)$.

Enfin, nous effectuons des tests numériques dans le cas 1-D pour valider notre résultats théoriques obtenus ici et au chapitre 2. En fait, les résultats numériques montrent un meilleur comportement que celui attendu par les résultats théoriques.

1.2.4 Chapitre 4 : Stabilité d'un système de Bresse avec amortissement local Kelvin-Voigt et coefficient non régulière à l'interface

Ce chapitre est consacré à l'étude de la stabilité d'un système de Bresse élastique avec amortissement local de type Kelvin-Voigt et de coefficient non-régulier à l'interface dans des conditions de Dirichlet ou Dirichlet-Neumann-Neumann aux bords. Le système défini sur $(0, L) \times (0, +\infty)$ est donné par :

$$\begin{cases} \rho_1 \varphi_{tt} - [k_1(\varphi_x + \psi + lw) + D_1(\varphi_{xt} + \psi_t + lw_t)]_x - lk_3(w_x - l\varphi) - lD_3(w_{xt} - l\varphi_t) = 0, \\ \rho_2 \psi_{tt} - [k_2\psi_x + D_2\psi_{xt}]_x + k_1(\varphi_x + \psi + lw) + D_1(\varphi_{xt} + \psi_t + lw_t) = 0, \\ \rho_1 w_{tt} - [k_3(w_x - l\varphi) + D_3(w_{xt} - l\varphi_t)]_x + lk_1(\varphi_x + \psi + lw) + lD_1(\varphi_{xt} + \psi_t + lw_t) = 0, \end{cases} \quad (1.2.22)$$

Les coefficients $\rho_1, \rho_2, k_1, k_2, k_3$ et l sont des constantes positives. D_1, D_2 et D_3 sont des fonctions positives sur $(0, L)$.

Résultat antérieur :

Le matériel Kelvin-Voigt est une structure viscoélastique ayant des propriétés d'élasticité et de viscosité. L'amortissement Kelvin-Voigt peut être distribué globalement ou localement. Mais le cas qui nous intéresse est quand l'amortissement est localisé sur un intervalle quelconque du domaine. Les propriétés de régularité et de stabilité d'une solution dépendent des propriétés des coefficients d'amortissement. En effet, le système est plus efficacement contrôlé par l'amortissement local Kelvin-Voigt lorsque le coefficient est plus régulier à l'interface.

Récemment, X. Tian et Q. Zhang dans [66] ont considéré le système de Timoshenko suivant défini sur $(0, L) \times (0, +\infty)$ avec des conditions de Dirichlet aux bords :

$$\begin{cases} \rho_1 \varphi_{tt} - [k_1 (\varphi_x + \psi)_x + D_1 (\varphi_{xt} - \psi_t)]_x = 0, \\ \rho_2 \psi_{tt} - (k_2 \psi_x + D_2 \psi_{xt})_x + k_1 (\varphi_x + \psi)_x + D_1 (\varphi_{xt} - \psi_t) = 0. \end{cases} \quad (1.2.23)$$

Ils ont étudié ce système avec un amortissement Kelvin-Voigt distribué localement ou globalement et les coefficients $D_1, D_2 \in C([0, L])$. D'abord, lorsque les amortissements sont globalement distribués, ils ont montré que le système (1.2.23) avec des conditions aux bords entièrement Dirichlet est analytique. Ensuite, quand les amortissements sont localisés au voisinage du bord, ils ont analysé la stabilité exponentielle et polynomiale dépendant des propriétés des coefficients D_1, D_2 . Contrairement aux résultats de [66], dans ce chapitre, nous étudions le système de Bresse (1.2.22) avec des conditions aux bords de type Dirichlet ou de Dirichlet-Neumann-Neumann et dans le cas où les amortissements sont localisés sur un intervalle quelconque du domaine.

Principaux résultats obtenus :

D'abord, nous commençons à étudier l'existence, l'unicité et la régularité de la solution de notre système en utilisant la théorie de semigroupe. Soit $(\varphi, \varphi_t, \psi, \psi_t, w, w_t)$ une solution régulière du système (1.2.22), son énergie associée est définie par :

$$\begin{aligned} E(t) = \frac{1}{2} \left\{ \int_0^L (\rho_1 |\varphi_t|^2 + \rho_2 |\psi_t|^2 + \rho_1 |w_t|^2 + k_1 |\varphi_x + \psi + lw|^2) dx \right. \\ \left. + \int_0^L (k_2 |\psi_x|^2 + k_3 |w_x - l\varphi|^2) dx \right\}. \end{aligned} \quad (1.2.24)$$

Par un calcul direct, on obtient

$$E'(t) = - \int_0^L (D_1 |\varphi_{xt} + \psi_t + lw_t|^2 + D_2 |\psi_{xt}|^2 + D_3 |w_{xt} - l\varphi_t|^2) dx \leq 0. \quad (1.2.25)$$

Alors, le système (1.2.22) est dissipatif dans le sens que son énergie est décroissante par rapport au temps t . Ensuite, nous définissons les espaces d'énergie suivants :

$$\mathcal{H}_1 = (H_0^1 \times L^2)^3 \quad \text{et} \quad \mathcal{H}_2 = H_0^1 \times L^2 \times (H_*^1 \times L_*^2)^2,$$

où

$$L_*^2 = \{f \in L^2(0, L) : \int_0^L f(x)dx = 0\} \text{ et } H_*^1 = \{f \in H^1(0, L) : \int_0^L f(x)dx = 0\}.$$

Nous définissons les opérateurs linéaires non bornés \mathcal{A}_j dans $\mathcal{H}_j, j = 1, 2$ par

$$D(\mathcal{A}_1) = \left\{ U \in \mathcal{H}_1 \mid v^2, v^4, v^6 \in H_0^1(0, L), \right. \\ \left. [k_1(v_x^1 + v^3 + lv^5) + D_1(v_x^2 + v^4 + lv^6)]_x \in L^2(0, L), \right. \\ \left. [k_2v_x^3 + D_2v_x^4]_x \in L^2(0, L), [k_3(v_x^5 - lv^1) + D_3(v_x^6 - lv^2)]_x \in L^2(0, L) \right\},$$

$$D(\mathcal{A}_2) = \left\{ U \in \mathcal{H}_2 \mid v^2 \in H_0^1(0, L), v^4, v^6 \in H_*^1(0, L), v_x^3|_{(0,L)} = v_x^5|_{(0,L)} = 0, \right. \\ \left. [k_1(v_x^1 + v^3 + lv^5) + D_1(v_x^2 + v^4 + lv^6)]_x \in L^2(0, L), \right. \\ \left. [k_2v_x^3 + D_2v_x^4]_x \in L_*^2(0, L), [k_3(v_x^5 - lv^1) + D_3(v_x^6 - lv^2)]_x \in L_*^2(0, L) \right\},$$

et

$$\mathcal{A}_j \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^5 \\ v^6 \end{pmatrix} = \begin{pmatrix} v^2 \\ \rho_1^{-1} ([k_1(v_x^1 + v^3 + lv^5) + D_1(v_x^2 + v^4 + lv^6)]_x + lk_3(v_x^5 - lv^1) + lD_3(v_x^6 - lv^2)) \\ v^4 \\ \rho_2^{-1} ((k_2v_x^3 + D_2v_x^4)_x - k_1(v_x^1 + v^3 + lv^5) - D_1(v_x^2 + v^4 + lv^6)) \\ v^6 \\ \rho_1^{-1} ([k_3(v_x^5 - lv^1) + D_3(v_x^6 - lv^2)]_x - lk_1(v_x^1 + v^3 + lv^5) - lD_1(v_x^2 + v^4 + lv^6)) \end{pmatrix}$$

pour tout $U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(\mathcal{A}_j)$. Soit $U = (\varphi, \varphi_t, \psi, \psi_t, w, w_t)^T$, système (1.2.22) peut se réécrire de la forme

$$U_t = \mathcal{A}_j U, j = 1, 2 \quad U = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1)^T.$$

Ensuite, nous démontrons que l'opérateur \mathcal{A}_j est m-dissipatif dans l'espace de l'énergie \mathcal{H}_j . Par conséquent, grâce au théorème de Lumer-Phillips, nous déduisons que \mathcal{A}_j engendre un C_0 -semigroupe de contractions $e^{t\mathcal{A}_j}$ dans \mathcal{H}_j , et que le problème est bien posé dans \mathcal{H}_j . En plus, en utilisant une critère générale de Arendt et Batty [20], nous montrons que le C_0 -semigroupe $e^{t\mathcal{A}_j}$ est fortement stable en l'absence de la compacité du résolvant de \mathcal{A}_j et la présence d'au moins un amortissement local de Kelvin-Voigt.

Maintenant, notre but est d'étudier la décroissance de l'énergie en utilisant la méthode fréquentielle combinée avec la technique des multiplicateurs dans plusieurs cas dépendant de la régularité des coefficients D_1, D_2 et D_3 , de la localisation de leurs supports et de leurs nombres.

Dans le cas où les trois amortissements sont globalement distribués, nous démontrons une stabilité analytique de l'énergie.

Ensuite, dans le cas où les fonctions D_i , $i = 1, 2, 3$ sont continues et satisfaisantes la condition suivante :

$$\exists d_0 > 0 \text{ tel que } D_i \geq d_0 > 0 \text{ pour tout } x \in (\alpha, \beta), \quad 0 < \alpha < \beta < L, \quad (1.2.26)$$

nous établissons un taux de décroissance exponentiel de l'énergie donnée par le théorème suivant :

Théorème 1.2.14. (*Taux de décroissance exponentiel*) *Supposons que (1.2.26) est satisfait et D_1, D_2 et $D_3 \in W^{1,\infty}(0, L)$. Le C_0 -semigroupe $e^{t\mathcal{A}_j}$ est exponentiellement stable, i.e., il existe des constantes $M \geq 1$ et $\epsilon > 0$ indépendant de U_0 tel que*

$$\|e^{t\mathcal{A}_j}U_0\|_{\mathcal{H}_j} \leq Me^{-\epsilon t} \|U_0\|_{\mathcal{H}_j}, \quad t \geq 0, \quad j = 1, 2.$$

□

Cependant dans le cas où $D_i \in L^\infty(0, L)$, $i = 1, 2, 3$ satisfait la condition suivante :

$$\omega = \text{supp}D_1 \cap \text{supp}D_2 \cap \text{supp}D_3 = (\alpha, \beta) \subset (0, L) \text{ tel que } \text{mes}(\omega) > 0, \quad (1.2.27)$$

nous établissons un taux de décroissance polynomial donné par le théorème suivant :

Théorème 1.2.15. (*Taux de décroissance polynomial*) *Supposons que la condition (1.2.27) est satisfaite. Supposons de plus D_1, D_2 et $D_3 \in L^\infty(0, L)$. Alors, il existe une constante positive $c > 0$ tel que pour tout $U_0 \in D(\mathcal{A}_j)$, $j = 1, 2$, l'énergie du système satisfait le taux de décroissance suivant :*

$$E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A}_j)}^2. \quad (1.2.28)$$

□

En revanche, nous démontrons que notre système (1.2.22) avec des conditions aux bords de type Dirichlet-Neumann-Neumann et sous la condition suivante

$$D_1 = 0 \quad \text{et} \quad D_2 = D_3 = 1 \quad \text{dans} \quad (0, L), \quad (1.2.29)$$

n'est pas uniformément stable. De même, nous démontrons aussi que le système n'est pas exponentiellement stable sous la condition suivante :

$$D_1 = D_3 = 0 \quad \text{et} \quad D_2 = 1 \quad \text{dans} \quad (0, L). \quad (1.2.30)$$

En effet, pour démontrer la stabilité non uniforme, nous construisons une suite (V_n) des éléments dans $D(\mathcal{A}_2)$ et une suite réelle (λ_n) tel que $\|V_n\| \rightarrow +\infty$ et $\|(i\lambda_n I - \mathcal{A}_2)U_n\|_{\mathcal{H}_2}$ est borné quand $n \rightarrow +\infty$. Par conséquent, la résolvante de \mathcal{A}_2 n'est pas uniformément bornée sur l'axe imaginaire. D'après Huang [36] et Pruss [60] nous concluons que la semigroupe $e^{t\mathcal{A}_2}$ n'est pas uniformément stable dans \mathcal{H}_2 .

Finalement, dans le cas d'un seul amortissement local, i.e. supposons qu'il existe $d_2 > 0$ tel que

$$D_1 = D_3 = 0 \text{ dans } (0, L) \quad \text{et} \quad D_2 \geq d_2 > 0 \text{ dans } (\alpha, \beta) \subset (0, L), \quad (1.2.31)$$

nous démontrons le théorème suivant :

Théorème 1.2.16. (*Taux de décroissance polynomial*) *Supposons que (1.2.31) est satisfait. Supposons de plus que $D_2 \in L^\infty(0, L)$. Alors, il existe une constant positive $c > 0$ tel que pour tout $U_0 \in D(\mathcal{A}_j)$, $j = 1, 2$, l'énergie du système (1.2.22) satisfait le taux de décroissance suivant :*

$$E(t) \leq \frac{c}{\sqrt{t}} \|U_0\|_{D(\mathcal{A}_j)}^2. \quad (1.2.32)$$

□

Chapitre 2

Local indirect stabilization of N-d system of two coupled wave equations under geometric conditions

Abstract : The purpose of this chapter is to investigate the stabilization of a system of two wave equations coupled by velocities with only one localized damping. The main novelty in this chapter is that the waves are not necessarily propagating at same speed and the coupling coefficient is not assumed to be positive and small. Assume that the coupling region and the damping region intersect. We prove that our system is strongly stable without geometric conditions. We then study the energy decay rate by distinguishing two cases. The first one is when the waves propagate at the same speed. In this case, under appropriate geometric conditions, we establish an exponential energy decay estimate for usual initial data. For the other case, we first show that our system is not uniformly stable. Next, under the same geometric conditions, we establish a polynomial energy decay of type $\frac{1}{t}$ for smooth initial data. Finally, in one space dimension, using the real part of the asymptotic expansion of eigenvalues of the system, we prove that the obtained polynomial decay rate is optimal.

2.1 Introduction

Let Ω be a nonempty bounded open set of \mathbb{R}^N having a boundary Γ of class C^2 . In [12], F. Alabau-Boussouira et al. considered the energy decay of a system of two wave equations coupled by velocities :

$$u_{tt} - a\Delta u + b(x)y_t + \rho(x, u_t) = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (2.1.1)$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (2.1.2)$$

$$u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*, \quad (2.1.3)$$

with the following initial data :

$$u(x, 0) = u_0, \quad y(x, 0) = y_0, \quad u_t(x, 0) = u_1 \text{ and } y_t(x, 0) = y, \quad x \in \Omega$$

where $a > 0$ constant and $b \in C^0(\bar{\Omega}; \mathbb{R})$ is a non-zero function. The damping term ρ is applied at the first equation and the second equation is indirectly damped through the coupling between the two equations. In [12], using an approach based on multiplier techniques, weighted nonlinear inequalities and the optimal-weight convexity method (developed in [7]), the authors established an explicit energy decay formula in terms of the behavior of the nonlinear feedback close to the origin. Their results are obtained in the case when the following three conditions are satisfied : the waves propagate at the same speed ($a = 1$), the coupling coefficient $b(x)$ is small positive ($0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ where b^* is a constant depending on Ω and on the control region) and both the coupling and the damping regions satisfying an appropriated geometric conditions named Piecewise Multipliers Geometric Conditions (introduced in [48], used in [7] and denoted by PMGC, in short). Then the stabilization of system (2.1.1)-(2.1.3) in the case where the waves are not assumed to be propagated with equal speeds (a is not necessarily equal to 1) and/or when the coupling coefficient $b(x)$ is not assumed to be positive and small has been left as an open problem even when the damping term ρ is linear with respect to the second variable. In this chapter, we are interested to answer this open question and to provide a stability analysis for system (2.1.1)-(2.1.3) when the damping term ρ is linear with respect to the second variable *i.e.* $\rho(x, u_t) = c(x)u_t$ where $c \in C^0(\bar{\Omega}; \mathbb{R}_+)$. So, we consider the stability of the following system :

$$u_{tt} - a\Delta u + b(x)y_t + c(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (2.1.4)$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (2.1.5)$$

$$u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*, \quad (2.1.6)$$

with the following initial data :

$$u(x, 0) = u_0, \quad y(x, 0) = y_0, \quad u_t(x, 0) = u_1 \quad \text{and} \quad y_t(x, 0) = y, \quad x \in \Omega.$$

The notion of indirect damping mechanisms has been introduced by D.L. Russell in [63], and since then, it attracted the attention of many authors. In particular, the stabilization of systems of two second order equations coupled through displacements when only one equation is effectively damped by internal or boundary feedback, has been initiated and studied in [2, 8, 4], and further studied by many authors, for instance [9, 50, 14]. Recall that the exponential or polynomial energy decay rate occurs in many control problems, we quote [23, 65] for the Timoshenko system in bounded or unbounded domains. Here, we focus our attention only on the literature of the indirect internal stability of coupled wave equations. In [37], B. Kapitonov studied the stabilization of a system of two coupled hyperbolic equations involving (2.1.4)-(2.1.6). He established an exponential energy decay rate for usual initial data in the case that the waves propagate at same speed, ($a = 1$) and the damping and coupling coefficients have the same support. For the other cases, when $a \neq 1$ and/or support of b does not coincide with that of c no energy decay rate has been discussed. In [8], F. Alabau et al. studied the indirect stabilization of a system of two evolution equations coupled through displacements where the damping is effective in the whole domain. Using the method of higher order energies initiated in [2], they established polynomial energy decay depending on the smoothness of initial data. These results have been generalized by F. Alabau and M. Léautaud in [9] to the case when the

coupling and the damping coefficients are localized in Ω and both satisfying the PMGC conditions. In addition, without geometric conditions, using an interpolation inequality for elliptic system (see Proposition 5.1 in [42]) together with the resolvent estimates of G. Lebeau in [43], the authors proved that the energy decay of smooth initial data is at least logarithmic when the coupling and the damping regions intersect in a nonempty subdomain $\omega \subset \Omega$. However, when $\omega = \emptyset$, the question of the stability or the null controllability of the system, is still an open problem. Indeed, F. Alabau and M. Léautaud in [10], solved partially this problem by proving that the system is null controllable provided that both the coupling and the damping regions satisfy the optimal geometric condition named Geometric Control Condition introduced by Bardos et al. in [21]. Finally, we refer to [2, 8, 6, 18, 22, 68, 65, 67, 57, 37, 54, 16] for the indirect stabilization and the indirect exact controllability of distributed systems with different kinds of damping.

In this chapter, we study the stability of system (2.1.4)-(2.1.6) when the coupling region and the damping region intersect in $\omega \subset \Omega$. First, we establish the strong stability without geometric conditions. We then study the energy decay rate of our system by distinguishing two cases. The first one is when the waves propagate at same speed, i.e. $a = 1$. In this case, under the hypothesis that ω satisfies the geometric conditions PMGC (see below), we establish an exponential energy decay rate for usual initial data. Next, in the general case, when $a \neq 1$, we prove the non uniform (exponential) stability and under the same geometric conditions, we establish a polynomial energy decay rate of type $\frac{1}{t}$ for smooth initial data. Finally, in one space dimension, using the real part of the asymptotic expansion of the eigenvalues of the system, we show that the obtained polynomial decay is optimal.

2.2 Well posedness and strong stability

In this Section, we will study the strong stability of system (2.1.4)-(2.1.6) without additional geometric conditions. First, we will study the existence, uniqueness, and regularity of the solution of our system.

2.2.1 Well posedness of the problem

Let $U = (u, u_t, y, y_t)$ be a regular solution of (2.1.4)-(2.1.6), its associated energy is defined by

$$E(t) = \frac{1}{2} \int_{\Omega} (|u_t|^2 + a|\nabla u|^2 + |y_t|^2 + |\nabla y|^2) dx. \quad (2.2.1)$$

So, a direct computation gives

$$\frac{d}{dt} E(t) = - \int_{\Omega} c(x) |u_t|^2 dx \leq 0. \quad (2.2.2)$$

Consequently, system (2.1.4)-(2.1.6) is dissipative in the sense that its energy is non-increasing.

First, we define the energy space $\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2$ equipped, for all $U = (u, v, y, z)$, $\tilde{U} =$

$(\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in \mathcal{H}$, by the scalar product :

$$(U, \tilde{U})_{\mathcal{H}} = a \int_{\Omega} (\nabla u \cdot \nabla \tilde{u}) dx + \int_{\Omega} v \tilde{v} dx + \int_{\Omega} (\nabla y \cdot \nabla \tilde{y}) dx + \int_{\Omega} z \tilde{z} dx.$$

Next, we define the unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}$ by :

$$D(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega))^2, \quad \mathcal{A}U = (v, a\Delta u - bz - cv, z, \Delta y + bv).$$

Note that, using the fact that $c(x) \geq 0$, then \mathcal{A} is m-dissipatif and generates a C_0 semi-group of contractions $e^{t\mathcal{A}}$ on the energy space \mathcal{H} . As the system (2.1.4)-(2.1.6) is equivalent to

$$U_t = \mathcal{A}U \text{ in } \mathcal{H}, \quad t > 0, \quad U(0) = U_0 \quad (2.2.3)$$

with $U = (u, u_t, y, y_t)$, we deduce its well posedness character. So, we have the following existence results :

Theorem 2.2.1. *Let $U_0 \in \mathcal{H}$ then, problem (2.2.3) admits a unique weak solution U satisfies*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}).$$

Moreover, if $U_0 \in D(\mathcal{A})$ then, problem (2.2.3) admits a unique strong solution U satisfies

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$

□

Secondly, we will study the strong stability of our system.

2.2.2 Strong stability

In this subsection we study the asymptotic behavior of $E(t)$. For this aim, we assume that there exists a nonempty open $\omega_{c_+} \subset \Omega$ satisfying the following condition :

$$\{x \in \Omega : c(x) > 0\} \supset \bar{\omega}_{c_+} \quad (\text{LH1})$$

On the other side, as $b(x)$ is a non-zero continuous function, then there exists a nonempty open $\omega_{b_+} \cup \omega_{b_-} \subset \Omega$ such that

$$\{x \in \Omega : b(x) > 0\} \supset \bar{\omega}_{b_+} \quad \text{and} \quad \{x \in \Omega : b(x) < 0\} \supset \bar{\omega}_{b_-} \quad (\text{LH2})$$

Our main result in this part is the following

Theorem 2.2.2. *(Strong Stability) Assume that $a > 0$, condition (LH1) holds and that $\omega = \omega_{c_+} \cap \omega_{b_+} \neq \emptyset$ or $\omega_{c_+} \cap \omega_{b_-} \neq \emptyset$. Then the semi group of contractions $e^{t\mathcal{A}}$ is strongly stable on the energy space \mathcal{H} , i.e. for any $U_0 \in \mathcal{H}$, we have*

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}}U_0\|_{\mathcal{H}} = 0. \quad (2.2.4)$$

In [9], the authors considered the stabilization of a system of two wave equations coupled in displacements with one localized internal damping. They showed that, under the assumption that the damping region and the coupling region have a non-empty intersection in Ω i.e. $\omega = \omega_{c_+} \cap \omega_{b_+} \neq \emptyset$ (or $\omega_{c_+} \cap \omega_{b_-} \neq \emptyset$), the energy of smooth solutions decays logarithmically to zero as t goes to infinity. This result still holds in the case where the two wave equations are coupled through the velocities. Indeed, following the method introduced by G. Lebeau and L. Robbiano in [44], M. Léautaud in [42] established an interpolation inequality for the associated elliptic system. This interpolation inequality implies the resolvent estimates of G. Lebeau in [43] (see also [45]) that provide the logarithmic energy decay rate for smooth initial data. So, using the density of $D(\mathcal{A})$ in \mathcal{H} and the contraction property of the C_0 semigroup $e^{t\mathcal{A}}$, we deduce that the energy of system (2.1.4)-(2.1.6) decays asymptotically to zero as t goes to infinity for all usual initial data.

Then we are interested, in this chapter, to study the energy decay rate by distinguishing two cases.

2.3 Exponential stability, the case $a = 1$

This section is devoted to the study of the exponential stability of system (2.1.4)-(2.1.6) in case the waves propagate at same speed (in the case $a = 1$) and under appropriated geometric conditions. For that purpose, we will use a frequency domain approach combined with piecewise multiplier technique.

Before presenting our main result of this section, we recall the Piecewise Multipliers Geometric Condition introduced by K. Liu in [48].

Definition 2.3.1. *We say that ω satisfies the Piecewise Multipliers Geometric Condition (PMGC in short) if there exist $\Omega_j \subset \Omega$ having Lipschitz boundary $\Gamma_j = \partial\Omega_j$ and $x_j \in \mathbb{R}^N$, $j = 1, \dots, J$ such that $\Omega_j \cap \Omega_i = \emptyset$ for $j \neq i$ and ω contains a neighborhood in Ω of the set $\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)$ where $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ and ν_j is the outward unit normal vector to Γ_j .*

Remark 2.3.2. *The PMGC is the generalization of the Multipliers Geometric Condition (MGC in short) introduced by Lions in [46], saying that ω contains a neighborhood in Ω of the set $\{x \in \Gamma : (x - x_0) \cdot \nu(x) > 0\}$, for some $x_0 \in \mathbb{R}^N$, where ν is the outward unit normal vector to $\Gamma = \partial\Omega$. □*

Now, we are in position to present the main result of this section.

Theorem 2.3.3. *(Exponential decay rate) Let $a = 1$. Assume that condition (LH1) holds. Assume also that the nonempty open set $\omega = \omega_{c_+} \cap \omega_{b_+}$ (or $\omega = \omega_{c_+} \cap \omega_{b_-}$) satisfies the geometric conditions PMGC and that $b, c \in W^{1,\infty}(\Omega)$. Then there exist positive constants $M \geq 1$, $\theta > 0$ such that for all initial data $(u_0, u_1, y_0, y_1) \in \mathcal{H}$ the energy of the system (2.1.4)-(2.1.6) satisfies the following decay rate :*

$$E(t) \leq Me^{-\theta t} E(0), \quad \forall t > 0. \tag{2.3.1}$$

Remark 2.3.4. Note that in Theorem 2.3.3 we have no restriction on the upper bound and the sign of the function b . This theorem is then a generalization in the linear case of the result of [12] where the coupling coefficient considered have to satisfy $0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ where b^* is a constant depending on Ω and on the control region. Nevertheless, the problem still be open in the nonlinear case. \square

In order to prove the above theorem, we apply a result of F. L. Huang [36] and J. Pruss [60] : A C_0 - semigroup of contraction $(e^{t\mathcal{A}})_{t \geq 0}$ in a Hilbert space \mathcal{H} is uniformly stable if and only if

$$i\mathbb{R} \subseteq \rho(\mathcal{A}) \quad (\text{H1})$$

and

$$\sup_{\beta \in \mathbb{R}} \|(i\beta I - \mathcal{A})^{-1}\| < +\infty \quad (\text{H2})$$

hold.

Since the resolvent of \mathcal{A} is compact and $0 \in \rho(\mathcal{A})$, then from Theorem 2.2.2, we deduce that condition (H1) is satisfied. We now prove that condition (H2) holds, using an argument of contradiction. For this aim, we suppose that there exist a real sequence β_n with $\beta_n \rightarrow +\infty$ and a sequence $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ such that

$$\|(u_n, v_n, y_n, z_n)\|_{\mathcal{H}} = 1, \quad (2.3.2)$$

and

$$\lim_{n \rightarrow \infty} \|(i\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} = 0. \quad (2.3.3)$$

Now, detailing equation (2.3.3), we get

$$i\beta_n u_n - v_n = f_n^1 \rightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (2.3.4)$$

$$i\beta_n v_n - \Delta u_n + b(x)z_n + c(x)v_n = g_n^1 \rightarrow 0 \quad \text{in } L^2(\Omega), \quad (2.3.5)$$

$$i\beta_n y_n - z_n = f_n^2 \rightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (2.3.6)$$

$$i\beta_n z_n - \Delta y_n - b(x)v_n = g_n^2 \rightarrow 0 \quad \text{in } L^2(\Omega). \quad (2.3.7)$$

Eliminating v_n and z_n from the previous system, we obtain the following reduced system

$$\beta_n^2 u_n + \Delta u_n - i\beta_n b(x)y_n - i\beta_n c(x)u_n = -g_n^1 - b(x)f_n^2 - i\beta_n f_n^1 - c(x)f_n^1, \quad (2.3.8)$$

$$\beta_n^2 y_n + \Delta y_n + i\beta_n b(x)u_n = -i\beta_n f_n^2 + b(x)f_n^1 - g_n^2. \quad (2.3.9)$$

On the other side, using equation (2.3.2) we deduce that z_n and v_n are uniformly bounded in $L^2(\Omega)$. It follows, from equations (2.3.4) and (2.3.6), that

$$\int_{\Omega} |y_n|^2 dx = \frac{O(1)}{\beta_n^2} \quad \text{and} \quad \int_{\Omega} |u_n|^2 dx = \frac{O(1)}{\beta_n^2}. \quad (2.3.10)$$

Lemma 2.3.5. The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (2.3.4)-(2.3.7) satisfies the following estimation

$$\int_{\omega_{c+}} |\beta_n u_n|^2 dx = o(1). \quad (2.3.11)$$

Proof: First, since U_n is uniformly bounded in \mathcal{H} , then from (2.3.3), we get

$$\operatorname{Re} \{i\beta_n \|U_n\|^2 - (\mathcal{A}U_n, U_n)\} = \int_{\Omega} c(x)|v_n|^2 dx = o(1). \quad (2.3.12)$$

Under condition (LH1), it follows that

$$\int_{\omega_{c_+}} |v_n|^2 dx = o(1). \quad (2.3.13)$$

So, using equations (2.3.12) and (2.3.4), we get

$$\int_{\Omega} c(x)|\beta_n u_n|^2 dx = o(1). \quad (2.3.14)$$

Consequently, we have

$$\int_{\omega_{c_+}} |\beta_n u_n|^2 dx = o(1).$$

The proof is thus complete. □

Now, the subset ω satisfies the PMGC. Hence, denoting by Ω_j and x_j , $j = 1, \dots, J$ the sets and the points given by the PMGC, we have $\omega \supset \mathcal{N}_{\varepsilon} (\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)) \cap \Omega$. In this expression, $\mathcal{N}_{\varepsilon}(\mathcal{O}) = \{x \in \mathbb{R}^N : d(x, \mathcal{O}) < \varepsilon\}$ with $d(\cdot, \mathcal{O})$ is the usual euclidean distance to the subset \mathcal{O} of \mathbb{R}^N and $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ where ν_j is the outward unit normal vector to $\Gamma_j = \partial\Omega_j$. Let the reals $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon$ and define

$$\mathcal{V}_i = \mathcal{N}_{\varepsilon_i} (\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)), \quad i = 1, 2, 3.$$

Since $(\overline{\Omega} \setminus \mathcal{V}_3) \cap \overline{\mathcal{V}_2} = \emptyset$, then we may define the function $\eta \in C_0^{\infty}(\mathbb{R}^N)$ by

$$\eta(x) = 0 \quad \text{if } x \in \Omega \setminus \mathcal{V}_3, \quad 0 \leq \eta(x) \leq 1, \quad \eta(x) = 1 \quad \text{if } x \in \mathcal{V}_2.$$

Lemma 2.3.6. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (2.3.4)-(2.3.7) satisfies the following estimation*

$$\int_{\Omega} \eta(x) |\nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\nabla u_n|^2 dx = o(1). \quad (2.3.15)$$

Proof: Multiplying equation (2.3.8) by $\eta \bar{u}_n$. Then, using Green's formula and the fact that $u_n = 0$ on Γ , we obtain

$$\begin{aligned} & \int_{\Omega} \eta(x) |\beta_n u_n|^2 dx - \int_{\Omega} \eta(x) |\nabla u_n|^2 dx - \int_{\Omega} \bar{u}_n (\nabla \eta \cdot \nabla u_n) dx \\ & - i\beta_n \int_{\Omega} b(x) \eta y_n \bar{u}_n dx - i\beta_n \int_{\Omega} c(x) \eta(x) |u_n|^2 dx \\ & = \int_{\Omega} (-g_n^1 - b(x) f_n^2 - i\beta_n f_n^1 - c(x) f_n^1) \eta \bar{u}_n dx. \end{aligned} \quad (2.3.16)$$

As f_n^1 and f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^1 converges to zero in $L^2(\Omega)$, the sequences $(\beta_n u_n)$, $(\beta_n y_n)$, (∇u_n) are uniformly bounded in $L^2(\Omega)$ and $\|u_n\| = o(1)$, we get

$$\int_{\Omega} \eta(x) |\beta_n u_n|^2 dx - \int_{\Omega} \eta(x) |\nabla u_n|^2 dx = o(1). \quad (2.3.17)$$

By using the definition of η and equations (2.3.11) and (2.3.17), we deduce

$$\int_{\Omega} \eta(x) |\nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\nabla u_n|^2 dx = o(1).$$

The proof is thus complete \square

Lemma 2.3.7. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (2.3.4)-(2.3.7) satisfies the following estimation*

$$\int_{\Omega} \eta(x) |\nabla y_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\nabla y_n|^2 dx = o(1). \quad (2.3.18)$$

Proof: The proof contains three points.

(i) Notice that, from equation (2.3.9), $\frac{1}{\beta_n} \Delta \bar{y}_n$ is uniformly bounded in $L^2(\Omega)$. So, multiplying equation (2.3.8) by $\frac{1}{\beta_n} \eta \Delta \bar{y}_n$. Using Green's formula and the fact that $u_n = y_n = f_n^1 = 0$ on Γ , we get

$$\begin{aligned} & - \int_{\Omega} \beta_n \eta(x) (\nabla \bar{y}_n \cdot \nabla u_n) dx - \int_{\Omega} \beta_n u_n (\nabla \eta \cdot \nabla \bar{y}_n) dx + \int_{\Omega} \frac{1}{\beta_n} \eta(x) \Delta \bar{y}_n \Delta u_n dx \\ & + i \int_{\Omega} b(x) \eta(x) |\nabla y_n|^2 dx + i \int_{\Omega} \eta(x) y_n (\nabla b \cdot \nabla \bar{y}_n) dx + i \int_{\Omega} b(x) y_n (\nabla \eta \cdot \nabla \bar{y}_n) dx \\ & + i \int_{\Omega} c(x) \eta(x) (\nabla u_n \cdot \nabla \bar{y}_n) dx + i \int_{\Omega} c(x) u_n (\nabla \eta \cdot \nabla \bar{y}_n) dx \\ & + i \int_{\Omega} \eta(x) u_n (\nabla c \cdot \nabla \bar{y}_n) dx = + \int_{\Omega} (-g_n^1 - b(x) f_n^2 - c(x) f_n^1) \left(\frac{1}{\beta_n} \eta(x) \Delta \bar{y}_n \right) dx \\ & + i \int_{\Omega} \eta(x) (\nabla f_n^1 \cdot \nabla \bar{y}_n) dx + i \int_{\Omega} f_n^1 (\nabla \eta \cdot \nabla \bar{y}_n) dx. \end{aligned} \quad (2.3.19)$$

First, since f_n^1 , f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^1 converges to zero in $L^2(\Omega)$ and (∇y_n) , $(\frac{1}{\beta_n} \Delta y_n)$ are uniformly bounded in $L^2(\Omega)$, then we have

$$\begin{aligned} & \int_{\Omega} (-g_n^1 - b f_n^2 - c f_n^1) \left(\frac{1}{\beta_n} \eta \Delta \bar{y}_n \right) dx + i \int_{\Omega} \eta (\nabla f_n^1 \cdot \nabla \bar{y}_n) dx \\ & + i \int_{\Omega} f_n^1 (\nabla \eta \cdot \nabla \bar{y}_n) dx = o(1). \end{aligned} \quad (2.3.20)$$

Next, using the definition of η , the equations (2.3.11), (2.3.15), and the fact that $\|u_n\| = o(1)$, $\|y_n\| = o(1)$ and (∇y_n) is uniformly bounded $L^2(\Omega)$, we get

$$\begin{aligned}
 & - \int_{\Omega} \beta_n u_n (\nabla \eta \cdot \nabla \bar{y}_n) dx + i \int_{\Omega} \eta(x) y_n (\nabla b \cdot \nabla \bar{y}_n) dx \\
 & + i \int_{\Omega} b(x) y_n (\nabla \eta \cdot \nabla \bar{y}_n) dx + i \int_{\Omega} c(x) \eta(x) (\nabla u_n \cdot \nabla \bar{y}_n) dx \\
 & + i \int_{\Omega} c(x) u_n (\nabla \eta \cdot \nabla \bar{y}_n) dx + i \int_{\Omega} \eta(x) u_n (\nabla c \cdot \nabla \bar{y}_n) dx = o(1).
 \end{aligned} \tag{2.3.21}$$

Finally, inserting (2.3.20) and (2.3.21) into (2.3.19), we get

$$- \int_{\Omega} \beta_n \eta (\nabla \bar{y}_n \cdot \nabla u_n) dx + \int_{\Omega} \frac{1}{\beta_n} \eta \Delta \bar{y}_n \Delta u_n dx + i \int_{\Omega} b \eta |\nabla y_n|^2 dx = o(1). \tag{2.3.22}$$

(ii) Multiplying (2.3.9) by the bounded sequence $\frac{1}{\beta_n} \eta \Delta \bar{u}_n$, integrating over Ω and use the fact that $u_n = y_n = f_n^2 = 0$ on Γ , we get

$$\begin{aligned}
 & - \int_{\Omega} \beta_n \eta(x) (\nabla y_n \cdot \nabla \bar{u}_n) dx - \beta_n \int_{\Omega} y_n (\nabla \eta \cdot \nabla \bar{u}_n) dx + \int_{\Omega} \frac{1}{\beta_n} \eta(x) \Delta y_n \Delta \bar{u}_n dx \\
 & - i \int_{\Omega} \eta(x) u_n (\nabla b \cdot \nabla \bar{u}_n) dx - i \int_{\Omega} \eta(x) b(x) |\nabla u_n|^2 dx - i \int_{\Omega} b(x) u_n (\nabla \bar{u}_n \cdot \nabla \eta) dx = \\
 & i \int_{\Omega} (\eta(x) (\nabla f_n^2 \cdot \nabla \bar{u}_n) + f_n^2 (\nabla \eta \cdot \nabla \bar{u}_n)) dx + \int_{\Omega} (b(x) f_n^1 - g_n^2) \frac{1}{\beta_n} \eta(x) \Delta \bar{u}_n dx.
 \end{aligned} \tag{2.3.23}$$

First, since f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^2 converges to zero in $L^2(\Omega)$ and $\nabla u_n, \frac{1}{\beta_n} \Delta u_n$ are uniformly bounded in $L^2(\Omega)$, then we have

$$\begin{aligned}
 & i \int_{\Omega} (\eta(x) (\nabla f_n^2 \cdot \nabla \bar{u}_n) + f_n^2 (\nabla \eta \cdot \nabla \bar{u}_n)) dx \\
 & + \int_{\Omega} (b(x) f_n^1 - g_n^2) \frac{1}{\beta_n} \eta(x) \Delta \bar{u}_n dx = o(1).
 \end{aligned} \tag{2.3.24}$$

Next, using equation (2.3.15) and the fact that $\|u_n\| = o(1)$ and $(\beta_n y_n)$ is uniformly bounded in $L^2(\Omega)$, we deduce that

$$\begin{aligned}
 & - \beta_n \int_{\Omega} y_n (\nabla \eta \cdot \nabla \bar{u}_n) dx - i \int_{\Omega} \eta(x) u_n (\nabla b \cdot \nabla \bar{u}_n) dx \\
 & - i \int_{\Omega} \eta(x) b(x) |\nabla u_n|^2 dx - i \int_{\Omega} b(x) u_n (\nabla \bar{u}_n \cdot \nabla \eta) dx = o(1).
 \end{aligned} \tag{2.3.25}$$

Finally, inserting (2.3.24) and (2.3.25) into (2.3.23), we get

$$- \int_{\Omega} \beta_n \eta(x) (\nabla y_n \cdot \nabla \bar{u}_n) dx + \int_{\Omega} \frac{1}{\beta_n} \eta(x) \Delta y_n \Delta \bar{u}_n dx = o(1). \quad (2.3.26)$$

(iii) Summing the imaginary parts of equations (2.3.22) and (2.3.26), we obtain

$$\int_{\Omega} b(x) \eta(x) |\nabla y_n|^2 dx = o(1). \quad (2.3.27)$$

Using the definition of function η and condition (LH2), we deduce that

$$\int_{\Omega} \eta(x) |\nabla y_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\nabla y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 2.3.8. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (2.3.4)-(2.3.7) satisfies the following estimation*

$$\int_{\Omega} \eta(x) |\beta_n y_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\beta_n y_n|^2 dx = o(1). \quad (2.3.28)$$

Proof: Multiplying equation (2.3.9) by $\eta \bar{y}_n$ and integrating over Ω . Then, using Green's formula and the fact that $\|y_n\| = o(1)$ and $y_n = 0$ on Γ , we get

$$\begin{aligned} & \int_{\Omega} \eta(x) |\beta_n y_n|^2 dx - \int_{\Omega} \bar{y}_n (\nabla y_n \cdot \nabla \eta) dx \\ & - \int_{\Omega} \eta(x) |\nabla y_n|^2 dx + i \beta_n \int_{\Omega} b(x) \eta(x) y_n \bar{u}_n dx = o(1). \end{aligned} \quad (2.3.29)$$

Combining equations (2.3.11), (2.3.18) and (2.3.29) and using the fact that $\|y_n\| = o(1)$, we get

$$\int_{\Omega} \eta(x) |\beta_n y_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\beta_n y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Now, since $(\bar{\Omega}_j \setminus \mathcal{V}_2) \cap \bar{\mathcal{V}}_1 = \emptyset$, then we may define the function $\psi_j \in C_0^\infty(\mathbb{R}^N)$ by :

$$\psi_j(x) = 0 \quad \text{if } x \in \mathcal{V}_1, \quad 0 \leq \psi_j \leq 1, \quad \psi_j(x) = 1 \quad \text{if } x \in \bar{\Omega}_j \setminus \mathcal{V}_2.$$

For $m_j(x) = (x - x_j)$, we define $h_j(x) = \psi_j(x) m_j(x)$.

Lemma 2.3.9. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (2.3.4)-(2.3.7) satisfies the following estimation*

$$\begin{aligned} & N \int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} |\beta_n u_n|^2 dx + (2 - N) \int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} |\nabla u_n|^2 dx \\ & + 2\text{Re} \left\{ i \sum_{j=1}^J \int_{\Omega_j \setminus (\mathcal{V}_2 \cap \Omega_j)} \beta_n b(x) y_n (m_j \cdot \nabla \bar{u}_n) dx \right\} \leq o(1). \end{aligned} \quad (2.3.30)$$

Proof: Multiplying equation (2.3.8) by $2(h_j \cdot \nabla \bar{u}_n)$ and integrating over Ω_j , we obtain

$$\begin{aligned} & 2\beta_n^2 \int_{\Omega_j} u_n(h_j \cdot \nabla \bar{u}_n) dx + 2 \int_{\Omega_j} \Delta u_n(h_j \cdot \nabla \bar{u}_n) dx - 2i \int_{\Omega_j} \beta_n b(x) y_n(h_j \cdot \nabla \bar{u}_n) dx = \\ & 2 \int_{\Omega_j} (-g_n^1 - b(x) f_n^2 - c(x) f_n^1)(h_j \cdot \nabla \bar{u}_n) dx - 2i \int_{\Omega_j} \beta_n f_n^1(h_j \cdot \nabla \bar{u}_n) dx. \end{aligned} \quad (2.3.31)$$

(i) **Estimation of the second member of (2.3.31).** First, using Green's formula, the fact that $u_n = 0$ on $(\Gamma_j \setminus \gamma_j) \cap \Gamma$ and that $h_j = 0$ on γ_j , we get

$$-2i \int_{\Omega_j} \beta_n f_n^1(h_j \cdot \nabla \bar{u}_n) dx = 2i \int_{\Omega_j} \beta_n \bar{u}_n(h_j \cdot \nabla f_n^1) dx + 2i \int_{\Omega_j} \beta_n \bar{u}_n f_n^1(\operatorname{div} h_j) dx. \quad (2.3.32)$$

It follows, from the convergence of f_n^1 to zero in $H_0^1(\Omega)$ and the uniformly boundedness in $L^2(\Omega)$ of $\beta_n u_n$, that

$$-2i \int_{\Omega_j} \beta_n f_n^1(h_j \cdot \nabla \bar{u}_n) dx = o(1). \quad (2.3.33)$$

Next, as f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^1 converges to zero in $L^2(\Omega)$ and (∇u_n) is uniformly bounded in $L^2(\Omega)$, we deduce that

$$2 \int_{\Omega_j} (-g_n^1 - b(x) f_n^2 - c(x) f_n^1)(h_j \cdot \nabla \bar{u}_n) dx = o(1). \quad (2.3.34)$$

Finally, we deduce that the second member of equation (2.3.31) is $o(1)$.

(ii) **Estimation of the first integral of equation (2.3.31).** Using Green's formula, we get

$$\operatorname{Re} \left\{ 2 \int_{\Omega_j} \beta_n^2 u_n(h_j \cdot \nabla \bar{u}_n) dx \right\} = - \int_{\Omega_j} (\operatorname{div} h_j) |\beta_n u_n|^2 dx + \int_{\Gamma_j} (h_j \cdot \nu_j) |\beta_n u_n|^2 d\Gamma_j. \quad (2.3.35)$$

Since $\Psi_j = 0$ on γ_j and $u_n = 0$ on $(\Gamma_j \setminus \gamma_j) \cap \Gamma$, then we have

$$\operatorname{Re} \left\{ 2 \int_{\Omega_j} \beta_n^2 u_n(h_j \cdot \nabla \bar{u}_n) dx \right\} = - \int_{\Omega_j} (\operatorname{div} h_j) |\beta_n u_n|^2 dx. \quad (2.3.36)$$

(iii) **Estimation of the second integral of equation (2.3.31).** Using Green's formula, we get

$$\begin{aligned} \operatorname{Re} \left\{ 2 \int_{\Omega_j} \Delta u_n(h_j \cdot \nabla \bar{u}_n) \right\} &= -2 \operatorname{Re} \left\{ \sum_{i,k=1}^N \int_{\Omega_j} \partial_i h_j^k \partial_i u_n \partial_k \bar{u}_n dx \right\} + \int_{\Omega_j} (\operatorname{div} h_j) |\nabla u_n|^2 dx \\ & - \int_{\Gamma_j} (h_j \cdot \nu_j) |\nabla u_n|^2 d\Gamma_j + 2 \operatorname{Re} \left\{ \int_{\Gamma_j} (\partial_{\nu_j} u_n)(h_j \cdot \nabla \bar{u}_n) d\Gamma_j \right\}. \end{aligned} \quad (2.3.37)$$

According to the choice of Ψ_j , only the boundary terms over $(\Gamma_j \setminus \gamma_j) \cap \Gamma$ are non vanishing in (2.3.37). But on this part of the boundary $u_n = 0$, and consequently $\nabla u_n = (\partial_\nu u_n) \cdot \nu = (\partial_{\nu_j} u_n) \nu_j$. Then, we have

$$\begin{aligned} & - \int_{\Gamma_j} (h_j \cdot \nu_j) |\nabla u_n|^2 d\Gamma_j + 2\operatorname{Re} \left\{ \int_{\Gamma_j} (\partial_{\nu_j} u_n) (h_j \cdot \nabla \bar{u}_n) d\Gamma_j \right\} = \\ & \int_{(\Gamma_j \setminus \gamma_j) \cap \Gamma} (\psi_j m_j \cdot \nu_j) |\partial_{\nu_j} u_n|^2 d\Gamma_j \leq 0. \end{aligned} \quad (2.3.38)$$

Inserting (2.3.38) in (2.3.37), we get

$$\operatorname{Re} \left\{ 2 \int_{\Omega_j} \Delta u_n (h_j \cdot \nabla \bar{u}_n) \right\} \leq -2\operatorname{Re} \left\{ \sum_{i,k=1}^N \int_{\Omega_j} \partial_i h_j^k \partial_i u_n \partial_k \bar{u}_n dx \right\} + \int_{\Omega_j} (\operatorname{div} h_j) |\nabla u_n|^2 dx. \quad (2.3.39)$$

(iv) **The main estimation.** Inserting equations (2.3.33), (2.3.34), (2.3.36) and (2.3.39) in (2.3.31) and using the fact that $\psi_j = 0$ on \mathcal{V}_1 , we get

$$\begin{aligned} & \int_{\Omega_j \setminus (\mathcal{V}_1 \cap \Omega_j)} \operatorname{div}(\psi_j m_j) (|\beta_n u_n|^2 - |\nabla u_n|^2) dx \\ & + 2\operatorname{Re} \int_{\Omega_j \setminus (\mathcal{V}_1 \cap \Omega_j)} \sum_{i,k=1}^N \partial_i (\psi_j m_j^k) \partial_i u_n \partial_k \bar{u}_n dx \\ & + 2i \operatorname{Re} \int_{\Omega_j \setminus (\mathcal{V}_1 \cap \Omega_j)} \beta_n b(x) y_n (\psi_j m_j \cdot \nabla \bar{u}_n) dx \leq o(1). \end{aligned} \quad (2.3.40)$$

Thus, summing over j and using the fact that $\psi_j = 1$ on $\bar{\Omega}_j \setminus \mathcal{V}_2$, we get

$$\begin{aligned} & N \int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} |\beta_n u_n|^2 dx + (2 - N) \int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} |\nabla u_n|^2 dx \\ & + 2\operatorname{Re} \left\{ i \sum_{j=1}^J \int_{\Omega_j \setminus (\mathcal{V}_2 \cap \Omega_j)} \beta_n b(x) y_n (m_j \cdot \nabla \bar{u}_n) dx \right\} \\ & \leq - \sum_{j=1}^J \int_{\mathcal{V}_2 \cap \Omega_j} \left[\operatorname{div}(\psi_j m_j) (|\beta_n u_n|^2 - |\nabla u_n|^2) dx + 2 \sum_{i,k=1}^N \partial_i (\psi_j m_j^k) \partial_i u_n \partial_k \bar{u}_n \right] dx \\ & - 2i \sum_{j=1}^J \int_{\mathcal{V}_2 \cap \Omega_j} \beta_n b(x) y_n (\psi_j m_j \cdot \nabla \bar{u}_n) dx + o(1). \end{aligned} \quad (2.3.41)$$

Using equations (2.3.11), (2.3.15) and (2.3.28), we deduce

$$\begin{aligned} & - \sum_{j=1}^J \int_{\mathcal{V}_2 \cap \Omega_j} \left[\operatorname{div}(\psi_j m_j) (|\beta_n u_n|^2 - |\nabla u_n|^2) dx + 2 \sum_{i,k=1}^N \partial_i (\psi_j m_j^k) \partial_i u_n \partial_k \bar{u}_n \right] dx \\ & - 2i \sum_{j=1}^J \int_{\mathcal{V}_2 \cap \Omega_j} \beta_n b(x) y_n (\psi_j m_j \cdot \nabla \bar{u}_n) dx = o(1). \end{aligned} \quad (2.3.42)$$

Finally, inserting (2.3.42) into (2.3.41), we obtain the desired equation (2.3.30) and the proof is thus complete. \square

Lemma 2.3.10. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (2.3.4)-(2.3.7) satisfies the following estimation*

$$\begin{aligned} & N \int_{\Omega \setminus \mathcal{V}_2 \cap \Omega} |\beta_n y_n|^2 dx + (2 - N) \int_{\Omega \setminus \mathcal{V}_2 \cap \Omega} |\nabla y_n|^2 dx \\ & + 2\operatorname{Re} \left\{ i \sum_{j=1}^J \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b(x) \bar{y}_n (m_j \cdot \nabla u_n) dx \right\} \leq o(1). \end{aligned} \quad (2.3.43)$$

Proof: Multiplying equation (2.3.9) by $2(h_j \cdot \nabla \bar{y}_n)$, we obtain

$$\begin{aligned} & 2\beta_n^2 \int_{\Omega_j} y_n (h_j \cdot \nabla \bar{y}_n) dx + 2 \int_{\Omega_j} \Delta y_n (h_j \cdot \nabla \bar{y}_n) dx + 2i \int_{\Omega_j} \beta_n b(x) u_n (h_j \cdot \nabla \bar{y}_n) dx = \\ & -2i \int_{\Omega_j} \beta_n f_n^2 (h_j \cdot \nabla \bar{y}_n) dx + 2 \int_{\Omega_j} (b(x) f_n^1 - g_n^2) (h_j \cdot \nabla \bar{y}_n) dx. \end{aligned} \quad (2.3.44)$$

(i) **Estimation of the seconde member in (2.3.44).** Using Green's formula and the fact that $y_n = 0$ on $(\Gamma_j \setminus \gamma_j) \cap \Gamma$ and $h_j = 0$ on $\gamma_j \cap \Gamma$, we obtain

$$-2i \int_{\Omega_j} \beta_n f_n^2 (h_j \cdot \nabla \bar{y}_n) dx = 2i \int_{\Omega_j} \beta_n \bar{y}_n (h_j \cdot \nabla f_n^2) dx + 2i \int_{\Omega_j} \beta_n \bar{y}_n f_n^2 (\operatorname{div} h_j) dx.$$

It follows, since f_n^2 converges to zero in $H_0^1(\Omega)$ and $\beta_n y_n$ is uniformly bounded in $L^2(\Omega)$, that

$$-2i \int_{\Omega_j} \beta_n f_n^2 (h_j \cdot \nabla \bar{y}_n) dx = o(1). \quad (2.3.45)$$

As f_n^1 converges to zero in $H_0^1(\Omega)$, g_n^2 converges to zero in $L^2(\Omega)$ and (∇y_n) is uniformly bounded in $L^2(\Omega)$, we deduce that

$$2 \int_{\Omega_j} (b(x) f_n^1 - g_n^2) (h_j \cdot \nabla \bar{y}_n) dx = o(1). \quad (2.3.46)$$

So, combining (2.3.45) and (2.3.46) with (2.3.44), we get

$$2\beta_n^2 \int_{\Omega_j} y_n (h_j \cdot \nabla \bar{y}_n) dx + 2 \int_{\Omega_j} \Delta y_n (h_j \cdot \nabla \bar{y}_n) dx + 2i \int_{\Omega_j} \beta_n b(x) u_n (h_j \cdot \nabla \bar{y}_n) dx = o(1). \quad (2.3.47)$$

(ii) **Estimation of first member of (2.3.44).** Using Green's formula in (2.3.47) and following the same technique used in Lemma (2.3.9), we get

$$\begin{aligned} & N \int_{\Omega \setminus \mathcal{V}_2 \cap \Omega} |\beta_n y_n|^2 dx + (2 - N) \int_{\Omega \setminus \mathcal{V}_2 \cap \Omega} |\nabla y_n|^2 dx \\ & - 2\operatorname{Re} \left\{ i \sum_{j=1}^J \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b(x) u_n (m_j \cdot \nabla \bar{y}_n) dx \right\} \leq o(1). \end{aligned} \quad (2.3.48)$$

(iii) **Estimation of the third integral of (2.3.48).** Integrating by parts and using the fact that $u_n = y_n = 0$ on $\partial(\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j) \subset \Gamma$, we obtain

$$\begin{aligned} 2i \sum_{j=1}^J \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b(x) u_n (m_j \cdot \nabla \bar{y}_n) dx &= -2i \sum_{j=1}^J \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b \bar{y}_n (m_j \cdot \nabla u_n) dx \\ &- 2i \sum_{j=1}^J \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n u_n \bar{y}_n (m_j \cdot \nabla b) dx - 2i \sum_{j=1}^J \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} b(x) \operatorname{div}(m_j) \beta_n u_n \bar{y}_n dx. \end{aligned} \quad (2.3.49)$$

Using the fact that $\beta_n u_n$ is uniformly bounded in $L^2(\Omega)$ and $\|y_n\| = o(1)$ in the right hand side of (2.3.49), we deduce

$$2i \sum_{j=1}^J \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b(x) u_n (m_j \cdot \nabla \bar{y}_n) dx = -2i \sum_{j=1}^J \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b(x) \bar{y}_n (m_j \cdot \nabla u_n) dx + o(1). \quad (2.3.50)$$

(iii) **The main estimation.** Combining equations (2.3.48) and (2.3.50), we obtain

$$\begin{aligned} &N \int_{\Omega \setminus \mathcal{V}_2 \cap \Omega} |\beta_n y_n|^2 dx + (2 - N) \int_{\Omega \setminus \mathcal{V}_2 \cap \Omega} |\nabla y_n|^2 dx \\ &+ 2\operatorname{Re} \left\{ i \sum_{j=1}^J \int_{\Omega_j \setminus \mathcal{V}_2 \cap \Omega_j} \beta_n b(x) \bar{y}_n (m_j \cdot \nabla u_n) dx \right\} \leq o(1). \end{aligned}$$

The proof is thus complete. \square

Lemma 2.3.11. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (2.3.4)-(2.3.7) satisfies the following estimation*

$$\int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} (|\nabla u_n|^2 + |\beta_n u_n|^2 + |\nabla y_n|^2 + |\beta_n y_n|^2) dx = o(1). \quad (2.3.51)$$

Proof: By combining equations (2.3.30) and (2.3.43), we conclude

$$N \int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} (|\beta_n u_n|^2 + |\beta_n y_n|^2) dx + (2 - N) \int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} (|\nabla u_n|^2 + |\nabla y_n|^2) dx \leq o(1). \quad (2.3.52)$$

Now, multiplying (2.3.8) by $(1 - N)\bar{u}_n$, integrating on Ω , using Green's formula, the fact that $(\beta_n u_n)$ is uniformly bounded in $L^2(\Omega)$, $\|u_n\| = o(1)$ and $\|y_n\| = o(1)$, we obtain

$$(1 - N) \int_{\Omega} |\beta_n u_n|^2 dx - (1 - N) \int_{\Omega} |\nabla u_n|^2 dx = o(1). \quad (2.3.53)$$

Using (2.3.11) and (2.3.15) in (2.3.53), we deduce

$$(1 - N) \int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} |\beta_n u_n|^2 dx - (1 - N) \int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} |\nabla u_n|^2 dx = o(1). \quad (2.3.54)$$

Similarly, multiplying (2.3.9) by $(1 - N)\bar{y}_n$, integrating on Ω , using Green's formula, the fact that $(\beta_n y_n)$ is uniformly bounded in $L^2(\Omega)$, $\|u_n\| = o(1)$ and $\|y_n\| = o(1)$, we obtain

$$(1 - N) \int_{\Omega} |\beta_n y_n|^2 dx - (1 - N) \int_{\Omega} |\nabla y_n|^2 dx = o(1). \quad (2.3.55)$$

Using (2.3.18) and (2.3.28) in (2.3.55), we deduce

$$(1 - N) \int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} |\beta_n y_n|^2 dx - (1 - N) \int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} |\nabla y_n|^2 dx = o(1). \quad (2.3.56)$$

Finally, combining (2.3.52), (2.3.54) and (2.3.56), we obtain

$$\int_{\Omega \setminus (\mathcal{V}_2 \cap \Omega)} (|\nabla u_n|^2 + |\beta_n u_n|^2 + |\nabla y_n|^2 + |\beta_n y_n|^2) dx = o(1).$$

The proof is thus complete. □

Remark 2.3.12. *It is easy to see that, the condition $a = 1$ is only used in the proof of estimation (2.3.18) in Lemma 2.3.7. So, if one may get this estimation in the case $a \neq 1$, then the results of Lemmas 2.3.8, 2.3.9, 2.3.10 and 2.3.11 still also true for $a \neq 1$.* □

Proof of Theorem 2.3.1 It follows, from (2.3.11), (2.3.15), (2.3.18), (2.3.28) and (2.3.51), that $\|U_n\|_{\mathcal{H}} = o(1)$ which is a contradiction with (2.3.2). Consequently, condition (H2) holds and the energy of the system (2.1.4)-(2.1.6) decays exponentially to zero as t goes to infinity. The proof has been completed. □

2.4 Non uniform stability in the case $a \neq 1$

The aim of this section is to show that system (2.1.4)-(2.1.6) is not uniformly (*i.e.* not exponentially) stable when the waves propagate with different speeds (*i.e.* $a \neq 1$), since it is already the case for c and b are constants in the whole domain, as shown below. Our result is the following

Theorem 2.4.1. *Assume that $c = c_0 > 0$, $b = b_0 \neq 0$ in Ω and that $a \neq 1$. Then the energy of system (2.1.4)-(2.1.6) does not decrease exponentially to zero as t goes to infinity.*

Proof: We will show that the resolvent of the operator \mathcal{A} is not uniformly bounded on the imaginary axis *i.e.* condition (H2) does not hold. So, to prove the non uniform stability, it suffices to construct a sequence $\beta_n \in \mathbb{R}$ and a sequence $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ such that

$$\beta_n \rightarrow +\infty, \quad (2.4.1)$$

$$\|U_n\|_{\mathcal{H}} \rightarrow +\infty \quad (2.4.2)$$

and

$$\|(i\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} \leq C < +\infty. \quad (2.4.3)$$

For this aim, let $\mu_n^2 > 0$ be an eigenvalue of the Laplacian with Dirichlet boundary condition and φ_n its associated eigenfunction :

$$\begin{cases} -\Delta\varphi_n = \mu_n^2\varphi_n, & \text{in } \Omega, \\ \varphi_n = 0, & \text{on } \Gamma. \end{cases} \quad (2.4.4)$$

Set $\beta_n = \mu_n$ and $U_n = (u_n, v_n, y_n, z_n) = (b_n\varphi_n, i\beta_n b_n\varphi_n, a_n\varphi_n, i\beta_n a_n\varphi_n)$ where

$$a_n = \frac{(a-1)}{b^2} + \frac{ic}{b^2\mu_n} + \frac{i}{b\mu_n} \quad \text{and} \quad b_n = \frac{1}{i\mu_n b}. \quad (2.4.5)$$

This implies that $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ and

$$v_n - i\mu_n u_n = 0, \quad (2.4.6)$$

$$a\Delta u_n - bz_n - cv_n - i\mu_n v_n = \varphi_n, \quad (2.4.7)$$

$$z_n - i\mu_n y_n = 0, \quad (2.4.8)$$

$$\Delta y_n + bv_n - i\mu_n z_n = \varphi_n. \quad (2.4.9)$$

It follows that U_n is a solution of the equation

$$\mathcal{A}U_n - i\mu_n U_n = V_n, \quad (2.4.10)$$

where $V_n = (0, \varphi_n, 0, \varphi_n) \in \mathcal{H}$.

Finally, from (2.4.10), we deduce that

$$\|i\mu_n U_n - \mathcal{A}U_n\|_{\mathcal{H}}^2 = \|(0, \varphi_n, 0, \varphi_n)\|_{\mathcal{H}}^2 = 2. \quad (2.4.11)$$

On the other side, we have

$$\|U_n\|_{\mathcal{H}}^2 = 2(|a_n|^2 + |b_n|^2)\mu_n^2 \sim \frac{(a-1)^2}{b^4}\mu_n^2 \rightarrow +\infty. \quad (2.4.12)$$

Consequently, the sequences $\beta_n = \mu_n$ and $U_n = (b_n\varphi_n, i\beta_n b_n\varphi_n, a_n\varphi_n, i\beta_n a_n\varphi_n)$ satisfy the conditions (2.4.1)-(2.4.3). So, using Huang [36] and Pruss [60], system (2.1.4)-(2.1.6) is not uniformly stable in the energy space \mathcal{H} . The proof is thus complete. \square

2.5 Polynomial stability in the case $a \neq 1$

The condition of equal speed is then a necessary and sufficient condition for the exponential stability of our system. Therefore, we look for a polynomial energy decay rate. Our second main result when the waves propagate at different speeds ($a \neq 1$) can be stated as follows.

Theorem 2.5.1. *(Polynomial decay rate) Let $a \neq 1$. Assume that all assumptions of Theorem 2.3.3 are satisfied. Then there exists a positive constant $C > 0$ independent of U_0 such that for all initial data $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$ the energy of the system (2.1.4)-(2.1.6) satisfies the following decay rate :*

$$E(t) \leq C \frac{1}{t} \|U(0)\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \quad (2.5.1)$$

According to Theorem 2.4 of Borichev-Tomilov in [25], a C_0 -semigroup of contractions $e^{t\mathcal{A}}$ on a Hilbert space \mathcal{H} verifies (2.5.1) if the following conditions

$$i\mathbb{R} \subseteq \rho(\mathcal{A}) \quad (\text{H1})$$

and

$$\sup_{|\beta| \geq 1} \frac{1}{\beta^2} \| (i\beta I - \mathcal{A})^{-1} \| < +\infty \quad (\text{H3})$$

hold. As condition (H1) was already checked in Theorem 2.2.2, we now prove that condition (H3) holds, using an argument of contradiction. For this aim, we suppose that there exist a real sequence (β_n) , with $\beta_n \rightarrow +\infty$ and a sequence $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ such that

$$\| U_n \|_{\mathcal{H}} = 1 \quad (2.5.2)$$

and

$$\lim_{n \rightarrow \infty} \| \beta_n^2 (i\beta_n I - \mathcal{A}) U_n \|_{\mathcal{H}} = 0. \quad (2.5.3)$$

Next, by detailing equation (2.5.3), we obtain

$$i\beta_n^3 u_n - \beta_n^2 v_n = f_n^1 \rightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (2.5.4)$$

$$i\beta_n^3 v_n - a\beta_n^2 \Delta u_n + b(x)\beta_n^2 z_n + c(x)\beta_n^2 v_n = g_n^1 \rightarrow 0 \quad \text{in } L^2(\Omega), \quad (2.5.5)$$

$$i\beta_n^3 y_n - \beta_n^2 z_n = f_n^2 \rightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (2.5.6)$$

$$i\beta_n^3 z_n - \beta_n^2 \Delta y_n - b(x)\beta_n^2 v_n = g_n^2 \rightarrow 0 \quad \text{in } L^2(\Omega). \quad (2.5.7)$$

Eliminating v_n and z_n from (2.5.4)-(2.5.7), we get

$$\beta_n^2 u_n + a\Delta u_n - i\beta_n b(x)y_n - i\beta_n c(x)u_n = \frac{-g_n^1 - bf_n^2 - i\beta_n f_n^1 - cf_n^1}{\beta_n^2}, \quad (2.5.8)$$

$$\beta_n^2 y_n + \Delta y_n + i\beta_n b(x)u_n = \frac{-i\beta_n f_n^2 + bf_n^1 - g_n^2}{\beta_n^2}. \quad (2.5.9)$$

In addition, from equation (2.5.2), (2.5.4) and (2.5.6), we deduce that

$$\| u_n \|_{L^2(\Omega)} = \frac{O(1)}{\beta_n} \quad \text{and} \quad \| y_n \|_{L^2(\Omega)} = \frac{O(1)}{\beta_n}. \quad (2.5.10)$$

Lemma 2.5.2. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (2.5.4)-(2.5.7) satisfies the following estimation*

$$\int_{\omega_{c_+}} |u_n|^2 dx = \frac{o(1)}{\beta_n^4}. \quad (2.5.11)$$

Proof: Multiplying equation (2.5.3) by U_n in \mathcal{H} , we get

$$\operatorname{Re} \left\{ i\beta_n^3 \| U_n \|^2 - \beta_n^2 (\mathcal{A}U_n, U_n) \right\} = \beta_n^2 \int_{\Omega} c(x)|v_n|^2 dx = o(1). \quad (2.5.12)$$

Under condition (LH1), it follows

$$\int_{\omega_{c_+}} |v_n|^2 dx = \frac{o(1)}{\beta_n^2}. \quad (2.5.13)$$

So, using equations (2.5.4) and (2.5.12), we get

$$\int_{\Omega} c(x)|u_n|^2 dx = \frac{o(1)}{\beta_n^4}. \quad (2.5.14)$$

This yields

$$\int_{\omega_{c_+}} |u_n|^2 dx = \frac{o(1)}{\beta_n^4}.$$

The proof is thus complete. \square

Lemma 2.5.3. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (2.5.4)-(2.5.7) satisfies the following estimation*

$$\int_{\Omega} \eta |\beta_n \nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\beta_n \nabla u_n|^2 dx = o(1). \quad (2.5.15)$$

Proof: First, multiplying equation (2.5.8) by $\beta_n^2 \eta \bar{u}_n$. Later, using Green's formula, (2.5.10), (2.5.11) and the fact that the sequences f_n^1, f_n^2, g_n^1 converge to zero, respectively, in $H_0^1(\Omega)$, $H_0^1(\Omega)$ and $L^2(\Omega)$, we get

$$\begin{aligned} & \int_{\Omega} \eta |\beta_n^2 u_n|^2 dx - a \beta_n^2 \int_{\Omega} \eta |\nabla u_n|^2 dx - \beta_n^2 \int_{\Omega} a \bar{u}_n (\nabla \eta \cdot \nabla u_n) dx \\ & - i \beta_n^3 \int_{\Omega} \eta b y_n \bar{u}_n dx = o(1). \end{aligned} \quad (2.5.16)$$

Using (2.5.11) and the fact that the sequences $(\beta_n y_n)$ and (∇u_n) are uniformly bounded in $L^2(\Omega)$, we deduce from (2.5.16) that

$$\int_{\Omega} \eta |\beta_n \nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\beta_n \nabla u_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 2.5.4. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (2.5.4)-(2.5.7) satisfies the following estimation*

$$\int_{\Omega} \eta |\beta_n y_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\beta_n y_n|^2 dx = o(1). \quad (2.5.17)$$

Proof: Multiplying equation (2.5.8) by $\beta_n \eta \bar{y}_n$. Using Green's formula and the fact that $y_n = 0$ on Γ , we obtain

$$\begin{aligned} & \int_{\Omega} \beta_n^3 \eta \bar{y}_n u_n dx - a \int_{\Omega} \beta_n \eta (\nabla u_n \cdot \nabla \bar{y}_n) dx - a \int_{\Omega} \beta_n \bar{y}_n (\nabla u_n \cdot \nabla \eta) dx \\ & - i \int_{\Omega} b(x) \eta |\beta_n y_n|^2 dx - i \beta_n^2 \int_{\Omega} c(x) \eta u_n \bar{y}_n dx = o(1). \end{aligned} \quad (2.5.18)$$

Using (2.5.10), (2.5.11), (2.5.15) and the fact that the sequences $(\beta_n y_n)$ and (∇y_n) are uniformly bounded in $L^2(\Omega)$ in (2.5.18), we deduce

$$\int_{\Omega} b(x)\eta|\beta_n y_n|^2 dx = o(1). \quad (2.5.19)$$

Using the definition of function η and condition (LH2), we deduce

$$\int_{\Omega} \eta|\beta_n y_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\beta_n y_n|^2 dx = o(1).$$

The proof is thus complete. □

Lemma 2.5.5. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (2.5.4)-(2.5.7) satisfies the following estimation*

$$\int_{\Omega} \eta |\nabla y_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\nabla y_n|^2 dx = o(1). \quad (2.5.20)$$

Proof: Multiplying equation (2.5.9) by $\eta \bar{y}_n$. Then, using Green's formula, the condition $y_n = 0$ on Γ , we get

$$\int_{\Omega} \eta|\beta_n y_n|^2 dx - \int_{\Omega} \eta|\nabla y_n|^2 dx - \int_{\Omega} (\nabla \eta \cdot \nabla y_n) \bar{y}_n dx + i\beta_n \int_{\Omega} b u_n \eta \bar{y}_n = o(1) \quad (2.5.21)$$

Using (2.5.10) and the fact that $(\beta_n y_n)$ and (∇y_n) are bounded in $L^2(\Omega)$ in (2.5.21), we get

$$\int_{\Omega} \eta|\beta_n y_n|^2 dx - \int_{\Omega} \eta |\nabla y_n|^2 dx = o(1). \quad (2.5.22)$$

Finally, from (2.5.17), we deduce

$$\int_{\Omega} \eta|\nabla y_n|^2 dx = o(1) \quad \text{and} \quad \int_{\mathcal{V}_2 \cap \Omega} |\nabla y_n|^2 dx = o(1).$$

The proof is thus complete. □

Proof of Theorem 2.5.1 As we mention in Remark 2.3.12, using Lemmas 2.5.2, 2.5.3, 2.5.4 and 2.5.5, we deduce that estimation (2.3.51) is also true in the case $a \neq 1$. It follows, from the estimations (2.5.11), (2.5.15), (2.5.17), (2.5.20) and (2.3.51) that $\|U_n\|_{\mathcal{H}} = o(1)$ which is a contradiction with (2.5.2). Consequently, condition (H3) holds and the energy of smooth solution of system (2.1.4)-(2.1.6) decays polynomially to zero as t goes to infinity. □

Remark 2.5.6. *Note that our results in Theorem 2.5.1 might be more general because the waves are not assumed to be propagated with the same speed. So, this theorem generalize the results of [12] and [37].* □

2.6 Optimality of the polynomial energy decay rate

We study here the optimality of the polynomial decay rate obtained for the N -dimensional coupled wave system in Theorem 2.5.1. For this aim, we will study the asymptotic behavior of the eigenvalues of the operator \mathcal{A} in the one dimensional case for b and c are constants. Indeed, we consider the 1-dimensional version of system (2.1.4)-(2.1.6) :

$$u_{tt} - au_{xx} + by_t + cu_t = 0, \quad \text{in } (0, 1) \times (0, +\infty), \quad (2.6.1)$$

$$y_{tt} - y_{xx} - bu_t = 0, \quad \text{in } (0, 1) \times (0, +\infty), \quad (2.6.2)$$

$$u(0, t) = u(1, t) = y(0, t) = y(1, t) = 0, \quad \text{in } (0, +\infty) \quad (2.6.3)$$

with the following initial data :

$$u(x, 0) = u_0, \quad y(x, 0) = y_0, \quad u_t(x, 0) = u_1 \text{ and } y_t(x, 0) = y_1, \quad x \in (0, 1)$$

where $1 \neq a > 0$, $c > 0$ and $b \in \mathbb{R}^*$. From subsection 2.1, we express system (2.6.1)-(2.6.3) as an evolution equation of type (2.2.3) with $\mathcal{H} = (H_0^1(0, 1) \times L^2(0, 1))^2$ and $\mathcal{A} : (H^2(0, 1) \cap H_0^1(0, 1) \times H_0^1(0, 1))^2 \rightarrow \mathcal{H}$, defined by $\mathcal{A}(u, v, y, z) := (v, au_{xx} - bz - cv, z, y_{xx} + bv)$. The aim of this section is the following result :

Theorem 2.6.1. *Assume that $N = 1$, $a \neq 1$, $b = b_0 \neq 0$ and $c = c_0 > 0$. The energy decay rate (2.5.1) is optimal in the sense that for any $\varepsilon > 0$, we can not expect the decay rate $\frac{1}{t^{1+\varepsilon}}$ for all initial data $U_0 \in D(\mathcal{A})$.*

For the proof of the Theorem 2.6.1, we first study the asymptotic behavior of the eigenvalues of the operator \mathcal{A} . Since \mathcal{A} is dissipative, we fix $\alpha_0 > 0$ small enough and we study the asymptotic behavior of the eigenvalues λ of \mathcal{A} in the strip

$$S = \{\lambda \in \mathbb{C} : -\alpha_0 \leq \text{Re}(\lambda) \leq 0\}.$$

So, let $\lambda \in \mathbb{C}^*$ be an eigenvalue of \mathcal{A} with associated eigenvector $U = (u, v, y, z) \in D(\mathcal{A})$. Then $\mathcal{A}U = \lambda U$ and equivalently

$$\begin{cases} v = \lambda u, \\ au_{xx} - bz - cv = \lambda v, \\ z = \lambda y, \\ y_{xx} + bv = \lambda z, \\ u(0) = u(1) = y(0) = y(1) = 0. \end{cases} \quad (2.6.4)$$

Eliminating v and z from (2.6.4), we get

$$\begin{cases} au_{xx} - \lambda(\lambda + c)u - b\lambda y = 0, \\ y_{xx} - \lambda^2 y + b\lambda u = 0, \\ u(0) = u(1) = 0, \\ y(0) = y(1) = 0. \end{cases} \quad (2.6.5)$$

From the second equation of (2.6.5), we have

$$u = \frac{1}{b\lambda}[\lambda^2 y - y_{xx}]. \quad (2.6.6)$$

Substituting (2.6.6) in the first equation of (2.6.5), we get

$$\begin{cases} ay_{xxxx} - [\lambda^2(a+1) + c\lambda]y_{xx} + \lambda^2(\lambda^2 + c\lambda + b^2)y & = 0, \\ y(0) = y(1) & = 0, \\ y_{xx}(0) = y_{xx}(1) & = 0. \end{cases} \quad (2.6.7)$$

The characteristic equation associated with system (2.6.7) is given by

$$Q(r) := ar^4 - [\lambda^2(a+1) + c\lambda]r^2 + \lambda^2(\lambda^2 + c\lambda + b^2) = 0.$$

In order to proceed, we set the following notation. Here and below, in the case where z is a non zero non-real number, we define (and denote) by \sqrt{z} the square root of z ; *i.e.*, the unique complex number with positive real part whose square is equal to z .

The general solution of the first equation of (2.6.7) is given by

$$y(x) = \sum_{i=1}^4 c_i e^{r_i(\lambda)x},$$

where

$$r_1(\lambda) = \frac{1}{\sqrt{2a}} \sqrt{\lambda \left[\lambda(a+1) + c + \sqrt{(a-1)^2 \lambda^2 - 2c(a-1)\lambda - 4ab^2 + c^2} \right]}, \quad r_2(\lambda) = -r_1(\lambda), \quad (2.6.8)$$

$$r_3(\lambda) = \frac{1}{\sqrt{2a}} \sqrt{\lambda \left[\lambda(a+1) + c - \sqrt{(a-1)^2 \lambda^2 - 2c(a-1)\lambda - 4ab^2 + c^2} \right]}, \quad r_4(\lambda) = -r_3(\lambda). \quad (2.6.9)$$

For simplicity, here and below, we denote $r_i(\lambda)$ by r_i .

Hence, the general solution is given by

$$y(x) = A_1 \sinh(r_1 x) + A_2 \cosh(r_1 x) + A_3 \sinh(r_3 x) + A_4 \cosh(r_3 x).$$

Using the boundary condition $y(0) = y_{xx}(0) = 0$, we get

$$\begin{cases} A_2 + A_4 & = 0, \\ A_2 r_1^2 + A_4 r_3^2 & = 0, \end{cases} \quad (2.6.10)$$

which implies $A_2 = A_4 = 0$, since $r_3^2 - r_1^2 \neq 0$.

Therefore

$$y(x) = A_1 \sinh(r_1 x) + A_3 \sinh(r_3 x).$$

The boundary conditions $y(1) = y_{xx}(1) = 0$ may be written as the following system

$$M(\lambda)c(\lambda) = \begin{pmatrix} \sinh(r_1) & \sinh(r_3) \\ r_1^2 \sinh(r_1) & r_3^2 \sinh(r_3) \end{pmatrix} \begin{pmatrix} A_1 \\ A_3 \end{pmatrix} = 0. \quad (2.6.11)$$

So, set $F(\lambda) = \det(M(\lambda))$. We have the following results :

Proposition 2.6.2. *Assume that $N = 1$, $a \neq 1$, $b = b_0 \neq 0$ and $c = c_0 > 0$. Then, there exist $n_0 \in \mathbb{N}$ sufficiently large and two sequences $(\lambda_n^{(0)})$ and $(\lambda_{n'}^{(1)})$ of simple roots of F (that are also simple eigenvalues of \mathcal{A}) satisfying the following asymptotic behavior*

$$\lambda_n^{(0)} = in\pi - \frac{ib^2}{2(a-1)n\pi} - \frac{cb^2}{2(a-1)^2n^2\pi^2} + O\left(\frac{1}{n^3}\right), \quad \forall |n| \geq n_0 \quad (2.6.12)$$

and

$$\lambda_{n'}^{(1)} = in'\pi\sqrt{a} - \frac{c}{2} + O\left(\frac{1}{n'}\right), \quad \forall |n'| \geq n_0. \quad (2.6.13)$$

Proof: It is easy to see that system (2.6.11) has a non-trivial solution $(A_1, A_3) \neq (0, 0)$ if and only if λ is solution of the following equation

$$F(\lambda) = (r_3^2 - r_1^2) \sinh(r_1) \sinh(r_3) = 0.$$

Since $r_3^2 - r_1^2 \neq 0$ then $\sinh(r_1) = 0$ or $\sinh(r_3) = 0$. Thus

$$r_1 = in\pi \quad \text{or} \quad r_3 = in'\pi, \quad n, n' \in \mathbb{Z}.$$

It follows, from the asymptotic expansion in (2.6.8) and (2.6.9), that

$$\lambda - \frac{b^2}{2(a-1)\lambda} - \frac{cb^2}{2(a-1)^2\lambda^2} + O\left(\frac{1}{\lambda^3}\right) = in\pi,$$

or

$$\frac{\lambda}{\sqrt{a}} + \frac{c}{2\sqrt{a}} + O\left(\frac{1}{\lambda}\right) = in'\pi.$$

Hence, since $n \sim n' \sim \lambda$, we obtain two branches $\lambda_n^{(0)}$ and $\lambda_{n'}^{(1)}$ of eigenvalues of the operator \mathcal{A} which satisfy the asymptotic behavior (2.6.12) and (2.6.13). The proof is thus complete. \square

Remark 2.6.3. *The operator \mathcal{A} has two branches of eigenvalues. The energy corresponding to the first branch $\lambda_n^{(0)}$ decays polynomially while the energy corresponding to the second branch of eigenvalues $\lambda_{n'}^{(1)}$ decays exponentially.* \square

Proof of Theorem 2.6.1 Let $\epsilon > 0$ and set $\hat{l} = \frac{\epsilon}{1+\epsilon}$. First, let $\lambda_n^{(0)}$, with $n \geq n_0$, be the sequence of eigenvalues of the operator \mathcal{A} described in Proposition 2.6.2 and let $U_n \in D(\mathcal{A})$ be the associated normalized eigenfunction. Moreover, we introduce the following sequence

$$\beta_n = \Im(\lambda_n^{(0)}), \quad \forall n \geq n_0.$$

Next, using (2.6.12) we have

$$(i\beta_n I - \mathcal{A})U_n = \left(\frac{cb^2}{2(a-1)^2n^2\pi^2} + O\left(\frac{1}{n^3}\right) \right) U_n, \quad \forall n \geq n_0,$$

and therefore

$$\beta_n^{2-2i} \|(i\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} \sim \frac{cb^2}{2(a-1)^2\pi^2} \times \frac{1}{n^{\frac{2\epsilon}{1+\epsilon}}}, \quad \forall n \geq n_0.$$

Thus, we deduce that

$$\lim_{n \rightarrow +\infty} \beta_n^{2-2i} \|(i\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} = 0.$$

Thanks to Theorem 2.4 of Borichev-Tomilov in [25], we deduce that for $U_0 \in D(\mathcal{A})$, $\|e^{t\mathcal{A}}U_0\|_{\mathcal{H}}$ decays slower than $\frac{1}{t^{\frac{1}{2-2i}}}$ as the time $t \rightarrow +\infty$. The proof is thus complete. \square

Remark 2.6.4. *We can use Theorem 3.4.1 in [56] and equation (2.6.12), to deduce the optimality of the polynomial energy decay rate (2.5.1) in the case $N = 1$.* \square

Chapitre 3

Exact controllability and stabilization of locally coupled wave equations

Abstract : In this chapter, we study the exact controllability and stabilization of a system of two wave equations coupled by velocities with an internal locally control acted at only one equation. We distinguish two cases. In the first one, when the waves propagate at the same speed : using frequency domain approach combined with multiplier technique, we prove that the system is exponentially stable when the coupling region satisfies the geometric control condition GCC. Following a result of Haraux ([31]), we establish the main indirect observability inequality. This results leads, by the HUM method, to prove that the total system is exactly controllable by means of locally distributed control. On the contrary case, when the waves propagate at different speed, we establish an exponential decay rate in the weak energy space. Consequently, the system is exactly controllable using a result of [31]. Finally, numerically, we provide results that insure the theoretical results of [38].

3.1 Introduction

3.1.1 Motivation and aims.

Let Ω be an non empty connected open subset of \mathbb{R}^N having a boundary Γ of class C^2 . In [12], F. Alabau et al. considered the energy decay of a system of two wave equations coupled by velocities

$$u_{tt} - a\Delta u + \rho(x, u_t) + b(x)y_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (3.1.1)$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (3.1.2)$$

$$u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*, \quad (3.1.3)$$

where $a > 0$ constant, $b \in C^0(\overline{\Omega}, \mathbb{R})$ and $\rho(x, u_t)$ is a non linear damping. Using an approach based on multiplier techniques, weighted nonlinear inequalities and the optimal-weight convexity method (developed in [7]), the authors established an explicit energy

decay formula in terms of the behavior of the nonlinear feedback close to the origin. Their results are obtained in the case when the following three conditions are satisfied : the waves propagate at the same speed ($a = 1$), the coupling coefficient $b(x)$ is small positive ($0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ where b^* is a constant depending on Ω and on the control region) and both the coupling and the damping regions satisfying an appropriated geometric conditions named Piecewise Multipliers Geometric Conditions (introduced in [48] and denoted by PMGC in short). In their work, the case where the waves are not assumed to be propagated with equal speeds (a is not necessarily equal to 1) and/or the coupling coefficient $b(x)$ is not assumed to be positive and small has been left as an open problem even when the damping term ρ is linear with respect to the second variable. Recently, C. Kassem et al. in [38], answered this important open question by studying the stabilization of the following linear system :

$$u_{tt} - a\Delta u + c(x)u_t + b(x)y_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (3.1.4)$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (3.1.5)$$

$$u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*, \quad (3.1.6)$$

in the case where the waves propagate with equal or different speeds and the coupling coefficient is not assumed to be positive and small. Indeed, they distinguished two cases. The first one is when the waves propagate at the same speed (i.e. $a = 1$), but unlike the works of [12], the coupling coefficient function b is not necessarily assumed to be positive and small. In this case, under the condition that the coupling region and the damping region have non empty intersection satisfying the PMGC conditions, they established an exponential energy decay rate for weak initial data. On the contrary (i.e. $a \neq 1$) they first proved the lack of the exponential stability of the system. However, under the same geometric condition, an optimal energy decay rate of type $\frac{1}{t}$ was established for smooth initial data. The aim of this chapter is to investigate the exact controllability of the following system :

$$u_{tt} - a\Delta u + b(x)y_t = c(x)v(t) \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (3.1.7)$$

$$y_{tt} - \Delta y - b(x)u_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+^*, \quad (3.1.8)$$

$$u = y = 0 \quad \text{on } \Gamma \times \mathbb{R}_+^*, \quad (3.1.9)$$

with the following initial data

$$u(x, 0) = u_0, \quad y(x, 0) = y_0, \quad u_t(x, 0) = u_1 \text{ and } y_t(x, 0) = y_1, \quad x \in \Omega, \quad (3.1.10)$$

under appropriate geometric conditions. Here, $a > 0$ constant, $b \in C^0(\overline{\Omega}, \mathbb{R})$, $c \in C^0(\overline{\Omega}, \mathbb{R}^+)$ and v is an appropriate control. The idea is to use a result of A. Haraux in [31] for which the observability of the homogeneous system associated to (3.1.7)-(3.1.9) is equivalent to the exponential stability of system (3.1.4)-(3.1.6) in an appropriate Hilbert space. So, we provide a complete analysis for the exponential stability of system (3.1.4)-(3.1.6) in different Hilbert spaces. First, when the waves propagate at the same speed (i.e., $a = 1$), under the condition that the coupling region includes in the damping region and satisfying the called Geometric Control Condition (GCC in Short), we establish the exponential stability of system (3.1.4)-(3.1.6). Consequently, an observability inequality of the solution of the homogeneous system associated to (3.1.7)-(3.1.9) in the space $(H_0^1(\Omega) \times L^2(\Omega))^2$ is established.

This leads, by the HUM method introduced by J. L. Lions in [46], to the exact controllability of system (3.1.7)-(3.1.9) in the space $(H^{-1}(\Omega) \times L^2(\Omega))^2$. Noting that, the geometric situations covered here are richer than those considered in [12] and [38]. Furthermore, on the contrary when the waves propagate at different speeds, (i.e., $a \neq 1$), we establish the exponential stability of system (3.1.4)-(3.1.6) in the space $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$ provided that the damping region satisfies the PMGC condition while the coupling region includes in the damping region and satisfying the GCC conditions. Consequently, an observability inequality of the solution of the homogeneous system associated to (3.1.7)-(3.1.9) is established. This leads, by the HUM method, to the exact controllability of system (3.1.7)-(3.1.9) in the space $L^2(\Omega) \times H^{-1}(\Omega) \times H^{-1}(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))'$, where the duality is according to $L^2(\Omega)$. Finally, we perform numerical tests in the 1-D case to insure the theoretical results obtained here and in [38]. In fact, the numerical results show a better behavior than the one expected by the theoretical results.

3.1.2 Literature

Since the work of J. L. Lions in [46], the observability and controllability of coupled wave equations have been studied by an intensive number of publications. In [46], J. L. Lions studied the complete and partial observability and controllability of coupled systems of either hyperbolic-hyperbolic type or hyperbolic-parabolic type. These results assume that the coupling parameter is sufficiently small. In [3] and [5], F. Alabau studied the indirect boundary observability of an abstract system of two weakly coupled second order evolution equations where the coupling coefficient is strictly positive in the whole domain. In particular, using a piecewise multiplier method, she proved that, for a sufficiently large time T , the observation of the trace of the normal derivative of the first component of the solution on a part of the boundary allows us to get back a weakened energy of the initial data. Consequently, using Hilbert Uniqueness Method, she proved that the system is exactly controllable for small coupling parameter by means of one boundary control. Noting that, the situation where the waves propagate with different speeds is not covered. Later, the indirect boundary controllability of a system of two weakly coupled one-dimensional wave equations has been studied by Z. Liu and B. Rao in [52]. Using the non harmonic analysis, they established several weak observability inequalities which depend on the ratio of the wave propagation speeds and proved the indirect exact controllability. The null controllability of the reaction diffusion system has been studied by F. Ammar-Khodja et al. in [15], by deriving an observability estimate for the linearized problem. The exact controllability of a system of weakly coupled wave equations with an internal locally control acted on only one equation has been studied by A. Wehbe and W. Youssef in [69] and [70]. They showed that, for sufficiently large time, the observation of the velocity of the first component of the solution on a neighborhood of a part of the boundary allows us to get back a weakened energy of initial data of the second component, this if the coupling parameter is sufficiently small, but non- vanishing and by the HUM method, they proved that the total system is exactly controllable. F. Alabau and M. Léautaud in [10], considered a symmetric systems of two wave-type equations only one of them being controlled. The two equations are coupled by zero order terms, localized in part of the domain. They obtained an internal and a boundary controllability result in any space dimension, provided that

both the coupling and the control regions satisfy the Geometric Control Condition.

3.1.3 Description of the chapter

This chapter is organized as follows : In section (3.2), first, we show that the system (3.1.4)-(3.1.6) can be reformulated into a first order evolution equation and we deduce the well posedness property of the problem by the semi group approach. Second, by using Theorem 2.2 of [38], we show that our problem is strongly stable without geometric conditions. In section 3.3, we show the exponential decay rate of system (3.1.4)-(3.1.6) when the coupling region b is a subset of the damping region c and satisfying the geometric control condition GCC. After that, we show that our system is exactly controllable by using Proposition 2 of A. Haraux in [31]. In section 3.4, we show the exponential decay rate of system (3.1.4)-(3.1.6) in the weak energy space provided that the damping region satisfies the PMGC condition while the coupling region is a subset of the damping region and satisfying the GCC condition. Section 5 is devoted to the numerical approximation of the problem by a finite difference discretization and to the validation of the theoretical results stated in the previous sections.

3.2 Well posedness and strong stability

Let us define the energy space $\mathcal{H} = \left(H_0^1(\Omega) \times L^2(\Omega) \right)^2$ equipped with the following inner product and norm, respectively

$$(U, \tilde{U})_{\mathcal{H}} = a \int_{\Omega} (\nabla u \cdot \nabla \tilde{u}) dx + \int_{\Omega} v \tilde{v} dx + \int_{\Omega} (\nabla y \cdot \nabla \tilde{y}) dx + \int_{\Omega} z \tilde{z} dx, \quad \|U\|_{\mathcal{H}} = \sqrt{(U, U)_{\mathcal{H}}},$$

for all $U = (u, v, y, z)$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in \mathcal{H}$.

Let (u, u_t, y, y_t) be a regular solution of the system (3.1.4)-(3.1.6). Its associated energy is defined by

$$E(t) = \frac{1}{2} \int_{\Omega} (|u_t|^2 + a|\nabla u|^2 + |y_t|^2 + |\nabla y|^2) dx.$$

A straight forward computations gives

$$E'(t) = - \int_{\Omega} c(x)|u_t|^2 dx \leq 0.$$

Consequently, system (3.1.4)-(3.1.6) is dissipative in the sense that its energy is non-increasing with respect to t . Setting $U = (u, u_t, y, y_t)$, system (3.1.4)-(3.1.6) may be recast as :

$$U_t = \mathcal{A}U, \text{ in } (0, +\infty), U(0) = (u_0, u_1, y_0, y_1),$$

where the unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is given by :

$$D(\mathcal{A}) = \left((H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \right)^2 \quad (3.2.1)$$

and

$$\mathcal{A}U = (v, a\Delta u - bz - cv, z, \Delta y + bv), \quad \forall U = (u, v, y, z) \in D(\mathcal{A}). \quad (3.2.2)$$

Noting that due to the fact that $c(x) \geq 0$, the operator \mathcal{A} is m-dissipative in \mathcal{H} and generates a C_0 -semigroup of contractions $(e^{t\mathcal{A}})_{t \geq 0}$. So, system (3.1.4)-(3.1.6) is well posed in \mathcal{H} .

We need now to study the asymptotic behavior of $E(t)$. For this aim, we suppose that there exists a non empty open $\omega_{c_+} \subset \Omega$ satisfying the following condition

$$\{x \in \Omega : c(x) > 0\} \supset \bar{\omega}_{c_+}. \quad (\text{LH1})$$

On the other hand, as $b(x)$ is not identically null and continuous, then there exists a non empty open $\omega_b \subset \Omega$ such that

$$\{x \in \Omega : b(x) \neq 0\} \supset \bar{\omega}_b. \quad (\text{LH2})$$

If $\omega = \omega_{c_+} \cap \omega_b \neq \emptyset$ and condition (LH1) holds, then system (3.1.4)-(3.1.6) is strongly stable using Theorem 2.2 in [38], i.e.

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}}(u_0, u_1, y_0, y_1)\|_{\mathcal{H}} = 0 \quad \forall (u_0, u_1, y_0, y_1) \in \mathcal{H}.$$

3.3 Exponential stability and exact controllability in the case $a = 1$

3.3.1 Exponential stability

This subsection is devoted to study the exponential stability of system (3.1.4)-(3.1.6) in the case when the waves propagate at the same speed, i.e., $a = 1$ under an appropriate geometric conditions.

Before we state our results, we recall the Geometric Control Conditions GCC introduced by Rauch and Taylor in [61] for manifolds without boundaries and by Bardos, Lebeau and Rauch in [21] for domains with boundaries.

Definition 3.3.1. *We say that a subset ω of Ω satisfies the **GCC** if every ray of the geometrical optics starting at any point $x \in \Omega$ at $t = 0$ enters the region ω in finite time T .*

We recall also the Piecewise Multipliers Geometric Condition introduced by K. Liu in [48].

Definition 3.3.2. *We say that ω satisfies the Piecewise Multipliers Geometric Condition (PMGC in short) if there exist $\Omega_j \subset \Omega$ having Lipschitz boundary $\Gamma_j = \partial\Omega_j$ and $x_j \in \mathbb{R}^N$, $j = 1, \dots, J$ such that $\Omega_j \cap \Omega_i = \emptyset$ for $j \neq i$ and ω contains a neighborhood in Ω of the set $\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)$ where $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ and ν_j is the outward unit normal vector to Γ_j .*

Remark 3.3.3. *The PMGC is the generalization of the Multipliers Geometric Condition (MGC in short) introduced by Lions in [46], saying that ω contains a neighborhood in Ω of the set $\{x \in \Gamma : (x - x_0) \cdot \nu(x) > 0\}$, for some $x_0 \in \mathbb{R}^N$, where ν is the outward unit normal vector to $\Gamma = \partial\Omega$. \square*

Now, we are in position to state our first main result by the following theorem :

Theorem 3.3.4. *(Exponential decay rate) Let $a = 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that $\omega_b \subset \omega_{c_+}$ satisfies the geometric control conditions GCC and that $b, c \in W^{1,\infty}(\Omega)$. Then there exist positive constants $M \geq 1$, $\theta > 0$ such that for all initial data $(u_0, u_1, y_0, y_1) \in \mathcal{H}$ the energy of the system (3.1.4)-(3.1.6) satisfies the following decay rate :*

$$E(t) \leq Me^{-\theta t} E(0), \quad \forall t > 0. \quad (3.3.1)$$

Remark 3.3.5. *The geometric situations covered by Theorem 3.3.4 are richer than that considered in [38] and [12]. Indeed, in the previous references, the authors consider the PMGC geometric conditions that are more restrictive than GCC. On the other hand, unlike the results in [12], we have no restriction in Theorem 3.3.4 on the upper bound and the sign of the coupling function coefficient b . This theorem is then a generalization in the linear case of the result of [12] where the coupling coefficient considered have to satisfy $0 \leq b(x) \leq b_0$, $b_0 \in (0, b^*]$ where b^* is a constant depending on Ω and on the control region. \square*

In order to prove Theorem 3.3.4, we apply a result of Huang [36] and Prüss [60]. A C_0 -semigroup of contraction $(e^{t\mathcal{A}})_{t \geq 0}$ in a Hilbert space \mathcal{H} is exponentially stable if and only if

$$i\mathbb{R} \subseteq \rho(\mathcal{A}) \quad (\text{H1})$$

and

$$\limsup_{\beta \in \mathbb{R}, |\beta| \rightarrow +\infty} \| (i\beta I - \mathcal{A})^{-1} \|_{\mathcal{L}(\mathcal{H})} < \infty \quad (\text{H2})$$

hold.

Since the resolvent of \mathcal{A} is compact and $0 \in \rho(\mathcal{A})$, then from the fact that our system is strongly stable, we deduce that condition (H1) is satisfied. We now prove that condition (H2) holds, using an argument of contradiction. For this aim, we suppose that there exist a real sequence β_n with $\beta_n \rightarrow +\infty$ and a sequence $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ such that

$$\| U_n \|_{\mathcal{H}} = 1, \quad (3.3.2)$$

and

$$\lim_{n \rightarrow \infty} \| (i\beta_n I - \mathcal{A})U_n \|_{\mathcal{H}} = 0. \quad (3.3.3)$$

Next, detailing equation (3.3.3), we get

$$i\beta_n u_n - v_n = f_n^1 \rightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (3.3.4)$$

$$i\beta_n v_n - \Delta u_n + bz_n + cv_n = g_n^1 \rightarrow 0 \quad \text{in } L^2(\Omega), \quad (3.3.5)$$

$$i\beta_n y_n - z_n = f_n^2 \rightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (3.3.6)$$

$$i\beta_n z_n - \Delta y_n - bv_n = g_n^2 \rightarrow 0 \quad \text{in } L^2(\Omega). \quad (3.3.7)$$

Eliminating v_n and z_n from the previous system, we obtain the following system

$$\beta_n^2 u_n + \Delta u_n - i\beta_n b y_n - i\beta_n c u_n = -g_n^1 - b f_n^2 - i\beta_n f_n^1 - c f_n^1, \quad (3.3.8)$$

$$\beta_n^2 y_n + \Delta y_n + i\beta_n b u_n = -i\beta_n f_n^2 + b f_n^1 - g_n^2. \quad (3.3.9)$$

On the other side, we notice that v_n and z_n are uniformly bounded in $L^2(\Omega)$. It follows, from equations (3.3.4) and (3.3.6), that

$$\int_{\Omega} |y_n|^2 dx = \frac{O(1)}{\beta_n^2} \quad \text{and} \quad \int_{\Omega} |u_n|^2 dx = \frac{O(1)}{\beta_n^2}. \quad (3.3.10)$$

For clarity, we divide the proof into several Lemmas.

Lemma 3.3.6. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.3.4)-(3.3.7) satisfies the following estimation*

$$\int_{\Omega} c |\beta_n u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\omega_{c_+}} |\beta_n u_n|^2 dx = o(1). \quad (3.3.11)$$

Proof: First, since U_n is uniformly bounded in \mathcal{H} , then from (3.3.3), we get

$$\operatorname{Re} \{ i\beta_n \|U_n\|_{\mathcal{H}}^2 - (\mathcal{A}U_n, U_n)_{\mathcal{H}} \} = \int_{\Omega} c(x) |v_n|^2 dx = o(1). \quad (3.3.12)$$

Under condition (LH1), it follows that

$$\int_{\omega_{c_+}} |v_n|^2 dx = o(1). \quad (3.3.13)$$

Then, using equations (3.3.12) and (3.3.4), we get

$$\int_{\Omega} c |\beta_n u_n|^2 dx = o(1). \quad (3.3.14)$$

Consequently, we have

$$\int_{\omega_{c_+}} |\beta_n u_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.3.7. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.3.4)-(3.3.7) satisfies the following estimation*

$$\int_{\Omega} c |\nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\omega_{c_+}} |\nabla u_n|^2 dx = o(1). \quad (3.3.15)$$

Proof: Multiplying equation (3.3.8) by $c\bar{u}_n$, integrating by parts and using the fact that $u_n = 0$ on Γ , we get

$$\begin{aligned} & \int_{\Omega} c|\beta_n u_n|^2 dx - \int_{\Omega} c|\nabla u_n|^2 dx - \int_{\Omega} (\nabla c \cdot \nabla u_n) \bar{u}_n dx - i \int_{\Omega} \beta_n b y_n c \bar{u}_n dx \\ & - i \int_{\Omega} \beta_n c u_n \bar{u}_n dx = \int_{\Omega} (-g_n^1 - b f_n^2 - i \beta_n f_n^1 - c f_n^1) c \bar{u}_n dx. \end{aligned} \quad (3.3.16)$$

Using the fact that f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^1 converges to zero in $L^2(\Omega)$ and $\beta_n \bar{u}_n$ is uniformly bounded in $L^2(\Omega)$, we obtain

$$\int_{\Omega} (-g_n^1 - b f_n^2 - i \beta_n f_n^1 - c f_n^1) c \bar{u}_n dx = o(1). \quad (3.3.17)$$

Using the fact that $\nabla u_n, \beta_n y_n, \beta_n u_n$ are uniformly bounded in $L^2(\Omega)$ and $\|u_n\| = o(1)$, we get

$$- \int_{\Omega} (\nabla c \cdot \nabla u_n) \bar{u}_n dx - i \int_{\Omega} \beta_n b y_n c \bar{u}_n dx - i \int_{\Omega} \beta_n c u_n \bar{u}_n dx = o(1). \quad (3.3.18)$$

Inserting (3.3.17) and (3.3.18) into (3.3.16), we get

$$\int_{\Omega} c|\beta_n u_n|^2 dx - \int_{\Omega} c|\nabla u_n|^2 dx = o(1). \quad (3.3.19)$$

Finally, using estimation (3.3.11) in (3.3.19), we deduce

$$\int_{\omega_{c_+}} |\nabla u_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.3.8. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.3.4)-(3.3.7) satisfies the following estimation*

$$\int_{\omega_b} |\nabla y_n|^2 dx = o(1). \quad (3.3.20)$$

Proof: The proof contains three points.

i) First, multiplying equation (3.3.8) by $\frac{1}{\beta_n} \Delta \bar{y}_n$, then using Green's formula and the fact that $u_n = f_n^1 = 0$ on Γ , we obtain

$$\begin{aligned} & - \int_{\Omega} \beta_n (\nabla u_n \cdot \nabla \bar{y}_n) dx + \frac{1}{\beta_n} \int_{\Omega} \Delta u_n \Delta \bar{y}_n dx + i \int_{\Omega} (\nabla b \cdot \nabla \bar{y}_n) y_n dx \\ & + i \int_{\Omega} b |\nabla y_n|^2 dx + i \int_{\Omega} (\nabla c \cdot \nabla \bar{y}_n) u_n dx + i \int_{\Omega} c (\nabla u_n \cdot \nabla \bar{y}_n) dx \\ & = \int_{\Omega} (-g_n^1 - b f_n^2 - c f_n^1) \frac{1}{\beta_n} \Delta \bar{y}_n dx + i \int_{\Omega} (\nabla f_n^1 \cdot \nabla \bar{y}_n) dx. \end{aligned} \quad (3.3.21)$$

As f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^1 converges to zero in $L^2(\Omega)$ and the fact that $\frac{1}{\beta_n}\Delta y_n, \nabla y_n$ are uniformly bounded in $L^2(\Omega)$, we have

$$\int_{\Omega} (-g_n^1 - bf_n^2 - cf_n^1) \frac{1}{\beta_n} \Delta \bar{y}_n dx + i \int_{\Omega} (\nabla f_n^1 \cdot \nabla \bar{y}_n) dx = o(1). \quad (3.3.22)$$

Using the fact that ∇y_n is uniformly bounded in $L^2(\Omega)$, $\|u_n\|_{L^2(\Omega)} = o(1)$, $\|y_n\|_{L^2(\Omega)} = o(1)$ and using the estimation (3.3.15), we get

$$i \int_{\Omega} (\nabla b \cdot \nabla \bar{y}_n) y_n dx + i \int_{\Omega} (\nabla c \cdot \nabla \bar{y}_n) u_n dx + i \int_{\Omega} c (\nabla u_n \cdot \nabla \bar{y}_n) dx = o(1). \quad (3.3.23)$$

Inserting now (3.3.22) and (3.3.23) into (3.3.21), we get

$$- \int_{\Omega} \beta_n (\nabla u_n \cdot \nabla \bar{y}_n) dx + \frac{1}{\beta_n} \int_{\Omega} \Delta u_n \Delta \bar{y}_n dx + i \int_{\Omega} b |\nabla y_n|^2 dx = o(1). \quad (3.3.24)$$

ii) Similarly, multiplying equation (3.3.9) by $\frac{1}{\beta_n} \Delta \bar{u}_n$, then using Green's formula and the fact that $y_n = f_n^2 = 0$ on Γ , we obtain

$$\begin{aligned} & - \int_{\Omega} \beta_n (\nabla y_n \cdot \nabla \bar{u}_n) dx + \frac{1}{\beta_n} \int_{\Omega} \Delta y_n \Delta \bar{u}_n dx - i \int_{\Omega} (\nabla b \cdot \nabla \bar{u}_n) u_n dx \\ & - i \int_{\Omega} b |\nabla u_n|^2 dx = \int_{\Omega} (bf_n^1 - g_n^2) \frac{1}{\beta_n} \Delta \bar{u}_n dx + i \int_{\Omega} (\nabla f_n^2 \cdot \nabla \bar{u}_n) dx. \end{aligned} \quad (3.3.25)$$

Using the fact that f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^2 converges to zero in $L^2(\Omega)$ and the fact that $\frac{1}{\beta_n} \Delta u_n, \nabla u_n$ are uniformly bounded in $L^2(\Omega)$, we get

$$\int_{\Omega} (bf_n^1 - g_n^2) \frac{1}{\beta_n} \Delta \bar{u}_n dx + i \int_{\Omega} (\nabla f_n^2 \cdot \nabla \bar{u}_n) dx = o(1). \quad (3.3.26)$$

Also, using the fact that ∇u_n is uniformly bounded in $L^2(\Omega)$, $\|u_n\|_{L^2(\Omega)} = o(1)$, we have

$$- i \int_{\Omega} (\nabla b \cdot \nabla \bar{u}_n) u_n dx = o(1). \quad (3.3.27)$$

Inserting (3.3.26) and (3.3.27) into (3.3.25), we get

$$- \int_{\Omega} \beta_n (\nabla y_n \cdot \nabla \bar{u}_n) dx + \frac{1}{\beta_n} \int_{\Omega} \Delta y_n \Delta \bar{u}_n dx - i \int_{\Omega} b |\nabla u_n|^2 dx = o(1). \quad (3.3.28)$$

iii) Finally, by combining (3.3.24) and (3.3.28) and taking the imaginary part, we obtain

$$\int_{\Omega} b |\nabla y_n|^2 dx = \int_{\Omega} b |\nabla u_n|^2 dx + o(1). \quad (3.3.29)$$

Since $\omega_b \subset \omega_{c+}$, it follows from (3.3.15) and (3.3.29) that

$$\int_{\omega_b} |\nabla y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.3.9. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.3.4)-(3.3.7) satisfies the following estimation*

$$\int_{\omega_b} |\beta_n y_n|^2 dx = o(1). \quad (3.3.30)$$

Proof: Multiplying equation (3.3.9) by $b\bar{y}_n$. Then using Green's formula and the fact that $y_n = 0$ on Γ , we obtain

$$\begin{aligned} & \int_{\Omega} b|\beta_n y_n|^2 dx - \int_{\Omega} b|\nabla y_n|^2 dx - \int_{\Omega} (\nabla b \cdot \nabla y_n) \bar{y}_n dx \\ & + i \int_{\Omega} b^2 \beta_n u_n \bar{y}_n dx = \int_{\Omega} (-i\beta_n f_n^2 + b f_n^1 - g_n^2) b \bar{y}_n dx. \end{aligned} \quad (3.3.31)$$

As f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$, g_n^2 converges to zero in $L^2(\Omega)$ and $\beta_n y_n$ is uniformly bounded in $L^2(\Omega)$, we get

$$\int_{\Omega} (-i\beta_n f_n^2 + b f_n^1 - g_n^2) b \bar{y}_n dx = o(1). \quad (3.3.32)$$

Using the fact that $\beta_n u_n$ and ∇y_n are uniformly bounded in $L^2(\Omega)$ and $\|y_n\|_{L^2(\Omega)} = o(1)$, we get

$$\int_{\Omega} (\nabla b \cdot \nabla y_n) \bar{y}_n + i \int_{\Omega} b^2 \beta_n u_n \bar{y}_n dx = o(1). \quad (3.3.33)$$

Inserting (3.3.32), (3.3.33) into (3.3.31), we obtain

$$\int_{\Omega} b|\beta_n y_n|^2 dx - \int_{\Omega} b|\nabla y_n|^2 dx = o(1).$$

Using the estimation (3.3.20) in the previous equation, we get

$$\int_{\Omega} b|\beta_n y_n|^2 dx = o(1).$$

This yields

$$\int_{\omega_b} |\beta_n y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.3.10. *Let f_n be a bounded sequence in $L^2(\Omega)$. Then the solution $\phi_n \in H_0^1(\Omega) \cap H^2(\Omega)$ of the following system*

$$\begin{cases} \beta_n^2 \phi_n + \Delta \phi_n - i b \beta_n \phi_n & = f_n & \text{in } \Omega, \\ \phi_n & = 0 & \text{on } \Gamma, \end{cases} \quad (3.3.34)$$

verifies the following estimation

$$\int_{\Omega} (|\beta_n \phi_n|^2 + |\nabla \phi_n|^2) dx \leq C \int_{\Omega} |f_n|^2 dx, \quad (3.3.35)$$

where C is a constant independent of n .

Proof: Consider the following wave equation

$$\begin{cases} \phi_{tt} - \Delta\phi + b\phi_t = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma. \end{cases} \quad (3.3.36)$$

System (3.3.36) is well posed in the space $H = H_0^1(\Omega) \times L^2(\Omega)$ and since ω_b verifies GCC condition then it is exponentially stable (see [21]). Therefore, following Huang [36] and Pruss [60], we deduce that the resolvent of its corresponding operator

$$\mathcal{A}_{aux} : D(\mathcal{A}_{aux}) \longrightarrow H_0^1(\Omega) \times L^2(\Omega)$$

defined by $D(\mathcal{A}_{aux}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ and $\mathcal{A}_{aux}(\phi, \tilde{\phi}) = (\tilde{\phi}, \Delta\phi - b\tilde{\phi})$ is uniformly bounded on the imaginary axis.

On the other hand, system (3.3.34) can be rewritten in the form :

$$\begin{cases} i\beta_n\phi_n - \tilde{\phi}_n = 0, \\ i\beta_n\tilde{\phi}_n - \Delta\phi_n + b\tilde{\phi}_n = -f_n. \end{cases} \quad (3.3.37)$$

So,

$$(i\beta_n - \mathcal{A}_{aux}) \begin{pmatrix} \phi_n \\ \tilde{\phi}_n \end{pmatrix} = \begin{pmatrix} 0 \\ -f_n \end{pmatrix}. \quad (3.3.38)$$

Equivalently,

$$\begin{pmatrix} \phi_n \\ \tilde{\phi}_n \end{pmatrix} = (i\beta_n - \mathcal{A}_{aux})^{-1} \begin{pmatrix} 0 \\ -f_n \end{pmatrix}. \quad (3.3.39)$$

This yields

$$\begin{aligned} \|(\phi_n, \tilde{\phi}_n)\|_H^2 &\leq \|(i\beta_n - \mathcal{A}_{aux})^{-1}\|_{\mathcal{L}(H)}^2 \|(0, -f_n)\|_H^2 \\ &\leq C \int_{\Omega} |f_n|^2 dx, \end{aligned} \quad (3.3.40)$$

where C is a constant independent of n . Consequently, we deduce

$$\int_{\Omega} (|\beta_n\phi_n|^2 + |\nabla\phi_n|^2) dx \leq C \int_{\Omega} |f_n|^2 dx.$$

The proof is thus complete. \square

Lemma 3.3.11. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.3.4)-(3.3.7) satisfies the following estimation*

$$\int_{\Omega} |\beta_n u_n|^2 dx = o(1). \quad (3.3.41)$$

Proof: Taking $f_n = u_n$ in Lemma 3.3.10 and multiplying equation (3.3.8) by $\beta_n^2 \bar{\phi}_n$ where ϕ_n is a solution of (3.3.34). Then using Green's formula and the fact that $u_n = \phi_n = 0$ on Γ , we obtain

$$\begin{aligned} &\int_{\Omega} \beta_n^2 u_n (\beta_n^2 \bar{\phi}_n + \Delta \bar{\phi}_n) dx - i \int_{\Omega} b \beta_n y_n \beta_n^2 \bar{\phi}_n dx - i \int_{\Omega} c \beta_n u_n \beta_n^2 \bar{\phi}_n dx \\ &= \int_{\Omega} (-g_n^1 - b f_n^2 - c f_n^1) \beta_n^2 \bar{\phi}_n dx - i \int_{\Omega} \beta_n f_n^1 \beta_n^2 \bar{\phi}_n dx. \end{aligned} \quad (3.3.42)$$

Substituting the first equation of system (3.3.34) into the first term of (3.3.42), we get

$$\begin{aligned} & \int_{\Omega} |\beta_n u_n|^2 dx - i \int_{\Omega} \beta_n^2 \bar{\phi}_n b \beta_n u_n dx - i \int_{\Omega} b \beta_n y_n \beta_n^2 \bar{\phi}_n dx - i \int_{\Omega} c \beta_n u_n \beta_n^2 \bar{\phi}_n dx \\ &= \int_{\Omega} (-g_n^1 - b f_n^2 - c f_n^1) \beta_n^2 \bar{\phi}_n dx - i \int_{\Omega} \beta_n f_n^1 \beta_n^2 \bar{\phi}_n dx. \end{aligned} \quad (3.3.43)$$

As f_n^1, f_n^2 converge to zero in $H_0^1(\Omega)$ and $\beta_n^2 \phi_n$ is uniformly bounded in $L^2(\Omega)$ due to (3.3.35), we get

$$\int_{\Omega} (-g_n^1 - b f_n^2 - c f_n^1) \beta_n^2 \bar{\phi}_n dx = o(1). \quad (3.3.44)$$

From the first equation of (3.3.34), we have $\beta_n^2 \bar{\phi}_n = \bar{u}_n - \Delta \bar{\phi}_n - i b \beta_n \bar{\phi}_n$. Consequently, we have

$$\begin{aligned} -i \int_{\Omega} \beta_n f_n^1 \beta_n^2 \bar{\phi}_n dx &= -i \int_{\Omega} \beta_n f_n^1 (\bar{u}_n - \Delta \bar{\phi}_n - i b \beta_n \bar{\phi}_n) dx \\ &= -i \int_{\Omega} \beta_n f_n^1 \bar{u}_n dx - i \int_{\Omega} \beta_n (\nabla \bar{\phi}_n \cdot \nabla f_n^1) dx - \int_{\Omega} b f_n^1 \beta_n^2 \bar{\phi}_n, \end{aligned} \quad (3.3.45)$$

which yields

$$-i \int_{\Omega} \beta_n f_n^1 \beta_n^2 \bar{\phi}_n dx = o(1), \quad (3.3.46)$$

because f_n^1 converges to zero in $H_0^1(\Omega)$ and $\beta_n u_n, \beta_n^2 \phi_n, \beta_n \nabla \phi_n$ are uniformly bounded in $L^2(\Omega)$.

Substituting now (3.3.44) and (3.3.46) into (3.3.43)

$$\int_{\Omega} |\beta_n u_n|^2 dx - i \int_{\Omega} \beta_n^2 \bar{\phi}_n b \beta_n u_n dx - i \int_{\Omega} b \beta_n y_n \beta_n^2 \bar{\phi}_n dx - i \int_{\Omega} c \beta_n u_n \beta_n^2 \bar{\phi}_n dx = o(1). \quad (3.3.47)$$

Finally, using estimations (3.3.11), (3.3.30) and the fact that $\beta_n^2 \bar{\phi}_n$ is uniformly bounded in $L^2(\Omega)$ into the previous equation, we obtain

$$\int_{\Omega} |\beta_n u_n|^2 dx = o(1). \quad (3.3.48)$$

The proof is thus complete. \square

Lemma 3.3.12. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.3.4)-(3.3.7) satisfies the following estimation*

$$\int_{\Omega} |\beta_n y_n|^2 dx = o(1). \quad (3.3.49)$$

Proof: Taking $f_n = y_n$ in Lemma 3.3.10. Multiplying equation (3.3.9) by $\beta_n^2 \bar{\phi}_n$ where ϕ_n is a solution of (3.3.34). Then using Green's formula and the fact that $y_n = \phi_n = 0$ on Γ , we obtain

$$\begin{aligned} & \int_{\Omega} \beta_n^2 y_n (\beta_n^2 \bar{\phi}_n + \Delta \bar{\phi}_n) dx + i \int_{\Omega} b \beta_n u_n \beta_n^2 \bar{\phi}_n dx \\ &= -i \int_{\Omega} \beta_n f_n^2 \beta_n^2 \bar{\phi}_n + \int_{\Omega} (b f_n^1 - g_n^2) \beta_n^2 \bar{\phi}_n dx. \end{aligned} \quad (3.3.50)$$

Then, substituting the first equation of problem (3.3.34) into the first term of (3.3.50), we get

$$\begin{aligned} & \int_{\Omega} |\beta_n y_n|^2 dx - i \int_{\Omega} b \beta_n^2 \bar{\phi}_n \beta_n y_n dx + i \int_{\Omega} b \beta_n u_n \beta_n^2 \bar{\phi}_n dx \\ &= -i \int_{\Omega} \beta_n f_n^2 \beta_n^2 \bar{\phi}_n + \int_{\Omega} (b f_n^1 - g_n^2) \beta_n^2 \bar{\phi}_n dx. \end{aligned} \quad (3.3.51)$$

Since $\beta_n^2 \bar{\phi}_n$ is uniformly bounded in $L^2(\Omega)$, f_n^1 converges to zero in $H_0^1(\Omega)$ and g_n^2 converges to zero in $L^2(\Omega)$, we have

$$\int_{\Omega} (-b f_n^1 - g_n^2) \beta_n^2 \bar{\phi}_n dx = o(1). \quad (3.3.52)$$

Moreover, using the first equation of problem (3.3.34) and integrating by parts yields

$$\begin{aligned} -i \int_{\Omega} \beta_n f_n^2 \beta_n^2 \bar{\phi}_n dx &= -i \int_{\Omega} (\bar{y}_n - \Delta \bar{\phi}_n - i b \beta_n \bar{\phi}_n) \beta_n f_n^2 dx \\ &= -i \int_{\Omega} f_n^2 \beta_n \bar{y}_n dx - i \int_{\Omega} \beta_n (\nabla \bar{\phi}_n \cdot \nabla f_n^2) dx - \int_{\Omega} b f_n^2 \beta_n^2 \bar{\phi}_n dx. \end{aligned} \quad (3.3.53)$$

Using the fact that $\beta_n y_n$, $\beta_n^2 \phi_n$ and $\beta_n \nabla \phi_n$ are uniformly bounded in $L^2(\Omega)$ and f_n^2 converges to zero in $H_0^1(\Omega)$ in (3.3.53), we get

$$-i \int_{\Omega} \beta_n f_n^2 \beta_n^2 \bar{\phi}_n dx = o(1). \quad (3.3.54)$$

Inserting (3.3.52), (3.3.54) into (3.3.51), we obtain

$$\int_{\Omega} |\beta_n y_n|^2 dx - i \int_{\Omega} b \beta_n^2 \bar{\phi}_n \beta_n y_n dx + i \int_{\Omega} b \beta_n u_n \beta_n^2 \bar{\phi}_n dx = o(1). \quad (3.3.55)$$

Finally, using (3.3.30), (3.3.41) and the fact that $\beta_n^2 \bar{\phi}_n$ is uniformly bounded in $L^2(\Omega)$, we deduce

$$\int_{\Omega} |\beta_n y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.3.13. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A})$ of system (3.3.4)-(3.3.7) satisfies the following estimation*

$$\int_{\Omega} |\nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\nabla y_n|^2 dx = o(1). \quad (3.3.56)$$

Proof: Multiplying equation (3.3.8) by \bar{u}_n , applying Green's formula and using the fact that $u_n = 0$ on Γ , we get

$$\int_{\Omega} |\beta_n u_n|^2 dx - \int_{\Omega} |\nabla u_n|^2 dx - i \int_{\Omega} \beta_n b y_n \bar{u}_n dx - i \int_{\Omega} \beta_n c u_n \bar{u}_n dx = o(1). \quad (3.3.57)$$

Using the fact that $\beta_n u_n, \beta_n y_n$ are uniformly bounded in $L^2(\Omega)$, $\|u_n\| = o(1)$ and the estimation (3.3.41) in (3.3.57), we obtain

$$\int_{\Omega} |\nabla u_n|^2 dx = o(1). \quad (3.3.58)$$

Similarly, multiplying equation (3.3.9) by \bar{y}_n and applying Green's formula and using the fact that $y_n = 0$ on Γ , we get

$$\int_{\Omega} |\beta_n y_n|^2 dx - \int_{\Omega} |\nabla y_n|^2 dx + i \int_{\Omega} \beta_n b u_n \bar{y}_n dx = o(1). \quad (3.3.59)$$

Using the fact that $\beta_n u_n$ is uniformly bounded in $L^2(\Omega)$, $\|y_n\| = o(1)$ and (3.3.49) in (3.3.59), we obtain

$$\int_{\Omega} |\nabla y_n|^2 dx = o(1). \quad (3.3.60)$$

The proof is thus complete. \square

Proof of Theorem 3.3.4 It follows from (3.3.41), (3.3.49) and (3.3.56) that $\|U_n\|_{\mathcal{H}} = o(1)$ which is a contradiction with (3.3.2). Consequently, condition (H2) holds and the energy of system (3.1.4)-(3.1.6) decays exponentially to zero. The proof is thus complete. \square

3.3.2 Observability and exact controllability

First, we consider the following homogeneous system associated to (3.1.7)-(3.1.9) for $a = 1$ by :

$$\psi_{tt} - \Delta \psi + b(x)\varphi_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \quad (3.3.61)$$

$$\varphi_{tt} - \Delta \varphi - b(x)\psi_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \quad (3.3.62)$$

$$\psi = \varphi = 0 \quad \text{on } \Gamma \times \mathbb{R}_+, \quad (3.3.63)$$

$$\psi(\cdot, 0) = \psi_0, \psi_t(\cdot, 0) = \psi_1, \varphi(\cdot, 0) = \varphi_0, \varphi_t(\cdot, 0) = \varphi_1 \quad \text{in } \Omega. \quad (3.3.64)$$

Let $\Phi = (\psi, \psi_t, \varphi, \varphi_t)$ be a regular solution of system (3.3.61)-(3.3.63), its associated total energy is given by :

$$E(t) = \frac{1}{2} \int_{\Omega} (|\psi_t|^2 + |\nabla \psi|^2 + |\varphi_t|^2 + |\nabla \varphi|^2) dx. \quad (3.3.65)$$

A direct computation gives

$$\frac{d}{dt}E(t) = 0. \quad (3.3.66)$$

Thus, system (3.3.61)-(3.3.63) is conservative in the sense that its energy $E(t)$ is constant. It is also well posed and admits a unique solution in the energy space \mathcal{H} .

Now, we establish the direct and indirect inequality given by the following theorem :

Theorem 3.3.14. *Let $a = 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that $\omega_b \subset \omega_{c_+}$ satisfying the geometric control condition GCC and that $b, c \in W^{1,\infty}(\Omega)$. Then there exists a time T_0 such that for all $T > T_0$, there exist two constants $M_1 > 0$, $M_2 > 0$ such that the solution of system (3.3.61)-(3.3.63) satisfies the following observability inequalities :*

$$M_1 \|\Phi_0\|_{\mathcal{H}}^2 \leq \int_0^T \int_{\Omega} c(x) |\psi_t|^2 dx dt \leq M_2 \|\Phi_0\|_{\mathcal{H}}^2, \quad (3.3.67)$$

for all $\Phi_0 = (\psi_0, \psi_1, \varphi_0, \varphi_1) \in \mathcal{H}$.

Proof: The direct inequality follows from the definition of the total energy for all $T > 0$. While the proof of the inverse inequality is a direct consequence of Proposition 2 of Haraux in [31] for which the exponential stability (3.3.1) implies the existence of a time $T_0 > 0$ such that for all $T > T_0$ there exist two constants $M_1 > 0$ and $M_2 > 0$ such that (3.3.67) holds. The proof is thus complete. \square

Now, we are ready to study the exact controllability of a system (3.1.7)-(3.1.9) by using the HUM method. First, thanks to the direct inequality, the solution of the system (3.1.7)-(3.1.9) can be obtained as usual by the method of transposition (see [46]). Let $v_0 \in L^2(0, T; L^2(\omega_{c_+}))$, we choose the control

$$v(t) = -\frac{d}{dt}v_0(t) \in [H^1(0, T; L^2(\omega_{c_+}))]', \quad (3.3.68)$$

where the derivative $\frac{d}{dt}$ is not taken within the meaning of the distributions but within the meaning of the duality between $H^1(0, T; L^2(\omega_{c_+}))$ and its dual $[H^1(0, T; L^2(\omega_{c_+}))]'$, i.e.,

$$-\int_0^T \frac{d}{dt}v_1(t)\mu(t)dt = \int_0^T v_1(t)\frac{d}{dt}\mu(t)dt, \quad \forall \mu \in H^1(0, T; L^2(\omega_{c_+})).$$

Then we have the following result :

Theorem 3.3.15. *Let $T > 0$ and $a = 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that $\omega_b \subset \omega_{c_+}$ satisfying the geometric control condition GCC and that $b, c \in W^{1,\infty}(\Omega)$. Given*

$$U_0 = (u_0, u_1, y_0, y_1) \in (L^2(\Omega) \times H^{-1}(\Omega))^2, \quad v = -\frac{d}{dt}v_0 \in [H^1(0, T; L^2(\omega_{c_+}))]',$$

the controlled system (3.1.7)-(3.1.9) has a unique weak solution

$$U = (u, u_t, y, y_t) \in C^0([0, T], (L^2(\Omega) \times H^{-1}(\Omega))^2).$$

Proof: Let $(\psi, \psi_t, \varphi, \varphi_t)$ be the solution of (3.3.61)-(3.3.63) associated to $\Phi_0 = (\psi_0, \psi_1, \varphi_0, \varphi_1)$. Multiplying the first equation of (3.1.7)-(3.1.9) by ψ and the second by φ and integrating by parts, we obtain

$$\left\{ \begin{array}{l} \int_{\Omega} y_t(T)\varphi(T)dx + \int_{\Omega} u_t(T)\psi(T)dx - \int_{\Omega} y(T)\varphi_t(T)dx \\ - \int_{\Omega} u(T)\psi_t(T)dx - \int_{\Omega} bu(T)\varphi(T)dx + \int_{\Omega} by(T)\psi(T)dx = \\ \int_{\Omega} y_t(0)\varphi(0)dx + \int_{\Omega} u_t(0)\psi(0)dx - \int_{\Omega} \varphi_t(0)y(0)dx \\ - \int_{\Omega} \psi_t(0)u(0)dx - \int_{\Omega} bu(0)\varphi(0)dx + \int_{\Omega} by(0)\psi(0)dx + \int_0^T \int_{\Omega} c(x)v(t)\psi dxdt. \end{array} \right.$$

Noting that $\mathcal{H}' = (H^{-1}(\Omega) \times L^2(\Omega))^2$. Then we have

$$\left\{ \begin{array}{l} \langle (u_t(T, x), -u(T, x), y_t(T, x), -y(T, x)), \Phi(T) \rangle_{\mathcal{H}' \times \mathcal{H}} = \\ \langle (u_1, -u_0, y_1, -y_0), \Phi_0 \rangle_{\mathcal{H}' \times \mathcal{H}} + \int_0^T \int_{\Omega} cv(t)\psi dxdt = L(\Phi_0). \end{array} \right. \quad (3.3.69)$$

Using the direct observability inequality (3.3.67), we deduce that

$$\|L\|_{\mathcal{L}(\mathcal{H}, \mathbb{R})} \leq \|v_0\|_{L^2(0, T; L^2(\omega_{c_+}))} + \|U_0\|_{\mathcal{H}'}. \quad (3.3.70)$$

Using Riesz representation theorem, there exists an element $\mathcal{Z}(x, t) \in \mathcal{H}'$ solution of

$$L(\Phi_0) = \langle \mathcal{Z}, \Phi_0 \rangle_{\mathcal{H}' \times \mathcal{H}}, \quad \forall \Phi_0 \in \mathcal{H}. \quad (3.3.71)$$

Then, define the weak solution $U(x, t)$ of system (3.1.7)-(3.1.9) by $S_{\mathcal{A}}^*(t)U(x, t) = \mathcal{Z}(x, t)$. The proof is thus complete. \square

Next, we consider the indirect locally internal exact controllability problem : For given $T > 0$ (sufficiently large) and initial data U_0 , does there exists a suitable control v that brings back the solution to equilibrium at time T , that is such the solution of (3.1.7)-(3.1.9) satisfies $u(T) = u_t(T) = y(T) = y_t(T) = 0$. Indeed, applying the HUM method, we obtain the following result.

Theorem 3.3.16. *Let $a = 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that $\omega_b \subset \omega_{c_+}$ satisfying the geometric control condition GCC and that $b, c \in W^{1, \infty}(\Omega)$. For every $T > M_1$, where M_1 is given in Theorem 3.3.14 and for every*

$$U_0 \in (L^2(\Omega) \times H^{-1}(\Omega))^2,$$

there exists a control

$$v(t) \in [H^1(0, T; L^2(\omega_{c_+}))]',$$

such that the solution of the controlled system (3.1.7)-(3.1.9) satisfies

$$u(T) = u_t(T) = y(T) = y_t(T) = 0.$$

Proof: We will apply the HUM method. Thanks to the indirect observability inequalities (3.3.67), we consider the seminorme defined by

$$\|\Phi_0\|_{\mathcal{H}}^2 = \int_0^T \int_{\omega_b} |\psi_t|^2 dx dt,$$

where $\Phi = (\psi, \psi_t, \varphi, \varphi_t)$ designate the solution of the homogeneous problem (3.3.61)-(3.3.63).

Take the control $v = \frac{d}{dt}\psi_t$. Now, we solve the following retrograde problem :

$$\begin{cases} \zeta_{tt} - \Delta\zeta + b\chi_t & = c\frac{d}{dt}\psi_t & \text{in } (0, T) \times \Omega, \\ \chi_{tt} - \Delta\chi - b\zeta_t & = 0 & \text{in } (0, T) \times \Omega, \\ \chi(T) = \chi_t(T) = \zeta(T) = \zeta_t(T) & = 0. \end{cases} \quad (3.3.72)$$

By Theorem 3.3.15, the system (3.3.72) admits a solution

$$\Psi(x, t) = (\zeta, \zeta_t, \chi, \chi_t) \in C^0([0, T], H').$$

We define the linear operator Λ by :

$$\Lambda : \mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2 \rightarrow (H^{-1}(\Omega) \times L^2(\Omega))^2,$$

where

$$\Lambda\Phi_0 = (\zeta_t(0), -\zeta(0), \chi_t(0), -\chi(0)) \quad \forall \Phi_0 \in (H_0^1(\Omega) \times L^2(\Omega)).$$

In addition, we define the following linear form

$$\langle \Lambda\Phi_0, \tilde{\Phi}_0 \rangle = \int_0^T \int_{\omega_c} \psi_t \tilde{\psi}_t dx dt = (\Phi_0, \tilde{\Phi}_0)_{\mathcal{H}}, \quad \forall \tilde{\Phi}_0 \in \mathcal{H}, \quad (3.3.73)$$

where $(\cdot, \cdot)_{\mathcal{H}}$ is the scalar product associated to the norm $\|\cdot\|_{\mathcal{H}}$.

Using Cauchy-Schwarz in (3.3.73) , we deduce that

$$|\langle \Lambda\Phi_0, \tilde{\Phi}_0 \rangle_{\mathcal{H} \times \mathcal{H}'}| \leq \|\Phi_0\|_{\mathcal{H}} \|\tilde{\Phi}_0\|_{\mathcal{H}}, \quad \forall \Phi_0, \tilde{\Phi}_0 \in \mathcal{H}. \quad (3.3.74)$$

In particular, we have

$$|\langle \Lambda\Phi_0, \Phi_0 \rangle_{\mathcal{H} \times \mathcal{H}'}| = \|\Phi_0\|_{\mathcal{H}}^2 \quad \forall \Phi_0 \in \mathcal{H}.$$

Then the inverse inequality in Theorem 3.3.14 implies that the operator Λ is coercive and continuous over \mathcal{H} . Thanks to Lax-Milgram theorem, we have Λ is an isomorphism from \mathcal{H} into \mathcal{H}' . In particular, for every $U_0 \in (L^2(\Omega) \times H^{-1}(\Omega))^2$, there exists a solution $\Phi_0 \in \mathcal{H}$, such that

$$\Lambda(\Phi_0) = -U_0 = (\zeta_t(0), -\zeta(0), \chi_t(0), -\chi(0)).$$

It follows from the uniqueness of the solution of problem (3.3.72) that

$$U = \Psi.$$

Consequently, we have

$$u(T) = u_t(T) = y(T) = y_t(T) = 0.$$

The proof is thus complete. \square

3.4 Exponential stability and exact controllability in the case $a \neq 1$

3.4.1 Exponential stability in the weak energy space

The aim of this subsection is to show the exponential stability of system (3.1.4)-(3.1.6) in a weak energy space in the case when the waves are not assumed to propagate with same speed, i.e., $a \neq 1$. For this, we define the weak energy space

$$D = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$$

equipped with the scalar product

$$(U, \tilde{U}) = \int_{\Omega} (a \nabla u \cdot \nabla \tilde{u} + v \tilde{v} + y \tilde{y} + (-\Delta)^{-1/2} z (-\Delta)^{-1/2} \tilde{z}) dx,$$

for all $U = (u, v, y, z) \in D$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in D$.

Next, we define the unbounded linear operator $\mathcal{A}_d : D(\mathcal{A}_d) \subset D \rightarrow D$ by

$$\mathcal{A}_d U = (v, a \Delta u - bz - cv, z, \Delta y + bv),$$

$$D(\mathcal{A}_d) = ((H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)), \quad \forall U = (u, v, y, z) \in D(\mathcal{A}_d).$$

We define the partial energy associated to a solution $U = (u, u_t, y, y_t)$ of (3.1.4)-(3.1.6) by

$$e_1(t) = \frac{1}{2} (a \|\nabla u\|_{L^2(\Omega)}^2 + \|u_t\|_{L^2(\Omega)}^2).$$

We define also the weakened partial energy by

$$\tilde{e}_2(t) = \frac{1}{2} (\|y_t\|_{H^{-1}(\Omega)}^2 + \|y\|_{L^2(\Omega)}^2)$$

and the total mixed energy by

$$E_m(t) = e_1(t) + \tilde{e}_2(t).$$

In order to study the exponential decay rate, we need to assume that ω_{c_+} satisfies the geometric conditions PMGC, then there exist $\varepsilon > 0$, subsets $\Omega_j \subset \Omega$, $j = 1, \dots, J$, with Lipschitz boundary $\Gamma_j = \partial\Omega_j$ and points $x_j \in \mathbb{R}^N$ such that $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$ and $\omega_c^+ \supset \mathcal{N}_\varepsilon(\cup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \cup_{j=1}^J \Omega_j)) \cap \Omega$ with $\mathcal{N}_\varepsilon(\mathcal{O}) = \{x \in \mathbb{R}^N : d(x, \mathcal{O}) < \varepsilon\}$ where $\mathcal{O} \subset \mathbb{R}^N$, $\gamma_j(x_j) = \{x \in \Gamma_j : (x - x_j) \cdot \nu_j(x) > 0\}$ where ν_j is the outward unit normal vector to Γ_j and that ω_b satisfies the GCC condition and

$$\omega_b \subset (\Omega \setminus \cup_{j=1}^J \Omega_j). \quad (\text{LH3})$$

Now, we are ready to establish the following main theorem of this section :

Theorem 3.4.1. (*Exponential decay rate*) Let $a \neq 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that ω_{c_+} satisfies the geometric conditions PMGC, ω_b satisfies GCC condition and (LH3) and $b, c \in L^\infty(\Omega)$. Then there exist positive constants $M \geq 1$, $\theta > 0$ such that for all initial data $(u_0, u_1, y_0, y_1) \in D$ the energy of system (3.1.4)-(3.1.6) satisfies the following decay rate :

$$E_m(t) \leq M e^{-\theta t} E_m(0), \quad \forall t > 0. \quad (3.4.1)$$

In order to prove the above theorem, we apply a result of Huang [36] and Prüss [60]. A C_0 - semigroup of contraction $(e^{tA})_{t \geq 0}$ in a Hilbert space \mathcal{H} is exponentially stable if and only if

$$i\mathbb{R} \subseteq \rho(\mathcal{A}_d) \quad (H1)$$

and

$$\limsup_{\beta \in \mathbb{R}, |\beta| \rightarrow +\infty} \|(i\beta I - \mathcal{A}_d)^{-1}\|_{\mathcal{L}(D)} < \infty \quad (H2)$$

hold.

Condition (H1) was already proved. We now prove that condition (H2) holds, using an argument of contradiction. For this aim, we suppose that there exist a real sequence β_n with $\beta_n \rightarrow +\infty$ and a sequence $U_n = (u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ such that

$$\|U_n\|_D = 1, \quad (3.4.2)$$

and

$$\lim_{n \rightarrow \infty} \|(i\beta_n I - \mathcal{A}_d)U_n\|_D = 0. \quad (3.4.3)$$

Next, detailing equation (3.4.3), we get

$$i\beta_n u_n - v_n = f_n^1 \rightarrow 0 \quad \text{in } H_0^1(\Omega), \quad (3.4.4)$$

$$i\beta_n v_n - a\Delta u_n + bz_n + cv_n = g_n^1 \rightarrow 0 \quad \text{in } L^2(\Omega), \quad (3.4.5)$$

$$i\beta_n y_n - z_n = f_n^2 \rightarrow 0 \quad \text{in } L^2(\Omega), \quad (3.4.6)$$

$$i\beta_n z_n - \Delta y_n - bv_n = g_n^2 \rightarrow 0 \quad \text{in } H^{-1}(\Omega). \quad (3.4.7)$$

Eliminating v_n and z_n from the previous system, we obtain the following system

$$\beta_n^2 u_n + a\Delta u_n - i\beta_n b y_n - i\beta_n c u_n = -g_n^1 - b f_n^2 - i\beta_n f_n^1 - c f_n^1 \quad \text{in } L^2(\Omega), \quad (3.4.8)$$

$$\beta_n^2 y_n + \Delta y_n + i\beta_n b u_n = -i\beta_n f_n^2 + b f_n^1 - g_n^2 \quad \text{in } H^{-1}(\Omega). \quad (3.4.9)$$

From (3.4.2), we have ∇u_n , v_n and y_n are uniformly bounded in $L^2(\Omega)$ and z_n is uniformly bounded in $H^{-1}(\Omega)$.

Using now (3.4.2) and (3.4.4), we deduce that $\beta_n u_n$ is uniformly bounded in $L^2(\Omega)$. In addition, using (3.4.2) and (3.4.6), we deduce that $\beta_n y_n$ is uniformly bounded in $H^{-1}(\Omega)$. More precisely,

$$\|u_n\|_{L^2(\Omega)} = \frac{O(1)}{\beta_n} = o(1) \quad \text{and} \quad \|y_n\|_{H^{-1}(\Omega)} = \frac{O(1)}{\beta_n} = o(1).$$

Lemma 3.4.2. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (3.4.4)-(3.4.7) satisfies the following estimation*

$$\int_{\Omega} c|\beta_n u_n|^2 dx = o(1) \quad \text{and} \quad \int_{\omega_{c+}} |\beta_n u_n|^2 dx = o(1). \quad (3.4.10)$$

Proof: First, since U_n is uniformly bounded in D and using (3.4.3), we get

$$\operatorname{Re} \{i\beta_n \|U_n\|^2 - (\mathcal{A}_d U_n, U_n)\} = \int_{\Omega} c(x)|v_n|^2 dx = o(1). \quad (3.4.11)$$

Next, using equations (3.4.11) and (3.4.4), we get

$$\int_{\Omega} c|\beta_n u_n|^2 dx = o(1). \quad (3.4.12)$$

Under condition (LH1), it follows

$$\int_{\omega_{c+}} |\beta_n u_n|^2 dx = o(1).$$

The proof is thus complete. \square

Now as ω_{c+} satisfies the PMGC condition, let the reals $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon$ and define

$$Q_i = \mathcal{N}_{\varepsilon_i} \left(\bigcup_{j=1}^J \gamma_j(x_j) \cup (\Omega \setminus \bigcup_{j=1}^J \Omega_j) \right), \quad i = 1, 2.$$

Since $\overline{\mathbb{R}^N \setminus \omega_{c+}} \cap \overline{Q_2} = \emptyset$, we can construct a function $\hat{\eta} \in C_0^\infty(\Omega)$ defined by

$$\hat{\eta}(x) = 0 \quad \text{if } x \in \Omega \setminus \omega_{c+}, \quad 0 \leq \hat{\eta}(x) \leq 1, \quad \hat{\eta}(x) = 1 \quad \text{if } x \in Q_2.$$

Lemma 3.4.3. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (3.4.4)-(3.4.7) satisfies the following estimation*

$$\int_{\Omega} \hat{\eta} |\nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{Q_2 \cap \Omega} |\nabla u_n|^2 dx = o(1). \quad (3.4.13)$$

Proof: First, multiplying equation (3.4.8) by $\hat{\eta} \bar{u}_n$. Then, using Green's formula and the fact that $u_n = 0$ on Γ , we obtain

$$\begin{aligned} & \int_{\Omega} \hat{\eta} |\beta_n u_n|^2 dx - a \int_{\Omega} \hat{\eta} |\nabla u_n|^2 dx - a \int_{\Omega} \bar{u}_n (\nabla \hat{\eta} \cdot \nabla u_n) dx - i\beta_n \int_{\Omega} b \hat{\eta} y_n \bar{u}_n dx \\ & - i\beta_n \int_{\Omega} c \hat{\eta} |u_n|^2 dx = \int_{\Omega} (-g_n^1 - b f_n^2 - i\beta_n f_n^1 - c f_n^1) \hat{\eta} \bar{u}_n dx. \end{aligned} \quad (3.4.14)$$

As f_n^1 converges to zero in $H_0^1(\Omega)$, f_n^2 , g_n^1 converge to zero in $L^2(\Omega)$ and $\beta_n u_n$ is uniformly bounded in $L^2(\Omega)$, we get

$$\int_{\Omega} (-g_n^1 - b f_n^2 - i\beta_n f_n^1 - c f_n^1) \hat{\eta} \bar{u}_n dx = o(1). \quad (3.4.15)$$

Using the fact that $\nabla u_n, y_n$ are uniformly bounded in $L^2(\Omega)$, $\|u_n\|_{L^2(\Omega)} = o(1)$ and estimation (3.4.10), we will have

$$\int_{\Omega} \hat{\eta} |\beta_n u_n|^2 dx - a \int_{\Omega} \bar{u}_n (\nabla \hat{\eta} \cdot \nabla u_n) dx - i \beta_n \int_{\Omega} b \hat{\eta} y_n \bar{u}_n dx - i \beta_n \int_{\Omega} c \hat{\eta} |u_n|^2 dx = o(1). \quad (3.4.16)$$

Finally, inserting (3.4.15) and (3.4.16) into (3.4.14), we deduce

$$\int_{\Omega} \hat{\eta} |\nabla u_n|^2 dx = o(1) \quad \text{and} \quad \int_{Q_2 \cap \Omega} |\nabla u_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.4.4. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (3.4.4)-(3.4.7) satisfies the following estimation*

$$\int_{\omega_b} |y_n|^2 dx = o(1). \quad (3.4.17)$$

Proof: The proof contains two steps.

Step 1. (Boundedness of $\frac{1}{\beta_n} \nabla y_n$). Multiplying equation (3.4.9) by $\frac{1}{\beta_n^2} \bar{y}_n$, we obtain

$$\begin{aligned} \int_{\Omega} |y_n|^2 dx + \langle \Delta y_n, \frac{1}{\beta_n^2} \bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} &= -i \int_{\Omega} \frac{1}{\beta_n} f_n^2 \bar{y}_n dx + \int_{\Omega} b f_n^1 \frac{1}{\beta_n^2} \bar{y}_n dx \\ &- \langle g_n^2, \frac{1}{\beta_n^2} \bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}. \end{aligned} \quad (3.4.18)$$

Since f_n^1 converges to zero in $H_0^1(\Omega)$, f_n^2 converges to zero in $L^2(\Omega)$ and y_n is uniformly bounded in $L^2(\Omega)$, we get

$$-i \int_{\Omega} \frac{1}{\beta_n} f_n^2 \bar{y}_n dx + \int_{\Omega} b f_n^1 \frac{1}{\beta_n^2} \bar{y}_n dx = o(1). \quad (3.4.19)$$

Inserting (3.4.19) into (3.4.18), we will have after integrating by parts

$$\int_{\Omega} \left| \frac{\nabla y_n}{\beta_n} \right|^2 dx = \int_{\Omega} |y_n|^2 dx + \langle g_n^2, \frac{1}{\beta_n^2} \bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + o(1).$$

Using Cauchy-Schwartz and Young inequalities in the previous equation, we obtain that

$$\frac{1}{2} \left\| \frac{\nabla y_n}{\beta_n} \right\|_{L^2(\Omega)}^2 \leq \|y_n\|_{L^2(\Omega)}^2 + \frac{1}{2} \|g_n^2\|_{H^{-1}(\Omega)}^2 + o(1).$$

It follows, from the uniform boundedness of y_n in $L^2(\Omega)$ and g_n^2 in $H^{-1}(\Omega)$, that

$$\left\| \frac{\nabla y_n}{\beta_n} \right\|_{L^2(\Omega)}^2 = O(1). \quad (3.4.20)$$

Step 2. (Main asymptotic estimation). Multiplying equation (3.4.8) by $\hat{\eta} \frac{1}{\beta_n} \bar{y}_n$. Later, using Green's formula and the fact that $y_n = 0$ on Γ , we get

$$\begin{aligned} & \int_{\Omega} \hat{\eta} \beta_n u_n \bar{y}_n dx - a \int_{\Omega} \frac{1}{\beta_n} \hat{\eta} (\nabla u_n \cdot \nabla \bar{y}_n) dx - a \int_{\Omega} \frac{1}{\beta_n} (\nabla \hat{\eta} \cdot \nabla u_n) \bar{y}_n dx \\ & - i \int_{\Omega} b \hat{\eta} |y_n|^2 dx - i \int_{\Omega} c u_n \hat{\eta} \bar{y}_n dx = \int_{\Omega} (-g_n^1 - b f_n^2 - i \beta_n f_n^1 - c f_n^1) \frac{\hat{\eta}}{\beta_n} \bar{y}_n dx. \end{aligned} \quad (3.4.21)$$

Next, using the definition of $\hat{\eta}$ and equations (3.4.13) and (3.4.20), we get

$$- a \int_{\Omega} \frac{1}{\beta_n} \hat{\eta} (\nabla u_n \cdot \nabla \bar{y}_n) dx = o(1). \quad (3.4.22)$$

Using (3.4.10), (3.4.13) and the fact that y_n is uniformly bounded in $L^2(\Omega)$, we obtain

$$- a \int_{\Omega} \frac{1}{\beta_n} (\nabla \hat{\eta} \cdot \nabla u_n) \bar{y}_n dx - \int_{\Omega} \hat{\eta} \beta_n u_n \bar{y}_n dx - i \int_{\Omega} c u_n \hat{\eta} y_n dx = o(1). \quad (3.4.23)$$

Using the fact that f_n^1 converges to zero in $H_0^1(\Omega)$, f_n^2 , g_n^1 converge to zero in $L^2(\Omega)$ and y_n is uniformly bounded in $L^2(\Omega)$, we will have

$$\int_{\Omega} (-g_n^1 - b f_n^2 - i \beta_n f_n^1 - c f_n^1) \frac{\hat{\eta}}{\beta_n} \bar{y}_n dx = o(1). \quad (3.4.24)$$

Finally, inserting (3.4.22)-(3.4.24) into (3.4.21), we get

$$\int_{\Omega} b \hat{\eta} |y_n|^2 dx = o(1).$$

It follows, from condition (LH3), that

$$\int_{\omega_b} |y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.4.5. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (3.4.4)-(3.4.7) satisfies the following estimation*

$$\int_{\Omega} |y_n|^2 dx = o(1). \quad (3.4.25)$$

Proof: Noting that ω_b satisfies the GCC condition, so we can taking $f_n = y_n$ in Lemma 3.3.10. Multiplying equation (3.4.9) by $\bar{\phi}_n$. Then, we have

$$\begin{aligned} & \int_{\Omega} \beta_n^2 \bar{\phi}_n y_n dx - \langle \Delta y_n, \bar{\phi}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + i \int_{\Omega} \beta_n b u_n \bar{\phi}_n dx = -i \int_{\Omega} \beta_n f_n^2 \bar{\phi}_n dx \\ & + \int_{\Omega} b f_n^1 \bar{\phi}_n dx - \langle g_n^2, \bar{\phi}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}. \end{aligned} \quad (3.4.26)$$

Using the fact that $\phi_n \in H^2(\Omega) \cap H_0^1(\Omega)$ and $y_n \in H_0^1(\Omega)$, then we have

$$- \langle \Delta y_n, \bar{\phi}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \int_{\Omega} y_n \Delta \bar{\phi}_n dx. \quad (3.4.27)$$

It follows, from the first equation of (3.3.34) and (3.4.26), that

$$\begin{aligned} \int_{\Omega} |y_n|^2 dx &= i \int_{\Omega} b \beta_n \bar{\phi}_n y_n dx - i \int_{\Omega} \beta_n b u_n \bar{\phi}_n dx - i \int_{\Omega} \beta_n f_n^2 \bar{\phi}_n dx \\ &+ \int_{\Omega} b f_n^1 \bar{\phi}_n dx - \langle g_n^2, \bar{\phi}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}. \end{aligned} \quad (3.4.28)$$

Using the fact that $\beta_n \phi_n$ is uniformly bounded in $L^2(\Omega)$, f_n^1 converges to zero in $H_0^1(\Omega)$, f_n^2 converges to zero in $L^2(\Omega)$, g_n^2 converges to zero in H^{-1} , (3.4.17) and $\|u_n\| = o(1)$ in equation (3.4.28), we obtain

$$\int_{\Omega} |y_n|^2 dx = o(1). \quad (3.4.29)$$

The proof is thus complete. \square

Lemma 3.4.6. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (3.4.4)-(3.4.7) satisfies the following estimation*

$$\int_{\Omega} |\beta_n (-\Delta)^{-1/2} y_n|^2 dx = o(1). \quad (3.4.30)$$

Proof: Multiplying equation (3.4.9) by $(-\Delta)^{-1} \bar{y}_n$, then integrating by parts and using the fact that $y_n = 0$ on Γ , we get

$$\begin{aligned} \int_{\Omega} |\beta_n (-\Delta)^{-1/2} y_n|^2 dx &= \int_{\Omega} |y_n|^2 dx - i \int_{\Omega} \beta_n b u_n (-\Delta)^{-1} \bar{y}_n dx \\ &- i \int_{\Omega} \beta_n (-\Delta)^{-1/2} f_n^2 (-\Delta)^{-1/2} \bar{y}_n dx + \int_{\Omega} b f_n^1 (-\Delta)^{-1} \bar{y}_n dx \\ &- \langle g_n^2, (-\Delta)^{-1} \bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \end{aligned} \quad (3.4.31)$$

Using Cauchy-Schwartz and Poincaré inequalities, we get

$$\begin{aligned} \left| \int_{\Omega} \beta_n (-\Delta)^{-1/2} f_n^2 (-\Delta)^{-1/2} \bar{y}_n dx \right| &\leq \|(-\Delta)^{-1/2} f_n^2\|_{L^2(\Omega)} \|\beta_n (-\Delta)^{-1/2} \bar{y}_n\|_{L^2(\Omega)} \\ &\leq c_0 \|f_n^2\|_{L^2(\Omega)} \|\beta_n \bar{y}_n\|_{H^{-1}(\Omega)}. \end{aligned} \quad (3.4.32)$$

It follows, from the convergence to zero of f_n^2 in $L^2(\Omega)$ and the boundedness of $\beta_n y_n$ in $H^{-1}(\Omega)$, that

$$\int_{\Omega} \beta_n (-\Delta)^{-1/2} f_n^2 (-\Delta)^{-1/2} \bar{y}_n dx = o(1). \quad (3.4.33)$$

Similarly, we have

$$\begin{aligned} |\langle g_n^2, (-\Delta)^{-1}\bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}| &= \int_{\Omega} (-\Delta)^{-1/2} g_n^2 (-\Delta)^{-1/2} \bar{y}_n dx \\ &\leq \|(-\Delta)^{-1/2} g_n^2\|_{L^2(\Omega)} \|(-\Delta)^{-1/2} \bar{y}_n\|_{L^2(\Omega)} \\ &\leq \|g_n^2\|_{H^{-1}(\Omega)} \|\bar{y}_n\|_{H^{-1}(\Omega)}. \end{aligned} \quad (3.4.34)$$

It follows, from the convergence of g_n^2 and y_n to zero in $H^{-1}(\Omega)$, that

$$\langle g_n^2, (-\Delta)^{-1}\bar{y}_n \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = o(1). \quad (3.4.35)$$

Noting that $(-\Delta)^{-1}$ is compact operator from L^2 to L^2 , then $(-\Delta)^{-1}y_n$ is uniformly bounded in L^2 . Finally, using (3.4.10), (3.4.25), (3.4.33), (3.4.35) and the fact that f_n^1 converges to zero in $H_0^1(\Omega)$ into equation (3.4.31), we deduce

$$\int_{\Omega} |\beta_n (-\Delta)^{-1/2} y_n|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 3.4.7. *The solution $(u_n, v_n, y_n, z_n) \in D(\mathcal{A}_d)$ of system (3.4.4)-(3.4.7) satisfies the following estimation*

$$\int_{\Omega \setminus (Q_2 \cap \Omega)} (|\nabla u_n|^2 + |\beta_n u_n|^2) dx = o(1). \quad (3.4.36)$$

Proof: Since $(\bar{\Omega}_j \setminus Q_2) \cap \bar{Q}_1 = \emptyset$, we define the function $\psi_j \in C_0^\infty(\mathbb{R}^N)$ by :

$$\psi_j(x) = 0 \quad \text{if } x \in Q_1, \quad 0 \leq \psi_j \leq 1, \quad \psi_j(x) = 1 \quad \text{if } x \in \bar{\Omega}_j \setminus Q_2.$$

For $m_j(x) = (x - x_j)$, we define $h_j(x) = \psi_j(x)m_j(x)$.

Multiplying equation (3.4.8) by $2(h_j \cdot \nabla \bar{u}_n)$ and integrating over Ω_j , using the dissipation (3.4.10) and the fact that ∇u_n is uniformly bounded in $L^2(\Omega)$, we obtain

$$\begin{aligned} 2\beta_n^2 \int_{\Omega_j} u_n (h_j \cdot \nabla \bar{u}_n) dx + 2a \int_{\Omega_j} \Delta u_n (h_j \cdot \nabla \bar{u}_n) dx - 2i \int_{\Omega_j} \beta_n b y_n (h_j \cdot \nabla \bar{u}_n) dx = \\ 2 \int_{\Omega_j} (-g_n^1 - b f_n^2 - c f_n^1) (h_j \cdot \nabla \bar{u}_n) dx - 2i \int_{\Omega_j} \beta_n f_n^1 (h_j \cdot \nabla \bar{u}_n) dx. \end{aligned} \quad (3.4.37)$$

i) **Estimation of the second member of** (3.4.37). First, using Green's formula and the fact that $u_n = 0$ on $(\Gamma_j \setminus \gamma_j) \cap \Gamma$ and $h_j = 0$ on γ_j , we get

$$-2i \int_{\Omega_j} \beta_n f_n^1 (h_j \cdot \nabla \bar{u}_n) dx = 2i \int_{\Omega_j} \beta_n \bar{u}_n (h_j \cdot \nabla f_n^1) dx + 2i \int_{\Omega_j} \beta_n \bar{u}_n f_n^1 (\operatorname{div} h_j) dx. \quad (3.4.38)$$

So, from the fact that f_n^1 converges to zero in $H_0^1(\Omega)$ and $\beta_n u_n$ is uniformly bounded in $L^2(\Omega)$, we obtain

$$-2i \int_{\Omega_j} \beta_n f_n^1 (h_j \cdot \nabla \bar{u}_n) dx = o(1). \quad (3.4.39)$$

Next, as f_n^1 converges to zero in $H_0^1(\Omega)$, f_n^2, g_n^1 converge to zero in $L^2(\Omega)$ and the sequence (∇u_n) is uniformly bounded in $L^2(\Omega)$, we deduce

$$2 \int_{\Omega_j} (-g_n^1 - b f_n^2 - c f_n^1)(h_j \cdot \nabla \bar{u}_n) dx = o(1). \quad (3.4.40)$$

Finally, we deduce that the second member of (3.4.37) is $o(1)$.

ii) **Estimation of the first integral of equation (3.4.37).** Using Green's formula, we get

$$\operatorname{Re} \left\{ 2 \int_{\Omega_j} \beta_n^2 u_n (h_j \cdot \nabla \bar{u}_n) dx \right\} = - \int_{\Omega_j} (\operatorname{div} h_j) |\beta_n u_n|^2 dx + \int_{\Gamma_j} (h_j \cdot \nu_j) |\beta_n u_n|^2 d\Gamma_j. \quad (3.4.41)$$

Since $\Psi_j = 0$ on γ_j and $u_n = 0$ on $(\Gamma_j \setminus \gamma_j) \cap \Gamma$, then we have

$$\operatorname{Re} \left\{ 2 \int_{\Omega_j} \beta_n^2 u_n (h_j \cdot \nabla \bar{u}_n) dx \right\} = - \int_{\Omega_j} (\operatorname{div} h_j) |\beta_n u_n|^2 dx. \quad (3.4.42)$$

iii) **Estimation of the second integral of equation (3.4.37).** Using Green's formula, we get

$$\begin{aligned} \operatorname{Re} \left\{ 2a \int_{\Omega_j} \Delta u_n (h_j \cdot \nabla \bar{u}_n) \right\} &= -2a \operatorname{Re} \left\{ \sum_{i,k=1}^N \int_{\Omega_j} \partial_i h_j^k \partial_i u_n \partial_k u_n dx \right\} + \\ &a \int_{\Omega_j} (\operatorname{div} h_j) |\nabla u_n|^2 dx - a \int_{\Gamma_j} (h_j \cdot \nu_j) |\nabla u_n|^2 d\Gamma_j + 2a \operatorname{Re} \left\{ \int_{\Gamma_j} \partial_{\nu_j} u_n (h_j \cdot \nabla \bar{u}_n) d\Gamma_j \right\}. \end{aligned} \quad (3.4.43)$$

According to the choice of ψ_j , only the boundary terms over $(\Gamma_j \setminus \gamma_j) \cap \Gamma$ are non vanishing in (3.4.43). But on this part of the boundary $u_n = 0$, and consequently $\nabla u_n = (\partial_\nu u_n) \cdot \nu = (\partial_{\nu_j} u_n) \nu_j$. Then, we have

$$\begin{aligned} -a \int_{\Gamma_j} (h_j \cdot \nu_j) |\nabla u_n|^2 d\Gamma_j + 2a \operatorname{Re} \left\{ \int_{\Gamma_j} (\partial_{\nu_j} u_n) (h_j \cdot \nabla \bar{u}_n) d\Gamma_j \right\} &= \\ a \int_{(\Gamma_j \setminus \gamma_j) \cap \Gamma} (\psi_j m_j \cdot \nu_j) |\partial_{\nu_j} u_n|^2 d\Gamma_j &\leq 0. \end{aligned} \quad (3.4.44)$$

Inserting (3.4.44) into (3.4.43), we get

$$\begin{aligned} \operatorname{Re} \left\{ 2a \int_{\Omega_j} \Delta u_n (h_j \cdot \nabla \bar{u}_n) \right\} &\leq -2a \operatorname{Re} \left\{ \sum_{i,k=1}^N \int_{\Omega_j} \partial_i h_j^k \partial_i u_n \partial_k u_n dx \right\} \\ &+ a \int_{\Omega_j} (\operatorname{div} h_j) |\nabla u_n|^2 dx. \end{aligned} \quad (3.4.45)$$

iv) **The main estimation.** Inserting equations (3.4.39), (3.4.40), (3.4.42) and (3.4.45) into (3.4.37) and using the fact that $\psi_j = 0$ on Q_1 , we get

$$\begin{aligned} & \int_{\Omega_j \setminus (Q_1 \cap \Omega_j)} \left[\operatorname{div}(\psi_j m_j) (|\beta_n u_n|^2 - a |\nabla u_n|^2) dx + 2a \sum_{i,k=1}^N \partial_i(\psi_j m_j^k) \partial_i u_n \partial_k u_n \right] dx \\ & + 2i \int_{\Omega_j \setminus (Q_1 \cap \Omega_j)} \beta_n b y_n (\psi_j m_j \cdot \nabla \bar{u}_n) dx \leq o(1). \end{aligned}$$

Thus, summing over j and using the fact that $\psi_j = 1$ on $\bar{\Omega}_j \setminus Q_2$, we get

$$\begin{aligned} & N \int_{\Omega \setminus (Q_2 \cap \Omega)} |\beta_n u_n|^2 dx + (2 - N)a \int_{\Omega \setminus (Q_2 \cap \Omega)} |\nabla u_n|^2 dx \\ & + 2\operatorname{Re} \left\{ i \sum_{j=1}^J \int_{\Omega_j \setminus (Q_1 \cap \Omega_j)} \beta_n b y_n (\psi_j m_j \cdot \nabla \bar{u}_n) dx \right\} \tag{3.4.46} \\ & \leq - \sum_{j=1}^J \int_{Q_2 \cap \Omega_j} \left[\operatorname{div}(\psi_j m_j) (|\beta_n u_n|^2 - a |\nabla u_n|^2) dx + 2a \sum_{i,k=1}^N \partial_i(\psi_j m_j^k) \partial_i u_n \partial_k u_n \right] dx + o(1). \end{aligned}$$

Using (3.4.10) and (3.4.13), we deduce

$$\begin{aligned} & - \sum_{j=1}^J \int_{Q_2 \cap \Omega_j} \left[\operatorname{div}(\psi_j m_j) (|\beta_n u_n|^2 - a |\nabla u_n|^2) dx \right. \\ & \left. + 2a \sum_{i,k=1}^N \partial_i(\psi_j m_j^k) \partial_i u_n \partial_k u_n \right] dx = o(1). \tag{3.4.47} \end{aligned}$$

Inserting (3.4.47) in (3.4.46), we obtain

$$\begin{aligned} & N \int_{\Omega \setminus (Q_2 \cap \Omega)} |\beta_n u_n|^2 dx + (2 - N)a \int_{\Omega \setminus (Q_2 \cap \Omega)} |\nabla u_n|^2 dx + \tag{3.4.48} \\ & 2\operatorname{Re} \left\{ i \sum_{j=1}^J \int_{\Omega_j \setminus (Q_1 \cap \Omega_j)} \beta_n b y_n (\psi_j m_j \cdot \nabla \bar{u}_n) dx \right\} \leq o(1). \end{aligned}$$

Under condition (LH3) and the definition of ψ_j , we will have

$$2\operatorname{Re} \left\{ i \sum_{j=1}^J \int_{\Omega_j \setminus (Q_1 \cap \Omega_j)} \beta_n b y_n (\psi_j m_j \cdot \nabla \bar{u}_n) dx \right\} = 0.$$

Inserting the previous estimation into (3.4.48), we get

$$N \int_{\Omega \setminus (Q_2 \cap \Omega)} |\beta_n u_n|^2 dx + (2 - N)a \int_{\Omega \setminus (Q_2 \cap \Omega)} |\nabla u_n|^2 dx \leq o(1). \tag{3.4.49}$$

Multiplying (3.4.8) by $(1 - N)\bar{u}_n$. Then integrating on Ω , using Green's formula, the fact that y_n and $\beta_n u_n$ are bounded in $L^2(\Omega)$ and the estimation (3.4.10), we obtain

$$(1 - N) \int_{\Omega} |\beta_n u_n|^2 dx - (1 - N)a \int_{\Omega} |\nabla u_n|^2 dx = o(1). \quad (3.4.50)$$

Using (3.4.10) and (3.4.13) in (3.4.50), we deduce

$$(1 - N) \int_{\Omega \setminus (Q_2 \cap \Omega)} |\beta_n u_n|^2 dx - (1 - N)a \int_{\Omega \setminus (Q_2 \cap \Omega)} |\nabla u_n|^2 dx = o(1). \quad (3.4.51)$$

Finally, combining (3.4.49) and (3.4.51), we get the following estimation

$$\int_{\Omega \setminus (Q_2 \cap \Omega)} (a|\nabla u_n|^2 + |\beta_n u_n|^2) dx = o(1).$$

The proof is thus complete. \square

Proof of Theorem 3.3.14 It follows from (3.4.10) (3.4.13), (3.4.25), (3.4.30) and (3.4.36) that $\|U_n\| = o(1)$ which is a contradiction with (3.4.2). Consequently, condition (H2) holds and the energy of system (3.1.4)-(3.1.6) decays exponentially to zero in the weak energy space D . The proof is thus complete. \square

3.4.2 Observability and exact controllability

First, we consider the following homogeneous system associated to (3.1.4)-(3.1.6) for $a \neq 1$ by :

$$\psi_{tt} - a\Delta\psi + b(x)\varphi_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \quad (3.4.52)$$

$$\varphi_{tt} - \Delta\varphi - b(x)\psi_t = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \quad (3.4.53)$$

$$\psi = \varphi = 0 \quad \text{on } \Gamma \times \mathbb{R}_+, \quad (3.4.54)$$

$$\psi(\cdot, 0) = \psi_0, \psi_t(\cdot, 0) = \psi_1, \varphi(\cdot, 0) = \varphi_0, \varphi_t(\cdot, 0) = \varphi_1 \quad \text{in } \Omega. \quad (3.4.55)$$

Let $\Phi = (\psi, \psi_t, \varphi, \varphi_t)$ be a regular solution of system (3.3.61)-(3.3.63), its associated total energy is given by :

$$E_m(t) = \frac{1}{2} \left(a\|\nabla\psi\|_{L^2(\Omega)}^2 + \|\psi_t\|_{L^2(\Omega)}^2 + \|\varphi_t\|_{H^{-1}(\Omega)}^2 + \|\varphi\|_{L^2(\Omega)}^2 \right). \quad (3.4.56)$$

A direct computation gives

$$\frac{d}{dt} E_m(t) = 0. \quad (3.4.57)$$

Thus, system (3.4.52)-(3.4.54) is conservative in the sense that its energy $E_m(t)$ is constant. It is also well posed and admits a unique solution in the energy space D .

Now, we establish the direct and indirect inequality given by the following theorem :

Theorem 3.4.8. *Let $0 < a \neq 1$. Assume that conditions (LH1) and (LH2) hold. Assume also that ω_{c_+} satisfies the PMGC, ω_b satisfies GCC condition and (LH3) and $b, c \in L^\infty(\Omega)$. Then there exists a time T_0 such that for all $T > T_0$, there exist two constants $C_1 > 0$, $C_2 > 0$ such that the solution of system (3.4.52)-(3.4.54) satisfies the following observability inequalities :*

$$C_1 \|\Phi_0\|_D^2 \leq \int_0^T \int_\Omega c(x) |\psi_t|^2 dx dt \leq C_2 \|\Phi_0\|_D^2, \quad (3.4.58)$$

for all $\Phi_0 = (\psi_0, \psi_1, \phi_0, \phi_1) \in D$.

Proof: The direct inequality follows from the definition of the total energy for all $T > 0$. While the proof of the inverse inequality is a direct consequence of Proposition 2 of Haraux in [31] for which the exponential stability (3.4.1) implies the existence of a time $T_0 > 0$ such that for all $T > T_0$ there exist two constants $C_1 > 0$ and $C_2 > 0$ such that (3.4.58) holds. \square

It is well known that observability of the homogeneous system (3.4.52)-(3.4.54) implies the exact controllability of the associated system to (3.1.7)-(3.1.9).

3.5 Numerical approximation : Validation of the theoretical results

This section is devoted to the numerical approximation of the problem by a finite difference discretization and to the validation of the theoretical results stated in the previous sections. We will firstly construct in detail a discretization in the 1D case and we will define its corresponding discrete energy. Numerical experiments are performed to validate the theoretical results. In fact, the numerical results in 1D show an exponential stabilization in any case when $a = 1$ and a polynomial stabilization in any case in the case $a \neq 1$. They are better than expected.

3.5.1 Finite difference scheme in one dimensional space

We firstly introduce the finite difference scheme we will work on then we will construct the corresponding energy and finally we will perform numerical experiments. Let us firstly recall the problem we are considered.

Consider $\Omega = [0, 1]$. We are interested to study the controllability of the following coupled wave equations by velocities :

$$\begin{cases} u_{tt} - au_{xx} + b(x)y_t + c(x)u_t & = 0 & x \in (0, 1), t > 0 \\ y_{tt} - y_{xx} - b(x)u_t & = 0 & x \in (0, 1), t > 0 \\ u(0, t) = u(1, t) = y(0, t) = y(1, t) & = 0 & t > 0, \end{cases} \quad (3.5.1)$$

with the following initial data

$$u(x, 0) = u_0(x), \text{ and } y(x, 0) = y_0(x) \quad x \in (0, 1) \quad (3.5.2)$$

and

$$u_t(x, 0) = u_1(x) \text{ and } y_t(x, 0) = y_1(x), \quad x \in (0, 1) \quad (3.5.3)$$

where $a > 0$ constant, $b \in C^0([0, 1], \mathbb{R})$ and $c \in C^0([0, 1], \mathbb{R}^+)$. We will study the two cases $a = 1$ and $a \neq 1$.

Construction of the numerical scheme

Let N be a non negative integer. Consider the subdivision of $[0, 1]$ given by

$$0 = x_0 < x_1 < \dots < x_N < x_{N+1} = 1, \quad \text{i.e. } x_j = j\Delta x, \quad j = 0, \dots, N + 1.$$

Set $t^{n+1} - t^n = \Delta t$ for all $n \in \mathbb{N}$. For $j = 0, \dots, N + 1$, we denote $b_j = b(x_j)$, $c_j = c(x_j)$. The explicit finite-difference discretization of system (3.5.1) is thus, for $n \in \mathbb{N}$ and $j = 1, \dots, N$:

$$\left\{ \begin{array}{l} \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} - a \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + b_j \frac{y_j^{n+1} - y_j^{n-1}}{2\Delta t} + c_j \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = 0 \\ \frac{y_j^{n+1} - 2y_j^n + y_j^{n-1}}{\Delta t^2} - \frac{y_{j+1}^n - 2y_j^n + y_{j-1}^n}{\Delta x^2} - b_j \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} = 0 \\ u_0^n = u_{N+1}^n = 0 \\ y_0^n = y_{N+1}^n = 0 \end{array} \right. \quad (3.5.4)$$

According to the initial conditions given by equations (3.5.2), we have firstly : for $j = 1, \dots, N$,

$$u_j^0 = u_0(x_j) \quad (3.5.5)$$

$$y_j^0 = y_0(x_j). \quad (3.5.6)$$

We can use the second initial conditions (3.5.3) to find the values of u and y at time $t^1 = \Delta t$, by employing a “ghost” time-boundary (i.e. $t^{-1} = -\Delta t$) and the second-order central difference formula for $j = 1, \dots, N$:

$$u_1(x_j) = \frac{\partial u}{\partial t} \Big|_{x_j, 0} = \frac{u_j^1 - u_j^{-1}}{2\Delta t} + O(\Delta t^2). \quad (3.5.7)$$

Thus we have for $j = 1, \dots, N$:

$$u_j^{-1} = u_j^1 - 2\Delta t u_1(x_j). \quad (3.5.8)$$

We use the same discrete form of the initial conditions for y , for $j = 1, \dots, N$:

$$y_j^{-1} = y_j^1 - 2\Delta t y_1(x_j). \quad (3.5.9)$$

Setting $n = 0$, in the numerical scheme (3.5.4), the two previous equalities permits to compute $(u_j^1, y_j^1)_{j=0, N}$. Finally, the solution (u, y) can be computed at any time t^n .

Practical implementation and CFL condition

Let us denote $\lambda = \frac{\Delta t^2}{\Delta x^2}$. We easily remark that the discrete scheme (3.5.4) is composed of N linear systems of two equations which can be written under the form :

$$\text{for } j = 1, \dots, N, M_j \cdot \begin{pmatrix} u_j^{n+1} \\ y_j^{n+1} \end{pmatrix} = \begin{pmatrix} A_j \\ B_j \end{pmatrix} \quad (3.5.10)$$

where

$$M_j = \begin{pmatrix} 1 + \frac{c_j \Delta t}{2} & \frac{b_j \Delta t}{2} \\ \frac{-b_j \Delta t}{2} & 1 \end{pmatrix}$$

$$A_j = 2(1 - a\lambda)u_j^n + \left(\frac{c_j}{2}\Delta t - 1\right)u_j^{n-1} + a\lambda(u_{j+1}^n + u_{j-1}^n) + \frac{b_j}{2}\Delta t y_j^{n-1}$$

and

$$B_j = 2(1 - \lambda)y_j^n + \lambda(y_{j+1}^n + y_{j-1}^n) - y_j^{n-1} - \frac{b_j}{2}\Delta t u_j^{n-1}.$$

Thanks to the hypothesis $\forall x \in (0, 1), c(x) \geq 0$, for $j = 1, \dots, N$ the determinant of M_j given by

$$|M_j| = 1 + \frac{c_j \Delta t}{2} + \left(\frac{b_j \Delta t}{2}\right)^2,$$

is a strictly positive quantity.

Consequently, system (3.5.10) admits a unique solution given by : for $j = 1, \dots, N$,

$$u_j^{n+1} = (1 - a\lambda)\alpha_j u_j^n + \lambda\beta_j(u_{j+1}^n + u_{j-1}^n) + \gamma_j u_j^{n-1} - (1 - \lambda)\varrho_j y_j^n - \lambda\xi_j(y_{j+1}^n + y_{j-1}^n) + \kappa_j y_j^{n-1} \quad (3.5.11)$$

$$y_j^{n+1} = (1 - \lambda)\tilde{\alpha}_j y_j^n + \lambda\tilde{\beta}_j(y_{j+1}^n + y_{j-1}^n) + \tilde{\gamma}_j y_j^{n-1} + (1 - a\lambda)\tilde{\varrho}_j u_j^n + \lambda\tilde{\xi}_j(u_{j+1}^n + u_{j-1}^n) + \tilde{\kappa}_j u_j^{n-1} \quad (3.5.12)$$

where we have set :

$$\begin{aligned} \alpha_j &= \frac{2}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j \Delta t}{2}\right)^2}, & \beta_j &= \frac{a}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j \Delta t}{2}\right)^2}, \\ \gamma_j &= \frac{\frac{c_j}{2}\Delta t + \left(\frac{b_j \Delta t}{2}\right)^2 - 1}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j \Delta t}{2}\right)^2}, & \varrho_j &= \frac{b_j \Delta t}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j \Delta t}{2}\right)^2}, \\ \xi_j &= \frac{b_j \Delta t}{2 \left(1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j \Delta t}{2}\right)^2\right)}, & \kappa_j &= \frac{b_j \Delta t}{1 + \frac{c_j}{2}\Delta t + \left(\frac{b_j \Delta t}{2}\right)^2}, \end{aligned}$$

$$\begin{aligned}
 \tilde{\alpha}_j &= 2 - \frac{(b_j \Delta t)^2}{2 \left(1 + \frac{c_j}{2} \Delta t + \left(\frac{b_j \Delta t}{2} \right)^2 \right)} , & \tilde{\beta}_j &= 1 - \frac{(b_j \Delta t)^2}{4 \left(1 + \frac{c_j}{2} \Delta t + \left(\frac{b_j \Delta t}{2} \right)^2 \right)} , \\
 \tilde{\gamma}_j &= \frac{(b_j \Delta t)^2}{2 \left(1 + \frac{c_j}{2} \Delta t + \left(\frac{b_j \Delta t}{2} \right)^2 \right)} - 1 , & \tilde{\varrho}_j &= \frac{b_j \Delta t}{1 + \frac{c_j}{2} \Delta t + \left(\frac{b_j \Delta t}{2} \right)^2} , \\
 \tilde{\xi}_j &= \frac{ab_j \Delta t}{2 \left(1 + \frac{c_j}{2} \Delta t + \left(\frac{b_j \Delta t}{2} \right)^2 \right)} , & \tilde{\kappa}_j &= \left[\frac{\frac{c_j}{2} \Delta t + \left(\frac{b_j \Delta t}{2} \right)^2 - 1}{1 + \frac{c_j}{2} \Delta t + \left(\frac{b_j \Delta t}{2} \right)^2} - 1 \right] \frac{b_j \Delta t}{2} .
 \end{aligned}$$

The implementation of the numerical discretization of the problem (3.5.1) consists finally of equations (3.5.5), (3.5.6), (3.5.11), (3.5.12) where (u^{-1}, y^{-1}) used for $n = 0$, are defined by (3.5.8), (3.5.9).

By a standard Von Neumann stability analysis (that is a discrete Fourier analysis, see for instance [13]), the numerical scheme is stable if and only if, the following Courant-Friedrichs-Lewy, CFL, condition holds :

$$\Delta t^2 \leq \Delta x^2 \text{ and } a \Delta t^2 \leq \Delta x^2$$

which is equivalent to

$$\Delta t \leq \min \left(1, \frac{1}{\sqrt{a}} \right) \Delta x \quad . \quad (3.5.13)$$

The number $\min \left(1, \frac{1}{\sqrt{a}} \right)$ is called the CFL number and is denoted in the following by *CFL*.

Discrete energy : definition and dissipation.

The aim of this section is to design a discrete energy that might be preserved in the case $c = 0$ and to obtain the dissipation of the discrete energy in the case $c > 0$. To this end, let us define :

- the discrete kinetic energy for u as : $E_{k,u}^n = \frac{1}{2} \sum_{j=1}^N \left(\frac{u_j^{n+1} - u_j^n}{\Delta t} \right)^2$
- the discrete potential energy for u as : $E_{p,u}^n = \frac{a}{2} \sum_{j=0}^N \left(\frac{u_{j+1}^n - u_j^n}{\Delta x} \right) \left(\frac{u_{j+1}^{n+1} - u_j^{n+1}}{\Delta x} \right)$
- the discrete kinetic energy for y as : $E_{k,y}^n = \frac{1}{2} \sum_{j=1}^N \left(\frac{y_j^{n+1} - y_j^n}{\Delta t} \right)^2$
- the discrete potential energy for y as : $E_{p,y}^n = \frac{1}{2} \sum_{j=0}^N \left(\frac{y_{j+1}^n - y_j^n}{\Delta x} \right) \left(\frac{y_{j+1}^{n+1} - y_j^{n+1}}{\Delta x} \right)$

The total discrete energy is then defined as

$$\mathcal{E}^n = E_{k,u}^n + E_{p,u}^n + E_{k,y}^n + E_{p,u}^n. \quad (3.5.14)$$

Let us prove now that this definition of the energy fulfill the two properties stated above. For this sake, we multiply the first equation of (3.5.4) by $(u_j^{n+1} - u_j^{n-1})$ and we sum over $j = 1, \dots, N$. We obtain :

$$\begin{aligned} & \sum_{j=1}^N \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} (u_j^{n+1} - u_j^{n-1}) - a \sum_{j=1}^N \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} (u_j^{n+1} - u_j^{n-1}) \\ & + \sum_{j=1}^N b_j \frac{y_j^{n+1} - y_j^{n-1}}{2\Delta t} (u_j^{n+1} - u_j^{n-1}) + \sum_{j=1}^N c_j \frac{(u_j^{n+1} - u_j^{n-1})^2}{2\Delta t} = 0. \end{aligned} \quad (3.5.15)$$

Estimation of the first term of (3.5.15) We firstly have :

$$\begin{aligned} \sum_{j=1}^N \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} (u_j^{n+1} - u_j^{n-1}) &= \sum_{j=1}^N \frac{u_j^{n+1} - u_j^n - (u_j^n - u_j^{n-1})}{\Delta t^2} (u_j^{n+1} - u_j^n + u_j^n - u_j^{n-1}) \\ &= \sum_{j=1}^N \left(\frac{u_j^{n+1} - u_j^n}{\Delta t} \right)^2 - \sum_{j=1}^N \left(\frac{u_j^{n+1} - u_j^{n-1}}{\Delta t} \right)^2 \\ &= 2(E_{k,u}^n - E_{k,u}^{n-1}). \end{aligned} \quad (3.5.16)$$

Estimation of the second term of (3.5.15). Using the same trick we have :

$$\begin{aligned} -a \sum_{j=1}^N \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} (u_j^{n+1} - u_j^{n-1}) &= -a \sum_{j=1}^N \frac{u_{j+1}^n - u_j^n - (u_j^n - u_{j-1}^n)}{\Delta x^2} (u_j^{n+1} - u_j^{n-1}) \\ &= -a \sum_{j=1}^N \frac{(u_{j+1}^n - u_j^n)(u_j^{n+1} - u_j^{n-1})}{\Delta x^2} \\ &+ a \sum_{j=1}^{N+1} \frac{(u_j^n - u_{j-1}^n)(u_j^{n+1} - u_j^{n-1})}{\Delta x^2}. \end{aligned}$$

So, by translation of index in the second term in the previous sum, we will have :

$$\begin{aligned}
 -a \sum_{j=1}^N \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} (u_j^{n+1} - u_j^{n-1}) &= -a \sum_{j=0}^N \frac{(u_{j+1}^n - u_j^n)(u_j^{n+1} - u_j^{n-1})}{\Delta x^2} \\
 &+ a \sum_{j=0}^N \frac{(u_{j+1}^n - u_j^n)(u_{j+1}^{n+1} - u_{j+1}^{n-1})}{\Delta x^2} \\
 &= a \sum_{j=0}^N \frac{(u_{j+1}^{n+1} - u_j^{n+1})(u_{j+1}^n - u_j^n)}{\Delta x^2} \\
 &- a \sum_{j=0}^N \frac{(u_{j+1}^{n-1} - u_j^{n-1})(u_{j+1}^n - u_j^n)}{\Delta x^2} \\
 &= 2(E_{p,u}^n - E_{p,u}^{n-1}). \tag{3.5.17}
 \end{aligned}$$

Substituting (3.5.16) and (3.5.17) into (3.5.15), we get

$$2(E_{k,u}^n + E_{p,u}^n - E_{k,u}^{n-1} - E_{p,u}^{n-1}) + 2\Delta t \sum_{j=1}^N c_j \left(\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} \right)^2 + \sum_{j=1}^N b_j \frac{y_j^{n+1} - y_j^{n-1}}{2\Delta t} (u_j^{n+1} - u_j^{n-1}) = 0. \tag{3.5.18}$$

Similarly, by multiplying the second equation of (3.5.4) by $(y_j^{n+1} - y_j^{n-1})$, and using the same algebraic tricks, we will get :

$$2(E_{k,y}^n + E_{p,y}^n - E_{k,y}^{n-1} - E_{p,y}^{n-1}) - \sum_{j=1}^N b_j \frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} (y_j^{n+1} - y_j^{n-1}) = 0. \tag{3.5.19}$$

Using the definition of the total discrete energy, (3.5.14), and the two equations (3.5.18), (3.5.19) leads to :

$$(\mathcal{E}^n - \mathcal{E}^{n-1}) + \Delta t \sum_{j=1}^N c_j \left(\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} \right)^2 = 0. \tag{3.5.20}$$

Consequently, the total discrete energy of system (3.5.4) is decreasing along time.

3.5.2 Numerical experiments : validation of the theoretical results

In every experiments, we have chosen :

$$u_0(x) = x(x-1), \quad u_1(x) = x(x-1), \quad y_0(x) = -x(x-1), \quad y_1(x) = -x(x-1).$$

The mesh size is chosen as $N = 100$ so that $\Delta x = 0.01$ and the time step is chosen as $\frac{\Delta t}{\Delta x} = CFL$.

In order to validate the different theoretical results, we have chosen different functions b and c synthesized in the list below :

- No coupling : $b_1(x) = 0$ or no dissipation $c_1(x) = 0$,
- Full coupling $b_2(x) = 1 \mathbb{1}_{(0,1)}(x)$ or full dissipation $c_2(x) = 1 \mathbb{1}_{(0,1)}(x)$,
- Partial coupling $b_3(x) = \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$ or partial dissipation $c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$,
- Partial coupling $b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ or partial dissipation $c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$,
- Partial coupling $b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ or partial dissipation $c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

Combining the different choices of the coupling and damping functions in order to have or not $\omega_b \cap \omega_{c_+} \neq \emptyset$ will permit us to validate the theoretical results.

Let us notice that in the special case of the dimension 1, the geometric control conditions GCC holds as soon as $\omega_{c_+} \neq \emptyset$.

Same propagation speed : $a = 1$

For every numerical simulation, the final time T is chosen as $T = 500$.

No damping : conservation of the total energy Firstly, let us verify that when no damping are present, the discrete energy is conserved. We present in figure 3.2 the numerical experiment when $c = c_1 = 0$ and $b = b_3 = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$. Indeed, the total energy is conserved along time.

Remark 3.5.1. *This numerical test where no damping is applied shows that without a damping term, the total energy is completely conserved. This fact suggests that the numerical scheme does not produce numerical dissipation. So the numerical behavior observed thereafter is only due to the considered model.* \square

$\omega_b \cap \omega_{c_+} \neq \emptyset$. **Exponential stability.** Let us now verify the theoretical results when we suppose that $\omega_b \cap \omega_{c_+} \neq \emptyset$. For this sake, we present in figure 3.3, the total energy and the quantity $-\ln(E(t))/t$ versus time t for large time, where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$. This choice verifies the assumption that $\omega_b \cap \omega_{c_+} \neq \emptyset$ and in figure 3.3, it is shown that the energy is decreasing and an exponential decay is observed since it seems that $-\ln(E(t))/t$ tends to a constant as $t \rightarrow +\infty$. The final time profile confirms that u and y are small and the final profiles of u and y are smooth as expected (high frequency oscillations are exponentially dissipated).

$\omega_b \cap \omega_{c_+} = \emptyset$. **Unpredicted behavior.** At the numerical level, we are interested in the long time behavior of the solution (u, y) when we suppose that $\omega_b \cap \omega_{c_+} = \emptyset$. For this sake, we present in figure 3.4, the total energy and the quantity $-\ln(E(t))/t$ versus time t for large time, where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$. This choice verifies the assumption that $\omega_b \cap \omega_{c_+} = \emptyset$. In figure 3.4, it is shown that the energy is decreasing and an exponential decay is observed since it seems that $-\ln(E(t))/t$ tends to a constant as $t \rightarrow +\infty$. The final time profile confirms that u and y are small and again the couple of solution (u, y) is smooth .

This result is surprising since it is not predicted by the theoretical results.

So we decided to confirm this strange behavior by choosing $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$. This choice verifies also the assumption that $\omega_b \cap \omega_{c_+} = \emptyset$. In figure 3.5, it is shown that the energy is decreasing and an exponential decay is observed since it seems that $-\ln(E(t))/t$ tends to a constant as $t \rightarrow +\infty$. The final time profile confirms that u and y are small and again the couple of solution (u, y) is smooth.

Remark 3.5.2. *Let us notice that when the propagation speeds are the same for u and y , the final profiles of the solution u , y presented in figure 3.3(c), figure 3.4(c) and in figure 3.5(c) have the same form as the initial one, that is no spurious oscillations due to high frequency are present.* \square

Different propagation speed : $a = 2$

We investigate now the long time behavior of (u, y) when the propagation speeds are different. We firstly investigate the case when the propagation speed for u is greater than the one of y namely $a > 1$. We have chosen $a = 2$.

$\omega_b \cap \omega_{c_+} \neq \emptyset$. **Polynomial stability** Let us now verify the theoretical results when we suppose that $\omega_b \cap \omega_{c_+} \neq \emptyset$. For this sake, we present in figure 3.6, the total energy where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$. When taking as final time $T = 500$, it seems that the energy does not tend to zero as shown in figure 3.6(a). This is the reason why we have chosen for the case when $a \neq 1$ take as final time $T = 500\,000$ and figure 3.6(b) shows that the energy suddenly goes to zero.

To explore the speed of convergence to zero, we have plotted in figure 3.7 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 3.7(a) shows clearly that $-\ln(E(t))/t$ tends to zero and it permits to conclude that $E(t)$ tends to zero slower than an exponential. Figure 3.7(b) permits to conclude that $E(t)$ tends to zero faster than $1/t$. Finally figure 3.7(c) shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 1.4$.

The final time profile presented in figure 3.7(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.

$\omega_b \cap \omega_{c_+} = \emptyset$. **Unpredicted behavior** At the numerical level, we are interested in the long time behavior of the solution (u, y) when we suppose that $\omega_b \cap \omega_{c_+} = \emptyset$. For this sake, we present in figure 3.8, the total energy where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

Again, when taking as final time $T = 500$, it seems that the energy does not tend to zero as shown in figure 3.8(a). Taking as final time $T = 500\,000$, figure 3.8(b) shows that the energy suddenly goes to zero.

To explore the speed of convergence to zero, we have plotted in figure 3.9 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 3.9(a) shows clearly that $-\ln(E(t))/t$ tends to zero and it permits to conclude that $E(t)$ tends to zero slower than an exponential but figure 3.9(b) shows that $E(t)$ tends to zero slower than $1/t$. This fact is confirmed by figure 3.9(c) which shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 0.9$. Eventually, taking

a larger time could conclude that the convergence is like $1/t$.

Again, the final time profile presented in figure 3.9(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.

As for the case when the two propagation speed were identical this results is surprising since it is not predicted by the theoretical results.

So we decided to confirm this strange behavior by choosing $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$. Again, when taking as final time $T = 500$, it seems that the energy does not tends to zero as shown in figure 3.10(a). Taking as final time $T = 500\,000$, figure 3.8(b) shows that the energy suddenly goes to zero.

To explore the speed of convergence to zero, we have plotted in figure 3.11 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 3.11(a) shows clearly that $-\ln(E(t))/t$ tends to zero and it permits to conclude that $E(t)$ tends to zero slower than an exponential and figure 3.11(b) permits to conclude that the convergence is faster than $1/t$. Finally figure 3.11(c) shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 1.19$.

Remark 3.5.3. *The final time profile presented in figure 3.7(d), figure 3.9(d) and figure 3.11(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.* \square

Different propagation speed : $a = 0.5$

When $a \neq 1$, in order to see if the same behavior occurs no matter if a is greater or less than 1, we investigate now the long time behavior of (u, y) when the propagation speeds is less than the one of y namely $a > 1$. We have chosen $a = 0.5$.

$\omega_b \cap \omega_{c_+} \neq \emptyset$. **Polynomial stability** Let us now verify the theoretical results when we suppose that $\omega_b \cap \omega_{c_+} \neq \emptyset$. For this sake, we present in figure 3.12(a), the total energy where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.

When taking as final time $T = 500$, it seems that the energy does not tend to zero as shown in figure 3.12(a). Taking as final time $T = 500\,000$, figure 3.12(b) shows that the energy suddenly goes to zero.

To explore the speed of convergence to zero, we have plotted in figure 3.13 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 3.13(a) shows clearly that $-\ln(E(t))/t$ tends to zero slower than an exponential. Figure 3.13(b) permits to conclude that $E(t)$ tends to zero faster than $1/t$. Finally figure 3.13(c) shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 1.5$.

The final time profile confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.

$\omega_b \cap \omega_{c_+} = \emptyset$: **Unpredicted behavior** Again, the numerical level, we are interested in the long time behavior of the solution (u, y) when we suppose that $\omega_b \cap \omega_{c_+} = \emptyset$. For this sake, we present in figure 3.14(a), the total energy where we have chosen $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

Again, when taking as final time $T = 500$, it seems that the energy does not tend to zero as shown in figure 3.14(a). Taking as final time $T = 500\,000$, figure 3.14(b) shows that the energy suddenly goes to zero.

To explore the speed of convergence to zero, we have plotted in figure 3.15 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 3.15(a) shows clearly that $-\ln(E(t))/t$ tends to zero slower than an exponential. But figure 3.15(b) shows that $E(t)$ tends to zero faster than $1/t$. Finally figure 3.15(c) shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 1.25$. Again, the final time profile presented in figure 3.15(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled. This result is surprising since it is not predicted by the theoretical results.

So we decided to confirm this strange behavior by choosing $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$. Again, when taking as final time $T = 500$, it seems that the energy does not tend to zero as shown in figure 3.16(a). Taking as final time $T = 500\,000$, figure 3.16(b) shows that the energy suddenly goes to zero.

To explore the speed of convergence to zero, we have plotted in figure 3.17 $-\ln(E(t))/t$, $t \cdot E(t)$ and finally $-\ln(E(t))/\ln(t)$ versus t . Figure 3.17(a) shows clearly that $-\ln(E(t))/t$ tends to zero and it permits to conclude that $E(t)$ tends to zero slower than an exponential but figure 3.17(b) shows that $E(t)$ tends to zero faster than $1/t$. Finally figure 3.17(c) shows that $E(t)$ tends to zero as $1/t^\alpha$ with $\alpha \simeq 1.15$.

Again, the final time profile presented in figure 3.17(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.

Remark 3.5.4. *The final time profile presented in figure 3.13(d), figure 3.15(d) and figure 3.17(d) confirms that u and y are small but it shows also that high frequencies for the unknown y are not completely controlled.* \square

Remark 3.5.5. *When the propagation speeds are not equal, the solution (u, y) has the same behavior no matter if $a > 1$ or $a < 1$. The polynomial convergence is numerically better than $1/t$ but it will be probably be $1/t$ for greater time. For reason of computation time, we did not perform very long simulation to confirm.* \square

3.6 General conclusion

In this work, we have obtain theoretical results for waves equations coupled by an order one term and a dissipation term of order one. We have completely investigate the two cases when the propagation speed are the same and when they are different. In the case when they are the same, we have obtain theoretical results no matter the size of the damping and the coupling term. This result is up to our knowledge new. Moreover when the propagation speeds are different, we have investigate the convergence behavior and proved a polynomial decay.

Numerical simulations in the 1D case confirm the theoretical results and are even better since when the intersection between the support of the coupling and damping are empty, the same long time behavior is observed. This fact will be investigated in the 2D dimension.

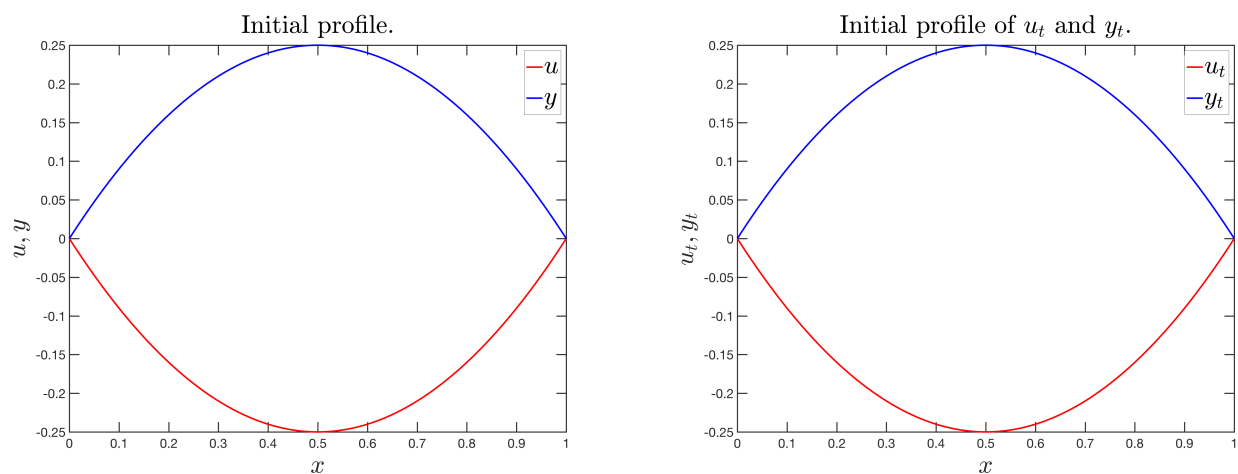
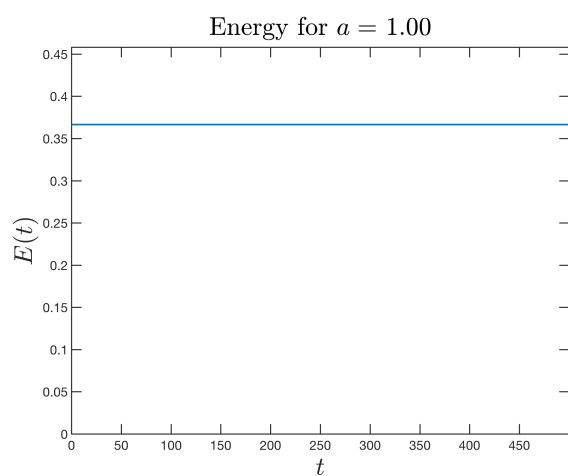


FIGURE 3.1 – Initial profiles

FIGURE 3.2 – No damping : $c = c_1 = 0$, partial coupling $b = b_3 = 1 \mathbf{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$

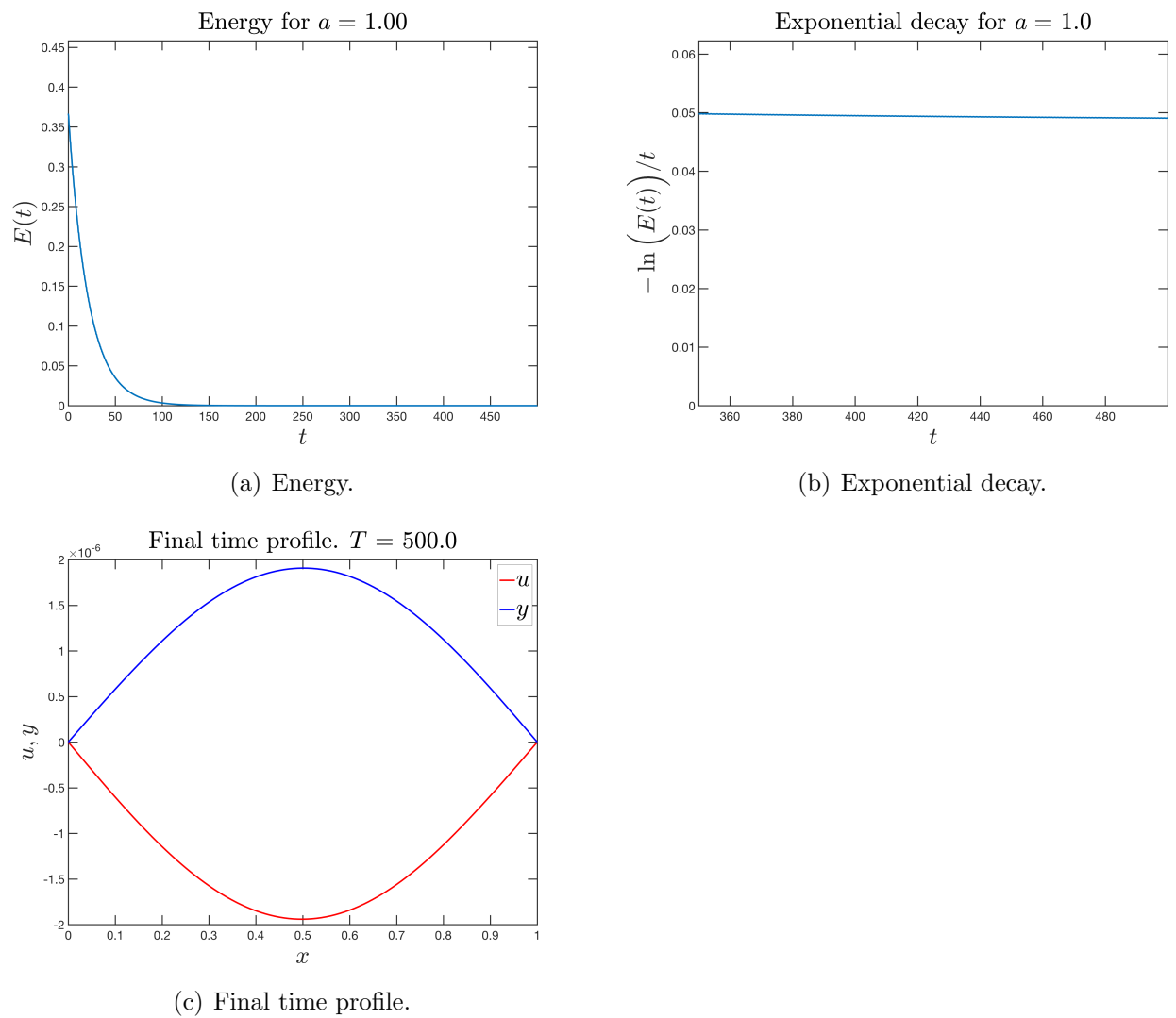


FIGURE 3.3 – Long time behavior when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.

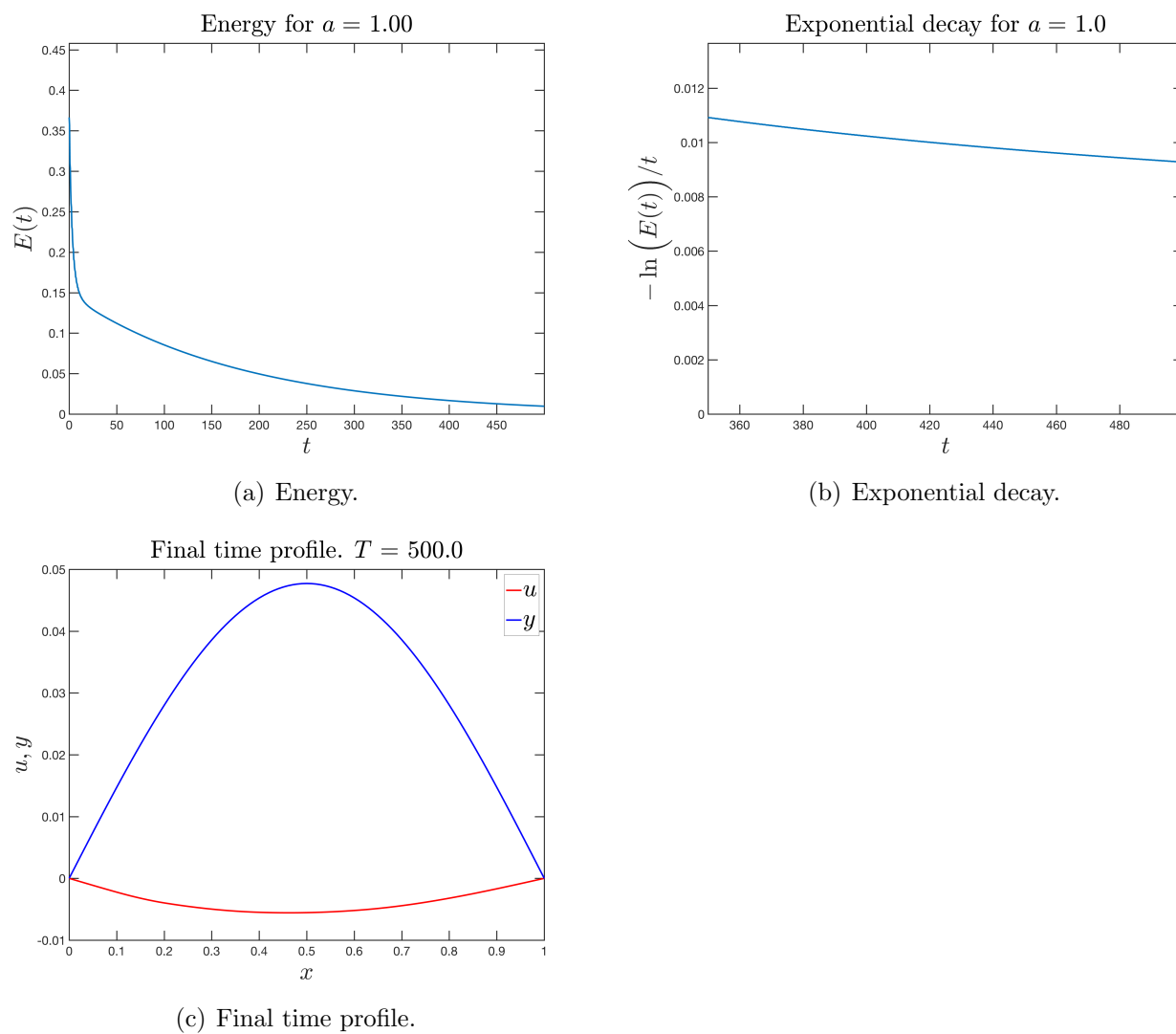


FIGURE 3.4 – Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

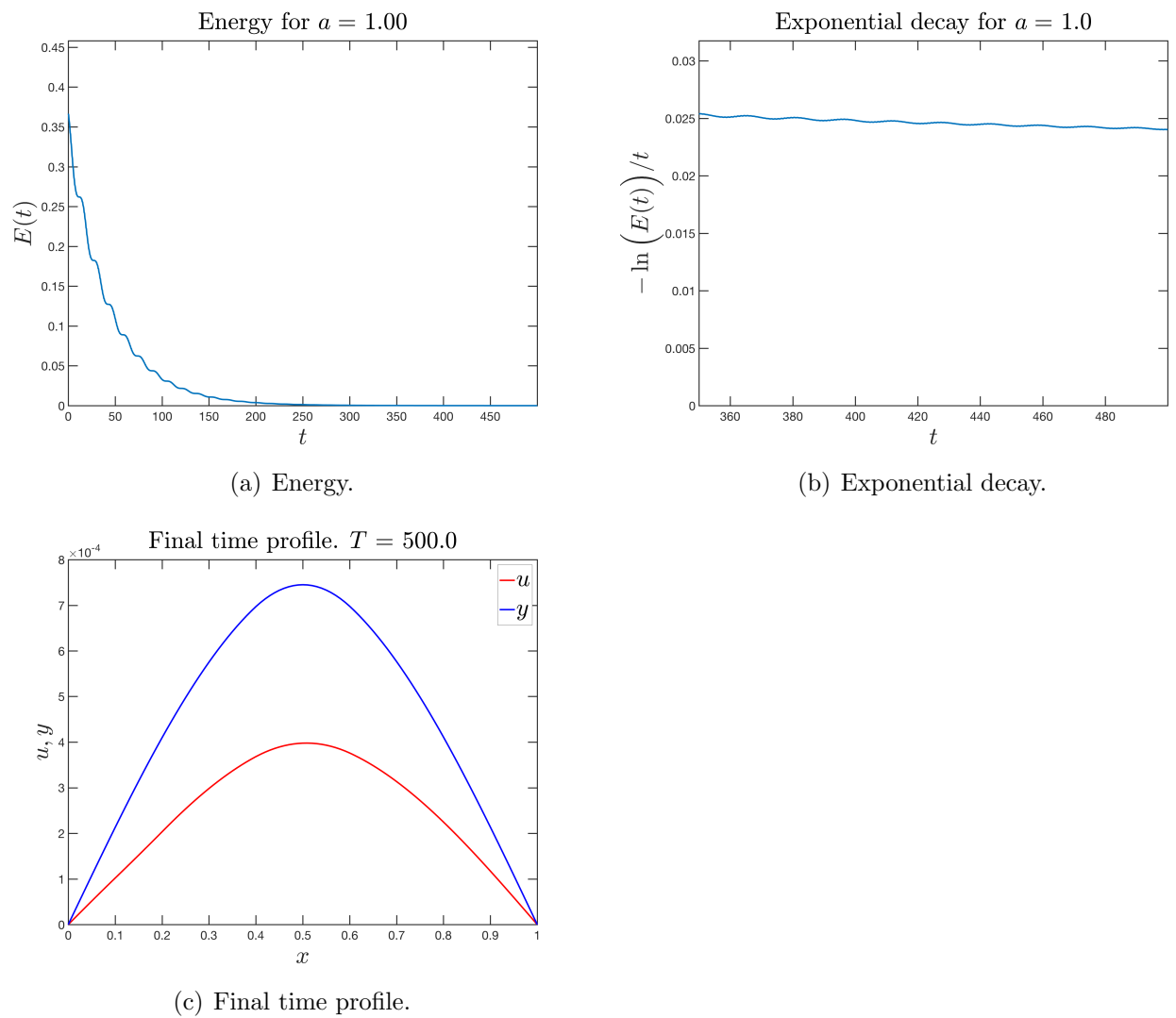


FIGURE 3.5 – Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$.

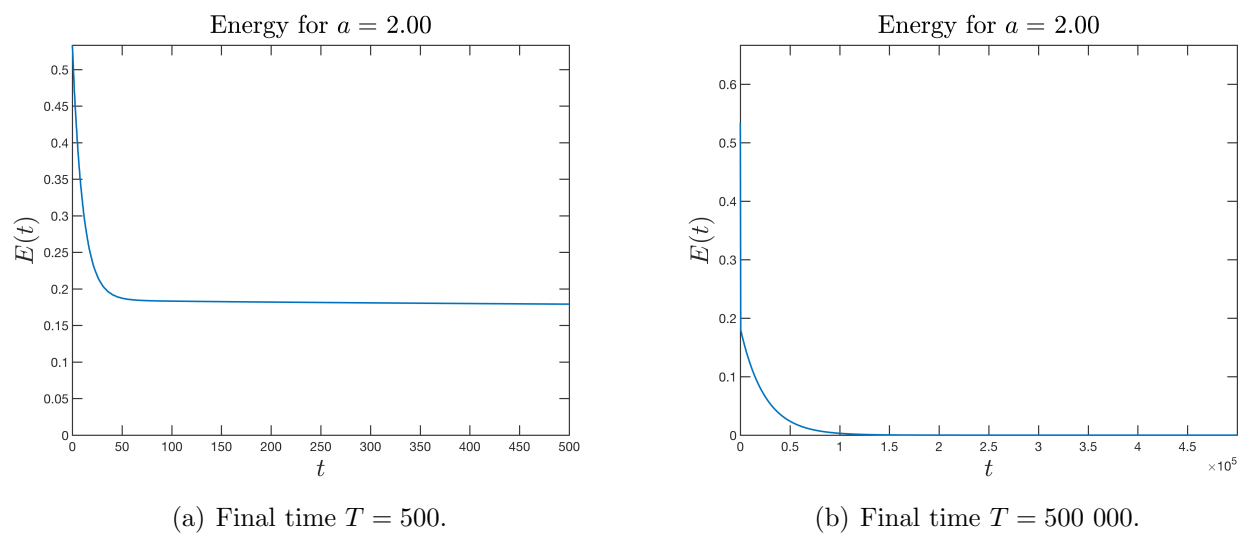
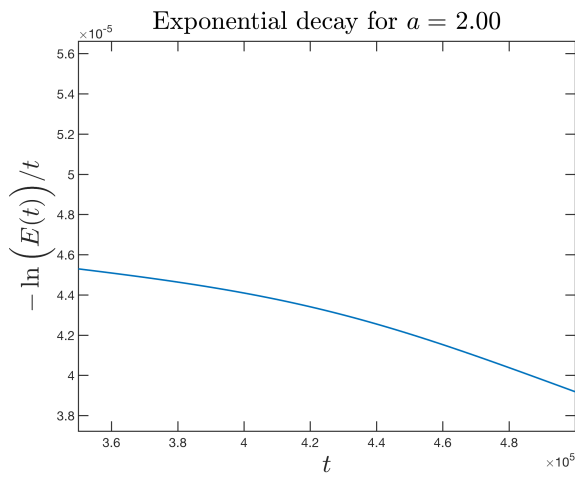
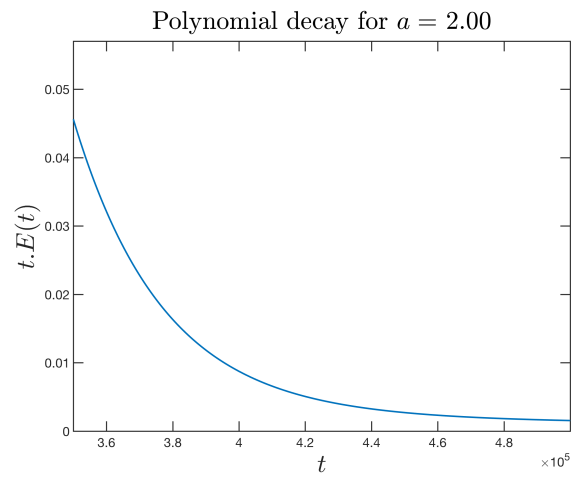


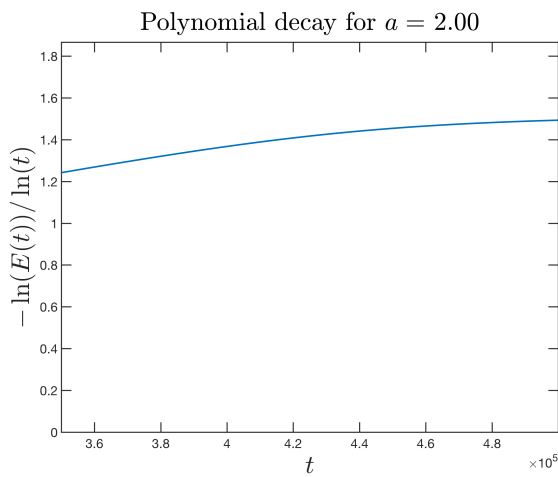
FIGURE 3.6 – Energy when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.



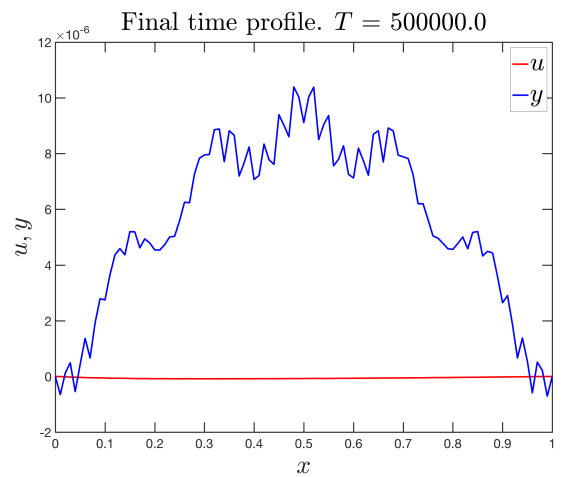
(a) Exponential decay?



(b) Polynomial decay in 1/t?



(c) Which exponent if polynomial decay?



(d) Final time profile.

FIGURE 3.7 – Long time behavior when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.

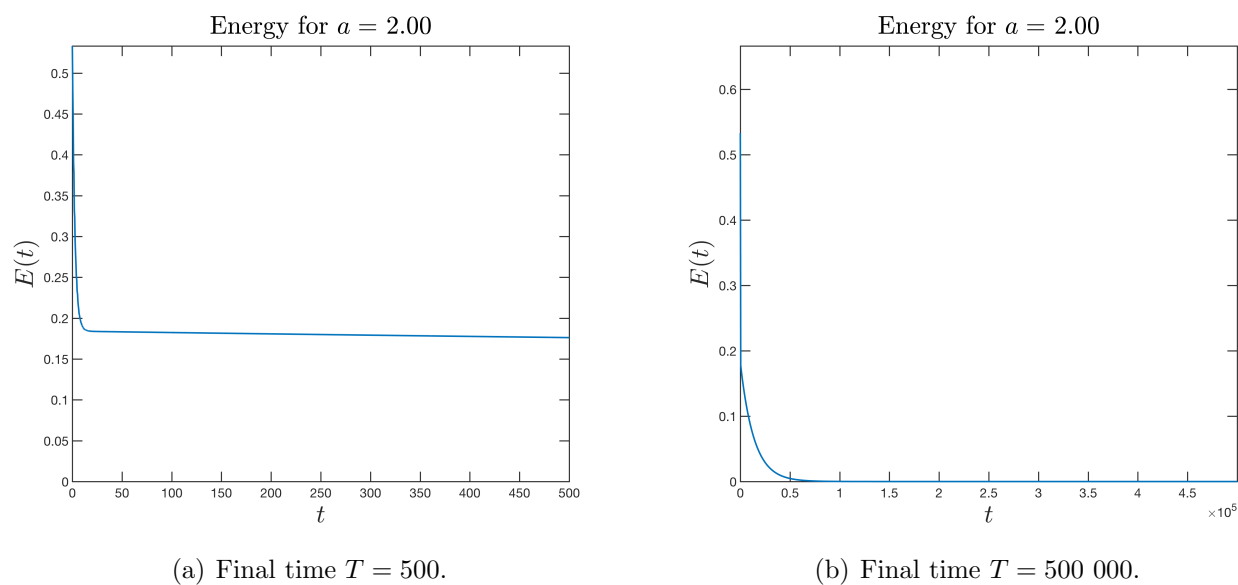
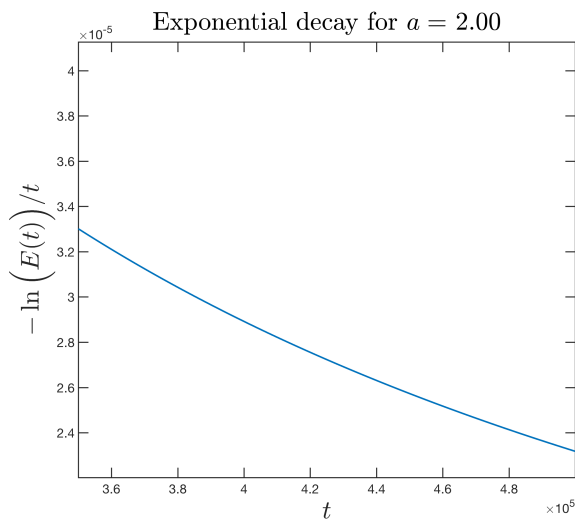
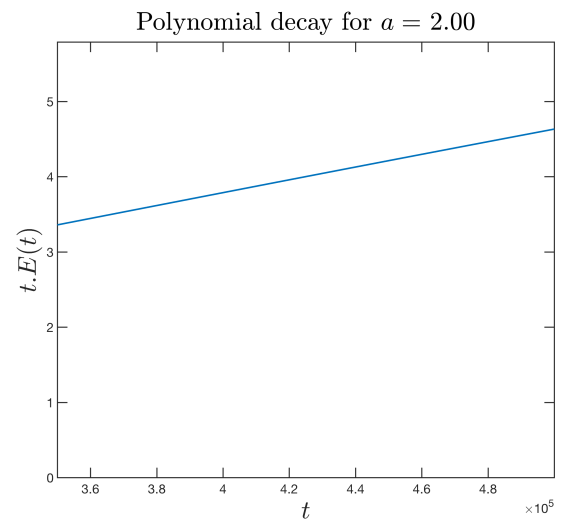


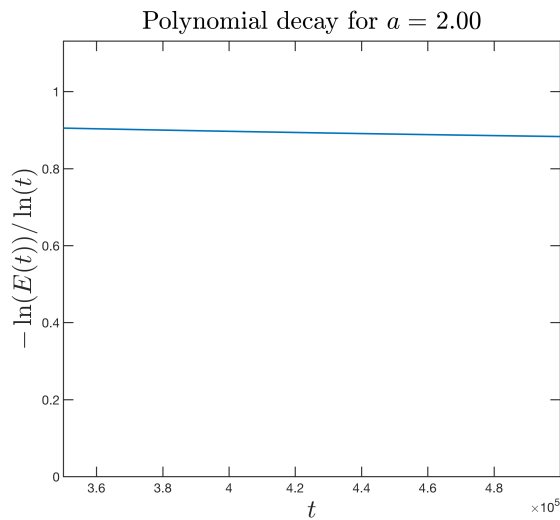
FIGURE 3.8 – Energy when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.



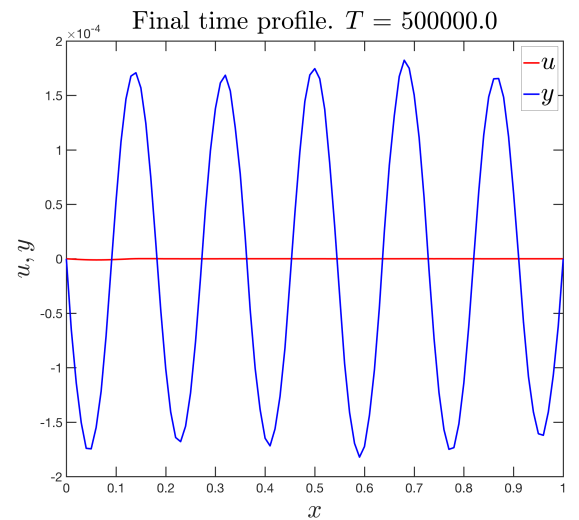
(a) Exponential decay ?



(b) Polynomial decay in 1/t ?



(c) Which exponent if polynomial decay ?



(d) Final time profile.

FIGURE 3.9 – Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

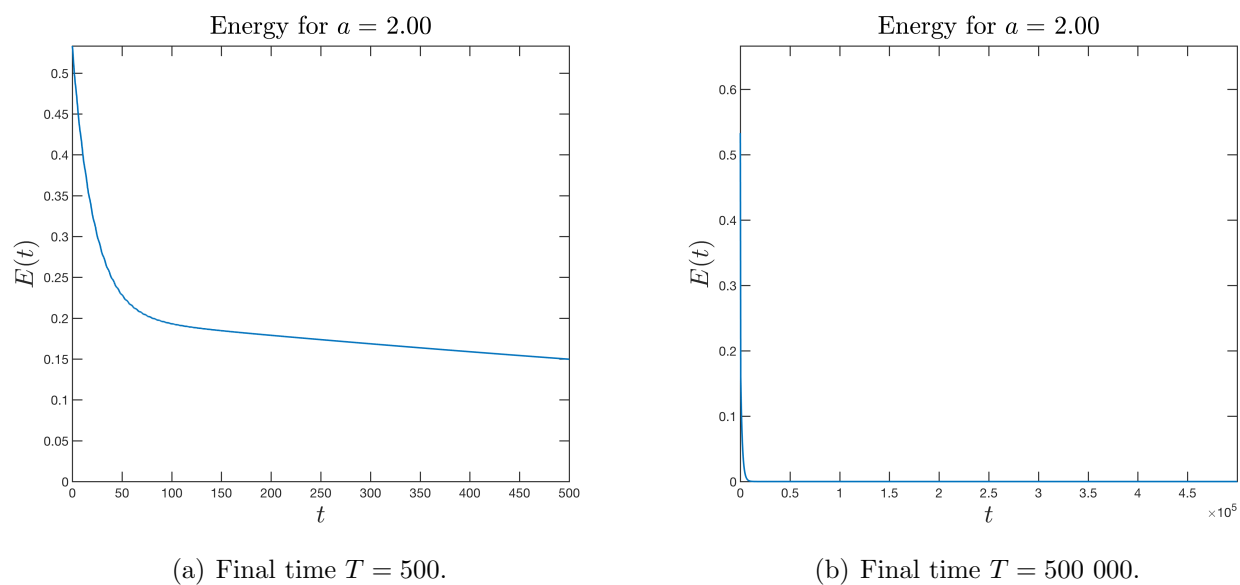
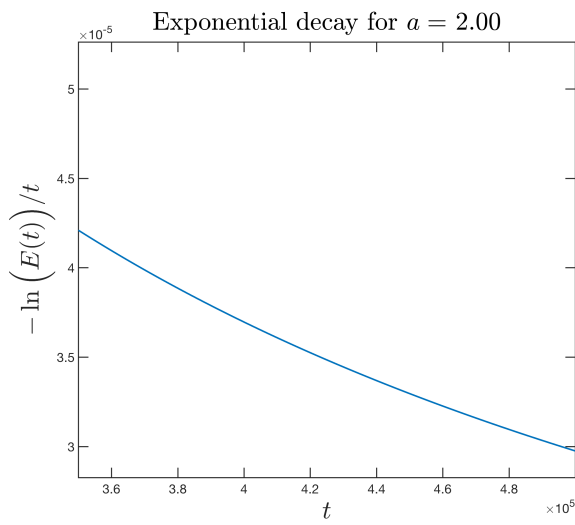
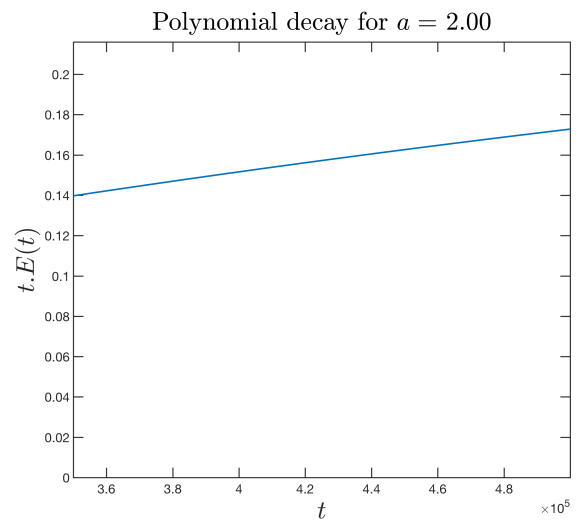


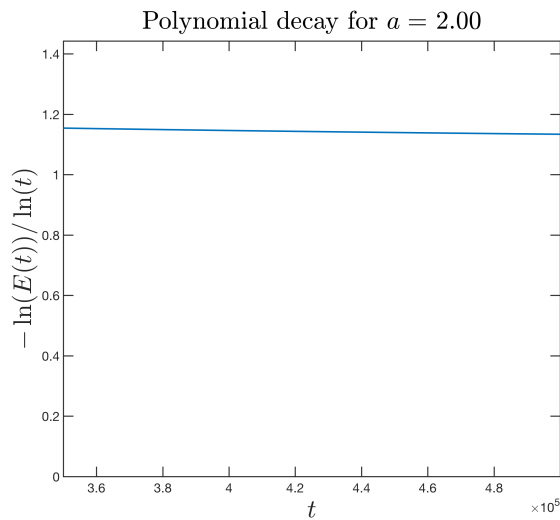
FIGURE 3.10 – Energy when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$.



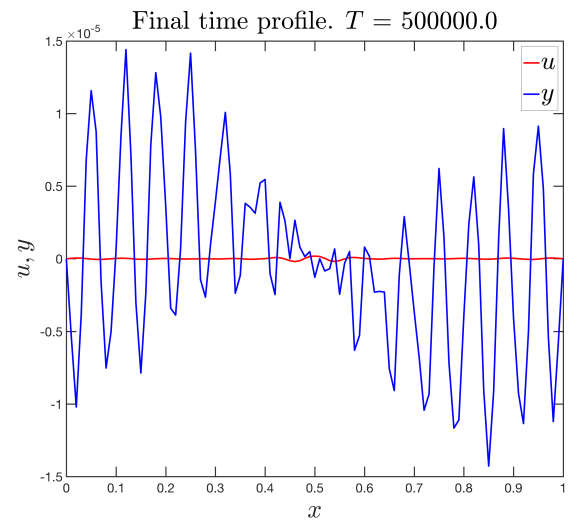
(a) Exponential decay?



(b) Polynomial decay in 1/t?



(c) Which exponent if polynomial decay?



(d) Final time profile.

FIGURE 3.11 – Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$.

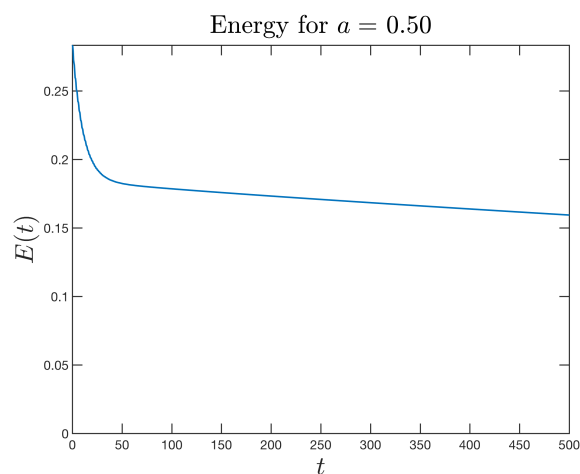
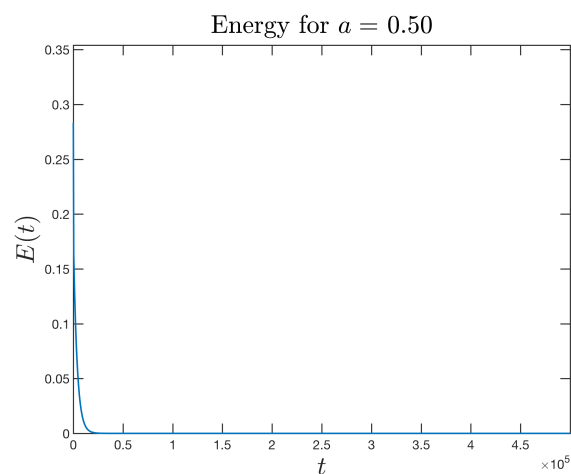
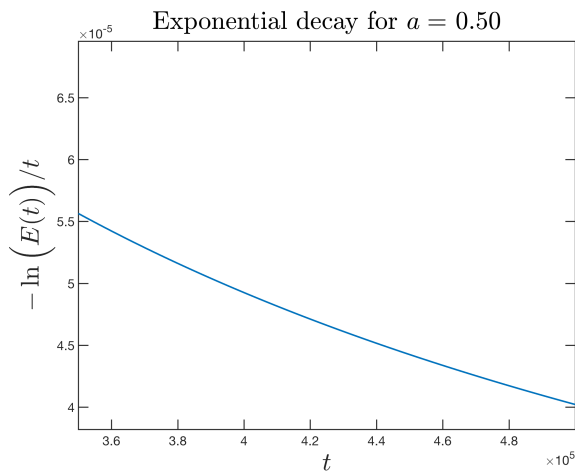
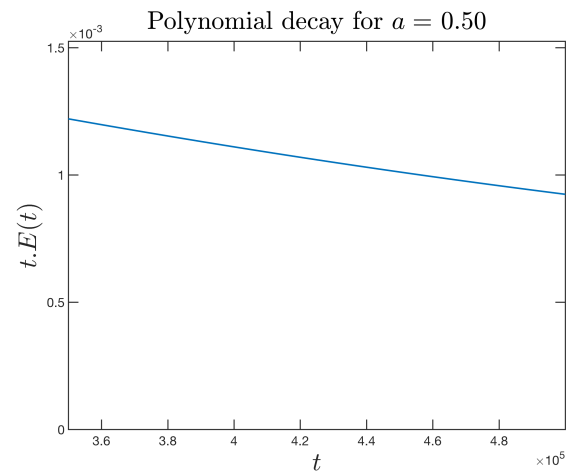
(a) Final time $T = 500$.(b) Final time $T = 500\,000$.

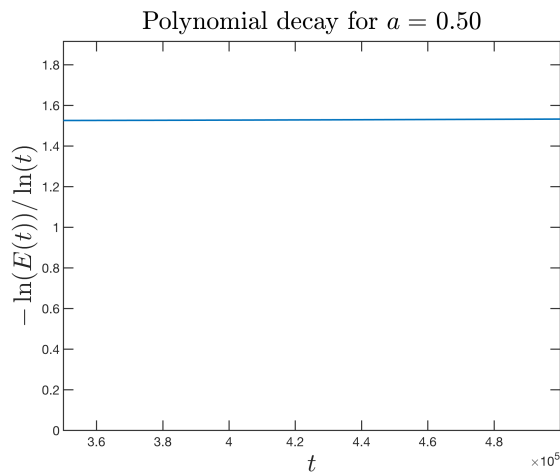
FIGURE 3.12 – Energy when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.



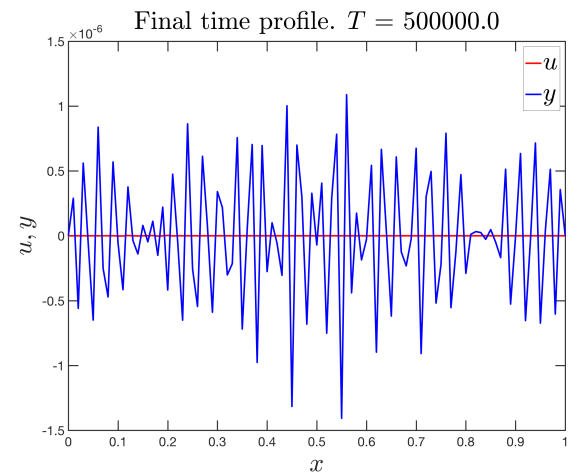
(a) Exponential decay?



(b) Polynomial decay in 1/t?



(c) Which exponent if polynomial decay?



(d) Final time profile.

FIGURE 3.13 – Long time behavior when $\omega_b \cap \omega_{c_+} \neq \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_3(x) = 1 \mathbb{1}_{[0.1,0.2] \cup [0.8,0.9]}(x)$.

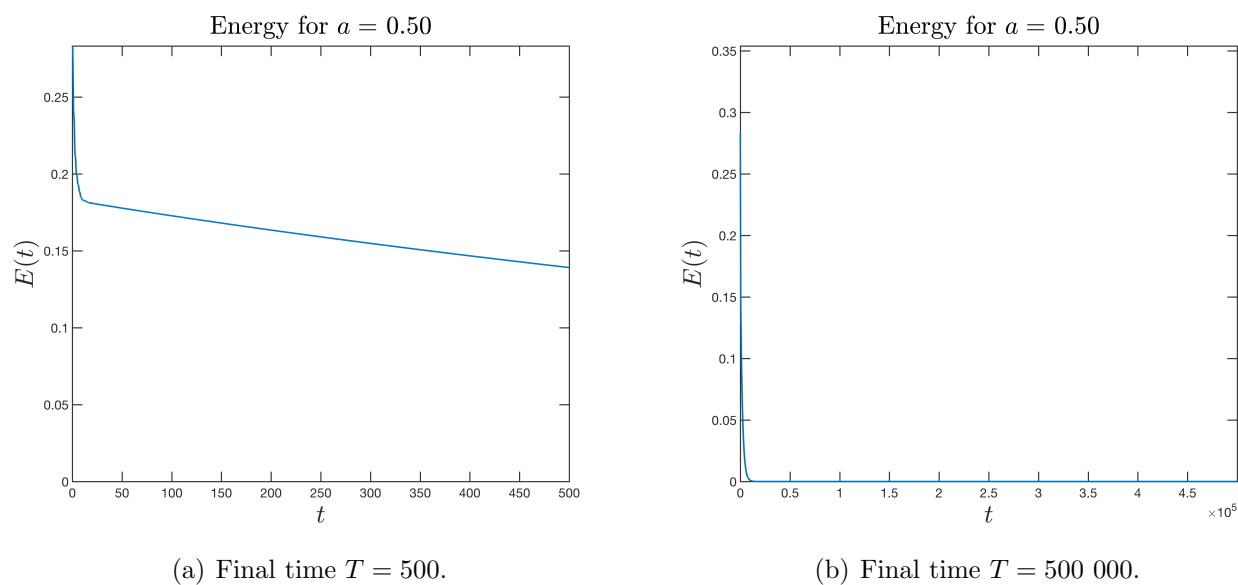


FIGURE 3.14 – Energy when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

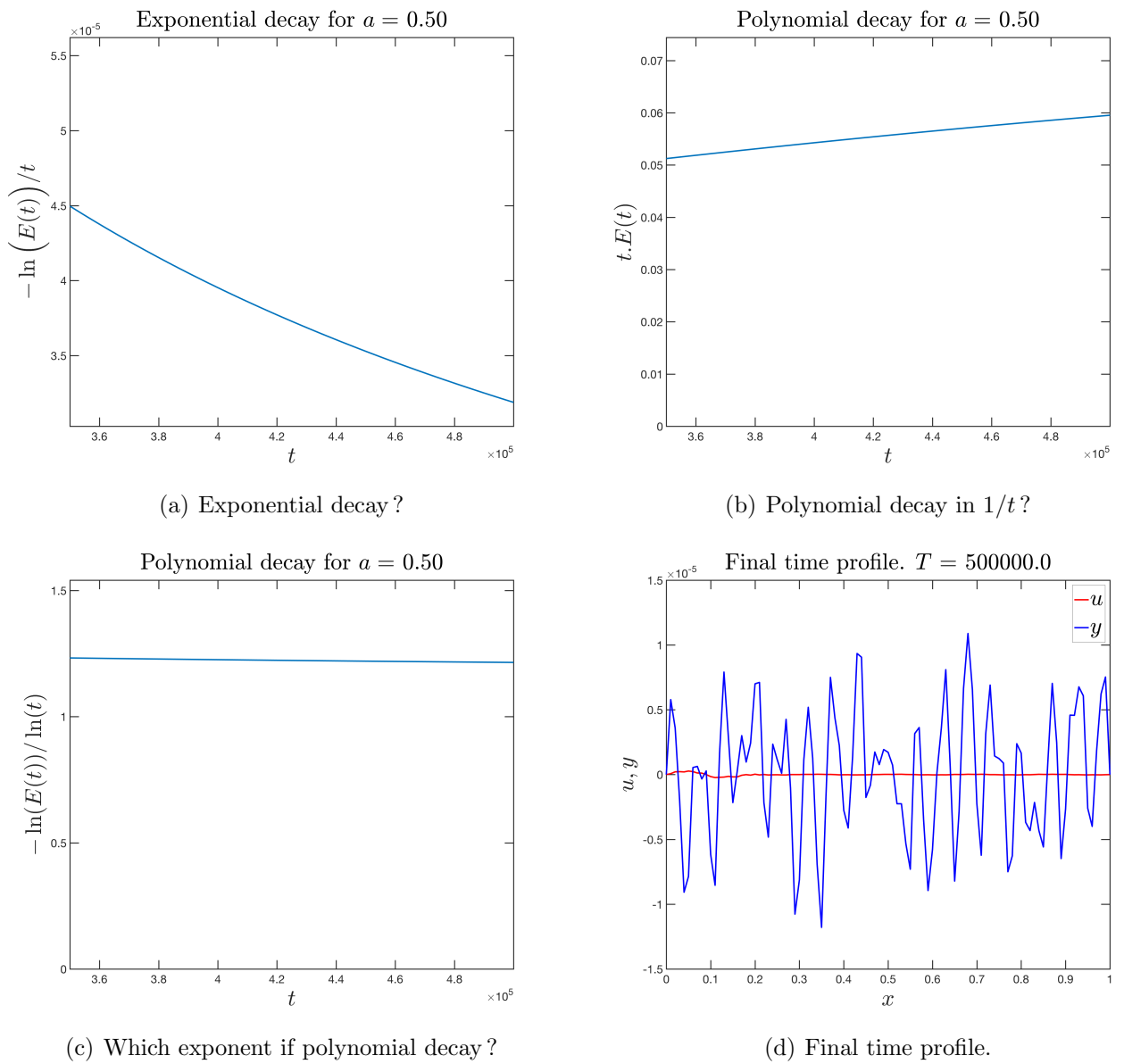


FIGURE 3.15 – Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$.

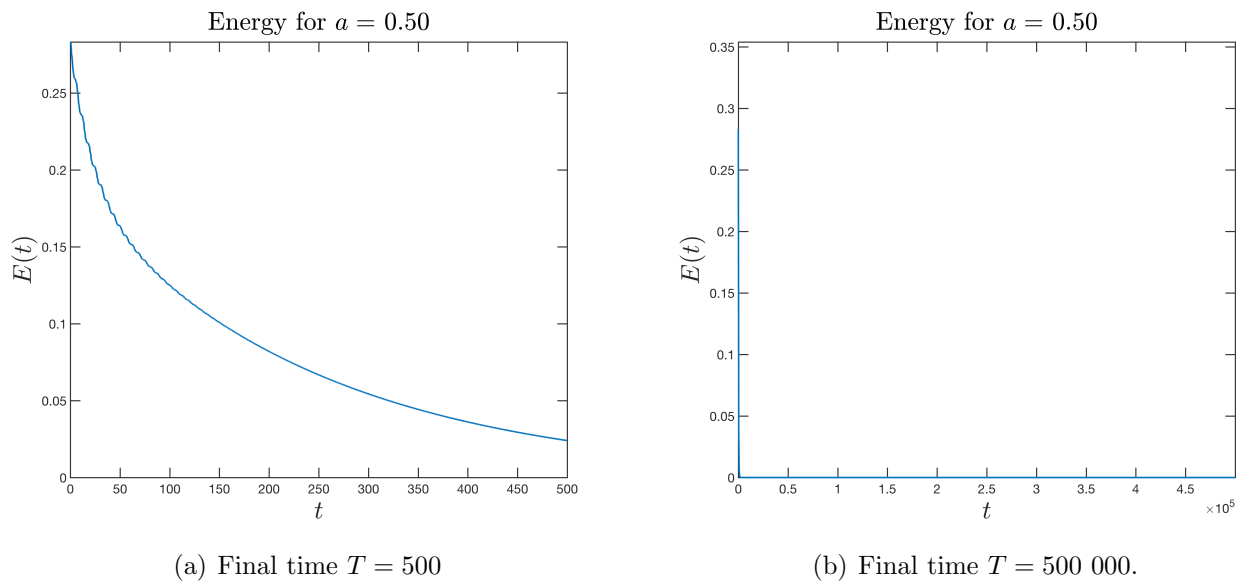


FIGURE 3.16 – Energy when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_5(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$.

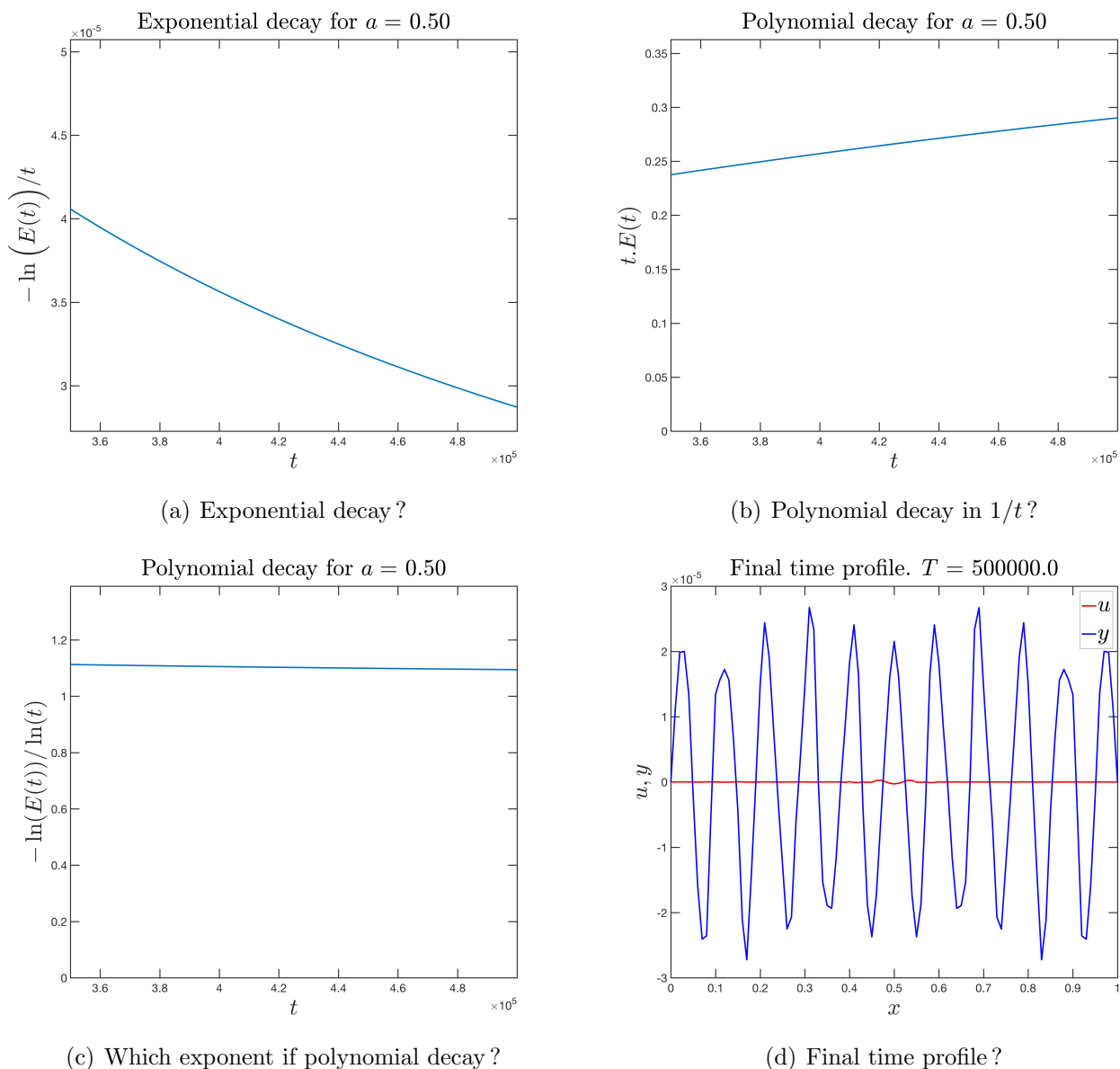


FIGURE 3.17 – Long time behavior when $\omega_b \cap \omega_{c_+} = \emptyset$. $b = b_5(x) = 1 \mathbb{1}_{[0.4,0.6]}(x)$ and $c = c_4(x) = 1 \mathbb{1}_{[0.1,0.2]}(x)$.

Chapitre 4

Stability of a Bresse system with local Kelvin-Voigt damping and non-smooth coefficient at interface

Abstract : In this chapter, we study the stability of an elastic Bresse system with local Kelvin-Voigt damping with non-smooth coefficient at interface and with fully Dirichlet or Dirichlet-Neumann-Neumann boundary conditions. The physical model consists of three waves equations damped by three global or local Kelvin-Voigt dampings. In the case of three dampings globally distributed, we prove an analytic stability. While in the important case when the three dampings are locally distributed, we prove an exponential decay or a polynomial decay of type $\frac{1}{t}$ (following the properties of the coefficients). On the other hand, when the waves are only subjected to one or two global dampings and under Dirichlet-Neumann-Neumann boundary conditions, non exponential decay is proved. Then a polynomial one of type $\frac{1}{\sqrt{t}}$ is obtained in the case of one local distributed dissipative. The frequency domain method and the multiplier technique are used.

4.1 Introduction

In this chapter, we study the stability of an elastic Bresse system with local Kelvin-Voigt damping and non-smooth coefficient at interface. This system defined on $(0, L) \times (0, +\infty)$ takes the following form

$$\left\{ \begin{array}{l} \rho_1 \varphi_{tt} - [k_1 (\varphi_x + \psi + lw) + D_1 (\varphi_{xt} + \psi_t + lw_t)]_x - lk_3 (w_x - l\varphi) - lD_3 (w_{xt} - l\varphi_t) = 0, \\ \rho_2 \psi_{tt} - [k_2 \psi_x + D_2 \psi_{xt}]_x + k_1 (\varphi_x + \psi + lw) + D_1 (\varphi_{xt} + \psi_t + lw_t) = 0, \\ \rho_1 w_{tt} - [k_3 (w_x - l\varphi) + D_3 (w_{xt} - l\varphi_t)]_x + lk_1 (\varphi_x + \psi + lw) + lD_1 (\varphi_{xt} + \psi_t + lw_t) = 0, \end{array} \right. \quad (4.1.1)$$

with fully Dirichlet boundary conditions

$$\varphi(0, \cdot) = \varphi(L, \cdot) = \psi(0, \cdot) = \psi(L, \cdot) = w(0, \cdot) = w(L, \cdot) = 0 \quad \text{in } \mathbb{R}_+, \quad (4.1.2)$$

or with Dirichlet-Neumann-Neumann boundary conditions

$$\varphi(0, \cdot) = \varphi(L, \cdot) = \psi_x(0, \cdot) = \psi_x(L, \cdot) = w_x(0, \cdot) = w_x(L, \cdot) = 0 \quad \text{in } \mathbb{R}_+, \quad (4.1.3)$$

in addition to the following initial conditions

$$\begin{aligned} \varphi(\cdot, 0) &= \varphi_0(\cdot), \quad \psi(\cdot, 0) = \psi_0(\cdot), \quad w(\cdot, 0) = w_0(\cdot), \\ \varphi_t(\cdot, 0) &= \varphi_1(\cdot), \quad \psi_t(\cdot, 0) = \psi_1(\cdot), \quad w_t(\cdot, 0) = w_1(\cdot), \quad \text{in } (0, L). \end{aligned} \quad (4.1.4)$$

The functions φ , ψ , and w model the vertical, shear angle, and longitudinal displacements of the filament. The coefficients ρ_1 , ρ_2 , k_1 , k_2 , k_3 , l are positive constants. D_1 , D_2 and D_3 are positive functions over $(0, L)$.

The Bresse system is usually considered in studying elastic structures of the arcs type (see [41]). It can be expressed by the equations of motion

$$\begin{aligned} \rho_1 \varphi_{tt} &= Q_x + lN \\ \rho_2 \psi_{tt} &= M_x - Q \\ \rho_1 w_{tt} &= N_x - lQ \end{aligned}$$

where

$$\begin{aligned} N &= k_3(w_x - l\varphi) + D_3(w_{xt} - l\varphi_t) \\ Q &= k_1(\varphi_x + \psi + lw) + D_1(\varphi_{xt} + \psi_t + lw_t) \\ M &= k_2\psi_x + D_2\psi_{xt} \end{aligned}$$

are the stress strain relations for elastic behavior. Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k_1 = k'GA$, $k_3 = EA$, $k_2 = EI$, $l = R^{-1}$ where ρ is the density of the material, E is the modulus of elasticity, G is the shear modulus, k' is the shear factor, A is the cross-sectional area, I is the second moment of area of the cross-section, and R is the radius of curvature. φ , ψ , and w are the vertical, shear angle, and longitudinal displacements. We note that when $R \rightarrow \infty$, then $l \rightarrow 0$ and the Bresse model reduces to well-known Timoshenko beam equations.

The stability of the elastic Bresse system with different types of dissipative has been intensively studied. We start by recall some results. In [30], Guesmia et al. considered Bresse system with infinite memories acting in the three equations of the system. They established asymptotic stability results under some conditions on the relaxation functions regardless the speeds of propagation. Wehbe and Youssef in [68] considered an elastic Bresse system subject to two locally internal dissipation laws. They proved that the system is exponentially stable if and only if the waves propagate at the same speed. Otherwise, a polynomial decay holds. Alabau et al in [11] considered the same system with one globally distributed dissipation law. The authors proved the existence of polynomial decays with rates that depend on some particular relation between the coefficients. Moreover, for the thermoelastic Bresse system, we quote [27], [51] and [57]. In [51], Liu and Rao considered the Bresse system with two thermal dissipation laws. The authors proved an exponential decay rate when the wave speed of the vertical displacement coincides with the wave speed of longitudinal displacement or of the shear angle displacement. Otherwise, they showed polynomial decays depending on the boundary conditions. These results are improved by Fatori and Rivera in [27] where they considered the case of one thermal dissipation law

globally distributed on the displacement equation. Wehbe and Najdi in [57] extended and improved the results of [27], when the thermal dissipation is locally distributed.

The purpose of this chapter is to study the Bresse system in the presence of local Kelvin-Voigt damping with non-smooth coefficient at interface and under fully Dirichlet boundary conditions or Dirichlet-Neumann-Neumann boundary conditions. First, we study the strong stability of the Bresse system under the condition of the existence of at least one local damping. Note that we give the proof only in the case of one local damping applied at the shear angle displacement under, either the boundary condition (4.1.2) or (4.1.3) since the other cases are similar to prove it. Next, we consider the case when the Kelvin-Voigt damping are globally distributed. Here, we show that the Bresse system (4.1.1) is analytic stable. Later, in the presence of three local Kelvin-Voigt dampings, we analyse the exponential and polynomial stability according to the properties of coefficient functions D_1 , D_2 and D_3 . Indeed, if $D_1, D_2, D_3 \in W^{1,\infty}(0, L)$ and their supports coincide, using the frequency domain approach combined with multiplier technique, we prove that the Bresse system (4.1.1) is exponentially stable (see Theorem 4.5.1). Otherwise, if $D_1, D_2, D_3 \in L^\infty(0, L)$ and their supports have a non-empty intersection, we establish a polynomial stability of type $\frac{1}{t}$ (see Theorem 4.6.1). Moreover, in the absence of at least one damping, we prove the lack of uniform stability for the system (4.1.1) subjected to (4.1.3). Finally, in the presence of only one local damping D_2 acting on the shear angle displacement, we establish a polynomial decay rate of type $\frac{1}{\sqrt{t}}$ (see Theorem 4.8.1). In these cases, we conjecture the optimality of the obtained decay rate.

Furthermore, in addition to the previously cited papers, we rapidly recall some previous studies done on the Timoshenko system. The stability of the Timoshenko system with different kinds of dampings has been studied in [28], [26], [64] and [66]. Hago and al. in [28] proved that the Timoshenko system with history type damping is not exponentially stable under Cattaneo's law, while under Fourier's law, an exponential decay can be obtained if the speeds are equal. Moreover, if the speeds are different, no decay rate has been discussed. This result has been improved by Fatori and al in [26], where an exponential decay can be attained with Cattaneo's law if and only if a few condition on the wave speed of propagation is verified. In [66], Tian and Zhang studied the Timoshenko system with Kelvin-Voigt damping. They proved that the energy decays exponentially or polynomially and the decay rate depends on material of coefficient function. Last but not least, we exhibit some studies done on systems with Kelvin-Voigt damping. In [32], Hassine considered the longitudinal and transversal vibrations of the transmission Euler-Bernoulli beam with Kelvin-Voigt damping distributed locally on any subinterval of the region occupied by the beam and only in one side of the transmission point. He proved that the semigroup associated with the equation for the transversal motion of the beam is exponentially stable, although the semigroup associated with the equation for the longitudinal motion of the beam is polynomially stable. Hassine in [33] discussed the asymptotic behavior of the transmission Euler-Bernoulli plate and wave equation with a localized Kelvin-Voigt damping. He proved that sufficiently smooth solutions decay logarithmically at infinity even the feedback affects a small open subset of the interior. Also, in [34], Hassine considered a beam and a wave equations coupled by an elastic beam through transmission condition. The damping which is locally distributed acts only at one equation. First, he considered the case where the dissipation acts through the beam equation, he showed a

precise polynomial energy decay rate. Second, in the case where damping acts through the wave equation, he provided a precise polynomial energy decay rate. In both cases, he proved the lack of exponential stability. In [35], Hassine studied the asymptotic behavior of the energy decay of a transmission plate equation with locally distributed Kelvin-Voigt damping. More precisely, he proved that the energy decay at least logarithmically over the time. Recently, Ammari et al in [17] considered the wave equation with Kelvin-Voigt damping in a bounded domain. In their work, they proposed to deal with the geometrical condition by considering a singular Kelvin-Voigt damping which is localized far away from the boundary. In this particular case, they showed that the energy of the wave equation decreases logarithmically to zero as time goes to infinity.

This chapter is organized as follows : In Section 4.2, we prove the well-posedness of system (4.1.1) with either the boundary conditions (4.1.2) or (4.1.3). Next, in Section 4.3, we prove the strong stability of the system in the lack of the compactness of the resolvent of the generator. In Section 4.4, we prove the analytic stability when the three Kelvin-Voigt dampings are globally distributed. Later, Sections 4.5 and 4.6 are devoted to analyze the stability of the system provided the existence of three local dampings by distinguishing two cases : in the first one, when the coefficient functions D_1 , D_2 , and D_3 are smooth, in Section 4.5, we prove the exponential stability of the system. In the second one, when the coefficient functions D_1 , D_2 , and D_3 are non smooth, in Section 4.6, we prove the polynomial stability of type $\frac{1}{t}$. Last but not least, in Section 4.7, under boundary conditions (4.1.3), we prove the lack of uniform stability of the system in the absence of at least one damping. Finally, in section 4.8, we prove the polynomial energy decay rate of type $\frac{1}{\sqrt{t}}$ for the system in the case of one local non-smooth damping D_2 acting on the shear angle displacement.

4.2 Well-posedness of the problem

In this part, using a semigroup approach, we establish well-posedness result for the system (4.1.1). Its energy is given by

$$E(t) = \frac{1}{2} \left\{ \int_0^L (\rho_1 |\varphi_t|^2 + \rho_2 |\psi_t|^2 + \rho_3 |w_t|^2 + k_1 |\varphi_x + \psi + lw|^2) dx + \int_0^L (k_2 |\psi_x|^2 + k_3 |w_x - l\varphi|^2) dx \right\}, \quad (4.2.1)$$

and it is dissipated according to the following law

$$E'(t) = - \int_0^L (D_1 |\varphi_{xt} + \psi_t + lw_t|^2 + D_2 |\psi_{xt}|^2 + D_3 |w_{xt} - l\varphi_t|^2) dx \leq 0. \quad (4.2.2)$$

Now, we define the following energy spaces :

$$\mathcal{H}_1 = (H_0^1 \times L^2)^3 \quad \text{and} \quad \mathcal{H}_2 = H_0^1 \times L^2 \times (H_*^1 \times L_*^2)^2,$$

where

$$L_*^2 = \{f \in L^2(0, L) : \int_0^L f(x)dx = 0\} \text{ and } H_*^1 = \{f \in H^1(0, L) : \int_0^L f(x)dx = 0\}.$$

Both spaces \mathcal{H}_1 and \mathcal{H}_2 are equipped with the inner product which induces the energy norm

$$\begin{aligned} \|U\|_{\mathcal{H}_j}^2 &= \|(v^1, v^2, v^3, v^4, v^5, v^6)\|_{\mathcal{H}_j}^2, \quad j = 1, 2, \\ &= \rho_1 \|v^2\|^2 + \rho_2 \|v^4\|^2 + \rho_1 \|v^6\|^2 + k_1 \|v_x^1 + v^3 + lv^5\|^2 \\ &\quad + k_2 \|v_x^3\|^2 + k_3 \|v_x^5 - lv^1\|^2. \end{aligned} \quad (4.2.3)$$

Here and after $\|\cdot\|$ denotes the norm of $L^2(0, L)$.

Remark 4.2.1. In the case of boundary condition (4.1.2), it is easy to see that expression (4.2.3) defines a norm on the energy space \mathcal{H}_1 . But in the case of boundary condition (4.1.3) the expression (4.2.3) define a norm on the energy space \mathcal{H}_2 if $L \neq \frac{n\pi}{1}$ for all positive integer n . Then, here and after, we assume that there exists no $n \in \mathbb{N}$ such that $L = \frac{n\pi}{1}$ when $j = 2$.

Next, we define the linear operator \mathcal{A}_j in \mathcal{H}_j by

$$D(\mathcal{A}_1) = \left\{ U \in \mathcal{H}_1 \mid v^2, v^4, v^6 \in H_0^1(0, L), \right. \\ \left. \begin{aligned} &[k_1(v_x^1 + v^3 + lv^5) + D_1(v_x^2 + v^4 + lv^6)]_x \in L^2(0, L), \\ &[k_2v_x^3 + D_2v_x^4]_x \in L^2(0, L), [k_3(v_x^5 - lv^1) + D_3(v_x^6 - lv^2)]_x \in L^2(0, L) \end{aligned} \right\},$$

$$D(\mathcal{A}_2) = \left\{ U \in \mathcal{H}_2 \mid v^2 \in H_0^1(0, L), v^4, v^6 \in H_*^1(0, L), v_x^3|_{(0,L)} = v_x^5|_{(0,L)} = 0, \right. \\ \left. \begin{aligned} &[k_1(v_x^1 + v^3 + lv^5) + D_1(v_x^2 + v^4 + lv^6)]_x \in L^2(0, L), \\ &[k_2v_x^3 + D_2v_x^4]_x \in L_*^2(0, L), [k_3(v_x^5 - lv^1) + D_3(v_x^6 - lv^2)]_x \in L_*^2(0, L) \end{aligned} \right\}$$

and

$$\mathcal{A}_j \begin{pmatrix} v^1 \\ v^2 \\ v^3 \\ v^4 \\ v^5 \\ v^6 \end{pmatrix} = \begin{pmatrix} v^2 \\ \rho_1^{-1} ([k_1(v_x^1 + v^3 + lv^5) + D_1(v_x^2 + v^4 + lv^6)]_x + lk_3(v_x^5 - lv^1) + lD_3(v_x^6 - lv^2)) \\ v^4 \\ \rho_2^{-1} ((k_2v_x^3 + D_2v_x^4)_x - k_1(v_x^1 + v^3 + lv^5) - D_1(v_x^2 + v^4 + lv^6)) \\ v^6 \\ \rho_1^{-1} ([k_3(v_x^5 - lv^1) + D_3(v_x^6 - lv^2)]_x - lk_1(v_x^1 + v^3 + lv^5) - lD_1(v_x^2 + v^4 + lv^6)) \end{pmatrix}$$

for all $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A}_j)$. If $U = (\varphi, \varphi_t, \psi, \psi_t, w, w_t)^\top$ is the state of (4.1.1), then the Bresse beam system is transformed into a first order evolution equation on the Hilbert space \mathcal{H}_j :

$$\begin{cases} U_t(x, t) = \mathcal{A}_j U(x, t), \\ U(x, 0) = U_0(x), \end{cases} \quad (4.2.4)$$

where

$$U_0(x) = (\varphi_0(x), \varphi_1(x), \psi_0(x), \psi_1(x), w_0(x), w_1(x))^T.$$

Remark 4.2.2. It is easy to see that there exists a positive constant c_0 such that

$$k_1 \|\varphi_x + \psi + lw\|^2 + k_2 \|\psi_x\|^2 + k_3 \|w_x - l\varphi\|^2 \leq c_0 (\|\varphi_x\|^2 + \|\psi_x\|^2 + \|w_x\|^2). \quad (4.2.5)$$

On the other hand, we can show by a contradiction argument the existence of a positive constant c_1 such that, for any $(\varphi, \psi, w) \in (H_0^1(0, L))^3$ for $j = 1$ and for any $(\varphi, \psi, w) \in H_0^1(0, L) \times (H_*^1(0, L))^2$ for $j = 2$,

$$c_1 (\|\varphi_x\|^2 + \|\psi_x\|^2 + \|w_x\|^2) \leq k_1 \|\varphi_x + \psi + lw\|^2 + k_2 \|\psi_x\|^2 + k_3 \|w_x - l\varphi\|^2. \quad (4.2.6)$$

Therefore the norm on the energy space \mathcal{H}_j given in (4.2.3) is equivalent to the usual norm on \mathcal{H}_j . \square

Proposition 4.2.3. Assume that coefficients functions D_1 , D_2 and D_3 are non negative. Then, the operator \mathcal{A}_j is m-dissipative in the energy space \mathcal{H}_j , for $j = 1, 2$.

Proof: For all $U \in D(\mathcal{A}_j)$, by a straight forward calculation, we have

$$\operatorname{Re}(\mathcal{A}_j U, U)_{\mathcal{H}_j} = - \int_0^L \left(D_1 |v_x^2 + v^4 + lv^6|^2 + D_2 |v_x^4|^2 + D_3 |v_x^6 - lv^2|^2 \right) dx. \quad (4.2.7)$$

As $D_1 \geq 0$, $D_2 \geq 0$ and $D_3 \geq 0$, we get that \mathcal{A}_j is dissipative. Now, let

$$F = (f^1, f^2, f^3, f^4, f^5, f^6)^T \in \mathcal{H}_j,$$

we prove the existence of

$$U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(\mathcal{A}_j)$$

unique solution of the equation

$$-\mathcal{A}_j U = F.$$

Equivalently, we have the following system

$$-v^2 = f^1, \quad (4.2.8)$$

$$- \left[k_1 (v_x^1 + v^3 + lv^5) + D_1 (v_x^2 + v^4 + lv^6) \right]_x - lk_3 (v_x^5 - lv^1) - lD_3 (v_x^6 - lv^2) = \rho_1 f^2, \quad (4.2.9)$$

$$-v^4 = f^3, \quad (4.2.10)$$

$$- (k_2 v_x^3 + D_2 v_x^4)_x + k_1 (v_x^1 + v^3 + lv^5) + D_1 (v_x^2 + v^4 + lv^6) = \rho_2 f^4, \quad (4.2.11)$$

$$-v^6 = f^5, \quad (4.2.12)$$

$$- [k_3(v_x^5 - lv^1) + D_3(v_x^6 - lv^2)]_x + lk_1(v_x^1 + v^3 + lv^5) + lD_1(v_x^2 + v^4 + lv^6) = \rho_1 f^6. \quad (4.2.13)$$

Let $(\varphi^1, \varphi^3, \varphi^5) \in (H_0^1(0, L))^3$ for $j = 1$ and $(\varphi^1, \varphi^3, \varphi^5) \in (H_0^1(0, L) \times (H_*^1(0, L))^2)$ for $j = 2$ be a test function. Multiplying (4.2.9), (4.2.11) and (4.2.13) by φ^1 , φ^3 and φ^5 respectively. Consequently, (4.2.8) - (4.2.13) can be written after integrating by parts in the following form

$$\begin{cases} k_1(v_x^1 + v^3 + lv^5) \varphi_x^1 - lk_3(v_x^5 - lv^1) \varphi^1 = h^1, \\ k_2 v_x^3 \varphi_x^3 + k_1(v_x^1 + v^3 + lv^5) \varphi^3 = h^3, \\ k_3(v_x^5 - lv^1) \varphi_x^5 + lk_1(v_x^1 + v^3 + lv^5) \varphi^5 = h^5, \end{cases} \quad (4.2.14)$$

where

$$\begin{aligned} h^1 &= \rho_1 f^2 \varphi^1 + D_1(f_x^1 + f^3 + lf^5) \varphi_x^1 - lD_3(f_x^5 - lf^1) \varphi^1, \\ h^3 &= \rho_2 f^4 \varphi^3 + D_2 f_x^3 \varphi_x^3 + D_1(f_x^1 + f^3 + lf^5) \varphi^3, \end{aligned}$$

and

$$h^5 = \rho_1 f^6 \varphi^5 + D_3(f^5 - lf^1) \varphi_x^5 + lD_1(f_x^1 + f^3 + lf^5) \varphi^5.$$

Using Lax-Milgram Theorem (see [59]), we deduce that (4.2.14) admits a unique solution in $(H_0^1(0, L))^3$ for $j = 1$ and in $(H_0^1(0, L) \times (H_*^1(0, L))^2)$ for $j = 2$. Thus, $-\mathcal{A}_j U = F$ admits a unique solution $U \in D(\mathcal{A}_j)$ and consequently $0 \in \rho(\mathcal{A}_j)$. Then, \mathcal{A}_j is closed and consequently $\rho(\mathcal{A}_j)$ is open set of \mathbb{C} (see Theorem 6.7 in [39]). Hence, we easily get $R(\lambda I - \mathcal{A}_j) = \mathcal{H}_j$ for sufficiently small $\lambda > 0$. This, together with the dissipativeness of \mathcal{A}_j , imply that $D(\mathcal{A}_j)$ is dense in \mathcal{H}_j and that \mathcal{A}_j is m-dissipative in \mathcal{H}_j (see Theorems 4.5, 4.6 in [59]). Thus, the proof is complete. \square

Thanks to Lumer-Phillips Theorem (see [53, 59]), we deduce that \mathcal{A}_j generates a C_0 -semigroup of contraction $e^{t\mathcal{A}_j}$ in \mathcal{H}_j and therefore problem is well-posed. Then, we have the following result.

Theorem 4.2.4. For any $U_0 \in \mathcal{H}_j$, problem (4.2.4) admits a unique weak solution

$$U \in C(\mathbb{R}_+; \mathcal{H}_j).$$

Moreover, if $U_0 \in D(\mathcal{A}_j)$, then

$$U \in C(\mathbb{R}_+; D(\mathcal{A}_j)) \cap C^1(\mathbb{R}_+; \mathcal{H}_j).$$

4.3 Strong stability of the system

In this part, we use a general criteria of Arendt-Batty in [20] to show the strong stability of the C_0 -semigroup $e^{t\mathcal{A}_j}$ associated to the Bresse system (4.1.1) in the absence of the compactness of the resolvent of \mathcal{A}_j . Our main result is the following Theorem.

Theorem 4.3.1. Assume that there exists at least one localized damping D_i for $i = 1, 2$ or 3 in $(0, L)$. Then the C_0 -semigroup $e^{t\mathcal{A}_j}$ is strongly stable in \mathcal{H}_j , $j = 1, 2$, i.e., for all $U_0 \in \mathcal{H}_j$, the solution of (4.2.4) satisfies

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_j} U_0\|_{\mathcal{H}_j} = 0.$$

For the proof of Theorem 4.3.1, we need the following two lemmas.

Lemma 4.3.2. Under the same condition of Theorem 4.3.1, we have

$$\ker(i\lambda I - \mathcal{A}_j) = \{0\}, \quad j = 1, 2, \quad \text{for all } \lambda \in \mathbb{R}. \quad (4.3.1)$$

Proof: We will prove Lemma 4.3.2 in the case $D_1 = D_3 = 0$ on $(0, L)$ and $D_2 \geq a_0 > 0$ on $(\alpha, \beta) \subset (0, L)$ and the other cases are similar to prove.

First, from Proposition 4.2.3, we claim that $0 \in \rho(\mathcal{A}_j)$. We still have to show the result for $\lambda \in \mathbb{R}^*$. Suppose that there exist a real number $\lambda \neq 0$ and $0 \neq U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A}_j)$ such that

$$\mathcal{A}_j U = i\lambda U. \quad (4.3.2)$$

Our goal is to find a contradiction by proving that $U = 0$. Taking the real part of the inner product in \mathcal{H}_j of $\mathcal{A}_j U$ and U , we get

$$\operatorname{Re}(\mathcal{A}_j U, U)_{\mathcal{H}_j} = - \int_0^L D_2 |v_x^4|^2 dx = 0. \quad (4.3.3)$$

Since by assumption $D_2 \geq a_0 > 0$ on (α, β) , it follows from equality (4.3.3) that

$$v_x^4 = 0 \quad \text{in } (\alpha, \beta). \quad (4.3.4)$$

Detailing (4.3.2) we get

$$v^2 = i\lambda v^1, \quad (4.3.5)$$

$$k_1 (v_x^1 + v^3 + lv^5)_x + lk_3 (v_x^5 - lv^1) = i\rho_1 \lambda v^2, \quad (4.3.6)$$

$$v^4 = i\lambda v^3, \quad (4.3.7)$$

$$(k_2 v_x^3 + D_2 v_x^4)_x - k_1 (v_x^1 + v^3 + lv^5) = i\rho_2 \lambda v^4, \quad (4.3.8)$$

$$v^6 = i\lambda v^5, \quad (4.3.9)$$

$$k_3 (v_x^5 - lv^1)_x - lk_1 (v_x^1 + v^3 + lv^5) = i\rho_1 \lambda v^6. \quad (4.3.10)$$

Next, inserting (4.3.4) in (4.3.7) and using the fact that $\lambda \neq 0$, we get

$$v_x^3 = 0 \quad \text{in } (\alpha, \beta). \quad (4.3.11)$$

Moreover, substituting equations (4.3.5), (4.3.7) and (4.3.9) into equations (4.3.6), (4.3.8) and (4.3.10), we get

$$\begin{cases} \rho_1 \lambda^2 v^1 + k_1 (v_x^1 + v^3 + lv^5)_x + lk_3 (v_x^5 - lv^1) = 0, \\ \rho_2 \lambda^2 v^3 + (k_2 v_x^3 + iD_2 \lambda v_x^3)_x - k_1 (v_x^1 + v^3 + lv^5) = 0, \\ \rho_1 \lambda^2 v^5 + k_3 (v_x^5 - lv^1)_x - lk_1 (v_x^1 + v^3 + lv^5) = 0. \end{cases} \quad (4.3.12)$$

Now, we introduce the functions \widehat{v}^i , for $i = 1, \dots, 6$ by

$$\widehat{v}^i = v_x^i,$$

where it is easy to see that $\widehat{v}^i \in H^1(0, L)$.

It follows from equations (4.3.4) and (4.3.11) that

$$\widehat{v}^3 = \widehat{v}^4 = 0 \quad \text{in } (\alpha, \beta) \quad (4.3.13)$$

and consequently system (4.3.12) will be after differentiating it with respect to x

$$\rho_1 \lambda^2 \widehat{v}^1 + k_1 (\widehat{v}_x^1 + l\widehat{v}^5)_x + lk_3 (\widehat{v}_x^5 - l\widehat{v}^1) = 0 \quad \text{in } (\alpha, \beta), \quad (4.3.14)$$

$$\widehat{v}_x^1 + l\widehat{v}^5 = 0 \quad \text{in } (\alpha, \beta), \quad (4.3.15)$$

$$\rho_1 \lambda^2 \widehat{v}^5 + k_3 (\widehat{v}_x^5 - l\widehat{v}^1)_x - lk_1 (\widehat{v}_x^1 + l\widehat{v}^5) = 0 \quad \text{in } (\alpha, \beta). \quad (4.3.16)$$

Furthermore, substituting equation (4.3.15) into (4.3.14) and (4.3.16), we get

$$\rho_1 \lambda^2 \widehat{v}^1 + lk_3 (\widehat{v}_x^5 - l\widehat{v}^1) = 0 \quad \text{in } (\alpha, \beta), \quad (4.3.17)$$

$$\widehat{v}_x^1 + l\widehat{v}^5 = 0 \quad \text{in } (\alpha, \beta), \quad (4.3.18)$$

$$\rho_1 \lambda^2 \widehat{v}^5 + k_3 (\widehat{v}_x^5 - l\widehat{v}^1)_x = 0 \quad \text{in } (\alpha, \beta). \quad (4.3.19)$$

Differentiating equation (4.3.17) with respect to x , a straightforward computation with equation (4.3.19) yields

$$\rho_1 \lambda^2 (\widehat{v}_x^1 - l\widehat{v}^5) = 0 \quad \text{in } (\alpha, \beta).$$

Equivalently

$$\widehat{v}_x^1 - l\widehat{v}^5 = 0 \quad \text{in } (\alpha, \beta). \quad (4.3.20)$$

Hence, from equations (4.3.18) and (4.3.20), we get

$$\widehat{v}^5 = 0 \quad \text{and} \quad \widehat{v}_x^1 = 0 \quad \text{in } (\alpha, \beta). \quad (4.3.21)$$

The fact that $\widehat{v}^5 = 0$, from (4.3.17), we get

$$(\rho_1 \lambda^2 - l^2 k_3) \widehat{v}^1 = 0. \quad (4.3.22)$$

To finish our proof, we are against two cases to discuss :

Case 1 : $\lambda \neq l\sqrt{\frac{k_3}{\rho_1}}$.

Using equation (4.3.22), we deduce that

$$\widehat{v}^1 = 0 \quad \text{in } (\alpha, \beta).$$

Setting $V = (\widehat{v}^1, \widehat{v}_x^1, \widehat{v}^3, \widehat{v}_x^3, \widehat{v}^5, \widehat{v}_x^5)^\top$. By continuity of \widehat{v}^i on $(0, L)$, we deduce that $V(\alpha) = 0$. Then system (4.3.12) could be given as

$$\begin{cases} V_x &= BV, & \text{in } (0, \alpha) \\ V(\alpha) &= 0, \end{cases} \quad (4.3.23)$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\lambda^2 \rho_1 + l^2 k_3 & 0 & 0 & -1 & \frac{-l(k_1 + k_3)}{k_1} & 0 \\ k_1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{k_1}{k_2 + i\lambda D_2} & \frac{k_1 - \lambda^2 \rho_2}{k_2 + i\lambda D_2} & 0 & \frac{lk_1}{k_2 + i\lambda D_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{l(k_3 + k_1)}{k_3} & \frac{lk_1}{k_3} & 0 & \frac{l^2 k_1 - \lambda^2 \rho_1}{k_3} & 0 \end{pmatrix}. \quad (4.3.24)$$

Using ordinary differential equation theory, we deduce that system (4.3.23) has the unique trivial solution $V = 0$ in $(0, \alpha)$. Same argument as above leads us to prove that $V = 0$ on (β, L) . Consequently, we obtain $\widehat{v}^1 = \widehat{v}^3 = \widehat{v}^5 = 0$ on $(0, L)$. It follows that $\widehat{v}^2 = \widehat{v}^4 = \widehat{v}^6 = 0$ on $(0, L)$, thus $\widehat{U} = 0$. This gives that $U = C$, where C is a constant. Finally, since $U(0) = 0$, we deduce that $U = 0$.

Case 2 : $\lambda = l\sqrt{\frac{k_3}{\rho_1}}$.

The fact that $\widehat{v}_x^1 = 0$ on (α, β) , we get $\widehat{v}^1 = c$ on (α, β) , where c is a constant. By continuity of \widehat{v}^1 on $(0, L)$, we deduce that $\widehat{v}^1(\alpha) = c$. We know also that $\widehat{v}^3 = \widehat{v}^5 = 0$ on (α, β) from (4.3.13) and (4.3.21). Hence, setting $V(\alpha) = (c, 0, 0, 0, 0, 0)^\top = V_0$, we can rewrite system (4.3.12) on $(0, \alpha)$ under the form

$$\begin{cases} V_x &= \widehat{B}V, \\ V(\alpha) &= V_0, \end{cases}$$

where

$$\widehat{B} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & \frac{-l(k_1 + k_3)}{k_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{k_1}{k_2 + il\sqrt{\frac{k_3}{\rho_1}}D_2} & \frac{k_1 - \lambda^2 \rho_2}{k_2 + il\sqrt{\frac{k_3}{\rho_1}}\lambda D_2} & 0 & \frac{lk_1}{k_2 + il\sqrt{\frac{k_3}{\rho_1}}D_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{l(k_3 + k_1)}{k_3} & \frac{lk_1}{k_3} & 0 & \frac{l^2(k_1 - k_3)}{k_3} & 0 \end{pmatrix}.$$

Introducing $\widetilde{V} = (\widehat{v}_x^1, \widehat{v}^3, \widehat{v}_x^3, \widehat{v}^5, \widehat{v}_x^5)^\top$ and

$$\widetilde{B} = \begin{pmatrix} 0 & 0 & -1 & \frac{-l(k_1 + k_3)}{k_1} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{k_1}{k_2 + il\sqrt{\frac{k_3}{\rho_1}}D_2} & \frac{k_1 - \lambda^2 \rho_2}{k_2 + il\sqrt{\frac{k_3}{\rho_1}}\lambda D_2} & 0 & \frac{lk_1}{k_2 + il\sqrt{\frac{k_3}{\rho_1}}D_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{l(k_3 + k_1)}{k_3} & \frac{lk_1}{k_3} & 0 & \frac{l^2(k_1 - k_3)}{k_3} & 0 \end{pmatrix}.$$

Then system (4.3.12) could be given as

$$\begin{cases} \tilde{V}_x &= \tilde{B}\tilde{V}, \text{ in } (0, \alpha), \\ \tilde{V}(\alpha) &= 0. \end{cases} \quad (4.3.25)$$

Using ordinary differential equation theory, we deduce that system (4.3.25) has the unique trivial solution $\tilde{V} = 0$ in $(0, \alpha)$. This implies that on $(0, \alpha)$, we have $\hat{v}^3 = \hat{v}^5 = 0$. Consequently, $v^3 = c_3$ and $v^5 = c_5$ where c_3 and c_5 are constants. But using the fact that $v^3(0) = v^5(0) = 0$, we deduce that $v^3 = v^5 = 0$ on $(0, \alpha)$.

Substituting v^3 and v^5 by their values in the second equation of system (4.3.12), we get that $v_x^1 = 0$. This yields $v^1 = c_1$, where c_1 is a constant. But as $v^1(0) = 0$, we get $v^1 = 0$ on $(0, \alpha)$. Thus $U = 0$ on $(0, \alpha)$. Same argument as above leads us to prove that $U = 0$ on (β, L) and therefore $U = 0$ on $(0, L)$. Thus the proof is complete. \square

Lemma 4.3.3. Under the same condition of Theorem 4.3.1, $(i\lambda I - \mathcal{A}_j), j = 1, 2$ is surjective for all $\lambda \in \mathbb{R}$.

Proof: We will prove Lemma 4.3.3 in the case $D_1 = D_3 = 0$ on $(0, L)$ and $D_2 \geq a_0 > 0$ on $(\alpha, \beta) \subset (0, L)$ and the other cases are similar to prove.

Since $0 \in \rho(\mathcal{A}_j)$, we still need to show the result for $\lambda \in \mathbb{R}^*$. For any

$$F = (f^1, f^2, f^3, f^4, f^5, f^6)^\top \in \mathcal{H}_j, \lambda \in \mathbb{R}^*,$$

we prove the existence of

$$U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A}_j)$$

solution of the following equation

$$(i\lambda I - \mathcal{A}_j)U = F.$$

Equivalently, we have the following system

$$i\lambda v^1 - v^2 = f^1, \quad (4.3.26)$$

$$\rho_1 i\lambda v^2 - k_1 (v_x^1 + v^3 + lv^5)_x - lk_3 (v_x^5 - lv^1) = \rho_1 f^2, \quad (4.3.27)$$

$$i\lambda v^3 - v^4 = f^3, \quad (4.3.28)$$

$$\rho_2 i\lambda v^4 - (k_2 v_x^3 + D_2 v_x^4)_x + k_1 (v_x^1 + v^3 + lv^5) = \rho_2 f^4, \quad (4.3.29)$$

$$i\lambda v^5 - v^6 = f^5, \quad (4.3.30)$$

$$\rho_1 i\lambda v^6 - k_3 (v_x^5 - lv^1)_x + lk_1 (v_x^1 + v^3 + lv^5) = \rho_1 f^6. \quad (4.3.31)$$

From (4.3.26),(4.3.28) and (4.3.30), we have

$$v^2 = i\lambda v^1 - f^1, \quad v^3 = i\lambda v^3 - f^3, \quad v^6 = i\lambda v^5 - f^5. \quad (4.3.32)$$

Inserting (4.3.32) in (4.3.27), (4.3.29) and (4.3.31), we get

$$\begin{cases} -\lambda^2 v^1 - k_1 \rho_1^{-1} (v_x^1 + v^3 + lv^5)_x - lk_3 \rho_1^{-1} (v_x^5 - lv^1) = h^1, \\ -\lambda^2 v^3 - \rho_2^{-1} (k_2 + i\lambda D_2) v_{xx}^3 + k_1 \rho_2^{-1} (v_x^1 + v^3 + lv^5) = h^3, \\ -\lambda^2 v^5 - k_3 \rho_1^{-1} (v_x^5 - lv^1)_x + lk_1 \rho_1^{-1} (v_x^1 + v^3 + lv^5) = h^5, \end{cases} \quad (4.3.33)$$

where

$$h^1 = f^2 + i\lambda f^1, \quad h^3 = f^4 + i\lambda f^3 - \rho_2^{-1} D_2 f_{xx}^3, \quad h^5 = f^6 + i\lambda f^5.$$

For all $v = (v^1, v^3, v^5)^\top \in (H_0^1(0, L))^3$ for $j = 1$ and $v = (v^1, v^3, v^5)^\top \in H_0^1(0, L) \times H_*^1(0, L)^2$ for $j = 2$, we define the linear operator \mathcal{L} by

$$\mathcal{L}v = \begin{pmatrix} -k_1 \rho_1^{-1} (v_x^1 + v^3 + lv^5)_x - lk_3 \rho_1^{-1} (v_x^5 - lv^1) \\ -\rho_2^{-1} (k_2 + i\lambda D_2) v_{xx}^3 + k_1 \rho_2^{-1} (v_x^1 + v^3 + lv^5) \\ -k_3 \rho_1^{-1} (v_x^5 - lv^1)_x + lk_1 \rho_1^{-1} (v_x^1 + v^3 + lv^5) \end{pmatrix}.$$

Using Lax-Milgram theorem, it is easy to show that \mathcal{L} is an isomorphism from $(H_0^1(0, L))^3$ onto $(H^{-1}(0, L))^3$. Let $v = (v^1, v^3, v^5)^\top$ and $h = (h^1, h^3, h^5)^\top$, then we transform system (4.3.33) into the following form

$$v - \lambda^2 \mathcal{L}^{-1} v = \mathcal{L}^{-1} h. \quad (4.3.34)$$

Using the compactness embeddings from $L^2(0, L)$ into $H^{-1}(0, L)$ and from $H_0^1(0, L)$ into $L^2(0, L)$, we deduce that the operator \mathcal{L}^{-1} is compact from $(L^2(0, L))^3$ into $(L^2(0, L))^3$. Consequently, by Fredholm alternative, proving the existence of v solution of (4.3.34) reduces to proving $\text{Ker}(I - \lambda^2 \mathcal{L}^{-1}) = 0$. Indeed, if $\tilde{v} = (\tilde{v}^1, \tilde{v}^3, \tilde{v}^5)^\top \in \text{Ker}(I - \lambda^2 \mathcal{L}^{-1})$, then we have $\lambda^2 \tilde{v} - \mathcal{L} \tilde{v} = 0$. It follows that

$$\begin{cases} -\rho_1 \lambda^2 \tilde{v}^1 - k_1 (\tilde{v}_x^1 + \tilde{v}^3 + l\tilde{v}^5)_x - lk_3 (\tilde{v}_x^5 - l\tilde{v}^1) = 0, \\ -\rho_2 \lambda^2 \tilde{v}^3 - (k_2 + i\lambda D_2) \tilde{v}_{xx}^3 + k_1 (\tilde{v}_x^1 + \tilde{v}^3 + l\tilde{v}^5) = 0, \\ -\rho_1 \lambda^2 \tilde{v}^5 - k_3 (\tilde{v}_x^5 - l\tilde{v}^1)_x + lk_1 (\tilde{v}_x^1 + \tilde{v}^3 + l\tilde{v}^5) = 0. \end{cases} \quad (4.3.35)$$

Now, it is easy to see that if $(\tilde{v}^1, \tilde{v}^3, \tilde{v}^5)$ is a solution of system (4.3.35), then the vector \tilde{V} defined by

$$\tilde{V} = (\tilde{v}^1, i\lambda \tilde{v}^1, \tilde{v}^3, i\lambda \tilde{v}^3, \tilde{v}^5, i\lambda \tilde{v}^5)^\top$$

belongs to $D(\mathcal{A}_j)$ and we have

$$i\lambda \tilde{V} - \mathcal{A}_j \tilde{V} = 0.$$

Therefore, by Lemma 4.3.2, we get $\tilde{V} = 0$ and so

$$\text{Ker}(I - \lambda^2 \mathcal{L}^{-1}) = \{0\}.$$

Thanks to Fredholm alternative, the equation (4.3.34) admits a unique solution $v = (v^1, v^3, v^5) \in (H_0^1(0, L))^3$. Thus, using (4.3.32) and a classical regularity arguments, we conclude that $(i\lambda I - \mathcal{A}_j)U = F$ admits a unique solution $U \in D(\mathcal{A}_j)$. Thus, the proof is complete. \square

Proof of Theorem 4.3.1. Following a general criteria of Arendt-Batty in [20], the C_0 -semigroup $e^{t\mathcal{A}_j}$ of contractions is strongly stable if \mathcal{A}_j has no pure imaginary eigenvalues and $\sigma(\mathcal{A}_j) \cap i\mathbb{R}$ is countable. By Lemma 4.3.2, the operator \mathcal{A}_j has no pure imaginary eigenvalues and by Lemma 4.3.3, $\text{R}(i\lambda - \mathcal{A}_j) = \mathcal{H}_j$ for all $\lambda \in \mathbb{R}$. Therefore the closed graph theorem of Banach implies that $\sigma(\mathcal{A}_j) \cap i\mathbb{R} = \emptyset$. Thus, the proof is complete. \square

4.4 Analytic stability in the case of three global dampings

In this part, we prove the analytic stability of the Bresse system (4.1.1) provided that there exists a positive constant a_0 such that

$$D_1, D_2 \text{ and } D_3 \geq a_0 > 0 \text{ for every } x \in (0, L). \quad (4.4.1)$$

Before we state our main result, we recall the following standard result see [36], [60] for part i), [25], [20] for ii) and [59] for iii).

Theorem 4.4.1. Let $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ generates a C_0 -semigroup of contractions $e^{t\mathcal{A}}$ on \mathcal{H} . Assume that $i\lambda \in \rho(\mathcal{A})$, for all $\lambda \in \mathbb{R}$. Then, the C_0 -semigroup $e^{t\mathcal{A}}$ is :

i) Exponentially stable if and only if

$$\limsup_{|\lambda| \rightarrow +\infty, \lambda \in \mathbb{R}} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

ii) Polynomially stable of order $\frac{1}{l}$ ($l > 0$) if and only if

$$\limsup_{|\lambda| \rightarrow +\infty, \lambda \in \mathbb{R}} |\lambda|^{-l} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

iii) Analytic stable if and only if

$$\limsup_{|\lambda| \rightarrow +\infty, \lambda \in \mathbb{R}} |\lambda| \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < +\infty.$$

Now, we are in a position to establish the main result of this part by the following stability estimate.

Theorem 4.4.2. Assume that (4.4.1) is satisfied and D_1, D_2 and $D_3 \in L^\infty(0, L)$. Then, the C_0 -semigroup $e^{t\mathcal{A}_j}$ is analytically stable for $j = 1, 2$.

To prove Theorem 4.4.2, we have to check if the following conditions

$$i\mathbb{R} \subseteq \rho(\mathcal{A}_j) \quad (\text{H1})$$

and

$$\limsup_{\lambda \rightarrow +\infty, \lambda \in \mathbb{R}} |\lambda| \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} = O(1), \quad (\text{H2})$$

hold.

Condition (H1) is already proved in Lemma 4.3.2 and Lemma 4.3.3. To prove condition (H2), we use a contradiction argument. For this aim, suppose that there exist a sequence of real numbers $(\lambda_n)_n$, with $|\lambda_n| \rightarrow +\infty$ and a sequence of vectors

$$U_n = (v_n^1, v_n^2, v_n^3, v_n^4, v_n^5, v_n^6)^\top \in D(\mathcal{A}_j) \text{ with } \|U_n\|_{\mathcal{H}_j} = 1 \quad (4.4.2)$$

such that

$$\lambda_n^{-1} (i\lambda_n U_n - \mathcal{A}_j U_n) = (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6)^\top \rightarrow 0 \text{ in } \mathcal{H}_j, \quad j = 1, 2 \quad (4.4.3)$$

detailed as

$$iv_n^1 - \lambda_n^{-1}v_n^2 = f_n^1, \quad (4.4.4)$$

$$i\rho_1v_n^2 - \lambda_n^{-1} \left[k_1 \left((v_n^1)_x + v_n^3 + lv_n^5 \right) + D_1 \left((v_n^2)_x + v_n^4 + lv_n^6 \right) \right]_x \\ - \lambda_n^{-1}k_3 \left[(v_n^5)_x - lv_n^1 \right] - l\lambda_n^{-1}D_3 \left[(v_n^6)_x - lv_n^2 \right] = \rho_1f_n^2, \quad (4.4.5)$$

$$iv_n^3 - \lambda_n^{-1}v_n^4 = f_n^3, \quad (4.4.6)$$

$$i\rho_2v_n^4 - \lambda_n^{-1} \left[k_2 \left((v_n^3)_x + D_2 \left(v_n^4 \right)_x \right) \right]_x + \lambda_n^{-1}k_1 \left[(v_n^1)_x + v_n^3 + lv_n^5 \right] \\ + \lambda_n^{-1}D_1 \left[(v_n^2)_x + v_n^4 + lv_n^6 \right] = \rho_2f_n^4, \quad (4.4.7)$$

$$iv_n^5 - \lambda_n^{-1}v_n^6 = f_n^5, \quad (4.4.8)$$

$$i\rho_1v_n^6 - \lambda_n^{-1} \left[k_3 \left((v_n^5)_x - lv_n^1 \right) + D_3 \left((v_n^6)_x - lv_n^2 \right) \right]_x + \lambda_n^{-1}k_1 \left[(v_n^1)_x + v_n^3 + lv_n^5 \right] \\ + \lambda_n^{-1}D_1 \left[(v_n^2)_x + v_n^4 + lv_n^6 \right] = \rho_1f_n^6. \quad (4.4.9)$$

In the following we will check the condition (H2) by finding a contradiction with (4.4.2) such as $\|U_n\|_{\mathcal{H}_j} = o(1)$. For clarity, we divide the proof into several lemmas. From now on, for simplicity, we drop the index n .

First, remark that using (4.4.2) and (4.4.4) (respectively (4.4.6) and (4.4.8)), we deduce that

$$\|v^1\| = o(1), \quad \|v^3\| = o(1), \quad \|v^5\| = o(1). \quad (4.4.10)$$

Lemma 4.4.3. Under condition (4.4.1), we have

$$\lambda^{-1/2}\|v_x^2\| = o(1), \quad \lambda^{-1/2}\|v_x^4\| = o(1) \quad \text{and} \quad \lambda^{-1/2}\|v_x^6\| = o(1). \quad (4.4.11)$$

Proof: Taking the inner product of (4.4.3) with U in \mathcal{H}_j , then using the fact that U is uniformly bounded in \mathcal{H}_j , we get

$$\lambda^{-1} \int_0^L \left(D_1 |v_x^2|^2 + D_2 |v_x^4|^2 + D_3 |v_x^6|^2 \right) dx = \lambda^{-1} \operatorname{Re} (\mathcal{A}_j U, U)_{\mathcal{H}_j} \\ = -\lambda^{-1} \operatorname{Re} (i\lambda U - \mathcal{A}_j U, U)_{\mathcal{H}_j} = o(1).$$

Thanks to condition (4.4.1), we obtain the desired asymptotic equation (4.4.11). Thus, the proof is complete.

Lemma 4.4.4. Under condition (4.4.1), we have

$$\|v_x^1\| = o(1), \quad \|v_x^3\| = o(1) \quad \text{and} \quad \|v_x^5\| = o(1). \quad (4.4.12)$$

Proof: Differentiating (4.4.4), (4.4.6) and (4.4.8) with respect to the variable x , we get

$$v_x^1 = \frac{\lambda^{-1/2}v_x^2}{i\lambda^{1/2}} - if_x^1, \quad v_x^3 = \frac{\lambda^{-1/2}v_x^4}{i\lambda^{1/2}} - if_x^3 \quad \text{and} \quad v_x^5 = \frac{\lambda^{-1/2}v_x^6}{i\lambda^{1/2}} - if_x^5. \quad (4.4.13)$$

Using the asymptotic estimation (4.4.11) and the fact that f^1 , f^3 and f^5 converge to zero in $H_0^1(0, L)$ in (4.4.13), we obtain the desired estimation (4.4.12).

Lemma 4.4.5. Under condition (4.4.1), we have

$$\|v^2\| = o(1), \quad \|v^4\| = o(1) \quad \text{and} \quad \|v^6\| = o(1). \quad (4.4.14)$$

Proof: i) Multiplying (4.4.5) by $-i\overline{v^2}$ in $L^2(0, L)$ and after integrating over x , we get

$$\begin{aligned} \rho_1 \int_0^L |v^2| dx - i\lambda^{-1}k_1 \int_0^L (v_x^1 + v^3 + lv^5) \overline{v_x^2} dx - i\lambda^{-1} \int_0^L D_1 (v_x^2 + v^4 + lv^6) \overline{v_x^2} dx \\ + i\lambda^{-1}k_3 \int_0^L (v_x^5 - lv^1) \overline{v^2} dx + i\lambda^{-1} \int_0^L D_3 (v_x^6 - lv^2) \overline{v^2} dx = -i \int_0^L \rho_1 f^2 \overline{v^2} dx. \end{aligned}$$

Using (4.4.10), the first asymptotic estimation of (4.4.11), (4.4.12), the fact that v^2 , v^4 and v^6 are uniformly bounded in $L^2(0, L)$ and f^2 converges to zero in $L^2(0, L)$ in the above equation, we obtain that $\|v^2\| = o(1)$.

ii) Similarly, multiplying (4.4.7) by $-i\overline{v^4}$ in $L^2(0, L)$, we get

$$\begin{aligned} \rho_2 \int_0^L |v^4| dx - i\lambda^{-1} \int_0^L (k_2 v_x^3 + D_2 v_x^4) \overline{v_x^4} dx - i\lambda^{-1}k_1 \int_0^L (v_x^1 + v^3 + lv^5) \overline{v^4} dx \\ - i\lambda^{-1} \int_0^L D_1 (v_x^2 + v^4 + lv^6) \overline{v^4} dx = -i\rho_2 \int_0^L f^4 \overline{v^4} dx. \end{aligned}$$

Using (4.4.10), (4.4.11), (4.4.12), the fact that v^4 and v^6 are uniformly bounded in $L^2(0, L)$ and f^4 converges to zero in $L^2(0, L)$ in the preceding equation, we deduce that $\|v^4\| = o(1)$.

iii) Finally, multiplying (4.4.9) by $-i\overline{v^6}$ in $L^2(0, L)$, we get

$$\begin{aligned} \rho_1 \int_0^L |v^6| dx - i\lambda^{-1}k_3 \int_0^L (v_x^5 - lv^1) \overline{v_x^6} dx - i\lambda^{-1} \int_0^L D_3 (v_x^6 - lv^2) \overline{v_x^6} dx \\ - i\lambda^{-1}k_1 \int_0^L (v_x^1 + v^3 + lv^5) \overline{v^6} dx - i\lambda^{-1} \int_0^L D_1 (v_x^2 + v^4 + lv^6) \overline{v^6} dx = -i\rho_1 \int_0^L f^6 \overline{v^6} dx. \end{aligned}$$

Using (4.4.10), (4.4.11), (4.4.12), the fact that v^2 , v^4 and v^6 are uniformly bounded in $L^2(0, L)$ and f^6 converges to zero in $L^2(0, L)$ in the previous equation, we deduce that $\|v^6\| = o(1)$. Thus the proof is complete. \square

Proof of Theorem 4.4.2. Using Lemma 4.4.4 and Lemma 4.4.5, we obtain $\|U\|_{\mathcal{H}_j} = o(1)$ which contradicts (4.4.2). Therefore, (H2) holds and consequently we deduce the analytic stability of the system (4.1.1) in the case of three global dampings. \square

4.5 Exponential stability in the case of three local smooth dampings

In this section, we will establish the uniform stability of the C_0 -semigroup $e^{t\mathcal{A}_j}$ in the case when the positive continuous functions D_i , $i = 1, 2, 3$ satisfy the following condition :

$$\exists d_0 > 0 \text{ such that } D_i \geq d_0 > 0 \text{ for every } x \in (\alpha, \beta), 0 < \alpha < \beta < L. \quad (4.5.1)$$

Our main result in this part is the following stability estimate.

Theorem 4.5.1. Assume that (4.5.1) is satisfied and D_1, D_2 and $D_3 \in W^{1,\infty}(0, L)$. The C_0 -semigroup $e^{t\mathcal{A}_j}$ is exponentially stable, i.e., there exist constants $M \geq 1$ and $\epsilon > 0$ independent of U_0 such that

$$\|e^{t\mathcal{A}_j}U_0\|_{\mathcal{H}_j} \leq Me^{-\epsilon t} \|U_0\|_{\mathcal{H}_j}, \quad t \geq 0, j = 1, 2.$$

According to [36] and [60], we have to check if the following conditions hold,

$$i\mathbb{R} \subseteq \rho(\mathcal{A}_j) \quad (H1)$$

and

$$\limsup_{|\lambda| \rightarrow +\infty, \lambda \in \mathbb{R}} \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} = O(1). \quad (H2)$$

Condition $i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$ is already proved in Lemma 4.3.2 and Lemma 4.3.3. We will establish (H2) by contradiction. Suppose that there exist a sequence of real numbers $(\lambda_n)_n$, with $|\lambda_n| \rightarrow +\infty$ and a sequence of vectors

$$U_n = (v_n^1, v_n^2, v_n^3, v_n^4, v_n^5, v_n^6)^\top \in D(\mathcal{A}_j) \text{ with } \|U_n\|_{\mathcal{H}_j} = 1 \quad (4.5.2)$$

such that

$$i\lambda_n U_n - \mathcal{A}_j U_n = (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6)^\top \rightarrow 0 \text{ in } \mathcal{H}_j, \quad j = 1, 2 \quad (4.5.3)$$

detailed as

$$i\lambda_n v_n^1 - v_n^2 = f_n^1, \quad (4.5.4)$$

$$i\rho_1 \lambda_n v_n^2 - k_1 \left[((v_n^1)_x + v_n^3 + lv_n^5) + \frac{D_1}{k_1} ((v_n^2)_x + v_n^4 + lv_n^6) \right]_x - lk_3 [(v_n^5)_x - lv_n^1] - lD_3 [(v_n^6)_x - lv_n^2] = \rho_1 f_n^2, \quad (4.5.5)$$

$$i\lambda_n v_n^3 - v_n^4 = f_n^3, \quad (4.5.6)$$

$$i\rho_2 \lambda_n v_n^4 - k_2 \left[(v_n^3)_x + \frac{D_2}{k_2} (v_n^4)_x \right]_x + k_1 [(v_n^1)_x + v_n^3 + lv_n^5] + D_1 [(v_n^2)_x + v_n^4 + lv_n^6] = \rho_2 f_n^4, \quad (4.5.7)$$

$$i\lambda_n v_n^5 - v_n^6 = f_n^5, \quad (4.5.8)$$

$$\begin{aligned} i\rho_1 \lambda_n v_n^6 - k_3 \left[((v_n^5)_x - lv_n^1) + \frac{D_3}{k_3} ((v_n^6)_x - lv_n^2) \right]_x \\ + lk_1 [(v_n^1)_x + v_n^3 + lv_n^5] + lD_1 [(v_n^2)_x + v_n^4 + lv_n^6] = \rho_1 f_n^6. \end{aligned} \quad (4.5.9)$$

In the following we will check the condition (H2) by finding a contradiction with (4.5.2) such as $\|U_n\|_{\mathcal{H}_j} = o(1)$. For clarity, we divide the proof into several lemmas. From now on, for simplicity, we drop the index n .

Lemma 4.5.2. Assume that condition (4.5.1) is verified. Then, we have

$$\|D_1^{1/2}(v_x^2 + v^4 + lv^6)\| = o(1), \quad \|D_2^{1/2}v_x^4\| = o(1), \quad \|D_3^{1/2}(v_x^6 - lv^2)\| = o(1) \quad (4.5.10)$$

and

$$\|(v_x^2 + v^4 + lv^6)\| = o(1), \quad \|v_x^4\| = o(1), \quad \|(v_x^6 - lv^2)\| = o(1) \quad \text{in } (\alpha, \beta). \quad (4.5.11)$$

Proof: Taking the inner product of (4.5.3) with U in \mathcal{H}_j , we get

$$\begin{aligned} \operatorname{Re}(i\lambda U - \mathcal{A}_j U, U)_{\mathcal{H}_j} &= -\operatorname{Re}(\mathcal{A}_j U, U)_{\mathcal{H}_j} \\ &= \int_0^L (D_1 |v_x^2 + v^4 + lv^6|^2 + D_2 |v_x^4|^2 + D_3 |v_x^6 - lv^2|^2) dx = o(1). \end{aligned} \quad (4.5.12)$$

Thanks to (4.5.1), we obtain the desired asymptotic equation (4.5.10) and (4.5.11). Thus the proof is complete. \square

Lemma 4.5.3. Assume that condition (4.5.1) is verified. Then, we have

$$\|v_x^1 + v^3 + lv^5\| = \frac{o(1)}{\lambda}, \quad \|v_x^3\| = \frac{o(1)}{\lambda}, \quad \|v_x^5 - lv^1\| = \frac{o(1)}{\lambda} \quad \text{in } (\alpha, \beta). \quad (4.5.13)$$

Proof: First, using equations (4.5.4), (4.5.6) and (4.5.8), we obtain

$$\lambda(v_x^1 + v^3 + lv^5) = -i(v_x^2 + f_x^1 + v^4 + f^3 + lv^6 + lf^5). \quad (4.5.14)$$

Consequently,

$$\int_\alpha^\beta \lambda^2 |v_x^1 + v^3 + lv^5|^2 dx \leq 2 \int_\alpha^\beta |v_x^2 + v^4 + lv^6|^2 dx + 2 \int_\alpha^\beta |f_x^1 + f^3 + lf^5|^2 dx. \quad (4.5.15)$$

Using the first estimation of (4.5.11) and the fact that f^1, f^3, f^5 converge to zero in $H_0^1(0, L)$ in (4.5.15), we deduce

$$\int_\alpha^\beta \lambda^2 |v_x^1 + v^3 + lv^5|^2 dx = o(1). \quad (4.5.16)$$

In a similar way, one can prove

$$\int_{\alpha}^{\beta} \lambda^2 |v_x^3|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha}^{\beta} \lambda^2 |v_x^5 - lv^1|^2 dx = o(1). \quad (4.5.17)$$

The proof is thus complete. \square

Here and after ϵ designate a fixed positive real number such that $0 < \alpha + \epsilon < \beta - \epsilon < L$. We define the cut-off function $\eta \in C_c^\infty(\mathbb{R})$ by

$$\eta = 1 \quad \text{on} \quad [\alpha + \epsilon, \beta - \epsilon], \quad 0 \leq \eta \leq 1, \quad \eta = 0 \quad \text{on} \quad (0, L) \setminus (\alpha, \beta).$$

Lemma 4.5.4. Assume that condition (4.5.1) is verified. Then, we have

$$\int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = o(1), \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = o(1), \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^5|^2 dx = o(1). \quad (4.5.18)$$

Proof: First, multiplying equation (4.5.4) by $i\lambda\eta v^{\bar{1}}$ in $L^2(0, L)$ and integrating by parts, we get

$$- \int_0^L \eta |\lambda v^1|^2 dx - i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = i \int_0^L \lambda f^1 \eta \bar{v}^1 dx. \quad (4.5.19)$$

As λv^1 is uniformly bounded in $L^2(0, L)$ and f^1 converges to zero in $H_0^1(0, L)$, we get that the term on the right hand side of (4.5.19) converges to zero and consequently

$$- \int_0^L \eta |\lambda v^1|^2 dx - i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = o(1). \quad (4.5.20)$$

Moreover, multiplying (4.5.5) by $\rho_1^{-1} \eta \bar{v}^1$ in $L^2(0, L)$ and then integrating by parts, we obtain

$$\begin{aligned} & i \int_0^L \lambda \eta v^2 \bar{v}^1 dx + k_1 \rho_1^{-1} \int_0^L \left((v_x^1 + v^3 + lv^5) + \frac{D_1}{k_1} (v_x^2 + v^4 + lv^6) \right) (\eta \bar{v}^1)_x dx \\ & - lk_3 \rho_1^{-1} \int_0^L (v_x^5 - lv^1) \eta \bar{v}^1 dx - l \rho_1^{-1} \int_0^L D_3 (v_x^6 - lv^2) \eta \bar{v}^1 dx = \int_0^L f^2 \eta \bar{v}^1 dx. \end{aligned} \quad (4.5.21)$$

Using (4.5.10), (4.5.13), the fact that f^2 converges to zero in $L^2(0, L)$ and λv^1 is uniformly bounded in $L^2(0, L)$ in (4.5.21), we get

$$i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = o(1). \quad (4.5.22)$$

Finally, using (4.5.22) in (4.5.20), we get

$$\int_0^L \eta |\lambda v^1|^2 dx = o(1), \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = o(1).$$

In a same way, we show

$$\int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = o(1), \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^5|^2 dx = o(1).$$

\square

Lemma 4.5.5. Assume that condition (4.5.1) is verified. Then, we have

$$\|\sqrt{D_2}\lambda v^4\| = O(1). \quad (4.5.23)$$

Proof: First, multiplying (4.5.7) by $iD_2\lambda\bar{v}^4$ and integrating by parts, we get

$$\begin{aligned} \rho_2 \int_0^L D_2 |\lambda v^4|^2 dx &= \operatorname{Re} \left\{ k_2 \int_0^L i\lambda v_x^3 \left(D_2 \bar{v}^4 \right)_x dx + \int_0^L D_2 v_x^4 i\lambda \left(D_2 \bar{v}^4 \right)_x dx \right. \\ &\quad + \int_0^L ik_1 (v_x^1 + v^3 + lv^5) D_2 \lambda \bar{v}^4 dx \\ &\quad \left. + \int_0^L iD_1 (v_x^2 + v^4 + lv^6) D_2 \lambda \bar{v}^4 dx - \int_0^L i\rho_2 f^4 D_2 \lambda \bar{v}^4 dx \right\}. \end{aligned} \quad (4.5.24)$$

(i) **Estimation of the second term of (4.5.24).** Using (4.5.6), we have

$$k_2 \int_0^L (i\lambda v^3)_x \left(D_2 \bar{v}^4 \right)_x dx = k_2 \int_0^L (v_x^4 + f^3) \left(D_2' \bar{v}^4 + D_2 \bar{v}_x^4 \right) dx. \quad (4.5.25)$$

Using the fact that v^4 is uniformly bounded in $L^2(0, L)$, f^3 converges to zero in $H_0^1(0, L)$ and $\|D_2 v_x^4\| = o(1)$ due to (4.5.10) in the above equation, we deduce that

$$k_2 \int_0^L (i\lambda v^3)_x \left(D_2 \bar{v}^4 \right)_x dx = o(1). \quad (4.5.26)$$

(ii) **Estimation of the third term of (4.5.24).** We have

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L D_2 v_x^4 i\lambda \left(D_2 \bar{v}^4 \right)_x dx \right\} &= \operatorname{Re} \left\{ \int_0^L D_2 v_x^4 i\lambda \left(D_2' \bar{v}^4 + D_2 \bar{v}_x^4 \right) dx \right\} \\ &= \operatorname{Re} \left\{ \int_0^L i\lambda D_2 v_x^4 \bar{v}^4 D_2' dx \right\} + \operatorname{Re} \left\{ \int_0^L |D_2|^2 i\lambda |v_x^4|^2 dx \right\} \\ &= \operatorname{Re} \left\{ \int_0^L i\lambda D_2 v_x^4 \bar{v}^4 D_2' dx \right\}. \end{aligned}$$

Let ϵ_1 be a positive constant. Using Young's inequality in the above equation and then using the second estimation of (4.5.10), we get

$$\operatorname{Re} \left\{ \int_0^L D_2 v_x^4 i\lambda \left(D_2 \bar{v}^4 \right)_x dx \right\} \leq \epsilon_1 \int_0^L D_2 |\lambda v^4|^2 dx + \frac{1}{\epsilon_1} \int_0^L D_2 |v_x^4|^2 |D_2'|^2 dx \quad (4.5.27)$$

$$\leq \epsilon_1 \int_0^L D_2 |\lambda v^4|^2 dx + o(1). \quad (4.5.28)$$

(iii) **Estimation of the fourth term of (4.5.24).** Let $\epsilon_2 > 0$. Using Young's inequality and the fact that $v_x^1 + v^3 + lv^5$ is uniformly bounded in $L^2(0, L)$ due to (4.5.2) in the fourth

term of (4.5.24), we obtain

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L i k_1 (v_x^1 + v^3 + lv^5) D_2 \lambda \bar{v}^4 dx \right\} &\leq \epsilon_2 \int_0^L D_2 |\lambda v^4|^2 dx + \frac{1}{\epsilon_2} \int_0^L k_1^2 D_2 |v_x^1 + v^3 + lv^5|^2 dx \\ &\leq \epsilon_2 \int_0^L D_2 |\lambda v^4|^2 dx + O(1). \end{aligned} \quad (4.5.29)$$

(iv) **Estimation of the fifth term of (4.5.24).** Let $\epsilon_3 > 0$. Using Young's inequality and the first estimation of (4.5.10) in the fifth term of (4.5.24), we obtain

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L i D_1 (v_x^2 + v^4 + lv^6) D_2 \lambda \bar{v}^4 dx \right\} &\leq \epsilon_3 \int_0^L D_2 |\lambda v^4|^2 dx + \frac{1}{\epsilon_3} \int_0^L |D_1|^2 |v_x^2 + v^4 + lv^6|^2 D_2 dx \\ &\leq \epsilon_3 \int_0^L D_2 |\lambda v^4|^2 dx + o(1). \end{aligned} \quad (4.5.30)$$

(V) **Estimation of the last term of (4.5.24).** Let $\epsilon_4 > 0$. Using Young's inequality and the fact that f^4 converges to zero in $L^2(0, L)$, we get

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L i \rho_2 f^4 D_2 \lambda \bar{v}^4 dx \right\} &\leq \epsilon_4 \int_0^L D_2 |\lambda v^4|^2 dx + \frac{1}{\epsilon_4} \int_0^L D_2 \rho_2^2 |f^4|^2 dx \\ &\leq \epsilon_4 \int_0^L D_2 |\lambda v^4|^2 dx + o(1). \end{aligned} \quad (4.5.31)$$

Main estimation. Finally, inserting (4.5.26), (4.5.28), (4.5.29), (4.5.30) and (4.5.31) into (4.5.24), we get

$$(\rho_2 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) \int_0^L D_2 |\lambda v^4|^2 dx \leq O(1). \quad (4.5.32)$$

Taking $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \frac{\rho_2}{8}$ in the above equation, we get the desired estimation (4.5.23). The proof is thus complete. \square

Lemma 4.5.6. Assume that condition (4.5.1) is verified. Then, we have

$$\|\sqrt{D_1} \lambda v^2\| = O(1). \quad (4.5.33)$$

Proof: Multiplying (4.5.5) by $i D_1 \lambda \bar{v}^2$ and integrating by parts, we get

$$\begin{aligned} \rho_1 \int_0^L D_1 |\lambda v^2|^2 dx &= \operatorname{Re} \left\{ k_1 \int_0^L ((i \lambda v^1)_x + i \lambda v^3 + li \lambda v^5) (D_1 \bar{v}^2)_x dx + \int_0^L D_1 (i \lambda v^2)_x (D_1 \bar{v}^2)_x dx \right. \\ &\quad + \int_0^L D_1 (v^4 + lv^6) i \lambda (D_1 \bar{v}^2)_x dx - \int_0^L i l k_3 (v_x^5 - lv^1) D_1 \lambda \bar{v}^2 dx \\ &\quad \left. - \int_0^L i l D_3 (v_x^6 - lv^2) D_1 \lambda \bar{v}^2 - \int_0^L \rho_1 f^2 i \lambda D_1 \bar{v}^2 dx \right\}. \end{aligned} \quad (4.5.34)$$

(i) **Estimation of the second term of (4.5.34).** Using equations (4.5.4), (4.5.6) and (4.5.8), we get

$$\begin{aligned} & \operatorname{Re} \left\{ k_1 \int_0^L ((i\lambda v^1)_x + i\lambda v^3 + li\lambda v^5)(D_1 \overline{v^2})_x dx \right\} = \\ & \operatorname{Re} \left\{ k_1 \int_0^L (v_x^2 + v^4 + lv^6 + f_x^1 + f^3 + lf^5)(D_1' \overline{v^2} + D_1 \overline{v_x^2}) dx \right\}. \end{aligned}$$

Consequently, using the fact that v^2 is uniformly bounded in $L^2(0, L)$, f^1, f^3, f^5 converge to zero in $H_0^1(0, L)$ and equation (4.5.10), we obtain

$$\operatorname{Re} \left\{ k_1 \int_0^L ((i\lambda v^1)_x + i\lambda v^3 + li\lambda v^5)(D_1 \overline{v^2})_x dx \right\} = o(1). \quad (4.5.35)$$

(ii) **Estimation of the third term of (4.5.34).** We have

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L D_1 v_x^2 i\lambda (D_1 \overline{v^2})_x dx \right\} &= \operatorname{Re} \left\{ \int_0^L D_1 v_x^2 i\lambda (D_1' \overline{v^2} + D_1 \overline{v_x^2}) dx \right\} \\ &= \operatorname{Re} \left\{ \int_0^L D_1 v_x^2 i\lambda \overline{v^2} D_1' dx \right\} + \operatorname{Re} \left\{ \int_0^L i\lambda |D_1|^2 |v_x^2|^2 dx \right\} \\ &= \operatorname{Re} \left\{ \int_0^L D_1 v_x^2 i\lambda \overline{v^2} D_1' dx \right\}. \end{aligned}$$

Let $\epsilon_1 > 0$. Using Young's inequality in the above equation and then using the first estimation of (4.5.10), we get

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L D_1 v_x^2 i\lambda (D_1 \overline{v^2})_x dx \right\} &\leq \epsilon_1 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_1} \int_0^L D_1 |v_x^2|^2 |D_1'|^2 dx \\ &\leq \epsilon_1 \int_0^L D_1 |\lambda v^2|^2 dx + o(1). \end{aligned} \quad (4.5.36)$$

(iii) **Estimation of the fourth term of (4.5.34).** We have

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L D_1 (v^4 + lv^6) i\lambda (D_1 \overline{v^2})_x dx \right\} &= \operatorname{Re} \left\{ \int_0^L D_1 (v^4 + lv^6) i\lambda (D_1' \overline{v^2} + D_1 \overline{v_x^2}) dx \right\} \\ &= \operatorname{Re} \left\{ \int_0^L D_1 (v^4 + lv^6) D_1' i\lambda \overline{v^2} dx \right\} \\ &\quad + \operatorname{Re} \left\{ \int_0^L |D_1|^2 (v^4 + lv^6) i\lambda \overline{v_x^2} dx \right\}. \end{aligned} \quad (4.5.37)$$

Now, we need to estimate each term of (4.5.37) as follows :

1) Let $\epsilon_2 > 0$ and using Young's inequality and the fact that v^4 and v^6 are uniformly bounded in $L^2(0, L)$, we get

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L D_1 (v^4 + lv^6) D_1' i\lambda \overline{v^2} dx \right\} &\leq \epsilon_2 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_2} \int_0^L D_1 |D_1'|^2 |v^4 + lv^6|^2 dx \\ &\leq \epsilon_2 \int_0^L D_1 |\lambda v^2|^2 dx + O(1). \end{aligned} \quad (4.5.38)$$

2) We have

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L |D_1|^2 (v^4 + lv^6) i \lambda \overline{v_x^2} dx \right\} &= \operatorname{Re} \left\{ \int_0^L D_1 i \lambda v^4 D_1 \overline{v_x^2} dx \right\} \\ &+ \operatorname{Re} \left\{ \int_0^L |D_1|^2 i \lambda v^6 \overline{v_x^2} dx \right\}. \end{aligned} \quad (4.5.39)$$

Hence, since $\operatorname{supp} D_1 = \operatorname{supp} D_2$ by using Lemma 4.5.5 and estimation (4.5.10), we get

$$\operatorname{Re} \left\{ \int_0^L D_1 i \lambda v^4 D_1 \overline{v_x^2} dx \right\} = o(1). \quad (4.5.40)$$

On the other hand, after integrating by parts, then using Young's inequality, the fact that v^6 is uniformly bounded in $L^2(0, L)$ and the estimation (4.5.10), we get for $\epsilon_3 > 0$, $\epsilon_4 > 0$

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L |D_1|^2 i \lambda v^6 \overline{v_x^2} dx \right\} &= - \operatorname{Re} \left\{ \int_0^L 2 D_1 D_1' l v^6 i \lambda \overline{v^2} dx \right\} - \operatorname{Re} \left\{ \int_0^L D_1 l v_x^6 D_1 i \lambda \overline{v^2} dx \right\} \\ &\leq \epsilon_3 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_3} \int_0^L 4 D_1 |D_1'|^2 |v^6|^2 dx \\ &\quad + \epsilon_4 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_4} \int_0^L |D_1|^3 |v_x^6|^2 dx \\ &\leq (\epsilon_3 + \epsilon_4) \int_0^L D_1 |\lambda v^2|^2 dx + O(1). \end{aligned} \quad (4.5.41)$$

Finally, inserting (4.5.38), (4.5.40), (4.5.41) into (4.5.37), we obtain

$$\operatorname{Re} \left\{ \int_0^L D_1 (v^4 + lv^6) i \lambda (D_1 \overline{v^2})_x dx \right\} \leq (\epsilon_2 + \epsilon_3 + \epsilon_4) \int_0^L D_1 |\lambda v^2|^2 dx + O(1). \quad (4.5.42)$$

(iv) **Estimation of the fifth term of (4.5.34).** For $\epsilon_5 > 0$, by using Young's inequality and the fact that $v_x^5 - lv^1$ is uniformly bounded in $L^2(0, L)$, we get

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L i l k_3 (v_x^5 - lv^1) D_1 \lambda \overline{v^2} dx \right\} &\leq \epsilon_5 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_5} \int_0^L l^2 k_3^2 D_1 |v_x^5 - lv^1|^2 dx \\ &\leq \epsilon_5 \int_0^L D_1 |\lambda v^2|^2 dx + O(1). \end{aligned} \quad (4.5.43)$$

(v) **Estimation of the sixth term of (4.5.34).** For $\epsilon_6 > 0$, by using Young's inequality and the third estimation of (4.5.10), we obtain

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L i l D_3 (v_x^6 - lv^2) D_1 \lambda \overline{v^2} dx \right\} &\leq \epsilon_6 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_6} \int_0^L l^2 D_1 |D_3|^2 |v_x^6 - lv^2|^2 dx \\ &\leq \epsilon_6 \int_0^L D_1 |\lambda v^2|^2 dx + o(1). \end{aligned} \quad (4.5.44)$$

(vi) **Estimation of the last term of (4.5.34).** For $\epsilon_7 > 0$, by using Young's inequality and the fact that f^2 converges to zero in $L^2(0, L)$, we get

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L \rho_1 f^2 i \lambda D_1 \overline{v^2} dx \right\} &\leq \epsilon_7 \int_0^L D_1 |\lambda v^2|^2 dx + \frac{1}{\epsilon_7} \int_0^L D_1 \rho_1^2 |f^2|^2 dx \\ &\leq \epsilon_7 \int_0^L D_1 |\lambda v^2|^2 dx + o(1). \end{aligned} \quad (4.5.45)$$

Main estimation. Inserting (4.5.35), (4.5.36), (4.5.42), (4.5.43), (4.5.44) and (4.5.45) into (4.5.34), we obtain

$$(\rho_1 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7) \int_0^L D_1 |\lambda v^6|^2 dx \leq O(1). \quad (4.5.46)$$

Taking $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon_5 = \epsilon_6 = \epsilon_7 = \frac{\rho_1}{14}$ in the above equation, we get the desired estimation (4.5.33). The proof is thus complete. \square

Lemma 4.5.7. Assume that condition (4.5.1) is verified. Then, we have

$$\|\sqrt{D_3} \lambda v^6\| = O(1). \quad (4.5.47)$$

Proof: First, multiplying (4.5.9) by $i D_3 \lambda \overline{v^6}$ and integrating by parts, we get

$$\begin{aligned} \rho_1 \int_0^L D_3 |\lambda v^6|^2 dx &= \operatorname{Re} \left\{ k_3 \int_0^L ((i \lambda v^5)_x - i \lambda v^1) (D_3 \overline{v^6})_x dx + \int_0^L D_3 v_x^6 i \lambda (D_3 \overline{v^6})_x dx \right. \\ &\quad - \int_0^L l v^2 D_3 i \lambda (D_3 \overline{v^6})_x dx + \int_0^L i l k_1 (v_x^1 + v^3 + l v^5) D_3 \lambda \overline{v^6} dx \\ &\quad \left. + \int_0^L i l D_1 (v_x^2 + v^4 + l v^6) D_3 \lambda \overline{v^6} dx - \int_0^L i \rho_1 f^6 D_3 \lambda \overline{v^6} dx \right\}. \end{aligned} \quad (4.5.48)$$

(i) **Estimation of the second term of (4.5.48).** Using (4.5.4) and (4.5.8), we get

$$\begin{aligned} &\operatorname{Re} \left\{ k_3 \int_0^L ((i \lambda v^5)_x - i \lambda v^1) (D_3 \overline{v^6})_x dx \right\} \\ &= \operatorname{Re} \left\{ k_3 \int_0^L (v_x^6 + f_x^5 - l v^2 - l f^1) (D_3' \overline{v^6} + D_3 \overline{v_x^6}) dx \right\}, \end{aligned} \quad (4.5.49)$$

consequently, by using the fact that v^6 is uniformly bounded in $L^2(0, L)$, f^1 , f^5 converge to zero in $H_0^1(0, L)$ and the third estimation of (4.5.10), we get

$$\operatorname{Re} \left\{ k_3 \int_0^L ((i \lambda v^5)_x - i \lambda v^1) (D_3 \overline{v^6})_x dx \right\} = o(1). \quad (4.5.50)$$

(ii) **Estimation of the third term of (4.5.48).** We have

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L D_3 v_x^6 i \lambda (D_3 \overline{v^6})_x dx \right\} &= \operatorname{Re} \left\{ \int_0^L D_3 v_x^6 i \lambda (D_3' \overline{v^6} + D_3 \overline{v_x^6}) dx \right\} \\ &= \operatorname{Re} \left\{ \int_0^L D_3 v_x^6 i \lambda \overline{v^6} D_3' dx \right\} + \operatorname{Re} \left\{ \int_0^L i \lambda |D_3|^2 |v_x^6|^2 dx \right\} \\ &= \operatorname{Re} \left\{ \int_0^L D_3 v_x^6 i \lambda \overline{v^6} D_3' dx \right\}. \end{aligned}$$

Consequently, using Young's inequality in the above equation and then using the third estimation of (4.5.10), we get for $\epsilon_1 > 0$

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L D_3 v_x^6 i \lambda \left(D_3 \overline{v^6} \right)_x dx \right\} &\leq \epsilon_1 \int_0^L D_3 |\lambda \overline{v^6}|^2 dx + \frac{1}{\epsilon_1} \int_0^L D_3 |v_x^6|^2 |D_3'|^2 dx \\ &\leq \epsilon_1 \int_0^L D_3 |\lambda \overline{v^6}|^2 dx + o(1). \end{aligned} \quad (4.5.51)$$

(iii) **Estimation of the fourth term of (4.5.48).** We have

$$\operatorname{Re} \left\{ \int_0^L l v^2 D_3 i \lambda (D_3 \overline{v^6})_x dx \right\} = \operatorname{Re} \left\{ \int_0^L l i \lambda v^2 D_3 (D_3' \overline{v^6} + D_3 \overline{v_x^6}) dx \right\}, \quad (4.5.52)$$

since $\operatorname{supp} D_3 = \operatorname{supp} D_1$, using Lemma 4.5.6, the fact that v^6 is uniformly bounded in $L^2(0, L)$ and the third estimation of (4.5.10), we deduce that

$$\operatorname{Re} \left\{ \int_0^L l v^2 D_3 i \lambda (D_3 \overline{v^6})_x dx \right\} = O(1). \quad (4.5.53)$$

(iv) **Estimation of the fifth term of (4.5.48).** Let $\epsilon_2 > 0$. Using Young's inequality and then the fact that $v_x^1 + v_3 + l v^5$ is uniformly bounded in $L^2(0, L)$, we get

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L i l k_1 (v_x^1 + v_3 + l v^5) D_3 \lambda \overline{v^6} dx \right\} &\leq \epsilon_2 \int_0^L D_3 |\lambda v^6|^2 dx + \frac{1}{\epsilon_2} \int_0^L l^2 k_1^2 D_3 |v_x^1 + v_3 + l v^5|^2 dx \\ &\leq \epsilon_2 \int_0^L D_3 |\lambda v^6|^2 dx + O(1). \end{aligned} \quad (4.5.54)$$

(v) **Estimation of the sixth term of (4.5.48).** Let $\epsilon_3 > 0$. Using Young's inequality and then the first estimation of (4.5.10), we obtain

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L i l D_1 (v_x^2 + v^4 + l v^6) D_3 \lambda \overline{v^6} dx \right\} &\leq \epsilon_3 \int_0^L D_3 |\lambda v^6|^2 dx + \frac{1}{\epsilon_3} \int_0^L l^2 D_3 |D_1|^2 |v_x^2 + v^4 + l v^6|^2 dx \\ &\leq \epsilon_3 \int_0^L D_3 |\lambda v^6|^2 dx + o(1). \end{aligned} \quad (4.5.55)$$

(vi) **Estimation of the last term of (4.5.48).** Let $\epsilon_4 > 0$. Using Young's inequality and then the fact that f^6 converges to zero in $L^2(0, L)$, we obtain

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L i \rho_1 f^6 D_3 \lambda \overline{v^6} dx \right\} &\leq \epsilon_4 \int_0^L D_3 |\lambda v^6|^2 dx + \frac{1}{\epsilon_4} \int_0^L D_3 \rho_2^2 |f^6|^2 dx \\ &\leq \epsilon_4 \int_0^L D_3 |\lambda v^6|^2 dx + o(1). \end{aligned} \quad (4.5.56)$$

Main estimation. Inserting (4.5.50), (4.5.51), (4.5.53), (4.5.54), (4.5.55) and (4.5.56) into (4.5.48), we get

$$(\rho_1 - \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) \int_0^L D_3 |\lambda v^6|^2 dx \leq O(1). \quad (4.5.57)$$

Taking $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \frac{\rho_1}{8}$ in the above equation, we get the desired estimation (4.5.47). The proof is thus complete. \square

Now, let $h \in H_0^1(0, L)$.

Lemma 4.5.8. Assume that condition (4.5.1) is verified. Then, we have

$$\begin{aligned} & \frac{1}{2}\rho_1 \int_0^L h'|v^2|^2 dx + \frac{k_1}{2} \int_0^L h'|v_x^1 + \frac{D_1}{k_1}(v_x^2 + v^4 + lv^6)|^2 dx \\ & + \operatorname{Re} \left\{ -k_1 \int_0^L h v_x^3 \overline{v_x^1} dx - lk_1 \int_0^L v_x^5 h \overline{v_x^1} dx - lk_3 \int_0^L v_x^5 h \overline{v_x^1} dx \right\} = o(1). \end{aligned} \quad (4.5.58)$$

Proof: First, let $M = v_x^1 + \frac{D_1}{k_1}(v_x^2 + v^4 + lv^6)$. Multiplying (4.5.5) by $h\overline{M}$ then integrating by parts and using (4.5.10), the fact that v_x^1 is uniformly bounded in $L^2(0, L)$ and f^2 converges to zero in $L^2(0, L)$, we get

$$\begin{aligned} & \operatorname{Re} \left\{ \int_0^L i\lambda\rho_1 v^2 h \overline{M} dx - k_1 \int_0^L v_x^3 h \overline{M} dx - lk_1 \int_0^L v_x^5 h \overline{M} dx \right. \\ & \left. + \frac{k_1}{2} \int_0^L h'|M|^2 dx - lk_3 \int_0^L (v_x^5 - lv^1) h \overline{M} dx \right\} = o(1). \end{aligned} \quad (4.5.59)$$

(i) **Estimation of the first term of (4.5.59).** First, we have

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L i\lambda\rho_1 v^2 h \overline{M} dx \right\} &= -\operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h (i\lambda v^1)_x dx \right\} \\ &+ \operatorname{Re} \left\{ \int_0^L i\lambda v^2 \rho_1 h \frac{D_1}{k_1} \overline{(v_x^2 + v^4 + lv^6)} dx \right\}. \end{aligned} \quad (4.5.60)$$

Next, we need to estimate each term of (4.5.60). For this, by using equation (4.5.4) we get

$$\begin{aligned} -\operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h (i\lambda v^1)_x dx \right\} &= -\operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h (\overline{v_x^2} + \overline{f_x^1}) dx \right\} \\ &= \frac{1}{2}\rho_1 \int_0^L h'|v^2|^2 dx - \operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h \overline{f_x^1} dx \right\}. \end{aligned} \quad (4.5.61)$$

In addition, using the fact that v^2 is uniformly bounded in $L^2(0, L)$ and f^1 converges to zero in $H_0^1(0, L)$, we obtain

$$\operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h \overline{f_x^1} dx \right\} = o(1). \quad (4.5.62)$$

Inserting (4.5.62) into (4.5.61), we get

$$-\operatorname{Re} \left\{ \int_0^L \rho_1 v^2 h (i\lambda v^1)_x dx \right\} = \frac{1}{2}\rho_1 \int_0^L h'|v^2|^2 dx + o(1). \quad (4.5.63)$$

Moreover, using Lemma 4.5.6 and the first estimation of (4.5.10), we deduce that

$$\operatorname{Re}\left\{\int_0^L i\lambda v^2 \rho_1 h \frac{D_1}{k_1} (\overline{v_x^2 + v^4 + lv^6}) dx\right\} = o(1). \quad (4.5.64)$$

Finally, inserting (4.5.63) and (4.5.64) into (4.5.60), we get

$$\operatorname{Re}\left\{\int_0^L i\lambda \rho_1 v^2 h \overline{M} dx\right\} = \frac{1}{2} \rho_1 \int_0^L h' |v^2|^2 dx + o(1). \quad (4.5.65)$$

(ii) **Estimation of the second and third terms of (4.5.59).** By using the fact that v_x^3 is uniformly bounded in $L^2(0, L)$ and the first estimation of (4.5.10), we get

$$\begin{aligned} \operatorname{Re}\left\{k_1 \int_0^L v_x^3 h \overline{M} dx\right\} &= \operatorname{Re}\left\{k_1 \int_0^L v_x^3 h \overline{v_x^1} dx + \int_0^L v_x^3 D_1 (\overline{v_x^2 + v^4 + lv^6}) dx\right\} \\ &= \operatorname{Re}\left\{k_1 \int_0^L v_x^3 h \overline{v_x^1} dx\right\} + o(1). \end{aligned} \quad (4.5.66)$$

Also, by using the fact that v_x^5 is uniformly bounded in $L^2(0, L)$ and the first estimation of (4.5.10), we get

$$\begin{aligned} \operatorname{Re}\left\{lk_1 \int_0^L v_x^5 h \overline{M} dx\right\} &= \operatorname{Re}\left\{l \int_0^L v_x^5 h \overline{v_x^1} dx + lk_1 \int_0^L v_x^5 D_1 (\overline{v_x^2 + v^4 + lv^6}) dx\right\} \\ &= \operatorname{Re}\left\{lk_1 \int_0^L v_x^5 h \overline{v_x^1} dx\right\} + o(1). \end{aligned} \quad (4.5.67)$$

(iii) **Estimation of the fifth term of (4.5.59).** Using the fact that v_x^1 and v_x^5 are uniformly bounded in $L^2(0, L)$, $v^1 = O(\frac{1}{\lambda})$ and the first estimation of (4.5.10), we get

$$\begin{aligned} \operatorname{Re}\left\{-lk_3 \int_0^L (v_x^5 - lv^1) h \overline{M} dx\right\} &= \operatorname{Re}\left\{-lk_3 \int_0^L (v_x^5 - lv^1) h \left(\overline{v_x^1} + \frac{D_1}{k_1} (\overline{v_x^2 + v^4 + lv^6})\right)\right\} \\ &= \operatorname{Re}\left\{-lk_3 \int_0^L v_x^5 h \overline{v_x^1} dx\right\} + o(1). \end{aligned} \quad (4.5.68)$$

(iv) **Main estimation.** Inserting (4.5.65), (4.5.66), (4.5.67) and (4.5.68) into (4.5.59), we obtain the desired estimation (4.5.58). Thus the proof is complete. \square

Lemma 4.5.9. Assume that condition (4.5.1) is verified. Then, we have

$$\frac{1}{2} \rho_2 \int_0^L h' |v^4|^2 dx + \frac{k_2}{2} \int_0^L h' |v_x^3 + \frac{D_2}{k_2} v^4|^2 dx + \operatorname{Re}\left\{k_1 \int_0^L v_x^1 h \overline{v_x^3} dx\right\} = o(1). \quad (4.5.69)$$

Proof: Let $N = v_x^3 + \frac{D_2}{k_2}v_x^4$. Multiplying (4.5.7) by $h\bar{N}$ then integrating by parts and using (4.5.10), the fact that v_x^3 is uniformly bounded in $L^2(0, L)$ and f^4 converges to zero in $L^2(0, L)$, we get

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L i\lambda\rho_2v^4h\bar{N}dx + \frac{k_2}{2} \int_0^L h'|N|^2dx \right. \\ \left. + k_1 \int_0^L (v_x^1 + v^3 - lv^5) h\bar{N}dx \right\} = o(1). \end{aligned} \quad (4.5.70)$$

(i) **Estimation of the first term of (4.5.70).** First, we have

$$\begin{aligned} \operatorname{Re} \left\{ \int_0^L i\lambda\rho_2v^4h\bar{N}dx \right\} = -\operatorname{Re} \left\{ \int_0^L \rho_2v^4h(i\lambda v^3)_x dx \right\} \\ + \operatorname{Re} \left\{ \int_0^L i\lambda v^4\rho_2h\frac{D_2}{k_2}\bar{v}_x^4 dx \right\}. \end{aligned} \quad (4.5.71)$$

Next, we need to estimate each term of (4.5.71). For this, by using equation (4.5.6) and then integrating by parts, we get

$$\begin{aligned} -\operatorname{Re} \left\{ \int_0^L \rho_1v^4h(i\lambda v^3)_x dx \right\} = -\operatorname{Re} \left\{ \int_0^L \rho_1v^4h(\bar{v}_x^4 + \bar{f}_x^3) dx \right\} \\ = \frac{1}{2}\rho_2 \int_0^L h'|v^4|^2 dx - \operatorname{Re} \left\{ \int_0^L \rho_2v^4h\bar{f}_x^3 dx \right\}. \end{aligned} \quad (4.5.72)$$

In addition, using the fact that v^4 is uniformly bounded in $L^2(0, L)$ and f^3 converges to zero in $H_0^1(0, L)$, we obtain

$$\operatorname{Re} \left\{ \int_0^L \rho_2v^4h\bar{f}_x^3 dx \right\} = o(1). \quad (4.5.73)$$

Inserting (4.5.73) into (4.5.72), we get

$$-\operatorname{Re} \left\{ \int_0^L \rho_2v^4h(i\lambda v^3)_x dx \right\} = \frac{1}{2}\rho_2 \int_0^L h'|v^4|^2 dx + o(1). \quad (4.5.74)$$

Moreover, using Lemma 4.5.5 and the second estimation of (4.5.10), we deduce that

$$\operatorname{Re} \left\{ \int_0^L i\lambda v^4\rho_2h\frac{D_2}{k_2}\bar{v}_x^4 dx \right\} = o(1). \quad (4.5.75)$$

Finally, inserting (4.5.74) and (4.5.75) into (4.5.71), we get

$$\operatorname{Re} \left\{ \int_0^L i\lambda\rho_2v^4h\bar{N}dx \right\} = \frac{1}{2}\rho_2 \int_0^L h'|v^4|^2 dx + o(1). \quad (4.5.76)$$

(ii) **Estimation of the third term of (4.5.70).** Using the fact that v_x^1, v_x^3 are uniformly bounded in $L^2(0, L)$, $v^3 = O(\frac{1}{\lambda})$, $v^5 = O(\frac{1}{\lambda})$ and the second estimation of (4.5.10), we get

$$\begin{aligned} \operatorname{Re}\left\{k_1 \int_0^L (v_x^1 + v^3 - lv^5) h \bar{N} dx\right\} &= \operatorname{Re}\left\{k_1 \int_0^L (v_x^1 + v^3 - lv^5) h \left(\bar{v}_x^3 + \frac{D_2}{k_2} \bar{v}_x^4\right) dx\right\} \\ &= \operatorname{Re}\left\{k_1 \int_0^L v_x^1 h \bar{v}_x^3 dx\right\} + o(1). \end{aligned} \quad (4.5.77)$$

(iii) **Main estimation.** Inserting (4.5.76) and (4.5.77) into (4.5.70) we get the desired estimation (4.5.69). Thus the proof is complete.

Lemma 4.5.10. Assume that condition (4.5.1) is verified. Then, we have

$$\begin{aligned} &\frac{1}{2}\rho_1 \int_0^L h' |v^6|^2 dx + \frac{k_3}{2} \int_0^L h' |v_x^5 + \frac{D_3}{k_3} (v_x^6 - lv^2)|^2 dx \\ &+ \operatorname{Re}\left\{lk_1 \int_0^L v_x^1 h \bar{v}_x^5 dx + lk_3 \int_0^L v_x^1 h \bar{v}_x^5 dx\right\} = o(1). \end{aligned} \quad (4.5.78)$$

Proof: Let $T = v_x^5 + \frac{D_3}{k_3} (v_x^6 - lv^2)$. Multiplying (4.5.9) by $h\bar{T}$ then integrating by parts and using (4.5.10), the fact that v_x^5 is uniformly bounded in $L^2(0, L)$ and f^6 converges to zero in $L^2(0, L)$, we get

$$\begin{aligned} \operatorname{Re}\left\{\int_0^L i\lambda\rho_1 v^6 h \bar{T} dx + \frac{k_3}{2} \int_0^L h' |T|^2 dx + lk_3 \int_0^L v_x^1 h \bar{T} dx\right. \\ \left.+ lk_1 \int_0^L (v_x^1 + v^3 + lv^5) h \bar{T} dx\right\} = o(1). \end{aligned} \quad (4.5.79)$$

(i) **Estimation of the first term of (4.5.79).** First, we have

$$\begin{aligned} \operatorname{Re}\left\{\int_0^L i\lambda\rho_1 v^6 h \bar{T} dx\right\} &= -\operatorname{Re}\left\{\int_0^L \rho_1 v^6 h (i\lambda v^5)_x dx\right\} \\ &+ \operatorname{Re}\left\{\int_0^L i\lambda v^6 \rho_1 h \frac{D_3}{k_3} (\overline{v_x^6 - lv^2}) dx\right\}. \end{aligned} \quad (4.5.80)$$

Next, we need to estimate each term of (4.5.80). For this, by using equation (4.5.8) we get

$$\begin{aligned} -\operatorname{Re}\left\{\int_0^L \rho_1 v^6 h (i\lambda v^5)_x dx\right\} &= -\operatorname{Re}\left\{\int_0^L \rho_1 v^6 h (\bar{v}_x^6 + \bar{f}_x^5) dx\right\} \\ &= \frac{1}{2}\rho_1 \int_0^L h' |v^6|^2 dx - \operatorname{Re}\left\{\int_0^L \rho_1 v^6 h \bar{f}_x^5 dx\right\}. \end{aligned} \quad (4.5.81)$$

Moreover, using the fact that v^6 is uniformly bounded in $L^2(0, L)$ and f^5 converges to zero in $H_0^1(0, L)$, we obtain

$$\operatorname{Re}\left\{\int_0^L \rho_1 v^6 h \bar{f}_x^5 dx\right\} = o(1). \quad (4.5.82)$$

Inserting (4.5.82) into (4.5.81), we get

$$- \operatorname{Re} \left\{ \int_0^L \rho_1 v^6 h (\overline{i\lambda v^5})_x dx \right\} = \frac{1}{2} \rho_1 \int_0^L h' |v^6|^2 dx + o(1). \quad (4.5.83)$$

Furthermore, using Lemma 4.5.7 and the third estimation of (4.5.10), we deduce that

$$\operatorname{Re} \left\{ \int_0^L i\lambda v^6 \rho_1 h \frac{D_3}{k_3} (\overline{v_x^6 - lv^2}) dx \right\} = o(1). \quad (4.5.84)$$

Finally, inserting (4.5.83) and (4.5.84) into (4.5.80), we get

$$\operatorname{Re} \left\{ \int_0^L i\lambda \rho_1 v^6 h \overline{T} dx \right\} = \frac{1}{2} \rho_1 \int_0^L h' |v^6|^2 dx + o(1). \quad (4.5.85)$$

(ii) **Estimation of the third and fourth**

terms of (4.5.79). By using the fact that v_x^1 is uniformly bounded in $L^2(0, L)$ and the third estimation of (4.5.10), we get

$$\begin{aligned} \operatorname{Re} \left\{ lk_3 \int_0^L v_x^1 h \overline{T} dx \right\} &= \operatorname{Re} \left\{ lk_3 \int_0^L v_x^1 h \overline{v_x^5} dx + l \int_0^L v_x^1 h D_3 (\overline{v_x^6 - lv^2}) dx \right\} \\ &= \operatorname{Re} \left\{ lk_3 \int_0^L v_x^1 h \overline{v_x^5} dx \right\} + o(1). \end{aligned} \quad (4.5.86)$$

Using the fact that v_x^1, v_x^5 are uniformly bounded in $L^2(0, L)$, $v^3 = O(\frac{1}{\lambda}), v^5 = O(\frac{1}{\lambda})$ and the third estimation of (4.5.10), we get

$$\begin{aligned} \operatorname{Re} \left\{ lk_1 \int_0^L (v_x^1 + v^3 + lv^5) h \overline{T} dx \right\} &= \operatorname{Re} \left\{ lk_1 \int_0^L (v_x^1 + v^3 + lv^5) h \left(\overline{v_x^5 + \frac{D_3}{k_3} (\overline{v_x^6 - lv^2})} \right) dx \right\} \\ &= \operatorname{Re} \left\{ lk_1 \int_0^L v_x^1 h \overline{v_x^5} dx \right\} + o(1). \end{aligned} \quad (4.5.87)$$

(iii) **Main estimation.** Inserting (4.5.85), (4.5.86) and (4.5.87) into (4.5.79) we get the desired estimation (4.5.78). Thus the proof is complete. \square

Lemma 4.5.11. Assume that condition (4.5.1) is verified. Then, we have

$$\begin{aligned} &\frac{1}{2} \rho_1 \left(\int_0^{\alpha+\epsilon} |v^2|^2 dx + \int_{\beta-\epsilon}^L |v^2|^2 dx \right) + \frac{k_1}{2} \left(\int_0^{\alpha+\epsilon} |v_x^1|^2 dx + \int_{\beta-\epsilon}^L |v_x^1|^2 dx \right) \\ &\frac{1}{2} \rho_2 \left(\int_0^{\alpha+\epsilon} |v^4|^2 dx + \int_{\beta-\epsilon}^L |v^4|^2 dx \right) + \frac{k_2}{2} \left(\int_0^{\alpha+\epsilon} |v_x^3|^2 dx + \int_{\beta-\epsilon}^L |v_x^3|^2 dx \right) \\ &\frac{1}{2} \rho_1 \left(\int_0^{\alpha+\epsilon} |v^6|^2 dx + \int_{\beta-\epsilon}^L |v^6|^2 dx \right) + \frac{k_3}{2} \left(\int_0^{\alpha+\epsilon} |v_x^5|^2 dx + \int_{\beta-\epsilon}^L |v_x^5|^2 dx \right) = o(1). \end{aligned} \quad (4.5.88)$$

Proof: First, combining Lemma 4.5.8, Lemma 4.5.9 and Lemma 4.5.10, we get

$$\begin{aligned} & \frac{1}{2}\rho_1 \int_0^L h'|v^2|^2 dx + \frac{k_1}{2} \int_0^L h'|v_x^1 + \frac{D_1}{k_1} (v_x^2 + v^4 + lv^6)|^2 dx \\ & \frac{1}{2}\rho_2 \int_0^L h'|v^4|^2 dx + \frac{k_2}{2} \int_0^L h'|v_x^3 + \frac{D_2}{k_2} v_x^4|^2 dx \\ & \frac{1}{2}\rho_1 \int_0^L h'|v^6|^2 dx + \frac{k_3}{2} \int_0^L h'|v_x^5 + \frac{D_3}{k_3} (v_x^6 - lv^2)|^2 dx = o(1). \end{aligned} \quad (4.5.89)$$

Next, we define the cut-off function $\tilde{\eta} \in C_0^\infty(0, L)$ by

$$\tilde{\eta} = 1 \text{ on } (0, \alpha + \epsilon), \quad 0 \leq \tilde{\eta} \leq 1, \quad \tilde{\eta} = 0 \text{ on } (\alpha + 2\epsilon, L)$$

and the cut-off function $\hat{\eta} \in C_0^\infty(0, L)$ by

$$\hat{\eta} = 0 \text{ on } (0, \beta - 2\epsilon), \quad 0 \leq \hat{\eta} \leq 1, \quad \hat{\eta} = 1 \text{ on } (\beta - \epsilon, L).$$

Taking $h = x\tilde{\eta} + (x - L)\hat{\eta}$ in (4.5.89). Then using estimations (4.5.10), (4.5.13), and (4.5.18), we get the desired estimation (4.5.88). Thus the proof is complete. \square

Proof of Theorem 4.5.1 By using (4.5.13), (4.5.18) and (4.5.88), we get $\|U\|_{\mathcal{H}_j} = o(1)$ on $(0, L)$ which is a contradiction with (4.5.2). Therefore (H2) holds and so, by [36] and [60], we deduce the exponential stability of the system (4.1.1) in the case of three local smooth dampings. The proof is thus complete. \square

4.6 Polynomial stability in the case of three local non smooth dampings

In this section, we will establish the polynomial stability of the C_0 -semigroup $e^{t\mathcal{A}_j}$ in the case when the positive functions $D_i \in L^\infty(0, L)$, $i = 1, 2, 3$ satisfy the following condition :

$$\omega = \text{supp}D_1 \cap \text{supp}D_2 \cap \text{supp}D_3 = (\alpha, \beta) \subset (0, L) \text{ such that } \text{mes}(\omega) > 0. \quad (4.6.1)$$

Our main result in this section can be given by the following theorem :

Theorem 4.6.1. Assume that condition (4.6.1) is satisfied. Assume also that D_1, D_2 and $D_3 \in L^\infty(0, L)$. Then, there exists a positive constant $c > 0$ such that for all $U_0 \in D(\mathcal{A}_j)$, for $j = 1, 2$, the energy of the system satisfies the following decay rate :

$$E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A}_j)}^2. \quad (4.6.2)$$

Referring to [25], (4.6.2) is verified if the following conditions

$$i\mathbb{R} \subseteq \rho(\mathcal{A}_j) \quad (\text{H1})$$

and

$$\limsup_{\lambda \rightarrow +\infty, \lambda \in \mathbb{R}} \frac{1}{\lambda^2} \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} = O(1) \quad (\text{H3})$$

hold.

Condition $i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$ is already proved in Lemma 4.3.2 and Lemma 4.3.3. We will establish (H3) by contradiction. Suppose that there exist a sequence of real numbers $(\lambda_n)_n$, with $|\lambda_n| \rightarrow +\infty$ and a sequence of vectors

$$U_n = (v_n^1, v_n^2, v_n^3, v_n^4, v_n^5, v_n^6)^\top \in D(\mathcal{A}_j) \quad \text{with} \quad \|U_n\|_{\mathcal{H}_j} = 1 \quad (4.6.3)$$

such that

$$\lambda_n^2 (i\lambda_n U_n - \mathcal{A}_j U_n) = (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6)^\top \rightarrow 0 \quad \text{in} \quad \mathcal{H}_j, \quad j = 1, 2 \quad (4.6.4)$$

detailed as

$$i\lambda_n v_n^1 - v_n^2 = \frac{f_n^1}{\lambda_n^2}, \quad (4.6.5)$$

$$\begin{aligned} i\rho_1 \lambda_n v_n^2 - [k_1 ((v_n^1)_x + v_n^3 + lv_n^5) + D_1 ((v_n^2)_x + v_n^4 + lv_n^6)]_x \\ - lk_3 [(v_n^5)_x - lv_n^1] - lD_3 [(v_n^6)_x - lv_n^2] = \rho_1 \frac{f_n^2}{\lambda_n^2}, \end{aligned} \quad (4.6.6)$$

$$i\lambda_n v_n^3 - v_n^4 = \frac{f_n^3}{\lambda_n^2}, \quad (4.6.7)$$

$$\begin{aligned} i\rho_2 \lambda_n v_n^4 - [k_2 (v_n^3)_x + D_2 (v_n^4)_x]_x + k_1 [(v_n^1)_x + v_n^3 + lv_n^5] \\ + D_1 [(v_n^2)_x + v_n^4 + lv_n^6] = \rho_2 \frac{f_n^4}{\lambda_n^2}, \end{aligned} \quad (4.6.8)$$

$$i\lambda_n v_n^5 - v_n^6 = \frac{f_n^5}{\lambda_n^2}, \quad (4.6.9)$$

$$\begin{aligned} i\rho_1 \lambda_n v_n^6 - [k_3 ((v_n^5)_x - lv_n^1) + D_3 ((v_n^6)_x - lv_n^2)]_x + lk_1 [(v_n^1)_x + v_n^3 + lv_n^5] \\ + lD_1 [(v_n^2)_x + v_n^4 + lv_n^6] = \rho_1 \frac{f_n^6}{\lambda_n^2}. \end{aligned} \quad (4.6.10)$$

From (4.6.3), (4.6.5), (4.6.7) and (4.6.9), we deduce that

$$\|v_n^1\| = O\left(\frac{1}{\lambda_n}\right), \quad \|v_n^3\| = O\left(\frac{1}{\lambda_n}\right), \quad \|v_n^5\| = O\left(\frac{1}{\lambda_n}\right). \quad (4.6.11)$$

In the following we will check the condition (H3) by finding a contradiction with (4.6.3) such as $\|U_n\|_{\mathcal{H}_j} = o(1)$. For clarity, we divide the proof into several lemmas. From now on, for simplicity, we drop the index n .

Lemma 4.6.2. Assume that condition (4.6.1) is verified. Then, we have

$$\|D_1^{1/2}(v_x^2 + v^4 + lv^6)\| = \frac{o(1)}{\lambda}, \quad \|D_2^{1/2}v_x^4\| = \frac{o(1)}{\lambda}, \quad \|D_3^{1/2}(v_x^6 - lv^2)\| = \frac{o(1)}{\lambda} \quad (4.6.12)$$

and

$$\|v_x^2 + v^4 + lv^6\| = \frac{o(1)}{\lambda}, \quad \|v_x^4\| = \frac{o(1)}{\lambda}, \quad \|v_x^6 - lv^2\| = \frac{o(1)}{\lambda} \quad \text{in } (\alpha, \beta). \quad (4.6.13)$$

Proof: Taking the inner product of (4.6.4) with U in \mathcal{H}_j , we get

$$\begin{aligned} \operatorname{Re} (i\lambda^3 \|U\|^2 - \lambda^2 (\mathcal{A}_j U, U))_{\mathcal{H}_j} &= -\lambda^2 \operatorname{Re} (\mathcal{A}_j U, U)_{\mathcal{H}_j} \\ &= \lambda^2 \int_0^L (D_1 |v_x^2 + v^4 + lv^6|^2 + D_2 |v_x^4|^2 + D_3 |v_x^6 - lv^2|^2) dx \\ &= o(1). \end{aligned}$$

Thanks to (4.6.1), we obtain the desired asymptotic equation (4.6.12) and (4.6.13). Thus the proof is complete. \square

Lemma 4.6.3. Assume that condition (4.5.1) is verified. Then, we have

$$\|v_x^1 + v^3 + lv^5\| = \frac{o(1)}{\lambda^2}, \quad \|v_x^3\| = \frac{o(1)}{\lambda^2}, \quad \|v_x^5 - lv^1\| = \frac{o(1)}{\lambda^2} \quad \text{in } (\alpha, \beta). \quad (4.6.14)$$

Proof: First, using equations (4.6.5), (4.6.7) and (4.6.9), we obtain

$$\lambda (v_x^1 + v^3 + lv^5) = -i(v_x^2 + \frac{f_x^1}{\lambda^2} + v^4 + \frac{f_x^3}{\lambda^2} + lv^6 + l\frac{f_x^5}{\lambda^2}). \quad (4.6.15)$$

Consequently,

$$\int_{\alpha}^{\beta} \lambda^2 |v_x^1 + v^3 + lv^5|^2 dx \leq 2 \int_{\alpha}^{\beta} |v_x^2 + v^4 + lv^6|^2 dx + 2 \int_{\alpha}^{\beta} \frac{|f_x^1 + f_x^3 + lf_x^5|^2}{\lambda^4} dx. \quad (4.6.16)$$

Using the first estimation of (4.6.13) and the fact that f^1, f^3, f^5 converge to zero in $H_0^1(0, L)$ in (4.6.16), we deduce

$$\int_{\alpha}^{\beta} \lambda^2 |v_x^1 + v^3 + lv^5|^2 dx = \frac{o(1)}{\lambda^2}. \quad (4.6.17)$$

In a similar way, one can prove

$$\int_{\alpha}^{\beta} \lambda^2 |v_x^3|^2 dx = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_{\alpha}^{\beta} \lambda^2 |v_x^5 - lv^1|^2 dx = \frac{o(1)}{\lambda^2}. \quad (4.6.18)$$

The proof is thus complete. \square

Lemma 4.6.4. Assume that condition (4.6.1) is verified. Then, we have

$$\int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = \frac{o(1)}{\lambda}, \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = \frac{o(1)}{\lambda}, \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^5|^2 dx = \frac{o(1)}{\lambda}. \quad (4.6.19)$$

Proof: First, multiplying equation (4.6.5) by $i\lambda\eta\bar{v}^1$ in $L^2(0, L)$ and integrating by parts, we get

$$-\int_0^L \eta |\lambda v^1|^2 dx - i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = i \int_0^L \frac{f^1}{\lambda^2} \eta \lambda \bar{v}^1 dx. \quad (4.6.20)$$

As λv^1 is uniformly bounded in $L^2(0, L)$ and f^1 converges to zero in $H_0^1(0, L)$, we get that the term on the right hand side of (4.6.20) converges to zero and consequently

$$-\int_0^L \eta |\lambda v^1|^2 dx - i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = \frac{o(1)}{\lambda^2}. \quad (4.6.21)$$

Moreover, multiplying (4.6.6) by $\rho_1^{-1}\eta\bar{v}^1$ in $L^2(0, L)$, then integrating by parts we obtain

$$\begin{aligned} & i \int_0^L \lambda \eta v^2 \bar{v}^1 dx + \rho_1^{-1} \int_0^L (k_1 (v_x^1 + v^3 + lv^5) + D_1 (v_x^2 + v^4 + lv^6)) (\eta \bar{v}^1)_x dx \\ & - lk_3 \rho_1^{-1} \int_0^L (v_x^5 - lv^1) \eta \bar{v}^1 dx - l\rho_1^{-1} \int_0^L D_3 (v_x^6 - lv^2) \eta \bar{v}^1 dx = \int_0^L \frac{f^2}{\lambda^2} \eta \bar{v}^1 dx. \end{aligned} \quad (4.6.22)$$

Using (4.6.12), (4.6.14), the fact that f^2 converges to zero in $L^2(0, L)$ and $\lambda v^1, v_x^1$ are uniformly bounded in $L^2(0, L)$ in (4.6.22), we get

$$i \int_0^L \lambda \eta v^2 \bar{v}^1 dx = \frac{o(1)}{\lambda}. \quad (4.6.23)$$

Finally, using (4.6.23) in (4.6.20), we get

$$\int_0^L \eta |\lambda v^1|^2 dx = \frac{o(1)}{\lambda}, \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = \frac{o(1)}{\lambda}.$$

In a same way, we show

$$\int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = \frac{o(1)}{\lambda}, \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^5|^2 dx = \frac{o(1)}{\lambda}.$$

□

Lemma 4.6.5. Let $a_1(x), a_2(x)$ and $a_3(x) \in L^\infty(0, L)$. Then the solution $(u, y, z) \in (H_0^1(0, L) \cap H^2(0, L))^3$ for $j = 1$ and $(u, y, z) \in (H_0^1(0, L) \cap H^2(0, L)) \times (H_*^1(0, L) \cap H^2(0, L))^2$ for $j = 2$ of the following system

$$\begin{cases} \rho_1 \lambda^2 u + k_1 (u_x + y + lz)_x + lk_3 (z_x - lu) - a_1(x) i \lambda u & = v^1, \\ \rho_2 \lambda^2 y + k_2 y_{xx} - k_1 (u_x + y + lz) - a_2(x) i \lambda y & = v^3, \\ \rho_1 \lambda^2 z + k_3 (z_x - lu)_x - lk_1 (u_x + y + lz) - a_3(x) i \lambda z & = v^5, \\ u(0) = u(L) = y(0) = y(L) = z(0) = z(L) & = 0, \quad \text{for } j = 1, \\ u(0) = u(L) = y_x(0) = y_x(L) = z_x(0) = z_x(L) & = 0, \quad \text{for } j = 2, \end{cases} \quad (4.6.24)$$

verifies the following estimation

$$\int_0^L \left(\rho_1 |\lambda u|^2 + \rho_2 |\lambda y|^2 + \rho_1 |\lambda z|^2 + k_2 |y_x|^2 + k_1 |u_x + y + lz|^2 + k_3 |z_x - lu|^2 \right) dx < C \int_0^L (|v^1|^2 + |v^3|^2 + |v^5|^2) dx, \quad (4.6.25)$$

where C is a constant independent of n .

Proof: Consider the system

$$\begin{cases} \rho_1 u_{tt} - k_1 (u_x + y + lz)_x - lk_3 (z_x - lu) + a_1(x)u_t = 0, \\ \rho_2 y_{tt} - k_2 y_{xx} + k_1 (u_x + y + lz) + a_2(x)y_t = 0, \\ \rho_1 z_{tt} - k_3 (z_x - lu)_x + lk_1 (u_x + y + lz) + a_3(x)z_t = 0. \end{cases} \quad (4.6.26)$$

System (4.6.26) is well posed in the space $H_1 = (H_0^1(0, L) \times L^2(0, L))^3$ and it is exponentially stable. Therefore, following Huang [36] and Pruss [60], we deduce that the resolvent of our operator

$$\mathcal{A}_{aux} : D(\mathcal{A}_{aux}) \subset H_1 \rightarrow H_1$$

defined by

$$\mathcal{A}_{aux} \begin{pmatrix} u \\ \tilde{u} \\ y \\ \tilde{y} \\ z \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ \rho_1^{-1} [k_1 (u_x + y + lz)_x + lk_3 (z_x - lu) - a_1(x)\tilde{u}] \\ \tilde{y} \\ \rho_2^{-1} [k_2 y_{xx} - k_1 (u_x + y + lz) - a_2(x)\tilde{y}] \\ \tilde{z} \\ \rho_1^{-1} [k_3 (z_x - lu)_x - lk_1 (u_x + z + lz) - a_3(x)\tilde{z}] \end{pmatrix}$$

is uniformly bounded on the imaginary axis.

On the other hand, system (4.6.24) can be rewritten in the following form :

$$\begin{cases} i\lambda u - \tilde{u} = 0, \\ i\rho_1 \lambda \tilde{u} - k_1 (u_x + y + lz)_x - lk_3 (z_x - lu) + a_1(x)\tilde{u} = -v^1, \\ i\lambda y - \tilde{y} = 0, \\ i\rho_2 \lambda \tilde{y} - k_2 y_{xx} + k_1 (u_x + y + lz) + a_2(x)\tilde{y} = -v^3, \\ i\lambda z - \tilde{z} = 0, \\ i\rho_1 \lambda \tilde{z} - k_3 (z_x - lu)_x + lk_1 (u_x + z + lz) + a_3(x)\tilde{z} = -v^5. \end{cases} \quad (4.6.27)$$

So,

$$(i\lambda - \mathcal{A}_{aux}) \begin{pmatrix} u \\ \tilde{u} \\ y \\ \tilde{y} \\ z \\ \tilde{z} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-1}{\rho_1}v^1 \\ 0 \\ \frac{-1}{\rho_2}v^3 \\ 0 \\ \frac{-1}{\rho_1}v^1 \end{pmatrix}.$$

Equivalently,

$$\begin{pmatrix} u \\ \tilde{u} \\ y \\ \tilde{y} \\ z \\ \tilde{z} \end{pmatrix} = (i\lambda - \mathcal{A}_{aux})^{-1} \begin{pmatrix} 0 \\ \frac{-1}{\rho_1}v^1 \\ 0 \\ \frac{-1}{\rho_2}v^3 \\ 0 \\ \frac{-1}{\rho_1}v^1 \end{pmatrix}.$$

This yields

$$\begin{aligned} \|(u, \tilde{u}, y, \tilde{y}, z, \tilde{z})\|_{H_1}^2 &\leq \|(i\lambda - \mathcal{A}_{aux})^{-1}\|_{\mathcal{L}(H_1)} \|(0, \frac{-1}{\rho_1}v^1, 0, \frac{-1}{\rho_2}v^3, 0, \frac{-1}{\rho_1}v^1)\|_{H_1} \\ &\leq C \int_0^L (|v^1|^2 + |v^3|^2 + |v^5|^2) dx, \end{aligned} \quad (4.6.28)$$

where C is a constant independent of n . Consequently, (4.6.25) holds. The proof is thus complete. \square

Lemma 4.6.6. Assume that condition (4.6.1) is satisfied. Then we have

$$\int_0^L |\lambda v^1|^2 dx = o(1), \quad \int_0^L |\lambda v^3|^2 dx = o(1), \quad \int_0^L |\lambda v^5|^2 dx = o(1). \quad (4.6.29)$$

Proof: For clarity of the proof, we divide the proof into several steps.

Step 1. First, multiplying (4.6.5) by $i\rho_1\lambda\bar{u}$, where u is a solution of system (4.6.24), we get

$$- \int_0^L \rho_1 \lambda^2 \bar{u} v^1 dx - i \int_0^L \rho_1 \lambda \bar{u} v^2 dx = \rho_1 \int_0^L \frac{if^1}{\lambda} \bar{u} dx. \quad (4.6.30)$$

Moreover, multiplying (4.6.6) by \bar{u} and integrating by parts, we obtain

$$\begin{aligned} &i \int_0^L \rho_1 \lambda \bar{u} v^2 dx - \int_0^L k_1 \bar{u}_{xx} v^1 dx - \int_0^L k_3 (-l\bar{u}) v^1 dx + \int_0^L k_1 \bar{u}_x v^3 dx \\ &+ \int_0^L k_1 \bar{u}_x v^5 dx + \int_0^L k_3 \bar{u}_x v^5 dx + \int_0^L D_1 (v_x^2 + v^4 + lv^6) \bar{u}_x dx \\ &- \int_0^L l D_3 (v_x^6 - lv^2) \bar{u} dx = \rho_1 \int_0^L \frac{f^2}{\lambda^2} \bar{u} dx. \end{aligned} \quad (4.6.31)$$

Now, combining (4.6.30) and (4.6.31), we get

$$\begin{aligned}
& \int_0^L [\rho_1 \lambda^2 \bar{u} + k_1 \bar{u}_{xx} + lk_3(-l\bar{u})] v^1 dx - \int_0^L k_1 \bar{u}_x v^3 dx \\
& - \int_0^L lk_1 \bar{u}_x v^5 dx - \int_0^L lk_3 \bar{u}_x v^5 dx - \int_0^L D_1(v_x^2 + v^4 + lv^6) \bar{u}_x dx \\
& + \int_0^L lD_3(v_x^6 - lv^2) \bar{u} dx = -\rho_1 \int_0^L \left(\frac{if^1}{\lambda} + \frac{f^2}{\lambda^2} \right) \bar{u} dx. \tag{4.6.32}
\end{aligned}$$

Step 2. Similarly to Step 1, multiplying (4.6.7) by $i\rho_2 \lambda \bar{y}$ and (4.6.8) by \bar{y} , where y is a solution of system (4.6.24), we get

$$\begin{aligned}
& \int_0^L [\rho_2 \lambda^2 \bar{y} + k_2 \bar{y}_{xx} - k_1 \bar{y}] v^3 dx + \int_0^L k_1 \bar{y}_x v^1 dx - \int_0^L lk_1 \bar{y} v^5 dx \\
& - \int_0^L D_2 v_x^4 \bar{y}_x dx - \int_0^L D_1(v_x^2 + v^4 + lv^6) \bar{y} dx = -\rho_2 \int_0^L \left(\frac{if^3}{\lambda} + \frac{f^4}{\lambda^2} \right) \bar{y} dx. \tag{4.6.33}
\end{aligned}$$

Step 3. As in step1 and step2, by multiplying (4.6.9) by $i\rho_1 \lambda \bar{z}$ and (4.6.10) by \bar{z} , where z is a solution of system (4.6.24), we get

$$\begin{aligned}
& \int_0^L [\rho_1 \lambda^2 \bar{z} + k_3 \bar{z}_{xx} - lk_1(l\bar{z})] v^5 dx + \int_0^L lk_3 \bar{z}_x v^1 dx + \int_0^L lk_1 \bar{z}_x v^1 dx \\
& - \int_0^L lk_1 \bar{z} v^3 dx - \int_0^L D_3(v_x^6 - lv^2) \bar{z}_x dx - l \int_0^L D_1(v_x^2 + v^4 + lv^6) \bar{z} dx \\
& = -\rho_1 \int_0^L \left(\frac{if^5}{\lambda} + \frac{f^6}{\lambda^2} \right) \bar{z} dx. \tag{4.6.34}
\end{aligned}$$

Step 4. First, combining (4.6.32), (4.6.33) and (4.6.34), we obtain

$$\begin{aligned}
& \int_0^L [\rho_1 \lambda^2 \bar{u} + k_1 (\bar{u}_x + \bar{y} + l\bar{z})_x + lk_3(\bar{z}_x - l\bar{u})] v^1 dx + \int_0^L [\rho_2 \lambda^2 \bar{y} + k_2 \bar{y}_{xx} - k_1 (\bar{u}_x + \bar{y} + l\bar{z})] v^3 dx \\
& + \int_0^L [\rho_1 \lambda^2 \bar{z} + k_3 (\bar{z}_x - l\bar{u})_x - lk_1 (\bar{u}_x + \bar{y} + l\bar{z})] v^5 dx - \int_0^L D_1(v_x^2 + v^4 + lv^6) \bar{u}_x dx \\
& + \int_0^L lD_3(v_x^6 - lv^2) \bar{u} dx - \int_0^L D_2 v_x^4 \bar{y}_x dx - \int_0^L D_1(v_x^2 + v^4 + lv^6) \bar{y} dx - \int_0^L D_3(v_x^6 - lv^2) \bar{z}_x dx \\
& - l \int_0^L D_1(v_x^2 + v^4 + lv^6) \bar{z} dx = -\rho_1 \int_0^L \left(\frac{if^1}{\lambda} + \frac{f^2}{\lambda^2} \right) \bar{u} dx - \rho_2 \int_0^L \left(\frac{if^3}{\lambda} + \frac{f^4}{\lambda^2} \right) \bar{y} dx \\
& - \rho_1 \int_0^L \left(\frac{if^5}{\lambda} + \frac{f^6}{\lambda^2} \right) \bar{z} dx. \tag{4.6.35}
\end{aligned}$$

Then, substituting the first three terms of (4.6.35) by the equivalent terms using the

equations of system (4.6.24), we get

$$\begin{aligned}
 & \int_0^L |v^1|^2 dx - i \int_0^L \lambda \bar{u} a_1(x) v^1 dx + \int_0^L |v^3|^2 dx - i \int_0^L \lambda \bar{y} a_2(x) v^3 dx + \int_0^L |v^5|^2 dx \\
 & - i \int_0^L \lambda \bar{z} a_3(x) v^5 dx - \int_0^L D_1(v_x^2 + v^4 + lv^6) \bar{u}_x dx + \int_0^L l D_3(v_x^6 - lv^2) \bar{u} dx - \int_0^L D_2 v_x^4 \bar{y}_x dx \\
 & - \int_0^L D_1(v_x^2 + v^4 + lv^6) \bar{y} dx - \int_0^L D_3(v_x^6 - lv^2) \bar{z}_x dx - l \int_0^L D_1(v_x^2 + v^4 + lv^6) \bar{z} dx \\
 & = -\rho_1 \int_0^L \left(\frac{if^1}{\lambda} + \frac{f^2}{\lambda^2} \right) \bar{u} dx - \rho_2 \int_0^L \left(\frac{if^3}{\lambda} + \frac{f^4}{\lambda^2} \right) \bar{y} dx - \rho_1 \int_0^L \left(\frac{if^5}{\lambda} + \frac{f^6}{\lambda^2} \right) \bar{z} dx.
 \end{aligned} \tag{4.6.36}$$

Moreover, multiplying (4.6.36) by λ^2 , we obtain

$$\begin{aligned}
 & \int_0^L |\lambda v^1|^2 dx - i \int_0^L \lambda^2 \bar{u} a_1(x) \lambda v^1 dx + \int_0^L |\lambda v^3|^2 dx - i \int_0^L \lambda^2 \bar{y} a_2(x) \lambda v^3 dx \\
 & + \int_0^L |\lambda v^5|^2 dx - i \int_0^L \lambda^2 \bar{z} a_3(x) \lambda v^5 dx - \int_0^L \lambda D_1(v_x^2 + v^4 + lv^6) \lambda \bar{u}_x dx + \int_0^L l D_3(v_x^6 - lv^2) \lambda^2 \bar{u} dx \\
 & - \int_0^L \lambda D_2 v_x^4 \lambda \bar{y}_x dx - \int_0^L D_1(v_x^2 + v^4 + lv^6) \lambda^2 \bar{y} dx - \int_0^L \lambda D_3(v_x^6 - lv^2) \lambda \bar{z}_x dx \\
 & - l \int_0^L D_1(v_x^2 + v^4 + lv^6) \lambda^2 \bar{z} dx = -\rho_1 \int_0^L (if^1 \lambda + f^2) \bar{u} dx - \rho_2 \int_0^L (if^3 \lambda + f^4) \bar{y} dx \\
 & - \rho_1 \int_0^L (if^5 \lambda + f^6) \bar{z} dx.
 \end{aligned} \tag{4.6.37}$$

Next, we need to estimate each term of (4.6.37). Assuming now that for $i = 1, 2, 3$, there exists $a_0 > 0$ such that

$$a_i(x) \geq a_0 > 0 \quad \text{in } (\alpha + \epsilon, \beta - \epsilon). \tag{4.6.38}$$

It follows under condition (4.6.38) and estimations (4.6.19) combined with the fact that $\lambda^2 u$, $\lambda^2 y$ and $\lambda^2 z$ are uniformly bounded in $L^2(0, L)$ due to (4.6.25) that

$$i \int_0^L \lambda^2 \bar{u} a_1(x) \lambda v^1 dx + i \int_0^L \lambda^2 \bar{y} a_2(x) \lambda v^3 dx + i \int_0^L \lambda^2 \bar{z} a_3(x) \lambda v^5 dx = \frac{o(1)}{\lambda}. \tag{4.6.39}$$

In addition, using (4.6.12) and the fact that λu_x , λy_x and λz_x are uniformly bounded in $L^2(0, L)$ due to (4.6.25). we get

$$\begin{aligned}
 & \int_0^L \lambda D_1(v_x^2 + v^4 + lv^6) \lambda \bar{u}_x dx + \int_0^L \lambda D_2 v_x^4 \lambda \bar{y}_x dx \\
 & - \int_0^L \lambda D_3(v_x^6 - lv^2) \lambda \bar{z}_x dx = o(1).
 \end{aligned} \tag{4.6.40}$$

Also, by using (4.6.12) and the fact that $\lambda^2 u$, $\lambda^2 y$ and $\lambda^2 z$ are uniformly bounded in $L^2(0, L)$ due to (4.6.25), we obtain

$$\begin{aligned} & \int_0^L 1D_3(v_x^6 - lv^2)\lambda^2 \bar{u} dx + \int_0^L D_1(v_x^2 + v^4 + lv^6)\lambda^2 \bar{y} dx \\ & + 1 \int_0^L D_1(v_x^2 + v^4 + lv^6)\lambda^2 \bar{z} dx = \frac{o(1)}{\lambda}. \end{aligned} \quad (4.6.41)$$

Moreover, we have

$$\begin{aligned} & -\rho_1 \int_0^L (if^1 \lambda + f^2) \bar{u} dx - \rho_2 \int_0^L (if^3 \lambda + f^4) \bar{y} dx \\ & - \rho_1 \int_0^L (if^5 \lambda + f^6) \bar{z} dx = o(1), \end{aligned} \quad (4.6.42)$$

since f^1, f^3, f^5 converges to zero in $H_0^1(0, L)$, f^2, f^4, f^6 converges to zero in $L^2(0, L)$, and $\lambda^2 u, \lambda^2 y, \lambda^2 z$ are uniformly bounded in $L^2(0, L)$.

Finally, inserting (4.6.39) - (4.6.42) into (4.6.37), we get the desired estimation (4.6.29). Thus the proof is complete. \square

Lemma 4.6.7. Assume that condition (4.6.1) is satisfied. Then we have

$$\int_0^L |v_x^1|^2 dx = o(1), \quad \int_0^L |v_x^3|^2 dx = o(1), \quad \int_0^L |v_x^5|^2 dx = o(1). \quad (4.6.43)$$

Proof: First, multiplying (4.6.6) by \bar{v}^1 and then integrating by parts, we get

$$\begin{aligned} & i \int_0^L \rho_1 \lambda v^2 \bar{v}^1 dx + k_1 \int_0^L |v_x^1|^2 dx + k_1 \int_0^L (v^3 + lv^5) \bar{v}_x^1 dx + \int_0^L D_1(v_x^2 + v^4 + lv^6) \bar{v}_x^1 dx \\ & - lk_3 \int_0^L (v_x^5 - lv^1) \bar{v}^1 dx - 1 \int_0^L D_3(v_x^6 - lv^2) \bar{v}^1 dx = \rho_1 \int_0^L \frac{f^2}{\lambda^2} \bar{v}^1 dx. \end{aligned} \quad (4.6.44)$$

Then, using (4.6.11), (4.6.12) and the fact that $v_x^1, (v_x^5 - lv^1)$ are uniformly bounded in $L^2(0, L)$ due to (4.6.3), we obtain

$$\begin{aligned} & k_1 \int_0^L (v^3 + lv^5) \bar{v}_x^1 dx + \int_0^L D_1(v_x^2 + v^4 + lv^6) \bar{v}_x^1 dx \\ & - lk_3 \int_0^L (v_x^5 - lv^1) \bar{v}^1 dx - 1 \int_0^L D_3(v_x^6 - lv^2) \bar{v}^1 dx = o(1). \end{aligned} \quad (4.6.45)$$

As f^2 converges to zero in $L^2(0, L)$ and λv^1 is uniformly bounded in $L^2(0, L)$, we have

$$\rho_1 \int_0^L \frac{f^2}{\lambda^2} \bar{v}^1 dx = o(1). \quad (4.6.46)$$

Next, inserting (4.6.45) and (4.6.46) into (4.6.44), we get

$$i \int_0^L \rho_1 \lambda v^2 \overline{v^1} dx + k_1 \int_0^L |v_x^1|^2 dx = o(1). \quad (4.6.47)$$

Using Lemma 4.6.6 and the fact that v^2 is uniformly bounded in $L^2(0, L)$ in (4.6.47), we deduce

$$\int_0^L |v_x^1|^2 dx = o(1).$$

Similarly, one can prove that

$$\int_0^L |v_x^3|^2 dx = o(1), \quad \int_0^L |v_x^5|^2 dx = o(1).$$

Thus, the proof is complete. □

Proof of Theorem 4.6.1 Using Lemma 4.6.6 and Lemma 4.6.7, we get that $\|U\|_{\mathcal{H}_j} = o(1)$. Therefore, we get a contradiction with (4.6.3) and consequently (H3) holds. Thus the proof is complete. □

4.7 Lack of exponential stability

In this section, we consider the system (4.1.1) subject to the initial data (4.1.4) and to the boundary conditions (4.1.3). We prove first that system (4.1.1), subject to (4.1.3), under the following hypothesis

$$D_1 = 0 \quad \text{and} \quad D_2 = D_3 = 1 \quad \text{on} \quad (0, L) \quad (4.7.1)$$

is not exponentially stable. Then, we prove that this system is also not exponentially stable under the following hypothesis

$$D_1 = D_3 = 0 \quad \text{and} \quad D_2 = 1 \quad \text{on} \quad (0, L). \quad (4.7.2)$$

Theorem 4.7.1. Under hypothesis (4.7.1), system (4.1.1), subject to the boundary conditions (4.1.3), is not exponentially stable in the energy space \mathcal{H}_2 .

Proof: For the proof of Theorem 4.7.1, it suffices to show the existence of sequences $(\lambda_n) \subset \mathbb{R}$ with $\lim_{n \rightarrow +\infty} |\lambda_n| = +\infty$, $(V_n) \subset D(\mathcal{A}_2)$, such that $(i\lambda_n I - \mathcal{A}_2)V_n$ is bounded in \mathcal{H}_2 and $\lim_{n \rightarrow +\infty} \|V_n\| = +\infty$. Let $F = (0, 0, 0, f_4, 0, 0) \in \mathcal{H}_2$ with

$$f_4(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \lambda = \frac{n\pi\sqrt{\rho_2 k_2}}{L\rho_2}, \quad n \in \mathbb{N}.$$

We solve the following equations :

$$i\lambda v^1 - v^2 = 0, \quad (4.7.3)$$

$$i\lambda\rho_1v^2 - k_1(v_{xx}^1 + v_x^3 + lv_x^5) - lk_3(v_x^5 - lv^1) - l(v_x^6 - lv^2) = 0, \quad (4.7.4)$$

$$i\lambda v^3 - v^4 = 0, \quad (4.7.5)$$

$$i\lambda\rho_2v^4 - k_2v_{xx}^3 + k_1(v_x^1 + v^3 + lv^5) = \rho_2f_4, \quad (4.7.6)$$

$$i\lambda v^5 - v^6 = 0, \quad (4.7.7)$$

$$i\lambda\rho_1v^6 - k_3(v_{xx}^5 - lv_x^1) + lv_x^2 + lk_1(v_x^1 + v^3 + lv^5) = 0. \quad (4.7.8)$$

Eliminating v^2 , v^4 and v^6 in (4.7.4), (4.7.6) and (4.7.8) by (4.7.3), (4.7.5) and (4.7.7), we get

$$\lambda^2\rho_1v^1 + k_1(v_{xx}^1 + v_x^3 + lv_x^5) + l(k_3 + i\lambda)(v_x^5 - lv^1) = 0, \quad (4.7.9)$$

$$\lambda^2\rho_2v^3 + k_2v_{xx}^3 - k_1(v_x^1 + v^3 + lv^5) = -\rho_2f^4, \quad (4.7.10)$$

$$\lambda^2\rho_1v^5 + k_3(v_{xx}^5 - lv_x^1) - i\lambda lv_x^1 - lk_1(v_x^1 + v^3 + lv^5) = 0. \quad (4.7.11)$$

This can be solved by the ansatz

$$v^1 = A \sin\left(\frac{n\pi x}{L}\right), \quad v^3 = B \cos\left(\frac{n\pi x}{L}\right), \quad v^5 = C \cos\left(\frac{n\pi x}{L}\right) \quad (4.7.12)$$

where A , B and C depend on λ to be determined. Notice that $k_2\left(\frac{n\pi}{L}\right)^2 - \rho_2\lambda^2 = 0$, and inserting (4.7.12) in (4.7.9)-(4.7.11) we obtain that

$$\left(\left(\frac{n\pi}{L}\right)^2 k_1 - \lambda^2\rho_1 + (k_3 + i\lambda)l^2\right) A + k_1\left(\frac{n\pi}{L}\right) B + (k_1 + k_3 + i\lambda)l\left(\frac{n\pi}{L}\right) C = 0, \quad (4.7.13)$$

$$k_1\left(\frac{n\pi}{L}\right) A + k_1B + lk_1C = \rho_2, \quad (4.7.14)$$

$$(k_1 + k_3 + i\lambda)l\left(\frac{n\pi}{L}\right) A + lk_1B + \left[k_3\left(\frac{n\pi}{L}\right)^2 - \lambda^2\rho_1 + l^2k_1\right] C = 0. \quad (4.7.15)$$

Equivalently,

$$\begin{pmatrix} \left(\frac{n\pi}{L}\right)^2 k_1 - \lambda^2\rho_1 + (k_3 + i\lambda)l^2 & k_1\left(\frac{n\pi}{L}\right) & (k_1 + k_3 + i\lambda)l\left(\frac{n\pi}{L}\right) \\ k_1\left(\frac{n\pi}{L}\right) & k_1 & lk_1 \\ (k_1 + k_3 + i\lambda)l\left(\frac{n\pi}{L}\right) & lk_1 & k_3\left(\frac{n\pi}{L}\right)^2 - \lambda^2\rho_1 + l^2k_1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ \rho_2 \\ 0 \end{pmatrix}.$$

This implies that

$$A = \frac{(k_2\rho_1 - \rho_2k_3)\rho_2^2L}{\pi(k_2\rho_1^2 - k_3\rho_1\rho_2 + \rho_2l^2)k_2n} + O(n^{-2}), \quad (4.7.16)$$

$$B = \frac{\rho_2(k_1k_3\rho_2^2 + ((-k_1 - k_3)\rho_1 + l^2)k_2\rho_2 + k_2^2\rho_1^2)}{k_1((-k_3\rho_1 + l^2)\rho_2 + k_2\rho_1^2)k_2} + O(n^{-1}), \quad (4.7.17)$$

$$C = \frac{i\rho_2^2L\sqrt{\rho_2k_2}}{\pi((-k_3\rho_1 + l^2)\rho_2 + k_2\rho_1^2)k_2n} + O(n^{-2}). \quad (4.7.18)$$

Now, let $V_n = (v^1, i\lambda v^1, v^3, i\lambda v^3, v^5, i\lambda v^5)$, where v^1 , v^3 and v^5 are given by (4.7.12) and (4.7.16)-(4.7.18). It is easy to check that

$$\|V_n\|_{\mathcal{H}_2} \geq \sqrt{\rho_2}\|\lambda v^3\| \sim |B\lambda| \sim |n| \rightarrow +\infty \text{ as } n \rightarrow +\infty.$$

On the other hand, using (4.7.3)-(4.7.8), we deduce that

$$\|(i\lambda I - \mathcal{A}_2)V_n\|_{\mathcal{H}_2}^2 = \|(0, 0, 0, \rho^2 f^4 - i\lambda D_2 v_{xx}^3, 0, i\lambda D_3 v_{xx}^5)\|_{\mathcal{H}_2}^2 \leq c.$$

Consequently, $\|(i\lambda I - \mathcal{A}_2)V_n\|_{\mathcal{H}_2}^2$ is bounded as n tends to $+\infty$. Thus the proof is complete. \square

Theorem 4.7.2. Under hypothesis (4.7.2), system (4.1.1), subject to the boundary conditions (4.1.3), is not exponentially stable in the energy space \mathcal{H}_2 .

Proof: For the proof of Theorem 4.7.2, we will follow the same argument used in Theorem 4.7.1. It suffices to show that the existence of sequences $(\lambda_n) \subset \mathbb{R}$ with $\lim_{n \rightarrow +\infty} |\lambda_n| = +\infty$, $(V_n) \subset D(\mathcal{A}_2)$, such that $(i\lambda_n I - \mathcal{A})V_n$ is bounded in \mathcal{H}_2 and $\lim_{n \rightarrow +\infty} \|V_n\| = +\infty$. We distinguish several cases :

case 1. If $\frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}$. Let $F = (0, 0, 0, f_4, 0, 0) \in \mathcal{H}_2$ with

$$f_4(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \lambda = \frac{n\pi\sqrt{\rho_2 k_2}}{\rho_2 L}, \quad n \in \mathbb{N}.$$

We solve the following equations :

$$i\lambda v^1 - v^2 = 0, \tag{4.7.19}$$

$$i\lambda\rho_1 v^2 - k_1 (v_{xx}^1 + v_x^3 + lv_x^5) - lk_3 (v_x^5 - lv^1) = 0, \tag{4.7.20}$$

$$i\lambda v^3 - v^4 = 0, \tag{4.7.21}$$

$$i\lambda\rho_2 v^4 - k_2 v_{xx}^3 + k_1 (v_x^1 + v^3 + lv^5) = \rho_2 f_4, \tag{4.7.22}$$

$$i\lambda v^5 - v^6 = 0, \tag{4.7.23}$$

$$i\lambda\rho_1 v^6 - k_3 (v_{xx}^5 - lv_x^1) + lk_1 (v_x^1 + v^3 + lv^5) = 0. \tag{4.7.24}$$

Eliminating v^2, v^4 and v^6 in (4.7.20), (4.7.22) and (4.7.24) by (4.7.19), (4.7.21) and (4.7.23) we get

$$\lambda^2 \rho_1 v^1 + k_1 (v_{xx}^1 + v_x^3 + lv_x^5) + lk_3 (v_x^5 - lv^1) = 0, \tag{4.7.25}$$

$$\lambda^2 \rho_2 v^3 + k_2 v_{xx}^3 - k_1 (v_x^1 + v^3 + lv^5) = -\rho_2 f_4, \tag{4.7.26}$$

$$\lambda^2 \rho_1 v^5 + k_3 (v_{xx}^5 - lv_x^1) - lk_1 (v_x^1 + v^3 + lv^5) = 0. \tag{4.7.27}$$

This can be solved by the ansatz

$$v^1 = A \sin\left(\frac{n\pi x}{L}\right), \quad v^3 = B \cos\left(\frac{n\pi x}{L}\right), \quad v^5 = C \cos\left(\frac{n\pi x}{L}\right) \tag{4.7.28}$$

where A, B and C depend on λ to be determined. Noting that $k_2 \left(\frac{n\pi}{L}\right)^2 - \lambda^2 \rho_2 = 0$ and inserting (4.7.28) in (4.7.25)-(4.7.27) we obtain that

$$\left(\left(\frac{n\pi}{L}\right)^2 k_1 - \lambda^2 \rho_1 + l^2 k_3 \right) A + k_1 \left(\frac{n\pi}{L}\right) B + (k_1 + k_3) l \left(\frac{n\pi}{L}\right) C = 0, \tag{4.7.29}$$

$$k_1 \left(\frac{n\pi}{L} \right) A + k_1 B + lk_1 C = \rho_2, \quad (4.7.30)$$

$$(k_1 + k_3)l \left(\frac{n\pi}{L} \right) A + lk_1 B + \left[l^2 k_1 + k_3 \left(\frac{n\pi}{L} \right)^2 - \lambda^2 \rho_1 \right] C = 0. \quad (4.7.31)$$

Equivalently,

$$\begin{pmatrix} \left(\frac{n\pi}{L} \right)^2 k_1 - \lambda^2 \rho_1 + l^2 k_3 & k_1 \left(\frac{n\pi}{L} \right) & (k_1 + k_3)l \left(\frac{n\pi}{L} \right) \\ k_1 \left(\frac{n\pi}{L} \right) & k_1 & lk_1 \\ (k_1 + k_3)l \left(\frac{n\pi}{L} \right) & lk_1 & l^2 k_1 + k_3 \left(\frac{n\pi}{L} \right)^2 - \lambda^2 \rho_1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ \rho_2 \\ 0 \end{pmatrix}.$$

This implies that

$$A = \frac{\rho_2^2 L}{\pi \rho_1 k_2 n} + O(n^{-3}), \quad (4.7.32)$$

$$B = \frac{-k_1 \rho_2^2 + k_2 \rho_1 \rho_2}{\rho_1 k_1 k_2} + O(n^{-2}), \quad (4.7.33)$$

$$C = \frac{l \rho_2^2 (k_2 \rho_1 + k_3 \rho_2) L^2}{\pi^2 \rho_1 (k_2 \rho_1 - k_3 \rho_2) k_2 n^2} + O(n^{-4}). \quad (4.7.34)$$

Now, let $V_n = (v^1, i\lambda v^1, v^3, i\lambda v^3, v^5, i\lambda v^5)$, where v^1, v^3 and v^5 are given by (4.7.28) and (4.7.32)-(4.7.34). It is easy to check that

$$\|V_n\|_{\mathcal{H}_2} \geq \sqrt{\rho_2} \|\lambda v^3\| \sim |B\lambda| \sim |n| \rightarrow +\infty \text{ as } n \rightarrow +\infty.$$

On the other hand, using (4.7.19)-(4.7.24), we deduce that

$$\|(i\lambda I - \mathcal{A}_2)V_n\|_{\mathcal{H}_2}^2 = \|(0, 0, 0, -i\lambda D_2 v_{xx}^3 + \rho_2 f_4, 0, 0)\|_{\mathcal{H}_2}^2 \leq c.$$

Consequently, we deduce that $\|(i\lambda I - \mathcal{A}_2)V_n\|_{\mathcal{H}_2}^2$ is bounded as n tends to $+\infty$. Thus the proof is complete.

case 2. If $k_1 \neq k_3$. Let $F = (0, 0, 0, 0, 0, f_6) \in \mathcal{H}_2$ with

$$f_6(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \lambda = \frac{n\pi\sqrt{\rho_1 k_3}}{\rho_1 L}, \quad n \in \mathbb{N}.$$

We solve the following equations :

$$i\lambda v^1 - v^2 = 0, \quad (4.7.35)$$

$$i\lambda \rho_1 v^2 - k_1 (v_{xx}^1 + v_x^3 + lv_x^5) - lk_3 (v_x^5 - lv^1) = 0, \quad (4.7.36)$$

$$i\lambda v^3 - v^4 = 0, \quad (4.7.37)$$

$$i\lambda \rho_2 v^4 - k_2 v_{xx}^3 + k_1 (v_x^1 + v^3 + lv^5) = 0, \quad (4.7.38)$$

$$i\lambda v^5 - v^6 = 0, \quad (4.7.39)$$

$$i\lambda \rho_1 v^6 - k_3 (v_{xx}^5 - lv_x^1) + lk_1 (v_x^1 + v^3 + lv^5) = \rho_1 f_6. \quad (4.7.40)$$

Eliminating v^2 , v^4 and v^6 in (4.7.36), (4.7.38) and (4.7.40) by (4.7.35), (4.7.37) and (4.7.39) we get

$$\lambda^2 \rho_1 v^1 + k_1 (v_{xx}^1 + v_x^3 + lv_x^5) + lk_3 (v_x^5 - lv^1) = 0, \quad (4.7.41)$$

$$\lambda^2 \rho_2 v^3 + k_2 v_{xx}^3 - k_1 (v_x^1 + v^3 + lv^5) = 0, \quad (4.7.42)$$

$$\lambda^2 \rho_1 v^5 + k_3 (v_{xx}^5 - lv_x^1) - lk_1 (v_x^1 + v^3 + lv^5) = -\rho_1 f_6. \quad (4.7.43)$$

This can be solved by the ansatz

$$v^1 = A \sin\left(\frac{n\pi x}{L}\right), \quad v^3 = B \cos\left(\frac{n\pi x}{L}\right), \quad v^5 = C \cos\left(\frac{n\pi x}{L}\right) \quad (4.7.44)$$

where A , B and C depend on λ to be determined. Noting that $k_3 \left(\frac{n\pi}{L}\right)^2 - \lambda^2 \rho_1 = 0$ and inserting (4.7.44) in (4.7.41)-(4.7.43) we obtain that

$$\left(\left(\frac{n\pi}{L}\right)^2 k_1 - \lambda^2 \rho_1 + l^2 k_3 \right) A + k_1 \left(\frac{n\pi}{L}\right) B + (k_1 + k_3) l \left(\frac{n\pi}{L}\right) C = 0, \quad (4.7.45)$$

$$k_1 \left(\frac{n\pi}{L}\right) A + \left[k_1 + k_2 \left(\frac{n\pi}{L}\right)^2 - \lambda^2 \rho_2 \right] B + lk_1 C = 0, \quad (4.7.46)$$

$$(k_1 + k_3) l \left(\frac{n\pi}{L}\right) A + lk_1 B + l^2 k_1 C = \rho_1. \quad (4.7.47)$$

Equivalently,

$$\begin{pmatrix} \left(\frac{n\pi}{L}\right)^2 k_1 - \lambda^2 \rho_1 + l^2 k_3 & k_1 \left(\frac{n\pi}{L}\right) & (k_1 + k_3) l \left(\frac{n\pi}{L}\right) \\ k_1 \left(\frac{n\pi}{L}\right) & k_1 + k_2 \left(\frac{n\pi}{L}\right)^2 - \lambda^2 \rho_2 & lk_1 \\ (k_1 + k_3) l \left(\frac{n\pi}{L}\right) & lk_1 & l^2 k_1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \rho_1 \end{pmatrix}.$$

This yields

$$A = \frac{L \rho_1 (k_1 + k_3)}{\pi k_3 (3k_1 + k_3) l n} + O(n^{-3}), \quad (4.7.48)$$

$$B = -\frac{2k_1 \rho_1^2 L^2}{l(3k_1 + k_3)(k_2 \rho_1 - k_3 \rho_2) \pi^2 n^2} + O(n^{-4}), \quad (4.7.49)$$

$$C = -\frac{(k_1 - k_3) \rho_1}{(3k_1 + k_3) k_3 l^2} + O(n^{-2}). \quad (4.7.50)$$

Now, let $V_n = (v^1, i\lambda v^1, v^3, i\lambda v^3, v^5, i\lambda v^5)$, where v^1 , v^3 and v^5 are given by (4.7.44) and (4.7.48)-(4.7.50). It is easy to check $\|V_n\| \rightarrow +\infty$ and $\|(i\lambda I - \mathcal{A}_2)V_n\|_{\mathcal{H}_2}^2$ is bounded as n tends to $+\infty$.

case 3. If $k_1 = k_3$ and $\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}$. Let $F = (0, 0, 0, f_4, 0, 0) \in \mathcal{H}_2$ with

$$f_4(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \lambda = \frac{n\pi \sqrt{\rho_2 k_2}}{\rho_2 L}, \quad n \in \mathbb{N}.$$

We solve the following equations :

$$i\lambda v^1 - v^2 = 0, \quad (4.7.51)$$

$$i\lambda\rho_1 v^2 - k_3 (v_{xx}^1 + v_x^3 + lv_x^5) - lk_3 (v_x^5 - lv^1) = 0, \quad (4.7.52)$$

$$i\lambda v^3 - v^4 = 0, \quad (4.7.53)$$

$$i\lambda\rho_2 v^4 - k_2 v_{xx}^3 + k_3 (v_x^1 + v^3 + lv^5) = \rho_2 f_4, \quad (4.7.54)$$

$$i\lambda v^5 - v^6 = 0, \quad (4.7.55)$$

$$i\lambda\rho_1 v^6 - k_3 (v_{xx}^5 - lv_x^1) + lk_3 (v_x^1 + v^3 + lv^5) = 0. \quad (4.7.56)$$

Eliminating v^2 , v^4 and v^6 in (4.7.52), (4.7.54) and (4.7.56) by (4.7.51), (4.7.53) and (4.7.55) we get

$$\lambda^2\rho_1 v^1 + k_3 (v_{xx}^1 + v_x^3 + lv_x^5) + lk_3 (v_x^5 - lv^1) = 0, \quad (4.7.57)$$

$$\lambda^2\rho_2 v^3 + k_2 v_{xx}^3 - k_3 (v_x^1 + v^3 + lv^5) = -\rho_2 f_4, \quad (4.7.58)$$

$$\lambda^2\rho_1 v^5 + k_3 (v_{xx}^5 - lv_x^1) - lk_3 (v_x^1 + v^3 + lv^5) = 0. \quad (4.7.59)$$

This can be solved by the ansatz

$$v^1 = A \sin\left(\frac{n\pi x}{L}\right), \quad v^3 = B \cos\left(\frac{n\pi x}{L}\right), \quad v^5 = C \cos\left(\frac{n\pi x}{L}\right), \quad (4.7.60)$$

where A , B and C depend on λ to be determined. Noting that $\left(\frac{n\pi}{L}\right)^2 k_3 - \lambda^2\rho_1 = 0$ and $k_2\left(\frac{n\pi}{L}\right)^2 - \lambda^2\rho_2 = 0$. Inserting (4.7.60) in (4.7.57)-(4.7.59) we obtain that

$$l^2 k_3 A + \frac{k_3 n\pi}{L} B + \frac{2k_3 l n\pi}{L} C = 0, \quad (4.7.61)$$

$$\frac{k_3 n\pi}{L} A + k_3 B + lk_3 C = \rho_2, \quad (4.7.62)$$

$$\frac{2k_3 l n\pi}{L} A + lk_3 B + l^2 k_3 C = 0. \quad (4.7.63)$$

Equivalently,

$$\begin{pmatrix} l^2 k_3 & \frac{k_3 n\pi}{L} & \frac{2k_3 l n\pi}{L} \\ \frac{k_3 n\pi}{L} & k_3 & lk_3 \\ \frac{2k_3 l n\pi}{L} & lk_3 & l^2 k_3 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ \rho_2 \\ 0 \end{pmatrix}. \quad (4.7.64)$$

This implies that

$$A = \frac{-\rho_2 L}{n\pi k_3}, \quad B = \frac{4\rho_2}{k_3} - \frac{\rho_2 L^2 l^2}{k_3 \pi^2 n^2}, \quad C = \frac{-2\rho_2}{lk_3} + \frac{L^2 l \rho_2}{k_3 \pi^2 n^2}. \quad (4.7.65)$$

Now, let $V_n = (v^1, i\lambda v^1, v^3, i\lambda v^3, v^5, i\lambda v^5)$, where v^1, v^3 and v^5 are given by (4.7.60) and (4.7.65). It is easy to check $\|V_n\| \rightarrow +\infty$ and $\|(i\lambda I - \mathcal{A}_2)V_n\|_{\mathcal{H}_2}^2$ is bounded as n tends to $+\infty$. The proof is thus complete. \square

Remark 4.7.3. Unlike the results in the static case (see [68] and [1]), where the energy decays exponentially provided that the wave speeds are equal, in the case of Kelvin-Voigt damping the energy does not decay uniformly even if the waves speed are equal. \square

4.8 Polynomial stability in the case of one local damping

This section is devoted to show the polynomial stability of system (4.1.1) under the following hypothesis

$$D_1 = D_3 = 0 \text{ in } (0, L) \quad \text{and} \quad \exists d_2 > 0 \text{ such that } D_2 \geq d_2 > 0 \text{ in } (\alpha, \beta) \subset (0, L). \quad (4.8.1)$$

The main result of this section is given by the following theorem :

Theorem 4.8.1. Assume that condition (4.8.1) is satisfied. Assume also that $D_2 \in L^\infty(0, L)$. Then, there exists a positive constant $c > 0$ such that for all $U_0 \in D(\mathcal{A}_j)$, $j = 1, 2$, the energy of system (4.1.1) satisfies the following decay rate :

$$E(t) \leq \frac{c}{\sqrt{t}} \|U_0\|_{D(\mathcal{A}_j)}^2. \quad (4.8.2)$$

Referring to [25], (4.8.2) is verified if the following conditions

$$i\mathbb{R} \subseteq \rho(\mathcal{A}_j) \quad (H1)$$

and

$$\limsup_{|\lambda| \rightarrow +\infty, \lambda \in \mathbb{R}} \frac{1}{\lambda^4} \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} = O(1) \quad (H4)$$

hold.

Condition $i\mathbb{R} \subseteq \rho(\mathcal{A}_j)$ is already proved in Lemma 4.3.2 and Lemma 4.3.3. We will establish (H4) by contradiction. Suppose that there exist a sequence of real numbers $(\lambda_n)_n$, with $|\lambda_n| \rightarrow +\infty$ and a sequence of vectors

$$U_n = (v_n^1, v_n^2, v_n^3, v_n^4, v_n^5, v_n^6)^\top \in D(\mathcal{A}_j) \quad \text{with} \quad \|U_n\|_{\mathcal{H}_j} = 1 \quad (4.8.3)$$

such that

$$\lambda_n^4 (i\lambda_n U_n - \mathcal{A}_j U_n) = (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6)^\top \rightarrow 0 \text{ in } \mathcal{H}_j, \quad j = 1, 2 \quad (4.8.4)$$

detailed as

$$i\lambda_n v_n^1 - v_n^2 = \frac{f_n^1}{\lambda_n^4}, \quad (4.8.5)$$

$$i\rho_1 \lambda_n v_n^2 - k_1 [(v_n^1)_x + v_n^3 + lv_n^5]_x - lk_3 [(v_n^5)_x - lv_n^1] = \rho_1 \frac{f_n^2}{\lambda_n^4}, \quad (4.8.6)$$

$$i\lambda_n v_n^3 - v_n^4 = \frac{f_n^3}{\lambda_n^4}, \quad (4.8.7)$$

$$i\rho_2 \lambda_n v_n^4 - [k_2 (v_n^3)_x + D_2 (v_n^4)_x]_x + k_1 [(v_n^1)_x + v_n^3 + lv_n^5] = \rho_2 \frac{f_n^4}{\lambda_n^4}, \quad (4.8.8)$$

$$i\lambda_n v_n^5 - v_n^6 = \frac{f_n^5}{\lambda_n^4}, \quad (4.8.9)$$

$$i\rho_1 \lambda_n v_n^6 - [k_3 ((v_n^5)_x - lv_n^1)]_x + lk_1 [(v_n^1)_x + v_n^3 + lv_n^5] = \rho_1 \frac{f_n^6}{\lambda_n^4}. \quad (4.8.10)$$

From (4.8.5), (4.8.7), (4.8.9) and (4.8.3), we deduce that

$$\|v_n^1\| = O\left(\frac{1}{\lambda_n}\right), \quad \|v_n^3\| = O\left(\frac{1}{\lambda_n}\right), \quad \|v_n^5\| = O\left(\frac{1}{\lambda_n}\right). \quad (4.8.11)$$

In the following we will check the condition (H4) by finding a contradiction with (4.8.3) such as $\|U_n\|_{\mathcal{H}_j} = o(1)$. For clarity, we divide the proof into several lemmas. From now on, for simplicity, we drop the index n .

Lemma 4.8.2. Assume that condition (4.8.1) is verified. Then, we have

$$\|D_2^{1/2} v_x^4\| = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \|v_x^4\| = \frac{o(1)}{\lambda^2} \quad \text{in} \quad (\alpha, \beta). \quad (4.8.12)$$

Proof: Taking the inner product of (4.8.4) with U in \mathcal{H}_j , we get

$$\operatorname{Re} \left(i\lambda^5 \|U\|^2 - \lambda^4 (\mathcal{A}_j U, U) \right)_{\mathcal{H}_j} = -\lambda^4 \operatorname{Re} (\mathcal{A}_j U, U)_{\mathcal{H}_j} = \lambda^4 \int_0^L D_2 |v_x^4|^2 dx = o(1).$$

Thanks to (4.8.1), we obtain the desired asymptotic equation (4.8.12). Thus the proof is complete.

Lemma 4.8.3. Assume that condition (4.8.1) is verified. Then, we have

$$\|v_x^3\| = \frac{o(1)}{\lambda^3} \quad \text{in} \quad (\alpha, \beta). \quad (4.8.13)$$

Proof: Differentiating equation (4.8.7), we get

$$i\lambda v_x^3 = v_x^4 + \frac{f_x^3}{\lambda^4},$$

and consequently

$$\int_{\alpha}^{\beta} |\lambda v_x^3|^2 dx \leq 2 \int_{\alpha}^{\beta} |v_x^4|^2 dx + 2 \int_{\alpha}^{\beta} \frac{|f_x^3|^2}{\lambda^8} dx.$$

Using (4.8.12) and the fact that f_3 converges to zero in $H_0^1(0, L)$ in the above equation, we get the desired estimation (4.8.13). Thus the proof is complete. \square

Let ϵ be a positive constant such that $0 < \alpha + \epsilon < \beta - \epsilon$. We define the cut-off function η by

$$\eta(x) = 1 \text{ in } (\alpha + \epsilon, \beta - \epsilon), \quad 0 \leq \eta(x) \leq 1, \quad \eta(x) = 0 \text{ in } (0, L) \setminus (\alpha, \beta).$$

Lemma 4.8.4. Assume that condition (4.8.1) is verified. Then, we have

$$\|\sqrt{\eta}\lambda v^4\| = O(1). \quad (4.8.14)$$

Proof: First, multiplying (4.8.8) by $i\rho_2^{-1}\eta\lambda\bar{v}^4$ and after integrating by parts, we get

$$\begin{aligned} \int_0^L \eta|\lambda v^4|^2 dx = \operatorname{Re} \left\{ \rho_2^{-1} \int_0^L k_2 i \lambda v_x^3 (\eta \bar{v}_x^4 + \eta' \bar{v}^4) dx + \rho_2^{-1} \int_0^L D_2 v_x^4 i \lambda (\eta \bar{v}_x^4 + \eta' \bar{v}^4) dx \right. \\ \left. + \rho_2^{-1} \int_0^L k_1 (v_x^1 + v^3 + lv^5) \eta i \lambda \bar{v}^4 dx - i \int_0^L \frac{f^4}{\lambda^3} \eta \bar{v}^4 dx \right\}. \end{aligned} \quad (4.8.15)$$

Now, using (4.8.12), (4.8.13) and the fact that v^4 is uniformly bounded in $L^2(0, L)$, we get

$$\operatorname{Re} \left\{ \rho_2^{-1} \int_0^L k_2 i \lambda v_x^3 (\eta \bar{v}_x^4 + \eta' \bar{v}^4) dx \right\} = \frac{o(1)}{\lambda^2}. \quad (4.8.16)$$

Moreover, using (4.8.12) and the fact that v^4 is uniformly bounded in $L^2(0, L)$, we obtain

$$\operatorname{Re} \left\{ \rho_2^{-1} \int_0^L D_2 v_x^4 i \lambda (\eta \bar{v}_x^4 + \eta' \bar{v}^4) dx \right\} = \frac{o(1)}{\lambda}. \quad (4.8.17)$$

Then, using Young's inequality and the fact that $v_x^1 + v^3 + lv^5$ is uniformly bounded in $L^2(0, L)$, we get for $\epsilon_1 > 0$

$$\begin{aligned} \operatorname{Re} \left\{ \rho_2^{-1} \int_0^L k_1 (v_x^1 + v^3 + lv^5) \eta i \lambda \bar{v}^4 dx \right\} &\leq \frac{1}{\epsilon_1} \int_0^L k_1^2 \eta |v_x^1 + v^3 + lv^5|^2 dx + \epsilon_1 \int_0^L \eta |\lambda v^4|^2 dx \\ &\leq \epsilon_1 \int_0^L \eta |\lambda v^4|^2 dx + O(1). \end{aligned} \quad (4.8.18)$$

Also, using the fact that f^4 converges to zero in $L^2(0, L)$ and v^4 is uniformly bounded in $L^2(0, L)$, we get

$$\operatorname{Re} \left\{ \int_0^L \frac{f^4}{\lambda^3} \eta \bar{v}^4 dx \right\} = \frac{o(1)}{\lambda^3}. \quad (4.8.19)$$

Finally, inserting (4.8.16), (4.8.17), (4.8.18) and (4.8.19) into (4.8.15), we get

$$(1 - \epsilon_1) \int_0^L \eta |\lambda v^4|^2 dx \leq O(1).$$

Consequently for $\epsilon_1 = \frac{1}{2}$, we get the desired estimation (4.8.14). Thus the proof is complete. \square

Lemma 4.8.5. Assume that condition (4.8.1) is verified. Then, we have

$$\int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = o(1). \quad (4.8.20)$$

Proof: First, multiplying (4.8.7) by $i\eta\lambda\overline{v^3}$, we get

$$-\int_0^L \eta |\lambda v^3|^2 dx - i \int_0^L \eta \lambda v^4 \overline{v^3} dx = \int_0^L i \eta \frac{f^3}{\lambda^3} \overline{v^3} dx. \quad (4.8.21)$$

Multiplying (4.8.8) by $\rho_2^{-1}\eta\overline{v^3}$, we obtain after integrating by parts

$$\begin{aligned} & i \int_0^L \eta \lambda v^4 \overline{v^3} dx + \rho_2^{-1} \int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta' \overline{v^3} + \eta \overline{v_x^3}) dx \\ & + \rho_2^{-1} \int_0^L k_1 (v_x^1 + v^3 + lv^5) \eta \overline{v^3} dx = \int_0^L \frac{f^4}{\lambda^4} \eta \overline{v^3} dx. \end{aligned} \quad (4.8.22)$$

Now, combining (4.8.21) and (4.8.22), we get

$$\begin{aligned} \int_0^L \eta |\lambda v^3|^2 dx &= \rho_2^{-1} \int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta' \overline{v^3} + \eta \overline{v_x^3}) dx + \rho_2^{-1} \int_0^L k_1 (v_x^1 + v^3 + lv^5) \eta \overline{v^3} dx \\ & - \int_0^L i \eta \frac{f^3}{\lambda^3} \overline{v^3} dx - \int_0^L \frac{f^4}{\lambda^4} \eta \overline{v^3} dx. \end{aligned} \quad (4.8.23)$$

Then, using (4.8.12) and (4.8.13), $\|v^3\| = O(\frac{1}{\lambda})$, the fact that $(v_x^1 + v^3 + lv^5)$ is uniformly bounded in $L^2(0, L)$ and the fact that f^3, f^4 converge to zero in $H_0^1(0, L), L^2(0, L)$ respectively, we deduce that

$$\begin{aligned} & \rho_2^{-1} \int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta' \overline{v^3} + \eta \overline{v_x^3}) dx + \rho_2^{-1} \int_0^L k_1 (v_x^1 + v^3 + lv^5) \eta \overline{v^3} dx \\ & - \int_0^L i \eta \frac{f^3}{\lambda^3} \overline{v^3} dx - \int_0^L \frac{f^4}{\lambda^4} \eta \overline{v^3} dx = o(1). \end{aligned} \quad (4.8.24)$$

Finally, inserting (4.8.24) into (4.8.23) and using the definition of η , we deduce

$$\int_0^L \eta |\lambda v^3|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^3|^2 dx = o(1).$$

The proof is thus complete. \square

Lemma 4.8.6. Assume that condition (4.8.1) is verified. Then, we have

$$\int_0^L \eta |v_x^1|^2 dx = \frac{o(1)}{\lambda^2}, \quad \int_0^L \eta |\lambda v^1|^2 dx = \frac{o(1)}{\lambda^2} \quad (4.8.25)$$

and

$$\int_{\alpha+\epsilon}^{\beta-\epsilon} |v_x^1|^2 dx = \frac{o(1)}{\lambda^2}, \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = \frac{o(1)}{\lambda^2}. \quad (4.8.26)$$

Proof: For the clarity of the proof, we divide the proof into several steps.

Step 1. Our first aim here is to prove

$$\int_0^L \eta |v_x^1|^2 dx = o(1). \quad (4.8.27)$$

For this, multiplying (4.8.8) by $\overline{\eta v_x^1}$ and integrating by parts, we get

$$\begin{aligned} & -i \int_0^L \lambda \rho_2 v^4 \eta' \overline{v^1} dx - i \int_0^L \lambda \rho_2 v_x^4 \eta \overline{v^1} dx + \int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta \overline{v_{xx}^1}) dx + \int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta' \overline{v_x^1}) dx \\ & + \int_0^L \eta k_1 |v_x^1|^2 dx + \int_0^L \eta k_1 v^3 \overline{v_x^1} dx + \int_0^L k_1 \eta v^5 \overline{v_x^1} dx = \int_0^L \rho_2 \frac{f^4}{\lambda^4} \eta \overline{v_x^1} dx. \end{aligned} \quad (4.8.28)$$

Now, we need to estimate each term of (4.8.28) :

- Using (4.8.11) and (4.8.14), we get

$$-i \int_0^L \lambda \rho_2 v^4 \eta' \overline{v^1} dx = o(1). \quad (4.8.29)$$

- Using (4.8.12) and the fact that λv^1 is uniformly bounded in $L^2(0, L)$, we obtain

$$-i \int_0^L \lambda \rho_2 v_x^4 \eta \overline{v^1} dx = \frac{o(1)}{\lambda^2}. \quad (4.8.30)$$

- From (4.8.6), we remark that $\frac{1}{\lambda} v_{xx}^1$ is uniformly bounded in $L^2(0, L)$. This fact combined with (4.8.12) and (4.8.13) yields

$$\int_0^L (k_2 \lambda v_x^3 + D_2 \lambda v_x^4) (\eta \frac{\overline{v_{xx}^1}}{\lambda}) dx = \frac{o(1)}{\lambda}. \quad (4.8.31)$$

- Using (4.8.12), (4.8.13) and the fact that v_x^1 is uniformly bounded in $L^2(0, L)$, we get

$$\int_0^L (k_2 v_x^3 + D_2 v_x^4) (\eta' \overline{v_x^1}) dx = \frac{o(1)}{\lambda^2}. \quad (4.8.32)$$

- Using (4.8.11) and the fact that v_x^1 is uniformly bounded in $L^2(0, L)$, we obtain

$$\int_0^L \eta k_1 v^3 \overline{v_x^1} dx + \int_0^L k_1 \eta v^5 \overline{v_x^1} dx = o(1). \quad (4.8.33)$$

- Using the fact that f^4 converges to zero in $L^2(0, L)$ and v_x^1 is uniformly bounded in $L^2(0, L)$, we get

$$\int_0^L \rho_2 \frac{f^4}{\lambda^4} \eta \overline{v_x^1} dx = \frac{o(1)}{\lambda^4}. \quad (4.8.34)$$

Finally, inserting equations (4.8.29)-(4.8.34) into (4.8.28) we get the desired estimate (4.8.27).

Next, our second aim is to prove

$$\int_0^L \eta |\lambda v^1|^2 dx = o(1). \quad (4.8.35)$$

Multiplying (4.8.5) by $i\eta\lambda\bar{v}^1$, we get

$$-\int_0^L \eta|\lambda v^1|^2 dx - i \int_0^L \eta\lambda\bar{v}^1 v^2 dx = i \int_0^L \frac{f^1}{\lambda^3} \eta\bar{v}^1 dx. \quad (4.8.36)$$

Then, multiplying (4.8.6) by $\rho_1^{-1}\eta\bar{v}^1$ and integrating by parts, we get

$$\begin{aligned} & i \int_0^L \eta\lambda\bar{v}^1 v^2 dx + \rho_1^{-1} \int_0^L k_1(v_x^1 + v^3 + lv^5)(\eta' \bar{v}^1 + \eta\bar{v}_x^1) dx \\ & - \rho_1^{-1} \int_0^L k_3(v_x^5 - lv^1)\eta\bar{v}^1 dx = \int_0^L \frac{f^2}{\lambda^4} \eta\bar{v}^1 dx. \end{aligned} \quad (4.8.37)$$

Combining (4.8.36) and (4.8.37), we get

$$\begin{aligned} \int_0^L \eta|\lambda v^1|^2 dx &= \rho_1^{-1} \int_0^L k_1(v_x^1 + v^3 + lv^5)(\eta' \bar{v}^1 + \eta\bar{v}_x^1) dx - \rho_1^{-1} \int_0^L k_3(v_x^5 - lv^1)\eta\bar{v}^1 dx \\ & - \int_0^L \left(\frac{f^2}{\lambda^4} + i \frac{f^1}{\lambda^3} \right) \eta\bar{v}^1 dx. \end{aligned} \quad (4.8.38)$$

Finally, using (4.8.11), (4.8.27), the fact that $(v_x^1 + v^3 + lv^5)$, $(v_x^5 - lv^1)$ are uniformly bounded in $L^2(0, L)$ and f^1 , f^2 converge respectively to zero in $H_0^1(0, L)$, $L^2(0, L)$ in the right hand side of the above equation, we deduce that

$$\int_0^L \eta|\lambda v^1|^2 dx = o(1).$$

Step 2. Our aim here is to prove

$$\int_0^L \eta|v_x^1|^2 dx = \frac{o(1)}{\lambda} \quad (4.8.39)$$

and

$$\int_0^L \eta|\lambda v^1|^2 dx = \frac{o(1)}{\lambda}. \quad (4.8.40)$$

To prove (4.8.39), multiplying (4.8.28) by λ , we get

$$\begin{aligned} & -i \int_0^L \lambda\rho_2 v^4 \eta' \lambda\bar{v}^1 dx - i \int_0^L \lambda\rho_2 v_x^4 \eta\lambda\bar{v}^1 dx + \int_0^L (k_2\lambda v_x^3 + D_2\lambda v_x^4)(\eta\bar{v}_{xx}^1) dx \\ & + \int_0^L (k_2\lambda v_x^3 + D_2\lambda v_x^4)(\eta' \bar{v}_x^1) dx + \int_0^L \eta k_1 \lambda |v_x^1|^2 dx + \int_0^L \eta k_1 \lambda v^3 \bar{v}_x^1 dx \\ & + \int_0^L k_1 \eta \lambda v^5 \bar{v}_x^1 dx = \int_0^L \rho_2 \frac{f^4}{\lambda^3} \eta\bar{v}_x^1 dx. \end{aligned} \quad (4.8.41)$$

Now, we need to estimate each term of (4.8.41) as follows :

- Using (4.8.14) and (4.8.35), we get

$$-i \int_0^L \lambda\rho_2 v^4 \eta' \lambda\bar{v}^1 dx = o(1). \quad (4.8.42)$$

- Using (4.8.12) and the fact that λv^1 is uniformly bounded in $L^2(0, L)$, we obtain

$$-i \int_0^L \lambda \rho_2 v_x^4 \eta \lambda \overline{v^1} dx = \frac{o(1)}{\lambda}. \quad (4.8.43)$$

- Using the fact $\frac{1}{\lambda} v_{xx}^1$ is uniformly bounded in $L^2(0, L)$ due to (4.8.6) combined with (4.8.12) and (4.8.13) yields

$$\int_0^L (k_2 \lambda^2 v_x^3 + D_2 \lambda^2 v_x^4) (\eta \frac{\overline{v_{xx}^1}}{\lambda}) dx = o(1). \quad (4.8.44)$$

- Using (4.8.12), (4.8.13) and the fact that v_x^1 is uniformly bounded in $L^2(0, L)$, we get

$$\int_0^L (k_2 \lambda v_x^3 + D_2 \lambda v_x^4) (\eta \overline{v_x^1}) dx = \frac{o(1)}{\lambda}. \quad (4.8.45)$$

- Using (4.8.27) and the fact that λv^3 and λv^5 are uniformly bounded in $L^2(0, L)$, we obtain

$$\int_0^L \eta k_1 \lambda v^3 \overline{v_x^1} dx + \int_0^L k_1 \eta \lambda v^5 \overline{v_x^1} dx = o(1). \quad (4.8.46)$$

- Using the fact that f^4 converges to zero in $L^2(0, L)$ and v_x^1 is uniformly bounded in $L^2(0, L)$, we get

$$\int_0^L \rho_1 \frac{f^4}{\lambda^3} \eta \overline{v_x^1} dx = \frac{o(1)}{\lambda^3}. \quad (4.8.47)$$

Finally, inserting equations (4.8.42)-(4.8.47) into (4.8.41), we deduce that

$$\int_0^L \eta |v_x^1|^2 dx = \frac{o(1)}{\lambda}.$$

On the other hand, our target now is to prove (4.8.40). For this, multiplying (4.8.38) by λ , we get

$$\begin{aligned} \int_0^L \eta \lambda |\lambda v^1|^2 dx &= \rho_1^{-1} \int_0^L k_1 (v_x^1 + v^3 + l v^5) (\eta' \lambda \overline{v^1} + \eta \lambda \overline{v_x^1}) dx - \rho_1^{-1} \int_0^L k_3 (v_x^5 - l v^1) \eta \lambda \overline{v^1} \\ &\quad - \int_0^L \left(\frac{f^2}{\lambda^3} + i \frac{f^1}{\lambda^2} \right) \eta \overline{v^1} dx. \end{aligned} \quad (4.8.48)$$

Using (4.8.11), (4.8.35), (4.8.39), the fact that $(v_x^1 + v^3 + l v^5)$, $(v_x^5 - l v^1)$ are uniformly bounded in $L^2(0, L)$ and the fact that f^1 , f^2 converge to zero respectively in $H_0^1(0, L)$, $L^2(0, L)$ in the right hand side of the above equation, we deduce that

$$\int_0^L \eta \lambda |\lambda v^1|^2 dx = o(1).$$

Step 3. Our target is to prove

$$\left\| \eta \frac{v_{xx}^1}{\sqrt{\lambda}} \right\| = O(1). \quad (4.8.49)$$

So, multiplying (4.8.5) by $\eta\sqrt{\lambda}$, we get

$$\eta\sqrt{\lambda}v^2 = i\eta\sqrt{\lambda}\lambda v^1 - \eta\sqrt{\lambda}\frac{f^1}{\lambda^4}. \quad (4.8.50)$$

Then, integrating (4.8.50) over $(0, L)$, we get

$$\int_0^L \eta^2 \lambda |v^2|^2 dx \leq 2 \int_0^L \eta^2 \lambda |\lambda v^1|^2 dx + 2 \int_0^L \eta^2 \frac{|f^1|^2}{\lambda^7} dx. \quad (4.8.51)$$

Using (4.8.40) and the fact that f^1 converges to zero in $H_0^1(0, L)$ in the previous equation, we deduce

$$\|\eta\sqrt{\lambda}v^2\| = o(1). \quad (4.8.52)$$

Next, multiplying (4.8.6) by $\frac{\eta}{\sqrt{\lambda}}$, we get

$$k_1 \eta \frac{v_{xx}^1}{\sqrt{\lambda}} = i\rho_1 \eta \sqrt{\lambda} v^2 - k_1 \frac{\eta}{\sqrt{\lambda}} (v_x^3 + lv_x^5) - lk_3 \frac{\eta}{\sqrt{\lambda}} (v_x^5 - lv^1) - \rho_1 \eta \frac{f^2}{\lambda^4 \sqrt{\lambda}}. \quad (4.8.53)$$

Finally, using (4.8.52), the fact that $v_x^3, v_x^5, (v_x^5 - lv^1)$ are uniformly bounded in $L^2(0, L)$ and f^2 converges to zero in $L^2(0, L)$ in the previous equation, we get the desired estimation (4.8.49).

Step 4. Our aim is to prove

$$\int_0^L \eta |v_x^1|^2 dx = \frac{o(1)}{\lambda^{1+1/2}} \quad (4.8.54)$$

and

$$\int_0^L \eta |\lambda v^1|^2 dx = \frac{o(1)}{\lambda^{1+1/2}}. \quad (4.8.55)$$

To prove (4.8.54), multiplying (4.8.28) by $\lambda^{1+1/2}$, we get

$$\begin{aligned} & -i \int_0^L \lambda \rho_2 v^4 \eta' \lambda^{1+1/2} \overline{v^1} dx - i \int_0^L \lambda^2 \rho_2 v_x^4 \eta \lambda^{1/2} \overline{v^1} dx + \int_0^L (k_2 \lambda^{1+1/2} v_x^3 + D_2 \lambda^{1+1/2} v_x^4) (\eta \overline{v_{xx}^1}) dx \\ & + \int_0^L (k_2 \lambda v_x^3 + D_2 \lambda v_x^4) (\eta' \lambda^{1/2} \overline{v^1}) dx + \int_0^L \eta k_1 \lambda^{1+1/2} |v_x^1|^2 dx + \int_0^L \eta k_1 \lambda v^3 \lambda^{1/2} \overline{v_x^1} dx \\ & + \int_0^L lk_1 \eta \lambda v^5 \lambda^{1/2} \overline{v_x^1} dx = \int_0^L \rho_2 \frac{f^4}{\lambda^3} \eta \lambda^{1/2} \overline{v_x^1} dx. \end{aligned} \quad (4.8.56)$$

Next, we need to estimate each term of (4.8.56) as follows

- Using (4.8.14) and (4.8.40), we get

$$-i \int_0^L \lambda \rho_2 v^4 \eta' \lambda^{1+1/2} \overline{v^1} dx = o(1). \quad (4.8.57)$$

- Using (4.8.12) and λv^1 is uniformly bounded in $L^2(0, L)$, we obtain

$$-i \int_0^L \lambda^2 \rho_2 v_x^4 \eta \lambda^{1/2} \overline{v^1} dx = \frac{o(1)}{\lambda^{1/2}}. \quad (4.8.58)$$

- Using (4.8.12), (4.8.13) and (4.8.49), we obtain

$$\int_0^L (k_2 \lambda^{1+1/2} v_x^3 + D_2 \lambda^{1+1/2} v_x^4) (\eta \overline{v_x^1}) dx = \int_0^L (k_2 \lambda^2 v_x^3 + D_2 \lambda^2 v_x^4) \left(\eta \frac{\overline{v_x^1}}{\sqrt{\lambda}} \right) = o(1). \quad (4.8.59)$$

- Using (4.8.12), (4.8.13) and (4.8.39), we get

$$\int_0^L (k_2 \lambda v_x^3 + D_2 \lambda v_x^4) (\eta' \lambda^{1/2} \overline{v_x^1}) dx = \frac{o(1)}{\lambda}. \quad (4.8.60)$$

- Using (4.8.39) and the fact that λv^3 and λv^5 are uniformly bounded in $L^2(0, L)$, we obtain

$$\int_0^L \eta k_1 \lambda v^3 \lambda^{1/2} \overline{v_x^1} dx + \int_0^L k_1 \eta \lambda v^5 \lambda^{1/2} \overline{v_x^1} dx = o(1). \quad (4.8.61)$$

- Using (4.8.39) and the fact that f^4 converges to zero in $H_0^1(0, L)$, we get

$$\int_0^L \rho_1 \frac{f^4}{\lambda^3} \eta \lambda^{1/2} \overline{v_x^1} dx = \frac{o(1)}{\lambda^3}. \quad (4.8.62)$$

Finally, inserting equations (4.8.57)-(4.8.62) into (4.8.56), we deduce that

$$\int_0^L \eta |v_x^1|^2 dx = \frac{o(1)}{\lambda^{1+1/2}}.$$

On the other side, our aim now is to prove (4.8.55). For this aim, multiplying (4.8.38) by $\lambda^{1+1/2}$, we get

$$\begin{aligned} \int_0^L \eta \lambda^{1+1/2} |\lambda v^1|^2 dx &= \rho_1^{-1} \int_0^L k_1 (v_x^1 + v^3 + l v^5) (\eta' \lambda^{1+1/2} \overline{v^1} + \eta \lambda^{1+1/2} \overline{v_x^1}) dx \\ &\quad - \rho_1^{-1} \int_0^L k_3 (v_x^5 - l v^1) \eta \lambda^{1+1/2} \overline{v^1} - \int_0^L \left(\frac{f^2}{\lambda^{5/2}} + i \frac{f^1}{\lambda^{3/2}} \right) \eta \overline{v^1} dx. \end{aligned} \quad (4.8.63)$$

Using (4.8.11), (4.8.40), (4.8.54), the fact that $(v_x^1 + v^3 + l v^5)$, $(v_x^5 - l v^1)$ are uniformly bounded in $L^2(0, L)$ and the fact that f^1 , f^2 converge to zero respectively in $H_0^1(0, L)$, $L^2(0, L)$ in the right hand side of the above equation, we deduce that

$$\int_0^L \eta \lambda^{1+1/2} |\lambda v^1|^2 dx = o(1).$$

Step 5. Using (4.8.27), (4.8.39), (4.8.54) and the definition of η , we deduce

$$\int_0^L \eta |v_x^1|^2 dx = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |v_x^1|^2 dx = \frac{o(1)}{\lambda^2}.$$

Using (4.8.35), (4.8.40), (4.8.55) and the definition of η , we deduce

$$\int_0^L \eta |\lambda v^1|^2 dx = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^1|^2 dx = \frac{o(1)}{\lambda^2}.$$

The proof is thus complete. □

Lemma 4.8.7. Assume that condition (4.8.1) is verified. Then, we have

$$\int_0^L \eta |v_x^5|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+\epsilon}^{\beta-\epsilon} |v_x^5|^2 dx = o(1) \quad (4.8.64)$$

Proof: First, substituting (4.8.5) into (4.8.6), we get

$$-\lambda^2 \rho_1 v^1 - k_1 (v_x^1 + v^3 + lv^5)_x - lk_3 (v_x^5 - lv^1) = \rho_1 \left(\frac{f^2}{\lambda^4} + i \frac{f^1}{\lambda^3} \right). \quad (4.8.65)$$

Multiplying (4.8.65) by $\overline{\eta v_x^5}$ and integrating over $(0, L)$, we get

$$\begin{aligned} (lk_1 + lk_3) \int_0^L \eta |v_x^5|^2 &= -\rho_1 \int_0^L \eta \lambda^2 v^1 \overline{v_x^5} dx + k_1 \int_0^L v_x^1 \eta' \overline{v_x^5} dx + k_1 \int_0^L \eta \lambda v_x^1 \frac{\overline{v_{xx}^5}}{\lambda} dx \\ &\quad - k_1 \int_0^L \eta v_x^3 \overline{v_x^5} dx + l^2 k_3 \int_0^L \eta v^1 \overline{v_x^5} dx + \frac{o(1)}{\lambda^3}. \end{aligned} \quad (4.8.66)$$

Finally, using (4.8.13), (4.8.25), the fact that v_x^5 is uniformly bounded in $L^2(0, L)$ and $\frac{1}{\lambda} v_{xx}^5$ is uniformly bounded in $L^2(0, L)$ due to (4.8.10) in the right hand side of the previous equation, we get the desired estimation (4.8.64). The proof is thus complete.

Lemma 4.8.8. Assume that condition (4.8.1) is verified. Then, we have

$$\int_{\alpha+\epsilon}^{\beta-\epsilon} |\lambda v^5|^2 dx = o(1). \quad (4.8.67)$$

Proof: Multiplying (4.8.9) by $i\eta\lambda\overline{v^5}$, we get

$$-\int_0^L \eta |\lambda v^5|^2 dx - i \int_0^L \eta \lambda \overline{v^5} v^6 dx = i \int_0^L \frac{f^5}{\lambda^3} \eta \overline{v^5} dx. \quad (4.8.68)$$

Then, multiplying (4.8.10) by $\eta \rho_1^{-1} \overline{v^5}$ and integrating by parts, we get

$$\begin{aligned} i \int_0^L \eta \lambda \overline{v^5} v^6 dx + \rho_1^{-1} \int_0^L k_3 (v_x^5 - lv^1) (\eta' \overline{v^5} + \eta \overline{v_x^5}) dx \\ + \rho_1^{-1} \int_0^L lk_1 (v_x^1 + v^3 + lv^5) \eta \overline{v^5} dx = \int_0^L \frac{f^6}{\lambda^4} \eta \overline{v^5} dx. \end{aligned} \quad (4.8.69)$$

Combining (4.8.68) and (4.8.69), we get

$$\begin{aligned} \int_0^L \eta |\lambda v^5|^2 dx &= \rho_1^{-1} \int_0^L k_3 (v_x^5 - lv^1) (\eta' \overline{v^5} + \eta \overline{v_x^5}) dx + \rho_1^{-1} \int_0^L lk_1 (v_x^1 + v^3 + lv^5) \eta \overline{v^5} dx \\ &\quad - \int_0^L \left(\frac{f^6}{\lambda^4} + i \frac{f^5}{\lambda^3} \right) \eta \overline{v^5} dx. \end{aligned} \quad (4.8.70)$$

Using (4.8.11), (4.8.64), the fact that $(v_x^1 + v^3 + lv^5)$, $(v_x^5 - lv^1)$ are uniformly bounded in $L^2(0, L)$, f^6 , f^5 converge to zero respectively in $H_0^1(0, L)$, $L^2(0, L)$ in the right hand side of the above equation, we deduce

$$\int_0^L \eta |\lambda v^5|^2 dx = o(1).$$

Finally, using the definition of η , we get the desired estimation (4.8.67). The proof is thus complete. \square

Remark 4.8.9. It is easy to see that the results of Lemmas 4.5.8, 4.5.9, 4.5.10 still hold here, and consequently one may get the estimation (4.5.88) of Lemma 4.5.11. \square

Proof of Theorem 4.8.1 As we mention in Remark 4.8.9, the estimation (4.5.88) is also true here. It follows from estimations (4.8.13), (4.8.20), (4.8.25), (4.8.64), (4.8.67) and (4.5.88) that $\|U_n\|_{\mathcal{H}_j} = o(1)$ which is a contradiction with (4.8.3). Consequently, condition (H4) holds and the energy of smooth solutions of system (4.1.1) decays polynomially as t goes to infinity. \square

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