



Asymptotic analysis of some partial differential equations in domains becoming unbounded

Adrien Ceccaldi

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THÈSE

Pour obtenir le diplôme de doctorat

Spécialité Mathématiques

Préparée au sein de l'Université de Rouen Normandie

Analyse asymptotique de quelques problèmes d'EDP dans des domaines devenant infinis

Présentée et soutenue par
Adrien CECALDI

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M. Michel Marie CHIPOT	Professeur, Université de Zurich	Rapporteur
M. Grigory PANASENKO	Professeur, Université Jean-Monnet-Saint-Étienne	Rapporteur
Mme Renata BUNOIU-SCHILTZ	Maître de Conférences, Université de Lorraine	Examineur
M. Olivier GUIBÉ	Maître de Conférences, Université de Rouen Normandie	Examineur
Mme Patrizia DONATO	Professeur, Université de Rouen Normandie	Directeur de thèse
M. Sorin MARDARE	Maître de Conférences, Université de Rouen Normandie	Codirecteur de thèse

Thèse dirigée par Patrizia DONATO (Directeur) et Sorin MARDARE (Codirecteur),
Laboratoire de Mathématiques Raphaël Salem, UMR-CNRS 6085, Université de Rouen Normandie



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Abstract

This thesis has for aim the study of some elliptic problems in some domains becoming unbounded in one or several directions.

In the first part of the thesis, we study the problem

$$\begin{cases} -\operatorname{div}(A\nabla u_\ell) = f & \text{in } \Omega_\ell \\ u_\ell = g & \text{on } \partial\Omega_\ell, \end{cases}$$

where Ω_ℓ is the cylinder $\ell\omega_1 \times \omega_2$ with ω_1 and ω_2 two bounded domains of \mathbb{R}^k and \mathbb{R}^{n-k} respectively (with $1 \leq k \leq n-1$). We denote by Ω_∞ the infinite cylinder $\mathbb{R}^k \times \omega_2$ and we take $f \in H_{loc}^{-1}(\bar{\Omega}_\infty)$ and $g \in H_{loc}^1(\bar{\Omega}_\infty)$, so that $f \in H^{-1}(\Omega_\ell)$ and $g \in H^1(\Omega_\ell)$ for any $\ell > 0$. This work is based on the methods developed in [23] and [19]. We show that it is possible to indifferently pass to the limit in the sequence of cylinders and then to solve the problem on the infinite cylinder, or to first solve the problem on the cylinder Ω_ℓ and then to pass to the limit. The limit here is to be understood in the sense of a Saint-Venant type principle, which is to say that the convergence takes place for the restrictions of u_ℓ to smaller domains $\Omega_{\ell'}$ (with $0 < \ell' < \ell$) contained in Ω_ℓ . After that, we give some optimality results concerning the domain in which the sequence of solutions u_ℓ converges to u_∞ .

In the second chapter, we construct some correctors that enable us to extend the convergence on the whole cylinder. The construction of these correctors is inspired from the ones made in [17] and [18].

In the third chapter of the thesis we prove that, under some decreasing conditions at infinity for the data f , it is possible to recover the same convergence on the whole cylinder, without the adjunction of correctors.

In the last part of the thesis, we study the Stokes problem

$$\begin{cases} -\mu\Delta u_\ell + \nabla p_\ell = f & \text{in } \Omega_\ell \\ \operatorname{div} u = 0 & \text{in } \Omega_\ell \\ u = 0 & \text{on } \partial\Omega_\ell \end{cases}$$

on a domain $\Omega_\ell = B_\ell \times \omega$, where $B_\ell \subset \mathbb{R}^k$ ($1 \leq k \leq n-1$) is the ball of radius

ℓ centered at the origin. Here, an important hypothesis is that f has some radial invariant properties with respect to the first k coordinates. One of the major tools in the proof of the result of this chapter (concerning especially the case $k \geq 2$) is a result of the divergence-problem type. More precisely, based on a construction inspired by [9], we prove the following result (here below, $D_\ell = \Omega_{\ell+1} \setminus \overline{\Omega}_\ell$):

If $g \in W^{1,p}(D_\ell)$ ($1 < p < \infty$) is a radial function along X_1 such that $\int_{D_\ell} g = 0$ and that $g = 0$ on $(B_{\ell+1} \setminus \overline{B}_\ell) \times \partial\omega$, then there exists $u \in (W_0^{1,p}(D_\ell))^n$ such that:

$$\begin{cases} \operatorname{div} u = g & \text{in } D_\ell \\ \|\nabla u\|_{L^p(D_\ell)} \leq C(\|g\|_{L^p(D_\ell)} + \|\nabla_{X_2} g\|_{L^p(D_\ell)}), \end{cases}$$

the constant C being independent of ℓ for $\ell \geq 1$ (C depends on k, n, p and ω).

Thanks to this result, used in the case $p = 2$, we finally prove that in this case, we also have an exponential rate of convergence of $u_{\ell|\Omega_{\frac{\ell}{2}}}$ to u_∞ .

Keywords: Cylinders, Long domains, Asymptotic analysis, Stokes problem, correctors.

Résumé

Cette thèse a pour objet l'étude de quelques problèmes elliptiques dans des domaines qui deviennent infinis dans une ou plusieurs directions.

Dans la première partie de la thèse, nous étudions le problème

$$\begin{cases} -\operatorname{div}(A\nabla u_\ell) = f & \text{dans } \Omega_\ell \\ u_\ell = g & \text{sur } \partial\Omega_\ell, \end{cases}$$

où Ω_ℓ est le cylindre $\ell\omega_1 \times \omega_2$, avec ω_1 et ω_2 deux domaines bornés de \mathbb{R}^k et \mathbb{R}^{n-k} respectivement (avec $1 \leq k \leq n-1$). On note Ω_∞ le cylindre infini $\mathbb{R}^k \times \omega_2$, et l'on prend $f \in H_{loc}^{-1}(\bar{\Omega}_\infty)$ et $g \in H_{loc}^1(\bar{\Omega}_\infty)$, de sorte que $f \in H^{-1}(\Omega_\ell)$ et $g \in H^1(\Omega_\ell)$ pour tout $\ell > 0$. Ce travail se fonde sur les méthodes développées dans [23] et [19]. On démontre qu'il est possible de passer indifféremment à la limite dans la suite de cylindres puis de résoudre le problème, ou de d'abord résoudre le problème sur le cylindre Ω_ℓ puis de passer à la limite. Ici, la limite doit être comprise dans le sens d'un principe de Saint-Venant, c'est-à-dire que la convergence a lieu pour des restrictions de u_ℓ à des cylindres $\Omega_{\ell'}$ plus petits (avec $0 < \ell' < \ell$). Dans la suite de ce travail sont présentés des résultats sur l'optimalité du domaine dans lequel la suite de solutions u_ℓ converge vers u_∞ .

Dans le deuxième chapitre, nous montrons comment il est possible d'étendre le domaine de convergence à l'ensemble du cylindre Ω_ℓ par la construction de correcteurs. La construction de ces correcteurs se fonde sur celles présentées dans [17] et [18].

Dans la troisième partie de la thèse, nous prouvons que sous des hypothèses de décroissance à l'infini de la donnée f , il est possible de retrouver la même convergence sur l'entière du cylindre, sans passer par le biais de correcteurs.

La dernière partie de cette thèse est consacrée à l'étude du problème de Stokes

$$\begin{cases} -\mu\Delta u_\ell + \nabla p_\ell = f & \text{dans } \Omega_\ell \\ \operatorname{div} u = 0 & \text{dans } \Omega_\ell \\ u = 0 & \text{sur } \partial\Omega_\ell \end{cases}$$

sur le domaine $\Omega_\ell = B_\ell \times \omega$, où $B_\ell \subset \mathbb{R}^k$ est la boule de rayon ℓ centrée à l'origine. Ici, une hypothèse primordiale est celle qui concerne les propriétés d'invariance radiale de f par rapport aux k premières coordonnées. L'un des outils principaux dans la preuve du résultat de cette partie (en particulier dans le cas où $k \geq 2$) est un résultat sur un problème de divergence. Plus précisément, en se fondant sur une construction inspirée par [9], nous montrons le résultat suivant (ci-dessous, $D_\ell = \Omega_{\ell+1} \setminus \overline{\Omega}_\ell$) :

Soit $g \in W^{1,p}(D_\ell)$ une fonction radiale par rapport aux k premières coordonnées, et qui vérifie $g = 0$ sur $(B_{\ell+1} \setminus \overline{B}_\ell) \times \partial\omega$ et $\int_{D_\ell} g \, dx = 0$. Alors, il existe $u \in (W_0^{1,p}(D_\ell))^n$ telle que

$$\begin{cases} \operatorname{div} u = g \text{ in } D_\ell \\ \|\nabla u\|_{L^p(D_\ell)} \leq C(\|g\|_{L^p(D_\ell)} + \|\nabla_{X_2} g\|_{L^p(D_\ell)}), \end{cases}$$

avec C une constante qui, pourvu que $\ell \geq 1$, dépend seulement de k, n, p et ω .

À l'aide de ce résultat, appliqué au cas $p = 2$, nous démontrons finalement que dans ce cas aussi, il y a convergence exponentielle de $u_\ell|_{\Omega_{\frac{\ell}{2}}}$ vers u_∞ .

Mots clés : Cylindres, Domaines longs, Analyse asymptotique, Problème de Stokes, Correcteurs.

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Introduction

The aim of this thesis is to study the behaviour of some elliptic problems in domains becoming unbounded in one or several directions. In order to do so, we consider a sequence of domains $\Omega_\ell \subset \mathbb{R}^n$, constructed as follows:

$$\Omega_\ell = \ell\omega_1 \times \omega_2,$$

with ℓ a real number, $\omega_1 \subset \mathbb{R}^k$ with $1 \leq k < n$ and $\omega_2 \subset \mathbb{R}^{n-k}$. Along the thesis, ω_1 varies from an interval of \mathbb{R} to a ball of \mathbb{R}^k centered at the origin, passing by a subdomain of \mathbb{R}^k star-shaped with respect to a ball centered at the origin. To each of these domains Ω_ℓ corresponds a solution u_ℓ of a given elliptic problem. This leads to the construction of a sequence of solutions u_ℓ . Furthermore, we define the infinite cylinder Ω_∞ as

$$\Omega_\infty = \mathbb{R}^k \times \omega_2,$$

and we show that the limit of the solution u_ℓ (as ℓ goes to ∞) is the solution u_∞ of the elliptic problem on the infinite cylinder Ω_∞ .

The convergence results are very intuitive, since they can be seen as the commutativity of two processes. More precisely, the limit u_∞ is obtained by first solving the problem in the cylinder Ω_ℓ and then get the limit by making ℓ go to infinity in the sequence of solutions u_ℓ . But u_∞ is also the solution of the same type of problem in the infinite cylinder Ω_∞ , which is in fact the limit of the bounded domains Ω_ℓ as ℓ goes to infinity.

Let us now be more specific regarding the sense in which the convergence of u_ℓ takes place. The convergence that we consider describes in fact a Saint-Venant-type principle, meaning that u_∞ is a good approximation of u_ℓ as long as we are far enough from the “lateral boundary” $\partial(\ell\omega_1) \times \omega_2$. More specifically, under very weak growth assumptions (at infinity) on f , we prove results of the type $\|u_\ell - u_\infty\|_{H^1(\Omega_{\ell'})} \rightarrow 0$ for some $\ell' < \ell$. The minimal case that presents interest here is the one where ℓ' is a constant ($\ell' = \ell_0$), but as

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we will see in the first chapter, one can also consider ℓ' depending of ℓ , as for instance $\ell' = \frac{\ell}{2}$.

The approach and the techniques used in this thesis are inspired by the ones developed by Michel Chipot and his collaborators, see for instance [13]-[23]. However, we would like to mention here the earlier work of O.A. Oleinik and G.A. Yosifian (see [55]) concerning the classical Saint-Venant principle in elasticity.

With respect to the asymptotic analysis, one important remark here is that, at least intuitively, this study can be related to the study of PDE problems in thin domains. Indeed, in order to do this, it suffices to make the change of variables $y = \frac{1}{\ell}x$, the domain Ω_ℓ becoming then a thin domain Ω_ε , with $\varepsilon = \frac{1}{\ell}$. An important literature related to this topic was produced. We mention here only a few of these works, starting with some papers written almost 60 years ago, by M.G. Dzhavadov, who in [32] studied the behaviour of an elliptic operator in a thin rectangle, by A.L. Goldenveizer, who in [38]-[40] considered the elasticity problem for thin structures, giving in this way a justification of the two-dimensional models used for plates and shells. Later, P.G. Ciarlet and P. Destuynder (see [25]) also justified the plate model by means of asymptotical analysis.

More recently, G.P. Panasenko considered in [57] the stationary Navier-Stokes equations in a network of thin cylinders. Note that this paper contains very fine results, since an asymptotic expansion of the solution is provided. Later, G.P. Panasenko and K. Pileckas considered the non-steady Navier-Stokes system in a tube structure (see [63] and [64]). The asymptotic behaviour of a biharmonic problem in a thin multi-structure is studied in [36] by A. Gaudiello, G.P. Panasenko and A. Piatnitski. The study of Stokes and Navier-Stokes problems in domains with outlets is also the subject of several books, such as [34], [35] or [51].

The interested reader can also consult the papers [29] and [58], that are also treating subjects similar to the ones described here above.

Also related to the study of the behaviour of the solutions of PDE problems in thin domains, the method of asymptotic partial decomposition of a domain is an important tool developed by Grigory P. Panasenko (see, e.g., [59]-[61]). This method allows to give a good approximation on the whole domain by separating it in two parts, one (the main part) where a reduction of the dimension can be made, and a small part where the full dimension is preserved, this part being the one where the boundary layers are located.

As a proof of the large amount of work that have been done on the topic

of the asymptotic analysis in thin domains, many books have been written gathering the results of these works. We only mention here several of these monographs, such as [5], [24], [26], [28], [45], [51].

Let us now come back to the approach adopted in this thesis and make some comments with respect to the parallels that can be made between the two approaches (long domains and thin structures). First of all, notice that when doing the change of variables $y = \frac{1}{\ell}x$, we transform our problems - where the parameter ℓ intervenes only in the domain Ω_ℓ - in a problem in thin domains, where the parameter ε intervenes three times, since now the operator, the domain and the data (the applied forces) depend on ε . But most importantly, with the possible exception of the third chapter, the general hypotheses on f are not transformed into hypotheses that are adequate to the kind of problems in thin domains as treated in the works cited here above. Also, from the point of view of the reduction of dimension - which is treated here in the first two chapters - the boundary conditions are not of the same type. We want to emphasize here that, in our case, the reduction of dimension is made exactly with respect to the complementary directions as the ones usually considered for the thin domains. Another problem is that, because of the change of variables formula, a good approximation - in a Sobolev norm - for the solution to the problem in the thin domain does not necessary translate into a good approximation in Ω_ℓ , unless the rate of convergence is high enough.

However, under some specific hypotheses on the data, the connection between the two kinds of problems could be tighter. This could be a very interesting theme for future researches. This is not the topic of this thesis, our approach being the following: keeping in mind that what we are looking for is to prove a Saint-Venant-type principle, what can we say with respect to the most obvious limit u_∞ , that is the solution of the same problem in the infinite domain? We prove that not only this natural limit is the good one, but moreover, the rate of convergence of u_ℓ towards u_∞ (on some smaller domain $\Omega_{\ell'} \subset \Omega_\ell$) is very high, being in many cases exponential.

Before giving a more detailed presentation of the kind of results that we obtain, let us point out another related problem. We refer here to the PDE anisotropic singular perturbation problems in a fixed, bounded domain. Indeed, making the change of variables $y = (Y_1, Y_2) = (\frac{1}{\ell}X_1, X_2)$ (where $x = (X_1, X_2) \in \Omega_\ell$, with $X_1 = (x_1, \dots, x_k) \in \ell\omega_1$ and $X_2 = (x_{k+1}, \dots, x_n) \in \omega_2$) transforms the variable domain Ω_ℓ into the fixed domain $\Omega = \omega_1 \times \omega_2$. The

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price to pay is that now the operator and the applied forces f will depend on $\varepsilon = \frac{1}{\ell}$. However, this dependence only takes place with respect to the first k coordinates, hence the anisotropic character of the problem. This type of problems have also been studied by Michel Chipot and his collaborators, see for instance [4], [14] and [42].

Let us now give some more specific details on the type of problems considered in this text. We study two different kinds of problems. First, we consider the equation

$$-\operatorname{div}(A\nabla u) = f.$$

To start with, by using some techniques developed by M. Chipot, S. Mardare and K. Yeressian, we prove the following convergence result

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq Ce^{-\alpha\ell},$$

for the case of nonhomogeneous Dirichlet boundary conditions. For homogeneous Dirichlet conditions, this estimate was proved in [23].

Once proved this inequality, we study the optimality of the domain in which this convergence holds by making use of some theoretical computations, as well as some specific examples. Indeed, we prove it is not possible to have a convergence on the whole cylinder Ω_ℓ even if it is possible to get a domain closer to Ω_ℓ than $\Omega_{\frac{\ell}{2}}$.

Then, we consider again this problem, but this time we focus our study on the construction of some correctors. These correctors have two aims.

The first one is to correct the limit u_∞ with a suitable corrector that provides a good estimate of u_ℓ on the whole cylinder Ω_ℓ . Therefore, adding a well chosen corrector w_ℓ enables us to have a convergence of the type

$$\|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_\ell)} \rightarrow 0.$$

Even better, we obtain the same exponential rate of convergence as for u_ℓ to u_∞ on $\Omega_{\frac{\ell}{2}}$ mentioned above.

The second aim is to give another proof of the optimality for the domain of convergence in the case studied by M. Chipot and K. Yeressian.

In the third chapter, we find another case where still the convergence on the whole cylinder holds: assuming that the data f satisfies some decreasing

properties at the infinity, and more specifically that $f \in H^{-1}(\mathbb{R}^k \times \omega_2)$, we prove that the convergence of u_ℓ to u_∞ takes place once again in the whole cylinder, the speed being the slowest between the exponential and the decreasing speed of f at infinity.

We would like to notice here that one of the main ingredients in proving the results of the third chapter is the study of the solution u_∞ at infinity. This kind of study is an important domain of research that has a great interest on its own. An extensive literature on this topic was written, in particular by the Russian school. A pioneering work was done in 1963 by E.M. Landis (see [48]) for second order elliptic equations. In 1977, E.M. Landis and G.P. Panasenko studied the behaviour of solutions to elliptic equations with coefficients which are periodic with respect to all variables except one (see [49]). In 1982, O.A. Oleinik and G.A. Yosifian described in [56] the asymptotic behaviour at infinity for the solutions of the linear elasticity problem. For the Stokes and Navier-Stokes problems, the same kind of study was realized by numerous authors such as L.V. Kapitanskii [44], O.A. Ladyzhenskaya and V.A. Solonnikov [47], S.A. Nazarov and K. Pileckas (see, e.g., [54], [65],[66]).

However, the objectives of our study are not exactly the same as in the papers cited here above. We are interested in determining the rate of decay versus zero (as the space variable goes to infinity) of the solution, under the hypothesis that f satisfies a similar decay property at infinity. Also, the tools used for proving it are similar to the ones used in the previous chapters. In particular, we do not make use of weighted Sobolev spaces, as in many of the works cited above.

The second equation, studied in the last part of this thesis, is another classical elliptic problem: the Stokes problem. In a domain of type $B_\ell \times \omega_2$, where $B_\ell \subset \mathbb{R}^k$ is a ball of radius ℓ centered at the origin, we do the same study of the possibility to swap the two limit processes. Here we adapt the methods used for the previous problem, but we encounter some additional difficulties. This is due to the fact that we need to solve a problem of the form

$$\begin{cases} \operatorname{div} u = g \text{ in } D_\ell \\ \|\nabla u\|_{L^p(D_\ell)} \leq C_p(\|g\|_{L^p(D_\ell)} + \|\nabla_{X_2} g\|_{L^p(D_\ell)}), \end{cases}$$

which represents the main theoretical aspect of the study. Thanks to the result we prove for this divergence problem and to some results about radial

functions that we prove along the paper, we finally can make a good use of the two methods from [23] and [19] to get the same exponential convergence as in the previous papers.

We would like to emphasize here that the novelty with respect to the previous works - dedicated to the Stokes problem in long cylinders - arises in the case $k \geq 2$. Indeed, concerning the case $k = 1$, M. Specovius-Neugebauer [67] proved the convergence of u_ℓ towards u_∞ on $\Omega_{\ell-\varepsilon}$ (for any $\varepsilon > 0$), with an exponential rate, provided that the applied forces f satisfy a similar property. Also in the case $k = 1$, M. Chipot and S. Mardare [19] proved a Saint-Venant type result under the same general growth assumptions on f than the ones made in the forth chapter of this thesis.

This thesis is divided in four chapters as follows:

- Chapter 1: Elliptic problems in long cylinders revisited.
- Chapter 2: Correctors to elliptic problems in long cylinders.
- Chapter 3: Asymptotic analysis of some elliptic problems in long cylinders with data decreasing at infinity.
- Chapter 4: Asymptotic analysis for the Stokes problem in domains becoming unbounded in several directions.

In the remaining part of this introduction we give a brief overview of our work, introducing the topic and the problem for each chapter. We also present a short review of some related works, and describe our contribution to each problem.

0.1 Elliptic problems in cylinders

In the first chapter of the thesis, we study the very classical problem

$$-\operatorname{div}(A\nabla u) = f$$

on some cylinder Ω_ℓ . The study of this problem has been widely developed, since the first article

- Chipot, M.; Rougirel, A., *On the asymptotic behaviour of the solution of parabolic problems in cylindrical domains of large size in some directions*, Discrete Contin. Dyn. Syst. Ser. B 1 (2001), no. 3, 319-338.

There are two articles that greatly influenced the evolution of this field of research:

- Chipot, M.; Yeressian, K., *Exponential rates of convergence by an iteration technique*, C. R. Acad. Sci. Paris, Ser. I 346 (2008), 21-26.
- Chipot, M.; Mardare, S., *Asymptotic behaviour of the Stokes problem in cylinders becoming unbounded in one direction*, J. Math. Pure et Appliquées, Vol. 90, Issue 2 (2008), 133-159.

The article of 2001 gives a primary convergence result, for some parabolic problems of the same kind: for a fixed ℓ_0 , u_ℓ converges to u_∞ on the cylinder Ω_{ℓ_0} as ℓ goes to ∞ .

The two articles published in 2008 are more directly related to the work we present in the first chapter. In the first one, under some assumptions that we describe in the next section, the authors prove the following convergence:

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \rightarrow 0 \text{ as } \ell \rightarrow +\infty.$$

This is what is now considered to be the classical expression of this kind of convergence. Even more important is a technique introduced in this article and which is nowadays very usual approach.

It consists in an iteration technique: we control the norm of $\nabla(u_\ell - u_\infty)$ on any cylinder Ω_{ℓ_1} ($\ell_1 \leq \ell - 1$) by its norm on the cylinder Ω_{ℓ_1+1} , the quotient between the two being less than 1. Then, we iterate this inequality a suitable amount of times, in order to get an inequality of the type

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq Ce^{-\alpha\ell}.$$

The second article, although it concerns the Stokes problem, provides a general method used in many following works. The idea developed there is to consider the sequence of solutions (u_ℓ) as a "Cauchy sequence". The use of this method in the first chapter, allows us to get rid of the restrictive hypotheses on f and A made by M. Chipot and K. Yeressian in their article.

Finally, in 2014, M. Chipot proved the same convergence in the homogeneous Dirichlet problem in the laplacian case (A being the identity matrix) with some general hypotheses, and with a general matrix field A in the book "*Asymptotic Issues for Some Partial Differential Equations*" in 2016.

Our contribution

In this work, we study the following non homogeneous Dirichlet problem in the cylinder Ω_ℓ :

$$\begin{cases} -\operatorname{div}(A\nabla u_\ell) = f & \text{in } \Omega_\ell \\ u_\ell = g & \text{on } \partial\Omega_\ell, \end{cases}$$

where $f \in H_{loc}^{-1}(\bar{\Omega}_\infty)$ and $g \in H_{loc}^1(\bar{\Omega}_\infty)$, so that $f \in H^{-1}(\Omega_\ell)$ and $g \in H^1(\Omega_\ell)$ for any $\ell > 0$.

The first different point here with respect to the results quoted above is that the Dirichlet condition we consider is nonhomogeneous. In order to deal with this condition, we make a good use of the two techniques previously introduced.

Furthermore, there are a few improvements that we describe below: first of all (see also *Asymptotic Issues for Some Partial Differential Equations* by M. Chipot in 2016) we get rid of the hypothesis made in [23] that some part of the data do not depend on X_1 . Indeed, in [23] the authors suppose that f depends only on X_2 and that the matrix A , decomposed as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

verifies that $A_{12}(x) = A_{12}(X_2)$ and $A_{22}(x) = A_{22}(X_2)$. We also make a slight improvement on the hypothesis concerning the domain ω_1 by allowing it to be only star-shaped (instead of convex) with respect to an open ball centered at the origin. Furthermore, we prove (see Section 1.2) that the result of M. Chipot and K. Yeressian can then be derived as a particular case of the one presented in this chapter.

In the last part of the chapter, we provide some optimality results on the domain in which u_ℓ converges to u_∞ . These results, given in the case of homogeneous boundary conditions, detail some conditions on the parameters for the convergence to take place. After that, we present some examples where we retrieve the theoretical parameters for optimality computed before.

0.2 Correctors for elliptic problems in cylinders

The topic of correctors for the kind of problems in which we are interested have been very few explored in the literature. More precisely, to our best knowledge, there are only two works on this subject:

- Chipot, M.; Mardare, S., *On correctors for the Stokes problem in cylinders*, Proceedings of the conference on nonlinear phenomena with energy dissipation, Chiba, November 2007, Gakuto International Series, Mathematical Sciences and Applications, Vol. 29, Gakkotosho (2008), 37-52.
- Chipot, M.; Guesmia, S., *Correctors for some asymptotic problems*, Proceedings of the Steklov Institute of Mathematics, Vol 270, (2010), 263-277.

There are mainly two motivations for the work presented in this second chapter. The first one is the search of a way to extend the domain in which the convergence from the Chapter 1 takes place. Indeed, as shown in the first chapter, the convergence result is classically given on $\Omega_{\frac{\ell}{2}}$, but can be obtained on any $\Omega_{\gamma\ell}$ for $0 \leq \gamma < 1$, in the case where f has polynomial growth at infinity. Nevertheless, it is not possible in general to directly obtain a convergence result on the whole cylinder. Therefore, one of the main aims of this chapter is to construct a corrector, that is to say a function w_ℓ in $H^1(\Omega_\ell)$ such that, added to u_∞ we get a good approximation of u_ℓ . More specifically, we look for a function w_ℓ such that

$$\|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_\ell)} \rightarrow 0$$

with the best rate of convergence we can find. Actually, we prove that it is possible to have the same exponential rate of convergence as in the results given in the first chapter.

The end of the chapter is devoted to the second motivation, that is to prove the optimality of the result given by [23]. Here, it is important to note that the optimality is to be understood in a quite unusual way: we do not study the optimality of the convergence speed, but instead we look for the largest domain in which the convergence takes place. Therefore, this justifies

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the use of correctors since we prove that it is not possible, under the general conditions of this chapter, to get a convergence of u_ℓ to u_∞ on the whole cylinder.

Our contribution

The work we have done in the second chapter is essentially a generalisation of the article by M. Chipot and S. Guesmia cited in the first part of this section. Nevertheless, this generalisation is far from being straightforward. Indeed, in their article the authors are considering the Laplacian (that is, the matrix A is the identity matrix of \mathbb{M}^n , the space of square matrices of order n), which has some convenient symmetry properties, used along the article. Furthermore, they assume the data f to depend only on X_2 . Under these assumptions, they construct a corrector $w_\ell \in H^1(\Omega_\ell)$ such that

$$\|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_\ell)} \leq Ce^{-\alpha\ell}.$$

Let us point out that their result is self-contained, which is to say that it can be obtained independently of some convergence results such as the one from [23] or from the previous chapter. Even more, the convergence result of [23] can be retrieved from this result.

Here, we work with the problem

$$\begin{cases} -\operatorname{div}(A\nabla u_\ell) = f & \text{in } \Omega_\ell \\ u_\ell = 0 & \text{on } \partial\Omega_\ell, \end{cases}$$

with f a $H_{loc}^{-1}(\bar{\Omega}_\infty)$ -distribution satisfying, for some constant $\beta > 0$,

$$\|f\|_{H^{-1}(\Omega_\ell)} \leq Ce^{\beta\ell} \quad \text{for all } \ell > 0,$$

and $A = (a_{ij})_{1 \leq i,j \leq n} \in L^\infty(\Omega_\infty; \mathcal{M}_n(\mathbb{R}))$ a matrix field satisfying the properties:

$$\begin{aligned} \lambda|\xi|^2 &\leq A(x)\xi \cdot \xi \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega_\infty \\ |A(x)\xi| &\leq \Lambda|\xi| \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega_\infty. \end{aligned}$$

The difference with the first chapter is that we take the specific case $k = 1$ (with $\omega_1 = (-1, 1)$ and $\omega_2 = \omega \subset \mathbb{R}^{n-1}$), meaning that the domain goes to infinity in only one direction.

Since our assumptions are weaker than those in [17], as the general elliptic operator does not verify the same symmetry properties as the Laplacian, we cannot use the construction done in [17].

Instead, the corrector is constructed separately on the right-hand side and the left-hand side of the cylinder Ω_ℓ . This is why the corrector we construct is in $H^1(\Omega_\ell \setminus (\{0\} \times \omega))$ and not in $H^1(\Omega_\ell)$. However, we show (see Section 2.1) that this corrector allows to very easily construct a new corrector belonging to $H^1(\Omega_\ell)$ and having the same convergence properties with respect to u_ℓ .

In the second part of the chapter, we assume the same conditions on the data as in [23]. Under these conditions, we show that for every ℓ , the right and left correctors w_ℓ^+ and w_ℓ^- can be obtained as a translation of the same functions w^+ and w^- defined on $(-\infty, 0) \times \omega$ and $(0, +\infty) \times \omega$, respectively. We then can use the properties of w^+ and w^- to prove the optimality of the result from [23] (the optimality being understood in the sense given in the first part of this section). More precisely, we prove that in general, for a constant $a > 0$, we have that

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-a})} \not\rightarrow 0.$$

0.3 Elliptic problems in cylinders with data decreasing at infinity

This smaller chapter comes in the same spirit as the previous one. More specifically, the aim of this part remains the extension of the domain of convergence of u_ℓ to u_∞ . Nevertheless, this time instead of the adjunction of a new correcting function, we achieve this goal by putting some stronger hypothesis on the data f . More precisely, as pointed out in a remark of the second chapter, we reach the convergence result

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_\ell)} \rightarrow 0$$

by assuming that $f \in H^{-1}(\mathbb{R}^k \times \omega_2)$. We would like to emphasize here the fact that in this chapter, we go back to the general setting concerning the domains Ω_ℓ , i.e. that they become unbounded in several directions, not only in one direction as in the second chapter. Another important difference with the previous chapter is that this time, we do not necessarily have the same

exponential rate of convergence as we usually do, but here the speed is the slowest between the exponential one and the decreasing speed of f to 0 at infinity. Finally, in the very last part of this chapter we prove (see Section 3.2) that in the case where $k = 1$, the same result can be obtained more directly by making a good use of some results proved along the second chapter.

0.4 The Stokes problem in domains becoming unbounded in several directions

In this last chapter, we study a different elliptic problem, the Stokes problem

$$\begin{cases} -\mu\Delta u_\ell + \nabla p_\ell = f & \text{in } \Omega_\ell \\ \operatorname{div} u = 0 & \text{in } \Omega_\ell \\ u = 0 & \text{on } \partial\Omega_\ell \end{cases}$$

on a domain $\Omega_\ell = B_\ell \times \omega$, where $B_\ell \subset \mathbb{R}^k$ is a ball of radius ℓ centered at the origin.

This work is very different from the others presented in the rest of the thesis. Indeed, the difficulties to overcome here do not come from a direct application of the methods described above, but they come from some side problems arising when trying to make the computations.

In a more precise way, the two principal tools that we use in order to overcome the difficulties - generated by the fact that k might be greater than 1 - are a particular form of the divergence problem described below, and some results about functions which are radial with respect to the first k coordinates.

Our contribution

In this chapter, we do not necessarily suppose that the domain becomes unbounded in only one direction, but instead we allow it to go to infinity in several directions.

In the case where $k = 1$, it has already been proved in [19] that for some positive constants a and α , the solution (u_ℓ, p_ℓ) of the Stokes problem in Ω_ℓ

satisfies the estimate

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} + \|\nabla(p_\ell - p_\infty)\|_{\dot{L}^2(\Omega_{\frac{\ell}{2}})} \leq \alpha e^{-a\ell} \|f\|_{L^2(Q)},$$

where $Q = (0, 1) \times \omega$ and $\hat{L}^2(\Omega_{\frac{\ell}{2}}) = L^2(\Omega_{\frac{\ell}{2}})/\mathbb{R}$.

One of the key ingredients in the proof of the main result of this chapter (concerning the case $k \geq 1$) is a result of the divergence-problem type. Let us remind here the classical divergence problem:

If $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain and $f \in L^2(\Omega)$ is such that $\int_{\Omega} f \, dx = 0$, then there exists $u \in (H_0^1(\Omega))^n$ such that

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega \\ \|\nabla u\|_{L^2(\Omega)} \leq C \|f\|_{L^2(\Omega)} \end{cases}$$

with C independent of f .

In our case, we need to prove a similar result in a domain $D_\ell = \Omega_{\ell+1} \setminus \overline{\Omega}_\ell = (B_{\ell+1} \setminus \overline{B}_\ell) \times \omega$ (where $B_\ell = B(0, \ell) \subset \mathbb{R}^k$ and $\omega \subset \mathbb{R}^{n-k}$), but with a constant C that does not depend on ℓ .

Actually, we obtain the following result:

If $g \in W^{1,p}(D_\ell)$ ($1 < p < \infty$) is a radial function along X_1 such that $\int_{D_\ell} g = 0$ and that $g = 0$ on $(B_{\ell+1} \setminus \overline{B}_\ell) \times \partial\omega$, then there exists $u \in (W_0^{1,p}(D_\ell))^n$ such that:

$$\begin{cases} \operatorname{div} u = g & \text{in } D_\ell \\ \|\nabla u\|_{L^p(D_\ell)} \leq C(\|g\|_{L^p(D_\ell)} + \|\nabla_{X_2} g\|_{L^p(D_\ell)}), \end{cases}$$

the constant C being independent of ℓ for $\ell \geq 1$ (C depends on k, n, p and ω).

In the proof of the main result of the chapter, we use this result in the specific case of $p = 2$. A first important remark is that here, we control the H^1 -norm of ∇u by the H^1 -norm of g and not as usually by the L^2 -norm of g . Nevertheless, this weaker inequality still remains sufficient as a key tool in proving the main result of this chapter concerning the asymptotic properties,

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i.e. the one of u_ℓ and p_ℓ as ℓ goes to infinity.

The second remark concerns the radiality assumption on g . Indeed, without this assumption, the constant C appearing in the inequality $\|\nabla u\|_{H^1(D_\ell)} \leq C\|g\|_{H^1(D_\ell)}$ depends on ℓ , i.e. the constant C is in fact a C_ℓ . Hence, in order to obtain a constant independent of ℓ , we make the additional assumption that g is radial with respect to $X_1 = (x_1, \dots, x_k)$. Then, we prove that it is possible to construct a vector field u on D_ℓ satisfying the properties described above, and in particular the inequality involving a constant C that is independent on ℓ . The construction of the vector field u is inspired by the one made by Bourgain and Brezis in [9], for the divergence problem in hypercubes of \mathbb{R}^n .

Chapter 1

Elliptic problems in long cylinders revisited

The asymptotic study of partial differential equations in cylinders becoming unbounded in one or several directions has known important developments in the last years, especially thanks to the works of Michel Chipot and his collaborators, see for example [13]-[23], [41] and [69]. In this paper, we prove the convergence to the solution of a linear elliptic problem on an infinite cylinder of the solutions of the same problem taken on larger and larger truncations of the cylinder. Following the methods introduced in [23] and [19], we generalize the result of convergence found in [23] for the case where the data is not necessarily independent of the coordinate along the axis of the cylinder. We also consider the non-homogenous Dirichlet problem instead of the homogenous one.

1.1 Introduction and Notation

On a generalized cylinder $\Omega_\ell = \ell\omega_1 \times \omega_2 \subset \mathbb{R}^n$, where $\omega_1 \subset \mathbb{R}^k$ ($1 \leq k \leq n-1$) and $\omega_2 \subset \mathbb{R}^{n-k}$ are bounded open sets, we consider a linear elliptic problem with non-homogenous Dirichlet boundary conditions. The data is given by two functions f and g defined on the infinite cylinder $\mathbb{R}^k \times \omega_2$. Under very weak assumptions on f and g , we can prove that the restriction to a bounded subset of $\mathbb{R}^k \times \omega_2$ of the solution u_ℓ of the Dirichlet problem in Ω_ℓ converges, as ℓ goes to infinity, to the solution of a similar Dirichlet problem on the

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infinite cylinder. Moreover, the rate of this convergence is exponential. We would like to precise here that the asymptotic analysis for the homogeneous Dirichlet problem is presented in [15], under a stronger assumption on the domain ω_1 .

Let us now give the notation that will be used in this chapter.

The notation $|\cdot|$ stands for the Euclidean norm in \mathbb{R}^k . For $x \in \mathbb{R}^k$ and $r > 0$, we use the notation $B(x, r)$ for the open ball of \mathbb{R}^k , centered at x of radius r :

$$B(x, r) = \{y \in \mathbb{R}^k ; |y - x| < r\}.$$

We denote by $|A|$ the k -dimensional Lebesgue measure of a measurable subset A of \mathbb{R}^k .

If U is an unbounded open set of \mathbb{R}^n , we define

$$\begin{aligned} L_{loc}^2(\bar{U}) &= \{v \in L_{loc}^2(U) \mid v \in L^2(\mathcal{O}), \text{ for any bounded open set } \mathcal{O} \subset U\}, \\ H_{loc}^1(\bar{U}) &= \{v \in H_{loc}^1(U) \mid v \in H^1(\mathcal{O}), \text{ for any bounded open set } \mathcal{O} \subset U\}, \\ H_{loc}^{-1}(\bar{U}) &= \{v \in H_{loc}^{-1}(U) \mid v \in H^{-1}(\mathcal{O}), \text{ for any bounded open set } \mathcal{O} \subset U\}. \end{aligned}$$

If \mathcal{O} is an open subset of \mathbb{R}^n and $v \in H_0^1(\mathcal{O})$, then by convention, we will also denote by v its extension by 0 outside of \mathcal{O} . Note that in this case, we have $v \in H_0^1(U)$ for any open set $U \supset \mathcal{O}$.

For a bounded Lipschitz domain $\mathcal{O} \subset \mathbb{R}^k$ ($1 \leq k \leq n$), we denote by $\gamma(v)$ the trace on $\partial\mathcal{O}$ of the function $v \in H^1(\mathcal{O})$.

Let k be an integer such that $1 \leq k \leq n-1$ and ω_1 be a bounded domain (i.e., open and connected) of \mathbb{R}^k , verifying

$$\omega_1 \text{ is star-shaped with respect to an open ball of } \mathbb{R}^k \text{ centered at } 0. \quad (1.1.1)$$

Note that in particular, any bounded open convex set containing 0 satisfies the property (1.1.1).

Let ω_2 be a bounded Lipschitz domain of \mathbb{R}^{n-k} . We set

$$\Omega_\ell = \ell\omega_1 \times \omega_2, \quad \Omega_\infty = \mathbb{R}^k \times \omega_2.$$

For a point $x \in \mathbb{R}$, we set

$$X_1 = (x_1, \dots, x_k), \quad X_2 = (x_{k+1}, \dots, x_n),$$

hence we can write $x = (X_1, X_2)$, and we use the notation

$$\nabla_{X_1} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right), \quad \nabla_{X_2} = \left(\frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_n} \right).$$

The hypothesis of ω_1 satisfying the property (1.1.1) is fundamental in the proof of the main result of this paper, since this property allows us to construct a function

$$\rho = \rho(X_1) \in W^{1,\infty}(\mathbb{R}^p \times \omega_2)$$

such that

$$0 \leq \rho \leq 1, \quad \rho = 1 \text{ on } \ell\omega_1 \times \omega_2, \quad \rho = 0 \text{ on } (\mathbb{R}^k \setminus (\ell+1)\omega_1) \times \omega_2,$$

and $|\nabla_{X_1}\rho| \leq c_0$ in $\mathbb{R}^k \times \omega_2$, with c_0 a constant depending only on ω_1 . The construction of such functions ρ is possible thanks to the following result:

Lemma 1.1.1. *Let $\omega_1 \subsetneq \mathbb{R}^k$ an open set which is star-shaped with respect to the open ball $B_\delta = B(0, \delta) = \{x \in \mathbb{R}^k ; |x| < \delta\}$. Then for all $\ell > 0$, one has*

$$\text{dist}(\ell\omega_1, \mathbb{R}^k \setminus (\ell+1)\omega_1) \geq \delta. \quad (1.1.2)$$

Proof. Proving inequality (1.1.2) is equivalent to proving the inclusions

$$B(X_1, \delta) \subset (\ell+1)\omega_1 \quad \text{for all } X_1 \in \ell\omega_1, \quad (1.1.3)$$

since (1.1.3) implies $|X_1 - Y_1| \geq \delta$ for all $X_1 \in \ell\omega_1$ and all $Y_1 \in \mathbb{R}^k \setminus (\ell+1)\omega_1$.

So let $X_1 \in \ell\omega_1$ be fixed. Thus, $X_1 = \ell\tilde{X}_1$ for some $\tilde{X}_1 \in \omega_1$. Then for any $Y_1 \in B(X_1, \delta)$, we have $Y_1 = (\ell+1)\frac{1}{\ell+1}Y_1$ and

$$\frac{1}{\ell+1}Y_1 = \frac{1}{\ell+1}(Y_1 - \ell\tilde{X}_1 + \ell\tilde{X}_1) = \frac{1}{\ell+1}(Y_1 - \ell\tilde{X}_1) + \frac{\ell}{\ell+1}\tilde{X}_1.$$

But $|Y_1 - \ell\tilde{X}_1| = |Y_1 - X_1| < \delta$, hence $Y_1 - \ell\tilde{X}_1 \in B_\delta$. Consequently, since $\tilde{X}_1 \in \omega_1$ and ω_1 is star-shaped with respect to the ball B_δ , we deduce that $\frac{1}{\ell+1}Y_1 \in \omega_1$, which implies $Y_1 \in (\ell+1)\omega_1$ for any $Y_1 \in B(X_1, \delta)$. The argument above being valid for any $X_1 \in \ell\omega_1$, this ends the proof of the inclusion (1.1.3). \square

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Note that the condition of ω_1 being star-shaped with respect to the origin is necessary in order to have the inclusion $\ell_1\omega_1 \subset \ell_2\omega_1$ for $\ell_1 \leq \ell_2$ and the hypothesis (1.1.1) is the minimal assumption that insures

$$\overline{\ell_1\omega_1} \subset \ell_2\omega_1 \quad \text{if} \quad \ell_1 < \ell_2. \quad (1.1.4)$$

Indeed, if ω_1 is only star-shaped with respect to a convex set of dimension $k-1$, (1.1.4) may no longer be satisfied. Note also that the assumption that the origin belongs to the open set ω_1 implies $\mathbb{R}^k = \bigcup_{\ell>0} \ell\omega_1$, thus $\bigcup_{\ell>0} \Omega_\ell = \Omega_\infty$.

Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a field of $n \times n$ matrices defined on $\mathbb{R}^k \times \omega_2$, such that $a_{ij} \in L^\infty(\mathbb{R}^k \times \omega_2)$ for all $i, j \in \{1, \dots, n\}$, and for some constants $\lambda > 0$ and $\Lambda > 0$,

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \mathbb{R}^k \times \omega_2, \quad (1.1.5)$$

$$|A(x)\xi| \leq \Lambda|\xi| \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \mathbb{R}^k \times \omega_2. \quad (1.1.6)$$

For $\beta \in \mathbb{R}$, we define

$$\begin{aligned} V_\beta(\Omega_\infty) &= \{f \in L^2_{loc}(\bar{\Omega}_\infty) \mid \exists C_0 \geq 0 \text{ such that } \|f\|_{L^2(\Omega_\ell)} \leq C_0 e^{\beta\ell} \quad \forall \ell > 0\}, \\ W_\beta(\Omega_\infty) &= \{f \in H^{-1}_{loc}(\bar{\Omega}_\infty) \mid \exists \tilde{C}_0 \geq 0 \text{ such that } \|f\|_{H^{-1}(\Omega_\ell)} \leq \tilde{C}_0 e^{\beta\ell} \quad \forall \ell > 0\}. \end{aligned}$$

For any $\ell > 0$, we consider the following non-homogenous Dirichlet problem in Ω_ℓ :

$$\begin{cases} -\operatorname{div}(A\nabla u_\ell) = f & \text{in } \Omega_\ell \\ u_\ell = g & \text{on } \partial\Omega_\ell, \end{cases} \quad (1.1.7)$$

where $f \in H^{-1}_{loc}(\bar{\Omega}_\infty)$ and $g \in H^1_{loc}(\bar{\Omega}_\infty)$, so that $f \in H^{-1}(\Omega_\ell)$ and $g \in H^1(\Omega_\ell)$ for any $\ell > 0$.

In particular, the fact that $f \in H^{-1}_{loc}(\bar{\Omega}_\infty)$ implies that $\langle f, v \rangle$ is defined for any $v \in \bigcup_{\ell>0} H^1_0(\Omega_\ell)$. This remark is useful when defining the problem in the infinite cylinder Ω_∞ (see Section 1.2).

For the simplicity of the presentation, we consider here a function g defined on the whole infinite cylinder Ω_∞ in order to define the boundary condition in problem (1.1.7) for any $\ell > 0$. However, the results of this paper are easily adapted to the case where we only consider $g \in H^{\frac{1}{2}}_{loc}(\partial\Omega_\infty)$ and where the boundary condition in (1.1.7) is replaced by

$$u_\ell = g_\ell \text{ on } \partial\Omega_\ell,$$

for some functions $g_\ell \in H^{\frac{1}{2}}(\partial\Omega_\ell)$ satisfying

$$g_\ell = g \text{ on } \partial\Omega_\ell \cap \partial\Omega_\infty \text{ for all } \ell > 0.$$

Thanks to the Lax-Milgram theorem, there exists a unique weak solution to problem (1.1.7), i.e. there exists a unique solution $u_\ell \in H^1(\Omega_\ell)$ to the variational problem:

$$\begin{cases} \int_{\Omega_\ell} A \nabla u_\ell \cdot \nabla v \, dx = \langle f, v \rangle & \text{for all } v \in H_0^1(\Omega_\ell), \\ u_\ell = g & \text{on } \partial\Omega_\ell. \end{cases} \quad (1.1.8)$$

Notice that the last equality is to be taken in the sense of the trace theory in $H^1(\Omega_\ell)$, i.e. u_ℓ satisfies $\gamma(u_\ell) = g$ on $\partial\Omega_\ell$. The aim of this paper is to study the behaviour of u_ℓ as ℓ goes to infinity.

Let us first recall a classical result that we prove below, for the sake of completeness. In its proof, we use a technique that have been used for example by Morrey in [53].

Lemma 1.1.2. *Let $D_1 \subset \mathbb{R}^k$ be a bounded Lipschitz domain and $v \in H^1(D_1 \times \omega_2)$ such that $v = 0$ on $D_1 \times \partial\omega_2$. Then, for almost every X_1 in D_1 , we have:*

$$v(X_1, \cdot) \in H_0^1(\omega_2) \text{ and } [\nabla_{X_2} v](X_1, \cdot) = \nabla_{X_2}[v(X_1, \cdot)]. \quad (1.1.9)$$

Proof. Since D_1 and ω_2 are bounded Lipschitz domains in \mathbb{R}^k , respectively \mathbb{R}^{n-k} , their Cartesian product $D_1 \times \omega_2$ is a bounded Lipschitz domain in \mathbb{R}^n . Then there exists a sequence $\{v_k\} \subset C^1(\overline{D_1 \times \omega_2})$ such that $v_k \rightarrow v$ in $H^1(D_1 \times \omega_2)$ (see, e.g, [2]). Therefore,

$$\int_{D_1 \times \omega_2} (|v_k - v|^2 + |\nabla(v_k - v)|^2) \, dx \xrightarrow{k \rightarrow +\infty} 0.$$

Using the Fubini theorem, we can write

$$\int_{D_1} \left(\int_{\omega_2} (|v_k - v|^2 + |\nabla(v_k - v)|^2) \, dX_2 \right) dX_1 \xrightarrow{k \rightarrow +\infty} 0.$$

It follows that, up to a subsequence,

$$\int_{\omega_2} (|(v_k - v)(X_1, \cdot)|^2 + |\nabla(v_k - v)(X_1, \cdot)|^2) \, dX_2 \xrightarrow{k \rightarrow +\infty} 0 \quad \text{a.e. } X_1 \in D_1$$

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and more particularly

$$\int_{\omega_2} (|(v_k - v)(X_1, \cdot)|^2 + |\nabla_{X_2}(v_k - v)(X_1, \cdot)|^2) dX_2 \xrightarrow{k \rightarrow +\infty} 0 \quad \text{a.e. } X_1 \in D_1.$$

This last convergence implies that for almost every $X_1 \in D_1$, $v_k(X_1, \cdot) \rightarrow v(X_1, \cdot)$ in $H^1(\omega_2)$ and $\nabla_{X_2}[v(X_1, \cdot)] = [\nabla_{X_2} v](X_1, \cdot)$, since this equality is valid for every $v_k \in C^1(\overline{D_1 \times \omega_2})$. By definition of the trace on $\partial(D_1 \times \omega_2)$, we have that $v_k|_{\partial(D_1 \times \omega_2)} \rightarrow \gamma(v)$ in $L^2(\partial(D_1 \times \omega_2))$, so particularly in $L^2(D_1 \times \partial\omega_2)$. By hypothesis, $\gamma(v) = 0$ on $D_1 \times \partial\omega_2$. Thus,

$$\int_{D_1 \times \partial\omega_2} v_k^2(X_1, \cdot) d\sigma_x = \int_{D_1} \left(\int_{\partial\omega_2} v_k^2(X_1, \cdot) d\sigma_{X_2} \right) dX_1 \rightarrow 0$$

We deduce that, up to a subsequence,

$$\int_{\partial\omega_2} v_k^2(X_1, \cdot) d\sigma_{X_2} \rightarrow 0 \quad \text{a.e. } X_1 \in D_1.$$

This implies that $v_k(X_1, \cdot)|_{\partial\omega_2} \rightarrow 0$ in $L^2(\partial\omega_2)$ for a.e. $X_1 \in D_1$. Since we know that $v_k(X_1, \cdot) \rightarrow v(X_1, \cdot)$ in $H^1(\omega_2)$ for a.e. $X_1 \in D_1$, we have that for almost every $X_1 \in D_1$, $\gamma(v(X_1, \cdot)) = 0$ in the sense of the trace theory in $H^1(\omega_2)$. Since ω_2 is Lipschitz, this finally implies that $v(X_1, \cdot) \in H_0^1(\omega_2)$ for a.e. $X_1 \in D_1$. \square

The previous lemma helps us to establish the following Poincaré inequality:

Lemma 1.1.3. *Let $v \in H^1(\Omega_\ell)$ such that $v = 0$ on $\ell\omega_1 \times \partial\omega_2$ and $\tilde{\omega}_1 \subset \ell\omega_1$ a measurable set. Then there exists a constant c_{ω_2} depending only on ω_2 such that:*

$$\|v\|_{L^2(\tilde{\omega}_1 \times \omega_2)} \leq c_{\omega_2} \|\nabla_{X_2} v\|_{L^2(\tilde{\omega}_1 \times \omega_2)}. \quad (1.1.10)$$

Proof. Since ω_1 is a bounded domain satisfying (1.1.1), it is also a bounded Lipschitz domain (see, e.g., Teorema 1.I in [33]) and the same is true for $\ell\omega_1$. Therefore, thanks to Lemma 1.1.2, we know that for almost every $X_1 \in \ell\omega_1$, $v(X_1, \cdot) \in H_0^1(\omega_2)$ and $[\nabla_{X_2} v](X_1, \cdot) = \nabla_{X_2}[v(X_1, \cdot)]$, which allows us to apply the usual Poincaré inequality on $H_0^1(\omega_2)$ to $v(X_1, \cdot)$, to get

$$\int_{\omega_2} v^2(X_1, X_2) dX_2 \leq c_{\omega_2}^2 \int_{\omega_2} |\nabla_{X_2} v(X_1, \cdot)|^2 dX_2,$$

where $c_{\omega_2}^2$ is a constant depending only on ω_2 . To obtain (1.1.10), it is enough to integrate the previous inequality in X_1 over $\tilde{\omega}_1$. \square

1.2 The main result

As announced in the Introduction, the aim of this paper is to prove that the solutions u_ℓ defined in the previous section converge, in a sense that will be precised in the next theorem, to the solution of a non-homogeneous Dirichlet problem in the infinite cylinder Ω_∞ . This problem is introduced in the statement of our main result here below.

Theorem 1.2.1. *Let $f \in W_\beta(\Omega_\infty)$ and $g \in H_{loc}^1(\bar{\Omega}_\infty)$ such that $\nabla g \in (V_\beta(\Omega_\infty))^n$ for some small enough $\beta > 0$ and let $u_\ell \in H^1(\Omega_\ell)$ be the solution of the variational problem (1.1.8). Then for all $\ell_0 > 0$,*

$$u_\ell \rightarrow u_\infty \text{ strongly in } H^1(\Omega_{\ell_0})$$

as $\ell \rightarrow \infty$, where $u_\infty \in H_{loc}^1(\bar{\Omega}_\infty)$ is the weak solution to the following non-homogenous Dirichlet problem in the cylinder Ω_∞ :

$$\begin{cases} -\operatorname{div}(A\nabla u_\infty) = f & \text{in } \Omega_\infty \\ u_\infty = g & \text{on } \partial\Omega_\infty \\ \|\nabla u_\infty\|_{L^2(\Omega_\ell)} \leq C_\infty e^{2\beta\ell} & \forall \ell > 0, \end{cases} \quad (1.2.1)$$

with $C_\infty \geq 0$ not depending on ℓ .

Furthermore, we have the estimate

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-\alpha\ell} \quad \text{for all } \ell > 0,$$

where $C \geq 0$ and $\alpha > 0$ are constants depending only on $\omega_1, \omega_2, \lambda, \Lambda, C_0, \tilde{C}_0$ (the constants appearing in the definitions of $V_\beta(\Omega_\infty)$ and $W_\beta(\Omega_\infty)$) and β .

Let first notice that the variational formulation corresponding to the first equation of problem (1.2.1) is

$$\int_{\Omega_\infty} A\nabla u_\infty \cdot \nabla v \, dx = \langle f, v \rangle \quad \text{for all } v \in \bigcup_{\ell > 0} H_0^1(\Omega_\ell) \quad (1.2.2)$$

and the inequality satisfied by ∇u_∞ in (1.2.1) can be reformulated as

$$\nabla u_\infty \in (V_{2\beta}(\Omega_\infty))^n.$$

Notice also that, thanks to the Poincaré inequality (1.1.10), if $f \in V_\beta(\Omega_\infty)$, then $f \in W_\beta(\Omega_\infty)$. More generally, any linear combination of functions and

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partial derivatives (in the distributional sense) of functions in $V_\beta(\Omega_\infty)$ belongs to $W_\beta(\Omega_\infty)$.

In the case where the boundary condition in problem (1.1.8) is replaced by $u_\ell = g_\ell$ on $\partial\Omega_\ell$ (with $g_\ell \in H^{\frac{1}{2}}(\partial\Omega_\ell)$ satisfying $g_\ell = g$ on $\partial\Omega_\ell \cap \partial\Omega_\infty$), the condition $\nabla g \in (V_\beta(\Omega_\infty))^n$ should be replaced by

$$\|g_\ell\|_{H^{\frac{1}{2}}(\partial\Omega_\ell)} \leq C_0 e^{\beta\ell} \text{ for all } \ell > 0,$$

where the norm on $H^{\frac{1}{2}}(\partial\Omega_\ell)$ is given by

$$\|w\|_{H^{\frac{1}{2}}(\partial\Omega_\ell)} = \inf\{\|h\|_{H^1(\Omega_\ell)} ; h \in H^1(\Omega_\ell) \text{ and } \gamma(h) = w \text{ on } \partial\Omega_\ell\}.$$

Note however, that in the statement of Theorem 1.2.1, one does not need to have an estimate for the whole H^1 -norm of g , but only for the L^2 -norm of its gradient.

Finally, an important remark is that the existence and uniqueness of the solution u_∞ to problem (1.2.1) is a result of the theorem. We would like to emphasize the fact that a growth condition at infinity for u_∞ is necessary in order to have uniqueness of the solution to problem (1.2.1). To see that, it is enough to consider the particular case of the Laplacian (where the field A is constant and equal to the identity matrix), $k = 1$, $f = 0$ and $g = 0$. Then the functions 0 and $(X_1, X_2) \mapsto e^{\sqrt{\mu}X_1}\psi(X_2)$ are both solutions to the problem given by the first two equations of (1.2.1), where $-\mu < 0$ is an eigenvalue and $\psi \in H_0^1(\omega_2)$ is an eigenfunction (associated to $-\mu$) of the Laplacian on ω_2 .

Proof of Theorem 1.2.1. The proof of the theorem is divided into six steps. The first three ones will show that u_ℓ is a Cauchy “sequence” for the norm of $H^1(\Omega_{\ell_0})$ for all $\ell_0 > 0$, then we prove that the limit of (u_ℓ) is the solution to the problem (1.2.1) and finally we prove the uniqueness of the solution to this problem.

Step I. *There exists a constant $a \in (0, 1)$ only depending on ω_1 , ω_2 , λ and Λ such that*

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq a \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx$$

for all $\ell > 1$, for all $\ell_1 \leq \ell - 1$ and for all $r \geq 0$.

In this step of the proof, we consider $\ell > 1$ and $r \geq 0$ fixed.

Let $v \in H_0^1(\Omega_\ell)$. In particular $v \in H_0^1(\Omega_{\ell+r})$, using the convention we made regarding the extensions of functions in $H_0^1(\Omega_\ell)$. Then we can use v as a test function for the variational equations satisfied by u_ℓ and $u_{\ell+r}$. Subtracting the two equations, we derive

$$\int_{\Omega_\ell} A \nabla(u_\ell - u_{\ell+r}) \cdot \nabla v \, dx = 0 \quad \text{for all } v \in H_0^1(\Omega_\ell), \quad (1.2.3)$$

given that v and ∇v are zero on $\Omega_{\ell+r} \setminus \Omega_\ell$. Taking a good test function $v \in H_0^1(\Omega_\ell)$ in equation (1.2.3) allows us to prove the estimate in (I). In what follows, we describe the construction of such a test function.

Let $\ell_1 \in \mathbb{R}$ be such that $0 < \ell_1 \leq \ell - 1$. We define a function $\rho \in W^{1,\infty}(\mathbb{R}^k \times \omega_2)$ only depending on X_1 such that $0 \leq \rho \leq 1$, $\rho = 1$ on Ω_{ℓ_1} , $\rho = 0$ on $(\mathbb{R}^k \times \omega_2) \setminus \Omega_{\ell_1+1}$, and $|\nabla_{X_1} \rho| \leq c_0$ in Ω_ℓ , with c_0 a constant depending only on ω_1 , and therefore independent of ℓ_1 or ℓ . Since $\ell_1 + 1 \leq \ell$, we have

$$\rho = 0 \quad \text{on} \quad \partial(\ell\omega_1) \times \bar{\omega}_2. \quad (1.2.4)$$

Note that this construction is possible thanks to Lemma 1.1.1.

On the other hand, $\partial\Omega_\ell = (\partial(\ell\omega_1) \times \bar{\omega}_2) \cup (\ell\omega_1 \times \partial\omega_2)$ and, since $u_\ell = g$ on $\partial\Omega_\ell$ and $u_{\ell+r} = g$ on $\partial\Omega_{\ell+r}$, we derive

$$u_\ell - u_{\ell+r} = 0 \quad \text{on} \quad \ell\omega_1 \times \partial\omega_2, \quad (1.2.5)$$

given that $\ell\omega_1 \times \partial\omega_2 \subset (\ell+r)\omega_1 \times \partial\omega_2 \subset \partial\Omega_{\ell+r}$. Therefore, it is possible to take $v = \rho(u_\ell - u_{\ell+r}) \in H_0^1(\Omega_\ell)$ (since Ω_ℓ is Lipschitz and $\rho(u_\ell - u_{\ell+r}) = 0$ on $\partial\Omega_\ell$, thanks to (1.2.4) and (1.2.5)) as a test function in the equation (1.2.3).

Observing that $\nabla_{X_1} \rho = 0$ in $\Omega_{\ell_1} \cup (\Omega_\ell \setminus \Omega_{\ell_1+1})$, we have

$$\begin{aligned} & \int_{\Omega_\ell} A \nabla(u_\ell - u_{\ell+r}) \cdot \rho \nabla(u_\ell - u_{\ell+r}) \, dx \\ &= - \int_{\Omega_\ell} A \nabla(u_\ell - u_{\ell+r}) \cdot \begin{pmatrix} \nabla_{X_1} \rho \\ 0 \end{pmatrix} (u_\ell - u_{\ell+r}) \, dx \\ &\leq \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |A \nabla(u_\ell - u_{\ell+r})| |\nabla_{X_1} \rho| |u_\ell - u_{\ell+r}| \, dx. \end{aligned}$$

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Using the Cauchy-Schwartz inequality and properties (1.1.5)-(1.1.6) satisfied by A , this leads to

$$\begin{aligned} \lambda \int_{\Omega_\ell} \rho |\nabla(u_\ell - u_{\ell+r})|^2 dx &\leq \int_{\Omega_\ell} \rho A \nabla(u_\ell - u_{\ell+r}) \cdot \nabla(u_\ell - u_{\ell+r}) dx \\ &\leq c_0 \Lambda \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})| |u_\ell - u_{\ell+r}| dx \\ &\leq c_0 \Lambda \|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\ell_1+1} \setminus \Omega_{\ell_1})} \|u_\ell - u_{\ell+r}\|_{L^2(\Omega_{\ell_1+1} \setminus \Omega_{\ell_1})}. \end{aligned}$$

Then we use the following Poincaré inequality:

$$\|(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\ell_1+1} \setminus \Omega_{\ell_1})} \leq c_{\omega_2} \|\nabla_{X_2}(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\ell_1+1} \setminus \Omega_{\ell_1})},$$

which is a consequence of Lemma 1.1.3, since $u_\ell - u_{\ell+r} \in H^1(\Omega_\ell)$,

$$\Omega_{\ell_1+1} \setminus \Omega_{\ell_1} = ((\ell_1 + 1)\omega_1 \setminus \ell_1\omega_1) \times \omega_2, \quad (\ell_1 + 1)\omega_1 \setminus \ell_1\omega_1 = \tilde{\omega}_1 \subset \ell\omega_1$$

and $u_\ell - u_{\ell+r} = 0$ on $\ell\omega_1 \times \partial\omega_2$.

Consequently,

$$\begin{aligned} \lambda \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx &\leq c_0 \Lambda c_{\omega_2} \int_{\Omega_{\ell_1+1} \setminus \Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \\ &= c_0 \Lambda c_{\omega_2} \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \\ &\quad - c_0 \Lambda c_{\omega_2} \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx, \end{aligned}$$

which is

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq \frac{C}{1+C} \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx,$$

with $C = c_0 c_{\omega_2} \frac{\Lambda}{\lambda}$. This is exactly the inequality we were looking for, with $a = \frac{C}{1+C} \in (0, 1)$.

Step II. *There exists constants $C \geq 0$ and $\alpha > 0$, depending only on $\omega_1, \omega_2, \lambda, \Lambda, C_0, \tilde{C}_0$ and β , such that*

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-\alpha \ell} \quad \text{for all } \ell > 0 \text{ and all } r \in [0, 1]. \quad (1.2.6)$$

Let $\ell > 0$ and $r \in [0, 1]$. Starting with $\ell_1 = \frac{\ell}{2}$, we iterate the inequality

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq a \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx$$

$[\frac{\ell}{2}]$ times, where $[\frac{\ell}{2}]$ is the integer part of $\frac{\ell}{2}$. As a result, we have

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq a^{[\frac{\ell}{2}]} \int_{\Omega_{\frac{\ell}{2}+[\frac{\ell}{2}]}} |\nabla(u_\ell - u_{\ell+r})|^2 dx.$$

Noticing that

$$\frac{\ell}{2} - 1 < [\frac{\ell}{2}] \leq \frac{\ell}{2} \quad \text{et} \quad 0 < a < 1,$$

it follows that $\Omega_{\frac{\ell}{2}+[\frac{\ell}{2}]} \subset \Omega_\ell$ and then

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq a^{\frac{\ell}{2}-1} \int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 dx.$$

Hence,

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq ce^{-\tilde{\alpha}\ell} \|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_\ell)}, \quad (1.2.7)$$

with $c = a^{-\frac{1}{2}}$ and $\tilde{\alpha} = \frac{1}{4} \ln(\frac{1}{a}) > 0$.

Now, in order to establish the step **II**, we only need to estimate $\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_\ell)}$. Taking $z_\ell = u_\ell - g$, we have that $z_\ell \in H_0^1(\Omega_\ell)$. Then, we can use z_ℓ as a test function in (1.1.8), leading to

$$\int_{\Omega_\ell} A \nabla u_\ell \cdot \nabla z_\ell dx = \int_{\Omega_\ell} A \nabla(z_\ell + g) \cdot \nabla z_\ell dx = \langle f, z_\ell \rangle.$$

Consequently, using properties (1.1.5)-(1.1.6),

$$\begin{aligned} \lambda \int_{\Omega_\ell} |\nabla z_\ell|^2 dx &\leq \int_{\Omega_\ell} A \nabla z_\ell \cdot \nabla z_\ell dx = \langle f, z_\ell \rangle - \int_{\Omega_\ell} A \nabla g \cdot \nabla z_\ell dx \\ &\leq \langle f, z_\ell \rangle + \int_{\Omega_\ell} |A \nabla g| |\nabla z_\ell| dx \\ &\leq \|f\|_{H^{-1}(\Omega_\ell)} \|\nabla z_\ell\|_{L^2(\Omega_\ell)} + \Lambda \|\nabla g\|_{L^2(\Omega_\ell)} \|\nabla z_\ell\|_{L^2(\Omega_\ell)} \\ &\leq Ce^{\beta\ell} \|\nabla z_\ell\|_{L^2(\Omega_\ell)}, \end{aligned}$$

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thanks to the fact that $f \in W_\beta(\Omega_\infty)$ and $\nabla g \in (V_\beta(\Omega_\infty))^n$. Therefore,

$$\|\nabla z_\ell\|_{L^2(\Omega_\ell)} \leq C e^{\beta\ell},$$

for a constant C only depending on λ , Λ and on the constants C_0 , \tilde{C}_0 appearing in the definition of the spaces $V_\beta(\Omega_\infty)$ and $W_\beta(\Omega_\infty)$. Consequently,

$$\|\nabla u_\ell\|_{L^2(\Omega_\ell)} = \|\nabla(z_\ell + g)\|_{L^2(\Omega_\ell)} \leq \|\nabla z_\ell\|_{L^2(\Omega_\ell)} + \|\nabla g\|_{L^2(\Omega_\ell)} \leq C e^{\beta\ell}. \quad (1.2.8)$$

In the same way

$$\|\nabla z_{\ell+r}\|_{L^2(\Omega_\ell)} \leq \|\nabla z_{\ell+r}\|_{L^2(\Omega_{\ell+r})} \leq C e^{\beta(\ell+r)} \leq C e^\beta e^{\beta\ell},$$

since $r \in [0, 1]$, with C being the same constant as previously.

Since $u_\ell - u_{\ell+r} = z_\ell - z_{\ell+r}$ on Ω_ℓ , we have

$$\begin{aligned} \|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_\ell)} &= \|\nabla(z_\ell - z_{\ell+r})\|_{L^2(\Omega_\ell)} \leq \|\nabla z_\ell\|_{L^2(\Omega_\ell)} + \|\nabla z_{\ell+r}\|_{L^2(\Omega_\ell)} \\ &\leq C e^{\beta\ell}, \end{aligned}$$

with the last constant still not depending on $\ell > 0$ or on $r \in [0, 1]$.

Combined with the inequality (1.2.7), the last inequality implies

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-(\tilde{\alpha}-\beta)\ell}.$$

Finally, if β from the definitions of $V_\beta(\Omega_\infty)$ and $W_\beta(\Omega_\infty)$ verifies $\beta < \tilde{\alpha}$ (where $\tilde{\alpha} = \frac{1}{4} \ln(\frac{1}{a})$) only depends on ω_1 , ω_2 , λ and Λ , since this is the case for the constant a found in step **I**), we have

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-\alpha\ell},$$

for all positive ℓ and all $r \in [0, 1]$, with $\alpha \in \mathbb{R}$ given by $\alpha = \tilde{\alpha} - \beta > 0$.

Step III. *There exists two constants $C \geq 0$ and $\alpha > 0$ depending only on ω_1 , ω_2 , λ , Λ , C_0 , \tilde{C}_0 and β such that*

$$\|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-\alpha\ell}, \quad (1.2.9)$$

for all positive ℓ and all non-negative t .

This is a simple consequence of inequality (1.2.6):

$$\begin{aligned}
 \|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\frac{\ell}{2}})} &\leq \sum_{i=0}^{[t]-1} \|\nabla(u_{\ell+i} - u_{\ell+i+1})\|_{L^2(\Omega_{\frac{\ell}{2}})} \\
 &\quad + \|\nabla(u_{\ell+[t]} - u_{\ell+t})\|_{L^2(\Omega_{\frac{\ell}{2}})} \\
 &\leq \sum_{i=0}^{[t]-1} \|\nabla(u_{\ell+i} - u_{\ell+i+1})\|_{L^2(\Omega_{\frac{\ell+i}{2}})} \\
 &\quad + \|\nabla(u_{\ell+[t]} - u_{\ell+t})\|_{L^2(\Omega_{\frac{\ell+[t]}{2}})} \\
 &\leq \sum_{i=0}^{[t]} C e^{-\alpha(\ell+i)} = C e^{-\alpha\ell} \sum_{i=0}^{[t]} e^{-\alpha i} \\
 &\leq C \frac{1}{1 - e^{-\alpha}} e^{-\alpha\ell},
 \end{aligned}$$

with α being the same as in step **II** and C being different but depending on the same parameters.

Step IV. *There exists $u_\infty \in H_{loc}^1(\bar{\Omega}_\infty)$ such that for all $\ell_0 > 0$, $u_\ell \rightarrow u_\infty$ in $H^1(\Omega_{\ell_0})$, and $u_\ell - u_\infty$ verifies*

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-\alpha\ell} \quad \text{for all } \ell > 0, \quad (1.2.10)$$

for some constants $C \geq 0$ and $\alpha > 0$, depending only on ω_1 , ω_2 , λ , Λ , C_0 , \tilde{C}_0 and β .

A consequence of the Poincaré inequality (1.1.10) (since $u_\ell - u_{\ell+t} \in H^1(\Omega_\ell)$ and $u_\ell - u_{\ell+t} = 0$ on $\ell\omega_1 \times \partial\omega_2$) and of inequality (1.2.9) is that for a fixed $\ell_0 > 0$,

$$\|u_\ell - u_{\ell+t}\|_{H^1(\Omega_{\ell_0})} \leq C \|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\ell_0})} \leq C e^{-\alpha\ell},$$

for all $\ell \geq 2\ell_0$ and all $t \geq 0$. This implies that $(u_\ell)_{\ell \geq 0}$ is a Cauchy “sequence” for the norm of the space $H^1(\Omega_{\ell_0})$. Since $H^1(\Omega_{\ell_0})$ is a Banach space, there exists $u_\infty^{\ell_0} \in H^1(\Omega_{\ell_0})$ such that $u_\ell \rightarrow u_\infty^{\ell_0}$ in $H^1(\Omega_{\ell_0})$.

defined Making ℓ_0 vary in \mathbb{N}^* gives us a sequence of limits u_∞^k in Ω_k . For all non-zero natural integer k , we have $u_\ell \rightarrow u_\infty^k$ in $H^1(\Omega_k)$. Since $\Omega_{k_1} \subset \Omega_{k_2}$

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for all $k_1 < k_2$, we have $u_\infty^{k_1} = u_\infty^{k_2}$ a.e. in Ω_{k_1} , since $u_\ell \rightarrow u_\infty^{k_1}$ and $u_\ell \rightarrow u_\infty^{k_2}$ in $H^1(\Omega_{k_1})$.

We can then construct a function $u_\infty \in H_{loc}^1(\bar{\Omega}_\infty)$ such that $u_\infty = u_\infty^k$ in Ω_k for all positive integer k . It is enough to set

$$u_\infty = \begin{cases} u_\infty^1 & \text{in } \Omega_1 \\ u_\infty^k & \text{in } \Omega_k \setminus \Omega_{k-1} \quad \text{for all } k \geq 2. \end{cases}$$

This function verifies

$$u_\ell \rightarrow u_\infty \text{ in } H^1(\Omega_{\ell_0}) \quad \text{for all } \ell_0 > 0.$$

Estimate (1.2.10) is finally obtained by taking $\ell > 0$ fixed and making t go to infinity in inequality (1.2.9).

Step V. *The limit u_∞ from the previous step is a solution to problem (1.2.1).*

For a fixed $\ell_0 > 0$, let $v \in H_0^1(\Omega_{\ell_0})$. Then, for all $\ell \geq \ell_0$, $v \in H_0^1(\Omega_\ell)$. It follows that v is a good test function for the variational problem verified by u_ℓ , for all $\ell \geq \ell_0$. Since v is 0 outside of Ω_{ℓ_0} , we have that for all $\ell \geq \ell_0$,

$$\int_{\Omega_{\ell_0}} A \nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_\ell} A \nabla u_\ell \cdot \nabla v \, dx = \langle f, v \rangle.$$

Since $\nabla u_\ell \rightarrow \nabla u_\infty$ strongly in $(L^2(\Omega_{\ell_0}))^n$, letting ℓ go to infinity leads to

$$\int_{\Omega_\infty} A \nabla u_\infty \cdot \nabla v \, dx = \int_{\Omega_{\ell_0}} A \nabla u_\infty \cdot \nabla v \, dx = \langle f, v \rangle.$$

This shows that u_∞ satisfies the variational equation (1.2.2), since ℓ_0 is taken arbitrarily. So u_∞ satisfies the first equation of the problem (1.2.1).

On the other hand, for any fixed $\ell_0 > 0$, we have that, for any $\ell \geq \ell_0$, $\gamma(u_\ell) = g$ on $\partial\Omega_\ell$, hence $\gamma(u_\ell) = g$ on $\ell\omega_1 \times \partial\omega_2$. Since $\ell_0\omega_1 \subset \ell\omega_1$, we get $\gamma(u_\ell) = g$ on $\ell_0\omega_1 \times \partial\omega_2 \subset \partial\Omega_{\ell_0}$. Remembering that $u_\ell \rightarrow u_\infty$ in $H^1(\Omega_{\ell_0})$ and using the continuity of the trace operator, we deduce $\gamma(u_\ell) \rightarrow \gamma(u_\infty)$ in $L^2(\partial\Omega_{\ell_0})$ and particularly in $L^2(\ell_0\omega_1 \times \partial\omega_2)$. Thus, $\gamma(u_\infty) = g$ on $\ell_0\omega_1 \times \partial\omega_2$. Since ℓ_0 was arbitrarily taken, we derive $\gamma(u_\infty) = g$ on $\partial\Omega_\infty = \mathbb{R}^k \times \partial\omega_2 = \bigcup_{\ell_0 > 0} (\ell_0\omega_1 \times \partial\omega_2)$.

Finally, using the estimate (1.2.10) and the inequality (1.2.8), we have

$$\begin{aligned}
\|\nabla u_\infty\|_{L^2(\Omega_\ell)} &\leq \|\nabla(u_\infty - u_{2\ell})\|_{L^2(\Omega_\ell)} + \|\nabla u_{2\ell}\|_{L^2(\Omega_\ell)} \\
&\leq \|\nabla(u_\infty - u_{2\ell})\|_{L^2(\Omega_\ell)} + \|\nabla u_{2\ell}\|_{L^2(\Omega_{2\ell})} \\
&\leq C(e^{-2\alpha\ell} + e^{2\beta\ell}) \\
&\leq C_\infty e^{2\beta\ell}.
\end{aligned}$$

Step VI. *There exists a unique solution to problem (1.2.1).*

The existence of a solution being already established, we only need to prove its uniqueness. Let $u_\infty, \tilde{u}_\infty$ be two solutions of the problem. Then for any $\ell_1 > 0$, the computations of step **I** remain valid for u_∞ and \tilde{u}_∞ replacing u_ℓ and $u_{\ell+r}$. We finally get the inequality

$$\int_{\Omega_{\ell_1}} |\nabla(u_\infty - \tilde{u}_\infty)|^2 dx \leq a \int_{\Omega_{\ell_1+1}} |\nabla(u_\infty - \tilde{u}_\infty)|^2 dx,$$

for all $\ell_1 > 0$, where a is the same constant as in step **I**. Then,

$$\|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} \leq a^{\frac{1}{2}} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1+1})}.$$

Iterating k times the previous inequality leads to

$$\begin{aligned}
\|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} &\leq a^{\frac{k}{2}} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1+k})} \\
&= e^{-2\tilde{\alpha}k} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1+k})},
\end{aligned}$$

with $\tilde{\alpha} = \frac{1}{4} \ln(\frac{1}{a})$ found in step **II**. Combining the last inequality with the ones satisfied by u_∞ and \tilde{u}_∞ in problem (1.2.1), we deduce that,

$$\begin{aligned}
\|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} &\leq 2C_\infty e^{2\beta(\ell_1+k)} e^{-2\tilde{\alpha}k} \\
&= 2C_\infty e^{2\beta\ell_1} e^{-2(\tilde{\alpha}-\beta)k}.
\end{aligned}$$

Fixing ℓ_1 and making k go to infinity, we have that $\|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} = 0$, since $\beta < \tilde{\alpha}$.

On the other hand, $u_\infty = \tilde{u}_\infty = g$ on $\partial\Omega_\infty$, implying $u_\infty - \tilde{u}_\infty = 0$ on $\partial\Omega_\infty$. More particularly, for any $\ell_1 > 0$, we have $u_\infty - \tilde{u}_\infty \in H^1(\Omega_{\ell_1})$ and $u_\infty - \tilde{u}_\infty = 0$ on $\ell_1\omega_1 \times \partial\omega_2$. Thanks to the Poincaré inequality (1.1.10), this implies $u_\infty - \tilde{u}_\infty = 0$ a.e. in Ω_{ℓ_1} . Since ℓ_1 was arbitrarily chosen, this leads to $u_\infty - \tilde{u}_\infty = 0$ a.e. in Ω_∞ . \square

The following generalization of Theorem 1.2.1 is straightforward:

1.2. THE MAIN RESULT

Theorem 1.2.2. *Let f and g satisfy the assumptions of Theorem 1.2.1 and $\{f_\ell\}_{\ell>0}$, $\{g_\ell\}_{\ell>0}$ such that $f_\ell \in H^{-1}(\Omega_\ell)$, $g_\ell \in H^{\frac{1}{2}}(\partial\Omega_\ell)$ for all $\ell > 0$ and*

$$\begin{aligned} \|f_\ell - f\|_{H^{-1}(\Omega_\ell)} &\leq C_1 e^{-\alpha_1 \ell} \quad \text{for all } \ell > 0, \\ \|g_\ell - g\|_{H^{\frac{1}{2}}(\partial\Omega_\ell)} &\leq C_2 e^{-\alpha_2 \ell} \quad \text{for all } \ell > 0, \end{aligned}$$

for some constants $C_1 \geq 0$, $C_2 \geq 0$, $\alpha_1 > 0$ and $\alpha_2 > 0$. If $\tilde{u}_\ell \in H^1(\Omega_\ell)$ is the weak solution to the problem

$$\begin{cases} -\operatorname{div}(A\nabla \tilde{u}_\ell) = f_\ell & \text{in } \Omega_\ell \\ \tilde{u}_\ell = g_\ell & \text{on } \partial\Omega_\ell, \end{cases}$$

then the following estimate take place:

$$\|\tilde{u}_\ell - u_\infty\|_{H^1(\Omega_{\frac{\ell}{2}})} \leq C e^{-\gamma \ell} \quad \text{for all } \ell > 0, \quad (1.2.11)$$

where u_∞ is the solution to problem (1.2.1) and $C \geq 0$, $\gamma > 0$ are constants independent of ℓ .

This result is an immediate consequence of Theorem 1.2.1 and of the well-known estimate (obtained by considering the problem satisfied by $\tilde{u}_\ell - u_\ell$):

$$\|\tilde{u}_\ell - u_\ell\|_{H^1(\Omega_\ell)} \leq C (\|f_\ell - f\|_{H^{-1}(\Omega_\ell)} + \|g_\ell - g\|_{H^{\frac{1}{2}}(\partial\Omega_\ell)}),$$

where C is a constant depending only on λ , Λ and ω_2 , thanks to the Poincaré inequality (1.1.10). Hence the estimate (1.2.11) is obtained by taking $\gamma = \min\{\alpha_1, \alpha_2, \alpha\}$, where α is the constant given by Theorem 1.2.1.

Remark. We can obtain the result considered in [23] and corresponding to a data that is constant in the direction of the cylinder's axis as a particular case of Theorem 1.2.1.

More specifically, consider that A is divided into four blocks

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is a $p \times p$ matrix and A_{22} a $(n-p) \times (n-p)$ matrix.

Then if the data of the problem satisfy the conditions

$$A_{12} = A_{12}(X_2), \quad A_{22} = A_{22}(X_2), \quad f = f(X_2), \quad g = g(X_2),$$

with $f \in H^{-1}(\omega_2)$ and $g \in H^1(\omega_2)$, we retrieve the result in [23]. Here, the equality $f = f(X_2)$ for some $f \in H^{-1}(\omega_2)$ is to be understood in the following sense: for a given $f \in H^{-1}(\omega_2)$, the element $f \in H_{loc}^{-1}(\bar{\Omega}_\infty)$ such that $f = f(X_2)$ is defined by the formula

$$\langle f, v \rangle = \int_{\mathbb{R}^k} \langle f, v(X_1, \cdot) \rangle dX_1 \quad \text{for all } v \in \bigcup_{R>0} H_0^1(B_R \times \omega_2),$$

where B_R is the open ball of \mathbb{R}^k of radius R , centered at the origin. We can see that this is a valid definition, thanks to Lemma 1.1.2. In fact, the less obvious part here is the measurability of the integrand, but this can be obtained (using the same technique as in the proof of Lemma 1.1.2) by proving that this integrand is the a.e.-limit of the sequence of continuous functions $X_1 \in \mathbb{R}^k \mapsto \langle f, v_k(X_1, \cdot) \rangle$, where, for $v \in H_0^1(B_R \times \omega_2)$, $\{v_k\} \subset \mathcal{D}(B_R \times \omega_2)$ is a sequence satisfying $v_k \rightarrow v$ in $H^1(B_R \times \omega_2)$.

Using Lemma 1.1.2, we can verify that under these assumptions, the solution u_∞ of problem (1.2.1) can be written as $u_\infty = u_\infty(X_2)$, where $u_\infty \in H^1(\omega_2)$ is the weak solution to the problem

$$\begin{cases} -\operatorname{div}_{X_2}(A_{22}\nabla u_\infty) = f & \text{in } \omega_2 \\ u = g & \text{on } \partial\omega_2, \end{cases}$$

i.e. u_∞ is the solution in $H^1(\omega_2)$ to the variational problem

$$\begin{cases} \int_{\omega_2} A_{22}\nabla_{X_2} u_\infty \cdot \nabla_{X_2} v \, dx = \langle f, v \rangle & \forall v \in H_0^1(\omega_2) \\ u = g & \text{on } \partial\omega_2. \end{cases}$$

Note that, under these particular assumptions, f and g satisfy the hypothesis of Theorem 1.2.1, since $f \in W_\beta(\Omega_\infty)$ and $\nabla g \in (V_\beta(\Omega_\infty))^n$ for any $\beta > 0$. Indeed, we have $\|f\|_{H^{-1}(\Omega_\ell)} \leq (\ell^k |\omega_1|)^{\frac{1}{2}} \|f\|_{H^{-1}(\omega_2)} \leq C_0 e^{\beta\ell}$ for all $\ell > 0$, by taking

$$C_0 = \sup_{\ell \in (0, +\infty)} \frac{|\omega_1|^{\frac{1}{2}} \|f\|_{H^{-1}(\omega_2)} \ell^{\frac{k}{2}}}{e^{\beta\ell}}.$$

A similar computation gives $\|\nabla g\|_{L^2(\Omega_\ell)} = (\ell^k |\omega_1|)^{\frac{1}{2}} \|\nabla_{X_2} g\|_{L^2(\omega_2)} \leq C_0 e^{\beta\ell}$ for all $\ell > 0$.

1.3 Precisions and optimality results

In this section we give some optimality results concerning the domain of convergence in which the convergence previously proven of u_ℓ to u_∞ takes place. For the sake of simplicity, we only consider here the case of homogeneous boundary conditions.

To start with, we recall that from the first step of the proof of the main theorem of the previous section comes the result:

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq \frac{C}{C+1} \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx$$

which is valid for every $r > 0$. Instead of following the second step by taking $0 \leq r \leq 1$, we then can iterate this inequality directly. Therefore, for all $r > 0$ and for all $\ell_1 \leq \ell - 1$, we have

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\ell_1})} \leq C e^{-\tilde{\alpha}(\ell-\ell_1)} \|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_\ell)}. \quad (1.3.1)$$

Then, assuming that $0 \leq r \leq 1$, by following the second step of the proof of the main theorem in the previous section we can derive the inequality

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\ell_1})} \leq C e^{-\tilde{\alpha}(\ell-\ell_1)} \|f\|_{H^{-1}(\Omega_{\ell+1})}, \quad (1.3.2)$$

where $\tilde{\alpha} = \frac{1}{2} \ln \frac{1}{a}$ with $a = \frac{c_0 c_{\omega_2} \frac{\Lambda}{\lambda}}{c_0 c_{\omega_2} \frac{\Lambda}{\lambda} + 1}$.

Assuming that $\|f\|_{H^{-1}(\Omega_\ell)} = O(e^{\beta\ell})$ we have the following cases:

- If $\beta < \tilde{\alpha}$, then for any fixed $a > 0$, it follows from (1.3.2) that

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_a)} \leq C' e^{-\tilde{\alpha}(\ell-a)} e^{\beta\ell} = C e^{-(\tilde{\alpha}-\beta)\ell}.$$

Then, for $t > 0$, the triangle inequality leads to

$$\|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_a)} \leq C e^{-\alpha\ell}$$

for $\alpha = \tilde{\alpha} - \beta > 0$ with $\beta < \tilde{\alpha}$. Therefore, following the arguments in the previous section, we get the convergence of u_ℓ towards u_∞ and the estimate

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_a)} \leq C e^{-\alpha\ell}. \quad (1.3.3)$$

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- Taking $0 \leq \gamma < 1$, if $\|f\|_{H^{-1}(\Omega_\ell)} = O(e^{\beta\ell})$ with $\beta < (1 - \gamma)\tilde{\alpha}$, we have that

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\ell\gamma})} \leq Ce^{-\tilde{\alpha}(\ell-\gamma\ell)}e^{\beta\ell} = Ce^{-(\tilde{\alpha}(1-\gamma)-\beta)\ell}.$$

Then, for $t > 0$, using the triangle inequality we get

$$\|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\gamma\ell})} \leq Ce^{-\alpha\ell}, \text{ with}$$

$\alpha = \tilde{\alpha}(1 - \gamma) - \beta$. Following again the arguments in the previous section, we get the convergence of u_ℓ towards u_∞ and the estimate

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\gamma\ell})} \leq Ce^{-\alpha\ell}. \quad (1.3.4)$$

Remark 1.3.1. Since the hypothesis $\beta < (1 - \gamma)\tilde{\alpha}$ insures the validity of estimate (1.3.4), we can see that, under the hypothesis $\beta < \tilde{\alpha}$, we can always obtain more than estimate (1.3.3), i.e. an estimate of the type (1.3.4). Indeed, for any γ small enough, more specifically for γ satisfying $\gamma < 1 - \frac{\beta}{\tilde{\alpha}}$, one can derive the estimate (1.3.4) on $\Omega_{\gamma\ell}$.

Let us now assume that $\|f\|_{H^{-1}(\Omega_\ell)} = O(\ell^\gamma)$ for some $\gamma > 0$. Using (1.3.2), it follows that, for a positive constant η ,

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\ell-\eta\ln\ell})} \leq Ce^{-\tilde{\alpha}\eta\ln\ell}\ell^\gamma = C\ell^{-(\tilde{\alpha}\eta-\gamma)}.$$

We can now see that for η large enough we obtain convergence for u_ℓ on $\Omega_{\ell-\eta\ln\ell}$. Indeed, if $\alpha' = \tilde{\alpha}\eta - \gamma > 1$, we can derive that, since $\ell - \eta\ln\ell \leq (\ell + r) - \eta\ln(\ell + r)$ for $\ell \geq \eta$,

$$\begin{aligned} \|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\ell-\eta\ln\ell})} &\leq \sum_{0 \leq i \leq [t]-1} \|\nabla(u_{\ell+i} - u_{\ell+i+1})\|_{L^2(\Omega_{(\ell+i)-\eta\ln(\ell+i)})} \\ &\quad + \|\nabla(u_{\ell+[t]} - u_{\ell+t})\|_{L^2(\Omega_{(\ell+[t])-\eta\ln(\ell+[t])})} \\ &\leq C' \sum_{0 \leq k \leq [t]} (\ell + k)^{-\alpha'} \\ &\leq C' \int_{\ell-1}^{+\infty} x^{-\alpha'} dx \\ &\leq C(\ell - 1)^{-(\alpha'-1)}, \end{aligned}$$

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where $[\cdot]$ stands for the integer part and $C = \frac{C'}{\alpha' - 1}$.

Therefore we can directly derive that

$$\|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\ell-\eta \ln \ell})} \leq C\ell^{-\alpha}$$

with $\alpha = (\alpha' - 1) = (\tilde{\alpha}\eta - \gamma) - 1$. So we finally have that, since α has to be positive, that η has to be larger than $\frac{1+\gamma}{\tilde{\alpha}}$. Moreover, we can obtain polynomial convergence of any order provided that η is large enough.

We now suppose that $k = 1$ and that f and A verify $f(x_1, X_2) = f(X_2)$ and $A_{12} = A_{12}(X_2)$, $A_{22} = A_{22}(X_2)$. First of all, we prove that $\int_{\Omega_\ell} \|\nabla(u_\ell - u_\infty)\|^2 dx \leq C\|u_\infty\|_{H^1(\omega)}^2$. To do so, we first of all define a function $\rho = \rho(x_1)$ such that $\rho = \begin{cases} \rho(x_1) = 0 & \text{in } \Omega_{\ell-1} \\ \rho(x_1) = 1 & \text{in } \Omega_\ell^c \\ \rho(x_1) = |x_1| - \ell + 1 & \text{in } \Omega_\ell \setminus \Omega_{\ell-1}. \end{cases}$ The function $u_\ell - u_\infty + \rho u_\infty$ is then in $H_0^1(\Omega_\ell)$ and can be used as a test function in

$$\int_{\Omega_\ell} A \nabla(u_\ell - u_\infty) \cdot \nabla v \, dx = 0.$$

Using the Cauchy-Schwarz inequality and the properties of A , it follows that

$$\begin{aligned} \lambda \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx &= - \int_{\Omega_\ell} A \nabla(u_\ell - u_\infty) \cdot \nabla(\rho u_\infty) \, dx \\ &\leq \int_{\Omega_\ell} A \nabla(u_\ell - u_\infty) \cdot \nabla(u_\ell - u_\infty) \, dx \\ &\leq \Lambda \left(\int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega_\ell} |\nabla(\rho u_\infty)|^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Taking the square of this inequality and making use of the properties of ρ ,

$$\begin{aligned} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx &\leq \frac{\Lambda^2}{\lambda^2} \int_{\Omega_\ell} |\nabla(\rho u_\infty)|^2 dx \\ &\leq \frac{\Lambda^2}{\lambda^2} \int_{\Omega_\ell \setminus \Omega_{\ell-1}} |\rho \nabla_{X_2} u_\infty|^2 + |u_\infty \nabla_{x_1} \rho|^2 dx \\ &\leq C \int_{\omega} |u_\infty|^2 + |\nabla_{X_2} u_\infty|^2 dx_2, \end{aligned}$$

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which is the equality we were looking for. Coming back to inequality (1.3.1), we derive that, since r in this inequality can be chosen arbitrarily,

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell_1})} \leq C e^{-\tilde{\alpha}(\ell - \ell_1)} \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_\ell)}$$

by letting r go to $+\infty$. Using the previously proven inequality,

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell_1})} \leq C e^{-\tilde{\alpha}(\ell - \ell_1)} \|u_\infty\|_{H^1(\omega)}.$$

Therefore, in order to ensure the convergence of u_ℓ towards u_∞ , we can in this case replace ℓ_1 by $\ell - k(\ell)$ with $k(\ell) < \ell$ and $k(\ell) \rightarrow +\infty$. Thus,

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell - k(\ell)})} \leq C e^{-\tilde{\alpha}k(\ell)} \xrightarrow{\ell \rightarrow +\infty} 0,$$

with a constant C that does not depend on ℓ .

Remark 1.3.2. *In what follows, we would like to give a method due to N. Bruyère and which, in the case where $\Omega_\ell = B_\ell \times \omega_2$ (with $B_\ell \subset \mathbb{R}^k$ the open ball of center 0 and of radius ℓ), gives some better constants for the first two cases previously treated. In fact, in the case of the laplacian ($A = I_n$), these constants are optimal, as can be seen in the examples given at the end of this chapter.*

We place ourselves in the particular case of the laplacian, with the data f satisfying the growth condition

$$\|f\|_{H^{-1}(\Omega_\ell)} = O(e^{\beta\ell}) \text{ for some } \beta > 0,$$

and the solution u_∞ in the infinite domain Ω_∞ satisfying the condition

$$\|\nabla u_\infty\|_{L^2(\Omega_\ell)} = O(e^{\beta\ell}) \tag{1.3.5}$$

We start from the equation

$$\int_{\Omega_\ell} \nabla(u_\ell - u_\infty) \cdot \nabla v = 0 \text{ for all } v \in H_0^1(\Omega_\ell).$$

Then, we define the function

$$\rho(X_1) = e^{-\sigma|X_1|} - e^{-\sigma\ell},$$

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with σ to be chosen later on. The function $v = \rho(u_\ell - u_\infty)$ belongs to $H_0^1(\Omega_\ell)$, and we therefore can derive that

$$\int_{\Omega_\ell} \rho |\nabla(u_\ell - u_\infty)|^2 dx = - \int_{\Omega_\ell} \nabla_{X_1}(u_\ell - u_\infty) \cdot \begin{pmatrix} \nabla_{X_1} \rho \\ 0 \end{pmatrix} (u_\ell - u_\infty) dx.$$

Since

$$\nabla_{X_1} \rho = -\sigma e^{-\sigma|X_1|} \frac{X_1}{|X_1|},$$

we therefore have that

$$\int_{\Omega_\ell} \rho |\nabla(u_\ell - u_\infty)|^2 dx \leq \sigma \int_{\Omega_\ell} e^{-\sigma|X_1|} |\nabla_{X_1}(u_\ell - u_\infty)| |u_\ell - u_\infty| dx.$$

We now use the inequality

$$ab \leq \frac{a^2 r}{2} + \frac{b^2}{2r} \quad (\text{for any } r > 0)$$

to get

$$\int_{\Omega_\ell} \rho |\nabla(u_\ell - u_\infty)|^2 dx \leq \sigma \int_{\Omega_\ell} e^{-\sigma|X_1|} \left(\frac{r}{2} |\nabla_{X_1}(u_\ell - u_\infty)|^2 + \frac{1}{2r} |u_\ell - u_\infty|^2 \right) dx.$$

Then, using the Poincaré inequality on the section ω_2

$$\int_{\omega_2} (u_\ell - u_\infty)^2(X_1, \cdot) dX_2 \leq c_{\omega_2}^2 \int_{\omega_2} (\nabla_{X_2}(u_\ell - u_\infty))^2(X_1, \cdot) dX_2$$

leads to

$$\begin{aligned} \int_{\Omega_\ell} e^{-\sigma|X_1|} (u_\ell - u_\infty)^2 dx &= \int_{-\ell}^{\ell} e^{-\sigma|X_1|} \int_{\omega_2} (u_\ell - u_\infty)^2 dX_2 dX_1 \\ &\leq c_{\omega_2}^2 \int_{\Omega_\ell} e^{-\sigma|X_1|} (\nabla_{X_2}(u_\ell - u_\infty))^2 dx. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\Omega_\ell} \rho |\nabla(u_\ell - u_\infty)|^2 dx &\leq \frac{\sigma}{2} \int_{\Omega_\ell} e^{-\sigma|X_1|} \left(r |\nabla_{X_1}(u_\ell - u_\infty)|^2 + \frac{c_{\omega_2}^2}{r} |\nabla_{X_2}(u_\ell - u_\infty)|^2 \right) dx \\ &\leq \frac{\sigma}{2} \max\left\{r, \frac{c_{\omega_2}^2}{r}\right\} \int_{\Omega_\ell} e^{-\sigma|X_1|} \|\nabla(u_\ell - u_\infty)\|^2 dx. \end{aligned} \tag{1.3.6}$$

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Since

$$\max\{r, \frac{c_{\omega_2}^2}{r}\} \geq \frac{1}{2}(r + \frac{c_{\omega_2}^2}{r}) \geq \sqrt{r \frac{c_{\omega_2}^2}{r}} = c_{\omega_2}$$

we can see that the best constant in inequality (1.3.6) is $\frac{\sigma c_{\omega_2}}{2}$, reached for $r = c_{\omega_2}$. Therefore,

$$\int_{\Omega_\ell} (e^{-\sigma|X_1|} - e^{-\sigma\ell}) |\nabla_{X_1}(u_\ell - u_\infty)|^2 dx \leq \frac{\sigma c_{\omega_2}}{2} \int_{\Omega_\ell} e^{-\sigma|X_1|} |\nabla(u_\ell - u_\infty)|^2 dx.$$

It follows that

$$(1 - \frac{\sigma c_{\omega_2}}{2}) \int_{\Omega_\ell} e^{-\sigma|X_1|} |\nabla(u_\ell - u_\infty)|^2 dx \leq e^{-\sigma\ell} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx.$$

Let now $\ell_1 > \ell$. Using the fact that $\Omega_{\ell_1} \subset \Omega_\ell$, it follows that, for $\sigma < \frac{2}{c_{\omega_2}}$,

$$\int_{\Omega_{\ell_1}} e^{-\sigma|X_1|} |\nabla(u_\ell - u_\infty)|^2 dx \leq \frac{e^{-\sigma\ell}}{(1 - \frac{\sigma c_{\omega_2}}{2})} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx.$$

Furthermore, for $|X_1| \leq \ell_1$, we have that $e^{-\sigma\ell_1} \leq e^{-\sigma|X_1|}$ and therefore,

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2 dx \leq \frac{e^{-\sigma\ell} e^{\sigma\ell_1}}{(1 - \frac{\sigma c_{\omega_2}}{2})} \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 dx.$$

Finally, using (1.3.5) and the estimate for u_ℓ , we derive that

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell_1})}^2 \leq \frac{c}{(1 - \frac{\sigma c_{\omega_2}}{2})} e^{-\sigma(\ell - \ell_1)} e^{2\beta\ell} \quad (1.3.7)$$

in the case where $\|f\|_{H^{-1}(\Omega_\ell)} = O(e^{\beta\ell})$, c being a constant independant of ℓ and ℓ_1 . Let us now consider the two cases $\ell_1 = a = \text{constant}$, and $\ell_1 = \gamma\ell$ with $\gamma \in (0, 1)$.

- Case $\ell_1 = a > 0$.

The inequality (1.3.7) then becomes

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_a)}^2 \leq \frac{c}{(1 - \frac{\sigma c_{\omega_2}}{2})} e^{-\sigma(\ell - a)} e^{2\beta\ell} = \frac{ce^{\sigma a}}{(1 - \frac{\sigma c_{\omega_2}}{2})} e^{(\sigma - 2\beta)\ell}.$$

Therefore the convergence takes place if we can chose σ satisfying

$$2\beta < \sigma < \frac{2}{c_{\omega_2}},$$

which is possible provided that $\beta < \frac{1}{c_{\omega_2}}$.

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- Case $\ell_1 = \gamma\ell$. The existence of σ is then ensured if β satisfies the inequality $\beta < \frac{1}{c_{\omega_2}}$. The inequality (1.3.7) then rewrites

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\gamma\ell})}^2 \leq \frac{c}{\left(1 - \frac{\sigma c_{\omega_2}}{2}\right)} e^{-\ell(\sigma(1-\gamma)-2\beta)}.$$

To get the convergence to 0 as ℓ goes to infinity, we therefore need γ to verify $\sigma(1-\gamma) - 2\beta > 0$, i.e.

$$\gamma < 1 - \frac{2\beta}{\sigma}.$$

Then, we have

$$\frac{2\beta}{1-\gamma} < \sigma < \frac{2}{c_{\omega_2}},$$

implying $\frac{\beta}{1-\gamma} < \frac{1}{c_{\omega_2}}$, i.e.

$$\gamma < 1 - \beta c_{\omega_2}.$$

Let us now take some examples to show the optimality of the results established above. The optimality is to be understood with respect to the largest type of domain $\Omega_{\ell'} \subset \Omega_\ell$ such that $\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell'})} \rightarrow 0$ as ℓ goes to $+\infty$. In this part, we take $\Omega_\ell = (-\ell, \ell) \times (0, \pi) \subset \mathbb{R}^2$, we assume A to be the identity matrix (the equation therefore becomes $-\Delta u_\ell = f$), and f to be of the form

$$f(x_1, x_2) = \varphi(x_1) \sin(x_2)$$

(note that $x_2 \mapsto \sin x_2$ is the first eigenfunction of $-\Delta$). Then we can see that the solutions u_ℓ are also of the type

$$u_\ell(x_1, x_2) = \phi(x_1) \sin x_2,$$

the function ϕ being solution to the problem

$$\begin{cases} \phi'' - \phi = -\varphi & \text{in } (-\ell, \ell) \\ \phi(-\ell) = \phi(\ell) = 0, \end{cases}$$

taking into account the fact that

$$\Delta(u_\ell)(x_1, x_2) = (\phi''(x_1) - \phi(x_1)) \sin x_2.$$

Finally, let us note that in this case, the Poincaré constant (which is equal to $\frac{1}{\sqrt{\lambda_1}}$, λ_1 being the first eigenvalue of $-\Delta$ on $\omega_2 = (0, \pi)$) is equal to 1.

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- *First example.* We prove that the sequence of u_ℓ is not a Cauchy sequence in $H^1(\Omega_a)$ for $a > 0$ fixed, if $\|f\|_{L^2(\Omega_\ell)} = O(e^\ell)$.

Remark 1.3.3. From the previous computations, we know that the estimate (1.3.3) takes place provided that $\|f\|_{L^2(\Omega_\ell)} = O(e^{\beta\ell})$ with $\beta < \frac{1}{c_{\omega_2}} = 1$. Therefore, the example below proves the optimality of the hypothesis $\beta < \frac{1}{c_{\omega_2}}$ ensuring the validity of estimate (1.3.3).

Let f be defined by $f(x) = -2\sinh(x_1)\sin(x_2)$. Then,

$$\Delta(u_\ell) = (\phi(x_1)'' - \phi(x_1))\sin(x_2)$$

and we look for ϕ solution of the equation

$$\phi''(x_1) - \phi(x_1) = 2\sinh(x_1).$$

Since a particular solution to this equation is $\phi(x_1) = x_1\cosh(x_1)$ and that general solutions of the homogeneous equation are of the type $C_1\cosh(x_1) + C_2\sinh(x_1)$, we have that general solutions of the equation $\phi''(x_1) - \phi(x_1) = 2\sinh(x_1)$ are of the type

$$C_1\cosh(x_1) + C_2\sinh(x_1) + x_1\cosh(x_1).$$

Then, we make use of the boundary conditions $\phi(-\ell) = \phi(\ell) = 0$ to compute the unique solution u_ℓ . The conditions actually are giving that

$$C_1 = 0 \text{ and } C_2 = -\frac{\ell\cosh\ell}{\sinh\ell}.$$

Then,

$$u_\ell(x_1, x_2) = \left(x_1\cosh(x_1) - \frac{\ell\cosh\ell}{\sinh\ell}\sinh(x_1) \right) \sin(x_2).$$

It follows that for any ℓ ,

$$(u_{2\ell} - u_\ell)(x_1, x_2) = \sinh(x_1)\sin(x_2) \left(\frac{\ell\cosh(\ell)}{\sinh(\ell)} - \frac{2\ell\cosh(2\ell)}{\sinh(2\ell)} \right).$$

We now can compute

$$\|\nabla(u_{2\ell} - u_\ell)\|_{L^2(\Omega_a)} = \left| \frac{\ell\cosh(\ell)}{\sinh(\ell)} - \frac{2\ell\cosh(2\ell)}{\sinh(2\ell)} \right| \|\nabla(\sinh(x_1)\sin(x_2))\|_{L^2(\Omega_a)}.$$

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Since $\|\nabla(\sinh(x_1)\sin(x_2))\|_{L^2(\Omega_a)}$ does not depend on ℓ we just have to find the limit when ℓ goes to infinity of $(\frac{\ell \cosh(\ell)}{\sinh(\ell)} - \frac{2\ell \cosh(2\ell)}{\sinh(2\ell)})$. We have that, using the equivalence $\cosh(\ell) \sim \sinh(\ell)$ as $\ell \rightarrow \infty$

$$\begin{aligned} \left| \frac{\ell \cosh(\ell)}{\sinh(\ell)} - \frac{2\ell \cosh(2\ell)}{\sinh(2\ell)} \right| &= |\ell| \left| \frac{\cosh(\ell)\sinh(2\ell) - 2\sinh(\ell)\cosh(2\ell)}{\sinh(\ell)\sinh(2\ell)} \right| \\ &= |\ell| \left| \frac{\cosh(\ell)\sinh(2\ell)(1 - 2(\frac{\sinh(\ell)}{\cosh(\ell)} \frac{\sinh(2\ell)}{\cosh(2\ell)}))}{\sinh(\ell)\sinh(2\ell)} \right| \rightarrow +\infty \text{ as } \ell \rightarrow +\infty. \end{aligned}$$

Since the limit is not 0, we deduce that u_ℓ is not a Cauchy sequence.

- *Second example.* We have seen in the beginning of this section that if $\|f\|_{H^{-1}(\Omega_\ell)} = O(e^{\beta\ell})$ with $\beta < \tilde{\alpha}$, then $\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\gamma\ell})} \rightarrow 0$ for $\gamma \in (0, 1)$ satisfying $\beta < (1 - \gamma)\tilde{\alpha}$. We show in the following example that the convergence on $\Omega_{\gamma\ell}$ is lost if γ is not sufficiently small.

For this, let f be defined by $f(x) = \cosh(\frac{x_1}{2})\sin(x_2)$. A simple computation shows that $\|f\|_{L^2(\Omega_\ell)} = O(e^{\frac{\ell}{2}})$. Hence, here $\beta = \frac{1}{2}$. Following the same method as in the first example, we look for ϕ solution of the equation

$$\phi''(x_1) - \phi(x_1) = -\frac{3}{4}\cosh(\frac{x_1}{2}).$$

A particular solution of the equation is $\phi(x_1) = \cosh(\frac{x_1}{2})$ and therefore, the homogeneous equation remaining the same, the general solutions of the equation are of the type

$$C_1 \cosh(x_1) + C_2 \sinh(x_1) + \cosh(\frac{x_1}{2}).$$

Using the boundary conditions, we have that

$$C_2 = 0 \text{ and } C_1 = -\frac{\cosh(\frac{\ell}{2})}{\cosh(\ell)}.$$

Then

$$u_\ell = \left(-\frac{\cosh(\frac{\ell}{2})}{\cosh(\ell)} \cosh(x_1) + \cosh(\frac{x_1}{2}) \right) (\sin(x_2)),$$

and since

$$u_\infty = \left(\cosh(\frac{x_1}{2}) \right) (\sin(x_2))$$

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(note that $\|\nabla u_\infty\|_L^2(\Omega_\ell) = O(e^{\frac{\ell}{2}})$, we have that

$$u_\ell - u_\infty = -\frac{\cosh(\frac{\ell}{2})}{\cosh(\ell)} \cosh(x_1) \sin(x_2).$$

Then, we have that

$$\nabla(u_\ell - u_\infty) = \begin{pmatrix} -\frac{\cosh(\frac{\ell}{2})}{\cosh(\ell)} \sinh(x_1) \sin(x_2) \\ -\frac{\cosh(\frac{\ell}{2})}{\cosh(\ell)} \cosh(x_1) \cos(x_2) \end{pmatrix},$$

and it follows that

$$|\nabla(u_\ell - u_\infty)|^2 = \frac{\cosh^2(\frac{\ell}{2})}{\cosh^2(\ell)} \left(\sinh^2(x_1) \sin^2(x_2) + \cosh^2(x_1) \cos^2(x_2) \right),$$

which can be written

$$|\nabla(u_\ell - u_\infty)|^2 = \frac{\cosh^2(\frac{\ell}{2})}{\cosh^2(\ell)} (\cosh^2(x_1) - \sin^2(x_2)).$$

Finally, we deduce that

$$\begin{aligned} \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\gamma\ell})}^2 &= \int_0^\pi \int_{-\gamma\ell}^{\gamma\ell} \frac{\cosh^2(\frac{\ell}{2})}{\cosh^2(\ell)} (\cosh^2(x_1) - \sin^2(x_2)) dx_1 dx_2 \\ &= \frac{\pi \cosh^2(\frac{\ell}{2})}{\cosh^2(\ell)} \int_{-\gamma\ell}^{\gamma\ell} \cosh^2(x_1) dx_1 - \frac{\gamma\ell \cosh^2(\frac{\ell}{2})}{\cosh^2(\ell)} \int_0^\pi \sin^2(x_2) dx_2. \end{aligned}$$

We now compute $-\frac{\pi \cosh(\frac{\ell}{2})}{\cosh(\ell)} \int_{-\gamma\ell}^{\gamma\ell} \cosh^2(x_1) dx_1$. Using the formula

$$\cosh^2(x_1) = \frac{1 + \cosh(2x_1)}{2}$$

, we derive

$$\int_{-\gamma\ell}^{\gamma\ell} \cosh^2(x_1) dx_1 = \int_{-\gamma\ell}^{\gamma\ell} \frac{1}{2} + \frac{\cosh(2x_1)}{2} dx_1 = \gamma\ell + \frac{1}{4} (\sinh(2\gamma\ell) - \sinh(-2\gamma\ell)).$$

It follows that, when ℓ goes to $+\infty$, since

$$\frac{\gamma\ell \cosh^2(\frac{\ell}{2})}{\cosh^2(\ell)} \int_0^\pi \sin^2(x_2) dx_2 \rightarrow 0$$

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and

$$-\frac{\gamma\pi\ell\cosh^2(\frac{\ell}{2})}{\cosh^2(\ell)} \rightarrow 0,$$

in order to have $\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\gamma\ell})} \rightarrow 0$ we need to have

$$-\frac{\pi\sinh(2\gamma\ell)\cosh^2(\frac{\ell}{2})}{\cosh^2(\ell)} \rightarrow 0,$$

which implies that γ must be less than $\frac{1}{2}$.

- *Third example.* In this example we show that if we have that $\|f\|_{H^{-1}(\Omega_\ell)} = O(\ell^\gamma)$ for some $\gamma > 0$, then the convergence $\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-\eta\ln\ell})} \rightarrow 0$ can be lost if η is too small. Consequently, we cannot replace $\eta\ln\ell$ by a quantity having a slower growth at infinity than $\ln\ell$.

In this example, we suppose that $f(x) = x_1 \sin(x_2)$. Hence $\|f\|_{L^2} = O(\ell^{\frac{3}{2}})$. Still following the same method, we look for a solution of the equation

$$\phi''(x_1) - \phi(x_1) = -x_1.$$

A particular solution of this equation is $\phi(x_1) = x_1$ and it follows that the general solutions of the equation are of the type

$$C_1 \cosh(x_1) + C_2 \sinh(x_1) + x_1.$$

We derive from the boundary conditions that

$$C_1 = 0 \text{ and } C_2 = \frac{-\ell}{\sinh(\ell)}.$$

Then,

$$u_\ell = \left(\frac{-\ell}{\sinh(\ell)} \sinh(x_1) + x_1 \right) \sin(x_2)$$

and since

$$u_\infty = x_1 \sin(x_2)$$

(note that $\|\nabla u_\infty\|_{L^2(\Omega_\ell)} = O(\ell^{\frac{3}{2}})$, we have that

$$u_\ell - u_\infty = \frac{-2\ell}{\sinh(\ell)} \sinh(x_1) \sin(x_2).$$

From that comes

$$\nabla(u_\ell - u_\infty) = \begin{pmatrix} \frac{-\ell}{\sinh(\ell)} \cosh(x_1) \sin(x_2) \\ \frac{-\ell}{\sinh(\ell)} \sinh(x_1) \cos(x_2) \end{pmatrix}$$

and

$$\begin{aligned} |\nabla(u_\ell - u_\infty)|^2 &= \frac{\ell^2}{\sinh^2(\ell)} \left(\cosh^2(x_1) \sin^2(x_2) + \sinh^2(x_1) \cos^2(x_2) \right) \\ &= \frac{\ell^2}{\sinh^2(\ell)} \left(ch^2(x_1) - \cos^2(x_2) \right) \\ &= \frac{\ell^2}{\sinh^2(\ell)} \left(\frac{1 + ch(2x_1)}{2} - \cos^2(x_2) \right). \end{aligned}$$

We deduce that

$$\begin{aligned} \|\nabla(u_\ell - u_\infty)\|_{L^2(\ell - \eta \ln \ell)} &= \frac{\pi \ell^2}{\sinh^2(\ell)} \int_{-\ell + \eta \ln \ell}^{\ell - \eta \ln \ell} \frac{1 + \cosh(2x_1)}{2} dx_1 \\ &\quad - \frac{\pi \ell^2}{\sinh^2(\ell)} (2\ell - 2\eta \ln \ell) \int_0^\pi \cos^2(x_2) dx_2. \end{aligned}$$

In order to have $\|\nabla(u_\ell - u_\infty)\|_{L^2(\ell - \eta \ln \ell)} \rightarrow 0$, we need to have

$$\frac{\pi \ell^2}{\sinh^2(\ell)} \int_{-\ell + \eta \ln \ell}^{\ell - \eta \ln \ell} \cosh(2x_1) dx_1 \rightarrow 0.$$

We have that

$$\begin{aligned} \frac{\pi \ell^2}{\sinh^2(\ell)} \int_{-\ell + \eta \ln \ell}^{\ell - \eta \ln \ell} \cosh(2x_1) dx_1 &= \frac{\pi \ell^2}{\sinh^2(\ell)} \sinh(2\ell - 2\eta \ln \ell) \\ &\underset{+\infty}{\sim} \frac{\pi \ell^2 e^{2\ell - 2\eta \ln \ell}}{e^{2\ell}} = 2\pi e^{2(1-\eta)\ell}, \end{aligned}$$

since $\sinh(t) \underset{+\infty}{\sim} \frac{e^t}{2}$. So if we want $\|\nabla(u_\ell - u_\infty)\|_{L^2(\ell - \eta \ln \ell)}$ to go to 0 as ℓ goes to ∞ , we need η to be greater than 1.

Chapter 2

Correctors to elliptic problems in long cylinders

The last years have seen the development of the asymptotic study of partial differential equations in cylinders becoming unbounded in one or several directions, particularly under the impetus of Michel Chipot and his collaborators. In this paper, we aim to improve some results that have already been shown about the convergence to the solution of a linear elliptic problem on an infinite cylinder of the solutions of the same problem taken on larger and larger truncations of the cylinder. This aim will be realized by the construction of well-adjusted correctors. Thanks to our main results established in Section 2 of this paper, we conclude by an application in a particular case (by taking data that does not depend on the coordinate along the cylinder's axis) that the convergence results that can be obtained using the methods introduced by Chipot and Yeressian (in their paper of 2008) are optimal. The particularity here is that the optimality is taken in the sense of “the largest domain where the convergence takes place” instead of the classical optimality of the speed of convergence itself.

2.1 Introduction

The asymptotic behavior of PDE problems in cylinders of the form $\Omega_\ell = \ell\omega_1 \times \omega_2 \subset \mathbb{R}^n$ (with $\omega_1 \subset \mathbb{R}^k$ and $\omega_2 \subset \mathbb{R}^{n-k}$, where $1 \leq k \leq n-1$) have been heavily studied during the last twenty years by Michel Chipot and his collaborators, see for instance [13] - [23], [41], [69]. However, few works were

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dedicated to the construction of correctors allowing to have good estimates in the whole cylinder Ω_ℓ . To the best of authors' knowledge, this type of problem has only been addressed in [17] and [18]. In this paper, we will construct correctors for elliptic problems in cylinders becoming unbounded in one direction (hence $k = 1$ in the definition of Ω_ℓ).

On a cylinder $\Omega_\ell = (-\ell, \ell) \times \omega \subset \mathbb{R}^n$, where $\omega \subset \mathbb{R}^{n-1}$ is a bounded Lipschitz open set, we consider a linear elliptic problem with homogeneous Dirichlet boundary conditions. Thanks to the results in [16] (see also [15]), we already know that the “sequence” of the solutions u_ℓ of the problem in a larger and larger cylinder (i.e., as ℓ goes to infinity) converges to the unique solution u_∞ of the problem in the infinite cylinder. Since the functions u_ℓ are not all defined on the same domain, this convergence takes place on any fixed cylinder Ω_{ℓ_0} . However, one can ask the question of an estimate of $u_\ell - u_\infty$ on the whole cylinder Ω_ℓ and we are in particular interested in estimates giving $\|u_\ell - u_\infty\|_{H^1(\Omega_\ell)} \rightarrow 0$ as $\ell \rightarrow +\infty$. In [16] it was shown that

$$\|u_\ell - u_\infty\|_{H^1(\Omega_{\frac{\ell}{2}})} \rightarrow 0 \quad (2.1.1)$$

exponentially fast and it is easy to see that, adapting the methods presented in this paper, one can replace $\Omega_{\frac{\ell}{2}}$ by cylinders that are closer to Ω_ℓ (see section 3 for more details). But, under the very weak assumptions made on the data, it is not possible to replace $\Omega_{\frac{\ell}{2}}$ by Ω_ℓ in convergence (2.1.1), see for instance [15]. In order to have a good approximation of u_∞ on the whole cylinder Ω_ℓ , we need to add a corrector to the solution u_∞ of the problem in the infinite cylinder. The construction of this corrector and the proof of the resulting estimate on Ω_ℓ is the main purpose of this paper and is the subject of the second section. We would like to mention here the fact that the correctors problem have been considered in [17] (see also [15]) for the case of the laplacian and for data not depending on the coordinate along the cylinder's axis. However, in this work, the symmetry properties of the laplacian (implying the symmetry of the solutions) plays an important role in the proofs. While we follow here some ideas from [17], we also need to find new strategies allowing us to treat the case of general elliptic operators. One of the main differences is that the results in [17] are obtained independently of the previous results concerning the convergence (2.1.1) and one can derive a new proof of the estimates leading to (2.1.1) as a consequence of the result on correctors. In our paper, this is not possible, since the convergence (2.1.1)

is used in our proof of the estimate involving the correctors.

Finally, the corrector constructed in Section 2 allows us to prove the optimality of the results that can be obtained with the methods introduced in [23] regarding the convergence of u_ℓ towards u_∞ on cylinders smaller than Ω_ℓ . This argument is applied to a particular case in Section 3.

Let us now give the notation that will be used in this chapter.

The notation $|\cdot|$ will designate the Euclidean norm in \mathbb{R}^k . The space of $n \times n$ square matrices with real entries will be denoted by $\mathcal{M}_n(\mathbb{R})$.

For an unbounded domain $\Omega \subset \mathbb{R}^n$, we define the spaces

$$\begin{aligned} L_{loc}^2(\bar{\Omega}) &= \{v \in L_{loc}^2(\Omega) \mid v \in L^2(\mathcal{O}), \text{ for any bounded open set } \mathcal{O} \subset \Omega\}, \\ H_{loc}^1(\bar{\Omega}) &= \{v \in H_{loc}^1(\Omega) \mid v \in H^1(\mathcal{O}), \text{ for any bounded open set } \mathcal{O} \subset \Omega\}, \\ H_{loc}^{-1}(\bar{\Omega}) &= \{v \in H_{loc}^{-1}(\Omega) \mid v \in H^{-1}(\mathcal{O}), \text{ for any bounded open set } \mathcal{O} \subset \Omega\}. \end{aligned}$$

By convention, any function u in $H_0^1(\Omega)$ is extended by 0 outside of Ω and will still be denoted by u .

Let ω be a bounded Lipschitz domain of \mathbb{R}^{n-1} . For any $\ell > 0$, we define

$$\Omega_\ell = (-\ell, \ell) \times \omega,$$

and we set

$$\Omega_\infty = \mathbb{R} \times \omega.$$

We denote by Ω_ℓ^+ and Ω_ℓ^- the domains

$$\Omega_\ell^+ = (0, \ell) \times \omega \text{ and } \Omega_\ell^- = (-\ell, 0) \times \omega.$$

A point x in \mathbb{R}^n is denoted

$$x = (x_1, x'), \text{ with } x' = (x_2, \dots, x_n).$$

Similarly, we use the notation $\nabla' = (\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$ and div' where, for $v = (v_2, v_3, \dots, v_n) : \omega \rightarrow \mathbb{R}^{n-1}$,

$$\text{div}' v = \frac{\partial v_2}{\partial x_2} + \dots + \frac{\partial v_n}{\partial x_n}.$$

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Let f be a $H_{loc}^{-1}(\bar{\Omega}_\infty)$ distribution satisfying, for some constant $\beta > 0$,

$$\|f\|_{H^{-1}(\Omega_\ell)} \leq Ce^{\beta\ell} \quad \text{for all } \ell > 0. \quad (2.1.2)$$

We consider a matrix field $A = (a_{ij})_{1 \leq i, j \leq n} \in L^\infty(\Omega_\infty; \mathcal{M}_n(\mathbb{R}))$, i.e. $a_{ij} \in L^\infty(\Omega_\infty)$ for all $i, j \in \{1, \dots, n\}$, satisfying the following properties:

$$\begin{aligned} \lambda|\xi|^2 &\leq A(x)\xi \cdot \xi \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega_\infty, \\ |A(x)\xi| &\leq \Lambda|\xi| \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega_\infty, \end{aligned} \quad (2.1.3)$$

for some constants $\lambda > 0$ and $\Lambda > 0$. To the matrix field A , we associate the matrix field $A' \in L^\infty(\Omega_\infty; \mathcal{M}_{n-1}(\mathbb{R}))$ defined by $A' = (a_{ij})_{2 \leq i, j \leq n}$. It is then easy to see that A' also satisfies the properties (2.1.3) for all $\xi \in \mathbb{R}^{n-1}$, a.e. $x \in \Omega_\infty$.

We consider the following problem in the cylinder Ω_ℓ :

$$\begin{cases} -\operatorname{div}(A\nabla u_\ell) = f & \text{in } \Omega_\ell \\ u_\ell = 0 & \text{on } \partial\Omega_\ell. \end{cases} \quad (2.1.4)$$

This problem has for weak formulation

$$\begin{cases} \text{Find } u_\ell \in H_0^1(\Omega_\ell) \text{ such that} \\ \int_{\Omega_\ell} A\nabla u_\ell \nabla v \, dx = \langle f, v \rangle \quad \text{for all } v \in H_0^1(\Omega_\ell). \end{cases} \quad (2.1.5)$$

Thanks to the Lax-Milgram theorem, there exists a unique solution u_ℓ of this problem.

We also consider the following elliptic problem in the infinite cylinder Ω_∞ :

$$\begin{cases} -\operatorname{div}(A\nabla u_\infty) = f & \text{in } \Omega_\infty \\ u_\infty = 0 & \text{on } \partial\Omega_\infty \\ \|\nabla u_\infty\|_{L^2(\Omega_\ell)} \leq C_0 e^{2\beta\ell} & \text{for all } \ell > 0, \text{ for some constant } C_0 \geq 0, \end{cases} \quad (2.1.6)$$

with $\beta > 0$ a constant which is small enough.

More precisely, u_∞ is the solution of the variational problem

$$\begin{cases} \text{Find } u_\infty \in H_{loc}^1(\bar{\Omega}_\infty) \text{ such that} \\ \int_{\Omega_\infty} A\nabla u_\infty \cdot \nabla v \, dx = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega_\ell), \text{ for some } \ell > 0 \\ u_\infty = 0 & \text{on } \partial\Omega_\infty \\ \|\nabla u_\infty\|_{L^2(\Omega_\ell)} \leq C_0 e^{2\beta\ell} & \text{for all } \ell > 0, \text{ for some constant } C_0 \geq 0. \end{cases} \quad (2.1.7)$$

Notice that u_ℓ and u_∞ share the property of being 0 on $(-\ell, \ell) \times \partial\omega$. The following result gives a useful Poincaré inequality for functions vanishing on the lateral boundary of a cylinder.

Theorem 2.1.1. *There exists a constant C_ω depending only on ω such that for any $a, b \in \overline{\mathbb{R}}$, $a < b$, we have*

$$\|v\|_{L^2((a,b) \times \omega)} \leq C_\omega \|\nabla v\|_{L^2((a,b) \times \omega)} \quad (2.1.8)$$

for all $v \in H^1((a,b) \times \omega)$ satisfying $v = 0$ on $(a,b) \times \partial\omega$.

Proof. The hypothesis on v implies that $v(x_1, \cdot)$ belongs to $H_0^1(\omega)$ for a.e. $x_1 \in (a,b)$. We then derive inequality (2.1.8) from the Poincaré inequality on ω . For a detailed proof of Lemma 2.1.1, see e.g. [11]. \square

The following result, established in [16] (see also [15] and [11]), states that as ℓ goes to infinity, the restrictions to a fixed cylinder of the solutions u_ℓ (of problem (2.1.5)) converge to the solution u_∞ of problem (2.1.7). Moreover, this convergence is exponential.

Theorem 2.1.2. *Let f verify inequality (2.1.2) for $\beta > 0$ small enough, and let $u_\ell \in H_0^1(\Omega_\ell)$ be the solution of the variational problem (2.1.5). Then for all $\ell_0 > 0$,*

$$u_\ell \rightarrow u_\infty \text{ strongly in } H^1(\Omega_{\ell_0})$$

as $\ell \rightarrow +\infty$, where $u_\infty \in H_{loc}^1(\bar{\Omega}_\infty)$ is the weak solution to the homogeneous Dirichlet problem (2.1.7) in the cylinder Ω_∞ .

Furthermore, we have the estimate

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C_1 e^{-\alpha_1 \ell} \quad \text{for all } \ell > 0, \quad (2.1.9)$$

where $C_1 \geq 0$ and $\alpha_1 > 0$ are constants depending only on ω , λ , Λ , C_0 (the constant appearing in (2.1.7)) and β .

Note that the estimate (2.1.9) is a stronger result than the one stating that u_ℓ converges exponentially to u_∞ in $H^1(\Omega_{\ell_0})$ for any fixed cylinder Ω_{ℓ_0} . On the other hand, (2.1.9) is weaker than having an estimate in the whole cylinder Ω_ℓ . In order to obtain an estimate in the whole cylinder, we need to introduce a corrector that, added to the solution u_∞ of problem (2.1.7),

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can give a good approximation of u_ℓ on Ω_ℓ . The main purpose of this paper is to construct such a corrector, introduced below.

Observe first that for a given ℓ , the perfect corrector would be $w_\ell = u_\ell - u_\infty$, verifying

$$\begin{cases} w_\ell \in H^1(\Omega_\ell) \\ \int_{\Omega_\ell} A \nabla w_\ell \cdot \nabla v \, dx = 0 \text{ for all } v \in H_0^1(\Omega_\ell) \\ w_\ell = -u_\infty \text{ on } \{-\ell\} \times \omega \text{ and on } \{\ell\} \times \omega \\ w_\ell = 0 \text{ on } (-\ell, \ell) \times \partial\omega. \end{cases} \quad (2.1.10)$$

Nevertheless, we do not use this corrector since the dependence on the parameter ℓ is too strong. For practical reasons, we would like to have a corrector that can be easily derived by some computation: here, we construct correctors w_ℓ that, in some particular cases (see Section 4), can be derived only in terms of two functions w^+ and w^- which are independent of ℓ .

Let us now describe the construction of our corrector: we build it separately on Ω_ℓ^+ and Ω_ℓ^- . Thus, w_ℓ will be defined as

$$w_\ell = \begin{cases} w_\ell^- & \text{on } \Omega_\ell^- \\ w_\ell^+ & \text{on } \Omega_\ell^+ \end{cases} \quad (2.1.11)$$

where w_ℓ^+ is the solution to the problem

$$\begin{cases} w_\ell^+ \in H^1((-\infty, \ell) \times \omega) \\ \int_{(-\infty, \ell) \times \omega} A \nabla w_\ell^+ \cdot \nabla v \, dx = 0 \quad \forall v \in H_0^1((-\infty, \ell) \times \omega) \\ w_\ell^+ = -u_\infty(\ell, \cdot) \text{ on } \{\ell\} \times \omega \\ w_\ell^+ = 0 \text{ on } (-\infty, \ell) \times \partial\omega, \end{cases} \quad (2.1.12)$$

while w_ℓ^- is the solution of

$$\begin{cases} w_\ell^- \in H^1((-\ell, +\infty) \times \omega) \\ \int_{(-\ell, +\infty) \times \omega} A \nabla w_\ell^- \cdot \nabla v \, dx = 0 \quad \forall v \in H_0^1((-\ell, +\infty) \times \omega) \\ w_\ell^- = -u_\infty(-\ell, \cdot) \text{ on } \{-\ell\} \times \omega \\ w_\ell^- = 0 \text{ on } (-\ell, +\infty) \times \partial\omega. \end{cases} \quad (2.1.13)$$

As one could notice, this corrector w_ℓ belongs to $H^1(\Omega_\ell \setminus (\{0\} \times \omega))$ but does not necessarily belong to $H^1(\Omega_\ell)$. To get a $H^1(\Omega_\ell)$ -corrector, it is enough to take a Lipschitz-continuous function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\psi(x_1) = \begin{cases} 1 & \text{if } |x_1| > 1 \\ 0 & \text{if } |x_1| < \frac{1}{2}, \end{cases}$$

and then to set $\tilde{w}_\ell = \psi w_\ell \in H^1(\Omega_\ell)$, where $\psi(x) = \psi(x_1)$.

Nevertheless, in all that follows, we consider the corrector defined in (2.1.11) and the gradient of w_ℓ will be considered in the sense of $\mathcal{D}'(\Omega_\ell \setminus (\{0\} \times \omega))$. Since the n -dimensional Lebesgue measure of $\{0\} \times \omega$ is 0, we can consider the gradient ∇w_ℓ as a measurable function on the whole set Ω_ℓ .

2.2 Main results

We begin this section by establishing the existence of the corrector introduced in the previous section, then we give a first estimate related to this corrector, which will be used in the final step of the proof of the result justifying the pertinence of the corrector.

Lemma 2.2.1. *There exists a unique solution w_ℓ^+ to problem (2.1.12).*

Proof. Let $\rho : (-\infty, \ell) \times \omega \rightarrow \mathbb{R}$ be the Lipschitz function defined by

$$\begin{cases} \rho = \rho(x_1), & \rho(x_1) = x_1 + 1 - \ell \text{ if } x_1 \in (\ell - 1, \ell), \\ \rho(x_1) = 0 \text{ for } x_1 \leq \ell - 1. \end{cases} \quad (2.2.1)$$

Then, $\rho u_\infty \in H^1((-\infty, \ell) \times \omega)$ and, thanks to the Lax-Milgram theorem, there exists a unique solution \hat{w}_ℓ to the following problem:

$$\begin{cases} \hat{w}_\ell \in H_0^1((-\infty, \ell) \times \omega) \\ \int_{(-\infty, \ell) \times \omega} A \nabla \hat{w}_\ell \cdot \nabla v \, dx = \int_{(-\infty, \ell) \times \omega} A \nabla(\rho u_\infty) \cdot \nabla v \, dx \\ \forall v \in H_0^1((-\infty, \ell) \times \omega) \end{cases} \quad (2.2.2)$$

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Note that the coerciveness of the bilinear form

$$(u, v) \in \left(H_0^1((-\infty, \ell) \times \omega) \right)^2 \mapsto \int_{(-\infty, \ell) \times \omega} A \nabla u \cdot \nabla v \, dx$$

is an immediate consequence of Poincaré inequality (2.1.8). Then we can easily verify that the function $w_\ell^+ = \hat{w}_\ell - \rho u_\infty$ is a solution to problem (2.1.12). The uniqueness of w_ℓ^+ is a simple consequence of the linearity of the problem and of the coerciveness of the bilinear form defined here above. \square

Lemma 2.2.2. *We have*

$$\|\nabla w_\ell^+\|_{L^2((-\infty, \ell) \times \omega)} \leq C_3 e^{2\beta\ell}, \quad (2.2.3)$$

where β is the constant appearing in (2.1.7) and $C_3 > 0$ is a constant depending only on λ, Λ, ω and C_0 (the constant appearing in (2.1.7)).

Proof. Since, (see the proof of Lemma 2.2.1),

$$w_\ell^+ = \hat{w}_\ell - \rho u_\infty,$$

we have

$$\|\nabla w_\ell^+\|_{L^2((-\infty, \ell) \times \omega)} \leq \|\nabla \hat{w}_\ell\|_{L^2((-\infty, \ell) \times \omega)} + \|\nabla(\rho u_\infty)\|_{L^2((-\infty, \ell) \times \omega)} \quad (2.2.4)$$

Using $\hat{w}_\ell \in H_0^1((-\infty, \ell) \times \omega)$ as test function in the variational equation (2.2.2), we deduce the inequality

$$\|\nabla \hat{w}_\ell\|_{L^2((-\infty, \ell) \times \omega)} \leq \frac{\Lambda}{\lambda} \|\nabla(\rho u_\infty)\|_{L^2((-\infty, \ell) \times \omega)} \quad (2.2.5)$$

thanks to properties (2.1.3) satisfied by A and the Cauchy-Schwarz inequality.

Note that

$$\nabla(\rho u_\infty) = \rho \nabla u_\infty + u_\infty(\rho', 0, \dots, 0).$$

Hence, since ρu_∞ vanishes outside of $(\ell - 1, \ell) \times \omega$, we get

$$\begin{aligned} \|\nabla(\rho u_\infty)\|_{L^2((-\infty, \ell) \times \omega)} &= \|\nabla(\rho u_\infty)\|_{L^2((\ell-1, \ell) \times \omega)} \\ &\leq \|\nabla u_\infty\|_{L^2((\ell-1, \ell) \times \omega)} + \|u_\infty\|_{L^2((\ell-1, \ell) \times \omega)}. \end{aligned} \quad (2.2.6)$$

Using (2.2.5), (2.2.6) and the inequality in (2.1.7), we obtain

$$\begin{aligned}
 \|\nabla w_\ell^+\|_{L^2((-\infty, \ell) \times \omega)} &\leq \left(\frac{\Lambda}{\lambda} + 1\right) \|\nabla(\rho u_\infty)\|_{L^2((-\infty, \ell) \times \omega)} \\
 &\leq \left(\frac{\Lambda}{\lambda} + 1\right) (\|\nabla u_\infty\|_{L^2((\ell-1, \ell) \times \omega)} + \|u_\infty\|_{L^2((\ell-1, \ell) \times \omega)}) \\
 &\leq \left(\frac{\Lambda}{\lambda} + 1\right) (C_\omega + 1) \|\nabla u_\infty\|_{L^2((\ell-1, \ell) \times \omega)} \\
 &\leq \left(\frac{\Lambda}{\lambda} + 1\right) (C_\omega + 1) \|\nabla u_\infty\|_{L^2(\Omega_\ell)} \\
 &\leq C_3 e^{2\beta\ell},
 \end{aligned}$$

where C_ω is the Poincaré constant on $\omega \subset \mathbb{R}^{n-1}$ and C_3 is given by $C_3 = C_0 \left(\frac{\Lambda}{\lambda} + 1\right) (C_\omega + 1)$. \square

Remark 2.2.1. *Note that in the previous proof we established an even more refined estimate for ∇w_ℓ^+ , since we proved that*

$$\|\nabla w_\ell^+\|_{L^2((-\infty, \ell) \times \omega)} \leq \tilde{C} \|\nabla u_\infty\|_{L^2((\ell-1, \ell) \times \omega)}, \quad (2.2.7)$$

with $\tilde{C} = \left(\frac{\Lambda}{\lambda} + 1\right) (C_\omega + 1)$. Hence the constant \tilde{C} is only depending on Λ , λ and ω . This more precise inequality cannot be exploited under the very weak assumptions (2.1.2) on the data. However, if the function f satisfies stronger conditions with respect to its growth at infinity, inequality (2.2.7) can be used in order to derive a very interesting result, as it will be seen at the end of this section.

Now comes an important result giving an upper estimate for the L^2 -norm of $\nabla(u_\ell - u_\infty - w_\ell^+)$:

Theorem 2.2.1. *There exists a constant $C \geq 0$, depending only on λ , Λ and ω , such that*

$$\|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_\ell^+)} \leq C(\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_1^+)} + \|\nabla w_\ell^+\|_{L^2(\Omega_1)}). \quad (2.2.8)$$

Proof. Noting that $H_0^1(\Omega_\ell^+)$ is included in both $H_0^1(\Omega_\ell)$ and $H_0^1((-\infty, \ell) \times \omega)$, we derive from the equations (2.1.5), (2.1.7) and (2.1.12) satisfied by u_ℓ , u_∞ and w_ℓ^+ that

$$\int_{\Omega_\ell^+} A \nabla(u_\ell - u_\infty - w_\ell^+) \nabla v \, dx = 0 \quad \forall v \in H_0^1(\Omega_\ell^+) \quad (2.2.9)$$

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Let ρ be the function defined by

$$\begin{cases} \rho = \rho(x_1) \\ \rho(x_1) = 1 - x_1 & \text{for } 0 \leq x_1 \leq 1 \\ \rho(x_1) = 0 & \text{for } x_1 > 1. \end{cases} \quad (2.2.10)$$

Noticing that $v = (1 - \rho)(u_\ell - u_\infty - w_\ell^+)$ is in $H_0^1(\Omega_\ell^+)$, we have

$$\begin{aligned} 0 &= \int_{\Omega_\ell^+} A \nabla(u_\ell - u_\infty - w_\ell^+) \nabla v \, dx \\ &= \int_{\Omega_\ell^+} A \nabla(u_\ell - u_\infty - w_\ell^+) \nabla(u_\ell - u_\infty - w_\ell^+) \, dx \\ &\quad - \int_{\Omega_\ell^+} A \nabla(u_\ell - u_\infty - w_\ell^+) \nabla(\rho(u_\ell - u_\infty - w_\ell^+)) \, dx \end{aligned}$$

and by (2.1.3),

$$\begin{aligned} \lambda \int_{\Omega_\ell^+} |\nabla(u_\ell - u_\infty - w_\ell^+)|^2 \, dx &\leq \int_{\Omega_\ell^+} A \nabla(u_\ell - u_\infty - w_\ell^+) \nabla(u_\ell - u_\infty - w_\ell^+) \, dx \\ &= \int_{\Omega_\ell^+} A \nabla(u_\ell - u_\infty - w_\ell^+) [\rho'(x_1)(u_\ell - u_\infty - w_\ell^+, 0, \dots, 0) \\ &\quad + \rho \nabla(u_\ell - u_\infty - w_\ell^+)] \, dx. \end{aligned} \quad (2.2.11)$$

By taking into account the fact that $\rho = 0$ outside of Ω_1^+ , we derive

$$\begin{aligned} \lambda \int_{\Omega_\ell^+} |\nabla(u_\ell - u_\infty - w_\ell^+)|^2 \, dx &\leq \int_{\Omega_1^+} A \nabla(u_\ell - u_\infty - w_\ell^+) (-1)(u_\ell - u_\infty - w_\ell^+, 0, \dots, 0) \, dx \\ &\quad + \int_{\Omega_1^+} \rho A \nabla(u_\ell - u_\infty - w_\ell^+) \nabla(u_\ell - u_\infty - w_\ell^+) \, dx. \end{aligned}$$

Since $0 \leq \rho \leq 1$ on Ω_1 , we derive the following inequality, using the properties

(2.1.3) of A :

$$\begin{aligned}
 \lambda \|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_\ell^+)}^2 &= \lambda \int_{\Omega_\ell^+} |\nabla(u_\ell - u_\infty - w_\ell^+)|^2 dx \\
 &\leq \Lambda \left(\int_{\Omega_1^+} |\nabla(u_\ell - u_\infty - w_\ell^+)| |u_\ell - u_\infty - w_\ell^+| dx \right. \\
 &\quad \left. + \int_{\Omega_1^+} |\nabla(u_\ell - u_\infty - w_\ell^+)|^2 dx \right) \\
 &\leq \Lambda (\|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_1^+)} \|u_\ell - u_\infty - w_\ell^+\|_{L^2(\Omega_1^+)} \\
 &\quad + \|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_1^+)}^2) \\
 &\leq \Lambda(C_\omega + 1) \|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_1^+)}^2,
 \end{aligned}$$

where C_ω is the Poincaré constant on $\omega \subset \mathbb{R}^{n-1}$.

Setting $C = \sqrt{\frac{\Lambda}{\lambda}}(C_\omega + 1)$, we get (2.2.8) thanks to the triangle inequality. \square

The following result will help us to estimate the quality of the approximation of the solution u_ℓ of (2.1.5) by the corrected function $u_\infty + w_\ell$.

Theorem 2.2.2. *Let $k \in (-\infty, \ell - 1]$ be fixed. Then there exists $C_2 \geq 0$ depending only on Λ , λ , k and ω ; and $\alpha_2 > 0$ depending only on Λ , λ and ω such that*

$$\|\nabla w_\ell^+\|_{L^2((-\infty, k) \times \omega)} \leq C_2 e^{-\alpha_2 \ell} \|\nabla w_\ell^+\|_{L^2((-\infty, \ell) \times \omega)}. \quad (2.2.12)$$

Proof. Let us first denote by $h = h(x_1)$ the function defined by

$$h(x_1) = \begin{cases} 1 & \text{for } x_1 \leq k \\ 0 & \text{for } x_1 \geq k + 1 \\ -x_1 + k + 1 & \text{for } k < x_1 < k + 1. \end{cases} \quad (2.2.13)$$

Then, for $k + 1 \leq \ell$,

$$hw_\ell^+ \in H_0^1((-\infty, \ell) \times \omega), \quad (2.2.14)$$

so that we have

$$\int_{(-\infty, \ell) \times \omega} A \nabla w_\ell^+ \nabla (hw_\ell^+) dx = 0.$$

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This implies

$$\begin{aligned}
\lambda \int_{(-\infty, k) \times \omega} |\nabla w_\ell^+|^2 dx &\leq \int_{(-\infty, k) \times \omega} A \nabla w_\ell^+ \cdot \nabla w_\ell^+ dx \\
&\leq \int_{(-\infty, \ell) \times \omega} h A \nabla w_\ell^+ \cdot \nabla w_\ell^+ dx \\
&= - \int_{(-\infty, \ell) \times \omega} w_\ell^+ A \nabla w_\ell^+ \cdot \nabla h dx \\
&\leq \Lambda \int_{(k, k+1) \times \omega} |w_\ell^+| |\nabla w_\ell^+| dx \\
&\leq \Lambda \|w_\ell^+\|_{L^2((k, k+1) \times \omega)} \|\nabla w_\ell^+\|_{L^2((k, k+1) \times \omega)} \\
&\leq \Lambda C_\omega \|\nabla w_\ell^+\|_{L^2((k, k+1) \times \omega)}^2
\end{aligned}$$

by (2.1.3), the Cauchy-Schwarz and the Poincaré inequalities. Consequently

$$\begin{aligned}
\int_{(-\infty, k) \times \omega} |\nabla w_\ell^+|^2 dx &\leq \frac{\Lambda C_\omega}{\lambda} \int_{(k, k+1) \times \omega} |\nabla w_\ell^+|^2 dx \\
&= \frac{\Lambda C_\omega}{\lambda} \int_{(-\infty, k+1) \times \omega} |\nabla w_\ell^+|^2 dx - \frac{\Lambda C_\omega}{\lambda} \int_{(-\infty, k) \times \omega} |\nabla w_\ell^+|^2 dx
\end{aligned}$$

which, for $C = \frac{\Lambda C_\omega}{\lambda}$, reads

$$\int_{(-\infty, k) \times \omega} |\nabla w_\ell^+|^2 dx \leq \left(\frac{C}{C+1} \right) \int_{(-\infty, k+1) \times \omega} |\nabla w_\ell^+|^2 dx.$$

Iterating $[\ell - k]$ times this formula, we get

$$\int_{(-\infty, k) \times \omega} |\nabla w_\ell^+|^2 dx \leq \left(\frac{C}{C+1} \right)^{[\ell - k]} \int_{(-\infty, \ell) \times \omega} |\nabla w_\ell^+|^2 dx.$$

Since $\ell - k - 1 \leq [\ell - k] \leq \ell - k$ this leads to

$$\int_{(-\infty, k) \times \omega} |\nabla w_\ell^+|^2 dx \leq e^{-2\alpha_2(\ell - k - 1)} \int_{(-\infty, \ell) \times \omega} |\nabla w_\ell^+|^2 dx,$$

with $\alpha_2 = \frac{1}{2} \ln \frac{1+C}{C}$. This completes the proof by taking $C_2 = e^{\alpha_2(k+1)}$. \square

This last result finally implies our main result which is

Theorem 2.2.3. *Assuming that the constant β appearing in inequalities (2.1.2) and (2.1.7) is small enough, there exists C depending only on Λ , λ , C_0 and ω and α depending only on Λ , λ , ω and β two positive constants such that*

$$\|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_\ell^+)} \leq Ce^{-\alpha\ell} \quad (2.2.15)$$

for all $\ell > 0$.

Proof. By Theorem 2.2.1, we know that

$$\|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_\ell^+)} \leq C(\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_1^+)} + \|\nabla w_\ell^+\|_{L^2(\Omega_1^+)}).$$

Combining this inequality with (2.1.9), (2.2.3) and (2.2.12), we directly have, since $\Omega_1^+ \subset (-\infty, 1) \times \omega$,

$$\begin{aligned} \|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_\ell^+)} &\leq C(C_1 e^{-\alpha_1 \ell} + C_2 e^{-\alpha_2 \ell} \|\nabla w_\ell^+\|_{L^2((-\infty, \ell) \times \omega)}) \\ &\leq C(C_1 e^{-\alpha_1 \ell} + C_2 C_3 e^{-\alpha_2 \ell} e^{2\beta \ell}) \leq C' e^{-\alpha \ell} \end{aligned} \quad (2.2.16)$$

with $\alpha = \min(\alpha_1, \alpha_2 - 2\beta)$.

The constant α is positive, provided that $2\beta < \alpha_2$. This is indeed the case, since the assumption of smallness that β must satisfy in Theorem 2.1.2 is exactly the same (i.e. $2\beta < \alpha_2$, with the α_2 defined at the end of the proof of Theorem 2.2.2). In order to see this, it is enough to follow the proof of Theorem 1.2.1, for instance in [11], since in our particular case, the constant c_0 appearing in the proof of Theorem 2.1 of [11] (corresponding to the Theorem 1.2.1 of the present paper) is equal to 1. \square

Remark 2.2.2. 1. *It is important to notice that the construction of the constant α in (2.2.15) is well done, since it is not cyclical. Indeed, this is due to the fact that the constant α_2 of (2.2.12) depends only on Λ , λ and ω , and therefore is independent of the constants C_0 and β appearing in (2.1.7).*

2. *Note also that in order to obtain the estimate (2.2.15), we only need an estimate of the H^1 -norm of $u_\ell - u_\infty$ on Ω_1 . But this kind of estimate can be obtained even under the weaker assumption that $\beta < \alpha_2$. However, in this case, in order to preserve the uniqueness of the solution u_∞ to problem (2.1.7), one must replace 2β by β in the growth condition satisfied by u_∞ .*

By the same arguments used to prove Theorem 2.2.3, we get

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Theorem 2.2.4. *Assuming that the constant β appearing in inequalities (2.1.2) and (2.1.7) is small enough, there exists C depending only on Λ , λ , C_0 and ω ; and α depending only on Λ , λ , ω and β two positive constants such that*

$$\|\nabla(u_\ell - u_\infty - w_\ell^-)\|_{L^2(\Omega_\ell^-)} \leq Ce^{-\alpha\ell}. \quad (2.2.17)$$

Observing that

$$\|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_\ell)}^2 = \|\nabla(u_\ell - u_\infty - w_\ell^-)\|_{L^2(\Omega_\ell^-)}^2 + \|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_\ell^+)}^2 \quad (2.2.18)$$

where ∇w_ℓ is to be understood as the distributional derivative in the space $\mathcal{D}'(\Omega_\ell \setminus (\{0\} \times \omega))$, we obtain the main result of this paper:

Theorem 2.2.5. *There exists positive constants C and α independent of ℓ such that*

$$\|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_\ell)} \leq Ce^{-\alpha\ell} \quad (2.2.19)$$

for all $\ell > 0$, where w_ℓ is the corrector introduced in (2.1.11).

As mentioned in the introduction, the corrector defined in (2.1.11) does not belong to $H^1(\Omega_\ell)$, since the traces of w_ℓ^+ and w_ℓ^- on $\{0\} \times \omega$ may not coincide. However, we can overcome this difficulty by multiplying w_ℓ by a Lipschitz function vanishing in the neighbourhood of $\{0\} \times \omega$. More precisely, we have the following:

Theorem 2.2.6. *Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz-continuous function such that*

$$\psi(x_1) = \begin{cases} 1 & \text{if } |x_1| > 1 \\ 0 & \text{if } |x_1| < \frac{1}{2}, \end{cases}$$

and let w_ℓ be the corrector introduced in (2.1.11). Then $\tilde{w}_\ell = \psi w_\ell$ belongs to $H^1(\Omega_\ell)$ and satisfies

$$\|\nabla(u_\ell - u_\infty - \tilde{w}_\ell)\|_{L^2(\Omega_\ell)} \leq Ce^{-\alpha\ell} \text{ for all } \ell > 0,$$

where $C, \alpha > 0$ are constants independent of ℓ .

Proof. Notice that with such a definition, we have that $\tilde{w}_\ell = w_\ell$ outside of Ω_1 . In order to get the desired inequality, let us start by estimating ψw_ℓ^+ in Ω_1^+ , a symmetrical argument being used to give an estimate of ψw_ℓ^- in

Ω_1^- . Since ψ is a Lipschitz-continuous function, there exists two constants $K_1 > 0$ and $K_2 > 0$ such that $\|\psi\|_{L^\infty(\Omega_1^+)} \leq K_1$ and $\|\nabla\psi\|_{L^\infty(\Omega_1^+)} \leq K_2$. Then we have, using the Cauchy-Schwarz and the Poincaré inequalities combined with (2.2.12) and (2.2.3),

$$\begin{aligned}
 \|\nabla(\psi w_\ell^+)\|_{L^2(\Omega_1^+)} &\leq \|\psi \nabla w_\ell^+ + w_\ell^+ \nabla \psi\|_{L^2(\Omega_1^+)} \\
 &\leq \|\psi \nabla w_\ell^+\|_{L^2(\Omega_1^+)} + \|w_\ell^+ \nabla \psi\|_{L^2(\Omega_1^+)} \\
 &\leq \|\psi\|_{L^\infty(\Omega_1^+)} \|\nabla w_\ell^+\|_{L^2(\Omega_1^+)} + \|\nabla \psi\|_{L^\infty(\Omega_1^+)} \|w_\ell^+\|_{L^2(\Omega_1^+)} \\
 &\leq \|\psi\|_{L^\infty(\Omega_1^+)} \|\nabla w_\ell^+\|_{L^2(\Omega_1^+)} + \|\nabla \psi\|_{L^\infty(\Omega_1^+)} C_\omega \|\nabla w_\ell^+\|_{L^2(\Omega_1^+)} \\
 &\leq (\|\psi\|_{L^\infty(\Omega_1^+)} + C_\omega \|\nabla \psi\|_{L^\infty(\Omega_1^+)}) \|\nabla w_\ell^+\|_{L^2((-\infty, 1) \times \omega)} \\
 &\leq (K_1 + C_\omega K_2) C_2 e^{-\alpha_2 \ell} \|\nabla w_\ell^+\|_{L^2((-\infty, \ell) \times \omega)} \\
 &\leq C_4 e^{-(\alpha_2 - 2\beta)\ell}
 \end{aligned}$$

with $C_4 = C_2 C_3 (K_1 + K_2 C_\omega)$ a positive constant depending on ω , λ , Λ , C_0 and ψ .

It follows that there exists C_5 depending on the same parameters as C_4 such that $\|\nabla \tilde{w}_\ell\|_{L^2(\Omega_1)} \leq C_5 e^{-(\alpha_2 - 2\beta)\ell}$.

Then we have, using (2.1.9), (2.2.19) and the previous inequality,

$$\begin{aligned}
 \|\nabla(u_\ell - u_\infty - \tilde{w}_\ell)\|_{L^2(\Omega_\ell)} &\leq \|\nabla(u_\ell - u_\infty - \tilde{w}_\ell)\|_{L^2(\Omega_\ell \setminus \Omega_1)} + \|\nabla(u_\ell - u_\infty - \tilde{w}_\ell)\|_{L^2(\Omega_1)} \\
 &= \|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_\ell \setminus \Omega_1)} + \|\nabla(u_\ell - u_\infty - \tilde{w}_\ell)\|_{L^2(\Omega_1)} \\
 &\leq \|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_\ell)} + \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_1)} \\
 &\quad + \|\nabla \tilde{w}_\ell\|_{L^2(\Omega_1)} \\
 &\leq C e^{-\alpha \ell} + C_1 e^{-\alpha_1 \ell} + C_5 e^{-(\alpha_2 - 2\beta)\ell} \\
 &\leq C' e^{-\alpha \ell}
 \end{aligned}$$

which is exactly the result we were looking for, with $\alpha = \min\{\alpha_1, \alpha_2 - 2\beta\}$ (see proof of Theorem 2.2.3). \square

Remark 2.2.3. Thanks to inequality (2.2.7) of Remark 2.2.1, if $f \in H^{-1}(\Omega_\infty)$ (hence in particular, if $f \in L^2(\Omega_\infty)$), we have that

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_\ell)} \rightarrow 0, \quad (2.2.20)$$

since $f \in H^{-1}(\Omega_\infty)$ implies $u_\infty \in H^1(\Omega_\infty)$ and therefore,

$$\|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell-1})}^2 = \|\nabla u_\infty\|_{L^2((-\infty, -\ell+1) \times \omega)}^2 + \|\nabla u_\infty\|_{L^2((\ell-1, +\infty) \times \omega)}^2 \rightarrow 0$$

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as ℓ goes to infinity.

The convergence (2.2.20) then follows from inequality (2.2.7), (2.2.8) and their similar counterparts involving w_ℓ^- . Nevertheless, even in this case, using a corrector remains interesting in order to improve the rate of convergence.

2.3 An important particular case

In this section, we study the case in which both a_{ij} , $1 \leq i, j \leq n$ and f are not depending on x_1 .

First of all, we make the following remark: assuming that instead of condition (2.1.2), f satisfies the stronger condition

$$\|f\|_{H^{-1}(\Omega_\ell)} = O(\ell^\gamma), \quad (2.3.1)$$

where $\gamma > 0$ is a constant, and using the methods introduced in [23], we can derive the stronger convergence result

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-\eta \ln \ell})} \xrightarrow{\ell \rightarrow +\infty} 0,$$

for some large enough constant $\eta > 0$. This is a general result that is true even if the data is not necessarily independent of x_1 and is also valid in the framework of domains Ω_ℓ that goes to infinity in several directions (for a rigorous definition of these domains, see e.g., [23] or [11]). Note that in the particular case of this section, f satisfies the hypothesis (2.3.1) with $\gamma = \frac{1}{2}$.

However, in the case of cylinders going to infinity only in one direction and where the data is independent of x_1 (and also in the case where the data is periodic in the x_1 -direction), one can even prove the following stronger convergence:

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-\kappa(\ell)})} \xrightarrow{\ell \rightarrow +\infty} 0, \quad (2.3.2)$$

where $\kappa(\ell) \leq \ell$ and $\lim_{\ell \rightarrow +\infty} \kappa(\ell) = +\infty$.

Thanks to the results shown in the previous section applied to this particular case, we are now able to prove that the convergence result (2.3.2) is optimal. This optimality is to be understood in the following sense: we are looking for the largest domains that can be considered in the L^2 -norm appearing in (2.3.2) such that the convergence of u_ℓ to u_∞ described in (2.3.2)

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remains valid. In the end of this section we give two examples proving that, if we want the convergence (2.3.2) to be preserved, we cannot consider bigger domains than these of the type $\Omega_{\ell-k(\ell)}$ with $\lim_{x \rightarrow +\infty} k(\ell) = +\infty$.

Let us first remind that, since A and f are independent of x_1 , it follows from the definition of u_∞ that u_∞ has the same property. Indeed, it is easy to see that if $u_\infty \in H_0^1(\omega)$ is the weak solution to the problem (see Section 1 for the definitions of A' and div')

$$\begin{cases} -\text{div}'(A'\nabla u_\infty) = f & \text{in } \omega \\ u_\infty = 0 & \text{on } \partial\omega, \end{cases} \quad (2.3.3)$$

then $u_\infty = u_\infty(x')$ is the solution to problem (2.1.7).

Hence, the functions w_ℓ^+ and w_ℓ^- are solutions of the variational problems

$$\begin{cases} w_\ell^+ \in H^1((-\infty, \ell) \times \omega) \\ \int_{(-\infty, \ell) \times \omega} A \nabla w_\ell^+ \cdot \nabla v \, dx = 0 \quad \forall v \in H_0^1((-\infty, \ell) \times \omega) \\ w_\ell^+ = -u_\infty \quad \text{on } \{\ell\} \times \omega \\ w_\ell^+ = 0 \quad \text{on } (-\infty, \ell) \times \partial\omega \end{cases} \quad (2.3.4)$$

and

$$\begin{cases} w_\ell^- \in H^1((-\ell, +\infty) \times \omega) \\ \int_{(-\ell, +\infty) \times \omega} A \nabla w_\ell^- \cdot \nabla v \, dx = 0 \quad \forall v \in H_0^1((-\ell, +\infty) \times \omega) \\ w_\ell^- = -u_\infty \quad \text{on } \{-\ell\} \times \omega \\ w_\ell^- = 0 \quad \text{on } (-\ell, +\infty) \times \partial\omega. \end{cases} \quad (2.3.5)$$

Since A and u_∞ are independent of x_1 , it is obvious that these functions can be obtained by translations from the solution to problems set on the fixed domains $(-\infty, 0) \times \omega$ and $(0, +\infty) \times \omega$. More specifically, we have

$$w_\ell^+(x_1, x') = w^+(x_1 - \ell, x'), \quad (2.3.6)$$

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where w^+ is defined by

$$\begin{cases} w^+ \in H^1((-\infty, 0) \times \omega) \\ \int_{(-\infty, 0) \times \omega} A \nabla w^+ \nabla v \, dx = 0 \quad \forall v \in H_0^1((-\infty, 0) \times \omega) \\ w^+ = -u_\infty \quad \text{on } \{0\} \times \omega \\ w^+ = 0 \quad \text{on } (-\infty, 0) \times \partial\omega \end{cases} \quad (2.3.7)$$

and

$$w_\ell^-(x_1, x') = w^-(x_1 + \ell, x')$$

with w^- defined by

$$\begin{cases} w^- \in H^1((0, +\infty) \times \omega) \\ \int_{(0, +\infty) \times \omega} A \nabla w^- \nabla v \, dx = 0 \quad \forall v \in H_0^1((0, +\infty) \times \omega) \\ w^- = -u_\infty \quad \text{on } \{0\} \times \omega \\ w^- = 0 \quad \text{on } (0, +\infty) \times \partial\omega. \end{cases} \quad (2.3.8)$$

Remark 2.3.1. *Using the properties of w^+ and w^- , we can justify that in general the following negative result takes place:*

For a constant $a > 0$, we have that

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-a})} \not\rightarrow 0.$$

Let us suppose that, to the contrary,

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-a})} \rightarrow 0.$$

Then, since

$$\|\nabla w_\ell\|_{L^2(\Omega_{\ell-a})} \leq \|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_{\ell-a})} + \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-a})}$$

and

$$\|\nabla(u_\ell - u_\infty - w_\ell)\|_{L^2(\Omega_{\ell-a})} \rightarrow 0 \quad (\text{thanks to (2.2.19)}),$$

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\ell-a})} \rightarrow 0,$$

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as ℓ goes to $+\infty$, we have

$$\|\nabla w_\ell\|_{L^2(\Omega_{\ell-a})} \rightarrow 0. \quad (2.3.9)$$

Obviously $\|\nabla w_\ell^+\|_{L^2(\Omega_{\ell-a}^+)} = \|\nabla w^+\|_{L^2((-\ell, -a) \times \omega)}$. Since the function $\ell \mapsto \|\nabla w^+\|_{L^2((-\ell, -a) \times \omega)}$ is positive and increasing, it follows from (2.3.9) that $\|\nabla w^+\|_{L^2((-\infty, -a) \times \omega)} = 0$. Using the Poincaré inequality, we deduce that

$$w^+ = 0 \quad \text{a.e. in } (-\infty, -a) \times \omega. \quad (2.3.10)$$

Let us now give two examples where this is impossible:

- If the coefficients of A are analytic, then it follows from its definition that w^+ is analytic on $(-\infty, 0) \times \omega$, see for instance [43] or [52]. Using (2.3.10) we therefore derive $w^+ = 0$ in $(-\infty, 0) \times \omega$, which contradicts $w^+ = -u_\infty$ on $\{0\} \times \omega$ in the case where $u_\infty \neq 0$, i.e. if $f \neq 0$.
- If $f \leq 0$ on ω , then by the weak maximum principle $u_\infty \leq 0$ a.e. in ω , hence a.e. in Ω_∞ . Therefore $w^+ = -u_\infty \geq 0$ a.e. on $\{0\} \times \omega$ and again, by the weak maximum principle applied to w^+ , one has $w^+ \geq 0$ a.e. in $(-\infty, 0) \times \omega$. But $w^+ = 0$ a.e. in $(-\infty, -a) \times \omega$, hence w^+ “achieves its minimum” inside $(-\infty, 0) \times \omega$, in the sense that there exists an open ball $B \subset \subset (-\infty, 0) \times \omega$ such that $\text{ess inf}_B w^+ = \text{ess inf}_{(-\infty, 0) \times \omega} w^+$. Therefore, thanks to the strong maximum principle (see, e.g. [37]), we derive $w^+ = 0$ on $(-\infty, 0) \times \omega$. This is in contradiction with $w^+ = -u_\infty$ on $\{0\} \times \omega$, if $u_\infty \neq 0$. Obviously, a similar argument is valid in the case $f \geq 0$ in ω .

The result described in Remark 2.3.1 shows the optimality of the convergence result (2.3.2). We would like to mention here the fact that the non-convergence result of Remark 2.3.1 have been proved in [17] (see also [15]), in the particular case of the laplacian.

Chapter 3

Asymptotic analysis of some elliptic problems in long cylinders with data decreasing at infinity

The asymptotic study of PDEs in cylinders becoming unbounded in one or several directions have been fairly developed since the beginning of our century, starting from Michel Chipot and his collaborators. In this paper we give some estimation about the rate of the convergence to the solution of a linear elliptic problem on an infinite cylinder of the solutions of the same problem taken on larger and larger truncations of the cylinder when the data is in $H^{-1}(\Omega_\infty)$, Ω_∞ being the infinite cylinder. The difference with the previous articles of Chipot and co-authors is that the convergence takes place in the whole cylinder Ω_ℓ (see here below for its definition) instead of only a portion of it, that usually is $\Omega_{\ell/2}$. As expected, this is possible thanks to more restrictive assumptions on the data.

3.1 Introduction

We put ourselves in a generalized cylinder $\Omega_\ell = \ell\omega_1 \times \omega_2 \subset \mathbb{R}^n$, where $\omega_1 \subset \mathbb{R}^k$ ($1 \leq k \leq n-1$) and $\omega_2 \subset \mathbb{R}^{n-k}$ are bounded open sets, and we consider a linear elliptic problem with homogeneous Dirichlet boundary conditions, where the data is given f , an element belonging to $H^{-1}(\mathbb{R}^k \times \omega_2)$.

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The aim here is to prove that the solutions u_ℓ of the Dirichlet problem in Ω_ℓ are converging, as ℓ goes to infinity, to the solution of a similar Dirichlet problem on the infinite cylinder, at a speed which is either exponential or equivalent to the decreasing speed of f at infinity.

In the case $k = 1$, these results can also be obtained as a consequence of the estimates obtained in the previous chapter, where the use of correctors allowed us to obtain good estimates in the whole domain Ω_ℓ .

Let us now give the notation that will be used in this chapter.

The notation $|\cdot|$ stands for the Euclidean norm in \mathbb{R}^m ($m \in \mathbb{N}^*$). For $x \in \mathbb{R}^m$ and $r > 0$, we use the notation $B(x, r)$ for the open ball of \mathbb{R}^m , centered at x of radius r :

$$B(x, r) = \{y \in \mathbb{R}^m ; |y - x| < r\}.$$

We denote by $|A|$ the m -dimensional Lebesgue measure of a measurable subset A of \mathbb{R}^m .

If \mathcal{O} is an open subset of \mathbb{R}^n and $v \in H_0^1(\mathcal{O})$, then by convention, we will also denote by v its extension by 0 outside of \mathcal{O} . Note that in this case, we have $v \in H_0^1(U)$ for any open set $U \supset \mathcal{O}$.

Let k be an integer such that $1 \leq k \leq n - 1$ and ω_1 be a bounded domain (i.e., open and connected) of \mathbb{R}^k , verifying

$$\omega_1 \text{ is star-shaped with respect to an open ball of } \mathbb{R}^k \text{ centered at } 0. \quad (3.1.1)$$

Note that in particular, any bounded open convex set containing 0 satisfies the property (3.1.1).

Let ω_2 be a bounded Lipschitz domain of \mathbb{R}^{n-k} . We set

$$\Omega_\ell = \ell\omega_1 \times \omega_2, \quad \Omega_\infty = \mathbb{R}^k \times \omega_2.$$

For a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set

$$X_1 = (x_1, \dots, x_k), \quad X_2 = (x_{k+1}, \dots, x_n),$$

hence we can write $x = (X_1, X_2)$, and we use the notation

$$\nabla_{X_1} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right), \quad \nabla_{X_2} = \left(\frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_n} \right).$$

We will also use the notation ∂_{x_i} for the partial derivative $\frac{\partial}{\partial x_i}$.

The hypothesis of ω_1 satisfying the property (3.1.1) is fundamental in the proof of the main result of this paper, since this property allows us to construct a function

$$\rho_\ell = \rho_\ell(X_1) \in W^{1,\infty}(\Omega_\infty)$$

such that

$$0 \leq \rho_\ell \leq 1, \quad \rho_\ell = 1 \text{ in } \Omega_\ell, \quad \rho_\ell = 0 \text{ in } \Omega_\infty \setminus \Omega_{\ell+1},$$

and $|\nabla_{X_1} \rho_\ell| \leq c_0$ in Ω_∞ , with c_0 a constant depending only on ω_1 . The construction of such functions ρ_ℓ is possible thanks to the following result, established in the first chapter (see Lemma 1.1.1).

Lemma 3.1.1. *Let $\omega_1 \subsetneq \mathbb{R}^k$ be an open set which is star-shaped with respect to the open ball $B_\delta = B(0, \delta) = \{x \in \mathbb{R}^k ; |x| < \delta\}$. Then for all $\ell > 0$, one has*

$$\text{dist}(\ell\omega_1, \mathbb{R}^k \setminus (\ell+1)\omega_1) \geq \delta. \quad (3.1.2)$$

Using classical methods, as for instance the convolution with a regularizing sequence of the indicator function of the set $E_{\ell, \frac{\delta}{2}} = \bigcup_{X_1 \in \ell\omega_1} B(X_1, \frac{\delta}{2})$ (with $\ell = \ell_1 - 1$ for the following lemma), we can now construct the function ρ_ℓ announced above, which plays a major role in the proof of the main result of this chapter.

Lemma 3.1.2. *For any $\ell_1 > 1$, there exists a function ρ in $W^{1,\infty}(\Omega_\infty)$ satisfying $\rho(x) = \rho(X_1)$,*

$$\rho = \begin{cases} 0 & \text{in } \Omega_{\ell_1-1} \\ 1 & \text{in } \Omega_\infty \setminus \Omega_{\ell_1} \\ 0 \leq \rho(x) \leq 1 & \text{in } \Omega_{\ell_1} \setminus \Omega_{\ell_1-1} \end{cases}$$

and

$$\|\nabla \rho\|_{L^\infty(\Omega_\infty)} \leq c_\delta, \quad (3.1.3)$$

where c_δ is a constant independent of ℓ_1 .

Another important tool that is used along this chapter is the following Poincaré inequality, which was already introduced in the first chapter (see Lemma 1.1.3).

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Lemma 3.1.3. *Let $v \in H^1(\Omega_\ell)$ such that $v = 0$ on $\ell\omega_1 \times \partial\omega_2$ and $\tilde{\omega}_1 \subset \ell\omega_1$ a measurable set. Then there exists a constant c_{ω_2} depending only on ω_2 such that:*

$$\|v\|_{L^2(\tilde{\omega}_1 \times \omega_2)} \leq c_{\omega_2} \|\nabla_{X_2} v\|_{L^2(\tilde{\omega}_1 \times \omega_2)}. \quad (3.1.4)$$

Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a field of $n \times n$ matrices defined on Ω_∞ , such that $a_{ij} \in L^\infty(\Omega_\infty)$ for all $i, j \in \{1, \dots, n\}$, and for some constants $\lambda > 0$ and $\Lambda > 0$,

$$\lambda|\xi|^2 \leq A(x)\xi \cdot \xi \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega_\infty, \quad (3.1.5)$$

$$|A(x)\xi| \leq \Lambda|\xi| \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega_\infty. \quad (3.1.6)$$

For any $\ell > 0$, we consider the following homogenous Dirichlet problem in Ω_ℓ :

$$\begin{cases} -\operatorname{div}(A\nabla u_\ell) = f & \text{in } \Omega_\ell \\ u_\ell = 0 & \text{on } \partial\Omega_\ell, \end{cases} \quad (3.1.7)$$

where $f \in H^{-1}(\Omega_\infty)$, so that $f \in H^{-1}(\Omega_\ell)$ for any $\ell > 0$.

Thanks to the Lax-Milgram theorem, there exists a unique weak solution to problem (3.1.7), i.e. there exists a unique solution $u_\ell \in H_0^1(\Omega_\ell)$ to the variational problem:

$$\int_{\Omega_\ell} A\nabla u_\ell \cdot \nabla v \, dx = \langle f, v \rangle \quad \text{for all } v \in H_0^1(\Omega_\ell). \quad (3.1.8)$$

Let $u_\infty \in H_0^1(\Omega_\infty)$ the weak solution to the following non-homogeneous Dirichlet problem in the cylinder Ω_∞ :

$$\begin{cases} -\operatorname{div}(A\nabla u_\infty) = f & \text{in } \Omega_\infty \\ u_\infty = 0 & \text{on } \partial\Omega_\infty \end{cases} \quad (3.1.9)$$

Since f belongs to $H^{-1}(\Omega_\infty)$, the variational equation associated to (3.1.9) simply writes as

$$\int_{\Omega_\infty} A\nabla u_\infty \cdot \nabla v \, dx = \langle f, v \rangle \quad \text{for all } v \in H_0^1(\Omega_\infty). \quad (3.1.10)$$

We now recall this classical result on the characterization of the H^{-1} -spaces, that can be found, e.g., in [10] or [27].

Lemma 3.1.4. *If Ω is an open subset of \mathbb{R}^n and f is an element of $H^{-1}(\Omega)$, then there exists some functions f_0, f_1, \dots, f_n in $L^2(\Omega)$, such that*

$$f = f_0 + \sum_{i=1}^n \partial_{x_i} f_i.$$

The following result is easily derived from the previous lemma.

Lemma 3.1.5. *If f belongs to $H^{-1}(\Omega_\infty)$, then we have*

$$\lim_{\ell \rightarrow \infty} \|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_\ell)} = 0$$

Proof. If $f \in H^{-1}(\Omega_\infty)$, then by Lemma 3.1.4, there exists some functions f_0, f_1, \dots, f_n in $L^2(\Omega_\infty)$ such that

$$f = f_0 + \sum_{i=1}^n \partial_{x_i} f_i.$$

Then we just have to estimate the norm

$$\|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_\ell)} = \sup_{\|v\|_{H_0^1(\Omega_\infty \setminus \overline{\Omega}_\ell)}=1} \langle f, v \rangle$$

In what follows, we consider the space $H_0^1(\Omega_\infty \setminus \overline{\Omega}_\ell)$ endowed with the L^2 -norm of the gradient. Note that the result of Lemma 3.1.3 remains valid - with the corresponding modifications - if one considers v in the Sobolev space $H_0^1(\Omega_\infty \setminus \overline{\Omega}_\ell)$ (see the proof of Lemma 1.1.3).

For any v in $H_0^1(\Omega_\infty \setminus \overline{\Omega}_\ell)$ we have that

$$\langle f, v \rangle = \int_{\Omega_\infty \setminus \overline{\Omega}_\ell} f_0 v \, dx - \sum_{i=1}^n \int_{\Omega_\infty \setminus \overline{\Omega}_\ell} f_i \partial_{x_i} v \, dx,$$

and then, by applying the Cauchy-Schwarz inequality,

$$|\langle f, v \rangle| \leq \|f_0\|_{L^2(\Omega_\infty \setminus \overline{\Omega}_\ell)} \|v\|_{L^2(\Omega_\infty \setminus \overline{\Omega}_\ell)} + \sum_{i=1}^n \|f_i\|_{L^2(\Omega_\infty \setminus \overline{\Omega}_\ell)} \|\partial_{x_i} v\|_{L^2(\Omega_\infty \setminus \overline{\Omega}_\ell)}.$$

Hence, for any v satisfying $\|v\|_{H_0^1(\Omega_\infty \setminus \overline{\Omega}_\ell)} = 1$, one has

$$|\langle f, v \rangle| \leq c_{\omega_2} \|f_0\|_{L^2(\Omega_\infty \setminus \overline{\Omega}_\ell)} + \sum_{i=1}^n \|f_i\|_{L^2(\Omega_\infty \setminus \overline{\Omega}_\ell)},$$

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using the Poincaré inequality (3.1.4). Therefore,

$$\|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_\ell)} \leq c_{\omega_2} \|f_0\|_{L^2(\Omega_\infty \setminus \overline{\Omega}_\ell)} + \sum_{i=1}^n \|f_i\|_{L^2(\Omega_\infty \setminus \overline{\Omega}_\ell)}$$

and the result follows, since $\|f_i\|_{L^2(\Omega_\infty \setminus \overline{\Omega}_\ell)} \rightarrow 0$ as $\ell \rightarrow +\infty$ for any $0 \leq i \leq n$, taking into account the fact that $f_i \in L^2(\Omega_\infty)$. \square

3.2 Main result

To start this section, we recall this result from [16] and [11] (see Theorem 1.2.1 of the first chapter):

Theorem 3.2.1. *Let $f \in H^{-1}(\Omega_\infty)$. Then the functions u_ℓ and u_∞ introduced in the previous section verify the following inequality:*

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq Ce^{-\alpha\ell} \quad \text{for all } \ell > 0, \quad (3.2.1)$$

where $C \geq 0$ and $\alpha > 0$ are constants depending only on ω_1 , ω_2 , λ and Λ .

We will see in what follows that, in order to obtain a good estimate on the remaining part $\Omega_\ell \setminus \Omega_{\frac{\ell}{2}}$, it suffices to estimate separately u_ℓ and u_∞ on this part, in other words, to describe the decay of u_ℓ and u_∞ at infinity.

Note that in the case of u_∞ , this means that we study the behaviour at infinity of the solution to an elliptic equation in an infinite domain. We would like to emphasize that this kind of analysis is an important research topic on its own. Let us first remind the 1963 paper of E.M. Landis [48], and later, in 1977, the work of E.M. Landis and G.P. Panasenko [49], where the subject was the behaviour at infinity of solutions to elliptic equations. For the linear elasticity problem, the interested reader can consult the 1982 paper of O.A. Oleinik and G.A. Yosifian [56]. The asymptotic behavior at infinity of solutions to Stokes or Navier-Stokes problems in domains with infinite cylindrical outlets was the subject of many articles. To mention only a few, see for instance the works of L.V. Kapitanskii [44], O.A. Ladyzhenskaya and V.A. Solonnikov [47], or S.A. Nazarov and K. Pileckas ([54], [65],[66]).

For the sake of completeness, we include here the proof of the result describing the behaviour at infinity of u_∞ , which is very similar to the one

concerning u_ℓ . In many of the works cited above, the asymptotic behaviour at infinity is described by means of weighted Sobolev spaces. Here, we only use elementary tools, resembling to the ones introduced in Chapter 1.

Theorem 3.2.2. *Assume that $f \in H^{-1}(\Omega_\infty)$. Then for any ℓ positive, the following inequality stands:*

$$\|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\frac{\ell}{2}})} \leq C \left(e^{-\tilde{\alpha}\ell} + \|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_{\frac{\ell}{4}})} \right), \quad (3.2.2)$$

where $C \geq 0$ and $\tilde{\alpha} > 0$ are constants depending only on ω_1 , ω_2 , λ and Λ .

Proof. The proof is divided into two steps.

Step I. We will first prove that: for any $1 < \ell_1 < \ell$, we have

$$\|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1})}^2 \leq b \|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_{\ell_1-1})}^2 + a \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})}^2,$$

where the constants $b \geq 0$ and $0 < a < 1$ depend only on ω_1 , ω_2 , λ and Λ (hence they are independent of ℓ and ℓ_1).

So let $1 < \ell_1 < \ell$. We use the function ρu_ℓ , which belongs to $H_0^1(\Omega_\ell)$, as a test function in (3.1.8) (ρ being the function introduced in Lemma 3.1.2), leading us to the following computations:

$$\begin{aligned} \int_{\Omega_\ell} A \nabla u_\ell \nabla(\rho u_\ell) dx &= \int_{\Omega_{\ell_1-1}} A \nabla u_\ell \nabla(\rho u_\ell) dx + \int_{\Omega_{\ell_1} \setminus \Omega_{\ell_1-1}} A \nabla u_\ell \nabla(\rho u_\ell) dx \\ &\quad + \int_{\Omega_\ell \setminus \Omega_{\ell_1}} A \nabla u_\ell \nabla(\rho u_\ell) dx \\ &= \langle f, \rho u_\ell \rangle. \end{aligned}$$

Since the function ρ is equal to 0 on Ω_{ℓ_1-1} , we have that

$$\int_{\Omega_{\ell_1-1}} A \nabla u_\ell \nabla(\rho u_\ell) dx = 0.$$

Secondly, since by definition ρ is equal to 1 in $\Omega_\ell \setminus \overline{\Omega}_{\ell_1}$, one has that $\rho u_\ell = u_\ell$ in the open set $\Omega_\ell \setminus \overline{\Omega}_{\ell_1}$. Hence $\nabla(\rho u_\ell) = \nabla u_\ell$ in $\Omega_\ell \setminus \overline{\Omega}_{\ell_1}$. Using

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the fact that $|\Omega_\ell \cap \partial\Omega_{\ell_1}| = 0$ (thanks to the assumptions on ω_1 , which imply that ω_1 is a Lipschitz domain; see e.g., Teorema 1.I in [33]), it follows that

$$\begin{aligned} \int_{\Omega_\ell \setminus \Omega_{\ell_1}} A \nabla u_\ell \nabla(\rho u_\ell) dx &= \int_{\Omega_\ell \setminus \overline{\Omega}_{\ell_1}} A \nabla u_\ell \nabla(\rho u_\ell) dx = \int_{\Omega_\ell \setminus \overline{\Omega}_{\ell_1}} A \nabla u_\ell \nabla u_\ell dx \\ &= \int_{\Omega_\ell \setminus \Omega_{\ell_1}} A \nabla u_\ell \nabla u_\ell dx. \end{aligned}$$

Using this two observations combined with the property (3.1.5) verified by A , we have

$$\begin{aligned} \lambda \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1})}^2 &\leq \int_{\Omega_\ell \setminus \Omega_{\ell_1}} A \nabla u_\ell \nabla u_\ell dx = \langle f, \rho u_\ell \rangle - \int_{\Omega_{\ell_1} \setminus \Omega_{\ell_1-1}} A \nabla u_\ell \nabla(\rho u_\ell) dx \\ &= \langle f, \rho u_\ell \rangle - \int_{\Omega_{\ell_1} \setminus \Omega_{\ell_1-1}} A \nabla u_\ell (\rho \nabla u_\ell + u_\ell \nabla \rho) dx. \end{aligned}$$

We now use (3.1.6) in order to derive

$$\begin{aligned} \lambda \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1})}^2 &\leq \langle f, \rho u_\ell \rangle + \Lambda \|\nabla u_\ell\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})}^2 \\ &\quad + \Lambda \|\nabla \rho\|_{L^\infty(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})} \|\nabla u_\ell\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})} \|u_\ell\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})}. \end{aligned}$$

Notice that the function ρu_ℓ belongs to the space $H_0^1(\Omega_\ell \setminus \overline{\Omega}_{\ell_1-1})$. We use now the definition of the H^{-1} -norm and the inequalities (3.1.3) and (3.1.4), in order to deduce

$$\begin{aligned} \lambda \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1})}^2 &\leq \langle f, \rho u_\ell \rangle + \Lambda(1 + c_\delta c_{\omega_2}) \|\nabla u_\ell\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})}^2 \\ &\leq \|f\|_{H^{-1}(\Omega_\ell \setminus \overline{\Omega}_{\ell_1-1})} \|\nabla(\rho u_\ell)\|_{L^2(\Omega_\ell \setminus \overline{\Omega}_{\ell_1-1})} \\ &\quad + \Lambda(1 + c_\delta c_{\omega_2}) \|\nabla u_\ell\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})}^2 \\ &= \|f\|_{H^{-1}(\Omega_\ell \setminus \overline{\Omega}_{\ell_1-1})} \|\nabla(\rho u_\ell)\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})} \\ &\quad + \Lambda(1 + c_\delta c_{\omega_2}) \|\nabla u_\ell\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})}^2, \end{aligned}$$

where in the last equality we have used the fact that the n -dimensional Lebesgue measure of $\Omega_\ell \cap \partial\Omega_{\ell_1-1}$ vanishes.

Taking into account the properties of ρ and the Poincaré inequality (3.1.4), we obtain as in the previous computations,

$$\begin{aligned} \|\nabla(\rho u_\ell)\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})} &\leq \|\rho\|_{L^\infty(\Omega_\ell \setminus \Omega_{\ell_1-1})} \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})} \\ &\quad + \|\nabla \rho\|_{L^\infty(\Omega_\ell \setminus \Omega_{\ell_1-1})} \|u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})} \\ &\leq (1 + c_\delta c_{\omega_2}) \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})}. \end{aligned}$$

Combining the last two inequalities, we get

$$\begin{aligned} \lambda \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1})}^2 &\leq (1 + c_\delta c_{\omega_2}) \|f\|_{H^{-1}(\Omega_\ell \setminus \bar{\Omega}_{\ell_1-1})} \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})} \\ &\quad + \Lambda(1 + c_\delta c_{\omega_2}) \|\nabla u_\ell\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})}^2. \end{aligned}$$

To continue, we notice that

$$\Omega_{\ell_1} \setminus \Omega_{\ell_1-1} = (\Omega_\ell \setminus \Omega_{\ell_1-1}) \setminus (\Omega_\ell \setminus \Omega_{\ell_1}).$$

Therefore,

$$\|\nabla u_\ell\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})}^2 = \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})}^2 - \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1})}^2,$$

which leads to the following inequality:

$$\begin{aligned} (\lambda + \Lambda(1 + c_\delta c_{\omega_2})) \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1})}^2 &\leq (1 + c_\delta c_{\omega_2}) \|f\|_{H^{-1}(\Omega_\ell \setminus \bar{\Omega}_{\ell_1-1})} \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})} \\ &\quad + \Lambda(1 + c_\delta c_{\omega_2}) \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})}^2. \end{aligned}$$

We now use the inequality

$$|xy| \leq \frac{x^2}{4\varepsilon} + \varepsilon y^2$$

which is valid for any $\varepsilon > 0$, in order to derive

$$\begin{aligned} (\lambda + \Lambda(1 + c_\delta c_{\omega_2})) \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1})}^2 &\leq (1 + c_\delta c_{\omega_2}) \left(\frac{1}{4\varepsilon} \|f\|_{H^{-1}(\Omega_\ell \setminus \bar{\Omega}_{\ell_1-1})}^2 + \varepsilon \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})}^2 \right) \\ &\quad + \Lambda(1 + c_\delta c_{\omega_2}) \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})}^2 \end{aligned}$$

and therefore,

$$\begin{aligned} \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1})}^2 &\leq \frac{1 + c_\delta c_{\omega_2}}{4\varepsilon(\lambda + \Lambda(1 + c_\rho c_{\omega_2}))} \|f\|_{H^{-1}(\Omega_\ell \setminus \bar{\Omega}_{\ell_1-1})}^2 \\ &\quad + \frac{\varepsilon(1 + c_\delta c_{\omega_2}) + \Lambda(1 + c_\delta c_{\omega_2})}{\lambda + \Lambda(1 + c_\delta c_{\omega_2})} \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})}^2 \end{aligned}$$

Let us set

$$a = \frac{\varepsilon(1 + c_\delta c_{\omega_2}) + \Lambda(1 + c_\delta c_{\omega_2})}{\lambda + \Lambda(1 + c_\delta c_{\omega_2})} \quad \text{and} \quad b = \frac{1 + c_\delta c_{\omega_2}}{4\varepsilon(\lambda + \Lambda(1 + c_\rho c_{\omega_2}))},$$

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so that the previous inequality rewrites

$$\|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1})}^2 \leq b \|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\ell_1-1})}^2 + a \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\ell_1-1})}^2. \quad (3.2.3)$$

The goal of the first step is now attained, provided that the constant a satisfies $0 < a < 1$. Or this is obtained by choosing an $\varepsilon > 0$ such that $\varepsilon(1 + c_\delta c_{\omega_2}) < \lambda$.

Step II. We deduce the inequality (3.2.2) by an iteration technique. We first take $\ell_1 = \frac{\ell}{2}$ in inequality (3.2.3) and then iterate it $[\frac{\ell}{4}]$ times, with $[\cdot]$ denoting the integer part. Keeping in mind that $\sum_{j=0}^{+\infty} a^j = \frac{1}{1-a}$ (since $0 < a < 1$, this series is convergent), this leads to

$$\begin{aligned} \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\frac{\ell}{2}})}^2 &\leq b \sum_{i=1}^{[\frac{\ell}{4}]} \left(a^{i-1} \|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell}{2}-i})}^2 \right) + a^{[\frac{\ell}{4}]} \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\frac{\ell}{2}-[\frac{\ell}{4}]})}^2 \\ &\leq b \left(\sum_{i=1}^{[\frac{\ell}{4}]} a^{i-1} \right) \|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell}{4}})}^2 + a^{[\frac{\ell}{4}]} \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\frac{\ell}{4}})}^2 \\ &\leq \frac{b}{1-a} \|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell}{4}})}^2 + a^{\frac{\ell}{4}-1} \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\frac{\ell}{4}})}^2, \end{aligned}$$

using the fact that

$$\|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell}{2}-i})} \leq \|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell}{4}})} \quad \text{for all } i \in \left\{ 1, \dots, \left[\frac{\ell}{4} \right] \right\}.$$

Finally, we obtain, by taking $\tilde{\alpha} = \frac{1}{8} \ln(\frac{1}{a})$,

$$\begin{aligned} \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\frac{\ell}{2}})}^2 &\leq C \left(\|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell}{4}})}^2 + e^{-2\tilde{\alpha}\ell} \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\frac{\ell}{4}})}^2 \right) \\ &\leq C \left(\|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell}{4}})}^2 + e^{-2\tilde{\alpha}\ell} \|\nabla u_\ell\|_{L^2(\Omega_\ell)}^2 \right) \end{aligned}$$

and so we retrieve the result of the theorem, thanks to the inequality

$$\|\nabla u_\ell\|_{L^2(\Omega_\ell)} \leq \frac{1}{\lambda} \|f\|_{H^{-1}(\Omega_\ell)} \leq \frac{1}{\lambda} \|f\|_{H^{-1}(\Omega_\infty)},$$

obtained by taking u_ℓ as test function in (3.1.8). □

In the same spirit as the previous one, we have the following result:

Theorem 3.2.3. *Assume that $f \in H^{-1}(\Omega_\infty)$. Then for any ℓ positive, the following inequality stands:*

$$\|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\frac{\ell}{2}})} \leq C \left(e^{-\tilde{\alpha}\ell} + \|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell}{4}})} \right), \quad (3.2.4)$$

where $C \geq 0$ and $\tilde{\alpha} > 0$ are constants depending only on ω_1 , ω_2 , λ and Λ .

Proof. The proof is very similar to the one of Theorem 3.2.2. Therefore, we will outline it by insisting only on the most noteworthy aspects.

As in the proof of Theorem 3.2.2, we begin by establishing the following result:

There exists a constant $b \geq 0$ and a constant $a \in (0, 1)$ depending only on λ , Λ , ω_1 and ω_2 such that

$$\|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1})}^2 \leq b \|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\ell_1-1})}^2 + a \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1-1})}^2.$$

for any $\ell_1 > 1$.

So let $\ell_1 > 1$. The function ρu_ℓ belongs to $H_0^1(\Omega_\infty)$, hence we can use it as a test function in the variational equation (3.1.10):

$$\begin{aligned} \int_{\Omega_\infty} A \nabla u_\infty \nabla(\rho u_\infty) dx &= \int_{\Omega_{\ell_1-1}} A \nabla u_\infty \nabla(\rho u_\infty) dx + \int_{\Omega_{\ell_1} \setminus \Omega_{\ell_1-1}} A \nabla u_\infty \nabla(\rho u_\infty) dx \\ &\quad + \int_{\Omega_\infty \setminus \Omega_{\ell_1}} A \nabla u_\ell \nabla(\rho u_\infty) dx \\ &= \langle f, \rho u_\infty \rangle. \end{aligned}$$

As in the proof of Theorem 3.2.2, we deduce from the properties of ρ that $\nabla(\rho u_\infty) = 0$ in Ω_{ℓ_1-1} and

$$\int_{\Omega_\infty \setminus \Omega_{\ell_1}} A \nabla u_\infty \nabla(\rho u_\infty) dx = \int_{\Omega_\infty \setminus \Omega_{\ell_1}} A \nabla u_\infty \nabla u_\infty dx.$$

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Consequently, combining this equalities with the property (3.1.5) verified by A , we derive

$$\begin{aligned} \lambda \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1})}^2 &\leq \int_{\Omega_\infty \setminus \Omega_{\ell_1}} A \nabla u_\infty \nabla u_\infty \, dx \\ &= \langle f, \rho u_\infty \rangle - \int_{\Omega_{\ell_1} \setminus \Omega_{\ell_1-1}} A \nabla u_\infty \nabla(\rho u_\infty) \, dx \\ &= \langle f, \rho u_\infty \rangle - \int_{\Omega_{\ell_1} \setminus \Omega_{\ell_1-1}} A \nabla u_\infty (\rho \nabla u_\infty + u_\infty \nabla \rho) \, dx. \end{aligned}$$

Using the same arguments as in the proof of Theorem 3.2.2 and taking into account the fact that $\rho u_\infty \in H_0^1(\Omega_\infty \setminus \overline{\Omega}_{\ell_1-1})$, we get

$$\begin{aligned} \lambda \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1})}^2 &\leq \|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_{\ell_1-1})} \|\nabla(\rho u_\infty)\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1-1})} \\ &\quad + \Lambda(1 + c_\delta c_{\omega_2}) \|\nabla u_\infty\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})}^2 \\ &\leq (1 + c_\delta c_{\omega_2}) \|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_{\ell_1-1})} \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1-1})} \\ &\quad + \Lambda(1 + c_\delta c_{\omega_2}) \|\nabla u_\infty\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})}^2. \end{aligned}$$

Since

$$\Omega_{\ell_1} \setminus \Omega_{\ell_1-1} = (\Omega_\infty \setminus \Omega_{\ell_1-1}) \setminus (\Omega_\infty \setminus \Omega_{\ell_1}).$$

we deduce

$$\|\nabla u_\infty\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})}^2 = \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1-1})}^2 - \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1})}^2.$$

Thus, the last inequality becomes

$$\begin{aligned} (\lambda + \Lambda(1 + c_\delta c_{\omega_2})) \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1})}^2 &\leq (1 + c_\delta c_{\omega_2}) \|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_{\ell_1-1})} \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1-1})} \\ &\quad + \Lambda(1 + c_\delta c_{\omega_2}) \|\nabla u_\infty\|_{L^2(\Omega_{\ell_1} \setminus \Omega_{\ell_1-1})}^2. \end{aligned}$$

We then make the same computations as in the previous proof in order to obtain the inequality

$$\|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1})}^2 \leq b \|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_{\ell_1-1})}^2 + a \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\ell_1-1})}^2, \quad (3.2.5)$$

with $a \in (0, 1)$ and $b \geq 0$ given by the same formulas as in the proof of Theorem 3.2.2.

Now we use an iteration technique: we start with $\ell_1 = \frac{\ell}{2}$ in inequality (3.2.5) and then we iterate this inequality $[\frac{\ell}{4}]$ times. With the same arguments as in the previous proof, this leads us to the following inequality:

$$\begin{aligned} \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\frac{\ell}{2}})}^2 &\leq C \left(\|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_{\frac{\ell}{4}})}^2 + e^{-2\tilde{\alpha}\ell} \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\frac{\ell}{4}})}^2 \right) \\ &\leq C \left(\|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_{\frac{\ell}{4}})}^2 + e^{-2\tilde{\alpha}\ell} \|\nabla u_\infty\|_{L^2(\Omega_\infty)}^2 \right), \end{aligned}$$

where $\tilde{\alpha} = \frac{1}{8} \ln(\frac{1}{a})$. Combining the last inequality with the estimate

$$\|\nabla u_\infty\|_{L^2(\Omega_\infty)} \leq \frac{1}{\lambda} \|f\|_{H^{-1}(\Omega_\infty)},$$

obtained by taking u_∞ as test function in (3.1.10), allows us to establish the inequality (3.2.4) and thus to end the proof of the theorem. \square

Remark 3.2.1. *Note that the constant $\tilde{\alpha}$ appearing in (3.2.4) is exactly the same as the one in (3.2.2).*

Combining the previous results, we can now easily obtain the main result of this chapter.

Theorem 3.2.4. *Let $f \in H^{-1}(\Omega_\infty)$. Then for any $\ell > 0$, we have the estimate:*

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_\ell)} \leq C \left(e^{-\alpha'\ell} + \|f\|_{H^{-1}(\Omega_\infty \setminus \overline{\Omega}_{\frac{\ell}{4}})} \right), \quad (3.2.6)$$

for some constants $C \geq 0$ and $\alpha' > 0$ depending only on λ , Λ , ω_1 and ω_2 .

Proof. To start with, we make the following splitting:

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_\ell)}^2 = \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})}^2 + \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_\ell \setminus \Omega_{\frac{\ell}{2}})}^2$$

Therefore, using the triangular inequality,

$$\begin{aligned} \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_\ell)} &\leq \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} + \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_\ell \setminus \Omega_{\frac{\ell}{2}})} \\ &\leq \|\nabla(u_\ell - u_\infty)\|_{H^1(\Omega_\ell)} + \|\nabla u_\ell\|_{H^1(\Omega_\ell \setminus \Omega_{\frac{\ell}{2}})} + \|\nabla u_\infty\|_{L^2(\Omega_\ell \setminus \Omega_{\frac{\ell}{2}})} \\ &\leq \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_\ell)} + \|\nabla u_\ell\|_{L^2(\Omega_\ell \setminus \Omega_{\frac{\ell}{2}})} + \|\nabla u_\infty\|_{L^2(\Omega_\infty \setminus \Omega_{\frac{\ell}{2}})}. \end{aligned}$$

The inequality (3.2.6) is then derived from inequalities (3.2.1), (3.2.2) and (3.2.4), by taking $\alpha' = \min\{\alpha, \tilde{\alpha}\}$. \square

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Remark 3.2.2. *The inequality (3.2.6) implies that the rate of convergence to zero of the difference between u_ℓ and u_∞ on Ω_ℓ is the slowest between the one of $e^{-\alpha\ell}$ and the one of $\|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell}{4}})}$. We recall that, thanks to Theorem 3.1.5, we have that $\|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell}{4}})} \rightarrow 0$, as ℓ goes to infinity.*

In particular, if f decays exponentially to zero at infinity, with respect to the H^{-1} -norm, i.e.

$$\|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_\ell)} \leq Ce^{-\alpha''\ell} \quad \text{for all } \ell > 0$$

(C and α'' being positive constants independent of ℓ), then we obtain once again an exponential rate of convergence of u_ℓ towards u_∞ , but this time on the whole cylinder Ω_ℓ .

Remark 3.2.3. *In the case where $k = 1$ and $\omega_1 = (-1, 1)$, the same result can be recovered using the correctors constructed in the previous chapter. Indeed, the following estimate was established in Theorem 2.2.3:*

$$\|\nabla(u_\ell - u_\infty - w_\ell^+)\|_{L^2(\Omega_\ell^+)} \leq Ce^{-\alpha\ell}, \quad (3.2.7)$$

where w_ℓ is a corrector and w_ℓ^+ its restriction to $\Omega_\ell^+ = (0, \ell) \times \omega_2$.

On the other hand, we have seen in Remark 2.2.1 that

$$\|\nabla w_\ell^+\|_{L^2((-\infty, \ell) \times \omega_2)} \leq \tilde{C} \|\nabla u_\infty\|_{L^2((\ell-1, \ell) \times \omega)}, \quad (3.2.8)$$

Combining inequalities (3.2.7) and (3.2.8) with (3.2.4) (with $2(\ell-1)$ in the role of ℓ) leads us to the inequality

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_\ell^+)} \leq C \left(e^{-\alpha\ell} + \|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell-1}{2}})} \right).$$

A similar inequality takes place on the other semi-cylinder $\Omega_\ell^- = (-\ell, 0) \times \omega_2$. Combining the inequalities on Ω_ℓ^+ and Ω_ℓ^- , we derive the estimate

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_\ell)} \leq C \left(e^{-\alpha\ell} + \|f\|_{H^{-1}(\Omega_\infty \setminus \bar{\Omega}_{\frac{\ell-1}{2}})} \right),$$

which gives in this case an improved version of (3.2.6).

Chapter 4

Asymptotic analysis for the Stokes problem in long domains becoming unbounded

4.1 Introduction

On a cylinder $\Omega_\ell = B_\ell \times \omega$ where $B_\ell \subset \mathbb{R}^k$ is the ball centered at the origin and of radius ℓ and $\omega \subset \mathbb{R}^{n-k}$ is a bounded Lipschitz domain, we study the Stokes problem

$$\begin{cases} -\mu\Delta u_\ell + \nabla p_\ell = f & \text{in } \Omega_\ell \\ \operatorname{div} u = 0 & \text{in } \Omega_\ell \\ u = 0 & \text{on } \partial\Omega_\ell \end{cases} \quad (4.1.1)$$

of unknowns $u_\ell \in (H_0^1(\Omega_\ell))^n$ and $p_\ell \in L^2(\Omega_\ell)/\mathbb{R}$, where $f \in L^2(\Omega_\ell)$ (for any $\ell > 0$) and μ is a positive constant describing the dynamic viscosity of the fluid. The aim of the chapter is to describe the behaviour of the solutions (u_ℓ, p_ℓ) as ℓ goes to ∞ while assuming a radiality hypothesis on f which, as we prove along the chapter, involves the same kind of properties for u_ℓ .

We remind the following classical result (see e.g. [34] and [68]): there exists a unique weak solution to the Stokes problem (4.1.1). More specifically,

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there exists a unique solution (u_ℓ, p_ℓ) to the variational problem

$$\begin{cases} (u_\ell, p_\ell) \in \hat{\mathbb{H}}_0^1(\Omega_\ell) \times \hat{L}^2(\Omega_\ell), \\ \mu \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v \, dx - \int_{\Omega_\ell} p_\ell \operatorname{div} v \, dx = \int_{\Omega_\ell} f v \, dx \text{ for all } v \in \mathbb{H}_0^1(\Omega_\ell). \end{cases} \quad (4.1.2)$$

In the case where $k = 1$, it have been proved in [19] that, provided that $\|f\|_{L^2(\Omega_\ell)} = O(e^{\beta\ell})$ for $\beta > 0$ small enough, then for some positive constants C and α , the solution (u_ℓ, p_ℓ) to the problem (4.1.1) satisfies

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} + \|p_\ell - p_\infty\|_{\hat{L}^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-\alpha\ell}$$

as ℓ goes to ∞ , where $(u_\infty, p_\infty) \in (H_{\text{loc}}^1(\overline{\Omega}_\infty))^n \times L_{\text{loc}}^2(\overline{\Omega}_\infty)/\mathbb{R}$ is the solution to the following problem:

$$\begin{cases} -\mu \Delta u_\infty + \nabla p_\infty = f & \text{in } (\mathcal{D}'(\Omega_\infty))^n \\ \operatorname{div} u_\infty = 0 & \text{in } \Omega_\infty \\ u_\infty = 0 & \text{on } \partial\Omega_\infty \\ \int_{\omega} u_{\infty,1}(X_1, X_2) \, dX_2 = 0 & \text{for a.e. } X_1 \in \mathbb{R} \\ \|\nabla u_\infty\|_{L^2(\Omega_\ell)} = O(e^{2\beta\ell}), \end{cases} \quad (4.1.3)$$

where $u_{\infty,1}$ is the first component of u_∞ .

In our case, if $k > 1$, the main difficulty comes from the emergence of the following divergence problem (in what follows, $D_\ell = \Omega_{\ell+1} \setminus \overline{\Omega}_\ell$):

Let $g \in H^1(D_\ell)$ verifying $g = 0$ on $(B_{\ell+1} \setminus \overline{B}_\ell) \times \partial\omega$ and $\int_{D_\ell} g \, dx = 0$. Then there exists a vector field $u \in (H_0^1(D_\ell))^n$ such that

$$\begin{cases} \operatorname{div} u = g & \text{in } D_\ell \\ \|\nabla u\|_{L^2(D_\ell)} \leq C \|g\|_{H^1(D_\ell)}, \end{cases}$$

with the constant C being independent of ℓ .

Usually, the constant C depends on the domain D_ℓ , hence it depends on ℓ if $k \geq 2$. Nevertheless, under some specific hypotheses on g , it is possible to

get a constant that does not depend on ℓ . As an example, if g is radial along X_1 , there exists a function $u \in H_0^1(D_\ell)$ solution to the previous problem, with C a constant independent of ℓ . Note that a difference arising between this specific problem and the classic results for the equation $\operatorname{div} u = g$ is that here, the H^1 -norm of u is controlled by the H^1 -norm of g while usually, it is possible to have the L^2 -norm of g instead. Nevertheless, this weaker inequality is still sufficient for our purposes, since it plays a key role in the proof of the main result given in the last section of this paper.

We now introduce the notation that we use in the chapter:

The notation $|\cdot|$ stands for the Euclidean norm in \mathbb{R}^m , $m \in \mathbb{N}^*$. The same notation is used for the Frobenius norm for matrices. Let us remind here that this norm is a matrix norm, i.e. that it satisfies $|AB| \leq |A||B|$ whenever the matrix product AB is possible.

The space of orthogonal matrices of order m is denoted by \mathbb{O}^m , i.e.

$$\mathbb{O}^m = \{Q \in \mathbb{M}^m, Q^T Q = I_m\}$$

where \mathbb{M}^m is the space of square matrices of order m , Q^T is the transpose of Q and I_m is the identity matrix of \mathbb{M}^m .

For a vector field $v = (v_1, \dots, v_m) : O \rightarrow \mathbb{R}^m$, with $O \subset \mathbb{R}^l$ an open set, we consider ∇v as a matrix field taking its values in $\mathbb{M}^{m \times l}$:

$$\nabla v = ((\nabla v)_{i,j})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq l}} = \left(\frac{\partial v_i}{\partial x_j} \right)_{\substack{1 \leq i \leq m \\ 1 \leq j \leq l}}.$$

Therefore, ∇v_i is the i -th line of ∇v . We also use the convention that each time a vector of \mathbb{R}^l appears in a matrix product, it is by default considered as a column vector.

For $x \in \mathbb{R}^k$ and $r > 0$, we use the notation B_r for the open ball of \mathbb{R}^k , centered at 0 of radius r :

$$B_r = \{y \in \mathbb{R}^k ; |y| < r\}.$$

We say that $f \in C^1(\bar{\Omega})$ if f is the restriction of a C^1 function on an open set O verifying $\bar{\Omega} \subset O$.

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We denote by $|A|$ the k -dimensional Lebesgue measure of a measurable subset A of \mathbb{R}^k . We also use the notation σ_k for the area of the unit sphere of \mathbb{R}^k . We note, for U and O open sets of \mathbb{R}^n , U being unbounded,

$$\begin{aligned} L^2_{loc}(\bar{U}) &= \{v \in L^2_{loc}(U) \mid v \in L^2(\mathcal{O}), \text{ for any bounded open set } \mathcal{O} \subset U\} \\ H^1_{loc}(\bar{U}) &= \{v \in H^1_{loc}(U) \mid v \in H^1(\mathcal{O}), \text{ for any bounded open set } \mathcal{O} \subset U\} \\ \mathbb{H}^1(O) &= (H^1(O))^n \\ \mathbb{H}^1_0(O) &= (H^1_0(O))^n \\ \hat{\mathbb{H}}^1_0(O) &= \{v \in \mathbb{H}^1_0(O) \mid \operatorname{div} v = 0 \text{ in } O\} \\ \hat{L}^2(O) &= L^2(O)/\mathbb{R}. \end{aligned}$$

In all the chapter, for $v = (v_1, \dots, v_m)$ a vector field in $(W^{1,p}(O))^m$ ($1 \leq p \leq +\infty$), with $O \subset \mathbb{R}^l$ an open set, we define the L^p -norm of ∇v by

$$\|\nabla v\|_{L^p(O)} = \sum_{i=1}^m \sum_{j=1}^l \|\partial_{x_j} v_i\|_{L^p(O)}.$$

Note that this norm is equivalent to the norm

$$v \mapsto \|\nabla v\|_{L^p(O)} = \left(\int_O |\nabla v|^p dx \right)^{\frac{1}{p}},$$

where $|\nabla v|$ is the Frobenius norm of ∇v , i.e.

$$|\nabla v| = \left(\sum_{i=1}^m \sum_{j=1}^l |\partial_{x_j} v_i|^2 \right)^{\frac{1}{2}}$$

where $\partial_{x_j} v_i$ denotes the partial derivative $\frac{\partial v_i}{\partial x_j}$. Indeed, on the one hand we have

$$|\nabla v| \leq \sum_{i=1}^m \sum_{j=1}^l |\partial_{x_j} v_i|$$

hence

$$\begin{aligned} \|\nabla v\|_{L^p(O)} &\leq \left\| \sum_{i=1}^m \sum_{j=1}^l |\partial_{x_j} v_i| \right\|_{L^p(O)} \leq \sum_{i=1}^m \sum_{j=1}^l \|\partial_{x_j} v_i\|_{L^p(O)} = \|\nabla v\|_{L^p(O)}. \end{aligned} \tag{4.1.4}$$

On the other hand, $|\partial_{x_j} v_i| \leq |\nabla v|$ for all i and j . Thus

$$\|\nabla v\|_{L^p(O)} = \sum_{i=1}^m \sum_{j=1}^l \|\partial_{x_j} v_i\|_{L^p(O)} \leq \sum_{i=1}^m \sum_{j=1}^l \|\nabla v\|_{L^p(O)} = ml \|\nabla v\|_{L^p(O)}. \quad (4.1.5)$$

For ω a bounded Lipschitz domain of \mathbb{R}^{n-k} , we set

$$\Omega_\ell = B_\ell \times \omega, \quad \Omega_\infty = \mathbb{R}^k \times \omega.$$

We furthermore set

$$A_\ell = B_{\ell+1} \setminus \overline{B}_\ell \quad \text{and} \quad D_\ell = \Omega_{\ell+1} \setminus \overline{\Omega}_\ell = A_\ell \times \omega.$$

For a point $x \in \mathbb{R}$, we set

$$X_1 = (x_1, \dots, x_k) \quad \text{and} \quad X_2 = (x_{k+1}, \dots, x_n).$$

Hence, we can write $x = (X_1, X_2)$, and we use the notation

$$\nabla_{X_1} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \right), \quad \nabla_{X_2} = \left(\frac{\partial}{\partial x_{k+1}}, \dots, \frac{\partial}{\partial x_n} \right).$$

Also, if $u = (u_1, \dots, u_n)$ is a \mathbb{R}^n -valued function, then we note

$$u' = (u_1, \dots, u_k) \quad \text{and} \quad u'' = (u_{k+1}, \dots, u_n).$$

Hence $u = (u', u'')$, with u' and u'' being respectively \mathbb{R}^k and \mathbb{R}^{n-k} -valued functions.

For $\beta \in \mathbb{R}$, we define

$$V_\beta(\Omega_\infty) = \{f \in L^2_{loc}(\overline{\Omega}_\infty) \mid \exists C_0 \geq 0 \text{ such that } \|f\|_{L^2(\Omega_\ell)} \leq C_0 e^{\beta \ell} \quad \forall \ell > 0\}$$

and we give the following definition:

Definition 4.1.1. *Let $f : (B_R \setminus \overline{B}_r) \times \omega \rightarrow \mathbb{R}$, with $0 \leq r < R \leq +\infty$. One says that f is radial along X_1 if there exists a function $g : (r, R) \times \omega \rightarrow \mathbb{R}$ such that*

$$f(X_1, X_2) = g(|X_1|, X_2)$$

for all $(X_1, X_2) \in (B_R \setminus \overline{B}_r) \times \omega$.

We now give some primary results.

4.2 Preliminaries

We start by reminding three useful general results.

We begin by a first result that can be seen as a spherical Fubini formula. Indeed, this result can be obtained by first passing in spherical coordinates, then using the classical Fubini formula.

Theorem 4.2.1. *Let a and b be two positive constants such that $0 \leq a < b \leq +\infty$, and f a function such that $f \in L^1(B_b \setminus \overline{B}_a)$. Then, we have that*

$$\int_{B_b \setminus \overline{B}_a} f(x) \, dx = \int_a^b \left(\int_{\partial B_s} f(x) \, d\sigma_x \right) \, ds.$$

The second result concerns the change of variables in Sobolev spaces.

Theorem 4.2.2. *Let Ω and Ω' be two open subsets of \mathbb{R}^n and $H : \Omega' \rightarrow \Omega$ a bijection such that*

$$H \in C^1(\Omega'; \mathbb{R}^n), \quad H^{-1} \in C^1(\Omega; \mathbb{R}^n)$$

and

$$J(H) \in (L^\infty(\Omega'))^{n \times n} \text{ and } J(H^{-1}) \in (L^\infty(\Omega))^{n \times n},$$

where $J(H)$ and $J(H^{-1})$ are respectively the Jacobian matrix of H and the Jacobian matrix of H^{-1} . Then for $u \in W^{1,p}(\Omega)$ ($1 \leq p < +\infty$), $u \circ H \in W^{1,p}(\Omega')$ and

$$\frac{\partial}{\partial y_j}(u \circ H)(y) = \sum_{i=1}^n \frac{\partial u}{\partial x_i}(H(y)) \frac{\partial H_i}{\partial y_j}(y) \quad \forall 1 \leq j \leq n.$$

This result and its proof can be found in, as an example, [10].

We also give this Poincaré inequality which is an adaptation of Lemma 1.1.3:

Lemma 4.2.1. *Let $v \in H^1(\Omega_\ell)$ such that $v = 0$ on $B_\ell \times \partial\omega$ and $\tilde{\omega}_1 \subset B_\ell$ a measurable set. Then there exists a constant C_ω depending only on ω such that:*

$$\|v\|_{L^2(\tilde{\omega}_1 \times \omega)} \leq C_\omega \|\nabla_{X_2} v\|_{L^2(\tilde{\omega}_1 \times \omega)}. \quad (4.2.1)$$

We now give some results about radially.

Theorem 4.2.3. *If $g : \overline{\Omega}_\ell \rightarrow \mathbb{R}$ satisfies $g(QX_1, X_2) = g(X_1, X_2)$ for all $Q \in \mathbb{O}^k$ and all $(X_1, X_2) \in \Omega_\ell$, then g is radial along X_1 , which is to say that there exists a function $h : [0, \ell] \times \omega \rightarrow \mathbb{R}$ such that*

$$g(X_1, X_2) = h(r, X_2) \text{ for all } (X_1, X_2) \in \Omega_\ell, \quad (4.2.2)$$

where $r = |X_1|$.

Theorem 4.2.4. *If $g : \overline{\Omega}_\ell \rightarrow \mathbb{R}^k$ satisfies $g(QX_1, X_2) = Qg(X_1, X_2)$ for all $Q \in \mathbb{O}^k$ and all $(X_1, X_2) \in \Omega_\ell$, then there exists $R : [0, \ell] \times \omega \rightarrow \mathbb{R}$ such that $g(X_1, X_2) = R(r, X_2)X_1$.*

Proof. For $X_1 \in \mathbb{R}^k$, we set $r = |X_1|$. Then we have that $|re_1| = |X_1|$ where e_1 is the first element the canonical basis of \mathbb{R}^k . Therefore, there exists $Q \in \mathbb{O}^k$ such that $Q(re_1) = X_1$. It follows that $g(X_1, X_2) = g(Q(re_1), X_2) = Qg(re_1, X_2)$.

Let now $R = \text{diag}(1, -1, \dots, -1)$, the diagonal matrix of order k in which the elements of the diagonal coincide with the ones of the vector $(1, -1, \dots, -1) \in \mathbb{R}^k$. Then we have that $R \in \mathbb{O}^k$ and $Re_1 = e_1$. Therefore, we have:

$$g(re_1, X_2) = g(rRe_1, X_2) = g(R(re_1), X_2) = Rg(re_1, X_2).$$

It follows that

$$(I_k - R)g(re_1, X_2) = 0$$

and therefore $g_i(re_1, X_2) = 0$ for all $2 \leq i \leq k$, which is to say that $g(re_1, X_2) = g_1(re_1, X_2)e_1$. Combining all the equalities on g , we have that

$$g(X_1, X_2) = Qg(re_1, X_2) = Qg_1(re_1, X_2)e_1 = g_1(re_1, X_2)Qe_1 = \frac{g_1(re_1, X_2)}{r}X_1.$$

This finally implies that $g(X_1, X_2) = R(r, X_2)X_1$ with $R(r, X_2) = \frac{g_1(re_1, X_2)}{r}$. \square

Theorem 4.2.5. *Let $f \in L^1(\mathbb{R}^n)$ such that for all $Q \in \mathbb{O}^k$, one has $f(QX_1, X_2) = f(X_1, X_2)$ for a.e. $(X_1, X_2) \in \mathbb{R}^n$, and let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a radial function along X_1 . Then, the convolution product $f * \rho$ is radial along X_1 .*

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Proof. Let f and ρ be two functions verifying the hypotheses of the theorem. According to Theorem 4.2.3, it is enough to prove that

$$(f * \rho)(QX_1, X_2) = (f * \rho)(X_1, X_2)$$

for all $Q \in \mathbb{O}^k$ and all $(X_1, X_2) \in \mathbb{R}^n$. We now fix $Q \in \mathbb{O}^k$. Then for all $(X_1, X_2) \in \mathbb{R}^n$, we have

$$(f * \rho)(QX_1, X_2) = \int_{\mathbb{R}^n} f(QX_1 - Y_1, X_2 - Y_2) \rho(Y_1, Y_2) dY.$$

Taking $QZ_1 = Y_1$ and $Z_2 = Y_2$, we have that $dZ = dZ_1 dZ_2 = dY_1 dY_2 = dY$ since Q is orthogonal. Therefore, using the assumptions on f and ρ , we obtain for any $(X_1, X_2) \in \mathbb{R}^n$,

$$\begin{aligned} (f * g)(QX_1, X_2) &= \int_{\mathbb{R}^n} f(QX_1 - QZ_1, X_2 - Z_2) \rho(QZ_1, Z_2) dZ \\ &= \int_{\mathbb{R}^n} f(Q(X_1 - Z_1), X_2 - Z_2) \rho(QZ_1, Z_2) dZ \\ &= \int_{\mathbb{R}^n} f(X_1 - Z_1, X_2 - Z_2) \rho(Z_1, Z_2) dZ \\ &= (f * \rho)(X_1, X_2). \end{aligned}$$

□

Theorem 4.2.6. *Let $g \in L^1(\Omega_\ell)$ such that for all $Q \in \mathbb{O}^k$, one has $g(QX_1, X_2) = g(X_1, X_2)$ for a.e. $(X_1, X_2) \in \Omega_\ell$. Then g has a representative \tilde{g} , satisfying*

$$\tilde{g}(X_1, X_2) = h(r, X_2) \text{ for all } (X_1, X_2) \in \Omega_\ell,$$

where $r = |X_1|$ and h is a function defined on $[0, \ell) \times \omega$.

Proof. To start with, we extend g outside of Ω_ℓ by 0. For convenience, we also denote this extend function by g . Note that this extension verifies $g \in L^1(\mathbb{R}^k \times \omega)$ and that for every $(X_1, X_2) \in (\mathbb{R}^k \setminus \Omega_\ell) \times \omega$ and $Q \in \mathbb{O}^k$, since $QX_1 \in (\mathbb{R}^k \setminus \Omega_\ell)$, $g(X_1, X_2) = g(QX_1, X_2)$.

We now take a sequence $\{\rho_m\}$ of mollifiers in $C^\infty(\mathbb{R}^n)$ such that for all m , ρ_m verifies that for every $(X_1, X_2) \in \mathbb{R}^k \times \omega$ and $Q \in \mathbb{O}^k$, $\rho_m(QX_1, X_2) = \rho_m(X_1, X_2)$. Then, the convolution product also verifies this property and therefore, by using Theorem 4.2.5, g_m is radial along X_1 for every m . Consequently, there exists a function $h_m : [0, +\infty) \times \omega$ such that $g_m(X_1, X_2) =$

$h_m(|X_1|, X_2)$.

We also know that g_m converges to g in $L^1(\mathbb{R}^k \times \omega)$. Therefore, g_m is a Cauchy sequence in $L^2(\mathbb{R}^k \times \omega)$. We have that

$$\|g_m - g\|_{L^2(\mathbb{R}^k \times \omega)} = \int_{\omega} \int_0^{+\infty} \int_{\partial B_r} (g_m - g)^2 d\sigma_{X_1} dr dX_2 \rightarrow 0,$$

and then, up to a subsequence,

$$\|g_m(\cdot, X_2) - g(\cdot, X_2)\|_{L^2(\partial B_r)} \rightarrow 0 \text{ in } L^2((0, +\infty) \times \omega).$$

Therefore, for a.e. $r \in (0, +\infty)$ and a.e. $X_2 \in \omega$,

$$\int_{\partial B_r} (g_m - g)^2 d\sigma_{X_1} \rightarrow 0,$$

and it follows that

$$\|g_m(\cdot, X_2) - g(\cdot, X_2)\|_{L^2(\partial B_r)} \rightarrow 0 \text{ for a.e. } (r, X_2) \in (0, +\infty) \times \omega.$$

It follows that there exists a set $\Gamma \subset (0, +\infty) \times \omega$ of zero measure such that $\{g_m(\cdot, X_2)\}$ is a Cauchy sequence in $L^1(\partial B_r)$ for all $(r, X_2) \in ((0, +\infty) \times \omega) \setminus \Gamma$. This implies that, for any fixed $(r, X_2) \in ((0, +\infty) \times \omega) \setminus \Gamma$ and for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}^*$ such that for all m_1 and m_2 greater than n_0 , $\|g(\cdot, X_2)_{m_1} - g(\cdot, X_2)_{m_2}\|_{L^2(\partial B_r)} \leq \epsilon \sigma_k r^{k-1}$. Therefore, since

$$\|g_{m_1} - g_{m_2}\|_{L^1(\partial B_r)} = \sigma_k r^{k-1} |h(\cdot, X_2)_{m_1} - h(\cdot, X_2)_{m_2}|,$$

it follows that $\{h_m(r, X_2)\}$ is a Cauchy sequence for all $(r, X_2) \in ((0, +\infty) \times \omega) \setminus \Gamma$. Then, for all $(r, X_2) \in ((0, +\infty) \times \omega) \setminus \Gamma$, we have that there exists $\tilde{h}(r, X_2)$ defined by

$$\tilde{h}(r, X_2) = \lim_{m \rightarrow +\infty} h_m(r, X_2).$$

Let $h(r, X_2)$ be defined as

$$h(r, X_2) = \begin{cases} \tilde{h}(r, X_2) & \text{if } (r, X_2) \in ((0, +\infty) \times \omega) \setminus \Gamma \\ 0 & \text{if } (r, X_2) \in \Gamma. \end{cases}$$

Finally, we set $\tilde{g}(X_1, X_2) = h(|X_1|, X_2)$, and we want to prove that $g_m \rightarrow \tilde{g}$ a.e. in $\mathbb{R}^k \times \omega$. We have that

$$g_m(X_1, X_2) = h_m(|X_1|, X_2) \rightarrow h(|X_1|, X_2) = h(r, X_2)$$

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for all (X_1, X_2) such that $(|X_1|, X_2) \in ((0, +\infty) \times \omega) \setminus \Gamma$. Since the set Γ is of zero measure in $(0, +\infty) \times \omega$, the set

$$\Lambda\{(X_1, X_2) \in \mathbb{R}^k \times \omega; (|X_1|, X_2) \in \Gamma\}$$

is of zero measure in $\mathbb{R}^k \times \omega$. Indeed, we have that $g_m \rightarrow \tilde{g}$ a.e. in $\mathbb{R}^k \times \omega$, and therefore

$$g = \tilde{g} \text{ a.e. in } \mathbb{R}^k \times \omega,$$

since, up to a subsequence, we also have

$$g_m \rightarrow g \text{ in } \mathbb{R}^k \times \omega$$

$$g_m \rightarrow g \text{ a.e.}$$

□

Definition 4.2.1. *A function g is satisfying the hypotheses of Theorem 4.2.6 is said to have radial symmetry along X_1 .*

Convention. *In order to simplify the notation, we will use the following convention for functions radial along X_1 : we will write*

$$g(X_1, X_2) = g(r, X_2), \text{ where } r = |X_1|,$$

instead of (4.2.2). Moreover, the notation r will be used to denote both a simple variable (for instance when considering the partial derivative $\frac{\partial}{\partial r}$) and the function of X_1 defined by

$$r(X_1) = |X_1|$$

as in (4.2.2). The different meanings of this notation will be clear from the context.

In a similar manner as for Theorem 4.2.6, we can prove the following result:

Theorem 4.2.7. *If $g \in L^1(\Omega_\ell, \mathbb{R}^k)$ is such that for all $Q \in \mathbb{O}^k$, one has $g(QX_1, X_2) = Qg(X_1, X_2)$ for a.e. (X_1, X_2) , then g has a representative, for convenience also denoted by g , satisfying*

$$g(X_1, X_2) = R(r, X_2)X_1 \text{ for all } (X_1, X_2) \in \Omega_\ell,$$

where $r = |X_1|$ and $R : [0, \ell) \times \omega$.

The following result states that it is possible to approach a X_1 -radial symmetric function in $W^{1,p}(D_\ell)$ (where $D_\ell = A_\ell \times \omega = (B_{\ell+1} \setminus \overline{B}_\ell) \times \omega$) by regular X_1 -radial symmetric functions.

Theorem 4.2.8. *Let $g \in W^{1,p}(D_\ell)$ ($1 \leq p < +\infty$) be a function with radial symmetry along X_1 such that $g = 0$ on $A_\ell \times \partial\omega$. Then, there exists a sequence $\{g_m\}$ of $C^1(\overline{D}_\ell)$ functions radial along X_1 such that*

$$g_m \xrightarrow{W^{1,p}(D_\ell)} g.$$

Proof. The idea of the proof is to extend g on a domain larger than D_ℓ , in order to approximate g by a sequence (g_m) of C^1 -functions, using the convolution with a regularizing sequence. To simplify the presentation, we consider for g a representative also denoted by g , satisfying

$$g(X_1, X_2) = h(r, X_2)$$

for all $(X_1, X_2) \in D_\ell$ (see Theorem 4.2.6).

To start with, we extend g to a $W^{1,p}$ -function on $A_\ell \times \mathbb{R}^{n-k}$ (also denoted by g) by defining $g(X_1, X_2) = 0$ for all $(X_1, X_2) \in A_\ell \times (\mathbb{R}^{n-k} \setminus \omega)$. This function belongs to $W^{1,p}(A_\ell \times \mathbb{R}^{n-k})$, since $g = 0$ on $A_\ell \times \partial\omega$. Then, we extend this function by \tilde{g} in the following way:

$$\tilde{g}(X_1, X_2) = \begin{cases} g(X_1, X_2) & \text{if } \ell < |X_1| < \ell + 1 \\ g\left(\left(\frac{2(\ell+1)}{|X_1|} - 1\right)X_1, X_2\right) & \text{if } \ell + 1 < |X_1| < \ell + 2 \\ g\left(\left(\frac{2\ell}{|X_1|} - 1\right)X_1, X_2\right) & \text{if } \ell - \varepsilon < |X_1| < \ell, \end{cases} \quad (4.2.3)$$

with $\varepsilon = \min(1, \frac{\ell}{2})$. Note that for $X_1 \in (B_{\ell+2} \setminus B_\ell)$, the point $Y_1 = \left(\frac{2(\ell+1)}{|X_1|} - 1\right)X_1$ is the “radial” symmetric of X_1 with respect to $\partial B_{\ell+1}$, i.e. the symmetric of X_1 with respect to the point at the intersection between $\partial B_{\ell+1}$ and the ray $[OX_1)$. In the same manner, $Z_1 = \left(\frac{2\ell}{|X_1|} - 1\right)X_1$ is the “radial” symmetric of X_1 with respect to ∂B_ℓ . We set

$$C_\ell = (B_{\ell+2} \setminus \overline{B}_{\ell-\varepsilon}) \times \tilde{\omega},$$

where $\tilde{\omega} \subset \mathbb{R}^{n-k}$ is a fixed bounded Lipschitz domain such that $\omega \subset \subset \tilde{\omega}$. It is clear from the radially of g on D_ℓ that \tilde{g} is radial with respect to X_1 on C_ℓ .

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We have that

$$(X_1, X_2) \in (B_{\ell+2} \setminus \overline{B_{\ell+1}}) \times \tilde{\omega} \mapsto \left(\left(\frac{2(\ell+1)}{|X_1|} - 1 \right) X_1, X_2 \right) \in A_\ell \times \tilde{\omega}$$

and

$$(X_1, X_2) \in (B_\ell \setminus \overline{B_{\ell-\varepsilon}}) \times \tilde{\omega} \mapsto \left(\left(\frac{2\ell}{|X_1|} - 1 \right) X_1, X_2 \right) \in (B_{\ell+\varepsilon} \setminus \overline{B_\ell}) \times \tilde{\omega} \subset A_\ell \times \tilde{\omega}.$$

are C^1 -diffeomorphisms satisfying the hypotheses the diffeomorphism H in Theorem 4.2.2 (note that their inverses are given by exactly the same formulas). Therefore, $\tilde{g} \in W^{1,p}((B_{\ell+2} \setminus \overline{B_{\ell+1}}) \times \tilde{\omega})$ and $\tilde{g} \in W^{1,p}((B_\ell \setminus \overline{B_{\ell-\varepsilon}}) \times \tilde{\omega})$. Consequently, $\tilde{g} \in W^{1,p}(C_\ell \setminus \overline{A_\ell \times \tilde{\omega}})$ since $C_\ell \setminus \overline{A_\ell \times \tilde{\omega}} = ((B_{\ell+2} \setminus \overline{B_{\ell+1}}) \times \tilde{\omega}) \cup ((B_\ell \setminus \overline{B_{\ell-\varepsilon}}) \times \tilde{\omega})$.

Since $\tilde{g} \in W^{1,p}(A_\ell \times \tilde{\omega})$, there exists a sequence $\{\bar{g}_m\}$ of functions $C^1(\overline{A_\ell \times \tilde{\omega}})$ such that $\bar{g}_m \rightarrow \tilde{g}$ in $W^{1,p}(A_\ell \times \tilde{\omega})$. Therefore,

$$\bar{g}_m|_{(\partial B_{\ell+1}) \times \tilde{\omega}} \xrightarrow{L^p((\partial B_{\ell+1}) \times \tilde{\omega})} \text{Tr}_{(\partial B_{\ell+1}) \times \tilde{\omega}} \tilde{g}.$$

Next, we extend \bar{g}_m on $(B_{\ell+2} \setminus \overline{B_{\ell+1}}) \times \tilde{\omega}$ in the same manner as in (4.2.3). Then the extensions \tilde{g}_m belong to $C^1(\overline{(B_{\ell+2} \setminus \overline{B_{\ell+1}}) \times \tilde{\omega}})$ and satisfy

$$\tilde{g}_m \rightarrow \tilde{g} \text{ in } W^{1,p}((B_{\ell+2} \setminus \overline{B_{\ell+1}}) \times \tilde{\omega})$$

using the formula of Theorem 4.2.2 and the fact that $\tilde{g}_m \rightarrow \tilde{g}$ in $W^{1,p}(A_\ell \times \tilde{\omega})$. Consequently,

$$\tilde{g}_m|_{(\partial B_{\ell+1}) \times \tilde{\omega}} \xrightarrow{L^p((B_{\ell+2} \setminus \overline{B_{\ell+1}}) \times \tilde{\omega})} \text{Tr}_{(\partial B_{\ell+1}) \times \tilde{\omega}} \tilde{g}.$$

Since $\bar{g}_m|_{(\partial B_{\ell+1}) \times \tilde{\omega}} = \tilde{g}_m|_{(\partial B_{\ell+1}) \times \tilde{\omega}}$, it follows that $\tilde{g} \in W^{1,p}((B_{\ell+2} \setminus \overline{B_\ell}) \times \tilde{\omega})$. Following the same arguments on $(\partial B_\ell) \times \tilde{\omega}$, we have that $\tilde{g} \in W^{1,p}((B_{\ell+1} \setminus \overline{B_{\ell-\varepsilon}}) \times \tilde{\omega})$. Thus, $\tilde{g} \in W^{1,p}(C_\ell)$.

Since $D_\ell \subset\subset C_\ell$ we therefore can take a sequence of mollifiers $\{\rho_m\}$, radial along X_1 , verifying that

$$g_m = \tilde{g} * \rho_m \xrightarrow{W^{1,p}(D_\ell)} g.$$

Note that, thanks to Theorem 4.2.5, g_m is radial along X_1 , since both \tilde{g} and ρ_m are radial along X_1 . \square

The following result will be very useful in the proof of the main result of this chapter (see section 4.4). It states that if f has some radial symmetry properties with respect to X_1 , then the weak solution of the Stokes problem (4.1.1) has some similar properties.

Theorem 4.2.9. *Let $(u, p) \in \mathbb{H}_0^1(\Omega_\ell) \times \hat{L}^2(\Omega_\ell)$ be a weak solution to the Stokes problem in $\Omega_\ell = B_\ell \times \omega$, i.e. satisfying the variational equation in (4.1.2) with u_ℓ and p_ℓ replaced by u and p . If f verifies*

$$f = (f', f'') = \begin{cases} f'(QX_1, X_2) = Qf'(X_1, X_2) \\ f''(QX_1, X_2) = f''(X_1, X_2) \end{cases}$$

for all $Q \in \mathbb{O}^k$ and for a.e. $(X_1, X_2) \in \Omega_\ell$, then (u, p) satisfies

$$\begin{cases} u'(QX_1, X_2) = Qu'(X_1, X_2) \\ u''(QX_1, X_2) = u''(X_1, X_2) \\ p(QX_1, X_2) = p(X_1, X_2) \end{cases}$$

for all $Q \in \mathbb{O}^k$ and for a.e. $(X_1, X_2) \in \Omega_\ell$.

Proof. Let (u, p) be a weak solution of the Stokes problem (4.1.1), which is to say $u \in \mathbb{H}_0^1(\Omega_\ell)$, and

$$\begin{cases} \mu \int_{\Omega_\ell} \nabla u \cdot \nabla v \, dx - \int_{\Omega_\ell} p \operatorname{div} v \, dx = \int_{\Omega_\ell} f v \, dx \text{ for all } v \in \mathbb{H}_0^1(\Omega_\ell) \\ \operatorname{div} u = 0 \text{ in } \Omega_\ell. \end{cases}$$

Let $Q \in \mathbb{O}^k$. We take (\tilde{u}, \tilde{p}) with $\tilde{u} = (\tilde{u}', \tilde{u}'')$ and

$$\begin{cases} \tilde{u}'(X_1, X_2) = Q^T u'(QX_1, X_2) \\ \tilde{u}''(X_1, X_2) = u''(QX_1, X_2) \\ \tilde{p}(X_1, X_2) = p(QX_1, X_2), \end{cases} \quad (4.2.4)$$

that we can rewrite

$$\begin{cases} \begin{pmatrix} \tilde{u}' \\ \tilde{u}'' \end{pmatrix} (X_1, X_2) = \begin{pmatrix} Q^T & 0 \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} u' \\ u'' \end{pmatrix} (QX_1, X_2) \\ \tilde{p}(X_1, X_2) = p(QX_1, X_2). \end{cases}$$

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We will prove that (\tilde{u}, \tilde{p}) is also a weak solution to the Stokes problem. The formulas (4.2.4) will then follow from the uniqueness of the solution to the Stokes problem. Using the formula $\nabla(g \circ \phi) = (\nabla g \circ \phi)\nabla\phi$, we derive

$$\nabla \begin{pmatrix} \tilde{u}' \\ \tilde{u}'' \end{pmatrix} (X_1, X_2) = \begin{pmatrix} Q^T & 0 \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} \nabla u' \\ \nabla u'' \end{pmatrix} (QX_1, X_2) \begin{pmatrix} Q & 0 \\ 0 & I_{n-k} \end{pmatrix}$$

by taking $\phi(X_1, X_2) = (QX_1, X_2) = \begin{pmatrix} Q & 0 \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ and $\nabla\phi = \begin{pmatrix} Q & 0 \\ 0 & I_{n-k} \end{pmatrix}$, which is to say

$$\nabla \begin{pmatrix} \tilde{u}' \\ \tilde{u}'' \end{pmatrix} (X_1, X_2) = \begin{pmatrix} Q^T & 0 \\ 0 & I_{n-k} \end{pmatrix} \begin{pmatrix} \nabla_{X_1} u'(QX_1, X_2)Q & \nabla_{X_2} u'(QX_1, X_2) \\ \nabla_{X_1} u''(QX_1, X_2) & \nabla_{X_2} u''(QX_1, X_2) \end{pmatrix}$$

and finally

$$\nabla \begin{pmatrix} \tilde{u}' \\ \tilde{u}'' \end{pmatrix} (X_1, X_2) = \begin{pmatrix} Q^T \nabla_{X_1} u'(QX_1, X_2)Q & Q^T \nabla_{X_2} u'(QX_1, X_2) \\ \nabla_{X_1} u''(QX_1, X_2) & \nabla_{X_2} u''(QX_1, X_2) \end{pmatrix}.$$

We therefore have that

$$\begin{aligned} \operatorname{div} \tilde{u}'(X_1, X_2) &= \operatorname{Tr}(\nabla \tilde{u})(X_1, X_2) \\ &= \operatorname{Tr}(Q^T \nabla_{X_1} u'(QX_1, X_2)Q) + \operatorname{Tr}(\nabla_{X_2} u''(QX_1, X_2)) \\ &= \operatorname{Tr}(\nabla_{X_1} u'(QX_1, X_2)) + \operatorname{Tr}(\nabla_{X_2} u''(QX_1, X_2)) \\ &= \operatorname{Tr}(\nabla u(QX_1, X_2)) \\ &= (\operatorname{div} u)(QX_1, X_2) = 0. \end{aligned}$$

Let now $v \in \mathbb{H}_0^1(\Omega_\ell)$. We have that $\mu \int_{\Omega_\ell} \nabla \tilde{u} \cdot \nabla v \, dx - \int_{\Omega_\ell} \tilde{p} \operatorname{div} v \, dx$ can be written

$$\begin{aligned} \mu \int_{\Omega_\ell} \begin{pmatrix} Q^T \nabla_{X_1} u'(QX_1, X_2)Q & Q^T \nabla_{X_2} u'(QX_1, X_2) \\ \nabla_{X_1} u''(QX_1, X_2) & \nabla_{X_2} u''(QX_1, X_2) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} v' & \nabla_{X_2} v' \\ \nabla_{X_1} v'' & \nabla_{X_2} v'' \end{pmatrix} (QX_1, X_2) \, dx \\ - \int_{\Omega_\ell} p(QX_1, X_2) \operatorname{div} v(X_1, X_2) \, dx. \end{aligned}$$

Using then the change of variable $\tilde{x} = (\tilde{X}_1, \tilde{X}_2) = (QX_1, X_2)$, which gives $(X_1, X_2) = (Q^T \tilde{X}_1, \tilde{X}_2)$ and $dx = d\tilde{x}$, it then have that the previous expres-

sion equals

$$\begin{aligned}
 & \mu \int_{\Omega_\ell} \left(Q^T \nabla_{X_1} u'(\tilde{X}_1, \tilde{X}_2) Q \cdot (\nabla_{X_1} v')(Q^T X_1, X_2) \right. \\
 & \quad + Q^T \nabla_{X_2} u'(\tilde{X}_1, \tilde{X}_2) \cdot (\nabla_{X_2} v')(Q^T \tilde{X}_1, \tilde{X}_2) \\
 & \quad + \nabla_{X_1} u''(\tilde{X}_1, \tilde{X}_2) Q \cdot (\nabla_{X_1} v'')(Q^T \tilde{X}_1, \tilde{X}_2) \\
 & \quad \left. + \nabla_{X_2} u''(\tilde{X}_1, \tilde{X}_2) \cdot \nabla_{X_2} v''(Q^T \tilde{X}_1, \tilde{X}_2) \right) d\tilde{x} \\
 & - \int_{\Omega_\ell} p(\tilde{X}_1, \tilde{X}_2) (\operatorname{div} v)(Q^T \tilde{X}_1, \tilde{X}_2) d\tilde{x}
 \end{aligned}$$

Let us now compute each term separately after recalling that $A \cdot B = \operatorname{Tr}(AB^T) = \operatorname{Tr}(A^T B)$:

- First of all,

$$\begin{aligned}
 & Q^T \nabla_{X_1} u'(\tilde{X}_1, \tilde{X}_2) Q \cdot (\nabla_{X_1} v')(Q^T X_1, X_2) \\
 & = \operatorname{Tr}(Q^T \nabla_{X_1} u'(\tilde{X}_1, \tilde{X}_2) Q \cdot (\nabla_{X_1} v')^T(Q^T X_1, X_2)) \\
 & = \operatorname{Tr}(\nabla_{X_1} u'(\tilde{X}_1, \tilde{X}_2) Q \cdot (\nabla_{X_1} v')^T(Q^T X_1, X_2) Q^T) \\
 & = \operatorname{Tr}(\nabla_{X_1} u'(X_1, X_2) (Q(\nabla_{X_1} v')(Q^T X_1, X_2))^T) \\
 & = \nabla_{X_1} u'(\tilde{X}_1, \tilde{X}_2) \cdot Q(\nabla_{X_1} v')(Q^T \tilde{X}_1, \tilde{X}_2) Q^T
 \end{aligned}$$

-Then,

$$\begin{aligned}
 & Q^T \nabla_{X_2} u'(\tilde{X}_1, \tilde{X}_2) \cdot \nabla_{X_2} v'(Q^T \tilde{X}_1, \tilde{X}_2) \\
 & = \operatorname{Tr}(Q^T \nabla_{X_2} u'(\tilde{X}_1, \tilde{X}_2) (\nabla_{X_2} v')^T(Q^T \tilde{X}_1, \tilde{X}_2)) \\
 & = \operatorname{Tr}(\nabla_{X_2} u'(\tilde{X}_1, \tilde{X}_2) (\nabla_{X_2} v')^T(Q^T \tilde{X}_1, \tilde{X}_2) Q^T) \\
 & = \operatorname{Tr}(\nabla_{X_2} u'(\tilde{X}_1, \tilde{X}_2) (Q(\nabla_{X_2} v')(Q^T \tilde{X}_1, \tilde{X}_2))^T) \\
 & = \nabla_{X_2} u'(\tilde{X}_1, \tilde{X}_2) \cdot Q(\nabla_{X_2} v')(Q^T \tilde{X}_1, \tilde{X}_2).
 \end{aligned}$$

-Finally,

$$\begin{aligned}
 & \nabla_{X_1} u''(\tilde{X}_1, \tilde{X}_2) Q \cdot (\nabla_{X_1} v'')(Q^T \tilde{X}_1, \tilde{X}_2) \\
 & = \operatorname{Tr}(\nabla_{X_1} u''(\tilde{X}_1, \tilde{X}_2) Q (\nabla_{X_1} v'')^T(Q^T \tilde{X}_1, \tilde{X}_2)) \\
 & = \operatorname{Tr}(\nabla_{X_1} u''(\tilde{X}_1, \tilde{X}_2) ((\nabla_{X_1} v'')(Q^T \tilde{X}_1, \tilde{X}_2))^T) \\
 & = \nabla_{X_1} u''(\tilde{X}_1, \tilde{X}_2) \cdot \nabla_{X_1} v''(Q^T \tilde{X}_1, \tilde{X}_2) Q^T.
 \end{aligned}$$

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Therefore, we have that $\mu \int_{\Omega_\ell} \nabla \tilde{u} \cdot \nabla v \, d\tilde{x} - \int_{\Omega_\ell} \tilde{p} \operatorname{div} v \, d\tilde{x}$ is

$$\begin{aligned} \mu \int_{\Omega_\ell} \begin{pmatrix} \nabla_{X_1} u' & \nabla_{X_2} u' \\ \nabla_{X_1} u'' & \nabla_{X_2} u'' \end{pmatrix} (\tilde{X}_1, \tilde{X}_2) \cdot \begin{pmatrix} Q \nabla_{X_1} v'(Q^T \tilde{X}_1, \tilde{X}_2) & Q \nabla_{X_2} v'(Q^T \tilde{X}_1, \tilde{X}_2) \\ \nabla_{X_1} v''(Q^T \tilde{X}_1, \tilde{X}_2) Q^T & \nabla_{X_2} v''(Q^T \tilde{X}_1, \tilde{X}_2) \end{pmatrix} d\tilde{x} \\ - \int_{\Omega_\ell} p(\tilde{X}_1, \tilde{X}_2) (\operatorname{div} v)(Q^T \tilde{X}_1, \tilde{X}_2) d\tilde{x}. \end{aligned}$$

By setting $\tilde{v}(\tilde{X}_1, \tilde{X}_2) = (\tilde{v}'(\tilde{X}_1, \tilde{X}_2), \tilde{v}''(\tilde{X}_1, \tilde{X}_2) = (Q v'(Q^T \tilde{X}_1, \tilde{X}_2), v''(Q^T \tilde{X}_1, \tilde{X}_2))$, and since $(\operatorname{div} \tilde{v})(\tilde{X}_1, \tilde{X}_2) = (\operatorname{div} v)(Q^T \tilde{X}_1, \tilde{X}_2)$ we have

$$\begin{aligned} \mu \int_{\Omega_\ell} \nabla \tilde{u} \cdot \nabla v \, d\tilde{x} - \int_{\Omega_\ell} \tilde{p} \operatorname{div} v \, d\tilde{x} &= \mu \int_{\Omega_\ell} \begin{pmatrix} \nabla_{X_1} u' & \nabla_{X_2} u' \\ \nabla_{X_1} u'' & \nabla_{X_2} u'' \end{pmatrix} (\tilde{X}_1, \tilde{X}_2) \cdot \nabla \tilde{v}(\tilde{X}_1, \tilde{X}_2) d\tilde{x} \\ &\quad - \int_{\Omega_\ell} p \operatorname{div} \tilde{v} \, d\tilde{x} \\ &= \mu \int_{\Omega_\ell} \nabla u \cdot \nabla \tilde{v} \, d\tilde{x} - \int_{\Omega_\ell} p \operatorname{div} \tilde{v} \, d\tilde{x} \\ &= \int_{\Omega_\ell} f \tilde{v} \, d\tilde{x} \\ &= \int_{\Omega_\ell} (f' \cdot \tilde{v}' + f'' \cdot \tilde{v}'') \, d\tilde{x} \\ &= \int_{\Omega_\ell} f'(\tilde{X}_1, \tilde{X}_2) \cdot Q v'(Q^T \tilde{X}_1, \tilde{X}_2) \\ &\quad + f''(\tilde{X}_1, \tilde{X}_2) \cdot v''(Q^T \tilde{X}_1, \tilde{X}_2) \, d\tilde{x} \\ &= \int_{\Omega_\ell} f'(Q X_1, X_2) \cdot Q v'(X_1, X_2) \\ &\quad + f''(Q X_1, X_2) \cdot v''(X_1, X_2) \, dx \\ &= \int_{\Omega_\ell} f v \, dx, \end{aligned}$$

where the last inequality comes from the fact that

$$\begin{aligned} f'(Q X_1, X_2) \cdot Q v'(X_1, X_2) &= (Q v'(X_1, X_2))^T f'(Q X_1, X_2) \\ &= (v'(X_1, X_2))^T Q^T f'(Q X_1, X_2) \\ &= Q^T f'(Q X_1, X_2) \cdot v'(X_1, X_2) \\ &= f'(X_1, X_2) \cdot v'(X_1, X_2). \end{aligned}$$

From the uniqueness of the weak solution of the Stokes problem follows that $\tilde{u} = u$ a.e in Ω_ℓ and since the pressure is unique up to an additive constant, there exists a constant $c_Q \in \mathbb{R}$ such that $\tilde{p} = p + c_Q$ a.e. in Ω_ℓ . Then we have that

$$\int_{\Omega_\ell} (p(X_1, X_2) + c_Q) dx = \int_{\Omega_\ell} \tilde{p}(X_1, X_2) dx = \int_{\Omega_\ell} p(QX_1, X_2) dx = \int_{\Omega_\ell} p(X_1, X_2) dx,$$

and therefore the constant c_Q equals 0, which is to say $\tilde{p} = p$ a.e. in Ω_ℓ . \square

Let now $g \in W^{1,p}(D_\ell)$ satisfying $g = 0$ on $A_\ell \times \partial\omega$ and $\int_{D_\ell} g(x) dx = 0$. In order to construct a function $u \in W_0^{1,p}(D_\ell)$ satisfying:

$$\begin{cases} \operatorname{div} u = g & \text{in } D_\ell \\ \|\nabla u\|_{L^p(D_\ell)} \leq C \|g\|_{W^{1,p}(D_\ell)}, \end{cases}$$

we use a construction inspired by the one in [9]. More precisely, we look for solutions of the type

$$u(X_1, X_2) = (h(r, X_2)X_1, \phi(r)v(X_2)),$$

where $r = |X_1|$ and $\phi : A_\ell \rightarrow \mathbb{R}$ is a radial function to be determined later. For the sake of simplicity, we use the same notation for the radial function defined on A_ℓ and for the function defined on $(\ell, \ell+1)$ describing the radially, i.e. we write $\phi(r) = \phi(X_1)$ if $r = |X_1|$. Finally, $v \in (W_0^{1,p}(\omega))^{n-k}$ is a solution to the problem

$$\begin{cases} \operatorname{div}_{X_2} v = \bar{g} & \text{in } \omega \\ \|\nabla_{X_2} v\|_{L^p(\omega)} \leq C \|\bar{g}\|_{L^p(\omega)}. \end{cases}$$

where $C = C(p, \omega)$ and $\bar{g} \in L^p(\omega)$ is defined by

$$\bar{g}(X_2) = \int_{A_\ell} g(X_1, X_2) dX_1.$$

Note that such a function v exists since

$$\int_{\omega} \bar{g} dX_2 = \int_{\omega} \int_{A_\ell} g(X_1, X_2) dX_1 dX_2 = \int_{D_\ell} g dx = 0.$$

The computations that follow are in a first time made formally (as if all the functions were regular), their justifications under our regularity hypotheses

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being done later. In order to simplify the notation, we set $\partial_r h(\cdot, X_2) = \frac{\partial h}{\partial r}(\cdot, X_2)$. Then,

$$\begin{aligned}
(\operatorname{div} u)(X_1, X_2) &= \operatorname{div}_{X_1}(h(r, X_2)X_1) + \operatorname{div}_{X_2}(\phi(r)v(X_2)) \\
&= \sum_{i=1}^k \left(\partial_r h(r, X_2) \frac{\partial r}{\partial x_i} x_i + h(r, X_2) \right) + \phi(r)(\operatorname{div}_{X_2} v)(X_2) \\
&= \partial_r h(r, X_2) \frac{\sum_{i=1}^k x_i^2}{r} + \sum_{i=1}^k h(r, X_2) + \phi(r)(\operatorname{div}_{X_2} v)(X_2) \\
&= \partial_r h(r, X_2)r + kh(r, X_2) + \phi(r)\bar{g}(X_2) \\
&= g(r, X_2).
\end{aligned}$$

It follows that

$$\partial_r h(r, X_2)r + kh(r, X_2) = g(r, X_2) - \phi(r)\bar{g}(X_2).$$

Multiplying by r^{k-1} , we therefore have

$$\partial_r h(r, X_2)r^k + kr^{k-1}h(r, X_2) = r^{k-1}g(r, X_2) - r^{k-1}\phi(r)\bar{g}(X_2).$$

This previous inequality then rewrites

$$\partial_r (h(r, X_2)r^k) = r^{k-1}(g(r, X_2) - r^{k-1}\phi(r)\bar{g}(X_2)) \text{ in } (\ell, \ell + 1).$$

For X_2 fixed, a solution to this differential equation is then given by the formula

$$\begin{aligned}
h(r, X_2) &= \frac{1}{r^k} \left[\int_{\ell}^r s^{k-1} g(s, X_2) \, ds - \bar{g}(X_2) \int_{\ell}^r s^{k-1} \phi(s) \, ds \right] \\
&= \frac{1}{r^k \sigma_k} \left[\int_{\ell}^r \int_{\partial B_s} g(X_1, X_2) \, d\sigma_{X_1} \, ds - \bar{g}(X_2) \int_{\ell}^r \int_{\partial B_s} \phi(s) \, d\sigma_{X_1} \, ds \right] \\
&= \frac{1}{r^k \sigma_k} \left[\int_{A_{\ell, r}} g(X_1, X_2) \, dX_1 - \bar{g}(X_2) \int_{A_{\ell, r}} \phi(X_1) \, dX_1 \right].
\end{aligned} \tag{4.2.5}$$

where we denote by $A_{\ell, r}$ the set $B_r \setminus \bar{B}_{\ell}$. Note that $h = 0$ on $A_{\ell} \times \partial\omega$ comes from the fact that $g = 0$ on $A_{\ell} \times \partial\omega$, which implies $\bar{g} = 0$ on $\partial\omega$.

Since we want our solution u to be in $(H_0^1(D_{\ell}))$, we need u' to verify $u(X_1, X_2) = 0$ on ∂D_{ℓ} , which is to say

$$\begin{cases} h(\ell, X_2) = h(\ell + 1, X_2) = 0 \\ h(r, X_2) = 0 \quad \forall X_2 \in \partial\omega. \end{cases}$$

Computing $h(\ell + 1, X_2)$ gives

$$\begin{aligned} h(\ell + 1, X_2) &= \frac{1}{(\ell + 1)^k \sigma_k} \left[\int_{A_\ell} g(X_1, X_2) dX_1 - \bar{g}(X_2) \int_{A_\ell} \phi(X_1) dX_1 \right] \\ &= \frac{1}{(\ell + 1)^k \sigma_k} \bar{g}(X_2) \left(1 - \int_{A_\ell} \phi(X_1) dX_1 \right). \end{aligned}$$

Combined with the fact that $u'' \in (H^1(D_\ell))^{n-k}$ must satisfy $u'' = 0$ on ∂D_ℓ , we derive the following conditions for ϕ :

$$\begin{cases} \phi \in W^{1,\infty}(A_\ell) \\ \phi(\ell) = \phi(\ell + 1) = 0 \\ \int_{A_\ell} \phi(X_1) dX_1 = 1. \end{cases} \quad (4.2.6)$$

We therefore set

$$\phi(r) = \frac{\Phi(r)}{\sigma_k r^{k-1}} \quad (4.2.7)$$

with $\Phi : [\ell, \ell + 1] \rightarrow \mathbb{R}$ a Lipschitz-continuous function satisfying

$$\begin{cases} \Phi(\ell) = \Phi(\ell + 1) = 0 \\ \int_\ell^{\ell+1} \Phi(s) ds = 1. \end{cases} \quad (4.2.8)$$

In order to fix the ideas, let us take

$$\Phi(r) = \begin{cases} 4(r - \ell) & \text{if } \ell \leq r \leq \ell + \frac{1}{2} \\ 4(\ell + 1 - r) & \text{if } \ell + \frac{1}{2} \leq r \leq \ell + 1. \end{cases} \quad (4.2.9)$$

Note that $0 \leq \Phi(r) \leq 2$ for all $r \in [\ell, \ell + 1]$. Consequently, $\phi \geq 0$ and

$$\|\phi\|_{L^\infty(A_\ell)} \leq \frac{2}{\sigma_k \ell^{k-1}}.$$

Theorem 4.2.10. *Let $g \in C^1(\overline{D_\ell})$ be a radial function, and $\phi \in W^{1,\infty}(A_\ell)$ be the radial function defined in (4.2.7). Then the function*

$$(X_1, X_2) \mapsto u'(X_1, X_2) = h(|X_1|, X_2)X_1,$$

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with h defined by (4.2.5), belongs to $(C^1(\overline{D}_\ell))^k$ and satisfies

$$\begin{cases} (\operatorname{div}_{X_1} u')(x) = g(X_1, X_2) - \phi(r) \bar{g}(X_2) \text{ in } D_\ell \\ u' = 0 \text{ on } (\partial A_\ell) \times \omega \\ \|\nabla u'\|_{L^p(D_\ell)} \leq C(\|g\|_{L^p(D_\ell)} + \|\nabla_{X_2} g\|_{L^p(D_\ell)}), \end{cases} \quad (4.2.10)$$

for all $p \in [1, +\infty]$, where the constant C depends only on k and n (for $\ell \geq 1$). Moreover, the mapping associating u' to g is linear.

Proof. We start by proving that $u' \in (C^1(\overline{D}_\ell))^k$. From the definition of the space $C^1(\overline{D}_\ell)$, it follows that g is the restriction to \overline{D}_ℓ of a function of class C^1 on an open set $(B_{\ell+1+\varepsilon} \setminus \overline{B}_{\ell-\varepsilon}) \times \tilde{\omega}$ containing \overline{D}_ℓ . For convenience, we also denote this function by g . We extend ϕ by 0 outside of A_ℓ , and we define $h : (\ell - \varepsilon, \ell + 1 + \varepsilon) \times \tilde{\omega} \rightarrow \mathbb{R}$ by

$$\begin{aligned} h(r, X_2) &= \frac{1}{r^k \sigma_k} \left[\int_\ell^r \int_{\partial B_s} g(X_1, X_2) d\sigma_{X_1} ds - \bar{g}(X_2) \int_\ell^r \int_{\partial B_s} \phi(X_1) d\sigma_{X_1} ds \right] \\ &= \frac{1}{r^k \sigma_k} H(r, X_2). \end{aligned} \quad (4.2.11)$$

for all $r \in (\ell - \varepsilon, \ell + 1 + \varepsilon)$ and all $X_2 \in \tilde{\omega}$, where

$$\bar{g}(X_2) = \int_{A_\ell} g(X_1, X_2) dX_1.$$

Since g is of class C^1 on $(B_{\ell+1+\varepsilon} \setminus \overline{B}_{\ell-\varepsilon}) \times \tilde{\omega}$, we have

$$(\nabla_{X_2} \bar{g})(X_2) = \int_{A_\ell} (\nabla_{X_2} g)(X_1, X_2) dX_1.$$

for all $X_2 \in \tilde{\omega}$.

Note that the mapping associating the function h to the function g is linear. Consequently, the application giving u' is linear with respect to g . Another remark is that \bar{g} is of class C^1 on $\tilde{\omega}$, since g is C^1 on $(B_{\ell+1+\varepsilon} \setminus \overline{B}_{\ell-\varepsilon}) \times \tilde{\omega}$.

Since $X_1 \mapsto |X_1|$ is of class C^∞ on $\mathbb{R}^k \setminus \{0\}$ and $r \mapsto \frac{1}{r^k \sigma_k}$ is of class C^∞ on $(0, +\infty)$, it suffices to prove that H is of class C^1 on $(\ell - \varepsilon, \ell + 1 + \varepsilon) \times \tilde{\omega}$.

Therefore, we will prove that $\partial_r H$ and $\nabla_{X_2} H$ exist and are continuous on $(\ell - \varepsilon, \ell + 1 + \varepsilon) \times \tilde{\omega}$. For X_2 fixed, we have that

$$s \mapsto \int_{\partial B_s} g(X_1, X_2) d\sigma_{X_1} \text{ and } s \mapsto \int_{\partial B_s} \phi(X_1) d\sigma_{X_1}$$

are continuous on $(\ell - \varepsilon, \ell + 1 + \varepsilon)$ thanks to the continuity of g and ϕ . Therefore, the partial derivative with respect to r exists and

$$\partial_r H(r, X_2) = \int_{\partial B_r} g(X_1, X_2) d\sigma_{X_1} - \bar{g}(X_2) \int_{\partial B_r} \phi(X_1) d\sigma_{X_1}. \quad (4.2.12)$$

Since \bar{g} depends only on X_2 and is continuous on $\tilde{\omega}$ and $r \mapsto \int_{\partial B_r} \phi(X_1) d\sigma_{X_1}$ depends only on r and is continuous on $(\ell - \varepsilon, \ell + 1 + \varepsilon)$, the product $\bar{g}(X_2) \int_{\partial B_r} \phi(X_1) d\sigma_{X_1}$ is continuous on $(\ell - \varepsilon, \ell + 1 + \varepsilon) \times \tilde{\omega}$. All is left to do is to prove that the function $(r, X_2) \mapsto \int_{\partial B_r} g(X_1, X_2) d\sigma_{X_1}$ is continuous. To do so, let us take $(r, X_2) \in (\ell - \varepsilon, \ell + 1 + \varepsilon) \times \tilde{\omega}$ and $(r', X'_2) \in ([r - \delta, r + \delta] \times \overline{B}(X_2, \gamma)) \subset ((\ell - \varepsilon, \ell + 1 + \varepsilon) \times \tilde{\omega})$. We then use the equality

$$\int_{\partial B_r} g(X_1, X_2) d\sigma_{X_1} = r^{k-1} \int_{\partial B_1} g(rY_1, X_2) d\sigma_{Y_1}.$$

We derive that

$$\begin{aligned} \int_{\partial B_{r'}} g(X_1, X'_2) d\sigma_{X_1} - \int_{\partial B_r} g(X_1, X_2) d\sigma_{X_1} &= r'^{k-1} \int_{\partial B_1} g(r'Y_1, X'_2) d\sigma_{Y_1} - r^{k-1} \int_{\partial B_1} g(rY_1, X_2) d\sigma_{Y_1} \\ &= r'^{k-1} \left(\int_{\partial B_1} g(r'Y_1, X'_2) - g(rY_1, X_2) d\sigma_{Y_1} \right) \\ &\quad + (r'^{k-1} - r^{k-1}) \int_{\partial B_1} g(rY_1, X_2) d\sigma_{Y_1}. \end{aligned}$$

Since

$$\begin{aligned} |(r'Y_1, X'_2) - (rY_1, X_2)| &= |((r - r')Y_1, X'_2 - X_2)| \\ &= \sqrt{(r - r')^2 |Y_1|^2 + |X'_2 - X_2|^2} \\ &= \sqrt{(r - r')^2 + |X'_2 - X_2|^2} \\ &= |(r', X'_2) - (r, X_2)| \end{aligned}$$

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and $(r')^{k-1}$ is bounded as $r' \rightarrow r$, using the uniform continuity of g on the compact $(\overline{B}_{r+\delta} \setminus B_{r-\delta}) \times \overline{B}(X_2, \gamma)$, we derive

$$r'^{k-1} \left(\int_{\partial B_1} g(r'Y_1, X_2) - g(rY_1, X_2) d\sigma_{Y_1} \right) \rightarrow 0 \text{ as } (r', X_2') \rightarrow (r, X_2).$$

Since $(r'^{k-1} - r^{k-1}) \rightarrow 0$ as $r' \rightarrow r$, we have that (recall that r and X_2 are fixed here)

$$(r'^{k-1} - r^{k-1}) \int_{\partial B_1} g(rY_1, X_2) d\sigma_{Y_1} \rightarrow 0 \text{ as } (r', X_2') \rightarrow (r, X_2),$$

and so finally

$$\int_{\partial B_{r'}} g(X_1, X_2') d\sigma_{X_1} - \int_{\partial B_r} g(X_1, X_2') d\sigma_{X_1} \rightarrow 0 \text{ as } (r', X_2') \rightarrow (r, X_2),$$

which implies that $\partial_r H(r, X_2)$ is continuous.

Let us now prove the existence and the continuity of $\nabla_{X_2} H$. Since for any s fixed, the function

$$X_2 \in \tilde{\omega} \mapsto \int_{\partial B_s} g(X_1, X_2) d\sigma_{X_1}$$

is in $C^1(\tilde{\omega})$ and its gradient on X_2 is given by

$$\int_{\partial B_s} \nabla_{X_2} g(X_1, X_2) d\sigma_{X_1},$$

we deduce that for all $r \in (\ell - \varepsilon, \ell + 1 + \varepsilon)$ and all $X_2 \in \tilde{\omega}$, $\nabla_{X_2} H$ exists and is defined by

$$\nabla_{X_2} H(r, X_2) = \int_{\ell}^r \int_{\partial B_s} \nabla_{X_2} g(X_1, X_2) d\sigma_{X_1} ds - \nabla_{X_2} \bar{g}(X_2) \int_{\ell}^r \int_{\partial B_s} \phi(X_1) d\sigma_{X_1} ds.$$

To start with, the expression

$$\nabla_{X_2} \bar{g}(X_2) \int_{\ell}^r \int_{\partial B_s} \phi(X_1) d\sigma_{X_1} ds$$

is continuous as the product of $\nabla_{X_2} \bar{g}(X_2)$, which is continuous since \bar{g} is of class C^1 , with the function $r \mapsto \int_{\ell}^r \int_{\partial B_s} \phi(X_1) d\sigma_{X_1} ds$ which is continuous thanks to the local integrability of

$$s \mapsto \int_{\partial B_s} \phi(X_1) d\sigma_{X_1} \text{ on } (\ell - \varepsilon, \ell + 1 + \varepsilon).$$

Therefore, the only part that remains to study is

$$\int_{\ell}^r \int_{\partial B_s} \nabla_{X_2} g(X_1, X_2) d\sigma_{X_1} ds.$$

Let $\psi(s, X_2) = \int_{\partial B_s} \nabla_{X_2} g(X_1, X_2) d\sigma_{X_1}$. Using the same kind of computations as previously, one can prove that ψ is continuous on $(\ell - \varepsilon, \ell + 1 + \varepsilon) \times \tilde{\omega}$. Then, we have that

$$\int_{\ell}^{r'} \psi(s, X_2') ds - \int_{\ell}^r \psi(s, X_2) ds = \int_{\ell}^r (\psi(s, X_2') - \psi(s, X_2)) ds + \int_{r'}^r \psi(s, X_2) ds,$$

which goes to 0 as $(r', X_2') \rightarrow (r, X_2)$ by using the uniform continuity of ψ on $[\ell, r] \times \overline{B}(X_2, \gamma)$ for the first term and its boundedness on $[r - \delta, r + \delta] \times \overline{B}(X_2, \gamma)$ for the second term. Therefore, the function

$$(r, X_2) \mapsto \int_{\ell}^r \psi(s, X_2) ds,$$

is continuous and we deduce that $\nabla_{X_2} H(r, X_2)$ is continuous.

We have, from the construction, that

$$\begin{aligned} (\nabla u')(X_1, X_2) &= (\nabla_{X_1} u' \quad \nabla_{X_2} u') (X_1, X_2) \\ &= \left(\frac{X_1 X_1^T}{r} \partial_r h(r, X_2) + h(r, X_2) I_k \quad X_1 \nabla_{X_2} h(r, X_2) \right). \end{aligned}$$

Therefore, for any $x = (X_1, X_2) \in D_{\ell} = A_{\ell} \times \omega$, we have

$$\begin{aligned} (\operatorname{div}_{X_1} u')(x) &= \operatorname{Tr} \left(\frac{X_1 X_1^T}{r} \partial_r h(r, X_2) + h(r, X_2) I_k \right) \\ &= \sum_{i=1}^k \left(\frac{x_i^2}{r} \partial_r h(r, X_2) + h(r, X_2) \right) \\ &= r \partial_r h(r, X_2) + k h(r, X_2) \\ &= \frac{1}{r^{k-1}} (r^k \partial_r h(r, X_2) + k r^{k-1} h(r, X_2)) \\ &= \frac{1}{r^{k-1}} \partial_r (r^k h(r, X_2)) \\ &= \frac{1}{r^{k-1}} \partial_r \left(\frac{H(r, X_2)}{\sigma_k} \right) \\ &= \frac{1}{\sigma_k r^{k-1}} \left(\int_{\partial B_r} g(X_1, X_2) dX_1 - \bar{g}(X_2) \int_{\partial B_r} \phi(X_1) d\sigma_{X_1} \right) \\ &= g(X_1, X_2) - \bar{g}(X_2) \phi(r) \end{aligned} \tag{4.2.13}$$

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thanks to (4.2.12) and to the radially of g and ϕ on D_ℓ . Then, all that remains to do is to prove the estimate

$$\|\nabla u'\|_{L^p(D_\ell)} \leq C(\|g\|_{L^p(D_\ell)} + \|\nabla_{X_2} g\|_{L^p(D_\ell)}).$$

- We start by estimating $\|\nabla_{X_1} u'\|_{L^p(D_\ell)}$.
We have that, for any $x = (X_1, X_2) \in D_\ell$,

$$\begin{aligned} |(\nabla_{X_1} u')(x)| &= \left| \frac{X_1 X_1^T}{r} \partial_r h(r, X_2) + h(r, X_2) I_k \right| \\ &\leq |r \partial_r h(r, X_2)| \left| \frac{X_1 X_1^T}{r^2} \right| + \sqrt{k} |h(r, X_2)| |I_k| \\ &\leq |r \partial_r h(r, X_2)| + \sqrt{k} |h(r, X_2)| \end{aligned}$$

since $\left| \frac{X_1 X_1^T}{r^2} \right| = 1$ and $|I_k| = \sqrt{k}$. Using the equality (see (4.2.13))

$$r \partial_r h(r, X_2) + k h(r, X_2) = g(r, X_2) - \phi(r) \bar{g}(X_2),$$

we therefore can derive that

$$|(\nabla_{X_1} u')(x)| \leq |g(r, X_2)| + |\phi(r) \bar{g}(X_2)| + (\sqrt{k} + k) |h(r, X_2)|.$$

It follows that

$$\|\nabla_{X_1} u'\|_{L^p(D_\ell)} \leq \|g\|_{L^p(D_\ell)} + \|\phi\|_{L^\infty(D_\ell)} \|\bar{g}\|_{L^p(D_\ell)} + (\sqrt{k} + k) \|h\|_{L^p(D_\ell)}.$$

Futhermore, we have that

$$\begin{aligned} \|\bar{g}\|_{L^p(D_\ell)}^p &= \int_{D_\ell} \left| \int_{A_\ell} g(X_1, X_2) dX_1 \right|^p d\tilde{X}_1 dX_2 \\ &\leq \int_{D_\ell} \left[\left(\int_{A_\ell} dX_1 \right)^{\frac{1}{p'}} \left(\int_{A_\ell} |g(X_1, X_2)|^p dX_1 \right)^{\frac{1}{p}} \right]^p d\tilde{X}_1 dX_2 \\ &= \int_{D_\ell} |A_\ell|^{\frac{p}{p'}} \left(\int_{A_\ell} |g(X_1, X_2)|^p dX_1 \right) d\tilde{X}_1 dX_2 \\ &= |A_\ell|^{\frac{p}{p'}} \int_{A_\ell} \int_{\omega} \int_{A_\ell} |g(X_1, X_2)|^p dX_1 dX_2 d\tilde{X}_1 \\ &= |A_\ell|^{\frac{p}{p'}} \int_{A_\ell} \|g\|_{L^p(D_\ell)}^p d\tilde{X}_1 \\ &\leq |A_\ell|^{\frac{p}{p'}+1} \|g\|_{L^p(D_\ell)}^p = |A_\ell|^p \|g\|_{L^p(D_\ell)}^p, \end{aligned}$$

and therefore

$$\begin{aligned} \|\nabla_{X_1} u'\|_{L^p(D_\ell)} &\leq \|g\|_{L^p(D_\ell)} + \|\phi\|_{L^\infty(D_\ell)} |A_\ell| \|g\|_{L^p(D_\ell)} + (\sqrt{k} + k) \|h\|_{L^p(D_\ell)} \\ &\leq \|g\|_{L^p(D_\ell)} + |A_\ell| \frac{2}{\ell^{k-1} \sigma_k} \|g\|_{L^p(D_\ell)} + (k + \sqrt{k}) \|h\|_{L^p(D_\ell)}. \end{aligned}$$

On the one hand, we have that

$$\begin{aligned} |A_\ell| &= \int_\ell^{\ell+1} \left(\int_{\partial B_r} 1 \, d\sigma \right) dr \\ &\leq \sigma_k \int_\ell^{\ell+1} r^{k-1} dr \\ &\leq \sigma_k \int_\ell^{\ell+1} (\ell+1)^{k-1} dr \\ &\leq \sigma_k \int_\ell^{\ell+1} (2\ell)^{k-1} dr = 2^{k-1} \sigma_k \ell^{k-1}. \end{aligned} \tag{4.2.14}$$

Using this inequalities gives

$$\|\nabla_{X_1} u'\|_{L^p(D_\ell)} \leq \|g\|_{L^p(D_\ell)} + 2^k \|g\|_{L^p(D_\ell)} + (k + \sqrt{k}) \|h(r, X_2)\|_{L^p(D_\ell)}.$$

We now have to estimate $\|h\|_{L^p(D_\ell)}$. Using the expression of h form (4.2.11), we have that

$$\begin{aligned} \|h\|_{L^p(D_\ell)} &\leq \left\| \frac{1}{\sigma_k r^k} \right\|_{L^\infty(D_\ell)} \|H\|_{L^p(D_\ell)} \\ &\leq \frac{1}{\sigma_k \ell^k} \|H\|_{L^p(D_\ell)}. \end{aligned}$$

For all $r \in (\ell, \ell+1)$ and for all $X_2 \in \omega$, we furthermore have

$$\begin{aligned} |H(r, X_2)| &= \left| \int_{A_{\ell,r}} g(X_1, X_2) \, dX_1 - \bar{g}(X_2) \int_{A_{\ell,r}} \phi(X_1) \, dX_1 \right| \\ &\leq \int_{A_\ell} |g(X_1, X_2)| \, dX_1 + |\bar{g}(X_2)| \int_{A_\ell} \phi(X_1) \, dX_1 \\ &\leq 2 \int_{A_\ell} |g(X_1, X_2)| \, dX_1 \\ &\leq 2 |A_\ell|^{\frac{1}{p'}} \|g(\cdot, X_2)\|_{L^p(A_\ell)}, \end{aligned}$$

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using that $A_{\ell,r} \subset A_\ell$ and that $0 \leq \int_{A_{\ell,r}} \phi(X_1) dX_1 \leq \int_{A_\ell} \phi(X_1) dX_1 = 1$ (since ϕ is positive). It follows that

$$\begin{aligned}
\|h\|_{L^p(D_\ell)} &\leq \frac{2}{\sigma_k \ell^k} \left\| |A_\ell|^{\frac{1}{p'}} \|g(\cdot, X_2)\|_{L^p(A_\ell)} \right\|_{L^p(D_\ell)} \\
&\leq \frac{2}{\sigma_k \ell^k} \left(\int_{D_\ell} |A_\ell|^{\frac{p}{p'}} \int_{A_\ell} |g(X_1, X_2)|^p dX_1 dx \right)^{\frac{1}{p}} \\
&\leq \frac{2}{\sigma_k \ell^k} \left(|A_\ell|^{\frac{p}{p'}} \int_{A_\ell} \int_{\omega} \int_{A_\ell} |g(X_1, X_2)|^p dX_1 dX_2 d\tilde{X}_1 \right)^{\frac{1}{p}} \\
&\leq \frac{2}{\sigma_k \ell^k} \left(|A_\ell|^{\frac{p}{p'}} \int_{A_\ell} \|g\|_{L^p(D_\ell)}^p d\tilde{X}_1 \right)^{\frac{1}{p}} \\
&\leq \frac{2}{\sigma_k \ell^k} |A_\ell|^{\frac{1}{p'}} |A_\ell|^{\frac{1}{p}} \|g\|_{L^p(D_\ell)} \\
&\leq \frac{2}{\sigma_k \ell^k} |A_\ell| \|g\|_{L^p(D_\ell)} \leq \frac{2^k}{\ell} \|g\|_{L^p(D_\ell)},
\end{aligned}$$

thanks to equality (4.2.14).

Combining all these estimates, we finally have that

$$\| |\nabla_{X_1} u'| \|_{L^p(D_\ell)} \leq \left(1 + 2^k + \frac{2^k(k + \sqrt{k})}{\ell} \right) \|g\|_{L^p(D_\ell)}.$$

- We now deal with $\|\nabla_{X_2} u'\|_{L^p(D_\ell)}$.
We have that, since $|\frac{X_1}{r}| = 1$,

$$\begin{aligned}
|(\nabla_{X_2} u')(X_1, X_2)| &= |X_1 \nabla_{X_2} h(r, X_2)| \\
&= |r \nabla_{X_2} h(r, X_2)| \\
&= \frac{1}{\sigma_k r^{k-1}} \left| \int_{A_{\ell,r}} \nabla_{X_2} g(X_1, X_2) dX_1 - \nabla_{X_2} \bar{g}(X_2) \int_{A_{\ell,r}} \phi(X_1) dX_1 \right| \\
&\leq \frac{1}{\sigma_k r^{k-1}} \left(\int_{A_{\ell,r}} |\nabla_{X_2} g| dX_1 + |\nabla_{X_2} \bar{g}(X_2)| \int_{A_\ell} |\phi(X_1)| dX_1 \right)
\end{aligned}$$

using again that $A_{\ell,r} \subset A_\ell$, that ϕ is positive and that $\int_{A_\ell} \phi dX_1 = 1$.
Therefore, using the equality

$$\nabla_{X_2} \bar{g}(X_2) = \int_{A_\ell} (\nabla_{X_2} g)(X_1, X_2) dX_1,$$

we derive that

$$\begin{aligned} |(\nabla_{X_2} u')(X_1, X_2)| &\leq \frac{1}{\sigma_k r^{k-1}} \left(\int_{A_\ell} |\nabla_{X_2} g(X_1, X_2)| dX_1 + \int_{A_\ell} |\nabla_{X_2} g(X_1, X_2)| dX_1 \right) \\ &\leq \frac{2}{\sigma_k r^{k-1}} \int_{A_\ell} |\nabla_{X_2} g(X_1, X_2)| dX_1 \end{aligned}$$

Using the Hölder inequality, it follows that

$$\begin{aligned} \|\nabla_{X_2} u'\|_{L^p(D_\ell)} &\leq \left\| \frac{1}{\sigma_k r^{k-1}} \right\|_{L^\infty(D_\ell)} \left\| \int_{A_\ell} |\nabla_{X_2} g(X_1, X_2)| dX_1 \right\|_{L^p(D_\ell)} \\ &\leq \frac{1}{\sigma_k \ell^{k-1}} \left(\int_{D_\ell} \left| \int_{A_\ell} |\nabla_{X_2} g(X_1, X_2)| dX_1 \right|^p dx \right)^{\frac{1}{p}} \\ &\leq \frac{1}{\sigma_k \ell^{k-1}} \left(\int_{D_\ell} |A_\ell|^{\frac{p}{p'}} \int_{A_\ell} |\nabla_{X_2} g(X_1, X_2)|^p dX_1 dx \right)^{\frac{1}{p}} \\ &\leq \frac{1}{\sigma_k \ell^{k-1}} \left(|A_\ell|^{\frac{p}{p'}} \int_{A_\ell} \int_{\omega} \int_{A_\ell} |\nabla_{X_2} g(X_1, X_2)|^p dX_1 dX_2 d\tilde{X}_1 \right)^{\frac{1}{p}} \\ &\leq \frac{1}{\sigma_k \ell^{k-1}} |A_\ell|^{\frac{1}{p'}} \left(\int_{A_\ell} \|\nabla_{X_2} g\|_{L^p(D_\ell)}^p d\tilde{X}_1 \right)^{\frac{1}{p}} \\ &\leq \frac{1}{\sigma_k \ell^{k-1}} |A_\ell|^{\frac{1}{p'}} |A_\ell|^{\frac{1}{p}} \|\nabla_{X_2} g\|_{L^p(D_\ell)} = \frac{|A_\ell|}{\sigma_k \ell^{k-1}} \|\nabla_{X_2} g\|_{L^p(D_\ell)} \\ &\leq 2^{k-1} \|\nabla_{X_2} g\|_{L^p(D_\ell)}, \end{aligned}$$

by using equality (4.2.14).

Combining the estimates satisfied by $\nabla_{X_1} u'$ and $\nabla_{X_2} u'$ we get, for $\ell \geq 1$,

$$\begin{aligned} \|\nabla_{X_1} u'\|_{L^p(D_\ell)} + \|\nabla_{X_2} u'\|_{L^p(D_\ell)} &\leq \left(1 + 2^k + \frac{2^k(k + \sqrt{k})}{\ell} \right) \|g\|_{L^p(D_\ell)} + 2^{k-1} \|\nabla_{X_2} g\|_{L^p(D_\ell)} \\ &\leq \left(1 + 2^k(1 + k + \sqrt{k}) \right) \|g\|_{L^p(D_\ell)} + 2^{k-1} \|\nabla_{X_2} g\|_{L^p(D_\ell)} \\ &\leq C_k (\|g\|_{L^p(D_\ell)} + \|\nabla_{X_2} g\|_{L^p(D_\ell)}) . \end{aligned}$$

The result then follows thanks to the equivalence between the norms $\|\nabla(\cdot)\|_{L^p(D_\ell)}$ and $\| |\nabla(\cdot)| \|_{L^p(D_\ell)}$ (see (4.1.5)).

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For the case $p = +\infty$, while it is possible to give a direct proof, the easiest way at this point is to let p go to infinity in (4.2.10)₃, using the fact that the constant is independent of p . \square

4.3 Some results on the divergence problem

In its simplest form, the divergence problem - which plays a crucial role in the mathematical approach of fluid mechanics theory - is stated in the following way:

Let Ω be a Lipschitz domain of \mathbb{R}^n and $g \in L^p(\Omega)$, with $1 < p < +\infty$, satisfying $\int_{\Omega} g \, dx = 0$. Then there exists a vector field $u \in (W_0^{1,p}(\Omega))^n$ such that

$$\begin{cases} \operatorname{div} u = g \text{ in } \Omega \\ \|\nabla u\|_{L^p(\Omega)} \leq C \|g\|_{L^p(\Omega)}, \end{cases} \quad (4.3.1)$$

where $C > 0$ is a constant depending only on Ω .

Despite the simplicity of its statement, this problem is far from being trivial. The difficulty comes in particular from the boundary condition. Indeed, without this condition, it is very easy to construct a solution to the divergence problem (at least for regular domains) by considering a solution $w \in H^2(\Omega)$ to the Laplace equation $\Delta w = g$ in Ω (with some boundary conditions, as for instance homogeneous Dirichlet or Neumann conditions) and then taking $u = \nabla w$. Note that if we consider the homogeneous Neumann conditions, only a small part of the boundary conditions will be satisfied by u , since in this case, only the normal part of the gradient of w -hence of u - vanishes on $\partial\Omega$, not the whole gradient.

Considering the important role that this problem plays in the study of the equations of fluid mechanics, there is no wonder that many authors took interest in finding original solutions to it. To mention only a few, we send the reader to the pioneering works of O.A. Ladyzhenskaya [46] and Bogovskii ([6] and [7]) and to the more recent papers of Acosta et al. [1], Amrouche & Girault [3], Borchers & Sohr [8], Bourgain & Brezis [9], and Dacorogna [30]. For the same problem, we also recommend the books of Dacorogna [31], Galdi [34] and Temam [68].

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For general domains, the manner in which the constant C in (4.3.1) depends on Ω is quite complex, see for instance [6] or [34]. In this section, we are interested in solving the problem (4.3.1) in tubular annuli of the type $D_\ell = A_\ell \times \omega = (B_{\ell+1} \setminus \overline{B}_\ell) \times \omega$ (i.e., we take $\Omega = D_\ell$ in (4.3.1)), and we need the constant C to be independent of ℓ . If we consider the constants appearing in the works cited above, we can clearly see that they depend on ℓ . However, the constructions made in these works do not necessarily lead to the best constants for the inequality (4.3.1)₂. Therefore, there is still a chance for the best constants in (4.3.1) to be independent of ℓ , when considering the problem in the domains D_ℓ , with ℓ going to infinity.

In fact, we know now that without any additional assumptions on g , this is not possible. Indeed, in a recent paper, C. Mardare [50] proved that the best constant C of (4.3.1)₂ is at least of order $\ell^{1/p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$.

The following theorem plays a key role in the proof of the main result of this chapter and it states that one can have on D_ℓ a result similar to the one in (4.3.1), with a constant C independent of ℓ , provided that g is radial with respect to X_1 . In fact, the inequality that we obtain is weaker than the one appearing in the classical divergence problem, since one also needs the L^p -norm of $\nabla_{X_2} g$ - not only the L^p -norm of g - in order to control the L^p -norm of ∇u . This new inequality is however sufficient for our purpose, which is the proof of Theorem 4.4.1 in the next section.

Theorem 4.3.1. *If $g \in W^{1,p}(D_\ell)$ ($1 < p < \infty$) is a radial function along X_1 such that $\int_{D_\ell} g = 0$ and that $g = 0$ on $A_\ell \times \partial\omega$, then there exists $u \in (W_0^{1,p}(D_\ell))^n$ such that:*

$$\begin{cases} \operatorname{div} u = g \text{ in } D_\ell \\ \|\nabla u\|_{L^p(D_\ell)} \leq C(\|g\|_{L^p(D_\ell)} + \|\nabla_{X_2} g\|_{L^p(D_\ell)}), \end{cases} \quad (4.3.2)$$

the constant C being independent of ℓ for $\ell \geq 1$ (C depends on k , n , p and ω). Moreover, u can be chosen depending linearly on g .

Remark 4.3.1. *We would like to emphasize here the fact that, even with the weaker inequality in (4.3.2), it is still not possible to drop the radially assumption on g , if we want the constant C of (4.3.2) to be independent of ℓ . Indeed, without the radially assumption, the best constant in (4.3.2) is at least of order $\ell^{1/p'}$ (see [50]).*

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Proof. To start with, we take once again u defined in the following way:

$$u(X_1, X_2) = (h(r, X_2)X_1, \phi(r)v(X_2)) = (u', u'')(X_1, X_2)$$

with h being defined by the formula

$$h(r, X_2) = \frac{1}{r^k \sigma_k} \left[\int_{A_{\ell, r}} g(X_1, X_2) dX_1 - \bar{g}(X_2) \int_{A_{\ell, r}} \phi(X_1) dX_1 \right],$$

where $A_{\ell, r} = B_r \setminus \bar{B}_\ell$ (for $\ell < r < \ell + 1$), $v \in \left(W_0^{1,p}(\omega)\right)^{n-k}$ is a solution to the problem

$$\begin{cases} \operatorname{div}_{X_2} v = \bar{g} \text{ in } \omega \\ \|\nabla_{X_2} v\|_{L^p(\omega)} \leq C_{p,\omega} \|\bar{g}\|_{L^p(\omega)} \end{cases} \quad (4.3.3)$$

and ϕ is given by (4.2.7) and (4.2.9). Recall that the existence of v is guaranteed by the condition $\int_{D_\ell} g dx = 0$, which implies $\int_\omega \bar{g}(X_2) dX_2 = 0$.

We now are going to justify this formula. First of all, since $g \in H^1(D_\ell)$ is a radial function along X_1 and $g = 0$ on $A_\ell \times \partial\omega$, it follows from Theorem 4.2.8 that there exists a sequence $\{g_m\}$ of $C^1(\bar{D}_\ell)$ -functions, radial along X_1 such that $g_m \xrightarrow{W^{1,p}(D_\ell)} g$. Therefore, we can construct a sequence of functions $\{u'_m\}$, with

$$u'_m(X_1, X_2) = h_m(r, X_2)X_1$$

and

$$h_m(r, X_2) = \frac{1}{r^k \sigma_k} \left[\int_{A_{\ell, r}} g_m(X_1, X_2) dX_1 - \bar{g}_m(X_2) \int_{A_{\ell, r}} \phi(X_1) dX_1 \right]$$

in the exact same way as we did before Theorem 4.2.10. Therefore, according to Theorem 4.2.10, u'_m satisfies

$$\begin{cases} (\operatorname{div}_{X_1} u'_m)(x) = g_m(X_1, X_2) - \phi(r)\bar{g}_m(X_2) \text{ in } D_\ell \\ u'_m = 0 \text{ on } (\partial A_\ell) \times \omega \\ \|\nabla u'_m\|_{L^p(D_\ell)} \leq C(\|g_m\|_{L^p(D_\ell)} + \|\nabla_{X_2} g_m\|_{L^p(D_\ell)}). \end{cases}$$

We now prove that $u'_m \rightarrow u'$ in $L^p(D_\ell)$. Since the function

$$(X_1, X_2) \mapsto \frac{X_1}{\sigma_k r^k}$$

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is in $(L^\infty(D_\ell))^k$, it is enough to prove that

$$(X_1, X_2) \mapsto H_m(|X_1|, X_2) \xrightarrow{L^p(D_\ell)} (X_1, X_2) \mapsto H(|X_1|, X_2)$$

where

$$H(r, X_2) = \int_{A_{\ell,r}} g(X_1, X_2) dX_1 - \bar{g}(X_2) \int_{A_{\ell,r}} \phi(X_1) dX_1.$$

To simplify the computations, we prove that

$$\left((X_1, X_2) \mapsto \int_{A_{\ell,r}} g_m(X_1, X_2) dX_1 \right) \xrightarrow{L^p(D_\ell)} \left((X_1, X_2) \mapsto \int_{A_{\ell,r}} g(X_1, X_2) dX_1 \right).$$

Indeed, using the Hölder inequality (with $\frac{1}{p} + \frac{1}{p'} = 1$),

$$\begin{aligned} \int_{D_\ell} \left| \int_{A_{\ell,r}} (g_m(X_1, X_2) - g(X_1, X_2)) dX_1 \right|^p dx \\ &\leq \int_{D_\ell} |A_{\ell,r}|^{\frac{p}{p'}} \int_{A_{\ell,r}} |g_m(X_1, X_2) - g(X_1, X_2)|^p dX_1 dx \\ &\leq |A_\ell|^{\frac{p}{p'}} \int_{D_\ell} \int_{A_{\ell,r}} |g_m(X_1, X_2) - g(X_1, X_2)|^p dX_1 dx \\ &\leq |A_\ell|^{\frac{p}{p'}} \int_{D_\ell} \int_{A_\ell} |g_m(X_1, X_2) - g(X_1, X_2)|^p dX_1 dx \\ &= |A_\ell|^{\frac{p}{p'}} \int_{A_\ell} \int_{D_\ell} |g_m(X_1, X_2) - g(X_1, X_2)|^p dx dX_1 \\ &= |A_\ell|^{\frac{p}{p'}+1} \|g_m - g\|_{L^p(D_\ell)}^p \\ &= |A_\ell|^p \|g_m - g\|_{L^p(D_\ell)}^p \rightarrow 0, \end{aligned}$$

as $m \rightarrow +\infty$.

The same kind of computations, associated with the fact that

$$\int_{A_\ell} \phi(X_1) dX_1 = 1,$$

lead to

$$(X_1, X_2) \mapsto \bar{g}_m(X_2) \int_{A_{\ell,r}} \phi(X_1) dX_1 \xrightarrow{L^p(D_\ell)} (X_1, X_2) \mapsto \bar{g}(X_2) \int_{A_{\ell,r}} \phi(X_1) dX_1.$$

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and therefore,

$$(X_1, X_2) \mapsto H_m(|X_1|, X_2) \xrightarrow{L^p(D_\ell)} (X_1, X_2) \mapsto H(|X_1|, X_2).$$

Since the mapping giving u' in Theorem 4.2.10 is linear with respect to g , by noting $u' = u'_g$, one has

$$u'_{m_1} - u'_{m_2} = u'_{g_{m_1}} - u'_{g_{m_2}} = u'_{g_{m_1} - g_{m_2}}.$$

Therefore, using Theorem 4.2.10,

$$\|\nabla u'_{m_1} - \nabla u'_{m_2}\|_{L^p(D_\ell)} = \|\nabla u'_{g_{m_1} - g_{m_2}}\|_{L^p(D_\ell)} \leq C\|g_{m_1} - g_{m_2}\|_{W^{1,p}(D_\ell)},$$

which implies, since $\{g_m\}$ is a Cauchy sequence, that $\{\nabla u'_m\}$ is a Cauchy sequence and therefore converges in $(L^p(D_\ell))^n$. We then have that

$$u'_m \xrightarrow{W^{1,p}(D_\ell)} u' \in W^{1,p}(D_\ell). \quad (4.3.4)$$

and since

$$\|\nabla u'_m\|_{L^p(D_\ell)} \leq C(\|g_m\|_{L^p(D_\ell)} + \|\nabla_{X_2} g_m\|_{L^p(D_\ell)}),$$

thanks to Theorem 4.2.10, we derive

$$\|\nabla u'\|_{L^p(D_\ell)} \leq C(\|g\|_{L^p(D_\ell)} + \|\nabla_{X_2} g\|_{L^p(D_\ell)}) \quad (4.3.5)$$

by letting $m \rightarrow +\infty$. Furthermore, we have, for every $x \in D_\ell$,

$$\begin{aligned} (\operatorname{div} u)(x) &= (\operatorname{div}_{X_1} u')(x) + (\operatorname{div}_{X_2} u'')(x) \\ &= (\operatorname{div}_{X_1} u')(x) + \phi(r)(\operatorname{div}_{X_2} v)(X_2) \\ &= (\operatorname{div}_{X_1} u')(x) + \phi(r)\bar{g}(X_2). \end{aligned} \quad (4.3.6)$$

Thanks to the convergence (4.3.4), we know that

$$\operatorname{div}_{X_1} u'_m \rightarrow \operatorname{div}_{X_1} u' \text{ in } L^p(D_\ell).$$

Since $g_m \xrightarrow{L^p(D_\ell)} g$, it follows that

$$(\operatorname{div}_{X_1} u'_m)(x) = g_m(X_1, X_2) - \phi(r)\bar{g}_m(X_2) \rightarrow g(X_1, X_2) - \phi(r)\bar{g}(X_2).$$

Therefore, $(\operatorname{div}_{X_1} u')(x) = g(X_1, X_2) - \phi(r)\bar{g}(X_2)$ and then we deduce from (4.3.6) that

$$\operatorname{div} u = g \text{ in } D_\ell.$$

We now estimate $\|\nabla_{X_1} u''\|_{L^p(D_\ell)}$ and $\|\nabla_{X_2} u''\|_{L^p(D_\ell)}$.

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- To start with, we study the term $\|\nabla_{X_1} u''\|_{L^p(D_\ell)}$. We have

$$|(\nabla_{X_1} u'')(X_1, X_2)| = \left| \frac{\phi'(r)}{r} v(X_2) X_1^T \right|,$$

and since $\left| \frac{X_1^T}{r} \right| = 1$,

$$|(\nabla_{X_1} u'')(X_1, X_2)| \leq |\phi'(r) v(X_2)| = |\phi'(r)| |v(X_2)|.$$

Since $\phi(r) = \frac{\Phi(r)}{\sigma_k r^{k-1}}$ with $\Phi(r) = \begin{cases} 4(r - \ell) & \text{if } \ell \leq r \leq \ell + \frac{1}{2} \\ 4(\ell + 1 - r) & \text{if } \ell + \frac{1}{2} \leq r \leq \ell + 1, \end{cases}$

we have

$$\phi'(r) = \left(\frac{\Phi(r)}{\sigma_k r^{k-1}} \right)' = \frac{1}{\sigma_k} \left(\frac{\Phi'(r)}{r^{k-1}} - \frac{(k-1)\Phi(r)}{r^k} \right)$$

and then

$$|\phi'(r)| \leq \frac{1}{\sigma_k} \left(\frac{4}{\ell^{k-1}} + \frac{2(k-1)}{\ell^k} \right) \leq \frac{2(k+1)}{\sigma_k \ell^{k-1}}$$

if $\ell \geq 1$. Using the Poincaré inequality (4.2.1) for v and the Hölder inequality, we obtain

$$\begin{aligned} \left(\int_{D_\ell} |\nabla_{X_1} u''|^p dx \right)^{\frac{1}{p}} &\leq \left(\int_{D_\ell} |\phi'(r)|^p |v(X_2)|^p dx \right)^{\frac{1}{p}} \\ &= \left(\int_{A_\ell} |\phi'(r)|^p dX_1 \right)^{\frac{1}{p}} \left(\int_{\omega} |v(X_2)|^p dX_2 \right)^{\frac{1}{p}} \\ &\leq \left(\int_{A_\ell} \left(\frac{2(k+1)}{\sigma_k \ell^{k-1}} \right)^p dX_1 \right)^{\frac{1}{p}} \|v\|_{L^p(\omega)} \\ &\leq \frac{2(k+1)}{\sigma_k \ell^{k-1}} C_\omega |A_\ell|^{\frac{1}{p}} \|\nabla_{X_2} v\|_{L^p(\omega)} \\ &\leq \frac{2(k+1)}{\sigma_k \ell^{k-1}} C_\omega C_{p,\omega} |A_\ell|^{\frac{1}{p}} \|\bar{g}\|_{L^p(\omega)} \end{aligned} \tag{4.3.7}$$

4.3. SOME RESULTS ON THE DIVERGENCE PROBLEM

We now estimate the term $\|\bar{g}\|_{L^p(\omega)}$:

$$\begin{aligned}
\|\bar{g}\|_{L^p(\omega)} &\leq \left(\int_{\omega} \left| \int_{A_\ell} g(X_1, X_2) \, dX_1 \right|^p \, dX_2 \right)^{\frac{1}{p}} \\
&\leq \left(\int_{\omega} \left(\int_{A_\ell} |g(X_1, X_2)| \, dX_1 \right)^p \, dX_2 \right)^{\frac{1}{p}} \\
&\leq \left(\int_{\omega} \left(|A_\ell|^{\frac{1}{p'}} \|g(\cdot, X_2)\|_{L^p(A_\ell)} \right)^p \, dX_2 \right)^{\frac{1}{p}} \quad (4.3.8) \\
&\leq |A_\ell|^{\frac{1}{p'}} \left(\int_{\omega} \|g(\cdot, X_2)\|_{L^p(A_\ell)}^p \, dX_2 \right)^{\frac{1}{p}} \\
&\leq |A_\ell|^{\frac{1}{p'}} \|g\|_{L^p(D_\ell)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\left(\int_{D_\ell} |\nabla_{X_1} u''|^p \, dx \right)^{\frac{1}{p}} &\leq \frac{2(k+1)}{\sigma_k \ell^{k-1}} C_\omega C_{p,\omega} |A_\ell|^{\frac{1}{p}} \|\bar{g}\|_{L^p(\omega)} \\
&\leq \frac{2(k+1)}{\sigma_k \ell^{k-1}} C_\omega C_{p,\omega} |A_\ell|^{\frac{1}{p}} |A_\ell|^{\frac{1}{p'}} \|g\|_{L^p(D_\ell)} \\
&\leq \frac{2(k+1)}{\sigma_k \ell^{k-1}} C_\omega C_{p,\omega} |A_\ell| \|g\|_{L^p(D_\ell)} \\
&\leq 2^k (k+1) C_\omega C_{p,\omega} \|g\|_{L^p(D_\ell)}
\end{aligned}$$

by combining inequalities (4.3.7) and (4.3.8) with equality (4.2.14).

- We finally take care of the last term $\|\nabla_{X_2} u''\|_{L^p(D_\ell)}$. First of all,

$$|(\nabla_{X_2} u'')(X_1, X_2)| = |\phi(r)(\nabla_{X_2} v)(X_2)|.$$

Then,

$$\begin{aligned}
 \left(\int_{D_\ell} |\nabla_{X_2} u''|^p dx \right)^{\frac{1}{p}} &\leq \left(\int_{D_\ell} |\phi(r)(\nabla_{X_2} v)(X_2)|^p dx \right)^{\frac{1}{p}} \\
 &= \left(\int_{A_\ell} |\phi(r)|^p dX_1 \int_{\omega} |(\nabla_{X_2} v)(X_2)|^p dX_2 \right)^{\frac{1}{p}} \\
 &\leq \|\phi\|_{L^p(A_\ell)} \|\nabla_{X_2} v\|_{L^p(\omega)} \\
 &\leq |A_\ell|^{\frac{1}{p}} \|\phi\|_{L^\infty(A_\ell)} \|\nabla_{X_2} v\|_{L^p(\omega)} \\
 &\leq \frac{2}{\sigma_k \ell^{k-1}} C_{p,\omega} |A_\ell|^{\frac{1}{p}} \|\bar{g}\|_{L^p(\omega)} \\
 &\leq \frac{2}{\sigma_k \ell^{k-1}} C_{p,\omega} |A_\ell|^{\frac{1}{p}} |A_\ell|^{\frac{1}{p'}} \|g\|_{L^p(D_\ell)} \\
 &\leq \frac{2}{\sigma_k \ell^{k-1}} C_{p,\omega} |A_\ell| \|g\|_{L^p(D_\ell)} \\
 &\leq 2^k C_{p,\omega} \|g\|_{L^p(D_\ell)},
 \end{aligned}$$

by using inequalities (4.3.8) and (4.2.14).

Putting together the estimates for the L^p -norms of $|\nabla_{X_1} u''|$ and $|\nabla_{X_2} u''|$, we get

$$\|\nabla_{X_1} u''\|_{L^p(D_\ell)} + \|\nabla_{X_2} u''\|_{L^p(D_\ell)} \leq 2^k C_{p,\omega} ((k+1)C_\omega + 1) \|g\|_{L^p(D_\ell)}. \quad (4.3.9)$$

The estimate (4.3.2) follows from equations (4.3.5) and (4.3.9) combined with the equivalence of the norm $\|\nabla(\cdot)\|_{L^p(D_\ell)}$ and $\| |\nabla(\cdot)| \|_{L^p(D_\ell)}$ (see (4.1.5)).

We now verify that $u' = 0$ on $\partial D_\ell = (\partial A_\ell) \times \omega \cup A_\ell \times \partial\omega$. To start with, we have that

$$(u'_m)_{|\partial D_\ell} \xrightarrow{L^p(\partial D_\ell)} \text{Tr}(u')$$

by using the convergence (4.3.4). Since, thanks to Theorem 4.2.10, we know that $(u'_m)_{|(\partial A_\ell) \times \omega} = 0$ for all m , it follows that $\text{Tr}(u') = 0$ on $(\partial A_\ell) \times \omega$. The remaining part is to prove that $u' = 0$ on $A_\ell \times \partial\omega$ and to do so, we look to the traces of u'_m on this set. We have that

$$(u'_m)_{|A_\ell \times \partial\omega} = h_m(r, X_2) X_{1|A_\ell \times \partial\omega}.$$

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Since $\frac{1}{r^k \sigma_k} X_1 \in L^\infty(A_\ell \times \partial\omega)$, it suffices to prove that

$$H_m|_{A_\ell \times \omega} \rightarrow 0 \text{ in } L^p(A_\ell \times \omega).$$

We will only give the computations proving the convergence of the first term in the definition of H_m , the second can be treated in a similar manner. We have:

$$\begin{aligned} \int_{A_\ell \times \partial\omega} \left| \int_{A_{\ell,r}} g_m(X_1, X_2) dX_1 \right|^p d\sigma &\leq \int_{A_\ell \times \partial\omega} |A_{\ell,r}|^{\frac{p}{p'}} \int_{A_{\ell,r}} |g_m(X_1, X_2)|^p dX_1 d\sigma \\ &\leq |A_\ell|^{\frac{p}{p'}} \int_{A_\ell \times \partial\omega} \int_{A_\ell} |g_m(X_1, X_2)|^p dX_1 d\sigma \\ &= |A_\ell|^{\frac{p}{p'}+1} \int_{A_\ell \times \partial\omega} |g_m(X_1, X_2)|^p d\sigma \rightarrow 0, \end{aligned}$$

using the fact that

$$\|(g_m)|_{A_\ell \times \partial\omega}\|_{L^p(A_\ell \times \partial\omega)} \rightarrow \|\text{Tr}(g)\|_{L^p(A_\ell \times \partial\omega)} = 0.$$

We therefore can conclude that

$$\text{Tr}(u') = 0 \text{ on } A_\ell \times \partial\omega.$$

We therefore have that

$$u' = 0 \text{ on } \partial D_\ell.$$

It remains to prove that $u'' \in W_0^{1,p}(D_\ell)$. Since $v \in W_0^{1,p}(\omega)$, there exists a sequence $\{v_m\} \subset \mathbb{D}(\omega)$ of functions converging to v in $W^{1,p}(\omega)$. Therefore, $\phi v_m \in W^{1,p}(D_\ell) \cap C(\overline{D}_\ell)$ and $\phi v_m = 0$ on $\partial\omega$, which is to say that $\phi v_m \in W_0^{1,p}(D_\ell)$, and therefore $\phi v \in W_0^{1,p}(D_\ell)$ using the fact that $W_0^{1,p}$ is a closed space and $\phi v_m = 0$ on ∂D_ℓ . \square

Remark 4.3.2. Note that in the case $k = n - 1$, hence $n - k = 1$, we have an obvious solution to problem (4.3.3), with $C_{p,\omega} = 1$, that is even true for $p \in [1, +\infty]$. Thus, in this case, we have a relatively simple constructive solution for the problem (4.3.2). Moreover, the result of Theorem 4.3.1 is this time valid for $1 \leq p \leq +\infty$, with the constant in (4.3.2)₂ depending only on k and ω . In order to see that, it suffices to remark that, on the one hand, the arguments in the proof of Theorem 4.3.1 remain valid for any

$p \in [1, +\infty)$ (since the Theorem 4.2.8 and the inequality (4.2.10)₃ of Theorem 4.2.10 are valid for $p \in [1, +\infty)$); and on the other hand, for $p = +\infty$, since the domain D_ℓ is bounded, one has $g \in W^{1,s}(D_\ell)$ for any $s \in [1, +\infty)$, hence we can apply the result of Theorem 4.3.1 with $p = s$ and then let s go to $+\infty$ and use the fact that the constant in (4.3.2)₂ is independent of p .

Remark 4.3.3. We would also like to remark here that one obtains a different solution u satisfying (4.3.2) for each choice of $\Phi \in W^{1,\infty}((\ell, \ell + 1))$ that intervenes in the construction of the function ϕ . Indeed, we made the special choice (4.2.9) in order to fix the ideas, but any positive function satisfying the properties (4.2.8) is valid, since what is important here is that the function ϕ defined in (4.2.7) satisfies the properties (4.2.6).

Furthermore, we have the following lemma, which can be easily obtained by a scaling argument.

Lemma 4.3.1. *Let $\ell \geq 1$ and $g \in L^p(\Omega_\ell)$ ($1 < p < +\infty$) such that $\int_{\Omega_\ell} g \, dx = 0$. Then there exists $u \in (W_0^{1,p}(\Omega_\ell))^n$ such that*

$$\begin{cases} \operatorname{div} u = g & \text{in } \Omega_\ell \\ \|\nabla u\|_{L^p(\Omega_\ell)} \leq C\ell \|g\|_{L^p(\Omega_\ell)}, \end{cases}$$

where C is a constant depending only on k , p and ω .

4.4 The main result

Before giving the main result of this paper, we provide here a last Lemma that gives a useful property of solenoidal vector fields in Ω_ℓ , vanishing on $B_\ell \times \partial\omega$.

Lemma 4.4.1. *If a function $v \in \mathbb{H}^1(\Omega_\ell)$ satisfies*

$$\begin{cases} \operatorname{div} v = 0 & \text{in } \Omega_\ell \\ v = 0 & \text{on } B_\ell \times \partial\omega, \end{cases}$$

then

$$\int_{(\partial B_r) \times \omega} v \cdot \nu \, d\sigma = 0$$

for all $r \in (0, \ell]$.

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Proof. We have that

$$0 = \int_{B_r \times \omega} \operatorname{div} v \, dx = \int_{\partial(B_r \times \omega)} v \cdot \nu \, d\sigma = \int_{(\partial B_r) \times \omega} v \cdot \nu \, d\sigma + \int_{B_r \times \partial\omega} v \cdot \nu \, d\sigma,$$

thanks to the divergence theorem. Since $B_r \times \partial\omega \subset B_\ell \times \partial\omega$, it follows that $\int_{B_r \times \partial\omega} v \cdot \nu \, d\sigma = 0$, and therefore $\int_{(\partial B_r) \times \omega} v \cdot \nu \, d\sigma = 0$. \square

We now give our main result:

Theorem 4.4.1. *Let $f \in V_\beta(\Omega_\infty)$ for some small enough $\beta > 0$ verify*

$$f = (f', f'') = \begin{cases} f'(QX_1, X_2) = Qf'(X_1, X_2) \\ f''(QX_1, X_2) = f''(X_1, X_2), \end{cases} \quad (4.4.1)$$

and let $u_\ell \in \hat{\mathbb{H}}^1(\Omega_\ell)$ be the solution of the variational problem (4.1.2). Then for all $\ell_0 > 0$,

$$(u_\ell, p_\ell) \rightarrow (u_\infty, p_\infty) \text{ strongly in } \mathbb{H}^1(\Omega_{\ell_0}) \times \hat{L}^2(\Omega_{\ell_0})$$

as $\ell \rightarrow \infty$, where $(u_\infty, p_\infty) \in (H_{\text{loc}}^1(\bar{\Omega}_\infty))^n \times L_{\text{loc}}^2(\bar{\Omega}_\infty)/\mathbb{R}$ is weak solution to the following Dirichlet problem in the cylinder Ω_∞ :

$$\begin{cases} u_\infty \text{ satisfies the same radial properties as } f \text{ (see (4.4.1))} \\ -\mu \Delta u_\infty + \nabla p_\infty = f \quad \text{in } (\mathcal{D}'(\Omega_\infty))^n \\ \operatorname{div} u_\infty = 0 \text{ in } \Omega_\infty \\ u_\infty = 0 \text{ on } \partial\Omega_\infty \\ \|\nabla u_\infty\|_{L^2(\Omega_\ell)} = O(e^{2\beta\ell}). \end{cases} \quad (4.4.2)$$

Furthermore, we have the estimates

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-\alpha\ell} \quad \text{for all } \ell > 1$$

and

$$\|p_\ell - p_\infty\|_{\hat{L}^2(\Omega_{\ell/2})} \leq C \ell e^{-\alpha\ell} \quad \text{for all } \ell > 1,$$

where $C \geq 0$ and $\alpha > 0$ are constants depending only on k, n, ω, μ, C_0 (the constant appearing in the definition of $V_\beta(\Omega_\infty)$) and β .

Remark 4.4.1. *The radial properties of u_∞ are used to prove the uniqueness of u_∞ , and it is highly probable that without this condition, the uniqueness of the solution to problem (4.4.2) would be lost. This fact is suggested by the case $k = 1$, where one can construct different solutions to problem (4.4.2) (minus the radial properties) having non-zero fluxes (see e.g. [19]).*

Proof. The proof of the theorem is divided in seven steps. The first three ones will show that u_ℓ is a Cauchy “sequence” for the norm of $H^1(\Omega_{\ell_0})$ for all $\ell_0 > 0$, then we prove that the limit of (u_ℓ) is the solution to the problem (4.4.2) and the uniqueness of the solution to this problem. Finally, we treat the estimate of the pressure.

Step I. *There exists a constant $a \in (0, 1)$ only depending on k , n and ω such that*

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq a \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \quad (4.4.3)$$

for all $\ell > 1$, for all $\ell_1 \leq \ell - 1$ and for all $r \geq 0$.

We start here by recalling the variational equalities satisfied by u_ℓ and $u_{\ell+t}$ on Ω_ℓ and $\Omega_{\ell+t}$ respectively:

$$\mu \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v dx = \int_{\Omega_\ell} f v dx \text{ for all } v \in \hat{\mathbb{H}}_0^1(\Omega_\ell) \quad (4.4.4)$$

and

$$\mu \int_{\Omega_{\ell+r}} \nabla u_{\ell+r} \cdot \nabla v dx = \int_{\Omega_{\ell+r}} f v dx \text{ for all } v \in \hat{\mathbb{H}}_0^1(\Omega_{\ell+r}).$$

Since a function $v \in \hat{\mathbb{H}}_0^1(\Omega_\ell)$ also belongs to $\hat{\mathbb{H}}_0^1(\Omega_{\ell+r})$ (by extending it by 0 outside of Ω_ℓ), one can use the same function v as a test function for both equations. Subtracting the two equations leads to

$$\int_{\Omega_\ell} \nabla(u_\ell - u_{\ell+r}) \cdot \nabla v dx = 0 \text{ for all } v \in \hat{\mathbb{H}}_0^1(\Omega_\ell). \quad (4.4.5)$$

Inspired by the methods from numerous previous works (see e.g. [11] or [23]), we look for a test function of the type $\rho(u_\ell - u_{\ell+r})$, with $\rho = \rho(X_1)$ and $\rho(\ell) = 0$. More precisely, we define ρ in the following way:

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$$\rho = \rho(|X_1|) = \begin{cases} 0 & \text{if } |X_1| > \ell_1 + 1 \\ 1 & \text{if } |X_1| < \ell_1 \\ \ell_1 + 1 - |X_1| & \text{if } \ell_1 \leq |X_1| \leq \ell_1 + 1. \end{cases}$$

This implies that

$$\nabla_{X_1} \rho = -\frac{X_1}{|X_1|} \text{ on } D_{\ell_1}.$$

However, this kind of functions cannot be considered as test functions in (4.4.5). Indeed, while $\rho(u_\ell - u_{\ell+r}) \in \mathbb{H}_0^1(\Omega_\ell)$, its divergence do not vanish on Ω_ℓ since

$$\begin{aligned} \operatorname{div}(\rho(u_\ell - u_{\ell+r})) &= \sum_{i=1}^k (\partial_{x_i} \rho)(u_{\ell,i} - u_{\ell+r,i}) + \sum_{i=1}^k \rho \partial_{x_i}(u_\ell - u_{\ell+r})_i \\ &\quad + \sum_{i=k+1}^n \rho \partial_{x_i}(u_\ell - u_{\ell+r})_i \\ &= (\nabla_{X_1}) \cdot \rho(u_\ell - u_{\ell+r})' + \rho \operatorname{div}(u_\ell - u_{\ell+r}). \end{aligned}$$

Since $\operatorname{div}(u_\ell - u_{\ell+r}) = 0$, in order to have $\operatorname{div}(\rho(u_\ell - u_{\ell+r})) = 0$ in D_{ℓ_1} , we therefore have to subtract from $\rho(u_\ell - u_{\ell+r})$ a $\mathbb{H}_0^1(D_{\ell_1})$ -function β such that

$$\begin{cases} \operatorname{div} \beta = (\nabla_{X_1} \rho) \cdot (u_\ell - u_{\ell+r})' \text{ in } D_{\ell_1} \\ \|\nabla \beta\|_{L^2(D_{\ell_1})} \leq C \left(\|(\nabla_{X_1} \rho) \cdot (u_\ell - u_{\ell+r})'\|_{L^2(D_{\ell_1})} \right. \\ \quad \left. + \|(\nabla_{X_1} \rho) \cdot \nabla_{X_2}(u_\ell - u_{\ell+r})'\|_{L^2(D_{\ell_1})} \right). \end{cases} \quad (4.4.6)$$

Note that here, we use the equality

$$\nabla_{X_2}((\nabla_{X_1} \rho) \cdot (u_\ell - u_{\ell+r})') = (\nabla_{X_1} \rho) \cdot \nabla_{X_2}(u_\ell - u_{\ell+r})'.$$

In order to ensure the existence of such a function, we need to verify that $(\nabla_{X_1} \rho) \cdot (u_\ell - u_{\ell+r})'$ verifies the hypotheses of Theorem 4.3.1.

First of all, $(\nabla_{X_1} \rho) \cdot (u_\ell - u_{\ell+r})' \in H^1(D_{\ell_1})$, since $\nabla_{X_1} \rho = \frac{X_1}{|X_1|} \in (W^{1,\infty}(D_{\ell_1}))^k$.

Then, we have that (recall that $D_{\ell_1} = A_{\ell_1} \times \omega$)

$$\begin{aligned} \int_{D_{\ell_1}} (\nabla_{X_1} \rho) \cdot (u_\ell - u_{\ell+r})' dx &= \int_{A_{\ell_1} \times \omega} -\frac{X_1}{|X_1|} \cdot (u_\ell - u_{\ell+r})' dx \\ &= \int_{A_{\ell_1}} \int_{\omega} -\frac{X_1}{|X_1|} \cdot (u_\ell - u_{\ell+r})' dx \\ &= \int_{\ell_1}^{\ell_1+1} \int_{(\partial B_r) \times \omega} -\frac{X_1}{|X_1|} \cdot (u_\ell - u_{\ell+r})' d\sigma_{X_1} dX_2 dr. \end{aligned}$$

Since $\nu(x) = \left(\frac{X_1}{|X_1|}, 0 \right)$ on $(\partial B_r) \times \omega$, then

$$\frac{X_1}{|X_1|} \cdot (u_\ell - u_{\ell+r})' = (u_\ell - u_{\ell+r}) \cdot \nu(x)$$

on $(\partial B_r) \times \omega$.

Moreover, we have that $\partial\Omega_\ell = ((\partial B_\ell) \times \omega) \cup (B_\ell \times \partial\omega)$ and, since $u_\ell = 0$ on $\partial\Omega_\ell$ and $u_{\ell+r} = 0$ on $\partial\Omega_{\ell+r}$, we derive

$$u_\ell - u_{\ell+r} = 0 \quad \text{on} \quad B_\ell \times \partial\omega,$$

given that $B_\ell \times \partial\omega \subset B_{\ell+r} \times \partial\omega \subset \partial\Omega_{\ell+r}$.

It follows that

$$\int_{D_{\ell_1}} (\nabla_{X_1} \rho) \cdot (u_\ell - u_{\ell+r})' dx = - \int_{\ell_1}^{\ell_1+1} \int_{(\partial B_r) \times \omega} (u_\ell - u_{\ell+r}) \cdot \nu d\sigma dr = 0.$$

thanks to Lemma 4.4.1. Note that the fact that $u_\ell - u_{\ell+r} = 0$ on $B_\ell \times \partial\omega$ also implies that

$$(\nabla_{X_1} \rho) \cdot (u_\ell - u_{\ell+r})' = 0 \quad \text{on} \quad A_{\ell_1} \times \partial\omega \subset B_\ell \times \partial\omega.$$

We now have to prove that $(\nabla_{X_1} \rho) \cdot (u_\ell - u_{\ell+r})'$ is radial along X_1 .

Since f verifies that

$$f = (f', f'') = \begin{cases} f'(QX_1, X_2) = Qf'(X_1, X_2) \\ f''(QX_1, X_2) = f''(X_1, X_2), \end{cases}$$

we have by Theorem 4.2.9 that u_ℓ and $u_{\ell+r}$ verify

$$u'_\ell(QX_1, X_2) = Qu'_\ell(X_1, X_2) \quad \text{and} \quad u'_{\ell+r}(QX_1, X_2) = Qu'_{\ell+r}(X_1, X_2).$$

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Then, we know by Theorem 4.2.7 that there exists two functions

$$R_\ell = R_\ell(r, X_2) \text{ and } R_{\ell+r} = R_{\ell+r}(r, X_2),$$

with $r = |X_1|$, such that

$$u'_\ell(X_1, X_2) = R_\ell(r, X_2)X_1 \text{ for all } (X_1, X_2) \in \Omega_\ell$$

and

$$u'_{\ell+r}(X_1, X_2) = R_{\ell+r}(r, X_2)X_1 \text{ for all } (X_1, X_2) \in \Omega_\ell.$$

It follows that, on Ω_ℓ , $(u_\ell - u_{\ell+r})'$ can be written as $R(r, X_2)X_1$. Since $\nabla_{X_1}\rho = -\frac{X_1}{|X_1|}$, we therefore have that

$$(\nabla_{X_1}\rho) \cdot (u_\ell - u_{\ell+r})' = -\frac{X_1}{|X_1|} \cdot R(r, X_2)X_1 = -R(r, X_2)r$$

which is radial along X_1 . Consequently, we can apply Theorem 4.3.1 and there exists a function β verifying the properties (4.4.6). We extend this function outside D_{ℓ_1} by 0, the function obtained in this way satisfying the properties

$$\beta \in \mathbb{H}_0^1(\Omega_\ell) \text{ and } \operatorname{div} \beta = \operatorname{div}(\rho(u_\ell - u_{\ell+r})) \text{ in } \Omega_\ell.$$

Therefore,

$$v = \rho(u_\ell - u_{\ell+r}) - \beta \in \hat{\mathbb{H}}_0^1(\Omega_\ell).$$

Then,

$$\int_{\Omega_\ell} \nabla(u_\ell - u_{\ell+r}) \cdot \nabla(\rho(u_\ell - u_{\ell+r}) - \beta) \, dx = 0.$$

This implies, taking into account that $|\nabla_{X_1}\rho| = 1$,

$$\begin{aligned} \int_{\Omega_\ell} \rho |\nabla(u_\ell - u_{\ell+r})|^2 \, dx &= - \int_{D_{\ell_1}} \nabla_{X_1}(u_\ell - u_{\ell+r}) \cdot (u_\ell - u_{\ell+r})(\nabla_{X_1}\rho)^T \, dx \\ &\quad + \int_{D_{\ell_1}} \nabla(u_\ell - u_{\ell+r}) \cdot \nabla\beta \, dx \\ &\leq \int_{D_{\ell_1}} |\nabla_{X_1}(u_\ell - u_{\ell+r})| |(u_\ell - u_{\ell+r})| \, dx \\ &\quad + \int_{D_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})| |\nabla\beta| \, dx. \end{aligned}$$

Using the Cauchy-Schwarz inequality, (4.4.6) and the fact that on D_{ℓ_1} , $\nabla_{X_1}\rho = -\frac{X_1}{|X_1|}$ and hence $\|\nabla_{X_1}\rho\|_{L^\infty(D_{\ell_1})} \leq 1$, we deduce that

$$\begin{aligned} \int_{\Omega_\ell} \rho |\nabla(u_\ell - u_{\ell+r})|^2 dx &\leq \|\nabla_{X_1}(u_\ell - u_{\ell+r})\|_{L^2(D_{\ell_1})} \|u_\ell - u_{\ell+r}\|_{L^2(D_{\ell_1})} \\ &\quad + \|\nabla(u_\ell - u_{\ell+r})\|_{L^2(D_{\ell_1})} \|\nabla\beta\|_{L^2(D_{\ell_1})} \\ &\leq \|\nabla_{X_1}(u_\ell - u_{\ell+r})\|_{L^2(D_{\ell_1})} \|u_\ell - u_{\ell+r}\|_{L^2(D_{\ell_1})} \\ &\quad + C \|\nabla(u_\ell - u_{\ell+r})\|_{L^2(D_{\ell_1})} \left(\|(u_\ell - u_{\ell+r})'\|_{L^2(D_{\ell_1})} \right. \\ &\quad \left. + \|\nabla_{X_2}(u_\ell - u_{\ell+r})'\|_{L^2(D_{\ell_1})} \right). \end{aligned}$$

Since $(u_\ell - u_{\ell+r})$ vanishes on $B_\ell \times \partial\omega$, we can use the Poincaré inequality (4.2.1) to deduce

$$\int_{\Omega_\ell} \rho |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq C \|\nabla(u_\ell - u_{\ell+r})\|_{L^2(D_{\ell_1})}^2 = C \int_{D_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx$$

and, since ρ is nonnegative, this leads to

$$\begin{aligned} \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx &= \int_{\Omega_{\ell_1}} \rho |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq \int_{\Omega_\ell} \rho |\nabla(u_\ell - u_{\ell+r})|^2 dx \\ &\leq C \left\{ \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx - \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \right\} \end{aligned}$$

which is to say that

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq \frac{C}{1+C} \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx.$$

which proves inequality (4.4.3) with $a = \frac{C}{1+C} \leq 1$.

Step II. *There exists constants $C \geq 0$ and $\alpha > 0$, depending only on ω , C_0 , β and μ such that*

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-\alpha\ell} \quad \text{for all } \ell > 0 \text{ and all } r \in [0, 1]. \quad (4.4.7)$$

Let $\ell > 0$ and $r \in [0, 1]$. Starting with $\ell_1 = \frac{\ell}{2}$, we iterate the inequality

$$\int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq a \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_{\ell+r})|^2 dx$$

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$[\frac{\ell}{2}]$ times, where $[\frac{\ell}{2}]$ is the integer part of $\frac{\ell}{2}$. As a result, we have

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq a^{[\frac{\ell}{2}]} \int_{\Omega_{\frac{\ell}{2} + [\frac{\ell}{2}]}} |\nabla(u_\ell - u_{\ell+r})|^2 dx.$$

Noticing that

$$\frac{\ell}{2} - 1 < \left[\frac{\ell}{2}\right] \leq \frac{\ell}{2} \quad \text{et} \quad 0 < a < 1,$$

it follows that $\Omega_{\frac{\ell}{2} + [\frac{\ell}{2}]} \subset \Omega_\ell$ and then

$$\int_{\Omega_{\frac{\ell}{2}}} |\nabla(u_\ell - u_{\ell+r})|^2 dx \leq a^{\frac{\ell}{2}-1} \int_{\Omega_\ell} |\nabla(u_\ell - u_{\ell+r})|^2 dx.$$

Hence,

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq ce^{-\tilde{\alpha}\ell} \|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_\ell)}, \quad (4.4.8)$$

with $c = a^{-\frac{1}{2}}$ and $\tilde{\alpha} = \frac{1}{4} \ln(\frac{1}{a}) > 0$.

Now, in order to establish the step **II**, we only need to estimate $\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_\ell)}$. We have that $u_\ell \in \hat{\mathbb{H}}_0^1(\Omega_\ell)$, and we therefore can use u_ℓ as a test function in (4.4.4). This writes

$$\mu \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla u_\ell dx = \int_{\Omega_\ell} f u_\ell dx.$$

Consequently, using the Poincaré inequality (4.2.1),

$$\begin{aligned} \mu \int_{\Omega_\ell} |\nabla u_\ell|^2 dx &\leq \|f\|_{L^2(\Omega_\ell)} \|u_\ell\|_{L^2(\Omega_\ell)} \\ &\leq C_\omega \|f\|_{L^2(\Omega_\ell)} \|\nabla u_\ell\|_{L^2(\Omega_\ell)} \\ &\leq Ce^{\beta\ell} \|\nabla u_\ell\|_{L^2(\Omega_\ell)}, \end{aligned}$$

thanks to the fact that $f \in V_\beta(\Omega_\infty)$. Therefore,

$$\|\nabla u_\ell\|_{L^2(\Omega_\ell)} \leq Ce^{\beta\ell}, \quad (4.4.9)$$

for a constant C only depending on μ , C_ω and on the constant C_0 appearing in the definition of the space $V_\beta(\Omega_\infty)$.

In the same way

$$\|\nabla u_{\ell+r}\|_{L^2(\Omega_\ell)} \leq \|\nabla u_{\ell+r}\|_{L^2(\Omega_{\ell+r})} \leq C e^{\beta(\ell+r)} \leq C e^\beta e^{\beta\ell},$$

since $r \in [0, 1]$, with C being the same constant as previously.

We therefore have

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_\ell)} \leq \|\nabla u_\ell\|_{L^2(\Omega_\ell)} + \|\nabla u_{\ell+r}\|_{L^2(\Omega_\ell)} \leq C e^{\beta\ell},$$

with the last constant still not depending on $\ell > 0$ or on $r \in [0, 1]$.

Combined with the inequality (4.4.8), the last inequality implies

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-(\tilde{\alpha}-\beta)\ell}.$$

Finally, if β from the definition of $V_\beta(\Omega_\infty)$ verifies $\beta < \tilde{\alpha}$ (where $\tilde{\alpha} = \frac{1}{4} \ln(\frac{1}{a})$ only depends on k, n and ω since this is the case for the constant a found in step **I**), we have

$$\|\nabla(u_\ell - u_{\ell+r})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-\alpha\ell},$$

for all positive ℓ and all $r \in [0, 1]$, with $\alpha \in \mathbb{R}$ given by $\alpha = \tilde{\alpha} - \beta > 0$.

Step III. *There exists two constants $C \geq 0$ and $\alpha > 0$ depending only on μ, k, n, ω, C_0 , and β such that*

$$\|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-\alpha\ell}, \quad (4.4.10)$$

for all positive ℓ and all non-negative t .

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This is a simple consequence of inequality (4.4.7):

$$\begin{aligned}
\|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\frac{\ell}{2}})} &\leq \sum_{i=0}^{[t]-1} \|\nabla(u_{\ell+i} - u_{\ell+i+1})\|_{L^2(\Omega_{\frac{\ell}{2}})} \\
&\quad + \|\nabla(u_{\ell+[t]} - u_{\ell+t})\|_{L^2(\Omega_{\frac{\ell}{2}})} \\
&\leq \sum_{i=0}^{[t]-1} \|\nabla(u_{\ell+i} - u_{\ell+i+1})\|_{L^2(\Omega_{\frac{\ell+i}{2}})} \\
&\quad + \|\nabla(u_{\ell+[t]} - u_{\ell+t})\|_{L^2(\Omega_{\frac{\ell+[t]}{2}})} \\
&\leq \sum_{i=0}^{[t]} C e^{-\alpha(\ell+i)} = C e^{-\alpha\ell} \sum_{i=0}^{[t]} e^{-\alpha i} \\
&\leq C \frac{1}{1 - e^{-\alpha}} e^{-\alpha\ell},
\end{aligned}$$

with α being the same as in step **II** and C being different but depending on the same parameters.

Step IV. *There exists $u_\infty \in \mathbb{H}_{\text{loc}}^1(\bar{\Omega}_\infty)$ such that for all $\ell_0 > 0$, $u_\ell \rightarrow u_\infty$ in $\mathbb{H}^1(\Omega_{\ell_0})$, and $u_\ell - u_\infty$ verifies*

$$\|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C e^{-\alpha\ell} \quad \text{for all } \ell > 0, \quad (4.4.11)$$

for some constants $C \geq 0$ and $\alpha > 0$, depending only on ω , μ , C_0 and β .

A consequence of the Poincaré inequality (since $u_\ell - u_{\ell+t} \in \mathbb{H}^1(\Omega_\ell)$ and $u_\ell - u_{\ell+t} = 0$ on $B_\ell \times \partial\omega$) and of inequality (4.4.10) is that for a fixed $\ell_0 > 0$,

$$\|u_\ell - u_{\ell+t}\|_{H^1(\Omega_{\ell_0})} \leq C \|\nabla(u_\ell - u_{\ell+t})\|_{L^2(\Omega_{\ell_0})} \leq C e^{-\alpha\ell},$$

for all $\ell \geq 2\ell_0$ and all $t \geq 0$. This implies that $(u_\ell)_{\ell \geq 0}$ is a Cauchy “sequence” for the norm of the space $\mathbb{H}^1(\Omega_{\ell_0})$. Since $\mathbb{H}^1(\Omega_{\ell_0})$ is a Banach space, there exists $u_\infty^{\ell_0} \in \mathbb{H}^1(\Omega_{\ell_0})$ such that $u_\ell \rightarrow u_\infty^{\ell_0}$ in $\mathbb{H}^1(\Omega_{\ell_0})$.

Making ℓ_0 vary in \mathbb{N}^* gives us a sequence of limits u_∞^k defined in Ω_k . For all non-zero natural integer k , we have $u_\ell \rightarrow u_\infty^k$ in $\mathbb{H}^1(\Omega_k)$. Since $\Omega_{k_1} \subset \Omega_{k_2}$ for all $k_1 < k_2$, we have $u_\infty^{k_1} = u_\infty^{k_2}$ a.e. in Ω_{k_1} , since $u_\ell \rightarrow u_\infty^{k_1}$ and $u_\ell \rightarrow u_\infty^{k_2}$ in $\mathbb{H}^1(\Omega_{k_1})$.

Then, we can construct a function $u_\infty \in (H_{\text{loc}}^1(\bar{\Omega}_\infty))^n$ such that $u_\infty = u_\infty^k$ in Ω_k for all positive integer k . It is enough to set

$$u_\infty = \begin{cases} u_\infty^1 & \text{in } \Omega_1 \\ u_\infty^k & \text{in } \Omega_k \setminus \Omega_{k-1} \quad \text{for all } k \geq 2. \end{cases}$$

This function verifies

$$u_\ell \rightarrow u_\infty \text{ in } \mathbb{H}^1(\Omega_{\ell_0}) \quad \text{for all } \ell_0 > 0.$$

Estimate (4.4.11) is finally obtained by taking $\ell > 0$ fixed and making t go to infinity in inequality (4.4.10).

Step V. *The limit u_∞ from the previous step is a solution to problem (4.4.2)*

First of all, u_∞ satisfies the same radial properties as f , since for any fixed $\ell_0 > 0$,

$$u_\ell \rightarrow u_\infty \text{ in } \mathbb{H}^1(\Omega_{\ell_0})$$

as $\ell \rightarrow +\infty$, and u_ℓ satisfies the following property: for all $Q \in \mathbb{O}^k$, one has

$$\begin{cases} u'_\ell(QX_1, X_2) = Qu'_\ell(X_1, X_2) \\ u''_\ell(QX_1, X_2) = u''_\ell(X_1, X_2) \end{cases} \quad \text{for a.e. } (X_1, X_2) \in \Omega_{\ell_0}.$$

The properties for u_∞ are then obtained by proving that we also have

$$u_\ell \rightarrow \tilde{u}_\infty \text{ in } (L^2(\Omega_{\ell_0}))^n$$

with

$$\tilde{u}_\infty(X_1, X_2) = (\tilde{u}'_\infty, \tilde{u}''_\infty)(X_1, X_2) = (Q^T u'_\infty(QX_1, X_2), u''_\infty(QX_1, X_2)).$$

Indeed,

$$\begin{aligned} \|u'_\ell - \tilde{u}'_\infty\|_{L^2(\Omega_{\ell_0})}^2 &= \int_{\Omega_{\ell_0}} |u'_\ell(X_1, X_2) - Q^T u'_\infty(QX_1, X_2)|^2 dx \\ &= \int_{\Omega_{\ell_0}} |Q^T(u'_\ell(QX_1, X_2) - u'_\infty(QX_1, X_2))|^2 dx \\ &= \int_{\Omega_{\ell_0}} |(u'_\ell - u'_\infty)(QX_1, X_2)|^2 dx \\ &= \int_{\Omega_{\ell_0}} |(u'_\ell - u'_\infty)(Y_1, Y_2)|^2 dy \end{aligned}$$

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since $dx = dy$ for $y = (Y_1, Y_2) = (QX_1, X_2)$ (recall that $Q \in \mathbb{O}^k$). A similar computation can be made in order to prove the convergence $u_\ell'' \rightarrow \tilde{u}_\infty''$ in $(L^2(\Omega_{\ell_0}))^{n-k}$.

For a fixed $\ell_0 > 0$, let $v \in \hat{\mathbb{H}}_0^1(\Omega_{\ell_0})$. Then, for all $\ell \geq \ell_0$, $v \in \mathbb{H}_0^1(\Omega_\ell)$. It follows that v is a good test function for the variational problem verified by u_ℓ , for all $\ell \geq \ell_0$. Since v is 0 outside of Ω_{ℓ_0} , we have that for all $\ell \geq \ell_0$,

$$\mu \int_{\Omega_{\ell_0}} \nabla u_\ell \cdot \nabla v \, dx = \mu \int_{\Omega_\ell} \nabla u_\ell \cdot \nabla v \, dx = \int_{\Omega_\ell} f v \, dx = \int_{\Omega_{\ell_0}} f v \, dx.$$

Since $\nabla u_\ell \rightarrow \nabla u_\infty$ strongly in $(L^2(\Omega_{\ell_0}))^n$, letting ℓ go to infinity leads to

$$\mu \langle -\Delta u_\infty, v \rangle = \mu \int_{\Omega_{\ell_0}} \nabla u_\infty \cdot \nabla v \, dx = \int_{\Omega_{\ell_0}} f v \, dx.$$

This shows that u_∞ satisfies the variational equation associated with (4.4.2), since ℓ_0 is taken arbitrarily. In particular, u_∞ satisfies

$$\langle -\mu \Delta u_\infty - f, \phi \rangle = 0$$

for all $\phi \in (\mathcal{D}(\Omega_\infty))^n$ such that $\operatorname{div} \phi = 0$. Therefore, thanks to the de Rham Theorem, there exists $\tilde{p}_\infty \in \mathcal{D}'(\Omega_\infty)$ satisfying

$$-\mu \Delta u_\infty - f = \nabla \tilde{p}_\infty.$$

Hence,

$$-\mu \Delta u_\infty + \nabla p_\infty = f \text{ in } (\mathcal{D}'(\Omega_\infty))^n$$

with $p_\infty = -\tilde{p}_\infty$. Since $\mu \Delta u_\infty + f \in (H^{-1}(\Omega_\ell))^n$ for all $\ell > 0$, this implies that $p_\infty \in L_{\text{loc}}^2(\overline{\Omega}_\infty)$, using the fact that every Ω_ℓ is a bounded Lipschitz domain (see e.g., [3]). As usual, we consider the class associated to p_∞ in $L_{\text{loc}}^2(\overline{\Omega}_\infty)/\mathbb{R}$ (for simplicity also denoted by p_∞) in order to retrieve the uniqueness for p_∞ . Therefore, the couple (u_∞, p_∞) satisfies the first equation of problem (4.4.2).

On the other hand, for any fixed $\ell_0 > 0$, we have that, for any $\ell \geq \ell_0$, $\gamma(u_\ell) = 0$ on $\partial\Omega_\ell$, hence $\gamma(u_\ell) = 0$ on $B_\ell \times \partial\omega$, where γ is the trace operator. Since $B_{\ell_0} \subset B_\ell$, we get $\gamma(u_\ell) = 0$ on $B_{\ell_0} \times \partial\omega \subset \partial\Omega_{\ell_0}$. Remembering that $u_\ell \rightarrow u_\infty$ in $\mathbb{H}^1(\Omega_{\ell_0})$ and using the continuity of the trace operator, we deduce $\gamma(u_\ell) \rightarrow \gamma(u_\infty)$ in $L^2(\partial\Omega_{\ell_0})$ and particularly in $L^2(B_{\ell_0} \times \partial\omega)$. Thus,

$\gamma(u_\infty) = 0$ on $B_{\ell_0} \times \partial\omega$. Since ℓ_0 is arbitrarily taken, we derive $\gamma(u_\infty) = 0$ on $\partial\Omega_\infty = \mathbb{R}^k \times \partial\omega = \bigcup_{\ell_0 > 0} (B_{\ell_0} \times \partial\omega)$.

Finally, using the estimate (4.4.11) and the inequality (4.4.9), we have

$$\begin{aligned} \|\nabla u_\infty\|_{L^2(\Omega_\ell)} &\leq \|\nabla(u_\infty - u_{2\ell})\|_{L^2(\Omega_\ell)} + \|\nabla u_{2\ell}\|_{L^2(\Omega_\ell)} \\ &\leq \|\nabla(u_\infty - u_{2\ell})\|_{L^2(\Omega_\ell)} + \|\nabla u_{2\ell}\|_{L^2(\Omega_{2\ell})} \\ &\leq C(e^{-2\alpha\ell} + e^{2\beta\ell}) \\ &\leq C_\infty e^{2\beta\ell}. \end{aligned}$$

Step VI. *There exists a unique solution to problem (4.4.2).*

The existence of a solution being already established, we only need to prove its uniqueness. Let $u_\infty, \tilde{u}_\infty$ be two solutions of the problem. Then for any $\ell_1 > 0$, the computations of step **I** remain valid for u_∞ and \tilde{u}_∞ replacing u_ℓ and $u_{\ell+r}$. We finally get the inequality

$$\int_{\Omega_{\ell_1}} |\nabla(u_\infty - \tilde{u}_\infty)|^2 dx \leq a \int_{\Omega_{\ell_1+1}} |\nabla(u_\infty - \tilde{u}_\infty)|^2 dx,$$

for all $\ell_1 > 0$, where a is the same constant as in step **I**. Then,

$$\|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} \leq a^{\frac{1}{2}} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1+1})}.$$

Iterating k times the previous inequality leads to

$$\begin{aligned} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} &\leq a^{\frac{k}{2}} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1+k})} \\ &= e^{-2\tilde{\alpha}k} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1+k})}, \end{aligned}$$

with $\tilde{\alpha} = \frac{1}{4} \ln(\frac{1}{a})$ found in step **II**. Combining the last inequality with the ones satisfied by u_∞ and \tilde{u}_∞ in problem (4.4.2), we deduce that,

$$\begin{aligned} \|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} &\leq 2C_\infty e^{2\beta(\ell_1+k)} e^{-2\tilde{\alpha}k} \\ &= 2C_\infty e^{2\beta\ell_1} e^{-2(\tilde{\alpha}-\beta)k}. \end{aligned}$$

Fixing ℓ_1 and making k go to infinity, we have that $\|\nabla(u_\infty - \tilde{u}_\infty)\|_{L^2(\Omega_{\ell_1})} = 0$, since $\beta < \tilde{\alpha}$.

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On the other hand, $u_\infty = \tilde{u}_\infty = 0$ on $\partial\Omega_\infty$, implying $u_\infty - \tilde{u}_\infty = 0$ on $\partial\Omega_\infty$. More particularly, for any $\ell_1 > 0$, we have $u_\infty - \tilde{u}_\infty \in \mathbb{H}^1(\Omega_{\ell_1})$ and $u_\infty - \tilde{u}_\infty = 0$ on $B_{\ell_1} \times \partial\omega$. Thanks to the Poincaré inequality, this implies $u_\infty - \tilde{u}_\infty = 0$ a.e. in Ω_{ℓ_1} . Since ℓ_1 was arbitrarily chosen, this leads to $u_\infty - \tilde{u}_\infty = 0$ a.e. in Ω_∞ .

Step VII. *Estimate for the pressure: there exists a constant $C \geq 0$ independent of ℓ such that*

$$\|p_\ell - p_\infty\|_{\dot{L}^2(\Omega_{\frac{\ell}{2}})} \leq C\ell e^{-\alpha\ell} \quad (4.4.12)$$

for any $\ell \geq 2$.

By subtracting the equations (4.1.1) and (4.1.3) we obtain in $\mathbb{H}^{-1}(\Omega_\ell)$:

$$-\nabla(p_\ell - p_\infty) = -\mu\Delta(u_\ell - u_\infty).$$

This is equivalent to

$$\int_{\Omega_\ell} (p_\ell - p_\infty) \operatorname{div} v \, dx = \mu \int_{\Omega_\ell} \nabla(u_\ell - u_\infty) \cdot \nabla v \, dx \quad \text{for all } v \in \mathbb{H}_0^1(\Omega_\ell). \quad (4.4.13)$$

For $p_\ell - p_\infty$ belonging to $\hat{L}^2(\Omega_\ell)$, we choose the representative, for the simplicity also denoted by $p_\ell - p_\infty$, which satisfies

$$\int_{\Omega_{\frac{\ell}{2}}} (p_\ell - p_\infty) \, dx = 0.$$

Then, by Lemma 4.3.1, there exists $v \in \mathbb{H}_0^1(\Omega_{\frac{\ell}{2}})$ satisfying

$$\begin{cases} \operatorname{div} v = p_\ell - p_\infty & \text{in } \Omega_{\frac{\ell}{2}} \\ \|\nabla v\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C\ell \|p_\ell - p_\infty\|_{L^2(\Omega_{\frac{\ell}{2}})} \end{cases}$$

for some constant C depending only on k and ω (and independent of ℓ).

Extending v by 0 outside $\Omega_{\frac{\ell}{2}}$ and using it as a test function in (4.4.13), we get

$$\begin{aligned} \int_{\Omega_{\frac{\ell}{2}}} (p_\ell - p_\infty)^2 \, dx &= \int_{\Omega_\ell} (p_\ell - p_\infty) \operatorname{div} v \, dx = \mu \int_{\Omega_\ell} \nabla(u_\ell - u_\infty) \cdot \nabla v \, dx \\ &= \mu \int_{\Omega_{\frac{\ell}{2}}} \nabla(u_\ell - u_\infty) \cdot \nabla v \, dx \end{aligned}$$

This leads to

$$\begin{aligned} \|p_\ell - p_\infty\|_{L^2(\Omega_{\frac{\ell}{2}})}^2 &\leq \mu \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \|\nabla v\|_{L^2(\Omega_{\frac{\ell}{2}})} \\ &\leq C\ell \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \|p_\ell - p_\infty\|_{L^2(\Omega_{\frac{\ell}{2}})}. \end{aligned}$$

Finally, from the definition of the \hat{L}^2 -norm, we have that

$$\|p_\ell - p_\infty\|_{\hat{L}^2(\Omega_{\frac{\ell}{2}})} \leq \|p_\ell - p_\infty\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C\ell \|\nabla(u_\ell - u_\infty)\|_{L^2(\Omega_{\frac{\ell}{2}})} \leq C\ell e^{-\alpha\ell}.$$

□

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