Some aspects of the central role of financial market microstructure : Volatility dynamics, optimal trading and market design

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LE GRADE DE DOCTEUR EN SCIENCES DE L'ECOLE POLYTECHNIQUE

Spécialité : Mathématiques

par

Paul JUSSELIN

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Jean-Philippe BOUCHAUD	rapporteur
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Nicole EL KAROUI	examinateur
Marc HOFFMANN	examinateur
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Résumé

Cette thèse est organisée en trois parties. Dans la première on examine les relations entre la dynamique microscopique et macroscopique du marché en se concentrant sur les propriétés de la volatilité. Tout d'abord on montre que le principe d'absence d'opportunité d'arbitrage implique que la volatilité suit un modèle *rough Heston*, puis on étudie des extensions du modèle rough Heston permettant de reproduire l'effet *Zumbach*. On utilise ensuite ces extensions pour la calibration jointe des nappes de volatilité du SPX et du VIX. Dans la deuxième partie on s'intéresse au contrôle optimal stochastique de processus ponctuels. On commence par proposer une méthode pour résoudre des problèmes de contrôle stochastique de processus de Hawkes. On examine ensuite les limites d'échelle de problèmes de contrôle de dynamique de population. Finalement dans la troisième partie on étudie deux problématiques de *market design*: tout d'abord la question de l'organisation des marchés de produits dérivés puis la celle de la durée optimale d'enchère pour un marché organisé enchères séquentielles.

On commence cette thèse par l'étude des liens entre le principe d'absence d'opportunité d'arbitrage et l'irrégularité de la volatilité. A l'aide d'une méthode de changement d'échelle on montre que l'on peut effectivement connecter ces deux notions par l'analyse du *market impact* des métaordres. Plus précisément on modélise le flux des ordres marchés en utilisant des procesus de Hawkes linéaires. Puis on montre que le principe d'absence d'opportunité d'arbitrage ainsi que l'existence d'un market impact non trivial impliquent que la volatilité est rugueuse et plus précisément qu'elle suit un modèle rough Heston. On examine ensuite une classe de modèles microscopiques où le flux d'ordre est un processus de Hawkes quadratique. L'objectif est d'étendre le modèle rough Heston à des modèles continus permettant de reproduire l'effet Zumbach. Finalement on utilise un de ces modèles, le modèle rough Heston quadratique, pour la calibration jointe des nappes de volatilité du SPX et du VIX.

Motivé par l'usage intensif de processus ponctuels dans la première partie, on s'intéresse dans la deuxième au contrôle stochastique de processus ponctuels. Notre objectif est de fournir des résultats théoriques en vue d'applications en finance. On commence par considérer le cas du contrôle de processus de Hawkes. On prouve l'existence d'une solution puis l'on propose une méthode permettant d'appliquer ce contrôle en pratique. On examine ensuite les limites d'échelles de problèmes de contrôles stochastiques dans le cadre de modèles de dynamique de population. Plus exactement on considère une suite de modèles de dynamique d'une population discrète qui converge vers un modèle pour une population continue. Pour chacun des modèles on considère un problème de contrôle. On prouve que la suite des contrôles optimaux associés aux modèles discrets converge vers le contrôle optimal associé au modèle continu. Ce résultat repose sur la continuité, par rapport à différents paramètres, de la solution d'une équation différentielle schostatique rétrograde.

Dans la dernière partie on s'intéresse à deux problèmatiques de market design. On examine d'abord la question de l'organisation d'un marché liquide de produits dérivés. En se concentrant

sur un marché d'options, on propose une méthode en deux étapes pouvant facilement être appliquée en pratique. La première étape consiste à choisir les options qui seront listées sur le marché. Pour cela on utilise un algorithme de quantification qui permet de sélectionner les options les plus demandées par les investisseurs. On propose ensuite une méthode d'incitation tarifaire visant à encourager les *market makers* à proposer des prix attractifs. On formalise ce problème comme un problème de type principal-agent que l'on résoud explicitement. Finalement, on cherche la durée optimale d'une enchère pour les marchés organisés en enchères séquentielles, le cas de la durée nulle correspondant à celui d'une double enchère continue. On utilise un modèle où les *market takers* sont en compétition et on considère que la durée optimale est celle correspondant au processus de découverte du prix le plus efficace. Après avoir prouvé l'existence d'un équilibre de Nash pour la compétition entre les market takers, on applique nos résultats sur des données de marchés. Pour la plupart des actifs, la durée optimale se trouve entre 2 et 10 minutes.

Abstract

This thesis is made of three parts. In the first one, we study the connections between the dynamics of the market at the microscopic and macroscopic scales, with a focus on the properties of the volatility. We begin with showing that the no-arbitrage principle implies that volatility follows a rough Heston model. Then we study extensions of this model reproducing the so-called Zumbach effect and use them for simultaneous calibration of SPX and VIX options smiles. In the second part we deal with optimal control for point processes. We first propose a method to solve stochastic control problems driven by Hawkes processes. Then we consider the scaling limits of stochastic control problems in the framework of population modeling. Finally in the third part we study two questions of market design. We start by analyzing the organization of a derivatives exchange. Then we look for the optimal auction duration in sequential auctions markets.

We begin this thesis with studying the links between the no-arbitrage principle and the (ir)regularity of volatility. Using a microscopic to macroscopic approach, we show that we can connect those two notions through the market impact of metaorders. We model the market order flow using linear Hawkes processes and show that the no-arbitrage principle together with the existence of a non-trivial market impact imply that the volatility process has to be rough, more precisely a rough Heston model. Then we study a class of microscopic models where order flows are driven by quadratic Hawkes processes. The objective is to extend the rough Heston model building continuous models that reproduce the feedback of price trends on volatility: the so-called Zumbach effect. We show that using appropriate scaling procedures the microscopic models converge towards price dynamics where volatility is rough and that reproduce the Zumbach effect. Finally we use one of those models, the quadratic rough Heston model, to solve the longstanding problem of joint calibration of SPX and VIX options smiles.

Motivated by the extensive use of point processes in the first part of our work we focus in the second part on stochastic control for point processes. Our aim is to provide theoretical guarantees for applications in finance. We begin with considering a general stochastic control problem driven by Hawkes processes. We prove the existence of a solution and more importantly provide a method to implement the optimal control in practice. Then we study the scaling limits of solutions to stochastic control problems in the framework of population modeling. More precisely we consider a sequence of models for the dynamics of a discrete population converging to a model with continuous population. For each model we consider a stochastic control problem. We prove that the sequence of optimal controls associated to the discrete models converges towards the optimal control associated to the continuous model. This result relies on the continuity of the solution to a backward stochastic differential equation with respect to the driving martingale and terminal value.

In the last part we address two questions of market design. We are first interested in designing a liquid electronic market of derivatives. We focus on options and propose a two steps method that can be easily applied in practice. The first step is to select the listed options. For this we

use a quantization algorithm enabling us to pick the options capturing most of market demand. The second step is to design a make-take fees policy for market makers to incentivize them to set attractive quotes. We formalize this issue as a principal agent problem that we explicitly solve. Finally we look for the optimal auction duration that should be used on a market organized in sequential auctions, the case of auctions with 0 second duration corresponding to the continuous double auctions situation. To do so, we use an agent based model where market takers are competing. We consider that the optimal auction duration is the one leading to the best quality of price formation process. After proving existence of a Nash equilibrium for the competition between market takers we apply our results on stocks market data. We find that for most of the stocks, the optimal auction duration lies between 2 and 10 minutes.

List of papers being part of this thesis

- P. Jusselin and M. Rosenbaum, *No-arbitrage implies power-law market impact and rough volatility*, to appear in Mathematical Finance, 2019.
- A. Dandapani, P. Jusselin and M. Rosenbaum, *From quadratic Hawkes processes to super-Heston rough volatility models with Zumbach effect*, under revision, 2019.
- J. Gatheral, P. Jusselin and M. Rosenbaum, *The quadratic rough Heston model: fitting simultaneously historical volatility together with SPX and VIX smiles with a continuous model*, to appear in Risk Magazine, 2020.
- P. Jusselin, *Optimal market making under persistent order flow*, submitted, 2020.
- P. Jusselin, T. Mastrolia, *Scaling limit for stochastic control problems in population dynamics*, submitted, 2019.
- B. Baldacci, P. Jusselin, M. Rosenbaum, *How to design a derivatives market?*, submitted, 2019.
- P. Jusselin, T. Mastrolia, M. Rosenbaum, *Optimal auction duration: A price formation viewpoint*, under revision, 2019.

Contents

Introduction

This thesis is split into three parts. In the first one, our goal is to find microstructural foundations for some important properties of volatility. The second part deals with problems related to stochastic control for point processes. Our objective here is to provide mathematical guarantees for some original and relevant control problems with applications in finance in mind. The third part is devoted to market design. More precisely we aim at building solutions to practical market design issues based on a quantitative analysis.

Motivations

Many recent studies have shown that historical volatility is very irregular and shares the scaling properties of a fractional Brownian motion with Hurst index close to 0.1. To sum up those stylized facts one says that *volatility is rough*. This roughness property has been measured on more than 6000 assets and therefore seems to be universal. Hence it must be related to the only concept which is somehow universal in finance: the no-arbitrage principle. The first question we consider in this thesis is therefore:

Question 1. *Can we explain the roughness of volatility based on some no-arbitrage principle?*

To tackle this question we consider a very general tick-by-tick model based on Hawkes processes for market order flow. Then we show that under no-statistical arbitrage assumption, rough volatility arises at the scaling limit: more precisely the price follows a rough Heston model. Furthermore we also show that in this setting market impact of metaorders is necessarily a power-law with exponent related to the Hurst index of volatility. Though it reproduces many empirical properties of volatility both under the historical and risk neutral measures the rough Heston model has some limitations. In particular it does not generate any feedback of price trends on the volatility, which is also referred to as the *Zumbach effect*. So,

Question 2. *Can we extend the rough Heston model to a rough volatility model with Zumbach effect?*

To address this issue, we go beyond linear Hawkes processes and use quadratic Hawkes processes to model the market order flow. As a matter of fact, those processes naturally reproduce the Zumbach effect at the microscopic level. Then using different scaling procedures we obtain three new volatility models. All of them generate Zumbach effect and can be viewed as extensions of the rough Heston model. A next step is to wonder whether they can

outperform the rough Heston model into fitting options market data. In particular we are interested *in the holy grail of volatility modeling:* the joint calibration of SPX and VIX options smiles. This is a very challenging task that no model with continuous paths has been able to complete so far. Hence we are concerned with the following question:

Question 3. *Can we find a (rough) continuous model fitting simultaneously SPX and VIX options smiles?*

In order to answer this question we consider a specific model among the ones obtained for Question 2: the quadratic rough Heston model. We show that it is possible to achieve a very accurate joint calibration of SPX and VIX options smiles using this model.

The microscopic models used for Questions 1, 2 and 3 all rely on Hawkes processes. Hawkes processes allow us to reproduce many properties of market order flow such as its self-exciting behavior and long memory. These are important stylized facts that must be taken into account by market participants while designing their trading strategies. Since Hawkes processes are a very convenient tool to reproduce those properties we want to address the following question:

Question 4. *How to deal with stochastic control problems driven by Hawkes processes?*

To tackle this question we consider the problem of market making when the market order flow is driven by Hawkes processes. We formalize it as a stochastic control problem for which we prove existence of a solution. Then we propose a methodology to approach numerically the optimal control and implement it in practice. Our experiments suggest that taking into account the self exciting and long memory properties of market order flow in a trading strategy leads to great improvements of the risk management. Another interesting point is to understand how trading strategies scale with respect to their time horizon or their size. To address this issue we consider the following question:

Question 5. *Are stochastic control problems for point processes robust to scaling?*

We address this question in the specific framework of population modeling. More precisely we consider a sequence of models for the dynamics of a discrete population (population with values in $\mathbb{N}$) converging at the scaling limit towards a continuous model. To each of these models we associate a stochastic control problem modeling the problem faced by a resource manager controlling the population. This gives an optimal control for each of the model with discrete population. We show that this sequence converges towards the optimal control associated to the continuous model. To obtain this result, we prove that solution to a backward stochastic differential equation (BSDE for short) is continuous with respect to the BSDE's diving martingale and terminal value.

BSDEs are also one the key ingredients to deal with finite horizon principal agent problems. Such approaches have been recently used, in the purpose of improving markets design, to build incentives policy for market participants. More precisely using a principal-agent model one can derive an optimal fees policy to incentivize market makers to quote attractive spreads.

However results have only been obtained for stocks and options have not been investigated. So,

Question 6. *How to use principal-agent approaches to help design an options market?*

To deal with this question, we take the point of view of an exchange willing to design its options market in a way that attracts liquidity, in order to collect transaction fees. We proceed into two steps. First we select the options that are going to be listed on the platform. For this we show that selecting the set of most suitable options is equivalent to solving a quantization problem. In a second step we design a make-take fees policy to incentivize market makers, that have to deal with all the listed options simultaneously, to provide competitive liquidity. To do this we view the exchange problem as a principal-agent problem. We fully solve this problem and illustrate numerically that our methodology enables us to significantly reduce the spreads of all the listed options.

Question 6 addresses a very specific part of the general topic of market design. Another interesting issue, rarely investigated, is the trading organization of exchanges. On most of them trading is organized as a continuous double auction system. We refer to this type of organization as continuous limit order book (CLOB for short) markets . Traders can send and cancel limit orders and fill the limit order book (LOB for short) or send market orders. However on some venues trading is organized in sequential auctions. This means that during a given period traders can send or cancel buy or sell limit and market orders, which are matched at the end of the auction period. Auction markets represent an increasing part of the trading activity and even in CLOB markets, a large fraction of the traded volume is actually made during the opening and closing auctions. Therefore it is important to consider the following question:

Question 7. *Is CLOB optimal compared to sequential auctions and what is the optimal auction duration?*

To provide an answer to this question we propose a model for a sequential auctions market with constant auction duration, such that the zero seconds duration case corresponds to a CLOB market. We consider two types of agents: market makers that are non strategic and provide liquidity in the LOB based on their knowledge of the efficient price and market takers that are strategic and seek for the lowest possible transaction costs. This generates a competition between buyers and sellers market takers for which we prove existence of a Nash equilibrium. Assuming that we are in the Nash situation we consider that the optimal duration is the one maximizing the quality of price formation process. Finally we calibrate our model on intra day market data for French stocks traded on the Euronext platform. For most of the assets we obtain that the optimal duration lies between 2 and 10 minutes and that CLOB are always not so far from being optimal.

Outline

This thesis is split into three parts. In the first part we deal with the connections between market microstructure and volatility: Questions 1, 2 and 3. Then in the second part we tackle Questions 4 and 5 on stochastic control problems for point processes. Finally in the last part we address market design issues: Questions 6 and 7.

In Chapter I we answer Question 1 by drawing a link between the no-arbitrage principle and the roughness of volatility. For this we use the statistical version of the no-arbitrage principle operating at the microstructural level: the absence of round trip strategies with positive expected profit. It has been shown that this property implies that the permanent market impact of metaorders is linear. This allows us to write the dynamics of the price under a very general form, valid for any order flow model. Then to reproduce the long memory and self exciting property of the order flow we consider that market order flow is driven by a Hawkes process without any assumption on its parameters. We finally investigate the scaling limit of the model under the constraint that transient market impact is not trivial. We show that asymptotically the only possibility is that market impact is a power-law function and that volatility follows a rough Heston model. Moreover we get a relationship between the roughness of the volatility and the shape of the market impact. If the volatility is rough with Hurst index equal to H, then the market impact must be a power-law function with exponent $H+1/2$.

We turn to Question 2 in Chapter II where we look for an extension of the rough Heston model reproducing the Zumbach effect. We recall that Zumbach effect refers to the feedback of price trends on volatility and more precisely to the fact that volatility depends on the past behavior of the price. In order to address this question we study the possible scaling limits of a tick-by-tick model describing the arrival times of buy and sell market orders. The model is a natural extension of the one considered in Chapter I since we consider that market order flow is now driven by quadratic Hawkes processes. We use those processes because they naturally reproduce the Zumbach effect at the microscopic scale since the intensity of event arrival is partially driven by a squared moving average of past price returns. So if price is trending, the intensity of market order arrival increases. We find three different possible scaling limits for the model, depending on the way we rescale its parameters. All these models are rough volatility models reproducing the Zumbach effect and can be considered as extensions of the rough Heston model.

In Chapter III we tackle Question 3. Following an intuition that came out from a discussion with Julien Guyon, we define a new rough volatility model in the spirit of the ones considered in Chapter II: the quadratic rough Heston model. We then illustrate that this model manages to fit simultaneously SPX and VIX options smiles remarkably well.

We answer to Question 4 in Chapter IV. We consider a general order flow model driven by two Hawkes processes, one representing the buy order flow and the other the sell order flow. Then we focus on solving the problem of a market maker controlling the bid and ask

spreads, which affect the market order intensities. His goal is to maximize its profit while managing its inventory risk. We start by building a Markovian representation of the model in infinite dimension. We then prove existence and uniqueness of a viscosity solution to the Hamilton-Jacobi-Bellman (HJB for short) equation associated to the control problem and the Markovian representation. This characterizes an optimal control solving the problem. The drawback of this approach is that it does not allow the development of classical numerical methods since we use an infinite dimensional representation. To circumvent this issue we first show that we can find a sequence of approximating controls converging towards the optimal control. The approximating controls are actually solution to the same control problem when we consider specific kernels for the Hawkes processes, allowing for finite dimensional Markovian representations. Hence we can approach numerically the approximating controls and therefore the optimal control.

In Chapter V we deal with Question 5. We consider a sequence of models for the dynamics of a discrete population that converges in law towards a continuous model. By construction, a converging sequence of martingales is associated to the converging sequence of discrete models. We first prove a convergence result for martingale representation relatively to this sequence. We extend this property by considering a sequence of BSDEs driven by those martingales. We show that, under some natural assumptions on the behavior of the sequences of generators and terminal values, the solutions to those BSDEs converge towards the solution of a BSDE driven by the limiting martingale. Based on this property we then associate a non Markovian control problem to each of the models. We finally prove that the sequence of controls solving the discrete control problems converges towards the control solving the continuous control problem.

We address Question 6 in Chapter VI where we propose a tractable methodology to help exchanges design a derivatives market. Our goal is to design the market in order to satisfy investors' demand and provide high quality liquidity. We first focus on selecting the options that are going to be listed on our market. To do this we use a quantization algorithm on historical market data. This allows us to select a finite number of options that concentrate most of client's demand. We then work on improving liquidity. The interest of market takers for a product is naturally decreasing with respect to the quoted spread. Therefore in order to be competitive a platform must offer small spreads. However the exchange cannot directly decide the spreads that are set by market makers. It can only try to influence a market maker, what we formalize as a principal-agent problem. More precisely we consider a market with a single market maker dealing with all the listed options simultaneously. In order to incentivize the market maker to quote small spreads, the exchange offers him a contract whose payoff depends on the whole market order flow. Consequently the market maker adapts its behavior to the contract. The exchange problem then boils down to find the contract maximizing a certain measure of the liquidity on the platform. We are able to exhibit an optimal contract for the exchange problem. Thanks to it, one gets significantly lower spreads on any of the listed options.

Finally we study Question 7 in Chapter VII. In order to compare sequential auctions markets

with CLOB markets, we build a model for sequential auctions market such that when the auction duration is zero the market is equivalent to a CLOB market. We consider that during and in between auctions, market makers send limit orders in the LOB according to a linear supply function centered on their view on the efficient price at the next clearing time. We assume that market makers do not optimize their behavior and that their arrivals are given by a Poisson process. On the aggressive side of the market, we consider two agents, a buyer and a seller that can only send market orders. The first market order sent after the closing of an auction triggers a new auction. During auctiond, the buyer and seller adapt their trading intensities in order to minimize their transaction costs at the clearing time. We show that there exists a Nash equilibrium for the competition occurring between the buyer and the seller. Finally we consider that the optimal auction duration is the one maximizing the quality of the price formation process. Formally it corresponds to the duration minimizing the expected squared difference between the clearing price and the efficient price at the clearing time. We calibrate the parameters of our model on intraday market data provided by Euronext. Assuming that market takers are in the Nash equilibrium, we then compute the optimal auction duration for each asset in our database. For most of them we find that the optimal auction duration lies between 2 and 10 minutes. For some assets the CLOB system is optimal. We also observe numerically that CLOB are always not far from being optimal.

Let us now rapidly review the main results of this thesis.

1 Part I: Microstructural foundations of volatility properties

In this first part we study microstructural foundations of some essential properties of the volatility. We start with the rough behavior of volatility that we prove to be a consequence of the no-arbitrage principle. Then we deal with the Zumbach effect. Finally based on these properties, we build a model fitting simultaneously SPX and VIX options smiles.

1.1 Chapter I: No-arbitrage implies power-law market impact and rough volatility

Rough volatility is a universal phenomenon in finance. Therefore one gets the intuition that it must related to the most fundamental property in quantitative finance: the no-arbitrage principle. We start this thesis by drawing the link between these two concepts.

We first study the consequences of the no-statistical arbitrage principle on the price dynamics. Based on previous works we know that this principle implies that long term market impact of metaorders is linear. This property allows us to fully specify the dynamics of the price from market order flow, independently of the model used.

We then consider a model for the order flow based on Hawkes processes. Using our relationship between price and order flow, we obtain closed form formulas for the price dynamics and the market impact. From several empirical evidences, we know that market impact is non-trivial at

the macroscopic level. By non-trivial we mean that market impact has a transient part. Thus we investigate the possible scaling limits of the model under the constraint that market impact is non-trivial at the limit. We find that necessarily, at the scaling limit, the transient part of the market impact is an increasing concave power-law function and that volatility follows a rough Heston model.

1.1.1 No-statistical arbitrage and permanent market impact

To understand the consequences of the no-statistical arbitrage principle at the microstructural level, we need to introduce two notions: round trip strategy and permanent market impact of a metaorder. A round trip strategy is a trading strategy starting and finishing with null inventory. The no-statistical arbitrage principle is the absence of round trip strategies with positive average profit and loss (P&L for short). A metaorder is a large order which is not executed at once, but split by a broker over a time horizon that can go from a few minutes to several days. The permanent market impact of a metaorder with volume V, denoted by $PMI(V)$, is defined as the long time average difference between the price before and after the execution of the metaorder (independently of the trading strategy),

$$PMI(V) = \lim_{t\to+\infty} \mathbb{E}[P_t - P_0|V].$$

One can show using various approaches that the absence of round trip strategies with positive P&L implies that permanent market impact is a linear function of the volume, or equivalently $PMI(V) = \kappa V$, for some positive constant κ, that we take equal to one without loss of generality. We furthermore assume that the price is a martingale, then from the linearity of permanent market impact we get

$$P_t = \lim_{S\to+\infty} \mathbb{E}[V_S^a - V_S^b|\mathscr{F}_t], \tag{1}$$

where V_t^a (resp. V_t^b) is the total number of buy (resp. sell) market orders sent between time 0 and time t and $(\mathscr{F})_{t\geq 0}$ the natural filtration of the process $(V_t^a, V_t^b)_{t\geq 0}$. According to Equation (1), when an order is executed, price moves because market participants reconsider their expectation about the long term accumulated volume imbalance. Note that the relation between price and order flow given in Equation (1) is always valid under no-statistical arbitrage and does not depend on the model for the order flow. We use this dynamic for the price in the rest of this chapter.

1.1.2 A Hawkes based model for market order flow

We now specify a model for the order flow. We respectively denote by N^a and N^b the total number of buy and sell market orders (assumed to be of unit size) and consider that they are independent. More precisely for $i \in \{a, b\}$ the intensity of the process N^i is given by

$$\lambda_t^i = \mu + \int_0^t \phi(t-s)\mathrm{d}N_s^i,$$

where μ is a positive constant and ϕ a positive function in $L^1(\mathbb{R}_+)$ such that $\|\phi\|_1$ is smaller than one. From Equation (1), we can show that

$$P_t = \int_0^t \xi(t-s)(\mathrm{d}N_s^a - \mathrm{d}N_s^b), \tag{2}$$

where

$$\xi(t) = 1 + \left(1 - \|\phi\|_1\right)^{-1} \int_t^{+\infty} \phi(u)\mathrm{d}u.$$

The above kernel ξ quantifies how market digests the order flow. A single market order has an instantaneous impact $\xi(0)$. Then its impact decays towards a permanent impact, corresponding to the limit of ξ at infinity.

We write T for the horizon of our model, that corresponds to the duration of a metaorder. The time-length of metaorders is typically large compared to the inter-arrivals of individual market orders. Hence it is natural to investigate the possible scaling limits of Model (2) when T goes to infinity. To do this, we rescale the different processes involved in the model from $[0, T]$ onto $[0, 1]$ using a proper scaling factor and let T goes to infinity. In order to be very general, we even allow the parameters of the model to depend on T. Hence we use a superscript T to stress the dependence of the parameters with T. We make no assumptions on the parameters, except for the kernel of the Hawkes processes that we assume to be of the form $\phi^T = a^T\phi$ where ϕ is a positive function in $L^1(\mathbb{R}_+)$, with norm equal to one and $(a^T)_{T\geq 0}$ a sequence with values in $[0, 1)$.

From Equation (1), we see that the price is homogeneous to the volume. Hence the natural scaling factor for the price process is the average number of market orders sent between time 0 and time T. In the stationary version of the model, it is equal to $T\beta^T$ where $\beta^T = \frac{\mu^T}{1-a^T}$. The rescaled price process is therefore given by

$$\overline{P}_t^T = P_{tT}^T \frac{1}{T\beta^T}.$$

In the next step, we take the point of view of an agent executing a metaorder and focus on its market impact.

1.1.3 Market impact and its scaling limit in the model

We consider a market participant executing a buy metaorder. Its trading flow is added to the global market order flow given by the processes N^a and N^b. We assume that the metaorder is split through unitary market orders sent according to a non-homogenous Poisson process with intensity

$$\nu^T(t) = \gamma\beta^T f(t/T),$$

where $\gamma \in (0, 1)$ and f is a positive function in $L^1(\mathbb{R}_+)$ with norm equal to 1 and supported on $[0, 1]$. This specification of the intensity corresponds to a metaorder executed on the interval

$[0,T]$ and whose size is non-negligible compared to the global order flow. Using Equation (2), we can write the market impact function of this metaorder as

$$MI_t^T = \mathbb{E}[P_t - P_0] = \int_0^t \xi^T(t-s)\nu^T(s)\mathrm{d}s.$$

From the definition of ξ^T, we get that market impact can be decomposed as $MI^T = TMI^T + PMI^T$ with

$$PMI_t^T = \int_0^t \nu^T(s)\mathrm{d}s \text{ and } TMI_t^T = \int_0^t \Gamma^T(t-s)\nu^T(s)\mathrm{d}s,$$

where

$$\Gamma^T(t) = (1-a_T)^{-1}\int_t^{+\infty}\phi^T(u)\mathrm{d}u.$$

Hence market impact is the sum of a vanishing term TMI^T: the transient market impact, and a non-vanishing term, PMI^T, that is linear in the metaorder volume: the permanent market impact. The rescaled transient and permanent market impact functions are therefore

$$\overline{PMI}_t^T = PMI_{tT}^T/T\beta^T = \gamma\int_0^t f(u)\mathrm{d}u,$$
$$\overline{TMI}_t^T = TMI_{tT}^T/T\beta^T = \gamma\frac{a^T(1-a^T)^{-1}}{T}\int_0^{tT} f(t-u/T)\int_u^{+\infty}\phi(v)\mathrm{d}v\mathrm{d}u.$$

Our goal is to derive the possible scaling limits of the price process under the constraint that market impact is non-trivial at the limit. We recall that by trivial market impact we mean a market impact with no transient part. So our constraint is that $(\overline{TMI}^T)_{T\geq 0}$ is non-vanishing at the limit. We can actually characterize the limit of $(\overline{TMI}^T)_{T\geq 0}$ under the assumption that this limit exists and is non-trivial for any homogeneous metaorder. Formally we consider the following assumption.

Assumption 1. *For constant execution rate, that is $f = \mathbf{1}_{[0,s]}$ for some $s \in (0,1]$, the scaling limit of the market impact function exists pointwise and is non-increasing after time s. Furthermore, there exists $t > s$ such that the value of this limiting function at time t is strictly smaller than that at time s.*

This assumption is very natural since metaorders do generate transient market impact at the macroscopic scale in practice. Under this assumption, we obtain the following result.

Result 1. *For any measurable non-negative function f defined on $\mathbb{R}_+$, continuous on $[0,1]$ and supported on $[0,1]$, the corresponding macroscopic market impact function and its transient part exist. More precisely, there exist parameters $\alpha \in (0,1]$ and $K > 0$ such that for any f and $t > 0$, when $\alpha < 1$,*

$$\lim_{T\to+\infty}\overline{TMI}^T(f,t) = \gamma K(1-\alpha)\int_0^t f(t-u)u^{-\alpha}\mathrm{d}u,$$

and when $\alpha = 1$

$$\lim_{T\to+\infty}\overline{TMI}^T(f,t) = \gamma K f(t).$$

Furthermore, the Hawkes kernel ϕ has to satisfy

$$\int_0^t \int_s^{+\infty} \phi(u)\mathrm{d}u\mathrm{d}s = t^{1-\alpha} L(t),$$

where L is a slowly varying function. Finally we necessarily have

$$(1-a^T)^{-1} T^{-\alpha} L(T) \to K,$$

so necessarily $a^T \to 1$.

Therefore, at the scaling limit the transient market impact of an homogeneous metaorder is a power-law function, see Figure .1 for an illustration. Remark that we obtain that a necessary condition to get a non-trivial market impact at the limit is that $(a^T)_{T\geq 0}$ converges to 1. Since a_T measures the degree of endogeneity of the market order flow, this means that a non-trivial market impact can only exist in a highly endogenous market.

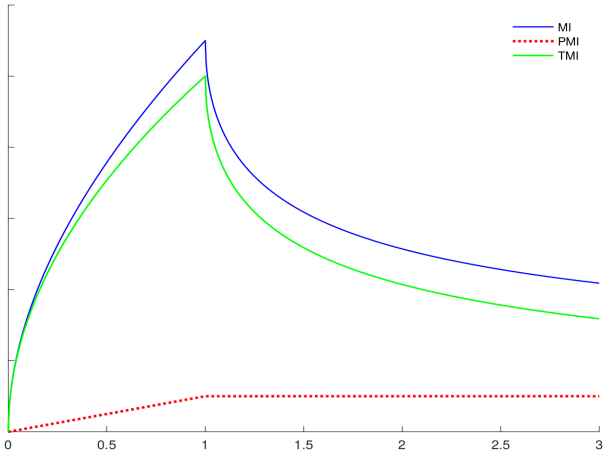

Figure .1 – Illustration of the decomposition of the macroscopic market impact function for a metaorder executed uniformly over $[0,1]$, with $\alpha = 0.5$. Time is on the $x-$axis.

1.1.4 Scaling limit of the price

We finally study the consequences of Assumption 1 on the scaling limit of the price. We need an additional technical assumption.

Assumption 2. *For some $\delta > 0$, we have*

$$(1-a^T)\mu^T T \underset{T\to+\infty}{\to} \delta.$$

Then under Assumptions 1 and 2 we obtain the convergence of the sequence $(\overline{P}^T)_{T\geq 0}$.

Result 2. *The sequence of rescaled price processes $(\overline{P}^T)_{T\geq 0}$ converges in law for the Skorokhod topology towards a martingale $\widehat{P}$ with bracket X solution of following stochastic Volterra equation*

$$X_t = \frac{2}{\delta}\int_0^t F^{\alpha,\lambda}(s)\mathrm{d}s + \frac{1}{\delta\sqrt{\lambda}}\int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}M_s,$$

where M is a continuous martingale with bracket equal to X, α is given in Result 1, λ is a positive constant and $F^{\alpha,\lambda}$ is the Mittag-Leffler cumulative distribution function, see [HMS11] for more details.

We can say more on the process X, notably when $\alpha > 1/2$.

Result 3.

(i) For any $\varepsilon > 0$, the process X has Hölder regularity $1 \wedge (2\alpha - \varepsilon)$. It is not continuously differentiable when $\alpha \leq 1/2$ and it is when $\alpha > 1/2$. In this case its derivative Y is solution of the stochastic Volterra equation

$$Y_t = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\lambda(\frac{2}{\delta} - Ys) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\frac{\sqrt{\lambda}}{\delta}\sqrt{Y_s}\mathrm{d}W_s,$$

where W is a Brownian motion. The process Y has Hölder regularity $\alpha - 1/2 - \varepsilon$ for any positive ε.

(ii) For any continuously differentiable function h from $\mathbb{R}_+$ into $\mathbb{R}$ such that $h(0) = 0$, we have

$$K(h,t) = \mathbb{E}\big[\exp\big(\int_0^t ih(t-s)\mathrm{d}X_s\big)\big] = \exp\big(\int_0^t g(s)\mathrm{d}s\big),$$

with g the unique continuous solution to the Volterra Riccati equation

$$g(t) = \int_0^t f^{\alpha,\lambda}(t-s)\big(\delta^{-1}\frac{1}{4}g(s)^2 + \delta^{-1}2ih(s)\big)\mathrm{d}s,$$

where $f^{\alpha,\lambda}$ is the Mittag-Leffler density function, see [HMS11] for more details.

Thus assuming only no-statistical arbitrage and non-trivial market impact, we manage to prove that necessarily market impact is a power-law and volatility is rough. Moreover we have obtained a relationship between the Hurst index of volatility, $\alpha - 1/2$, the market order flow memory and the shape of market impact: power-law with exponent $1 - \alpha$. This is consistent with empirical studies that suggest that in practice the parameter α is close to 0.5. In the case where $\alpha > 1/2$ the volatility follows a rough Heston model. We have also identified a class of hyper-rough volatility models, corresponding to $\alpha \leq 1/2$. We have shown that for those models instantaneous variance does not exist and only cumulative variance, here X, makes sense.

The results of Chapter I are yet other elements demonstrating the fundamental aspect of rough volatility models in finance. However the rough Heston model we obtained in the limit in Chapter I may have limitations in practice. Hence in the next chapter, we look for an extension of this model that allows us to reproduce even more stylized facts of volatility.

1.2 Chapter II: From quadratic Hawkes processes to super-Heston rough volatility models with Zumbach effect

In this chapter we want to improve the rough Heston model in two directions. First we focus on the tail of volatility, which is known to be quite heavy. So, even though the rough Heston model manages to almost perfectly fit SPX options smile and does produce heavy volatility tails, we may wonder how to generate even fatter distribution queues. The other aspect, brought to our attention by Jean-Philippe Bouchaud, is the existence of so-called Zumbach effect. Essentially Zumbach effect is the fact that volatility is correlated to price trends. More precisely if price is trending down then volatility is going to be higher than in standard situations. This is also referred to as the feedback of price on volatility. It is a somehow vague notion which is hard to define and measure. To quantify this effect, econophysicists usually consider the difference between the correlation of past squared price returns with future volatility and the correlation of past volatility with future squared price returns. This quantity is typically positive on market data. The rough Heston model does enjoy this property when $H < 1/2$. Nevertheless this is not because of a causal link between price paths and volatility, as described in the original idea by Zumbach. It is only due to leverage effect together with the fact that $H < 1/2$. Hence we call this property of correlations weak Zumbach effect and introduce the strong Zumbach effect. It is defined as the fact that conditional law of future volatility is not fully determined by past volatility but by past prices and past volatility together. The rough Heston model does not reproduce the strong Zumbach effect because of its Markovian structure, see [EER18].

Our goal in this chapter is therefore to propose an extension of the rough Heston model that reproduces the Zumbach effect. We also wish that the instantaneous variance of volatility is equal to the classical square root term of Heston like models multiplied by a non-trivial process, in order to enhance volatility tails. Any model satisfying the latter property will be called a super-Heston model.

We proceed with the same methodology as that used in Chapter I. We first consider a sequence of models indexed by T. Each model is then rescaled from $[0, T]$ onto $[0, 1]$. Inspired by [BDB17], we model the market order flow using quadratic Hawkes processes which naturally generate Zumbach effect at the microscopic scale. We then investigate the possible limits of the sequence. We distinguish two types of limit depending on whether or not we are in a highly endogenous regime as in Chapter I. Each of those limits corresponds to a super-Heston rough volatility model and reproduces the Zumbach effect.

1.2.1 Quadratic Hawkes based model for the order flow

We start by defining our model. For a given T we model the arrival of market orders moving the price by the event times of a counting process N^T. Each market order is either a buy or a sell market order of unit volume with probability 1/2. The price P^T is then the difference between the total number of buy orders, $N^{T,a}$ and the total number of sell orders, $N^{T,b}$. Finally

we assume that the intensity of N^T is that of a quadratic Hawkes process and given by

$$\lambda_t^T = \mu_T + H_t^T + |Z_t^T|^2, \text{ with } H_t^T = \int_0^t \phi_T(t-s)\mathrm{d}N_s^T \text{ and } Z_t^T = \int_0^t k_T(t-s)\mathrm{d}P_s^T, \tag{3}$$

where $(\mu_T)_{T\geq 0}$ is a positive sequence, $(\phi_T)_{T\geq 0}$ a sequence of positive $L^1(\mathbb{R}_+)$ functions and $(k_T)_{T\geq 0}$ a sequence of positive $L^2(\mathbb{R}_+)$ functions such that $a_T = \|\phi_T\| + \|k_T\|_2^2 < 1$ for any T. This is a stability condition playing the same role as $\|\phi_T\| < 1$ in the linear case.

Compared with the model of Chapter I, market order intensity is driven by two components. A linear self-exciting term H^T, which corresponds to the intensity of a linear Hawkes process and a quadratic term, $|Z^T|^2$. Notice that Z^T is a moving average of the price returns. Therefore when the past price is trending, Z^T is large in absolute value and therefore λ^T is large, which will eventually lead to high volatility. Hence Model (3) basically generates Zumbach effect.

To be as general as possible in the scaling procedure, we use a generic sequence of scaling factors $(\omega_T)_{T\geq 0}$ for the intensity. The rescaled intensity is therefore given by $\lambda_t^{*T} = \omega_T \lambda_{tT}^T$ and the rescaled price process by $P_t^{*T} = \sqrt{\omega_T/T} P_{tT}^T$.

1.2.2 The stable case

In this section we assume that for any T, $\|\phi_T\|_1 = \beta$ and $\|k_T\|_2^2 = \gamma$ where β and γ are positive constants such that $\beta + \gamma < 1$. In this case, the magnitude of average market order intensity does not depend on T and we can take $\omega_T = 1$. So the rescaled intensity is

$$\lambda_t^{*T} = \mu_T + H_t^{*T} + |Z_t^{*T}|^2,$$

with

$$H_t^{*T} = \int_0^t T\phi_T\big(T(t-s)\big)\frac{\mathrm{d}N_{sT}^T}{T} \text{ and } Z_t^{*T} = \int_0^t \sqrt{T}k_T\big(T(t-s)\big)\mathrm{d}P_s^{*T}.$$

Note that up to a martingale term, we have $\mathrm{d}N_{tT}^T/T = \lambda_t^{*T}\mathrm{d}t$ and $\mathrm{d}\langle P^{*T}\rangle_t = \lambda_t^{*T}\mathrm{d}t$. Since the order of magnitude of the intensity essentially does not depend on T, the sequences $(N_{\cdot T}^T/T)_{T\geq 0}$ and $(P^{*T})_{T\geq 0}$ can be considered tight. Hence to ensure that $(Z^{*T})_{T\geq 0}$ is tight it is natural to assume that the sequence of integrands converges. Thus we consider the following specification:

$$\mu_T = \mu,\ \phi_T(t) = \frac{\beta}{T}\phi(t/T) \text{ and } k_T(t) = \sqrt{\frac{\gamma}{T}}k(t/T),$$

for $\mu \geq 0$, ϕ a positive $L^1(\mathbb{R}_+)$ function with norm equal to 1 and k a positive $L^2(\mathbb{R}_+)$ function with norm equal to 1. We formalize here this assumption together with some technical ones.

Assumption 3.
i) The sequence of kernels $(\phi_T, k_T)_{T\geq 0}$ is defined by

$$\phi_T = \frac{\beta}{T}\phi(\frac{\cdot}{T}) \text{ and } k_T = \sqrt{\frac{\gamma}{T}}k(\frac{\cdot}{T})$$

with $0 < \gamma + \beta < 1$ *and* ϕ *and* k *two non-negative measurable functions such that* $\|\phi\|_1 = \|k\|_2 = 1$. *Furthermore we take* $\mu_T = \mu > 0$.

ii) The function k *belongs to* $L^{2+\varepsilon}$ *for some* $\varepsilon > 0$ *and for any* $0 \le t < t' \le 1$,

$$\int_0^t |k(t'-s) - k(t-s)|^2 \mathrm{d}s < C|t'-t|^r,$$

for some $r > 0$ *and* $C > 0$ *and*

$$\frac{1}{\eta}\int_0^1 |k(t)|^2 t^{-2\eta}\mathrm{d}t + \int_0^1\int_0^1 \frac{|k(t)-k(s)|^2}{|t-s|^{1+2\eta}}\mathrm{d}s\mathrm{d}t < +\infty,$$

for some $\eta \in (0,1)$.

Under this assumption we prove the following result.

Result 4. *The process* $(P^{*T})_{T\ge 0}$ *is C-tight for the Skorokhod topology on* $[0,1]$ *and for any limit point* P *there exists a Brownian motion* B *such that*

$$P_t = \int_0^t \sqrt{V_s}\mathrm{d}B_s,$$

with V *solution on* $[0,1]$ *of*

$$V_t = \mu + \beta H_t + \gamma Z_t^2, \tag{4}$$

where

$$H_t = \int_0^t \phi(t-s)V_s\mathrm{d}s \text{ and } Z_t = \int_0^t k(t-s)\sqrt{V_s}\mathrm{d}B_s.$$

If k *is equal to the Mittag Leffler density function* $f^{H+1/2,\lambda}$ *for* $H \in (0,1/2)$ *and* $\lambda > 0$, *the process* V *is almost surely* $H-\varepsilon$ *Hölder for any* $\varepsilon > 0$.

This result extends the one in [BDB17] where the authors focused on exponential kernels for ϕ and k. We then check that the model defined by Equation (4) is a super-Heston model reproducing the strong Zumbach effect.

If k is continuously differentiable and continuous in 0, up to finite variation term we have

$$\mathrm{d}V_t = 2k(0)Z_t\sqrt{V_s}\mathrm{d}B_s.$$

Hence Model (4) is a super-Heston volatility model. To investigate the Zumbach effect we consider that $\phi(t) = \kappa e^{-\kappa t}$ and $k(t) = \sqrt{2\nu}e^{-\nu t}$. With this setting, we can write

$$V_{t_0+h} = \mu + \tilde{H}_h + \tilde{Z}_h^2 + Z_{t_0}e^{-2\nu h} + 2Z_{t_0}e^{-\nu h}\tilde{Z}_h + H_{t_0}e^{-\kappa h},$$

where

$$\tilde{H}_t = \int_0^t \phi(t-s)V_{t_0+s}\mathrm{d}s \text{ and } \tilde{Z}_t = \int_0^t k(t-s)\mathrm{d}P_{t_0+s}.$$

The process Z cannot be deduced from observation of V because of the square, so Z_{t_0} is not $\sigma\big((V_s)_{s\leq t_0}\big)$ measurable. So the law of the future volatility depends both on the past of volatility and price. This shows that the process Z generates Zumbach effect in Model (4).

It is important to remark that here there is no need to consider nearly unstable models ($a_T \to 1$) to obtain stochastic volatility at the limit. This is in contrast with the situation described in Chapter I. However in Model (4), the Brownian motion driving the volatility is the same as that driving the price. We will see in the next section that when the model is nearly unstable at the limit, we get an extra source of randomness in the volatility. We also remark that in Model (4) roughness of volatility is a consequence of the behavior in 0 of the function k whereas in Chapter I it is generated by the tail of the Hawkes kernel.

1.2.3 Nearly unstable case

We now investigate the situation where the model is nearly unstable at the scaling limit. More precisely we assume that a_T converges to 1 as T goes to infinity. So we naturally take $\omega_T = (1-a_T)/\mu_T$. This is very similar to the case of nearly unstable linear Hawkes processes studied in Chapter I. Here also we take $\phi_T = \beta_T \phi$ with $(\beta_T)_{T\geq 0}$ a positive sequence and ϕ a $L^1(\mathbb{R}_+)$ positive function with norm equal to 1 satisfying $\phi(x) \sim x^{-(1+\alpha)}$ for some $\alpha \in (0,1)$. Such shape for the tail will generate rough volatility at the scaling limit, as for purely linear Hawkes. The main difference with the stable case is that we have to use the resolvent of the kernel ϕ_T to derive the limiting model. More precisely we write

$$\lambda_t^{*T} = (1-a_T) + \omega_T |Z_{tT}^T|^2 + \int_0^t T\psi_T\big(T(t-s)\big)\omega_T\big(\mu_T + |Z_{sT}^T|^2\big)\mathrm{d}s + \int_0^t \psi_T\big(T(t-s)\big)\omega_T \mathrm{d}M_{sT}^T$$

where $M_t^T = N_t^T - \int_0^t \lambda_s^T \mathrm{d}s$ and $\psi_T = \sum_{i=1}^{+\infty} \phi_T^{*i}$. The following assumptions are key to ensure the tightness of $(P^{*T})_{T\geq 0}$.

Assumption 4.

i) The sequence $(\beta_T)_{T\geq 0}$ satisfies $\beta_T = (2a_T - 1)$ with $(a_T)_{T\geq 0}$ a sequence in $(0,1)$ and ϕ a non-negative measurable function such that $\|\phi\|_1 = 1$. Furthermore for some $K > 0$ and $\alpha \in (0,1)$,

$$\lim_{x\to+\infty} \alpha x^\alpha \int_x^{+\infty} \phi(s)\mathrm{d}s = K.$$

ii) The sequence of kernels $(k_T)_{T\geq 0}$ satisfies $k_T = k(\cdot/T)\sqrt{\frac{1-a_T}{T}}$ with k a non-negative continuously differentiable function (in particular $k(0) < +\infty$) such that $\|k\|_2 = 1$.

iii) Set $\delta = K\Gamma(1-\alpha)/\alpha$, there are two positive constants λ and μ^* such that

$$\lim_{T\to+\infty} (1-a_T)T^\alpha = \lambda\delta \text{ and } \lim_{T\to+\infty} T^{1-\alpha}\mu_T = \mu^*\delta^{-1}.$$

Under Assumption 4 we have the following result.

Result 5. *The sequence $(P^{*T})_{T\geq 0}$ is C-tight for the Skorohod topology on $[0,1]$. Any of its limit point P is a continuous martingale with brackets X satisfying*

$$X_t = \int_0^t \frac{1}{2} F^{\alpha,\lambda}(t-s)\big(1+Z_s^2\big)\mathrm{d}s + \int_0^t \frac{1}{2} f^{\alpha,\lambda}(t-s)\frac{1}{\sqrt{\lambda\mu^*}} M_s \mathrm{d}s, \tag{5}$$

where M is a continuous martingale with bracket equal to X such that $\langle P, M\rangle = 0$ and

$$Z_t = \int_0^t k(t-s)\mathrm{d}P_s.$$

If $\alpha \in (1/2, 1)$, the process X is almost surely continuously differentiable with derivative V solution of

$$V_t = \int_0^t \frac{1}{2} f^{\alpha,\lambda}(t-s)\big(1+Z_s^2\big)\mathrm{d}s + \int_0^t \frac{1}{2} f^{\alpha,\lambda}(t-s)\frac{1}{\sqrt{\lambda\mu^*}} \sqrt{V_s}\mathrm{d}B_s,$$

where B is a Brownian motion. For any $\varepsilon > 0$, V has almost surely $\alpha - \frac{1}{2} - \varepsilon$ Hölder regularity for any $\varepsilon > 0$.

From the last point, we get that the limiting model (5) is a rough volatility model. Here roughness is generated by the tail of the kernel ϕ as in Chapter I. Moreover up to a finite variation term we have

$$V_t = \int_0^t f^{\alpha,\lambda}(t-s)\frac{1}{\sqrt{\lambda\mu^*}} \sqrt{V_s}\mathrm{d}B_s + \int_0^t F^{\alpha,\lambda}(t-s)k(0)Z_s\sqrt{V_s}\mathrm{d}W_s$$

where W is a Brownian motion independent of B. Hence Model (5) is a super-Heston rough volatility model. Finally similarly as in the stable case we can show that the process Z generates Zumbach effect in Model (5).

Since quadratic Hawkes processes generalize linear Hawkes processes, that lead to the rough Heston model, Models (4) and (5) can be seen as extensions of the rough Heston model. Hence we expect them to outperform the rough Heston model in fitting market data. In particular we know that the rough Heston model, even though fitting very well the SPX options smile, fails to reproduce the concave shape of VIX options smile. Hence in the next chapter we investigate the joint calibration of SPX and VIX options smiles using a model inspired from (4) and (5).

1.3 Chapter III: The quadratic rough Heston model: fitting simultaneously historical volatility together with SPX and VIX smiles with a continuous model

In this chapter we address a longstanding question brought to our knowledge by Julien Guyon, which he refers to as the *grail of volatility modeling* : can we find a continuous model that allows for simultaneous fitting of SPX and VIX options smiles? We focus here on a specific model, the quadratic rough Heston model, which is an extension of Model (4). We show that this parsimonious model can achieve an almost perfect fit of SPX and VIX options smiles at the same time.

1.3.1 The quadratic rough Heston model

We define the quadratic rough Heston model as the following joint dynamic for a stock price S and its volatility V:

$$\mathrm{d}S_t = S_t\sqrt{V_t}\mathrm{d}W_t, \quad V_t = a(Z_t - b)^2 + c, \tag{6}$$

with

$$Z_t = \int_0^t (t-s)^{\alpha-1}\frac{\lambda}{\Gamma(\alpha)}\big(\theta_0(s) - Z_s\big)\mathrm{d}s + \int_0^t (t-s)^{\alpha-1}\frac{\lambda}{\Gamma(\alpha)}\sqrt{V_s}\mathrm{d}W_s,$$

where $\lambda > 0$, $\alpha \in (1/2, 1)$ and θ_0 is a deterministic function. The existence of this model is ensured by an extension of Result 4. It is a path dependent volatility model. This means here that volatility is a deterministic functional of the price paths without any additional source of randomness. The variable Z is essentially a moving average of price log-returns since we can write

$$Z_t = \int_0^t f^{\alpha,\lambda}(t-s)\theta_0(s)\mathrm{d}s + \int_0^t f^{\alpha,\lambda}(t-s)\sqrt{V_s}\mathrm{d}W_s.$$

Thus in the quadratic rough Heston model, volatility is simply a polynomial of degree two of an appropriately chosen weighted moving average of log-returns. Note that due to the property of the kernel $f^{\alpha,\lambda}$ the process Z, and consequently V, is $\alpha - 1/2 - \varepsilon$ Hölder for any $\varepsilon > 0$. So volatility is indeed rough. This will allow the model to reproduce the short term asymptotic of at the money (ATM for short) skew.

Note that when $b = 0$, this model corresponds to Model (4) defined in Chapter II with $\beta = 0$. Thus we have just introduced an additional asymmetry in the feedback of price trends on volatility. Indeed since b is positive, for a same absolute value of Z, the volatility is higher when Z is negative. This simply reflects the fact that market reacts more to downward trends than to upward ones. This phenomenon is observable on historical data where we can see that the VIX index spikes when the SPX index tumbles down. This is also illustrated by the asymmetry of the SPX options smiles. Finally we highlight the fact that, as Model (4), the quadratic rough Heston model reproduces the Zumbach effect.

1.3.2 Numerical results

For calibration of the model, we have used market data from a randomly chosen day with typical shapes for the SPX and VIX options smiles. We have focused on short maturities, from 2 to 8 weeks, since we know from the literature that short maturities are the ones which are the hardest to fit. We restrict our model to the following form for Z:

$$Z_t = Z_0 - \int_0^t (t-s)^{\alpha-1}\frac{\lambda}{\Gamma(\alpha)}Z_s\mathrm{d}s + \int_0^t (t-s)^{\alpha-1}\frac{\lambda}{\Gamma(\alpha)}\eta\sqrt{V_s}\mathrm{d}W_s,$$

which is equivalent to take $\theta_0(t) = Z_0 t^{-\alpha}/\lambda\Gamma(1-\alpha)$. To calibrate the model we minimize the mean squared error between market implied volatilities and implied volatilities given by the quadratic rough Heston model. To price options we simply use Monte-Carlo simulations. The corresponding SPX and VIX options smiles are plotted in Figures .2 and .3 with the market smiles. The fits obtained are almost perfect for both SPX and VIX options. In particular we

manage to conciliate the ATM skew of SPX options with the ATM implied volatility of VIX options.

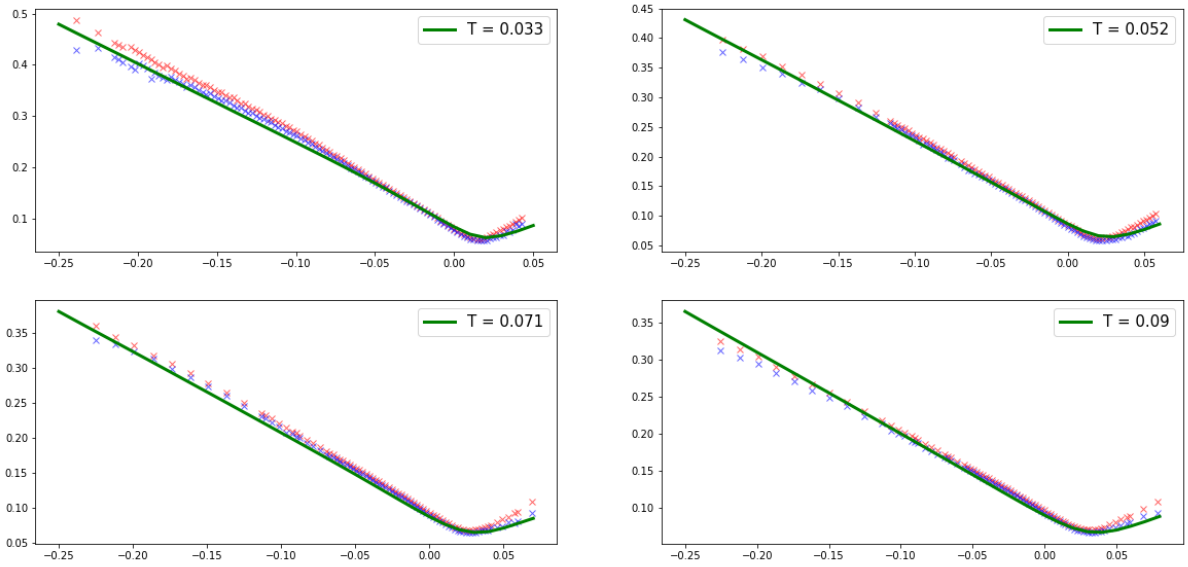

Figure .2 – Implied volatility of SPX options for the 19^{th} May 2017. The blue and red points are respectively the market bid and ask implied volatilities and in green is the implied volatility curve given by the quadratic rough Heston model. The strikes are in log-moneyness and T is the maturity in years.

In this first part we have extensively used point processes for financial modeling at the microscopic scale. Using such models allows for accurate representation of many stylized facts of market order flows. It is therefore natural to take into account these properties when optimizing trading strategies. This is why we aim at giving in the next part theoretical guarantees for stochastic control problems for point processes.

2 Part II: Optimal control for point processes

In the second part of this thesis, we first provide theoretical basis for problems related to market making when the order flow is driven by Hawkes processes. Then we investigate the scaling limits of solutions of control problems when the underlying models are point processes converging in law towards a continuous process.

2.1 Chapter IV: Optimal market making with persistent order flow

Hawkes processes are at the heart of Chapters I and II. Those processes have been introduced in finance because they reproduce remarkably well the self exciting property of order flows.

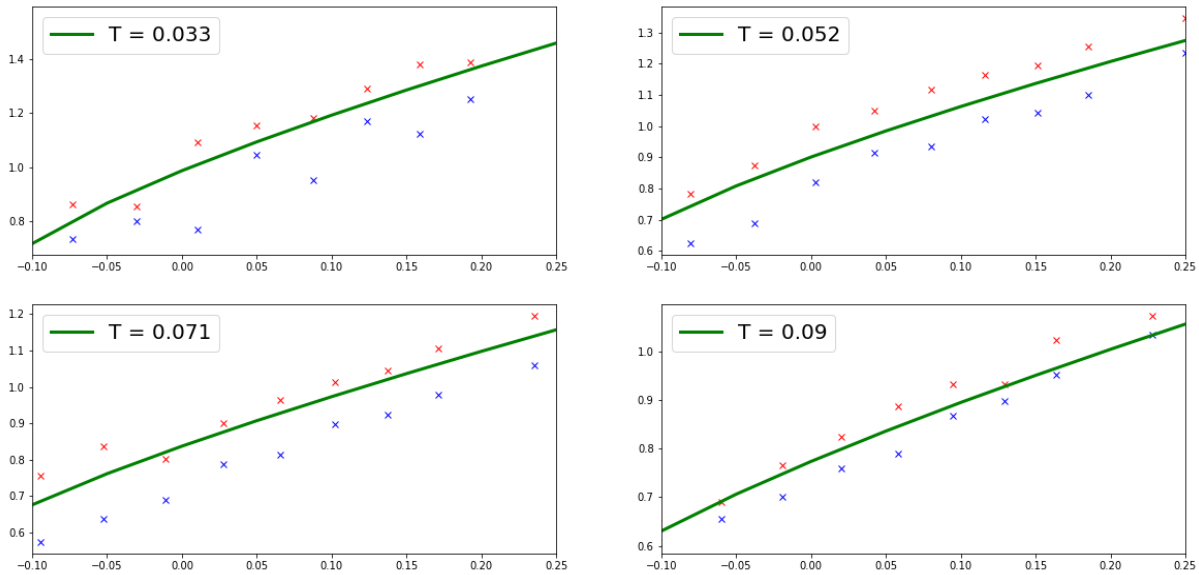

Figure .3 – Implied volatility of VIX options for the 19^{th} May 2017. The blue and red points are respectively the bid and ask implied volatilities and in green is the implied volatility curve given by the quadratic Heston model. The strikes are in log-moneyness and T is the maturity in years.

This refers to various empirical evidences showing that order flow dynamics strongly depends on their own past. One of these stylized facts is the long memory property of the sign of market order flow (+1 for a buy order −1 for a sell order). Formally it means that the autocorrelation function of market orders sign is slowly decreasing. It is crucial for high frequency market participants, for example market makers, to be able to adapt their strategies to such key property of the order flow. It is therefore natural to model the market making problem as a stochastic control problem driven by Hawkes processes. Hence in this chapter we take the point of view of a market maker and study a general class of stochastic control problems driven by Hawkes processes. Our goal is to derive an explicit optimal control. To do so, we study the HJB equation associated to the problem and characterize an optimal control by proving existence and uniqueness of a viscosity solution to the HJB equation. Finally we propose a tractable method to approach and implement in practice the optimal control.

2.1.1 The control problem

We assume that there is a single asset and a single market maker in the market. We consider N^a and N^b two independent counting processes that respectively denotes the total number of buy and sell market orders. The market maker's inventory is therefore $i = N^b - N^a$. The market maker controls the bid and ask spreads: δ^a and δ^b. The set of admissible controls is

therefore

$$\mathscr{A} = \{\delta = (\delta^a, \delta^b) \in \mathbb{R}_+^2, \text{ s.t. } \delta \text{ is predictable}\},$$

where predictability refers to the filtration generated by N^a, N^b and P, the fundamental price of the asset. When the market maker control is δ, the intensities of the processes N^a and N^b are respectively

$$\lambda_t^{a,\delta} = e^{-\frac{k}{\sigma}\delta_t^a}\Phi\Big(\int_0^t K(t-s)\mathrm{d}N_s^a\Big) \text{ and } \lambda_t^{b,\delta} = e^{-\frac{k}{\sigma}\delta_t^b}\Phi\Big(\int_0^t K(t-s)\mathrm{d}N_s^b\Big),$$

where K is a $L^1(\mathbb{R}_+)$ positive completely monotone function[1], Φ a continuous positive function and σ and k two positive constants. Hence market order intensities are decreasing with respect to the spreads and depend on a self exciting term. The constant σ is the volatility of P. The dynamic of the price is not affected by the control and is given by

$$\mathrm{d}P_t = d(t, P_t)\mathrm{d}t + \sigma \mathrm{d}W_t,$$

where d is a Lipschitz function. Finally we consider that the market maker wishes to solve

$$\sup_{\delta\in\mathscr{A}} \mathbb{E}^\delta\Big[G(i_T, P_T)e^{-rT} + \int_0^T e^{-rs}\Big(g(i_s, P_s)\mathrm{d}s + \delta_s^a \mathrm{d}N_s^a + \delta_s^b \mathrm{d}N_s^b\Big)\Big], \tag{7}$$

where r is a positive constant and g and G are two continuous functions with at most quadratic growth. The former represents a continuous reward received by the market maker (besides its P&L) and the latter is a final lump sum payment paid to the market maker at the end of its trading. We have two goals: prove the existence of a solution to the problem (7) and provide a tractable method to implement this solution in practice.

2.1.2 Existence of a solution

To get an explicit control, it is convenient to study the HJB equation associated to (7). In order to do this we need a Markovian representation of the model. For this purpose we consider the following processes which associate a random function to any given t,

$$\theta_t^a = \int_0^t K(\cdot - s)\mathrm{d}N_s^a \text{ and } \theta_t^b = \int_0^t K(\cdot - s)\mathrm{d}N_s^b \in L^1(\mathbb{R}_+)^2.$$

Then the process $X_t = (P_t, i_t, \theta_t^a, \theta_t^b)$ is Markovian since for $j \in \{a, b\}$ we have

$$\mathrm{d}\theta_t^j = K(\cdot - t)\mathrm{d}N_t^j \text{ and } \lambda_t^{j,\delta} = e^{-\frac{k}{\sigma}\delta_t^j}\Phi\big(\theta_t^j(t)\big).$$

We could use $L^1(\mathbb{R}_+)$ as state space for the processes θ^a and θ^b but to apply the theory of viscosity solutions it is convenient to work on a locally compact domain. We notice that for $j \in \{a, b\}$ we have almost surely for any $t \in [0, T]$,

$$\theta_t^j \in \Theta_t^K = \{\sum_{i=1}^n K(\cdot - T_i),\ n \in \mathbb{N},\ T_1 \le \cdots \le T_n \le T\} \subset \Theta_T^K.$$

[1] A completely monotone function of $\mathbb{R}_+$ is an infinitely differentiable function of $\mathbb{R}_+$ such that, $(-1)^p f^{(p)} \ge 0$, for all $p \in \mathbb{N}$.

[2] We consider that K is defined on $\mathbb{R}$ with zero value on $\mathbb{R}_-^*$.

Endowed with the topology of $L^1(\mathbb{R}_+)$ the set Θ_T^K is locally compact. Hence a convenient state space for the process $(t, X_t)_{t\geq 0}$ is

$$\mathscr{E}^K = \{(t, p, i, \theta^a, \theta^b) \in [0, T] \times \mathbb{R} \times \mathbb{Z} \times \Theta_t^K \times \Theta_t^K\},$$

which is locally compact. From now on, when we consider $x \in \mathscr{E}^K$ we always assume the decomposition $x = (t, p, i, \theta^a, \theta^b)$. Under this representation the HJB equation associated to Problem (7) is

$$(\mathbf{HJB})_K : \begin{cases} F\big(x, U(x), \nabla U(x), \partial^2_{pp} U(x), D^K U(x)\big) = 0, \text{ for } x \in \mathscr{E}^K, \\ U(x) = G(i, p), \text{ for } x = (T, p, i, \theta^a, \theta^b) \in \mathscr{E}^K, \end{cases}$$

with $\nabla U = \big(\partial_t U, \partial_p U\big)$, $D^K U = \big(D_a^K U, D_b^K U\big)$, where D_a^K and D_b^K are the infinitesimal generators associated toN^a and N^b, defined for any $x \in \mathscr{E}^K$ by

$$D_a^K U(x) = U\big(t, p, i-1, \theta^a + \theta(\cdot - t), \theta^b\big) - U(x) \text{ and } D_b^K U(x) = U\big(t, p, i+1, \theta^a, \theta^b + \theta(\cdot - t)\big) - U(x).$$

The function F is defined for $(x, u, q, A, I) \in \mathscr{E}^K \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ by

$$\begin{aligned} F(x, u, q, A, I) = & r u - q_1 - d(t, p) q_2 - \frac{1}{2}\sigma^2 A - g(i, p) \\ & - \Phi\big(\theta^a(t)\big) \sup_{\delta \in \mathbb{R}_+} e^{-\frac{a}{k}\delta}(\delta + I_1) - \Phi\big(\theta^b(t)\big) \sup_{\delta \in \mathbb{R}_+} e^{-\frac{a}{k}\delta}(\delta + I_2). \end{aligned}$$

We have the following result of existence and uniqueness for $(\mathbf{HJB})_K$.

Result 6. *(i) There exists a unique viscosity solution U^K with polynomial growth to* $(\mathbf{HJB})_K$.

(ii) Problem (7) *admits a solution given by $\delta_t^* = \delta^K(t, X_t)$, with $\delta^K = (\delta_a^K, \delta_b^K)$ where*

$$\delta_a^K = \big(\sigma/k - D_a^K U^K\big)_+ \text{ and } \delta_b^K = \big(\sigma/k - D_b^K U^K\big)_+.$$

Thus we have succeeded in finding an expression for an optimal control solving Problem (7).

2.1.3 How to approximate the optimal control

To make the above result useful in practice, we need a way to implement the optimal control δ^K, or at least an approximation of it. Since the set $\mathscr{E}^K$ is a subset of an infinite dimensional space we cannot use classical finite differences methods to numerically solve $(\mathbf{HJB})_K$. Our strategy is therefore to build approximating controls that are optimal with respect to approximated versions of Problem (7) and admitting a Markovian representation in finite dimension. We proceed in two steps.

- We show the continuity of δ^K with respect to K.

  Result 7. *Consider a sequence $(K_n)_{n\geq 0}$ of completely monotone functions in $L^1(\mathbb{R}_+)$ that converges towards K (also completely monotone) in $L^1(\mathbb{R}_+)$ and uniformly on $[0, T]$, then we have*

  $$\delta^K(x) = \lim_{(y,n) \in \bar{\mathscr{E}} \to (x, +\infty)} \delta^{K_n}(y) \text{ where } \bar{\mathscr{E}} = \mathscr{E}^K \times \{\infty\} \cup \Big(\bigcup_{n\geq 0} \mathscr{E}^{K_n} \times \{n\} \Big).$$

- Inspired by [AJEE19] we show that for any completely monotone function K in $L^1(\mathbb{R}_+)$ we can find a sequence $(K_n)_{n\geq 0}$ that converges towards K in $L^1(\mathbb{R}_+)$ and uniformly on $[0,T]$ and such that for any n, K_n is a positive linear combination of decreasing exponential functions. We then prove the following result.

  Result 8. *For any n, up to a change of variable, the value function U^{K_n} is the unique viscosity solution of a finite dimensional PDE. Thus we can approximate numerically δ^{K_n} using standard methods.*

Combining Results 7 and 8 gives a way to approach the optimal control δ^K for any kernel K. The remaining issue with this methodology is that the dimension of the approximating PDE is increasing with n. So when n is large it is very unlikely that finite differences methods can be applied. To tackle this problem we use a probabilistic representation for semi-linear PDEs introduced in [HLTT14]. This approach being probabilistic it has the advantage to circumvent the curse of dimensionality.

2.2 Chapter V: Scaling limit for stochastic control problems in population dynamics

In Chapters I and II we focus on scaling limits of microscopic financial models and in Chapter IV we study stochastic control for such type of models. We now naturally address the question of scaling to stochastic control problems. This issue is particularly interesting since it is often less convenient to deal with point processes based microscopic models than with their continuous counterpart, obtained passing to the scaling limit.

More precisely, we consider a sequence of population models appropriately rescaled in size. Under appropriate assumptions, we prove that it converges to a continuous limiting model. We then take the point of view of a resource manager controlling the population. For each discrete model the manager's problem is equivalent to a non-Markovian control problem for which we prove existence of a solution. This defines a sequence of controls. To investigate the convergence of this sequence we prove a general continuity result for solutions to BSDEs with respect to the driving martingale and terminal value. Using this result, we show that under reasonable scaling assumptions the sequence of optimal controls converges. The limit corresponds to a control solving the manager's problem in the continuous limiting model. We also show that the law of the population dynamic driven by the optimal control converges in law at the scaling limit.

2.2.1 Scaling limit of a population model

We start by defining a sequence of population models. Each model is driven by the same dynamic but for different initial sizes. We note N^b (resp. N^d) the number of birth (resp. death) in the population and fix $n \in \mathbb{N}$. The population size is indexed by K. We consider that for the $K-th$ model the population size is given by $N^K_t = nK + N^b_t - N^d_t$. The rescaled

population process is $X^K = N_t^K/K$. We denote by $\mathbb{P}^K$ the law of the $K-th$ model, under which the intensities of N^b and N^d are respectively

$$\lambda_t^{K,b} = Kf^b(X_t^K) + K^2\sigma^2 X_t^K/2 \text{ and } \lambda_t^{K,d} = Kf^d(X_t^K) + K^2\sigma^2 X_t^K/2,$$

where f^b, f^d are positive functions and σ is a positive constant. The process

$$M_t^K = (M_t^{K,b}, M_t^{K,d}), \text{ where } M_t^{K,b} = N_t^b - \int_0^t \lambda_s^{K,b}\mathrm{d}s \text{ and } M_t^{K,d} = N_t^d - \int_0^t \lambda_s^{K,d}\mathrm{d}s,$$

is a $\mathbb{P}^K$ local martingale. The natural scaled version of M^K is $\overline{M}^K = M^K/K$. We have the following result.

Result 9. *Under suitable regularity assumptions, the sequence* $(X^K, \overline{M}^{K,b}, \overline{M}^{K,d})_{K\geq 0}$ *converges in law for the Skorohod topology towards a process* (X, M^b, M^d) *such that* $M = (M^b, M^d)$ *is a bi-dimensional martingale. Moreover we have*

$$\langle M^b, M^d\rangle = 0,\ \langle M^b\rangle_t = \langle M^d\rangle_t = \int_0^t \sigma^2 X_s\mathrm{d}s/2$$

and

$$X_t = \int_0^t f(X_s)\mathrm{d}s + \mathrm{d}M_s^X,$$

with $f = f^b - f^d$ *and* $M_t^X = M_t^b - M_t^d$.

Using the Skorohod representation theorem, we can actually assume that the processes $(X^K, \overline{M}^{K,b}, \overline{M}^{K,d})_{K\geq 0}$ and (X, M^a, M^b) are defined on the same probability space, $(\Omega, \mathbb{P}, \mathbb{F})$ and that the convergence actually holds in $\mathscr{S}_3^2$, where $\mathscr{S}_d^p$ is the set of $\mathbb{F}$-predictable $\mathbb{R}^d$ valued process X such that

$$\|X\|_p^p = \mathbb{E}[\sup_{t\in[0,T]} \|X_t\|^p] < +\infty.$$

We write $\mathbb{F}^K = (\mathscr{F}_t^K)_{t\in[0,T]}$ the natural filtration associated to the process X^K and $\mathbb{F}^X = (\mathscr{F}_t^X)_{t\in[0,T]}$ the natural filtration associated to the process X. We now investigate the convergence of a sequence of BSDEs driven by the martingales $(M^K)_{K\geq 0}$. We start by a result on the continuity of martingale representation.

2.2.2 Continuity of martingale representations

For any K, we consider ξ^K a $\mathscr{F}_T^K-$measurable $L^2(\Omega)$ random variable. From [CF13] we know that ξ^K has the representation property relative to M^K. So there exists Z^K, a $\mathbb{F}^K$-predictable $\mathbb{R}^2$ valued process, such that

$$\mathbb{E}[\xi^K|\mathscr{F}_t^K] = \mathbb{E}[\xi^K] + \int_0^t Z_s^K\mathrm{d}M_s^K.$$

We also consider ξ a $\mathscr{F}_T^X$-measurable $L^2(\Omega)$ random variable and prove that there exists a $\mathbb{F}^X$-predictable real valued process Z such that

$$\mathbb{E}[\xi|\mathscr{F}_t^X] = \mathbb{E}[\xi] + \int_0^t Z_s \mathrm{d}M_s^X.$$

We have the following continuity result.

Result 10. *If the sequence $(\xi^K)_{K\geq 0}$ converges towards ξ in $L^{2+\varepsilon}$ for some $\varepsilon > 0$, we have the convergence*

$$\Big(Q^K, \langle Q^K, Q^K\rangle, \langle Q^K, \overline{M}^K\rangle\Big) \to \Big(Q, \langle Q, Q\rangle, \langle Q, M\rangle\Big), \text{ as } K \to +\infty,$$

in $\mathscr{S}_1^{2+\varepsilon'} \times \mathscr{S}_1^{1+\varepsilon'/2} \times \mathscr{S}_2^{2+\varepsilon'}$ for any $\varepsilon' \in [0, \varepsilon)$, where the processes Q and Q^K are defined by

$$Q_t^K = \int_0^t Z_s^K \mathrm{d}M_s^K \text{ and } Q_t = \int_0^t Z_s \mathrm{d}M_s^X.$$

We then extend this result to the continuity of solutions to BSDEs.

2.2.3 Continuity of solutions to BSDEs

For any K, let ξ^K a $\mathscr{F}_T^K-$measurable random variable and $g^{K,b}$ and $g^{K,d}$ two continuous functions from $\mathbb{R}^3$ into $\mathbb{R}$. For any $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$ we write

$$g^K(x, y, z) = \big(g^{K,b}(x, y, z^b), g^{K,d}(x, y, z^d)\big),$$

where have implicitly used the decomposition $z = (z^b, z^d)$. We consider the BSDE driven by M^K with terminal value ξ^K and generator g^K:

$$(\mathbf{B})_K: \; Y_t = \xi^K + \int_t^T g^K(X_s^K, Y_s, Z_s) \cdot \phi_s^K \mathrm{d}A_s^K - \int_t^T Z_s \cdot \mathrm{d}M_s^K,$$

where

$$A_t^K = \int_0^t K^{-2}(\lambda_s^{K,b} + \lambda_s^{K,d})\mathrm{d}s \text{ and } \phi_t^K = K^2\big(\frac{\lambda_t^{K,b}}{\lambda_t^{K,b} + \lambda_t^{K,d}}, \frac{\lambda_t^{K,d}}{\lambda_t^{K,b} + \lambda_t^{K,d}}\big).$$

We also consider ξ a $\mathscr{F}_T^X-$measurable random variable and g a function from $\mathbb{R}^3$ into $\mathbb{R}$. The BSDE driven by M^X with generator g and terminal value ξ is then

$$(\mathbf{B}): \; Y_t = \xi + \int_t^T g(X_s, Y_s, Z_s)\mathrm{d}A_s - \int_t^T Z_s \mathrm{d}M_s^X,$$

where $\mathrm{d}A_t = \sigma^2 X_t \mathrm{d}t$. For any K, under suitable assumptions, the BSDE $(\mathbf{B})_K$ admits a unique solution, (Y^K, Z^K), and $(\mathbf{B})$ has a unique solution, (Y, Z). The notion of solution is made precise in Chapter V Section 4.2. Moreover under appropriate convergence of $(\xi^K, g^K)_{K\geq 0}$ towards (ξ, g) we show the following result.

Result 11. *We have in $\mathscr{S}_1^2 \times \mathscr{S}_1^2 \times \mathscr{S}_2^1 \times \mathscr{S}_1^1$,*

$$\lim_{K\to+\infty} \Big(Y^K, \int_0^\cdot Z_t^K \cdot \mathrm{d}M_t^K, \langle Y^K, \overline{M}^K\rangle, \langle Y^K\rangle\Big) = \Big(Y, \int_0^\cdot Z_t \mathrm{d}M_t^X, \langle Y, M\rangle, \langle Y\rangle\Big).$$

We can now deal with the scaling limit of stochastic control problems.

2.2.4 Application to stochastic control and convergence of optimal controls

For any K, we consider a manager controlling the death intensity of the population, for say a ranger dealing with a population of boars. When the control is α the law of the population is denoted by $\mathbb{P}^{K,\alpha}$, under which the death intensity of the population is $\lambda_t^{K,d,\alpha} = \lambda_t^{K,d} + h^K(X_t^K, \alpha_t)$ where h^K is a real function defined on $\mathbb{R}^2$. The set of admissible controls is given by

$$\mathscr{A}^K = \{\alpha - \mathbb{F}^K \text{ predictable s.t. } \alpha \in [-\underline{a}, \overline{a}]\}.$$

We consider that the manager receives a continuous reward c^K related to his effort and a lump sum payment at time T, depending on the whole history of the population. In the case of a ranger dealing with boars, ξ^K can for example be proportional to the time spent by the population size in a given interval. We formalize the manager's goal as the following stochastic control problem

$$(\mathbf{P})_K : \; V_0^K = \sup_{\alpha \in \mathscr{A}^K} J_0^{K,\alpha} \text{ with } J_0^{K,\alpha} = \mathbb{E}^{K,\alpha}\big[\xi^K + \int_0^T c^K(X_s^K, \alpha_s)\mathrm{d}s\big],$$

where $\mathbb{E}^{K,\alpha}$ denotes the expectation under the probability $\mathbb{P}^{K,\alpha}$. We also consider the equivalent problem in the continuous framework. The law of the population size under the control α is given by $\mathbb{P}^\alpha$ under which the process X is a strong solution of

$$X_t = x_0 + \int_0^t \big(f(X_s) - h(X_s, \alpha_s)\big)\mathrm{d}s + M_t^\alpha,$$

where $M_t^\alpha = M_t^X + \int_0^t h(X_s, \alpha_s)ds$ is a $\mathbb{P}^\alpha-$martingale. The set of admissible control is

$$\mathscr{A} = \{\alpha - \mathbb{F}^X \text{ predictable s.t. } \alpha \in [-\underline{a}, \overline{a}]\}.$$

As for the discrete framework, we assume that the manager's problem is to solve the maximization problem

$$(\mathbf{P}) : \; V_0 = \sup_{\alpha \in \mathscr{A}} J_0^\alpha \text{ with } J_0^\alpha = \mathbb{E}^\alpha\big[\xi + \int_0^T c(X_s, \alpha_s)\mathrm{d}s\big],$$

where $\mathbb{E}^\alpha$ denotes the expectation under the probability $\mathbb{P}^\alpha$, c is a reward function related to the manager's effort and ξ is a lump sum received by the manager at the end of its work.

Using a technical assumption we show that for any K, $(\mathbf{P})_K$ admits a solution $\alpha^{K,*}$ and that $(\mathbf{P})$ has a solution α^*. These optimal controls are characterized via the BSDEs associated to the control problems. Finally denoting $\mathbb{P}^{K,*}$ the law of X^K under the control $\alpha^{K,*}$ and $\mathbb{P}^*$ the law of X under the control α^*, we prove that under suitable convergence of $(\xi^K, c^K, h^K)_{K\geq 0}$ towards (ξ, c, h) the following result holds.

Result 12. *(i) The sequence $(V_0^K)_{K\geq 0}$ converges towards V_0,*

(ii) we have in $\mathscr{S}_1^1 \times \mathscr{S}_1^1$

$$\lim_{K\to+\infty} \Big(\int_0^\cdot \alpha_s^{K,*}\lambda_s^{K,d}K^{-2}\mathrm{d}s, \int_0^\cdot (\alpha_s^{K,*})^2\lambda_s^{K,d}K^{-2}\mathrm{d}_s\Big) = \Big(\int_0^\cdot \alpha_s \mathrm{d}A_s/2, \int_0^\cdot \alpha_s^2 \mathrm{d}A_s/2\Big),$$

(iii) the sequence $(\mathbb{P}^{K,*})_{K\geq 0}$ *converges (for the Skorohod topology) towards* $\mathbb{P}^*$.

In this part we have focused on providing theoretical basis for stochastic control problems driven by point processes. This is of important practical interest for market participants aiming at improving their risk management and their understanding of trading strategies. In the last part of this thesis we now take the point of view of other important actors of the financial ecosystem, the exchanges and regulators, and deal with issues related to market design.

3 Part III: Market design

This part is dedicated to market design problems. First we consider the organization of a derivatives exchange. Then we address the question of finding the optimal duration for markets operating in sequential auctions.

3.1 Chapter VI: How to design a derivatives market?

In this chapter we answer to Question 6 and study the issue of market design for derivatives exchanges. This work has been motivated by discussions with collaborators from Euronext, one of the largest European exchanges.

To tackle this question, we take the point of view of an exchange that wants to create an options market. We consider that the goal of the exchange is to attract liquidity towards its platform and to be attractive for market takers, which boils down to having small quoted spreads. Our method goes in two steps: first we select according to market takers demand the options that are going to be listed on the platform, then we design a make-take fees policy for market makers in order to encourage them to quote attractive spreads.

3.1.1 Market driven selection of listed options

Theoretically all strikes and maturities could be displayed, but offering the possibility to trade them all will surely generate heavy technical difficulties and scarce liquidity. Hence the exchange needs to select a finite set of options to list on its platform. Therefore he wishes to choose the options that are the most relevant for market takers. We focus on selecting the strikes for call options at a given maturity. To address this problem we model the trading behavior of market takers as follows.

If a market taker would ideally buy a call option with strike K and that the listed strikes are given by $K_1 < \cdots < K_n$, then we assume that he is going to buy the option with strike K_i where i is such that

$$i = \underset{1\leq j\leq n}{\arg\min} |K - K_j|.$$

The regret of the market maker for not being able to buy exactly the strike K is measured by $|K - K_i|^p$ where $p \geq 2$. We finally assume that market takers demand is represented by a random variable K distributed according to some distribution $\mathbb{P}^{mkt}$ which can be estimated

using historical market data. In order to be attractive, the exchange looks for the set of strikes $(K_j)_{1\leq j\leq N}$ that minimizes the average regret of market makers:

$$\underset{K_1\leq\cdots\leq K_n}{\operatorname{argmin}}\ \mathbb{E}^{mkt}[\min_{1\leq j\leq n}|K-K_j|^p]. \tag{8}$$

This is a so-called quantization problem for which we prove the following result.

Result 13. *If $\mathbb{P}^{mkt}$ is compactly supported, then for any $p\geq 2$ Problem* (8) *admits a unique solution of the form $K_1<\cdots<K_n$ which is a fixed point of the Lloyd's algorithm. In practice we then iterate the Lloyd's algorithm to approximate the solutions. This is illustrated in Figure .4 where we use historical data provided by Euronext.*

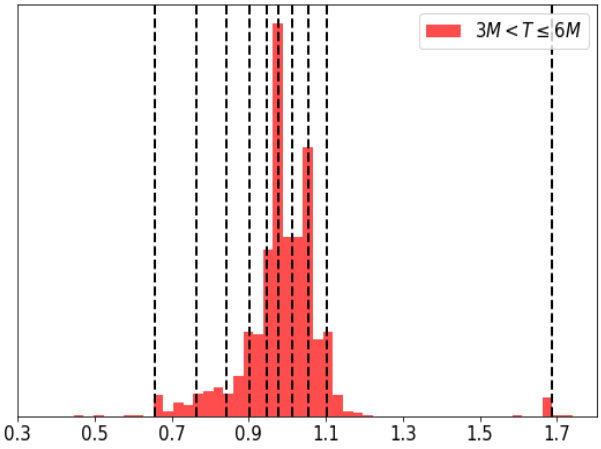

Figure .4 – The Lloyd's algorithm applied to options with maturities between 3 and 6 months: In red is the empirical distribution of market demand over the strikes (in % of the spot value). The black dotted lines correspond to the strikes selected by the Lloyd's algorithm.

3.1.2 Incentive policy of the exchange

We then design a make-take fees policy in order to encourage market makers to quote attractive spreads on each of the listed options. Since the exchange wants to influence the market spreads that are set by market makers, which he cannot control, we interpret this issue as a principal-agent problem. For the sake of simplicity, we assume that there is only one market maker. The exchange offers a contract to this market maker. The contract's payoff depends on the whole market order flow. The market maker adapts its strategy to the payoff according to its utility function. Thus knowing the market maker's response to any given contract, the problem of the exchange boils down to design the contract that maximizes its own utility function.

The market model: We note $\mathcal{K}$ the set of listed strikes and $\mathcal{T}$ the set of listed maturities. The underlying asset price is denoted by S and we assume that

$$\mathrm{d}S_t = \sigma \mathrm{d}W_s,$$

where σ is a positive constant. The price of the option with maturity τ and strike k is written $C^{k,\tau}$ and has dynamic

$$\mathrm{d}C_t^{k,\tau} = \sigma \Delta^{k,\tau} \mathrm{d}W_t,$$

where $\Delta^{k,\tau}$ is the Bachelier delta of the option at the beginning of the trading day. The set of admissible controls of the market maker is

$$\mathcal{A} = \big\{(\delta_t)_{t\in[0,T]} = (\delta_t^{k,\tau,i})_{t\in[0,T]}, k \in \mathcal{K}, \tau \in \mathcal{T}, i \in \{a,b\}, \text{ predictable and s.t } |\delta_t^{k,\tau,i}| \leq \delta_\infty\big\},$$

where $\delta_\infty > 0$ is a constant. The quantities $\delta^{k,\tau,b}$ and $\delta^{k,\tau,a}$ represent respectively the bid and ask spread on the option $C^{k,\tau}$. Hence when the control is δ, the best bid and ask prices on the option $C^{k,\tau}$ are respectively

$$P_t^{k,\tau,b} = C_t^{k,\tau} - \delta_t^{k,\tau,b} \text{ and } P_t^{k,\tau,a} = C_t^{k,\tau} + \delta_t^{k,\tau,a}, \ t \in [0,T].$$

We write $Q^{k,\tau}$ for the number of options $C^{k,\tau}$ held by the market maker and define its aggregated delta weighted inventory as

$$\mathcal{Q}_t = \sum_{k,\tau \in \mathcal{K}\times\mathcal{T}} \Delta^{k,\tau} Q^{k,\tau}.$$

We assume that all market orders have unit size and denote by $N^{k,\tau,a}$ (resp. $N^{k,\tau,b}$) the number of buy (resp. sell) market orders sent towards the option $C^{k,\tau}$. Since large spreads lead to smaller trading intensities we assume that when the control is δ the intensity of $N^{k,\tau,i}$ (for $i \in \{a,b\}$) is given by

$$\lambda^{k,\tau}(\delta_t^{k,\tau,i}, \mathcal{Q}_t) = A \exp\Big(-\frac{C}{\sigma}(\delta_t^{k,\tau,i} + f^{k,\tau})\Big) \mathbf{1}_{\phi(i)\mathcal{Q}_{t^-} > -\overline{q}},$$

where $\phi(a) = 1$, $\phi(b) = -1$, A, C are positive constants, $f^{k,\tau}$ is a transaction fee fixed by the exchange on the option $C^{k,\tau}$ and $\overline{q}$ is the maximal delta weighted inventory admissible by the market maker. We now define the market maker's maximization problem.

The market maker's problem: The market maker's profit and loss is

$$PL_t^\delta = \mathcal{W}_t^\delta + \mathcal{Q}_t S_t,$$

where $\mathcal{W}^\delta$ is the cash process of the market maker given by

$$\mathcal{W}_t^\delta = \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \int_0^t P_u^{k,\tau,a} \mathrm{d}N_u^{k,\tau,a} - \int_0^t P_u^{k,\tau,b} \mathrm{d}N_u^{k,\tau,b}.$$

Additionally to its trading P&L, the market maker receives at time T the payoff of the contract offered by the exchange, ξ which is a $\mathscr{F}_T$-measurable random variable, where $(\mathscr{F}_t)_{0\leq t\leq T}$ is the natural filtration associated to the canonical space of our model. Thus the market maker problem boils down to the following maximization problem

$$V_{MM}(\xi) = \sup_{\delta\in\mathscr{A}} \mathbb{E}^{\delta}\Big[-\exp\Big(-\gamma(\xi + PL_T^{\delta})\Big)\Big],$$

where γ is a positive constant and $\mathbb{E}^{\delta}$ denotes the expectation under the probability associated to the control δ. To make sure that the above quantity is well defined we need to assume some integrability conditions on ξ. More precisely we assume that for some $\gamma' > \gamma$ and $\eta' > \eta$ (the constant η is related to the exchange's problem, see below) we have

$$\sup_{\delta\in\mathscr{A}} \mathbb{E}^{\delta}[e^{-\gamma'\xi}] < +\infty \text{ and } \sup_{\delta\in\mathscr{A}} \mathbb{E}^{\delta}[e^{\eta'\xi}] < +\infty.$$

We also need to assume that the payoff ensures a minimum reservation utility $R < 0$ to the market maker. More precisely we require ξ to satisfy $V_{MM}(\xi) \geq R$. We say that any $\mathscr{F}_T$ measurable random variable satisfying those conditions is an admissible contract and note $\mathscr{C}$ the set of admissible contracts.

There is a special class of contract for which we can compute explicitly the market maker optimal strategy. Those contracts are indexed by $Y_0 > 0$ and

$$Z = (Z^{C^{k,\tau}}, Z^{k,\tau,i})_{k\in\mathscr{K},\tau\in\mathscr{T},i\in\{a,b\}} \in \mathscr{Z},$$

a definition of $\mathscr{Z}$ is given Chapter VI. The corresponding payoff is written $Y_T^{Y_0,Z}$ and is defined by

$$\begin{aligned} Y_T^{Y_0,Z} = Y_0 + \int_0^T \Big(\sum_{i=a,b} \sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} Z_r^{k,\tau,i} \mathrm{d}N_r^{k,\tau,i} + Z_r^{C^{k,\tau}} \mathrm{d}C_r^{k,\tau} \Big) \\ + \Big(\frac{1}{2}\gamma\sigma^2 \Big(\sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} \Delta^{k,\tau} (Z_r^{C^{k,\tau}} + Q_r^{k,\tau}) \Big)^2 - H(Z_r, Q_r) \Big) \mathrm{d}r, \end{aligned}$$

where for $(z,q) \in \mathbb{R}^{2\times\#\mathscr{K}\times\#\mathscr{T}} \times \mathbb{R}$ with $z = (z^{i,k,\tau})_{(i,k,\tau)\in\{a,b\}\times\mathscr{K}\times\mathscr{T}}$, the function H is given by

$$H(z,q) = \sup_{\delta\in\mathbb{R}^{2\times\#\mathscr{K}\times\#\mathscr{T}}} \sum_{i=a,b} \sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} \gamma^{-1}\Big(1 - \exp\Big(-\gamma\big(z^{k,\tau,i} + \delta^{k,\tau,i}\big)\Big)\Big) \lambda^{k,\tau}(\delta^{k,\tau,i}, q).$$

We prove that this class of contract is exactly the set of admissible contracts.

Result 14. *Any $\xi \in \mathscr{C}$ has a unique representation $\xi = Y_T^{Y_0,Z}$ for some $(Y_0, Z) \in \mathbb{R}\times\mathscr{Z}$ with $Y_0 \geq \hat{Y}_0 = -\log(-R)/\gamma$.*

When the exchange offers the payoff $Y_T^{Y_0,Z}$ to the market maker, we have the following result characterizing the market maker's optimal response.

Result 15. *The market maker's utility is* $V_{\mathrm{MM}}(Y_T^{Y_0,Z}) = -\exp(-\gamma Y_0)$. *It is associated to the optimal bid-ask policy* $\hat{\delta}_t^{k,\tau,i}(\xi) = \Delta^i(Z_t^{k,\tau,i})$, *where*

$$\Delta^i(z) = (-\delta_\infty) \vee \Big(-z + \frac{1}{\gamma}\log\Big(1 + \frac{\sigma\gamma}{C}\Big)\Big) \wedge \delta_\infty, \text{ for } i \in \{a,b\}.$$

We now specify and solve the exchange's problem.

The exchange's problem: Recall that the first aim of the exchange is to maximize the liquidity or equivalently to maximize the numbers of market orders sent towards the listed options. He also wishes to encourage market makers to quote attractive spreads for commercial reasons. Since the exchange's objective in terms of liquidity may not be the same for all the listed options, we consider the weighted total number of trades

$$\mathscr{N}_t = \sum_{i=a,b} \sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} c^{k,\tau} N_t^{k,\tau,i},$$

where $c^{k,\tau}$ is a positive constant measuring the relative importance that the exchange attributes to the option $C^{k,\tau}$. To quantify the second objective we consider

$$\mathscr{L}_T^\delta = \sum_{i=a,b} \sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} \int_0^T \omega\big(\delta_t^{k,\tau,i} - \delta_\infty^{k,\tau}\big)\mathrm{d}N_t^{k,\tau,i},$$

where $\omega \in [0,1)$ and $\delta_\infty^{k,\tau}$ is a spread threshold the exchange would like to impose to the market maker for the option $C^{k,\tau}$. Thus using an exponential utility function the exchange objective is to maximize

$$\mathbb{E}^{\hat{\delta}(\xi)}\Big[-\exp\Big(-\eta\Big(\mathscr{N}_T - \mathscr{L}_T^{\delta(\xi)} - \xi\Big)\Big)\Big],$$

where η is a positive constant and $\hat{\delta}(\xi)$ is the market maker's control associated to the contract ξ which is defined in Result 15. From Result 14 the exchange's problem boils down to the following optimization problem,

$$V_0^E = \sup_{Z\in\mathscr{Z}} \mathbb{E}^{\Delta(Z)}\Big[-\exp\Big(-\eta\Big(\mathscr{N}_T^Z - \mathscr{L}_T^Z - Y_T^{\hat{Y}_0,Z}\Big)\Big)\Big]. \tag{9}$$

We prove that if we take δ_∞ large enough, then this problem admits an explicit solution.

Result 16. *The maximization problem* (9) *admits a solution* $Z^\star$ *given by*

$$Z^{\star k,\tau,i}(t,\mathscr{Q}_{t^-}) = \frac{1}{\alpha-\beta}\log\Big(\frac{\beta x_2 U\big(t,\mathscr{Q}_{t^-}\big)}{\alpha x_1^{k,\tau} U\big(t,\mathscr{Q}_{t^-} - \Delta^{k,\tau}\phi(i)\big)}\Big) \text{ and } Z^{\star C^{k,\tau}}(t,Q_t^{k,\tau}) = -\frac{\gamma}{\gamma+\eta}Q_t^{k,\tau},$$

for any $(k,\tau,i) \in \mathscr{K}\times\mathscr{T}\times\{a,b\}$, *where* α, β, $(x_1^{k,\tau})_{k\in\mathscr{K},\tau\in\mathscr{T}}$ *and* x_2 *are constants and* U *is such that* $\tilde{U} = (-U)^{-\frac{C}{\sigma\eta(1-\omega)}}$ *is the unique solution of the following linear PDE on* $[0,T]\times\mathbb{R}$*:*

$$(\mathbf{E})_{ex}: \begin{cases} 0 = \partial_t\tilde{U}(t,\mathscr{Q}) - \tilde{U}(t,\mathscr{Q})\frac{C\gamma\eta}{\gamma+\eta}\frac{\sigma}{2(1-\omega)}\mathscr{Q}^2 + \sum_{i=a,b}\sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} \hat{C}^{k,\tau}\tilde{U}\big(t,\mathscr{Q}-\Delta^{k,\tau}\phi(i)\big)\mathbf{1}_{\phi(i)\mathscr{Q}>-\overline{q}}, \\ \tilde{U}(T,\mathscr{Q}) = 1, \end{cases}$$

where the $\hat{C}^{k,\tau}$*s are positive constants.*

From this result we get that the contract the exchange should offer to the market maker is

$$\xi^\star = Y^{\hat{Y}_0, Z^*} = \hat{Y}_0 + \int_0^T \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \Big(\sum_{i=a,b} Z_r^{\star k,\tau,i} \mathrm{d}N_r^{k,\tau,i} + Z_r^{\star C^{k,\tau}} \mathrm{d}C_r^{k,\tau} \Big) + \int_0^T \Big(\frac{1}{2}\gamma\sigma^2 \Big(\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \Delta^{k,\tau} (Z_r^{\star C^{k,\tau}} + Q_r^{k,\tau}) \Big)^2 - H(Z_r^\star, Q_r) \Big) \mathrm{d}r,$$

which is explicit up to solving the linear PDE $(\mathbf{E})_{ex}$. Note that even though the exchange's problem had initially $2 \times \#\mathcal{T} \times \#\mathcal{K}$ dimensions, $(\mathbf{E})_{ex}$ is only a two dimensional PDE. So we have made the exchange's problem tractable. To solve $(\mathbf{E})_{ex}$ it is possible to use a Monte-Carlo method based on the following representation.

$$\tilde{U}(t,q) = \mathbb{E}\Big[\exp\Big(\int_t^T -\tilde{C}\big(\mathcal{Q}_s^{t,q}\big)^2 + \sum_{i=a,b} \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \overline{\lambda}_s^{k,\tau,i} \mathrm{d}s \Big) \Big], \tag{10}$$

with

$$\mathcal{Q}_s^{t,q} = q + \int_t^s \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \Delta^{k,\tau} \mathrm{d}\big(\overline{N}_u^{k,\tau,b} - \overline{N}_u^{k,\tau,a} \big),$$

where for any $(k,\tau) \in \mathcal{K}\times\mathcal{T}$ and $i \in \{a,b\}$, $\overline{N}^{k,\tau,i}$ is a point process with intensity $\overline{\lambda}_s^{k,\tau,i} = \hat{C}^{k,\tau} \mathbf{1}_{\{\phi(i)\mathcal{Q}_{s^-}^{t,q} > -\overline{q}\}}$.

Numerical experiments: We illustrate the efficiency of our method to lower spreads and attract liquidity. We consider the following parameters setting

$$A = 1.5 s^{-1},\ \sigma = C = 0.3 s^{-1/2},\ \eta = 1,\ \gamma = 0.001,\quad T = 100 s \text{ and } \overline{q} = 40$$

and that there are 3 options on the market with characteristics

$$f^{k,\tau} = [0.5, 0.8, 0.8],\ \delta_\infty^{k,\tau} = [2,3,3],\ \Delta^{k,\tau} = [0.5, 0.8, 0.2].$$

We plot in Figures .5 and .6 the optimal spreads at time $t = 0$ for each option as a function of the market maker's initial inventory. The situation without the exchange intervention corresponds to the case $c = \omega = 0$. As expected, raising ω reduces the spread on all the listed options and raising c for an option induces a decrease of the associated spread.

3.2 Chapter VII: Optimal auction duration: A price formation viewpoint

In this last chapter of this thesis we deal with Question 7.

Recent studies about market design have raised many arguments suggesting that a CLOB system may not be the optimal way to organize electronic markets. One of the main points is that continuous trading promotes speed so that fast traders can systematically benefit from arbitrage opportunities due to latency advantages. It is suggested in [BCS15] that designing markets in sequential batch auctions, where settlement takes place after an auction period,

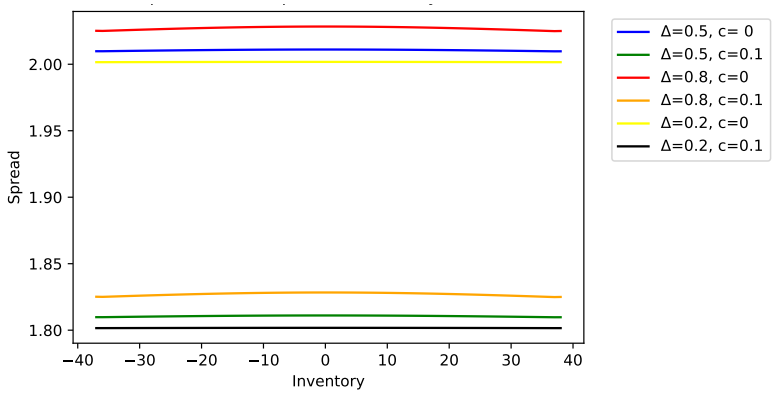

Figure .5 – Spread at initial time with respect to the market maker inventory, $\omega = 0$.

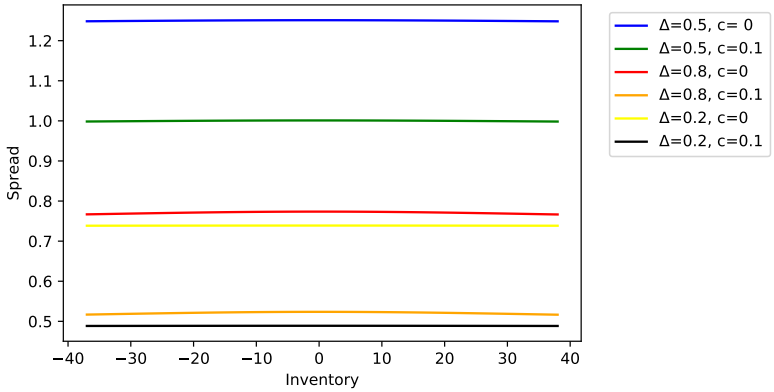

Figure .6 – Spread at initial time with respect to the market maker inventory, $\omega = 0.2$.

eliminates a large part of the advantages due to speed. Auctions represent an increasing part of the daily exchanged volume. Even in CLOB markets, a large fraction of the volume is executed during the opening and closing auctions and some exchanges, such as BATS-Cboe, already offer to their clients to trade in a sequential auctions market. Therefore designing those markets is an important issue.

In this chapter, we focus on one key aspect of this problematic, finding the optimal auction duration for a sequential auctions market. Furthermore we want to investigate whether auctions are optimal or suboptimal compared to CLOB. To tackle this question we begin by designing a class of models that encompasses CLOB and sequential auctions markets. Then we define our price discovery based criterion to determine the optimal duration. In order to take into account the competition between market participants, we consider that market takers are competing. We model this competition and show it has a Nash equilibrium. Finally, assuming that we are in the Nash equilibrium situation, we compute the optimal auction duration for

different French stocks after calibration of the model on market data. We find that for most of the stocks, CLOB are suboptimal compared to sequential auctions and more precisely that the optimal auction duration lies between 2 and 10 minutes depending on the stock.

3.2.1 The model

We describe our model for a sequential auctions market.

Auction rules: An auction opens after the end of the previous auction when a first market order is sent, it ends h seconds later. With this convention the case $h = 0$ is equivalent to a CLOB market since every market order is instantaneously matched. The starting times of auctions are given by the sequence $(\tau_i^{op})_{i\geq 0}$ and their closing times by $(\tau_i^{cl} = \tau_i^{op} + h)_{i\geq 0}$. During and after an auction, market participants can send limit and market orders. We distinguish two types of participants, market makers that only use limit orders and market takers that only send market orders.

Market makers: During the $i-th$ market phase, market makers arrival times are given by $(\tau_{i-1}^{cl} + \tau_k^{i,mm})_{k\geq 0}$ where $\tau_k^{i,mm}$ is the k-th event time of a counting process $N^{i,mm}$. When he arrives on the market, the $k-$th market maker sends a sequence of limit orders according to a supply function S_k. More precisely $S_k(p)$ represents the number of shares the $k-$th market maker is willing to sell at a price greater than p. We assume that

$$S_k(p) = K(p - \tilde{P}_k),$$

where K is a positive constant and $\tilde{P}_k$ is the value of fair price at the clearing according to the k-th market maker. We assume it has the following form

$$\tilde{P}_k = P_{\tau_{i-1}^{cl} + \tau_k^{i,mm}} + g_k,$$

where $(g_k)_{k\geq 0}$ is a sequence of independent identically distributed random normal variables with variance σ^2, independent of all the other processes. The process P represents the fundamental price of the asset. We assume it has the form $P_t = P_0 + \sigma_f W_t$ where σ_f a positive constant and W is a Brownian motion independent of all the other processes. We finally assume that after the auction clearing, the unmatched limit orders are canceled.

Market takers: During the i-th market phase the arrival time of the k-th buy (resp. sell) market order is $\tau_{i-1}^{cl} + \tau_k^{i,a}$ (resp. $\tau_k^{i,b}$) where $\tau_k^{i,a}$ (resp. $\tau_k^{i,b}$) is the k-th event time of a counting process $N^{i,a}$ (resp. $N^{i,b}$). Note that by definition we have $\tau_i^{op} = \tau_{i-1}^{cl} + \tau_k^{i,a} \vee \tau_k^{i,b}$. We assume that market orders have a constant volume v and we write $I^i = v(N_t^{i,a} - N_t^{i,b})$ the cumulated imbalance of market orders for the i-th auction.

Clearing rule: At the end of the i-th auction, a clearing price $P^{cl}_{\tau_i^{cl}}$ is set and sellers who are willing to sell below it sell their shares at price $P^{cl}_{\tau_i^{cl}}$ to buyers ready to buy above it. The

clearing price is chosen in order to maximize the exchanged volume. It is therefore the solution of

$$\sum_{i=1}^{N_{\Delta_i}^{i,mm}} S_k(p) - I_{\Delta_i}^i = 0,$$

where $\Delta_i = \tau_{i-1}^{cl} - \tau_i^{cl}$. We thus obtain

$$P_{\tau_i^{cl}}^{cl} = \frac{1}{N_{\Delta_i}^{i,mm}} \sum_{k=1}^{N_{\Delta_i}^{i,mm}} \tilde{P}_k + \frac{1}{K} \frac{I_{\Delta_i}^i}{N_{\Delta_i}^{i,mm}}.$$

In order to get a regenerative model we make the following assumption.

Assumption 5.

i) After each auction clearing, the market regenerates: the processes $(N^{i,mm}, N^{i,a}, N^{i,b}, I^i)_{i\geq 0}$ *are independent and identically distributed.*

ii) The random variables $(\tau_1^{i,a} \wedge \tau_1^{i,b})_{i\geq 0}$ *are i.i.d. exponential random variables with parameter* ν.

iii) The random variables $N_{\tau_1^{cl}}^{1,a}$ *and* $N_{\tau_1^{cl}}^{1,b}$ *are squared integrable.*

Since it is very unlikely that a market taker would send a market order if the LOB is empty, we make the following assumption ensuring that some limit orders are sent before the auction clearing.

Assumption 6. *Let* $\mu > 0$. *The density of* $\left(\tau_1^{1,mm}, (\tau_1^{1,a} \wedge \tau_1^{1,b})\right)$ *at point* $(s,t) \in \mathbb{R}^2$ *is given by*

$$\mathbf{1}_{0\leq s\leq t+h} \frac{\mu e^{-\mu s}}{1 - e^{-\mu(t+h)}} \mathrm{d}s \; \nu e^{-\nu t} \mathbf{1}_{t\geq 0} \mathrm{d}t \tag{11}$$

and $(N_{s+\tau_1^{1,mm}}^{1,mm} - 1)_{0\leq s\leq h}$ *is a Poisson process with intensity* μ, *independent of* P *and* $(N_s^{1,a}, N_s^{1,b})_{s\geq \tau_1^{1,mm}}$.

Thanks to this assumption, the LOB is never empty at clearing times. In particular when $h = 0$ we are in the situation of a CLOB market.

3.2.2 A metric for the quality of the price formation process

We want to select the auction duration that leads to the best price discovery. Equivalently we look for the duration that minimizes the cumulated difference between the clearing price and the efficient price. We measure this cumulated error as

$$Z_t^h = \int_0^t (\overline{P}_s^{cl} - \overline{P}_s)^2 \mathrm{d}s,$$

where h is the auction duration and $\overline{P}_t^{cl}$ and $\overline{P}_t$ are respectively the last clearing price before time t and the efficient price at the last clearing time before time t. The optimal auction duration is h^* such that Z^{h^*} is asymptotically smaller than any other Z^h. We show the following result.

Result 17. *We have almost surely*

$$\lim_{t\to+\infty}\frac{Z_t^h}{t}=\mathbb{E}[(P^{cl}_{\tau_1^{cl}}-P_{\tau_1^{cl}})^2].$$

Hence the optimal auction duration is the minimizer of

$$E(h)=\mathbb{E}[(P^{cl}_{\tau_1^{cl}}-P_{\tau_1^{cl}})^2].$$

Some computations leads to the following semi-explicit formula

Result 18.

$$E(h)=E^{mid}(h)+\frac{\mathbb{E}[I^2_{\tau_1^{cl}}]}{K^2}(1-e^{-\mu h}\frac{\nu}{\nu+\mu})^{-1}e^{\nu h}\int_h^{+\infty}\nu e^{-\nu t}e^{-\mu t}\int_0^{\mu t}\frac{1}{s}\int_0^s\frac{e^u-1}{u}\mathrm{d}u\mathrm{d}s\mathrm{d}t,$$

with $E^{mid}(h)$ *given by*

$$(1-e^{-\mu h}\frac{\nu}{\nu+\mu})^{-1}e^{\nu h}\int_h^{+\infty}\nu e^{-\nu t}\Big((\sigma_f^2\frac{t}{6}+\sigma^2)e^{-\mu t}\int_0^{\mu t}\frac{e^s-1}{s}\mathrm{d}s+\sigma_f^2\frac{t}{3}(1-e^{-\mu t})\Big)\mathrm{d}t.$$

The function E^{mid} corresponds to the situation where there is no market takers. We see from Result 18 that the market takers squared imbalance plays a central role in the determination of the optimal auction duration. A first possibility is to consider that market orders arrival times are given by a Poisson processes with intensity ν, in this case we get

$$\mathbb{E}[I^2_{\tau_1^{cl}}]=v^2(\nu h+1).$$

However this model is probably too naive and may lead to a non reliable optimal duration. Hence we propose a finer model allowing market takers to optimize their trading intensity.

3.2.3 The market takers competition

We consider that market takers aim at minimizing their asymptotic transaction costs by controlling their trading intensities. For example if a seller observes that a large number of buy market orders have been sent, moving the clearing price up to his advantage, he is more likely to send sell market orders. We assume that there are only two market takers, a buyer, denoted by a, and a seller, b. We do not consider that market makers are strategic.

The accumulated trading cost of player a at time t is

$$C_t^a=\sum_{i=1}^{N_t^{cl}}N^{i,a}_{\Delta_i}(P^{cl}_{\tau_i^{cl}}-P_{\tau_i^{cl}}).$$

Assuming that the market takers strategies reset after each auction, we can characterize their asymptotic trading costs.

Result 19. *We have almost surely*

$$\lim_{t\to+\infty}\frac{C_t^a}{t}=\frac{\nu}{K}\mathbb{E}[N^{1,a}_{\tau_1^{cl}}(N^{1,a}_{\tau_1^{cl}}-N^{1,b}_{\tau_1^{cl}})]\mathbb{E}[\frac{1}{N^{1,mm}_{\Delta_i}}].$$

Since $N^{1,a}_{\tau_1^{op}}\vee N^{1,b}_{\tau_1^{op}}$ is equal to one by assumption, the problem of a boils down to minimize

$$\mathbb{E}[N_h^{1,a}(N_h^{1,a}-N_h^{1,b})],$$

when $(N_0^{1,a},N_0^{1,b})=(1,0)$ or when $(N_0^{1,a},N_0^{1,b})=(0,1)$. Symmetrically the objective of b is to minimize $\mathbb{E}[N_h^{1,b}(N_h^{1,b}-N_h^{1,a})]$. The market takers control their trading intensity: a controls λ^a, the intensity of $N^{1,a}$ and b controls λ^b, the intensity of $N^{1,b}$. The set of admissible controls of market takers is

$$\mathscr{A}=\{\lambda,\text{ predictable process with values in }[\lambda_-,\lambda_+]\},$$

where λ_- and λ_+ are two positive constants and predictability is relative to the natural filtration of $(N^{1,a},N^{1,b})$. This a non zero-sum game situation for which we prove existence of a Nash equilibrium.

Result 20. *There exists a Nash equilibrium to the simultaneous optimization problem faced by players a and b given by some Markovian controls* $(\lambda_a^\star,\lambda_b^\star)$ *satisfying*

$$\inf_{\lambda^a\in\mathscr{A}}V_h^{a,\alpha,\beta}(\lambda_a,\lambda_b^\star)=\mathbb{E}^{\mathbb{P}^{\lambda_a^\star,\lambda_b^\star}}[N_h^a(N_h^a-N_h^b)|(N_0^a,N_0^b)=(\alpha,\beta)],$$

and

$$\inf_{\lambda^b\in\mathscr{A}}V_h^{b,\alpha,\beta}(\lambda_a^\star,\lambda_b)=\mathbb{E}^{\mathbb{P}^{\lambda_a^\star,\lambda_b^\star}}[N_h^b(N_h^b-N_h^a)|(N_0^a,N_0^b)=(\alpha,\beta)].$$

This result is not only theoretical since its proof also provides a natural method to compute the associated value functions. This enables us to get the optimal durations since we have the following result.

Result 21. *At the Nash equilibrium* $(\lambda_a^\star,\lambda_b^\star)$*, we have*

$$\mathbb{E}[I^2_{\tau_1^{op}+h}]=V_h^{a,1,0}(\lambda_a^\star,\lambda_b^\star)+V_h^{b,1,0}(\lambda_a^\star,\lambda_b^\star).$$

3.2.4 Application on market data

For numerical applications we use intra-day market data from 77 of the most liquid stocks traded on Euronext for all trading days of September 2018. For any of the stocks we calibrate the model and compute the minimizer of the function E in two situations:

1. non optimizing market takers with Poisson trading intensity ν,
2. competing market takers at the Nash equilibrium.

	DurationPoisson	DiffrelPoisson	DurationNash	DiffrelNash
Bouygues	228 [226;230]	1%	152 [150;153]	20%
Alstom	0 [0;0]	0%	180 [178;181]	14%
Orange	382 [379;385]	21%	274 [273.6;278]	42%
Veolia Environ.	350 [346;353]	3%	253 [251;256]	21%
Credit Agricole	87.7 [87.2;88.5]	2%	58.6 [58;59.4]	22%

Table .1 – Optimal auction durations (in seconds) Part 1 with a 90% confidence interval.

Results are partially reported in Table .1. The optimal durations for the Poisson case are reported in column DurationPoisson and in the competitive case in the column DurationNash. We have also reported the relative difference between the utility function for the optimal duration and for duration 0, $(E(0) - E(h^*))/E(h^*)$, in order to investigate the quality of the CLOB mechanism. We observe for most of the stocks that the optimal duration is between 2 and 10 minutes. The optimal duration when market takers are strategic is smaller than when they are naive, except for the stocks where CLOB market is optimal for naive market takers. When market takers are strategic CLOB is always suboptimal. We also notice that the CLOB is always not so far from being optimal.

Part I

Microstructural foundations of volatility properties

CHAPTER I

No-arbitrage implies power-law market impact and rough volatility

Abstract

Market impact is the link between the volume of a (large) order and the price move during and after the execution of this order. We show that in a quite general framework, under no-arbitrage assumption, the market impact function can only be of power-law type. Furthermore, we prove that this implies that the macroscopic price is diffusive with rough volatility, with a one-to-one correspondence between the exponent of the impact function and the Hurst parameter of the volatility. Hence we simply explain the universal rough behavior of the volatility as a consequence of the no-arbitrage property. From a mathematical viewpoint, our study relies in particular on new results about hyper-rough stochastic Volterra equations.

Keywords: No-arbitrage property, market impact, rough volatility, rough Heston model, hyper-rough Heston model, Hawkes processes.

1 Introduction

It is now well-admitted that volatility is rough. This stylized fact first established in [GJR18] and confirmed in [BLP16, LMPR18] means that the (log-)volatility process of an asset essentially behaves as a fractional Brownian motion (fBm for short) with Hurst parameter of order 0.1. Recall that a fBm $(W_t^H)_{t\geq 0}$ with Hurst parameter $H \in (0,1)$ is a Gaussian process that can be written under the Mandelbrot-van Ness representation as

$$W_t^H = \int_{-\infty}^0 \left((t-s)^{H-\frac{1}{2}} - (-s)^{\frac{1}{2}}\right) \mathrm{d}B_s + \int_0^t (t-s)^{H-\frac{1}{2}} \mathrm{d}B_s,$$

with $(B_t)_{t\geq 0}$ a classical Brownian motion. For any $\varepsilon > 0$, the sample paths of $(W_t^H)_{t\geq 0}$ are almost surely $H-\varepsilon$ Hölder (and not H Hölder). Therefore the trajectories are very rough when H is small.

Various rough volatility models have been recently introduced in the literature, notably for the purpose of risk management of derivatives, such as the rough Heston model of [EER19] where the asset price $(P_t)_{t\geq 0}$ satisfies

$$\frac{\mathrm{d}P_t}{P_t} = \sqrt{V_t}\big(\rho \mathrm{d}B_t^2 + \sqrt{1-\rho^2}\mathrm{d}B_t^1\big)$$

with

$$V_t = V_0 + \frac{\lambda}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-\frac{1}{2}}\big(\theta(t) - V_s\big)\mathrm{d}s + \frac{\nu}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-\frac{1}{2}}\sqrt{V_s}\mathrm{d}B_s^1, \tag{1}$$

where B_1 and B_2 are independent Brownian motions, λ and ν two positive constants, θ a deterministic non-negative function and $\rho \in (-1,1)$ a correlation factor. The particular interest of this model is that, as for the classical Heston model, semi-explicit pricing and hedging formulas can be obtained, see [EER19, EER18].

A puzzling question is the origin of the universal rough volatility property of financial assets. A first explanation is proposed in [EEFR18]. In this work, the authors place themselves in a highly endogenous market, meaning that most orders are sent in reaction to other orders and without economic motivation. They show that in this context, the widely used trading practice of metaorders splitting (see below for definition of a metaorder) leads to the rough Heston dynamic (1) for the macroscopic price. However, this result is found using a quite specific parametric model for the high frequency price.

In this paper, we wish to obtain a fundamental explanation underlying the rough volatility property. In fact we prove that in a quite general framework, rough volatility is simply a consequence of the no-arbitrage principle together with the existence of market impact.

Market impact is the fact that on average, a buy order moves the price up and a sell order moves the price down. There are two main viewpoints on market impact. The first one is to consider that agents receive information and reveal their information through transactions, hence impacting the market. The other one is purely mechanical, not referring to any notion of information, and considering that prices move up and down only through volume pressures, following supply and demand. We adopt the mechanical paradigm in this work. The impact of a single order being very difficult to assess, one usually considers large sets of orders split by brokers, so-called metaorders. Empirical studies of market impact have shown that for a buy metaorder (and symmetrically for a sell metaorder) market impact can be decomposed in two phases: a transient phase with a concave rise of the price during the metaorder execution, and a decay phase, where the price decreases towards a long-term level after the execution is completed, see [BILL15, TLD$^+$11, DB15, LFM03].

Let us consider a buy (say) metaorder and let $(q_t)_{t\geq 0}$ be the cumulative volume of this metaorder executed between the initial time 0 and time t. The market impact function of this metaorder is defined as

$$MI(t) = \mathbb{E}[P_t^{(q_s)_{s\leq t}} - P_0],$$

where we put the superscript $(q_s)_{s\leq t}$ on P to insist on the fact that the price dynamic depends on the execution process of the metaorder. Of course the above formula only makes sense in a model where $P_t^{(q_s)_{s\leq t}}$ is a well-defined stochastic process, as will be the case in the next sections.

The permanent market impact (PMI for short) of this metaorder is given by the quantity

$$PMI = \lim_{t\to+\infty} MI(t).$$

Intuitively, it is quite clear that in the long run, the permanent impact of a metaorder with volume Q should be the same as that of two consecutive orders of volume $Q/2$. This is formalized in Theorem 1 in [HS04] and further developed in [Gat10] where it is shown that under mild modeling assumptions, the absence of price manipulation on a market[1] implies that the permanent market impact is proportional to the total volume of the metaorder. In particular it does not depend on the metaorder execution strategy. From now on, we take this linear permanent market impact property as granted. This has consequences for the price dynamics. In particular, we now assume that the price P is a martingale. We take this martingale hypothesis as a simplifying and convenient version of the classical mathematical finance condition of no almost sure arbitrage which states that price should be a semi-martingale. In this setting, it is shown in Theorem 2.1 in [Jai15] that under the purely mechanical view for market impact,

$$P_t = \lim_{s\to+\infty} \mathbb{E}\big[V_s^a - V_s^b | \mathscr{F}_t\big], \tag{2}$$

where V^a (resp. V^b) is the cumulated volume of buy (resp. sell) market orders since the initial time 0 and $(\mathscr{F}_t)_{t\geq 0}$ corresponds to the filtration generated by the order flow process. Hence the price moves when orders arrive on the market because market participants revise their anticipation about the long term cumulative imbalance of the order flow. Remark that to derive (2), we only use the set of assumptions that we consider here as our no arbitrage conditions: martingale price and linear permanent market impact.

As for the transient part of the market impact, empirical measurements show that provided the execution rate of the metaorder is relatively constant, the function MI is close to a power-law with respect to time, that is $MI(t) \sim t^{1-\alpha}$ with $\alpha \in (0,1)$, see [BILL15, Bou10, LFM03, TLD+11]. More precisely, the coefficient α is found to be about $1/2$ so that the so-called square root law is approximately satisfied. Actually, it is proved in [PRST17] that under some leverage neutrality assumption, the square root law can be simply derived from dimensional analysis.

We show in this work that under no-arbitrage assumption (represented by the linear permanent impact, the martingale price and thus (2)), the market impact function has indeed to be a power-law of the form $MI(t) \sim t^{1-\alpha}$. Then we prove that for any $\alpha \in (0,1)$, the scaling limit of the price (2) exists and satisfies

$$\widehat{P}_t = B_{X_t}$$

[1] A price manipulation is a round-trip (strategy starting and finishing with null inventory) whose expected cost is negative.

with

$$X_t = \frac{2}{\delta}\int_0^t F^{\alpha,\lambda}(s)\mathrm{d}s + \frac{1}{\delta\sqrt{\lambda}}\int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}W_{X_s}, \tag{3}$$

where W and B are two Brownian motions, δ and λ two positive constants and $F^{\alpha,\lambda}$ is the Mittag-Leffler cumulative distribution function, see Appendix I.A.1 for definition. The correlation between the Brownian motions B and W is stochastic and related to the order flow imbalance. The above equation is a generalization of the rough Heston model (1). Indeed we can show that when $\alpha > 1/2$, after differentiation, Equation (3) can be rewritten under the form of (1) (up to a stochastic correlation factor) with associated Hurst parameter $H = \alpha - 1/2$. For $\alpha \leq 1/2$, we prove that X is not continuously differentiable but has Hölder regularity $2\alpha - \varepsilon$ for any $\varepsilon > 0$. Therefore we call (3) the *hyper-rough Heston model* when $\alpha \leq 1/2$. Hence we are able to define rough Heston models for Hurst parameter in $(-\frac{1}{2}, \frac{1}{2}]$.

To obtain our results, we only need to specify a model for the order flow dynamics. We indeed see from Equation (2) that we do not need to model metaorders individually. Only the aggregated order flow matters in order to derive the price dynamic. More precisely, we consider for buy and sell market order arrivals two independent Hawkes processes N^a and N^b and assume that each order is of unit size, see [EEFR18, JR16]. Recall that a Hawkes process N is a self-exciting point process whose intensity $(\lambda_t)_{t\geq 0}$ is defined by

$$\lambda_t = \mu + \int_0^t \phi(t-s)\mathrm{d}N_s,$$

with μ a positive constant and ϕ a non-negative locally integrable function. Such dynamic is a generalization of the Poisson process which is usually considered when modeling order flows, see among others [CDL13, CST10, SFGK03]. It is non-parametric and very flexible so that it is really reasonable to assume that the actual order flow can be well approximated by a Hawkes based model, see [BMM15, BM16]. Note that we will not put any restriction on the Hawkes parameters μ and ϕ, except the non-negativity of μ, the local integrability of ϕ and the fact that they are the same for the buy and sell flows. In this case, it is shown in [Jai15] that the price process (2) satisfies

$$P_t = P_0 + \int_0^t \xi(t-s)\mathrm{d}(N^a - N^b)_s, \tag{4}$$

with

$$\xi(t) = 1 + \left(1 + \int_0^{+\infty} \psi(u)\mathrm{d}u\right)\int_t^{+\infty} \phi(u)\mathrm{d}u \tag{5}$$

and

$$\psi = \sum_{i\geq 1} (\phi)^{*i},$$

where $(\phi)^{*1} = \phi$ and for $k \geq 2$, $(\phi)^{*k}$ denotes the convolution product of $(\phi)^{*(k-1)}$ with ϕ.

Using a rescaling procedure to describe the macroscopic behavior of (4), we show that only one very subtle specification of the Hawkes processes can lead to a non-trivial market impact,

which has to be power-law. Furthermore, it implies that the market is highly endogenous. In addition, depending on the market impact shape, the scaling limit of the price is a rough or hyper-rough Heston model (3), with a one-to-one correspondence between the exponent of the impact function and the Hurst parameter of the volatility.

The paper is organized as follows. In Section 2, we show that under the assumption that the market impact function is not degenerate, it can only be a power-law with parameter $1-\alpha$ for some $\alpha \in (0,1)$. Then in Section 3 we explain that the macroscopic limit of (4) is a rough or hyper-rough Heston model with Hurst parameter $H = \alpha - 1/2$.

2 Market impact is power-law

In this section, we show that if there exists a non-degenerate market impact function, it has to be a power-law. Moreover we will see that it implies a highly endogenous market. By non-degenerate we essentially mean a market impact function which is ultimately decreasing for buy metaorders (and conversely for sell metaorders), see Assumption 1. This is the formalization of the two phases behavior of market impact discussed in the introduction.

2.1 Asymptotic framework and metaorders modeling

Let T be our final horizon time for the metaorders we will define in the sequel. Recall that the market order flow on $[0,T]$ (and after T) is given by two Hawkes processes with the same parameters, N^a for the buy market orders and N^b for the sell orders, with respective intensities λ^a and λ^b. As the time-length of a metaorder is typically large compared to the inter-arrivals of individual market orders, it is natural to consider that T goes to infinity.

We want to work in a general setting which enables us to be compatible with empirical studies showing that markets are highly endogenous. In the Hawkes process context, the degree of endogeneity of the market is measured by the L^1 norm of ϕ, denoted by $\|\phi\|_1$, see [FS15, HBB13, JR15, JR16]. Therefore a highly endogenous market corresponds to the case where $\|\phi\|_1$ is close but smaller than unity. So we allow the model parameters to possibly depend on T. Thus, from now on, we use the superscript T for all quantities that could depend on T. In particular $\|\phi^T\|_1$ may go to one as T tends to infinity. We also write $N^{a,T}$, $N^{b,T}$, μ^T, ϕ^T to describe the market order flow and model parameters corresponding to the time-horizon T, and we set $\phi^T = a^T\phi$ for ϕ a non-negative function such that $\|\phi\|_1 = 1$ and $(a^T)_{T\geq 0}$ a real sequence in $(0,1)$. Note that we do not impose that a^T goes to one. In fact we will show that one does need to have a^T tends to one for the existence of a non-degenerate market impact function.

We finally need to define a formalism for a sequence of buy (say) metaorders which will be added to the global order flow. We assume that a metaorder is split through market orders of size one over $[0,T]$. In the spirit of [Jai15], we consider that the arrival times of the market

orders are given by a non-homogenous Poisson process with intensity

$$\nu^T(t) = I^T f(\frac{t}{T}),$$

f a non-negative continuous function on $[0,1]$ with integral one and I^T a sequence of non-negative real numbers such that the expected total volume of the metaorder is $I^T T$. The order of magnitude of its duration is T. Note that this is slightly smaller than T as the metaorder will end after the last jump time of the Poisson process before T. We allow f to be different from a constant to get more realistic splitting schemes than those given by constant rate Poisson processes. Suitable choices for f may be exponentially decaying (arrival price benchmark) or linearly decaying (VWAP benchmark), see for example [AC01, HLR15].

To compute the market impact function in practice, one typically considers the empirical mean of the price movements over many metaorders with similar durations and volumes counted in proportion of the total traded volume. So in our setting, it is natural to take $I^T \times T$ (the order of magnitude of the total volume of our metaorder) essentially proportional to the total number of other orders executed over $[0,T]$. To do so we take the intensities proportional

$$I^T = \gamma\beta^T, \text{ with } \beta^T = \mu^T(1-a^T)^{-1},$$

where $\gamma < 1$ and β^T is the long-term average intensity of the Hawkes process $N^{a,T}$ ($\beta^T = \lim_{t\to+\infty}(1/t)\int_0^t \lambda_s^a \mathrm{d}s$, see [BDHM13]). Thus, the proportion of the order flow which is due to the considered metaorder is essentially $\gamma/(1+\gamma)$ and γ will be considered reasonably small.

2.2 Market impact in the Hawkes setting

In this section the parameter T is fixed. Assuming that the volume of our metaorder is small enough, the total order flow is not deeply modified by it. Hence other agents do not observe significant changes in the order flow dynamics. So the way the market reacts to the incoming orders remains unchanged. Recall that in our model, the market reaction to the order flow (without our metaorder) is given by (4).

We work under the setting of the previous section assuming that the number of shares bought through our metaorder is a non-homogenous Poisson process $(n_t^T)_{t\geq 0}$. Therefore we obtain

$$P_t^T = P_0 + \int_0^t \xi^T(t-s)\mathrm{d}(N^{a,T} - N^{b,T} + n^T)_s,$$

where $(N_t^{a,T}, N_t^{b,T})_{t\geq 0}$ corresponds to the aggregated order flows of all other agents. Indeed all orders being anonymous in the market, our metaorder cannot be distinguished from the global order flow. Thus the market digests the order flow through the kernel ξ, as if it is a bivariate Hawkes process with parameters μ and ϕ (γ being small).

Now we are in the position to properly compute the market impact function of our metaorder. We have

$$MI^T(t) = \mathbb{E}[P_t^T - P_0] = \int_0^t \xi^T(t-s)\mathbb{E}[\mathrm{d}n_s^T].$$

This equation together with (5) shows that for any $t \geq 0$, the market impact function can be decomposed into two terms as follows

$$MI^T(t) = PMI^T(t) + TMI^T(t),$$

where

$$PMI^T(t) = \mathbb{E}[n_t^T]$$

and

$$TMI^T(t) = \int_0^t \Gamma^T(t-s)\mathbb{E}[\mathrm{d}n_s^T],$$

with

$$\Gamma^T(s) = (1-a^T)^{-1}\int_s^{+\infty} \phi^T(u)\mathrm{d}u,$$

where we have used the fact that

$$\int_0^{+\infty} \psi^T(u)\mathrm{d}u = \sum_{k\geq 1}^{+\infty} \Big(\int_0^{+\infty} \phi^T(u)\mathrm{d}u\Big)^k = \frac{a^T}{1-a^T}.$$

Note that the definition of $PMI^T(t)$ is compatible with that of the permanent market impact PMI given in the introduction. Indeed the order intensity from our metaorder being eventually null and because $\Gamma^T(t)$ tends to zero as t goes to infinity, we get

$$\lim_{t\to+\infty} TMI^T(t) = 0.$$

The effect of the term TMI^T is thus only temporary. That is why this term is called transient part of the market impact.

2.3 Scaling limit of the market impact

We now rescale the market impact function as the horizon time T goes to infinity. If the sequence of rescaled market impact functions converges, we call its limit macroscopic market impact function.

First we reparametrize in time and consider $(MI^T(f, tT))_{t\in\mathbb{R}^+}$ (we put the function f as parameter of MI^T to insist on the fact that the market impact function depends on the metaorder strategy). Thus $t = 1$ corresponds essentially to the end of the metaorder. Regarding the scaling in space, because in our framework the size of a metaorder is measured relatively to the total volume, which is of order $T\beta^T$ on $[0, T]$, we finally define our rescaled market impact function $\overline{MI}^T$ on $\mathbb{R}^+$ by

$$\overline{MI}^T(f,t) = \frac{MI^T(f,tT)}{T\beta^T} = \overline{PMI}^T(f,t) + \overline{TMI}^T(f,t),$$

with

$$\overline{PMI}^T(f,t) = \gamma \int_0^t f(x)\mathrm{d}x$$

the rescaled permanent impact and

$$\overline{TMI}^T(f,t) = \gamma \frac{a^T(1-a^T)^{-1}}{T} \int_0^{Tt} f(t-x/T) \int_x^{+\infty} \phi(u)\mathrm{d}u\mathrm{d}x$$

the rescaled transient impact. Remark that the permanent impact term does not depend on T. Thus there always exists a macroscopic permanent market impact function and the convergence of the sequence $(\overline{MI}^T(f,\cdot))_{T\geq 0}$ is equivalent to that of $(\overline{TMI}^T(f,\cdot))_{T\geq 0}$. Motivated by the empirical results on market impact [BILL15, Bou10, GW15, LFM03, PB03] discussed in the introduction, we make the following natural assumption.

Assumption 1. *For constant execution rate, that is $f = \mathbf{1}_{[0,s]}$ for some $s \in (0,1]$, the scaling limit of the market impact function exists pointwise and is non-increasing after time s. Furthermore, there exists $t > s$ such that the value of this limiting function at time t is smaller than that at time s.*

There do exist some model parameters such that Assumption 1 is satisfied. For example any kernel ϕ such that $\phi(t) \sim_{+\infty} c\,t^{-\alpha-1}$ with $c > 0$. Assumption 1 implies that for $f = \mathbf{1}_{[0,s]}$ with $s \in (0,1]$, we can define the macroscopic market impact function $\widehat{MI}(f,t)$ and its transient and permanent components $\widehat{TMI}(f,t)$ and $\widehat{PMI}(f,t)$ as

$$\widehat{MI}(f,t) = \lim_{T\to+\infty} \overline{MI}^T(f,t), \quad \widehat{TMI}(f,t) = \lim_{T\to+\infty} \overline{TMI}^T(f,t), \quad \widehat{PMI}(f,t) = \lim_{T\to+\infty} \overline{PMI}^T(f,t).$$

Using Tauberian theorems, see Appendix I.A.2, we obtain the following result.

Theorem 1. *Under Assumption 1, for any non-negative function f defined on $\mathbb{R}^+$, continuous on $[0,1]$ and supported on $[0,1]$, the macroscopic market impact function and its transient part exist. More precisely, there exists a parameter $\alpha \in (0,1]$ such that for any $t > 0$, when $\alpha < 1$,*

$$\lim_{T\to+\infty} \overline{TMI}^T(f,t) = \gamma K(1-\alpha) \int_0^t f(t-u)u^{-\alpha}\mathrm{d}u, \tag{6}$$

for some $K > 0$, and when $\alpha = 1$

$$\lim_{T\to+\infty} \overline{TMI}^T(f,t) = \gamma K f(t). \tag{7}$$

Furthermore, the Hawkes kernel ϕ necessarily satisfies

$$\int_0^t \int_s^{+\infty} \phi(u)\mathrm{d}u\mathrm{d}s = t^{1-\alpha}L(t),$$

where L is a slowly varying function (see definition in Appendix I.A.2). Finally we necessarily have

$$(1-a^T)^{-1}T^{-\alpha}L(T) \to K,$$

and consequently $a^T \to 1$ (see Proposition 7 in Appendix I.A.2).

Considering for example $f = \mathbf{1}_{[0,1]}$, Theorem 1 shows that under no-arbitrage together with the assumption of the existence of the macroscopic market impact function, the transient part of the market impact is power-law while the permanent part is linear. Moreover Equation (6) gives that the decay of the market impact is essentially a power-law with exponent $-\alpha$, see Figure I.1 for illustration.

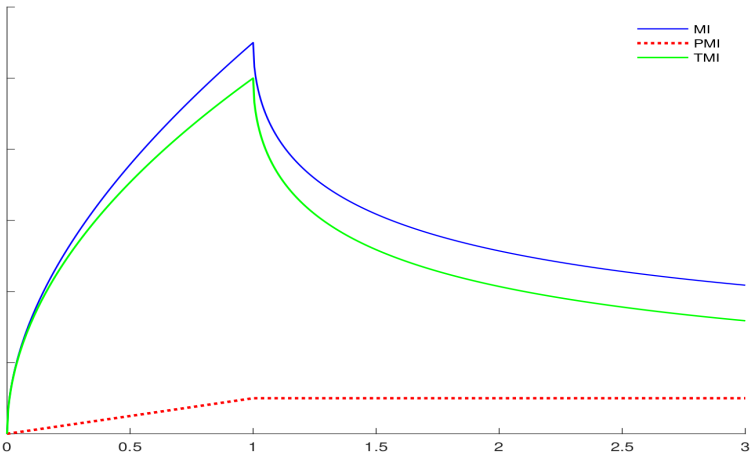

Figure I.1 – Illustration of the decomposition of the macroscopic market impact function for a metaorder executed uniformly over $[0,1]$, with $\alpha = 0.5$. Time is on the $x-$axis.

We see that the connection between a_T and T is completely specified in Theorem 1. For given α, there is only one asymptotic regime leading to a non-trivial limit. Note that the fact that a_T goes to one implies that the non-linear transient part of the market impact (case $\alpha < 1$) can arise only in a highly endogenous market. This is actually very natural for the following reason. A non-linear transient impact means that the market reacts differently to a child order which is in the end of the metaorder compared to a child order in the beginning of the metaorder. For this to be possible, one needs that correlations in the order flow to survive all over the time-length of our metaorder. In our probabilistic setting, using the population approach to Hawkes processes, see [BM16, FS15, JR15, JR16], it is easily seen that such property can hold only provided a_T goes to one.

In this regard the case $\alpha = 1$ is quite degenerate because the market has somehow no memory and reacts the same way to market orders, independently of their position within the metaorder. Even more, the price instantaneously decreases to its permanent level when the metaorder is completed. This means that the market is able to detect instantaneously the end of a metaorder, which seems unrealistic and incompatible with empirical measurements.

3 Macroscopic limit of the price

We finally show in this section that under Assumption 1, the macroscopic price, that is the limit as time goes to infinity of the properly rescaled microscopic price (4), is diffusive with rough or hyper-rough volatility. Moreover, we make explicit the link between the market impact shape exponent and the Hurst parameter of the volatility.

3.1 Scaling limit of the price process

We start with an assumption which is necessary to get a non-trivial long term limit for the price (4).

Assumption 2. *For some $\delta > 0$, we have*

$$(1 - a^T)\mu^T T \underset{T\to+\infty}{\to} \delta.$$

Assumption 2 is classical in the context of Hawkes processes with kernel whose L^1 norm tends to one, see [JR16]. Indeed, it ensures that the number of events does not explode asymptotically.

According to Equation (2), price and volume are homogenous. Therefore we rescale the price the same way as the metaorders. Taking for simplicity and without loss of generality $P_0 = 0$, we define for $t \in [0,1]$

$$\overline{P}_t^T = \frac{1}{T\beta^T} P_{tT}^T = \frac{1 - a^T}{T\mu_T} \int_0^t \xi^T\big(T(t-s)\big)\mathrm{d}\big(N_{tT}^{a,T} - N_{tT}^{b,T}\big),$$

where

$$\xi^T(t) = \big(1 + \int_0^{+\infty} \psi^T(u)\mathrm{d}u\big)\big(1 - \int_0^t \phi^T(u)\mathrm{d}u\big).$$

Let α be the parameter of the market impact function in Theorem 1, K the constant introduced in Equations (6) and (7) and $\lambda = (K\Gamma(2-\alpha))^{-1}$. We have the following result for the macroscopic limit of the price process, whose proof is given in Section 4.2.

Theorem 2. *Under Assumptions 1 and 2, the sequence of rescaled price processes $(\overline{P}^T)_{T\geq 0}$ converges in law for the Skorokhod topology towards a process $\widehat{P}$ such that for $t \in [0,1]$*

$$\widehat{P}_t = \frac{1}{\sqrt{\delta}}\big(B^a_{X^a_t} - B^b_{X^b_t}\big),$$

where B^a and B^b are two independent Brownian motions such that $B^a_{X^a_t}$ and $B^b_{X^b_t}$ are two martingales, X^a is increasing and satisfies

$$X^a_t = \int_0^t F^{\alpha,\lambda}(s)\mathrm{d}s + \frac{1}{\sqrt{\delta\lambda}} \int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}B^a_{X^a_s}$$

and X^b is increasing and solution of the same equation as above replacing the superscript a by b.

In particular, there exists a Brownian motion W such that the integrated variance $X = \left(X^a + X^b\right)/\delta$ of $\widehat{P}$ is solution of the stochastic rough Volterra equation

$$X_t = \frac{2}{\delta}\int_0^t F^{\alpha,\lambda}(s)\mathrm{d}s + \frac{1}{\delta\sqrt{\lambda}}\int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}W_{X_s}. \tag{8}$$

Moreover, for any $\varepsilon > 0$, the process X has Hölder regularity $1 \wedge (2\alpha - \varepsilon)$. It is continuously differentiable for $\alpha > 1/2$ and not continuously differentiable for $\alpha \leq 1/2$.

Theorem 2 shows that the no-arbitrage principle together with the existence of market impact imply that the macroscopic price[2] is a diffusive process whose cumulative variance is solution of a stochastic rough Volterra equation (except when $\alpha = 1$ which corresponds to the classical Heston model, see Corollary 1). Note that X plays the role of an integrated variance and that when $\alpha \leq 1/2$ it is not continuously differentiable. Thus, in that case, the spot variance is not well-defined and only its integrated version makes sense. This is why for $\alpha \leq 1/2$, we call this model hyper-rough volatility model (more precisely hyper-rough Heston model, see below).

From Theorem 3.2 in [JR16], we have that for $\alpha > 1/2$, the process X^a is almost surely differentiable and its derivative Y^a is the unique solution of

$$Y_t^a = \frac{\lambda}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1}(1 - Y_s^a)\mathrm{d}s + \frac{1}{\sqrt{\delta\lambda}}\int_0^t (t-s)^{\alpha-1}\sqrt{Y_s^a}\mathrm{d}B_s^a\Big).$$

The same result holds for Y^b replacing the superscript a by b. We deduce that when $\alpha > 1/2$, the integrated volatility admits a derivative and the macroscopic limit of the price follows a rough Heston model. More precisely, we have the following corollary.

Corollary 1. *When $\alpha > 1/2$, the process X is differentiable almost surely and its derivative Y is the unique solution of the stochastic rough Volterra equation*

$$Y_t = (Y_t^a + Y_t^b)/\delta = \frac{\lambda}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1}(\frac{2}{\delta} - Y_s)\mathrm{d}s + \frac{1}{\delta\sqrt{\lambda}}\int_0^t (t-s)^{\alpha-1}\sqrt{Y_s}\mathrm{d}W_s\Big),$$

with W a Brownian motion. Furthermore the dynamic of the price $\widehat{P}$ is

$$\mathrm{d}\widehat{P}_t = \frac{1}{\sqrt{\delta}}\Big(\sqrt{Y_t^a}\mathrm{d}B_t^a - \sqrt{Y_t^b}\mathrm{d}B_t^b\Big).$$

This result highlights the fact that at the macroscopic limit, the correlation ρ_t between the two Brownian motions driving price and volatility is stochastic. More precisely we have

$$\rho_t = \frac{Y_t^a - Y_t^b}{Y_t^a + Y_t^b}.$$

[2]Remark that under our completely symmetric setting, price can become negative. This is however obviously not very important for our purpose here.

Hence the correlation sign depends on that of $Y_t^a - Y_t^b$. The process Y^a (resp. Y^b) corresponding to the volatility of the ask (resp. bid) side of the market (see Step 4 in Section 4.2), this can be interpreted in terms of order flow dynamics. Indeed suppose that $Y^a \gg Y^b$ and that price is increasing. Then the instantaneous imbalance has the same sign as price returns. Thus the volatility increases as the order flow excites the price dynamic. Conversely, if the price increases and $Y^a \ll Y^b$, the volatility decreases as the order flow tends to compensate the upward price variation.

To prove the convergence in law in Theorem 2, we show that $(\overline{P}^T)_{T\geq 0}$ is tight and that all limit points have the same law. This is done using the characteristic function of Hawkes processes in the spirit of [EER19]. A direct proof would consist in obtaining uniqueness in law for solutions of Equation (8) as done in [AJLP19] for $\alpha > 1/2$. However, such approach seems quite intricate to adapt for $\alpha \leq \frac{1}{2}$. We have the following result whose proof is given in Section 4.3.

Theorem 3. *Let X be the cumulated variance process given in Theorem 2 and h a continuously differentiable function from $\mathbb{R}^+$ to $\mathbb{R}$ such that $h(0) = 0$. The function*

$$\psi(h,t) = \mathbb{E}\big[\exp\big(\int_0^t ih(t-s)\mathrm{d}X_s\big)\big]$$

satisfies

$$\psi(h,t) = \exp\big(\int_0^t g(s)\mathrm{d}s\big),$$

with g the unique continuous solution of the Volterra Riccati equation

$$g(t) = \int_0^t f^{\alpha,\lambda}(t-s)\big(\delta^{-1}\frac{1}{4}g(s)^2 + \delta^{-1}2ih(s)\big)\mathrm{d}s, \tag{9}$$

where $f^{\alpha,\lambda}$ is the Mittag-Leffler density function, see Appendix I.A.1.

Theorem 3 extends some already known results about characteristic functions related to rough Heston models for $\alpha > \frac{1}{2}$, see [EEGR19, EER19, AJLP19]. Note that the characteristic function of the macroscopic price process $\widehat{P}_t$ can also be obtained using the same type of proof as that for Theorem 3.

3.2 Conclusion

We have considered three main assumptions in this work:

- No arbitrage, in the sense that the price process is a martingale and permanent market impact is linear.
- Existence of market impact with a transient component,
- The order flow can be fitted by a Hawkes process (with no restriction on the Hawkes parameters).

We have shown that in this quite general framework, the market impact function can only be a power-law with exponent $1-\alpha$ for some $\alpha \in (0,1)$ (we drop here the case $\alpha = 1$ which leads to a somehow degenerate market impact function). The parameter α also appears necessarily in the tail of the kernel of the Hawkes process driving the order flow: $\phi(x) \sim x^{-(1+\alpha)}$ as x goes to infinity. Furthermore, this also implies that the market is highly endogenous. Even more interestingly, we obtain that the macroscopic behavior of the price is that of a rough or hyper-rough Heston model with Hurst parameter $H = \alpha - 1/2$.

The relationship between market impact, tail of Hawkes kernel and volatility Hurst parameter enables us to confront our results to empirical measurements. In [BILL15] it is found that the market impact function fits a power-law with exponent 0.45. In [GJR18] it is shown that volatility is rough with a Hurst parameter of order 0.1. Finally in [HBB13], the authors calibrate a Hawkes process on market orders arrival and obtain that the kernel decays as a power-law function with exponent around -1.45. All these measurements are compatible with our results (and suggest that market impact is close to square root).

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4 Proofs

4.1 Proof of Theorem 1

Let $f = \mathbf{1}_{[0,s]}$, $s \in (0,1]$. From Assumption 1, we have the pointwise convergence of $(\overline{MI}^T(f,\cdot))_{T\geq 0}$. As previously explained, this is equivalent to the convergence of $(\overline{TMI}^T(f,\cdot))_{T\geq 0}$. Moreover, $(\overline{PMI}^T(f,\cdot))_{T\geq 0}$ being independent of T, Assumption 1 implies that the sequence of functions

$$\overline{TMI}^T(f,t) = \gamma \int_0^t a^T (1-a^T)^{-1} \int_{yT}^{+\infty} \phi(u)\mathrm{d}u f(t-y)\mathrm{d}y, \tag{10}$$

converges pointwise. The function ϕ being non-negative and integrable, $\overline{TMI}^T(f,\cdot)$ is non-negative, non-decreasing and concave on $[0,s]$ and then non-increasing. Hence $\overline{TMI}^T(f,\cdot)$ reaches its maximum in s. By pointwise convergence, $\widehat{TMI}(f,\cdot)$ has the same properties. Because we have assumed that $\widehat{MI}(f,t) < \widehat{MI}(f,s)$ for some $t > s$ and $\widehat{PMI}(f,\cdot)$ is non-decreasing, we deduce that $\widehat{TMI}(f,s) > 0$.

Let $g(t) = \gamma^{-1}\widehat{TMI}(\mathbf{1}_{[0,t]}, t)$ for $t \in (0,1]$ and consider

$$R(t) = \int_0^t \int_y^{+\infty} \phi(u)\mathrm{d}u\mathrm{d}y > 0.$$

According to Equation (10), we have for $t \in (0,1]$

$$\frac{R(Tt)}{R(T)} \underset{T\to+\infty}{\to} \frac{g(t)}{g(1)} > 0. \tag{11}$$

By the characterisation theorem, see Theorem 4 in Appendix I.A.2, we deduce that the previous limit holds for all $t > 0$ with some suitable extension of the function g. Moreover there exist some $\beta \in \mathbb{R}$, $K > 0$ and L a slowly varying function such that for $t > 0$

$$g(t) = Kt^\beta,\ R(t) = L(t)t^\beta.$$

Remark that for $t \in (0,1]$, we have $g(t) = \widehat{TMI}(\mathbf{1}_{[0,1]}, t)$, which is concave. Thus $\beta \in [0,1]$. Taking $s = t = 1$ in the pointwise convergence (10), we get

$$a^T \frac{(1-a^T)^{-1}}{T} \int_0^T \int_y^{+\infty} \phi(u)\mathrm{d}u\mathrm{d}y = a^T(1-a^T)^{-1}T^{\beta-1}L(T) \underset{T\to+\infty}{\to} g(1) = K > 0. \tag{12}$$

Consider now the sequence of functions

$$\overline{\Gamma}^T(y) = a^T(1-a^T)^{-1}\int_{Ty}^{+\infty} \phi(u)\mathrm{d}u.$$

We get from (11), (12) and property of slowly varying function that for any $t > 0$

$$\lim_{T\to+\infty} \int_0^t \overline{\Gamma}^T(y)\mathrm{d}y = Kt^\beta.$$

Suppose that $\beta \neq 0$. Let $0 \le a < b$ and $y \in [a,b]$. We have

$$\lim_{T\to+\infty} \int_a^y \frac{\overline{\Gamma}^T(u)}{\int_a^b \overline{\Gamma}^T(v)\mathrm{d}v}\mathrm{d}u = \frac{y^\beta - a^\beta}{b^\beta - a^\beta}.$$

The right hand side is the cumulative distribution function of a random variable with support on $[a,b]$ whose law is denoted by $m^\beta_{a,b}$. Hence we have the convergence in law

$$\mathbf{1}_{[a,b]} \frac{\overline{\Gamma}^T(u)\mathrm{d}u}{\int_a^b \overline{\Gamma}^T(v)\mathrm{d}v} \underset{T\to+\infty}{\to} m^\beta_{a,b}(\mathrm{d}u).$$

So for any bounded continuous function g on $[a,b]$, we get

$$\lim_{T\to+\infty} \int_a^b \frac{\overline{\Gamma}^T(u)}{\int_a^b \overline{\Gamma}^T(v)\mathrm{d}v} g(u)\mathrm{d}u = \int_a^b g(u) m^\beta_{a,b}(\mathrm{d}u).$$

Consequently,

$$\lim_{T\to+\infty} \int_a^b \overline{\Gamma}^T(u)g(u)\mathrm{d}u = K\int_a^b g(u)m^\beta_{a,b}(\mathrm{d}u)\left(b^\beta - a^\beta\right) = K\beta\int_a^b g(u)u^{\beta-1}\mathrm{d}u.$$

Now let h be a non-negative measurable function defined on $\mathbb{R}^+$, continuous on $[0,1]$ and supported on $[0,1]$. For $t \leq 1$ we have

$$\int_0^t h(t-u)\overline{\Gamma}^T(u)\mathrm{d}u \underset{T\to+\infty}{\to} K\beta \int_0^t h(t-u)u^{\beta-1}\mathrm{d}u$$

and for $t > 1$

$$\int_0^t h(t-u)\overline{\Gamma}^T(u)\mathrm{d}u = \int_{t-1}^t h(t-u)\overline{\Gamma}^T(u)\mathrm{d}u \underset{T\to+\infty}{\to} K\beta \int_{t-1}^t h(t-u)u^{\beta-1}\mathrm{d}u.$$

Finally for any $t \geq 0$

$$\widehat{TMI}(h,t) = \lim_{T\to+\infty} \overline{TMI}^T(h,t) = \gamma K\beta \int_0^t h(t-u)u^{\beta-1}\mathrm{d}u.$$

Thus when $\beta \in (0,1]$, we have the existence of a macroscopic limit for the transient part of the market impact function (and therefore for the market impact function). Remark that for $\beta = 1$

$$\widehat{TMI}(f,t) = \gamma K \int_0^t f(u)\mathrm{d}u.$$

Consequently, in that case, $\widehat{TMI}(\mathbf{1}_{[0,1]},\cdot)$ is a non-decreasing function. This is in contradiction with Assumption 1, hence β cannot be equal to 1.

Suppose that $\beta = 0$. For any $t > 0$ we have

$$\mathbf{1}_{[0,t]} \frac{\overline{\Gamma}^T(u)\mathrm{d}u}{\int_0^t \overline{\Gamma}^T(v)\mathrm{d}v} \underset{T\to+\infty}{\to} \delta_0(\mathrm{d}u),$$

where δ_0 is the Dirac measure in 0. Then for any bounded continuous function g

$$\lim_{T\to+\infty} \int_0^t \frac{\overline{\Gamma}^T(u)}{\int_0^t \overline{\Gamma}^T(v)\mathrm{d}v} g(u)\mathrm{d}u = g(0).$$

Now let f be a non-negative measurable function defined on $\mathbb{R}^+$, continuous on $[0,1]$ and supported on $[0,1]$. For $t \leq 1$ we have

$$\int_0^t f(t-s)\overline{\Gamma}^T(s)\mathrm{d}s \underset{T\to+\infty}{\to} Kf(t)$$

and for $t > 1$

$$0 \leq \int_0^t f(t-s)\overline{\Gamma}^T(s)\mathrm{d}s \leq \int_0^t \tilde{f}(t-s)\overline{\Gamma}^T(s)\mathrm{d}s \underset{T\to+\infty}{\to} 0,$$

with $\tilde{f}$ is a non-negative continuous extension of $f\mathbf{1}_{[0,1]}$ on $\mathbb{R}^+$ supported on $[0, 1+\frac{t-1}{2}]$. Finally for any $t \geq 0$

$$\widehat{TMI}(f,t) = \gamma K f(t).$$

Consequently for $\beta = 0$, we also have the existence of a macroscopic limit for the transient part of the market impact function (and therefore of the market impact function). We obtain the result letting $\alpha = 1 - \beta$.

4.2 Proof of Theorem 2

We proceed in five steps.

1. Step 1: We first prove a preliminary result on the characteristic function of Hawkes processes that we use later in Step 3.
2. Step 2: We rewrite the sequence $(\overline{P}^T)_{T\geq 0}$ in a convenient way.
3. Step 3: We adapt results from [EER19] and [JR16] on scaling limits of nearly unstable heavy-tailed Hawkes processes to our more general framework.
4. Step 4: We deduce from the previous steps the convergence in law for the Skorokhod topology of the sequence $(\overline{P}^T)_{T\geq 0}$ and make explicit the equation satisfied by the limit.
5. Step 5: We prove the results on the regularity of solutions of Equation (8).

For simplicity and without loss of generality we take $P_0 = 0$.

4.2.1 Step 1

We derive a result on the characteristic function of Hawkes processes using similar arguments as those introduced in [EER19]. Recall that the notation $*$ stands for the convolution product on $\mathbb{R}^+$. More precisely for f and g suitable measurable functions and m a measure

$$(f * g)(t) = \int_0^t f(t-s)g(s)\mathrm{d}s$$

and

$$(f * \mathrm{d}m)(t) = \int_0^t f(t-s)m(\mathrm{d}s).$$

We have the following proposition.

Proposition 1. *Let N be a Hawkes process with parameters (ν, ϕ), with ν a locally integrable non-negative function and ϕ a non-negative measurable function such that $\|\phi\|_1 < 1$. For any continuous function h from $\mathbb{R}^+$ into $\mathbb{R}$,*

$$L(h,t) = \mathbb{E}[\exp((ih * \mathrm{d}N)(t))]$$

satisfies

$$L(h,t) = \exp\Big(\int_0^t (C(h,s)-1)\nu(t-s)\mathrm{d}s\Big),$$

where C is solution of the equation

$$C(h,\cdot) = \exp\big(ih + (C(h,\cdot)-1) * \phi\big).$$

Proof. Let $\tilde{N}$ be a Hawkes process with parameters (ϕ,ϕ) and N^0 a Poisson process with intensity ν. Let $(\tilde{N}^j)_{j\in\mathbb{N}^\star}$ be independent copies of $\tilde{N}$, also independent of N^0. Using the population interpretation of Hawkes processes, see Appendix C.1 in [EER18], we deduce the following equality in law

$$N_t \overset{\mathcal{L}}{=} N_t^0 + \sum_{j=1}^{N_t^0} \tilde{N}^j_{t-T_j},$$

where $(T_j)_{j\in\mathbb{N}^\star}$ are the jump times of the process N^0. Consequently

$$(ih * \mathrm{d}N)(t) \overset{\mathcal{L}}{=} (ih * \mathrm{d}N^0)(t) + \sum_{j=1}^{N_t^0} (ih * \mathrm{d}\tilde{N}^j)(t - T_j).$$

Then taking the exponential and conditional expectation with respect to N^0 we get

$$\begin{aligned}\mathbb{E}\big[\exp\big((ih * \mathrm{d}N)(t)\big)|N^0\big] &= \exp\big((ih * \mathrm{d}N^0)_t\big) \prod_{j=1}^{N_t^0} \tilde{L}(h, t - T_j) \\ &= \exp\Big(\big(\big(ih + \log(\tilde{L}(h,\cdot))\big) * \mathrm{d}N^0\big)(t)\Big),\end{aligned}$$

where $\tilde{L}$ is defined as L with $\tilde{N}$ instead of N. Remark that

$$\big(\big(ih + \log(\tilde{L}(h,\cdot))\big) * \mathrm{d}N^0\big)(t) = \sum_{j=1}^{N_t^0} ih(t - T_j) + \log(\tilde{L}(h, t - T_j))$$

and that $Re\big(\log\big(\tilde{L}(h,\cdot)\big)\big) \le 0$ as $|\tilde{L}(h,\cdot)| \le 1$. Thus using Proposition 8 in Appendix I.A.4, we get

$$L(h,t) = \exp\Big(\int_0^t \big(e^{ih(t-s)}\tilde{L}(h, t-s) - 1\big)\nu(s)\mathrm{d}s\Big).$$

In the same way, we have

$$\tilde{L}(h,t) = \exp\Big(\int_0^t \big(e^{ih(t-s)}\tilde{L}(h, t-s) - 1\big)\phi(s)\mathrm{d}s\Big).$$

Thus setting

$$C(h,t) = e^{ih(t)}\tilde{L}(h,t),$$

we obtain

$$L(h,t) = \exp\Big(\int_0^t \big(C(h,s) - 1\big)\nu(t-s)\mathrm{d}s\Big)$$

and

$$C(h,\cdot) = \exp\big(ih + \big(C(h,\cdot) - 1\big) * \phi\big).$$

□

4.2.2 **Step** 2

We consider the price model (4). Let $M^{a,T}$ be defined by

$$M_t^{a,T} = N_t^{a,T} - \int_0^t \lambda_s^{a,T} \mathrm{d}s.$$

We define $M^{b,T}$ the same way replacing the superscript a by b in the above equation. We have the following result.

Lemma 1. *The price process* (4) *can be written as*

$$P_t^T = \big(1 + \int_0^{+\infty} \psi^T(v)\mathrm{d}v\big)(M_t^{a,T} - M_t^{b,T}).$$

Proof. We have

$$P_t^T = \int_0^t \big(1 + \int_0^{+\infty} \psi^T(v)\mathrm{d}v\big)\big(1 - \int_0^{t-u} \phi^T(v)\mathrm{d}v\big)\mathrm{d}(N^{a,T} - N^{b,T})_u.$$

We first deal with the term T_1 defined by

$$\begin{aligned} T_1 &= \int_0^t \big(1 + \int_0^{+\infty} \psi^T(v)\mathrm{d}v\big) \int_0^{t-u} \phi^T(v)\mathrm{d}v\mathrm{d}(N^{a,T} - N^{b,T})_u \\ &= \big(1 + \int_0^{+\infty} \psi^T(v)\mathrm{d}v\big) \int_0^t \int_u^t \phi^T(v-u)\mathrm{d}v\mathrm{d}(N^{a,T} - N^{b,T})_u. \end{aligned}$$

Using Fubini-Tonelli theorem we get

$$T_1 = \big(1 + \int_0^{+\infty} \psi^T(v)\mathrm{d}v\big) \int_0^t \int_0^v \phi^T(v-u)\mathrm{d}(N^{a,T} - N^{b,T})_u \mathrm{d}v.$$

Thus we deduce

$$T_1 = \big(1 + \int_0^{+\infty} \psi^T(v)\mathrm{d}v\big) \int_0^t \big(\lambda_v^{a,T} - \mu - \lambda_v^{b,T} + \mu\big)\mathrm{d}v.$$

Finally

$$\begin{aligned} P_t^T &= \big(1 + \int_0^{+\infty} \psi^T(v)\mathrm{d}v\big) \int_0^t \big(\mathrm{d}N_v^{a,T} - \lambda_v^{a,T}\mathrm{d}v - \mathrm{d}N_v^{b,T} + \lambda_v^{b,T}\mathrm{d}v\big) \\ &= \big(1 + \int_0^{+\infty} \psi^T(v)\mathrm{d}v\big)(M_t^{a,T} - M_t^{b,T}). \end{aligned}$$

□

Lemma 1 leads to

$$\overline{P}_t^T = \frac{1 - a^T}{T\mu^T}\big(1 + \int_0^{+\infty} \psi^T(v)\mathrm{d}v\big)(M_{tT}^{a,T} - M_{tT}^{b,T}).$$

4.2.3 Step 3

We temporarily drop the superscripts a and b. Indeed, the results are valid both for buy and sell order flows. Consider the sequences

$$X_t^T = \frac{1-a^T}{T\mu^T} N_{tT}^T,\ \Lambda_t^T = \frac{1-a^T}{T\mu^T}\int_0^{tT} \lambda_s^T \mathrm{d}s,\ Z_t^T = \sqrt{\frac{T\mu^T}{1-a^T}}\left(X_t^T - \Lambda_t^T\right) \tag{13}$$

defined for $t \in [0,1]$. The following result is borrowed from [JR16].

Proposition 2. *The sequence (Λ^T, X^T, Z^T) is tight. Furthermore, for any limit point (Λ, X, Z) of (Λ^T, X^T, Z^T), Z is a continuous martingale, $[Z,Z] = X$ and $\Lambda = X$.*

In addition, we have the following proposition which extends Theorem 3.1 in [JR16].

Proposition 3. *Under Assumptions 1 and 2, for any limit point (X, Z) of (X^T, Z^T), there exists a Brownian motion B on $(\Omega, \mathscr{A}, \mathbb{P})$ (up to extension of the space) such that*

$$Z_t = B_{X_t}$$

and X is a solution of the stochastic rough Volterra equation

$$X_t = \int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}s + \frac{1}{\sqrt{\delta\lambda}}\int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}B_{X_s}. \tag{14}$$

Moreover, for any $\varepsilon > 0$, the process X has Hölder regularity $1 \wedge (2\alpha - \varepsilon)$.

Note that we are here under more general assumptions than in Theorem 3.1 in [JR16]. Indeed in [JR16] we have

$$\int_t^{+\infty} \phi(s)\mathrm{d}s = Kt^{-\alpha},$$

while we only know that

$$\int_0^t \int_s^{+\infty} \phi(u)\mathrm{d}u\mathrm{d}s = L(t)t^{1-\alpha},$$

with L a slowly varying function. To prove Proposition 3, it is enough to get the following lemma. The rest of the proof is similar to that in [JR16].

Lemma 2. *The sequence of functions $\rho^T(t) = \frac{1-a^T}{a^T}\psi^T(Tt)T$ converges weakly towards $f^{\alpha,\lambda}$. Furthermore $\int_0^t \rho^T(s)\mathrm{d}s$ converges uniformly towards $F^{\alpha,\lambda}$.*

Proof. The function ρ^T is non-negative with integral equal to one. So it can be interpreted as the density of a random variable. Hence it is enough to show that its Laplace transform converges pointwise to get weak convergence. We have for $z > 0$

$$\hat{\rho}^T(z) = \frac{1-a^T}{a^T}\hat{\psi}^T\left(\frac{z}{T}\right) = \frac{\hat{\phi}(\frac{z}{T})}{1 - a^T(1-a^T)^{-1}(\hat{\phi}(\frac{z}{T}) - 1)}.$$

Let

$$R(t) = \int_0^t \int_s^{+\infty} \phi(u) \mathrm{d}u \mathrm{d}s.$$

Recall that from Theorem 1, $R(t) = t^{1-\alpha} L(t)$. By Karamata's Tauberian theorem, see Theorem 5 in Appendix I.A.2, we have

$$\hat{R}(z) \sim_{0^+} z^{\alpha-2} L(\frac{1}{z}) \Gamma(2-\alpha).$$

Integrating by parts twice we obtain

$$\hat{R}(z) = \int_0^{+\infty} e^{-zs} R(s) \mathrm{d}s = \frac{1}{z^2}\big(1 - \hat{\phi}(z)\big).$$

So we get

$$a^T (1-a^T)^{-1} (1 - \hat{\phi}(\frac{z}{T})) \sim_{T\to+\infty} a^T (1-a^T)^{-1} T^{-\alpha} L(T) \frac{L(\frac{T}{z})}{L(T)} z^\alpha \Gamma(2-\alpha).$$

We have shown in Theorem 1 that

$$a^T (1-a^T)^{-1} T^{-\alpha} L(T) \underset{T\to+\infty}{\to} K.$$

The function L being slowly varying, see Appendix I.A.2, we deduce

$$\lim_{T\to+\infty} a^T (1-a^T)^{-1} (1 - \hat{\phi}(\frac{z}{T})) = z^\alpha \Gamma(2-\alpha) K,$$

and finally

$$\lim_{T\to+\infty} \hat{\rho}^T(z) = \frac{1}{1 + K\Gamma(2-\alpha) z^\alpha} = \frac{\lambda}{\lambda + z^\alpha} = \hat{f}^{\alpha,\lambda},$$

with $\lambda = (K\Gamma(2-\alpha))^{-1}$. The uniform convergence in Lemma 2 is obviously deduced from Dini's theorem. □

We finally show that the sequence $(X^T, Z^T)_{T\geq 0}$ converges in law for the Skorokhod topology. We already know that it is tight, so it is enough to prove that all the limit points have the same law.

Let (X, Z) be a limit point of $(X^T, Z^T)_{T\geq 0}$. Using Proposition 3 together with the stochastic Fubini theorem, see [Ver12], we have

$$X_t = \int_0^t f^{\alpha,\lambda}(t-s)\big(s + \frac{1}{\sqrt{\delta\lambda}} Z_s\big) \mathrm{d}s.$$

From Example 42.2 in [SKM93], this leads to

$$D^\alpha X_t + \lambda X_t - \lambda t = \sqrt{\frac{\lambda}{\delta}} Z_t,$$

where D^α is the fractional derivative operator defined in Appendix I.A.3. Thus the law of (X, Z) is uniquely determined by the law of X. Consequently it is enough to prove uniqueness in law for limit points of $(X^T)_{T\geq 0}$ to get convergence in law of $(X^T, Z^T)_{T\geq 0}$. For this we prove that the characteristic function of any limit point X of the sequence $(X^T)_{T\geq 0}$ is a functional of the solution of a fractional Riccati equation. Uniqueness in law is then a consequence from the uniqueness of the solution of this equation.

Proposition 4. *Let X be a limit point of $(X^T)_{T\geq 0}$ and h a continuously differentiable function from $\mathbb{R}^+$ to $\mathbb{R}$ such that $h(0) = 0$. The function*

$$\psi(h, t) = \mathbb{E}[\exp(ih * \mathrm{d}X)_t]$$

satisfies

$$\psi(h, t) = \exp(\int_0^t g(s)\mathrm{d}s),$$

with g the unique continuous solution of the rough Volterra Riccati equation

$$g = f^{\alpha,\lambda} * \big(\delta^{-1}\frac{1}{2}g^2 + ih\big). \tag{15}$$

To show this result, we are inspired by the methodology of [EER19]. However, note again that we are in a more general setting.

Proof. Recall that

$$X_t^T = \frac{1-a^T}{T\mu^T} N_{tT}^T.$$

We introduce the following quantities

$$h^T(t) = \frac{1-a^T}{T\mu^T} h(\frac{t}{T}),\ L^T(h^T, t) = \mathbb{E}[\exp(ih^T * \mathrm{d}N^T)(t)] \text{ and } \psi^T = L^T(h^T, tT).$$

For every T, according to Proposition 1, there exists a function C^T solution of

$$C^T = \exp\big(ih^T + \big(C^T - 1\big) * \phi^T\big)$$

such that

$$L^T(h^T, t) = \exp\Big(\int_0^t \big(C^T(s) - 1\big)\mu^T \mathrm{d}s\Big).$$

Now define the sequence g^T

$$g^T(s) = C^T(sT) - 1.$$

We have

$$\psi^T = \exp\Big(\frac{g^T}{1-a^T} * (T(1-a^T)\mu^T \mathbf{1}_{\mathbb{R}^+})\Big) \text{ and } g^T + 1 = \exp\big(\frac{1-a^T}{T\mu^T} ih + g^T * (T\phi^T(\cdot T))\big). \tag{16}$$

An immediate adaptation of Proposition 6.4. in [EER19] gives that for any $s \in [0, t]$

$$|g^T(s)| \leq c(h)(1-a^T), \tag{17}$$

with $c(h)$ a positive constant depending only on h. Hence for T large enough we have

$$\log(1+g^T) = g^T - \frac{1}{2}(g^T)^2 - \epsilon^T(h,\cdot), \tag{18}$$

with $|\epsilon^T(h,\cdot)| \le c(h)(1-a^T)^3$. According to Equations (16) and (18), we get for every $s \in [0,t]$

$$g^T(s) = \frac{1}{2}g^T(s)^2 + \epsilon^T(h,s) + \frac{1-a^T}{T\mu^T} ih(s) + g^T * \big(T\phi^T(\cdot T)\big)(s).$$

Using that

$$\sum_{i\ge 1} (T\phi^T(\cdot T))^{*i} = T\psi^T(\cdot T),$$

we deduce from Lemma 4.1 in [JR15] that

$$g^T(s) = \big(T\psi^T(\cdot T)\big) * \big(\frac{1}{2}(g^T)^2 + \epsilon^T(h,\cdot) + \frac{1-a^T}{T\mu^T} ih\big)(s) + \frac{1}{2}g^T(s)^2 + \epsilon^T(h,s) + \frac{1-a^T}{T\mu^T} ih(s).$$

Consequently, letting $\theta_T = (1-a^T)^{-1} g^T$

$$\theta_T(s) = \big(T(1-a^T)\psi^T(\cdot T)\big) * \big(\frac{1}{2}\theta_T^2 + \frac{1}{\delta} ih\big)(s) + r_1^T(s),$$

with

$$\begin{aligned} r_1^T(s) &= \big(T(1-a^T)\psi^T(\cdot T)\big) * \big(\epsilon^T(h,\cdot)(1-a^T)^{-2} + (\frac{1}{T(1-a^T)\mu^T} - \delta^{-1}) ih\big)(s) \\ &+ (1-a^T)^{-1}\frac{1}{2}(g^T(s))^2 + (1-a^T)^{-1}\epsilon^T(h,s) + \frac{1}{T\mu^T} ih(s). \end{aligned}$$

Because a^T goes to 1, we know from Lemma 2 that in the sense of weak convergence

$$T(1-a^T)\psi^T(\cdot T) \underset{T\to+\infty}{\to} f^{\alpha,\lambda}.$$

Finally we have

$$\theta_T = f^{\alpha,\lambda} * \big(\frac{1}{2}\theta_T^2 + \frac{1}{\delta} ih\big) + r_1^T + r_2^T,$$

where

$$r_2^T = \big(T(1-a^T)\psi^T(\cdot T) - f^{\alpha,\lambda}\big) * \big(\frac{1}{2}\theta_T^2 + \frac{1}{\delta} ih\big).$$

We now prove that $(r_1^T)_{T\ge 0}$ and $(r_2^T)_{T\ge 0}$ goes to 0 in $C^0([0,t],\mathbb{R})$ for the sup-norm.

Using Assumption 2, the second part of Lemma 2 and Equation (17), we get that $(r_1^T)_{T\ge 0}$ goes to zero in $C^0([0,t],\mathbb{R})$. The sequence $(\theta_T)_{T\ge 0}$ is bounded for the sup-norm according to Equation (17). Moreover according to Lemma 3 (see after the proof) θ_T is differentiable, and $(\theta_T')_{T\ge 0}$ is bounded for the sup-norm. By integration by parts we have

$$r_2^T(t) = \big(\int_0^{\cdot} T(1-a^T)\psi^T(sT)\mathrm{d}s - F^{\alpha,\lambda}\big) * \big(\theta_T'\theta_T + \frac{1}{\delta} ih'\big)(t),$$

where we have used the fact that $\theta_T(0) = 0$ and $h(0) = 0$. We then conclude that $(r_2^T)_{T\geq 0}$ converges towards 0 in $C^0([0,t],\mathbb{R})$ using dominated convergence. Lemma 3 together with the Ascoli theorem gives that the sequence $(\theta_T)_{T\geq 0}$ is relatively compact in $\big(C^0([0,t],\mathbb{R}), \|\ \|_\infty\big)$. Moreover for any limit point θ of the sequence $(\theta_T)_{T\geq 0}$, we have that θ is solution of

$$\theta = f^{\alpha,\lambda} * \big(\frac{1}{2}\theta^2 + \frac{1}{\delta} ih\big).$$

The above equation has a unique continuous solution in $C^0([0,t],\mathbb{R})$, see Section 6.2.4 in [EER19]. Thus the sequence $(\theta_T)_{T\geq 0}$ converges toward this solution.

Finally remark that

$$\psi^T(t) = \mathbb{E}[\exp(ih * \mathrm{d}X^T)(t)].$$

Thus convergence in law of $(X^T)_{T\geq 0}$ towards X implies that $(\psi^T)_{T\geq 0}$ converges pointwise towards the function ψ. Passing to the limit in (16) we get

$$\psi(t) = \exp\Big((\theta * (\delta \mathbf{1}_{\mathbb{R}^+}))_t\Big) = \exp\Big(\delta \int_0^t \theta(s)\mathrm{d}s\Big).$$

Letting $g = \delta\theta$, we have the result. □

It is enough to characterize the law of X to know $\psi(h,t)$ for any $t \in \mathbb{R}^+$ and $h \in C_0^1([0,t],\mathbb{R})$. Therefore uniqueness in law for the limit points of $(X^T)_{T\geq 0}$ is a corollary from the uniqueness of continuous solution for the Volterra Riccati Equation (15), see Section 6.2.4 in [EER19].

We now give the lemma we used in the proof of Proposition 4.

Lemma 3. *The functions $(\theta_T)_{T\geq 0}$ are continuously differentiable and $(\theta_T')_{T\geq 0}$ is bounded in $C^0([0,t],\mathbb{R})$.*

Proof. Using the proof of Proposition 1 we have

$$\theta_T = (1-a^T)^{-1}\Big(\mathbb{E}[\exp\big((ih + ih * \mathrm{d}\tilde{N}_{\cdot T}^T)\frac{1-a^T}{T\mu^T}\big)] - 1\Big),$$

with $\tilde{N}$ a Hawkes processes with parameters (ϕ^T, ϕ^T) where $\phi^T = a^T\phi$. Because $h(0) = 0$, $h * \mathrm{d}\tilde{N}_{\cdot T}^T$ admits a derivative and for any $s \in [0,t]$

$$(h * \mathrm{d}\tilde{N}_{\cdot T}^T)'(s) = (h' * \mathrm{d}\tilde{N}_{\cdot T}^T)(s).$$

Furthermore we have

$$|(h' * \mathrm{d}\tilde{N}_{\cdot T}^T)(s)| \leq \|h'\|_\infty \tilde{N}_{tT}^T.$$

Using that

$$\tilde{\lambda}_s^T = \psi^T(s) + \int_0^t \psi^T(t-s)\mathrm{d}\tilde{M}_s^T,$$

we get

$$(1-a^T)\mathbb{E}[\tilde{N}^T_{tT}] \le (1-a^T)\mathbb{E}[\int_0^{tT} \tilde{\lambda}^T_s \mathrm{d}s] \le \int_0^{tT} (1-a^T)\psi^T(s)\mathrm{d}s \le 1.$$

Consequently using derivation for integral with parameters, θ^T is differentiable and

$$\theta'_T = (1-a^T)^{-1}\mathbb{E}[\big(ih' + ih' * \mathrm{d}\tilde{N}^T_{\cdot T}\big)\frac{1-a^T}{T\mu^T}\exp\big((ih + ih * \mathrm{d}\tilde{N}^T_{\cdot T})\frac{1-a^T}{T\mu^T}\big)].$$

Thus we have for all $s \in [0,t]$

$$|\theta'_T(s)| \le \frac{1}{T\mu^T(1-a^T)}(1-a^T)\mathbb{E}[\|h'\|_\infty + \|h'\|_\infty \tilde{N}^T_{tT}].$$

The right hand side is finite and independent of s, consequently the sequence $(\theta'_T)_{T\ge 0}$ is bounded in $C^0([0,t],\mathbb{R})$. □

Finally we have proved that the sequence $(X^T, Z^T)_{T\ge 0}$ converges in law for the Skorokhod topology.

4.2.4 Step 4

Consider the sequence $(X^{a,T}, Z^{a,T})_{T\ge 0}$ (resp. $(X^{b,T}, Z^{b,T})_{T\ge 0}$) defined the same way as in Equation (13) with $(N^{a,T})_{T\ge 0}$ (resp. $(N^{b,T})_{T\ge 0}$) instead of $(N^T)_{T\ge 0}$. According to Lemma 1 we have

$$\overline{P}^T_t = \frac{1-a^T}{T\mu^T}\big(1 + \int_0^{+\infty} \psi^T(u)\mathrm{d}u\big)\big(M^{a,T}_{tT} - M^{b,T}_{tT}\big) = \frac{1}{T\mu^T}\big(M^{a,T}_{tT} - M^{b,T}_{tT}\big).$$

Thus,

$$\overline{P}^T_t = \frac{1}{\sqrt{T\mu^T(1-a^T)}}\big(Z^{a,T}_t - Z^{b,T}_t\big).$$

Using Step 3, we get that $(Z^{a,T})_{T\ge 0}$, and $(Z^{b,T})_{T\ge 0}$ converge for the Skorohod topology. These sequences being independent, $(\overline{P}^T)_{T\ge 0}$ converges towards a process $\widehat{P}$ in the Skorokhod topology. Furthermore we deduce from Proposition 3 together with Assumption 2 that there exist two independent Brownian motions B^a and B^b such that

$$\widehat{P}_t = \frac{1}{\sqrt{\delta}}\big(B^a_{X^a_t} - B^b_{X^b_t}\big),$$

where X^a (resp. X^b) is the limit of the sequence $(X^{a,T})_{T\ge 0}$ (resp. $(X^{a,T})_{T\ge 0}$) and is solution of Equation (14) with Brownian motion B^a (resp. B^b). Hence $X = \frac{X^a + X^b}{\delta}$ is solution of

$$X_t = \frac{2}{\delta}\int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}s + \frac{1}{\delta\sqrt{\lambda}}\int_0^t F^{\alpha,\lambda}(t-s)\frac{1}{\sqrt{\delta}}\mathrm{d}\big(B^a_{X^a_s} + B^b_{X^b_s}\big).$$

Moreover there exists a Brownian motion W such that $W_{X_t} = \frac{1}{\sqrt{\delta}}(B^a_{X^a_t} + B^b_{X^b_t})$. Consequently

$$X_t = \frac{2}{\delta}\int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}s + \frac{1}{\delta\sqrt{\lambda}}\int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}W_{X_s}.$$

4.2.5 Step 5

We first recall a result from [JR16].

Proposition 5. *Let X be a solution of the stochastic Volterra equation* (8). *Then for any $\varepsilon > 0$, almost surely, X has Hölder regularity $1 \wedge (2\alpha - \varepsilon)$. And if $\alpha > 1/2$, X is almost surely differentiable.*

We now give a new result on the regularity of the solution of Equation (8).

Proposition 6. *Let $\alpha \leq \frac{1}{2}$. Let X be a solution of the stochastic Volterra equation* (8). *Then, almost surely, X is not continuously differentiable.*

Proof. As already seen in Step 3, X satisfies

$$D^\alpha X_t = -\lambda X_t + \frac{2\lambda}{\delta} t + \frac{\sqrt{\lambda}}{\delta} W_{X_t}. \tag{19}$$

Applying the law of iterated logarithm we get for $0 \leq t \leq 1$

$$\limsup_{s \to t^-} \frac{D^\alpha X_t - D^\alpha X_s - \frac{2\lambda}{\delta}(t-s)}{\sqrt{2(X_t - X_s)\log\log\left((X_t - X_s)^{-1}\right)}} = \frac{\sqrt{\lambda}}{\delta}.$$

Assume that X is continuously differentiable. According to Appendix I.A.3 we have

$$D^\alpha X_t = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} X'_s \mathrm{d}s.$$

Let t be such that $X'_t \neq 0$. Such a point almost surely exists because X is not constant. Indeed suppose it is constant, as $X_0 = 0$ it implies that $X = 0$. But obviously the null function is not solution of Equation (19). For such t using that

$$X_t - X_s \sim_{s \to t} (t-s) X'_t,$$

we have

$$\lim_{s \to t^-} \frac{t-s}{\sqrt{2(X_t - X_s)\log\log\left[(X_t - X_s)^{-1}\right]}} = 0.$$

Hence

$$\limsup_{s \to t^-} \frac{D^\alpha X_t - D^\alpha X_s}{\sqrt{2(X_t - X_s)\log\log\left[(X_t - X_s)^{-1}\right]}} = \frac{\sqrt{\lambda}}{\delta}. \tag{20}$$

We now give a bound on $|D^\alpha X_t - D^\alpha X_s|$, for $s < t$, where $\|X'\|_\infty$ denotes the supremum norm of X'

$$\begin{aligned}
|D^\alpha X_t - D^\alpha X_s| &= |\int_0^s \big((t-u)^{-\alpha} - (s-u)^{-\alpha}\big) X'_s \mathrm{d}u + \int_s^t (t-u)^{-\alpha} X'_u \mathrm{d}u| \\
&\leq \int_0^s \big|(t-u)^{-\alpha} - (s-u)^{-\alpha}\big| \, \|X'\|_\infty \mathrm{d}u + \frac{\|X'\|_\infty}{1-\alpha}(t-s)^{1-\alpha} \\
&\leq \left|\int_0^s (t-u)^{-\alpha} - (s-u)^{-\alpha}\right| \mathrm{d}u \|X'\|_\infty + \frac{\|X'\|_\infty}{1-\alpha}(t-s)^{1-\alpha} \\
&\leq \big(\int_0^{t-s} u^{-\alpha}\mathrm{d}u + \int_s^t u^{-\alpha}\mathrm{d}u\big)\|X'\|_\infty + \frac{\|X'\|_\infty}{1-\alpha}(t-s)^{1-\alpha} \\
&\leq \big(\frac{1}{1-\alpha}(t-s)^{1-\alpha} + (t-s)s^{-\alpha}\big)\|X'\|_\infty + \frac{\|X'\|_\infty}{1-\alpha}(t-s)^{1-\alpha}.
\end{aligned}$$

We get

$$\lim_{s\to t^-} \frac{D^\alpha X_t - D^\alpha X_s}{\sqrt{2(X_t - X_s)\log\log\big[(X_t - X_s)^{-1}\big]}} = 0.$$

This is in contradiction with Equation (20), hence X cannot be continuously differentiable. □

4.3 Proof of Theorem 3

We have seen in Section 4.2.4 that $X = (X^a + X^b)/\delta$, with X^a and X^b independent copies of the limit of the sequence $(X^T)_{T\geq 0}$. From Proposition 4, we immediately obtain Theorem 3.

I.A Appendix

I.A.1 Mittag-Leffler functions

Let $(\alpha, \beta) \in (\mathbb{R}_\star^+)^2$. The Mittag-Leffler function $E_{\alpha,\beta}$ is defined for $z \in \mathbb{C}$ by

$$E_{\alpha,\beta}(z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(\alpha n + \beta)}.$$

For $(\alpha, \lambda) \in (0,1] \times \mathbb{R}^+$, we also define

$$\begin{aligned}
f^{\alpha,\lambda}(t) &= \lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha),\ t > 0, \\
F^{\alpha,\lambda}(t) &= \int_0^t f^{\alpha,\lambda}(s)\mathrm{d}s,\ t \geq 0.
\end{aligned}$$

The function $f^{\alpha,\lambda}$ is a density function on $\mathbb{R}_+$ called the Mittag-Leffler density function. Its Laplace transform is

$$\hat{f}^{\alpha,\lambda}(z) = \frac{\lambda}{\lambda + z^\alpha}.$$

When $\alpha = 1$, the Mittag-Leffler density simply corresponds to the exponential law with parameter λ.

I.A.2 Tauberian theorems

The following results can be found in [BGT89].

Definition 1. *A measurable function $L:\mathbb{R}^+ \to \mathbb{R}$ is slowly varying if for all $s>0$*

$$\frac{L(st)}{L(t)} \underset{t\to+\infty}{\to} 1.$$

Proposition 7. *Let L be a slowly varying function and $\alpha>0$, then*

$$t^{-\alpha}L(t) \underset{t\to+\infty}{\to} 0.$$

Theorem 4. *(Characterisation theorem) Let U be a positive measurable function on $\mathbb{R}_+$ such that for all $s\in C$, with C a set with positive Lebesgue measure*

$$\frac{U(ts)}{U(t)} \underset{t\to+\infty}{\to} g(s)>0,$$

for some function g. Then the previous limit can be extended for all $s>0$. Let $\tilde{g}$ be this limiting function extending g. There exist $\alpha\in\mathbb{R}$ such that $g(t)=t^\alpha$ and a slowly varying function L such that $U(t)=t^\alpha L(t)$.

Theorem 5. *(Karamata's Tauberian theorem) Let U be a measurable non-negative function, $c\geq 0$, $\rho>-1$ and assume $\hat{U}(z)=\int_0^{+\infty} e^{-zs}U(s)\mathrm{d}s$ is finite for any $z>0$. Then*

$$U(t)\sim_{+\infty} ct^\rho \frac{L(t)}{\Gamma(1+\rho)}$$

for L a slowly varying function implies

$$\hat{U}(z)\sim_{0^+} cz^{-\rho-1}L(\frac{1}{z}).$$

I.A.3 Fractional derivative

For $\alpha\in[0,1)$, the fractional derivative operator D^α is defined for f λ-Hölder function (with $\lambda>\alpha$) by

$$D^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha}f(s)\mathrm{d}s.$$

Note that if the function f is continuously differentiable and $f(0)=0$. The derivation for integral with parameters gives

$$D^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f'(s)\mathrm{d}s.$$

More information on fractional differential operator can be found in [SKM93].

I.A.4 A result on inhomogenous Poisson process

We recall the following well known result.

Proposition 8. *(Exponential formula) Let N be an inhomogenous Poisson process with intensity ν and f be a complex measurable function defined on $\mathbb{R}^+$ such that $Re(f) \leq 0$. Consider the function*

$$N_f(t) = \sum_{i=1}^{N_t} f(T_i),$$

where $(T_i)_{i\in\mathbb{N}}$ are the jump times of N. For any $t \geq 0$ we have

$$\mathbb{E}[\exp\big(N_f(t)\big)] = \exp\Big(\int_0^t (e^{f(s)} - 1)\nu(s)\mathrm{d}s\Big).$$

CHAPTER II

From quadratic Hawkes processes to super-Heston rough volatility models with Zumbach effect

Abstract

Using microscopic price models based on Hawkes processes, it has been shown that under some no-arbitrage condition, the high degree of endogeneity of markets together with the phenomenon of metaorders splitting generate rough Heston-type volatility at the macroscopic scale. One additional important feature of financial dynamics, at the heart of several influential works in econophysics, is the so-called feedback or Zumbach effect. This essentially means that past trends in returns convey significant information on future volatility. A natural way to reproduce this property in microstructure modeling is to use quadratic versions of Hawkes processes. We show that after suitable rescaling, the long term limits of these processes are refined versions of rough Heston models where the volatility coefficient is enhanced compared to the square root characterizing Heston-type dynamics. Furthermore the Zumbach effect remains explicit in these limiting rough volatility models.

1 Introduction

Since the paper [GJR18], it has been well accepted that volatility is rough. This means that log-volatility essentially behaves as fractional Brownian motion with Hurst parameter of order 0.1, see also for example [BLP16, DFZ19, GH18, LMPR18]. There are microstructural foundations for rough volatility that use Hawkes processes to create a microscopic model for asset prices. In this vein, the authors in [EEFR18] consider four stylized facts concerning market microstructure: the high degree of endogeneity of markets, the no-arbitrage property, buying/selling asymmetry and the long memory of the market order flow generated by metaorders. They show that when only the three first stylized facts are taken into account, one obtains the Heston model for the scaling limit of the price process. When the long memory property of the flow is added, the limit is the rough Heston model introduced and developed in [EER18, EER19]. In the rough

Heston model, the spot variance V_t can be written as follows:

$$V_t = V_0 + \frac{\lambda}{\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} \big(\theta_0(s) - V_s\big) \mathrm{d}s + \nu \sqrt{V_s} \mathrm{d}B_s, \tag{1}$$

where λ and ν are some positive constants, θ_0 is a deterministic function, $\alpha \in (1/2, 1)$ and B is a Brownian motion. The rough behavior is due to the singular kernel $(t-s)^{\alpha-1}$ which is the same as that appearing in the Mandelbrot-van Ness representation of a fractional Brownian motion with Hurst parameter $\alpha - 1/2$. More recently, assuming only that the order flow is driven by a linear Hawkes process and that there is no statistical arbitrage on the market, it is shown in [JR18] that the price necessarily follows a rough Heston model. In fact, as far as we know, all the works on microstructural foundations of rough volatility have hitherto produced a rough Heston model.

However, in the context of rough models, there are other aspects of volatility that one could wish to understand from a microstructural perspective. A first point is to go beyond the square root associated to the dynamic of the volatility in the rough Heston model (1). A particularly interesting case is when an additional additive or multiplicative factor appears, enhancing the square root and leading to fatter volatility tails, see [AJEE19, BDB17]. We call such models *super-Heston rough volatility models*.

Another important stylized fact of financial time series is the feedback of price returns on volatility. This phenomenon is introduced by Zumbach in [Zum10] where he measures the impact of price trends on future volatility, see also [LZ03, Zum09]. It is demonstrated that price trends induce an increase of volatility. We refer to this property as Zumbach effect. In the literature, see notably [CB12], a way to reinterpret the Zumbach effect is to consider that the predictive power of past squared returns on future volatility is stronger than that of past volatility on future squared returns. To check this on data, one typically shows that the covariance between past squared price returns and future realized volatility (over a given duration) is larger than that between past realized volatility and future squared price returns, see [BDB17, CB12, EEGRR20] for more details. We refer to this version of Zumbach effect as *weak Zumbach effect*.

It has been proved in [EEGRR20] that the rough Heston model reproduces the weak form of Zumbach effect. However, it is not obtained through feedback effect, which is the motivating phenomenon in the original paper by Zumbach [Zum10]. It is only due to the dependence between price and volatility created by the correlation of the Brownian motions driving their dynamics. In particular in the rough Heston model, the conditional law of the volatility depends on the past dynamic of the price only through the past volatility, see [EER18]. From now on, we speak about *strong Zumbach effect* when the conditional law of future volatility depends not only on past volatility trajectory but also on past returns.

Inspired by the methodology of [EEFR18], our goal in this paper is to propose microstructural foundations for the strong Zumbach effect. We also wish to obtain models such that the

instantaneous volatility of variance is equal to the classical square root term of Heston like models multiplied by a non-trivial process, in order to enhance volatility tails. Any model satisfying the latter property will be called a super-Heston model.

A convenient way to build a microscopic model, encoding Zumbach effect and leading naturally to super-Heston rough volatility, is to use a quadratic Hawkes based price process, in the spirit of [BDB17]. More precisely, we consider the following microstructural model for the price $(P_t)_{t\geq 0}$: it is piecewise constant with sizes of price jumps independent and identically distributed taking values 1 or -1 with probability $1/2$. The jump times are those of a counting process N. We assume that N is a quadratic Hawkes type process as introduced in [BDB17, Oga81]. This means the intensity $(\lambda_t)_{t\geq 0}$ of N is given by

$$\lambda_t = \mu + \int_0^t \phi(t-s)\mathrm{d}N_s + Z_t^2, \text{ with } Z_t = \int_0^t k(t-s)\mathrm{d}P_s, \tag{2}$$

where ϕ and k are two non-negative measurable functions supported on $\mathbb{R}_+$ and $\mu > 0$. In the definition of the intensity, the linear term with kernel ϕ enables us to model the self-exciting nature of the order flow. The component Z_t is a moving average of past returns. It can be thought of as a proxy for price return over a given time horizon. If the price has been essentially trending in the past, Z_t is large leading to high intensity. On the contrary if it has been oscillating, Z_t is close to zero and there is low feedback from the returns on the intensity. Hence Z_t can obviously be understood as a (strong) Zumbach term. Note that of course one can think that positive and negative price trends have different impact on the volatility. However for simplicity we neglect this asymmetry in this paper. Finally recall that the stability condition for Model (2) is $\|\phi\|_1 + \|k\|_2^2$ being strictly smaller than one, see [BDB17].

Remark that if we forget the quadratic term Z_t^2 in the intensity, we are left with a linear Hawkes process just as in [JR16]. In this case, at the scaling limit, if the kernel ϕ is heavy tailed and if we are near instability, meaning $\|\phi\|_1$ tends to one with the time parameter driving the asymptotic, the rescaled intensity process converges in law to a rough dynamic similar to (1), see [EEFR18, JR16]. When the kernel norm $\|\phi\|_1$ is fixed and strictly smaller than one, a deterministic limiting model is obtained. Thus we see that being in the near instability regime is crucial so that roughness can arise from the kernel ϕ. Recall that this regime corresponds to a high degree of endogeneity of the market, see [FS12b, HBB13, JR15, JR16]

In [BDB17], the authors study the long term behavior of the intensity of quadratic Hawkes processes. That is, on the time horizon $[0, T]$, letting T tend to infinity, they are interested in the limiting dynamic of $(\lambda_{tT})_{t\in[0,1]}$, which can be viewed as the macroscopic (squared) volatility. They work in a setting where $\|\phi\|_1 + \|k\|_2^2 = 2\gamma < 1$ is fixed, not depending on T. Based on PDE techniques, they obtain a diffusion process with power-law marginal distributions and strong Zumbach effect for the asymptotic volatility. More precisely, their limiting model $(\hat{P}_t, V_t)$ for

price and volatility writes as follows: $\mathrm{d}\hat{P}_t = \sqrt{V_t}\mathrm{d}B_t$ with

$$V_t = \mu + (Z_t)^2 + \int_0^t \gamma\beta e^{-\beta(t-s)} V_s \mathrm{d}s$$
$$Z_t = \int_0^t \sqrt{\gamma\alpha} e^{-(t-s)\alpha/2} \mathrm{d}\hat{P}_s,$$

with B a Brownian motion and α, β some positive parameters defining the functions ϕ and k taken exponential in [BDB17].

In this paper, we wish to go beyond the case treated in [BDB17] from which we draw inspiration. We describe further relevant limiting price dynamics that can be generated from quadratic Hawkes processes. We focus on finding microscopic basis for super-Heston rough volatility processes with strong Zumbach effect. Our goal is to establish connections between micro-parameters of the quadratic Hawkes dynamic and macro-phenomena such as the roughness of the volatility and the strong Zumbach effect.

We first focus in Section 2 on the purely quadratic case, that is when ϕ is equal to zero. Choosing appropriate scaling parameters, we obtain the following limiting model: $\mathrm{d}\hat{P}_t = \sqrt{V_t}\mathrm{d}B_t$ with

$$V_t = \mu + Z_t^2 \tag{3}$$
$$Z_t = \sqrt{\gamma}\int_0^t k(t-s)\mathrm{d}\hat{P}_s,$$

where $\gamma \in (0,1)$ is related to the scaling of the kernel k. In contrast to the purely linear case, we do not need any sort of near instability here so that a stochastic volatility model arises at the scaling limit. In (3) the strong Zumbach effect is naturally encoded since the volatility is a functional of past price returns through Z. We also have that the quadratic feedback of price returns on volatility implies that V_t is of super-Heston type (essentially log-normal here). This can be seen for instance when $\mu = 0$ where we get

$$Z_t = \sqrt{\gamma}\int_0^t k(t-s)|Z_s|\mathrm{d}B_s.$$

Moreover taking for example $k = f^{H+1/2,\lambda}$ for $H \in (0,1/2)$ and $\lambda > 0$ with $f^{\alpha,\lambda}$ the Mittag-Leffler function[1], we get that the volatility has Hölder regularity $H-\varepsilon$ for any $\varepsilon > 0$. Thus, from a natural microscopic dynamic, we are able to obtain a super-Heston rough volatility model with strong Zumbach effect at the macroscopic limit.

We then investigate the limiting models arising from quadratic Hawkes processes with non-vanishing linear part. Knowing that roughness can be obtained from the linear part only in the near instability regime, we treat separately this case and the stable one. We consider in

[1]See [EER19] for a reminder and connections with the Mandelbrot-van Ness representation of fractional Brownian motion.

Section 3 the situation where the stability condition is not asymptotically violated. The result is similar to (3) up to the addition of a drift term $\beta \int_0^t \phi(t-s) V_s \mathrm{d}s$ in the dynamic of V_t, where β is a constant related to the scaling procedure.

We study the nearly unstable case where the L^1[2] norm of the kernel driving the linear part tends to one in Section 4. Assuming $\phi(x)$ behaves as $x^{-(1+\alpha)}$, $\alpha \in (1/2, 1)$, when x goes to infinity, we prove that the following dynamic arises at the scaling limit: $\mathrm{d}\hat{P}_t = \sqrt{V_t}\mathrm{d}B_t^{(1)}$ with

$$\begin{aligned} V_t &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda \big(\theta^0(s) + Z_s^2 - V_s\big) \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda \eta \sqrt{V_s} \mathrm{d}B_s^{(2)} \\ Z_t &= \int_0^t k(t-s) \sqrt{V_s} \mathrm{d}B_s^{(1)}, \end{aligned} \tag{4}$$

with λ, η some positive constants, θ^0 a deterministic function and $(B^{(1)}, B^{(2)})$ two independent Brownian motions. As in the linear case, the near instability condition leads to appearance of a second Brownian motion driving a rough Heston type term. We see that the strong Zumbach effect is still reproduced thanks to the Z_t^2 term which is here convolved with a power-law kernel. Interestingly, we also show that when k is regular, the $\mathrm{d}s$ term is proportional (up to a finite variation term) to $\int_0^t h(t-s) Z_s \mathrm{d}Z_s$, where h is a deterministic function with $h(0) < +\infty$. This can be interpreted as an essentially log-normal (non-rough) component, allowing us to view (4) as a super-Heston rough volatility model.

2 Asymptotic behavior of purely quadratic Hawkes models

In this section we investigate the possible scaling limits of purely quadratic Hawkes based price processes. This corresponds to (2) with $\phi = 0$. We devote a specific section to this case since it enables us to convey some of our main ideas in a simplified setting. More precisely, we consider $(N^T)_{T \geq 0}$ with intensity given by

$$\lambda_t^T = \mu_T + (Z_t^T)^2, \text{ with } Z_t^T = \int_0^t k_T(t-s) \mathrm{d}P_s^T. \tag{5}$$

For any T, the existence of the process (N^T, P^T) can be obtained from [Jac75]. We are interested in the long time behavior of the price P^T and of its intensity λ^T. Before stating the main result of this section we first discuss (in a non-rigorous manner) our scaling procedure.

2.1 Scaling procedure

The scaling procedure consists in finding appropriate factors ω_T so that the sequence $\omega_T \lambda_{tT}^T$ converges towards a non-degenerate limit. Assume $\omega_T \lambda_{tT}^T$ converges towards some process V_t. Since $[P^T]_t = N_t^T$, we have

$$\langle P^T \rangle_{Tt} = \int_0^t T \lambda_{Ts}^T \mathrm{d}s.$$

[2] We use the notation L^p without reference to the underlying domain when no confusion is possible.

Thus we expect the martingale $P_t^{*T} = \sqrt{\frac{\omega_T}{T}} P_{tT}^T$ to converge since its bracket does. Let P be its limit. Since we wish to get P continuous, we need ω_T/T to go to zero. From the convergence of $(\sqrt{\omega_T} Z_{tT}^T)^2$, we expect that of

$$\sqrt{\omega_T} Z_{tT}^T = \int_0^t k_T\big(T(t-s)\big)\sqrt{T}\mathrm{d}P_s^{*T}$$

too, which requires $k_T(T\cdot)\sqrt{T}$ to converge. This leads us to consider, as in [BDB17], a sequence of kernels k_T of the form

$$k_T = k(\cdot/T)\sqrt{\gamma/T}$$

for some $\gamma > 0$ and $\omega_T = 1$ (since we observe that ω_T plays eventually no role). Finally passing to the limit in (5) we obtain the following candidate for our limiting process:

$$V_t = \mu + Z_t^2, \text{ with } Z_t = \int_0^t k(t-s)\mathrm{d}P_s.$$

2.2 Assumptions and results in the purely quadratic case

We now give our exact assumptions, the second of them being purely technical.

Assumption 1.

i) The sequence of kernels $(k_T)_{T\geq 0}$ is given by

$$k_T = \sqrt{\frac{\gamma}{T}} k(\frac{\cdot}{T}),$$

with $\gamma \in (0,1)$ and k a non-negative measurable function such that $\|k\|_2 = 1$. Furthermore $\mu_T = \mu > 0$.

ii) The function k belongs to $L^{2+\varepsilon}$ for some $\varepsilon > 0$ and for any $0 \leq t < t' \leq 1$,

$$\int_0^t |k(t'-s) - k(t-s)|^2 \mathrm{d}s < C|t'-t|^r,$$

for some $r > 0$ and $C > 0$ and

$$\frac{1}{\eta}\int_0^1 |k(t)|^2 t^{-2\eta}\mathrm{d}t + \int_0^1\int_0^1 \frac{|k(t)-k(s)|^2}{|t-s|^{1+2\eta}}\mathrm{d}s\mathrm{d}t < +\infty$$

for some $\eta \in (0,1)$.

Note that for $\alpha \in (1/2, 1)$ and $\lambda > 0$, the Mittag-Leffler function $f^{\alpha,\lambda}$ satisfies Assumption 1 *ii)* for any $\varepsilon \in \big(0, (2\alpha-1)/(1-\alpha)\big)$, $\eta \in (0, \alpha - 1/2)$ and $r = 2\alpha - 1$.

Under Assumption 1, for any T, we have $\|k_T\|_2^2 = \gamma < 1$. So the stability condition is not violated at the limit. We now state the main result of this section. Consider the rescaled processes

$$X_t^T = \frac{N_{tT}^T}{T} \text{ and } P_t^{*T} = \frac{1}{\sqrt{T}} P_{tT}^T.$$

We have the following theorem.

Theorem 1. *Under Assumption 1, as T goes to infinity, the sequence $(X^T, P^{*T})_{T\geq 0}$ converges in law for the Skorohod topology on $[0,1]$ towards some processes (X,P) satisfying the following properties:*

- *X is almost surely continuously differentiable.*
- *There exists a Brownian motion B such that*

$$P_t = \int_0^t \sqrt{V_s} \mathrm{d}B_s,$$

where V is the derivative of X and the unique continuous solution of

$$V_t = \mu + Z_t^2, \quad Z_t = \int_0^t \sqrt{\gamma} k(t-s) \sqrt{V_s} \mathrm{d}B_s, \text{ on } [0,1]. \tag{6}$$

- *For any $\varepsilon > 0$, if $k = f^{H+1/2,\lambda}$ with $H \in (0,1/2)$ and $\lambda > 0$, V has almost surely $H-\varepsilon$ Hölder regularity.*

Theorem 1 will be generalized in Section 3 and its proof is given in Section 5.1.

2.3 Discussion of Theorem 1

• From Theorem 1, we see that we do not need to be in the near instability regime $\|k_T\|_2^2 + \|\phi_T\|_1 \to 1$ in order to obtain a stochastic model at the scaling limit. This is actually not very surprising since quadratic Hawkes models share many similarities with GARCH and QARCH models, see [Eng82, EB86, Sen95]. It is well known that GARCH like processes lead to stochastic volatility at the scaling limit without any degeneracy in their parameters, see [Nel90].

• However, in the limiting model (6), volatility and price are driven by the same Brownian motion B. This is in contrast to the GARCH case or to that of nearly unstable Hawkes processes where a new Brownian motion appears in the volatility dynamic, see [EEFR18]. Compared to the GARCH situation, the difference essentially lies in the very constrained law of the returns here.

• The Zumbach effect is obviously present in the limiting model: the volatility is purely driven by the returns via the term Z_t.

• The use of Mittag-Leffler type kernels as in the last point of Theorem 1 is very standard in the rough volatility literature, see for example [JR16]. It enables us to obtain at the limit a

rough behavior for the sample paths of the volatility process.

• When $k(t) = \sqrt{2\nu}e^{-\nu t}$, Model (6) is that of [BDB17] with $\phi = 0$. Therefore Theorem 1 extends the results of [BDB17] to any kernel k with suitable integrability conditions. In the next section we provide an even more general extension that encompasses the case $\phi \neq 0$ and clearly shows the super-Heston nature of the dynamic (6).

3 General quadratic Hawkes models: the stable case

We now study the asymptotic behavior of a sequence of general quadratic Hawkes models for which the stability condition is not violated at the limit. We consider $(N^T)_{T\geq 0}$ with intensity given by (2) (with parameters depending on T) where $\|\phi_T\|_1 + \|k_T\|_2^2$ is a fixed constant strictly smaller than one. As in the previous section, we first give intuitions about our scaling procedure.

3.1 Suitable scaling in the general case

Using a scaling factor ω_T, the rescaled intensity becomes

$$\begin{aligned}\omega_T\lambda_{tT}^T = \mu_T\omega_T &+ \int_0^t \phi_T\big(T(t-s)\big)T\omega_T\lambda_{Ts}^T\mathrm{d}s \\ &+ \int_0^t \phi_T\big(T(t-s)\big)\sqrt{\omega_T T}\mathrm{d}(\sqrt{\frac{\omega_T}{T}}M_{Ts}^T) + (\sqrt{\omega_T}Z_{tT}^T)^2, \end{aligned} \tag{7}$$

where

$$M_t^T = N_t^T - \int_0^t \lambda_s^T \mathrm{d}s.$$

Assume that $(\omega_T\lambda_{tT}^T)_{T\geq 0}$ converges and consider the processes $M_t^{*T} = M_{tT}^T\sqrt{\frac{\omega_T}{T}}$ and $P_t^{*T} = P_{tT}^T\sqrt{\frac{\omega_T}{T}}$. We have

$$\langle P^{*T}\rangle_t = \langle M^{*T}\rangle_t = \int_0^t \omega_T\lambda_{Ts}^T\mathrm{d}s \text{ and } \langle P^{*T}, M^{*T}\rangle_t = 0.$$

Thus we expect P^{*T} and M^{*T} to converge towards two martingales M and P such that $\langle M, P\rangle = 0$. As in the previous section, to obtain continuous martingales M and P, we pick ω_T such that ω_T/T tends to zero.

One of our goals being to preserve Zumbach effect in the limit of (7), we need a non-degenerate behavior for the feedback term $\sqrt{\omega_T}Z_{tT}^T$. We have

$$\sqrt{\omega_T}Z_{tT}^T = \int_0^t k_T\big(T(t-s)\big)\sqrt{T}\mathrm{d}P_t^{*T},$$

which leads us again to the specification

$$k_T = k(\cdot/T)\sqrt{\gamma/T}$$

for some positive γ. Now, if $\sqrt{\omega_T} Z^T_{tT}$ converges, according to (7) we should also obtain convergence of

$$\mu_T \omega_T + \int_0^t \phi_T\big(T(t-s)\big) T \omega_T \lambda^T_{Ts} \mathrm{d}s + \int_0^t \phi_T\big(T(t-s)\big)\sqrt{\omega_T T}\mathrm{d}M^{*T}_s.$$

Thus, since both $\omega_T \lambda^T_{tT}$ and M^{*T} are expected to converge, we set $\mu_T = \mu/\omega_T$ and must ensure the convergence of both $\phi_T(Tt)T$ and $\phi_T(Tt)\sqrt{\omega_T T}$. Because ω_T/T tends to zero, the first integral dominates the second one. Consequently we only need to take care of the first integral and again we can take $\omega_T = 1$. A logical specification is therefore

$$\phi_T = \phi(\cdot/T)(\beta/T)$$

for some positive β. Passing to the limit in Equation (7) we expect the following limiting model:

$$V_t = \mu + \int_0^t \beta\phi(t-s)V_s \mathrm{d}s + Z^2_t, \text{ with } Z_t = \int_0^t \sqrt{\gamma} k(t-s)\mathrm{d}P_s.$$

3.2 Assumptions and results in the presence of a linear component in the stable case

We now give our exact assumptions which are quite similar to those in the previous section.

Assumption 2.

i) The sequence of kernels is given by

$$k_T(t) = \sqrt{\frac{\gamma}{T}} k(\frac{t}{T}), \quad \phi_T(t) = \frac{\beta}{T}\phi(\frac{t}{T}),$$

with $0 < \gamma + \beta < 1$ *and* k *and* ϕ *non-negative measurable such that* $\|k\|_2^2 = \|\phi\|_1 = 1$. *Furthermore* $\mu_T = \mu > 0$.

ii) Assumption 1 ii) holds.

Assumption 2 implies that the stability condition is not violated at the limit. Nevertheless, from a rescaling perspective, the choice of kernels ϕ_T and k_T does not seem really natural at first sight. It would be probably more satisfactory to consider kernel sequences of the form $a_T\phi$ and $a'_T k$ (with ϕ and k not depending on T) and then investigate the limit of $\omega_T \lambda^T_{tT}$ as in [EEFR18, JR15, JR16]. This would imply here $\phi_T(Tt)T = a_T\phi(Tt)T$. According to Tauberian theorems, see for example [BGT89], $a_T\phi(Tt)T$ can only converge in that case towards a power-law function of the form $t^{-\delta}$ for some positive δ and ϕ has to be such that $\phi(t) \sim_{+\infty} t^{-\delta}$ up to a slowly varying function. But recall that ϕ must be integrable and so we need $\delta \geq 1$. However such choice would lead to difficulties for defining properly the limit of the integral

$$\int_0^t T a_T \phi\big(T(t-s)\big)\omega_T \lambda^T_{Ts}\mathrm{d}s.$$

To be able to consider such types of natural but technically more intricate rescaling procedures, we will drop the stability assumption in Section 4 where we work in the nearly unstable case.

Let us define the rescaled process $X_t^T = N_{tT}/T$. We have the following theorem whose proof is given in Section 5.2.

Theorem 2. *Under Assumption 2, the sequence $(X^T, P^{*T})_{T\geq 0}$ is C-tight for the Skorohod topology on $[0,1]$ as T goes to infinity, with the following properties for any limit point (X,P):*

- *X is almost surely continuously differentiable.*
- *There exists a Brownian motion B such that*

$$P_t = \int_0^t \sqrt{V_s}\mathrm{d}B_s,$$

where V is the derivative of X and the unique continuous solution of

$$V_t = \mu + H_t + Z_t^2, \text{ with } H_t = \int_0^t \beta\phi(t-s)V_s\mathrm{d}s \tag{8}$$

and

$$Z_t = \int_0^t \sqrt{\gamma}k(t-s)\sqrt{V_s}\mathrm{d}B_s, \text{ on } [0,1].$$

- *For any $\varepsilon > 0$, if $k = f^{H+1/2,\lambda}$ with $H \in (0,1/2)$ and $\lambda > 0$, V has almost surely $H-\varepsilon$ Hölder regularity.*

3.3 Discussion of Theorem 2

• Compared to Theorem 1, only one new term appears in the volatility equation (8). It comes from the self-exciting part in the Hawkes dynamic. Thus the elements in the discussion of the purely quadratic case in Section 2.3 remain valid here.

• Let us consider the case where k is a continuously differentiable kernel with $0 < k(0) < +\infty$. Using integration by parts and Fubini's theorem we can write

$$\begin{aligned} Z_t &= \int_0^t \sqrt{\gamma}k(0)\sqrt{V_s}\mathrm{d}B_s + \int_0^t\int_u^t \sqrt{\gamma}k'(s-u)\mathrm{d}s\sqrt{V_u}\mathrm{d}B_u \\ &= \int_0^t \sqrt{\gamma}k(0)\sqrt{V_s}\mathrm{d}B_s + \int_0^t\int_0^s \sqrt{\gamma}k'(s-u)\sqrt{V_u}\mathrm{d}B_u\mathrm{d}s. \end{aligned}$$

Therefore Z is a semi-martingale and up to a finite variation term we have

$$Z_t^2 = \int_0^t 2\sqrt{\gamma}k(0)Z_s\sqrt{V_s}\mathrm{d}B_s.$$

We see that the quadratic feedback term in the Hawkes dynamic induces a super-Heston type volatility because of the multiplicative term Z_s in front of the $\sqrt{V_s}$ in the equation above.

• Let us take the kernel k as the Mittag-Leffler function $f^{\alpha,\lambda}$ with $\alpha \in (1/2,1)$ and $\lambda > 0$ and $\phi(t) = \kappa e^{-\kappa t}$ for some $\kappa > 0$. Adapting Theorem 2.1 in [EER18] we get for any h and t_0 positive

$$Z_{t_0+h} = \xi_{t_0}(h) + \tilde{Z}_h, \text{ with } \tilde{Z}_h = \int_0^h \sqrt{\gamma} f^{\alpha,\lambda}(h-s)\mathrm{d}P_{s+t_0}$$

and

$$\xi_{t_0}(h) = Z_{t_0} + \int_0^h f^{\alpha,\lambda}(h-s)\theta_{t_0}(s)\mathrm{d}s$$

where

$$\theta_{t_0}(h) = -Z_{t_0} + \frac{\alpha}{\lambda\Gamma(1-\alpha)}\int_0^{t_0}(t_0-s+h)^{-1-\alpha}(Z_s - Z_{t_0})\mathrm{d}s - \frac{(h+t_0)^{-\alpha}}{\lambda\Gamma(1-\alpha)}Z_{t_0}.$$

Then we can write the forward volatility as

$$V_{t_0+h} = H_{t_0}e^{-\kappa h} + \left(\xi_{t_0}(h)\right)^2 + 2\xi_{t_0}(h)\tilde{Z}_h + \mu + \tilde{H}_h + (\tilde{Z}_h)^2 \tag{9}$$

with

$$\tilde{H}_h = \int_0^h \phi(h-s)V_{t_0+s}\mathrm{d}s.$$

The function ξ_{t_0} only depends on $(Z_t)_{0\le t\le t_0}$ and cannot be expressed as a function of $(V_t)_{0\le t\le t_0}$. So we get from (9) that conditional on the history of the market from time 0 to t_0, the law of $(V_{t_0+h})_{h\ge 0}$ does depend on past returns and not only through past volatility. It means Models (6) and (8) can reproduce the strong Zumbach effect. So when k is a Mittag-Leffler function, Model (8) is a super-Heston type rough volatility model with strong Zumbach effect.

In the case of exponential kernels $k(t) = \sqrt{2\nu}e^{-\nu t}$ and $\phi(t) = \kappa e^{-\kappa t}$ using similar computations we prove that

$$V_{t_0+h} = \mu + \tilde{Z}_h + \tilde{H}_t + e^{-2\nu h}Z_{t_0}^2 + 2Z_{t_0}e^{-\nu h} + e^{-\kappa h}H_{t_0}.$$

• Finally remark that we do not prove uniqueness in law of the limit points (X,P) in general. However we can show uniqueness in the special case $\phi = 0$. This particular case can be treated because Z is the solution of a stochastic Volterra equation which admits a unique strong solution, see Section 5.1 for details and [AJLP19] for more results about uniqueness of rough equations.

4 Nearly unstable quadratic Hawkes models

We now focus on the case where the instability condition becomes almost violated at the limit. Let us consider a sequence of quadratic Hawkes processes $(N^T)_{T\ge 0}$ such that

$$\|\phi_T\|_1 + \|k_T\|_2^2 \to 1.$$

Contrary to the sections before, we wish to work here with a natural renormalization (at least for ϕ, see comments below Assumption 2) and therefore take ϕ_T of the form $\phi_T = \beta_T\phi$ with $\beta_T \in (0,1)$ and $\|\phi\|_1 = 1$. We also assume that ϕ is heavy-tailed ($\phi(x) \sim x^{-(1+\alpha)}$ with $\alpha \in (0,1)$ as x tends to infinity) since this type of kernels leads to rough volatility in the case of linear Hawkes processes, see [EEFR18, JR16]. Again, we start with insights about the suitable scaling procedure.

4.1 An adapted scaling procedure in the nearly unstable case

Let $a_T = \|k_T\|_2^2 + \|\phi_T\|_1$. We have

$$\mathbb{E}[\lambda_t^T] = \mu_T + \int_0^t (k_T^2 + \phi_T)(t-s)\mathbb{E}[\lambda_s^T]\mathrm{d}s$$

and therefore

$$\mathbb{E}[\lambda_t^T] \leq \frac{\mu_T}{1-a_T}.$$

So we naturally define the following renormalized processes:

$$\lambda_t^{*T} = \frac{1-a_T}{\mu_T}\lambda_{tT}^T, \quad \Lambda_t^{*T} = \frac{1}{T}\int_0^{tT} \lambda_s^{*T}\mathrm{d}s \text{ and } X_t^T = \frac{1-a^T}{T\mu_T}N_{tT}^T.$$

Let us assume that λ^{*T} converges to some V. We can then expect that Λ^{*T} and X^{*T} converge to some Λ and X. Consider the rescaled martingales

$$M_t^{*T} = \sqrt{\frac{1-a_T}{T\mu_T}}M_{tT}^T \text{ and } P_t^{*T} = \sqrt{\frac{1-a_T}{T\mu_T}}P_{tT}^T,$$

where $M_t^T = N_t^T - \int_0^t \lambda_s^T \mathrm{d}s$. Since $[M^T]_t = [P^T]_t = N_t^T$, we have $[M^{*T}]_t = [P^{*T}]_t = X_t^T$. Moreover $\langle M^{*T}, P^{*T}\rangle = 0$ and so M^{*T} and P^{*T} are likely to converge towards some martingales M and P with same bracket X and such that $\langle M, P\rangle = 0$.

Let

$$\psi_T = \sum_{i\geq 1} \phi_T^{*i}.$$

Using Proposition 2.1 in [JR15], we deduce from (2) that

$$\lambda_t^T = \mu_T + (Z_t^T)^2 + \int_0^t \psi_T(t-s)(\mu_T + (Z_s^T)^2)\mathrm{d}s + \int_0^t \psi_T(t-s)\mathrm{d}M_s^T.$$

So we have

$$\begin{aligned}\lambda_t^{*T} =& (1-a_T) + \frac{1-a_T}{\mu_T}(Z_{tT}^T)^2 + \int_0^t (1-a_T)T\psi_T\big(T(t-s)\big)\Big(1 + \frac{1}{\mu_T}(Z_{sT}^T)^2\Big)\mathrm{d}s \\ &+ \int_0^t T(1-a_T)\psi_T\big(T(t-s)\big)\frac{1}{\sqrt{T\mu_T(1-a_T)}}\mathrm{d}M_s^{*T}.\end{aligned} \tag{10}$$

The function $T\psi_T(T\cdot)$ has L^1 norm equal to $(1-\beta_T)^{-1}$. Therefore $T(1-a_T)\psi_T(\cdot T)$ is non-vanishing only provided $1-\beta_T$ is of order $1-a_T$. Consequently we set $\beta_T = 2a_T - 1$ (so that $\beta_T < a_T$). Since $\|\phi_T\|_1 = \beta_T \to 1$ then $\|k_T\|_2^2 \to 0$. However we will see that the sequence k_T still plays a role in the limit.

In (10) the first integral is

$$\int_0^t T(1-a_T)\psi_T\big(T(t-s)\big)\mathrm{d}s.$$

It already appears in the case of a purely linear Hawkes process. We know from [JR15, JR18] that this term is crucial in the limiting behavior of the intensity and that a necessary condition to obtain a non-trivial scaling limit for it is that $T^\alpha(1-a_T)$ tends to a positive constant. Under this specification, we need to impose additionally that $T\mu_T(1-a_T)$ converges in order to obtain a non-degenerate asymptotic limit for the last integral in (10).

We now study the terms containing the quadratic feedback:

$$\frac{1-a_T}{\mu_T}(Z_{tT}^T)^2 \text{ and } \frac{1-a_T}{\mu_T}\int_0^t T\psi_T\big(T(t-s)\big)(Z_{sT}^T)^2\mathrm{d}s.$$

Since $\|T\psi_T(\cdot T)\|_1 = (1-\beta_T)^{-1}$ which tends to infinity, the second term dominates the first one. To make the second term converge, we need a proper behavior of $Z_t^{*T} = Z_{tT}^T/\sqrt{\mu_T}$. We have

$$Z_t^{*T} = \sqrt{\frac{T}{1-a_T}}\int_0^t k_T\big(T(t-s)\big)\mathrm{d}P_s^{*T}. \tag{11}$$

Thus we wish $\sqrt{\frac{T}{1-a_T}}k_T(Tt)$ to converge and are naturally lead to assume that k_T is of the form

$$k_T = k(\cdot/T)\sqrt{\frac{1-a_T}{T}}.$$

4.2 Assumptions and results in the nearly unstable case

We now summarize the conditions derived in the above discussion into the following assumption.

Assumption 3.

i) The sequence of kernels $(\phi_T)_{T\geq 0}$ *satisfies* $\phi_T = (2a_T-1)\phi$ *with* $(a_T)_{T\geq 0}$ *a sequence in* $(0,1)$ *and* ϕ *a non-negative measurable function such that* $\|\phi\|_1 = 1$. *Furthermore for some* $K>0$ *and* $\alpha\in(0,1)$,

$$\lim_{x\to+\infty}\alpha x^\alpha\int_x^{+\infty}\phi(s)\mathrm{d}s = K.$$

ii) The sequence of kernels $(k_T)_{T\geq 0}$ *satisfies* $k_T = k(\cdot/T)\sqrt{\frac{1-a_T}{T}}$ *with* k *a non-negative continuously differentiable function such that* $\|k\|_2 = 1$ *(in particular* $k(0)<+\infty$*).*

iii) Let $\delta = K\frac{\Gamma(1-\alpha)}{\alpha}$. *There are two positive constants* λ *and* μ^* *such that*

$$\lim_{T\to+\infty}(1-a_T)T^\alpha = \lambda\delta \text{ and } \lim_{T\to+\infty}T^{1-\alpha}\mu_T = \mu^*\delta^{-1}.$$

The choice of δ in Point $iii)$ is just for convenience of notation in the results and proofs.

Recall that from Lemma 4.3 in [JR16], under Assumption 3, the function

$$F^T(t) = \int_0^t T(1-a_T)\psi_T(Ts)\mathrm{d}s$$

converges towards $\frac{1}{2}F^{\alpha,\lambda}(t)$ where

$$F^{\alpha,\lambda}(t) = \int_0^t f^{\alpha,\lambda}(s)\mathrm{d}s.$$

So

$$T(1-a_T)\psi_T(Ts) \sim \frac{1}{2}f^{\alpha,\lambda}(s).$$

This provides us intuition for the form of the limit (V,Z) of (10)-(11):

$$V_t = \int_0^t \frac{1}{2}f^{\alpha,\lambda}(t-s)(1+Z_s^2)\mathrm{d}s + \int_0^t \frac{1}{2}f^{\alpha,\lambda}(t-s)\frac{1}{\sqrt{\lambda\mu^*}}\mathrm{d}M_s,$$

with $Z_t = \int_0^t k(t-s)\mathrm{d}P_s$ and where M and P are martingales such that $\langle M,P\rangle = 0$ and $\langle M\rangle_t = \langle P\rangle_t = \int_0^t V_s\mathrm{d}s$.

We eventually state the main result of this section whose proof is given in Section 5.3.

Theorem 3. *Under Assumption 3, the sequence $(X^T, M^{*T}, P^{*T})_{T\geq 0}$ is C-tight for the Skorohod topology on $[0,1]$ as T goes to infinity, with the following properties for any limit point (X,M,P):*

- *We have $\langle M\rangle = \langle P\rangle = X$ and*

$$X_t = \int_0^t \frac{1}{2}F^{\alpha,\lambda}(t-s)\big(1+Z_s^2\big)\mathrm{d}s + \int_0^t \frac{1}{2}f^{\alpha,\lambda}(t-s)\frac{1}{\sqrt{\lambda\mu^*}}M_s\mathrm{d}s \tag{12}$$

  with

$$Z_t = \int_0^t k(t-s)\mathrm{d}P_s.$$

- *If $\alpha \in (1/2,1)$, the process X is almost surely continuously differentiable with derivative V and up to an enlargement of the filtration there exists two Brownian motions $B^{(1)}$ and $B^{(2)}$ such that V is solution of*

$$V_t = \int_0^t \frac{1}{2}f^{\alpha,\lambda}(t-s)\Big(\big(1+Z_s^2\big)\mathrm{d}s + \frac{1}{\sqrt{\lambda\mu^*}}\sqrt{V_s}\mathrm{d}B_s^{(1)}\Big)$$

  with

$$Z_t = \int_0^t k(t-s)\sqrt{V_s}\mathrm{d}B_s^{(2)}.$$

  Moreover, for any $\varepsilon > 0$, V has almost surely $\alpha - \frac{1}{2} - \varepsilon$ Hölder regularity.

4.3 Discussion of Theorem 3

• The form of the feedback is not the same in Model (12) as in Model (8). In (8) it is instantaneous through the Z_t^2 term while in (12) it is digested via a convolution with a fractional kernel.

• In Model (12) price and volatility are driven by two different Brownian motions. This additional Brownian motion comes from the rescaling of the linear part of the intensity, as already observed for example in [JR15].

• Rough volatility appears for very different reasons in Model (8) and Model (12). In (12) the origin of rough volatility is the fat tail of the kernel ϕ while in (8) it arises from the behavior of the kernel k in zero. Moreover it is clear from the proof of the last point of Theorem 3 that the regularity of Z has no influence on that of V.

• As computed in the previous section we can write

$$Z_t = \int_0^t k(0)\sqrt{V_s}\mathrm{d}B_s^{(2)} + \int_0^t \int_0^s k'(s-u)\sqrt{V_u}\mathrm{d}B_u^{(2)}\mathrm{d}s.$$

Therefore Z is a semi-martingale and up to a finite variation term we have

$$\mathrm{d}Z_t^2 = 2k(0)Z_t\sqrt{V_s}\mathrm{d}B_s^{(2)}.$$

Furthermore using integration by parts we get

$$\int_0^t f^{\alpha,\lambda}(t-s)Z_s^2\mathrm{d}s = \int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}Z_s^2.$$

So up to a finite variation term, we have in Model (12)

$$V_t = \int_0^t f^{\alpha,\lambda}(t-s)\frac{1}{\sqrt{\lambda\mu^*}}\sqrt{V_s}\mathrm{d}B_s^{(1)} + \int_0^t F^{\alpha,\lambda}(t-s)k(0)Z_s\sqrt{V_s}\mathrm{d}B_s^{(2)}.$$

Thus as in Model (6) and (8), the quadratic feedback term in the volatility dynamic induces that Model (12) is a super-Heston type rough volatility model. Note however that in that case, the super-Heston and rough components are not the same.

• Using Lemma A.2 in [EER19], when $\alpha \in (1/2, 1)$, we get that Equation (12) is equivalent to

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(Z_s^2 + \theta_0(s) - V_s)\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sqrt{V_s}\frac{1}{\sqrt{\lambda\mu^*}}\mathrm{d}B_s^{(1)}$$

with θ_0 a deterministic function. In the case $k(t) = \sqrt{2\nu}e^{-\nu t}$ with $\nu > 0$, direct adaptation of Theorem 2.1 in [EER18] gives that

$$\begin{aligned} V_{t_0+h} = V_{t_0} &+ \frac{1}{\Gamma(\alpha)}\int_0^h (h-s)^{\alpha-1}\lambda(\tilde{Z}_s^2 + 2\tilde{Z}_s Z_{t_0}e^{-\nu s} + \theta_{t_0}(s) - V_{t_0+s})\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^h (h-s)^{\alpha-1}\sqrt{V_{t_0+s}}\frac{1}{\sqrt{\lambda\mu^*}}\mathrm{d}B_s^{(1)}, \end{aligned}$$

where $\theta_{t_0}(h)$ is equal to

$$\theta_0(t_0+h)+\frac{\alpha}{\Gamma(1-\lambda)}\int_0^{t_0}(t_0-v+h)^{-1-\alpha}(V_v-V_{t_0})\mathrm{d}v+\frac{(t+t_0)^{-\alpha}}{\lambda\Gamma(1-\alpha)}(V_0-V_{t_0})+Z_{t_0}^2e^{-2\nu h}$$

and

$$\tilde{Z}_h=\int_0^h k(h-s)\mathrm{d}P_{t_0+s}.$$

The term Z_{t_0} cannot be written as a function of $(V_t)_{0\le t\le t_0}$. So Model (12) reproduces the strong Zumbach effect. Finally Model (12) is a super-Heston type rough volatility model with strong Zumbach effect.

5 Proofs

We gather all the proofs in this section. We first show Theorem 1 assuming that Theorem 2 holds. Then we give the proof of Theorem 2 and finally that of Theorem 3.

5.1 Proof of Theorem 1

Using the results of Theorem 2, we only need to prove that when $\phi=0$, the limiting process (X,P) in Theorem 2 $ii)$ is unique in law.

Consider (X,P) a limit point of $(X^T,P^{*T})_{T\ge0}$. Then V the derivative of X satisfies

$$V_t=\mu+Z_t^2$$

and there exists a Brownian motion B such that $P_t=\int_0^t\sqrt{V_s}\mathrm{d}B_s$. Thus we can write

$$Z_t=\int_0^t k(t-s)\sqrt{\mu+Z_s^2}\ \mathrm{d}B_s. \tag{13}$$

From Assumption 1 $ii)$ together with Theorems 3.1 and 3.3 in [Zha10], there is a unique process Z satisfying (13) and it is continuous. Since P is a continuous martingale satisfying $[P]=X$, $X=\int_0^t V_s\mathrm{d}s$ and $V_t=\mu+Z_t^2$, the limiting process (X,P) is fully determined by the only solution of (13). So we get convergence of (X^T,P^{*T}) for the Skorohod topology.

5.2 Proof of Theorem 2

We proceed in three steps. First we prove that the sequence $(X^T,P^{*T})_{T\ge0}$ is C-tight for the Skorohod topology. Then we show the results about the dynamics of the limit points. Finally we establish the regularity properties of the limit points.

5.2.1 Tightness of the sequence $(X^T, P^{*T})_{T\geq 0}$

We consider the processes

$$\Lambda_t^{*T} = \int_0^t \lambda_s^{*T} \mathrm{d}s \text{ and } Z_t^{*T} = \int_0^t k(t-s)\mathrm{d}P_s^{*T}$$

defined for $t \in [0,1]$. Remark that Λ^{*T} is the predictable compensator of X^T. We have the following equality:

$$\mathbb{E}[\lambda_t^T] = \mu_T + \int_0^t \big(k_T^2 + \phi_T\big)(t-s)\mathbb{E}[\lambda_s^T]\mathrm{d}s.$$

Thus

$$\mathbb{E}[\lambda_t^T] \leq \frac{\mu}{1 - \|\phi_T\|_1 - \|k_T\|_2^2}$$

and consequently

$$\mathbb{E}[X_1^T] = \mathbb{E}[\Lambda_1^{*T}] \leq \frac{\mu}{1-\gamma-\beta}.$$

Since the processes X^T and Λ^{*T} are increasing for any T, using the last inequality, we deduce from Theorem VI-3.21 together with Proposition VI-3.35 in [JS13] that $(X^T)_{T\geq 0}$ and $(\Lambda^{*T})_{T\geq 0}$ are tight. Moreover since $|\Delta X^T| + |\Delta \Lambda^{*T}| \leq 1/T$ almost surely on $[0,1]$, according to Proposition VI-3.26 in [JS13], $(X^T)_{T\geq 0}$ and $(\Lambda^{*T})_{T\geq 0}$ are C-tight. The tightness of $(M^{*T})_{T\geq 0}$ and $(P^{*T})_{T\geq 0}$ follows from Theorem VI-4.13 in [JS13] using that $\langle M^{*T}\rangle_t = \langle P^{*T}\rangle_t = \Lambda_t^{*T}$. We then get C-tightness because $|\Delta M^{*T}| + |\Delta P^{*T}| \leq 2/T$. Finally $(X^T, \Lambda^{*T}, M^{*T}, P^{*T})_{T\geq 0}$ is C-tight for the Skorohod topology on $[0,1]$.

We also show that the sequence $(Z^{*T})_{T\geq 0}$ is tight for the $L^2([0,1])$ topology. For this, inspired by [AJCLP19], we consider the Sobolev-Slobodeckij norm defined for any measurable function f by

$$\|f\|_{W^{\eta,2}([0,1])} = \Big(\int_0^1 f(s)^2 \mathrm{d}s + \int_0^1 \int_0^1 \frac{|f(t)-f(s)|^2}{|t-s|^{1+2\eta}} \mathrm{d}s\mathrm{d}t\Big)^{1/2}.$$

We recall that the closed balls of $\|\cdot\|_{W^{\eta,2}([0,1])}$ are relatively compact in $L^2([0,1])$, see [FG95]. Therefore it is enough to show that $\big(\mathbb{E}[\|Z^{*T}\|^2_{W^{\eta,2}([0,1])}]\big)_{T\geq 0}$ is uniformly bounded to conclude the tightness of $(Z^{*T})_{T\geq 0}$ in $L^2([0,1])$.

For any $t \in [0,1]$, we have using Ito's formula

$$\mathbb{E}[(Z_t^{*T})^2] = \int_0^t k^2(t-s)\mathbb{E}[\lambda_{Ts}^T]\mathrm{d}s \leq \frac{\mu}{1-\gamma-\beta}\|k\|_2^2$$

and for $0 \leq s \leq t \leq 1$

$$Z_t^{*T} - Z_s^{*T} = \int_s^t k(t-u)\mathrm{d}P_u^{*T} + \int_0^s \big(k(t-u) - k(s-u)\big)\mathrm{d}P_u^{*T}.$$

Then we get

$$\mathbb{E}[(Z_t^{*T} - Z_s^{*T})^2] = \int_s^t k^2(t-u)\mathbb{E}[\lambda_{Tu}^T]\mathrm{d}u + \int_0^s \big(k(t-u) - k(s-u)\big)^2\mathbb{E}[\lambda_{Tu}^T]\mathrm{d}u.$$

Using that $\mathbb{E}[\lambda_u^T] \leq \frac{\mu}{1-\gamma-\beta}$ we obtain

$$\mathbb{E}[(Z_t^{*T} - Z_s^{*T})^2] \leq \frac{\mu}{1-\gamma-\beta}\Big(\int_s^t k^2(t-u)\mathrm{d}u + \int_0^s \big(k(t-u)-k(s-u)\big)^2\mathrm{d}u\Big).$$

According to [AJCLP19] we have

$$\int_0^1\int_0^1\int_{s\wedge t}^{s\vee t} \frac{k(s\vee t-u)^2}{|t-s|^{1+2\eta}}\mathrm{d}u\mathrm{d}s\mathrm{d}t \leq \frac{1}{\eta}\int_0^1 |k(t)|^2 t^{-2\eta}\mathrm{d}t$$

and

$$\int_0^1\int_0^1\int_{s\wedge t}^{s\vee t} \frac{|k(t-u)-k(s-u)|^2}{|t-s|^{1+2\eta}}\mathrm{d}u\mathrm{d}s\mathrm{d}t \leq \int_0^1\int_0^1 \frac{|k(t)-k(s)|^2}{|t-s|^{1+2\eta}}\mathrm{d}s\mathrm{d}t,$$

which is bounded from Assumption 2 ii). Finally using Fubini's theorem we deduce that $(\mathbb{E}[\|Z^{*T}\|^2_{W^{\eta,2}([0,1])}])_{T\geq 0}$ is bounded. So $(Z^{*T})_{T\geq 0}$ is tight in $L^2([0,1])$.

Before going to the next step we prove the following lemma.

Lemma 1. *The sequence of martingales $X^T - \Lambda^{*T}$ converges to 0 uniformly in probability on $[0,1]$.*

Proof. Since $N_t^T - \int_0^t \lambda_s^T \mathrm{d}s$ is a true martingale, from Doob's inequality we get

$$\mathbb{E}[\sup_{t\in[0,1]}(X_t^T - \Lambda_t^{T*})^2] \leq \frac{1}{T^2}\mathbb{E}[N_T^T].$$

Using that $\mathbb{E}[N_T^T] = T\mathbb{E}[\Lambda_1^{*T}]$ we deduce

$$\mathbb{E}[\sup_{t\in[0,1]}(X_t^T - \Lambda_t^{*T})^2] \leq \frac{1}{T}\frac{\mu}{1-\gamma-\beta},$$

which concludes the proof. □

5.2.2 Dynamic of the limit points

We now consider (X, X, M, P, Z) a limit point of $(X^T, \Lambda^{*T}, M^{*T}, P^{*T}, Z^{*T})_{T\geq 0}$. Using Skorohod representation theorem and the fact that (X, X, M, P) is continuous, we may consider that almost surely $(X^T, \Lambda^{*T}, M^{*T}, P^{*T})_{T\geq 0}$ converges uniformly on $[0,1]$ towards (X, X, M, P) and $(Z^{*T})_{T\geq 0}$ converges in $L^2([0,1])$ towards Z:

$$\begin{aligned} &\sup_{t\in[0,1]}|X_t^T - X_t| \underset{T\to+\infty}{\to} 0, \quad \sup_{t\in[0,1]}|M_t^{*T} - M_t| \underset{T\to+\infty}{\to} 0, \\ &\sup_{t\in[0,1]}|P_t^{*T} - P_t| \underset{T\to+\infty}{\to} 0 \text{ and } \int_0^1 (Z_s^{*T} - Z_s)^2\mathrm{d}s \underset{T\to+\infty}{\to} 0. \end{aligned} \tag{14}$$

From Corollary IX-1.19 in [JS13] we have that M and P are local martingales. Moreover $[M^{*T}] = [P^{*T}] = X^T$ so Corollary VI-6.29 in [JS13] gives that $[M] = [P] = X$. Since M and P are continuous we have

$$\langle M\rangle = [M] = \langle P\rangle = [P] = X.$$

We also note that $\mathbb{E}[X_1^T]$ is uniformly bounded in T. So from Fatou's lemma X_1 is integrable and M and P are true martingales. Moreover up to a subsequence $(Z^{*T})_{T\geq 0}$ converges almost surely towards Z. We deduce that Z is adapted. Moreover since $\mathbb{E}[\sup_{t\in[0,1]}(Z_t^{*T})^2]$ is bounded we get

$$\sup_{t\in[0,1]}\mathbb{E}[Z_t^2] < +\infty.$$

We show that Λ^{*T} converges almost surely uniformly on $[0,1]$ towards

$$\int_0^t \left(\mu + Z_s^2\right)\mathrm{d}s + \int_0^t F_1(t-s)\mathrm{d}X_s, \tag{15}$$

where $F_1(t) = \int_0^t \beta\phi(s)\mathrm{d}s$. We have

$$\Lambda_t^{*T} = \int_0^t \left(\mu + (Z_s^{*T})^2\right)\mathrm{d}s + \int_0^t\int_0^t \beta\phi(s-u)\mathrm{d}X_u^T\mathrm{d}s.$$

The almost sure convergence of $(Z^{*T})_{T\geq 0}$ in L^2 towards Z implies that almost surely, uniformly in $t\in[0,1]$,

$$\int_0^t (Z_s^{*T})^2\mathrm{d}s \underset{T\to+\infty}{\to} \int_0^t Z_s^2\mathrm{d}s.$$

Moreover using Ito's formula together with Fubini's theorem we get

$$\int_0^t\int_0^s \beta\phi(s-u)\mathrm{d}X_u^T\mathrm{d}s = \int_0^t \beta\phi(t-s)X_s^T\mathrm{d}s.$$

From Equation (14), we deduce that this quantity converges almost surely uniformly towards

$$\int_0^t \beta\phi(t-s)X_s\mathrm{d}s.$$

Again Ito's formula together with Fubini's theorem give

$$\int_0^t \beta\phi(t-s)X_s\mathrm{d}s = \int_0^t F_1(t-s)\mathrm{d}X_s.$$

So we obtain the almost sure uniform convergence of $(\Lambda^{*T})_{T\geq 0}$ towards (15). Consequently, using Lemma 1, we deduce

$$X_t = \int_0^t \mu + Z_s^2\mathrm{d}s + \int_0^t F_1(t-s)\mathrm{d}X_s$$

and eventually

$$X_t = \int_0^t \mu + Z_s^2 + \int_0^s \beta\phi(s-u)\mathrm{d}X_u\mathrm{d}s.$$

Thus X is absolutely continuous with respect to the Lebesgue measure with derivative V given by

$$V_t = \mu + Z_t^2 + \int_0^t \beta\phi(t-s)V_s\mathrm{d}s.$$

Letting $\psi = \sum_{i\geq 1} (\beta\phi)^{*i}$ we have

$$V_t = \mu + Z_t^2 + \int_0^t \psi(t-s) Z_s^2 \mathrm{d}s. \tag{16}$$

The boundedness of $(\mathbb{E}[Z_t^2])_{t\in[0,1]}$ gives that $(V_t)_{t\in[0,1]}$ is uniformly bounded in L^1.

We now prove that

$$Z_t = \int_0^t k(t-s)\mathrm{d}P_s.$$

Using Cauchy-Schwarz inequality, the convergence of $(Z^{*T})_{T\geq 0}$ implies that almost surely, uniformly in $t \in [0,1]$,

$$\int_0^t Z_s^{*T}\mathrm{d}s \underset{T\to+\infty}{\to} \int_0^t Z_s \mathrm{d}s.$$

From Ito's formula we get

$$\int_0^t Z_s^{*T}\mathrm{d}s = \int_0^t k(t-s) P_s^{*T}\mathrm{d}s$$

and using Equation (14) we deduce that it converges almost surely uniformly towards

$$\int_0^t k(t-s) P_s \mathrm{d}s.$$

Since $F_2(t) = \int_0^t \gamma k^2(s)\mathrm{d}s < 1$ we have

$$\int_0^t \int_0^s k(s-u)^2 \mathrm{d}X_u \mathrm{d}s = \int_0^t F_2(t-s)\mathrm{d}X_s < X_t < +\infty.$$

So we can use the stochastic Fubini theorem and show that

$$\int_0^t \int_0^s k(s-u)\mathrm{d}P_u \mathrm{d}s = \int_0^t k(t-s)P_s \mathrm{d}s.$$

Thus almost surely, for any $t \in [0,1]$,

$$\int_0^t Z_s \mathrm{d}s = \int_0^t \int_0^s k(s-u)\mathrm{d}P_u \mathrm{d}s$$

and

$$Z_t = Z_0 + \int_0^t \sqrt{\gamma} k(t-s)\mathrm{d}P_s.$$

Moreover, from Theorem V-3.9 in [RY13], there exists a Brownian motion B such that

$$P_t = \int_0^t \sqrt{V_s}\mathrm{d}B_s$$

and finally we get

$$Z_t = \int_0^t \sqrt{\gamma} k(t-s)\sqrt{V_s}\mathrm{d}B_s.$$

We recall that $(\mathbb{E}[V_t])_{t\in[0,1]}$ is bounded in L^2. So using Assumption 2 $ii)$ together with Theorems 3.1 and 3.3 in [Zha10] we obtain that the process Z is continuous. Therefore using (16) V is also continuous. This concludes this part of the proof.

5.2.3 Regularity property

We now consider that k is given by $f^{H+1/2,\lambda}$ for $H \in (0,1/2)$ and $\lambda > 0$. We can write

$$\int_0^t Z_s \mathrm{d}s = \int_0^t \sqrt{\gamma} f^{H+1/2,\lambda}(t-s) P_s \mathrm{d}s.$$

Since $P_t = \int_0^t \sqrt{V_s}\mathrm{d}B_s$, P has the same regularity as a Brownian motion. Thus we can use the same arguments as in Section 4.4 in [JR16] to deduce that Z, and therefore V, are $H-\varepsilon$ Hölder for any $\varepsilon > 0$.

5.3 Proof of Theorem 3

We proceed again in three steps. First we show that the sequence $(X^T, M^{*T}, P^{*T})_{T\geq 0}$ is C-tight for the Skorohod topology. Then we prove the results about the dynamics of the limit points and finally those on the regularity of the limit points.

5.3.1 Tightness of $(X^T, M^{*T}, P^{*T})_{T\geq 0}$

Recall the definition of the renormalized processes

$$\lambda_t^{*T} = \frac{1-a_T}{\mu_T}\lambda_{tT}^T, \quad \Lambda_t^{*T} = \frac{1-a^T}{T\mu_T}\int_0^{tT} \lambda_s^T \mathrm{d}s, \quad X_t^T = \frac{1-a^T}{T\mu_T} N_{tT}^T, \quad Z_t^{*T} = Z_{tT}^T/\sqrt{\mu_T}$$

$$M_t^{*T} = \sqrt{\frac{1-a_T}{T\mu_T}} M_{tT}^T \text{ and } P_t^{*T} = \sqrt{\frac{1-a_T}{T\mu_T}} P_{tT}^T.$$

We have

$$\mathbb{E}[\lambda_t^T] \leq \mu_T + \int_0^t \left(k_T^2(t-s) + \phi_T(t-s)\right)\mathbb{E}[\lambda_s^T]\mathrm{d}s.$$

Thus

$$\mathbb{E}[\lambda_t^T] \leq \frac{\mu_T}{1 - \|\phi_T\|_1 - \|k_T\|_2^2}$$

and consequently

$$\mathbb{E}[\lambda_t^{*T}] \leq \frac{1-a_T}{1-\beta_T - \|k_T\|_2^2} = 1.$$

So

$$\mathbb{E}[X_1^T] = \mathbb{E}[\Lambda_1^{*T}] \leq 1,$$

which gives the tightness of the sequences $\left((X_t^T)_{t\in[0,1]}\right)_{T\geq 0}$ and $\left((\Lambda_t^{*T})_{t\in[0,1]}\right)_{T\geq 0}$, both of them being increasing. Actually we get C-tightness since $|\Delta X^T| + |\Delta \Lambda^{*T}| \leq \frac{1-a_T}{T\mu_T}$ that goes to zero as T goes to infinity. Remark that Lemma 1 still holds under Assumption 3.

The tightness of $(M^{*T})_{T\geq 0}$ and $(P^{*T})_{T\geq 0}$ follows from Theorem VI-4.13 in [JS13] because $\langle M^{*T}\rangle_t = \langle P^{*T}\rangle_t = \Lambda_t^{*T}$ and $\langle M^{*T}, P^{*T}\rangle = 0$. We then obtain C-tightness since $|\Delta M^{*T}| + |\Delta P^{*T}| \leq 2/T$. Finally $(X^T, \Lambda^{*T}, M^{*T}, P^{*T})_{T\geq 0}$ is C-tight for the Skorohod topology on $[0,1]$.

5.3.2 Dynamics of the limit points

We now take (X, X, M, P) a limit point of $(X^T, \Lambda^{*T}, M^{*T}, P^{*T})_{T\geq 0}$. Since (X, X, M, P) is continuous, according to the Skorohod representation theorem, we can consider that $(X^T, \Lambda^{*T}, M^{*T}, P^{*T})_{T\geq 0}$ converges almost surely uniformly towards (X, X, M, P):

$$\sup_{t\in[0,1]} |X_t^T - X_t| \underset{T\to+\infty}{\to} 0, \quad \sup_{t\in[0,1]} |\Lambda_t^{*T} - X_t| \underset{T\to+\infty}{\to} 0,$$

and

$$\sup_{t\in[0,1]} |M_t^{*T} - M_t| \underset{T\to+\infty}{\to} 0, \quad \sup_{t\in[0,1]} |P_t^{*T} - P_t| \underset{T\to+\infty}{\to} 0. \tag{17}$$

From Corollary IX-1.19 in [JS13], we have that M and P are local martingales. Moreover since $[M^{*T}] = [P^{*T}] = X^T$, we have $[M] = [P] = X$ and $\langle M, P\rangle = 0$ using Corollary VI-6.29 in [JS13]. Because M and P are continuous, we deduce

$$\langle M\rangle = [M] = \langle P\rangle = [P] = X.$$

Because $\mathbb{E}[X_1^T]$ is uniformly bounded in T, we get that X_1 is in L^1 and so M and P are true martingales. In addition, the Dambis-Dubin-Schwarz theorem gives the existence of two independent Brownian motions $B^{(1)}$ and $B^{(2)}$ such that

$$M_t = B^{(1)}_{X_t} \text{ and } P_t = B^{(2)}_{X_t}.$$

Recall that for $F^T(t) = \int_0^t T(1-a_T)\psi_T(Ts)\mathrm{d}s$ we have

$$\begin{aligned}\Lambda_t^{*T} &= t(1-a_T) + \int_0^t F^T(t-s)ds + \int_0^t \frac{F^T(t-s)}{\sqrt{T(1-a_T)\mu_T}} dM_s^{*T} \\ &\quad + \int_0^t F^T(t-s)(Z_s^{*T})^2 ds + \int_0^t (1-a_T)(Z_s^{*T})^2 \mathrm{d}s.\end{aligned}$$

According to Lemma 4.3 in [JR16], we have the uniform convergence

$$\int_0^t F^T(t-s)ds \underset{T\to+\infty}{\to} \int_0^t \frac{1}{2} F^{\alpha,\lambda}(t-s)ds.$$

Using integration by parts we obtain

$$Z_t^{*T} = k(0)P_t^{*T} + \int_0^t k'(t-s)P_s^{*T}\mathrm{d}s.$$

Assumption 3 *i*) implies that k' is bounded on $[0,1]$. As a consequence of (17) we have that almost surely, Z^{*T} converges uniformly on $[0,1]$ towards

$$k(0)P_t + \int_0^t k'(t-s)\mathrm{d}P_s\mathrm{d}s = \int_0^t k(t-s)\mathrm{d}P_s$$

which is continuous. This convergence together with Lemma 4.3 in [JR16] implies that almost surely, uniformly in $t \in [0,1]$,

$$\int_0^t F^T(t-s)(Z_s^{*T})^2 ds \underset{T\to+\infty}{\to} \int_0^t F^{\alpha,\lambda}(t-s) Z_s^2 ds$$

and

$$(1-a_T)\int_0^t (Z_s^{*T})^2 \mathrm{d}s \underset{T\to+\infty}{\to} 0.$$

We now prove that $\int_0^t \frac{F^T(t-s)}{\sqrt{T(1-a_T)\mu_T}} dM_s^{*T}$ converges uniformly in probability towards

$$\int_0^t \frac{f^{\alpha,\lambda}(t-s)}{2\sqrt{\lambda\mu^*}} M_s \mathrm{d}s.$$

Using integration by parts we have

$$\int_0^t \frac{F^T(t-s)}{\sqrt{T(1-a_T)\mu_T}} \mathrm{d}M_s^{*T} = \int_0^t \frac{f^T(t-s)}{\sqrt{T(1-a_T)\mu_T}} M_s^{*T} \mathrm{d}s.$$

Remark that

$$\int_0^t f^T(t-s) M_s^{*T} \mathrm{d}s - \int_0^t f^{\alpha,\lambda}(t-s)\frac{1}{2} M_s \mathrm{d}s$$

can be written

$$\int_0^t \frac{1}{2} f^{\alpha,\lambda}(t-s)(M_s^{*T} - M_s)\mathrm{d}s + \int_0^t \big(f^T(t-s) - \frac{1}{2} f^{\alpha,\lambda}(t-s)\big) M_s^{*T} \mathrm{d}s. \tag{18}$$

The first term in (18) goes almost surely uniformly to zero using (17) and the fact that $f^{\alpha,\lambda} \in L^1$. Applying integration by parts again we obtain

$$\int_0^t \big(f^T(t-s) - f^{\alpha,\lambda}(t-s)\big) M_s^{*T} \mathrm{d}s = \int_0^t \big(F^T(t-s) - \frac{1}{2} F^{\alpha,\lambda}(t-s)\big) \mathrm{d}M_s^{*T}$$

and using Burkholder-Davis-Gundy inequality we get (C denotes here a positive constant that varies from line to line)

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t\in[0,1]} \Big(\int_0^t \big(F^T(t-s) - \frac{1}{2} F^{\alpha,\lambda}(t-s)\big) \mathrm{d}M_s^{*T}\Big)^2\Big] & \\
&\le C\mathbb{E}\Big[\int_0^t \big(F^T(t-s) - \frac{1}{2} F^{\alpha,\lambda}(t-s)\big)^2 \mathrm{d}X_s^T\Big] \\
&\le C\int_0^t \big(F^T(t-s) - \frac{1}{2} F^{\alpha,\lambda}(t-s)\big)^2 \frac{1-a_T}{\mu_T} \mathbb{E}[\lambda_{Ts}^T] \mathrm{d}s \\
&\le C\int_0^t \big(F^T(t-s) - \frac{1}{2} F^{\alpha,\lambda}(t-s)\big)^2 \mathrm{d}s.
\end{aligned}$$

This converges to zero according to Lemma 4.3 in [JR16]. So we have proved that

$$X_t = \int_0^t \frac{1}{2} F^{\alpha,\lambda}(t-s)(1+Z_s^2)\mathrm{d}s + \int_0^t f^{\alpha,\lambda}(t-s)\frac{1}{2\sqrt{\lambda\mu^*}} M_s \mathrm{d}s$$

with

$$Z_t^2 = \int_0^t k(t-s)\mathrm{d}P_s.$$

5.3.3 Regularity property

We can write X as

$$X_t = \int_0^t \frac{1}{2} f^{\alpha,\lambda}(t-s)\big(s + \int_0^s Z_u^2 \mathrm{d}u + M_s\big)\mathrm{d}s.$$

Since Z is continuous, $\int_0^t Z_s^2 \mathrm{d}s$ is continuously differentiable. So using the same arguments as in Sections 4.3 and 4.4 in [JR16] replacing s by $s + \int_0^s Z_u^2 \mathrm{d}u$, we obtain that almost surely, X is differentiable with derivative V satisfying

$$V_t = \int_0^t \frac{1}{2} f^{\alpha,\lambda}(t-s)(1+Z_s^2)\mathrm{d}s + \int_0^t \frac{1}{2} f^{\alpha,\lambda}(t-s)\frac{1}{\sqrt{\lambda\mu^*}}\mathrm{d}M_s.$$

We get the stated result using Theorem V-3.9 in [RY13] which gives the existence of two independent Brownian motions $B^{(1)}$ and $B^{(2)}$ such that

$$M_t = \int_0^t \sqrt{V_s}\mathrm{d}B_s^{(1)} \text{ and } P_t = \int_0^t \sqrt{V_s}\mathrm{d}B_s^{(2)}.$$

The regularity property of V is also deduced using the same arguments as in Sections 4.3 and 4.4 in [JR16].

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CHAPTER III

The quadratic rough Heston model: fitting simultaneously historical volatility together with SPX and VIX smiles with a continuous model

Abstract

Fitting simultaneously SPX and VIX smiles is known to be one of the most challenging problems in volatility modeling. A long-standing conjecture due to Julien Guyon is that it may not be possible to calibrate jointly these two quantities with a model with continuous sample-paths. We present the quadratic rough Heston model as a counterexample to this conjecture. The key idea is the combination of rough volatility together with a price-feedback (Zumbach) effect.

Keywords: SPX smiles, VIX smiles, rough Heston model, Zumbach effect, quadratic rough Heston model, Guyon's conjecture.

1 Introduction

The Volatility Index, or VIX, was introduced in 1993 by the Chicago Board Options Exchange (CBOE for short) and was originally designed according to [CBO03] to "measure the market's expectation of 30-day volatility implied by at-the-money S&P 100 Index option price". Since 2003, the VIX has been redefined as the square root of the price of a specific basket of options on the S&P 500 Index (SPX) with maturity 30 days. The basket coefficients are chosen so that at any time t, the VIX represents the annualized square root of the price of a contract with payoff equal to $\log(S_{t+\Delta}/S_t)$ where $\Delta = 30$ days and S denotes the value of the SPX. Consequently, it can be formally written via risk-neutral expectation under the form

$$\mathrm{VIX}_t = \sqrt{-\tfrac{2}{\Delta}\,\mathbb{E}[\log(S_{t+\Delta}/S_t)|\mathscr{F}_t]} \times 100, \tag{1}$$

where $(\mathscr{F}_t)_{t\geq 0}$ is the natural filtration of the market.

Since 2004, investors have been able to trade VIX futures. Quoting the CBOE white paper, they "provide market participants with a variety of opportunities to implement their view using volatility trading strategies, including risk management, alpha generation and portfolio diversification"[1]. Subsequently in 2006, CBOE introduced VIX options "providing market participants with another tool to manage volatility. VIX options enable market participants to hedge portfolio volatility risk distinct from market price risk and trade based on their view of the future direction or movement of volatility"[2]. Those products are now among the most liquid financial instruments in the world. There are indeed more than 500,000 VIX options traded each day, with most of the liquidity concentrated on the first three monthly contracts.

Nevertheless, despite that more vega is now traded in the VIX market than in the SPX market, the wide bid-ask spreads in the VIX options market betray its lack of maturity. One of the reasons underlying these wide spreads is that the market lacks a reliable pricing methodology for VIX options; since the VIX is by definition a derivative of the SPX, any reasonable methodology must necessarily be consistent with the pricing of SPX options. Designing a model that jointly calibrates SPX and VIX options prices is known to be extremely challenging. Indeed, this problem is sometimes considered to be *the holy grail of volatility modeling*. We will simply refer to it as the *joint calibration problem*.

The joint calibration problem has been extensively studied by Julien Guyon who provides a review of various approaches in [Guy19b]. We can split the different attempts to solve it into three categories. In probably the most technical and original proposal, and the first to have succeeded in obtaining a perfect joint calibration, the joint calibration problem is interpreted as a model-free constrained martingale transport problem, as initially observed in [DMHL15]. In his recent paper [Guy19b], using this viewpoint, Guyon manages to get a perfect calibration of VIX options smile at time T_1 and SPX options smiles at dates T_1 and $T_2 = T_1 + 30$ days. As noticed by the author, although this methodology can theoretically be extended to any set of maturities, it is much more intricate in practice because of the computational complexity.

This drawback is avoided in the second and third types of approach where models are in continuous-time. Continuous-time models have the advantage that they rely on observable properties of assets and so allow for practical intuition on their dynamics. The second approach is to attempt joint calibration with models where SPX trajectories are continuous, see in particular [GIP17]. Unfortunately, for now, continuous models have not been completely successful in this task. An interpretation for this failure is given in [Guy19b] where the author explains that "the very large negative skew of short-term SPX options, which in continuous models implies a very large volatility of volatility, seems inconsistent with the comparatively low levels of VIX implied volatilities". To circumvent this issue, it is then natural to think of rough volatility models as recently introduced in [GJR18]. However, these models also appear unsuccessful thus far, see [Guy18].

[1] https://cfe.cboe.com/cfe-products/vx-cboe-volatility-index-vix-futures

[2] http://www.cboe.com/products/vix-index-volatility/vix-options-and-futures/vix-options

The last approach is to allow for jumps in the dynamic of the SPX, see [BB14, CK13, KS15, PPR18, PS14]. Doing so, one can reconcile the skewness of SPX options with the level of VIX implied volatilities. Nevertheless, probably besides those in [CK13] and [PPR18], existing models with jumps do not really achieve a satisfying accuracy for the joint calibration problem. Specifically, most of them fail to reproduce VIX smiles for maturities shorter than one month.

As an aside, even though some models with jumps may satisfactorily resolve the joint calibration problem, such models are unsatisfactory in other respects. For example, perfect hedging is not possible in such models whereas in contrast, under rough volatility, derivatives hedging is fully understood as shown in [EER18, EER19], see also [AJLP19, CT20]. Moreover, jumps are conventionally modeled as Lévy jumps, giving rise to unrealistic model time series properties that are at odds with those observed empirically, specifically the clustering of large moves in the underlying. One could imagine trying to fix the latter problem by modeling with self-exciting jump processes. But that would lead in the end back to rough volatility models, which can be regarded as special limits of self-exciting jump processes.

In summary, according to Guyon in [Guy19b], despite the many efforts "so far all the attempts at solving the joint SPX/VIX smile calibration problem [using a continuous time model] only produced imperfect, approximate fits". In particular, regarding continuous models, Guyon concludes that "joint calibration seems out of the reach of continuous-time models with continuous SPX paths". In this paper, we provide a counterexample to Guyon's conjecture, namely a model with continuous SPX and VIX paths that enables us to fit SPX and VIX options smiles simultaneously.

2 Rough volatility and the Zumbach effect

Recently rough volatility models, where volatility trajectories, though continuous, are very irregular, have generated a lot of attention. The reason for this success is the ability of these very parsimonious models to reproduce all the main stylized facts of historical volatility time series and to fit SPX options smiles, see [BFG16, EEGR19, GJR18]. One particularly interesting rough volatility model is the rough Heston model introduced in [EER19] which as its name suggests, is a rough version of the classical Heston model. This model arises as limit of natural Hawkes process-based models of price and order flow, see [EEFR18, JR16, JR18]. Moreover, there is a quasi-closed form formula for the characteristic function of the rough Heston model, just as in the classical case. So fast pricing of European options is possible, see [CGP18, GR19].

Despite these successes, a subtle question raised by Jean-Philippe Bouchaud remains: can a rough volatility model reproduce the so-called *Zumbach effect*, the observation originally due to Gilles Zumbach, see [LZ03, Zum09, Zum10], that financial time series are not time-reversal invariant? To answer this question, we introduce two notions, each of which corresponds to different aspects of the Zumbach effect:

- The *weak Zumbach effect* (typically considered in the econophysics literature, see [Zum09]): Past squared returns forecast better future integrated volatilities than past integrated volatilities forecast future squared returns. This property is not satisfied in classical stochastic volatility models. However, rough stochastic volatility models are consistent with the weak Zumbach effect, see [EEGRR20] for explicit computations in the rough Heston model.
- The *strong Zumbach effect*: Conditional dynamics of volatility with respect to the past depend not only on the past volatility trajectory but also on the historical price path; specifically, price trends tend to increase volatility, see [Zum10]. Such feedback of the historical price path on volatility also occurs on implied volatility as illustrated in Figure 1 of [FP08] and in [Zum10]. Rough stochastic volatility models such as the rough Heston model are not consistent with the strong Zumbach effect, see [EER18].

The quest for a rough volatility model consistent with the strong Zumbach effect and the empirical success of quadratic Hawkes process-based models documented in [BDB17] led to the development of super-Heston rough volatility models in [DJR19]. These extensions of the rough Heston model arise as limits of quadratic Hawkes process-based microstructural models just as the rough Heston model arises as the continuous-time limit of a linear Hawkes process-based microstructural model.

The idea of using super-Heston rough volatility models to solve the joint calibration problem came after a presentation of Julien Guyon at École Polytechnique in March 2019. In this talk, he gave a necessary condition for a continuous model to fit simultaneously SPX and VIX smiles: The inversion of convex ordering between volatility and the local volatility implied by option prices, see [AG19, Guy19a]. The intuition behind this condition could be reinterpreted as some kind of strong Zumbach effect. It was then natural for us to investigate the ability of super-Heston rough volatility models to solve the joint calibration problem.

3 The quadratic rough Heston model

The quadratic rough Heston model that we consider is essentially a special case of the super-Heston rough volatility models of [DJR19]. The joint dynamics of the asset S (here the SPX), and its spot variance V satisfy

$$\mathrm{d}S_t = S_t\sqrt{V_t}dW_t,\ V_t = a(Z_t - b)^2 + c,$$

where W is a Brownian motion, a, b and c some positive constants and Z_t is of rough Heston type, in the sense that weighted past price returns are drivers of the volatility dynamics. More precisely,

$$Z_t = \int_0^t (t-s)^{\alpha-1}\frac{\lambda}{\Gamma(\alpha)}(\theta_0(s) - Z_s)\mathrm{d}s + \int_0^t (t-s)^{\alpha-1}\frac{\lambda}{\Gamma(\alpha)}\eta\sqrt{V_s}\mathrm{d}W_s, \tag{2}$$

with $\alpha \in (1/2, 1)$, $\lambda > 0$, $\eta > 0$ and θ_0 a deterministic function. In this special case of a super-Heston rough volatility model, the asset S and its volatility depend on the history of

only one Brownian motion. The model is thus a pure feedback model; volatility is driven only by the price dynamics, with no additional source of randomness. In general of course, the volatility process does not need to depend only on the Brownian motion driving the asset price S. For simplicity, we will refer to (2), a pure feedback version of a super-Heston rough volatility model, as the *quadratic rough Heston model*.

As in the general case of super-Heston rough volatility models, because the effect of past returns on Z cannot be reduced to an influence of past volatility dynamics on Z, the quadratic rough Heston model also exhibits the strong Zumbach effect (see [DJR19] for more details).

3.1 The quadratic rough Heston process

The process Z_t may be understood as a weighted moving average of past price log returns. Indeed from Lemma A.1 in [EER18], we have that

$$Z_t = \int_0^t f^{\alpha,\lambda}(t-s)\theta_0(s)\mathrm{d}s + \int_0^t f^{\alpha,\lambda}(t-s)\eta\sqrt{V_s}\mathrm{d}W_s,$$

where $f^{\alpha,\lambda}(t)$ is the Mittag-Leffler density function defined for $t \geq 0$ as

$$f^{\alpha,\lambda}(t) = \lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}),$$

with

$$E_{\alpha,\beta}(z) = \sum_{n\geq 0} \frac{z^n}{\Gamma(\alpha n + \beta)}.$$

The variable Z_t is therefore *path-dependent*, a weighted average of past returns of the type typically considered in path-dependent volatility models, see [HR98]. As explained in [Guy14], modeling with path-dependent variables is a natural way to reproduce the fact that volatility depends on recent price changes. However the kernels used to model this dependency are typically exponential, see for example [HR98]. Here a crucial idea, motivated by our previous work [DJR19], is to use a rough kernel, more precisely the Mittag-Leffler density function. Thanks to this kernel, the "memory" of Z decays as a power law and Z is highly sensitive to recent returns since

$$f^{\alpha,\lambda}(t) \underset{t\to+\infty}{\sim} \frac{\alpha}{\lambda\Gamma(1-\alpha)} t^{-\alpha-1} \text{ and } f^{\alpha,\lambda}(t) \underset{t\to 0^+}{\sim} \frac{\lambda}{\Gamma(\alpha)} t^{\alpha-1}.$$

This essentially means that long periods of trends or sudden upwards or downwards moves of the price generate large values for $|Z|$ and so high volatility, in particular when Z is negative. Such link is clearly observed on data, see Figure III.1 where the VIX index spikes almost instantaneously after large negative returns of the SPX and then decreases slowly afterwards. We plot in Figure III.2 an example of sample paths of SPX and VIX indexes in our model. The feedback of negative price trends on volatility is very well reproduced. Finally the choice of $f^{\alpha,\lambda}$ as kernel ensures that the volatility process is rough, with Hurst parameter equal to $H = \alpha - 1/2$. As shown in [GJR18], this enables us to reproduce the behavior of historical

volatility time series provided H is taken of order 0.1.

As explained above, an immediate consequence of the feedback effect is that negative price trends generate high volatility levels. But such trends also impact the instantaneous variance of volatility in our model. To see this, consider the classical case with $\alpha = 1$. In that case, an application of Itô's Formula gives that up to a drift term,

$$\mathrm{d}V_t = 2a(Z_t - b)\lambda\eta\sqrt{V_t}\mathrm{d}W_t.$$

Thus the "variance of instantaneous variance" coefficient is proportional to $a(Z_t - b)^2$ which, up to c, is equal to the variance of $\log S$. Thus when volatility is high, volatility of volatility is also high. In particular, conditional on a large downwards move in SPX, we would expect V to be high and so also the volatility of V. This explains why our model generates upward sloping VIX smiles.

We remark that incorporating the influence of price trends on volatility and instantaneous variance of volatility is the main motivation underlying the model of [GIP17]. That model, although not solving the joint calibration problem, is probably the best of the continuous models introduced so far. In this switching model, the price follows a classical Heston dynamic where the parameters can change depending on the value of an hidden Markov chain with three states. It is motivated by a 100-days rolling calibration of the classical Heston model performed by the authors, see Figure 2 in [GIP17]. This rolling calibration suggests that volatility, leverage and volatility of volatility are higher in period of crisis. Hence they introduce a Markov chain to trigger crisis phases and switch the parameters of the Heston model depending on the situation. The three possible states of the chain can therefore be interpreted as corresponding to the following situations:

- Flat or increasing SPX.
- Transition phase between flat SPX and crisis.
- Crisis with dramatically decreasing SPX.

The Markov chain in [GIP17] can therefore somehow be seen as an *ad hoc* version of the process Z in the quadratic rough Heston model.

3.2 Parameter interpretation

The parameters a, b and c in the specification

$$V_t = a(Z_t - b)^2 + c$$

can be interpreted in the following way.

- c represents the minimal instantaneous variance. When calibrating the model, we use c to shift upward or downward the smiles of SPX options.

- $b > 0$ encodes the asymmetry of the feedback effect. Indeed for the same absolute value of Z, volatility is higher when Z is negative than when it is positive. Such asymmetry aims at reproducing the empirical behavior of the VIX. This is illustrated in Figure III.1 where we observe that the VIX spikes when the SPX tumbles down, but not after it goes up. From a calibration viewpoint, the higher b the more SPX options smiles are shifted to the right.

- a is the sensitivity of the volatility to the feedback of price returns. The greater a, the greater the role of feedback in the model and the higher is volatility of volatility. Consistent with this SPX smiles become more extreme as a increases.

3.3 Infinite dimensional Markovian representation

Though the quadratic rough Heston model is not Markovian in the variables (S, V), it does admit an infinite dimensional Markovian representation. Inspired by the computations in [EER18], we obtain that for any t and t_0 positive

$$Z_{t_0+t} = \int_0^t (t-s)^{\alpha-1} \frac{\lambda}{\Gamma(\alpha)} (\theta_{t_0}(s) - Z_{t_0+s}) \mathrm{d}s + \int_0^t (t-s)^{\alpha-1} \frac{\lambda}{\Gamma(\alpha)} \eta \sqrt{V_{t_0+s}} \mathrm{d}W_{t_0+s}, \tag{3}$$

with θ_{t_0} a $\mathscr{F}_{t_0}$-measurable function. More precisely θ_{t_0} is given by

$$\theta_{t_0}(u) = \theta_0(t_0+u) + \frac{\alpha}{\lambda \Gamma(1-\alpha)} \int_0^{t_0} (t_0 - v + u)^{-1-\alpha} Z_v \mathrm{d}v.$$

Equation (3) implies that the law of $(S_t, V_t)_{t \geq t_0}$ only depends on S_{t_0} and θ_{t_0}. In view of (1) and using the same methodology as in [EER18], it means that we can express the VIX at time t as a function of θ_t and S_t. Consequently we can write the price of any European option with pay-off depending on SPX and VIX as a function of time, S and θ.

4 Numerical results

In this section, we illustrate how successfully we can fit both SPX and VIX smiles on May 19, 2017[3], one of the dates considered in [EEGR19], an otherwise randomly chosen date. We focus on short expirations, from 2 to 5 weeks, where the bulk of VIX liquidity is. Moreover, short-dated smiles are the ones that are typically poorly fitted by conventional models.

In the quadratic rough Heston model, the function $\theta_0(\cdot)$ needs to be calibrated to market data. In the rough Heston model there is a simple bijection between $\theta_0(\cdot)$ and the forward variance curve. In the quadratic rough Heston model, this connection is more intricate and so for simplicity we choose the following restrictive parametric form for Z:

$$Z_t = Z_0 - \int_0^t (t-s)^{\alpha-1} \frac{\lambda}{\Gamma(\alpha)} Z_s \mathrm{d}s + \int_0^t (t-s)^{\alpha-1} \frac{\lambda}{\Gamma(\alpha)} \eta \sqrt{V_s} \mathrm{d}W_s,$$

[3]Market data is from OptionMetrics via Wharton Data Research Services (WRDS).

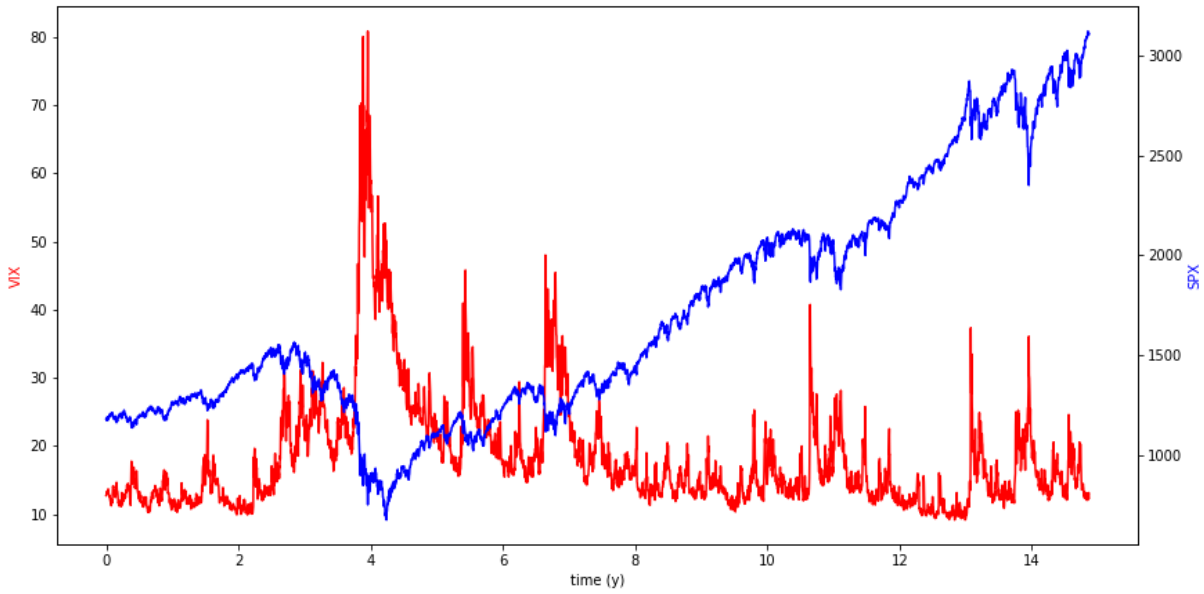

Figure III.1 – SPX (in blue) and VIX (in red) indexes from 25 November 2004 to 25 November 2019.

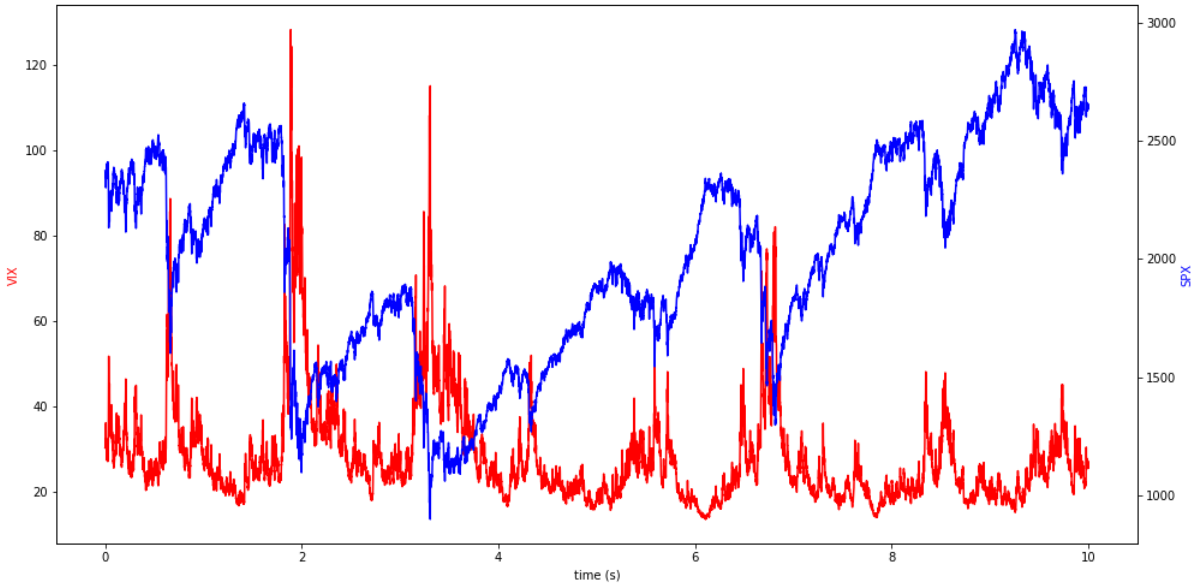

Figure III.2 – SPX (in blue) and VIX (in red) indexes from simulation of the quadratic rough Heston model.

which is equivalent to taking

$$\theta_0(t) = \frac{Z_0}{\lambda\Gamma(1-\alpha)}\, t^{-\alpha}.$$

Allowing $\theta_0(\cdot)$ to belong to a larger space would obviously lead to even better results, but would require a more complex calibration methodology. Thus we are left to calibrate the parameters $\nu = (\alpha, \lambda, a, b, c, Z_0)$. We use the following objective function:

$$F(\nu) = \frac{1}{\#\mathcal{O}^{SPX}} \sum_{o \in \mathcal{O}^{SPX}} (\sigma^{o,mid} - \sigma^{o,\nu})^2 + \frac{1}{\#\mathcal{O}^{VIX}} \sum_{o \in \mathcal{O}^{VIX}} (\sigma^{o,mid} - \sigma^{o,\nu})^2,$$

where $\mathcal{O}^{SPX}$ is the set of SPX options, $\mathcal{O}^{VIX}$ the set of VIX options, $\sigma^{o,mid}$ denotes the market mid implied volatility for the option o and $\sigma^{o,\nu}$ is the implied volatility of the option o in the quadratic rough Heston model with parameter ν obtained by Monte-Carlo simulations. To calibrate the model, we minimize the function F over a grid centered around an initial guess ν_0.

We obtain the following parameters:

$$\alpha = 0.51;\ \lambda = 1.2;\ a = 0.384;\ b = 0.095;\ c = 0.0025,\ Z_0 = 0.1. \tag{4}$$

The corresponding SPX and VIX options smiles are plotted in Figures .2 and .3.

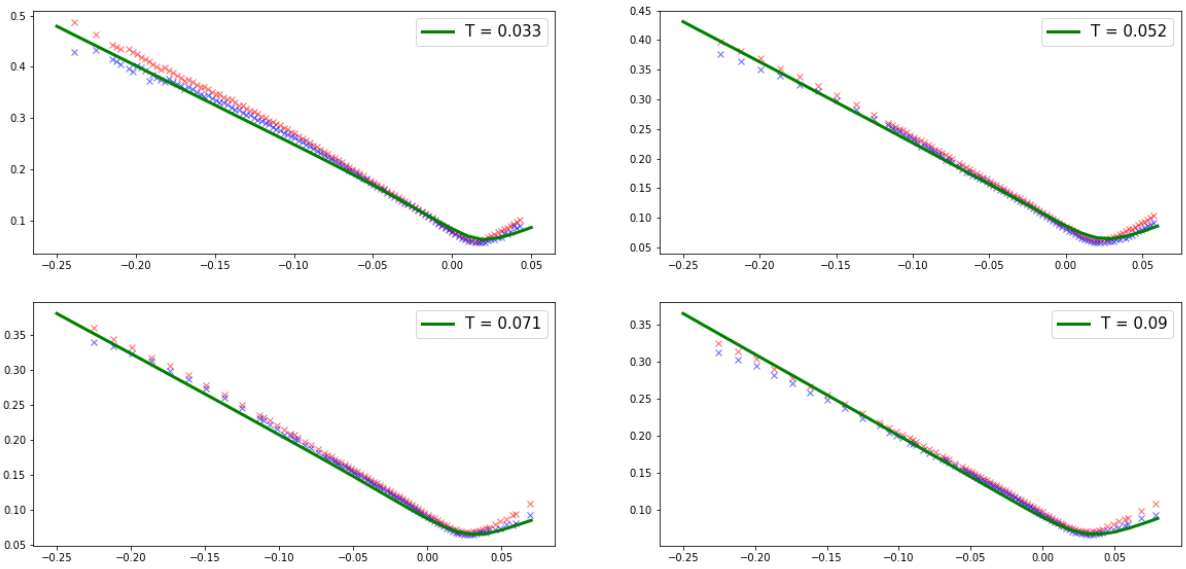

Figure III.3 – Implied volatility on SPX options for 19 May 2017. The blue and red points are respectively the bid and ask of market implied volatilities. The implied volatility smiles from the model are in green. The strikes are in log-moneyness and T is time to expiry in years.

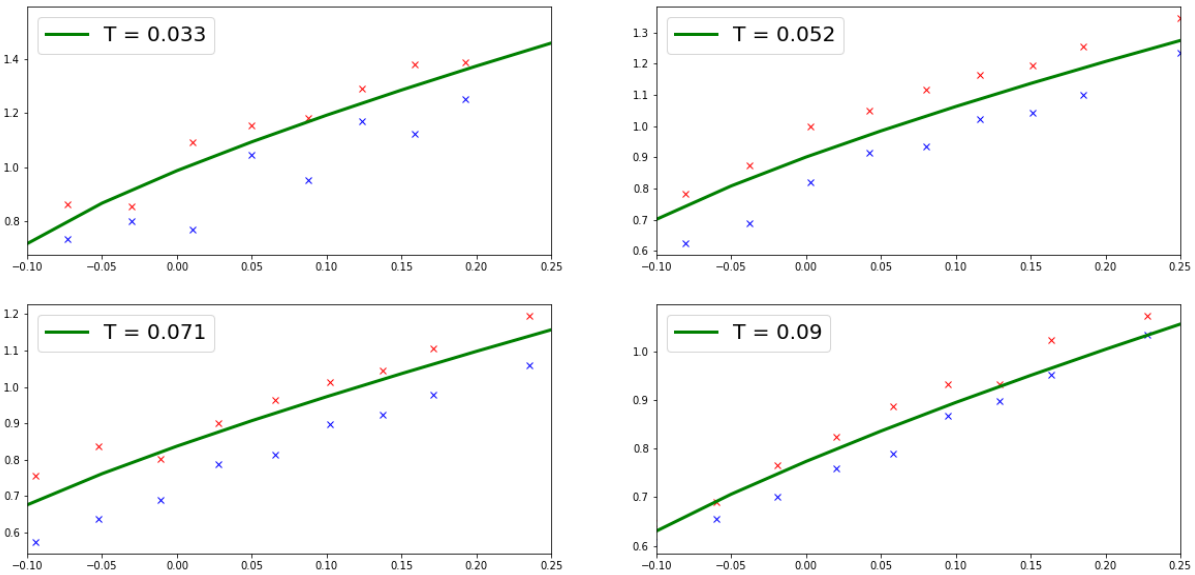

Figure III.4 – Implied volatility on VIX options for 19 May 2017. The blue and red points are respectively the bid and ask of market implied volatilities. The implied volatility smiles from the model are in green. The strikes are in log-moneyness and T is time to expiry in years.

Despite that our calibration methodology is suboptimal and we only have six parameters, VIX smiles generated by the model with parameters (4) fall systematically within market bid-ask spreads. The overall shape of the shorter-dated SPX smiles shown in Figure 3, are well reproduced. Obviously fits can be made even greater by reducing the range of strikes of interest or by fine tuning the calibration, notably through improving the $\theta_0(\cdot)$ function. We are currently working on a fast calibration approach, inspired by recent works on neural networks, see for example [HMT19].

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Part II

Optimal control with point processes and application to finance

CHAPTER IV

Optimal market making with persistent order flow

Abstract

We address the issue of market making on electronic markets when taking into account the self exciting property of market order flow. We consider a market with order flows driven by Hawkes processes where one market maker operates, aiming at optimizing its profit. We characterize an optimal control solving this problem by proving existence and uniqueness of a viscosity solution to the associated Hamilton Jacobi Bellman equation. Finally we propose a methodology to approximate the optimal strategy.

Keywords: Hawkes processes, market making, high frequency trading, stochastic control, viscosity solutions.

1 Introduction

Most electronic exchanges are organized as an anonymous continuous double auction system. Market participants can send limit orders to a central limit order book (LOB for short) displaying a volume of shares and a price at which they are ready to buy or sell. Limit orders in the LOB can be canceled (cancellation order). Market participants can also use market orders specifying a volume in order to buy or sell instantaneously at the best available price. In a very stylized view we can consider that there are two types of market participants: market takers seeking to buy or sell shares for strategic purposes using market orders and market makers filling the LOB with limit orders so that they play the role of intermediaries between buyer and seller market takers.

In practice one of the main risk faced by a market maker is the inventory risk. For example if he has a large positive inventory, price may decrease to his disadvantage. Market makers thus adapt their strategies to mitigate this risk. Basically we expect a market maker with a large positive inventory to set attractive ask prices and less competitive bid prices, in order to attract more buy than sell market orders. To do so in a relevant way he must therefore adapt

his strategy to the main statistical features of the order flow.

Some key aspects that market makers should incorporate in their trading strategies are the clustering and long memory properties of order flow. The clustering property refers to the fact that buy and sell market orders are not distributed homogeneously in time but tend to be clustered, see [Hew06]. In practice it means that after a buy (for say) market order it is likely that a new one is going to be sent shortly. Long memory of order flow designates the fact that the autocorrelation function of trade sign (+1 for a buy order and −1 for a sell order) exhibits a power-law tail, see [LF04]. These two properties imply that market order flow is very persistent. Our goal in this paper is to propose a method to design market making strategies that take into account those two features. For this purpose we consider a market with one market maker controlling the best bid and ask prices and with market takers using only market orders (of unit volume).

The issue of market making while managing an inventory risk has been notably addressed in [AS08, GLFT13] where market order flow is modeled using Poisson processes, see also the books [CJP15, Gué16]. However these processes neither reproduce the clustering nor the long memory property of order flow. To take them into account, the authors of [CJR14, CWZZ20] use a refined model based on Hawkes processes with exponential kernels. Such modeling is also used in [AB16, Hew06] to design optimal liquidation strategies. In this work we consider generalized Hawkes processes. More precisely N is a generalized Hawkes process with intensity λ_t if

$$\lambda_t = \Phi\Big(\int_0^t K(t-s)\mathrm{d}N_s\Big),$$

where Φ is a continuous function and K a completely monotone L^1 function[1]. In [AB16, CJR14] the authors consider exponential functions for K. For such kernel Hawkes processes manage to reproduce the clustering property of the order flow but not its long memory. However when the kernel K has a power law tail: $K(t) \sim t^{-\beta}$ for some $\beta > 1$, both properties are reproduced, see [BJM16, JR16]. So in this paper we extend the works [AS08, CJR14, GLFT13] to market order flows driven by Hawkes processes with general kernels.

We denote by N_t^a (resp. N_t^b) the total number of buy (resp. sell) market orders sent between time 0 and time t and write $i_t := N_t^b - N_t^a$ for the market maker's inventory, which is null at time 0. As in [AS08] the market maker controls the bid and ask spreads, denoted by δ^a and δ^b. The corresponding best ask and bid prices are $P + \delta^a$ and $P - \delta^b$, where P is the fundamental price of the underlying asset. The set of admissible controls is then

$$\mathscr{A} = \{\delta = (\delta^a, \delta^b) \in \mathbb{R}_+^2, \text{ s.t. } \delta \text{ is predictable}\},$$

where predictability is relative to the natural filtration generated by (P, N^a, N^b), see Section 5.1 for more details. Since market takers are seeking for low transaction costs, their trading intensity is decreasing with the spreads. More precisely we know from classical financial

[1] In this paper we consider complete monotony on $\mathbb{R}_+$.

economics results, see [DR15, MRR97, WBK+08], that the average number of trades per unit of time is a decreasing function of the ratio between spread and volatility. To model this we consider that when the spreads are δ^a and δ^b market order intensities are given by

$$\lambda_t^{a,\delta} = e^{-\frac{k}{\sigma}\delta_t^a}\lambda_t^{a,0} \text{ and } \lambda_t^{b,\delta} = e^{-\frac{k}{\sigma}\delta_t^b}\lambda_t^{b,0},$$

where k is a positive constant and σ is the volatility of price and

$$\lambda_t^{a,0} = \Phi\Big(\int_0^t K(t-s)\mathrm{d}N_s^a\Big),\ \lambda_t^{b,0} = \Phi\Big(\int_0^t K(t-s)\mathrm{d}N_s^b\Big).$$

Regarding the dynamic of P we assume it is given by

$$\mathrm{d}P_t = d(t,P_t)\mathrm{d}t + \sigma\mathrm{d}W_t \tag{1}$$

where d is a Lipschitz function. Finally we formalize the market maker problem as a general stochastic control problem

$$\sup_{\delta\in\mathscr{A}} \mathbb{E}^\delta\Big[G(i_T,P_T)e^{-rT} + \int_0^T e^{-rs}\big(g(i_s,P_s)\mathrm{d}s + \delta_s^a\mathrm{d}N_s^a + \delta_s^b\mathrm{d}N_s^b\big)\Big], \tag{2}$$

where r is a positive constant and g and G are two continuous functions with at most quadratic growth. The former represents a continuous reward received by the market maker (besides its P&L) and the latter is a final lump sum payment paid to the market maker at the end of its trading. Typical choices would be $G(x,y) = xy$ and $g(x,y) = x^2$. The notation $\mathbb{E}^\delta$ denotes the expectation under the law corresponding to the control δ (see Section 5.1 for details).

In [AS08], to solve the market maker's optimization problem in a Poisson context, the authors study the associated Hamilton Jacobi Bellman equation (HJB for short). They can use this method since (P,i,N^a,N^b) is Markovian in this case. However when N^a and N^b are Hawkes processes, notably when the kernel is not exponential, (P,i,N^a,N^b) is not Markovian[2]. In order to circumvent this difficulty we consider auxiliary state variables enabling us to work in a Markovian setting. More precisely we consider the process $X = (P,i,\theta^a,\theta^b)$ where

$$\theta_t^a = \int_0^t K(\cdot - s)\mathrm{d}N_s^a \text{ and } \theta_t^b = \int_0^t K(\cdot - s)\mathrm{d}N_s^b.$$

Note that here θ_t^a and θ_t^a are random functions from $\mathbb{R}_+$ into $\mathbb{R}_+$[3]. The process $(t,X_t)_{t\geq 0}$ is Markovian. Moreover, studying the HJB equation associated with this representation, we prove in Section 2 that the optimal control problem (2) admits a solution of the form

$$\delta_t^* = \delta^K(t,X_t),$$

where δ^K is a feedback control function. In our approach the HJB equation is defined on a subset of an infinite dimensional vector space. So in general we cannot rely on classical numerical methods to approximate δ^K. To tackle this issue we propose the following strategy.

[2] In the exponential case the process $(P,i,N^a,N^b,\lambda^b,\lambda^a)$ is actually Markovian.

[3] To define θ_t^a and θ_t^b we consider that K is extended to $\mathbb{R}$ with value 0 on $\mathbb{R}_-^*$.

- We show that if $(K_n)_{n\geq 0}$ converges towards K in L^1 and uniformly on $[0,T]$ then $(\delta^{K_n})_{n\geq 0}$ converges point-wise towards δ^K.
- We show that when $K(t)=\sum_{i=1}^n \alpha_i e^{-\gamma_i t}$ then there exists a Markovian representation of the model in dimension $2n+2$. Therefore in this case the optimal control δ^K can be approximated numerically.
- Inspired by [AJ19], we prove that for any completely monotone kernel K in L^1, we can find a sequence $(K_n)_{n\geq 0}$, converging towards K in L^1 and uniformly on $[0,T]$ and such that for any n, K_n is a linear combination of n decreasing exponential functions.

Those three points give a simple methodology to approximate δ^K. However when n is large we cannot rely on finite differences methods to compute δ^{K_n} since the dimension is too large. So for numerical experiments we use the probabilistic representation of semi-linear partial differential equations (PDEs for short) introduced in [HLOT+19].

The paper is organized as follows. In Section 2 we prove existence of a solution to Problem (2) based on the study of its associated HJB equation. In Section 3 we explain how to approximate the optimal control obtained in Section 2. Finally in Section 4 we present some numerical experiments. The proofs are relegated to the Appendix.

2 Solving the market maker problem using viscosity solutions

In this section we prove existence of a solution to Problem (2). First we define an appropriate set for the process X. Then we show that the associated HJB equation has a unique viscosity solution with polynomial growth and prove the existence of an optimal control solving (2).

2.1 Appropriate domain for the process X

To study the uniqueness of solution to a PDE in the sense of viscosity, it is convenient to deal with locally compact domain. We have $X=(P,i,\theta^a,\theta^b)\in\mathbb{R}\times\mathbb{Z}\times L^1\times L^1$, but since L^1 is not locally compact we need to specify more precisely the set in which the processes θ^a and θ^b belong. Obviously we have for $j=a$ or b

$$\theta^j_t\in\Theta^K_t=\{\sum_{i=1}^n K(\cdot-T_i),\ n\in\mathbb{N},\ T_1\leq\cdots\leq T_n\leq t\}\subset\Theta^K_T.$$

We naturally endow Θ^K_T with the topology of L^1 and prove in Appendix IV.A that it enjoys the following topological properties.

Lemma 1.

(i) The set Θ^K_T is a locally compact closed subset of L^1.

(ii) For any sequence $(s_n,\theta_n)_{n\geq 0}$ with values in $[0,T]\times\Theta^K_T$ such that for any n, $\theta_n\in\Theta^K_{s_n}$, if $(s_n,\theta_n)_{n\geq 0}$ converges towards (s,θ) then we have $\theta\in\Theta^K_s$ and $\theta_n(s_n)\to\theta(s)$ when $n\to+\infty$.

(iii) Moreover if K is a sum of exponential functions then we have for any $k \geq 0$,

$$\theta_n^{(k)}(T) \to \theta^{(k)}(T), \text{ when } n \to +\infty.$$

From point (i) in Lemma 1 the set θ_T^K is adapted to our purpose. Points (ii) and (iii) are purely technical and are used in Section 3. Based on the sets $(\Theta_t^K)_{t\in[0,T]}$ we define a locally compact domain for X. More precisely for any $t \in [0,T]$ we consider

$$\mathcal{Z}_t^K = \{(i,\theta^a,\theta^b) \in \mathbb{Z}\times\Theta_t^K\times\Theta_t^K\} \text{ and } \mathcal{X}_t^K = \{(p,i,\theta^a,\theta^b) \text{ s.t. } p\in\mathbb{R} \text{ and } (i,\theta^a,\theta^b)\in\mathcal{Z}_t^K\}.$$

The set $\mathcal{Z}_t^K$ (resp. $\mathcal{X}_t^K$) is a locally compact closed subset of $\mathbb{Z}\times L^1\times L^1$ (resp. $\mathbb{R}\times\mathbb{Z}\times L^1\times L^1$). We finally define

$$\mathcal{E}^K = \{(t,x)\in[0,T]\times\mathcal{X}_T^K \text{ s.t } x\in\mathcal{X}_t^K\}$$

which is a locally compact closed subset of $[0,T]\times\mathbb{R}\times\mathbb{Z}\times L^1\times L^1$. Obviously we have $(t,X_t)\in\mathcal{E}^K$ for any $t\in[0,T]$. To lighten the notations when we consider $x\in\mathcal{E}^K$ (resp. $x\in\mathcal{X}_t^K$, $z\in\mathcal{Z}_t^K$) we implicitly assume that $x=(t,p,i,\theta^a,\theta^b)$ (resp. $x=(p,i,\theta^a,\theta^b)$, $z=(i,\theta^a,\theta^b)$). We also define for any $x=(p,i,\theta^a,\theta^b)\in\mathbb{R}\times L^1\times L^1\times\mathbb{Z}$ the norm $\|x\| = \sqrt{p^2+i^2+\|\theta^a\|_1^2+\|\theta^b\|_1^2}$ and for any non-negative R the set,

$$\mathcal{E}_R^K = \{(t,x)\in\mathcal{E}^K, \text{ s.t. } \|x\|\leq R\},$$

which is a compact subset of $\mathcal{E}^K$ as consequence of Lemma 1 (i).

Now that we have defined a set adapted to PDE analysis we derive in the next section the HJB equation related to the stochastic control problem (2).

2.2 Hamilton-Jacobi-Bellman equation associated to the control problem

The stochastic control problem (2) is written in an unconventional way because of the integrals

$$\int_0^T \delta_s^a \mathrm{d}N_s^a \text{ and } \int_0^T \delta_s^b \mathrm{d}N_s^b.$$

However up to a $\mathbb{P}^\delta$-local martingale those terms are respectively equal to $\int_0^T \delta_s^a\lambda_s^{a,\delta}\mathrm{d}s$ and $\int_0^T \delta_s^b\lambda_s^{b,\delta}\mathrm{d}s$. So as consequence of Appendix IV.B.3 for any $\delta\in\mathcal{A}$ we have

$$\begin{aligned}\mathbb{E}^\delta[G(i_T,P_T)e^{-rT} &+ \int_0^T e^{-rs}\Big(g(i_s,P_s)\mathrm{d}s+\delta_s^a\mathrm{d}N_s^a+\delta_s^b\mathrm{d}N_s^b\Big)] \\ &= \mathbb{E}^\delta[G(i_T,P_T)e^{-rT}+\int_0^T e^{-rs}\Big(g(i_s,P_s)+\delta_s^a\lambda_s^{a,\delta}+\delta_s^b\lambda_s^{b,\delta}\Big)\mathrm{d}s].\end{aligned}$$

Thus (2) is equivalent to the stochastic control problem

$$\sup_{\delta\in\mathcal{A}} \mathbb{E}^\delta[G(i_T,P_T)e^{-rT}+\int_0^T e^{-rs}\Big(g(i_s,P_s)+\delta_s^a\lambda_s^{a,\delta}+\delta_s^b\lambda_s^{b,\delta}\Big)\mathrm{d}s]. \tag{3}$$

In order to give intuition on the HJB equation related to this stochastic control problem we write the Ito formula related to X. We consider a function φ defined on $[0,T]\times\mathbb{R}\times\mathbb{Z}\times L^1\times L^1$ that is $C^{2,2,0,0,0}$. We call any function with such regularity a *test function*. For any $s<t\in[0,T]$ we have

$$\begin{aligned}\varphi(t,X_t)-\varphi(s,X_s)=\int_s^t\Big(\partial_t\varphi(u,X_{u-})+\mathscr{L}^P\varphi(u,X_{u-})+\sum_{j=a,b}D_j^K\varphi(u,X_{u-})e^{-\frac{k}{\sigma}\delta_u^j}\Phi\big(\theta_u^j(u)\big)\Big)\mathrm{d}u\\ +\partial_p\varphi(u,X_{u-})\sigma\mathrm{d}W_u+D_a^K\varphi(u,X_{u-})\mathrm{d}M_u^{a;\delta}+D_b^K\varphi(u,X_{u-})\mathrm{d}M_u^{b;\delta},\end{aligned}$$

where

$$M_t^{a;\delta}=N_t^a-\int_0^t\lambda_s^{a,\delta}\mathrm{d}s\text{ and }M_t^{b;\delta}=N_t^b-\int_0^t\lambda_s^{b,\delta}\mathrm{d}s$$

are $\mathbb{P}^\delta$-uniformly integrable martingales, see Appendix IV.B.1 for details. The operator $\mathscr{L}^P$ is the infinitesimal generator related to the diffusion of P and is defined for any test function φ and $(t,x)\in\mathscr{E}^K$ by

$$\mathscr{L}^P\varphi(t,x)=d(t,p)\partial_p\varphi(t,x)+\frac{1}{2}\sigma^2\partial_{pp}^2\varphi(t,x).$$

The operators D_a^K and D_b^K correspond to the infinitesimal generators related to the diffusion of N^a and N^b. They are defined for $(t,x)\in\mathscr{E}^K$ by

$$\begin{aligned}D_a^K\varphi(t,x)&=\varphi\big(t,p,i-1,\theta^a+K(\cdot-t),\theta^b\big)-\varphi(t,p,i,\theta^a,\theta^b),\\ D_b^K\varphi(t,x)&=\varphi\big(t,p,i+1,\theta^a,\theta^b+K(\cdot-t)\big)-\varphi(t,p,i,\theta^a,\theta^b).\end{aligned}$$

Hence the HJB equation associated to the control problem (3) is

$$(\mathbf{HJB})_K:\begin{cases}F\big(x,U(x),\nabla U(x),\partial_{pp}^2U(x),D^KU(x)\big)=0\text{ for }x\in\mathscr{E}^K,\\ U(T,y)=G(i,p)\text{ for }y\in\mathscr{X}_T^K\end{cases}$$

with $\nabla U=\big(\partial_tU,\partial_pU\big)$, $D^KU=\big(D_a^KU,D_b^KU\big)$ and where the function F is defined for $(x,u,q,A,I)\in\mathscr{E}_t^K\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}\times\mathbb{R}^2$ by

$$\begin{aligned}F(x,u,q,A,I)=&ru-q_1-d(t,p)q_2-\frac{1}{2}\sigma^2A-g(i,p)\\ &-\sup_{\delta\in\mathbb{R}_+}\Phi\big(\theta^a(t)\big)e^{-\frac{\alpha}{k}\delta}(\delta+I_1)-\sup_{\delta\in\mathbb{R}_+}\Phi\big(\theta^b(t)\big)e^{-\frac{\alpha}{k}\delta}(\delta+I_2).\end{aligned}$$

A simple computation gives the maximizers

$$\delta^{*a}=\big(\sigma/k-I_1\big)_+\text{ and }\delta^{*b}=\big(\sigma/k-I_2\big)_+.\tag{4}$$

Note that the dependence in K of $(\mathbf{HJB})_K$ lies in the operator D^K. It seems hard to prove existence of a smooth solution to the integro-partial differential equation (IPDE for short) $(\mathbf{HJB})_K$. Therefore in the next section we look for viscosity solutions.

2.3 Viscosity solutions: some definitions

Since we are dealing with an IPDE defined on a non usual set and in order to make things precise we define the notion of viscosity solution in our framework. First we give the classical definition and then its counterparts based on semi jets.

Definition 1.

- *A locally bounded function $U \in USC(\mathcal{E}^K)$ (the set of upper semi-continuous function on $\mathcal{E}^K$) is a viscosity sub-solution of $(\mathbf{HJB})_K$ if for all $x \in \mathcal{E}^K$ and test function ϕ such that x is a maximum on $\mathcal{E}^K$ of $U - \phi$ we have*

$$F\big(x, \phi(x), \nabla\phi(x), \partial^2_{pp}\phi(x), D^K U(x)\big) \leq 0.$$

- *A locally bounded function $U \in LSC(\mathcal{E}^K)$ (the set of lower semi-continuous function on $\mathcal{E}^K$) is a viscosity super-solution of $(\mathbf{HJB})_K$ if for all $x \in \mathcal{E}^K$ and test function ϕ such that x is a minimum on $\mathcal{E}^K$ of $U - \phi$ we have*

$$F\big(x, \phi(x), \nabla\phi(x), \partial^2_{pp}\phi(x), D^K U(x)\big) \geq 0.$$

- *A continuous function U defined on $\mathcal{E}^K$ is a viscosity solution of $(\mathbf{HJB})_K$ if it is a viscosity super-solution and a viscosity sub-solution.*

In the above definition it is equivalent to consider local (or local strict) extrema. We have not replaced U by ϕ for the last operator D^K. This is because $D^K U$ do not require regularity assumption on the function U to be defined. However it is equivalent to replace $D^K U$ by $D^K \phi$ in Definition 1. Indeed in the case of sub-solution for say, we can always build a sequence of test functions $(\phi_n)_{n\geq 0}$ satisfying $U \leq \phi_n$ with equality at point x and such that

$$(\nabla\phi_n(x), \partial^2_{pp}\phi_n(x)) = (\nabla\phi(x), \partial^2_{pp}\phi(x)) \text{ with } D^K\phi_n(x) \underset{n\to+\infty}{\to} D^K U(x).$$

By continuity of F we get the equivalence. This also holds for super-solution.

We now introduce the notions of semi super and sub-jets in our framework. For U a USC function on $\mathcal{E}^K$ and $x = (t, p, z) \in \mathcal{E}^K$, the super-jet of U at point x is the set

$$\begin{aligned}\mathcal{J}^+U(x) = \{&(g, A, h) \in \mathbb{R}^2 \times \mathbb{R} \times C^0(\mathcal{Z}_T^K), \text{ s.t. for any } y = (s, q, v) \in \mathcal{E}^K \text{ we have}\\ &U(s, y) \leq U(t, x) + g_1(t - s) + g_2(p - q) + \frac{1}{2}A(p - q)^2 + h(z - v) + o(|t - s| + |p - q|^2)\\ &\text{and } h(0) = 0\}\end{aligned}$$

and the semi super-jet of U at point x is

$$\begin{aligned}\overline{\mathcal{J}}^+U(x) = \{&(g, A, h) \in \mathbb{R}^2 \times \mathbb{R} \times C^0(\mathcal{Z}_T^K) \text{ s.t. there exists a sequence}\\ &(x_n, g_n, A_n, h_n)_{n\geq 0} \text{ with for any } n \geq 0 \ (g_n, A_n, h_n) \in \mathcal{J}^+U(x_n)\\ &\text{and such that} \big(x_n, U(x_n), g_n, A_n, h_n\big) \underset{n\to+\infty}{\to} \big(x, U(x), g, A, h\big)\}.\end{aligned}$$

In the above definition the convergence of h_n is taken in the sense of locally uniform convergence around 0. By analogy for a LSC function U we define the sub-jet $\mathscr{J}^- U(x)$ and the semi sub-jet $\overline{\mathscr{J}}^- U(x)$. We can now give another characterization of viscosity sub and super-solutions relying on the notions of semi jets.

Definition 2.

- *A locally bounded function $U \in USC(\mathscr{E}^K)$ is a viscosity sub-solution of* $(\mathbf{HJB})_K$ *if for all $x \in \mathscr{E}^K$, and $(g, A, h) \in \overline{\mathscr{J}}^+ U(x)$ we have*

$$F\big(x, U(x), g, A, D^K U(x)\big) \leq 0.$$

- *A locally bounded function $U \in LSC(\mathscr{E}^K)$ is a viscosity super-solution of* $(\mathbf{HJB})_K$ *if for all $x \in \mathscr{E}^K$, and $(g, A, h) \in \overline{\mathscr{J}}^- U(x)$ we have*

$$F\big(x, U(x), g, A, D^K U(x)\big) \geq 0.$$

We show that Definition 1 and 2 are equivalent in Appendix IV.C.

In the next section based on the study of $(\mathbf{HJB})_K$ we prove that the control problem (3) admits a solution.

2.4 Existence of an optimal control

In this section we prove existence of a solution to Problem (3). Before stating the result we present a sketch of the proof.

We start by proving uniqueness of a viscosity solution with polynomial growth to $(\mathbf{HJB})_K$ using a comparison result. The main difficulty is to adapt the Crandall-Ishi's lemma to our framework, which is done in Appendix IV.D. Using a verification argument we then check that the continuation utility function U^K associated to the problem (3) is actually this unique solution. The maximizers of the Hamiltonian given in Equation (4) then naturally provide a control solving Problem (3). The full proof is given in Section 5.2.

Theorem 1.

(i) There exists a unique viscosity solution U^K with polynomial growth to $(\mathbf{HJB})_K$.

(ii) This solution satisfies

$$U^K(0) = \sup_{\delta \in \mathscr{A}} \mathbb{E}^\delta [G(i_T, P_T) e^{-rT} + \int_0^T e^{-rs} \Big(g(i_s, P_s) + \delta_s^a \lambda_s^{a,\delta} + \delta_s^b \lambda_s^{b,\delta} \Big) \mathrm{d}s].$$

(iii) The problem (3) *admits a solution given by*

$$\delta_t^* = \delta^K(t, X_t), \text{ with } \delta^K = (\delta_a^K, \delta_b^K),$$

where

$$\delta_a^K = \big(\sigma/k - D_b^K U^K\big)_+ \text{ and } \delta_b^K = \big(\sigma/k - D_b^K U^K\big)_+. \tag{5}$$

It is important to remark that to obtain existence of an admissible optimal control we have benefited from the fact that we are controlling counting processes, whose infinitesimal generators are defined for any finite functions. From a practical point of view Theorem 1 implies that if we manage to compute U^K it is possible to implement the optimal control by monitoring the processes θ^a and θ^b. Note that to do this it is sufficient to monitor the arrival times of buy and sell market orders. However since $\mathscr{E}^K$ is a subset of an infinite dimensional vector space, we cannot approximate U^K using classic numerical methods. Therefore we need to find another way to approach the control δ^K. We tackle this problem in the next section.

3 How to approach the optimal control

In this section we explain how to approach numerically the feedback control δ^K. We proceed in three steps:

- We show that if $(K_n)_{n\geq 0}$ converges towards K in L^1 and uniformly on $[0,T]$ then $(\delta^{K_n})_{n\geq 0}$ converges point-wise towards δ^K.
- We prove that when $K(t) = \sum_{i=1}^{n} \alpha_i e^{-\gamma_i t}$ there exists a Markovian representation of the model in dimension $2n+2$.
- Inspired by [AJ19] we show that for any completely monotone function K in L^1 we can find a sequence $(K_n)_{n\geq 0}$ converging towards K in L^1 and uniformly on $[0,T]$ such that for any n, K_n is a linear combination of n decreasing exponential functions.

Those three points give a simple method to implement an approximate version of the control δ^K: choose $\tilde{K}$, written as sum of positive decreasing exponential functions, close enough to K. Use the finite dimensional representation to compute $U^{\tilde{K}}$ and implement $\delta^{\tilde{K}}$ instead of δ^K. We make precise this method in the last part of this section.

3.1 Convergence of solutions and optimal controls

Consider a completely monotone function K in L^1. We show that if a sequence of continuous L^1 functions $(K_n)_{n\geq 0}$ converges towards K in L^1 and uniformly on $[0,T]$ then the sequence $(\delta^{K_n})_{n\geq 0}$ converges point-wise towards δ^K. With respect to Equation (5) it is sufficient to prove that the sequence $(U^{K_n})_{n\geq 0}$ converges point-wise towards U^K.

From Theorem 5.8 in [Tou12] we observe that the notion of viscosity solution is perfectly adapted to prove the convergence of solutions to a sequence of IPDEs. Hence we prove in Section 5.3 the following result which is an adaptation of Theorem 5.8 in [Tou12] to our framework.

Proposition 1. *Consider a sequence $(K_n)_{n\geq 0}$ of continuous L^1 functions converging towards a completely monotone function K in L^1 and uniformly on $[0,T]$, then for any $x \in \mathscr{E}^K$ we have*

$$U^K(x) = \lim_{(y,n)\in\tilde{\mathscr{E}}\to(x,+\infty)} U^{K_n}(y) \tag{6}$$

where

$$\bar{\mathscr{E}} = \mathscr{E}^K \times \{\infty\} \cup \Big(\bigcup_{n\geq 0} \mathscr{E}^{K_n} \times \{n\} \Big).$$

The main technical difficulty in the proof of Proposition 1, compared to Theorem 5.8 in [Tou12], is that the functions $(U^{K_n})_{n\geq 0}$ are defined on different domains. From now on, when we consider a limit as in Equation (6) we forget to write $\bar{\mathscr{E}}$ to lighten notations. Proposition 1 perfectly fits our purpose of approaching U^K for any K. Indeed suppose we manage to find a dense[4] subset of the completely monotone L^1 functions such that for any K in this subset, the function U^K can be approximated numerically. Then Proposition 1 guarantees that for any completely monotone function K in L^1 we can approach numerically U^K. We show in the next two sections that the set

$$\mathscr{SE} = \bigcup_{n\geq 0} \{ \sum_{i=1}^{n} \alpha_i e^{-\gamma_i \cdot} \mathbf{1}_{\mathbb{R}_+} \text{ s.t. } \alpha \in \mathbb{R}_+^n \text{ and } \gamma \in \mathbb{R}_+^n \}$$

satisfies those two conditions. Note that $\mathscr{SE}$ is simply the set of positive linear combination of decreasing exponential functions.

We start by studying Problem (3) when the function K is in $\mathscr{SE}$. Then we show that $\mathscr{SE}$ is dense in the set of completely monotone functions in L^1.

3.2 Solving the optimal control for $K \in \mathscr{SE}$

We explain in this section how to solve the stochastic control problem (3) when the function K belongs in $\mathscr{SE}$.

We consider that the kernel of the Hawkes processes N^a and N^b is given by $K_{\alpha,\gamma}(t) = \sum_{i=1}^{n} \alpha_i e^{-\gamma_i t} \mathbf{1}_{\mathbb{R}_+}(t)$, where n is a positive integer, $\alpha \in \mathbb{R}_+^n$ and $\gamma \in \mathbb{R}_+^n$. For $i \in \{1, \dots, n\}$ and $j = a$ or b we define the process

$$c_t^{j,i} = \int_0^t \alpha_i e^{-\gamma_i (t-s)} \mathrm{d}N_s^j.$$

Then $Y_t^{\alpha,\gamma} = \big(t, P_t, i_t, (c_t^{a,i})_{1\leq i\leq n}, (c_t^{b,i})_{1\leq i\leq n}\big)$ is a Markovian process since

$$\lambda_t^{j,0} = \Phi(\sum_{i=1}^{n} c_t^{j,i}) \text{ and } \mathrm{d}c_t^{j,i} = -\gamma_i c_t^{j,i} \mathrm{d}t + \alpha_i \mathrm{d}N_t^j \text{ for } j = a \text{ or } b.$$

The domain associated to this representation is $\mathscr{E}^n = [0,T] \times \mathbb{R} \times \mathbb{Z} \times \mathbb{R}_+^n \times \mathbb{R}_+^n$, which is locally compact. As for $\mathscr{E}^K$, when we have $(t,x) \in \mathscr{E}^n$ we implicitly consider that $x = (p, i, c^a, c^b)$. We can naturally go from the first representation to the second one. More precisely we prove in Appendix IV.E that there exists a continuous function $R^{\alpha,\gamma}$ from $\mathscr{E}^{K_{\alpha,\gamma}}$ into $\mathscr{E}^n$ such that for any $t > 0$ we have $R^{\alpha,\gamma}(t, X_t) = (t, Y_t^{\alpha,\gamma})$. However notice that the second representation is

[4] Here dense is intended in the sense of convergence in L^1 together with uniform convergence on $[0,T]$.

somehow larger than the first one: if we consider $y = (t, p, i, c^a, c^b) \in \mathscr{E}^n$ there is a priori no $x \in \mathscr{E}^{K_{\alpha,\gamma}}$ such that

$$y = \mathscr{R}^{\alpha,\gamma}(x).$$

This is because in general there does not exist $m \geq 0$ and $(T_i)_{1\leq i\leq m}$ in $[0, T]$ such that for any $i \in \{1, \dots, n\}$

$$c^{a,i} = \alpha_i \sum_{j=1}^{m} e^{-\gamma_i(t-T_j)}.$$

The infinitesimal generators associated to the processes N^a and N^b for the new representation are denoted by D_a^α and D_b^α. They are defined for any function U on $\mathscr{E}^n$ and $x \in \mathscr{E}^n$ by

$$\begin{aligned} D_a^\alpha U(x) &= U(t, p, i-1, c^a + \alpha, c^b) - U(t, p, i, c^a, c^b), \\ D_b^\alpha U(x) &= U(t, p, i+1, c^a, c^b + \alpha) - U(t, p, i, c^a, c^b). \end{aligned}$$

The HJB equation related to Problem (3) in this new representation is therefore

$$(\mathbf{HJB})_{\alpha,\gamma} : \begin{cases} G_{\alpha,\gamma}\big(x, U(x), \nabla^c U(x), \nabla U(x), \partial^2_{pp} U(x), D^\alpha U(x)\big) = 0, \text{ for } x \in \mathscr{E}^n, \\ U(T, y) = G(i, p) \text{ for } (T, y) \in \mathscr{E}^n \end{cases}$$

with $\nabla^c U = (\nabla_a^c U, \nabla_b^c U)$ where for $j = a$ or b, $\nabla_j^c U = (\partial_{c^{j,i}} U)_{1\leq i\leq n}$, $\nabla U(t, x) = \big(\partial_t U(t, x), \partial_p U(t, x)\big)$,

$$D^\alpha U(t, x) = \big(D_a^\alpha U(t, x), D_b^\alpha U(t, x)\big)$$

and where the function $G_{\alpha,\gamma}$ is defined for $(x, u, h, q, A, I) \in \mathscr{E}^n \times \mathbb{R} \times (\mathbb{R}^n)^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ by

$$\begin{aligned} G_{\alpha,\gamma}\big(x, u, q, h, A, I\big) = {} & ru - h_1 - d(t, p)h_2 - \frac{1}{2}\sigma^2 A - \langle \gamma, q_1 \rangle - \langle \gamma, q_2 \rangle - g(i, p) \\ & - \sup_{\delta\in\mathbb{R}_+} \Phi\Big(\sum_{i=1}^{n} c^{a,i}\Big) e^{-\frac{k}{\sigma}\delta}(\delta + I_1) - \sup_{\delta\in\mathbb{R}_+} \Phi\Big(\sum_{i=1}^{n} c^{b,i}\Big) e^{-\frac{k}{\sigma}\delta}(\delta + I_2). \end{aligned}$$

We can easily adapt the proof of Theorem 1 to $(\mathbf{HJB})_{\alpha,\gamma}$ and prove the following result.

Theorem 2.

(i) There exists a unique continuous viscosity solution with polynomial growth $U^{\alpha,\gamma}$ to $(\mathbf{HJB})_{\alpha,\gamma}$.

(ii) The solution $U^{\alpha,\gamma}$ satisfies

$$U^{\alpha,\gamma}(0) = \sup_{\delta\in\mathscr{A}} \mathbb{E}^\delta[G(i_T, P_T)e^{-rT} + \int_0^T e^{-rs}\big(g(i_s, P_s) + \delta_s^a \lambda_s^{a,\delta} + \delta_s^b \lambda_s^{b,\delta}\big)\mathrm{d}s].$$

(iii) The stochastic control problem (3) admits a solution that is written

$$\delta^{\alpha,\gamma}(t, Y_t^{\alpha,\gamma}), \text{ with } \delta^{\alpha,\gamma} = (\delta_a^{\delta,\gamma}, \delta_a^{\delta,\gamma})$$

where

$$\delta_a^{\alpha,\gamma} = \big(\sigma/k - D_a^\alpha U^{\alpha,\gamma}\big)_+ \text{ and } \delta_b^{\alpha,\gamma} = \big(\sigma/k - D_b^\alpha U^{\alpha,\gamma}\big)_+.$$

(iv) We have $U^{K_{\alpha,\gamma}} = U^{\alpha,\gamma} \circ \mathscr{R}^{\alpha,\gamma}$.

The proof of the three first points is exactly the same as the proof of Theorem 1. We deal with point (iv) in Section 5.4. Points (iii) and (iv) of Theorem 2 imply that for any α and γ in $\mathbb{R}_+^n$ we can approach numerically $\delta^{K_{\alpha,\gamma}}$. We just need to approximate $U^{\alpha,\gamma}$ using any numerical method, which is possible because the domain of $(\mathbf{HJB})_{\alpha,\gamma}$ is a subset of a finite dimensional vector space. Then using the change of variable $\mathscr{R}^{\alpha,\gamma}$ one gets

$$\delta^{K_{\alpha,\gamma}} = \delta^{\alpha,\gamma} \circ \mathscr{R}^{\alpha,\gamma}.$$

This shows that the optimal control processes given in Theorem 1 (iii) and Theorem 2 (iii) are actually the same.

3.3 Density of $\mathscr{SE}$ in the set of completely monotone function

In this section we show that $\mathscr{SE}$ is dense in the set of completely monotone functions in L^1. Before giving the result we present a sketch of the proof. The main point is that any completely monotone function can be written as the Laplace transform of a positive measure m, see Lemma 2.3 in [Mer14]:

$$K(x) = \int_0^{+\infty} e^{-ux} m(\mathrm{d}u). \tag{7}$$

Moreover if $K(0) < +\infty$ then m is L^1 and if K is in L^1 then $\int_0^{+\infty} \frac{m(\mathrm{d}u)}{u} < \infty$. Hence using a Riemann sum to approach the integral in (7) we get a natural approximation of K by a function in $\mathscr{SE}$. Based on this idea we prove the following result in Appendix IV.F.

Lemma 2. *For any completely monotone function K in L^1 we can find a sequence $(\alpha_n, \gamma_n)_{n\geq 0}$, where for any n $(\alpha_n, \gamma_n) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$, such that the sequence $(K_{\alpha_n,\gamma_n})_{n\geq 0}$ converges towards K in L^1 and uniformly on every compact set of $\mathbb{R}_+$. Moreover we may choose $(\alpha_n, \gamma_n)_{n\geq 0}$ such that*

$$\|K_{\alpha_n,\gamma_n}\|_1 = \|K\|_1 \text{ and } K_{\alpha_n,\gamma_n}(0) = K(0).$$

Lemma 2 concludes on the existence of a procedure to approach δ^K. In the next section we sum up our results and explain how one may implement in practice an approximation of the optimal control.

3.4 Conclusion on approaching the optimal control

We fix a completely monotone function K in L^1 and a sequence $(\alpha_n, \gamma_n)_{n\geq 0}$ such that $(K_{\alpha_n,\gamma_n})_{n\geq 0}$ converges towards K in L^1 and uniformly on $[0, T]$. We write K_n instead of K_{α_n,γ_n} to lighten notations. The existence of such sequence is given by Lemma 2. Moreover from Proposition 1, for any $x \in \mathcal{E}^K$ we have

$$U^K(x) = \lim_{(y,n)\to(x,+\infty)} U^{K_n}(y).$$

Therefore using point (iii) of Theorems 1 and 2 we get

$$\delta^K(x) = \lim_{(y,n)\to(x,+\infty)} \delta^{K_n}(y),$$

and Theorem 2 (iv) gives that for any $x \in \mathscr{E}^K$

$$\delta^K(x) = \lim_{(y,n)\to(x,+\infty)} \delta^{\alpha_n,\gamma_n} \circ \mathscr{R}^{\alpha_n,\gamma_n}(y).$$

For a given $x = (t, p, i, \theta^a, \theta^b) \in \mathscr{E}^K$ we now explicit a sequence $(x_n)_{n\geq 0}$ converging towards x and such that $x_n \in \mathscr{E}^{K_n}$ for any n. By definition of Θ_t^K there exists m_a and m_b two non-negative integers and two sequences $(T_i^a)_{1\leq i\leq m_a}$ and $(T_i^b)_{1\leq i\leq m_a}$ in $[0, t]$ such that

$$\theta^a = \sum_{i=1}^{m_a} K(\cdot - T_i^a) \text{ and } \theta^b = \sum_{i=1}^{m_b} K(\cdot - T_i^b).$$

Consequently for any $n > 0$ we naturally define :

$$\theta^{n,a} = \sum_{i=1}^{m_a} K_n(\cdot - T_i^a) \text{ and } \theta^{n,b} = \sum_{i=1}^{m_b} K_n(\cdot - T_i^b) \in \Theta_t^{K_n},$$

and $x_n = (t, p, i, \theta^{n,a}, \theta^{n,b})$ which obviously belongs in $\mathscr{E}^{K_n}$. Because of Lemma 2 the sequence $(\theta^{n,a})_{n\geq 0}$ (resp. $(\theta^{n,b})_{n\geq 0}$) converges in L^1 towards θ^a (resp. θ^b). Therefore we get (x_n, n) converges towards $(x, +\infty)$ as n goes to infinity, consequently

$$\delta^K(x) = \lim_{n\to+\infty} \delta^{\alpha_n,\gamma_n} \circ \mathscr{R}^{\alpha_n,\gamma_n}(x_n).$$

Hence for n large enough we can consider that for any $t \in [0, T]$:

$$\delta_t^* = \delta^K(X_t) \approx \delta^{\alpha_n,\gamma_n}(Y_t^{\alpha_n,\gamma_n}).$$

In conclusion to implement an approached version of the optimal control δ^K one must:

1. Fix n positive and find $\alpha, \gamma \in \mathbb{R}_+^n$ such that $K_{\alpha,\gamma}$ is close to K. See Appendix IV.F for a method to build $K_{\alpha,\gamma}$.

2. Approach numerically $U^{\alpha,\gamma}$, the solution of $(\mathbf{HJB})_{\alpha,\gamma}$ which is equivalent to approach numerically the feedback $\delta^{\alpha,\gamma}$.

3. Monitor $Y^{\alpha,\gamma}$ and apply the control $\delta^{\alpha,\gamma}(Y^{\alpha,\gamma})$.

The only flaw of this method is that the set $\mathscr{E}^n$ is a subset of a vector space of dimension $2n+2$. Hence when n is larger than 2 it is very unlikely that simple finite differences methods can be used to solve numerically $(\mathbf{HJB})_{\alpha,\gamma}$. To tackle this issue one has to use other numerical methods such as neural networks, see [BHLP18, HPBL18] for example, or probabilistic method, see [HLOT+19]. In this article we propose to use the later method for numerical applications.

4 Numerical applications

In this section we present some numerical experiments illustrating our results.

We consider a simplified version of the market maker's problem:

$$(N) : \sup_{\delta \in \mathcal{A}} \mathbb{E}^{\delta}[\int_0^T \delta_s^a \mathrm{d}N_s^a + \delta_s^b \mathrm{d}N_s^b - \mu i_s^2].$$

This corresponds to $G = 0$ and $g(i,p) = -\mu i^2$. We take $k/\sigma = 50$ and $\mu = 0.02$. We note U_K the unique viscosity solution (with polynomial growth) of the HJB equation associated to (N) when the Hawkes processes' kernel is K. In all this section we discard the price variable from the IPDEs since it does not appear in the optimization problem.

We first consider in Section 4.1 the cases of kernels in $\mathcal{SE}$ with $n = 2$. We use a finite differences method to solve the IPDEs. Then in Section 4.2 we deal with more complex functions K. To solve the IPDEs we use the probabilistic representation introduced in [HLOT+19] which is described in Appendix IV.G.

4.1 The small dimension case

We consider three control functions δ^0, δ^1 and δ^2 computed in the following way:

- The control δ^0 is computed by a market maker that believes buy and sell order flows are Poisson processes with intensity μ_0.
- The control δ^1 is computed by a market maker that believes order flows are driven by Hawkes processes with intensity μ_1 and kernel $K_1(t) = \alpha^1 e^{-\gamma^1 t}$.
- The control δ^2 is computed by a market maker that believes order flows are driven by Hawkes processes with intensity μ_2 and kernel $K_2(t) = \alpha_1^2 e^{-\gamma_1^2 t} + \alpha_2^2 e^{-\gamma_2^2 t}$.

We use the following parameters settings:

- $\mu_0 = 0.01$
- $\mu_1 = 0.001$, $\gamma^1 = 1$ and $\alpha^1 = 0.9$
- $\mu_2 = 0.001$, $\gamma^2 = (1,1)$ and $\alpha^2 = (0.45, 0.45)$.

These parameters are consistent with respect to the average intensity of market orders (in a stationary version):

$$\mu_0 = \frac{\mu_1}{1 - \|K_1\|} = \frac{\mu_2}{1 - \|K_2\|}.$$

In order to estimate the gain made by market makers using refined strategies we compute the value function associated to each control when the order flows actually follows the modeling of the third market maker, see Figures IV.1, IV.2 and IV.3. As expected the control δ^2 is optimal

and δ^0 is sub-optimal compared δ^1. We observe in Figure IV.1 that considering a one factor model for the order flows leads to a 10% gain compared with a strategy considering that market order flows is a Poisson process. Using two factors leads to another 10% gain compared to the one factor case.

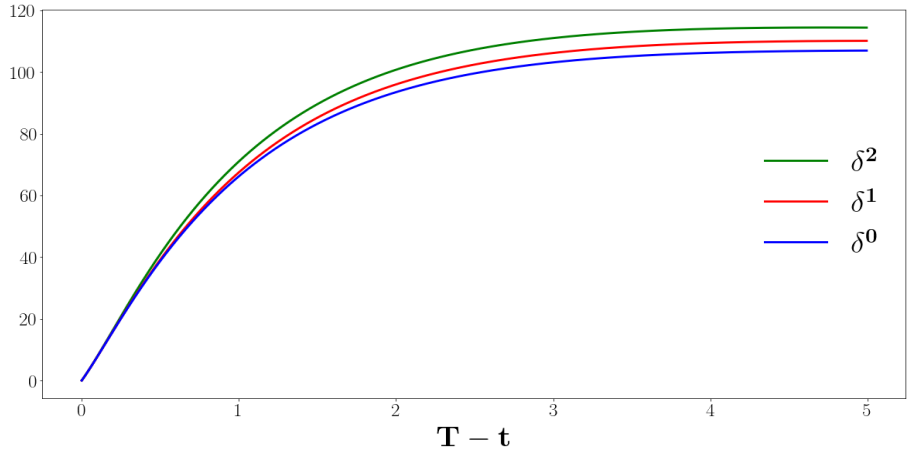

Figure IV.1 – Value function along the time for controls δ^0 and δ^1 with initial condition $c^a = (0, 10)$, $c^b = (0, 10)$ and $i = -10$.

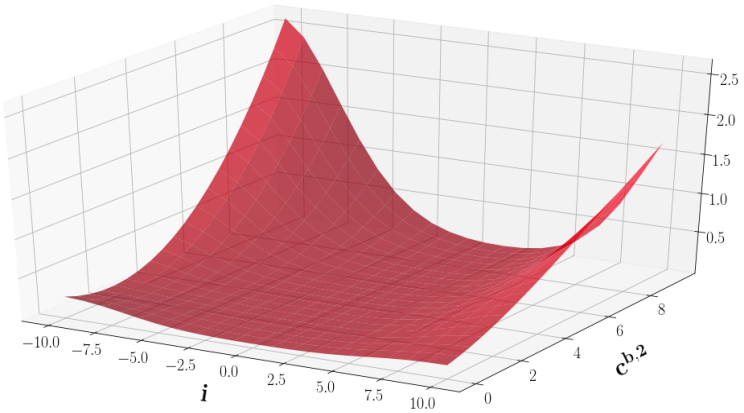

Figure IV.2 – Difference between the value function associated to control δ^2 and δ^1 for $c^a = (10, 0)$, $c^{b,1} = 10$.

4.2 The large dimension case

In this section we apply the method presented in Section 3.4 to estimate U_K at several points when the function K is a positive linear combination of n decreasing exponential functions

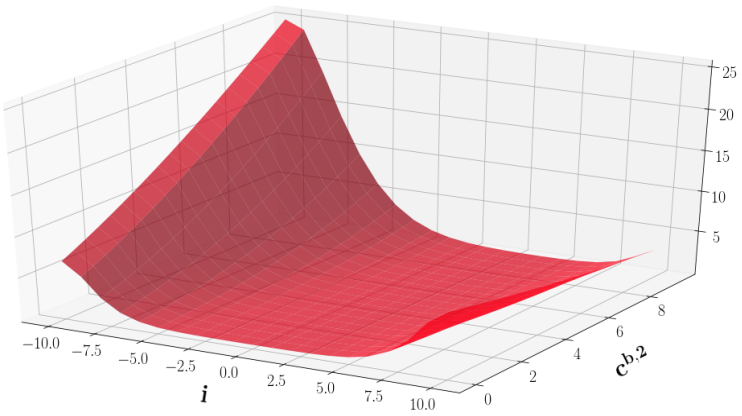

Figure IV.3 – Difference between the value function associated to control δ^2 and δ^0 for $c^a = (10, 0)$, $c^{b,1} = 10$.

and n is large.

More precisely for $n \in \{1, \dots, 200\}$ we consider the kernel K_n given by $K_n(t) = \sum_{i=1}^{n} \alpha_i e^{-\gamma_i t}$. We write $K = K_{200}$ and for any $n \leq 200$ set $\alpha^n = (\alpha_i)_{1 \leq i \leq n}$. The parameters $(\alpha_i)_{1 \leq i \leq 200}$ and $(\gamma_i)_{1 \leq i \leq 200}$ are given in Figure IV.5. For any n using the probabilistic representation of [HLOT+19], see Appendix IV.G for more details, we estimate U_{K_n} at the points

$$x_0^n = (0, 0, K^n, 0), \; x_1^n = (0, 10, K^n, 0) \text{ and } x_2^n = (0, -10, K^n, 0) \text{ in } \mathscr{E}^{K_n}.$$

We consider $x_0 = (0, 0, K, 0)$, $x_1 = (0, 10, K, 0)$ and $x_2 = (0, -10, K, 0)$ in $\mathscr{E}^K$. According to Proposition 1 we have for any $i \in \{1, \dots, 2\}$

$$U_{K_n}(x_i^n) \underset{K \to +\infty}{\to} U_K(x_i).$$

This convergence is clearly illustrated in Figure IV.4. This prove the tractability of our approach to take into account the self exciting properties of market order flow into market making strategies.

Acknowledgments

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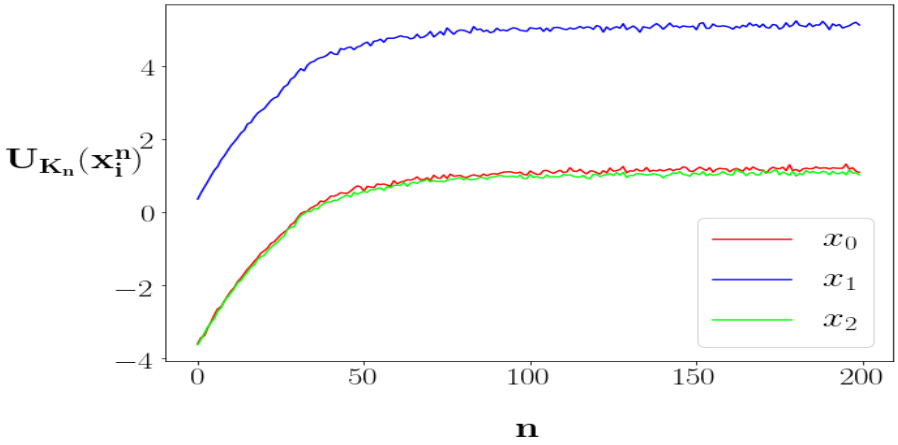

Figure IV.4 – Estimation of the value function U_{K_n} at points x_i^n for $n \in \{1,\dots,200\}$ and $i \in \{0,1,2\}$.

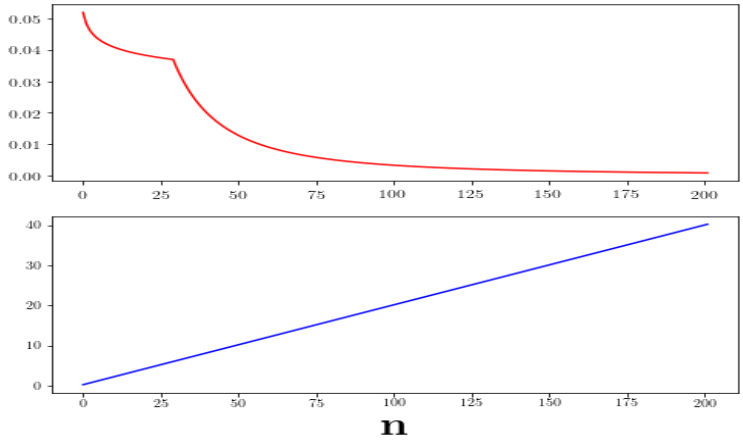

Figure IV.5 – Parameters $(\alpha_i)_{1\leq i\leq 200}$ (in blue) and $(\gamma_i)_{1\leq i\leq 200}$ (in red).

5 Proofs

5.1 Formal definition of the probability space

In this section we make precise the probability space we are working on. In particular we give a proper definition to $\mathbb{E}^\delta$. First we define the canonical process and the probability space associated to our stochastic control problem.

- Consider Ω_d the set of increasing piecewise constant càdlàg functions from $[0,T]$ into $\mathbb{N}$ with jumps equal to 1 and Ω_p the set of continuous functions from $[0,T]$ into $\mathbb{R}$. We define $\Omega = \Omega_p \times \Omega_d^2$.

- We let $(W_t, N_t^a, N_t^b)_{t\in[0,T]}$ be the canonical process on Ω.

- The associated filtration is $\mathbb{F} = (\mathscr{F}_t^p \otimes \mathscr{F}_t^d \otimes \mathscr{F}_t^d)_{t\in[0,T]}$ where $(\mathscr{F}_t^d)_{t\in[0,T]}$ (resp. $(\mathscr{F}_t^p)_{t\in[0,T]}$) is the right continuous completed filtration associated with N^a (or N^b) (resp. W).
- We denote by $\mathbb{P}_0$ the probability measure on $(\Omega,\mathbb{F})$ such that $\big(M_s^a = N_s^a - s\lambda_0,\ M_s^b = N_s^b - s\lambda_0\big)_{s\in[0,T]}$, for $\lambda_0 > 0$, are local martingales and $(W_s)_{s\in[0,T]}$ is a Brownian motion .

We now introduce some process related to our model. For a fixed $(t,x) \in \mathscr{E}^K$ we define $X^{t,x} = (P^{t,x}, i^{t,x}, \theta^{t,x;a}, \theta^{t,x;b})$ that is the state of the system after time t when starting from point (t,x). The dynamic of $X^{t,x}$ is given on $[t,T]$ by

$$\begin{aligned}
&\mathrm{d}P_s^{t,x} = d(s,P_s^{t,x})\mathrm{d}s + \sigma \mathrm{d}W_s, \quad P_t^{t,x} = p,\\
&\mathrm{d}i_s^{t,x} = \mathrm{d}N_s^a - \mathrm{d}N_s^b, \quad i_t^{t,x} = i,\\
&\mathrm{d}\theta_s^{t,x;a} = K(\cdot - s)\mathrm{d}N_s^a, \quad \theta_t^{t,x;a} = \theta^a,\\
&\mathrm{d}\theta_s^{t,x;b} = K(\cdot - s)\mathrm{d}N_s^b, \quad \theta_t^{t,x;b} = \theta^b.
\end{aligned}$$

Using those processes we explicit the change of measure associated to each control process. For this we consider the functions

$$\lambda^a(t,x,\delta) = e^{-\frac{k}{\sigma}\delta^a}\Phi\big(\theta^a(t)\big) \text{ and } \lambda^b(t,x,\delta) = e^{-\frac{k}{\sigma}\delta^b}\Phi\big(\theta^b(t)\big),$$

that represent the ask and bid intensity in the state $(t,x) \in \mathscr{E}^K$ when the control is δ. For any $\delta \in \mathscr{A}$ we define $\mathbb{P}^{t,x;\delta}$ by

$$\frac{\mathrm{d}\mathbb{P}^{t,x;\delta}}{\mathrm{d}\mathbb{P}_0} = L_T^{t,x;\delta}$$

where $L_T^{t,x;\delta}$ is the Doléans-Dade exponential of

$$Z_s^{t,x;\delta} = \int_0^s \frac{\lambda^a(s,X_s^{t,x},\delta_s) - \lambda_0}{\lambda_0}\mathbf{1}_{s\geq t}\mathrm{d}M_s^a + \frac{\lambda^b(s,X_s^{t,x},\delta_s) - \lambda_0}{\lambda_0}\mathbf{1}_{s\geq t}\mathrm{d}M_s^b.$$

Since $\lambda^a(t,x,\delta) \leq C(1+\|x\|)$ and $\lambda^b(t,x,\delta) \leq C(1+\|x\|)$, by the Corrolary 2.6 in [Sok13], for any $(t,x) \in \mathscr{E}^K$, $(L_s^{t,x;\delta})_{s\in[t,T]}$ is a true $\mathbb{P}_0$ martingale. Moreover by Theorem III-3.11 in [JS13] the processes

$$M^{t,x;a,\delta} = N^a - \int_t^{\cdot} \lambda^a(u,\delta_u,X_u^{t,x})\mathrm{d}u \text{ and } M^{t,x;b,\delta} = N^b - \int_t^{\cdot} \lambda^b(u,\delta_u,X_u^{t,x})\mathrm{d}u$$

are $\mathbb{P}^{t,x;\delta}$-local martingales on $[t,T]$. Actually they are true martingales, see Appendix IV.B.1.

For $(t,x) \in \mathscr{E}^K$ and $\delta \in \mathscr{A}$ we note $\mathbb{E}_{t,x}^{\delta}$ the expectation under the law $\mathbb{P}^{t,x;\delta}$ and note $\mathbb{E}^{\delta}$ instead of $\mathbb{E}_{0,0}^{\delta}$.

Finally, for any F bounded continuous function, $\delta \in \mathscr{A}$ and θ stopping time with values in $[t,T]$ we have:

$$\mathbb{E}_{t,x}^{\delta}[F(X_T^{t,x})|\mathscr{F}_\theta] = \mathbb{E}_{\theta,X_\theta^{t,x}}^{\delta^\theta}[F(X_T^{\theta,X_\theta^{t,x}})] \tag{8}$$

where, δ^θ is the restriction to $[\theta,T]$ of δ. This prove that for any $(t,x) \in \mathscr{E}^K$ the process $(s,X_s^{t,x})_{s\geq t}$ is Markovian.

5.2 Proof of Theorem 1

We proceed in 5 steps.

1. Section 5.2.1: Using a comparison result we show that $(\mathbf{HJB})_K$ admits a unique viscosity solution with polynomial growth.
2. Section 5.2.2: For any K we define U^K the continuation utility function associated to (3).
3. Section 5.2.3: We prove a dynamic programming principle for U^K.
4. Section 5.2.4: Using a verification argument we show that U^K is the unique viscosity solution (with polynomial growth) of $(\mathbf{HJB})_K$.
5. Section 5.2.5: We show that the control given in Equation (5) solves the control problem (3).

5.2.1 Comparison result for $(\mathbf{HJB})_K$

We start by proving a comparison result for bounded solutions, then we extend it to functions with polynomial growth.

Proposition 2. *Let $U \in USC(\mathscr{E}^K)$ be a bounded from above viscosity sub-solution of $(\mathbf{HJB})_K$ and $V \in LSC(\mathscr{E}^K)$ be a bounded from below viscosity super-solution of $(\mathbf{HJB})_K$ such that $U(T,\cdot) \leq V(T,\cdot)$ then*

$$U \leq V \text{ on } \mathscr{E}^K.$$

Proof. We suppose that there exists some $(t_0, x_0) \in \mathscr{E}^K$ such that

$$U(t_0, x_0) - V(t_0, x_0) = \delta > 0.$$

By hypothesis necessarily $t_0 \in [0, T)$. We show that this implies a contradiction. We consider the following quantities

$$N_\varepsilon = \sup_{(t,x)\in\mathscr{E}^K} U(t,x) - V(t,x) - 2\varepsilon\|x\|^2$$

and

$$N_\varepsilon^\alpha = \sup_{(t,x),(t,y)\in\mathscr{E}^K} U(t,x) - V(t,y) - \varepsilon(\|x\|^2 + \|y\|^2) - \alpha\|x-y\|^2.$$

The function U and $-V$ being bounded from above we have

$$\lim_{\|x\|+\|y\|\to+\infty} U(t,x) - V(t,y) - \alpha\|x-y\|^2 - \varepsilon\|x\|^2 - \varepsilon\|y\|^2 = -\infty$$

uniformly in t. Thus we can restrict the supremums to bounded sets that depends only on ε. More precisely

$$N_\varepsilon = \sup_{(t,x)\in\mathscr{E}_R^K} U(t,x) - V(t,x) - 2\varepsilon\|x\|^2 \tag{9}$$

$$N_\varepsilon^\alpha = \sup_{(t,x),(t,y)\in\mathscr{E}_R^K} U(t,x) - V(t,y) - \varepsilon(\|x\|^2 + \|y\|^2) - \alpha\|x-y\|^2 \tag{10}$$

where R only depends on ε. We remind that the set $\mathscr{E}_R^K$ is compact. Hence the supremum N_ε^α is achieved at some $(t_\varepsilon^\alpha, x_\varepsilon^\alpha, y_\varepsilon^\alpha)$. We show at the end of the proof that when $\alpha \to +\infty$, up to a subsequence, we have

$$\lim_{\alpha\to+\infty}(t_\varepsilon^\alpha, x_\varepsilon^\alpha, y_\varepsilon^\alpha) = (t_\varepsilon, x_\varepsilon, x_\varepsilon) \tag{11}$$

where $(t_\varepsilon, x_\varepsilon)$ achieves the supremum N_ε. We also prove that

$$\lim_{\alpha\to+\infty} \alpha\|x_\varepsilon^\alpha - y_\varepsilon^\alpha\|^2 = 0,\ \lim_{\alpha\to 0} N_\varepsilon^\alpha = N_\varepsilon,\ \lim_{\varepsilon\to 0} \varepsilon\|x_\varepsilon\|^2 = 0 \tag{12}$$

and that

$$\lim_{\varepsilon\to 0} N_\varepsilon = N = \sup_{(t,x)\in\mathscr{E}^K} U(t,x) - V(t,x). \tag{13}$$

A consequence of Equation (13) is that

$$\lim_{\alpha\to+\infty}\big(U(t_\varepsilon^\alpha, x_\varepsilon^\alpha), V(t_\varepsilon^\alpha, x_\varepsilon^\alpha)\big) = \big(U(t_\varepsilon, x_\varepsilon), V(t_\varepsilon, x_\varepsilon)\big). \tag{14}$$

We use the notations $x_\varepsilon^\alpha = (P_\varepsilon^\alpha, i_\varepsilon^\alpha, \theta_\varepsilon^{a,\alpha}, \theta_\varepsilon^{b,\alpha})$ and $y_\varepsilon^\alpha = (Q_\varepsilon^\alpha, j_\varepsilon^\alpha, \beta_\varepsilon^{a,\alpha}, \beta_\varepsilon^{b,\alpha})$.

With respect to Lemma 5, which is an adaptation of the Crandall-Ishi's lemma to our framework, for any $\beta > 0$ there exists $\big((\lambda_\varepsilon^\alpha, p_\varepsilon^\alpha), A_\varepsilon^{\beta,\alpha}, h\big) \in \overline{\mathscr{J}}^+ U(t_\varepsilon^\alpha, x_\varepsilon^\alpha)$ and $\big((\hat\lambda_\varepsilon^\alpha, q_\varepsilon^\alpha), B_\varepsilon^{\beta,\alpha}, g\big) \in \overline{\mathscr{J}}^- V(t_\varepsilon^\alpha, y_\varepsilon^\alpha)$ such that

$$-(\beta^{-1} + 2\varepsilon + 4\alpha) I_2 \le \begin{pmatrix} A_\varepsilon^{\beta,\alpha} & 0 \\ 0 & -B_\varepsilon^{\beta,\alpha} \end{pmatrix} \le (2\varepsilon + \beta 4\varepsilon^2) I_2 + \big(2\alpha + 8\beta(\alpha\varepsilon + \alpha^2)\big)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

with

$$p_\varepsilon^\alpha = 2\varepsilon P_\varepsilon^\alpha + 2\alpha(P_\varepsilon^\alpha - Q_\varepsilon^\alpha),\ q_\varepsilon^\alpha = -2\varepsilon Q_\varepsilon^\alpha - 2\alpha(Q_\varepsilon^\alpha - P_\varepsilon^\alpha),\ \lambda_\varepsilon^\alpha = 0 \text{ and } \hat\lambda_\varepsilon^\alpha = 0.$$

Remark that for ε small enough

$$U(t_\varepsilon^\alpha, x_\varepsilon^\alpha) - V(t_\varepsilon^\alpha, y_\varepsilon^\alpha) \ge \delta - \varepsilon\|x_0\|^2 > \frac{\delta}{2}$$

We now walk towards a contradiction by showing that

$$\limsup_{\varepsilon\to 0} \limsup_{\alpha\to+\infty} U(t_\varepsilon^\alpha, x_\varepsilon^\alpha) - V(t_\varepsilon^\alpha, y_\varepsilon^\alpha) \le 0.$$

According to the definition of sub-solution and super-solution we have

$$F\big(t_\varepsilon^\alpha, x_\varepsilon^\alpha, U(t_\varepsilon^\alpha, x_\varepsilon^\alpha), (\lambda_\varepsilon^\alpha, p_\varepsilon^\alpha), A_\varepsilon^{\beta,\alpha}, D^K U(t_\varepsilon^\alpha, x_\varepsilon^\alpha)\big) \le 0$$

and

$$F\big(t_\varepsilon^\alpha, y_\varepsilon^\alpha, V(t_\varepsilon^\alpha, y_\varepsilon^\alpha), (\hat\lambda_\varepsilon^\alpha, q_\varepsilon^\alpha), B_\varepsilon^{\beta,\alpha}, D^K V(t_\varepsilon^\alpha, y_\varepsilon^\alpha)\big) \ge 0.$$

By definition of F:

$$\begin{aligned} r\big(U(t_\varepsilon^\alpha, x_\varepsilon^\alpha) - V(t_\varepsilon^\alpha, y_\varepsilon^\alpha)\big) \le\ & F\big(t_\varepsilon^\alpha, x_\varepsilon^\alpha, U(t_\varepsilon^\alpha, x_\varepsilon^\alpha), (\lambda_\varepsilon^\alpha, p_\varepsilon^\alpha), A_\varepsilon^{\beta,\alpha}, D^K U(t_\varepsilon^\alpha, x_\varepsilon^\alpha)\big) \\ & - F\big(t_\varepsilon^\alpha, x_\varepsilon^\alpha, V(t_\varepsilon^\alpha, y_\varepsilon^\alpha), (\lambda_\varepsilon^\alpha, p_\varepsilon^\alpha), A_\varepsilon^{\beta,\alpha}, D^K U(t_\varepsilon^\alpha, x_\varepsilon^\alpha)\big), \end{aligned}$$

thus

$$\begin{aligned} r\big(U(t_\varepsilon^\alpha, x_\varepsilon^\alpha) - V(t_\varepsilon^\alpha, y_\varepsilon^\alpha)\big) &\le F(t_\varepsilon^\alpha, y_\varepsilon^\alpha, V(t_\varepsilon^\alpha, y_\varepsilon^\alpha), \hat{\lambda}_\varepsilon^\alpha, q_\varepsilon^\alpha, B_\varepsilon^{\beta,\alpha}, D^K V(t_\varepsilon^\alpha, y_\varepsilon^\alpha)) \\ &\quad - F(t_\varepsilon^\alpha, x_\varepsilon^\alpha, V(t_\varepsilon^\alpha, y_\varepsilon^\alpha), \lambda_\varepsilon^\alpha, p_\varepsilon^\alpha, A_\varepsilon^{\beta,\alpha}, D^K U(t_\varepsilon^\alpha, x_\varepsilon^\alpha)) \\ &\le d(t_\varepsilon^\alpha, P_\varepsilon^\alpha) p_\varepsilon^\alpha - d(t_\varepsilon^\alpha, Q_\varepsilon^\alpha) q_\varepsilon^\alpha + \frac{1}{2}\sigma^2 A_\varepsilon^\alpha - \frac{1}{2}\sigma^2 B_\varepsilon^\alpha \\ &\quad + H(t_\varepsilon^\alpha, y_\varepsilon^\alpha, D^K V(t_\varepsilon^\alpha, y_\varepsilon^\alpha)) - H(t_\varepsilon^\alpha, x_\varepsilon^\alpha, D^K U(t_\varepsilon^\alpha, x_\varepsilon^\alpha)) \end{aligned}$$

where

$$H(t,x,I) = -\sup_{\delta\in\mathbb{R}_+} \Phi\big(\theta^a(t)\big) e^{-\frac{\sigma}{k}\delta}(\delta + I_1) - \sup_{\delta\in\mathbb{R}_+} \Phi\big(\theta^b(t)\big) e^{-\frac{\sigma}{k}\delta}(\delta + I_2) + g(i,p).$$

Note that the function H is Lipschitz continuous. Taking $\beta = \alpha^{-1}$ we get

$$\sigma^2 A_\varepsilon^{\beta,\alpha} - \sigma^2 B_\varepsilon^{\beta,\alpha} \le 2(2\varepsilon + \alpha^{-1} 4\varepsilon^2)\sigma^2.$$

The RHS can be taken arbitrarly small when $\alpha \to +\infty$ and $\varepsilon \to 0$ by Equation (12). Using the Lipschitz property of d we have

$$\begin{aligned} d(t_\varepsilon^\alpha, P_\varepsilon^\alpha) p_\varepsilon^\alpha - d(t_\varepsilon^\alpha, Q_\varepsilon^\alpha) q_\varepsilon^\alpha &= 2\varepsilon\big(d(t_\varepsilon^\alpha, P_\varepsilon^\alpha) P_\varepsilon^\alpha + d(t_\varepsilon^\alpha, Q_\varepsilon^\alpha) Q_\varepsilon^\alpha\big) + 2\alpha(P_\varepsilon^\alpha - Q_\varepsilon^\alpha)\big(d(t_\varepsilon^\alpha, P_\varepsilon^\alpha) - d(t_\varepsilon^\alpha, q_\varepsilon^\alpha)\big) \\ &\le 2\varepsilon C(1 + \|y_\varepsilon^\alpha\|^2 + \|x_\varepsilon^\alpha\|^2) + C\alpha\|x_\varepsilon^\alpha - y_\varepsilon^\alpha\|^2. \end{aligned}$$

Here again the RHS goes to zero when $\alpha \to +\infty$ and then $\varepsilon \to 0$ because of Equation (12). Finally by Equation (14) and since U (resp. V) is a USC (resp. LSC) function and H is continuous and decreasing with respect to its last variable we have

$$\limsup_{\alpha\to+\infty} H(t_\varepsilon^\alpha, y_\varepsilon^\alpha, D^K V(t_\varepsilon^\alpha, y_\varepsilon^\alpha)) - H(t_\varepsilon^\alpha, x_\varepsilon^\alpha, D^K U(t_\varepsilon^\alpha, x_\varepsilon^\alpha)) \le H\big(t_\varepsilon, x_\varepsilon, D^K V(t_\varepsilon, x_\varepsilon)\big) - H\big(t_\varepsilon, x_\varepsilon, D^K U(t_\varepsilon, x_\varepsilon)\big).$$

Remark that for any z such that $(t_\varepsilon, x_\varepsilon + z) \in \mathscr{E}^K$ we have by definition of $(t_\varepsilon, x_\varepsilon)$

$$U(t_\varepsilon, x_\varepsilon) - V(t_\varepsilon, x_\varepsilon) - 2\varepsilon\|x_\varepsilon\|^2 \ge U(t_\varepsilon, x_\varepsilon + z) - V(t_\varepsilon, x_\varepsilon + z) - 2\varepsilon\|x_\varepsilon + z\|^2.$$

Consequently we have

$$V(t_\varepsilon, x_\varepsilon + z) - V(t_\varepsilon, x_\varepsilon) \ge U(t_\varepsilon, x_\varepsilon + z) - U(t_\varepsilon, x_\varepsilon) - 2\varepsilon\big(\|x_\varepsilon + z\|^2 - \|x_\varepsilon\|^2\big)$$

and so

$$D^K V(t_\varepsilon, x_\varepsilon) \le D^K(U - 2\varepsilon\|\cdot\|)(t_\varepsilon, x_\varepsilon).$$

The monotony and Lipschitz regularity of H implies

$$\begin{aligned} &\limsup_{\alpha\to+\infty} H(t_\varepsilon^\alpha, y_\varepsilon^\alpha, D^K V(t_\varepsilon^\alpha, y_\varepsilon^\alpha)) - H(t_\varepsilon^\alpha, x_\varepsilon^\alpha, D^K U(t_\varepsilon^\alpha, x_\varepsilon^\alpha)) \\ &\qquad \le H\big(x_\varepsilon, D^K(U - 2\varepsilon\|\cdot\|)(t_\varepsilon, x_\varepsilon)\big) - H\big(x_\varepsilon, D^K U(t_\varepsilon, x_\varepsilon)\big) \\ &\qquad \le C\varepsilon\|x_\varepsilon\|\Big\|D^K\|\cdot\|^2(t_\varepsilon, x_\varepsilon)\Big\|. \end{aligned}$$

Notice that for any $x \in \mathscr{E}^K$

$$\begin{aligned} D^K \|\cdot\|^2(x) &= \begin{pmatrix} \|\theta^a + K(\cdot - t)\|_1^2 - \|\theta^a\|_1^2 + |i+1|^2 - |i|^2 \\ \|\theta^b + K(\cdot - t)\|_1^2 - \|\theta^b\|_1^2 + |i-1|^2 - |i|^2 \end{pmatrix} \\ &= \begin{pmatrix} \|K(\cdot - t)\|_1^2 + 2\|K(\cdot - t)\|_1 \|\theta^a\|_1 + 1 + 2i \\ \|K(\cdot - t)\|_1^2 + 2\|K(\cdot - t)\|_1 \|\theta^b\|_1 + 1 - 2i \end{pmatrix}, \end{aligned}$$

thus there exists $C > 0$ such that $\|D^K \|\cdot\|^2(t,x)\| \leq C(1 + \|x\|)$. Consequently we get

$$\limsup_{\alpha \to +\infty} H(t_\varepsilon^\alpha, y_\varepsilon^\alpha, D^K V(t_\varepsilon^\alpha, y_\varepsilon^\alpha)) - H(t_\varepsilon^\alpha, x_\varepsilon^\alpha, D^K U(t_\varepsilon^\alpha, x_\varepsilon^\alpha)) \leq C\varepsilon(1 + \|x_\varepsilon\|^2)$$

that goes to zero when taking the limit $\varepsilon \to 0$. Finally we have shown that

$$\limsup_{\varepsilon \to 0} \limsup_{\alpha \to +\infty} U(t_\varepsilon^\alpha, x_\varepsilon^\alpha) - V(t_\varepsilon^\alpha, y_\varepsilon^\alpha) \leq 0.$$

We get a contradiction.

We finally prove the statements (11), (12) and (13). We consider $(t_\varepsilon, x_\varepsilon, y_\varepsilon) \in \overline{(t_\varepsilon^\alpha, x_\varepsilon^\alpha, y_\varepsilon^\alpha)_{\alpha \geq 0}}$ that exists since $\mathscr{E}^K$ is compact. Since $N_\varepsilon^\alpha \geq N_\varepsilon$ then necessarily $x_\varepsilon = y_\varepsilon$. We now prove the first limit of (12) and that $(t_\varepsilon, x_\varepsilon)$ corresponds to a point where the supremum N_ε is achieved. Passing to the lower limit we get

$$U(t_\varepsilon, x_\varepsilon) - V(t_\varepsilon, x_\varepsilon) - 2\varepsilon\|x_\varepsilon\|^2 - \limsup_{\alpha \to +\infty} \alpha \|x_\varepsilon^\alpha - y_\varepsilon^\alpha\|^2 \geq N_\varepsilon.$$

Hence by definition of N_ε we necessarily have that $\lim_{\alpha \to +\infty} \alpha \|x_\varepsilon^\alpha - y_\varepsilon^\alpha\|^2 = 0$ and that

$$N_\varepsilon = U(t_\varepsilon, x_\varepsilon) - V(t_\varepsilon, x_\varepsilon) - 2\varepsilon\|x_\varepsilon\|^2.$$

To conclude we show that $N_\varepsilon \to N$ and that $\varepsilon\|x_\varepsilon\|^2 \to 0$. For $\xi > 0$ consider (t,x) that is ξ-optimal in the definition of N:

$$U(t,x) - V(t,x) \geq N - \xi.$$

For ε small enough $2\varepsilon\|x\|^2$ is lower than ξ, and we get

$$N \geq N_\varepsilon \geq U(t,x) - V(t,x) - 2\varepsilon\|x\|^2 \geq N - 2\xi.$$

Therefore we get convergence of N_ε towards N and as consequence

$$U(t_\varepsilon, x_\varepsilon) - V(t_\varepsilon, x_\varepsilon) - 2\varepsilon\|x_\varepsilon\|^2 \to N.$$

Since for any ε we have

$$N \geq U(t_\varepsilon, x_\varepsilon) - V(t_\varepsilon, x_\varepsilon) \geq U(t_\varepsilon, x_\varepsilon) - V(t_\varepsilon, x_\varepsilon) - 2\varepsilon\|x_\varepsilon\|^2 = N_\varepsilon,$$

we get that $\varepsilon\|x_\varepsilon\|^2 \to 0$. This concludes the proof.

□

Now we extend Proposition 2 to the case of functions with polynomial growth.

Proposition 3. *Let $U \in USC(\mathscr{E}^K)$ with polynomial growth be a viscosity sub-solution of Equation $(\mathbf{HJB})_K$ and $V \in LSC(\mathscr{E}^K)$ with polynomial growth be a viscosity super-solution of Equation $(HJB)_K$ such that $U(T,\cdot) \leq V(T,\cdot)$. Then*

$$U \leq V \text{ on } \mathscr{E}^K.$$

Proof. There exists $k > 0$ such that

$$\lim_{\|x\| \to +\infty} \frac{|U(t,x)| + |V(t,x)|}{1 + \|x\|^k} = 0.$$

We introduce the following function

$$w(t,x) = e^{K(T-t)}(1 + \|x\|^{2k}).$$

We have

$$D^K w(t,x) = e^{K(T-t)} \begin{pmatrix} P^{11}_{2k-1}(\|\theta^a\|_1) & P^{12}_{2k-1}(i) \\ P^{21}_{2k-1}(\|\theta^b\|_1) & P^{22}_{2k-1}(i) \end{pmatrix}$$

with $(P^{ij}_{2k-1})_{i,j\in\{1,2\}}$ polynomials with degree $2k-1$. Consequently for some $C > 0$

$$\|x\| \|D^K w(t,x)\| \leq C w(t,x).$$

We have

$$\sigma^2 \partial^2_P w(t,x) \leq C(1 + \|x\|^2) e^{K(T-t)} Q_{2k-2}(\|x\|) \leq C w(t,x)$$

and

$$d(t,x) \partial_P w(t,x) \leq e^{K(T-t)} C(1 + \|x\|) Q_{2k-1}(\|x\|) \leq C w(t,x)$$

where Q_{2k-2} and Q_{2k-1} are two polynomials with respective degree $2k-2$ and $2k-1$. Consequently for any constant B

$$-\partial_t w(t,x) - d(t,x)\partial_P w(t,x) - \frac{1}{2}\sigma^2 \partial^2_P w(t,x) - B\|x\| \|D^K w(t,x)\| \geq w(t,x)(K - C)$$

which is positive for K large enough. Hence for any $\varepsilon > 0$ the function $U - \varepsilon w$ is a bounded from above viscosity sub-solution of Equation $(\mathbf{HJB})_K$. Indeed if $U - \varepsilon w < \phi$ then $U < \phi + \varepsilon w$ consequently

$$F\big(t,x,U(t,x), \nabla(\phi + \varepsilon w)(t,x), \partial^2_{pp}(\phi + \varepsilon w)(t,x), D^K U(t,x)\big) \leq 0.$$

We have for K large enough

$$\begin{aligned}
&F\big(t,x,U(t,x) - \varepsilon w(t,x), \nabla\phi(t,x), \partial^2_{pp}\phi(t,x), D^K(U - \varepsilon w)(t,x)\big) \\
&-F\big(t,x,U(t,x), (\nabla(\phi + \varepsilon w)(t,x), \partial^2_{pp}(\phi + \varepsilon w)(t,x), D^K U(t,x)\big) \\
&\quad \leq \varepsilon\big(-r w(t,x) + (\partial_t + d\partial_P + \frac{1}{2}\sigma^2 \partial^2_{pp}) w(t,x) + C\|x\| \|D^K w(t,x)\|\big) \\
&\quad < 0.
\end{aligned}$$

It implies that

$$F\big(t,x,U(t,x)-\varepsilon w(t,x),(\partial_t\phi(t,x),\partial_p\phi(t,x)),\partial^2_{pp}\phi(t,x),D^K(U-\varepsilon w)(t,x)\big)\leq 0.$$

We show in the same way that $V+\varepsilon w$ is a bounded from below viscosity super-solution. Then from Proposition 2 we have

$$U-\varepsilon w\leq V+\varepsilon w$$

and taking ε to 0 we get the stated result. □

An immediate consequence from Proposition 3 is that there exists a unique viscosity solution with polynomial growth to $(\mathbf{HJB})_K$. We now prove the existence of such solution using a verification argument.

5.2.2 Definition of the continuation utility function

For $(t,x)\in\mathcal{E}^K$ and $\delta\in\mathcal{A}$ we define

$$J^K(t,x;\delta)=\mathbb{E}^{\delta}_{t,x}[G(i_T^{t,x},P_T^{t,x})e^{-r(T-t)}+\int_t^T e^{-r(s-t)}\tilde{g}(s,X_s^{t,x},\delta_s)\mathrm{d}s]$$

where

$$\tilde{g}(s,x,\delta)=g(i,P)+\delta^a\lambda^a(s,x,\delta)+\delta^b\lambda^b(s,x,\delta).$$

We also define

$$U^K(t,x)=\sup_{\delta\in\mathcal{A}}J^K(t,x;\delta) \tag{15}$$

that is the maximal utility than can expect a market maker starting its trading from time t with initial market condition given by x. By Lemma 3 we get that U^K has polynomial growth. More precisely there exists a positive constant κ such that

$$U^K(t,x)\leq\kappa(1+\|x\|^2).$$

We also define

$$\mathcal{A}_t=\{\delta\in\mathcal{A}\text{ s.t. }\delta\text{ is independent of }\mathcal{F}_t\},$$

the set of controls starting from t and independent from the past. Since under $\mathbb{P}_0$ the processes N^a and N^b have independent increments, using the same arguments than in Remark 2.2-(iv) in [Tou12] we get

$$U^K(t,x)=\sup_{\delta\in\mathcal{A}_t}J^K(t,x;\delta).$$

In the next sections we show that the function U^K is the unique viscosity solution with polynomial growth to $(\mathbf{HJB})_K$. For this we prove a dynamic programming principle for U^K and then conclude using a verification argument.

5.2.3 Dynamic programming principle

Consider the lower and upper semi-continuous version of U^K:

$$U^K_*(x) = \liminf_{y\to x} U^K(y) \text{ and } U^{K*}(x) = \limsup_{y\to x} U^K(y).$$

Inspired by [Toul2] we prove the following dynamic programming principle.

Theorem 3. *Let $(t,x) \in \mathscr{E}^K$ be fixed and $\{\theta^\delta,\ \delta \in \mathscr{A}_t\}$ be a family of finite stopping times with values in $[t,T]$. Assume that for any δ, $(X^{t,x}_s \mathbf{1}_{s\in[t,\theta^\delta]})_{s\in[0,T]}$ is L^∞-bounded. Then we have*

$$U^K(t,x) \geq \sup_{\delta\in\mathscr{A}_t} \mathbb{E}^\delta[e^{-r(\theta^\delta - t)} U^K_*(\theta^\delta, X^{t,x}_{\theta^\delta}) + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s]$$

and

$$U^K(t,x) \leq \sup_{\delta\in\mathscr{A}_t} \mathbb{E}^\delta[e^{-r(\theta^\delta - t)} U^{K*}(\theta^\delta, X^{t,x}_{\theta^\delta}) + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s].$$

The proof of Theorem 3 is the same as the one of Theorem 2.3 in [Toul2]. However since we are working on non-usual domains we write the proof for the sake of completeness.

Proof. We first show the first inequality. We consider a continuous function ψ such that $U^K \geq \psi$. By definition of U^K for any $(t,x) \in \mathscr{E}^K$ there is an admissible control $\delta^{t,x,\varepsilon} \in \mathscr{A}_t$ that is ε optimal:

$$J^K(t,x;\delta^{t,x,\varepsilon}) \geq U^K(t,x) - \varepsilon.$$

The function G and $\tilde{g}$ being lower semi-continuous, the function $J^K(\cdot;\delta^{t,x,\varepsilon})$ is also lower semi-continuous by Fatou's lemma. Then ψ being upper semi continuous we can find a family of positive real $\left(r_{t,x}\right)_{t,x\in\mathscr{E}^K}$ such that for any $(t,x) \in \mathscr{E}^K$ we have

$$\psi(t,x) - \psi(s,y) \geq -\varepsilon \text{ and } J^K(t,x;\delta^{t,x,\varepsilon}) - J^K(s,y;\delta^{t,x,\varepsilon}) \leq \varepsilon, \text{ for } (s,y) \in B(t,x;r_{t,x})$$

where

$$B(t,x;r) = \{(s,y) \in \mathscr{E}^K \text{ s.t. } s \in (t-r,t),\ \|x-y\| < r\}.$$

The system $(B(t,x;r_{t,x}))_{t,x\in\mathscr{E}}$ forms an open covering of $\mathscr{E}^K$. With the topology it is endowed $\mathscr{E}^K$ is second countable since $[0,T]\times\mathbb{R}\times\mathbb{Z}\times\mathbb{L}_1\times\mathbb{L}_1$ is second countable. So by the Lindelöf covering Theorem we can extract from $\left(B(t,x;r(t,x))\right)_{(t,x)\in\mathscr{E}^K}$ a countable subfamily that covers $\mathscr{E}^K$. Thus we have $(t_i,x_i,r_i)_{i\in\mathbb{N}}$ such that

$$\mathscr{E}^K \subset \bigcup_{i\in\mathbb{N}} B(t_i,x_i;r_i).$$

Set $A^n = \bigcup_{0\leq i\leq n} A_i$. Consider $A_0 = \{T\}\times\mathscr{X}^K_T$, $C_{-1} = \emptyset$ and define the sequence

$$A_{i+1} = B(t_{i+1},x_{i+1};r_{i+1})\backslash C_i, \text{ where } C_i = C_{i-1}\cup A_i,\ i \geq 0.$$

Now fix $\delta \in \mathscr{A}_t$. With the above construction, we have $(\theta^\delta, X^{t,x}_{\theta^\delta}) \in \cup_{i\geq 0} A_i$ and for $i \geq 1$, we have

$$J^K(\cdot;\delta^{t_i,x_i,\varepsilon}) \geq \psi - 3\varepsilon \text{ on } A_i.$$

We define the control process $\delta^{\varepsilon,n}$ by

$$\delta^{\varepsilon,n}_s = \mathbf{1}_{[t,\theta^\delta]}(s)\delta_s + \mathbf{1}_{(\theta^\delta,T]}(s)\big(\mathbf{1}_{A^{nc}}(\theta^\delta, X^{t,x}_{\theta^\delta})\delta_s + \sum_{i=1}^n \mathbf{1}_{A_i}(\theta^\delta, X^{t,x}_{\theta^\delta})\delta^{t_i,x_i,\varepsilon}_s\big).$$

The control $\delta^{\varepsilon,n}$ is in $\mathscr{A}_t$. By Equation (8) we have

$$\begin{aligned}
&\mathbb{E}^{\delta^{\varepsilon,n}}_{t,x}[G(i^{t,x}_T, P^{t,x}_T)e^{-r(T-t)} + \int_t^T e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s|\mathscr{F}_{\theta^\delta}]\mathbf{1}_{A^n}(\theta, X^{t,x}_{\theta^\delta}) \\
&\qquad = \big(U^K(T, X^{t,x}_T)e^{-r(T-t)} + \int_t^T e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s\big)\mathbf{1}_{A_0}(\theta^\delta, X^{t,x}_{\theta^\delta}) \\
&\qquad\quad + \sum_{i=1}^n \big(e^{-r(\theta^\delta - t)}J^K(\theta^\delta, X^{t,x}_{\theta^\delta}; \delta^{t_i,x_i,\varepsilon}) + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s\big)\mathbf{1}_{A_i}(\theta^\delta, X^{t,x}_{\theta^\delta}) \\
&\qquad \geq \sum_{i=0}^n \big(e^{-r(\theta^\delta - t)}\psi(\theta^\delta, X^{t,x}_{\theta^\delta}) - 3\varepsilon + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s\big)\mathbf{1}_{A_i}(\theta^\delta, X^{t,x}_{\theta^\delta}) \\
&\qquad \geq \big(e^{-r(\theta^\delta - t)}\psi(\theta^\delta, X^{t,x}_{\theta^\delta}) - 3\varepsilon + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s\big)\mathbf{1}_{A^n}(\theta^\delta, X^{t,x}_{\theta^\delta}).
\end{aligned}$$

Thus we get

$$\begin{aligned}
U^K(t,x) &\geq J^K(t,x;\delta^{\varepsilon,n}) \\
&\geq \mathbb{E}^{\delta^{\varepsilon,n}}_{t,x}[\mathbb{E}^{\delta^{\varepsilon,n}}_{t,x}[G(i^{t,x}_T, P^{t,x}_T)e^{-r(T-t)} + \int_t^T e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta^{\varepsilon,n}_s)\mathrm{d}s|\mathscr{F}_{\theta^\delta}]] \\
&\geq \mathbb{E}^{\delta^{\varepsilon,n}}_{t,x}[\big(e^{-r(\theta^\delta - t)}\psi(\theta^\delta, X^{t,x}_{\theta^\delta}) - 3\varepsilon + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s\big)\mathbf{1}_{A_n}(\theta^\delta, X^{t,x}_{\theta^\delta})] \\
&\quad + \mathbb{E}^{\delta^{\varepsilon,n}}_{t,x}[\big(G(i^{t,x}_T, P^{t,x}_T)e^{-r(T-t)} + \int_t^T e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s\big)\mathbf{1}_{A^{nc}}(\theta^\delta, X^{t,x}_{\theta^\delta})].
\end{aligned}$$

Since $L^{t,x;\delta^{\varepsilon,n}}$ is a true martingale and $L^{t,x;\delta^{\varepsilon,n}}_s = L^{t,x;\delta}_s$ for $s \in [t, \theta^\delta]$ we have

$$\begin{aligned}
U^K(t,x) &\geq \mathbb{E}^{\delta}_{t,x}[\big(e^{-r(\theta^\delta - t)}\psi(\theta^\delta, X^{t,x}_{\theta^\delta}) - 3\varepsilon + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s\big)\mathbf{1}_{A_n}(\theta^\delta, X^{t,x}_{\theta^\delta})] \\
&\quad + \mathbb{E}^{\delta^{\varepsilon,n}}_{t,x}[\big(G(i^{t,x}_T, P^{t,x}_T)e^{-r(T-t)} + \int_t^T e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s\big)\mathbf{1}_{A^{nc}}(\theta^\delta, X^{t,x}_{\theta^\delta})].
\end{aligned}$$

By dominated convergence letting $n \to +\infty$ we get

$$U^K(t,x) \geq -3\varepsilon + \mathbb{E}^{\delta}_{t,x}[e^{-r(\theta^\delta - t)}\psi(\theta^\delta, X^{t,x}_{\theta^\delta}) + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s].$$

Since ε is any positive real we have

$$U^K(t,x) \geq \mathbb{E}^{\delta}_{t,x}[e^{-r(\theta^\delta - t)}\psi(\theta^\delta, X^{t,x}_{\theta^\delta}) + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s)].$$

We now explain how to pass from ψ dominated by U^K to U^K_*. By hypothesis for any δ we can find r such that almost surely $\|X^{t,x}_s\| \leq r$ for any $s \in [t, \theta^\delta]$. Then we can find an increasing sequence of continuous functions on $\mathscr{E}^K$, $(\Phi_n)_{n\geq 0}$ such that $\Phi_n \leq U^K_* \leq U^K$ and such that Φ_n converges pointwise towards U^K_* on $\big([0,T] \times B_r(x)\big) \cap \mathscr{E}^K$ (see Lemma 3.5. in [Ren99]), where

$$B_r(x) = \{y \in \mathbb{R} \times \mathbb{Z} \times L^1 \times L^1 \text{ s.t. } \|y - x\| \leq r\}.$$

Consequently from monotone convergence Theorem we have

$$U^K(t,x) \geq \ \mathbb{E}^{\delta}_{t,x}[e^{-r(\theta^\delta - t)}U^K_*(\theta^\delta, X^{t,x}_{\theta^\delta}) + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s)].$$

Then we can pass to the supremum in $\delta \in \mathscr{A}_t$ to get the result.

Now we show the first inequality. Take $\delta \in \mathscr{A}_t$ and consider $\tilde{\delta}$ the controlled process obtained after freezing the trajectory of δ up to time θ^δ. By definition of U^K we have

$$U^{K*}(\theta^\delta, X^{t,x}_{\theta^\delta}) \geq \mathbb{E}^{\tilde{\delta}}_{\theta, X^{t,x}_\theta}[e^{-r(T-\theta^\delta)}G(i_T^{\theta^\delta, X^{t,x}_{\theta^\delta}}, P_T^{\theta^\delta, X^{t,x}_{\theta^\delta}}) + \int_{\theta^\delta}^T e^{-r(s-\theta^\delta)}\tilde{g}(s, X_s^{\theta^\delta, X^{t,x}_{\theta^\delta}}, \delta_s)\mathrm{d}s].$$

Using Equation (8) this gives

$$\begin{aligned} U^{K*}(\theta, X^{t,x}_\theta)e^{-r(\theta^\delta - t)} + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s \\ \geq \mathbb{E}^{\delta}_{t,x}[e^{-r(T-t)}G(i^{t,x}_T, P^{t,x}_T) + \int_t^T e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s|\mathscr{F}_{\theta^\delta}]. \end{aligned}$$

Now taking the average, by arbitrariness of δ we get the second inequality

$$\sup_{\delta \in \mathscr{A}_t} \mathbb{E}^{\delta}_{t,x}[U^{K*}(\theta^\delta, X^{t,x}_{\theta^\delta})e^{-r(\theta^\delta - t)} + \int_t^{\theta^\delta} e^{-r(s-t)}\tilde{g}(s, X^{t,x}_s, \delta_s)\mathrm{d}s] \geq U^K(t,x).$$

□

In the next section we show that U^K is a viscosity solution of $(\mathbf{HJB})_K$ using a verification argument based on Theorem 3.

5.2.4 Verification

In this section using the dynamic programming principle proved previously we prove that U^{K*} (resp. U^K_*) is a viscosity super (resp. sub)-solution of $(\mathbf{HJB})_K$. The proof is inspired from the proof of Propositions 6.2 and 6.3 in [Toul2].

Proposition 4. *The function U_*^K (resp. U^{K*}) is a viscosity sub (resp. super)-solution of* $(\mathbf{HJB})_K$.

Proof. We first show that U_*^K is a viscosity super-solution and then that U^{K*} is a viscosity sub-solution.

Let $(t,x) \in \mathscr{E}$ and ϕ be a test function such that

$$(U_*^K - \phi)(t,x) = \min_{\mathscr{E}^K} U_*^K - \phi = 0$$

and (t_n, x_n) a sequence in $\mathscr{E}^K$ such that

$$(t_n, x_n) \to (t,x) \text{ and } U^K(t_n,x_n) \to U_*^K(t,x).$$

Since ϕ is continuous we have

$$\eta_n = U^K(t_n,x_n) - \phi(t_n,x_n) \to 0.$$

Let $\delta \in \mathbb{R}_+^2$ and consider the constant control process equal to δ. We use the notation $X^n = X^{t_n,x_n}$ and $\mathbb{E}_n^\delta = \mathbb{E}_{t_n,x_n}^\delta$. Finally, for all $n>0$ we define the stopping time:

$$\tau_n = \inf\{s > t_n \text{ s.t. } (s-t_n, X_s^n - x_n) \notin [0,h_n) \times B_\alpha\},$$

where B_α the ball for $\|\cdot\|$, centered in 0 with radius α positive and small enough such that if a jump occurs then the stopping time τ_n is immediatly reached. We take

$$h_n = \sqrt{\eta_n}\mathbf{1}_{\eta_n \neq 0} + n^{-1}\mathbf{1}_{\eta_n = 0}.$$

Notice that $\tau_n \to t$ almost surely.

From the first inequality in the dynamic programming principle, we have

$$0 \le \mathbb{E}_n^\delta\big[U^K(t_n,x_n) - e^{-r(\tau_n - t_n)}U_*^K(\tau_n, X_{\tau_n}^n) - \int_{t_n}^{\tau_n} e^{-r(s-t_n)}\tilde{g}(s,X_s^n,\delta)\mathrm{d}s\big].$$

Now using that $U_*^K \ge \phi$ we get

$$0 \le \eta_n + \mathbb{E}_n^\delta\big[\phi(t_n,x_n) - e^{-r(\tau_n - t_n)}\phi(\tau_n, X_{\tau_n}^n) - \int_{t_n}^{\tau_n} e^{-r(s-t_n)}\tilde{g}(s,X_s^n,\delta_s)\mathrm{d}s\big].$$

We can use the Ito formula since ϕ is smooth. Thus we get

$$0 \le \eta_n - \mathbb{E}_n^\delta\big[\int_{t_n}^{\tau_n} e^{-r(s-t_n)}\big((-r\phi + \partial_t\phi + \mathscr{L}^\delta\phi)(s,X_s^n) + \tilde{g}(s,X_s^n,\delta_s)\big)\mathrm{d}s\big] - \mathbb{E}_n^\delta[M_{\tau_n}^n]$$

where

$$\mathscr{L}^\delta\phi(s,x) = \mathscr{L}^P\phi(s,x) + \sum_{j=a,b} D_j^K\varphi(s,x)e^{-\frac{k}{\sigma}\delta_u^j}\Phi\big(\theta^j(s)\big)$$

and with

$$M_s^n = \int_{t_n}^{s} e^{-r(s-t_n)}\big(D_K^a\phi(s,X_s^n)\mathrm{d}M_s^{\delta,a} + D_K^b\phi(s,X_s^n)\mathrm{d}M_s^{\delta,b} + \sigma\partial_p\phi(s,X_s^n)\mathrm{d}W_s\big)$$

The function ϕ being continuous, the integrands in the term M^n are all bounded so the expectation of M^n under $\mathbb{P}_n^\delta$ is 0. Consequently we have

$$0 \le \frac{\eta_n}{h_n} - \mathbb{E}_n^\delta\big[\frac{1}{h_n}\int_{t_n}^{\tau_n} e^{-r(s-t_n)}\big((-r\phi + \partial_t\phi + \mathscr{L}^\delta\phi)(s,X_s^n) + \tilde{g}(s,X_s^n,\delta)\big)\mathrm{d}s\big].$$

Taking $n \to +\infty$ using dominated convergence and arbitrariness of δ we get

$$0 \le (r\phi - \partial_t\phi - \mathscr{L}^\delta\phi)(t,x) - \tilde{g}(t,x,\delta).$$

The control δ being arbitrary we finally have that

$$F(t,x,\phi(t,x),\nabla\phi(t,x),\partial_{pp}^2\phi(t,x),D^K\phi(t,x)) \ge 0.$$

Thus U_*^K is a viscosity supersolution of $(\mathbf{HJB})_K$.

Now we suppose that U^{K*} is not a viscosity subsolution of $(\mathbf{HJB})_K$ and exhibit a contradiction. According to the definition of viscosity subsolution we can find ϕ a test function and (t_0,x_0) such that

$$0 = (U^{K*} - \phi)(t_0,x_0) > (U^{K*} - \phi)(t,x), \ \forall\ (t,x) \in \mathscr{E}^K\backslash\{(t_0,x_0)\}$$

and that

$$F(t_0,x_0,\phi(t_0,x_0),\nabla\phi(t_0,x_0),\partial_{pp}^2\phi(t_0,x_0),D^K\phi(t_0,x_0)) > 0. \tag{16}$$

By continuity of ϕ and F we have existence of a $r>0$ small enough such that on $B_r(t_0,x_0)\backslash\{(t_0,x_0)\}$ we have

$$h = -F(\cdot,\phi,\nabla\phi,\partial_{pp}^2\phi,D^K\phi) < 0.$$

Moreover we can find some $\eta > 0$ (up to a change of r), such that

$$\sup_{\partial B_r(t_0,x_0)\cup\mathscr{J}(t_0,x_0)} U^{K*} - \phi = -2\eta e^{rT}$$

where $\mathscr{J}(t_0,x_0)$ is the set of all values that can be reached if a jump occurs inside $B_r(t_0,x_0)$. Note that it is a compact set. We consider a sequence $(t_n,x_n)_{n\ge0} \in \mathscr{E}^K$ such that

$$\lim_{n\to+\infty}(t_n,x_n) = (t_0,x_0) \text{ and } \lim_{n\to+\infty} U^K(t_n,x_n) = U^{K*}(t_0,x_0).$$

Since $U^K(t_n,x_n) - \phi(t_n,x_n) \to 0$ we can assume that

$$|U^K(t_n,x_n) - \phi(t_n,x_n)| \le \eta \text{ for any } n \ge 1.$$

For a fixed control $\delta \in \mathscr{A}_{t_n}$ We define the stopping time

$$\tau_n = \inf\{t > t_n \text{ s.t } X_t^{t_n,x_n} \notin B_r(t_0,x_0)\}.$$

At the stopping time, either the process X^{t_n,x_n} has not jumped and so is on $\partial B_r(t_0,x_0)$ or has jumped and is in $\mathscr{J}(t_0,x_0)$. Thus

$$e^{-r(\tau_n-t_n)}\phi(\tau_n,X^{t_n,x_n}_{\tau_n}) \geq 2\eta + e^{-r(\tau_n-t_n)}U^K(\tau_n,X^{t_n,x_n}_{\tau_n}).$$

We derive from the Ito formula

$$\begin{aligned} U^K(t_n,x_n) &\geq -\eta+\phi(t_n,x_n) \\ &= -\eta+\mathbb{E}^\delta_n\big[e^{-r(\tau_n-t_n)}\phi(\tau_n,X^n_{\tau_n}) - \int_{t_n}^{\tau_n} e^{-r(s-t_n)}(-r+\partial_t+\mathscr{L}^\delta)\phi(s,X^n_{s^-})\mathrm{d}s\big]. \end{aligned}$$

So by to Equation (16) we have

$$\begin{aligned} U^K(t_n,x_n) &\geq -\eta+\mathbb{E}^\delta_n\big[e^{-r(\tau_n-t_n)}\phi(\tau_n,X^n_{\tau_n}) + \int_{t_n}^{\tau_n} e^{-r(s-t_n)}\tilde{g}(s,X^n_s,\delta)\mathrm{d}s\big] \\ &\geq \eta+\mathbb{E}^\delta_n\big[e^{-r(\tau_n-t_n)}U^{K*}(\tau_n,X^n_{\tau_n}) + \int_{t_n}^{\tau_n} e^{-r(s-t_n)}\tilde{g}(s,X^n_s,\delta)\mathrm{d}s\big]. \end{aligned}$$

Since δ is any control and η is positive this contradict the second equation of Theorem 3. Thus U^{K*} is a viscosity sub-solution of $(\mathbf{HJB})_K$.

□

A direct consequence of Proposition 4 together with Proposition 3 is that

$$U^K_* \geq U^{K*}.$$

But obviously we have $U^K_* \leq U^{K*}$, therefore $U^K_* = U^{K*} = U^K$. In particular U^K is continuous and therefore is the unique continuous viscosity solution with polynomial growth to $(\mathbf{HJB})_K$.

5.2.5 Proof of Theorem 1 (iii)

To prove Theorem 1 (iii) we must show that $J^K(\cdot;\delta^*) = U^K$.

As we did previously we can show that $J^K(\cdot;\delta^*)$ is the unique viscosity solution with polynomial growth of

$$(\mathbf{LHJB})_K: \begin{cases} rU-\partial_t U-\mathscr{L}^P U-g-\sum_{j=a,b} e^{-\frac{k}{\sigma}\delta^K_j}\Phi\big(\theta^j(u)\big)(D^K_j U+\delta^K_j)=0, \text{ on } \mathscr{E}^K \\ U(T,x)=G(i,P) \text{ for } x\in\mathscr{X}^K_T \end{cases}.$$

But since U^K is a viscosity solution of $(\mathbf{HJB})_K$ and by definition of δ^K, U^K is also a viscosity solution with polynomial growth of $(\mathbf{LHJB})_K$. So we get the result.

5.3 Proof of Proposition 1.

We define the following functions on $\mathscr{E}^K$:

$$\overline{U}(x) = \limsup_{(y,n)\in\bar{\mathscr{E}}\to(x,+\infty)} U^{K_n}(y) \text{ and } \underline{U}(x) = \liminf_{(y,n)\in\bar{\mathscr{E}}\to(x,+\infty)} U^{K_n}(y),$$

We show that $\underline{U}$ and $\overline{U}$ are respectively a viscosity super-solution and a viscosity sub-solution of $(\mathbf{HJB})_K$.

Consider ϕ a test function and $\underline{x}\in\mathscr{E}^K$ a strict minimizer of $\underline{U}-\phi$. We have existence of a sequence $(x_n,\sigma_n)_{n\in\mathbb{N}}$ in $\overline{\mathscr{E}}$ such that

$$(x_n,\sigma_n)\to(+\infty,\underline{x}) \text{ and } U^{K_n}(x_n)\to\underline{U}(\underline{x}).$$

Consider $B_r(\underline{x})$ the closed ball of $[0,T]\times\mathbb{R}\times\mathbb{Z}\times L^1\times L^1$ with radius $r>0$ centered in $\underline{x}$. Then we can always suppose that $x_n\in B_r(\underline{x}),\ \forall n\geq 0$. Let $\underline{x}_n$ be a minimizer of the difference $U^{K_n}-\phi$ on $\mathscr{E}^{K_n}\cap B_r(\underline{x})$ (exists because $\mathscr{E}^{K_n}$ is locally compact). We note $\underline{x}_n=(t_n,p_n,i_n,\theta^{n,a},\theta^{n,b})$. We show at the end of the proof that there exists $x\in\mathscr{E}^K$ such that $(x,+\infty)$ is the limit of a subsequence of $(\underline{x}_n,\sigma_n)_{n\geq 0}$ and that $\theta^{n,j}(t_n)\to\theta^j(t)$ for $j=a$ and b. So we can write

$$\begin{aligned}\underline{U}(\underline{x})-\phi(\underline{x}) &= \lim_{n\to+\infty} U^{K_n}(x_n)-\phi(x_n)\\ &\geq \liminf_{n\to+\infty} U^{K_n}(\underline{x}_n)-\phi(\underline{x}_n)\\ &\geq \underline{U}(x)-\phi(x).\end{aligned}$$

Thus by definition of $\underline{x}$ we get that $(\underline{x}_n)_{n\geq 0}$ converges towards $\underline{x}$ and that

$$U^{K_n}(\underline{x}_n)\underset{n\to+\infty}{\to}\underline{U}(\underline{x}).$$

As a consequence when n is large enough $\underline{x}_n$ is a local minimizer of $U^{K_n}-\phi$ (because it is in the interior of $\overline{B}$) hence by definition of viscosity solutions

$$F\big(\underline{x}_n,U^{K_n}(\underline{x}_n),\nabla\phi(\underline{x}_n),\partial^2_{pp}\phi(\underline{x}_n),D^{K_n}U^{K_n}(\underline{x}_n)\big)\geq 0.$$

Then by definition of $\underline{U}$ and since $U^{K_n}(\underline{x}_n)\to\underline{U}(\underline{x})$:

$$\liminf_{n\to+\infty} D^{K_n}U^{K_n}(\underline{x_n})\geq D^K\underline{U}(\underline{x}).$$

Finally since F is decreasing with respect to the last variable and since $\theta^{n,j}(t_n)$ converges towards $\theta^j(t)$ for $j=a$ and b we have

$$F\big(\underline{x},\underline{U}(\underline{x}),\nabla\phi(\underline{x}),\partial^2_{pp}\phi(\underline{x}),D_K\underline{U}(\underline{x})\big)\geq\limsup_{n\to+} F\big(\underline{x}_n,U^{K_n}(\underline{x}_n),\nabla\phi(\underline{x}_n),\partial^2_{pp}\phi(\underline{x}_n),D^{K_n}U^{K_n}(\underline{x}_n)\big)\geq 0.$$

So by Definition 1 $\underline{U}$ is a viscosity super-solution of $(\mathbf{HJB})_K$. In the same way we can show that $\overline{U}$ is a viscosity sub-solution of $(\mathbf{HJB})_K$. Moreover since the a priori inequalities on U^{K_n} can

be chosen uniform in n (because $\|K_n\|_1 \to \|K\|_1$) they are true for $\underline{U}$ and $\overline{U}$. So Proposition 3 implies that $\underline{U} \geq \overline{U}$. Because we have the other inequality by definition we get $\overline{U} = \underline{U} = U^K$, the unique viscosity solution with polynomial growth of $(\mathbf{HJB})_K$.

To complete the proof we show that $(\underline{x_n})_{n\geq 0}$ admits a subsequence converging towards some $x \in \mathscr{E}^K$ and that for $j = a$ and b, $\theta^{n,a}(t_n)$ converges $\theta^a(t)$.

We have $\underline{x}_n = (t_n, p_n, i_n, \theta^{n,a}, \theta^{n,b})$ with

$$\theta^{n,a} = \sum_{j=1}^{m^{n,a}} K_n(-T_j^{n,a}) \text{ and } \theta^{n,b} = \sum_{j=1}^{m^{n,b}} K_n(-T_j^{n,b})$$

where $m^{n,a}$ and $m^{n,b}$ are non-negative integers, $(T_j^{n,a})_{1\leq j\leq m^{n,a}}$ and $(T_j^{n,b})_{1\leq j\leq m^{n,b}}$ are in $[0, t_n]$. We recall that $(\|x_n\|)_{n\geq 0}$ is bounded. Hence up to a subsequence $(t_n, p_n, i_n, \|\theta^{n,a}\|_1, \|\theta^{n,b}\|_1)_{n\geq 0}$ converges towards some (t, p, i, l^a, l^b). Since we have assumed that $\|K\|_1$ is positive the convergence of $(\|\theta^{n,a}\|_1)_{n\geq 0}$ and $(\|\theta^{n,b}\|_1)_{n\geq 0}$ imply those of $(m^{n,a})_{n\geq}$ and $(m^{n,b})_{n\geq 0}$. Consequently those sequences are eventually constant equal to m^a and m^b for n large enough. Then up to a subsequence we have convergence of $\left((T_j^{n,a})_{1\leq j\leq m^a}\right)_{n\geq 0}$ and $\left((T_j^{n,b})_{1\leq j\leq m^b}\right)_{n\geq 0}$ since they take their values in $[0, T]^{m^a}$ and $[0, T]^{m^b}$ which are compact sets. We consider $(T_j^a)_{1\leq j\leq m^a}$ and $(T_j^b)_{1\leq j\leq m^b}$ their limits. We now show that $(\theta^{n,a})_{n\geq 0}$ converges in L^1 towards

$$\theta^a = \sum_{j=1}^{m^a} K(\cdot - T_j^a) \in \Theta_t^K.$$

Since by comparison theorem $T_j^a \leq t$ it is enough to show that $K_n(\cdot - T_1^{n,a})$ converges in L^1 towards $K(\cdot - T_1^a)$ to conclude. We write

$$\begin{aligned}\|K(\cdot - T_1^a) - K_n(\cdot - T_1^{n,a})|_1 &\leq \|K_n(\cdot - T_1^{n,a}) - K(\cdot - T_1^{n,a})\|_1 + \|K(\cdot - T_1^{n,a}) - K(\cdot - T_1^a)\|_1 \\ &\leq \|K_n - K\|_1 + \|K(\cdot - T_1^{n,a}) - K(\cdot - T_1^a)\|_1.\end{aligned}$$

The first term goes to 0 by hypothesis, the second by dominated convergence. Same results holds for $(\theta^{n,b})_{n\geq 0}$ and θ^b. Consequently we have proved the convergence of $(x_n)_{n\geq 0}$ towards

$$x = (t, p, i, \theta^a, \theta^b) \in \mathscr{E}^K.$$

We finally show that $\theta^{n,a}(t_n)$ converges towards $\theta^a(t)$, the same methodology holds for b. We have for n large enough

$$|\theta^{n,a}(t_n) - \theta^a(t)| \leq \sum_{j=1}^{m^a} |K_n(t_n - T_j^{n,a}) - K(t - T_j^a)|.$$

The uniform convergence of K_n towards K implies that $K_n(t_n - T_j^{n,a})$ converges towards $K(t - T_j^a)$. This concludes the proof.

5.4 Proof of point (iv) of Theorem 2

We recall that the proof of Theorem 2 is exactly the same of Theorem 1. So for any $(t,y) \in \mathscr{E}^n$ we define for $(t,y) \in \mathscr{E}^n$ the process $Y^{t,y} = (i^{t,y}, P^{t,y}, c^{t,y;a}, c^{t,y;b}) \in \mathscr{E}^n$ by analogy with the process $X^{t,x}$ defined in Section 5.2.2. Note that by construction for any $(t,x) \in \mathscr{E}^{K_{\alpha,\gamma}}$ and for any $s \in [t,T]$ we have for $(t,y) = \mathscr{R}^{\alpha,\gamma}(t,x)$

$$(s, Y_s^{t,y}) = \mathscr{R}^{\alpha,\gamma}(s, X_s^{t,x}). \tag{17}$$

Then as in Section 5.2.4 we prove that the function

$$U^{\alpha,\gamma}(t,y) = \sup_{\delta \in \mathscr{A}_t} \mathbb{E}^{\delta}[G(i_T^{t,y}, P_T^{t,y}) + \int_t^T \Big(g(i_s^{t,y}, P_s^{t,y}) + \delta_s^a \lambda_s^{a,\delta} + \delta_s^b \lambda_s^{b,\delta} \Big) \mathrm{d}s]$$

is the unique viscosity solution with polynomial growth of $(\mathbf{HJB})_{\alpha,\gamma}$. Moreover for any $(t,x) \in \mathscr{E}^{K_{\alpha,\gamma}}$ and $(t,y) = \mathscr{R}^{\alpha,\gamma}(t,x)$ by Equation (17) we have:

$$U^{\alpha,\gamma}(t,y) = \sup_{\delta \in \mathscr{A}_t} \mathbb{E}^{\delta}[G(i_T^{t,x}, P_T^{t,x}) + \int_t^T \Big(g(i_s^{t,x}, P_s^{t,x}) + \delta_s^a \lambda_s^{a,\delta} + \delta_s^b \lambda_s^{b,\delta} \Big) \mathrm{d}s] = U^{K_{\alpha,\gamma}}(t,x).$$

Therefore for any $(t,x) \in \mathscr{E}^{K_{\alpha,\gamma}}$ we have $U^{K_{\alpha,\gamma}}(t,x) = U^{\alpha,\gamma} \circ \mathscr{R}^{\alpha,\gamma}(t,x)$. This concludes on the proof of point (iv) of Theorem 2.

IV.A Proof of Lemma 1

We first prove (i). Consider $(\theta_k)_{k\geq 0}$ a sequence with values in Θ_t^K that converges towards some θ in L^1. We have

$$\theta_k = \sum_{j=1}^{N_k} K(\cdot - T_j^k).$$

The convergence of $\|\theta_k\|_1$ towards $\|\theta\|_1$ gives that N_k is constant equal to some N up to a certain rank. Finally for any subsequence $\big((T_j^{\sigma(k)})_{1\leq j\leq N}\big)_{k\geq 0}$ converging to some $(T_j)_{1\leq j\leq N}$ we have :

$$\theta_{\sigma(k)} \to \sum_{j=1}^{N} K(\cdot - T_j) = \theta, \text{ in } L^1.$$

so Θ_t^K is closed. Now with the same notation we consider a bounded sequence $(\theta_k)_{k\geq 0}$. We can find an extraction σ such that $N_{\sigma(k)}$ is constant equal to some N and such that for $j = 1 \dots N$, $T_j^{\sigma(k)} \to T_j$. This implies that

$$\theta_{\sigma(k)} \to \sum_{j=1}^{N} K(\cdot - T_j).$$

This show that that Θ_t^K is locally compact.

Now we prove (ii). Consider a converging sequence $(s_k,\theta_k)_{k\geq 0}$ such that $\theta_k \in \Theta^K_{s_k}$ for any k and let $\theta = \sum_i^N K(\cdot - T_j)$ be the limit of $(\theta_k)_{k\geq 0}$. Then necessarily $((T_j^k)_{1\leq j\leq n})_{k\geq 0}$ converges towards $(T_j)_{1\leq j\leq n}$. Moreover by comparison we have $T_j \leq s$ and by continuity of K that

$$\theta_k(s_k) \to \sum_{j=1}^N K(s - T_j).$$

Finally consider now that $K(t) = \alpha e^{-\gamma t}$, for $l \in \mathbb{N}$ we have

$$(\theta_k^K)^{(l)}(T) = \sum_{i=1}^N \alpha(-\gamma)^l e^{-\gamma(T-T_i^k)}.$$

The convergence of $(T_j^k)_{k\geq 0}$ thus imply that $\theta_k^{(l)}(T) \to \theta^{(l)}(T)$.

IV.B A priori inequalities

In this section we prove some a priori inequalities.

IV.B.1 Hawkes processes

Consider a Hawkes process N with kernel $K = c\mathbf{1}_{\mathbb{R}_+}$ and exogenous intensity μ. The intensity of N is given by

$$\lambda_t = \mu + N_t c.$$

The existence of such process is proved in [Jac75]. Consider $T_p = \inf\{s \text{ s.t. } N_s > p\}$, by to [Jac75], $T_\infty = \lim_{n\to+\infty} T_n = +\infty$. Then let $N^p = N^{T_p}$. We have for any $t \in [0,T]$

$$\mathbb{E}[N_t^p] = \mathbb{E}[\int_0^{t\wedge T_p} \lambda_s \mathrm{d}s] \leq \mathbb{E}[\int_0^t C(1+N_s^p)\mathrm{d}s].$$

thus using a Gronwall lemma we get $\mathbb{E}[N_T^p] \leq CTe^{CT}$. The RHS being independent of p and using monotonous convergence we get

$$\mathbb{E}[N_T] < +\infty.$$

We also have

$$\begin{aligned}\mathbb{E}[(N_t^p)^2] &= \mathbb{E}[\int_0^{t\wedge T_p} (2N_{s^-}^p + 1)\mathrm{d}N_s] \\ &= \mathbb{E}[\int_0^{t\wedge T_p} (2N_{s^-}^p + 1)\lambda_s \mathrm{d}s] \leq \mathbb{E}[\int_0^{t\wedge T_p} C(2N_s^p + 1)(N_s^p + 1)s\mathrm{d}s].\end{aligned}$$

Using again a Gronwall lemma we deduce that $\mathbb{E}[(N_T^p)^2] \leq CT^2e^{CT}$ with C independent of p, so

$$\mathbb{E}[(N_T)^2] < +\infty.$$

Now consider a Hawkes process N with kernel K bounded and intensity given by

$$\lambda_t = \Phi\Big(\int_0^t K(t-s)\Big)\mathrm{d}N_s$$

with Φ non decreasing in its last variable and such that $|\Phi(x)| \leq C(1+|x|)$ for some $C > 0$. By thining we can see N as dominated by some Hawkes process $\tilde{N}$ with kernel $C\mathbf{1}_{\mathbb{R}_+}$ and exogenous intensity C. Remark that as a consequence $\tilde{\lambda}$ dominates λ. So we get

$$\mathbb{E}[\tilde{N}_T + \int_0^T \tilde{\lambda}_s] < +\infty$$

then consequently

$$\mathbb{E}[N_T] < +\infty,\ \mathbb{E}[N_T^2] < +\infty \text{ and } \mathbb{E}[\int_0^T \delta e^{-k\delta}\lambda_s \mathrm{d}s] < +\infty.$$

This ensures that the function U^K defined in Equation (15) is well defined. This also implies that the martingales $M^{t,x;a,\delta}$ and $M^{t,x;b,\delta}$ are uniformly integrable martingales.

IV.B.2 A priori estimates on X

We prove here that the value function U^K defined in Equation (15) has polynomial growth in x. For this we show some inequalities on the norm of $(X^{t,x})_{(t,x)\in\mathscr{E}^K}$. More precisely we prove the following result:

Lemma 3. *There exists some positive constant C depending only on T and on the regularity constants of G and g such that for any $(t,x) \in \mathscr{E}$*

$$\mathbb{E}^{\delta}_{t,x}[\sup_{s\in[t,T]} \|X_s^{t,x}\|^2] \leq C(1+\|x\|^2)$$

and

$$|U^K(t,x)| \leq C(1+\|x\|^2).$$

To prove Lemma 3, consider $(t,x) \in \mathscr{E}^K$ and $\delta \in \mathscr{A}_t$ with $x = (P, i, \theta^a, \theta^b)$. We show differents a priori estimates on the subprocesses composing $X^{t,x} = (P^{t,x}, i^{t,x}, \theta^{t,x;a}, \theta^{t,x;b})$ under the probability $\mathbb{P}^{t,x;\delta}$.

A priori estimates on $\theta^{t,x;a}$ and $\theta^{t,x;b}$: We have

$$N_s^a - N_t^a = M_s^{t,x;a,\delta} + \int_t^s \lambda^a(u, X_u^{t,x}, \delta_u)\mathrm{d}u$$

since $\lambda^a(t,x,\delta) \leq C(1+\|\theta^a\|_1)$ and $\|\theta_u^a\|_1 = \|\theta^a\|_1 + (N_u^a - N_t^a)\|K\|_1$ we have

$$N_s^a - N_t^a \leq M_s^{t,x;a,\delta} + \int_t^s C\big(1 + \|\theta_u^{t,x;a}\|_1 + \|K\|_1(N_u^a - N_t^a)\big)\mathrm{d}u.$$

Passing to the average under $\mathbb{P}^{t,x;\delta}$ using the fact that $M^{t,x;d,\delta}$ is a true martingale and a Grönwall lemma we get

$$\mathbb{E}^{\delta}_{t,x}[N^a_T - N^a_t] \leq C(1 + \|\theta^a\|_1) \tag{18}$$

where C only depends on T and on the model constants. Consequently we have for some positive constant C

$$\mathbb{E}^{\delta}_{t,x}[\|\theta^{t,x;a}_s\|_1] \leq C(1 + \|\theta^a\|_1).$$

We now give an a priori estimate for the second order moment.

$$\begin{aligned}(N^a_s - N^a_t)^2 &= \int_t^s \big(2(N^a_{u^-} - N^a_t) + 1\big)\lambda_u \mathrm{d}u + \int_t^s \big(2(N^a_{u^-} - N^a_t) + 1\big)\mathrm{d}M^{t,x;a,\delta}_u \\ &\leq \int_t^s \big(2(N^a_{u^-} - N^a_t) + 1\big)C(1 + \|\theta^{t,x;a}_u\|_1 + N^a_u - N^a_t)\mathrm{d}u + \int_t^s \big(2(N^a_{u^-} - N^a_t) + 1\big)\mathrm{d}M^{t,x;a,\delta}_u \\ &\leq \int_t^s C(N^a_{u^-} - N^a_t)^2 + (N^a_{u^-} - N^a_t)C(1 + \|\theta^{t,x;a}_u\|_1)\mathrm{d}u + \int_t^s \big(2(N^a_{u^-} - N^a_t) + 1\big)\mathrm{d}M^{t,x;a,\delta}_u.\end{aligned}$$

The average of the last term of the right hand side is 0 as consequence of Appendix IV.B.1. Thus taking the average and using a Grönwall lemma we get

$$\mathbb{E}^{\delta}_{t,x}[(N^a_T - N^a_t)^2] \leq C(1 + \|\theta\|_1^2). \tag{19}$$

A priori estimates on $P^{t,x}$: We have

$$\mathrm{d}P^{t,x}_s = d(s, P^{t,x}_s)\mathrm{d}s + \sigma \mathrm{d}W_s, \text{ with } P^{t,x}_t = P.$$

By assumptions there exists $k > 0$ such that :$|d(t,p) - d(t,q)| \leq k|p - q|$. We have the classic apriori estimates (see for exemple Theorem 1.2 in [Tou12]).

$$\mathbb{E}^{\delta}_{t,x}[\sup_{s \leq T} P^2_s] \leq C(1 + P^2). \tag{20}$$

Where C only depends only on the Lipshitz constant k and on T.

A priori estimates on $X^{t,x}$: We have

$$\begin{aligned}&i_s = i + N^a_s - N^a_t + N^b_s - N^b_t \\ &\|\theta^j_s\|_1 = \|\theta^j\|_1 + \|K\|_1(N^j_s - N^j_t), \text{ for } j = a,\ b.\end{aligned}$$

Thus we have

$$\|X^{t,x}_s\|^2 \leq C(1 + i^2 + \|\theta^a\|_1^2 + \|\theta^b\|_1^2 + (N^a_s - N^a_t)^2 + (N^b_s - N^b_t)^2 + P^2_s).$$

Taking the average and using Equations (18), (19) and (20) we get

$$\mathbb{E}^{\delta}_{t,x}\big[\sup_{s \leq T}\|X^{t,x}_s\|^2\big] \leq C(1 + i^2 + \|\theta^a\|_1^2 + \|\theta^b\|_1^2 + P^2) = C(1 + \|x\|^2) \tag{21}$$

where C is independent of δ.

A priori estimates on the value function: By the quadratic growth of G and $\tilde{g}$ we get

$$|J^K(t,x;\delta)| \leq \mathbb{E}^{\delta}_{t,x}\big[e^{-r(T-t}C(1+\|X_T^{t,x}\|^2) + \int_t^T e^{-r(s-t)}C(1+\|X_s^{t,x}\|^2)\big].$$

Because of the a priori estimates (18), (19), (20) and (21) we have

$$|J^K(t,x;\delta)| \leq C(1+i^2+\|\theta^a\|_1^2+\|\theta^b\|_1^2+P^2) \leq C(1+\|x\|^2)$$

where C only depends on T and the regularity constants. We conclude by arbitrariness of δ.

IV.B.3 Rewriting of the utility

We show that for any $\delta \in \mathscr{A}$ we have

$$\mathbb{E}^{\delta}[\int_0^T e^{-rs}\delta_s^a \mathrm{d}N_s^a] = \mathbb{E}^{\delta}[\int_0^T e^{-rs}\delta_s^a \lambda_s^{a,\delta}\mathrm{d}s].$$

The same result for b holds by the same arguments. To conclude it is enough to show that

$$\overline{M}_t = \int_0^t e^{-rs}\delta_s^a \mathrm{d}M_s^a$$

is a true martingale. We have

$$[\overline{M}]_t = \int_0^t e^{-2rs}(\delta_s^a)^2 \mathrm{d}N_s^a \text{ and } \langle\overline{M}\rangle_t = \int_0^t e^{-2rs}(\delta_s^a)^2\lambda_s^{a,\delta}\mathrm{d}s$$

and since $(\delta_t^a)^2\lambda_t^{a,\delta} \leq C(1+\|X_t\|_1)$ we get

$$\langle\overline{M}\rangle_T \leq \int_0^T e^{-2rs}C(1+\|X_s\|_1)\mathrm{d}s \leq TC(1+\sup_{s\in[0,T]}\|X_s\|_1).$$

The last term of the RHS is integrable by Lemma 3. So by the monotone convergence $[\overline{M}]_T$ is also integrable so $\overline{M}$ is a uniformly integrable martingale. As consequence we get

$$\begin{aligned}\mathbb{E}^{\delta}[G(i_T,P_T)e^{-rT} + \int_0^T e^{-rs}\Big(g(i_s,P_s)\mathrm{d}s + \delta_s^a\mathrm{d}N_s^a + \delta_s^b\mathrm{d}N_s^b\Big)] \\ = \mathbb{E}^{\delta}[G(i_T,P_T)e^{-rT} + \int_0^T e^{-rs}\Big(g(i_s,P_s) + \delta_s^a\lambda_s^{a,\delta} + \delta_s^b\lambda_s^{b,\delta}\Big)\mathrm{d}s].\end{aligned}$$

IV.C Equivalence between the two definitions of viscosity solutions

Lemma 4. *Definition 1 and Definition 2 are equivalent.*

Proof. We show it for sub-solutions, the demonstration is the same for super-solutions.

Consider U a USC function sub-solution of $(\mathbf{HJB})_K$ in the sense of Definition 2. Now consider ϕ a test function such that $0 = U(t_0,x_0) - \phi(t_0,x_0) = \sup_{\mathcal{V}} U - \phi$ for $\mathcal{V}$ a neighborhood of (t_0,x_0) in $\mathscr{E}^K$. We show that

$$F\big(t_0,x_0,U(t_0,x_0),\nabla\phi(t_0,x_0),\partial^2_{pp}\phi(t_0,x_0),D^K\phi(t_0,x_0)\big)\leq 0.$$

Writing $x=(p,z)\in\mathbb{R}\times\mathcal{Z}_T^K$ we have $\phi(t,p,z)=\phi(t_0,p_0,z_0)+\partial_t\phi(t_0,x_0)(t-t_0)+\partial_p\phi(t_0,x_0)(p-p_0)+\partial^2_{pp}\phi(t_0,x_0)\frac{(p-p_0)^2}{2}+o(|p-p_0|^2+|t-t_0|^2)+h(z-z_0)$. where h is a modulus of continuity of ϕ. Thus we have

$$(\nabla\phi(t_0,x_0),\partial^2_{pp}\phi(t_0,x_0),h)\in\mathscr{J}^+u(t_0,x_0).$$

Consequently

$$F(t_0,x_0,U(t_0,x_0),\nabla\phi(t_0,x_0),\partial^2_{pp}\phi(t_0,x_0),D^KU(t_0,x_0))\geq 0.$$

So U is a viscocity sub-solution of $(\mathbf{HJB})_K$ in the sense of Definition 1.

Now we show the opposite implication. Consider U a USC function sub-solution of $(\mathbf{HJB})_K$ in the sense of Definition 1. Consider $(d,A,h)\in\mathscr{J}^+U(t_0,x_0)$, we built a test function ϕ dominating U with equality at point (t_0,x_0) and such that

$$\big(\nabla\phi(x_0),\partial^2_{pp}\phi(x_0)\big)=(d,A).$$

We will then get the expected inequality that will extend directly to $\overline{\mathscr{J}}^+U(t_0,x_0)$ by continuity of F.

Using the notation $(t,x)=(t,p,z)\in[0,T]\times\mathbb{R}\times\mathcal{Z}_t^K$ we have

$$U(t,x)\leq U(t_0,x_0)+d_1(t-t_0)+d_2(p-p_0)+\frac{1}{2}A(p-p_0)^2+h(z-z_0)+o(|p-p_0|^2)+o(|t-t_0|).$$

hence

$$U(t,p,z)-h(z-z_0)\leq U(t_0,x_0)+d_1(t-t_0)+d_2(p-p_0)+\frac{1}{2}A(p-p_0)^2+o(|p-p_0|^2)+o(|t-t_0|).$$

We take the supremum on z over a compact neighborhood of z_0, and consider

$$\tilde{U}(t,p)=\sup_{z\in B_r(z_0)\cap\mathcal{Z}_t^K}U(t,p,z)-h(z-z_0).$$

Since $\tilde{U}(t_0,p_0)=U(t_0,x_0)$ we get

$$\tilde{U}(t,p)\leq\tilde{U}(t_0,p_0)+d_1(t-t_0)+d_2(p-p_0)+\frac{1}{2}A(p-p_0)^2+o(|p-p_0|^2)+o(|t-t_0|).$$

We prove at the end that $\tilde{U}$ is a USC function and assume this is true. The last equation means that $(d, A) \in \mathscr{J}^+\tilde{U}(t_0, p_0)$. Then by an argument developped for the analysis of viscosity solutions on $\mathbb{R}^d$ (see for example [FS06] Lemma 4.1.) we have existence of a function $\phi \in C^{1,2}$ such that

$$\tilde{U}(t,p) - \phi(t,p) \leq \tilde{U}(t_0,p_0) - \phi(t_0,p_0) \text{ with } (\nabla\phi(t_0,p_0), \partial^2_{pp}\phi(t_0,p_0)) = (d,A).$$

So finally we have on a compact neighborhood of x_0:

$$U(t,p,e) - \phi(t,p) - h(e-e_0) \leq U(t_0,p_0,e_0) - \phi(t_0,p_0) - h(e_0-e_0).$$

This local domination can then be extended to the whole domain $\mathscr{E}^K$.

Finally we show that $\tilde{U}$ is a USC function. Fix $\varepsilon > 0$ and (t,p). Since U is USC and h continuous, for any $e \in B_r(e_0)$ we can find r_e such that on $B_{r_e}(t,p,e)$ we have

$$U + h(\cdot - e_0) \leq U(t,p,e) + h(e-e_0) + \varepsilon.$$

The collection $\left(B_{\frac{r_e}{2}}(t,p,e)\right)_{e\in B_r(e_0)}$ forms an open covering of $\{t\}\times\{p\}\times B_r(e_0)$ which is a compact set by Lemma 1. Thus we may find a finite sequence $\left(B_{\frac{r_{e_i}}{2}}(t,p,e_i)\right)_{1\leq i\leq N}$ that covers $\{t\}\times\{p\}\times B_r(e_0)$. Consider $r_* = \min\limits_{1\leq i\leq N} \frac{r_{e_i}}{2}$. Now take any $(s,q) \in B_{r_*}(t,p)$, then fo any $e \in B_r(e_0)$ there is some $i \in \{1,\dots,N\}$ such that $(t,p,e) \in B_{r_{e_i}/2}(t,p,e_i)$. Hence we get

$$\|(s,q,e) - (t,p,e_i)\| \leq \frac{r_{e_i}}{2} + r^* \leq r_{e_i}$$

so $(s,q,e) \in B_{r_{e_i}}(t,p,e_i)$ and consequently

$$U(s,q,e) - h(e-e_0) \leq U(t,p,e_i) + h(e_i-e_0) + \varepsilon \leq \tilde{U}(t,p) + \varepsilon.$$

Passing to the supremum in $e \in B_r(e_0)$ in the LHS we get that $\tilde{U}$ is USC. □

IV.D Crandall Ishi's lemma

The most crucial point to prove comparison result for viscosity solutions is the Crandall-Ishi's lemma that allows to deal with the second order terms. In the general case the Crandall Ishi's lemma is proved for subset of $\mathbb{R}^n$, see [CIL92]. Hence our particular domain requires an adaptation of the classic version of the Lemma.

Lemma 5. *Let $\phi_1 \in C^2([0,T]^2)$, $\phi_2 \in C^1(\mathbb{R}^2)$ and $\phi_3 \in C^0\left((\mathcal{Z}_T^K)^2\right)$, $u \in USC(\mathscr{E}^K)$ and $v \in LSC(\mathscr{E}^K)$. Suppose we have $(t_0,p_0,z_0) \in \mathbb{R}^2\times\mathbb{R}^2\times(\mathcal{Z}_T^K)^2$ such that*

$$\begin{aligned} &u(t_0^1,p_0^1,z_0^1) - v(t_0^2,p_0^2,z_0^2) - \phi_1(t_0) - \phi_2(p_0) - \phi_3(z_0) \\ &= \sup_{t,p,z\in\mathbb{R}^2\times[0,T]^2\times(\mathcal{Z}_T^K)^2} u(t^1,p^1,z^1) - v(t^2,p^2,z^2) - \phi_1(t) - \phi_2(p) - \phi_3(z). \end{aligned} \tag{22}$$

Then for any ε there is (A_ε, h) and (B_ε, h) in $\mathbb{R} \times C^0(\mathcal{Z}_T^K)$ such that

$$\big((\nabla_1\phi_1(t_0), \nabla_1\phi_2(p_0)), A_\varepsilon, h\big) \in \overline{\mathscr{J}}^+ u(t_0^1, p_0^1, z_0^1), \quad \big((-\nabla_2\phi_1(t_0), -\nabla_2\phi_2(p_0)), B_\varepsilon, h\big) \in \overline{\mathscr{J}}^- v(t_0^2, p_0^2, z_0^2)$$

and that

$$-(\varepsilon^{-1} + |H\phi_2(p_0)|) I_2 \leq \begin{pmatrix} A_\varepsilon & 0 \\ 0 & -B_\varepsilon \end{pmatrix} \leq H\phi_2(p_0) + \varepsilon H\phi_2(p_0)^2 \tag{23}$$

where H is the Hessian operator and $|A|$ denotes the spectral radius of the matrix A.

Note that even thought this extension is not straightforward we benefit from the fact that in $(\mathbf{HJB})_K$ the second order derivative is related to a real variable. Therefore the strategy of the proof is to bring back the problem in the classic framework.

Proof. We can consider that there exists $\mathcal{V}$ a compact neighborhood of (t_0, p_0, z_0) in $\mathscr{E}^K$ such that on $\mathcal{V} \setminus \{t_0, p_0, z_0\}$ we have

$$\begin{aligned}
(u-v)(t_0, p_0, z_0) &\geq (u-v)(t, p, z) - \phi_2(p) + \phi_2(p_0) - \phi_1(t) + \phi_1(t_0) - \phi_3(z) + \phi_3(z_0) \\
&\geq (u-v)(t, p, z) - \phi_2(p) + \phi_2(p_0) + \nabla\phi_1(t_0)(t_0 - t) + \mathcal{O}(\|t - t_0\|^2) - \phi_3(z) + \phi_3(z_0) \\
&> (u-v)(x, y, z) - \phi_2(p) + \phi_2(p_0) + \nabla\phi_1(t_0)(t_0 - t) - C\|t - t_0\|^2 - h(z_0^1 - z^1) - h(z_0^2 - z^2)
\end{aligned}$$

where h is any modulus of continuity of the function ϕ_3 and C a positive constant. For $x \in \mathbb{R}$ consider $g_j(x) = \partial_j\phi_1(t_0)x - Cx^2$ for $j = 1$ or 2. So on $\mathcal{V} \setminus (t_0, p_0, z_0)$ we have:

$$\begin{aligned}
(u-v)(t_0, p_0, z_0) > \quad & u(t^1, p^1, z^1) - v(t^2, p^2, z^2) - \phi_2(p) + \phi_2(p_0) \\
& - h(z_0^1 - z^1) - h(z_0^2 - z^2) + g_1(t_0^1 - t^1) + g_2(t_0^2 - t^2)
\end{aligned} \tag{24}$$

with equality at (t_0, p_0, z_0) and with $h(0) = g_i(0) = 0, \quad g_j'(0) = \partial_j\phi_1(t_0)$.

We can always assume that there exists $r > 0$ so that $\mathcal{V}$ is of the form

$$\mathcal{V} = \big(B_r(t_0) \times B_r(p_0) \times B_r(z_0)\big) \cap \mathscr{E}^K$$

where $B_r(x)$ denotes the closed ball with center x and radius r. We define the following functions

$$\begin{aligned}
\tilde{u}(p^1) &= \sup_{t,z \in B_r(t_0) \times B_r(z_0)} u(t^1, p^1, z^1) - h(z_0^1 - z^1) + g_1(t_0^1 - t^1) \\
\tilde{v}(p^2) &= \inf_{t,z \in B_r(t_0) \times B_r(z_0)} v(t^2, p^2, z^2) - h(z_0^2 - z^2) + g_2(t_0^2 - t^2)
\end{aligned}$$

where the supremums above are taken for (t, z) such that $(t, p, z) \in \mathscr{E}^K$. The functions $\tilde{u}$ and $\tilde{v}$ are respectively USC and LSC functions since the supremums are taken over compact subsets (see the proof of Lemma 4). And we have

$$\tilde{u}(p^1) - \tilde{v}(p^2) - \phi_2(p) \leq \tilde{u}(p_0^1) - \tilde{v}(p_0^2) - \phi_2(p_0).$$

Thus by the Crandall Ishi's lemma (see for exemple Theorem 6.1. in [FS06]) there exists $(A_\varepsilon, B_\varepsilon)$ satisfying (23) such that

$$(\partial_1\phi_2(p_0), A_\varepsilon) \in \overline{\mathscr{J}}^+ \tilde{u}(p_0^1) \text{ and } (-\partial_2\phi_2(p_0), B_\varepsilon) \in \overline{\mathscr{J}}^- \tilde{v}(p_0^2).$$

Consequently there exist a sequence $(q_n, A_n, p_n^1, \tilde{u}(p_n^1))_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty}(q_n, A_n, p_n^1, \tilde{u}(p_n^1)) = (\partial_1\phi_2(p_0), A_\varepsilon, p_0^1, \tilde{u}(p_0^1)), \text{ and } \forall\ n\geq 0,\ (q_n, A_n, p_n^1, \tilde{u}(p_n^1)) \in \mathscr{J}^+\tilde{u}(p_n^1).$$

So for any n we have

$$\tilde{u}(p^1) \leq \tilde{u}(p_n^1) + q_n(p^1 - p_n^1) + \frac{1}{2}A_n(p^1 - p_n^1)^2 + o(|p^1 - p_n^1|^2).$$

Consider t_n^1 and z_n^1 such that

$$\tilde{u}(p_n^1) = u(t_n^1, p_n^1, z_n^1) - h(z_0^1 - z_n^1) + g_1(t_0^1 - t_n^1)$$

such maximizers exist by compacity. We show that (t_n^1, p_n^1, z_n^1) converges towards (t_0^1, p_0^1, z_0^1), we assume it for now. Equation (24) implies that for any (t, p, z) we have

$$\begin{aligned} u(t^1, p^1, z^1) \leq\ & u(t_n^1, p_n^1, z_n^1) + q_n(p^1 - p_n^1) + \frac{1}{2}A_n(p^1 - p_n^1)^2 + o(|p^1 - p_n^1|^2) \\ & - h(z_0^1 - z_n^1) + h(z_0^1 - z^1) + g_1(t_0^1 - t_n^1) - g_1(t_0^1 - t^1). \end{aligned}$$

Consider the function $h_n(z^1) = -h(z_0^1 - z_n^1) + h(z_0^1 - z_n^1 - z^1)$ such that $h_n(0) = 0$ and

$$h_n(z^1 - z_n^1) = -h(z_0^1 - z_n^1) + h(z_0^1 - z^1).$$

Since z_n^1 converges towards z^1 the sequence $(h_n)_{n\geq 0}$ converges uniformly towards h because h is continuous and because we are working on compact neighborhood. Consider $q_n^1 = \partial_1\phi_1(t_0) - 2C(t_0^1 - t_n^1)$ that converges towards $\partial_1\phi_1(t_0)$

$$g_1(t_0^1 - t_n^1) - g_1(t_0^1 - t^1) = q_n^1(t^1 - t_n^1) + C(t_n^1 - t^1)^2.$$

Thus we have

$$\begin{aligned} u(t^1, p^1, z^1) \leq & u(t_n^1, p_n^1, z_n^1) + q_n(p^1 - p_n^1) + \frac{1}{2}A_n(p^1 - p_n^1)^2 + o(|p^1 - p_n^1|^2) \\ & + q_n^1(t^1 - t_n^1) + o(|t^1 - t_n^1|) + h_n(z^1 - z_n^1) \end{aligned}$$

hence $\big((q_n^1, q_n), A_n, h_n\big) \in \mathscr{J}^+u(t_n^1, p_n^1, z_n^1)$ and

$$\big((q_n^1, q_n), A_n, h_n\big) \to \big((\partial_1\phi_1(t_0), \partial_1\phi_2(p_0)), A_\varepsilon, h\big)$$

Finally we show that $(t_n^1, p_n^1, z_n^1) \underset{n\to+\infty}{\to} (t_0^1, p_0^1, z_0^1)$ which will imply the conclusion that

$$\big((\partial_1\phi_1(t_0), \partial_2\phi_2(p_0)), A_\varepsilon, h\big) \in \overline{\mathscr{J}}^+u(t_0^1, p_0^1, z_0^1).$$

We have for any $n \geq 0$:

$$\tilde{u}(p_n^1) = u(t_n^1, p_n^1, z_n^1) - h(z_0^1 - z_n^1) - g_1(t_0^1 - t_n^1).$$

Consider any $(t^1, z^1) \in \overline{(t_n^1, z_n^1)}_{n\geq 0}$. Since $\tilde{u}(p_n^1) \to \tilde{u}(p_0^1)$, by upper semi-continuity of u and by the definition of $\tilde{u}$ we get

$$u(t^1, p_0^1, z^1) - h(z_0^1 - z^1) + g_1(t_0^1 - t^1) \geq u(t_0^1, p_0^1, z_0^1).$$

Which implies that $(t^1, z^1) = (t_0^1, z_0^1)$ since everywhere else the above inequality is false because of Equation (24). So we get

$$\big((q_n^1, q_n), A_n, h_n, (x_n^1, y_n^1, z_n^1)\big) \underset{n\to\infty}{\to} \big((\partial_1\phi_1(t_0), \partial_1\phi_2(p_0)), A_\varepsilon, h, (t_0^1, p_0^1, z_0^1)\big)$$

and so

$$\big((\partial_1\phi_1(t_0), \partial_1\phi_2(p_0)), A_\varepsilon, h\big) \in \overline{\mathscr{J}}^+ u(t_0^1, p_0^1, z_0^1).$$

Similarly we get

$$\big((-\partial_2\phi_1(t_0), -\partial_2\phi_2(p_0)), B_\varepsilon, h\big) \in \overline{\mathscr{J}}^- v(t_0^2, p_0^2, z_0^2).$$

This concludes the proof. □

IV.E Existence of $\mathscr{R}^{\alpha,\gamma}$

Consider $t \geq 0$ we have for any $j \geq 0$

$$\theta_t^{a(j)}(T) = \sum_{i=1}^n c_t^{a,i}(-\gamma_i)^j e^{-\gamma_i(T-t)} \text{ and } \theta_t^{b(j)}(T) = \sum_{i=1}^n c_t^{b,i}(-\gamma_i)^j e^{-\gamma_i(T-t)}. \tag{25}$$

So let A be the matrix with coefficient $A_{ij} = (-\gamma_i)^j$. This is a Vandermonde matrix which is invertible. By Equation (25) we have

$$c_t^{a,i} = e^{\gamma_i(T-t)} \sum_{j=1}^n (A^{-1})_{ij}\theta_t^{a(j)}(T) \text{ and } c_t^{b,i} = e^{\gamma_i(T-t)} \sum_{j=1}^n (A^{-1})_{ij}\theta_t^{b(j)}(T).$$

So we define $\mathscr{R}^{\alpha,\gamma}$ for $(t,x) \in \mathscr{E}^{K_{\alpha,\gamma}}$ by

$$\mathscr{R}^{\alpha,\gamma}(t,x) = \big(t, p, i, c^a(t,x), c^b(t,x)\big)$$

where

$$c^a(t,x) = \big(e^{\gamma_i(T-t)} \sum_{j=1}^n (A^{-1})_{ij}\theta^{a(j)}(T)\big)_{1\leq j\leq n} \text{ and } c^b(t,x) = \big(e^{\gamma_i(T-t)} \sum_{j=1}^n (A^{-1})_{ij}\theta^{b(j)}(T)\big)_{1\leq j\leq n}.$$

By Lemma 1 the map $\mathscr{R}^{\alpha,\gamma}$ is continuous and by construction we have for any $t \geq 0$

$$\mathscr{R}^{\alpha,\gamma}(t, X_t) = (t, Y_t^{\alpha,\gamma}).$$

IV.F Proof of Lemma 2

We are going to approximate the integral in Equation (7) by Riemann sum. We take $A_n = \sqrt{n}$ and $(a_i)_{0\le i\le n-1}$ a regular grid of $[0, A_n]$ with mesh $\frac{1}{\sqrt{n}}$. We set

$$K_n(t) = \sum_{i=0}^{n-1} e^{-a_{i+1}t} \int_{a_i}^{a_{i+1}} m(\mathrm{d}u) \le K(t).$$

For $t \in \mathbb{R}_+$, we have

$$K(t) - K_n(t) = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} m(\mathrm{d}u) \int_u^{a_{i+1}} te^{-tv}\mathrm{d}v - \int_{A_n}^{+\infty} e^{-tu} m(\mathrm{d}u).$$

Hence for any T and $t \le T$:

$$\begin{aligned}
|K_n(t) - K(t)| &\le \sum_{i=0}^{n-1} T \int_{a_i}^{a_{i+1}} m(\mathrm{d}u)(a_{i+1} - a_i) + \int_{A_n}^{+\infty} m(\mathrm{d}u) \\
&\le \frac{T}{\sqrt{n}} \int_0^n m(\mathrm{d}u) + \int_{A_n}^{+\infty} m(\mathrm{d}u) \\
&\le \frac{T}{\sqrt{n}} \int_0^{+\infty} m(\mathrm{d}u) + \int_{A_n}^{+\infty} m(\mathrm{d}u)
\end{aligned}$$

which goes to 0 when n goes to infinity, uniformly on $t \in [0, T]$. Hence the sequence K_n converges almost surely towards K and is dominated by K so K_n converges in $\mathbb{L}_1$ towards K.

Set $\alpha_n = K(0) - K_n(0)$ and $\beta_n = \frac{\alpha_n}{\|K\|_1 - \|K_n\|_1}$ and consider $\tilde{K}_n = K_n + \alpha_n e^{-\beta_n \cdot}$, we have for any n

$$\tilde{K}_n(0) = K(0) \text{ and } \|\tilde{K}_n\|_1 = \|K\|_1$$

and $\tilde{K}_n \to K$ in $\mathbb{L}_1$. Thus the sequence $(\tilde{K}_n)_{n\ge 0}$ gives the result.

IV.G Probabilistic representation of IPDE in high dimension

We are going to use a probabilistic representation based on branching processes. This method is insensitive to the dimension of the domain of the IPDE. Theoretically the method works for any semi-linear IPDE admitting a strong solution and with a generator that can be written as a power serie. Thought this is not the case for $(\mathbf{HJB})_{\alpha,\gamma}$, in order to implement this method we approximate the generator of the IPDE by a second order polynomial and assume that the approached IPDE have a strong solution. Thus we are left with an IPDE of the form

$$(\mathbf{HJB})'_{\alpha,\gamma}: \quad -\partial_t U - \mathscr{L}U - f(U, D^\alpha U) = 0, \quad u(T,\cdot) = 0 \text{ on } \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n$$

where

$$\begin{aligned}
f(U, D^\alpha U)(t,x) = &f_0(t,x) + f_1(t,x)U(t,x) \\
&+ f^a_{2,1}(t,x) D^\alpha_a U(t,x) + f^b_{2,1}(t,x) D^\alpha_b U(t,x) \\
&+ f^a_{2,2}(t,x) D^\alpha_a U(t,x)^2 + f^b_{2,2}(t,x) D^\alpha_b U(t,x)^2.
\end{aligned}$$

The operator $\mathscr{L}$ is defined by $\mathscr{L}U(t,x) = -\langle \Gamma, \nabla_a^c U(t,x)\rangle - \langle \Gamma, \nabla_b^c U(t,x)\rangle$.

Consider a process $\tilde{X}^{t,x}$ starting at time t with initial state x such that $(t,x) \in \mathscr{E}^n$ and which dynamic is driven by the infinitesimal generator $\mathscr{L}$. The Feynman-Kac formula gives

$$U(t,x) = \mathbb{E}[\frac{f(U, D^\alpha U)(\tau, \tilde{X}_{t\tau}^{t,x})}{\rho(\tau)} \mathbf{1}_{t+\tau<T} \mathrm{d}s] \tag{26}$$

where τ is a positive random variable with density ρ.

We show in Appendix IV.G.1 that there exists an appropriate probability measure $\mathbb{P}_{\mathscr{T}}$ on the set

$$\mathscr{T} = \left\{0, 1, (2, j, d, \varepsilon), \text{ with } d \in \{0,1,2\},\ j \in \{a,b\},\ \varepsilon \in \{0,1\}^d\right\}$$

and a set of functions $(g_\tau)_{\tau\in\mathscr{T}}$ from $[0,T] \times \mathbb{Z} \times \mathbb{R}_+^n \times \mathbb{R}_+^n$ such that for any random variable ξ with law $\mathbb{P}_{\mathscr{T}}$ we have

$$f(U, D^\alpha U)(t,x) = \mathbb{E}[g_\xi(U, D^\alpha U)(t,x)]. \tag{27}$$

The set $(g_\tau)_{\tau\in\mathscr{T}}$ is defined by

$$g_0(U, D^\alpha U)(t,x) = f_0(t,x)\mathbb{P}(l=0)^{-1},\ g_1(U, D^\alpha U)(t,x) = f_1(t,x)\mathbb{P}(l=1)^{-1}U(t,x)$$

and

$$g_{(2,j,d,\varepsilon)}(U, D^\alpha U)(t,x) = f_{2,d}^j(t,x)\mathbb{P}\big(l=(2,j,d,\varepsilon)\big)^{-1} \prod_{k=1}^d U(t, x+\Delta^j \varepsilon_i)(-1)^{1-\varepsilon_k},$$

where Δ^a (resp. Δ^b) is the jump corresponding to a ask (resp. bid) market order, namely $\Delta^a = (-1, \alpha, 0)$ and $\Delta^b = (1, 0, \alpha)$ (We recall that the price variable is no longer part of the domain).

We now define a branching process in the following way: any particle is noted by $(t, x, l_0, l_1, \ldots, l_n)$ where $(x,t) \in \mathscr{E}^n$ and the l_i's belong in $\mathscr{T}$. The variable x denotes the initial position of the particle and t its birth time, l_n is the label of the particle, l_{n-1} the label of its parent, and so on. The lifetime of the particles are i.i.d random variables with density ρ

We now describe the evolution of the particle. Consider a particle born at time s at the state x with lifetime τ. During its lifetime the particle state is described by its position: $\left((i_t^{s,x}, c_t^{s,x;a}, c_t^{s,x;b})\right)_{s\le t\le s+\tau}$ in $\mathbb{Z} \times \mathbb{R}_+^n \times \mathbb{R}_+^n$. The dynamic of the particle position is given by

$$\mathrm{d}c_t^{j,i} = -\gamma_i c_t^{j,i} \mathrm{d}t, \text{ for } i \in \{1, \ldots, n\} \text{ and } j = a \text{ or } b.$$

The other components are constants. Note that this dynamic corresponds to the infinitesimal generator $\mathscr{L}$. When the particle dies it gave birth to independent particles. The number and type of child particles depend on the label l_n of the particle:

- if $l_n = 0$: 0 child

- if $l = 1$: 1 child
- if $l_n = (2, d, j, \varepsilon)$: d children
  - if $j = a$ the initial state of the $i-th$ child particle is $X_{s+\tau}^{s;x} + \Delta^a \varepsilon_i$
  - if $j = b$ the initial state of the $i-th$ child particle is $X_{s+\tau}^{s;x} + \Delta^b \varepsilon_i$

The labels of the child particles are i.i.d. random variables with law $\mathbb{P}_{\mathcal{T}}$. We note $\mathscr{C}_p$ the set of the child particles.

Considering a particle starting at point (t, x), Equations (27) and (26) give

$$U(t,x) = \mathbb{E}[\frac{a(l, t+\tau, X_{t+\tau}^{t;x})}{\rho(\tau)} \prod_{c \in \mathscr{C}_p} U(t+\tau, X_c) \mathbf{1}_{t+\tau<T}]$$

where X_c denotes the initial position of the child particle c and where a is defined by

$$a(i,t,x) = f_0(t,x)\mathbb{P}(l=i)^{-1}, \text{ for } i = 1 \text{or } 2$$
$$a\big((2,j,d,\varepsilon),t,x\big) = f_{2,d}^{j}(t,x)\mathbb{P}\big(l=(2,j,d,\varepsilon)\big)^{-1} \prod_{k=1}^{d} (-1)^{1-\varepsilon_k}$$

By iterating the above equality to the descendents of the particle and assuming that the number of descendent particles born before the time horizon T is almost surely finite we can evaluate $U(t,x)$ using Monte Carlo simulation. For more details on this method we refer to [HLOT+19].

IV.G.1 Existence of a measure for the particle method

We have

$$f(u,Du)(t,x) = \mathbb{E}[f_I(u,Du)(t,x)]$$

where I is a random variable with values in $\{0,1,2\}$, and

$$f_0(u,Du)(x) = f_0(t,x)\mathbb{P}(I=0)^{-1}$$
$$f_1(u,Du)(x) = f_1(t,x)u(t,x)\mathbb{P}(I=1)^{-1}$$
$$f_2(u,Du)(x) = \mathbb{E}[f_l(u,Du)(t,x)]\mathbb{P}(I=2)^{-1}$$

where l is a random variable with values in $\{(a,1),(b,1),(a,2),(b,2)\}$ and

$$f_{(j,d)}(u,Du)(t,x) = f_{2,d}^{j}(t,x)D^j u(t,x)^d \mathbb{P}(l=(j,d))^{-1}.$$

Finally we have

$$D^j u(t,x)^d = 2^d \ \mathbb{E}[\prod_{k=1}^{d} u(t, x+\Delta^j \varepsilon_k)(-1)^{1-\varepsilon_k}]$$

with $(\varepsilon_i)_{1\leq i\leq d}$ i.i.d. random variables with law $Ber(\frac{1}{2})$. Thus finally

$$f(u,Du)(t,x) = \mathbb{E}[g_l(u,Du)(t,x)]$$

with l is a random variable whose law is the uniform probability on the set $\mathscr{L} = \big\{0, 1, (2, j, d, \varepsilon),\ \text{with } d \in \{0,1,2\},\ j \in \{a,b\},\ \varepsilon \in \{0,1\}^d\big\}$ and where

$$\begin{aligned}
g_0(u,Du)(t,x) &= f_0(t,x)\ \mathbb{P}(l=0)^{-1}\\
g_1(u,Du)(t,x) &= f_1(t,x)u(t,x)\ \mathbb{P}(l=1)^{-1}\\
g_{(2,j,d,\varepsilon)}(u,Du)(t,x) &= f^j_{2,d}(t,x)\mathbb{P}\big(l=(2,j,d,\varepsilon)\big)^{-1}\prod_{k=1}^{d} u(t,x+\Delta^j\varepsilon_k)(-1)^{1-\varepsilon_k}.
\end{aligned}$$

CHAPTER V

Scaling limit for stochastic control problems in population dynamics

Abstract

Going from a scaling approach for birth/death processes, we investigate the convergence of solutions to BSDEs driven a sequence of converging martingales. We apply our results to non-Markovian stochastic control problems for discrete population models. In particular we describe how the values and optimal controls of control problems converge when the models converge towards a continuous population model.

Key words: stochastic control, population models, birth and death processes, backward stochastic differential equation (BSDE), stability of BSDEs, martingale properties.

1 Introduction

The sustainability of natural resources has become a major subject of interest in the last decades for public institutions. For instance, in 1983 the European Union has launched its common fisheries policy to manage European fish stocks. In August 2010, a report of European commission named *Water Scarcity and Drought in the European Union,* has emphasized that "an adequate supply of good-quality water is a pre-requisite for economic and social progress, so we need to do two things: we must learn to save water, and also to manage our available resources more efficiently". A large part of academic literature has dealt with such issues. For example, Reed in [Ree79] or Clarke and Kirkwood in [CK86] have studied the exploitation of a natural resource under uncertainty on its evolution in a multi-period model. May, Beddington, Horwood and Sherpherd in [MBHS78] have considered the problem by assuming that the intrinsic population growth rate is given by the difference between recruitment and mortality for general recruitment functions. These models have been extended to stochastic differential equations driven by a Brownian motion (see for instance the work of Saphores [Sap03]). Evans, Hening and Shreiber in [EHS15] or more recently Kharroubi, Lim and Ly Vath in [LKLV18] have modelled the dynamic of the natural resource as the solution of the logistic stochastic

differential equation to solve a control problem under interaction between species and delayed renewal of the resource. All the models mentioned above use a Brownian motion to describe the uncertainty of the system evolution. We refer to this class of model as *continuous models*. On the other side of the literature, Getz in [Get75] has studied control problems related to a birth/death process. This work has been extended more recently by Claisse in [Cla18] to branching processes. We refer to those models as *discrete models*.

It is well known that some continuous population models can be seen as scaling limits of discrete models, see for example the work of Bansaye and Méléard in [BM15]. Hence continuous models can be considered as good approximations of the macroscopic evolution of a population size. Therefore it is relevant to consider continuous models for resources management purposes. Moreover those models are attractive from a tractability viewpoint compared to discrete models. Indeed solving control problems in Brownian driven model essentially boils down to solve a partial differential equation. Whereas for discrete models it leads to integral-partial differential equation, which are often more complex to solve. Yet the remaining question is the relevancy of designing a management policy based on a continuous modeling while the controlled population (or resource) is naturally discrete.

To try to give an answer to this question we are going to consider a sequence of discrete population models that converges towards a continuous population model. For each of those models we consider a control problem. Each of them are the natural adaptations of the same control problem to the different models. Therefore we expect the solutions of the discrete control problems to converge towards the solution of the continuous limit problem. From Γ-convergence results adapted to stochastic control problems as in for instance the articles of Buttazzo and Del Maso [BDM82] and Belloni, Buttazzo and Freddi [BBF93], we expect to have convergence of value functions (see also for instance [DM12, Theorem 10.22]) and a kind of weak convergence of optimal controls (see for instance [BBF93, Proposition 2.8]). This is emphasized in a toy model (see Section 2) where besides convergence of the value functions we also get convergence in law of the state process under the optimal control. In this paper we prove the convergence of the controls as sequence of processes. This is stronger than $\Gamma-$convergence. Since we aim at dealing with non-Markovian stochastic control problems our problematic is to prove the convergence of solutions to a sequence of Backward Stochastic Differential Equations (BSDE for short) driven by a sequence of converging martingales.

We know from the seminal paper of Donsker [Don51] that a scaling in time procedure leads to the weak convergence of a random walk to a Brownian motion. Extending this result to the theory of BSDE with fixed time horizon $T > 0$, Briand, Delyon and Mémin in [BDM01] have provided a time discretization of the Brownian motion to get the convergence of a time discretized BSDE. More precisely, they consider a sequence of random walks $(W^n)_{n\geq 0}$ converging towards a Brownian motion W. Then they prove the convergence of the solutions of a sequence of BSDEs driven the $(W^n)_{n\geq 0}$ towards the solution of a BSDE driven by W. The main idea is to prove the convergence of the terms involved in the martingale representation with respect to W^n when n goes to infinity. For this they use the convergence, in the sense of

Coquet, Mémin and Slominski [CMS01], of the filtrations associated to each of the $(W^n)_{n\geq 0}$ towards the natural filtration associated to W. Those results have been extended in [BDM02] to a more general situation, without assuming that W^n has a predictable representation property, but assuming that the brackets of the martingales $(W^n)_{n\geq 0}$ are uniformly bounded.

In the present paper we aim to extend the results of [BDM01] to a wider class of martingale convergence. Starting from a scaling result in [BM15] showing that a sequence of scaled birth/death processes $(X^K)_{K\geq 0}$ with scaling parameter $K > 0$ converges in law to the solution X of a stochastic Feller diffusion, we begin to extend it to more general birth and death intensities. We then consider a sequence of BSDEs of the form

$$(\mathbf{B})_K: \; Y_t^K = \xi^K + \int_t^T g^K(X_s^K, Y_s^K, Z_s^K)\cdot \phi_s^K \mathrm{d}A_s^K - \int_t^T Z_s^K \cdot \mathrm{d}M_s^K,$$

where ξ^K is some general terminal random condition, g^K the generator of the BSDE, M^K a two dimensional martingale associated to the population model X^K and $\phi^K \mathrm{d}A^K$ denotes the measure associated to the angle bracket of M^K. We also consider the continuous counterpart of $(\mathbf{B})_K$,

$$(\mathbf{B}): \; Y_t = \xi + \int_t^T g(X_s, Y_s, Z_s)\mathrm{d}A_s - \int_t^T Z_s \mathrm{d}M_s^X,$$

where ξ is some terminal condition, g is the generator of the BSDE, M^X is a one dimensional martingale related to the diffusion term of X and A is its angle bracket. The existence and uniqueness of solutions to such BSDEs driven by general martingales have been studied, for instance, by El Karoui and Huang in [EKH97], Confortola and Fuhrman in [CF13] or more recently by Papapantoleon, Possamaï and Saplaouras in [PPS18] in a general framework. Inspired by [BDM02], we prove that the solution of $(\mathbf{B})_K$ converges to the solution of $(\mathbf{B})$ when $(\xi^K)_{K\geq 0}$ converges toward ξ and $(g^K)_{K\geq 0}$ towards g. The difficulty pointed at this step, compared to [BDM02], is that the brackets of the $(M^K)_{K\geq 0}$ are not bounded. Therefore we need a stronger assumption on the convergence of the sequence of terminal conditions. The methods used are related to the so-called martingale problem as stated by Jacod and Shiryaev in [JS13] and to the double-Picard iterations craftily used in [BDM02].

Finally we would like to point that after having published the first version of this paper online, we have learnt that a paper of Papapantoleon, Possamaï and Saplaouras dealing with convergence of BSDES for a very general class of martingales is in progress and preliminary results in the one dimensional case are written in the PhD Thesis of Alexandros Saplaouras. Our study, made independently of their, has to be seen as a particular case of their works more focused on applications in stochastic control theory for population models.

The structure of the paper is the following. In Section 2 we study the convergence of a rescaled birth/death process to the solution of a stochastic Feller type SDE by extending [BM15] to more general dynamics (see Theorem 1). We also provide fundamental properties of our state processes such as exponential moments (see Proposition 1 and Corollary 1). Section 3 introduces a toy model motivating our study and illustrated with numerical simulations. In Section 4 we

first provide a convergence result for a sequence of martingale representations (see Proposition 2). Then in Theorem 2 we extend the convergence result of [BDM02] by showing that the solutions to $(\mathbf{B})_K$ converge to the solution of $(\mathbf{B})$. In Section 5 from our BSDE approach we deduce convergence of the values and optimal controls to a sequence of control problems. Our results go beyond $\Gamma-$convergence since we obtain a strong form of convergence for the optimal controls. Section 6 gives the main proofs of our results. Minor proofs are given in the appendix.

The technical spaces considered related to discrete and continuous models are defined in Appendix V.A. We provide below the notations for classical spaces used in this paper.

Classical spaces.

- L^p the set of real valued random variable Z such that

$$\|Z\|_{L_p}^p = \mathbb{E}[|Z|^p] < +\infty$$

- $\mathscr{S}_d^p$ is the set of $\mathbb{F}$-predictable $\mathbb{R}^d$ valued process X such that

$$\|X\|_p^p = \mathbb{E}[\sup_{t\in[0,T]} \|X_t\|^p] < +\infty.$$

2 From a discrete to a continuous population model

In this section we define a sequence of discrete population models. We show that this sequence converges in law towards a continuous Feller population model by extending [BM15, Theorem III-3.2] to more general population dynamic models.

2.1 Definition of the discrete population models

We consider positive continuous functions f^b, f^d and σ defined from $\mathbb{R}$ into $\mathbb{R}^+$ that satisfy the following standing assumption.

Assumption 1.

(i) The functions f^b, f^d and σ are null on $\mathbb{R}^-$ and there exists non negative constants ν, μ, η and $\underline{\eta}$ such that for any $x \in \mathbb{R}^+$

$$f^b(x) \le \nu x,\ f^d(x) \le \mu x,\ \underline{\eta} \le \sigma^2(x) \le \eta(1+x),$$

(ii) $f := f^b - f^d$ and σ^2 are Lipschitz continuous.

We note Ω_d the set of piecewise continuous increasing positive functions with jumps equal to 1. We consider $\mathbb{F}$ the natural filtration associated to the canonical process (N^b, N^d) of Ω_d^2.

For a fixed $K \geq 0$ and $n \geq 0$ we define a population model on the stochastic basis $(\Omega_d^2, \mathbb{F})$. The initial population is Kn and the processes N^b and N^d represent respectively the number of birth and death in the population. This means that when the process N^b jumps there is a new individual in the population and when N^d jumps there is one individual less in the population. Therefore at time t the population size is $Kn + N_t^b - N_t^d$. As we are interested in the large population limit (which corresponds to K large) we consider the rescaled population process

$$X^{K,n} = n + \frac{N^b - N^d}{K}.$$

We define the birth intensity in the model with parameter K and initial population n as

$$\lambda_t^{K,n,b} = \lambda^{K,b}(X_{t-}^{K,n}) := f^b(X_{t-}^{K,n})K + \frac{\sigma^2(X_{t-}^{K,n})}{2}K^2$$

and the intensity of death

$$\lambda_t^{K,n,d} = \lambda^{K,d}(X_{t-}^{K,n}) := f^d(X_{t-}^{K,n})K + \frac{\sigma^2(X_{t-}^{K,n})}{2}K^2.$$

Remark 1. *Note that $f^b(x) = \mu x$, $f^d(x) = \nu x$ and $\sigma^2(x) = \sigma^2 x$ satisfy Assumption 1. Consequently, the model studied in Theorem III-3.2 in [BM15] is included in the scope of this paper.*

Following Theorem 3.6 in [Jac75] there exists a unique probability measure $\mathbb{P}^{K,n}$ on $(\Omega_d^2, \mathbb{F})$ such that the processes

$$M_t^{K,n,i} = N_t^i - \int_0^t \lambda_s^{K,n,i} \mathrm{d}s, \text{ for } i \in \{b, d\}$$

are local martingales. It means that under the probability $\mathbb{P}^{K,n}$ the process N^b (resp. N^d) has intensity $\lambda^{K,n,b}$ (resp. $\lambda^{K,n,d}$). Note that if $m > n \geq 0$ then $\mathbb{P}^{K,n}$ is absolutely continuous with respect to $\mathbb{P}^{K,m}$ and we have:

$$\frac{\mathrm{d}\mathbb{P}^{K,n}}{\mathrm{d}\mathbb{P}^{K,m}} = L_T^{n,m} \tag{1}$$

where

$$\mathrm{d}L_t^{n,m} = L_{t-}^{n,m}\Big(\sum_{i\in\{b,d\}} \frac{\lambda_t^{K,n,i} - \lambda_t^{K,m,i}}{\lambda_t^{K,m,i}} \mathbf{1}_{X_{t-}^{K,m}>0} \mathrm{d}M_t^{K,m,i}\Big) \text{ with } L_0^{n,m} = 1.$$

We justify this change of measure in Appendix V.B.

For the rest of this work we fix an initial population x_0 and do not write anymore the superscript x_0 to lighten the notations. We write $\mathbb{E}$ instead of $\mathbb{E}^{\mathbb{P}^K}$ when there is no ambiguity on the probability used. For any K we consider the processes $M^K = (M^{K,b}, M^{K,d})$, $\lambda_t^K = \lambda_t^{K,b} - \lambda_t^{K,d}$ and for $i \in \{b, d\}$

$$\overline{\Lambda}_t^{K,i} = \int_0^t \lambda_s^{K,i} K^{-2} \mathrm{d}s, \ \overline{N}^{K,i} = N^i K^{-2} \text{ and } \overline{M}^{K,i} = M^{K,i} K^{-1}.$$

We note $\overline{M}^K = (\overline{M}^{K,b}, \overline{M}^{K,d})$. The rescaled population process is now noted

$$X^K = x_0 + \frac{N^b - N^d}{K}.$$

2.2 Scaling limit of the sequence $(X^K)_{K\geq 0}$

Intuitively, and having in mind [BM15, Theorem 3.2], a continuous version of the processes $(X^K)_{K\geq 0}$, denoted by X, would be an Ito diffusion with drift equal to $f(X)$ and volatility given by $\sigma^2(X)$. We formalize this intuition in the following result which extends [BM15, Theorem 3.2]. The proof is given in Section 6.1.

Theorem 1. *The sequence* $\left(X^K, \overline{M}^K, \overline{N}^{K,b}, \overline{N}^{K,d}, \overline{\Lambda}^{K,b}, \overline{\Lambda}^{K,d}\right)_{K\geq 0}$ *converges in law for the Skorokhod topology towards* (X, M, A, A, A, A) *such that*

(i) there exists a bi-dimensional Brownian motions (B^b, B^d) *satisfying*

$$M_t = \int_0^t \frac{\sigma(X_s)}{\sqrt{2}} \mathrm{d}(B_s^b, B_s^d),$$

(ii) with $B = (B^b + B^d)/\sqrt{2}$*, the process* X *is the unique strong solution of*

$$(\mathbf{S}): \; X_t = x_0 + \int_0^t f(X_s)\mathrm{d}s + \sigma(X_s)\mathrm{d}B_s,$$

(iii) $A = \int_0^{\cdot} \sigma^2(X_s)\mathrm{d}s$.

Moreover, there exists a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ *and a sequence* $(N^{K,d}, N^{K,b})_{K\geq 0}$ *such that for any* $K \geq 0$, $(N^{K,d}, N^{K,b})$ *has the law of* (N^d, N^b) *under* $\mathbb{P}^K$*. Moreover on this space the sequence*

$$(X^K, \overline{M}^K, \overline{N}^{K,b}, \overline{N}^{K,d}, \overline{\Lambda}^{K,b}, \overline{\Lambda}^{K,d})_{K\geq 0}$$

converges in $\mathscr{S}_1^2 \times \mathscr{S}_2^2 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1$ *to* (X, M, A, A, A, A) *when* K *goes to* $+\infty$.

According to the last point of Theorem 1 from now on we work under the probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and we consider that the processes $(X^K, \overline{M}^K, \overline{N}^{K,b}, \overline{N}^{K,d}, \overline{\Lambda}^{K,b}, \overline{\Lambda}^{K,d})_{K\geq 0}$ and (X, M, A, A, A, A) are defined on this space. For any K we note $\mathbb{F}^K$ the natural filtration associated to X^K and $\mathbb{F}^X$ the natural filtration associated to X.

Before going to the next section, we define some processes that we will extensively use in the rest of the paper. We note for $i \in \{b, d\}$

$$M_t^i := \int_0^t \frac{\sigma(X_s)}{\sqrt{2}} \mathrm{d}B_s^i,$$

so that $M = (M^d, M^b)$ and $M^X := M^b + M^d$ that is written

$$M_t^X = \int_0^t \sigma(X_s)\mathrm{d}B_s.$$

We also consider the processes

$$A_t^K := \int_0^t \frac{\lambda_s^{K,b} + \lambda_s^{K,d}}{K^2}\mathrm{d}s, \; p_t^{K,b} := \frac{\lambda_t^{K,b}}{\lambda_t^{K,b} + \lambda_t^{K,d}}, \text{ and } p_t^{K,d} := \frac{\lambda_t^{K,d}}{\lambda_t^{K,b} + \lambda_t^{K,d}}.$$

Note that under the probability $\mathbb{P}^K$ the random measure m associated to the process (N^b, N^d) interpreted as a compound jump process with values in $E = \{b, d\}$ admits as predictable compensator measure

$$\pi^K(\mathrm{d}e, \mathrm{d}t) = \big(\phi_t^{K,b}\delta_b(\mathrm{d}e) + \phi_t^{K,d}\delta_d(\mathrm{d}e)\big)\mathrm{d}A_t^K$$

with $\phi_t^K = (\phi_t^{K,b}, \phi_t^{K,d}) = (p_t^{K,b}, p_t^{K,d})K^2$ and where δ_i denotes the Dirac measure at point $i \in \{b, d\}$. This point of view is introduced in order to draw a parallel with the framework of [CF13] to which we will refer extensively in Section 4.2.

2.3 Uniform exponential moments

Finally we show that the sequence of processes $(X^K, \int_0^{\cdot} X_s^K)$ admits exponential moments uniformly in K if σ^2 is linear. The proof of this result is postponed in Appendix V.F.

Proposition 1. *If there exists a positive constant σ such that $\sigma^2(x) = \sigma^2 x$ there exists some positive constants β_0, K_0 and T such that for any $s \le T$ we have*

$$\sup_{K \ge K_0} \mathbb{E}[\exp(\beta_0 \int_0^s X_u^K \mathrm{d}u + \beta_0 X_s^K)] < \infty.$$

Without loss of of generality we assume that $K_0 = 0$. From now we fix a positive constant β strictly smaller than β_0. As a consequence of Proposition 1, for any integer q we have for any $s \le T$

$$\sup_{K \ge 0} \mathbb{E}[\exp(\beta \int_0^s X_u^K \mathrm{d}u + \beta X_s^K)(1 + |X_s^K|^q)] < +\infty$$

and

$$\sup_{K \ge 0} \mathbb{E}[\int_0^T \exp(\beta \int_0^s X_u^K \mathrm{d}u + \beta X_s^K)(1 + |X_s^K|^q)\mathrm{d}s] < +\infty.$$

We deduce from Fatou's Lemma together with Proposition 1 that X inherits from the exponential moments of X^K as stated in the following corollary.

Corollary 1. *If there exists σ positive such that $\sigma^2(x) = \sigma^2 x$ there exists some positive constants β_0 and T such that for any $s \le T$ we have*

$$\mathbb{E}[\exp(\beta_0 \int_0^s X_u \mathrm{d}u + \beta_0 X_s)] < \infty.$$

In order to benefit from those exponential moments we now assume that $\sigma^2(x) = \sigma^2 x$ for some positive constant σ fixed.

3 Illustration of the study on a toy model

In this section, we illustrate the $\Gamma-$convergence result applied to optimization problems in population dynamics. We consider specific parameters f^d, f^b and a sequence of control problems for which we are able to make explicit computations. Then, we show that the sequence of optimal controls converges in law to the optimal control of a continuous problem. In this section, we aim at providing the general main ideas of the paper rather than being perfectly accurate. Rigorous statements will be given in Section 5.

3.1 Discrete populations models

We consider a discrete birth/death model as studied in [BM15] by choosing:

- the initial population $x_0 \in \mathbb{N}$,
- the birth rate $f^b(x) = \nu x$ for some $\nu > 0$,
- the death rate $f^d(x) = \mu x$ for some $\mu > 0$.

Recall from Remark 1 and from [BM15, Theorem 3.2] that $(X_t^K)_{t\in[0,T]}$ converges in law for the Skorokhod topology towards the continuous diffusion process $(X_t)_{t\in[0,T]}$ solution of the Feller stochastic differential equation

$$\mathrm{d}X_t = (\nu - \mu)X_t\mathrm{d}t + \sigma\sqrt{X_s}\mathrm{d}W_t, \tag{2}$$

for W a Brownian motion.

In this toy model, we assume that a resource manager regulates the population X^K through an $\mathbb{F}^K$-predictable control α. A control α is admissible if

- there exists a unique law $\mathbb{P}^{K,\alpha}$ under which the death intensity of the population is

$$\lambda_t^{K,d,\alpha} := KX_t^K(\mu + K\frac{\sigma^2}{2}) + KX_t^K\alpha_t,$$

  and the birth intensity is $\lambda^{K,b}$. When this probability exists, it is the law of the population under the control α.
- $\lambda^{K,d,\alpha}$ is a non negative process $\mathbb{P}^{K,\alpha}$ almost surely.

We denote by $\mathscr{A}^K$ the set of admissible controls.

The agent is assumed to be penalized if he fails at reaching a fixed level $\tilde{x} > 0$ of the resource at time T determined by a regulator. We model this penalization by the square of the difference between the effective population size at time T and the target $\tilde{x}$. So that the manager pays $\gamma(X_T^K - \tilde{x})^2$ at time T where γ is a positive constant. The manager payoff is also assumed to

be penalized by the instantaneous amount $\frac{|\alpha_t X_t^K|^2}{2}$ per unit of time when its effort is α. The problem of the resource manager is thus to solve

$$(\mathbf{TM})_K: \; V_0^K = \sup_{\alpha \in \mathscr{A}^K} \mathbb{E}^{K,\alpha}[-\gamma(X_T^K - \tilde{x})^2 - \int_0^T \frac{(\alpha_s X_s^K)^2}{2} \mathrm{d}s]$$

where $\mathbb{E}^{K,\alpha}$ denotes the expectation taken under the probability $\mathbb{P}^{K,\alpha}$. We assume that $\sigma^2 > 2\gamma\tilde{x}$ and $\gamma < \mu$.

To solve this problem, as usual in stochastic control theory, we study the corresponding Hamilton-Jacobi-Bellman (HJB for short) equation and use a verification argument. The HJB equation associated to the control problem $(\mathbf{TM})_K$ is

$$(\mathbf{HJB})_K \begin{cases} \partial_t U^K(t,x) + H^K\big(x, D_+^K U^K(t,x), D_-^K U^K(t,x)\big) = 0, & (t,x) \in [0,T) \times (\mathbb{N}^*/K), \\ U^K(T,x) = -\gamma(x-\tilde{x})^2, & x \in (\mathbb{N}^*/K), \end{cases}$$

with Hamiltonian H^K given by

$$H^K(x,p^+,p_-) = \sup_\alpha \Big\{ Kx(\nu + \frac{\sigma^2}{2}K)p_+ + Kx(\mu + \alpha + \frac{\sigma^2}{2}K)p_- - \frac{(\alpha x)^2}{2} \Big\},$$

and where

$$D_+^K U^K(t,x) = U^K(t,x+1/K) - U^K(t,x) \text{ and } \; D_-^K U^K(t,x) = U^K(t,x-1/K) - U^K(t,x).$$

The maximizer of the Hamiltonian is $\alpha^{K,*} = \frac{Kp_-}{x}\mathbf{1}_{x>0}$, hence

$$H^K(x,p^+,p_-) = Kx(\nu + \frac{\sigma^2}{2}K)p_+ + Kx(\mu + \frac{\sigma^2}{2}K)p_- + \frac{(Kp_-)^2}{2}\mathbf{1}_{x>0}.$$

Note that we do not actually care about the value of the control when $x = 0$ since if the population reaches 0, it is stuck at this value. The partial differential equation (PDE for short) $(\mathbf{HJB})_K$ is quadratic, so we search for a solution under the form

$$U^K(t,x) = a_K(t)x^2 + b_K(t)x + c_K(t).$$

Identifying the monomials, we get that U^K is solution of $(\mathbf{HJB})_K$ if and only if (a_K, b_K, c_K) is solution of the following systems of ODEs:

$$\mathbf{(ODE)}_K: \begin{cases} a_K'(t) + 2a_K(t)(\nu-\mu) + 2a_K^2(t) = 0, & a_K(T) = -\gamma, \\ b_K'(t) - 2a_K(t)\big(\frac{a_K(t)}{K} - b_K(t)\big) + a_K(t)(\sigma^2 + \frac{\mu+\nu}{K}) + b_K(t)(\nu-\mu) = 0, & b_K(T) = 2\gamma\tilde{x}, \\ c_K'(t) + \frac{1}{2}\big(\frac{a_K(t)}{K} - b_K(t)\big)^2 = 0, & c_K(T) = -\gamma\tilde{x}^2. \end{cases}$$

By Cauchy-Lipschitz theorem this system admits a unique solution. Thus the optimal effort of the agent is

$$\alpha_t^{K,*} = \frac{1}{X_t^K}\Big(\frac{a_K(t)}{K} - 2X_t^K a_K(t) - b_K(t)\Big)\mathbf{1}_{X_t^K>0}$$

and the corresponding death intensity is given by

$$\lambda_t^{K,d,\alpha^{K,*}} = KX_t^K(\mu + K\frac{\sigma^2}{2}) + \big(-2X_t^K a_K(t) + \frac{a_K(t)X_t^K}{K} - b_K(t)\big)\mathbf{1}_{X_t^K>0}.$$

Note that in view of $(\mathbf{ODE})_K$ and since $a_K(T)$ is negative and $b_K(T)$ positive, there exists a T small enough, independent of K, such that for any K the control $\alpha^{K,*}$ is in $\mathscr{A}^K$. We refer to Appendix V.C for more details on this point. We assume that we are considering such short enough time horizon here.

3.2 Continuous populations model

We now turn to the continuous version of the control problem. We assume that the manager controls the drift term in (2) through an $\mathbb{F}^X-$predictable process α. We say that α is an admissible control when the following SDE admits a unique weak solution

$$\mathrm{d}X_t = (\nu - \mu - \alpha_t)X_t\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}W_t.$$

When such solution exists we note $\mathbb{P}^\alpha$ its law that is the law of the population under the control α. We denote by $\mathscr{A}$ the set of admissible controls.

The control problem in the continuous framework is written

$$(\mathbf{TM}):\ V_0 = \sup_{\alpha\in\mathscr{A}} \mathbb{E}^\alpha[-\gamma(X_T - \widetilde{x})^2 - \int_0^T \frac{(\alpha_s X_s)^2}{2}\mathrm{d}s].$$

The associated HJB equation is given by

$$(\mathbf{HJB})\begin{cases}\partial_t U(t,x) + H(x, DU(t,x), \Delta U(t,x)) = 0, & (t,x)\in[0,T)\times\mathbb{R}^+,\\ U(T,x) = -\gamma(x-\widetilde{x})^2, & x\in\mathbb{R}^+,\end{cases}$$

where the Hamiltonian H is

$$H(x,p,q) = \sup_\alpha\Big\{(\nu-\mu-\alpha)xp - \frac{|\alpha x|^2}{2} + \frac{1}{2}x\sigma^2 q\Big\} = (\nu-\mu)xp + \frac{1}{2}x\sigma^2 q + \frac{p^2}{2}\mathbf{1}_{x>0}.$$

The maximizer of the Hamiltonian is

$$\alpha^*(x,p) = \frac{-p}{x}\mathbf{1}_{x>0}$$

As previously, we are looking for a quadratic solution of the form

$$U(t,x) = a(t)x^2 + b(t)x + c(t).$$

Identifying the monomials, we get that U is solution of (**HJB**) if and only if (a,b,c) is solution of the following system of ODEs.

$$\mathbf{(ODE)}:\ \begin{cases} a'(t) + 2a(t)(\nu-\mu) + 2a^2(t) = 0, & a(T) = -\gamma,\\ b'(t) + 2a(t)b(t) + a(t)\sigma^2 + b(t)(\nu-\mu) = 0, & b(T) = 2\gamma\widetilde{x},\\ c'(t) + \frac{|b(t)|^2}{2} = 0, & c(T) = -\gamma\widetilde{x}^2.\end{cases}$$

Hence, the optimal control is given by

$$\alpha_t^* = -\frac{DU(t,X_t)}{X_t}\mathbf{1}_{X_t>0} = -\frac{2a(t)X_t + b(t)}{X_t}\mathbf{1}_{X_t>0}.$$

Note that in view of (**ODE**) and since $a(T)$ is negative and $b(T)$ positive, there exists a T small enough such that the control α^* is in $\mathscr{A}$, see Appendix V.C for details. We assume that we are considering such time horizon here. We also note that $a_K = a$ and as consequence of Grönwall Lemma $(b_K, c_K)_{K\geq 0}$ converges to (b,c) when K goes to $+\infty$. Consequently we get the convergence of the value of the control problems, $\lim_{K\to+\infty} V_0^K = V_0$. Moreover a direct adaption of the proof of Proposition 1 gives the convergence in law of the optimally controlled population:

$$\lim_{K\to+\infty} \mathbb{P}^{K,\alpha^{K,*}} = \mathbb{P}^{\alpha^*}.$$

Those convergences are illustrated in Figures V.1 and V.2 respectively.

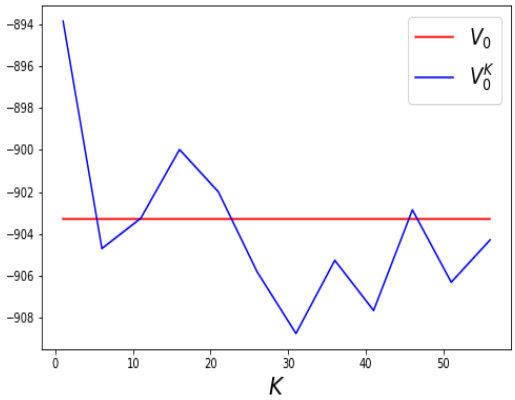

Figure V.1 – Convergence of $(V_0^K)_{K\geq 0}$ towards V_0 with $\sigma^2 = 0.3$, $\mu = 0.1$, $\nu = 0.2$, $T = 0.1$, $x_0 = 50$, $\tilde{x} = 20$ and $\gamma = 1$.

4 Convergence of BSDEs

In this section we prove the main results of this paper. We first prove[1] a result on the convergence of a sequence of martingale representations. Then we extend it to the convergence of a sequence of BSDEs driven by the sequence of martingales $(M^K)_{K\geq 0}$.

[1]After having finished the paper, it has been raised to our attention that a recent paper [PPS19] extends this result to the convergence of general martingales.

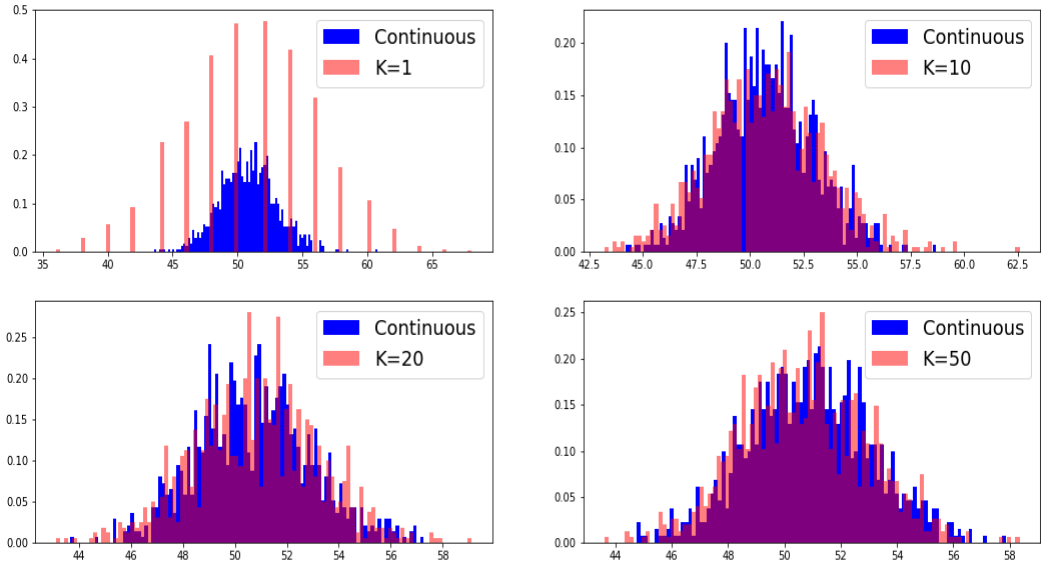

Figure V.2 – Empirical distribution of the discrete optimal controls at time $t = 0.1$ for different values of K (in red) compared to the distribution of the continuous optimal control. The parameters are the same than in Figure V.1

.

4.1 Convergence of martingale representations

From Theorem 2 in [Dav76] we know any $\mathbb{F}^K-$martingale has the representation property with respect to M^K (in the sense of Definition III-4.22 in [JS13]). Moreover we prove in Appendix V.E that any $\mathbb{F}^X-$ martingale has the representation property relative to M^X.

For any $K \geq 0$ we consider $\xi^K \in L^2$ an $\mathscr{F}_T^K$-measurable real random variable and $\xi \in L^2$ an $\mathscr{F}_T^X$-measurable real random variable. We define the closed martingale Q^K by $Q_t^K = \mathbb{E}[\xi^K|\mathscr{F}_t^K]$, $\mathbb{P}-$a.s.. Since Q^K is an $\mathbb{F}^K$-martingale and $\xi^K \in L^2$, we know that there exists a unique process $Z^K \in L^2(M^K)$ such that

$$Q_t^K = \mathbb{E}[\xi^K|\mathscr{F}_t^K] = Q_0^K + \int_0^t Z_s^K \cdot \mathrm{d}M_s^K.$$

Similarly considering the $\mathbb{F}^X$-martingale Q defined by $Q_t = \mathbb{E}[\xi|\mathscr{F}_t]$, $\mathbb{P}-a.s.$ since $\xi \in \mathbb{L}^2$ we have existence and uniqueness of $Z \in L^2(M^X)$ such that

$$Q_t = \mathbb{E}[\xi|\mathscr{F}_t^X] = Q_0 + \int_0^t Z_s \mathrm{d}M_s^X.$$

We prove the following fundamental convergence result in Appendix 6.2

Proposition 2 (**martingale representations convergence**). *If the sequence* $(\xi^K)_{K\geq 0}$ *and* ξ *are in* $L^{2+\varepsilon}$ *and* $(\xi^K)_{K\geq 0}$ *converges towards* ξ *in* $L^{2+\varepsilon}$ *for* $\varepsilon > 0$ *then*

$$\left(Q^K, \langle Q^K, Q^K\rangle, \langle Q^K, \overline{M}^K\rangle\right) \to \left(Q, \langle Q, Q\rangle, \langle Q, M\rangle\right) \text{ as } K \to +\infty$$

in $\mathscr{S}_1^{2+\varepsilon'} \times \mathscr{S}_1^{1+\varepsilon'/2} \times \mathscr{S}_2^{2+\varepsilon'}$ *for any* $\varepsilon' \in [0, \varepsilon)$.

Compared to Theorem 5 in [BDM02] we have assumed that the convergence of $(\xi^K)_{K\geq 0}$ takes place in $L^{2+\varepsilon}$ instead in L^2. This is in order to extend the convergence of $(\langle Q^K, \overline{M}^K\rangle)_{K\geq 0}$ beyond $\mathscr{S}_2^1$. Indeed if we only assume that $\langle Q^K\rangle \in \mathscr{S}_1^1$, $\langle Q^K, \overline{M}^K\rangle$ is not squared integrable *a priori*. In [BDM02] the authors do not face this issue since they assume that the brackets of the martingales they consider are bounded, see Hypothesis $(H1)$. In our framework the sequence $(\langle \overline{M}^K\rangle_T)_{K\geq 0}$ is not bounded in general. However, if we instead consider a sequence of models with a bounded population then $(\langle \overline{M}^K\rangle_T)_{K\geq 0}$ would be bounded and we could get the same result assuming only the convergence of $(\xi^K)_{K\geq 0}$ in L^2 only.

4.2 Convergence of BSDEs

We now extend the previous result to convergence of a sequence of BSDEs driven by M^K.

For any $K \geq 0$ we consider an $\mathscr{F}_T^K$ random variable ξ^K and two continuous functions g_b^K and g_d^K from $\mathbb{R}^3$ into $\mathbb{R}$. We write for $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2$

$$g^K(x, y, z) = \left(g_b^K(x, y, z^b), g_d^K(x, y, z^d)\right).$$

Note that in the above equation, we implicitly use the decomposition $z = (z^b, z^d)$. We will always assume such convention when we are dealing with a pair of elements such that one element of the pair is related to the birth in the population and the other is related to the death.

We introduce the BSDE with generator g^K and terminal value ξ^K by setting

$$(\mathbf{B})_K: \; Y_t^K = \xi^K + \int_t^T g^K(X_s^K, Y_s^K, Z_s^K) \cdot \phi_s^K \mathrm{d}A_s^K - \int_t^T Z_s^K \cdot \mathrm{d}M_s^K.$$

Definition 1. *A solution to BSDE* $(\mathbf{B})_K$ *is a pair of processes* $(Y, Z) \in \mathbb{S}^K$ *such that the relation* $(\mathbf{B})_K$ *holds* $\mathbb{P} - a.s.$

As a consequence of Theorem 3.4 in [CF13] we have the following result.

Lemma 1. *Assume that*

(i) $\xi^K \in \mathbb{T}^K$,

(ii) there exists a positive constant L *such that* $\beta > L^2 + 2L$ *and for any* x, y, y', z, z' *and* $K \geq 0$ *we have for* $j \in \{b, d\}$

$$K^2 |g_j^K(x, y, z/K) - g_j^K(x, y', z'/K)| \leq L\left(|y - y'| + |z - z'|\right),$$

(iii) $g_j(X_t^K, 0, 0) \in \mathbb{H}_1^K$ *for* $j \in \{b, d\}$,

then the BSDE $(\mathbf{B})_K$ *has a unique solution* $(Y^K, Z^K) \in \mathbb{S}^K$.

We also introduce a class of BSDE driven by the martingale M^X. For an $\mathscr{F}_T^X$ real valued random variable ξ and a continuous function g from $\mathbb{R}^3$ into $\mathbb{R}$ we consider the BSDE

$$(\mathbf{B}): \; Y_t = \xi + \int_t^T g(X_s, Y_s, Z_s)\mathrm{d}A_s - \int_t^T Z_s \mathrm{d}M_s^X.$$

Definition 2. *A solution to BSDE* (**B**) *is a pair of processes* $(Y, Z) \in \mathbb{S}$ *such that the relation* (**B**) *holds* $\mathbb{P} - a.s.$

We get the following result on existence and uniqueness of solution to (**B**) which is a consequence of Theorem 6.1 in [EKH97] or Theorem 2.1 in [CFS08].

Lemma 2. *Assume that*

(i) $\xi \in \mathbb{T}$,

(ii) there exists a positive constant L *such that* $\beta > L^2 + 2L$ *and for any* x, y, y', z, z' *and* $K \geq 0$ *we have:*

$$|g(x, y, z) - g(x, y', z')| \leq L\big(|y - y'| + |z - z'|\big),$$

(iii) $g(X_t, 0, 0) \in \mathbb{H}$,

then the BSDE (**B**) *has a unique solution* $(Y, Z) \in \mathbb{S}$.

We are interested in the convergence of the solutions to $(\mathbf{B})_K$ when $(\xi^K)_{K \geq K}$ and $(g^K)_{K \geq 0}$ converge. Therefore we make the following converging assumptions on the drivers of the BSDEs $(\mathbf{B})_K$.

Assumption 2.

(i) The sequence $(\xi^K)_{K \geq 0}$ *converges towards* $\xi \in \mathbb{T}$ *in* $L^{2+\varepsilon}$ *for* $\varepsilon > 0$,

(ii) there exists a positive constant C *such that for any* x, x', y, z, $K \geq 0$ *and* $j \in \{b, d\}$

$$K^2 |g_j^K(x, y, z) - g_j^K(x', y, z)| \leq C|x - x'|,$$

(iii) there exists a pair of continuous functions (g_b, g_d) *from* $\mathbb{R}^3$ *and a positive sequence* $(\upsilon_K)_{K \geq 0}$ *converging towards* 0 *such that for any* $K \geq 0$, x, y *and* z *we have for* $j \in \{b, d\}$*:*

$$|K^2 g_j^K(x, y, z/K) - g_j(x, y, z)| \leq \upsilon_K(1 + x^2 + y^2 + \|z\|^2).$$

Remark 2. *Under Assumption 2 (iii) if for any* K *the pair* (g_b^K, g_d^K) *satisfies assumptions (ii) and (iii) in Lemma 1 then the function* $g = (g_b + g_d)/2$ *satisfies the assumptions (ii) and (iii) in Lemma 2.*

For any $K \geq 0$ we consider $(Y^K, Z^K) \in \mathbb{S}^K$ the unique solution of $(\mathbf{B})_K$. We have the following convergence result for the sequence $(Y^K, Z^K)_{K\geq 0}$ whose proof is given in Section 6.3.

Theorem 2. *Under Assumption 2 if the assumptions of Lemma 1 are satisfied for any K then the BSDE driven by M^X with generator $g := \frac{g_b+g_d}{2}$ and terminal value ξ has a unique solution (Y, Z) and we have the following convergence:*

$$\Big(Y^K, \int_0^\cdot Z_t^K \cdot \mathrm{d}M_t^K, \langle Y^K, \overline{M}^K\rangle, \langle Y^K\rangle\Big) \to \Big(Y, \int_0^\cdot Z_t \mathrm{d}M_t^X, \langle Y, M\rangle, \langle Y\rangle\Big) \text{ as } K \to +\infty$$

in $\mathscr{S}_1^2 \times \mathscr{S}_1^2 \times \mathscr{S}_2^1 \times \mathscr{S}_1^1$.

The convergence in Theorem 2 implies the following convergence

$$\Big(\int_0^\cdot \frac{Z_t^{K,b}}{K}\lambda_t^{K,b}\mathrm{d}t, \int_0^\cdot \frac{Z_t^{K,d}}{K}\lambda_t^{K,d}\mathrm{d}t, \int_0^\cdot |Z_t^K|^2 \cdot \phi_t^K \mathrm{d}A_t^K\Big) \to \Big(\int_0^\cdot Z_t \mathrm{d}A_t/2, \int_0^\cdot Z_t \mathrm{d}A_t/2, \int_0^\cdot Z_t^2 \mathrm{d}A_t\Big),$$

in $\mathscr{S}_1^1 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1$ when $K \to +\infty$.

5 Application to a control problem

In this section we apply the results of Section 4 to the convergence of a sequence of controls problems.

5.1 The discrete problem

We first focus on the discrete control problem in the same spirit than Section 3. We consider that a resource manager monitors his harvesting intensity through a control α, which is assumed to be bounded with bounds $\underline{a}, \overline{a} > 0$. We assume that his harvesting modifies the death rate of the natural resource according to a continuous function $h^K : \mathbb{R}^+ \times [-\underline{a}, \overline{a}] \longmapsto \mathbb{R}$ which satisfies the following assumption.

Assumption 3. *There exists a positive constant $C < 2\beta$ such that for any $(x, \alpha) \in \mathbb{R}^+ \times [-\underline{a}, \overline{a}]$*

$$\frac{|h^K(x,\alpha)|^2}{\lambda^{K,d}(x)} \leq Cx \text{ and } Kf^{K,d}(x) + h^K(x,\alpha) \geq 0$$

with equality if $x = 0$.

The set of admissible controls is defined by

$$\mathscr{A}^K = \{\alpha - \mathbb{F}^K \text{ predictable s.t. } \alpha \in [-\underline{a}, \overline{a}]\}.$$

For any $\alpha \in \mathscr{A}^K$ we define

$$L_t^{K,\alpha} = \mathscr{E}\Big(\int_0^\cdot \frac{h^K(X_s^K, \alpha_s)}{\lambda_s^{K,d}} \mathrm{d}M_s^{K,d}\Big)_t,$$

where $\mathscr{E}$ denotes the Doleans-Dade exponential process. We deduce from Assumption 3, Proposition 1 together with [Sok13, Corollary 2.6] that $(L_t^{K,\alpha})_{t\in[0,T]}$ is a true martingale. Hence the law of the population process under the control α is given by $\mathbb{P}^{K,\alpha}$ characterized by

$$\frac{\mathrm{d}\mathbb{P}^{K,\alpha}}{\mathrm{d}\mathbb{P}} = L_T^{K,\alpha}.$$

Under the probability $\mathbb{P}^{K,\alpha}$ the death intensity of the population becomes

$$\lambda_t^{K,d,\alpha} = \lambda_t^{K,d} + h^K(X_t^K, \alpha_t)$$

and the birth intensity is unchanged.

We assume that the manager receives at maturity T a lump sum random compensation $\xi^K \in \mathbb{T}^K$ for his action. In addition, the manager receives continuous incomes along the time depending on the size of the population and on his control that is given by a function c^K from $\mathbb{R}^+ \times [-\underline{a}, \overline{a}]$ into $\mathbb{R}$. This gain can be negative which corresponds to a cost related to the effort of the manager. This is what we have considered in Section 3. Therefore, the goal of the manager is to solve the following maximization problem

$$\textbf{(P)}_K: \; V_0^K = \sup_{\alpha\in\mathscr{A}^K} J_0^{K,\alpha} \text{ with } J_0^{K,\alpha} := \mathbb{E}^{K,\alpha}[\xi^K + \int_0^T c^K(X_s^K, \alpha_s)\mathrm{d}s],$$

where $\mathbb{E}^{K,\alpha}$ denotes the expectation taken under the probability $\mathbb{P}^{K,\alpha}$. Using the notations of Section 4.2 the BSDE associated to this control problem is

$$\textbf{(BSDE)}_K: \; Y_t^K = \xi^K + \int_t^T g^K(X_s^K, Z_s^K)\cdot\phi_s^K \mathrm{d}A_s^K - \int_t^T Z_s^K \cdot \mathrm{d}M_s^K$$

with for any $(x, z) \in \mathbb{R}^+ \times \mathbb{R}^2$

$$g^K(x,z) = (0, g^{K,d}(x, z^d)) \text{ where } g^{K,d}(x, z^d) = \sup_{\alpha\in[-\underline{a},\overline{a}]} \Big(c^K\big(x,\alpha\big) + z^d h^K\big(x,\alpha\big)\Big)\frac{1}{\lambda^{K,d}(x)}.$$

We need to assume that the functions c^K and h^K are chosen such that g^K satisfies the assumptions (ii) and (iii) of Lemma 1 and that the maximizer in the above equation is unique. Formally we make the following assumption.

Assumption 4.

(i) $g^{K,d}(X^K, 0) \in \mathbb{H}_1^K$,

(ii) there exists a positive L satisfying $\beta > L^2 + 2L$ and such that for any x, z, z', α and $K \geq 0$ we have

$$\frac{|Kzh^K(x,\alpha) - Kz'h^K(x,\alpha)|}{\lambda^{K,d}(x)} \leq L|z - z'|.$$

(iii) For any x, z there exists a unique $\alpha^{K,*}(x,z)$ such that

$$g^{K,d}(x,z) = \frac{c^K\big(x, \alpha^{K,*}(x,z)\big) + z h^K(x, \alpha^{K,*}(x,z))}{\lambda^{K,d}(x)}.$$

We thus have the following characterization of the optimal control (we refer to Appendix 6.4 for the proof).

Theorem 3 (Verification for $\mathbf{(P)}_K$). *Let $(Y^K, Z^K) \in \mathbb{S}^K$ be the unique solution of $\mathbf{(BSDE)}_K$. Then $V_0^K = Y_0^K$ and $\alpha_t^{K,\star} := \alpha^{K,\star}(X_t^K, Z_t^{K,d})$ solves the problem $\mathbf{(P)}_K$.*

We now define the continuous version of this control problem.

5.2 The continuous problem

As previously the resource manager monitors his harvesting intensity through a control α, assumed to be bounded with bounds $\underline{a}, \overline{a} > 0$. We assume that his harvesting modified the death rate of the natural resource according to a continuous function $h : \mathbb{R}^+ \times [-\underline{a}, \overline{a}] \longmapsto \mathbb{R}$ which is assumed to satisfy the following assumption.

Assumption 5. *There exists a positive constant $C < 2\beta$ such that for any $(x,\alpha) \in \mathbb{R}^+ \times [-\underline{a}, \overline{a}]$*

$$\frac{h^2(x,\alpha)}{\sigma^2 x} \le Cx \text{ and } h(x,\alpha) + f^d(x) \ge 0.$$

with equality if $x = 0$.

The set of admissible control is

$$\mathscr{A} = \{\alpha - \mathbb{F}^X \text{ predictable s.t. } \alpha \in [-\underline{a}, \overline{a}]\}.$$

Considering the process

$$L_t^\alpha = \mathscr{E}\Big(\int_0^\cdot -\frac{h(X_s,\alpha_s)}{\sigma^2 X_s} \mathrm{d}M_s^X\Big)_t,$$

where we recall that X is given by (**S**). We deduce from Assumption 5 and Corollary 2.6 in [Sok13] that L^α is a true martingale. Hence, we define the probability $\mathbb{P}^\alpha$ by

$$\frac{\mathrm{d}\mathbb{P}^\alpha}{\mathrm{d}\mathbb{P}} = L_T^\alpha$$

which is the probability measure corresponding to the control α. Under $\mathbb{P}^\alpha$ the process X is a strong solution of

$$X_t = x_0 + \int_0^t \big(f(X_s) - h(X_s,\alpha_s)\big)\mathrm{d}s + \int_0^t \sigma\sqrt{X_s}\mathrm{d}B_s^\alpha,$$

where $B^\alpha := B + \int_0^\cdot \frac{h(X_s,\alpha_s)}{\sigma\sqrt{X_s}} ds$ is a $\mathbb{P}^\alpha-$Brownian motion.

As in the discrete case we assume that the manager receives at maturity T a lump sum random compensation $\xi \in \mathbb{T}$ for his action. In addition, the manager receives continuous incomes term depending on the size of the population and his control. This term is given by a function c from $\mathbb{R}^+ \times [-\underline{a}, \overline{a}]$ into $\mathbb{R}$. Therefore, the goal of the manager is to solve the following maximization problem

$$\textbf{(P)}: \; V_0 = \sup_{\alpha \in \mathscr{A}} J_0^\alpha \text{ with } J_0^\alpha := \mathbb{E}^\alpha[\xi + \int_0^T c(X_s, \alpha_s)\mathrm{d}s],$$

where $\mathbb{E}^\alpha$ denotes the expectation taken under the probability $\mathbb{P}^\alpha$. The BSDE associated to this control problem is

$$\textbf{(BSDE)}: \; Y_t = \xi + \int_t^T g(X_s, Z_s)\mathrm{d}A_s - \int_t^T Z_s \mathrm{d}M_s^X$$

with for any $(x, z) \in \mathbb{R}^+ \times \mathbb{R}$

$$g(x, z) = \sup_{\alpha \in [-\underline{a}, \overline{a}])} \Big(c\big(x, \alpha\big) + zh\big(x, \alpha\big)\Big)\frac{1}{\sigma^2 x}.$$

We need to assume that the functions c and h are chosen such that g satisfies the assumptions (ii) and (iii) of Lemma 2 and that the maximizer in the above equation is unique. Formally we make the following assumption.

Assumption 6.

(i) $g(X, 0) \in \mathbb{H}_1$,

(ii) there exists a positive L *satisfying* $\beta > L^2 + 2L$ *such that for any* x, z, z' *and* α *we have*

$$\frac{|zh(x, \alpha) - z'h(x, \alpha)|}{\sigma^2 x} \leq L|z - z'|.$$

(iii) For any x, z *there exists a unique* $\alpha^*(x, z)$ *such that*

$$g(x, z) = \frac{c\big(x, \alpha^*(x, z)\big) + zh(x, \alpha^*(x, z))}{\sigma^2 x}$$

We thus have the following characterization of the optimal control (we refer to Appendix 6.5 for the proof).

Theorem 4 (Verification for **(P)**). *Let* $(Y, Z) \in \mathbb{S}$ *be the unique solution of* ***(BSDE)****. Then* $V_0 = Y_0$ *and* $\alpha_t^\star := \alpha^\star(X_t, Z_t)$ *solves the problem* ***(P)****.*

5.3 Convergence of the value functions and of the optimal controls

In this section, we show that under some natural assumptions the sequences of value functions $(V_0^K)_{K\geq 0}$ and of controls $(\alpha^{K,*})_{K\geq 0}$ converge respectively towards V_0 and α^*. More precisely we consider the following assumptions.

Assumption 7.

(i) $(\xi^K)_{K\geq 0}$ *converges to* ξ *in* $L^{2+\varepsilon}$ *for some* $\varepsilon > 0$,

(ii) there exists a positive sequence $(\eta_K)_{K\geq 0}$ *that converges towards* 0 *such that for any* x, α, z *and* K *we have*

$$|K^2 \frac{c^K(x,\alpha)}{\lambda^{K,d}(x)} - \frac{c(x,\alpha)}{\sigma^2 x/2}| + |K \frac{h^K(x,\alpha)}{\lambda^{K,d}(x)} - \frac{h(x,\alpha)}{\sigma^2 x/2}| \leq \eta_K(1+|x|)$$

and

$$|\alpha^{K,*}(x, z/K) - \alpha^*(x,z)| + \Big|\Big(\frac{Kh^K(x,\alpha)}{\lambda^{K,d}(x)}\Big)^2 - \Big(\frac{h(x,\alpha)}{\sigma^2 x/2}\Big)^2\Big| \leq \eta_K(1+|x|+|z|).$$

(iii) There exists a positive constant $C > 0$ *such that for any* x, x', z *and* K *we have*

$$K^2 |\frac{c^K(x,\alpha) - zK^{-1}h^K(x,\alpha)}{\lambda^{K,d}(x)} - \frac{c^K(x',\alpha) - zK^{-1}h^K(x',\alpha)}{\lambda^{K,d}(x')}| \leq C|x-x'|$$

and for any $x, x', z, z', \alpha, \alpha'$ *and* K *we have*

$$|\alpha^{K,*}(x, \frac{z}{K}) - \alpha^{K,*}(x', \frac{z'}{K})| + \Big|\Big(\frac{Kh^K(x,\alpha)}{\lambda^{K,d}(x)}\Big)^2 - \Big(\frac{Kh^K(x',\alpha')}{\lambda^{K,d}(x')}\Big)^2\Big| \leq C(|x-x'| + |\alpha-\alpha'| + |z-z'|).$$

Assumption 7 contains the natural assumptions ensuring that the problem (**P**) is the version of the problems $(\mathbf{P})_K$ in the framework of the continuous population model X.

Using a slight abuse we note $\mathbb{P}^{K,*}$ the law of X^K under the control $\alpha^{K,*}$ and $\mathbb{P}^*$ the law of X under the control α^*. We have the following convergence result which proof is given in Appendix 6.6.

Theorem 5. *(i) We have in* $\mathscr{S}_1^2 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1$*:*

$$\lim_{K\to+\infty} \Big(Y^K, \int_0^\cdot \alpha_s^{K,*}\lambda_s^{K,d}K^{-2}\mathrm{d}s, \int_0^\cdot (\alpha_s^{K,*})^2\lambda_s^{K,d}K^{-2}\mathrm{d}_s\Big) = \Big(Y, \int_0^\cdot \alpha_s \mathrm{d}A_s/2, \int_0^\cdot \alpha_s^2 \mathrm{d}A_s/2\Big).$$

(ii) The sequence $(\mathbb{P}^{K,*})_{K\geq 0}$ *converges for the Skorohod topology towards* $\mathbb{P}^*$.

Since $Y_0^K = V_0^K$ and $Y_0 = V_0$ a consequence of Theorem 5 (i) is that $(V_0^K)_{K\geq 0}$ converges towards V_0. The point (i) also implies that the sequence of controls $(\alpha^K)_{K\geq 0}$ converges towards the control α. But this convergence is in a weak sense and we do not get the convergence of $(\alpha^{K,*})_{K\geq 0}$ towards α^* in law for the Skorohod topology.

Remark 3. *Note that the sequence of control problems considered in Section 3 when* $\alpha \in [-\nu, \overline{a}]$ *(for* $\overline{a}$ *positive) satisfy any of the assumptions of Section 5.*

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6 Proofs

6.1 Proof of Theorem 1

We introduce the process

$$Y_t^K = \int_0^t f(X_s^K)\mathrm{d}s.$$

The proof is divided in four main steps detailed below.

1. We prove that (**S**) admits a unique strong solution.
2. We show that the sequence $(Y^K, \overline{M}^K, \overline{N}^{K,b}, \overline{N}^{K,d}, \overline{\Lambda}^{K,b}, \overline{\Lambda}^{K,d})_{K\geq 0}$ is C-tight.
3. We show that for any limit point $(Y, M, N^b, N^d, \Lambda^b, \Lambda^d)$ of the above sequence we have $\Lambda^b = N^b = \Lambda^d = N^d$ and Y is almost surely differentiable with derivative X weak solution of (**S**).
4. Finally, we prove that up to a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ such that the above convergence holds in probability, the process $(X^K, \overline{M}^K, \overline{N}^{K,b}, \overline{N}^{K,d}, \overline{\Lambda}^{K,b}, \overline{\Lambda}^{K,d})_{K\geq 0}$ actually converges to (X, M, A, A, A, A) when K goes to $+\infty$ in $\mathscr{S}_1^2 \times \mathscr{S}_2^2 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1$.

Step 1: Pathwise uniqueness under existence. The uniqueness result is a direct consequence of [RY13, IX-Theorem 3.5 (ii)] under Assumption **(A)**$(ii)-(iii)$.

Step 2: Tightness property.
In order to show tightness we first show that the sequence $\big(\sup_{t\in[0,1]} \mathbb{E}[X_t^K]\big)_{K\geq 0}$ is bounded uniformly with respect to K. We have

$$\mathbb{E}[X_t^K] = n_0 + \int_0^t \mathbb{E}[f^b(X_s^K) - f^d(X_s^K)]\mathrm{d}s.$$

Hence, according to **(A)**$-(iii)$, there exists a positive constant C (independent of K) such that

$$\mathbb{E}[X_t^K] \leq n_0 + \int_0^t C\mathbb{E}[X_s^K]\mathrm{d}s.$$

By using Grönwall's inequality we deduce that $\big(\sup_{t\in[0,1]} \mathbb{E}[X_t^K]\big)_{K\geq 0}$ is bounded uniformly with respect to K.

We have

$$\mathbb{E}[\overline{N}_t^{K,b}] \leq \int_0^t \mathbb{E}\big[f^b(X_t^K)/K + \frac{\sigma^2(X_t^K)}{2}\big]\mathrm{d}s,$$

therefore $\big(\mathbb{E}[\overline{N}_T^{K,b}]\big)_{K\geq 0}$ is bounded and since $\overline{N}^{K,d} \leq \overline{N}^{K,b}$ then $\big(\mathbb{E}[\overline{N}_1^{K,d}]\big)_{K\geq 0}$ is also bounded. Moreover since $f(X^K) \leq CX^K$ the sequence $\big(\mathbb{E}[Y_T^K]\big)_{K\geq 0}$ bounded. Using Theorem VI-3.21 in [JS13] and that the processes Y^K, $\overline{N}^{K,i}$ and $\overline{\Lambda}^{K,i}$ for $i \in \{b,d\}$ are nondecreasing for any K we get that the sequences $(Y^K)_{K\geq 0}$, $(\overline{N}^{K,b})_{K\geq 0}$, $(\overline{N}^{K,d})_{K\geq 0}$, $(\overline{\Lambda}^{K,b})_{K\geq 0}$ and $(\overline{\Lambda}^{K,d})_{K\geq 0}$ are tight.

Moreover since $|\Delta\overline{N}^{K,i}| = 1/K^2$ for $i \in \{b,d\}$ and the processes Y^K, $\Lambda^{K,b}$ and $\Lambda^{K,d}$ are continuous for any K following Proposition VI-3.26 in [JS13] we get that the sequences $(Y^K)_{K\geq 0}$, $(\overline{N}^{K,b})_{K\geq 0}$, $(\overline{N}^{K,d})_{K\geq 0}$, $(\overline{\Lambda}^{K,b})_{K\geq 0}$ and $(\overline{\Lambda}^{K,d})_{K\geq 0}$ C-are tight.

The tightness of $(\overline{M}^{K,b}, \overline{M}^{K,d})_{K\geq 0}$ then follows from Theorem VI-4.13 in [JS13] since $\langle \overline{M}^{K,i} \rangle = \overline{\Lambda}^{K,i}$. We then get C-tightness since $|\Delta\overline{M}^{K,i}| \leq K^{-1}$. Since marginal tightness implies tightness (Corollary IV-3.33 in [Jac75]) we get that $(Y^K, \overline{M}^K, \overline{N}^{K,b}, \overline{N}^{K,d}, \overline{\Lambda}^{K,b}, \overline{\Lambda}^{K,d})_{K\geq 0}$ is C-tight.
Step 3: convergence of the processes and existence of a solution to (**S**)
We first show the following lemma:

Lemma 3. *For $i \in \{b,d\}$ the process $|\overline{N}^{K,i} - \overline{\Lambda}^{K,i}|$ converges uniformly towards 0 in probability.*

Proof of Lemma 3. Obviously we have $\overline{N}_t^{K,i} - \overline{\Lambda}_t^{K,i} = \overline{M}_t^{K,i}/K$ so using the BDG inequality we have

$$\mathbb{E}\big[\sup_{t\in[0,T]} \frac{|\overline{M}_t^{K,i}|^2}{K^2}\big] \leq \frac{\mathbb{E}[\overline{N}_T^{K,i}]}{K^2}.$$

that converges towards 0. We conclude using Markov inequality. □

In view of the tightness result obtained in Step 1, we denote by $(Y, M, N^b, N^d, \Lambda^b, \Lambda^d)$ a limit point of $(Y^K, \overline{M}^K, \overline{N}^{K,b}, \overline{N}^{K,d}, \overline{\Lambda}^{K,b}, \overline{\Lambda}^{K,d})_{K\geq 0}$ with $M = (M^b, M^d)$. By the Skorohod representation theorem since the limit of each marginal is continuous we can consider that $(Y^K, \overline{M}^K, \overline{N}^{K,b}, \overline{N}^{K,d})_{K\geq 0}$ converges almost surely and uniformly on $[0,T]$ towards (Y, M, N^b, N^d), *i.e.*

$$\sup_{t\in[0,1]} |Y_t^K - Y_t| \underset{K\to+\infty}{\to} 0,$$

and for any $i \in \{b,d\}$

$$\sup_{t\in[0,1]} |\overline{M}_t^{K,i} - M_t^i| \underset{K\to+\infty}{\to} 0, \quad \sup_{t\in[0,1]} |\overline{N}_t^{K,i} - N_t^i| \underset{K\to+\infty}{\to} 0, \quad \sup_{t\in[0,1]} |\overline{\Lambda}_t^{K,i} - \Lambda_t^i| \underset{K\to+\infty}{\to} 0.$$

According to Corollary IX-1.19 in [JS13] we have that M is a local martingales. Moreover we have $[\overline{M}^{K,i}] = \overline{N}^{K,i}$ and $[\overline{M}^{K,b}, \overline{M}^{K,d}] = 0$ so Corollary VI-6.29 in [JS13] gives $[M^i] = N^i = \Lambda^i$ and $[M^b, M^d] = 0$. Since M^i is a continuous martingale we get $\langle M^i \rangle = [M^i] = \Lambda^i$. We also notice that $\mathbb{E}[\overline{\Lambda}_T^{K,i}]$ is uniformly bounded in K, so according to Fatou's lemma Λ^i is integrable,

therefore M^i is a true martingale.

We recall that

$$X_t^K = n_0 + \int_0^t f(X_s^K)\mathrm{d}s + \overline{M}_t^{K,b} - \overline{M}_t^{K,d}.$$

Then, X^K converges almost surely and uniformly on $[0,T]$ towards

$$X_t := n_0 + Y_t + M_t^b - M_t^d$$

and Y^K converges almost surely uniformly on $[0,T]$ towards

$$\int_0^{\cdot} f(X_s)\mathrm{d}s.$$

Since we have

$$\langle M^b\rangle_t = \langle M^d\rangle_t = \int_0^t \frac{\sigma^2(X_s)}{2}\mathrm{d}s \text{ and } \langle M^b, M^d\rangle = 0$$

we get from Theorem V-3.9 in [RY13] that there exists a bi-dimensional Brownian motion (B^b, B^d) such that

$$(M_t^b, M_t^d) = \int_0^t \frac{\sigma(X_s)}{\sqrt{2}}\mathrm{d}(B_s^b, B_s^d).$$

So finally we have shown that

$$X_t = n_0 + \int_0^t f(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}(\frac{B_s^b + B_s^d}{\sqrt{2}}).$$

This concludes the proof of the first part of Theorem 1 since $(X^K, \overline{M}^K, \overline{N}^{K,b}, \overline{N}^{K,d}, \overline{\Lambda}^{K,b}, \overline{\Lambda}^{K,d})$ converges in law for the Skorohod topology to (X, M, A, A, A, A).

Step 4: convergence of a copy $(X^K, \overline{M}^K, \overline{N}^{K,b}, \overline{N}^{K,d}, \overline{\Lambda}^{K,b}, \overline{\Lambda}^{K,d})_{K\geq 0}$ *in* $\mathscr{S}_1^2 \times \mathscr{S}_2^2 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1$.
In view of the conclusion of Step 3, and by the Skorohod representation theorem, there exists a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and a copy in law of the sequence of processes $(X^K, \overline{M}^K, \overline{N}^{K,b}, \overline{N}^{K,d}, \overline{\Lambda}^{K,b}, \overline{\Lambda}^{K,d})_{K\geq 0}$ that converges in probability toward a copy of (X, M, A, A, A, A) when K goes to $+\infty$. To prove that the convergence actually holds in $\mathscr{S}_1^2 \times \mathscr{S}_2^2 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1$ we show that:

(*i*) $(\overline{N}^{K,b})_{K\geq 0}$ and $(\overline{N}^{K,b})_{K\geq 0}$ are bounded in $\mathscr{S}_1^2$,

(*ii*) $(\overline{\Lambda}^{K,b})_{K\geq 0}$ and $(\overline{\Lambda}^{K,d})_{K\geq 0}$ are bounded in $\mathscr{S}_1^2$,

(*iii*) $(\overline{M}^K)_{K\geq 0}$ is bounded in $\mathscr{S}^4$,

(*iv*) $(X^K)_{K\geq 0}$ is bounded in $\mathscr{S}_1^4$.

Then we will get the convergence using dominated convergence.
Proof of (i). We write

$$\sup_{t\in[0,T]}(\overline{N}_s^{K,b})^2 = (\overline{N}_T^{K,b})^2 = \int_0^T (2\frac{\overline{N}_s^{K,b}}{K^2} + K^{-4})\mathrm{d}N_s^{K,b}.$$

Therefore we have for a positive constant C independent of K such that

$$\mathbb{E}[\sup_{t\in[0,T]}(\overline{N}_s^{K,b})^2] = \mathbb{E}[\int_0^T (2\overline{N}_s^{K,b} + K^{-2})\lambda_s^{K,b}\mathrm{d}s] \leq \mathbb{E}[\int_0^T (2\overline{N}_s^{K,b} + K^{-2})CX_s^K\mathrm{d}s].$$

Hence to conclude it is enough to show that $(\mathbb{E}[\overline{N}_t^{K,b}X_t^K])_{t\in[0,T]}$ is bounded. We have

$$\begin{aligned}\overline{N}_t^{K,b}X_t^K &= \int_0^t (X_s^K K^{-2} + \overline{N}_s^{K,b}K^{-1} + K^{-3})\mathrm{d}N_s^{K,b} - \overline{N}_s^{K,b}K^{-1}\mathrm{d}N_s^{K,d} \\ &= \int_0^t (X_s^K K^{-2} + K^{-3})\mathrm{d}N_s^{K,b} + \int_0^t \overline{N}_s^{K,b}K^{-1}\mathrm{d}(N_s^{K,b} - N_s^{K,d}).\end{aligned}$$

So we get

$$\begin{aligned}\mathbb{E}[\overline{N}_t^{K,b}X_t^K] &= \mathbb{E}[\int_0^t (X_s^K + K^{-1})K^{-2}\lambda_s^{K,b}\mathrm{d}s + \int_0^t \overline{N}_s^{K,b}K^{-1}(\lambda_s^{K,b} - \lambda_s^{K,d})\mathrm{d}s] \\ &\leq \mathbb{E}[\int_0^t (X_s^K + K^{-1})CX_s^K\mathrm{d}s + \int_0^t \overline{N}_s^{K,b}CX_s^K\mathrm{d}s].\end{aligned}$$

Therefore by Proposition 1 and Grönwall lemma we get point (i) (since the same results follows for $N^{K,d}$).

Proof of (ii). We have

$$\sup_{t\in[0,T]}\overline{\Lambda}_t^{K,d} \leq C\int_0^T X_s^K\mathrm{d}s,$$

therefore point (ii) follows from Proposition 1. Same proof holds for $\overline{N}^{K,d}$.

Proof of (iii). Using the Burkholder-Davis-Gundy inequality we get

$$\mathbb{E}[\sup_{t\in[0,T]}|\overline{M}_t^{K,b}|^4] \leq C\mathbb{E}[(\overline{N}_T^{K,b})^2].$$

Therefore because of point (i) we get point (iii). Same proof holds for $\overline{M}^{K,d}$.
Proof of (iv). We write

$$\begin{aligned}X_t^K &= x_0^K + \int_0^t (\lambda_s^{K,b} - \lambda_s^{K,d})K^{-1}\mathrm{d}s + \overline{M}_t^{K,b} - \overline{M}_t^{K,d} \\ &\leq x_0^K + \int_0^t CX_s^K\mathrm{d}s + \overline{M}_t^{K,b} - \overline{M}_t^{K,d}.\end{aligned}$$

So we have

$$|X_t^K|^4 \leq C\Big((x_0^K)^4 + \big(\int_0^T X_s^K \mathrm{d}s\big)^4 + |\overline{M}_t^{K,b}|^4 + |\overline{M}_t^{K,d}|^4\Big)$$

taking the supremum over $t \in [0,T]$ and then the expectation we obtain point (iv) as corollary of Proposition 1 and point (iii).

6.2 Proof of Proposition 2.

The proof of Proposition 2 is inspired by that of Theorem 5 in [BDM02].

It is easy to check using (1) that for any K the processes X^K is a càd-làg Feller process. For details on this point see Appendix V.D. Moreover, by Theorem 1.1 in [Küh18] X is a continuous Feller process. Hence from Proposition 4.B) in [CMS01] we get the weak convergence of $\mathbb{F}^K$ towards $\mathbb{F}^X$. So by Remark 1.2) in [CMS01] we have the convergence in probability of $(Q^K)_{K\geq 0}$ towards Q for the Skorohod topology. Since Q is a Brownian martingale, it is a continuous martingale and therefore $(Q^K)_{K\geq 0}$ converges in probability towards Q for the topology of the uniform convergence.

Since $(|\overline{M}^K|^2)_{K\geq 0}$ is uniformly integrable it converges in $\mathscr{S}_1^2$ towards M. From Proposition 2 in [BDM02], we obtain that

$$\Big(\langle Q^K, Q^K\rangle, \langle Q^K, \overline{M}^K\rangle\Big) \to \Big(\langle Q, Q\rangle, \langle Q, M\rangle\Big) \text{ as } K \to +\infty$$

in $\mathscr{S}_1^1 \times \mathscr{S}_2^1$. We then extend this convergence and the one of $(Q^K)_{K\geq 0}$ using dominated convergence theorem. For this we show that for any $\varepsilon' < \varepsilon$ we have:

(i) $(Q^K)_{K\geq 0}$ is bounded in $\mathscr{S}_1^{2+\varepsilon'}$,

(ii) $\big(\langle Q^K\rangle\big)_{K\geq 0}$ is bounded in $\mathscr{S}_1^{1+\varepsilon'}$,

(iii) $\big(\langle Q^K, \overline{M}^K\rangle\big)_{K\geq 0}$ is bounded in $\mathscr{S}_1^{2+\varepsilon'}$.

Proof of (i): By Doob's maximal inequality we have for a positive constant C

$$\mathbb{E}[\sup_{t\in[0,T]} |Q_t^K|^{2+\varepsilon'}] \leq C\mathbb{E}[|Q_T^K|^{2+\varepsilon'}] \leq C\mathbb{E}[|\xi^K|^{2+\varepsilon'}].$$

Therefore we get (i) since $(\xi^K)_{K\geq 0}$ is bounded in $L^{2+\varepsilon}$.

Proof of (ii): By Equation (100.2) p.183 in [DM80] and using BDG inequality we have

$$\mathbb{E}[\langle Q^K\rangle_T^{1+\varepsilon'/2}] \leq C\mathbb{E}[\sup_{t\in[0,T]} |Q_t^K|^{2+\varepsilon'}],$$

thus we get (ii).

Proof of (iii): Using Kunita-Watanabe inequality we have

$$\|\langle Q^K, \overline{M}^K\rangle_T\|^2 \leq \langle Q^K\rangle_T \langle A^K\rangle_T.$$

Therefore by Hölder inequality we get for any $p > 1$ such that $p(1+\varepsilon'/2) < (1+\varepsilon/2)$:

$$\mathbb{E}[\sup_{t\in[0,T]} \|\langle Q^K, \overline{M}^K\rangle_T\|_1^{2+\varepsilon'}] \leq \left(\mathbb{E}[\langle Q^K\rangle_T^{p(1+\varepsilon'/2)}]\right)^{1/p} \left(\mathbb{E}[\langle A^K\rangle_T^{p*(1+\varepsilon/2)}]\right)^{1/p*}$$

where $p* = (1-p^{-1})^{-1}$ so (iii) follows from (ii) and Proposition 1.

6.3 Proof of Theorem 2.

The proof of Theorem 2 is inspired from the proof of Theorem 12 in [BDM02].

We proceed in 3 steps:

(i) We show that there exists $\alpha \in (0,1)$ and some $\alpha-$contracting functions $(F^K)_{K\geq 0}$ and F such that for any K, the unique solution of $(\mathbf{B})_K$ is the fixed point of F^K and the fixed point of F is solution to $(\mathbf{B})$.

(ii) We introduce a double indexed sequence and prove a convergence result by induction.

(iii) We conclude.

6.3.1 Step (i)

For any K we define the function

$$F^K : \left| \begin{array}{ccc} \mathbb{S}^K & \longrightarrow & \mathbb{S}^K \\ (Y,Z) & \longmapsto & (Y',Z') \end{array} \right.$$

where (Y',Z') is the unique solution of the BSDE:

$$Y'_t = \xi^K + \int_t^T g^K(X_s^K, Y_s, Z_s)\cdot \phi_s^K \mathrm{d}A_s^K - \int_t^T Z'_s \cdot \mathrm{d}M_s^K.$$

Since $(Y,Z) \in \mathbb{S}^K$ and because of assumptions (ii) and (iii) in Lemma 1 we have $g^K(X^K, Y, Z) \in \mathbb{H}_2^K$. So we can properly define

$$Y'_t = \mathbb{E}[\xi^K + \int_t^T g^K(X_s^K, Y_s, Z_s)\cdot \phi_s^K \mathrm{d}A_s^K | \mathscr{F}_t^K]$$

and Z' is the unique process in $\mathbb{H}_2^K$ satisfying

$$\mathbb{E}[\xi^K + \int_0^T g^K(X_s^K, Y_s, Z_s)\cdot \phi_s^K \mathrm{d}A_s^K | \mathscr{F}_t^K] = Y_0'^K + \int_0^t Z'_s \cdot \mathrm{d}M_s^K.$$

Consider two pairs (Y^1, Z^1), $(Y^2, Z^2) \in \mathbb{S}^K$ and noting $(\overline{Y}^1, \overline{Z}^1) = F^K(Y^1, Z^1)$ (resp. $(\overline{Y}^2, \overline{Z}^2) = F^K(Y^2, Z^2)$). Using Ito's formula on $e^{\beta A_t^K}|\overline{Y}_t^1 - \overline{Y}_t^2|^2$ between 0 and T we get

$$\begin{aligned}
-|\overline{Y}_0^1 - \overline{Y}_0^2|^2 = &\int_0^T e^{\beta A_t^K}\Big(\beta|\overline{Y}_t^1 - \overline{Y}_t^2|^2 - 2(\overline{Y}_t^1 - \overline{Y}_t^2)\big(g_K(X_t^K, Y_t^1, Z_t^1) - g_K(X_t^K, Y_t^2, Z_t^2)\big)\cdot\phi_t^K\Big)\mathrm{d}A_t^K \\
&+ \int_0^T e^{\beta A_t^K} 2(\overline{Y}_t^1 - \overline{Y}_t^2)(\overline{Z}_t^1 - \overline{Z}_t^2)\cdot \mathrm{d}M_t^K \\
&+ \int_0^T e^{\beta A_t^K}|\overline{Z}_t^{1,b} - \overline{Z}_t^{2,b}|^2\mathrm{d}N_t^b + \int_0^T e^{\beta A_t^K}|\overline{Z}_t^{1,d} - \overline{Z}_t^{2,d}|^2\mathrm{d}N_t^d.
\end{aligned}$$

Taking the expectation we get

$$\begin{aligned}
&|\overline{Y}_0^1 - \overline{Y}_0^2|^2 + \mathbb{E}[\int_0^T e^{\beta A_t^K}\beta|\overline{Y}_t^1 - \overline{Y}_t^2|^2\mathrm{d}A_t^K] + \mathbb{E}[\int_0^T e^{\beta A_t^K}|\overline{Z}_t^1 - \overline{Z}_t^2|^2\cdot\phi_t^K\mathrm{d}A_t^K] \\
&\qquad \le \mathbb{E}[\int_0^T e^{\beta A_t^K} 2|\overline{Y}_t^1 - \overline{Y}_t^2||g^K(X_t^K, Y_t^1, Z_t^1) - g^K(X_t^K, Y_t^2, Z_t^2)|\cdot\phi_t^K\mathrm{d}A_t^K].
\end{aligned}$$

Therefore using the assumptions of Lemma 1 together with Young's inequality we get for any positive α and γ that

$$\beta\|\overline{Y}^1 - \overline{Y}^2\|_{\mathbb{H}_1^K}^2 + \|\overline{Z}^1 - \overline{Z}^2\|_{\mathbb{H}_2^K}^2 \le (\frac{L}{\gamma} + \frac{L}{\alpha})\|\overline{Y}^1 - \overline{Y}^2\|_{\mathbb{H}_1^K}^2 + L\gamma\|Z^1 - Z^2\|_{\mathbb{H}_2^K}^2 + L\alpha\|Y^1 - Y^2\|_{\mathbb{H}_1^K}^2$$

or equivalently

$$(\beta - \frac{L}{\alpha} - \frac{L}{\gamma})\|\overline{Y}^1 - \overline{Y}^2\|_{\mathbb{H}_1^K}^2 + \|\overline{Z}^1 - \overline{Z}^2\|_{\mathbb{H}_2^K}^2 \le L\gamma\|Z^1 - Z^2\|_{\mathbb{H}_2^K}^2 + L\alpha\|Y^1 - Y^2\|_{\mathbb{H}_1^K}^2.$$

Inspired by the proof of Theorem 3.4 in [CF13] we choose $\gamma = \alpha/L$ and $\alpha \in (0,1)$ such that

$$\beta - \frac{L^2 + L}{\alpha} > L.$$

We can make such choice since $\beta - L^2 - L > L$. Therefore we obtain

$$L\|\overline{Y}^1 - \overline{Y}^2\|_{\mathbb{H}_1^K}^2 + \|\overline{Z}^1 - \overline{Z}^2\|_{\mathbb{H}_2^K}^2 \le \alpha\big(L\|Y^1 - Y^2\|_{\mathbb{H}_1^K}^2 + \|Z^1 - Z^2\|_{\mathbb{H}_2^K}^2\big).$$

Therefore for any K the function F^K is an $\alpha-$contraction on $\mathbb{S}^K$ for the norm equivalent to $\|\cdot\|_{\mathbb{S}^K}$ and defined by

$$\|(Y,Z)\|'_{\mathbb{S}^K} = \Big(L\|Y\|_{\mathbb{H}_1^K}^2 + \|Z\|_{\mathbb{H}_2^K}^2\Big)^{1/2}$$

In the continuous case we consider

$$F: \left|\begin{array}{ccc} \mathbb{S} & \longrightarrow & \mathbb{S} \\ (Y,Z) & \longmapsto & (Y',Z') \end{array}\right.$$

where (Y', Z') is the unique solution of the BSDE:

$$Y'_t = \xi + \int_t^T g(X_s, Y_s, Z_s)\mathrm{d}As - \int_t^T Z'_s\cdot\mathrm{d}M_s^X.$$

Since $(Y,Z)\in\mathbb{S}$ and because of Remark 2 we have $g(X,Y,Z)\in\mathbb{H}_2$. So we can properly define

$$Y'_t=\mathbb{E}[\xi+\int_t^T g(X_s,Y_s,Z_s)\cdot\phi_s\mathrm{d}As|\mathscr{F}_t]$$

and Z' is the unique process in $\mathbb{H}_2$ satisfying

$$\mathbb{E}[\xi+\int_t^T g(X_s,Y_s,Z_s)\cdot\phi_s^K\mathrm{d}As|\mathscr{F}_t]=\int_0^t Z'_s\cdot\mathrm{d}M_s.$$

Similarly we obtain that F is an $\alpha-$contraction for the equivalent norm on $\mathbb{S}$:

$$\|(Y,Z)\|'_{\mathbb{S}}=\left(L\|Y\|^2_{\mathbb{H}}+\|Z\|^2_{\mathbb{H}}\right)^{1/2}.$$

6.3.2 Step (ii)

For any $K\geq 0$ we define the sequence $(Y^{K,p},Z^{K,p})_{p\geq 0}$ satisfying

$$(Y^{K,0},Z^{K,0})=0 \text{ and } (Y^{K,p+1},Z^{K,p+1})=F^K(Y^{K,p},Z^{K,p}).$$

We similarly consider the sequence $(Y^p,Z^p)_{p\geq 0}$ defined by

$$(Y^0,Z^0)=0 \text{ and } (Y^{p+1},Z^{p+1})=F(Y^p,Z^p).$$

Since for any $K\geq 0$, F^K is a contraction. For any $K\geq 0$ the sequence $(Y^{K,p},Z^{K,p})_{p\geq 0}$ converges towards (Y^K,Z^K) in $\mathbb{S}^K$. In the same way $(Y^p,Z^p)_{p\geq 0}$ converges towards (Y,Z) in $\mathbb{S}$.

We use the following notation:

$$Q_t^{K,p+1}=\int_0^t Z_s^{K,p+1}\cdot\mathrm{d}M_s^K,\ \chi_t^{K,p}:=\int_0^t g_K(X_s^K,Y_s^{K,p},Z_s^{K,p})\cdot\phi_s^K\mathrm{d}A_s^K,$$

$$Q_t^{p+1}=\int_0^t Z_s^{p+1}\mathrm{d}M_s^X \text{ and } \chi_t^p:=\int_0^t g(X_s,Y_s^p,Z_s^p)\mathrm{d}A_s.$$

So that we can write:

$$Y_t^{K,p+1}=\xi^K+\chi_T^{K,p}-\chi_t^{K,p}-Q_T^{K,p+1}+Q_t^{K,p+1} \tag{3}$$

and

$$Y_t^{p+1}=\xi+\chi_T^p-\chi_t^p-Q_T^{p+1}+Q_t^{p+1}. \tag{4}$$

We prove by induction that the following convergence holds for any p:

$$\left(Y^{K,p},Q^{K,p},\langle Q^{K,p},\overline{M}^K\rangle,\langle Q^{K,p}\rangle\right)\to\left(Y^p,Q^p,\langle Q^p,M\rangle,\langle Q^p\rangle\right) \text{ as } K\to+\infty$$

in $\mathscr{S}_1^{2+\varepsilon_p}\times\mathscr{S}_1^{2+\varepsilon_p}\times\mathscr{S}_2^{2+\varepsilon_p}\times\mathscr{S}_1^{1+\varepsilon_p/2}$ where $\varepsilon_p=\varepsilon/2^p$.

Obviously the result holds for $p=0$. We assume that the converge holds for p and show that it implies the convergence for $p+1$.

We write

$$\mathbb{E}[\xi^K + \chi_T^{K,p} | \mathscr{F}_t^K] = Y_0^{K,p+1} + Q_t^{K,p+1} \text{ and } \mathbb{E}[\xi + \chi_T^p | \mathscr{F}_t^X] = Y_0^{p+1} + Q_t^{p+1}.$$

We prove in Appendix 6.3.4 that the induction hypothesis implies that $(\chi^{K,p})_{K\geq 0}$ converges towards χ^p in $\mathscr{S}_1^{2+\tilde{\varepsilon}_p}$ where $\tilde{\varepsilon}_p = (\varepsilon_p + \varepsilon_{p+1})/2$. Therefore $(\xi^K + \chi_T^{p,K})_{K\geq 0}$ converges towards $(\xi + \chi^p)$ in $L^{2+\tilde{\varepsilon}_p}$. Since $\tilde{\varepsilon}_p > \varepsilon_{p+1}$ using Proposition 2 we get

$$\left(Q^{K,p+1}, \langle Q^{K,p+1}, \overline{M}^K\rangle, \langle Q^{K,p+1}\rangle\right) \to \left(Q^{p+1}, \langle Q^{p+1}, M\rangle, \langle Q^{p+1}\rangle\right)$$

in $\mathscr{S}_1^{2+\varepsilon_{p+1}} \times \mathscr{S}_2^{2+\varepsilon_{p+1}} \times \mathscr{S}_1^{1+\varepsilon_{p+1}/2}$. From equations (3) and (4) we immediatly get that $(Y^{K,p+1})_{K\geq 0}$ converges towards Y^p in $\mathscr{S}_1^{2+\varepsilon_{p+1}}$. Therefore we get the convergence for $p+1$.

6.3.3 Step (iii)

Note that a consequence of Step (i) is that for a certain positive constant C we have

$$\|(Y^{K,p}, Z^{K,p}) - (Y^K, Z^K)\|_{\mathbb{S}^K} + \|(Y^p, Z^p) - (Y, Z)\|_{\mathbb{S}} \leq C\alpha^p. \tag{5}$$

We write

$$\|Q^K - Q\|_2 \leq \|Q^p - Q\|_2 + \|Q^K - Q^{K,p}\|_2 + \|Q^{K,p} - Q^p\|_2.$$

Notice that according to the BDG inequality there exists a positive constant C such that for any K

$$\|Q^{K,p} - Q^K\|_2^2 + \|Q^p - Q\|_2^2 \leq C\big(\|Z^{K,p} - Z^K\|_{\mathbb{H}^K}^2 + \|Z^p - Z\|_{\mathbb{H}}^2\big)$$

which converges towards 0 uniformly in K when $p \to +\infty$ by Equation (5). Hence $(Q^K)_{K\geq 0}$ converges in $\mathscr{S}_1^2$ towards Q.

Similarly we write

$$\|Y^K - Y\|_2 \leq \|Y^p - Y\|_2 + \|Y^{K,p} - Y^K\|_2 + \|Y^{K,p} - Y^p\|_2.$$

We proved in the previous section that the last term goes to 0 when $K \to +\infty$. Remark that we have

$$Y_t^K - Y_t^{K,p} = \mathbb{E}[\int_t^T \left(g^K(X_s^K, Y_s^K, Z_s^K) - g^K(X_s^K, Y_s^{K,p-1}, Z_s^{K,p-1})\right) \cdot \phi_s^K \mathrm{d}A_s^K | \mathscr{F}_t^K]$$

so using Jensen and Doob's inequality we get

$$\begin{aligned}\mathbb{E}[\sup_{t\in[0,T]} e^{\beta A_t^K} |Y_t^K - Y_t^{K,p-1}|^2] &\leq L^2 \mathbb{E}[\int_0^T e^{\beta A_s^K} |Y_s^K - Y_s^{K,p-1}|^2 \mathrm{d}A_s^K + \int_0^T e^{\beta A_s^K} |Z_s^K - Z_s^{K,p-1}|^2 \cdot \phi_s^K \mathrm{d}A_s^K] \\ &\leq L \|(Y^{K,p-1}, Z^{K,p-1}) - (Y^K, Z^K)\|_{\mathbb{S}^K}^2.\end{aligned}$$

Therefore $\|Y^K - Y^{K,p}\|_{\mathbb{K}^K}$ goes to 0 when $p \to +\infty$. In the same way we get that $\|Y - Y^p\|_{\mathbb{K}}$ goes to 0 when $p \to +\infty$. So $\|Y^K - Y\|_2$ converges towards 0.

Finally notice that,

$$\langle Y^K, \overline{M}^K\rangle = \langle Q^K, \overline{M}^K\rangle,\ \langle Y^K\rangle = \langle Q^K\rangle,\ \langle Y,\ M\rangle = \langle Q, M\rangle \text{ and } \langle Y\rangle = \langle Q\rangle.$$

So the convergence

$$\left(\langle Y^K, \overline{M}^K\rangle, \langle Y^K\rangle\right) \to \left(\langle Y, M\rangle, \langle Y\rangle\right) \text{ as } K \to +\infty$$

in $\mathscr{S}_2^1 \times \mathscr{S}_1^1$ follows from Proposition 2 in [BDM02] and from the convergence of $(Q^K, \overline{M}^K)_{K\geq 0}$ in $\mathscr{S}_1^2 \times \mathscr{S}_2^2$ towards (Q, M).

6.3.4 Convergence of $(\chi^{p,K})_{K\geq 0}$ towards χ^p

To prove the convergence we first prove that $(\chi^{p,K})_{K\geq 0}$ converges towards χ^p in probability for the uniform topology. Then we show that $(|\chi^{p,K}| + |\chi^p|)_{K\geq 0}$ is bounded in $\mathscr{S}^{2+\hat{\varepsilon}_p}$ where $\hat{\varepsilon}_p = (\varepsilon_p + \tilde{\varepsilon}_p)/2 > \tilde{\varepsilon}_p$. We conclude by dominated convergence.

For any n we note $Z^{p,n} = Z^p \mathbf{1}_{|Z^p|<n}$. We write

$$\sup_{t\in[0,T]} |\phi_t^K - \phi_t| \leq \sum_{i=b,d} T_{i,1}^{n,K,p} + T_{i,2}^{n,K,p} + T_{i,3}^{n,K,p} + T_{i,4}^{n,p}/2$$

where for $i \in b, d$, we recall that $g = (g_b + g_d)/2$,

$$\begin{aligned}
T_{i,1}^{n,K,p} &= \sup_{t\in[0,T]} \left| \int_0^t g_i^K(X_s^K, Y_s^{K,p}, Z_s^{K,p})\phi_s^{K,i} \mathrm{d}A_s^K - \int_0^t g_i^K(X_s^K, Y_s^{K,p}, Z_s^{p,n}/K)\phi_s^{K,i}\mathrm{d}A_s^K \right| \\
T_{i,2}^{n,K,p} &= \sup_{t\in[0,T]} \left| \int_0^t g_i^K(X_s^K, Y_s^{K,p}, Z_s^{p,n}/K)\phi_s^{K,i} \mathrm{d}A_s^K - \int_0^t g_i(X_s, Y_s^p, Z_s^{p,n})K^{-2}\phi_s^{K,i}\mathrm{d}A_s^K \right| \\
T_{i,3}^{n,K,p} &= \sup_{t\in[0,T]} \left| \int_0^t g_i(X_s, Y_s^p, Z_s^{p,n})K^{-2}\phi_s^{K,i} \mathrm{d}A_s^K - \int_0^t g_i(X_s, Y_s^p, Z_s^{p,n})/2\mathrm{d}A_s \right| \\
T_{i,4}^{n,p} &= \sup_{t\in[0,T]} \left| \int_0^t g_i(X_s, Y_s^p, Z_s^{p,n}) - g_i(X_s, Y_s^p, Z_s^p)\mathrm{d}A_s \right|.
\end{aligned}$$

For $i \in \{b, d\}$ the sequence $(T_{i,4}^{n,p})_{n\geq 0}$ obviously converges to 0 in probability by almost sure convergence as n goes to infinity.

The sequence $(\int_0^{\cdot} K^{-2}\phi_s^{K,i}\mathrm{d}A_s^K)_{K\geq 0}$ converges towards $A/2$ in probability for the Skorohod topology and satisfy the P-UT condition by Proposition VI-6.12 in [JS13]. So for any n, $(T_{i,3}^{n,K,p})_{K\geq 0}$ converges towards 0 in probability (for the Skorohod topology) as a consequence of Theorem VI-6.22 in [JS13].

For the second term we write

$$\begin{aligned}|K^2 g_b^K(X^K, Y^{K,p}, Z^{p,n}/K) - g_b(X, Y^p, Z^{p,n})| \\ \leq |K^2 g_b^K(X^K, Y^{K,p}, Z^{p,n}/K) - K^2 g_b^K(X, Y^p, Z^{p,n}/K)| \\ + |K^2 g_b^K(X, Y^p, Z^{p,n}/K) - g_b(X, Y^p, Z^{p,n})|.\end{aligned}$$

So by assumptions of Lemma 1 and Assumption 2 we get:

$$\begin{aligned}|K^2 g_b^K(X^K, Y^{K,p}, Z^{p,n}/K) - g_b(X, Y^p, Z^{p,n})| \\ \leq C|X^K - X| + L|Y^{K,p} - Y^p| + \upsilon_K(|X|^2 + |Y^p|^2 + n^2).\end{aligned}$$

Thus there exits $\tilde{C} > 0$

$$\begin{aligned}|T_{b,2}^{n,K,p}| + |T_{d,2}^{n,K,p}| \leq \tilde{C} A_T^K \Big(\sup_{t\in[0,T]} |X_t^K - X_t| + \sup_{t\in[0,T]} |Y_t^{K,p} - Y_t^p| \\ + \upsilon_K(\sup_{t\in[0,T]} |X_t|^2 + \sup_{t\in[0,T]} |Y_t^p|^2 + n^2) \Big)\end{aligned}$$

which obviously goes to 0 in probability when $K \to +\infty$ according to Slutsky's theorem, in view of the induction hypothesis and since $(\upsilon_K)_{K\geq 0}$ goes to 0.

Finally we write:

$$|K^2 g_b^K(X_s^K, Y_s^{K,p}, Z_s^{K,p}) - K^2 g_b^K(X_s^K, Y_s^{K,p}, Z_s^{p,n}/K)|^2 \leq L^2 |K Z_s^{K,p,b} - Z_s^{p,n}|^2.$$

So we have

$$|T_{b,1}^{n,K,p}| + |T_{d,1}^{n,K,p}| \leq \sup_{t\in[0,T]} L^2 \int_0^t \big((K Z_s^{K,p})^2 + (Z_s^{p,n})^2 - 2 Z_s^{p,n} K Z_s^{K,p} \big) \cdot K^{-2} \phi_s^K \mathrm{d}A_s^K$$

Taking the average and going to the upper limit in K we get by induction hypothesis and from Theorem VI-6.22 in [JS13] that

$$\begin{aligned}\limsup_{K\to+\infty} \mathbb{E}[|T_{b,1}^{n,K,p}| + |T_{d,1}^{n,K,p}|] \leq L^2 \mathbb{E}\big[\sup_{t\in[0,T]} \int_0^t \big((Z_s^p)^2 + (Z_s^{p,n})^2 - 2 Z_s^{p,n} Z_s^p \big) \mathrm{d}A_s \big] \\ \leq L^2 \mathbb{E}\big[\sup_{t\in[0,T]} \int_0^t (Z_s^p - Z_s^{p,n})^2 \mathrm{d}A_s \big].\end{aligned}$$

The RHS converges to 0 when $n \to +\infty$ by dominated convergence. Hence we have shown that $(\chi^{K,p})_{K\geq 0}$ converges to χ^p in probability for the uniform convergence.

To conclude we show that $(|\chi^{K,p}| + |\chi^p|)_{K\geq 0}$ is bounded in $S^{2+\hat{\varepsilon}_p}$. We write

$$\sup_{t\in[0,T]} |\chi_t^{K,p}| \leq C A_T^K \Big(1 + \sup_{t\in[0,T]} |X_t^K| + \sup_{t\in[0,T]} |Y_t^K| \Big) + C \int_0^T |K Z_s^K| \cdot \frac{\phi_s^K}{K^2} \mathrm{d}A_s^K.$$

Using Kunita-Watanabe it is easy to see that the last term is bounded in $L^{2+\hat{\varepsilon}_p}$. The other terms are bounded in $L^{2+\hat{\varepsilon}_p}$ by induction assumption and Proposition 1. So $(\chi^{K,p})_{K\geq 0}$ is bounded in $\mathscr{S}_1^{2+\hat{\varepsilon}_p}$. In the same way we show that $\chi^p \in \mathscr{S}_1^{2+\hat{\varepsilon}_p}$. Therefore we obtain the convergence of $(\chi^{K,p})_{K\geq 0}$ towards χ^p in $\mathscr{S}_1^{2+\tilde{\varepsilon}_p}$ by dominated convergence.

6.4 Proof of Theorem 3

From Assumption 4 (i)-(ii) we get that the generator g^K satisfies the conditions of Lemma 1. Therefore $(\mathbf{BSDE})_K$ admits a unique solution $(Y^K, Z^K) \in \mathbb{S}^K$. We consider $\alpha_t^{K,*} = \alpha^{K,*}(X_t^K, Z_t^K)$, and show that $\alpha^{K,*}$ solve the optimal control problem $(\mathbf{C})_K$. Since $\alpha^{K,*}$ is admissible according to Assumption 4 (iii) we have $J_0^{K,\alpha^{K,*}} = Y_0^K$.

We now take any $\alpha \in \mathscr{A}^K$ and show that

$$J_0^{K,\alpha^{K*}} \geq J_0^{K,\alpha}.$$

We write:

$$\begin{aligned} J_0^{K,\alpha^{K*}} = \xi^K + \int_0^T \big(c^K(X_t^K, \alpha_t^{K*}) + Z_t^{K,d} h^K(X_t^K, \alpha_t^{K*}) - c^K(X_t^K, \alpha_t) - Z_t^{K,d} h^K(X_t^K, \alpha_t)\big)\mathrm{d}s \\ + \int_0^T \big(c^K(X_t^K, \alpha_t) + Z_t^{K,d} h(X_t^K, \alpha_t)\big)\mathrm{d}s - \int_0^T Z_s^K \cdot \mathrm{d}M_s^K. \end{aligned}$$

By definition the first integrand term is almost surely non negative and therefore we have

$$J_0^{K,\alpha^{K*}} \geq \xi^K + \int_0^T \big(c^K(X_t^K, \alpha_t) + Z_t^{K,d} h^K(X_t^K, \alpha_t)\big)\mathrm{d}s - \int_0^T Z_s^K \cdot \mathrm{d}M_s^K,$$

or equivalently

$$J_0^{K,\alpha^{K*}} \geq \xi^K + \int_0^T c^K(X_t^K, \alpha_t)\mathrm{d}s - \int_0^T Z_s^K \cdot \mathrm{d}M_s^{K,\alpha}.$$

Taking the expectation with respect to $\mathbb{P}^{K,\alpha}$ we get the result.

6.5 Proof of Theorem 4

From Assumption 6 (i)-(ii) we get that the generator g satisfies the conditions of Lemma 2. Therefore (**BSDE**) admits a unique solution $(Y, Z) \in \mathbb{S}$. We consider $\alpha_t^* = \alpha^*(X_t, Z_t)$, and show that α^* solve the optimal control problem (**C**). Since α^* is admissible according to Assumption 6 (iii) we have $J_0^{\alpha^*} = Y_0$.

We now take any $\alpha \in \mathscr{A}$ and show that

$$J_0^{\alpha^*} \geq J_0^{\alpha}.$$

We write:

$$\begin{aligned} J_0^{\alpha^*} = \xi + \int_0^T \big(c(X_t, \alpha_t^*) + Z_t h(X_t, \alpha_t^*) - c(X_t, \alpha_t) - Z_t h(X_t, \alpha_t)\big)\mathrm{d}s \\ + \int_0^T \big(c(X_t, \alpha_t) + Z_t h(X_t, \alpha_t)\big)\mathrm{d}s - \int_0^T Z \mathrm{d}M_s^X. \end{aligned}$$

By definition the first integrand term is almost surely non negative and therefore we have

$$J_0^{\alpha^*} \geq \xi + \int_0^T \big(c(X_t, \alpha_t) + Z_t^d h(X_t, \alpha_t)\big)\mathrm{d}s - \int_0^T Z_s \cdot \mathrm{d}M_s^X,$$

or equivalently

$$J_0^{\alpha^*} \geq \xi + \int_0^T c(X_t, \alpha_t)\mathrm{d}s - \int_0^T Z_s \mathrm{d}M_s^\alpha.$$

Taking the expectation with respect to $\mathbb{P}^\alpha$ we get the result.

6.6 Proof of Theorem 5

According to Assumption 7 the sequences $(\xi^K)_{K\geq 0}$ and $(g^K)_{K\geq 0}$ satisfy the assumptions of Lemma 4 for any K and Assumption 2. So from Theorem 2 we have in $\mathscr{S}_1^2 \times \mathscr{S}_1^1 \times \mathscr{S}_1^1$

$$\Big(Y^K, \int_0^\cdot Z_s^{K,d} \lambda_s^{k,d} K^{-1}\mathrm{d}s, \int_0^\cdot |KZ_s^{K,d}|^2 \lambda_s^{k,d} K^{-2}\mathrm{d}s\Big) \\ \to \Big(Y, \int_0^\cdot Z_s \sigma^2 X_s/2\mathrm{d}s, \int_0^\cdot Z_s^2 \sigma^2 X_s/2\mathrm{d}s\Big) \text{ as } K \to +\infty. \tag{6}$$

6.6.1 Proof of point (i)

We write

$$\Big|\int_0^T \alpha_t^{K,*} \lambda_s^{K,d} K^{-2}\mathrm{d}s - \int_0^T \alpha_t^* \mathrm{d}A_s/2\Big| \leq \int_0^T |\alpha_t^{K,*} - \alpha^{K,*}(X_t, Z_t/K)|\lambda_s^{K,d} K^{-2}\mathrm{d}s \\ + \Big|\int_0^T \alpha^{K,*}(X_t, Z_t/K)\lambda_s^{K,d} K^{-2}\mathrm{d}s - \int_0^T \alpha^{K,*}(X_t, Z_t/K)\mathrm{d}A_s/2\Big| \\ + \int_0^T |\alpha^{K,*}(X_t, Z_t/K) - \alpha_t^*|\mathrm{d}A_t/2.$$

The second term converges towards 0 by Theorem 1. The last one terms goes to 0 from Assumption 7 (ii). Using Assumption 7 (iii) we can dominate the first term by

$$\int_0^T |\alpha_t^{K,*} - \alpha^{K,*}(X_t, Z_t/K)|\lambda_s^{K,d} K^{-2}\mathrm{d}s \leq A_T^K \int_0^T |\alpha_t^{K,*} - \alpha^{K,*}(X_t, Z_t/K)|^2 \lambda_s^{K,d} K^{-2}\mathrm{d}s \\ \leq A_T^K \int_0^T \big(|X_t^K - X_t|^2 + |KZ_t^K - Z_t|^2\big)\lambda_s^{K,d} K^{-2}\mathrm{d}s.$$

that goes to 0 according to Theorem 1 and to the convergence (6).

In the same way, using that the control is bounded, we get that in probability

$$\int_0^T |\alpha_t^{K,*}|^2 \lambda_s^{K,d} K^{-2}\mathrm{d}s \to \int_0^T |\alpha_t^*|^2 \mathrm{d}A_s/2 \text{ as } K \to +\infty.$$

We then extend the convergences to $\mathscr{S}_2^1$ by uniform integrability since the control is bounded. Thus we get the first statement of Theorem 5.

6.6.2 Proof of point (ii)

We consider $(t_i)_{1\le i\le n}\in[0,T]^n$ and a bounded continuous function f defined from $\mathbb{R}^n$ into $\mathbb{R}$. We show that

$$\mathbb{E}^{K,*}[f(X^K_{t_1},\dots,X^K_{t_n})]\to\mathbb{E}^*[f(X_{t_1},\dots,X_{t_n})]\text{ as }K\to+\infty$$

where $\mathbb{E}^{K,*}$ (resp. $\mathbb{E}^*$) denotes the expectation under the control $\alpha^{K,*}$ (resp. α^*). We write

$$\mathbb{E}^{K,*}[f(X^K_{t_1},\dots,X^K_{t_n})]=\mathbb{E}[f(X^K_{t_1},\dots,X^K_{t_n})L_T^{K,\alpha^{K,*}}]$$

and

$$\mathbb{E}^*[f(X^K_{t_1},\dots,X^K_{t_n})]=\mathbb{E}[f(X_{t_1},\dots,X_{t_n})L_T^{\alpha^*}].$$

Suppose we have shown that $(L_T^{K,\alpha^{K,*}})_{K\ge0}$ converges in probability surely towards L_T^*. Then writing

$$|L_T^{\alpha^*}-L_T^{k,\alpha^{K,*}}|=2(L_T^{\alpha^*}-L_T^{k,\alpha^{K,*}})_+-(L_T^{\alpha^*}-L_T^{k,\alpha^{K,*}})$$

we get that $(L_T^{k,\alpha^{K,*}})_{K\ge0}$ converges towards $L_T^{\alpha^*}$ in L^1 by dominated converges and since

$$\mathbb{E}[L_T^{\alpha^*}]=\mathbb{E}[L_T^{K,\alpha^{K,*}}]=1.$$

Then we conclude noticing that:

$$|f(X_{t_1},\dots,X_{t_n})L_T^{\alpha^*}-f(X^K_{t_1},\dots,X^K_{t_n})L_T^{K,\alpha^{K,*}}|\le|f(X_{t_1},\dots,X_{t_n})-f(X^K_{t_1},\dots,X^K_{t_n})|L_T^{\alpha^*}+\|f\|_\infty|L_T^{\alpha^*}-L_T^{k,\alpha^{K,*}}|$$

We finally prove the convergence of $(L_T^{K,\alpha^{K,*}})_{K\ge0}$ towards L_T^* in probability. We introduce the following sequences

$$\begin{aligned}
\varepsilon_1^K&=\int_0^T\log\Big(1+\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}\Big)\mathrm{d}N_s^{K,d}-\int_0^T\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}-\frac12\Big(\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}\Big)^2\mathrm{d}N_s^{K,d},\\
\varepsilon_2^K&=\int_0^T\Big(\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}\Big)^2\mathrm{d}N_s^{K,d}-\int_0^T\Big(\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}\Big)^2\lambda_s^{K,d}\mathrm{d}s,\\
\varepsilon_3^K&=\int_0^T\Big(\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}\Big)^2\lambda_s^{K,d}\mathrm{d}s-\int_0^T\Big(\frac{h(X_s,\alpha_s^*)}{\sigma^2X_s/2}\Big)^2\mathrm{d}A_s/2,\\
\varepsilon_4^K&=\int_0^T\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}\mathrm{d}M_s^{K,d}-\int_0^T\frac{h(X_s,\alpha_s^*)}{\sigma^2X_s/2}\mathrm{d}M_s^d
\end{aligned}$$

and show that they all converges to 0 in probability.

For some $C>0$ independent of K we have

$$|\varepsilon_1^K|\le C\int_0^T\Big(\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}\Big)^3\mathrm{d}N_s^{K,d}\mathbf{1}_{\sup_{s\in[0,T]}|\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}|<1}+|\varepsilon_1^K|\mathbf{1}_{\sup_{s\in[0,T]}|\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}|>1}.$$

The first term of the RHS converges towards 0 in probability according to Markov inequality. The second one since $\sup_{s\in[0,T]} |\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}|$ converges almost surely towards 0 from Assumption 7-(i). Remark that

$$\varepsilon_2^K = \int_0^T \Big(\frac{h^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}\Big)^2 \mathrm{d}M_s^{K,d}$$

Consequently using Assumption 7 (iii) and Tchebychev inequality we get that $(\varepsilon_2^K)_{K\geq 0}$ converges towards 0 in probability. Notice that we have

$$\begin{aligned}
\varepsilon_3^K = &\int_0^T \Big[\Big(\frac{Kh^K(X_s^K,\alpha_s^{K,*})}{\lambda_s^{K,d}}\Big)^2 - \Big(\frac{Kh^K(X_s,\alpha^{K,*}(X_s,Z_s/K))}{\lambda^{K,d}(X_s)}\Big)^2\Big] K^{-2}\lambda_s^{K,d}\mathrm{d}s \\
&+ \int_0^T \Big(\frac{Kh^K(X_s,\alpha^{K,*}(X_s,Z_s/K))}{\lambda^{K,d}(X_s)}\Big)^2 K^{-2}\lambda_s^{K,d}\mathrm{d}s - \int_0^T \Big(\frac{Kh^K(X_s,\alpha^{K,*}(X_s,Z_s/K))}{\lambda^{K,d}(X_s)}\Big)^2 \mathrm{d}A_s/2 \\
&+ \int_0^T \Big(\frac{Kh^K(X_s,\alpha^{K,*}(X_s,Z_s/K))}{\lambda^{K,d}(X_s)}\Big)^2 - \Big(\frac{h(X_s,\alpha_s^*)}{\sigma^2 X_s/2}\Big)^2 \mathrm{d}A_s/2.
\end{aligned}$$

The second and last terms go to 0 in probability by Theorem 1, Assumption 7 (ii) and from Proposition VI-6.12 and Theorem VI-6.22 in [JS13]. As we did in the proof of Theorem 5 (i), the first term goes to 0 from Cauchy Schwarz inequality Assumption 7 (iii) together with the convergence (6).

Finally we write

$$\begin{aligned}
\varepsilon_4^K \leq &\Big|\int_0^T \frac{Kh^K(X_s^K,\alpha_s^{K,*})}{\lambda^{K,d}(X_s^K)} - \frac{Kh^K(X_s,\alpha_s^*)}{\lambda^{K,d}(X_s)}\mathrm{d}M_s^{K,d}\Big| \\
&+ \Big|\int_0^T \frac{Kh^K(X_s,\alpha_s^*)}{\lambda^{K,d}(X_s)}\mathrm{d}\overline{M}_s^K - \int_0^T \frac{Kh^K(X_s,\alpha_s^*)}{\lambda^{K,d}(X_s)}\mathrm{d}M_s\Big| \\
&+ \Big|\int_0^T \frac{h(X_s,\alpha_s^*)}{\sigma^2 X_s/2} - \frac{Kh^K(X_s,\alpha_s^*)}{\lambda^{K,d}(X_s)}\mathrm{d}M_s\Big|
\end{aligned}$$

The second and last terms converge towards 0 by Assumption 7 (ii), Theorem 1, Proposition VI-6.12, Theorem VI-6.22 in [JS13] and Theorem 5. Using Ito's isometry, Cauch-Schwarz inequality, Assumption 7 (iii) together with Theorem 1 and Theorem 5 (i) we get that the first term goes to 0 in probability. Therefore $(\varepsilon_4^K)_{K\geq 0}$ converges towards 0 in probability.

Thus we conclude that $(L_T^{K,\alpha^{K,*}})_{K\geq 0}$ converges toward $L_T^{\alpha^*}$ in probability.

V.A Spaces and notations

- For any $K \geq 0$ we consider the sets.
  - $L^2(M^K)$ is the set of $\mathscr{F}^K$ predictable process $\mathbb{R}^2$ valued Z such that
$$\|Z\|^2_{L^2(M^K)} = \mathbb{E}[\int_0^T |Z_s|^2 \cdot \phi_s^K \mathrm{d}A_s^K] < +\infty.$$
  - $\mathbb{T}^K$ is the set of $\mathscr{F}_T^K$ measurable $\mathbb{R}$ valued random variable ξ such that
$$\|\xi\|^2_{\mathbb{T}^K} = \mathbb{E}[e^{\beta A_T^K}|\xi|^2] < +\infty.$$
  - $\mathbb{K}^K$ is the set of $\mathbb{F}^K$-optional $\mathbb{R}$ valued process Y such that
$$\|Y\|^2_{\mathbb{K}^K} = \mathbb{E}[e^{\beta A_T^K} \sup_{t\in[0,T]} |Y_t|^2] < +\infty.$$
  - $\mathbb{H}_2^K$ is the set of $\mathbb{F}^K$-predictable $\mathbb{R}^2$ valued process Z such that
$$\|Z\|^2_{\mathbb{H}^K} = \mathbb{E}[\int_0^T e^{\beta A_t^K} Z_s^2 \cdot \phi_t^K \mathrm{d}A_t^K] < +\infty \text{ with } Z^2 = (Z_1^2, Z_2^2).$$
  - $\mathbb{H}_1^K$ is the set of $\mathbb{F}^K$-predictable $\mathbb{R}$ valued process Y such that
$$\|Y\|^2_{\mathbb{H}_1^K} = \mathbb{E}[\int_0^T e^{\beta A_t^K} |Y_s^t|^2 \mathrm{d}A_t^K] < +\infty.$$
  - $\mathbb{S}^K$ is the set of pair $(Y,Z) \in \mathbb{H}_1^K \times \mathbb{H}_2^K$, we note $\|(Y,Z)\|^2_{\mathbb{S}^K} = \|Y\|^2_{\mathbb{H}_1^K} + \|Z\|^2_{\mathbb{H}_2^K}$.
- We also consider the sets related to the continuous model.
  - $L^2(M^X)$ is the set of $\mathscr{F}^X$ predictable process $\mathbb{R}$ valued Z such that
$$\|Z\|^2_{L^2(M^X)} = \mathbb{E}[\int_0^T |Z_s|^2 \mathrm{d}A_s] < +\infty.$$
  - $\mathbb{T}$ is the set of $\mathscr{F}_T^X$ measurable $\mathbb{R}$ valued random variable ξ such that
$$\|\xi\|^2_{\mathbb{T}} = \mathbb{E}[e^{\beta A_T}|\xi|^2] < +\infty.$$
  - $\mathbb{K}$ is the set of $\mathbb{F}^X$-optional $\mathbb{R}$ valued process Y such that
$$\|Y\|^2_{\mathbb{K}} = \mathbb{E}[e^{\beta A_T} \sup_{t\in[0,T]} |Y_t|^2] < +\infty.$$
  - $\mathbb{H}$ is the set of $\mathbb{F}^X$-predictable $\mathbb{R}$ valued process Z such that
$$\|Z\|^2_{\mathbb{H}} = \mathbb{E}[\int_0^T e^{\beta A_t} Z_s^2 \mathrm{d}A_t] < +\infty.$$

  - $\mathbb{S}$ is the set of pair $(Y,Z) \in \mathbb{K} \times \mathbb{H}$, we note $\|(Y,Z)\|_{\mathbb{S}}^2 = \|Y\|_{\mathbb{K}}^2 + \|Z\|_{\mathbb{H}}^2$.

- Finally we consider the sets:

  - L^p the set of real valued random variable Z such that

$$\|Z\|_{L_p}^p = \mathbb{E}[|Z|^p] < +\infty$$

  - $\mathscr{S}_d^p$ is the set of $\mathbb{F}$-predictable $\mathbb{R}^d$ valued process X such that

$$\|X\|_p^p = \mathbb{E}[\sup_{t\in[0,T]} \|X_t\|^p] < +\infty.$$

V.B Change of measure for initial population

We consider $m \in \mathbb{R}_+^*$ and $n \in [0,m)$ and define the process

$$Q_t^{K,n,m} = \int_0^t \sum_{i\in\{b,d\}} \frac{\lambda_s^{n,K,i} - \lambda_s^{m,K,i}}{\lambda_s^{m,K,i}} \mathbf{1}_{X_s^{m,k}>0} \mathrm{d}M_s^{K,i}.$$

We have $|\Delta Q^{K,n,m}| \leq 1$ and therefore $\Delta Q^{K,n,m} \geq 1$. Moreover from Assumption 1 we have for some constant C positive

$$\begin{aligned}
\langle Q^{K,n,m}\rangle_t &= \int_0^t \sum_{i\in\{b,d\}} \frac{|\lambda_s^{n,K,i} - \lambda_s^{m,K,i}|^2}{\lambda_s^{m,K,i}} \mathbf{1}_{X_s^{m,K}>0} \mathrm{d}s \\
&\leq \int_0^t \sum_{i\in\{b,d\}} \frac{C(K^2+K)|n-m|^2}{K^2\underline{\eta} X_s^{m,K}} \mathbf{1}_{X_s^{m,K}>0} \mathrm{d}s \\
&\leq \int_0^t \sum_{i\in\{b,d\}} \frac{C(K^2+K)|n-m|^2}{K^2\underline{\eta} X_{min}^{m,K}} \mathrm{d}s,
\end{aligned}$$

where $X_{min}^{m,K} > 0$ is the lowest positive value that the process $X^{m,K}$ can take. Therefore by Theorem 2.4 in [Sok13] the process $L^{n,m}$ is a uniformly integrable martingale.

Moreover according to Theorem III-3.11 in [JS13] under the probability $\mathbb{P}^{K,m}$ for $i \in \{b,d\}$ the processes

$$M^{K,m,i} - \langle Q^{K,n,m}, M^{K,m,i}\rangle$$

are local martingales. Finally we conclude since

$$\begin{aligned}
M_t^{K,m,i} - \langle Q^{K,n,m}, M^{K,m,i}\rangle_t &= N_t^i - \int_0^t \lambda_s^{m,K,i} - (\lambda_s^{n,K,i} - \lambda_s^{m,K,i}) \mathbf{1}_{X_s^{m,K}>0} \mathrm{d}s \\
&= N_t^i - \int_0^t \lambda_s^{n,K,i} \mathbf{1}_{X_s^{m,K}>0} + \lambda_s^{m,K,i} \mathbf{1}_{X_s^{m,K}=0} \mathrm{d}s \\
&= N_t^i - \int_0^t \lambda_s^{n,K,i} \mathrm{d}s \\
&= M_t^{K,n,i}.
\end{aligned}$$

V.C Admissibility of the controls in the toy model

V.C.1 Discrete models

We show that the control $\alpha^{K,*}$ is admissible. We have

$$\lambda_t^{K,d,\alpha^{K,*}} = KX_{t-}^K(\mu + K\sigma^2) + KX_{t-}^K\big(-2a_K(t) + \frac{a_K(t)}{X_{t-}^K K} - \frac{b_K(t)}{X_{t-}^K}\big)\mathbf{1}_{X_{t-}^K>0}.$$

By [Jac75] the probability $\mathbb{P}^{K,\alpha^{K,*}}$ exists. We recall that we have chosen T small enough such that a_K is negative and b_K positive on $[0,T]$. Hence we have

$$\begin{aligned}\lambda_t^{K,d,\alpha^{K,*}} &\geq KX_{t-}^K(\mu + K\sigma^2) + K\big(-2a_K(t)X_{t-}^K + \frac{a_K(t)}{K} - b_K(t)\big)\mathbf{1}_{X_{t-}^K>0}\\ &\geq X_{t-}^K K\Big(\mu - a_K(t)\mathbf{1}_{X_{t-}^K>0} + K\sigma^2 - Kb_K(t)\Big).\end{aligned}$$

We can always assume that T is small enough so that we can assume that for any $t\in[0,T]$, $\sigma^2 - b_K(t) > 0$. So $\lambda^{K,d,\alpha^{K,*}}$ is $\mathbb{P}^{K,\alpha^{K,*}}$ almost surely non negative and the control $\alpha^{K,*}$ is admissible.

V.C.2 Continuous models

We have

$$X_t\alpha_t^* = \big(-2a(t)X_t - b(t)\big)\mathbf{1}_{X_t>0}.$$

So the SDE

$$\mathrm{d}X_t = X_t(\nu - \mu - \alpha_t^*)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}W_t$$

writes

$$\mathrm{d}X_t = \Big(X_t(\nu-\mu) - \big(-2a(t)X_t - b(t)\big)\mathbf{1}_{X_t>0}\Big)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}W_t,\ X_0 = x_0.$$

Obviously this SDE admits a unique strong solution given by $Y_t\mathbf{1}_{\inf_{s\in[0,t]}Y_s>0}$ where Y is the unique strong solution of

$$\mathrm{d}Y_t = \big(Y_t(\nu-\mu) + 2a(t)Y_t + b(t)\big)\mathrm{d}t + \sigma\sqrt{Y_t}\mathrm{d}W_t,\ Y_0 = x_0.$$

V.D Feller property of the model

We consider a non negative real x. We obviously have that when $t\to 0$ the $X_t^{K,x}$ converges almost surely towards x. Now we consider a non negative sequence $(x_n)_{n\geq 0}$ that converges towards x and show that for any $t>0$, $(X_t^{K,x_n})_{n\geq 0}$ converges in law towards $X_t^{K,x}$. We fix x_0 larger than x and any of the x_n and f a bounded continuous function on $\mathbb{R}_+$.

We write

$$\mathbb{E}^{K,x_n}[f(X_t^{K,x_n})] = \mathbb{E}^{K,x_0}[f(X_t^{K,x_n})L_t^{K,x_n,x_0}] \text{ and } \mathbb{E}^{K,x}[f(X_t^{K,x})] = \mathbb{E}^{K,x_0}[f(X_t^{K,x})L_t^{K,x,x_0}],$$

and

$$|f(X_t^{K,x_n})L_t^{K,x_n,x_0} - f(X_t^{K,x})L_t^{K,x,x_0}| \leq |f(X_t^{K,x_n}) - f(X_t^{K,x})|L_t^{K,x,x_0} + |f(X_t^{K,x_n})||L_t^{K,x_n,x_0} - L_t^{K,x,x_0}|.$$

The first term of the right hand side (RHS for short) goes to 0 by dominated convergence. We can dominate the second on by

$$\|f\|_\infty |L_t^{K,x_n,x_0} - L_t^{K,x,x_0}|$$

that converges towards 0 according to Scheffé's lemma. Therefore our model has the Feller property.

V.E Martingale representation with respect to M^X

We show in this section that any $(\mathbb{F}^X, \mathbb{P})-$martingale has the representation property relative to M^X.

We set $\mathscr{H} = \mathscr{F}_0^X$ and $\mathbb{P}_0 = \varepsilon_{X_0=x_0}$, *i.e.* the probability measure on $\mathscr{H}$ such that that $\mathbb{P}_0(X_0 = x_0) = 1$. For $\widetilde{X}$ a càdlàg process adapted to the filtration $\mathbb{F}^X$ and B and C two $\mathbb{F}^X$-predictable processes with finite variation such that $B_0 = C_0 = 0$ we recall the definition of the martingale problem associated with $(\mathscr{H}, \widetilde{X})$ and $(\mathbb{P}_0, B, C)$.

Definition 3 (Definition III-2.6 in [JS13]). *A solution to the martingale problem associated with $(\mathscr{H}, \widetilde{X})$ and $(\mathbb{P}_0, \widetilde{B}, C)$ is a probability measure $\mathbb{Q}$ on $(\Omega, \mathbb{F}^X)$ such that*

- *the restriction of $\mathbb{Q}$ to $\mathscr{H}$ equals $\mathbb{P}_0$,*
- *$\widetilde{X}$ is a semi-martingale on $(\Omega, \mathbb{F}^X, \mathbb{Q})$ with characteristics (B, C).*

We denote by $\mathbf{s}(\mathscr{H}, \widetilde{X}|\mathbb{P}_0; B, C)$ the set of solutions to this martingale problem.

From this definition we see that the projection of $\mathbb{P}$ on $\mathbb{F}^K$ is a solution of $\mathbf{s}(\mathscr{H}, X|\mathbb{P}_0; D, A)$ where

$$D_t = \int_0^t f(X_s)\mathrm{d}s \text{ and } A_t = \int_0^t \sigma^2 X_s \mathrm{d}s.$$

We have $M^X = X_t - D_t$ so that M^X is a $\mathbb{F}^X$-adapted process and it makes sense to consider the set $\mathbf{s}(\mathscr{H}, M^X|\mathbb{P}_0; 0, A)$. We show that

$$\mathbf{s}(\mathscr{H}, M|\mathbb{P}_0; 0, A) = \mathbf{s}(\mathscr{H}, X|\mathbb{P}_0; D, A) \tag{7}$$

and that $\mathbf{s}(\mathscr{H}, X|\mathbb{P}_0; D, A)$ is reduced to a singleton. This will be enough to conclude according to Theorem III-4.29 in [JS13].

Consider $\mathbb{Q} \in \mathbf{s}(\mathscr{H}, M|\mathbb{P}_0; 0, C)$. We have $X = M^X + A$ and since D and C are continuous process with finite variation, we deduce that under $\mathbb{Q}$ the characteristics of X are (A, C). Conversely, if $\mathbb{Q} \in \mathbf{s}(\mathscr{H}, X|\mathbb{P}_0; D, A)$ then by recalling that $M^X = X - D$ we obtain $\mathbb{Q} \in \mathbf{s}(\mathscr{H}, M|\mathbb{P}_0; 0, A)$. Hence,

(7) holds.

Since (**S**) admits a unique strong solution it admits a unique solution in law (see Theorem IX-1.7 [RY13]). Therefore from Theorem III-2.26 in [JS13] the set $\mathbf{s}(\mathscr{H}, X|\mathbb{P}_0; D, C)$ is reduced to a singleton. As a consequence of (7), the set $\mathbf{s}(\mathscr{H}, M^X|\mathbb{P}_0; 0, C)$ is also reduced to a singleton. Therefore, we deduce from Theorem III-4.29 that all $(\mathbb{F}^X, \mathbb{P})$-martingales have the representation property relative to M^X.

V.F Proof of Proposition 1

For two non negative reals ν and μ we say that N is a linear branching process with birth rate ν and intensity μ if it can be written as $N = N^b - N^d$ where N^b and N^d are two counting processes with respective intensity νN and μN. This corresponds to a branching process as defined in Section III-3.3.1 in [Mél16] with parameters $a = \nu + \mu$, $p_0 = \frac{\mu}{\nu+\mu}$ and $p_2 = \frac{\nu}{\nu+\mu}$.

To prove Proposition 1 we proceed in two steps:

- Step 1: We prove a result similar to Proposition 1 for linear branching process.
- Step 2: We show that a under some assumption a population processes is almost surely dominated by a linear branching process.
- Step 3: We conclude using the previous steps.

V.F.1 Step 1: exponential moments for linear branching processes

We consider N a linear branching process with birth and death rate given by ν and μ.

We define the function F from $\mathbb{N} \times (\mathbb{R}_+^*)^2 \times \mathbb{R}_+$ into $\overline{\mathbb{R}_+}$ by

$$F(n, \beta, t) = \mathbb{E}_n[e^{\beta_1 \int_0^t N_s \mathrm{d}s + \beta_2 N_t}]$$

where $\mathbb{E}_n$ is the expectation taker under the probability law that corresponds to initial condition population of size n. We have the following lemma:

Lemma 4. *For any $\beta \in \mathbb{R}_+^2$ consider*

$$\gamma_{\nu,\mu,\beta} = \frac{\nu + \mu - \beta_1}{2\nu},\ \phi_{\nu,\mu,\beta} = \nu\gamma_{\nu,\mu,\beta}^2 - \mu,\ \Delta_{\nu,\mu,\beta} = \sqrt{\frac{\nu}{\phi_{\nu,\mu,\beta}}}(e^{\beta_2} - \gamma_{\nu,\mu,\beta}),$$

$$\alpha_{\nu,\mu,\beta} = \log\Big(\Big|\frac{\Delta_{\nu,\mu,\beta} - 1}{\Delta_{\nu,\mu,\beta} + 1}\Big|\Big) \text{ and } t^*_{\nu,\mu,\beta} = \frac{-\alpha_{\nu,\mu,\beta}}{2\sqrt{\nu\phi_{\nu,\mu,\beta}}}.$$

With those notations if β satisfies $\phi_{\nu,\mu,\beta} > 0$ and $\Delta_{\nu,\mu,\beta} > 1$, we have for any $t \in [0, t^*_{\nu,\mu,\beta})$

$$F(n, \beta, t) = \Big(\sqrt{\frac{\phi_{\nu,\mu,\beta}}{\nu}}\Big(\frac{2}{1 - exp(\alpha_{\nu,\mu,\beta} + 2\sqrt{\nu\phi_{\nu,\mu,\beta}}t)} - 1\Big) + \gamma_{\nu,\mu,\beta}\Big)^n.$$

Note that since $(0,0)$ satisfies the above conditions, they also hold for β_1 and β_2 small enough.

Proof of Lemma 4. Consider a population starting with one individual. We call τ the lifetime of this particle and $C = 1$ or 2 the size his offspring. Since all particles are independent and follow the same law we can consider that:

$$e^{\beta_1 \int_0^t N_s \mathrm{d}s + \beta_2 N_t} = \mathbf{1}_{\tau > t} e^{\beta_1 t + \beta_2} + \mathbf{1}_{\tau \le t} e^{\beta_1 \tau} \prod_{i=1}^{C} e^{\beta_1 \int_\tau^t N^{(i)}_{s-\tau} \mathrm{d}s + \beta_2 N^{(i)}_{t-\tau}} \tag{8}$$

where $(N^{(i)})_{1 \le i \le 2}$ are independent copies of N.

Consider the stopping times

$$T_n = \inf\{s > 0 \text{ s.t. } N_s = n\} \text{ and } T_n^{(i)} = \inf\{s > 0 \text{ s.t. } N_s^{(i)} = n\} \text{ for } i = 1, 2.$$

From Equation (8) we get

$$e^{\beta_1 \int_0^t N_s^{T_n} \mathrm{d}s + \beta_2 N_t^{T_n}} \le \mathbf{1}_{\tau > t} e^{\beta_1 t + \beta_2} + \mathbf{1}_{\tau \le t} e^{\beta_1 \tau} \prod_{i=1}^{C} e^{\beta_1 \int_\tau^t N^{(i) T_n^{(i)}}_{s-\tau} \mathrm{d}s + \beta_2 N^{(i) T_n^{(i)}}_{t-s}}$$

and taking the average we have

$$F_n(\beta, t) \le e^{-(\nu+\mu)t} e^{\beta_1 t + \beta_2} + \int_0^t (\nu + \mu) e^{-(\nu+\mu)s} e^{\beta_1 s} \Big(\frac{\nu}{\nu + \mu} F_n^2(\beta, t - s) + \frac{\mu}{\nu + \mu} \Big) \mathrm{d}s.$$

where $F_n(\beta, t) = \mathbb{E}_1[e^{\beta_1 \int_0^t N_s^{T_n} \mathrm{d}s + \beta_2 N_t^{T_n}}]$. We therefore consider the following ODE:

$$(R)_{\nu,\mu,\beta} : f' = \nu f^2 - (\nu + \mu - \beta_1) f + \mu, \ f(0) = e^{\beta_2}.$$

We show that $(R)_{\nu,\mu,\beta}$ has a unique maximal solution defined on $t \in [0, t^*_{\nu,\mu,\beta})$ by

$$f_{\mu,\nu,\beta}(t) = \sqrt{\frac{\phi_{\nu,\mu,\beta}}{\nu}} \Big(\frac{2}{1 - \exp(\alpha_{\nu,\mu,\beta} + 2\sqrt{\nu \phi_{\nu,\mu,\beta}} t)} - 1 \Big) + \gamma_{\nu,\mu,\beta}.$$

Using the change of variable $g = f - \gamma_{\nu,\mu,\beta}$, the ODE $(R)_{\nu,\mu,\beta}$ is equivalent to

$$(R)'_{\nu,\mu,\beta} : \ g' = \phi_{\nu,\mu,\beta} \Big(\frac{\nu}{\phi_{\nu,\mu,\beta}} g^2 - 1 \Big).$$

By Cauchy-Lispchitz theorem this ODE admits a maximal solution g. By hypothesis on β we have

$$\phi_{\nu,\mu,\beta} \Big(\frac{\nu}{\phi_{\nu,\mu,\beta}} g^2(0) - 1 \Big) = \phi_{\nu,\mu,\beta} (\Delta^2_{\nu,\mu,\beta} - 1) > 0.$$

So for all t such that $\frac{\nu}{\phi_{\nu,\mu,\beta}} g^2(t) - 1 > 0$ we can write

$$\frac{g'(t)}{\frac{\nu}{\phi_{\nu,\mu,\beta}} g^2(t) - 1} = \phi_{\nu,\mu,\beta}. \tag{9}$$

We recognize the derivative of

$$x \to \frac{1}{2}\sqrt{\frac{\phi_{\nu,\mu,\beta}}{\nu}}\log\Big(\frac{\sqrt{\frac{\nu}{\phi_{\nu,\mu,\beta}}}x-1}{\sqrt{\frac{\nu}{\phi_{\nu,\mu,\beta}}}x+1}\Big).$$

So integrating on both sides of (9) we have

$$\log\Big(\frac{\sqrt{\frac{\nu}{\phi_{\nu,\mu,\beta}}}g(t)-1}{\sqrt{\frac{\nu}{\phi_{\nu,\mu,\beta}}}g(t)+1}\Big) = \alpha_{\nu,\mu,\beta} + 2\sqrt{\nu\phi_{\nu,\mu,\beta}}\,t.$$

Therefore it is then easy to show that for any $t < t^*_{\nu,\mu,\beta}$ we have

$$g(t) + \gamma_{\nu,\mu,\beta} = \sqrt{\frac{\phi_{\nu,\mu,\beta}}{\nu}}\Big(\frac{2}{1-\exp(\alpha_{\nu,\mu,\beta}+2\sqrt{\nu\phi_{\nu,\mu,\beta}}\,t)} - 1\Big) + \gamma_{\nu,\mu,\beta}.$$

Reciprocally it is easy to show that this function is a maximal solution of $(R)_{\nu,\mu,\beta}$ defined on $[0, t^*_{\nu,\mu,\beta})$.

The function $F_n(\beta,\cdot)$ being continuous a direct application of the Grönwall lemma gives that for any $t \in [0, t^*_{\mu,\nu,\beta})$, $F_n(\beta,t) \le f_{\mu,\nu,\beta}(t)$. By monotone convergence we obtain that $F(1,\beta,t)$ is finite and taking the average in Equation (8) we obtain that $F(1,\beta,\cdot)$ is solution of $(R)_{\mu,\nu,\beta}$ therefore we have $F(1,\beta,t) = f_{\mu,\nu,\beta}(t)$.

Finally if we consider a population N starting with n individual we can consider that

$$N = \sum_{i=1}^{n} N^{(i)}$$

where $(N^{(i)})_{1\le i\le n}$ are independent copies of the branching process starting with one individuals. Therefore for $t < t^*_{\nu,\mu,\beta}$ we get $F(n,\beta,t) = F(1,\beta,t)^n$ which concludes the proof. □

We now consider a sequence of branching process $(N^K)_{K\ge 0}$ with initial condition Kn and parameters

$$\mu^K = \mu + aK \text{ and } \nu^K = \nu + aK.$$

We consider $\beta_K = (\beta_1/K, \beta_2/K)$ such that $(\nu-\mu)^2 > 4a\beta_1$ and note

$$\Lambda_\infty = \Big(\frac{\nu-\mu}{2a} + \beta_2\Big)\frac{2a}{\sqrt{(\nu-\mu)^2-4a\beta_1}},\ \eta = \frac{\sqrt{(\nu-\mu)^2-4a\beta_1}}{2},$$

$$\alpha_\infty := \log(|\frac{\Lambda_\infty - 1}{\Lambda_\infty + 1}|) \text{ and } t^*_\infty = -\frac{\alpha_\infty}{2\eta}.$$

We assume that β_1 and β_2 satisfy $\Lambda_\infty > 1$. Those conditions imply that for K large enough β_K satisfies the assumption in Lemma 4.

To lighten the notations we use the under-script K instead of (ν_K, μ_K, β_K). One can easily show the following convergence or equivalence:

$$1-\gamma_K \sim \frac{\nu-\mu}{2a}\frac{1}{K},\ \frac{\phi_K}{\nu_K} \sim \frac{1}{K^2}\frac{(\nu-\mu)^2-4a\beta_1}{4a^2}, \tag{10}$$

$$\lim_{K\to+\infty}\Lambda_K=\Lambda_\infty,\ \lim_{K\to+\infty}\sqrt{\nu_K\phi_K}=\eta,\ \lim_{K\to+\infty}\alpha_K=\alpha_\infty \text{ and } \lim_{K\to+\infty}t_K^*=t_\infty^*. \tag{11}$$

The convergence of the sequence $(t_K)_{K\geq 0}$ implies that for any $t<t_\infty^*$ and K large enough $F(nK,\beta_K,t)$ is finite. Moreover from (10) and (11) we get that the sequence $\big(F(nK,\beta_K,t)\big)_{K\geq 0}$ converges. More precisely it is easy to check that for any $t<t_\infty^*$ we have

$$\lim_{K\to+\infty}F(nK,\beta_K,t)=e^{n\Psi(\beta,t)}$$

where

$$\Psi(\beta,t)=\frac{\mu-\nu}{2a}+\frac{\eta}{a}\Big(\frac{2}{1-\exp(\alpha_\infty+2\eta t)}-1\Big).$$

Therefore we deduce that there exists $K_0\in\mathbb{N}$, $T>0$ and $\beta\in\mathbb{R}_+^2$ such that for any $s\in[0,t)$ we have

$$\sup_{K\geq K_0}\mathbb{E}_{nK}[e^{\frac{\beta_1}{K}\int_0^s N_u^K\mathrm{d}u+\frac{\beta_2}{K}N_s^K}]<+\infty. \tag{12}$$

V.F.2 Step 2: domination of X^K by linear process

We begin by showing the following lemma.

Lemma 5. *Consider two functions g_d and g_b from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that*

$$g_b(x)\leq \nu x,\ g_d(x)\geq \mu x,\ g_d(0)=0.$$

We consider two counting processes N^d and N^b with respective intensity $g_d(N)$ and $g_b(N)$ where $N=N^b-N^d$. Then up to an extension of the probability space there exists a linear branching process $\tilde{N}$ with birth rate ν and death rate μ.

Proof of Lemma 5. We proceed by thinning. We consider a multivariate point process X with values in $\mathscr{E}=\{b_1,b_2,d_1,d_2\}$ and let p be its corresponding random measure. For any $e\in\mathscr{E}$ we define:

$$N^e=\int_0^t\int_{\mathscr{E}}\mathbf{1}_{x=e}p(\mathrm{d}x,\mathrm{d}t).$$

For $i=1$ and 2 we note $N^i=N^{b_i}-N^{d_i}$ and

$$\lambda^{b_1}=\mu N^1,\ \lambda^{d_1}=\nu N^1,\ \lambda^{b_2}=g_b(N^2) \text{ and } \lambda^{d_2}=g_d(N^2).$$

We set $p(\mathrm{d}x,\mathrm{d}t)=m_t(\mathrm{x})\lambda_t\mathrm{d}t$ where $\lambda_t=\lambda_t^{b_2}+\lambda_t^{d_1}$. The measure m_t is defined by:

$$m_t(b_1)=\varepsilon_t^1\delta_{b_1},\ m_t(b_2)=\varepsilon_t^1\varepsilon_t^2\delta_{b_2},\ m_t(d_2)=(1-\varepsilon_t^1)\delta_{d_2},\ m_t(d_1)=(1-\varepsilon_t^1)\varepsilon_t^3\delta_{d_1}$$

where $(\varepsilon_t^i)_{1\leq i\leq 3}$ are Bernoulli random variable with parameters

$$p_t^1 = \frac{\lambda_t^{b_1}}{\lambda_t},\ p_t^2 = \frac{\lambda_t^{b_2}}{\lambda_t^{b_1}} \text{ and } p_t^3 = \frac{\lambda_t^{d_1}}{\lambda_t^{d_2}}.$$

For existence of the process X see [Jac75]. Basically we get that the when there is an event either N^{b_1} N^{d_2} jump. If N^{b_1} has jumped, then N^{b_2} may jump or not and If N^{d_2} has jumped, then N^{d1} may jump or not. So almost surely we have $N^1 \geq N^2$. According to Proposition 1. in [Oga81] for any $e \in \mathscr{E}$ the process N^e is a counting process with intensity λ^e. This concludes the proof of the Lemma. □

V.F.3 Step 3: conclusion

As consequence of Lemma 5 for any K up to an extension of the probability space we can consider that there exists a branching process with birth and death rate given by

$$\nu_K = \nu + \frac{\sigma^2}{2}K \text{ and } \mu_K = \frac{\sigma^2}{2}K$$

that dominates X^K almost surely. So according to Equation (12) in Step 2, there exists some positive constants β_0, T and K_0 such that for any $s \leq T$

$$\sup_{K\geq K_0} \mathbb{E}[\exp(\beta_0 \int_0^s X_u^K \mathrm{d}u + \beta_0 X_s^K)] < +\infty.$$

This conclude the proof of the proposition.

Part III

Market design

CHAPTER VI

How to design a derivatives market?

Abstract

We consider the problem of designing a derivatives exchange aiming at addressing clients needs in terms of listed options and providing suitable liquidity. We proceed into two steps. First we use a quantization method to select the options that should be displayed by the exchange. Then, using a principal-agent approach, we design a make take fees contract between the exchange and the market maker. The role of this contract is to provide incentives to the market maker so that he offers small spreads for the whole range of listed options, hence attracting transactions and meeting the commercial requirements of the exchange.

Key words: Make take fees, market making, derivatives, market design, quantization, Lloyd's algorithm, financial regulation, high frequency trading, principal-agent problem, stochastic control

1 Introduction

Nowadays a typical role of an exchange is to give the possibility to investors to buy or sell financial products on electronic platforms, in sufficiently large quantity and at a reasonable price. Therefore exchanges have to set up their markets in a relevant way in order to achieve this goal. The issues related to market design cover a wide range of topics, from the microstructure of electronic trading platforms to the basic question of selecting the products that will be traded on the exchange.

Recently many papers have focused on the microstructural aspects of market design. For example the way of choosing an optimal tick size is addressed in [DR16], where the authors study the relations between tick size, volatility and bid-ask bounce frequency. In [BCS15, JMR19], the relevance of continuous trading and its comparison with a frequent batch auction system is discussed, while market fragmentation is analyzed in [LL18]. Macroscopic features have also been investigated, see for example [KP04], where different market structures are classified with respect to several criteria such as matching mechanism, information feedback

and bid structure.

Most of the research on market design focuses on stock markets. However, even if exchanges concentrate a large part of their activities on simple products such as stocks or futures, many also offer to their clients the possibility to trade more complex financial instruments such as derivatives. Actually there is very few academic literature on derivatives market design, mostly addressing the relationship between stock and option markets. For example in [MM04] the authors investigate the factors influencing the selection of stocks for option listing. However, they neither question the optimality of those factors, nor search for more relevant ones. The papers dealing with market design can in fact be separated into two groups: the ones that review and try to understand market practice and those proposing a theoretical framework in order to help exchanges improve their market design. Surprisingly, to our knowledge, there is no paper of the last kind dealing with derivatives market. In this article we propose a first contribution in that direction.

We take the realistic point of view of an exchange who wants to organize, or reorganize, its derivatives market. We consider that the market is made of vanilla European options only, that we view as independent of the underlying. By this, we mean that we deal with options that are used as hedging instruments and whose prices are essentially fixed by supply and demand. Finally we suppose that the exchange has access to data allowing for the estimation of the distribution of options market demand. For example, if the exchange already has a derivatives market it can use its own data, otherwise that of other exchanges. We focus on two issues: selecting the options that are going to be traded and attracting liquidity on those options.

The first issue faced by the exchange is the choice of the derivatives offered to the clients. Obviously it is impossible for the exchange to propose all maturities and strikes on its platform. This would be very hard to manage from a technical point of view and it would be impossible to guarantee liquidity on each option. As the maturities are quite standardized, the main challenge relies in strikes selection satisfying clients needs. Therefore, we consider that the exchange's problem is to select n call options (or equivalently n strikes), with fixed maturity, with the aim of maximizing the clients satisfaction. We define a measurement of this satisfaction and write the exchange objective under the form of a quantization problems. We refer to [GL07, PPP04] for an introduction to quantization. Such approach allows the exchange to select automatically a set of options based only on the data at its disposal.

The next goal of the exchange is to attract liquidity on its platform in order to increase the amount of executed orders. To do so, one way is to use a make take fees system: the exchange typically associates a fee rebate to executed limit orders, while charging a transaction fee for market orders. This enables it to subsidise liquidity provision and tax liquidity consumption. In [EEMRT18] the authors design the optimal make take fees policy for a market with one market maker and a single undeying asset. This work has been extended in [BPR19] to the case of multiple market makers. The general principle of the approach in [BPR19, EEMRT18] is to consider that the exchange offers a contract to the market maker whose pay-off depends

on the market order flow he generates. The problem of the exchange then boils down into designing the optimal contract in order to optimize the number of transactions.

However, in our setting the problem faced by the exchange is more complex to several extents. The main difference with the framework of [BPR19, EEMRT18] is that the exchange has to manage several assets simultaneously, namely the different options quoted on the platform. In order to focus on this issue we assume that there is only one market maker setting bid and ask quotes for all available options. Another challenge for a derivatives exchange is the possible absence of quotations for far from the money options (or quotations with a too wide spread). Such issue arises essentially for commercial reasons. Indeed, an exchange does not wish to display to its clients a product with scarse liquidity. It wants to make sure that there is sufficient available volume on the market for the whole range of listed options. Therefore, the design of an optimal make take fees policy for options market must aim at providing incentives to the market maker to lower the spreads, notably for far from the money options.

To do so, we are inspired by [BPR19, EEMRT18], using a principal-agent framework. The exchange (the principal) has to design a contract towards the market maker (the agent) that maximizes a certain utility that depends on the behavior of the market maker. The main point is that the market maker's behavior, here the quoted spread on every available option, cannot be dictated by the exchange and depends on the contract. For example if the contract offers high incentives for every executed ask market order, then it is likely that the ask price quoted by the market maker will be close to the mid price. Formally, for a given contract, the market maker determines its behavior by solving a stochastic control problem. Then in order to find the optimal contract, the exchange maximizes its expected utility over the set of admissible contracts, knowing the market maker's response to each contract.

The paper is organized as follows. In Section 2 we explain how an exchange can select the options that will be traded on its platform using only market data. Then in Section 3 we design the optimal contract that the exchange should offer to the market maker in order to maximize liquidity. Proofs and technical results are relegated to the Appendix.

2 Market driven selection of the listed options

In this section we build a method for the exchange to select the strikes that are going to be traded on its platform. This approach uses only data from trades volume reports and is based on a quantization algorithm. We illustrate this method by numerical experiments using data provided by Euronext.

2.1 How to choose the strikes in order to match market demand?

We consider European call options with strikes expressed in percentage of the spot price (in moneyness) and that the exchange wishes to select n strikes.[1] Choosing relevant strikes, the exchange's objective is to maximize the satisfaction of the investors. So, we focus in this section on the market taking side of the trading flow. Section 3 will be rather devoted to market makers.

We measure the regret of a market taker associated to the execution of a market order as a function of the difference between the strike he would have ideally bought (or sold) and the strike he actually bought (or sold). More precisely, for a given maturity, consider strikes $K_1 < \cdots < K_n$ that represent the options listed by the exchange. When a market taker wants to buy an option with strike K he sends a market order on the option whose strike is the nearest from K. Hence he buys (or sells) the option with strike K_i where i is such that

$$K_i = \underset{1\leq j\leq n}{\arg\min} |K - K_j|.$$

We consider that the regret associated to this market order is $\rho(|K - K_i|)$ where ρ is an increasing function. Note that the regret of the market order can be written

$$\min_{1\leq j\leq n} \rho(|K - K_j|).$$

We finally assume that the strike K is randomly chosen according to the distribution $\mathbb{P}^{mkt}$. This probability measure represents the law of market demand. Thus the higher the demand for a given strike the higher the probability that K is close to this strike. The exchange can easily estimate the distribution $\mathbb{P}^{mkt}$ using data from its own options market or from other exchanges. The average regret of a market order is therefore written

$$\mathbb{E}^{mkt}[\min_{1\leq j\leq n} \rho(|K - K_j|)], \tag{1}$$

where $\mathbb{E}^{mkt}$ denotes the expectation when $K \sim \mathbb{P}^{mkt}$. The problem of the exchange is then to find the n-uplet $(K_i)_{1\leq i\leq n}$ that minimizes (1). Formally this corresponds to the following minimization problem:

$$\underset{K_1\leq\cdots\leq K_n}{\arg\min}\, \mathbb{E}^{mkt}[\min_{1\leq j\leq n} \rho(|K - K_j|)]. \tag{2}$$

This type of optimization is classical in the field of signal or image processing and is called *quantization* problem. The main idea of quantization is to summarize the information contained in a complex probability measure into a uniform probability with finite support. As an example, it allows to compress a signal (or an image) by selecting among its spectrum a given number of frequencies that summarizes the signal with the smallest possible loss of information. For an introduction to quantization problem see [GL07, PPP04].

[1] We do not address here the problem of choosing the number of strikes to propose. This point is left for further research.

In this article we consider the quantization problem (2) when ρ is a power-law function of the form $\rho(x) = |x|^p$ with $p \geq 2$. The power-law function has the advantage to be symmetric and convex. Therefore greater errors are increasingly penalized. As a consequence we expect the solution of (2) to capture the features of the tails of $\mathbb{P}^{mkt}$. Moreover the greater p, the more large errors are penalized. Hence for a large p, the $(K_i)_{1\leq i\leq n}$ solution of (2) are likely to be more spread towards large strikes and contain more extreme values of the distribution $\mathbb{P}^{mkt}$.

2.2 Solving the quantization problem

In this section we give some sufficient conditions that ensure that (2) has a unique solution. We also explain how (2) can be solved.

To get existence of a solution to the problem (2) we need to make the following assumption.

Assumption 1. *The probability $\mathbb{P}^{mkt}$ is absolutely continuous with respect to the Lebesgue measure with density that is log-concave and compactly supported in $[0, \overline{K}], \overline{K} > 0$.*

The assumption on the support of the probability is very reasonable since strikes between 0 and 200% of the spot price basically cover all the possible strikes of traded options. The log-concavity assumption is not really restrictive since it allows us to consider a wide class of probability distributions such as exponential type and Gaussian laws. It is shown in [GL07, Theorem I-5.1] that under Assumption 1, Problem (2) admits a unique non degenerate solution. The term non degenerate simply means that the optimal set of strikes satisfies $K_1 < \cdots < K_n$.

We now present a way to approximate numerically the solution of (2). The idea behind the algorithm is that the solution $(K_i)_{1\leq i\leq n}$ can be seen as the fixed point of a function. This provides us a numerical method to approximate the $(K_i)_{1\leq i\leq n}$ that consists in iterating this function. This is known as the Lloyd's algorithm, which is a very intuitive approach that searches step by step the solution of (2). A very convenient aspect of this algorithm is that it is automatic and easy to implement.

The Lloyd's algorithm starts with an initial set of strikes $(K_i)_{1\leq i\leq n}$ and is made of three steps:

1. For any i, identify A_i the set of "wished" strikes that corresponds to market orders sent to the strike K_i. Equivalently A_i contains all the strikes K which are closer to K_i than from any other K_j
$$A_i = \{K, \text{ s.t } i = \underset{1\leq j\leq n}{\arg\min} |K - K_j|\}.$$
2. Set K_i' as the unique strike in A_i that minimizes the average regret of market orders sent with ideal strike in A_i. More precisely K_i' is given by
$$K_i' = \underset{k\in A_i}{\arg\min}\ \mathbb{E}^{mkt}[|K-k|^p \mathbf{1}_{K\in A_i}].$$
3. Go back to Step 1 with $(K_i)_{1\leq i\leq n} = (K_i')_{1\leq i\leq n}$ (or stop if a certain stopping criterion is reached and consider $(K_i')_{1\leq i\leq n}$ as the approximate solution of (2)).

The Lloyd's algorithm has a very clear interpretation in terms of selecting the optimal set of strikes: first it identifies the area "controlled" by the $i-th$ strike and then improves the choice of the strikes. It is then intuitive that the solution of (2) is a fixed point of the Lloyd's algorithm. The sets $(A_i)_{1\leq i\leq n}$ form a covering of $\mathbb{R}_+$ that is often called the Voronoï tesselation associated to the $(K_i)_{1\leq i\leq n}$. It is easy to show that, for Step 1

$$A_1 = [0, K_1], \quad A_n = [K_n, \overline{K}] \text{ and for } i \in \{2, \dots, n-1\}: \quad A_i = [\frac{K_i + K_{i-1}}{2}, \frac{K_{i+1} + K_i}{2}].$$

A usual stopping criterion for Step 3 is when $(K_i')_{1\leq i\leq n}$ is too close from $(K_i)_{1\leq i\leq n}$. More precisely the algorithm stops if

$$\sum_{i=1}^{n} |K_i' - K_i| < \varepsilon,$$

for a certain $\varepsilon > 0$. Note that, starting from a discrete valued $\mathbb{P}^{mkt}$ (as will be the case here), when $p = 2$, Step 2 of the Lloyd's algorithm boils down to compute the average realization of $\mathbb{P}^{mkt}$ conditional on being in A_i. This can be obtained instantaneously. However when $p > 2$, Step 2 is not straightforward to compute in general. Yet the objective function being convex and taking the derivative with respect to k, a necessary and sufficient condition for k to be solution of Step 2 is

$$\mathbb{E}[|K-k|^{p-2}(K-k)\mathbf{1}_{K\in A_i}] = 0$$

or equivalently

$$k = \frac{\mathbb{E}^{mkt}[K|K-k|^{p-2}\mathbf{1}_{K\in A_i}]}{\mathbb{E}^{mkt}[|K-k|^{p-2}\mathbf{1}_{K\in A_i}]}.$$

This characterizes the solution of Step 2 as a fixed point. Thus one usually replaces Step 2 by its iterative version:

$$K_i' = \frac{\mathbb{E}^{mkt}[K|K-K_i|^{p-2}\mathbf{1}_{K\in A_i}]}{\mathbb{E}^{mkt}[|K-K_i|^{p-2}\mathbf{1}_{K\in A_i}]}.$$

From now on, we call Lloyd's algorithm the initial algorithm where we replace Step 2 by its approximate version. We prove in Appendix 1 that $(K_i)_{1\leq i\leq n}$ is solution of (2) if and only if it is a fixed point of the Lloyd's algorithm. The great strength of this method is that it is easy to implement, transparent, and completely automatic. Note also that if $\mathbb{P}^{mkt}$ has a discrete support, say 10 strikes, then the Lloyd's algorithm will not necessarily select those strikes as solution of (2).

We now turn to numerical experiments illustrating the efficiency of our method.

2.3 Application

In this section we apply our methodology to market data. First we describe the data and then present our numerical results.

2.3.1 Description of the data

We use data from Euronext, one of the main stock and option exchanges in Europe. The dataset contains for every trading day from the 3-rd of December 2018 to the 24-th of May 2019 and for every available options the total number of trades (buy and sell) during the day. Our dataset is only made of transactions that occurred on the Euronext platform. In particular we neither use OTC data nor data from another exchange. We choose for our example the most standard call options in terms of underlying on Euronext, namely options on the CAC 40 index. We report in Table VI.1 the number of call options traded each month for different ranges of maturity and in Table VI.2 the number of call options traded each month for each strike.

In Figure VI.1, we display the empirical distribution of traded option strikes (for all maturities) and the quantile plot of the maturity distribution in log-scale. The distribution of the strikes is unimodal, concentrated near the money and skewed towards in the money strikes. In Figure VI.2, we provide the empirical distribution of traded options strikes for different ranges of maturity. We see that the distribution of the strikes depends on the maturity. In particular, the variance of the distribution is increasing with the maturity. The skewness towards in the money strikes is present for any maturity.

Maturity	December	January	February	March	April	May
T≤ 1M	135951	99202	96323	191357	161937	108491
1M<T≤ 3M	79016	61651	30371	117400	58914	121267
3M<T≤ 6M	10990	13279	15979	33901	11227	11779
6M < T	71977	30278	14197	17158	25354	21330

Table VI.1 – Number of options traded by maturity and month.

2.3.2 Numerical results

We now present our numerical results. Since the distribution of the strikes depends on the maturity and because short maturities are over-represented in our data, we split our dataset into four subsets depending on the maturity:

- maturity less than 1 month,
- maturity between 1 and 3 months,
- maturity between 3 and 6 months,
- maturity larger than 6 months.

For any of those subsets we approximate the solution of the quantization problem (2) using the Lloyd's algorithm for $n = 10$ and with stopping parameter $\varepsilon = 10^{-8}$. As initial value, we use n points $(K_i)_{1\leq i\leq n}$ generated with uniform law between the 10-th and 90-th percentile of the

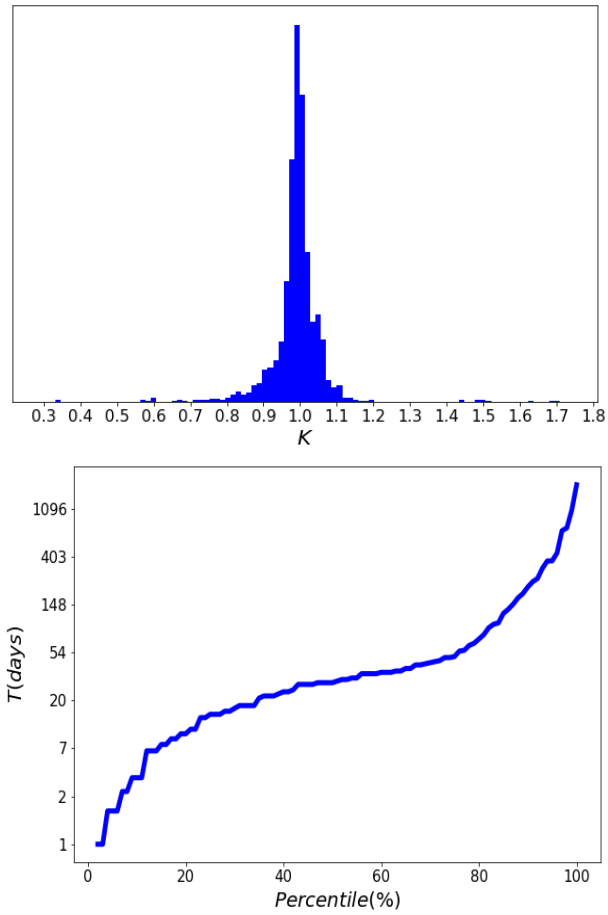

Figure VI.1 – Empirical distribution of traded option strikes (left). Quantile plot in log-scale of traded option maturities for the whole sample set (right).

dataset. In Figures VI.3 and VI.4 we plot a visualization of the quantization of the different sets obtained for $p = 2$ and $p = 8$.

The strikes selected by the Lloyd's algorithm manage to reproduce some of the statistical properties of the demand distribution $\mathbb{P}^{mkt}$. In particular, for any range of maturity, the distribution of the $(K_i)_{1\leq i\leq n}$ is skewed towards in the money strikes. Also the variance of the selected strikes is increasing with the maturity as for market data.

We observe that for $p = 8$ the strikes selected by the quantization method are more spread towards large strikes than for $p = 2$. This is not surprising since the penalization of large errors is increasing with p for the regret function $|\cdot|^p$. Therefore, as expected, the larger p, the more the solution of the quantization problem (2) contains extreme values of the distribution $\mathbb{P}^{mkt}$. We also note that the selected strikes for $p = 8$ exhibit some kind of redundancy: some of

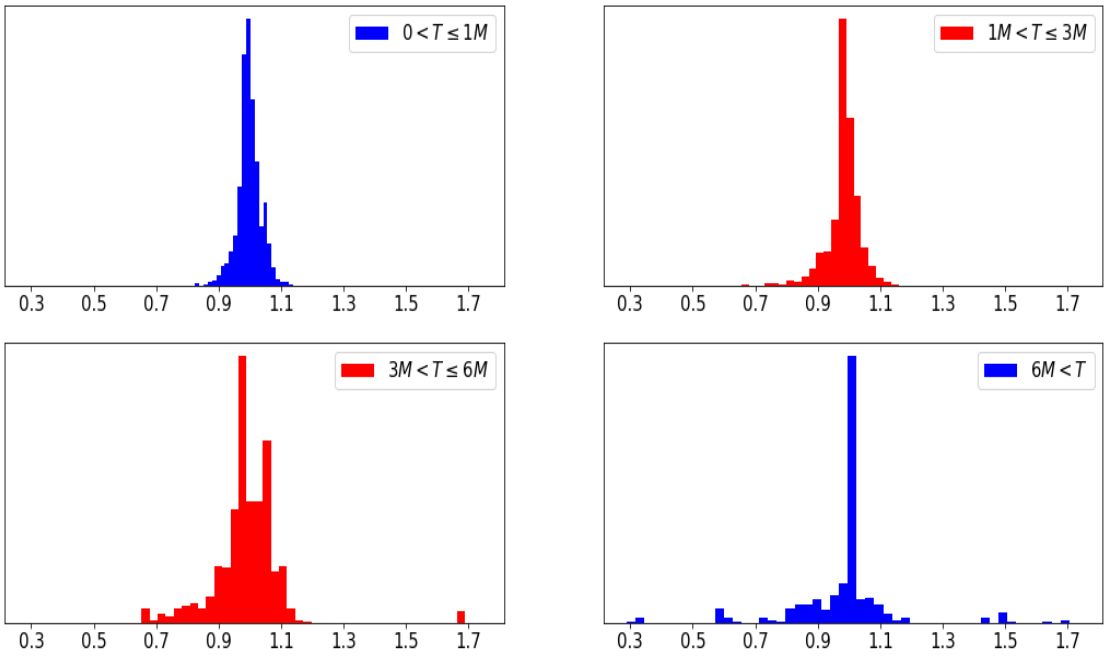

Figure VI.2 – Empirical distribution of the strikes for different maturities.

Strike (%)	December	January	February	March	April	May
20	0	0	0	0	55	10
30	1	1692	2	381	0	0
40	0	77	0	80	3	41
50	58	417	0	328	2031	1948
60	1933	152	31	323	691	2092
70	1402	1928	653	3837	2412	2956
80	12814	12952	3400	10118	14689	12147
90	113210	114463	10465	247877	184835	147362
100	159075	68747	130002	94714	50621	90528
110	5811	3586	12253	1766	83	2205
120	869	94	64	11	0	16
130	1	11	0	0	0	0
140	0	0	0	0	2012	1960
150	0	0	0	381	0	1602
160	1720	271	0	0	0	0
170	1040	20	0	0	0	0

Table VI.2 – Number of options traded by strike and month.

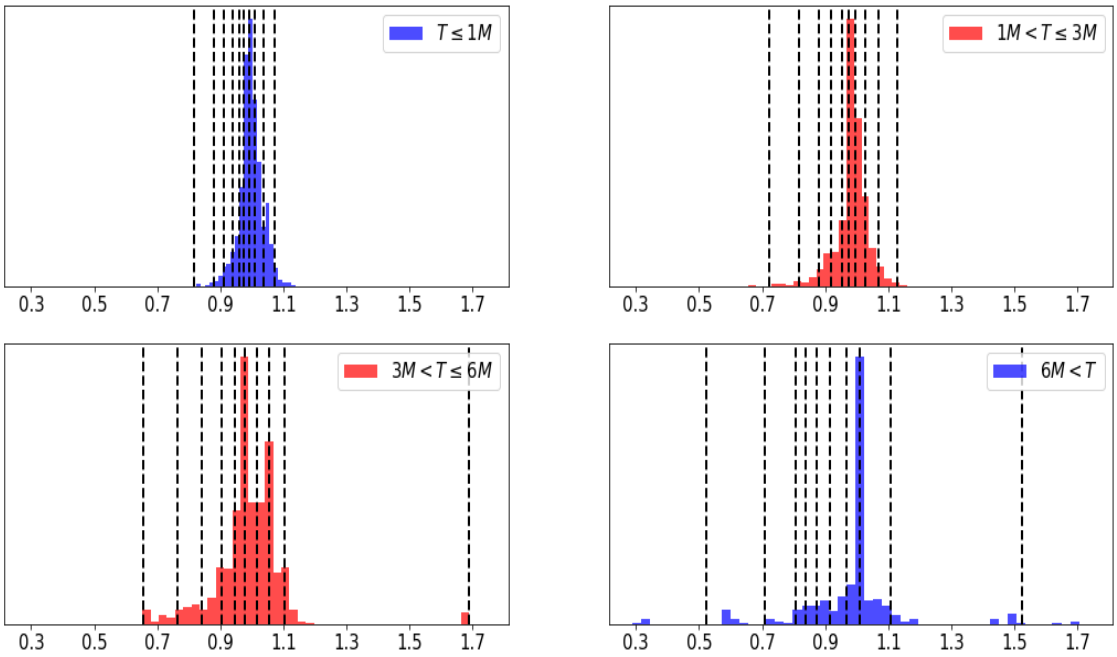

Figure VI.3 – Quantization of the option strikes using $p = 2$ and $\varepsilon = 10^{-8}$. Empirical distribution of traded strikes is plotted in blue or red. The dotted lines correspond to the optimal quantization of $\mathbb{P}^{mkt}$.

them are very close to each other. In practice, one would of course discard one of two strikes being very close (it may then be interesting to take a smaller n). For practical applications, the easiest approach is probably to use $p = 2$. With this choice, the Lloyd's algorithm is very fast and easy to implement. It also corresponds to the most documented case.

Finally we insist on the fact that when an exchange uses our methodology for strikes selection, it is interesting, if possible, to include transactions from other exchanges and from the OTC market in the dataset. This is because using only its own trade data may induce a bias in the strikes selection. For example if for some reasons clients of an exchange go on other venues to buy (or sell) out of the money options, then, in the exchange dataset, there will be very few transactions reported on out of the money options. This will lead to inaccuracies since the demand for out of the money options will be underestimated.

We now turn to the problem of providing incentives to the market maker to quote attractive spreads in order to attract liquidity towards the selected options.

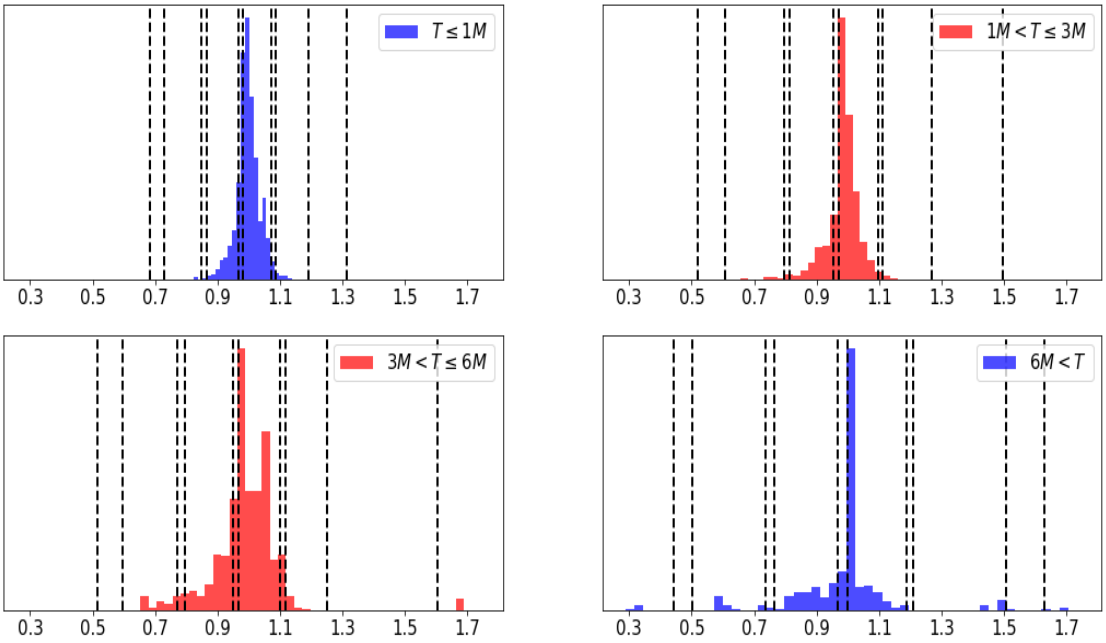

Figure VI.4 – Quantization of the option strikes using $p = 8$ and $\varepsilon = 10^{-8}$. Empirical distribution of traded strikes is plotted in blue or red. The dotted lines correspond to the optimal quantization of $\mathbb{P}^{mkt}$.

3 Incentive policy of the exchange

In this section, we assume that the exchange has already selected a list of options. The goal is to design a contract between the exchange and the market maker so that the latter receives incentives to provide suitable liquidity on all the options. We first describe the market and assumptions. In particular, due to the short time horizon we are working on, we can assume a Bachelier model for the underlying asset and constant delta for the options. Then, we introduce a class of tractable admissible contracts proposed to the market maker. These contracts are indexed on the transactions induced by the behavior of the market maker. We show that there is no loss of generality in considering such class of contracts. For a given contract, the market maker solves an optimization problem to deduce its optimal quotes for each option. Then, the exchange maximizes his expected utility over the set of admissible contracts, knowing the response of the market maker to a given contract.

The utility of the exchange is made of two parts: one component related to the actual Profit and Loss (PnL for short) due to transactions, and one aiming at ensuring that enough liquidity is constantly posted on every option. As explained in the introduction, this second component addresses commercial constraints in order to make the exchange competitive. In particular,

our model is flexible and can be designed so that the exchange has more interest in reducing the spreads for far from the money options, although not very traded, than for near the money options. We derive explicitly the optimal incentives that should be offered, up to the resolution of a two-dimensional linear PDE.

We conclude this section with numerical results showing the impact of the incentive policy on the spread of the listed options.

3.1 The market

This section is devoted to the description of the market model.

We consider a finite trading horizon time $T > 0$ and a probability space $(\Omega, \mathscr{F}, \mathbb{P}^0)$ under which all stochastic processes are defined. Following Section 2, we work on a market where European call options with strike $k \in \mathscr{K} := \{K_1, \dots, K_n\}$ and maturity $\tau \in \mathscr{T} := \{T_1, \dots, T_m\}$ can be traded. We focus on call options but our results can be extended to put options in a straightforward manner. The price of the underlying, observable by all market participants, has a dynamic given by

$$\mathrm{d}S_t = \sigma \mathrm{d}W_t, \tag{3}$$

where $\sigma > 0$ is the volatility of the asset and W is a one-dimensional Brownian motion. The choice of an arithmetic Brownian motion is motivated by the fact that we use a reasonably short time horizon T (less than one day). On such scale, Bachelier and Black-Scholes type dynamics are quite indistinguishable.

Assuming zero interest rate, we write the price at time t of the call option with maturity τ and strike k as $C_t^{k,\tau}$. Its dynamic is given by

$$\mathrm{d}C_t^{k,\tau} = \sigma \Delta_t^{k,\tau} \mathrm{d}W_t, \tag{4}$$

where $\Delta_t^{k,\tau} := \mathscr{N}(d_t)$ is the Bachelier delta of the call option $C^{k,\tau}$ at time t, $\mathscr{N}(\cdot)$ is the cumulative distribution function of the standard Gaussian law and $d_t := \frac{S_t - k}{\sigma\sqrt{\tau}}$.

As we work over a short time horizon, the delta of the quoted options does not vary significantly. Hence, throughout the paper, we assume it to be constant.

Assumption 2. *We consider that*

$$\Delta_t^{k,\tau} = \Delta^{k,\tau}.$$

This assumption, which can be relaxed, leads to technical simplifications. Note that in our problem setting, as the considered time horizon is one trading day, it is very reasonable to assume a constant delta (by taking the one at the opening of the market) and recalibrate it at the end of the day. This will lead to a different pay-off of the contract for the market maker in

case of a significant price move from one day to another.

The market maker displays bid and ask quotes on the listed options. The market maker best bid price and best ask price at time t on the option with maturity τ and strike k are

$$P_t^{k,\tau,b} := C_t^{k,\tau} - \delta_t^{k,\tau,b},\ P_t^{k,\tau,a} = C_t^{k,\tau} + \delta_t^{k,\tau,a},\ t \in [0,T],$$

where the superscript b (resp. a) stands for bid (resp. ask). So we consider that the market maker controls the spreads $\delta^{k,\tau} := (\delta^{k,\tau,a}, \delta^{k,\tau,b})$ on each option. The set of admissible controls for the market maker is therefore defined as

$$\mathcal{A} := \left\{(\delta_t)_{t\in[0,T]} = (\delta_t^{k,\tau,i})_{t\in[0,T]}, k \in \mathcal{K}, \tau \in \mathcal{T}, i \in \{a,b\},\ \text{predictable and s.t } |\delta_t^{k,\tau,i}| \le \delta_\infty\right\}, \quad (5)$$

where $\delta_\infty > 0$ is a constant, assumed to be large enough to satisfy technical conditions (see Appendix VI.A.7). In practice it is of course not restrictive to assume that the spreads are bounded.

We now describe the dynamics of the market order flow. For every listed option, the arrival of ask (resp. bid) market orders is modeled by a point process $N^{k,\tau,a}$ (resp. $N^{k,\tau,b}$). We expect the intensity of buy (resp. sell) market order arrivals to be a decreasing function of both the spread quoted by the market maker $\delta^{k,\tau}$ and the transaction cost $f^{k,\tau}$ collected by the exchange. This has quite natural interpretation as a wider spread or higher fee decreases the number of transactions on the considered option. Moreover, we know from the literature (see [DR16], [MRR97] and [WBK$^+$08]) that the average number of trades per unit of time for single assets is a decreasing function of the ratio between spread and volatility. Assuming same kind of behavior for the options, this leads to the following form of the intensity function:

$$\lambda^{k,\tau}(\delta_t^{k,\tau,i}) := A\exp\Big(-\frac{C}{\sigma}(\delta_t^{k,\tau,i} + f^{k,\tau})\Big),$$

where A and C are positive constants that can be calibrated using market data, and $f^{k,\tau}$ represents the fee fixed by the exchange for each market order. Furthermore, we assume that all market orders are of unit size.

The main difficulty in our framework is that the market maker is dealing with multiple derivatives. If the market maker strategy depends on its inventory on each option, then the problem lies in dimension n, which becomes intricate for large n. However, we will see that we can circumvent this issue since in our case we can aggregate the risk factors related to the inventories through the delta weighted cumulated inventory:

$$\mathcal{Q}_t := \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \Delta^{k,\tau} Q_t^{k,\tau}, \quad (6)$$

where $Q_t^{k,\tau} := N_t^{k,\tau,b} - N_t^{k,\tau,a}$ is the number of options $C^{k,\tau}$ held by the market maker at time t. Each inventory is weighted by the corresponding Δ (see Section 3.2 for details). Thus, the quantity $\mathcal{Q}$ represents the marked-to-market value of the market maker's portfolio. It therefore

contains the market risk carried by the market maker. For example an out of the money option will account for a small part of the total risk, and conversely for in the money options. Finally we consider that the market maker has a critical absolute inventory $\overline{q} \in \mathbb{N}$. The intensity of the orders arrival is then

$$\lambda^{k,\tau,i} := \lambda^{k,\tau}(\delta_t^{k,\tau,i})\mathbb{1}_{\{\phi(i)\mathcal{Q}_{t^-} > -\overline{q}\}} \text{ with } \phi(i) := \begin{cases} 1 \text{ if } i = a \\ -1 \text{ if } i = b. \end{cases}$$

Remark 1. *Note that there is a direct link between the spread quoted by the market maker and his inventory process. Indeed a lower spread $\delta^{k,\tau,b}$ (resp. $\delta^{k,\tau,a}$) on the bid (resp. ask) side of the listed option $C^{k,\tau}$ increases the intensity of orders arrival $\lambda^{k,\tau,b}$ (resp. $\lambda^{k,\tau,a}$). This leads to an increase (resp. decrease) of the inventory process $Q^{k,\tau}$. In other words, the market maker skews his quotes depending on the level of its aggregated inventory.*

3.2 Market maker's problem and contract representation

In this section we exhibit the class of contracts used by the exchange. We also explain and solve the market maker's problem for any admissible contract.

The PnL of the market maker is defined as the sum of the cash earned from his executed orders and of the value of his inventory on each traded option. Thus, using that $\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} Q_t^{k,\tau}\Delta^{k,\tau}S_t = \mathcal{Q}_t S_t$, it writes

$$PL_t^\delta := \mathcal{W}_t^\delta + \mathcal{Q}_t S_t, \tag{7}$$

where

$$\mathcal{W}_t^\delta := \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \int_0^t P_u^{k,\tau,a}\mathrm{d}N_u^{k,\tau,a} - \int_0^t P_u^{k,\tau,b}\mathrm{d}N_u^{k,\tau,b}$$

stands for his cash process at time $t \in [0,T]$. This expression shows the relevance of the variable $\mathcal{Q}$ for the market maker. It represents the volatility of the market maker's PnL with respect to the underlying price movements. Using (4), a direct integration by parts leads to the following form of the PnL process:

$$PL_t^\delta := \sum_{i\in\{a,b\}} \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \int_0^t \delta_u^{k,\tau,i}\mathrm{d}N_u^{k,\tau,i} + \mathcal{Q}_u\mathrm{d}S_u.$$

Moreover, the exchange offers to the market maker a contract ξ, namely an $\mathcal{F}_T$-measurable random variable, which is added to his PnL at the end of the trading period. This contract aims at incentivizing the market maker to reduce the spread quoted for each option. More details will be given in Section 3.3. The contract depends on all the transactions occuring between time 0 and time T, as well as on the efficient price moves.

Thus taking an exponential utility function, the market maker maximizes the following functional of his wealth:

$$V_{\mathrm{MM}}(\xi) := \sup_{\delta\in\mathcal{A}} \mathbb{E}^\delta\Big[-\exp\Big(-\gamma\big(\xi + PL_T^\delta\big)\Big)\Big], \tag{8}$$

where $\gamma > 0$ denotes the market maker's risk aversion parameter and $\mathbb{E}^{\delta}$ the probability measure associated to a given control process $\delta \in \mathscr{A}$, see Appendix VI.A.2.2 for details. For the well-posedness of Equation (8), we need integrability conditions on the contract ξ, see Appendix VI.A.3 for details.

Finally we consider that the market maker accepts a contract ξ only if its associated optimal expected utility $V_{\mathrm{MM}}(\xi)$ is above some fixed threshold $R < 0$. This threshold, called reservation utility of the agent, is the critical utility value under which the market maker has no interest in the contract. This quantity has to be taken into account carefully by the exchange before proposing a contract to the market makers.

We now introduce the class of contracts proposed to the market maker. Given $Y_0 > 0$, and predictable processes $Z := (Z^{C^{k,\tau}}, Z^{k,\tau,i})_{k\in\mathcal{K}\ \tau\in\mathcal{T}\ i\in\{a,b\}} \in \mathcal{Z}$ (see Appendix VI.A.3 for a definition of $\mathcal{Z}$), we introduce a special class of remuneration $\xi = Y_T^{Y_0,Z}$ of the form

$$Y_T^{Y_0,Z} := Y_0 + \int_0^T \Big(\sum_{i=a,b} \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} Z_r^{k,\tau,i}\,\mathrm{d}N_r^{k,\tau,i} + Z_r^{C^{k,\tau}}\mathrm{d}C_r^{k,\tau}\Big) + \Big(\frac{1}{2}\gamma\sigma^2\Big(\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \Delta^{k,\tau}(Z_r^{C^{k,\tau}} + Q_r^{k,\tau})\Big)^2 - H(Z_r, Q_r)\Big)\mathrm{d}r, \tag{9}$$

where for $(z,q) \in \mathbb{R}^{2\times\#\mathcal{K}\times\#\mathcal{T}} \times \mathbb{R}$ with $z := (z^{k,\tau})_{(k,\tau)\in\mathcal{K}\times\mathcal{T}}$, the function H, called Hamiltonian of the market maker, is defined by[2]

$$H(z,q) := \sup_{\delta\in\mathbb{R}^{2\times\#\mathcal{K}\times\#\mathcal{T}}} h(\delta, z, q)$$

with

$$h(\delta, z, q) := \sum_{i=a,b} \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \gamma^{-1}\Big(1 - \exp\Big(-\gamma\big(z^{k,\tau,i} + \delta^{k,\tau,i}\big)\Big)\Big)\lambda^{k,\tau}(\delta^{k,\tau,i})\,\mathbb{1}_{\{\phi(i)\mathcal{Q}>-\overline{q}\}}.$$

Actually, it turns out that it is enough to consider contracts of the form (9). More precisely, we show that any admissible contract (in the sense of the integrability conditions specified in Appendix 20), is of this form. We have the following lemma proved in Appendix VI.A.5.

Lemma 1. *Any contract ξ satisfying* (20) *has a unique representation $\xi = Y_T^{Y_0,Z}$ for some $(Y_0, Z) \in \mathbb{R}\times\mathcal{Z}$.*

Furthermore, the terms defining (9) have natural interpretation.

- The compensation Y_0 is calibrated by the exchange to ensure the reservation utility constraint with level R of the market maker.[3]
- The term $Z^{C^{k,\tau}}$ is the compensation given to the market maker with respect to the volatility risk induced by the option $C^{k,\tau}$.

[2]This Hamiltonian term appears naturally when applying the dynamic programming principle for the market maker's problem.

[3]From Theorem 1, we see that taking $Y_0 = -\log(-R)$ ensures the reservation utility of the market maker.

- Each time a trade is executed on the ask (resp. bid) side for the option $C^{k,\tau}$, the market maker is compensated by the term $Z^{k,\tau,a}$ (resp. $Z^{k,\tau,b}$).
- The term $\frac{1}{2}\gamma\sigma^2\Big(\sum_{k=1}^{\mathcal{K}}\sum_{\tau=1}^{\mathcal{T}}\Delta(Z^{C^{k,\tau}}+Q^{k,\tau})\Big)^2 - H(Z,Q)$ is a continuous coupon given to the market maker.

When the market maker remuneration is $Y^{Y_0,Z}$, its optimal response can be computed explicitly as a functional of Z.

Theorem 1. *For $\xi = Y^{Y_0,Z}$, the market maker utility is*

$$V_{\mathrm{MM}}(Y_T^{Y_0,Z}) = -\exp(-\gamma Y_0),$$

associated to the optimal bid-ask policy $\hat{\delta}_t^{k,\tau,i}(\xi) := \Delta^i(Z_t^{k,\tau,i})$, where

$$\Delta^i(Z_t^{k,\tau,i}) := (-\delta_\infty) \vee \Big(-Z_t^{k,\tau,i} + \frac{1}{\gamma}\log\Big(1+\frac{\sigma\gamma}{C}\Big)\Big) \wedge \delta_\infty \text{ for } (k,\tau,i) \in \mathcal{K}\times\mathcal{T}\times\{a,b\}. \tag{10}$$

Theorem 1 provides the optimal response of the market maker to any contract of the form (9), see Appendix VI.A.6 for the proof. Moreover from Equation (10), we get that the exchange can anticipate the optimal behavior of the market maker. It is therefore easy for the platform to compute its own utility for a given contract.

3.3 Solving the exchange's problem

In this section we formalize the goal of the exchange and solve the problem of designing the optimal contract.

3.3.1 Description of the exchange's problem

We recall that the exchange has two objectives. The first one is to receive a high number of trades to collect the associated fees. The second is to have small spreads on its platform, in particular for far from the money options for which spreads are typically large. This is because the clients want to have sufficient liquidity on the whole list of options.

In order to quantify the first objective, we introduce a weighted version of the total number of trades:

$$\mathcal{N}_t = \sum_{i=a,b}\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} c^{k,\tau} N_t^{k,\tau,i},$$

where for any $(k,\tau)\in\mathcal{K}\times\mathcal{T}$, $c^{k,\tau}\geq 0$ represents the value attributed to a trade on the option $C^{k,\tau}$ by the exchange.[4] Hence the more the exchange wants to attract liquidity on the option $C^{k,\tau}$, the higher $c^{k,\tau}$ has to be. If the considered option is very liquid (at the money options

[4] One can for example take $c^{k,\tau} = f^{k,\tau}$. In this case, $\mathcal{N}_T$ represents the total amount of fees collected by the exchange.

for example), the exchange may choose a rather small $c^{k,\tau}$.

To take into account the second objective, we consider the following quantity

$$\mathscr{L}_T^{\delta} := \sum_{i=a,b} \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \int_0^T \omega\big(\delta_t^{k,\tau,i} - \delta_\infty^{k,\tau}\big)\mathrm{d}N_t^{k,\tau,i}, \tag{11}$$

where $\omega \in [0,1)$,[5] and $\delta_\infty^{k,\tau}$ can be seen as a spread threshold the exchange would like to impose to the market maker. The more important the second objective for the exchange, the closer to one ω has to be chosen.

We thus consider that the exchange is looking for the contract ξ that maximizes the following quantity:

$$\mathbb{E}^{\hat{\delta}(\xi)}\Big[-\exp\Big(-\eta\Big(\mathscr{N}_T - \mathscr{L}_T^{\delta(\xi)} - \xi\Big)\Big)\Big], \tag{12}$$

where $\eta > 0$ is the risk aversion of the exchange and $\hat{\delta}(\xi)$ denotes the optimal response of the market maker given the contract ξ.

According to Lemma 1, we know that it is enough for the exchange to consider contracts of the form $Y_T^{Y_0,Z}$ with $(Y_0, Z) \in \mathbb{R}\times\mathcal{Z}$. So, (12) becomes

$$\mathbb{E}^{\hat{\delta}(Y^{Y_0,Z})}\Big[-\exp\Big(-\eta\Big(\mathscr{N}_T - \mathscr{L}_T^{\delta(Y^{Y_0,Z})} - Y_T^{Y_0,Z}\Big)\Big)\Big]. \tag{13}$$

Moreover for a contract of the form $Y^{Y_0,Z}$, from Theorem 1, the exchange knows the best response $\hat{\delta}(Y^{Y_0,Z})$ of the market maker. Indeed we recall that the optimal controls are given by

$$\hat{\delta}^{k,\tau,i}(Y^{Y_0,Z}) = \Delta^i(Z_t^{k,\tau,i}).$$

It implies that

$$\mathscr{L}_T^{\hat{\delta}(Y^{Y_0,Z})} = \mathscr{L}_T^{Z} := \sum_{i=a,b} \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \int_0^T \omega\big(\Delta^i(Z_t^{k,\tau,i}) - \delta_\infty^{k,\tau}\big)\mathrm{d}N_t^{k,\tau,i}.$$

As in [EEMRT18], we notice that for a given contract $Y^{Y_0,Z}$, the market maker's optimal response does not depend on Y_0. The exchange objective function (13) being decreasing in Y_0, the maximization with respect to Y_0 is achieved at the level $\hat{Y}_0 = -\log(-R)$.[6] Finally, the exchange problem becomes

$$V_0^E := \sup_{Z\in\mathcal{Z}} \mathbb{E}^{\Delta(Z)}\Big[-\exp\Big(-\eta\Big(\mathscr{N}_T^Z - \mathscr{L}_T^Z - Y_T^{\hat{Y}_0,Z}\Big)\Big)\Big]. \tag{14}$$

[5] The choice of $\omega \in [0,1)$ is for technical reasons only.

[6] Note that $-\exp(-\hat{Y}_0) = R$.

3.3.2 Stochastic control approach for the reduced exchange problem

In this section we solve the reduced exchange problem (14). We characterize the optimal contract components $Z^\star$ and explain how to compute them in practice.

To solve this stochastic control problem, we study the associated Hamilton-Jacobi-Bellman (HJB for short) equation. This approach characterizes an optimal $Z^\star$ solving (14) under the form of a feedback function. The following result is proved in Appendix VI.A.7.

Theorem 2. *The maximization problem* (14) *admits a solution $Z^\star$ given by*

$$Z^{\star k,\tau,i}(t,\mathcal{Q}_{t^-}) := \frac{1}{a-b}\log\Big(\frac{bx_2U\big(t,\mathcal{Q}_{t^-}\big)}{ax_1^{k,\tau}U\big(t,\mathcal{Q}_{t^-}-\Delta^{k,\tau}\phi(i)\big)}\Big) \text{ and } Z^{\star C^{k,\tau}}(t,Q_t^{k,\tau}) := -\frac{\gamma}{\gamma+\eta}Q_t^{k,\tau}, \tag{15}$$

for $(k,\tau,i)\in\mathcal{K}\times\mathcal{T}\times\{a,b\}$, where a, b, $(x_1^{k,\tau})_{k\in\mathcal{K},\tau\in\mathcal{T}}$ and x_2 are constants defined in Appendix VI.A.7 and where $\tilde{U} := (-U)^{-\frac{C}{\sigma\eta(1-\omega)}}$ is the unique solution of the following linear PDE on $[0,T]\times\mathbb{R}$:

$$\begin{cases} 0 = \partial_t\tilde{U}(t,\mathcal{Q}) - \tilde{U}(t,\mathcal{Q})\frac{C\gamma\eta}{\gamma+\eta}\frac{\sigma}{2(1-\omega)}\mathcal{Q}^2 + \sum\limits_{i=a,b}\sum\limits_{(k,\tau)\in\mathcal{K}\times\mathcal{T}}\hat{C}^{k,\tau}\tilde{U}\big(t,\mathcal{Q}-\Delta^{k,\tau}\phi(i)\big)\mathbf{1}_{\phi(i)\mathcal{Q}>-\overline{q}}, \\ \tilde{U}(T,\mathcal{Q}) = 1, \end{cases} \tag{16}$$

where $\hat{C}^{k,\tau}$ are defined in Apppendix VI.A.7.

Theorem 2 provides the incentives $Z^\star$ that maximize the exchange expected utility function, see Appendix VI.A.7 for the proof. The optimal contract is therefore given by

$$\begin{aligned}\xi^\star = Y^{\hat{Y}_0,Z^*} = \hat{Y}_0 &+ \int_0^T \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}}\Big(\sum_{i=a,b} Z_r^{\star k,\tau,i}\mathrm{d}N_r^{k,\tau,i} + Z_r^{\star C^{k,\tau}}\mathrm{d}C_r^{k,\tau}\Big) \\ &+ \int_0^T\Big(\frac{1}{2}\gamma\sigma^2\Big(\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}}\Delta^{k,\tau}(Z_r^{\star C^{k,\tau}}+Q_r^{k,\tau})\Big)^2 - H(Z_r^\star,Q_r)\Big)\mathrm{d}r.\end{aligned} \tag{17}$$

We now provide some comments on the interpretation of the optimal incentives.

- The term $\int_0^T Z_u^{\star C^{k,\tau}}\mathrm{d}C_u^{k,\tau}$ in the optimal contract corresponds to part of the inventory risk process of the market maker $(Q_t^{k,\tau}C_t^{k,\tau})_{t\in[0,T]}$ that is supported by the exchange. As in [EEMRT18], the proportion of risk handled by the platform on each option is $\frac{\gamma}{\gamma+\eta}$. Hence, the more risk averse the exchange, the smallest this proportion.
- An application of Ito's formula gives the following approximation:

$$\log\Big(\frac{\tilde{U}(t,\mathcal{Q})}{\tilde{U}\big(t,\mathcal{Q}-\Delta^{k,\tau}\phi(i)\big)}\Big) \approx \phi(i)2\frac{\sigma}{C}(T-t)\tilde{C}\Delta^{k,\tau}\mathcal{Q}, \tag{18}$$

  where $\tilde{C} := \frac{C\gamma\eta}{\gamma+\eta}\frac{\sigma}{2(1-\omega)}$. Thus, when the aggregated inventory is highly positive, the exchange provides incentives to the market maker so that it attracts buy market orders and tries to dissuade him to accept more sell market orders, and conversely for a negative inventory.

- Numerically, we show that the incentive $Z^{\star k,\tau,a}$ and $Z^{\star k,\tau,b}$ given by (15) are increasing functions of the value $c^{k,\tau}$ that the principal associates to the option $C^{k,\tau}$. Hence, he logically provides higher incentives to an option he is more interested in.
- Although the principal manages a large number of listed options, we circumvent the curse of dimensionality by working with the aggregated inventory process. Note that the pay-off of the optimal contract depends only on t and $\mathcal{Q}$. Thus it is very easy to compute for the exchange at the end of the trading day.

In practice to implement the above methodology, one needs to compute the function $\tilde{U}$ in order to design the optimal contract. A first way to do this is to use a classical finite difference scheme on the PDE (16). In Section 3.4 we use this technique for some numerical experiments on our method.

Moreover, as PDE (16) is linear, we can also resort to a probabilistic representation to compute $\tilde{U}$ using a Monte-Carlo method. More precisely we have the following result which is a direct consequence of the Feynman-Kac formula.

Lemma 2. *We have the following representation:*

$$\tilde{U}(t,q) := \mathbb{E}\Big[\exp\Big(\int_t^T -\tilde{C}\big(\mathcal{Q}_s^{t,q}\big)^2 + \sum_{i=a,b}\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \overline{\lambda}_s^{k,\tau,i}\,\mathrm{d}s\Big)\Big], \tag{19}$$

where

$$\mathcal{Q}_s^{t,q} = q + \int_t^s \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \Delta^{k,\tau}\,\mathrm{d}\big(\overline{N}_u^{k,\tau,b} - \overline{N}_u^{k,\tau,a}\big),$$

where for any $(k,\tau)\in\mathcal{K}\times\mathcal{T}$ and $i = a$ or b, $\overline{N}^{k,\tau,i}$ is a point process with intensity $\overline{\lambda}_s^{k,\tau,i} := \hat{C}^{k,\tau}\mathbf{1}_{\{\phi(i)\mathcal{Q}_{s^-}^{t,q} > -\overline{q}\}}$, with $\hat{C}^{k,\tau}$ defined in Appendix VI.A.7.

The proof is in the same vein as [EEMRT18, Proposition 4.1]. We now turn to numerical illustrations of our make take fees policy.

3.4 Numerical results

For numerical experiments, we consider three options which are characterized by their delta. We fix the following parameters: $A = 1.5s^{-1}, \sigma = C = 0.3s^{-1/2}, f^{k,\tau} = [0.5, 0.8, 0.8]$ the vector of fees, and $\delta_\infty^{k,\tau} = [2,3,3]$ the set of quotation thresholds. The first option is at the money, the second one is in the money and the third is out of the money, hence the following set of deltas $[0.5, 0.8, 0.2]$. Moreover, we take $\eta = 1, \gamma = 0.01, T = 100s, \overline{q} = 40$.

We analyze the impact of the penalty ω and the weight associated to each options $c^{k,\tau}$ in the value function of the exchange.

In Figure VI.5, we display the average bid-ask spread at initial time on each option for $\omega = 0$, and $c^{k,\tau}$ being equal either to 0 or 0.1. We see that a higher $c^{k,\tau}$ leads to a decrease of the

spread for the option $C^{k,\tau}$. This result is in line with the form of the incentives in Theorem 2. Indeed $Z^{\star k,\tau,i}$ is an increasing function of $c^{k,\tau}$ and $\hat{\delta}^{k,\tau,i}$ is a decreasing function of $Z^{\star k,\tau,i}$. Thus, increasing the interest of the principal for the option $C^{k,\tau}$ leads to a decrease of the spread proposed by the market maker on this option. This shows that the exchange has a direct control on each option he is interested in.

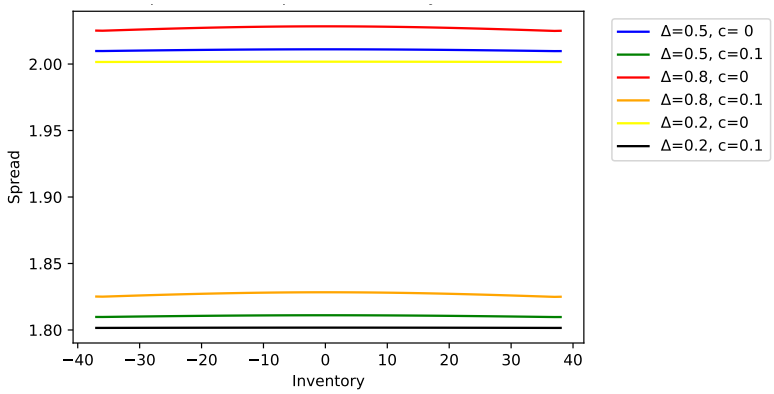

Figure VI.5 – Spread at initial time with respect to the market maker inventory, $\omega = 0$.

In Figure VI.6, we focus on the role of ω, equal to 0.1 on the spreads proposed by the market maker. As expected, a non-vanishing value of ω leads to a decrease of the spread for all the quoted options. This agrees with Theorem 2, where we see that the incentives are an increasing function of $\omega \in [0, 1)$. Thus, the exchange can influence the whole set of spreads proposed on the quoted options.

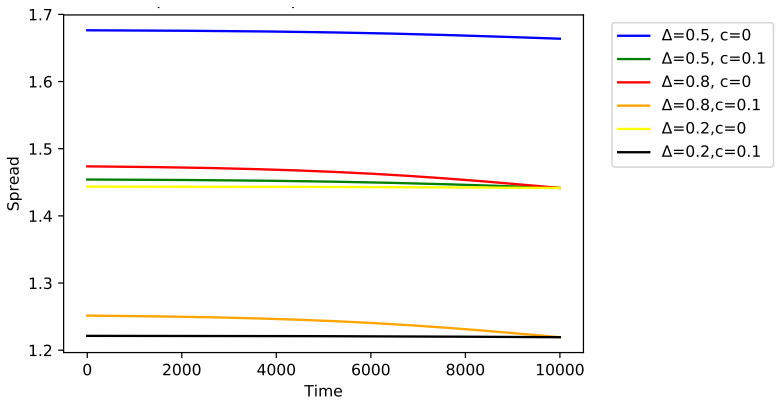

Figure VI.6 – Optimal ask spread for a null inventory with respect to the remaining trading time (in seconds), $\omega = 0.1$.

We conclude by showing in Figure VI.7 the behavior of the average spread with a higher ω, equal to 0.2. We obtain similar effects as in Figure VI.6, namely a decrease of the spread on all quoted options for a higher ω.

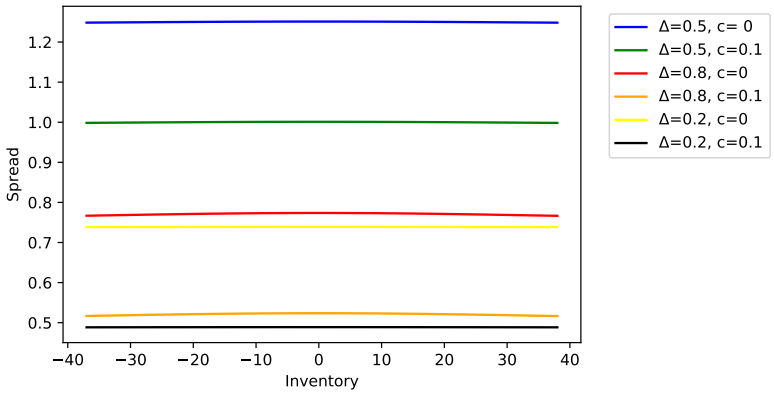

Figure VI.7 – Spread at initial time with respect to the market maker inventory, $\omega = 0.2$.

3.5 Conclusion

This work is, to our knowledge, the first to address the problem of designing a derivatives exchange, based solely on market data. In the first part, a simple market driven methodology enables us to choose which options the exchange should select to attract market takers. In the second part, we provide a make take fees policy between the exchange and the market maker which ensures a high quality of liquidity for the listed options.

VI.A Appendix

VI.A.1 Proof of the convergence of the Lloyd's algorithm

According to Paragraph 5.2 in [GL07], the set $(K_i)_{1\leq i\leq n}$ is a solution of (2) if and only if for any i, A_i has positive Lebesgue measure and

$$\int_{A_i} |K_i - x|^{p-1} \mathrm{sgin}(x - K_i) \mathbb{P}^{mkt}(\mathrm{d}x) = 0,$$

where sgin is the sign function. This is equivalent to

$$K_i = \frac{\int_{A_i} |K_i - x|^{p-2} x \mathbb{P}^{mkt}(\mathrm{d}x)}{\int_{A_i} |K_i - x|^{p-2} \mathbb{P}^{mkt}(\mathrm{d}x)} = \frac{\mathbb{E}^{mkt}[|K_i - K|^{p-2} K \mathbf{1}_{K\in A_i}]}{\mathbb{E}^{mkt}[|K_i - K|^{p-2} \mathbf{1}_{K\in A_i}]}.$$

Thus $(K_i)_{1\leq i\leq N}$ is the solution of (2) if and only if it is a fixed point of the Lloyd's algorithm.

We now give proofs and technical results for Section 3. They are mostly inspired by [EEMRT18]. However, for sake of completeness, we provide rigorous derivations.

VI.A.2 Stochastic basis

VI.A.2.1 Canonical process

In this section, we give an accurate definition of the probability space defined in Section 3.1. We consider a final horizon time $T > 0$ and the space $\Omega =: \Omega_c \times \Omega_d^{2\times\#\mathcal{T}\times\#\mathcal{K}}$, with Ω_c the set of continuous functions from $[0,T]$ into $\mathbb{R}$ and Ω_d the set of piecewise constant càdlàg functions from $[0,T]$ into $\mathbb{N}$. We consider Ω as a subspace of the Skorokhod space $\mathscr{D}([0,T],\mathbb{R}^{2\times\#\mathcal{T}\times\#\mathcal{K}+1})$ of càdlàg functions from $[0,T]$ into $\mathbb{R}^{2\times\#\mathcal{T}\times\#\mathcal{K}+1}$ and $\mathscr{F}$ the trace Borel $\sigma-$algebra on Ω, where the topology is the one associated to the usual Skorokhod distance on $\mathscr{D}([0,T],\mathbb{R}^{2\times\#\mathcal{T}\times\#\mathcal{K}+1})$.

We define $(\mathscr{X}_t)_{t\in[0,T]} := \left(W_t, (N_t^{k,\tau,i})_{i=a,b;k\in\mathcal{K};\tau\in\mathcal{T}}\right)$ as the canonical process on Ω, that is for any $\omega = (w, n^{k,\tau,i}) \in \Omega$

$$W_t(\omega) = w(t),\ N_t^{k,\tau,i}(\omega) = n^{k,\tau,i}(t).$$

VI.A.2.2 Probability measure

We now properly define $\mathbb{P}^0$ and the associated change of measure. We set the probability $\mathbb{P}^0$ on $(\Omega,\mathscr{F})$ such that under $\mathbb{P}^0$, W, $N^{k,\tau,i}$ are independent, W is a one-dimensional Brownian motion and the $N^{k,\tau,i}, k\in\mathcal{K},\tau\in\mathcal{T}, i=a,b$ are Poisson processes with intensity $\lambda^{k,\tau,i}(0)$.[7] Finally, we endow the space $(\Omega,\mathscr{F})$ with the ($\mathbb{P}^0-$completed) canonical filtration $\mathbb{F} := (\mathscr{F}_t)_{t\in[0,T]}$ generated by $(\mathscr{X}_t)_{t\in[0,T]}$.

By (5), the control process must be predictable and uniformly bounded. The last assumption is required to define the associated probability measure. So for $\delta\in\mathscr{A}$ we introduce the corresponding probability measure $\mathbb{P}^\delta$ under which $S_t = S_0 + \sigma W_t$ follows (3) and for $k\in\mathcal{K},\tau\in\mathcal{T},i\in\{a,b\}$ the

$$N_t^{\delta,k,\tau,i} := N_t^{k,\tau,i} - \int_0^t \lambda^{k,\tau}(\delta_r^{k,\tau,i})\mathbb{1}_{\{\phi(i)Q_{r^-}>-\overline{q}\}}\mathrm{d}r$$

are martingales. This probability measure is defined by the corresponding Doléans-Dade exponential:

$$L_t^\delta := \exp\left(\sum_{i=a,b}\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}}\int_0^t \mathbb{1}_{\{\phi(i)Q_{r^-}>-\overline{q}\}}\left(\log\left(\frac{\lambda^{k,\tau}(\delta_r^{k,\tau,i})}{A}\right)\mathrm{d}N_r^{k,\tau,i} - \left(\lambda^{k,\tau}(\delta_r^{k,\tau,i}) - A\right)\mathrm{d}r\right)\right),$$

which is a true martingale by the uniform boundedness of $\delta_t^{k,\tau,i}$.[8] We can therefore define the Girsanov change of measure $\frac{\mathrm{d}\mathbb{P}^\delta}{\mathrm{d}\mathbb{P}^0}|_{\mathscr{F}_t} = L_t^\delta$, for all $t\in[0,T]$. In particular, all the probability measures $\mathbb{P}^\delta$ indexed by $\delta\in\mathscr{A}$ are equivalent. We shall write $\mathbb{E}_t^\delta$ for the conditional expectation with respect to $\mathscr{F}_t$ under the probability measure $\mathbb{P}^\delta$.

[7] In other words, $\mathbb{P}^0$ is simply the product measure of the Wiener measure on Ω_c and the unique measure on $\Omega_d^{2\times\#\mathcal{T}\times\#\mathcal{K}}$ that makes the canonical process an homogeneous Poisson process with the prescribed intensity.

[8] The associated Novikov criterion is given in [Sok13].

VI.A.3 Well-posedness of the optimization problems

We give in this section the necessary integrability conditions ensuring that both exchange and market maker's problems are well defined. We consider the following assumptions:

$$\sup_{\delta\in\mathscr{A}}\mathbb{E}^{\delta}\Big[\exp\Big(-\gamma'\xi\Big)\Big]<+\infty,\ \text{for some } \gamma'>\gamma,\quad \sup_{\delta\in\mathscr{A}}\mathbb{E}^{\delta}\Big[\exp\Big(\eta'\xi\Big)\Big]<+\infty,\ \text{for some } \eta'>\eta. \tag{20}$$

Moreover, the next technical assumption is required in order to derive the best response of the market maker in Theorem 1:

$$\sup_{\delta\in\mathscr{A}}\sup_{t\in[0,T]}\mathbb{E}^{\delta}\Big[\exp\Big(-\gamma' Y_t^{0,Z}\Big)\Big]<+\infty,\ \text{for some } \gamma'>\gamma. \tag{21}$$

Finally, we define $\mathscr{Z}$ as the set of predictable processes $(Z_t)_{t\in[0,T]}$ such that Conditions (20) and (21) are satisfied. This is the set of admissible contract components of the exchange.

VI.A.4 Dynamic programming principle

In the spirit of [EEMRT18], we provide a proof of a dynamic programming principle for the market maker's problem. Note that a same type of dynamic programming principle exists for the exchange's problem.

For any $\mathbb{F}$ stopping time $\tau\in[t,T]$ and $\mu\in\mathscr{A}_\tau$, we define

$$J_T(\tau,\mu)=\mathbb{E}_\tau^{\mu}\Big[-\exp\Big(-\gamma\Big(\xi+\sum_{i=a,b}\sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}}\int_\tau^T \mu_u^{k,\tau,i}\mathrm{d}N_u^{k,\tau,i}+Q_u^{k,\tau}\mathrm{d}C_u^{k,\tau}\Big)\Big)\Big]$$

where $\mathscr{A}_\tau$ denotes the restriction of $\mathscr{A}$ to controls on $[\tau,T]$. We also define the set $\mathscr{J}_{\tau,T}=(J_T(\tau,\mu))_{\mu\in\mathscr{A}_\tau}$. The continuation utility of the market maker is defined for any $\mathscr{F}$-stopping time τ by

$$V_\tau=\operatorname*{ess\,sup}_{\mu\in\mathscr{A}_\tau}J_T(\tau,\mu).$$

We first prove the following technical lemma.

Lemma 3. *Let τ be a stopping time with values in $[t,T]$. Then there exists an increasing sequence $(\mu^n)_{n\in\mathbb{N}}$ in $\mathscr{A}_\tau$ such that $V_\tau=\lim_{n\to+\infty}J_T(\tau,\mu^n)$.*

Proof. For $\mu,\mu'\in\mathscr{A}_\tau$ we define

$$\hat{\mu}:=\mu\mathbb{1}_{\{J_T(\tau,\mu)\geq J_T(\tau,\mu')\}}+\mu'\mathbb{1}_{\{J_T(\tau,\mu)\leq J_T(\tau,\mu')\}}.$$

We have $\hat{\mu}\in\mathscr{A}_\tau$ and by definition of $\hat{\mu}$, $J_T(\tau,\hat{\mu})\geq\max(J_T(\tau,\mu),J_T(\tau,\mu'))$. Thus $\mathscr{J}_{\tau,T}$ is increasing, and we obtain the same result as in [EEMRT18]. The conclusion follows. □

We set

$$\mathscr{D}_{t,T}(\delta) := \exp\Big(-\gamma\Big(\sum_{i=a,b}\sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}}\int_t^T \delta_u^{k,\tau,i}\mathrm{d}N_u^{k,\tau,i} + Q_u^{k,\tau}\mathrm{d}C_u^{k,\tau}\Big)\Big).$$

Given Lemma 3, we can now prove the dynamic programming principle associated to (8).

Lemma 4. *Let $t\in[0,T]$ and τ be an $\mathbb{F}$ stopping time with values in [t,T]. Then*

$$V_t = \operatorname*{ess\,sup}_{\delta\in\mathscr{A}} \mathbb{E}_t^\delta\Big[-\mathscr{D}_{t,\tau}(\delta)V_\tau\Big].$$

Proof. Let $t\in[0,T]$ and τ be a stopping time with values in $[t,T]$. First, by tower property, we have

$$\begin{aligned} V_t &= \operatorname*{ess\,sup}_{\delta\in\mathscr{A}} \mathbb{E}_t^\delta\Big[-\mathscr{D}_{t,T}(\delta)\exp\big(-\gamma\xi\big)\Big] \\ &= \operatorname*{ess\,sup}_{\delta\in\mathscr{A}} \mathbb{E}_t^\delta\Big[\mathscr{D}_{t,\tau}(\delta)\mathbb{E}_\tau^\delta\Big[-\mathscr{D}_{\tau,T}(\delta)\exp\big(-\gamma\xi\big)\Big]\Big]. \end{aligned}$$

Then, Bayes rule yields

$$\begin{aligned} \mathbb{E}_\tau^\delta\Big[-\mathscr{D}_{\tau,T}(\delta)\exp\big(-\gamma\xi\big)\Big] &= \mathbb{E}_\tau^0\Big[-\frac{L_T^\delta}{L_\tau^\delta}\mathscr{D}_{\tau,T}(\delta)\exp\big(-\gamma\xi\big)\Big] \\ &\le \operatorname*{ess\,sup}_{\delta\in\mathscr{A}} \mathbb{E}_\tau^\delta\Big[\mathscr{D}_{\tau,T}(\delta)\exp\big(-\gamma\xi\big)\Big] \\ &= V_\tau. \end{aligned}$$

Finally we obtain

$$V_t \le \operatorname*{ess\,sup}_{\mu\in\mathscr{A}} \mathbb{E}_t^\mu\Big[V_\tau\mathscr{D}_{t,\tau}(\delta)\Big].$$

We next prove the reverse inequality. Let $\delta\in\mathscr{A}$ and $\mu\in\mathscr{A}_\tau$. We define $(\delta\otimes_\tau\mu)_u = \delta_u 1_{\{0\le u\le\tau\}} + \mu_u 1_{\{\tau\le u\le T\}}$. Then $\delta\otimes_\tau\mu\in\mathscr{A}$ and by tower property

$$V_t \ge \mathbb{E}_t^{\delta\otimes_\tau\mu}\Big[-\mathscr{D}_{\tau,T}(\mu)\mathscr{D}_{t,\tau}(\delta)\exp\big(-\gamma\xi\big)\Big] = \mathbb{E}_t^{\delta\otimes_\tau\mu}\Big[\mathbb{E}_\tau^{\delta\otimes_\tau\mu}\Big[-\mathscr{D}_{\tau,T}(\mu)\exp(-\gamma\xi)\Big]\mathscr{D}_{t,\tau}(\delta)\Big].$$

Using Bayes formula and noting that $\frac{L_T^{\delta\otimes_\tau\mu}}{L_\tau^{\delta\otimes_\tau\mu}} = \frac{L_T^\mu}{L_\tau^\mu}$, we have

$$\begin{aligned} \mathbb{E}_\tau^{\delta\otimes_\tau\mu}\Big[-\mathscr{D}_{\tau,T}(\mu)\exp(-\gamma\xi)\Big] &= \mathbb{E}_\tau^0\Big[-\frac{L_T^\mu}{L_\tau^\mu}\mathscr{D}_{\tau,T}(\mu)\exp(-\gamma\xi)\Big] \\ &= J_T(\tau,\mu). \end{aligned}$$

This implies

$$V_t \ge \mathbb{E}_t^{\delta\otimes_\tau\mu}\Big[\mathscr{D}_{t,\tau}(\delta)J_T(\tau,\mu)\Big].$$

We can therefore use again Bayes rule and the fact that $\frac{L_T^{\delta\otimes_\tau\mu}}{L_t^{\delta\otimes_\tau\mu}} = \frac{L_T^\delta}{L_t^\delta}$ to obtain

$$\begin{aligned}
V_t \geq \mathbb{E}_t^0\Big[\frac{L_T^{\delta\otimes_\tau\mu}}{L_t^{\delta\otimes_\tau\mu}}\mathscr{D}_{t,\tau}(\delta)J_T(\tau,\mu)\Big] &= \mathbb{E}_t^0\Big[\mathbb{E}_\tau^0\Big[\frac{L_T^{\delta\otimes_\tau\mu}}{L_\tau^{\delta\otimes_\tau\mu}}\frac{L_\tau^{\delta\otimes_\tau\mu}}{L_t^{\delta\otimes_\tau\mu}}\mathscr{D}_{t,\tau}(\delta)J_T(\tau,\mu)\Big]\Big] \\
&= \mathbb{E}_t^0\Big[\mathbb{E}_\tau^0\Big[\frac{L_T^{\delta\otimes_\tau\mu}}{L_\tau^{\delta\otimes_\tau\mu}}\Big]\frac{L_\tau^{\delta\otimes_\tau\mu}}{L_t^{\delta\otimes_\tau\mu}}\mathscr{D}_{t,\tau}(\delta)J_T(\tau,\mu)\Big] \\
&= \mathbb{E}_t^0\Big[\frac{L_\tau^{\delta\otimes_\tau\mu}}{L_t^{\delta\otimes_\tau\mu}}\mathscr{D}_{t,\tau}(\delta)J_T(\tau,\mu)\Big] \\
&= \mathbb{E}_t^\delta\Big[\mathscr{D}_{t,\tau}(\delta)J_T(\tau,\mu)\Big].
\end{aligned}$$

Since the previous inequality holds for any $\mu\in\mathscr{A}_\tau$, we deduce from monotone convergence theorem together with Lemma 3 that there exists a sequence $(\mu^n)_{n\in\mathbb{N}}$ of controls in $\mathscr{A}_\tau$ such that

$$\begin{aligned}
V_t \geq \lim_{n\to+\infty}\mathbb{E}_t^\delta\Big[\mathscr{D}_{t,\tau}(\delta)J_T(\tau,\mu^n)\Big] &= \mathbb{E}_t^\delta\Big[\mathscr{D}_{t,\tau}(\delta)\lim_{n\to+\infty}J_T(\tau,\mu^n)\Big] \\
&= \mathbb{E}_t^\delta\Big[\mathscr{D}_{t,\tau}(\delta)V_\tau\Big].
\end{aligned}$$

This concludes the proof. □

VI.A.5 Proof of Lemma 1

We divide the proof into six steps.

Step 1: **Derivation of the martingale representation**.

For $\delta\in\mathscr{A}$, it follows from the dynamic programming principle of Lemma 4 that the process

$$U_t^\delta = V_t\mathscr{D}_{0,t}(\delta)$$

defines a $\mathbb{P}^\delta$-supermartingale for any $\delta\in\mathscr{A}$. By standard analysis, we may then consider it in its càdlàg version (by taking right limits along rationals). By the Doob-Meyer decomposition, we can write $U_t^\delta = M_t^\delta - A_t^\delta$ where M^δ is a $\mathbb{P}^\delta$-martingale and $A_t^\delta = A_t^{\delta,c} + A_t^{\delta,d}$ is an integrable non-decreasing predictable process such that $A_0^{\delta,c} = A_0^{\delta,d} = 0$ with pathwise continuous component $A^{\delta,c}$ and with $A^{\delta,d}$ a piecewise constant predictable process.

From the martingale representation theorem under $\mathbb{P}^\delta$, see Appendix A.1 in [EEMRT18], there exists $\tilde{Z}^\delta = (\tilde{Z}^{\delta,S}, \tilde{Z}^{\delta,k,\tau,i})_{k\in\mathscr{K},\tau\in\mathscr{T},i=a,b}$ predictable, such that

$$M_t^\delta = V_0 + \int_0^t \tilde{Z}_r^{\delta,S}\mathrm{d}S_r + \sum_{i=a,b}\sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}}\int_0^t \tilde{Z}_r^{\delta,k,\tau,i}\mathrm{d}N_r^{\delta,k,\tau,i}.$$

Step 2: **Boundedness of the value function**.

We show that V is a negative process. In fact, thanks to the uniform boundedness of $\delta \in \mathscr{A}$, we have that

$$\frac{L_T^\delta}{L_t^\delta} \geq \alpha_{t,T} = \exp\Big(- \sum_{i=a,b} \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \frac{k}{\sigma} N_T^{k,\tau,i} - 2 \times \#\mathcal{T} \times \#\mathcal{K}\, A \mathrm{e}^{-\frac{kc_\infty}{\sigma}} (\mathrm{e}^{\frac{k}{\sigma}} + 1)(T-t) \Big),$$

where $c_\infty := \max_{k,\tau} c^{k,\tau}$. Therefore

$$V_t \leq \mathbb{E}_t^0 \Big[- \alpha_{t,T} \exp\Big(- \gamma(\delta_\infty \sum_{i=a,b} \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} N_T^{k,\tau,i} + \int_t^T Q_u^{k,\tau} \mathrm{d}C_u^{k,\tau}) \Big) \mathrm{e}^{-\gamma\xi} \Big] < 0.$$

Step 3: **Identification of the coefficients (1/2)**.

Let Y be the process defined for any $t \in [0,T]$ by $V_t = -\mathrm{e}^{-\gamma Y_t}$. As $A^{\delta,d}$ is a predictable point process and the jumps of $N^{k,\tau,i}, i = a,b$ are totally inaccessible stopping times under $\mathbb{P}^0$, we have $\langle N^{k,\tau,i}, A^{\delta,d} \rangle_t = 0$ a.s. Using Ito's formula, we obtain that

$$Y_T = \xi, \text{ and } \mathrm{d}Y_t = \sum_{i=a,b} \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} Z_t^{k,\tau,i} \mathrm{d}N_t^{k,\tau,i} + Z_t^S \mathrm{d}S_t - \mathrm{d}I_t - \mathrm{d}\tilde{A}_t^d,$$

with

$$\begin{aligned}
Z_t^{k,\tau,a} &= -\frac{1}{\gamma} \log\Big(1 + \frac{\tilde{Z}_t^{\delta,k,\tau,a}}{U_{t^-}^\delta}\Big) - \delta_t^{k,\tau,a} \\
Z_t^{k,\tau,b} &= -\frac{1}{\gamma} \log\Big(1 + \frac{\tilde{Z}_t^{\delta,k,\tau,b}}{U_{t^-}^\delta}\Big) - \delta_t^{k,\tau,b} \\
Z_t^S &= -\frac{\tilde{Z}_t^{\delta,S}}{\gamma U_{t^-}^\delta} - \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} Q_{t^-}^{k,\tau} \Delta^{k,\tau} \\
I_t &= \int_0^t \Big(\overline{h}(\delta_r, Z_r, Q_r)\mathrm{d}r - \frac{1}{\gamma U_r^\delta} \mathrm{d}A_r^{\delta,c} \Big) \\
\overline{h}(\delta, Z_t, Q_t) &= h(\delta, Z_t, Q_t) - \frac{1}{2}\gamma\sigma^2 (Z_t^S + Q_t)^2 \\
\tilde{A}_t^d &= \frac{1}{\gamma} \sum_{s\leq t} \log\Big(1 - \frac{\Delta A_t^{\delta,d}}{U_{t^-}^\delta}\Big).
\end{aligned}$$

In particular, the last relation between $\tilde{A}^d$ and $A^{\delta,d}$ shows that $\Delta a_t \geq 0$ is independent of $\delta \in \mathscr{A}$, with $a_t = -\frac{A_t^{\delta,d}}{U_{t^-}^\delta}$ and abusing notations slightly, $\Delta a_t = -\frac{\Delta A_t^{\delta,d}}{U_{t^-}^\delta}$.

In order to complete the proof, we argue in the subsequent steps that $Z \in \mathcal{Z}$ and that, for $t \in [0,T]$, $A_t^{\delta,d} = -\sum_{s\leq t} U_{s^-}^\delta \Delta a_s = 0$ so that $\tilde{A}_t^d = 0$ and $I_t = \int_0^t \overline{H}(Z_r, Q_r)\mathrm{d}r$, where

$$\overline{H}(Z_t, Q_t) = H(Z_t, Q_t) - \frac{1}{2}\gamma\sigma^2 (Z_t^S)^2.$$

Step 4: **Identification of the coefficients (2/2).**

Since $V_T = -1$, we get that

$$\begin{aligned}0 &= \sup_{\delta\in\mathscr{A}} \mathbb{E}^\delta[U_T^\delta] - V_0\\ &= \sup_{\delta\in\mathscr{A}} \mathbb{E}^\delta[U_T^\delta - M_T^\delta]\\ &= \gamma \sup_{\delta\in\mathscr{A}} \mathbb{E}^0\Big[L_T^\delta \int_0^T U_{r^-}^\delta (\mathrm{d}I_r - \overline{h}(\delta, Z_r, Q_r)\mathrm{d}r + \frac{\mathrm{d}a_r}{\gamma})\Big].\end{aligned}$$

Moreover, the controls being uniformly bounded, we have

$$U_t^\delta \le -\beta_t = V_t \exp\Big(-\gamma\Big(\delta_\infty \sum_{i=a,b}\sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} N_T^{k,\tau,i} + \int_0^t Q_u^{k,\tau}\mathrm{d}C_u^{k,\tau}\Big)\Big) < 0.$$

Then, using $A^{\delta,d} \ge 0, U^\delta \le 0$ and $\mathrm{d}I_t - \overline{h}(\delta, Z_t, Q_t)\mathrm{d}t \ge 0$, we obtain

$$\begin{aligned}0 &\le \sup_{\delta\in\mathscr{A}} \mathbb{E}^0\Big[\alpha_{0,T}\int_0^T -\beta_{r^-}\big(\mathrm{d}I_r - \overline{h}(\delta, Z_r, Q_r)\mathrm{d}r + \frac{\mathrm{d}a_r}{\gamma}\big)\Big]\\ &= -\mathbb{E}^0\Big[\alpha_{0,T}\int_0^T \beta_{r^-}\big(\mathrm{d}I_r - \overline{H}(Z_r, Q_r)\mathrm{d}r + \frac{\mathrm{d}a_r}{\gamma}\big)\Big].\end{aligned}$$

The quantities $\alpha_{0,T}\int_0^T \beta_{r^-}(\mathrm{d}I_r - \overline{H}(Z_r, Q_r))\mathrm{d}r$ and $\alpha_{0,T}\int_0^T \beta_{r^-}\frac{\mathrm{d}a_r}{\gamma}$ being non-negative random variables, the result follows.

Step 5: **Admissibility of the process** Z.

As ξ satisfies the conditions in (20), to prove that $Z \in \mathscr{Z}$, it is enough to show that for some $p > 0$

$$\sup_{\delta\in\mathscr{A}}\sup_{t\in[0,T]} \mathbb{E}^\delta[\exp(-\gamma(p+1)Y_t)] < +\infty.$$

Using Hölder inequality together with the boundedness of the intensities of the $N^{k,\tau,i}$, we have that $\sup_{\delta\in\mathscr{A}} \mathbb{E}^\delta[|U_T^\delta|^{p'+1}] < +\infty$ for some $p' > 0$. We deduce

$$\sup_{\delta\in\mathscr{A}}\sup_{t\in[0,T]} \mathbb{E}^\delta[|U_t^\delta|^{p'+1}] = \sup_{\delta\in\mathscr{A}} \mathbb{E}^\delta[|U_T^\delta|^{p'+1}] < +\infty$$

because U^δ is a $\mathbb{P}^\delta$-negative supermartingale. The conclusion follows using again Hölder inequality, the uniform boundedness of the intensities of the $N^{k,\tau,i}$ and the fact that

$$\exp(-\gamma Y_t) = U_t^\delta \exp\Big(\gamma\Big(\sum_{i=a,b}\sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} \int_0^t \delta_u^{k,\tau,i}\mathrm{d}N_u^{k,\tau,i} + Q_u^{k,\tau}\mathrm{d}C_u^{k,\tau}\Big)\Big).$$

Step 6: **Uniqueness of the representation**.

Let $(Y_0, Z), (Y_0^{'}, Z^{'}) \in \mathbb{R} \times \mathscr{Z}$ be such that $\xi = Y_T^{Y_0,Z} = Y_T^{Y_0^{'},Z^{'}}$. By following the lines of the verification argument in the proof of Theorem 1, we obtain the equality $Y_t^{Y_0,Z} = Y_t^{Y_0^{'},Z^{'}}$ using the fact that the value of the continuation utility of the market maker satisfies

$$-\mathrm{e}^{-\gamma Y_t^{Y_0,Z}} = -\mathrm{e}^{-\gamma Y_t^{Y_0^{'},Z^{'}}} = \operatorname*{ess\,sup}_{\delta \in \mathscr{A}} \mathbb{E}_t^{\delta}\Big[-\mathrm{e}^{-\gamma(PL_T^{\delta} - PL_t^{\delta} + \xi)} \Big].$$

This in turn implies that for $t \in [0,T]$ $Z_t^{k,\tau,i}\mathrm{d}N_t^{k,\tau,i} = Z_t^{'k,\tau,i}\mathrm{d}N_t^{k,\tau,i}$ and $Z_t^S \sigma^2 \mathrm{d}t = Z_t^{'S}\sigma^2\mathrm{d}t = \mathrm{d}\langle Y, S\rangle_t$. Consequently, $(Y_0, Z) = (Y_0^{'}, Z^{'})$.

VI.A.6 Proof of Theorem 1

Let $\xi = Y_T^{Y_0,Z}$ with $(Y_0, Z) \in \mathbb{R} \times \mathscr{Z}$. We first prove that for an arbitrary set of controls $\delta \in \mathscr{A}$, we have $J_{\mathrm{MM}}(\delta, \xi) \leq -\mathrm{e}^{-\gamma Y_0}$, where $J_{\mathrm{MM}}(\delta, \xi)$ is such that $V_{\mathrm{MM}}(\xi) = \sup_{\delta \in \mathscr{A}} J_{\mathrm{MM}}(\delta, \xi)$. Then, we will see that this inequality is in fact an equality when the corresponding Hamiltonian $h(\delta, z, q)$ is maximized. Denote

$$\overline{Y}_t := Y_t^{Y_0,Z} + \sum_{i=a,b} \sum_{(k,\tau) \in \mathscr{K} \times \mathscr{T}} \int_0^t \delta_u^{k,\tau,i}\mathrm{d}N_u^{k,\tau,i} + Q_u^{k,\tau}\mathrm{d}C_u^{k,\tau}$$

with $t \in [0,T]$. A direct application of Ito's formula leads to

$$\begin{aligned}\mathrm{d}\mathrm{e}^{-\gamma\overline{Y}_t} =& \gamma \mathrm{e}^{-\gamma\overline{Y}_{t^-}}\Bigg(-\Big(\sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} Q_t^{k,\tau}\Delta^{k,\tau} + Z_t^S\Big)\mathrm{d}S_t + \big(H(Z_t, Q_t) - h(\delta, Z_t, Q_t)\big)\mathrm{d}t \\ &- \sum_{i=a,b}\sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} \gamma^{-1}\Big(1 - \exp\Big(-\gamma\big(Z_t^{k,\tau,i} + \delta_t^{k,\tau,i}\big)\Big)\Big)\mathrm{d}N_t^{\delta,k,\tau,i}\Bigg).\end{aligned}$$

Thus, $\mathrm{e}^{-\gamma\overline{Y}_\cdot}$ is a $\mathbb{P}^{\delta}$-local submartingale. Thanks to Condition (21), the uniform boundedness of the intensities of the $N^{k,\tau,i}$ and Hölder inequality, $\left(\mathrm{e}^{-\gamma\overline{Y}_t}\right)_{t\in[0,T]}$ is uniformly integrable and hence is a true submartingale. Doob-Meyer decomposition theorem gives us that

$$\int_0^{\cdot} \gamma \mathrm{e}^{-\gamma\overline{Y}_{t^-}}\Bigg(-\Big(\sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} Q_t^{k,\tau}\Delta^{k,\tau} + Z_t^S\Big)\mathrm{d}S_t - \sum_{i=a,b}\sum_{(k,\tau)\in\mathscr{K}\times\mathscr{T}} \gamma^{-1}\Big(1 - \exp\Big(-\gamma\big(Z_t^{k,\tau,i} + \delta_t^{k,\tau,i}\big)\Big)\Big)\mathrm{d}N_t^{\delta,k,\tau,i}\Bigg)$$

is a true martingale. This implies that

$$\begin{aligned}J_{\mathrm{MM}}(\delta,\xi) &= \mathbb{E}^{\delta}\Big[-\mathrm{e}^{-\gamma\overline{Y}_T}\Big] \\ &= -\mathrm{e}^{-\gamma Y_0} - \mathbb{E}^{\delta}\Big[\int_0^T \gamma \mathrm{e}^{-\gamma\overline{Y}_{t^-}}\big(H(Z_t,Q_t) - h(\delta, Z_t, Q_t)\big)\mathrm{d}t\Big] \\ &\leq -\mathrm{e}^{-\gamma Y_0}.\end{aligned}$$

In addition to this, the previous inequality becomes an equality if and only if δ is chosen as the maximizer of the Hamiltonian h thus leading to the optimal quotes provided in Theorem 1. So we deduce $J_{\mathrm{MM}}(\delta,\xi) = -\mathrm{e}^{-\gamma Y_0}$. Finally we have $V_{\mathrm{MM}}(\xi) = -\mathrm{e}^{-\gamma Y_0}$ with optimal response $(\hat{\delta}_t)_{t\in[0,T]}$.

VI.A.7 Proof of Theorem 2

We define for any map $v : [0,T] \times \mathbb{Z}^{\#\mathcal{K}\times\#\mathcal{T}} \longrightarrow (-\infty, 0)$, $x \in \mathbb{R}$, $(k,\tau) \in \mathcal{K}\times\mathcal{T}$ and $(t,q) \in [0,T]\times\mathbb{Z}^{\#\mathcal{K}\times\#\mathcal{T}}$

$$v(t, q\ominus_{K_i,T_j} x) := v(t, q^{K_1,T_1}, q^{K_1,T_2}, \dots, q^{K_i,T_{j+1}}, q^{K_i,T_j} - x, q^{K_i,T_{j+2}}, \dots, q^{K_n,T_m}).$$

The Hamilton-Jacobi-Bellman equation of the stochastic control problem (14) is given by

$$0 = \partial_t v(t,q) + \mathscr{H}_E\big(t,q,v(t,\cdot)\big), \quad v(T,q) = -1, \tag{22}$$

with

$$\mathscr{H}_E\big(t,q,v(t,\cdot)\big) = \sup_{z\in\mathcal{Z}} h_E\big(t,q,s,z,v(t,\cdot)\big),$$

$$\begin{aligned} h_E\big(t,q,s,z,v(t,\cdot)\big) = v(t,q)\Big(&\frac{\eta}{2}\gamma\sigma^2\Big(\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \Delta^{k,\tau}(z^{C^{k,\tau}} + q^{k,\tau})\Big)^2 + \frac{\eta^2}{2}\sigma^2\Big(\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} \Delta^{k,\tau} z^{C^{k,\tau}}\Big)^2\Big) \\ &+ \sum_{i=a,b}\ \sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}} h^i_{k,\tau}\Big(t, z^{k,\tau,i}, v(t,q), v\big(t, q\ominus_{k,\tau}\phi(i)\big)\Big)\mathbf{1}_{\phi(i)\mathcal{Q}>-\overline{q}} \end{aligned}$$

and

$$h^i_{k,\tau}(t,z,y,y') = \big(y' x_1^{k,\tau} e^{az} - y x_2 e^{bz}\big) O_{k,\tau}$$

where

$$x_1^{k,\tau} = e^{-\eta(c^{k,\tau}+\omega\left(\delta_\infty^{k,\tau} - \gamma^{-1}\log(1+\frac{\sigma\gamma}{C})\right)}, \quad x_2 = \big(1 + \eta\frac{1-(1+\frac{\sigma\gamma}{C})^{-1}}{\gamma}\big), \quad O_{k,\tau} = (1+\frac{\sigma\gamma}{C})^{-\frac{C}{\gamma\sigma}} e^{-\frac{C}{\sigma}f^{k,\tau}},$$

and

$$a = \eta(1-\omega) + \frac{C}{\sigma}, \quad b = \frac{C}{\sigma}.$$

Tedious but straightforward computations lead to the following optimizers:

$$\begin{aligned} z^{\star k,\tau,i} &:= \frac{1}{a-b}\log\big(\frac{b x_2 v(t,q)}{a x_1^{k,\tau} v\big(t, q\ominus_{k,\tau}\phi(i)\big)}\big), \\ z^{\star C^{k,\tau}} &:= -\frac{\gamma}{\gamma+\eta} q^{k,\tau}. \end{aligned}$$

Note that from these computations, we get that this above optimization makes sense only if we assume that there exists δ_∞ large enough so that for $i = a$ or b, $k \in \mathcal{K}$, $\tau \in \mathcal{T}$ and any t, q:

$$|-z_t^{\star k,\tau,i}(t,q) + \frac{1}{\gamma}\log\Big(1+\frac{\sigma\gamma}{C}\Big)| < \delta_\infty. \tag{23}$$

We will check that we can make such choice at the end of the verification argument. Equation (22) is rewritten as

$$0=\partial_t v(t,q)+v(t,q)\frac{\gamma\eta^2}{\gamma+\eta}\frac{\sigma^2}{2}\Big(\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}}\Delta^{k,\tau}q^{k,\tau}\Big)^2-v(t,q)\sum_{i=a,b}\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}}\tilde{C}^{k,\tau}\Big(\frac{v(t,q)}{v(t,q\ominus_{k,\tau}\phi(i))}\Big)^{\frac{C}{\sigma\eta(1-\omega)}}\mathbf{1}_{\phi(i)\mathcal{Q}>-\overline{q}}, \tag{24}$$

where

$$\tilde{C}^{k,\tau}=x_2\big(\frac{x_2}{x_1^{k,\tau}}\big)^{\frac{a}{a-b}}O_{k,\tau}\Big(\big(\frac{b}{a}\big)^{\frac{b}{a-b}}-\big(\frac{b}{a}\big)^{\frac{a}{a-b}}\Big)>0.$$

We now make the ansatz $v(t,q)=u(t,\mathcal{Q})$. We derive the following PDE

$$0=\partial_t u(t,\mathcal{Q})+u(t,\mathcal{Q})\frac{\gamma\eta^2}{\gamma+\eta}\frac{\sigma^2}{2}\mathcal{Q}^2-u(t,\mathcal{Q})\sum_{i=a,b}\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}}\tilde{C}^{k,\tau}\Big(\frac{u(t,\mathcal{Q})}{u(t,\mathcal{Q}-\Delta^{k,\tau}\phi(i))}\Big)^{\frac{C}{\sigma\eta(1-\omega)}}\mathbf{1}_{\phi(i)\mathcal{Q}>-\overline{q}}, \tag{25}$$

with terminal condition $u(T,\mathcal{Q})=-1$.

Using the classical change of variable $\tilde{u}:=(-u)^{-\frac{C}{\sigma\eta(1-\omega)}}$, PDE (25) becomes

$$0=\partial_t\tilde{u}(t,\mathcal{Q})-\tilde{u}(t,\mathcal{Q})\frac{C\gamma\eta}{\gamma+\eta}\frac{\sigma}{2(1-\omega)}\mathcal{Q}^2+\sum_{i=a,b}\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}}\hat{C}^{k,\tau}\tilde{u}(t,\mathcal{Q}-\Delta^{k,\tau}\phi(i))\mathbf{1}_{\phi(i)\mathcal{Q}>-\overline{q}}, \tag{26}$$

where $\hat{C}^{k,\tau}:=\tilde{C}^{k,\tau}\frac{C}{\sigma\eta(1-\omega)}$. Eventually Cauchy-Lipschitz theorem provides existence and uniqueness of a bounded solution to (26) and so to (24).

For the verification argument, we first introduce a technical lemma.

Lemma 5. *Let $Z\in\mathcal{Z},\xi=Y_T^{\hat{Y}_0,Z}$. We define*

$$K_t^Z:=\exp\Big(-\eta\Big(\sum_{i=a,b}\sum_{(k,\tau)\in\mathcal{K}\times\mathcal{T}}c^{k,\tau}N_t^{k,\tau,i}-\int_0^t\omega\big(\Delta^i(Z_s^{k,\tau,i})-\delta_\infty^{k,\tau}\big)\mathrm{d}N_s^{k,\tau,i}-Y_t^{Y_0,Z}\Big)\Big),\quad t\in[0,T].$$

There exists $\epsilon>0$ such that

$$\sup_{t\in[0,T]}\mathbb{E}^{\hat{\delta}(Z)}\left[|K_t^Z|^{1+\epsilon}\right]<+\infty,$$

where $\hat{\delta}(Z)$ is defined in Theorem 1.

The proof is borrowed from [EEMRT18]. We now verify that the unique solution v of Equation (22) coincides at any point $(0,Q_0)$ with the value v_0^E of the reduced problem (14). We also prove that in (14), the maximum is achieved for feedback controls issued from (15).

Using Itô's formula we get

$$\begin{aligned}\mathrm{d}[v(t,Q_t)K_t^Z] = K_{t^-}^Z\Big[\Big(h_E\big(t,Q_{t^-},S_t,Z_t,v(t,\cdot)\big)-\mathscr{H}_E\big(t,Q_{t^-},v(t,\cdot)\big)\Big)\mathrm{d}t \\ + v(t,Q_{t^-})\eta\sum_{k=1}^{\mathscr{K}}\sum_{\tau=1}^{\mathscr{T}} Z_t^{C^{k,\tau}}\mathrm{d}C_t^{k,\tau} \\ + \sum_{i=a,b}\sum_{k=1}^{\mathscr{K}}\sum_{\tau=1}^{\mathscr{T}}\big(e^{-\eta(c^{k,\tau}-Z_t^{k,\tau,i})}v(t,Q_{t^-}^{k,\tau}-\phi(i))-v(t,Q_{t^-})\big)\mathrm{d}N_t^{\hat{\delta}(Z),k,\tau,i}\Big].\end{aligned}$$

The process K^Z is uniformly integrable on $[0,T]$ according to (21), Hölder inequality and the boundedness of the intensity of the processes $N^{k,\tau,i}$. Moreover v being uniformly bounded as a consequence of the Cauchy-Lipschitz theorem, the process $\big(v(t,Q_t)K_t^Z\big)_{t\in[0,T]}$ is a $\mathbb{P}^{\hat{\delta}(Z)}$ supermartingale and the local martingale term in the above equation is a true martingale. Hence

$$v(0,Q_0)\geq \mathbb{E}^{\hat{\delta}(Z)}[v(T,Q_T)K_T^Z] = -\mathbb{E}^{\hat{\delta}(Z)}[K_T^Z]. \tag{27}$$

Since $Z\in\mathscr{Z}$ is arbitrary, we get

$$v(0,Q_0)\geq \sup_{Z\in\mathscr{Z}} -\mathbb{E}^{\hat{\delta}(Z)}[K_T^Z] = v_0^E.$$

The feedback form of Z, issued from (15), being bounded according to Equation (23), it is admissible. Considering the process $Z^\star$, we get an equality instead of an inequality in the above equation.

For consistency we now check that there does exist some constant δ_∞ such that (23) is satisfied. In the same vein as in Step 2 of the proof of Theorem 1, we can show that for any t and q, $v(t,q)$ is negative. Because of the compactness of the domain of v, the function is uniformly negative: we can find ε such that $v<-\varepsilon$ on $[0,T]\times\mathscr{D}$. Consequently $\log\Big(\frac{v(t,q)}{v\left(t,q^{k,\tau}\ominus_{k,\tau}\phi(i)\right)}\Big)$ is uniformly bounded in i, k, τ, t and q. Thus we can always choose a δ_∞ satisfying (23).

CHAPTER VII

Optimal auction duration: A price formation viewpoint

Abstract

We consider an auction market in which market makers fill the order book during a given time period while some other investors send market orders. We define the clearing price of the auction as the price maximizing the exchanged volume at the clearing time according to the supply and demand of each market participants. Then we derive in a semi-explicit form the error made between this clearing price and the efficient price as a function of the auction duration. We study the impact of the behavior of market takers on this error. To do so we consider the case of naive market takers and that of rational market takers playing a Nash equilibrium to minimize their transaction costs. We compute the optimal duration of the auctions for 77 stocks traded on Euronext and compare the quality of price formation process under this optimal value to the case of a continuous limit order book. Continuous limit order books are found to be usually sub-optimal. However, in term of our metric, they only moderately impair the quality of price formation process. Order of magnitude of optimal auction durations is from 2 to 10 minutes.

Keywords: Microstructure, market design, auctions, limit order books, continuous trading, market making, Nash equilibrium, BSDEs.

1 Introduction

In most historical (lit) markets, trading operates through a continuous-time double auction system: the continuous limit order book (CLOB). This mechanism allows market participants to buy or sell shares at any time point at the quoted prices. However market orders systematically pay (at least) the spread as transaction cost. Moreover volumes impact prices as market makers readjust their positions in reaction to the order flow, resulting in additional trading costs. Beyond this, it has been argued that some mechanical flaws are inherent to CLOBs, particularly in the context of high frequency trading. The debate started in the academic literature notably with the very influential paper [BCS15], see also [FS12a, WW13]. In this work, the authors explain that CLOBs lead to obvious mechanical arbitrage and generate a competition in speed

rather than in price between high frequency market makers, to the detriment of final investors. They convincingly show that frequent batch auctions could be a way to remedy these flaws.

The idea that auctions could be a suitable mechanism for the functioning of financial markets is not new. For example, in the important paper [Mad92], see also [GS79], the interest of auctions compared to CLOBs is already investigated. Of course the discussion in this work is not about high frequency arbitrage opportunities, but rather on the fact that auctions could be beneficial for the price formation process by enabling investors to trade directly between each others, avoiding to pay spread costs to market makers.

In [BCS15], the authors provide the order of magnitude of a lower bound for auction period leading to elimination of the high frequency flaws of CLOBs (about 100 milliseconds). However, the mentioned earlier literature suggests that longer auction times could be suitable, but usually without giving figures. This is why, quoting [BCS15], *developing a richer understanding of the costs of lengthening the time between auctions is an important topic.* This is exactly what we do in this paper. We provide a sound and operational quantitative analysis of the optimal auction duration on a financial market, and compare the efficiency of this mechanism with that of a CLOB. We work with a criterion based on quality of the price formation process as in [Mad92], but in the context of modern high frequency markets as in [BCS15]. Thus we hope to bridge the gap between these two seminal papers.

Actually, there seems to be a growing interest in practice for trading outside standard CLOBs. For example, a very important fraction of trading activity is still made over the counter and a rising part of market participants turns to new forms of market structures such as dark pools or auctions. Some auctions are already organized regularly in many markets where the main mechanism is a CLOB, typically at the beginning and at the end of the trading day. There also exist auction markets where auctions take place one after the other all along the day, and without continuous trading phase between two auctions. During an auction, market participants can send and cancel limit or market orders. Then at a certain time (possibly random), a clearing price is fixed in order to maximize the exchanged volume and matched orders are executed at this price. This is for example the case in the BATS-Cboe periodic auctions market for European equities. In this market, auctions are triggered when a first order is sent (limit or market). Then settlement takes place at a random time such that the auction cannot last more than a pre-fixed duration (100 milliseconds)[1].

In an auction context, the key issue for a regulator or an exchange is to set a proper time period for the auction, and to compare the relevance of this mechanism with that of a CLOB. In [DZ17], the authors study the efficiency of an auction market with respect to the duration of the auction. They propose a microscopic agent-based model with deterministic or stochastic arrival of private and public information. Agents optimize their demand schedules with respect to their information and some personal characteristics. The average utility over all agents is used as a criterion to prove that the optimal auction duration is related to the law of exogenous

[1] https://markets.cboe.com/europe/equities/trading/periodic_auctions_book

information arrival. The authors also study the impact of heterogenous speeds of agents. They show that fast agents prefer short auction durations while slow ones prefer long ones. However, in the case of heterogenous agents, they do not give any results on what the optimal auction duration should be.

Most other works on this topic use a price formation point of view to assess the quality of the specification of an auction. This is the case in [GS79] where the authors propose a simple model for price formation in an auction market. The average squared difference between an efficient price and the clearing price is used as a metric to show that a positive optimal auction duration always exists. The suggested optimal duration is a trade-off between averaging effect (a long duration allows a large number of agents to take part in the auction, hence reducing uncertainty about the efficient price) and volatility risk (a short duration leads to small volatility risk). This model has been refined in [FG18]. In this paper, the authors investigate several generalizations of this framework such as the multi-assets case or the presence of a market maker using filtering techniques and observing correlated assets to infer the efficient price at the clearing time.

In our work, the same driving forces as in [FG18, GS79] will be key to define our optimal durations: averaging effect *versus* volatility risk. However, an important limitation in [FG18, GS79] is that no market orders are considered so that all the agents can be seen as liquidity providers. It is necessary to relax this assumption since one observes a large part of market orders in the trading flows of actual auctions, see [BDLM17]. For example, market participants having a marked to market benchmark or seeking for priority in execution may typically use market orders. This type of orders will have a crucial role when computing optimal auction durations. This is because long durations induce large variance in the imbalance of the market order flow leading to less accurate fixing prices.

Another important remark is that in [DZ17, FG18, GS79], no comparison between the auction and CLOB markets can be made. This is because the CLOB structure is not included in the range of their models. They obtain optimal durations for auction markets, but cannot say wether CLOB markets are sub-optimal or not. In our modeling, CLOBs exactly correspond to auctions with duration equal to zero, making the comparison between auctions and CLOBs possible.

In this paper, inspired by the cited earlier literature, we take price discovery as our criterion to compute an optimal auction duration. Our approach extends in several directions those in [FG18, GS79, Mad92] and goes as follows. We consider a regenerative auction market with auctions starting when a market order is sent and with constant duration h. More precisely, we assume that after the $(i-1)-$th auction clearing (ended at time τ_{i-1}^{cl}) the limit order book is emptied and a new market phase starts independently of the past. A new auction opens at time τ_i^{op} when a first market order is sent. This new auction ends at time $\tau_i^{cl} = \tau_i^{op} + h$. Our model encompasses both CLOB and auction market structures since CLOB corresponds to an auction with duration 0 (because auctions are triggered by the arrival of a market order, as in

several actual markets, and we assume that the LOB is never empty at the auction clearing).

We now describe briefly our model for liquidity and price formation with two kinds of agents. During the $i-$th market phase the arrival of the $k-th$ market maker sending limit orders is given by $\tau^{cl}_{i-1}+\tau^{i,mm}_k$ where $\tau^{i,mm}_k$ is the $k-$th event time of a counting process $(N^{i,mm}_s)_{s\geq 0}$. The limit orders sent by the $k-$th market maker are represented by a supply function $S_k(p)$ that gives, when positive, the number of shares the market maker is willing to sell at price p or above (when $S_k(p)$ is negative, it means that the market maker wants to buy shares at price p or below). The supply function S_k depends on the market maker's view on the efficient price at the clearing time when he sends his limit orders. The efficient price process is $(P_s)_{s\geq 0}$. It can be seen as the average of market makers' opinions at a given time on the "fair" value of the underlying asset. More precisely, we take

$$S_k(p)=K(p-\tilde{P}_k),\ \text{with}\ \tilde{P}_k=\mathbb{E}[P_{\tau^{cl}_i}|\mathscr{F}_{\tau^{cl}_{i-1}+\tau^{i,mm}_k}]+g_k,$$

where K is a positive constant, $(g_k)_{k>0}$ a sequence of i.i.d random variables with variance σ^2 representing the estimation noise in the inference of the efficient price by the market maker and $(\mathscr{F}_t)_{t\geq 0}$ the natural filtration associated to the efficient price process. Hence the market maker is willing to sell (resp. buy) shares above (resp. below) the price $\tilde{P}_k$, which is his view on the efficient price at the clearing time. In the sequel, we assume that $P_s=P_0+\sigma_f W_s$, where W is a Brownian motion and σ_f a positive constant. Note that in this case, we simply have

$$\tilde{P}_k=P_{\tau^{cl}_{i-1}+\tau^{i,mm}_k}+g_k.$$

Then we consider market takers who send market orders with fixed volume v. During the $i-$th market phase the $k-$th buy (resp. sell) market order is sent at time $\tau^{cl}_{i-1}+\tau^{i,a}_k$ (resp. $\tau^{i,b}_k$), where $\tau^{i,a}_k$ and $\tau^{i,b}_k$ are respectively the $k-$th event times of two counting processes $(N^{i,a}_s)_{s\geq 0}$ and $(N^{i,b}_s)_{s\geq 0}$. The auction beginning time is then given by

$$\tau^{op}_i=\tau^{cl}_{i-1}+\tau^{i,a}_1\wedge\tau^{i,b}_1.$$

Consequently at time $\tau^{cl}_i=\tau^{op}_i+h$, a clearing price $P^{cl}_{\tau^{cl}_i}$ is set in order to maximize the exchanged volume. This clearing rule is used in most of electronic markets for the opening and clearing auctions. This is also the rule considered in the academic literature (see for instance [DZ17]). We have that this price corresponds to the equilibrium between supply and demand curves and is the only solution to

$$\sum_{k=1}^{N^{i,mm}_{\Delta_i}} S_k(P^{cl}_{\tau^{cl}_i})=v(N^{i,a}_{\Delta_i}-N^{i,b}_{\Delta_i})$$

where $\Delta_i=\tau^{cl}_i-\tau^{cl}_{i-1}$ is the duration of the $i-$th market phase.

To compare the different auction durations (0 corresponding to the CLOB market), we consider as a metric the integrated quadratic error between the efficient price and the clearing price, denoted by Z_t:

$$Z_t = \sum_{i=1}^{N_t^{cl}-1} \Delta_{i+1}(P_{\tau_i^{cl}} - P^{cl}_{\tau_i^{cl}})^2 + (t - \tau^{cl}_{N_t^{cl}})(P_{\tau^{cl}_{N_t^{cl}}} - P^{cl}_{\tau^{cl}_{N_t^{cl}}})^2,$$

with N_t^{cl} the number of cleared auctions before time t. Thus Z_t measures the realized error of the clearing price and therefore the quality of the price formation process[2]. We define our optimal auction duration as the one leading to the smallest value for Z_t, asymptotically as t goes to infinity. Using the regeneration property for the marker order flow, we can show that

$$\frac{Z_t}{t} \underset{t\to+\infty}{\to} E(h) = \mathbb{E}[(P^{cl}_{\tau_1^{cl}} - P_{\tau_1^{cl}})^2].$$

The optimal auction duration is therefore a minimizer of $E(h)$. We are able to provide a semi-explicit expression for $E(h)$ in a very general setting for the market order flow counting processes (N^a, N^b). In the Poisson case, we even obtain a closed-form formula.

In CLOB markets, there is competition between market makers optimizing their quotes and market takers search for suitable execution times. In auction markets, market takers have an additional possibility to access cheap liquidity: they can try to match their orders with other market orders sent in the opposite direction. For example if a large volume of buy market orders is sent before the auction clearing, it is a good opportunity for selling market takers to execute their orders at a good price. In this context, a new form of competition between buying and selling market takers may arise, with market makers playing a side role. We also investigate this situation where market takers are strategic and act optimally in order to reduce their trading costs. We notably show that there exists a Nash equilibrium for this game. In this framework, we can compute the function E too, and thus find an optimal auction duration. From a mathematical point of view, the existence of a Nash equilibrium in this type of problems is up to our knowledge new and extends the results of [HM14] to the case of counting processes.

Finally, based on Euronext exchange data, we use our model to compute the optimal auction duration according to our price discovery criterion for 77 European stocks traded on Euronext. The first striking result is that the suggested durations are much larger than a few milliseconds, rather of order of 1 to 5 minutes. The second one is that in term of our metric, CLOB are indeed sub-optimal. However, the quality of the price formation process in CLOB market is not very far from that of the auction with optimal duration. Of course this work is only a first step towards a full analysis of the auction issue since we focus here on one specific (but crucial) criterion. Other aspects such as liquidity, tick size effects and fees or potential arbitrage opportunities should certainly be addressed in future works. We also neglect potential optimization of market makers strategies who could for example revise their quotes during

[2]Note that we of course have in mind cases with relatively frequent auctions during the day, so that Z_t is close to the integral of the quadratic error between the clearing price and the efficient price over the day.

the auction according to the current market orders imbalance. Nevertheless, we believe our results are original and striking enough to help exchanges and financial authorities rethink their policies in terms of market design.

The paper is organized as follows. In Section 2 we describe the auction mechanism and our model. We also provide our first main result on the computation of the function E. The case of strategic market takers optimizing their trading cost is considered in Section 3. Our calibration methodology and numerical results on equity data can be found in Section 4. Proofs are relegated to an Appendix.

2 The model

In this section, we introduce our model for auction market. We describe the organization of the market and the behavior of the two types of agents: market makers filling the limit order book (LOB) with limit orders and market takers sending market orders. Then we explain the clearing rule and compute the clearing price. Finally we provide a semi-explicit expression for the quality of the price formation process.

2.1 Auction market design

We consider an auction market organized in independent sequential auctions triggered by market orders. More precisely, after the opening of the market or after the clearing of an auction, a new auction starts when a first market order is sent. We write $(\tau_i^{op})_{i\in\mathbb{N}^*}$ for the sequence of opening times of the auctions and $(\tau_i^{cl})_{i\in\mathbb{N}}$, with $\tau_0^{cl}=0$, for the sequence of clearing times. An auction has a duration of h seconds and allows market takers to meet. When $h=0$, our model corresponds to a CLOB market since any market order is matched against the limit orders present in the LOB.

For a given auction starting at some time τ_i^{op}, market participants can send market or limit orders. At the auction clearing time $\tau_i^{cl}=\tau_i^{op}+h$, a clearing price, denoted by $P^{cl}_{\tau_i^{cl}}$, is set to maximize the exchanged volume. More precisely, sellers who are willing to sell below the price $P^{cl}_{\tau_i^{cl}}$ sell their shares to buyers who are willing to buy above $P^{cl}_{\tau_i^{cl}}$. Each cleared share is sold at the clearing price, independently of the posted price of the associated limit order if any (to the benefit of participants sending limit orders).

2.2 Market makers and market takers

Along the day, market makers arrive randomly in the market and send limit orders to fill the LOB. During the $i-$th market phase market makers arrival times are given by $(\tau_{i-1}^{cl}+\tau_k^{i,mm})_{k\geq 0}$ where $\tau_k^{i,mm}$ is the $k-$th event time of a counting process $(N_s^{i,mm})_{s\geq 0}$. We describe the liquidity provided by the $k-$th market maker by its supply function S_k. When positive, the quantity $S_k(p)$ represents the number of shares the $k-$th market maker is willing to sell at price p or

above. Negative values correspond to shares he is willing to buy at price p or below. We assume that

$$S_k(p) = K(p - \tilde{P}_k),$$

where $\tilde{P}_k$ is the view on the price of the asset by the $k-$th market maker when he sends his orders and K a positive constant. Linear supply functions are also considered in [DZ17, FG18]. This is equivalent to assume that each market maker sends uniform limit sell order above price $\tilde{P}_k$ and uniform limit buy orders to price level below $\tilde{P}_k$. We suppose that

$$\tilde{P}_k = P_{\tau^{cl}_{i-1}+\tau^{i,mm}_k} + g_k,$$

where $(g_k)_{k\geq 0}$ is a sequence of i.i.d centered random variables with variance σ^2, independent of all other processes, and $(P_s)_{s\geq 0}$ is the efficient price of the asset. It satisfies $P_s = P_0 + \sigma_f W_s$ with W a Brownian motion and σ_f a positive constant.

In practice there are different kinds of market makers and we could have assumed that each market maker has its own noise. That said, there are typically not so many market makers in the market and since they basically have the same technology, it is reasonable to assume that they have the same noise parameter. Note also that a model with different variance parameters would be very hard to calibrate because of the anonymity of our data.

Consequently, market makers inject information in the LOB since they reveal the knowledge they have on the price through their supply function. However, the longer the auction duration, the less reliable the views of market makers arrived early in term of the estimation of the efficient price P at the clearing time[3,4]. Finally to obtain a regenerative market we consider that after the auction clearing time τ^{cl}_i market makers cancel their unmatched limit orders. Since sequential auctions markets with sufficiently large durations do not really exist, it is hard to have an idea of what would be the market maker's behavior. Of course, total cancellation after the clearing is not so realistic. However, note that the times of interest of our analysis are the auctions closing times, where the model is very reasonable. For example, when $h = 0$, which corresponds to a CLOB market, at each time a market order is sent, the LOB is already filled thanks to Assumption 2 below. By recalling that $\Delta_i = \tau^{cl}_i - \tau^{cl}_{i-1}$ we deduce that at the clearing there is $N^{i,mm}_{\Delta_i}$ market makers in the LOB.

During the $i-$th market phase the arrival time of the $k-th$ buy (resp. sell) market order is given by $\tau^{cl}_{i-1} + \tau^{i,a}_k$ (resp. $\tau^{i,b}_k$) where $\tau^{i,a}_k$ (resp. $\tau^{i,b}_k$) is the $k-$th event time of the counting process $(N^{i,a}_s)_{s\geq 0}$ (resp. $(N^{i,b}_s)_{s\geq 0}$). Consequently the opening time of the $i-$th auction is $\tau^{op}_i = \tau^{cl}_{i-1} + \tau^{i,a}_1 \wedge \tau^{i,b}_1$. We suppose that each market taker sends market orders with constant volume v. Moreover we assume that $(N^{i,mm}, N^{i,a}, N^{i,b})$ is independent of the efficient price P. We define I^i as the cumulated imbalance of the market takers: $I^i_t = vN^{i,a}_t - vN^{i,b}_t$. The

[3]To partially address this issue we extend our model allowing market makers to revise their position by canceling their orders in Appendix 4.

[4]Note that a possible extension would be to consider that $\tilde{P}_k$ also depends on recently observed clearing prices, see [FG18].

aggregated demand of the market takers at the clearing of the $i-$th auction is thus given by $I^i_{\Delta_i}$.

We now make the following natural assumption, which states that market is regenerative.

Assumption 1. *The market dynamics satisfy:*

i) After each auction clearing the market regenerates: the processes $(N^{i,mm}, N^{i,a}, N^{i,b}, I^i)_{i\geq 0}$ *are independent and identically distributed.*

ii) The random variables $(\tau_1^{i,a} \wedge \tau_1^{i,b})_{i\geq 0}$ *are i.i.d. with exponential law with parameter* ν.

iii) The random variables $N^{1,a}_{\tau_1^{cl}}$ *and* $N^{1,b}_{\tau_1^{cl}}$ *are squared integrable.*

Points i) and ii) of Assumption 1 imply that market order flow is basically a Poisson process, which is the most standard dynamic used in the microstructure literature, see [AS08, Gué17]. This assumption is not perfectly realistic, in particular it does not enable us to reproduce the long memory property of market order flow, see for example [BFL09]. A possible way to relax this assumption would be to consider Hawkes-type intensities. However this would make the model much more intricate in terms of computation and calibration. Point iii) is a classical technical assumption.

Note that Points i) and ii) of Assumption 1 mean that for any $i \geq 0$, $\tau^{op}_{i+1} - \tau^{cl}_i$ follows an exponential random variable with parameter ν. We also consider (N^{mm}, N^a, N^b, I) a random variable with the law of $(N^{1,mm}, N^{1,a}, N^{1,b}, I^1)$. This will be useful to lighten some notations.

In practice it is very unlikely that a market taker sends a market order if there is no liquidity in the LOB and a situation with empty LOB is very unrealistic. A way to adapt the non empty LOB assumption setting is to consider that the first market maker always arrives before the auction clearing occurs. It means that almost surely for any i we have $\tau_1^{i,mm} < (\tau_1^{i,a} \wedge \tau_1^{i,b}) + h$. Hence we consider the following assumption

Assumption 2. *Let* $\mu > 0$. *The density of* $\left(\tau_1^{1,mm}, (\tau_1^{1,a} \wedge \tau_1^{1,b})\right)$ *at point* $(s,t) \in \mathbb{R}^2$ *is given by*

$$\mathbf{1}_{0\leq s\leq t+h} \frac{\mu e^{-\mu s}}{1 - e^{-\mu(t+h)}} \mathrm{d}s \; \nu e^{-\nu t} \mathbf{1}_{t\geq 0} \mathrm{d}t.$$

Finally we assume that $(N^{1,mm}_{s+\tau_1^{1,mm}} - 1)_{0\leq s\leq h}$ *is a Poisson process with intensity* μ *that is independent of* P *and* $(N^{1,a}_s, N^{1,b}_s)_{s\geq \tau_1^{1,mm}}$.

Assumption 2 means that $(N^{1,mm}_s)_{0\leq s\leq \tau_1^{op}+h}$ has the law of a Poisson process with intensity μ conditional on the fact that its first event occurs before time τ_1^{cl}. This assumption[5] also allows to obtain a non-degenerate CLOB at the limit $h = 0$.

[5] An alternative idea leading to a very different approach would be to endogenize the market behavior of market makers, see [DZ17]

2.3 Clearing rule

We now explain how the clearing price is settled at the end of an auction. We denote by $F^-(p)$ (resp. $F^+(p)$) the total number of shares that buyers (resp. sellers) are willing to buy (resp. sell) at price p. The function F^- (resp. F^+) is decreasing (resp. increasing). Assume that a clearing price P^{cl} is set. The total volume exchanged is then $F^-(P^{cl}) \wedge F^+(P^{cl})$. Now suppose that F^- and F^+ are continuous at point P^{cl} and $F^-(P^{cl}) < F^+(P^{cl})$. If there is still remaining liquidity on the bid side of the book (formally if F^- is not bounded by $F^-(P^{cl})$), the exchanged volume is not optimal as it may be improved by decreasing the price. Conversely, assume that $F^-(P^{cl}) > F^+(P^{cl})$ and if there is liquidity on the ask side (formally, if F^+ is not bounded by $F^+(P^{cl})$), then the exchanged volume is not optimal as it may be improved by increasing the price. Thus, when such equality is possible and in order to maximize volume at the clearing time, the optimal clearing price has to satisfied

$$F^-(P^{cl}) - F^+(P^{cl}) = 0. \tag{1}$$

Note that the value $F^-(+\infty)$ (resp. $F^+(-\infty)$) is the number of shares to be bought (resp. to be sold) at any price.

The function $F^- - F^+$ is the algebraic supply-demand function of all market participants together (market makers and market takers). Thus we have obtained that the clearing price is a zero of the aggregated supply-demand of the agents. Consequently, in our framework, the clearing price $P^{cl}_{\tau_i^{cl}}$ of the $i-th$ auction, defined as a solution of (1), can be found solving the following equation:

$$\sum_{k=1}^{N^{i,mm}_{\Delta_i}} S_k(p) - I^i_{\Delta_i} = 0.$$

The $i-$th clearing price is then given by

$$P^{cl}_{\tau_i^{cl}} = \frac{1}{N^{i,mm}_{\Delta_i}} \sum_{k=1}^{N^{i,mm}_{\Delta_i}} \tilde{P}_k + \frac{1}{K} \frac{I^i_{\Delta_i}}{N^{i,mm}_{\Delta_i}}. \tag{2}$$

Finally, we define the mid-price P^{mid} of the LOB as the obtained clearing price without taking into account market orders in the auction clearing:

$$P^{mid}_{\tau_i^{cl}} = \frac{1}{N^{i,mm}_{\Delta_i}} \sum_{k=1}^{N^{i,mm}_{\Delta_i}} \tilde{P}_k. \tag{3}$$

2.4 A metric for the quality of the price formation process

One of the main roles of a financial market is to reveal with accuracy the price of the underlying asset, guaranteeing fair transaction prices to market participants. In our framework, this is equivalent to have a clearing price close to the efficient price. Therefore a natural criterion to compare different auction durations is to assess, with respect to the auctions duration, the

accumulated error between the efficient price and the clearing prices over the day. To do so, we consider the following time-weighted quadratic error:

$$Z_t^h = \sum_{i=1}^{N_t^{cl}-1} \Delta_{i+1}(P_{\tau_i^{cl}} - P^{cl}_{\tau_i^{cl}})^2 + (t - \tau^{cl}_{N_t^{cl}})(P_{\tau^{cl}_{N_t^{cl}}} - P^{cl}_{\tau^{cl}_{N_t^{cl}}})^2, \tag{4}$$

where N_t^{cl} denotes the number of auctions cleared before time t. Thus, for each auction, we consider the quadratic deviation between the clearing price and the efficient price and weight this deviation by the time to wait until a new price is set. Note that (4) may be rewritten

$$Z_t^h = \int_0^t (\overline{P}_s^{cl} - \overline{P}_s)^2 \mathrm{d}s,$$

where the processes $\overline{P}_s^{cl}$ and $\overline{P}_s$ are respectively the clearing and efficient price at the last auction clearing time before time s, that is

$$(\overline{P}_s^{cl}, \overline{P}_s) = (P^{cl}_{\tau_i^{cl}}, P_{\tau_i^{cl}}), \text{ where } i = \sup\{j \geq 1, \text{ s.t } \tau_j^{cl} \leq s\}.$$

We define an auction duration h^* as optimal if almost surely, $Z_t^{h^*}$ is asymptotically smaller than Z_t^h for any $h \geq 0$. Using the fact that $\left((\overline{P}_s^{cl} - \overline{P}_s)^2\right)_{s\geq 0}$ is a regenerative process we obtain, see Appendix 5, the following important result for our asymptotic computations.

Lemma 1. *The following convergence holds almost surely:*

$$\lim_{t\to+\infty} \frac{Z_t^h}{t} = \mathbb{E}[(P^{cl}_{\tau_1^{cl}} - P_{\tau_1^{cl}})^2].$$

In light of Lemma 1, a duration h^* is optimal if it is a minimizer of the function E given by

$$E(h) = \mathbb{E}[(P^{cl}_{\tau_1^{cl}} - P_{\tau_1^{cl}})^2].$$

We also consider the efficiency of the mid-price defined in (3), denoted by E^{mid}:

$$E^{mid}(h) = \mathbb{E}[(P^{mid}_{\tau_1^{op}+h} - P_{\tau_1^{op}+h})^2].$$

We now give our first main theorem. It provides a semi-explicit expression for the function E. Its proof is given in Appendix 1.

Theorem 1. *The quality of price formation process metric satisfies:*

$$E(h) = E^{mid}(h) + \frac{\mathbb{E}[I^2_{\tau_1^{cl}}]}{K^2}(1 - e^{-\mu h}\frac{\nu}{\nu+\mu})^{-1} e^{\nu h} \int_h^{+\infty} \nu e^{-\nu t} e^{-\mu t} \int_0^{\mu t} \frac{1}{s} \int_0^s \frac{e^u - 1}{u} \mathrm{d}u\mathrm{d}s\mathrm{d}t,$$

with $E^{mid}(h)$ given by

$$(1 - e^{-\mu h}\frac{\nu}{\nu+\mu})^{-1} e^{\nu h} \int_h^{+\infty} \nu e^{-\nu t} \Big((\sigma_f^2 \frac{t}{6} + \sigma^2) e^{-\mu t} \int_0^{\mu t} \frac{e^s - 1}{s} \mathrm{d}s + \sigma_f^2 \frac{t}{3}(1 - e^{-\mu t}) \Big) \mathrm{d}t. \tag{5}$$

Remark 1. *Note that we can simplify the double integrals by using the so-called Exponential Integral function* $E_1 : \mathbb{R}_+^* \longrightarrow \mathbb{R}_+$ *defined by* $E_1(x) = \int_x^{+\infty} \frac{e^{-u}}{u} du$. *We thus get*

$$E(h) = E^{mid}(h) + \frac{\mathbb{E}[I^2_{\tau_1^{cl}}]}{K^2}(1 - e^{-\mu h}\frac{\nu}{\nu+\mu})^{-1} e^{\nu h}\Big(\int_h^{+\infty} \frac{e^{-\mu u - 1}}{u} \frac{\nu}{\nu+\mu} E_1\big((\nu+\mu)u\big)\mathrm{d}u$$
$$+ \int_0^h \frac{e^{-\mu u - 1}}{u} \frac{\nu}{\nu+\mu}\big(\log(\frac{h}{u})e^{-(\nu+\mu)h} + E_1((\nu+\mu)h)\big)\mathrm{d}u\Big),$$

where E^{mid} *is given by* (5).

We remark from Theorem 1 that for given $h > 0$, $E(h) > E^{mid}(h)$. This is quite intuitive: the presence of market orders induces here additional deviations of clearing prices which are not directly driven by information, rather by imbalance between supply and demand. Of course when $\mu = 0$, we get $E(h) = E^{mid}(h)$. We also see that the price formation process is of higher quality when K is large. In that case a large amount of liquidity is already present close to the efficient price, leading to better transaction prices. Finally note that a similar expression as the one in Theorem 1 can be obtained when we allow market makers to cancel their orders, see Appendix 4.

If we have access to the quantity $\mathbb{E}[I^2_{\tau_1^{op}+h}]$, which depends on the market takers behavior, Theorem 1 enables us to compute the function E and therefore to find the optimal auction duration by minimizing E. We can for example consider the standard assumption that N^a and N^b are independent Poisson processes with intensity $\nu/2$ along the auction (this is consistent with Assumption 1). In this case we get

$$\mathbb{E}[I^2_{\tau_1^{op}+h}] = v^2(\nu h + 1),$$

see Appendix 2. Therefore the function $h \to E(h)$ of Theorem 1 becomes fully explicit and we can obtain numerically the optimal duration. We refer to Section 4 for numerical details, empirical results and statistical methodology to estimate the parameters appearing in the expression of $E(h)$.

The Poisson assumption for the market order flow is very classical and leads to easy computations and simple results. However, in an auction setting, market orders play a crucial role and one should also investigate the possibility of strategic placements, taking into account the auction environment. We deal with this situation in the next section.

3 Strategic market takers

In practice, market orders are sent through algorithms optimizing transaction times. So, in this section, we consider that market takers aim at minimizing their trading costs by adapting their trading intensities to the market state. We formalize this into a competitive game between buying and selling market takers. We show that this game admits a Nash equilibrium.

Moreover, when market takers follow the strategies corresponding to this Nash equilibrium, we can compute the key quantity $\mathbb{E}[I^2_{\tau_1^{op}+h}]$ appearing in the expression of $E(h)$. Note that it would of course be interesting to also consider that market makers are also strategic alter their behaviors in response to changing duration of the auction, see [BCS15, DZ17]. However we left this case for further research and focus here on the specific feature of auction markets from a market taker viewpoint.

3.1 Trading costs of market takers

We model the aggregated group of buying (resp. selling) market takers as a single player called Player a (resp. b). During the auctions, Player a (resp. b) controls the intensity λ^a (resp. λ^b) of the arrival process N^a (resp. N^b), wishing to get minimal costs. In practice, market orders are often send to execute large metaorders over a specified time-interval. Consequently, whatever the market design, market takers are usually required to buy or sell a certain volume on a given period. To reproduce the fact that market takers intensities can neither be too high nor too low, we assume that λ^a and λ^b are bounded from above and below by two positive constants λ_+ and λ_-.

The aggregated total trading cost at time t of buying market takers, denoted by C^a_t, satisfies

$$C^a_t = \sum_{i=1}^{N^{cl}_t} N^{i,a}_{\Delta_i}(P^{cl}_{\tau^{cl}_i} - P_{\tau^{cl}_i}).$$

From Theorem 3.1 in Chap VI in [Asm08] together with the fact that the market is regenerative we obtain the following lemma on the asymptotic behavior of the trading costs.

Lemma 2. *We have the following almost sure convergence:*

$$\lim_{t\to+\infty} \frac{C^a_t}{t} = \mathbb{E}[N^a_{\tau^{cl}_1}(P^{cl}_{\tau^{cl}_1} - P_{\tau^{cl}_1})]\frac{\nu}{1+\nu h}.$$

Therefore, in the long run, the average trading cost of buying market takers is a multiple of

$$\mathbb{E}[N^a_{\tau^{cl}_1}(P^{cl}_{\tau^{cl}_1} - P_{\tau^{cl}_1})] = \frac{\nu}{K}\mathbb{E}[\frac{1}{N^{mm}_{\Delta_i}}]\mathbb{E}[N^a_{\tau^{cl}_1}(N^a_{\tau^{cl}_1} - N^b_{\tau^{cl}_1})].$$

Now writing $N^a_{\tau^{cl}_1} = N^a_{\tau^{op}_1+h} - N^a_{\tau^{op}_1} + N^a_{\tau^{op}_1}$ and using the fact that $N^a_{\tau^{op}_1}$ is either equal to one or zero, solving the problem of Player a is equivalent to be able to minimize

$$\mathbb{E}[N^a_h(N^a_h - N^b_h)]$$

when $(N^a_0, N^b_0) = (1,0)$ and when $(N^a_0, N^b_0) = (0,1)$. Consequently, for any $(\alpha,\beta) \in \mathbb{N}^2$, we consider the more general problem for Player a minimizing

$$\mathbb{E}[N^a_h(N^a_h - N^b_h)|(N^a_0, N^b_0) = (\alpha,\beta)].$$

In the same way, Player b minimizes $\mathbb{E}[N_h^b(N_h^b - N_h^a)|(N_0^a, N_0^b) = (\alpha, \beta)]$. Each player aims at deriving its own trading intensity which will lead to the smallest possible trading costs for him.

Note that in our setting, Assumption 1 implies that market takers reset their strategies at the beginning of each auction. We could have considered the case where market takers optimize their behavior all along the day. However, since we are interested in the effects of auction durations in a stationary context, our framework remains reasonable.

3.2 Nash equilibrium

We now give our result on the existence of a Nash equilibrium in this game of competing market takers. We consider that market takers control their trading intensities. The set of admissible controls is denoted by $\mathscr{U}$ and defined as the set of $\mathbb{F}-$predictable processes with values in $[\lambda_-, \lambda_+]$ for fixed $0 < \lambda_- \leq \lambda_+$. Any couple of strategies $(\lambda_a, \lambda_b) \in \mathscr{U}^2$ of Player a and b induces a probability measure $\mathbb{P}^{\lambda_a, \lambda_b}$ such that

$$N_\cdot^a - \int_0^\cdot \lambda_s^a \mathrm{d}s \text{ and } N_\cdot^b - \int_0^\cdot \lambda_s^b \mathrm{d}s$$

are martingales under $\mathbb{P}^{\lambda_a, \lambda_b}$. In order to minimize its costs, Player a solves

$$\inf_{\lambda^a \in \mathscr{U}} V_h^{a,\alpha,\beta}(\lambda_a, \lambda_b), \tag{6}$$

with $V_h^{a,\alpha,\beta}(\lambda_a, \lambda_b) = \mathbb{E}^{\mathbb{P}^{\lambda_a,\lambda_b}}[N_h^a(N_h^a - N_h^b)|(N_0^a, N_0^b) = (\alpha, \beta)]$, for fixed λ_b chosen by the selling market taker, Player b. Symmetrically, Player b solves

$$\inf_{\lambda^b \in \mathscr{U}} V_h^{b,\alpha,\beta}(\lambda_a, \lambda_b), \tag{7}$$

with $V_h^{b,\alpha,\beta}(\lambda_a, \lambda_b) = \mathbb{E}^{\mathbb{P}^{\lambda_a,\lambda_b}}[N_h^b(N_h^b - N_h^a)|(N_0^a, N_0^b) = (\alpha, \beta)]$ for fixed λ_a chosen by the buying market taker, Player a. A Nash equilibrium is obtained if the two optimization problems (6) and (7) can be addressed simultaneously.

Note that this framework is realistic regarding the information observable by market takers. Indeed we only assume that market takers observe market orders imbalance. This information is for example available on the Euronext platform for the opening and closing auctions and on the auctions market of BATS-Cboe.

We prove that this game indeed admits a (non-necessarily unique) Nash equilibrium with corresponding optimal controls $(\lambda_a^\star, \lambda_b^\star)$. More precisely using these notations we have the following result.

Theorem 2. *There exists a Nash equilibrium to the simultaneous optimization problem* (6)-(7) *given by some Markovian controls*[6] $(\lambda_a^\star, \lambda_b^\star)$ *satisfying*

$$\inf_{\lambda^a \in \mathscr{U}} V_h^{a,\alpha,\beta}(\lambda_a, \lambda_b^\star) = \mathbb{E}^{\mathbb{P}^{\lambda_a^\star, \lambda_b^\star}}[N_h^a(N_h^a - N_h^b)|(N_0^a, N_0^b) = (\alpha, \beta)]$$

[6]The notion of Markovian control has to be understood in the sense of [CD18, Definition 2.10]

and

$$\inf_{\lambda^b \in \mathcal{U}} V_h^{b,\alpha,\beta}(\lambda_a^\star, \lambda_b) = \mathbb{E}^{\mathbb{P}^{\lambda_a^\star, \lambda_b^\star}}[N_h^b(N_h^b - N_h^a) | (N_0^a, N_0^b) = (\alpha, \beta)].$$

The proof of Theorem 2 is provided in Appendix 3. The HJB equation related to the optimization problem is somehow degenerated. This prevents us from using classical arguments to obtain a solution. In order to give intuition about it we give here a short sketch of the proof.

Step 0. We first consider a smoothed version of the HJB equation associated with our problem. Hence, the proof of Theorem 2 is reduced to the existence and then convergence of the solutions of a (smooth) system of HJB equations (see Theorem 3).

Step 1. We consider the backward stochastic differential equation (BSDE for short) associated to the smoothed HJB equation. The existence of a Nash equilibrium is then related to the existence of a solution to this (Lipschitz) BSDE.

Step 2. We prove that the sequence of BSDEs converges in suitable spaces towards a solution of a degenerate BSDE.

Step 3. We conclude by showing that the solution we obtain at the limit corresponds to a Nash equilibrium of the competition between market takers.

Note that we do not get uniqueness of the Nash equilibrium, only the existence. Since the generator of the BSDE associated to this problem has discontinuities there is almost no chance that a uniqueness result can be found by classical methods. Moreover, even if the method used give uniqueness of the limit Nash equilibrium, this limit will be strongly dependent of the smoothing procedure.

The proof of Theorem 2 also provides a numerical method to approximate $V_h^{a,1,0}(\lambda_a^\star, \lambda_b^\star)$ and $V_h^{b,1,0}(\lambda_a^\star, \lambda_b^\star)$ using solutions of some integro-differential equations, see Appendix 3.3. It is particularly important since it enables us to compute optimal auction durations when market takers are playing the Nash equilibrium. This is because the function E of Theorem 1 explicitly depends on $V_h^{a,1,0}(\lambda_a^\star, \lambda_b^\star)$ and $V_h^{b,1,0}(\lambda_a^\star, \lambda_b^\star)$, as stated in the following corollary.

Corollary 1. *Under the Nash equilibrium* $(\lambda_a^\star, \lambda_b^\star)$*, we have*

$$\mathbb{E}[I^2_{\tau_1^{op}+h}] = V_h^{a,1,0}(\lambda_a^\star, \lambda_b^\star) + V_h^{b,1,0}(\lambda_a^\star, \lambda_b^\star).$$

4 Optimal auction durations for some European stocks

We give here the results obtained on real data when applying our methodology to derive optimal auction durations. We consider both situations of non-strategic and strategic market takers and compare with the CLOB case.

4.1 Description of the data

We have access to intra-day market data for 77 of the most liquid stocks traded on Euronext exchange, for all trading days of September 2018. For each stock, every trade is reported with the following information:

- Timestamp of the trade.
- Traded volume.
- Execution price.
- Best bid and ask prices just before the transaction.
- Volumes at best bid and best ask just before the transaction.

We discard from our study trades related to 1% upper and lower quantiles in term of volume in order to remove some outliers.

4.2 Calibration of model parameters

Our market data are CLOB data and not auction data. Still, we are able to calibrate the parameters of our model as explained below.

4.2.1 Market takers parameters

The behavior of market takers is characterized by three parameters:

- Their intensity of arrival between two auctions ν.
- The volume of market orders v.
- The upper and lower bounds for their trading intensity λ_- and λ_+.

CLOB corresponds to the case where auctions last zero second. Consequently, in our framework, the market order flow in a CLOB market is given by two Poisson processes N^a and N^b with intensity ν. Thus we estimate ν by the average number of market orders per day divided by the duration of a trading day and v by the average volume of a market order. Finally we set $\lambda_+ = 2\nu$ and $\lambda_- = \nu/4$. This choice seems reasonable since the market order flow should have similar order of magnitude irrespectively of the market design so that agents can complete execution of their metaorders.

4.2.2 Market makers parameters and calibration of price volatility

The behavior of market makers is characterized by three parameters:

- The variance σ of the $(g_i)_{i\geq 0}$. We assume that σ is equal to the implicit spread of the asset that we estimate from the uncertainty zones model of [DR15].

- The intensity of market makers arrivals μ.
- The slope of their supply function K.

Let α be the tick value of the asset. According to our model, in the CLOB case, the average volume available in the first limit of the LOB when a market taker arrives, denoted by e, satisfies

$$e = K\alpha\mathbb{E}[N^{mm}_{\tau_1^{op}}] = K\alpha\frac{\nu+\mu}{\nu}$$

and the average squared volume of the first limit, denoted by ς, satisfies

$$\varsigma = K^2\alpha^2\mathbb{E}[(N^{mm}_{\tau_1^{op}})^2] = K^2\alpha^2\frac{\nu+\mu}{\nu}(1+2\frac{\mu}{\nu}).$$

Those results are a direct consequence of Assumption 2 and of some computations. Consequently we have

$$K = (2e - \frac{\varsigma}{e})\alpha^{-1} \text{ and } \mu = \nu(\frac{e}{\alpha K} - 1).$$

So we can estimate μ and K from empirical measurements of e and ς. Finally, we estimate the volatility σ_f of the efficient price from the five minutes sampling based realized volatility of the traded price.

4.3 Numerical results

Using our approach, we provide in Table VII.1 and VII.2 the optimal auction durations for 77 stocks traded on Euronext. We give the results when assuming Poisson arrivals for the market takers and when considering they optimize their trading costs, leading to a Nash equilibrium (see Appendix 3.3 for numerical aspects in this case).

The first column is the stock name. In the Poisson (resp. Nash) case, the second (resp. fourth) column is the optimal duration in seconds. The third (resp. fourth) one is the relative difference of quality of the price formation process between the optimal duration case and the CLOB situation: $(E(0) - E(h^*))/E(h^*)$. In the optimal durations columns we provide estimated optimal durations together with 90% confidence interval (with respect to the estimated value for the parameter ν).

The optimal duration range is essentially between 0 and 10 minutes and our results are very robust to the parameter ν. For all the assets such that the optimal auction duration for Poisson market takers is positive, the optimal duration in the Nash case is smaller. Some assets have the CLOB structure as optimal in the Poisson case. However, when considering the Nash case, CLOB become always suboptimal. We also remark that no straightforward structural explanation (sector, capitalization, ...) seems to explain the difference in optimal duration between assets. Finding microstructural foundations for these results is left for further work.

As explained in Section 3, we constrain market takers trading intensities to the range $[\lambda_-, \lambda_+]$. From numerical experiments, by testing several ranges of controls $[\lambda_-, \lambda_+]$, we have observed

	DurationPoisson	DiffrelPoisson	DurationNash	DiffrelNash
Bouygues	228 [226;230]	1%	152 [150;153]	20%
Arkema	397 [392;400]	23 %	268 [265; 272]	19 %
Michelin	1053 [1046;1060]	60%	763 [757;768]	89%
Eurofins Scient.	761 [749;773]	18%	554 [546;563]	37%
Engie	866 [857;875]	104%	866 [857;875]	158%
Stmicroelectronics	177 [176;179]	2%	123 [122;124]	21%
Alstom	0 [0;0]	0%	180 [178;181]	14%
Legrand SA	325 [322;329]	0%	216 [214;221]	19%
Eiffage	0 [0;0]	0%	149 [147;150]	12%
Eramet	1086 [1074;1098]	30%	812 [803;822]	50%
SES Sa	0 [0;0]	0%	81 [80;83]	6%
Pernod Ricard	427 [423;430]	22%	301 [298;304]	45%
Iliad	163 [162;164]	0%	109 [108;110]	18%
Faurecia	0 [0;0]	0%	36 [35;37]	4%
Orange	382 [379;385]	21%	274 [273.6;278]	42%
Sodexo	0 [0;0]	0%	49 [51;47]	1%
Air France - KLM	295 [292;297]	17%	218 [216;220]	35%
Teleperformance	1241 [1224;1259]	27 %	881 [868;894]	50 %
Hermes	295 [292;298]	1%	205 [203;207]	19%
Eutelsat Com.	0 [0; 0]	0 %	40 [39; 42]	2 %
Nexans	487 [480;494]	8%	360 [356;365]	23%
Ingenico Group	0 [0;0]	0%	143 [142;144]	15%
Unibail - Wfd Unibai	187 [186;188]	19%	142 [141;143]	36%
Plastic Omnium	0 [0;0]	0%	176.5 [176.3;176.8]	9%
Veolia Environ.	350 [346;353]	3%	253 [251;256]	21%
Schneider Electric	246 [245;248]	39%	171 [170;172]	65%
Peugeot	386 [383;389]	10%	282 [280;285]	29%
Vinci	350 [348;353]	39%	252 [250;253]	64%
CGG	837 [827;847]	15%	605 [597;613]	36%
Atos	962 [954;969]	66%	700 [694;706]	95%
Suez Environnement	0 [0;0]	0%	311 [308;315]	14%
Danone	204 [203;206]	15%	146 [145;147]	35%
Kering	133 [132;134]	19%	93.4 [93.1;94]	42%

Table VII.1 – Optimal auction durations (in seconds) Part 1 with a 90% confidence interval.

	DurationPoisson	DiffrelPoisson	DurationNash	DiffrelNash
EssilorLuxottica	342 [339;345]	30%	238 [236;240]	55%
Lagardere	0 [0;0]	0 %	42 [39; 44]	3 %
Credit Agricole	87.7 [87.2;88.5]	2%	58.6 [58;59.4]	22%
CapGemini	502 [497;508]	20%	354 [350;358]	43%
Lvmh	121 [120;122]	6%	87.3 [87;88]	25%
Valeo	0 [0;0]	0%	98 [97;98.2]	16%
Air Liquide	627 [622;632]	35%	459 [456;463]	58%
Total	359 [357;360]	60%	261 [260;263]	89%
Vivendi	1023 [1014;1031]	42%	750 [743;756]	67%
Casino Guichard	158 [157;159]	15%	119 [118;120]	28%
Societe Generale	104 [104;105]	18%	74.1 [74;74.3]	40%
Klepierre	0 [0;0]	0%	219 [217;221]	14%
Publicis Groupe	601 [595;606]	32%	428 [424;432]	56%
Sanofi	124 [123;124]	12%	88.2 [88;89]	32%
Thales	644 [637;652]	23%	454 [449;460]	46%
TechnipFMC	331 [327;334]	7%	234 [232;236]	27%
Bnp Paribas	104.3 [104.2;104.8]	18%	73.4 [73.2;74]	41%
Safran	0 [0;0]	0%	107 [106;108]	16%
Saint Gobain	0 [0;0]	0%	58.2 [58;59]	11%
Orpea	834 [822;846]	29%	578 [569;587]	55%
Carrefour	410 [407;413]	34%	293 [291;295]	58%
Ipsen	827 [817;838]	65%	551 [544;559]	101%
Natixis	351 [348;354]	9%	253 [251;255]	28%
EDF	341 [338;344]	15%	246 [244;248]	35%
Axa	252 [251;254]	36%	182 [181;183]	60%
Dassault Systemes	316 [312;319]	7%	222 [220;224]	27%
Accor Hotels	0 [0;0]	0%	105.3 [105.8;104.7]	6%
Airbus	210 [209;211]	34%	146 [145;147]	60%
Ubi Soft Entertain	0 [0;0]	0%	43.4 [43;44]	1%
Renault	0 [0;0]	0%	41.7 [41;42.2]	3%
Solvay	528 [522;534]	11%	375 [371;380]	32%
Edenred	313 [309;316]	8%	210 [208;212]	29%

Table VII.2 – Optimal auction durations (in seconds) Part 2 with a 90% confidence interval.

that the optimal duration is quite robust to those parameters. Still we remark the following sensitivities: if we allow for a smaller λ_- without modifying λ_+, the optimal auction duration becomes larger. This is because having a small λ_- means that market takers can send less market orders when the situation is not in their favor. This implies that $\mathbb{E}[I^2_{\tau^{op}+h}]$ increases slowlier with h. This moves the minimum of E to a higher level. For symmetric reasons, if we raise λ_+ without changing λ_-, the optimal auction duration becomes smaller.

We notice that CLOBs are sometimes optimal in the Poisson case. When they are not, the difference in the values of the metric for $h = 0$ and $h = h^*$ is typically not very large. Therefore even though CLOB markets are usually sub-optimal, they are in general leading to a fairly satisfactory market microstructure. On BATS-Cboe the auction duration is approximately 100ms which is very small compared to the typical optimal auction durations we find. Moreover according to the empirical study [BLF19], there is essentially only one market order involved in each auction. This means that the duration of auctions chosen by BATS-Cboe does not allow buyers and sellers market takers to match their orders, to the profit of market makers. Indeed a larger auction duration may lead to smaller gains for market makers. For example if market takers always match their orders with other market takers, market makers never collect the spread. Hence it is possible that BATS-Cboe chose this short duration in order to keep its platform attractive for market makers, which guarantee its liquidity. This is actually another possible point of view on this problematic that we have not considered in this paper. It is also likely that exchanges may be reluctant to change drastically their market design so that clients are not too surprised. This could also explain why they decided to move only slightly from the CLOB system.

1 Proof of Theorem 1

We are reduced to compute :

$$E(h) = \mathbb{E}[(P_{\tau_1^{cl}} - P^{cl}_{\tau_1^{cl}})^2 | \widetilde{\Omega}]$$

When N^{mm} be a Poisson process with intensity μ and $\widetilde{\Omega} = \{N^{mm}_{\tau_1^{cl}} > 0\}$. We are reduced to compute

$$E(h) = \mathbb{E}[(P_{\tau_1^{cl}} - P^{cl}_{\tau_1^{cl}})^2 | \Omega].$$

Thus, recalling that $\tau_1^{op} + h = \tau_1^{cl}$, we get

$$\begin{aligned}
E(h) &= \mathbb{E}\Big[\Big(\sum_{k=1}^{N^{mm}_{\tau_1^{cl}}} \frac{P_{\tau_k^{mm}} - P_{\tau_1^{cl}}}{N^{mm}_{\tau_1^{cl}}}\Big)^2 | \widetilde{\Omega}\Big] + \mathbb{E}\Big[\Big(\sum_{k=1}^{N^{mm}_{\tau_1^{cl}}} \frac{g_k}{N^{mm}_{\tau_1^{cl}}}\Big)^2 | \widetilde{\Omega}\Big] + \frac{1}{K^2}\mathbb{E}\Big[\frac{I^2_{\tau_1^{cl}}}{N^{mm2}_{\tau_1^{cl}}} | \widetilde{\Omega}\Big] \\
&= \mathbb{P}(N^{mm}_{\tau_1^{cl}} > 0)^{-1}\Big(\mathbb{E}\Big[\mathbf{1}_{N^{mm}_{\tau_1^{cl}} > 0}\Big\{\Big(\sum_{k=1}^{N^{mm}_{\tau_1^{cl}}} \frac{P_{\tau_k^{mm}} - P_{\tau_1^{cl}}}{N^{mm}_{\tau_1^{cl}}}\Big)^2 + \Big(\sum_{k=1}^{N^{mm}_{\tau_1^{cl}}} \frac{g_k}{N^{mm}_{\tau_1^{cl}}}\Big)^2 + \frac{1}{K^2}\frac{I^2_{\tau_1^{cl}}}{N^{mm2}_{\tau_1^{cl}}}\Big\}\Big]\Big) \\
&= \mathbb{P}(N^{mm}_{\tau_1^{cl}} > 0)^{-1} e^{\nu h}\int_h^{+\infty} \nu e^{-\nu t}\big(g(t) + \sigma^2 f(t) + \frac{1}{K^2}\ell(t)\big)\mathrm{d}t
\end{aligned} \tag{8}$$

with

$$g(t) = \mathbb{E}[\mathbf{1}_{N_t^{mm}>0}(\sum_{k=1}^{N_t^{mm}} \frac{P_{\tau_k^{mm}} - P_t}{N_t^{mm}})^2],\ f(t) = \mathbb{E}[\frac{\mathbf{1}_{N_t^{mm}>0}}{N_t^{mm}}],\ \text{ and } \ell(t) = \mathbb{E}[I^2_{\tau_1^{cl}}]\mathbb{E}[\frac{\mathbf{1}_{N_t^{mm}>0}}{N_t^{mm2}}].$$

A direct computation gives

$$\mathbb{P}(N^{mm}_{\tau_1^{cl}} > 0) = 1 - e^{-\mu h}\frac{\nu}{\nu + \mu}. \tag{9}$$

We now turn to the computation of the function g. We have the following lemma.

Lemma 3. *We have for any $t > 0$*

$$g(t) = \sigma_f^2 \frac{t^2}{2}\mu\mathbb{E}[\frac{1}{(N_t^{mm}+1)^2}] + \sigma_f^2\frac{t^3}{3}\mu^2\mathbb{E}[\frac{1}{(N_t^{mm}+2)^2}]. \tag{10}$$

Proof. Note that

$$g(t) = \sigma_f^2\mathbb{E}\Big[\mathbf{1}_{N_t^{mm}>0}\sum_{k=1}^{N_t^{mm}}\Big(\frac{W_{\tau_k^{mm}} - W_t}{N_t^{mm} - 1 + 1}\Big)^2\Big] + \sigma_f^2\mathbb{E}\Big[\mathbf{1}_{N_t^{mm}>0}\sum_{k,l=1\ \text{s.t.}\ k\neq l}^{N_t^{mm}} \frac{(W_{\tau_k^{mm}} - W_t)(W_{\tau_l^{mm}} - W_t)}{(N_t^{mm} - 2 + 2)^2}\Big].$$

Consider X_t the Poisson scatter made of the event times of N^{mm} between time 0 and t. Then we have

$$g(t) = \sigma_f^2\mathbb{E}\Big[\sum_{x\in X_t}\frac{(W_x - W_t)^2}{(\#\{X_t\backslash\{x\}\}+1)^2}\Big] + \sigma_f^2\mathbb{E}\Big[\sum_{x,y\in X_t\ \text{s.t.}\ x\neq y}\frac{(W_x - W_t)(W_y - W_t)}{(\#\{X_t\backslash\{x,\ y\}\}+2)^2}\Big].$$

Since $P_t = \sigma_f W_t$ is independent of N^{mm}, we get

$$g(t) = \sigma_f^2 \mathbb{E}\Big[\sum_{x \in X_t} \frac{(t-x)^2}{(\#\{X_t \backslash \{x\}\}+1)^2} \Big] + \sigma_f^2 \mathbb{E}\Big[\sum_{x,y \in X_t \text{ s.t. } x \neq y} \frac{(t-x)\wedge(t-y)}{(\#\{X_t \backslash \{x,\ y\}\}+2)^2} \Big].$$

Finally using Palm's Formula, see for example [CMW17], we get

$$g(t) = \sigma_f^2 \mathbb{E}\big[\frac{1}{(N_t^{mm}+1)^2}\big] \int_0^t (t-u)\mu \mathrm{d}u + \sigma_f^2 \mathbb{E}\big[\frac{1}{(N_t^{mm}+2)^2}\big] \int_0^t \int_0^t (t-u)\wedge(t-v)\mu^2 \mathrm{d}u\mathrm{d}v,$$

and (10) follows.

□

To compute explicitly f, ℓ and g from Lemma 3, we need the following additional results.

Lemma 4. *Let N be a general inhomogeneous Poisson process with intensity measure λ. The following equalities hold:*

$$\mathbb{E}[\frac{\mathbf{1}_{N_t>0}}{N_t}] = e^{-m_t} \int_0^{m_t} \frac{e^s-1}{s} \mathrm{d}s, \ \textit{and}\ \mathbb{E}[\frac{\mathbf{1}_{N_t>0}}{N_t^2}] = e^{-m_t} \int_0^{m_t} \frac{1}{s} \int_0^s \frac{e^u-1}{u} \mathrm{d}u\mathrm{d}s, \tag{11}$$

$$\mathbb{E}[\frac{1}{(1+N_t)^2}] = \frac{e^{-m_t}}{m_t} \int_0^{m_t} \frac{e^s-1}{s} \mathrm{d}s, \ \textit{and}\ \mathbb{E}[\frac{1}{(2+N_t)^2}] = \frac{1}{m_t^2}\big(1 - e^{-m_t} - e^{-m_t} \int_0^{m_t} \frac{e^s-1}{s} \mathrm{d}s\big), \tag{12}$$

with $m_t = \int_0^t \lambda(\mathrm{d}s)$.

Proof of (11). Note that

$$\mathbb{E}[\frac{\mathbf{1}_{N_t>0}}{N_t}] = \sum_{n=1}^{+\infty} \frac{1}{n} \frac{m_t^n}{n!} e^{-m_t} \text{ and } \mathbb{E}[\frac{\mathbf{1}_{N_t>0}}{N_t^2}] = \sum_{n=1}^{+\infty} \frac{1}{n^2} \frac{m_t^n}{n!} e^{-m_t}.$$

The functions e_1 and e_2 defined by

$$e_1(x) = \sum_{n=1}^{+\infty} \frac{1}{n} \frac{x^n}{n!} \text{ and } e_2(x) = \sum_{n=1}^{+\infty} \frac{1}{n^2} \frac{x^n}{n!}$$

are continuously differentiable function, so that

$$e_1'(x) = \sum_{n=1}^{+\infty} \frac{x^{n-1}}{n!} = \frac{e^x-1}{x} \text{ and } x e_2'(x) = \sum_{n=1}^{+\infty} \frac{1}{n} \frac{x^n}{n!} = e_1(x).$$

By integrating these functions, we get (11).

Proof of (12). Note that

$$\mathbb{E}[\frac{1}{(1+N_t)^2}] = \sum_{n=0}^{+\infty} \frac{1}{(1+n)^2} \frac{m_t^n}{n!} e^{-m_t} \text{ and } \mathbb{E}[\frac{1}{(2+N_t)^2}] = \sum_{n=0}^{+\infty} \frac{1}{(2+n)^2} \frac{m_t^n}{n!} e^{-m_t}.$$

Consider, for $i > 0$, the functions

$$r_i(x) = \sum_{n=0}^{+\infty} \frac{1}{(i+n)^2} \frac{x^{n+i}}{n!} \text{ and } s_i(x) = \sum_{n=0}^{+\infty} \frac{1}{i+n} \frac{x^{n+i}}{n!}.$$

We have

$$r_i'(x) = \sum_{n=0}^{+\infty} \frac{1}{i+n} \frac{x^{n+i-1}}{n!} \text{ hence } r_i(x) = \int_0^x \frac{s_i(s)}{s} \mathrm{d}s.$$

Since

$$s_i'(x) = \sum_{n=0}^{+\infty} \frac{x^{n+i-1}}{n!} = x^{i-1}e^x \text{ we get } r_i(x) = \int_0^x \frac{1}{s} \int_0^s u^{i-1}e^u \mathrm{d}u\mathrm{d}s.$$

Taking $i = 1$ and $i = 2$ we get (12). □

Injecting Equations (11) and (9) into f and ℓ and Equation (12) into g in view of (10), using (8) we obtain the formulas stated in Theorem 1.

2 Computation of the expected square imbalance in the Poisson case

We want to compute $\mathbb{E}[I^2_{\tau_1^{op}+h}]$ when N^a and N^b are independent Poisson processes with intensity $\nu/2$. We have

$$\mathbb{E}[I^2_{\tau_1^{op}+h}] = \upsilon^2 \mathbb{E}[\big((N^a_{\tau_1^{op}+h} - N^a_{\tau_1^{op}} + N^a_{\tau_1^{op}}) - (N^b_{\tau_1^{op}+h} - N^b_{\tau_1^{op}} + N^b_{\tau_1^{op}})\big)^2].$$

Using the strong Markov property of Poisson process and taking conditional expectation with respect to τ_1^{op} we get

$$\mathbb{E}[I^2_{\tau_1^{op}+h}] = \upsilon^2(\nu h + 1),$$

where we use $\mathbb{E}[N^a_{\tau_1^{op}}] = \mathbb{E}[(N^a_{\tau_1^{op}})^2] = 1/2$.

3 Existence of a Nash equilibrium

In this section, we set $h > 0$ as a terminal time of the auction to investigate the game played by the market takers.

3.1 Nash equilibrium

We are interested in finding a Nash equilibrium to the game between buyers and sellers. Starting at $(N_0^a, N_0^b) = (\alpha, \beta) \in \mathbb{N}^2$, we set[7]

$$V_h^{a,\alpha,\beta}(\lambda_a, \lambda_b) = \mathbb{E}^{\mathbb{P}^{\lambda_a,\lambda_b}}[N_h^a(N_h^a - N_h^b)] \tag{13}$$

[7]Rigorously speaking we should write $V_0^{i,\alpha,\beta}(\lambda_a, \lambda_b, h)$ instead of $V_h^{i,\alpha,\beta}(\lambda_a, \lambda_b)$ with $i \in \{a, b\}$, since we define here the value function of each market taker at time 0 and h is a time horizon. Since we consider only value functions of market takers at time 0, we make this slight abuse of notation.

$$V_h^{b,\alpha,\beta}(\lambda_a,\lambda_b)=\mathbb{E}^{\mathbb{P}^{\lambda_a,\lambda_b}}[N_h^b(N_h^b-N_h^a)]. \tag{14}$$

Formally, we can thus compute the optimal P&L of market takers for buy orders and sell orders by solving the following coupled system

$$\begin{cases} \inf\limits_{\lambda^a\in\mathscr{U}} V_h^{a,\alpha,\beta}(\lambda_a,\lambda_b^\star) &= \mathbb{E}^{\mathbb{P}^{\lambda_a^\star,\lambda_b^\star}}[N_h^a(N_h^a-N_h^b)] \\ \inf\limits_{\lambda^b\in\mathscr{U}} V_h^{b,\alpha,\beta}(\lambda_a^\star,\lambda_b) &= \mathbb{E}^{\mathbb{P}^{\lambda_a^\star,\lambda_b^\star}}[N_h^b(N_h^b-N_h^a)] \end{cases}, \tag{15}$$

where $\lambda_b^\star$ and $\lambda_b^\star$ are simultaneous optimizers of (13) and (14) respectively (depending on the action of market takers having the opposite behavior).

We now investigate theoretically the existence of a Nash equilibrium associated with (15). First we introduce some notations.

- Let Ω be the set of piece-wise constant functions with jumps of size 1. Consider[8] $X=(N^a,N^b)^\top$ be the canonical processes in Ω^2 and $\mathbb{F}=(\mathscr{F}_s)_{0\le s\le h}$ the smallest filtration for which X is adapted.

- Let $\mathbb{P}$ be a probability measure on $(\Omega^2,\mathscr{F}_h)$ such that

  $$M_s=X_s-s\mathscr{L}_0,\ \text{with}\ \mathscr{L}_0:=(\lambda_0,\lambda_0)^\top,\ 0<\lambda_0<\lambda_+,\ s\in[0,h],$$

  is a local martingale. A proof of the existence of such measure $\mathbb{P}$ is given in [Jac75]. We set $M_r^a:=M_{1,r}$ (resp. $M_r^b:=M_{2,r}$) the first (resp. the second) component of M. Moreover to any pair $(\lambda^a,\lambda^b)\in\mathscr{U}^2$ of admissible controls we associate $\mathbb{P}^{\lambda^a,\lambda^b}$ the measure defined by

  $$\frac{\mathrm{d}\mathbb{P}^{\lambda^a,\lambda^b}}{\mathrm{d}\mathbb{P}}=\exp\Big(\int_0^h\log\big(\frac{\lambda_s^a}{\lambda_0}\big)\mathrm{d}N_s^a-\big(\lambda_s^a-\lambda_0\big)\mathrm{d}s+\log\big(\frac{\lambda_s^b}{\lambda_0}\big)\mathrm{d}N_s^b-\big(\lambda_s^b-\lambda_0\big)\mathrm{d}s\Big).$$

  Hence, under the measure $\mathbb{P}^{\lambda^a,\lambda^b}$,

  $$\Big(X_s-\int_0^s(\lambda_u^a,\lambda_u^b)^\top\mathrm{d}u\Big)_{0\le s\le h}$$

  is a martingale.

- For $(E,\|\cdot\|)$ a normed space, any $0\le s\le t\le h$ and $p>1$, we define

  $$\mathscr{H}_{s,t}^p(E)=\{Y,\ E-\text{valued and }\mathbb{F}-\text{adapted process s.t.},\mathbb{E}[(\int_s^t\|Y_r\|^2\mathrm{d}r)^{\frac{p}{2}}]<+\infty\}$$

  $$\mathscr{S}_{s,t}^p(E)=\{Y,\ E-\text{valued and }\mathbb{F}-\text{adapted process s.t.},\mathbb{E}[\sup_{s\le t}\|Y_r\|^p\mathrm{d}r]<+\infty\}$$

[8] Here for the notation $\top$ denotes the transposition of a vector to identify as usual any element of $\mathbb{N}^2$ with a column vector.

$$\mathbb{L}^p(E) = \{\xi,\ E-\text{valued } \mathscr{F}_h-\text{measurable random variable, s.t. } \mathbb{E}[\|\xi\|^p] < +\infty\}.$$

When $s = 0$ we omit the index s in the previous definitions. If $E = \mathbb{R}^2$, we set $\|\cdot\|_2$ and $\|\cdot\|_1$ the classical Manhattan norm and Euclidean norm on $\mathbb{R}^2$ respectively. For any $\mathbb{R}^2-$valued process $Y := (Y_r)_{0\le r\le h}$, we denote by $Y_{r,1}$ and $Y_{r,2}$ its first and second coordinates respectively for any time $r \in [0,h]$.

- For any $z \in \mathbb{R}^2$ and $\varepsilon^a, \varepsilon^b \in [\lambda_-, \lambda_+]$, we set

$$\textbf{(L)} \begin{cases} \lambda_a^\star(z,\varepsilon^a) &= \mathbf{1}_{z_1>0}\lambda_- + \mathbf{1}_{z_1<0}\lambda_+ + \varepsilon^a \mathbf{1}_{z_1=0} \\ \lambda_b^\star(z,\varepsilon^b) &= \mathbf{1}_{z_2>0}\lambda_- + \mathbf{1}_{z_2<0}\lambda_+ + \varepsilon^b \mathbf{1}_{z_2=0}. \end{cases}$$

Note that both $z_1\lambda_a^\star(z,\varepsilon^a)$ and $z_2\lambda_b^\star(z,\varepsilon^b)$ do not depend on ε^a and ε^b. To alleviate notations, when one of these products appears, we will denote it simply by $z_1\lambda_a^\star(z)$ and $z_2\lambda_b^\star(z)$ respectively.

- For any $z, \tilde{z} \in \mathbb{R}^2$ and any $\varepsilon \in [\lambda_-, \lambda_+]$, we set $H^{a,\star}(z,\tilde{z},\varepsilon) = z_1\lambda_a^\star(z) + z_2\lambda_b^\star(\tilde{z},\varepsilon)$ and $H^{b,\star}(z,\tilde{z},\varepsilon) = z_2\lambda_b^\star(z) + z_1\lambda_a^\star(\tilde{z},\varepsilon)$.

- for $x \in \mathbb{N}^2$ we define $g^a(x) = x_1(x_1 - x_2)$ and $g^b(x) = x_2(x_2 - x_1)$.

- Let U be a map from $[0,h]\times\mathbb{N}^2$ into $\mathbb{R}$. For any $(s,\alpha,\beta) \in [0,h]\times\mathbb{N}^2$ we set

$$(\textbf{D}) \begin{cases} D_aU(s,\alpha,\beta) = & U(s,\alpha+1,\beta) - U(s,\alpha,\beta) \\ D_bU(s,\alpha,\beta) = & U(s,\alpha,\beta+1) - U(s,\alpha,\beta)\big) \\ DU(s,\alpha,\beta) = & (D_aU(s,\alpha,\beta), D_bU(s,\alpha,\beta))^\top. \end{cases}$$

We first provide a very general result by associated to the existence of a Nash equilibrium for (15) a system of coupled ODE on $\mathbb{N}^2$, as a direct extension of [DJVLS00, Theorem 8.5].

Proposition 1. *Assume that there exist two maps $\varepsilon^a, \varepsilon^b$ from $[0,h]\times\mathbb{N}^2$ into $[\lambda_-,\lambda_+]$ such that the following coupled system*

$$(\textbf{S}) \begin{cases} \partial_s V^a + H^{a,\star}(DV^a, DV^b, \varepsilon^b) = 0, \quad s \in [0,h),\ (\alpha,\beta) \in \mathbb{N}^2 \\ V^a(h,\alpha,\beta) = g^a(\alpha,\beta), \\ \partial_s V^b + H^{b,\star}(DV^b, DV^a, \varepsilon^a) = 0, \quad s \in [0,h),\ (\alpha,\beta) \in \mathbb{N}^2 \\ V^b(h,\alpha,\beta) = g^b(\alpha,\beta), \end{cases}$$

has a continuously differentiable (in time) solution denoted by (V^a, V^b) on $[0,h]\times\mathbb{N}^2$ and assume moreover that

$$DV^i(\cdot, N_\cdot^a, N_\cdot^b) \in \mathscr{H}_h^2(\mathbb{R}^2),\ i = a,b.$$

Then, $(\lambda_a^\star(DV^a,\varepsilon^a), \lambda_b^\star(DV^b,\varepsilon^b))$ is a Nash equilibrium for (15).

Proof. The proof follows a standard verification argument. Notice however that we need feedback control for the thresholds $(\varepsilon^a,\varepsilon^b)$ in order to have classical HJB equations. See for instance [DJVLS00, Theorem 8.5]. □

Although the previous result provides sufficient conditions to get a Nash equilibrium for the stochastic differential game (15), it is quite hard to justify such existence in practice. Note indeed that the optimizers $\lambda_a^\star$ and $\lambda_b^\star$ are singular in view of their definition **(L)**. Thus, the main difficulty encountered in this proposition is to solve the bang-bang type system **(S)** of ODEs on $\mathbb{N}^2$ for relevant thresholds $\varepsilon^a, \varepsilon^b$. As far as we now, we have no PDE results ensuring the existence of a solution to **(S)**.

Inspired by [HM14], we thus propose to study a smooth approximation of **(S)** and then to build a sequence of processes converging (up to a subsequence) to a Nash equilibrium for the game (15).

Let $n \in \mathbb{N}$. We consider the smoothed control functions for any $z \in \mathbb{R}$

$$\lambda^n(z) = \begin{cases} \lambda_+ & \text{if } z \leq -\frac{1}{n} \\ \lambda_- & \text{if } z \geq \frac{1}{n} \\ n\frac{\lambda_- - \lambda_+}{2} z + \frac{\lambda_+ + \lambda_-}{2} & \text{if } z \in (-\frac{1}{n}, \frac{1}{n}). \end{cases}$$

The functions λ^n and $z \longmapsto z\lambda^n(z)$ are Lipschitz continuous. Also consider Φ_n, the truncation function defined for any $x \in \mathbb{R}$ by

$$\Phi_n(x) = (x \wedge n) \vee (-n).$$

Hence, we introduction the smoother of $H^\star$ denoted by $H^{\star,n}$ and defined by for any $(z_1, z_2, \tilde{z}) \in \mathbb{R}^3$ by

$$H^{\star,n}(z_1, z_2, \tilde{z}) = \Phi_n(z_1\lambda^n(z_1)) + \Phi_n(z_2)\lambda^n(\tilde{z}).$$

Theorem 3. *For any $n \in \mathbb{N}$, there exists a unique (viscosity) solution denoted by $V^{a,n}$ to the following system of integro-PDEs*

$$(\mathbf{S^n}) \begin{cases} \partial_s V^{a,n} + H^{\star,n}(D_a V^{a,n}, D_b V^{a,n}, D_b V^{b,n}) = 0,\ s \in [0,h),\ (\alpha,\beta) \in \mathbb{N}^2, \\ V^{a,n}(h,\alpha,\beta) = g^a(\alpha,\beta), \\ \partial_s V^{b,n} + H^{\star,n}(D_b V^{b,n}, D_a V^{b,n}, D_a V^{a,n}) = 0,\ s \in [0,h),\ (\alpha,\beta) \in \mathbb{N}^2, \\ V^{b,n}(h,\alpha,\beta) = g^b(\alpha,\beta). \end{cases}$$

Moreover,

- *The system $(\mathbf{S^n})$ admits a unique viscosity solution.*
- *There exists a subsequence $(n_k)_{k\geq 0}$ and two measurable applications V^a, V^b from $[0,h] \times \mathbb{N}^2$ into $\mathbb{R}$ such that for any $(s,\alpha,\beta) \in [0,h] \times \mathbb{N}^2$*

$$\lim_{k \to +\infty} V^{i,n_k}(s,\alpha,\beta) = V^i(s,\alpha,\beta),\ i \in \{a,b\}$$

and

$$\lim_{n \to +\infty} DV^{i,n}(s,\alpha,\beta) = DV^i(s,\alpha,\beta),\ i \in \{a,b\}.$$

- *Moreover* $\lambda^{n_k}(D_a V^{a,n_k}(\cdot, N^a, N^b))\mathbf{1}_{D_a V^a(\cdot,N^a,N^b)=0}$ *and* $\lambda^{n_k}(D_b V^{b,n_k}(\cdot, N^a, N^b))\mathbf{1}_{D_b V^b(\cdot,N^a,N^b)=0}$ *converges weakly in* $\mathcal{H}_h^2(\mathbb{R}^2)$ *to some progressively measurable and* $[\lambda_-, \lambda^+]$*-valued processes denoted respectively by* θ *and* ϑ.

Thus, $(\lambda_a^\star, \lambda_b^\star) = (\lambda_a^\star(DV^a(s, N_s^a, N_s^b), \theta_s), \lambda_b^\star(DV^b(s, N_s^a, N_s^b), \vartheta_s))_{0\le s\le t}$ *is a Nash equilibrium for the game* (15) *and* $V_h^{i,\alpha,\beta}(\lambda_a^\star, \lambda_b^\star) = V^i(0, \alpha, \beta)$, $i \in \{a, b\}$.

We give here the sketch of the proof of this result. The details are postponed to Appendix 3.2.

Sketch of the proof of Theorem 3 The proof will be divided in three steps. The main tool used is the theory of BSDE with jumps (see [TL94, BP94, BBP97]) and their representations through integro-partial differential equations.

Step 1. We associated to the system ($\mathbf{S^n}$) a two dimensional BSDE for which it is well-known that there exists a unique solutions in appropriate spaces.

Step 2. By mimicking the proof of Theorem 2.5 in [HM14] extended to the case of counting processes, we prove that the solution of the BSDE associated to ($\mathbf{S^n}$) converges up to a subsequence to a solution of a two-dimensional BSDE associated with the system ($\mathbf{S}$).

Step 3. We prove that this approximation provides a Nash equilibrium for the game (15) with well-chosen thresholds obtained in Step 2 as limits of functions of the components of the solution to the approached BSDE considered, see Proposition 2 below.

We conclude thanks to semi-linear Feynman-Kac formula for BSDEs and the system ($\mathbf{S^n}$) established in Step 1, together with convergence results.

3.2 Proof of Theorem 3

For the proof we follow the methodology of [HM14]. First we introduce a series of smoothed BSDE with Lipschitz generator by smoothing the controls $\lambda_a^\star$, $\lambda_b^\star$. Then we show that the solution of the smoothed BSDE converges (up to a subsequence) almost surely towards a solution of Equation (28).

We have the following a priori estimates results which is a consequence of the BDG inequalities and of the Gronwall Lemma.

Lemma 5. *For* $(s, x) \in [0, h] \times \mathbb{N}^2$ *let* $X^{s,x}$ *be the process in* Ω *defined onto* $[s, h]$ *by*

$$X_u^{s,x} = x + X_u - X_s.$$

We have for any $s \in [0, h]$ *and* $\rho > 0$

$$\mathbb{E}[\sup_{s\le u\le h} \|X_u^{s,x}\|_1^\rho] \le C_\rho(1 + |x_1|^\rho + |x_2|^\rho)$$

and for any $(\lambda_a, \lambda_b) \in \mathscr{U}^2$

$$\mathbb{E}^{\mathbb{P}^{\lambda_a,\lambda_b}}[\sup_{s\le u\le h} \|X_u^{s,x}\|_1^\rho] \le C_\rho(1+|x_1|^\rho+|x_2|^\rho)$$

We now turn to the proof of Theorem 3.

3.2.1 Step 1: Approximation, existence and uniqueness

From now, $s \in [0,h)$. We recall the definition of smoothed control functions

$$\lambda^n(z) = \begin{cases} \lambda_+ & \text{if } z \le -\frac{1}{n} \\ \lambda_- & \text{if } z \ge \frac{1}{n} \\ n\frac{\lambda_- - \lambda_+}{2} z + \frac{\lambda_+ + \lambda_-}{2} & \text{if } z \in (-\frac{1}{n}, \frac{1}{n}) \end{cases}$$

Consider Φ_n, the truncation function

$$\Phi_n(x) = (x \wedge n) \vee (-n).$$

Now we define the system of smoothed BSDEs for any $u \in [s,h]$:

$$(\mathbf{J^n}) \begin{cases} -\mathrm{d}Y_u^{a,n;s,x} &= (H^{\star,n}(Z_{1,u}^{a,n;s,x}, Z_{2,u}^{a,n;s,x}, Z_{2,u}^{b,n;s,x}) - \mathscr{L}_0 \cdot Z_u^{a,n;s,x})\mathrm{d}u - Z_u^{a,n;s,x} \cdot \mathrm{d}M_u, \\ Y_h^{a,n;s,x} &= g^a(X_h^{s,x}) \\ -\mathrm{d}Y_u^{b,n;s,x} &= (H^{\star,n}(Z_{2,u}^{b,n;s,x}, Z_{1,u}^{b,n;s,x}, Z_{1,u}^{a,n;s,x}) - \mathscr{L}_0 \cdot Z_u^{b,n;s,x})\mathrm{d}u - Z_u^{b,n;s,x} \cdot \mathrm{d}M_u, \\ Y_h^{b,n;s,x} &= g^b(X_h^{s,x}), \end{cases}$$

with $Z_u^{i,n;s,x} = (Z_{1,u}^{i,n;s,x}, Z_{2,u}^{i,n;s,x})^\top$ for any $i \in \{a,b\}$.

From Proposition 2.1. in [BP94] since Φ_n is Lipschitz continuous there exists a unique solution to $(\mathbf{J^n})$ such that

$$\left((Y^{a,n;s,x}, Z^{a,n;s,x}), (Y^{b,n;s,x}, Z^{b,n;s,x})\right) \in \left(\mathscr{S}_{s,h}^2(\mathbb{R}) \times \mathscr{H}_{s,h}^2(\mathbb{R}^2)\right)^2.$$

Moreover (Proposition 3.8. in [BP94]) there exist measurable deterministic functions $V^{a,n}$, $V^{b,n}$ defined on $[s,h] \times \mathbb{N}^2$ with values in $\mathbb{R}$ such that:

$$\forall u \in [s,h],\ Y_u^{i,n;s,x} = V^{i,n}(s, X_u^{s,x}) \text{ and } Z_u^{i,n;s,x} = DV^{i,n}(u, X_{u^-}^{s,x}),\ \text{for } i = a,b. \tag{16}$$

From Theorem 3.4. in [BBP97], we know that the unique solution of $(\mathbf{J^n})$ provides a unique viscosity solution denoted by $(V^{a,n}, V^{b,n})$ to $(\mathbf{S^n})$ and given by (16).

Before going to the convergence of $Y^{i,n}$ and $Z^{i,n}$, notice that by considering the generator functions

$$\begin{cases} H^{a,n}(u,x) &= \left(\Phi_n(D_a V^{a,n}(u,x)\lambda^n(D_a V^{a,n}(u,x))) + \Phi_n(D_b V^{a,n}(u,x))\lambda^n(D_b V^{b,n}(u,x))\right) \\ H^{b,n}(u,x) &= \left(\Phi_n(D_b V^{b,n}(u,x)\lambda^n(D_b V^{b,n}(u,x))) + \Phi_n(D_a V^{b,n}(u,x))\lambda^n(D_a V^{a,n}(u,x))\right), \end{cases}$$

we deduce from (16) that

$$H^{a,n}(u, X^{s,x}_{u^-}) = H^{\star,n}(Z^{a,n;s,x}_{1,u}, Z^{a,n;s,x}_{2,u}, Z^{b,n;s,x}_{2,u}),$$

and

$$H^{b,n}(u, X^{s,x}_{u^-}) = H^{\star,n}(Z^{b,n;s,x}_{2,u}, Z^{b,n;s,x}_{1,u}, Z^{a,n;s,x}_{1,u}),$$

so that $(\mathbf{J^n})$ becomes

$$(\widetilde{\mathbf{J}^{\mathbf{n}}}) \begin{cases} -\mathrm{d}Y^{a,n;s,x}_u &= (H^{a,n}(s, X^{s,x}_{u^-}) - \mathscr{L}_0 \cdot Z^{a,n;s,x}_u)\mathrm{d}u - Z^{a,n;s,x}_u \cdot \mathrm{d}M_u,\ Y^{a,n;s,x}_h = g^a(X^{s,x}_h) \\ -\mathrm{d}Y^{b,n;s,x}_u &= (H^{b,n}(u, X^{s,x}_{u^-}) - \mathscr{L}_0 \cdot Z^{b,n;s,x}_u)\mathrm{d}u - Z^{b,n;s,x}_u \cdot \mathrm{d}M_u,\ Y^{b,n;s,x}_h = g^b(X^{s,x}_h). \end{cases}$$

3.2.2 Step 2: Convergence to the solution of a bang-bang system of BSDEs

From now, we consider any index i equals to a or b, we set $x \in \mathbb{N}^2$ and $s \in [0, h]$.

Step 2a. Uniform estimates.

In order to use dominated convergence we give some uniform a priori estimates for processes $(Y^{i,n;s,x}, Z^{i,n;s,x})$.

We first aim at using a comparison principle to control the upper bound of $Y^{i,n}$ and introduce the following BSDE

$$\overline{Y}^{i,n;s,x}_u = g^i(X^{s,x}_h) + \int_u^h 4\lambda^+ \|\overline{Z}^{i,n;s,x}_r\|_1 \mathrm{d}r - \int_u^h \overline{Z}^{i,n;s,x}_r \cdot \mathrm{d}M_r,\ s \le u \le h. \tag{17}$$

Once again according to [BP94] there exists a unique solution $(\overline{Y}^{i,n;s,x}, \overline{Z}^{i,n;s,x})$ of the above BSDE in the space $\mathscr{S}^2_{s,h}(\mathbb{R}) \times \mathscr{H}^2_{s,h}(\mathbb{R}^2)$ and there exists deterministic measurable functions $\overline{V}^{i,n}$ such that for any $u \in [s, h]$:

$$\overline{Y}^{i,n;s,x}_u = \overline{V}^{i,n}(u, X^{s,x}_u).$$

By comparison theorem for BSDE (see for instance[9] Theorem 2.5 in [Roy06]), for any time $s \le u \le h$ we get

$$Y^{i,n;s,x}_u \le \overline{Y}^{i,n;s,x}_u, \mathbb{P} - a.s. \tag{18}$$

We now give a uniform estimates of $\overline{Y}^{i,n;s,x}$ to get a uniform estimates for $Y^{i,n;s,x}$ in view of the previous relation. Consider the bi-dimensional process:

$$M^{i,n}_u = M_u - 4\lambda_+ \mathrm{sign}(\overline{Z}^{i,n;s,x}_u),$$

where the sign is taken coordinate by coordinate. The process $M^{i,n} = (M^{i,n}_1, M^{i,n}_2)$ is a bi-dimensional martingale under the probability $\mathbb{P}^{i,n}$ equivalent to $\mathbb{P}$ with density given by

$$\mathscr{E}^{i,n}_h = \exp\Big(\int_0^h \log(\frac{\gamma^{i,n}_{t,1}}{\lambda_0})\mathrm{d}N^a_t - (\gamma^{i,n}_{t,1} - \lambda_0)\mathrm{d}t + \log(\frac{\gamma^{i,n}_{t,2}}{\lambda_0})\mathrm{d}N^b_t - (\gamma^{i,n}_{t,2} - \lambda_0)\mathrm{d}t\Big)$$

[9] To be more accurate, we identify our pair of processes as a compound Poisson process with jumps in $\{-1, 1\}$, so that we are in the framework of [Roy06] for a compensator $\lambda(dx) = \lambda_0(\delta_1(dx) + \delta_{-1}(dx))$.

with

$$\gamma^{i,n}_{t,j} = \lambda_0 + 4\lambda_+ \text{sign}(\overline{Z}^{i,n;t,x}_{j,t}).$$

Consequently we get

$$\overline{V}^{i,n}(s,x) = \mathbb{E}^{\mathbb{P}^{i,n}}[g_i(X^{s,x}_h)].$$

By polynomial growth of g_i we deduce that there exists a positive constant $\tilde{C}$ such that

$$|\overline{V}^{i,n}(s,x)| \leq \tilde{C}\mathbb{E}^{\mathbb{P}^{i,n}}[\|X^{s,x}_h\|_2^2].$$

Note that there exists a positive constant $\tilde{\kappa}$ such that

$$\mathbb{E}^{\mathbb{P}^{i,n}}[\|X^{s,x}_h\|_2^2] \leq \tilde{\kappa}(\|x\|_2^2 + 1).$$

The previous equation implies the following polynomial growth bound

$$|\overline{V}^{i,n}(s,x)| \leq C(1 + \|x\|_2^2),$$

where $C := \tilde{C}\tilde{\kappa} > 0$.
According to the comparison result (18) together with (16), we deduce that there exists some positive constant C, which does not depend on n, such that

$$V^{i,n}(s,x) \leq C(1 + |x_1|^2 + |x_2|^2).$$

Similarly, by considering a BSDE similar to (17) but with a minus sign in the generator, we get

$$V^{i,n}(s,x) \geq -C(1 + |x_1|^2 + |x_2|^2).$$

We thus deduce that for any $(s,x) \in [0,h] \times \mathbb{N}^2$ and $p \geq 1$ the following estimate holds for some positive constant C_p

$$\mathbb{E}[\sup_{s\leq u\leq h} |Y^{i,n;s,x}_u|^p] \leq C_p(1 + |x_1|^{2p} + |x_2|^{2p}). \tag{19}$$

Moreover, the characterization (16) allows to transfer the prior estimates of $Y^{i,n;s,x}$ to $Z^{i,n;s,x}$. In particular we get that for any $p \geq 1$

$$\mathbb{E}[\sup_{s\leq u\leq h} |Z^{i,n;s,x}_u|^p] \leq C_p(1 + |x_1|^{2p} + |x_2|^{2p}). \tag{20}$$

Note that the constant C_p does not depend on n, so that Estimates (19) and (20) are uniform with respect to n.

Step 2b. Convergence of the solutions of the smoothed BSDE.

We now turn to the convergence of $(Y^{i,n;s,x}, Z^{i,n;s,x})$, in $\mathscr{S}^2_{s,h}(\mathbb{R}) \times \mathscr{H}^2_{s,h}(\mathbb{R}^2)$. For any $q \leq 2$, there exists a positive constant $\tilde{C}$ which does not depend on n such that

$$\mathbb{E}[\int_0^h |H^{i,n}(r, X^{0,0}_{r^-})|^q \mathrm{d}r \leq \mathbb{E}[\int_0^h 2\lambda_+ \|Z^{i,n;0,0}_r\|_1^q \mathrm{d}r] \leq \tilde{C}.$$

The sequence $(H^{i,n})_{n\geq 0}$ is bounded in $\mathbb{L}^2\big([0,h]\times\mathbb{N}^2, \mathrm{d}r\times\mu(0,0;r,\mathrm{d}x)\big)$ where $\mu(0,0;r,\mathrm{d}x)$ is the law of $X_{r^-}^{0,0}$ under $\mathbb{P}$. Thus there exists a subsequence $(n_k)_{k\geq 0}$ such that $(H^{i,n_k})_{k\geq 0}$ converges weakly in $\mathbb{L}^2([0,h]\times\mathbb{R}, \mu(0,0;r,\mathrm{d}x)\mathrm{d}r)$. We omit the index k and still write n instead of n_k to reduce the notations.

We now prove that for any $(s,x)\in[0,h]\times\mathbb{N}^2$, $(V^{i,n}(s,x))_{n\geq 0}$ is a Cauchy sequence. We set the function $\Delta^{i,n,m}(t,x,z_n,z_m) := H^{i,n}(t,x) - H^{i,m}(t,x) - \mathscr{L}_0\cdot(z_n - z_m)$ with $(n,m)\in\mathbb{N}$ and $(t,x,z_n,z_m)\in[0,T]\times\mathbb{N}^2\times\mathbb{R}^2\times\mathbb{R}^2$. Let $\delta\in[0,h-s]$ and $k\in\mathbb{N}$, we have

$$\begin{aligned}|V^{i,n}(s,x) - V^{i,m}(s,x)| &= |\mathbb{E}[\int_s^h \Delta^{i,n,m}(r, X_{r^-}^{s,x}, Z_r^{i,n;s,x}, Z_r^{i,m;s,x})\mathrm{d}r]| \\ &\leq E_-^{s+\delta,h} + E_+^{s+\delta,h} + E^{s,s+\delta},\end{aligned} \tag{21}$$

with

$$E_-^{s+\delta,h} := |\mathbb{E}[\int_{s+\delta}^h \mathbf{1}_{\|X_{r^-}^{s,x}\|_\infty\leq k}\Delta^{i,n,m}(r, X_{r^-}^{s,x}, Z_r^{i,n;s,x}, Z_r^{i,m;s,x})\mathrm{d}r]|,$$

$$E_+^{s+\delta,h} := |\mathbb{E}[\int_{s+\delta}^h \mathbf{1}_{\|X_{r^-}^{s,x}\|_\infty> k}\Delta^{i,n,m}(r, X_{r^-}^{s,x}, Z_r^{i,n;s,x}, Z_r^{i,m;s,x})\mathrm{d}r]|,$$

and

$$E^{s,s+\delta} := |\mathbb{E}[\int_s^{s+\delta} \Delta^{i,n,m}(r, X_{r^-}^{s,x}, Z_r^{i,n;s,x}, Z_r^{i,m;s,x})\mathrm{d}r]|.$$

We obtain from (20) that there exists some constant C independent of n and m such that

$$E^{s,s+\delta}\leq C\delta.$$

We now turn to $E_+^{s+\delta,h}$. By using Cauchy Schwarz and Markov inequalities together with the prior inequalities (19) and (20), there exists a positive constant $\hat{C}$ again independent of n and m such that for any positive integer k

$$\begin{aligned}E_+^{s+\delta,h} &\leq |\mathbb{E}[\int_{s+\delta}^h \mathbf{1}_{\|X_{r^-}^{s,x}\|_\infty> k}\mathrm{d}r]|^{\frac{1}{2}}|\mathbb{E}[\int_{s+\delta}^h \Delta^{i,n,m}(r, X_{r^-}^{s,x}, Z_r^{i,n;s,x}, Z_r^{i,m;s,x})^2\mathrm{d}r]|^{\frac{1}{2}} \\ &\leq \frac{\hat{C}}{\sqrt{k}}.\end{aligned}$$

Finally, we note that

$$E_-^{s+\delta,h} = |\sum_{(p,q)\in\mathbb{N}^2}\int_s^h \Delta^{i,n,m}(r,p,q,DV^{i,n}(t,p,q), DV^{i,m}(t,p,q))\mathbb{P}\big(X_r^{t,(0,0)} = (p,q)\big)\phi_{s,x}(r,p,q)\mathrm{d}r|$$

with

$$\phi_{s,x}(r,p,q) = \mathbf{1}_{p\leq k}\mathbf{1}_{q\leq k}\mathbf{1}_{r\geq s+\delta}\frac{\mathbb{P}\big(X_r^{t,x} = (p,q)\big)}{\mathbb{P}\big(X_r^{t,(0,0)} = (p,q)\big)}.$$

Since

$$\mathbb{P}\big(X_r^{t,(0,0)} = (p,q)\big)^{-1} = e^{2\lambda_0 r}\frac{p!q!}{(\lambda_0 r)^{p+q}}$$

is bounded for p and q lower than k and r lower than h. The function $\phi_{s,x}$ is bounded and thus in $\mathbb{L}^2([0,h]\times\mathbb{N}^2,\mu(0,0;s,\mathrm{d}x)\times\mathrm{d}s))$ consequently by weak convergence of $H^{i,n}$, we have that $E_-^{s+\delta,h}$ goes to 0 when m,n go to infinity. Hence, taking the limit when δ goes to 0 and k,n,m go to infinity, we deduce from (21) that $(V^{i,n}(s,x))_{n\geq 0}$ is a Cauchy sequence. We thus denote by $V^i(s,x)$ the limit of $(V^{i,n}(s,x))_{n\geq 0}$. We recall that V^i depends on the subsequence $(n_k)_{k\geq 0}$

We have the $\mathbb{P}$-almost sure convergence (up to the subsequence) of $Y_u^{i,n;s,x}$ since $Y_u^{i,n;s,x} = V^{i,n}(u,X_u^{s,x})$. We denote by $Y^{i;s,x}$ the almost sure limit of $Y^{i,n;s,x}$. Notice moreover that in view of (**D**), we have

$$\lim_{n\to+\infty} DV^{i,n}(s,x) = DV^i(s,x),\ (s,x)\in[0,h]\times\mathbb{N}^2. \tag{22}$$

By Equation (19) and Lebesgue dominated convergence theorem we have for any $\rho\geq 1$

$$\mathbb{E}[\int_s^h |Y_r^{i,n;s,x} - Y_r^{i;s,x}|^\rho \mathrm{d}r] \underset{n\to+\infty}{\to} 0. \tag{23}$$

Let now n,m be two positive integers. From Ito's formula applied to $(Y^{i,n;s,x}-Y^{i,m;s,x})^2$ we get for any $s\leq u\leq h$

$$\begin{aligned}
&|Y_u^{i,n;s,x} - Y_u^{i,m;s,x}|^2 \\
&= -\int_u^h |Z_{1,r}^{i,n;s,x} - Z_{1,r}^{i,m;s,x}|^2 \mathrm{d}(M_r^a + \lambda_0 r) - \int_u^h |Z_{2,r}^{i,n;s,x} - Z_{2,r}^{i,m;s,x}|^2 \mathrm{d}(M_r^b + \lambda_0 r) \\
&\quad + 2\int_u^h (Y_r^{i,n;s,x} - Y_r^{i,m;s,x})\big((H^{i,n} - H^{i,m})(r, X_{r^-}^{t,x}) - \mathscr{L}_0\cdot(Z_r^{i,n;s,x} - Z_r^{i,m;s,x})\big)\mathrm{d}r \\
&\quad - 2\int_u^h (Y_r^{i,n;s,x} - Y_r^{i,m;s,x})(Z_r^{i,n;s,x} - Z_r^{i,m;s,x})\cdot \mathrm{d}M_r.
\end{aligned} \tag{24}$$

Using Young's inequality and the definitions of H^n and H^m we deduce that there exists a positive constant $\tilde{c}$ (independent of n and m) such that for any $\varepsilon>0$

$$\begin{aligned}
&|Y_u^{i,n;s,x} - Y_u^{i,m;s,x}|^2 + \int_u^h \lambda_0 \|Z_r^{i,n;s,x} - Z_r^{i,m;s,x}\|_2^2 \mathrm{d}r \\
&\leq \tilde{c}\varepsilon|\lambda_+|^2 \int_u^h \big(\|Z_r^{i,n;s,x}\|_2^2 + \|Z_r^{i,m;s,x}\|_2^2\big)\mathrm{d}r + \frac{1}{\varepsilon}\int_u^h |Y_r^{i,n;s,x} - Y_r^{i,m;s,x}|^2 \mathrm{d}r \\
&\quad - 2\int_u^h (Y_r^{i,n;s,x} - Y_r^{i,m;s,x})(Z_r^{i,n;s,x} - Z_r^{i,m;s,x})\cdot \mathrm{d}M_r \\
&\quad - \int_u^h |Z_{1,r}^{i,n;s,x} - Z_{1,r}^{i,m;s,x}|^2 \mathrm{d}M_r^a - \int_u^h |Z_{2,r}^{i,n;s,x} - Z_{2,r}^{i,m;s,x}|^2 \mathrm{d}M_r^b.
\end{aligned}$$

For $u=s$, by taking the expectation and by choosing n,m large enough, we obtain from (20) and (22), (23) and the fact that ε is arbitrary small that the following convergence holds

$$\limsup_{n,m\to+\infty} \mathbb{E}[\int_s^h \|Z_r^{i,n;s,x} - Z_r^{i,m;s,x}\|_2^2 \mathrm{d}r] = 0. \tag{25}$$

Hence, $(Z^{i,n;s,x})_{n\in\mathbb{N}}$ is a Cauchy sequence (along the subsequence) and thus converges in $\mathscr{H}^2_{s,h}(\mathbb{R}^2)$ to some process $(Z^{i;s,x}_u)_{s\le u\le h}$.

Similarly, by using (24) and by noting that $-\int_u^h |Z^{i,n;s,x}_{1,r} - Z^{i,m;s,x}_{1,r}|^2 \mathrm{d}(M^a_r + \lambda_0 r) \le 0$ and $-\int_u^h |Z^{i,n;s,x}_{2,r} - Z^{i,m;s,x}_{2,r}|^2 \mathrm{d}(M^b_r + \lambda_0 r) \le 0$ since $M^\alpha_\cdot + \lambda_0 \cdot = X_\cdot$ is a non decreasing process for $\alpha \in \{a, b\}$, we have

$$
\begin{aligned}
&\mathbb{E}[\sup_{u\in[s,h]} |Y^{i,n;s,x}_u - Y^{i,m;s,x}_u|^2] \\
&\le \tilde{c}\varepsilon|\lambda_+|^2\mathbb{E}[\int_s^h \big(\|Z^{i,n;s,x}_r\|_2^2 + \|Z^{i,m;s,x}_r\|_2^2\big)\mathrm{d}r] + \frac{1}{\varepsilon}\mathbb{E}[\int_s^h |Y^{i,n;s,x}_r - Y^{i,m;s,x}_r|^2 \mathrm{d}r] \\
&+ 2\mathbb{E}[\int_0^h |Y^{i,n;s,x}_r - Y^{i,m;s,x}_r||Z^{i,n;s,x}_{1,r} - Z^{i,m;s,x}_{1,r}|(dN^a_r + \lambda_0 dr)] \\
&+ 2\mathbb{E}[\int_0^h |Y^{i,n;s,x}_r - Y^{i,m;s,x}_r||Z^{i,n;s,x}_{2,r} - Z^{i,m;s,x}_{2,r}|(dN^b_r + \lambda_0 dr)] \\
&\le \tilde{c}\varepsilon|\lambda_+|^2\mathbb{E}[\int_s^h \big(\|Z^{i,n;s,x}_r\|_2^2 + \|Z^{i,m;s,x}_r\|_2^2\big)\mathrm{d}r] + \frac{1}{\varepsilon}\mathbb{E}[\int_s^h |Y^{i,n;s,x}_r - Y^{i,m;s,x}_r|^2 \mathrm{d}r] \\
&+ 2\mathbb{E}[\int_0^h |Y^{i,n;s,x}_r - Y^{i,m;s,x}_r||Z^{i,n;s,x}_{1,r} - Z^{i,m;s,x}_{1,r}|(dM^a_r + 2\lambda_0 dr)] \\
&+ 2\mathbb{E}[\int_0^h |Y^{i,n;s,x}_r - Y^{i,m;s,x}_r||Z^{i,n;s,x}_{2,r} - Z^{i,m;s,x}_{2,r}|(dM^b_r + 2\lambda_0 dr)] \\
&\le \tilde{c}\varepsilon|\lambda_+|^2\mathbb{E}[\int_s^h \big(\|Z^{i,n;s,x}_r\|_2^2 + \|Z^{i,m;s,x}_r\|_2^2\big)\mathrm{d}r] + \frac{1}{\varepsilon}\mathbb{E}[\int_s^h |Y^{i,n;s,x}_r - Y^{i,m;s,x}_r|^2 \mathrm{d}r] \\
&+ 4\lambda_0\mathbb{E}[\int_0^h |Y^{i,n;s,x}_r - Y^{i,m;s,x}_r|\|Z^{i,n;s,x}_r - Z^{i,m;s,x}_r\|_1 dr].
\end{aligned}
$$

By using again Young inequality for the last term in the previous inequality with the same ε, we deduce that there exists a positive constant $c > 0$ independent of n, m and ε such that

$$
\begin{aligned}
&\mathbb{E}[\sup_{u\in[s,h]} |Y^{i,n;s,x}_u - Y^{i,m;s,x}_u|^2] \\
&\le c\big(\varepsilon|\lambda_+|^2\mathbb{E}[\int_s^h \big(\|Z^{i,n;s,x}_r\|_2^2 + \|Z^{i,m;s,x}_r\|_2^2\big)\mathrm{d}r] + \frac{1}{\varepsilon}\mathbb{E}[\int_s^h |Y^{i,n;s,x}_r - Y^{i,m;s,x}_r|^2 \mathrm{d}r]\big).
\end{aligned}
$$

Since ε is arbitrary and because of Equations (19), (20) and (23) we deduce

$$
\limsup_{n,m\to+\infty} \mathbb{E}[\sup_{u\in[s,t]} |Y^{i,n;s,x}_u - Y^{i,m;s,x}_u|^2] = 0.
$$

So we have the convergence of $(Y^{i,n;s,x})_{n\ge0}$ in $\mathscr{S}^2_{s,h}(\mathbb{R})$ towards a process $(Y^{i;s,x}_u)_{s\le u\le h}$ up to a subsequence.

Step 2c. Convergence of the generator

We study the convergence of $(H^{i,n})_{n\geq 0}$, for $i \in \{a,b\}$ (still along the subsequence introduced in Step 2b.). We focus on $(H^{a,n})_{n\geq 0}$, the proof is identical for $(H^{b,n})_{n\geq 0}$. Recall that

$$H^{a,n}(u, X^{s,x}_{u^-}) = \Phi_n(Z^{a,n;s,x}_{1,u}\lambda^\star_a(Z^{a,n;s,x}_{1,u})) + \Phi_n(Z^{a,n;s,x}_{2,u})\lambda^n(Z^{b,n;s,x}_{2,u}).$$

First note that

$$\Phi_n(Z^{a,n;s,x}_{1,u}\lambda^\star_a(Z^{a,n;s,x}_{1,u})) \underset{n\to+\infty}{\to} Z^{a;s,x}_{1,u}\lambda^\star_a(Z^{a;s,x}_{1,u})$$

with convergence taking place $\mathbb{P}$-a.s. and in $\mathscr{H}^2_{s,u}(\mathbb{R}^2)$ by dominated convergence and uniform integrability of $(\|Z^{a,n;s,x}\|_2^2)_{n\geq 0}$. We split the remaining part in a continuous and a non continuous parts

$$\Phi_n(Z^{a,n;s,x}_{2,u})\lambda^n(Z^{b,n;s,x}_{2,u}) = \Phi_n(Z^{a,n;s,x}_{2,u})\lambda^n(Z^{b,n;s,x}_{2,u})\mathbf{1}_{Z^{b;s,x}_u\neq 0} + \Phi_n(Z^{a,n;s,x}_{2,u})\lambda^n(Z^{b,n;s,x}_{2,u})\mathbf{1}_{Z^{b;s,x}_{2,u}=0}.$$

We have the convergence of $\Phi_n(Z^{a,n;s,x}_{2,u})\lambda^n(Z^{b,n;s,x}_{2,u})\mathbf{1}_{Z^{b;s,x}_{2,u}\neq 0}$, $\mathrm{d}s\times\mathrm{d}\mathbb{P}$ *a.e* and the convergence also holds in $\mathscr{H}^2_{s,h}(\mathbb{R}^2)$. Moreover, $(\lambda^n(Z^{b,n;s,x}_{2,u})\mathbf{1}_{Z^{b;s,x}_u=0})_{n\geq 0}$ being bounded we denote by ϑ a weak limit in $\mathscr{H}^2_{s,h}(\mathbb{R}^2)$.

Now we show that for any stopping time $\tau \in [s,h]$ we have in the sense of weak convergence in $\mathbb{L}^2(\mathbb{R})$:

$$\int_s^\tau \Phi_n(Z^{a,n;s,x}_{2,u})\lambda^n(Z^{b,n;s,x}_{2,u})\mathbf{1}_{Z^{b;s,x}_{2,u}=0}\mathrm{d}u \underset{n\to+\infty}{\to} \int_s^\tau Z^{a;s,x}_{2,u}\vartheta_u\mathbf{1}_{Z^{b;s,x}_{2,u}}\mathrm{d}u. \tag{26}$$

We have

$$\begin{aligned}\int_s^\tau \Phi_n(Z^{a,n;s,x}_{2,u})\lambda^n(Z^{b,n;s,x}_{2,u})\mathbf{1}_{Z^{b;s,x}_{2,u}=0}\mathrm{d}u &= \int_s^\tau (\Phi_n(Z^{a,n;s,x}_{2,u}) - Z^{a;s,x}_{2,u})\lambda^n(Z^{b,n;s,x}_{2,u}))\mathbf{1}_{Z^{b,s,x}_{2,u}=0}\mathrm{d}u \\ &\quad + \int_s^\tau Z^{a;s,x}_{2,u}\lambda^n(Z^{b,n;s,x}_{2,u})\mathbf{1}_{Z^{b;s,x}_{2,u}=0}\mathrm{d}s.\end{aligned}$$

The first term in the previous equality converges to 0 in $\mathbb{L}^2(\mathbb{R})$ by dominated convergence therefore it converges weakly. Now we show that the second term converges weakly. We prove that for any random variable $\xi \in \mathbb{L}^2(\mathbb{R})$ and $\mathscr{F}_h$-measurable the following convergence holds

$$\mathbb{E}[\xi\int_s^\tau Z^{a;s,x}_{2,u}\lambda^n(Z^{b,n;s,x}_{2,u})\mathbf{1}_{Z^{b;s,x}_{2,u}=0}\mathrm{d}u] \underset{n\to+\infty}{\to} \mathbb{E}[\xi\int_s^\tau Z^{a;s,x}_{2,u}\vartheta_u\mathbf{1}_{Z^{b;s,x}_{2,u}=0}\mathrm{d}u]. \tag{27}$$

Using a martingale decomposition result for martingales associated to jump processes, see [Dav76], to the conditional expectation of ξ with respect to the filtration $\mathscr{F}$ we have

$$\mathbb{E}[\xi|\mathscr{F}_\tau] = \mathbb{E}[\xi] + \int_s^\tau \Lambda_u\cdot\mathrm{d}M_u$$

for some $\Lambda \in \mathscr{H}^2_{s,h}(\mathbb{R}^2)$. Consequently

$$\begin{aligned}\mathbb{E}[\xi\int_s^\tau Z^{a;s,x}_{2,u}\lambda^n(Z^{b,n;s,x}_{2,u})\mathbf{1}_{Z^{b;s,x}_{2,u}=0}\mathrm{d}u] &= \mathbb{E}[\int_s^\tau \Lambda_u\cdot\mathrm{d}M_u\int_s^\tau Z^{a;s,x}_{2,u}\lambda^n(Z^{b,n;s,x}_{2,u})\mathbf{1}_{Z^{b;s,x}_{2,u}=0}\mathrm{d}u] \\ &\quad + \mathbb{E}[\xi]\mathbb{E}[\int_s^\tau Z^{a;s,x}_{2,u}\lambda^n(Z^{b,n;s,x}_{2,u})\mathbf{1}_{Z^{b;s,x}_{2,u}=0}\mathrm{d}u].\end{aligned}$$

Notice moreover that

$$\mathbb{E}[\xi]\mathbb{E}[\int_s^\tau Z_{2,u}^{a;s,x}\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u] \underset{n\to+\infty}{\to} \mathbb{E}[\xi]\mathbb{E}[\int_s^\tau Z_{2,u}^{a;s,x}\vartheta_u\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u]$$

since $\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}$ converges to $\vartheta_u\mathbf{1}_{Z_{2,u}^{b;s,x}=0}$ and since $Z^{a;s,x}\in\mathscr{H}^2_{s,h}(\mathbb{R}^2)$. Using Ito's formula, we get

$$\begin{aligned}
&\mathbb{E}[\int_s^\tau \Lambda_u\cdot\mathrm{d}M_u\int_s^\tau Z_{2,u}^{a;s,x}\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u]\\
&=\mathbb{E}\Big[\int_s^\tau\big(\int_s^u Z_{r,2}^{a;s,x}\lambda^n(Z_{r,2}^{b,n;s,x})\mathbf{1}_{Z_{r,2}^{b;s,x}=0}\mathrm{d}r\big)\Lambda_u\cdot\mathrm{d}M_u\Big]\\
&+\mathbb{E}\Big[\int_s^\tau\int_s^u \Lambda_r\cdot\mathrm{d}M_r Z_{2,u}^{a;s,x}\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u\Big].
\end{aligned}$$

The first term is equal to zero. Concerning the second term, we set $\psi_r=\int_s^r\Lambda_u\cdot\mathrm{d}M_u$. Hence, for any $\kappa\geq 0$

$$\begin{aligned}
&\mathbb{E}[\int_s^\tau \psi_u Z_{2,u}^{a;s,x}(\lambda^n(Z_{2,u}^{b,n;s,x})\mathbf{1}_{Z_{2,u}^{b;s,x}=0}-\vartheta_u)\mathrm{d}u]\\
&=\mathbb{E}[\int_s^\tau \psi_u Z_{2,u}^{a;s,x}\mathbf{1}_{|\psi_u Z_{2,u}^{a;s,x}|<\kappa}(\lambda^n(Z_{2,u}^{b,n;s,x})-\vartheta_u)\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u]\\
&+\mathbb{E}[\int_s^\tau \psi_u Z_{2,u}^{a;s,x}\mathbf{1}_{|\psi_u Z_{2,u}^{a;s,x}|\geq\kappa}(\lambda^n(Z_{2,u}^{b,n;s,x})\ -\vartheta_u)\mathbf{1}_{Z_{2,u}^{b;s,x}=0}\mathrm{d}u].
\end{aligned}$$

The first term in the previous expression converges to 0 since $\lambda^n(Z_{2,\cdot}^{b,n;s,x})\mathbf{1}_{Z_{2,\cdot}^{b;s,x}=0}$ converges weakly towards ϑ. The second one goes to zero when κ goes to infinity as $\psi\|Z^{a;s,x}\|_2$ is in $\mathscr{H}^2_{s,h}(\mathbb{R})$. We have proved the convergence (27). Hence, the convergence (26) holds weakly in $\mathbb{L}^2(\mathbb{R})$.

We deduce that $\int_s^\tau H^{a,n}(u,X_u^{s,x})\mathrm{d}u$ converges weakly to $\int_0^\tau H^{a,\star}(Z_u^{a;s,x},Z_u^{b;s,x},\vartheta_u)\mathrm{d}u$ in $\mathbb{L}^2(\mathbb{R})$ along the subsequence $(n_k)_{k\geq 0}$.

Step 2d. Convergence to the solution of a bang-bang BSDE

If we write the first BSDE in the system $(\mathbf{J^n})$ in a forward way, we get

$$Y_\tau^{a,n;s,x}=Y_s^{a,n;s,x}-\int_s^\tau H^{a,n}(u,X_{u^-}^{s,x})\mathrm{d}u+\int_s^\tau Z_u^{a,n;s,x}\mathrm{d}M_u.$$

We recall that we write n instead of n_k so that all the convergence that we obtain has to be understood up to a subsequence. Thus, from the almost sure and $\mathscr{S}^2_{s,h}(\mathbb{R})$ convergence of $(Y^{a,n;s,x})_{n\geq 0}$ to $Y^{a;s,x}$ together with

$$\int_s^\tau Z_u^{a,n;s,x}\cdot\mathrm{d}M_u\underset{n\to+\infty}{\to}\int_s^\tau Z_u^{a;s,x}\cdot\mathrm{d}M_u,\ \text{in}\ \mathbb{L}^2(\mathbb{R}),$$

and the convergence of the generator $H^{a,n}$ proved in Step 2c, we deduce that

$$Y_\tau^{a;s,x} = Y_s^{a;s,x} - \int_s^\tau H^{a,\star}(Z_u^{a;s,x}, Z_u^{b;s,x}, \vartheta_u)\mathrm{d}u + \int_s^\tau Z_u^{a;s,x}\mathrm{d}M_u, \mathbb{P}-a.s.$$

This result being true for any stopping time $\tau \in [s,h]$, the processes on both sides are indistinguishable and we have

$$\mathbb{P}-a.s.\ Y_u^{a;s,x} = Y_s^{a;s,x} - \int_s^u H^{a,\star}(Z_r^{a;s,x}, Z_r^{b;s,x}, \vartheta_r)\mathrm{d}r + \int_s^u Z_r^{a;s,x}\mathrm{d}M_r,\ \forall u \in [s,h].$$

Finally we have

$$\mathbb{P}-a.s.\ Y_u^{a;s,x} = g^a(X_h^{s,x}) + \int_u^h H^{a,\star}(Z_r^{a;s,x}, Z_r^{b;s,x}, \vartheta_r)\mathrm{d}r - \int_u^h Z_r^{a;s,x}\mathrm{d}M_r,\ \forall u \in [s,h].$$

with $Y^{a;s,x} \in \mathscr{S}^2_{s,h}(\mathbb{R})$ and $Z^{a;s,x} \in \mathscr{H}^2_{s,h}(\mathbb{R}^2)$. We have the same result by considering the index b and by denoting θ_u the almost sure limit of $(\lambda^n(Z_u^{a,n;s,x})\mathbf{1}_{Z_u^{a,n;s,x}=0})_{n\geq 0}$ which holds also in $\mathscr{H}^2_{s,h}$ by the dominated convergence theorem.

Step 3: **Nash equilibrium and conclusion**.

We have seen in the previous step that we can build ϑ and θ, which are functions of (u, N_u^a, N_u^b) ensuring the existence of a solution a solution $(Y^a, Y^b, Z^a, Z^b) \in (\mathscr{S}^2_{s,h}(\mathbb{R}))^2 \times (\mathscr{H}^2_{s,h}(\mathbb{R}^2))^2$ to the following coupled BSDE (by taking $s=0$),

$$\begin{cases} -\mathrm{d}Y_u^a = & H^{a,\star}(Z_u^a, Z_u^b, \vartheta_u) - Z_u^a \cdot \mathrm{d}M_u,\ Y_h^a = g^a(X_t^{0,0}) \\ -\mathrm{d}Y_u^b = & H^{b,\star}(Z_u^a, Z_u^b, \theta_u) - Z_u^b \cdot \mathrm{d}M_u,\ Y_h^b = g^b(X_t^{0,0}). \end{cases} \tag{28}$$

We could rely this BSDE to the system (**S**) and use Proposition 1. However, we are not able to prove the continuous differentiability of the functions V^i with respect to the time variable. It is why we use the theory of BSDEs similarly to [HM14] with the proposition below to conclude.

Proposition 2 (Extension of Theorems 2.5 and 2.6 in [HM14]). *There exist a pair of deterministic functions V^a, V^b and some adapted processes ϑ and θ with values in $[\lambda_-, \lambda_+]$ such that*

- *BSDE* (28) *admits a solution* $(Y^a, Y^b, Z^a, Z^b) \in (\mathscr{S}^2_h(\mathbb{R}))^2 \times (\mathscr{H}^2_h(\mathbb{R}^2))^2$,
- *V^a and V^b are two deterministic measurable functions with polynomial growth from* $[0,h] \times \mathbb{R}^2$ *to* $\mathbb{R}$ *such that* $\mathbb{P}-as,\ \forall u \leq h,\ Y_u^a = V^a(u, X_u)$ *and* $Y_u^b = V^b(u, X_u)$.
- *The pair of controls $(\lambda_a^\star(Z_u^a, \theta_u), \lambda_b^\star(Z_u^b, \vartheta_u))_{u\leq t}$ defined by* (**L**) *where ϑ and θ are obtained as an almost sure (up to a subsequence) and $\mathscr{H}^2_h(\mathbb{R}^2)$ limits of $\lambda^n(Z_u^{b,n})\mathbf{1}_{Z_u^b=0}$ and $\lambda^n(Z_u^{a,n})\mathbf{1}_{Z_u^a=0}$ respectively is a bang-bang type Nash equilibrium point of the non zero-sum stochastic differential game* (15).

Proof. Properties 1. and 2. are direct consequences of the proof made in Step 2. Property 3. is obtained by adapting the proof of Proposition 2.4 in [HM14] to the jump case, with minimizations instead of maximizations. □

Hence, Step 1 provides that the system $(\mathbf{S^n})$ admits a unique viscosity solution given by the unique solution of $(\widetilde{\mathbf{J}^\mathbf{n}})$ which approaches the solution of (28) so that $\lambda^n(Z_u^{b,n})\mathbf{1}_{Z_u^b=0}$ and $\lambda^n(Z_u^{a,n})\mathbf{1}_{Z_u^a=0}$ converge almost surely up to a subsequence (and in fact in $\mathscr{H}_h^2(\mathbb{R}^2)$) to a Nash equilibrium for the game (15) by using Proposition 2. This concludes the proof of Theorem 3.

3.3 Proof of Corollary 1 and numerical method

In Theorem 3 we only get convergence results up to a subsequence. However numerically we observe that the sequence $(V^{i,n})_{n\geq 0}$ converges for $i = a$ or b. Therefore to approach the solution of the system $(\mathbf{S})$ we solve the approached system $(\mathbf{S^n})$ for n large. To implement the numerical method we need to bound the domain. In practice this means that there is only a limited number of orders in auctions. Thus we consider the new system

$$(\mathbf{S^n_Q})\begin{cases} \partial_s V^{a,n} + H^{a,n}(D_a^Q V^{a,n}, D_b^Q V^{a,n}, D_b^Q V^{b,n}) = 0,\ s\in[0,h),\ (\alpha,\beta)\in\{0,\dots,Q\}^2,\\ V^{a,n}(h,\alpha,\beta) = g^a(\alpha,\beta),\\ \partial_s V^{b,n} + H^{b,n}(D_b^Q V^{b,n}, D_a^Q V^{b,n}, D_a^Q V^{a,n})) = 0,\ s\in[0,h),\ (\alpha,\beta)\in\{0,\dots,Q\}^2,\\ V^{b,n}(h,\alpha,\beta) = g^b(\alpha,\beta), \end{cases}$$

on the domain $[0,h]\times\{0,\dots,Q\}^2$. The operators (D_a^Q, D_b^Q) are defined similarly to (D_a, D_b) with the following boundary conditions

$$D_a^Q V(s,Q,m) = 0 \text{ and } D_b^Q V(s,n,Q) = 0 \text{ for any } (s,n,m)\in[0,h]\times\{0,\dots,Q\}^2.$$

Interpreting $(\mathbf{S^n_b})$ as an ordinary differential equation in $\mathbb{R}^{(Q+1)^2}$ according to Cauchy-Lipschitz Theorem we have existence of a solution $(V_Q^{a,n}, V_Q^{b,n})$ for the system $(\mathbf{S^n_Q})$ which is unique.

Remember that in our model the auction starts at time $\tau = \inf\{s>0 \text{ s.t. } N_s^a + N_s^b > 0\}$. Consequently market takers optimize their behavior by controlling the processes $(N^a_{\tau+\cdot}, N^b_{\tau+\cdot})$. Now remark that

$$I^2_{\tau+h} = N^a_{\tau+h}(N^a_{\tau+h} - N^b_{\tau+h}) + N^b_{\tau+h}(N^b_{\tau+h} - N^a_{\tau+h}).$$

Consequently, the symmetry of the problem with respect to a and b leads to

$$\mathbb{E}[I^2_{\tau+h}] = \mathbb{P}(N^a_\tau = 1)\big(V^a(0,1,0) + V^b(0,1,0)\big) + \mathbb{P}(N^b_\tau = 1)\big(V^a(0,0,1) + V^b(0,0,1)\big).$$

Now we assume that market takers controls their intensities using a pair of Nash Equilibrium controls $(\lambda_a^\star, \lambda_b^\star)$ obtained in Theorem 3 as limit of the smoothed problem. According to the first point of Theorem 3 and since $V^a(0,0,1) = V^b(0,1,0)$ and $V^b(0,0,1) = V^a(0,1,0)$, we get Corollary 1 so that

$$\mathbb{E}[I_h^2] = \lim_{n\to+\infty} V^{a,n}(0,1,0) + V^{b,n}(0,1,0) = V^a(0,1,0) + V^b(0,1,0).$$

Let $\bar{V}^{a,n}$ (resp. $\bar{V}^{b,n}$) be defined as the backward form of the solutions $V^{a,n}$ (resp. $V^{b,n}$) of $(\mathbf{S^n})$, more precisely

$$\bar{V}^{i,n}(s,\cdot,\cdot) = V^{i,n}(h-s,\cdot,\cdot),\ s\in[0,h],\ \text{for } i\in\{a,b\}.$$

In the same way, we denote by $(\bar{V}_Q^{a,n}, \bar{V}_Q^{b,n})$ the backward versions of the solution $(V_Q^{a,n}, V_Q^{b,n})$ of $(\mathbf{S_Q^n})$. The functions $(\bar{V}_Q^{a,n}, \bar{V}_Q^{b,n})$ are computed by solving the backward system $(\mathbf{S_Q^n})$.

Finally note that

$$\mathbb{E}[I_h^2] = \lim_{n\to+\infty} \bar{V}^{a,n}(h,1,0) + \bar{V}^{b,n}(h,1,0) \approx \bar{V}_Q^{a,n}(h,1,0) + \bar{V}_Q^{b,n}(h,1,0).$$

Hence we use the quantity $\bar{V}_Q^{a,n}(h,1,0) + \bar{V}_Q^{b,n}(h,1,0)$ for $n = 1000$ and $Q = 100$ to approach more accurately $\mathbb{E}[I_h^2]$.

4 Model extension: Market makers can cancel their limit orders

We can extend our model and allow market makers to revise their position before the auction clearing by cancelling their limit orders. Formally a market maker arrived at time $\tau \le \tau_i^{cl}$ will maintain its position until the auction clearing at time t with a probability $\theta(t-\tau_i^{cl})$, where θ is a $[0,1]$-valued decreasing function such that $\theta(0) = 1$. Hence, the number of market makers present at the $i-th$ auction clearing is

$$\tilde{N}_{\tau_{i-1}^{cl}} - \tilde{N}_{\tau_i^{cl}}\ , \quad \text{with} \quad \tilde{N}_s = \sum_{j=N^{mm}_{\tau_{i-1}^{cl}}+1}^{N^{mm}_{\tau_i^{cl}}} \mathbf{1}_{X_k \le \theta(\tau_k - \tau_i^{cl})},$$

where $(X_j)_{j\ge 0}$ is a sequence of i.i.d. random variables with uniform law on $[0,1]$. We can show that during auction time $(\overline{N}_s)_{0\le s\le h} = (\tilde{N}_{\tau_i^{op}+s})_{s\ge 0}$ has the same law than an inhomogeneous Poisson process with intensity

$$\lambda(s) = \mu\theta(t-s).$$

Moreover we still have an explicit formula for E.

$$E^{mid}(h) = (1 - e^{-m_h}\frac{\nu}{\nu+\mu})^{-1} e^{\nu h} \int_h^{+\infty} \nu e^{-\nu t}\Big((\sigma_f^2\frac{t}{6} + \sigma^2)e^{-m_t}\int_0^{m_t}\frac{e^s-1}{s}\mathrm{d}s + \sigma_f^2\frac{t}{3}(1-e^{-m_t})\Big)\mathrm{d}t$$

and

$$E(h) = E^{mid}(h) + \frac{\mathbb{E}[I^2_{\tau_1^{op}+h}]}{K^2}(1 - e^{-m_h}\frac{\nu}{\nu+\mu})^{-1} e^{\nu h}\int_h^{+\infty}\nu e^{-m_t}e^{-m_t}\int_0^{m_t}\frac{1}{s}\int_0^s\frac{e^u-1}{u}\mathrm{d}u\mathrm{d}s\mathrm{d}t$$

with

$$m_t = \int_0^t \mu\theta(s)\mathrm{d}s.$$

5 Proof of Lemma 1

Consider for any $s > \tau_1^{cl}$, $X_s = (\overline{P}_s^{cl} - \overline{P}_s)^2$. We show that $(X_s)_{s>\tau_1^{cl}}$ is a regenerative process with renewal times given by $(\tau_i^{cl})_{i\geq 1}$.

Consider $\tau_i^{cl} \leq s < \tau_{i+1}^{cl}$ we have

$$\overline{P}_s^{cl} - \overline{P}_s = \frac{1}{N_{\Delta_i}^{i,mm}} \sum_{k=1}^{N_{\Delta_i}^{i,mm}} (P_{\tau_i^{cl}} - P_{\tau_{i-1}^{cl}+\tau_k^{i,mm}}) + \frac{1}{N_{\Delta_i}^{i,mm}} \sum_{k=1}^{N_{\Delta_i}^{i,mm}} g_k + \frac{I_{\Delta_i}^i}{K N_{\Delta_i}^{i,mm}}. \tag{29}$$

According to Assumption 1 the process $(N_t^{i,mm}, I_t^i)_{t\geq 0}$ is independent from $\mathscr{F}_{\tau_{i-1}^{cl}}$ with same law as $(N_t^{mm}, I_t)_{t\geq 0}$. Same results holds for $(P_{\tau_{i-1}^{cl}+t} - P_{\tau_{i-1}^{cl}})_{t\geq 0}$ and $(P_t - P_0)_{t\geq 0}$ since P is a Brownian motion. Consequently $N_{\Delta_i}^{i,mm}$, $I_{\Delta_i}^i$ and $(P_{\tau_{i-1}^{cl}+t} - P_{\tau_{i-1}^{cl}})_{t\geq 0}$ are independent from $(X_s)_{s<\tau_i^{cl}}$ with same law as $N_{\tau_1^{cl}}^{mm}$, $I_{\tau_1^{cl}}$ and $(P_t - P_0)_{t\geq 0}$.

Thus according to (29) and since X is piecewise continuous with jump at times $(\tau_i^{cl})_{i\geq 1}$, for any $\tau_i^{cl} \leq s < \tau_{i+1}^{cl}$, X_s is independent of $(X_s)_{s<\tau_i^{cl}}$ and has the same distribution than $X_{\tau_1^{cl}}$. Thus X is regenerative with renewal times equal to $(\tau_i^{cl})_{i\geq>1}$

Thus according to Theorem 3.1 Chap VI in [Asm08] we have the almost sure convergence

$$\begin{aligned}
\frac{\int_0^t X_s \mathrm{d}s}{t} \underset{t\to+\infty}{\to} & \frac{\mathbb{E}[\int_{\tau_1^{cl}}^{\tau_2^{cl}} X_s \mathrm{d}s]}{\mathbb{E}[\tau_2^{cl} - \tau_1^{cl}]} \\
= & \frac{\mathbb{E}[\tau_2^{cl} - \tau_1^{cl}]\mathbb{E}[X_{\tau_1^{cl}}]}{\mathbb{E}[\tau_2^{cl} - \tau_1^{cl}]} \\
= & \mathbb{E}[X_{\tau_1^{cl}}] = \mathbb{E}[(P_{\tau_1^{cl}} - P_{\tau_1^{cl}}^{cl})^2].
\end{aligned}$$

Thus we get the stated result.

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Titre : Quelques aspects du rôle central de la microstructure des marchés financiers : volatilité, exécution optimale et organisation des marchés

Mots clés : rough volatility, optimal control, market design, market microstructure

Résumé : Cette thèse est organisée en trois parties. Dans la première on examine les relations entre la dynamique microscopique et macroscopique du marché en se concentrant sur les propriétés de la volatilité. Dans la deuxième partie on s'intéresse au contrôle optimal stochastique de processus ponctuels. Finalement dans la troisième partie on étudie deux problématiques de *market design*. On commence cette thèse par l'étude des liens entre le principe d'absence d'opportunité d'arbitrage et l'irrégularité de la volatilité. A l'aide d'une méthode de changement d'échelle on montre que l'on peut effectivement connecter ces deux notions par l'analyse du *market impact* des métaordres. Plus précisément on modélise le flux des ordres marchés en utilisant des procesus de Hawkes linéaires. Puis on montre que le principe d'absence d'opportunité d'arbitrage ainsi que l'existence d'un market impact non trivial impliquent que la volatilité est rugueuse et plus précisément qu'elle suit un modèle rough Heston. On examine ensuite une classe de modèles microscopiques où le flux d'ordre est un processus de Hawkes quadratique. L'objectif est d'étendre le modèle rough Heston à des modèles continus permettant de reproduire l'effet Zumbach. Finalement on utilise un de ces modèles, le modèle rough Heston quadratique, pour la calibration jointe des nappes de volatilité du SPX et du VIX. Motivé par l'usage intensif de processus ponctuels dans la première partie, on s'intéresse dans la deuxième au contrôle stochastique de processus ponctuels. Notre objectif est de fournir des résultats théoriques en vue d'applications en finance. On commence par considérer le cas du contrôle de processus de Hawkes. On prouve l'existence d'une solution puis l'on propose une méthode permettant d'appliquer ce contrôle en pratique. On examine ensuite les limites d'échelles de problèmes de contrôles stochastiques dans le cadre de modèles de dynamique de population. Plus exactement on considère une suite de modèles de dynamique d'une population discrète qui converge vers un modèle pour une population continue. Pour chacun des modèles on considère un problème de contrôle. On prouve que la suite des contrôles optimaux associés aux modèles discrets converge vers le contrôle optimal associé au modèle continu. Ce résultat repose sur la continuité, par rapport à différents paramètres, de la solution d'une équation différentielle schostatique rétrograde. Dans la dernière partie on s'intéresse à deux problèmatiques de market design. On examine d'abord la question de l'organisation d'un marché liquide de produits dérivés. En se concentrant sur un marché d'options, on propose une méthode en deux étapes pouvant facilement être appliquée en pratique. La première étape consiste à choisir les options qui seront listées sur le marché. Pour cela on utilise un algorithme de quantification qui permet de sélectionner les options les plus demandées par les investisseurs. On propose ensuite une méthode d'incitation tarifaire visant à encourager les *market makers* à proposer des prix attractifs. On formalise ce problème comme un problème de type principal-agent que l'on résoud explicitement. Finalement, on cherche la durée optimale d'une enchère pour les marchés organisés en enchères séquentielles, le cas de la durée nulle correspondant à celui d'une double enchère continue. On utilise un modèle où les *market takers* sont en compétition et on considère que la durée optimale est celle correspondant au processus de découverte du prix le plus efficace. Après avoir prouvé l'existence d'un équilibre de Nash pour la compétition entre les market takers, on applique nos résultats sur des données de marchés. Pour la plupart des actifs, la durée optimale se trouve entre 2 et 10 minutes.

Title : Some aspects of the central role of financial markets microstructure : volatility dynamics, optimal trading and market design

Keywords : volatilité rugueuse, contrôle optimal, organisation des marchés, microstructure des marchés

Abstract : This thesis is made of three parts. In the first one, we study the connections between the dynamics of the market at the microscopic and macroscopic scales, with a focus on the properties of the volatility. In the second part we deal with optimal control for point processes. Finally in the third part we study two questions of market design. We begin this thesis with studying the links between the no-arbitrage principle and the (ir)regularity of volatility. Using a microscopic to macroscopic approach, we show that we can connect those two notions through the market impact of metaorders. We model the market order flow using linear Hawkes processes and show that the no-arbitrage principle together with the existence of a non-trivial market impact imply that the volatility process has to be rough, more precisely a rough Heston model. Then we study a class of microscopic models where order flows are driven by quadratic Hawkes processes. The objective is to extend the rough Heston model building continuous models that reproduce the feedback of price trends on volatility: the so-called Zumbach effect. We show that using appropriate scaling procedures the microscopic models converge towards price dynamics where volatility is rough and that reproduce the Zumbach effect. Finally we use one of those models, the quadratic rough Heston model, to solve the longstanding problem of joint calibration of SPX and VIX options smiles. Motivated by the extensive use of point processes in the first part of our work we focus in the second part on stochastic control for point processes. Our aim is to provide theoretical guarantees for applications in finance. We begin with considering a general stochastic control problem driven by Hawkes processes. We prove the existence of a solution and more importantly provide a method to implement the optimal control in practice. Then we study the scaling limits of solutions to stochastic control problems in the framework of population modeling. More precisely we consider a sequence of models for the dynamics of a discrete population converging to a model with continuous population. For each model we consider a stochastic control problem. We prove that the sequence of optimal controls associated to the discrete models converges towards the optimal control associated to the continuous model. This result relies on the continuity of the solution to a backward stochastic differential equation with respect to the driving martingale and terminal value. In the last part we address two questions of market design. We are first interested in designing a liquid electronic market of derivatives. We focus on options and propose a two steps method that can be easily applied in practice. The first step is to select the listed options. For this we use a quantization algorithm enabling us to pick the options capturing most of market demand. The second step is to design a make-take fees policy for market makers to incentivize them to set attractive quotes. We formalize this issue as a principal agent problem that we explicitly solve. Finally we look for the optimal auction duration that should be used on a market organized in sequential auctions, the case of auctions with 0 second duration corresponding to the continuous double auctions situation. To do so, we use an agent based model where market takers are competing. We consider that the optimal auction duration is the one leading to the best quality of price formation process. After proving existence of a Nash equilibrium for the competition between market takers we apply our results on stocks market data. We find that for most of the stocks, the optimal auction duration lies between 2 and 10 minutes.

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