The charged Klein-Gordon equation in the De Sitter-Reissner-Nordström metric

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L'équation chargée de Klein-Gordon en métrique de De Sitter-Reissner-Nordström

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Abstract/Résumé

In this thesis, we study the charged Klein-Gordon equation in the exterior De Sitter-Reissner-Nordström spacetime. We first show decay in time of the local energy by means of a resonance expansion of the local propagator. Then we construct a scattering theory for the equation and give a geometric interpretation in an extended spacetime of asymptotic completeness in terms of traces at horizons. Exponential decay of local energy for solutions of the wave equation in this extension up to and through horizons is obtained harmonic by harmonic. We next turn to a numerical study of an abstract Klein-Gordon type equation and introduce a scheme which approximate solutions up to an error we can control. Finally, we propose a numerical method to localize low frequency resonances.

Many results in the thesis are prerequisite to the construction of the Unruh state satisfying the Hadamard property for the charged Klein-Gordon equation in the De Sitter-Reissner-Nordström spacetime.

Dans cette thèse, nous étudions l'équation chargée de Klein-Gordon dans l'espace-temps extérieur de De Sitter-Reissner-Nordström. Nous montrons tout d'abord la décroissance en temps de l'énergie locale au moyen d'une expansion en termes de résonances du propagateur local. Nous construisons ensuite une théorie de la diffusion pour l'équation et donnons une interprétation géométrique dans un espace-temps étendu de la complétude asymptotique en termes de traces aux horizons. La décroissance exponentielle de l'énergie locale pour les solutions de l'équation d'onde dans cette extension jusqu'aux et à travers les horizons est obtenue harmonique par harmonique. Nous nous intéressons ensuite à l'étude numérique d'une équation abstraite de type Klein-Gordon et introduisons un schéma qui approche les solutions avec une erreur que l'on peut contrôler. Finalement, nous proposons une méthode numérique pour localiser les résonances à basse fréquence.

Plusieurs résultats de la thèse sont des prérequis à la construction de l'état de Unruh satisfaisant la propriété de Hadamard pour l'équation chargée de Klein-Gordon dans l'espace-temps de De Sitter-Reissner-Nordström.

Contents

Acknowledgments

I remember a talk about Riemannian geometry (or something like this) where the speaker started with this deep definition:

Definition 0.0.1. *A thesis advisor is the person whom teaches you techniques to solve problems you would have never met without her/him.*

As this is definitively true, I have to say that learning new problems and methods to solve them is very rewarding. Many people actually contributed to my education and I would like to warmly thank them all now.

Starting with my first undergraduate year, I thank my mathematics teacher B. Dudin for his kindness, my chemistry teacher M. Casida for is benevolence. Second year: my mathematics teacher J.-P. Demailly who gave me my very first lecture of mathematical physics (about heat equation and Fourier series, I remember perfectly well!), this has triggered inside me the irrevocable wish to learn mathematics; I thank P. Brulard for his humor when teaching electromagnetism, my physics teacher N. Boudjada for her patience and kindness when I came after almost surely each lecture with N questions (let N goes to infinity...). Third year: J. Meyer who taught mathematics for physicists, F. Hekking and J. Ferreira for the analytical mechanics lectures, B. Grenier for the crystallography lectures as well as L. Canet and V. Rossetto for the statistical mechanics lecture. Second third year: Th. Gallay for his kind help when I attempted to obtain my mathematics Bachelor degree, E. Russ to whom I do apologize for the overwhelming C^∞ flux of questions and C. Leuridan for his help and patience during algebra lectures. Fourth year: I thank E. Dumas, D. Spehner, H. Pajot, R. Rossignol. Fifth year: O. Druet and A. Mikelić. Thanks to P. Millet for some useful discussions during the thesis, particularly for his help for the geometric interpretation of the scattering.

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I thank Dietrich Häfner for having accepted to be my advisor and letting me discovering this wonderful topic that General Relativity is. He also convinced me to attempt a Bachelor degree in mathematics after my education in physics: I owe him a lot for his advices. Although I remember the day when he drew on the black board a well with a ball and said "See? The equilibrium is unstable because the ball can roll downward..." as an *answer* to a question about normally hyperbolic trapping in black hole type spacetimes, I have really enjoyed these three years under his teaching. Thanks you very much!

Introduction

The theory of general relativity is geometric: gravity is curvature, manifestation of the intrinsic property of the spacetime to interact with its own content. The rigid Newtonian framework containing particles and interactions is gone, letting the Einstein's dynamical container governing and being governed by the contained density of energy in the universe. From the mathematical point of view, this is encoded by Einstein's equations which can be written as

$$\mathrm{Ric}_g - \frac{1}{2}\mathrm{R}_g g + \Lambda g = T \tag{1}$$

where Ric_g is the Ricci tensor associated to the (unknown) metric g, R_g is the scalar curvature, Λ is the cosmological constant and T is the energy-momentum tensor describing the energy density in the model of universe we wish to study. They form a system of coupled non-linear partial differential equations whose the most famous explicit solution is the (De Sitter-)Kerr-Newman family of charged rotating black holes, parametrized by the mass $M > 0$ of the black hole, its electric charge Q and its angular momentum a (and the cosmological constant $\Lambda > 0$ in the De Sitter- case). A natural question we may ask is whether these spacetimes are stable as solutions of the Einstein equations or not. Full non-linear stability has been shown for the De Sitter-Kerr spacetime by Hintz-Vasy [HiVa18] for small angular momentum, the De Sitter-Kerr-Newman spacetime by Hintz [Hi18] for small angular momentum as well as for the Schwarzschild spacetime for polarized perturbations by Klainerman-Szeftel [KlSz18]. Such results are based on the so-called linear stability results, meaning a precise description of the solutions of the linearized Einstein equations around the given spacetime as a stationary part plus a part for which one obtains precise decay estimates. Linear stability results have been obtained by Dafermos-Holzegel-Rodnianski [DHR16] for the Schwarzschild spacetime (see also Hung-Keller-Wang [HKW17]), by Giorgi [Gi19] for the sub-extremal Reissner-Nordström spacetime and by Andersson-Bäckdahl-Blue-Ma [ABBM19] as well as by Häfner-Hintz-Vasy [HaHiVa19] for the Kerr spacetime. We also mention the work of Finster-Smoller [FS16] for the Kerr spacetime but which does not contain precise decay rates.

Einstein's theory has successfully passed all the tests to be considered as one of the most reliable theory we dispose of today. Yet, it fails to describe quantum particles in strong gravitational fields (that is, not considered "infinitely far" from the source of gravitation). No satisfactory quantum theory of gravity exists, but even the construction of a non-interacting quantum field theory on a fixed curved background still faces some open questions. A fundamental problem is the lack of symmetries: in this context, it is even not obvious how to construct an acceptable

equivalent of the vaccuum state. We can try to construct the so-called *Hadamard states* which are possible physical states of the non-interacting quantum field theory on a curved spacetime. One can construct Hadamard states in quite general geometric situations, see *e.g.* the works of Gérard-Wrochna [GW13] and [GW16]. Nevertheless not all of these Hadamard states are natural and an important question is how to choose the phyiscally most meaningful. While black hole spacetimes themselves do not admit enough symmetries, a lot of symmetries exist at infinity. One can therefore construct states invariant by certain symmetries at the horizons or null infinity and then "send" them inside by scattering theory: these are the so-called *Unruh states*. Construction of these states thus requires classical scattering theory. The questions whether these states fufill the Hadamard condition and to which part of the maximal extension of the spacetime they extend turn out to be rather difficult. The Hadamard property of the Unruh state is known for bosons in the Schwarzschild spacetime, see [DMP11], and for massless fermions in the Kerr case, see ungoing work by Gérard-Häfner-Wrochna [GHW]. The boson case on Kerr spacetime is still open. Note that very important obstructions to the construction of such states exist in Kerr spacetime due to the absence of a global timelike Killing vector field, see [KW91]. Many of these obstructions already exist for the charged Klein-Gordon field on the De Sitter-Reissner-Nordström spacetime.

One can also consider a dynamical situation of the collapse of a star. The states then "evolve" when considered by a far away observer who will see the emergence of a thermal state even when starting with a vacuum state: this is the famous *Hawking effect*, see [Haw75]. Mathematical rigorous descriptions of this effect exist now, see the series of works of Bachelot [Ba97], [Ba99] and [Ba00] for the spherically symmetric case and Häfner [Ha09] for the rotating case. For both problems, the construction of the Unruh state and the mathematically rigorous description of the Hawking effect, a fundamental ingredient is scattering theory for the classical field. In this sense, understanding the classical equation is a first step in understanding the quantization of the field.

In this thesis, we will consider the charged Klein-Gordon equation

$$\big((\nabla_\mu + \mathrm{i}qA_\mu)(\nabla^\mu + \mathrm{i}qA^\mu) + m^2\big)u = 0 \tag{2}$$

where q is the charge of the field, $m > 0$ is its mass and $A = \frac{Q}{r}\mathrm{d}t$ is the Coulombian 1-form encoding the electrostatic interaction with the charged black hole (here t is a time coordinate). The natural spacetime to study equation (2) is the De Sitter-Reissner-Nordström (DSRN in the sequel) spacetime $(\mathcal{M}, g)$ introduced in the paragraph 1.1.1; the couple (g, A) solves (1) with T the Maxwell energy-momentum tensor associated to $\mathrm{d}A$. The ultimate goal is to construct a Unruh state having the Hadamard property in a black hole context with no positive conserved energy. Except for the the maximally symmetric Minkowski spacetime, there is no canonical way to define the vacuum state; the Hadamard property is an extension of this concept in curved spacetimes. In Schwarzschild spacetime, such a state has been constructed in [DMP11] In our context, the strategy is first using the symmetry of the horizons for the construction of a Unruh state then "propagating" it by back-scattering in the outer communication region of the spacetime. The prerequisite is therefore the study of the decay as well as the scattering properties of the equation (2).

The spherical symmetry of the problem makes its study easier. However, the coupling with A creates issues that doe not exist otherwise. For example, the existence of a global timelike Killing vector field in the exterior DSRN spacetime is no longer enough to define conserved quantities associated to solutions of (2) because of negative contributions of A_t near horizons. All happens as if there was no such global timelike Killing vector field anymore after the coupling.

The situation is in this regard similar to the Klein-Gordon equation in the exterior De Sitter-Kerr spacetime where the coupling of the field with the black hole is created by the non-zero angular momentum of the black hole. In this latter case, the coupling has a geometric origin contrary to the situation we will care of in this thesis. To understand this phenomenon, we will need to add by hand some geometric tools using an extension of the original spacetime, viewing then the electrostatic interaction as a rotation in an extra dimension. This enriched background will allow us to interpret scattering for the charged Klein-Gordon equation as transport along principal null geodesics in the extended spacetime. We emphasize here that this interpretation only holds in dimension $1+4$ as no geodesics in the exterior DSRN spacetime contains the charge and the mass of the field. Notice also that only the charge prevents us to give a geometric interpretation of the scattering in the original spacetime as the mass term vanishes near the horizons (so that the scattering process "does not see" it).

Equation (2) enters the general framework of [GGH17] that we recall now. Let $\mathcal{H}$ be a Hilbert space (typically a L^2 space). An abstract Klein-Gordon equation is an equation of the form

$$(\partial_t^2 - 2\mathrm{i}k\partial_t + h)u = 0 \tag{3}$$

where h and k two self-adjoint operators acting on $\mathcal{H}$. Letting $v := \mathrm{e}^{-\mathrm{i}kt}u$, (3) reads

$$(\partial_t^2 + h(t))u = 0$$

with $h(t) = \mathrm{e}^{-\mathrm{i}kt}(h-k^2)\mathrm{e}^{\mathrm{i}kt}$. Therefore equation (3) is hyperbolic if and only if $h_0 := h - k^2 \geq 0$. The natural conserved energy associated to a solution u of (3) is given by

$$\|u\|^2 := \langle hu, u\rangle_{\mathcal{H}} + \|\partial_t u\|^2_{\mathcal{H}}.$$

However, this energy is not positive if h is not positive. In this situation, we define

$$\|u\|^2_{\dot{\mathcal{E}}} := \langle h_0 u, u\rangle_{\mathcal{H}} + \|\partial_t u - \mathrm{i}ku\|^2_{\mathcal{H}}$$

which is nonnegative, positive if $h_0 > 0$, but in general not conserved. Indeed,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\dot{\mathcal{E}}} = \langle [\mathrm{i}k, h]u, u\rangle_{\mathcal{H}} \tag{4}$$

which does not cancel if h and k does not commute. This means that this energy can grow in time. This phenomenon is called *superradiance*. It happens for example in the euclidean case when the scalar field interacts with an electric potential; the Hamiltonian associated to (3) is then not self-adjoint on the underlying Hilbert space but can be realized as a self-adjoint operator acting on a Krein space, see [Ge12]. In a quite general setting, boundary value of the resolvent for selfadjoint operators on Krein spaces as well as propagation estimates for the Klein–Gordon equation have been obtained in [GGH13] and [GGH15]. A new difficulty occurs when the operator k as different formal limits at the end of considered manifold. This time the Hamiltonian can no longer be realized as a self-adjoint operator on a Krein space and the results of both the latter works are not applicable. This situation can already be encountered in the case of the one-dimensional charged Klein-Gordon equation, when coupling the Klein-Gordon field with a step-like electrostatic potential, see [Ba04].

Superradiance can occur in black hole type spacetimes when no global timelike Killing field exists. This is the case in the (De Sitter-)Kerr spacetimes. In addition the operator k (which is

linked to the rotation of the black hole) has different limits at the ends of the spacetime. This also happens in our context where the coupling between the charged scalar field with the black hole comes from the equation itself and not from the geometry.

Let us review the principal parts of this thesis.

Chapter 1: Decay of the local energy. The first question we will address to is the asymptotic-in-time behavior of solutions of the superradiant charged Klein-Gordon equation (2). To handle this problem, we will use the theory of resonances. This is a powerful tool that allows us to establish decay and non-decay results as well as asymptotic for solutions using resolvent estimates. Assume that $[\mathrm{i}k, h] \lesssim h_0$ (this hypothesis is satisfied in the present case as well as for the Klein-Gordon equation in the exterior De Sitter-Kerr spacetime). Then (4) entails

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\dot{\mathcal{E}}}^2 \lesssim \langle h_0 u, u\rangle_{\mathcal{H}} \lesssim \|u\|_{\dot{\mathcal{E}}}^2$$

which means by Grönwall's inequality that there exists $C, \kappa > 0$ such that $\|u\|_{\dot{\mathcal{E}}} \leq C\mathrm{e}^{\kappa|t|}$ for all $t \in \mathbb{R}$. As an introduction example, we can consider the forward forcing problem:

$$\begin{cases} (\partial_t^2 - 2\mathrm{i}k\partial_t + h)u &= f \in \mathcal{H} \\ u(t,\cdot) &= 0 \qquad \forall t < 0 \end{cases}.$$

Let (t, x) be local coordinates. Taking the time-dependent Fourier transform (denoted by the symbol $\hat{}$), we get

$$(h_0 - (z-k)^2)\hat{u} = \hat{f}.$$

Call $p(z, k)$ the *quadratic pencil* $h_0 - (z-k)^2$. Assuming that $p(z, k)^{-1}$ is well-defined for $\Im(z) \gg 0$, we can write

$$\hat{u} = p(z, k)^{-1}\hat{f}.$$

We know that u can not grow exponentially too fast, so that we have the inversion formula

$$u(t, x) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\nu}^{+\infty+\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}zt} p(z, k)^{-1}\hat{f}(z, x)\mathrm{d}z$$

for some $\nu > \kappa > 0$. If f is compactly supported in x, then

$$\chi u(t, x) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\nu}^{+\infty+\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}zt}\chi p(z, k)^{-1}\chi\hat{f}(z, x)\mathrm{d}z \tag{5}$$

for any cut-off χ such that $\chi \equiv 1$ on $\mathrm{Supp}\, f(t, \cdot)$. Assume now that $\chi p(z, k)^{-1}\chi$ can be meromorphically extended to a strip in $\{\Im(z) > -\nu\}$ for some $\nu > 0$. Then contour deformations can be performed to obtain integrals in $\mathbb{C}^-$ providing exponential decay. In the meanwhile, the residue theorem makes appear poles of the meromorphic extension of $\chi p(z, k)^{-1}\chi$: they are called *resonances*. Let $\mathrm{Res}(p)$ be the set of resonances; then the above procedure ultimately yields the asymptotic expansion

$$\chi u(t, x) = \sum_{\substack{z\in\mathrm{Res}(p) \\ \Im(z)>-\nu}} \sum_{k=0}^{m(z)} \mathrm{e}^{-\mathrm{i}zt} t^k \Pi_{z,k}^{\chi}\hat{f} + E(t)u \tag{6}$$

where $m(z)$ is the multiplicity (as a pole of a meromorphic function) of z, $\Pi^{\chi}_{z,k}$ are cut-off projectors onto the resonant states associated to z and $E(t) = \mathcal{O}(\mathrm{e}^{-\nu t})$. The error term is estimated thanks to resolvent estimates. See Theorem 1.3.2 for the exact statement of this result in the DSRN context. Let us make some comments:

1. In the most elementary case of the wave equation in $\mathbb{R}$, there is only one non-vanishing term in (6) corresponding to the projection on the *resonant state* $x \mapsto 1$ associated to a resonance at $z = 0$. The existence of the resonance 0 and the resonant state 1 hold in the De Sitter-Schwarzschild case (see Bony-Häfner [BoHa08], formula (1.9); observe that the resonant state is r therein because of the transformation $r\hat{P}r^{-1}$ below equation (1.3)) as well as in the De Sitter-Kerr metric (see Dyatlov [DQNM11], formula (1.5)).

2. A detailed analysis of the long-time behavior of linear and non-linear waves on metrics solving the vacuum Einstein's equations with positive cosmological constant has been carried out in [Hi15].

3. Since the work of Ralston [R69], we know that there is a loss of regularity in presence of obstacles in local energy estimates. For the wave equation on the De Sitter-Schwarzschild and De Sitter-Kerr metrics where there exist trapping sets (the so-called *photon sphere* in the spherically symmetric case), we lose angular derivatives, *cf.* respectively [BoHa08] and [DQNM11].

4. Expansion (5) provides the rate of decay or growth in time of solutions of (3) depending of the localization of resonances. Any resonance in $\mathbb{C}^+$ gives exponentially growing terms (the corresponding resonant state is called growing mode); conversely, any resonance in $\mathbb{C}^-$ gives exponentially decaying terms. The existence of real resonances has more subtle consequences: it leads to polynomially growing terms if the multiplicity of the resonance (as a pole of the meromorphic extension) is greater than 1, or to a stationary term with no growth or decay in time for a resonance of multiplicity equal to 1 (this happens in [BoHa08] and [DQNM11]).

5. Formula (5) strongly relies on the existence of the meromorphic extension of the cut-off quadratic pencil. Besides, the exponential weight in contour deformations are usable because of the exponential decay of some metric coefficients near horizons. When $\Lambda = 0$, only a polynomial decay holds near $+\infty$ and we only expect a polynomial decay in time of solutions (the Fourier transform in the above argument is replaced by a Mellin transform).

6. In [Va13], a general setting has been developped for the wave equation on asymptotically De Sitter-Kerr spacetimes. The key point is a general microlocal framework for the Fredholm analysis of non-elliptic problems. We think that this framework could also be applied in the present setting.

We will apply this scheme to our context. We will assume that the charge product $s := qQ$ is sufficiently small in order to use perturbation arguments from results in [BoHa08].

Chapter 2: Scattering theory. Having established the decay of local energy and localized resonances (at least excluded their existence near and above the real axis), we will turn to the scattering theory for equation (3) assuming the charge product s small enough. Time dependent

scattering theory describes large time scale interactions between a physical system (particles, waves) and its environment. The fundamental result that one may wish to establish is then the so-called *asymptotic completeness* which compares dynamics one is interested in to a simpler and well-known one, a "free" dynamics.

Many works have considered the case when the Hamiltonian associated to the system is self-adjoint with respect to a Hilbert space structure. In the case when the naturally conserved energy of solutions of the field equations is not positive along the flow of the dynamics, it is not possible to realize the Hamiltonian as self-adjoint operator on the underlying Hilbert space. The generator can have real and complex eigenfrequencies and the energy $\|\cdot\|_{\dot{\mathcal{E}}}$ can grow in time at a polynomial or exponential rate. As explained above, this is the case for the superradiant Klein-Gordon equation (3). This also happens in the euclidean case, when the scalar field interacts with a strong electromagnetic potential. In this situation however, the Hamiltonian is self-adjoint on a Krein space, see [Ge12]; in a quite general setting, boundary value of the resolvent for self-adjoint operators on Krein spaces as well as propagation estimates for the Klein–Gordon equation have been obtained in [GGH13] and [GGH15].

Another difficulty can occur however when the coupling term have two different limits at different ends of the considered manifold. If no conserved energy is continuous with respect to the energy $\|\cdot\|_{\dot{\mathcal{E}}}$, then it is no longer possible to use a self-adjoint realization with respect to a Krein structure, *cf.* [GGH17]. This situation can already be encountered in the case of the one-dimensional charged Klein-Gordon equation, when coupling the Klein-Gordon field with a step-like electrostatic potential, see [Ba04]. This issue is also present in our context: the operator k in (3) has two formal distinct limit operators $k_{\pm}$ near horizons. Asymptotic completeness in Kerr spacetimes has been obtained in absence of cosmological constant in [DRSR18] using geometric techniques, and for positive cosmological constant and bounded angular momenta of the field in [GGH17] using spectral methods. We will use methods in [GGH17] to construct our scattering theory.

As already emphasized, coupling "by hand" the field with the black hole removes any geometric meaning of the coupling as the charged field is not taken into account in Einstein's equations. Superradiance's origin is then not clear as we dispose of the global timelike Killing vector field ∂_t in the exterior DSRN spacetime (*i.e.* no ergoregion or dyadosphere exists *a priori*). It turns out that scattering itself loses any geometric meaning in comparison to the Kerr case. Indeed, consider the non-rotating exterior Kerr spacetime (that is the Schwarzschild solution). Scattering theory in this context has been built in [N15] and is also provided with a geometric interpretation as transport along *principal null geodesics*. These geodesics are used to construct the event horizon and the conformal infinity and carry energy spaces thereon (the obtained energy on the horizon is then the flux through this hypersurface of its Killing generator). In absence of such geometric background, it is no longer possible to give a geometric interpretation of scattering for equation (3).

In order to encode the electrostatic interaction in the geometry, we add a fifth dimension which represents this interaction. As the charge and the mass of the Klein-Gordon field are not in Einstein's equation, we "remove" it from the Klein-Gordon operator to produce the gauge-invariant wave operator for an extended metric: this is what we have called the *neutralization procedure*. More precisely, we use the symbol of the operator in (3) to build a $(1+4)$-dimensional Lorentzian manifold for which the latter operator can be seen as a wave operator. This consists somehow in quantizing on the unit circle the charge in k and the mass in h. The procedure however fails

when $m = 0$. We obtain in this way a *Kaluza-Klein* extension of the DSRN spacetime. About one century ago, Kaluza proposed to encode the electromagnetic interaction in an extra dimension in his paper [Kal21] where he tried to unify gravitation and electromagnetism. He proposed that electromagnetism could be encoded in a fifth dimension and formulated his theory using the *cylinder condition*, stating that no component of the new five-dimensional metric depends on the fifth dimension (actually, this condition makes equations easier to handle and avoid extra degrees of freedom). With the then outbreaks of quantum mechanics, Klein interpreted the cylinder condition as a microscopic curling of the electromagnetic field along the extra dimension (see [Kle26])

The (very simple) neutralization procedure has several consequences. First of all, it turns the original black hole into a *black ring*, the equivalent to black holes in 5 and more dimension, the difference lying in the topology of the horizon ($\mathbb{S}^1 \times \mathbb{S}^2$ in our situation); the black ring solves a new Einstein equation whose energy-momentum tensor is the sum of a Maxwell tensor with effective charge $Q\sqrt{1 - \frac{q^2}{2m^2}}$ and a perfect fluid tensor acting in the plane generated by the time and the extra variables. Next, with respect to the extended metric, ∂_t is no longer timelike near horizons meaning that *dyadorings* exist in the extended spacetime (the equivalent of ergoregions in Kerr's terminology); the right future direction is then given by $\nabla_t \neq \partial_t$. Finally, it provides us with principal null geodesics which can be used to interpret scattering as transport towards horizons. More precisely, inverse wave operators are (up to unitary transforms) traces on horizons. This can be reformulated as stating that an abstract Goursat problem can be solved in energy spaces on horizons, the latter being obtained by transport along principal null geodesics. The corresponding energy is then nothing but the flux through the horizons of the corresponding Killing generator. Issues caused by decoupling the charged field to Einstein's equation are then fixed by this method. However, this interpretation can not be "projected" onto the original spacetime due to the absence of appropriate geodesics; existence and unicity of the abstract Goursat problem is suspected to be false in $1 + 3$ dimensions.

Chapter 3: Decay of the energy through the horizons in the Kaluza-Klein extension. Chapter 3 establish an exponential decay up to and through the horizons of the extended spacetime using the decay of the local energy of Theorem 1.3.2 as well as the red-shift effect near the horizons introduced in [Da05]. We essentially follow [Dy11] which shows the same result for waves in the exterior De Sitter-Kerr spacetime. The spherical symmetry simplifies some computations but is not necessary; the situation of an original slowly rotating black hole would require only tiny modifications in the proof.

Chapter 4: Numerical approximations of solutions of the charged Klein-Gordon equation in the exterior DSRN spacetime. The fourth chapter of this thesis concerns numerical approximations of solutions of (3) as well as low frequency resonances. Numerical approximation consists in two steps: the first one is a control of the error committed when using the approximated objects, the second one is the optimization of the code used for numerical computations. The purpose of this chapter is to introduce a method for approximations with a careful control of the error. A good error estimate allows us to give precise statement about the considered problem from numerical computations.

In Chapter 4, we propose a numerical scheme which allows us to estimate the error of approximation of solutions for an abstract Klein-Gordon type equation with Dirichlet conditions.

Of course, equation (3) enters this setting, and approximated solutions can give some insights on what happens when the charge product qQ is not small with respect to the mass of the field m. Let $\left(\mathscr{W}_1, \|\cdot\|_{\mathscr{W}_1}\right)$ and $\left(\mathscr{W}_0, \|\cdot\|_{\mathscr{W}_0}\right)$ be two functional spaces. Consider the equation $Lu = 0$ with $u \in \mathscr{W}_1$ and $L : \mathscr{W}_1 \to \mathscr{W}_0$ a (possibly unbounded) linear operator. A *scheme* for this problem consists in approximated spaces $\left(\mathscr{W}_j^\star, \|\cdot\|_{\mathscr{W}_j^\star}\right)$ containing the original ones, as well a family of projectors $\mathscr{P}_{\bullet,j}$ and lifts $\mathscr{R}_{\bullet,j}$ indexed on some (possibly not discrete) set satisfying $\mathscr{W}_{\bullet,j}^\star := \mathscr{P}_{\bullet,j}\mathscr{W}_j^\star \subset \mathbb{C}^{N_\bullet}$ with $N_\bullet \in \mathbb{N} \setminus \{0\}$ and $\mathscr{R}_{\bullet,j} : \mathscr{W}_{\bullet,j}^\star \hookrightarrow \mathscr{W}_j^\star$. We then define the approximated operator

$$L_\bullet := \mathscr{P}_{\bullet,0} \circ L \circ \mathscr{R}_{\bullet,1} : \mathscr{W}_{\bullet,1}^\star \to \mathscr{W}_{\bullet,0}^\star$$

and the approximated solution $u_\bullet$ of u as the solution of the discrete linear equation $L_\bullet u_\bullet = 0$. The point is that computers understand what elements of $\mathscr{W}_{\bullet,j}^\star \subset \mathbb{C}^{N_\bullet}$ are: this makes numerical computations possible.

$$\begin{array}{ccccccccc}
\mathscr{W}_1 & \hookrightarrow & \mathscr{W}_1^\star & = & \mathscr{W}_1^\star & \xrightarrow{L} & \mathscr{W}_0^\star & = & \mathscr{W}_0^\star & \hookleftarrow & \mathscr{W}_0 \\
 & & \downarrow{\scriptstyle \mathscr{P}_{\bullet,1}} & & \uparrow{\scriptstyle \mathscr{R}_{\bullet,1}} & & \uparrow{\scriptstyle \mathscr{R}_{\bullet,0}} & & \downarrow{\scriptstyle \mathscr{P}_{\bullet,0}} & & \\
 & & \mathscr{W}_{\bullet,1}^\star & = & \mathscr{W}_{\bullet,1}^\star & \xrightarrow[L_\bullet]{} & \mathscr{W}_{\bullet,0}^\star & = & \mathscr{W}_{\bullet,0}^\star & &
\end{array}$$

Figure 1: Example of an abstract scheme.

We say that the scheme *converges* if the inductive limits of $\mathscr{W}_{\bullet,j}$ exist and are $\mathscr{W}_j$ (see Subsection 4.1.2 for more details). The approximated solution $u_\bullet$ then converges to the solution u if and only if

$$\|u - \mathscr{R}_{\bullet,1}u_\bullet\|_{\mathscr{W}_{\bullet,1}} \underset{\bullet}{\longrightarrow} 0.$$

It is then natural to ask for *consistency* of the norms, that is

$$\left|\|v\|_{\mathscr{W}_{\bullet,1}} - \|v\|_{\mathscr{W}_1}\right| \underset{\bullet}{\longrightarrow} 0 \qquad\qquad \forall v \in \mathscr{W}_1.$$

We stress here that in general $\mathscr{W}_{\bullet,1} \not\subset \mathscr{W}_1$ so that $\|v_\bullet\|_{\mathscr{W}_1}$ make *a priori* no sense. The geometric meaning of the above convergence is the following one: the affine cones $C_{\bullet,1} := \left\{\|u - v_\bullet\|_{\mathscr{W}_{\bullet,1}} \mid v_\bullet \in \mathscr{W}_{\bullet,1}\right\}$ converges in the sense of sets in $\mathbb{R}_+$ to the cone $C_1 := \left\{\|u - v\|_{\mathscr{W}_1} \mid v \in \mathscr{W}_1\right\}$, and in each cone $C_{\bullet,1}$, the approximated solution $u_\bullet$ approach u in the sense that $\|u - v_\bullet\|_{\mathscr{W}_{\bullet,1}} \to 0$ at the inductive limit. The convergence is shown in the following manner: write

$$\|u - \mathscr{R}_{\bullet,1}u_\bullet\|_{\mathscr{W}_{\bullet,1}} \leq \|u - \mathscr{F}_{\bullet,1}u\|_{\mathscr{W}_{\bullet,1}} + \|\mathscr{F}_{\bullet,1}u - \mathscr{R}_{\bullet,1}u_\bullet\|_{\mathscr{W}_{\bullet,1}} \tag{7}$$

where $\mathscr{F}_{\bullet,j} := \mathscr{R}_{\bullet,j}\mathscr{P}_{\bullet,j} : \mathscr{W}_j \to \mathscr{W}_j$ are the *filters* of the scheme. The first term in (7) is shown to go to 0 at the inductive limit using properties of the projector $\mathscr{P}_{\bullet,1}$ and lift $\mathscr{R}_{\bullet,1}$ (we have to wisely choose them considering the original space $\mathscr{W}_1$); the second term (7) is controlled using properties of the discrete operator $L_\bullet$. Behind all of this are hidden some regularity issues: the smoother the solution u is, the better the control of (7) is.

Chapter 5: Approximation of low frequency resonances. Chapter 5 is devoted to approximation of low frequency resonances of the charged Klein-Gordon equation in the exterior DSRN spacetime. Localization of resonances is an interesting but difficult task. High frequency resonances can be localized in the (De Sitter-)Schwarzschild spacetime (see [SaZw97]); in Theorem 1.3.1, we have the same result in the DSRN case for small charge product. Purely numerical approximations of these high frequency resonances have been carried out in [CCDHJa] and [CCDHJb] based on Chebyshev type polynomial interpolation. In these references, the authors compute approximated low frequency resonances in function of the charge product qQ: the resonance 0 of the uncharged and non-massive case is shown to move into the upper complex half plane when qQ is not small in regard of m (Theorem 1.3.2 shows that resonances are repelled to the lower complex half plane otherwise) then go down to the lower half plane for higher values of qQ. That this resonance eventually reaches a certain line $\Im z = -\kappa/2$ with $\kappa > 0$ is linked to the modern formulation of the Strong Cosmic Censorship (see [CCDHJb, Section I]). In this thesis, we adopt another method based on complex analysis. We show in Subsection 1.2.3 that the kernel of the quadratic pencil $p(z,s) := h_0 - (z - sV)^2$ is inversely proportional to an analytic function $\mathscr{W}(z)$, called the *Wronskian*. This provides an explicit characterization of resonances as being the zeros of $\mathscr{W}$. Using a scheme similar to the one introduced above, we can defined an approximated Wronskian $\mathscr{W}_\bullet(z)$ which is analytic in z. Assume that we have a control of the error in the following sense: there exists $C, h_\bullet > 0$ such that for all $z \in \Gamma \subset \mathbb{C}$ where Γ is a positively oriented contour,

$$|\mathscr{W}(z) - \mathscr{W}_\bullet(z)| \leq C h_\bullet. \tag{8}$$

The term $h_\bullet$ tends to zero at the inductive limit of the scheme. Then Rouché's theorem implies that $\mathscr{W}_\bullet(z)$ has as many zeros as $\mathscr{W}(z)$ inside Γ as the scheme converges. Hence, we can have the number $N(\Gamma)$ of resonances (counted with their multiplicity) inside Γ at the inductive limit using the argument principle

$$N_\bullet(\Gamma) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{\mathscr{W}_\bullet'(z)}{\mathscr{W}_\bullet(z)} \mathrm{d}z.$$

The effectiveness of this method lies in the fact that the above formula gives an integer, so that a sufficiently small error of approximation provides the *exact* number of resonances enclosed by Γ. Furthermore, if a resonance z_0 is isolated inside Γ, then an approximated resonance $z_\bullet(\Gamma)$ is given by

$$z_\bullet(\Gamma) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma z \frac{\mathscr{W}_\bullet'(z)}{\mathscr{W}_\bullet(z)} \mathrm{d}z.$$

The counterpart of such an ambitious precision is that the error can be very difficult to estimate in practice when $\mathscr{W}$ is not analytic below a line $\Im(z) = \epsilon$, $\epsilon \in \mathbb{R}$: estimates for the error then worsen as the contour Γ approaches this line.

Chapter 6: Discussions and perspectives. The last part of this manuscript is devoted to some concluding discussions and remarks about the work carried out during the thesis.

Contents

Notations and conventions

We present here the notations that we will use throughout this document.

The set $\{z \in \mathbb{C} \mid \Im z \gtrless 0\}$ will be denoted by $\mathbb{C}^{\pm}$. For any complex number $\lambda \in \mathbb{C}$, we will write $\langle \lambda \rangle := \left(1 + |\lambda|^2\right)^{1/2}$, $D(\lambda, R)$ will be the disc centered at $\lambda \in \mathbb{C}$ of radius $R > 0$ and $D(\lambda, R)^{\complement}$ its complementary set. For all $\omega = |\omega| \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \backslash]-\infty, 0]$, $\theta \in \mathbb{R}$, we will use the branch of the square root defined by $\sqrt{\omega} := \sqrt{|\omega|} \mathrm{e}^{\mathrm{i}\theta/2}$. To emphasize some important dependences, the symbol $\equiv$ will be used: for example, $a \equiv a(b)$ means "a depends on b". We will write $u \lesssim v$ to mean $u \leq Cv$ for some constant $C > 0$ independant of u and v.

The notation $\mathcal{C}^k_{\mathrm{c}}$ will be used to denote the space of compactly supported $\mathcal{C}^k$ functions. Also, the Schwartz space on $\mathbb{R}$ will be denoted by $\mathscr{S}$. If V, W are complex vector spaces, then $\mathcal{L}(V, W)$ will be the space of bounded linear operators $V \to W$. We will denote by $\mathcal{B}(V)$ (respectively $\mathcal{B}_\infty(V)$) the space of all bounded (respectively compact) operators acting on V. If $W \subset V$ is a subspace and $\|.\|$ is a norm on V, then $\overline{W}^{\|.\|}$ denotes the completion of W for the norm $\|.\|$.

All the scalar products $\langle \cdot, \cdot \rangle$ will be antilinear with respect to their first component and linear with respect to their second component. For any function f, the support of f will be denoted by $\operatorname{Supp} f$. If A is an operator, we will denote by $\mathscr{D}(A)$ its domain, $\sigma(A)$ its spectrum and $\rho(A)$ its resolvent set. $A \geq 0$ will mean that $\langle Au, u \rangle \geq 0$ for all $u \in \mathscr{D}(A)$, and $A > 0$ will mean that $A \geq 0$ and $\ker(A) = \{0\}$.

Now we define the symbol classes on $\mathbb{R}^{2d}$

$$S^{m,n} := \left\{ a \in \mathcal{C}^\infty(\mathbb{R}^{2d}, \mathbb{C}) \mid \forall (\alpha, \beta) \in \mathbb{N}^{2d}, \exists\, C_{\alpha,\beta} > 0, |\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|} \langle x \rangle^{n-|\beta|} \right\}$$

for any $(m, n) \in \mathbb{Z}^{2d}$ (here $\mathbb{N}$ and $\mathbb{Z}$ both include 0). We then define the semiclassical pseudodifferential operators classes

$$\Psi^{m,n} := \{ a^{\mathrm{w}}(x, h\mathrm{D}) \mid a \in S^{m,n} \}, \qquad \Psi^{-\infty,n} := \bigcap_{m \in \mathbb{Z}} \Psi^{m,n}$$

with $a^{\mathrm{w}}(x, h\mathrm{D})$ the Weyl quantization of the symbol a. For any $c > 0$, the notation $P \in c\Psi^{m,n}$ means that $P \in \Psi^{m,n}$ and the norm of P is bounded by a positive multiple of c.

When using the standard spherical coordinates $(\theta, \varphi) \in]0, \pi[\times]0, 2\pi[$ on $\mathbb{S}^2$, we will always ignore the singularities $\{\theta = 0\}$, $\{\theta = \pi\}$. We refer to [ON95, Lemma 2.2.2] to properly fix it.

Let $I \subset \mathbb{R}$ be a Lebesgue measurable set and let $\gamma \in [0,1]$. We denote by $\mathcal{C}^{k,\gamma}(I,\mathbb{K})$ the space of k times continuously differentiable functions from I to $\mathbb{K}$ whose k-th derivative is γ-Hölder, and by $\mathcal{C}^{k,\gamma}_{\rm c}(I,\mathbb{K})$ the space of all such functions with compact support in I. These spaces can be endowed with the following norm:

$$\|u\|_{\mathcal{C}^{k,\gamma}(I,\mathbb{K})} := \sum_{j=0}^{k} \|\partial_r^j u\|_{L^\infty(I,\mathbb{K})} + \sup_{r,r'\in I} \frac{|(\partial_r^k u)(r) - (\partial_r^k u)(r')|}{|r-r'|^\gamma}.$$

The space of piece-wise $\mathcal{C}^k$ functions on I will be denoted by $\mathcal{C}^k_{\rm piece}(I,\mathbb{K})$; any element u will be defined everywhere on I by putting $u(r_0) := 0$ at any discontinuity point r_0.

Given $\rho : I \to]0,+\infty[$ a Lebesgue measurable function, we define the weighted space

$$L^2_\rho := \left\{u : I \to \mathbb{C} \mid u \in L^2(I,\rho(r)\mathrm{d}r)\right\}$$

and write $\langle .\,,.\rangle_{L^2_\rho}$ and $\|\,.\,\|_{L^2_\rho}$ for the associated weighted L^2 right linear scalar product and norm, respectively. The standard norm on $L^p(I,\mathrm{d}r)$ (respectively on $H^p(I,\mathrm{d}r)$) will be denoted by $\|.\|_{L^p}$ (respectively by $\|.\|_{H^p}$) for all $1 \le p \le +\infty$, and we will note $\langle .\,,.\rangle_{H^p}$ the standard scalar product on $H^p(I,\mathrm{d}r)$. Given a (pseudo-)differential operator P acting on L^2_ρ which is self-adjoint and such that $P > 0$ (that is $P \ge 0$ and $\ker P = \{0\}$), we define for $k \ge 0$ the scales of Sobolev spaces

$$\begin{aligned}
\dot{\mathcal{H}}^{2k+1}_\rho &:= \left\{u : I \to \mathbb{C} \mid \left\langle P^{2k+1}u, u\right\rangle_{L^2_\rho} < +\infty\right\},\\
\dot{\mathcal{H}}^{2k}_\rho &:= \left\{u : I \to \mathbb{C} \mid \|P^k u\|_{L^2_\rho} < +\infty\right\},\\
\mathcal{H}^k_\rho &:= \dot{\mathcal{H}}^k_\rho \cap L^2_\rho
\end{aligned}$$

with the corresponding norms

$$\begin{aligned}
\|u\|_{\dot{\mathcal{H}}^{2k+1}_\rho} &:= \left\langle P^{2k+1}u, u\right\rangle^{1/2}_{L^2_\rho},\\
\|u\|_{\dot{\mathcal{H}}^{2k}_\rho} &:= \|P^k u\|_{L^2_\rho},\\
\|u\|_{\mathcal{H}^k_\rho} &:= \sqrt{\|u\|^2_{L^2_\rho} + \|u\|^2_{\dot{\mathcal{H}}^k_\rho}}.
\end{aligned}$$

(Scale of) Sobolev spaces with negative exponent are then defined as the dual spaces of the corresponding (scale of) Sobolev spaces with positive exponent. When $\rho \equiv 1$ (i.e. $\rho(r)\mathrm{d}r$ is the Lebesgue measure), we will simply use the subscript L^p instead of L^p_1 in the norms and scalar products (when $p = 2$). We will denote by $H^k_{\ell\rm oc}$ and $H^k_{\rm c}$ respectively the space of locally or compactly supported H^k functions.

If $u : \mathbb{R} \to \mathbb{C}$ is Lebesgue integrable for the measure μ, then for all Lebesgue measurable set $S \subset \mathbb{R}$ such that $\mu(S) \neq 0$, we define

$$⨍_S u(x)\,\mathrm{d}\mu(x) := \frac{1}{\mu(S)} \int_S u(x)\,\mathrm{d}\mu(x).$$

The set of Lebesgue points of u (in I) will be denoted by $\mathscr{L}(u)$.

Chapter 1

Decay of the Local Energy for the Charged Klein-Gordon Equation in the Exterior De Sitter-Reissner-Nordström Spacetime

In this chapter, we show a resonance expansion for the solutions of the charged Klein-Gordon equation on the DSRN metric. As a corollary we obtain exponential decay of the local energy for these solutions. We restrict our study to the case where the product of black hole charge and the scalar field charge is small. Such a resonance expansion for the solutions of the wave equation has been obtained first by Bony-Häfner for the wave equation on the De Sitter-Schwarzschild metric [BoHa08]. This result has been generalized to much more complicated situations which include perturbations of the De Sitter-Kerr metric by Vasy [Va13]. This last paper has developed new methods including a Fredholm theory for non elliptic problems. These methods could probably also be applied to the present case. In this chapter however we use the more elementary methods of Bony-Häfner [BoHa08] and Georgescu-Gérard-Häfner [GGH17].

The smallness of the charge product is non-quantitative as it is determined from a compactness argument. It allows us at many places to use perturbation arguments with respect to the non-charged case. As far as we are aware the absence of growing modes for the present system is not known for general charge products. In contrast to that absence of growing modes is known for the wave equation on the Kerr metric for general angular momentum of the black hole, see [Wh89]. The question of the existence or not of such modes is a very subtle question and growing modes appear for example for the Klein-Gordon equation on the Kerr metric, see [SR14].

Organization of the chapter. Chapter 1 is organized as follows. In Section 1.1 we give an introduction to the DSRN metric and the charged Klein-Gordon equation on it. In Section 1.2, a meromorphic extension result is shown for the cut-off resolvent and resonances are introduced. The resonance expansion as well as the exponential decay through the horizons are presented in Section 1.3. Suitable resolvent-type estimates are obtained in Section 1.4. In section 1.5 we prove the main theorems by a suitable contour deformation and using the resolvent-type estimates of Section 1.4. The appendix contains a semiclassical limiting absorption principle for a class of generalized resolvents which might have some independent interest.

1.1 Functional framework

1.1.1 The charged Klein-Gordon equation on the DSRN metric

Let

$$F(r) := 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}$$

with $M > 0$ the mass of the black hole, $Q \in \mathbb{R} \setminus \{0\}$ its electric charge and $\Lambda > 0$ the cosmological constant. We assume that

$$\Delta := 9M^2 - 8Q^2 > 0, \qquad \max\left\{0, \frac{6\left(M - \sqrt{\Delta}\right)}{\left(3M - \sqrt{\Delta}\right)^3}\right\} < \Lambda < \frac{6\left(M + \sqrt{\Delta}\right)}{\left(3M + \sqrt{\Delta}\right)^3} \tag{1.1}$$

so that F has four distinct zeros $-\infty < r_n < 0 < r_c < r_- < r_+ < +\infty$ and is positive for all $r \in]r_-, r_+[$ (see [Hi18, Proposition 3.2]; see also [Mok17, Proposition 1] with Λ replaced by $\Lambda/3$ for a different statement of the condition). We also assume that $9\Lambda M^2 < 1$ so that we can use the work of Bony-Häfner [BoHa08]. The exterior DSRN spacetime is the Lorentzian manifold $(\mathcal{M}, g)$ with

$$\mathcal{M} = \mathbb{R}_t \times]r_-, r_+[_r \times \mathbb{S}^2_\omega, \qquad g = F(r)\,\mathrm{d}t^2 - F(r)^{-1}\,\mathrm{d}r^2 - r^2\mathrm{d}\omega^2$$

where $\mathrm{d}\omega^2$ is the standard metric on the unit sphere $\mathbb{S}^2$.

Let $A := \frac{Q}{r}\mathrm{d}t$. Then the charged wave operator on $(\mathcal{M}, g)$ is

$$\boxtimes_g = (\nabla_\mu - \mathrm{i}qA_\mu)(\nabla^\mu - \mathrm{i}qA^\mu) = \frac{1}{F(r)}\left(\left(\partial_t - \mathrm{i}\frac{qQ}{r}\right)^2 - \frac{F(r)}{r^2}\partial_r r^2 F(r)\partial_r - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2}\right)$$

and the corresponding charged Klein-Gordon equation reads

$$\boxtimes_g u + m^2 u = 0 \qquad\qquad m > 0.$$

We set $s := qQ \in \mathbb{R}$ the charge product (which appears in the perturbation term of the standard wave operator), $X :=]r_-, r_+[_r \times \mathbb{S}^2_\omega$ and $V(r) := r^{-1}$ so that the above equation reads

$$(\partial_t - \mathrm{i}sV)^2 u + \hat{P}u = 0 \tag{1.2}$$

with

$$\begin{aligned}\hat{P} &= -\frac{F(r)}{r^2}\partial_r\left(r^2F(r)\partial_r\right) - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} + m^2F(r)\\ &= -F(r)^2\partial_r^2 - F\left(\frac{2F(r)}{r} + \frac{\partial F}{\partial r}(r)\right)\partial_r - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} + m^2F(r)\end{aligned} \tag{1.3}$$

defined on $\mathscr{D}(\hat{P}) := \left\{u \in L^2\left(X, F(r)^{-1}r^2\mathrm{d}r\mathrm{d}\omega\right) \middle| \hat{P}u \in L^2\left(X, F(r)^{-1}r^2\mathrm{d}r\mathrm{d}\omega\right)\right\}$ (this is the spatial operator in [BoHa08] with the additional mass term $m^2F(r)$). In the sequel, we will use the following notations:

$$V_\pm := \lim_{r \to r_\pm} V(r) = r_\pm^{-1}.$$

It turns out that the positive mass makes the study of the equation easier. Besides the fact that massless charged particles do not exist in physics, it is not excluded that the resonance 0 for the case $s = 0$ (see [BoHa08]) can move to $\mathbb{C}^+$ in the case $s \neq 0$ and $m = 0$.

1.1.2 The Regge-Wheeler coordinate

We introduce the Regge-Wheeler coordinate $x \equiv x(r)$ defined by the differential relation

$$\frac{\mathrm{d}x}{\mathrm{d}r} := \frac{1}{F(r)}. \tag{1.4}$$

Using the four roots r_α of F, $\alpha \in I := \{n, c, -, +\}$, we can write

$$\frac{1}{F(r)} = -\frac{3r^2}{\Lambda} \sum_{\alpha \in I} \frac{A_\alpha}{r - r_\alpha}$$

where $A_\alpha = \prod_{\beta \in I \setminus \{\alpha\}} (r_\alpha - r_\beta)^{-1}$ for all $\alpha \in I$, and $\pm A_\pm > 0$. Integrating (1.4) then yields

$$x(r) = -\frac{3}{\Lambda} \sum_{\alpha \in I} A_\alpha r_\alpha^2 \ln \left| \frac{r - r_\alpha}{\mathfrak{r} - r_\alpha} \right| \tag{1.5}$$

with $\mathfrak{r} := \frac{1}{2}\left(3M + \sqrt{9M^2 - 8Q^2}\right)$ (we will explain this choice below); observe that $|Q| < \frac{3}{\sqrt{8}}M$ if (**??**) holds (see the discussion below (17) in [Mok17]). Therefore, we have

$$|r - r_\alpha| = |\mathfrak{r} - r_\alpha| \prod_{\beta \in I \setminus \{\alpha\}} \left| \frac{r - r_\beta}{\mathfrak{r} - r_\beta} \right|^{-A_\beta r_\beta^2 / \left(A_\alpha r_\alpha^2\right)} \exp\left(-\frac{\Lambda}{3 A_\alpha r_\alpha^2} x \right) \qquad \forall \alpha \in I$$

which entails the asymptotic behaviours

$$F(r(x)) + |r(x) - r_\pm| \lesssim \exp\left(-\frac{\Lambda}{3 A_\pm r_\pm^2} x \right) \qquad x \to \pm\infty. \tag{1.6}$$

Note here that

$$-\frac{\Lambda}{3 A_\pm r_\pm^2} = F'(r_\pm) = 2\kappa_\pm \tag{1.7}$$

where $\kappa_- > 0$ is the surface gravity at the event horizon and $\kappa_+ < 0$ is the surface gravity at the cosmological horizon. Recall that $\kappa_\pm$ is defined by the relation

$$X^\mu \nabla_\mu X^\nu = -2\kappa_\pm X^\nu \qquad\qquad X = \partial_t$$

where the above equation is to be considered at the corresponding horizon.

In Appendix 1.6.1, we follow [BaMo93, Proposition IV.2] to show the extension result:

Proposition 1.1.1. *There exists a constant $\mathscr{A} > 0$ such that the function $x \mapsto r(x)$ extends analytically to $\{\lambda \in \mathbb{C} \mid |\Re\lambda| > \mathscr{A}\}$.*

On $L^2(X, \mathrm{d}x\mathrm{d}\omega)$, define the operator $P := r\hat{P}r^{-1}$, given in the coordinates (x, ω) by the expression

$$P = -r^{-1}\partial_x r^2 \partial_x r^{-1} - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} + m^2 F(r) = -\partial_x^2 - W_0 \Delta_{\mathbb{S}^2} + W_1 \tag{1.8}$$

where

$$W_0(x) := \frac{F(r(x))}{r(x)^2}, \quad W_1(x) := \frac{F(r(x))}{r(x)} \frac{\partial F}{\partial r}(r(x)) + m^2 F(r(x)). \tag{1.9}$$

It will happen in the sequel that we write $F(x)$ for $F(r(x))$ and also $V(x)$ for $V(r(x))$. Observe that the potentials W_0 and W_1 satisfy the same estimate as in (1.6).

As

$$\frac{\mathrm{d}W_0}{\mathrm{d}x} = F(r)\frac{\mathrm{d}W_0}{\mathrm{d}r} = \frac{2F(r)}{r^5}\left(3Mr - 2Q^2 - r^2\right),$$

we see that the (unstable) maximum of W_0 occurs when $x = 0$, *i.e.* $r = \mathfrak{r} = \frac{1}{2}\left(3M + \sqrt{9M^2 - 8Q^2}\right)$: this is the radius of the photon sphere. It is the only trapping set outside the black hole for null geodesics (see [Mok17, Proposition 1]). The trapping will have a consequence on some resolvent type estimates, see the paragraph 1.4.2.

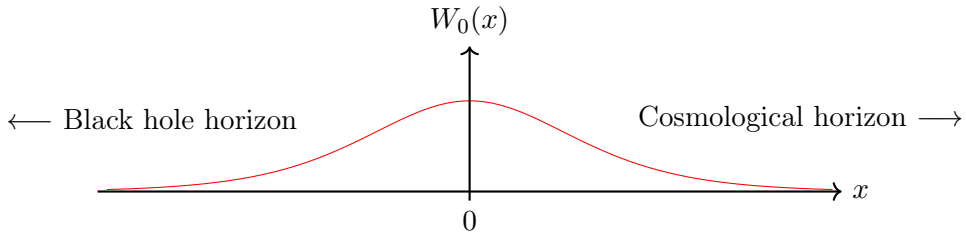

Figure 1.1: The potential W_0 in the Regge-Wheeler coordinates.

1.1.3 The charge Klein-Gordon operator

Taking advantage of the spherical symmetry, we write

$$L^2\left(\mathbb{R}\times\mathbb{S}^2, \mathrm{d}x\mathrm{d}\omega\right) \simeq \bigoplus_{\ell\in\mathbb{N}} \left(L^2(\mathbb{R}, \mathrm{d}x)\otimes Y_\ell\right) =: \bigoplus_{\ell\in\mathbb{N}} \mathcal{V}_\ell$$

where for all $\ell \in \mathbb{N}$, Y_ℓ is the $(2\ell+1)$-dimensional eigenspace of the operator $\left(-\Delta_{\mathbb{S}^2}, H^2(\mathbb{S}^2, \mathrm{d}\omega)\right)$ associated to the eigenvalue $\ell(\ell+1)$. On each $\mathcal{V}_\ell$, we define P_ℓ as the restriction of P onto $\mathcal{V}_\ell$ which will be identified with an operator acting on $L^2(\mathbb{R}, \mathrm{d}x)$, i.e.

$$P_\ell = -\partial_x^2 + \ell(\ell+1)W_0 + W_1 \tag{1.10}$$

and we set $\mathscr{D}(P_\ell) := H^2(\mathbb{R}, \mathrm{d}x)$ so that P_ℓ is self-adjoint. In the sequel, we will use the following (self-adjoint) realization of the total operator P:

$$P := \bigoplus_{\ell\in\mathbb{N}} P_\ell, \qquad \mathscr{D}(P) := \left\{u = (u_\ell)_{\ell\in\mathbb{N}} \in \bigoplus_{\ell\in\mathbb{N}} \mathcal{V}_\ell \mid \forall \ell\in\mathbb{N},\ u_\ell \in \mathscr{D}(P_\ell)\right\}.$$

Now the charged Klein-Gordon equation reads

$$(\partial_t - \mathrm{i}sV)^2 u + Pu = 0. \tag{1.11}$$

The point is to see that if u is a solution of (1.11), then $v := (u, -\mathrm{i}\partial_t u - sVu)$ solves the first order equation

$$-\mathrm{i}\partial_t v = \hat{K}(s)v \tag{1.12}$$

where

$$\hat{K}(s) := \begin{pmatrix} sV & \mathbb{1} \\ P & sV \end{pmatrix} \tag{1.13}$$

is the charge Klein-Gordon operator. Conversely, if $v = (v_0, v_1)$ solves (1.12), then v_0 solves (1.11). We also define $\hat{K}_\ell \equiv \hat{K}_\ell(s)$[1] with P_ℓ in place of P for any $\ell \in \mathbb{N}$. Following [GGH17, Section 3], we realize $\hat{K}_\ell$ with the domain

$$\mathscr{D}(\hat{K}_\ell) := \left\{ u \in P_\ell^{-1/2} L^2(\mathbb{R}, \mathrm{d}x) \oplus L^2(\mathbb{R}, \mathrm{d}x) \mid \hat{K}_\ell u \in P_\ell^{-1/2} L^2(\mathbb{R}, \mathrm{d}x) \oplus L^2(\mathbb{R}, \mathrm{d}x) \right\}$$

and realize the operator $\hat{K}$ as the direct sum on $\mathbb{N} \ni \ell$ of the $\hat{K}_\ell$.

Let $\dot{\mathcal{E}}_\ell$ be the completion of $P_\ell^{-1/2} L^2(\mathbb{R}, \mathrm{d}x) \oplus L^2(\mathbb{R}, \mathrm{d}x)$ for the norm[2]

$$\|u\|^2_{\dot{\mathcal{E}}_\ell} := \langle u_0, P_\ell u_0 \rangle_{L^2(\mathbb{R}, \mathrm{d}x)} + \|u_1 - sVu_0\|^2_{L^2(\mathbb{R}, \mathrm{d}x)} \qquad u = (u_0, u_1) \in \dot{\mathcal{E}}_\ell$$

and define $\left(\dot{\mathcal{E}}, \|.\|_{\dot{\mathcal{E}}}\right)$ as the direct sum of the spaces $\dot{\mathcal{E}}_\ell$. [GGH17, Lemma 3.19] shows that $\hat{K}_\ell$ generates a continuous one-parameter group[3] $(\mathrm{e}^{-\mathrm{i}t\hat{K}_\ell})_{t\in\mathbb{R}}$ on $(\dot{\mathcal{E}}_\ell, \|.\|_{\dot{\mathcal{E}}_\ell})$. We similarly construct the spaces $\left(\mathcal{E}_\ell, \|.\|_{\mathcal{E}_\ell}\right)$ and $\left(\mathcal{E}, \|.\|_{\mathcal{E}}\right)$ with $\langle P_\ell \rangle$ instead of P_ℓ. Let us mention here that for any $n \in \mathbb{R}$ the quantity

$$\langle v \,|\, v \rangle_n := \langle v_1 - nv_0, v_1 - nv_0 \rangle_{L^2(\mathbb{R}, \mathrm{d}x)} + \langle (P - (sV - n)^2) v_0, v_0 \rangle_{L^2(\mathbb{R}, \mathrm{d}x)} \tag{1.14}$$

is formally conserved if $v = (u, -\mathrm{i}\partial_t u)$ with u solution of (1.11) and is continuous with respect to the norm $\|.\|_{\mathcal{E}}$. However, it is in general not positive nor continuous with respect to the norm $\|.\|_{\dot{\mathcal{E}}}$ (see [GGH17, paragraph 3.4.3] for more details): this is superradiance. When $\Lambda = 0$ (that is, when the cosmological horizon is at infinity), the natural energy $\langle . \,|\, . \rangle_{sV_-}$ is positive for s small enough and it can be used to define a Hilbert space framework.

An important observation is the fact that the norms $\|.\|_{\dot{\mathcal{E}}_\ell}$ and $\|.\|_{\mathcal{E}_\ell}$ are locally equivalent, meaning that for any $v \in \dot{\mathcal{E}}$ and any cut-off $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}, \mathbb{R})$, we have

$$\|\chi v\|_{\dot{\mathcal{E}}} \lesssim \|\chi v\|_{\mathcal{E}} \lesssim \|\chi v\|_{\dot{\mathcal{E}}}. \tag{1.15}$$

The first inequality is obvious, and the second one is established with the Hardy type estimate $\|\chi v\|_{L^2} \lesssim \|P^{1/2} v\|_{L^2}$ (see [GGH17, Lemma 9.5]; the validity of this result in our setting is discussed in Subsection 5.1 below).

[1] We will often drop the dependence in s.

[2] Note that the norm $\|.\|^2_{\dot{\mathcal{E}}_\ell}$ is conserved if $[P_\ell, sV] = 0$; it is the case if $s = 0$.

[3] The notation is abusive here as the group should be denoted by $(\mathrm{e}^{\mathrm{i}t\hat{K}_\ell})_{t\in\mathbb{R}}$. We will however keep this convention in this chapter.

1.1.4 The quadratic pencil

Let u be a solution of (1.11). If we look for u of the form $u = \mathrm{e}^{\mathrm{i}zt}v$ with $z \in \mathbb{C}$ for some v, then v satisfies the equation $(P - (z - sV)^2)v = 0$. We define the harmonic quadratic pencil

$$p_\ell(z,s) := P_\ell - (z - sV)^2, \qquad \mathscr{D}(p_\ell(z,s)) := \langle P_\ell \rangle^{-1} L^2(\mathbb{R}, \mathrm{d}x) = H^2(\mathbb{R}, \mathrm{d}x)$$

and realize the total quadratic pencil as

$$p(z,s) := \bigoplus_{\ell \in \mathbb{N}} p_\ell(z,s),$$
$$\mathscr{D}(p(z,s)) := \Big\{ u = (u_\ell)_{\ell \in \mathbb{N}} \in \bigoplus_{\ell \in \mathbb{N}} \mathcal{V}_\ell \;\Big|\; \forall \ell \in \mathbb{N},\, u_\ell \in \mathscr{D}(p_\ell(z,s)),\, \sum_{\ell \in \mathbb{N}} \|p_\ell(z,s)u_\ell\|^2_{\mathcal{V}_\ell} < +\infty \Big\}.$$

[GGH17, Proposition 3.15] sets the useful relations

$$\rho(\hat{K}_\ell) \cap \mathbb{C} \setminus \mathbb{R} = \big\{ z \in \mathbb{C} \setminus \mathbb{R} \;\big|\; p_\ell(z,s) : H^2(\mathbb{R}, \mathrm{d}x) \to L^2(\mathbb{R}, \mathrm{d}x) \text{ is bijective} \big\} \tag{1.16}$$

and

$$\hat{R}_\ell(z,s) := (\hat{K}_\ell(s) - z)^{-1} = \begin{pmatrix} p_\ell(z,s)^{-1}(z - sV) & p_\ell(z,s)^{-1} \\ \mathbb{1} + (z - sV)\, p_\ell(z,s)^{-1}(z - sV) & (z - sV)\, p_\ell(z,s)^{-1} \end{pmatrix} \tag{1.17}$$

for all $z \in \rho(\hat{K}_\ell) \cap \mathbb{C} \setminus \mathbb{R}$. In comparison, the relation (1.7) in [BoHa08] involves the resolvent of P_ℓ, which corresponds to the case $s = 0$ for us. [GGH17, Proposition 3.12] shows that (1.17) is also valid for $z \in \rho(\hat{K}_\ell) \cap \mathbb{R}$ when we work on $(\mathcal{E}_\ell, \|.\|_{\mathcal{E}_\ell})$; by using the local equivalence (1.15) of the norms $\|.\|_{\dot{\mathcal{E}}_\ell}$ and $\|.\|_{\mathcal{E}_\ell}$, we can use (1.17) for $z \in \rho(\hat{K}_\ell) \cap \mathbb{R}$ if we consider the cut-off resolvent $\chi \hat{R}_\ell(z,s)\chi$ with $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}, \mathbb{R})$. In the sequel, we will simply call $p_\ell(z,s)$ the quadratic pencil when $\ell \in \mathbb{N}$ will be fixed.

1.2 Meromorphic extension and resonances

We construct in this Section a meromorphic extension for the weighted resolvent of $\hat{K}(s)$. The main result of this chapter, Theorem 1.3.2, which provides asymptotic decay (in time) for solutions of the charged Klein-Gordon equation (4.37), relies on such a construction. The presence of the mixed term $sV\partial_t$ in (4.37) prevents us to directly use Mazzeo-Melrose result [MaMe87]. However, $p(z,s)^{-1}$ formally tends to $(P - z^2)^{-1}$ as $s \to 0$ for which [MaMe87] applies; moreover, the case $s = 0$ is very similar (even easier) to the case treated in [BoHa08]. We will therefore obtain results for small s using perturbation arguments. Our strategy is the following one:

(*i*) Define first suitable "asymptotic" energy spaces by removing the troublesome negative contributions from the electromagnetic potential sV near $r_\pm$ and define "asymptotic" selfadjoint Hamiltonians $\hat{H}_\pm(s)$ (see the paragraph 1.2.1 below).

(*ii*) For $s = 0$, the situation is really similar to the Klein-Gordon equation on De Sitter-Schwarzschild metric: using the standard results [BoHa08] and [MaMe87], we can meromorphically extend the weighted resolvent of $\hat{H}_\pm(0)$ from $\mathbb{C}^+$ to $\mathbb{C}$ with no poles on and above the real axis (see Lemma 1.2.1 and Lemma 1.2.2).

(*iii*) If s remains small, we can use analytic Fredholm theory to get a meromorphic extension for the weighted resolvents of the asymptotic Hamiltonians $\hat{H}_\pm(s)$ into a strip in $\mathbb{C}^-$ (the perturbation argument entails a bound on the width of this strip which is directly linked to the rate of decay of the potentials W_0 and W_1 in P near $r_\pm$). We will also get the absence of poles near the real axis (see Lemma 1.2.1 and Lemma 1.2.3).

(*iv*) Finally, we construct a parametrix for the resolvent of an equivalent operator to $\hat{K}(s)$ by gluing together the resolvent of $\hat{H}_\pm(s)$ (see (1.25)). Using again the analytic Fredholm theory for s sufficiently small, we show the existence of the weighted resolvent and also that the poles can only lie below the real axis (see Theorem 1.2.8).

The sequel of this Section is organized as follows: Subsection 1.2.1 introduces notations and tools (operators, functional spaces) from [GGH17] which will be used for the construction of the meromorphic extension of the weighted resolvent of $\hat{K}(s)$. Subsection 1.2.2 aims to show that results obtained in [GGH17] are available for us. Then Subsection 1.2.3 establishes the announced results for the asymptotic Hamiltonians $\hat{H}_\pm(s)$. Subsection 5.1 eventually gives the proof of the existence of the meromorphic extension of the weighted resolvent of $\hat{K}(s)$ and also shows that the poles in any compact neighbourhood of 0 lie below the real axis.

1.2.1 Notations

We introduce some notations following [GGH17, Section 2.1]. First observe that if u solves (4.37), then $v := \mathrm{e}^{-\mathrm{i}sV_+t}u$ satisfies

$$(\partial_t^2 - 2\mathrm{i}s(V(r) - V_+)\partial_t - s^2(V(r) - V_+)^2 + P)v = 0.$$

We can therefore work with the potential[4] $\tilde{V} := V - V_+ = \mathcal{O}_{r\to r_+}(r_+ - r)$ in this Section. In order not to overload notations, we will still denote $\tilde{V}$ by V and $\lim_{r\to r_\pm} \tilde{V}(r) = V_\pm$.

Let us define $\mathcal{H} := L^2(X, \mathrm{d}r\mathrm{d}\omega)$ and

$$\mathscr{P} := rF(r)^{-1/2}\hat{P}r^{-1}F(r)^{1/2} = -r^{-1}F(r)^{1/2}\partial_r r^2 F(r)\partial_r r^{-1}F(r)^{1/2} - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} + m^2F(r) \tag{1.18}$$

with $\hat{P}$ given by (1.3). Since $u \mapsto r^{-1}F^{1/2}u$ is an unitary isomorphism from $\mathcal{H}$ to $L^2\left(X, F^{-1}r^2\mathrm{d}r\mathrm{d}\omega\right)$, the results obtained below on $\mathscr{P}$ will also apply to $\hat{P}$ (and thus to P). Observe that the space $\dot{\mathcal{E}}$ has been defined in our setting with the operator P which is $r\hat{P}r^{-1}$ expressed with the Regge-Wheeler coordinate, and $\hat{P}$ is equivalent to $\mathscr{P}$ as explained above; in the sequel, we will denote by $\dot{\mathscr{E}}$ the completion of $\mathscr{P}^{-1/2}\mathcal{H} \oplus \mathcal{H}$ for the norm $\|(u_0, u_1)\|^2_{\dot{\mathscr{E}}} := \langle u_0, \mathscr{P}u_0\rangle_{\mathcal{H}} + \|u_1 - sVu_0\|^2_{\mathcal{H}}$. Let $i_\pm, j_\pm \in \mathcal{C}^\infty(]r_-, r_+[, \mathbb{R})$ such that

$$i_\pm = j_\pm = 0 \text{ close to } r_\mp, \qquad i_\pm = j_\pm = 1 \text{ close to } r_\pm,$$
$$i_-^2 + i_+^2 = 1, \qquad i_\pm j_\pm = j_\pm, \qquad i_- j_+ = i_+ j_- = 0.$$

[4] From a geometrical point of view, we are changing the gauge. Namely, $\frac{Q}{r}\mathrm{d}t$ is replaced by $\left(\frac{Q}{r} - \frac{Q}{r_+}\right)\mathrm{d}t$ which does not degenerate anymore at $r = r_+$. To see this, we use the standard Eddington-Finkelstein advanced and retarded coordinates $u = t - x$, $v = t + x$ to define the horizons: we have locally near the cosmological horizon $\mathrm{d}t = \mathrm{d}u + \mathrm{d}x$, $\mathrm{d}t = \mathrm{d}v - \mathrm{d}x$ and then $\frac{\mathrm{d}t}{\mathrm{d}r} = \pm F(r)^{-1}$. We eventually use that $\left(\frac{1}{r} - \frac{1}{r_+}\right)F(r)^{-1}$ remains bounded and does not vanish at $r = r_+$.

We then define the operators

$$k_\pm := s(V \mp j_\mp^2 V_-), \qquad \mathscr{P}_\pm := \mathscr{P} - k_\pm^2, \qquad \tilde{\mathscr{P}}_- := \mathscr{P} - (sV_- - k_-)^2.$$

We now define the isomorphism on $\dot{\mathscr{E}}$ (see comments above Lemma 3.13 in [GGH17])

$$\Phi(sV) := \begin{pmatrix} \mathbb{1} & 0 \\ sV & \mathbb{1} \end{pmatrix}$$

and we introduce the energy Klein-Gordon operator

$$\hat{H}(s) = \Phi(sV)\hat{K}(s)\Phi^{-1}(sV) = \begin{pmatrix} 0 & \mathbb{1} \\ \mathscr{P} - s^2V^2 & 2sV \end{pmatrix}$$

with domain

$$\mathscr{D}(\hat{H}(s)) = \left\{ u \in \mathscr{P}^{-1/2}\mathcal{H} \oplus \mathcal{H} \mid \hat{H}(s)u \in \mathscr{P}^{-1/2}\mathcal{H} \oplus \mathcal{H} \right\}$$

as well as the asymptotic Hamiltonians

$$\hat{H}_\pm(s) = \begin{pmatrix} 0 & \mathbb{1} \\ \mathscr{P}_\pm & 2k_\pm \end{pmatrix}$$

with domains

$$\mathscr{D}(\hat{H}_+(s)) = \left(\mathscr{P}_+^{-1/2}\mathcal{H} \cap \mathscr{P}_+^{-1}\mathcal{H}\right) \oplus \langle \mathscr{P}_+ \rangle^{-1/2}\mathcal{H},$$
$$\mathscr{D}(\hat{H}_-(s)) = \Phi(sV_-)\left(\tilde{\mathscr{P}}_-^{-1/2}\mathcal{H} \cap \tilde{\mathscr{P}}_-^{-1}\mathcal{H}\right) \oplus \langle \tilde{\mathscr{P}}_- \rangle^{-1/2}\mathcal{H}.$$

These operators are self-adjoint on the following spaces (see the beginning of the paragraph 5.2 in [GGH17]):

$$\dot{\mathscr{E}}_+ := \mathscr{P}_+^{-1/2}\mathcal{H} \oplus \mathcal{H},$$
$$\dot{\mathscr{E}}_- := \Phi(sr_-^{-1})\left(\tilde{\mathscr{P}}_-^{-1/2}\mathcal{H} \oplus \mathcal{H}\right).$$

In the sequel, we will also use the spaces $\mathscr{E}_\pm$ defined as above but with the operators $\langle \mathscr{P}_\pm \rangle$ instead of $\mathscr{P}_\pm$. Finally, we define the weight $w(r) := \sqrt{(r - r_-)(r_+ - r)}$.

1.2.2 Abstract setting

Meromorphic extensions in our setting follow from the works of Mazzeo-Melrose [MaMe87] and Guillarmou [Gu04], as stated in [GGH17, Proposition 5.3]. The abstract setting in which this result can be used is recalled in this paragraph.

We first recall for the reader convenience the Abstract assumptions (A1)-(A3), the Meromorphic Extensions assumptions (ME1)-(ME2) as well as the "Two Ends" assumptions (TE1)-(TE3)

of [GGH17]:

$$\mathscr{P} > 0, \tag{A1}$$

$$\begin{cases} sV \in \mathcal{B}(\mathscr{P}^{-1/2}L^2) > 0, \\ \text{if } z \neq \mathbb{R} \text{ then } (z - sV)^{-1} \in \mathcal{B}(\mathscr{P}^{-1/2}L^2) \text{ and there exists } n > 0 \\ \text{such that } \|(z - sV)^{-1}\|_{\mathcal{B}(\mathscr{P}^{-1/2}L^2)} \lesssim |\Im z|^{-n}, \\ \text{there exists } c > 0 \text{ such that } \|(z - sV)^{-1}\|_{\mathcal{B}(\mathscr{P}^{-1/2}L^2)} \lesssim \big||z| - \|sV\|_{L^\infty}\big| \\ \text{if } |z| \geq c\|sV\|_{L^\infty} \end{cases}, \tag{A2}$$

$$\begin{cases} \text{(a) } wVw \in L^\infty, \\ \text{(b) } [V, w] = 0 \\ \text{(c) } \mathscr{P}^{-1/2}[\mathscr{P}, w^{-\epsilon}]w^{\epsilon/2} \in \mathcal{B}(L^2) \text{ for all } 0 < \epsilon \leq 1, \\ \text{(d) if } \epsilon > 0 \text{ then } \|w^{-\epsilon}u\|_{L^2} \lesssim \|\mathscr{P}^{1/2}u\|_{L^2} \text{ for all } u \in \mathscr{P}^{-1/2}L^2, \\ \text{(e) } w^{-1}\langle\mathscr{P}\rangle^{-1} \in \mathcal{B}(L^2) \text{ is compact} \end{cases}, \tag{ME1}$$

$$\begin{cases} \text{For all } \epsilon > 0 \text{ there exists } \delta_\epsilon > 0 \text{ such that } w^{-\epsilon}(\mathscr{P} - z^2)^{-1}w^{-\epsilon} \text{ extends from } \mathbb{C}^+ \\ \text{to } \{z \in \mathbb{C} \mid \Im z > -\delta_\epsilon\} \text{ as a finite meromorphic function with values} \\ \text{in compact operators acting on } L^2 \end{cases}, \tag{ME2}$$

$$\begin{cases} [x, sV] = 0, \\ x \mapsto w(x) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}), \\ \chi_1(x)\mathscr{P}\chi_2(x) = 0 \text{ for all } \chi_1, \chi_2 \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) \text{ bounded with all their derivatives} \\ \text{and such that } \operatorname{Supp}\chi_1 \cap \operatorname{Supp}\chi_2 = \emptyset \end{cases}, \tag{TE1}$$

$$\text{There exists } \ell_- \in \mathbb{R} \text{ such that } (\mathscr{P}_+, k_+) \text{ and } (\tilde{\mathscr{P}}_-, (k_- - \ell_-)) \text{ satisfy (A2)}, \tag{TE2}$$

$$\begin{cases} \text{(a) } wi_+sVi_+w, wi_-(sV - \ell_-)i_-w \in L^\infty, \\ \text{(b) } [\mathscr{P} - s^2V^2, i_\pm] = \tilde{i}[\mathscr{P} - s^2V^2, i_\pm]\tilde{i} \text{ for some } \tilde{i} \in \mathcal{C}_c^\infty(]-2, 2[, \mathbb{R}) \text{ such that } \tilde{i}_{|[-1,1]} \equiv 1 \\ \text{(c) } (\mathscr{P}_+, k_+, w) \text{ and } (\mathscr{P}_-, (k_- - \ell_-), w) \text{ fulfill (ME1) and (ME2)}, \\ \text{(d) } \mathscr{P}_\pm^{1/2}i_\pm\mathscr{P}_\pm^{-1/2}, \mathscr{P}^{1/2}i_\pm\mathscr{P}^{-1/2} \in \mathcal{B}(L^2), \\ \text{(e) } w[(\mathscr{P} - s^2V^2), i_\pm]w\mathscr{P}_\pm^{-1/2}, w[(\mathscr{P} - s^2V^2), i_\pm]w\mathscr{P}^{-1/2}, [(\mathscr{P} - s^2V^2), i_\pm]\mathscr{P}_\pm^{-1/2}, \\ \quad [(\mathscr{P} - s^2V^2), i_\pm]\mathscr{P}^{-1/2}, \mathscr{P}^{-1/2}[w^{-1}, \mathscr{P}]w \text{ are bounded operators on } L^2, \\ \text{(e) if } \epsilon > 0 \text{ then } \|w^{-\epsilon}u\|_{L^2} \lesssim \|\mathscr{P}^{1/2}u\|_{L^2} \text{ for all } u \in \mathscr{P}^{-1/2}L^2 \end{cases}. \tag{TE3}$$

[GGH17, Section 9] shows that all the above hypotheses actually follow from some geometric assumptions (the assumptions (G1)-(G7) of [GGH17, paragraph 2.1.1]). We show here that the charged Klein-Gordon equation in the exterior DSRN spacetime can be dealt within this geometric setting:

(G1) The operator P in [GGH17] is $-\Delta_{\mathbb{S}^2}$ for us, and satisfies of course $[\Delta_{\mathbb{S}^2}, \partial_\phi] = 0$.

(G2) The operator $h_{0,s}$ in [GGH17] is $\mathscr{P}$ for us, that is $\alpha_1(r) = \alpha_3(r) = r^{-1}F(r)^{1/2}$, $\alpha_2(r) = rF(r)^{1/2}$ and $\alpha_4(r) = mF(r)^{1/2}$. These last coefficients are clearly smooth in r. Furthermore,

since we can write $F(r) = g(r)w(r)^2$ with $g(r) = \frac{\Lambda}{3r^2}(r-r_n)(r-r_c) \gtrsim 1$ for all $r \in]r_-, r_+[$, it comes for all $j \in \{1,2,3,4\}$ as $r \to r_\pm$

$$\alpha_j(r) - w(r)\left(i_-(r)\,\alpha_j^- + i_+(r)\,\alpha_j^+\right) = w(r)\left(g(r)^{1/2} - \alpha_j^\pm\right) = \mathcal{O}_{r\to r_\pm}\big(w(r)^2\big),$$

$$\alpha_1^\pm = \alpha_3^\pm = \frac{1}{r_\pm^2}\sqrt{\frac{\Lambda(r_\pm - r_n)(r_\pm - r_c)}{3}},$$

$$\alpha_2^\pm = \sqrt{\frac{\Lambda(r_\pm - r_n)(r_\pm - r_c)}{3}},$$

$$\alpha_4^\pm = \frac{m}{r_\pm}\sqrt{\frac{\Lambda(r_\pm - r_n)(r_\pm - r_c)}{3}}.$$

Also, we clearly have $\alpha_j(r) \gtrsim w(r)$. Direct computations show that

$$\partial_r^m\partial_\omega^n\left(\alpha_j - w\left(i_-\,\alpha_j^- + i_+\,\alpha_j^+\right)\right)(r) = \mathcal{O}_{r\to r_\pm}\big(w(r)^{2-2m}\big)$$

for all $m, n \in \mathbb{N}$.

(G3) The operator k_s in [GGH17] is $sV(r)$ for us, so $k_s = k_{s,v}$ and $k_{s,r} = 0$. We have $V(r) - V_\pm = \mathcal{O}_{r\to r_\pm}(|r_\pm - r|) = \mathcal{O}_{r\to r_\pm}\big(w(r)^2\big)$ (with $V_+ = 0$, recall the discussion at the beginning of Subsection 1.2.1) and $\partial_r^m\partial_\omega^n V(r)$ is bounded for any $m, n \in \mathbb{N}$.

(G4) The perturbation k in [GGH17] is simply $k = k_s = sV$ for us, so that this assumption is trivially verified.

(G5) The operator h_0 in [GGH17] is simply $h_0 = h_{0,s} = \mathscr{P}$ for us, and we have

$$\begin{aligned}\mathscr{P} &= -\alpha_1(r)\partial_r w(r)^2 r^2 g(r)\partial_r\alpha_1(r) - \alpha_1(r)^2\Delta_{\mathbb{S}^2} + \alpha_1(r)^2 m^2 r^2\\ &= \alpha_1(r)\left(-\partial_r w(r)^2 r^2 g(r)\partial_r - \Delta_{\mathbb{S}^2} + m^2 r^2\right)\alpha_1(r)\\ &\gtrsim \alpha_1(r)\left(-\partial_r w(r)^2\partial_r - \Delta_{\mathbb{S}^2} + 1\right)\alpha_1(r).\end{aligned}$$

(G6) This assumption is trivial in our setting.

(G7) We check that $(\mathscr{P}_+, k_+)$ and $(\tilde{\mathscr{P}}_-, k_- - sV_-)$ satisfy (G5). Since $\alpha_1(r), k_+(r) = \mathcal{O}_{r\to r_\pm}(|r_\pm - r|)$, we can write for $|s| < mr_-$

$$\begin{aligned}\mathscr{P}_+ &= -\alpha_1(r)\partial_r w(r)^2 r^2 g(r)\partial_r\alpha_1(r) - \alpha_1(r)^2\Delta_{\mathbb{S}^2} + \alpha_1(r)^2 m^2 r^2 - k_+(r)^2\\ &= \alpha_1(r)\left(-\partial_r w(r)^2 r^2 g(r)\partial_r - \Delta_{\mathbb{S}^2} + m^2 r^2 - \frac{k_+(r)^2}{\alpha_1(r)^2}\right)\alpha_1(r)\\ &\gtrsim \alpha_1(r)\left(-\partial_r w(r)^2\partial_r - \Delta_{\mathbb{S}^2} + 1\right)\alpha_1(r).\end{aligned}$$

As $k_-(r) - sV_- = \mathcal{O}_{r\to r_\pm}(|r_\pm - r|)$ too, we get the same conclusion with $\tilde{\mathscr{P}}_-$.

To end this Subsection, we recall from [GGH17, Section 9] that

$$\text{(G3)} \implies \text{(A1)-(A3)}, \qquad \text{(G3)} \implies \text{(ME1)}, \qquad \text{(G3)-(G5)} \implies \text{(TE1)-(TE3)}$$

and (ME2) is satisfied by assumptions (G1), (G2) and (G7) on the form of the operator $\mathscr{P}$ using Mazzeo-Melrose standard result (see [GGH17, paragraph 9.2.2] and also [MaMe87] for the original work of Mazzeo-Melrose).

1.2.3 Study of the asymptotic Hamiltonians

The aim of this paragraph is to show the existence of a meromorphic continuation of the weighted resolvent $w^\delta(\hat{H}_\pm(s)-z)^{-1}w^\delta$ from $\mathbb{C}^+$ into a strip in $\mathbb{C}^-$ which is analytic in z in a tight box near 0. We start with the meromorphic extension.

Lemma 1.2.1. *For all $\delta > \delta' > 0$ and all $s \in \mathbb{R}$, $w^\delta(\hat{H}_\pm(s)-z)^{-1}w^\delta$ has a meromorphic extension from $\mathbb{C}^+$ to $\{\omega \in \mathbb{C} \mid \Im\omega > -\delta'\}$ with values in compact operators acting on $\dot{\mathscr{E}}_\pm$.*

Proof. Since hypotheses (G) are satisfied, we can apply [GGH17, Lemma 9.3] which shows that we can apply Mazzeo-Melrose result: the meromorphic extension of $w^\delta(\mathscr{P}_\pm - z^2)^{-1}w^\delta$ exists from $\mathbb{C}^+$ to a strip $\mathcal{O}_\delta$. This strip is explicitly given in the work of Guillarmou (*cf.* [Gu04, Theorem 1.1]):

$$\mathcal{O}_\delta = \left\{ z \in \mathbb{C} \mid z^2 = \lambda(3-\lambda),\ \Re\lambda > \frac{3}{2} - \delta \right\}.$$

The absence of essential singularity is due to the fact that the metric g is even (see Theorem 1.4 and also Definition 1.2 in [Gu04]). We have to check that the set $\mathcal{O}_\delta$ contains a strip in $\overline{\mathbb{C}^-}$. To see this, write $\lambda = \alpha + \mathrm{i}\beta$ and $z = a + \mathrm{i}b$ with $\alpha, \beta, a, b \in \mathbb{R}$, $b \leq 0$ and $z^2 = \lambda(3-\lambda)$. Solving for

$$\begin{cases} a^2 - b^2 &= \alpha(3-\alpha) + \beta^2 \\ 2ab &= (3-2\alpha)\beta \end{cases} \tag{1.19}$$

we find

$$\begin{cases} \beta &= \pm\sqrt{\frac{1}{2}(a^2-b^2-9/4) + \frac{1}{2}\sqrt{(a^2-b^2-9/4)^2 + 4a^2b^2}} \\ \alpha &= \frac{3}{2} - \frac{ab}{\beta} \end{cases}$$

and these expressions make sense since $\beta = 0$ can happen only if $ab = 0$, and

$$\beta = \pm\frac{|a||b|}{\sqrt{b^2+9/4}} + \mathcal{O}_{a\to 0}(a), \qquad\qquad \beta = \mathcal{O}_{b\to 0}(b).$$

If $b = 0$ then $\alpha = 3/2$ and β solves $a^2 = 9/4 + \beta^2$, and conversely $\alpha = 3/2$ implies $b = 0$. Hence $\alpha = 3/2$ allows all $z \in \mathbb{R}$. We may now assume $b < 0$ (hence $\alpha \neq 0$). The condition $\Re\lambda = \alpha > 3/2 - \delta$ reads $\frac{ab}{\beta} < \delta$, and this condition is trivially satisfied if $\alpha \geq 3/2$ since (1.19) implies that $\frac{ab}{\beta} \leq 0 < \delta$. Otherwise, if $\alpha < 3/2$ then (1.19) implies that $\frac{ab}{\beta} > 0$ and

$$b > -\left|\frac{\beta}{a}\right|\delta.$$

We compute

$$\left(\frac{\beta}{a}\right)' = \frac{a\beta' - \beta}{a^2}$$

where $'$ denotes here the derivative with respect to a, and

$$\beta' = \frac{a}{2\beta}\left(1 + \frac{(a^2-b^2-9/4)+2b^2}{\sqrt{(a^2-b^2-9/4)^2+4a^2b^2}}\right)$$

so that

$$\begin{aligned} a\beta'-\beta=0 &\iff a^2\left(1+\frac{(a^2-b^2-9/4)+2b^2}{\sqrt{(a^2-b^2-9/4)^2+4a^2b^2}}\right)=2\beta^2 \\ &\iff a^2(a^2-b^2-9/4)+2a^2b^2 \\ &\qquad = -(b^2+9/4)\sqrt{(a^2-b^2-9/4)^2+4a^2b^2}+(a^2-b^2-9/4)^2+4a^2b^2 \\ &\iff (b^2+9/4)^2((a^2-b^2-9/4)^2+4a^2b^2)=(b^4+81/16+a^2b^2-9a^2/4+9b^2/2)^2. \end{aligned}$$

After some tedious simplifications, we obtain the very simple condition

$$a\beta'-\beta=0 \iff 9a^4b^2=0.$$

Thus $a=0$ is the only possible extremum of β when $b<0$. One can check that $\beta\to 1$ as $a\to\pm\infty$, whence

$$\{z\in\mathbb{C}\mid 0\geq \Im z>-\delta\}\subset\mathcal{O}_\delta.$$

From there, we deduce the existence of the meromorphic extension of $w^\delta(\hat{H}_\pm(s)-z)^{-1}w^\delta$ for $z\in\{\omega\in\mathbb{C}\mid \Im\omega>-\delta'\}$ thanks to Lemma 4.3 and Proposition 4.4 in [GGH17] (the parameters ϵ and δ_ϵ therein are identical in our situation, and $\delta_{\epsilon/2}$ can be replaced by any $\delta'<\delta_\epsilon$). □

Before proving the analyticity near 0 of the weighted resolvent, we need to prove the following result:

Lemma 1.2.2. *For all $\delta>0$, $w^\delta(\mathscr{P}-z^2)^{-1}w^\delta$ has no pole in $\mathbb{R}$.*

Proof. We can work with the operator P expressed in the Regge-Wheeler coordinate since $P\mapsto\mathscr{P}$ is an unitary transform (as explained at the beginning of Subsection 1.2.1).

For all $\ell\in\mathbb{N}$, P_ℓ is selfadjoint and the potential $W_0+\ell(\ell+1)W_1$ (with W_0 and W_1 as in (1.9)) is bounded on $\mathscr{D}(P_\ell)$ and tends to 0 at infinity exponentially fast; as a result, the Kato-Agmon-Simon theorem (*cf.* [RS4, Theorem XIII.57]) implies that P_ℓ has no positive eigenvalue. As $P_\ell\geq 0$, we deduce that there is no eigenvalue on $\mathbb{R}\setminus\{0\}$. Furthermore, [BaMo93, Proposition II.1] shows that 0 is not an eigenvalue for P_ℓ thanks to the exponential decay of $W_0+\ell(\ell+1)W_1$. Finally, P_ℓ verifies the limiting absorption principle

$$\sup_{\mu>0}\left\|\langle x\rangle^{-\alpha}(P_\ell-(\lambda+\mathrm{i}\mu))^{-1}\langle x\rangle^{-\alpha}\right\|_{L^2}<+\infty \qquad \forall\lambda\in\mathbb{R}\setminus\{0\},\ \forall\alpha>1,$$

see Mourre [Mou80]. The only issue then is $z=0$ which could be a pole.

We introduce then the Jost solutions following [Ba04, Section 2] (recall that we are considering the case $s=0$ in this Lemma so that the potential sV vanishes). Fix $\ell\in\mathbb{N}$ and set $\tilde{W}_\ell:=\ell(\ell+1)W_0+W_1$. Set

$$\kappa:=\min\{\kappa_-,|\kappa_+|\} \tag{1.20}$$

where $\kappa_\pm$ are the surface gravity at the event and cosmological horizons (cf. (1.7)). For any $\alpha \in]0, 2\kappa[$,

$$\int_{-\infty}^{+\infty} \left|\tilde{W}_\ell(x)\right| \mathrm{e}^{\alpha|x|} \mathrm{d}x < +\infty.$$

The convergence of the above integral comes from the exponential decay of $\tilde{W}_\ell$ at infinity. For all $z \in \mathbb{C}$ such that $\Im z > -\kappa$, [Ba04, Proposition 2.1] shows that there exist two unique $\mathcal{C}^2$ functions $x \mapsto e^\pm(x, z, \ell)$, that we will simply write $e_\pm(x)$ or $e_\pm(x, z)$, satisfying the Schrödinger equation

$$(\partial_x^2 + z^2 - \tilde{W}_\ell(x))e_\pm(x) = 0 \qquad\qquad \forall x \in \mathbb{R}$$

with $\partial_x e_\pm \in L^\infty_{\ell\mathrm{oc}}(\mathbb{R}_x, \mathbb{C})$, and such that if $\Im z > -\kappa$, then $\partial_x^j e_\pm$ is analytic in z for all $0 \leq j \leq 1$. Moreover, they satisfy

$$\lim_{x \to \pm\infty} \left(\left| e_\pm(x) - \mathrm{e}^{\pm \mathrm{i}zx} \right| + \left| \partial_x e_\pm(x) \mp \mathrm{i}z \mathrm{e}^{\pm \mathrm{i}zx} \right| \right) = 0. \tag{1.21}$$

By checking the formula on $\mathcal{C}^2_c(\mathbb{R}, \mathbb{C})$ first and then extending it on $H^2(\mathbb{R}, \mathrm{d}x)$ by density, one easily shows that the kernel K of $(P_\ell - (z - sV)^2)^{-1} : H^2_{\mathrm{c}}(\mathbb{R}, \mathrm{d}x) \to L^2_{\ell\mathrm{oc}}(\mathbb{R}, \mathrm{d}x)$ for $\Im z > -\kappa$ is given by[5]

$$K(z; x, y) = \frac{1}{\mathscr{W}(z)} \big(e_+(x, z) e_-(y, z) \mathbb{1}_{x \geq y}(x, y) + e_+(y, z) e_-(x, z) \mathbb{1}_{y \geq x}(x, y) \big)$$

where $\mathscr{W}(z) = e_+(x)(e_-)'(x) - (e_+)'(x)e_-(x)$ is the Wronskian between e_+ and e_-. Since $\mathscr{W}$ is independent of $x \in \mathbb{R}$ (as shown at the very beginning of the proof of [Ba04, Proposition 2.1]), we see using the non trivial limits for $e_\pm$ in (5.4) that a pole z of order $n > 0$ for $w^\delta(P_\ell - (z - sV)^2)^{-1} w^\delta$ with $\Im z > -\kappa$ is a zero of order n of the Wronskian $\mathscr{W}$, and $e_+(\cdot, z)$ and $e_-(\cdot, z)$ are then collinear.

We now reproduce the computation (2.14) in [BoHa08]. Assume that $z = 0$ is a pole; for all $\ell \in \mathbb{N}$, $e_+(\cdot, 0, \ell) \in L^2_{\ell\mathrm{oc}}(\mathbb{R}, \mathrm{d}x)$ and satisfies $P_\ell e_+ = 0$, so that

$$\begin{aligned} 0 &= \int_{-R_0}^{R_0} (P_\ell e_+) \, \overline{e_+} \mathrm{d}x \\ &= \left[r \overline{e_+} \partial_x \left(r^{-1} e_+ \right) \right]_{-R_0}^{R_0} + \int_{-R_0}^{R_0} \left| r \partial_x \left(r^{-1} e_+ \right) \right|^2 \mathrm{d}x + \ell \left(\ell + 1 \right) \int_{-R_0}^{R_0} F\left(r \right) \left| r^{-1} e_+ \right|^2 \mathrm{d}x \\ &\quad + m^2 \int_{-R_0}^{R_0} F\left(r \right) \left| e_+ \right|^2 \mathrm{d}x. \end{aligned}$$

Letting $R_0 \to +\infty$ and using the decay of the derivative of e_+ in (5.4) for $z = 0$ show that $e_+ = 0$, a contradiction[6]. □

We are now ready to prove the analyticity. In the next result, we will consider s as a complex number.

[5] Observe that in the example at the beginning of Section 1.2, the kernel $R(z; x, y)$ is also given in terms of the Jost functions $e_\pm(x, z) = \mathrm{e}^{\pm \mathrm{i}zx}$.

[6] We may notice here that the positive mass term m^2 allowed us to conclude that $z = 0$ is not a pole. For the wave equation as in [BoHa08], we do not have any positivity for $\ell = 0$ and $z = 0$ is shown to be a pole.

Proposition 1.2.3. *Let $0 < \delta < \kappa$ and $R > 0$. There exist $\varepsilon_0 \equiv \varepsilon_0(\delta) > 0$ and $\sigma \equiv \sigma(\mathscr{P}, \tilde{k}_\pm)$ such that the extension of $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$ is holomorphic in (s,z) for $s \in D(0,\sigma)$ and $z \in \,]-R,R[+ \mathrm{i}\,]-\varepsilon_0,\varepsilon_0[$.*

The restriction $\delta < \kappa$ comes from the fact that the extension of $(\hat{H}_\pm(s) - z)^{-1}$ depends itself on δ (see formula (1.22) in the proof). One can increase the exponent of the weight w but the width of the strip in $\mathbb{C}^-$ in which the result of Proposition 1.2.3 holds is bounded by κ.

Proof. Observe that $(\hat{H}_-(s) - z)^{-1} = (\Phi(-sV_-)\hat{H}_-(s)\Phi(sV_-) - z)^{-1}$ on $\dot{\mathscr{E}}_-$, so it is sufficient to prove the announced results for $w^\delta(\hat{H}_+(s) - z)^{-1}w^\delta$ and $w^\delta(\widetilde{\hat{H}}_-(s) - z)^{-1}w^\delta$ with

$$\widetilde{\hat{H}}_-(s) := \Phi(-sV_-)\hat{H}_-(s)\Phi(sV_-) = \begin{pmatrix} sV_- & \mathbb{1} \\ \tilde{\mathscr{P}}_- & 2k_- - sV_- \end{pmatrix}.$$

Proceeding as in the proof of [GGH17, Proposition 4.4], we can work on the operators $\mathscr{P} - (z - \tilde{k}_\pm)^2$ with

$$\tilde{k}_+ := k_+, \qquad\qquad \tilde{k}_- := k_- - sV_-$$

so that $\tilde{k}_\pm$ are now exponentially decaying potentials at infinity (and are polynomial in s).

We reproduce the perturbation argument of [GGH17, Lemma 4.3]. Choose $\varepsilon_0 \in \,]0,\delta[$ sufficiently small and pick $z \in \,]-R,R[+ \mathrm{i}\,]-\varepsilon_0,\varepsilon_0[$ so that $w^\delta(\mathscr{P} - z^2)^{-1}w^\delta$ is holomorphic (it is possible since there is no pole in $\mathbb{R}$ by Lemma 1.2.2) and $w^\delta(\mathscr{P} - (z - \tilde{k}_\pm)^2)^{-1}w^\delta$ is meromorphic[7] in z. Write then in $\mathcal{H}$

$$w^\delta(\mathscr{P} - z^2)^{-1}w^\delta = w^\delta(\mathscr{P} - (z - \tilde{k}_\pm)^2)^{-1}w^\delta\big(\mathbb{1} + \mathcal{K}_\pm(s,z)\big) \tag{1.22}$$

with

$$\mathcal{K}_\pm(s,z) := w^{-\delta}\tilde{k}_\pm(2z - \tilde{k}_\pm)w^{-\delta}w^\delta(\mathscr{P} - z^2)^{-1}w^\delta.$$

$\mathcal{K}_\pm(s,z)$ is clearly analytic in $s \in D(0,1)$ and in $z \in \,]-R,R[+ \mathrm{i}\,]-\varepsilon_0,\varepsilon_0[$. Since $\tilde{k}_\pm = \mathcal{O}_{r\to r_\pm}\big(w^{2\kappa}\big)$ by (1.6) and (1.20), $\delta < \kappa$ and $w^\delta(\mathscr{P} - z^2)^{-1}w^\delta$ is compact by Lemma 1.2.1, we see that $\mathcal{K}_\pm(s,z)$ is compact. By two-dimensional analytic Fredholm theory, there exists a subvariety $S \subset D(0,1) \times \big(\,]-R,R[+ \mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\big)$ such that $(\mathbb{1} + \mathcal{K}_\pm(s,z))^{-1}$ exists and is holomorphic in $(s,z) \in \Big(D(0,1) \times \big(\,]-R,R[+ \mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\big)\Big) \setminus S$. We then get the representation formula for the extension:

$$w^\delta(\mathscr{P} - (z - \tilde{k}_\pm)^2)^{-1}w^\delta = w^\delta(\mathscr{P} - z^2)^{-1}w^\delta\big(\mathbb{1} + \mathcal{K}_\pm(s,z)\big)^{-1}. \tag{1.23}$$

We claim that for $\sigma > 0$ sufficiently small, we have

$$\Big(D(0,\sigma) \times \big(\,]-R,R[+ \mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\big)\Big) \cap S = \emptyset. \tag{1.24}$$

Otherwise, for every $n \in \mathbb{N} \setminus \{0\}$, there is a couple $(s_n, z_n) \in D(0,1/n) \times \big(\,]-R,R[+ \mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\big)$ such that $\mathbb{1} + \mathcal{K}(s_n,z_n)$ is not invertible. By compactness, we can assume that $(s_n,z_n) \to (0,z_0)$

[7] Lemma 1.2.1 of course applies if we replace k_- by $\tilde{k}_-$.

as $n \to +\infty$ for some $z_0 \in [-R,R] + \mathrm{i}[-\varepsilon_0, \varepsilon_0]$. But $\mathbb{1} + \mathcal{K}(0, z_0) = \mathbb{1}$ is invertible for all $z \in \mathbb{C}$, so $\mathbb{1} + \mathcal{K}_\pm(s,z)$ must be invertible too for all (s,z) in a small neighbourhood of $(0,z_0)$, a contradiction.

Assuming now that s is sufficiently small so that (1.24) is true, we deduce by (1.23) that the poles of $w^\delta(\mathscr{P} - (z - \tilde{k}_\pm)^2)^{-1} w^\delta$ are exactly the poles of $w^\delta(\mathscr{P} - z^2)^{-1} w^\delta$. Since for $z \in]-R, R[+ \mathrm{i}]-\varepsilon_0, \varepsilon_0[$, $w^\delta(\mathscr{P} - z^2)^{-1} w^\delta$ has no pole, the same conclusion applies for $w^\delta(\mathscr{P} - (z - \tilde{k}_\pm)^2)^{-1} w^\delta$. □

1.2.4 Construction of the meromorphic extension of the weighted resolvent

The aim of this paragraph is to show the existence of a meromorphic extension for $w^\delta(\hat{K}(s) - z)^{-1} w^\delta$ in a strip near 0 of width uniform in s. Since the operators $\hat{K}(s)$ and $\hat{H}(s)$ are equivalent modulo the isomorphism $\Phi(sV)$ (by (3.19) in [GGH17]), we will work with the latter one.

We first need some preliminary results. The starting point is the following result:

Proposition 1.2.4 (Proposition 5.5 in [GGH17]). *There is a finite set $Z \in \mathbb{C} \setminus \mathbb{R}$ with $\bar{Z} = Z$ such that the spectrum of $\hat{H}(s)$ is included in $\mathbb{R} \cup Z$ and the resolvent has a meromorphic extension to $\mathbb{C} \setminus \mathbb{R}$. Moreover, the set Z consists of eigenvalues of finite multiplicity of $\hat{H}(s)$.*

An important fact is that [GGH17, Proposition 3.6] shows that $Z \equiv Z(s)$ is contained in the disc $D(0, C|s|)$ for some constant $C > 0$ (we can take $C = 2\|V\|_{L^\infty}$). We show below that $Z(s) \cap \mathbb{C}^+ = \emptyset$ for s sufficiently small.

We henceforth use the Regge-Wheeler coordinate x introduced in Subsection 1.1.2. We will still denote by $\mathscr{P}$ the operator defined in (1.18) expressed in the coordinates (x, ω):

$$\mathscr{P} = -F(r(x))^{-1/2}\, \partial_x^2\, F(r(x))^{-1/2} - W_0(x)\Delta_{\mathbb{S}^2} + W_1(x)$$

Let also $\mathcal{H} = L^2(\mathbb{R} \times \mathbb{S}^2_\omega, F(r(x))\mathrm{d}x\mathrm{d}\omega)$.

Lemma 1.2.5. *For all $\delta > 0$, w^δ sends $\dot{\mathscr{E}}$ into $\dot{\mathscr{E}}_\pm$ and $\dot{\mathscr{E}}$ into itself.*

Proof. Let $u = (u_0, u_1) \in \dot{\mathscr{E}}$. We only show that $w^\delta \dot{\mathscr{E}} \subset \dot{\mathscr{E}}_-$, the proof of the other statements being slightly easier. We thus look for $v = (v_0, v_1) \in \tilde{\mathscr{P}}_-^{-1/2}\mathcal{H} \oplus \mathcal{H}$ such that $(w^\delta u_0, w^\delta u_1) = (v_0, sV_- v_0 + v_1)$. Since w^δ is bounded on $\mathbb{R}$, $w^\delta u_1 \in \mathcal{H}$. Next, using the facts that $(w^\delta)' u_0$, $V_+ w^\delta u_0$ and $W_j^{1/2} w^\delta u_0$ are in $\mathcal{H}$ thanks to (ME1) (d) $(0 \le j \le 1)$, we compute

$$\|\tilde{\mathscr{P}}_-^{1/2} w^\delta u_0\|_{\mathcal{H}}^2 = \big\langle \mathscr{P} w^\delta u_0, w^\delta u_0 \big\rangle_{\mathcal{H}} - \underbrace{\|s^2 (V_- - k_-)^2 w^\delta u_0\|_{\mathcal{H}}^2}_{\lesssim \|\mathscr{P}^{1/2} u_0\|_{\mathcal{H}}^2}$$

and working with the operators $\mathscr{P}_\ell$ defined as P_ℓ $(\ell \in \mathbb{N})$, we get

$$\big\langle \mathscr{P}_\ell w^\delta u_0, w^\delta u_0 \big\rangle_{\mathcal{H}} = \|\partial_x w^\delta u_0\|_{\mathcal{H}}^2 + \underbrace{\big\langle (-\ell(\ell+1) W_0 + W_1) w^\delta u_0, w^\delta u_0 \big\rangle_{\mathcal{H}}}_{\lesssim \|\mathscr{P}_\ell^{1/2} u_0\|_{\mathcal{H}}^2},$$

$$\|\partial_x w^\delta u_0\|_{\mathcal{H}}^2 \lesssim \|(w^\delta)' u_0\|_{\mathcal{H}}^2 + \|w^\delta u_0'\|_{\mathcal{H}}^2 \lesssim \|\mathscr{P}_\ell^{1/2} u_0\|_{\mathcal{H}}^2.$$

This proves that $w^\delta u_0 \in \tilde{\mathscr{P}}_-^{-1/2}\mathcal{H}$. Hence $v_0 := w^\delta u_0 \in \tilde{\mathscr{P}}_-^{1/2}\mathcal{H}$, and the problem boils down to show that $v_1 := w^\delta u_1 - sV_- v_0 = w^\delta u_1 - sV_- w^\delta u_0$ is in $\mathcal{H}$; this is a consequence of (ME1) (d) which implies that $sV_- w^\delta u_0 \in \mathcal{H}$. □

For all $z \in \mathbb{C}^+$ and $s \in \mathbb{R}$, we introduce for the operator

$$Q(s,z) := i_-^2(\hat{H}_-(s) - z)^{-1} + i_+^2(\hat{H}_+(s) - z)^{-1} = \sum_{\pm} i_\pm^2(\hat{H}_\pm(s) - z)^{-1}.$$

In Subsection 1.2.3 above, we have studied the resolvents of the asymptotic Hamiltonians. In particular, we know that $Q(s,z)$ meromorphically extends into a strip in $\mathbb{C}^-$ and is analytic in a small neighbourhood of $\mathbb{R}$. We wish to show that $(\hat{H}(s) - z)^{-1}$ has the same properties. To do this, we show that $Q(s,z)$ is a parametrix for the resolvent on the energy space.

By Lemma 1.2.5, $Q(s,z)w^\delta$ is well-defined in $\dot{\mathscr{E}}$. Using the potentials $k_\pm = s(V \mp j_\mp^2 V_-)$ introduced in Subsection 1.2.1 as well as the relations $i_\pm j_\mp = 0$, we compute:

$$i_\pm^2(\hat{H}(s) - z)(\hat{H}_\pm(s) - z)^{-1} = i_\pm^2 \mathbb{1} - i_\pm^2 \begin{pmatrix} 0 & 0 \\ s^2(V^2 - k_\pm^2) & 2s(V - k_\pm) \end{pmatrix} (\hat{H}_\pm(s) - z)^{-1} = i_\pm^2 \mathbb{1}.$$

Since $i_+^2 + i_-^2 = 1$, we obtain in $\dot{\mathscr{E}}$:

$$(\hat{H}(s) - z)Q(s,z)w^\delta = \left(\mathbb{1} + \sum_{\pm} [\hat{H}(s), i_\pm^2](\hat{H}_\pm(s) - z)^{-1} \right) w^\delta. \tag{1.25}$$

It follows that for all $z \notin Z$ (see Proposition 1.2.4)

$$w^\delta Q(s,z) w^\delta = w^\delta (\hat{H}(s) - z)^{-1} w^\delta \big(\mathbb{1} + \hat{\mathcal{K}}_\pm(s,z) \big) \tag{1.26}$$

with

$$\hat{\mathcal{K}}_\pm(s,z) := w^{-\delta} \sum_{\pm} [\hat{H}(s), i_\pm^2](\hat{H}_\pm(s) - z)^{-1} w^\delta.$$

Lemma 1.2.6. *The operators on the left and right-hand sides of* (1.26) *send $\dot{\mathscr{E}}$ into itself.*

Proof. For the left-hand side of (1.26), we successively use Lemma 1.2.5, the facts that $(\hat{H}_\pm(s) - z)^{-1}$ sends $\dot{\mathscr{E}}_\pm$ into $\mathscr{D}(\hat{H}_\pm(s)) \subset \dot{\mathscr{E}}_\pm$ and $i_\pm$ sends $\dot{\mathscr{E}}_\pm$ into $\dot{\mathscr{E}}$ by [GGH17, Lemma 5.4], and again Lemma 1.2.5.

We now deal with the right-hand side of (1.26). By Lemma 1.2.5, we only have to show that $w^{-\delta}[\hat{H}(s), i_\pm](\hat{H}_\pm(s) - z)^{-1}$ sends $\dot{\mathscr{E}}_\pm$ into $\dot{\mathscr{E}}$. Let $u \in \dot{\mathscr{E}}_\pm$ and write $v = (v_0, v_1) : (\hat{H}_\pm(s) - z)^{-1} u \in \dot{\mathscr{E}}_\pm$. We have

$$w^{-\delta}[H(s), i_\pm](H_\pm(s) - z)^{-1} u = w^{-\delta} \begin{pmatrix} 0 & 0 \\ [\mathscr{P}, i_\pm] & 0 \end{pmatrix} (v_0, v_1) = \begin{pmatrix} 0 \\ w^{-\delta} [\mathscr{P}, i_\pm] v_0 \end{pmatrix} = \begin{pmatrix} 0 \\ w^{-\delta} [\mathscr{P}, i_\pm] w^{-\delta} w^\delta v_0 \end{pmatrix}.$$

Since $w^\delta \dot{\mathscr{E}}_\pm \subset \dot{\mathscr{E}}_\pm$, we can use (TE3) (e) to conclude that the second component is in $\mathcal{H}$, whence $w^{-\delta}[\hat{H}(s), i_\pm](\hat{H}_\pm(s) - z)^{-1} \dot{\mathscr{E}}_\pm \subset \dot{\mathscr{E}}$ (when $\pm = -$, we use that $\mathscr{P}^{1/2} \tilde{\mathscr{P}}_-^{-1/2}$ is bounded on $\mathcal{H}$). □

In the following, we will consider s as a complex number lying in a small neighbourhood of 0.

Lemma 1.2.7. *Let $0 < \delta < \kappa$ and $R > R_0$. $\mathbb{1} + \hat{\mathcal{K}}_\pm(s,z)$ is a holomorphic family of Fredholm operators acting on $\dot{\mathscr{E}}$ for $(s,z) \in D(0,\sigma) \times \big( ]-R, R[+ \mathrm{i}\,]-\varepsilon_0, \varepsilon_0[\big)$, with $\sigma > 0$ sufficiently small and $\varepsilon_0 > 0$ as in Proposition 1.2.3.*

Proof. Write

$$\hat{\mathcal{K}}_\pm(s,z) = \sum_\pm w^{-\delta}[\hat{H}(s), i_\pm^2]w^{-\delta} w^\delta (\hat{H}_\pm(s) - z)^{-1} w^\delta.$$

By Lemma 1.2.1, $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$ is compact on $\dot{\mathscr{E}}_\pm$ and Proposition 1.2.3 shows that the extension is holomorphic in (s,z). Furthermore,

$$w^{-\delta}[\hat{H}(s), i_\pm^2]w^{-\delta} = \begin{pmatrix} 0 & 0 \\ w^{-\delta}\left[\mathscr{P}, i_\pm^2\right] w^{-\delta} & 0 \end{pmatrix}$$

is bounded on $\dot{\mathscr{E}}_\pm$ (as a consequence of (TE3) (e), see the end of the proof of Lemma 1.2.6). Hence $\hat{\mathcal{K}}_\pm(s,z)$ is compact and thus $\mathbb{1} + \hat{\mathcal{K}}_\pm(s,z)$ is Fredholm. □

We are now ready to construct the meromorphic extension of the weighted resolvent. For all $s_0 > 0$, define $R_0 := 2Cs_0$. Proposition 1.2.4 and the remark below then show that $Z(s) \subset D(0, R/2)$ for all $s \in]-s_0, s_0[$ and all $R > R_0$.

Theorem 1.2.8. *Let $0 < \delta < \kappa$ and $s \in]-s_0, s_0[$.*

1. *For s small enough, $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$ has a meromorphic extension from $\mathbb{C}^+ \setminus Z$ to $\{\omega \in \mathbb{C} \mid \Im\omega > -\delta'\}$ for all $0 < \delta' < \delta$ with values in compact operators acting on $\dot{\mathscr{E}}$.*
2. *For all $R > R_0$, there exists $0 < s_1 < s_0$ such that for all $s \in]-s_1, s_1[$, the extension of $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$ is analytic in $z \in]-R, R[+ \mathrm{i}]-\varepsilon_0, \varepsilon_0[$ with $\varepsilon_0 \equiv \varepsilon_0(\delta) > 0$ as in Proposition 1.2.3.*

Proof. We first show Part 1. Let $s \in \mathbb{C}$ small enough and let $z \in]-R, R[+ \mathrm{i}]-\delta', \delta'[$. Since $\hat{H}_\pm(0) = \hat{H}(0)$, we observe that $\hat{\mathcal{K}}_\pm(0,z) = 0$ and $Q(0,z) = (\hat{H}(0) - z)^{-1}$. Hence the operator $\mathbb{1} + \hat{\mathcal{K}}_\pm(0,z) = \mathbb{1}$ is invertible for all $z \in \mathbb{C}$. Finally, Lemma 1.2.1 shows that $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$ is meromorphic in z. We can therefore use the meromorphic Fredholm theory to invert $\mathbb{1} + \hat{\mathcal{K}}_\pm(s,z)$ on $\dot{\mathscr{E}}$. Using (1.26), we have the representation formula

$$w^\delta(\hat{H}(s) - z)^{-1}w^\delta = w^\delta Q(s,z) w^\delta \big(\mathbb{1} + \hat{\mathcal{K}}_\pm(s,z)\big)^{-1} \tag{1.27}$$

which is valid for $z \in]-R, R[+ \mathrm{i}]-\delta', \delta'[$. This shows that $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$ has a meromorphic extension in this strip and Part 1 is settled.

Let us show Part 2. of the theorem. We pick this time $(s,z) \in D(0,\sigma) \times \big(]-R, R[+ \mathrm{i}]-\varepsilon_0, \varepsilon_0[\big)$ with $\sigma, \varepsilon_0 > 0$. Lemma 1.2.7 shows that, if σ is very small, $\mathbb{1} + \hat{\mathcal{K}}_\pm(s,z)$ is a holomorphic family of Fredholm operators acting on $\dot{\mathscr{E}}$. We can thus use the two-dimensional analytic Fredholm theory which implies that there is a meromorphic extension $D(0,\sigma) \times \big(]-R, R[+ \mathrm{i}]-\varepsilon_0, \varepsilon_0[\big) \ni (s,z) \mapsto \big(\mathbb{1} + \hat{\mathcal{K}}_\pm(s,z)\big)^{-1}$, and (1.27) is valid for $(s,z) \in D(0,\sigma) \times \big(]-R, R[+ \mathrm{i}]-\varepsilon_0, \varepsilon_0[\big)$ with σ small. This shows that the poles of $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$ are the poles of $(\mathbb{1} + \hat{\mathcal{K}}_\pm(s,z))^{-1}$ and $w^\delta Q(s,z) w^\delta$, the last ones being the poles of $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$.

The multidimensional analytic Fredholm theory also implies that there exists a (possibly empty) subvariety $S \subset D(0,\sigma) \times \big(]-R, R[+ \mathrm{i}]-\varepsilon_0, \varepsilon_0[\big)$ such that $\mathbb{1} + \hat{\mathcal{K}}(s,z)$ is invertible for $(s,z) \notin S$. We claim that we can take $\sigma > 0$ small enough so that

$$\Big(D(0,\sigma) \times \big(]-R, R[+ \mathrm{i}]-\varepsilon_0, \varepsilon_0[\big)\Big) \cap S = \emptyset.$$

Otherwise, for every $n \in \mathbb{N} \setminus \{0\}$, there is a couple $(s_n, z_n) \in D(0,1/n) \times \big(\,]-R,R[+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[\big)$ such that $\mathbb{1} + \hat{\mathcal{K}}(s_n, z_n)$ is not invertible. By compactness, we can assume that $(s_n, z_n) \to (0, z_0)$ as $n \to +\infty$ for some $z_0 \in [-R,R] + \mathrm{i}[-\varepsilon_0,\varepsilon_0]$. But $\mathbb{1} + \hat{\mathcal{K}}(0,z_0) = \mathbb{1}$ is invertible for all $z \in \mathbb{C}$, so $\mathbb{1} + \hat{\mathcal{K}}_\pm(s,z)$ must be invertible too for all (s,z) in a small neighbourhood of $(0,z_0)$, a contradiction.

We now assume $|s| < s_1$ where s_1 is so small that $\mathbb{1} + \hat{\mathcal{K}}_\pm(s,z)$ is invertible on $\dot{\mathscr{E}}$ for $z \in]-R,R[+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[$. Using then the formula (1.27), we conclude that the poles of $w^\delta(\hat{H}(s)-z)^{-1}w^\delta$ are precisely the poles of $w^\delta Q(s,z)w^\delta$, which are the poles of $w^\delta(\hat{H}_\pm(s)-z)^{-1}w^\delta$. We then use Proposition 1.2.3 to conclude that there is no pole for $z \in]-R,R[+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[$. This completes the proof. □

As a first consequence, we deduce a holomorphy result for the resolvent.

Corollary 1.2.9. *Let $\varepsilon_0 > 0$ as in Proposition 1.2.3. Then for all $s \in \mathbb{R}$ such that $|s| < s_1$ with $Cs_1 < \varepsilon_0$, the resolvent $(\hat{H}(s)-z)^{-1}$ is holomorphic in $z \in \mathbb{C}^+$. Furthermore, the spectrum of $\hat{H}(s)$ is contained in $\mathbb{R}$.*

Proof. We know by Theorem 1.2.8 that the weighted resolvent $w^\delta(\hat{H}(s)-z)^{-1}w^\delta$ is holomorphic in $z \in (]-R,R[+\mathrm{i}\,]-\varepsilon_0,\varepsilon_0[) \cap \mathbb{C}^+ \subset D(0,R/2) \subset \mathbb{C}^+$ if we assume $s < s_1$ by Part 2. of Theorem 1.2.8.
By Proposition 1.2.4, $(\hat{H}(s)-z)^{-1}$ is holomorphic in $\mathbb{C}^+ \setminus Z$. Assume then that $z_0 \in \mathbb{C}^+ \cap Z$ is a pole of order $m_0 \in \mathbb{N}$: there exist some finite rank operators $A_1, \dots, A_{m_0} : \mathcal{H} \to \mathcal{H}$ such that

$$(\hat{H}(s)-z)^{-1} = \sum_{j=1}^{m_0} \frac{A_j}{(z-z_0)^j} + \text{holomorphic term} \qquad \forall z \in \mathbb{C}^+ \text{ near } z_0.$$

Since $R > R_0$, $Z \cap \mathbb{C}^+ \subset D(0,R/2) \cap \mathbb{C}^+$ and then

$$w^\delta(\hat{H}(s)-z)^{-1}w^\delta = \sum_{j=1}^{m_0} \frac{w^\delta A_j w^\delta}{(z-z_0)^j} + \text{holomorphic term}$$

is holomorphic in z near z_j, so that $A_1 = \ldots = A_{m_0} = 0$ and $(\hat{H}(s)-z)^{-1}$ is holomorphic in $z \in \mathbb{C}^+$. By Proposition 1.2.4, this implies that the spectrum of $\hat{H}(s)$ in $Z \cap \mathbb{C}^+$ is empty; by symmetry, we deduce that $\bar{Z} \cap \mathbb{C}^- = \emptyset$ too. □

Remark 1.2.10. *Theorem 1.2.8 and Corollary 1.2.9 answer Bachelot's open question of the nature of the sets σ_p (the eigenvalues in $\mathbb{C}^+$) and σ_{ss} (the real resonances, also called hyperradiant modes) defined in [Ba04] by equations (2.35) and (2.36), when the charge product s is sufficiently small: both are empty as he conjectured at the end of his paper.*

We finally deduce the existence of the cut-off inverse of the quadratic pencil and define resonances.

Corollary 1.2.11. *Let $s \in]-s_0,s_0[$ small enough. The operator $\chi p(z,s)^{-1}\chi : L^2(\mathbb{R},\mathrm{d}x) \to H^2(\mathbb{R},\mathrm{d}x)$ defines for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$ a meromorphic function of $z \in \{\omega \in \mathbb{C} \mid \Im\omega > -\kappa\}$ and analytic if $\Im z > -\varepsilon_0$ with $\varepsilon_0 > 0$ given by Proposition 1.2.3.*
If χ is not identically 0, then the poles z of this extension are exactly the poles of the cut-off resolvent $\chi(\hat{H}(s)-z)^{-1}\chi$ and are independent of the choice of χ. We call them resonances *of p and write $z \in \mathrm{Res}(p)$. Similarly, we define $\mathrm{Res}(p_\ell)$ as the poles of $\chi p_\ell(z,s)^{-1}\chi$ for all $\ell \in \mathbb{N}$.*

Proof. Let $R > 0$ and let $z \in \mathbb{C}$ with $-R \leq \Re z \leq R$ and $\Im z > -\kappa$. The meromorphic extension $w^\delta(\hat{H}(s) - z)^{-1}w^\delta : \dot{\mathscr{E}} \to \dot{\mathscr{E}}$ (with $0 < \delta < \kappa$) entails the meromorphic extension $w^\delta(\hat{K}(s) - z)^{-1}w^\delta : \dot{\mathcal{E}} \to \dot{\mathcal{E}}$ since $\hat{H}(s)$ and $\hat{K}(s)$ are equivalent on $\dot{\mathcal{E}}$ modulo the isomorphism $\Phi(sV)$ introduced in Subsection 1.2.1. Since $w(x)$ is exponentially decaying by (1.6), we can write for any cut-off $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}, \mathbb{R})$

$$\chi(\hat{K}(s) - z)^{-1}\chi = (\chi w^{-\delta})w^\delta(\hat{K}(s) - z)^{-1}w^\delta(w^{-\delta}\chi) : L^2(\mathbb{R}, \mathrm{d}x) \to H^2(\mathbb{R}, \mathrm{d}x)$$

using that $\chi w^{-\delta} \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}, \mathbb{R})$. In particular, if χ is not identically 0, the poles of $\chi(\hat{K}(s) - z)^{-1}\chi$ and $w^\delta(\hat{K}(s) - z)^{-1}w^\delta$ coincide.

By formula (1.17) and the discussion below, we see that we can define the operator $\chi p(z, s)^{-1}\chi : L^2(\mathbb{R}, \mathrm{d}x) \to H^2(\mathbb{R}, \mathrm{d}x)$ for any $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}, \mathbb{R})$ as a meromorphic function of z, and its poles are precisely the poles of $\chi(\hat{K}(s) - z)^{-1}\chi$.

To conclude the proof, it remains to prove the analyticity in the whole strip $\{z \in \mathbb{C} \,|\, \Im z > -\varepsilon_0\}$ (which excludes a possible accumulation of resonances to $\mathbb{R}$ at infinity): this follows from Theorem 1.3.1 below. □

1.3 Resonance expansion for the charged Klein-Gordon equation

We present in this section the main result of this chapter which is an extension of [BoHa08, Theorem 1.3]. By using the formula (1.16) and (1.17) as well as (3.21) in [GGH17] and the local equivalence (1.15) of the norms $\|.\|_{\dot{\mathcal{E}}_\ell}$ and $\|.\|_{\mathcal{E}_\ell}$ if $z \in \mathbb{R}$, we can define for $\Im z > -\kappa$ the meromorphic extension of the cut-off resolvent $\hat{R}_{\chi,\ell}(z) := \chi(\hat{K}_\ell - z)^{-1}\chi$. For all resonance $z_0 \in \mathrm{Res}(p_\ell)$, denote by $m(z_0) \in \mathbb{N}$ its multiplicity and set

$$\Pi^{\chi,\ell}_{j,k} := \frac{1}{2\pi \mathrm{i}} \oint_{\partial\gamma} \frac{(-\mathrm{i})^k}{k!} \hat{R}_{\chi,\ell}(z)(z - z_0)^k \mathrm{d}z$$

defined for all integer $k \geq -(m(z_0) + 1)$ with γ a small positively oriented circle enclosing z_0 and no other resonance. We will denote by $\hat{R}_\chi(z)$ and $\Pi^\chi_{j,k}$ the cut-off resolvent of the full operator $\hat{K}$ and the corresponding generalized projector, respectively. Recall that $\mathrm{Res}(p)$ is introduced in Corollary 1.2.11.

We first introduce the set of pseudo-poles of P whose points approximate high frequency resonances. The proof is given in Appendix 1.6.2.

Theorem 1.3.1. *There exist $K > 0$ and $\theta > 0$ such that, for any $C > 0$, there exists an injective map $\tilde{b} : \Gamma \to \mathrm{Res}(p)$ with*

$$\Gamma = \frac{\sqrt{F(\mathfrak{r})}}{\mathfrak{r}} \left(\pm\mathbb{N} \setminus \{0\} \pm \frac{1}{2} \pm \frac{qQ}{\sqrt{F(\mathfrak{r})}} - \frac{\mathrm{i}}{2}\sqrt{\left|3 - \frac{12M}{\mathfrak{r}} + \frac{10Q^2}{\mathfrak{r}^2}\right|} \left(\mathbb{N} + \frac{1}{2}\right) \right)$$

the set of pseudo-poles, such that all the poles in

$$\Omega_C = \left\{\lambda \in \mathbb{C} \;\middle|\; |\lambda| > K, \Im\lambda > -\max\{C, \theta|\Re\lambda|\}\right\}$$

are in the image of $\tilde{b}$. Furthermore, if $\mu \in \Gamma$ and $\tilde{b}(\mu) \in \Omega_C$, then

$$\lim_{|\mu| \to +\infty} (\tilde{b}(\mu) - \mu) = 0.$$

If $\Re\mu = \frac{\sqrt{F(\mathfrak{r})}}{\mathfrak{r}}\left(\pm\ell \pm \frac{1}{2} \pm \frac{qQ}{\sqrt{F(\mathfrak{r})}}\right)$ *for* $\ell \in \mathbb{N}\setminus\{0\}$, *then the corresponding pole* $\tilde{b}(\mu)$ *has multiplicity* $2\ell+1$.

We can now state our main result (the proof is given in Section 1.5):

Theorem 1.3.2 (Decay of the local energy)**.** *Let* $\chi \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$.

(i) Let $\nu > 0$ *such that* $\nu \notin \Gamma$ *(*Γ *is the set of pseudo-poles as in Theorem 1.3.1),* $\nu < \kappa$ *and* $\mathrm{Res}(p) \cap \{\lambda \in \mathbb{C} \mid \Im\lambda = -\nu\} = \emptyset$. *There exists* $N > 0$ *such that, for all* $u \in \dot{\mathcal{E}}$ *with* $\langle -\Delta_{\mathbb{S}^2}\rangle^N u \in \dot{\mathcal{E}}$ *and* s *small enough, we have*

$$\chi \mathrm{e}^{-\mathrm{i}t\hat{K}}\chi u = \sum_{\substack{z_j \in \mathrm{Res}(p)\\ \Im z_j > -\nu}} \sum_{k=0}^{m(z_j)} \mathrm{e}^{-\mathrm{i}z_j t} t^k \Pi_{j,k}^{\chi} u + E(t)u \tag{1.28}$$

for $t > 0$ *sufficiently large, with*

$$\|E(t)u\|_{\dot{\mathcal{E}}} \lesssim \mathrm{e}^{-\nu t}\|\langle -\Delta_{\mathbb{S}^2}\rangle^N u\|_{\dot{\mathcal{E}}}$$

and the sum is absolutely convergent in the sense that

$$\sum_{\substack{z_j \in \mathrm{Res}(p)\\ \Im z_j > -\nu}} \sum_{k=0}^{m(z_j)} \|\Pi_{j,k}^{\chi}\langle -\Delta_{\mathbb{S}^2}\rangle^{-N}\|_{\dot{\mathcal{E}}\to\dot{\mathcal{E}}} < +\infty.$$

(ii) There exists $\varepsilon > 0$ *such that, for any increasing positive function* g *with* $\lim_{x\to+\infty} g(x) = +\infty$ *and* $g(x) \le x$ *for* $x \gg 0$, *for all* $u \in \dot{\mathcal{E}}$ *with* $g(-\Delta_{\mathbb{S}^2})u \in \dot{\mathcal{E}}$ *and* s *small enough, we have*

$$\|\chi \mathrm{e}^{-\mathrm{i}t\hat{K}}\chi u\|_{\dot{\mathcal{E}}} \lesssim (g(\mathrm{e}^{\varepsilon t}))^{-1}\|g(-\Delta_{\mathbb{S}^2})u\|_{\dot{\mathcal{E}}}$$

for $t > 0$ *sufficiently large.*

Remark 1.3.3.

1. *Formula (1.28) provides a physical interpretation of resonances: they are the frequencies and dumping rates of charged Klein-Gordon field in presence of the charged black hole (see Chapter 4.35 in [Ch92] for a discussion on the interpretation of resonances).*

2. *Part (ii) of Theorem 1.3.2 shows that a logarithmic derivative loss in the angular direction* $(\ln\langle -\Delta_{\mathbb{S}^2}\rangle)^\alpha u \in \dot{\mathcal{E}}$ *with* $\alpha > 1$ *entails the integrability of the local energy:*

$$\left\| \int_0^{+\infty} \chi \mathrm{e}^{-\mathrm{i}t(\hat{K}-z)}\chi\, u\, \mathrm{d}t \right\|_{\dot{\mathcal{E}}} \lesssim \|(\ln\langle -\Delta_{\mathbb{S}^2}\rangle)^\alpha u\|_{\dot{\mathcal{E}}}.$$

3. *In the limits* $9\Lambda M^2 \to 1^-$ *and* $Q \to 0$, *the expansion in part (i) of Theorem 1.3.2 is not empty has (infinitely many) pseudo-poles of* Γ *(introduced in Theorem 1.3.1) lie in the strip* $\{z \in \mathbb{C} \mid \Im(z) > -\kappa\}$. *To see this, it suffices to consider the case* $Q = 0$. *Then* $\mathfrak{r} = 3M$ *and*

$$\min\{|\mathrm{Im}\lambda| \mid \lambda \in \Gamma\} = \frac{\sqrt{F(\mathfrak{r})}}{4\mathfrak{r}}\sqrt{\left|3 - \frac{12M}{\mathfrak{r}}\right|} = \frac{\sqrt{1-9\Lambda M^2}}{12\sqrt{3}M}.$$

We show that

$$\frac{\sqrt{1-9\Lambda M^2}}{12\sqrt{3}M} < |\kappa_\pm|. \tag{1.29}$$

Observe that for $Q=0$, we have

$$F'(r) = \frac{2M}{r^2} - \frac{2\Lambda r}{3} = \frac{1-F(r)}{r} - \Lambda r$$

so that

$$F'(r_\pm) = \frac{1}{r_\pm} - \Lambda r_\pm.$$

Thus (1.29) becomes

$$\frac{\sqrt{1-9\Lambda M^2}}{6\sqrt{3}M} < \frac{|1-\Lambda r_\pm^2|}{r_\pm}. \tag{1.30}$$

Set $\alpha := 3\sqrt{\Lambda}M < 1$. The footnote page 6 in [SaZw97] shows that

$$r_\pm = \frac{1}{\sqrt{\Lambda}} \operatorname{Im}\left(\left(\mp\sqrt{1-\alpha^2} + \mathrm{i}\alpha \right)^{1/3} \right). \tag{1.31}$$

As $\mp\sqrt{1-\alpha^2}+\mathrm{i}\alpha$ has modulus one, we can write $r_\pm = \frac{\sin\theta_\pm}{\sqrt{\Lambda}}$ for some $\theta_\pm \in]0,\pi[$ (the roots are positive) and thus (1.30) reads

$$\frac{\sqrt{1-\alpha^2}}{6\sqrt{3}M} < \frac{\cos^2\theta\pm}{\sin\theta_\pm}\sqrt{\Lambda}.$$

We eventually show that

$$\frac{\sqrt{1-\alpha^2}}{2\sqrt{3}\alpha} < \frac{\cos^2\theta\pm}{\sin\theta_\pm}.$$

When $\alpha \to 1^-$, the left-hand side above goes to 0 whereas the right-hand side remains positive: this last assertion can be checked in (1.31) as then

$$\operatorname{Im}\left(\left(\mp\sqrt{1-\alpha^2} + \mathrm{i}\alpha \right)^{1/3} \right) = \operatorname{Im}\left(\mathrm{i}^{1/3} \right) = \operatorname{Im}\left(\mathrm{e}^{\mathrm{i}\pi/6} \right) = \frac{1}{2} \neq 0.$$

1.4 Estimates for the cut-off inverse of the quadratic pencil

In this section, we show some estimates on the cut-off inverse of the quadratic pencil. We can work with $\ell \in \mathbb{N}$ fixed but our estimates have to be uniform in ℓ. Since $\big(\chi p(-\bar{z}+2sV,s)\chi\big)^* = \chi p(z,s)\chi$, we can restrict ourselves to consider $z \in \mathbb{C}$ with $\Re z > -2s_0\|V\|_{L^\infty}$ for some fixed $s_0 > 0$ such that $0 < |s| < s_0$. In the following, we are simply denoting by L^2 the space $L^2(\mathbb{R}, \mathrm{d}x)$. For some real numbers $R, C_0, C_1 > 0$ (determined by Theorem 1.4.1 below), we define the

- zone I as $[-R,R]+\mathrm{i}\,[-C_0,C_0]$,
- zone II as $[R,\ell/R]+\mathrm{i}\,[-C_0,C_0]$,
- zone III as $[\ell/R,R\ell]+\mathrm{i}\,[-C_0,C_0]$,
- zone IV as $([R\ell,+\infty[+\mathrm{i}\,]-\infty,C_0])\cap\{\lambda\in\mathbb{C}\mid \Im\lambda\geq -C_0-C_1\ln\langle\lambda\rangle\}\cap\Omega_\kappa$

with $\Omega_\kappa:=\{\omega\in\mathbb{C}\mid \Im\omega>-\kappa\}$ (recall that $\kappa:=\min\{\kappa_-,|\kappa_+|\}$).

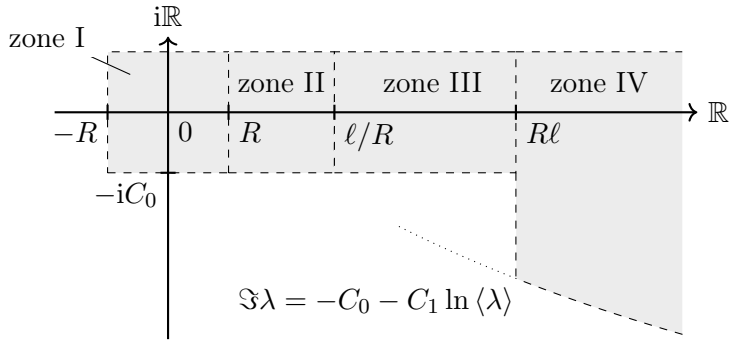

Figure 1.2: The four zones.

We quote here all the estimates that we are going to show in this Section in the following theorem (which is an extension of [BoHa08, Theorem 2.1] to our setting):

Theorem 1.4.1. *Let $\chi\in\mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$, $s\in\mathbb{R}$ and Ω_κ as above. If s is small enough, then the following estimates hold uniformly in $\ell\in\mathbb{N}$:*

1. *For all $R>0$, $C>0$ and $0<C_0<\varepsilon$, $\mathrm{Res}(p)\cap([-R,R]+\mathrm{i}\,[-C_0,C])=\emptyset$ and the operator*
$$\chi p(z,s)^{-1}\chi : L^2\to L^2 \tag{1.32}$$
exists and is bounded uniformly in $z\in[-R,R]+\mathrm{i}\,[-C_0,C]$. Moreover, we have
$$\|\chi p_\ell(z,s)^{-1}\chi\|_{L^2\to L^2}\leq\|\chi p(z,s)^{-1}\chi\|_{L^2\to L^2}\lesssim\prod_{\substack{z_j\in\mathrm{Res}(p)\\|z_j|<2R}}\frac{1}{|z-z_j|}. \tag{1.33}$$

2. *There exist $R>0$ and $0<C_0<\varepsilon$ such that there is no resonance in $[R,\ell/R]+\mathrm{i}\,[-C_0,C_0]$. Furthermore, for all $z\in[R,\ell/R]+\mathrm{i}\,[-C_0,C_0]$, we have*
$$\|\chi p_\ell(z,s)^{-1}\chi\|_{L^2\to L^2}\lesssim\frac{1}{\langle z\rangle^2}. \tag{1.34}$$

3. *Let $R>0$ and $0<C_0<\varepsilon$ be fixed and suppose that $\ell\gg 0$. The number of resonances of p_ℓ in $[\ell/R,R\ell]+\mathrm{i}[-C_0,C_0]$ is bounded uniformly in ℓ and there exists $C>0$ such that, for all $z\in[\ell/R,R\ell]+\mathrm{i}[-C_0,C_0]$,*
$$\|\chi p_\ell(z,s)^{-1}\chi\|_{L^2\to L^2}\lesssim\langle z\rangle^C\prod_{\substack{z_j\in\mathrm{Res}(p_\ell)\\|z-z_j|<1}}\frac{1}{|z-z_j|}. \tag{1.35}$$

Furthermore, there exists $\varepsilon > 0$ such that there is no resonance in $[\ell/R, R\ell] + \mathrm{i}[-\varepsilon, 0]$ and we have for all $z \in [\ell/R, R\ell] + \mathrm{i}[-\varepsilon, 0]$

$$\|\chi p_\ell(z,s)^{-1}\chi\|_{L^2\to L^2} \lesssim \frac{\ln\langle z\rangle}{\langle z\rangle}\,\mathrm{e}^{|\Im z|\ln\langle z\rangle}. \tag{1.36}$$

4. *Let $R \gg 0$, $C_0 > 0$ and $C_1 > 0$. Set*

$$\tilde\Omega_\ell := \big([R\ell, +\infty[+ \mathrm{i}\,] - \infty, C_0]\big) \cap \big\{\lambda \in \mathbb{C} \mid \Im\lambda \geq -C_0 - C_1 \ln\langle\lambda\rangle\big\} \cap \Omega_\kappa.$$

There is no resonance in $\tilde\Omega_\ell$ and there exists $C > 0$ such that for all z in this set,

$$\|\chi p_\ell(z,s)^{-1}\chi\|_{L^2\to L^2} \leq C\langle z\rangle^{-1}\mathrm{e}^{C|\Im z|}. \tag{1.37}$$

Remark 1.4.2. *High frequency resonances of the zone III (i.e. resonances whose real part are of order $\ell \gg 0$) are localized in Theorem 1.3.1.*

The announced estimate in the zone I is a direct application of results of Section 1.2 (see Theorem 1.2.8). We thus show the estimates for the zones II, III and IV.

1.4.1 Estimates in the zone II

We prove part 2. of Theorem 1.4.1 using the complex scaling introduced in [Zw99, Section 4]. Observe that the zone II does not exist if $\ell = 0$, so that we can assume that $\ell \geq 1$. Let $z \in [R, \ell/R] + \mathrm{i}\,[-C_0, C_0]$ and choose $N \in [R, \ell/R]$ such that $z \in [N, 2N] + \mathrm{i}\,[-C_0, C_0]$. We introduce the semiclassical parameter

$$h := N^{-1}$$

and the new spectral parameter

$$\lambda := h^2 z^2 \in [1/4, 4] + \mathrm{i}\,[-4C_0h, 4C_0h]\,.$$

In this setting, we define the operator

$$\tilde p_h(\sqrt\lambda, s) := h^2 p_\ell\,(z,s) = \underbrace{-h^2\partial_x^2 + \alpha^2 W_0\,(x)}_{=:Q_h} - \lambda + \underbrace{h^2 W_1\,(x) + 2h\sqrt\lambda s V\,(x) - h^2 s^2 V\,(x)^2}_{=:R_h(\lambda)} \tag{1.38}$$

where $\alpha := h(\ell(\ell+1))^{1/2} \gg 2\mathscr{A} > 0$, $\mathscr{A}$ as in Proposition 1.1.1.

We now use the (α-dependent) contour $\Gamma_\theta := \Gamma_\theta^- \cup \Gamma_\theta^+$ for $0 < \theta < \pi/2$, with[8]

$$\Gamma_\theta^\pm := \big\{x + \mathrm{i} f_\theta^\pm(x, \ln(g_\infty^\pm)/\kappa_\pm) \mid x \in \mathbb{R}_\pm\big\}$$

where (using estimate (1.6) for W_0)

$$g_\infty^\pm := \lim_{x\to\pm\infty} \mathrm{e}^{2\kappa_\pm x} W_0(x)$$

[8] The factor $1/\kappa_\pm$ in the second argument of $f_\theta^\pm$ comes from the fact that $\kappa_\pm x$ corresponds to Zworski's variable r.

and

$$f_\theta^\pm(x,\beta) := \begin{cases} 0 & \text{if } |x| \leq \beta/2 - C_1 \\ \theta(x-\beta/2) & \text{if } |x| \geq \beta/2 + C_2 \end{cases}$$

with constants $C_1, C_2 > 0$ as in (4.4) in [Zw99, Section 4] (see Figure 1.3 for the behaviour of Γ_θ). Define next $L^2(\Gamma_\theta)$ and $H^2(\Gamma_\theta)$ as the associated Lebesgue and Sobolev spaces. Using the analytic extension of $x \mapsto r(x)$ on the set $\Sigma := \{\eta \in \mathbb{C} \mid |\Re\eta| > \mathscr{A}\}$, we extend V, W_0 and W_1 on Σ (and still denote them V, W_0 and W_1). We then define the distorted operators

$$\tilde{p}_{h,\theta}(\sqrt{\lambda}, s) = \tilde{p}_h(\sqrt{\lambda}, s) \restriction_{\Gamma_\theta}, \qquad Q_{h,\theta} := Q_h \restriction_{\Gamma_\theta}, \qquad R_{h,\theta}(\lambda) = R_h(\lambda) \restriction_{\Gamma_\theta}.$$

If $q_{h,\theta}$ denotes the symbol of $Q_{h,\theta}$, then [Zw99, Lemma 4.3] shows that there exists $0 < c < 1$

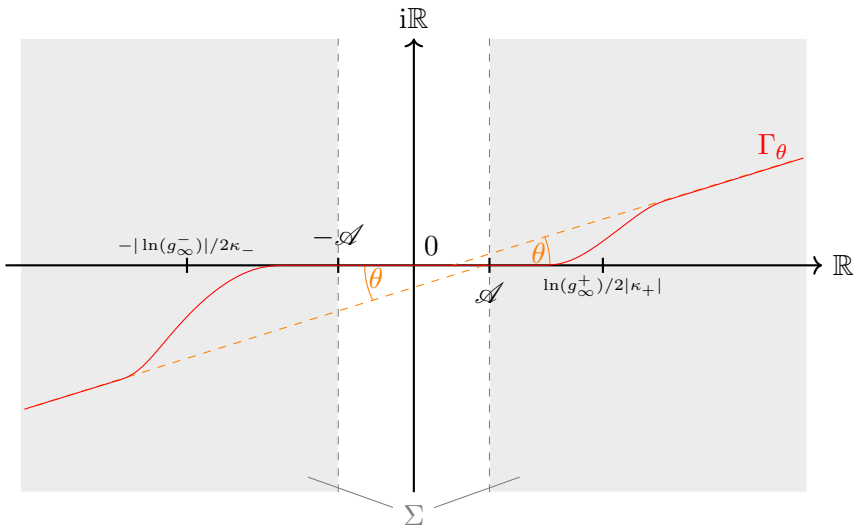

Figure 1.3: The contour Γ_θ.

and $\theta_0 > 0$ such that

$$|q_{h,\theta}(x,\xi) - \lambda| \gtrsim \theta(\langle\xi\rangle^2 + \mathrm{e}^{\kappa_\pm x}\langle\alpha\rangle^2) \qquad\qquad \pm x \geq 0$$

provided that $\Re(\lambda) > 0$, $\Im(\lambda) < c$ and $0 < \theta < \theta_0$ (recall that $\pm\kappa_\pm < 0$). For h small enough (that is R large enough), we can apply [Zw99, Proposition 4.1] to get

$$\|(Q_{h,\theta} - \lambda)^{-1}\|_{L^2(\Gamma_\theta)\to H^2(\Gamma_\theta)} = \mathcal{O}(\theta^{-1}).$$

In order to invert the distorted quadratic pencil, we use a Neumann series argument by showing that $R_{h,\theta}(\lambda) = \mathcal{O}_{L^2(\Gamma_\theta)}(h)$ (as a multiplication operator). In view of the form of $R_{h,\theta}$ in (1.38) and because the extension of r is analytic, it is enough to bound $x \mapsto r(x + \mathrm{i}f_\theta^\pm(x))$ below and above for $|x| > 2\mathscr{A}$. By Lagrange inversion formula (1.57), we can write

$$|r(x + \mathrm{i}f_\theta^\pm(x)) - r_\pm| \leq \sum_{k=1}^{+\infty} c_k \frac{\mathrm{e}^{2k\kappa_\pm x}}{k!}$$

for some coefficients $c_k > 0$ (recall from Subsection 1.1.2 that $2\kappa_\pm = \frac{\Lambda}{3A_\pm r_\pm^2}$), and the series converges (because (1.57) converges uniformly when $|\Re x| > \mathscr{A}$). Since the sum is decreasing with respect to x, we deduce $r < +\infty$ on Γ_θ. On the other hand,

$$|r - r_\pm| = C(r - r_n)^{-\frac{A_n r_n^2}{A_\pm r_\pm^2}} (r - r_c)^{-\frac{A_c r_c^2}{A_\pm r_\pm^2}} |r - r_\mp|^{-\frac{A_\mp r_\mp^2}{A_\pm r_\pm^2}} \mathrm{e}^{2\kappa_\pm x}$$

with $C \in \mathbb{R}$. Since no terms on the right-hand side can blow up when restricted on Γ_θ and since the exponential goes to zero when $|x| \to +\infty$, it follows that $r \to r_\pm > 0$ as $x \to \pm\infty$. We therefore conclude that the restriction of r on $\Gamma_\theta \cap D(0, R_0)^\complement$ is bounded from below and above for $R_0 \gg 0$, giving $R_{h,\theta}(\lambda) = \mathcal{O}_{L^2(\Gamma_\theta)}(h)$. Thus,

$$\tilde{p}_{h,\theta}(\sqrt{\lambda}, s)^{-1} = \big(1 + (Q_{h,\theta} - \lambda)^{-1} R_{h,\theta}(\lambda)\big)^{-1} (Q_{h,\theta} - \lambda)^{-1}.$$

We finally choose $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ and increase if necessary the value of the number $\mathscr{A}$ so that $\mathrm{Supp}\chi \subset [-\mathscr{A}, \mathscr{A}]$. From [SjwZ91, Lemma 3.5], we have in the L^2 sense

$$\chi \tilde{p}_{h,\theta}(\sqrt{\lambda}, s)^{-1} \chi = \chi \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \chi$$

whence

$$\|\chi p_\ell(z, s)^{-1} \chi\|_{L^2 \to L^2} = h^2 \|\chi \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \chi\|_{L^2 \to L^2} \lesssim \langle z \rangle^{-2}.$$

1.4.2 Estimates in the zone III

We turn to the proof of part 3. of Theorem 1.4.1. We define the semiclassical parameter

$$h := (\ell(\ell + 1))^{-1/2}$$

with again $\ell > 0$ since the zone does not exist for $\ell = 0$. For $z \in [\ell/R, R\ell] + \mathrm{i}\,[-C_0, C_0]$, we define a new spectral parameter

$$\lambda := h^2 z^2 \in \left[\frac{1}{3R^2}, R^2\right] + \mathrm{i}[-\sqrt{2}\, C_0 R h, \sqrt{2}\, C_0 R h] \subset [a, b] + \mathrm{i}[-ch, ch]$$

for some $0 < a < b$ and $c > 0$. Finally, we set

$$\tilde{P}_h := h^2 P_\ell = -h^2 \partial_x^2 + W_0 + h^2 W_1, \qquad \tilde{p}_h(\sqrt{\lambda}, s) := \tilde{P}_h - (\sqrt{\lambda} - hsV)^2$$

and write $\tilde{P}_\theta$ and $\tilde{p}_\theta$ for the corresponding distorted operators on the contour Γ_θ as we did in the paragraph 1.4.1. We are still using the subscript L^2 when we work with the distorted operators.

As W_0 admits a non-degenerate maximum at $x = 0$ (see Section 1.1, Figure 1.1), $(x, \xi) = (0, 0)$ is a trivial solution of the Hamilton equations associated to the principal symbol of $\tilde{P}_h$:

$$\begin{cases} \dot{x} &= 2\xi \\ \dot{\xi} &= -W_0'(x) \end{cases}.$$

Therefore the energy level $\mathfrak{E}_0 := W_0(0)$ is trapping. For this reason, the zone III is called the trapping zone.

We first show an adaptation of [BoMi04, Lemma 6.5] to our setting.

Proposition 1.4.3. *For $\theta = Nh$ with $N > 0$ large enough and $s \in \mathbb{R}$ sufficiently small, there exist $C \equiv C(N) > 0$ and $\varepsilon > 0$ such that, for all $E \in [\mathfrak{E}_0 - \varepsilon, \mathfrak{E}_0 + \varepsilon]$ and $|\lambda - E| \leq \varepsilon\theta/2$, it holds*

$$\|(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \to L^2} = \mathcal{O}(h^{-C}) \prod_{\substack{\lambda_j \in \mathrm{Res}(\tilde{p}) \\ |\lambda - \lambda_j| < \varepsilon\theta}} \frac{h}{|\lambda - \lambda_j|}.$$

Proof. The announced estimate is known for the resolvent $(\tilde{P}_\theta - \lambda)^{-1} = \tilde{p}_\theta(\sqrt{\lambda}, 0)^{-1}$ corresponding to the case $s = 0$. The argument can be found in [TZ98] which uses techniques developed in [Sj96], and the authors of [BoMi04] adapted it for the one dimensional case of a non degenerate trapping energy level $\mathfrak{E}_0$. More precisely, for $\theta = Nh$ with $N \gg 0$ large enough, one can construct a bounded operator $\tilde{K} \in \mathcal{L}\left(L^2, L^2\right)$ (see (6.15) in [BoMi04]) satisfying the following properties:

(*i*) $\|\tilde{K}\|_{L^2 \to L^2} = \mathcal{O}(1)$,

(*ii*) $r := \operatorname{rank} \tilde{K} \leq \mathcal{O}(\theta h^{-1} \ln(1/\theta))$,

(*iii*) for h small enough, there exists $\varepsilon > 0$ such that, for all $E \in [\mathfrak{E}_0 - \varepsilon, \mathfrak{E}_0 + \varepsilon]$ and $\lambda \in [E - \varepsilon\theta, E + \varepsilon\theta]$,

$$\|(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - \lambda)^{-1}\|_{L^2 \to \mathscr{D}} \leq \mathcal{O}(\theta^{-1}), \qquad \mathscr{D} := \mathscr{D}(\tilde{P}_\theta).$$

In [SjwZ91, Lemma 3.2], it is shown that $\tilde{P}_\theta - \lambda$ is a Fredholm operator from its domain $\mathscr{D}$ to L^2, so we can construct a well-posed Grushin problem

$$\mathcal{P}(\lambda) := \begin{pmatrix} \tilde{P}_\theta - \lambda & R_- \\ R_+ & 0_{\mathbb{C}^r \to \mathbb{C}^r} \end{pmatrix} : \mathscr{D} \oplus \mathbb{C}^r \to L^2 \oplus \mathbb{C}^r \tag{1.39}$$

where R_- and R_+ are constructed with $\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - \lambda$ (see [Sj96], page 401, below (6.12) for the construction).

Now consider $s \neq 0$. If s is small enough, $(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}$ is invertible by pseudodifferential calculus[9] as for the case $s = 0$. By the resolvent identity, one can show that

$$\|(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \to \mathscr{D}} \leq \mathcal{O}(\theta^{-1}) + \mathcal{O}(h|s|)\|(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \to \mathscr{D}} \mathcal{O}(\theta^{-1})$$

since $|\lambda| \leq h(|\Re z| + |\Im z|) \leq \mathcal{O}(1) + \mathcal{O}(h)$. Hence for s sufficiently small, we have

$$\|(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2 \to \mathscr{D}} \leq \mathcal{O}(\theta^{-1}) \qquad \lambda \in [E - \varepsilon\theta, E + \varepsilon\theta].$$

Because the quadratic pencil remains a Fredholm operator provided that $\|hsV\|_{L^\infty}$ is sufficiently small[10], we can write a new well-posed Grushin problem

$$\mathcal{P}(\lambda) := \begin{pmatrix} \tilde{p}_\theta(\sqrt{\lambda}, s) & R_- \\ R_+ & 0_{\mathbb{C}^r \to \mathbb{C}^r} \end{pmatrix} : \mathscr{D} \oplus \mathbb{C}^r \to L^2 \oplus \mathbb{C}^r$$

[9] See the definition of the operator K at the beginning of the proof of Lemma 6.5 in [BoMi04].
[10] Recall that the set of Fredholm operators in $\mathcal{L}(\mathscr{D}, L^2)$ is open for the norm topology.

where this time R_+ and R_- are constructed with $\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2$. If we set

$$\mathcal{E}(\lambda) := \mathcal{P}(\lambda)^{-1} = \begin{pmatrix} E(\lambda) & E_+(\lambda) \\ E_-(\lambda) & E_0(\lambda) \end{pmatrix} : L^2 \oplus \mathbb{C}^r \to \mathscr{D} \oplus \mathbb{C}^r,$$

then the relations $\mathcal{E}(\lambda)\mathcal{P}(\lambda) = \mathcal{P}(\lambda)\mathcal{E}(\lambda) = \mathbb{1}$ as well as the following estimate (which is a consequence of properties (i) and (iii) above)

$$\big\|(\tilde{P}_\theta - \mathrm{i}\theta\tilde{K} - (\sqrt{\lambda} - hsV)^2)^{-1}(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2)\big\|_{L^2 \to L^2} = \mathcal{O}(1)$$

imply as in [BoMi04] that $\|E(\lambda)\|_{L^2\to\mathscr{D}}, \|E_-(\lambda)\|_{L^2\to\mathbb{C}^r} = \mathcal{O}(\theta^{-1})$ and $\|E_+(\lambda)\|_{\mathbb{C}^r\to\mathscr{D}}, \|E_0(\lambda)\|_{\mathbb{C}^r\to\mathbb{C}^r} = \mathcal{O}(1)$. Applying formula (8.11) in [Sj96], we obtain

$$(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2)^{-1} = E(\lambda) - E_+(\lambda)E_0(\lambda)^{-1}E_-(\lambda)$$

which implies

$$\|(\tilde{P}_\theta - (\sqrt{\lambda} - hsV)^2)^{-1}\|_{L^2\to\mathscr{D}} = \mathcal{O}(\theta^{-1})\big(1 + \|E_0(\lambda)^{-1}\|_{\mathbb{C}^r\to\mathbb{C}^r}\big)$$

as in [BoMi04, Lemma 6.5], and we then follow the end of its proof to conclude. □

We can now follow the arguments below [BoHa08, Lemma 2.2]. The set of pseudo-poles (2.28) and the injective map (2.29) in this reference exist in our setting by Theorem 1.3.1 (but are quite different). This implies that there is no resonance in $\Omega(h) := [a/2, 2b] + \mathrm{i}[-\varepsilon h, ch]$ provided that h and s are small enough. As a result, (1.35) holds true. As for the estimate (1.36), we use Burq's Lemma:

Lemma 1.4.4 (Lemma 2.3 in [BoHa08]). *Suppose that $f(\lambda, h)$ is a family of holomorphic functions defined for $0 < h < 1$ in a neighbourhood of $\Omega(h) := [a/2, 2b] + \mathrm{i}[-ch, ch]$ with $0 < a < b$ and $c > 0$, such that*

$$|f(\lambda, h)| \lesssim \begin{cases} h^{-C'} & \text{in } \Omega(h) \\ \dfrac{1}{|\Im\lambda|} & \text{in } \Omega(h) \cap \mathbb{C}^+ \end{cases}$$

for some $C' > 0$. Then there exists h_0, $C > 0$ such that, for any $0 < h < h_0$ and any $\lambda \in [a, b] + \mathrm{i}[-ch, 0]$,

$$|f(\lambda, h)| \le C\frac{|\ln h|}{h}\mathrm{e}^{C|\Im\lambda||\ln h|/h}.$$

We apply this result to the function $f(\lambda, h) := \|\chi(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\chi\|$, observing that for all $\lambda \in \Omega(h) \cap \mathbb{C}^+$ the resolvent identity gives

$$\|(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\| \le \frac{1}{|\Im\lambda|} + \|(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\|\frac{\mathcal{O}(h|s|)}{|\Im\lambda|} \lesssim \frac{1}{|\Im\lambda|}$$

because $\|(\tilde{P}_h - (\sqrt{\lambda} - hsV)^2)^{-1}\|$ is uniformly bounded on this set for h and s small enough.

1.4.3 Estimates in the zone IV

This last paragraph is devoted to the proof of part 4. of Theorem 1.4.1. For $z \in ([R\ell, +\infty[+ \mathrm{i}\,]-\infty, C_0]) \cap \{\lambda \in \mathbb{C} \mid \Im\lambda \geq -C_0 - C_1 \ln\langle\lambda\rangle\}$, there exists a number $N > R\ell > 0$ such that $z \in [N, 2N] + \mathrm{i}\,[-C \ln N, C_0]$. We introduce the semiclassical parameters

$$h := \frac{1}{N}, \qquad \mu := \ell\,(\ell+1)\,h^2, \qquad \nu := h^2.$$

Observe that these parameters are very small when $N \gg 0$. Moreover, we can consider that $h \leq 1$ even if $\ell = 0$, simply by taking $R \geq 1$ in the zone I if it was not the case (R as in Theorem 1.4.1). We then define a new spectral parameter

$$\lambda := z^2h^2 \in [1,2] + \mathrm{i}\,[Ch \ln h, C_0 h] \subset [a,b] + \mathrm{i}\,[-ch\,|\ln h|\,, ch]$$

where $0 < a \leq 1 < 2 \leq b < +\infty$ and $\max\{C, C_0\} < c < +\infty$ (observe that a and b do not depend on h). Let $J := [a,b]$ and set

$$J^+ := \{\eta \in \mathbb{C}^+ \mid \Re(\eta) \in J\}.$$

Define then

$$\tilde{P}_h := h^2 P_\ell = -h^2\partial_x^2 + \mu W_0 + \nu W_1, \qquad \tilde{p}_h(\sqrt{\lambda}, s) := h^2 p_\ell\,(z,s) = \tilde{P}_h - (\sqrt{\lambda} - hsV)^2.$$

Semiclassical limiting absorption principle for the quadratic pencil. As in [BoHa08], we first get a control until the real line by using a semiclassical limiting absorption principle for the semiclassical quadratic pencil. The appendix 1.6.3 provides a proof, close to the idea developed by Gérard [Ge08], of such a result for a class of perturbed resolvents, so we only have to check if the required abstract assumptions are satisfied.

Introduce the generator of dilations $A := -\mathrm{i}h\,(x\partial_x + \partial_x x)$ with domain $\mathscr{D}(A) := \left\{u \in L^2 \mid Au \in L^2\right\}$. We then pick $\rho \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}, [0,1])$ such that $\mathrm{Supp}\,\rho \subset [a/3, 3b]$ and $\rho \equiv 1$ on $I := [a/2, 2b]$, and we define $\mathcal{A}$ as the closure of the operator $\rho(P)A\rho(P)$. In this setting, $\rho(P)A\rho(P)$ is well-defined on $\mathscr{D}(A)$, $\mathcal{A}$ is self-adjoint and we have $P \in \mathcal{C}^2(\mathcal{A})$ (*cf.* [BoHa08, Section 2.4]) so that (I) holds. A direct computation shows that

$$\mathrm{i}h^{-1}[P, A] = 4P - 4\mu W_0 - 4\nu W_1 - 2\mu x W_0' - 2\nu x W_1'$$

so that, for μ and ν sufficiently small, we get the Mourre estimate (M) (uniform in μ, ν)

$$\mathbb{1}_I(P)[P, \mathrm{i}\mathcal{A}]\mathbb{1}_I(P) \geq ah\mathbb{1}_I(P).$$

Since $V \in \mathcal{B}(\mathscr{D}(P), L^2)$ it is clear that $V \in L^\infty_{\ell\mathrm{oc}}(\tilde{P}_h)$. Moreover, assumption (C) is fulfilled for $f(z,B) := (\sqrt{z} - sB)^2$.

It remains to show that assumption (A) is satisfied for $B = hV$. Observe that this abstract assumption is particularly well adapted to semiclassical pseudodifferential calculus framework, especially the commutator estimate which provides the supplementary term h. In [Ha01], it is shown that $\mathcal{A} \in \Psi^{-\infty,1}$ ($\mathcal{A}$ is the operator $c_{\tilde{\chi}}$ in [Ha01], see above Lemma 3.3). We will use it to show the following result:

Lemma 1.4.5. *Let $\sigma \in [0,1]$. Then $V \in \mathcal{B}(\mathscr{D}(\langle\mathcal{A}\rangle^\sigma))$ and $[V, \chi(\tilde{P}_h)] \in h\mathcal{B}(\mathscr{D}(\langle\mathcal{A}\rangle^\sigma))$.*

Proof. Let $\Omega := [0,1] + \mathrm{i}\mathbb{R}$ and let $z \in \Omega$. On $\mathscr{D}(\langle\mathcal{A}\rangle^2) \times \mathscr{D}(\langle\mathcal{A}\rangle^2)$, we define the sesquilinear form

$$Q_z(\varphi,\psi) := \langle V\langle\mathcal{A}\rangle^{-2z}\varphi, \langle\mathcal{A}\rangle^{2z}\psi\rangle \qquad \forall \varphi,\psi \in \mathscr{D}(\langle\mathcal{A}\rangle^2).$$

By functional calculus, Q_z is well-defined and analytic in $z \in \Omega$. When $z \in \{0\}+\mathrm{i}\mathbb{R}$, $|(1+\lambda^2)^{z/2}| = 1$ for all $\lambda \in \mathbb{R}$ so that functional calculus first applied to $\langle\mathcal{A}\rangle^{2z}$ and then to $\langle\mathcal{A}\rangle^{-2z}$ gives

$$\begin{aligned} |Q_z(\varphi,\psi)| &\leq |\langle V\langle\mathcal{A}\rangle^{-2z}\varphi,\psi\rangle| \\ &= |\langle\langle\mathcal{A}\rangle^{-2z}\varphi, V\psi\rangle| \\ &\leq |\langle\varphi, V\psi\rangle| \\ &\leq \|V\|_{L^\infty}\|\varphi\|_{L^2}\|\psi\|_{L^2}. \end{aligned}$$

When $z = 1$, pseudodifferential calculus shows that $\langle\mathcal{A}\rangle^2 V\langle\mathcal{A}\rangle^{-2} \in \Psi^{0,0}$, so that for all $z \in \{1\}+\mathrm{i}\mathbb{R}$ (using again functional calculus for $\langle\mathcal{A}\rangle^{\pm 2\mathrm{i}\Im z}$),

$$\begin{aligned} |Q_z(\varphi,\psi)| &\leq |\langle\langle\mathcal{A}\rangle^2 V\langle\mathcal{A}\rangle^{-2}\langle\mathcal{A}\rangle^{-2\mathrm{i}\Im z}\varphi,\psi\rangle| \\ &\leq \|\langle\mathcal{A}\rangle^2 V\langle\mathcal{A}\rangle^{-2}\|_{L^2\to L^2}\|\langle\mathcal{A}\rangle^{-2\mathrm{i}\Im z}\varphi\|_{L^2}\|\psi\|_{L^2} \\ &\leq \|\langle\mathcal{A}\rangle^2 V\langle\mathcal{A}\rangle^{-2}\|_{L^2\to L^2}\|\varphi\|_{L^2}\|\psi\|_{L^2}. \end{aligned}$$

By the maximum principle, there exists a constant $C > 0$ such that Q_z is bounded by C for all $0 \leq \Re z \leq 1$. In particular, we can extend $Q_{\sigma/2}$ on $L^2 \times \mathscr{D}(\langle\mathcal{A}\rangle^2)$ as a bounded sesquilinear form and for $\sigma \in [0,1]$ and $\varphi \in L^2$, we have

$$|Q_{\sigma/2}(\varphi,\psi)| \leq C\|\varphi\|_{L^2}\|\psi\|_{L^2}.$$

This means that the map $\mathscr{D}(\langle\mathcal{A}\rangle^\sigma) \ni \psi \mapsto \langle V\langle\mathcal{A}\rangle^{-\sigma}\varphi, \langle\mathcal{A}\rangle^\sigma\psi\rangle$ is continuous. By definition of the adjoint operator and because $\langle\mathcal{A}\rangle^\sigma$ is self-adjoint, this implies that $V\langle\mathcal{A}\rangle^{-\sigma}\varphi \in \mathscr{D}(\langle\mathcal{A}\rangle^\sigma)$ for all $\varphi \in L^2$.

Consider now the sesquilinear form

$$\tilde{Q}_z(\varphi,\psi) := \langle[V,\chi(\tilde{P}_h)]\langle\mathcal{A}\rangle^{-2z}\varphi, \langle\mathcal{A}\rangle^{2z}\psi\rangle \qquad \forall \varphi,\psi \in \mathscr{D}(\langle\mathcal{A}\rangle^2).$$

By semiclassical pseudodifferential calculus, we have (see e.g. (4.4.19) in [**?**])

$$\begin{aligned} [V,\chi(\tilde{P}_h)] &= \frac{h}{\mathrm{i}}\{V(x), \chi(\xi^2 + \mu W_0(x) + \nu W_1(x))\}^{\mathrm{w}} + h^3\Psi^{-\infty,0} \\ &= h\Psi^{-\infty,-\infty} + h^3\Psi^{-\infty,0} \end{aligned}$$

because $V(x) \in \Psi^{0,0}$, $V'(x) \in \Psi^{0,-\infty}$ and $\chi(\tilde{P}_h) \in \Psi^{-\infty,0}$. Despite the fact that the error term above looks less regular than the main term, it is in fact more regular as it can be shown using expansion (4.4.15) in [**?**] (but we will not need such a regularity). Now we can proceed as above with Q_z and V to conclude. □

Now that all assumptions in appendix 1.6.3 have been checked, we can use Theorem 1.6.3 as well as the fact that $\|\langle x\rangle^{-\sigma}\langle\mathcal{A}\rangle^\sigma\| \lesssim 1$ for all $\sigma \leq 1$[11]: for $\sigma \in]1/2, 1]$ and h small enough, we have uniformly in μ,ν

$$\sup_{\lambda\in J^+}\left\|\langle x\rangle^{-\sigma}\tilde{p}_h(\sqrt{\lambda},s)^{-1}\langle x\rangle^{-\sigma}\right\| \leq \|\langle x\rangle^{-\sigma}\langle\mathcal{A}\rangle^\sigma\|\left(\sup_{\lambda\in J^+}\left\|\langle\mathcal{A}\rangle^{-\sigma}\tilde{p}_h(\sqrt{\lambda},s)^{-1}\langle\mathcal{A}\rangle^{-\sigma}\right\|\right)\|\langle\mathcal{A}\rangle^\sigma\langle x\rangle^{-\sigma}\| \lesssim h^{-1}.$$

[11] We show it using the sesquilinear form $(\varphi,\psi) \mapsto \langle\langle x\rangle^{-\sigma}\langle\mathcal{A}\rangle^\sigma\varphi,\psi\rangle$ first well-defined on $\mathscr{D}(\mathcal{A}^2) \times \mathscr{D}(\mathcal{A}^2)$ because $\langle x\rangle^{-2} \in \Psi^{0,-2}$, and then extended to $L^2 \times L^2$ by maximum principle.

Estimates below the real axis. Next, we can use the work of Martinez [Ma02] to get a bound under the real line. Indeed, Section 4 of the last reference applies in our setting because $\tilde{p}_h(\sqrt{\lambda}, s)$ is a differential operator (so that Proposition 3.1 and Corollary 3.2 of [Ma02] apply) and because $(\lambda - hsV(x))^2 \in [\lambda - \delta, \lambda + \delta] + \mathrm{i}[ch \ln h, 0]$ for all λ in the zone IV and all $x \in \mathbb{R}$ if s is small enough (so that the estimate (4.6) in [Ma02] still holds). It follows that equation (4.13) holds with $\tilde{p}_h(\sqrt{\lambda}, s)$ instead of $P_\theta - \rho$[12]. In our setting, this reads

$$\|\chi \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \chi\| \leq C h^{-C} \tag{1.40}$$

for some $C > 0$.

To get (1.37), we reproduce the argument at the end of the proof of [BoHa08, Lemma 2.4]. Choose f holomorphic satisfying the following conditions:

$$\begin{cases} |f| < 1 & \text{for } \lambda \in [a/2, 2b] + \mathrm{i}[ch \ln h, 0], \\ |f| \geq 1 & \text{for } \lambda \in [a, b] + \mathrm{i}[ch \ln h, 0], \\ |f| \leq h^C & \text{for } \lambda \in [a/2, 2b] \setminus [2a/3, 3b/2] + \mathrm{i}[ch \ln h, 0] \end{cases}$$

where $C > 0$ is the constant in (1.40). Since f is holomorphic, the function

$$g(\lambda) := \ln \|\chi \tilde{p}(\sqrt{\lambda}, s)^{-1} \chi\|_{L^2 \to L^2} + \ln |f(\lambda)| + \frac{C}{ch} \Im \lambda$$

is subharmonic. We can check that $g(\lambda) \lesssim \ln(h^{-1})$ on the boundary of $[a/2, 2b] + \mathrm{i}[ch \ln h, 0]$. By the maximum principle, this estimate holds for all $\lambda \in [a/2, 2b] + \mathrm{i}[ch \ln h, 0]$, whence

$$\|\chi \tilde{p}_h(\sqrt{\lambda}, s)^{-1} \chi\|_{L^2 \to L^2} \lesssim h^{-1} \mathrm{e}^{\frac{C}{ch} |\Im \lambda|}.$$

The desired estimate (1.37) then follows.

1.5 Proof of the main theorem

We prove in this Section Theorem 1.3.2. The resonance expansion (1.28) follows from the theory of resonances as presented in [BoHa08, Section 3], and we can follow the proof of this paper. We only have to adapt [BoHa08, Proposition 3.1] to get an estimate for the resolvent $\hat{R}_{\chi,\ell}(z)$:

Proposition 1.5.1. *Let $\ell \in \mathbb{N}$ and let $\chi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}, \mathbb{R})$. There exists $\tilde{\chi} \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}, \mathbb{R})$ satisfying $\tilde{\chi}\chi = \chi$ such that for all $z \in \mathbb{C} \setminus \mathrm{Res}(p_\ell)$, the cut-off resolvent $\chi(\hat{K}_\ell - z)^{-1}\chi$ is a bounded operator on $\dot{\mathcal{E}}_\ell$ and satisfies uniformly in ℓ*

$$\|\hat{R}_{\chi,\ell}(z)\|_{\dot{\mathcal{E}}_\ell \to \dot{\mathcal{E}}_\ell} \lesssim \langle z \rangle \|\tilde{\chi} p_\ell(z, s)^{-1} \tilde{\chi}\|_{L^2 \to L^2}.$$

Proof. Since the norms $\|.\|_{\mathcal{E}_\ell}$ and $\|.\|_{\dot{\mathcal{E}}_\ell}$ are locally equivalent thanks to the Hardy type estimate $\|\chi .\|_{L^2} \lesssim \|P_\ell^{1/2} .\|_{L^2}$ uniformly in ℓ (*cf.* [GGH17, Lemma 9.5]), we can work on $(\mathcal{E}_\ell, \|.\|_{\mathcal{E}_\ell})$. For $(u_0, u_1) \in \mathcal{E}_\ell$, we have

$$\hat{R}_{\chi,\ell}(z) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} \chi p_\ell(z, s)^{-1} \chi((z - sV)u_0 + u_1) \\ \chi(1 + (z - sV)p_\ell(z, s)^{-1}(z - sV))\chi u_0 + (z - sV)\chi p_\ell(z, s)^{-1} \chi u_1 \end{pmatrix} \tag{1.41}$$

[12] We can in fact insert any pseudodifferential operator here provided that hypotheses of Section 2 in [Ma02] are verified.

and since it holds

$$\|(z-sV)\chi p_\ell(z,s)^{-1}\chi u_1\|_{L^2} \le (1+|s|\|V\|_{L^\infty})\langle z\rangle \|\chi p_\ell(z,s)^{-1}\chi\|_{L^2\to L^2}\|u_1\|_{L^2},$$

the $\mathcal{E}_\ell$-norm of (1.41) can be bounded if we show the following estimates:

$$\|P_\ell^{1/2}\chi p_\ell(z,s)^{-1}\chi(z-sV)u_0\|_{L^2} \le C_{\mathrm{a}}\langle z\rangle \|\tilde\chi p_\ell(z,s)^{-1}\tilde\chi\|_{L^2\to L^2}\|P_\ell^{1/2}u_0\|_{L^2}, \tag{1.42a}$$

$$\|P_\ell^{1/2}\chi p_\ell(z,s)^{-1}\chi u_1\|_{L^2} \le C_{\mathrm{b}}\langle z\rangle \|\tilde\chi p_\ell(z,s)^{-1}\tilde\chi\|_{L^2\to L^2}\|u_1\|_{L^2}, \tag{1.42b}$$

$$\|\chi(1+(z-sV)p_\ell(z,s)^{-1}(z-sV))\chi u_0\|_{L^2} \le C_{\mathrm{c}}\langle z\rangle \|\tilde\chi p_\ell(z,s)^{-1}\tilde\chi\|_{L^2\to L^2}\|P_\ell^{1/2}u_0\|_{L^2}. \tag{1.42c}$$

We use complex interpolation.

Estimate (1.42a)**.** Let us define

$$\Lambda_{\mathrm{a}}(\theta) := \langle z\rangle^{-2\theta}P_\ell^\theta \chi p_\ell(z,s)^{-1}\chi P_\ell^{-\theta}.$$

By functional calculus, Λ_{a} is analytic from $[0,1]+\mathrm{i}\mathbb{R}$ to $\mathcal{L}(L^2,L^2)$ because $P_\ell>0$ and $\langle z\rangle>0$. We want to show that

$$\|\Lambda_{\mathrm{a}}(1/2)u\|_{L^2} \le C_{\mathrm{a}}\|\tilde\chi p_\ell(z,s)^{-1}\tilde\chi\|_{L^2\to L^2}\|u\|_{L^2} \qquad \forall u\in L^2$$

for some $C_{\mathrm{a}}>0$. By the maximum principle, it is sufficient to bound $\Lambda_{\mathrm{a}}(\theta)$ for $\theta\in\{0,1\}+\mathrm{i}\mathbb{R}$, and since P_ℓ is self-adjoint, it is sufficient by functional calculus to restrict ourselves to $\Im\theta=0$. If $\theta=0$, there is nothing to do. Now for $\theta=1$, we put $u=(z-sV)u_0$ and try to show that

$$\|P_\ell\chi p_\ell(z,s)^{-1}\chi u\|_{L^2\to L^2} \le C_{\mathrm{a}}\langle z\rangle \|\tilde\chi p_\ell(z,s)^{-1}\tilde\chi\|_{L^2\to L^2}\|P_\ell u\|_{L^2}. \tag{1.43}$$

Write

$$P_\ell\chi p_\ell(z,s)^{-1}\chi = \underbrace{[P_\ell,\chi]p_\ell(z,s)^{-1}\chi}_{=:A} + \underbrace{\chi P_\ell p_\ell(z,s)^{-1}\chi}_{=:B}. \tag{1.44}$$

We first deal with A. Pick $z_0\in\rho(\hat K_\ell)\cap\mathbb{C}^+$ so that $p_\ell(z_0,s)^{-1}$ exists (*cf.* (1.16)). Then

$$\big[P_\ell,\chi\big]p_\ell(z,s)^{-1} = p_\ell(z_0,s)^{-1}\big[p_\ell(z_0,s),\big[P_\ell,\chi\big]\big]p_\ell(z,s)^{-1} + p_\ell(z_0,s)^{-1}\big[P_\ell,\chi\big]p_\ell(z_0,s)p_\ell(z,s)^{-1}$$

with

$$\begin{aligned}\big[P_\ell,\chi\big] &= -\chi\partial_x - \chi',\\ \big[p_\ell(z_0,s),\big[P_\ell,\chi\big]\big] &= 2\chi'\partial_x^2 + (\chi'+\chi'')\partial_x + 2z_0sV'\chi - 2s^2VV'\chi.\end{aligned}$$

By pseudodifferential calculus, we get:

$$p_\ell(z_0,s)^{-1}\in\Psi^{-2,0}, \qquad \big[P_\ell,\chi\big]\in\Psi^{1,-\infty}, \qquad \big[p_\ell(z_0,s),\big[P_\ell,\chi\big]\big]\in\Psi^{2,-\infty}.$$

On the other hand, we have

$$\begin{aligned} p_\ell(z_0,s)p_\ell(z,s)^{-1} &= \big(p_\ell(z,s)+(z^2-z_0^2)-2(z-z_0)sV\big)p_\ell(z,s)^{-1} \\ &= 1+p_\ell(z,s)^{-1}(z^2-z_0^2)-2(z-z_0)sVp_\ell(z,s)^{-1} \\ &= p_\ell(z,s)^{-1}(P_\ell-(z_0^2-2zsV+s^2V^2))-2(z-z_0)sVp_\ell(z,s)^{-1}. \end{aligned} \tag{1.45}$$

Using the identity

$$\chi p_\ell(z,s)^{-1}P_\ell\chi = \chi p_\ell(z,s)^{-1}\big[P_\ell,\chi\big]+\chi p_\ell(z,s)^{-1}\chi P_\ell$$

and the uniform bound in ℓ

$$\|\chi' u'\|_{L^2} \lesssim \|\chi_1 u\|_{L^2}+\|\chi_2 u''\|_{L^2} \qquad \chi_j\in\mathcal{C}^\infty_{\rm c}(\mathbb{R},\mathbb{R}),\ \operatorname{Supp}\chi_j=\operatorname{Supp}\chi, \tag{1.46}$$

we obtain from (1.44)

$$\|Au\|_{L^2} \le \tilde{C}_{\rm a}\langle z\rangle\|\tilde\chi p_\ell(z,s)^{-1}\tilde\chi\|_{L^2\to L^2}\|u\|_{L^2} \tag{1.47}$$

where the constant $\tilde{C}_{\rm a}$ only depends on $z_0,s,V,V',\chi,\chi',\chi'',\chi_1$ and χ_2.

We now turn to B. Using again (1.45), we see that

$$\begin{aligned} \|\chi P_\ell p_\ell(z,s)^{-1}\chi u\|_{L^2} &\le \|\chi p_\ell(z_0,s)p_\ell(z,s)^{-1}\chi u\|_{L^2} \\ &+ \|\chi(z_0^2-2z_0sV+s^2V^2)p_\ell(z,s)^{-1}\chi u\|_{L^2} \\ &\le \|\chi p_\ell(z,s)^{-1}(P_\ell-(z_0^2-2zsV+s^2V^2))\chi u\|_{L^2}+2\big(|z|+|z_0|\big)|s|\|V\|_{L^\infty}\|\chi p_\ell(z,s)^{-1}\chi u\|_{L^2} \\ &+ \big(|z_0|^2+2|z_0||s|\|V\|_{L^\infty}+s^2\|V\|_{L^\infty}^2\big)\|\chi p_\ell(z,s)^{-1}\chi u\|_{L^2} \\ &\le \|\chi p_\ell(z,s)^{-1}P_\ell\chi v\|_{L^2} \\ &+ \underbrace{2\big\langle|z_0|+2|s|\|V\|_{L^\infty}\big\rangle^2}_{=:\tilde{\tilde{C}}_{\rm a}}\langle z\rangle\|\chi p_\ell(z,s)^{-1}\chi\|_{L^2\to L^2}\|u\|_{L^2}. \end{aligned}$$

Commuting P_ℓ with χ and using (1.46), we get (1.43) with $C_{\rm a}=\max\big\{\tilde{C}_{\rm a},1+\tilde{\tilde{C}}_{\rm a}\big\}$.

Estimate (1.42b)**.** Let us define

$$\Lambda_{\rm b}(\theta)=\langle z\rangle^{-2\theta}P_\ell^\theta\chi p_\ell(z,s)^{-1}\chi \qquad\qquad \theta\in[0,1]+{\rm i}\mathbb{R}.$$

$\Lambda_{\rm b}$ is analytic from $[0,1]+{\rm i}\mathbb{R}$ to $\mathcal{L}(L^2,L^2)$. As the above estimate, it is sufficient to show a bound on $\Lambda_{\rm b}(1)$, the imaginary part of θ playing no role and the case $\theta=0$ being trivial. We get (1.42b) if we show that

$$\|P_\ell\chi p_\ell(z,s)^{-1}\chi u\|_{L^2} \le C_{\rm b}\langle z\rangle^2\|\tilde\chi p_\ell(z,s)^{-1}\tilde\chi\|_{L^2}\|u\|_{L^2}\forall u\in L^2 \tag{1.48}$$

for some $C_{\rm b}>0$. Using the identity (1.44) and the estimate (1.47), we obtain

$$\|P_\ell\chi p_\ell(z,s)^{-1}\chi u\|_{L^2\to L^2} \le \tilde{C}_{\rm a}\langle z\rangle\|\tilde\chi p_\ell(z,s)^{-1}\tilde\chi\|_{L^2\to L^2}\|u\|_{L^2}+\|\chi p_\ell(z,s)^{-1}P_\ell\chi u\|_{L^2}$$

but this time we ask for the L^2 norm of u. Hence, we use that

$$p_\ell(z,s)^{-1}P_\ell = 1+p_\ell(z,s)^{-1}(z-sV)^2$$

which yields (1.48) with $C_{\rm b}=\max\big\{\tilde{C}_{\rm a},2\big\langle s\|V\|_{L^\infty}\big\rangle^2\big\}$.

Estimate (1.42c)**.** Let us define

$$\Lambda_{\rm c}(\theta) := \langle z\rangle^{2(\theta-1)}\chi(1+(z-(sV)^{2(1-\theta)})p_\ell(z,s)^{-1}(z-2^{2\theta-1}sV))\chi P_\ell^{-\theta}.$$

Once again, $\Lambda_{\rm c}$ is analytic from $[0,1]+\mathrm{i}\mathbb{R}$ to $\mathcal{L}(L^2,L^2)$ and (dropping the imaginary part)

$$\|\Lambda_{\rm c}(0)\|_{L^2\to L^2} \le (2+|s|\|V\|_{L^\infty})^3\|\chi p_\ell(z,s)^{-1}\chi\|_{L^2\to L^2}.$$

We then get a bound on $\Lambda_c(1)$: we prove

$$\|\chi(1+zp_\ell(z,s)^{-1}(z-2sV))\chi u\|_{L^2} \le C_{\rm c}\|\tilde\chi p_\ell(z,s)^{-1}\tilde\chi\|_{L^2\to L^2}\|P_\ell u\|_{L^2} \qquad \forall u\in L^2.$$

We have

$$\|\chi(1+p_\ell(z,s)^{-1}z(z-2sV))\chi u\|_{L^2} \le \|\chi(1+p_\ell(z,s)^{-1}(z-sV)^2\chi)u\|_{L^2} + \|\chi p_\ell(z,s)^{-1}s^2V^2\chi u\|_{L^2}$$

and

$$\chi(1+p_\ell(z,s)^{-1}(z-sV)^2)\chi = \chi p_\ell(z,s)^{-1}P_\ell\chi.$$

Commuting P_ℓ with χ and using (1.46) gives us

$$\|\chi p_\ell(z,s)^{-1}z(z-2sV)\chi v\|_{L^2} \le C_{\rm c}\|\tilde\chi p_\ell(z,s)^{-1}\tilde\chi\|_{L^2\to L^2}\|\langle P_\ell\rangle v\|_{L^2}$$

with $C_{\rm c} = \max\left\{(1+|s|\|V\|_{L^\infty})^3, \|\chi\|_{L^\infty}, \|\chi_1\|_{L^\infty}+\|\chi_2\|_{L^\infty}\right\}$. □

The proof is now the same as in [BoHa08, Section 3.2]. For $\nu>0$ fixed and for $\ell\in\mathbb{N}$, we define $L^2_\nu(\mathbb{R},\dot{\mathcal{E}}_\ell)$ as the class of functions $t\mapsto v(t)$ with values in $\dot{\mathcal{E}}_\ell$ such that $t\mapsto \mathrm{e}^{-\nu t}v(t)\in L^2(\mathbb{R},\dot{\mathcal{E}}_\ell)$. For $u\in\dot{\mathcal{E}}_\ell$, the componentwise defined function

$$v(t) = \begin{cases} \mathrm{e}^{-\mathrm{i}t\hat K_\ell}u & \text{for } t\ge 0\\ 0 & \text{for } t<0\end{cases}$$

is in $L^2_\nu(\mathbb{R},\dot{\mathcal{E}}_\ell)$ if ν is sufficiently large and thus

$$\tilde v(z) = \int_0^{+\infty}\mathrm{e}^{\mathrm{i}zt}v(t)\mathrm{d}t \tag{1.49}$$

is well-defined as soon as $\Im z\ge\nu$. For all $t\ge 0$, we have the inversion formula

$$v(t) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\nu}^{+\infty+\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}zt}\tilde v(z)\mathrm{d}z \tag{1.50}$$

that is

$$\mathrm{e}^{-\mathrm{i}t\hat K_\ell}u = \frac{1}{2\pi\mathrm{i}}\int_{-\infty+\mathrm{i}\nu}^{+\infty+\mathrm{i}\nu}\mathrm{e}^{-\mathrm{i}zt}(\hat K_\ell - z)^{-1}u\mathrm{d}z$$

in the $L^2_\nu(\mathbb{R},\dot{\mathcal{E}}_\ell)$ sense. We then use the following result:

Lemma 1.5.2 (Lemma 3.2 in [BoHa08]). *Let $N \in \mathbb{N}$, $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R},\mathbb{R})$ and define for all $j \in \mathbb{N}$ the spaces $\dot{\mathcal{E}}_\ell^{\ -j} := (\hat{K}_\ell - \mathrm{i})^j \dot{\mathcal{E}}_\ell$. Then for all $k \in \{0,\dots,N\}$, there exist bounded operators $B_j \in \mathcal{L}(\dot{\mathcal{E}}_\ell^{-k}, \dot{\mathcal{E}}_\ell^{-k-j})$ and $B \in \mathcal{L}(\dot{\mathcal{E}}_\ell^{-k}, \dot{\mathcal{E}}_\ell^{-k-N-1})$ such that*

$$\hat{R}_{\chi,\ell}(z) = \sum_{j=0}^{N} \frac{B_j}{(z - \mathrm{i}(\nu+1))^{j+1}} + \frac{B\hat{R}_{\tilde{\chi},\ell}(z)\chi}{(z-\mathrm{i}(\nu+1))^{N+1}}$$

for some $\tilde{\chi} \in \mathcal{C}^\infty_{\rm c}(\mathbb{R},\mathbb{R})$ satisfying $\tilde{\chi}\chi = \chi$.

Now define

$$\tilde{R}_{\chi,\ell}(z) := \hat{R}_{\chi,\ell}(z) - \sum_{0\le j\le 1} \frac{B_j}{(z-\mathrm{i}(\nu+1))^{j+1}} \tag{1.51}$$

with $B_j \in \mathcal{L}(\dot{\mathcal{E}}_\ell, \dot{\mathcal{E}}_\ell^{-j})$ as in Lemma 1.5.2; we thus have[13]

$$\|\tilde{R}_{\chi,\ell}(z)\|_{\mathcal{L}(\dot{\mathcal{E}}_\ell, \dot{\mathcal{E}}_\ell^{-2})} \lesssim \langle z\rangle^{-2} \|\hat{R}_{\chi,\ell}(z)\|_{\mathcal{L}(\dot{\mathcal{E}}_\ell,\dot{\mathcal{E}}_\ell)}. \tag{1.52}$$

We can show that

$$\int_{-\infty+\mathrm{i}\nu}^{+\infty+\mathrm{i}\nu} \frac{B_j}{(z-\mathrm{i}(\nu+1))^{j+1}} \mathrm{d}z = 0 \tag{1.53}$$

for all $j \in \mathbb{N}$ using a contour deformation (integrate first along the square with apexes $\pm C_0 + \mathrm{i}\nu$, $\pm C_0 - \mathrm{i}\mu$ and let $\mu \to -\infty$ then $C_0 \to +\infty$). We thus obtain

$$\chi \mathrm{e}^{-\mathrm{i}t\hat{K}_\ell}\chi u = \frac{1}{2\pi\mathrm{i}} \int_{-\infty+\mathrm{i}\nu}^{+\infty+\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}zt} \tilde{R}_{\chi,\ell}(z) u \mathrm{d}z \tag{1.54}$$

and the integral absolutely converges in $\mathcal{L}(\dot{\mathcal{E}}_\ell, \dot{\mathcal{E}}_\ell^{-2})$. We then integrate $\mathrm{e}^{-\mathrm{i}zt}\tilde{R}_{\chi,\ell}\, u$ over the

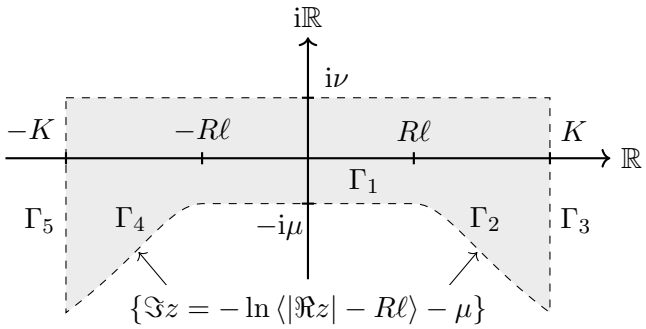

Figure 1.4: The contour used for the derivation of the resonance expansion.

[13] The purpose of Lemma 1.5.2 is to provide us with integrability in z at the prize of using the weaker spaces $\dot{\mathcal{E}}_\ell^{-2}$. The task then consists in showing that all the terms in $\dot{\mathcal{E}}_\ell^{-2}$ vanish after deformation of contours and the remaining terms are in $\dot{\mathcal{E}}_\ell$.

(positively oriented) contour described in Figure 1.4 defined for $K, \mu > 0$. Setting

$$I_j := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_j} \mathrm{e}^{-\mathrm{i}zt} \tilde{R}_{\chi,\ell}(z) u \mathrm{d}z,$$

one obtains by the residue theorem:

$$\frac{1}{2\pi \mathrm{i}} \int_{-K+\mathrm{i}\nu}^{K+\mathrm{i}\nu} \mathrm{e}^{-\mathrm{i}zt} \tilde{R}_{\chi,\ell}(z) u \mathrm{d}z = \sum_{\substack{z_j \in \mathrm{Res}(p_\ell) \\ \Im z_j > -\mu}} \sum_{k=0}^{m(z_j)} \mathrm{e}^{-\mathrm{i}z_j t} t^k \Pi_{j,k}^{\chi} u + \sum_{1 \leq j \leq 5} I_j. \tag{1.55}$$

Using the estimate (1.37) in the Zone IV as well as Proposition 1.5.1 and (1.52) above, we compute for t large enough:

$$\begin{aligned} \|I_3\|_{\dot{\mathcal{E}}_\ell^{-2}} &\lesssim \int_{K-\mathrm{i}\ln\langle K\rangle}^{K+\mathrm{i}\nu} \|\mathrm{e}^{-\mathrm{i}\lambda t} \tilde{R}_{\chi,\ell}(\lambda) u\|_{\dot{\mathcal{E}}_\ell^{-2}} \mathrm{d}\lambda \\ &\lesssim \|u\|_{\dot{\mathcal{E}}_\ell} \int_{\ln\langle K\rangle}^{\nu} \langle z\rangle^{-2} \mathrm{e}^{\lambda t + C|\lambda|} \mathrm{d}\lambda \\ &\lesssim \langle K\rangle^{-2} \frac{\mathrm{e}^{\nu t}}{t} \|u\|_{\dot{\mathcal{E}}_\ell}. \end{aligned}$$

We let $K \to +\infty$: the integrals I_3 and I_5 then vanish. We still denote by I_2 and I_4 the integrals over Γ_2 and Γ_4 which now go to infinity. As for (1.53), we can show that

$$\int_{\Gamma_4 \cup \Gamma_1 \cup \Gamma_2} \frac{B_j}{(z - \mathrm{i}(\nu+1))^{j+1}} \mathrm{d}z = 0. \tag{1.56}$$

Now, using (1.35) for I_1 and (1.37) for I_2 and I_4, we get for t large enough:

$$\begin{aligned} \|I_1\|_{\dot{\mathcal{E}}_\ell} &\lesssim \int_{-R\ell}^{R\ell} \|\mathrm{e}^{-\mu t} \hat{R}_{\chi,\ell}(r - \mathrm{i}\mu) u\|_{\dot{\mathcal{E}}_\ell} \mathrm{d}r \\ &\lesssim \mathrm{e}^{-\mu t} \|u\|_{\dot{\mathcal{E}}_\ell} \int_{-R\ell}^{R\ell} \langle r\rangle^C \mathrm{d}r \\ &\lesssim \mathrm{e}^{-\mu t} \ell^{C+1} \|u\|_{\dot{\mathcal{E}}_\ell}, \end{aligned}$$

$$\begin{aligned} \|I_2\|_{\dot{\mathcal{E}}_\ell} &\lesssim \int_0^{+\infty} \left\| \mathrm{e}^{-\mathrm{i}\left(R\ell + \lambda - \mathrm{i}(\mu + \ln\langle\lambda\rangle)\right)t} \hat{R}_{\chi,\ell}\left(R\ell + \lambda - \mathrm{i}(\mu + \ln\langle\lambda\rangle)\right) u \right\|_{\dot{\mathcal{E}}_\ell} \mathrm{d}\lambda \\ &\lesssim \mathrm{e}^{-\mu t} \|u\|_{\dot{\mathcal{E}}_\ell} \int_0^{+\infty} \mathrm{e}^{-\ln\langle\lambda\rangle t + C(\ln\langle\lambda\rangle + \mu)} \mathrm{d}\lambda \\ &\lesssim \mathrm{e}^{-\mu t} \|u\|_{\dot{\mathcal{E}}_\ell}. \end{aligned}$$

All these estimates hold in $\dot{\mathcal{E}}_\ell$, hence we have established part (i) of Theorem 1.3.2 with $N = (C+1)/2$.

Let us turn to part (ii). For $\mu < \varepsilon_0$ with ε_0 as in part 2. of Theorem 1.2.8, we know that there is no resonance in formula 1.55. If $\ell < \mathrm{e}^{\varepsilon' t}$ for some $\varepsilon' > 0$, then

$$\|I_1\|_{\dot{\mathcal{E}}_\ell} \lesssim \mathrm{e}^{((C+1)\varepsilon' - \mu)t} \|u\|_{\dot{\mathcal{E}}_\ell}.$$

Otherwise, if $\ell \geq \mathrm{e}^{\varepsilon' t}$, then the exponential decay of the local energy as well as the hypotheses on g imply together:

$$\|\chi \mathrm{e}^{-\mathrm{i}t\hat{K}_\ell} \chi u\|_{\dot{\mathcal{E}}_\ell} \lesssim 1 \lesssim \frac{g(\ell(\ell+1))}{g(\mathrm{e}^{2\varepsilon' t})}.$$

It remains to take ε' small enough and $\varepsilon := \min\{2\varepsilon', \mu - (C+1)\varepsilon'\}$ to conclude the proof.

1.6 Appendix

1.6.1 Analytic extension of the coordinate r

In this Appendix, we prove Proposition 1.1.1 which is analogous to [BaMo93, Proposition IV.2]. Let $r \in]r_-, r_+[$. By equation (1.5), we have

$$\exp\left(-\frac{\Lambda}{3A_\pm r_\pm^2} x\right) = \prod_{\alpha \in I} \left|\frac{r - r_\alpha}{\mathfrak{r} - r_\alpha}\right|^{\frac{A_\alpha r_\alpha^2}{A_\pm r_\pm^2}}.$$

Call the left-hand side z and the right-hand side $g_\pm(r)$. Observe that $g_\pm(r_\pm) = 0$. Since $r \mapsto x(r)$ is increasing and analytic, we can apply the Lagrange's inversion theorem (see for example [De81], paragraph 2.2 and references therein) to write

$$r = r_\pm + \sum_{\ell=1}^{+\infty} \frac{z^\ell}{\ell!} \left[\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}} \left(\frac{r - r_\pm}{g_\pm(r)}\right)^\ell\right]_{r=r_\pm}. \tag{1.57}$$

Let us introduce Kronecker's symbol

$$\delta_{\alpha,\pm} := \begin{cases} 1 & \text{if } \alpha = \pm \\ 0 & \text{otherwise} \end{cases}$$

and the notation

$$B_{\pm,\alpha} := \frac{A_\alpha r_\alpha^2}{A_\pm r_\pm^2} - \delta_{\alpha,\pm}.$$

Observe that $B_{-,-} = B_{+,+} = 0$. We then have

$$\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}} \left(\frac{r - r_\pm}{g_\pm(r)}\right)^\ell = \left(\prod_{\alpha \in I \setminus \{\pm\}} |\mathfrak{r} - r_\alpha|^{\ell B_{\pm,\alpha}}\right) \frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}} \left(\prod_{\alpha \in I \setminus \{\pm\}} |r - r_\alpha|^{-\ell B_{\pm,\alpha}}\right).$$

We now fix $\pm = +$ (the conclusion will not be changed if we choose $\pm = -$). Then

$$\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}}\left(\prod_{\alpha\in I\setminus\{+\}}(r-r_\alpha)^{-\ell B_{+,\alpha}}\right)=\sum_{0\le k_2\le k_1\le\ell}C_{\ell,k_1,k_2}\left(\frac{\mathrm{d}^{\ell-k_1}}{\mathrm{d}r^{\ell-k_1}}(r-r_n)^{-\ell B_{+,n}}\right)\times \\ \times\left(\frac{\mathrm{d}^{k_1-k_2}}{\mathrm{d}r^{k_1-k_2}}(r-r_c)^{-\ell B_{+,c}}\right)\left(\frac{\mathrm{d}^{k_2}}{\mathrm{d}r^{k_2}}(r-r_-)^{-\ell B_{+,-}}\right)$$

where

$$C_{\ell,k_1,k_2}=\binom{\ell}{k_1}\binom{k_1}{k_2}.$$

Direct computation shows that

$$\frac{\mathrm{d}^p}{\mathrm{d}r^p}(r-r_\alpha)^{-\ell B_{+,\alpha}}=(-1)^p(\ell B_{+,\alpha})(\ell B_{+,\alpha}+1)\dots(\ell B_{+,\alpha}+p-1)(r-r_\alpha)^{-\ell B_{+,\alpha}-p}.$$

If we let

$$K:=\prod_{\alpha\in I\setminus\{+\}}(\mathfrak{r}-r_\alpha)^{B_{+,\alpha}},\qquad\qquad B_+:=\max_{\alpha\in I\setminus\{+\}}\{|B_{+,\alpha}|\},$$

then it follows that

$$\begin{aligned}\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}}\left(\frac{r-r_+}{g_+(r)}\right)^\ell &= K^\ell\sum_{0\le k_2\le k_1\le\ell}C_{\ell,k_1,k_2}(-1)^\ell\times\\ &\times(\ell B_{+,n})(\ell B_{+,n}+1)\dots(\ell B_{+,n}+(\ell-k_1)-1)(r-r_n)^{-\ell B_{+,n}-(\ell-k_1)}\times\\ &\times(\ell B_{+,c})(\ell B_{+,c}+1)\dots(\ell B_{+,c}+(k_1-k_2)-1)(r-r_c)^{-\ell B_{+,c}-(k_1-k_2)}\times\\ &\times(\ell B_{+,-})(\ell B_{+,-}+1)\dots(\ell B_{+,-}+k_2-1)(r-r_\alpha)^{-\ell B_{+,-}-k_2}\end{aligned}$$

and thus

$$\begin{aligned}\left|\frac{\mathrm{d}^{\ell-1}}{\mathrm{d}r^{\ell-1}}\left(\frac{r-r_+}{g_+(r)}\right)^\ell\right| &\le K^\ell\ell^\ell(B_++1)^\ell\left(\prod_{\alpha\in I\setminus\{+\}}(r_+-r_\alpha)^{-B_{+,\alpha}}\right)^\ell\times\\ &\times\sum_{0\le k_2\le k_1\le\ell}C_{\ell,k_1,k_2}(r_+-r_n)^{-(\ell-k_1)}(r_+-r_c)^{-(k_1-k_2)}(r_+-r_-)^{-k_2}\\ &=K^\ell\ell^\ell(B_++1)^\ell\left(\prod_{\alpha\in I\setminus\{+\}}(r_+-r_\alpha)^{-B_{+,\alpha}}\right)^\ell\left(\sum_{\alpha\in I\setminus\{+\}}(r_+-r_\alpha)^{-1}\right)^\ell\\ &=\left(K(B_++1)\prod_{\alpha\in I\setminus\{+\}}(r_+-r_\alpha)^{-B_{+,\alpha}}\sum_{\alpha\in I\setminus\{+\}}(r_+-r_\alpha)^{-1}\right)^\ell\ell^\ell\\ &=:\tilde{K}^\ell\ell^\ell.\end{aligned}$$

Therefore, the convergence of the original series is absolute for $z \in \mathbb{C}$ if

$$\frac{(|z|\ell\tilde{K})^\ell}{\ell!} < \ell^{-(1+\varepsilon)}$$

for any $\varepsilon > 0$. Using Stirling approximation $\ell! \sim \sqrt{2\pi}\ell^{\ell+1/2}$ for large values of ℓ, we see that it is sufficient to have

$$\tilde{K}|z| < \frac{\mathrm{e}^{-(1/2+\varepsilon)\ln \ell/\ell}}{\sqrt{2\pi}^\ell} < 1.$$

This condition is fulfilled if

$$\Re x > \frac{3A_+ r_+^2}{\Lambda} \ln \tilde{K}.$$

1.6.2 Localization of high frequency resonances

We provide in this Appendix an asymptotic approximation of resonances near the maximal energy $W_0(0) = \max_{x\in\mathbb{R}}\{W_0(x)\}$ as $h \to 0$. This a generalization of the main Theorem in [SaZw97] to the case $Q \neq 0$. More precisely, we show that the resonances associated to the meromorphic extension of $p(z,s)^{-1}$ are close to the ones associated with the extension of $(P - z^2)^{-1}$, provided that Q is sufficiently small. This is a direct consequence of the fact that the extra term hsV in the semiclassical quadratic pencil is $\mathcal{O}(hs)$.

As in the paragraph 1.4.2, we set $h := (\ell(\ell+1))^{-1/2}$ with $\ell > 0$ and consider $z \in [\ell/R, R\ell] + \mathrm{i}\,[-C_0, C_0]$. We then define the spectral parameter $\lambda := h^2 z^2$ and also $\tilde{P}_h$ the semiclassical operator associated to P_ℓ. Recall also that $\mathfrak{r} = \frac{3M}{2}\left(1 + \sqrt{1 - \frac{8Q^2}{9M^2}}\right)$ is the radius of the photon sphere and $W_0(0) = F(\mathfrak{r})/\mathfrak{r}^2$ with our definition of the Regge-Wheeler coordinate x (see (1.5)).

Theorem 1.6.1. *Let*

$$\Gamma_0(h) := \left\{ W_0(0) + h\left(2\sqrt{W_0(0)}sV(0) + \mathrm{i}^{-1}\sqrt{W_0''(0)/2}\left(k + \frac{1}{2}\right)\right) \mid k \in \mathbb{N}\right\}.$$

For all $C_0 > 0$ such that $\partial D(W_0(0), C_0 h) \cap \Gamma_0(h) = \emptyset$, there is a bijection $b \equiv b(h)$ from $\Gamma_0(h)$ onto the set of resonances of $\tilde{P}_h$ in $D(W_0(0), C_0 h)$ (counted with their natural multiplicity) such that

$$b(h)(\mu) - \mu = o_{h\to 0}(h) \qquad \text{uniformly for } \mu \in \Gamma_0(h).$$

Proof. This is a direct application of the results of Sá Barreto-Zworski [SaZw97] which are based on the work of Sjöstrand [Sj87] (see Theorem 0.1), the latter dealing with resonances generated by non-degenerate critical points when the trapping set is reduced to a single point (the difference for us is $W_0(0) \neq 0$).

We recall that in the zone III the symbol of the semiclassical quadratic pencil is the function $(x,\xi) \mapsto \xi^2 + W_0(x) + h^2 W_1(x) - (\sqrt{\lambda} - hsV(x))^2 =: p(x,\xi) - \lambda$. We also recall the hypothesis in [Sj87] for the case of a Schrödinger operator of the form (0.1) in the reference:

- The trapping set is reduced to the point $\{(0,0)\}$ ((0.3) in [Sj87]),

- 0 is a non-degenerate critical point ((0.4) in [Sj87], which implies in the Schrödinger case the more general assumptions (0.7) and (0.9) in the reference).

Although the symbol p depends on λ, its principal part p_0 and subprincipal part p_{-1} do not: indeed, for $\lambda \in D\left(W_0(0), C_0 h\right)$ with $C_0 > 0$, we can write when $h \ll 1$

$$p(x,\xi) = \underbrace{\xi^2 + W_0(x)}_{p_0(x,\xi)} + h \underbrace{2\sqrt{W_0(0)} s V(x)}_{p_{-1}(x,\xi)} + \text{lower order terms in } h.$$

This is enough to apply [Sj87], Theorem 0.1: using formula (0.14) in the reference, we get the result for the set

$$\left\{ p_0(0,0) + h\left(p_{-1}(0,0) + \mathrm{i}^{-1}\sqrt{W_0''(0)/2}\left(k + \frac{1}{2}\right)\right) \mid k \in \mathbb{N} \right\}$$

which is $\Gamma_0(h)$. □

Approximation of high frequency resonances $\Gamma_0(h) \ni z^2 = \lambda/h^2$ is obtained as in [SaZw97], by taking the square root of any element of $\Gamma_0(h)$ and using Taylor expansion for $0 < h \ll 1$ (corresponding to $\ell \gg 0$) as well as symmetry with respect to the imaginary axis (for the choice of the sign of the square root). In our setting, we obtain the set Γ of Theorem 1.3.1.

Remark 1.6.2. *1. Let Γ_{DSS} be the set of pseudo-poles in the De Sitter-Schwarzschild case (see the Theorem at the end of [SaZw97]). Then Γ_{DSS} is the limit of Γ as $Q \to 0$ in the sense of the sets, i.e. for all $z \in \Gamma$, there exists $z_0 \in \Gamma_{\mathrm{DSS}}$ such that $z \to z_0$ as $Q \to 0$.*

2. The pseudo-poles in the charged case are shifted with respect to the uncharged case. If the charges of the Klein-Gordon field and the black hole have the same sign (that is if $qQ > 0$), then all the pseudo-poles go to infinity with a real part which never vanishes. However, if the charges have opposite sign ($qQ < 0$), then all the pseudo-poles real part cancels precisely when $qQ = -(k+1/2)\sqrt{F(\mathfrak{r})}$, $k \in \mathbb{N} \setminus \{0\}$, before going to infinity. Notice that no pseudo-pole goes to $\mathbb{C}^+$ as $|s| \to +\infty$.

3. We can provide a physical interpretation of the set of pseudo-poles. First observe that $\sqrt{F(\mathfrak{r})}/\mathfrak{r}$ is nothing but the inverse of the impact parameter $b = |E/L|$ of trapped null geodesics with energy E and angular momentum L. Theorem 1.3.1 shows that resonances near the real line in the zone III are qQ-dependent multiples of this quantity: they thus correspond to impact parameters of trapped photons with high energy and angular momentum.

4. Observe that in Newtonian mechanics, the electromagnetism and gravitation do not interact with chargeless and massless photons. As a consequence, photons are not deviated and only ones with impact parameter $|b| \leq r_-$ can "fall" in the black hole. Hence, high frequency resonances in zone III are expected to be multiple of r_-^{-1}. As $r_- \to 0$, all resonances go to infinity: the trajectory are now classical straight lines as there is no obstacle anymore.

1.6.3 Abstract Semiclassical Limiting Absorption Principle for a class of Generalized Resolvents

We show in this Appendix an abstract semiclassical limiting absorption principle for perturbed resolvents.

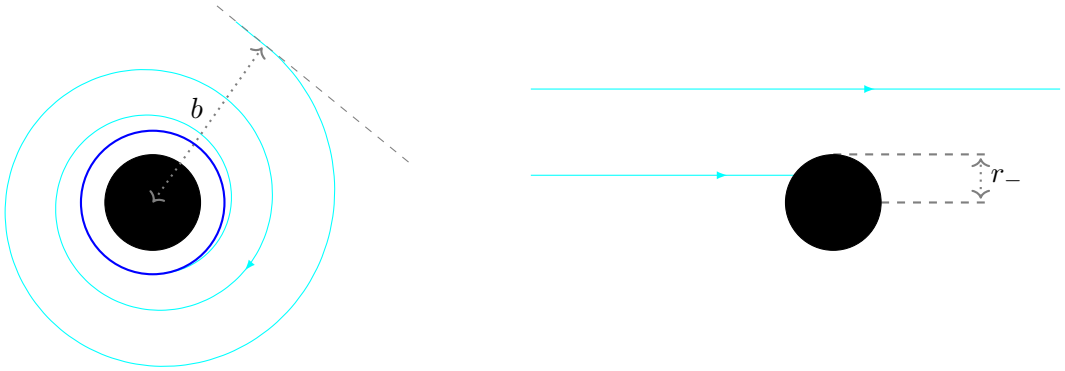

Figure 1.5: On the left: a relativistic trapped null geodesic. On the right: classical null geodesic trajectories.

Abstract setting. Let $\big(\mathcal{H}, \langle\cdot,\cdot\rangle\big)$ be a Hilbert space, $J := [a,b] \subset \mathbb{R}$, $J^+_\mu := \{\omega \in \mathbb{C}^+ \mid \Re\omega \in J,\ \Im\omega < \mu\}$ for some $\mu > 0$ fixed and $h_0 > 0$. The norm associated to $\langle\cdot,\cdot\rangle$ will be denoted by $\|\cdot\|$. We consider families of self-adjoint operators $P \equiv P(h)$ and $\mathcal{A} \equiv \mathcal{A}(h)$ acting on $\mathcal{H}$ for $0 < h < h_0$. We set

$$L^\infty_{\ell\mathrm{oc}}(P) := \big\{A : \mathcal{H} \to \mathcal{H} \text{ linear} \mid \forall \chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R},\mathbb{R}),\ \forall u \in \mathscr{D}(P),\ \|\chi(P)Au\| < +\infty\big\}$$

and $\|.\|_{P}$ will be the operator norm on $\mathcal{B}(\mathscr{D}(P),\mathcal{H})$. We also define the local version of the operator P:

$$P_\tau := \tau(P)P \qquad\qquad \forall \tau \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R},\mathbb{R}).$$

Let then $f : \mathbb{C} \times L^\infty_{\ell\mathrm{oc}}(P) \to L^\infty_{\ell\mathrm{oc}}(P)$ satisfying the following continuity type relation near $0_{L^\infty_{\ell\mathrm{oc}}(P)}$: there exist $\delta_{J,\mu} : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\delta_{J,\mu}(r) \to 0$ as $r \to 0$ and $\varepsilon_{J,\mu} : L^\infty_{\ell\mathrm{oc}}(P) \to L^\infty_{\ell\mathrm{oc}}(P)$ such that, for all $(z,A) \in J^+_\mu \times L^\infty_{\ell\mathrm{oc}}(P)$ with $\|A\|_{P}$ small,

$$f(z,A) = z + \delta_{J,\mu}(\|A\|_{P})\,\varepsilon_{J,\mu}(A). \tag{C}$$

We make the following assumptions:

$$(P - f(z,hA))^{-1} \text{ exists for all } z \in J^+_\mu \text{ and } A \in L^\infty_{\ell\mathrm{oc}}(P) \text{ if } h \le h_0 \tag{I}$$

$$P \in \mathcal{C}^2(\mathcal{A}) \tag{P}$$

$$\mathbb{1}_I(P)[P, i\mathcal{A}]\mathbb{1}_I(P) \ge ch\mathbb{1}_I(P) \qquad \text{for some } c > 0 \text{ and } J \Subset I :=]\alpha,\beta[\subset \mathbb{R} \tag{M}$$

$$\mathrm{ad}^k_{\chi(P)}(\varepsilon_{J,\mu}(A)) \in h^k\mathcal{B}(\mathscr{D}(\mathcal{A})) \qquad \text{for all } k \in \{0,1\},\ \chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R},\mathbb{R}) \text{ and } \|A\|_{P} < c' \text{ for } c' > 0. \tag{A}$$

Recall that $P \in \mathcal{C}^2(\mathcal{A})$ means for all $z \in \mathbb{C} \setminus \sigma(P)$ that the map

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{\mathrm{i}t\mathcal{A}}(P-z)^{-1}\mathrm{e}^{-\mathrm{i}t\mathcal{A}}$$

is $\mathcal{C}^2$ for the strong topology of L^2. Recall also that for all linear operators L_1, L_2 acting on $\mathcal{H}$, $\mathrm{ad}^0_{L_1}(L_2) := L_2$ and $\mathrm{ad}^{k+1}_{L_1}(L_2) := [L_1, \mathrm{ad}^k_{L_1}(L_2)]$. Our goal is to show the following result:

Theorem 1.6.3. *Assume hypotheses (C), (I), (P), (M) and (A). Then for all $\sigma > 1/2$,*

$$\sup_{z\in J_\mu^+} \|\langle\mathcal{A}\rangle^{-\sigma}(P - f(z,hB))^{-1}\langle\mathcal{A}\rangle^{-\sigma}\| \lesssim h^{-1}. \tag{1.58}$$

In the sequel, we will write $R(z,hB) := (P - f(z,hB))^{-1}$ and call it the generalized resolvent (of P). Also, since J and μ are now fixed, we will simply write J, δ and ε instead of J_μ, $\delta_{J,\mu}$ and $\varepsilon_{J,\mu}$.

Preliminary results. The purpose of this paragraph is to show preliminary results used to prove Theorem 1.6.3. We first prove an adapted version of [Ge08, Lemma 2.1] to our situation.

Lemma 1.6.4. *Let $0 \le \sigma \le 1$, $z \in J^+$ and let $\chi \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$. If h is small enough, then $R(z,hB)$ and $\chi(P)$ are bounded on $\mathscr{D}(\langle\mathcal{A}\rangle^\sigma)$.*

Proof. The result is true for $(P-z)^{-1}$ and $\chi(P)$ by [Ge08, Lemma 2.1]. Let us show that $R(z,hB)\mathscr{D}(\langle\mathcal{A}\rangle^\sigma) \subset \mathscr{D}(\langle\mathcal{A}\rangle^\sigma)$:

$$\begin{aligned}\|\langle\mathcal{A}\rangle^\sigma R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}\| &\le \|\langle\mathcal{A}\rangle^\sigma(P-z)^{-1}\langle\mathcal{A}\rangle^{-\sigma}\| + \|\langle\mathcal{A}\rangle^\sigma(R(z,hB)-(P-z)^{-1})\langle\mathcal{A}\rangle^{-\sigma}\| \\ &\lesssim 1 + \|\langle\mathcal{A}\rangle^\sigma R(z,hB)(z-f(z,hB))(P-z)^{-1}\langle\mathcal{A}\rangle^{-\sigma}\|\end{aligned}$$

and (using that $\varepsilon(hB) \in \mathcal{B}(\mathscr{D}(\mathcal{A}))$ by Assumption (A) for $k=0$)

$$\begin{aligned}&\|\langle\mathcal{A}\rangle^\sigma R(z,hB)(z-f(z,hB))(P-z)^{-1}\langle\mathcal{A}\rangle^{-\sigma}\| \\ &\le \|\langle\mathcal{A}\rangle^\sigma R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}\|\|\langle\mathcal{A}\rangle^\sigma(z-f(z,hB))\langle\mathcal{A}\rangle^{-\sigma}\|\|\langle\mathcal{A}\rangle^\sigma(P-z)^{-1}\langle\mathcal{A}\rangle^{-\sigma}\| \\ &\lesssim \delta(h\|B\|_P)\|\langle\mathcal{A}\rangle^\sigma\varepsilon(hB)\langle\mathcal{A}\rangle^{-\sigma}\|\|\langle\mathcal{A}\rangle^\sigma R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}\|.\end{aligned}$$

We then use the uniformity in assumption (A) for $k=0$ to write for h very small

$$\delta(h\|B\|_P)\|\langle\mathcal{A}\rangle^\sigma\varepsilon(hB)\langle\mathcal{A}\rangle^{-\sigma}\|\|\langle\mathcal{A}\rangle^\sigma R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}\| < \frac{1}{2}\|\langle\mathcal{A}\rangle^\sigma R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}\|.$$

The proof is complete. □

Corollary 1.6.5. *Let $0 \le \sigma \le 1$, $z \in J^+$ and $\tau, \chi \in \mathcal{C}_c^\infty(\mathbb{R},[0,1])$ such that $\chi \equiv 1$ on I and $\tau\chi = \chi$. If h is small enough, then $(P_\tau - f(z,hB))\chi(P)$, $(P_\tau - f(z,hB))\chi(P)(P+\mathrm{i})^{-1}$ and $(P-f(z,hB))(P+\mathrm{i})^{-1}$ preserve $\mathscr{D}(\langle\mathcal{A}\rangle^\sigma)$.*

Proof. We have

$$\begin{aligned}\langle\mathcal{A}\rangle^\sigma(P_\tau - f(z,hB))\chi(P)\langle\mathcal{A}\rangle^{-\sigma} &= \langle\mathcal{A}\rangle^\sigma(P_\tau - z)\chi(P)\langle\mathcal{A}\rangle^{-\sigma} \\ &\quad + \langle\mathcal{A}\rangle^\sigma(z-f(z,hB))\langle\mathcal{A}\rangle^{-\sigma}\langle\mathcal{A}\rangle^\sigma\chi(P)\langle\mathcal{A}\rangle^{-\sigma}\end{aligned}$$

which is bounded by assumption (A) for $k=0$, Lemma 1.6.4 and the fact that $P_\tau\chi(P) = \varphi(P)$ with $\varphi \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$ by functional calculus. Next, [Ge08, Lemma 2.1] implies that $(P+\mathrm{i})^{-1}$ preserves $\mathscr{D}(\mathcal{A})$, so we can write

$$\langle\mathcal{A}\rangle^\sigma(P_\tau - f(z,hB))\chi(P)(P+\mathrm{i})^{-1}\langle\mathcal{A}\rangle^{-\sigma} = \langle\mathcal{A}\rangle^\sigma(P_\tau - f(z,hB))\chi(P)\langle\mathcal{A}\rangle^{-\sigma}\langle\mathcal{A}\rangle^\sigma(P+\mathrm{i})^{-1}\langle\mathcal{A}\rangle^{-\sigma}$$

which is clearly bounded thanks to the above computation. Finally,

$$\begin{aligned}\langle\mathcal{A}\rangle^{\sigma}(P-f(z,hB))(P+\mathrm{i})^{-1}\langle\mathcal{A}\rangle^{-\sigma} &= \langle\mathcal{A}\rangle^{\sigma}(P+\mathrm{i}-\mathrm{i}-z+z-f(z,hB))(P+\mathrm{i})^{-1}\langle\mathcal{A}\rangle^{-\sigma}\\ &= \mathbb{1}-(\mathrm{i}+z)\langle\mathcal{A}\rangle^{\sigma}(P+\mathrm{i})^{-1}\langle\mathcal{A}\rangle^{-\sigma}\\ &\quad+\langle\mathcal{A}\rangle^{\sigma}(z-f(z,hB))\langle\mathcal{A}\rangle^{-\sigma}\langle\mathcal{A}\rangle^{\sigma}(P+\mathrm{i})^{-1}\langle\mathcal{A}\rangle^{-\sigma}\end{aligned}$$

and we again use [Ge08, Lemma 2.1] and assumption (A) for $k=0$. □

The next result is an adaptation of [Ge08, Lemma 3.1] to our setting.

Lemma 1.6.6. *Let $0<\sigma\leq 1$ and let $\tau,\chi\in\mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R},[0,1])$ such that $\chi\equiv 1$ on I and $\tau\chi=\chi$. Consider the following three statements:*

(i) $$\sup_{z\in J^{+}}\|\langle\mathcal{A}\rangle^{-\sigma}R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}\|\lesssim h^{-1};$$

(ii) For all $z\in J^{+}$ and all $u\in(P+\mathrm{i})^{-1}\mathscr{D}(\langle\mathcal{A}\rangle^{\sigma})$,

$$\|\langle\mathcal{A}\rangle^{-\sigma}u\|\lesssim h^{-1}\|(P-f(z,hB))u\|+h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}(P-f(z,hB))\chi(P)u\|;$$

(iii) For all $z\in J^{+}$ and all $u\in\mathscr{D}(\langle\mathcal{A}\rangle^{\sigma})$,

$$\|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\|\lesssim h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}(P_{\tau}-f(z,hB))\chi(P)u\|.$$

If h is sufficiently small, then (iii) implies (ii) and (ii) implies (i).

Proof. First of all, observe that (i) makes sense by Lemma (1.6.4), and (ii), (iii) make sense by Corollary 1.6.5 and because $P\chi(P)=P_{\tau}\chi(P)$.

- We show that (ii) implies (i). Let $u\in\mathcal{H}$ and let $v:=R(z,hB)\langle A\rangle^{-\sigma}u$. Then

  $$w:=u-\langle\mathcal{A}\rangle^{\sigma}(f(z,hB)-\mathrm{i})R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}u\in\mathcal{H}.$$

  This makes sense if h is small enough because $R(z,hB)$ preserves $\mathscr{D}(\langle\mathcal{A}\rangle^{\sigma})$ by Lemma 1.6.4 and because

  $$\langle\mathcal{A}\rangle^{\sigma}(f(z,hB)-\mathrm{i})\langle\mathcal{A}\rangle^{-\sigma}=\langle\mathcal{A}\rangle^{\sigma}(f(z,hB)-z)\langle\mathcal{A}\rangle^{-\sigma}+(z-\mathrm{i})$$

  is bounded by assumption (A) for $k=0$. Next, using the resolvent identity $(P+\mathrm{i})^{-1}-R(z,hB)=(P+\mathrm{i})^{-1}(f(z,hB)-\mathrm{i})R(z,hB)$, we see that

  $$\begin{aligned}(P+\mathrm{i})^{-1}\langle A\rangle^{-\sigma}w &= \big((P+\mathrm{i})^{-1}-(P+\mathrm{i})^{-1}(f(z,hB)-\mathrm{i})\,R(z,hB)\big)\langle\mathcal{A}\rangle^{-\sigma}u\\ &= R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}u\\ &= v\end{aligned}$$

  so that $v\in(P+\mathrm{i})^{-1}\mathscr{D}(\langle\mathcal{A}\rangle^{\sigma})$. Hence, applying (ii) to v yields

  $$\begin{aligned}\|\langle\mathcal{A}\rangle^{-\sigma}R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}u\| &= \|\langle\mathcal{A}\rangle^{-\sigma}v\|\\ &\lesssim h^{-1}\|\langle\mathcal{A}\rangle^{-\sigma}u\|+h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}(P-f(z,hB))\chi(P)R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}u\|\\ &\lesssim h^{-1}\|\langle\mathcal{A}\rangle^{-\sigma}u\|+h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}[P-f(z,hB),\chi(P)]R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}u\|\\ &\quad+h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}\chi(P)\langle\mathcal{A}\rangle^{-\sigma}u\|.\end{aligned}$$

By assumption (A) for $k = 1$ and Lemma 1.6.4, we have

$$\begin{aligned}
&\|\langle\mathcal{A}\rangle^{\sigma}[P - f(z,hB),\chi(P)]R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}u\| \\
&= \|\langle\mathcal{A}\rangle^{\sigma}[z - f(z,hB),\chi(P)]R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}u\| \\
&\leq \delta(h\|B\|_{P})\|\langle\mathcal{A}\rangle^{\sigma}[\varepsilon(hB),\chi(P)]\langle\mathcal{A}\rangle^{-\sigma}\|\|\langle\mathcal{A}\rangle^{\sigma}R(z,hB)\langle\mathcal{A}\rangle^{-\sigma}u\| \\
&\lesssim h\delta(h\|B\|_{P}).
\end{aligned}$$

Therefore, (i) follows from (ii) if h is small enough.

- We show that (iii) implies (ii). Let $\tilde{\chi} := 1 - \chi$ and let $u \in (P+\mathrm{i})^{-1}\mathscr{D}(\langle\mathcal{A}\rangle^{\sigma})$. We write

$$\|\langle\mathcal{A}\rangle^{-\sigma}u\| \leq \|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\| + \|\langle\mathcal{A}\rangle^{-\sigma}\tilde{\chi}(P)u\| \tag{1.59}$$

and (iii) implies that

$$\|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\| \lesssim h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}(P - f(z,hB))\chi(P)u\|$$

because $\tau \equiv 1$ on $\operatorname{Supp}\chi$. In order to control the term involving $\tilde{\chi}(P)$ in (1.59), we write $\tilde{\chi} = \psi_{-}+\psi_{+}$ with $\psi_{\pm} \in \mathcal{C}^{\infty}(\mathbb{R},[0,1])$ such that $\operatorname{Supp}\psi_{-} \subset]-\infty,\alpha]$ and $\operatorname{Supp}\psi_{+} \subset [\beta,+\infty[$. We also pick $\rho \in \mathcal{C}^{\infty}_{\mathrm{c}}(\mathbb{R},\mathbb{R})$ such that $\rho\psi_{-} = \psi_{-}$. Since $B \in L^{\infty}_{\ell\mathrm{oc}}(P)$, we have for any $v \in \mathscr{D}(P)$

$$\begin{aligned}
&\Re\big\langle\psi_{-}(P)^2(f(z,hB) - P)v, v\big\rangle \\
&= \Re\big\langle\psi_{-}(P)^2 zv, v\big\rangle + \Re\big\langle\psi_{-}(P)^2\delta(h\|B\|_{P})\varepsilon(hB)v, v\big\rangle - \Re\big\langle\psi_{-}(P)^2 Pv, v\big\rangle \\
&\geq a\|\psi_{-}(P)v\|^2 - \delta(h\|B\|_{P})\|\rho(P)\varepsilon(hB)\|_{P}\|\psi_{-}(P)v\|^2 - \alpha\|\psi_{-}(P)^2 v\|^2 \\
&\geq c_{-}\|\psi_{-}(P)v\|^2
\end{aligned} \tag{1.60}$$

where $c_{-} > 0$ if h is sufficiently small. Using Cauchy-Schwarz inequality, we get $\|\psi_{-}(P)(P - f(z,hB))v\| \geq c_{-}\|\psi_{-}(P)v\|$ and thus $\|\psi_{-}(P)R(z,hB)v\| \lesssim \|\psi_{-}(P)v\|$. Similarly, one can show $\|\psi_{+}(P)R(z,hB)v\| \lesssim \|\psi_{+}(P)v\|$. These inequalities and $\tilde{\chi}^2 = (\psi_{-}+\psi_{+})^2 = \psi_{-}^2 + \psi_{+}^2$ then imply

$$\|\tilde{\chi}(P)R(z,hB)v\| \lesssim \|\tilde{\chi}(P)v\|$$

which in turn implies for $u \in \mathscr{D}(P)$

$$\begin{aligned}
\|\langle\mathcal{A}\rangle^{-\sigma}\tilde{\chi}(P)u\| &\lesssim \|\tilde{\chi}(P)u\| \\
&= \|\tilde{\chi}(P)R(z,hB)(P - f(z,hB))u\| \\
&\lesssim \|\tilde{\chi}(P)(P - f(z,hB))u\| \\
&\lesssim \|(P - f(z,hB))u\|.
\end{aligned}$$

□

Proof of Theorem 1.6.3. We show that the regularity (P) and the Mourre estimate (M) are enough to establish (1.58). As pointed out at the beginning of [Ge08], the key point is the following energy estimate: for any self-adjoint operators H acting on $\mathcal{H}$, $u \in \mathscr{D}(H)$, $\tau \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}, [0,1])$ and $P_\tau := \tau(P)P$, we have

$$2\Im\langle Hu, (P_\tau - f(z,hB))u\rangle = \langle u, [P_\tau, \mathrm{i}H]u\rangle - 2\Im\langle u, f(z,hB)Hu\rangle \tag{1.61}$$

where the commutator must be understood as a quadratic form on $\mathscr{D}(H)$.

We follow the proof of [Ge08, Theorem 1]. Let $\tau, \chi \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}, [0,1])$ such that $\chi \equiv 1$ on I and $\tau\chi = \chi$ and let

$$F(\xi) := -\int_{\xi}^{+\infty} g(\zeta)^2 \mathrm{d}\zeta$$

with $g \in \mathcal{C}^{\infty}(\mathbb{R}, [0,1])$ satisfying $g(\xi) = 0$ for $\xi \geq 2$ and $g(\xi) = 1$ for $\xi \leq 1$. By Lemma 1.6.6, it is sufficient to prove the following estimate: for any $z \in J^+$ and $u \in \mathscr{D}(\langle\mathcal{A}\rangle^{\sigma})$,

$$\|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\| \lesssim h^{-1}\|\langle\mathcal{A}\rangle^{\sigma}(P_\tau - f(z,hB))\chi(P)u\|.$$

As $P \in \mathcal{C}^2(\mathcal{A})$, P and $\mathcal{A}$ are self-adjoint and satisfy the Mourre estimate (M) on I, we can apply the estimate (3.30) in the proof of [Ge08, Theorem 1]:

$$\chi(P)[P_\tau, \mathrm{i}F(\mathcal{A})]\chi(P) \gtrsim h\chi(P)\langle\mathcal{A}\rangle^{-2\sigma}\chi(P). \tag{1.62}$$

Now we apply the identity (1.61) with $H = F(\mathcal{A})$: for all $u \in \mathscr{D}(\mathcal{A})$,

$$2\Im\langle F(\mathcal{A})u, (P_\tau - f(z,hB))u\rangle = \langle u, [P_\tau, \mathrm{i}F(\mathcal{A})]u\rangle + 2\Im\langle f(z,hB)u, F(\mathcal{A})u\rangle.$$

Since $F < 0$ is bounded and $\Im z > 0$, we can write for all h sufficiently small

$$\begin{aligned}
&2\Im\langle F(\mathcal{A})u, (P_\tau - f(z,hB))u\rangle \\
&= \langle u, [P_\tau, \mathrm{i}F(\mathcal{A})]u\rangle - 2(\Im z)\langle u, F(\mathcal{A})u\rangle - 2\delta(h\|B\|_{P})\Im\langle u, \varepsilon(hB)F(\mathcal{A})u\rangle \\
&> \langle u, [P_\tau, \mathrm{i}F(\mathcal{A})]u\rangle - 2\delta(h\|B\|_{P})\|\varepsilon(hB)u\|\|F(\mathcal{A})u\|
\end{aligned}$$

where we used that that $\varepsilon(hB) \in \mathcal{B}(\mathscr{D}(\mathcal{A}))$ by Assumption (A). It thus follows

$$2\Im\langle F(\mathcal{A})u, (P_\tau - f(z,hB))u\rangle \geq \langle u, [P_\tau, \mathrm{i}F(\mathcal{A})]u\rangle. \tag{1.63}$$

Plugging the estimate (1.62) into inequality (1.63) and putting $\chi(P)u$ instead of u yield

$$\begin{aligned}
\|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\|^2 &= \langle u, \chi(P)\langle\mathcal{A}\rangle^{-2\sigma}\chi(P)u\rangle \\
&\lesssim h^{-1}\langle u, \chi(P)[P_\tau, \mathrm{i}F(\mathcal{A})]\chi(P)u\rangle \\
&\leq h^{-1}\big|\langle F(\mathcal{A})\chi(P)u, (P_\tau - f(z,hB))\chi(P)u\rangle\big|.
\end{aligned}$$

Using again the boundedness of F, we get

$$\|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\|^2 \lesssim h^{-1}\|\langle\mathcal{A}\rangle^{-\sigma}\chi(P)u\|\|\langle\mathcal{A}\rangle^{\sigma}(P_\tau - f(z,hB))\chi(P)u\|$$

which establishes the point (iii) and thus the point (i) in Lemma 1.6.6.

Chapter 2

Scattering Theory for the Charged Klein-Gordon Equation in the Exterior De Sitter-Reissner-Nordström Spacetime

This chapter deals with scattering results associated to the charged Klein-Gordon equation (4.37). The mass term $mF(r)$ vanishes at the horizons but not the electrostatic potential $sV(r) \to sr_{\pm}^{-1} \neq 0$. This means that the comparison dynamics have to contain the charge term too. It turns out that scattering can not be interpreted as a transport along principal null geodesics as it can be in the (De Sitter-)Kerr spacetime, as the metric is independent of the charge of the field. This is just a consequence of the fact that the Klein-Gordon field does not appear in Einstein equation that yields the DSRN spacetime $(\mathcal{M}, g)$.

This motivates us to use an extended spacetime which encodes the charge and the mass of the field. There are *a priori* many ways to proceed. The one we choose, yet very simple, leads to a rich extension of the original manifold in which all the information carried on by the scalar field is caught. The extended spacetime, which is nothing but a Kaluza-Klein extension of the original one, verifies a non-trivial Einstein equation in $1+4$ dimension and provides a natural interpretation of the asymptotic completeness. We stress out here that this interpretation is *false* in the original spacetime as the principal null geodesics used in the extension do not exist there.

Plan of the chapter. Chapter 2 is organized as follows. In Section 2.1, we construct and present some important properties of the Kaluza-Klein extension of the DSRN spacetime. In Section 2.2, we build the Killing horizons with the principal null geodesics as well as the conformal boundary at infinity of the extension. Section 2.3 is devoted to the statement of the analytic scattering results whose proofs are given in Section 2.4. Finally, in Section 2.5, we interpret the wave operators as transports along the principal null geodesics. Asymptotic completeness is used to define traces on the horizons and solve an abstract Goursat problem.

2.1 The extended spacetime

This Section introduces the basic notions and objects used throughout this chapter. Subsection 2.1.1 presents the neutralization procedure which extends the original spacetime using the charged Klein-Gordon operator. We next show in Subsection 2.1.2 how the neutralization affects Einstein-Maxwell equations. We then deduce in Subsection 2.1.3 a dominant energy condition that fulfill the energy-momentum tensor associated to $(\widetilde{\mathcal{M}}, \widetilde{g})$ under some assumption on the cosmological constant Λ.

2.1.1 The neutralization procedure

In this chapter, we will use a slightly different definition of the horizon function: we let

$$F(r) := 1 - \frac{2M}{r} + \frac{Q^2}{2r^2} - \frac{\Lambda r^2}{3}.$$

with $M > 0$ the mass of the black hole, $Q \in \mathbb{R} \setminus \{0\}$ its electric charge and $\Lambda > 0$ the cosmological constant. We then assume (1.1) with $Q^2/2$ instead of Q^2 (so that now $\Delta = 9M^2 - 4Q^2$) which ensures that F has four distinct zeros $-\infty < r_n < 0 < r_c < r_- < r_+ < +\infty$ and is positive for all $r \in]r_-, r_+[$. We also assume that $9\Lambda M^2 < 1$. With these conventions, $(g, A = (Q/r)\mathrm{d}t)$ solves the Einstein-Maxwell field equation

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = -T_{\mu\nu}, \qquad T_{\mu\nu} = \mathfrak{F}_{\mu\sigma}\mathfrak{F}_{\nu}{}^{\sigma} - \frac{1}{4}g_{\mu\nu}\mathfrak{F}^{\sigma\rho}\mathfrak{F}_{\sigma\rho} \tag{2.1}$$

where Ric is the Ricci tensor, R the scalar curvature and $\mathfrak{F} = \mathrm{d}A$ the electromagnetic tensor.

We now introduce a fifth dimension labeled $z \in \mathbb{S}^1$ in order to reinterpret the charged Klein-Gordon operator in the De Sitter-Reissner-Nordström spacetime as a wave operator in a $1+4$ spacetime. Define

$$L := \frac{1}{F(r)}(\partial_t - sV(r)\partial_z)^2 - \frac{1}{r^2}\partial_r r^2 F(r)\partial_r - \frac{1}{r^2}\Delta_{\mathbb{S}^2} - m^2\partial_z^2.$$

Diagonalizing $-\mathrm{i}\partial_z$ on the unit circle, we can recover P by restriction to the harmonic $-\mathrm{i}\partial_z = 1$. We construct a new metric $\widetilde{g}$ such that $\Box_{\widetilde{g}} = L$: the *extended metric* in the *extended Boyer-Lindquist coordinates* (t, z, r, ω) with signature $(+,-,-,-,-)$ is defined as

$$\widetilde{g} := \left(F(r) - \frac{s^2V(r)^2}{m^2}\right)\mathrm{d}t^2 - \frac{sV(r)}{m^2}(\mathrm{d}t\mathrm{d}z + \mathrm{d}z\mathrm{d}t) - \frac{1}{m^2}\mathrm{d}z^2 - \frac{1}{F(r)}\mathrm{d}r^2 - r^2\mathrm{d}\omega^2.$$

It is non-degenerate since the determinant is equal to $-r^4\sin^2\theta/m^2 < 0$ (notice here the crucial hypothesis $m \neq 0$). The inverse extended metric is given by

$$\begin{aligned}\widetilde{g}^{-1} = {} & \frac{1}{F(r)}\partial_t \otimes \partial_t - \frac{sV(r)}{F(r)}(\partial_t \otimes \partial_z + \partial_z \otimes \partial_t) + \left(\frac{s^2V(r)^2}{F(r)} - m^2\right)\partial_z \otimes \partial_z \\ & - F(r)\partial_r \otimes \partial_r - \frac{1}{r^2}\partial_\theta \otimes \partial_\theta - \frac{1}{r^2\sin^2\theta}\partial_\varphi \otimes \partial_\varphi.\end{aligned}$$

Let us define the four blocks

$$\begin{aligned}\widetilde{\mathcal{M}}_1 &:= \mathbb{R}_t \times \mathbb{S}^1_z \times]0, r_c[_r \times \mathbb{S}^2_\omega, & \widetilde{\mathcal{M}}_2 &:= \mathbb{R}_t \times \mathbb{S}^1_z \times]r_c, r_-[_r \times \mathbb{S}^2_\omega, \\ \widetilde{\mathcal{M}}_3 &:= \mathbb{R}_t \times \mathbb{S}^1_z \times]r_-, r_+[_r \times \mathbb{S}^2_\omega, & \widetilde{\mathcal{M}}_4 &:= \mathbb{R}_t \times \mathbb{S}^1_z \times]r_+, +\infty[_r \times \mathbb{S}^2_\omega.\end{aligned}$$

The *extended spacetime* is then the $(1+4)$-dimensional manifold $\left(\widetilde{\mathcal{M}}_{\text{ext}}, \widetilde{g}\right)$ with

$$\widetilde{\mathcal{M}}_{\text{ext}} := \bigcup_{j=1}^{4} \widetilde{\mathcal{M}}_j.$$

In the sequel, we will consider more carefully the *outer space*

$$\widetilde{\mathcal{M}} := \mathbb{R}_t \times \mathbb{S}^1_z \times]r_-, r_+[_r \times \mathbb{S}^2_\omega$$

which we will simply call the extended spacetime when no confusion can occur. In $\widetilde{\mathcal{M}}$, we will use the timelike vector field $\nabla t = F(r)^{-1}(\partial_t - sV(r)\partial_z)$ to define an orientation on $\widetilde{\mathcal{M}}$: any causal vector field $X \in T\widetilde{\mathcal{M}}$ will be said *future-pointing* if and only if

$$\widetilde{g}(\nabla t, X) > 0.$$

We can easily check that the outer space is a globally hyperbolic spacetime. Besides, we can check that $X_\alpha := \partial_t - sV_\alpha \partial_z$ are timelike Killing vector fields for $\alpha \in \{c, -, +\}$ (they will be useful in Subsection 2.2.2).

Remark 2.1.1 (Dyadorings)**.** *For s small enough, the shifted horizon function*

$$\mathbf{F}(r) := F(r) - \frac{s^2V^2}{m^2} = 1 - \frac{2M}{r} + \frac{Q^2}{2r^2}\left(1 - \frac{2q^2}{m^2}\right) - \frac{\Lambda r^2}{3}$$

has four roots with two[1] *inside $]r_-, r_+[$; call them r_1, r_2 with $r_- < r_1 < r_2 < r_+$. We can check that ∂_t becomes spacelike when $r \in]r_-, r_1[\cup]r_2, r_+[$. We then define the* dyadorings

$$\mathcal{D}_- := \mathbb{R}_t \times \mathbb{S}^1_z \times]r_-, r_1[_r \times \mathbb{S}^2_\omega, \qquad \mathcal{D}_+ := \mathbb{R}_t \times \mathbb{S}^1_z \times]r_2, r_+[_r \times \mathbb{S}^2_\omega.$$

We may observe that if s is too large, namely if

$$|s| \geq mrF(r)^{1/2} \qquad \forall r \in]r_-, r_+[,$$

then the dyadorings cover the entire outer space $\widetilde{\mathcal{M}}$.

The wave equation on $(\widetilde{\mathcal{M}}, \widetilde{g})$ reads

$$\Box_{\widetilde{g}} u = \frac{1}{F(r)}\left(\left(\partial_t - sV(r)\partial_z\right)^2 - \frac{F(r)}{r^2}\partial_r r^2 F(r)\partial_r - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} - m^2F(r)\partial_z^2\right)u = 0$$

with $u \in L^2(\mathbb{R}_t \times \mathbb{S}^1_z \times]r_-, r_+[_r \times \mathbb{S}^2_\omega, r^2F(r)^{-1}\mathrm{d}t\mathrm{d}z\mathrm{d}r\mathrm{d}\omega)$. It will be convenient for Section 2.3 to rewrite this equation as

$$\left(\partial_t^2 - 2sV(r)\partial_z\partial_t + \hat{P}\right)u = 0 \tag{2.2}$$

where

$$\hat{P} = -\frac{F(r)}{r^2}\partial_r r^2 F(r)\partial_r - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} - \left(m^2F(r) - s^2V(r)^2\right)\partial_z^2 \tag{2.3}$$

acts on $L^2(\mathbb{S}^1_z \times]r_-, r_+[_r \times \mathbb{S}^2_\omega, r^2F(r)^{-1}\mathrm{d}z\mathrm{d}r\mathrm{d}\omega)$. Restricting u to $\ker(\mathrm{i}\partial_z + \mathbf{z})$, we get back the original charge Klein-Gordon operator with the modified mass $m\mathbf{z}$ and charge $s\mathbf{z}$.

[1] It is a simple consequence of the intermediate value theorem since $\mathbf{F} < F$, $F > 0$ inside $]r_-, r_+[$ and F cancels at $r = r_\pm$.

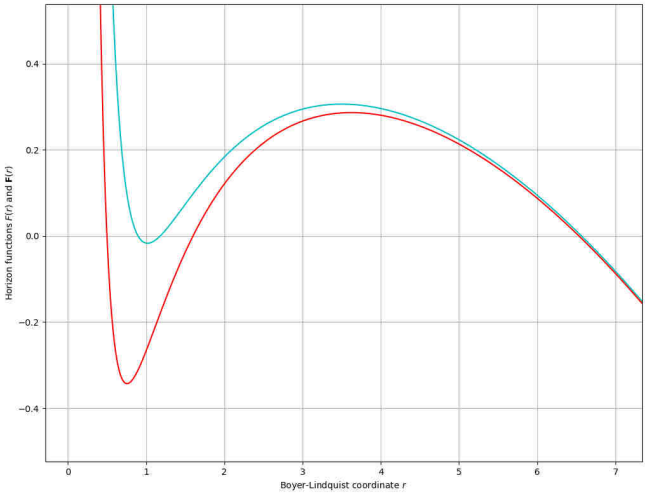

Figure 2.1: The horizon functions F (in cyan) and $\mathbf{F}$ (in red). Data used for the plot are $M = Q = 1$, $\Lambda = 0.05$, $m = 0.2$ and $s = 0.1$.

2.1.2 Extended Einstein-Maxwell equations

The neutralization procedure has modified the Einstein-Maxwell equations (2.1) as follows. Let $\widetilde{A} = \frac{Q}{r}\sqrt{1 - \frac{q^2}{2m^2}}\mathrm{d}t$. Tedious but direct computations show that $(\widetilde{g}, \widetilde{A})$ solves the Einstein field equations

$$\widetilde{\mathrm{Ric}} - \frac{1}{2}\widetilde{R}\widetilde{g} - \Lambda\widetilde{g} = -\widetilde{T} \tag{2.4}$$

with $\widetilde{\mathrm{Ric}}$ the Ricci tensor and $\widetilde{\mathrm{R}}$ the scalar curvature associated to $\widetilde{g}$. We refer to Appendix 2.6 for a complete list of the tensors involved in the computation of the right-hand side of (2.4). The extended stress-energy tensor $\widetilde{T}$ is given by $\widetilde{T} = \widetilde{T}_{\mathrm{Maxwell}} + \widetilde{T}_{\mathrm{fluid}}$ with in matrix notations

$$\widetilde{T}_{\mathrm{Maxwell}} = \frac{Q^2}{2r^4}\left(1 - \frac{q^2}{2m^2}\right)\begin{pmatrix} -F(r) - \frac{s^2V(r)^2}{m^2} & -\frac{sV(r)}{m^2} & 0 & 0 & 0 \\ -\frac{sV(r)}{m^2} & -\frac{1}{m^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{F(r)} & 0 & 0 \\ 0 & 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix},$$

$$\widetilde{T}_{\mathrm{fluid}} = \left(\Lambda + \frac{Q^2}{2r^4}\left(1 + \frac{q^2}{m^2}\right)\right)\begin{pmatrix} \frac{s^2V(r)^2}{m^2} & \frac{sV(r)}{m^2} & 0 & 0 & 0 \\ \frac{sV(r)}{m^2} & \frac{1}{m^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\widetilde{T}_{\text{Maxwell}}$ is nothing but the Maxwell electromagnetic tensor associated to $\widetilde{g}$ and $\widetilde{A}$; $\widetilde{T}_{\text{fluid}}$ describes as for itself a perfect fluid:

$$\widetilde{T}_{\text{fluid}} = \rho(r) u \otimes u$$

with $\rho(r) = \Lambda + \frac{Q^2}{2r^4}\left(1 + \frac{q^2}{m^2}\right)$ the energy density in the fluid and $u = \frac{1}{m}\big(sV(r)\mathrm{d}t + \mathrm{d}z\big)$ the dual vector field of the fluid's velocity which is given by

$$v^\mu = \widetilde{g}^{\mu\nu} u_\nu = -m\partial_z = \frac{1}{m}\big(sV(r)\nabla t + \nabla z\big). \tag{2.5}$$

Observe that the energy density of the fluid as measured by an observer at rest is zero:

$$(\widetilde{T}_{\text{fluid}})_{\mu\nu}\nabla^\mu t \nabla^\nu t = 0.$$

Observe also that taking the divergence on both sides of (2.4) yields

$$\mathrm{div}(\rho(r) u \otimes u) = \frac{\rho(r)F(r)W(r)W'(r)}{2m^2}\partial_r =: -\nabla P$$

with

$$P(r) = -\frac{q^2Q^2}{4m^2r^2}\left(\Lambda + \frac{Q^2(1+\frac{q^2}{m^2})}{6r^4}\right) \leq 0.$$

Moreover, we can see that

$$\mathrm{div}(\rho u) = 0.$$

Thus u obeys compressible Euler law for a static fluid with *mass density* ρ, *pressure* P and no internal source term:

$$\begin{cases} \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) &= -\nabla P \\ \partial_t \rho + \mathrm{div}(\rho u) &= 0 \end{cases}.$$

The second equation above is the conservation of the mass.

2.1.3 Dominant energy condition

In all this Subsection, we will assume that $|q| < 2m$. Let us rewrite

$$-\widetilde{T} = \mathfrak{Q}(r)\Big(F(r)\mathrm{d}t^2 - F(r)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\omega^2\Big) - \mathfrak{D}(r)\big(sV(r)\mathrm{d}t + \mathrm{d}z\big)^2$$

where

$$\mathfrak{Q}(r) = \frac{Q^2}{2r^4}\left(1 - \frac{q^2}{2m^2}\right), \qquad \mathfrak{D}(r) = \left(\frac{\Lambda}{m^2} + \frac{3q^2Q^2}{4m^4r^4}\right).$$

Consider then the following condition:

$$m^2\mathfrak{D}(r) \leq \mathfrak{Q}(r) \tag{2.6}$$

which is equivalent to $\Lambda \leq \frac{Q^2}{2r_+^4}\left(1 - \frac{2q^2}{m^2}\right)$.

Lemma 2.1.2. *The condition (2.6) implies that for all timelike vector field X, we have*

$$-\widetilde{T}_{\mu\nu}X^\mu X^\nu \geq 0. \tag{2.7}$$

If $Q \neq 0$, then both the above conditions are equivalent.

Proof. Let $X = X^t\partial_t + X^z\partial_z + X^r\partial_r + X^\theta\partial_\theta + X^\varphi\partial_\varphi$ be a timelike vector field:

$$\left(F(r) - \frac{s^2V(r)^2}{m^2}\right)(X^t)^2 - 2\frac{sV(r)}{m^2}X^tX^z - \frac{1}{m^2}(X^z)^2 - F(r)^{-1}(X^r)^2 - r^2(X^\theta)^2 - r^2\sin^2\theta(X^\varphi)^2 > 0.$$

Assume first (2.6). Then

$$\begin{aligned}-\widetilde{T}_{\mu\nu}X^\mu X^\nu &= \mathfrak{Q}(r)\left(\widetilde{g}(X,X) + \left(\frac{1}{m^2} - \frac{\mathfrak{D}(r)}{\mathfrak{Q}(r)}\right)\left(s^2V(r)^2(X^t)^2 + 2sV(r)X^tX^z + (X^z)^2\right)\right.\\ &\qquad\left. + 2r^2(X^\theta)^2 + 2r^2\sin^2\theta(X^\varphi)^2\right)\\ &\geq \mathfrak{Q}(r)\left(\frac{1}{m^2} - \frac{\mathfrak{D}(r)}{\mathfrak{Q}(r)}\right)\left(s^2V(r)^2(X^t)^2 + 2sV(r)X^tX^z + (X^z)^2\right).\end{aligned}$$

Furthermore,

$$\left(\frac{1}{m^2} - \frac{\mathfrak{D}(r)}{\mathfrak{Q}(r)}\right)\begin{pmatrix}s^2V(r)^2 & sV(r)\\ sV(r) & 1\end{pmatrix} \geq 0$$

as quadratic form since the spectrum of the matrix on the left-hand side is $\{0, 1 + s^2V(r)^2\}$ (the eigenvalue 0 is associated to ∇t). It follows that $-\widetilde{T}_{\mu\nu}X^\mu X^\nu \geq 0$.

Assume now that (2.6) is not verified and $Q \neq 0$ (so that $\mathfrak{Q}(r) \neq 0$). Put

$$\alpha(r) := \frac{1}{m^2} - \frac{\mathfrak{D}(r)}{\mathfrak{Q}(r)} < 0.$$

Let $\delta, \varepsilon \in \mathcal{C}^\infty(]r_-, r_+[,]0, +\infty[)$ with ε supported far away from the dyadorings and set $X := \delta(r)\big(\partial_t - sV(r)(1+\varepsilon(r))\partial_z\big)$. Taking ε sufficiently small, we get

$$\widetilde{g}(X,X) = \delta(r)\left(F(r) - \frac{s^2V(r)^2\varepsilon(r)^2}{m^2}\right) > 0.$$

Taking now δ large enough, we find

$$\begin{aligned}-\widetilde{T}_{\mu\nu}X^\mu X^\nu &= \mathfrak{Q}(r)\Big(\widetilde{g}(X,X) + \alpha(r)\varepsilon(r)^2\delta(r)^2s^2V(r)^2\Big)\\ &= \mathfrak{Q}(r)\delta(r)\left(F(r) - \frac{s^2V(r)^2\varepsilon(r)^2}{m^2} + \alpha(r)\varepsilon(r)^2\delta(r)s^2V(r)^2\right)\\ &< 0\end{aligned}$$

for $r \in \operatorname{Supp}\varepsilon$. This completes the proof. □

Proposition 2.1.3 (Dominant energy condition)**.** *Assume $Q \neq 0$. Then the condition (2.6) is verified if and only if*

$$\begin{cases} -\widetilde{T}_{\mu\nu}X^\mu X^\nu \geq 0 & \text{for all timelike vector field } X, \\ \text{If } X \text{ is a future-pointing causal vector field, then so is for } -\widetilde{T}^\mu{}_\nu X^\nu \geq 0 \end{cases}. \tag{2.8}$$

Proof. Lemma 2.1.2 shows that (2.6) is equivalent to the first condition in (2.8) provided that $Q \neq 0$. Next, a direct computation shows that

$$\begin{aligned} -\widetilde{T}^\mu{}_\nu = -\widetilde{g}^{\mu\sigma}\widetilde{T}_{\sigma\nu} = \mathfrak{Q}(r)\partial_t \otimes \mathrm{d}t + \big(m^2\mathfrak{D}(r) - \mathfrak{Q}(r)\big)sV(r)\partial_z \otimes \mathrm{d}t + m^2\mathfrak{D}(r)\partial_z \otimes \mathrm{d}z \\ + \mathfrak{Q}(r)\partial_r \otimes \mathrm{d}r - \mathfrak{Q}(r)\partial_\theta \otimes \mathrm{d}\theta - \mathfrak{Q}(r)\partial_\varphi \otimes \mathrm{d}\varphi. \end{aligned}$$

Let $X = X^t\partial_t + X^z\partial_z + X^r\partial_r + X^\theta\partial_\theta + X^\varphi\partial_\varphi$ be a future-pointing causal vector field:

$$\widetilde{g}(\nabla t, X) = X^t > 0, \qquad \widetilde{g}(X, X) \geq 0.$$

We compute:

$$\begin{aligned} -\widetilde{T}^\mu{}_\nu X^\nu = \mathfrak{Q}(r)X^t\partial_t + \big((m^2\mathfrak{D}(r) - \mathfrak{Q}(r))sV(r)X^t + m^2\mathfrak{D}(r)X^z\big)\partial_z \\ + \mathfrak{Q}(r)X^r\partial_r - \mathfrak{Q}(r)X^\theta\partial_\theta - \mathfrak{Q}(r)X^\varphi\partial_\varphi. \end{aligned}$$

Then the vector field $-\widetilde{T}^\mu{}_\nu X^\nu$ is future-pointing if and only if

$$\widetilde{g}(\nabla_t, -\widetilde{T}^\mu{}_\nu X^\nu) = -\widetilde{T}^t_\nu X^\nu = \mathfrak{Q}(r)X^t > 0$$

which is always true when $Q \neq 0$. Furthermore:

$$\begin{aligned} \mathfrak{Q}(r)^{-2}\widetilde{g}(\widetilde{T}^\mu{}_\nu X^\nu, \widetilde{T}^\mu{}_\nu X^\nu) &= \left(F(r) - \frac{s^2V(r)^2}{m^2}\left(\frac{m^2\mathfrak{D}(r)}{\mathfrak{Q}(r)}\right)^2\right)(X^t)^2 - 2\frac{sV(r)}{m^2}\left(\frac{m^2\mathfrak{D}(r)}{\mathfrak{Q}(r)}\right)^2 X^tX^z \\ &\quad - \frac{1}{m^2}\left(\frac{m^2\mathfrak{D}(r)}{\mathfrak{Q}(r)}\right)^2(X^z)^2 - F(r)^{-1}(X^r)^2 \\ &= \tilde{g}(X, X) \\ &\quad + \left(1 - \left(\frac{m^2\mathfrak{D}(r)}{\mathfrak{Q}(r)}\right)^2\right)\left(\frac{s^2V(r)^2}{m2}(X^t)^2 + 2\frac{sV(r)}{m^2}X^tX^z + \frac{1}{m^2}(X^z)^2\right). \end{aligned}$$

Using $\widetilde{g}(X, X) \geq 0$, we get:

$$\mathfrak{Q}(r)^{-2}\widetilde{g}(\widetilde{T}^\mu{}_\nu X^\nu, \widetilde{T}^\mu{}_\nu X^\nu) \geq \left(1 - \left(\frac{m^2\mathfrak{D}(r)}{\mathfrak{Q}(r)}\right)^2\right)\left(\frac{s^2V(r)^2}{m2}(X^t)^2 + 2\frac{sV(r)}{m^2}X^tX^z + \frac{1}{m^2}(X^z)^2\right)$$

and this quantity is nonnegative if and only if $m^2\mathfrak{D}(r) \leq \mathfrak{Q}(r)$ as in the proof of Lemma 2.1.2 (the same vector field as therein shows the necessity of the condition). This completes the proof. □

Remark 2.1.4. *The condition (2.6) is called the* dominant energy condition. *It will allow us to define constant surface gravities in Subsection 2.2.2. In the De Sitter-Reissner-Nordström case,*

Proposition 2.1.3 is always true. Indeed, denoting by T the corresponding stress-energy tensor as well as $\mathfrak{Q}(r) = \frac{Q^2}{2r^4}$, we have

$$\begin{aligned} T_{\mu\nu}X^\mu X^\nu &= \mathfrak{Q}(r)\left(F(r)(X^t)^2 - F(r)^{-1}(X^r)^2 + r^2(X^\theta)^2 + r^2\sin^2\theta(X^\varphi)^2\right), \\ \widetilde{g}(\partial_t, T^\mu{}_\nu X^\nu) &= \mathfrak{Q}(r)X^t, \\ \widetilde{g}(T^\mu{}_\nu X^\nu, T^\mu{}_\nu X^\nu) &= \mathfrak{Q}(r)X^t g(X,X) \end{aligned}$$

for any vector field X.

Remark 2.1.5. *We may notice that condition (1.1) implies a lower bound to Λ if $|Q| \geq M$. In this situation and for a small charge q, the dominant energy condition (2.6) is not satisfied.*

2.2 Global geometry of the extended spacetime

This Section is devoted to the study of the geometry of the extended spacetime. The goal is two-fold: first, the study of the geometric objects is interesting in its own right as it describes in details a new solution to Einstein-Maxwell equations in 5 dimensions. On the other hand, the construction of some geometric objects (such as principal null geodesics and horizons) are prerequisites to formulate the scattering theory developed in Section 2.3. We will henceforth assume the dominant energy condition (2.6).

Let us outline here the plan of this Section. Subsection 2.2.1 introduces first special null geodesics and local coordinates used to build horizons. In Subsection 2.2.2, we define surface gravities. In Subsection 2.2.3, we add crossing rings to complete the construction of the horizons. Finally, we define the conformal infinity in Subsection 2.2.4 and show that the extended spacetime contains black rings.

2.2.1 Principal null geodesics

We introduce in this Subsection a family of null geodesics which send data from the blocks $\widetilde{\mathcal{M}}_j$ to the horizons located at the roots of F. They will be used for the geometric interpretation of the scattering in Section 2.5. We follow the standard procedure (see for example the construction in [ON95, Section 2.5] in the Kerr spacetime).

Let $\gamma : \mathbb{R} \ni \mu \mapsto (t(\mu), z(\mu), r(\mu), \omega(\mu)) \in \widetilde{\mathcal{M}}$ be a null geodesic. Denoting by $\dot{}$ the derivative with respect to μ, we have

$$\left(F(r) - \frac{s^2V(r)^2}{m^2}\right)\dot{t}^2 - \frac{2sV(r)}{m^2}\dot{t}\dot{z} - \frac{1}{m^2}\dot{z}^2 - \frac{1}{F(r)}\dot{r}^2 = 0 \tag{2.9}$$

if $\dot{\omega} = r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi} = 0$. Since ∂_z is Killing[2], there exists a constant $\mathfrak{Z} \in \mathbb{R}$ such that

$$\widetilde{g}(\dot{\gamma}, \partial_z) = sV(r)\dot{t} + \dot{z} = -m^2\mathfrak{Z}. \tag{2.10}$$

Plugging (2.10) into (2.9) then yields

$$F(r)\dot{t}^2 - \frac{1}{F(r)}\dot{r}^2 = m^2\mathfrak{Z}^2. \tag{2.11}$$

[2]This is a reformulation of the cylinder condition in the Kaluza-Klein theory.

We build our geodesics so that $\mathfrak{Z} = 0$. This is not only a convenient choice that makes computations easier and explicit: the geometric interpretation is that

$$\widetilde{g}(\dot{\gamma}, \partial_z) = 0$$

that is, the principal null geodesics will be $\widetilde{g}$-orthogonal to the velocity vector field of the perfect fluid appearing in the stress-energy tensor after neutralization (see (2.5)). Solving (2.10) and (2.11) for $\mathfrak{Z} = 0$, we get

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \pm F(r)^{-1}, \qquad \frac{\mathrm{d}z}{\mathrm{d}r} = \mp sV(r)F(r)^{-1}.$$

In particular, we can parametrize our geodesic by $\mu = \pm r$.

The *incoming principal null geodesic* $\gamma_{\mathrm{in}}(\mu) = (t(\mu), z(\mu), r(\mu), \omega(\mu))$ is the null geodesic parametrized by $\mu = -r$ and defined by

$$\begin{cases} \dot{t}(\mu) &= F(r)^{-1} \\ \dot{z}(\mu) &= -sV(r)F(r)^{-1} \\ \dot{r}(\mu) &= -1 \\ \dot{\omega}(\mu) &= 0 \end{cases}.$$

We then define the *extended-star* coordinates[3] $(t^\star, z^\star, r, \omega)$ with

$$t^\star := t + T(r), \qquad T(r) := \int_{\mathfrak{r}}^{r} \frac{\mathrm{d}\rho}{F(\rho)} = \sum_{\alpha \in \{n,c,-,+\}} \frac{1}{2\kappa_\alpha} \ln \left| \frac{r - r_\alpha}{\mathfrak{r} - r_\alpha} \right|, \tag{2.12}$$

$$z^\star := z + Z(r), \qquad Z(r) := -\int_{\mathfrak{r}}^{r} \frac{sV(\rho)\mathrm{d}\rho}{F(\rho)} = -\sum_{\alpha \in \{n,c,-,+\}} \frac{s}{2r_\alpha\kappa_\alpha} \ln \left| \frac{\mathfrak{r}}{r} \frac{r - r_\alpha}{\mathfrak{r} - r_\alpha} \right| \tag{2.13}$$

for some $\mathfrak{r} \in \widetilde{\mathcal{M}}_{\mathrm{ext}}$. Here we have used the factorization

$$F(r) = \frac{\Lambda}{3r^2}(r - r_n)(r - r_c)(r - r_-)(r_+ - r) \tag{2.14}$$

as well as the *surface gravities* $\kappa_n := F'(r_n)/2 > 0$, $\kappa_c := F'(r_c)/2 < 0$, $\kappa_- := F'(r_-)/2 > 0$ and $\kappa_+ := F'(r_+)/2 < 0$ (we will give a more precise meaning of them in Subsection 2.2.2). The function T is the so-called *Regge-Wheeler* (or *tortoise*) coordinate, and will be denoted by x in Section 2.3 and Section 2.5. The expression of the extended metric in these coordinates is given by

$$\widetilde{g}_\star = \left(F(r) - \frac{s^2 V(r)^2}{m^2} \right) (\mathrm{d}t^\star)^2 - \frac{sV(r)}{m^2} (\mathrm{d}t^\star \mathrm{d}z^\star + \mathrm{d}z^\star \mathrm{d}t^\star) - \frac{1}{m^2} (\mathrm{d}z^\star)^2 - (\mathrm{d}t^\star \mathrm{d}r + \mathrm{d}r\mathrm{d}t^\star) - r^2 \mathrm{d}\omega^2$$

with inverse

$$\begin{aligned} (\widetilde{g}_\star)^{-1} = &-m^2 \partial_{z^\star}^{\otimes 2} - (\partial_{t^\star} \otimes \partial_r + \partial_r \otimes \partial_{t^\star}) + sV(r) (\partial_{z^\star} \otimes \partial_r + \partial_r \otimes \partial_{z^\star}) \\ &- F(r) \partial_r^{\otimes 2} - \frac{1}{r^2} \partial_\theta^{\otimes 2} - \frac{1}{r^2 \sin^2 \theta} \partial_\varphi^{\otimes 2}. \end{aligned}$$

[3] The map $(t, z, r, \omega) \mapsto (t^\star, z^\star, r, \omega)$ is one-to-one in each block $\widetilde{\mathcal{M}}_2$, $\widetilde{\mathcal{M}}_3$ and $\widetilde{\mathcal{M}}_4$ since T and Z are (their derivative are respectively F^{-1} and $-(sVF)^{-1}$ which have constant signs) and its Jacobian determinant is 1; it therefore defines a coordinates chart.

Observe that, by construction, we have $\dot{t}^\star = \dot{z}^\star = 0$: this shows that $t \to \pm\infty$ as $r \to (r_\pm)^\mp$ (but $t^\star$ remains smooth because $\dot{t}^\star = 0$). The same conclusion can be drawn for z.

We similarly define the *outgoing principal null geodesic* $\gamma_{\text{out}}(\mu) = (t(\mu), z(\mu), r(\mu), \omega(\mu))$ as the null geodesic parametrized by $\mu = r$ and defined by

$$\begin{cases} \dot{t}(\mu) &= F(r)^{-1} \\ \dot{z}(\mu) &= -sV(r)F(r)^{-1} \\ \dot{r}(\mu) &= 1 \\ \dot{\omega}(\mu) &= 0 \end{cases}.$$

We then define the *star-extended* coordinates $({}^\star t, {}^\star z, r, \omega)$ with

$${}^\star t := t - T(r), \qquad {}^\star z := z - Z(r)$$

with T and Z as in (2.12) and (2.13). The expression of the extended metric in these coordinates is given by

$${}_\star\widetilde{g} = \left(F(r) - \frac{s^2V(r)^2}{m^2}\right)(\mathrm{d}{}^\star t)^2 - \frac{sV(r)}{m^2}(\mathrm{d}{}^\star t\mathrm{d}{}^\star z + \mathrm{d}{}^\star z\mathrm{d}{}^\star t) - \frac{1}{m^2}(\mathrm{d}{}^\star z)^2 + (\mathrm{d}{}^\star t\mathrm{d}r + \mathrm{d}r\mathrm{d}{}^\star t) - r^2\mathrm{d}\omega^2$$

with inverse

$$\begin{aligned}({}_\star\widetilde{g})^{-1} = &-m^2\partial_{z^\star}^{\otimes 2} + (\partial_{t^\star}\otimes\partial_r + \partial_r\otimes\partial_{t^\star}) - sV(r)\,(\partial_{z^\star}\otimes\partial_r + \partial_r\otimes\partial_{z^\star}) \\ &- F(r)\partial_r^{\otimes 2} - \frac{1}{r^2}\partial_\theta^{\otimes 2} - \frac{1}{r^2\sin^2\theta}\partial_\varphi^{\otimes 2}.\end{aligned}$$

We have ${}^\star\dot{t} = {}^\star\dot{z} = 0$ which entails $t \to \mp\infty$ and $z \to \mp\infty$ as $r \to (r_\pm)^\mp$.

2.2.2 Surface gravities and Killing horizons

Surface gravities are accelerations felt in the incoming direction locally near a hypersurface by an unit test mass due to the gravitational force (measured infinitely far away from the hypersurface). We compute in this Subsection the surface gravities κ_α at $r = r_\alpha$ for all $\alpha \in \{c, -, +\}$.

Consider the following normalization of the velocity vector field ∇t associated to a static observer:

$$F(r)\nabla t = \partial_t - sV(r)\partial_z. \tag{2.15}$$

For De Sitter Kerr black holes, $\frac{\nabla t}{\sqrt{\nabla^i t\nabla_i t}}$ is the velocity vector field which follows the rotation of the black hole; it tends to $\partial_t - \frac{a}{r_\pm^2+a^2}\partial_\varphi$ as $r \to r_\pm$ which provides two Killing vector fields in the ergoregions near the horizons. The equivalent in our setting of the rotations at the horizons $\frac{a}{r_\pm^2+a^2}$ are the terms sV_α, $\alpha \in \{c, -, +\}$. We thus consider the Killing vector fields

$$X_c := \partial_{t^\star} - sV_c\partial_{z^\star}, \qquad X_- := \partial_{t^\star} - sV_-\partial_{z^\star}, \qquad X_+ := \partial_{t^\star} - sV_+\partial_{z^\star} \tag{2.16}$$

which are null at $r = r_\alpha$, $\alpha \in \{c, -, +\}$:

$$\widetilde{g}_\star(X_\alpha, X_\alpha) = F - \frac{s^2(V - V_\alpha)^2}{m^2}.$$

Remark 2.2.1. *We used the extended-starr coordinates to define surface gravities. It is perfectly fine doing it with the star-extended coordinates, the only difference being the opposite sign we have to put in formula (2.17) below.*

The physical interpretation of the surface gravities leads to the following Newtonian forces equilibrium[4]:

$$(\nabla_{X_\alpha} X_\alpha)\,|_{\{r=r_\alpha\}} = -\kappa_\alpha X_\alpha\,|_{\{r=r_\alpha\}} \qquad \kappa_\alpha \in \mathbb{R}. \tag{2.17}$$

The constants κ_α are well defined since the surface gravities are constant on $\{r = r_\alpha\}$ by the dominant energy condition (2.6) (see *e.g.* [Wa84, equation (12.5.31)]). It is easy to see that $\nabla^\mu\big((X_\alpha)^\nu(X_\alpha)_\nu\big) = -2\kappa_\alpha(X_\alpha)^\mu$ on the corresponding horizon (*cf.* [Wa84, equation (12.5.2)]). We then compute:

$$\nabla^\mu\big((X_\alpha)^\nu(X_\alpha)_\nu\big)\,|_{\{r=r_\alpha\}} = \left((\widetilde{g}_\star)^{\mu r}\partial_r\left(F - \frac{s^2(V-V_\alpha)^2}{m^2}\right)\right)\Big|_{\{r=r_\alpha\}}.$$

Since $(\widetilde{g}_\star)^{tr} = -1$ and $(\widetilde{g}_\star)^{zr} = sV(r)$ are the only non-zero coefficients of the form $(\widetilde{g}_\star)^{\mu r}$, we get in all cases:

$$\kappa_\alpha = \frac{F'(r_\alpha)}{2} = \frac{(3r_\alpha - 3M - 2\Lambda r_\alpha^2)}{3r_\alpha^2}.$$

Observe that κ_α is nothing but the De Sitter-Reissner-Nordström surface gravity at the corresponding horizon (we show in the next Subsection that $\{r = r_\alpha\}$ are still Killing horizons in the extended spacetime). Using the signs of F' at $r = r_\alpha$, we get:

$$\kappa_c < 0, \qquad \kappa_- > 0, \qquad \kappa_+ < 0.$$

The surface gravities κ_α provide the rate of convergence of the Boyer-Lindquist coordinate r to r_α in terms of the Regge-Wheeler coordinate $T(r)$ introduced in equation (2.12): indeed, using the factorization (4.40), we find

$$|r - r_\alpha| = |\mathfrak{r} - r_\alpha|\left(\prod_{\substack{\beta\in\{n,c,-,+\}\\ \beta\neq\alpha}}\left|\frac{\mathfrak{r}-r_\beta}{r-r_\beta}\right|^{\frac{\kappa_\alpha}{\kappa_\beta}}\right)\mathrm{e}^{2\kappa_\alpha T(r)} = \mathcal{O}_{r\to r_\pm}\big(\mathrm{e}^{2\kappa_\alpha T(r)}\big). \tag{2.18}$$

Let now $r_0 > 0$ and define the hypersurface $\Sigma_{r_0} := \mathbb{R}_t \times \mathbb{S}^1_z \times \{r = r_0\} \times \mathbb{S}^2_\omega$. Using then the inverse metric expression in the extended-star coordinates, we compute

$$n_{\Sigma_{r_0}} := \nabla r = -\partial_{t^\star} + sV(r)\partial_{z^\star} - F(r)\left(\frac{\partial}{\partial r}\right)_{E\star} \tag{2.19}$$

where $\left(\frac{\partial}{\partial r}\right)_{E\star}$ is the vector field ∂_r in the extended-star coordinates. Since

$$\widetilde{g}_\star\left(n_{\Sigma_{r_0}}, -\left(\frac{\partial}{\partial r}\right)_{E\star}\right) = 1,$$

[4]Using that orthogonal null vector fields are collinear, we only know in general that κ_α is at least a smooth function $\{r = r_\alpha\} \to \mathbb{R}$.

$n_{\Sigma_{r_0}}$ is a incoming normal vector field to Σ_{r_0}. Besides,

$$\widetilde{g}_\star(n_{\Sigma_{r_0}}, n_{\Sigma_{r_0}}) = -F(r)$$

so that the incoming normal $n_{\Sigma_{r_\alpha}}$ is also tangent to Σ_{r_α} and this hypersurface is null (we could have also seen this by observing that $\det(\widetilde{g}_{\star\,|\Sigma_{r_\alpha}}) = 0$). As

$$\det(\widetilde{g}_\star) = \frac{r^4 \sin^2\theta}{m^2} = \det({}_\star\widetilde{g})$$

does not vanish at $r = r_\alpha$, Σ_{r_α} is not degenerate. Since $\nabla r \to X_\alpha$ as $r \to r_\alpha$, $\alpha \in \{c, -, +\}$, this hypersurface $\{r = r_\alpha\}$ is a *Killing horizon*, that is a non-degenerate null hypersurface generated by a Killing vector field. We then define:

$$\begin{aligned}
\mathscr{H}_c^+ &:= \mathbb{R}_{t^\star} \times \mathbb{S}^1_{z^\star} \times \{r_c\}_r \times \mathbb{S}^2_\omega && (\textit{future Cauchy horizon}),\\
\mathscr{H}_c^- &:= \mathbb{R}_{\star t} \times \mathbb{S}^1_{\star z} \times \{r_c\}_r \times \mathbb{S}^2_\omega && (\textit{past Cauchy horizon}),\\
\mathscr{H}^+ &:= \mathbb{R}_{t^\star} \times \mathbb{S}^1_{z^\star} \times \{r_-\}_r \times \mathbb{S}^2_\omega && (\textit{future event horizon}),\\
\mathscr{H}^- &:= \mathbb{R}_{\star t} \times \mathbb{S}^1_{\star z} \times \{r_-\}_r \times \mathbb{S}^2_\omega && (\textit{past event horizon}),\\
\mathscr{I}^+ &:= \mathbb{R}_{t^\star} \times \mathbb{S}^1_{z^\star} \times \{r_+\}_r \times \mathbb{S}^2_\omega && (\textit{future cosmological horizon}),\\
\mathscr{I}^- &:= \mathbb{R}_{\star t} \times \mathbb{S}^1_{\star z} \times \{r_+\}_r \times \mathbb{S}^2_\omega && (\textit{past cosmological horizon}).
\end{aligned}$$

Observe that the construction of the horizons is not complete so far: we need to add two 3-surfaces where $\mathscr{H}^+$ and $\mathscr{H}^-$ on the one hand, $\mathscr{I}^+$ and $\mathscr{I}^-$ on the other hand, meet. This will be done in Subsection 2.2.3.

2.2.3 Crossing rings

In the previous Subsections, we have constructed Killing horizons. We now build sets of codimension 2 which will complete the construction of the hypersurfaces $\{r = r_\alpha\}$ for all $\alpha \in \{c, -, +\}$. As only the event and cosmological horizons will be concerned in the next Sections, we will only detail computations related to them.

We start with the event horizon. Following (8.19) and (8.20) in [HaNi04], we define the *Kruskal-Boyer-Lindquist coordinates*

$$u := \mathrm{e}^{-\kappa_- {}^\star t}, \quad v := \mathrm{e}^{\kappa_- t^\star}, \quad z^\sharp := z - \left(\frac{\dot{z}^\star}{\dot{t}^\star}\right)\Big|_{r=r_-} t = z + sV_- t.$$

The variable $z^\sharp$ must be understood as an element of $\mathbb{S}^1$ (*i.e.* it is defined modulo 2π), the rotation direction on the circle being imposed by the sign of the charge product s. It is introduced to follow the "rotation" of the event horizon (by analogy with the Kerr case); concretely, it cancels the "rotation" term $\frac{s^2V(r)^2}{m^2}$ in front of $\mathrm{d}t^2$ in the extended metric as $r \to r_-$. The extended metric $\widetilde{g}$ now reads in these coordinates

$$\begin{aligned}
\widetilde{g} = {} & \frac{s^2\big(V(r) - V_-\big)^2}{4m^2\kappa_-^2 u^2 v^2}\Big(uv\big(\mathrm{d}u\mathrm{d}v + \mathrm{d}v\mathrm{d}u\big) - u^2\mathrm{d}v^2 - v^2\mathrm{d}u^2\Big)\\
& - \frac{s\big(V(r) - V_-\big)}{2m^2\kappa_- uv}\Big(\big(u\mathrm{d}v - v\mathrm{d}u\big)\mathrm{d}z^\sharp + \mathrm{d}z^\sharp\big(u\mathrm{d}v - v\mathrm{d}u\big)\Big) - \frac{1}{m^2}(\mathrm{d}z^\sharp)^2 - r^2\mathrm{d}\omega^2.
\end{aligned}$$

We have to ensure that the metric is well defined and non degenerate at $r = r_-$ (that is when $u = v = 0$). Set

$$G_-(u,v,r) := \frac{r - r_-}{uv} = (r - r_-)\mathrm{e}^{-2\kappa_- T(r)}.$$

The function G_- is analytic and non vanishing near $r = r_-$ because of (2.18). It follows that the extended metric

$$\begin{aligned}\widetilde{g} = &-\frac{s^2 G_-(u,v,r)^2}{4m^2 r^2 r_-^2 \kappa_-^2}\big(u^2\mathrm{d}v^2 + v^2\mathrm{d}u^2\big) - \frac{s^2 G_-(u,v,r)(r - r_-)}{4m^2 r^2 r_-^2 \kappa_-^2}\big(\mathrm{d}u\mathrm{d}v + \mathrm{d}v\mathrm{d}u\big)\\ &+ \frac{sG_-(u,v,r)}{2m^2 r r_- \kappa_-}\Big(\big(u\mathrm{d}v - v\mathrm{d}u\big)\mathrm{d}z^\sharp + \mathrm{d}z^\sharp\big(u\mathrm{d}v - v\mathrm{d}u\big)\Big) - \frac{1}{m^2}(\mathrm{d}z^\sharp)^2 - r^2\mathrm{d}\omega^2\end{aligned}$$

extends smoothly on a neighbourhood of $\big\{u = v = 0\big\}$. Call the set

$$\mathscr{R}_c^{\mathscr{H}} := \{u = 0\} \times \{v = 0\} \times \mathbb{S}^1_{z^\sharp} \times \mathbb{S}^2_\omega$$

the *crossing ring at the event horizon* (this is the equivalent of the crossing sphere in the usual case of Kerr-Newman black holes).

We now turn to the construction of the complete cosmological horizon. Computations are so similar that we will omit most of them. Define the new coordinates

$$\tilde{u} := \mathrm{e}^{-\kappa_+ {}^\star t}, \quad \tilde{v} := \mathrm{e}^{\kappa_+ t^\star}, \quad {}^\sharp z := z - \left(\frac{{}^\star\dot{z}}{{}^\star\dot{t}}\right)\Big|_{r=r_+} t = z + sV_+ t.$$

Recall that $\kappa_+ < 0$. The extended metric $\widetilde{g}$ reads in these coordinates

$$\begin{aligned}\widetilde{g} = &-\frac{s^2 G_+(\tilde{u},\tilde{v},r)^2}{4m^2 r^2 r_+^2 \kappa_+^2}\big(\tilde{u}^2\mathrm{d}\tilde{v}^2 + \tilde{v}^2\mathrm{d}\tilde{u}^2\big) + \frac{s^2 G_+(\tilde{u},\tilde{v},r)(r - r_+)}{4m^2 r^2 r_+^2 \kappa_+^2}\big(\mathrm{d}\tilde{u}\mathrm{d}\tilde{v} + \mathrm{d}\tilde{v}\mathrm{d}\tilde{u}\big)\\ &- \frac{sG_+(\tilde{u},\tilde{v},r)^2}{2m^2 r r_+ \kappa_+}\Big(\big(\tilde{u}\mathrm{d}\tilde{v} - \tilde{v}\mathrm{d}\tilde{u}\big)\mathrm{d}^\sharp z + \mathrm{d}^\sharp z\big(\tilde{u}\mathrm{d}\tilde{v} - \tilde{v}\mathrm{d}\tilde{u}\big)\Big) - \frac{1}{m^2}(\mathrm{d}^\sharp z)^2 - r^2\mathrm{d}\omega^2\end{aligned}$$

with

$$G_+(\tilde{u},\tilde{v},r) := \frac{r_+ - r}{\tilde{u}\tilde{v}} = (r_+ - r)\mathrm{e}^{-2\kappa_+ T(r)}.$$

which is analytic and non vanishing near $r = r_+$ because of (2.18). Then $\widetilde{g}$ extends smoothly on a neighbourhood of $\big\{\tilde{u} = \tilde{v} = 0\big\}$. Call the set

$$\mathscr{R}_c^{\mathscr{I}} := \{\tilde{u} = 0\} \times \{\tilde{v} = 0\} \times \mathbb{S}^1_{\sharp_z} \times \mathbb{S}^2_\omega$$

the *crossing ring at the cosmological horizon.*

Now the construction is complete. The event horizon is defined as

$$\begin{aligned}\mathscr{H} &:= \mathscr{H}^- \cup \mathscr{R}_c^{\mathscr{H}} \cup \mathscr{H}^+\\ &= \big([0,+\infty[_u \times \{0\}_v \times \mathbb{S}^1_{z^\sharp} \times \mathbb{S}^2_\omega\big) \cup \big(\{0\}_u \times [0,+\infty[_v \times \mathbb{S}^1_{z^\sharp} \times \mathbb{S}^2_\omega\big)\end{aligned}$$

and the cosmological horizon is

$$\begin{aligned}\mathscr{I} &:= \mathscr{I}^- \cup \mathscr{R}_c^{\mathscr{I}} \cup \mathscr{I}^+ \\ &= \big(\{0\}_{\tilde{u}} \times [0, +\infty[_{\tilde{v}} \times \mathbb{S}^1_{\sharp_z} \times \mathbb{S}^2_\omega\big) \cup \big([0, +\infty[_{\tilde{u}} \times \{0\}_{\tilde{v}} \times \mathbb{S}^1_{\sharp_z} \times \mathbb{S}^2_\omega\big).\end{aligned}$$

The outer space can be extended itself as the *global outer space*

$$\overline{\mathcal{M}} := \widetilde{\mathcal{M}} \cup \mathscr{H} \cup \mathscr{I}.$$

As discussed at the beginning of this Section, the compactification shown in Figure 2.2 is not really complete as some points still lie at infinity whatever the coordinate system is. They are the future timelike infinity i^+ and the past timelike infinity i^-.

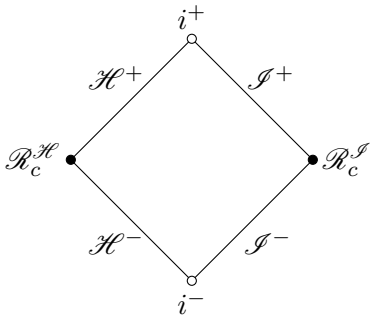

Figure 2.2: The global outer space $\overline{\mathcal{M}}$ (each point in the diagram is a copy of $\mathbb{S}^1 \times \mathbb{S}^2$). Only the points i^+ and i^- are at infinity.

2.2.4 Black rings

This Subsection is devoted to the construction of a global structure of $\widetilde{\mathcal{M}}_{\text{ext}}$. The procedure is similar to the standard one for De Sitter-Reissner-Nordström spacetime and will reveal the black rings structure of $\widetilde{\mathcal{M}}_{\text{ext}}$.

Conformal infinity. We start by including the infinity to the region $\{r \geq r_+\}$. Recall here that $V(r) = 1/r$. Let us set the *conformal factor* $\Omega := 1/r$. In the coordinates $(t^\star, z^\star, R, \omega)$ with $R := 1/r \in [0, 1/r_+]$, we define the conformal extended-star metric

$$\begin{aligned}\hat{g}_\star &:= \Omega^2 \widetilde{g}_\star \\ &= R^2 \left(F(1/R) - \frac{s^2 R^2}{m^2}\right) (\mathrm{d}t^\star)^2 - \frac{sR^3}{m^2} (\mathrm{d}t^\star \mathrm{d}z^\star + \mathrm{d}z^\star \mathrm{d}t^\star) - \frac{R^2}{m^2} (\mathrm{d}z^\star)^2 + (\mathrm{d}t^\star \mathrm{d}R + \mathrm{d}R \mathrm{d}t^\star) - \mathrm{d}\omega^2\end{aligned}$$

and similarly, in the coordinates $({}^\star t, {}^\star z, R, \omega)$, the conformal star-extended metric is

$$\begin{aligned}{}_\star\hat{g} &:= \Omega^2 {}_\star\widetilde{g} \\ &= R^2 \left(F(1/R) - \frac{s^2 R^2}{m^2}\right) (\mathrm{d}^\star t)^2 - \frac{sR^3}{m^2} (\mathrm{d}^\star t \mathrm{d}^\star z + \mathrm{d}^\star z \mathrm{d}^\star t) - \frac{R^2}{m^2} (\mathrm{d}^\star z)^2 - (\mathrm{d}^\star t \mathrm{d}R + \mathrm{d}R \mathrm{d}^\star t) - \mathrm{d}\omega^2.\end{aligned}$$

We can check that $\det \hat{g}_\star = \det {}_\star\hat{g} = \frac{R^2}{m^2}$. Hence, the following hypersurfaces are null:

$$\begin{aligned}
&\mathscr{I}^+_\infty := \mathbb{R}_{\star t} \times \mathbb{S}^1_{\star z} \times \{0\}_R \times \mathbb{S}^2_\omega && (\textit{future null infinity}),\\
&\mathscr{I}^-_\infty := \mathbb{R}_{t^\star} \times \mathbb{S}^1_{z^\star} \times \{0\}_R \times \mathbb{S}^2_\omega && (\textit{past null infinity}),\\
&\mathscr{I}_\infty := \mathscr{I}^+_\infty \cup \mathscr{I}^-_\infty && (\textit{null infinity}).
\end{aligned}$$

They are the sets of "end points" at infinity of the principal null geodesics; observe that they do not intersect (the spacelike infinity still lies at infinity as for the De Sitter-Kerr-Newman family). The conformal metrics then restrict to

$$\hat{g}_\star{}_{|\mathscr{I}^-_\infty} = -\frac{\Lambda}{3}(\mathrm{d}t^\star)^2 - \mathrm{d}\omega^2 = {}_\star\hat{g}_{|\mathscr{I}^+_\infty}.$$

Time orientation. We now define a global time function in $\widetilde{\mathcal{M}}_{\text{ext}}$; recall that a *time function* τ is a $\mathcal{C}^1$ function such that $\nabla\tau$ is timelike. The extended spacetime $\widetilde{\mathcal{M}}_{\text{ext}}$ is not connected as we remove the hypersurfaces Σ_{r_α} for $\alpha \in \{c, -, +\}$. As a result, there is no canonical way of defining a time-orientation on it. Since

$$\begin{aligned}
&\nabla t = F(r)^{-1}\big(\partial_t - sV(r)\partial_z\big), && \widetilde{g}(\nabla t, \nabla t) = F(r)^{-1} > 0 \quad \text{in } \widetilde{\mathcal{M}}_1 \text{ and } \widetilde{\mathcal{M}}_3,\\
&\nabla r = -F(r)\partial_r, && \widetilde{g}(\nabla r, \nabla r) = -F(r) > 0 \quad \text{in } \widetilde{\mathcal{M}}_2 \text{ and } \widetilde{\mathcal{M}}_4,
\end{aligned}$$

we see that $\pm t$ is a time orientation in the blocks $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_3$ whereas $\pm r$ is a time orientation in the blocks $\widetilde{\mathcal{M}}_2$ and $\widetilde{\mathcal{M}}_4$. From now on, $\widetilde{\mathcal{M}}_j$ denotes the corresponding block endowed with the time orientation t if $j \in \{1,3\}$ or r if $j \in \{2,4\}$, and $\widetilde{\mathcal{M}}'_j$ denotes the same block but with the time orientation $-t$ if $j \in \{1,3\}$ or $-r$ if $j \in \{2,4\}$. We still write $\widetilde{\mathcal{M}}_{\text{ext}} := \bigcup_{j=1}^4 \widetilde{\mathcal{M}}_j$ and set $\widetilde{\mathcal{M}}'_{\text{ext}} := \bigcup_{j=1}^4 \widetilde{\mathcal{M}}'_j$.

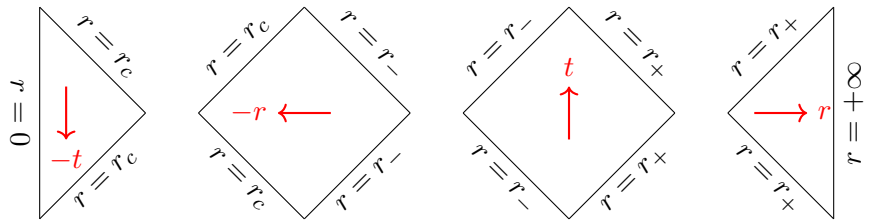

Figure 2.3: The four blocks $\widetilde{\mathcal{M}}_j$ and their respective time orientation.

Carter-Penrose diagram of the extended spacetime. We now construct a global structure $\overline{\mathcal{M}}_{\text{ext}}$ which respects the time orientation of each block $\widetilde{\mathcal{M}}_j$ and $\widetilde{\mathcal{M}}'_j$.

Let $\overline{\mathcal{M}}_{\text{DSRN}}$ be the maximal analytic extension of the De Sitter-Reissner-Nordström spacetime (see *e.g.* Subsection 1.2 and particularly the paragraph 1.2.5 in [Mok17]). This extension essentially consists in defining appropriate coordinates near the positive[5] roots of F in which the metric

[5] $r = 0$ being a genuine singularity at which the metric can not be $\mathcal{C}^2$, we can not build an analytic extension near the negative root.

is analytic, so that all the work boils down to build it considering the quotient of the original spacetime by the action of the rotations group on $\mathbb{S}^2$. $\overline{\mathcal{M}}_{\text{DSRN}}$ also satisfies a local inextendibility property: there is no open non-empty subset whose closure is non-compact and can be embedded in an analytic manifold with a relatively compact image (see [Mok17, Subsection 1.2]). We then define

$$\overline{\mathcal{M}}_{\text{ext}} := \overline{\mathcal{M}}_{\text{DSRN}} \times \mathbb{S}^1$$

which is nothing but the orbit of $\overline{\mathcal{M}}_{\text{DSRN}}$ under the action of the rotation on $\mathbb{S}^1$. It contains infinitely many isometric copies of $\widetilde{\mathcal{M}}_{\text{ext}} \sqcup \mathscr{H}_c \sqcup \mathscr{H} \sqcup \mathscr{I} \sqcup \mathscr{I}_\infty$ and $\widetilde{\mathcal{M}}'_{\text{ext}} \sqcup \mathscr{H}_c \sqcup \mathscr{H} \sqcup \mathscr{I} \sqcup \mathscr{I}_\infty$.

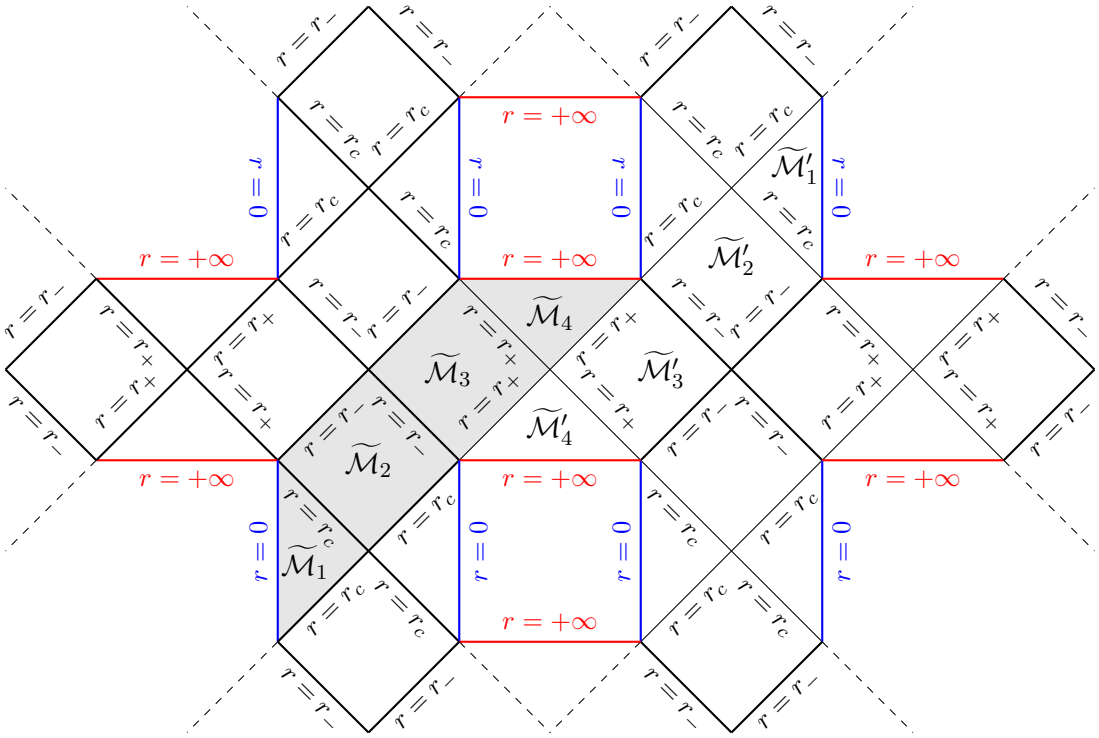

Figure 2.4: The global geometry of the extended spacetime. Time orientation goes from the bottom to the top of the figure.

Black rings, white rings and worm rings The neutralization procedure presented in Subsection 2.1.1 has turned the original De Sitter-Reissner-Nordström black hole into a stranger object, a *black ring*. It is the equivalent of a black hole except that the topology of the horizon is $\mathbb{S}^1 \times \mathbb{S}^2$. As the we will see, the maximal extension $\overline{\mathcal{M}}_{\text{ext}}$ contains infinitely many of them so we have to distinguish the different copies of $\widetilde{\mathcal{M}}_{\text{ext}}$ from each others.

A *piece of universe* $\mathscr{P}$ is any copy of $\widetilde{\mathcal{M}}_{\text{ext}} \sqcup \mathscr{H}_c \sqcup \mathscr{H} \sqcup \mathscr{I} \sqcup \mathscr{I}_\infty$. We exclude in this definition the reverse time-oriented blocks $\widetilde{\mathcal{M}}'_j$ as the physical block (that containing the Earth) is any copy of the outer space $\widetilde{\mathcal{M}}_3$. Adapting the definition of Wald [Wa84] (see Subsection 12.1 therein which deals with the Kerr case), we will say that:

- a non-empty closed subset $\mathcal{B} \subset \overline{\mathcal{M}}_{\rm ext}$ is a *black ring* if $\partial\mathcal{B} = \mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^2$ and if for all inextendible causal future-pointing geodesic $\gamma : \mathbb{R} \to \overline{\mathcal{M}}_{\rm ext}$ starting in $\widetilde{\mathcal{M}}_3$ and entering $\mathcal{B}$ at some μ_0, then
$$\bigcup_{\mu \geq \mu_0} \{\gamma(\mu)\} \cap \widetilde{\mathcal{M}}_4 \cap \mathscr{P} = \emptyset;$$
- a non-empty subset $\mathcal{W} \subset \overline{\mathcal{M}}_{\rm ext}$ is a *white ring* if it is a black ring for the reverse time orientation;
- a non-empty subset $\mathcal{S} \subset \overline{\mathcal{M}}_{\rm ext}$ is a *worm ring* $\mathcal{S}$ if there exist a black ring $\mathcal{B}$ and a white ring $\mathcal{W}$ such that $\mathcal{S} = \mathcal{B} \cap \mathcal{W}$.

A black ring thus prevents any inextendible causal future-pointing geodesic entering inside it to escape at infinity in the same piece of universe: it can escape only in another copy of the universe. Observe that $\partial\mathcal{B}$ and $\partial\mathcal{W}$ are necessarily null.

We can similarly define the more standard notions of black/white/worm holes by requiring the topology $\mathbb{R} \times \mathbb{S}^3$ for the boundary. It turns that the extended spacetime contains only black/white/worm rings.

Lemma 2.2.2. *1. All the copies of $\widetilde{\mathcal{M}}'_2 \sqcup \mathscr{H}_c \sqcup \mathscr{H}$ are black rings.*

2. All the copies of $\widetilde{\mathcal{M}}_2 \sqcup \mathscr{H}_c \sqcup \mathscr{H}$ are white rings.

3. All the copies of $\widetilde{\mathcal{M}}_1 \sqcup \mathscr{H}_c \sqcup \mathscr{H}$ and $\widetilde{\mathcal{M}}'_1 \sqcup \mathscr{H}_c \sqcup \mathscr{H}$ are worm rings.

4. There is no other black/white/worm ring in $\overline{\mathcal{M}}_{\rm ext}$. Furthermore, $\overline{\mathcal{M}}_{\rm ext}$ contains no black-/white/worm hole.

Proof. 1. Let $\gamma = (t, z, r, \omega)$ be an inextendible causal future-pointing geodesic which starts in $]r_-, r_+[$ and such that $r(\mu_0) \in]r_c, r_-[$ for some $\mu_0 \in I$ (such a curve exists, take *e.g.* the incoming principal null geodesics $\gamma_{\rm in}$). Since $r(\mu) < r_-$ for some $\mu < \mu_0$, there exists a proper time $\underline{\mu} \in I$, $\underline{\mu} \leq \mu_0$, such that $\dot{r}(\underline{\mu}) < 0$. Using that $-r$ is a time orientation in $\widetilde{\mathcal{M}}'_2$, we have
$$\dot{r} = \nabla_{\dot\gamma} r = \widetilde{g}_{\mu\nu} \dot\gamma^\mu \nabla^\nu r \leq 0.$$
As a result, $\dot{r} \leq 0$ along the flow of γ with equality as long as γ lies on $\{r = r_-\}$. Thus γ will stay in the block $\widetilde{\mathcal{M}}_2$ or will cross $\{r = r_c\}$, entailing that it can escape to $r = +\infty$ only in another piece of universe.

2. and 3. follow from 1.

4. Removing all the black/white/worm rings from $\overline{\mathcal{M}}_{\rm ext}$, it only remains copies of $\widetilde{\mathcal{M}}_3$, $\widetilde{\mathcal{M}}'_3$, $\widetilde{\mathcal{M}}_4$ and $\widetilde{\mathcal{M}}'_4$ (with the cosmological horizons at $r = r_+$). Starting in any of these blocks, the outgoing principal null geodesics $\gamma_{\rm out}$ can escape to $r = +\infty$, meaning that no black hole (and thus no white and worm hole) lies there.

□

Remark 2.2.3. *All the black and white rings contained in the global extended spacetime $\overline{\mathcal{M}}_{\rm ext}$ are delimited by horizons. In particular, the timelike singularity $\{r = 0\}$ lying in all the worm ring copies is always hidden by horizons. The weak Cosmic Censorship is therefore respected.*

2.3 Analytic scattering theory

This section is devoted to the scattering theory associated to the extended wave equation (2.2). We will show existence and completeness of wave operators associated to several comparison dynamics for fixed momenta $\mathrm{i}\partial_z = \mathbf{z} \in \mathbb{Z} \neq \{0\}$ of the scalar field. The case $\mathbf{z} = 1$ corresponds to the charged Klein-Gordon equation. Most of the results below follows from the work of Georgescu-Gérard-Häfner [GGH17].

In Subsection 2.3.1, we introduce the Hamiltonian formalism associated to equation (2.2). The different comparison dynamics used for the scattering are introduced in Subsection 2.3.2. In Subsection 2.3.3, we explicit the action of the geometric dynamics in term of transport along principal null geodesics The structure of the energy spaces for the comparison dynamics related to transport is analyzed in Subsection 2.3.4 in order to prepare the proof of the scattering results. The analytic scattering results are then presented in Subsection 2.3.5 and proved in Section 2.4.

2.3.1 Hamiltonian formulation of the extended wave equation

We introduce in this Subsection the full dynamics associated to (2.2).

The full dynamics. Observe first of all that if u solves (2.2) then $v := \mathrm{e}^{-\mathrm{i}sV_+ t\partial_z} u$ satisfies

$$\left(\partial_t^2 - 2s\tilde{V}(r)\partial_z\partial_t + \tilde{P}\right) u = 0, \qquad \tilde{V}(r) := V(r) - V_+. \tag{2.20}$$

We can therefore work with the potential[6] $\tilde{V}$ in this Section and use the results in [GGH17]. In order not to overload notations, we will still denote $\tilde{V}$ by V; to keep track of the potential V, we will often write V_+ in the sequel even though it is the constant 0. Note here that we can again compare sV_+ to the Kerr rotation $\Omega_+ = \frac{a}{r_+^2+a^2}$: the unitary transform performed above is the equivalent operation in our setting to the variable change $\varphi \mapsto \varphi - \Omega_+ t$ at the very beginning of Section 13 in [GGH17].

We introduce the Regge-Wheeler coordinate

$$\frac{\mathrm{d}x}{\mathrm{d}r} := T'(r) = \frac{1}{F(r)}$$

so that $x \equiv x(r)$ is equal to the function T up to an additive constant; in particular, $x \in \mathbb{R}$ and $x \to \pm\infty$ if and only if $r \to r_\pm$. In the sequel, we will denote by $'$ the derivative with respect to the variable r.

Next, let us define $\mathcal{H} := L^2\left(\mathbb{S}^1_z \times]r_-, r_+[_r \times \mathbb{S}^2_\omega, T'(r)\mathrm{d}z\mathrm{d}r\mathrm{d}\omega\right) = L^2\left(\mathbb{S}^1_z \times \mathbb{R}_x \times \mathbb{S}^2_\omega, \mathrm{d}z\mathrm{d}x\mathrm{d}\omega\right)$ and

$$\begin{aligned} h &:= r\hat{P}r^{-1} \\ &= -r^{-1}\partial_x r^2 \partial_x r^{-1} - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} - \left(m^2F(r) - s^2V(r)^2\right)\partial_z^2 \\ &= -\partial_x^2 + F(r)\left(-\frac{1}{r^2}\Delta_{\mathbb{S}^2} + \frac{F'(r)}{r} - m^2\partial_z^2\right) + s^2V(r)^2\partial_z^2. \end{aligned}$$

[6] Working with this potential is actually equivalent to use the gauge $A_\mu = \left(\frac{Q}{r} - \frac{Q}{r_+}\right)\mathrm{d}t$.

with $\hat{P}$ given by (2.3). Here the variable r has to be understood as a function $r(x)$ of the Regge-Wheeler coordinate. The operator ∂_z plays a similar role as ∂_φ for the De Sitter-Kerr case (*cf.* [GGH17, equation (13.3)]). Observe here that h is not positive in the dyadorings $\mathcal{D}_\pm$ (because $sV - m^2F < 0$ near $r = r_\pm$). This problem is very similar to the failure of ∂_t to be timelike in the extended spacetime (or in Kerr spacetime). Considering the timelike (but not Killing) vector field $\nabla t = F(r)^{-1}(\partial_t - sV(r)\partial_z)$ instead, we add the extra "rotating" term $-sV(r)\partial_z$ which cancels the negative parts of $\widetilde{g}(\partial_t, \partial_t)$ near $r = r_\pm$.

Introduce now cut-offs $i_\pm, j_\pm \in \mathcal{C}^\infty(]r_-, r_+[\,, \mathbb{R})$ satisfying

$$\begin{aligned} &\operatorname{Supp} \mathrm{i}_- =]-\infty, 1]\,, \qquad \operatorname{Supp} \mathrm{i}_+ = [-1, +\infty[\,, \\ &i_-^2 + i_+^2 = 1, \qquad i_\pm j_\pm = j_\pm, \qquad i_\pm j_\mp = 0. \end{aligned}$$

They will be useful to separate incoming and outgoing parts of solutions for the scattering. We next introduce the following operators:

$$\begin{aligned} k &:= -\mathrm{i}sV(r)\partial_z, \\ k_\pm &:= -\mathrm{i}sV_\pm\partial_z, \\ \widetilde{k}_\pm &:= k \mp j_\mp^2 k_-, \\ h_0 := h + k^2 &= -r^{-1}\partial_x r^2 \partial_x r^{-1} - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} - m^2F(r)\partial_z^2 \\ &= -\partial_x^2 + F(r)\left(-\frac{1}{r^2}\Delta_{\mathbb{S}^2} + \frac{F'(r)}{r} - m^2\partial_z^2\right), \\ \widetilde{h}_\pm &:= h_0 - (\widetilde{k} - k_\pm)^2. \end{aligned}$$

Observe that $h_0 \geq 0$ which witnesses of the hyperbolic nature of the equation (2.2) (*cf.* [GGH17, Remark 2.2]). Observe also that if s is sufficiently small (that we will always assume in the sequel), then $\widetilde{h}_\pm \geq 0$ to since $\widetilde{k} - k_\pm$ as the same exponential decay at infinity as F, *cf.* (2.18).

Using the spherical symmetries of the problem, that is, using the diagonalizations $-\Delta_{\mathbb{S}^2} = \ell(\ell+1)$ and $-\mathrm{i}\partial_z = \mathbf{z}$ with $(\ell, \mathbf{z}) \in \mathbb{N} \times \mathbb{Z}$, we define

$$h_0^{\ell,\mathbf{z}} := -r^{-1}\partial_x r^2 \partial_x r^{-1} - \frac{F(r)}{r^2}\ell(\ell+1) + m^2F(r)\mathbf{z}^2 = -\partial_x^2 + F(r)\left(\frac{\ell(\ell+1)}{r^2} + \frac{F'(r)}{r} + m^2\mathbf{z}^2\right),$$

$$\mathscr{D}(h_0^{\ell,\mathbf{z}}) := \left\{u \in \mathcal{H} \,\middle|\, h_0^{\ell,\mathbf{z}} u \in \mathcal{H}\right\}.$$

In their work [BaMo93], Bachelot and Motet-Bachelot show in Proposition II.1 that $-\partial_x^2 + \mathcal{V}(x)$ has no 0 eigenvalue if the potential $\mathcal{V}(x)$ has some polynomial decay at infinity; this is the case for $h_0^{\ell,\mathbf{z}}$ which is then an elliptic second order differential operator, thus self-adjoint on $\mathcal{H}$. We realize h_0 as the direct sum of the harmonic operators $h_0^{\ell,\mathbf{z}}$ *i.e.*

$$\mathscr{D}(h_0) := \left\{u = \sum_{\ell,\mathbf{z}} u_{\ell,\mathbf{z}} \;\middle|\; u_{\ell,\mathbf{z}} \in \mathscr{D}(h_0^{\ell,\mathbf{z}}),\ \sum_{\ell,\mathbf{z}} \left\|h_0^{\ell,\mathbf{z}} u_{\ell,\mathbf{z}}\right\|_{\mathcal{H}}^2 < +\infty\right\}$$

which is in turn elliptic and self-adjoint. Finally, as $k \in \mathcal{B}(\mathcal{H})$, we can also realize h as a self-adjoint operator on the domain $\mathscr{D}(h) = \mathscr{D}(h_0)$. In the sequel, we will use the elliptic self-adjoint realizations $(\widetilde{h}_\pm, \mathscr{D}(\widetilde{h}_\pm))$ defined as above for h_0.

Energy spaces for the full dynamics. We now turn to the definition of the energy spaces associated to the Hamiltonian $\dot{H}$ following [GGH17] (see the paragraphs 3.4.1 and 3.4.2 therein).

We define the *inhomogeneous energy spaces*

$$\mathcal{E} := \langle h_0 \rangle^{-1/2} \mathcal{H} \times \mathcal{H}$$

equipped with the norm[7]

$$\|(u_0, u_1)\|_{\mathcal{E}}^2 := \langle (1 + h_0) u_0, u_0 \rangle_{\mathcal{H}} + \|u_1 - k u_0\|_{\mathcal{H}}^2.$$

Using that $h_0 \geq 0$ has no kernel in $\mathcal{H}$, we can also define the *homogeneous energy space* $\dot{\mathcal{E}}$ as the completion of $\mathcal{E}$ for the norm[8]

$$\|(u_0, u_1)\|_{\dot{\mathcal{E}}}^2 := \langle h_0 u_0, u_0 \rangle_{\mathcal{H}} + \|u_1 - k u_0\|_{\mathcal{H}}^2.$$

Observe that, as explained in paragraph 3.4.3 of [GGH17], this energy is not conserved in general. The natural conserved energies $\langle \cdot \,|\, \cdot \rangle_\ell$ defined for all $\ell \in \mathbb{R}$ by

$$\langle (u_0, u_1) \mid (u_0, u_1) \rangle_\ell := \langle (h_0 - (k - \ell)^2) u_0, u_0 \rangle_{\mathcal{H}} + \|u_1 - \ell u_0\|_{\mathcal{H}}^2$$

are not in general positive (because of the existence of the dyadorings). In contrast, the (positive) energy $\| \cdot \|_{\dot{\mathcal{E}}}$ is not conserved along the flow of ∂_t and may grow in time: this is *superradiance*. From the geometric point of view, superradiance occurs because of the existence of the dyadorings and we are using the timelike vector field ∇t of (2.15) instead of ∂_t to get a positive quantity near the dyadorings; the cost to pay for this is the non conservation of the energy since ∇t is not Killing.

Define next the *asymptotic energy spaces*[9]

$$\dot{\widetilde{\mathcal{E}}}_\pm := \Phi(\widetilde{k}_\pm) \left(\widetilde{h}_\pm^{-1/2} \mathcal{H} \times \mathcal{H} \right)$$

where

$$\Phi(\widetilde{k}_\pm) := \begin{pmatrix} \mathbb{1} & 0 \\ \widetilde{k}_\pm & \mathbb{1} \end{pmatrix}.$$

The spaces $\dot{\widetilde{\mathcal{E}}}_\pm$ are equipped with the norms

$$\|(u_0, u_1)\|_{\dot{\widetilde{\mathcal{E}}}_\pm}^2 := \langle \widetilde{h}_\pm u_0, u_0 \rangle_{\mathcal{H}} + \|u_1 - \widetilde{k}_\pm u_0\|_{\mathcal{H}}^2.$$

As discussed above Lemma 3.13 in [GGH17], the operators $\Phi(\widetilde{k}_\pm) : \widetilde{h}_\pm^{-1/2} \mathcal{H} \times \mathcal{H} \to \dot{\widetilde{\mathcal{E}}}_\pm$ are isomorphisms with inverses $\Phi(-\widetilde{k}_\pm)$.

The Hamiltonian form of (2.20) is given by $-\mathrm{i}\partial_t u = \dot{H} u$ with

$$\dot{H} = \begin{pmatrix} 0 & \mathbb{1} \\ h & 2k \end{pmatrix}, \qquad \mathscr{D}(\dot{H}) = \left\{ u \in \dot{\mathcal{E}} \mid \dot{H} u \in \dot{\mathcal{E}} \right\}$$

[7] We use here that $h_0 \geq 0$.
[8] We use here that $h_0 > 0$.
[9] Now we use that $\widetilde{h}_\pm > 0$.

is the *energy Klein-Gordon operator*. We will also need to use the *asymptotic Hamiltonians*

$$\dot{\widetilde{H}}_\pm = \begin{pmatrix} 0 & \mathbb{1} \\ \widetilde{h}_\pm & 2\widetilde{k}_\pm \end{pmatrix}, \qquad \mathscr{D}(\dot{\widetilde{H}}_\pm) := \Phi(k_\pm)\left(\widetilde{h}_\pm^{-1/2}\mathcal{H} \cap \widetilde{h}_\pm^{-1}\mathcal{H} \times \langle \widetilde{h}_\pm\rangle^{-1/2}\mathcal{H}\right).$$

Given now $\mathbf{z} \in \mathbb{Z}$, we define on $\ker(\mathrm{i}\partial_z + \mathbf{z})$ the restricted operators $h^{\mathbf{z}}$, $h_0^{\mathbf{z}}$, $k^{\mathbf{z}}$, $\dot{H}^{\mathbf{z}}$, etc. as well as the spaces $\mathcal{H}^{\mathbf{z}}$, $\mathcal{E}^{\mathbf{z}}$, $\dot{\mathcal{E}}^{\mathbf{z}}$ in the obvious way. The operator $h_0^{\mathbf{z}}$ is the one in (1.8) with (s, m) replaced by $(s\mathbf{z}, m\mathbf{z})$. Furthermore, if $\mathbf{z} \neq 0$, then $\langle h_0\rangle^{-1/2}\mathcal{H}^{\mathbf{z}} = h_0^{-1/2}\mathcal{H}^{\mathbf{z}} = (h_0^{\mathbf{z}})^{-1/2}\mathcal{H}^{\mathbf{z}}$.

[Ha01, Lemma 2.2] shows[10] that $(\dot{\widetilde{H}}_\pm^{\mathbf{z}}, \mathscr{D}(\dot{\widetilde{H}}_\pm^{\mathbf{z}}))$ are self-adjoint. The difference with $(\dot{H}^{\mathbf{z}}, \mathscr{D}(\dot{H}^{\mathbf{z}}))$ is that for the asymptotic operators, we have $\|\widetilde{k}_\pm^{\mathbf{z}} u\|_{\mathcal{H}^{\mathbf{z}}} \lesssim \langle \widetilde{h}_\pm^{\mathbf{z}} u, u\rangle_{\mathcal{H}^{\mathbf{z}}}$ for all $u \in \mathscr{D}\big((\widetilde{h}_\pm^{\mathbf{z}})^{1/2}\big)$ (as required by (2.22) in [Ha01]); such an estimate is false with $k^{\mathbf{z}}$ and $h^{\mathbf{z}}$.

[GGH17, Lemma 3.19] shows that $\dot{H}$ is the generator of a continuous group $(\mathrm{e}^{\mathrm{i}t\dot{H}})_{t\in\mathbb{R}}$ on $\dot{\mathcal{E}}$. If $u = (u_0, u_1)$ solves $-\mathrm{i}\partial_t u = \dot{H}u$, then u_0 is a solution of (2.2) and conversely, if u solves (2.2) then $v = (u, -\mathrm{i}\partial_t u)$ satisfies $-\mathrm{i}\partial_t v = \dot{H}v$.

2.3.2 Comparison dynamics

We present in this Subsection the comparison dynamics we will use for the scattering.

Separable comparison dynamics. The first dynamics we will compare $(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}})_{t\in\mathbb{R}}$ to are the following *separable comparison dynamics*. Define the operators

$$h_{\pm\infty} := -\partial_x^2 - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} - \big(m^2F(r) - s^2V_\pm^2\big)\partial_z^2, \qquad k_{\pm\infty} := -\mathrm{i}sV_\pm\partial_z.$$

The associated second order equation reads

$$\big(\partial_t^2 - 2\mathrm{i}k_{\pm\infty}\partial_t + h_{\pm\infty}\big)u = 0. \tag{2.21}$$

The corresponding Hamiltonian is given by

$$\dot{H}_{\pm\infty} = \begin{pmatrix} 0 & \mathbb{1} \\ h_{\pm\infty} & 2k_{\pm\infty} \end{pmatrix}, \qquad \mathscr{D}(\dot{H}_{\pm\infty}) = \big\{u \in \dot{\mathcal{E}}_{\pm\infty} \mid \dot{H}_{\pm\infty}u \in \dot{\mathcal{E}}_{\pm\infty}\big\}$$

where the spaces $\big(\dot{\mathcal{E}}_{\pm\infty}, \|\cdot\|_{\dot{\mathcal{E}}_{\pm\infty}}\big)$ are defined in the canonical way *i.e.* $\dot{\mathcal{E}}_{\pm\infty}$ is the completion of smooth compactly supported functions with respect to the norm

$$\|(u_0, u_1)\|_{\dot{\mathcal{E}}_{\pm\infty}}^2 := \big\langle \big(h_{\pm\infty} + k_{\pm\infty}^2\big)u_0, u_0\big\rangle_{\mathcal{H}} + \|u_1 - k_{\pm\infty}u_0\|_{\mathcal{H}}^2.$$

Denote by $h_{\pm\infty}^{\mathbf{z}}, k_{\pm\infty}^{\mathbf{z}}, \dot{H}_{\pm\infty}^{\mathbf{z}}$ and $\dot{\mathcal{E}}_{\pm\infty}^{\mathbf{z}}$ the restrictions on $\ker(\mathrm{i}\partial_z + \mathbf{z})$ for $\mathbf{z} \in \mathbb{Z}$ of the above operators and spaces. If $u \in \dot{\mathcal{E}}^{\mathbf{z}}$ solves (2.21), then

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u, -\mathrm{i}\partial_t u)\|_{\dot{\mathcal{E}}_{\pm\infty}^{\mathbf{z}}}^2 &= \Re\Big(\big\langle \big(h_{\pm\infty}^{\mathbf{z}} + (k_{\pm\infty}^{\mathbf{z}})^2\big)u, \partial_t u\big\rangle_{\mathcal{H}} + \big\langle \partial_t^2 u - \mathrm{i}k_{\pm\infty}^{\mathbf{z}}\partial_t u, \partial_t u - \mathrm{i}k_{\pm\infty}^{\mathbf{z}} u\big\rangle_{\mathcal{H}}\Big) \\
&= \Re\Big(\big\langle \big(h_{\pm\infty} + (k_{\pm\infty}^{\mathbf{z}})^2\big)u, \partial_t u\big\rangle_{\mathcal{H}} + \big\langle \mathrm{i}k_{\pm\infty}^{\mathbf{z}}\partial_t u - h_{\pm\infty}u, \partial_t u - \mathrm{i}k_{\pm\infty}^{\mathbf{z}} u\big\rangle_{\mathcal{H}}\Big) \\
&= \Re\Big(\big\langle \mathrm{i}k_{\pm\infty}^{\mathbf{z}}\partial_t u, \partial_t u\big\rangle_{\mathcal{H}} + \big\langle h_{\pm\infty}u, \mathrm{i}k_{\pm\infty}^{\mathbf{z}} u\big\rangle_{\mathcal{H}}\Big) \\
&= 0
\end{aligned}$$

[10] $\mathcal{H}^2$ must be replaced by its completion with respect to the homogeneous norm as pointed out by the author of this paper in [Ha03], above Lemma 2.1.1.

because

$$\begin{aligned}\operatorname{sign}(s\mathbf{z}) \times \big\langle h_{\pm\infty} u, \mathrm{i} k^{\mathbf{z}}_{\pm\infty} u \big\rangle_{\mathcal{H}} &= \mathrm{i} \big\| |k^{\mathbf{z}}_{\pm\infty}|^{1/2} \partial_x u \big\|^2_{\mathcal{H}} + \mathrm{i} \big\| r^{-1} F(r)^{1/2} |k^{\mathbf{z}}_{\pm\infty}|^{1/2} \nabla_{\mathbb{S}^2} u \big\|^2_{\mathcal{H}} \\ &\quad + \mathrm{i} \big\| m F(r)^{1/2} |k^{\mathbf{z}}_{\pm\infty}|^{1/2} u \big\|^2_{\mathcal{H}} - \mathrm{i} \big\| |k^{\mathbf{z}}_{\pm\infty}|^{3/2} \partial_z u \big\|^2_{\mathcal{H}}.\end{aligned}$$

Notice here that the above conservation law strongly relies on the fact that $[\partial_x, k_{\pm\infty}] = 0$ (in comparison, $[\partial_x, k] \neq 0$). It results that the associated dynamics $(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\pm\infty}})_{t\in\mathbb{R}}$ are unitary on $\mathcal{H}^{\mathbf{z}} \times \mathcal{H}^{\mathbf{z}}$ and the infinitesimal generators $(\dot{H}^{\mathbf{z}}_{\pm\infty}, \mathscr{D}(\dot{H}^{\mathbf{z}}_{\pm\infty}))$ are self-adjoint by Stone's theorem.

Theorem 2.3.9 states that the operators $\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_\pm \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\pm\infty}}$ and $\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\pm\infty}} i_\pm \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$ have strong limits in $\mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}_{\pm\infty}, \dot{\mathcal{E}}^{\mathbf{z}})$ and $\mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{E}}^{\mathbf{z}}_{\pm\infty})$ as $|t| \to +\infty$.

Asymptotic profiles. We next introduce the *asymptotic profiles*: they consist in the simplest possible asymptotic comparison dynamics obtained by formally taking the limit $x \to \pm\infty$ in h and k. Set

$$h_{-/+} := -\partial_x^2 + s^2 V_{-/+}^2 \partial_z^2, \qquad k_{-/+} := -\mathrm{i}s V_{-/+} \partial_z.$$

The associated second order equation reads

$$\big(\partial_t^2 - 2\mathrm{i} k_{-/+} \partial_t + h_{-/+}\big) u = 0. \tag{2.22}$$

Notice the following factorization :

$$\partial_t^2 - 2s V_{-/+} \partial_z \partial_t + h_{-/+} = (\partial_t - \partial_x - s V_{-/+} \partial_z)(\partial_t + \partial_x - s V_{-/+} \partial_z).$$

We call *incoming* respectively *outgoing* solutions of (2.22) are the solutions of $(\partial_t - \partial_x - s V_{-/+} \partial_z) u = 0$ respectively $(\partial_t + \partial_x - s V_{-/+} \partial_z) u = 0$. Define the corresponding Hamiltonians

$$\dot{H}_{-/+} = \begin{pmatrix} 0 & \mathbb{1} \\ h_{-/+} & 2k_{-/+} \end{pmatrix}, \qquad \mathscr{D}(\dot{H}_{-/+}) = \big\{ u \in \dot{\mathcal{E}}_{-/+} \mid \dot{H}_{-/+} u \in \dot{\mathcal{E}}_{-/+} \big\}$$

with the canonical energy spaces $\big(\dot{\mathcal{E}}_{-/+}, \|\cdot\|_{\dot{\mathcal{E}}_{-/+}}\big)$ and the homogeneous norms

$$\|(u_0, u_1)\|^2_{\dot{\mathcal{E}}_{-/+}} := \big\langle \big(h_{-/+} + k^2_{-/+}\big) u_0, u_0 \big\rangle_{\mathcal{H}} + \|u_1 - k_{-/+} u_0\|^2_{\mathcal{H}}.$$

Put $h^{\mathbf{z}}_{-/+}, k^{\mathbf{z}}_{-/+}, \dot{H}^{\mathbf{z}}_{-/+}$ and $\dot{\mathcal{E}}^{\mathbf{z}}_{-/+}$ for the restrictions of the above operators and spaces on $\ker(\mathrm{i}\partial_z + \mathbf{z})$. Then $(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{-/+}})_{t\in\mathbb{R}}$ is unitary and $(\dot{H}^{\mathbf{z}}_{-/+}, \mathscr{D}(\dot{H}^{\mathbf{z}}_{-/+}))$ self-adjoint (the argument is the same as for the separable comparison dynamics above).

Theorem 2.3.10 below states that there exists a dense subspace $\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{-/+} \subset \dot{\mathcal{E}}^{\mathbf{z}}_{-/+}$ such that $\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i^2_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{-/+}}$ and $\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{-/+}} i^2_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$ have strong limits in $\mathcal{B}(\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{-/+}, \dot{\mathcal{E}}^{\mathbf{z}})$ and $\mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{E}}^{\mathbf{z}}_{-/+})$ as $|t| \to +\infty$.

Geometric profiles. Finally, let us introduce two last profiles we will refer to later on as the *geometric profiles*. We want them to describe a transport along principal null geodesics $\gamma_{\text{in/out}}$ introduced in Subsection 2.2.1.

The generators of time-parametrized[11] principal null geodesics are given by

$$\begin{aligned} \boldsymbol{v}_{\text{in}} &= \partial_t + L_{\mathscr{H}}, & L_{\mathscr{H}} &:= -\partial_x - sV(r)\partial_z, \\ \boldsymbol{v}_{\text{out}} &= \partial_t + L_{\mathscr{I}}, & L_{\mathscr{I}} &:= \partial_x - sV(r)\partial_z. \end{aligned}$$

The natural equation one may want to consider to describe transport along $\gamma_{\text{in/out}}$ would be

$$\frac{1}{2}\big(\boldsymbol{v}_{\text{in}}\boldsymbol{v}_{\text{out}} + \boldsymbol{v}_{\text{out}}\boldsymbol{v}_{\text{in}}\big) = 0$$

However, $[L_{\mathscr{H}}, L_{\mathscr{I}}] \neq 0$ (because $[\partial_x, V] \neq 0$) and no conserved positive energy can be associated to this equation.

The idea is to use instead new dynamics with natural conserved energies whose incoming and outgoing parts are $\boldsymbol{v}_{\text{in}}$ and $\boldsymbol{v}_{\text{out}}$. Let us define

$$\begin{aligned} L_+ &:= -L_{\mathscr{H}} - 2sV_-\partial_z = \partial_x + s(V(r) - 2V_-)\partial_z, \\ L_- &:= -L_{\mathscr{I}} - 2sV_+\partial_z = -\partial_x + s(V(r) - 2V_+)\partial_z \end{aligned}$$

and set

$$h_{\mathscr{H}/\mathscr{I}} := -\big(L_{\mathscr{H}/\mathscr{I}} + sV_{-/+}\partial_z\big)^2 = -\big(\partial_x +/- s(V(r) - V_{-/+})\partial_z\big)^2, \qquad k_{\mathscr{H}/\mathscr{I}} := -\mathrm{i}sV_{-/+}\partial_z.$$

The associated second order equation reads

$$\big((\partial_t - \mathrm{i}k_{\mathscr{H}/\mathscr{I}})^2 + h_{\mathscr{H}/\mathscr{I}}\big)u = \partial_t^2 u - 2\mathrm{i}k_{\mathscr{H}/\mathscr{I}}\partial_t u + L_{\mathscr{H}/\mathscr{I}}L_{+/-}u = 0 \tag{2.23}$$

so that we have the factorizations

$$\begin{aligned} (\partial_t - \mathrm{i}k_{\mathscr{H}})^2 + h_{\mathscr{H}} &= \big(\partial_t + L_{\mathscr{H}}\big)\big(\partial_t + L_+\big), \\ (\partial_t - \mathrm{i}k_{\mathscr{I}})^2 + h_{\mathscr{I}} &= \big(\partial_t + L_{\mathscr{I}}\big)\big(\partial_t + L_-\big) \end{aligned}$$

with this time $[L_{\mathscr{H}}, L_+] = [L_{\mathscr{I}}, L_-] = 0$. The *incoming part* $\big(\partial_t + L_{\mathscr{H}}\big)$ describes a transport towards $\mathscr{H}$ along γ_{in} whereas the *outgoing part* $\big(\partial_t + L_{\mathscr{I}}\big)$ describes a transport towards $\mathscr{I}$ along γ_{out}. The artificial parts $\big(\partial_t + L_{+/-}\big)$ describe transports towards $\mathscr{I}/\mathscr{H}$ along modified principal null geodesics; they will disappear when we will send data on the horizons later on.

The corresponding Hamiltonians

$$\dot{H}_{\mathscr{H}/\mathscr{I}} = \begin{pmatrix} 0 & \mathbb{1} \\ h_{\mathscr{H}/\mathscr{I}} - k_{\mathscr{H}/\mathscr{I}}^2 & 2k_{\mathscr{H}/\mathscr{I}} \end{pmatrix}, \qquad \mathscr{D}(\dot{H}_{\mathscr{H}/\mathscr{I}}) = \big\{u \in \dot{\mathcal{E}}_{\mathscr{H}/\mathscr{I}} \mid \dot{H}_{\mathscr{H}/\mathscr{I}}u \in \dot{\mathcal{E}}_{\mathscr{H}/\mathscr{I}}\big\}$$

act on their energy spaces $\big(\dot{\mathcal{E}}_{\mathscr{H}/\mathscr{I}}, \|\cdot\|_{\dot{\mathcal{E}}_{\mathscr{H}/\mathscr{I}}}\big)$ with

$$\|(u_0, u_1)\|^2_{\dot{\mathcal{E}}_{\mathscr{H}/\mathscr{I}}} := \big\langle h_{\mathscr{H}/\mathscr{I}}u_0, u_0\big\rangle_{\mathcal{H}} + \|u_1 - k_{\mathscr{H}/\mathscr{I}}u_0\|^2_{\mathcal{H}}.$$

[11] We can parametrize the principal null geodesics by the time variable t as $\frac{\mathrm{d}t}{\mathrm{d}r} = \pm F(r)^{-1}$, *cf.* Subsection 2.2.1).

Notice here that $h_{\mathscr{H}/\mathscr{I}}$ plays the role of h_0 for the full dynamics, so that we have to subtract $k^2_{\mathscr{H}/\mathscr{I}}$ in the Hamiltonians $\dot{H}_{\mathscr{H}/\mathscr{I}}$; besides, we can check that $h_{\mathscr{H}/\mathscr{I}} > 0$ since $\ker_{\mathcal{H}^{\mathbf{z}}}(h_{\mathscr{H}/\mathscr{I}}) = \{0\}$ for all $\mathbf{z} \in \mathbb{Z}$. By construction, the energies $\|\cdot\|_{\dot{\mathcal{E}}_{\mathscr{H}/\mathscr{I}}}$ are conserved and the evolutions $(\mathrm{e}^{\mathrm{i}t\dot{H}_{\mathscr{H}/\mathscr{I}}})_{t\in\mathbb{R}}$ are unitary on $\mathcal{H}\times\mathcal{H}$.

Using as above the notation $^{\mathbf{z}}$ for the restriction on $\ker(\mathrm{i}\partial_z + \mathbf{z})$, Theorem 2.3.11 below states that that there exists a dense subspace $\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}/\mathscr{I}} \subset \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}$ such that $\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}}$ and $\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}} -_{/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$ have strong limits in $\mathcal{B}(\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}/\mathscr{I}}, \dot{\mathcal{E}}^{\mathbf{z}})$ and $\mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}})$. These limits will be given a geometric sense in Subsection 2.5.4.

Remark 2.3.1. *1. We implicitly used [GGH17, Lemma 3.19] to define all the above dynamics. We refer to Section 3 therein for a detailed discussion about the energy spaces and the Hamiltonian formalism associated to abstract Klein-Gordon equations.*

2. The self-jointedness of the different Hamiltonians (and thus the unitarity of the associated dynamics) have different origins:

(i) For the asymptotic Hamiltonians $\dot{\widetilde{H}}^{\mathbf{z}}_{\pm}$, self-adjointness is due to the decay rate $\widetilde{k}^{\mathbf{z}}_{\pm} = \mathcal{O}_{x\to\pm\infty}\big(\mathrm{e}^{2\kappa_{\pm}x}\big)$ which ensures that $\widetilde{k}^{\mathbf{z}}_{\pm} \lesssim \widetilde{h}^{\mathbf{z}}_{\pm}$.

(ii) For the comparison Hamiltonians $\dot{H}^{\mathbf{z}}_{\pm\infty}$, $\dot{H}^{\mathbf{z}}_{-/+}$ and $\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}$, self-adjointness is entailed by the conservation of the associated homogeneous energies, which relies on the commutation of ∂_x with $k^{\mathbf{z}}_{\pm\infty} = k^{\mathbf{z}}_{-/+} = k^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}$.

3. Let $\Sigma_{t'} := \{t = t'\} \times \mathbb{S}^1_z \times \mathbb{R}_x \times \mathbb{S}^2_\omega$. In all the above statements, we implicitly identify the slices Σ_0 and Σ_t for all $t \in \mathbb{R}$ using the curves $\{t = \text{constant}\}$. All the energy spaces $\dot{\mathcal{E}}, \dot{\mathcal{E}}_{\pm\infty}$ and $\dot{\mathcal{E}}_{-/+}$ use coordinates (z, x, ω) in Σ_0.

2.3.3 Transport along principal null geodesics

The purpose of this paragraph is to explicit the action of the Hamiltonians $\dot{H}_{\mathscr{H}/\mathscr{I}}$. The same considerations hold for $\dot{H}_{-/+}$, but we omit details here as this case has been treated in [GGH17], paragraph 13.3.

Let $\dot{\mathcal{H}}^1_{\mathscr{H}/\mathscr{I}}$ be the completions of $\mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0)$ with respect to the norms

$$\|u\|_{\dot{\mathcal{H}}^1_{\mathscr{H}/\mathscr{I}}} := \big\|(L_{\mathscr{H}/\mathscr{I}} + \mathrm{i}k_{\mathscr{H}/\mathscr{I}})u\big\|_{\mathcal{H}} = \big\|\big(\partial_x +/- s(V(r) - V_{-/+})\partial_z\big)u\big\|_{\mathcal{H}}.$$

When $s = 0$, we have $h_{\mathscr{H}/\mathscr{I}} = -\partial_x^2$ and $\dot{\mathcal{H}}^1_{\mathscr{H}/\mathscr{I}}$ are nothing but the standard homogeneous Sobolev space $\dot{H}^1(\mathbb{R}, \mathrm{d}x)$ naturally associated to the one-dimensional wave equation. Consider the following canonical transformations:

$$\Psi_{\mathscr{H}} := \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathrm{i}L_{\mathscr{H}} & \mathrm{i}L_+ \end{pmatrix} : \dot{\mathcal{H}}^1_{\mathscr{H}} \times \dot{\mathcal{H}}^1_{\mathscr{H}} \longrightarrow \dot{\mathcal{E}}_{\mathscr{H}},$$
$$\Psi_{\mathscr{I}} := \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathrm{i}L_- & \mathrm{i}L_{\mathscr{I}} \end{pmatrix} : \dot{\mathcal{H}}^1_{\mathscr{I}} \times \dot{\mathcal{H}}^1_{\mathscr{I}} \longrightarrow \dot{\mathcal{E}}_{\mathscr{I}}.$$

While the second column of $\Psi_{\mathscr{H}}$ is artificial, the first one is related to the principal null geodesics γ_{in}: for an incoming/outgoing solution u^{in}, that is $(\partial_t + L_{\mathscr{H}})u^{\mathrm{in}} = 0$, we have $\Psi_{\mathscr{H}}(u^{\mathrm{in}}, 0) =$

$(u^{\mathrm{in}}, -\mathrm{i}\partial_t u^{\mathrm{in}})$ so $\Psi_{\mathscr{H}}$ prepare initial data for evolution along γ_{in}. Similarly, the second column of $\Psi_{\mathscr{I}}$ is related to transport along γ_{out}.

Lemma 2.3.2. *The transformations $\Psi_{\mathscr{H}/\mathscr{I}}$ are invertible isometries with inverses*

$$\Psi_{\mathscr{H}}^{-1} = \sqrt{2}\begin{pmatrix} L_+ & \mathrm{i} \\ -L_{\mathscr{H}} & -\mathrm{i}\end{pmatrix}(L_+ - L_{\mathscr{H}})^{-1}, \qquad \Psi_{\mathscr{I}}^{-1} = \sqrt{2}\begin{pmatrix} L_{\mathscr{I}} & \mathrm{i} \\ -L_- & -\mathrm{i}\end{pmatrix}(L_{\mathscr{I}} - L_-)^{-1}.$$

Proof. We only treat the $\mathscr{H}$ case. Let $(u_0, u_1) \in \dot{\mathcal{H}}^1_{\mathscr{H}} \times \dot{\mathcal{H}}^1_{\mathscr{H}}$. Using the parallelogram law $\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2$, we compute:

$$\begin{aligned}\big\|\Psi_{\mathscr{H}}(u_0,u_1)\big\|^2_{\dot{\mathcal{E}}_{\mathscr{H}}} &= \frac{1}{2}\big\|(L_{\mathscr{H}} + \mathrm{i}k_{\mathscr{H}})(u_0+u_1)\big\|^2_{\mathcal{H}} + \frac{1}{2}\big\|\mathrm{i}(L_{\mathscr{H}} + \mathrm{i}k_{\mathscr{H}})u_0 + \mathrm{i}(L_+ + \mathrm{i}k_{\mathscr{H}})u_1\big\|^2_{\mathcal{H}} \\ &= \frac{1}{2}\big\|(L_{\mathscr{H}} + \mathrm{i}k_{\mathscr{H}})(u_0+u_1)\big\|^2_{\mathcal{H}} + \frac{1}{2}\big\|(L_{\mathscr{H}} + \mathrm{i}k_{\mathscr{H}})(u_0-u_1)\big\|^2_{\mathcal{H}} \\ &= \|(u_0,u_1)\|^2_{\dot{\mathcal{H}}^1_{\mathscr{H}}\times\dot{\mathcal{H}}^1_{\mathscr{H}}}.\end{aligned}$$

Next, the expression for $\Psi_{\mathscr{H}}^{-1}$ makes sense since $L_+ - L_{\mathscr{H}} = -2(L_{\mathscr{H}} + \mathrm{i}k_{\mathscr{H}})$ have a trivial kernel in $\mathcal{H}$ and we have

$$\big((L_{\mathscr{H}} + \mathrm{i}k_{\mathscr{H}})^{-1}\phi\big)(z,x,\omega) = -\int_0^x \mathrm{e}^{-\mathrm{i}s\mathbf{z}\int_y^x (V(y')-V_-)\mathrm{d}y'}\phi(z,y,\omega)\mathrm{d}y \qquad \forall \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0)\cap\ker(\mathrm{i}\partial_z + \mathbf{z}).$$

To check that $\Psi_{\mathscr{H}}^{-1}$ indeed inverts $\Psi_{\mathscr{H}}$, we use that $[L_{\mathscr{H}}, L_+] = 0$. □

Lemma 2.3.3. *The transformations $\Psi_{\mathscr{H}/\mathscr{I}}$ diagonalize the Hamiltonians $\dot{H}_{\mathscr{H}/\mathscr{I}}$:*

$$\Psi_{\mathscr{H}}^{-1}\dot{H}_{\mathscr{H}}\Psi_{\mathscr{H}} = \dot{\mathbb{H}}_{\mathscr{H}} := \mathrm{i}\begin{pmatrix} L_{\mathscr{H}} & 0 \\ 0 & L_+\end{pmatrix} : \Psi_{\mathscr{H}}^{-1}\mathscr{D}(\dot{H}_{\mathscr{H}}) \longrightarrow \dot{\mathcal{H}}^1_{\mathscr{H}} \times \dot{\mathcal{H}}^1_{\mathscr{H}},$$
$$\Psi_{\mathscr{I}}^{-1}\dot{H}_{\mathscr{I}}\Psi_{\mathscr{I}} = \dot{\mathbb{H}}_{\mathscr{I}} := \mathrm{i}\begin{pmatrix} L_{\mathscr{I}} & 0 \\ 0 & L_-\end{pmatrix} : \Psi_{\mathscr{I}}^{-1}\mathscr{D}(\dot{H}_{\mathscr{I}}) \longrightarrow \dot{\mathcal{H}}^1_{\mathscr{I}} \times \dot{\mathcal{H}}^1_{\mathscr{I}}.$$

The dynamics $(\mathrm{e}^{\mathrm{i}t\dot{\mathbb{H}}_{\mathscr{H}/\mathscr{I}}})_{t\in\mathbb{R}}$ have therefore unitary extensions on $\dot{\mathcal{H}}^1_{\mathscr{H}/\mathscr{I}} \times \dot{\mathcal{H}}^1_{\mathscr{H}/\mathscr{I}}$.

Proof. We only treat the $\mathscr{H}$ case. Recall that $L_+ = -L_{\mathscr{H}} - 2\mathrm{i}k_{\mathscr{H}}$ and $h_{\mathscr{H}} - k^2_{\mathscr{H}} = L_{\mathscr{H}}L_+$. We compute:

$$\dot{H}_{\mathscr{H}}\Psi_{\mathscr{H}} = \begin{pmatrix} 0 & \mathbb{1} \\ L_{\mathscr{H}}L_+ & 2k_{\mathscr{H}}\end{pmatrix}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathrm{i}L_{\mathscr{H}} & \mathrm{i}L_+\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i}L_{\mathscr{H}} & \mathrm{i}L_+ \\ -L^2_{\mathscr{H}} & -L^2_+\end{pmatrix},$$
$$\Psi_{\mathscr{H}}\dot{\mathbb{H}}_{\mathscr{H}} = \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathrm{i}L_{\mathscr{H}} & \mathrm{i}L_+\end{pmatrix}\begin{pmatrix} \mathrm{i}L_{\mathscr{H}} & 0 \\ 0 & \mathrm{i}L_+\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i}L_{\mathscr{H}} & \mathrm{i}L_+ \\ -L^2_{\mathscr{H}} & -L^2_+\end{pmatrix}.$$

This gives the desired formula. □

We can now interpret the dynamics $(\mathrm{e}^{\mathrm{i}t\dot{H}_{\mathscr{H}/\mathscr{I}}})_{t\in\mathbb{R}}$ as mixed transports towards the horizons. Introduce the time-parametrized curves γ_+ and γ_- defined in the (z, r, ω) coordinates by

$$\begin{cases}\dot{\gamma}_+(t) := \big(1, s(V(r(t)) - 2V_-), F(r(t)), 0\big), \\ \dot{\gamma}_-(t) := \big(1, s(V(r(t)) - 2V_+), -F(r(t)), 0\big)\end{cases}. \tag{2.24}$$

The curve γ_+ carries data to $\mathscr{I}^+$ whereas γ_- carries data to $\mathscr{H}^+$. By Lemma 2.3.3, we have for all $(\phi_0, \phi_1) \in \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0) \times \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0)$ and all $t \in \mathbb{R}$:

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}t\dot{\mathbb{H}}_{\mathscr{H}/\mathscr{I}}} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} &= \begin{pmatrix} \mathrm{e}^{-tL_{\mathscr{H}/\mathscr{I}}} & 0 \\ 0 & \mathrm{e}^{-tL_{+/-}} \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} \phi_0 \circ \gamma_{\mathrm{in/out}}(t) \\ \phi_1 \circ \gamma_{+/-}(t) \end{pmatrix}, \\
\mathrm{e}^{\mathrm{i}t\dot{H}_{\mathscr{H}/\mathscr{I}}} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} -L_{+/-}\widetilde{\phi}_0 - \mathrm{i}\widetilde{\phi}_1 \\ \mathrm{i}L_{\mathscr{H}/\mathscr{I}}\big(-L_{+/-}\widetilde{\phi}_0 - \mathrm{i}\widetilde{\phi}_1\big) \end{pmatrix} \circ \gamma_{\mathrm{in/out}}(t) + \frac{1}{2} \begin{pmatrix} L_{\mathscr{H}/\mathscr{I}}\widetilde{\phi}_0 + \mathrm{i}\widetilde{\phi}_1 \\ \mathrm{i}L_{+/-}\big(L_{\mathscr{H}/\mathscr{I}}\widetilde{\phi}_0 + \mathrm{i}\widetilde{\phi}_1\big) \end{pmatrix} \circ \gamma_{+/-}(t) \\
&= \frac{1}{2} \begin{pmatrix} \phi_0 - \mathrm{i}\big(\widetilde{\phi}_1 - k_{\mathscr{H}/\mathscr{I}}\widetilde{\phi}_0\big) \\ \mathrm{i}L_{\mathscr{H}/\mathscr{I}}\Big[\phi_0 - \mathrm{i}\big(\widetilde{\phi}_1 - k_{\mathscr{H}/\mathscr{I}}\widetilde{\phi}_0\big)\Big] \end{pmatrix} \circ \gamma_{\mathrm{in/out}}(t) \\
&\quad + \frac{1}{2} \begin{pmatrix} \phi_0 + \mathrm{i}\big(\widetilde{\phi}_1 - k_{\mathscr{H}/\mathscr{I}}\widetilde{\phi}_0\big) \\ \mathrm{i}L_{+/-}\Big[\phi_0 + \mathrm{i}\big(\widetilde{\phi}_1 - k_{\mathscr{H}/\mathscr{I}}\widetilde{\phi}_0\big)\Big] \end{pmatrix} \circ \gamma_{+/-}(t)
\end{aligned} \tag{2.25}
$$

where $\widetilde{\phi}_j := (L_{\mathscr{H}/\mathscr{I}} + \mathrm{i}k_{\mathscr{H}/\mathscr{I}})^{-1}\phi_j$. At $t = 0$, we get (ϕ_0, ϕ_1) back on the right-hand side above.

2.3.4 Structure of the energy spaces for the comparison dynamics

We analyze in this Subsection the structure of the energy spaces associated to the asymptotic and geometric comparison dynamics introduced in Subsection 2.3.2. We will obtain explicit representation formulas on dense subspaces in smooth compactly supported functions. This will help us to show existence and completeness of the wave operators in Theorem 2.3.10 and Theorem 2.3.11. In all this Section, we will restrict ourselves to $\ker(\mathrm{i}\partial_z + \mathbf{z})$ with $\mathbf{z} \in \mathbb{Z}$.

Structure of the energy spaces for the asymptotic profiles

The solutions of the initial value problem associated to (2.22)

$$
\begin{cases} \big(\partial_t^2 - 2\mathrm{i}k^{\mathbf{z}}_{-/+}\partial_t + h^{\mathbf{z}}_{-/+}\big)u = 0 \\ u_{|\Sigma_0} = u_0 \in \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0) \\ (-\mathrm{i}\partial_t u)_{|\Sigma_0} = u_1 \in \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0) \end{cases}
$$

are given by the Kirchhoff type formula:

$$
\begin{aligned}
u(t, z, x, \omega) = &\frac{\mathrm{e}^{\mathrm{i}s\mathbf{z}tV_{-/+}}}{2} \left(u_0(z, x + t, \omega) + \mathrm{i}\int_0^{x+t} \big(u_1 - sV_{-/+}\mathbf{z}u_0\big)(z, y, \omega) \right) \\
&+ \frac{\mathrm{e}^{\mathrm{i}s\mathbf{z}tV_{-/+}}}{2} \left(u_0(z, x - t, \omega) + \mathrm{i}\int_{x-t}^{0} \big(u_1 - sV_{-/+}\mathbf{z}u_0\big)(z, y, \omega) \right).
\end{aligned} \tag{2.26}
$$

They are a linear combination of incoming and outgoing solutions of (2.22).

The simplicity of the asymptotic profiles has a cost: all the angular information has been lost in the construction of $\dot{H}^{\mathbf{z}}_{-/+}$. As a result, as explained in [GGH17], there is no chance that the limits

$$
\boldsymbol{W}_{l/r}u = \lim_{|t|\to+\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{-/+}} u
$$

exist for all $u \in \dot{\mathcal{E}}^{\mathbf{z}}_{-/+}$ since $\dot{H}^{\mathbf{z}}$ is built from $h^{\mathbf{z}}$ which acts on the angular part of u. The strategy is to define the limits first on a suitable dense subspace then to extend the corresponding operator by continuity on the whole space $\dot{\mathcal{E}}^{\mathbf{z}}_{-/+}$. We define the spaces

$$\mathcal{E}^{\mathrm{fin},\mathbf{z}}_{l/r} := \left\{ u \in \mathcal{E}^{\mathbf{z}}_{l/r} \;\middle|\; \exists \ell_0 > 0 \,;\; u \in \big(L^2(\mathbb{S}^1 \times \mathbb{R}, \mathrm{d}z\mathrm{d}x) \otimes \bigoplus_{\ell \leq \ell_0} Y_\ell\big) \times \big(L^2(\mathbb{S}^1 \times \mathbb{R}, \mathrm{d}z\mathrm{d}x) \otimes \bigoplus_{\ell \leq \ell_0} Y_\ell\big) \right\}$$

where Y_ℓ is the eigenspace associated to the eigenvalue $\ell(\ell+1)$ of the self-adjoint realization $(-\Delta_{\mathbb{S}^2}, H^2(\mathbb{S}^2, \mathrm{d}\omega))$. In order to exploit (2.26) and decompose elements of $\mathcal{E}^{\mathrm{fin},\mathbf{z}}_{l/r}$ into incoming and outgoing solutions of (2.22), we will use the following spaces:

$$\dot{\mathcal{E}}^{L,\mathbf{z}}_{-/+} := \Big\{ (u_0, u_1) \in \dot{\mathcal{E}}_{-/+} \;\Big|\; u_1 - sV_{-/+}\mathbf{z}u_0 \in L^1\big(\mathbb{R}, \mathrm{d}x\,; (\mathbb{S}^1 \times \mathbb{S}^2, \mathrm{d}z\mathrm{d}\omega)\big), \\ \int_{\mathbb{R}} \big(u_1 - sV_{-/+}\mathbf{z}u_0\big)(z,x,\omega)\mathrm{d}x = 0 \text{ a.e. in } z, \omega \Big\}.$$

[GGH17, Lemma 13.3] shows that[12]

$$\dot{\mathcal{E}}^{L,\mathbf{z}}_{-/+} = \dot{\mathcal{E}}^{\mathrm{in},\mathbf{z}}_{-/+} \oplus \dot{\mathcal{E}}^{\mathrm{out},\mathbf{z}}_{-/+} \tag{2.27}$$

with the spaces of incoming and outgoing initial data

$$\dot{\mathcal{E}}^{\mathrm{in},\mathbf{z}}_{-/+} = \Big\{ (u_0, u_1) \in \dot{\mathcal{E}}^{L,\mathbf{z}}_{-/+} \;\Big|\; u_1 = -\mathrm{i}\partial_x u_0 - sV_{-/+}\mathbf{z}u_0 \Big\},$$
$$\dot{\mathcal{E}}^{\mathrm{out},\mathbf{z}}_{-/+} = \Big\{ (u_0, u_1) \in \dot{\mathcal{E}}^{L,\mathbf{z}}_{-/+} \;\Big|\; u_1 = \mathrm{i}\partial_x u_0 - sV_{-/+}\mathbf{z}u_0 \Big\}.$$

Solutions in $\dot{\mathcal{E}}^{L,\mathbf{z}}_{-/+}$ verify a Huygens principle (*cf.* [GGH17, Remark 13.4]). The cancellation of the integral in particular removes the resonance at 0 of the wave equation; the non-vanishing term as $t \to \pm\infty$ is the projection of compactly supported data on the resonant state which is nothing but the constant solution $1 \notin \mathcal{H}$ (see the De Sitter-Schwarzschild case for $s = 0$ in [BoHa08], Proposition 1.2 and Theorem 1.3; here the resonant state is $r \otimes \omega_0$ with $\omega_0 \in Y_0$ the fundamental spherical harmonic because we use the conjugated spatial operator $r\widehat{P}r^{-1}$).

Finally, set

$$\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{l/r} := \big(\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^2) \times \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^2)\big) \cap \mathcal{E}^{\mathrm{fin},\mathbf{z}}_{l/r} \cap \dot{\mathcal{E}}^{L}_{-/+}.$$

Then [GGH17, Lemma 13.5] shows that $\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{l/r}$ is dense in $\big(\mathcal{E}^{\mathrm{fin},\mathbf{z}}_{l/r}, \|\cdot\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{-/+}}\big)$ (and thus in $\big(\dot{\mathcal{E}}^{\mathbf{z}}_{-/+}, \|\cdot\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{-/+}}\big)$). We will show similar results for the geometric profiles in the next paragraph.

Structure of the energy spaces for the geometric profiles

The solutions of the initial value problem associated to (2.23)

$$\begin{cases} \big(\partial_t^2 - 2\mathrm{i}k^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}\partial_t + L^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}L^{\mathbf{z}}_{+/-}\big)u = 0 \\ u_{|\Sigma_0} = u_0 \in \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0) \\ (-\mathrm{i}\partial_t u)_{|\Sigma_0} = u_1 \in \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0) \end{cases}$$

[12] Observe the different gauge used therein (the initial time-derivative is $\partial_t u$). To obtain the incoming/outgoing spaces of [GGH17], we need to replace u_1 by $\mathrm{i}u_1$ below; this of course does not modify the results.

are given by the Kirchhoff type formula (we drop the dependence in $(z,\omega)\in\mathbb{S}^1\times\mathbb{S}^2$):

$$u_{\mathscr{H}}(t,x)=\frac{1}{2}\mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^{x+t}V(x')\mathrm{d}x'}\left(u_0(x+t)+\mathrm{i}\int_0^{x+t}\mathrm{e}^{-\mathrm{i}s\mathbf{z}\int_y^{x+t}(V(y')-V_-)\mathrm{d}y'}\big(u_1-s\mathbf{z}V_-u_0\big)(y)\mathrm{d}y\right)$$
$$+\frac{1}{2}\mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^{x-t}(V(x')-2V_-)\mathrm{d}x'}\left(u_0(x-t)-\mathrm{i}\int_0^{x-t}\mathrm{e}^{-\mathrm{i}s\mathbf{z}\int_y^{x-t}(V(y')-V_-)\mathrm{d}y'}\big(u_1-s\mathbf{z}V_-u_0\big)(y)\mathrm{d}y\right), \tag{2.28}$$

$$u_{\mathscr{I}}(t,x)=\frac{1}{2}\mathrm{e}^{-\mathrm{i}s\mathbf{z}\int_x^{x+t}(V(x')-2V_+)\mathrm{d}x'}\left(u_0(x+t)+\mathrm{i}\int_0^{x+t}\mathrm{e}^{\mathrm{i}s\mathbf{z}\int_y^{x+t}(V(y')-V_+)\mathrm{d}y'}\big(u_1-s\mathbf{z}V_+u_0\big)(y)\mathrm{d}y\right)$$
$$+\frac{1}{2}\mathrm{e}^{-\mathrm{i}s\mathbf{z}\int_x^{x-t}V(x')\mathrm{d}x'}\left(u_0(x-t)-\mathrm{i}\int_0^{x-t}\mathrm{e}^{\mathrm{i}s\mathbf{z}\int_y^{x-t}(V(y')-V_+)\mathrm{d}y'}\big(u_1-s\mathbf{z}V_+u_0\big)(y)\mathrm{d}y\right). \tag{2.29}$$

The solution $u_{\mathscr{H}}$ is a linear combination of an incoming solution transported along the principal null geodesic γ_{in} and and outgoing solution transported along the artificial curve γ_+; the solution $u_{\mathscr{I}}$ is a linear combination of an incoming solution transported along the artificial curve γ_- and and outgoing solution transported along the principal null geodesic γ_{out}. Formulas (2.28) and (2.29) display the first components of $\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}}u$: we can easily check them using (2.25) with (we omit the dependence in $(z,\omega)\in\mathbb{S}^1\times\mathbb{S}^2$)

$$\big((L^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}+\mathrm{i}k^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}})^{-1}(u_1-k^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}u_0)\big)(x)=-/+\int_0^x\mathrm{e}^{-/+\mathrm{i}s\mathbf{z}\int_y^x(V(y')-V_{-/+})\mathrm{d}y'}\big(u_1-s\mathbf{z}V_{-/+}u_0\big)(y)\mathrm{d}y.$$

It will be useful below using the following simplified forms:

$$u_{\mathscr{H}}(t,x)=\frac{\mathrm{e}^{\mathrm{i}s\mathbf{z}tV_-}}{2}\left(\sum_{\pm}\mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^{x\pm t}(V(x')-V_-)\mathrm{d}x'}u_0(x\pm t)+\mathrm{i}\int_{x-t}^{x+t}\mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^y(V(x')-V_-)\mathrm{d}x'}\big(u_1-s\mathbf{z}V_-u_0\big)(y)\mathrm{d}y\right), \tag{2.30}$$

$$u_{\mathscr{I}}(t,x)=\frac{\mathrm{e}^{\mathrm{i}s\mathbf{z}tV_+}}{2}\left(\sum_{\pm}\mathrm{e}^{-\mathrm{i}s\mathbf{z}\int_x^{x\pm t}(V(x')-V_+)\mathrm{d}x'}u_0(x\pm t)+\mathrm{i}\int_{x-t}^{x+t}\mathrm{e}^{-\mathrm{i}s\mathbf{z}\int_x^y(V(x')-V_+)\mathrm{d}x'}\big(u_1-s\mathbf{z}V_+u_0\big)(y)\mathrm{d}y\right). \tag{2.31}$$

As for the asymptotic profiles of the paragraph 2.3.4, we need to control the angular directions on $\mathbb{S}^2_\omega$ if we wish to compare the geometric dynamics with the full one. We thus define

$$\mathcal{E}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}/\mathscr{I}}:=\left\{u\in\mathcal{E}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}\;\middle|\;\exists\ell_0>0\,;\;u\in\big(L^2(\mathbb{S}^1\times\mathbb{R},\mathrm{d}z\mathrm{d}x)\otimes\bigoplus_{\ell\leq\ell_0}Y_\ell\big)\times\big(L^2(\mathbb{S}^1\times\mathbb{R},\mathrm{d}z\mathrm{d}x)\otimes\bigoplus_{\ell\leq\ell_0}Y_\ell\big)\right\}.$$

Formulas (2.28) and (2.29) makes *a priori* no sense in $\mathcal{E}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}/\mathscr{I}}$ and the integral terms are not controlled in $\mathcal{H}$. We thus introduce the following spaces:

$$\dot{\mathcal{E}}^{L,\mathbf{z}}_{\mathscr{H}/\mathscr{I}}:=\Big\{(u_0,u_1)\in\dot{\mathcal{E}}_{\mathscr{H}/\mathscr{I}}\;\Big|\;u_1-s\mathbf{z}V_{-/+}u_0\in L^1\big(\mathbb{R},\mathrm{d}y\,;(\mathbb{S}^1\times\mathbb{S}^2,\mathrm{d}z\mathrm{d}\omega)\big),$$
$$\int_{\mathbb{R}}\mathrm{e}^{+/-\mathrm{i}s\mathbf{z}\int_0^y(V(x')-V_{-/+})\mathrm{d}x'}\big(u_1-s\mathbf{z}V_{-/+}u_0\big)(z,y,\omega)\mathrm{d}y=0\text{ a.e. in }z,\omega\Big\}.$$

We now establish a result similar to [GGH17, Lemma 13.3]: it gives a deeper meaning to incoming and outgoing data.

Lemma 2.3.4. *Let*

$$\dot{\mathcal{E}}^{\mathrm{in},\mathbf{z}}_{\mathscr{H}} = \Big\{(u_0,u_1)\in\dot{\mathcal{E}}^{L,\mathbf{z}}_{\mathscr{H}}\;\Big|\;u_1=\mathrm{i}L_{\mathscr{H}}u_0\Big\},\qquad \dot{\mathcal{E}}^{\mathrm{out},\mathbf{z}}_{\mathscr{H}} = \Big\{(u_0,u_1)\in\dot{\mathcal{E}}^{L,\mathbf{z}}_{\mathscr{H}}\;\Big|\;u_1=\mathrm{i}L_{+}u_0\Big\},$$
$$\dot{\mathcal{E}}^{\mathrm{in},\mathbf{z}}_{\mathscr{I}} = \Big\{(u_0,u_1)\in\dot{\mathcal{E}}^{L,\mathbf{z}}_{\mathscr{I}}\;\Big|\;u_1=\mathrm{i}L_{-}u_0\Big\},\qquad \dot{\mathcal{E}}^{\mathrm{out},\mathbf{z}}_{\mathscr{I}} = \Big\{(u_0,u_1)\in\dot{\mathcal{E}}^{L,\mathbf{z}}_{\mathscr{I}}\;\Big|\;u_1=\mathrm{i}L_{\mathscr{I}}u_0\Big\}.$$

The following decompositions into incoming and outgoing solutions of (2.23) *hold:*

$$\dot{\mathcal{E}}^{L,\mathbf{z}}_{\mathscr{H}/\mathscr{I}} = \dot{\mathcal{E}}^{\mathrm{in},\mathbf{z}}_{\mathscr{H}/\mathscr{I}}\oplus\dot{\mathcal{E}}^{\mathrm{out},\mathbf{z}}_{\mathscr{H}/\mathscr{I}}.$$

Furthermore, if $u=u^{\mathrm{in}}+u^{\mathrm{out}}\in\dot{\mathcal{E}}^{L,\mathbf{z}}_{\mathscr{H}/\mathscr{I}}$ *is supported in* $\mathbb{S}^1_z\times[R_1,R_2]_x\times\mathbb{S}^2_\omega$ *for some* $R_1,R_2\in\mathbb{R}$*, then*

$$\operatorname{Supp}u^{\mathrm{in}}\subset\mathbb{S}^1_z\times\,]-\infty,R_2]_x\times\mathbb{S}^2_\omega,\qquad \operatorname{Supp}u^{\mathrm{out}}\subset\mathbb{S}^1_z\times[R_1,+\infty[_x\times\mathbb{S}^2_\omega.\tag{2.32}$$

Proof. We only show the lemma for $\dot{\mathcal{E}}^{L,\mathbf{z}}_{\mathscr{H}}$. Recall that

$$\mathrm{i}L_{\mathscr{H}}u_0=-\mathrm{i}\partial_x u_0+s\mathbf{z}Vu_0,\qquad \mathrm{i}L_{+}u_0=\mathrm{i}\partial_x u_0-s\mathbf{z}(V-2V_-)u_0.$$

Let $u=(u_0,u_1)\in\dot{\mathcal{E}}^{L,\mathbf{z}}_{\mathscr{H}}$ and put $\tilde{u}(x):=\mathrm{e}^{\mathrm{i}s\mathbf{z}\int_0^x(V(y)-V_-)\mathrm{d}y}u(x)$. Then $u\in\dot{\mathcal{E}}^{\mathrm{in},\mathbf{z}}_{\mathscr{H}}$ if and only if

$$\tilde{u}_1=-\mathrm{i}\partial_x\tilde{u}_0+s\mathbf{z}V_-\tilde{u}_0$$

whereas $u\in\dot{\mathcal{E}}^{\mathrm{out},\mathbf{z}}_{\mathscr{H}}$ if and only if

$$\tilde{u}_1=\mathrm{i}\partial_x\tilde{u}_0+s\mathbf{z}V_-\tilde{u}_0.$$

These conditions define incoming and outgoing states for the asymptotic profiles (see [GGH17], above Lemma 13.3 with u_1 therein being $\mathrm{i}u_1$ for us). We then define[13] (omitting the dependence in $(z,\omega)\in\mathbb{S}^1\times\mathbb{S}^2$):

$$\tilde{u}_0^{\mathrm{in}}(x)=\frac{1}{2}\int_x^{+\infty}\big[-\partial_y\tilde{u}_0-\mathrm{i}(\tilde{u}_1-s\mathbf{z}V_-\tilde{u}_0)\big](y)\mathrm{d}y,$$
$$\tilde{u}_1^{\mathrm{in}}(x)=\frac{1}{2}\big[-\mathrm{i}\partial_x\tilde{u}_0+(\tilde{u}_1-s\mathbf{z}V_-\tilde{u}_0)\big](x)+\frac{s\mathbf{z}V_-}{2}\int_x^{+\infty}\big[-\partial_y\tilde{u}_0-\mathrm{i}(\tilde{u}_1-s\mathbf{z}V_-\tilde{u}_0)\big](y)\mathrm{d}y,$$

$$\tilde{u}_0^{\mathrm{out}}(x)=\frac{1}{2}\int_{-\infty}^{x}\big[\partial_y\tilde{u}_0-\mathrm{i}(\tilde{u}_1-s\mathbf{z}V_-\tilde{u}_0)\big](y)\mathrm{d}y,$$
$$\tilde{u}_1^{\mathrm{out}}(x)=\frac{1}{2}\big[\mathrm{i}\partial_x\tilde{u}_0+(\tilde{u}_1-s\mathbf{z}V_-\tilde{u}_0)\big](x)+\frac{s\mathbf{z}V_-}{2}\int_{-\infty}^{x}\big[\partial_y\tilde{u}_0-\mathrm{i}(\tilde{u}_1-s\mathbf{z}V_-\tilde{u}_0)\big](y)\mathrm{d}y.$$

Direct computations show that

$$\tilde{u}_1^{\mathrm{in}}=-\mathrm{i}\partial_x\tilde{u}_0^{\mathrm{in}}+s\mathbf{z}V_-\tilde{u}_0^{\mathrm{in}},\qquad \tilde{u}_1^{\mathrm{out}}=\mathrm{i}\partial_x\tilde{u}_0^{\mathrm{out}}+s\mathbf{z}V_-\tilde{u}_0^{\mathrm{out}},\qquad (\tilde{u}_0,\tilde{u}_1)=(\tilde{u}_0^{\mathrm{in}}+\tilde{u}_0^{\mathrm{out}},\tilde{u}_1^{\mathrm{in}}+\tilde{u}_1^{\mathrm{out}}).$$

[13] We use formula (13.8) of [GGH17, Lemma 13.3].

Here we use that

$$\int_{-\infty}^{+\infty} (\tilde{u}_1 - s\mathbf{z}V_-\tilde{u}_0)(y)\mathrm{d}y = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}s\mathbf{z}\int_0^y (V(y')-V_-)\mathrm{d}y'} \big(u_1 - s\mathbf{z}V_-u_0\big)(y)\mathrm{d}x = 0.$$

It follows that

$$u^{\mathrm{in/out}} := \mathrm{e}^{-\mathrm{i}s\mathbf{z}\int_0^x (V(y)-V_-)\mathrm{d}y}\big(\tilde{u}_0^{\mathrm{in/out}}, \tilde{u}_1^{\mathrm{in/out}}\big)$$

satisfy $u^{\mathrm{in/out}} \in \dot{\mathcal{E}}_{\mathscr{H}}^{\mathrm{in/out},\mathbf{z}}$ and $u = u^{\mathrm{in}} + u^{\mathrm{out}}$.

The support condition (2.32) for $\tilde{u}^{\mathrm{in/out}}$ directly reads on the above expressions and is in turn verified for $u^{\mathrm{in/out}}$ as multiplication by $\mathrm{e}^{\mathrm{i}s\mathbf{z}\int_0^x (V(y)-V_-)\mathrm{d}y}$ does not modify supports.

It remains to show that $\dot{\mathcal{E}}_{\mathscr{H}}^{\mathrm{in},\mathbf{z}} \cap \dot{\mathcal{E}}_{\mathscr{H}}^{\mathrm{out},\mathbf{z}} = \{0\}$. If u lies in the intersection, then $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_1 &= -\mathrm{i}\partial_x\tilde{u}_0 + s\mathbf{z}V_-\tilde{u}_0 \\ \tilde{u}_1 &= \mathrm{i}\partial_x\tilde{u}_0 + s\mathbf{z}V_-\tilde{u}_0 \end{cases}.$$

Adding and subtracting both the conditions, we see that $\tilde{u}_0$ is constant and $\tilde{u}_1 = s\mathbf{z}V_-\tilde{u}_0$ in $\mathcal{H}$, whence $\tilde{u}_0 = \tilde{u}_1 = 0$. This entails $u = 0$ and we are done. □

Remark 2.3.5. *Using the definitions of $\Psi_{\mathscr{H}/\mathscr{J}}$ in the paragraph 2.3.3, we easily show that*

$$\dot{\mathcal{E}}_{\mathscr{H}}^{\mathrm{in},\mathbf{z}} = \Psi_{\mathscr{H}}\big(\dot{\mathcal{H}}_{\mathscr{H}}^{1,\mathbf{z}} \times \{0\}\big) \cap \dot{\mathcal{E}}_{\mathscr{H}}^{L,\mathbf{z}}, \qquad \dot{\mathcal{E}}_{\mathscr{J}}^{\mathrm{out},\mathbf{z}} = \Psi_{\mathscr{J}}\big(\{0\} \times \dot{\mathcal{H}}_{\mathscr{J}}^{1,\mathbf{z}}\big) \cap \dot{\mathcal{E}}_{\mathscr{J}}^{L,\mathbf{z}}.$$

We then have the following adaptation of [GGH17, Lemma 13.5] to our framework:

Lemma 2.3.6. *Let*

$$\mathcal{D}_{\mathscr{H}/\mathscr{J}}^{\mathrm{fin},\mathbf{z}} := \big(\mathcal{C}_{\mathrm{c}}^\infty(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^2) \times \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{S}^1 \times \mathbb{R} \times \mathbb{S}^2)\big) \cap \mathcal{E}_{\mathscr{H}/\mathscr{J}}^{\mathrm{fin},\mathbf{z}} \cap \dot{\mathcal{E}}_{\mathscr{H}/\mathscr{J}}^{L,\mathbf{z}}.$$

Then $\mathcal{D}_{\mathscr{H}/\mathscr{J}}^{\mathrm{fin},\mathbf{z}}$ is dense in $\big(\dot{\mathcal{E}}_{\mathscr{H}/\mathscr{J}}^{\mathbf{z}}, \|\cdot\|_{\dot{\mathcal{E}}_{\mathscr{H}/\mathscr{J}}^{\mathbf{z}}}\big)$.

Proof. First of all, $\mathcal{E}_{\mathscr{H}/\mathscr{J}}^{\mathrm{fin},\mathbf{z}}$ is dense in $\dot{\mathcal{E}}_{\mathscr{H}/\mathscr{J}}^{\mathbf{z}}$ by definition of the homogeneous spaces and because the Hilbert direct sum of the eigenspaces Y_ℓ is $L^2(\mathbb{S}^2, \mathrm{d}\omega)$.

Next, $\big(\mathcal{C}_{\mathrm{c}}^\infty(\mathbb{S}^1_z \times \mathbb{R}_x \times \mathbb{S}^2_\omega) \times \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{S}^1_z \times \mathbb{R}_x \times \mathbb{S}^2_\omega)\big) \cap \mathcal{E}_{\mathscr{H}/\mathscr{J}}^{\mathrm{fin},,\mathbf{z}}$ is dense in $\mathcal{E}_{\mathscr{H}/\mathscr{J}}^{\mathrm{fin},,\mathbf{z}}$ by standard regularization arguments.

Finally, let $\psi \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{S}^1_z \times \mathbb{R}_x \times \mathbb{S}^2_\omega) \times \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{S}^1_z \times \mathbb{R}_x \times \mathbb{S}^2_\omega)$ such that $\psi \geq 0$, $\|\psi\|_{L^1(\mathbb{S}^1\times\mathbb{R}\times\mathbb{S}^2)} = 1$ and put $\psi_n(z,x,\omega) := \psi(z, n^{-1}x, \omega)$ for each $n \in \mathbb{N}\setminus\{0\}$. Pick $u \in \big(\mathcal{C}_{\mathrm{c}}^\infty(\mathbb{S}^1_z \times \mathbb{R}_x \times \mathbb{S}^2_\omega) \times \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{S}^1_z \times \mathbb{R}_x \times \mathbb{S}^2_\omega)\big) \cap \mathcal{E}_{\mathrm{in}}^{\mathrm{fin},\mathbf{z}}$ and define

$$\begin{cases} u_0^n := u_0, \\ u_1^n := u_1 - n^{-1}\psi_n \mathrm{e}^{-/+\mathrm{i}s\mathbf{z}\int_0^x (V(x')-V_{-/+})\mathrm{d}x'} \displaystyle\int_{\mathbb{R}} \mathrm{e}^{+/-\mathrm{i}s\mathbf{z}\int_0^y (V(x')-V_{-/+})\mathrm{d}x'} \big(u_1 - s\mathbf{z}V_{-/+}u_0\big)(z,y,\omega)\mathrm{d}y \end{cases}.$$

Then $u^n := (u_0^n, u_1^n) \in \big(\mathcal{C}_{\mathrm{c}}^\infty(\mathbb{S}^1_z \times \mathbb{R}_x \times \mathbb{S}^2_\omega) \times \mathcal{C}_{\mathrm{c}}^\infty(\mathbb{S}^1_z \times \mathbb{R}_x \times \mathbb{S}^2_\omega)\big) \cap \mathcal{E}_{\mathrm{in}}^{\mathrm{fin},\mathbf{z}} \cap \dot{\mathcal{E}}_{\mathrm{in}}^{L,\mathbf{z}}$ and we have

$$\begin{aligned} \|u_1^n - u_1\|_{\mathcal{H}^z} &\leq n^{-1/2}\|n^{-1/2}\psi_n\|_{\mathcal{H}^{\mathbf{z}}}\|u_1 - s\mathbf{z}V_{-/+}u_0\|_{L^1(\mathbb{S}^1\times\mathbb{R}\times\mathbb{S}^2)} \\ &\leq Cn^{-1/2}\|u_1 - s\mathbf{z}V_{-/+}u_0\|_{L^1(\mathbb{S}^1\times\mathbb{R}\times\mathbb{S}^2)} \end{aligned}$$

for some constant $C > 0$ independent of n. It remains to let $n \to +\infty$ to conclude the proof. □

Remark 2.3.7 (Minimal propagation speed)**.** *Let* $u = u^{\mathrm{in}} + u^{\mathrm{out}} \in \mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}}$ *and* $v = v^{\mathrm{in}} + v^{\mathrm{out}} \in \mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{J}}$ *such that* $\operatorname{Supp} u, \operatorname{Supp} v \subset \mathbb{S}^1_z \times [R_1, R_2]_x \times \mathbb{S}^2_\omega$ *for some* $R_1, R_2 \in \mathbb{R}$*. Using the relations*

$$u_1^{\mathrm{in}} = -\mathrm{i}\partial_x u_0^{\mathrm{in}} + s\mathbf{z}V u_0^{\mathrm{in}}, \qquad u_1^{\mathrm{out}} = \mathrm{i}\partial_x u_0^{\mathrm{out}} - s\mathbf{z}(V - 2V_-)u_0^{\mathrm{out}},$$
$$v_1^{\mathrm{in}} = -\mathrm{i}\partial_x v_0^{\mathrm{in}} - s\mathbf{z}(V - 2V_+)v_0^{\mathrm{in}}, \qquad v_1^{\mathrm{out}} = \mathrm{i}\partial_x v_0^{\mathrm{out}} + s\mathbf{z}V v_0^{\mathrm{out}},$$

we can integrate by parts in the Kirchhoff formulas (2.30) *and* (2.31) *to obtain*

$$\left(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u^{\mathrm{in}}\right)_0(z,x,\omega) = \mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^{x+t} V(x')\mathrm{d}x'} u_0^{\mathrm{in}}(z, x+t, \omega),$$
$$\left(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u^{\mathrm{out}}\right)_0(z,x,\omega) = \mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^{x-t} (V(x')-2V_-)\mathrm{d}x'} u_0^{\mathrm{out}}(z, x-t, \omega),$$
$$\left(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}} v^{\mathrm{in}}\right)_0(z,x,\omega) = \mathrm{e}^{-\mathrm{i}s\mathbf{z}\int_x^{x+t} (V(x')-2V_+)\mathrm{d}x'} v_0^{\mathrm{in}}(z, x+t, \omega),$$
$$\left(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}} v^{\mathrm{out}}\right)_0(z,x,\omega) = \mathrm{e}^{-\mathrm{i}s\mathbf{z}\int_x^{x-t} V(x')\mathrm{d}x'} v_0^{\mathrm{out}}(z, x-t, \omega).$$

It follows:

$$\operatorname{Supp}\left(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u^{\mathrm{in}}\right), \operatorname{Supp}\left(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}} v^{\mathrm{in}}\right) \subset \mathbb{S}^1_z \times]-\infty, R_2 - t]_x \times \mathbb{S}^2_\omega,$$
$$\operatorname{Supp}\left(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u^{\mathrm{out}}\right), \operatorname{Supp}\left(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}} v^{\mathrm{out}}\right) \subset \mathbb{S}^1_z \times [R_1 + t, +\infty[_x \times \mathbb{S}^2_\omega.$$

The dynamics $(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{J}}})_{t\in\mathbb{R}}$ *thus verifies the* Huygens principle *on* $\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}/\mathscr{J}}$.

2.3.5 Analytic scattering results

We state in this Subsection the scattering results we will prove in Section 2.4. For all of them, we need to first restrict to $\ker(\mathrm{i}\partial_z + \mathbf{z})$ with $\mathbf{z} \in \mathbb{Z}$ in order to apply the abstract theory of [GGH17] for Klein-Gordon equations with a non zero mass term, then to $|s|$ sufficiently small to use [**?**, Theorem 3.8].

The first result concerns the uniform boundedness of the propagator $(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}})_{t\in\mathbb{R}}$. The proof is given in Subsection 2.4.2.

Theorem 2.3.8 (Uniform boundedness of the evolution)**.** *Let* $\mathbf{z} \in \mathbb{Z} \setminus \{0\}$*. There exists* $s_0 > 0$ *such that for all* $s \in]-s_0, s_0[$*, there exists a constant* $C \equiv C(\mathbf{z}, s_0) > 0$ *such that*

$$\left\|\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \leq C\|u\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \qquad \forall t \in \mathbb{R},\ \forall u \in \dot{\mathcal{E}}^{\mathbf{z}}.$$

The next result concerns the asymptotic completeness between the full dynamics and the separable comparison dynamics. The proof is given in Subsection 2.4.3.

Theorem 2.3.9 (Asymptotic completeness, separable comparison dynamics)**.** *Let* $\mathbf{z} \in \mathbb{Z} \setminus \{0\}$*. There exists* $s_0 > 0$ *such that for all* $s \in]-s_0, s_0[$*, the following strong limits*

$$\boldsymbol{W}^f_\pm := \mathrm{s} - \lim_{t\to+\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_\pm \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\pm\infty}},$$
$$\boldsymbol{W}^p_\pm := \mathrm{s} - \lim_{t\to-\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_\pm \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\pm\infty}},$$
$$\boldsymbol{\Omega}^f_\pm := \mathrm{s} - \lim_{t\to+\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\pm\infty}} i_\pm \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}},$$
$$\boldsymbol{\Omega}^p_\pm := \mathrm{s} - \lim_{t\to-\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\pm\infty}} i_\pm \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$$

exist as bounded operators $\boldsymbol{W}^{f/p}_\pm \in \mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}_{\pm\infty}, \dot{\mathcal{E}}^{\mathbf{z}})$ *and* $\boldsymbol{\Omega}^{f/p}_\pm \in \mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{E}}^{\mathbf{z}}_{\pm\infty})$.

We now state an asymptotic completeness result with the asymptotic profiles. The proof is given in Subsection 2.4.4.

Theorem 2.3.10 (Asymptotic completeness, asymptotic profiles)**.** *Let* $\mathbf{z} \in \mathbb{Z} \setminus \{0\}$*. There exists* $s_0 > 0$ *such that for all* $s \in]-s_0, s_0[$*, the following holds:*

1. *For all* $u \in \mathcal{D}^{\mathrm{fin},\mathbf{z}}_{l/r}$*, the limits*

$$\boldsymbol{W}^{f}_{l/r} u = \lim_{t \to +\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{-/+}} u,$$
$$\boldsymbol{W}^{p}_{l/r} u = \lim_{t \to -\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{-/+}} u$$

   exist in $\dot{\mathcal{E}}^{\mathbf{z}}$*. The operators* $\boldsymbol{W}^{f/p}_{l/r}$ *extend to bounded operators* $\boldsymbol{W}^{f/p}_{l/r} \in \mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}_{-/+}, \dot{\mathcal{E}}^{\mathbf{z}})$*.*

2. *The inverse wave operators*

$$\boldsymbol{\Omega}^{f}_{l/r} u = \mathrm{s} - \lim_{t \to +\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{-/+}} i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}},$$
$$\boldsymbol{\Omega}^{p}_{l/r} u = \mathrm{s} - \lim_{t \to -\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{-/+}} i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$$

   exist in $\mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{E}}^{\mathbf{z}}_{-/+})$*.*

Finally, we state a last asymptotic completeness result using the geometric profiles. We will prove the following theorem in Subsection 2.4.5 and give a geometric interpretation in Subsection 2.5.4 (see Remark 2.5.8).

Theorem 2.3.11 (Asymptotic completeness, geometric profiles)**.** *Let* $\mathbf{z} \in \mathbb{Z} \setminus \{0\}$*. There exists* $s_0 > 0$ *such that for all* $s \in]-s_0, s_0[$*, the following holds:*

1. *For all* $u \in \mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}/\mathscr{I}}$*, the limits*

$$\boldsymbol{W}^{f}_{\mathscr{H}/\mathscr{I}} u = \lim_{t \to +\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}} u,$$
$$\boldsymbol{W}^{p}_{\mathscr{H}/\mathscr{I}} u = \lim_{t \to -\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}} u$$

   exist in $\dot{\mathcal{E}}^{\mathbf{z}}$*. The operators* $\boldsymbol{W}^{f/p}_{\mathscr{H}/\mathscr{I}}$ *extend to bounded operators* $\boldsymbol{W}^{f/p}_{\mathscr{H}/\mathscr{I}} \in \mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}, \dot{\mathcal{E}}^{\mathbf{z}})$*.*

2. *The inverse future/past wave operators*

$$\boldsymbol{\Omega}^{f}_{\mathscr{H}/\mathscr{I}} = \mathrm{s} - \lim_{t \to +\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}} i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}},$$
$$\boldsymbol{\Omega}^{p}_{\mathscr{H}/\mathscr{I}} = \mathrm{s} - \lim_{t \to -\infty} \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}} i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$$

   exist in $\mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}})$*.*

Remark 2.3.12. *Because of the cut-offs* $i_{\pm}$*, wave operators and inverse wave operators are not inverse. We will however justify this designation for the geometric profiles in Subsection 2.5.2.*

The geometric wave and inverse wave operators satisfy the following properties (the proof is given in Subsection 2.4.6):

Proposition 2.3.13. *Under the hypotheses of Theorem 2.3.11, it holds:*

$$\mathbf{\Omega}^{f/p}_{\mathscr{H}}\dot{\mathcal{E}}^{\mathbf{z}} \subset \Psi_{\mathscr{H}}\big(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \{0\}\big), \qquad \mathbf{\Omega}^{f/p}_{\mathscr{H}}\boldsymbol{W}^{f/p}_{\mathscr{H}} = \mathbb{1}_{\Psi_{\mathscr{H}}(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}\times\{0\})},$$
$$\mathbf{\Omega}^{f/p}_{\mathscr{I}}\dot{\mathcal{E}}^{\mathbf{z}} \subset \Psi_{\mathscr{I}}\big(\{0\} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{I}}\big), \qquad \mathbf{\Omega}^{f/p}_{\mathscr{I}}\boldsymbol{W}^{f/p}_{\mathscr{I}} = \mathbb{1}_{\Psi_{\mathscr{I}}(\{0\}\times\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{I}})}.$$

2.4 Proof of the analytic results

This Section is devoted to the proofs of the scattering results stated in Subsection 2.3.5. Theorems 2.3.8, 2.3.9 and 2.3.10 are direct consequences of the results obtained in [GGH17] once some geometric hypotheses are checked. Theorem 2.3.11 follows from standard arguments as well as propagation estimates of the full dynamics showed in [GGH17].

2.4.1 Geometric hypotheses

In this paragraph, we check that the geometric hypotheses (G) of [GGH17] are verified in our setting on $\ker(\mathrm{i}\partial_z + \mathbf{z})$, $\mathbf{z} \in \mathbb{Z} \setminus \{0\}$ ($\mathbf{z}$ must not be zero to conserve a mass term). We use the weight $w(r) := \sqrt{(r-r_-)(r_+-r)}$ defined for all $r \in]r_-, r_+[$.

(G1) The operator P in [GGH17] is[14] $-\Delta_{\mathbb{S}^2} - m^2r^2\partial_z^2$ for us, and satisfies of course $[P, \partial_z] = 0$.

(G2) Set

$$h^{\mathbf{z}}_{0,s} := F(r)^{-1/2}h^{\mathbf{z}}_0F(r)^{1/2} = -r^{-1}F(r)^{1/2}\partial_r r^2F(r)\partial_r r^{-1}F(r)^{1/2} - r^{-2}F(r)\Delta_{\mathbb{S}^2} + m^2F(r)\mathbf{z}^2.$$

Then $\alpha_1(r) = \alpha_3(r) = r^{-1}F(r)^{1/2}$, $\alpha_2(r) = rF(r)^{1/2}$ and $\alpha_4(r) = mF(r)^{1/2}\mathbf{z}$. These coefficients are clearly smooth in r. Furthermore, since we can write $F(r) = g(r)w(r)^2$ with $g(r) = \frac{\Lambda}{3r^2}(r-r_n)(r-r_c) \gtrsim 1$ for all $r \in]r_-, r_+[$, it comes for all $j \in \{1,2,3,4\}$

$$\alpha_j(r) - w(r)\left(i_-(r)\,\alpha_j^- + i_+(r)\,\alpha_j^+\right) = w(r)\left(g(r)^{1/2} - \alpha_j^{\pm}\right) = \mathcal{O}_{r\to r_\pm}\big(w(r)^2\big)$$

for

$$\alpha_1^{\pm} = \alpha_3^{\pm} = \frac{\alpha_2^{\pm}}{r_\pm^2} = \frac{r_\pm\alpha_4^{\pm}}{m\mathbf{z}} = \frac{1}{r_\pm^2}\sqrt{\frac{\Lambda(r_\pm - r_n)(r_\pm - r_c)}{3}}.$$

Also, we clearly have $\alpha_j(r) \gtrsim w(r)$. Direct computations show that

$$\partial_r^m\partial_\omega^n\left(\alpha_j - w\left(i_-\,\alpha_j^- + i_+\,\alpha_j^+\right)\right)(r) = \mathcal{O}_{r\to r_\pm}\big(w(r)^{2-2m}\big)$$

for all $m, n \in \mathbb{N}$.

[14] It is assumed in [GGH17] that the coefficients of P are independent of r; we can check however that this restriction can be relaxed to a broader class of operators including the one we use.

(G3) Set $k^{\mathbf{z}}_{s,v} := k^{\mathbf{z}}_s := k^{\mathbf{z}}$ and $k^{\mathbf{z}}_{s,r} := 0$, and put $k^{\mathbf{z},-}_{s,v} := sV_-$ and $k^{\mathbf{z},+}_{s,r} := sV_+$. We have $V(r) - V_\pm = \mathcal{O}_{r\to r_\pm}(|r_+ - r_\pm|) = \mathcal{O}_{r\to r_\pm}\big(w(r)^2\big)$ and $\partial_r^m \partial_\omega^n V(r)$ is bounded for any $m, n \in \mathbb{N}$.

(G4) & (G6) Our perturbed operator $h^{\mathbf{z}}_{0,s}$ is the same as in [GGH17] at the beginning of the paragraph 13.1 with $\Delta_r = F$, but without the bounded term $\lambda(r^2 + a^2)$. We then copy the proof, details are omitted.

(G5) Because $\mathbf{z} \neq 0$, we clearly have

$$\begin{aligned} h^{\mathbf{z}}_0 &= -\alpha_1(r)\partial_r w(r)^2 r^2 g(r) \partial_r \alpha_1(r) - \alpha_1(r)^2 \Delta_{\mathbb{S}^2} + \alpha_1(r)^2 m^2 r^2 \mathbf{z}^2 \\ &= \alpha_1(r)\left(-\partial_r w(r)^2 r^2 g(r)\partial_r - \Delta_{\mathbb{S}^2} + m^2 r^2 \mathbf{z}^2\right)\alpha_1(r) \\ &\gtrsim \alpha_1(r)\left(-\partial_r w(r)^2 \partial_r - \Delta_{\mathbb{S}^2} + 1\right)\alpha_1(r). \end{aligned}$$

(G7) We check that $(h_0 - \widetilde{k}_+^2, \widetilde{k}_+)$ and $(h_0 - (\widetilde{k}_- - k_-)^2, \widetilde{k}_- - k_-)$ satisfy (G5). Since $\alpha_1(r), k_\pm(r) - sV_\pm = \mathcal{O}_{r\to r_\pm}(|r_\pm - r|)$, we can write for $|s| < mr_-$:

$$\begin{aligned} \widetilde{h}_\pm &= -\alpha_1(r)\partial_r w(r)^2 r^2 g(r)\partial_r \alpha_1(r) - \alpha_1(r)^2\Delta_{\mathbb{S}^2} + \alpha_1(r)^2 m^2 r^2 \mathbf{z}^2 - (k_\pm(r) - sV_\pm)^2 \\ &= \alpha_1(r)\left(-\partial_r w(r)^2 r^2 g(r)\partial_r - \Delta_{\mathbb{S}^2} + m^2 r^2 \mathbf{z}^2 - \frac{(k_\pm(r) - V_\pm)^2}{\alpha_1(r)^2}\right)\alpha_1(r) \\ &\gtrsim \alpha_1(r)\left(-\partial_r w(r)^2 \partial_r - \Delta_{\mathbb{S}^2} + 1\right)\alpha_1(r). \end{aligned}$$

The geometric hypotheses are thus satisfied and we can apply the scattering results of Section 10 in [GGH17].

2.4.2 Proof of Theorem 2.3.8

Let us show Theorem 2.3.8. This result is similar to [GGH17, Theorem 12.1].

First, pick $\delta > 0$ and set $w(r) := \sqrt{(r - r_-)(r_+ - r)}$ defined for all $r \in]r_-, r_+[$. In [**?**], Subsections 3.3 and 3.4, it is shown that for a positive mass $m^2 > 0$ (which correspond to any $\mathbf{z} \neq 0$ here), there exists $\varepsilon > 0$ such that for all $s \in \mathbb{R}$ sufficiently small, the weighted resolvents

$$w^\delta(\dot{\widetilde{H}}^{\mathbf{z}}_\pm - z)^{-1} w^\delta : \dot{\widetilde{\mathcal{E}}}_\pm \to \dot{\widetilde{\mathcal{E}}}_\pm, \qquad\qquad w^\delta(\dot{H}^{\mathbf{z}} - z)^{-1} w^\delta : \dot{\mathcal{E}} \to \dot{\mathcal{E}}$$

extend form $\mathbb{C}^+$ into the strip $\{\lambda \in \mathbb{C} \mid \Im\lambda > -\varepsilon\}$ as meromorphic operators. The poles are called *resonances*. Then Proposition 3.3 and Theorem 3.8 state that for both the above operators, there is no resonance in a tighter strip $\{\lambda \in \mathbb{C} \mid \Im\lambda > -\varepsilon'\}$ for some $\varepsilon' \in]0, \varepsilon[$. Furthermore, [**?**, Corollary 3.9] shows that the spectrum of $\dot{H}$ is real provided that s is small enough.

We can now follow the proof of [GGH17, Theorem 12.1] in Subsection 13.2 therein. It follows from [GGH17, Theorem 7.1] since $\dot{H}^{\mathbf{z}}$ has no eigenvalue and both $\dot{\widetilde{H}}^{\mathbf{z}}_\pm$ and $\dot{H}^{\mathbf{z}}$ have no resonance on $\mathbb{R}$.

2.4.3 Proof of Theorem 2.3.9

Let us show Theorem 2.3.9. It is similar to [GGH17, Theorem 12.2] for $n \neq 0$ which follows from Theorem 10.5 therein once we have showed the absence of complex pure point spectrum of $\dot{H}$ and also the absence of real resonances of $\dot{H}_{\pm\infty}$. These conditions follow again from [GGH17, Proposition 13.1] for s small enough and $\mathbf{z} \neq 0$.

2.4.4 Proof of Theorem 2.3.10

Theorem 2.3.10 is similar to [GGH17, Theorem 12.3] and the proof therein applies to our setting. As we will show below the same result for the slightly more complicated geometric profiles (*cf.* Subsection 2.4.5), we omit the details here.

2.4.5 Proof of Theorem 2.3.11

The proof of Theorem 2.3.11 uses the following fact:

Lemma 2.4.1. *The operators $i_{-/+} : \dot{\mathcal{E}}^{\mathbf{z}} \to \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}$ are bounded.*

Proof. We only show the result for i_-. Let $u = (u_0, u_1) \in \dot{\mathcal{E}}$ and write

$$\begin{aligned}\|i_- u\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} &= \big\|(\partial_x - \mathrm{i}(k^z - k^z_{\mathscr{H}}))i_- u_0\big\|^2_{\mathcal{H}^{\mathbf{z}}} + \big\|i_-(u_1 - \mathrm{i}k^{\mathbf{z}}_{\mathscr{H}} u_0)\big\|^2_{\mathcal{H}^{\mathbf{z}}} \\ &\lesssim \big\|[\partial_x, i_-]u_0\big\|^2_{\mathcal{H}^{\mathbf{z}}} + \|\partial_x u_0\|^2_{\mathcal{H}^{\mathbf{z}}} + \big\|(k^z - k^z_{\mathscr{H}})i_- u_0\big\|^2_{\mathcal{H}^{\mathbf{z}}} + \|u_1 - \mathrm{i}k^{\mathbf{z}} u_0\|^2_{\mathcal{H}^{\mathbf{z}}} + \big\|i_-(k^{\mathbf{z}} - k^{\mathbf{z}}_{\mathscr{H}})u_0\big\|^2_{\mathcal{H}^{\mathbf{z}}} \\ &\lesssim \|u\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}} + \|f u_0\|^2_{\mathcal{H}^{\mathbf{z}}}\end{aligned}$$

where $f \in \mathcal{C}^\infty(\mathbb{R}_x, \mathbb{R})$ is exponentially decaying at infinity. Using Hardy type inequality (ii) of [GGH17, Lemma 9.5], we get

$$\|f u_0\|_{\mathcal{H}^{\mathbf{z}}} \lesssim \|h_0^{1/2} u_0\|_{\mathcal{H}^{\mathbf{z}}} \lesssim \|u\|_{\dot{\mathcal{E}}^{\mathbf{z}}}$$

and the lemma follows. □

Remark 2.4.2. *We may notice that $i_{-/+}\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}} \to \dot{\mathcal{E}}^{\mathbf{z}}$ are not bounded operators. However, the proof of the existence of the direct future wave operator $\boldsymbol{W}^f_{\mathscr{H}}$ below will show that*

$$i_{-/+}\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}} : \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}} \to \dot{\mathcal{E}}^{\mathbf{z}}$$

are bounded at the limits $t \to \pm\infty$ as extensions of bounded operators defined on the dense subspaces $\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}/\mathscr{I}}$.

Proof of Theorem 2.3.11. We will only show the theorem for the future operators in the $\mathscr{H}$ case. We closely follow the proof of [GGH17, Theorem 12.3].

Existence of the future direct wave operator $\boldsymbol{W}^f_{\mathscr{H}}$. Let $u = u^{\mathrm{in}} + u^{\mathrm{out}} \in \mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}}$. By Remark 2.3.7,

$$\begin{aligned}(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u^{\mathrm{in}})_0(z, x, \omega) &= \mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^{x+t} V(x')\mathrm{d}x'} u^{\mathrm{in}}_0(z, x+t, \omega), \\ \big(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u^{\mathrm{out}}\big)_0(z, x, \omega) &= \mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^{x-t}(V(x')-2V_-)\mathrm{d}x'} u^{\mathrm{out}}_0(z, x-t, \omega)\end{aligned}$$

and then $\boldsymbol{W}^f_{\mathscr{H}} u^{\mathrm{out}} = 0$ because of the support of i_-. From now on, we write $u = u^{\mathrm{in}}$. We use Cook's method: a sufficient condition for the limit to exist in $\dot{\mathcal{E}}^{\mathbf{z}}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u\right) = \mathrm{i}\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}}\big(\dot{H}^{\mathbf{z}} i_- - i_- \dot{H}^{\mathbf{z}}_{\mathscr{H}}\big)\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u \in L^1(\mathbb{R}^+_t, \mathrm{d}t; \dot{\mathcal{E}}^{\mathbf{z}}).$$

Put $v = (v_0, v_1) := \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u$. The above expression makes sense since v is smooth and compactly supported. Recalling the definition of the operators $h^{\mathbf{z}}$ and $k^{\mathbf{z}}$ in Subsection 2.3.1 as well as $h^{\mathbf{z}}_{\mathscr{H}}$ and $k^{\mathbf{z}}_{\mathscr{H}}$ Subsection 2.3.2, we compute

$$\begin{aligned}
&\dot{H}^{\mathbf{z}} i_- - i_- \dot{H}^{\mathbf{z}}_{\mathscr{H}} \\
&= \begin{pmatrix} 0 & 0 \\ h^{\mathbf{z}} i_- - i_-(h^{\mathbf{z}}_{\mathscr{H}} - (k^{\mathbf{z}}_{\mathscr{H}})^2) & 2i_-(k^{\mathbf{z}} - k^{\mathbf{z}}_{\mathscr{H}}) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ -[\partial_x^2, i_-] + i_-\Big(\mathcal{V}_\ell(r) - \mathrm{i}s\mathbf{z}\big(\partial_x \mathcal{W}(r) + \mathcal{W}(r)\partial_x\big) - 2s^2\mathbf{z}^2 V(r)^2 \mathcal{W}(r)\Big) & -2i_- \mathrm{i}s\mathbf{z}\mathcal{W}(r) \end{pmatrix}.
\end{aligned} \tag{2.33}$$

where

$$\mathcal{V}_\ell(r) := F(r)\left(r^{-1}F'(r) + r^{-2}\sum_{0\leq\ell\leq\ell_0} \ell(\ell+1) + m^2\mathbf{z}^2\right), \qquad \mathcal{W}(r) := V(r) - V_-.$$

Using the minimal speed of v_0 (*cf.* Remark 2.3.7), the uniform boundedness of $\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$ (*cf.* Theorem 2.3.8) as well as the exponential decay of F and $V - V_-$ (*cf.* (2.18)), we get:

$$\begin{aligned}
\left\| \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u\right)\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} &\lesssim (1+\ell_0)^3 \mathrm{e}^{-2\kappa_- t}\big(\|v_0\|_{\mathcal{H}^{\mathbf{z}}} + \|\partial_x v_0\|_{\mathcal{H}^{\mathbf{z}}} + \|v_1\|_{\mathcal{H}^{\mathbf{z}}}\big) \\
&\lesssim (1+\ell_0)^3 \mathrm{e}^{-2\kappa_- t}\big(\|v_0\|_{\mathcal{H}^{\mathbf{z}}} + \|(\partial_x - \mathrm{i}s\mathbf{z}(V - V_-))v_0\|_{\mathcal{H}^{\mathbf{z}}} \\
&\qquad + \|v_1 - \mathrm{i}s\mathbf{z}V_- v_0\|_{\mathcal{H}^{\mathbf{z}}}\big) \\
&\lesssim (1+\ell_0)^3 \mathrm{e}^{-2\kappa_- t}\big(\|v_0\|_{\mathcal{H}^{\mathbf{z}}} + \|v\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}}\big).
\end{aligned} \tag{2.34}$$

Here the symbol $\lesssim$ contains no dependence in ℓ_0 (but depends on $\mathbf{z}$). We have $\|v\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} = \|u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}}$ and $\|v_0(t)\|_{\mathcal{H}^{\mathbf{z}}} = \|u_0^{\mathrm{in}}(\cdot + t)\|_{\mathcal{H}^{\mathbf{z}}} = \|u_0^{\mathrm{in}}\|_{\mathcal{H}^{\mathbf{z}}}$ as translations are unitary on $\mathcal{H}^{\mathbf{z}}$. In particular, these norms are uniformly bounded in time. This provides a sufficient decay as $t \to +\infty$ in (2.34) and proves the existence of the limit $\boldsymbol{W}^f_{\mathscr{H}}$ on $\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}}$.

To extend the existence of the future wave operator to $\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}$, we use a density argument: we show that there exists a constant $C > 0$ such that for all $u \in \mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}}$,

$$\|\boldsymbol{W}^f_{\mathscr{H}} u\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \leq C\|u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}}.$$

Using again the uniform boundedness of $\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$, we can write:

$$\begin{aligned}
\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}} &\lesssim \|i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}} \\
&\lesssim \big\langle h^{\mathbf{z}}_0 i_- v_0, i_- v_0\big\rangle_{\mathcal{H}^{\mathbf{z}}} + \|i_-(v_1 - k^{\mathbf{z}} v_0)\|^2_{\mathcal{H}^{\mathbf{z}}} \\
&\lesssim \big\langle i_- h^{\mathbf{z}}_0 v_0, i_- v_0\big\rangle_{\mathcal{H}^{\mathbf{z}}} + \big\langle [h^{\mathbf{z}}_0, i_-] v_0, i_- v_0\big\rangle_{\mathcal{H}^{\mathbf{z}}} + \|i_-(v_1 - k^{\mathbf{z}} v_0)\|^2_{\mathcal{H}^{\mathbf{z}}} \\
&\lesssim \big\langle i_-(h^{\mathbf{z}}_0 - h^{\mathbf{z}}_{\mathscr{H}}) v_0, i_- v_0\big\rangle_{\mathcal{H}^{\mathbf{z}}} + \big\langle h^{\mathbf{z}}_{\mathscr{H}} v_0, v_0\big\rangle_{\mathcal{H}^{\mathbf{z}}} + \big\langle [h_0, i_-] v_0, i_- v_0\big\rangle_{\mathcal{H}^{\mathbf{z}}} \\
&\quad + \|i_-(k^{\mathbf{z}}_{\mathscr{H}} - k^{\mathbf{z}}) v_0)\|^2_{\mathcal{H}^{\mathbf{z}}} + \|v_1 - k^{\mathbf{z}}_{\mathscr{H}} v_0\|^2_{\mathcal{H}^{\mathbf{z}}} \\
&\lesssim \|v\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + r(t) \\
&\lesssim \|u\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + r(t)
\end{aligned}$$

where $r(t) = \mathcal{O}_{t\to+\infty}(\mathrm{e}^{-4\kappa_- t}\|v_0\|^2_{\mathcal{H}^{\mathbf{z}}})$ is a rest similar to the right-hand side of (2.34). Letting $t \to +\infty$ thus gives the desired result.

Existence of the future inverse wave operator $\boldsymbol{\Omega}^f_{\mathscr{H}}$. The existence of $\boldsymbol{\Omega}^f_{\mathscr{H}}$ necessarily relies on a propagation estimate for the full dynamics $(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}})_{t\in\mathbb{R}}$. In [GGH17, Proposition 6.8], it is shown that for all $\delta > 0$ and all $u \in \dot{\mathcal{E}}^{\mathbf{z}}$, we have

$$\int_{\mathbb{R}} \left\| w^{\delta} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} \chi(\dot{H}^{\mathbf{z}}) u \right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \mathrm{d}t \lesssim \|u\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \tag{2.35}$$

where $w(r(x)) = \sqrt{(r(x)-r_-)(r_+-r(x))}$ and $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ (χ must cancel in a neighborhood of the real resonances of $\dot{H}$ in [GGH17], but [**?**, Theorem 3.8] shows that no resonance lies on $\mathbb{R}$ for us). The operator $\chi(\dot{H}^{\mathbf{z}})$ is defined with a Helffer-Sjöstrand type formula and is bounded on $\dot{\mathcal{E}}^{\mathbf{z}}$, see [GGH17, Subsection 5.5].

To prove that $\boldsymbol{\Omega}^f_{\mathscr{H}}$ exists, we show that the sequence $\left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u\right)_{t>0}$ is Cauchy in $\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}$. Let $\varepsilon > 0$ and $u \in \dot{\mathcal{E}}^{\mathbf{z}}$. First of all, the uniform boundedness of $\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}}$ and $\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$ as well as Lemma 2.4.1 imply that $\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} \in \mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{E}}^{\mathbf{z}}_{\mathcal{H}})$ for all $t \in \mathbb{R}$. The spherical symmetry allows us to use the decomposition

$$\dot{\mathcal{E}}^{\mathbf{z}} = \bigoplus_{\ell\in\mathbb{N}} \dot{\mathcal{E}}^{\mathbf{z}}_{\ell}, \qquad \dot{\mathcal{E}}^{\mathbf{z}}_{\ell} := \left\{ v = \sum_{\ell\in\mathbb{N}} v_\ell \;\middle|\; \|v\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}} = \sum_{\ell\in\mathbb{N}} \|v_\ell\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\ell}} < +\infty \right\}.$$

The harmonic v_ℓ are such that $\omega \mapsto v_\ell(z,x,\omega) \in H^2(\mathbb{S}^2, \mathrm{d}\omega)$ and $-\Delta_{\mathbb{S}^2} v_\ell = \ell(\ell+1)v_\ell$ on $\dot{\mathcal{E}}^{\mathbf{z}}_{\ell}$ for almost every $(z,x) \in \mathbb{S}^1_z \times \mathbb{R}_x$. Then we have by dominated convergence

$$\begin{aligned}
\left\| \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u - \mathrm{e}^{-\mathrm{i}t'\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t'\dot{H}^{\mathbf{z}}} \sum_{0\le\ell\le\ell_0} u_\ell \right\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} &= \left\| \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} \chi(\dot{H}^{\mathbf{z}}) \sum_{\ell>\ell_0} u_\ell \right\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} \\
&\le C \left\| \sum_{\ell>\ell_0} u_\ell \right\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}} \\
&= C \sum_{\ell>\ell_0} \|u_\ell\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\ell}} \\
&< (\varepsilon/6)^2
\end{aligned}$$

for ℓ_0 large enough as the remainder of a convergent series. Fix such a ℓ_0 and call again u the truncated sum $\sum_{0\le\ell\le\ell_0} u_\ell$. Next,

$$\begin{aligned}
\left\| \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u - \mathrm{e}^{-\mathrm{i}t'\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t'\dot{H}^{\mathbf{z}}} u \right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} &\le \left\| \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} (1-\chi(\dot{H}^{\mathbf{z}}))u \right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} \\
&\quad + \left\| \mathrm{e}^{-\mathrm{i}t'\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t'\dot{H}^{\mathbf{z}}} (1-\chi(\dot{H}^{\mathbf{z}}))u \right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} \\
&\quad + \left\| \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} \chi(\dot{H}^{\mathbf{z}})u - \mathrm{e}^{-\mathrm{i}t'\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t'\dot{H}^{\mathbf{z}}} \chi(\dot{H}^{\mathbf{z}})u \right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}}.
\end{aligned}$$

We will show that the right-hand side above is lesser than $2\varepsilon/3$ for t, t' sufficiently large.

Using again the uniform boundedness of $\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}}$ and $\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$ as well as Lemma 2.4.1, we can write

$$\left\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}(1-\chi(\dot{H}^{\mathbf{z}}))u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + \left\|\mathrm{e}^{-\mathrm{i}t'\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t'\dot{H}^{\mathbf{z}}}(1-\chi(\dot{H}^{\mathbf{z}}))u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} \leq 2C\left\|(1-\chi(\dot{H}^{\mathbf{z}}))u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}}.$$

By [GGH17, Proposition 5.11] combined with the fact that $\dot{H}^{\mathbf{z}}$ has no eigenvalue if s is small enough (see the proof of Theorem 2.3.8 in Subsection 2.4.2),

$$\mathrm{s}-\lim_{L\to+\infty} \chi(\dot{H}^{\mathbf{z}}/L) = 1. \tag{2.36}$$

Fix χ so that $\left\|(1-\chi(\dot{H}^{\mathbf{z}}))u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} < \varepsilon/6C$. Thus

$$\left\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}(1-\chi(\dot{H}^{\mathbf{z}}))u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + \left\|\mathrm{e}^{-\mathrm{i}t'\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t'\dot{H}^{\mathbf{z}}}(1-\chi(\dot{H}^{\mathbf{z}}))u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} < \frac{\varepsilon}{3}.$$

It remains to show that

$$\left\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u - \mathrm{e}^{-\mathrm{i}t'\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t'\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} < \frac{\varepsilon}{3}$$

for t, t' large enough.

Pick $\delta \in]0,2[$ and compute:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right) = \mathrm{i}\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}}\left(\dot{H}^{\mathbf{z}}_{\mathscr{H}} i_- - i_-\dot{H}^{\mathbf{z}}\right) w^{-\delta} w^{\delta}\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u$$

Then (2.33) above shows that for all $v = (v_0, v_1) \in \dot{\mathcal{E}}^{\mathbf{z}}$,

$$\left\|\left(\dot{H}^{\mathbf{z}}_{\mathscr{H}} i_- - i_-\dot{H}^{\mathbf{z}}\right) w^{-\delta} v\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} \lesssim (1+\ell_0)^3\left(\|f v_0\|_{\mathcal{H}^{\mathbf{z}}} + \|g v_1\|_{\mathcal{H}^{\mathbf{z}}}\right)$$

for some smooth functions $f, g \in \mathcal{O}_{|x|\to+\infty}\left(\mathrm{e}^{(\delta-2\kappa_-)|x|}\right)$. Using the Hardy type inequality (ii) of [GGH17, Lemma 9.5], it follows

$$\left\|\left(\dot{H}^{\mathbf{z}}_{\mathscr{H}} i_- - i_-\dot{H}^{\mathbf{z}}\right) w^{-\delta} v\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} \lesssim (1+\ell_0)^3\left(\|h_0^{1/2} v_0\|_{\mathcal{H}^{\mathbf{z}}} + \|v_1 - k^{\mathbf{z}} v_0\|_{\mathcal{H}^{\mathbf{z}}}\right) \lesssim (1+\ell_0)^3\|v\|_{\dot{\mathcal{E}}^{\mathbf{z}}}.$$

With Lemma 2.4.1, this gives

$$\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}}\left(\dot{H}^{\mathbf{z}}_{\mathscr{H}} i_- - i_-\dot{H}^{\mathbf{z}}\right) w^{-\delta} \in \mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}, \dot{\mathcal{E}}^{\mathbf{z}}).$$

The propagation estimate (2.35) then implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right) \in L^1(\mathbb{R}, \mathrm{d}t)$$

whence

$$\left\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u - \mathrm{e}^{-\mathrm{i}t'\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t'\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} < \varepsilon/3$$

for t, t' large enough. The proof is complete. □

2.4.6 Proof of Proposition 2.3.13

We only show the f, $\mathscr{H}$ case:

$$\boldsymbol{\Omega}^f_{\mathscr{H}}\dot{\mathcal{E}}^{\mathbf{z}} \subset \Psi_{\mathscr{H}}\big(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \{0\}\big), \tag{2.37}$$

$$\boldsymbol{\Omega}^f_{\mathscr{H}}\boldsymbol{W}^f_{\mathscr{H}} = \mathbb{1}_{\Psi_{\mathscr{H}}(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}\times\{0\})}. \tag{2.38}$$

Proof of (2.37). We follow the proof of a similar proposition in [Mi]. Let

$$\Upsilon_{\mathscr{H}} : \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}} \ni u = (u_0, u_1) \longmapsto u_1 - \mathrm{i}L_{\mathscr{H}}u_0 \in \mathcal{H}^{\mathbf{z}}.$$

It defines a continuous operator as

$$\|\Upsilon_{\mathscr{H}}u\|_{\mathcal{H}^{\mathbf{z}}} = \|u_1 - \mathrm{i}L_{\mathscr{H}}u_0\|_{\mathcal{H}^{\mathbf{z}}} \leq \|(L_{\mathscr{H}} + \mathrm{i}k_{\mathscr{H}})u_0\|_{\mathcal{H}^{\mathbf{z}}} + \|u_1 - k_{\mathscr{H}}u_0\|_{\mathcal{H}^{\mathbf{z}}} = \|u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}}.$$

Clearly $\ker\Upsilon_{\mathscr{H}} = \Psi_{\mathscr{H}}\big(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \{0\}\big)$ so we will show that

$$\Upsilon_{\mathscr{H}}\boldsymbol{\Omega}^f_{\mathscr{H}} = 0.$$

We claim that

$$\big\|\Upsilon_{\mathscr{H}}\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}}u\big\|_{\mathcal{H}^{\mathbf{z}}} = \sqrt{2}\big\|\Upsilon_{\mathscr{H}}u\big\|_{\mathcal{H}^{\mathbf{z}}} \qquad \forall u \in \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}. \tag{2.39}$$

To see this, write $\Upsilon_{\mathscr{H}} = \frac{\mathrm{i}}{\sqrt{2}}(L_+ - L_{\mathscr{H}})\pi_1\Psi^{-1}_{\mathscr{H}}$ with $\pi_1(u_0, u_1) := u_1$. Now (2.39) follows from $\Psi^{-1}_{\mathscr{H}}\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} = \mathrm{e}^{\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{\mathscr{H}}}\Psi^{-1}_{\mathscr{H}}$ and the unitarity of $\mathrm{e}^{\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{\mathscr{H}}}\Psi^{-1}_{\mathscr{H}} : \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}} \to \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}$ and $L_{\mathscr{H}} + \mathrm{i}k_{\mathscr{H}} = -\frac{1}{2}(L_+ - \mathrm{i}L_{\mathscr{H}}) : \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \to \mathcal{H}^{\mathbf{z}}$.

Let now $u \in \dot{\mathcal{E}}^{\mathbf{z}}$. In view of (2.39), we are boiled to show that[15]

$$\lim_{t\to+\infty}\big\|\Upsilon_{\mathscr{H}}i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}u\big\|_{\mathcal{H}^{\mathbf{z}}} = 0. \tag{2.40}$$

Fix $\varepsilon > 0$ and pick $\ell_0 \in \mathbb{N}$ so that

$$\begin{aligned}\big\|\Upsilon_{\mathscr{H}}i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}u\big\|_{\mathcal{H}^{\mathbf{z}}} &\leq \Big\|\Upsilon_{\mathscr{H}}i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\sum_{0\leq\ell\leq\ell_0}u_\ell\Big\|_{\mathcal{H}^{\mathbf{z}}} + \Big\|\Upsilon_{\mathscr{H}}i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\Big(u - \sum_{0\leq\ell\leq\ell_0}u_\ell\Big)\Big\|_{\mathcal{H}^{\mathbf{z}}}\\ &\leq \Big\|\Upsilon_{\mathscr{H}}i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\sum_{0\leq\ell\leq\ell_0}u_\ell\Big\|_{\mathcal{H}^{\mathbf{z}}} + \varepsilon/2\end{aligned} \tag{2.41}$$

where $-\Delta_{\mathbb{S}^2}u_\ell = \ell(\ell+1)u_\ell$ as in the proof of Theorem 2.3.11 (we used the continuity of $\Upsilon_{\mathscr{H}}i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} : \dot{\mathcal{E}}^{\mathbf{z}} \to \mathcal{H}^{\mathbf{z}}$ as well as dominated convergence). We will write again u for $\sum_{0\leq\ell\leq\ell_0}u_\ell$. Set $v(t) := \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}u$ and compute:

$$\begin{aligned}\mathrm{i}(\partial_t + L_+)\Upsilon_{\mathscr{H}}i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}u &= (\partial_t + L_+)(\partial_t + L_{\mathscr{H}})i_-v_0(t)\\ &= \big((\partial_t - \mathrm{i}k_{\mathscr{H}})^2 + h_{\mathscr{H}}\big)i_-v_0(t)\\ &= \Big([h_{\mathscr{H}}, i_-] - i_-(k^2_{\mathscr{H}} - k^2) + i_-(h_{\mathscr{H}} - h)\Big)v_0(t) + 2i_-(k_{\mathscr{H}} - k)v_1(t)\\ &=: \Xi(t).\end{aligned} \tag{2.42}$$

[15] Recall that $i_- \in \mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}})$, *cf.* Lemma 2.4.1.

This expression makes sense in $\mathcal{H}^{\mathbf{z}}$: we have

$$\begin{aligned}\|\Xi(t)\|_{\mathcal{H}^{\mathbf{z}}} &\lesssim \|f_1(x)v_0\|_{\mathcal{H}^{\mathbf{z}}} + \|f_2(x)\partial_x v_0\|_{\mathcal{H}^{\mathbf{z}}} + \|f_3(x)v_1\|_{\mathcal{H}^{\mathbf{z}}}\\ &\lesssim \|g(x)v_0\|_{\mathcal{H}^{\mathbf{z}}} + \|\partial_x v_0\|_{\mathcal{H}^{\mathbf{z}}} + \|v_1 - kv_0\|_{\mathcal{H}^{\mathbf{z}}}\end{aligned}$$

with $|f_j(x)|, |g(x)| \leq C_j \ell_0^3 \mathrm{e}^{-2\kappa|x|}$, then Hardy type inequality [GGH17, Lemma 9.5] as well as uniform boundedness of $\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$ yield

$$\|\Xi(t)\|_{\mathcal{H}^{\mathbf{z}}} \lesssim \|\tilde{g}(x)w^{\delta}v(t)\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \lesssim \|\tilde{g}(x)w^{\delta}u\|_{\dot{\mathcal{E}}^{\mathbf{z}}}$$

with $\delta \in]0, 2\kappa[$ and $|\tilde{g}(x)| \leq \tilde{C}\ell_0^3 \mathrm{e}^{-2\kappa|x|-\delta}$. By [GGH17, Proposition 6.8], we have $\Xi \in L^1(\mathbb{R}_t, \mathcal{H}^{\mathbf{z}})$. From (2.42) and the fact that L_+ generates a strongly continuous group on $\mathcal{H}^{\mathbf{z}}$, we then deduce the following Kirchhoff type formula

$$\Upsilon_{\mathscr{H}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u = \mathrm{i}\mathrm{e}^{-tL_+}\Upsilon_{\mathscr{H}} i_- u + \mathrm{i}\int_0^t \mathrm{e}^{-(t-t')L_+}\Xi(t')\mathrm{d}t'$$

which we can rewritten as

$$\Upsilon_{\mathscr{H}} i_- \mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} u = \mathrm{i}\mathrm{e}^{-tL_+}\left(\Upsilon_{\mathscr{H}} i_- u + \mathrm{i}\int_0^{+\infty} \mathrm{e}^{t'L_+}\Xi(t')\mathrm{d}t'\right) + o_{t\to+\infty}(1) \tag{2.43}$$

in the $L^1(\mathbb{R}_t, \mathcal{H}^{\mathbf{z}})$ sense. Finally, write $i_- = j_- i_-$ so that

$$\Upsilon_{\mathscr{H}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u = \mathrm{i}j_-' i_- (\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u)_0 + j_- \Upsilon_{\mathscr{H}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u. \tag{2.44}$$

Since j_-' is supported near 0, [GGH17, Proposition 6.5] shows that the first term above goes to 0 as $t \to +\infty$; the second term above also falls off at the limit using (2.43) since $j_-\mathrm{e}^{-tL_+}\phi \to 0$ as $t \to +\infty$ for any $\phi \in \mathcal{H}^{\mathbf{z}}$. This shows that the expression in (2.44) is smaller than $\varepsilon/2$ for $t \gg 0$; back into (2.41), this gives (2.40). □

Proof of (2.38). Notice that we can define $\boldsymbol{\Omega}^f_{\mathscr{H}}$ using j_- instead of i_- in part 2. of Theorem 2.3.11; this will immediately cancel mixed terms $j_- i_+$ below[16].

Let first $u \in \Psi_{\mathscr{H}}(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \{0\}) \cap \mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}}$ so that $\boldsymbol{W}^f_{\mathscr{H}} u$ is the limit of $\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u$ as $t \to +\infty$. Since $u_1 = \mathrm{i}L_{\mathscr{H}}$, we obtain as in Remark 2.3.7

$$\left(\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u\right)_0(x) = \mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^{x+t} V(x')\mathrm{d}x'} u_0(x+t) = \mathrm{e}^{-tL_{\mathscr{H}}} u_0(x) \tag{2.45}$$

where we omit the dependence in $(z, \omega) \in \mathbb{S}^1 \times \mathbb{S}^2$. Set then $\widetilde{u} := \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u$; integrating by parts in Kirchhoff formula (2.30), we get:

$$\begin{aligned}&\left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} j_- \widetilde{u}\right)_0(x)\\ &= \frac{\mathrm{e}^{-\mathrm{i}s\mathbf{z}tV_-}}{2}\left(\sum_{\pm} \mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^{x\pm t}(V(x')-V_-)\mathrm{d}x'} j_-(x\pm t)\widetilde{u}_0(x\pm t)\right.\\ &\qquad\left. +\mathrm{i}\int_{x+t}^{x-t} \mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^y (V(x')-V_-)\mathrm{d}x'} j_-(y)\Big(\widetilde{u}_1 - s\mathbf{z}V_-\widetilde{u}_0\Big)(y)\mathrm{d}y\right)\\ &= j_-(x-t)u_0(x) - \frac{\mathrm{e}^{-\mathrm{i}s\mathbf{z}tV_-}}{2}\int_{x+t}^{x-t} \mathrm{e}^{\mathrm{i}s\mathbf{z}\int_x^y (V(x')-V_-)\mathrm{d}x'}\mathrm{e}^{\mathrm{i}s\mathbf{z}\int_y^{y+t} V(x')\mathrm{d}x'} j_-'(y)u_0(y+t)\mathrm{d}y.\end{aligned}$$

[16] Otherwise, we have to involve a propagation estimate to make $i_- i_+ \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathcal{H}}}$ vanish at the limit $t \to +\infty$.

Since $j_-(x-t) \to 1$ as $t \to +\infty$, $j_-(x-t)u_0(x) = u_0(x)$ for $t \gg 0$. The integral term vanishes for large t because j' is supported near 0 whereas $\operatorname{Supp} u_0(x+t)$ leaves any neighborhood of 0. This means that $\left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} j_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u\right)_0 = u_0$ in finite time and then $\left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} j_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} u\right)_1 = \mathrm{i}L_{\mathscr{H}} u_0$ using (2.45).

By Theorem 2.3.11, $\boldsymbol{W}^f_{\mathscr{H}}$ and $\boldsymbol{\Omega}^f_{\mathscr{H}}$ are bounded operators so that

$$s - \lim_{t\to+\infty} \left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} j_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\right)\left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\right) = \boldsymbol{\Omega}^f_{\mathscr{H}} \boldsymbol{W}^f_{\mathscr{H}}.$$

The above computations show that the left-hand side above is $\mathbb{1}_{\Psi_{\mathscr{H}}(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}\times\{0\})}$ (first proved on a dense subspace then extended to $\Psi_{\mathscr{H}}(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \{0\})$ by continuity) which entails (2.38). □

2.5 Geometric interpretation

We provide the geometric interpretation of the scattering associated to the dynamics $\dot{H}_{\mathscr{H}/\mathscr{I}}$ of Subsection 2.3.2. We will show how the inverse wave operators of Theorem 2.3.11 are related to traces onto the horizons $\mathscr{H}$ and $\mathscr{I}$. We adapt [HaNi04] which deals with Dirac equation in Kerr spacetime.

This section is organized as follows: in Subsection 2.5.1, we define energy spaces on horizons using the principal null geodesics; we construct in Subsection 2.5.2 full wave and inverse wave operators using Theorem 2.3.11; Subsection 2.5.3 then shows that they are indeed inverse; in Subsection 2.5.4, we extend the trace operators from smooth compactly supported data to energy spaces as bounded invertible operators; finally, we solve an abstract Goursat problem in Subsection 2.5.5.

2.5.1 Energy spaces on the horizons

We define in this Subsection the energy spaces on the horizons obtained by transport of the principal null geodesics. This will allow us to define traces in Subsection 2.5.4 and extend them as abstract operators acting on energy spaces.

First, we explicit the correspondence between horizons and the initial data slice Σ_0 using principal null geodesics. Recall from Subsection 2.2.1 the extended-star $(t^\star, z^\star, \omega^\star)$ and star extended $({}^\star t, {}^\star z, {}^\star\omega)$ coordinates which describe the horizons:

$$\begin{aligned} t^\star &= t + T(r), & {}^\star t &= t - T(r), \\ z^\star &= z + Z(r), & {}^\star z &= z - Z(r), \\ \omega^\star &= {}^\star\omega = \omega \end{aligned}$$

with $T = x$ and

$$T'(r) = F(r)^{-1}, \qquad Z'(r) = -sV(r)F(r)^{-1}.$$

Then

$$\begin{aligned} \mathscr{H}^+ &= \mathbb{R}_{t^\star} \times \mathbb{S}^1_{z^\star} \times \{-r_-\}_r \times \mathbb{S}^2_\omega, & \mathscr{I}^+ &= \mathbb{R}_{{}^\star t} \times \mathbb{S}^1_{{}^\star z} \times \{r_+\}_r \times \mathbb{S}^2_\omega \\ \mathscr{H}^- &= \mathbb{R}_{{}^\star t} \times \mathbb{S}^1_{{}^\star z} \times \{r_-\}_r \times \mathbb{S}^2_\omega, & \mathscr{I}^- &= \mathbb{R}_{t^\star} \times \mathbb{S}^1_{z^\star} \times \{-r_+\}_r \times \mathbb{S}^2_\omega. \end{aligned}$$

Let $(t_0, z_0, r_0, \omega_0) \in \widetilde{\mathcal{M}}$. Since $t^\star$ and $z^\star$ (respectively ${}^\star t$ and ${}^\star z$) are constant along γ_{in} (respectively along γ_{out}), we find

$$\begin{aligned}
\lim_{r\to r_-} \gamma_{\text{in}}(r) &= (t_0 + T(r_0), z_0 + Z(r_0), -r_-, \omega_0) \in \mathscr{H}^+,\\
\lim_{r\to r_+} \gamma_{\text{out}}(r) &= \big(t_0 - T(r_0), z - Z(r_0), r_+, \omega_0\big) \in \mathscr{I}^+,\\
\lim_{r\to r_-} \gamma_{\text{out}}(r) &= \big(t_0 - T(r_0), z - Z(r_0), r_-, \omega_0\big) \in \mathscr{H}^-,\\
\lim_{r\to r_+} \gamma_{\text{in}}(r) &= \big(t_0 + T(r_0), z + Z(r_0), -r_+, \omega_0\big) \in \mathscr{I}^-
\end{aligned}$$

for the end points of the principal null geodesics intersecting $(t_0, z_0, r_0, \omega_0)$ at $r = r_0$ (the explicit expressions of $T(r) = x(r)$ and $Z(r)$ are given in (2.12) and (2.13)). As $T, Z :]r_-, r_+[\to \mathbb{R}$ are smooth diffeomorphisms, the applications $\mathscr{F}^\pm_{\mathscr{H}} : \mathscr{H}^\pm \to \Sigma_0$ and $\mathscr{F}^\pm_{\mathscr{I}} : \mathscr{I}^\pm \to \Sigma_0$ defined by

$$\mathscr{F}^{-/+}_{\mathscr{H}}\left(\lim_{r\to r_-} \gamma_{\text{out/in}}(r)(p_0)\right) := p_0, \qquad \mathscr{F}^{-/+}_{\mathscr{I}}\left(\lim_{r\to r_+} \gamma_{\text{in/out}}(r)(p_0)\right) := p_0$$

for all $p_0 \in \Sigma_0$ are well-defined diffeomorphisms which identify end points on the future/past horizons to the initial point on Σ_0.

Let us turn to the definition of the energy spaces on the horizons. We define the *asymptotic future/past energy spaces* $\dot{\mathscr{E}}^{\mathbf{z}}_\pm := ((\mathscr{F}^\pm_{\mathscr{H}})^{-1})^* \dot{\mathcal{H}}^1_{\mathscr{H}} \times ((\mathscr{F}^\pm_{\mathscr{I}})^{-1})^* \dot{\mathcal{H}}^1_{\mathscr{I}}$ and their restrictions $\dot{\mathscr{E}}^{\mathbf{z}}_\pm$ to $\ker(\mathrm{i}\partial_z + \mathbf{z})$ endowed with the norms

$$\|(\xi,\zeta)\|^2_{\dot{\mathscr{E}}_\pm} := \big\|(L_{\mathscr{H}} + \mathrm{i}k_{\mathscr{H}})\big((\mathscr{F}^\pm_{\mathscr{H}})^{-1}\big)^*\xi\big\|^2_{\mathcal{H}} + \big\|(L_{\mathscr{I}} + \mathrm{i}k_{\mathscr{I}})\big((\mathscr{F}^\pm_{\mathscr{I}})^{-1}\big)^*\zeta\big\|^2_{\mathcal{H}}.$$

Using the coordinates recalled in Subsection 2.5.1, we can write

$$\begin{aligned}
\partial_x &= \partial_{t^\star} - sV(r)\partial_{z^\star}, \qquad & \partial_z &= \partial_{z^\star},\\
\partial_x &= -\partial_{{}^\star t} + sV(r)\partial_{{}^\star z}, & \partial_z &= \partial_{{}^\star z}.
\end{aligned}$$

Using $L_{\mathscr{H}} = -\partial_x - sV\partial_z$ and $L_{\mathscr{I}} = \partial_x - sV\partial_z$, we explicitly get:

$$\begin{aligned}
\big\|(L_{\mathscr{H}} + \mathrm{i}k_{\mathscr{H}})\big((\mathscr{F}^+_{\mathscr{H}})^{-1}\big)^*\xi\big\|^2_{\mathcal{H}} &= \int_{\mathbb{R}_{t^\star}\times\mathbb{S}^1_{z^\star}\times\mathbb{S}^2_\omega} \Big(\partial_{t^\star}\xi - sV_-\partial_{z^\star}\xi\Big)^2 (t^\star, z^\star, \omega)\mathrm{d}t^\star\mathrm{d}z^\star\mathrm{d}\omega,\\
\big\|(L_{\mathscr{I}} + \mathrm{i}k_{\mathscr{I}})\big((\mathscr{F}^+)^{-1}\big)^*\zeta\big\|^2_{\mathcal{H}} &= \int_{\mathbb{R}_{{}^\star t}\times\mathbb{S}^1_{{}^\star z}\times\mathbb{S}^2_\omega} \Big(\partial_{{}^\star t}\zeta - sV_+\partial_{{}^\star z}\zeta\Big)^2 ({}^\star t, {}^\star z, \omega)\mathrm{d}{}^\star t\mathrm{d}{}^\star z\mathrm{d}\omega.
\end{aligned}$$

Similar formulas hold on $\mathscr{H}^-$ and $\mathscr{I}^-$.

Remark 2.5.1. *The asymptotic energies on the horizons are nothing but the flux of the Killing generators $X_- := \partial_{t^\star} - sV_-\partial_{z^\star}$, $X_+ := \partial_{{}^\star t} - sV_+\partial_{{}^\star z}$ defined in (2.16) through the corresponding horizon.*

2.5.2 The full wave operators

The operators in Theorem 2.3.11 are not inverse despite their name because the cut-offs $i_\pm$ cancel outgoing/incoming data. In this Subsection, we construct full wave and inverse wave operators which encode scattering in both the ends of the spacetime.

Pick $\mathbf{z} \in \mathbb{Z} \setminus \{0\}$. Let $\Pi_0 : \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}} \ni (u_0, u_1) \mapsto (u_0, 0) \in \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}$, $\Pi_1 : \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}} \ni (u_0, u_1) \mapsto (0, u_1) \in \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}$ and $\widehat{\Pi}_0 : \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}} \ni (u_0, u_1) \mapsto (u_0, 0) \in \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \{0\}$, $\widehat{\Pi}_1 : \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{J}} \ni (u_0, u_1) \mapsto (0, u_0) \in \{0\} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}$. For all $t \in \mathbb{R}$, we define the following operators:

$$W(t) := \sqrt{2}\mathrm{e}^{-\mathrm{i}t\dot{H}} i_- \Psi_{\mathscr{H}} \mathrm{e}^{\mathrm{i}t\dot{\mathbb{H}}_{\mathscr{H}}} \Pi_0 + \sqrt{2}\mathrm{e}^{-\mathrm{i}t\dot{H}} i_+ \Psi_{\mathscr{J}} \mathrm{e}^{\mathrm{i}t\dot{\mathbb{H}}_{\mathscr{J}}} \Pi_1, \tag{2.46}$$

$$\Omega(t) := \mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}_{\mathscr{H}}} \widehat{\Pi}_0 j_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} + \mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}_{\mathscr{J}}} \widehat{\Pi}_1 j_+ \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}. \tag{2.47}$$

Recall that $j_{-/+} \in \mathcal{B}\big(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{J}}\big)$ by Lemma 2.4.1 so that (2.47) makes sense.

Lemma 2.5.2. *Let $\mathbf{z} \in \mathbb{Z} \setminus \{0\}$. There exists $s_0 > 0$ such that for all $s \in]-s_0, s_0[$, the following holds:*

1. *For all $u = (u_0, u_1)$ such that $(u_0, 0) \in \Psi^{-1}_{\mathscr{H}}\big(\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}}\big)$ and $(0, u_1) \in \Psi^{-1}_{\mathscr{J}}\big(\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{J}}\big)$, the limits*

$$\mathbb{W}^{\pm} u := \lim_{t \to \pm\infty} W(t) u = \boldsymbol{W}^{f/p}_{\mathscr{H}} \Psi_{\mathscr{H}} \Pi_0 u + \boldsymbol{W}^{f/p}_{\mathscr{J}} \Psi_{\mathscr{J}} \Pi_1 u$$

   exist in $\dot{\mathcal{E}}^{\mathbf{z}}$. The operators $\mathbb{W}^{\pm}$ extend to bounded operators $\boldsymbol{W}_{\pm} \in \mathcal{B}\big(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}, \dot{\mathcal{E}}^{\mathbf{z}}\big)$. We call $\mathbb{W}^{\pm}$ the full future/past wave operators.

2. *The following strong limits exist:*

$$\mathbb{\Omega}^{\pm} := \mathrm{s} - \lim_{t \to \pm\infty} \Omega(t) = \Psi^{-1}_{\mathscr{H}} \boldsymbol{\Omega}^{f/p}_{\mathscr{H}} + \Psi^{-1}_{\mathscr{J}} \boldsymbol{\Omega}^{f/p}_{\mathscr{J}} \in \mathcal{B}\big(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}\big).$$

   The operators $\mathbb{\Omega}^{\pm}$ are the full future/past inverse wave operators.

Proof. 1. Let $\pi_j : (u_0, u_1) \mapsto u_j$ be the projection onto the j-th component, $j \in \{0, 1\}$. The existence of the strong limits on $\pi_0\left(\Psi^{-1}_{\mathscr{H}}\big(\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}}\big)\right) \times \pi_1\left(\Psi^{-1}_{\mathscr{J}}\big(\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{J}}\big)\right)$ follows from part 1. of Theorem 2.3.11 as $\Psi_{\mathscr{H}/\mathscr{J}} \mathrm{e}^{\mathrm{i}t\dot{\mathbb{H}}_{\mathscr{H}/\mathscr{J}}} = \mathrm{e}^{\mathrm{i}t\dot{H}_{\mathscr{H}/\mathscr{J}}} \Psi_{\mathscr{H}/\mathscr{J}}$. To define the operator on any $u \in \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}$, it suffices to observe that

$$\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}} = \overline{\pi_0\left(\Psi^{-1}_{\mathscr{H}}\big(\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}}\big)\right) \times \pi_1\left(\Psi^{-1}_{\mathscr{J}}\big(\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{J}}\big)\right)}^{\|\cdot\|_{\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}}}. \tag{2.48}$$

   This follows from the facts that $\mathcal{D}^{\mathrm{fin},\mathbf{z}}_{\mathscr{H}/\mathscr{J}}$ are dense in $\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{J}}$ (*cf.* Lemma 2.3.6), that $\Psi_{\mathscr{H}/\mathscr{J}}$ are homeomorphisms (*cf.* Lemma 2.3.2) and that projections are continuous with respect to the product topology. We then conclude using that $\boldsymbol{W}^{f/p}_{\mathscr{H}}$ and $\boldsymbol{W}^{f/p}_{\mathscr{J}}$ have continuous extensions by part 1. of Theorem 2.3.11.

2. Let us write

$$\Omega(t) = \mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{\mathscr{H}}} \widehat{\Pi}_0 \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} \left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} j_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\right) + \mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{\mathscr{J}}} \widehat{\Pi}_1 \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}} \left(\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}} j_+ \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\right)$$

   where

$$\mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{\mathscr{H}}} \widehat{\Pi}_0 \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} \in \mathcal{B}\big(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}\big), \qquad \mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{\mathscr{J}}} \widehat{\Pi}_1 \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}} \in \mathcal{B}\big(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}} \times \dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}\big)$$

uniformly in $t \in \mathbb{R}$. Theorem 2.3.11 then implies that

$$\Omega(t)\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix} = \mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{\mathscr{H}}}\widehat{\Pi}_0\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}}\boldsymbol{\Omega}^f_{\mathscr{H}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix} + \mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{\mathscr{J}}}\widehat{\Pi}_1\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}}\boldsymbol{\Omega}^f_{\mathscr{J}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix} + o_{t\to+\infty}(1).$$

Next, Proposition 2.3.13 shows that $\boldsymbol{\Omega}^f_{\mathscr{H}}\dot{\mathcal{E}}^{\mathbf{z}} \subset \Psi_{\mathscr{H}}\big(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}\times\{0\}\big)$ and $\boldsymbol{\Omega}^f_{\mathscr{J}}\dot{\mathcal{E}}^{\mathbf{z}} \subset \Psi_{\mathscr{J}}\big(\{0\}\times\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}\big)$. Since $\Psi_{\mathscr{H}}\widehat{\Pi}_0$ projects onto states u satisfying $u_1 = \mathrm{i}L_{\mathscr{H}}u_0$, it comes:

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{\mathscr{H}}}\widehat{\Pi}_0\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}}\boldsymbol{\Omega}^f_{\mathscr{H}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix} &= \mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{\mathscr{H}}}\widehat{\Pi}_0\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}}\Psi_{\mathscr{H}}\widehat{\Pi}_0\Psi^{-1}_{\mathscr{H}}\boldsymbol{\Omega}^f_{\mathscr{H}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}\\
&= \mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{\mathscr{H}}}\widehat{\Pi}_0\Psi_{\mathscr{H}}\mathrm{e}^{\mathrm{i}t\dot{\mathbb{H}}^{\mathbf{z}}_{H}}\widehat{\Pi}_0\Psi^{-1}_{\mathscr{H}}\boldsymbol{\Omega}^f_{\mathscr{H}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}\\
&= \begin{pmatrix}\mathrm{e}^{tL^{\mathbf{z}}_{\mathscr{H}}} & 0\\ 0 & \mathrm{e}^{tL^{\mathbf{z}}_{\mathscr{H}}}\end{pmatrix}\widehat{\Pi}_0\Psi_{\mathscr{H}}\begin{pmatrix}\mathrm{e}^{-tL^{\mathbf{z}}_{\mathscr{H}}} & 0\\ 0 & \mathrm{e}^{-tL^{\mathbf{z}}_{\mathscr{H}}}\end{pmatrix}\widehat{\Pi}_0\Psi^{-1}_{\mathscr{H}}\boldsymbol{\Omega}^f_{\mathscr{H}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}\\
&= \widehat{\Pi}_0\Psi_{\mathscr{H}}\widehat{\Pi}_0\Psi^{-1}_{\mathscr{H}}\boldsymbol{\Omega}^f_{\mathscr{H}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}\\
&= \Psi^{-1}_{\mathscr{H}}\boldsymbol{\Omega}^f_{\mathscr{H}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}.
\end{aligned}$$

Similarly, we have

$$\Psi^{-1}_{\mathscr{J}}\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}}\Psi_{\mathscr{J}}\widehat{\Pi}_1\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}}\boldsymbol{\Omega}^f_{\mathscr{J}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix} = \Psi^{-1}_{\mathscr{J}}\boldsymbol{\Omega}^f_{\mathscr{J}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}$$

whence finally:

$$\Omega(t)\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix} = \Psi^{-1}_{\mathscr{H}}\boldsymbol{\Omega}^f_{\mathscr{H}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix} + \Psi^{-1}_{\mathscr{J}}\boldsymbol{\Omega}^f_{\mathscr{J}}\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix} + o_{t\to+\infty}(1).$$

□

Remark 2.5.3. *We have*

$$\ker \mathbb{\Omega}^{\pm} = \ker \boldsymbol{\Omega}^{f/p}_{\mathscr{H}} \cap \ker \boldsymbol{\Omega}^{f/p}_{\mathscr{J}}. \tag{2.49}$$

Indeed, let $u = (u_0, u_1) \in \dot{\mathcal{E}}^{\mathbf{z}}$. Using that $\Psi^{-1}_{\mathscr{H}/\mathscr{J}}$ are isometries by Lemma 2.3.2, we can write

$$\big\|\mathbb{\Omega}^{\pm}u\big\|^2_{\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}\times\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}} = \Big\|\big(\boldsymbol{\Omega}^{f/p}_{\mathscr{H}}u\big)_0\Big\|^2_{\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}} + \Big\|\big(\boldsymbol{\Omega}^{f/p}_{\mathscr{J}}u\big)_0\Big\|^2_{\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}} \le \Big\|\boldsymbol{\Omega}^{f/p}_{\mathscr{H}}u\Big\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + \Big\|\boldsymbol{\Omega}^{f/p}_{\mathscr{J}}u\Big\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{J}}} \tag{2.50}$$

so that $\ker \mathbb{\Omega}^{\pm} \supset \ker \boldsymbol{\Omega}^{f/p}_{\mathscr{H}} \cap \ker \boldsymbol{\Omega}^{f/p}_{\mathscr{J}}$. Since $\boldsymbol{\Omega}^{f/p}_{\mathscr{H}}\dot{\mathcal{E}}^{\mathbf{z}} \subset \Psi_{\mathscr{H}}\big(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}\times\{0\}\big)$ and $\boldsymbol{\Omega}^{f/p}_{\mathscr{J}}\dot{\mathcal{E}}^{\mathbf{z}} \subset \Psi_{\mathscr{J}}\big(\{0\}\times\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}\big)$ by Proposition 2.3.13, $\big(\boldsymbol{\Omega}^{f/p}_{\mathscr{H}/\mathscr{J}}u\big)_0 = 0$ entails $\big(\boldsymbol{\Omega}^{f/p}_{\mathscr{H}/\mathscr{J}}u\big)_1 = 0$. This means that the left-hand side in (2.50) vanishes if and only if the right-hand side does, and (2.49) follows.

We will show in the proof of Theorem 2.5.5 that both the sets in (2.49) are actually trivial.

2.5.3 Inversion of the full wave operators

In this Subsection, we show that the full wave operators and the full inverse wave operators are indeed inverses in the energy spaces.

Lemma 2.5.4. *Let* $\mathbf{z} \in \mathbb{Z} \setminus \{0\}$ *and pick*[17] $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$. *For all* $u \in \dot{\mathcal{E}}^{\mathbf{z}}$,

$$\lim_{|t|\to+\infty} \left| \left\| i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} \chi(\dot{H}^{\mathbf{z}}) u \right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} - \left\| i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} \chi(\dot{H}^{\mathbf{z}}) u \right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}} \right| = 0.$$

Proof. We only treat the $-, \mathscr{H}$ case. As in the proof of the existence of the future inverse wave operator in Theorem 2.3.11 (*cf.* Subsection 2.4.5), we can assume that $-\Delta_{\mathbb{S}^2} u = \sum_{0\leq\ell\leq\ell_0} \ell(\ell+1)u$. Set $v := \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u \in \dot{\mathcal{E}}^{\mathbf{z}}$. Then

$$\begin{aligned}
\left| \|i_- v\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} - \|i_- v\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}} \right| &\leq \left| \left\langle (h_{\mathscr{H}} - h_0) i_- v_0, i_- v_0 \right\rangle_{\mathcal{H}^{\mathbf{z}}} \right| + \left| \left\langle i_-(v_1 - \mathrm{i}s\mathbf{z}V_- v_0), i_- \mathrm{i}(V - V_-) v_0 \right\rangle_{\mathcal{H}^{\mathbf{z}}} \right| \\
&\quad + \left| \left\langle i_- \mathrm{i}(V - V_-) v_0, i_-(v_1 - \mathrm{i}s\mathbf{z}V v_0) \right\rangle_{\mathcal{H}^{\mathbf{z}}} \right| \\
&\lesssim \left| \left\langle (f(x) + (V - V_-)\partial_x) i_- v_0, i_- v_0 \right\rangle_{\mathcal{H}^{\mathbf{z}}} \right| + \left\| v_1 - \mathrm{i}s\mathbf{z}V_- v_0 \right\|_{\mathcal{H}^{\mathbf{z}}} \left\| i_-(V - V_-) v_0 \right\|_{\mathcal{H}^{\mathbf{z}}} \\
&\quad + \left\| i_-(V - V_-) v_0 \|_{\mathcal{H}^{\mathbf{z}}} \right\| v_1 - \mathrm{i}s\mathbf{z}V v_0 \right\|_{\mathcal{H}^{\mathbf{z}}} \\
&\lesssim \left| \left\langle g(x) i_- v_0, i_- v_0 \right\rangle_{\mathcal{H}^{\mathbf{z}}} \right| + \|\partial_x v_0\|_{\mathcal{H}^{\mathbf{z}}} \left\| i_-(V - V_-) v_0 \right\|_{\mathcal{H}^{\mathbf{z}}} \\
&\quad + \left\| i_-(V - V_-) v_0 \|_{\mathcal{H}^{\mathbf{z}}} \right\| v_1 - \mathrm{i}s\mathbf{z}V v_0 \right\|_{\mathcal{H}^{\mathbf{z}}}
\end{aligned}$$

with $|f(x)|, |g(x)| \lesssim (1 + \ell_0^3)\mathrm{e}^{-2\kappa|x|}$ and $\kappa := \min\{\kappa_-, |\kappa_+|\}$ (as in (2.34), $\lesssim$ is ℓ_0-independent but depends on $\mathbf{z}$). It follows

$$\left| \|i_- v\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} - \|i_- v\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}} \right| \lesssim \left| \left\langle w^2 v_0, v_0 \right\rangle_{\mathcal{H}^{\mathbf{z}}} \right| + \|v\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \left\| w^2 v_0 \right\|_{\mathcal{H}^{\mathbf{z}}}$$

where $w(r(x)) = \sqrt{(r(x) - r_-)(r_+ - r(x))} = \mathcal{O}_{|x|\to+\infty}(\mathrm{e}^{-2\kappa|x|})$. The $\mathcal{H}^{\mathbf{z}}$ norms make sense thanks to Hardy type inequality [GGH17, Lemma 9.5]:

$$\begin{aligned}
\left| \|i_- v\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} - \|i_- v\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}} \right| &\lesssim \|h_0^{1/2} w^{1/2} v_0\|^2_{\mathcal{H}^{\mathbf{z}}} + \|v\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \left\| h_0^{1/2} w^{1/2} v_0 \right\|_{\mathcal{H}^{\mathbf{z}}} \\
&\lesssim \|w^{1/2} v\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}} + \|v\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \|w^{1/2} v\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \\
&\lesssim \|w^{1/2} v\|_{\dot{\mathcal{E}}^{\mathbf{z}}}.
\end{aligned}$$

Fix now $\varepsilon > 0$ arbitrarily small and pick $(\phi_n)_{\in\mathbb{R}}$ a sequence of smooth compactly supported functions such that $\phi_n \to u$ in $\dot{\mathcal{E}}^{\mathbf{z}}$ as $n \to +\infty$. We have

$$\|w^{1/2} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \lesssim \|w^{1/2} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} (\phi_n - u)\|_{\dot{\mathcal{E}}^{\mathbf{z}}} + \|w^{1/2} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} \phi_n\|_{\dot{\mathcal{E}}^{\mathbf{z}}}.$$

Fix $N \gg 0$ so that

$$\|w^{1/2} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} (\phi_N - u)\|_{\dot{\mathcal{E}}^{\mathbf{z}}} < \varepsilon/2$$

[17] Recall as in the proof of Theorem 2.3.11 that there is no restriction on the support of χ because we assume s small enough so that [**?**, Theorem 3.8] implies that there is no resonance on $\mathbb{R}$; in the general case, χ must cancel in a neighborhood of the real resonances.

then apply [GGH17, Proposition 6.7] to get

$$\left\|w^{1/2}\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})\phi_N\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \leq \left\|w^{1/2}\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})w^{1/2}\right\|_{\mathcal{B}(\dot{\mathcal{E}}^{\mathbf{z}})}\left\|w^{-1/2}\phi_N\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \leq C_N\langle t\rangle^{-1}$$

with $C_N > 0$ depending on the support of ϕ_N. Choose $t \gg 0$ (depending on N) so that $C_N\langle t\rangle^{-1} < \varepsilon/2$. Hence, for t large enough, we have

$$\left|\left\|i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} - \left\|i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}}\right| < C\varepsilon$$

for some $C > 0$ (independent of ϕ_N). This completes the proof. □

Theorem 2.5.5 (Inversion of the full wave operators)**.** *Let $\mathbf{z} \in \mathbb{Z}\setminus\{0\}$. There exists $s_0 > 0$ such that for all $s \in]-s_0, s_0[$,*

$$\mathbb{\Omega}^{\pm}\mathbb{W}^{\pm} = \mathbb{1}_{\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}\times\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}}, \tag{2.51}$$

$$\mathbb{W}^{\pm}\mathbb{\Omega}^{\pm} = \mathbb{1}_{\dot{\mathcal{E}}^{\mathbf{z}}}. \tag{2.52}$$

Proof. Let us show (2.51). Since $\sqrt{2}\Psi_{\mathscr{H}}\Pi_0$ (respectively $\sqrt{2}\Psi_{\mathscr{J}}\Pi_1$) is the projection onto $\Psi_{\mathscr{H}}\big(\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{H}}\times\{0\}\big)$ (respectively onto $\Psi_{\mathscr{H}}\big(\{0\}\times\dot{\mathcal{H}}^{1,\mathbf{z}}_{\mathscr{J}}\big)$), this identity directly follows from Proposition 2.3.13.

Let us show (2.52). It is sufficient to show that $\mathbb{\Omega}^{\pm}$ is one-to-one, since then the right-inverse is also a left-inverse[18]. By Remark 2.5.3, it is sufficient to show that

$$\ker \mathbf{\Omega}^{f/p}_{\mathscr{H}/\mathscr{J}} = \{0\}. \tag{2.53}$$

Let $u \in \dot{\mathcal{E}}^{\mathbf{z}}$ such that

$$\mathbf{\Omega}^{f/p}_{\mathscr{H}}u = \mathbf{\Omega}^{f/p}_{\mathscr{J}}u = 0. \tag{2.54}$$

Pick $\varepsilon > 0$, $t \in \mathbb{R}$ and $\chi \in \mathcal{C}^{\infty}_{\mathrm{c}}(\mathbb{R})$ as in Lemma 2.5.4 then write:

$$\|u\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \lesssim \left\|\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} + \left\|(1-\chi(\dot{H}^{\mathbf{z}}))u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}}, \tag{2.55}$$

$$\begin{aligned}\left\|\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} &\lesssim \left\|\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}}\\ &\lesssim \left\|(i_-^2+i_+^2)\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}}\\ &\lesssim \left\|i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} + \left\|i_+\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}}.\end{aligned}$$

We have used above the boundedness on $\dot{\mathcal{E}}^{\mathbf{z}}$ of $\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}$ (*cf.* Theorem 2.3.8) and $i_\pm$ (*cf.* [GGH17, Lemma 5.4]). Next, using Lemma 2.5.4 as well as the unitarity of $\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{J}}}$, we get for $|t| \gg 0$:

$$\begin{aligned}\left\|\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}} &\lesssim \left\|i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + \left\|i_+\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{J}}} + \frac{\varepsilon}{6}\\ &\lesssim \left\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}}i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + \left\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}}i_+\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}\chi(\dot{H}^{\mathbf{z}})u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{J}}} + \frac{\varepsilon}{6}\\ &\lesssim \left\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}}i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + \left\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}}i_+\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{J}}}\\ &+ \left\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}}i_-\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}(1-\chi(\dot{H}^{\mathbf{z}}))u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + \left\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{J}}}i_+\mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}(1-\chi(\dot{H}^{\mathbf{z}}))u\right\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{J}}} + \frac{\varepsilon}{6}.\end{aligned} \tag{2.56}$$

[18] Indeed, assume that $AB = \mathbb{1}$ and A has been shown to be one-to-one. Then $A = (AB)A$ and the injectivity allows us to simplify the equation on its left, that is $\mathbb{1} = BA$.

By part 2. of Theorem 2.3.11, we have for t sufficiently large:

$$\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}}} i_- \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} \leq \|\mathbf{\Omega}^{f/p}_{\mathscr{H}} u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + \frac{\varepsilon}{6}, \qquad \|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{I}}} i_+ \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{I}}} \leq \|\mathbf{\Omega}^{f/p}_{\mathscr{I}} u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{I}}} + \frac{\varepsilon}{6}.$$

Furthermore,

$$\|\mathrm{e}^{-\mathrm{i}t\dot{H}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}} i_{-/+} \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}} (1-\chi(\dot{H}^{\mathbf{z}}))u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}} \lesssim \|(1-\chi(\dot{H}^{\mathbf{z}}))u\|_{\dot{\mathcal{E}}^{\mathbf{z}}}$$

by uniform boundedness of the evolutions as well as boundedness of $i_{-/+} : \dot{\mathcal{E}}^{\mathbf{z}} \to \dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}/\mathscr{I}}$ (*cf.* Lemma 2.4.1). Back into (2.56), we obtain:

$$\|\chi(\dot{H}^{\mathbf{z}})u\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \lesssim \|\mathbf{\Omega}^{f/p}_{\mathscr{H}} u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + \|\mathbf{\Omega}^{f/p}_{\mathscr{I}} u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{I}}} + 2\|(1-\chi(\dot{H}^{\mathbf{z}}))u\|_{\dot{\mathcal{E}}^{\mathbf{z}}} + \frac{\varepsilon}{2}. \tag{2.57}$$

Plugging (2.57) into (2.55) and letting the support of χ widespread enough in order to use (2.36), we obtain with assumption (2.54):

$$\|u\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \lesssim \|\mathbf{\Omega}^{f/p}_{\mathscr{H}} u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{H}}} + \|\mathbf{\Omega}^{f/p}_{\mathscr{I}} u\|_{\dot{\mathcal{E}}^{\mathbf{z}}_{\mathscr{I}}} + \varepsilon \lesssim \varepsilon.$$

As ε was arbitrary, we have shown (2.53). This completes the proof. □

2.5.4 Traces on the energy spaces

Let $(\phi_0, \phi_1) \in \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0) \times \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0)$. By Leray's theorem (*cf.* [Le53]), there exists an unique solution $\phi \in \mathcal{C}^\infty(\widetilde{\mathcal{M}})$ of

$$\begin{cases} \Box_{\tilde{g}}\phi = 0 \\ \phi_{|\Sigma_0} = \phi_0 \\ (-\mathrm{i}\partial_t\phi)_{|\Sigma_0} = \phi_1 \end{cases}. \tag{2.58}$$

Moreover, ϕ extends to a smooth function $\hat{\phi} \in \mathcal{C}^\infty(\overline{\mathcal{M}})$. In particular, $\hat{\phi}$ has traces $(\xi^\pm, \zeta^\pm) \in \mathcal{C}^\infty(\mathscr{H}^\pm) \times \mathcal{C}^\infty(\mathscr{I}^\pm)$. The *future/past trace operators* are then defined on smooth compactly supported data by

$$\mathscr{T}^\pm : \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0) \times \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0) \ni (\phi_0, \phi_1) \longmapsto (\xi^\pm, \zeta^\pm) \in \mathcal{C}^\infty(\mathscr{H}^\pm) \times \mathcal{C}^\infty(\mathscr{I}^\pm). \tag{2.59}$$

The purpose of this Subsection is to extend the traces on the asymptotic energy spaces $\dot{\mathscr{E}}^{\mathbf{z}}_\pm$. To do so, we will use completeness of wave operators.

Using the identification diffeomorphisms of Subsection 2.5.1, we first link traces on the horizons to the operators $\mathbb{\Omega}^\pm$:

Lemma 2.5.6 (Pointwise traces)**.** *Define the isometries*

$$\mathcal{U}^\pm := \begin{pmatrix} (\mathscr{F}^\pm_{\mathscr{H}})^* & 0 \\ 0 & (\mathscr{F}^\pm_{\mathscr{I}})^* \end{pmatrix} \in \mathcal{B}\big(\dot{\mathcal{H}}^1_{\mathscr{H}} \times \dot{\mathcal{H}}^1_{\mathscr{I}}, \dot{\mathscr{E}}_\pm\big).$$

For all $\phi = (\phi_0, \phi_1) \in \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0) \times \mathcal{C}^\infty_{\mathrm{c}}(\Sigma_0)$,

$$\mathscr{T}^\pm \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = \mathcal{U}^\pm \mathbb{\Omega}^\pm \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}.$$

Proof. We only treat the + case. Let $\phi = (\phi_0, \phi_1) \in \mathcal{C}_c^\infty(\Sigma_0) \times \mathcal{C}_c^\infty(\Sigma_0)$ and set $(v_0(t), v_1(t)) := \mathrm{e}^{\mathrm{i}t\dot{H}^{\mathbf{z}}}(\phi_0, \phi_1)$. The operators $(\mathscr{F}^+_{\mathscr{H}/\mathscr{I}})^* \mathrm{e}^{-\mathrm{i}t\dot{\mathbb{H}}_{\mathscr{H}/\mathscr{I}}}$ carry data onto the future horizons along principal null geodesics, so we have

$$\begin{aligned}\Omega(t)\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix} &= \begin{pmatrix}(\mathscr{F}^+_{\mathscr{H}})^*(j_- v_0(t)) \circ \gamma_{\mathrm{in}}(-t)\\ (\mathscr{F}^+_{\mathscr{I}})^*(j_+ v_0(t)) \circ \gamma_{\mathrm{out}}(-t)\end{pmatrix}\\ &= \begin{pmatrix}(v_0)_{|\mathscr{H}^+}\\ (v_0)_{|\mathscr{I}^+}\end{pmatrix}\\ &= \mathscr{T}^+\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}\end{aligned}$$

for $t \gg 0$ since $i_{-/+} \equiv 1$ near $\mathscr{H}/\mathscr{I}$. □

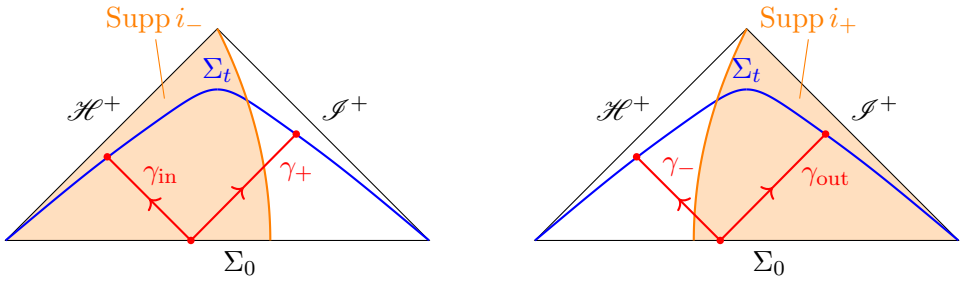

Figure 2.5: Transports from Σ_0 onto Σ_t along the principal null geodesics $\gamma_{\mathrm{in/out}}$ and the curves $\gamma_{+/-}$. Data reaching horizons are carried only by $\gamma_{\mathrm{in/out}}$.

Combining Lemma 2.5.6 with Theorem 2.5.5, we obtain:

Theorem 2.5.7 (Extension of the traces)**.** *Let $\mathbf{z} \in \mathbb{Z} \setminus \{0\}$. There exists $s_0 > 0$ such that for all $s \in]-s_0, s_0[$, the traces extend to energy spaces as bounded invertible operators:*

$$\begin{aligned}\mathscr{T}^\pm &= \mathcal{U}^\pm \mathbb{\Omega}^\pm \in \mathcal{B}\big(\dot{\mathcal{E}}^{\mathbf{z}}, \dot{\mathscr{E}}^{\mathbf{z}}_\pm\big),\\ (\mathscr{T}^\pm)^{-1} &= \mathbb{W}^\pm (\mathcal{U}^\pm)^{-1} \in \mathcal{B}\big(\dot{\mathscr{E}}^{\mathbf{z}}_\pm, \dot{\mathcal{E}}^{\mathbf{z}}\big).\end{aligned}$$

Remark 2.5.8. *The regularity of elements in the energy spaces does* a priori *not ensure the existence of the traces for general solutions of the extended wave equation. Theorem 2.5.7 shows that they exist thanks to the completeness of the wave operators.*

Theorem 2.5.7 also provides the geometric interpretation of the full wave operators as inverses of the traces on horizons. Both are linked by the transformations $\mathcal{U}^\pm$ which identify points on horizons and Σ_0 via transport along principal null geodesics.

2.5.5 Solution of the Goursat problem

The Goursat problem consists in an inverse problem on the global outer space $(\overline{\mathcal{M}}, \widetilde{g})$ constructed in Subsection 2.2.3. Given boundary data $(\xi^\pm, \zeta^\pm) \in \mathcal{C}_c^\infty(\mathscr{H}^\pm) \times \mathcal{C}_c^\infty(\mathscr{I}^\pm)$, we are asked to find $\phi : \overline{\mathcal{M}} \to \mathbb{C}^2$ solving the wave equation (2.2) and such that

$$\phi_{|\mathscr{H}^\pm} = \xi^\pm, \qquad \phi_{|\mathscr{I}^\pm} = \zeta^\pm. \tag{2.60}$$

The Goursat problem is linked to the trace operators as being the inverse procedure of taking the trace of a solution of equation (2.2). The analytic scattering theory solves this problem by constructing the inverse wave operators. See the paper of Nicolas [N15, Remark 1.1 & Section 4] for some discussions about the different points of view of the scattering.

Theorem 2.5.7 allows us to solve the following abstract Goursat problem:

Theorem 2.5.9. *Let $\mathbf{z} \in \mathbb{Z} \setminus \{0\}$. There exists $s_0 > 0$ such that for all $s \in]-s_0, s_0[$ the following property: there exist homeomorphisms*

$$\mathbb{T}^{\pm} : \dot{\mathcal{E}}^{\mathbf{z}} \longrightarrow \dot{\mathscr{E}}^{\mathbf{z}}_{\pm}$$

solving the Goursat problem (2.60) in the energy spaces, that is, for all $(\xi^{\pm}, \zeta^{\pm}) \in \dot{\mathscr{E}}^{\mathbf{z}}_{\pm}$, there exists a unique $\phi \in \mathcal{C}^0(\mathbb{R}_t; \dot{\mathcal{E}}^{\mathbf{z}})$ solving the wave equation on $(\widetilde{\mathcal{M}}, \widetilde{g})$ with initial data $\phi(0) = (\phi_0, \phi_1)$ such that

$$(\xi^{\pm}, \zeta^{\pm}) = \mathbb{T}^{\pm}(\phi_0, \phi_1).$$

Remark 2.5.10. *In the standard case of the wave equation on De Sitter-Schwarzschild spacetime [N15], the traces extend as unitary operators between energy spaces defined on Σ_0 and on the horizons. Here, we only have bounded extensions because of the superradiance. In particular, we have the following control of the energies:*

$$\frac{1}{C}\|\phi(0)\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}} \leq \|(\xi^{\pm}, \zeta^{\pm})\|^2_{\dot{\mathscr{E}}^{\mathbf{z}}_{\pm}} \leq C\|\phi(0)\|^2_{\dot{\mathcal{E}}^{\mathbf{z}}}$$

for some constant $C > 0$.

2.6 Appendix: Remarkable tensors

In this Appendix, we explicitly give the components in the Boyer-Lindquist coordinates $(t, z, r, \theta, \varphi)$ of the tensors associated to the extended spacetime $(\widetilde{\mathcal{M}}, \widetilde{g})$. We will consider the case of an electromagnetic potential $A = W(r)\mathrm{d}t$ with W smooth and put $W(r) := sV(r)$ only for the scalar curvature and the energy-momentum tensor.

Christoffel symbols. The components of Christoffel symbols $\widetilde{\Gamma}$ are given in matrix notations by:

$$\widetilde{\Gamma}^0_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{m^2F' - WW'}{2m^2F} & 0 & 0 \\ 0 & 0 & -\frac{W'}{2m^2F} & 0 & 0 \\ \frac{m^2F' - WW'}{2m^2F} & -\frac{W'}{2m^2F} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\Gamma}^1_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{m^2(FW' - F'W) + W^2W'}{2m^2F} & 0 & 0 \\ 0 & 0 & \frac{WW'}{2m^2F} & 0 & 0 \\ \frac{m^2(FW' - F'W) + W^2W'}{2m^2F} & \frac{WW'}{2m^2F} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\Gamma}^2_{\mu\nu} = \begin{pmatrix} \frac{F(m^2F'-2WW')}{2m^2} & -\frac{FW'}{2m^2} & 0 & 0 & 0 \\ -\frac{FW'}{2m^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{F'}{2F} & 0 & 0 \\ 0 & 0 & 0 & -rF & 0 \\ 0 & 0 & 0 & 0 & -rF\sin^2\theta \end{pmatrix},$$

$$\widetilde{\Gamma}^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix}, \qquad \widetilde{\Gamma}^4_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & 0 & \cot\theta \\ 0 & 0 & \frac{1}{r} & \cot\theta & 0 \end{pmatrix}.$$

Riemann curvature tensor. The components of the Riemann curvature tensor $\widetilde{\mathrm{Riem}}$ are given in matrix notations by:

$$\widetilde{\mathrm{Riem}}^0_{0\mu\nu} = \begin{pmatrix} 0 & -\frac{WW'^2}{4m^4} & 0 & 0 & 0 \\ \frac{WW'^2}{4m^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{\mathrm{Riem}}^0_{1\mu\nu} = \begin{pmatrix} 0 & -\frac{W'^2}{4m^4} & 0 & 0 & 0 \\ \frac{W'^2}{4m^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^0_{2\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{-2m^2F''+3W'^2+2WW''}{4m^2F} & 0 & 0 \\ 0 & 0 & \frac{W''}{2m^2F} & 0 & 0 \\ -\frac{-2m^2F''+3W'^2+2WW''}{4m^2F} & -\frac{W''}{2m^2F} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^0_{3\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{r(-m^2F'+WW')}{2m^2} & 0 \\ 0 & 0 & 0 & \frac{rW'}{2m^2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{r(-m^2F'+WW')}{2m^2} & -\frac{rW'}{2m^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^0_{4\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{r(-m^2F'+WW')}{2m^2}\sin^2\theta \\ 0 & 0 & 0 & 0 & \frac{rW'}{2m^2}\sin^2\theta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{r(-m^2F'+WW')}{2m^2}\sin^2\theta & -\frac{rW'}{2m^2}\sin^2\theta & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{1}_{0\mu\nu} = \begin{pmatrix} 0 & \frac{-m^2FW'^2+W^2W'^2}{4m^4} & 0 & 0 & 0 \\ -\frac{-m^2FW'^2+W^2W'^2}{4m^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{1}_{1\mu\nu} = \begin{pmatrix} 0 & \frac{WW'^2}{4m^4} & 0 & 0 & 0 \\ -\frac{WW'^2}{4m^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{1}_{2\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{m^2(F''W-FW'')-W(2W'^2+WW'')}{2m^2F} & 0 & 0 \\ 0 & 0 & -\frac{W'^2+2WW''}{4m^2F} & 0 & 0 \\ -\frac{m^2(F''W-FW'')-W(2W'^2+WW'')}{2m^2F} & \frac{W'^2+2WW''}{4m^2F} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{1}_{3\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{r(m^2(F'W-FW')-W^2W')}{2m^2} & 0 \\ 0 & 0 & 0 & -\frac{rWW'}{2m^2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{r(m^2(F'W-FW')-W^2W')}{2m^2} & \frac{rWW'}{2m^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{1}_{4\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{r(m^2(F'W-FW')-W^2W')}{2m^2}\sin^2\theta \\ 0 & 0 & 0 & 0 & -\frac{rWW'}{2m^2}\sin^2\theta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{r(m^2(F'W-FW')-W^2W')}{2m^2}\sin^2\theta & \frac{rWW'}{2m^2}\sin^2\theta & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{2}_{0\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{m^2(-2m^2FF''+3FW'^2+4FWW'')+W^2W'^2}{4m^4} & 0 & 0 \\ 0 & 0 & \frac{2m^2FW''+WW'^2}{4m^4} & 0 & 0 \\ -\frac{m^2(-2m^2FF''+3FW'^2+4FWW'')+W^2W'^2}{4m^4} & -\frac{2m^2FW''+WW'^2}{4m^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{2}_{1\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{2m^2FW''+WW'^2}{4m^4} & 0 & 0 \\ 0 & 0 & \frac{W'^2}{4m^4} & 0 & 0 \\ -\frac{2m^2FW''+WW'^2}{4m^4} & -\frac{W'^2}{4m^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{2}_{2\mu\nu} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \qquad \widetilde{\mathrm{Riem}}^{2}_{3\mu\nu} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&-\frac{rF'}{2}&0 \\ 0&0&\frac{rF'}{2}&0&0 \\ 0&0&0&0&0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{2}_{4\mu\nu} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&-\frac{rF'}{2}\sin^2\theta \\ 0&0&0&0&0 \\ 0&0&\frac{rF'}{2}\sin^2\theta&0&0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{3}_{0\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{F(-m^2F'+2WW')}{2m^2r} & 0 \\ 0 & 0 & 0 & \frac{FW'}{2m^2r} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{F(-m^2F'+2WW')}{2m^2r} & -\frac{FW'}{2m^2r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{3}_{1\mu\nu} = \begin{pmatrix} 0&0&0&\frac{FW'}{2m^2r}&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ -\frac{FW'}{2m^2r}&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \qquad \widetilde{\mathrm{Riem}}^{3}_{2\mu\nu} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&\frac{F'}{2rF}&0 \\ 0&0&-\frac{F'}{2rF}&0&0 \\ 0&0&0&0&0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{3}_{3\mu\nu} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \quad \widetilde{\mathrm{Riem}}^{3}_{4\mu\nu} = \begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&-(F-1)\sin^2\theta \\ 0&0&0&(F-1)\sin^2\theta&0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{4}_{0\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{F(-m^2F'+2WW')}{2m^2r} \\ 0 & 0 & 0 & 0 & \frac{FW'}{2m^2r} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{F(-m^2F'+2WW')}{2m^2r} & -\frac{FW'}{2m^2r} & 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{4}_{1\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{FW'}{2m^2r} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{FW'}{2m^2r} & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{\mathrm{Riem}}^{4}_{2\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{F'}{2rF} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{F'}{2rF} & 0 & 0 \end{pmatrix},$$

$$\widetilde{\mathrm{Riem}}^{4}_{3\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & F-1 \\ 0 & 0 & 0 & -(F-1) & 0 \end{pmatrix}, \qquad \widetilde{\mathrm{Riem}}^{4}_{4\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Ricci curvature tensor. The non-zero components of the Ricci curvature tensor $\widetilde{\mathrm{Ricc}}$ are given by:

$$\begin{aligned}
\widetilde{\mathrm{Ricc}}_{00} &= \frac{F\left(2F'+rF''\right)}{2r} - \left(\frac{2FWW'}{m^2r} + \frac{FWW''}{m^2} + \frac{FW'^2}{2m^2} + \frac{W^2W'^2}{2m^4}\right), \\
\widetilde{\mathrm{Ricc}}_{01} &= \widetilde{\mathrm{Ricc}}_{10} = -\left(\frac{FW'}{m^2r} + \frac{FW''}{2m^2} + \frac{WW'^2}{2m^4}\right), \\
\widetilde{\mathrm{Ricc}}_{11} &= -\frac{W'^2}{2m^4}, \\
\widetilde{\mathrm{Ricc}}_{22} &= -\frac{2F'+rF''}{2rF} + \frac{W'^2}{2m^2F}, \\
\widetilde{\mathrm{Ricc}}_{33} &= 1-F-rF', \\
\widetilde{\mathrm{Ricc}}_{44} &= \left(1-F-rF'\right)\sin^2\theta.
\end{aligned}$$

Scalar curvature. Put now $W(r) := sV(r)$. The scalar curvature $\widetilde{\mathrm{R}}$ is given by

$$\begin{aligned}
\widetilde{\mathrm{R}} &= \left(F'' + \frac{4F'}{r} + \frac{2F}{r^2} - \frac{2}{r^2}\right) - \frac{W'^2}{2m^2} \\
&= \mathrm{R}_{\mathrm{DSRN}} - \frac{q^2Q^2}{2m^2r^4}
\end{aligned}$$

with $\mathrm{R}_{\mathrm{DSRN}} = -4\Lambda$ the Ricci curvature associated to the DSRN metric g. We see here that $r=0$ is still a genuine singularity in the extended spacetime.

Energy-momentum tensor. The extended energy-momentum tensor $\widetilde{T}$ is given by $\widetilde{T} = \widetilde{T}_{\text{Maxwell}} + \widetilde{T}_{\text{fluid}}$ with in matrix notations

$$\widetilde{T}_{\text{Maxwell}} = \frac{Q^2}{2r^4}\left(1 - \frac{q^2}{2m^2}\right)\begin{pmatrix} -F(r) - \frac{s^2V(r)^2}{m^2} & -\frac{sV(r)}{m^2} & 0 & 0 & 0 \\ -\frac{sV(r)}{m^2} & -\frac{1}{m^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{F(r)} & 0 & 0 \\ 0 & 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix},$$

$$\widetilde{T}_{\text{fluid}} = \left(\Lambda + \frac{Q^2}{2r^4}\left(1 + \frac{q^2}{m^2}\right)\right)\begin{pmatrix} \frac{s^2V(r)^2}{m^2} & \frac{sV(r)}{m^2} & 0 & 0 & 0 \\ \frac{sV(r)}{m^2} & \frac{1}{m^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Chapter 3

Decay of the Local Energy through the Horizons for the Wave Equation in the Exterior Extended Spacetime

Having established the decay of the local energy Theorem 1.3.2 in the exterior DSRN spacetime, it is possible to obtain decay estimates through the horizons in the Kaluza-Klein extension of Chapter 2 following Dyatlov paper [Dy11]. The main idea is to use the so-called red-shift vector field method introduced by Dafermos and Rodnianski in order to obtain decay near horizons by the red-shif effect, see *e.g.* [Da05], [DaRo07] and [DaRo09]. This method is applicable in dimension $1+4$ even for extensions of non-spherical rotating black holes, provided that the rotation remains small.

In all this chapter, we will use the notations of Chapter 2, particularly those in Subsection 2.3.1. For example, $\dot{\mathcal{E}}$ is the inhomogeneous energy space defined in the extended spacetime (and not in the original DSRN spacetime).

3.1 Statement of the main result

Introduce a new time coordinate τ smooth near $r = r_\pm$ and set for all $\delta > 0$

$$X_\delta := \mathbb{S}^1_z \times]r_- - \delta, r_+ + \delta[_r \times \mathbb{S}^2_\omega.$$

Denote by $H^\sigma(X_\delta)$, $\sigma \in \mathbb{R}$, the inhomogeneous Sobolev spaces associated to $L^2(\mathbb{S}^1_\zeta \times]r_-, r_+[_r \times \mathbb{S}^2_\omega, F(r)^{-1}\mathrm{d}\zeta\mathrm{d}r\mathrm{d}\omega)$ and wet $H^\sigma_{\mathbf{z}}(X_\delta)$ for the restriction to $\ker(\mathrm{i}\partial_z + \mathbf{z})$, $\mathbf{z} \in \mathbb{Z}$. As we are interested in large time estimate, *we will implicitly use the convention that Sobolev spaces in time are defined on* $]0, +\infty[$.

From the decay of the local energy of Theorem 1.3.2 and the fact that the inhomogeneous energy $\|\cdot\|_{\dot{\mathcal{E}}^{\mathbf{z}}}$ coincide with the inhomogeneous energy defined in the DSRN spacetime (with a charge term $s\mathbf{z}$), we immediatly deduce the following result:

Lemma 3.1.1. *Let $\mathbf{z} \in \mathbb{Z} \neq \{0\}$ and $\chi \in X_\delta$. For s sufficiently small, there exists $\nu > 0$ such that for all $u \in \dot{\mathcal{E}}^{\mathbf{z}}$ solving $\Box_{\widetilde{g}} u$ with initial data $u(0) = u_0$ compactly supported in $\mathbb{S}^1_z \times]r_-, r_+[_r \times \mathbb{S}^2_\omega$,*

we have for $t > 0$ large enough:

$$\|\chi u\|_{\dot{\mathcal{E}}^{\mathbf{z}}} \lesssim \mathrm{e}^{-\nu t}\|u_0\|_{\dot{\mathcal{E}}^{\mathbf{z}}}.$$

We then have the following result (the proof is given in Subsection 3.3):

Theorem 3.1.2 (Decay of the energy through the horizons)**.** *Let $\mathbf{z} \in \mathbb{Z} \neq \{0\}$, $\delta > 0$, $\varepsilon > 0$, $\sigma \geq 0$ and let u be the solution of the initial value problem:*

$$\begin{cases} \Box_{\widetilde{g}} u &= 0 \\ u(0,\cdot) &= u_0 \in H^{\sigma+2+\varepsilon}_{\mathbf{z}}(X_\delta) \\ (\partial_\tau u)(0,\cdot) &= u_1 \in H^{\sigma+1+\varepsilon}_{\mathbf{z}}(X_\delta) \end{cases} \tag{3.1}$$

If δ is sufficiently small, then there exist $\nu_0 \equiv \nu_0(\delta,\varepsilon) > 0$ and $C > 0$ (independent of u) such that, for all $\nu \in]0,\nu_0[$ and $s \in \mathbb{R}$ sufficiently small, the following estimate holds:

$$\|u(\tau,\cdot)\|_{H^{\sigma+1}_{\mathbf{z}}(X_\delta)} \leq C\mathrm{e}^{-\nu\tau}\Big(\|u_0\|_{H^{\sigma+2+\varepsilon}_{\mathbf{z}}(X_\delta)} + \|u_1\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(X_\delta)}\Big) \qquad \forall \tau > 0. \tag{3.2}$$

One may notice that (3.2) is not a global energy decay as estimates are restricted on a part of the extended spacetime $(\widetilde{\mathcal{M}},\widetilde{g})$, *cf.* Remark 3.3.1.

3.2 Construction of the red-shift vector field

The extension of decay estimates beyond the horizons relies on the construction of the red-shift vector field. We first need to extend $\widetilde{g}$ near $r = r_\pm$. Define the new coordinates

$$\tau := t - \mathcal{T}(r), \qquad \zeta := z - \mathcal{Z}(r)$$

where $\mathcal{T},\mathcal{Z} \in \mathcal{C}^\infty(]r_-,r_+[_r,\mathbb{R})$ satisfy

$$\mathcal{T}'(r) = \frac{\rho(r)}{F(r)} + \alpha(r), \qquad \mathcal{Z}'(r) = -sV(r)\mathcal{T}'(r) \qquad \text{near } r_\pm.$$

Here $\rho \in \mathcal{C}^\infty(]0,+\infty[,\mathbb{R})$, $\rho \equiv \pm 1$ near $r_\pm$ and $\alpha \in \mathcal{C}^\infty(]0,+\infty[,\mathbb{R})$ are two bounded functions to be chosen later on. In order to simplify the proof of Lemma 3.2.1 below, we let α depend on the charge product $s \in \mathbb{R}$ so that $\|\alpha\|_{L^\infty(]0,+\infty[,\mathbb{R})} = \mathcal{O}_{s\to 0}(|s|)$. Observe that $\partial_t = \partial_\tau$ and $\partial_z = \partial_\zeta$.

The extended metric $\widetilde{g}$ reads in these new coordinates:

$$\widetilde{g} = \left(F(r) - \frac{s^2V(r)^2}{m^2}\right)\mathrm{d}\tau^2 - \frac{sV(r)}{m^2}(\mathrm{d}\tau\mathrm{d}\zeta + \mathrm{d}\zeta\mathrm{d}\tau) - \frac{1}{m^2}\mathrm{d}\zeta^2 + \big(\alpha(r)F(r) + \rho(r)\big)(\mathrm{d}\tau\mathrm{d}r + \mathrm{d}r\mathrm{d}\tau)$$
$$+ \left(\frac{\rho^2-1}{F(r)} + \alpha(r)\big(\alpha(r)F(r) + 2\rho(r)\big)\right)\mathrm{d}r^2 - r^2\mathrm{d}\omega^2.$$

The determinant is still $r^4 \sin^2\theta/m^2$ so that the metric is smooth and does not degenerate as long as α remains smooth. The inverse metric is given by

$$\begin{aligned}\widetilde{g}^{-1} = &-\left(\frac{\rho(r)^2-1}{F(r)} + \alpha(r)\big(\alpha(r)F(r)+2\rho(r)\big)\right)\partial_\tau^{\otimes 2}\\ &+ sV(r)\left(\frac{\rho(r)^2-1}{F(r)} + \alpha(r)\big(\alpha(r)F(r)+2\rho(r)\big)\right)(\partial_\tau\otimes\partial_\zeta + \partial_\zeta\otimes\partial_\tau)\\ &-\left(m^2 + s^2V(r)^2\left(\frac{\rho(r)^2-1}{F(r)} + \alpha(r)\big(\alpha(r)F(r)+2\rho(r)\big)\right)\right)\partial_\zeta^{\otimes 2}\\ &+\big(\alpha(r)F(r)+\rho(r)\big)(\partial_\tau\otimes\partial_r + \partial_r\otimes\partial_\tau) - sV(r)\big(\alpha(r)F(r)+\rho(r)\big)(\partial_\zeta\otimes\partial_r+\partial_r\otimes\partial_\zeta)\\ &-F(r)\partial_r^{\otimes 2} - \frac{1}{r^2}\partial_\theta^{\otimes 2} - \frac{1}{r^2\sin^2\theta}\partial_\varphi^{\otimes 2}.\end{aligned}$$

We introduced the function α to make the hypersurfaces $\Sigma_{\tau_0} := \{\tau = \tau_0\}$ spacelike near $r_\pm$ for all $\tau_0 \in \mathbb{R}$:

$$\begin{aligned}\nabla\tau = &-\left(\frac{\rho(r)^2-1}{F(r)} + \alpha(r)\big(\alpha(r)F(r)+2\rho(r)\big)\right)\partial_\tau\\ &+ sV(r)\left(\frac{\rho(r)^2-1}{F(r)} + \alpha(r)\big(\alpha(r)F(r)+2\rho(r)\big)\right)\partial_\zeta + \big(\alpha(r)F(r)+\rho(r)\big)\partial_r\end{aligned}$$

and

$$\widetilde{g}(\nabla\tau,\nabla\tau) = -\left(\frac{\rho(r)^2-1}{F(r)} + \alpha(r)\big(\alpha(r)F(r)+2\rho(r)\big)\right).$$

Since $F(r_\pm) = 0$ and $\rho(r)^2 - 1 = 0$ for r near $r_\pm$, $\nabla\tau$ is timelike in $\mathbb{R}_\tau \times]r_\pm - \delta, r_\pm + \delta[\times \mathbb{S}^2_\omega$ for $\delta > 0$ small enough if $\alpha\rho < 0$ there; asking for $|\rho| < 1$ as well as $\alpha F + 2\rho < 0$ for $r \in [r_- + \delta, r_+ - \delta]$, we obtain that $\nabla\tau$ is timelike in the whole $\mathbb{R}_\tau \times]r_- - \delta, r_+ + \delta[\times \mathbb{S}^2_\omega$. This result is independent of s in α as long as it is non-zero and small enough.

For $\delta > 0$ sufficiently small, the metric $\widetilde{g}$ thus smoothly extends to

$$\mathscr{M}_\delta := \mathbb{R}_\tau \times \mathbb{S}^1_\zeta \times \widetilde{\mathcal{M}}_\delta, \qquad \widetilde{\mathcal{M}}_\delta :=]r_- - \delta, r_+ + \delta[_r \times \mathbb{S}^2_\omega.$$

Let now

$$\mathscr{K}_\delta := \mathbb{R}_\tau \times \mathbb{S}^1_\zeta \times \mathcal{K}_\delta \times \mathbb{S}^2_\omega, \qquad \mathcal{K}_\delta :=]r_- + \delta, r_+ - \delta[_r \subset \widetilde{\mathcal{M}}_\delta.$$

For all vector field $X \in T\mathscr{M}_\delta$, we define

$$K^X_{ij} := \frac{1}{2}\left(\mathcal{L}_X\widetilde{g}\right)_{ij} - \frac{1}{4}\mathrm{Tr}_{\widetilde{g}}(\mathcal{L}_X\widetilde{g})\,\widetilde{g}_{ij}.$$

The red-shift vector field can now be constructed:

Lemma 3.2.1. *For $\delta > 0$ and $s \in \mathbb{R}$ sufficiently small, there exits a vector field $X \in T\mathscr{M}_\delta$ such that:*

- $[X, \partial_\tau] = 0$;

- *On $\mathscr{M}_\delta \setminus \mathscr{K}_{2\delta}$, X is timelike and $X\tau > 0$, $\pm Xr > 0$ near $r_\pm$;*
- *The tensor field K^X is negative definite on $\mathscr{M}_\delta \setminus \mathscr{K}_{2\delta}$.*

Proof. As in [Dy11], we only show the proposition for $s = 0$ and at the horizons; the result will then follow by smoothness for $\delta > 0$ and $s \in \mathbb{R}$ sufficiently small.

For $s = 0$, the extended metric and its inverse become near the horizons

$$\begin{aligned}\widetilde{g} &= F(r)\mathrm{d}\tau^2 - \frac{1}{m^2}\mathrm{d}\zeta^2 \pm (\mathrm{d}\tau\mathrm{d}r + \mathrm{d}r\mathrm{d}\tau) - r^2\mathrm{d}\omega^2,\\ \widetilde{g}^{-1} &= -m^2\partial_\zeta^{\otimes 2} \pm (\partial_\tau\partial_r + \partial_r\partial_\tau) - F(r)\partial_r^{\otimes 2} - \frac{1}{r^2}\partial_\theta^{\otimes 2} - \frac{1}{r^2\sin^2\theta}\partial_\varphi^{\otimes 2}.\end{aligned}$$

Take now $X = X^\tau(r)\partial_\tau + X^r(r)\partial_r$ with X^τ, X^r smooth in $r \in \widetilde{\mathcal{M}}_\delta$. Clearly $[X, \partial_\tau] = 0$ and we have

$$\widetilde{g}(X, X) = X^\tau\left(F(r)X^\tau \pm 2X^r\right)$$

which is timelike near $r_\pm$ if $X^\tau \equiv 1$ and $\pm X^r \equiv 1$ there. Besides, this choice trivially implies that $X\tau = X^\tau > 0$ and $\pm Xr = \pm X(r) > 0$ near the horizons.

Next, using the formula

$$(\mathcal{L}_X T) = \Big(X^k\partial_k T_{ij} + T_{kj}\partial_i X^k + T_{ik}\partial_j X^k\Big)\mathrm{d}x^i\mathrm{d}x^j$$

for any bilinear form field T, we compute:

$$\begin{aligned}(\mathcal{L}_X\widetilde{g}) &= X^r\Big(F'(r)\mathrm{d}\tau^2 - 2r\mathrm{d}\omega^2\Big) + \big(F(r)\partial_r X^\tau \pm \partial_r X^r\big)(\mathrm{d}\tau\mathrm{d}r + \mathrm{d}r\mathrm{d}\tau) \pm 2\partial_r X^\tau\mathrm{d}r^2,\\ \gamma := \mathrm{Tr}_{\widetilde{g}}(\mathcal{L}_X\widetilde{g}) &= \pm 2F(r)\partial_r X^\tau + 2\partial_r X^r \mp 2F(r)\partial_r X^\tau + \frac{4}{r},\\ 2K^X &= \left(X^r F'(r) - \frac{\gamma}{2}F(r)\right)\mathrm{d}\tau^2 \pm \left(\partial_r X^r - \frac{\gamma}{2}\right)(\mathrm{d}\tau\mathrm{d}r + \mathrm{d}r\mathrm{d}\tau) + \frac{\gamma}{2m^2}\mathrm{d}\zeta^2\\ &\quad \pm 2\partial_r X^\tau\mathrm{d}r^2 - r\left(2 - \frac{\gamma r}{2}\right)\mathrm{d}\theta^2 - r\sin^2\theta\left(2 - \frac{\gamma r}{2}\right)\mathrm{d}\varphi^2\\ &\overset{r=r_\pm}{=} X^r F'(r_\pm)\mathrm{d}\tau^2 \mp \frac{2}{r_\pm}(\mathrm{d}\tau\mathrm{d}r + \mathrm{d}r\mathrm{d}\tau) + \frac{1}{m^2}\left(\partial_r X^r + \frac{2}{r_\pm}\right)\mathrm{d}\zeta^2\\ &\quad \pm 2\partial_r X^\tau\mathrm{d}r^2 + r_\pm^2\partial_r X^r\mathrm{d}\omega^2.\end{aligned}$$

The expression of K^X above looks like to the one in [Dy11] with a metric on $\mathbb{S}^1_z \times \mathbb{S}^2_\omega$ instead. We can use similar conditions: if $\mp\partial_r X^\tau \gg 0$ and $\partial_r X^r < -2/r_\pm$ at $r_\pm$, then K^X is indeed negative definite at the horizons. □

3.3 Proof of the decay through the horizons

As in [Dy11], we will obtain decay for solutions of the initial value problem (3.1) by dealing with the forward problem

$$\begin{cases}\Box_{\widetilde{g}} u &= f \in H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta) \qquad \text{for all } \sigma \geq 0\\ \mathrm{Supp}\, u \subset \{\tau > -T\} \qquad &\text{for some } T > 0\end{cases}. \tag{3.3}$$

Indeed, pick $\eta \in]0,T[$ and let $\chi_\eta \in \mathcal{C}^\infty(\mathbb{R}_\tau,[0,1])$ such that $\chi_\eta(\tau)=0$ if $\tau \le -\eta$, $\chi_\eta(\tau)=1$ if $\tau \ge \eta$, $\chi'(0)\neq 0$ and $\chi''(0)\neq 0$. Let $\tilde{u}$ be the solution of (3.1); then $u := \chi_\eta \tilde{u}$ solves (3.3) with $f = [\Box_{\widetilde{g}},\chi_\eta]\tilde{u}$ and $u=\tilde{u}$ for large $\tau>0$. As $\widetilde{g}$ is smooth in the new coordinates (τ,ζ,r,ω) and $sV\mathbf{z}$ is bounded in the Sobolev spaces $H_{\mathbf{z}}^{\sigma+1+\varepsilon}(\mathscr{M}_\delta)$, we have

$$\|f\|_{H_{\mathbf{z}}^{\sigma+1+\varepsilon}(\mathscr{M}_\delta)} \lesssim \|\chi_\eta'\partial_\tau\tilde{u}\|_{H_{\mathbf{z}}^{\sigma+1+\varepsilon}(\mathscr{M}_\delta)} + \|\chi_\eta''\tilde{u}\|_{H_{\mathbf{z}}^{\sigma+1+\varepsilon}(\mathscr{M}_\delta)}.$$

Since $u \in H_{\mathbf{z}}^{\sigma+2+\varepsilon}(\mathscr{M}_\delta)$ (see Remark 3.3.1 below), so is $\tilde{u}$. Thus, we have by continuity

$$\|\mathrm{e}^{\nu\tau}f\|_{H_{\mathbf{z}}^{\sigma+1+\varepsilon}(\mathscr{M}_\delta)} \lesssim \|\tilde{u}(0,\cdot)\|_{H_{\mathbf{z}}^{\sigma+2+\varepsilon}} + \|(\partial_\tau\tilde{u})(0,\cdot)\|_{H_{\mathbf{z}}^{\sigma+1+\varepsilon}} + \eta$$

for all $\nu>0$ (the exponential term being bounded on the support of χ_η' and χ_η'') and η arbitrarily small.

Remark 3.3.1. *Problem (3.3) has an unique solution in $H_{\mathbf{z}}^{\sigma+2+\varepsilon}(\mathscr{M}_\delta)$ by the standard theory of hyperbolic equation (see Hörmander [Ho94], Theorem 24.1.1)* if *we consider it on the maximal extension of $(\widetilde{\mathcal{M}},\widetilde{g})$. Indeed, $(\mathscr{M}_\delta,\widetilde{g})$ is not globally hyperbolic as we can check by a simple argument depicted on Figure 3.1. As a result, the energy estimates we will obtain on the restriction of the solution to $\mathscr{M}_\delta$ are actually local and not global, even though the entire region $[r_-,r_+]$ is covered.*

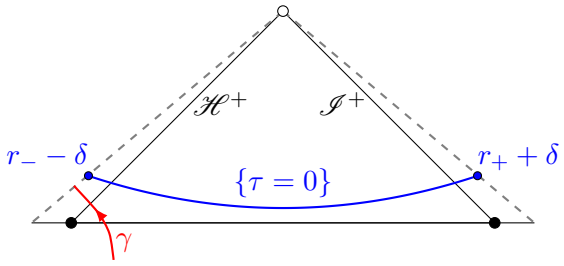

Figure 3.1: Example of a timelike geodesic γ in $\widetilde{\mathcal{M}}_\delta$ which does not meet the hypersurface $\{\tau=0\}$.

Theorem 3.1.2 will therefore follow from the next result:

Theorem 3.3.2 (Decay through the horizons)**.** *Let $\mathbf{z}\in\mathbb{Z}\setminus\{0\}$ and $\delta>0$ sufficiently small to apply Lemma 3.2.1. For all $\varepsilon>0$, there exist $\nu_0>0$ and $s_0>0$ sufficiently small such that for all $\nu\in]0,\nu_0[$ and $s\in]-s_0,s_0[$, the solution u of the problem (3.3) satisfies for all $\sigma\ge 0$ the following estimate:*

$$\|\mathrm{e}^{\nu\tau}u\|_{H_{\mathbf{z}}^{\sigma+1}(\mathscr{M}_\delta)} \lesssim \|\mathrm{e}^{\nu\tau}f\|_{H_{\mathbf{z}}^{\sigma+1+\varepsilon}(\mathscr{M}_\delta)}. \tag{3.4}$$

By convention, the right-hand side in (3.4) is $+\infty$ if $\mathrm{e}^{\nu\tau}f \notin H_{\mathbf{z}}^{\sigma+1+\varepsilon}(\mathscr{M}_\delta)$.

The proof of Theorem 3.3.2 needs some intermediary results. The arguments are really similar to the ones in [Dy11] using the smallness of the charge product s (which plays a role similar to the angular momentum a in [Dy11]); in many places, only trivial modifications are needed and we will then omit the details.

We first link energy estimates on $\mathscr{M}_\delta$ by energy estimates on $\mathcal{K}_\delta$. The proof is the same as in [Dy11]; the induction step is even easier as our extended metric has more symmetries that the metric of De Sitter-Kerr has[1].

Lemma 3.3.3 (Proposition 1.7 in [Dy11]). *Let $\delta > 0$ and $s \in \mathbb{R}$ sufficiently small to apply Lemma 3.2.1 and pick $\psi \in \mathcal{C}^\infty(]r_-, r_+[, \mathbb{R})$ such that $\psi \geq 0$ outside $]r_- + \delta, r_+ - \delta[$. There exists $\nu_0 > 0$ such that for all $\nu \in]0, \nu_0[$, the solution u of the problem*

$$\begin{cases} \big(\Box_{\tilde{g}} + \psi X\big)u \;=\; f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathscr{M}_\delta) \\ \operatorname{Supp} u \subset \{\tau > -T\} \quad \text{for some } T > 0 \end{cases}$$

satisfies for all $\sigma \geq 0$ the following estimate:

$$\|\mathrm{e}^{\nu\tau} u\|_{H^{\sigma+1}(\mathscr{M}_\delta)} \lesssim \|\mathrm{e}^{\nu\tau} f\|_{H^\sigma(\mathscr{M}_\delta)} + \|\mathrm{e}^{\nu\tau} u\|_{H^{\sigma+1}(\mathcal{K}_\delta)}. \tag{3.5}$$

We now turn to local estimates. We want to establish the equivalent of Proposition 2.1 in [Dy11] which we do using a slightly different proof.

Lemma 3.3.4. *Let $\mathbf{z} \in \mathbb{Z} \setminus \{0\}$, $\delta > 0$ and $\sigma \geq 0$. For all $\varepsilon > 0$, there exist $\nu_0 > 0$ and $s_0 > 0$ such that for all $\nu \in]0, \nu_0[$ and $s \in]-s_0, s_0[$, the solution u of the problem*

$$\begin{cases} \Box_{\tilde{g}} u \;=\; f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathscr{K}_\delta) \\ \operatorname{Supp} u \subset \{\tau > -T\} \quad \text{for some } T > 0 \end{cases} \tag{3.6}$$

satisfies the following estimate:

$$\|\mathrm{e}^{\nu t} u\|_{H^{\sigma+1}_{\mathbf{z}}(\mathscr{K}_\delta)} \lesssim \|\mathrm{e}^{\nu t} f\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{K}_\delta)}. \tag{3.7}$$

Here both the weights $\mathrm{e}^{\nu t}$ and $\mathrm{e}^{\nu\tau}$ are equivalent since $|t - \tau|$ is bounded in $\mathcal{K}_\delta$.

Proof. We show the result for $\sigma = 0$ and $\sigma \in 2\mathbb{N}+1$, the general case following from interpolations between Sobolev spaces. We will use the Hamiltonian form of the problem:

$$-\mathrm{i}\partial_t U = \hat{K}(s)U - F, \qquad U = \begin{pmatrix} u \\ -\mathrm{i}\partial_t u - sVu \end{pmatrix}, \; F = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

We will prove the result for each harmonic $\ell \in \mathbb{N}$ (with uniform bounds in ℓ); in order to not overload notations, we will omit the dependence in ℓ (so u will be u_ℓ, $\hat{K}(s)$ will be $\hat{K}_\ell(s)$ and so on).

Let us start with $\sigma = 0$. Since f is smooth and compactly supported, so is F and then U is smooth too. For $\nu' > 0$ large enough, we have $U \in L^2_{\nu'}(\mathbb{R}^*_+, \dot{\mathcal{E}}_{\mathbf{z}})$ (see the short argument proving Proposition 1.1 in [DQNM11] for the existence of such ν', or, equivalently, the proof of Lemma 3.19 in [GGH17]), hence we can define the time-dependent Fourier transforms of U and F for $\Im z \geq \nu'$ and $t > 0$:

$$\hat{U}(z, \cdot) := \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}zt} U(t, \cdot)\mathrm{d}t, \qquad \hat{F}(z, \cdot) := \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}zt} F(t, \cdot)\mathrm{d}t.$$

[1] To be really concrete, the proof uses that the term $[\Box_{\tilde{g}}, Y]$ is zero for any Killing vector field Y on $\mathbb{S}^2_\omega$ or $Y = \partial_\zeta$; in comparison, $[\Box_{\tilde{g}}, Y] = \mathcal{O}(a)$ in [Dy11] which is handled assuming the angular momentum a sufficiently small.

We then have:

$$(\hat{K}(s) - z)\hat{U} = \hat{F}.$$

The resolvent of $\hat{K}(s)$ is analytic in $\mathbb{C}^+$ for s sufficiently small by Corollary 1.2.9; moreover, $z \mapsto \hat{U}(z,\cdot)$ and $z \mapsto \hat{F}(z,\cdot)$ are in $L^2(\mathbb{R}, \dot{\mathcal{E}}_{\mathbf{z}})$ provided that $\Im z \geq \nu'$. Thus, for $t > 0$ (which means $\tau > -T$ for some $T > 0$), Fourier inversion formula yields

$$U(t,\cdot) = \frac{1}{2\pi} \int_{-\infty+\mathrm{i}\nu'}^{+\infty+\mathrm{i}\nu'} \mathrm{e}^{-\mathrm{i}zt}(\hat{K}(s) - z)^{-1}\hat{F}(z,\cdot)\mathrm{d}z$$

as elements of $L^2_{\nu'}(\mathbb{R}^*_+, \dot{\mathcal{E}}_{\mathbf{z}})$. Now let $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(]r_-, r_+[, \mathbb{R})$ such that $\chi \equiv 1$ on the spatial support of F (which is the same as for $\hat{F}$) and proceed as in the proof of Theorem 1.3.2. Using the modified cut-off resolvent $\tilde{R}_\chi(z)$ of (1.51), we can write

$$\chi U(t,\cdot) = \frac{1}{2\pi} \int_{-\infty+\mathrm{i}\nu'}^{+\infty+\mathrm{i}\nu'} \mathrm{e}^{-\mathrm{i}zt}\tilde{R}_\chi(z)\hat{F}(z,\cdot)\mathrm{d}z$$

this time as elements of $\dot{\mathcal{E}}^{-2}_{\mathbf{z}} := (\hat{K}(s) - \mathrm{i})^2\dot{\mathcal{E}}_{\mathbf{z}}$. This formula is the equivalent of formula (1.54) (and also of formula (2.1) in [Dy11]). After a contour deformation[2] using the contour in the proof of Theorem 1.3.2 but with $\Gamma_4 \cup \Gamma_1 \cup \Gamma_2$ replaced by $\{\Im z = -\mu\}$, we obtain

$$\chi U(t,\cdot) = \frac{1}{2\pi} \int_{-\infty-\mathrm{i}\mu}^{+\infty-\mathrm{i}\mu} \mathrm{e}^{-\mathrm{i}zt}\tilde{R}_\chi(z)\hat{F}(z,\cdot)\mathrm{d}z$$

if we choose $\mu < \varepsilon_0$, ε_0 of part 2. of Theorem 1.2.8 (so that we exclude all the resonances inside the contour). Using then estimate (1.36) for the zone III (which is worst than (1.37) for the zone IV) as well as (1.56), we get for all $\nu \in]0, \mu[$:

$$\begin{aligned}
\|(1+\partial_t)\mathrm{e}^{\nu t}\chi U(t,\cdot)\|_{\dot{\mathcal{E}}_{\mathbf{z}}} &\lesssim \int_{-\infty-\mathrm{i}\mu}^{+\infty-\mathrm{i}\mu} \mathrm{e}^{(\nu-\Im z)t}\langle z\rangle \|\hat{R}_\chi(z)\hat{F}(z,\cdot)\|_{\dot{\mathcal{E}}_{\mathbf{z}}}\mathrm{d}z \\
&\lesssim \int_{-\infty-\mathrm{i}\mu}^{+\infty-\mathrm{i}\mu} \mathrm{e}^{(\nu-\Im z)t}\langle z\rangle \ln\langle z\rangle \mathrm{e}^{|\Im z|\ln\langle z\rangle}\|\hat{F}(z,\cdot)\|_{\dot{\mathcal{E}}_{\mathbf{z}}}\mathrm{d}z \\
&\lesssim \mathrm{e}^{(\nu-\mu)t}\int_{-\infty}^{+\infty} \langle z\rangle^{1+2\mu}\|\hat{f}(z,\cdot)\|_{\mathcal{H}_{\mathbf{z}}}\mathrm{d}z \\
&\lesssim \mathrm{e}^{(\nu-\mu)t}\|\mathrm{e}^{\nu t}f\|_{H^{1+2\mu}_{\mathbf{z}}(\mathscr{K}_\delta)}.
\end{aligned} \tag{3.8}$$

On the other hand, since the norms $\|.\|_{\mathcal{E}}$ and $\|.\|_{\dot{\mathcal{E}}}$ are locally equivalent, we have

$$\|\chi u\|_{H^1_{\mathbf{z}}} \lesssim \|\chi U(t,\cdot)\|_{\mathcal{E}^{\mathbf{z}}} \lesssim \|\chi U(t,\cdot)\|_{\dot{\mathcal{E}}_{\mathbf{z}}}.$$

Thus, integrating (3.8) in time over $\mathbb{R}_+$ yields (3.11) with $\sigma = 0$ and $\varepsilon = 2\mu$.

[2]Here we use that $\hat{F}(z)$ is still well-defined for $\Im z < 0$ since F is smooth and compactly supported in time. This is not the case for $\hat{U}(z)$ which can be defined *a priori* only for $\Im z > 0$ sufficiently large (this is the only reason why we used the space $L^2_{\nu'}(\mathbb{R}^*_+, \dot{\mathcal{E}}_{\mathbf{z}})$).

Let now $\sigma = 2n+1$ with $n \in \mathbb{N} \setminus \{0\}$. Compute first

$$\left\|P^n \chi u\right\|_{\mathcal{H}_{\mathbf{z}}} \lesssim \left\|[P^n, \chi]u\right\|_{\mathcal{H}_{\mathbf{z}}} + \left\|\chi P^n u\right\|_{\mathcal{H}_{\mathbf{z}}}.$$

Since the commutator only contains the terms $\chi^{(n)}, \chi^{(n-1)}\partial_x, \dots, \chi'\partial_x^{n-1}$, we find

$$\|\chi u\|_{H^{2n}_{\mathbf{z}}} \lesssim \left\|P^n \chi u\right\|_{\mathcal{H}_{\mathbf{z}}} \lesssim \left\|\chi P^{n-1} u\right\|_{\mathcal{H}_{\mathbf{z}}} + \left\|\chi P^n u\right\|_{\mathcal{H}_{\mathbf{z}}}. \tag{3.9}$$

Next, let us write

$$\hat{K}(s)^{2n-1} = (\hat{K}(0) + sV\mathbb{1})^{2n-1} = \sum_{j=1}^{2^{2n-1}} sA_j$$

where $A_1 = \hat{K}(0)^{2n-1}/s$ and A_j a homogeneous polynomial in s with nonnegative degree for $j \neq 1$. For s small enough (depending on n), it turns that

$$\begin{aligned}
\left\|\chi \hat{K}(s)^{2n-1} U\right\|^2_{\dot{\mathcal{E}}_{\mathbf{z}}} &= \left\|\chi \sum_{j=1}^{2^{2n-1}} sA_j U\right\|^2_{\dot{\mathcal{E}}_{\mathbf{z}}} \\
&= \sum_{j=1}^{2^{2n-1}} \|\chi sA_j U\|^2_{\dot{\mathcal{E}}_{\mathbf{z}}} + 2\sum_{j\neq k} \Re\left\langle \chi sA_j U, \chi sA_k U\right\rangle_{\dot{\mathcal{E}}_{\mathbf{z}}} \\
&\geq \sum_{j=1}^{2^{2n-1}} \|\chi sA_j U\|^2_{\dot{\mathcal{E}}_{\mathbf{z}}} - 2|s| \sum_{j\neq k} \left|\left\langle \chi A_j U, \chi A_k U\right\rangle_{\dot{\mathcal{E}}_{\mathbf{z}}}\right| \\
&\gtrsim \sum_{j=1}^{2^{2n-1}} \|\chi sA_j U\|^2_{\dot{\mathcal{E}}_{\mathbf{z}}} \\
&\geq \|\chi \hat{K}(0)^{2n-1} U\|^2_{\dot{\mathcal{E}}_{\mathbf{z}}}.
\end{aligned} \tag{3.10}$$

As

$$\hat{K}(0)^{2n-1} = \begin{pmatrix} 0 & P^{n-1} \\ P^n & 0 \end{pmatrix},$$

it follows from (3.9) and (3.10)

$$\|\chi u\|_{H^{2n}_{\mathbf{z}}} \lesssim \left\|\chi P^n u\right\|_{\mathcal{H}_{\mathbf{z}}} \leq \|\chi \hat{K}(0)^{2n-1} U\|_{\dot{\mathcal{E}}_{\mathbf{z}}} \lesssim \|\chi \hat{K}(s)^{2n-1} U\|_{\dot{\mathcal{E}}_{\mathbf{z}}}.$$

We have

$$\hat{K}(s)^{2n-1} U(t,\cdot) = \frac{1}{2\pi} \int_{-\infty+\mathrm{i}\nu'}^{+\infty+\mathrm{i}\nu'} \mathrm{e}^{-\mathrm{i}zt} (\hat{K}(s) - z)^{-1} \hat{K}(s)^{2n-1} \hat{F}(z,\cdot)\mathrm{d}z$$

and thus

$$\partial_t^{2n} \chi \hat{K}(s)^{2n-1} U(t,\cdot) = \frac{1}{2\pi} \int_{-\infty+\mathrm{i}\nu'}^{+\infty+\mathrm{i}\nu'} \mathrm{e}^{-\mathrm{i}zt} (\mathrm{i}z)^{2n} \tilde{R}_\chi(z) \hat{K}(s)^{2n-1} \hat{F}(z,\cdot)\mathrm{d}z.$$

From there, we can proceed as in (3.8) with $\tilde{R}_\chi(z)\hat{K}(s)^{2n-1}\hat{F}(z,\cdot)$ instead of $\tilde{R}_\chi(z)\hat{F}(z,\cdot)$ and $\langle z\rangle^{2n+2\mu}$ instead of $\langle z\rangle^{1+2\mu}$. We need to use that

$$\left\|\hat{K}(s)^{2n-1}\hat{F}\right\|_{\dot{\mathcal{E}}_{\mathbf{z}}} = \left\|(\hat{K}(0)+sV\mathbb{1})^{2n-1}\hat{F}\right\|_{\mathcal{H}_{\mathbf{z}}} \lesssim \left\|P^n\hat{f}\right\|_{\mathcal{H}_{\mathbf{z}}} + \sum_{j=0}^{n-1}\left\|B_j\hat{f}\right\|_{\mathcal{H}_{\mathbf{z}}}$$

where B_j contains commutator terms of the form $\mathrm{ad}_P^{j+1}(sV)$, $\mathrm{ad}_{sV}^k(P^{j+1})$ which provides Sobolev norms controlled by $\left\|P^{n-1}\hat{f}\right\|_{\mathcal{H}_{\mathbf{z}}}$. This settles (3.11) with $\sigma = 2n-1$ for all $n \in \mathbb{N}\setminus\{0\}$. □

Corollary 3.3.5 (Proposition 2.2 in [Dy11]). *Let $\mathbf{z} \in \mathbb{Z}\setminus\{0\}$, $\delta > 0$ and $s \in \mathbb{R}$ sufficiently small to apply Lemma 3.2.1. For all $\varepsilon > 0$, there exists $\nu_0 > 0$ such that for all $\nu \in]0,\nu_0[$, the solution u of the problem (3.6) satisfies for all $\sigma \geq 0$ the following estimate:*

$$\|\mathrm{e}^{\nu\tau}u\|_{H^{\sigma+1}_{\mathbf{z}}(\mathscr{M}_\delta)} \lesssim \|\mathrm{e}^{\nu\tau}f\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{K}_\delta)}. \tag{3.11}$$

Proof. We show the result for $\sigma \geq 0$ integer, the general case following from interpolation between Sobolev spaces. By Lemma 3.3.3, it holds

$$\|\mathrm{e}^{\nu\tau}u\|_{H^{\sigma+1}_{\mathbf{z}}(\mathscr{M}_\delta)} \lesssim \|\mathrm{e}^{\nu\tau}f\|_{H^{\sigma}_{\mathbf{z}}(\mathscr{M}_\delta)} + \|\mathrm{e}^{\nu\tau}u\|_{H^{\sigma+1}_{\mathbf{z}}(\mathcal{K}_\delta)}.$$

We can use here the restrictions on the kernel of $\mathrm{i}\partial_z + \mathbf{z}$ because $\partial_z = \partial_\zeta$ commutes with $\Box_{\widetilde{g}}$. Applying Lemma 3.3.4 with τ instead of t, it comes

$$\|\mathrm{e}^{\nu\tau}u\|_{H^{\sigma+1}_{\mathbf{z}}(\mathcal{K}_\delta)} \lesssim \|\mathrm{e}^{\nu\tau}f\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathcal{K}_\delta)}$$

which provides the desired estimate. □

The last step is to show an exponential decay on a small neighborhood of the horizons.

Lemma 3.3.6 (Proposition 2.3 in [Dy11]). *Let $\delta > 0$ and $s \in \mathbb{R}$ sufficiently small to apply Lemma 3.2.1. There exists $\nu_0 > 0$ such that for all $\nu \in]0,\nu_0[$, the solution v of the problem*

$$\begin{cases} \Box_{\widetilde{g}}v \;=\; f \in \mathcal{C}^\infty_{\mathrm{c}}(\mathscr{M}_\delta\setminus\mathscr{K}_\delta) \\ \mathrm{Supp}\, v \subset \{\tau > -T\} \quad \text{for some } T > 0 \\ v_{|\partial\mathscr{K}_{2\delta}} \equiv 0 \end{cases} \tag{3.12}$$

satisfies for all $\sigma \geq 0$ the following estimate:

$$\|\mathrm{e}^{\nu\tau}v\|_{H^{\sigma+1}(\mathscr{M}_\delta\setminus\mathscr{K}_{2\delta})} \lesssim \|\mathrm{e}^{\nu\tau}f\|_{H^{\sigma}(\mathscr{M}_\delta\setminus\mathscr{K}_{2\delta})}. \tag{3.13}$$

We can now show the exponential decay through the horizons:

Proof of Theorem 3.3.2. Let u be the solution of (3.3) with first $f \in \mathcal{C}^\infty_{\mathrm{c}}(\widetilde{\mathcal{M}}_\delta)$ and v be the solution of (3.12). Set $w := u - \chi v$ and take $\chi \in \mathcal{C}^\infty(\mathcal{K}_\delta)$ such that $\chi \equiv 0$ near $\mathcal{K}_{2\delta}$ and $\chi \equiv 1$ outside $\mathcal{K}_\delta$. We have

$$\Box_{\widetilde{g}}w = (1-\chi)f - [\Box_{\widetilde{g}},\chi]v$$

which is supported in $\mathscr{K}_\delta$. We have

$$\begin{aligned}\|\mathrm{e}^{\nu\tau}\Box_{\widetilde{g}}w\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta)} &\lesssim \|\mathrm{e}^{\nu\tau}(1-\chi)fw\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta)} + \|\mathrm{e}^{\nu\tau}[\Box_{\widetilde{g}},\chi]w\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta)}\\ &\lesssim \|\mathrm{e}^{\nu\tau}(1-\chi)f\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{K}_\delta)} + \|\mathrm{e}^{\nu\tau}\chi'\partial_r w\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta)}\\ &\lesssim \|\mathrm{e}^{\nu\tau}f\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta)}\end{aligned}$$

where we applied Lemma 3.3.6 to the term χ'. By Corollary 3.3.5,

$$\|\mathrm{e}^{\nu\tau}w\|_{H^{\sigma+1}_{\mathbf{z}}(\mathscr{M}_\delta)} \lesssim \|\mathrm{e}^{\nu\tau}\Box_{\widetilde{g}}w\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta)} \lesssim \|\mathrm{e}^{\nu\tau}f\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta)}$$

since $\Box_{\widetilde{g}}w$ is supported in $\mathscr{K}_\delta$. Since χ vanishes on $\mathscr{K}_\delta$, we can apply Lemma 3.3.6 again and find:

$$\begin{aligned}\|\mathrm{e}^{\nu\tau}u\|_{H^{\sigma+1}_{\mathbf{z}}(\mathscr{M}_\delta)} &\lesssim \|\mathrm{e}^{\nu\tau}w\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta)} + \|\mathrm{e}^{\nu\tau}v\|_{H^{\sigma+1}_{\mathbf{z}}(\mathscr{M}_\delta)}\\ &\lesssim \|\mathrm{e}^{\nu\tau}f\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta)} + \|\mathrm{e}^{\nu\tau}f\|_{H^{\sigma+1}_{\mathbf{z}}(\mathscr{M}_\delta)}.\end{aligned}$$

Finally, the functional $\big(\mathcal{C}^\infty_{\mathrm{c}}(\widetilde{\mathcal{M}}_\delta), \|\mathrm{e}^{\nu\tau}\cdot\|_{H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta)}\big) \ni f \longmapsto u \in \big(H^{\sigma+1}_{\mathbf{z}}(\mathscr{M}_\delta), \|\mathrm{e}^{\nu\tau}\cdot\|_{H^{\sigma+1}_{\mathbf{z}}(\mathscr{M}_\delta)}\big)$ being continuous, we can extend it to $\mathrm{e}^{\nu\tau}H^{\sigma+1+\varepsilon}_{\mathbf{z}}(\mathscr{M}_\delta)$. This concludes the proof. □

Chapter 4

Numerical Study of an Abstract Klein-Gordon type Equation: Applications to the Charged Klein-Gordon Equation in the Exterior De Sitter-Reissner-Nordström Spacetime

Chapter 4 proposes a numerical scheme to investigate some important features linked to the charged Klein-Gordon equation in the exterior DSRN spacetime (such as localization of resonances, see Chapter 5). The first step is to approximate solutions of the equation using "discretized" operators, spaces and norms.

In a more abstract setting, a numerical scheme is a method which allows one to discretize a given problem *i.e.* transform data in some abstract normed functional space into data in $\mathbb{C}^N$ for some $N \in \mathbb{N}\setminus\{0\}$. The transformation is carried out by means of some projectors and embeddings which send data from a space to another one. To these operators are associated spaces and norms that approximate the initial ones. Given a set of indices ($\mathbb{N}$ in general, but this is not mandatory), we construct a family of such operators, spaces and norms. We are then interested in the inductive limit over the indices ($n \to +\infty$ for indices $n \in \mathbb{N}$). Given an abstract equation between normed spaces, we wish to find a family of approximated solutions to the discrete version of the equation (that is, taken in approximated spaces with approximated operators) and then show that this family converges at the inductive limit to the "true" solution of the "true" equation.

In this chapter, we will be interested in an abstract Klein-Gordon type equation. We propose a method based on polynomial interpolation. Convergence is then obtained and given a geometric interpretation in Theorem 4.3.1 and Theorem 4.3.4. A difficulty is to respect the structure of the spaces: we will *not* assume data to be smooth enough in order to facilitate estimates. We will also take care of the error we commit by following each constants in our estimates. As we will see, this will lead to painful computations. For the sake of clarity, we have postponed each technical computation to the Appendix, so that the main body of the chapter only contains important demonstrations. The error is controlled on some compact set in space and grows as the maximal

time T; the approximation becomes useful as we know that the local energy decays in time (or is at least bounded).

Organization of the chapter. Chapter 4 is organized as follows: Section 4.1 introduce the abstract problem we will deal with as well as the scheme and some notations. Consistency and stability are shown in Section 4.2: they amount of how good the approximation is. In Section 4.3, we prove that the approximated solution of the abstract problem converges to the "true" solution as the mesh converges (briefly: the scheme converges) using consistency and stability. We then apply these results to the charged Klein-Gordon equation in the exterior DSRN spacetime.

4.1 The abstract problem

In this Section, we first introduce the abstract problem we are interested in with some assumptions. In a second time, we introduce a discretization of the problem in order to numerically solve it.

4.1.1 Introduction of the abstract problem

General assumptions

For $0 < r_- < r_+ < +\infty$, let $I =]r_-, r_+[\subset \mathbb{R}$, $\rho_0 > 0$ and $\rho : I \to]\rho_0, +\infty[$ be a Lebesgue measurable function. We assume that

$$\begin{cases} \rho \in L^1_{\ell\mathrm{oc}}(I, \mathrm{d}r), \\ \rho(r) = \mathcal{O}_{r\to r_\pm}\left(|r - r_\pm|^{-(1+\zeta_\pm)}\right) \text{ with } \zeta_\pm \geq 0 \end{cases} . \tag{AP1}$$

We next consider a second order partial differential operator $P > 0$ of the form

$$P = a(r)\partial_r^2 + b(r)\partial_r + c(r)$$

self-adjoint on some domain in L^2_ρ and such that $\ker P = \{0\}$. We ask for the condition

$$\dot{\mathcal{H}}^k_\rho \subset H^k_{\ell\mathrm{oc}}(I, \mathbb{C}) \qquad \forall k \in \{0, 1, 2\}. \tag{AP2}$$

We shall refer to this Assumption for $k > 2$ if needed by explicitly specifying it. Also, the coefficients of P must satisfy

$$a, b, c \in L^\infty(I, \mathbb{R}) \tag{AP3}$$

Let $(H, \mathscr{D}(H))$ be an operator acting in $L^2_\rho \times L^2_\rho$ of the form

$$H = \begin{pmatrix} Q & \mathbb{1} \\ P & Q \end{pmatrix}, \qquad \mathscr{D}(H) = \left\{ u = (u_0, u_1) \in \dot{\mathcal{H}}^1_\rho \times L^2_\rho \mid Hu \in \dot{\mathcal{H}}^1_\rho \times L^2_\rho \right\}. \tag{AP4}$$

where Q is self-adjoint acting on L^2_ρ and $\mathbb{1}$ is the identity operator on L^2_ρ. We assume that

$$\begin{cases} Q \in \mathcal{B}(\dot{\mathcal{H}}^1_\rho), \\ Q \text{ is local: } \mathrm{Supp}\,(Qu) \subset \mathrm{Supp}\, u \text{ for all } u \in \dot{\mathcal{H}}^1_\rho \end{cases} . \tag{AP5}$$

We are interested in the numerical resolution of the abstract Dirichlet problem

$$\begin{cases} -\mathrm{i}\partial_t u = Hu \qquad \forall t \in [0,T] \\ u(0,\cdot) \in \mathscr{D}(H) \text{ is compactly supported in } I \end{cases} \tag{ADP}$$

with $u \in \mathcal{C}^0([0,T], \mathscr{D}(H)) \cap \mathcal{C}^1([0,T], \dot{\mathcal{H}}^1_\rho \times L^2_\rho)$ for some fixed $T > 0$. Finally, we define the norm

$$\|(u_0,u_1)\|^2_{\dot{\mathcal{E}}_\rho} := \|u_0\|^2_{\dot{\mathcal{H}}^1_\rho} + \|u_1 - Qu_0\|^2_{L^2_\rho}$$

and the space $\dot{\mathcal{E}}_\rho := \overline{\dot{\mathcal{H}}^1_\rho \times L^2_\rho}^{\|\cdot\|_{\dot{\mathcal{E}}_\rho}}$. We make two last assumptions:

H generates on $\dot{\mathcal{E}}_\rho$ a strongly continuous one parameter group $(\mathrm{e}^{\mathrm{i}tH})_{0\le t\le T}$, (AP6)

If φ is compactly supported in I, then $\cup_{0\le t\le T} \operatorname{Supp}(\mathrm{e}^{\mathrm{i}tH}\varphi)$ is a compact subset of I. (AP7)

Remark 4.1.1. *Assumption (AP1) means that the weight ρ can explode in the L^1 sense only at the boundary of I. It is of course possible to consider cases where this happens at finitely many points in I by splitting I at these points and applying the theory developed in the setting of Assumption (AP1).*

Remark 4.1.2. *The coefficients a and b are linked by the symmetry of P. To see how, pick $\chi_1, \chi_2 \in \mathcal{C}^\infty_{\mathrm{c}}(I,\mathbb{R})$. We have*

$$\big\langle P\chi_1, \chi_2\big\rangle_{L^2_\rho} = \big\langle \chi_1, P\chi_2\big\rangle_{L^2_\rho} + \int_I \chi_1(r)\Big(\chi_2(r)\big((a\rho)'' - (b\rho)'\big)(r) + 2\chi_2'(r)\big((a\rho)' - (b\rho)\big)(r)\Big)\mathrm{d}r.$$

The symmetry of P implies that

$$\int_I \chi_1(r)\Big(\chi_2(r)\big((a\rho)'' - (b\rho)'\big)(r) + 2\chi_2'(r)\big((a\rho)' - (b\rho)\big)(r)\Big)\mathrm{d}r.$$

This means that in the distribution sense, $w := (a\rho)' - (b\rho)$ satisfies:

$$\chi_2(r)w'(r) - \chi_2'(r)w(r) = 0. \tag{4.1}$$

Since χ_2 is smooth and has fast decay at the boundary of I, the operator $\chi_2\partial_r - \chi_2'$ is hypoelliptic, hence the above equality implies that $w \in \mathcal{C}^\infty(I,\mathbb{R})$. Now take $\chi_2 \equiv 1$ on a compact set $K \subset I$: this gives

$$\big((a\rho)' - (b\rho)\big)(r) = C \in \mathbb{R} \qquad \forall r \in K.$$

Back into (4.1), we discover $(a\rho)' = b\rho$ on K (that is $C = 0$). As K was arbitrary, this holds in I. Observe also that $a\rho \in W^{1,1}(I,\mathbb{R})$ because $b\rho \in L^1(I,\mathbb{R})$. Furthermore, for all $u, v \in \dot{\mathcal{H}}^1_\rho$ compactly supported, the integration by parts

$$\int_I a(r)\overline{(\partial_r^2 u)(r)}v(r)\rho(r)\mathrm{d}r = -\int_I a(r)\overline{(\partial_r u)(r)}(\partial_r v)(r)\rho(r)\mathrm{d}r - \int_I b(r)\overline{(\partial_r u)(r)}v(r)\rho(r)\mathrm{d}r$$

is legitimate thanks to Assumption (AP2).

In fact, it can be said a little bit more on the coefficient a: let $\tilde{a} := a/\rho$, $\tilde{b} := b/\rho$. Since $\rho > \rho_0 > 0$, $\tilde{a}, \tilde{b} \in L^\infty(I,\mathbb{R})$. But $\tilde{a}' = \tilde{b}$ so $\tilde{a} \in W^{1,k}(I,\mathbb{R})$ for all $k \ge 1$, whence $\tilde{a} \in \mathcal{C}^{0,1}(I,\mathbb{R})$ is locally Lipschitz in I. Note that in the case where ρ is constant (it is the case for the Lebesgue measure), this implies that $a' = b$ and then P is a divergence type operator:

$$P = \partial_r\big(a(r)\partial_r\big) + c(r).$$

Remark 4.1.3. *The positivity of P also implies $c > 0$ Lebesgue everywhere in I. Indeed, take $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(I, \mathbb{R})$ and compute:*

$$0 < \langle P\chi, \chi \rangle_{L^2_\rho} = \int_I \Big(a(r)\chi''(r) + b(r)\chi'(r) + c(r)\chi(r) \Big) \chi(r)\rho(r)\mathrm{d}r.$$

For any compact $K \subset I$, take χ so that $\chi \equiv 1$ on K: this entails

$$0 < \int_K c(r)\chi(r)^2\rho(r)\mathrm{d}r.$$

Since $\chi^2\rho > 0$, we conclude that $c > 0$ almost everywhere on K. As K was arbitrary, the claim is proved.

Remark 4.1.4. *The method developed in this chapter can also deal with Hamiltonian H defined with two different operators Q_1 and Q_2 in Assumption (AP4), provided they both satisfy Assumption (AP5). We yet consider the case $Q_1 = Q_2 = Q$ in order to make computations more readable. Moreover, we will consider multiplication operator for some concrete applications in this chapter, even if it is possible to adapt it for more general operators at the prize of increasing the complexity of the problem (for example, Q could be a differential operator, but then we would have to discretize it as it will be done for the operator P).*

Remark 4.1.5. *Assumption (AP7) is a (finite speed) propagation estimate which witnesses of the wave-type nature of the equation. This gives useful information on the support of the solution at any time provided that the initial data is compactly supported. The counter part is that the propagator $\mathrm{e}^{-\mathrm{i}tH}$ is not assumed to be regularizing. One can of course not assume this hypothesis and consider the truncated propagator $\chi\mathrm{e}^{-\mathrm{i}tH}$ for some cut-off $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(I, \mathbb{R})$.*

There is a priori no restriction on the existence of $\mathrm{e}^{-\mathrm{i}tH}$. We can consider that it exists for any time $t > 0$, for any time $t \leq T$, or that some microlocalization hypotheses have been made on the initial data in order to avoid singularities as t increases. We refer to [GGH17, Section 3] for the existence of the propagator associated to an abstract Klein-Gordon equation.

Remark 4.1.6. *Lemma 4.1.7 below implies some minimum regularity for initial data φ in (ADP): indeed, it must satisfy*

$$\varphi \in (\dot{\mathcal{H}}^1_\rho \cap \dot{\mathcal{H}}^2_\rho) \times \mathcal{H}^1_\rho, \qquad (\varphi_1 - Q\varphi_0) \in \mathcal{H}^1_\rho.$$

But φ is also compactly supported, so Assumption (AP2) implies that $\varphi \in H^2_{\mathrm{c}}(I, \mathrm{d}r) \times H^1_{\mathrm{c}}(I, \mathrm{d}r)$ and $\varphi_1 - Q\varphi_0 \in H^1_{\mathrm{c}}$ if Q since local by Assumption (AP5). By Sobolev embedding theorem, we deduce that $\varphi \in \mathcal{C}^{1,1/2}_{\mathrm{c}}(I, \mathbb{C}) \times \mathcal{C}^{0,1/2}_{\mathrm{c}}(I, \mathbb{C})$ and $\varphi_1 - Q\varphi_0 \in \mathcal{C}^{0,1/2}_{\mathrm{c}}(I, \mathbb{C})$.

Regularity of solutions

Let $t \geq 0$ and let $\varphi = (\varphi_0, \varphi_1) \in \mathscr{D}(H)$ and $\psi = (\psi_0, \psi_1) = \mathrm{e}^{\mathrm{i}tH}(\varphi_0, \varphi_1)$.

Lemma 4.1.7. *For all $t \geq 0$, we have*

$$\psi(t, \cdot) \in (\dot{\mathcal{H}}^1_\rho \cap \dot{\mathcal{H}}^2_\rho) \times \mathcal{H}^1_\rho, \qquad \psi_1(t, \cdot) - Q\psi_0(t, \cdot) \in \mathcal{H}^1_\rho.$$

Proof. Every element of $\mathscr{D}(H)$ is by definition in $\dot{\mathcal{H}}^1_\rho \times L^2_\rho$. Since $(\mathrm{e}^{\mathrm{i}tH})_{t\geq 0}$ is a continuous one parameter group by assumption (AP6), there exists $C > 0$ such that

$$\|H^k\psi\|_{\dot{\mathcal{E}}_\rho} \leq \mathrm{e}^{C|t|}\|H^k\varphi\|_{\dot{\mathcal{E}}_\rho} \qquad \forall k \in \mathbb{N},\, \forall t \in \mathbb{R} \tag{4.2}$$

provided that $H^k\varphi \in \dot{\mathcal{E}}_\rho$ (it is always true for $k \in \{0,1\}$). Then ψ satisfies

$$\begin{cases} \psi_0 \in \dot{\mathcal{H}}^1_\rho \\ \psi_1 - Q\psi_0 \in L^2_\rho \end{cases}, \qquad \begin{cases} Q\psi_0 + \psi_1 \in \dot{\mathcal{H}}^1_\rho \\ (P-Q^2)\psi_0 \in L^2_\rho \end{cases}.$$

Since $Q\psi_0 \in \dot{\mathcal{H}}^1_\rho$ by Assumption (AP5), the first line on the right hand side implies $\psi_1 \in \dot{\mathcal{H}}^1_\rho$. On the other hand, the second line on the right hand side implies $P\psi_0 \in \dot{\mathcal{H}}^{-1}_\rho \cap (L^2_\rho + \dot{\mathcal{H}}^1_\rho) = \mathcal{H}^{-1}_\rho + L^2_\rho \subset L^2_\rho$ whence $\psi_0 \in \dot{\mathcal{H}}^2_\rho$. Finally, we have $\psi_1 - Q\psi_0 \in \dot{\mathcal{H}}^1_\rho$ since $Q \in \mathcal{B}(\dot{\mathcal{H}}^1_\rho)$ and $\psi_0 \in \dot{\mathcal{H}}^1_\rho$. □

Corollary 4.1.8. *If ψ is a solution of (ADP) then for all $t \geq 0$, we have*

$$\psi(t,\cdot) \in \mathcal{C}^{1,1/2}_{\mathrm{c}}(I,\mathbb{C}) \times \mathcal{C}^{0,1/2}_{\mathrm{c}}(I,\mathbb{C}), \qquad \psi_1(t,\cdot) - Q\psi_0(t,\cdot) \in \mathcal{C}^{0,1/2}_{\mathrm{c}}(I,\mathbb{C}) \text{ if } Q \text{ is local.}$$

Proof. As φ is compactly supported, so is for $\psi(t,\cdot)$ by Assumption (AP7). It turns that $\psi \in H^2_{\mathrm{c}} \times H^1_{\mathrm{c}}$ by Assumption (AP2) and Sobolev embedding theorem then gives the announced result for ψ. The same argument holds for $\psi_1 - Q\psi_0$ if Q is local. □

Solutions regularity increases as does initial data's if we add some assumption on the operator Q:

Lemma 4.1.9. *Let $\varphi \in \mathscr{D}(H)$. If $H^k\varphi \in \mathscr{D}(H)$ and $Q \in \cap_{\ell=1}^{k+1}\mathcal{B}(\dot{\mathcal{H}}^\ell_\rho)$ for some $k \in \mathbb{N}\setminus\{0\}$, then for all $t \geq 0$, we have*

$$(\psi_0(t,\cdot),\psi_1(t,\cdot)) \in \left(\cap_{\ell=1}^{k+2}\dot{\mathcal{H}}^\ell_\rho\right) \times \left(\cap_{\ell=1}^{k+1}\mathcal{H}^\ell_\rho\right).$$

Proof. We show the result by induction. For $k = 0$ and $k = 1$, the semi-group estimate (4.2) gives

$$\begin{cases} Q\psi_0 + \psi_1 \in \dot{\mathcal{H}}^1_\rho \\ (P-Q^2)\psi_0 \in L^2_\rho \end{cases}, \qquad \begin{cases} Q(Q\psi_0+\psi_1) + (P\psi_0 + Q\psi_1) \in \dot{\mathcal{H}}^1_\rho \\ (P-Q^2)(Q\psi_0+\psi_1) \in L^2_\rho \end{cases}.$$

Since $\psi \in \mathscr{D}(H)$, we have $\psi \in \dot{\mathcal{H}}^1_\rho \times L^2_\rho$. The assumption $Q \in \mathcal{B}(\dot{\mathcal{H}}^1_\rho)$ combined the first line on the left hand side then gives $\psi \in \dot{\mathcal{H}}^1_\rho$ and the first line on the right hand side entails $P\psi_0 \in \dot{\mathcal{H}}^1_\rho$, so $\psi_0 \in \dot{\mathcal{H}}^3_\rho$. Back on the left hand side, the second line combined gives

$$P\psi_0 \in \dot{\mathcal{H}}^{-1}_\rho \cap (L^2_\rho + \dot{\mathcal{H}}^1_\rho) \subset L^2_\rho$$

so that $\psi_0 \in \dot{\mathcal{H}}^2_\rho$. The same argument applies in the second line on the right hand side and gives $Q\psi_0 + \psi_1 \in \dot{\mathcal{H}}^2_\rho$. If we assume $Q \in \mathcal{B}(\dot{\mathcal{H}}^2_\rho)$ then it follows that $\psi_1 \in \dot{\mathcal{H}}^2_\rho$.

Assume now the result settled for some $k \in \mathbb{N}\setminus\{0\}$. We know that $\psi \in \dot{\mathcal{H}}^1_\rho \times L^2_\rho$ because $\psi \in \mathscr{D}(H)$. We want to show that if $\varphi, H^{k+1}\varphi \in \mathscr{D}(H)$ then the evolution $\psi(t,\cdot)$ satisfies $\psi \in \left(\cap_{\ell=1}^{2k+2}\dot{\mathcal{H}}^\ell_\rho\right) \times \left(\cap_{\ell=1}^{2k+1}\mathcal{H}^\ell_\rho\right)$. Put then

$$\tilde{\varphi} := H\varphi, \qquad \tilde{\psi} := H\psi.$$

Then $H^k\tilde{\varphi} \in \mathscr{D}(H)$ and $\tilde{\psi}(t,\cdot)$ is the evolution of $\tilde{\varphi}$ at the time $t \geq 0$. Hence the induction hypothesis implies that

$$\tilde{\psi} = \begin{pmatrix} Q\psi_0 + \psi_1 \\ P\psi_0 + Q\psi_1 \end{pmatrix} \in \big(\cap_{\ell=1}^{2k+2} \dot{\mathcal{H}}_\rho^\ell\big) \times \big(\cap_{\ell=1}^{2k+1} \mathcal{H}_\rho^\ell\big).$$

Again we use that $\psi \in \dot{\mathcal{E}}_\rho$ so $\psi_0 \in \dot{\mathcal{H}}_\rho^1$. Using the assumption $Q \in \mathcal{B}(\dot{\mathcal{H}}_\rho^1)$, the first line above gives $\psi_1 \in \dot{\mathcal{H}}_\rho^1$ whereas the second one gives $P\psi_0 \in \dot{\mathcal{H}}_\rho^{-1} \cap \dot{\mathcal{H}}_\rho^1 = L_\rho^2$ i.e. $\psi_0 \in \dot{\mathcal{H}}_\rho^2$. Back to the first line, we then get $\psi_1 \in \dot{\mathcal{H}}_\rho^3$ using the assumption $Q \in \mathcal{B}(\dot{\mathcal{H}}_\rho^3)$, and the second line together with $Q \in \mathcal{B}(\dot{\mathcal{H}}_\rho^2)$ give $P\psi_0 \in \dot{\mathcal{H}}_\rho^2$, that is $\psi_0 \in \dot{\mathcal{H}}_\rho^4$. We can keep on increasing the regularity of ψ in the same way until getting $\psi_0 \in \dot{\mathcal{H}}_\rho^{2k+2}$ and $\psi_1 \in \dot{\mathcal{H}}_\rho^{2k+1}$. We finally use the same argument as above to increase one more time the regularity of ψ and we are done. □

Remark 4.1.10. *Solutions $u, -\mathrm{i}\partial_t u - Qu$ of (ADP) satisfy the abstract Klein-Gordon equation*

$$\big(\partial_t^2 - 2\mathrm{i}Q\partial_t + P\big)u = 0.$$

Assume $P + Q^2 \geq 0$ and the initial data smooth; then the above equation is hyperbolic and u is smooth by the standard theory of hyperbolic partial differential equations (see e.g. [Le53]).

4.1.2 Discretization of the abstract problem

Meshes

Temporal mesh. The discretization in time will consist in using a *regular mesh* $\widetilde{\mathscr{T}}([0,T])$. A sequence of elements of $\widetilde{\mathscr{T}}([0,T])$ will be denoted by $\widetilde{\mathscr{T}}_\bullet = (\widetilde{\mathscr{T}}_{\bullet,0}, \widetilde{\mathscr{T}}_{\bullet,1})$, where $\widetilde{\mathscr{T}}_{\bullet,0}$ is a points set of $[0,T]$ and $\widetilde{\mathscr{T}}_{\bullet,1}$ is the set of intervals of the form $]t,t'[$ with $t < t'$ two elements of $\widetilde{\mathscr{T}}_{\bullet,0}$. In the sequel, we will use the following notations: for all $s \in \widetilde{\mathscr{T}}_{\bullet,0}$,

$$\begin{aligned} |s| &:= \mathrm{card}\big\{s' \in \widetilde{\mathscr{T}}_{\bullet,0} \mid s' \leq s\big\}, \\ \underline{t} &:= \min\{s \mid s \in \widetilde{\mathscr{T}}_{\bullet,0}\} = 0, & \bar{t} &:= \max\{s \mid s \in \widetilde{\mathscr{T}}_{\bullet,0}\} = T, \\ s_- &:= \max\big\{s' \in \widetilde{\mathscr{T}}_{\bullet,0} \mid s' < s\big\}, & s_+ &:= \min\big\{s' \in \widetilde{\mathscr{T}}_{\bullet,0} \mid s' > s\big\}, \\ \eta_\bullet &:= \max\big\{s_+ - s_- \mid s \in \widetilde{\mathscr{T}}_{\bullet,0}\big\}. \end{aligned}$$

We will say that *the (temporal) mesh converges* if a sequence $(\widetilde{\mathscr{T}}_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} \eta_n = 0$ has been built.

Spatial mesh. The discretization in space will consist in using an *adapted mesh* $\mathscr{T}(I)$. A sequence of elements of $\mathscr{T}(I)$ will be denoted by $\mathscr{T}_\bullet = (\mathscr{T}_{\bullet,0}, \mathscr{T}_{\bullet,1})$, where $\mathscr{T}_{\bullet,0}$ is a points set of I and $\mathscr{T}_{\bullet,1}$ is the set of intervals of the form $]r,r'[$ with $r < r'$ two elements of $\mathscr{T}_{\bullet,0}$. In the sequel,

we will use the following notations: for all $p \in \mathscr{T}_{\bullet,0}$,

$$
\begin{aligned}
|p| &:= \operatorname{card}\{p' \in \mathscr{T}_{\bullet,0} \mid p' \leq p\}, \\
\underline{p} &:= \min\{p \mid p \in \mathscr{T}_{\bullet,0}\}, \qquad & \overline{p} &:= \max\{p \mid p \in \mathscr{T}_{\bullet,0}\}, \\
p_- &:= \begin{cases} \max\{p' \in \mathscr{T}_{\bullet,0} \mid p' < p\} & \text{if } p \neq \underline{p} \\ r_- & \text{if } p = \underline{p} \end{cases}, \\
p_+ &:= \begin{cases} \min\{p' \in \mathscr{T}_{\bullet,0} \mid p' > p\} & \text{if } p \neq \overline{p} \\ r_+ & \text{if } p = \overline{p} \end{cases} \\
\omega_p &:= \left]\frac{p_- + p}{2}, \frac{p + p_+}{2}\right[, \\
\omega_{\underline{p}} &:= \left]\frac{r_- + \underline{p}}{2}, \frac{\underline{p} + \underline{p}_+}{2}\right[, \qquad & \omega_{\overline{p}} &:= \left]\frac{\overline{p}_- + \overline{p}}{2}, \frac{\overline{p} + r_+}{2}\right[, \\
\omega_\bullet &:= \bigcup_{p \in \mathscr{T}_{\bullet,0}} \omega_p, \\
h_\bullet &:= \max\{p_+ - p_- \mid p \in \mathscr{T}_{\bullet,0}\}, \qquad & \tilde{h}_\bullet &:= \min\{p_+ - p_- \mid p \in \mathscr{T}_{\bullet,0}\}.
\end{aligned}
$$

For all $p \in \mathscr{T}_{\bullet,0}$, ω_p is the *control interval centered at* p. We will say that *the (spatial) mesh converges* if a sequence $(\mathscr{T}_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to+\infty} h_n = 0$ has been built.

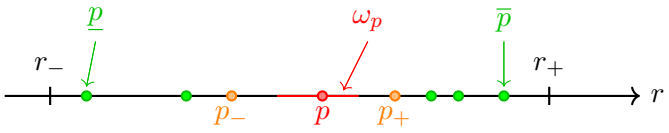

Figure 4.1: The spatial mesh.

In the sequel and unless it be specified otherwise (as for Lemma 4.2.3, Corollary 4.2.4 and Proposition 4.2.10), we will implicitly fix an arbitrary adapted mesh $\mathscr{T}_\bullet$ satisfying the following assumption:

$$
C_I := \sup_{\mathscr{T}_{\bullet,0} \subset \mathscr{T}_\bullet} \sup_{\substack{p \in \mathscr{T}_{\bullet,0} \\ r \in \omega_p}} \frac{|r - p|}{|p - p_\pm|} < +\infty. \tag{M}
$$

All the following operators and spaces will be defined with respect to a fixed mesh.

Discretized spaces and operators I

We thus define the *work spaces*

$$
\begin{aligned}
\mathscr{W} &:= \mathscr{D}(H), \\
\mathscr{W}_1 &:= \mathcal{C}^1([0,T], \mathscr{W}), \\
\mathscr{W}_0 &:= \mathcal{C}^0([0,T], \mathscr{W})
\end{aligned}
$$

and the *approximated spaces*

$$\begin{aligned}
\mathscr{W}^\star &:= \overline{\dot{\mathcal{H}}^1_\rho \times \mathcal{H}^1_\rho}^{\|\cdot\|_{\dot{\mathcal{E}}_\rho}}, \\
\mathscr{W}^\star_H &:= \overline{\dot{\mathcal{H}}^1_\rho \times L^2_\rho}^{\|\cdot\|_{\dot{\mathcal{E}}_\rho}} = \dot{\mathcal{E}}_\rho, \\
\mathscr{W}^\star_1 &:= \mathcal{C}^0\left([0,T], \mathscr{W}^\star\right), \\
\mathscr{W}^\star_{1,H} &:= \mathcal{C}^0\left([0,T], \mathscr{W}^\star_H\right), \\
\mathscr{W}^\star_0 &:= L^2([0,T], \mathscr{W}^\star_H).
\end{aligned}$$

Solutions of (ADP) are thus elements of $\mathscr{W}_1$. Observe that

$$\mathscr{W} \subset \mathscr{W}^\star \subset \mathscr{W}^\star_H \tag{4.3}$$

the first inclusion being a direct consequence of Lemma 4.1.7 (see also Remark 4.1.6). As a result, we have

$$\mathscr{W}_1 \subset \mathscr{W}^\star_1 \subset \mathscr{W}^\star_{1,H} \subset \mathscr{W}^\star_0. \tag{4.4}$$

Next, let

$$B := \mathrm{i}\partial_t + H$$

so that the equation in (ADP) boils down to solve $Bu = 0$ for $u \in \mathcal{C}^1([0,T], \mathscr{W})$ with compactly supported initial data in $\mathscr{W}$. For all $u \in \mathscr{W}^\star_1$ and all $(t,r) \in [0,T] \times I$, we define the *approximated operators*

$$(\partial^\star_t u)(t,\cdot) := \begin{cases} (\partial_t u)(t,.) & \text{if } u \text{ is } \mathcal{C}^1(\mathcal{U}, \mathscr{W}^\star) \text{ for some neighborhood } \mathcal{U} \ni t \\ 0 & \text{otherwise} \end{cases},$$

$$(P^\star u_0)(t,r) := \begin{cases} (Pu_0)(t,r) & \text{if } u_0 \in \dot{\mathcal{H}}^1_\rho(\mathcal{V}) \text{ for some neighborhood } \mathcal{V} \ni r \\ 0 & \text{otherwise} \end{cases}.$$

We finally define $H^\star$ as H with $P^\star$ instead of P, and we set

$$B^\star := \mathrm{i}\partial^\star_t + H^\star : \mathscr{W}^\star_1 \longrightarrow \mathscr{W}^\star_0.$$

Projectors, lifts and filters

Let $u : [0,T] \times I \to \mathbb{C}$ be a function. The *temporal Lagrange interpolation* of u over the set $\mathscr{A} = \{a_1, \ldots, a_m\} \subset [0,T]$ is defined where it makes sense by

$$\left(\mathscr{I}^{\mathrm{temp}}_{\mathscr{A}} u\right)(t,r) := \sum_{j=1}^{m} l^{\mathscr{A}}_j(t) u(a_j, r), \qquad l^{\mathscr{A}}_j(t) := \prod_{\substack{1 \le j' \le m \\ j' \neq j}} \frac{t - a_{j'}}{a_j - a_{j'}}.$$

Similarly, the *spatial Lagrange interpolation* of u over the set $\mathscr{B} = \{b_1, \ldots, b_n\} \subset I$ is defined by

$$\left(\mathscr{I}^{\mathrm{spat}}_{\mathscr{B}} u\right)(t,r) := \sum_{k=1}^{n} l^{\mathscr{B}}_k(r) u(t, b_k) \qquad l^{\mathscr{B}}_k(r) := \prod_{\substack{1 \le k' \le n \\ k' \neq k}} \frac{r - b_{k'}}{b_k - b_{k'}}.$$

The following operators are defined after having chosen a mesh $\mathscr{T}_{\bullet,0}$.

Temporal projectors. The projectors we will use in the temporal scheme are the following ones:

$$\left(\mathscr{P}^{\mathrm{temp}}_{\bullet,1}\, u\right)(t,r) := \sum_{s\in\widetilde{\mathscr{T}}_{\bullet,0}} \left(\mathscr{I}^{\mathrm{temp}}_{\{s,s_+\}} u\right)(t,r)\, \mathbb{1}_{[s,s_+[}(t),$$

$$\left(\mathscr{P}^{\mathrm{temp}}_{\bullet,0}\, u\right)(t,r) := \sum_{s\in\widetilde{\mathscr{T}}_{\bullet,0}} \left(⨍_{[s,s_+[,\mathscr{A}} u(\tau,r)\mathrm{d}\tau\right) \mathbb{1}_{[s,s_+[}(t)$$

where

$$⨍_{[s,s_+[,\mathscr{A}} u(\tau,r)\mathrm{d}\tau := \int_0^1 \left(\mathscr{I}^{\mathrm{temp}}_{\mathscr{A}} u\right)\big((s_+-s)\tau'+s,r\big)\mathrm{d}\tau', \qquad \mathscr{A}\subset[0,1].$$

When $\mathscr{A}=\{0\}$ one says the scheme is *explicit*; when $\mathscr{A}=\{1\}$, the scheme is *implicit*.

Spatial operators. The operators we will use in the spatial scheme are the following ones:

$$\left(\mathscr{P}^{\mathrm{spat}}_{\bullet,1}\, u\right)(t,r) := \sum_{p\in\mathscr{T}_{\bullet,0}} \left(\mathscr{I}^{\mathrm{spat}}_{S_p} u\right)(t,r)\, \mathbb{1}_{\omega_p}(r)$$

with

$$S_p := \begin{cases} \{p_-,p,p_+\} & \text{if } p\notin\{\underline{p},\overline{p}\} \\ \{r_-,\underline{p}\} & \text{if } p=\underline{p} \\ \{\overline{p},r_+\} & \text{if } p=\overline{p} \end{cases},$$

and

$$\left(\mathscr{P}^{\mathrm{spat}}_{\bullet,0}\, u\right)(t,r) := \begin{cases} \displaystyle\sum_{p\in\mathscr{T}_{\bullet,0}} \left(\mathscr{I}^{\mathrm{spat}}_{S_p} u\right)(t,r)\, \mathbb{1}_{\omega_p}(r) & \text{if } \mathscr{T}_{\bullet,0}\subset\mathscr{L}(u) \\ 0 & \text{otherwise} \end{cases}.$$

Remark 4.1.11. *The projector $\mathscr{P}^{\mathrm{spat}}_{\bullet,1}$ is well defined on $\mathscr{W}_1^\star$ since one evaluates functions only at interior points,which makes sense thanks to Assumption (AP2) and Sobolev embedding $H^1(I,\mathrm{d}r)\subset\mathcal{C}^{0,1/2}(I,\mathbb{C})$.*

As for the projector $\mathscr{P}^{\mathrm{spat}}_{\bullet,0}$, its action on the first component of elements of $\mathscr{W}_0^\star$ is well defined as explained above whereas its action on the second component is well defined by Lebesgue differentiation theorem for meshes defined on $\mathscr{L}(u)$ (which is a full measure subset of I). Indeed Lebesgue's theorem applies for locally L^1 functions, so if $u\in L^2_\rho$ then for all compact subset $K\subset I$,

$$\left|\int_K u(r)\mathrm{d}r\right| \le |K|^{1/2}\left(\int_K |u(r)|^2\mathrm{d}r\right)^{1/2} \le \left(\frac{|I|}{\rho_0}\right)^{1/2}\|u\|_{L^2_\rho} < +\infty.$$

$\mathscr{P}^{\mathrm{spat}}_{\bullet,0}$ filters "bad behaviors" with respect to the measure $\rho(r)\mathrm{d}r$.

If we assume the second component more regular than L^2_ρ, say continuous, then $\mathscr{P}^{\mathrm{spat}}_{\bullet,0}$ is perfectly well defined for any mesh since in this case $\mathscr{L}(u)=I$.

Remark 4.1.12. *The choice of the set S_p for the spatial interpolation can be adapted to the behaviour of the weight ρ near the boundary of I. Even though all the computations below are done for compactly supported functions, we could be concerned with more and more widespread supports (for long time numerical simulations for example). Here the linear behavior at the boundary has be chosen according to the concrete case presented in Chapter* **??** *where $\rho(r) = \mathcal{O}_{r\to r_\pm}\big(|r - r_\pm|\big)$.*

Remark 4.1.13. *Given a function $u \in L^2_\rho$, it is possible constructing a mesh $\mathscr{T}_\bullet$ contained in $\mathscr{L}(u)$ satisfying (M) and such that $\max\big\{|\omega| \mid \omega \in \mathscr{T}_{\bullet,1}\big\} \leq \delta_\bullet$ for some given decreasing family of positive numbers $(\delta_\bullet)_\bullet$.*

A possible construction is to split I into $3/h_\bullet$ parts of same length $h_\bullet/3$ with $h_\bullet = 1/\lceil 1/\delta_\bullet\rceil$. Then choose a Lebesgue point of u in the $(2k+1)$-th subinterval for all $k \in \{0, 1, \dots, \frac{3/h_\bullet - 1}{2}\}$ (this is always possible since these points form a full measure set in I). In this way, any two consecutive points of $\mathscr{T}_{\bullet,0}$ are separated by a distance of at least $h_\bullet/3$ and at most $h_\bullet \leq \delta_\bullet$ (see Figure 4.2). This gives a suitable mesh with $C_I = 3$.

We can ever do better: given any countable set of locally integrable functions $\big\{u_n \mid n \in \mathbb{N}\big\}$, we can arrange ourselves to get

$$\mathscr{T}_{\bullet,0} \subset \bigcap_{n\in\mathbb{N}} \mathscr{L}(u_n).$$

It is possible since a countable intersection of full measure sets still has a full measure, so we can repeat the above construction and choose the mesh points on each interval in the intersection of the Lebesgue's points of the u_n.

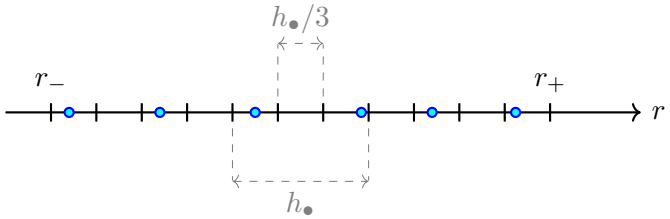

Figure 4.2: A possible choice for the spatial mesh (blue dots are points in $\mathscr{L}(u)$).

Remark 4.1.14. *If one does not want to deal with Lebesgue points of u to define $\mathscr{P}^{\mathrm{spat}}_{\bullet,0}u$ when one component of u is only in L^2_ρ, then it is possible to define it as the piecewise constant function on each control interval ω_p, $p \in \mathscr{T}_{\bullet,0}$ being equal to the mean value of u over ω_p. This provides a convergent scheme but computations are slightly more complicated.*

Lifts and filters. For $i \in \{0,1\}$ and $\clubsuit \in \{\mathrm{temp}, \mathrm{spat}\}$, we set $\mathscr{P}_{\bullet,i} := \mathscr{P}^{\mathrm{temp}}_{\bullet,i} \circ \mathscr{P}^{\mathrm{spat}}_{\bullet,i}$, $\mathscr{R}^{\clubsuit}_{\bullet,i} : \mathscr{P}^{\clubsuit}_{\bullet,i}\big(\mathscr{W}_i^\star\big) \hookrightarrow \mathscr{W}_i^\star$ the canonical embedding, and $\mathscr{R}_{\bullet,i} := \mathscr{R}^{\mathrm{spat}}_{\bullet,i} \circ \mathscr{R}^{\mathrm{temp}}_{\bullet,i}$. We then define the filters $\mathscr{F}^{\clubsuit}_{\bullet,i} := \mathscr{R}^{\clubsuit}_{\bullet,i} \circ \mathscr{P}^{\clubsuit}_{\bullet,i}$ and $\mathscr{F}_{\bullet,i} := \mathscr{R}_{\bullet,i} \circ \mathscr{P}_{\bullet,i} = \mathscr{R}^{\mathrm{spat}}_{\bullet,i} \circ \mathscr{R}^{\mathrm{temp}}_{\bullet,i} \circ \mathscr{P}^{\mathrm{temp}}_{\bullet,i} \circ \mathscr{P}^{\mathrm{spat}}_{\bullet,i}$.

Discretized spaces and operators II

As the operator B (and the corresponding approximated operators) has temporal and spatial parts, we introduce some other spaces in order to handle the problem more conveniently. For

$i \in \{0,1\}$, let

$$\begin{aligned}
\mathscr{W}_{\bullet,i} &:= \mathscr{P}_{\bullet,i}(\mathscr{W}_i^\star),\\
\mathscr{W}_{\bullet,1}^{\mathrm{spat}} &:= \mathscr{P}_{\bullet,1}^{\mathrm{spat}}(\mathscr{W}^\star),\\
\mathscr{W}_{\bullet,0}^{\mathrm{spat}} &:= \mathscr{P}_{\bullet,0}^{\mathrm{spat}}(\mathscr{W}_H^\star),\\
\mathscr{W}_{\bullet,1}^\star &:= \mathcal{C}^0([0,T],\mathscr{W}_{\bullet,1}^{\mathrm{spat}}),\\
\mathscr{W}_{\bullet,H}^\star &:= \mathcal{C}^0([0,T],\mathscr{W}_{\bullet,0}^{\mathrm{spat}}),\\
\mathscr{W}_{\bullet,0}^\star &:= L^2([0,T],\mathscr{W}_{\bullet,0}^{\mathrm{spat}}).
\end{aligned}$$

We also introduce another operators:

$$\begin{aligned}
M_\bullet &:= \mathscr{P}_{\bullet,0}^{\mathrm{spat}} \circ \mathscr{R}_{\bullet,1}^{\mathrm{spat}} : \mathscr{W}_{\bullet,1}^{\mathrm{spat}} \longrightarrow \mathscr{W}_{\bullet,0}^{\mathrm{spat}},\\
H_\bullet^\star &:= \mathscr{P}_{\bullet,0}^{\mathrm{spat}} \circ H^\star \circ \mathscr{R}_{\bullet,1}^{\mathrm{spat}} : \mathscr{W}_{\bullet,1}^\star \longrightarrow \mathscr{W}_{\bullet,0}^\star,\\
\partial_\bullet &:= \mathscr{P}_{\bullet,0}^{\mathrm{temp}} \circ \partial_t^\star \circ M_\bullet \circ \mathscr{R}_{\bullet,1}^{\mathrm{temp}} : \mathscr{W}_{\bullet,1} \longrightarrow \mathscr{W}_{\bullet,0},\\
H_\bullet &:= \mathscr{P}_{\bullet,0}^{\mathrm{temp}} \circ H_\bullet^\star \circ \mathscr{R}_{\bullet,1}^{\mathrm{temp}} : \mathscr{W}_{\bullet,1} \longrightarrow \mathscr{W}_{\bullet,0}.
\end{aligned}$$

The canonical embedding $\mathscr{W}^\star \hookrightarrow \mathscr{W}_H^\star$ (see (4.3)) has been implicitly used in the definition of the operator $M_\bullet$. In the sequel, we may also view this operator as acting from $\mathscr{W}_{\bullet,1}^\star$ to $\mathscr{W}_{\bullet,H}^\star$. Note furthermore that $H_\bullet^\star$ must act on spatio-temporal spaces as losing a time derivative for solutions of (ADP) is equivalent to lose spatial derivatives via the action of the Hamiltonian H.

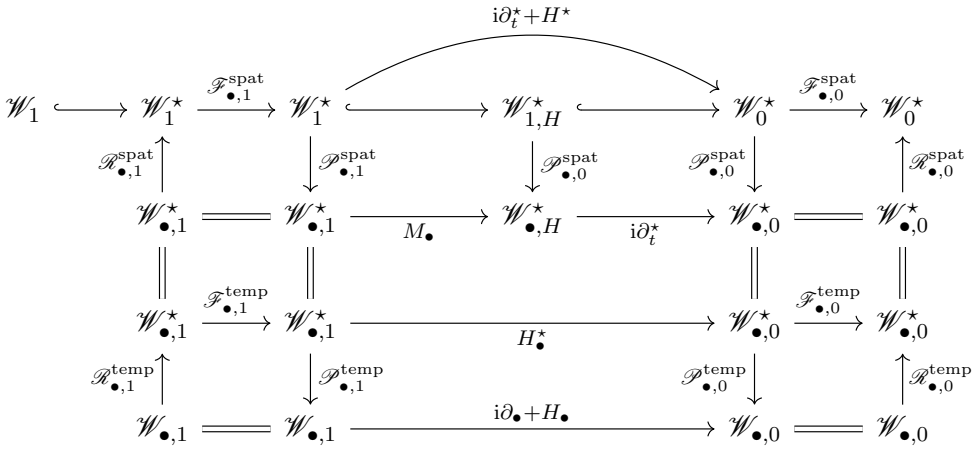

Figure 4.3: Summary of the discretization of spaces and operators.

Figure 4.3 above gives some insight about the numerical scheme: we look for a good approximation of solution of (ADP) in $\mathscr{W}_1$, so we include the latter space into the bigger one $\mathscr{W}_1^\star$ and then discretize the problem to get the discrete space $\mathscr{W}_{\bullet,1}$. It is in this latter one that approximated solutions lie, the advantage being that this space is actually some $\mathbb{C}^{N_\bullet}$ for some $N_\bullet \in \mathbb{N}$ (a similar but quite different property holds for the discrete target space $\mathscr{W}_{\bullet,0}$, even though it is only defined

to give sense to the operators defined above). In a short sentence, the numerical scheme provides *arrows* between the two main objects which are $\mathscr{W}_1$ (the space of solutions) and $\mathscr{W}_{\bullet,1}$ (the finite dimensional vector space used for the numerical programming).

Once an approximate solution in $\mathscr{W}_{\bullet,1}$ is exhibited, we still have to show how it consists in a *good* approximation of the real one, in the sens that we have to show that the difference between the solution and its approximation goes to 0 in some sense; in this purpose we have to construct an approximated norm on the whole space $\mathscr{W}_1^\star$. We will deal with all these points in the next Section.

4.1.3 Matrix representation

We investigate in this Section the matrix representations of the operators and the norms used for the discretization of the abstract problem. As concrete computations will be carried out in the sequel, we will sometimes need to specify Q. We thus add the following assumption:

$$Q \text{ is a } \mathcal{C}^2(I,\mathbb{R}) \text{ multiplication operator on } I \tag{Q}$$

that is: for all $u \in L^p(I,\mathrm{d}r)$, $(Qu)(r) = q(r)u(r)$ with $q \in \mathcal{C}^2(I,\mathbb{R})$. Since Q is assumed to be self-adjoint, q is necessarily real. Besides, for $k \in \{1,2\}$, $Q \in \mathcal{B}(\dot{\mathcal{H}}^k_\rho(J))$ for all open set $J \Subset I$ since then $\dot{\mathcal{H}}^k_\rho(J) \subset H^k(J)$ by Assumption (AP2). Therefore Assumption (AP5) is locally verified for multiplication operators.

Remark 4.1.15. *Actually, we will only need $q \in H^2_{\ell\mathrm{oc}}(I,\mathbb{R})$ in Proposition 4.2.10, but this requires to ask for $\mathscr{T}_{\bullet,0} \subset \mathscr{L}(q)$ for each considered sequence $\mathscr{T}_\bullet$. For the sake of simplicity, we will restrict ourselves to the case of Assumption (Q) above.*

Discrete vector spaces

We introduce in this paragraph the discrete spaces which are used for the numerical programming For all $u, v \in \mathscr{W}^{\mathrm{spat}}_{\bullet,1}$, we define

$$u \,\mathcal{R}\, v \qquad \Longleftrightarrow \qquad u(p) = v(p) \quad \forall p \in \mathscr{T}_{\bullet,0}$$

and similarly, if $u, v \in \mathscr{W}^{\mathrm{spat}}_{\bullet,0}$, we define

$$u \,\mathcal{R}_H\, v \qquad \Longleftrightarrow\ u(p) = v(p) \quad \forall p \in \mathscr{T}_{\bullet,0}.$$

We will note $[\cdot]$ and $[\cdot]_H$ the corresponding equivalence classes in the quotient spaces $\mathscr{W}^{\mathrm{spat}}_{\bullet,1}/\mathcal{R}$ and $\mathscr{W}^{\mathrm{spat}}_{\bullet,0}/\mathcal{R}_H$, respectively.

Remark 4.1.16. *If one define the projector $\mathscr{P}^{\mathrm{spat}}_{\bullet,0}$ as in Remark 4.1.14, then we can work on the entire work space $\mathscr{W}^{\mathrm{spat}}_{\bullet,0}$ and the equivalence relation $\mathcal{R}_H$ can be defined as follows:*

$$u \,\mathcal{R}_H\, v \qquad \Longleftrightarrow \qquad \fint_{\omega_p} u(\xi)\mathrm{d}\xi = \fint_{\omega_p} v(\xi)\mathrm{d}\xi \quad \forall p \in \mathscr{T}_{\bullet,0}.$$

It turns out that the spaces $\mathscr{W}^{\mathrm{spat}}_{\bullet,1}$ and $\mathscr{W}^{\mathrm{spat}}_{\bullet,0}$ can be identified respectively to $\mathscr{W}^{\mathrm{spat}}_{\bullet,1}/\mathcal{R}$ and $\mathscr{W}^{\mathrm{spat}}_{\bullet,0}/\mathcal{R}_H$, which basically means that we can "drop" all the information but the values at the mesh points p. In fact, we have a little bit more:

Lemma 4.1.17. *The spaces $\mathscr{W}^{\mathrm{spat}}_{\bullet,1}$, $\mathscr{W}^{\mathrm{spat}}_{\bullet,1}/\mathcal{R}$, $\mathscr{W}^{\mathrm{spat}}_{\bullet,0}$ and $\mathscr{W}^{\mathrm{spat}}_{\bullet,0}/\mathcal{R}_H$ are all isomorphic to $\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$.*

Proof. Define first $\Pi_{\bullet,1} : \mathscr{W}^{\mathrm{spat}}_{\bullet,1} \to \mathscr{W}^{\mathrm{spat}}_{\bullet,1}/\mathcal{R}$ the canonical projection:

$$\Pi_{\bullet,1} u := [u].$$

$\Pi_{\bullet,1}$ is clearly linear and surjective. If $u, v \in \mathscr{W}^{\mathrm{spat}}_{\bullet,1}$ are such that $\Pi_{\bullet,1} u = \Pi_{\bullet,1} v$, then $[u-v]_1 = [0]_1$ that is $(u-v)(p) = 0$ for all $p \in \mathscr{T}_{\bullet,0}$. But $(u_j - v_j)_{|\omega_p}$ is the restriction of the unique second order (respectively first order if $p = \underline{p}$ or $p = \overline{p}$) $\mathbb{C}$-valued polynomial which cancels at p_-, p and p_+ (respectively at r_- and $\underline{p}$ or $\overline{p}$ and r_+), thus $u_j - v_j \equiv 0$ on ω_p for all $p \in \mathscr{T}_{\bullet,0}$ and $j \in \{0,1\}$. This shows that $\Pi_\bullet$ is injective.

Next, define $\Psi_\bullet : \mathscr{W}^{\mathrm{spat}}_{\bullet,1} \to \mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$ the evaluation function:

$$\Psi_\bullet u := (u_0(p), u_1(p))_{p \in \mathscr{T}_{\bullet,0}}.$$

$\Psi_\bullet$ is clearly linear. It is surjective since smooth compactly supported regularizations of $\mathbb{1}_{\omega_p}$ are elements of $\mathscr{W}^\star$. Take now $u, v \in \mathscr{W}^{\mathrm{spat}}_{\bullet,1}$ such that $\Psi_\bullet u = \Psi_\bullet v$. Then $(u-v)(p) = 0$ for all $p \in \mathscr{T}_{\bullet,0}$ and we conclude as for $\Pi_{\bullet,1}$ above that $\Psi_\bullet$ is injective.

Finally, define

$$\Pi_{\bullet,0} : \mathscr{W}^{\mathrm{spat}}_{\bullet,0} \ni u := [u]_H \in \mathscr{W}^{\mathrm{spat}}_{\bullet,0}/\mathcal{R}_H$$

and $\Psi_{\bullet,H} : \mathscr{W}^{\mathrm{spat}}_{\bullet,0} \to \mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$ the evaluation function. As above, we show that $\Pi_{\bullet,0}$ is an isomorphism. As for $\Psi_{\bullet,H}$, the only issue may be the surjectivity as it may happen that all the second component of elements of $\tilde{\mathscr{V}}^\star_H$ cancel at the mesh points $p \in \mathscr{T}_{\bullet,0}$; however, the range of the functions $(0, \mathbb{1}_{\omega_p}) \in \mathscr{W}^\star_H$ by the composition $\Psi_{\bullet,H} \circ \mathscr{P}^{\mathrm{spat}}_{\bullet,0}$ is precisely the canonical basis of $\mathbb{C}^{|\mathscr{T}_{\bullet,0}|}$, ensuring the surjectivity of $\Psi_{\bullet,H}$. □

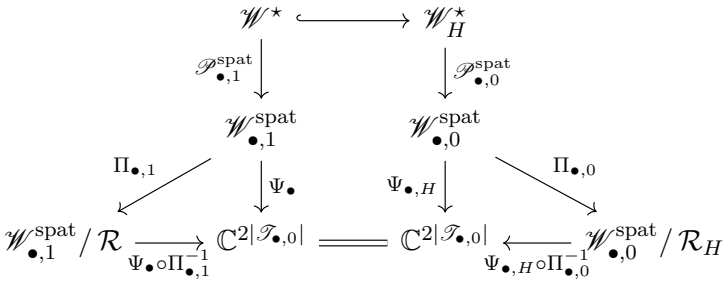

Figure 4.4: Discretization of spatial work spaces.

Remark 4.1.18. *1. Despite the somewhat singular definition of $\mathscr{P}^{\mathrm{spat}}_{\bullet,0}$, we managed to get equivalences modulo isomorphisms between the discrete space $\mathscr{W}^{\mathrm{spat}}_{\bullet,0}$ and the finite dimensional one $\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$. The key argument is that $\mathscr{P}^{\mathrm{spat}}_{\bullet,0}(\mathscr{W}^{\mathrm{spat}}_{\bullet,0})$ has enough elements so that the evaluation function spans $\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$; more precisely, given a mesh $\mathscr{T}_{\bullet,0}$, some elements of $\mathscr{W}^\star_H$ (the functions $(0, \mathbb{1}_{\omega_p})$ with $p \in \mathscr{T}_{\bullet,0}$ for us) are sent by $\Psi_{\bullet,H} \circ \mathscr{P}^{\mathrm{spat}}_{\bullet,0}$ on a generating family of the target finite dimensional space. The argument of Lemma 4.1.17 fails otherwise.*

2. *A similar commutative diagram as above can be drawn for the temporal discretization. For the sake of brevity, we will omit details in the sequel.*

Approximated norms

As we wish to evaluate the approximation error of a solution of (ADP), our task consists now in defining an adequate approximated norm on $\mathscr{W}_1^\star$ and then giving a natural matrix representation.

On $\mathscr{W}^\star$ and $\mathscr{W}_H^\star$, we define the spatial approximated norm

$$\mathcal{N}_\bullet^2(v) := \left\langle P^\star \mathfrak{R}_{\bullet,j}\mathfrak{P}_{\bullet,j} v_0, \mathfrak{R}_{\bullet,j}\mathfrak{P}_{\bullet,j} v_0 \right\rangle_{L^2_\rho} + \left\| \mathfrak{R}_{\bullet,j}\mathfrak{P}_{\bullet,j} v_1 - \mathfrak{R}_{\bullet,j}\mathfrak{P}_{\bullet,j} Q v_0 \right\|^2_{L^2_\rho}$$

where

$$\mathfrak{P}_{\bullet,j} v_k = \left(\mathscr{P}^{\mathrm{spat}}_{\bullet,j} v\right)_k, \qquad \mathfrak{R}_{\bullet,j} v_k = \left(\mathscr{R}^{\mathrm{spat}}_{\bullet,j} v\right)_k$$

for all $j,k \in \{0,1\}$. We placed Q in the definition of $\mathcal{N}_\bullet$ so that computations in Appendix 4.4.3 are easier. On $\mathscr{W}_1^\star$ and $\mathscr{W}_0^\star$, we then define the approximated norm

$$[\![u]\!]_\bullet := \sup_{t\in[0,T]} \mathcal{N}_\bullet(u(t,\cdot))$$

The corresponding approximated operator norm is

$$|||L_\bullet|||_\bullet := \sup_{u\neq 0} \frac{\mathcal{N}_\bullet(L_\bullet u)}{\mathcal{N}_\bullet(u)}.$$

Note that $\mathcal{N}_\bullet^2$ is an hermitian form on $\mathscr{W}^\star$ (because P is non-negative and self-adjoint), so we can associate to it the (right linear) sesquilinear form

$$\mathcal{N}_\bullet^2(u,v) := \left\langle P^\star \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0, \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} v_0 \right\rangle_{L^2_\rho} + \left\langle \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}\left(u_1 - Q u_0\right), \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}\left(v_1 - Q v_0\right) \right\rangle_{L^2_\rho}.$$

It turns out that the approximated norm $\mathcal{N}_\bullet$ is not convenient to show convergence. We thus introduce two other ones:

$$|||(u_0,u_1)|||_{\dot{\mathcal{E}}_\rho} := \|u_0\|_{\dot{\mathcal{H}}^1_\rho} + \|u_1 - Q u_0\|_{L^2_\rho} \qquad (u_0,u_1) \in \dot{\mathcal{E}}_\rho,$$

$$\widetilde{\mathcal{N}}_\bullet((u_0,u_1)) := \left\langle P^\star \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0, \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0 \right\rangle^{1/2}_{L^2_\rho} + \left\| \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_1 - \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} Q u_0 \right\|_{L^2_\rho} \qquad (u_0,u_1) \in \mathscr{W}^\star.$$

Certainly these norms are respectively equivalent to $\|\cdot\|_{\dot{\mathcal{E}}_\rho}$ and $\mathcal{N}_\bullet(\cdot)$:

$$|||\cdot|||_{\dot{\mathcal{E}}_\rho} \leq \sqrt{2}\,\|\cdot\|_{\dot{\mathcal{E}}_\rho} \leq \sqrt{2}\,|||\cdot|||_{\dot{\mathcal{E}}_\rho}, \tag{4.5}$$

$$\widetilde{\mathcal{N}}_\bullet(\cdot) \leq \sqrt{2}\,\mathcal{N}_\bullet(\cdot) \leq \sqrt{2}\,\widetilde{\mathcal{N}}_\bullet(\cdot). \tag{4.6}$$

Note however that these norms do not satisfy the parallelogram law (and thus do not define sesquilinear forms). Our goal is then to show that $\widetilde{\mathcal{N}}_\bullet(\cdot)$ converges to $|||\cdot|||_{\dot{\mathcal{E}}_\rho}$ is some sense (see Corollary 4.2.8). Then decay and non-decay results numerically computed with $\mathcal{N}_\bullet(\cdot)$ will apply to $\widetilde{\mathcal{N}}_\bullet(\cdot)$, hence to $|||\cdot|||_{\dot{\mathcal{E}}_\rho}$ modulo an error which vanishes as the mesh converges, and finally to $\|\cdot\|_{\dot{\mathcal{E}}_\rho}$ by equivalence of the norms.

We now turn to the matrix representation of the norm $\mathcal{N}_\bullet$ (the one we will use for the numerical programming). As the scheme will converge in the space $\mathscr{W}_1^\star$ (see Theorem 4.3.1) we only need to explicit it on the work space $\mathscr{W}_{\bullet,1}$. Let $(e_j)_{1\leq j\leq 2|\mathscr{T}_{\bullet,0}|}$ be the canonical basis of $\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$. By construction, we have

$$\begin{aligned}
\Psi_\bullet^{-1}e_1 &= \left(\frac{r-r_-}{\underline{p}-r_-}\mathbb{1}_{\omega_{\underline{p}}} + \frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\mathbb{1}_{\omega_{\underline{p}_+}}, 0\right),\\
\Psi_\bullet^{-1}e_2 &= \left(0, \frac{r-r_-}{\underline{p}-r_-}\mathbb{1}_{\omega_{\underline{p}}} + \frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\mathbb{1}_{\omega_{\underline{p}_+}}\right),\\
\Psi_\bullet^{-1}e_3 &= \left(\frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})}\mathbb{1}_{\omega_{\underline{p}_+}} + \frac{(r-\underline{p}_{++})(r-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})}\mathbb{1}_{\omega_{\underline{p}_{++}}}, 0\right),\\
\Psi_\bullet^{-1}e_4 &= \left(0, \frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})}\mathbb{1}_{\omega_{\underline{p}_+}} + \frac{(r-\underline{p}_{++})(r-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})}\mathbb{1}_{\omega_{\underline{p}_{++}}}\right),
\end{aligned}$$

$$\begin{aligned}
\Psi_\bullet^{-1}e_{2|\mathscr{T}_{\bullet,0}|-3} &= \left(\frac{(r-\overline{p}_{---})(r-\overline{p}_{--})}{(\overline{p}_- -\overline{p}_{---})(\overline{p}_- -\overline{p}_{--})}\mathbb{1}_{\omega_{\overline{p}_{--}}} + \frac{(r-\overline{p}_{--})(r-\overline{p})}{(\overline{p}_- -\overline{p}_{--})(\overline{p}_- -\overline{p})}\mathbb{1}_{\omega_{\overline{p}_-}}, 0\right),\\
\Psi_\bullet^{-1}e_{2|\mathscr{T}_{\bullet,0}|-2} &= \left(0, \frac{(r-\overline{p}_{---})(r-\overline{p}_{--})}{(\overline{p}_- -\overline{p}_{---})(\overline{p}_- -\overline{p}_{--})}\mathbb{1}_{\omega_{\overline{p}_{--}}} + \frac{(r-\overline{p}_{--})(r-\overline{p})}{(\overline{p}_- -\overline{p}_{--})(\overline{p}_- -\overline{p})}\mathbb{1}_{\omega_{\overline{p}_-}}\right),\\
\Psi_\bullet^{-1}e_{2|\mathscr{T}_{\bullet,0}|-1} &= \left(\frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)}\mathbb{1}_{\omega_{\overline{p}_-}} + \frac{r-r_+}{\overline{p}-r_+}\mathbb{1}_{\omega_{\overline{p}}}, 0\right),\\
\Psi_\bullet^{-1}e_{2|\mathscr{T}_{\bullet,0}|} &= \left(0, \frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)}\mathbb{1}_{\omega_{\overline{p}_-}} + \frac{r-r_+}{\overline{p}-r_+}\mathbb{1}_{\omega_{\overline{p}}}\right),
\end{aligned}$$

$$\begin{aligned}
\Psi_\bullet^{-1}e_{2|p|-1} &= \left(\frac{(r-p_{--})(r-p_-)}{(p-p_{--})(p-p_-)}\mathbb{1}_{\omega_{p_-}} + \frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)}\mathbb{1}_{\omega_p} + \frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})}\mathbb{1}_{\omega_{p_+}}, 0\right),\\
\Psi_\bullet^{-1}e_{2|p|} &= \left(0, \frac{(r-p_{--})(r-p_-)}{(p-p_{--})(p-p_-)}\mathbb{1}_{\omega_{p_-}} + \frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)}\mathbb{1}_{\omega_p} + \frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})}\mathbb{1}_{\omega_{p_+}}\right)
\end{aligned}$$

for all $p \in \mathring{\mathscr{T}}_{\bullet,0} \setminus \{\underline{p}_+, \overline{p}_-\}$. One may be tempted to write $\Psi_\bullet^{-1}(e_{2|p|-1}) = \left(\frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)}\mathbb{1}_{\omega_p}, 0\right)$ but the latter function is not an element of $\mathscr{W}_{\bullet,1}$.

Note the following orthogonality property: for all $(j,k) \in \{1,\dots,|\mathscr{T}_{\bullet,0}|\}^2$, for all $j' \in \{2j-1, 2j\}$ and $k' \in \{2k-1, 2k\}$, we have

$$\mathcal{N}_\bullet^2(\Psi_\bullet^{-1}e_{j'}, \Psi_\bullet^{-1}e_{k'}) = 0 \quad \Longleftrightarrow \quad |j-k| \geq 3 \tag{4.7}$$

that is the matrix representation of $\mathcal{N}_\bullet^2(\Psi_\bullet^{-1}\cdot, \Psi_\bullet^{-1}\cdot)$ in the canonical basis is block-hexagonal. In Appendix 4.4.3 we compute all the non-zero matrix coefficients $(\mathscr{N}_\bullet^2)_{ij} := \mathcal{N}_\bullet^2(\Psi_\bullet^{-1}e_i, \Psi_\bullet^{-1}e_j)$ of the sesquilinear form $\mathcal{N}_\bullet^2(\Psi_\bullet^{-1}\cdot, \Psi_\bullet^{-1}\cdot)$.

With the above construction, for all $u = (u_0, u_1)$, $v = (v_0, v_1) \in \mathscr{W}_{\bullet,1}$ and for all $t \in [0,T]$, we

have

$$\begin{aligned}
&\mathcal{N}_\bullet^2(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}u, \mathscr{R}^{\mathrm{spat}}_{\bullet,1}v)\\
&= \sum_{(p,p')\in(\mathscr{T}_{\bullet,0})^2} \mathcal{N}_\bullet^2\Big(\Psi_\bullet^{-1}(e_{2|p|-1})u_0(t,p) + \Psi_\bullet^{-1}(e_{2|p|})u_1(t,p), \Psi_\bullet^{-1}(e_{2|p'|-1})v_0(t,p') + \Psi_\bullet^{-1}(e_{2|p'|})v_1(t,p')\Big)\\
&= \sum_{(p,p')\in(\mathscr{T}_{\bullet,0})^2} \Big[(\mathcal{N}_\bullet^2)_{2|p|-1,2|p'|-1}\overline{u_0(t,p)}v_0(t,p') + (\mathcal{N}_\bullet)_{2|p|-1,2|p'|}\overline{u_0(t,p)}v_1(t,p')\\
&\qquad + (\mathcal{N}_\bullet^2)_{2|p|,2|p'|-1}\overline{u_1(t,p)}v_0(t,p') + (\mathcal{N}_\bullet)_{2|p|,2|p'|}\overline{u_1(t,p)}v_1(t,p')\Big]\\
&= \Big\langle U(t), \mathcal{N}_\bullet^2 V(t)\Big\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}}
\end{aligned}$$

where

$$U(t) = (u_0(t,p), u_1(t,p))_{p\in\mathscr{T}_{\bullet,0}}, \qquad V(t) = (v_0(t,p), v_1(t,p))_{p\in\mathscr{T}_{\bullet,0}}.$$

Approximated operators

We now explicit the matrix representation of the approximated operators.

Discrete norm $\mathcal{N}_\bullet$. Let $u \in \mathscr{W}_{\bullet,1}$ and $U := \Psi_\bullet u \in \mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$ its discretized version ($\Psi_\bullet$ acts on the spatial part of u). Then

$$\mathcal{N}_\bullet(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}u) = \sqrt{\langle U, \mathcal{N}_\bullet^2 U\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}}} = \sqrt{\sum_{i=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{j=1}^{2|\mathscr{T}_{\bullet,0}|}(\mathcal{N}_\bullet^2)_{ij}\overline{U}_i U_j}.$$

Matrix associated to $M_\bullet$. The discrete version of the operator $M_\bullet$ and is given by

$$\mathscr{M}_\bullet := \Psi_\bullet M_\bullet \Psi_{\bullet,H}^{-1} : \mathbb{C}^{2|\mathscr{T}_{\bullet,0}|} \to \mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$$

This makes sense since $M : \mathscr{W}^\star_{\bullet,1} \to \mathscr{W}^\star_{\bullet,H}$ and $\mathscr{W}^\star_{\bullet,1}$, $\mathscr{W}^\star_{\bullet,H}$ are isomorphic to $\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$ by Lemma 4.1.17. For all $j,k \in \{1,\dots,2|\mathscr{T}_{\bullet,0}|\}$ we have

$$(\mathscr{M}_\bullet)_{jk} = \langle e_j, \Psi_\bullet M_\bullet \Psi_{\bullet,H}^{-1} e_k\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}}.$$

In Appendix 4.4.1, we show that

$$(M_\bullet u)(p) = u(p) \qquad \forall p \in \mathscr{T}_{\bullet,0}$$

whence

$$(\mathscr{M}_\bullet)_{jk} = \langle e_j, e_k\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}} = \delta_{jk}$$

that is $\mathscr{M}_\bullet$ is the identity matrix. Since $\Psi_\bullet$ is an isomorphism, we deduce that $M_\bullet$ itself is invertible.

$$\begin{array}{ccc}
\mathscr{W}^\star_{\bullet,1} & \xrightarrow{\Psi_\bullet} & \mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}\\
{\scriptstyle M_\bullet}\downarrow & & \downarrow{\scriptstyle \mathscr{M}_\bullet}\\
\mathscr{W}^\star_{\bullet,H} & \xrightarrow{\Psi_{\bullet,H}} & \mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}
\end{array}$$

Matrix associated to $H^\star_\bullet$. The discrete version of the operator $H^\star_\bullet$ and is given by

$$\mathscr{H}^\star_\bullet := \Psi_\bullet H^\star_\bullet \Psi^{-1}_{\bullet,H}.$$

For all $j,k \in \{1,\dots,2|\mathscr{T}_{\bullet,0}|\}$ we have

$$(\mathscr{H}^\star_\bullet)_{jk} = \left\langle e_j, \Psi_\bullet H^\star_\bullet \Psi^{-1}_{\bullet,H} e_k \right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}}.$$

Assume (Q). In Appendix 4.4.2, we show that for all $u=(u_0,u_1) \in \mathscr{W}^{\mathrm{spat}}_{\bullet,1}$

$$\left(H^\star_\bullet u\right)_0(t,p) = q(p)u_0(t,p) + u_1(t,p),$$

$$\left(H^\star_\bullet u\right)_1(t,p) = \sum_{k\in K} \frac{2a(p)+b(p)\big(2p-(p_{k'}+p_{k''})\big)}{(p_k-p_{k'})(p_k-p_{k''})} u_0(t,p_k) + c(p)u_0(t,p) + q(p)u_1(t,p)$$

whence:

$$\begin{aligned}(\mathscr{H}^\star_\bullet)_{j1} &= \left\langle e_j, Q(\underline{p})e_1 + \frac{b(\underline{p})+c(\underline{p})(\underline{p}-r_-)}{\underline{p}-r_-}e_2 + \frac{2a(\underline{p}_+)+b(\underline{p}_+)(\underline{p}_+-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}e_4 \right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}} \\ &= Q(\underline{p})\delta_{j1} + \frac{b(\underline{p})+c(\underline{p})(\underline{p}-r_-)}{\underline{p}-r_-}\delta_{j2} + \frac{2a(\underline{p}_+)+b(\underline{p}_+)(\underline{p}_+-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\delta_{j4},\end{aligned}$$

$$(\mathscr{H}^\star_\bullet)_{j2} = \left\langle e_j, e_1 + Q(\underline{p})e_2 \right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}} = \delta_{j1} + Q(\underline{p})\delta_{j2},$$

$$\begin{aligned}(\mathscr{H}^\star_\bullet)_{j3} &= \left\langle e_j, Q(\underline{p}_+)e_3 + \left(\frac{2a(\underline{p}_+)+b(\underline{p}_+)(2\underline{p}_+-(\underline{p}+\underline{p}_{++}))}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})} + c(\underline{p}_+)\right)e_4 \right. \\ &\qquad \left. + \frac{2a(\underline{p}_{++})+b(\underline{p}_{++})(\underline{p}_{++}-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})}e_6 \right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}} \\ &= Q(\underline{p}_+)\delta_{j3} + \left(\frac{2a(\underline{p}_+)+b(\underline{p}_+)(2\underline{p}_+-(\underline{p}+\underline{p}_{++}))}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})} + c(\underline{p}_+)\right)\delta_{j4} \\ &\quad + \frac{2a(\underline{p}_{++})+b(\underline{p}_{++})(\underline{p}_{++}-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})}\delta_{j6},\end{aligned}$$

$$(\mathscr{H}^\star_\bullet)_{j4} = \left\langle e_j, e_3 + Q(\underline{p}_+)e_4 \right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}} = \delta_{j3} + Q(\underline{p}_+)\delta_{j4},$$

$$\begin{aligned}(\mathscr{H}^\star_\bullet)_{j,2|\mathscr{T}_{\bullet,0}|-3} &= \left\langle e_j, Q(\overline{p}_-)e_{2|\mathscr{T}_{\bullet,0}|-3} + \frac{2a(\overline{p}_{--})+b(\overline{p}_{--})(\overline{p}_{--}-\overline{p}_{---})}{(\overline{p}_--\overline{p}_{---})(\overline{p}_--\overline{p}_{--})}e_{2|\mathscr{T}_{\bullet,0}|-4} \right. \\ &\qquad \left. + \left(\frac{2a(\overline{p}_-)+b(\overline{p}_-)(2\overline{p}_-(\overline{p}_{--}+\overline{p}))}{(\overline{p}_--\overline{p}_{--})(\overline{p}_--\overline{p})} + c(\overline{p}_-)\right)e_{2|\mathscr{T}_{\bullet,0}|-2} \right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}} \\ &= Q(\overline{p}_-)\delta_{j,2|\mathscr{T}_{\bullet,0}|-3} + \frac{2a(\overline{p}_{--})+b(\overline{p}_{--})(\overline{p}_{--}-\overline{p}_{---})}{(\overline{p}_--\overline{p}_{---})(\overline{p}_--\overline{p}_{--})}\delta_{j,2|\mathscr{T}_{\bullet,0}|-4} \\ &\quad + \left(\frac{2a(\overline{p}_-)+b(\overline{p}_-)(2\overline{p}_--(\overline{p}_{--}+\overline{p}))}{(\overline{p}_--\overline{p}_{--})(\overline{p}_--\overline{p})} + c(\overline{p}_-)\right)\delta_{j,2|\mathscr{T}_{\bullet,0}|-2},\end{aligned}$$

$$(\mathscr{H}_\bullet^\star)_{j,2|\mathscr{T}_{\bullet,0}|-2} = \left\langle e_j, e_{2|\mathscr{T}_{\bullet,0}|-3} + Q(\overline{p}_-)e_{2|\mathscr{T}_{\bullet,0}|-2}\right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}} = \delta_{j,2|\mathscr{T}_{\bullet,0}|-3} + Q(\overline{p}_-)\delta_{j,2|\mathscr{T}_{\bullet,0}|-2},$$

$$\begin{aligned}(\mathscr{H}_\bullet^\star)_{j,2|\mathscr{T}_{\bullet,0}|-1} &= \left\langle e_j, Q(\overline{p})e_{2|\mathscr{T}_{\bullet,0}|-1} + \frac{2a(\overline{p}_-) + b(\overline{p}_-)(\overline{p}_- - \overline{p}_{--})}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)}e_{2|\mathscr{T}_{\bullet,0}|-2}\right.\\ &\qquad\left. + \frac{b(\overline{p}) + c(\overline{p})(\overline{p}-r_+)}{\overline{p}-r_+}e_{2|\mathscr{T}_{\bullet,0}|}\right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}}\\ &= Q(\overline{p})\delta_{j,2|\mathscr{T}_{\bullet,0}|-1} + \frac{2a(\overline{p}_-) + b(\overline{p}_-)(\overline{p}_- - \overline{p}_{--})}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)}\delta_{j,2|\mathscr{T}_{\bullet,0}|-2}\\ &\quad + \frac{b(\overline{p}) + c(\overline{p})(\overline{p}-r_+)}{\overline{p}-r_+}\delta_{j,2|\mathscr{T}_{\bullet,0}|},\end{aligned}$$

$$(\mathscr{H}_\bullet^\star)_{j,2|\mathscr{T}_{\bullet,0}|} = \left\langle e_j, e_{2|\mathscr{T}_{\bullet,0}|-1} + Q(\overline{p})e_{2|\mathscr{T}_{\bullet,0}|}\right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}} = \delta_{j,2|\mathscr{T}_{\bullet,0}|-1} + Q(\overline{p})\delta_{j,2|\mathscr{T}_{\bullet,0}|},$$

$$\begin{aligned}(\mathscr{H}_\bullet^\star)_{j,2|p|-1} &= \left\langle e_j, Q(p)e_{2|p|-1} + \frac{2a(p_-) + b(p_-)(p_- - p_{--})}{(p-p_{--})(p-p_-)}e_{2|p|-2}\right.\\ &\qquad + \left(\frac{2a(p) + b(p)(2p-(p_-+p_+))}{(p-p_-)(p-p_+)} + c(p)\right)e_{2|p|}\\ &\qquad\left. + \frac{2a(p_+) + b(p_+)(p_+ - p_{++})}{(p-p_+)(p-p_{++})}e_{2|p|+2}\right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}}\\ &= Q(p)\delta_{j,2|p|-1} + \frac{2a(p_-) + b(p_-)(p_- - p_{--})}{(p-p_{--})(p-p_-)}\delta_{j,2|p|-2}\\ &\quad + \left(\frac{2a(p) + b(p)(2p-(p_-+p_+))}{(p-p_-)(p-p_+)} + c(p)\right)\delta_{j,2|p|} + \frac{2a(p_+) + b(p_+)(p_+ - p_{++})}{(p-p_+)(p-p_{++})}\delta_{j,2|p|+2},\end{aligned}$$

$$(\mathscr{H}_\bullet^\star)_{j,2|p|} = \left\langle e_j, e_{2|p|-1} + Q(p)e_{2|p|}\right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}} = \delta_{j,2|p|-1} + Q(p)\delta_{j,2|p|}$$

for all $p \in \mathring{\mathscr{T}}_{\bullet,0} \setminus \{\underline{p}_+, \overline{p}_-\}$.

4.2 Consistency and stability

In this Section, we study the consistency of the operators and the stability of the scheme introduced in Chapter 4.1 under the hypotheses (AP) of Section 4.1.1. We will consider compactly supported solutions, motivated by the propagation result Proposition 4.3.5. Roughly speaking, consistency means that the approximated operators converges to the "true" ones as the mesh converges for the strong topology of the working spaces; stability says that solutions of a perturbed discrete problem will stay close to the solution of the unperturbed problem provided that both have same initial data and the perturbation is small.

The discrete problem consists in solving

$$B_\bullet = 0 \tag{4.8}$$

where (using definitions and notations of the paragraph 4.1.2)

$$\begin{aligned} B_\bullet &:= \mathscr{P}_{\bullet,0} \circ B^\star \circ \mathscr{R}_{\bullet,1} \\ &= \mathscr{P}^{\mathrm{temp}}_{\bullet,0} \circ \mathscr{P}^{\mathrm{spat}}_{\bullet,0} \circ (\mathrm{i}\partial_t^\star + H^\star) \circ \mathscr{R}^{\mathrm{spat}}_{\bullet,1} \circ \mathscr{R}^{\mathrm{temp}}_{\bullet,1} \\ &= \mathscr{P}^{\mathrm{temp}}_{\bullet,0} \circ \left(\mathrm{i}\mathscr{P}^{\mathrm{spat}}_{\bullet,0} \circ \partial_t^\star \circ \mathscr{R}^{\mathrm{spat}}_{\bullet,1} + \mathscr{P}^{\mathrm{spat}}_{\bullet,0} \circ H^\star \circ \mathscr{R}^{\mathrm{spat}}_{\bullet,1} \right) \circ \mathscr{R}^{\mathrm{temp}}_{\bullet,1} \\ &= \mathscr{P}^{\mathrm{temp}}_{\bullet,0} \circ (\mathrm{i}\partial_t^\star \circ M_\bullet + H_\bullet) \circ \mathscr{R}^{\mathrm{temp}}_{\bullet,1} \end{aligned}$$

with

$$M_\bullet = \mathscr{P}^{\mathrm{spat}}_{\bullet,0} \circ \mathscr{R}^{\mathrm{spat}}_{\bullet,1}$$

and

$$H_\bullet = \mathscr{P}^{\mathrm{spat}}_{\bullet,0} \circ H^\star \circ \mathscr{R}^{\mathrm{spat}}_{\bullet,1}.$$

We implicitly used the canonical embedding $\mathscr{W}_1^\star \hookrightarrow \mathscr{W}_0^\star$.

4.2.1 Error of interpolation

We investigate in this Section the error of interpolation with respect to the projector $\mathscr{P}^{\mathrm{spat}}_{\bullet,1}$ (considering $\mathscr{P}^{\mathrm{spat}}_{\bullet,0}$ as being the same). The obtained results will be used in Section 4.2.2 when we will show the consistency of the spatial operators of the scheme.

As one may expect, the error worsens as the regularity of the interpolated function decreases. Assumption (AP2) and Lemma 4.1.9 entail that Hölder spaces is a natural framework.

Lemma 4.2.1. *Assume (M). Let $p \in \mathring{\mathscr{T}}_{\bullet,0}$ and let $u \in \mathcal{C}^{k,\gamma}(I,\mathbb{C})$ for some $k \in \{0,1,2\}$ and $\gamma \in [0,1]$. There exists a constant $C_{I,u} > 0$ depending only on C_I and $\|u\|_{\mathcal{C}^{k,\gamma}(I,\mathbb{C})}$ such that*

$$\left\| \partial_r^\ell \left(\mathscr{I}^{\mathrm{spat}}_{S_p} u - u \right) \right\|_{L^\infty(I,\mathbb{C})} \leq C_{I,u} h_\bullet^{\max\{\min\{1,k-\ell\},\gamma\}} \qquad \forall\, 0 \leq \ell \leq k.$$

Proof. Assume $k = 0$ and let $r \in \omega_p$. We write

$$\begin{aligned} (\mathscr{I}^{\mathrm{spat}}_{S_p} u)(r) = &\left(\frac{(r-p)(r-p_+)}{(p_- - p)(p_- - p_+)} + \frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)} + \frac{(r-p_-)(r-p)}{(p_+-p_-)(p_+-p)} \right) u(p) \\ &+ \frac{(r-p)(r-p_+)}{(p_- - p)(p_- - p_+)} (u(p_-) - u(p)) + \frac{(r-p_-)(r-p)}{(p_+-p_-)(p_+-p)} (u(p_+) - u(p)). \end{aligned} \tag{4.9}$$

Since

$$\frac{(r-p)(r-p_+)}{(p_- - p)(p_- - p_+)} + \frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)} + \frac{(r-p_-)(r-p)}{(p_+-p_-)(p_+-p)} = 1$$

and $|u(r) - u(p)|$, $|u(p_\pm) - u(p)| \leq K_u h_\bullet^\gamma$ with $K_u > 0$ the Hölder constant of u, it comes

$$\left| ((\mathscr{I}_{\mathscr{B}} u) - u)(r) \right| \leq \left(1 + 2C_I\right) K_u h_\bullet^\gamma.$$

Assume now $k \geq 1$. Write for all $r \in \omega_p$

$$u(p_\pm) = \sum_{j=0}^{k} \frac{u^{(j)}(p)}{j!} (p_\pm - p)^j + (p_\pm - p)^k \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \Big(u^{(k)}\left(p + t\,(p_\pm - p)\right) - u^{(k)}(p) \Big) \mathrm{d}t$$

and

$$\left| \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \Big(u^{(k)}\left(p + t\,(p_\pm - p)\right) - u^{(k)}(p) \Big) \mathrm{d}t \right| \leq \frac{K_{u^{(k)}}}{(k+1)!} (p_\pm - p)^\gamma$$

with $K_{u^{(k)}} > 0$ the Hölder constant of $u^{(k)}$. Then

$$\begin{aligned}
(\mathscr{I}_{S_p}^{\mathrm{spat}} u)(r) &= \left(\frac{(r-p)(r-p_+)}{(p_- - p)(p_- - p_+)} + \frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)} + \frac{(r-p_-)(r-p)}{(p_+ - p_-)(p_+ - p)} \right) u(p) \\
&\quad + \sum_{j=1}^{k} \frac{1}{j!} \left(\frac{(r-p)(r-p_+)\,(p_- - p)^j}{(p_- - p_+)\,(p_- - p)} + \frac{(r-p_-)(r-p)\,(p_+ - p)^j}{(p_+ - p_-)\,(p_+ - p)} \right) u^{(j)}(p) \\
&\quad + \frac{(r-p)(r-p_+)(p_- - p)^k}{(p_- - p)(p_- - p_+)} \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \Big(u^{(k)}\left(p + t\,(p_- - p)\right) - u^{(k)}(p) \Big) \mathrm{d}t \\
&\quad + \frac{(r-p_-)(r-p)(p_+ - p)^k}{(p_+ - p_-)(p_+ - p)} \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \Big(u^{(k)}\left(p + t\,(p_+ - p)\right) - u^{(k)}(p) \Big) \mathrm{d}t \\
&= u(p) + \sum_{j=1}^{k} \frac{u^{(j)}(p)}{j!} (r-p)^j + \sum_{\pm} \pm \frac{(r-p)(r-p_\mp)(p_\pm - p)^k}{(p_\pm - p)(p_+ - p_-)} \times \text{integral term}.
\end{aligned} \tag{4.10}$$

For $\ell = 0$, we deduce

$$\begin{aligned}
\left| \big(\mathscr{I}_{S_p}^{\mathrm{spat}} u - u \big)(r) \right| &\leq \|u'\|_{L^\infty(I,\mathbb{C})} h_\bullet + \sum_{j=1}^{k} \|u^{(j)}\|_{L^\infty(I,\mathbb{C})} \frac{h_\bullet^j}{j!} + 2C_I K_{u^{(k)}} h_\bullet^{k+\gamma} \\
&\leq \|u\|_{\mathcal{C}^{k,\gamma}(I,\mathbb{C})} \left(\frac{5}{2} + 2C_I \right) h_\bullet.
\end{aligned}$$

whereas for $k = \ell = 1$, we get

$$\left| \partial_r \big(\mathscr{I}_{S_p}^{\mathrm{spat}} u - u \big)(r) \right| \leq K_{u'} \big(1 + 2C_I \big) h_\bullet^\gamma.$$

When $k = 2$ and $\ell = 1$, we find

$$\begin{aligned}
\left| \partial_r \big(\mathscr{I}_{S_p}^{\mathrm{spat}} u - u \big)(r) \right| &\leq 2\|u''\|_{L^\infty(I,\mathbb{C})} h_\bullet + \frac{2}{3} C_I K_{u''} h_\bullet^{1+\gamma} \\
&\leq 2\|u\|_{\mathcal{C}^{2,\gamma}(I,\mathbb{C})} \left(1 + \frac{C_I}{3} \right) h_\bullet
\end{aligned}$$

and finally, if $k = \ell = 2$, we have

$$\left| \partial_r^2 \big(\mathscr{I}_{S_p}^{\mathrm{spat}} u - u \big)(r) \right| \leq K_{u''} \left(1 + \frac{2C_I}{3} \right) h_\bullet^\gamma.$$

This completes the proof. □

We now turn to the general case of a L^2_ρ function. The main argument in the proof relies on a the construction of a suitable mesh which depends on the given function. This is the object of the next lemma.

Lemma 4.2.2. *Let $K = [a, b] \subset \mathbb{R}$, $\eta > 0$, $\eta \ll |K|$, and let $\mathscr{I}_\eta \subset K$ be a measurable subset such that $|\mathscr{I}_\eta| < \eta$. For all $\delta > 0$ (possibly much smaller than η), there exists a mesh $\mathscr{T}_\bullet$ depending only on K and $\mathscr{I}_\eta$ satisfying all the following properties:*

1. $K = \overline{\bigcup_{\omega \in \mathscr{T}_{\bullet,1}} \omega}$.
2. *For all $\omega \in \mathscr{T}_{\bullet,1}$, $\delta/9 \leq |\omega| \leq \delta$.*
3. *If $p \in \mathscr{T}_{\bullet,0} \cap \mathscr{I}_\eta$, then $\omega_p \subset \mathscr{I}_\eta$.*
4. $C_K := \max_{p \in \mathring{\mathscr{T}}_{\bullet,0}} \dfrac{p_+ - p_-}{|p_\pm - p|}$ *is uniformly bounded (in particular, it does not depend on η).*

Proof. We may assume that $\delta > 0$ is such that $\delta^{-1} =: N \in \mathbb{N}$ for we can always shrink it otherwise. Let $I_1 := [a, a + \delta/3]$, $I_2 := [a\delta/3, a + 2\delta/3], \ldots, I_{3N} := [b - \delta/3, b]$ so that

$$K = \bigcup_{j=1}^{3N} I_j, \qquad |I_j| = \delta/3.$$

For each I_j, define

$$I_j^- := \left[\min I_j, \frac{\min I_j + \max I_j}{2}\right], \qquad I_j^+ := \left[\frac{\min I_j + \max I_j}{2}, \max I_j\right].$$

We first choose the mesh points $\mathscr{T}_{\bullet,0}$. For all $j \in \{1, \ldots, 3N\}$ *odd*:

1. If $I_j \setminus \mathscr{I}_\eta \neq \emptyset$, then take any $p_j \in I_j \setminus \mathscr{I}_\eta$.
2. If $I_j \subset \mathscr{I}_\eta$, then take any point p_j in $\left[\frac{\min I_j + \max I_j}{3}, \frac{2(\min I_j + \max I_j)}{3}\right]$.

We now turn to the control intervals $\mathscr{T}_{\bullet,1}$. For all $p \in \mathscr{T}_{\bullet,0}$, we will write $\omega_p^\pm$ the left and right parts of ω_p, that is

$$\omega_p = \omega_p^- \cup \omega_p^+, \qquad \omega_p^\pm \subset \{p\} \cup \{r \in K \mid r \lesseqgtr p\}.$$

Now let $p, p_+ \in \mathscr{T}_{\bullet,0}$ with say $p \in I_j$ and $p_+ \in I_{j+2}$.

1. If $p, p_+ \notin \mathscr{I}_\eta$, then take $\omega_p^+ := \left[p, \frac{p+p_+}{2}\right[$ and $\omega_{p_+}^- := \left[\frac{p+p_+}{2}, p_+\right[$. Both sets ω_p^+ and $\omega_{p_+}^-$ have length greater or equal to $\delta/6$ (the equality occurring when $p = \max I_j$ and $p_+ = \min I_{j+2}$).
2. If $p \in \mathscr{I}_\eta$ and $p_+ \notin \mathscr{I}_\eta$, then take

$$\omega_p^+ := \left[p, \inf\{r \in [p, \max I_j + \delta/6] \mid [p, r[\in \mathscr{I}_\eta\}\right[,$$
$$\omega_{p_+}^- := \left]\inf\{r \in [p, \max I_j + \delta/6] \mid [p, r[\in \mathscr{I}_\eta\}, p_+\right].$$

   It follows that $\omega_p^+ \in \mathscr{I}_\eta$, $|\omega_p^+| \in [\delta/9, 5\delta/9]$ and $|\omega_{p_+}^-| \geq \delta/6$.

3. If $p \notin \mathscr{I}_\eta$ and $p_+ \in \mathscr{I}_\eta$, then take

$$\begin{aligned}\omega_p^+ &:= \big[p, \sup\{r \in [\min I_{j+2} - \delta/6, p_+] \;|]r, p_+] \in \mathscr{I}_\eta\}\big[,\\ \omega_{p_+}^- &:= \big]\sup\{r \in [\min I_{j+2} - \delta/6, p_+] \;|]r, p_+] \in \mathscr{I}_\eta\}, p_+\big].\end{aligned}$$

   It follows that $\omega_{p_+}^- \in \mathscr{I}_\eta$, $|\omega_{p_+}^-| \in [\delta/9, 5\delta/9]$ and $|\omega_p^+| \geq \delta/6$.

4. If both $p, p_+ \in \mathscr{I}_\eta$ (which implies that $I_j \subset \mathscr{I}_\eta$ and $I_{j+2} \subset \mathscr{I}_\eta$), then we distinguish two subcases:

   i) If $I_j^+ \cup I_{j+1} \cup I_{j+2}^- \subset \mathscr{I}_\eta$, then take again $\omega_p^+ := \Big[p, \frac{p+p_+}{2}\Big[$ and $\omega_{p_+}^- := \Big[\frac{p+p_+}{2}, p_+\Big[$; once again, both sets ω_p^+ and $\omega_{p_+}^-$ have length greater or equal to $\delta/6$.

   ii) Otherwise, there exists $p' \in I_{j+1} \setminus \mathscr{I}_\eta$: add p' to $\mathscr{T}_{\bullet,0}$. Then p, p' are in the case 2 and p', p_+ are in the case 3 above. We ensure that $|\omega_{p'}| \geq \delta/9$ by requiring

$$\begin{aligned}\omega_p^+ &:= \big[p, \sup\{r \geq p + \delta/18 \mid [p, r[\subset \mathscr{I}_\eta,\ |\omega_{p'}^-| \geq \delta/18\}\big[,\\ \omega_{p_+}^- &:= \big]\inf\{r \leq p_+ - \delta/18 \mid]r, p_+] \subset \mathscr{I}_\eta,\ |\omega_{p'}^+| \geq \delta/18\}, p_+\big],\\ \omega_{p'} &:=]p + \delta/18, p_+ - \delta/18[\setminus \big(\omega_p^+ \cup \omega_{p_+}^-\big).\end{aligned}$$

   This is possible since $\mathrm{dist}(p, I_{j+1}) \geq \delta/9$ and $\mathrm{dist}(p_+, I_{j+1}) \geq \delta/9$.

5. At last, we complete $\omega_{\underline{p}}$ and $\omega_{\overline{p}}$ by taking

$$\omega_{\underline{p}}^- :=]a, \underline{p}], \qquad \omega_{\overline{p}}^+ := [\underline{p}, b[.$$

   If $\underline{p} = a$ (respectively $\overline{p} = b$), then we take $[a', b'] \subset I$ containing K such that $\underline{p} > a'$ (respectively $\overline{p} < b'$); it is doable even if we have to decrease η.

By construction, the control intervals cover $\mathring{K}$ except at finite many points and if $p \in \mathscr{T}_{\bullet,0} \cap \mathscr{I}_\eta$, then $\omega_p \subset \mathscr{I}_\eta$. Furthermore, for each consecutive mesh points p, p', we have $|p, p'| \leq \delta$ and p, p' are separated by a distance greater or equal to $\delta/9$ (the worst situation occurring in the subcase 4. ii) above). This implies that

$$C_K := \sup_{\substack{p \in \mathscr{T}_{\bullet,0} \\ r \in \omega_p}} \frac{|r - p_\pm|}{|p - p_\pm|} = 9$$

uniformly in δ, η. Finally, it is clear that $|\omega_p| \leq \delta$. We also have $|\omega_p| \geq \delta/9$, the worst scenario again occurring in the case 4. ii), since p' can be so close to $\mathscr{I}_\eta$ that $|\omega_{p'}| = \delta/9 + \epsilon$ for some $\epsilon > 0$ arbitrarily small. This completes the proof.

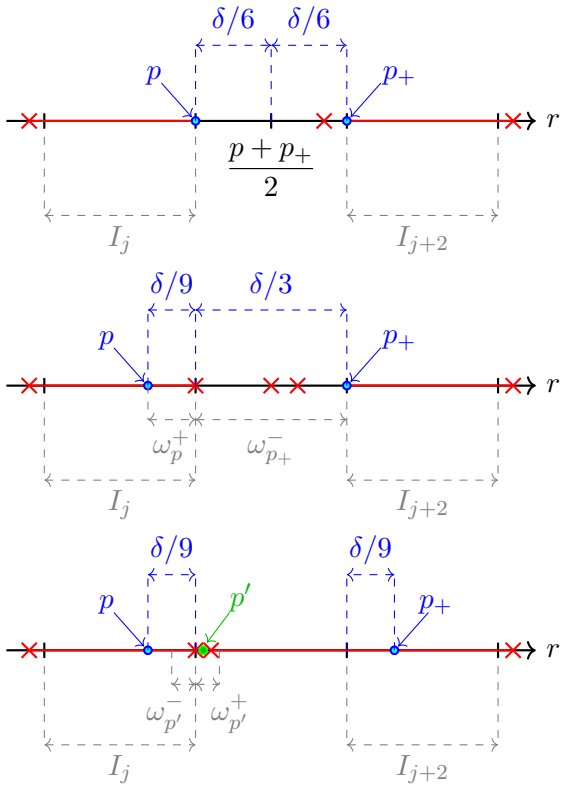

Figure 4.5: Examples of "bad" situations in the construction of $\mathscr{T}_\bullet$. The set $\mathscr{I}_\eta$ is drawn in red.

□

Lemma 4.2.3. *Let $u \in L^2_\rho$ and $K = [a,b] \subset I$. For all $\varepsilon > 0$, there exist a mesh $\mathscr{T}_\bullet$ depending on u and K, as well as a constant $C > 0$ depending on the mesh, $\|u\|_{L^2(K,\mathbb{C})}$ and $\|\rho\|_{L^1(K,\mathbb{C})}$ such that*

$$\left\| \mathscr{R}^{\mathrm{spat}}_{\bullet,1} \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u - u \right\|_{L^2_\rho(K,\mathbb{C})} \leq C\varepsilon.$$

Proof. Fix $\eta > 0$ such that $\eta \ll |K|$. We will show that there exists a subset $\mathscr{S}_\eta \subset K$ depending on K, u and η, satisfying $|\mathscr{S}_\eta| \leq 10\eta$ and such that

$$\left\| \mathscr{R}^{\mathrm{spat}}_{\bullet,1} \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u - u \right\|_{L^2_\rho(K,\mathbb{C})} \leq C \left\| \left((1 + |u|^2) \mathbb{1}_{\mathscr{S}_\eta} + \varepsilon^2 \right) \rho \right\|^{1/2}_{L^1(K,\mathbb{C})}$$

with $C > 0$ as in the hypotheses of the lemma. Letting $\eta \to 0$, we can make the right hand side above smaller than $C\varepsilon$ (if $\rho \in L^{1+\theta}(K,\mathbb{R})$ with $\theta > 0$, we can choose $\eta = \varepsilon^{\frac{1+\theta}{\theta}}/10$).

By standard measure theory arguments, we can construct a *continuous* function $v : I \to \mathbb{C}$ such that the set

$$\mathscr{I}_\eta := \left\{ r \in I \mid v(r) \neq u(r) \right\}$$

satisfies $|\mathscr{I}_\eta| \leq \eta$ (it is a consequence of Lusin's theorem, see *e.g.* the book of Evans & Gariepy [EvGa91], Theorem 1.15.). Since K is compact, Heine's theorem implies that there exists a constant $\delta > 0$ (possibly smaller than η) such that for all $r, r' \in K$

$$|r - r'| \leq \delta \qquad \Longrightarrow \qquad |v(r) - v(r')| \leq \varepsilon.$$

We now construct a mesh as in Lemma 4.2.2 (we can add to it any mesh point as in Remark 4.1.13 outside the compact set K, it does not matter since $u \equiv 0$ there). Observe that we can make it so that $\mathscr{T}_{\bullet,0} \subset \mathscr{L}(u)$ since Lebesgue's points of u form a full measure set in K. Moreover, the support of the interpolation of u is contained in K by the point 5. in the proof Lemma 4.2.2.

The strategy to get the announced estimate is to replace u by v when it is possible and use a continuity argument to get an estimate; otherwise, we use a rough bound for the integrand and integrate over ω_p which is sufficiently small in some sense. Using formula (4.9), we can write for all $p \in \mathscr{T}_{\bullet,0}$

$$(\mathscr{I}^{\mathrm{spat}}_{S_p} u)(r) = u(p) + \sum_{\pm} \pm \frac{(r-p)(r-p_\mp)}{(p_\pm - p)(p_+ - p_-)} \big(u(p_\pm) - u(p)\big).$$

It follows:

$$\begin{aligned}
\int_K \Big|\big(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}\mathscr{P}^{\mathrm{spat}}_{\bullet,1} u - u\big)(r)\Big|^2 \rho(r)\mathrm{d}r &= \sum_{p \in \mathscr{T}_{\bullet,0} \cap \mathscr{I}_\eta} \int_{\omega_p} \Big|\big(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}\mathscr{P}^{\mathrm{spat}}_{\bullet,1} u - u\big)(r)\Big|^2 \rho(r)\mathrm{d}r \\
&+ \sum_{p \in \mathscr{T}_{\bullet,0} \cap \mathscr{I}_\eta^{\complement}} \int_{\omega_p} \Big|\big(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}\mathscr{P}^{\mathrm{spat}}_{\bullet,1} u - u\big)(r)\Big|^2 \rho(r)\mathrm{d}r \\
&\leq 2 \sum_{p \in \mathscr{T}_{\bullet,0} \cap \mathscr{I}_\eta} \int_{\omega_p} |u(p) - u(r)|^2 \rho(r)\mathrm{d}r \\
&+ 4 \sum_{p \in \mathscr{T}_{\bullet,0} \cap \mathscr{I}_\eta} \int_{\omega_p} \sum_{\pm} \left|\frac{(r-p)(r-p_\mp)}{(p_\pm - p)(p_+ - p_-)} \big(u(p_\pm) - u(p)\big)\right|^2 \rho(r)\mathrm{d}r \\
&+ 2 \sum_{p \in \mathscr{T}_{\bullet,0} \cap \mathscr{I}_\eta^{\complement}} \int_{\omega_p} |u(p) - u(r)|^2 \rho(r)\mathrm{d}r \\
&+ 4 \sum_{p \in \mathscr{T}_{\bullet,0} \cap \mathscr{I}_\eta^{\complement}} \int_{\omega_p} \sum_{\pm} \left|\frac{(r-p)(r-p_\mp)}{(p_\pm - p)(p_+ - p_-)} \big(u(p_\pm) - u(p)\big)\right|^2 \rho(r)\mathrm{d}r \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

Estimate for I & II. The crucial point here is that since $p \in \mathscr{I}_\eta$, we have $\omega_p \subset \mathscr{I}_\eta$. Therefore

$$\begin{aligned}
\int_{\omega_p} |u(p) - u(r)|^2 \rho(r)\mathrm{d}r &= \int_{\omega_p \cap \mathscr{I}_\eta} |u(p) - u(r)|^2 \rho(r)\mathrm{d}r \\
&\leq 2\,|u(p)|^2 \int_{\omega_p \cap \mathscr{I}_\eta} \rho(r)\mathrm{d}r + 2 \int_{\omega_p \cap \mathscr{I}_\eta} |u(r)|^2 \rho(r)\mathrm{d}r \\
&\leq 2\big(1 + |u(p)|^2\big) \int_{\omega_p \cap \mathscr{I}_\eta} \big(1 + |u(r)|^2\big) \rho(r)\mathrm{d}r
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_{\omega_p} \sum_{\pm} \left| \frac{(r-p)(r-p_{\mp})}{(p_{\pm}-p)(p_+-p_-)} \big(u(p_{\pm})-u(p)\big) \right|^2 \rho(r)\mathrm{d}r \\
&= \sum_{\pm} \int_{\omega_p \cap \mathscr{I}_\eta} \left| \frac{(r-p)(r-p_{\mp})}{(p_{\pm}-p)(p_+-p_-)} \big(u(p_{\pm})-u(p)\big) \right|^2 \rho(r)\mathrm{d}r \\
&\leq 2C_K^2 \big(|u(p_-)|^2 + |u(p_+)|^2 + 2|u(p)|^2\big) \int_{\omega_p \cap \mathscr{I}_\eta} \rho(r)\mathrm{d}r.
\end{aligned}$$

Since $p \in \mathscr{L}(u)$, Lebesgue's differentiation theorem implies

$$\begin{aligned}
|u(p)|^2 &= \lim_{\epsilon \to 0^+} \fint_{]-\epsilon,\epsilon[} |u(p+\xi)|^2 \mathrm{d}\xi \\
&= \lim_{\epsilon \to 0^+} \fint_{]-1,1[} |u(p+t\epsilon)|^2 \mathrm{d}t \\
&\leq \frac{1}{2} \int_K |u(\xi)|^2 \mathbb{1}_{\omega_p \cap K}(\xi) \mathrm{d}\xi \\
&\leq \frac{1}{2} \int_K |u(\xi)|^2 \mathrm{d}\xi \\
&\leq \frac{1}{2} \|u\|^2_{L^2(K,\mathbb{C})}.
\end{aligned} \tag{4.11}$$

The above computation relies on the fact that for any $p \in \mathscr{T}_{\bullet,0}$, we have $|u(p+\bullet)|\mathbb{1}_{[-\epsilon,\epsilon]} \leq |u|\mathbb{1}_K$ for $\epsilon \ll 1$. We deduce that

$$\begin{aligned}
&2 \int_{\omega_p \cap \mathscr{I}_\eta} |u(p)-u(r)|^2 \rho(r)\mathrm{d}r + 4 \int_{\omega_p \cap \mathscr{I}_\eta} \sum_{\pm} \left| \frac{(r-p)(r-p_{\mp})}{(p_{\pm}-p)(p_+-p_-)} \big(u(p_{\pm})-u(p)\big) \right|^2 \rho(r)\mathrm{d}r \\
&\leq 2\big(2 + \|u\|^2_{L^2(K,\mathbb{C})}\big) \big\| \big(1+|u|^2\big) \rho \mathbb{1}_{\omega_p \cap \mathscr{I}_\eta} \big\|_{L^1(K,\mathbb{C})} + 16 C_K^2 \|u\|^2_{L^2(K,\mathbb{C})} \big\| \rho \mathbb{1}_{\omega_p \cap \mathscr{I}_\eta} \big\|_{L^1(K,\mathbb{C})}
\end{aligned}$$

whence

$$\begin{aligned}
&2 \sum_{p \in \mathscr{T}_{\bullet,0} \cap \mathscr{I}_\eta} \int_{\omega_p} |u(p)-u(r)|^2 \rho(r)\mathrm{d}r + 4 \sum_{p \in \mathscr{T}_{\bullet,0} \cap \mathscr{I}_\eta} \int_{\omega_p} \sum_{\pm} \left| \frac{(r-p)(r-p_{\mp})}{(p_{\pm}-p)(p_+-p_-)} \big(u(p_{\pm})-u(p)\big) \right|^2 \rho(r)\mathrm{d}r \\
&\leq 2\big(2 + \|u\|^2_{L^2(K,\mathbb{C})}\big) \big\| \big(1+|u|^2\big) \rho \mathbb{1}_{\mathscr{I}_\eta} \big\|_{L^1(K,\mathbb{C})} + 16 C_K^2 \|u\|^2_{L^2(K,\mathbb{C})} \big\| \rho \mathbb{1}_{\mathscr{I}_\eta} \big\|_{L^1(K,\mathbb{R})}.
\end{aligned} \tag{4.12}$$

Estimate for III. Since $p \in \mathscr{I}_\eta^{\complement}$, we can write

$$\int_{\omega_p} |u(p)-u(r)|^2 \rho(r)\mathrm{d}r = \int_{\omega_p \cap \mathscr{I}_\eta} |u(p)-u(r)|^2 \rho(r)\mathrm{d}r + \int_{\omega_p \cap \mathscr{I}_\eta^{\complement}} |v(p)-v(r)|^2 \rho(r)\mathrm{d}r.$$

On the one hand, we have

$$\int_{\omega_p \cap \mathscr{I}_\eta} |u(p)-u(r)|^2 \rho(r)\mathrm{d}r \leq 2\,|u(p)|^2 \int_{\omega_p \cap \mathscr{I}_\eta} \rho(r)\mathrm{d}r + 2 \int_{\omega_p \cap \mathscr{I}_\eta} |u(r)|^2 \rho(r)\mathrm{d}r$$

and thus (4.11) gives

$$\int_{\omega_p\cap\mathscr{I}_\eta} |u(p)-u(r)|^2\,\rho(r)\mathrm{d}r \le \big(2+\|u\|^2_{L^2(K,\mathbb{C})}\big)\big\|(1+|u|)\rho\mathbb{1}_{\omega_p\cap\mathscr{I}_\eta}\big\|_{L^1(K,\mathbb{C})}.$$

On the other hand, using the uniform continuity of v on K as well as the fact that $|p-r|<\delta$ for all $r\in\omega_p$, we find

$$\int_{\omega_p\cap\mathscr{I}_\eta^{\complement}} |v(p)-v(r)|^2\,\rho(r)\mathrm{d}r \le \varepsilon^2\|\rho\mathbb{1}_{\omega_p}\|_{L^1(K,\mathbb{C})}.$$

As a consequence, we obtain:

$$\sum_{p\in\mathscr{T}_{\bullet,0}\cap\mathscr{I}_\eta^{\complement}}\int_{\omega_p} |u(p)-u(r)|^2\,\rho(r)\mathrm{d}r \le \big(2+\|u\|^2_{L^2(K,\mathbb{C})}\big)\big\|\big(1+|u|^2\big)\rho\mathbb{1}_{\mathscr{I}_\eta}\big\|_{L^1(K,\mathbb{C})} + \|\rho\|_{L^1(K,\mathbb{C})}\varepsilon^2. \tag{4.13}$$

Estimate for IV**.** Write

$$\int_{\omega_p}\sum_{\pm}\left|\frac{(r-p)(r-p_\mp)}{(p_\pm-p)(p_+-p_-)}\big(u(p_\pm)-u(p)\big)\right|^2\rho(r)\mathrm{d}r \le C_K^2\sum_{\pm}\int_{\omega_p}\big|\big(u(p_\pm)-u(p)\big)\big|^2\rho(r)\mathrm{d}r.$$

There is a difficulty here because we do not know whether or not $p_\pm\in\mathscr{I}_\eta$. It can be overcome in the following way: let

$$\mathscr{A} := \big\{p\in\mathscr{T}_{\bullet,0}\mid p_-\in\mathscr{I}_\eta \text{ or } p_+\in\mathscr{I}_\eta\big\}, \qquad n_{\mathscr{A}} := \mathrm{card}(\mathscr{A}).$$

Since all the control intervals has a length greater or equal to $\delta/9$, for each $p\in\mathscr{A}$, a contribution of at least $\delta/9$ (and at most 2δ if both $\omega_{p_\pm}$ are in $\mathscr{I}_\eta$ and have a length equal to δ) in the length of $\mathscr{I}_\eta$ comes from ω_{p_-} or (and) ω_{p_+}. For $n_{\mathscr{A}}$ points in $\mathscr{A}$, we thus have a contribution of at least $2n_{\mathscr{A}}\delta/9$ (and at most $2n_{\mathscr{A}}\delta$). This means that

$$n_{\mathscr{A}} \le \frac{9\eta}{\delta}.$$

Therefore, if $p_\pm\in\mathscr{I}_\eta$, then we have using (4.11)

$$\int_{\omega_p}\big|\big(u(p_\pm)-v(p)\big)\big|^2\rho(r)\mathrm{d}r \le \|u\|^2_{L^2(K,\mathbb{C})}\|\rho\mathbb{1}_{\omega_p}\|_{L^1(K,\mathbb{C})}$$

and thus

$$\sum_{p\in\mathscr{T}_{\bullet,0}\cap\mathscr{I}_\eta^{\complement}\cap\mathscr{A}}\int_{\omega_p}\sum_{\pm}\left|\frac{(r-p)(r-p_\mp)}{(p_\pm-p)(p_+-p_-)}\big(u(p_\pm)-u(p)\big)\right|^2\rho(r)\mathrm{d}r \le 2C_K^2\|u\|^2_{L^2(K,\mathbb{C})}\|\rho\mathbb{1}_{\bigsqcup_{p\in\mathscr{A}}\omega_p}\|_{L^1(K,\mathbb{R})} \tag{4.14}$$

with

$$\left|\bigsqcup_{p\in\mathscr{A}}\omega_p\right| = \sum_{p\in\mathscr{A}}|\omega_p| \le n_{\mathscr{A}}\delta \le 9\eta.$$

In the simpler case where $p_\pm \in \mathscr{I}_\eta^{\complement}$, we get

$$\sum_{p\in\mathscr{T}_{\bullet,0}\cap\mathscr{I}_\eta^{\complement}\cap\mathscr{A}^{\complement}}\int_{\omega_p}\sum_{\pm}\left|\frac{(r-p)(r-p_\mp)}{(p_\pm-p)(p_+-p_-)}\big(u(p_\pm)-u(p)\big)\right|^2\rho(r)\mathrm{d}r \le 2C_K^2\varepsilon^2\|\rho\|_{L^1(K,\mathbb{C})}. \tag{4.15}$$

Combining (4.14) and (4.15) then yields

$$\begin{aligned}&\sum_{p\in\mathscr{T}_{\bullet,0}\cap\mathscr{I}_\eta^{\complement}}\int_{\omega_p}\sum_{\pm}\left|\frac{(r-p)(r-p_\mp)}{(p_\pm-p)(p_+-p_-)}\big(u(p_\pm)-u(p)\big)\right|^2\rho(r)\mathrm{d}r\\ &\le 2C_K^2\left(\|u\|_{L^2(K,\mathbb{C})}^2\|\rho\mathbb{1}_{\bigsqcup_{p\in\mathscr{A}}\omega_p}\|_{L^1(K,\mathbb{R})}+\varepsilon^2\|\rho\|_{L^1(K,\mathbb{C})}\right).\end{aligned} \tag{4.16}$$

Conclusion. Collecting the estimates (4.12), (4.13) and (4.16), we finally get:

$$\begin{aligned}\big\|\mathscr{R}_{\bullet,1}^{\mathrm{spat}}\mathscr{P}_{\bullet,1}^{\mathrm{spat}}u-u\big\|_{L^2_\rho(K,\mathbb{C})}^2 &\le 3\big(2+\|u\|_{L^2(K,\mathbb{C})}^2\big)\big\|\big(1+|u|^2\big)\rho\mathbb{1}_{\mathscr{I}_\eta}\big\|_{L^1(K,\mathbb{C})}\\ &\quad+16C_K^2\|u\|_{L^2(K,\mathbb{C})}^2\big\|\rho\mathbb{1}_{\mathscr{I}_\eta}\big\|_{L^1(K,\mathbb{R})}\\ &\quad+\big(1+2C_K^2\big)\varepsilon^2\|\rho\|_{L^1(K,\mathbb{C})}\\ &\quad+2C_K^2\|u\|_{L^2(K,\mathbb{C})}^2\|\rho\mathbb{1}_{\bigsqcup_{p\in\mathscr{A}}\omega_p}\|_{L^1(K,\mathbb{R})}.\end{aligned}$$

We then take:

$$C^2:=7+2C_K^2+\big(3+18C_K^2\big)\|u\|_{L^2(K,\mathbb{C})}^2,\qquad \mathscr{S}_\eta:=\mathscr{I}_\eta\cup\left(\bigsqcup_{p\in\mathscr{A}}\omega_p\right).$$

The proof is complete. □

We then deduce two results which will be useful for the proof of the consistency of the norms (Proposition 4.2.8) and the Hamiltonians (Proposition 4.2.2).

Corollary 4.2.4. *Let $T>0$ and $u=(u_0,u_1)\in\mathscr{W}_1$ be the solution of (ADP). Note $K_T:=\cup_{0\le t\le T}\operatorname{Supp}u(t,\cdot)$ (it is a compact subset of I by Assumption (AP7)) and pick*[1] *$\mathscr{U}_T\subset I$ containing K_T.*

For all $\varepsilon>0$, there exist a mesh $\mathscr{T}_\bullet$ depending on u and K, as well as a constant $C>0$ depending on the mesh, $\|u\|_{H^2(\mathscr{U}_T,\mathbb{C})}$ and $\|\rho\|_{L^1(\mathscr{U}_T,\mathbb{C})}$ such that

$$\big\|\mathscr{P}_{\bullet,0}^{\mathrm{spat}}\big(\partial_r^2u-(\partial_r^\star)^2\mathscr{R}_{\bullet,1}^{\mathrm{spat}}\mathscr{P}_{\bullet,1}^{\mathrm{spat}}u\big)\big\|_{L^2_\rho}\le C\varepsilon, \tag{4.17}$$

$$\big\|\mathscr{P}_{\bullet,0}^{\mathrm{spat}}\big(\partial_ru-(\partial_r^\star)\mathscr{R}_{\bullet,1}^{\mathrm{spat}}\mathscr{P}_{\bullet,1}^{\mathrm{spat}}u\big)\big\|_{L^2_\rho}\le C\varepsilon, \tag{4.18}$$

$$\big\|\partial_r^\star((Hu)_0-\mathscr{R}_{\bullet,0}^{\mathrm{spat}}\mathscr{P}_{\bullet,0}^{\mathrm{spat}}(Hu)_0\big)\big\|_{L^2_\rho}\le C\varepsilon. \tag{4.19}$$

[1] The open set $\mathscr{U}_T$ is only needed to use Sobolev and Morrey's embeddings in Section 4.2.2.

Proof. The argument is similar to the one used to prove Lemma (4.2.3). Fix $\eta > 0$ and build two continuous functions v_1 and v_2 using Theorem 1.15 of [EvGa91] such that

$$\mathscr{I}_\eta = \bigcup_{k=1}^{2} \mathscr{I}_\eta^k, \qquad \mathscr{I}_\eta^1 = \{r \in I \mid \partial_r^2 u(r) \neq v_1(r)\}, \quad \mathscr{I}_\eta^2 = \{r \in I \mid (Hu)_0(r) \neq v_2(r)\}$$

We may as well assume that $K_T = [a, b]$ for we can always find $a, b \in I$ such that $K_T \subset [a, b] \subset I$ otherwise. Furthermore, we can ask for $\mathscr{T}_{\bullet,0} \subset \mathscr{L}(u) \cap \mathscr{L}(u') \cap \mathscr{L}(u'') \cap \mathscr{L}((Hu)_0')$ as Lebesgue's points of measurable functions form a set of full measure.

We start with the estimate (4.17) (estimate (4.18) is obtained is a similar way). Using formula (4.10) available at the Lebesgue's points of u'', we can write for all $r \in I$

$$\begin{aligned}
&\mathscr{P}_{\bullet,0}^{\mathrm{spat}} \left(\partial_r^2 u - (\partial_r^\star)^2 \mathscr{R}_{\bullet,1}^{\mathrm{spat}} \mathscr{P}_{\bullet,1}^{\mathrm{spat}} u\right)(r) \\
&= \sum_{p \in \mathscr{T}_{\bullet,0}} \left[\sum_{\pm} \pm \frac{(r-p)(r-p_\mp)}{(p_\pm - p)(p_+ - p_-)} \big(u''(p_\pm) - u''(p)\big) \right. \\
&\qquad \left. + \sum_{\pm} \pm \frac{p_\pm - p}{p_+ - p_-} \int_0^1 (1-t) \Big(u''\big(p + t(p_\pm - p)\big) - u''(p)\Big) \mathrm{d}t \right] \mathbb{1}_{\omega_p}(r).
\end{aligned}$$

The first term on the right hand side is dealt in the same way as in the proof of Lemma 4.2.3 (see estimates (4.12) and (4.16)): we find

$$\begin{aligned}
\sum_{p \in \mathscr{T}_{\bullet,0}} \int_{\omega_p} \sum_{\pm} \left| \frac{(r-p)(r-p_\mp)}{(p_\pm - p)(p_+ - p_-)} \big(u(p_\pm) - u(p)\big) \right|^2 \rho(r) \mathrm{d}r &\leq 18 C_K^2 \|u''\|_{L^2(K,\mathbb{C})}^2 \big\|\rho \mathbb{1}_{\mathscr{S}_\eta}\big\|_{L^1(K,\mathbb{R})} \\
&\quad + 2 C_K^2 \varepsilon^2 \|\rho\|_{L^1(K,\mathbb{C})}
\end{aligned}$$

where $\mathscr{S}_\eta \subset K$ is a measurable set satisfying $|\mathscr{S}_\eta| \leq 10\eta$.

We now deal with the remaining terms. For all $p \in \mathscr{T}_{\bullet,0}$, Jensen's inequality applied to the convex function $x \mapsto x^2$ gives

$$\left| \int_0^1 (1-t) \Big(u''\big(p + t(p_\pm - p)\big) - u''(p)\Big) \mathrm{d}t \right|^2 \leq \int_0^1 \left| u''\big(p + t(p_\pm - p)\big) - u''(p) \right|^2 \mathrm{d}t.$$

If $p \in \mathscr{I}_\eta$ (which means by Lemma 4.2.2 that $\omega_p \subset \mathscr{I}_\eta$) then we can use (4.11) to get the rough bound

$$\int_0^1 \left| u''\big(p + t(p_\pm - p)\big) - u''(p) \right|^2 \mathrm{d}t \leq 2 \|u'' \mathbb{1}_{\pm[p,p_\pm]}\|_{L^2(K,\mathbb{C})}^2 + 2|u''(p)|^2 \leq 3 \|u''\|_{L^2(K,\mathbb{C})}^2$$

because $\|\rho \mathbb{1}_{\omega_p}\|_{L^1(K,\mathbb{R})} = \|\rho \mathbb{1}_{\omega_p \cap \mathscr{I}_\eta}\|_{L^1(K,\mathbb{R})}$ becomes small when η goes to 0 (by convention,

$\pm[c,d] := [d,c]$ for all real numbers c, d). Otherwise, if $p \in \mathscr{I}_\eta^{\complement}$, then write

$$\begin{aligned}\int_0^1 \Big|u''\big(p+t(p_\pm-p)\big)-u''(p)\Big|^2\mathrm{d}t &= \fint_{\pm[0,p_\pm-p]}\Big|u''(p+\xi)-u''(p)\Big|^2\mathrm{d}\xi\\ &= \fint_{\pm[0,p_\pm-p]}\Big|u''(p+\xi)-u''(p)\Big|^2\mathbb{1}_{\mathscr{I}_\eta}(\xi)\mathrm{d}\xi\\ &+ \fint_{\pm[0,p_\pm-p]}\Big|v(p+\xi)-v(p)\Big|^2\mathbb{1}_{\mathscr{I}_\eta^{\complement}}(\xi)\mathrm{d}\xi\\ &\le 2\|u''\mathbb{1}_{\pm[p,p_\pm]\cap\mathscr{I}_\eta}\|^2_{L^2(K,\mathbb{C})}+\|u''\|^2_{L^2(K,\mathbb{C})}\big|\pm[p,p_\pm]\cap\mathscr{I}_\eta\big|\\ &+\varepsilon^2\big|\pm[p,p_\pm]\cap\mathscr{I}_\eta^{\complement}\big|.\end{aligned}$$

We finally sum up all the above estimates over $p \in \mathscr{T}_{\bullet,0}$: it comes

$$\begin{aligned}&\big\|\mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big(\partial_r^2u_0-(\partial_r^\star)^2\mathscr{R}^{\mathrm{spat}}_{\bullet,1}\mathscr{P}^{\mathrm{spat}}_{\bullet,1}u_0\big)\big\|^2_{L^2_\rho}\\ &\le 18C_K^2\|u''\|^2_{L^2(K,\mathbb{C})}\big\|\rho\mathbb{1}_{\mathscr{S}_\eta}\big\|_{L^1(K,\mathbb{R})}+2C_K^2\varepsilon^2\|\rho\|_{L^1(K,\mathbb{C})}\\ &+2C_K^2\left(3\|u''\|^2_{L^2(K,\mathbb{C})}\|\rho\mathbb{1}_{\mathscr{I}_\eta}\|_{L^1(K,\mathbb{R})}+\Big(2\|u''\mathbb{1}_{\mathscr{I}_\eta}\|^2_{L^2(K,\mathbb{C})}+\|u''\|^2_{L^2(K,\mathbb{C})}\big|\mathscr{I}_\eta\big|+\varepsilon^2\Big)\|\rho\|_{L^1(K,\mathbb{R})}\right).\end{aligned}$$

Recall that the constant C_K is uniformly bounded by Lemma 4.2.2. As η can be chosen as small as desired to make all the terms containing $\mathscr{I}_\eta$ and $\mathscr{S}_\eta$ lesser than ε, we can get the announced estimate by taking

$$C := 4C_K^2\Big(6\|u''\|^2_{L^2(K,\mathbb{C})}+\|\rho\|_{L^1(K,\mathbb{R})}\Big). \tag{4.20}$$

We now turn to the estimate (4.19). Using again formula (4.10) for $w := (Hu)_0$, we can write for all $p \in \mathscr{T}_{\bullet,0}$ and $r \in \omega_p \cap \mathscr{L}(w')$:

$$\begin{aligned}\partial_r^\star\big(w-(\partial_r^\star)^2\mathscr{R}^{\mathrm{spat}}_{\bullet,1}\mathscr{P}^{\mathrm{spat}}_{\bullet,1}w\big)(r) &= w'(p)-w'(r)\\ &+\sum_{\pm}\pm\frac{(r-p)+(r-p_\mp)}{p_+-p_-}\int_0^1(1-t)\Big(w'\big(p+t(p_\pm-p)\big)-w'(p)\Big)\mathrm{d}t.\end{aligned}$$

Since $w' \in L^2_\rho$, we can apply Lemma 4.2.3 for the difference $w'(p)-w'(r)$. As for the other terms, we can imitate the proof of the estimate (4.17) above with u'' replaced by w'. The proof is complete. □

To end this Section, we show a simple yet useful identity involving the operator $M_\bullet^{-1}$.

Lemma 4.2.5. *Let $u = (u_0, u_1) \in \mathscr{W}_H^\star$. Then for any mesh $\mathscr{T}_\bullet$ such that $\mathscr{T}_{\bullet,0} \subset \mathscr{L}(u)$, we have*

$$\mathscr{R}^{\mathrm{spat}}_{\bullet,1}M_\bullet^{-1}\mathscr{P}^{\mathrm{spat}}_{\bullet,0}u = \mathscr{P}^{\mathrm{spat}}_{\bullet,0}u \qquad \text{in } \mathbb{C}[X].$$

Proof. Let $j \in \{0,1\}$ and choose a mesh $\mathscr{T}_\bullet$ as in Remark 4.1.13. Since $M_\bullet^{-1}\mathscr{P}^{\mathrm{spat}}_{\bullet,0}u_j \in \mathscr{P}^{\mathrm{spat}}_{\bullet,1}\mathscr{W}^\star$, there exists (at least one) $v \in \mathscr{W}^\star$ such that $M_\bullet\mathscr{P}^{\mathrm{spat}}_{\bullet,1}v_j = \mathscr{P}^{\mathrm{spat}}_{\bullet,0}u_j$ in $\mathscr{W}^{\mathrm{spat}}_{\bullet,0}$. Recalling that $\mathscr{W}^\star \subset \mathscr{W}_H^\star$ (so that v can be seen as an element of $\mathscr{W}_H^\star$) and $M_\bullet = \mathscr{P}^{\mathrm{spat}}_{\bullet,0}\mathscr{R}^{\mathrm{spat}}_{\bullet,1}$, we see that

$$\mathscr{P}^{\mathrm{spat}}_{\bullet,0}u_j = M_\bullet\mathscr{P}^{\mathrm{spat}}_{\bullet,1}v_j = \mathscr{P}^{\mathrm{spat}}_{\bullet,0}\mathscr{R}^{\mathrm{spat}}_{\bullet,1}\mathscr{P}^{\mathrm{spat}}_{\bullet,1}v_j = \mathscr{P}^{\mathrm{spat}}_{\bullet,0}v_j \qquad \text{in } \mathbb{C}[X].$$

It follows that $u_j = v_j$ on the mesh points $\mathscr{T}_{\bullet,0}$ (this is exactly what means $\mathscr{M} = \mathbb{1}_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}}$ in the paragraph 4.1.3). As $\mathscr{P}^{\mathrm{spat}}_{\bullet,j} v$ is determined by the values of v at the mesh points, we get

$$\mathscr{P}^{\mathrm{spat}}_{\bullet,0} u_j = \mathscr{P}^{\mathrm{spat}}_{\bullet,0} v_j = \mathscr{P}^{\mathrm{spat}}_{\bullet,1} v_j = M_\bullet^{-1} \mathscr{P}^{\mathrm{spat}}_{\bullet,0} u_j \qquad \text{in } \mathbb{C}[X]$$

which completes the proof. □

4.2.2 Consistency

We investigate in this Section the consistency of the spatial scheme for solutions of (ADP). All the bounds will implicitly depend on the maximal existence time T via the support of the considered solutions. If one wish to get uniformity, then one need to make more assumptions; asking solutions to be in $H^2(I,\mathbb{C}) \times H^1(I,\mathbb{C})$ for all time $t \geq 0$ is enough (see Corollary 4.2.8 and Corollary 4.2.11). Moreover, getting a convergence rate costs another derivative for the first component of a solution (see Corollary 4.2.11).

Consistency of the norms

We prove in this paragraph the consistency of $\widetilde{\mathcal{N}}_\bullet$ with $|||.|||_{\dot{\mathcal{E}}_\rho}$. As the speed of convergence depends on the regularity of the argument in the norm, we distinguish two cases.

Proposition 4.2.6. *Let $u = (u_0, u_1) \in \mathcal{C}_{\mathrm{c}}^{1,\gamma}(I,\mathbb{C}) \times \mathcal{C}_{\mathrm{c}}^{0,\gamma'}(I,\mathbb{C})$ for some $\gamma, \gamma' \in [0,1]$ such that that*

$$Qu_0 \in \mathcal{C}_{\mathrm{c}}^{0,\gamma}(I,\mathbb{C}). \tag{4.21}$$

There exist a mesh $\mathscr{T}_\bullet$ satisfying (M) and a constant $C_{I,u,Q} > 0$ depending only on the constant C_I of Assumption (M), $\|u\|_{\mathcal{C}_{\mathrm{c}}^{1,\gamma}(I,\mathbb{C})\times\mathcal{C}_{\mathrm{c}}^{0,\gamma'}(I,\mathbb{C})}$ and $\|Qu_0\|_{\mathcal{C}_{\mathrm{c}}^{0,\gamma}(I,\mathbb{C})}$ such that

$$\left|\widetilde{\mathcal{N}}_\bullet(u) - |||u|||_{\dot{\mathcal{E}}_\rho}\right| \leq C_{I,u,Q}\Big(h_\bullet^\gamma \|a\rho\mathbb{1}_{\omega_\bullet}\|_{L^1}^{1/2} + h_\bullet \|c\rho\mathbb{1}_{\omega_\bullet}\|_{L^1}^{1/2} + h_\bullet^{\min\{\gamma,\gamma'\}} \|\rho\mathbb{1}_{\omega_\bullet}\|_{L^1}^{1/2}\Big).$$

Proof. Choose $\mathscr{T}_\bullet$ as in Remark 4.1.13 and such that the supports of u_0 and u_1 are contained in $\omega_\bullet$. By definition, we have

$$\begin{aligned}
\widetilde{\mathcal{N}}_\bullet(u) - |||u|||_{\dot{\mathcal{E}}} &= \Big(\big\langle P^\star \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0, \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0\big\rangle_{L^2_\rho}^{1/2} - \langle Pu_0, u_0\rangle_{L^2_\rho}^{1/2}\Big) \\
&\quad + \Big(\big\|\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_1 - \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} Qu_0\big\|_{L^2_\rho} - \big\|u_1 - Qu_0\big\|_{L^2_\rho}\Big) \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Bound for the term I. Let us write

$$\begin{aligned}
\Big|\big\langle P^\star \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0, \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0\big\rangle_{L^2_\rho}^{1/2} - \langle Pu_0, u_0\rangle_{L^2_\rho}^{1/2}\Big| &\leq \sqrt{\Big|\big\langle P^\star \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0, \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0\big\rangle_{L^2_\rho} - \langle Pu_0, u_0\rangle_{L^2_\rho}\Big|} \\
&\leq \Big|\big\langle P^\star\big(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0 - u_0\big), \big(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0 - u_0\big)\big\rangle_{L^2_\rho}\Big|^{1/2} \\
&\quad + \Big|\big\langle P^\star u_0, \big(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0 - u_0\big)\big\rangle_{L^2_\rho}\Big|^{1/2} \\
&\quad + \Big|\big\langle P^\star\big(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1} u_0 - u_0\big), u_0\big\rangle_{L^2_\rho}\Big|^{1/2} \\
&=: A + B + C.
\end{aligned}$$

(i) We first deal with the term A. Since u_0 is compactly supported, we can integrate by parts using Remark 4.1.2 to get

$$A^2 = -\sum_{p\in\mathscr{T}_{\bullet,0}}\int_{\omega_p} a(r)\left|\partial_r\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0-u_0\right)(r)\right|^2(r)\rho(r)\mathrm{d}r + \sum_{p\in\mathscr{T}_{\bullet,0}}\int_{\omega_p} c(r)\left|\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0-u_0\right)(r)\right|^2\rho(r)\mathrm{d}r.$$

Using Hölder inequalities, Lemma 4.2.1 and Assumption (M), we obtain

$$\begin{aligned}\left|A^2\right| &\le \sum_{p\in\mathscr{T}_{\bullet,0}}\int_{\omega_p}|a(r)|\left|\partial_r\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0-u_0\right)(r)\right|^2\rho(r)\mathrm{d}r\\ &+\sum_{p\in\mathscr{T}_{\bullet,0}}\int_{\omega_p}|c(r)|\left|\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0-u_0\right)(r)\right|^2\rho(r)\mathrm{d}r\\ &\le \sup_{p\in\mathscr{T}_{\bullet,0}}\left(\left\|\left(\partial_r\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0-u_0\right)\right)\mathbb{1}_{\omega_p}\right\|_{L^\infty}\right)^2\sum_{p\in\mathscr{T}_{\bullet,0}}\|a\rho\mathbb{1}_{\omega_p}\|_{L^1}\\ &+\sup_{p\in\mathscr{T}_{\bullet,0}}\left(\left\|\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0-u_0\right)\mathbb{1}_{\omega_p}\right\|_{L^\infty}\right)^2\sum_{p\in\mathscr{T}_{\bullet,0}}\|c\rho\mathbb{1}_{\omega_p}\|_{L^1}\\ &\le C_{I,u}\left(h_\bullet^{2\gamma}\|a\rho\mathbb{1}_{\omega_\bullet}\|_{L^1(I,\mathbb{R})}+h_\bullet^2\|c\rho\mathbb{1}_{\omega_\bullet}\|_{L^1(I,\mathbb{R})}\right)\end{aligned}$$

where $C_{I,u}>0$ only depends on the Hölder norm of u_0 and of C_I. We thus have

$$|A|\le C_{I,u}\left(h_\bullet^{\gamma}\|a\rho\mathbb{1}_{\omega_\bullet}\|_{L^1(I,\mathbb{R})}^{1/2}+h_\bullet\|c\rho\mathbb{1}_{\omega_\bullet}\|_{L^1(I,\mathbb{R})}^{1/2}\right)$$

where $C_{I,u}>0$ is another constant depending on the Hölder norm of u_0 and of C_I.

(ii) We now deal with the term B. We have

$$B^2 = -\sum_{p\in\mathscr{T}_{\bullet,0}}\int_{\omega_p} a(r)\overline{\left(\partial_r u_0\right)(r)}\left(\partial_r\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0-u_0\right)\right)(r)\rho(r)\mathrm{d}r + \sum_{p\in\mathscr{T}_{\bullet,0}}\int_{\omega_p} c(r)\overline{u_0(r)}\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0-u_0\right)(r)\rho(r)\mathrm{d}r.$$

Using again Hölder inequalities, Lemma 4.2.1 and Assumption (M), one obtains

$$\begin{aligned}\left|B^2\right| &\le \sum_{p\in\mathscr{T}_{\bullet,0}}\int_{\omega_p}|a(r)|\left|\left(\partial_r u_0\right)(r)\right|\left|\left(\partial_r\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0-u_0\right)\right)(r)\right|\rho(r)\mathrm{d}r\\ &+\sum_{p\in\mathscr{T}_{\bullet,0}}\int_{\omega_p}|c(r)|\left|u_0(r)\right|\left|\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0-u_0\right)(r)\right|\rho(r)\mathrm{d}r.\end{aligned}$$

and thus

$$|B|\le C_{I,u}\|u\|_{\mathcal{C}^{1,\gamma}(I,\mathbb{C})}\left(h_\bullet^{\gamma/2}\|a\rho\mathbb{1}_{\omega_\bullet}\|_{L^1(I,\mathbb{R})}^{1/2}+h_\bullet^{1/2}\|c\rho\mathbb{1}_{\omega_\bullet}\|_{L^1(I,\mathbb{R})}^{1/2}\right).$$

(*iii*) We finally bound the term C. Since

$$C = \sum_{p\in\mathscr{T}_{\bullet,0}} \big\langle P\big(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0 - u_0\big), u_0\big\rangle_{L^2_\rho(\omega_p)} = \sum_{p\in\mathscr{T}_{\bullet,0}} \overline{\big\langle Pu_0, \big(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0 - u_0\big)\big\rangle_{L^2_\rho(\omega_p)}},$$

we clearly have $|C| = |B|$.

Bound for the term II. Let us write

$$\begin{aligned}\big\|\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_1 - \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}Qu_0\big\|_{L^2_\rho} \le \big\|\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_1 - u_1\big\|_{L^2_\rho} + \big\|u_1 - Qu_0\big\|_{L^2_\rho}\\ + \big\|\left(Qu_0 - \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}Qu_0\right)\big\|_{L^2_\rho}.\end{aligned}$$

By Lemma 4.2.1 and Assumption (M), we have

$$\big\|\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_1 - u_1\big\|_{L^2_\rho} \le C_{I,u}h_\bullet^{\gamma'}\|\rho\mathbb{1}_{\omega_\bullet}\|_{L^1}^{1/2}$$

and (using that $Qu_0 \in \mathcal{C}^{0,\gamma}_{\mathrm{c}}(I,\mathbb{C})$ by Assumption (4.21))

$$\big\|Qu_0 - \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}Qu_0\big\|_{L^2_\rho} \le C_{I,u,Q}h_\bullet^{\gamma'}\|\rho\mathbb{1}_{\omega_\bullet}\|_{L^1}^{1/2}$$

where $C_{I,u,Q} > 0$ only depends on the Hölder norm of Qu_0 and of C_I. It follows

$$\mathrm{II} \le C_{I,u,Q}h_\bullet^{\min\{\gamma,\gamma'\}}\|\rho\mathbb{1}_{\omega_\bullet}\|_{L^1}^{1/2}$$

where $C_{I,u,Q} > 0$ denotes another constant depending only on u, Q and C_I. Collecting all the above estimates and writing again $C_{I,u,Q}$ for the sum of all the previous constants $C_{I,u}$, $C_{I,u,Q}$, we get the result. □

Remark 4.2.7. *If $\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0$ had no discontinuity point, then we could have directly written*

$$\begin{aligned}\Big|\big\langle P^\star\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0, \mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0\big\rangle_{L^2_\rho}^{1/2} - \big\langle Pu_0, u_0\big\rangle_{L^2_\rho}^{1/2}\Big| &= \Big|\|P^{1/2}\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0\|_{L^2_\rho} - \|P^{1/2}u_0\|_{L^2_\rho}\Big|\\ &\le \|P^{1/2}\big(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0 - u_0\big)\|_{L^2_\rho}\\ &= \big\langle P\big(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0 - u_0\big), \big(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}u_0 - u_0\big)\big\rangle_{L^2_\rho}^{1/2}\\ &\le C_{I,u,Q}h_\bullet^{\gamma}.\end{aligned}$$

The discontinuity obliges us to use the linear forms $\langle\cdot, u_0\rangle_{L^2_\rho}$ and $\langle P^\star u_0, \cdot\rangle_{L^2_\rho}$ instead of the norm $\|\cdot\|_{\dot{\mathcal{H}}_\rho}^{1/2}$.

Proposition 4.2.8 (Consistency of the norms)**.** *Assume (AP1)-(AP7). Let $T > 0$ and $u = (u_0, u_1) \in \mathscr{W}_1$ be the solution of (ADP). Note $K_T := \cup_{0\le t\le T}\operatorname{Supp} u(t,\cdot)$ (it is a compact subset of I by Assumption (AP7)) and pick $\mathcal{U}_T \subset I$ containing K_T.*

1. *There exist a mesh $\mathscr{T}_\bullet$ satisfying (M) and a constant $C_1 \equiv C_1(I, K_T, u, Q) > 0$ depending only on the constant C_I of Assumption (M), $\|u(t,\cdot)\|_{H^2(\mathcal{U}_T,\mathbb{C})\times H^1(\mathcal{U}_T,\mathbb{C})}$ and $\|Qu_0(t,\cdot)\|_{H^1(\mathcal{U}_T,\mathbb{C})}$ such that for all $t \in [0,T]$, we have*

$$\left|\widetilde{\mathcal{N}}_\bullet(u(t,\cdot)) - |||u(t,\cdot)|||_{\dot{\mathcal{E}}_\rho}\right| \leq C_1\Big(h_\bullet^{1/2}\|a\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})} + h_\bullet\|c\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})} + h_\bullet^{1/2}\|\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})}\Big).$$

For all $\varepsilon > 0$, there exist a mesh $\mathscr{T}_\bullet$ depending on v, v'' and K_T, as well as a constant $C_2 \equiv C_2(I, K_T, Hu, Q) > 0$ depending on the mesh, $\|Hu(t,\cdot)\|_{H^1(\mathcal{U}_T,\mathbb{C})\times L^2(\mathcal{U}_T,\mathbb{C})}$, $\|Qu_0(t,\cdot)\|_{H^1(\mathcal{U}_T,\mathbb{C})}$ and $\|\rho\|_{L^1(\mathcal{U}_T,\mathbb{R})}$ such that

$$\left|\widetilde{\mathcal{N}}_\bullet(Hu(t,\cdot)) - |||Hu(t,\cdot)|||_{\dot{\mathcal{E}}_\rho}\right| \leq C_2\varepsilon.$$

Proof. 1. We choose $\mathscr{T}_\bullet$ as in Remark 4.1.13 and such that the supports of K_T is contained in $\omega_\bullet$. Assumption (AP2) combined with Lemma 4.1.7 imply that $(u_0(t,\cdot), u_1(t,\cdot)) \in H^2(I,\mathbb{C}) \times H^1(I,\mathbb{C})$. Sobolev embedding theorem then implies that $(u_0(t,\cdot), u_1(t,\cdot)) \in \mathcal{C}^{1,1/2}(I,\mathbb{C}) \times \mathcal{C}^{0,1/2}(I,\mathbb{C})$ with support included in K_T. Finally, Assumption (AP5) implies that $Qu_0 \in \dot{\mathcal{H}}^1_\rho$ and thus Sobolev embedding theorem implies that $Qu_0 \in \mathcal{C}^{0,1/2}(I,\mathbb{C})$ with support included in K_T by Assumption (AP5).

We can then apply Proposition 4.2.6. We get the announced norms in the constant $C_{I,K_T,u,Q}$ using $u \in \mathcal{C}^0([0,T], \mathscr{D}(H))$ and Morrey's inequality $\|\cdot\|_{\mathcal{C}^{k,1/2}(\mathcal{U}_T,\mathbb{C})} \leq C_{\mathcal{U}_T}\|\cdot\|_{H^{k+1}(\mathcal{U}_T,\mathbb{C})}$ (here $C_{\mathcal{U}_T} > 0$ is a constant which only depends on $\mathcal{U}_T$, and thus on K_T).

2. Since $Hu \in \mathcal{C}^0([0,T], \dot{\mathcal{E}}_\rho)$, we do not dispose of a sufficient regularity to directly apply Proposition 4.2.6. Nevertheless, by looking carefully at the proof of this lemma, we realize it is enough to estimate the following terms:

$$\begin{aligned}&\left\|\partial_r^\star\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}(Hu)_0 - (Hu)_0\right)\right\|_{L^2_\rho}, \qquad \|\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}(Hu)_0 - (Hu)_0\|_{L^2_\rho},\\ &\|\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}(Hu)_1 - (Hu)_1\|_{L^2_\rho}, \qquad \|\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}Q(Hu)_0 - Q(Hu)_0\|_{L^2_\rho}.\end{aligned}$$

The first two terms above come form the term I in the proof of Lemma 4.2.6 whereas the two others come from the term II. Since u is a solution of (ADP), $(Hu)_0 \in \dot{\mathcal{H}}^1_\rho \subset \mathcal{C}^{0,1/2}(I,\mathbb{C})$; furthermore, $Q(Hu)_0 \in \dot{\mathcal{H}}^1_\rho \subset \mathcal{C}^{0,1/2}(I,\mathbb{C})$ by Assumption (AP5). We therefore see that we only need to estimate both the terms

$$\left\|\partial_r^\star\left(\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}(Hu)_0 - (Hu)_0\right)\right\|_{L^2_\rho}, \qquad \|\mathfrak{R}_{\bullet,1}\mathfrak{P}_{\bullet,1}(Hu)_1 - (Hu)_1\|_{L^2_\rho}.$$

Applying (4.19) of Corollary 4.2.4 for the first term and Lemma 4.2.3 for the second one, we are done.

□

Corollary 4.2.9. *Assume (AP1)-(AP7). Let $T > 0$ and $u = (u_0, u_1) \in \mathscr{W}_1$ be the solution of (ADP).*

If $\zeta := \max\{\zeta_-, \zeta_+\} < 2$ and if $u(t,\cdot) \in H^2(I,\mathbb{C}) \times H^1(I,\mathbb{C})$ for all $t \in [0,T]$, then there exist a mesh $\mathscr{T}_\bullet$ satisfying (M) and a constant $C \equiv C(I,u,Q) > 0$ depending only on the constant

C_I of Assumption (M), $\|u\|_{L^\infty([0,T],H^2(I,\mathbb{C})\times H^1(I,\mathbb{C}))}$ and $\|Qu_0\|_{L^\infty([0,T],H^1(I,\mathbb{C}))}$ such that for all $t\in[0,T]$, we have

$$\left|\widetilde{\mathcal{N}}_\bullet(u(t,\cdot)) - |||u(t,\cdot)|||_{\dot{\mathcal{E}}_\rho}\right| \le C\begin{cases} h_\bullet^{1/2-\zeta/2} & \text{if } \zeta_\pm>0 \\ h_\bullet^{1/2}\ln^{1/2}(h_\bullet) & \text{if } \zeta_\pm=0 \end{cases} \underset{\bullet}{\longrightarrow} 0.$$

Proof. Apply part 1. of Proposition 4.2.8 with the mesh of Remark 4.1.13: it clearly verifies $h_\bullet = \tilde{h}_\bullet$ so that the announced estimate follows from Assumption (AP1) and Assumption (AP3). □

Consistency of the operator $H_\bullet^\star$

We show in this paragraph the show the consistency of the operator $H_\bullet^\star$ with the Hamiltonian H on the solutions of (ADP). The convergence rate will of course depend on the regularity of the considered solution, which depends itself on the initial data; for the general case, no rate can be exhibited.

Proposition 4.2.10 (Consistency of the Hamiltonians)**.** *Assume (AP1)-(AP7) and (Q). Let $T>0$ and $u=(u_0,u_1)\in\mathscr{W}_1$ be the solution of (ADP). Note $K_T := \cup_{0\le t\le T}\operatorname{Supp} u(t,\cdot)$ (it is a compact subset of I by Assumption (AP7)) and pick $\mathcal{U}_T\subset I$ containing K_T.*

1. *For all $\varepsilon>0$, there exist a mesh $\mathscr{T}_\bullet$ depending on u, u'' and K_T, as well as a constant $C_1\equiv C_1(I,K_T,u,a,b,c)>0$ depending on the mesh, $\|u(t,\cdot)\|_{H^2(\mathcal{U}_T,\mathbb{C})\times H^1(\mathcal{U}_T,\mathbb{C})}$, $\|\rho\|_{L^1(\mathcal{U}_T,\mathbb{R})}$ and $\|a\|_{L^\infty(I,\mathbb{R})}, \|b\|_{L^\infty(I,\mathbb{R})}, \|c\|_{L^\infty(I,\mathbb{R})}$ such that*

$$\mathcal{N}_\bullet\big(\mathscr{R}_{\bullet,1}^{\mathrm{spat}} M_\bullet^{-1}\big(\mathscr{P}_{\bullet,0}^{\mathrm{spat}} Hu(t,\cdot) - H_\bullet^\star \mathscr{P}_{\bullet,1}^{\mathrm{spat}} u(t,\cdot)\big)\big) \le C_1\varepsilon.$$

2. *If the initial data φ satisfies $H\varphi\in\mathscr{D}(H)$, then for any mesh satisfying (M), there exists a constant $C_2\equiv C_2(I,K_T,u)>0$ depending only on the constant C_I of Assumption (M), $\|u(t,\cdot)\|_{H^3(\mathcal{U}_T,\mathbb{C})\times H^1(\mathcal{U}_T,\mathbb{C})}$ and $\|\rho\|_{L^1(\mathcal{U}_T,\mathbb{R})}$ such that*

$$\begin{aligned}&\mathcal{N}_\bullet\big(\mathscr{R}_{\bullet,1}^{\mathrm{spat}} M_\bullet^{-1}\big(\mathscr{P}_{\bullet,0}^{\mathrm{spat}} Hu(t,\cdot) - H_\bullet^\star \mathscr{P}_{\bullet,1}^{\mathrm{spat}} u(t,\cdot)\big)\big)\\ &\le C_2\left(\|a^2\rho\mathbb{1}_{\omega_\bullet}\|_{L^1(\mathcal{U}_T,\mathbb{R})}^{1/2} + \|b^2\rho\mathbb{1}_{\omega_\bullet}\|_{L^1(\mathcal{U}_T,\mathbb{R})}^{1/2} + \|c^2\rho\mathbb{1}_{\omega_\bullet}\|_{L^1(\mathcal{U}_T,\mathbb{R})}^{1/2}\right) h_\bullet^{1/2}.\end{aligned}$$

Proof. If $u=(u_0,u_1)\in\mathscr{W}_1$, then (AP7) implies that $u_j(t,.)$ is compactly supported in I for $j\in\{0,1\}$. Fix a mesh $\mathscr{T}_\bullet$ as in Corollary 4.2.4 in the first case (and note C_I the uniform constant C_K given by Lemma 4.2.3) and any mesh satisfying (M) for the second case, both being such that the supports of u_0 and u_1 are contained in $\omega_\bullet$. First observe that

$$\begin{aligned}\big(\mathscr{P}_{\bullet,0}^{\mathrm{spat}} Hu - H_\bullet^\star \mathscr{P}_{\bullet,1}^{\mathrm{spat}} u\big)_0 &= \mathscr{P}_{\bullet,0}^{\mathrm{spat}}\big(Qu_0+u_1\big) - \mathscr{P}_{\bullet,0}^{\mathrm{spat}}\big(Q\mathscr{R}_{\bullet,1}^{\mathrm{spat}}\mathscr{P}_{\bullet,1}^{\mathrm{spat}}u_0 + \mathscr{R}_{\bullet,1}^{\mathrm{spat}}\mathscr{P}_{\bullet,1}^{\mathrm{spat}}u_1\big)\\ &= \mathscr{P}_{\bullet,0}^{\mathrm{spat}} Qu_0 - \mathscr{P}_{\bullet,0}^{\mathrm{spat}} Q\mathscr{R}_{\bullet,1}^{\mathrm{spat}}\mathscr{P}_{\bullet,1}^{\mathrm{spat}}u_0 \qquad (4.22)\end{aligned}$$

since $\mathscr{P}_{\bullet,0}^{\mathrm{spat}}\mathscr{R}_{\bullet,1}^{\mathrm{spat}}\mathscr{P}_{\bullet,1}^{\mathrm{spat}} = \mathscr{P}_{\bullet,0}^{\mathrm{spat}}$. But Assumption (Q) also implies

$$\mathscr{P}_{\bullet,0}^{\mathrm{spat}} Qu_0 = \mathscr{P}_{\bullet,0}^{\mathrm{spat}} Q\mathscr{R}_{\bullet,1}^{\mathrm{spat}}\mathscr{P}_{\bullet,1}^{\mathrm{spat}}u_0$$

so that (4.22) is identically 0. Next write

$$\begin{aligned}\big(\mathscr{P}^{\mathrm{spat}}_{\bullet,0} Hu - H^\star_\bullet \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u\big)_1 &= \mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big(Pu_0 + Qu_1\big) - \mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big(P^\star \mathscr{R}^{\mathrm{spat}}_{\bullet,1} \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u_0 + Q \mathscr{R}^{\mathrm{spat}}_{\bullet,1} \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u_1\big)\\ &= \mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big(Pu_0 - P^\star \mathscr{R}^{\mathrm{spat}}_{\bullet,1} \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u_0\big).\end{aligned}$$

By Lemma 4.2.5, we have pointwisely $\mathscr{R}^{\mathrm{spat}}_{\bullet,1} M_\bullet^{-1} \mathscr{P}^{\mathrm{spat}}_{\bullet,0} v = \mathscr{P}^{\mathrm{spat}}_{\bullet,0} v$ when v is an element of $\mathscr{W}^\star_H$. We are thus boiled down to estimate the L^2_ρ norm of the term above.

Since $u_0 \in \mathcal{C}^{1,1/2}(I,\mathbb{C})$ by Assumption (AP2) and Sobolev embedding theorem, we can sum up the estimates of Lemma 4.2.1 over $p \in \mathscr{T}_{\bullet,0}$ such that $\omega_p \in K_T$ to obtain

$$\big\| \mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big((b\partial_r^\star + c)(\mathscr{R}^{\mathrm{spat}}_{\bullet,1} \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u_0 - u_0)\big)\big\|_{L^2_\rho} \le C_I \|b^2\rho\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})} h_\bullet^{1/2} + C_I \|c^2\rho\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})} h_\bullet.$$

We used the obvious estimate $\|\mathscr{P}^{\mathrm{spat}}_{\bullet,0} w\|_{L^\infty(I,\mathbb{C})} \le C_I(r_+ - r_-)\|w\|_{L^\infty(I,\mathbb{C})}$. It remains to deal with the second derivative term.

1. If $\varepsilon > 0$ has been fixed, then (4.17) of Corollary 4.2.4 shows that

   $$\big\| \mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big(\partial_r^2 u - (\partial_r^\star)^2 \mathscr{R}^{\mathrm{spat}}_{\bullet,1} \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u\big)\big\|_{L^2_\rho} \le \|a\|_{L^\infty(I,\mathbb{C})} \big\| \mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big(\partial_r^2 u - (\partial_r^\star)^2 \mathscr{R}^{\mathrm{spat}}_{\bullet,1} \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u\big)\big\|_{L^2_\rho} \le C\varepsilon$$

   where $C > 0$ is a constant which depends on the mesh, $\|u\|_{H^2(\mathcal{U}_T,\mathbb{C})}$ and $\|\rho\|_{L^1(\mathcal{U}_T,\mathbb{C})}$. This gives the announced estimate for part 1.

2. If we allow a loss of derivative for the initial data, then Lemma 4.1.9 for $k = 1$ shows in particular that $u_0 \in \dot{\mathcal{H}}^3_\rho$ (note that Assumption (Q) implies that $Q \in \mathcal{B}(\dot{\mathcal{H}}^2_\rho)$). By Assumption (AP2) for $k = 3$, $u_0 \in H^3(K,\mathbb{C})$ and Sobolev embedding theorem then implies that $u_0 \in \mathcal{C}^{2,1/2}(K,\mathbb{C})$. It follows

   $$\|a\partial_r^2(\mathscr{R}^{\mathrm{spat}}_{\bullet,1} \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u_0 - u_0)\|_{L^2_\rho} \le \hat{C}_u \|a^2\rho \mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})} h_\bullet^{1/2}$$

   where $\hat{C}_u > 0$ depends on $\|u_0''\|_{\mathcal{C}^{0,1/2}(\mathcal{U}_T,\mathbb{C})}$ (giving the dependence in $\|u_0\|_{H^3(\mathcal{U}_T,\mathbb{C})}$ by Morrey's embedding) and on the mesh (see the proof of Lemma 4.2.1). Collecting all the obtained estimates gives the announced estimate for part 2.

□

Corollary 4.2.11. *Assume (AP1)-(AP7) and (Q). Let $T > 0$ and $u = (u_0, u_1) \in \mathscr{W}_1$ be the solution of (ADP).*

If $u(t,\cdot) \in H^2(I,\mathbb{C}) \times H^1(I,\mathbb{C})$ for all $t \in [0,T]$, then there exist a mesh $\mathscr{T}_\bullet$ satisfying (M) and a constant $C \equiv C(I,u) > 0$ depending only on the constant C_I of Assumption (M) and $\|u\|_{L^\infty([0,T],H^2(I,\mathbb{C})\times H^1(I,\mathbb{C}))}$ such that for all $t \in [0,T]$, we have

$$\mathcal{N}_\bullet\big(\mathscr{R}^{\mathrm{spat}}_{\bullet,1} M_\bullet^{-1}\big(\mathscr{P}^{\mathrm{spat}}_{\bullet,0} Hu(t,\cdot) - H^\star_\bullet \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u(t,\cdot)\big)\big) \le C\varepsilon.$$

If furthermore $\zeta := \max\{\zeta_-, \zeta_+\} < 2$ and $u_0 \in H^3(I,\mathbb{C})$, then

$$\mathcal{N}_\bullet\big(\mathscr{R}^{\mathrm{spat}}_{\bullet,1} M_\bullet^{-1}\big(\mathscr{P}^{\mathrm{spat}}_{\bullet,0} Hu(t,\cdot) - H^\star_\bullet \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u(t,\cdot)\big)\big) \le C \begin{cases} h_\bullet^{1/2-\zeta/2} & \text{if } \zeta_\pm > 0 \\ h_\bullet^{1/2} \ln^{1/2}(h_\bullet) & \text{if } \zeta_\pm = 0 \end{cases} \xrightarrow[\bullet]{} 0.$$

For the proof of the convergence of the scheme, we will also need a time consistency aresult:

Lemma 4.2.12 (Temporal consistency). *Assume* (AP1)-(AP7) *and* (Q). *Let* $T > 0$ *and* $v = (v_0, v_1) \in \mathscr{W}_1$ *be the solution of* (ADP). *Note* $K_T := \cup_{0\le t\le T} \operatorname{Supp} v(t,\cdot)$ *(it is a compact subset of* I *by Assumption* (AP7)*) and pick* $\mathcal{U}_T \subset I$ *containing* K_T.

For all $\varepsilon > 0$, *there exists a mesh* $\mathscr{T}_\bullet$ *as well as two constants* $C_1, C_2 > 0$ *which only depend on the mesh, on* $\|u(t,\cdot)\|_{H^2(\mathcal{U}_T,\mathbb{C})\times H^1(\mathcal{U}_T,\mathbb{C})}$, $\|\rho\|_{L^1(\mathcal{U}_T,\mathbb{R})}$, $\|a\|_{L^\infty(I,\mathbb{R})}$, $\|b\|_{L^\infty(I,\mathbb{R})}$, $\|c\|_{L^\infty(I,\mathbb{R})}$ *and on* $\|q\|_{L^\infty(I,\mathbb{R})}$ *for* C_2, *such that, for all* $t \in [0,T]$, *we have*

$$\mathcal{N}_\bullet\big(v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big) \le C_1\varepsilon, \tag{4.23}$$

$$\mathcal{N}_\bullet\big(\mathscr{F}^{\mathrm{spat}}_{\bullet,0}\,\mathscr{F}^{\mathrm{temp}}_{\bullet,0}\big(\partial_t^\star(v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v)\big)\big) \le C_2\varepsilon. \tag{4.24}$$

Proof. Fix a mesh $\mathscr{T}_\bullet$ as in Corollary 4.2.4 (and note C_I the uniform constant C_K of this corollary). Since $v \in \mathcal{C}^0([0,T],\mathscr{D}(H)) \cap \mathcal{C}^1([0,T],\dot{\mathcal{E}}_\rho)$, there exists $\delta > 0$ such that for all $t, t' \in [0,T]$, we have

$$|t - t'| \le \delta \qquad \Longrightarrow \qquad \|v(t,\cdot) - v(t',\cdot)\|_{\dot{\mathcal{E}}_\rho} + \|\partial_t v(t,\cdot) - \partial_t v(t',\cdot)\|_{\dot{\mathcal{E}}_\rho} \le \varepsilon.$$

Let then $t \in [0,T[$ and $s \in \widetilde{\mathscr{T}}_{\bullet,0}$ the unique element such that $t \in [s, s_+[$; we have

$$\begin{aligned} v(t,\cdot) - \big(\mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big)(t,\cdot) &= v(t,\cdot) - \left(\frac{t - s_+}{s - s_+} v(s,\cdot) + \frac{t - s}{s_+ - s} v(s_+,\cdot)\right) \\ &= \big(v(t,\cdot) - v(s,\cdot)\big) + \frac{t - s_+}{s - s_+}\big(v(s_+,\cdot) - v(s,\cdot)\big) \end{aligned}$$

so that

$$\Big\|v(t,\cdot) - \big(\mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big)(t,\cdot)\Big\|_{\dot{\mathcal{E}}_\rho} \le 2\varepsilon$$

as soon as $\eta_\bullet \le \delta$, which can be assumed for we can always choose $\widetilde{\mathscr{T}}_\bullet$ so that this conditions is realized. Using the norm equivalences (4.5) and (4.6), we get

$$\begin{aligned} \mathcal{N}_\bullet\big(v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big) &\le \widetilde{\mathcal{N}}_\bullet\big(v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big) \\ &\le \Big|\widetilde{\mathcal{N}}_\bullet\big(v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big) - \big|\big|\big|v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big|\big|\big|_{\dot{\mathcal{E}}_\rho}\Big| \\ &\quad + \big|\big|\big|v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big|\big|\big|_{\dot{\mathcal{E}}_\rho} \\ &\le \Big|\widetilde{\mathcal{N}}_\bullet\big(v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big) - \big|\big|\big|v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big|\big|\big|_{\dot{\mathcal{E}}_\rho}\Big| \\ &\quad + \sqrt{2}\big\|v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big\|_{\dot{\mathcal{E}}_\rho}. \end{aligned}$$

Now the consistency of the norms showed in Proposition 4.2.8 implies the existence of a constant $C_1 \equiv C_1(I, K_T, v, Q) > 0$ depending only on C_I, $\|v\|_{L^\infty([0,T],H^2(\mathcal{U}_T,\mathbb{C})\times H^1(\mathcal{U}_T,\mathbb{C}))}$ and $\|Qv_0\|_{L^\infty([0,T],H^1(\mathcal{U}_T,\mathbb{C}))}$ such that

$$\begin{aligned} &\Big|\widetilde{\mathcal{N}}_\bullet\big(v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big) - \big|\big|\big|v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big|\big|\big|_{\dot{\mathcal{E}}_\rho}\Big| \\ &\le C_1\Big(h_\bullet^{1/2}\|a\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})} + h_\bullet\|c\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})} + h_\bullet^{1/2}\|\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})}\Big) + 2\sqrt{2}\varepsilon. \end{aligned}$$

This settles (4.23) with another constant C_1 and for $h_\bullet$ sufficiently small.

Next, for all $t \in [0,T]$, we have (dropping the spatial component of v)

$$\mathscr{P}^{\mathrm{temp}}_{\bullet,1} v(t) = \sum_{s \in \widetilde{\mathscr{T}}_{\bullet,0}} \Big(v(s) + (t-s)\partial_t v(s) + (t-s) \int_0^1 \big(\partial_t v\big(s + \tau(s_+ - s)\big) - \partial_t v(s) \big) \mathrm{d}\tau \Big) \mathbb{1}_{[s,s_+[}(t),$$

$$\big(\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \partial_t v\big)(t) = \sum_{s \in \widetilde{\mathscr{T}}_{\bullet,0}} \Big(\big(\partial_t v(s) - \partial_t v(t)\big) + \int_0^1 \big(\partial_t v\big(s + \tau(s_+ - s)\big) - \partial_t v(s) \big) \mathrm{d}\tau \Big) \mathbb{1}_{[s,s_+[}(t)$$

whence

$$\begin{aligned}
&\mathscr{P}^{\mathrm{temp}}_{\bullet,0} \big(\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \partial_t v\big)(t) \\
&= \sum_{s \in \widetilde{\mathscr{T}}_{\bullet,0}} \int_{[s,s_+[,\{1\}} \big(\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \partial_t v\big)(\tau) \mathrm{d}\tau \, \mathbb{1}_{[s,s_+[}(t) \\
&= \sum_{s \in \widetilde{\mathscr{T}}_{\bullet,0}} \int_0^1 \Bigg[\partial_t v(s) - \Big(\frac{(s_+ - s)\tau + s - s_+}{s - s_+} \partial_t v(s) + \frac{(s_+ - s)\tau}{s_+ - s} \partial_t v(s_+) \Big) \Big|_{\tau=1} \\
&\qquad\qquad + \int_0^1 \big(\partial_t v\big(s + \tau'(s_+ - s)\big) - \partial_t v(s) \big) \mathrm{d}\tau' \Bigg] \mathrm{d}\tau \, \mathbb{1}_{[s,s_+[}(t) \\
&= \sum_{s \in \widetilde{\mathscr{T}}_{\bullet,0}} \Big[\big(\partial_t v(s) - \partial_t v(s_+)\big) + \int_0^1 \big(\partial_t v\big(s + \tau'(s_+ - s)\big) - \partial_t v(s) \big) \mathrm{d}\tau' \Big] \mathbb{1}_{[s,s_+[}(t). \qquad (4.25)
\end{aligned}$$

In order to prove (4.24), we can not use the consistency of the norms because of the temporal derivative. Using Assumption (AP3), Remark 4.1.2 and Assumption (Q), we see that it is sufficient to estimate the L^2_ρ norm of the following terms:

$$(\partial_r^\star)^k \big(\mathscr{F}^{\mathrm{temp}}_{\bullet,0} \mathscr{F}^{\mathrm{spat}}_{\bullet,0} \big(\partial_t v_0 - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v_0 \big) \big) \qquad \forall k \in \{0,1\}, \qquad (4.26)$$

$$\big(\mathscr{F}^{\mathrm{temp}}_{\bullet,0} \mathscr{F}^{\mathrm{spat}}_{\bullet,0} Q \big(\partial_t v_0 - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v_0 \big) \big), \qquad (4.27)$$

$$\mathscr{F}^{\mathrm{temp}}_{\bullet,0} \mathscr{F}^{\mathrm{spat}}_{\bullet,0} \big(\partial_t v_1 - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v_1 \big). \qquad (4.28)$$

We start with (4.26). Let $w_0 := \mathscr{F}^{\mathrm{temp}}_{\bullet,0} \big(\partial_t v_0 - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v_0 \big)$ and write

$$\begin{aligned}
\big\| (\partial_r^\star)^k \big(\mathscr{F}^{\mathrm{temp}}_{\bullet,0} \mathscr{F}^{\mathrm{spat}}_{\bullet,0} \big(\partial_t v_0 - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v_0 \big) \big) \big\|_{L^2_\rho} &= \big\| (\partial_r^\star)^k \mathscr{F}^{\mathrm{spat}}_{\bullet,0} w_0 \big\|_{L^2_\rho} \\
&\le \big\| (\partial_r^\star)^k \mathscr{F}^{\mathrm{spat}}_{\bullet,0} w_0 - \partial_r^k w_0 \big\|_{L^2_\rho} + \big\| \partial_r^k w_0 \big\|_{L^2_\rho}.
\end{aligned}$$

Since $w_0 \in \dot{\mathcal{H}}^1_\rho$ by (4.25), the first term on the right hand side is estimated by Lemma 4.2.1 for $k = 0$ and by Corollary 4.2.4 for $k = 1$, whereas the remaining term is estimated as follows:

$$\big\| \partial_r^k w_0 \big\|_{L^\infty([0,T],L^2_\rho)} \le \|w\|_{L^\infty([0,T],\dot{\mathcal{E}}_\rho)} \le 2\varepsilon$$

where we assumed again $\eta_\bullet \le \delta$ and used that $v \in \mathcal{C}^1([0,T],\dot{\mathcal{E}}_\rho)$. We thus get:

$$\big\| (\partial_r^\star)^k \big(\mathscr{F}^{\mathrm{temp}}_{\bullet,0} \mathscr{F}^{\mathrm{spat}}_{\bullet,0} \big(\partial_t v_0 - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v_0 \big) \big) \big\|_{L^2_\rho} \le \begin{cases} C_2 h_\bullet^{1/2} + 2\varepsilon & \text{if } k = 0 \\ C_3 \varepsilon + 2\varepsilon & \text{if } k = 1 \end{cases}$$

where the constants $C_2, C_3 > 0$ are as in Lemma 4.2.1 and Corollary 4.2.4, respectively. The term (4.27) is estimated in the same way since Q is a multiplication operator by a $\mathcal{C}^2(I,\mathbb{R})$ function q, the only difference being that the constants will also depend on $\|q\|_{L^\infty(I,\mathbb{R})}$. As for the term (4.28), let $w_1 := \mathscr{F}^{\mathrm{temp}}_{\bullet,0}\big(\partial_t v_1 - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v_1\big)$ and write

$$\begin{aligned}\big\|\mathscr{F}^{\mathrm{temp}}_{\bullet,0}\mathscr{F}^{\mathrm{spat}}_{\bullet,0}\big(\partial_t v_1 - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v_1\big)\big\|_{L^2_\rho} &\le \big\|\mathscr{F}^{\mathrm{spat}}_{\bullet,0} w_1 - w_1\big\|_{L^2_\rho} + \big\|w_1\big\|_{L^2_\rho}\\ &\le \big\|\mathscr{F}^{\mathrm{spat}}_{\bullet,0} w_1 - w_1\big\|_{L^2_\rho} + \big\|w\big\|_{\dot{\mathcal{E}}_\rho}.\end{aligned}$$

By Lemma 4.2.3, there exists a constant $C_4 > 0$ such that

$$\big\|\mathscr{F}^{\mathrm{temp}}_{\bullet,0}\mathscr{F}^{\mathrm{spat}}_{\bullet,0}\big(\partial_t v_1 - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v_1\big)\big\|_{L^2_\rho} \le C_4\varepsilon + 2\varepsilon.$$

This gives (4.24) with another constant C_2 and concludes the proof of the lemma. □

4.2.3 Stability

In this Section, we interest ourselves in the stability of the scheme in the special case of Assumption (Q). We recall here some notations introduced in the paragraph 4.1.2:

$$\begin{aligned}\eta_\bullet &:= \max\{s_+ - s_- \mid s \in \widetilde{\mathscr{T}}_{\bullet,0}\},\\ h_\bullet &:= \max\{p_+ - p_- \mid p \in \mathscr{T}_{\bullet,0}\}.\end{aligned}$$

Lemma 4.2.13. *Assume (Q) and*

$$\big(1 + 2\|q\|_{L^\infty} + \|c\|_{L^\infty}\big)(t_+ - t) < 1 \qquad \forall t \in \widetilde{\mathscr{T}}_{\bullet,0} \setminus \{T\}. \tag{4.29}$$

Then $\mathbb{1} - (t_+ - t)M_\bullet^{-1}H_\bullet$ is invertible for all $t \in \widetilde{\mathscr{T}}_{n,0} \setminus \{T\}$ and we have the uniform bound

$$\mathcal{N}_\bullet\big(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}(\mathbb{1} - (t_+ - t)M_\bullet^{-1}H_\bullet)^{-1}u\big) \le \mathcal{N}_\bullet(u) \qquad \forall u \in \mathscr{W}^\star_{\bullet,1}. \tag{4.30}$$

Equality occurs if and only if the operator is applied on $\ker(H^\star_\bullet)$. *In particular,*

$$|||\mathscr{R}^{\mathrm{spat}}_{\bullet,1}(\mathbb{1} - (t_+ - t)M_\bullet^{-1}H_\bullet)^{-1}|||_\bullet = 1. \tag{4.31}$$

Proof. Recall that we showed in the paragraph 4.1.3 that $M_\bullet$ is equivalent to $\mathscr{M}_\bullet$ (which is the identity on $\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$) modulo the invertible operator $\Psi_\bullet$, so is invertible too. As for the operator $\mathbb{1} - \mathrm{i}(t_+ - t)M_\bullet^{-1}H_\bullet$, we shall show that $\mathrm{i}(t_+ - t)M_\bullet^{-1}H_\bullet$ has a norm smaller than 1.

Let $u \in \mathscr{W}^\star_{\bullet,1}$ and write $U = \Psi_\bullet^{-1}u \in \mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$ its discrete version. Using that

$$\begin{aligned}\frac{2a(p)}{(p_+ - p-)(p_+ - p)} + \frac{2a(p)}{(p_- - p)(p_- - p_+)} + \frac{2a(p)}{(p - p_-)(p - p_+)} &= 0,\\ \frac{b(p)(p - p_-)}{(p_+ - p-)(p_+ - p)} + \frac{b(p)(p - p_+)}{(p_- - p)(p_- - p_+)} + \frac{b(p)(2p - (p_- + p_+)}{(p - p_-)(p - p_+)} &= 0\end{aligned}$$

as well as the discrete version of $M_\bullet$ is the identity operator (see paragraph 4.1.3), we compute:

$$\begin{aligned}\mathcal{N}_\bullet(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}M_\bullet^{-1}H^\star_\bullet u) &= \left\langle \mathscr{H}^\star_\bullet U, \mathcal{N}_\bullet\mathscr{H}^\star_\bullet U\right\rangle_{\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}}\\ &= \sum_{i=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{j=1}^{2|\mathscr{T}_{\bullet,0}|}(\mathcal{N}_\bullet)_{ij}\overline{(\mathscr{H}^\star_\bullet U)}_i(\mathscr{H}^\star_\bullet U)_j\\ &= \sum_{i=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{j=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{k=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{\ell=1}^{2|\mathscr{T}_{\bullet,0}|}(\mathcal{N}_\bullet)_{ij}\overline{(\mathscr{H}^\star_\bullet)_{ik}}(\mathscr{H}^\star_\bullet)_{j\ell}\overline{U}_kU_\ell,\end{aligned}$$

$$\begin{aligned}&\sum_{j=1}^{2|\mathscr{T}_{\bullet,0}|}(\mathcal{N}_\bullet)_{ij}\sum_{\ell=1}^{2|\mathscr{T}_{\bullet,0}|}(\mathscr{H}^\star_\bullet)_{j\ell}U_\ell\\ &= \sum_{j=1}^{2|\mathscr{T}_{\bullet,0}|}(\mathcal{N}_\bullet)_{ij}\sum_{\substack{\ell=2|p|-1\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}}U_\ell\left[Q(p)\delta_{j,2|p|-1}+\sum_{\pm}\frac{2a(p_\pm)+b(p_\pm)(p_\pm-p_{\pm\pm})}{(p-p_{\pm\pm})(p-p_\pm)}\delta_{j,2|p|\pm2}\right.\\ &\qquad\left.+\left(\frac{2a(p)+b(p)(2p-(p_-+p_+))}{(p-p_-)(p-p_+)}+c(p)\right)\delta_{j,2|p|}\right]\\ &+\sum_{j=1}^{2|\mathscr{T}_{\bullet,0}|}(\mathcal{N}_\bullet)_{ij}\sum_{\substack{\ell=2|p|\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}}U_\ell\Big[\delta_{j,2|p|-1}+Q(p)\delta_{j,2|p|}\Big]\\ &= \sum_{\substack{j=2|p|-1\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}}(\mathcal{N}_\bullet)_{ij}U_j\Big[1+Q(p)\Big]\\ &+\sum_{\substack{j=2|p|\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}}(\mathcal{N}_\bullet)_{ij}U_j\left[Q(p)+\frac{2a(p)+b(p)(p-p_-)}{(p_+-p_-)(p_+-p)}+\frac{2a(p)+b(p)(p-p_+)}{(p_--p)(p_--p_+)}\right.\\ &\qquad\left.+\left(\frac{2a(p)+b(p)(2p-(p_-+p_+))}{(p-p_-)(p-p_+)}+c(p)\right)\right]\\ &= \sum_{\substack{j=2|p|-1\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}}(\mathcal{N}_\bullet)_{ij}U_j\Big[1+Q(p)\Big]+\sum_{\substack{j=2|p|\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}}(\mathcal{N}_\bullet)_{ij}U_j\left[Q(p)+c(p)\right]\end{aligned}$$

and similarly

$$\sum_{i=1}^{2|\mathscr{T}_{\bullet,0}|}(\mathcal{N}_\bullet)_{ij}\sum_{k=1}^{2|\mathscr{T}_{\bullet,0}|}\overline{(\mathscr{H}^\star_\bullet)_{ik}}\,\overline{U_k}=\sum_{\substack{i=2|p|-1\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}}(\mathcal{N}_\bullet)_{ij}\overline{U_i}\Big[1+\overline{Q(p)}\Big]+\sum_{\substack{i=2|p|\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}}(\mathcal{N}_\bullet)_{ij}\overline{U_i}\left[\overline{Q(p)}+\overline{c(p)}\right].$$

Using now the symmetry and the positive semi definiteness of $\mathscr{N}_\bullet$, one gets:

$$\begin{aligned}
\mathscr{N}_\bullet(\mathscr{R}^{\mathrm{spat}}_{\bullet,1} M_\bullet^{-1} H_\bullet^\star u) = &\sum_{\substack{i=2|p|-1\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}} \sum_{\substack{j=2|p'|-1\\ p'\in\mathring{\mathscr{T}}_{\bullet,0}}} (\mathscr{N}_\bullet)_{ij}\overline{U_i}U_j\Big[1+\overline{Q(p)}\Big]\Big[1+Q(p')\Big]\\
&+\sum_{\substack{i=2|p|-1\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}} \sum_{\substack{j=2|p'|\\ p'\in\mathring{\mathscr{T}}_{\bullet,0}}} (\mathscr{N}_\bullet)_{ij}\overline{U_i}U_j\Big[1+\overline{Q(p)}\Big]\Big[Q(p')+c(p')\Big]\\
&+\sum_{\substack{i=2|p|\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}} \sum_{\substack{j=2|p'|-1\\ p'\in\mathring{\mathscr{T}}_{\bullet,0}}} (\mathscr{N}_\bullet)_{ij}\overline{U_i}U_j\Big[\overline{Q(p)}+\overline{c(p)}\Big]\Big[1+Q(p')\Big]\\
&+\sum_{\substack{i=2|p|\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}} \sum_{\substack{j=2|p'|\\ p'\in\mathring{\mathscr{T}}_{\bullet,0}}} (\mathscr{N}_\bullet)_{ij}\overline{U_i}U_j\Big[\overline{Q(p)}+\overline{c(p)}\Big]\Big[Q(p')+c(p')\Big]\\
&=\sum_{\substack{i=2|p|-1\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}} (\mathscr{N}_\bullet)_{ii}|U_i|^2\big|1+Q(p)\big|^2\\
&+2\Re\sum_{\substack{i=2|p|-1\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}} \sum_{\substack{j=2|p'|-1\\ p'\in\mathring{\mathscr{T}}_{\bullet,0}\\ j\neq i}} (\mathscr{N}_\bullet)_{ij}\overline{U_i}U_j\Big[1+\overline{Q(p)}\Big]\Big[1+Q(p')\Big]\\
&+2\Re\sum_{\substack{i=2|p|-1\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}} \sum_{\substack{j=2|p'|\\ p'\in\mathring{\mathscr{T}}_{\bullet,0}}} (\mathscr{N}_\bullet)_{ij}\overline{U_i}U_j\Big[1+\overline{Q(p)}\Big]\Big[Q(p')+c(p')\Big]\\
&+\sum_{\substack{i=2|p|\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}} (\mathscr{N}_\bullet)_{ii}|U_i|^2\big|Q(p)+c(p)\big|^2\\
&+2\Re\sum_{\substack{i=2|p|\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}} \sum_{\substack{j=2|p'|\\ p'\in\mathring{\mathscr{T}}_{\bullet,0}\\ j\neq i}} (\mathscr{N}_\bullet)_{ij}\overline{U_i}U_j\Big[\overline{Q(p)}+\overline{c(p)}\Big]\Big[Q(p')+c(p')\Big]\\
&\leq \big(1+2\|q\|_{L^\infty}+\|c\|_{L^\infty}\big)^2 \sum_{i=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{j=1}^{2|\mathscr{T}_{\bullet,0}|} (\mathscr{N}_\bullet)_{ij}\overline{U_i}U_j
\end{aligned}$$

that is $|||\mathscr{R}^{\mathrm{spat}}_{\bullet,1} M_\bullet^{-1} H_\bullet^\star|||_\bullet \leq \big(1+2\|q\|_{L^\infty}+\|c\|_{L^\infty}\big)$. The invertibility then follows from (4.29) by a Neumann series argument.

We now look for a bound to the inverse of $\mathbb{1} - \mathrm{i}(t_+ - t)H_\bullet^\star$. Let again $u \in \mathscr{W}^\star_{\bullet,1}$ and $U = \Psi_\bullet^{-1}u \in \mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$ its discrete version. Compute:

$$\begin{aligned}
\mathscr{N}_\bullet\big(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}(\mathbb{1}-\mathrm{i}(t_+-t)M_\bullet^{-1}H_\bullet^\star)u\big) = &\,\mathscr{N}_\bullet(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}u) + (t_+-t)^2\mathscr{N}_\bullet(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}M_\bullet^{-1}H_\bullet^\star u)\\
&+2(t_+-t)\Im\mathscr{N}_\bullet(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}u, \mathscr{R}^{\mathrm{spat}}_{\bullet,1}M_\bullet^{-1}H_\bullet^\star u).
\end{aligned}$$

Using again that $\mathscr{M}_\bullet$ is the identity matrix, one can write

$$\begin{aligned}
&\mathcal{N}_\bullet(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}u, \mathscr{R}^{\mathrm{spat}}_{\bullet,1}M_\bullet^{-1}H_\bullet^\star u)\\
&= \sum_{i=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{j=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{k=1}^{2|\mathscr{T}_{\bullet,0}|}(\mathscr{N}_\bullet)_{ij}(\mathscr{H}^\star_\bullet)_{jk}\overline{U}_i U_k\\
&= \sum_{i=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{j=1}^{2|\mathscr{T}_{\bullet,0}|}(\mathscr{N}_\bullet)_{ij}\overline{U}_i \sum_{\substack{k=2|p|-1\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}} U_k\Bigg[Q(p)\delta_{j,2|p|-1} + \sum_{\pm}\frac{2a(p_\pm)+b(p_\pm)(p_\pm - p_{\pm\pm})}{(p-p_{\pm\pm})(p-p_\pm)}\delta_{j,2|p|\pm 2}\\
&\qquad\qquad + \left(\frac{2a(p)+b(p)(2p-(p_-+p_+))}{(p-p_-)(p-p_+)} + c(p)\right)\delta_{j,2|p|}\Bigg]\\
&+ \sum_{i=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{j=1}^{2|\mathscr{T}_{\bullet,0}|}(\mathscr{N}_\bullet)_{ij}\overline{U}_i \sum_{\substack{k=2|p|\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}} U_k\Big[\delta_{j,2|p|-1} + Q(p)\delta_{j,2|p|}\Big]\\
&= \sum_{i=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{\substack{j=2|p|-1\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}}(\mathscr{N}_\bullet)_{ij}\overline{U}_i U_j\Big[1+Q(p)\Big] + \sum_{i=1}^{2|\mathscr{T}_{\bullet,0}|}\sum_{\substack{j=2|p|\\ p\in\mathring{\mathscr{T}}_{\bullet,0}}}(\mathscr{N}_\bullet)_{ij}\overline{U}_i U_j\left[Q(p)+c(p)\right].
\end{aligned}$$

As Q is a real multiplication operator, the above computation shows that $\Im\mathcal{N}_\bullet(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}u, \mathscr{R}^{\mathrm{spat}}_{\bullet,1}M_\bullet^{-1}H_\bullet^\star u) = 0$ and therefore

$$\mathcal{N}_\bullet\big(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}(\mathbb{1} - \mathrm{i}(t_+ - t)M_\bullet^{-1}H_\bullet^\star)u\big) = \mathcal{N}_\bullet(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}u) + (t_+-t)^2\mathcal{N}_\bullet(\mathscr{R}^{\mathrm{spat}}_{\bullet,1}M_\bullet^{-1}H_\bullet^\star u) \geq \mathcal{N}_\bullet((\mathscr{R}^{\mathrm{spat}}_{\bullet,1}u)$$

with equality if and only if $u \in \ker(H_\bullet^\star)$. But for any invertible linear operator L acting on some normed vector space $(E, \|\cdot\|)$, if the operator norm satisfies $|||L||| \geq \delta > 0$, then for all vector $E \ni y = Lx \neq 0$ we have

$$\frac{\|L^{-1}y\|}{\|y\|} = \frac{\|L^{-1}Lx\|}{\|Lx\|} = \frac{\|x\|}{\|Lx\|} \leq \frac{1}{\delta}.$$

Taking $L = (\mathbb{1} - \mathrm{i}(t_+ - t)H_\bullet^\star)$ and $\delta = 1$, the estimate (4.30) follows with equality on $(\mathbb{1} - \mathrm{i}(t_+ - t)H_\bullet^\star)\ker(H_\bullet^\star) = \ker(H_\bullet^\star)$. This completes the proof. □

Proposition 4.2.14 (Stability of the scheme)**.** *Assume* (Q) *and* (4.29)*. Consider the implicit scheme (that is $\mathscr{A} = \{1\}$ in the definition of $\mathscr{P}^{\mathrm{temp}}_{\bullet,0}$ in the paragraph 4.1.2). For all $u, v \in \mathscr{W}_{\bullet,1}$ and $\varepsilon_\bullet \in \mathscr{W}_{\bullet,0}$ satisfying*

$$\begin{cases}(\mathrm{i}\partial_\bullet + H_\bullet)u = 0\\ (\mathrm{i}\partial_\bullet + H_\bullet)v = \varepsilon_\bullet\\ v(\underline{t},\cdot) = u(\underline{t},\cdot)\end{cases},$$

we have

$$[\![\,\mathscr{R}_{\bullet,1}(u-v)\,]\!]_\bullet \leq (1+T)[\![\,\mathscr{R}_{\bullet,1}M_\bullet^{-1}\varepsilon_\bullet\,]\!]_\bullet.$$

Proof. Set $w := v - u$ so that

$$\begin{cases} (\mathrm{i}\partial_\bullet + H_\bullet)w = \varepsilon_\bullet \\ w(\underline{t}, \cdot) = \varepsilon_\bullet(\underline{t}, \cdot) \end{cases}. \tag{4.32}$$

We can write

$$w(t,r) = \sum_{s \in \widetilde{\mathscr{T}}_{\bullet,0}} \left(\frac{t - s_+}{s - s_+} \tilde{w}(s,r) + \frac{t-s}{s_+ - s} \tilde{w}(s_+, r) \right) \mathbb{1}_{[s,s_+[}(t) \qquad \forall (t,r) \in [0,T] \times I$$

for some $\tilde{w} \in \mathscr{W}_1^\star$. Compute:

$$(\partial_\bullet w)(t,r) = \sum_{s \in \widetilde{\mathscr{T}}_{\bullet,0}} \left(\frac{(M_\bullet \tilde{w})(s_+, r) - (M_\bullet \tilde{w})(s,r)}{s_+ - s} \right) \mathbb{1}_{[s,s_+[}(t),$$

$$\int_t^{t_+} (\partial_\bullet w)(\tau, r) \mathrm{d}\tau = (M_\bullet \tilde{w})(s_+, r) - (M_\bullet \tilde{w})(s,r),$$

$$\begin{aligned} &(\mathscr{P}_{\bullet,0}^{\mathrm{temp}} w)(t,r) \\ &= \sum_{s \in \widetilde{\mathscr{T}}_{\bullet,0}} \left(\int_0^1 \left(\frac{(s_+ - s)\tau + s - s_+}{s - s_+} (M_\bullet \tilde{w})(s,r) + \frac{(s_+ - s)\tau'}{s_+ - s} (M_\bullet \tilde{w})(s_+, r) \right)_{|\tau=1} \mathrm{d}\tau \right) \mathbb{1}_{[s,s_+[}(t), \\ &= \sum_{s \in \widetilde{\mathscr{T}}_{\bullet,0}} (M_\bullet \tilde{w})(s_+, r) \mathbb{1}_{[s,s_+[}(t), \end{aligned}$$

$$\int_t^{t_+} (H_\bullet w)(\tau, r) \mathrm{d}\tau = (t_+ - t) H_\bullet^\star \tilde{w}(t_+, r).$$

Let now $t \in \widetilde{\mathscr{T}}_{\bullet,0} \setminus \{T\}$. Integrating the first line of (4.32) over $[t, t_+[$ yields (we drop the dependence in r and view $w(t)$ and $w(t_+)$ as functions of r)

$$\mathrm{i} M_\bullet \big(w(t_+) - w(t) \big) + (t_+ - t) H_\bullet^\star w(t_+) = \int_t^{t_+} \varepsilon_\bullet(\tau) \mathrm{d}\tau.$$

Using that $M_\bullet$ and $\mathbb{1} - (t_+ - t) M_\bullet^{-1} H_\bullet^\star$ are invertible if $\eta_\bullet$ is small enough (see Lemma 4.2.13), we can write

$$w(t_+) = \big(\mathbb{1} - \mathrm{i}(t_+ - t) M_\bullet^{-1} H_\bullet^\star \big)^{-1} w(t) + \mathrm{i} \big(\mathbb{1} + (t_+ - t) M_\bullet^{-1} H_\bullet^\star \big)^{-1} \int_t^{t_+} M_\bullet^{-1} \varepsilon_\bullet(\tau) \mathrm{d}\tau.$$

By induction, we discover the general formula

$$\begin{aligned} w(t_+) = &\prod_{s \le t} \big(\mathbb{1} - \mathrm{i}(s_+ - s) M_\bullet^{-1} H_\bullet^\star \big)^{-1} w(\underline{t}) \\ &+ \mathrm{i} \sum_{s \le t} \left[\prod_{s < s' \le t} \big(\mathbb{1} - \mathrm{i}(s'_+ - s') M_\bullet^{-1} H_\bullet^\star \big)^{-1} \right] \big(\mathbb{1} + (s_+ - s) M_\bullet^{-1} H_\bullet^\star \big)^{-1} \int_s^{s_+} M_\bullet^{-1} \varepsilon_\bullet(\tau) \mathrm{d}\tau. \end{aligned}$$

where products over the empty set are equal to 1 by convention. We now use the estimate (4.31) to get

$$\mathcal{N}_\bullet(\mathscr{R}_{\bullet,1}w(t_+)) \leq C_\bullet^{|t|}\,\mathcal{N}_\bullet(\mathscr{R}_{\bullet,1}w(\underline{t})) + \tilde{C}_\bullet^{|t|}\int_{\underline{t}}^{t_+}\mathcal{N}_\bullet(\mathscr{R}_{\bullet,1}M_\bullet^{-1}\varepsilon_\bullet(\tau))\mathrm{d}\tau \tag{4.33}$$

where $0 < C_\bullet, \tilde{C}_\bullet \leq 1$ and $C_\bullet = 1$ if and only if $w(\underline{t}) \in \ker(H_\bullet^\star)$. Taking the supremum over $t \in [0,T]$ and using $w(\underline{t}) = \varepsilon_\bullet(\underline{t},\cdot)$ then yield

$$[\![\,\mathscr{R}_{\bullet,1}w\,]\!]_\bullet \leq (1+T)[\![\,\mathscr{R}_{\bullet,1}M_\bullet^{-1}\varepsilon_\bullet\,]\!]_\bullet.$$

The proof is complete. □

Remark 4.2.15. *Equation (4.33) shows that if u is a solution of $B_\bullet u = 0$ (that is $\varepsilon_\bullet = 0$ in (4.32)), then $\widetilde{\mathscr{T}}_{\bullet,0} \ni t \mapsto u(t,\cdot)$ is exponentially decreasing in the sense that $\mathcal{N}_\bullet(\mathscr{R}_{\bullet,1}u(t,\cdot)) \leq \alpha(t)^{|t|}\mathcal{N}_\bullet(\mathscr{R}_{\bullet,1}u(\underline{t},\cdot))$ for some positive function $\alpha : \mathbb{R}_+ \to [0,1]$. However, it may happen that $\alpha(t)^{|t|} \to \epsilon > 0$ as $t \to +\infty$ even if we are not in the unlikely case $u(t,\cdot) \in \ker(H_\bullet^\star)$ for all $t \in \widetilde{\mathscr{T}}_{\bullet,0}$: indeed, if we have uniformly in $n \gg 0$*

$$\mathcal{N}_n(\mathscr{R}_{n,1}M_n^{-1}H_n^\star u) \geq C\mathcal{N}_n(u)$$

for some constant $C > 0$, then Lemma 4.2.13 shows that

$$\alpha(t)^{|t|} = \big(1 + C(t_+ - t)^2\big)^{-t} = \exp\big(-|t|\ln(1 + C(t_+ - t)^2)\big)$$

provided that $u(t,\cdot) \notin \ker(H_n^\star)$. If we take a uniform mesh $\eta_n = \sqrt{T/N}$ for some $N \equiv N(n)$, then we get

$$\lim_{T\to+\infty}\alpha(T)^T = \exp(-C).$$

Recall that here $|T| = N$ is the position of T in the mesh point $\widetilde{\mathscr{T}}_{\bullet,0}$.

4.3 Convergence and application

This Section is devoted to the proof of the convergence of the scheme in the abstract setting. We then apply the obtained results to the charged Klein-Gordon equation in the exterior DSRN spacetime.

4.3.1 Convergence of the scheme

We can now prove the scheme convergence using the consistency and stability results showed above. **In the following, we will consider the implicit scheme** (that is $\mathscr{A} = \{1\}$ in the definition of $\mathscr{P}_{\bullet,0}^{\mathrm{temp}}$ in paragraph 4.1.2).

As usual, if more regularity is assumed, then one can get a precise speed for the convergence. We will distinguish both the cases of low regularity (the general case) and higher regularity (the particular case where we allow a loss of derivative for the initial data).

For the reader's convenience, we recall here some notations of Subsection 4.1.2: we have $B_\bullet = \mathrm{i}\partial_\bullet + H_\bullet$ with (see paragraph 4.1.2)

$$\begin{aligned} M_\bullet &:= \mathscr{P}_{\bullet,0}^{\mathrm{spat}} \circ \mathscr{R}_{\bullet,1}^{\mathrm{spat}} : \mathscr{W}_{\bullet,1}^{\mathrm{spat}} \longrightarrow \mathscr{W}_{\bullet,0}^{\mathrm{spat}}, \\ \partial_\bullet &:= \mathscr{P}_{\bullet,0}^{\mathrm{temp}} \circ \partial_t^\star \circ M_\bullet \circ \mathscr{R}_{\bullet,1}^{\mathrm{temp}} : \mathscr{W}_{\bullet,1} \longrightarrow \mathscr{W}_{\bullet,0}, \\ H_\bullet &:= \mathscr{P}_{\bullet,0}^{\mathrm{temp}} \circ \mathscr{P}_{\bullet,0}^{\mathrm{spat}} \circ H^\star \circ \mathscr{R}_{\bullet,1}^{\mathrm{spat}} \circ \mathscr{R}_{\bullet,1}^{\mathrm{temp}} : \mathscr{W}_{\bullet,1} \longrightarrow \mathscr{W}_{\bullet,0}. \end{aligned}$$

Furthermore, the filters $\mathscr{F}_{\bullet,i}^{\clubsuit} := \mathscr{R}_{\bullet,i}^{\clubsuit} \circ \mathscr{P}_{\bullet,i}^{\clubsuit}$ ($\clubsuit \in \{\mathrm{temp}, \mathrm{spat}\}$) and $\mathscr{F}_{\bullet,i} := \mathscr{R}_{\bullet,i} \circ \mathscr{P}_{\bullet,i} = \mathscr{R}_{\bullet,i}^{\mathrm{spat}} \circ \mathscr{R}_{\bullet,i}^{\mathrm{temp}} \circ \mathscr{P}_{\bullet,i}^{\mathrm{temp}} \circ \mathscr{P}_{\bullet,i}^{\mathrm{spat}}$ are defined in paragraph 4.1.2. As usual, we will omit the composition symbol $\circ$.

The idea of the proof of Theorem 4.3.1 below is quite simple: we compare the solution v of (ADP) to its filtered version $\mathscr{F}_{\bullet,1}v$ by splitting $\mathscr{F}_{\bullet,1}$ into its temporal and spatial parts. We use then consistency results of Section 4.2.2 to handle the spatial part, and we use the regularity $v \in \mathcal{C}^0([0,T], \mathscr{D}(H)) \cap \mathcal{C}^1([0,T], \dot{\mathcal{E}}_\rho)$ to deal with the temporal one. The main theorem of this chapter is the following:

Theorem 4.3.1 (Convergence of the scheme, low regularity)**.** *Assume* (AP1)-(AP7) *and* (Q). *Let $T > 0$ and $v = (v_0, v_1) \in \mathscr{W}_1$ be the solution of* (ADP)*:*

$$\begin{cases} (\mathrm{i}\partial_t + H)v = 0 & \forall t \in [0,T] \\ v(0,\cdot) = \varphi \text{ is compactly supported in } I \end{cases}.$$

Note $K_T := \cup_{0\le t\le T} \operatorname{Supp} v(t,\cdot)$ (it is a compact subset of I by Assumption (AP7)*) and pick $\mathcal{U}_T \subset I$ containing K_T.*

For all $\varepsilon > 0$, there exists a mesh $\mathscr{T}_\bullet$ as well as a constant $C > 0$ which only depend on the mesh, on $\|u(t,\cdot)\|_{H^2(\mathcal{U}_T,\mathbb{C})\times H^1(\mathcal{U}_T,\mathbb{C})}$, $\|\rho\|_{L^1(\mathcal{U}_T,\mathbb{R})}$, $\|a\|_{L^\infty(I,\mathbb{R})}, \|b\|_{L^\infty(I,\mathbb{R})}, \|c\|_{L^\infty(I,\mathbb{R})}$ and on $\|q\|_{L^\infty(I,\mathbb{R})}$, such that the unique solution $u_\bullet \in \mathscr{W}_{\bullet,1}^\star$ of the discrete problem

$$\begin{cases} (\mathrm{i}\partial_\bullet + H_\bullet)u_\bullet = 0 & \forall t \in \widetilde{\mathscr{T}}_{\bullet,0} \\ u_\bullet(0,\cdot) = \mathscr{P}_{\bullet,1}\varphi \end{cases}$$

verifies

$$[\![v - \mathscr{R}_{\bullet,1}u_\bullet]\!]_\bullet \le C_1(1+T)\varepsilon. \tag{4.34}$$

In particular, the approximated solution $u_\bullet$ converges to the continuous solution v in norm: there exists a constant $C_2 > 0$ depending on the same parameters as C_1 above such that, for all $t \in [0,T]$,

$$\left|\widetilde{\mathcal{N}}_\bullet\big(\mathscr{R}_{\bullet,1}u_\bullet(t,\cdot)\big) - |||v(t,\cdot)|||_{\dot{\mathcal{E}}_\rho}\right| \le C_2(1+T)\varepsilon. \tag{4.35}$$

Proof. Fix a mesh $\mathscr{T}_\bullet$ as in Corollary 4.2.4 for the spatial part and such that the stability condition (4.29) is satisfied (this last condition ensures that the discrete problem has a unique solution, see Proposition 4.2.14). We will settle uniform bounds in $t \in [0,T]$ using the spatial norm $\mathcal{N}_\bullet$. Let us write

$$\mathcal{N}_\bullet\big(v - \mathscr{R}_{\bullet,1}u_\bullet\big) \le \mathcal{N}_\bullet\big(v - \mathscr{F}_{\bullet,1}v\big) + \mathcal{N}_\bullet\big(\mathscr{F}_{\bullet,1}v - \mathscr{R}_{\bullet,1}u_\bullet\big) =: \mathrm{I} + \mathrm{II}.$$

Estimate for I. For the first term, we have

$$\mathcal{N}_\bullet\big(v - \mathscr{F}_{\bullet,1} v\big) \le \mathcal{N}_\bullet\big(v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big) + \mathcal{N}_\bullet\big(\mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \mathscr{F}_{\bullet,1} v\big).$$

On the one hand, (4.23) of Lemma 4.2.12 shows that there exists a constants $C_1 > 0$ such that

$$\Big|\widetilde{\mathcal{N}}_\bullet\big(v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big) - \big|\!\big|\!\big|v - \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big|\!\big|\!\big|_{\dot{\mathcal{E}}_\rho}\Big| \le C_1 \varepsilon.$$

On the other hand, since $\mathscr{F}^{\mathrm{spat}}_{\bullet,1}\mathscr{F}^{\mathrm{spat}}_{\bullet,1} = \mathscr{F}^{\mathrm{spat}}_{\bullet,1}$ on $\mathscr{W}_1^\star$, and since Q is a multiplication operator, we trivially have[2]

$$\mathcal{N}_\bullet\big(\mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \mathscr{F}_{\bullet,1} v\big) = 0.$$

This gives an estimate for the term I.

Estimate for II. We use the stability of the scheme in order to bound the term II: by Proposition 4.2.14, we have

$$\mathcal{N}_\bullet\big(\mathscr{F}_{\bullet,1} v - \mathscr{R}_{\bullet,1} u_\bullet\big) \le (1+T)\,\mathcal{N}_\bullet\big(\mathscr{R}_{\bullet,1} M_\bullet^{-1} B_\bullet(\mathscr{P}_{\bullet,1} v - u_\bullet)\big).$$

Since $B_\bullet u_\bullet = Bv = 0$, we can replace $B_\bullet u_\bullet$ by $\mathscr{P}_{\bullet,0} Bv$ on the right hand side:

$$\mathcal{N}_\bullet\big(\mathscr{F}_{\bullet,1} v - \mathscr{R}_{\bullet,1} u_\bullet\big) \le (1+T)\,\mathcal{N}_\bullet\big(\mathscr{R}_{\bullet,1} M_\bullet^{-1}(B_\bullet \mathscr{P}_{\bullet,1} v - \mathscr{P}_{\bullet,0} Bv)\big). \tag{4.36}$$

Consider first the temporal part of the difference in (4.36):

$$\partial_\bullet \mathscr{P}_{\bullet,1} v - \mathscr{P}_{\bullet,0}\partial_t v = \mathscr{P}^{\mathrm{temp}}_{\bullet,0}\mathscr{P}^{\mathrm{spat}}_{\bullet,0}\Big(\big(\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1}\mathscr{F}^{\mathrm{spat}}_{\bullet,1} v - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big) + \big(\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \partial_t v\big)\Big).$$

We use that $\partial_t^\star$ commutes with $\mathscr{F}^{\mathrm{spat}}_{\bullet,1}$ as well as $\partial_t^\star v = \partial_t v$ and $\mathscr{P}^{\mathrm{spat}}_{\bullet,0}\mathscr{F}^{\mathrm{spat}}_{\bullet,1} = \mathscr{P}^{\mathrm{spat}}_{\bullet,0}$ (using the inclusion $\mathscr{W}^\star \subset \mathscr{W}_H^\star$) to get

$$\begin{aligned}
&\mathcal{N}_\bullet\big(\mathscr{R}_{\bullet,1} M_\bullet^{-1}\mathscr{P}^{\mathrm{temp}}_{\bullet,0}\mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big(\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1}\mathscr{F}^{\mathrm{spat}}_{\bullet,1} v - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big)\big)\\
&= \mathcal{N}_\bullet\big(\mathscr{R}_{\bullet,1} M_\bullet^{-1}\mathscr{P}^{\mathrm{temp}}_{\bullet,0}\mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big(\mathscr{F}^{\mathrm{spat}}_{\bullet,1}\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big)\big)\\
&= \mathcal{N}_\bullet\big(\mathscr{R}_{\bullet,1} M_\bullet^{-1}\mathscr{P}^{\mathrm{temp}}_{\bullet,0}\mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big(\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v\big)\big)\\
&= 0.
\end{aligned}$$

As for the other term, we use Lemma 4.2.5 to write

$$\begin{aligned}
&\mathcal{N}_\bullet\big(\mathscr{R}_{\bullet,1} M_\bullet^{-1}\mathscr{P}^{\mathrm{temp}}_{\bullet,0}\mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big(\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \partial_t v\big)\big)\\
&= \mathcal{N}_\bullet\big(\mathscr{R}^{\mathrm{spat}}_{\bullet,0}\mathscr{P}^{\mathrm{spat}}_{\bullet,0}\mathscr{R}^{\mathrm{temp}}_{\bullet,0}\mathscr{P}^{\mathrm{temp}}_{\bullet,0}(\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \partial_t v)\big)\\
&= \mathcal{N}_\bullet\big(\mathscr{F}^{\mathrm{spat}}_{\bullet,0}\mathscr{F}^{\mathrm{temp}}_{\bullet,0}(\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \partial_t v)\big).
\end{aligned}$$

and conclude with (4.24) of Lemma 4.2.12 that

$$\mathcal{N}_\bullet\big(\mathscr{R}_{\bullet,1} M_\bullet^{-1}\mathscr{P}^{\mathrm{temp}}_{\bullet,0}\mathscr{P}^{\mathrm{spat}}_{\bullet,0}\big(\partial_t^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \partial_t v\big)\big) \le C_2\varepsilon$$

[2] This property is the consistency of the spatial filter $\mathscr{F}^{\mathrm{spat}}_{\bullet,1}$ with the identity operator on $\mathscr{W}^\star$.

for some constant $C_2 > 0$.

Consider now the spatial part of the difference in (4.36):

$$H_\bullet \mathscr{P}_{\bullet,1} v - \mathscr{P}_{\bullet,0} H v = \mathscr{P}^{\mathrm{temp}}_{\bullet,0} \mathscr{P}^{\mathrm{spat}}_{\bullet,0} \Big(\big(H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} \mathscr{F}^{\mathrm{spat}}_{\bullet,1} v - H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v \big) + \big(H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - H v \big) \Big).$$

The first term above is exactly the one estimated in Proposition 4.2.10 for $t \in \widetilde{\mathscr{T}}_{\bullet,0}$ (because of the temporal filtrations $\mathscr{F}^{\mathrm{temp}}_{\bullet,0} \mathscr{F}^{\mathrm{temp}}_{\bullet,1}$): there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
&\mathcal{N}_\bullet \big(\mathscr{R}_{\bullet,1} M_\bullet^{-1} \mathscr{P}^{\mathrm{temp}}_{\bullet,0} \big(\mathscr{P}^{\mathrm{spat}}_{\bullet,0} \big(H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} \mathscr{F}^{\mathrm{spat}}_{\bullet,1} v - H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v \big) \big) \\
&= \mathcal{N}_\bullet \big(\mathscr{R}^{\mathrm{spat}}_{\bullet,1} M_\bullet^{-1} \mathscr{P}^{\mathrm{spat}}_{\bullet,0} \big(\mathscr{F}^{\mathrm{temp}}_{\bullet,0} H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} \mathscr{F}^{\mathrm{spat}}_{\bullet,1} v - \mathscr{F}^{\mathrm{temp}}_{\bullet,0} H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v \big) \\
&\leq C_3 \varepsilon.
\end{aligned}$$

As for the remaining term, a direct computation shows for $w := Hv$ that

$$\big(\mathscr{F}^{\mathrm{temp}}_{\bullet,1} w - \mathscr{F}^{\mathrm{temp}}_{\bullet,0} \mathscr{F}^{\mathrm{temp}}_{\bullet,1} w \big)(t) = \sum_{s \in \widetilde{\mathscr{T}}_{\bullet,0}} \frac{t - s_+}{s - s_+} \big(w(s) - w(s_+) \big) \mathbb{1}_{[s, s_+[}(t).$$

We use then Lemma 4.2.5, the norm equivalences (4.5) and (4.6) as well as the consistency of the norms 4.2.8 (for $Hv = H^\star v$): this yields

$$\begin{aligned}
&\mathcal{N}_\bullet \big(\mathscr{R}^{\mathrm{spat}}_{\bullet,1} M_\bullet^{-1} \mathscr{P}^{\mathrm{spat}}_{\bullet,0} \big(\mathscr{F}^{\mathrm{temp}}_{\bullet,0} H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \mathscr{F}^{\mathrm{temp}}_{\bullet,0} H v \big) \big) \\
&= \mathcal{N}_\bullet \big(\mathscr{R}^{\mathrm{spat}}_{\bullet,0} \mathscr{P}^{\mathrm{spat}}_{\bullet,0} \big(\mathscr{F}^{\mathrm{temp}}_{\bullet,0} H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \mathscr{F}^{\mathrm{temp}}_{\bullet,0} H v \big) \big) \\
&\leq \Big| \widetilde{\mathcal{N}}_\bullet \big(\mathscr{F}^{\mathrm{temp}}_{\bullet,0} H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \mathscr{F}^{\mathrm{temp}}_{\bullet,0} H v \big) - \big| \big| \big| \mathscr{F}^{\mathrm{temp}}_{\bullet,0} H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \mathscr{F}^{\mathrm{temp}}_{\bullet,0} H v \big| \big| \big|_{\dot{\mathcal{E}}_\rho} \Big| \\
&+ \sqrt{2} \big\| \mathscr{F}^{\mathrm{temp}}_{\bullet,0} H^\star \mathscr{F}^{\mathrm{temp}}_{\bullet,1} v - \mathscr{F}^{\mathrm{temp}}_{\bullet,0} H v \big\|_{\dot{\mathcal{E}}_\rho} \\
&\leq C_4 \varepsilon + C_5 \sqrt{2} \varepsilon
\end{aligned}$$

for some constants $C_4, C_5 > 0$. This gives an estimate for the term II.

Convergence of the norms. We now turn to the convergence in norm of the approximated solution $u_\bullet$ to the continuous one v (estimate (4.35)). The convergence estimate (4.34) and the norm consistency 4.2.8 imply that there exist two constant $C_1, C_2 > 0$ such that

$$\Big| \widetilde{\mathcal{N}}_\bullet \big(\mathscr{R}_{\bullet,1} u_\bullet \big) - |||v||| _{\dot{\mathcal{E}}_\rho} \Big| \leq \widetilde{\mathcal{N}}_\bullet \big(\mathscr{R}_{\bullet,1} u_\bullet - v \big) + \Big| \widetilde{\mathcal{N}}_\bullet (v) - |||v||| _{\dot{\mathcal{E}}_\rho} \Big| \leq C_1 (1 + T) \varepsilon + C_2 \varepsilon$$

as the mesh converges. This completes the proof. □

Remark 4.3.2. *We can give a geometric interpretation of the notion of convergence (see Figure 4.6). A sequence $(u_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathscr{W}_{n,1}$ converges to a solution $u \in \dot{\mathcal{E}}_\rho$ of (ADP) means that the (affine) cone for the approximated spatial norm $\widetilde{\mathcal{N}}_n(\cdot)$ above u converges for solutions to the (affine) cone for the norm $|||\cdot|||_{\dot{\mathcal{E}}_\rho}$ above u in the sense that $\widetilde{\mathcal{N}}_n(\mathscr{R}_{n,1} u_n(t,\cdot)) \to |||u(t,\cdot)|||_{\dot{\mathcal{E}}_\rho}$ for all $t \in [0,T]$, and in each cone $\big\{ |||v - u|||_{\dot{\mathcal{E}}_\rho} \mid v \in \mathscr{W}_{n,1} \big\} \subset \mathscr{W}_{n,1}$, the number $\widetilde{\mathcal{N}}_n(\mathscr{R}_{n,1} u_n(t,\cdot) - u(t,\cdot))$ becomes smaller as $n \to +\infty$. In other words, we can build a sequence*

$$\Big(\widetilde{\mathcal{N}}_n \big(\mathscr{R}_{n,1} u_n(t,\cdot) - u(t,\cdot) \big) \Big)_{n \in \mathbb{N}} \in (\mathbb{R}_+)^{\mathbb{N}}$$

which goes to 0 as $n \to +\infty$.

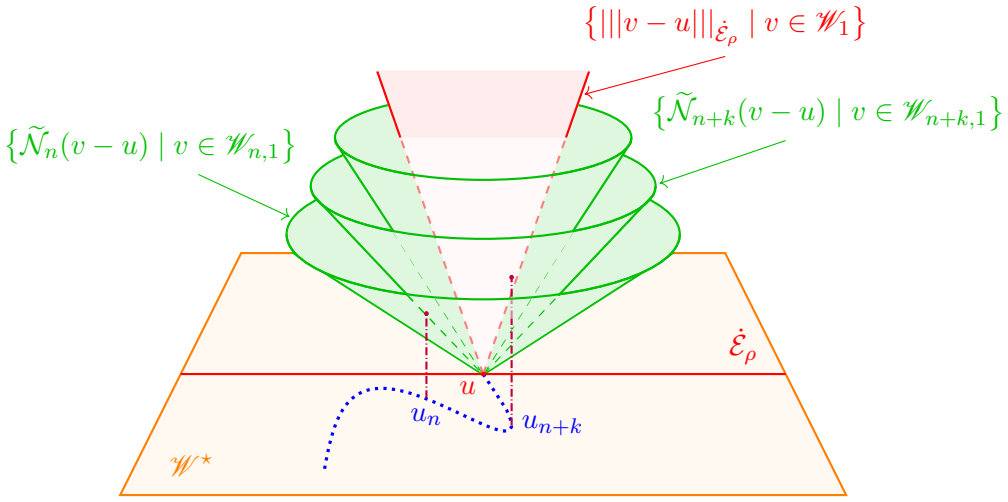

Figure 4.6: Geometric interpretation of convergence.

Remark 4.3.3. *The convergence of Theorem 4.3.1 implies a local convergence for the graph topology of* $\big([0,T]\times I\big)\oplus\mathcal{C}^0\big([0,T],\dot{\mathcal{E}}_\rho\big)$ *in the following sense. In each cell* $[s,s_+[\times\omega_p$ *of the mesh, the approximated solution* $u_\bullet$ *is smooth (it is a polynomial in each variable), so it belongs to the space* $\mathcal{C}^0\big([0,T],\dot{\mathcal{E}}_\rho\big)$*; on this cell, we can certainly compare up to a small error both the norms* $\mathcal{N}_\bullet$ *and* $\|\cdot\|_{\dot{\mathcal{E}}_\rho}$ *(the only difference lies in the position of the operator* Q *in the approximated norm* $\mathcal{N}_\bullet$*, but Assumption (Q) makes the comparison easy). We use then that the approximated norm of the difference* $u-\mathscr{R}_{\bullet,1}u_\bullet$ *is small for a choice of the mesh such that* s_+-s *and* $|\omega_p|$ *are small enough.*

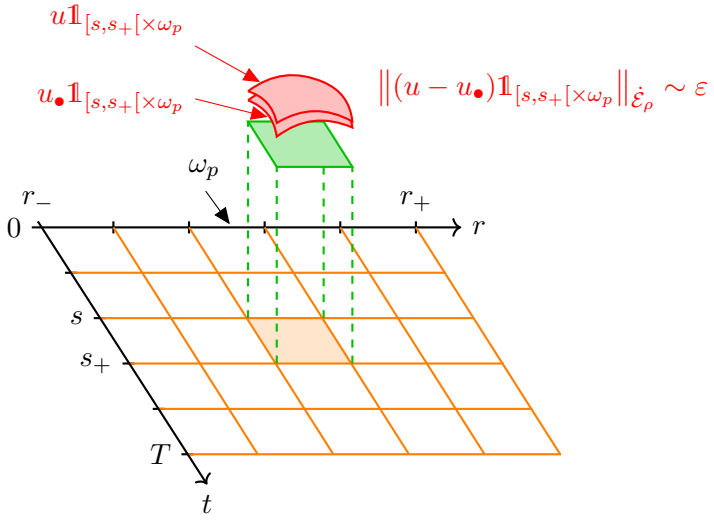

Figure 4.7: Graph convergence on the cells of the mesh.

Theorem 4.3.4 (Convergence of the scheme, high regularity)**.** *Assume* (AP1)-(AP7) *and* (Q).

Let $T > 0$ and $v = (v_0, v_1) \in \mathscr{W}_1$ be the solution of (ADP):

$$\begin{cases} (\mathrm{i}\partial_t + H)v = 0 & \forall t \in [0,T] \\ v(0,\cdot) = \varphi \text{ is compactly supported in } I \end{cases}.$$

Note $K_T := \cup_{0\le t\le T} \operatorname{Supp} v(t,\cdot)$ (it is a compact subset of I by Assumption (AP7)) and pick $\mathcal{U}_T \subset I$ containing K.

If $H\varphi \in \mathscr{D}(H)$, then for all mesh satisfying Assumption (M) as well as the stability condition (4.29), there exists a constant $C_1 > 0$ depending only on the constant C_I of Assumption (M), $\|u(t,\cdot)\|_{H^3(\mathcal{U}_T,\mathbb{C})\times H^1(\mathcal{U}_T,\mathbb{C})}$ and $\|q\|_{\mathcal{C}^1(I,\mathbb{R})}$ such that the unique solution $u_\bullet \in \mathscr{W}^\star_{\bullet,1}$ of the discrete problem

$$\begin{cases} (\mathrm{i}\partial_\bullet + H_\bullet)u_\bullet = 0 & \forall t \in \widetilde{\mathscr{T}}_{\bullet,0} \\ u_\bullet(0,\cdot) = \mathscr{P}_{\bullet,1}\varphi \end{cases}$$

verifies

$$[\![v - \mathscr{R}_{\bullet,1}u_\bullet]\!]_\bullet \le C_1(1+T)\Big(\|a\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})} + h_\bullet^{1/2}\|c\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})} + \|\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})}\Big)h_\bullet^{1/2}.$$

In particular, the approximated solution $u_\bullet$ converges to the continuous solution v in norm: there exists a constant $C_2 > 0$ depending on the same parameters as C_1 above such that, for all $t \in [0,T]$,

$$\begin{aligned} &\Big|\widetilde{\mathcal{N}}_\bullet\big(\mathscr{R}_{\bullet,1}u_\bullet(t,\cdot)\big) - |||v(t,\cdot)|||_{\dot{\mathcal{E}}_\rho}\Big| \\ &\le C_2(1+T)\Big(\|a\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})} + h_\bullet^{1/2}\|c\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})} + \|\rho\mathbb{1}_{\omega_\bullet}\|^{1/2}_{L^1(\mathcal{U}_T,\mathbb{R})}\Big)h_\bullet^{1/2}. \end{aligned}$$

If furthermore we assume $\zeta := \max\{\zeta_-, \zeta_+\} < 2$, then Assumption (AP1) implies

$$[\![v - \mathscr{R}_{\bullet,1}u_\bullet]\!]_\bullet \underset{\bullet}{\longrightarrow} 0.$$

Proof. Lemma 4.1.9 implies that the solution u of (ADP) satisfies $u \in \dot{\mathcal{H}}^3_\rho \times \mathcal{H}^2$. Using Assumption (AP2) and Sobolev embedding theorem, we see that $u \in \mathcal{C}^{2,1/2}(\mathcal{U}_T,\mathbb{C}) \times \mathcal{C}^{1,1/2}(\mathcal{U}_T,\mathbb{C})$ so that all the error of interpolation and consistency results obtained in Section 4.2.2 apply for a greater regularity, yielding the better convergence rate. We then just have to mimic the proof of Theorem 4.3.1 with these better estimates. □

4.3.2 Application to the charged Klein-Gordon equation in the DSRN space-time

We apply in this Section the numerical scheme introduced previously to the charged Klein-Gordon equation on the DSRN metric.

Notations

We recall some notations and results from Chapter 1. Let $V(r) := r^{-1}$. For $r \in]r_-, r_+[$, *the charged Klein-Gordon equation* reads

$$(\partial_t - \mathrm{i}qQV)^2 u + Pu = 0, \tag{4.37}$$

with

$$\begin{aligned} P &:= -r^{-2}F(r)\partial_r r^2 F(r)\partial_r - r^{-2}F(r)\Delta_{\mathbb{S}^2} + m^2 F(r) \\ &= -F(r)^2\partial_r^2 - F(r)\left(\frac{2F(r)}{r} + F'(r)\right)\partial_r - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} + m^2 F(r). \end{aligned}$$

Here, $(m, q) \in]0, +\infty[\times \mathbb{R}$ is the couple formed with the mass and the charge of the scalar field whose evolution is governed by the Klein-Gordon equation. We then define on $L^2\left(]r_-, r_+[\times \mathbb{S}^2, F^{-1}(r)\mathrm{d}r\mathrm{d}\omega\right)$ the new spatial operator

$$\begin{aligned} \hat{P} := rPr^{-1} &= -r^{-1}F(r)\partial_r r^2 F(r)\partial_r r^{-1} - r^{-2}F(r)\Delta_{\mathbb{S}^2} + m^2 F(r) \\ &= -F(r)\partial_r F(r)\partial_r - \frac{F(r)}{r^2}\Delta_{\mathbb{S}^2} + F(r)\left(\frac{F'(r)}{r} + m^2\right) \end{aligned} \tag{4.38}$$

We will denote by $\hat{P}_\ell$ the operator $\hat{P}$ restricted on $\ker(\Delta_{\mathbb{S}^2} + \ell(\ell+1))$ for all $\ell \in \mathbb{N}$. The Regge-Wheeler coordinate $x := \phi(r)$ is defined by

$$x \equiv \phi(r) = \sum_{\alpha \in \{n,c,-,+\}} \frac{1}{2\kappa_\alpha} \ln\left|\frac{r - r_\alpha}{\mathfrak{r} - r_\alpha}\right| \tag{4.39}$$

where $\kappa_\alpha := F'(r_\alpha)/2$ are the *surface gravities* and $\mathfrak{r} \in]r_-, r_+[$ (we do not necessarily choose $\mathfrak{r}$ as the radius of the photon sphere as in Subsection 1.1.2). Then $\phi :]r_-, r_+[\ni r \mapsto x \equiv x(r) \in \mathbb{R}$ is an increasing bijection. Using the factorization

$$F(r) = \frac{\Lambda}{3r^2}(r - r_n)(r - r_c)(r - r_-)(r_+ - r) \qquad \forall r \in]r_-, r_+[, \tag{4.40}$$

we see that

$$|r - r_\pm| = |\mathfrak{r} - r_\pm| \left(\prod_{\alpha \in \{n,c,\mp\}} \left|\frac{\mathfrak{r} - r_\alpha}{r - r_\alpha}\right|^{\frac{\kappa_\pm}{\kappa_\alpha}}\right) \mathrm{e}^{2\kappa_\pm x} = \mathcal{O}_{r \to \pm\infty}\left(\mathrm{e}^{2\kappa_\pm x}\right) \tag{4.41}$$

which is an exponential decay as $\kappa_- > 0$ and $\kappa_+ < 0$. In the coordinates $(x, \omega) \in \mathbb{R} \times \mathbb{S}^2$, the spatial operator $\hat{P}$ takes the form

$$\hat{P} = -r^{-2}\partial_x r^2 \partial_x r^{-1} - r^{-2}F(r)\Delta_{\mathbb{S}^2} + m^2 F(r)$$

where r is now to be read as a function of x, and $\hat{P}$ acts on $L^2\left(\mathbb{R} \times \mathbb{S}^2; \mathrm{d}x\mathrm{d}\omega\right)$.

We henceforth use the notation $s := qQ \in \mathbb{R}$ for the *charge product*. The charge Klein-Gordon operator is defined as

$$\hat{K}(s) := \begin{pmatrix} sV & \mathbb{1} \\ \hat{P} & sV \end{pmatrix}. \tag{4.42}$$

We can also define $\hat{K}_\ell(s)$ with P_ℓ instead of P. We can check that if u solves (4.37) then $v = (u, -\mathrm{i}\partial_t u - su)$ solves

$$-\mathrm{i}\partial_t v = \hat{K}(s)v. \tag{4.43}$$

Conversely, if $v = (v_0, v_1)$ solves (4.43) then v_0 solves (4.37). Theorem 1.3.2 shows that the solutions of (ADP) decay exponentially fast in time.

Application of the mumerical shceme

We easily check that the charged Klein-Gordon equation in the exterior DSRN spacetime enter the abstract setting introduced in Section 4.1. Only the propagation estimate for of solutions $v = (v_0, v_1) = (v_0, -\mathrm{i}\partial_t v_0 - sVv_0)$ of (4.43) is not immediate, so we give the argument in the following lemma. Since $v^\pm := \mathrm{e}^{-\mathrm{i}sr_\pm^{-1}t}v$ satisfies

$$\left(\partial_t^2 - 2\mathrm{i}sV_\pm\partial_t - s^2V_\pm^2 + P\right)v_0^\pm = 0$$

with $V_\pm := V - r_\pm^{-1}$ and $\mathrm{e}^{-\mathrm{i}sr_\pm^{-1}t}$ is unitary for all $t \geq 0$, the supports of v and $v^\pm$ are the same.

Lemma 4.3.5 (Propagation estimate)**.** *Assume* $\operatorname{Supp} v(0,.) \subset [R, R']$ *for some real constants* R, R' *such that* $R, R' \in [r_-, r_+]$*. If*

$$s^2 < \begin{cases} m^2\frac{\Lambda r_+}{3r}(r-r_n)(r-r_c)(r-r_-) & \text{for all } r \geq R' \\ m^2\frac{\Lambda r_-}{3r}(r-r_n)(r-r_c)(r_+-r) & \text{for all } r \leq R \end{cases}, \tag{4.44}$$

then

$$\operatorname{Supp} v(t,.) \subset \left[\phi^{-1}(\phi(R)-t)\,, \phi^{-1}\big(\phi(R')+t\big)\right] \qquad \forall t \geq 0.$$

Proof. We work on $\mathbb{R}$ with the Regge-Wheeler coordinate $x = \phi(r) \in \mathbb{R}$. For the sake of clarity, we will consider that $\ell = 0$ (one must add otherwise a non negative extra term $\ell(\ell+1)|v_0(t,x)|^2$ below which is easily handled).

Define for all $t \geq 0$ the function

$$E_+(t) := \int_{\phi(R')+t}^{+\infty} \Big[\, |\partial_t v_0^+(t,x)|^2 + |r\partial_x r^{-1} v_0^+(t,x)|^2 + \big(m^2F(r(x)) - s^2V_+(r(x))^2\big)|v_0^+(t,x)|^2 \,\Big]\mathrm{d}x.$$

The derivative is given by

$$\begin{aligned}
\dot{E}_+(t) = &-\Big(|\partial_t v_0^+(t,\phi(R')+t)|^2 + \left|r\partial_x r^{-1}v_0^+(t,\phi(R')+t)\right|^2 \\
&\quad + \left(m^2F(r(\phi(R')+t)) - s^2V_+(r(\phi(R')+t))^2\right)\left|v_0^+(t,\phi(R')+t)\right|^2\Big) \\
&+ 2\Re\int_{\phi(R')+t}^{+\infty}\Big[\,\overline{\partial_t v_0^+(t,x)}\partial_t^2 v_0^+(t,x) + \overline{r\partial_x r^{-1}v_0^+(t,x)}r\partial_x r^{-1}\partial_t v_0^+(t,x) \\
&\qquad + \big(m^2F(r(x)) - s^2V_+(r(x))^2\big)\overline{v_0^+(t,x)}\partial_t v_0^+(t,x)\,\Big]\mathrm{d}x
\end{aligned}$$

and an integration by part for the second term of the integrand yields

$$\begin{aligned}
&\Re\int_{\phi(R')+t}^{+\infty}\Big[\,\overline{\partial_t v_0^+(t,x)}\partial_t^2 v_0^+(t,x) + \overline{r\partial_x r^{-1}v_0^+(t,x)}r\partial_x r^{-1}\partial_t v_0^+(t,x) \\
&\qquad + \big(m^2F(r(x)) - s^2V_+(r(x))^2\big)\overline{v_0^+(t,x)}\partial_t v_0^+(t,x)\,\Big]\mathrm{d}x \\
&= \Re\int_{\phi(R')+t}^{+\infty}\overline{\partial_t v_0^+(t,x)}\Big[\,\partial_t^2 v_0^+(t,x) - r^{-1}\partial_x r^2\partial_x r^{-1}v_0^+(t,x) + \big(m^2F(r(x)) - s^2V_+(r(x))^2\big)v_0^+(t,x)\,\Big]\mathrm{d}x \\
&+ \Re\left(\partial_t v_0^+(t,\phi(R')+t)r\partial_x r^{-1}v_0^+(t,x)\right) \\
&= \Re\int_{\phi(R')+t}^{+\infty}\overline{\partial_t v_0^+(t,x)}\Big[\,2\mathrm{i}sV_+(r(x))\partial_t v_0^+(t,x)\,\Big]\mathrm{d}x + \Re\left(\partial_t v_0^+(t,\phi(R')+t)r\partial_x r^{-1}v_0^+(t,x)\right).
\end{aligned}$$

Since sV_+ is real, the last integral above is zero, whence

$$\dot{E}_+(t) = -\Big(\big|\partial_t v_0^+(t,\phi(R')+t) - r\partial_x r^{-1} v_0^+(t,\phi(R')+t)\big|^2 + \big(m^2 F(r(\phi(R')+t)) - s^2 V_+(r(\phi(R')+t))^2\big)\,\big|v_0^+(t,\phi(R')+t)\big|^2 \Big).$$

If now we assume (4.44), then

$$m^2 F(r(\phi(R')+t)) - s^2 V_+(r(\phi(R')+t))^2 > 0$$

for all $t \geq 0$ (we used that F and V_+ decay at r_+ at the same rate $r_+ - r$). In this condition, $\dot{E}(t) \leq 0$ for all $t \geq 0$, and thus

$$\begin{aligned} E_+(t) &\leq E_+(0) \\ &= \int_{\phi(R')+t}^{+\infty} \Big[\, |\partial_t v_0^+(t,x)|^2 + |r\partial_x r^{-1} v_0^+(t,x)|^2 + \big(m^2 F(r(x)) - s^2 V_+(r(x))^2\big)|v_0^+(t,x)|^2 \,\Big]\mathrm{d}x. \end{aligned}$$

Since by assumption, we have $v_0^+(0,x) = 0$ for all $x > \phi(R')$, it follows that $E_+(t) \leq E_+(0) = 0$ for all $t \geq 0$, whence $v_0^+(t,\phi(R')+t) = 0$ and thus $v^+(t,\phi(R')+t) = 0$.

In a similar way, we can show that for any $t \geq 0$, we have

$$E_-(t) := \int_{-\infty}^{\phi(R)-t} \Big[\, |\partial_t v_0^-(t,x)|^2 + |r\partial_x r^{-1} v_0^-(t,x)|^2 + \big(m^2 F(r(x)) - s^2 V_-(r(x))^2\big)|v_0^-(t,x)|^2 \,\Big]\mathrm{d}x = 0$$

so that $v^-(t,\phi(R')+t) = 0$ for all $t \geq 0$. This completes the proof. □

Remark 4.3.6. *If one wants to get a uniform upper bound for $|s|$ in (4.44) in order to apply Proposition 4.3.5, we can replace R and R' in the latter assumption by any $r_0 < r_1$ in I; however, the cost to pay is that the conclusion applies only for long times (long enough so that $\phi^{-1}\big(\phi(R')+t\big) > r_1$ and $\phi^{-1}\big(\phi(R)-t\big) < r_0$).*

4.4 Appendix

In this Appendix, we give compute the discrete operators introduced throughout this Chapter and in the particular case of the charged Klein-Gordon equation in the DSRN spacetime.

As all the operators we will be interested in are spatial ones, we will only work on the discrete spaces $\mathscr{W}^\star$ and $\mathscr{W}_H^\star$. We will assume (Q) and choose a mesh $\mathscr{T}_{\bullet,0}$ satisfying (M). Furthermore, for a given function $u \in \mathscr{W}_H^\star$, we will assume that the mesh is included in the Lebesgue points set $\mathscr{L}(u)$ (see Remark 4.1.13). In this setting, we have $\mathscr{P}^{\mathrm{spat}}_{\bullet,0} u = \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u$.

We introduce here some notations used in the sequel. Let $K := \{-,\ ,+\}$. We then define for all $k \in K$ and all $r \in I$

$$\sigma_k(p,r) = \frac{(r-p_{k'})(r-p_{k''})}{(p_k-p_{k'})(p_k-p_{k''})} \qquad\qquad k',k'' \in K\setminus\{k\}.$$

4.4.1 Computation of $M_\bullet$

Let $u \in \mathscr{W}^\star_{\bullet,1}$ and let $(t,r) \in [0,T] \times]r_-, r_+[$. Recall that

$$M_\bullet := \mathscr{P}^{\mathrm{spat}}_{\bullet,0} \circ \mathscr{R}^{\mathrm{spat}}_{\bullet,1} : \mathscr{W}^\star_{\bullet,1} \to \mathscr{W}^\star_{\bullet,H}.$$

Since $\mathscr{R}^{\mathrm{spat}}_{\bullet,1}$ is the canonical embedding $\mathscr{W}^\star_{\bullet,1} \hookrightarrow \mathscr{W}^\star_1$ and $\mathscr{P}^{\mathrm{spat}}_{\bullet,0} \circ \mathscr{P}^{\mathrm{spat}}_{\bullet,1} = \mathscr{P}^{\mathrm{spat}}_{\bullet,1}$, we simply have

$$(M_\bullet u)(t,r) = \sum_{p \in \mathring{\mathscr{T}}_{\bullet,0}} \sum_{k \in K} \sigma_k(p,r) u(t,p_k) \mathbb{1}_{\omega_p}(r) + \frac{r - r_-}{\underline{p} - r_-} u(t,\underline{p}) \mathbb{1}_{\omega_{\underline{p}}}(r) + \frac{r - r_+}{\overline{p} - r_+} u(t,\overline{p}) \mathbb{1}_{\omega_{\overline{p}}}(r).$$

Observe that $(M_\bullet u)(p) = u(p)$ for all $p \in \mathscr{T}_{\bullet,0}$.

4.4.2 Computation of $H^\star_\bullet$

Let $u = (u_0, u_1) \in \mathscr{W}^\star_{\bullet,1}$ and let $(t,r) \in [0,T] \times]r_-, r_+[$. Recall that

$$H_\bullet := \mathscr{P}^{\mathrm{spat}}_{\bullet,0} \circ H^\star \circ \mathscr{R}^{\mathrm{spat}}_{\bullet,1} : \mathscr{W}^\star_{\bullet,1} \to \mathscr{W}^\star_{\bullet,0}.$$

We will compute $\mathscr{P}^{\mathrm{spat}}_{\bullet,0} \circ \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u$, viewing $\mathscr{P}^{\mathrm{spat}}_{\bullet,1} u$ as an element of $\mathscr{W}^\star_{\bullet,1} \subset \mathscr{W}^\star_{\bullet,0}$. Write $v := \mathscr{P}^{\mathrm{spat}}_{\bullet,1} u$. Then

$$\begin{aligned}(\partial^\star_r v)(t,r) &= \partial^\star_r \left(\sum_{p \in \mathring{\mathscr{T}}_{\bullet,0}} \sum_{k \in K} \sigma_k(p,r) u(t,p_k) \mathbb{1}_{\omega_p}(r) + \frac{r - r_-}{\underline{p} - r_-} u(t,\underline{p}) \mathbb{1}_{\omega_{\underline{p}}}(r) + \frac{r - r_+}{\overline{p} - r_+} u(t,\overline{p}) \mathbb{1}_{\omega_{\overline{p}}}(r) \right) \\ &= \sum_{p \in \mathring{\mathscr{T}}_{\bullet,0}} \sum_{k \in K} \frac{2r - (p_{k'} + p_{k''})}{(p_k - p_{k'})(p_k - p_{k''})} u(t,p_k) \mathbb{1}_{\omega_p}(r) + \frac{u(t,\underline{p})}{\underline{p} - r_-} \mathbb{1}_{\omega_{\underline{p}}}(r) + \frac{u(t,\overline{p})}{\overline{p} - r_+} \mathbb{1}_{\omega_{\overline{p}}}(r)\end{aligned} \tag{4.45}$$

and

$$\big((\partial^\star_r)^2 v\big)(t,r) = 2 \sum_{p \in \mathring{\mathscr{T}}_{\bullet,0}} \sum_{k \in K} \frac{u(t,p_k)}{(p_k - p_{k'})(p_k - p_{k''})} \mathbb{1}_{\omega_p}(r) \tag{4.46}$$

where we used above the convention $k', k'' \in K \setminus \{k\}$ when $k \in K$ was fixed. We have

$$(H^\star v)(t,r) = \begin{pmatrix} (Qv_0)(t,r) + v_1(t,r) \\ (P^\star v_0)(t,r) + (Qv_1)(t,r) \end{pmatrix} = \begin{pmatrix} (Qv_0)(t,r) + v_1(t,r) \\ \big(a(r)(\partial^\star_r)^2 + b(r)\partial^\star_r + c(r)\big) v_0(t,r) + (Qv_1)(t,r) \end{pmatrix}.$$

We then componentwise apply the projector $\mathscr{P}^{\mathrm{spat}}_{\bullet,0}$:

$$\begin{aligned}\big(H^\star_\bullet u\big)_0(t,r) &= \sum_{p \in \mathring{\mathscr{T}}_{\bullet,0}} \sum_{k \in K} \sigma_k(p,r) \big(q(p_k) u_0(t,p_k) + u_1(t,p_k)\big) \mathbb{1}_{\omega_p}(r) \\ &\quad + \frac{r - r_-}{\underline{p} - r_-} \big(q(\underline{p}) u_0(t,\underline{p}) + u_1(t,\underline{p})\big) \mathbb{1}_{\omega_{\underline{p}}}(r) + \frac{r - r_+}{\overline{p} - r_+} \big(q(\overline{p}) u_0(t,\overline{p}) + u_1(t,\overline{p})\big) \mathbb{1}_{\omega_{\overline{p}}}(r),\end{aligned}$$

$$
\begin{aligned}
\left(H_\bullet^\star u\right)_1(t,r) &= 2\sum_{p\in\mathring{\mathscr{T}}_{\bullet,0}}\sum_{k\in K}\sigma_k(p,r)a(p_k)\left(\sum_{\ell\in K}\frac{u_0(t,p_{k\ell})}{(p_{k\ell}-p_{k\ell'})(p_{k\ell}-p_{k\ell''})}\right)\mathbb{1}_{\omega_p}(r)\\
&+\sum_{p\in\mathring{\mathscr{T}}_{\bullet,0}}\sum_{k\in K}\sigma_k(p,r)b(p_k)\left(\sum_{\ell\in K}\frac{\big(2p_k-(p_{k\ell'}+p_{k\ell''})\big)}{(p_{k\ell}-p_{k\ell'})(p_{k\ell}-p_{k\ell''})}u_0(t,p_{k\ell})\right)\mathbb{1}_{\omega_p}(r)\\
&+\frac{b(\underline{p})(r-r_-)}{(\underline{p}-r_-)^2}u_0(t,\underline{p})\mathbb{1}_{\omega_{\underline{p}}}(r)+\frac{b(\overline{p})(r-r_+)}{(\overline{p}-r_+)^2}u_0(t,\overline{p})\mathbb{1}_{\omega_{\overline{p}}}(r)\\
&+\sum_{p\in\mathring{\mathscr{T}}_{\bullet,0}}\sum_{k\in K}\sigma_k(p,r)c(p_k)u_0(t,p_k)\mathbb{1}_{\omega_p}(r)\\
&+\frac{c(\underline{p})(r-r_-)}{\underline{p}-r_-}u_0(t,\underline{p})\mathbb{1}_{\omega_{\underline{p}}}(r)+\frac{c(\overline{p})(r-r_+)}{\overline{p}-r_+}u_0(t,\overline{p})\mathbb{1}_{\omega_{\overline{p}}}(r)\\
&+\sum_{p\in\mathring{\mathscr{T}}_{\bullet,0}}\sum_{k\in K}\sigma_k(p,r)\big(q(p_k)u_1(t,p_k)\big)\mathbb{1}_{\omega_p}(r)\\
&+\frac{r-r_-}{\underline{p}-r_-}\big(q(p_k)u_1(t,p_k)\big)(t,\underline{p})\mathbb{1}_{\omega_{\underline{p}}}(r)+\frac{r-r_+}{\overline{p}-r_+}\big(q(p_k)u_1(t,p_k)\big)(t,\overline{p})\mathbb{1}_{\omega_{\overline{p}}}(r).
\end{aligned}
$$

Observe that for all $p\in\mathring{\mathscr{T}}_{\bullet,0}$, we have

$$
\begin{aligned}
\left(H_\bullet^\star u\right)_0(t,p) &= q(p)u_0(t,p)+u_1(t,p),\\
\left(H_\bullet^\star u\right)_1(t,p) &= \sum_{k\in K}\frac{2a(p)+b(p)\big(2p-(p_{k'}+p_{k''})\big)}{(p_k-p_{k'})(p_k-p_{k''})}u_0(t,p_k)+c(p)u_0(t,p)+q(p)u_1(t,p).
\end{aligned}
$$

4.4.3 Computation of the matrix coefficients of $\mathscr{N}_\bullet$

Abstract case

We use the notations of Subsection 4.1.3. Assume (Q). The non-zero matrix coefficients $(\mathscr{N}_\bullet^2)_{ij}:=\mathcal{N}_\bullet^2(\Psi_\bullet^{-1}e_i,\Psi_\bullet^{-1}e_j)$ of the sesquilinear form $\mathcal{N}_\bullet^2(\Psi_\bullet^{-1}\cdot,\Psi_\bullet^{-1}\cdot)$ acting on $\mathscr{W}_{\bullet,1}$ and expressed in the canonical basis $(e_j)_{1\le j\le 2|\mathscr{T}_{\bullet,0}|}$ of $\mathbb{C}^{2|\mathscr{T}_{\bullet,0}|}$ are given by:

$$
\begin{aligned}
(\mathscr{N}_\bullet^2)_{11} &= \int_{\omega_{\underline{p}}}\left[\overline{\left(\frac{b(r)+c(r)(r-r_-)}{\underline{p}-r_-}\right)}\frac{r-r_-}{\underline{p}-r_-}+\left|Q(\underline{p})\left(\frac{r-r_-}{\underline{p}-r_-}\right)\right|^2\right]\rho(r)\mathrm{d}r\\
&+\int_{\omega_{\underline{p}_+}}\left[\overline{\left(\frac{2a(r)+b(r)(2r-(\underline{p}_++\underline{p}_{++}))+c(r)(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)}\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right.\\
&\qquad\left.+\left|Q(\underline{p})\left(\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)\right|^2\right]\rho(r)\mathrm{d}r,
\end{aligned}
$$

$$(\mathscr{N}_\bullet^2)_{12} = \overline{(\mathscr{N}_\bullet^2)_{21}} = -\int_{\omega_{\underline{p}}} \overline{Q(\underline{p})\left(\frac{r-r_-}{\underline{p}-r_-}\right)}\frac{r-r_-}{\underline{p}-r_-}\rho(r)\mathrm{d}r$$
$$-\int_{\omega_{\underline{p}_+}} \overline{Q(\underline{p})\left(\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)}\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\,\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{13} = \overline{(\mathscr{N}_\bullet^2)_{31}}$$
$$= \int_{\omega_{\underline{p}_+}} \Bigg[\overline{\left(\frac{2a(r)+b(r)(2r-(\underline{p}_+ + \underline{p}_{++}))+c(r)(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)}\frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})}$$
$$+ \overline{Q(\underline{p})}Q(\underline{p}_+)\overline{\left(\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)}\frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})}\Bigg]\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{14} = \overline{(\mathscr{N}_\bullet^2)_{41}} = -\int_{\omega_{\underline{p}_+}} \overline{Q(\underline{p})\left(\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)}\frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})}\,\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{15} = \overline{(\mathscr{N}_\bullet^2)_{51}}$$
$$= \int_{\omega_{\underline{p}_+}} \Bigg[\overline{\left(\frac{2a(r)+b(r)(2r-(\underline{p}_+ + \underline{p}_{++}))+c(r)(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)}\frac{(r-\underline{p})(r-\underline{p}_+)}{(\underline{p}_{++}-\underline{p})(\underline{p}_{++}-\underline{p}_+)}$$
$$+ \overline{Q(\underline{p})}Q(\underline{p}_{++})\overline{\left(\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)}\frac{(r-\underline{p})(r-\underline{p}_+)}{(\underline{p}_{++}-\underline{p})(\underline{p}_{++}-\underline{p}_+)}\Bigg]\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{16} = \overline{(\mathscr{N}_\bullet^2)_{61}} = -\int_{\omega_{\underline{p}_+}} \overline{Q(\underline{p})\left(\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)}\frac{(r-\underline{p})(r-\underline{p}_+)}{(\underline{p}_{++}-\underline{p})(\underline{p}_{++}-\underline{p}_+)}\,\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{22} = \int_{\omega_{\underline{p}}}\left|\frac{r-r_-}{\underline{p}-r_-}\right|^2\rho(r)\mathrm{d}r + \int_{\omega_{\underline{p}_+}}\left|\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right|^2\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{23} = \overline{(\mathscr{N}_\bullet^2)_{32}} = -\int_{\omega_{\underline{p}_+}} \overline{\left(\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)} Q(\underline{p}_+) \frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})} \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{24} = \overline{(\mathscr{N}_\bullet^2)_{42}} = \int_{\omega_{\underline{p}_+}} \overline{\left(\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)} \frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})} \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{25} = \overline{(\mathscr{N}_\bullet^2)_{52}} = -\int_{\omega_{\underline{p}_+}} \overline{\left(\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)} Q(\underline{p}_{++}) \frac{(r-\underline{p})(r-\underline{p}_+)}{(\underline{p}_{++}-\underline{p})(\underline{p}_{++}-\underline{p}_+)} \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{26} = \overline{(\mathscr{N}_\bullet^2)_{62}} = \int_{\omega_{\underline{p}_+}} \overline{\left(\frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}-\underline{p}_+)(\underline{p}-\underline{p}_{++})}\right)} \frac{(r-\underline{p})(r-\underline{p}_+)}{(\underline{p}_{++}-\underline{p})(\underline{p}_{++}-\underline{p}_+)} \rho(r)\mathrm{d}r,$$

$$\begin{aligned}(\mathscr{N}_\bullet^2)_{33} = \int_{\omega_{\underline{p}_+}} \Bigg[& \overline{\left(\frac{2a(r)+b(r)(2r-(\underline{p}+\underline{p}_{++}))+c(r)(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})}\right)} \frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})} \\ & + \left| Q(\underline{p}_+) \left(\frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})}\right)\right|^2 \Bigg] \rho(r)\mathrm{d}r \\ + \int_{\omega_{\underline{p}_{++}}} \Bigg[& \overline{\left(\frac{2a(r)+b(r)(2r-(\underline{p}_{++}+\underline{p}_{+++}))+c(r)(r-\underline{p}_{++})(r-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})}\right)} \frac{(r-\underline{p}_{++})(r-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})} \\ & + \left| Q(\underline{p}_+) \left(\frac{(r-\underline{p}_{++})(r-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})}\right)\right|^2 \Bigg] \rho(r)\mathrm{d}r,\end{aligned}$$

$$\begin{aligned}(\mathscr{N}_\bullet^2)_{34} = \overline{(\mathscr{N}_\bullet^2)_{43}} = & -\int_{\omega_{\underline{p}_+}} \overline{Q(\underline{p}_+)} \left|\frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})}\right|^2 \rho(r)\mathrm{d}r \\ & -\int_{\omega_{\underline{p}_{++}}} \overline{Q(\underline{p}_+)} \left|\frac{(r-\underline{p}_{++})(r-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})}\right|^2 \rho(r)\mathrm{d}r,\end{aligned}$$

$$(\mathscr{N}_\bullet^2)_{35} = \overline{(\mathscr{N}_\bullet^2)_{53}}$$
$$= \int_{\omega_{\underline{p}_+}} \left[\overline{\left(\frac{2a(r) + b(r)(2r - (\underline{p} + \underline{p}_{++})) + c(r)(r - \underline{p})(r - \underline{p}_{++})}{(\underline{p}_+ - \underline{p})(\underline{p}_+ - \underline{p}_{++})} \right)} \frac{(r - \underline{p})(r - \underline{p}_+)}{(\underline{p}_{++} - \underline{p})(\underline{p}_{++} - \underline{p}_+)} \right.$$
$$\left. + \overline{Q(\underline{p}_+)} Q(\underline{p}_{++}) \overline{\left(\frac{(r - \underline{p})(r - \underline{p}_{++})}{(\underline{p}_+ - \underline{p})(\underline{p}_+ - \underline{p}_{++})} \right)} \frac{(r - \underline{p})(r - \underline{p}_+)}{(\underline{p}_{++} - \underline{p})(\underline{p}_{++} - \underline{p}_+)} \right] \rho(r)\mathrm{d}r$$
$$+ \int_{\omega_{\underline{p}_{++}}} \left[\overline{\left(\frac{2a(r) + b(r)(2r - (\underline{p}_{++} + \underline{p}_{+++})) + c(r)(r - \underline{p}_{++})(r - \underline{p}_{+++})}{(\underline{p}_+ - \underline{p}_{++})(\underline{p}_+ - \underline{p}_{+++})} \right)} \frac{(r - \underline{p}_+)(r - \underline{p}_{+++})}{(\underline{p}_{++} - \underline{p}_+)(\underline{p}_{++} - \underline{p}_{+++})} \right.$$
$$\left. + \overline{Q(\underline{p}_+)} Q(\underline{p}_{++}) \overline{\left(\frac{(r - \underline{p}_{++})(r - \underline{p}_{+++})}{(\underline{p}_+ - \underline{p}_{++})(\underline{p}_+ - \underline{p}_{+++})} \right)} \frac{(r - \underline{p}_+)(r - \underline{p}_{+++})}{(\underline{p}_{++} - \underline{p}_+)(\underline{p}_{++} - \underline{p}_{+++})} \right] \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{36} = \overline{(\mathscr{N}_\bullet^2)_{63}} = -\int_{\omega_{\underline{p}_+}} \overline{Q(\underline{p}_+) \left(\frac{(r - \underline{p})(r - \underline{p}_{++})}{(\underline{p}_+ - \underline{p})(\underline{p}_+ - \underline{p}_{++})} \right)} \frac{(r - \underline{p})(r - \underline{p}_+)}{(\underline{p}_{++} - \underline{p})(\underline{p}_{++} - \underline{p}_+)} \rho(r)\mathrm{d}r$$
$$- \int_{\omega_{\underline{p}_{++}}} \overline{Q(\underline{p}_+) \left(\frac{(r - \underline{p}_{++})(r - \underline{p}_{+++})}{(\underline{p}_+ - \underline{p}_{++})(\underline{p}_+ - \underline{p}_{+++})} \right)} \frac{(r - \underline{p}_+)(r - \underline{p}_{+++})}{(\underline{p}_{++} - \underline{p}_+)(\underline{p}_{++} - \underline{p}_{+++})} \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{37} = \overline{(\mathscr{N}_\bullet^2)_{73}}$$
$$= \int_{\omega_{\underline{p}_{++}}} \left[\overline{\left(\frac{2a(r) + b(r)(2r - (\underline{p}_{++} + \underline{p}_{+++})) + c(r)(r - \underline{p}_{++})(r - \underline{p}_{+++})}{(\underline{p}_+ - \underline{p}_{++})(\underline{p}_+ - \underline{p}_{+++})} \right)} \frac{(r - \underline{p}_+)(r - \underline{p}_{++})}{(\underline{p}_{+++} - \underline{p}_+)(\underline{p}_{+++} - \underline{p}_{++})} \right.$$
$$\left. + \overline{Q(\underline{p}_+)} Q(\underline{p}_{+++}) \overline{\left(\frac{(r - \underline{p}_{++})(r - \underline{p}_{+++})}{(\underline{p}_+ - \underline{p}_{++})(\underline{p}_+ - \underline{p}_{+++})} \right)} \frac{(r - \underline{p}_+)(r - \underline{p}_{++})}{(\underline{p}_{+++} - \underline{p}_+)(\underline{p}_{+++} - \underline{p}_{++})} \right] \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{38} = \overline{(\mathscr{N}_\bullet^2)_{83}} = -\int_{\omega_{\underline{p}_{++}}} \overline{Q(\underline{p}_+) \left(\frac{(r - \underline{p}_{++})(r - \underline{p}_{+++})}{(\underline{p}_+ - \underline{p}_{++})(\underline{p}_+ - \underline{p}_{+++})} \right)} \frac{(r - \underline{p}_+)(r - \underline{p}_{++})}{(\underline{p}_{+++} - \underline{p}_+)(\underline{p}_{+++} - \underline{p}_{++})} \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{44} = \int_{\omega_{\underline{p}_+}} \left| \frac{(r - \underline{p})(r - \underline{p}_{++})}{(\underline{p}_+ - \underline{p})(\underline{p}_+ - \underline{p}_{++})} \right|^2 \rho(r)\mathrm{d}r + \int_{\omega_{\underline{p}_{++}}} \left| \frac{(r - \underline{p}_{++})(r - \underline{p}_{+++})}{(\underline{p}_+ - \underline{p}_{++})(\underline{p}_+ - \underline{p}_{+++})} \right|^2 \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{45} = \overline{(\mathscr{N}_\bullet^2)_{54}} = -\int_{\omega_{\underline{p}_+}} \overline{\left(\frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})}\right)} Q(\underline{p}_{++}) \frac{(r-\underline{p})(r-\underline{p}_+)}{(\underline{p}_{++}-\underline{p})(\underline{p}_{++}-\underline{p}_+)}\rho(r)\mathrm{d}r$$
$$-\int_{\omega_{\underline{p}_{++}}} \overline{\left(\frac{(r-\underline{p}_{++})(r-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})}\right)} Q(\underline{p}_{++}) \frac{(r-\underline{p}_+)(r-\underline{p}_{+++})}{(\underline{p}_{++}-\underline{p}_+)(\underline{p}_{++}-\underline{p}_{+++})}\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{46} = \overline{(\mathscr{N}_\bullet^2)_{64}} = \int_{\omega_{\underline{p}_+}} \overline{\left(\frac{(r-\underline{p})(r-\underline{p}_{++})}{(\underline{p}_+-\underline{p})(\underline{p}_+-\underline{p}_{++})}\right)} \frac{(r-\underline{p})(r-\underline{p}_+)}{(\underline{p}_{++}-\underline{p})(\underline{p}_{++}-\underline{p}_+)}\rho(r)\mathrm{d}r$$
$$= \int_{\omega_{\underline{p}_{++}}} \overline{\left(\frac{(r-\underline{p}_{++})(r-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})}\right)} \frac{(r-\underline{p}_+)(r-\underline{p}_{+++})}{(\underline{p}_{++}-\underline{p}_+)(\underline{p}_{++}-\underline{p}_{+++})}\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{47} = \overline{(\mathscr{N}_\bullet^2)_{74}} = -\int_{\omega_{\underline{p}_{++}}} \overline{\left(\frac{(r-\underline{p}_{++})(r-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})}\right)} Q(\underline{p}_{+++}) \frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}_{+++}-\underline{p}_+)(\underline{p}_{+++}-\underline{p}_{++})}\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{48} = \overline{(\mathscr{N}_\bullet^2)_{84}} = \int_{\omega_{\underline{p}_{++}}} \overline{\left(\frac{(r-\underline{p}_{++})(r-\underline{p}_{+++})}{(\underline{p}_+-\underline{p}_{++})(\underline{p}_+-\underline{p}_{+++})}\right)} \frac{(r-\underline{p}_+)(r-\underline{p}_{++})}{(\underline{p}_{+++}-\underline{p}_+)(\underline{p}_{+++}-\underline{p}_{++})}\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|p|-1,2|p|-1} = \int_{\omega_{p_-}} \left[\overline{\left(\frac{2a(r)+b(r)(2r-(p_{--}+p_-))+c(r)(r-p_{--})(r-p_-)}{(p-p_{--})(p-p_-)}\right)} \frac{(r-p_{--})(r-p_-)}{(p-p_{--})(p-p_-)} \right.$$
$$\left. + \left| Q(p)\left(\frac{(r-p_{--})(r-p_-)}{(p-p_{--})(p-p_-)}\right)\right|^2 \right] \rho(r)\mathrm{d}r$$
$$+ \int_{\omega_p} \left[\overline{\left(\frac{2a(r)+b(r)(2r-(p_-+p_+))+c(r)(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)}\right)} \frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)} \right.$$
$$\left. + \left| Q(p)\left(\frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)}\right)\right|^2 \right] \rho(r)\mathrm{d}r$$
$$+ \int_{\omega_{p_+}} \left[\overline{\left(\frac{2a(r)+b(r)(2r-(p_++p_{++}))+c(r)(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})}\right)} \frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})} \right.$$
$$\left. + \left| Q(p)\left(\frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})}\right)\right|^2 \right] \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|p|-1,2|p|} = \overline{(\mathscr{N}_\bullet^2)_{2|p|,2|p|-1}} = -\int_{\omega_{p_-}} \overline{Q(p)} \left|\frac{(r-p_{--})(r-p_-)}{(p-p_{--})(p-p_-)}\right|^2 \rho(r)\mathrm{d}r$$
$$-\int_{\omega_p} \overline{Q(p)} \left|\frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)}\right|^2 \rho(r)\mathrm{d}r - \int_{\omega_{p_+}} \overline{Q(p)} \left|\frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})}\right|^2 \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|p|-1,2|p|+1} = \overline{(\mathscr{N}_\bullet^2)_{2|p|+1,2|p|-1}}$$
$$= \int_{\omega_p} \Bigg[\overline{\left(\frac{2a(r) + b(r)(2r - (p_- + p_+)) + c(r)(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)} \right)} \frac{(r-p_-)(r-p)}{(p_+ - p_-)(p_+ - p)}$$
$$+ \overline{Q(p)} Q(p_+) \overline{\left(\frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)} \right)} \frac{(r-p_-)(r-p)}{(p_+ - p_-)(p_+ - p)} \Bigg] \rho(r)\mathrm{d}r$$
$$+ \int_{\omega_{p_+}} \Bigg[\overline{\left(\frac{2a(r) + b(r)(2r - (p_+ + p_{++})) + c(r)(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})} \right)} \frac{(r-p)(r-p_{++})}{(p_+ - p)(p_+ - p_{++})}$$
$$+ \overline{Q(p)} Q(p_+) \overline{\left(\frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})} \right)} \frac{(r-p)(r-p_{++})}{(p_+ - p)(p_+ - p_{++})} \Bigg] \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|p|-1,2|p|+2} = \overline{(\mathscr{N}_\bullet^2)_{2|p|+2,2|p|-1}} = -\int_{\omega_p} \overline{Q(p) \left(\frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)} \right)} \frac{(r-p_-)(r-p)}{(p_+ - p_-)(p_+ - p)} \rho(r)\mathrm{d}r$$
$$- \int_{\omega_{p_+}} \overline{Q(p) \left(\frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})} \right)} \frac{(r-p)(r-p_{++})}{(p_+ - p)(p_+ - p_{++})} \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|p|-1,2|p|+3} = \overline{(\mathscr{N}_\bullet^2)_{2|p|+3,2|p|-1}}$$
$$= \int_{\omega_{p_+}} \Bigg[\overline{\left(\frac{2a(r) + b(r)(2r - (p_+ + p_{++})) + c(r)(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})} \right)} \frac{(r-p)(r-p_+)}{(p_{++} - p)(p_{++} - p_+)}$$
$$+ \overline{Q(p)} Q(p_{++}) \overline{\left(\frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})} \right)} \frac{(r-p)(r-p_+)}{(p_{++} - p)(p_{++} - p_+)} \Bigg] \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|p|-1,2|p|+4} = \overline{(\mathscr{N}_\bullet^2)_{2|p|+4,2|p|-1}} = -\int_{\omega_{p_+}} \overline{Q(p) \left(\frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})} \right)} \frac{(r-p)(r-p_+)}{(p_{++} - p)(p_{++} - p_+)} \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|p|,2|p|} = \int_{\omega_{p_-}} \left| \frac{(r-p_{--})(r-p_-)}{(p-p_{--})(p-p_-)} \right|^2 \rho(r)\mathrm{d}r + \int_{\omega_p} \left| \frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)} \right|^2 \rho(r)\mathrm{d}r$$
$$+ \int_{\omega_{p_+}} \left| \frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})} \right|^2 \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|p|,2|p|+1} = \overline{(\mathscr{N}_\bullet^2)_{2|p|+1,2|p|}} = -\int_{\omega_p} \overline{\left(\frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)} \right)} Q(p_+) \frac{(r-p_-)(r-p)}{(p_+ - p_-)(p_+ - p)} \rho(r)\mathrm{d}r$$
$$- \int_{\omega_{p_+}} \overline{\left(\frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})} \right)} Q(p_+) \frac{(r-p)(r-p_{++})}{(p_+ - p)(p_+ - p_{++})} \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|p|,2|p|+2} = \overline{(\mathscr{N}_\bullet^2)_{2|p|+2,2|p|}} = \int_{\omega_p} \overline{\left(\frac{(r-p_-)(r-p_+)}{(p-p_-)(p-p_+)}\right)} \frac{(r-p_-)(r-p)}{(p_+-p_-)(p_+-p)}\,\rho(r)\mathrm{d}r + \int_{\omega_{p_+}} \overline{\left(\frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})}\right)} \frac{(r-p)(r-p_{++})}{(p_+-p)(p_+-p_{++})}\,\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|p|,2|p|+3} = \overline{(\mathscr{N}_\bullet^2)_{2|p|+3,2|p|}} = -\int_{\omega_{p_+}} \overline{\left(\frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})}\right)} Q(p_{++}) \frac{(r-p)(r-p_+)}{(p_{++}-p)(p_{++}-p_+)}\,\rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|p|,2|p|+4} = \overline{(\mathscr{N}_\bullet^2)_{2|p|+4,2|p|}} = \int_{\omega_{p_+}} \overline{\left(\frac{(r-p_+)(r-p_{++})}{(p-p_+)(p-p_{++})}\right)} \frac{(r-p)(r-p_+)}{(p_{++}-p)(p_{++}-p_+)}\,\rho(r)\mathrm{d}r,$$

$$\begin{aligned}
&(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-3,2|\mathscr{T}_{\bullet,0}|-3} \\
&= \int_{\omega_{\overline{p}_{--}}} \left[\overline{\left(\frac{2a(r)+b(r)(2r-(\overline{p}_{---}+\overline{p}_{--}))+c(r)(r-\overline{p}_{---})(r-\overline{p}_{--})}{(\overline{p}_- - \overline{p}_{---})(\overline{p}_- - \overline{p}_{--})}\right)} \frac{(r-\overline{p}_{---})(r-\overline{p}_{--})}{(\overline{p}_- - \overline{p}_{---})(\overline{p}_- - \overline{p}_{--})} \right. \\
&\qquad \left. + \left| Q(\overline{p}_-) \left(\frac{(r-\overline{p}_{---})(r-\overline{p}_{--})}{(\overline{p}_- - \overline{p}_{---})(\overline{p}_- - \overline{p}_{--})}\right)\right|^2 \right] \rho(r)\mathrm{d}r \\
&+ \int_{\omega_{\overline{p}_-}} \left[\overline{\left(\frac{2a(r)+b(r)(2r-(\overline{p}_{--}+\overline{p}))+c(r)(r-\overline{p}_{--})(r-\overline{p})}{(\overline{p}_- - \overline{p}_{--})(\overline{p}_- - \overline{p})}\right)} \frac{(r-\overline{p}_{--})(r-\overline{p})}{(\overline{p}_- - \overline{p}_{--})(\overline{p}_- - \overline{p})} \right. \\
&\qquad \left. + \left| Q(\overline{p}_-) \left(\frac{(r-\overline{p}_{--})(r-\overline{p})}{(\overline{p}_- - \overline{p}_{--})(\overline{p}_- - \overline{p})}\right)\right|^2 \right] \rho(r)\mathrm{d}r,
\end{aligned}$$

$$(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-3,2|\mathscr{T}_{\bullet,0}|-2} = \overline{(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-2,2|\mathscr{T}_{\bullet,0}|-3}} = -\int_{\omega_{\overline{p}_{--}}} \overline{Q(\overline{p}_-)} \left|\left(\frac{(r-\overline{p}_{---})(r-\overline{p}_{--})}{(\overline{p}_- - \overline{p}_{---})(\overline{p}_- - \overline{p}_{--})}\right)\right|^2 \rho(r)\mathrm{d}r - \int_{\omega_{\overline{p}_-}} \overline{Q(\overline{p}_-)} \left|\left(\frac{(r-\overline{p}_{--})(r-\overline{p})}{(\overline{p}_- - \overline{p}_{--})(\overline{p}_- - \overline{p})}\right)\right|^2 \rho(r)\mathrm{d}r,$$

$$\begin{aligned}
&(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-3,2|\mathscr{T}_{\bullet,0}|-1} = \overline{(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-1,2|\mathscr{T}_{\bullet,0}|-3}} \\
&= \int_{\omega_{\overline{p}_-}} \left[\overline{\left(\frac{2a(r)+b(r)(2r-(\overline{p}_{--}+\overline{p}))+c(r)(r-\overline{p}_{--})(r-\overline{p})}{(\overline{p}_- - \overline{p}_{--})(\overline{p}_- - \overline{p})}\right)} \frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)} \right. \\
&\qquad \left. + \overline{Q(\overline{p}_-)} Q(\overline{p}) \overline{\left(\frac{(r-\overline{p}_{--})(r-\overline{p})}{(\overline{p}_- - \overline{p}_{--})(\overline{p}_- - \overline{p})}\right)} \frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)} \right] \rho(r)\mathrm{d}r,
\end{aligned}$$

$$\begin{aligned}
(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-3,2|\mathscr{T}_{\bullet,0}|} &= \overline{(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|,2|\mathscr{T}_{\bullet,0}|-3}} \\
&= -\int_{\omega_{\overline{p}_-}} \overline{Q(\overline{p}_-) \left(\frac{(r-\overline{p}_{--})(r-\overline{p})}{(\overline{p}_- - \overline{p}_{--})(\overline{p}_- - \overline{p})}\right)} \frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)}\,\rho(r)\mathrm{d}r,
\end{aligned}$$

$$(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-2,2|\mathscr{T}_{\bullet,0}|-2} = \int_{\omega_{\overline{p}_{--}}} \left|\left(\frac{(r-\overline{p}_{---})(r-\overline{p}_{--})}{(\overline{p}_- - \overline{p}_{---})(\overline{p}_- - \overline{p}_{--})}\right)\right|^2 \rho(r)\mathrm{d}r + \int_{\omega_{\overline{p}_-}} \left|\left(\frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)}\right)\right|^2 \rho(r)\mathrm{d}r,$$

$$\begin{aligned}(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-2,2|\mathscr{T}_{\bullet,0}|-1} &= \overline{(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-1,2|\mathscr{T}_{\bullet,0}|-2}} \\ &= -\int_{\omega_{\overline{p}_-}} \overline{\left(\frac{(r-\overline{p}_{--})(r-\overline{p})}{(\overline{p}_- - \overline{p}_{--})(\overline{p}_- - \overline{p})}\right)} Q(\overline{p}) \frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)} \rho(r)\mathrm{d}r,\end{aligned}$$

$$\begin{aligned}(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-2,2|\mathscr{T}_{\bullet,0}|} &= \overline{(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|,2|\mathscr{T}_{\bullet,0}|-2}} \\ &= \int_{\omega_{\overline{p}_-}} \overline{\left(\frac{(r-\overline{p}_{--})(r-\overline{p})}{(\overline{p}_- - \overline{p}_{--})(\overline{p}_- - \overline{p})}\right)} \frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)} \rho(r)\mathrm{d}r,\end{aligned}$$

$$\begin{aligned}&(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-1,2|\mathscr{T}_{\bullet,0}|-1} \\ &= \int_{\omega_{\overline{p}_-}} \left[\overline{\left(\frac{2a(r)+b(r)(2r-(\overline{p}_{--}+\overline{p}_-))+c(r)(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)}\right)} \frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)} \right. \\ &\qquad \left. + \left|Q(\overline{p})\left(\frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)}\right)\right|^2\right] \rho(r)\mathrm{d}r \\ &\quad + \int_{\omega_{\overline{p}}} \left[\overline{\left(\frac{b(r)+c(r)(r-r_+)}{\overline{p}-r_+}\right)} \frac{r-r_+}{\overline{p}-r_+} + \left|Q(\overline{p})\left(\frac{r-r_+}{\overline{p}-r_+}\right)\right|^2\right] \rho(r)\mathrm{d}r,\end{aligned}$$

$$(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|-1,2|\mathscr{T}_{\bullet,0}|} = \overline{(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|,2|\mathscr{T}_{\bullet,0}|-1}} = -\int_{\omega_{\overline{p}_-}} \overline{Q(\overline{p})} \left|\frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)}\right|^2 \rho(r)\mathrm{d}r - \int_{\omega_{\overline{p}}} \overline{Q(\overline{p})} \left|\frac{r-r_+}{\overline{p}-r_+}\right|^2 \rho(r)\mathrm{d}r,$$

$$(\mathscr{N}_\bullet^2)_{2|\mathscr{T}_{\bullet,0}|,2|\mathscr{T}_{\bullet,0}|} = \int_{\omega_{\overline{p}_-}} \left|\frac{(r-\overline{p}_{--})(r-\overline{p}_-)}{(\overline{p}-\overline{p}_{--})(\overline{p}-\overline{p}_-)}\right|^2 \rho(r)\mathrm{d}r + \int_{\omega_{\overline{p}}} \left|\frac{r-r_+}{\overline{p}-r_+}\right|^2 \rho(r)\mathrm{d}r,$$

for all $p \in \mathring{\mathscr{T}}_{\bullet,0}$.

DSRN case

Consider the operator P_ℓ with $\ell \in \mathbb{N}$. Then

$$\begin{cases} a(r) &= -F(r)^2, \\ b(r) &= -\frac{2F(r)^2}{r} - F(r)F'(r), \\ c(r) &= \frac{F(r)}{r^2}\big(\ell(\ell+1) + m^2 r^2\big), \\ \rho(r) &= \frac{r^2}{F(r)} \end{cases}.$$

Let next $J := \big\{n, c, -, +\big\}$. Put

$$\begin{aligned} &A_{\alpha,\beta,\gamma,\delta} = (\alpha+\beta) + 2(\gamma+\delta), && B_{\alpha,\beta,\gamma,\delta} = (\alpha+\beta)(\gamma+\delta) + 2\gamma\delta, \\ &C_{\alpha,\beta,\gamma,\delta} = \alpha+\beta+\gamma+\delta, && D_{\alpha,\beta,\gamma,\delta} = \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta, \\ &E_{\alpha,\beta,\gamma,\delta} = \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta. \end{aligned}$$

The coefficients of $\mathscr{N}_\bullet^2$ are computed by using the following antiderivatives:

$$\int (r-\alpha)(r-\beta)\,\rho(r)\mathrm{d}r = \frac{1}{2\Lambda^2}\left(9\sum_{\nu\in J}\frac{K_{\nu,\alpha,\beta}\ln|r-r_\nu|}{3(M-r_\nu)+2\Lambda r_\nu^3} - \Big(6(\Lambda\alpha\beta+3)r - 3\Lambda(\alpha+\beta)r^2 + 2\Lambda r^3\Big)\right) + C$$

where

$$\begin{aligned} K_{\nu,\alpha,\beta} = {} & \big(\Lambda Q^2(\alpha+\beta) + 2M(\Lambda\alpha\beta+3)\big)r_\nu - \big(\Lambda(Q^2+\alpha\beta+2M(\alpha+\beta))+3\big)r_\nu^2 \\ & + \big(2\Lambda M + (\alpha+\beta)\big)r_\nu^3 - Q^2(\Lambda\alpha\beta+3), \end{aligned}$$

$$\begin{aligned} &\int (r-\alpha)(r-\beta)(r-\gamma)(r-\delta)\,\rho(r)\mathrm{d}r \\ &= \frac{1}{2\Lambda^3}\left(9\sum_{\nu\in J}\frac{\tilde{K}_{\nu,\alpha,\beta,\gamma,\delta}\ln|r-r_\nu|}{3(M-r_\nu)+2\Lambda r_\nu^3} - 6\Big(3\Lambda Q^2 + 6\Lambda M C_{\alpha,\beta,\gamma,\delta} + \Lambda\alpha\beta\gamma\delta + 3\Lambda D_{\alpha,\beta,\gamma,\delta} + 9\Big)r\right. \\ &\qquad \left. +3\Lambda\Big(6M + \Lambda E_{\alpha,\beta,\gamma,\delta} + 3C_{\alpha,\beta,\gamma,\delta}\Big)r^2 - 2\Lambda\Big(\Lambda D_{\alpha,\beta,\gamma,\delta} + 3\Big)r^3 + \frac{3\Lambda^2 C_{\alpha,\beta,\gamma,\delta}\,r^4}{2} - \frac{6\Lambda^2 r^5}{5}\right) \end{aligned}$$

where

$$\begin{aligned} \tilde{K}_{\nu,\alpha,\beta,\gamma,\delta} = {} & -\Big(3\Lambda Q^2 + 6\Lambda M C_{\alpha,\beta,\gamma,\delta} + \Lambda^2\alpha\beta\gamma\delta + 3\big(\Lambda D_{\alpha,\beta,\gamma,\delta}+3\big)\Big)Q^2 \\ & + \Big(\big(12M + \Lambda E_{\alpha,\beta,\gamma,\delta} + 3C_{\alpha,\beta,\gamma,\delta}\big)\Lambda Q^2 + 12\Lambda M^2 C_{\alpha,\beta,\gamma,\delta} + 2M\big(\Lambda^2\alpha\beta\gamma\delta + 3\Lambda D_{\alpha,\beta,\gamma,\delta} + 9\big)\Big)r_\nu \\ & - \Big(\Lambda Q^2\big(\Lambda D_{\alpha,\beta,\gamma,\delta}+6\big) + 12\Lambda M^2 + 2\Lambda M\big(\Lambda E_{\alpha,\beta,\gamma,\delta} + 6C_{\alpha,\beta,\gamma,\delta}\big) + \Lambda^2\alpha\beta\gamma\delta + 3\big(\Lambda D_{\alpha,\beta,\gamma,\delta}+3\big)\Big)r_\nu^2 \\ & + \Lambda\Big(\Lambda Q^2 C_{\alpha,\beta,\gamma,\delta} + 2M\big(\Lambda D_{\alpha,\beta,\gamma,\delta}+6\big) + \Lambda E_{\alpha,\beta,\gamma,\delta} + 3C_{\alpha,\beta,\gamma,\delta}\Big)r_\nu^3, \end{aligned}$$

$$\begin{aligned} \int \overline{\big(b(r)+c(r)(r-\alpha)\big)}(r-\alpha)\,\rho(r)\mathrm{d}r = {} & \frac{4\Lambda r^5}{15} - \frac{\Lambda\alpha r^4}{3} - \frac{2r^3}{3} + (M+\alpha)r^2 \\ & - \frac{(r-\alpha)^3}{30}\big(m^2(6r^2+3\alpha r+\alpha^2) + 10\ell(\ell+1)\big) + C, \end{aligned}$$

$$
\begin{aligned}
&\int \overline{2a(r)+b(r)(2r-(\alpha+\beta))+c(r)(r-\alpha)(r-\beta)}(r-\gamma)(r-\delta)\,\rho(r)\mathrm{d}r\\
&= 2\left[-Q^2\gamma\delta r+\big(2M\gamma\delta+Q^2(\gamma+\delta)\big)\frac{r^2}{2}-\big(2M(\gamma+\delta)+Q^2+(\gamma+\delta)\big)\frac{r^3}{3}\right.\\
&\qquad\left.+\big(2M+(\gamma+\delta)\big)\frac{r^4}{4}+\left(\frac{\Lambda}{3}\gamma\delta-1\right)\frac{r^5}{5}-\frac{\Lambda}{3}(\gamma+\delta)\frac{r^6}{6}+\frac{\Lambda}{3}\frac{r^7}{7}\right]\\
&+2\left[-2M(\alpha+\beta)\gamma\delta r+\big(2MB_{\alpha,\beta,\gamma,\delta}+(\alpha+\beta)\gamma\delta\big)\frac{r^2}{2}-\big(2MA_{\alpha,\beta,\gamma,\delta}+B_{\alpha,\beta,\gamma,\delta}\big)\frac{r^3}{3}\right.\\
&\qquad\left.+\left(2M-\frac{2\Lambda}{3}(\alpha+\beta)\gamma\delta+A_{\alpha,\beta,\gamma,\delta}\right)\frac{r^4}{4}+2\left(\frac{\Lambda}{3}B_{\alpha,\beta,\gamma,\delta}-1\right)\frac{r^5}{5}-\frac{2\Lambda}{3}A_{\alpha,\beta,\gamma,\delta}\frac{r^6}{6}+\frac{4\Lambda}{3}\frac{r^7}{7}\right]\\
&+\left[\ell(\ell+1)\alpha\beta\gamma\delta r-\ell(\ell+1)E_{\alpha,\beta,\gamma,\delta}\frac{r^2}{2}+\big(\ell(\ell+1)D_{\alpha,\beta,\gamma,\delta}+m^2\alpha\beta\gamma\delta\big)\frac{r^3}{3}\right.\\
&\quad\left.-\big(\ell(\ell+1)C_{\alpha,\beta,\gamma,\delta}+m^2E_{\alpha,\beta,\gamma,\delta}\big)\frac{r^4}{4}+\big(\ell(\ell+1)+m^2D_{\alpha,\beta,\gamma,\delta}\big)\frac{r^5}{5}-m^2C_{\alpha,\beta,\gamma,\delta}\frac{r^6}{6}+m^2\frac{r^7}{7}\right]+C\\
&\\
&=\Big(-2M(\alpha+\beta)\gamma\delta-2Q^2\gamma\delta+\ell(\ell+1)\alpha\beta\gamma\delta\Big)r\\
&+\Big(2\big(2M(B_{\alpha,\beta,\gamma,\delta}+\gamma\delta)+Q^2(\gamma+\delta)\big)+2(\alpha+\beta)\gamma\delta-\ell(\ell+1)E_{\alpha,\beta,\gamma,\delta}\Big)\frac{r^2}{2}\\
&+\Big(-2\big(2M(A_{\alpha,\beta,\gamma,\delta}+(\gamma+\delta))+Q^2+B_{\alpha,\beta,\gamma,\delta}+(\gamma+\delta)\big)+\ell(\ell+1)D_{\alpha,\beta,\gamma,\delta}+m^2\alpha\beta\gamma\delta\Big)\frac{r^3}{3}\\
&+\left(2\left(4M-\frac{2\Lambda}{3}(\alpha+\beta)\gamma\delta+A_{\alpha,\beta,\gamma,\delta}+(\gamma+\delta)\right)-\big(\ell(\ell+1)C_{\alpha,\beta,\gamma,\delta}+m^2E_{\alpha,\beta,\gamma,\delta}\big)\right)\frac{r^4}{4}\\
&+\left(\frac{2\Lambda}{3}(2B_{\alpha,\beta,\gamma,\delta}+\gamma\delta)+\ell(\ell+1)+m^2B_{\alpha,\beta,\gamma,\delta}-6\right)\frac{r^5}{5}-\left(\frac{2\Lambda}{3}(2A_{\alpha,\beta,\gamma,\delta}+(\gamma+\delta))-m^2C_{\alpha,\beta,\gamma,\delta}\right)\frac{r^6}{6}\\
&+\left(\frac{10\Lambda}{3}+m^2\right)\frac{r^7}{7}+C.
\end{aligned}
$$

Chapter 5

Approximation of low frequency resonances

This chapter is devoted to localization and approximation of low frequency resonances of the charged Klein-Gordon operator on the DSRN metric. We propose a method based on complex analysis to precisely compute the number of resonances inside a positively oriented contour in the complex plane. It relies on a convenient characterization of resonances as zeros of an analytic function and allow us to give an estimate of the error of approximation.

As we have a good knowledge of the distribution of high frequency resonances in $\mathbb{C}^-$ (*cf.* Theorem 1.3.1), the low frequency ones are really hard to localize. Yet they could provide interesting information about our universe: as we do not dispose today of sufficiently precise detection devices to fully study signals such as gravitational waves, the exact localization of the nearest resonance to the real axis would be very helpful. It is also linked to the modern statement of the Strong Cosmic Censorship (*cf.* [CCDHJa] and [CCDHJb]).

Organization of the chapter. In Section 5.1, we use the Jost solutions as well as the Wronskian introduced in Chapter 1 in order to characterize resonances. Section 5.2 introduces the scheme used to approximate the Wronskian. Convergence and estimate for the error of approximation of the Wronskian are dealt with in Section 5.3.

we will focus in Section 5.2 on the construction of a scheme which provides a good approximation of the resonances. Obtained results will be presented in Section 4.3.2.

5.1 Characterization of resonances

Recall from Chapter 1 that there exists $\delta_0 > 0$ such that for a sufficiently small charge product $s = qQ$, the cut-off resolvent of the charge Klein-Gordon operator has a meromorphic extension from $\mathbb{C}^+$ to $\{z \in \mathbb{C} \mid \Im(z) > -\delta_0\}$ with value in compact operators acting on $L^2\left(]r_-, r_+[\times \mathbb{S}^2, F^{-1}(r)r^2 \mathrm{d}r\mathrm{d}\omega\right)$: for all $\chi \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$ and all $z \in \mathbb{C}$ such that $\Im(z) > -\delta_0$,

$$\chi(\hat{K}(s) - z)^{-1}\chi \in \mathcal{B}_\infty\big(L^2\left(]r_-, r_+[\times \mathbb{S}^2, F^{-1}(r)r^2 \mathrm{d}r\mathrm{d}\omega\right)\big). \tag{5.1}$$

The constant δ_0 is linked to the surface gravities κ_α:

$$\delta_0 < \kappa := \min\{\kappa_-, |\kappa_+|\} \qquad \kappa_\alpha := \frac{F'(r_\alpha)}{2}, \quad \forall \alpha \in \{n, c, -, +\}.$$

Recall also here that

$$\kappa_n, \kappa_- > 0, \qquad \kappa_c, \kappa_+ < 0.$$

We now recall the definition and properties of the Jost solutions (they have already been introduced in Chapter 1, *cf.* the proof of Lemma 1.2.2; see [**?**, Section 2] for the proof of their existence and uniqueness). Fix $(\ell, s) \in \mathbb{N} \times \mathbb{R}$ and put

$$W_\ell(r) := F(r)\left(\frac{\ell(\ell+1)}{r^2} + \frac{F'(r)}{r} + m^2\right). \tag{5.2}$$

For all $z \in \mathbb{C}$ such that $\Im(z) > -\kappa_-$, the (outgoing) Jost solutions $x \mapsto e^{\pm}(x, z, s, \ell)$, that we will simply write $e_\pm(x)$ or $e_\pm(x, z)$, are the two unique $\mathcal{C}^2$ functions satisfying the Schrödinger type equation

$$(\partial_x^2 + (z - sV)^2 - W_\ell(x))e_\pm(x) = 0 \qquad \forall x \in \mathbb{R} \tag{5.3}$$

and such that

$$\begin{aligned} e_+(x) &= \mathrm{e}^{\mathrm{i}(z-sV_+)x + \mathrm{i}s\int_x^{+\infty}(V(y)-V_+)\mathrm{d}y}\big(1 + \varepsilon_+(x)\big), \\ e_-(x) &= \mathrm{e}^{-\mathrm{i}(z-sV_-)x + \mathrm{i}s\int_{-\infty}^{x}(V(y)-V_-)\mathrm{d}y}\big(1 + \varepsilon_-(x)\big). \end{aligned}$$

Furthermore, $\partial_x e_\pm \in L^\infty_{\ell\mathrm{oc}}(\mathbb{R}_x, \mathbb{C})$ and if $\Im(z) > -\kappa_-$, then $\partial_x^j e_\pm$ is analytic in z for all $0 \leq j \leq 1$ and

$$\lim_{x \to \pm\infty} \big(|\varepsilon_\pm(x)| + |(\partial_x \varepsilon_\pm)(x)|\big) = 0. \tag{5.4}$$

The kernel K of $(P_\ell - (z - sV)^2)^{-1} : H^2_c(\mathbb{R}, \mathrm{d}x) \to L^2_{\ell\mathrm{oc}}(\mathbb{R}, \mathrm{d}x)$ for $\Im(z) > -\kappa_-$ is given by

$$K(x, y, z) = \frac{1}{\mathscr{W}(z)}\big(e_+(x, z)e_-(y, z)\mathbb{1}_{x \geq y}(x, y) + e_+(y, z)e_-(x, z)\mathbb{1}_{y \geq x}(x, y)\big)$$

where

$$\mathscr{W}(z) = e_+(x)e'_-(x) - e'_+(x)e_-(x)$$

is the *Wronskian* between e_+ and e_-. Since $\mathscr{W}$ is independent of $x \in \mathbb{R}$ (as shown at the very beginning of the proof of [Ba04, Proposition 2.1]), we see using the non trivial limits for $e_\pm$ in (5.4) that a pole z of order $n > 0$ for $(P_\ell - (z - sV)^2)^{-1}$ with $\Im(z) > -\kappa_-$ is a zero of order n of the Wronskian $\mathscr{W}$, and $e_+(\cdot, z)$ and $e_-(\cdot, z)$ are then collinear. We therefore have a convenient characterization of resonances

Proposition 5.1.1 (Characterization of resonances)**.** *The poles of the meromorphic extension* (5.1) *counted with their multiplicity are exactly the zeros of the wronskian* $\mathscr{W}$ *in the set* $\{\lambda \in \mathbb{C} \mid \Im(\lambda) > -\kappa\}$.

The numerical scheme therefore consists in approximating first the Wronskian using the Jost solutions at a given frequency $z \in \mathbb{C}$ with $\Im(z) > -\kappa$ then computing its zeros.

5.2 Approximation of the Wronskian

We present in this Section a numerical scheme to approximate the Wronskian $\mathscr{W}$.

5.2.1 Modified Jost solutions

The oscillations at infinity of Jost solutions prevent us to construct a well defined initial boundary values problem. We thus introduce an ansatz which remove the singular oscillating part at $r_\pm$ of $e_\pm$. Let us define the *modified Jost solutions*

$$f_\pm(x) := \mathrm{e}^{\mp \mathrm{i}(z - sV_\pm)x} e_\pm(x)$$

solving on $\mathbb{R}$

$$\Big[\partial_x^2 \pm 2\mathrm{i}(z - sV_\pm)\partial_x\Big] f_\pm = \Big(W_\ell(x) + s(V(x) - V_\pm)\big(2z - s(V(x) + V_\pm)\big)\Big) f_\pm. \tag{5.5}$$

Using (5.4), $\partial_x = F(r)\partial_r$ as well as $V(r) - V_\pm \to 0$ as $r \to r_\pm$, we obtain the following well-defined boundary values:

$$\lim_{r \to r_\pm} f_\pm(r) = 1, \qquad \lim_{r \to r_\pm} F(r)(\partial_r f_\pm)(r) = 0.$$

Let us derive integral representations for $f_\pm$. For all $\lambda \in \mathbb{C}$, the formal kernels $\mathcal{K}_\pm$ of the operators $(\partial_x^2 \pm 2\mathrm{i}\lambda\partial_x)^{-1}$ are given by

$$\mathcal{K}_\pm(x,y) := \begin{cases} \mathrm{D}_{+\lambda}(y-x)\mathbb{1}_{y \geq x}(y) & \text{if } \pm = + \\ \mathrm{D}_{-\lambda}(x-y)\mathbb{1}_{y \leq x}(y) & \text{if } \pm = - \end{cases}, \qquad \mathrm{D}_{\pm\lambda}(x) := \begin{cases} \dfrac{\sin(\lambda x)}{\lambda}\mathrm{e}^{\pm \mathrm{i}\lambda x} & \text{if } \lambda \in \mathbb{C} \setminus \{0\} \\ x & \text{if } \lambda = 0 \end{cases}.$$

By formal kernels, we mean that if $u_\pm$ solves

$$\Big[\partial_x^2 \pm 2\mathrm{i}\lambda\partial_x\Big] u_\pm = g_\pm$$

then

$$u_+(x) = \ell_+ + \int_x^{+\infty} \mathrm{D}_{+\lambda}(y-x) g_+(y)\mathrm{d}y, \qquad u_-(x) = \ell_- + \int_{-\infty}^{x} \mathrm{D}_{-\lambda}(x-y) g_-(y)\mathrm{d}y$$

provided that $|u_\pm - \ell_\pm| \to 0$ as $x \to \pm\infty$ and $g_\pm$ decays sufficiently fast at $\pm\infty$. For $z \in \mathbb{C}^+$ with $\Im(z) > -\kappa = \max\{-\kappa_-, \kappa_+\}$, we define from now on

$$\omega_\pm := z - sV_\pm.$$

Thus (5.5) entails the semi implicit forms for $\pm\lambda := \pm\omega_\pm$:

$$f_+(x) = 1 + \int_x^{+\infty} \mathrm{D}_{\omega_+}(y-x)\Big(W_\ell(y) + s(V(y) - V_+)\big(2z - s(V(y) + V_+)\big)\Big) f_+(y)\mathrm{d}y,$$
$$f_-(x) = 1 + \int_{-\infty}^{x} \mathrm{D}_{\omega_-}(x-y)\Big(W_\ell(y) + s(V(y) - V_-)\big(2z - s(V(y) + V_-)\big)\Big) f_-(y)\mathrm{d}y.$$

Formula (4.41) implies the exponential decay of W_ℓ and $V - V_\pm$ at $x \to \pm\infty$ which ensures the convergence of the above integrals: we have $|W_\ell(x)| + |V(x) - V_\pm| = \mathcal{O}_{x\to\pm\infty}(\mathrm{e}^{2\kappa_\pm|x|})$ and

$$|\mathrm{D}_{\omega_\pm}(x)| \leq C\mathrm{e}^{\mp 2\Im(\omega_\pm)x}$$

for some constant $C > 0$. Taking the derivative with respect to x then yields

$$(\partial_x f_+)(x) = -\int_x^{+\infty} (\partial_x \mathrm{D}_{\omega_+})(y-x)\Big(W_\ell(y) + s(V(y)-V_+)\big(2z - s(V(y)+V_+)\big)\Big) f_+(y)\mathrm{d}y,$$
$$(\partial_x f_-)(x) = \int_{-\infty}^{x} (\partial_x \mathrm{D}_{\omega_-})(x-y)\Big(W_\ell(y) + s(V(y)-V_-)\big(2z - s(V(y)+V_-)\big)\Big) f_-(y)\mathrm{d}y$$

with

$$(\partial_x \mathrm{D}_{\omega_\pm})(x) = \begin{cases} \mathrm{e}^{\pm 2\mathrm{i}\omega_\pm x} & \text{if } \omega_\pm \in \mathbb{C}\setminus\{0\} \\ 1 & \text{if } \omega_\pm = 0 \end{cases}.$$

Hence, back into the interval $I =]r_-, r_+[$ using the diffeomorphism $\phi : I \to \mathbb{R}$ ($\phi(r)$ is nothing but the Regge-Wheeler coordinate x), we can write for a fixed $r = \phi^{-1}(x) \in I$:

$$\begin{pmatrix} f_\pm(r) \\ F(r)(\partial_r f_\pm)(r) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \int_r^{r_\pm} A_\pm(r;\sigma) \begin{pmatrix} f_\pm(\sigma) \\ F(\sigma)(\partial_\sigma f_\pm)(\sigma) \end{pmatrix} \frac{\mathrm{d}\sigma}{F(\sigma)} \tag{5.6}$$

with

$$A_+(r;\sigma) := \begin{pmatrix} \mathrm{D}_{\omega_+}\big(\phi(\sigma)-\phi(r)\big)\mathcal{W}_{+,\ell}(\sigma) & 0 \\ -(\partial_x \mathrm{D}_{\omega_+})\big(\phi(\sigma)-\phi(r)\big)\mathcal{W}_{+,\ell}(\sigma) & 0 \end{pmatrix}, \quad A_-(r;\sigma) := \begin{pmatrix} \mathrm{D}_{\omega_-}\big(\phi(r)-\phi(\sigma)\big)\mathcal{W}_{-,\ell}(\sigma) & 0 \\ (\partial_x \mathrm{D}_{\omega_-})\big(\phi(r)-\phi(\sigma)\big)\mathcal{W}_{-,\ell}(\sigma) & 0 \end{pmatrix}$$

and $\mathcal{W}_{\pm,\ell}(r) := W_\ell(r) + s(V(r)-V_\pm)\big(2z - s(V(r)+V_\pm)\big)$.

5.2.2 Introduction of the scheme

We introduce the scheme we will use in order to solves the integral equations (5.6). Set

$$I_-^{\mathfrak{r}} :=]r_-, \mathfrak{r}], \qquad I_+^{\mathfrak{r}} := [\mathfrak{r}, r_+[$$

with $\mathfrak{r} \in]r_-, r_+[$ (it is not necessarily the radius of the photon sphere). Putting

$$U_\pm(r) := \begin{pmatrix} f_\pm(r) \\ F(r)(\partial_r f_\pm)(r) \end{pmatrix},$$

we can rewrite (5.6) as the following fixed point problem

$$\text{Find } U_\pm \in E_\pm \text{ such that } \begin{cases} U_\pm(r) = \mathcal{G}_\pm(U_\pm)(r) & \forall r \in I_\pm^{\mathfrak{r}} \\ U_\pm(r_\pm) = (1,0) \end{cases} \tag{5.7}$$

with

$$\mathcal{G}_\pm(U_\pm)(r) := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \int_r^{r_\pm} A_\pm(r;\sigma) U_\pm(\sigma) \frac{\mathrm{d}\sigma}{F(\sigma)}.$$

We introduce the spaces $\left(E_\pm, \|\cdot\|_{E_\pm}\right)$ defined by

$$E_\pm := \left\{(u, F\partial_r u) \mid u \in \mathcal{C}^2(I_\pm^{\mathfrak{r}}, \mathbb{C})\right\}, \qquad \|(u, F\partial_r u)\|_{E_\pm} := \|u\|_{\mathcal{C}^1(I_\pm^{\mathfrak{r}}, \mathbb{C})}.$$

Certainly $U_\pm \in E_\pm$. The space of bounded linear operators $\mathcal{B}(E_\pm)$ is equipped with the natural operator norm:

$$|||L|||_{E_\pm} := \sup_{\substack{u \in E_\pm \\ \|u\|_{E_\pm}=1}} \|Lu\|_{E_\pm} \qquad \forall L \in \mathcal{B}(E_\pm).$$

In this setting, $\mathcal{G}_\pm \in \mathcal{B}(E_\pm)$ for $\Im(z) > -\kappa$ thanks to the decay of $\mathcal{W}_{\pm,\ell}$ near $r_\pm$. We define next the work spaces and approximated norms. Let

$$\partial_r^\dagger u : I_\pm^{\mathfrak{r}} \ni r \longmapsto := \begin{cases} (\partial_r u)(r) & \text{if } u \in \mathcal{C}^1 \text{ on a neighborhood of } r \\ 0 & \text{otherwise} \end{cases}$$

where the neighborhoods on $I_\pm^{\mathfrak{r}}$ are defined by the topology inherited from $\mathbb{R}$. We define the Banach space $\left(E_\pm^\dagger, \|\cdot\|_{E_\pm^\dagger}\right)$ as

$$E_\pm^\dagger := \left\{(u, F\partial_r^\dagger u) \mid u \in \mathcal{C}^2_{\text{piece}}(\overline{I}_\pm^{\mathfrak{r}}, \mathbb{C})\right\}, \qquad \|(u, F\partial_r u)\|_{E_\pm^\dagger} := \|u\|_{\mathcal{C}^1(\overline{I}_\pm^{\mathfrak{r}}, \mathbb{C})}$$

and $\mathcal{B}(E_\pm^\dagger)$ is equipped with its natural operators norm. Recall our convention that $u \in \mathcal{C}^k_{\text{piece}}(\overline{I}_\pm^{\mathfrak{r}}, \mathbb{C})$ is defined everywhere on $I_\pm^{\mathfrak{r}}$ by defining $u(r_0) := 0$ at any point r_0 which admits no neighborhood on which u is $\mathcal{C}^k$. The above norms are thus well-defined modulo the identification of functions null almost everywhere on $I_\pm^{\mathfrak{r}}$.

We will use (spatial) meshes $\mathscr{T}_\bullet^\pm = (\mathscr{T}_{\bullet,0}^\pm, \mathscr{T}_{\bullet,1}^\pm)$ as in Subsection 4.1.2 except that $I_\pm^{\mathfrak{r}} \subset \mathscr{T}_{\bullet,0}^\pm$ (we include the endpoint $\mathfrak{r}$), $\min\left\{p_- \middle| p \in \mathscr{T}_{\bullet,0}^-\right\} = r_-$ and ω_p is replaced by $\omega_p^+ := [p, p_+[$ and $\omega_p^- =]p_-, p]$ for all $p \in \mathscr{T}_{\bullet,0}^\pm$. Such a mesh being fixed, we define the projectors[1]

$$\mathscr{P}_{\bullet,\pm} : E_\pm^\dagger \ni \begin{pmatrix} u \\ F\partial_r^\dagger u \end{pmatrix} \longmapsto \begin{pmatrix} u_{\bullet,\pm} \\ (F\partial_r^\dagger u)_{\bullet,\pm} \end{pmatrix} \in \mathscr{P}_{\bullet,\pm}\left(E_\pm^\dagger\right) =: E_{\bullet,\pm}^\dagger,$$

$$\begin{cases} u_{\bullet,\pm}(r) := \dfrac{1}{2} \displaystyle\sum_{p \in \mathscr{T}_{\bullet,0}^\pm} \big(u(p) + u(p_\pm)\big) \mathbb{1}_{\omega_p^\pm}(r) & \forall r \in I_\pm^{\mathfrak{r}} \\ u_{\bullet,\pm}(r_\pm) := u_\pm(r_\pm) \end{cases},$$

the canonical embeddings

$$\mathscr{R}_{\bullet,\pm} : E_{\bullet,\pm}^\dagger \hookrightarrow E_\pm^\dagger$$

[1] This choice of projection is unusual. There are two reasons that motivate it: first of all, we do not need to take derivatives of the functions which are sufficiently smooth anyway, so that the degree of the interpolation polynomial can be zero; on the other hand, adding terms as $\frac{r-p}{p_+-p}$ would make impossible explicit computations for the numerical programming as it would yield integrals without known antiderivative, see below.

as well as the filters

$$\mathscr{F}_{\bullet,\pm} : \mathscr{R}_{\bullet,\pm} \circ \mathscr{P}_{\bullet,\pm} : E^\dagger_\pm \longrightarrow E^\dagger_\pm.$$

Finally, we define the discrete version of the functional $\mathcal{G}_\pm$:

$$\mathcal{G}_{\bullet,\pm} := \mathscr{P}_{\bullet,\pm} \circ \mathcal{G}_\pm \circ \mathscr{R}_{\bullet,\pm} : E^\dagger_{\bullet,\pm} \longrightarrow E^\dagger_{\bullet,\pm}.$$

We summarize all these definitions in the commutative diagram 5.1. The numerical scheme

$$\begin{array}{ccccccc}
E^\dagger_\pm & \xrightarrow{\mathscr{F}_{\bullet,\pm}} & E^\dagger_\pm & \xrightarrow{\quad\mathcal{G}_\pm\quad} & E^\dagger_\pm & \xrightarrow{\mathscr{F}_{\bullet,\pm}} & E^\dagger_\pm \\
{\scriptstyle\mathscr{R}_{\bullet,\pm}}\uparrow & & {\scriptstyle\mathscr{P}_{\bullet,\pm}}\downarrow & & \downarrow{\scriptstyle\mathscr{P}_{\bullet,\pm}} & & \uparrow{\scriptstyle\mathscr{R}_{\bullet,\pm}} \\
E^\dagger_{\bullet,\pm} & = & E^\dagger_{\bullet,\pm} & \xrightarrow{\mathcal{G}_{\bullet,\pm}} & E^\dagger_{\bullet,\pm} & = & E^\dagger_{\bullet,\pm}
\end{array}$$

Figure 5.1: The work spaces and operators associated to (5.7).

consists in solving the discrete version of fixed point problem (5.7):

$$\text{Find } U_{\bullet,\pm} \in E^\dagger_{\bullet,\pm} \text{ such that} \qquad \begin{cases} U_{\bullet,\pm}(r) = \mathcal{G}_{\bullet,\pm}(U_{\bullet,\pm})(r) & \forall r \in I^{\mathfrak{r}}_\pm \\ U_{\bullet,\pm}(r_\pm) = (1,0) \end{cases}. \tag{5.8}$$

We do not know *a priori* whether $U_{\bullet,\pm} = (f_{\bullet,\pm}, (F\partial_r f)_{\bullet,\pm})$ or not.

It turns out that no antiderivative of $\big(A_{\bullet,\pm}(p,r)\big)_{j0}/F(r)$ is known. We thus need to modify the discrete operators $\mathcal{G}_{\bullet,\pm}$. Using the factorization (4.40) as well as formula (4.41), we can write $\mathcal{W}_{\pm,\ell}(r) =: g_{\pm,\ell}(r)k_\pm(r)\mathrm{e}^{2\kappa_\pm\phi(r)}$ for $\pm(r-\mathfrak{r}) \geq 0$ where

$$\begin{aligned}
g_{\pm,\ell}(r) &= \frac{\Lambda(r-r_n)(r-r_c)|r-r_\pm|}{3r^2}\widehat{W}_\ell(r) \pm \frac{s}{rr_\pm}\left(2z - \frac{s(r_\pm+r)}{rr_\pm}\right), \\
\widehat{W}_\ell(r) &= \frac{W_\ell(r)}{F(r)} = \frac{\ell(\ell+1)}{r^2} + \frac{F'(r)}{r} + m^2, \\
k_\pm(r) &= |\mathfrak{r}-r_\pm| \left(\prod_{\alpha\in\{n,c,\mp\}} \left(\frac{\mathfrak{r}-r_\alpha}{r-r_\alpha}\right)^{\frac{\kappa_\pm}{\kappa_\alpha}}\right).
\end{aligned}$$

We define the modified discrete operator

$$\widetilde{\mathcal{G}}_\pm(\widetilde{U}_{\bullet,\pm})(r) := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \int_r^{r_\pm} \widetilde{A}_\pm(r;\sigma)\widetilde{U}_{\bullet,\pm}(\sigma)\frac{\mathrm{d}\sigma}{F(\sigma)}$$

where

$$\widetilde{A}_{\bullet,+}(r;\sigma) := \begin{pmatrix} \mathrm{D}_{\omega_+}\big(\phi(\sigma)-\phi(r)\big) & 0 \\ -(\partial_x \mathrm{D}_{\omega_+})\big(\phi(\sigma)-\phi(r)\big) & 0 \end{pmatrix} \mathrm{e}^{2\kappa_+\phi(\sigma)} \sum_{p\in\mathscr{T}^+_{\bullet,0}} g_{+,\ell}(p)k_+(p)\mathbb{1}_{\omega_p^+}(\sigma),$$

$$\widetilde{A}_{\bullet,-}(r;\sigma) := \begin{pmatrix} \mathrm{D}_{\omega_-}\big(\phi(\sigma)-\phi(r)\big) & 0 \\ (\partial_x \mathrm{D}_{\omega_-})\big(\phi(\sigma)-\phi(r)\big) & 0 \end{pmatrix} \mathrm{e}^{2\kappa_-\phi(\sigma)} \sum_{p\in\mathscr{T}^-_{\bullet,0}} g_{-,\ell}(p)k_-(p)\mathbb{1}_{\omega_p^-}(\sigma).$$

We will therefore look for the solution of the following problem:

$$\text{Find } \widetilde{U}_{\bullet,\pm} \in E^{\dagger}_{\bullet,\pm} \text{ such that } \quad \begin{cases} \widetilde{U}_{\bullet,\pm}(r) = \widetilde{\mathcal{G}}_{\bullet,\pm}(\widetilde{U}_{\bullet,\pm})(r) & \forall r \in I^{\mathfrak{r}}_{\pm} \\ \widetilde{U}_{\bullet,\pm}(r_{\pm}) = (1,0) \end{cases} . \tag{5.9}$$

Computations for the numerical programming are carried out in Appendix 5.6.1.

Remark 5.2.1. *The scheme (5.9) gives an approximation of $(f_{\pm}, F\partial_r f_{\pm})$. It is not possible to get $\partial_r f_{\pm}$ alone, at least near $r_{\pm}$ because F cancels there. It is nonetheless not troublesome as we will never be interested only in the values of $\partial_r f_{\pm}(r)$ as the Wronskian uses $\partial_x f_{\pm} = F\partial_r f_{\pm}$.*

5.3 Error of approximation

In this Section, we prove the convergence of the numerical scheme introduced in Subsection 5.2.2 and give an estimate of the error.

5.3.1 Constants in the scheme

Control of the error of approximation of the modified Jost solutions (and thus of the Wronskian) involves many constants (which actually depends on z and s). We gather all of them in this paragraph to make the sequel clearer. For the sake of clarity, we give upper bounds for error terms appearing in the proof of the lemmas and proposition mentioned below. We stress out here that this does not yield an optimal estimate of the error; following each proof using optimal estimates presumably increases the final estimate.

Let $z \in \mathbb{C}$ such that $\Im(z) > -\kappa$ and set $\omega_+ := z - sV_{\pm}$. First of all, we define

$$K_{\pm} := |r_{\pm} - \mathfrak{r}| \left(\frac{\mathfrak{r} - r_n}{r_{\pm} - r_n} \right)^{\frac{\kappa_{\pm}}{\kappa_n}} \left| \frac{r_{\mp} - \mathfrak{r}}{r_+ - r_-} \right|^{\frac{\kappa_{\pm}}{\kappa_{\mp}}}, \tag{5.10}$$

$$C_{\pm}(z) := K_{\pm} \left[\frac{2|\kappa_+| r_+^2}{r_-^2} \left(\frac{\ell(\ell+1)}{r_-^2} + \frac{\|F'\|_{L^\infty(I,\mathbb{R})}}{r_-} + m^2 \right) + \frac{|s|}{r_- r_{\pm}} \left(2|z| + |s| \left(\frac{1}{r_-} + \frac{1}{r_{\pm}} \right) \right) \right], \tag{5.11}$$

$$\widetilde{G}_{\pm}(z) = \begin{cases} \dfrac{1}{2(|\kappa_{\pm}| + \Im(z)^-)|\omega_{\pm}|} & \text{if } |\omega_{\pm}| \geq |\kappa_{\pm}| \\ \dfrac{\sqrt{13}}{8(|\kappa_{\pm}| + \Im(z)^-)^2} & \text{if } 0 < |\omega_{\pm}| < |\kappa_{\pm}| \\ \dfrac{1}{4\kappa_{\pm}^2} & \text{if } \omega_{\pm} = 0 \end{cases} . \tag{5.12}$$

Let us briefly explain the role of these constants:

- $K_{\pm}$ is such that $|r - r_{\pm}| \leq K_{\pm} \mathrm{e}^{2\kappa_{\pm} x}$ as $\pm x \geq 0$, *cf.* the exponential decay (4.41).
- $C_{\pm}(z)$ is such that $\left| W_{\ell}(x) + s(V(x) - V_{\pm})\big(2z - s(V(x) + V_{\pm})\big) \right| \leq C_{\pm}(z) \mathrm{e}^{2\kappa_{\pm} x}$ whenever $\pm x \geq 0$; this uses the factorization (4.40) as well as $\frac{2|\kappa_+| r_+^2}{r_-^2} = \frac{\Lambda}{3r_-^2}(r_+ - r_n)(r_+ - r_c)(r_+ - r_-)$.

- $\widetilde{G}_{\pm}(z)$ is used to bound integrals of the form

$$\int_x^X \left|\mathrm{D}_{\omega_\pm}(y-x)\right|\Big(|W_\ell(x)|+|s||V(x)-V_\pm|\big(2|z|+|s||V(x)+V_\pm|\big)\Big)\mathrm{d}y.$$

From these constants, we define the following ones:

For Lemma 5.3.1:

$$\breve{C}_{\pm,0}(z)=\left(\frac{1}{4(|\kappa_\pm|+\Im(z))}+4\widetilde{G}_\pm(z)\max\left\{1,|\kappa_\pm|+\Im(z)^-\right\}\right)\breve{C}_\pm(z), \tag{5.13}$$

$$\breve{C}_{\pm,1}(z)=\left(1+\frac{\max\left\{3,|\omega_\pm|\right\}}{2(|\kappa_\pm|+\Im(z))}\right)\breve{C}_\pm(z) \tag{5.14}$$

with

$$\breve{C}_\pm(z)=C_\pm(z)\left(\sum_{\alpha\in\{n,\mp\}}\frac{1}{|\kappa_\alpha||\mathfrak{r}-r_\alpha|}+\max\left\{1,\frac{1}{|\kappa_\pm|}\right\}|r_\pm-\mathfrak{r}|^{\pm\frac{|\kappa_\pm|+\Im(z)^-}{\kappa_\pm}}\left(\frac{r_\pm-r_c}{\mathfrak{r}-r_c}\right)^{\mp\frac{|\kappa_\pm|+\Im(z)^-}{\kappa_c}}\right). \tag{5.15}$$

For Lemma 5.3.2:

$$\begin{aligned}\Psi_\pm(z)=\left\|(\widetilde{U}_{\bullet,\pm})_1\right\|_{L^\infty(I^{\mathfrak{r}}_\pm,\mathbb{C})}\Xi_\pm(z)+\frac{C_\pm(z)}{2(|\kappa_\pm|+\Im(z))}\left\|(\widetilde{U}_{\bullet,\pm})_0\right\|_{L^\infty(I^{\mathfrak{r}}_\pm,\mathbb{C})}\Xi_\pm(z)\\ \times\left(1+C_\pm(z)\widetilde{G}_\pm(z)\exp\left(C_\pm(z)\widetilde{G}_\pm(z)\right)\right),\end{aligned} \tag{5.16}$$

with

$$\begin{aligned}\left\|\left(\widetilde{U}_{\bullet,\pm}\right)_0\right\|_{L^\infty(I^{\mathfrak{r}}_\pm,\mathbb{C})}&\le 1+\Pi(z)K_\pm\widetilde{G}_\pm(z)\exp\left(\Pi(z)K_\pm\widetilde{G}_\pm(z)\right),\\ \left\|\left(\widetilde{U}_{\bullet,\pm}\right)_1\right\|_{L^\infty(I^{\mathfrak{r}}_\pm,\mathbb{C})}&\le\frac{\Pi(z)K_\pm}{2(|\kappa_\pm|+\Im(z))}\left(1+\Pi(z)K_\pm\widetilde{G}_\pm(z)\exp\left(\Pi(z)K_\pm\widetilde{G}_\pm(z)\right)\right),\end{aligned}$$

$$\Xi_\pm(z)=\left(\Upsilon_\pm(z)K_\pm+\Pi(z)\Theta_\pm\right)\max\left\{\widetilde{G}_\pm(z),\frac{1}{2(|\kappa_\pm|+\Im(z))}\right\},$$

$$\begin{aligned}\Upsilon_-(z)=&\frac{\Lambda}{3}\left\|\frac{(r-r_n)(r-r_c)\widehat{W}_\ell}{r^2}\right\|_{L^\infty(]r_-,\mathfrak{r}[,\mathbb{R})}+\frac{\Lambda(r_+-r_-)}{3}\left\|\frac{(r-r_n)\widehat{W}_\ell}{r^2}\right\|_{L^\infty(]r_-,\mathfrak{r}[,\mathbb{R})}\\ &+\frac{\Lambda(\mathfrak{r}-r_-)(\mathfrak{r}-r_c)}{3}\left\|\frac{\widehat{W}_\ell}{r^2}\right\|_{L^\infty(]r_-,\mathfrak{r}[,\mathbb{R})}+\frac{\Lambda}{3}\left\|\frac{(r-r_n)(r-r_c)(r_+-r)}{r}\right\|_{L^\infty(]r_-,\mathfrak{r}[,\mathbb{R})}\left\|\frac{\widehat{W}_\ell}{r^2}\right\|_{L^\infty(]r_-,\mathfrak{r}[,\mathbb{R})}\\ &+\frac{\Lambda}{3}\left\|\frac{(r-r_n)(r-r_c)(r_+-r)}{r^2}\right\|_{L^\infty(]r_-,\mathfrak{r}[,\mathbb{R})}\left(\left\|\frac{\widehat{W}_\ell}{r}\right\|_{L^\infty(]r_-,\mathfrak{r}[,\mathbb{R})}+\left\|\widehat{W}_\ell'\right\|_{L^\infty(]r_-,\mathfrak{r}[,\mathbb{R})}\right)\\ &+\frac{|s|}{r_-^3}\left(2|z|+\frac{3|s|}{r_-}\right),\end{aligned} \tag{5.17}$$

$$\begin{aligned}\Upsilon_+(z) = {} & \frac{\Lambda}{3}\left\|\frac{(r-r_n)(r-r_c)\widehat{W}_\ell}{r^2}\right\|_{L^\infty(]\mathfrak{r},r_+[,\mathbb{R})} + \frac{\Lambda(r_+-r_-)}{3}\left\|\frac{(r-r_n)\widehat{W}_\ell}{r^2}\right\|_{L^\infty(]\mathfrak{r},r_+[,\mathbb{R})} \\ & + \frac{\Lambda(r_+-r_-)(r_+-r_c)}{3}\left\|\frac{\widehat{W}_\ell}{r^2}\right\|_{L^\infty(]\mathfrak{r},r_+[,\mathbb{R})} + \frac{\Lambda}{3}\left\|\frac{(r-r_n)(r-r_c)(r-r_-)}{r}\right\|_{L^\infty(]\mathfrak{r},r_+[,\mathbb{R})}\left\|\frac{\widehat{W}_\ell}{r^2}\right\|_{L^\infty(]\mathfrak{r},r_+[,\mathbb{R})} \\ & + \frac{\Lambda}{3}\left\|\frac{(r-r_n)(r-r_c)(r-r_-)}{r^2}\right\|_{L^\infty(]\mathfrak{r},r_+[,\mathbb{R})}\left(\left\|\frac{\widehat{W}_\ell}{r}\right\|_{L^\infty(]\mathfrak{r},r_+[,\mathbb{R})} + \left\|\widehat{W}_\ell'\right\|_{L^\infty(]\mathfrak{r},r_+[,\mathbb{R})}\right) \\ & + \frac{|s|}{r_+\mathfrak{r}^2}\left(2|z| + |s|\left(\frac{1}{r_+}+\frac{2}{\mathfrak{r}}\right)\right), \end{aligned} \tag{5.18}$$

$$\widehat{W}_\ell(r) = \frac{\ell(\ell+1)}{r^2} + \frac{F'(r)}{r} + m^2,$$

$$\Pi(z) = \frac{\Lambda(r_+-r_n)(r_+-r_c)(r_+-r_-)}{3r_-^2}\|\widehat{W}_\ell\|_{L^\infty(I,\mathbb{R})} + \frac{|s|}{r_-^2}\left(2|z| + \frac{2|s|}{r_-}\right), \tag{5.19}$$

$$\Theta_\pm = |\mathfrak{r}-r_\pm| \sum_{\substack{(\alpha_1,\alpha_2,\alpha_3)\in\{n,c,\mp\}^3 \\ \alpha_1\neq\alpha_2\neq\alpha_3\neq\alpha_1}} (\vartheta^\pm)'_{\alpha_1}\vartheta^\pm_{\alpha_2}\vartheta^\pm_{\alpha_3} \tag{5.20}$$

and

$$\vartheta^\pm_n = \left(\frac{\mathfrak{r}-r_n}{r_\pm-r_n}\right)^{\frac{\kappa_\pm}{\kappa_n}}, \qquad \vartheta^\pm_c = 1, \qquad \vartheta^\pm_\mp = \left|\frac{\mathfrak{r}-r_\mp}{r_+-r_-}\right|^{\frac{\kappa_\pm}{\kappa_\mp}},$$

$$\begin{aligned}(\vartheta^\pm)'_n &= \left|\frac{\kappa_\pm}{\kappa_n}\right|\max\left\{1,\left(\frac{\mathfrak{r}-r_n}{r_\pm-r_n}\right)^{\frac{\kappa_\pm}{\kappa_n}-1}\right\}\frac{1}{(\mathfrak{r}-r_n)^2}, \\ (\vartheta^\pm)'_c &= \left|\frac{\kappa_\pm}{\kappa_c}\right|\max\left\{1,\left(\frac{\mathfrak{r}-r_c}{r_\pm-r_c}\right)^{\frac{\kappa_\pm}{\kappa_c}-1}\right\}\frac{1}{(\mathfrak{r}-r_c)^2}, \\ (\vartheta^\pm)'_\mp &= \left|\frac{\kappa_\pm}{\kappa_\mp}\right|\left|\frac{\mathfrak{r}-r_\mp}{r_+-r_-}\right|^{\frac{\kappa_\pm}{\kappa_\mp}-1}\frac{1}{(\mathfrak{r}-r_\mp)^2}.\end{aligned}$$

For Lemma 5.3.4:

$$\widehat{C}_\pm(z) = \max\left\{\widehat{C}_{\pm,0}(z), \widehat{C}_{\pm,1}(z)\right\} \tag{5.21}$$

with

$$\begin{aligned}\widehat{C}_{\pm,0}(z) &= 2C_\pm(z)\widetilde{G}_\pm(z), \\ \widehat{C}_{\pm,1}(z) &= \frac{C_\pm(z)}{(|\kappa_\pm| + \Im(z))}.\end{aligned}$$

For Proposition 5.3.7:

$$C_\bullet(z) = \sum_{\pm} \Bigg[\mathcal{Z}_{\pm,0}(z) \frac{C_\mp(z)}{2(|\kappa_\mp| + \Im(z))} \|f_\mp\|_{L^\infty(I^{\mathrm{r}}_\mp,\mathbb{C})} + \mathcal{Z}_{\pm,1}(z) \|f_\mp\|_{L^\infty(I^{\mathrm{r}}_\mp,\mathbb{C})} + \mathcal{Z}_{\pm,0}(z)\mathcal{Z}_{\mp,1}(z) + \frac{|s|(r_+ - r_-)}{r_+ r_-} \mathcal{Z}_{\pm,0}(z) \|f_\mp\|_{L^\infty(I^{\mathrm{r}}_\mp,\mathbb{C})} \Bigg] + \mathcal{Z}_{+,0}(z)\mathcal{Z}_{-,0}(z) \tag{5.22}$$

with

$$\mathcal{Z}_{\pm,j}(z) = \Psi_\pm(z) h_\bullet^{\frac{|\Im(z)^-|}{\kappa}} + \widetilde{C}_\pm(z) \|f_\pm\|_{L^\infty(I^{\mathrm{r}}_\pm,\mathbb{C})} \check{C}_{\pm,j}(z).$$

The constants $\check{C}_{\pm,j}(z)$ are defined in (5.13), (5.14) and $\Psi_\pm(z)$ is as (5.16). Upper bounds for $\widetilde{C}_\pm(z)$ and $f_\pm$ are respectively given in Lemma 5.3.4 and Lemma 5.3.5.

For Lemma 5.3.8:

$$\check{K}(z) = \sum_{\pm} \alpha_\pm(z)\big(1 + \beta_\pm(z)\mathrm{e}^{\beta_\pm(z)}\big) \left(\gamma_\mp(z) + \gamma_\pm(z) + \frac{|s|(r_+ - r_-)}{r_- r_+} \right) + \delta_\pm(z) \tag{5.23}$$

where

$$\alpha_\pm(z) = \frac{2|s|K_\pm}{r_- r_\pm} \widetilde{G}_\pm(z) + C_\pm(z)\check{G}_\pm(z),$$

$$\check{G}_\pm(z) = \begin{cases} \dfrac{1}{|\omega_\pm|} \left(\dfrac{1}{||\kappa_\pm| + \Im(z)|} + \widetilde{G}_\pm(z) \right) & \text{if } |\omega_+| \geq |\kappa_\pm| \\ \dfrac{5}{4|\kappa_\pm|^3} & \text{if } \Im(z) \geq 0 \text{ and } 0 \leq |\omega_+| < |\kappa_\pm| \\ \dfrac{1}{4||\kappa_\pm| + \Im(z)|^3} & \text{if } \Im(z) < 0 \text{ and } 0 < |\omega_+| < |\kappa_\pm| \end{cases} \tag{5.24}$$

$$\begin{aligned} \beta_\pm(z) &= C_\pm(z)\widetilde{G}_\pm(z), \\ \gamma_\pm(z) &= \frac{C_\pm(z)}{2||\kappa_\pm| + \Im(z)|}, \\ \delta_\pm(z) &= \frac{C_\pm(z)}{2(|\kappa_\pm| + \Im(z))^2} + \frac{|s|K_\pm}{||\kappa_\pm| + \Im(z)|r_- r_+}. \end{aligned}$$

Recall that $C_\pm(z)$ and $\widetilde{G}_\pm(z)$ are defined in (5.11) and (5.12) above.

5.3.2 Convergence of the scheme

We show in this paragraph the convergence of the above scheme using slightly different norms, the convergence in the spaces $E^\dagger_\pm$ being then a corollary of the results below. This will allow us to identify bounds for the error of approximation of $f_\pm$ and $f'_\pm$ alone.

Let $U_\pm := (f_\pm, F\partial_r f_\pm) \in E_\pm$ be the fixed point in (5.7) and let $U_{\bullet,\pm}, \widetilde{U}_{\bullet,\pm} \in E^\dagger_{\bullet,\pm}$ be the fixed points in (5.8) and (5.9). We start with the consistency of the filters with respect to the identity operator (the proof is given in Appendix 5.5.2).

Lemma 5.3.1. *Let $z \in \mathbb{C}$ such that $\Im(z) > -\kappa$ and set $\omega_\pm := z - sV_\pm$. Assume for simplicity that*[2]

$$\text{the mesh is regular,} \qquad \frac{1}{|r_\pm - \mathfrak{r}|}\left|\frac{r_\pm - r_c}{\mathfrak{r} - r_c}\right|^{\frac{\kappa_\pm}{\kappa_c}} h_\bullet \le 1. \tag{5.25}$$

Then for all $r \in I_\pm^{\mathfrak{r}}$ and, we have

$$\left|\left(U_\pm - \mathscr{F}_{\bullet,\pm}U_\pm\right)_j(r)\right| \le \|f_\pm\|_{L^\infty(I_\pm^{\mathfrak{r}},\mathbb{C})}\check{C}_{\pm,j}(z)h_\bullet^{1-\left|\frac{\Im(z)^-}{\kappa_\pm}\right|} \qquad \forall j \in \{0,1\} \tag{5.26}$$

where $\check{C}_{\pm,0}(z), \check{C}_{\pm,1}(z)$ are defined in (5.13) and (5.14).

As expected, approximations of the modified Jost solutions (5.26) worsen as $\Im(z) \to -\kappa$. The next result gives an estimate for the error committed whence replacing the solution of (5.8) by (5.9) (the proof is given in Appendix 5.5.3).

Lemma 5.3.2. *For all $r \in I_\pm^{\mathfrak{r}}$, we have*

$$\left|\mathscr{R}_{\bullet,\pm}\left(U_{\bullet,\pm} - \widetilde{U}_{\bullet,\pm}\right)_j(r)\right| \le \Psi_\pm(z)h_\bullet \qquad \forall j \in \{0,1\}$$

where $\Psi_\pm(z)$ is defined in (5.16).

From Lemma 5.3.1 and Lemma 5.3.2, we can deduce the convergence of the scheme (5.9):

Proposition 5.3.3. *Assume (5.25). For all $r \in I_\pm^{\mathfrak{r}}$, we have*

$$\left|\left(U_\pm - \mathscr{R}_{\bullet,\pm}\widetilde{U}_{\bullet,\pm}\right)_j(r)\right| \le \Psi_\pm(z)h_\bullet + \widetilde{C}_\pm(z)\|f_\pm\|_{L^\infty(I_\pm^{\mathfrak{r}},\mathbb{C})}\check{C}_{\pm,j}(z)h_\bullet^{1-\left|\frac{\Im(z)^-}{\kappa_\pm}\right|} \qquad \forall j \in \{0,1\}$$

where $\Psi_\pm(z)$, $\check{C}_{\pm,0}(z), \check{C}_{\pm,1}(z)$ are respectively defined in (5.16), (5.13) and (5.14), and with

$$\widetilde{C}_\pm(z) = \max_{j\in\{0,1\}}\left\{1 \pm 2\int_{\mathfrak{r}}^{r_\pm}\left|(A_\pm)_{j0}(r;\sigma)\right|\exp\left(\pm 2\int_\sigma^{r_\pm}\left|(A_\pm)_{j0}(r;\sigma')\right|\frac{\mathrm{d}\sigma'}{F(\sigma')}\right)\frac{\mathrm{d}\sigma}{F(\sigma)}\right\}. \tag{5.27}$$

Proof. We only treat the + case. Pick $r \in I_+^{\mathfrak{r}}$: there exists $p \in \mathscr{T}_{\bullet,0}^+$ such that $r \in \omega_p^+ = [p, p_+[$. By the triangle inequality, we first write

$$\left|\left(U_\pm - \mathscr{R}_{\bullet,\pm}\widetilde{U}_{\bullet,\pm}\right)_j(r)\right| \le \left|\mathscr{R}_{\bullet,\pm}\left(U_{\bullet,\pm} - \widetilde{U}_{\bullet,\pm}\right)_j(r)\right| + \left|\left(U_\pm - \mathscr{R}_{\bullet,\pm}U_{\bullet,\pm}\right)_j(r)\right|$$

and the first term on the right-hand side above is bounded by $\Psi_\pm(z)h_\bullet$ by in Lemma 5.3.2. Next, we compute:

$$\begin{aligned}(\mathscr{F}_{\bullet,+}\mathcal{G}_+(U_+ - \mathscr{R}_{\bullet,+}U_{\bullet,+}))(r) &= \frac{r - p_+}{p - p_+}\int_p^{r_+} A_+(r;\sigma)(U_+ - \mathscr{R}_{\bullet,+}U_{\bullet,+})(\sigma)\frac{\mathrm{d}\sigma}{F(\sigma)} \\ &\quad + \frac{r - p}{p_+ - p}\int_{p_+}^{r_+} A_+(r;\sigma)(U_+ - \mathscr{R}_{\bullet,+}U_{\bullet,+})(\sigma)\frac{\mathrm{d}\sigma}{F(\sigma)}.\end{aligned}$$

[2] Recall that the mesh is regular means that for all $p \in \mathscr{T}_{\bullet,0}^\pm$, $p_+ - p = h_\bullet$. This hypothesis is absolutely not necessary but is assumed to make clearer the proof of this lemma. The other hypothesis on the size of $h_\bullet$, satisfied for $h_\bullet$ sufficiently small, allows us to conveniently compare the term in (5.25) with similar ones to the power $1 - \left|\frac{\Im(z)^-}{\kappa_\pm}\right|$.

It follows for all $j \in \{0,1\}$:

$$\big|(\mathscr{F}_{\bullet,+}\mathcal{G}_+(U_+ - \mathscr{R}_{\bullet,+}U_{\bullet,+}))_j(r)\big| \le 2\int_p^{r_+} \big|(A_+)_{j0}(r;\sigma)\big|\big|(U_+ - \mathscr{R}_{\bullet,+}U_{\bullet,+})_j(\sigma)\big|\frac{\mathrm{d}\sigma}{F(\sigma)}. \tag{5.28}$$

Write then

$$\begin{aligned}\big|(U_+ - \mathscr{R}_{\bullet,+}U_{\bullet,+})(r)\big| &\le \big|(U_+ - \mathscr{F}_{\bullet,+}U_+)(r)\big| + \big|(\mathscr{F}_{\bullet,+}U_+ - \mathscr{R}_{\bullet,+}U_{\bullet,+})(r)\big| \\ &\le \big|(U_+ - \mathscr{F}_{\bullet,+}U_+)(r)\big| + \big|\mathscr{F}_{\bullet,+}\mathcal{G}_+(U_+ - \mathscr{R}_{\bullet,+}U_{\bullet,+})(r)\big|. \end{aligned}\tag{5.29}$$

Combining (5.28) and (5.29) with Grönwall's inequality, we obtain:

$$\begin{aligned}\big|(U_+ - \mathscr{R}_{\bullet,+}U_{\bullet,+})_j(r)\big| &\le \big|(U_+ - \mathscr{F}_{\bullet,+}U_+)_j(r)\big| + 2\int_p^{r_+} \big|(U_+ - \mathscr{F}_{\bullet,+}U_+)_j(\sigma)\big|\big|(A_+)_{j0}(r;\sigma)\big| \\ &\qquad \times \exp\left(2\int_\sigma^{r_+} \big|(A_+)_{j0}(r;\sigma')\big|\frac{\mathrm{d}\sigma'}{F(\sigma')}\right)\frac{\mathrm{d}\sigma}{F(\sigma)} \\ &\le \big\|(U_\pm - \mathscr{F}_{\bullet,\pm}U_\pm)_j\big\|_{L^\infty(I^{\mathfrak{r}}_\pm,\mathbb{C})} \\ &\quad \times \left(1 + 2\int_p^{r_+} \big|(A_+)_{j0}(r;\sigma)\big| \exp\left(2\int_\sigma^{r_+} \big|(A_+)_{j0}(r;\sigma')\big|\frac{\mathrm{d}\sigma'}{F(\sigma')}\right)\frac{\mathrm{d}\sigma}{F(\sigma)}\right).\end{aligned}$$

It remains to take $p = \sigma = \mathfrak{r}$ in the above integrals and apply (5.26) to complete the proof. □

It is worth for the sequel explicating the dependence in z of the constants $\widetilde{C}_\pm(z)$. This is the purpose of the next result.

Lemma 5.3.4. *Let $z \in \mathbb{C}$ such that $\Im(z) > -\kappa$ and set $\omega_+ := z - sV_\pm$. Then*

$$\widetilde{C}_\pm(z) \le 2\Big(1 + \widehat{C}_\pm(z)\exp\big(\widehat{C}_\pm(z)\big)\Big)$$

with $\widehat{C}_\pm(z)$ as in (5.21).

Proof. We only treat the + case. Consider first $j = 0$. Recalling the definition of W_ℓ in (5.2) and using (4.41) and (4.40), we compute:

$$2\int_\sigma^{r_+} \big|(A_+)_{00}(r;\sigma)\big|\frac{\mathrm{d}\sigma}{F(\sigma)} \le 2C_+(z)\int_x^{+\infty} \big|\mathrm{D}_{\omega_+}(y-x)\big|\mathrm{e}^{2\kappa_+ y}\mathrm{d}y$$

where the constant $C_+(z)$ is defined in (5.11). The integral on the right-hand side is dealt as in Lemma 5.5.1; we get:

$$\int_x^{+\infty} \big|\mathrm{D}_{\omega_+}(y-x)\big|\mathrm{e}^{2\kappa_+ y}\mathrm{d}y \le \begin{cases} \dfrac{1}{2|\kappa_+| + \Im(z)^-|\omega_+|} & \text{if } |\omega_+| \ge |\kappa_+| \\ \dfrac{\sqrt{13}}{8(|\kappa_+| + \Im(z)^-)^2} & \text{if } 0 < |\omega_+| < |\kappa_+| \\ \dfrac{1}{4\kappa_+^2} & \text{if } \omega_+ = 0 \end{cases}.$$

Next,

$$\int_\sigma^{r_+} \big|(A_+)_{01}(r;\sigma)\big| \frac{\mathrm{d}\sigma}{F(\sigma)} \leq C_+(z) \int_x^{+\infty} \big|(\partial_x \mathrm{D}_{\omega_+})(y-x)\big| \mathrm{e}^{2\kappa_+ y} \mathrm{d}y.$$

We have:

$$\int_x^{+\infty} \big|(\partial_x \mathrm{D}_{\omega_+})(y-x)\big| \mathrm{e}^{2\kappa_+ y} \mathrm{d}y = \frac{\mathrm{e}^{2\kappa_+ x}}{2|\kappa_+ - \Im(z)|}.$$

Recall here that $\kappa_+ - \Im(z) \neq 0$ since $\Im(z) > -\kappa \geq \kappa_+$. This concludes the proof. □

5.3.3 Bounds for the modified Jost solutions

While the functions $f_\pm$ are not exactly known, we can use their integral representations derived in the paragraph 5.2.1 in order to estimate their norms. This together with Proposition 5.3.3 will yield an estimate for the error of approximation of the Wronskian in the paragraph 5.3.4.

We start by showing the following result (the proof is given in Appendix 5.5.5):

Lemma 5.3.5. *Let $z \in \mathbb{C}$ with $\Im(z) > -\kappa$ and set $\omega_\pm := z - sV_\pm$. Then*

$$\|f_\pm\|_{L^\infty(I_\pm^{\mathfrak{r}},\mathbb{C})} \leq \begin{cases} 1 + \dfrac{C_\pm(z)}{2(|\kappa_\pm| + \Im(z)^-)|\omega_\pm|} \exp\left(\dfrac{C_\pm(z)}{2(|\kappa_\pm| + \Im(z)^-)|\omega_\pm|}\right) & \text{if } |\omega_\pm| \geq |\kappa_\pm| \\ 1 + \dfrac{\sqrt{13} C_\pm(z)}{8(|\kappa_\pm| + \Im(z)^-)^2} \exp\left(\dfrac{\sqrt{13} C_\pm(z)}{8(|\kappa_\pm| + \Im(z)^-)^2}\right) & \text{if } 0 < |\omega_\pm| < |\kappa_\pm| \\ 1 + \dfrac{C_\pm(z)}{4\kappa_\pm^2} \exp\left(\dfrac{C_\pm(z)}{4\kappa_\pm^2}\right) & \text{if } \omega_\pm = 0 \end{cases}.$$

We then deduce a bound for the first derivative in x using the following estimates (see the proof in Appendix 5.5.6):

Corollary 5.3.6. *Let $z \in \mathbb{C}$ with $\Im(z) > -\kappa$. Then*

$$\|\partial_x f_\pm\|_{L^\infty(I_\pm,\mathbb{C})} \leq \frac{C_\pm(z)}{2(|\kappa_\pm| + \Im(z))} \|f_\pm\|_{L^\infty(I_\pm^{\mathfrak{r}},\mathbb{C})}.$$

5.3.4 Error of approximation for the Wronskian

Proposition 5.3.3 entails a bound for the error of approximation for the Wronskian $\mathscr{W}$. Indeed, as $\mathscr{W}(z)$ is independent of $x \in \mathbb{R}$ for $\Im(z) > -\kappa$, we can choose $x = 0$ which corresponds to $r = \mathfrak{r}$. Using

$$e_\pm(x) := \mathrm{e}^{\pm \mathrm{i}(z - sV_\pm)x} f_\pm(x) \qquad \forall x \in \mathbb{R},$$

we see that $e_\pm(0) = f_\pm(\mathfrak{r})$ and $(\partial_x e_\pm)(0) = \mathrm{i}(z - sV_\pm) f_\pm(\mathfrak{r}) + F(\mathfrak{r}) f'_\pm(\mathfrak{r})$, whence:

$$\begin{aligned} \mathscr{W}(z) &= f_+(\mathfrak{r})\Big(\mathrm{i}(z - sV_-) f_-(\mathfrak{r}) + F(\mathfrak{r}) f'_-(\mathfrak{r})\Big) - \Big(\mathrm{i}(z - sV_+) f_+(\mathfrak{r}) + F(\mathfrak{r}) f'_+(\mathfrak{r})\Big) f_-(\mathfrak{r}) \\ &= F(\mathfrak{r})\Big(f_+(\mathfrak{r}) f'_-(\mathfrak{r}) - f'_+(\mathfrak{r}) f_-(\mathfrak{r})\Big) + \mathrm{i}s\big(V_+ - V_-\big) f_+(\mathfrak{r}) f_-(\mathfrak{r}). \end{aligned}$$

Letting $U_\pm(\mathfrak{r}) := (f_\pm(\mathfrak{r}), F(\mathfrak{r})(\partial_r f_\pm)(\mathfrak{r})$, we get (we omit the dependence in $\mathfrak{r}$):

$$\mathscr{W}(z) = (U_+)_0(U_-)_1 - (U_+)_1(U_-)_0 + \mathrm{i}s\big(V_+ - V_-\big)(U_+)_0(U_-)_0. \tag{5.30}$$

Formula (5.30) is exactly the one we will use to compute resonances. Set then

$$\mathscr{W}_\bullet(z) = (\mathscr{R}_{\bullet,+}\widetilde{U}_{\bullet,+})_0(\mathscr{R}_{\bullet,-}\widetilde{U}_{\bullet,-})_1 - (\mathscr{R}_{\bullet,+}\widetilde{U}_{\bullet,+})_1(\mathscr{R}_{\bullet,-}\widetilde{U}_{\bullet,-})_0 + \mathrm{i}s\big(V_+ - V_-\big)(\mathscr{R}_{\bullet,+}\widetilde{U}_{\bullet,+})_0(\mathscr{R}_{\bullet,-}\widetilde{U}_{\bullet,-})_0$$

evaluated at $r = \mathfrak{r}$. We have (we omit the dependence in $\mathfrak{r}$):

$$\begin{aligned}
\mathscr{W}(z) - \mathscr{W}_\bullet(z) = {} & \big(U_+ - \mathscr{R}_{\bullet,+}\widetilde{U}_{\bullet,+}\big)_0(U_-)_1 + (U_+)_0\big((U_-)_1 - (\mathscr{R}_{\bullet,-}\widetilde{U}_{\bullet,-})_1\big) \\
& - \big((U_+)_1 - (\mathscr{R}_{\bullet,+}\widetilde{U}_{\bullet,+})_1\big)(U_-)_0 - (U_+)_1\big(U_- - \mathscr{R}_{\bullet,-}\widetilde{U}_{\bullet,-}\big)_0 \\
& - \big(U_+ - \mathscr{R}_{\bullet,+}\widetilde{U}_{\bullet,+}\big)_0\big((U_-)_1 - (\mathscr{R}_{\bullet,-}\widetilde{U}_{\bullet,-})_1\big) + \big((U_+)_1 - (\mathscr{R}_{\bullet,+}\widetilde{U}_{\bullet,+})_1\big)\big(U_- - \mathscr{R}_{\bullet,-}\widetilde{U}_{\bullet,-}\big)_0 \\
& + \mathrm{i}s(V_+ - V_-)\Big[\big(U_+ - \mathscr{R}_{\bullet,+}\widetilde{U}_{\bullet,+}\big)_0(U_-)_0 + (U_+)_0\big(U_- - \mathscr{R}_{\bullet,-}\widetilde{U}_{\bullet,-}\big)_0 \\
& \qquad - \big(U_+ - \mathscr{R}_{\bullet,+}\widetilde{U}_{\bullet,+}\big)_0\big(U_- - \mathscr{R}_{\bullet,-}\widetilde{U}_{\bullet,-}\big)_0\Big]
\end{aligned}$$

whence

$$\begin{aligned}
\big|\mathscr{W}(z) - \mathscr{W}_\bullet(z)\big| \le \sum_\pm \Big[& \big|\big(U_\pm - \mathscr{R}_{\bullet,\pm}\widetilde{U}_{\bullet,\pm}\big)_0(\mathfrak{r})\big|\|\partial_x f_\mp\|_{L^\infty(I^{\mathfrak{r}}_\mp,\mathbb{C})} + \big|\big(U_\pm - \mathscr{R}_{\bullet,\pm}\widetilde{U}_{\bullet,\pm}\big)_1(\mathfrak{r})\big|\|f_\mp\|_{L^\infty(I^{\mathfrak{r}}_\mp,\mathbb{C})} \\
& + \big|\big(U_\pm - \mathscr{R}_{\bullet,\pm}\widetilde{U}_{\bullet,\pm}\big)_0(\mathfrak{r})\big|\big|\big(U_\mp - \mathscr{R}_{\bullet,\mp}\widetilde{U}_{\bullet,\mp}\big)_1(\mathfrak{r})\big|\Big] \\
& + \frac{|s|(r_+ - r_-)}{r_+ r_-}\Big[\sum_\pm \big|\big(U_\pm - \mathscr{R}_{\bullet,\pm}\widetilde{U}_{\bullet,\pm}\big)_0(\mathfrak{r})\big|\|f_\mp\|_{L^\infty(I^{\mathfrak{r}}_\mp,\mathbb{C})} \\
& \qquad + \big|\big(U_+ - \mathscr{R}_{\bullet,+}\widetilde{U}_{\bullet,+}\big)_0(\mathfrak{r})\big|\big|\big(U_- - \mathscr{R}_{\bullet,-}\widetilde{U}_{\bullet,-}\big)_0(\mathfrak{r})\big|\Big].
\end{aligned}$$

Using Proposition 5.3.3, we finally obtain:

Proposition 5.3.7 (Error of approximation for the Wronskian)**.** *Assume* (5.25)*. Let $z \in \mathbb{C}$ such that $\Im(z) > -\kappa$. Then*

$$|\mathscr{W}(z) - \mathscr{W}_\bullet(z)| \le C_\bullet(z) h_\bullet^{1 - \frac{|\Im(z)^-|}{\kappa}}$$

where $C_\bullet(z)$ is as in (5.22)*.*

We will also need a bound for $\mathscr{W}'(z) := (\partial_z \mathscr{W})(z)$ (see Appendix 5.5.7):

Lemma 5.3.8. *Let $z \in \mathbb{C}$ such that $\Im(z) > -\kappa$ and set $\omega_+ := z - sV_\pm$. Then*

$$|\mathscr{W}'(z)| \le \check{K}(z)\|f_+(\cdot, z)\|_{L^\infty(I^{\mathfrak{r}}_+,\mathbb{C})}\|f_-(\cdot, z)\|_{L^\infty(I^{\mathfrak{r}}_-,\mathbb{C})}$$

where $\check{K}(z)$ is defined in (5.23)*.*

5.4 Approximation of low frequency resonances

We present in this Section a method to approximate the zeros of the Wronskian $\mathscr{W}$ (which are the resonances). We will concentrate on low frequencies, that is, on the zeros contained in a compact neighborhood of 0 in the complex plane. As explained in the introduction **??**, we know that no resonance lies in the set $\{\lambda \in \mathbb{C} \mid \Im(\lambda) > -\delta_0\}$ for some $\delta_0 > 0$ (this number is not explicit as it comes from a compactness argument). We will be limited by the line $-\mathrm{i}\kappa$ as the meromorphic extension (5.1) is *a priori* not defined beyond this threshold. The research zone will be then of the form

$$\Gamma := [-R, R] + \mathrm{i}[-C_0, C_1] \tag{5.31}$$

with $R, C_0, C_1 > 0$ and $C_0 < \kappa$. Observe that for $z \in \mathbb{C}$ with $\Im(z) \leq -\kappa$, the estimates on the modified Jost solutions in Lemma 5.3.5 are no longer workable.

5.4.1 The method

In general, it is not easy to determine whether or not a function f cancels at some point because of the unavoidable error one does commit during a numerical approximation. How worth is it to say that f cancels at z_0 because $|f(z_0)| < 10^{-3}$? What if actually $|f(z_0)| = 10^{-999} \neq 0$? The situation is however far better if f is analytic. Indeed, we can in this situation *exactly* localize its zeros using the *argument principle*:

$$\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f'(z)}{f(z)} \mathrm{d}z = N(\Gamma) \in \mathbb{N}. \tag{5.32}$$

Here $\Gamma \subset \mathbb{C}$ is a positively oriented contour and $N(\Gamma)$ is the number of zeros of f counted with their multiplicity enclosed by Γ. This formula however requires that no zero of f lies on Γ. Of course, the exact position of the zeros are what we look for and it is not possible to know it *a priori*; however, if the zeros are not to close each other, then this difficulty is easily overcome if we modify the contour as soon as we detect a low value for $|f(z)|$ (a threshold is arbitrarily fixed here). The power of this method relies on the fact that (5.32) is an integer: if $N(\Gamma) = 3.026$, then even a precision of 0.5 is sufficient to determine that Γ contains exactly 3 zeros of f (counted with their multiplicity).

Another interesting point is that if Γ encloses (but does not intersect) $N(\Gamma)$ zeros $z_1, \ldots, z_{N(\Gamma)}$ of f, then

$$\frac{1}{2\pi\mathrm{i}} \oint_\Gamma z^k \frac{f'(z)}{f(z)} \mathrm{d}z = z_1^k + \ldots + z_{N(\Gamma)}^k \qquad \forall k \in \mathbb{N}. \tag{5.33}$$

In particular, if $N(\Gamma) = 1$ then it is possible to approximate the only zero z_0 inside Γ even if $\mathrm{dist}(z_0, \Gamma)$ is not small (which avoids some trouble caused by the term $1/f(z)$ in the contour integral). If we find $N(\Gamma) = 2$ and if the two zeros are too close to be separated into two different contours, then we can solve the algebraic system

$$\begin{cases} z_1 + z_2 &= \alpha \\ z_1^2 + z_2^2 &= \beta \end{cases}$$

where α and β are respectively the right-hand sides of (5.33) for $k = 1$ and $k = 2$. In this situation, we get

$$z_1 = \frac{\alpha \pm \sqrt{2\beta - \alpha^2}}{2}, \qquad z_2 = \frac{\alpha \mp \sqrt{2\beta - \alpha^2}}{2}.$$

More generally, if we have to deal with $N(\Gamma)$ close zeros, then we can solve for

$$\begin{cases} z_1 + z_2 + \ldots + z_{N(\Gamma)} &= \alpha_1 \\ z_1^2 + z_2^2 + \ldots + z_{N(\Gamma)}^2 &= \alpha_2 \\ \vdots & \\ z_1^{N(\Gamma)} + z_2^{N(\Gamma)} + \ldots + z_{N(\Gamma)}^{N(\Gamma)} &= \alpha_{N(\Gamma)} \end{cases}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{N(\Gamma)}$ are respectively the right-hand sides of (5.33) for $k = 1, 2, \ldots, N(\Gamma)$. While the system becomes tougher to deal with as $N(\Gamma)$ becomes large (as we look for the roots of polynomials of degrees $1, 2, \ldots, N(\Gamma)$), this alternative can be worthier than looking for an appropriate contour which separates very close zeros.

5.4.2 Application

We now apply the methods described in the paragraph 5.4.1. Fix $\Gamma \subset \mathbb{C}$ a positively oriented contour. Write

$$\Gamma = \bigcup_{\ell=1}^{4} \Gamma_\ell$$

and choose $M, N > 0$ as well as a precision $h_\bullet > 0$ (we will assume that it is the same as in Proposition 5.3.7 even if we can choose a different one) so that Γ can be discretized using four meshes $\mathscr{T}_{M,N}^{\Gamma_1}$, $\mathscr{T}_{M,N}^{\Gamma_2}$, $\mathscr{T}_{M,N}^{\Gamma_3}$ and $\mathscr{T}_{M,N}^{\Gamma_4}$ whose elements are respectively:

$$\begin{array}{llll} (x_{1,k})_{1\leq k\leq N} & \text{such that} & \Gamma_1 = [x_{1,1}, x_{1,N}] & \text{and} \quad \max\limits_{0\leq k\leq N}\{|x_{1,k} - x_{1,k+1}| \leq h_\bullet\}, \\ (y_{j,1})_{1\leq j\leq M} & \text{such that} & \Gamma_2 = [y_{1,1}, y_{M,1}] & \text{and} \quad \max\limits_{0\leq j\leq M}\{|y_{j,1} - y_{j+1,1}| \leq h_\bullet\}, \\ (x_{M,k})_{1\leq k\leq N} & \text{such that} & \Gamma_3 = [x_{M,1}, x_{M,N}] & \text{and} \quad \max\limits_{0\leq k\leq N}\{|x_{M,k} - x_{M,k+1}| \leq h_\bullet\}, \\ (y_{j,N})_{1\leq j\leq M} & \text{such that} & \Gamma_4 = [y_{1,N}, y_{M,N}] & \text{and} \quad \max\limits_{0\leq j\leq M}\{|y_{j,N} - y_{j+1,N}| \leq h_\bullet\}. \end{array}$$

The numbers $x_{1,k}$, $x_{M,k}$ are all real whereas the numbers $y_{j,1}$, $y_{j,N}$ are all purely imaginary. In practice, they will be the same as the one used in the paragraph 5.2 for the scheme (5.9).

For all $z \in \Gamma$, let

$$\widetilde{\mathscr{W}_\bullet}(z) := \sum_{\xi \in \bigcup_{\ell=1}^4 \mathscr{T}_{M^z,N^z}^{\Gamma_\ell}} \left(\frac{z - \xi_+}{\xi - \xi_+} \mathscr{W}_\bullet(\xi) + \frac{z - \xi}{\xi_+ - \xi} \mathscr{W}_\bullet(\xi_+) \right) \mathbb{1}_{[\xi, \xi_+[}(z).$$

and define

$$\widetilde{N}_\bullet(\Gamma) := \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{\widetilde{\mathscr{W}_\bullet'}(z)}{\widetilde{\mathscr{W}_\bullet}(z)} \mathrm{d}z.$$

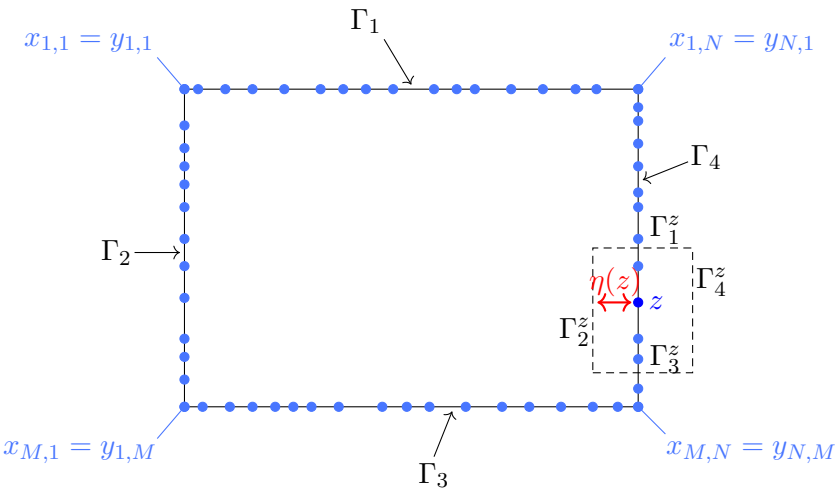

Figure 5.2: Discretization of the contour Γ.

The derivative has to be understood along the path Γ. Since $\mathscr{W}_\bullet$ is analytic in z, so is $\widetilde{\mathscr{W}_\bullet}$. Therefore, is the number of zeros inside $\widetilde{N}_\bullet(\Gamma)$ provided that no zero lies in Γ. This integral is explicitly computed in Appendix 5.6.2. We now show how good is this approximation of the number of zeros of $\mathscr{W}$ inside Γ:

Theorem 5.4.1. *Assume that*

$$\max_{z\in\Gamma}\left\{3C_\bullet(z)+\check{K}(z)\|f_+(\cdot,z)\|_{L^\infty(I_+^{\mathrm{r}},\mathbb{C})}\|f_-(\cdot,z)\|_{L^\infty(I_-^{\mathrm{r}},\mathbb{C})}\right\}h_\bullet^{1-\frac{|\Im(z)^-|}{\kappa}} < \min_{\zeta\in\bigcup_{\ell=1}^4\mathscr{T}_{M,N}^{\Gamma_\ell}}\left\{|\mathscr{W}_\bullet(\zeta)|\right\} \tag{5.34}$$

where $C_\bullet(z)$ and $\check{K}(z)$ are defined in (5.22) and (5.23) respectively. Then $\widetilde{N}_\bullet(\Gamma) = N_\bullet(\Gamma)$.

Proof. This is a direct consequence of Rouché's theorem: if

$$|\widetilde{\mathscr{W}_\bullet}(z)-\mathscr{W}(z)| < |\widetilde{\mathscr{W}_\bullet}(z)| \qquad \forall z\in\Gamma$$

then $\widetilde{\mathscr{W}_\bullet}$ and $\mathscr{W}$ have the same number of zeros (counted with their multiplicity) inside Γ. Using Proposition 5.3.7, we have

$$|\widetilde{\mathscr{W}_\bullet}(z)-\mathscr{W}(z)| \le C_\bullet(z)h_\bullet^{1-\frac{|\Im(z)^-|}{\kappa}} \qquad \forall z\in\Gamma.$$

Now pick $z\in\Gamma$: there exists $\xi\in\bigcup_{\ell=1}^4\mathscr{T}_{M,N}^{\Gamma_\ell}$ such that $z\in[\xi,\xi_+[$. Then:

$$\begin{aligned}
|\widetilde{\mathscr{W}_\bullet}(z)| &\ge \min_{\zeta\in\bigcup_{\ell=1}^4\mathscr{T}_{M,N}^{\Gamma_\ell}}\left\{|\mathscr{W}_\bullet(\zeta)|\right\} - |\widetilde{\mathscr{W}_\bullet}(z)-\mathscr{W}_\bullet(\xi)|,\\
|\widetilde{\mathscr{W}_\bullet}(z)-\mathscr{W}_\bullet(\xi)| &= |\mathscr{W}_\bullet(\xi_+)-\mathscr{W}_\bullet(\xi)|\\
&\le |\mathscr{W}_\bullet(\xi_+)-\mathscr{W}(\xi_+)| + |\mathscr{W}(\xi_+)-\mathscr{W}(\xi)| + |\mathscr{W}(\xi)-\mathscr{W}_\bullet(\xi)|\\
&\le \left(2C_\bullet(z)+\|\mathscr{W}'\|_{L^\infty(\Gamma,\mathbb{C})}\right)h_\bullet^{1-\frac{|\Im(z)^-|}{\kappa}}.
\end{aligned}$$

It remains to apply Lemma 5.3.8 to obtain the condition (5.34). □

Remark 5.4.2. *If $h_\bullet$ is sufficiently small, then formula (5.34) allows us to directly read on the minimal value of the approximated Wronskian $\mathscr{W}_\bullet$ on the mesh points whether or not $\widetilde{N}_\bullet(\Gamma) = N(\Gamma)$. As $h_\bullet \to 0$, the left-hand side goes to 0 so that the scheme necessarily gives the right number of resonances enclosed by Γ, unless we detect a zero on Γ (that is, an element $z \in \Gamma$ such that $|\widetilde{W}_\bullet(z)| \leq h_\bullet$). As for Proposition 5.3.7, the estimate worsens as $\Im(z) \to \kappa$.*

Note that it is possible to take maximum and minimum over each Γ_ℓ in (5.34) to improve the condition.

To end this Subsection, we show that the scheme also allows us to find a simple resonance up to an error which tends to 0 as $h_\bullet$ does. The purpose here is not giving a sharp estimate but only proving that we can approximate resonances with our scheme. In order to avoid estimates for $\mathscr{W}' - \mathscr{W}'_\bullet$ and $\mathscr{W}''_\bullet$, we use the analyticity of the functions as follows: we can obtain $\mathscr{W}'$ from $\mathscr{W}$ using Cauchy formula

$$W'(z) = \frac{1}{(2\pi\mathrm{i})^2} \oint_{\Gamma^z} \frac{\mathscr{W}(\omega)}{(\omega - z)^2} \mathrm{d}\omega \qquad \forall z \in \Gamma$$

where Γ^z is a positively oriented rectangle enclosing $z \in \Gamma$ (any other $\Gamma^{z'}$ is a translation of Γ^z). We write $\Gamma^z := \bigcup_{\ell=1}^4 \Gamma^z_\ell$ and discretize Γ^z as we did for Γ: we choose $M^z, N^z > 0$ and construct four meshes $\mathscr{T}^{\Gamma_1,z}_{M^z,N^z}$, $\mathscr{T}^{\Gamma_2,z}_{M^z,N^z}$, $\mathscr{T}^{\Gamma_3,z}_{M^z,N^z}$ and $\mathscr{T}^{\Gamma_4,z}_{M^z,N^z}$ whose elements are respectively $(x^z_{1,k})_{1\leq k\leq N^z}$, $(x^z_{M^z,k})_{1\leq k\leq N^z}$, $(y^z_{j,1})_{1\leq j\leq M^z}$ and $(y^z_{j,N^z})_{1\leq j\leq M^z}$. We construct them so that the larger distance between two mesh points is at most $h_\bullet^{1-\frac{|\Im(z)^-|}{\kappa}}$. Furthermore, we choose Γ^z so that (see Figure 5.2)

$$\eta(z) := \mathrm{dist}\big(z, \Gamma^z\big) > 0.$$

We also assume that

$$|\mathscr{W}(z)| > \epsilon > 2C_\bullet(\Gamma) h_\bullet^{1-\frac{|\Im(z)^-|}{\kappa}} > 0, \qquad C_\bullet(\Gamma) := \max_{z\in\Gamma} \big\{C_\bullet(z)\big\} \tag{5.35}$$

Having $|\mathscr{W}(z)| \leq C_\bullet(\Gamma) h_\bullet^{1-\frac{|\Im(z)^-|}{\kappa}}$ means that $|\mathscr{W}(z)|$ can be 0 in practice; the above condition can be checked by evaluating the minimum value of $W_\bullet(z)$, $z \in \Gamma$, then using Proposition 5.3.7. Let $z_0 \in \mathbb{C}$ be a *simple* resonance inside Γ and define

$$z_{\bullet,0} := \frac{1}{(2\pi\mathrm{i})^2} \oint_\Gamma \frac{z}{\widetilde{\mathscr{W}_\bullet}(z)} \left(\oint_{\Gamma^z} \frac{\widetilde{\mathscr{W}_\bullet}(\omega)}{(\omega - z)^2} \mathrm{d}\omega \right) \mathrm{d}z.$$

The explicit expression of $z_{\bullet,0}$ is given in Appendix 5.6.3.

Theorem 5.4.3. *Assume (5.25). Let $z \in \mathbb{C}$ such that $\Im(z) > -\kappa$. Then*

$$|z_0 - z_{\bullet,0}| \leq \max_{z\in\Gamma} \big\{|z|\big\} \widetilde{C}_\bullet(\Gamma) h_\bullet^{1-\frac{|\Im(z_0)^-|}{\kappa}}$$

where

$$\widetilde{C}_\bullet(\Gamma) = \frac{\max_{z\in\Gamma}\{|z|\}|\Gamma|}{(2\pi)^2\big(\epsilon - C_\bullet(\Gamma)h_\bullet\big)}\max_{z\in\Gamma}\left\{\frac{|\Gamma^z|}{\eta(z)^2}\big(\|W'\|_{L^\infty(\Gamma^z,\mathbb{C})} + 3C_\bullet(z)\big)\right\} \times\left(2+\frac{h_\bullet}{\big(\epsilon - C_\bullet(\Gamma)h_\bullet\big)}\max_{z\in\Gamma}\left\{\frac{3}{2}\|W'\|_{L^\infty(\Gamma^z,\mathbb{C})} + 3C_\bullet(z)\right\}\right).$$

The dependence in z of $C_\bullet(z)$ and $|W'(z)|$ are respectively shown in (5.22) and Lemma 5.3.8.

Proof. Form (5.33) for $k=1$, we can obviously get the term $\max_{z\in\Gamma}\{|z|\}$ out of the integral. We start by writing

$$\mathscr{W}(\omega)\widetilde{\mathscr{W}_\bullet}(z) - \mathscr{W}(z)\widetilde{\mathscr{W}_\bullet}(\omega) = \widetilde{\mathscr{W}_\bullet}(z)\Big(\mathscr{W}(\omega) - \widetilde{\mathscr{W}_\bullet}(\omega)\Big) + \Big(\widetilde{\mathscr{W}_\bullet}(\omega) - \mathscr{W}(\omega)\Big)\Big(\widetilde{\mathscr{W}_\bullet}(z) - \mathscr{W}(z)\Big) + \Big(\mathscr{W}(\omega) - \mathscr{W}(z)\Big)\Big(\widetilde{\mathscr{W}_\bullet}(z) - \mathscr{W}(z)\Big) + \mathscr{W}(z)\Big(\widetilde{\mathscr{W}_\bullet}(z) - \mathscr{W}(z)\Big).$$

It follows: Let $\xi \in \bigcup_{\ell=1}^{4}\mathscr{T}_{M,N}^{\Gamma_\ell}$ such that $z\in[\xi,\xi_+[$. We have

$$\begin{aligned}\mathscr{W}(z) - \widetilde{\mathscr{W}_\bullet}(z) &= \Big(\mathscr{W}(z) - \mathscr{W}_\bullet(\xi)\Big) + \frac{z-\xi}{\xi_+-\xi}\Big(\mathscr{W}_\bullet(\xi) - \mathscr{W}_\bullet(\xi_+)\Big)\\ &= \Big(\mathscr{W}(z) - \mathscr{W}(\xi)\Big) + \Big(\mathscr{W}(\xi) - \mathscr{W}_\bullet(\xi)\Big)\\ &\quad + \frac{z-\xi}{\xi_+-\xi}\Big(\Big(\mathscr{W}_\bullet(\xi) - \mathscr{W}(\xi)\Big) + \Big(\mathscr{W}(\xi) - \mathscr{W}(\xi_+)\Big) + \Big(\mathscr{W}(\xi_+) - \mathscr{W}_\bullet(\xi_+)\Big)\Big)\end{aligned}$$

with

$$\mathscr{W}(z) - \mathscr{W}(\xi) = (z-\xi)\int_0^1 \mathscr{W}'\big(\xi + t(z-\xi)\big)\mathrm{d}t$$

and a similar formula holds for $\mathscr{W}(\xi) - \mathscr{W}(\xi_+)$. Using that $\left|\frac{z-\xi}{\xi_+-\xi}\right| \le 1$, we obtain:

$$\Big|\mathscr{W}(z) - \widetilde{\mathscr{W}_\bullet}(z)\Big| \le 2|z-\xi|\|\mathscr{W}'\|_{L^\infty(\Gamma,\mathbb{C})} + 3\|\mathscr{W} - \mathscr{W}_\bullet\|_{L^\infty(\Gamma,\mathbb{C})}. \tag{5.36}$$

Likewise, let $\xi^z \in \bigcup_{\ell=1}^{4}\mathscr{T}_{M^z,N^z}^{\Gamma_{\ell,z}}$ such that $z\in[\xi^z,\xi_+^z[$; we have:

$$\Big|\mathscr{W}(\omega) - \widetilde{\mathscr{W}_\bullet}(\omega)\Big| \le 2|\omega-\xi^z|\|\mathscr{W}'\|_{L^\infty(\Gamma^z,\mathbb{C})} + 3\|\mathscr{W} - \mathscr{W}_\bullet\|_{L^\infty(\Gamma^z,\mathbb{C})}. \tag{5.37}$$

We also have:

$$\big|\mathscr{W}(\omega) - \mathscr{W}(z)\big| \le |\omega - z|\|\mathscr{W}'\|_{L^\infty(\mathring{\Gamma}^z,\mathbb{C})}. \tag{5.38}$$

Here $\mathring{\Gamma}^z$ denotes abusively the set of complex numbers enclosed by Γ^z.

Now integrate:

$$\begin{aligned}\left|N(\Gamma)-\widetilde{N}_\bullet(\Gamma)\right| &\le \frac{1}{(2\pi)^2}\oint_\Gamma\oint_{\Gamma^z}\left|\frac{\mathscr{W}(\omega)}{\mathscr{W}(z)}-\frac{\widetilde{\mathscr{W}_\bullet}(\omega)}{\widetilde{\mathscr{W}_\bullet}(z)}\right|\frac{|\mathrm{d}\omega||\mathrm{d}z|}{|\omega-z|^2}\\ &\le \frac{1}{(2\pi)^2}\oint_\Gamma\oint_{\Gamma^z}\frac{\left|\mathscr{W}(\omega)-\widetilde{\mathscr{W}_\bullet}(\omega)\right|}{|\mathscr{W}(z)|}\frac{|\mathrm{d}\omega||\mathrm{d}z|}{|\omega-z|^2}\\ &+\frac{1}{(2\pi)^2}\oint_\Gamma\oint_{\Gamma^z}\frac{\left|\widetilde{\mathscr{W}_\bullet}(z)-\mathscr{W}(z)\right|}{|\widetilde{\mathscr{W}_\bullet}(z)|}\frac{|\mathrm{d}\omega||\mathrm{d}z|}{|\omega-z|^2}\\ &+\frac{1}{(2\pi)^2}\oint_\Gamma\oint_{\Gamma^z}\frac{\left|\widetilde{\mathscr{W}_\bullet}(z)-\mathscr{W}(z)\right|\Big(\left|\widetilde{\mathscr{W}_\bullet}(\omega)-\mathscr{W}(\omega)\right|+\left|\mathscr{W}(\omega)-\mathscr{W}(z)\right|\Big)}{\left|\mathscr{W}(z)\widetilde{\mathscr{W}_\bullet}(z)\right|}\frac{|\mathrm{d}\omega||\mathrm{d}z|}{|\omega-z|^2}\\ &=: \mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

First, using (5.37), we find:

$$\begin{aligned}\mathrm{I} &\le \frac{1}{(2\pi)^2}\oint_\Gamma\oint_{\Gamma^z}\frac{\left|\mathscr{W}(\omega)-\widetilde{\mathscr{W}_\bullet}(\omega)\right|}{|\mathscr{W}(z)|}\frac{|\mathrm{d}\omega||\mathrm{d}z|}{|\omega-z|^2}\\ &\le \frac{1}{(2\pi)^2}\oint_\Gamma\sum_{\xi^z\in\bigcup_{\ell=1}^4\mathscr{T}^{\Gamma_\ell,z}_{M^z,N^z}}\int_{\xi^z}^{\xi^z_+}\frac{2|\omega-\xi^z|\|\mathscr{W}'\|_{L^\infty(\Gamma^z,\mathbb{C})}+3\|\mathscr{W}-\mathscr{W}_\bullet\|_{L^\infty(\Gamma^z,\mathbb{C})}}{|\mathscr{W}(z)|}\frac{|\mathrm{d}\omega||\mathrm{d}z|}{|\omega-z|^2}\\ &\le \frac{|\Gamma|}{(2\pi)^2\epsilon}\max_{z\in\Gamma}\left\{\frac{|\Gamma^z|}{\eta(z)^2}\Big(h_\bullet\|\mathscr{W}'\|_{L^\infty(\Gamma^z,\mathbb{C})}+3\|\mathscr{W}-\mathscr{W}_\bullet\|_{L^\infty(\Gamma^z,\mathbb{C})}\Big)\right\}.\end{aligned}$$

Next, (5.36) as well as the fact that $|\widetilde{\mathscr{W}_\bullet}(z)|\ge|\mathscr{W}(z)|-C_\bullet(\Gamma)h_\bullet$ (by Proposition 5.3.4 and assumption (5.35)) imply:

$$\begin{aligned}\mathrm{II} &\le \frac{1}{(2\pi)^2}\oint_\Gamma\oint_{\Gamma^z}\frac{\left|\widetilde{\mathscr{W}_\bullet}(z)-\mathscr{W}(z)\right|}{|\widetilde{\mathscr{W}_\bullet}(z)|}\frac{|\mathrm{d}\omega||\mathrm{d}z|}{|\omega-z|^2}\\ &\le \frac{1}{(2\pi)^2}\sum_{\xi\in\bigcup_{\ell=1}^4\mathscr{T}^{\Gamma_\ell}_{M,N}}\int_\xi^{\xi_+}\oint_{\Gamma^z}\frac{2|z-\xi|\|\mathscr{W}'\|_{L^\infty(\Gamma,\mathbb{C})}+3\|\mathscr{W}-\mathscr{W}_\bullet\|_{L^\infty(\Gamma,\mathbb{C})}}{|\widetilde{\mathscr{W}_\bullet}(z)|}\frac{|\mathrm{d}\omega||\mathrm{d}z|}{|\omega-z|^2}\\ &\le \frac{|\Gamma|}{(2\pi)^2\big(\epsilon-C_\bullet(\Gamma)h_\bullet\big)}\Big(h_\bullet\|\mathscr{W}'\|_{L^\infty(\Gamma,\mathbb{C})}+3\|\mathscr{W}-\mathscr{W}_\bullet\|_{L^\infty(\Gamma,\mathbb{C})}\Big)\max_{z\in\Gamma}\left\{\frac{|\Gamma^z|}{\eta(z)^2}\right\}.\end{aligned}$$

Finally, (5.36), (5.37), (5.38) yield:

$$\begin{aligned}\mathrm{III} &\le \frac{1}{(2\pi)^2}\oint_\Gamma\oint_{\Gamma^z}\frac{\left|\widetilde{\mathscr{W}_\bullet}(z)-\mathscr{W}(z)\right|\Big(\left|\widetilde{\mathscr{W}_\bullet}(\omega)-\mathscr{W}(\omega)\right|+\left|\mathscr{W}(\omega)-\mathscr{W}(z)\right|\Big)}{\left|\mathscr{W}(z)\widetilde{\mathscr{W}_\bullet}(z)\right|}\frac{|\mathrm{d}\omega||\mathrm{d}z|}{|\omega-z|^2}\\ &\le \frac{|\Gamma|}{(2\pi)^2\epsilon\big(\epsilon-C_\bullet(\Gamma)h_\bullet\big)}\Big(h_\bullet\|\mathscr{W}'\|_{L^\infty(\Gamma,\mathbb{C})}+3\|\mathscr{W}-\mathscr{W}_\bullet\|_{L^\infty(\Gamma,\mathbb{C})}\Big)\\ &\qquad\times\max_{z\in\Gamma}\left\{\frac{|\Gamma^z|}{\eta(z)^2}\left(h_\bullet\|\mathscr{W}'\|_{L^\infty(\Gamma^z,\mathbb{C})}+3\|\mathscr{W}-\mathscr{W}_\bullet\|_{L^\infty(\Gamma^z,\mathbb{C})}+\frac{h_\bullet}{2}\|\mathscr{W}'\|_{L^\infty(\mathring{\Gamma}^z,\mathbb{C})}\right)\right\}.\end{aligned}$$

By Proposition 5.3.7 and Lemma 5.3.8, we have:

$$|\mathscr{W}(z) - \mathscr{W}_\bullet(z)| \leq C_\bullet(z) h_\bullet^{1-\frac{|\Im(z)^-|}{\kappa}},$$
$$|\mathscr{W}'(z)| \leq \check{K}(z) \|f_+(\cdot, z)\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} \|f_-(\cdot, z)\|_{L^\infty(I_-^{\mathfrak{r}},\mathbb{C})}.$$

By checking the dependences in s, z of the constants $C_\bullet(z)$ (defined in (5.22)), $\check{K}(z)$ (defined in (5.23)) and $\|f_\pm(\cdot, z)\|_{L^\infty(I_\pm^{\mathfrak{r}},\mathbb{C})}$ (*cf.* Lemma 5.3.5), we see that the estimates increase as $\max\{1, |s||z|\}$ or $(|\kappa_\pm| - \Im(z)^-)^{-k}$ for some $k \in \mathbb{N}$. Now the theorem follows from

$$\|\mathscr{W} - \mathscr{W}_\bullet\|_{L^\infty(\Gamma,\mathbb{C})} \leq \|\mathscr{W} - \mathscr{W}_\bullet\|_{L^\infty(\Gamma^z,\mathbb{C})},$$
$$\|\mathscr{W}'\|_{L^\infty(\Gamma,\mathbb{C})} \leq \|\mathscr{W}'\|_{L^\infty(\mathring{\Gamma}^z,\mathbb{C})} = \|\mathscr{W}'\|_{L^\infty(\Gamma^z,\mathbb{C})}$$

since $\Gamma \subset \bigcup_{z\in\Gamma} \mathring{\Gamma}_z$. This completes the proof. □

Remark 5.4.4. *To get a better estimate, we can also start as follows:*

$$\frac{\mathscr{W}'(z)}{\mathscr{W}(z)} - \frac{\widetilde{\mathscr{W}_\bullet'}(z)}{\widetilde{\mathscr{W}_\bullet}(z)} = \frac{\widetilde{\mathscr{W}_\bullet}(z)\Big(\mathscr{W}'(z) - \widetilde{\mathscr{W}_\bullet'}(z)\Big) - \widetilde{\mathscr{W}_\bullet'}(z)\Big(\mathscr{W}(z) - \widetilde{\mathscr{W}_\bullet}(z)\Big)}{\mathscr{W}(z)\widetilde{\mathscr{W}_\bullet}(z)},$$

$$\mathscr{W}'(z) - \widetilde{\mathscr{W}_\bullet'}(z) = \mathscr{W}'(z) - \frac{\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)}{\xi_+ - \xi}$$
$$= \Big(\mathscr{W}'(z) - \mathscr{W}_\bullet'(\xi)\Big) + (\xi_+ - \xi)\int_0^1 (1-t)\mathscr{W}_\bullet''\big(\xi + t(\xi_+)\big)\mathrm{d}t.$$

But as explained above, this would require a control of $\mathscr{W}' - \mathscr{W}_\bullet'$ as well as $\mathscr{W}_\bullet''$, which in turn requires tedious computations and much time.

5.4.3 Discussion about the constants

We discuss in this Subsection the different values the constant $C_\bullet(z)$ can take depending on all the parameters M, Q, Λ, ℓ, m and s.

Computing the approximation $\mathscr{W}_\bullet(z)$ of the Wronskian, we discover that its modulus is small in practice: $|\mathscr{W}_\bullet(z)| \simeq 10^{-6}$ for $s \simeq 10^{-3}$, $|\mathscr{W}_\bullet(z)| \simeq 10^{-2}$ for $s \simeq 10^{-1}$. Besides, we need to more than 48 hours to compute $\mathscr{W}_\bullet(z)$ for z ranging over a regular mesh with $h_\bullet = 10^{-3}$ on the rectangle Γ of vertexes $0.5 + 0.5\mathrm{i}$, $-0.5 + 0.5\mathrm{i}$, -0.5 and 0.5. This means that "good" values for $C_\bullet(z)$ would satisfy $C_\bullet(z) \simeq 10^{-3}$ for $s \simeq 10^{-3}$, $C_\bullet(z) \simeq 10$ for $s \simeq 10^{-1}$.

Unfortunately, we did not found such satisfying values for $C_\bullet(z)$ as $C_\bullet(z) \geq 10^{10}$ in practice when z is near 0. It even happens that the value of $C_\bullet(z)$ is so huge that computer answers "inf" $(+\infty)$. We however noticed that more decent values are obtained near the extremal case $9\Lambda M^2 = 1 - \epsilon$ with $\epsilon > 0$ very small. The optimal configuration we have found so far is the following one:

$$M = 491, \qquad Q = 230, \qquad \Lambda = \frac{0.8175}{9M^2}, \qquad \ell = 0. \tag{5.39}$$

Of course, the smaller ℓ, m and s are, the better $C_\bullet(z)$ is. We would like to emphasize here the very complex dependence of $C_\bullet(z)$ in the above parameters. Modifying M, Q and Λ makes vary the four roots of the horizon function F, and thus the surface gravities κ_n, κ_c, κ_- and κ_+. For the values we have tested, it appeared that increasing M decreases the value of κ_-, and decreasing the value of Λ makes κ_- goes to 0. Q is linked as for itself to the Cauchy horizon and only κ_c seems to be sensible to variations of Q.

Having a careful look at the intermediary constants in Subsection 5.3.1, we discover that $C_\bullet(z)$ can easily blow up if $C_\pm(z)\widetilde{G}_\pm(z)$ becomes large because of the term $C_\pm(z)\widetilde{G}_\pm(z)\mathrm{e}^{C_\pm(z)\widetilde{G}_\pm(z)}$. The most favorable situation would occur if Λ, m, z and s are small and if $\kappa := \min\{\kappa_-, |\kappa_+|\}$ becomes large. It turns out that κ seems not to be able to exceed 0.5 and approach this value for very large Λ. Consequently, it is not easy to deduce the optimal value of $C_\pm(z)\widetilde{G}_\pm(z)$ is term of $\widetilde{\mathcal{M}}, Q$ and Λ.

The dependence in z creates undesirable effects near 0. Figure 5.3 shows the behaviour of $C_\bullet(z)$ when $z \in \mathbb{R}$ approaches 0 for M, Q, Λ, ℓ as in (5.39) and $m = s = 10^{-3}$. Quite surprisingly, $C_\bullet(z)$

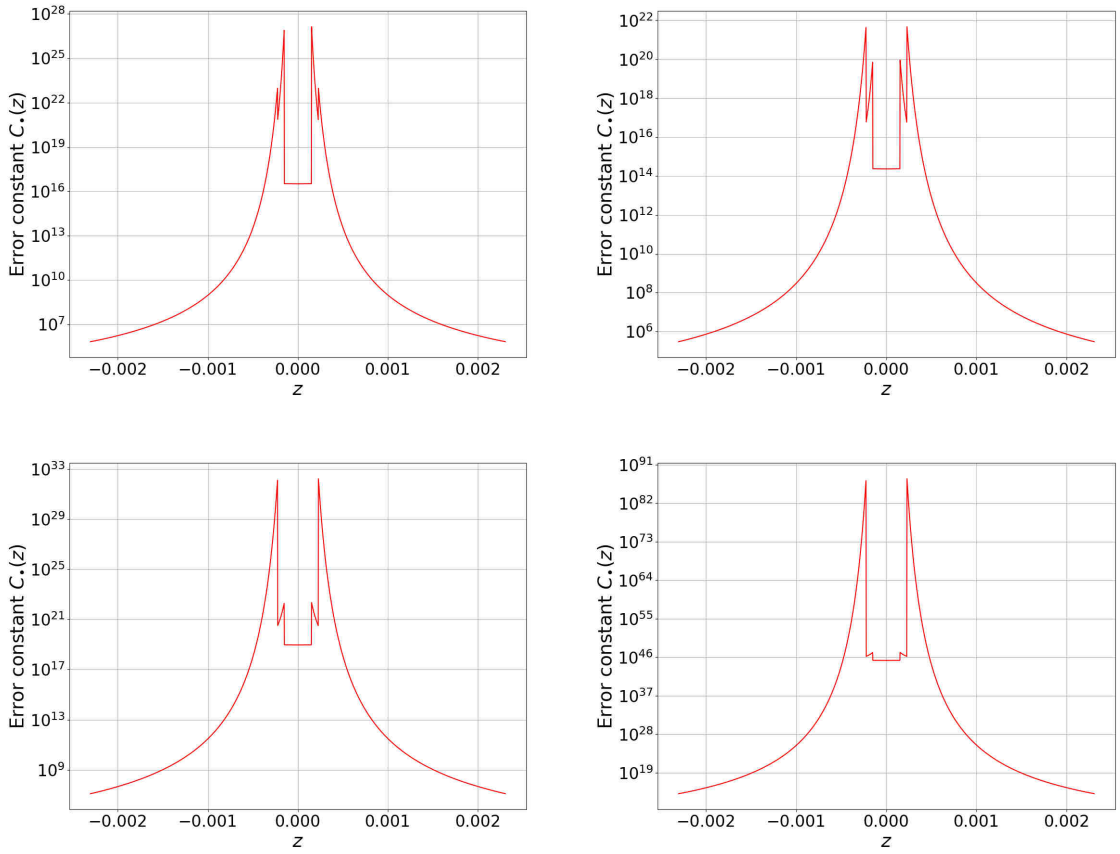

Figure 5.3: Error constant $C_\bullet(z)$ for z near 0 and for different values of $\mathfrak{r}$: at the top, $\mathfrak{r} = 0.8r_- + 0.2r_2$ on the left, $\mathfrak{r} = 0.6r_- + 0.4r_2$ on the right; in the bottom, $\mathfrak{r} = 0.4r_- + 0.6r_2$ on the left, $\mathfrak{r} = 0.2r_- + 0.8r_2$ on the right.

has maxima outside $[-\omega_-, \omega_-]$ contrary to what we could expect in view of the z dependence

of the constants in Subsection 5.3.1. We notice that the choice of $\mathfrak{r}$ modifies the estimates, the optimal choice lying somewhere near $\frac{r_-+r_+}{2}$.

While it would be interesting to understand how all the parameters can affect the error, a more reasonable objective is to improve the estimate of $C_\bullet(z)$ following the proofs in Appendix 5.5. Another room of improvement lies in the fact that we have separately estimated $f_\pm$ and $f_{\pm'}$ to obtain our estimates, so that some compensation effects in the formula $f_+f'_- - f'_+f_-$ could have possibly been canceled The ultimate purpose being an estimate for the ratio $\frac{\mathscr{W}'_\bullet(z)}{\mathscr{W}_\bullet(z)}$, a more subtle analysis of the functions could presumably lead to better estimates. The scheme used in this chapter has anyway nothing to do with these issues and furnishes a (improvable) control of the error.

As optimization of the scheme was not scheduled at the beginning of the thesis, we do not dispose of a sufficient amount of time to try to improve the above results. This will be the object of a future work.

5.5 Appendix: Proofs of technical results

This appendix is devoted to the prove of some useful estimates associated to the modified Jost solutions. More precisely, we prove Lemma 5.3.1, Lemma 5.3.5, Corollary 5.3.6 and Lemma 5.3.8.

Recall from Subsection 5.2.1 that smooth solutions of (5.5) have the following semi implicit forms for $\omega_\pm := z - sV_\pm$ with $\Im(z) > -\kappa$:

$$f_+(x) = 1 + \int_x^{+\infty} \mathrm{D}_{\omega_+}(y-x)\Big(W_\ell(y) + s(V(y)-V_+)\big(2z - s(V(y)+V_+)\big)\Big) f_+(y)\mathrm{d}y, \tag{5.40}$$

$$f_-(x) = 1 + \int_{-\infty}^{x} \mathrm{D}_{\omega_-}(x-y)\Big(W_\ell(y) + s(V(y)-V_-)\big(2z - s(V(y)+V_-)\big)\Big) f_-(y)\mathrm{d}y. \tag{5.41}$$

We can check that

$$|\mathrm{D}_{\omega_\pm}(x)| = \begin{cases} \dfrac{\sqrt{1 - 2\mathrm{e}^{\mp 2\Im(\omega_\pm)x}\cos(2\Re(\omega_\pm)x) + \mathrm{e}^{\mp 4\Im(\omega_\pm)x}}}{2|\omega_\pm|} & \text{if } \omega_\pm \in \mathbb{C}\setminus\{0\} \\ |x| & \text{if } \omega_\pm = 0 \end{cases}. \tag{5.42}$$

The derivatives $\partial_x f_\pm$ can be written as

$$(\partial_x f_+)(x) = -\int_x^{+\infty} (\partial_y \mathrm{D}_{\omega_+})(y-x)\Big(W_\ell(y) + s(V(y)-V_+)\big(2z - s(V(y)+V_+)\big)\Big) f_+(y)\mathrm{d}y, \tag{5.43}$$

$$(\partial_x f_-)(x) = \int_{-\infty}^{x} (\partial_y \mathrm{D}_{\omega_-})(x-y)\Big(W_\ell(y) + s(V(y)-V_-)\big(2z - s(V(y)+V_-)\big)\Big) f_-(y)\mathrm{d}y \tag{5.44}$$

with

$$(\partial_x \mathrm{D}_{\omega_\pm})(x) = \mathrm{e}^{\pm 2\mathrm{i}\omega_\pm x}, \qquad |\partial_x \mathrm{D}_{\omega_\pm}(x)| = \mathrm{e}^{\mp 2\Im(z)x}. \tag{5.45}$$

5.5.1 Preliminary estimates

Because of the semi-implicit form of the modified Jost solutions, we will need to use Grönwall's inequality. The following integrals will appear in our computations:

$$\mathcal{I}_+(x) := \int_x^X |\mathrm{D}_{\omega_+}(y-x)|G_+(y)\mathrm{d}y \qquad x \geq 0,$$

$$\mathcal{I}_-(x) := \int_{-X}^x |\mathrm{D}_{\omega_-}(x-y)|G_-(y)\mathrm{d}y \qquad x \leq 0$$

where $X \geq |x|$ and

$$G_\pm(x) := |W_\ell(x)| + |s||V(x) - V_\pm|\big(2|z| + |s||V(x) + V_\pm|\big) > 0 \qquad \forall x \in \mathbb{R}.$$

We clearly have $0 \leq G_\pm(x) \leq C_\pm(z)\mathrm{e}^{-2|\kappa_\pm||x|}$ for all $x \in \mathbb{R}$. The purpose of this paragraph is to provide the following x-independent bounds:

Lemma 5.5.1. *For all $x \in \mathbb{R}$ such that $\pm x \geq 0$, it holds*

$$\mathcal{I}_\pm(x) \leq C_\pm(z)\widetilde{G}_\pm(z)\mathrm{e}^{2\kappa_\pm x}\begin{cases} 1 - \mathrm{e}^{-2(|\kappa_\pm| + \Im(z)^-)(X \mp x)} & \text{if } |\omega_\pm| \geq |\kappa_\pm| \\ 1 - \big(2(|\kappa_\pm| + \Im(z)^-)(X \mp x) + 1\big)\mathrm{e}^{-2(|\kappa_\pm| + \Im(z)^-)(X \mp x)} & \text{if } 0 \leq |\omega_\pm| < |\kappa_\pm| \end{cases}$$

where $C_\pm(z)$ and $\widetilde{G}_\pm(z)$ are defined in (5.11) and (5.12). Recall that $\pm\kappa_\pm < 0$.

Proof. We only treat the $+$ case. Let us write

$$z = a + \mathrm{i}b, \qquad (a,b) \in \mathbb{R}^2.$$

We distinguish three cases.

First case: $b \geq 0$ and $z \neq sV_+$. Assume $|\omega_+| \geq |\kappa_+|$. By (5.42), we have for all $x \geq 0$

$$\begin{aligned}
&\int_x^X |\mathrm{D}_{\omega_+}(y-x)|G_+(y)\mathrm{d}y \\
&= \frac{C_+(z)}{2|\omega_+|}\int_x^X \mathrm{e}^{2\kappa_+ y}\sqrt{1 - 2\mathrm{e}^{-2b(y-x)}\cos(2(a - sV_+)(y-x)) + \mathrm{e}^{-4b(y-x)}}\mathrm{d}y \\
&\leq \frac{C_+(z)}{|\omega_+|}\int_x^X \mathrm{e}^{2\kappa_+ y}\mathrm{d}y \\
&= \frac{C_+(z)\left(\mathrm{e}^{2\kappa_+ x} - \mathrm{e}^{2\kappa_+ X}\right)}{2|\kappa_+||\omega_+|}.
\end{aligned}$$

Assume $|\omega_+| < |\kappa_+|$. Let us define for $y \geq x$

$$\begin{aligned}
A &:= \int_0^1 (1-t)\mathrm{e}^{-4tby}\mathrm{d}t, \\
B &:= \int_0^1 (1-t)\mathrm{e}^{-2tb(y-x)}\Big(3(a - sV_+)b\sin(2t(a - sV_+)(y-x)) \\
&\qquad + \big(2b^2 - (a - sV_+)^2\big)\cos(2t(a - sV_+)(y-x))\Big)\mathrm{d}t
\end{aligned}$$

which satisfy

$$|A| \leq \frac{1}{2}, \qquad |B| \leq \frac{5}{2}|\omega_+|^2.$$

Using Taylor's formula, it then comes:

$$1 - 2\mathrm{e}^{-2b(y-x)}\cos(2(a-sV_+)y) + \mathrm{e}^{-4b(y-x)} = 2\big(8b^2A + B\big)(y-x)^2 \leq 13|\omega_+|^2(y-x)^2.$$

Hence, for all $x \geq 0$,

$$\begin{aligned}\int_x^X |\mathrm{D}_{\omega_+}(y-x)|G_+(y)\mathrm{d}y &\leq \frac{\sqrt{13}}{2}C_+(z)\int_x^X (y-x)\mathrm{e}^{2\kappa_+ y}\mathrm{d}y \\ &= \frac{\sqrt{13}}{2}C_+(z)\mathrm{e}^{2\kappa_+ x}\int_0^{X-x} y\mathrm{e}^{2\kappa_+ y}\mathrm{d}y \\ &= \frac{\sqrt{13}}{2}C_+(z)\frac{\big(\mathrm{e}^{2\kappa_+ x} + \big(2\kappa_+(X-x)-1\big)\mathrm{e}^{2\kappa_+ X}\big)}{4\kappa_+^2}.\end{aligned}$$

Second case: $\omega_+ = 0$. Since $\mathrm{D}_0(x) = x$, we directly get

$$\int_x^X |\mathrm{D}_0(y-x)|G_+(y)\mathrm{d}y \leq C_+(z)\frac{\big(\mathrm{e}^{2\kappa_+ x} + \big(2\kappa_+(X-x)-1\big)\mathrm{e}^{2\kappa_+ X}\big)}{4\kappa_+^2}.$$

Third case: $0 > b > -\kappa$. If $|\omega_+| \geq |\kappa_+|$ then (5.42) implies for all $x \geq 0$ that

$$\begin{aligned}&\int_x^X |\mathrm{D}_{\omega_+}(y-x)|G_+(y)\mathrm{d}y \\ &= \frac{C_+(z)}{2|\omega_+|}\int_x^X \mathrm{e}^{2\kappa_+ y}\sqrt{1 - 2\mathrm{e}^{-2b(y-x)}\cos(2(a-sV_+)(y-x)) + \mathrm{e}^{-4b(y-x)}}\mathrm{d}y \\ &= \frac{C_+(z)}{2|\omega_+|}\int_x^X \mathrm{e}^{2\kappa_+ y}\mathrm{e}^{-2b(y-x)}\sqrt{\mathrm{e}^{4b(y-x)} - 2\mathrm{e}^{2b(y-x)}\cos(2(a-sV_+)(y-x)) + 1}\mathrm{d}y \\ &\leq \frac{C_+(z)}{|\omega_+|}\mathrm{e}^{2bx}\int_x^X \mathrm{e}^{2(\kappa_+ - b)y}\mathrm{d}y \\ &= \frac{C_+(z)\left(\mathrm{e}^{2\kappa_+ x} - \mathrm{e}^{2(\kappa_+ - b)X}\mathrm{e}^{2bx}\right)}{2(b-\kappa_+)|\omega_+|}.\end{aligned}$$

Assume now $|\omega_+| < |\kappa_+|$. Let us define for $y \geq 0$ (and $|a - sV_+|, |b| < |\omega_+|$)

$$\begin{aligned}A' &:= \int_0^1 (1-t)\mathrm{e}^{4tb(y-x)}\mathrm{d}t, \\ B' &:= \int_0^1 (1-t)\mathrm{e}^{2tb(y-x)}\Big(3(a-sV_+)b\sin(2t(a-sV_+)(y-x)) \\ &\qquad + \big((a-sV_+)^2 - 2b^2\big)\cos(2t(a-sV_+)(y-x))\Big)\mathrm{d}t\end{aligned}$$

which satisfy

$$|A'| \le \frac{1}{2}, \qquad |B'| \le \frac{5}{2}|\omega_+|^2.$$

By Taylor's formula, we have

$$\begin{aligned}
&1 - 2\mathrm{e}^{-2b(y-x)}\cos(2(a - sV_+)(y-x)) + \mathrm{e}^{-4b(y-x)} \\
&= \mathrm{e}^{-4b(y-x)}\Big(\mathrm{e}^{4b(y-x)} - 2\mathrm{e}^{2b(y-x)}\cos(2(a-sV_+)(y-x)) + 1\Big) \\
&= 2\mathrm{e}^{-4b(y-x)}\big(8b^2A' - B'\big)(y-x)^2 \\
&\le 13|\omega_+|^2(y-x)^2\mathrm{e}^{-4b(y-x)}.
\end{aligned}$$

Hence, for all $x \ge 0$,

$$\begin{aligned}
\int_x^X |\mathrm{D}_{\omega_+}(y-x)|G_+(y)\mathrm{d}y &\le \frac{\sqrt{13}}{2}C_+(z)\mathrm{e}^{2bx}\int_x^X (y-x)\mathrm{e}^{2(\kappa_+-b)y}\mathrm{d}y \\
&= \frac{\sqrt{13}}{2}C_+(z)\mathrm{e}^{2\kappa_+x}\int_0^{X-x} y\mathrm{e}^{2(\kappa_+-b)y}\mathrm{d}y \\
&= \frac{\sqrt{13}}{2}C_+(z)\frac{\big(\mathrm{e}^{2\kappa_+x} + \big(2(\kappa_+-b)(X-x)-1\big)\mathrm{e}^{2(\kappa_+-b)X}\mathrm{e}^{2bx}\big)}{4(b-\kappa_+)^2}.
\end{aligned}$$

This completes the proof. □

5.5.2 Proof of Lemma 5.3.1

We only treat the + case. Let $r \in I_+^{\mathfrak{r}}$; there exists an unique $p \in \mathscr{T}_{\bullet,0}^+$ such that $r \in \omega_p^+ = [p, p_+[$.

First component

We have

$$\big(f_+ - \mathscr{F}_{\bullet,+}f_+\big)(r) = \frac{f_+(r) - f_+(p)}{2} + \frac{f_+(r) - f_+(p_+)}{2}. \tag{5.46}$$

Having a bound for the first term on the right-hand side will be enough. The proof is different whether $p_+ = r_+$ or not.

1. First case: $p_+ \neq r_+$. In this case $r_+ - r \ge h_\bullet$. We have

$$\begin{aligned}
\big(f_+(r) - f_+(p)\big) &= \int_{\phi(r)}^{+\infty}\Big(\mathrm{D}_{\omega_+}(y-\phi(r)) - \mathrm{D}_{\omega_+}(y-\phi(p))\Big)\mathcal{W}_{+,\ell}(y)f_+(y)\mathrm{d}y \\
&\quad - \int_{\phi(p)}^{\phi(r)}\mathrm{D}_{\omega_+}(y-\phi(p))\mathcal{W}_{+,\ell}(y)f_+(y)\mathrm{d}y \\
&=: \mathrm{I} + \mathrm{II}
\end{aligned}$$

with $\mathcal{W}_{+,\ell}(x) := W_\ell(r(x)) + s(V(r(x)) - V_+)\big(2z - s(V(r(x)) + V_+)\big)$. Notice that $|\mathcal{W}_{+,\ell}(x)| \le C_+(z)\mathrm{e}^{2\kappa_+x}$.

Let us deal with I. When $\omega_+ \neq 0$, observe that

$$\begin{aligned}
\mathrm{D}_{\omega_+}(y-\phi(r)) - \mathrm{D}_{\omega_+}(y-\phi(p)) &= \frac{\mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(r))}-1}{2\mathrm{i}\omega_+} - \frac{\mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(p))}-1}{2\mathrm{i}\omega_+} \\
&= \begin{cases} \dfrac{\mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(r))}}{2\mathrm{i}\omega_+}\Big(1-\mathrm{e}^{2\mathrm{i}\omega_+(\phi(r)-\phi(p))}\Big) & \text{if } \Im(z)\geq 0 \\ \dfrac{\mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(p))}}{2\mathrm{i}\omega_+}\Big(\mathrm{e}^{2\mathrm{i}\omega_+(\phi(p)-\phi(r))}-1\Big) & \text{if } \Im(z)<0 \end{cases} \\
&= \begin{cases} -\mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(r))}(\phi(r)-\phi(p))\displaystyle\int_0^1 \mathrm{e}^{2\mathrm{i}\omega_+ t(\phi(r)-\phi(p))}\mathrm{d}t & \text{if } \Im(z)\geq 0 \\ \mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(p))}(\phi(p)-\phi(r))\displaystyle\int_0^1 \mathrm{e}^{2\mathrm{i}\omega_+ t(\phi(p)-\phi(r))}\mathrm{d}t & \text{if } \Im(z)<0 \end{cases}
\end{aligned}$$

and all the integral terms are bounded in norm by 1 since $0 \leq \phi(p) \leq \phi(r)$ (recall here that $\phi(r') \geq 0$ if and only if $r' \geq \mathfrak{r}$). It then follows:

$$|\mathrm{I}| \leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} C_+(z) \frac{(\phi(r)-\phi(p))}{2(|\kappa_+|+\Im(z))} \begin{cases} \mathrm{e}^{2\kappa_+\phi(r)} & \text{if } \Im(z)\geq 0 \\ \mathrm{e}^{2(\kappa_+-\Im(z))\phi(r)}\mathrm{e}^{2\Im(z)\phi(p)} & \text{if } \Im(z)<0 \end{cases}. \tag{5.47}$$

When $\omega_+ = 0$, we simply get

$$|\mathrm{I}| \leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} C_+(z) \frac{(\phi(r)-\phi(p))}{2|\kappa_+|} \mathrm{e}^{2\kappa_+\phi(r)}$$

which is contained in estimate (5.47). Next, formula (4.39) entails with the fact that $\kappa_c, \kappa_+ < 0$:

$$\begin{aligned}
\phi(r)-\phi(p) &= \sum_{\alpha\in\{n,c,-,+\}} \frac{1}{2\kappa_\alpha} \ln\left|\frac{r-r_\alpha}{\mathfrak{r}-r_\alpha}\right| - \sum_{\alpha\in\{n,c,-,+\}} \frac{1}{2\kappa_\alpha} \ln\left|\frac{p-r_\alpha}{\mathfrak{r}-r_\alpha}\right| \\
&= \sum_{\alpha\in\{n,c,-\}} \frac{1}{2\kappa_\alpha} \ln\left(1+\frac{r-p}{p-r_\alpha}\right) + \frac{1}{2\kappa_+}\ln\left(1-\frac{r-p}{r_+-p}\right) \\
&= \sum_{\alpha\in\{n,c,-\}} \frac{1}{2\kappa_\alpha}\frac{r-p}{p-r_\alpha}\int_0^1 \frac{\mathrm{d}t}{1+t\frac{r-p}{p-r_\alpha}} - \frac{1}{2\kappa_+}\frac{r-p}{r_+-p}\int_0^1 \frac{\mathrm{d}t}{1-t\frac{r-p}{r_+-p}} \\
&\leq \sum_{\alpha\in\{n,-\}} \frac{1}{2\kappa_\alpha}\frac{r-p}{p-r_\alpha} - \frac{1}{2\kappa_+}\frac{r-p}{r_+-r} \\
&\leq \sum_{\alpha\in\{n,-\}} \frac{1}{2\kappa_\alpha}\frac{r-p}{\mathfrak{r}-r_\alpha} + \frac{1}{2|\kappa_+|}\frac{r-p}{r_+-r}.
\end{aligned} \tag{5.48}$$

Using (4.41), we can write

$$\mathrm{e}^{2\kappa_+\phi(r)} = \frac{r_+-r}{r_+-\mathfrak{r}} \prod_{\alpha\in\{n,c,-\}} \left(\frac{r-r_\alpha}{\mathfrak{r}-r_\alpha}\right)^{\frac{\kappa_+}{\kappa_\alpha}} \tag{5.49}$$

so that

$$\frac{\mathrm{e}^{2(\kappa_+-\Im(z)^-)\phi(r)}}{r_+-r}=\frac{\left(\mathrm{e}^{2\kappa_+\phi(r)}\right)^{\frac{\kappa_+-\Im(z)^-}{\kappa_+}}}{r_+-r}=\frac{(r_+-r)^{-\frac{\Im(z)^-}{\kappa_+}}}{(r_+-\mathfrak{r})^{\frac{\kappa_+-\Im(z)^-}{\kappa_+}}}\prod_{\alpha\in\{n,c,-\}}\left(\frac{r-r_\alpha}{\mathfrak{r}-r_\alpha}\right)^{\frac{\kappa_+-\Im(z)^-}{\kappa_\alpha}}.$$

Since $r-p\le h_\bullet$ and $r_+-r\ge h_\bullet$, we eventually find:

$$\begin{aligned}|\mathrm{I}|&\le\|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}\frac{C_+(z)}{2(|\kappa_+|+\Im(z))}\left(\sum_{\alpha\in\{n,-\}}\frac{1}{2\kappa_\alpha}\frac{h_\bullet}{\mathfrak{r}-r_\alpha}+\frac{1}{2|\kappa_+|}\frac{(r-p)\mathrm{e}^{2(\kappa_+-\Im(z)^-)\phi(r)}}{r_+-r}\right)\\&\le\|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}\frac{C_+(z)}{4(|\kappa_+|+\Im(z))}\\&\times\left(\sum_{\alpha\in\{n,-\}}\frac{h_\bullet^{\frac{\Im(z)^-}{\kappa_+}}}{\kappa_\alpha(\mathfrak{r}-r_\alpha)}+\frac{(r_+-\mathfrak{r})^{\frac{|\kappa_+|+\Im(z)^-}{\kappa_+}}}{|\kappa_+|}\left(\frac{r_+-r_c}{\mathfrak{r}-r_c}\right)^{\frac{\kappa_+-\Im(z)^-}{\kappa_c}}\right)h_\bullet^{1-\frac{\Im(z)^-}{\kappa_+}}.\end{aligned}\tag{5.50}$$

Let us turn to II. Applying Lemma 5.5.1 with $X=\phi(r)$ and $x=\phi(p)$, we get

$$\begin{aligned}|\mathrm{II}|&\le\|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}C_+(z)\widetilde{G}_+(z)\mathrm{e}^{2\kappa_+\phi(p)}\\&\times\begin{cases}1-\mathrm{e}^{-2(|\kappa_+|+\Im(z)^-)(\phi(r)-\phi(p))} & \text{if } |\omega_+|\ge|\kappa_+|\\ 1-\big(2(|\kappa_+|+\Im(z)^-)(\phi(r)-\phi(p))+1\big)\mathrm{e}^{-2(|\kappa_+|+\Im(z)^-)(\phi(r)-\phi(p))} & \text{if } 0\le|\omega_+|<|\kappa_+|\end{cases}.\end{aligned}$$

We have

$$\mathrm{e}^{-2(|\kappa_+|+\Im(z)^-)(\phi(r)-\phi(p))}=1-2(|\kappa_+|+\Im(z)^-)(\phi(r)-\phi(p))\int_0^1\mathrm{e}^{-2(|\kappa_+|+\Im(z)^-)t(\phi(r)-\phi(p))}\mathrm{d}t$$

and the integral in bounded by 1. If $|\omega_+|\ge|\kappa_+|$, then

$$\begin{aligned}|\mathrm{II}|&\le2\|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}C_+(z)\widetilde{G}_+(z)(|\kappa_+|+\Im(z)^-)\mathrm{e}^{2\kappa_+\phi(p)}(\phi(r)-\phi(p))\\&\le2\|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}C_+(z)\widetilde{G}_+(z)(|\kappa_+|+\Im(z)^-)\mathrm{e}^{2\kappa_+\phi(p)}\left(\sum_{\alpha\in\{n,-\}}\frac{1}{2\kappa_\alpha}\frac{r-p}{\mathfrak{r}-r_\alpha}+\frac{1}{2|\kappa_+|}\frac{r-p}{r_+-r}\right).\end{aligned}$$

Using (5.49) with $r=p$ as well as $\frac{r_+-p}{r_+-r}\le2$ (because $p_+\neq r_+$), we can write

$$\frac{\mathrm{e}^{2\kappa_+\phi(p)}}{r_+-r}\le\frac{2}{r_+-\mathfrak{r}}\left(\frac{r_+-r_c}{\mathfrak{r}-r_c}\right)^{\frac{\kappa_+}{\kappa_c}}.$$

It follows for $|\omega_+|\ge|\kappa_+|$:

$$\begin{aligned}|\mathrm{II}|&\le2\|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}C_+(z)\widetilde{G}_+(z)(|\kappa_+|+\Im(z)^-)\\&\times\left(\sum_{\alpha\in\{n,-\}}\frac{1}{2\kappa_\alpha(\mathfrak{r}-r_\alpha)}+\frac{1}{|\kappa_+|(r_+-\mathfrak{r})}\left(\frac{r_+-r_c}{\mathfrak{r}-r_c}\right)^{\frac{\kappa_+}{\kappa_c}}\right)h_\bullet.\end{aligned}$$

Finally, assume $0 \leq |\omega_+| < |\kappa_+|$. Writing

$$\begin{aligned}
&\Big|1 - \big(2(|\kappa_+| + \Im(z)^-)(\phi(r) - \phi(p)) + 1\big)\mathrm{e}^{-2(|\kappa_+|+\Im(z)^-)(\phi(r)-\phi(p))}\Big| \\
&\leq \Big|1 - \mathrm{e}^{-2(|\kappa_+|+\Im(z)^-)(\phi(r)-\phi(p))}\Big| + 2(|\kappa_+| + \Im(z)^-)(\phi(r) - \phi(p))\mathrm{e}^{-2(|\kappa_+|+\Im(z)^-)(\phi(r)-\phi(p))} \\
&\leq 4(|\kappa_+| + \Im(z)^-)(\phi(r) - \phi(p)),
\end{aligned}$$

we obtain two times the precedent bound for |II|:

$$\begin{aligned}
|\mathrm{II}| \leq 4\|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} C_+(z)\widetilde{G}_+(z)(|\kappa_+| + \Im(z)^-) \\
\times \left(\sum_{\alpha\in\{n,-\}} \frac{1}{2\kappa_\alpha(\mathfrak{r} - r_\alpha)} + \frac{1}{|\kappa_+|(r_+ - \mathfrak{r})}\left(\frac{r_+ - r_c}{\mathfrak{r} - r_c}\right)^{\frac{\kappa_+}{\kappa_c}}\right) h_\bullet. \quad (5.51)
\end{aligned}$$

2. Second case: $p_+ = r_+$. This time $r_+ - p = h_\bullet$. As above, we only show the computations for[3] $f_+(r) - f_+(p)$ in (5.46). We have

$$\begin{aligned}
\big|f_+(r) - f_+(p)\big| &\leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} \left(\int_{\phi(r)}^{+\infty} \Big(\big|\mathrm{D}_{\omega_+}(y - \phi(r))\big| + \big|\mathrm{D}_{\omega_+}(y - \phi(p))\big|\Big)\big|\mathcal{W}_{+,\ell}(y)\big|\mathrm{d}y \right. \\
&\qquad \left. + \int_{\phi(p)}^{\phi(r)} \big|\mathrm{D}_{\omega_+}(y - \phi(p))\big|\big|\mathcal{W}_{+,\ell}(y)\big|\mathrm{d}y\right) \\
&\leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} \left(\int_{\phi(r)}^{+\infty} \big|\mathrm{D}_{\omega_+}(y - \phi(r))\big|\mathrm{e}^{2\kappa_+ y}\mathrm{d}y + 2\int_{\phi(p)}^{+\infty} \big|\mathrm{D}_{\omega_+}(y - \phi(p))\big|\mathrm{e}^{2\kappa_+ y}\mathrm{d}y\right).
\end{aligned}$$

Recalling that $\phi(r) \geq \phi(p)$, we can use Lemma 5.5.1 for $X = +\infty$ and $x = \phi(p)$:

$$\big|f_+(r) - f_+(p)\big| \leq 3\|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} C_+(z)\widetilde{G}_+(z)\mathrm{e}^{2\kappa_+\phi(p)}.$$

It remains to use (5.49) as well as $r_+ - p = h_\bullet$ to obtain

$$\big|f_+(r) - f_+(p)\big| \leq 3\|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} C_+(z)\widetilde{G}_+(z)\frac{1}{r_+ - \mathfrak{r}}\left(\frac{r_+ - r_c}{\mathfrak{r} - r_c}\right)^{\frac{\kappa_+}{\kappa_c}} h_\bullet. \quad (5.52)$$

3. Conclusion: comparing the bounds in (5.50), (5.51) and (5.52), we find with assumption (5.25):

$$\begin{aligned}
\big|f_+(r) - f_+(p)\big| &\leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} C_+(z)\left(\frac{1}{4(|\kappa_+| + \Im(z))} + 4\widetilde{G}_+(z)\max\big\{1, |\kappa_+| + \Im(z)^-\big\}\right) \\
&\quad \times \left(\sum_{\alpha\in\{n,-\}} \frac{1}{\kappa_\alpha(\mathfrak{r} - r_\alpha)} + \max\left\{1, \frac{1}{|\kappa_+|}\right\}(r_+ - \mathfrak{r})^{\frac{|\kappa_+|+\Im(z)^-}{\kappa_+}}\left(\frac{r_+ - r_c}{\mathfrak{r} - r_c}\right)^{\frac{\kappa_+ - \Im(z)^-}{\kappa_c}}\right) \\
&\quad \times h_\bullet^{1-\frac{\Im(z)^-}{\kappa_+}}. \quad (5.53)
\end{aligned}$$

This establishes (5.26) for $j = 0$.

[3] The other term is dealt using that $f_+(r_+) = 1$.

Second component

Let us write:

$$\big(F(r)\partial_r f_+ - \mathscr{F}_{\bullet,+}F(r)\partial_r u\big)(r) = \frac{F(r)f'_+(r) - F(p)f'_+(p)}{2} + \frac{F(r)f'_+(r) - F(p_+)f'_+(p_+)}{2}.$$

Once again, it is sufficient to get an estimate for the first term on the right-hand side. We have:

$$\begin{aligned}
F(r)f'_+(r) - F(p)f'_+(p) &= (\partial_x f_+)(\phi(r)) - (\partial_x f_+)(\phi(p)) \\
&= -\int_{\phi(r)}^{+\infty} (\partial_y \mathrm{D}_{\omega_+})(y-\phi(r))\mathcal{W}_{+,\ell}(y)f_+(y)\mathrm{d}y \\
&+ \int_{\phi(p)}^{+\infty} (\partial_y \mathrm{D}_{\omega_+})(y-\phi(p))\mathcal{W}_{+,\ell}(y)f_+(y)\mathrm{d}y \\
&= -\int_{\phi(r)}^{+\infty} \Big((\partial_y \mathrm{D}_{\omega_+})(y-\phi(r)) - (\partial_y \mathrm{D}_{\omega_+})(y-\phi(p))\Big)\mathcal{W}_{+,\ell}(y)f_+(y)\mathrm{d}y \\
&+ \int_{\phi(p)}^{\phi(r)} (\partial_y \mathrm{D}_{\omega_+})(y-\phi(p))\mathcal{W}_{+,\ell}(y)f_+(y)\mathrm{d}y \\
&=: A' + B'.
\end{aligned}$$

We distinguish two cases.

1. First case: $p_+ \neq r_+$. We start with A'. We have

$$\begin{aligned}
&(\partial_y \mathrm{D}_{\omega_+})(y-\phi(r)) - (\partial_y \mathrm{D}_{\omega_+})(y-\phi(p)) \\
&= \mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(r))} - \mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(p))} \\
&= \begin{cases} \mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(r))}\left(1 - \mathrm{e}^{2\mathrm{i}\omega_+(\phi(r)-\phi(p))}\right) & \text{if } \Im(z) \geq 0 \\ \mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(p))}\left(\mathrm{e}^{2\mathrm{i}\omega_+(\phi(p)-\phi(r))} - 1\right) & \text{if } \Im(z) < 0 \end{cases} \\
&= \begin{cases} -2\mathrm{i}\omega_+\mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(r))}(\phi(r)-\phi(p))\displaystyle\int_0^1 \mathrm{e}^{2\mathrm{i}\omega_+ t(\phi(r)-\phi(p))}\mathrm{d}t & \text{if } \Im(z) \geq 0 \\ 2\mathrm{i}\omega_+\mathrm{e}^{2\mathrm{i}\omega_+(y-\phi(p))}(\phi(p)-\phi(r))\displaystyle\int_0^1 \mathrm{e}^{2\mathrm{i}\omega_+ t(\phi(p)-\phi(r))}\mathrm{d}t & \text{if } \Im(z) < 0 \end{cases}
\end{aligned}$$

   and all the integral terms above have norm lesser than 1. It then follows

$$|A'| \leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} C_+(z) \frac{|\omega_+|(\phi(r)-\phi(p))}{|\kappa_+| + \Im(z)} \mathrm{e}^{2(\kappa_+ - \Im(z)^-)\phi(r)} \mathrm{e}^{2\Im(z)^-\phi(p)}$$

   which is a multiple of (5.47). The bound (5.50) then gives

$$\begin{aligned}
|A'| &\leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} \frac{C_+(z)|\omega_+|}{2(|\kappa_+| + \Im(z))} \\
&\times \left(\sum_{\alpha\in\{n,-\}} \frac{h_\bullet^{\frac{\Im(z)^-}{\kappa_+}}}{\kappa_\alpha(\mathfrak{r}-r_\alpha)} + \frac{(r_+-\mathfrak{r})^{\frac{|\kappa_+|+\Im(z)^-}{\kappa_+}}}{|\kappa_+|}\left(\frac{r_+-r_c}{\mathfrak{r}-r_c}\right)^{\frac{\kappa_+-\Im(z)^-}{\kappa_c}} \right) h_\bullet^{1-\frac{\Im(z)^-}{\kappa_+}}. \qquad (5.54)
\end{aligned}$$

For the term B', we compute:

$$\int_{\phi(p)}^{\phi(r)} \left|(\partial_y \mathrm{D}_{\omega_+})(y-\phi(p))\right| \left|\mathcal{W}_{+,\ell}(y)\right| \mathrm{d}y \leq C_+(z) \int_{\phi(p)}^{\phi(r)} \mathrm{e}^{-2\Im(z)(y-\phi(p))} \mathrm{e}^{2\kappa_+ y} \mathrm{d}y$$
$$= C_+(z) \frac{\mathrm{e}^{2\kappa_+\phi(p)}}{2(|\kappa_+|+\Im(z))} \left(1 - \mathrm{e}^{2(\kappa_+ - \Im(z))(\phi(r)-\phi(p))}\right).$$

By (5.48),

$$\begin{aligned} 1 - \mathrm{e}^{2(\kappa_+ - \Im(z))(\phi(r)-\phi(p))} &= 2(|\kappa_+| + \Im(z))(\phi(r)-\phi(p)) \int_0^1 \mathrm{e}^{2(\kappa_+ - \Im(z))t(\phi(r)-\phi(p))} \mathrm{d}t \\ &\leq 2(|\kappa_+| + \Im(z))(\phi(r)-\phi(p)) \\ &\leq 2(|\kappa_+| + \Im(z)) \left(\sum_{\alpha \in \{n,-\}} \frac{1}{2\kappa_\alpha} \frac{r-p}{\mathfrak{r} - r_\alpha} + \frac{1}{2|\kappa_+|} \frac{r-p}{r_+ - r} \right) \end{aligned}$$

and (5.49) for $r = p$ implies

$$\frac{\mathrm{e}^{2\kappa_+\phi(p)}}{r_+ - r} \leq \frac{1}{r_+ - \mathfrak{r}} \left(\frac{r_+ - r_c}{\mathfrak{r} - r_c} \right)^{\frac{\kappa_+}{\kappa_c}} \frac{r_+ - p}{r_+ - r}.$$

As a result, since $\frac{r_+ - p}{r_+ - r} \leq 2$, we find:

$$|B'| \leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} C_+(z) \left(\sum_{\alpha \in \{n,-\}} \frac{1}{2\kappa_\alpha(\mathfrak{r} - r_\alpha)} + \frac{1}{|\kappa_+|(r_+ - \mathfrak{r})} \left(\frac{r_+ - r_c}{\mathfrak{r} - r_c} \right)^{\frac{\kappa_+}{\kappa_c}} \right) h_\bullet. \tag{5.55}$$

2. Second case: $p_+ = r_+$. We compute:

$$\begin{aligned} &\left|\big(F(r)\partial_r f_+ - \mathscr{F}_{\bullet,+} F(r) \partial_r u\big)(r)\right| \\ &\leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} \left(\int_{\phi(r)}^{+\infty} \Big(\left|(\partial_y \mathrm{D}_{\omega_+})(y-\phi(r))\right| + \left|(\partial_y \mathrm{D}_{\omega_+})(y-\phi(p))\right| \Big) \left|\mathcal{W}_{+,\ell}(y)\right| \mathrm{d}y \right. \\ &\qquad \left. + \int_{\phi(p)}^{\phi(r)} \left|(\partial_y \mathrm{D}_{\omega_+})(y-\phi(p))\right| \left|\mathcal{W}_{+,\ell}(y)\right| \mathrm{d}y \right) \\ &\leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} C_+(z) \left(\int_{\phi(r)}^{+\infty} \left|(\partial_y \mathrm{D}_{\omega_+})(y-\phi(r))\right| \mathrm{e}^{2\kappa_+ y} \mathrm{d}y + 2 \int_{\phi(p)}^{+\infty} \left|(\partial_y \mathrm{D}_{\omega_+})(y-\phi(p))\right| \mathrm{e}^{2\kappa_+ y} \mathrm{d}y \right) \\ &= \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} \frac{C_+(z)}{2(|\kappa_+| + \Im(z))} \left(\mathrm{e}^{2\kappa_+\phi(r)} + 2\mathrm{e}^{2\kappa_+\phi(p)} \right) \\ &\leq 3\|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} \frac{C_+(z)}{2(|\kappa_+| + \Im(z))} \mathrm{e}^{2\kappa_+\phi(p)}. \end{aligned}$$

Using (5.49) for $r = p$ as well as $r_+ - p = h_\bullet$, we finally obtain:

$$\left|\big(F(r)\partial_r f_+ - \mathscr{F}_{\bullet,+} F(r) \partial_r u\big)(r)\right| \leq 3\|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} \frac{C_+(z)}{2(|\kappa_+| + \Im(z))} \frac{1}{r_+ - \mathfrak{r}} \left(\frac{r_+ - r_c}{\mathfrak{r} - r_c} \right)^{\frac{\kappa_+}{\kappa_c}} h_\bullet. \tag{5.56}$$

3. Conclusion: comparing all the estimates (5.54), (5.55) and (5.56) using assumption (5.25), we find

$$\begin{aligned}
&\big|\big(F(r)\partial_r f_+ - \mathscr{F}_{\bullet,+}F(r)\partial_r u\big)(r)\big| \\
&\leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} C_+(z)\left(1+\frac{\max\{3,|\omega_+|\}}{2(|\kappa_+|+\Im(z))}\right) \\
&\quad\times\left(\sum_{\alpha\in\{n,-\}}\frac{1}{\kappa_\alpha(\mathfrak{r}-r_\alpha)}+\frac{(r_+-\mathfrak{r})^{\frac{|\kappa_+|+\Im(z)^-}{\kappa_+}}}{|\kappa_+|}\left(\frac{r_+-r_c}{\mathfrak{r}-r_c}\right)^{\frac{\kappa_+-\Im(z)^-}{\kappa_c}}\right)h_\bullet^{1-\frac{\Im(z)^-}{\kappa_+}}.
\end{aligned}$$

This gives estimate (5.26) for $j=1$ and completes the proof.

5.5.3 Proof of Lemma 5.3.2

We only treat the + case and omit to write $\mathscr{R}_{\bullet,+}$ as it plays no role in the estimates. Let $r\in I_+^{\mathfrak{r}}$. We have

$$\begin{aligned}
\big(U_{\bullet,+}-\widetilde{U}_{\bullet,+}\big)(r) &= \big(\mathcal{G}_{\bullet,+}U_{\bullet,+}-\widetilde{\mathcal{G}}_{\bullet,+}\widetilde{U}_{\bullet,+}\big)(r) \\
&= \big(\mathcal{G}_{\bullet,+}-\widetilde{\mathcal{G}}_{\bullet,+}\big)\widetilde{U}_{\bullet,+}(r)+\mathcal{G}_{\bullet,+}\big(U_{\bullet,+}-\widetilde{U}_{\bullet,+}\big)(r). \qquad (5.57)
\end{aligned}$$

The left-hand side appears on the right-hand side, so we want to apply Grönwall's inequality. To do this, we need two intermediary results. Recall that the constants $K_\pm$, $\widetilde{G}_\pm(z)$, $\Upsilon_\pm(z)$, $\mathit{\Pi}(z)$ and $\Theta_\pm$ are respectively defined in (5.10), (5.12), (5.17), (5.18), (5.19) and (5.20).

Lemma 5.5.2. *For all $r\in I_\pm^{\mathfrak{r}}$, we have*

$$\big|\big((\mathcal{G}_{\bullet,\pm}-\widetilde{\mathcal{G}}_{\bullet,\pm})\widetilde{U}_{\bullet,\pm}\big)_j(r)\big| \leq \big\|(\widetilde{U}_{\bullet,\pm})_j\big\|_{L^\infty(I_\pm^{\mathfrak{r}},\mathbb{C})}\Xi_\pm(z)h_\bullet$$

where

$$\Xi_\pm(z)=\Big(\Upsilon_\pm(z)K_\pm+\mathit{\Pi}(z)\Theta_\pm\Big)\max\left\{\widetilde{G}_\pm(z),\frac{1}{2(|\kappa_\pm|+\Im(z))}\right\}.$$

The next result is proved in Appendix 5.5.4.

Lemma 5.5.3. *For all $r\in I_\pm^{\mathfrak{r}}$, we have*

$$\begin{aligned}
\big\|(\widetilde{U}_{\bullet,\pm})_0\big\|_{L^\infty(I_\pm^{\mathfrak{r}},\mathbb{C})} &\leq 1+\mathit{\Pi}(z)K_\pm\widetilde{G}_\pm(z)\exp\Big(\mathit{\Pi}(z)K_\pm\widetilde{G}_\pm(z)\Big), \\
\big\|(\widetilde{U}_{\bullet,\pm})_1\big\|_{L^\infty(I_\pm^{\mathfrak{r}},\mathbb{C})} &\leq \frac{\mathit{\Pi}(z)K_\pm}{2(|\kappa_\pm|+\Im(z))}\Big(1+\mathit{\Pi}(z)K_\pm\widetilde{G}_\pm(z)\exp\Big(\mathit{\Pi}(z)K_\pm\widetilde{G}_\pm(z)\Big)\Big).
\end{aligned}$$

Assume temporarily these lemmas proved (we give their proofs below). Then we deduce from (5.57) that

$$\begin{aligned}
\big|\big(U_{\bullet,+}-\widetilde{U}_{\bullet,+}\big)_j(r)\big| &\leq \big|\big((\mathcal{G}_{\bullet,+}-\widetilde{\mathcal{G}}_{\bullet,+})\widetilde{U}_{\bullet,+}\big)_j(r)\big|+\big|\big(\mathcal{G}_{\bullet,+}\big(U_{\bullet,+}-\widetilde{U}_{\bullet,+}\big)\big)_j(r)\big| \\
&\leq \big\|(\widetilde{U}_{\bullet,+})_j\big\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}\Xi_+(z)h_\bullet \\
&\quad+C_+(z)\int_r^{r_+}\big|(\partial_x^j\mathrm{D}_{\omega_+})\big(\phi(\sigma)-\phi(r)\big)\big|\mathrm{e}^{2\kappa_+\phi(\sigma)}\big|\big(U_{\bullet,+}-\widetilde{U}_{\bullet,+}\big)_0(\sigma)\big|\frac{\mathrm{d}\sigma}{F(\sigma)}
\end{aligned}$$

When $j = 0$, Grönwall's inequality implies

$$\begin{aligned}&\left|\left(U_{\bullet,+} - \widetilde{U}_{\bullet,+}\right)_0(r)\right| \\ &\leq \left\|(\widetilde{U}_{\bullet,+})_0\right\|_{L^\infty(I^{\mathfrak{r}}_+,\mathbb{C})}\Xi_+(z)h_\bullet\left(1 + C_+(z)\int_r^{r_+}\left|\mathrm{D}_{\omega_+}\left(\phi(\sigma) - \phi(r)\right)\right|\mathrm{e}^{2\kappa_+\phi(\sigma)}\right. \\ &\qquad\left.\times\exp\left(C_+(z)\int_\sigma^{r_+}\left|\mathrm{D}_{\omega_+}\left(\phi(\sigma') - \phi(r)\right)\right|\mathrm{e}^{2\kappa_+\phi(\sigma')}\frac{\mathrm{d}\sigma'}{F(\sigma')}\right)\frac{\mathrm{d}\sigma}{F(\sigma)}\right).\end{aligned}$$

We can then apply Lemma 5.5.1 with $X = +\infty$ and $x = 0 \leq \phi(r)$ to get

$$\left|\left(U_{\bullet,+} - \widetilde{U}_{\bullet,+}\right)_0(r)\right| \leq \left\|(\widetilde{U}_{\bullet,+})_0\right\|_{L^\infty(I^{\mathfrak{r}}_+,\mathbb{C})}\Xi_+(z)\left(1 + C_+(z)\widetilde{G}_+(z)\exp\left(C_+(z)\widetilde{G}_+(z)\right)\right)h_\bullet.$$

The case $j = 1$ easily follows. We have

$$\begin{aligned}&\left|\left(U_{\bullet,+} - \widetilde{U}_{\bullet,+}\right)_1(r)\right| \\ &\leq \left\|(\widetilde{U}_{\bullet,+})_1\right\|_{L^\infty(I^{\mathfrak{r}}_+,\mathbb{C})}\Xi_+(z)h_\bullet + C_+(z)\left\|(\widetilde{U}_{\bullet,+})_0\right\|_{L^\infty(I^{\mathfrak{r}}_+,\mathbb{C})}\Xi_+(z)\left(1 + C_+(z)\widetilde{G}_+(z)\exp\left(C_+(z)\widetilde{G}_+(z)\right)\right)h_\bullet \\ &\qquad\times\int_r^{r_+}\left|\left(\partial_x\mathrm{D}_{\omega_+}\right)\left(\phi(\sigma) - \phi(r)\right)\right|\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}\end{aligned}$$

with

$$\int_r^{r_+}\left|\left(\partial_x\mathrm{D}_{\omega_+}\right)\left(\phi(\sigma) - \phi(r)\right)\right|\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} = \int_{\phi(r)}^{+\infty}\mathrm{e}^{2(\kappa_+ - \Im(z))y}\mathrm{d}y = \frac{\mathrm{e}^{2\kappa_+\phi(r)}}{2(|\kappa_+| + \Im(z))}$$

whence

$$\begin{aligned}\left|\left(U_{\bullet,+} - \widetilde{U}_{\bullet,+}\right)_1(r)\right| \leq \left[\left\|(\widetilde{U}_{\bullet,+})_1\right\|_{L^\infty(I^{\mathfrak{r}}_+,\mathbb{C})}\Xi_+(z) + \frac{C_+(z)}{2(|\kappa_+| + \Im(z))}\left\|(\widetilde{U}_{\bullet,+})_0\right\|_{L^\infty(I^{\mathfrak{r}}_+,\mathbb{C})}\Xi_+(z)\right. \\ \left.\times\left(1 + C_+(z)\widetilde{G}_+(z)\exp\left(C_+(z)\widetilde{G}_+(z)\right)\right)\right]h_\bullet.\end{aligned}$$

This completes the proof of Lemma 5.3.2.

Proof of Lemma 5.5.2

Let $\sigma \in [r, r_+[$ and $p \in \mathscr{T}^+_{\bullet,0}$ such that $\sigma \in [p, p_+[$.

1. Estimate for $g_{+,\ell}$. Recall that

$$g_{+,\ell}(\sigma) = \frac{\Lambda(\sigma - r_n)(\sigma - r_c)(\sigma - r_-)}{3\sigma^2}\widehat{W}_\ell(\sigma) + \frac{s}{r_+\sigma}\left(2z - \frac{s(r_+ + \sigma)}{\sigma r_+}\right)$$

with

$$\widehat{W}_\ell(\sigma) = \frac{\ell(\ell+1)}{\sigma^2} + \frac{F'(\sigma)}{\sigma} + m^2$$

so that

$$\|g_{+,\ell}\|_{L^\infty(I,\mathbb{R})} \leq \frac{\Lambda(r_+ - r_n)(r_+ - r_c)(r_+ - r_-)}{3r_-^2}\|\widehat{W}_\ell\|_{L^\infty(I,\mathbb{R})} + \frac{|s|}{r_-^2}\left(2|z| + \frac{2|s|}{r_-}\right) =: \mathit{\Pi}(z).$$

We have

$$\begin{aligned}
&\frac{\Lambda(\sigma-r_n)(\sigma-r_c)(\sigma-r_-)}{3\sigma^2}\widehat{W}_\ell(\sigma)-\frac{\Lambda(p-r_n)(p-r_c)(p-r_-)}{3p^2}\widehat{W}_\ell(p)\\
&=\frac{\Lambda}{3\sigma^2}\widehat{W}_\ell(\sigma)\Big((\sigma-r_n)(\sigma-r_c)(\sigma-r_-)-(p-r_n)(p-r_c)(p-r_-)\Big)\\
&+\frac{\Lambda(p-r_n)(p-r_c)(p-r_-)}{3}\widehat{W}_\ell(\sigma)\left(\frac{1}{\sigma^2}-\frac{1}{p^2}\right)\\
&+\frac{\Lambda(p-r_n)(p-r_c)(p-r_-)}{3p^2}\big(\widehat{W}_\ell(\sigma)-\widehat{W}_\ell(p)\big)
\end{aligned}$$

and

$$\begin{aligned}
(\sigma-r_n)(\sigma-r_c)(\sigma-r_-)-(p-r_n)(p-r_c)(p-r_-)&=(\sigma-r_n)\big((\sigma-r_c)(\sigma-p)+(p-r_-)(\sigma-p)\big)\\
&\quad+(p-r_c)(p-r_-)(\sigma-p).
\end{aligned}$$

Furthermore,

$$\frac{s}{r_+\sigma}\big(2z-s(V(\sigma)+V_+)\big)-\frac{s}{r_+p}\big(2z-s(V(p)+V_+)\big)=\frac{s(\sigma-p)}{r_+p\sigma}\left(-2z+s\left(\frac{1}{r_+}+\frac{\sigma+p}{\sigma p}\right)\right).$$

It follows using the mean value theorem:

$$\begin{aligned}
\big|g_{+,\ell}(\sigma)-g_{+,\ell}(p)\big|\leq\Bigg[&\left(\frac{\Lambda(\sigma-r_n)(\sigma-r_c)}{3\sigma^2}+\frac{\Lambda(r_+-r_-)(\sigma-r_n)}{3\sigma^2}+\frac{\Lambda(r_+-r_c)(r_+-r_-)}{3\sigma^2}\right.\\
&\left.+\frac{\Lambda(p-r_n)(p-r_c)(p-r_-)}{3p^2\sigma}+\frac{\Lambda(p-r_n)(p-r_c)(p-r_-)}{3p\sigma^2}\right)\big|\widehat{W}_\ell(\sigma)\big|\\
&+\frac{\Lambda(p-r_n)(p-r_c)(p-r_-)}{3p^2}\big|\widehat{W}_\ell(\sigma)-\widehat{W}_\ell(p)\big|\\
&+\frac{|s|}{r_+p\sigma}\left(2|z|+|s|\left(\frac{1}{r_+}+\frac{\sigma+p}{\sigma p}\right)\right)\Bigg](\sigma-p)\\
\leq\Upsilon_+(z)h_\bullet.&
\end{aligned}\tag{5.58}$$

2. Estimate for k_+. Recall that

$$k_+(\sigma)=(r_+-\mathfrak{r})\left(\prod_{\alpha\in\{n,c,-\}}\left(\frac{\mathfrak{r}-r_\alpha}{\sigma-r_\alpha}\right)^{\frac{\kappa_+}{\kappa_\alpha}}\right).$$

We have

$$\|k_+\|_{L^\infty(I,\mathbb{R})}\leq(r_+-\mathfrak{r})\left(\frac{\mathfrak{r}-r_n}{r_+-r_n}\right)^{\frac{\kappa_+}{\kappa_n}}\left(\frac{\mathfrak{r}-r_-}{r_+-r_-}\right)^{\frac{\kappa_+}{\kappa_-}}=K_+$$

where K_+ is as in (5.10). Set $\theta_\alpha(r):=\left(\frac{\mathfrak{r}-r_\alpha}{r-r_\alpha}\right)^{\frac{\kappa_+}{\kappa_\alpha}}$ and compute:

$$\begin{aligned}
k_+(\sigma)-k_+(p)&=(r_+-\mathfrak{r})\Big(\theta_n(\sigma)\theta_c(\sigma)\theta_-(\sigma)-\theta_n(p)\theta_c(p)\theta_-(p)\Big)\\
&=(r_+-\mathfrak{r})\Big(\theta_n(\sigma)\Big[\theta_c(\sigma)\big(\theta_-(\sigma)-\theta_-(p)\big)+\big(\theta_c(\sigma)-\theta_c(p)\big)\theta_-(p)\Big]\\
&\qquad+\big(\theta_n(\sigma)-\theta_c(p)\big)\theta_c(p)\theta_-(p)\Big).
\end{aligned}$$

It comes with the mean value theorem:

$$\begin{aligned}\big|k_+(\sigma)-k_+(p)\big| \le (r_+-\mathfrak{r})\Big(&\|\theta_n'\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\|\theta_c\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\|\theta_-\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\\ &+\|\theta_n\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\|\theta_c'\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\|\theta_-\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\\ &+\|\theta_n\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\|\theta_c\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\|\theta_-'\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\Big)h_\bullet.\end{aligned}$$

Since $\kappa_n>0$, $\kappa_c<0$, $\kappa_->0$ and $\kappa_+<0$, we have:

$$\|\theta_n\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\le\left(\frac{\mathfrak{r}-r_n}{r_+-r_n}\right)^{\frac{\kappa_+}{\kappa_n}},\qquad \|\theta_c\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\le 1,\qquad \|\theta_-\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}\le\left(\frac{\mathfrak{r}-r_-}{r_+-r_-}\right)^{\frac{\kappa_+}{\kappa_-}},$$

$$\begin{aligned}\|\theta_n'\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}&\le\left|\frac{\kappa_+}{\kappa_n}\right|\left(\frac{\mathfrak{r}-r_n}{r_+-r_n}\right)^{\frac{\kappa_+}{\kappa_n}-1}\frac{1}{(\mathfrak{r}-r_n)^2},\\ \|\theta_c'\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}&\le\left|\frac{\kappa_+}{\kappa_c}\right|\max\left\{1,\left(\frac{\mathfrak{r}-r_c}{r_+-r_c}\right)^{\frac{\kappa_+}{\kappa_c}-1}\right\}\frac{1}{(\mathfrak{r}-r_c)^2},\\ \|\theta_-'\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{R})}&\le\left|\frac{\kappa_+}{\kappa_-}\right|\left(\frac{\mathfrak{r}-r_-}{r_+-r_-}\right)^{\frac{\kappa_+}{\kappa_-}-1}\frac{1}{(\mathfrak{r}-r_-)^2}.\end{aligned}$$

Whence:

$$\big|k_+(\sigma)-k_+(p)\big|\le\Theta_+h_\bullet. \tag{5.59}$$

3. Conclusion: combining (5.58) with (5.59), we find

$$\begin{aligned}\big|\mathfrak{W}_{+,\ell}(\sigma)-\mathfrak{W}_{+,\ell}(p)\big|&\le\Big(\big|g_{+,\ell}(\sigma)-g_{+,\ell}(p)\big|\|k_+\|_{L^\infty(I,\mathbb{R})}+\|g_{+,\ell}\|_{L^\infty(I,\mathbb{R})}\big|k_+(\sigma)-k_+(p)\big|\Big)\mathrm{e}^{2\kappa_+\phi(\sigma)}\\ &\le h_\bullet\Big(\Upsilon_+(z)K_++\mathit{\Pi}(z)\Theta_+\Big)\mathrm{e}^{2\kappa_+\phi(\sigma)}.\end{aligned}$$

We thus have:

$$\begin{aligned}\big|\big((\mathcal{G}_{\bullet,+}-\widetilde{\mathcal{G}}_{\bullet,+})\widetilde{U}_{\bullet,+}\big)_j(r)\big|&\le\int_r^{r_+}\big|(\partial_x^j\mathrm{D}_{\omega_+})\big(\phi(\sigma)-\phi(r)\big)\big|\big|\mathfrak{W}_{+,\ell}(\sigma)-\mathfrak{W}_{+,\ell}(p)\big|\big\|(\widetilde{U}_{\bullet,+})_j\big\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}\frac{\mathrm{d}\sigma}{F(\sigma)}\\ &\le\big\|(\widetilde{U}_{\bullet,+})_j\big\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}\Big(\Upsilon_+(z)K_++\mathit{\Pi}(z)\Theta_+\Big)h_\bullet\\ &\qquad\times\int_r^{r_+}\big|(\partial_x^j\mathrm{D}_{\omega_+})\big(\phi(\sigma)-\phi(r)\big)\big|\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}.\end{aligned}$$

When $j=0$, we can use Lemma 5.5.1 with $X=+\infty$ and $x=\phi(r)$ (but without the constant $C_+(z)$): we find

$$\big|\big((\mathcal{G}_{\bullet,+}-\widetilde{\mathcal{G}}_{\bullet,+})\widetilde{U}_{\bullet,+}\big)_0(r)\big|\le\big\|(\widetilde{U}_{\bullet,+})_0\big\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}\Big(\Upsilon_+(z)K_++\mathit{\Pi}(z)\Theta_+\Big)\widetilde{G}_+(z)h_\bullet.$$

When $j=1$, we can directly use that

$$\int_r^{r_+}\big|(\partial_x\mathrm{D}_{\omega_+})\big(\phi(\sigma)-\phi(r)\big)\big|\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}=\mathrm{e}^{2\Im(z)\phi(r)}\int_{\phi(r)}^{+\infty}\mathrm{e}^{2(\kappa_+-\Im(z))y}\mathrm{d}y=\frac{\mathrm{e}^{2\kappa_+\phi(r)}}{2(|\kappa_+|+\Im(z))}.$$

to get

$$\left|\left((\mathcal{G}_{\bullet,+}-\widetilde{\mathcal{G}}_{\bullet,+})\widetilde{U}_{\bullet,+}\right)_1(r)\right| \leq \left\|(\widetilde{U}_{\bullet,+})_1\right\|_{L^\infty(I^{\mathfrak{r}}_+,\mathbb{C})}\Big(\Upsilon_+(z)K_+ + \Pi(z)\Theta_+\Big)\frac{h_\bullet}{2(|\kappa_+|+\Im(z))}.$$

This completes the proof of the lemma.

5.5.4 Proof of Lemma 5.5.3

By definition of the fixed point $\widetilde{U}_{\bullet,+}$,

$$\widetilde{U}_{\bullet,+}(r) = \begin{pmatrix}1\\0\end{pmatrix} + \int_r^{r_+} \widetilde{A}_+(r;\sigma)\widetilde{U}_{\bullet,+}(\sigma)\frac{\mathrm{d}\sigma}{F(\sigma)}$$

and thus

$$\begin{aligned}\left|\left(\widetilde{U}_{\bullet,+}\right)_j(r)\right| &\leq \delta_{j0} + \sum_{p\in\mathscr{T}^+_{\bullet,0}} |g_{+,\ell}(p)||k_+(p)| \int_{]r,r_+[\cap\omega_p^+} \left|(\partial_x^j \mathrm{D}_{\omega_+})\big(\phi(\sigma)-\phi(r)\big)\right| \mathrm{e}^{2\kappa_+\phi(\sigma)}\left|\left(\widetilde{U}_{\bullet,+}\right)_0(\sigma)\right|\frac{\mathrm{d}\sigma}{F(\sigma)}\\ &\leq \delta_{j0} + \Pi(z)K_+\int_r^{r_+}\left|(\partial_x^j \mathrm{D}_{\omega_+})\big(\phi(\sigma)-\phi(r)\big)\right| \mathrm{e}^{2\kappa_+\phi(\sigma)}\left|\left(\widetilde{U}_{\bullet,+}\right)_0(\sigma)\right|\frac{\mathrm{d}\sigma}{F(\sigma)}\end{aligned}$$

where K_+ is as in (5.10) and $\Pi(z)$ as in (5.19). When $j=0$, we use Grönwall's inequality:

$$\begin{aligned}\left|\left(\widetilde{U}_{\bullet,+}\right)_0(r)\right| \leq 1 + \Pi(z)K_+\int_r^{r_+}&\left|\mathrm{D}_{\omega_+}\big(\phi(\sigma)-\phi(r)\big)\right|\mathrm{e}^{2\kappa_+\phi(\sigma)}\\ &\times\exp\left(\Pi(z)K_+\int_\sigma^{r_+}\left|\mathrm{D}_{\omega_+}\big(\phi(\sigma')-\phi(r)\big)\right|\mathrm{e}^{2\kappa_+\phi(\sigma')}\frac{\mathrm{d}\sigma'}{F(\sigma')}\right)\frac{\mathrm{d}\sigma}{F(\sigma)}.\end{aligned}$$

It remains to apply Lemma 5.5.1 with $X=+\infty$, $x=0\leq\phi(r)$ and $G_+(y)$ replaced by $\Pi(z)K_+\mathrm{e}^{2\kappa_+ y}$. From there we immediately deduce the case $j=1$:

$$\begin{aligned}\left|\left(\widetilde{U}_{\bullet,+}\right)_1(r)\right| &\leq \left\|\left(\widetilde{U}_{\bullet,+}\right)_0\right\|_{L^\infty(I^{\mathfrak{r}}_+,\mathbb{C})}\Pi(z)K_+\int_r^{r_+}\left|(\partial_x \mathrm{D}_{\omega_+})\big(\phi(\sigma)-\phi(r)\big)\right|\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}\\ &= \left\|\left(\widetilde{U}_{\bullet,+}\right)_0\right\|_{L^\infty(I^{\mathfrak{r}}_+,\mathbb{C})}\Pi(z)K_+\mathrm{e}^{2\Im(z)\phi(r)}\int_{\phi(r)}^{+\infty}\mathrm{e}^{2(\kappa_+-\Im(z))y}\mathrm{d}y\\ &= \left\|\left(\widetilde{U}_{\bullet,+}\right)_0\right\|_{L^\infty(I^{\mathfrak{r}}_+,\mathbb{C})}\Pi(z)K_+\frac{\mathrm{e}^{2\kappa_+\phi(r)}}{2(|\kappa_+|+\Im(z))}.\end{aligned}$$

This yields the announced formulas.

5.5.5 Proof of Lemma 5.3.5

We only treat the + case. By (5.40), we have:

$$|f_+(x)| \leq 1 + \int_x^{+\infty}|\mathrm{D}_{\omega_+}(y-x)|G_+(y)|f_+(y)|\mathrm{d}y.$$

By Grönwall's inequality, we get

$$|f_+(x)| \leq 1 + \int_x^{+\infty}|\mathrm{D}_{\omega_+}(y-x)|G_+(y)\exp\left(\int_y^{+\infty}|\mathrm{D}_{\omega_+}(y-y')|G_+(y')\mathrm{d}y'\right)\mathrm{d}y.$$

As $r=\mathfrak{r}$ corresponds to $x=0$, we are only needed to estimate the above terms for $x=y=0$. It remains to use Lemma 5.5.1 with $X=+\infty$ to conclude.

5.5.6 Proof of Corollary 5.3.6

We only treat the + case. Using equation (5.43), we can write for all $x \geq 0$:

$$\big|(\partial_x f_+)(x)\big| \leq \|f_+\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} \int_x^{+\infty} \big|(\partial_y \mathrm{D}_{\omega_+})(y-x)\big| G_+(y)\mathrm{d}y \tag{5.60}$$

with $|\partial_x \mathrm{D}_{\omega_+}(x)| = \mathrm{e}^{-2\Im(z)x}$ (*cf.* (5.45)). It follows:

$$\begin{aligned}\|\partial_x f_+\|_{L^\infty([0,+\infty[,\mathbb{C})} &\leq C_+(z)\|f_+\|_{L^\infty(I_+,\mathbb{C})} \int_x^{+\infty} \mathrm{e}^{-2\Im(z)(y-x)}\mathrm{e}^{2\kappa_+ y}\mathrm{d}y \\ &= \frac{C_+(z)\|f_+\|_{L^\infty(I_+,\mathbb{C})}\mathrm{e}^{2\kappa_+ x}}{2||\kappa_\pm| + \Im(z)|}.\end{aligned}$$

This completes the proof.

5.5.7 Proof of Lemma 5.3.8

To make appear the dependence in z, we will write $f_\pm(x) := f_\pm(x,z)$. Formula (5.30) reads in the Regge-Wheeler coordinate:

$$\mathscr{W}(z) = f_+(0,z)(\partial_x f_-)(0,z) - (\partial_x f_+)(0,z) f_-(0,z) + \mathrm{i}s\big(V_+ - V_-\big) f_+(0,z) f_-(0,z).$$

Taking the derivative with respect to z yields:

$$\begin{aligned}\mathscr{W}'(z) = {} & (\partial_z f_+)(0,z)(\partial_x f_-)(0,z) + f_+(0,z)(\partial_z\partial_x f_-)(0,z) \\ & - (\partial_z\partial_x f_+)(0,z) f_-(0,z) + (\partial_x f_+)(0,z)(\partial_z f_-)(0,z) \\ & + \mathrm{i}s\big(V_+ - V_-\big)\Big((\partial_z f_+)(0,z) f_-(0,z) + f_+(0,z)(\partial_z f_-)(0,z)\Big).\end{aligned}$$

We will thus get a bound on $\mathscr{W}'(z)$ if we can estimate $\|(\partial_z f_\pm)(\cdot,z)\|_{L^\infty(I_\pm^{\mathfrak{r}},\mathbb{C})}$ and $\|(\partial_z\partial_x f_\pm)(\cdot,z)\|_{L^\infty(I_\pm^{\mathfrak{r}},\mathbb{C})}$. We will only treat the + case. Lemma 5.3.8 then follows from (5.62)−(5.67) below, as well as Lemma 5.3.5 and Corollary 5.3.6.

Let $z \in \mathbb{C}$ such that $\Im(z) > -\kappa$. We will write $z = a + \mathrm{i}b$ with $(a,b) \in \mathbb{R}^2$ and use the constant $\widetilde{G}_+(z)$ defined in (5.12) so that

$$\int_x^{+\infty} \big|\mathrm{D}_{\omega_+}(y-x)\big|\mathrm{e}^{2\kappa_+ y}\mathrm{d}y \leq \widetilde{G}_+(z).$$

Estimate of the first derivatives

Let us compute

$$\begin{aligned}(\partial_z f_+)(x,z) = \int_x^{+\infty} & \Big((\partial_z \mathrm{D}_{\omega_+})(y-x) f_+(y,z) + \mathrm{D}_{\omega_+}(y-x)(\partial_z f_+)(y,z)\Big) \\ & \times \Big(W_\ell(y) + s(V(y)-V_+)\big(2z - s(V(y)+V_+)\big)\Big)\mathrm{d}y, \\ + 2s \int_x^{+\infty} & \mathrm{D}_{\omega_+}(y-x) f_+(y,z)(V(y)-V_+)\mathrm{d}y\end{aligned}$$

with

$$(\partial_z \mathrm{D}_{\omega_\pm})(x) = \begin{cases} \dfrac{1}{\omega_\pm}\left(x\mathrm{e}^{\pm 2\mathrm{i}\omega_\pm x} - \mathrm{D}_{\omega_\pm}(x)\right) & \text{if } \omega_\pm \in \mathbb{C}\setminus\{0\} \\ \pm \mathrm{i}x^2 & \text{if } \omega_\pm = 0 \end{cases}.$$

We have

$$\begin{aligned} \big|(\partial_z f_+)(x,z)\big| \le \|f_+(\cdot,z)\|_{L^\infty(I_+^{\mathrm{r}},\mathbb{C})} \int_x^{+\infty} \big|(\partial_z \mathrm{D}_{\omega_+})(y-x)\big| G_+(y,z)\mathrm{d}y \\ + 2|s|\|f_+(\cdot,z)\|_{L^\infty(I_+^{\mathrm{r}},\mathbb{C})} \int_x^{+\infty} \big|\mathrm{D}_{\omega_+}(y-x)(V(y)-V_+)\big|\mathrm{d}y \\ + \int_x^{+\infty} \big|\mathrm{D}_{\omega_+}(y-x)\big|\big|(\partial_z f_+)(y,z)\big| G_+(y,z)\mathrm{d}y. \end{aligned}$$

Recall that $G_+(x,z) := |W_\ell(x)| + |s||V(x)-V_+|\big(2|z| + |s||V(x)+V_+|\big) \le C_+(z)\mathrm{e}^{2\kappa_+ x}$ for all $x \ge 0$. Let

$$\begin{aligned} \mathcal{I}_1(x) &:= \int_x^{+\infty} \big|(\partial_z \mathrm{D}_{\omega_+})(y-x)\big| G_+(y,z)\mathrm{d}y, \\ \mathcal{I}_2(x) &:= \int_x^{+\infty} \big|\mathrm{D}_{\omega_+}(y-x)(V(y)-V_+)\big|\mathrm{d}y, \\ \mathcal{I}_3(x) &:= \int_x^{+\infty} \big|\mathrm{D}_{\omega_+}(y-x)\big| G_+(y,z)\mathrm{d}y. \end{aligned}$$

Then Grönwall's inequality implies:

$$\big|(\partial_z f_+)(x,z)\big| \le \|f_+(\cdot,z)\|_{L^\infty(I_+^{\mathrm{r}},\mathbb{C})}\big(\mathcal{I}_1(0) + 2|s|\mathcal{I}_2(0)\big)\Big(1 + \mathcal{I}_3(0)\exp\big(\mathcal{I}_3(0)\big)\Big). \tag{5.61}$$

It is therefore sufficient to get an x-independent bound for each $\mathcal{I}_j(x)$, $j \in \{1,2,3\}$. Observe that we already proved that $\mathcal{I}_3(x) \le C_+(z)\widetilde{G}_+(z)$ for all $x \ge 0$ in Subsection 5.5.5. Observe furthermore that, using $|r - r_+| \le K_+\mathrm{e}^{2\kappa_+ x}$ when $x \ge 0$, we directly get

$$\mathcal{I}_2(x) \le \frac{K_+}{r_- r_+}\int_x^{+\infty} \big|\mathrm{D}_{\omega_+}(y-x)\big|\mathrm{e}^{2\kappa_+ y}\mathrm{d}y \le \frac{K_+}{r_- r_+}\widetilde{G}_+(z).$$

We therefore only need to bound $\mathcal{I}_1(x)$ for all $x \ge 0$.

1. First case: $b \ge 0$ and $z \ne sV_+$. Assume first $|\omega_+| \ge |\kappa_+|$. We compute:

$$\begin{aligned} \mathcal{I}_1(x) &\le \frac{C_+(z)}{|\omega_+|}\int_x^{+\infty}\Big((y-x)\mathrm{e}^{2bx}\mathrm{e}^{2(\kappa_+ - b)y} + \big|\mathrm{D}_{\omega_+}(y-x)\big|\mathrm{e}^{2\kappa_+ y}\Big)\mathrm{d}y \\ &\le \frac{C_+(z)}{|\omega_+|}\left(\frac{\mathrm{e}^{2\kappa_+ x}}{b-\kappa_+} + \widetilde{G}_+(z)\right) \end{aligned}$$

We then deduce from (5.61):

$$\begin{aligned} \big|(\partial_z f_+)(x,z)\big| \le \|f_+(\cdot,z)\|_{L^\infty(I_+^{\mathrm{r}},\mathbb{C})}\left(\frac{C_+(z)}{|\omega_+|}\left(\frac{1}{b-\kappa_+} + \widetilde{G}_+(z)\right) + \frac{2|s|K_+}{r_- r_+}\widetilde{G}_+(z)\right) \\ \times\left(1 + \frac{C_+(z)}{2|\kappa_+||\omega_+|}\exp\left(\frac{C_+(z)}{2|\kappa_+||\omega_+|}\right)\right). \end{aligned} \tag{5.62}$$

Assume now $|\omega_+| < |\kappa_+|$. Let

$$A := \int_0^1 (1-t)\mathrm{e}^{2\mathrm{i}\omega_+ t(y-x)}\mathrm{d}t, \qquad B := \int_0^1 \frac{(1-t)^2}{2}\mathrm{e}^{2\mathrm{i}\omega_+ t(y-x)}\mathrm{d}t.$$

If $y - x \geq 0$ then

$$|A| \leq \frac{1}{2}, \qquad |B| \leq \frac{1}{6}.$$

It follows

$$\left|(\partial_z \mathrm{D}_{\omega_+})(y-x)\right| = \left|\mathrm{i}(y-x)^2 + 4\mathrm{i}\omega_+(y-x)^3(A-B)\right| \leq (y-x)^2 + \frac{8}{3}\omega_+(y-x)^3$$

whence

$$\begin{aligned}
\mathcal{I}_1(x) &\leq C_+(z)\int_x^{+\infty}\left((y-x)^2 + \frac{8}{3}\omega_+(y-x)^3\right)\mathrm{e}^{2\kappa_+ y}\mathrm{d}y \\
&= C_+(z)\mathrm{e}^{2\kappa_+ x}\int_0^{+\infty}\left(y^2 + \frac{8}{3}|\omega_+|y^3\right)\mathrm{e}^{2\kappa_+ y}\mathrm{d}y \\
&\leq C_+(z)\mathrm{e}^{2\kappa_+ x}\left(\frac{1}{4|\kappa_+|^3} + \frac{|\omega_+|}{\kappa_+^4}\right) \\
&\leq C_+(z)\mathrm{e}^{2\kappa_+ x}\frac{5}{4|\kappa_+|^3}.
\end{aligned}$$

Estimate (5.61) then gives:

$$\begin{aligned}
\left|(\partial_z f_+)(x,z)\right| \leq \|f_+(\cdot,z)\|_{L^\infty(I_+^{\mathrm{r}},\mathbb{C})}&\left(\frac{5C_+(z)}{4|\kappa_+|^3} + \frac{2|s|K_+}{r_- r_+}\widetilde{G}_+(z)\right) \\
&\times\left(1 + \frac{\sqrt{13}C_+(z)}{8\kappa_+^2}\exp\left(\frac{\sqrt{13}C_+(z)}{8\kappa_+^2}\right)\right).
\end{aligned} \tag{5.63}$$

2. Second case: $\omega_+ = 0$. We have

$$\begin{aligned}
\mathcal{I}_1(x) &\leq C_+(z)\int_x^{+\infty}\left((y-x)^2 + (y-x)\right)\mathrm{e}^{2\kappa_+ y}\mathrm{d}y \\
&= C_+(z)\mathrm{e}^{2\kappa_+ x}\int_0^{+\infty}\left(y^2 + y\right)\mathrm{e}^{2\kappa_+ y}\mathrm{d}y \\
&= C_+(z)\mathrm{e}^{2\kappa_+ x}\left(\frac{1}{4|\kappa_+|^3} + \frac{1}{4\kappa_+^2}\right).
\end{aligned}$$

It follows from (5.61):

$$\begin{aligned}
\left|(\partial_z f_+)(x,z)\right| \leq \|f_+(\cdot,z)\|_{L^\infty(I_+^{\mathrm{r}},\mathbb{C})}&\left(C_+(z)\left(\frac{1}{4|\kappa_+|^3} + \frac{1}{4\kappa_+^2}\right) + \frac{2|s|K_+}{r_- r_+}\widetilde{G}_+(z)\right) \\
&\times\left(1 + \frac{C_+(z)}{4\kappa_+^2}\exp\left(\frac{C_+(z)}{4\kappa_+^2}\right)\right).
\end{aligned} \tag{5.64}$$

3. Third case: $0 > b > -\kappa$. Assume first $|\omega_+| \geq |\kappa_+|$. The computation carried out for the first case shows that

$$\mathcal{I}_1(x) \leq \frac{C_+(z)}{|\omega_+|}\left(\frac{\mathrm{e}^{2\kappa_+ x}}{b-\kappa_+} + \widetilde{G}_+(z)\right)$$

hence estimate (5.61) becomes:

$$\left|(\partial_z f_+)(x,z)\right| \leq \|f_+(\cdot,z)\|_{L^\infty(I_+^{\mathrm{r}},\mathbb{C})}\left(\frac{C_+(z)}{|\omega_+|}\left(\frac{1}{b-\kappa_+} + \widetilde{G}_+(z)\right) + \frac{2|s|K_+}{r_- r_+}\widetilde{G}_+(z)\right) \times \left(1 + \frac{C_+(z)}{2|\kappa_+||\omega_+|}\exp\left(\frac{C_+(z)}{2|\kappa_+||\omega_+|}\right)\right). \tag{5.65}$$

Assume eventually $|\omega_+| < |\kappa_+|$. Define

$$A' := \int_0^1 (1-t)\mathrm{e}^{-2\mathrm{i}\omega_+ t(y-x)}\mathrm{d}t$$

with

$$|A'| \leq \frac{1}{2}$$

if $y - x \geq 0$. We then get

$$\begin{aligned}\left|(\partial_z \mathrm{D}_{\omega_+})(y-x)\right| &= \left|\frac{\mathrm{e}^{2\mathrm{i}\omega_+(y-x)}}{\omega_+}\left((y-x) - \frac{1-\mathrm{e}^{-2\mathrm{i}\omega_+(y-x)}}{2\mathrm{i}\omega_+}\right)\right| \\ &= \mathrm{e}^{-2b(y-x)}|2\mathrm{i}(y-x)^2 A'| \\ &\leq (y-x)^2\mathrm{e}^{-2b(y-x)}\end{aligned}$$

whence

$$\begin{aligned}\mathcal{I}_1(x) &\leq C_+(z)\mathrm{e}^{2bx}\int_x^{+\infty}(y-x)^2\mathrm{e}^{2(\kappa_+-b)y}\mathrm{d}y \\ &= C_+(z)\mathrm{e}^{2\kappa_+ x}\int_0^{+\infty} y^2\mathrm{e}^{2(\kappa_+-b)y}\mathrm{d}y \\ &\leq \frac{C_+(z)\mathrm{e}^{2\kappa_+ x}}{4|\kappa_+-b|^3}.\end{aligned}$$

From (5.61) we obtain:

$$\left|(\partial_z f_+)(x,z)\right| \leq \|f_+(\cdot,z)\|_{L^\infty(I_+^{\mathrm{r}},\mathbb{C})}\left(\frac{C_+(z)}{4|\kappa_+-b|^3} + \frac{2|s|K_+}{r_- r_+}\widetilde{G}_+(z)\right) \times \left(1 + \frac{\sqrt{13}C_+(z)}{8(b-\kappa_+)^2}\exp\left(\frac{\sqrt{13}C_+(z)}{8(b-\kappa_+)^2}\right)\right). \tag{5.66}$$

Estimate for the mixed derivatives

The mixed derivative is given by

$$\begin{aligned}(\partial_z\partial_x f_+)(x,z) = -\int_x^{+\infty} \Big((\partial_z\partial_y \mathrm{D}_{\omega_+})(y-x) f_+(y,z) + (\partial_y \mathrm{D}_{\omega_+})(y-x)(\partial_z f_+)(y,z)\Big) \\ \times \Big(W_\ell(y) + s(V(y)-V_+)\big(2z - s(V(y)+V_+)\big)\Big)\mathrm{d}y, \\ -2s\int_x^{+\infty} (\partial_y \mathrm{D}_{\omega_+})(y-x) f_+(y,z)(V(y)-V_+)\mathrm{d}y\end{aligned}$$

with

$$(\partial_z\partial_x \mathrm{D}_{\omega_+})(x) = 2\mathrm{i}x\mathrm{e}^{2\mathrm{i}\omega_+ x}.$$

Since $\|f_+(\cdot,z)\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}$ and $\|(\partial_z f_+)(\cdot,z)\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}$ have been already estimated, we only have 3 integrals to bound. First, we compute

$$\begin{aligned}\int_x^{+\infty} \big|(\partial_z\partial_y \mathrm{D}_{\omega_+})(y-x)\big| G_+(y,z)\mathrm{d}y &\le 2C_+(z)\mathrm{e}^{2\Im(z)x}\int_x^{+\infty}(y-x)\mathrm{e}^{2(\kappa_+-\Im(z))y}\mathrm{d}y \\ &= 2C_+(z)\mathrm{e}^{2\kappa_+ x}\int_0^{+\infty} y\mathrm{e}^{2(\kappa_+-\Im(z))y}\mathrm{d}y \\ &= \frac{C_+(z)\mathrm{e}^{2\kappa_+ x}}{2(\kappa_+-\Im(z))^2}.\end{aligned}$$

Next, we have

$$\int_x^{+\infty} \big|(\partial_y \mathrm{D}_{\omega_+})(y-x)\big| G_+(y,z)\mathrm{d}y \le C_+(z)\mathrm{e}^{2\Im(z)x}\int_x^{+\infty}\mathrm{e}^{2(\kappa_+-\Im(z))y}\mathrm{d}y = \frac{C_+(z)\mathrm{e}^{2\kappa_+ x}}{2|\kappa_+-\Im(z)|}.$$

Finally,

$$\int_x^{+\infty} \big|(\partial_y \mathrm{D}_{\omega_+})(y-x)(V(y)-V_+)\big|\mathrm{d}y \le \frac{K_+}{r_-r_+}\mathrm{e}^{2\Im(z)x}\int_x^{+\infty}\mathrm{e}^{2(\kappa_+-\Im(z))y}\mathrm{d}y = \frac{K_+\mathrm{e}^{2\kappa_+ x}}{2|\kappa_+-\Im(z)|r_-r_+}.$$

We eventually obtain the following estimate for all $x \ge 0$:

$$\begin{aligned}\big|(\partial_z\partial_x f_+)(x,z)\big| \le \left(\frac{C_+(z)}{2(\kappa_+-\Im(z))^2} + \frac{|s|K_+}{|\kappa_+-\Im(z)|r_-r_+}\right)\|f_+(\cdot,z)\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})} \\ + \frac{C_+(z)}{2|\kappa_+-\Im(z)|}\|(\partial_z f_+)(\cdot,z)\|_{L^\infty(I_+^{\mathfrak{r}},\mathbb{C})}.\end{aligned} \tag{5.67}$$

It remains to use the estimates collected in the above paragraph to conclude the proof.

5.6 Appendix: Approximations

This appendix gives explicit formulas for the discrete version of some operators associated to the scheme as well as the integral giving the number of resonances in a path $\Gamma \subset \mathbb{C}$.

5.6.1 Computation of the discretized operators for the scheme (5.9)

In this Section, we compute the coefficients of the discrete operators $\widetilde{\mathcal{G}}_{\bullet,\pm}$ for the numerical programming.

Let $\widetilde{U}_\pm \in E^\dagger_\pm$ such that $\mathscr{P}_{\bullet,\pm}\widetilde{U}_\pm = \widetilde{U}_{\bullet,\pm} \in E^\dagger_{\bullet,\pm}$ (all the functions in $E^\dagger_\pm$ satisfying this property coincide on the mesh points in $\mathscr{T}^\pm_{\bullet,0}$; as the mesh converges, we know by Proposition 5.3.3 that the only function with this property is the modified Jost solution $f_\pm \in E_\pm \subset E^\dagger_\pm$). Set

$$\mathbb{D}^+(p;\sigma) := \begin{pmatrix} \mathrm{D}_{\omega_+}\big(\phi(\sigma)-\phi(p)\big) \\ -(\partial_x \mathrm{D}_{\omega_+})\big(\phi(\sigma)-\phi(p)\big)\end{pmatrix}, \qquad \mathbb{D}^-(p;\sigma) := \begin{pmatrix} \mathrm{D}_{\omega_-}\big(\phi(p)-\phi(\sigma)\big) \\ (\partial_x \mathrm{D}_{\omega_-})\big(\phi(p)-\phi(\sigma)\big)\end{pmatrix}$$

and define the vectors of $\mathbb{C}^{2(|\mathscr{T}^\pm_{\bullet,0}|+1)}$:

$$\begin{aligned} &(\vec{U}_{\bullet,-})_0 = 1, && (\vec{U}_{\bullet,-})_1 = 0 && (\vec{U}_{\bullet,-})_{2|p|+\delta_{j1}} = \frac{1}{2}\big(\widetilde{U}_-(p)+\widetilde{U}_-(p_-)\big)_j, \\ &(\vec{U}_{\bullet,+})_{2|\mathscr{T}^\pm_{\bullet,0}|} = 1, && (\vec{U}_{\bullet,+})_{2|\mathscr{T}^\pm_{\bullet,0}|+1} = 0 && (\vec{U}_{\bullet,+})_{2(|p|-1)+\delta_{j1}} = \frac{1}{2}\big(\widetilde{U}_+(p)+\widetilde{U}_+(p_+)\big)_j. \end{aligned}$$

Recall the convention that $|p|$ is the number of elements $p' \in \mathscr{T}^\pm_{\bullet,0}$ such that $p' \le p$. For example, $|(r_-)_+| = 1$ and $|(r_+)_-| = |\mathscr{T}^+_{\bullet,0}|$. Notice that the boundary terms for $\vec{U}_{\bullet,-}$ are in the first components $0, 1$ whereas they are for $\vec{U}_{\bullet,+}$ is the last components $2|\mathscr{T}^\pm_{\bullet,0}|$, $2|\mathscr{T}^\pm_{\bullet,0}|+1$.

We identify $\vec{U}_{\bullet,\pm}$ with $\widetilde{U}_{\bullet,\pm}$ via the evaluation morphism $E^\dagger_\pm \to \mathbb{C}^{2(|\mathscr{T}^\pm_{\bullet,0}|+1)}$. For $p_0 \in \mathscr{T}^\pm_{\bullet,0}$ and $j \in \{0,1\}$, we have by definition of the problem (5.9):

$$\begin{aligned} \big(\widetilde{G}_{\bullet,\pm}\widetilde{U}_{\bullet,\pm}\big)_j(p_0) &= \delta_{j0} \pm \int_{p_0}^{r_\pm} \mathbb{D}^\pm_j(p_0;\sigma)\mathrm{e}^{2\kappa_\pm\phi(\sigma)} \left(\sum_{p\in\mathscr{T}^\pm_{\bullet,0}} g_{\pm,\ell}(p)k_\pm(p)\mathbb{1}_{\omega_{p_\pm}}(\sigma)\right) \\ &\qquad \times \left(\frac{1}{2}\sum_{p'\in\mathscr{T}^\pm_{\bullet,0}} \Big((\widetilde{U}_\pm)_0(p') + (\widetilde{U}_\pm)_0(p'_\pm)\Big)\mathbb{1}_{\omega^\pm_{p'}}(\sigma)\right)\frac{\mathrm{d}\sigma}{F(\sigma)} \\ &= \delta_{j0} \pm \sum_{\substack{p\in\mathscr{T}^\pm_{\bullet,0} \\ \pm(|p|-|p_0|)\ge 0}} \frac{1}{2} g_{\pm,\ell}(p)k_\pm(p)\Big((\widetilde{U}_\pm)_0(p) + (\widetilde{U}_\pm)_0(p_\pm)\Big) \\ &\qquad \times \int_p^{p_\pm} \mathbb{D}^\pm_j(p_0;\sigma)\mathrm{e}^{2\kappa_\pm\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}. \end{aligned}$$

The operators $\widetilde{\mathcal{G}}_{\bullet,\pm}$ can then be viewed as affine functions acting on $\mathbb{C}^{2(|\mathscr{T}^\pm_{\bullet,0}|+1)}$:

$$\begin{aligned} (\vec{U}_{\bullet,-})_{2|p_0|+\delta_{j1}} &= \big(\widetilde{U}_{\bullet,-}\big)_j(p_0) \\ &= \big(\widetilde{G}_{\bullet,-}\widetilde{U}_{\bullet,-}\big)_j(p_0) \\ &= \delta_{j0} + \sum_{\substack{p\in\mathscr{T}^-_{\bullet,0} \\ |p|\le|p_0|}} \left(g_{-,\ell}(p)k_-(p)\int_{p_-}^{p}\mathbb{D}^-_j(p_0;\sigma)\mathrm{e}^{2\kappa_-\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}\right)(\vec{U}_{\bullet,-})_{2|p|}, \end{aligned} \tag{5.68}$$

$$\begin{aligned}\big(\vec{U}_{\bullet,+}\big)_{2(|p_0|-1)+\delta_{j1}} &= \big(\widetilde{U}_{\bullet,+}\big)_j(p_0)\\ &= \big(\widetilde{G}_{\bullet,+}\widetilde{U}_{\bullet,+}\big)_j(p_0)\\ &= \delta_{j0} + \sum_{\substack{p\in\mathscr{T}^+_{\bullet,0}\\ |p|\geq|p_0|}} \left(g_{+,\ell}(p)k_+(p)\int_p^{p_+}\mathbb{D}_j^+(p_0;\sigma)\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}\right)\big(\vec{U}_{\bullet,+}\big)_{2(|p|-1)}.\end{aligned} \tag{5.69}$$

Recall here that $r_\pm \notin \mathscr{T}^\pm_{\bullet,0}$. In other words,

$$\widetilde{\mathcal{G}}_{\bullet,\pm} = (\delta_{j\,\mathrm{even}})_{0\leq j\leq 2|\mathscr{T}^\pm_{\bullet,0}|+1} + \widetilde{\mathcal{A}}_{\bullet,\pm}$$

where the non-zero coefficients of the matrices $\widetilde{\mathcal{A}}_{\bullet,\pm}$ are given by

$$\begin{aligned}\big(\widetilde{\mathcal{A}}_{\bullet,-}\big)_{2|p_0|,2|p|} &= \left(g_{-,\ell}(p)k_-(p)\int_{p_-}^{p}\mathbb{D}_0^-(p_0;\sigma)\mathrm{e}^{2\kappa_-\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}\right)\delta_{|p|\leq|p_0|}\\ \big(\widetilde{\mathcal{A}}_{\bullet,-}\big)_{2|p_0|+1,2|p|} &= \left(g_{-,\ell}(p)k_-(p)\int_{p_-}^{p}\mathbb{D}_1^-(p_0;\sigma)\mathrm{e}^{2\kappa_-\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}\right)\delta_{|p|\leq|p_0|}\\ \big(\widetilde{\mathcal{A}}_{\bullet,+}\big)_{2(|p_0|-1),2(|p|-1)} &= \left(g_{+,\ell}(p)k_+(p)\int_{p}^{p_+}\mathbb{D}_0^+(p_0;\sigma)\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}\right)\delta_{|p|\geq|p_0|}\\ \big(\widetilde{\mathcal{A}}_{\bullet,+}\big)_{2|p_0|-1,2(|p|-1)} &= \left(g_{+,\ell}(p)k_+(p)\int_{p}^{p_+}\mathbb{D}_1^+(p_0;\sigma)\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}\right)\delta_{|p|\geq|p_0|}\end{aligned}$$

for all $p_0, p \in \mathscr{T}^\pm_{\bullet,0}$. Notice that $\widetilde{\mathcal{A}}_{\bullet,-}$ (respectively $\widetilde{\mathcal{A}}_{\bullet,+}$) is lower triangular (respectively upper triangular) and has non-zero coefficients only in the columns with even index.

We can use (5.68) and (5.69) to inductively find $\vec{U}_{\bullet,\pm}$ starting from the known boundary values. From there it is straightforward deducing $U_\pm(p)$ for all $p \in \mathscr{T}^\pm_{\bullet,0}$:

$$\begin{aligned}(U_-)_j(p) &= 2\big(\vec{U}_{\bullet,-}\big)_{2|p|+\delta_{j1}} - (U_-)_j(p_-),\\ (U_+)_j(p) &= 2\big(\vec{U}_{\bullet,+}\big)_{2(|p|-1)+\delta_{j1}} - (U_+)_j(p_+).\end{aligned}$$

Let us compute the integrals in the coefficients of $\widetilde{\mathcal{A}}_{\bullet,\pm}$. Using $\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}(\sigma) = \frac{1}{F(\sigma)}$, we find for $\omega_\pm \neq 0$:

$$\begin{aligned}\int_p^{p_+}\mathbb{D}_0^+(p_0;\sigma)\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} &= \int_p^{p_+}\left(\frac{\mathrm{e}^{2\mathrm{i}\omega_+(\phi(\sigma)-\phi(p_0))}-1}{2\mathrm{i}\omega_+}\right)\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}\\ &= \left[\frac{1}{2\mathrm{i}\omega_+}\left(\frac{\mathrm{e}^{2(\kappa_++\mathrm{i}\omega_+)\phi(\sigma)}\mathrm{e}^{-2\mathrm{i}\omega_+\phi(p_0)}}{2(\kappa_++\mathrm{i}\omega_+)} - \frac{\mathrm{e}^{2\kappa_+\phi(\sigma)}}{2\kappa_+}\right)\right]_p^{p_+},\end{aligned}$$

$$\begin{aligned}\int_{p_-}^{p}\mathbb{D}_0^-(p_0;\sigma)\mathrm{e}^{2\kappa_-\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} &= \int_{p_-}^{p}\left(\frac{\mathrm{e}^{2\mathrm{i}\omega_-(\phi(p_0)-\phi(\sigma))}-1}{2\mathrm{i}\omega_-}\right)\mathrm{e}^{2\kappa_-\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)}\\ &= \left[\frac{1}{2\mathrm{i}\omega_-}\left(\frac{\mathrm{e}^{2(\kappa_--\mathrm{i}\omega_-)\phi(\sigma)}\mathrm{e}^{2\mathrm{i}\omega_-\phi(p_0)}}{2(\kappa_--\mathrm{i}\omega_-)} - \frac{\mathrm{e}^{2\kappa_-\phi(\sigma)}}{2\kappa_-}\right)\right]_{p_-}^{p},\end{aligned}$$

$$\int_p^{p_+} \mathbb{D}_1^+(p_0;\sigma)\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} = -\int_p^{p_+} \mathrm{e}^{2\mathrm{i}\omega_+(\phi(\sigma)-\phi(p_0))}\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} = -\left[\frac{\mathrm{e}^{2(\kappa_++\mathrm{i}\omega_+)\phi(\sigma)}\mathrm{e}^{-2\mathrm{i}\omega_+\phi(p_0)}}{2(\kappa_++\mathrm{i}\omega_+)}\right]_p^{p_+},$$

$$\int_{p_-}^{p} \mathbb{D}_1^-(p_0;\sigma)\mathrm{e}^{2\kappa_-\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} = \int_{p_-}^{p} \mathrm{e}^{2\mathrm{i}\omega_-(\phi(p_0)-\phi(\sigma))}\mathrm{e}^{2\kappa_-\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} = \left[\frac{\mathrm{e}^{2(\kappa_--\mathrm{i}\omega_-)\phi(\sigma)}\mathrm{e}^{2\mathrm{i}\omega_-\phi(p_0)}}{2(\kappa_--\mathrm{i}\omega_-)}\right]_{p_-}^{p}.$$

If $\omega_\pm = 0$, then we get:

$$\begin{aligned}\int_p^{p_+} \mathbb{D}_0^+(p_0;\sigma)\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} &= \int_p^{p_+} (\phi(\sigma)-\phi(p_0))\mathrm{e}^{2\kappa_+\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} \\ &= \left[\left(\frac{\phi(\sigma)-\phi(p_0)}{2\kappa_+} - \frac{1}{4\kappa_+^2}\right)\mathrm{e}^{2\kappa_+\phi(\sigma)}\right]_p^{p_+},\end{aligned}$$

$$\begin{aligned}\int_{p_-}^{p} \mathbb{D}_0^-(p_0;\sigma)\mathrm{e}^{2\kappa_-\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} &= \int_{p_-}^{p} (\phi(p_0)-\phi(\sigma))\mathrm{e}^{2\kappa_-\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} \\ &= \left[\left(\frac{\phi(p_0)-\phi(\sigma)}{2\kappa_-} + \frac{1}{4\kappa_-^2}\right)\mathrm{e}^{2\kappa_-\phi(\sigma)}\right]_{p_-}^{p},\end{aligned}$$

$$\pm\int_p^{p_\pm} \mathbb{D}_1^\pm(p_0;\sigma)\mathrm{e}^{2\kappa_\pm\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} = \mp\int_p^{p_\pm} \mathrm{e}^{2\kappa_\pm\phi(\sigma)}\frac{\mathrm{d}\sigma}{F(\sigma)} = \mp\left[\frac{\mathrm{e}^{2\kappa_\pm\phi(\sigma)}}{2\kappa_\pm}\right]_p^{p_\pm}.$$

5.6.2 Discrete version of the argument principle

In this Appendix, we give an explicit formula for

$$\widetilde{N}_\bullet(\Gamma) := \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\widetilde{\mathscr{W}_\bullet'}(z)}{\widetilde{\mathscr{W}_\bullet}(z)}\mathrm{d}z$$

where $\Gamma \subset \mathbb{C}$ is a positively oriented contour contained no zero of $\mathscr{W}_\bullet$, say $\Gamma = [R_0, R_1] + \mathrm{i}[C_0, C_1]$ with $R_0, R_1, C_0, C_1 > 0$ and $R_0 < R_1$, $C_0 < C_1$. Recall that this formula is used in the paragraph 5.4.2 to approximate the number of resonances inside Γ.

Let us write $\Gamma = \bigcup_{\ell=1}^4 \Gamma_\ell$ where $\Gamma_1 = [R_1, R_0] + \mathrm{i}C_1$, $\Gamma_2 = R_0 + \mathrm{i}[C_1, C_0]$, $\Gamma_3 = [R_0, R_1] + \mathrm{i}C_0$ and $\Gamma_3 = R_1 + \mathrm{i}[C_0, C_1]$. Call I_ℓ the integral over Γ_ℓ:

$$I_\ell := \int_{\Gamma_\ell} \frac{\widetilde{\mathscr{W}_\bullet'}(z)}{\widetilde{\mathscr{W}_\bullet}(z)}\mathrm{d}z = \sum_{\xi\in\mathscr{T}_{\bullet,0}^\ell} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi))\int_\xi^{\xi_+} \frac{\mathrm{d}z}{\mathscr{W}_\bullet(\xi_+)(z-\xi) - \mathscr{W}_\bullet(\xi)(z-\xi_+)}.$$

Put $\mathscr{W}_\bullet(\xi_+) := u_+ + \mathrm{i}v_+$ and $\mathscr{W}_\bullet(\xi) := u + \mathrm{i}v$ with $u, v, u_+, v_+ \in \mathbb{R}$. On Γ_1, $\xi_+ = \xi - h_\bullet$; on Γ_2, $\xi_+ = \xi - \mathrm{i}h_\bullet$; on Γ_3, $\xi_+ = \xi + h_\bullet$; on Γ_4, $\xi_+ = \xi + \mathrm{i}h_\bullet$. We compute[4]:

$$
\begin{aligned}
I_1 &= \sum_{\xi\in\mathscr{T}^1_{\bullet,0}} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)) \int_0^{-h_\bullet} \frac{\mathrm{d}t}{\mathscr{W}_\bullet(\xi_+)t - \mathscr{W}_\bullet(\xi)(t+h_\bullet)} \\
&= \sum_{\xi\in\mathscr{T}^1_{\bullet,0}} \big((u_+ - u) + \mathrm{i}(v_+ - v)\big) \int_0^{-h_\bullet} \frac{\big((u_+ - u)t - uh_\bullet\big) - \mathrm{i}\big((v_+ - v)t - vh_\bullet\big)}{\big((u_+ - u)t - uh_\bullet\big)^2 + \big((v_+ - v)t - vh_\bullet\big)^2}\mathrm{d}t \\
&= \sum_{\xi\in\mathscr{T}^1_{\bullet,0}} \frac{(u_+ - u) + \mathrm{i}(v_+ - v)}{2\left((v_+ - v)^2 + (u_+ - u)^2\right)} \\
&\times \left\{ (u_+ - u)\ln\left|\frac{u_+^2 + v_+^2}{u^2 + v^2}\right| - 2(v_+ - v)\arctan\left(\frac{u_+(u - u_+) + v_+(v - v_+)}{uv_+ - u_+v}\right)\right. \\
&\qquad + 2(v_+ - v)\arctan\left(\frac{u(u - u_+) + v(v - v_+)}{uv_+ - u_+v}\right) \\
&\quad - \mathrm{i}\left[(v_+ - v)\ln\left|\frac{u_+^2 + v_+^2}{u^2 + v^2}\right| + 2(u_+ - u)\arctan\left(\frac{u_+(u - u_+) + v_+(v - v_+)}{uv_+ - u_+v}\right)\right. \\
&\qquad \left.\left. - 2(u_+ - u)\arctan\left(\frac{u(u - u_+) + v(v - v_+)}{uv_+ - u_+v}\right)\right]\right\} \\
&= \sum_{\xi\in\mathscr{T}^1_{\bullet,0}} \left\{ \ln\left(\sqrt{\frac{u_+^2 + v_+^2}{u^2 + v^2}}\right) + \mathrm{i}\left[\arctan\left(\frac{u(u - u_+) + v(v - v_+)}{uv_+ - u_+v}\right)\right.\right. \\
&\qquad \left.\left. - \arctan\left(\frac{u_+(u - u_+) + v_+(v - v_+)}{uv_+ - u_+v}\right)\right]\right\}
\end{aligned}
$$

and same computations show that

$$
\begin{aligned}
I_\ell = \sum_{\xi\in\mathscr{T}^\ell_{\bullet,0}} \left\{ \ln\left(\sqrt{\frac{u_+^2 + v_+^2}{u^2 + v^2}}\right) + \mathrm{i}\left[\arctan\left(\frac{u(u - u_+) + v(v - v_+)}{uv_+ - u_+v}\right)\right.\right. \\
\left.\left. - \arctan\left(\frac{u_+(u - u_+) + v_+(v - v_+)}{uv_+ - u_+v}\right)\right]\right\}
\end{aligned}
$$

for all $\ell \in \{1, \dots, 4\}$. We therefore obtain the following formula:

$$
\begin{aligned}
\widetilde{N}_\bullet(\Gamma) &= \frac{1}{2\pi\mathrm{i}} \sum_{\ell=1}^4 I_\ell \\
&= \frac{1}{2\pi} \sum_{\ell=1}^4 \sum_{\xi\in\mathscr{T}^\ell_{\bullet,0}} \left(\arctan\left(\frac{u(u - u_+) + v(v - v_+)}{uv_+ - u_+v}\right) - \arctan\left(\frac{u_+(u - u_+) + v_+(v - v_+)}{uv_+ - u_+v}\right)\right).
\end{aligned}
$$

[4]The computation carried out assumes that $uv_+ - u_+v \neq 0$, that is $\mathscr{W}(\xi)$ and $\mathscr{W}(\xi_+)$ are not collinear as elements of $\mathbb{R}^2$. In practice, this never happens; it is anyway possible to pick $\xi' \in]\xi, \xi_+[$ then work on the intervals $\xi' \in [\xi, \xi'[$ and $\xi' \in [\xi', \xi_+[$.

5.6.3 Approximated resonances

In this Appendix, we give an explicit formula for

$$z_{\bullet,0} := \frac{1}{2\pi \mathrm{i}} \oint_\Gamma z \frac{\widetilde{\mathscr{W}_\bullet'}(z)}{\widetilde{\mathscr{W}_\bullet}(z)} \mathrm{d}z$$

using the notations of Appendix 5.6.2. Call J_ℓ the integral over Γ_ℓ:

$$J_\ell := \int_{\Gamma_\ell} z \frac{\widetilde{\mathscr{W}_\bullet'}(z)}{\widetilde{\mathscr{W}_\bullet}(z)} \mathrm{d}z = \sum_{\xi \in \mathscr{T}_{\bullet,0}^\ell} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)) \int_\xi^{\xi_+} \frac{z \mathrm{d}z}{\mathscr{W}_\bullet(\xi_+)(z-\xi) - \mathscr{W}_\bullet(\xi)(z-\xi_+)}.$$

Put $\mathscr{W}_\bullet(\xi_+) := u_+ + \mathrm{i}v_+$ and $\mathscr{W}_\bullet(\xi) := u + \mathrm{i}v$ with $u, v, u_+, v_+ \in \mathbb{R}$. Let us write

$$J_\ell = K_\ell + L_\ell$$

where

$$\begin{aligned}
K_\ell &= \sum_{\xi \in \mathscr{T}_{\bullet,0}^\ell} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)) \int_\xi^{\xi_+} \frac{(z-\xi)\mathrm{d}z}{\mathscr{W}_\bullet(\xi_+)(z-\xi) - \mathscr{W}_\bullet(\xi)(z-\xi_+)}, \\
L_\ell &= \sum_{\xi \in \mathscr{T}_{\bullet,0}^\ell} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi))\, \xi \int_\xi^{\xi_+} \frac{\mathrm{d}z}{\mathscr{W}_\bullet(\xi_+)(z-\xi) - \mathscr{W}_\bullet(\xi)(z-\xi_+)} \\
&= \sum_{\xi \in \mathscr{T}_{\bullet,0}^\ell} \xi \left\{ \ln\left(\sqrt{\frac{u_+^2 + v_+^2}{u^2+v^2}} \right) + \mathrm{i} \left[\arctan\left(\frac{u(u-u_+) + v(v-v_+)}{uv_+ - u_+ v} \right) \right.\right. \\
&\qquad \left.\left. - \arctan\left(\frac{u_+(u-u_+) + v_+(v-v_+)}{uv_+ - u_+ v} \right) \right] \right\}.
\end{aligned}$$

The integral in L_ℓ had already been computed in Appendix 5.6.2. Put

$$K_\ell := \sum_{\xi \in \mathscr{T}_{\bullet,0}^\ell} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)) \int_\xi^{\xi_+} \frac{(z-\xi)\mathrm{d}z}{\mathscr{W}_\bullet(\xi_+)(z-\xi) - \mathscr{W}_\bullet(\xi)(z-\xi_+)}.$$

We compute:

$$
\begin{aligned}
K_1 &= \sum_{\xi\in\mathscr{T}^1_{\bullet,0}} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)) \int_0^{-h_\bullet} \frac{t\mathrm{d}t}{\mathscr{W}_\bullet(\xi_+)t - \mathscr{W}_\bullet(\xi)(t+h_\bullet)} \\
&= \sum_{\xi\in\mathscr{T}^1_{\bullet,0}} \big((u_+-u) + \mathrm{i}(v_+-v)\big) \int_0^{-h_\bullet} \frac{\big((u_+-u)t - uh_\bullet\big)t - \mathrm{i}\big((v_+-v)t - vh_\bullet\big)t}{\big((u_+-u)t - uh_\bullet\big)^2 + \big((v_+-v)t - vh_\bullet\big)^2}\mathrm{d}t \\
&= -\sum_{\xi\in\mathscr{T}^1_{\bullet,0}} \frac{(u_+-u)+\mathrm{i}(v_+-v)}{2\left((u_+-u)^2+(v_+-v)^2\right)^2} \\
&\times \Bigg\{ \big(u(v_+^2-v^2) + 2u_+v(v-v_+) - u(u_+-u)^2\big)h_\bullet \ln\left|\frac{u_+^2+v_+^2}{u^2+v^2}\right| \\
&\quad + 2(u_+-u)\left((u_+-u)^2+(v_+-v)^2\right)h_\bullet \\
&\quad - 2\big(v(u_+^2-u^2) + 2uv_+(u-u_+) - v(v_+-v)^2\big)h_\bullet \left(\arctan\left(\frac{u_+(u-u_+)+v_+(v-v_+)}{uv_+-u_+v}\right)\right. \\
&\qquad\qquad \left. - \arctan\left(\frac{u(u-u_+)+v(v-v_+)}{uv_+-u_+v}\right)\right) \\
&\quad - \mathrm{i}\Bigg[\big(v(u_+^2-u^2) + 2uv_+(u-u_+) - v(v_+-v)^2\big)h_\bullet \ln\left|\frac{u_+^2+v_+^2}{u^2+v^2}\right| \\
&\qquad + 2(v_+-v)\left((u_+-u)^2+(v_+-v)^2\right)h_\bullet \\
&\qquad + 2\big(u(v_+^2-v^2) + 2u_+v(v-v_+) - u(u_+-u)^2\big)h_\bullet \left(\arctan\left(\frac{u_+(u-u_+)+v_+(v-v_+)}{uv_+-u_+v}\right)\right. \\
&\qquad\qquad \left. - \arctan\left(\frac{u(u-u_+)+v(v-v_+)}{uv_+-u_+v}\right)\right)\Bigg]\Bigg\} \\
&= \sum_{\xi\in\mathscr{T}^1_{\bullet,0}} h_\bullet \Bigg\{ \frac{(u_+-u)-\mathrm{i}(v_+-v)}{(u_+-u)^2+(v_+-v)^2}(u+\mathrm{i}v)\ln\left(\sqrt{\frac{u_+^2+v_+^2}{u^2+v^2}}\right) - 1 \\
&\qquad - \mathrm{i}\frac{(u_+-u)-\mathrm{i}(v_+-v)}{(u_+-u)^2+(v_+-v)^2}(u+\mathrm{i}v)\left[\arctan\left(\frac{u(u-u_+)+v(v-v_+)}{uv_+-u_+v}\right)\right. \\
&\qquad\qquad \left. - \arctan\left(\frac{u_+(u-u_+)+v_+(v-v_+)}{uv_+-u_+v}\right)\right]\Bigg\}.
\end{aligned}
$$

We then deduce:

$$
\begin{aligned}
K_2 &= \sum_{\xi\in\mathscr{T}^2_{\bullet,0}} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)) \int_0^{-\mathrm{i}h_\bullet} \frac{t\mathrm{d}t}{\mathscr{W}_\bullet(\xi_+)t - \mathscr{W}_\bullet(\xi)(t+\mathrm{i}h_\bullet)} \\
&= \mathrm{i} \sum_{\xi\in\mathscr{T}^2_{\bullet,0}} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)) \int_0^{-h_\bullet} \frac{t\mathrm{d}t}{\mathscr{W}_\bullet(\xi_+)t - \mathscr{W}_\bullet(\xi)(t+h_\bullet)} \\
&= \mathrm{i} \sum_{\xi\in\mathscr{T}^2_{\bullet,0}} h_\bullet \Bigg\{ \frac{(u_+-u)-\mathrm{i}(v_+-v)}{(u_+-u)^2+(v_+-v)^2}(u+\mathrm{i}v) \ln\left(\sqrt{\frac{u_+^2+v_+^2}{u^2+v^2}}\right) - 1 \\
&\qquad - \mathrm{i}\frac{(u_+-u)-\mathrm{i}(v_+-v)}{(u_+-u)^2+(v_+-v)^2}(u+\mathrm{i}v) \left[\arctan\left(\frac{u(u-u_+)+v(v-v_+)}{uv_+-u_+v}\right)\right. \\
&\qquad\qquad \left. - \arctan\left(\frac{u_+(u-u_+)+v_+(v-v_+)}{uv_+-u_+v}\right)\right] \Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
K_3 &= \sum_{\xi\in\mathscr{T}^3_{\bullet,0}} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)) \int_0^{h_\bullet} \frac{t\mathrm{d}t}{\mathscr{W}_\bullet(\xi_+)t - \mathscr{W}_\bullet(\xi)(t-h_\bullet)} \\
&= -\sum_{\xi\in\mathscr{T}^3_{\bullet,0}} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)) \int_0^{-h_\bullet} \frac{t\mathrm{d}t}{\mathscr{W}_\bullet(\xi_+)t - \mathscr{W}_\bullet(\xi)(t+h_\bullet)} \\
&= -\sum_{\xi\in\mathscr{T}^3_{\bullet,0}} h_\bullet \Bigg\{ \frac{(u_+-u)-\mathrm{i}(v_+-v)}{(u_+-u)^2+(v_+-v)^2}(u+\mathrm{i}v) \ln\left(\sqrt{\frac{u_+^2+v_+^2}{u^2+v^2}}\right) - 1 \\
&\qquad - \mathrm{i}\frac{(u_+-u)-\mathrm{i}(v_+-v)}{(u_+-u)^2+(v_+-v)^2}(u+\mathrm{i}v) \left[\arctan\left(\frac{u(u-u_+)+v(v-v_+)}{uv_+-u_+v}\right)\right. \\
&\qquad\qquad \left. - \arctan\left(\frac{u_+(u-u_+)+v_+(v-v_+)}{uv_+-u_+v}\right)\right] \Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
K_4 &= \sum_{\xi\in\mathscr{T}^4_{\bullet,0}} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)) \int_0^{\mathrm{i}h_\bullet} \frac{t\mathrm{d}t}{\mathscr{W}_\bullet(\xi_+)t - \mathscr{W}_\bullet(\xi)(t-\mathrm{i}h_\bullet)} \\
&= -\mathrm{i} \sum_{\xi\in\mathscr{T}^4_{\bullet,0}} (\mathscr{W}_\bullet(\xi_+) - \mathscr{W}_\bullet(\xi)) \int_0^{-h_\bullet} \frac{t\mathrm{d}t}{\mathscr{W}_\bullet(\xi_+)t - \mathscr{W}_\bullet(\xi)(t+h_\bullet)} \\
&= -\mathrm{i} \sum_{\xi\in\mathscr{T}^4_{\bullet,0}} h_\bullet \Bigg\{ \frac{(u_+-u)-\mathrm{i}(v_+-v)}{(u_+-u)^2+(v_+-v)^2}(u+\mathrm{i}v) \ln\left(\sqrt{\frac{u_+^2+v_+^2}{u^2+v^2}}\right) - 1 \\
&\qquad - \mathrm{i}\frac{(u_+-u)-\mathrm{i}(v_+-v)}{(u_+-u)^2+(v_+-v)^2}(u+\mathrm{i}v) \left[\arctan\left(\frac{u(u-u_+)+v(v-v_+)}{uv_+-u_+v}\right)\right. \\
&\qquad\qquad \left. - \arctan\left(\frac{u_+(u-u_+)+v_+(v-v_+)}{uv_+-u_+v}\right)\right] \Bigg\}.
\end{aligned}
$$

It remains to evaluate:

$$z_{\bullet,0} = \frac{1}{2\pi\mathrm{i}} \sum_{\ell=1}^{4} J_\ell = \frac{1}{2\pi\mathrm{i}} \sum_{\ell=1}^{4} \Big(K_\ell + L_\ell \Big).$$

Chapter 6

Discussions and Perspectives

This final chapter gathers conclusions, comments and perspectives for the work carried out during the thesis.

Let us start by summarizing the main results obtained in the previous chapters:

- Decay of local energy for solutions of the charged Klein-Gordon equation in the exterior DSRN spacetime has been proved in Chapter 1 (*cf.* Theorem 1.3.2). It is based on the exclusion of resonances from a neighborhood of $\overline{\mathbb{C}^+}$ which in turn uses a study at low frequency (*cf.* Theorem 1.2.8) as well as a precise localization of high frequency resonances (*cf.* Theorem 1.3.1).
- A scattering theory has been constructed in Chapter 2 for this equation. In particular, asymptotic completeness has been established for different comparison dynamics (*cf.* Subsection 2.3.5). Using a Kaluza-Klein extension of the original spacetime constructed from the symbol of the charged Klein-Gordon operator, we have introduced geometric profiles related to transport along principal null geodesics. This gave us a satisfying geometric interpretation of scattering as traces on horizons (*cf.* Theorem 2.5.7). Completeness of wave operators then allowed us to solve the Goursat problem on the horizons (*cf.* Theorem 2.5.9). These geometric results do however not hold (and are unlikely) in the original spacetime because of the absence of the (q, m)-dependent geodesics therein.
- Using the Kaluza-Klein extension, we can deduce the decay of the local energy proved in Chapter 1 in the extended spacetime until and through the horizons of the black ring (*cf.* theorem 3.1.2).
- Finally, in Chapter 5, we constructed a numerical scheme based on polynomial interpolations to approximate solutions of the charged Klein-Gordon equation and localize low frequency resonances. We paid attention to the error associated to the method and obtained convergence results with rates depending on initial data regularity (*cf.* Theorems 4.3.1, 4.3.4 and 5.4.1).

6.1 Discussions

We now comment some parts of the thesis.

Decay of the local energy. The results of Chapter 1 can be certainly extended to the more general setting of the charged Klein-Gordon equation in the exterior De Sitter-Kerr-Newman spacetime for small angular momentum a and charge of the field q. The heavy machinery developed by Vasy in [Va13] is likely powerful enough to tackle down this problem. However, perturbation techniques used in this manuscript, based on (multidimensional) Fredholm theory, may be sufficient to obtain satisfying precise decay estimates. Moreover, the red-shift vector field method introduced by Dafermos [Da05] may allow us to extend exponential decay results through horizons. This result in the extended spacetime does not "project" onto the DSRN spacetime, but similar arguments (using the work of Dyatlov [Dy11]) may lead to this result; the energy-momentum tensor associated to the wave equation has to be replaced by the one associated to the Klein-Gordon equation such as in [DiNi15], the smallness of the charge product probably appearing then as a perturbation we can absorb in the positive part of the tensor.

Scattering theory. It is possible to construct a scattering theory for the charged Klein-Gordon equation in the exterior De Sitter-Kerr-Newman spacetime (with a and q small) using similar methods to the one introduced in Chapter 2. Proving the absence of real resonances is likely the only result we need to proceed. Let us give some explanations.

The De Sitter-Kerr Newman metric is given in Boyer-Lindquist coordinates (t, r, θ, φ) by

$$g := \frac{\Delta_r}{\rho^2}\left(\frac{\mathrm{d}t - a\sin^2\theta \mathrm{d}\varphi}{\lambda}\right)^2 - \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(\frac{a\mathrm{d}t - (r^2+a^2)\mathrm{d}\varphi}{\lambda}\right)^2 - \frac{\rho^2}{\Delta_r}\mathrm{d}r^2 - \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2,$$

$$\begin{aligned} g^{-1} = {} & \frac{\sigma^2\lambda^2}{\rho^2\Delta_r\Delta_\theta}\partial_t\otimes\partial_t - \frac{a\lambda^2\left(\Delta_r - (r^2+a^2)\Delta_\theta\right)}{\rho^2\Delta_r\Delta_\theta}(\partial_t\otimes\partial_\varphi + \partial_\varphi\otimes\partial_t) \\ & - \frac{\lambda^2\left(\Delta_r - a^2\Delta_\theta\sin^2\theta\right)}{\rho^2\Delta_r\Delta_\theta\sin^2\theta}\partial_\varphi\otimes\partial_\varphi - \frac{\Delta_r}{\rho^2}\partial_r\otimes\partial_r - \frac{\Delta_\theta}{\rho^2}\partial_\theta\otimes\partial_\theta \end{aligned}$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta, \qquad \lambda = 1 + \frac{\Lambda a^2}{3}, \\ \Delta_r &= \left(1 - \frac{\Lambda r^2}{3}\right)(r^2+a^2) - 2Mr + \frac{Q^2}{2}, \\ \Delta_\theta &= 1 + \frac{\Lambda a^2}{3}\cos^2\theta, \\ \sigma^2 &= (r^2+a^2)^2\Delta_\theta - a^2\Delta_r\sin^2\theta. \end{aligned}$$

If a is sufficiently small, $g > 0$ as a bilinear form define on the tangent bundle of $\mathcal{M} := \mathbb{R}_t \times]r_-, r_+[_r \times \mathbb{S}^2_{\theta,\varphi}$, where $0 < r_- < r_+ < +\infty$ are the two largest roots of Δ_r. Let

$$\omega = \frac{qQr}{\rho^2}\left(\mathrm{d}t - \frac{a\sin^2\theta}{M}\mathrm{d}\varphi\right).$$

The charge Klein-Gordon operator on $(\mathcal{M}, g)$ then reads

$$\begin{aligned} \square_g + m^2 = {} & g^{tt}(\partial_t - \mathrm{i}\omega_t)^2 + g^{\varphi\varphi}(\partial_\varphi - \mathrm{i}\omega_\varphi)^2 + 2g^{t\varphi}\left(\partial_t\partial_\varphi - \mathrm{i}\omega_\varphi\partial_t - \mathrm{i}\omega_t\partial_\varphi - \omega_t\omega_\varphi\right) \\ & + \frac{1}{\sqrt{|g|}}\partial_r\sqrt{|g|}g^{rr}\partial_r + \frac{1}{\sqrt{|g|}}\partial_\theta\sqrt{|g|}g^{\theta\theta}\partial_\theta + m^2 \end{aligned} \tag{6.1}$$

with $|g| = \frac{\rho^4 \sin^2\theta}{\lambda^4}$. The neutralization procedure now yields the extended spacetime $\widetilde{\mathcal{M}} = \mathbb{R}_t \times \mathbb{S}^1_z \times]r_-, r_+[_r \times \mathbb{S}^2_{\theta,\varphi}$ endowed with the extended metric[1]

$$\begin{aligned}\widetilde{g} &= g^{\varphi\varphi}\left(g^{tt}\omega_t^2 - m^2\right)\frac{\mathrm{d}t^2}{\Xi} + \left(g^{tt}g^{\varphi\varphi} - (g^{t\varphi})^2\right)\frac{\mathrm{d}z^2}{\Xi} + g^{tt}\left(g^{\varphi\varphi}\omega_\varphi^2 - m^2\right)\frac{\mathrm{d}\varphi^2}{\Xi}\\ &+ g^{\varphi\varphi}\left(g^{tt}\omega_t - g^{t\varphi}\omega_\varphi\right)\frac{(\mathrm{d}t\mathrm{d}z + \mathrm{d}z\mathrm{d}t)}{\Xi} + g^{tt}\left(g^{\varphi\varphi}\omega_\varphi - g^{t\varphi}\omega_t\right)\frac{(\mathrm{d}z\mathrm{d}\varphi + \mathrm{d}\varphi\mathrm{d}z)}{\Xi}\\ &+ \left(g^{tt}g^{\varphi\varphi}\omega_t\omega_\varphi - g^{tt}g^{t\varphi}\omega_t^2 - g^{t\varphi}g^{\varphi\varphi}\omega_\varphi^2 - m^2 g^{t\varphi}\right)\frac{(\mathrm{d}t\mathrm{d}\varphi + \mathrm{d}\varphi\mathrm{d}t)}{\Xi} + g_{rr}\mathrm{d}r^2 + g_{\theta\theta}\mathrm{d}\theta^2\end{aligned}$$

where

$$\Xi = -g^{t\varphi}\left(g^{tt}g^{t\varphi}\omega_t^2 - 2g^{tt}g^{\varphi\varphi}\omega_t\omega_\varphi + g^{t\varphi}g^{\varphi\varphi}\omega_\varphi^2\right) - m^2\left(g^{tt}g^{\varphi\varphi} - (g^{t\varphi})^2\right) = \frac{m^2 r^4}{F(r)^2} + \mathcal{O}_{a\to 0}(a)$$

and F is the horizon function of the DSRN metric. The wave operator in $(\widetilde{\mathcal{M}}, \widetilde{g})$ is then given by (6.1) where now ω_t and ω_φ are multiplied by ∂_z and m^2 becomes $m^2\partial_z^2$.

Introduce the extended coordinates

$$t^\star := t + T(r), \qquad z^\star := z + Z(r), \qquad \varphi^\star := \varphi + \Phi(r)$$

where T, Z, Φ are such that

$$\begin{pmatrix} T' & Z' & \Phi'\end{pmatrix}\underbrace{\begin{pmatrix}\widetilde{g}_{tt} & \widetilde{g}_{tz} & \widetilde{g}_{t\phi}\\ \widetilde{g}_{tz} & \widetilde{g}_{zz} & \widetilde{g}_{z\phi}\\ \widetilde{g}_{t\phi} & \widetilde{g}_{z\phi} & \widetilde{g}_{\phi\phi}\end{pmatrix}}_{=:A}\begin{pmatrix}T'\\ Z'\\ \Phi'\end{pmatrix} = \widetilde{g}_{rr}.$$

For a small enough, tedious computations show that the real symmetric matrix A has one positive eigenvalue and two negative ones, so that T, Z, Φ are well-defined (and uniquely determined up to an integration constant). As in Subsection 2.2.1, we can then define the principal null geodesics of the Kaluza-Klein extension.

Following [GGH17, Subsection 11.2], we can write the operator (6.1) as an abstract Klein-Gordon operator as introduced in (3). To obtain the scattering results of [GGH17], we need to localize real resonances (if there exists any) and also to fit in the geometric hypotheses (G) introduced in [GGH17, Subsection 2.1]. The only issue here is Assumptions (G2)-(G3) which ask for θ-independent limit operators as $r \to r_\pm$, which is not the case here since $\omega_\varphi(r_\pm) = \frac{aqQr_\pm}{\rho(r_\pm)^2 M}\sin^2\theta$. To overcome this difficulty, we can use a new variable y defined by

$$\omega_t + \frac{g^{t\phi}}{g^{tt}}\omega_\phi =: \frac{qQ}{y}.$$

We can check that $y = r\left(1 + \mathcal{O}_{a\to 0}(a^2)\right)$ and $\frac{\partial y}{\partial r} = 1 + \mathcal{O}_{a\to 0}(a^2)$ so that $(r,\theta) \mapsto (y(r,\theta),\theta)$ is a well-defined change of variables. Now if u solves the charged Klein-Gordon equation, then

$$v = \mathrm{e}^{-\mathrm{i}\frac{2qQ}{y_+}\left(t + \frac{\phi}{\Omega_+}\right)}u, \qquad \Omega_\pm := \lim_{r\to r_\pm}\frac{g^{t\phi}}{g^{tt}}$$

[1] It would be interested to see how Einstein-Maxwell equation has been modified in this context and particularly how the angular momentum a is involved in the new equation.

solves another charged Klein-Gordon equation with $\tilde{\omega}_t := \omega_t - \frac{qQ}{2y_+}$ and $\tilde{\omega}_\varphi := \omega_\varphi - \frac{qQ}{2y_+\Omega_+}$; the limit operators at $r_\pm$ are this time θ-independent. Though computations are tougher than the DSRN case, it may be then possible to follow Chapter 2 to construct a scattering theory in this context.

Numerical approximations. There are many rooms of improvement in Chapter 4 and Chapter 5.

First of all, the error of approximation of a solution u in Theorems 4.3.1 and 4.3.4 depends on the local H^2 norm of the solution u. This is not a problem since $u \in \dot{\mathcal{H}}^2_\rho \subset H^2_{\ell\mathrm{oc}}(I, \mathbb{C})$, but such an error blows up as we approach the boundary of the interval I. A careful look at the different proofs of the error estimate makes us observe that we can replace the H^2 norm by the more natural $\dot{\mathcal{H}}^2_\rho$ norm (which does not worsen estimates near the boundary of I). This is certainly achievable by not splitting into two different parts the estimates for $(a(r)\partial_r^2 + b(r)\partial_r)u$.

The rate of convergence for general data could also be improved using a weak topology. Indeed, the local energy is measured using a cut-off χ; giving some derivatives of u to χ may increase the rate of decay of the error (that is, the exponent α of $h^\alpha_\bullet$). Modifications to carry out in the proof of the stability of the scheme (*cf.* Proposition 4.2.14) do however not seem straightforward. Besides, this may make impossible the use of the $\dot{\mathcal{H}}^2_\rho$ norm for the error of approximation, obliging us to stay far away from the boundary of I.

We should be able to add the spherical Laplacian $-\Delta_{\mathbb{S}^2}$ in the second order operator P of the abstract problem introduced in Section 4.1. As an example, we can consider

$$P := \partial_r(a(r)\partial_r) - b(r)\Delta_{\mathbb{S}^2} + c(r)$$

with $a < 0$, $b > 0$ and $c \geq 0$ almost everywhere. More generally, numerical approximation of solutions of the charged Klein-Gordon equation on De Sitter-Kerr(-Newman) metric is within easy reach as it only requires adding the angular derivative ∂_θ.

Finally, concerning the localization and approximation of resonances, the lake of analyticity of Jost solutions beyond the line $\{z \in \mathbb{C} \mid \Im(z) > -\kappa\}$ is the major difficulty when we wish do some numerical computations. We emphasize here that this issue has nothing to do with the scheme. The problem comes from the fact that Jost solutions are defined by an implicit integral, so that only Grönwall's inequality allowed us to bound them together with their derivatives. Furthermore, for the sake of readability, we drastically simplified the error of approximation associated to the scheme (and still many constants were needed to write the error, *cf.* Subsection 5.3.1). As a result, the error has been overestimated. Notice that the black hole and scalar field parameters M, Q, Λ, q, m play an important role in the estimation of the error: some combinations yield better estimates than others. The conclusion is that a numerical optimization seems to be needed in order to bring the error estimate back to decent reaches.

6.2 Perspectives

As explained in the introduction of this manuscript, the aim of the present work is the construction of the Unruh state satisfying the Haddamard property in the exterior DSRN spacetime. The strategy is first constructing it on the horizons, then sending it inside the Carter-Penrose diagram via the scattering channel and finally extending its existence beyond the event horizon. Such

a construction for Dirac equation in De Sitter-Kerr spacetime is in progress in [GHW]. In our context, the scattering properties of the charged Klein-Gordon equation for small charge product Chapter 1 and Chapter 2 are a prerequisite to proceed. Having shown satisfying approximation results, the method developed in Chapter 4 could also be helpful from the point of view of the quantum field theory.

All along this manuscript, we investigated several problems assuming the charge product s small as the general case seems out of reach for now. This allowed in many places to use perturbation arguments from the well-known situation of the Klein-Gordon equation on the De Sitter-Schwarzschild metric. In particular, exponential decay of the local energy is made possible thanks to the mass $m \gtrsim |s|$ which repels the resonance 0 of the wave equation into $\mathbb{C}^-$. However, perturbation of the wave equation on the De Sitter-Schwarzschild metric for s, m small enough reveals that growing modes exist when the charge product is greater to the mass term. This time the resonance 0 of the wave equation moves to $\mathbb{C}^+$. This work [BeHa] is still in preparation. This confirms the non-decay of the local energy we observed in Chapter 4 (*cf.* Subsection 4.3.2); this also confirms the numerical results in [CCDHJb] for small charge product s and mass of the field m. Knowing the full trajectory of this resonance 0 in function of s is a very difficult problem which will certainly not be overcome by simple perturbation arguments. The charged black hole background still has some interesting questions in supsense...

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