



# Boundary controllability of some coupled parabolic systems with Robin or Kirchhoff conditions

Kuntal Bhandari

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# THÈSE

En vue de l'obtention du

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**KUNTAL BHANDARI**

**Boundary controllability of some coupled parabolic systems with Robin  
or Kirchhoff conditions**

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## JURY

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**Résumé.** Dans cette thèse, on étudie la contrôlabilité à zéro par le bord de quelques systèmes paraboliques linéaires couplés par des termes de couplage intérieur et/ou au bord. Le premier chapitre est une introduction à l'ensemble du manuscrit. Dans le deuxième chapitre, on rappelle les principaux concepts et résultats autour des notions de contrôlabilité qui seront utilisés dans la suite. Dans le troisième chapitre, on étudie principalement la contrôlabilité par le bord d'un système couplé  $2 \times 2$  de type cascade avec des conditions au bord de Robin. En particulier, on prouve que les contrôles associés satisfont des bornes uniformes par rapport aux paramètres de Robin et convergent vers un contrôle de Dirichlet lorsque les paramètres de Robin tendent vers l'infini. Cette étude fournit une justification, dans le contexte du contrôle, de la méthode de pénalisation qui est couramment utilisée pour prendre en compte des données de Dirichlet peu régulières en pratique. Dans le quatrième et dernier chapitre, on étudie d'abord la contrôlabilité à zéro d'un système  $2 \times 2$  en dimension 1 contenant des termes de couplage à la fois à l'intérieur et au bord du domaine. Plus précisément, on considère une condition de type Kirchhoff sur l'un des bords du domaine et un contrôle de Dirichlet sur l'autre bord, dans l'une ou l'autre des équations. En particulier, on montre que les propriétés de contrôle du système diffèrent selon que le contrôle agisse sur la première ou sur la seconde équation, et selon les valeurs du coefficient de couplage intérieur et du paramètre de Kirchhoff. On étudie ensuite un modèle  $3 \times 3$  avec un ou deux contrôle(s) aux limites de Dirichlet à une extrémité et une condition de type Kirchhoff à l'autre extrémité ; ici la troisième équation est couplée (couplage intérieur) avec la première. Dans ce cas, on obtient ce qui suit : en considérant le contrôle sur la première équation, on a *contrôlabilité conditionnelle* dépendant des choix du coefficient de couplage intérieur et du paramètre de Kirchhoff, et en considérant le contrôle sur la deuxième équation, on obtient toujours une *contrôlabilité positive*. En revanche, considérer un contrôle sur la troisième équation conduit à un résultat de *contrôlabilité négative*. Dans cette situation, on a besoin de deux contrôles aux limites sur deux des trois équations pour retrouver la contrôlabilité. Enfin, on expose quelques études numériques basées sur l'approche pénalisée HUM pour illustrer les résultats théoriques, ainsi que pour tester d'autres exemples.

**Mots-clés.** Contrôlabilité par le bord, systèmes paraboliques, conditions de Robin, conditions de type Kirchhoff, inégalité Carleman, méthode des moments, analyse spectrale.

**Abstract.** In this thesis, we study the boundary null-controllability of some linear parabolic systems coupled through interior and/or boundary. We begin by giving an overall introduction of the thesis in Chapter 1 and we discuss some essentials about the notion of parabolic controllability in the second chapter. In Chapter 3, we investigate the boundary null-controllability of some  $2 \times 2$  coupled parabolic systems in the cascade form where the boundary conditions are of Robin type. This case is considered mainly in space dimension 1 and in the cylindrical geometry. We prove that the associated controls satisfy suitable uniform bounds with respect to the Robin parameters, which let us show that they converge towards a Dirichlet control when the Robin parameters go to infinity. This is a justification of the popular penalization method for dealing with Dirichlet boundary data in the framework of the controllability of coupled parabolic systems. Coming to the Chapter 4, we first discuss the boundary null-controllability of some  $2 \times 2$  parabolic systems in 1-D that contains a linear interior coupling with real constant coefficient and a Kirchhoff-type condition through which the boundary coupling enters in the system. The control is exerted on a part of the boundary through a Dirichlet condition on either one of the two state components. We show that the controllability properties vary depending on which component the control is being applied; the choices of interior coupling coefficient and the Kirchhoff parameter play a crucial role to deduce positive or negative controllability results. Thereafter, we study a  $3 \times 3$  model with *one* or *two* Dirichlet boundary control(s) at one end and a Kirchhoff-type boundary condition at the other; here the third equation is coupled (interior) through the first component. In this case we obtain the following: treating the control on the first component, we have *conditional controllability* depending on the choices of interior coupling coefficient and the Kirchhoff parameter, while considering a control on the second component always provides *positive result*. But in contrast, putting a control on the third entry yields a *negative controllability* result. In this situation, one must need two boundary controls on any two components to recover the controllability. Further in the thesis, we pursue some numerical studies based on the penalized Hilbert Uniqueness Method (HUM) to illustrate our theoretical results and test other examples in the framework of interior-boundary coupled systems.

**Key words.** Boundary controllability, parabolic systems, Robin conditions, Kirchhoff-type condition, Carleman estimate, moments method, spectral analysis.

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# Chapter I

## Introduction

### I.1 General context

In this chapter, we provide a general overview of the manuscript. We begin by introducing some existing results that are very much connected with the topics of this report and thereafter, we shall particularly give an overall description for each of the main chapters that are going to be covered. To conclude, I will present some perspectives related to my further interests to work in the field of control theory for PDEs.

To begin with, the manuscript is mainly based on studying the boundary null-controllability of some coupled parabolic systems with less number of controls than the equations. As soon as we use the terminology “coupled systems”, the immediate question that arises is what kind of coupling we are going to deal with: it can be an interior coupling, that is the coupling will be posed between the state equations, or some boundary coupling where we want to consider some coupling through the boundary conditions, or even in a more general setting we can have both. In this manuscript, we take into account the following two kinds of situations:

- **Dealing with an interior coupling.** The first one is dealt with the boundary null-controllability of some  $2 \times 2$  coupled parabolic systems in the *cascade form* where the boundary conditions are of *Robin* type. We mainly study the systems in 1D and some multi-D situation, namely in the cylindrical geometry.

In particular, we prove that the associated controls satisfy some suitable uniform bounds with respect to the positive Robin parameters which let us show that they converge towards a corresponding *Dirichlet control* when the Robin parameters go to infinity. This is a justification of the popular penalization method for dealing with non-smooth Dirichlet boundary data in the framework of coupled parabolic (boundary) control systems.

– This part of the manuscript is mainly based on the paper [26] published in *Evolution Equations and Control Theory (EECT)*.

- **Dealing with both interior and boundary couplings.** Next to this, we treat some  $2 \times 2$  and  $3 \times 3$  one-dimensional parabolic systems with both interior as well as boundary couplings. We first consider a  $2 \times 2$  parabolic system that contains a linear interior coupling with real constant coefficient and a Kirchhoff-type condition through which the boundary coupling enters into the game. The control is exerted on a part of the boundary through a Dirichlet condition on either one of the two state components, and in particular we show that the controllability property significantly changes depending on which component the control is being applied. Regarding this, we point out that the choices of *interior coupling coefficient* and the *Kirchhoff parameter* might play a crucial role to deduce the positive or negative controllability results.

Thereafter, we continue our investigation to a  $3 \times 3$  parabolic system (with a linear interior coupling of the first component to the third equation) where we consider *one* or *two* Dirichlet boundary control(s) at left and the Kirchhoff-type condition at right. One motivation to study a  $3 \times 3$  model along with an interior coupling, is coming from the point of considering a parabolic control system on a metric graph where some (or, all) of the edges correspond to vectorial parabolic equations. In particular, our  $3 \times 3$  system can be understood in a graph with two edges, where one edge corresponds to a *vectorial pde* and the other one to a *scalar pde*, coupled through their common node.

In this case, we briefly show the following. In one hand, when we exert *two* boundary controls on any two components, the system is null-controllable at any time. On the other hand, if we consider only *one* boundary control, then the controllability property changes depending on which component the control is being exerted. More precisely, by putting a control on the first entry, we see that the controllability depends sensitively on the choices of interior coupling coefficient and Kirchhoff parameter, while considering a control to the second entry always gives positive result regardless the choices of those quantities. But in contrast, when we put the control on the third entry, we immediately loose the hope of controllability.

Further to these, we pursue some numerical studies based on the well-known *penalized HUM approach*. In fact, we make some discretization (by means of finite differences) for a general interior-boundary coupled parabolic system to incorporate mainly the effects of boundary couplings to the discrete setting. This allows us to illustrate our theoretical results as well as to experiment some more examples which fit under the framework of the general system.

– The study of the  $2 \times 2$  systems and the numerical part are mainly based on the article [27] which is submitted for publication.

Before going into detail, we must say that the controllability of a system of  $n \geq 2$  partial differential equations with  $m < n$  number of controls is a challenging issue in the domain of control theory, we refer to [84, 51, 24, 29] among some very initial works on that. Next to this, we quote some other existing results concerning the *distributed controllability* issues for coupled parabolic systems, for instance [5, 7, 8, 65]. In particular, the authors of [5, 7] obtained a necessary and sufficient condition (more precisely a generalized Kalman rank condition) for the distributed null-controllability of  $n \times n$  parabolic systems with constant or time dependent coefficients. In [65], the authors analysed the controllability property of the so-called *cascade system* of  $n \geq 1$  coupled parabolic equations with only *one distributed control* which is really worth in the theory of control.

Beside these, we refer to the paper [35] where the authors discussed about the approximate controllability of some parabolic systems with space dependent interior coupling (zero-order) and with single control force in a subdomain of  $(0, 1)$ . The main concern of this paper is that the geometry of the control domain has an important influence to the controllability of their system, depending on the structure of the coupling terms.

Another fascinating result that has been proved in [6], is the null-controllability of a reaction-diffusion equation (some  $2 \times 2$  system which is semi-linear in nature and arises in mathematical biology) with single distributed control in a subdomain of a bounded smooth domain. It is also worth mentioning [66, 67], where the authors proved some distributed null-controllability results for linear or non-linear parabolic systems. What they actually proved is a *Carleman estimate* to obtain the controllability by means of a suitable observability inequality.

Coming to point of *boundary controllability* issues, it is by now well-understood to the control community that a parabolic boundary control system with less number of controls than the equations can be really intricate to study in various situations and that there is still no complete

theory in the literature. This is mainly due to the fact that the very powerful *Carleman type estimate* ([62]) is often inefficient in that context and we hereby recall that the boundary controllability for such systems is no longer equivalent with the distributed controllability as it has been proven for instance in [58]. This certainly indicates that the boundary controllability issues are more delicate to deal with. Among some really achieved works, we point out [58] where the authors proved a necessary and sufficient condition for boundary null-controllability of some  $2 \times 2$  coupled (linear and constant) parabolic system with single Dirichlet control. On the other hand, the reference [9] is dedicated to prove the controllability to trajectories of a 1D system of  $n \geq 2$  parabolic equations when  $m < n$  number of controls are exerted on a part of the boundary through Dirichlet conditions. What they actually proved is a general Kalman condition which is a necessary and sufficient controllability condition for their problem. A Kalman condition has also been developed in [16] for boundary controllability of  $n$  coupled wave equations (in 1D) by means of one control on a part of the boundary.

As per our knowledge, most of the boundary controllability results for a system with *less controls than the number of components* are in 1D because in that situation the spectral analysis of the corresponding adjoint elliptic operator plays a crucial role to develop the so-called *moments technique* (if applicable). But as we know that the drawback of the moments approach is that it cannot be implemented in higher dimensions straightforwardly. In this regard, we have some known multi-D results in the *cylindrical geometry* which has been developed in [21], see also [3]; which needs in one hand: a sharp estimate of the control cost for the associated 1D problem and in other hand: a Lebeau-Robbiano spectral inequality (see [83]) for higher dimensions. Indeed, in Chapter III, we shall make use of these tools to obtain a boundary controllability result for some coupled control system of type (I.1)–(I.2) in the cylindrical domain.

To this end, we mention some approximate boundary controllability result that has been drawn in [70] by means of a unique continuation principle, for a particular system of two parabolic coupled equations. In fact, this result is valid in several dimensions but only for a very particular kind of coupling. It is also worth here to recall the boundary controllability result for a system of two wave equations proved in [1]. In this context, we mention a very interesting work [2] where the null-controllability of some symmetric system of two wave-type equations has been analyzed in any space dimension with one control, provided that the control region satisfies the Geometric Control Condition. Indeed, the authors deduced similar results for some coupled parabolic and Schrödinger-type systems under the same geometric conditions. Finally, we refer to [10] where the authors made a very useful survey of several recent results concerning controllability of coupled parabolic systems.

### I.1.1 A model problem under study only with interior coupling

The main problem that we are interested in to study is the so-called cascade system as follows,

$$\begin{cases} \partial_t y_1 - \operatorname{div}(\gamma(x) \nabla y_1) = 0 & \text{in } (0, T) \times \Omega, \\ \partial_t y_2 - \operatorname{div}(\gamma(x) \nabla y_2) + y_1 = 0 & \text{in } (0, T) \times \Omega, \\ y_1(0, \cdot) = y_{0,1}(\cdot) & \text{in } \Omega, \\ y_2(0, \cdot) = y_{0,2}(\cdot) & \text{in } \Omega, \end{cases} \quad (\text{I.1})$$

with a Robin type of boundary control as follows:

$$\begin{cases} \frac{\partial y_1}{\partial \nu_\gamma} + \beta_1 y_1 = \mathbb{1}_{\Gamma_0} v & \text{in } (0, T) \times \Gamma, \\ \frac{\partial y_2}{\partial \nu_\gamma} + \beta_2 y_2 = 0 & \text{in } (0, T) \times \Gamma, \end{cases} \quad (\text{I.2})$$

where  $\Omega \subset \mathbb{R}^d$  for  $d \geq 1$ , is a  $\mathcal{C}^2$  bounded domain and  $\gamma : \Omega \rightarrow M_d(\mathbb{R})$  is a lipschitz-continuous bounded field of symmetric matrices which are uniformly coercive: there is some  $\gamma_{\min} > 0$  such

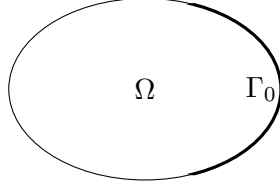


Figure I.1: A typical smooth, bounded domain  $\Omega$  where the control is supposed to be exerted on some part  $\Gamma_0$  of the boundary  $\Gamma := \partial\Omega$ .

that

$$(\gamma(x)\xi) \cdot \xi \geq \gamma_{\min}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad \forall x \in \Omega.$$

The co-normal derivative operator associated to the diffusion tensor  $\gamma$  is defined by

$$\frac{\partial}{\partial \nu_\gamma} = \nu \cdot (\gamma \nabla),$$

where  $\nu$  is the normal vector on the boundary  $\Gamma := \partial\Omega$ . We denote the control by  $v \in L^2(0, T; \mathbb{R})$  that is supposed to be applied only on the component  $y_1$  on some part  $\Gamma_0$  of the boundary  $\Gamma := \partial\Omega$  through a Robin type of boundary conditions characterised by two non-negative parameters  $\beta_1$  and  $\beta_2$ .

As per our knowledge, the usual studied case is the one with a Dirichlet boundary control, that is

$$\begin{cases} y_1 = \mathbf{1}_{\Gamma_0} v & \text{on } (0, T) \times \Gamma, \\ y_2 = 0 & \text{on } (0, T) \times \Gamma, \end{cases} \quad (\text{I.3})$$

and one of the main interest of studying a Robin control problem is coming from the point of a penalization approach to deal with non-smooth Dirichlet data and so far we know, in the framework of coupled parabolic control system, there are no such cases studied before.

### Motivations to study a Robin control system.

As mentioned above, one motivation to study the type of system (I.1)–(I.2) is coming from the point of a penalization approach. For instance, when we consider a Galerkin approximation of an elliptic or parabolic equation, the technique is to replace a Dirichlet boundary condition  $y = g$  by a Robin one  $\frac{1}{\beta} \frac{\partial y}{\partial \nu_\gamma} + y = g$ , with large penalization parameter  $\beta > 0$ . It generally induces more flexibility and robustness in the computational code. The penalization approach was initially studied in [17] for elliptic equations and in [20] for parabolic equations. In the framework of control theory, this was analyzed in [19, 43] for solving optimal control problems of elliptic equations.

Thus, motivated by those works we investigate the same kind of issue for a coupled parabolic system like (I.1)–(I.2) with a single boundary control and so far we know that this approach has not been studied before for coupled parabolic systems. In particular, we achieve in 1D and in some multi-D (the cylindrical domain) situations that the controls are satisfying some uniform estimates with respect to the non-negative Robin parameters and it follows that the controlled solutions converge to a controlled solution of the corresponding Dirichlet problem as the Robin parameters go to infinity.

Another interesting part of studying this Robin problem is that it has a regularising effect on the boundary data which is not in the case for a non-homogeneous Dirichlet boundary problem. To be more precise, for a Dirichlet problem if we start with  $v \in L^2((0, T) \times \Gamma_0)$ , we cannot expect the solution to exist in the usual energy space  $\mathcal{C}^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ , rather the solution belongs to a larger space  $\mathcal{C}^0([0, T]; H^{-1}(\Omega)) \cap L^2((0, T) \times \Omega)$  and in that case the boundary condition is being understood in a weak sense. But, as soon as we switch the Dirichlet

condition to a Robin one (or, in fact to a Neumann one), we recover the expected regularity for the weak solution even if the boundary data is taken from  $L^2$ .

In Section I.2.2, we present the main results which we obtain for the control system (I.1)–(I.2).

### I.1.2 A model problem with both the interior and boundary couplings

Let us come to the point of a parabolic system that contains both interior as well as boundary couplings and with less number of controls than the equations. As a model problem we present for instance, the following  $2 \times 2$  system:

$$\begin{cases} \partial_t y + \mathcal{A}y + \mathcal{M}_{coup} y = 0 & \text{in } (0, T) \times (0, 1), \\ \mathcal{D}_0 y(t, 0) + \mathcal{N}_0 \frac{\partial y}{\partial \nu_{\mathcal{A}}}(t, 0) = Bv(t) & \text{in } (0, T), \\ \mathcal{D}_1 y(t, 1) + \mathcal{N}_1 \frac{\partial y}{\partial \nu_{\mathcal{A}}}(t, 1) = 0 & \text{in } (0, T), \\ y(0, \cdot) = y_0(\cdot) & \text{in } (0, 1), \end{cases} \quad (\text{I.4})$$

where  $y(t, x) \in \mathbb{R}^2$  is the unknown,  $y_0$  is some given initial data and  $\mathcal{M}_{coup}, \mathcal{D}_j, \mathcal{N}_j \in M_2(\mathbb{R})$  ( $j = 0, 1$ ). The input  $v$  is supposed to act as a control on the boundary point  $x = 0$  through some non-zero vector  $B$ . One can take the operator  $\mathcal{A}$  for instance

$$\mathcal{A} = \begin{pmatrix} -\partial_x(\gamma_1 \partial_x) & 0 \\ 0 & -\partial_x(\gamma_2 \partial_x) \end{pmatrix}, \quad (\text{I.5})$$

where the diffusion coefficients  $\gamma_i \in \mathcal{C}^1([0, 1])$  with  $\gamma_{\min} := \inf_{[0, 1]} \{\gamma_i, i = 1, 2\} > 0$ , with its domain

$$D(\mathcal{A}) := \left\{ u \in (H^2(0, 1))^2 \mid \mathcal{D}_0 u(0) + \mathcal{N}_0 \frac{\partial u}{\partial \nu_{\mathcal{A}}}(0) = 0, \mathcal{D}_1 u(1) + \mathcal{N}_1 \frac{\partial u}{\partial \nu_{\mathcal{A}}}(1) = 0 \right\}, \quad (\text{I.6})$$

On the boundary points, we have the conormal derivative operator  $\frac{\partial}{\partial \nu_{\mathcal{A}}} := (\gamma_1 \frac{\partial}{\partial \nu}, \gamma_2 \frac{\partial}{\partial \nu})$ , where  $\nu$  is the normal vector.

**Remark I.1.1.** We must mention that the domain of the operator  $\mathcal{A}$  (with the formal expression as in (I.5)) appearing in (I.4) is not the same one as prescribed by (I.6) due to the presence of boundary term  $Bv \neq 0$ . But we keep the same notation if there is no confusion.

Let us make the following assumptions on the matrices encoding the boundary conditions.

**Assumption I.1.2.** For each  $j \in \{0, 1\}$ ,

1. The  $2 \times 4$  matrix  $(\mathcal{D}_j, \mathcal{N}_j)$  has the maximal rank.
2. The matrix  $\mathcal{D}_j \mathcal{N}_j^*$  is self-adjoint.

The first assumption ensures the sufficient number of boundary conditions in (I.4) whereas the second one is important for the differential operator  $(\mathcal{A}, D(\mathcal{A}))$  to be self-adjoint.

The main point here is that we consider the interior coupling by means of the matrix  $\mathcal{M}_{coup}$  and the boundary coupling(s) through the coefficient matrices  $\mathcal{D}_j, \mathcal{N}_j$ ,  $j = 0, 1$ , along with the Assumption I.1.2.

**Remark I.1.3.** One can think of a more general system like (I.4) with  $n$  differential equations instead of only 2. Indeed, in Section I.2.3.3, we shall introduce some  $3 \times 3$  coupled parabolic systems with one or two control force(s).

We first mention here that several systems with boundary couplings (let say without any control for the moment) arise when one considers some parabolic systems on a metric graph, see for instance [86, 79, 74, 77] (we refer to Section I.3–paragraph 2 for a notion of metric graph). In this context, we must say that there are diverse applications of the system of PDEs on metric graph in physics, chemistry, engineering or biology; we quote here very few of those, e.g., [80, 13, 18, 42, 41] and the references therein.

Coming to the controllability issues on metric graphs, we first address [50, Chapters 6, 8] where the authors discussed some controllability results of wave, heat and Schrödinger systems considered in some network in the case when some control(s) is (are) exerted on some of the vertices; see also the survey paper [15]. Addition to the foregoing, the authors in [44] proved some boundary null-controllability results for a linear Kuramoto–Sivashinsky equation ([81, 94]) on star-shaped trees with Dirichlet or Neumann boundary controls. We also refer to some very recent works [46] and [45] which contain some boundary controllability results for Korteweg de Vries equation ([30, 73]) on a tree-shaped and star-shaped network respectively. Last but not the least, it is worth to quote that a necessary and sufficient condition for approximate controllability of two 1D wave systems has been developed in [49].

In most of the known cases, one cannot arbitrarily impose an *interior coupling* to the system of differential equations that are considered on a metric graph. The reason being each edge corresponds to only one scalar differential equation with respect to one unknown, and thus, the only interaction between the unknowns occurs at the vertices.

Let us come to the case (I.4), where one observes that the main difference between this kind of systems and the systems that generally arise on graphs is: unlike the situation on graph, we have here the possibility of considering both the boundary and interior couplings.

From that angle, one can think of considering a system (of differential equations) on metric graph where each edge may correspond to a *vectorial differential equation* rather than a scalar one. With this abstract idea, one observes that our one-dimensional system (I.4) is the simplest version, where technically we have only one edge. Therewith, in Section I.2.3.3 we discuss some boundary controllability issues of a  $3 \times 3$  coupled parabolic system which can be identified on some metric graph with *two edges*, where one edge is associated with a vectorial pde and the other one with a scalar pde (see Fig. I.9).

In fact, in Section I.3 we shall briefly discuss about considering some vectorial differential equations on a metric graph, as a perspective linked with our present work.

Now, two types of difficulties may arise while dealing with the general system (I.4). As mentioned earlier, the *Carleman approach* is often incapable in the frame of a boundary control system (vectorial). Beside this obstruction, there occurs a change in the spectral analysis to the corresponding *adjoint elliptic operator* of the parabolic system (I.4), since this kind of operator is normally *non self-adjoint* in nature due to the presence of the interior coupling  $\mathcal{M}_{coup}$ . Thus, it is not so straightforward to apply a *moments technique* to the general system (I.4). Moreover, under the system (I.4), there are some examples of *negative controllability* results also, as given below.

**Some examples of negative controllability results.** It can be shown that a linear control system in *cascade* form with  $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is not even approximately controllable for either  $\mathcal{D}_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\mathcal{N}_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$  or  $\mathcal{D}_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\mathcal{N}_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $j = 0, 1$ ; we refer to Remark III.2.17, Ch. III for more detail. Basically, a careful look says that the above examples are actually the systems (I.1)–(I.2) in 1D with  $(\beta_1, \beta_2) = (+\infty, 0)$ , or  $(\beta_1, \beta_2) = (0, +\infty)$

Due to these indistinct phenomena, in this manuscript we mainly cope with some particular choices of control systems under the framework of (I.4), we shall precisely describe that in Section I.2.3.



In the forthcoming section, we briefly present the results that have been achieved during my PhD and some difficulties which arise related to the problems we considered. Let us go step by step into the detail.

## I.2 Brief overview of the manuscript

### I.2.1 Chapter II: Some crashes on parabolic control problems

Chapter II mainly contains some useful ingredients related to the parabolic control systems. More precisely, we start with a formal control problem and its well-posedness result in the general framework. Then, we briefly discuss about the so-called *moments method* (which has been used first by Fattorini and Russell in [56, 57]) for a scalar parabolic control problem (distributed or boundary) with Dirichlet boundary conditions. In fact, this technique will be exploited in Chapters III and IV to deduce some controllability results of parabolic systems with less number of control(s) than the equations. At the end, we present a short description about the well-known HUM approach (which is finding a control with minimal  $L^2$ -norm) and its penalized version ([63, 64, 31]) in a general functional setting, which will be used in Chapter IV for some numerical studies.

### I.2.2 Chapter III: The Robin control system with an interior coupling

In the third chapter, we investigate the boundary null-controllability of some  $2 \times 2$  coupled parabolic systems with a linear interior coupling and in this case, the control is exerted to the first component on some part of the boundary through Robin conditions. In the following section, we precisely state the main results of that chapter.

#### I.2.2.1 Main results that we obtain

In Chapter III, we first discuss the well-posedness result of the system (I.1)–(I.2) in any space dimension and also point out the fact that how the functional setting and regularity results change from the Dirichlet to Robin case. Then we cope with the boundary controllability issues depending on the parameters  $\beta_1, \beta_2$  and prove some convergence results towards the corresponding Dirichlet control problem.

**The 1D case.** We begin with the 1D controllability results that we prove.

- **The case when  $\beta_1 = \beta_2$ .** We first let the case when the two Robin parameters in both components are same, we denote  $\beta_1 = \beta_2 := \beta > 0$ . We further assume the diffusion coefficient  $\gamma \in \mathcal{C}^1([0, 1])$  which is scalar in this case and satisfying  $\gamma_{\min} := \inf_{[0, 1]} \gamma > 0$ . In such circumstances, we obtain the following controllability result.

**Controllability result using the classical moments method:** For any given initial data  $y_0 \in (L^2(0, 1))^2$ , time  $T > 0$  and positive parameter  $\beta > 0$ , there exists a null-control  $v_\beta \in L^2(0, T; \mathbb{R})$  for the system (I.1)–(I.2) that satisfies the following estimate:

$$\|v_\beta\|_{L^2(0, T)} \leq C e^{C/T} (1 + \beta) \|y_0\|_{(L^2(0, 1))^2}, \quad (\text{I.7})$$

where the constant  $C > 0$  may depend on  $\gamma$  but not on  $y_0$ ,  $T$  and  $\beta$ .

We prove this result, namely the point 1 of Theorem III.2.14 in Ch. III by the so-called moments method. Later, just after the next paragraph we shall see that the appearance of  $(1 + \beta)$  in the estimate (I.7) is essential to obtain a convergence of the Robin control system (I.1)–(I.2) to the corresponding Dirichlet one, i.e., the system (I.1)–(I.3), as  $\beta \rightarrow +\infty$ .



- **The case when  $\beta_1 \neq \beta_2$ .** In this situation, a *spectral condensation* occurs (of the associated adjoint elliptic operator) when the two parameters  $\beta_1 \neq \beta_2$  are close to each other and we will notice in Lemma III.5.3, Ch. III that one cannot have a uniform spectral gap which prevents us to obtain an uniform estimate like (I.7). More precisely, by using the classical moments technique, one obtains a constant  $C$  (appearing in the estimate of the controls) that depends on the parameters  $\beta_1, \beta_2$ . The drawback of this fact is that we cannot hope for a convergence result towards the associated Dirichlet control problem when  $\beta_1, \beta_2$  go to infinity.

To overcome this difficulty, and still prove a uniform controllability, we will use a very recent result: the *block moment technique* developed in [22] where a weak gap (in some sense) is sufficient to obtain the uniform estimate of the controls; we give more detail in Section III.5.2.2, Ch. III about that. In this case, we have the following result.

**Controllability result using the block moments method:** We fix the diffusion coefficient  $\gamma$  as a positive constant and let  $\beta^* > 0$  be given. We also let  $\beta := (\beta_1, \beta_2) \in (0, +\infty)^2$  be any couple of Robin parameters. Then, for any given  $y_0 \in (L^2(0, 1))^2$  and  $T > 0$ , there exists a null-control  $v_\beta \in L^2(0, T; \mathbb{R})$  for the problem (I.1)–(I.2) such that it satisfies

$$\|v_\beta\|_{L^2(0, T)} \leq C_{T, \beta^*}(1 + \beta_1)\|y_0\|_{(L^2(0, 1))^2},$$

as soon as we have either  $\beta_1, \beta_2 \in (0, \beta^*)$ , or  $\beta_1, \beta_2 \in [\beta^*, +\infty)$ , where  $C_{T, \beta^*} > 0$  does not depend on  $y_0$  and  $\beta$ .

- **A convergence result towards the Dirichlet control problem.** Let us provide a convergence result which we shall prove in Ch. III (namely, Theorem III.2.9 and Corollary III.2.15).

Let  $\beta_n := (\beta_{1,n}, \beta_{2,n}) \in (0, +\infty)^2$  be any sequence of Robin parameters such that  $\beta_{i,n} \rightarrow +\infty$  when  $n \rightarrow \infty$ , for  $i = 1, 2$ . If the diffusion coefficient  $\gamma$  is not a constant, we assume in addition that  $\beta_{1,n} = \beta_{2,n}$  for any  $n$ . For each  $n$ , let  $v_n$  (resp.  $y_n$ ) be the unique null-control of minimal  $L^2$  norm (resp. the associated trajectory) for the problem (I.1)–(I.2) with Robin parameters  $\beta_{1,n}$  and  $\beta_{2,n}$ .

Then, there exists a subsequence  $(n_k)_k$  such that

$$\begin{aligned} \frac{v_{n_k}}{\beta_{1, n_k}} &\xrightarrow[k \rightarrow +\infty]{} v_D \quad \text{weakly in } L^2(0, T), \\ y_{n_k} &\xrightarrow[k \rightarrow +\infty]{} y_D \quad \text{strongly in } (L^2(0, T; H^{-1}(0, 1)))^2 \quad \text{and} \\ y_{n_k} &\xrightarrow[k \rightarrow +\infty]{} y_D \quad \text{weakly in } (L^2((0, T) \times (0, 1)))^2, \end{aligned}$$

where  $v_D$  (resp.  $y_D$ ) is a null-control (resp. the associated trajectory) for the corresponding Dirichlet control problem (I.1)–(I.3).

Below, we clarify the above convergence result in Figures I.2 and I.3 for some fixed mesh size. From the numerical Fig. I.3, it seems that the convergence of the Robin controls towards a Dirichlet one is of *order 1*. Although, this does not provide any conclusion since we do not have any error estimate; theoretically we only know the weak convergence of the controls up to a subsequence.

**A multi-D case.** When both the Robin parameters are same ( $\beta_1 = \beta_2 = \beta$ ), we use the methodology described in [3, 21], to deduce a null-controllability result in a cylindrical domain in  $\mathbb{R}^d$ ,  $d \geq 2$ . More precisely, we consider a domain  $\Omega = (0, 1) \times \Omega_2$ , where  $\Omega_2$  is a smooth bounded connected domain in  $\mathbb{R}^{d-1}$ , see Figure I.4.

Let  $\omega_2 \subset \Omega_2$  be a non-empty open subset of  $\Omega_2$  and the control region is  $\Gamma_0 := \{0\} \times \omega_2$ . We have the following results in this case.

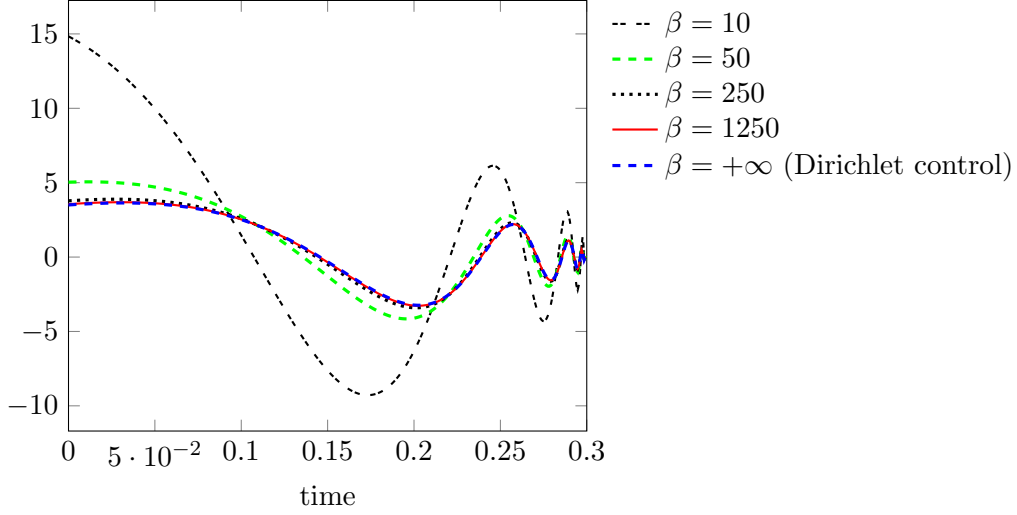


Figure I.2: Convergence of  $\frac{v_\beta}{\beta}$  (of minimal  $L^2$ -norm) to a Dirichlet control  $v_D$ .

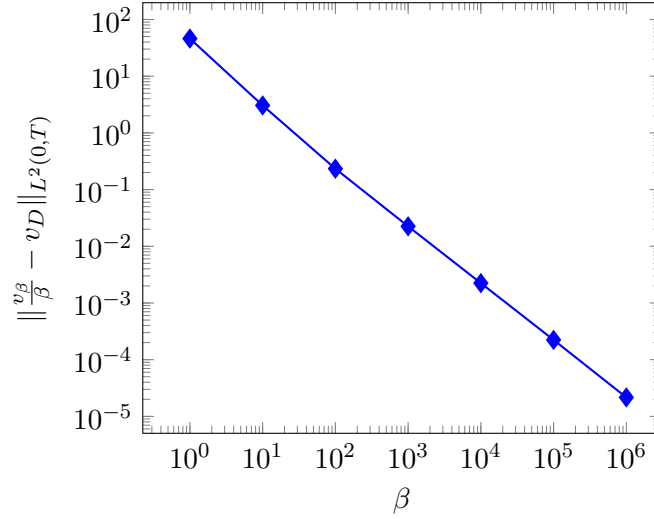


Figure I.3: The convergence in terms of the  $L^2$ -norm of the difference  $\left(\frac{v_\beta}{\beta} - v_D\right)$ .

- **A controllability result for  $\beta_1 = \beta_2$ .** Let  $y_0 \in (L^2(\Omega))^2$  be given. For any  $T > 0$  and any  $\beta \in (0, +\infty)$ , there exists a null-control  $v_\beta \in L^2((0, T) \times \Gamma_0)$  for the problem (I.1)–(I.2), that satisfies in addition the estimate

$$\|v_\beta\|_{L^2((0,T) \times \Gamma_0)} \leq C e^{C/T} (1 + \beta) \|y_0\|_{(L^2(\Omega))^2},$$

where the constant  $C > 0$  does not depend on  $y_0$ ,  $T$  and the parameter  $\beta$ .

The above result has been proved in Theorem III.2.18, Ch. III which needs a sharp estimate of the control cost for 1D problem (obtained by (I.7)) and the method of Lebeau-Robbiano [83]. In our context, we actually need a Lebeau-Robbiano spectral inequality which must be uniform in  $\beta \in (0, +\infty)$ , relative to the control region  $\omega_2$  for the eigenfunctions of the corresponding diffusion operator in  $\Omega_2$  (see the proof of Theorem III.2.18 and Section III.6).

- **A convergence result towards the Dirichlet control problem.** Let  $(\beta_n)_n$  be any sequence of positive Robin parameters such that  $\beta_n \rightarrow +\infty$  when  $n \rightarrow +\infty$ . For each  $n$  we define  $v_n$  (resp.  $y_n$ ) to be the null-control of minimal  $L^2$  norm (resp. the associated trajectory) for the problem (I.1)–(I.2) with Robin parameter  $\beta_n$ .

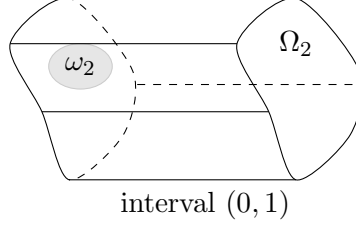


Figure I.4: The cylindrical geometry

Then, there exists a subsequence  $(n_k)_k$  such that

$$\frac{v_{n_k}}{\beta_{n_k}} \xrightarrow[k \rightarrow +\infty]{} v_D \quad \text{weakly in } L^2((0, T) \times \Gamma_0),$$

$$y_{n_k} \xrightarrow[k \rightarrow +\infty]{} y_D \quad \text{strongly in } (L^2(0, T; H^{-1}(\Omega)))^2 \quad \text{and}$$

$$y_{n_k} \xrightarrow[k \rightarrow +\infty]{} y_D \quad \text{weakly in } (L^2((0, T) \times \Omega))^2,$$

where  $v_D$  (resp.  $y_D$ ) is a null-control (resp. the associated trajectory) for the corresponding Dirichlet control problem (I.1)–(I.3).

### I.2.2.2 Some remarks to conclude Chapter III

Let us draw up some difficulties that are related to our work and still intricate to solve, we precise those as follows.

- Recall that, we have chosen the same (space dependent) diffusion coefficient in (I.1) for both the equations. It will be interesting if one treats two different diffusion coefficients (even if constants) for the two state equations, at least when  $\beta_1 = \beta_2$ . In this situation, a minimal time for the null-controllability might appear as it has been addressed in the paper [11] for abstract parabolic PDEs with Dirichlet boundary conditions.

When  $\beta_1 \neq \beta_2$ , we restricted to the constant coefficient case, i.e.,  $\gamma_1 = \gamma_2 = 1$ . For non-constant diffusion coefficient (even if they are equal for both components), again we do not have enough information about the spectral theory. Mainly, it is not that easy to determine a proper (weak) gap condition in this case.

Nevertheless, we perform a numerical simulation followed by a penalized HUM approach ([63, 64, 31]), for different diffusion coefficients (although constant) and different Robin parameters  $\beta_1, \beta_2$ .

We hereby choose the following quantities for our simulation,

$$\begin{aligned} T &= 0.3, \quad \gamma_1 = 1, \quad \gamma_2 = 2, \\ \beta_1 &= 2.5, \quad \beta_2 = 1.5, \\ y_{0,1}(x) &= 2 \sin(2\pi x), \quad y_{0,2}(x) = \mathbb{1}_{(0.3, 0.8)}(x). \end{aligned}$$

Using a general discretization method for a coupled parabolic system developed in Section IV.5.2, Ch. IV, we obtain the Fig. I.5, where one observes that as the mesh size/ penalization parameter is getting smaller, the size of the target data ( $\blacksquare$ ) converges to zero while the cost of the control ( $\blacklozenge$ ) and optimal energy ( $\bullet$ ) are approaching to some constant values (those notions have been introduced in Section IV.5.3 of Ch. IV). So, experimentally we can conclude that the problem with those above quantities are null-controllable at time  $T = 0.3$ .

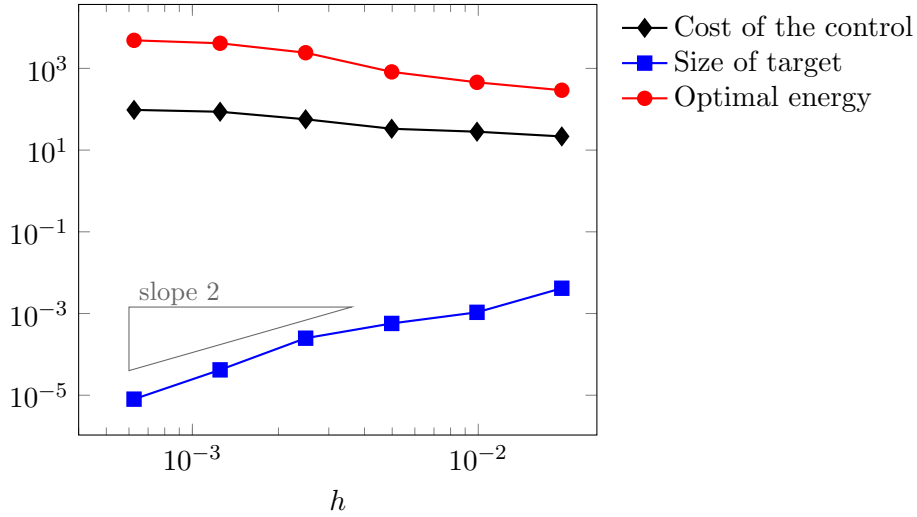


Figure I.5: Convergence properties of HUM control for coupled Robin system with  $\gamma_1 = 1, \gamma_2 = 2$  and  $\beta_1 = 2.5, \beta_2 = 1.5$ .

- Next, note that we strict to the case  $\beta_1 = \beta_2$  to obtain a null-controllability of (I.1)–(I.2) in the cylindrical geometry. But we could not prove this kind of result for any  $\beta_1 \neq \beta_2$ , since in this situation, we do not have the proper control cost  $Ce^{C/T}$  which is one of the two required ingredients.

### I.2.3 Chapter IV: Some parabolic systems with both interior and boundary couplings

In Chapter IV, we first deal with some  $2 \times 2$  coupled parabolic systems with only *one* boundary control, which fit into the framework of (I.4). To this end, we also study some  $3 \times 3$  parabolic system with *one* or *two* boundary control force(s). Apart from these, we pursue some numerical studies in accordance with the well-known penalized HUM approach.

Let us come to the main part of that chapter.

#### I.2.3.1 Main systems under study with Kirchhoff boundary condition in the $2 \times 2$ case

Let us consider the interior coupling as follows

$$\mathcal{M}_{coup} = \mathcal{M}_a := \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix},$$

for some  $a \in \mathbb{R}$  and we also take the boundary matrices

$$\begin{aligned} \mathcal{D}_0 &= I_{2 \times 2}, \quad \mathcal{N}_0 = O_{2 \times 2}, \\ \mathcal{D}_1 &= \begin{pmatrix} 1 & -1 \\ \alpha & 0 \end{pmatrix}, \quad \mathcal{N}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \end{aligned}$$

for some  $\alpha \geq 0$ .

Now taking into account the diffusion operator  $\mathcal{A}$  (introduced in I.5), the coupling matrix  $\mathcal{M}_a$  and the boundary matrices as above, the interior–boundary coupled system (I.4) reads as

follows

$$\begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ y_1(t, 1) = y_2(t, 1) & \text{in } (0, T), \\ \gamma_1(1) \frac{\partial y_1}{\partial x}(t, 1) + \gamma_2(1) \frac{\partial y_2}{\partial x}(t, 1) + \alpha y_1(t, 1) = 0 & \text{in } (0, T), \\ y_1(0, \cdot) = y_{0,1}(\cdot), \quad y_2(0, \cdot) = y_{0,2}(\cdot) & \text{in } (0, 1), \end{cases} \quad (\text{I.8})$$

with a Dirichlet control at the left end point either on the second or first component depending on the choices of  $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  or  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , that is to say

$$\text{either } y_1(t, 0) = 0, \quad y_2(t, 0) = v(t) \quad \text{in } (0, T), \quad (\text{I.9a})$$

$$\text{or } y_1(t, 0) = v(t), \quad y_2(t, 0) = 0 \quad \text{in } (0, T). \quad (\text{I.9b})$$

Observe that, we have chosen here a general *Kirchhoff condition* at the boundary point  $x = 1$  (the usual Kirchhoff condition is the one with  $\alpha = 0$ ). It is prominent that the Kirchhoff type condition appears widely in physics, electrical engineering and in various biological models. We quote here very few of those, for instance the papers [75, 76] cope with some mathematical studies on Kirchhoff's rule for quantum wires and the authors of [41] discussed a rigorous analysis on FitzHugh–Nagumo–Rall model of a neuronal network with a Kirchhoff type rule in axonal or dendrital ramification points.

In the framework of control theory, the usual type of system that has already been studied in the literature is the case when there is *no interior coupling* (i.e.,  $a = 0$ ) and with the standard Kirchhoff condition (i.e.,  $\alpha = 0$ ), see for instance [50, Ch. 8] and [15]. We also recall that, a controllability result of a  $2 \times 2$  parabolic system without any interior coupling, that is  $a = 0$ , has been addressed in [23, Remark 3.6], where a Kirchhoff condition with  $\alpha = 0$  has been considered. As it is mentioned in Section I.1.2, a non-zero interior coupling cannot be imposed to the chosen systems of most of the existing works, since those cases have been analyzed on a metric graph where each edge is associated with only one scalar equation with respect to one unknown.

But observe that when  $a = 0$ , the two control systems (I.8)–(I.9a) and (I.8)–(I.9b) are exactly the same.

Now, as soon as one consider some interior coupling coefficient  $a \neq 0$ , the control systems (I.8)–(I.9a) and (I.8)–(I.9b) are certainly different in nature and the choices of the component on which we exert the control, really has an influence to the controllability issues. In fact in the second situation (when we put the control on the first component), the choice of  $(\alpha, a)$  is very crucial to conclude the positive or negative controllability phenomena. We will precisely describe our main observations below.

## Main results.

- **The control on the second component: a Carleman approach.** The boundary controllability of the system (I.8)–(I.9a), that is when we consider a control on the *second component*  $y_2$ , can be established by means of a global Carleman estimate (and then to find an observability inequality) for any interior coupling coefficient  $a \in \mathbb{R}$  and boundary parameter  $\alpha \geq 0$ . The main result in this case is the following:

**Controllability result:** Let any  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}$  ( $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$ ) and  $T > 0$  be given. Then, for any  $y_0 \in \mathcal{H}_{-\alpha} \subset (H^{-1}(0, 1))^2$ , there exists a null-control  $v \in L^2(0, T)$  for the problem (I.8)–(I.9a), that satisfies the estimate

$$\|v\|_{L^2(0, T)} \leq C e^{C/T} \|y_0\|_{\mathcal{H}_{-\alpha}},$$

with the constant  $C := C(\gamma_1, \gamma_2, \alpha, a) > 0$  which does not depend on  $T > 0$  and  $y_0$ .

We will define the space  $\mathcal{H}_{-\alpha}$  in Section IV.2.1.1 and the above controllability result will be introduced in Theorem IV.2.7 of Ch. IV.

- **The control on the first component: a moments approach.** Interestingly, when we consider our control to be acted on the *first component*  $y_1$ , it appears that the Carleman approach cannot be applied to the system (I.8)–(I.9b) for  $a \neq 0$ . This is because the source integral due to the interior coupling in our Carleman estimate cannot be controlled by the boundary observation term with our choices of weight functions (defined later in Section IV.3.1); the exact technical point behind this will be specified in Remark IV.3.5, Ch. IV.

Due to this obstacle, the next immediate idea is to investigate the spectral analysis of the adjoint elliptic operator associated with our control system and try to develop the so-called moments method to construct a control by hand; indeed, by developing the spectral analysis, we observe that depending on the choices of the parameters  $(\alpha, a)$ , the controllability issues significantly changes. Moreover, we also find a class of parameters  $(\alpha, a)$  for which the controllability fails.

This is quite surprising as we know that, for a *cascade system* the usual business is to apply a boundary control on the first component  $y_1$  to study the controllability issue. In that case, there is no direct influence of the control to the equation of  $y_2$ , and here comes the role of the interior coupling: it acts as an indirect control to  $y_2$  (for instance, the system (I.1)–(I.2)). But in the case of (I.8)–(I.9), we have the opposite phenomena: when we consider the control to be acted on the *second entry*, the system is always controllable and in contrast, we have *conditional controllability* depending on the pairs  $(\alpha, a)$ , immediately after we put the control on the *first entry*.

The main results in this situation can be presented as follows:

**Controllability result:** Fix  $\gamma_1 = \gamma_2 = 1$  and let any  $T > 0$  be given. Then, there exists a non-empty set  $\mathcal{R} \subset \mathbb{R}_0^+ \times \mathbb{R}^*$  (where  $\mathbb{R}_0^+$  has been defined earlier and  $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ ) such that

1. for each pair  $(\alpha, a) \notin \mathcal{R}$ , there is a null-control to the problem (I.8)–(I.9b) for any given data  $y_0 \in \mathcal{H}_{-\alpha}$ ,
2. for each pair  $(\alpha, a) \in \mathcal{R}$ , there exists a subspace  $\mathcal{Y}_{\alpha,a} \subset \mathcal{H}_{-\alpha}$  of co-dimension 1, such that there exists a null-control to the problem (I.8)–(I.9b), if and only if  $y_0 \in \mathcal{Y}_{\alpha,a}$ .

In addition, in the controllable cases we can construct such a null-control  $v$  that satisfies the bound

$$\|v\|_{L^2(0,T)} \leq C_{\alpha,a} e^{C_{\alpha,a}/T} \|y_0\|_{\mathcal{H}_{-\alpha}},$$

where  $C_{\alpha,a} > 0$  is independent on  $T > 0$  and  $y_0$ .

We shall introduce this controllability result through the lens of Theorem IV.2.8 and the set  $\mathcal{R}$  and the space  $\mathcal{Y}_{\alpha,a}$  will be specified in Section IV.4.2.1 of Ch. IV. Sometimes, we shall call  $\mathcal{R}$  a critical set.

**Negative controllability:** In the case when  $(\alpha, a) \in \mathcal{R}$ , the problem (I.8)–(I.9b) is not even approximately controllable if we choose our initial data  $y_0 \notin \mathcal{Y}_{\alpha,a}$ .

This indeed proves that the *Carleman technique* is hopeless in the case when we exert the control on  $y_1$ .

### I.2.3.2 Some numerical experiments

To pursue some numerical experiments, we discretize (in terms of finite differences) the general system (I.4) under the Assumption I.1.2 in Section IV.5.2 of Ch. IV. This will allow us to investigate some more examples beside the Dirichlet ones, in the framework of our general system. We use the notable penalized Hilbert Uniqueness Method (HUM) ([63, 64, 31]) to pursue the numerical studies. In this regard, we must mention that several authors has utilized the penalized HUM technique to clarify various controllability issues related to the parabolic systems. For instance, the authors in [34] dealt with a numerical study of insensitizing control problems for parabolic semilinear equation and in [28], the controllability of 1D fractional heat equation has been analyzed from both numerical and theoretical aspects. We also quote [12], where some partial controllability (or, non-) results have been illustrated with help of penalized HUM technique.

Concerning the numerical studies for parabolic systems, in most of the known cases no boundary coupling has been taken into account. But in our case, we will introduce a general discretization methodology for the interior-boundary coupled parabolic system (I.4) (along with the Assumption I.1.2) mainly to incorporate the effect of the *boundary couplings* into our discrete setting.

To this end, we present several experiments to conclude the numerical part.

- **Dirichlet controls:** For the Dirichlet control problem (I.8)–(I.9b), the first thing which interests us is to find at least some pair  $(\alpha, a)$  that belongs to the critical set  $\mathcal{R}$ . Indeed, we find one such pair which is  $(\alpha, a) = (1, 3.1931469)$  (the value of  $a$  is approximate) and we have shown experimentally that there exists some eigenvalue  $\lambda_{\alpha, a}^c$  for which the associated eigenfunction produces the corresponding *observation term*  $\approx 0$ , which clarifies by means of Fattorini-Hautus test (see [55], [89]) that the system cannot be approximately controllable. Moreover, by an HUM penalization approach (see for instance, [31]) we have shown that the *cost of the control* is blowing up while the size of the target data  $y(T)$  remains almost constant when the mesh size/penalization parameter tends to 0 (see Fig. I.6a). In other words, we cannot expect approximate/null-controllability in this situation and that illustrates our theoretical result presented in the preceding section.

For the numerical simulation, we set

$$\begin{aligned} T &= 0.3, \quad \gamma_1 = \gamma_2 = 1, \\ \alpha &= 1, \quad a = 3.1931469, \\ y_{0,1}(x) &= 10 \sin^3(2\pi x), \quad y_{0,2}(x) = 5 \times \mathbf{1}_{(0.3, 0.8)}(x), \end{aligned} \tag{I.10}$$

and we obtain Fig. I.6a.

But, as soon as we exert the control in the *second component* that is the case (I.8)–(I.9a), we have mentioned before that regardless the choices of  $(\alpha, a)$  the null-controllability holds true by a *Carleman approach*. Thus, by choosing the same critical pair  $(\alpha, a) = (1, 3.1931469)$  and the quantities as in (I.10) to the system (I.8)–(I.9a), we obtain Fig. I.6b which shows the positive controllability of that system as expected.

- **Neumann controls:** Beside the Dirichlet control problems, one can also think about the Neumann boundary control system, that is the system (I.8) with the following two types of Neumann controls, depending on the choices of  $B = \gamma_2(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , or  $B = \gamma_1(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , that is

$$\text{either } \frac{\partial y_1}{\partial x}(t, 0) = 0, \quad \frac{\partial y_2}{\partial x}(t, 0) = v(t) \quad \text{in } (0, T), \tag{I.11a}$$

$$\text{or } \frac{\partial y_1}{\partial x}(t, 0) = v(t), \quad \frac{\partial y_2}{\partial x}(t, 0) = 0 \quad \text{in } (0, T). \tag{I.11b}$$

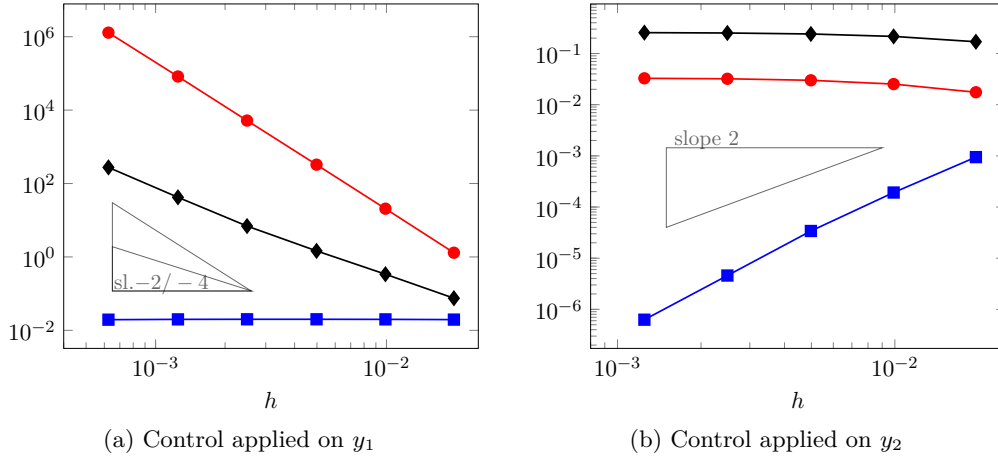


Figure I.6: Different controllability situations of the system (I.8) with the conditions (I.9b) and (I.9a) for  $(\alpha, a) = (1, 3.1931469)$ ; a penalized HUM approach. Same legend as in Fig. I.5.

We shall not pursue any theoretical study for this problem in the report. But numerically we find that the controllability properties are in the same spirit as of the Dirichlet ones. In fact, alike the Dirichlet case we will observe in Fig. I.7a that for some critical pair  $(\alpha, a) = (0.1, 1.2369289)$  (approximately determined) the approximate/null-controllability cannot hold when we apply the control to the first component, i.e., the case (I.11b). Interestingly, this situation will not appear for the same pair of  $(\alpha, a)$  when we consider the control to be acted on the second component, see Fig. I.7b.

For the numerical simulations, we set

$$\begin{aligned} T &= 0.5, \quad \gamma_1 = \gamma_2 = 1, \\ \alpha &= 0.1, \quad a = 1.2369289, \\ y_{0,1}(x) &= \sin(2\pi x), \quad y_{0,2}(x) = \mathbb{1}_{(0.3,0.8)}(x). \end{aligned}$$

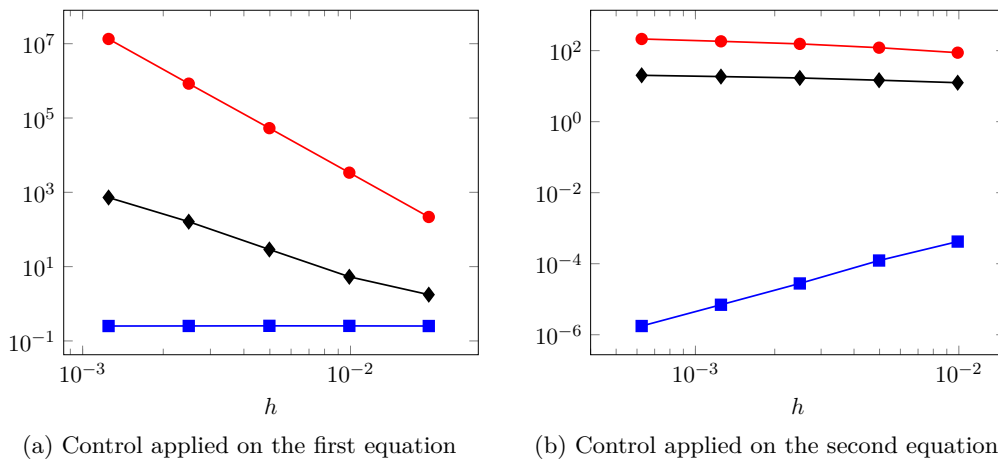


Figure I.7: Convergence properties due to the HUM approach with Neumann control applied in different equations with critical values  $(\alpha, a) = (0.1, 1.2369289)$ . Same legend as in Fig. I.5.

- **Control on the difference of two components:** Let us consider the following system



where we put some control in the difference of two components at point  $x = 0$ ,

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + a y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ y_1(t, 0) - y_2(t, 0) = v(t) & \text{in } (0, T), \\ \partial_x y_1(t, 0) + \partial_x y_2(t, 0) = 0 & \text{in } (0, T), \\ y_1(t, 1) = y_2(t, 1) & \text{in } (0, T), \\ \partial_x y_1(t, 1) + \partial_x y_2(t, 1) = 0 & \text{in } (0, T), \\ y_i(0, x) = y_{0i}(x) & \text{in } (0, 1), \ i = 1, 2. \end{cases} \quad (\text{I.12})$$

But, by a standard change of variables  $w_1 = y_1 - y_2$ ,  $w_2 = y_1 + y_2$ , we obtain the following system where there are only interior couplings but not boundary,

$$\begin{cases} \partial_t w_1 - \partial_x^2 w_1 - a \frac{w_1 + w_2}{2} = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t w_2 - \partial_x^2 w_2 + a \frac{w_1 + w_2}{2} = 0 & \text{in } (0, T) \times (0, 1), \\ w_1(t, 0) = v(t), \quad w_1(t, 1) = 0 & \text{in } (0, T), \\ \partial_x w_2(t, 0) = \partial_x w_2(t, 1) = 0, & \text{in } (0, T), \\ w_1(0, x) = y_{01}(x) - y_{02}(x), \quad w_2 = y_{01}(x) + y_{02}(x) & \text{in } (0, 1). \end{cases}$$

This leads the following two situations:

- **The case  $a = 0$ , never controllable:** Observe that, in absence of an interior coupling term (i.e., when  $a = 0$ ), we cannot hope for positive controllability of the above system since there is no effect of the control on  $w_2$ .
- **The case  $a \neq 0$ , possibly controllable:** On the other hand, the presence of an interior coupling ( $a \neq 0$ ) can give us a hope for positive controllability of the system (I.12), we refer to Section IV.5.3.3, Ch. IV for more detail.

Below, we put a figure (Fig. I.8) for  $a = 3$ , which ensures at least experimentally that the system (I.12) is supposed to be null-controllable. We set the following quantities for the simulation of our system (I.12),

$$\begin{aligned} T &= 0.4, \quad \gamma_1 = \gamma_2 = 1, \quad a = 3 \\ y_{0,1}(x) &= 2 \sin(2\pi x), \quad y_{0,2}(x) = \mathbb{1}_{(0.3, 0.8)}(x). \end{aligned}$$

### I.2.3.3 A $3 \times 3$ parabolic system with one or two control(s)

Next to the analysis of  $2 \times 2$  systems, we will treat some  $3 \times 3$  parabolic system, again with a linear interior coupling, the Kirchhoff type condition and the Dirichlet boundary control(s). The goal is to see what happens if we exert only *one* boundary control to that  $3 \times 3$  system. In this case, we realize that the null-controllability of the system depends on the choices of the components through which the control is being applied, and moreover, sometimes it depends significantly on the choices of the *interior coupling coefficient* and *Kirchhoff parameter*. But in the negative-controllability cases, if we consider an extra boundary control to some other component, then the null-controllability can be recovered again. In the subsequent paragraphs, we first present the main results of our system with two controls and then we move forward to the case of taking a single control.

The motivation to study a  $3 \times 3$  model along with an interior coupling, is coming from the point of considering a parabolic control system on a metric graph where some (or, all) of the edges correspond to vectorial parabolic equations. In this regard, we refer to Section I.3 where

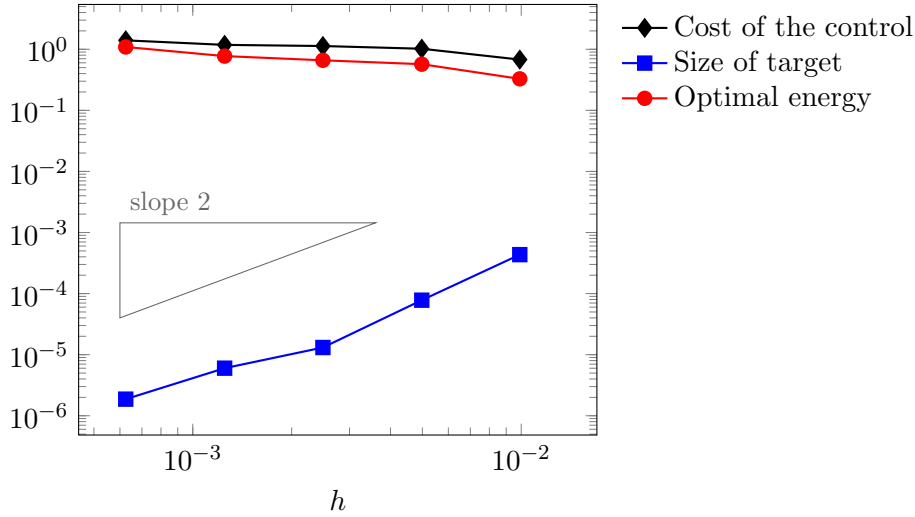


Figure I.8: Convergence properties of the HUM method with control on the difference and non-zero coupling term  $a = 3$ .

we shall discuss about a general parabolic system on metric graph and in particular, our  $3 \times 3$  case will be identified in a graph with two edges, see Fig I.9.

This part of the manuscript has been introduced in Section IV.6 of Chapter IV; we shall cover the following situations.

- **A system with two boundary controls.**

Let us consider the following system

$$\left\{ \begin{array}{ll} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) &= 0 \quad \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) &= 0 \quad \text{in } (0, T) \times (0, 1), \\ \partial_t y_3 - \partial_x(\gamma_3 \partial_x y_3) + a y_1 &= 0 \quad \text{in } (0, T) \times (0, 1), \\ \left\{ \begin{array}{l} y_1(t, 1) = y_2(t, 1) = y_3(t, 1), \\ \sum_{i=1}^3 \gamma_i(1) \partial_x y_i(t, 1) + \alpha y_1(t, 1) = 0 \end{array} \right. & \text{in } (0, T), \\ y_i(0, x) = y_{0,i}(x), \quad \text{for } i = 1, 2, 3, & \text{in } (0, 1), \end{array} \right. \quad (\text{I.13})$$

and in this case, two boundary controls will be applied through the Dirichlet conditions on any two components among the three (described below). In fact, one can prescribe the above system into a metric graph with two edges, where one edge corresponds the equations of  $y_1, y_3$  (as a vectorial equation of  $(y_1, y_3)$ ) and the other one to the scalar equation of  $y_2$ , see Fig. (I.9). In that figure, the Kirchhoff type condition is acting on the node  $n_2$ , the junction point of  $(y_1, y_3)$  and  $y_2$ , whereas the controls are exerted on  $n_1$  and/or  $n_3$  by means of the Dirichlet conditions. We have the three different situations as follows:

either the controls are applied on  $y_2$  and  $y_3$  (in the graph setting, on the nodes  $n_3$  and  $n_1$ )

$$y_1(t, 0) = 0, \quad y_2(t, 0) = v(t), \quad y_3(t, 0) = \tilde{v}(t) \quad \text{in } (0, T), \quad (\text{I.14a})$$

or, on  $y_1$  and  $y_2$  (i.e., on the nodes  $n_1$  and  $n_3$ )

$$y_1(t, 0) = v(t), \quad y_2(t, 0) = \tilde{v}(t), \quad y_3(t, 0) = 0 \quad \text{in } (0, T), \quad (\text{I.14b})$$

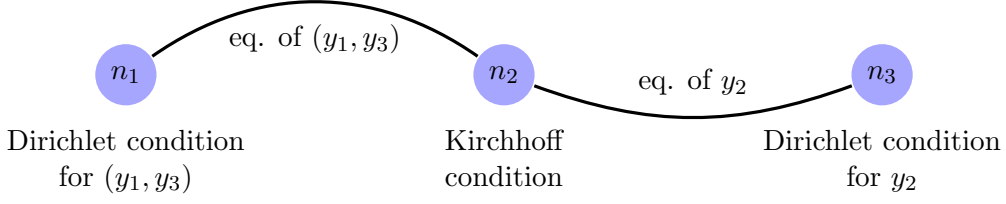


Figure I.9: The system (I.13)–(I.14), or (I.15) on a metric graph of two edges.

or, on  $y_1$  and  $y_3$  (i.e., only on the node  $n_1$ )

$$y_1(t, 0) = v(t), \quad y_2(t, 0) = 0, \quad y_3(t, 0) = \tilde{v}(t) \quad \text{in } (0, T). \quad (\text{I.14c})$$

As previous, we choose some interior coupling coefficient  $a \in \mathbb{R}^*$ , the Kirchhoff parameter  $\alpha \geq 0$  and the diffusion coefficients  $\gamma_i \in \mathcal{C}^1([0, 1])$  with  $\gamma_{\min} := \inf_{[0, 1]} \{\gamma_i, i = 1, 2, 3\} > 0$ .

**Controllability results.** Let any  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$  and  $T > 0$  be given. Then for any  $y_0 := (y_{0,1}, y_{0,2}, y_{0,3}) \in \tilde{\mathcal{H}}_{-\alpha} \subset (H^{-1}(0, 1))^3$ , we have the following controllability results (the space  $\mathcal{H}_{-\alpha}$  has been defined in Section IV.6, Ch. IV).

1. The case (I.13)–(I.14a), that is when we consider the controls on  $y_2$  and  $y_3$ , is null-controllable at any time  $T > 0$ , for any given data  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$  and any three different diffusion coefficients.

The proof is followed by a *Carleman approach* as in the  $2 \times 2$  case (I.8)–(I.9a).

2. For the cases (I.14b) and (I.14c), we could not succeed by the Carleman technique. The reason behind this is similar with the  $2 \times 2$  case (I.8)–(I.9b); the source integral due to the interior coupling in the Carleman estimate cannot be controlled by the boundary observation term with our choices of weight functions. Thus, we handle this situation by investigating the spectral analysis of the adjoint elliptic operator and then try to apply the moments technique. We hereby fix  $\gamma_1 = \gamma_2 = \gamma_3 = 1$  in order to have a concrete idea about the spectrum and in these cases we obtain the following.

Regardless the choices of  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$ , both the system (I.13)–(I.14b) and (I.13)–(I.14c) are null-controllable at any time  $T$ , for any given data  $y_0 \in \tilde{\mathcal{H}}_{\alpha, a}$ .

**Remark I.2.1.** In contrast, recall that the controllability of  $2 \times 2$  system (I.8) genuinely depends on the choices of the parameters  $(\alpha, a)$  when we change the position of the control from  $y_2$  to  $y_1$  (see Section I.2.3.1).

#### • A system with only one boundary control.

Next, we treat the same system (I.13) but with only one control exerted on one of the three components. More precisely, one can have the following three situations when we choose some interior coupling coefficient  $a \in \mathbb{R}^*$ .

either the control is applied on  $y_1$  (to the node  $n_1$  in the Fig. I.9)

$$y_1(t, 0) = v(t), \quad y_2(t, 0) = 0, \quad y_3(t, 0) = 0 \quad \text{in } (0, T), \quad (\text{I.15a})$$

or, on  $y_2$  (to the node  $n_3$ )

$$y_1(t, 0) = 0, \quad y_2(t, 0) = v(t), \quad y_3(t, 0) = 0 \quad \text{in } (0, T), \quad (\text{I.15b})$$

or, on  $y_3$  (to the node  $n_1$ )

$$y_1(t, 0) = 0, \quad y_2(t, 0) = 0, \quad y_3(t, 0) = v(t) \quad \text{in } (0, T). \quad (\text{I.15c})$$

**Controllability or non-controllability results.** In the above cases, the Carleman approach is essentially inefficient and the effective way is to deal with the moments method (if applicable). Here also we fix  $\gamma_1 = \gamma_2 = \gamma_3 = 1$  and by following the spectral analysis of the adjoint elliptic operator, we obtain three different situations. We let any  $T > 0$  and  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$ .

1. **Control on the first component– conditionally controllable:** This case is quite interesting in the sense that the controllability depends on the choices of the parameters  $(\alpha, a)$ . More precisely, there exists a non-empty set  $\tilde{\mathcal{R}} \subset \mathbb{R}_0^+ \times \mathbb{R}^*$  such that
  - (a) for each pair  $(\alpha, a) \notin \tilde{\mathcal{R}}$ , the system (I.13)–(I.15a) is null-controllable at any time  $T > 0$ , for any given data  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$ .
  - (b) for each pair  $(\alpha, a) \in \tilde{\mathcal{R}}$ , there exists a subspace  $\tilde{\mathcal{Y}}_{\alpha,a} \subset \tilde{\mathcal{H}}_{-\alpha}$  of co-dimension 1, such that our system (I.13)–(I.15a) is null-controllable at any time  $T > 0$ , for any given data  $y_0 \in \tilde{\mathcal{Y}}_{\alpha,a}$ .

The set  $\tilde{\mathcal{R}}$  and  $\tilde{\mathcal{Y}}_{\alpha,a}$  have been defined in Section IV.6, Ch. IV.

*Negative controllable case:* When  $(\alpha, a) \in \tilde{\mathcal{R}}$ , the system (I.13)–(I.15a) is not even approximately controllable if one chooses the initial data  $y_0 \notin \tilde{\mathcal{Y}}_{\alpha,a}$ .

2. **Control on the second component– always controllable:** For any  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$ , the system (I.13)–(I.15b), i.e., when we consider a control on  $y_2$ , is always null-controllable at any time  $T > 0$ , for any given data  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$ .
3. **Control on the third component– never controllable:** Unlike the previous cases, surprisingly we see that the system (I.13) with a control on  $y_3$ , i.e., the case (I.15c) is not even approximately controllable.

## I.3 Perspectives

We conclude this running chapter with some perspectives that are somehow linked with the PhD topic. Let us directly go into the details.

1. **Dealing with some purely non self-adjoint elliptic operator and non-linearity on the boundary.**

Recall that the cases we have considered, either (I.1)–(I.2) or (I.4) (along with the Assumption I.1.2), the diffusion operators were bounded perturbations of some densely defined self-adjoint operators in some Hilbert spaces. In such cases, one still has a good information about the spectrum of those operators, normally what we mean is that the spectrum is discrete (since these operators have compact resolvents, since the corresponding self-adjoint ones have). For more detail, we quote the book [69] by T. Kato where the perturbation theory for linear operator has been developed nicely. We also mention some recent results, for instance [92, 93], regarding this topic. In fact, for bounded perturbation one could expect that the real parts of the eigenvalues go to infinity while the imaginary parts remain bounded; indeed we shall deal with this kind of situation in Section IV.4.1 of Ch. IV. Other than this, it is well-known from a theory of Keldysh, see [71, 87] that the system of generalized eigenfunctions for this kind of perturbed operator generates a *complete family* in the associated Hilbert space,  $L^2$  for instance. All these information always let us allow at least to deal with the so-called moments method (of course, if all other requirements hold good) to construct a control by hand.

Hereafter, some idea that comes into mind is to deal with some parabolic control systems where the diffusion operator is *purely non self-adjoint* in nature, more precisely we want to say that the non self-adjointness could come only through the choices of boundary

conditions, not necessarily due to the bounded perturbations. And, indeed in several situations it used to appear: for instance in some biochemical model of localized two-enzyme kinetics, see [95], where the concerned model (where no control appears) is

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha y_2 = 0 & \text{in } (0, T) \times (0, 1), \\ y_1(0, \cdot) = y_{0,1}(\cdot) & \text{in } (0, 1), \\ y_2(0, \cdot) = y_{0,2}(\cdot) & \text{in } (0, 1), \end{cases} \quad (\text{I.16})$$

with the boundary conditions at  $x = 0$  as

$$\begin{cases} \frac{\partial y_1}{\partial x}(t, 0) + f(y_2(t, 0)) = 0 & \text{in } (0, T), \\ \frac{\partial y_2}{\partial x}(t, 0) = 0 & \text{in } (0, T), \end{cases} \quad (\text{I.17})$$

and at  $x = 1$  as

$$\begin{cases} \frac{\partial y_1}{\partial x}(t, 1) = 0 & \text{in } (0, T), \\ \frac{\partial y_2}{\partial x}(t, 1) - g(y_1(t, 1)) = 0 & \text{in } (0, T). \end{cases} \quad (\text{I.18})$$

In the model  $y_1$  and  $y_2$  denote the concentrations of the effectors of two enzymes and  $\alpha$  denotes a first order decay in  $(0, 1)$ . The two enzymes affect each other with their effectors  $y_1$  and  $y_2$  via some *activation* (or *inhibition*), namely by the functions  $f$  and  $g$ , which are actually non-linear in nature and satisfy some further assumptions that we do not take into account for the moment.

#### *Perspective of studying boundary controllability:*

One can think of putting a boundary control  $v$  ( $L^2$  for instance) to manage the *activation* or *inhibition* of the effectors  $y_1$  and  $y_2$ . Of course, the idea is to use one control function  $v \in L^2(0, T)$  to control both the effectors.

We hereby consider a simpler case say,  $f(w) = g(w) = w$  (possibly not very meaningful from a biological point of view) and formulate a control problem as follows: the system (I.16) along with the boundary condition at  $x = 1$  as

$$\begin{cases} \frac{\partial y_1}{\partial x}(t, 1) = 0 & \text{in } (0, T), \\ \frac{\partial y_2}{\partial x}(t, 1) - y_1(t, 1) = 0 & \text{in } (0, T). \end{cases}$$

and with a control  $v \in L^2(0, T)$  either exerted on the combination of two components like

$$\begin{cases} \frac{\partial y_1}{\partial x}(t, 0) + y_2(t, 0) = v(t) & \text{in } (0, T), \\ \frac{\partial y_2}{\partial x}(t, 0) = 0 & \text{in } (0, T), \end{cases}$$

or, with a control only on the second component as

$$\begin{cases} \frac{\partial y_1}{\partial x}(t, 0) + y_2(t, 0) = 0 & \text{in } (0, T), \\ \frac{\partial y_2}{\partial x}(t, 0) = v(t) & \text{in } (0, T). \end{cases}$$

In this case, it is not difficult to observe that the operator  $\mathcal{A} = \begin{pmatrix} -\partial_x^2 + \alpha & 0 \\ 0 & -\partial_x^2 + \alpha \end{pmatrix}$  with its domain

$$D(\mathcal{A}) := \left\{ u := (u_1, u_2) \in (H^2(0, 1))^2 \mid \begin{aligned} &u_1'(0) + u_2(0) = 0, \quad u_2'(0) = 0, \\ &u_1'(1) = 0, \quad u_2'(1) - u_1(1) = 0 \end{aligned} \right\},$$

is a non self-adjoint operator. This is the reason, it is not that easy to understand the spectral behavior this kind of operator (more precisely, its adjoint operator), even in this simpler situation.

Therefore, it will be really worth if we can tackle, at least partially, some such class of intricate systems and in fact, at some point it is one of my interest to pursue some further studies related to these types of parabolic control systems where the non self-adjointness of the elliptic operators takes place into the story.

## 2. Some abstract parabolic systems on compact metric graph.

We begin by introducing what it is meant by a metric graph and a quantum graph.

A *metric graph*, or network is a graph where each edge assigns a positive length, could be finite or infinite. It is then said to be compact if all the lengths are finite, i.e., each edge being equivalent to the finite interval  $(0, 1)$ . We hereby denote a metric graph by  $\mathcal{G}(\Sigma, \mathcal{I})$  where  $\mathcal{I}$  denotes the set of all edges and  $\Sigma$  the set of all vertices, for the moment assume that we have only finite number of edges. The Hilbert space structure on some metric graph can be defined as

$$E := \bigoplus_{j=1}^n L^2(I_j), \quad \mathcal{H}^k(\mathcal{G}) := \bigoplus_{j=1}^n H^k(I_j), \quad \text{for some } k \geq 1, \text{ say,}$$

with each edge  $I_j := (0, 1)$  in  $\mathcal{I}$  for  $j = 1, 2, \dots, n$  ( $n \geq 2$ ).

Finally, we mention that a *quantum graph* is some metric graph endowed with a differential operator, most likely one can think of the negative Laplace operator (see [25] for more details about quantum graph).

Now, we must say that the interest of studying partial differential equations on a metric graph is rising because it has wide variety of applications in physics, chemistry, engineering or biology, see for instance the review paper [79]. Beside this, in [80, 18], the authors discussed about some carbon nano-structures, in particular [80] contains some interesting studies on the spectra of carbon-nano structures. In the references [42, 41], some models have been addressed which are concerned with the signal transmission in biological neural network.

Let us come to the main point of attraction, that is the controllability issues on quantum graph. Recall that we already stated several references concerning parabolic controllability results on metric graphs in Section I.1.2. But as per our knowledge, most of the known controllability results have been dealt with a *scalar* differential equation in each edge of a graph. Thus, one idea that comes into mind is to tackle with some system that contains *vectorial* differential equation in each edge of a graph. To be more precise, we consider the following prototype system (without putting control let say)

$$\begin{cases} \partial_t y_j + \mathcal{A}y_j = 0 & \text{in } (0, T) \times I_j, \text{ for each } j \in \{1, 2, \dots, n\}, \\ \mathcal{D}_\sigma y(\sigma) + \mathcal{N}_\sigma \frac{\partial y}{\partial \nu}(\sigma) = 0 & \text{on } (0, T), \text{ and } \forall \sigma \in \Sigma, \\ y_j(0, \cdot) = y_{0,j}(\cdot) & \text{in } I_j, \text{ for each } j \in \{1, 2, \dots, n\}, \end{cases} \quad (\text{I.19})$$

where each unknown  $y_j$  corresponds to a vector of size  $m_j \geq 1$ , that is

$$y_j := (y_j^1, y_j^2, \dots, y_j^{m_j}), \quad \text{for } j = 1, 2, \dots, n,$$

so that  $y := (y_1, y_2, \dots, y_n)$  is the vector of size  $\tilde{m} := \sum_{j=1}^n m_j$ . In this general setting, we consider  $\mathcal{A}$  as some vectorial diffusion operator and the matrices  $\mathcal{D}_\sigma, \mathcal{N}_\sigma \in M_{\tilde{m}}(\mathbb{R})$  produce some boundary interactions between the unknowns on some (or, all) vertices  $\sigma \in \Sigma$ .

Recall that, a control system (I.13) (with (I.14), or (I.15)) under the above general framework has been already introduced in Section I.2.3.3, where we have considered the system in a graph of two edges: we prescribe a *vectorial* pde in one edge and a *scalar* pde in the other edge.

Let us now consider a typical example under the very general framework (I.19) and discuss about some possible control problems in this aspect. We write down the well-known *FitzHugh–Nagumo–Rall model* ([91, 61]) of a neuronal network where a vectorial differential equation used to appear: for  $t \geq 0$  and  $x \in (0, 1)$ ,

$$\left\{ \begin{array}{l} \text{axons} \quad \left\{ \begin{array}{l} \partial_t y_i - \partial_x (\gamma_i \partial_x y_i) + f_i(y_i) + R_i = 0, \\ \partial_t R_i + \alpha_i (R_i - y_i) = 0, \end{array} \right. \\ \text{dendrites} \quad \left\{ \partial_t y_j - \partial_x (\gamma_j \partial_x y_j) + \beta_j y_j = 0, \right. \\ \text{continuity at nodes} \quad \left\{ y_i(t, \sigma_k) = y_j(t, \sigma_k) := d_{\sigma_k}(t), \text{ if } i, j \in \Sigma(\sigma_k), \right. \\ \text{inactive nodes} \quad \left\{ \sum_{i \in \Sigma(\sigma_k)} \gamma_i \partial_x y_i + \sum_{j \in \Sigma(\sigma_k)} \gamma_j \partial_x y_j + c_k d_{\sigma_k} = 0, \right. \\ \text{active nodes} \quad \left\{ \sum_{i \in \Sigma(\sigma_k)} \gamma_i \partial_x y_i + \sum_{j \in \Sigma(\sigma_k)} \gamma_j \partial_x y_j + c_k d_{\sigma_k} + \partial_t d_{\sigma_k} = 0, \right. \end{array} \right. \quad (\text{I.20})$$

with suitable initial conditions.

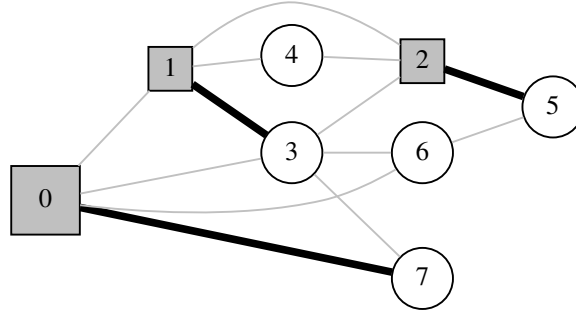


Figure I.10: Neuronal network: Boxes are Cell bodies (soma), Circles are synapses, black solid lines are axons and gray thin lines are dendrites.

- In the axons, each  $y_i := y_i(t, x)$  (electric potential) for  $i = 1, 2, \dots, n_0$  satisfies a FitzHugh–Nagumo (non-linear) PDE coupled with a linear cable equation of the ad-hoc variable  $R_i(t, x)$ . Normally, the functions  $f_i$  are non-linear in nature, but for simplicity one may choose linear functions.
- In the dendrites, each potential  $y_j$  for  $j = n_0, n_0 + 1, \dots, n$  satisfies a linear PDE (the so-called Rall's model) without any interior coupling.

- In the above system, each  $\sigma_k \in \Sigma$  for  $k \in \{1, 2, \dots, m\}$ , denotes the node of the network (cell) and by the notation  $i, j \in \Sigma(\sigma_k)$ , we mean that two edges  $I_i$  and  $I_j$  are incident with the common node  $\sigma_k$ .

It seems reasonable to assume the soma to be isopotential, and therefore a continuity condition at the nodes (soma and synapses) is important.

In the inactive nodes (synapses), a generalized Kirchhoff condition is imposed. We also see that some nodes (soma) are active such that a dynamic Kirchhoff law is considered.

For a complete well-posedness analysis of the associated system (I.20), we refer [41].

*Perspective of studying some controllability (or, non-controllability) results concerned with the system (I.20):*

Regarding the controllability issues, a few results are available in the literature mainly for the FHN model in the one-dimensional setting. We start by mentioning the work [36] where some *optimal control* problem has been addressed associated with the FHN system in 1D with homogeneous Dirichlet conditions and a distributed control in the equation of pde. We also quote [37] where some feedback control of the monodomain equations of FitzHugh-Nagumo type has been analyzed. A very recent result on the *approximate controllability* of 1D FHN system with Dirichlet boundary conditions has been achieved in [47]. The authors in [47] have proved that the linearized system is not null-controllable using a localized interior control since the spectrum of the concerned system has an accumulation point. Indeed, they have shown that although the global approximate controllability fails but it is possible to move from one steady-state to another arbitrarily close after some appropriate time by a localized interior control, provided that both steady states are in the same connected component of the set of steady states.

The model (I.20) is mainly described on a metric graph setting which is more meaningful from a biological point of view and moreover, a *Rall model* is also appeared in this case. But the controllability phenomena is really obscure associated with the model (I.20) since we have the boundary couplings as well which has not been appeared in [36, 37, 47] because of the absence of Rall equations. For the time being, we can drop the dynamic boundary conditions but still I suppose that the spectral analysis for the linearized model is far from obvious.

Motivating from the work [47], considering some localized interior control(s) in the axons (through the pdes) as well as in the dendrites (or, not?) can really be an interesting but non-trivial problem associated with the concerned model. Of course, the things are widely open and no more straightforward to tackle.





## Chapter II

# Some crashes on Parabolic control problems

### II.1 General settings

In this chapter, we shall give an overall description about controllability of parabolic pdes and present some important results that will be useful in the next chapters.

Let  $Y$  and  $U$  be two complex Hilbert spaces and we consider the following (formal) control problem

$$\begin{cases} \partial_t y + \mathcal{A}y = \mathcal{B}v, & \text{in } (0, T), \\ y(0) = y_0. \end{cases} \quad (\text{II.1})$$

Normally, the initial data  $y_0$  is assumed to exhaust in  $Y$  and  $v \in U$  is a control force. Let us now consider the following setting.

- Suppose that  $\mathcal{A} : D(\mathcal{A}) \subset Y \rightarrow Y$  is some unbounded operator such that  $-\mathcal{A}$  generates a strongly continuous semigroup in  $Y$ , denote it by  $(e^{-t\mathcal{A}})_{t \geq 0}$ .

Sometimes we need the semigroup to be analytic that means there exists an analytic extension  $z \mapsto e^{-z\mathcal{A}}$  in a sector in the complex plane of the form

$$S_\delta = \{z \in \mathbb{C} : \Re(z) \geq 0, |\Im(z)| \leq \delta \Re(z)\},$$

for some  $\delta > 0$ . This property always holds in the case of parabolic systems.

The semigroup for the adjoint operator  $-\mathcal{A}^*$  will be denoted by  $(e^{-t\mathcal{A}^*})_{t \geq 0}$ .

- $\mathcal{B} : U \rightarrow D(\mathcal{A}^*)'$  is the control operator. In the theory of parabolic control, actually it is easy and convenient to work with the operator  $\mathcal{B}^* : D(\mathcal{A}^*) \rightarrow U$  (we identify  $U$  with its dual).
- The control operator  $\mathcal{B}$  is admissible in the following sense: assume that there exists a Hilbert space  $X \hookrightarrow Y$ , continuously and densely embedded in  $Y$ , such that

$$(t \mapsto \mathcal{B}^* e^{-t\mathcal{A}^*} \zeta) \in L^2(0, T; U) \quad \forall \zeta \in X, \quad (\text{II.2})$$

and moreover, there exists a constant  $C > 0$  such that

$$\int_0^T \|\mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \zeta\|_U^2 dt \leq C \|\zeta\|_X^2 \quad \forall \zeta \in X. \quad (\text{II.3})$$

In that case, one may think  $X \hookrightarrow Y \hookrightarrow X'$ , that is we consider the dual space  $X'$  with its representation obtained by using  $Y$  as a pivot space.

Let us give a suitable meaning of the general control problem (II.1) in a dual sense as [48, 96].

**Theorem II.1.1 (Well-posedness in a dual sense).** *For any  $y_0 \in Y$  and  $v \in L^2(0, T; U)$ , there exists a unique  $y \in \mathcal{C}^0([0, T]; X') \cap L^2(0, T; Y)$ , solution to (II.1) in the following sense: for any  $t \in [0, T]$  and  $\zeta \in X$  we have*

$$\langle y(t), \zeta \rangle_{X', X} - \langle y_0, e^{-tA^*} \zeta \rangle_Y = \int_0^t \langle v(s), \mathcal{B}^* e^{-(t-s)A^*} \zeta \rangle_U ds. \quad (\text{II.4})$$

Moreover, if  $X$  is stable by the semigroup generated by  $\mathcal{A}^*$ , the above definition can be extended for any initial data  $y_0 \in X'$ .

One can observe here the importance of considering the adjoint problem (without control) to (II.1) which is backward in time,

$$\begin{cases} -\partial_t q + \mathcal{A}^* q = 0, & \text{in } (0, T) \\ q(T) = \zeta. \end{cases} \quad (\text{II.5})$$

**Examples: (scalar) distributed and boundary control.** Let  $\Omega \subset \mathbb{R}^d$  be a non-empty open, connected, and smooth domain. Also assume  $\omega \subset \Omega$  be a non-empty open set and  $\Gamma_0$  be a non-empty open subset of the boundary  $\partial\Omega$ . Of course, for one dimension, that is if one takes  $\Omega = (0, 1)$  then  $\Gamma_0 = \{0\}$ , or  $\{1\}$ . We further take the state space  $Y = L^2(\Omega)$  and the operator

$$\mathcal{A} = -\operatorname{div}(\gamma \nabla), \text{ with its domain } D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega),$$

where  $\gamma : \Omega \rightarrow M_d(\mathbb{R})$  be a lipschitz continuous bounded

field of symmetric matrices which are uniformly coercive: there is a  $\gamma_{\min} > 0$  such that

$$(\gamma(x)\xi, \xi) \geq \gamma_{\min} |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad \forall x \in \Omega.$$

- *A distributed parabolic control problem:* Consider the following problem

$$\begin{cases} \partial_t y + \mathcal{A}y = \mathbb{1}_\omega v, & \text{in } (0, T) \times \Omega, \\ y = 0, & \text{on } (0, T) \times \partial\Omega, \\ y(0) = y_0, & \text{in } \Omega. \end{cases} \quad (\text{II.6})$$

Here,  $U = L^2(\omega)$  and the control operator reads  $\mathcal{B} = \mathbb{1}_\omega$ . In fact, one has the observation operator  $\mathcal{B}^* = \mathbb{1}_\omega$  also.

- *A boundary parabolic control problem:* Let us now consider the following Dirichlet boundary control problem

$$\begin{cases} \partial_t y + \mathcal{A}y = 0, & \text{in } (0, T) \times \Omega, \\ y = \mathbb{1}_{\Gamma_0} v, & \text{on } (0, T) \times \partial\Omega, \\ y(0) = y_0, & \text{in } \Omega, \end{cases} \quad (\text{II.7})$$

Here, we have  $U = L^2(\Gamma_0)$ . In the case of a boundary control problem, we must say that it is not trivial to find the control operator  $\mathcal{B}$ , although we will discuss about it later. But multiplying the state equation by the solution of its adjoint problem and then through a formal integration by parts, it is not difficult to obtain the observation operator

$$\mathcal{B}^* = -\mathbb{1}_{\Gamma_0} \frac{\partial}{\partial \nu}, \quad (\text{II.8})$$

where  $\nu$  is the normal vector on  $\partial\Omega$ .

To define  $\mathcal{B}^*$  above and in fact, for the well-posedness of (II.7) in the sense of Theorem II.1.1, we take the space  $X = H_0^1(\Omega)$ , so that the solution to the adjoint system (II.5)  $q \in \mathcal{C}^0([0, T]; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$  (due to the standard regularity result), for any data  $\zeta \in H_0^1(\Omega)$ , which leads that

$$\mathcal{B}^* q = -\mathbb{1}_{\Gamma_0} \frac{\partial q}{\partial \nu} \in L^2(0, T; H^{\frac{1}{2}}(\partial\Omega)).$$

## II.2 Controllability issues

In this section, we shall discuss some controllability issues related to parabolic pdes. It is known to us that the exact controllability for parabolic pdes is not relevant due to the smoothing effects and we shall thus mainly focus on the cases of approximate and null-controllability.

**Definition II.2.1.**  $y_0 \in Y$  be an given initial data. We say that:

- The problem (II.1) is **approximately controllable from**  $y_0$ , if for any given time  $T > 0$ , a target data  $y_T \in Y$ , and  $\varepsilon > 0$ , there exists a control  $v : (0, T) \rightarrow U$  such the corresponding solution  $y$  to (II.1) satisfies

$$\|y(T) - y_T\|_Y \leq \varepsilon.$$

If this property holds for any  $y_0 \in Y$ , we simply say that the system is **approximately controllable**.

- (II.1) is **null-controllable from**  $y_0$ , if for any given time  $T > 0$ , there exists a control  $v : (0, T) \rightarrow U$ , such that the corresponding solution  $y$  to (II.1) satisfies

$$y(T) = 0.$$

If this property holds for any  $y_0$ , we simply say that the system is **null-controllable**.

We will now present some important criteria for a parabolic system to be approximately or null-controllable.

### II.2.1 Approximate controllability

**Theorem II.2.2 (Fattorini-Hautus test).** Assume that:

- $\mathcal{A}^*$  has a compact resolvent and a complete system of root vectors.
- $\mathcal{B}^*$  is a bounded operator from  $D(\mathcal{A}^*)$  (with the graph norm) into  $U$ .

We also assume that the semigroup generated by  $-\mathcal{A}^*$  is analytic. Then, our system (II.1) is approximately controllable at time  $T > 0$  if and only if we have

$$\text{Ker } \mathcal{B}^* \cap \text{Ker}(\mathcal{A}^* - \lambda I) = \{0\}, \quad \forall \lambda \in \mathbb{C}.$$

In particular, the approximate controllability property does not depend on  $T$ .

The above theorem is actually a general version of the original one proved by Fattorini (see [55]), and for the above version we refer to [89].

### II.2.2 Null-controllability

Let us now move forward to the null-controllability for our general system (II.1). In this context, we recall the definition (II.4) of solution  $y$  to (II.1) and we write the following proposition concerning null-controllability.

**Proposition II.2.3.** Let any  $y_0 \in Y$  and any finite time  $T > 0$  be given. Then a function  $v \in L^2(0, T; U)$  is said to be a null-control at time  $T$  for the problem (II.1) if and only if it satisfies: for any  $\zeta \in X$ , that

$$-\langle y_0, e^{-T\mathcal{A}^*} \zeta \rangle_Y = \int_0^T \langle v(s), \mathcal{B}^* e^{-(T-s)\mathcal{A}^*} \zeta \rangle_U ds. \quad (\text{II.9})$$

Let us now write the very well-known theorem regarding null-controllability in terms of observability criteria.

**Theorem II.2.4 (Null-controllability and Observability).** *The system (II.1) is null-controllable in  $Y$  at time  $T > 0$  if and only if the adjoint system (II.5) satisfies the following observability property with respect to the observation operator  $\mathcal{B}^*$ , namely: there exists a constant  $C > 0$  such that for any  $\zeta \in X$ , the corresponding solution  $q_\zeta$  to the adjoint system (II.5) satisfies*

$$\|q_\zeta(0)\|_Y^2 \leq C \int_0^T \|\mathcal{B}^* q_\zeta(t)\|_U^2 dt.$$

In semigroup notation, the observability inequality follows

$$\|e^{-T\mathcal{A}^*} \zeta\|_Y^2 \leq C \int_0^T \|\mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \zeta\|_U^2 dt, \quad \forall \zeta \in X.$$

### II.2.2.1 One dimensional case - the method of moments

In this section, we shall give an overall idea about the method of moments, mainly with the Dirichlet boundary conditions. This technique is quite useful to construct a null-control by hand mainly in one-dimensional setting. In fact, as we know that the *boundary null-controllability* of parabolic systems with less number of controls than the equations is troublesome to handle in various situations and the very powerful *Carleman technique* is often inefficient in that context. This is why the *moments approach* is really helpful to tackle the boundary control systems at least in 1D.

We give here a short description of the moments technique mainly for the scalar control problem. But in the next chapters we will indeed use this technique for some  $2 \times 2$  and  $3 \times 3$  parabolic systems, obviously with less number of control(s) than the equations.

Let us take the interval  $(0, 1) \subset \mathbb{R}$  and consider the following unbounded elliptic operator

$$\mathcal{A} = -\partial_x(\gamma \partial_x) : D(\mathcal{A}) = H^2(0, 1) \cap H_0^1(0, 1) \longrightarrow L^2(0, 1), \quad (\text{II.10})$$

with  $\gamma \in \mathcal{C}^1([0, 1])$  and  $\inf_{[0, 1]} \gamma = \gamma_{\min} > 0$ . We observe the following facts.

1.  $\mathcal{A}^* = \mathcal{A}$  has compact resolvent and the spectrum  $\Lambda$  of which is unbounded but locally finite and made of positive real eigenvalues. Moreover, one can show that there exists some constant  $C > 0$  independent on the eigenvalues such that:

- the set of eigenvalues satisfies a suitable gap condition

$$|\lambda - \tilde{\lambda}| \geq C\sqrt{\lambda}, \quad \forall \lambda, \tilde{\lambda} \in \Lambda, \quad \lambda > \tilde{\lambda}, \quad \text{and} \quad (\text{II.11})$$

- the counting function  $\mathcal{N} : (0, +\infty) \rightarrow (0, +\infty)$  corresponding to  $\Lambda$ , defined by

$$\mathcal{N}(r) = \#\{\lambda \in \Lambda : \lambda \leq r\}, \quad \forall r > 0,$$

satisfies

$$\mathcal{N}(r) \leq C\sqrt{r}, \quad \forall r > 0. \quad (\text{II.12})$$

For the proof of the above two facts we refer to [32, Theorem IV.1.3].

2. We further have that the set of eigenfunctions  $\{\phi_\lambda\}_{\lambda \in \Lambda}$  of  $\mathcal{A}^*$  defines an orthonormal basis in  $L^2(0, 1)$ .

**Remark II.2.5.** *In fact, for the moments approach what important is that the family of eigenfunctions  $\{\phi_\lambda\}_{\lambda \in \Lambda}$  of the adjoint elliptic operator (associated with the parabolic system) should at least define a complete family in  $L^2$  with some suitable upper bound of  $\|\phi_\lambda\|_{L^2}$ . In addition, we also need some proper lower bounds of the observation terms  $\mathcal{B}^* \phi_\lambda$ , for  $\lambda \in \Lambda$ .*

**Remark II.2.6** (1D Laplace operator). *For the case when  $\mathcal{A} = -\partial_x^2$  with the domain as defined in (II.10), the set of eigenvalues has explicit expressions as follows:*

$$\Lambda = \{(k+1)^2 \pi^2 : k \geq 0\},$$

*with the eigenfunctions  $\phi_\lambda(x) = \sin(\sqrt{\lambda}x)$ ,  $\forall \lambda \in \Lambda$ ,  $\forall x \in [0, 1]$ .*

– **The moments problem:** Let us now formulate the set of moments problem.

We suppose here that  $y_0 \in L^2(0,1)$  be any given initial data and  $\mathcal{A}$  is as prescribed in (II.10). Then, under the assumption in point 2 before, we can say that it is enough to check the equation (II.9) for  $\phi_\lambda, \forall \lambda \in \Lambda$ . This indeed tells us, for any given data  $y_0 \in L^2(0,1)$ , a function  $v \in L^2(0,T;U)$  is a null-control for the system (II.1) if and only if we have

$$-(y_0, e^{-T\lambda} \phi_\lambda)_{L^2} = \int_0^T \langle v(t), e^{-(T-t)\lambda} \mathcal{B}^* \phi_\lambda \rangle_U dt, \quad \forall \lambda \in \Lambda. \quad (\text{II.13})$$

The above set of equations is called the moments problem from which one needs to find a control  $v$ .

- *Distributed control case:* We consider the control in a non-empty open subset  $\omega \subset (0,1)$  so that in particular,  $\mathcal{B}^* = \mathbb{1}_\omega$  (as introduced in (II.6)) and  $U = L^2(\omega)$ . In this case, the moments equations (II.13) becomes

$$-(y_0, e^{-T\lambda} \phi_\lambda)_{L^2} = \int_0^T \int_\omega v(t, x) e^{-(T-t)\lambda} \phi_\lambda(x) dx dt, \quad \forall \lambda \in \Lambda, \quad (\text{II.14})$$

and we expect here a control  $v \in L^2((0,T) \times \omega)$ .

- *Boundary control case:* We now want to consider a control  $v$  on  $\Gamma_0 = \{0\}$  for the boundary control problem (II.7) in 1D. In this case  $U = \mathbb{R}$  and we are seeking for a control  $v \in L^2(0,T)$ . The observation operator  $\mathcal{B}^* = \mathbb{1}_{\{x=0\}} \frac{\partial}{\partial x}$  (since at the left end point  $\frac{\partial}{\partial v} = -\frac{\partial}{\partial x}|_{x=0}$ ), so that the moments equation read here

$$-(y_0, e^{-T\lambda} \phi_\lambda)_{L^2} = \int_0^T v(t) e^{-(T-t)\lambda} \phi'_\lambda(0) dt, \quad \forall \lambda \in \Lambda, \quad (\text{II.15})$$

Now, to obtain a control  $v$  with a suitable bound in  $L^2(0,T;U)$ , we need some useful facts which we present in terms of the following theorems.

**Theorem II.2.7 (Observation estimates).** *Let the operator  $\mathcal{A}$  be defined in (II.10). Also, recall the set of eigenfunctions  $\{\phi_\lambda\}_{\lambda \in \Lambda}$  of  $\mathcal{A}^* = \mathcal{A}$  and the set  $\omega \subset (0,1)$ , as introduced before. Then, there exists some constants  $C_1(\gamma, \omega) > 0$  and  $C_2(\gamma) > 0$  such that*

$$\|\phi_\lambda\|_{L^2(\omega)} \geq C_1(\gamma, \omega), \quad \forall \lambda \in \Lambda, \quad (\text{II.16a})$$

$$|\phi'_\lambda(0)| \geq C_2(\gamma) \sqrt{\lambda}, \quad \forall \lambda \in \Lambda. \quad (\text{II.16b})$$

For a proof of the above theorem, we refer to [32, Theorem IV.1.3]. In fact, for a more general boundary conditions: namely the Robin one, the above estimates (slightly different) have been improved in [26], and this will be detailed in Chapter III.

**Remark II.2.8 (Approximate controllability).** *Using the above theorem we have, for any  $\lambda \in \Lambda$ , that*

- $\mathcal{B}^* \phi_\lambda = \mathbb{1}_\omega \phi_\lambda \neq 0$  in  $\omega \subset (0,1)$ , for the distributed control problem (II.6) in 1D;
- also for the boundary control problem (II.7)  $\mathcal{B}^* \phi_\lambda = \phi'_\lambda(0) \neq 0$ .

This ensures that both the control problems (II.6) and (II.7) are approximately controllable in 1D setting.

– **Existence of a bi-orthogonal family to the exponentials:** The next crucial ingredient to solve the moments problem is the existence of a bi-orthogonal family in  $L^2(0, T)$  to the set of exponential functions  $\{e^{-t\lambda}\}_{\lambda \in \Lambda} \subset L^2(0, T)$ , for some  $T > 0$ .

Initially, Fattorini and Russell proved the existence of a bi-orthogonal family to  $\{e^{-tk^2\pi^2}\}_{k \geq 1}$  in [56, 57] with some suitable bounds. Later on in [58], it has been nicely developed to the family  $\{e^{-\sigma_k t}, te^{-\sigma_k t}\}_{k \geq 1}$  for a complex sequence  $\{\sigma_k\}_{k \geq 1} \subset \mathbb{C}$  with non-decreasing modulus and then a more general version has been achieved in [9] to the family  $\{t^j e^{-\sigma_k t}\}_{0 \leq j \leq \eta-1, k \geq 1}$  where  $\eta \geq 1$  is a positive integer.

In those above references, the estimates of the bi-orthogonal functions does not allow to obtain a control cost while solving the moments problem. But a new achievement has been drawn in [21, Theorem 1.5] where the authors proved a refined version of the bi-orthogonal estimates that allows us to attain an appropriate *control cost*. The proof is based on a proper gap condition of  $|\sigma_k - \sigma_n|$  for all  $k \neq n$  and some property of the counting function associated with  $(\sigma_k)_{k \geq 0}$  which has been introduced by point 5 and 6 of their proof. In fact, concerning this hypothesis on the counting function, slightly more general version has been introduced in [3, Remark 4.3].

In this context, we must mention that the authors in [39] and [38] proved the existence of some suitable bi-orthogonal families to obtain a *control cost*, respectively for weakly and strongly degenerate parabolic equations. Moreover, a very recent theory concerning the existence of bi-orthogonal family has been achieved by them in [40] when the pairs of eigenvalues condensate and their main goal is to obtain a sharp estimate of the control cost for a degenerate parabolic equation.

In this manuscript, we mainly use the following theorem which is a combined version of [21, Theorem 1.5] and [3, Remark 4.3].

**Theorem II.2.9 (Bi-orthogonal family to the exponentials for finite  $T > 0$ ).** *Let  $\{\sigma_k\}_{k \geq 0} \subset \mathbb{C}$  be a sequence of complex numbers fulfilling the following assumptions:*

1.  $\sigma_k \neq \sigma_n$  for all  $k, n \in \mathbb{N} \cup \{0\}$ , with  $k \neq n$ ;
2.  $\Re(\sigma_k) > 0$ , for all  $k \geq 0$  and there exists some constant  $c > 0$ , independent on  $k \geq 0$ , such that

$$|\Im(\sigma_k)| \leq c\sqrt{\Re(\sigma_k)}, \quad \forall k \geq 0;$$

3.  $\{\sigma_k\}_{k \geq 0}$  is non-decreasing in modulus, that is

$$|\sigma_{k+1}| \geq |\sigma_k|, \quad \forall k \geq 0;$$

4.  $\{\sigma_k\}_{k \geq 0}$  satisfies the following gap condition: there exists some  $\rho > 0$  and  $p \in \mathbb{N}$  such that

$$\begin{cases} |\sigma_k - \sigma_n| \geq \rho|k^2 - n^2|, & \forall k, n : |k - n| \geq p, \\ \inf_{k \neq n : |k - n| < p} |\sigma_k - \sigma_n| > \rho; \end{cases}$$

5. for some  $\theta > 0$  and  $c_{\min}, c_{\max} > 0$ , the counting function  $\mathcal{N} : (0, +\infty) \rightarrow (0, +\infty)$  corresponding to the set  $\{\sigma_k\}_{k \geq 0}$ ,

$$\mathcal{N}(r) = \#\{k : |\sigma_k| \leq r\}, \quad \forall r > 0,$$

satisfies the following inequality

$$-\theta + c_{\min}\sqrt{r} \leq \mathcal{N}(r) \leq \theta + c_{\max}\sqrt{r}, \quad \forall r > 0;$$

Then, there exists  $T_0 > 0$  such that, for every  $\eta \in \mathbb{N} \cup \{0\}$  and  $0 < T \leq T_0$ , we can find a family of complex valued functions

$$\{q_{k,j}\}_{k \geq 0, 0 \leq j \leq \eta} \subset L^2(0, T),$$

bi-orthogonal to  $\{(T-t)^j e^{-\sigma_k(T-t)}\}_{k \geq 0, 0 \leq j \leq \eta}$ , that is,

$$\int_0^T q_{k,j}(t)(T-t)^i e^{-\sigma_l(T-t)} dt = \delta_{k,l} \delta_{i,j}, \quad \forall k, l \geq 0, \quad \forall i, j \in \{0, \dots, \eta\},$$

with in addition,

$$\|q_{k,j}\|_{L^2(0,T)} \leq C e^{C\sqrt{\Re(\sigma_k)} + \frac{C}{T}}, \quad \text{for any } k \geq 0, \quad 0 \leq j \leq \eta,$$

with a constant  $C := C(c, \rho, \theta, c_{\min}, c_{\max}) > 0$  which does not depend on the choice of sequence  $\{\sigma_k\}_{k \geq 0}$ .

– **Null-controllability:** Using the above results we can now prove the null-controllability of the control problems (II.6) and (II.7) in 1D setting.

Before going to introduce the main theorems concerning null-controllability, we first observe that the set of eigenvalues  $\Lambda \subset \mathbb{R}$  of  $\mathcal{A}^*$  satisfies all the required assumptions in Theorem II.2.9.

But, we must say that the gap condition in point 4 is not so clear for the set  $\Lambda$ . For that, let us recall the Rayleigh quotient  $R$  associated with the eigenvalue problem  $\mathcal{A}\phi = \lambda\phi$ , with  $\mathcal{A}$  as in (II.10) ( $\lambda \in \mathbb{R}$ , since  $\mathcal{A}$  is self-adjoint),

$$R(\phi) = \frac{\int_0^1 \gamma(x) |\partial_x \phi(x)|^2 dx}{\int_0^1 |\phi(x)|^2 dx}, \quad \forall \phi \in H_0^1(0, 1),$$

Now, we have that the set of eigenvalues for the operator  $-\partial_x^2$  with homogeneous Dirichlet condition, is  $\{(k+1)^2 \pi^2 : k \geq 0\}$  (recall the Remark II.2.6), so that if we denote the set of eigenvalues of  $\mathcal{A}$  by  $\Lambda = \{\lambda_{k,\gamma}\}_{k \geq 0}$ , then one has

$$\gamma_{\max}(k+1)^2 \pi^2 = \lambda_{k,\gamma_{\max}} \geq \lambda_{k,\gamma} \geq \lambda_{k,\gamma_{\min}} = \gamma_{\min}(k+1)^2 \pi^2, \quad \forall k \geq 0, \quad (\text{II.17})$$

thanks to the fact that

$$\lambda_{k,\gamma} = \inf_{\substack{X_{k+1} \subset H_0^1(0,1) \\ \dim X_{k+1} = k+1}} \sup_{\phi \in X_{k+1}} R(\phi), \quad \forall k \geq 0.$$

Thus, using the fact (II.17) in the inequality (II.11), we obtain, for some  $C_\gamma > 0$ , that

$$|\lambda_{k+1,\gamma} - \lambda_{k,\gamma}| \geq C \sqrt{\lambda_{k+1,\gamma}} \geq C_\gamma(k+1), \quad \forall k \geq 0,$$

from which it is not difficult to obtain that

$$|\lambda_{k,\gamma} - \lambda_{n,\gamma}| \geq C_\gamma |k^2 - n^2|, \quad \forall k, n : |k - n| \geq 1.$$

1. The condition on counting functions is also fulfilled due to (II.12).
2. Other assumptions are trivially satisfied for the set  $\Lambda$ .

Hence, there is a  $T_0 > 0$  such that for every  $0 < T \leq T_0$ , there exists a family  $\{q_\lambda\}_{\lambda \in \Lambda} \subset L^2(0, T)$  bi-orthogonal to  $\{e^{-(T-t)\lambda}\}_{\lambda \in \Lambda} \subset L^2(0, T)$ , satisfying the following estimate

$$\|q_\lambda\|_{L^2(0,T)} \leq C e^{C\sqrt{\lambda} + \frac{C}{T}}, \quad \forall \lambda \in \Lambda, \quad \text{for } 0 < T \leq T_0. \quad (\text{II.18})$$

Let us prove the controllability theorems for both the distributed and boundary cases.



**Theorem II.2.10 (Distributed null-controllability).** *Let any non-empty open  $\omega \subset (0, 1)$ ,  $y_0 \in L^2(0, 1)$  and finite time  $T > 0$  be given. Then there exists a null-control  $v \in L^2((0, T) \times \omega)$  to the problem (II.6) in 1D and moreover, it enjoys the estimate*

$$\|v\|_{L^2((0,T) \times \omega)} \leq Ce^{C/T} \|y_0\|_{L^2(0,1)},$$

where the constant  $C > 0$  does not depend on  $T > 0$ .

*Proof.* We consider

$$\begin{aligned} v(t, x) &= \sum_{\lambda \in \Lambda} v_\lambda(t, x), \quad \forall (t, x) \in (0, T) \times \omega, \quad \text{with} \\ v_\lambda(t, x) &= -e^{-T\lambda} (y_0, \phi_\lambda)_{L^2(0,1)} \frac{\phi_\lambda(x)}{\|\phi_\lambda\|_{L^2(\omega)}^2} q_\lambda(t), \quad \forall (t, x) \in (0, T) \times \omega. \end{aligned}$$

With this above  $v$ , we see that the moments equation (II.14) is formally satisfied.

Define

$$\tilde{q}_\lambda(t, x) := \frac{\phi_\lambda(x)}{\|\phi_\lambda\|_{L^2(\omega)}^2} q_\lambda(t), \quad \forall (t, x) \in (0, T) \times (0, 1),$$

so that the family  $\{\tilde{q}_\lambda\}_{\lambda \in \Lambda}$  is bi-orthogonal to the family  $\{e^{-(T-t)\lambda} \mathbf{1}_\omega \phi_\lambda\}_{\lambda \in \Lambda}$  in  $L^2((0, T) \times \omega)$ .

Now, apply the following fact (using Young's inequality)

$$C\sqrt{\lambda} \leq \frac{T}{2}\lambda + \frac{C^2}{2T}, \quad (\text{II.19})$$

in the estimate of  $\|q_\lambda\|_{L^2(0,T)}$ , given by (II.18), we obtain the bound of  $v_\lambda$  as follows

$$\|v_\lambda\|_{L^2((0,T) \times \omega)} \leq Ce^{C/T} \frac{e^{-\frac{T}{2}\lambda}}{\|\phi_\lambda\|_{L^2(\omega)}} \|y_0\|_{L^2(0,1)} \|\phi_\lambda\|_{L^2(0,1)}, \quad \forall \lambda \in \Lambda, \text{ for } 0 < T \leq T_0.$$

But in our case,  $\|\phi_\lambda\|_{L^2(0,1)} = 1$ ; also thanks to the lower bound (II.16a) and the relation (II.17), we finally obtain

$$\begin{aligned} \|v\|_{L^2((0,T) \times \omega)} &= \sum_{\lambda \in \Lambda} \|v_\lambda\|_{L^2((0,T) \times \omega)} \leq Ce^{C/T} \|y_0\|_{L^2(0,1)} \sum_{k \geq 0} e^{-\gamma_{\min} \pi^2 (k+1)^2 \frac{T}{2}} \\ &\leq Ce^{C/T} \left(1 + \frac{1}{T}\right) \|y_0\|_{L^2(0,1)}, \quad \text{for } 0 < T \leq T_0. \end{aligned}$$

Thus, for  $0 < T \leq T_0$  we obtained a distributed null-control  $v$  to the system (II.6) in 1D, satisfying the above estimate. The case  $T > T_0$  is actually reduced to the previous one. Indeed, a continuation by zero of a control on  $(0, T_0)$  is a control on  $(0, T)$  for any  $T > T_0$  and the same estimate follows from the decrease of the control cost  $Ce^{C/T}$  with respect to time.

Hence, the proof is complete.  $\square$

**Theorem II.2.11 (Boundary null-controllability).** *Let any  $y_0 \in L^2(0, 1)$  and finite time  $T > 0$  be given. Then there exists a null-control  $v \in L^2(0, T)$  to the problem (II.7) in 1D, which is to be applied on the left boundary point of  $(0, 1)$ , that is on  $\Gamma_0 = \{0\}$ , and moreover, it enjoys the estimate*

$$\|v\|_{L^2(0,T)} \leq Ce^{C/T} \|y_0\|_{L^2(0,1)},$$

where the constant  $C > 0$  does not depend on  $T > 0$ .

*Proof.* We start by considering

$$v(t) = \sum_{\lambda \in \Lambda} v_\lambda(t), \quad \forall t \in (0, T) \quad \text{with}$$

$$v_\lambda(t) = -\frac{e^{-T\lambda}}{\phi'_\lambda(0)} (y_0, \phi_\lambda)_{L^2(0,1)} q_\lambda(t), \quad \forall t \in (0, T).$$

Observe that, the moments equation is formally satisfied with this choice of  $v$ .

The required  $L^2(0, T)$  bound of this control follows from the lower bound of the observation terms  $|\phi'_\lambda(0)|$  in (II.16b) and the bound of the bi-orthogonal estimates (then using the trick (II.19) as previous case); we are not providing all the detailed steps for this case.

The sketch of the proof is complete.  $\square$

**Remark II.2.12.** *We mainly use the moments method to solve some coupled parabolic systems with less number of boundary controls than the state equations. And in these cases, we shall see later in next chapters that it is no more straightforward to obtain all the required results for construction of a control via moments technique. This is mainly due to the complexity of the spectral analysis of the corresponding adjoint elliptic operator (vectorial). In fact, in Chapter IV we shall deal with some non self-adjoint operator that has certain number of complex eigenvalues without having any explicit formulation.*

**Remark II.2.13.** *We must mention that a very recent result has been developed in [22]: a “block moments technique” to tackle the situations where a spectral condensation occurs. Indeed in Chapter III, we will cope with some such situation where we will observe a lack of having uniform spectral gap for certain blocks of eigenvalues and in that case we will make use of the block moments technique to construct a control; we will go into detail in Section III.5.2.2 of Ch. III.*

### II.2.3 The penalized HUM approach

In this section, we give a short description of HUM and its penalized version (which has been initially achieved in [63] by R. Glowinski and J.-L. Lions) and we mainly make use of this in Section IV.5 of Chapter IV. Before going to the detail, we declare that the results will be incorporated here from the work of F. Boyer [31], but with a more general functional setting.

Let us recall the formal control system (II.1) and the spaces  $Y, X, X', U$  from Section II.1. For any  $\delta \geq 0$  and given  $y_0 \in Y$ , we define the closed, convex set (can be empty),

$$\text{Adm}(y_0, \delta) := \left\{ v \in L^2(0, T; U) : \|y_{v, y_0}(T)\|_{X'} \leq \delta \right\}.$$

where  $y_{v, y_0}$  is the weak solution to the system (II.1) with initial data  $y_0 \in Y$  and input  $v \in L^2(0, T; U)$  in a dual sense, see Theorem II.1.1.

We know that if one control (null or approximate) exists, then certainly it is not the unique one. But by the HUM approach (which is finding a control with minimal  $L^2$ -norm) one can get rid of that. To be more precise, we first define the following functional

$$F(v) := \frac{1}{2} \int_0^T \|v(t)\|_U^2 dt, \quad \forall v \in L^2(0, T; U). \quad (\text{II.20})$$

Then, the HUM approach is the following: for any  $\delta \geq 0$  such that  $\text{Adm}(y_0, \delta) \neq \emptyset$ , we suppose  $v_\delta$  be the unique minimizer of the above functional, that is to say

$$F(v_\delta) = \inf_{v \in \text{Adm}(y_0, \delta)} F(v).$$

**Definition II.2.14.** *Let us take  $\delta = 0$ . Then, if  $\text{Adm}(y_0, 0) \neq \emptyset$ , the unique minimizer  $v_0$  of the functional  $F(v)$  in  $\text{Adm}(y_0, 0)$ , is said to be the HUM (null) control of the system under study.*

Now, in practice we observe that even though the functional  $F$  has unique minimizer, it could be challenging to solve directly as it has been explained nicely in [31, Section 1]. In several situations, the functional  $F$  is used to be coercive in a very large abstract space which leads certain problems while using this approach for numerical purposes.

Because of the constraints, it is convenient to use the penalized version of the optimization problem (II.20). For any  $\epsilon > 0$ , we hereby define

$$F_\epsilon(v) := \frac{1}{2} \int_0^T \|v(t)\|_U^2 dt + \frac{1}{2\epsilon} \|y_{v,y_0}(T)\|_{X'}^2 \quad \forall v \in L^2(0, T; U) \quad (\text{II.21})$$

- Now, one can observe that for any  $\epsilon > 0$  the functional  $F_\epsilon$  is strictly convex, continuous and coercive. As a consequence there exists a unique minimizer say,  $v_\epsilon$  of the functional  $F_\epsilon$ .
- Moreover, the minimizer is characterized by the following Euler-Lagrange equation

$$\int_0^T \langle v_\epsilon(t), v(t) \rangle_U dt + \frac{1}{\epsilon} \langle y_{v_\epsilon, y_0}(T), y_{v,0}(T) \rangle_{X'} = 0 \quad \forall v \in L^2(0, T; U), \quad (\text{II.22})$$

where one can identify

$$\langle y_{v_\epsilon, y_0}(T), y_{v,0}(T) \rangle_{X'} = \langle y_{v_\epsilon, y_0}(T), S^{-1} y_{v,0}(T) \rangle_{X', X} \quad (\text{II.23})$$

and that operator  $S : X \rightarrow X'$  is defined as follows: for any given  $f \in X'$ , there exists unique element in  $X$ , we denote by  $S^{-1}f$  such that

$$\langle f, \zeta \rangle_{X', X} := \langle S^{-1}f, \zeta \rangle_X \quad \forall \zeta \in X. \quad (\text{II.24})$$

Using Fenchel-Rockafeller theory (see, for instance [52]), one can build an associated dual functional, say  $J_\epsilon$ , as follows: for any  $\epsilon > 0$ , we define

$$J_\epsilon(\zeta) = \frac{1}{2} \int_0^T \left\| \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \zeta \right\|_U^2 dt + \frac{\epsilon}{2} \|\zeta\|_X^2 + \langle y_0, e^{-T\mathcal{A}^*} \zeta \rangle_Y \quad \forall \zeta \in X. \quad (\text{II.25})$$

- The functional  $J_\epsilon$ , being strictly convex, continuous and coercive, has the unique minimizer in  $X$ , denote it by  $\zeta^\epsilon$ .
- The minimizer  $\zeta^\epsilon$  can be characterized by the following Euler-Lagrange equation

$$\int_0^T \left\langle \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \zeta^\epsilon, \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \zeta \right\rangle_U dt + \epsilon \langle \zeta^\epsilon, \zeta \rangle_X + \langle y_0, e^{-T\mathcal{A}^*} \zeta \rangle_Y = 0 \quad \forall \zeta \in X. \quad (\text{II.26})$$

We now explicitly write the relation between the two minimizers below.

**Proposition II.2.15.** *For any  $\epsilon > 0$ , the minimizers  $v_\epsilon$  and  $\zeta^\epsilon$  of the functionals  $F_\epsilon$  and  $J_\epsilon$  respectively, are related through the formula*

$$v_\epsilon(t) = \mathcal{B}^* e^{-(T-t)\mathcal{A}^*} \zeta^\epsilon \quad \text{for a.e. } t \in (0, T),$$

and

$$y_{v_\epsilon, y_0}(T) = -\epsilon S \zeta^\epsilon,$$

where the operator  $S : X \rightarrow X'$  has been defined in (II.24).

As a consequence, we have

$$\inf_{L^2(0, T; U)} F_\epsilon = F_\epsilon(v_\epsilon) = -J_\epsilon(\zeta^\epsilon) = -\inf_X J_\epsilon,$$

and finally,

$$\|y_{v_\epsilon, y_0}(T)\|_{X'} \leq \|y_{0, y_0}(T)\|_{X'}.$$

The proof of the above proposition can be deduced by following a similar approach as in [31, Proposition 1.5] and we omit that here. In fact, for a coupled parabolic system with a boundary control, we will rigorously prescribe the proof of this kind of result in Chapter IV, namely Proposition IV.5.1.

Hereafter, we write the approximate and null-controllability criteria for our control system in terms of the penalized HUM approach.

**Theorem II.2.16.** *Let  $v_\epsilon$  and  $y_{v_\epsilon, y_0}$  be as introduced earlier. Then, we have the following.*

- *Our control system (II.1) is approximately controllable at time  $T$  if and only if*

$$y_{v_\epsilon, y_0}(T) \rightarrow 0 \quad \text{in } X', \quad \text{as } \epsilon \rightarrow 0.$$

- *The system is null controllable at time  $T$  if and only if*

$$\mathcal{M}_{y_0}^2 := 2 \sup_{\epsilon > 0} \left( \inf_{L^2(0, T; U)} F_\epsilon \right) < +\infty.$$

*In that case, we have*

$$\|v_\epsilon\|_{L^2(0, T; U)} \leq \mathcal{M}_{y_0}, \quad \|y_{v_\epsilon, y_0}(T)\|_{X'} \leq \mathcal{M}_{y_0} \sqrt{\epsilon}.$$

*Moreover, the HUM control, denote by  $v_0$ , satisfies  $\|v_0\|_{L^2(0, T; U)} = \mathcal{M}_{y_0}$ , and*

$$v_\epsilon \xrightarrow{\epsilon \rightarrow 0} v_0 \quad \text{strongly in } L^2(0, T; U).$$

The sketch of the proof is similar to the reference [31, Theorem 1.7]. A version of Theorem II.2.16 for coupled parabolic system (with boundary control) will be proven in Chapter IV, namely Theorem IV.5.2.



## Chapter III

# Boundary null-controllability of $2 \times 2$ coupled parabolic systems with Robin conditions

### Abstract

The main goal of this chapter is to discuss the boundary controllability of some coupled parabolic systems in the cascade form in the case where the boundary conditions are of Robin type. In particular, we prove that the associated controls satisfy suitable uniform bounds with respect to the Robin parameters, that let us show that they converge towards a Dirichlet control when the Robin parameters go to infinity. This is a justification of the popular penalization method for dealing with Dirichlet boundary data in the framework of the controllability of coupled parabolic systems.

### III.1 Introduction

#### III.1.1 Statement of the problem

This chapter is concerned with the boundary null-controllability problem for linear coupled parabolic systems with less controls than equations. It is by now well-known that it can be a difficult problem in various situations and that there is still no complete theory in the literature. We will concentrate here on a particular case which is in the so-called cascade form and that can be written as follows

$$\begin{cases} \partial_t y_1 - \operatorname{div}(\gamma(x) \nabla y_1) = 0 & \text{in } (0, T) \times \Omega, \\ \partial_t y_2 - \operatorname{div}(\gamma(x) \nabla y_2) + y_1 = 0 & \text{in } (0, T) \times \Omega, \\ y_1(0, \cdot) = y_{0,1} & \text{in } \Omega, \\ y_2(0, \cdot) = y_{0,2} & \text{in } \Omega, \end{cases} \quad (\text{III.1})$$

where  $\Omega \subset \mathbb{R}^d$  be a  $\mathcal{C}^2$  bounded domain and  $\gamma : \Omega \rightarrow M_d(\mathbb{R})$  be a Lipschitz-continuous bounded field of symmetric matrices which are uniformly coercive: there is a  $\gamma_{\min} > 0$  such that

$$(\gamma(x)\xi) \cdot \xi \geq \gamma_{\min} |\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \quad \forall x \in \Omega.$$

We shall denote the control by  $v$  which will be acting only on the component  $y_1$  on some part  $\Gamma_0$  of the boundary  $\Gamma$  of the domain  $\Omega$ . Since we want to control both components of the system and  $v$  has no direct influence in the equation for  $y_2$ , the role of the coupling term  $y_1$  in the second equation is fundamental: it acts as an indirect control term. We refer for instance to the review paper [10] for a general presentation of different results on that topic.

The usually studied case is the one of a Dirichlet control, which means that the above system is supplemented with the following boundary conditions

$$\begin{cases} y_1 = \mathbb{1}_{\Gamma_0} v & \text{on } (0, T) \times \Gamma, \\ y_2 = 0 & \text{on } (0, T) \times \Gamma. \end{cases} \quad (\text{III.2})$$

In the present chapter, we would like to analyze the controllability properties of the same system when one considers, instead of the Dirichlet boundary conditions, a set of Robin boundary conditions with two non negative parameters  $\beta_1, \beta_2$

$$\begin{cases} \frac{\partial y_1}{\partial \nu_\gamma} + \beta_1 y_1 = \mathbb{1}_{\Gamma_0} v & \text{on } (0, T) \times \Gamma, \\ \frac{\partial y_2}{\partial \nu_\gamma} + \beta_2 y_2 = 0 & \text{on } (0, T) \times \Gamma, \end{cases} \quad (\text{III.3})$$

where the conormal derivative operator associated to the diffusion tensor  $\gamma$  is defined by

$$\frac{\partial}{\partial \nu_\gamma} = \nu \cdot (\gamma \nabla \cdot).$$

### III.1.2 Motivations and chapter organization

Our motivation for studying the above problem is two-fold.

The first one comes from the fact that it is an instance of the very popular penalization approach to deal with boundary condition that have never been studied, as far as we know, in the framework of the controllability of coupled parabolic systems. From a numerical point of view, for instance when considering a Galerkin approximation of an elliptic or parabolic equation, this approach consists in replacing a Dirichlet boundary condition  $y = g$  by a Robin boundary condition  $\frac{1}{\beta} \partial_{\nu_\gamma} y + y = g$ , with a large penalization parameter  $\beta$ . It generally induces more flexibility and robustness in the computational code. This approach is indeed proposed in many finite element libraries and software. Moreover, it is also a suitable way to deal with data  $g$  that have a regularity lower than the one expected to solve the problem in the usual energy spaces (typically if  $g \notin H^{1/2}(\Gamma)$  when solving the Laplace equation). This approach was initially studied in [17] for elliptic problems or in [20] for parabolic problems. In the framework of control theory, this penalization approach was for instance analyzed in [19, 43] for solving optimal control of elliptic equations. In each case, it is proven that the solution of the penalized problem actually converges to the one of the original problem, with some estimate of the rate of convergence.

Our motivation is thus to investigate the same kind of issues for the coupled parabolic system (III.1) with a single boundary control and in particular to show that, not only the problem (III.1)-(III.3) is null-controllable, but more importantly, that it is possible to prove estimates on the controls that are (in some sense that will be precised later) uniform with respect to the Robin (penalization) parameter. It will follow that the corresponding controlled solution converges towards a controlled solution of the Dirichlet problem when those parameters go to infinity.

Another motivation for this analysis, related to the discussion above, is that Robin boundary conditions have a regularizing effect on the boundary data. Indeed, as it will be recalled at the beginning of Section III.2, the functional analysis adapted to boundary controls in  $L^2$  for parabolic systems is a little intricate since, with such a low regularity of the data, we cannot expect solutions to exist in the usual energy space  $C^0([0, T], L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$  and to satisfy a standard weak formulation. Instead the solutions are known to live in a larger space  $C^0([0, T], H^{-1}(\Omega)) \cap L^2((0, T) \times \Omega)$ , the boundary condition being understood in a weak sense. When changing the Dirichlet boundary conditions into Robin (or Neumann) boundary conditions, the functional setting is more comfortable and we recover the expected regularity for weak solutions even if the boundary data is only in  $L^2$ .

Finally, we want to recall here that we lack of general mathematical techniques to deal with the controllability issue for those systems; applicability of the few available methods is very dependent on the structure of the underlying operators. Therefore, the analysis of each significantly new system needs to develop more elements (of spectral nature in our case) that are interesting by themselves and possibly useful in other situations. More precisely, there are not that many works regarding the controllability of coupled parabolic systems with less controls than equations, especially for boundary controls. This is mainly due to the fact that the very powerful *Carleman approach* is often inefficient in that context. In particular, most of the available controllability results concern the 1D setting since they are based on the moments method (that we will discuss below) which is not straightforward to implement in higher dimension. Among the few results available, we mention [9] where the authors proved the controllability to trajectories of a 1D system of  $n$  parabolic equations when  $m < n$  number of controls are exerted on a part of the boundary through Dirichlet conditions. They actually proved that a general Kalman condition is a necessary and sufficient controllability condition for this problem. In the multi-dimensional case, we quote [21, 3], where controllability results are obtained in particular cylindrical geometries by exploiting on the one hand a sharp estimate of the control for the associated 1D problem and on the other hand spectral Lebeau-Robbiano inequalities, see also the discussion in Section III.2.4.2. We also mention [2] where the null-controllability of some symmetric system of two wave-type equations has been analyzed in any space dimension with one control, provided that the control region satisfies the Geometric Control Condition. Indeed, the authors deduced similar results for some coupled parabolic and Schrödinger-type systems under the same geometric conditions.

**Chapter organization.** In Section III.2, we first recall the different notions of solutions for (III.1) with boundary conditions (III.2) or (III.3), that we will need in this chapter and we give the associated well-posedness and regularity results. In Section III.2.4, we give the precise statements of our main results. As mentioned above, those results essentially say that the coupled parabolic system with Robin boundary condition is null-controllable at any time  $T > 0$  and that we can find uniform bounds on the control that allow to justify the convergence towards a control for the Dirichlet problem when the Robin parameters are large. The proofs are given in Sections III.3, III.4 and III.5. They are based on the moments method [57] and on its recent extension called block moments method [22]; they require in particular a careful analysis of spectral properties of the underlying operators, with estimates uniform with respect to the parameters. Some of those spectral estimates are particularly difficult to obtain when the two Robin parameters are different, that is why in that case we restrict our analysis to a constant diffusion coefficient.

**Notations.** Throughout this chapter  $C$  or  $C'$  denotes a generic positive constant (that may vary from line to line) which does not depend on  $T$ ,  $y_0$  nor the parameters  $\beta_1, \beta_2$  but may depend on the diffusion coefficient  $\gamma$ . Sometimes, we will make emphasis on the dependence of a constant on some quantities  $\alpha_1, \alpha_2, \dots, \alpha_n$  ( $n \geq 1$ ) by  $C_{\alpha_1, \alpha_2, \dots, \alpha_n}$ .

By  $(\cdot, \cdot)_{L^2}$ , we denote the  $L^2$  inner product in  $\Omega$  and by  $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1}$ , we denote the duality pair between  $H^{-1}(\Omega)$  and  $H_0^1(\Omega)$ . As usual, the notation  $\| \cdot \|$  with in suffix the corresponding space, is used to denote the norm in the respective space. We choose  $\sigma$  as the trace variable on  $\Gamma$ .

Moreover, we shall use the following notation

$$((a, b)) := (\min\{a, b\}, \max\{a, b\}), \quad \text{for any } a, b \in \mathbb{R},$$

which is an open interval in  $\mathbb{R}$ .

The euclidean inner product in  $\mathbb{R}^d$ ,  $d \geq 1$ , will be simply denoted by  $\xi_1 \cdot \xi_2$  for any  $\xi_1, \xi_2 \in \mathbb{R}^d$ .



## III.2 General setting and main results

In this section, we will first discuss about the well-posedness for our parabolic system with Dirichlet and Robin boundary condition with  $L^2$  data. We will be particularly interested in estimates on the solutions that are uniform with respect to the Robin parameters. Then, we will give our main results concerning the associated control problems.

Let first study the scalar problem before discussing the coupled cascade system.

### III.2.1 The scalar problem

#### III.2.1.1 Dirichlet boundary data

We first recall the usual setting adapted to the analysis of the Dirichlet problem

$$\begin{cases} \partial_t y - \operatorname{div}(\gamma \nabla y) = f & \text{in } (0, T) \times \Omega, \\ y = g & \text{on } (0, T) \times \Gamma, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \end{cases} \quad (\text{III.4})$$

with non smooth data. In the case where  $g = 0$ , we can easily solve the above problem in a weak sense in  $\mathcal{C}^0([0, T], L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$  for given  $y_0 \in L^2(\Omega)$ . This can be done by using the continuous semigroup in  $L^2(\Omega)$  associated with the operator  $-A_D = \operatorname{div}(\gamma \nabla \cdot)$  with the domain  $D(A_D) = H^2(\Omega) \cap H_0^1(\Omega)$ . However, if one considers the case where  $g$  is any data in  $L^2((0, T) \times \Gamma)$  which is the usual framework in control theory, we cannot define as easily a good notion of weak solution because of a lack of regularity of the data. Instead, we have the following well-posedness result in a dual sense, see for instance [48, 96].

**Proposition III.2.1.** *For any  $y_0 \in L^2(\Omega)$ ,  $f \in L^2((0, T) \times \Omega)$ ,  $g \in L^2((0, T) \times \Gamma)$ , there exists a unique  $y \in \mathcal{C}^0([0, T], H^{-1}(\Omega)) \cap L^2((0, T) \times \Omega)$  solution of (III.4) in the following sense: for any  $t \in [0, T]$  and  $\zeta \in H_0^1(\Omega)$ , we have*

$$\begin{aligned} \langle y(t), \zeta \rangle_{H^{-1}, H_0^1} &= (y_0, e^{-tA_D^*} \zeta)_{L^2} + \int_0^t \int_{\Omega} f(s, x) (e^{-(t-s)A_D^*} \zeta)(x) dx ds \\ &\quad - \int_0^t \int_{\Gamma} g(s, \sigma) \frac{\partial}{\partial \nu_{\gamma}} (e^{-(t-s)A_D^*} \zeta)(\sigma) d\sigma ds. \end{aligned}$$

**Remark III.2.2.** *The operator  $A_D$  being self-adjoint, we could have replaced  $A_D^*$  by  $A_D$  in the previous statement but we prefer to keep it in order to be consistent with the non-scalar case that we will consider in Section III.2.3.*

#### III.2.1.2 Homogeneous Robin boundary data

For any  $\beta \in [0, +\infty)$ , we consider now the following parabolic problem

$$\begin{cases} \partial_t y - \operatorname{div}(\gamma \nabla y) = f & \text{in } (0, T) \times \Omega, \\ \frac{\partial y}{\partial \nu_{\gamma}} + \beta y = 0 & \text{on } (0, T) \times \Gamma, \\ y(0, \cdot) = y_0 & \text{in } \Omega, \end{cases} \quad (\text{III.5})$$

where the regularity of  $y_0$  and  $f$  will be precised below.

If the data are regular enough, the semigroup theory also gives a solution for this problem. Indeed, if one introduces the (self-adjoint) unbounded operator  $A_{\beta} = -\operatorname{div}(\gamma \nabla \cdot)$  in  $L^2(\Omega)$  associated with the domain

$$D(A_{\beta}) = \left\{ u \in H^2(\Omega) \mid \frac{\partial u}{\partial \nu_{\gamma}} + \beta u = 0 \text{ on } \Gamma \right\},$$

then we can prove that  $-A_{\beta}$  generates a continuous semigroup in  $L^2(\Omega)$ . Hence, the following result holds.

**Proposition III.2.3.** *Let  $\beta \in [0, +\infty)$  be given. For any  $y_0 \in D(A_\beta)$  and  $f \in \mathcal{C}^1([0, T], L^2(\Omega))$ , there exists a unique strong solution  $y \in \mathcal{C}^1([0, T], L^2(\Omega)) \cap \mathcal{C}^0([0, T], D(A_\beta))$  to (III.5), which is given by*

$$y(t) = e^{-tA_\beta} y_0 + \int_0^t e^{-(t-s)A_\beta} f(s) ds.$$

Moreover, this solution satisfies the energy estimates

$$\begin{aligned} \|y\|_{L^\infty(0,T,L^2(\Omega))} + \|y\|_{L^2(0,T,H^1(\Omega))} + \sqrt{\beta} \|y\|_{L^2((0,T) \times \Gamma)} \\ \leq C_T (\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T) \times \Omega)}), \end{aligned} \quad (\text{III.6})$$

and

$$\begin{aligned} \|y\|_{L^\infty(0,T,H^1(\Omega))} + \|\partial_t y\|_{L^2((0,T) \times \Omega)} + \|y\|_{L^2(0,T,H^2(\Omega))} + \sqrt{\beta} \|y\|_{L^\infty(0,T,L^2(\Gamma))} \\ \leq C_T (\|\nabla y_0\|_{L^2(\Omega)} + \sqrt{\beta} \|y_0\|_{L^2(\Gamma)} + \|f\|_{L^2((0,T) \times \Omega)}), \end{aligned}$$

where  $C_T > 0$  does not depend on  $\beta$ .

In particular, if  $y_0 \in D(A_\beta) \cap H_0^1(\Omega)$ , we have an estimate whose right-hand side does not depend on  $\beta$

$$\begin{aligned} \|y\|_{L^\infty(0,T,H^1(\Omega))} + \|\partial_t y\|_{L^2((0,T) \times \Omega)} + \|y\|_{L^2(0,T,H^2(\Omega))} + \sqrt{\beta} \|y\|_{L^\infty(0,T,L^2(\Gamma))} \\ \leq C_T (\|\nabla y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T) \times \Omega)}). \end{aligned} \quad (\text{III.7})$$

*Proof.* The existence of a unique strong solution is a standard result from semigroup theory, see for instance [53, Corollary 7.6]. We only sketch the proof of the estimates. The weak estimate (III.6) simply comes by multiplying the equation by  $y$  and using that

$$(A_\beta \zeta, \zeta)_{L^2} = \int_\Omega (\gamma \nabla \zeta)(x) \cdot \nabla \zeta(x) dx + \beta \int_\Gamma |\zeta(\sigma)|^2 d\sigma, \quad \forall \zeta \in D(A_\beta). \quad (\text{III.8})$$

To prove the stronger estimate, we first assume that  $y_0 \in D(A_\beta^2)$  and that  $f \in \mathcal{C}^1([0, T], D(A_\beta))$ , the final result being deduced by a density argument. With this regularity of the data we can justify that

$$\frac{1}{2} \frac{d}{dt} (A_\beta y(t), y(t))_{L^2} = (A_\beta y(t), \partial_t y(t))_{L^2} = -\|\partial_t y(t)\|_{L^2(\Omega)}^2 + (f(t), \partial_t y(t))_{L^2}, \quad \forall t \in [0, T].$$

Using the Cauchy-Schwarz inequality, and integrating in time, we get

$$(A_\beta y(t), y(t))_{L^2} + \int_0^t \|(\partial_t y)(s)\|_{L^2(\Omega)}^2 ds \leq (A_\beta y_0, y_0)_{L^2} + \int_0^t \|f(s)\|_{L^2(\Omega)}^2 ds.$$

By (III.8), it follows that

$$\begin{aligned} \|y\|_{L^\infty(0,T,H^1(\Omega))} + \|\partial_t y\|_{L^2((0,T) \times \Omega)} + \sqrt{\beta} \|y\|_{L^\infty(0,T,L^2(\Gamma))} \\ \leq C_T (\|\nabla y_0\|_{L^2(\Omega)} + \sqrt{\beta} \|y_0\|_{L^2(\Gamma)} + \|f\|_{L^2((0,T) \times \Omega)}). \end{aligned} \quad (\text{III.9})$$

It remains to prove the  $L^2(0, T, H^2(\Omega))$  estimate. To this end, we observe that

$$\|A_\beta y\|_{L^2((0,T) \times \Omega)} \leq \|f\|_{L^2((0,T) \times \Omega)} + \|\partial_t y\|_{L^2((0,T) \times \Omega)},$$

and thus the claim is just a consequence of (III.9) and of the following elliptic regularity property: there exists a  $C > 0$ , independent of  $\beta \in [0, +\infty)$ , such that

$$\|\zeta\|_{H^2(\Omega)} \leq C (\|\zeta\|_{L^2(\Omega)} + \|A_\beta \zeta\|_{L^2(\Omega)}), \quad \forall \zeta \in D(A_\beta).$$

This can be proved, for instance, as in [33, Theorems III.4.2 and III.4.3] and using the fact that  $\beta \geq 0$  to obtain a constant which is independent of  $\beta$ .  $\square$

### III.2.1.3 Non-homogeneous Robin boundary data

Let us now consider the same problem but with a non-homogeneous boundary data

$$\begin{cases} \partial_t y - \operatorname{div}(\gamma \nabla y) = f & \text{in } (0, T) \times \Omega, \\ \frac{\partial y}{\partial \nu_\gamma} + \beta y = g & \text{on } (0, T) \times \Gamma, \\ y(0, \cdot) = y_0 & \text{in } \Omega. \end{cases} \quad (\text{III.10})$$

The theory developed in [88] for this problem gives the following result concerning existence and uniqueness of a solution in the natural energy spaces.

**Proposition III.2.4.** *Let  $\beta \in [0, +\infty)$  be given. For any  $y_0 \in L^2(\Omega)$ ,  $f \in L^2((0, T) \times \Omega)$ ,  $g \in L^2((0, T) \times \Gamma)$ , there exists a unique weak solution  $y \in C^0([0, T], L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$  to (III.10) in the following sense:*

- $y(0) = y_0$ .
- For any test function  $\psi \in H^1(0, T, L^2(\Omega)) \cap L^2(0, T, H^1(\Omega))$ , and any  $t_1, t_2 \in [0, T]$  we have

$$\begin{aligned} & - \int_{t_1}^{t_2} \int_{\Omega} y \partial_t \psi \, dx \, dt + \int_{t_1}^{t_2} \int_{\Omega} (\gamma \nabla y) \cdot \nabla \psi \, dx \, dt + \beta \int_{t_1}^{t_2} \int_{\Gamma} y \psi \, d\sigma \, dt \\ & = \int_{\Omega} y(t_1) \psi(t_1) \, dx - \int_{\Omega} y(t_2) \psi(t_2) \, dx + \int_{t_1}^{t_2} \int_{\Omega} f \psi \, dx \, dt + \int_{t_1}^{t_2} \int_{\Gamma} g \psi \, d\sigma \, dt. \end{aligned} \quad (\text{III.11})$$

Moreover, it satisfies the estimate

$$\begin{aligned} & \|y\|_{C^0([0, T], L^2(\Omega))} + \|y\|_{L^2(0, T, H^1(\Omega))} + \|\partial_t y\|_{L^2(0, T, H^{-1}(\Omega))} \\ & \leq C_T (\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0, T) \times \Omega)} + \|g\|_{L^2((0, T) \times \Gamma)}), \end{aligned} \quad (\text{III.12})$$

for some  $C_T > 0$  independent of  $\beta$ .

**Remark III.2.5** (Strong estimates do not pass to the limit). *Note that if the boundary data for the Robin problem (III.10) is chosen in the form  $g = \beta g_D$  with  $g_D \in L^2((0, T) \times \Gamma)$ , then the boundary condition reads*

$$\frac{1}{\beta} \frac{\partial y}{\partial \nu_\gamma} + y = g_D, \quad \text{for } \beta \in (0, +\infty),$$

and we can formally expect the solution to converge, when  $\beta \rightarrow \infty$ , towards the one associated with the Dirichlet boundary condition  $y = g_D$ , that is to a solution of (III.4).

However, the estimate in the proposition above reads

$$\begin{aligned} & \|y\|_{C^0([0, T], L^2(\Omega))} + \|y\|_{L^2(0, T, H^1(\Omega))} + \|\partial_t y\|_{L^2(0, T, H^{-1}(\Omega))} \\ & \leq C_T (\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0, T) \times \Omega)} + \beta \|g_D\|_{L^2((0, T) \times \Gamma)}), \end{aligned}$$

which is not uniform with respect to  $\beta$  and therefore we cannot apriori prove that the associated solution  $y$  is bounded when  $\beta \rightarrow \infty$ . This is due to the fact that, considering only  $L^2$  boundary data, we cannot expect a uniform bound in  $L^2(0, T, H^1(\Omega))$  of the solution that would necessitate at least  $g_D$  to be in  $L^2(0, T, H^{1/2}(\Gamma))$ .

For the reasons above, we need to introduce a weaker formulation of the Robin problem that will allow to analyze the limit towards the Dirichlet problem with  $L^2$  data in a convenient way.

**Proposition III.2.6.** *We consider the same assumption as in Proposition III.2.4.*

1. The weak solution  $y$  to the problem (III.10) is the unique function belonging to  $C^0([0, T], L^2(\Omega))$  and satisfying, for any  $\zeta \in L^2(\Omega)$  and any  $t \in [0, T]$ ,

$$\begin{aligned} (y(t), \zeta)_{L^2} &= (y_0, e^{-tA_\beta^*} \zeta)_{L^2} + \int_0^t \int_\Omega f(s, x) (e^{-(t-s)A_\beta^*} \zeta)(x) dx ds \\ &\quad + \int_0^t \int_\Gamma g(s, \sigma) (e^{-(t-s)A_\beta^*} \zeta)(\sigma) d\sigma ds \end{aligned}$$

in addition with the estimate (III.12).

2. The weak solution  $y$  to the problem (III.10) with  $\beta \in (0, +\infty)$ , is also the unique function belonging to  $C^0([0, T], H^{-1}(\Omega))$  and satisfying, for any  $\zeta \in D(A_\beta^*) \cap H_0^1(\Omega)$  and any  $t \in [0, T]$ ,

$$\begin{aligned} \langle y(t), \zeta \rangle_{H^{-1}, H_0^1} &= (y_0, e^{-tA_\beta^*} \zeta)_{L^2} + \int_0^t \int_\Omega f(s, x) (e^{-(t-s)A_\beta^*} \zeta)(x) dx ds \\ &\quad - \int_0^t \int_\Gamma \frac{g(s, \sigma)}{\beta} \frac{\partial}{\partial \nu_\gamma} (e^{-(t-s)A_\beta^*} \zeta)(\sigma) d\sigma ds. \end{aligned}$$

Moreover, this weak solution satisfies the estimate

$$\begin{aligned} \|y\|_{C^0([0, T], H^{-1}(\Omega))} + \|y\|_{L^2((0, T) \times \Omega)} + \|\partial_t y\|_{L^2(0, T, H^{-2}(\Omega))} \\ \leq C_T \left( \|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0, T) \times \Omega)} + \left\| \frac{g}{\beta} \right\|_{L^2((0, T) \times \Gamma)} \right), \end{aligned} \quad (\text{III.13})$$

where  $C_T > 0$  does not depend on  $\beta$ .

**Remark III.2.7.** As explained in Remark III.2.2, we decided to keep the adjoint notation  $A_\beta^*$  instead of  $A_\beta$  in the previous statement, even though it is unnecessary.

*Proof.* 1. Let us first consider any  $\zeta \in D(A_\beta^*) = D(A_\beta)$  and let us choose as a test function in (III.11) the strong solution of the homogeneous backward problem  $t \mapsto \psi(t) = e^{-(T-t)A_\beta^*} \zeta$ . The integration by parts are well justified and naturally lead to the expected formula. By density of  $D(A_\beta)$  in  $L^2(\Omega)$  and by the estimate (III.6), we can extend the equality to any  $\zeta \in L^2(\Omega)$ .

2. In the case where  $\zeta \in D(A_\beta) \cap H_0^1(\Omega)$ , we know that  $\psi(t) \in D(A_\beta)$  for any  $t$ , and in particular we have the equality  $\frac{\partial \psi}{\partial \nu_\gamma} + \beta \psi = 0$  on  $(0, T) \times \Gamma$ , which gives the claimed equality. Now, applying the estimates (III.6) and (III.7) to  $\psi$ , we obtain for any  $t \in [0, T]$  and  $\beta \in (0, +\infty)$ ,

$$\begin{aligned} |\langle y(t), \zeta \rangle_{H^{-1}, H_0^1}| &\leq C_T (\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0, T) \times \Omega)}) \|\zeta\|_{L^2(\Omega)} \\ &\quad + C_T \left\| \frac{g}{\beta} \right\|_{L^2((0, T) \times \Gamma)} \|\nabla \zeta\|_{L^2(\Omega)}. \end{aligned}$$

Since  $D(A_\beta) \cap H_0^1(\Omega)$  is dense in  $H_0^1(\Omega)$ , we get the expected bound on  $\|y\|_{C^0([0, T], H^{-1}(\Omega))}$ . Let us show now the bound in  $L^2((0, T) \times \Omega)$ . Consider any  $h \in C_c^\infty((0, T) \times \Omega)$  and let  $\psi$  be the unique strong solution (as given by Proposition III.2.3) to the backward problem

$$\begin{cases} -\partial_t \psi - \operatorname{div}(\gamma \nabla \psi) = h & \text{in } (0, T) \times \Omega, \\ \frac{\partial \psi}{\partial \nu_\gamma} + \beta \psi = 0 & \text{on } (0, T) \times \Gamma, \\ \psi(T, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

By (III.7), we have the following estimate, uniformly with respect to the parameter  $\beta$

$$\|\psi\|_{L^\infty(0,T,H^1(\Omega))} + \|\partial_t \psi\|_{L^2((0,T)\times\Omega)} + \|\psi\|_{L^2(0,T,H^2(\Omega))} \leq C_T \|h\|_{L^2((0,T)\times\Omega)}. \quad (\text{III.14})$$

Putting this test function in (III.11) and integrating by parts lead to

$$\begin{aligned} \int_0^T \int_\Omega y h \, dx \, dt &= \int_\Omega y_0 \psi(0) \, dx + \int_0^T \int_\Omega f \psi \, dx \, dt \\ &\quad - \int_0^T \int_\Gamma \frac{g}{\beta} \frac{\partial \psi}{\partial \nu_\gamma} \, d\sigma \, dt, \quad \text{for } \beta \in (0, +\infty), \end{aligned}$$

where we have used the boundary condition satisfied by  $\psi$  at each time  $t$  in the boundary term. Using the Cauchy-Schwarz inequality and (III.14), we finally get

$$\begin{aligned} \left| \int_0^T \int_\Omega y h \, dx \, dt \right| &\leq C (\|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)}) \|\psi\|_{L^\infty(0,T,L^2(\Omega))} \\ &\quad + C \left\| \frac{g}{\beta} \right\|_{L^2((0,T)\times\Gamma)} \|\psi\|_{L^2(0,T,H^2(\Omega))} \\ &\leq C_T \left( \|y_0\|_{L^2(\Omega)} + \|f\|_{L^2((0,T)\times\Omega)} + \left\| \frac{g}{\beta} \right\|_{L^2((0,T)\times\Gamma)} \right) \|h\|_{L^2((0,T)\times\Omega)}. \end{aligned}$$

Since  $\mathcal{C}_c^\infty((0,T)\times\Omega)$  is dense in  $L^2((0,T)\times\Omega)$ , we obtain the expected estimate by duality.

Finally, we can easily see that the weak solution  $y$  satisfies, in the distribution sense, the equation  $\partial_t y - \operatorname{div}(\gamma \nabla y) = f$ , and the bound of  $\partial_t y$  in  $L^2(0,T,H^{-2}(\Omega))$  immediately follows.  $\square$

**Remark III.2.8** (Weak estimates pass to the limit). *Going back to the situation described in Remark III.2.5, that is if  $g = \beta g_D$ ,  $g_D \in L^2((0,T)\times\Gamma)$ , we deduce now a bound of the associated solution which is uniform when  $\beta \rightarrow +\infty$ , yet in weaker norms than above. We shall see in the next section that those estimates allow us to pass to the limit towards the Dirichlet problem.*

### III.2.2 Passing to the limit to the Dirichlet problem

With the above existence results and estimates, we can now state and prove a convergence result of the solutions of a suitable Robin problem to the one of a Dirichlet problem.

**Theorem III.2.9** (Convergence towards the Dirichlet problem). *Let  $y_0 \in L^2(\Omega)$  be a given initial data. For any  $\beta > 0$  we consider a source term  $f_\beta \in L^2((0,T)\times\Omega)$  and a boundary data  $g_\beta \in L^2((0,T)\times\Gamma)$ , and we denote by  $y_\beta$  the associated weak solution to (III.10).*

*We assume that, for some  $f_D \in L^2((0,T)\times\Omega)$ ,  $g_D \in L^2((0,T)\times\Gamma)$  we have the  $L^2$ -weak convergences*

$$f_\beta \xrightarrow{\beta \rightarrow +\infty} f_D, \quad \frac{g_\beta}{\beta} \xrightarrow{\beta \rightarrow +\infty} g_D, \quad (\text{III.15})$$

*Then  $y_\beta$  converges, when  $\beta \rightarrow +\infty$ , weakly in  $L^2((0,T)\times\Omega)$  and strongly in  $L^2(0,T,H^{-1}(\Omega))$  towards the unique solution  $y_D \in \mathcal{C}^0([0,T],H^{-1}(\Omega)) \cap L^2((0,T)\times\Omega)$  to the Dirichlet problem (III.4) associated to the data  $f_D$  and  $g_D$ .*

*Moreover, for any  $t \in [0,T]$ ,  $y_\beta(t) \xrightarrow{\beta \rightarrow +\infty} y_D(t)$  weakly in  $H^{-1}(\Omega)$ .*

*Proof.* From the hypothesis, we have a bound on the quantities  $\|g_\beta/\beta\|_{L^2((0,T)\times\Gamma)}$  and  $\|f_\beta\|_{L^2((0,T)\times\Omega)}$  uniform with respect to  $\beta \geq 1$ . Hence, from (III.13), we deduce that, for some  $C_{T,y_0} > 0$ , uniform in  $\beta$ , we have

$$\|y_\beta\|_{\mathcal{C}^0([0,T],H^{-1}(\Omega))} + \|y_\beta\|_{L^2((0,T)\times\Omega)} + \|\partial_t y_\beta\|_{L^2(0,T,H^{-2}(\Omega))} \leq C_{T,y_0}.$$

We can then find some  $y_D \in \mathcal{C}^0([0, T], H^{-1}(\Omega)) \cap L^2((0, T) \times \Omega)$  and a subsequence, still denoted by  $(y_\beta)_\beta$  such that

$$\begin{cases} y_\beta \xrightarrow{\beta \rightarrow +\infty} y_D & \text{weakly in } L^2((0, T) \times \Omega), \\ \partial_t y_\beta \xrightarrow{\beta \rightarrow +\infty} \partial_t y_D & \text{weakly in } L^2(0, T, H^{-2}(\Omega)), \\ y_\beta \xrightarrow[\beta \rightarrow +\infty]{*} y_D & \text{weakly-* in } \mathcal{C}^0([0, T], H^{-1}(\Omega)), \\ y_\beta \xrightarrow{\beta \rightarrow +\infty} y_D & \text{strongly in } L^2(0, T, H^{-1}(\Omega)). \end{cases} \quad (\text{III.16})$$

The last strong convergence comes from the compactness of the embeddings  $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$  and  $H^{-1}(\Omega) \hookrightarrow H^{-2}(\Omega)$  and the Aubin-Lions lemma.

All we need to show is that this limit  $y_D$  is indeed the solution to the corresponding Dirichlet problem. By uniqueness of the solution of Dirichlet problem (III.4) with the data  $f_D, g_D$ , the convergence of the whole family  $(y_\beta)_\beta$  will be established.

Let us consider a final data  $\zeta \in \mathcal{C}_c^\infty(\Omega) \subset D(A_\beta^*) \cap H_0^1(\Omega)$  for the adjoint homogeneous problem. The corresponding strong solution is given by  $\psi_\beta(t) = e^{-(T-t)A_\beta^*} \zeta$  and, thanks to (III.7), we have

$$\|\psi_\beta\|_{\mathcal{C}^0([0, T], H^1(\Omega))} + \|\psi_\beta\|_{L^2(0, T, H^2(\Omega))} + \|\partial_t \psi_\beta\|_{L^2((0, T) \times \Omega)} \leq C_T \|\zeta\|_{H_0^1(\Omega)}, \quad (\text{III.17})$$

where  $C_T$  is uniform in  $\beta$ .

We can then extract a subsequence, still denoted by  $(\psi_\beta)_\beta$ , such that

$$\begin{cases} \psi_\beta \xrightarrow{\beta \rightarrow +\infty} \psi_D & \text{weakly in } L^2(0, T, H^2(\Omega)), \\ \partial_t \psi_\beta \xrightarrow{\beta \rightarrow +\infty} \partial_t \psi_D & \text{weakly in } L^2((0, T) \times \Omega), \\ \psi_\beta \xrightarrow[\beta \rightarrow +\infty]{*} \psi_D & \text{weakly-* in } \mathcal{C}^0([0, T], H^1(\Omega)), \\ \psi_\beta \xrightarrow{\beta \rightarrow +\infty} \psi_D & \text{strongly in } L^2(0, T, H^1(\Omega)), \end{cases} \quad (\text{III.18})$$

for some  $\psi_D \in L^2(0, T, H^2(\Omega)) \cap \mathcal{C}^0([0, T], H^1(\Omega))$ . Here also we have used the Aubin-Lions lemma to obtain the last strong convergence.

Moreover, from the boundary condition satisfied by  $\psi_\beta$ , we have  $\psi_\beta = -\frac{1}{\beta} \frac{\partial \psi_\beta}{\partial \nu_\gamma}$  on the boundary where the quantity  $\left\| \frac{\partial \psi_\beta}{\partial \nu_\gamma} \right\|_{L^2((0, T) \times \Gamma)}$  is bounded for any large  $\beta$  (using (III.17)). Hence, it follows that

$$\psi_\beta \xrightarrow{\beta \rightarrow \infty} 0 \quad \text{in } L^2((0, T) \times \Gamma),$$

which actually implies that  $\psi_D = 0$  on the boundary  $(0, T) \times \Gamma$ . Moreover, by passing to the limit in the equation in the distribution sense, we finally find that  $\psi_D$  is the unique solution to the backward homogeneous Dirichlet problem, that is  $\psi_D(t) = e^{-(T-t)A_D^*} \zeta$ .

Now, using the trace theorem, we observe that

$$\begin{aligned} \int_0^T \int_\Gamma \left| \frac{\partial \psi_\beta}{\partial \nu_\gamma} - \frac{\partial \psi_D}{\partial \nu_\gamma} \right|^2 d\sigma ds &\leq C \int_0^T \|\psi_\beta(s) - \psi_D(s)\|_{H^1(\Omega)} \|\psi_\beta(s) - \psi_D(s)\|_{H^2(\Omega)} ds \\ &\leq C \|\psi_\beta - \psi_D\|_{L^2(0, T, H^1(\Omega))} \|\psi_\beta - \psi_D\|_{L^2(0, T, H^2(\Omega))}. \end{aligned}$$

By (III.18) we see that the first factor of the very right hand side of the above inequality converges to 0 as  $\beta \rightarrow +\infty$ , whereas the second factor is bounded. Thus, we have

$$\frac{\partial \psi_\beta}{\partial \nu_\gamma} \xrightarrow{\beta \rightarrow +\infty} \frac{\partial \psi_D}{\partial \nu_\gamma} \quad \text{in } L^2((0, T) \times \Gamma). \quad (\text{III.19})$$

In particular, from the uniform bound  $\|\psi_\beta(0)\|_{H^1(\Omega)} \leq C_T \|\zeta\|_{H_0^1(\Omega)}$  (recall (III.17)) and the fact that  $\psi_D \in \mathcal{C}^0([0, T], H^1(\Omega))$ , we have

$$\psi_\beta(0, \cdot) \xrightarrow{\beta \rightarrow +\infty} \psi_D(0, \cdot) \quad \text{weakly in } H^1(\Omega). \quad (\text{III.20})$$

With the convergence results (III.19) and (III.20) above together with (III.15) we get

$$\begin{aligned} (y_0, \psi_\beta(0))_{L^2} + \int_0^t \int_\Omega f_\beta \psi_\beta \, dx \, ds - \int_0^t \int_\Gamma \frac{g_\beta}{\beta} \frac{\partial \psi_\beta}{\partial \nu_\gamma} \, d\sigma \, ds \\ \xrightarrow{\beta \rightarrow +\infty} (y_0, \psi_D(0))_{L^2} + \int_0^t \int_\Omega f_D \psi_D \, dx \, ds - \int_0^t \int_\Gamma g_D \frac{\partial \psi_D}{\partial \nu_\gamma} \, d\sigma \, ds. \end{aligned}$$

Using the weak formulation (III.11) satisfied by  $y_\beta$  with  $\psi_\beta$  as a test function we see that have actually proved that

$$(y_\beta(t), \zeta)_{L^2} \xrightarrow{\beta \rightarrow +\infty} (y_0, \psi_D(0))_{L^2} + \int_0^t \int_\Omega f_D \psi_D \, dx \, ds - \int_0^t \int_\Gamma g_D \frac{\partial \psi_D}{\partial \nu_\gamma} \, d\sigma \, ds,$$

and at the same time, by (III.16), we have

$$(y_\beta(t), \zeta)_{L^2} = \langle y_\beta(t), \zeta \rangle_{H^{-1}, H_0^1} \xrightarrow{\beta \rightarrow +\infty} \langle y_D(t), \zeta \rangle_{H^{-1}, H_0^1}, \quad \forall t \in [0, T].$$

As a conclusion we have: for each  $t \in [0, T]$ ,  $y_D(t)$  satisfies

$$\begin{aligned} \langle y_D(t), \zeta \rangle_{H^{-1}, H_0^1} &= (y_0, e^{-tA_D^*} \zeta)_{L^2} + \int_0^t \int_\Omega f_D(s, x) (e^{-(t-s)A_D^*} \zeta)(x) \, dx \, ds \\ &\quad - \int_0^t \int_\Gamma g_D(s, \sigma) \frac{\partial}{\partial \nu_\gamma} (e^{-(t-s)A_D^*} \zeta)(\sigma) \, d\sigma \, ds, \end{aligned}$$

which is exactly the definition of the solution of (III.4) with the data  $f_D, g_D$ , see Proposition III.2.1.  $\square$

**Remark III.2.10** (Convergence towards the Neumann problem). *By similar, and in fact simpler, arguments one can prove that if  $(f_\beta)_\beta$  and  $(g_\beta)_\beta$  both weakly converge, when  $\beta \rightarrow 0$ , towards some  $f_N$  and  $g_N$  in  $L^2((0, T) \times \Omega)$  and  $L^2((0, T) \times \Gamma)$  respectively, then the corresponding solution  $y_\beta$  converges, when  $\beta \rightarrow 0$ , to the solution  $y_N$  of the corresponding non-homogeneous Neumann problem.*

### III.2.3 The coupled system

We can now move to the cascade coupled parabolic systems we are interested in, namely the one with Dirichlet boundary condition

$$\left\{ \begin{array}{ll} \partial_t y_1 - \operatorname{div}(\gamma \nabla y_1) &= f_1 \quad \text{in } (0, T) \times \Omega, \\ \partial_t y_2 - \operatorname{div}(\gamma \nabla y_2) + y_1 &= f_2 \quad \text{in } (0, T) \times \Omega, \\ y_1 &= g_1 \quad \text{on } (0, T) \times \Gamma, \\ y_2 &= g_2 \quad \text{on } (0, T) \times \Gamma, \\ y_1(0, \cdot) &= y_{0,1} \quad \text{in } \Omega, \\ y_2(0, \cdot) &= y_{0,2} \quad \text{in } \Omega, \end{array} \right. \quad (\text{III.21})$$



and the one with Robin boundary conditions

$$\left\{ \begin{array}{ll} \partial_t y_1 - \operatorname{div}(\gamma \nabla y_1) &= f_1 \quad \text{in } (0, T) \times \Omega, \\ \partial_t y_2 - \operatorname{div}(\gamma \nabla y_2) + y_1 &= f_2 \quad \text{in } (0, T) \times \Omega, \\ \frac{\partial y_1}{\partial \nu_\gamma} + \beta_1 y_1 &= g_1 \quad \text{on } (0, T) \times \Gamma, \\ \frac{\partial y_2}{\partial \nu_\gamma} + \beta_2 y_2 &= g_2 \quad \text{on } (0, T) \times \Gamma, \\ y_1(0, \cdot) &= y_{0,1} \quad \text{in } \Omega, \\ y_2(0, \cdot) &= y_{0,2} \quad \text{in } \Omega. \end{array} \right. \quad (\text{III.22})$$

We can obviously solve those two systems by simply using the results on the scalar case: we first solve the equation for  $y_1$  then we solve the scalar equation for  $y_2$  by considering the coupling term  $y_1$  as an additional  $L^2$  source term.

**Theorem III.2.11.** *We suppose given  $y_0 := (y_{0,1}, y_{0,2}) \in (L^2(\Omega))^2$ ,  $f := (f_1, f_2) \in (L^2((0, T) \times \Omega))^2$  and  $g := (g_1, g_2) \in (L^2((0, T) \times \Gamma))^2$ .*

1. *There exists a unique solution  $y = (y_1, y_2) \in (C^0([0, T], H^{-1}(\Omega))) \cap L^2((0, T) \times \Omega)^2$  of (III.21), that is, for any  $i = 1, 2$ ,  $y_i$  satisfies the corresponding scalar problem in the sense of Proposition III.2.1.*
2. *For any  $\beta_1, \beta_2 \in [0, +\infty)$ , there exists a unique solution  $y = (y_1, y_2) \in (C^0([0, T], L^2(\Omega))) \cap L^2(0, T, H^1(\Omega))^2$  of (III.22), that is, for any  $i = 1, 2$ ,  $y_i$  satisfies the corresponding scalar problem in the sense of Proposition III.2.4.*
3. *For any  $\beta = (\beta_1, \beta_2) \in (0, +\infty)^2$ , we suppose given  $f_{i,\beta} \in L^2((0, T) \times \Omega)$  and  $g_{i,\beta} \in L^2((0, T) \times \Gamma)$ , for  $i = 1, 2$  such that*

$$\frac{g_{i,\beta}}{\beta_i} \xrightarrow{\beta \rightarrow +\infty} g_i, \quad f_{i,\beta} \xrightarrow{\beta \rightarrow +\infty} f_i.$$

*Then, the solution  $y_\beta$  of (III.22) corresponding to the data  $f_\beta, g_\beta$  converges weakly in  $(L^2((0, T) \times \Omega))^2$  and strongly in  $(L^2(0, T, H^{-1}(\Omega)))^2$  towards the unique solution of the corresponding Dirichlet problem.*

For the analysis of the control problem, it is not convenient to make appear the component  $y_1$  of the solution as a source term in the equation for  $y_2$  since it breaks down the cascade structure of the system which is essential to prove its controllability with only one control. That is the reason why it is necessary to introduce the following unbounded operators in  $(L^2(\Omega))^2$ : let  $\mathcal{A}_{\beta_1, \beta_2}$  and  $\mathcal{A}_D$  be defined by the same formal expression

$$\begin{pmatrix} -\operatorname{div}(\gamma \nabla \cdot) & 0 \\ 1 & -\operatorname{div}(\gamma \nabla \cdot) \end{pmatrix}, \quad (\text{III.23})$$

but with the different domains

$$D(\mathcal{A}_{\beta_1, \beta_2}) := \left\{ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in (H^2(\Omega))^2 \mid \frac{\partial u_1}{\partial \nu_\gamma} + \beta_1 u_1 = 0, \frac{\partial u_2}{\partial \nu_\gamma} + \beta_2 u_2 = 0 \text{ on } \Gamma \right\},$$

and

$$D(\mathcal{A}_D) := (H^2(\Omega) \cap H_0^1(\Omega))^2,$$

respectively. Those operators are no more self-adjoint and we define their adjoints by  $D(\mathcal{A}_{\beta_1, \beta_2}^*) = D(\mathcal{A}_{\beta_1, \beta_2})$  and  $D(\mathcal{A}_D^*) = D(\mathcal{A}_D)$  and the same formal expression

$$\begin{pmatrix} -\operatorname{div}(\gamma \nabla \cdot) & 1 \\ 0 & -\operatorname{div}(\gamma \nabla \cdot) \end{pmatrix}.$$



Standard elliptic theory shows that  $-\mathcal{A}_{\beta_1, \beta_2}$ , and  $-\mathcal{A}_{\beta_1, \beta_2}^*$  as well as  $-\mathcal{A}_D$  and  $-\mathcal{A}_D^*$  generate continuous semigroups in  $(L^2(\Omega))^2$ . A similar analysis as in Section III.2.1.3 for the scalar case, leads to the following result.

**Proposition III.2.12.** *We suppose given any  $y_0 \in (L^2(\Omega))^2$ ,  $f \in (L^2((0, T) \times \Omega))^2$  and  $g \in (L^2((0, T) \times \Gamma))^2$ .*

1. *The solution to (III.21) is the unique element  $y \in (C^0([0, T], H^{-1}(\Omega)))^2$  satisfying, for any  $\zeta \in (H_0^1(\Omega))^2$  and any  $t \in [0, T]$*

$$\begin{aligned} \langle y(t), \zeta \rangle_{H^{-1}, H_0^1} &= (y_0, e^{-t\mathcal{A}_D^*} \zeta)_{L^2} + \int_0^t \int_{\Omega} f(s, x) \cdot (e^{-(t-s)\mathcal{A}_D^*} \zeta)(x) dx ds \\ &\quad - \int_0^t \int_{\Gamma} g(s, \sigma) \cdot \frac{\partial}{\partial \nu_{\gamma}} (e^{-(t-s)\mathcal{A}_D^*} \zeta)(\sigma) d\sigma ds. \end{aligned}$$

2. *For any  $\beta_1, \beta_2 \in [0, +\infty)$ , the solution to (III.22) is the unique element  $y \in (C^0([0, T], L^2(\Omega)))^2$  satisfying, for any  $\zeta \in (L^2(\Omega))^2$  and any  $t \in [0, T]$*

$$\begin{aligned} (y(t), \zeta)_{L^2} &= (y_0, e^{-t\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)_{L^2} + \int_0^t \int_{\Omega} f(s, x) \cdot (e^{-(t-s)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)(x) dx ds \\ &\quad + \int_0^t \int_{\Gamma} g(s, \sigma) \cdot (e^{-(t-s)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)(\sigma) d\sigma ds. \end{aligned}$$

### III.2.4 Main results in 1D and in some multi-D geometry

Let now  $\Gamma_0$  be a non empty open subset of  $\Gamma$ . Using the analysis presented in the previous sections we can now formulate the null-control problems we are interested in as follows.

**Proposition III.2.13.** *Let  $y_0 \in (L^2(\Omega))^2$  be given.*

1. *A function  $v \in L^2((0, T) \times \Gamma)$  is a null-control at time  $T$  for the Dirichlet problem (III.1)-(III.2), if and only if it satisfies: for any  $\zeta \in (H_0^1(\Omega))^2$*

$$(y_0, e^{-T\mathcal{A}_D^*} \zeta)_{L^2} = \int_0^T \int_{\Gamma} 1_{\Gamma_0} v(t, \sigma) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{\partial}{\partial \nu_{\gamma}} (e^{-(T-t)\mathcal{A}_D^*} \zeta)(\sigma) d\sigma dt. \quad (\text{III.24})$$

2. *A function  $v \in L^2((0, T) \times \Gamma)$  is a null-control at time  $T$  for the Robin problem (III.1)-(III.3), if and only if it satisfies: for any  $\zeta \in (L^2(\Omega))^2$*

$$- (y_0, e^{-T\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)_{L^2} = \int_0^T \int_{\Gamma} 1_{\Gamma_0} v(t, \sigma) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (e^{-(T-t)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)(\sigma) d\sigma dt. \quad (\text{III.25})$$

#### III.2.4.1 The 1D case

We start with a discussion of the 1D setting since, as we will see in the next section, we can deduce some multi-D results from the 1D analysis.

Hence, we particularize the above control problem to the 1D situation where  $\Omega = (0, 1)$ ,  $\Gamma_0 = \{0\}$  and the diffusion coefficient is simply a scalar function  $\gamma \in C^1([0, 1])$  with  $\gamma_{\min} = \inf_{[0, 1]} \gamma > 0$  and  $\gamma_{\max} = \sup_{[0, 1]} \gamma < +\infty$ . In that case the control we are looking for is just a scalar function  $v \in L^2(0, T)$  and the formulations (III.24), (III.25) just read

$$- (y_0, e^{-T\mathcal{A}_D^*} \zeta)_{L^2(0, 1)} = \gamma(0) \int_0^T v(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \frac{\partial}{\partial x} \Big|_{x=0} (e^{-(T-t)\mathcal{A}_D^*} \zeta) dt, \quad (\text{III.26})$$

for the Dirichlet problem and

$$- (y_0, e^{-T\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)_{L^2(0, 1)} = \int_0^T v(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot (e^{-(T-t)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta) \Big|_{x=0} dt, \quad (\text{III.27})$$

for the Robin problem respectively, with the same notations for the adjoint of the diffusion operators as in multi-D. It is convenient to introduce the observation operator  $\mathcal{B}^*$  (that does not depend on the Robin parameters  $\beta_1, \beta_2$ ) defined as follows

$$\mathcal{B}^* : \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in (H^1(0,1))^2 \mapsto \zeta_1(0), \quad (\text{III.28})$$

in such a way that (III.27) becomes

$$-(y_0, e^{-T\mathcal{A}_{\beta_1, \beta_2}^*} \zeta)_{L^2(0,1)} = \int_0^T v(t) \mathcal{B}^*(e^{-(T-t)\mathcal{A}_{\beta_1, \beta_2}^*} \zeta) dt. \quad (\text{III.29})$$

Most of the work in Sections III.4 and III.5 will consist in solving this problem with suitable estimates of the control with respect to the parameters  $\beta_1$  and  $\beta_2$ . Our main result in that direction is the following.

**Theorem III.2.14.** *Let  $y_0 \in (L^2(0,1))^2$  and  $T > 0$  be given.*

1. *Let  $\beta \in (0, +\infty)$  and set  $\beta_1 = \beta_2 = \beta$ . Then, there exists a null-control  $v_\beta \in L^2(0, T)$  for the 1D problem (III.29) that satisfies in addition the estimate*

$$\|v_\beta\|_{L^2(0,T)} \leq C e^{C/T} (1 + \beta) \|y_0\|_{L^2(0,1)},$$

where  $C > 0$  does not depend on  $\beta$  and  $T$ .

2. *Assume that  $\gamma$  is a positive constant and let  $\beta^* > 0$  be given. Let  $\beta := (\beta_1, \beta_2) \in (0, +\infty)^2$  be any couple of Robin parameters. Then, there exists a null-control  $v_\beta \in L^2(0, T)$  for the 1D problem (III.29) that satisfies in addition the estimate*

$$\|v_\beta\|_{L^2(0,T)} \leq C_{T, \beta^*} (1 + \beta_1) \|y_0\|_{L^2(0,1)},$$

as soon as either  $\beta_1, \beta_2 \in (0, \beta^*]$ , or  $\beta_1, \beta_2 \in [\beta^*, +\infty)$ , where  $C_{T, \beta^*} > 0$  does not depend on  $\beta$ .

**Corollary III.2.15** (Convergence towards Dirichlet control). *Let  $\beta_n = (\beta_{1,n}, \beta_{2,n}) \in (0, +\infty)^2$  be any sequence of Robin parameters such that  $\beta_{i,n} \rightarrow +\infty$  when  $n \rightarrow \infty$ , for  $i = 1, 2$ . If the diffusion coefficient  $\gamma$  is not a constant, we assume in addition that  $\beta_{1,n} = \beta_{2,n}$  for any  $n$ . For each  $n$ , let  $v_n$  (resp.  $y_n$ ) be the unique null-control of minimal  $L^2(0, T)$  norm (resp. the associated trajectory) for the problem (III.29) with Robin parameters  $\beta_{1,n}$  and  $\beta_{2,n}$ .*

*Then, there exists a subsequence  $(n_k)_k$  such that*

$$\frac{v_{n_k}}{\beta_{1, n_k}} \xrightarrow[k \rightarrow +\infty]{} v_D \text{ weakly in } L^2(0, T),$$

$$y_{n_k} \xrightarrow[k \rightarrow +\infty]{} y_D \text{ strongly in } (L^2(0, T, H^{-1}(0, 1)))^2 \text{ and}$$

$$y_{n_k} \xrightarrow[k \rightarrow +\infty]{} y_D \text{ weakly in } (L^2((0, T) \times (0, 1)))^2,$$

where  $v_D$  (resp.  $y_D$ ) is a null-control (resp. the associated trajectory) for the Dirichlet control problem (III.26). Below, we put the Figures III.1 and III.2 to clarify our result.

To obtain the numerical Figures III.1 and III.2, we follow the discrete setting developed in Section IV.5.2, Ch. IV (for a general  $2 \times 2$  coupled parabolic system) and choose the quantities below.

$$T = 0.3, \quad \gamma = 1, \\ y_{0,1}(x) = \sin(2\pi x), \quad y_{0,2}(x) = 3 \times \mathbf{1}_{(0.3, 0.8)}(x),$$

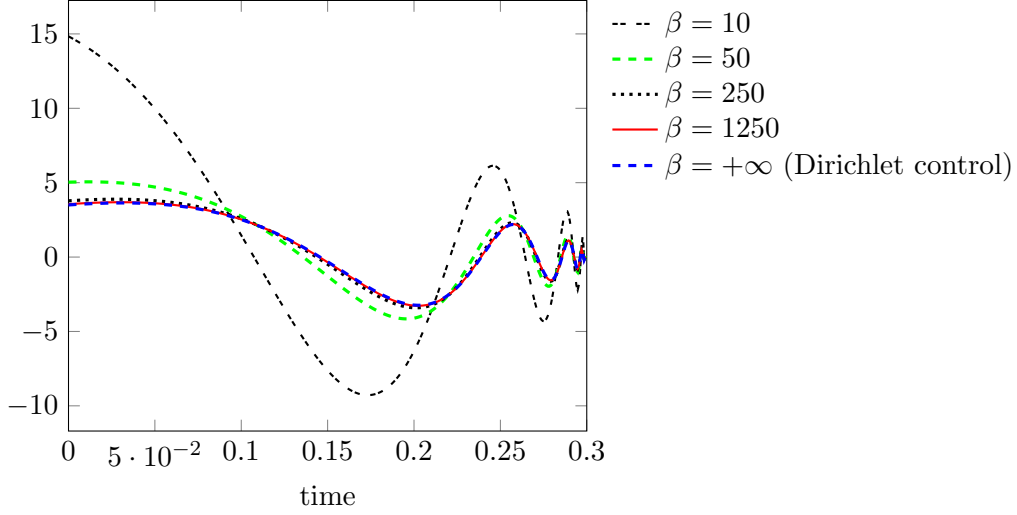


Figure III.1: Convergence of  $\frac{v_\beta}{\beta}$  (of minimal  $L^2$ -norm) to a Dirichlet control  $v_D$ .

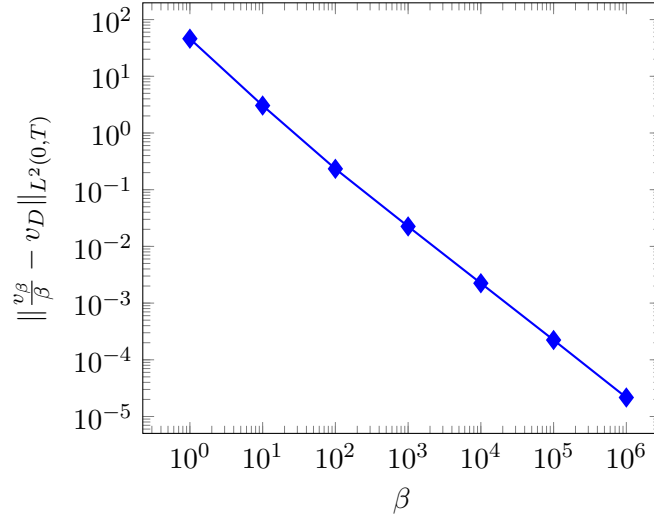


Figure III.2: The convergence in terms of the  $L^2$ -norm of the difference  $\left(\frac{v_\beta}{\beta} - v_D\right)$ .

with  $N = 200$ , the number of steps for space discretization and  $M = 400$ , the number of time steps.

From the Fig. III.2, it seems that the convergence of the Robin controls towards a Dirichlet one is of order 1. Although, this does not provide us any strong conclusion since we do not have any error estimate; theoretically we just know the weak convergence of the controls up to a subsequence.

**Remark III.2.16** (Convergence towards Neumann control). *With the same notation as in the previous corollary, if we assume that  $\beta_{i,n} \rightarrow 0$  when  $n \rightarrow \infty$ , for  $i = 1, 2$ , then we obtain the convergence, up to a subsequence, of the null-control  $v_n$  (resp. of the trajectory  $y_n$ ) towards a null-control  $v_N$  (resp. the trajectory  $y_N$ ) corresponding to the Neumann boundary conditions on both components.*

**Remark III.2.17** (The Dirichlet/Neumann case). *In point 2 of Theorem III.2.14, we needed to assume that either the two Robin parameters are both smaller than some  $\beta^*$  or that they are both higher than some  $\beta^*$ . It is worth noticing that we cannot expect to prove a similar result without those assumptions.*

*Indeed, if we were able to prove the estimate  $\|v_\beta\|_{L^2(0,T)} \leq C_T(1 + \beta_1)\|y_0\|_{L^2(0,1)}$ , for any*

couple of parameters  $\beta_1$  and  $\beta_2$ , then by following the same lines as in Corollary III.2.15, we would be able to prove the convergence, up to a subsequence, of  $v_n/\beta_{1,n}$  when  $\beta_{1,n} \rightarrow +\infty$  and  $\beta_{2,n} \rightarrow 0$  to some  $v_{DN}$  that would be a null-control for the Dirichlet/Neumann problem (that is system (III.1) in 1D, with a Dirichlet boundary condition for the first component  $y_1$  and a Neumann boundary condition for the second component  $y_2$ ). However, we know that this last problem is not even approximately controllable since the underlying operator  $\mathcal{A}_{\infty,0}^*$  has eigenspaces of dimension higher than 1, which prevents the Fattorini-Hautus criterion (see [55, 89]) from being satisfied.

The same remark holds for the Neumann/Dirichlet case, that is when  $\beta_{1,n} \rightarrow 0$  and  $\beta_{2,n} \rightarrow +\infty$ .

### III.2.4.2 A multi-D result

By using the methodology described in [3, 21] it is possible, starting from a suitable null-controllability result for the 1D problem, at least when both Robin parameters are the same, to deduce the corresponding result in any cylinder of  $\mathbb{R}^d$  for  $d \geq 2$ , see Figure III.3.

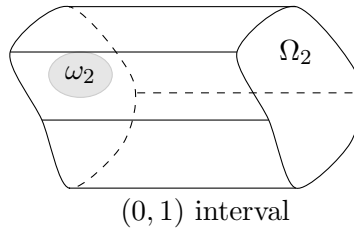


Figure III.3: The cylindrical geometry

More precisely, we consider a domain  $\Omega = (0, 1) \times \Omega_2$  in  $\mathbb{R}^d$  where  $\Omega_2$  is a bounded smooth connected domain in  $\mathbb{R}^{d-1}$ . The variable in  $\Omega$  will be denoted by  $(x, \bar{x})$ , with  $x \in (0, 1)$  and  $\bar{x} \in \Omega_2$  and we assume that the diffusion tensor has the following form

$$\gamma(x, \bar{x}) = \begin{pmatrix} \gamma(x) & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & \begin{pmatrix} \bar{\gamma}(\bar{x}) \end{pmatrix} & & \\ 0 & & & \end{pmatrix},$$

with  $\gamma : (0, 1) \rightarrow \mathbb{R}$  and  $\bar{\gamma} : \Omega_2 \rightarrow M_{d-1}(\mathbb{R})$ . Note that the cylindrical domain  $\Omega$  is not of class  $\mathcal{C}^2$  but since the  $H^2$  elliptic regularity property holds for our operator in this particular geometry, all the general material exposed before is still valid in this setting.

Let  $\omega_2 \subset \Omega_2$  be a non empty open subset of  $\Omega_2$ . The control region we will consider is  $\Gamma_0 = \{0\} \times \omega_2$  so that the control problem is the following

$$\begin{cases} \partial_t y_1 - \partial_x(\gamma(x)\partial_x y_1) - \operatorname{div}_{\bar{x}}(\bar{\gamma}(\bar{x})\nabla_{\bar{x}} y_1) &= 0 & \text{in } (0, T) \times \Omega, \\ \partial_t y_2 - \partial_x(\gamma(x)\partial_x y_2) - \operatorname{div}_{\bar{x}}(\bar{\gamma}(\bar{x})\nabla_{\bar{x}} y_2) + y_1 &= 0 & \text{in } (0, T) \times \Omega, \\ y_1(0, \cdot) &= y_{0,1} & \text{in } \Omega, \\ y_2(0, \cdot) &= y_{0,2} & \text{in } \Omega, \end{cases} \quad (\text{III.30})$$

associated with either Dirichlet boundary conditions

$$\begin{cases} y_1 = \mathbb{1}_{\{0\} \times \omega_2} v & \text{on } (0, T) \times \Gamma, \\ y_2 = 0 & \text{on } (0, T) \times \Gamma, \end{cases} \quad (\text{III.31})$$

or Robin boundary conditions with the same parameter

$$\begin{cases} \frac{\partial y_1}{\partial \nu_\gamma} + \beta y_1 = \mathbf{1}_{\{0\} \times \omega_2} v & \text{on } (0, T) \times \Gamma, \\ \frac{\partial y_2}{\partial \nu_\gamma} + \beta y_2 = 0 & \text{on } (0, T) \times \Gamma. \end{cases} \quad (\text{III.32})$$

Note that the result below is restricted to the case  $\beta_1 = \beta_2$  for two reasons. The main one is that when the two parameters are different, the problem has no more a suitable tensor product structure that is crucial in the analysis (see [3]). The second one is that the constant  $C_{T, \beta^*}$  in point 2 of Theorem III.2.14 is not explicit enough with respect to  $T$ ; we would need an exponential dependence of the constant for the analysis to apply directly.

**Theorem III.2.18.** *Let  $y_0 \in (L^2(\Omega))^2$  be given. For any  $T > 0$  and any  $\beta \in (0, +\infty)$ , there exists a null-control  $v_\beta \in L^2((0, T) \times \Gamma)$  for the multi-D problem (III.30)-(III.32) that satisfies in addition the estimate*

$$\|v_\beta\|_{L^2((0, T) \times \Gamma)} \leq C e^{C/T} (1 + \beta) \|y_0\|_{L^2(\Omega)},$$

where  $C > 0$  neither depends on  $\beta$  nor on  $T$ .

*Proof.* The proof is mainly based on the strategy developed in [3, 21] which needs the sharp estimate with respect to  $T$  of the 1D control cost given by point 1 of Theorem III.2.14 and a Lebeau-Robbiano spectral inequality, uniform in  $\beta \in (0, +\infty)$ , relative to our control region  $\omega_2$ , for the eigenfunctions of the diffusion operator  $-\operatorname{div}_{\bar{x}}(\bar{\gamma}(\bar{x}) \nabla_{\bar{x}} \cdot)$  with homogeneous Robin boundary condition in  $\Omega_2$ .

The required Lebeau-Robbiano inequality can be obtained by a similar approach as given by [85, Theorem 1.2]; the author proved here the inequality for the eigenfunctions of Laplace-Beltrami operator in a multi-dimensional connected compact  $\mathcal{C}^1$ -smooth Riemannian manifold  $M$  with the boundary condition  $\tilde{l} \frac{\partial u}{\partial \bar{x}} + lu = 0$  ( $u \in H^2(M)$ ) for  $\tilde{l} \equiv 1$  and any  $l \in L^\infty(\Gamma)$  with  $l \geq 0$ . Although they did not mention it in the paper, a careful look at their computations ensures us that the Lebeau-Robbiano inequality in this reference is in fact uniform with respect to any parameter  $l \geq 0$ . Thus, the required inequality holds true for our operator  $-\operatorname{div}_{\bar{x}}(\bar{\gamma}(\bar{x}) \nabla_{\bar{x}} \cdot)$  in  $\Omega_2$  with homogeneous Robin boundary condition with any parameter  $\beta \in (0, +\infty)$ .

We give a sketch of the proof for the concerned Lebeau-Robbiano inequality in Section III.6.  $\square$

**Corollary III.2.19** (Convergence towards Dirichlet control). *Let  $(\beta_n)_n$  be any sequence of positive Robin parameters such that  $\beta_n \rightarrow +\infty$  when  $n \rightarrow +\infty$ . For each  $n$  we define  $v_n$  (resp.  $y_n$ ) to be the null-control of minimal  $L^2$  norm (resp. the associated trajectory) for the problem (III.30)-(III.32) with Robin parameter  $\beta_n$ .*

*There exists a subsequence  $(n_k)_k$  such that*

$$\begin{aligned} \frac{v_{n_k}}{\beta_{n_k}} &\xrightarrow[k \rightarrow +\infty]{} v_D \quad \text{weakly in } L^2((0, T) \times \Gamma), \\ y_{n_k} &\xrightarrow[k \rightarrow +\infty]{} y_D \quad \text{strongly in } (L^2(0, T, H^{-1}(\Omega)))^2 \text{ and} \\ y_{n_k} &\xrightarrow[k \rightarrow +\infty]{} y_D \quad \text{weakly in } (L^2((0, T) \times \Omega))^2, \end{aligned}$$

where  $v_D$  (resp.  $y_D$ ) is a null-control (resp. the associated trajectory) for the Dirichlet control problem (III.30)-(III.31) for the same initial data.

**Remark III.2.20** (Convergence towards Neumann control). *When  $(\beta_n)_n$  goes to 0, we obtain the convergence towards a null-control for the Neumann problem as in Remark III.2.16.*

### III.2.5 Outline

The rest of this chapter is dedicated to the proof of our main theorem for the 1D case, namely Theorem III.2.14. First of all, we establish useful spectral properties for the 1D Robin eigenvalue problem in Section III.3. Then, we prove in Section III.4 the controllability result in the case of an arbitrary diffusion coefficient but for the same Robin parameter for both components (point 1 of Theorem III.2.14). Finally, in Section III.5, we investigate the case of a constant diffusion coefficient with two different Robin parameters (point 2 of Theorem III.2.14).

## III.3 Some spectral properties of 1D Robin eigenvalue problem

In this section, we develop some properties of the eigenvalue-eigenfunctions of the 1D scalar operator  $A_\beta$  as introduced in Section III.2.1.2. Note that we use same notation as for the general higher dimension case. Those results will be used to draw some spectral properties of our main operator  $\mathcal{A}_{\beta_1, \beta_2}^*$ .

### III.3.1 The case of a non-constant diffusion coefficient

We begin with the following scalar eigenvalue problem

$$\begin{cases} -\partial_x(\gamma(x)\partial_x\varphi) = \lambda\varphi & \text{in } (0, 1), \\ -\gamma(0)\partial_x\varphi(0) + \beta\varphi(0) = 0, \\ \gamma(1)\partial_x\varphi(1) + \beta\varphi(1) = 0, \end{cases} \quad (\text{III.33})$$

where  $\beta$  is any non-negative parameter and  $\gamma$  is chosen as in Section III.2.4.1. Let us denote the eigenvalue-eigenfunction pairs of the Sturm-Liouville problem (III.33) as  $(\lambda_{k,\gamma}^\beta, \varphi_{k,\gamma}^\beta)_{k \geq 0}$ . We recall that the eigenvalues are simple and real and can be numbered in such a way that

$$0 \leq \lambda_{0,\gamma}^\beta < \lambda_{1,\gamma}^\beta < \dots < \lambda_{k,\gamma}^\beta < \lambda_{k+1,\gamma}^\beta \nearrow +\infty, \quad (\text{III.34})$$

see for instance [14, Theorem 8.4.5]. Also it is well-known that the family  $(\varphi_{k,\gamma}^\beta)_{k \geq 0}$  is a Hilbert basis of  $L^2(0, 1)$ , as soon as they are normalized, and indeed each of  $\varphi_{k,\gamma}^\beta$  belongs to the domain of the corresponding differential operator in (III.33).

**Remark III.3.1.** Observe that for  $\beta = 0$ , the problem (III.33) reduces to a Neumann eigenvalue problem where we denote the Neumann eigenvalues by  $\lambda_{k,\gamma}^N := \lambda_{k,\gamma}^0$  for  $k \geq 0$ .

On the other hand, for  $\beta = +\infty$ , (III.33) degenerates into a Dirichlet eigenvalue problem and we denote by  $\lambda_{k,\gamma}^D := \lambda_{k,\gamma}^\infty$  for  $k \geq 0$ , the Dirichlet eigenvalues.

For any  $\beta \in [0, +\infty)$  and any  $\varphi \in H^1(0, 1)$ ,  $\varphi \neq 0$ , we define the Rayleigh quotient associated with (III.33),

$$R_\beta(\varphi) := \frac{\int_0^1 \gamma(x) |\partial_x \varphi(x)|^2 dx + \beta(|\varphi(0)|^2 + |\varphi(1)|^2)}{\int_0^1 |\varphi(x)|^2 dx}.$$

For  $\beta = +\infty$ , we set

$$R_\infty(\varphi) := \begin{cases} \frac{\int_0^1 \gamma(x) |\partial_x \varphi(x)|^2 dx}{\int_0^1 |\varphi(x)|^2 dx}, & \text{if } \varphi \in H_0^1(0, 1), \varphi \neq 0, \\ +\infty, & \text{if } \varphi \notin H_0^1(0, 1). \end{cases}$$

Conventionally we set  $R_\beta(0) = 0$  for any  $\beta \in [0, +\infty]$ .

We recall that for any  $\beta \in [0, +\infty]$  the eigenvalues of our problem can be characterised by the min-max formula

$$\lambda_{k,\gamma}^\beta = \inf_{\substack{X_{k+1} \subset H^1(0,1) \\ \dim X_{k+1} = k+1}} \sup_{\varphi \in X_{k+1}} R_\beta(\varphi). \quad (\text{III.35})$$

**Remark III.3.2.** A first consequence of the above formula is that, for any  $0 < \beta < \beta^*$ , we can bound from below the smallest eigenvalue  $\lambda_{0,\gamma}^\beta$  as follows

$$\lambda_{0,\gamma}^\beta = \inf_{\substack{\varphi \in H^1(0,1) \\ \varphi \neq 0}} R_\beta(\varphi) \geq \frac{\beta}{\beta^*} \inf_{\substack{\varphi \in H^1(0,1) \\ \varphi \neq 0}} R_{\beta^*}(\varphi) = \frac{\beta}{\beta^*} \lambda_{0,\gamma}^{\beta^*}.$$

**Lemma III.3.3.** For any two parameters  $0 \leq \alpha < \beta \leq +\infty$ , we have the following strict inequality

$$\lambda_{k,\gamma}^\alpha < \lambda_{k,\gamma}^\beta, \quad \forall k \geq 0$$

In particular, for any parameter  $0 < \beta < +\infty$ , we have

$$\lambda_{k,\gamma}^N < \lambda_{k,\gamma}^\beta < \lambda_{k,\gamma}^D, \quad \forall k \geq 0.$$

*Proof.* From (III.35), we write

$$\begin{aligned} \lambda_{k,\gamma}^\alpha &= \inf_{\substack{X_{k+1} \subset H^1(0,1) \\ \dim X_{k+1} = k+1}} \sup_{\varphi \in X_{k+1}} R_\alpha(\varphi) \\ &\leq \sup_{\varphi \in \text{Span}\{\varphi_{0,\gamma}^\beta, \dots, \varphi_{k,\gamma}^\beta\}} R_\alpha(\varphi) \\ &\leq \sup_{\varphi \in \text{Span}\{\varphi_{0,\gamma}^\beta, \dots, \varphi_{k,\gamma}^\beta\}} R_\beta(\varphi) \\ &= R_\beta(\varphi_{k,\gamma}^\beta) = \lambda_{k,\gamma}^\beta. \end{aligned} \quad (\text{III.36})$$

Let us show that the inequality is in fact strict.

Assume first that  $\beta < +\infty$  and that there exists some  $k \geq 0$  such that  $\lambda_{k,\gamma}^\alpha = \lambda_{k,\gamma}^\beta$ . This implies that all the inequalities above in (III.36) are, in fact, equalities. Thus, there is some  $\tilde{\varphi} = \sum_{j=0}^k a_j \varphi_{j,\gamma}^\beta$  with  $\sum_{j=0}^k |a_j|^2 = 1$ , such that

$$\lambda_{k,\gamma}^\alpha = \sup_{\varphi \in \text{Span}\{\varphi_{0,\gamma}^\beta, \dots, \varphi_{k,\gamma}^\beta\}} R_\alpha(\varphi) = R_\alpha(\tilde{\varphi}),$$

which yields that

$$\int_0^1 \gamma(x) |\partial_x \tilde{\varphi}(x)|^2 dx + \alpha (|\tilde{\varphi}(0)|^2 + |\tilde{\varphi}(1)|^2) = \lambda_{k,\gamma}^\alpha = \lambda_{k,\gamma}^\beta. \quad (\text{III.37})$$

On the other hand, since each  $\varphi_{j,\gamma}^\beta$  is a  $L^2$ -normalized eigenfunction of the operator  $-\partial_x(\gamma(x)\partial_x)$  with the Robin boundary condition with parameter  $\beta$ , corresponding to eigenvalue  $\lambda_{j,\gamma}^\beta$  for  $0 \leq j \leq k$ , we see that  $\tilde{\varphi}$  enjoys the following

$$\int_0^1 \gamma(x) |\partial_x \tilde{\varphi}(x)|^2 dx + \beta (|\tilde{\varphi}(0)|^2 + |\tilde{\varphi}(1)|^2) = \sum_{j=0}^k \lambda_{j,\gamma}^\beta |a_j|^2 \leq \lambda_{k,\gamma}^\beta \sum_{j=0}^k |a_j|^2 = \lambda_{k,\gamma}^\beta. \quad (\text{III.38})$$

Since  $\beta < +\infty$ , and  $\alpha < \beta$ , we can compare (III.37) and (III.38) to deduce that  $\tilde{\varphi}(0) = \tilde{\varphi}(1) = 0$  and moreover

$$\sum_{j=0}^k \lambda_{j,\gamma}^\beta |a_j|^2 = \lambda_{k,\gamma}^\beta \sum_{j=0}^k |a_j|^2.$$



By (III.34), this equality implies that  $a_j = 0$  for any  $0 \leq j \leq k-1$  and then that  $\tilde{\varphi}$  is proportional to  $\varphi_{k,\gamma}^\beta$ . However, such an eigenfunction cannot vanish at  $x = 0$  (see for instance Theorem III.3.5 below) which is a contradiction.

In the case where  $\beta = +\infty$ , we use the previous results to simply write

$$\lambda_{k,\gamma}^\alpha < \lambda_{k,\gamma}^{\alpha+1} \leq \lambda_{k,\gamma}^\infty,$$

and the proof of the lemma is complete.  $\square$

**Remark III.3.4.** Let  $0 \leq \beta \leq +\infty$ . We denote by  $\lambda_{k,\gamma_{\min}}^\beta$  and  $\lambda_{k,\gamma_{\max}}^\beta$  for  $k \geq 0$ , the eigenvalues to the operator  $-\gamma_{\min}\partial_x^2$  and  $-\gamma_{\max}\partial_x^2$  respectively, with Robin boundary conditions with parameter  $\beta$ . Then from (III.35), one has the following inequality

$$\lambda_{k,\gamma_{\min}}^\beta \leq \lambda_{k,\gamma}^\beta \leq \lambda_{k,\gamma_{\max}}^\beta, \quad \forall k \geq 0 \text{ and } \forall \beta \in [0, +\infty].$$

Let us observe now that, for any non-trivial eigenfunction  $\varphi_{k,\gamma}^\beta$  of our problem (III.33), the quantity  $\varphi_{k,\gamma}^\beta(0)$  (and hence  $(\varphi_{k,\gamma}^\beta)'(0)$ ) is non-zero for any  $k \geq 0$  and  $\beta \in (0, +\infty)$ . In fact, we prove the following theorem that give bounds from below for those quantities.

**Theorem III.3.5.** *There exists a constant  $C > 0$  depending only on the diffusion coefficient  $\gamma$  such that we have*

$$|\varphi_{k,\gamma}^\beta(0)|^2 \left( 1 + \frac{\beta^2}{\gamma(0)\lambda_{k,\gamma}^\beta} \right) \geq C, \quad \forall k \geq 0, \quad \beta \in (0, +\infty), \quad (\text{III.39})$$

$$|(\varphi_{k,\gamma}^\beta)'(0)|^2 \left( \frac{(\gamma(0))^2}{\beta^2} + \frac{\gamma(0)}{\lambda_{k,\gamma}^\beta} \right) \geq C, \quad \forall k \geq 0, \quad \beta \in (0, +\infty), \quad (\text{III.40})$$

$$\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \geq C\sqrt{\lambda_{k+1,\gamma}^\beta}, \quad \forall k \geq 0, \quad \beta \in [0, +\infty]. \quad (\text{III.41})$$

We first state the following lemma which is a straightforward consequence of [4, Lemma 2.2 and Lemma 2.3].

**Lemma III.3.6.** *Let  $f : [0, 1] \rightarrow \mathbb{R}$  be a continuous function and  $\lambda > 0$ . Suppose that  $u : [0, 1] \rightarrow \mathbb{R}$  is smooth and satisfies the following second-order differential equation (without any assumptions on the boundary conditions):*

$$-\partial_x(\gamma\partial_x u)(x) = \lambda u(x) + f(x), \quad \forall x \in (0, 1), \quad (\text{III.42})$$

*then there exists  $C' > 0$ , depending only on  $\gamma$ , such that for all  $x, y \in (0, 1)$ , we have*

$$|u(y)|^2 + \frac{\gamma(y)}{\lambda} |u'(y)|^2 \leq C' \left( |u(x)|^2 + \frac{\gamma(x)}{\lambda} |u'(x)|^2 + \frac{1}{\lambda} \int_0^1 \frac{1}{\gamma(w)} |f(w)|^2 dw \right). \quad (\text{III.43})$$

**Remark III.3.7.** We recall that in [4, Lemma 2.2], the authors have assumed that  $\lambda \geq 1$  but this was related to the fact that they considered the slightly more general second-order differential equation

$$-\partial_x(\gamma\partial_x u)(x) + q(x)u(x) = \lambda u(x) + f(x), \quad \forall x \in (0, 1).$$

In our case we have  $q(x) \equiv 0$  in (III.42), and so having a careful look at the proof of [4, Lemma 2.3], one can observe that we simply need  $\lambda > 0$  for the constant  $C'$  in Lemma III.3.6 to be uniform with respect to  $\lambda$ .



*Proof of Theorem III.3.5.*

• We recall the eigenvalue problem (III.33) and apply Lemma III.3.6 to  $u = \varphi_{k,\gamma}^\beta$ ,  $\lambda = \lambda_{k,\gamma}^\beta$  and  $f = 0$  to obtain for each  $k \geq 0$  and  $\beta \in (0, +\infty)$  that

$$|\varphi_{k,\gamma}^\beta(x)|^2 + \frac{\gamma(x)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(x)|^2 \geq \frac{1}{C'} \left( |\varphi_{k,\gamma}^\beta(y)|^2 + \frac{\gamma(y)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(y)|^2 \right), \quad \forall x, y \in (0, 1).$$

Putting  $x = 0$  above and integrating over  $y \in (0, 1)$ , we obtain

$$|\varphi_{k,\gamma}^\beta(0)|^2 + \frac{\gamma(0)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(0)|^2 \geq \frac{1}{C'} \left( \|\varphi_{k,\gamma}^\beta\|_{L^2(0,1)}^2 + \int_0^1 \frac{\gamma(y)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(y)|^2 dy \right).$$

Thanks to the normalizing condition  $\|\varphi_{k,\gamma}^\beta\|_{L^2(0,1)} = 1$  and due to the positivity of the second integral in the right hand side of the last inequality, we have

$$|\varphi_{k,\gamma}^\beta(0)|^2 + \frac{\gamma(0)}{\lambda_{k,\gamma}^\beta} |(\varphi_{k,\gamma}^\beta)'(0)|^2 \geq \frac{1}{C'}.$$

- In one hand, we use the boundary condition of  $\varphi_{k,\gamma}^\beta$  at  $x = 0$  to express  $(\varphi_{k,\gamma}^\beta)'(0)$  as a function of  $\varphi_{k,\gamma}^\beta(0)$  and obtain (III.39).
- On the other hand, we use the same boundary condition to express  $\varphi_{k,\gamma}^\beta(0)$  as a function of  $(\varphi_{k,\gamma}^\beta)'(0)$  and obtain (III.40).
- Secondly, for any  $k \geq 0$  and  $\beta \in [0, +\infty]$ , we define

$$u(x) := \varphi_{k+1,\gamma}^\beta(x) \varphi_{k,\gamma}^\beta(0) - \varphi_{k,\gamma}^\beta(x) \varphi_{k+1,\gamma}^\beta(0), \quad \forall x \in (0, 1),$$

which satisfies

$$-\partial_x(\gamma \partial_x u)(x) = \lambda_{k+1,\gamma}^\beta u(x) + f(x), \quad \forall x \in (0, 1)$$

with

$$f(x) = (\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta) \varphi_{k,\gamma}^\beta(x) \varphi_{k+1,\gamma}^\beta(0).$$

Moreover, we observe that  $u(0) = 0$  and  $u'(0) = 0$ , from the construction of  $u$ . So, by taking  $x = 0$  in the inequality (III.43), we see

$$|u(y)|^2 + \frac{\gamma(y)}{\lambda_{k+1,\gamma}^\beta} |u'(y)|^2 \leq C' \frac{(\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta)^2}{\lambda_{k+1,\gamma}^\beta} |\varphi_{k+1,\gamma}^\beta(0)|^2 \int_0^1 \frac{1}{\gamma(w)} |\varphi_{k,\gamma}^\beta(w)|^2 dw,$$

for all  $y \in (0, 1)$ . Thanks to the normalizing condition  $\|\varphi_{k,\gamma}^\beta\|_{L^2(0,1)} = 1$  and the definition of  $u$  together implies

$$|\varphi_{k+1,\gamma}^\beta(y) \varphi_{k,\gamma}^\beta(0) - \varphi_{k,\gamma}^\beta(y) \varphi_{k+1,\gamma}^\beta(0)|^2 \leq C' \frac{(\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta)^2}{\gamma_{\min} \lambda_{k+1,\gamma}^\beta} |\varphi_{k+1,\gamma}^\beta(0)|^2,$$

for all  $y \in (0, 1)$ . Now integrating the left hand side over  $y \in (0, 1)$  and using the  $L^2$ -orthonormality condition of  $(\varphi_{k,\gamma}^\beta)_{k \geq 0}$  we have

$$|\varphi_{k,\gamma}^\beta(0)|^2 + |\varphi_{k+1,\gamma}^\beta(0)|^2 \leq C' \frac{(\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta)^2}{\gamma_{\min} \lambda_{k+1,\gamma}^\beta} |\varphi_{k+1,\gamma}^\beta(0)|^2,$$

which yields that

$$\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \geq C \sqrt{\lambda_{k+1,\gamma}^\beta}, \quad \forall k \geq 0 \text{ and } \beta \in [0, +\infty].$$

where the constant  $C$  depends only on  $\gamma$ .

The proof is complete. □

### III.3.2 The case of a constant diffusion coefficient

In this case, without loss of generality we can assume that  $\gamma \equiv 1$  on  $[0, 1]$  and so we can find a more explicit form of the eigenfunctions to the following problem

$$\begin{cases} -\partial_x^2 \varphi = \lambda \varphi & \text{in } (0, 1), \\ -\partial_x \varphi(0) + \beta \varphi(0) = 0, \\ \partial_x \varphi(1) + \beta \varphi(1) = 0. \end{cases} \quad (\text{III.44})$$

Let us first assume that  $\beta \in (0, +\infty)$ . Using the boundary condition at  $x = 0$ , and solving explicitly the differential equation, we shall look for  $\varphi_k^\beta$  in the following form

$$\varphi_k^\beta(x) = \frac{\sqrt{\lambda_k^\beta}}{\beta} \cos \sqrt{\lambda_k^\beta} x + \sin \sqrt{\lambda_k^\beta} x, \quad \forall x \in (0, 1), \quad \forall \beta > 0 \text{ and } \forall k \geq 0, \quad (\text{III.45})$$

where the eigenvalue  $\lambda_k^\beta$  will be required to satisfy the following transcendental equation

$$2\beta \sqrt{\lambda_k^\beta} \cos \sqrt{\lambda_k^\beta} + (\beta^2 - \lambda_k^\beta) \sin \sqrt{\lambda_k^\beta} = 0, \quad \forall \beta > 0 \text{ and } \forall k \geq 0. \quad (\text{III.46})$$

This equation is obtained from the boundary condition that  $\varphi_k^\beta$  should satisfy at  $x = 1$ .

Notice that, in order to simplify the formulas, we do not assume here that  $\varphi_k^\beta$  is normalised in  $L^2$ . This will not be a problem in the sequel since we will only use the fact that this family is complete in  $L^2$ .

**Remark III.3.8.** 1. We know that the family of eigenvalue-eigenfunctions of the operator  $-\partial_x^2$  with Dirichlet and Neumann boundary conditions are

$$\begin{aligned} \varphi_k^D(x) &= \sin((k+1)\pi x), \quad x \in [0, 1] \text{ with } \lambda_k^D = (k+1)^2\pi^2, \quad \forall k \geq 0, \text{ and} \\ \varphi_k^N(x) &= \cos(k\pi x), \quad x \in [0, 1] \text{ with } \lambda_k^N = k^2\pi^2, \quad \forall k \geq 0. \end{aligned}$$

From above, our first obvious observation is  $\lambda_k^D = \lambda_{k+1}^N$ ,  $\forall k \geq 0$ .

2. Secondly, one has  $\lambda_k^\beta \in (k^2\pi^2, (k+1)^2\pi^2)$ ,  $\forall k \geq 0$  and  $\beta \in (0, +\infty)$ , thanks to Lemma III.3.3. To be more precise,  $\lambda_k^\beta$  is the unique solution of (III.46) in the interval  $(k^2\pi^2, (k+1)^2\pi^2)$  for each  $k \geq 0$ .

**Remark III.3.9.** For any fixed  $\lambda \in (k^2\pi^2, (k+1)^2\pi^2)$ ,  $k \geq 0$ , the following quadratic equation

$$2\beta \sqrt{\lambda} \cos \sqrt{\lambda} + (\beta^2 - \lambda) \sin \sqrt{\lambda} = 0,$$

for the unknown  $\beta$ , has one and only one positive solution. Indeed, we see that the solutions of the above equation are given by

$$\beta' = \sqrt{\lambda} \tan \frac{\sqrt{\lambda}}{2} \text{ and } \beta'' = -\sqrt{\lambda} \cot \frac{\sqrt{\lambda}}{2},$$

that clearly have different signs. More precisely, since  $\sqrt{\lambda} \in (k\pi, (k+1)\pi)$  we can see that

- for  $k$  even we have  $\beta' > 0$  and  $\beta'' < 0$ ,
- for  $k$  odd, we have  $\beta' < 0$  and  $\beta'' > 0$ .

**Remark III.3.10.** One can obtain from the transcendental equation (III.46) that

$$\sqrt{\lambda_k^\beta} = k\pi + \frac{2\beta}{k\pi} + O_\beta\left(\frac{1}{k^3}\right), \quad \text{for } k \text{ large enough, say } k \geq k_\beta,$$

for some  $k_\beta \in \mathbb{N}$ , possibly depending on  $\beta$ . See, for instance [68, Problem Ib].

We have seen in Remark III.3.8 that the sequence of eigenvalues for the Dirichlet boundary condition and the one for the Neumann boundary condition almost coincide. The following lemma shows that, for other pairs of Robin parameters the corresponding sequences of eigenvalues are in fact disjoint.

**Lemma III.3.11.** *Consider two parameters  $\beta_1, \beta_2 \in [0, +\infty]$ , such that  $\beta_1 < \beta_2$ . If for some  $k, l$  we have*

$$\lambda_k^{\beta_2} = \lambda_l^{\beta_1},$$

*then we necessarily have*

$$\beta_1 = 0, \beta_2 = +\infty, \quad \text{and} \quad l = k + 1.$$

*Proof.* If  $\beta_1 > 0$ , then  $\lambda_l^{\beta_1} \in (l^2\pi^2, (l+1)^2\pi^2)$  and thus  $\sqrt{\lambda_k^{\beta_2}} \notin \pi\mathbb{N}^*$  and thus  $\beta_2 < +\infty$ . Similarly, if we assume  $\beta_2 < +\infty$  then we necessarily have  $\beta_1 > 0$ .

Therefore, there are now two cases:

- First case  $(\beta_1, \beta_2) = (0, +\infty)$ : the result follows from Remark III.3.8.
- Second case  $0 < \beta_1 < \beta_2 < +\infty$ : the common value  $\lambda$  of  $\lambda_l^{\beta_1}$  and  $\lambda_k^{\beta_2}$  simultaneously belongs to  $(l^2\pi^2, (l+1)^2\pi^2)$  and  $(k^2\pi^2, (k+1)^2\pi^2)$ , which implies that  $k = l$  and thus we have a contradiction with Lemma III.3.3.

□

### III.4 Boundary controllability of the 1D problem with single Robin parameter

This section is devoted to establish the one-dimensional boundary null-controllability of our cascade system with same non negative Robin parameter on both components and for any diffusion coefficient  $\gamma$  as defined in Section III.2.4.1. In that case, the system (III.1)-(III.3) simply reads as

$$\left\{ \begin{array}{ll} \partial_t y_1 - \partial_x(\gamma(x)\partial_x y_1) &= 0 \quad \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x(\gamma(x)\partial_x y_2) + y_1 &= 0 \quad \text{in } (0, T) \times (0, 1), \\ \gamma(x)\frac{\partial y_1}{\partial \nu}(t, x) + \beta y_1(t, x) &= \mathbf{1}_{\{x=0\}}v(t) \quad \text{on } (0, T) \times \{0, 1\}, \\ \gamma(x)\frac{\partial y_2}{\partial \nu}(t, x) + \beta y_2(t, x) &= 0 \quad \text{on } (0, T) \times \{0, 1\}, \\ y_1(0, \cdot) &= y_{0,1} \quad \text{in } (0, 1), \\ y_2(0, \cdot) &= y_{0,2} \quad \text{in } (0, 1), \end{array} \right. \quad (\text{III.47})$$

which is associated with the operator  $\mathcal{A}_{\beta, \beta}$  as introduced in (III.23), but specialized here to the one-dimensional setting, that is for  $\Omega = (0, 1)$ . To simplify the notation, we will simply denote this operator by  $\mathcal{A}_\beta$ , since the two Robin parameters are equal.

#### III.4.1 Spectrum of $\mathcal{A}_\beta^*$

We consider the eigenvalue problem

$$\mathcal{A}_\beta^* u = \lambda u, \quad \lambda \in \mathbb{C},$$

for a complex-valued function  $u$ , that is

$$\begin{cases} -\partial_x(\gamma(x)\partial_x u_1) + u_2 &= \lambda u_1 & \text{in } (0, 1), \\ -\partial_x(\gamma(x)\partial_x u_2) &= \lambda u_2 & \text{in } (0, 1), \\ \gamma(x)\frac{\partial u_1}{\partial \nu}(x) + \beta u_1(x) &= 0 & \text{for } x \in \{0, 1\}, \\ \gamma(x)\frac{\partial u_2}{\partial \nu}(x) + \beta u_2(x) &= 0 & \text{for } x \in \{0, 1\}. \end{cases} \quad (\text{III.48})$$

- Assume first that  $u_2 \neq 0$ . Multiplying the second equation of (III.48) by  $\overline{u_2}$  and integrating by parts, we obtain that  $\lambda = \lambda_{k,\gamma}^\beta$  for some  $k$  and that we can assume that  $u_2 = \varphi_{k,\gamma}^\beta$ . Moreover, taking the real or imaginary part, we can assume that  $u_1$  is real-valued, then multiplying the first equation by  $u_2$  and integrating by parts, we obtain that  $\int_0^1 |u_2|^2 = 0$  which is a contradiction.
- We have proven that, necessarily,  $u_2 = 0$  and so from the first equation of (III.48), we deduce that  $\lambda = \lambda_{k,\gamma}^\beta$  for some  $k$  and that, up to a multiplicative constant, we have  $u_1 = \varphi_{k,\gamma}^\beta$ .

Hence, the eigenfunctions of  $\mathcal{A}_\beta^*$  are

$$\Phi_{k,\gamma}^\beta := \begin{pmatrix} \varphi_{k,\gamma}^\beta \\ 0 \end{pmatrix} \text{ corresponding to the eigenvalues } \lambda_{k,\gamma}^\beta, \forall k \geq 0. \quad (\text{III.49})$$

We observe that the set  $\{\Phi_{k,\gamma}^\beta\}_{k \geq 0}$  is not sufficient generate the whole space  $(L^2(0, 1))^2$  because the second component of  $\Phi_{k,\gamma}^\beta$  is 0 for each  $k \geq 0$ . Hence, we need to look for the generalized eigenfunctions by solving the following problem

$$\mathcal{A}_\beta^* u = \lambda_{k,\gamma}^\beta u + \Phi_{k,\gamma}^\beta, \quad \forall k \geq 0,$$

that is

$$\begin{cases} -\partial_x(\gamma(x)\partial_x u_1) + u_2 &= \lambda_{k,\gamma}^\beta u_1 + \varphi_{k,\gamma}^\beta & \text{in } (0, 1), \\ -\partial_x(\gamma(x)\partial_x u_2) &= \lambda_{k,\gamma}^\beta u_2 & \text{in } (0, 1), \\ \gamma(x)\frac{\partial u_1}{\partial \nu}(x) + \beta u_1(x) &= 0 & \text{for } x \in \{0, 1\}, \\ \gamma(x)\frac{\partial u_2}{\partial \nu}(x) + \beta u_2(x) &= 0 & \text{for } x \in \{0, 1\}. \end{cases} \quad (\text{III.50})$$

The second equation shows that  $u_2 = a\varphi_{k,\gamma}^\beta$  for some  $a \in \mathbb{R}$ . But multiplying the first equation by  $(u_2 - \varphi_{k,\gamma}^\beta)$ , i.e.,  $(a - 1)\varphi_{k,\gamma}^\beta$  and performing an integration by parts yields us that the only admissible value for  $a$  is 1.

Now its enough to take  $u_1 = 0$ , which is by default an admissible solution of the system (III.50) and hence the generalized eigenfunctions can be interpreted as

$$\Psi_{k,\gamma}^\beta = \begin{pmatrix} 0 \\ \varphi_{k,\gamma}^\beta \end{pmatrix}, \quad \forall k \geq 0. \quad (\text{III.51})$$

We observe now that, the family  $\{\Phi_{k,\gamma}^\beta, \Psi_{k,\gamma}^\beta\}_{k \geq 0}$  is a Riesz basis of  $(L^2(0, 1))^2$ , made of eigenfunctions and generalized eigenfunctions of the operator  $\mathcal{A}_\beta^*$ . By construction we simply have

$$\begin{cases} e^{-t\mathcal{A}_\beta^*} \Phi_{k,\gamma}^\beta = e^{-t\lambda_{k,\gamma}^\beta} \Phi_{k,\gamma}^\beta, & \forall t \in [0, T], \\ e^{-t\mathcal{A}_\beta^*} \Psi_{k,\gamma}^\beta = e^{-t\lambda_{k,\gamma}^\beta} (\Psi_{k,\gamma}^\beta - t\Phi_{k,\gamma}^\beta), & \forall t \in [0, T]. \end{cases} \quad (\text{III.52})$$

**Remark III.4.1** (Approximate controllability). *We observe that the eigenfunctions of  $\mathcal{A}_\beta^*$  are observable, in the sense that*

$$\mathcal{B}^* \Phi_{k,\gamma}^\beta = \varphi_{k,\gamma}^\beta(0) \neq 0, \quad \forall k \geq 0,$$

where  $\mathcal{B}^*$  is given by (III.28).

By using the Fattorini-Hautus test (the hypothesis of which are fulfilled in our case, see for instance [55, 89]), we deduce that the control system (III.47), with any  $\beta > 0$ , is approximately controllable at any time  $T > 0$ .

### III.4.2 Null-controllability

We are now in position to prove the null-controllability of our system, with a precise bound of the control with respect to  $\beta$ , that is the point 1 of Theorem III.2.14.

#### III.4.2.1 The moments problem

We recall that  $\{\Phi_{k,\gamma}^\beta, \Psi_{k,\gamma}^\beta\}_{k \geq 0}$  (defined by (III.49)-(III.51)) forms a complete family in  $(L^2(0, 1))^2$ , so it is enough to check the controllability equation (III.29) (with the operator  $\mathcal{A}_\beta^*$  here) for  $\Phi_{k,\gamma}^\beta$  and  $\Psi_{k,\gamma}^\beta$  for each  $k \geq 0$ . This indeed tells us, for any  $y_0 \in (L^2(0, 1))^2$ , that the input  $v \in L^2(0, T)$  is a null control for (III.47) if and only if we have

$$\begin{cases} -e^{-T\lambda_{k,\gamma}^\beta} (y_0, \Phi_{k,\gamma}^\beta)_{L^2(0,1)} = \int_0^T v(t) e^{-(T-t)\lambda_{k,\gamma}^\beta} \mathcal{B}^* \Phi_{k,\gamma}^\beta dt, & \forall k \geq 0, \\ -e^{-T\lambda_{k,\gamma}^\beta} (y_0, \Psi_{k,\gamma}^\beta - T\Phi_{k,\gamma}^\beta)_{L^2(0,1)} \\ \quad = \int_0^T v(t) e^{-(T-t)\lambda_{k,\gamma}^\beta} \mathcal{B}^* (\Psi_{k,\gamma}^\beta - (T-t)\Phi_{k,\gamma}^\beta) dt, & \forall k \geq 0, \end{cases}$$

using the formulas given by (III.52). Now since  $\mathcal{B}^* \Phi_{k,\gamma}^\beta = \varphi_{k,\gamma}^\beta(0) \neq 0$  and  $\mathcal{B}^* \Psi_{k,\gamma}^\beta = 0$  for each  $k \geq 0$ , we can simplify the above set of equations as

$$\begin{cases} -\frac{e^{-T\lambda_{k,\gamma}^\beta}}{\varphi_{k,\gamma}^\beta(0)} (y_{0,1}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} = \int_0^T v(t) e^{-\lambda_{k,\gamma}^\beta(T-t)} dt, & \forall k \geq 0, \\ -\frac{e^{-T\lambda_{k,\gamma}^\beta}}{\varphi_{k,\gamma}^\beta(0)} \left( T(y_{0,1}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} - (y_{0,2}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} \right) \\ \quad = \int_0^T v(t) (T-t) e^{-\lambda_{k,\gamma}^\beta(T-t)} dt, & \forall k \geq 0. \end{cases} \quad (\text{III.53})$$

The above set of equations is the moments problem that we shall solve in our case.

#### III.4.2.2 Existence of a bi-orthogonal family to real exponentials

To construct our control  $v$  by solving the moments problem above, the existence of a suitable bi-orthogonal family to time-dependent exponential functions is one the most important ingredient.

We deal with the real sequence  $(\lambda_{k,\gamma}^\beta)_{k \geq 0}$  and we show that this sequence satisfies all the assumptions (uniform in  $\beta \geq 0$ ) of Theorem II.2.9 prescribed in Chapter II. We mainly focus on proving the points 4, 5 and the other points are trivial to observe.

##### 1. The gap condition:

Without loss of generality we assume that  $k > n$  and therefore,  $k = n + m$  for some  $m \in \mathbb{N}$ .

We recall (III.41) and Lemma III.3.3 to observe that

$$\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \geq C \sqrt{\lambda_{k+1,\gamma}^\beta} \geq C \sqrt{\lambda_{k+1,\gamma}^N}, \quad \forall k \geq 0, \quad \forall \beta \geq 0. \quad (\text{III.54})$$

Also, by Remark III.3.4, we have  $\lambda_{k+1,\gamma}^N \geq \lambda_{k+1,\gamma_{\min}}^N$ , and since it is easy to observe from Remark III.3.8 that  $\lambda_{k,\gamma_{\min}}^N = \gamma_{\min} k^2 \pi^2$  for each  $k \geq 0$ , so the inequality (III.54) is simplified as

$$\lambda_{k+1,\gamma}^\beta - \lambda_{k,\gamma}^\beta \geq C\sqrt{\gamma_{\min}}(k+1)\pi, \quad \forall k \geq 0, \quad \forall \beta \geq 0, \quad (\text{III.55})$$

which gives us

$$\begin{aligned} \lambda_{n+m,\gamma}^\beta - \lambda_{n,\gamma}^\beta &\geq C\sqrt{\gamma_{\min}} \pi \sum_{j=n+1}^{n+m} j = C\sqrt{\gamma_{\min}} \pi \left[ mn + \frac{m(m+1)}{2} \right] \\ &= \frac{C}{2} \sqrt{\gamma_{\min}} \pi \left[ (m+n)^2 - n^2 + m \right]. \end{aligned}$$

Thus for any  $k, n$  with  $k \geq n+1$ , and for any  $\beta \geq 0$ , we have

$$\lambda_{k,\gamma}^\beta - \lambda_{n,\gamma}^\beta \geq \rho(k^2 - n^2), \quad \forall k, n : k - n \geq 1.$$

with  $\rho := \frac{C}{2} \sqrt{\gamma_{\min}} \pi$ , that proves point 4 of Theorem II.2.9.

## 2. The counting function:

Let  $\mathcal{N}$  be the counting function associated with the sequence  $(\lambda_{k,\gamma}^\beta)_{k \geq 0}$ , defined by

$$\mathcal{N}(r) = \#\{k : \lambda_{k,\gamma}^\beta \leq r\}, \quad \forall r > 0.$$

We observe that, the function  $\mathcal{N}$  is piecewise constant and non-decreasing in the interval  $[0, +\infty)$ . Also for every  $r \in [0, +\infty)$  we have  $\mathcal{N}(r) < +\infty$  and  $\lim_{r \rightarrow +\infty} \mathcal{N}(r) = +\infty$ . Moreover,

$$\mathcal{N}(r) = k \iff \lambda_{k,\gamma}^\beta \leq r \text{ and } \lambda_{k+1,\gamma}^\beta > r,$$

so that, in particular, if  $\mathcal{N}(r) = k$ , we have

$$\sqrt{\lambda_{k,\gamma}^\beta} \leq \sqrt{r} < \sqrt{\lambda_{k+1,\gamma}^\beta},$$

which yields, by Lemma III.3.3 and Remark III.3.4, that

$$\sqrt{\lambda_{k,\gamma_{\min}}^N} \leq \sqrt{\lambda_{k,\gamma}^N} \leq \sqrt{r} < \sqrt{\lambda_{k+1,\gamma}^D} \leq \sqrt{\lambda_{k+1,\gamma_{\max}}^D}$$

But we have  $\lambda_{k,\gamma_{\min}}^N = \gamma_{\min} k^2 \pi^2$  and  $\lambda_{k+1,\gamma_{\max}}^D = \gamma_{\max} (k+2)^2 \pi^2$ , hence

$$\sqrt{\gamma_{\min}} k \pi \leq \sqrt{r} < \sqrt{\gamma_{\max}} (k+2) \pi.$$

Replacing  $k$  by  $\mathcal{N}(r)$ , we determine that

$$\frac{1}{\sqrt{\gamma_{\max}} \pi} \sqrt{r} - 2 < \mathcal{N}(r) \leq \frac{1}{\sqrt{\gamma_{\min}} \pi} \sqrt{r} < \frac{1}{\sqrt{\gamma_{\min}} \pi} \sqrt{r} + 2,$$

which is the point 5 of Theorem II.2.9 with  $\theta = 2$ ,  $c_{\min} = \frac{1}{\sqrt{\gamma_{\max}} \pi}$  and  $c_{\max} = \frac{1}{\sqrt{\gamma_{\min}} \pi}$ .

From the discussion above, we can ensure for some  $T_0 > 0$  that there exist a bi-orthogonal family in  $L^2(0, T)$ , denoted by  $(q_{k,j}^\beta)_{k \geq 0, 0 \leq j \leq 1}$ , to the family of exponential functions  $((T-t)^i e^{-\lambda_{k,\gamma}^\beta (T-t)})_{k \geq 0, 0 \leq i \leq 1}$  for any  $0 < T < T_0$ . Moreover, this family satisfies the following estimates

$$\|q_{k,j}^\beta\|_{L^2(0,T)} \leq C e^{C\sqrt{\lambda_{k,\gamma}^\beta + \frac{C}{T}}}, \quad \forall k \geq 0, j = 0, 1, \quad 0 < T < T_0, \quad (\text{III.56})$$

where the constant  $C > 0$  is independent on  $T \in (0, T_0)$  and uniform with respect to  $k \geq 0$  and to the parameter  $\beta \geq 0$  since all the quantities  $\rho$ ,  $\theta$ ,  $c_{\min}$  and  $c_{\max}$  introduced above do not depend on the Robin parameter  $\beta \geq 0$ .

### III.4.2.3 The controllability result

We can now proceed to the proof of the null-controllability result in that case.

*Proof of point 1 of Theorem III.2.14.*

For  $0 < T < T_0$ , we consider

$$v_\beta(t) = \sum_{k \geq 0} v_k^\beta(t), \quad \forall t \in (0, T), \quad \text{where} \quad (\text{III.57})$$

$$\begin{aligned} v_k^\beta(t) = & -\frac{e^{-T\lambda_{k,\gamma}^\beta}}{\varphi_{k,\gamma}^\beta(0)} (y_{0,1}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} q_{k,0}^\beta(t) \\ & - \frac{e^{-T\lambda_{k,\gamma}^\beta}}{\varphi_{k,\gamma}^\beta(0)} \left( T(y_{0,1}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} - (y_{0,2}, \varphi_{k,\gamma}^\beta)_{L^2(0,1)} \right) q_{k,1}^\beta(t). \end{aligned}$$

With this choice of  $v = v_\beta$ , one can observe that the set of moment equations in (III.53) are formally satisfied. Now, all we have to check is the convergence of the series (III.57) in  $L^2(0, T)$ , with explicit bounds with respect to  $\beta \in (0, +\infty)$ . To this end, for each  $k \geq 0$ , we compute

$$\begin{aligned} \|v_k^\beta\|_{L^2(0,T)} \leq & \frac{\|y_{0,1}\|_{L^2(0,1)}}{|\varphi_{k,\gamma}^\beta(0)|} e^{-T\lambda_{k,\gamma}^\beta} \|q_{k,0}^\beta\|_{L^2(0,T)} \\ & + \frac{T\|y_{0,1}\|_{L^2(0,1)} + \|y_{0,2}\|_{L^2(0,1)}}{|\varphi_{k,\gamma}^\beta(0)|} e^{-T\lambda_{k,\gamma}^\beta} \|q_{k,1}^\beta\|_{L^2(0,T)}, \end{aligned}$$

thanks to the normalizing condition  $\|\varphi_{k,\gamma}^\beta\|_{L^2(0,1)} = 1$ . Moreover, the result (III.39) gives us

$$\frac{1}{|\varphi_{k,\gamma}^\beta(0)|} \leq C \left( 1 + \frac{\beta}{\sqrt{\lambda_{k,\gamma}^\beta}} \right), \quad \forall k \geq 0, \quad \beta \in (0, +\infty), \quad (\text{III.58})$$

where  $C$  depends only on  $\gamma$ .

Now, using (III.58) and the bounds on bi-orthogonal functions in (III.56), we deduce for each  $k \geq 0$  and for any finite  $0 < T < T_0$  that

$$\|v_k^\beta\|_{L^2(0,T)} \leq C(T+2) \left( 1 + \frac{\beta}{\sqrt{\lambda_{k,\gamma}^\beta}} \right) e^{-T\lambda_{k,\gamma}^\beta} e^{C\sqrt{\lambda_{k,\gamma}^\beta} + \frac{C}{T}} \|y_0\|_{L^2(0,1)}, \quad (\text{III.59})$$

since  $\gamma(0) > \gamma_{\min} > 0$ . Now, Young's inequality gives us

$$C\sqrt{\lambda_{k,\gamma}^\beta} \leq \frac{T}{2}\lambda_{k,\gamma}^\beta + \frac{C^2}{2T}, \quad \forall k \geq 0.$$

Thus, we see

$$e^{-T\lambda_{k,\gamma}^\beta + C\sqrt{\lambda_{k,\gamma}^\beta} + \frac{C}{T}} \leq e^{-\frac{T}{2}\lambda_{k,\gamma}^\beta + \frac{C}{T}}, \quad \forall k \geq 0. \quad (\text{III.60})$$

But we have,  $\lambda_{k,\gamma}^\beta \geq \lambda_{k,\gamma}^N \geq \lambda_{k,\gamma_{\min}}^N = \gamma_{\min} k^2 \pi^2$  for  $k \geq 0$ , and so, in one hand, we have

$$\sum_{k \geq 0} e^{-\frac{T}{2}\lambda_{k,\gamma}^\beta} \leq \sum_{k \geq 0} e^{-Ck^2 \frac{T}{2}} \leq \frac{1}{1 - e^{-CT/2}} \leq C \left( 1 + \frac{1}{T} \right). \quad (\text{III.61})$$

On the other hand, we see

$$\begin{aligned}
\sum_{k \geq 0} \frac{\beta}{\sqrt{\lambda_{k,\gamma}^\beta}} e^{-\frac{T}{2} \lambda_{k,\gamma}^\beta} &\leq \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} e^{-\frac{T}{2} \lambda_{0,\gamma}^\beta} + \frac{\beta}{\pi \sqrt{\gamma_{\min}}} \sum_{k \geq 1} \frac{1}{k} e^{-\pi^2 \gamma_{\min} k^2 \frac{T}{2}} \\
&\leq \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} + C\beta \sum_{k \geq 1} e^{-Ck^2 \frac{T}{2}}.
\end{aligned} \tag{III.62}$$

The second quantity in the right hand side of (III.62) has the bound  $C\beta \left(1 + \frac{1}{T}\right)$  where the bound of first quantity is not so obvious because we see that  $\lambda_{0,\gamma}^\beta$  is getting smaller with respect to smaller  $\beta > 0$ . But, we have

$$\begin{cases} \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} \leq \frac{\sqrt{\beta}}{\sqrt{\lambda_{0,\gamma}^1}} \leq C(1 + \beta), \text{ for } 0 < \beta < 1, \text{ by Remark III.3.2, and} \\ \frac{\beta}{\sqrt{\lambda_{0,\gamma}^\beta}} \leq C\beta, \text{ for } \beta \geq 1, \text{ which is easy to observe.} \end{cases}$$

Now, using this in (III.62), one can deduce that

$$\sum_{k \geq 0} \frac{\beta}{\sqrt{\lambda_{k,\gamma}^\beta}} e^{-\frac{T}{2} \lambda_{k,\gamma}^\beta} \leq C(1 + \beta) \left(1 + \frac{1}{T}\right). \tag{III.63}$$

Now, we take summation over  $k \geq 0$  in (III.59), and using the inequality (III.60), and then the estimates (III.61), (III.63), we get for any  $\beta \in (0, +\infty)$  and finite  $0 < T < T_0$  that

$$\sum_{k \geq 0} \|v_k^\beta\|_{L^2(0,T)} \leq C(1 + \beta) e^{C/T} \|y_0\|_{L^2(0,1)}.$$

Finally, for any time  $T \geq T_0$ , the proof actually reduces to the previous one; since any continuation by 0 (in  $(T_0, T)$ ) of a control in  $(0, T_0)$  does also act as a control in  $(0, T)$  and the estimate follows from the cost of the control  $e^{C/T}$ .

This completes the proof.  $\square$

### III.5 Boundary controllability result of the 1D problem with different Robin parameters

In this section, we discuss about the boundary controllability of the system (III.1)-(III.3) in 1D with two different parameters  $\beta_1 \neq \beta_2$  with  $\beta_1, \beta_2 \in (0, +\infty)$  for the boundary conditions of the two components  $y_1$  and  $y_2$  respectively and as mentioned in the introduction of this chapter, we assume now that  $\gamma$  is a positive constant that we can choose to be equal to 1. Taking into account these facts, we write the system below

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \frac{\partial y_1}{\partial \nu}(t, x) + \beta_1 y_1(t, x) = \mathbb{1}_{\{x=0\}} v(t) & \text{on } (0, T) \times \{0, 1\}, \\ \frac{\partial y_2}{\partial \nu}(t, x) + \beta_2 y_2(t, x) = 0 & \text{on } (0, T) \times \{0, 1\}, \\ y_1(0, \cdot) = y_{0,1} & \text{in } (0, 1), \\ y_2(0, \cdot) = y_{0,2} & \text{in } (0, 1). \end{cases} \tag{III.64}$$

In this case, we recall that the associated operator is  $\mathcal{A}_{\beta_1, \beta_2}$ , as defined in (III.23), specified here for  $\Omega = (0, 1)$  and for  $\gamma \equiv 1$ .



The main difference between the present section and Section III.4 concerns the spectral properties of the adjoint operators. Unlike the previous case, we will have here a possible condensation of eigenvalues with two different sets of eigenfunctions that form a complete family of the state space, instead of having well-separated eigenvalues and associated generalized eigenfunctions.

### III.5.1 Spectrum of $\mathcal{A}_{\beta_1, \beta_2}^*$

In the present situation, the eigenvalue problem associated with  $\mathcal{A}_{\beta_1, \beta_2}^*$  is explicitly given by

$$\begin{cases} -\partial_x^2 u_1 + u_2 &= \lambda u_1 & \text{in } (0, 1), \\ -\partial_x^2 u_2 &= \lambda u_2 & \text{in } (0, 1), \\ \frac{\partial u_1}{\partial \nu}(x) + \beta_1 u_1(x) &= 0 & \text{for } x \in \{0, 1\}, \\ \frac{\partial u_2}{\partial \nu}(x) + \beta_2 u_2(x) &= 0 & \text{for } x \in \{0, 1\}. \end{cases} \quad (\text{III.65})$$

*First case:* Assume that  $u_2 = 0$ , then our system (III.65) reduces to the Robin eigenvalue problem (III.44) with positive parameter  $\beta_1$  and this gives us the solution  $u_1 = \varphi_k^{\beta_1}$  corresponding  $\lambda = \lambda_k^{\beta_1}$  which is real for any  $k \geq 0$  (recall that,  $\varphi_k^{\beta}$  has already been given by (III.45) for all  $k \geq 0$  and  $\beta \in (0, +\infty)$ ). This gives us the following set of eigenfunctions (without normalizing) of  $\mathcal{A}_{\beta_1, \beta_2}^*$

$$\Phi_{k,1} := \begin{pmatrix} \varphi_k^{\beta_1} \\ 0 \end{pmatrix} \text{ corresponding to the eigenvalues } \lambda_k^{\beta_1}, \quad \forall k \geq 0. \quad (\text{III.66})$$

*Second case:* Assume now that  $u_2 \neq 0$ , then we first solve the second set of equations of (III.65), i.e.,

$$\begin{cases} -\partial_x^2 u_2 &= \lambda u_2 & \text{in } (0, 1), \\ \frac{\partial u_2}{\partial \nu}(x) + \beta_2 u_2(x) &= 0 & \text{for } x \in \{0, 1\}, \end{cases}$$

which gives  $u_2 = \varphi_k^{\beta_2}$ , up to a multiplicative constant (which we can take 1), corresponding to  $\lambda = \lambda_k^{\beta_2}$  for all  $k \geq 0$ . Now by implementing  $u_2 = \varphi_k^{\beta_2}$  for each  $k \geq 0$  to the first equation of (III.65) address us the following problem

$$\begin{cases} -\partial_x^2 u_1 + \varphi_k^{\beta_2} &= \lambda_k^{\beta_2} u_1 & \text{in } (0, 1), \\ -\partial_x u_1(0) + \beta_1 u_1(0) &= 0, \\ \partial_x u_1(1) + \beta_1 u_1(1) &= 0. \end{cases} \quad (\text{III.67})$$

The existence and uniqueness of the solution to (III.67) follows from the Fredholm alternative theorem and to the fact that  $\lambda_k^{\beta_2} \notin (\lambda_l^{\beta_1})_{l \geq 0}$  for any  $k \geq 0$  and  $\beta_1 \neq \beta_2$  (by Lemma III.3.11). Let us denote the unique solution  $u_1$  of (III.67) by  $\psi_k^{\beta_1, \beta_2}$ , for  $k \geq 0$  and hence the second set of eigenfunctions (without normalizing) of  $\mathcal{A}_{\beta_1, \beta_2}^*$  is given by

$$\Phi_{k,2} := \begin{pmatrix} \psi_k^{\beta_1, \beta_2} \\ \varphi_k^{\beta_2} \end{pmatrix} \text{ corresponding to the eigenvalues } \lambda_k^{\beta_2}, \quad \forall k \geq 0. \quad (\text{III.68})$$

The family  $\{\Phi_{k,1}, \Phi_{k,2}\}_{k \geq 0}$  is complete in  $(L^2(0, 1))^2$ , and we observe that

$$\begin{cases} e^{-t\mathcal{A}_{\beta_1, \beta_2}^*} \Phi_{k,1} = e^{-t\lambda_k^{\beta_1}} \Phi_{k,1}, & \forall k \geq 0, \\ e^{-t\mathcal{A}_{\beta_1, \beta_2}^*} \Phi_{k,2} = e^{-t\lambda_k^{\beta_2}} \Phi_{k,2}, & \forall k \geq 0. \end{cases} \quad (\text{III.69})$$

### III.5.1.1 More on spectral properties and approximate controllability

This section is devoted to show some properties of the first component  $\psi_k^{\beta_1, \beta_2}$  of the eigenfunction  $\Phi_{k,2}$  and how the spectral gap  $|\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|$  depends on the parameters  $\beta_1, \beta_2$  (for any  $k \geq 0$ ). We need all these to find a proper bound of our null-control.

Proving the estimates of this section for any non constant diffusion coefficient  $\gamma$  is still an open problem, that is why we restrict here our attention to the constant coefficient case.

**Lemma III.5.1.** *Let  $\beta_1 \neq \beta_2$  be any two real parameters with  $\beta_1, \beta_2 \in (0, +\infty)$  and  $(\psi_k^{\beta_1, \beta_2})_{k \geq 0}$  be the set of solutions to (III.67) as introduced before. Then, we have*

$$|\psi_k^{\beta_1, \beta_2}(0)| \geq \frac{\lambda_k^{\beta_2} + \beta_2^2}{4\beta_2 \sqrt{\lambda_k^{\beta_2}} |\beta_1 - \beta_2|}, \quad \forall k \geq 0.$$

*Proof.*

- Observe that  $\psi_k^{\beta_1, \beta_2}$  satisfies the second order ordinary differential equation (III.67), i.e.,

$$\frac{d^2 \psi_k^{\beta_1, \beta_2}}{dx^2} + \lambda_k^{\beta_2} \psi_k^{\beta_1, \beta_2} = \varphi_k^{\beta_2}, \quad \text{for each } k \geq 0. \quad (\text{III.70})$$

To solve this, recall the explicit form of  $\varphi_k^{\beta_2}$  from (III.45) (with  $\beta = \beta_2$ ), and hence for each  $k \geq 0$ , we are looking for  $\psi_k^{\beta_1, \beta_2}$  in the following form

$$\psi_k^{\beta_1, \beta_2}(x) = (A_k x + B_k) \cos(\sqrt{\lambda_k^{\beta_2}} x) + (C_k x + D_k) \sin(\sqrt{\lambda_k^{\beta_2}} x), \quad \forall x \in [0, 1], \quad (\text{III.71})$$

where  $A_k, B_k, C_k, D_k$  for  $k \geq 0$ , are real constants to determine.

Substituting (III.71) into the equation (III.70), we get

$$A_k = -\frac{1}{2\sqrt{\lambda_k^{\beta_2}}} \quad \text{and} \quad C_k = \frac{1}{2\beta_2}, \quad \forall k \geq 0. \quad (\text{III.72})$$

Then, from the boundary conditions at  $x = 0$  and 1 satisfying by  $\psi_k^{\beta_1, \beta_2}$  for  $k \geq 0$ , one can obtain

$$D_k \sqrt{\lambda_k^{\beta_2}} = \beta_1 B_k - A_k, \quad \text{and}$$

$$\begin{aligned} B_k & \left( 2\beta_1 \sqrt{\lambda_k^{\beta_2}} \cos \sqrt{\lambda_k^{\beta_2}} + (\beta_1^2 - \lambda_k^{\beta_2}) \sin \sqrt{\lambda_k^{\beta_2}} \right) \\ &= \left( \frac{\beta_1 \beta_2 - \lambda_k^{\beta_2}}{2\beta_2} \right) \cos \sqrt{\lambda_k^{\beta_2}} - \left( \frac{(1 + \beta_1 + \beta_2) \lambda_k^{\beta_2} + \beta_1 \beta_2}{2\beta_2 \sqrt{\lambda_k^{\beta_2}}} \right) \sin \sqrt{\lambda_k^{\beta_2}} \end{aligned} \quad (\text{III.73})$$

respectively. We must mention here that the coefficient of  $B_k$  in the left hand side of (III.73) never vanishes due to Remark III.3.9.

- By (III.46), it is known that the eigenvalue  $\lambda_k^{\beta_2}$  is the unique solution to

$$2\beta_2 \sqrt{\lambda_k^{\beta_2}} \cos \sqrt{\lambda_k^{\beta_2}} + (\beta_2^2 - \lambda_k^{\beta_2}) \sin \sqrt{\lambda_k^{\beta_2}} = 0, \quad \text{for each } k \geq 0. \quad (\text{III.74})$$

Now by substituting the expression of  $\cos \sqrt{\lambda_k^{\beta_2}}$  from (III.74) into (III.73) and replacing  $B_k$  by  $\psi_k^{\beta_1, \beta_2}(0)$ , we get

$$\begin{aligned} & \frac{1}{\beta_2} \psi_k^{\beta_1, \beta_2}(0) \left[ \beta_1 (\lambda_k^{\beta_2} - \beta_2^2) + \beta_2 (\beta_1^2 - \lambda_k^{\beta_2}) \right] \\ &= \frac{(\beta_1 \beta_2 - \lambda_k^{\beta_2})(\lambda_k^{\beta_2} - \beta_2^2)}{4\beta_2^2 \sqrt{\lambda_k^{\beta_2}}} - \frac{(1 + \beta_1 + \beta_2) \lambda_k^{\beta_2} + \beta_1 \beta_2}{2\beta_2 \sqrt{\lambda_k^{\beta_2}}}, \end{aligned}$$

where we omitted  $\sin \sqrt{\lambda_k^{\beta_2}}$  from both sides since  $\sin \sqrt{\lambda_k^{\beta_2}} \neq 0$  for all  $k \geq 0$  and  $\beta_2 \in (0, +\infty)$ . Now by simplifying the above equality provides us

$$\psi_k^{\beta_1, \beta_2}(0) = \frac{-(\lambda_k^{\beta_2})^2 - \beta_1 \beta_2 \lambda_k^{\beta_2} - \beta_2^2 \lambda_k^{\beta_2} - 2\beta_2 \lambda_k^{\beta_2} - 2\beta_1 \beta_2^2 - \beta_1 \beta_2^3}{4\beta_2 \sqrt{\lambda_k^{\beta_2}} [\beta_1 (\lambda_k^{\beta_2} - \beta_2^2) + \beta_2 (\beta_1^2 - \lambda_k^{\beta_2})]} =: \frac{I_k}{J_k}. \quad (\text{III.75})$$

Here one can rewrite the quantity  $J_k$  as

$$J_k = 4\beta_2 \sqrt{\lambda_k^{\beta_2}} (\beta_1 - \beta_2) (\lambda_k^{\beta_2} + \beta_1 \beta_2), \quad \forall k \geq 0, \quad (\text{III.76})$$

whereas,  $I_k$  enjoys the following

$$|I_k| \geq (\lambda_k^{\beta_2})^2 + \beta_1 \beta_2 \lambda_k^{\beta_2} + \beta_2^2 \lambda_k^{\beta_2} + \beta_1 \beta_2^3 = (\lambda_k^{\beta_2} + \beta_2^2) (\lambda_k^{\beta_2} + \beta_1 \beta_2), \quad \forall k \geq 0. \quad (\text{III.77})$$

To this end, we use (III.76) and (III.77) in the expression (III.75) to deduce that

$$|\psi_k^{\beta_1, \beta_2}(0)| \geq \frac{(\lambda_k^{\beta_2} + \beta_2^2)}{4\beta_2 \sqrt{\lambda_k^{\beta_2}} |\beta_1 - \beta_2|}, \quad \forall k \geq 0,$$

and this concludes the lemma.  $\square$

**Remark III.5.2** (Approximate controllability). *The control problem (III.64) is approximately controllable for any finite time  $T > 0$ .*

*To prove this, we will again use Fattorini-Hautus test as mentioned in Remark III.4.1, that is to show  $\mathcal{B}^* \Phi_{k,1} \neq 0$  and  $\mathcal{B}^* \Phi_{k,2} \neq 0$  for each  $k \geq 0$  (where  $\mathcal{B}^*$  has been defined in (III.28)). But (III.45) and Lemma III.5.1 respectively ensure us*

$$\mathcal{B}^* \Phi_{k,1} = \varphi_k^{\beta_1}(0) \neq 0 \text{ and } \mathcal{B}^* \Phi_{k,2} = \psi_k^{\beta_1, \beta_2}(0) \neq 0, \quad \forall k \geq 0,$$

*which gives the claim.*

**Lemma III.5.3.** *Let  $\beta_1 \neq \beta_2$  be any two parameters such that  $\beta_1, \beta_2 \in (0, +\infty)$  and  $\lambda_k^{\beta_1}, \lambda_k^{\beta_2}$ ,  $k \geq 0$ , be defined as before. Let  $\beta^* > 0$  be a fixed finite number. Then*

1. *for  $0 < \beta_1, \beta_2 < \beta^*$ , we have*

$$|\beta_1 - \beta_2| \leq C_{\beta^*} |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|, \quad \forall k \geq 0,$$

2. *for  $\beta_1, \beta_2 \geq \beta^*$ , we have*

$$\left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| \leq C_{\beta^*} |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|, \quad \forall k \geq 0,$$

*where the constant  $C_{\beta^*}$  does not depend on  $k$ .*

*Proof.* We begin with the fact from Remark III.3.9 that any  $\beta \in (0, +\infty)$  can be represented as

$$\begin{cases} \beta = \sqrt{\lambda_k^\beta} \tan \frac{\sqrt{\lambda_k^\beta}}{2}, & \text{for } k \text{ even and} \\ \beta = -\sqrt{\lambda_k^\beta} \cot \frac{\sqrt{\lambda_k^\beta}}{2}, & \text{for } k \text{ odd.} \end{cases} \quad (\text{III.78})$$

Also, since we have  $\sqrt{\lambda_k^\beta} \in (k\pi, (k+1)\pi)$  by point 2 of Remark III.3.8, so one can write

$$\sqrt{\lambda_k^{\beta_j}} = k\pi + \delta_k^{\beta_j} \quad \text{and} \quad \sqrt{\lambda_k^{\beta^*}} = k\pi + \delta_k^{\beta^*}, \quad \forall k \geq 0, \quad j = 1, 2, \quad (\text{III.79})$$

where  $\delta_k^{\beta_j}, \delta_k^{\beta^*} \in (0, \pi)$ .

1. Now, we assume that  $0 < \beta_1, \beta_2 < \beta^*$  which implies  $0 < \delta_k^{\beta_1}, \delta_k^{\beta_2} < \delta_k^{\beta^*} (< \pi)$ ,  $\forall k \geq 0$ . We denote

$$\delta_{\beta^*} := \sup_{k \geq 0} \delta_k^{\beta^*} \in (0, \pi), \quad (\text{III.80})$$

where  $\delta_{\beta^*} < \pi$  since the quantity  $\delta_k^{\beta^*}$  is getting smaller as  $k \in \mathbb{N}$  getting larger due to the asymptotic behavior of  $\sqrt{\lambda_k^{\beta^*}}$  given by Remark III.3.10.

We discuss the proof for  $k$  even, for odd  $k$  the steps will be similar. We have

$$\begin{aligned} \beta_1 - \beta_2 &= (k\pi + \delta_k^{\beta_1}) \tan \left( \frac{k\pi}{2} + \frac{\delta_k^{\beta_1}}{2} \right) - (k\pi + \delta_k^{\beta_2}) \tan \left( \frac{k\pi}{2} + \frac{\delta_k^{\beta_2}}{2} \right) \\ &= k\pi \left( \tan \frac{\delta_k^{\beta_1}}{2} - \tan \frac{\delta_k^{\beta_2}}{2} \right) + \delta_k^{\beta_1} \tan \frac{\delta_k^{\beta_1}}{2} - \delta_k^{\beta_2} \tan \frac{\delta_k^{\beta_2}}{2}. \end{aligned} \quad (\text{III.81})$$

Applying Mean value theorem to the functions  $\tan \frac{\mu}{2}$  and  $\mu \tan \frac{\mu}{2}$  on  $\mu \in ((\delta_k^{\beta_1}, \delta_k^{\beta_2}))$ , we have for some  $\delta'_k$  and  $\delta''_k$  in  $((\delta_k^{\beta_1}, \delta_k^{\beta_2}))$ , that

$$\left| \tan \frac{\delta_k^{\beta_1}}{2} - \tan \frac{\delta_k^{\beta_2}}{2} \right| \leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| \sec^2 \frac{\delta'_k}{2} \leq C_{\beta^*} |\delta_k^{\beta_1} - \delta_k^{\beta_2}|,$$

and

$$\begin{aligned} \left| \delta_k^{\beta_1} \tan \frac{\delta_k^{\beta_1}}{2} - \delta_k^{\beta_2} \tan \frac{\delta_k^{\beta_2}}{2} \right| &\leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| \left| \frac{\delta''_k}{2} \sec^2 \frac{\delta''_k}{2} + \tan \frac{\delta''_k}{2} \right| \\ &\leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| \frac{\delta''_k}{2} \left| \sec^2 \frac{\delta''_k}{2} + \frac{\sin(\delta''_k/2)}{\delta''_k/2} \sec \frac{\delta''_k}{2} \right| \\ &\leq C_{\beta^*} (\delta_k^{\beta_1} + \delta_k^{\beta_2}) |\delta_k^{\beta_1} - \delta_k^{\beta_2}|, \end{aligned}$$

where we make use of the fact that the quantities  $\sec \frac{\delta'_k}{2}$  and  $\sec \frac{\delta''_k}{2}$  can be bounded by  $\sec \frac{\delta_{\beta^*}}{2}$  which is some constant  $C_{\beta^*}$ . Now, we turn back to (III.81) to deduce that

$$\begin{aligned} |\beta_1 - \beta_2| &\leq C_{\beta^*} (k\pi + \delta_k^{\beta_1} + \delta_k^{\beta_2}) |\delta_k^{\beta_1} - \delta_k^{\beta_2}| \\ &= C_{\beta^*} \left( \sqrt{\lambda_k^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}} \right) \left| \sqrt{\lambda_k^{\beta_1}} - \sqrt{\lambda_k^{\beta_2}} \right| \\ &\leq C_{\beta^*} |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|, \quad \forall k \geq 0 \text{ even,} \end{aligned}$$

which is the proof for point 1.

2. Here also, we demonstrate the result for  $k$  even. We recall (III.78) to observe that

$$\begin{aligned} \frac{1}{\beta_1} - \frac{1}{\beta_2} &= \frac{1}{k\pi + \delta_k^{\beta_1}} \cot \left( \frac{k\pi}{2} + \frac{\delta_k^{\beta_1}}{2} \right) - \frac{1}{k\pi + \delta_k^{\beta_2}} \cot \left( \frac{k\pi}{2} + \frac{\delta_k^{\beta_2}}{2} \right) \\ &= \frac{1}{k\pi + \delta_k^{\beta_1}} \cot \frac{\delta_k^{\beta_1}}{2} - \frac{1}{k\pi + \delta_k^{\beta_2}} \cot \frac{\delta_k^{\beta_2}}{2}, \end{aligned} \quad (\text{III.82})$$

Let us define the function

$$g(\mu) = \frac{1}{k\pi + \mu} \cot \frac{\mu}{2}, \quad \text{for } \mu \in ((\delta_k^{\beta_1}, \delta_k^{\beta_2})) \subset (0, \pi).$$

Consequently,

$$g'(\mu) = -\frac{1}{2(k\pi + \mu)} \csc^2 \frac{\mu}{2} - \frac{1}{(k\pi + \mu)^2} \cot \frac{\mu}{2},$$

and  $|g'|$  is monotonically decreasing function in  $(0, \pi)$ . Now, applying Mean value theorem on  $g(\mu)$ , we have from (III.82)

$$\left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| \leq |\delta_k^{\beta_1} - \delta_k^{\beta_2}| |g'(\tilde{\delta}_k)|, \quad \text{for some } \tilde{\delta}_k \in ((\delta_k^{\beta_1}, \delta_k^{\beta_2})). \quad (\text{III.83})$$

But we have  $\delta_k^{\beta_j}, \tilde{\delta}_k \geq \delta_k^{\beta^*}$  ( $j = 1, 2$ ), since  $\beta_1, \beta_2 \geq \beta^*$ , and hence

$$\begin{aligned} |g'(\tilde{\delta}_k)| &\leq \frac{1}{k\pi + \tilde{\delta}_k} \left[ \frac{1}{2 \sin^2(\tilde{\delta}_k/2)} + \frac{1}{k\pi + \tilde{\delta}_k} \cot \frac{\tilde{\delta}_k}{2} \right] \\ &\leq \frac{1}{k\pi + \delta_k^{\beta^*}} \left[ \frac{1}{2 \sin^2(\delta_k^{\beta^*}/2)} + \frac{1}{k\pi + \delta_k^{\beta^*}} \frac{1}{\sin(\delta_k^{\beta^*}/2)} \right]. \end{aligned} \quad (\text{III.84})$$

Let us recall the asymptotic formula of  $\sqrt{\lambda_k^{\beta^*}}$  from Remark III.3.10 to observe for  $k \geq k_{\beta^*}$  (for some  $k_{\beta^*} \in \mathbb{N}$ ), that

$$\delta_k^{\beta^*} = \frac{2\beta^*}{k\pi} + O_{\beta^*} \left( \frac{1}{k^3} \right) \geq \frac{\bar{\beta}}{k\pi},$$

for some  $\bar{\beta} > 0$  depending only on  $\beta^*$ . As a consequence,  $\sin \frac{\delta_k^{\beta^*}}{2} \geq \sin \frac{\bar{\beta}}{2k\pi}$ ,  $\forall k \geq k_{\beta^*}$ , since  $\sin x$  is monotonically increasing in  $(0, \pi/2)$ . Also, since  $\sin x \geq \frac{2}{\pi}x$  for all  $x \in (0, \pi/2)$ , eventually we have

$$\sin \frac{\bar{\beta}}{2k\pi} \geq \frac{\bar{\beta}}{k\pi^2}, \quad \forall k \geq k_{\beta^*}.$$

So, coming back to (III.84), we obtain for any  $k \geq k_{\beta^*}$  even, that

$$\begin{aligned} |g'(\tilde{\delta}_k)| &\leq \frac{1}{\sqrt{\lambda_k^{\beta^*}}} \left[ \frac{k^2\pi^4}{2\bar{\beta}^2} + \frac{k\pi^2}{\bar{\beta}\sqrt{\lambda_k^{\beta^*}}} \right] \\ &\leq C_{\beta^*} \sqrt{\lambda_k^{\beta^*}} \leq C_{\beta^*} \left( \sqrt{\lambda_k^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}} \right). \end{aligned}$$

On the other hand, for each  $0 \leq k < k_{\beta^*}$  even, the quantity in the very right hand side of (III.84) is obviously bounded, and so for those  $k$ ,  $|g'(\tilde{\delta}_k)|$  can definitely be bounded by  $C_{\beta^*}(\sqrt{\lambda_k^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}})$  for some constant  $C_{\beta^*} > 0$ .

Now, implementing this estimate into (III.83), we obtain that

$$\begin{aligned} \left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| &\leq C_{\beta^*} \left| \sqrt{\lambda_k^{\beta_1}} - \sqrt{\lambda_k^{\beta_2}} \right| \left( \sqrt{\lambda_k^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}} \right) \\ &= C_{\beta^*} |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|, \quad \forall k \geq 0 \text{ even,} \end{aligned}$$

which is the point 2 of our lemma and hence, the proof is complete.  $\square$

Lemma III.5.3 now helps us to prove the following proposition which is the key point to obtain a uniform  $L^2(0, T)$ -bound of a control that we construct in next section, with respect to the parameters  $\beta_1, \beta_2$ .

**Proposition III.5.4.** *Let  $\beta_1 \neq \beta_2$  be any two parameters with  $\beta_1, \beta_2 \in (0, +\infty)$  and  $(\lambda_k^{\beta_1}, \Phi_{k,1})$ ,  $(\lambda_k^{\beta_2}, \Phi_{k,2})$  be the eigenvalue-eigenfunction pairs of the operator  $\mathcal{A}_{\beta_1, \beta_2}^*$  for each  $k \geq 0$ . Also, assume that  $\beta^* > 0$  be any fixed finite number. Then, for any  $k \geq 0$ , we have*

$$\frac{1}{|\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|} \left\| \frac{\Phi_{k,1}}{\mathcal{B}^* \Phi_{k,1}} - \frac{\Phi_{k,2}}{\mathcal{B}^* \Phi_{k,2}} \right\|_{L^2(0,1)} \leq \begin{cases} C_{\beta^*}, & \text{if } 0 < \beta_1, \beta_2 < \beta^*, \\ C_{\beta^*} \beta_1 \sqrt{\lambda_k^{\beta_2}}, & \text{if } \beta_1, \beta_2 \geq \beta^*, \end{cases}$$

where  $\mathcal{B}^*$  is defined in (III.28), and the constant  $C_{\beta^*}$  does not depend on  $k$ .

*Proof.* Since  $\mathcal{B}^* = \mathbb{1}_{\{x=0\}} \begin{pmatrix} 1 & 0 \end{pmatrix}$  (introduced in (III.28)), and using the definitions (III.66) and (III.68), the quantity we want to estimate can be denoted by

$$\Theta_k := \begin{pmatrix} \Theta_k^1 \\ \Theta_k^2 \end{pmatrix},$$

where the two components are

$$\begin{aligned} \Theta_k^1(x) &:= \frac{1}{(\lambda_k^{\beta_1} - \lambda_k^{\beta_2})} \left( \frac{\varphi_k^{\beta_1}(x)}{\varphi_k^{\beta_1}(0)} - \frac{\psi_k^{\beta_1, \beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)} \right), \quad x \in (0, 1), \\ \Theta_k^2(x) &:= \frac{1}{(\lambda_k^{\beta_1} - \lambda_k^{\beta_2})} \frac{\varphi_k^{\beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)}, \quad x \in (0, 1). \end{aligned}$$

In order to ease the computations, we denote by  $\mu_i = \sqrt{\lambda_k^{\beta_i}}$  for  $i = 1, 2$ , the dependence in  $k$  being now implicit.

1. We first assume that  $0 < \beta_1, \beta_2 < \beta^*$ . Using this assumption, and the fact that  $\lambda_0^{\beta_i} \geq \frac{\beta_i}{\beta^*} \lambda_0^{\beta^*}$  (by Remark III.3.2) and simply  $\lambda_k^{\beta_i} \geq k^2 \pi^2$ ,  $\forall k \geq 1$  (by point 2 of Remark III.3.8), we obtain uniformly in  $k \geq 0$ , that

$$\mu_i \geq C_{\beta^*} \sqrt{\beta_i}, \quad i = 1, 2. \quad (\text{III.85})$$

- *Estimate of the first component  $\Theta_k^1$ :* Recall the expression  $\varphi_k^{\beta_1}$  (with  $\beta = \beta_1$ ) and  $\psi_k^{\beta_1, \beta_2}$  from (III.45) and (III.71) respectively and following some steps of computations we obtain

$$\begin{aligned} \left( \frac{\varphi_k^{\beta_1}(x)}{\varphi_k^{\beta_1}(0)} - \frac{\psi_k^{\beta_1, \beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)} \right) &= \cos(\mu_1 x) - \cos(\mu_2 x) + \beta_1 \left[ \frac{\sin \mu_1 x}{\mu_1} - \frac{\sin \mu_2 x}{\mu_2} \right] \\ &\quad - \frac{A_k x}{B_k} \cos(\mu_2 x) - \frac{C_k x}{B_k} \sin(\mu_2 x) + \frac{A_k}{B_k \mu_2} \sin(\mu_2 x). \end{aligned} \quad (\text{III.86})$$

Let us bound the contribution of each term in the  $L^2$  norm of  $\Theta_k^1$ , recalling that  $(\lambda_k^{\beta_1} - \lambda_k^{\beta_2}) = \mu_1^2 - \mu_2^2$ .

- First, it is easy to deduce that for any  $\beta_1 \neq \beta_2$  and any  $x \in (0, 1]$ , we have

$$\left| \frac{\cos(\mu_1 x) - \cos(\mu_2 x)}{\mu_1^2 - \mu_2^2} \right| = \frac{x^2}{2} \left| \frac{\sin\left(\frac{\mu_1 - \mu_2}{2} x\right) \sin\left(\frac{\mu_1 + \mu_2}{2} x\right)}{\left(\frac{\mu_1 - \mu_2}{2} x\right) \left(\frac{\mu_1 + \mu_2}{2} x\right)} \right| \leq \frac{1}{2}. \quad (\text{III.87})$$

- For  $x \in (0, 1]$  fixed, let us define the function  $\mu \mapsto f(\mu) := \frac{\sin(\mu x)}{\mu}$ , whose derivative is  $f'(\mu) = \frac{1}{\mu} x \cos(\mu x) - \frac{1}{\mu^2} \sin(\mu x)$ . Applying Mean value theorem, we have

$$|f(\mu_1) - f(\mu_2)| \leq |\mu_1 - \mu_2| |f'(\tilde{\mu})|, \quad \text{for some } \tilde{\mu} \in ((\mu_1, \mu_2)).$$

Now, if  $0 < \tilde{\mu} < 1$  (consequently  $0 < \tilde{\mu} x < 1$  for  $x \in (0, 1]$ ), then we have

$$\begin{aligned} |f'(\tilde{\mu})| &= \left| \frac{1}{\tilde{\mu}} x \cos(\tilde{\mu} x) - \frac{1}{\tilde{\mu}^2} \sin(\tilde{\mu} x) \right| \\ &= \left| \tilde{\mu} x^3 \left( \frac{\cos(\tilde{\mu} x) - 1}{(\tilde{\mu} x)^2} - \frac{\sin(\tilde{\mu} x) - \tilde{\mu} x}{(\tilde{\mu} x)^3} \right) \right| \\ &\leq \tilde{\mu} x^3 \leq 1, \quad \forall x \in (0, 1], \end{aligned}$$

since  $\left| \frac{\cos(\tilde{\mu} x) - 1}{(\tilde{\mu} x)^2} \right| \leq C$  and  $\left| \frac{\sin(\tilde{\mu} x) - \tilde{\mu} x}{(\tilde{\mu} x)^3} \right| \leq C$ , for  $0 < \tilde{\mu} x < 1$ .

On the other hand, for  $\tilde{\mu} \geq 1$ , it is quite obvious to see that  $|f'(\tilde{\mu})| \leq C$ , and so finally we have uniformly in  $x$ ,

$$|f(\mu_1) - f(\mu_2)| \leq C |\mu_1 - \mu_2|, \quad \forall \beta_1 \neq \beta_2 \text{ positive.}$$

This implies that for  $0 < \beta_1 < \beta^*$  one has

$$\frac{\beta_1}{|\mu_1^2 - \mu_2^2|} \left| \frac{\sin(\mu_1 x)}{\mu_1} - \frac{\sin(\mu_2 x)}{\mu_2} \right| \leq \frac{C\beta_1}{\mu_1 + \mu_2} \leq C_{\beta^*}, \quad (\text{III.88})$$

where the last inequality follows from (III.85).

- For estimating the remaining three terms, we use the values of  $A_k$  and  $C_k$  from (III.72) and Lemma III.5.1, that gives the bound from below

$$|B_k| \geq \frac{\mu_2}{4\beta_2|\beta_1 - \beta_2|}.$$

With the inequality proved in the first point of Lemma III.5.3 and (III.85) we see that

$$\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{A_k x}{B_k} \cos(\mu_2 x) \right| \leq \frac{2\beta_2 |\beta_1 - \beta_2|}{\mu_2^2 |\mu_1^2 - \mu_2^2|} \leq \frac{C_{\beta^*} \beta_2}{\mu_2^2} \leq C_{\beta^*}. \quad (\text{III.89})$$

Similarly, we have

$$\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{C_k x}{B_k} \sin(\mu_2 x) \right| \leq \frac{|\beta_1 - \beta_2|}{|\mu_1^2 - \mu_2^2|} \left| \frac{x^2 \sin(\mu_2 x)}{\mu_2 x} \right| \leq C_{\beta^*}, \quad (\text{III.90})$$

and

$$\begin{aligned} \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{A_k x}{B_k \mu_2} \sin(\mu_2 x) \right| &\leq \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{A_k}{B_k} \right| \leq \frac{2\beta_2 |\beta_1 - \beta_2|}{\mu_2^2 |\mu_1^2 - \mu_2^2|} \\ &\leq \frac{C_{\beta^*} \beta_2}{\mu_2^2} \leq C_{\beta^*}. \end{aligned} \quad (\text{III.91})$$

Hence, gathering all the estimates from (III.87), (III.88), (III.89), (III.90) and (III.91), one can deduce that  $\|\Theta_k^1\|_{L^2(0,1)} \leq C_{\beta^*}$  for any  $k \geq 0$ .

- *Estimate of the second component  $\Theta_k^2$ :* By using the expression of  $\varphi_k^{\beta_2}$  from (III.45), we have for each  $k \geq 0$  that

$$\begin{aligned} \left| \frac{\varphi_k^{\beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)} \right| &\leq \frac{\mu_2 |\cos(\mu_2 x)|}{\beta_2 |\psi_k^{\beta_1, \beta_2}(0)|} + \frac{|\sin(\mu_2 x)|}{|\psi_k^{\beta_1, \beta_2}(0)|} \\ &\leq 4|\beta_1 - \beta_2| + 4\beta_2 |\beta_1 - \beta_2| \left| \frac{x \sin(\mu_2 x)}{\mu_2 x} \right| \\ &\leq 4(1 + \beta_2) |\beta_1 - \beta_2| \leq C_{\beta^*} (1 + \beta_2) |\mu_1^2 - \mu_2^2|, \end{aligned}$$

where we make use the facts that  $|\psi_k^{\beta_1, \beta_2}(0)| \geq \frac{\mu_2}{4\beta_2|\beta_1 - \beta_2|}$  and the estimate in the first point of Lemma III.5.3. Consequently, we deduce that  $\|\Theta_k^2\|_{L^2(0,1)} \leq C_{\beta^*}$  for any  $k \geq 0$ .

This completes the proof of the uniform estimate of  $\Theta_k$  in  $L^2$  for  $0 < \beta_1, \beta_2 < \beta^*$ ,  $\beta_1 \neq \beta_2$ .

2. Let us assume now that  $\beta_1, \beta_2 \geq \beta^* > 0$ . Using this assumption, and the fact that  $\lambda_0^{\beta_i} \geq \lambda_0^{\beta^*}$  and  $\lambda_k^{\beta_i} \geq k^2 \pi^2$ ,  $\forall k \geq 1$ , we obtain uniformly in  $k \geq 0$ , that

$$\mu_i \geq C_{\beta^*}, \quad i = 1, 2. \quad (\text{III.92})$$

We need to prove that  $\frac{1}{\beta_1} \|\Theta_k\|_{L^2(0,1)}$  is bounded uniformly in  $k$ ,  $\beta_1$  and  $\beta_2$ .

- *Estimate of the first component  $\Theta_k^1$ :* We still start from (III.86) and we estimate each term as follows.

– Analogous to (III.87) and (III.88), we respectively have

$$\left| \frac{\cos(\mu_1 x) - \cos(\mu_2 x)}{\mu_1^2 - \mu_2^2} \right| \leq \frac{1}{2}, \quad (\text{III.93})$$

and

$$\frac{\beta_1}{|\mu_1^2 - \mu_2^2|} \left[ \frac{\sin(\mu_1 x)}{\mu_1} - \frac{\sin(\mu_2 x)}{\mu_2} \right] \leq \frac{\beta_1}{\mu_1} \leq C_{\beta^*} \beta_1, \quad (\text{III.94})$$

by (III.92).

– Now using the fact  $|B_k| \geq \frac{\mu_2^2 + \beta_2^2}{4\beta_2\mu_2|\beta_1 - \beta_2|}$  from Lemma III.5.1, and the second point of Lemma III.5.3, we can bound the remaining three terms as follows.

$$\begin{aligned} \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{A_k x}{B_k} \cos(\mu_2 x) \right| &\leq \frac{2|\beta_1 - \beta_2|}{\beta_2 |\mu_1^2 - \mu_2^2|} \frac{\beta_2^2}{(\mu_2^2 + \beta_2^2)} \\ &\leq \frac{2\beta_1}{|\mu_1^2 - \mu_2^2|} \left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| \leq C_{\beta^*} \beta_1. \end{aligned} \quad (\text{III.95})$$

Similarly, we have

$$\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{C_k x}{B_k} \sin(\mu_2 x) \right| \leq \frac{|\beta_1 - \beta_2|}{\beta_2 |\mu_1^2 - \mu_2^2|} \frac{2\beta_2 \mu_2}{(\mu_2^2 + \beta_2^2)} \leq C_{\beta^*} \beta_1, \quad (\text{III.96})$$

and finally

$$\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{A_k x}{B_k \mu_2} \sin(\mu_2 x) \right| \leq \frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{A_k}{B_k} \right| \leq C_{\beta^*} \beta_1, \quad (\text{III.97})$$

which is obtained by a similar type of computations as in (III.95).

Gathering all the estimates (III.93), (III.94), (III.95), (III.96) and (III.97), we get that  $\|\Theta_k^1\|_{L^2(0,1)} \leq C_{\beta^*} \beta_1$  for any  $k \geq 0$ .

• *Estimate of the second component  $\Theta_k^2$ :* Using the same ingredients as before, we compute

$$\begin{aligned} \left| \frac{\varphi_k^{\beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)} \right| &\leq \frac{\mu_2}{\beta_2} \left| \frac{\cos(\mu_2 x)}{\psi_k^{\beta_1, \beta_2}(0)} \right| + \left| \frac{\sin(\mu_2 x)}{\psi_k^{\beta_1, \beta_2}(0)} \right| \\ &\leq 2\mu_2 \frac{2\beta_1\beta_2\mu_2}{(\mu_2^2 + \beta_2^2)} \left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| + 4\mu_2 \frac{\beta_1\beta_2^2}{(\mu_2^2 + \beta_2^2)} \left| \frac{1}{\beta_1} - \frac{1}{\beta_2} \right| \\ &\leq C_{\beta^*} \beta_1 \mu_2 \left| \mu_1^2 - \mu_2^2 \right|, \end{aligned}$$

by the second point of Lemma III.5.3. This implies

$$\frac{1}{|\mu_1^2 - \mu_2^2|} \left| \frac{\varphi_k^{\beta_2}(x)}{\psi_k^{\beta_1, \beta_2}(0)} \right| \leq C_{\beta^*} \beta_1 \mu_2,$$

and thus the expected bound  $\|\Theta_k^2\|_{L^2(0,1)} \leq C_{\beta^*} \beta_1 \mu_2$  for any  $k \geq 0$ .

This completes the proof of the uniform estimate of  $\Theta_k$  in  $L^2$  for  $\beta_1, \beta_2 \geq \beta^*$ ,  $\beta_1 \neq \beta_2$ .  $\square$

### III.5.2 Null-controllability

We can now prove the null-controllability of our system, with a precise bound of the control with respect to  $\beta_1$  and  $\beta_2$ , that is the point 2 of Theorem III.2.14.



### III.5.2.1 The moments problem

In the present context, we recall that the family  $\{\Phi_{k,1}, \Phi_{k,2}\}_{k \geq 0}$  (defined by (III.66)-(III.68)) is complete in  $(L^2(0,1))^2$  and so, by checking the equation (III.27) for  $\Phi_{k,1}$  and  $\Phi_{k,2}$  for each  $k \geq 0$ , indeed tells us that for any  $y_0 \in (L^2(0,1))^2$  the input  $v \in L^2(0,T)$  is a null control for (III.64) if and only if one has

$$\begin{cases} -e^{-T\lambda_k^{\beta_1}} \frac{(y_0, \Phi_{k,1})_{L^2(0,1)}}{\mathcal{B}^* \Phi_{k,1}} = \int_0^T v(t) e^{-\lambda_k^{\beta_1}(T-t)} dt, & \forall k \geq 0, \\ -e^{-T\lambda_k^{\beta_2}} \frac{(y_0, \Phi_{k,2})_{L^2(0,1)}}{\mathcal{B}^* \Phi_{k,2}} = \int_0^T v(t) e^{-\lambda_k^{\beta_2}(T-t)} dt, & \forall k \geq 0, \end{cases} \quad (\text{III.98})$$

where we used the formulas given in (III.69).

### III.5.2.2 The block moment method

It is known that for any  $k \geq 0$  the eigenvalue  $\lambda_k^\beta$  is continuous with respect to the parameter  $\beta \in [0, +\infty]$ , see for instance [72, Theorem 3.1] and as a consequence, it may occur that the two eigenvalues  $\lambda_k^{\beta_1}$  and  $\lambda_k^{\beta_2}$  are arbitrarily close if  $\beta_1$  and  $\beta_2$  are close. This phenomenon is called *spectral condensation* and may, in general, prevent us from obtaining uniform bounds on the controls when  $\beta_1$  and  $\beta_2$  are getting closer (see for instance a discussion on the influence of the condensation index on controllability properties of parabolic systems in [11]).

Indeed, the classic way to solve the moments problem, as we did in Section III.4 is inadequate. More precisely, it is not true anymore that any bi-orthogonal family to  $(e^{-\lambda_k^{\beta_j}(T-t)})_{k \geq 0, j=0,1}$  will satisfy uniform  $L^2(0,T)$ -bound with respect to the parameters  $\beta_1, \beta_2$  since the gap  $\inf_k |\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|$  may be arbitrary small when  $|\beta_1 - \beta_2|$  is small (see Lemma III.5.3).

To overcome this situation, and still prove uniform controllability result, we will use the block moment approach developed in [22] to solve problems like (III.98) when a weak gap condition holds, instead of a usual uniform gap condition. This method let us take benefit of the condensation of eigenfunctions that actually compensate the condensation of the eigenvalues. Let us go into the details.

We first define  $\Lambda^{\beta_i} := \{\lambda_k^{\beta_i}, k \geq 0\}$  for  $i = 1, 2$ , the two families of eigenvalues we are concerned with and we set  $\Lambda^{\beta_1, \beta_2} = \Lambda^{\beta_1} \cup \Lambda^{\beta_2}$ .

As we have seen in (III.55), each of the two families satisfies a uniform spectral gap property

$$\inf_k |\lambda_{k+1}^{\beta_i} - \lambda_k^{\beta_i}| \geq C\pi, \quad i = 1, 2, \quad (\text{III.99})$$

and their reciprocal values are uniformly summable in the sense that, there exists a function  $\mathcal{N} : (0, +\infty) \rightarrow (0, +\infty)$  that does not depend on  $\beta_1$  and  $\beta_2$ , such that

$$\sum_{\substack{\lambda \in \Lambda^{\beta_i} \\ \lambda > \mathcal{N}(\varepsilon)}} \frac{1}{\lambda} \leq \varepsilon, \quad (\text{III.100})$$

for any  $\varepsilon > 0$  and any  $i = 1, 2$ .

Therefore, by [22, Lemma 2.1], we know that the union family  $\Lambda^{\beta_1, \beta_2}$  satisfies a weak-gap property : for any  $\rho > 0$  (independent of  $\beta_1$  and  $\beta_2$ ) such that

$$\rho < C\pi, \quad (\text{III.101})$$

we have that  $\Lambda^{\beta_1, \beta_2} \cap [\mu, \mu + \rho)$ , contains at most 2 elements for any  $\mu > 0$ . Moreover, the reciprocal values of  $\Lambda^{\beta_1, \beta_2}$  are also uniformly summable as in (III.100) but with a possibly different function  $\mathcal{N}$ .

By [22, Proposition 7.1] we know that, for each value of  $\beta_1$  and  $\beta_2$ , we can find a family of disjoint non empty groups  $(G_n)_n$  each of them having a cardinal less or equal than 2 and such that

$$\begin{aligned}\Lambda^{\beta_1, \beta_2} &= \bigcup_n G_n, \\ (\min G_{n+1}) - (\max G_n) &\geq \rho/2, \\ \text{diam}(G_n) &< \rho.\end{aligned}$$

Let us prove now that, for  $\rho$  small enough, the structure of those groups is actually simple.

**Lemma III.5.5.** *Let  $\beta^* > 0$  be fixed and for  $\beta_1 \neq \beta_2$  assume that either  $\beta_1, \beta_2 < \beta^*$  or  $\beta_1, \beta_2 \geq \beta^*$ .*

*There exists  $\rho^*$  depending only on  $\beta^*$  such that, if we assume that  $\rho < \rho^*$  in the above construction in addition to (III.101), then for any group  $G_n$  of cardinal 2, there exists an integer  $k$  such that*

$$G_n = \{\lambda_k^{\beta_1}, \lambda_k^{\beta_2}\}.$$

*Proof.* Without loss of generality, we assume that  $\beta_1 < \beta_2$ . Since the diameter of  $G_n$  is less than  $\rho$  and using (III.99) and (III.101) we know that  $G_n$  contains exactly one element from  $\Lambda^{\beta_1}$  and one element from  $\Lambda^{\beta_2}$ , that is

$$G_n = \{\lambda_j^{\beta_1}, \lambda_k^{\beta_2}\},$$

for some integers  $k$  and  $j$ . We want to show that  $j = k$ .

By Lemma III.3.3 and Lemma III.3.11 we know that

$$\lambda_k^{\beta_1} < \lambda_k^{\beta_2} < \lambda_{k+1}^{\beta_1},$$

thus the only possibilities are  $j = k$  or  $j = k + 1$ .

- First, we treat the case when  $0 < \beta_1 < \beta_2 < \beta^*$ . We have

$$\sqrt{\lambda_{k+1}^{\beta_1}} - \sqrt{\lambda_k^{\beta_2}} \geq \sqrt{\lambda_{k+1}^N} - \sqrt{\lambda_k^{\beta^*}} = \pi - \delta_{\beta^*} > 0,$$

since  $\lambda_{k+1}^N = (k+1)^2\pi^2$  by point 1 of Remark III.3.8;  $\lambda_k^{\beta^*}$  and  $\delta_{\beta^*}$  have been introduced in (III.79) and (III.80) respectively. Hence, for all  $k \geq 0$ , we have

$$\begin{aligned}\lambda_{k+1}^{\beta_1} - \lambda_k^{\beta_2} &= \left(\sqrt{\lambda_{k+1}^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}}\right) \left(\sqrt{\lambda_{k+1}^{\beta_1}} - \sqrt{\lambda_k^{\beta_2}}\right) \\ &\geq \pi(\pi - \delta_{\beta^*}) > 0.\end{aligned}$$

We choose  $\rho^* = \pi(\pi - \delta_{\beta^*})$ . From the computation above we see that if  $\rho < \rho^*$ , then  $\lambda_{k+1}^{\beta_1}$  and  $\lambda_k^{\beta_2}$  cannot belong to the same group  $G_n$  and thus we necessarily have  $j = k$  and the claim is proved.

- Assume now that  $\beta_2 > \beta_1 \geq \beta^* > 0$ . Then, from the asymptotic formula given by Remark III.3.10, we see

$$\sqrt{\lambda_{k+1}^{\beta_1}} - \sqrt{\lambda_k^{\beta_2}} \geq \sqrt{\lambda_{k+1}^{\beta^*}} - \sqrt{\lambda_k^D} = \frac{2\beta^*}{(k+1)\pi} + O_{\beta^*}\left(\frac{1}{k^2}\right), \quad \forall k \geq k_{\beta^*}$$

(for some  $k_{\beta^*} \in \mathbb{N}$ ), since  $\lambda_k^D = (k+1)^2\pi^2$  by point 1 of Remark III.3.8. Now, it is obvious that  $\sqrt{\lambda_{k+1}^{\beta_1}} + \sqrt{\lambda_k^{\beta_2}} \geq (k+1)\pi$ , for all  $k \geq 0$  and so, there exists some  $\tilde{k}_{\beta^*} \in \mathbb{N}$ , depending only on  $\beta^*$  such that

$$\lambda_{k+1}^{\beta_1} - \lambda_k^{\beta_2} \geq \beta^* > 0, \quad \forall k \geq \tilde{k}_{\beta^*}.$$

It remains to deal with the other values of  $k = 0, 1, \dots, \tilde{k}_{\beta^*} - 1$ . We simply use the fact that

$$\lambda_{k+1}^{\beta_1} - \lambda_k^{\beta_2} \geq \lambda_{k+1}^{\beta^*} - \lambda_k^D,$$

to define

$$\rho^* := \min \left( \beta^*, \min_{0 \leq k < \tilde{k}_{\beta^*}} (\lambda_{k+1}^{\beta^*} - \lambda_k^D) \right) > 0.$$

Here also we conclude that if  $\rho < \rho^*$ ,  $\lambda_{k+1}^{\beta_1}$  and  $\lambda_k^{\beta_2}$  cannot be in the same group  $G_n$  and thus  $j = k$ , and the proof is complete.  $\square$

We can now proceed to the proof of our main result concerning the uniform null-controllability of the system with two different Robin parameters.

*Proof of point 2 of Theorem III.2.14.* We proved above that the sequence of eigenvalues  $\Lambda^{\beta_1, \beta_2}$  satisfy the good weak-gap and summability conditions required by the block moment method. More precisely, we can apply [22, Theorem 2.1] to find a solution to the set of equations (III.98) as an infinite sum of terms, each of them corresponding to the resolution of the contribution of the group  $G_n$ . In our case, we can observe that, by Lemma III.5.5, the set  $\{\lambda_k^{\beta_1}, \lambda_k^{\beta_2}\}$  for any  $k \geq 0$ , is either exactly one of the groups  $G_n$  or the union  $G_n \cup G_{n+1}$  of two distinct groups of cardinal 1.

It follows that, the result of [22, Theorem 2.1] can be reformulated as follows: there exist functions  $v_k^{\beta_1, \beta_2} \in L^2(0, T)$  for each  $k \geq 0$ , which satisfy the following

$$\begin{cases} \int_0^T v_k^{\beta_1, \beta_2}(t) e^{-\lambda_k^{\beta_1}(T-t)} dt = -e^{-T\lambda_k^{\beta_1}} \frac{(y_0, \Phi_{k,1})_{L^2(0,1)}}{\mathcal{B}^* \Phi_{k,1}}, \\ \int_0^T v_k^{\beta_1, \beta_2}(t) e^{-\lambda_k^{\beta_2}(T-t)} dt = -e^{-T\lambda_k^{\beta_2}} \frac{(y_0, \Phi_{k,2})_{L^2(0,1)}}{\mathcal{B}^* \Phi_{k,2}}, \\ \int_0^T v_k^{\beta_1, \beta_2}(t) e^{-\lambda_l^{\beta_i}(T-t)} dt = 0, \quad \forall l \neq k, \forall i = 1, 2, \end{cases}$$

and along with the following bound, for any  $\varepsilon > 0$ ,

$$\begin{aligned} \|v_k^{\beta_1, \beta_2}\|_{L^2(0, T)} &\leq C_{T, \varepsilon, \mathcal{N}, \rho^*} \|y_0\|_{L^2(0, 1)} \\ &\times e^{(\varepsilon - T)\lambda_k^{\beta_1}} \max \left\{ \left\| \frac{\Phi_{k,1}}{\mathcal{B}^* \Phi_{k,1}} \right\|_{L^2(0, 1)}, \frac{\left\| \frac{\Phi_{k,1}}{\mathcal{B}^* \Phi_{k,1}} - \frac{\Phi_{k,2}}{\mathcal{B}^* \Phi_{k,2}} \right\|_{L^2(0, 1)}}{|\lambda_k^{\beta_1} - \lambda_k^{\beta_2}|} \right\}. \end{aligned}$$

Note that in [22] it is assumed that all the eigenvalues in the system are greater than 1, whereas in our case we only know that they are non-negative (we recall that  $\lambda_0^\beta$  goes to 0 when  $\beta \rightarrow 0$ ). However, one can check that this does not change significantly the result since it simply amounts to add a factor  $e^T$  in front of the constant  $C_{T, \varepsilon, \mathcal{N}, \rho^*}$  in the estimate.

We now define  $v_\beta$  as

$$v_\beta(t) := \sum_{k \geq 0} v_k^{\beta_1, \beta_2}(t), \quad \forall t \in [0, T], \quad (\text{III.102})$$

so that  $v_\beta$  formally satisfies the set of moments problem (III.98), it remains to show that the series converges and to obtain the expected bound on  $v_\beta$ .

- In the case when  $\beta_1 = \beta_2$ , the result is just a particular case of point 2 Theorem III.2.14.

- Assume that  $\beta_1 \neq \beta_2$ . We observe that  $\left\| \frac{\Phi_{k,1}}{\mathcal{B}^* \Phi_{k,1}} \right\|_{L^2(0,1)} = \left\| \frac{\varphi_k^{\beta_1}}{\varphi_k^{\beta_1}(0)} \right\|_{L^2(0,1)}$  which can be bounded by  $C(1 + \beta_1)$  for any  $\beta_1 \in (0, +\infty)$  (recall the expression of  $\varphi_k^{\beta_1}$  from (III.45)). We can then choose  $\varepsilon = T/2$  and apply Proposition III.5.4 to obtain that, for  $0 < \beta_1, \beta_2 < \beta^*$ ,

$$\|v_k^{\beta_1, \beta_2}\|_{L^2(0,T)} \leq C_{T, \beta^*} (1 + \beta_1) e^{-\frac{T}{2} \lambda_k^{\beta_1}} \|y_0\|_{L^2(0,1)}, \quad (\text{III.103})$$

and for  $\beta_1, \beta_2 \geq \beta^*$ ,

$$\|v_k^{\beta_1, \beta_2}\|_{L^2(0,T)} \leq C_{T, \beta^*} (1 + \beta_1) \sqrt{\lambda_k^{\beta_2}} e^{-\frac{T}{2} \lambda_k^{\beta_1}} \|y_0\|_{L^2(0,1)}. \quad (\text{III.104})$$

From (III.102), it follows that

$$\|v_\beta\|_{L^2(0,T)} \leq \sum_{k \geq 0} \|v_k^{\beta_1, \beta_2}\|_{L^2(0,T)},$$

in which we can plug (III.103) or (III.104) to finally obtain

$$\|v_\beta\|_{L^2(0,T)} \leq C_{T, \beta^*} (1 + \beta_1) \|y_0\|_{L^2(0,1)},$$

where  $C_{T, \beta^*} > 0$  does not depend explicitly on the parameters  $\beta_1, \beta_2$  because the two series  $\sum_{k \geq 0} e^{-\frac{T}{2} \lambda_k^{\beta_1}}$  and  $\sum_{k \geq 0} \sqrt{\lambda_k^{\beta_2}} e^{-\frac{T}{2} \lambda_k^{\beta_1}}$  converges uniformly with respect to the parameters due to the fact  $\lambda_k^N < \lambda_k^{\beta_i} < \lambda_k^D$ ,  $\forall k \geq 0$ ,  $i = 1, 2$ , by Lemma III.3.3.

The proof of the theorem is complete.  $\square$

## III.6 A uniform Lebeau-Robbiano spectral inequality for Robin boundary problem

This section is devoted to present a Lebeau-Robbiano spectral inequality that is uniform with respect to the Robin parameters (positive) and we remind that this is essential to obtain a uniform estimate of the controls in the cylindrical geometry (see Theorem III.2.18).

As introduced in Section III.2.4.2, let  $\Omega_2 \subset \mathbb{R}^{d-1}$ ,  $d \geq 2$  be a non-empty open bounded smooth domain and  $\omega_2 \Subset \Omega_2$  be any non-empty open set. Also, recall from Section III.30 that, one of the main ingredient to prove the null-controllability of the system (III.30)–(III.32) in the cylindrical domain (Theorem III.2.18), we need a Lebeau-Robbiano spectral inequality relative to the control set  $\omega_2$ , for the eigenfunctions of the operator  $-\text{div}_{\bar{x}}(\bar{\gamma} \nabla_{\bar{x}})$  ( $\bar{x} \in \Omega_2$ ) with the Robin boundary conditions. To simplify the computations, we will write the proof for  $-\Delta_{\bar{x}}$ , and the same will be true for the operator  $-\text{div}_{\bar{x}}(\bar{\gamma} \nabla_{\bar{x}})$ .

We mainly provide here some results (more simplified) from the work [85] by Qi Lü, where the author has proved a Lebeau-Robbiano spectral inequality for Laplace-Beltrami operator with a general lateral boundary condition in a multi-D connected compact  $\mathcal{C}^1$ -smooth Riemannian manifold with  $\mathcal{C}^2$ -smooth boundary. In our setting, we shall restrict ourselves to the operator  $-\Delta_{\bar{x}}$  with Robin boundary conditions in the domain  $\Omega_2$ .

Before going into details, let us consider the following eigenvalue problem,

$$\begin{cases} -\Delta_{\bar{x}} \Phi = \lambda \Phi & \text{in } \Omega_2, \\ \frac{\partial \Phi}{\partial \nu_{\bar{x}}} + \beta \Phi = 0 & \text{on } \partial \Omega_2, \end{cases}$$

where  $\nu_{\bar{x}}$  is the normal vector on the boundary  $\partial \Omega_2$  and  $\beta \geq 0$  is the Robin parameter.

Since the operator here is self-adjoint, the spectrum is discrete, unbounded and it consists of real eigenvalues. Let us denote the set of eigenvalues by  $\Lambda_\beta$  and the eigenfunctions by  $\{\Phi_\lambda\}_{\lambda \in \Lambda_\beta}$ ,

which is indeed an orthonormal basis in  $L^2(\Omega_2)$ . Also we consider the following space for any  $r > 0$ ,

$$E_r := \text{Span} \{ \Phi_\lambda : \forall \lambda \in \Lambda_\beta \text{ with } \lambda \leq r \}. \quad (\text{III.105})$$

Below, we present the main theorem concerning the spectral inequality.

**Theorem III.6.1** (Lebeau-Robbiano spectral inequality). *Let  $\Omega_2 \subset \mathbb{R}^{d-1}$  ( $d \geq 2$ ), be any non-empty open bounded smooth domain and  $\omega_2 \Subset \Omega_2$  be any non-empty open set. Then, there exists a constant  $C := C(\omega_2) > 0$  independent on  $\beta \geq 0$  such that for any  $r > 0$ , we have*

$$\|\Phi\|_{L^2(\Omega_2)} \leq C e^{C\sqrt{r}} \|\Phi\|_{L^2(\omega_2)}, \quad \forall \Phi \in E_r,$$

where  $E_r$  has been defined in (III.105).

The above spectral inequality can be proved by means of an *augmented elliptic Carleman type estimate* that we will prove in the following section.

### III.6.1 A uniform Carleman estimate for augmented elliptic operator with lateral Robin boundary condition

Let  $T^* > 0$  be any positive quantity and denote  $Q = (0, T^*) \times \Omega_2$ , where  $\Omega_2$  has been introduced as before. Assume that  $u \in \mathcal{C}^2(\bar{Q})$  satisfies the following set of equations:

$$\begin{cases} \partial_\tau^2 u + \Delta_{\bar{x}} u = 0 & \text{in } Q, \\ \frac{\partial u}{\partial \nu_{\bar{x}}} + \beta u = 0 & \text{on } (0, T^*) \times \partial\Omega_2, \end{cases} \quad (\text{III.106})$$

where  $\beta \geq 0$ .

In this section, we shall prove a Carleman type estimate for the system (III.106) that is *uniform in  $\beta \geq 0$* .

**Remark III.6.2.** *The author in [85] proved a Lebeau-Robbiano spectral inequality for Laplace-Beltrami operator with the function  $\beta \in L^\infty(\partial\Omega_2)$ ,  $\beta \geq 0$ . Although it is not mentioned in that paper, a careful look on the computations tells that the estimate is uniform with respect to  $\|\beta\|_\infty$  in his setting. In particular, the same is true for any **parameter**  $\beta \geq 0$ . In this section, we shall in fact provide a **simpler proof** of an **augmented elliptic Carleman estimate** which is important to obtain a Lebeau-Robbiano spectral inequality.*

In this context, some suitable weight functions need to be constructed, we begin with the following lemma.

**Lemma III.6.3.** *Let  $\omega_2$  and  $\Omega_2$  be introduced as before. Then, there exists a function  $\eta \in \mathcal{C}^2(\bar{\Omega}_2)$  such that*

$$\begin{cases} \eta > 0 & \text{in } \Omega_2 \text{ and } \eta = 0 \text{ on } \partial\Omega_2, \\ |\nabla_{\bar{x}} \eta| \geq c > 0, \quad \forall \bar{x} \in \overline{\Omega_2 \setminus \omega_0} \text{ and } \frac{\partial \eta}{\partial \nu_{\bar{x}}} < 0 & \text{on } \partial\Omega_2, \end{cases}$$

where  $\omega_0$  is an nonempty open subset in  $\Omega_2$  such that  $\bar{\omega}_0 \subset \omega_2$ .

A proof of the above lemma has been sketched in [96, Chapter 14].

– **Construction of weight functions.** We will use the following weight functions which have been prescribed in [85], possibly with a more sharp choice of  $\delta > 0$  as in the next line.

Let  $0 < \delta < \frac{T^*}{6}$  and  $2\delta < T' < T'' < T^* - \delta$ . One may choose  $T' + T'' < T^*$ . We let

$$a_0 = \frac{T^* - T'}{2} - \delta, \quad a = \frac{T^*}{2} - 2\delta, \quad a_1 = \frac{T^*}{2} - \delta. \quad (\text{III.107})$$

With these, it is not difficult to observe that

$$\frac{T^*}{2} - T' < a_0 < a < a_1 < \frac{T^*}{2}.$$

Now, we choose

$$\begin{aligned} \varphi(\tau, \bar{x}) &= (c_1 - c_2) \frac{\eta(\bar{x})}{\|\eta\|_{L^\infty}} + a^2 - \left(\tau - \frac{T^*}{2}\right)^2 + \kappa, \quad \forall (\tau, \bar{x}) \in Q \quad \text{and} \\ \tilde{\varphi}(\tau, \bar{x}) &= -(c_1 - c_2) \frac{\eta(\bar{x})}{\|\eta\|_{L^\infty}} + a^2 - \left(\tau - \frac{T^*}{2}\right)^2 + \kappa, \quad \forall (\tau, \bar{x}) \in Q, \end{aligned}$$

where  $c_1 = a^2 - \left(\frac{T^*}{2} - T'\right)^2$ ,  $c_2 = a^2 - \frac{1}{2}\left(\frac{T^*}{2} - T'\right)^2 - a_0^2$  and  $\kappa$  is chosen large enough so that we have  $0 < \tilde{\varphi} \leq \varphi$  in  $Q$ . Also, since  $\eta = 0$  (see Lemma III.6.3) on  $\partial\Omega_2$ , we have  $\varphi = \tilde{\varphi}$  on  $(0, T^*) \times \partial\Omega_2$ . Moreover, we have

$$\begin{cases} |(\partial_\tau, \nabla_{\bar{x}})\varphi| = |\partial_\tau\varphi| + |\nabla_{\bar{x}}\varphi| = 2\left|\tau - \frac{T^*}{2}\right| + \frac{(c_1 - c_2)}{\|\eta\|_{L^\infty}} |\nabla_{\bar{x}}\eta| > 0, \quad \forall (\tau, \bar{x}) \in (0, T^*) \times \overline{(\Omega_2 \setminus \omega_0)}, \\ \frac{\partial\varphi}{\partial\nu_{\bar{x}}} = \frac{(c_1 - c_2)}{\|\eta\|_{L^\infty}} \frac{\partial\eta}{\partial\nu_{\bar{x}}} < 0, \quad \text{on } (0, T^*) \times \partial\Omega_2. \end{cases} \quad (\text{III.108})$$

Similarly,  $\tilde{\varphi}$  satisfies

$$\begin{cases} |(\partial_\tau, \nabla_{\bar{x}})\tilde{\varphi}| > 0, & \forall (\tau, \bar{x}) \in (0, T^*) \times \overline{(\Omega_2 \setminus \omega_0)}, \\ \frac{\partial\tilde{\varphi}}{\partial\nu_{\bar{x}}} = \frac{-(c_1 - c_2)}{\|\eta\|_{L^\infty}} \frac{\partial\eta}{\partial\nu_{\bar{x}}} > 0, & \text{on } (0, T^*) \times \partial\Omega_2. \end{cases} \quad (\text{III.109})$$

Now, we define the weight functions,

$$\alpha(\tau, \bar{x}) = e^{\lambda\varphi(\tau, \bar{x})} \quad \text{and} \quad \tilde{\alpha}(\tau, \bar{x}) = e^{\lambda\tilde{\varphi}(\tau, \bar{x})} \quad \text{for } \lambda \geq 1. \quad (\text{III.110})$$

It is then obvious that  $1 < \tilde{\alpha} \leq \alpha$ .

– Next, we consider a function  $\xi \in C_c^\infty(\delta, T^* - \delta)$  such that

$$\begin{cases} 0 \leq \xi(\tau) \leq 1, & \text{in } (\delta, T^* - \delta) = (T^*/2 - a_1, T^*/2 + a_1), \\ \xi(\tau) = 1, & \text{in } [a, T^* - a] \subset (\delta, T^* - \delta). \end{cases}$$

With this  $\xi$  we define  $u_1 = \xi u$ , so that  $\text{supp}(u_1) \subseteq Q_0 := (\delta, T^* - \delta) \times \Omega_2 \subset Q$ . Moreover,  $u_1$  satisfies the following set of equations (recall that  $u$  satisfies (III.106))

$$\begin{cases} \partial_\tau^2 u_1 + \Delta_{\bar{x}} u_1 = u \partial_\tau^2 \xi + 2\partial_\tau u \partial_\tau \xi & \text{in } Q_0, \\ \frac{\partial u_1}{\partial\nu_{\bar{x}}} + \beta u_1 = 0 & \text{on } (\delta, T^* - \delta) \times \partial\Omega_2, \\ u_1 = 0 & \text{on } \{\delta, T^* - \delta\} \times \Omega_2. \end{cases} \quad (\text{III.111})$$

– Let us write now the main theorem of this section.

**Theorem III.6.4.** *Let  $\omega_2$ ,  $\Omega_2$  and the functions  $\alpha$ ,  $\xi$  be as defined before. Then there exists  $\lambda_0 := \lambda_0(\omega_2, \Omega_2, T^*) > 0$ ,  $s_0 := s_0(\omega_2, \Omega_2, T^*) > 0$  and a constant  $C := C(\omega_2, \Omega_2, T^*) > 0$ , such that  $u_1 \in \mathcal{C}^2(\overline{Q_0})$  in (III.111) satisfies*

$$\begin{aligned} & s^3 \lambda^4 \int_{Q_0} \alpha^3 e^{2s\alpha} |u_1|^2 d\bar{x} d\tau + s \lambda^2 \int_{Q_0} \alpha e^{2s\alpha} (|\partial_\tau u_1|^2 + |\nabla_{\bar{x}} u_1|^2) d\bar{x} d\tau \\ & \leq C \left[ s \lambda^2 \int_{Q_0} e^{2s\alpha} |u \partial_\tau^2 \xi + 2 \partial_\tau u \partial_\tau \xi|^2 d\bar{x} d\tau + s^3 \lambda^4 \int_0^{T^*} \int_{\omega_2} \alpha^3 e^{2s\alpha} |u_1|^2 d\bar{x} d\tau \right], \end{aligned} \quad (\text{III.112})$$

for any  $\lambda \geq \lambda_0$  and  $s \geq s_0$ , where  $C$  does not depend on the parameter  $\beta \geq 0$ .

Note that, this is not like the usual Carleman type estimate which generally appears for the augmented elliptic operator, see for instance [82, Theorem 3.A.1]. But, the above estimate (III.112) is sufficient to prove a Lebeau-Robbiano spectral inequality in the Robin case. Actually, the estimate (III.112) is the key point to deduce some interpolation inequalities which will be helpful to obtain the required Lebeau-Robbiano spectral inequality.

Before going to the proof, let us first denote the independent variable in  $Q$  simply by  $X := (X_0, X_1, \dots, X_{d-1})$ , with  $X_0 := \tau \in (0, T^*)$  and  $(X_1, \dots, X_{d-1}) := \bar{x} \in \Omega_2$ ; we also declare the standard notation

$$\nabla_X = \left( \frac{\partial}{\partial X_0}, \frac{\partial}{\partial X_1}, \dots, \frac{\partial}{\partial X_{d-1}} \right) := (\partial_\tau, \nabla_{\bar{x}}), \quad \Delta_X = \sum_{i=0}^{d-1} \frac{\partial^2}{\partial^2 X_i} := (\partial_\tau^2 + \Delta_{\bar{x}}).$$

We shall use the symbol  $\nu_X$  to express the normal vector on  $\partial Q$ .

*Proof of Theorem III.6.4.* With the weight functions  $\alpha$  and  $\tilde{\alpha}$  defined in (III.110), we consider, for any  $s > 0$ ,

$$w = e^{s\alpha} u_1, \quad \tilde{w} = e^{s\tilde{\alpha}} u_1, \quad \text{in } Q_0. \quad (\text{III.113})$$

◆ We first observe that  $w$  satisfies

$$M_1 w + M_2 w = F \quad \text{in } Q_0, \quad \text{with}$$

$$\begin{cases} M_1 w = \Delta_X w + s^2 \lambda^2 \alpha^2 |\nabla_X \varphi|^2 w, \\ M_2 w = -2s\lambda \alpha (\nabla_X \varphi \cdot \nabla_X w) - 2s\lambda^2 \alpha |\nabla_X \varphi|^2 w, \\ F = e^{s\alpha} f + s\lambda \alpha (\Delta_X \varphi) w - s\lambda^2 \alpha |\nabla_X \varphi|^2 w, \end{cases} \quad (\text{III.114})$$

with  $f = (\partial_\tau^2 u_1 + \Delta_{\bar{x}} u_1) = (u \partial_\tau^2 \xi + 2 \partial_\tau u \partial_\tau \xi)$  in  $Q_0$ . We have

$$\|M_1 w\|_{L^2(Q_0)}^2 + \|M_2 w\|_{L^2(Q_0)}^2 + 2(M_1 w, M_2 w)_{L^2(Q_0)} = \|F\|_{L^2(Q_0)}^2. \quad (\text{III.115})$$

The quantity  $(M_1 w, M_2 w)_{L^2(Q_0)}$  can be expressed as the sum of four terms  $I_{ij}$  for  $1 \leq i, j \leq 2$ .

1. Let us consider the term  $I_{11}$ ,

$$I_{11} = -2s\lambda \int_{Q_0} \alpha (\nabla_X \varphi \cdot \nabla_X w) \Delta_X w dX = -2s\lambda \sum_{i,j=0}^{d-1} \int_{Q_0} \left( \alpha \frac{\partial \varphi}{\partial X_i} \frac{\partial w}{\partial X_i} \right) \frac{\partial^2 w}{\partial^2 X_j} dX.$$

Integrating by parts, we see

$$\begin{aligned} I_{11} &= 2s\lambda \sum_{i,j=0}^{d-1} \int_{Q_0} \frac{\partial}{\partial X_j} \left( \alpha \frac{\partial \varphi}{\partial X_i} \frac{\partial w}{\partial X_i} \right) \frac{\partial w}{\partial X_j} dX \\ &\quad - 2s\lambda \int_{\partial Q_0} \alpha (\nabla_X \varphi \cdot \nabla_X w) (\nabla_X w \cdot \nu_X) d\nu_X. \end{aligned} \quad (\text{III.116})$$

Recall that  $\alpha = e^{\lambda\varphi}$ , so that  $\frac{\partial\alpha}{\partial X_j} = \lambda\alpha \frac{\partial\varphi}{\partial X_j}$ , for  $0 \leq j \leq d-1$  and we concentrate on the first integral above, we have

$$\begin{aligned} 2s\lambda \sum_{i,j=0}^{d-1} \int_{Q_0} \frac{\partial}{\partial X_j} \left( \alpha \frac{\partial\varphi}{\partial X_i} \frac{\partial w}{\partial X_i} \right) \frac{\partial w}{\partial X_j} dX &= 2s\lambda^2 \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha \frac{\partial\varphi}{\partial X_i} \frac{\partial w}{\partial X_i} \frac{\partial\varphi}{\partial X_j} \frac{\partial w}{\partial X_j} dX \\ &+ 2s\lambda \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha \frac{\partial^2\varphi}{\partial X_i \partial X_j} \frac{\partial w}{\partial X_i} \frac{\partial w}{\partial X_j} dX + s\lambda \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha \frac{\partial\varphi}{\partial X_i} \frac{\partial}{\partial X_i} \left| \frac{\partial w}{\partial X_j} \right|^2 dX, \end{aligned} \quad (\text{III.117})$$

where one can express the first integral in the right hand side in a compact way as  $2s\lambda^2 \int_{Q_0} \alpha (\nabla_X \varphi \cdot \nabla_X w)^2 dX$ .

Now, applying an integrating by parts technique to the last integral in the right hand side of (III.117), we see

$$\begin{aligned} s\lambda \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha \frac{\partial\varphi}{\partial X_i} \frac{\partial}{\partial X_i} \left| \frac{\partial w}{\partial X_j} \right|^2 dX &= -s\lambda^2 \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha \left| \frac{\partial\varphi}{\partial X_i} \right|^2 \left| \frac{\partial w}{\partial X_j} \right|^2 dX \\ &- s\lambda \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha \frac{\partial^2\varphi}{\partial X_i^2} \left| \frac{\partial w}{\partial X_j} \right|^2 dX + s\lambda \int_{\partial Q_0} \alpha |\nabla_X w|^2 (\nabla_X \varphi \cdot \nu_X) d\nu_X, \end{aligned} \quad (\text{III.118})$$

where the first integral in the right hand side could be expressed simply as  $-s\lambda^2 \int_{Q_0} \alpha |\nabla_X \varphi|^2 |\nabla_X w|^2 dX$ .

Now, combining (III.117) and (III.118), we obtain from (III.116) the following

$$\begin{aligned} I_{11} &= 2s\lambda^2 \int_{Q_0} \alpha (\nabla_X \varphi \cdot \nabla_X w)^2 dX - s\lambda^2 \int_{Q_0} \alpha |\nabla_X \varphi|^2 |\nabla_X w|^2 dX \\ &- 2s\lambda \int_{\partial Q_0} \alpha (\nabla_X \varphi \cdot \nabla_X w) (\nabla_X w \cdot \nu_X) d\nu_X + s\lambda \int_{\partial Q_0} \alpha |\nabla_X w|^2 (\nabla_X \varphi \cdot \nu_X) d\nu_X \\ &+ R_{11}, \end{aligned} \quad (\text{III.119})$$

with

$$R_{11} = 2s\lambda \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha \frac{\partial^2\varphi}{\partial X_i \partial X_j} \frac{\partial w}{\partial X_i} \frac{\partial w}{\partial X_j} dX - s\lambda \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha \frac{\partial^2\varphi}{\partial X_i^2} \left| \frac{\partial w}{\partial X_j} \right|^2 dX.$$

Note that, the first integral of  $I_{11}$  in (III.119) is non-negative and at the end we shall simply ignore this term in the left hand side of the final estimate.

2. Let us focus on the term  $I_{12}$ ,

$$I_{12} = -2s\lambda^2 \int_{Q_0} \alpha w |\nabla_X \varphi|^2 \Delta_X w dX = -2s\lambda^2 \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha w \left| \frac{\partial\varphi}{\partial X_i} \right|^2 \frac{\partial^2 w}{\partial X_j^2} dX.$$

Integrating by parts, we have

$$\begin{aligned} I_{12} &= 2s\lambda^2 \sum_{i,j=0}^{d-1} \int_{Q_0} \frac{\partial}{\partial X_j} \left( \alpha w \left| \frac{\partial\varphi}{\partial X_i} \right|^2 \right) \frac{\partial w}{\partial X_j} dX - 2s\lambda^2 \int_{\partial Q_0} \alpha w |\nabla_X \varphi|^2 (\nabla_X w \cdot \nu_X) d\nu_X \\ &= 2s\lambda^2 \int_{Q_0} \alpha |\nabla_X \varphi|^2 |\nabla_X w|^2 dX + R_{12} - 2s\lambda^2 \int_{\partial Q_0} \alpha w |\nabla_X \varphi|^2 (\nabla_X w \cdot \nu_X) d\nu_X, \end{aligned} \quad (\text{III.120})$$

where

$$R_{12} = 2s\lambda^3 \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha w \frac{\partial w}{\partial X_j} \frac{\partial\varphi}{\partial X_j} \left| \frac{\partial\varphi}{\partial X_i} \right|^2 dX + 4s\lambda^2 \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha w \frac{\partial w}{\partial X_j} \frac{\partial\varphi}{\partial X_i} \frac{\partial^2\varphi}{\partial X_i \partial X_j} dX.$$



3. Next, we see

$$\begin{aligned} I_{21} &= -2s^3\lambda^3 \int_{Q_0} \alpha^3 |\nabla_X \varphi|^2 w (\nabla_X \varphi \cdot \nabla_X w) dX \\ &= -2s^3\lambda^3 \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha^3 \left| \frac{\partial \varphi}{\partial X_i} \right|^2 \frac{\partial \varphi}{\partial X_j} w \frac{\partial w}{\partial X_j} dX \\ &= -s^3\lambda^3 \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha^3 \left| \frac{\partial \varphi}{\partial X_i} \right|^2 \frac{\partial \varphi}{\partial X_j} \frac{\partial}{\partial X_j} |w|^2 dX, \end{aligned}$$

and again performing an integration by parts, we obtain

$$I_{21} = 3s^3\lambda^4 \int_{Q_0} \alpha^3 |\nabla_X \varphi|^4 |w|^2 - s^3\lambda^3 \int_{\partial Q_0} \alpha^3 |\nabla_X \varphi|^2 |w|^2 (\nabla_X \varphi \cdot \nu_X) d\nu_X + R_{21}, \quad (\text{III.121})$$

with

$$R_{21} = s^3\lambda^3 \sum_{i,j=0}^{d-1} \int_{Q_0} \alpha^3 \frac{\partial}{\partial X_j} \left( \left| \frac{\partial \varphi}{\partial X_i} \right|^2 \frac{\partial \varphi}{\partial X_j} \right) |w|^2 dX.$$

4. Finally, the fourth term is

$$I_{22} = -2s^3\lambda^4 \int_{Q_0} \alpha^3 |\nabla_X \varphi|^4 |w|^2 dX. \quad (\text{III.122})$$

– Now, upon an addition of all four terms  $I_{ij}$ ,  $1 \leq i, j \leq 4$ , given by (III.119), (III.120), (III.121) and (III.122), we observe that the leading integrals are

$$s^3\lambda^4 \int_{Q_0} \alpha^3 |\nabla_X \varphi|^4 |w|^2 dX \quad \text{and} \quad s\lambda^2 \int_{Q_0} \alpha |\nabla_X \varphi|^2 |\nabla_X w|^2 dX.$$

Now, from the properties of  $\varphi$  given by (III.108), we observe that the quantity  $|\nabla_X \varphi| > 0$  in  $(0, T^*) \times \overline{\Omega_2} \setminus \omega_0$ , so we eventually have from (III.115),

$$\begin{aligned} &s^3\lambda^4 \int_{Q_0} \alpha^3 |w|^2 dX + s\lambda^2 \int_{Q_0} \alpha |\nabla_X w|^2 dX \\ &- 2s\lambda^2 \int_{\partial Q_0} \alpha w |\nabla_X \varphi|^2 (\nabla_X w \cdot \nu_X) d\nu_X - 2s\lambda \int_{\partial Q_0} \alpha (\nabla_X \varphi \cdot \nabla_X w) (\nabla_X w \cdot \nu_X) d\nu_X \\ &+ s\lambda \int_{\partial Q_0} \alpha |\nabla_X w|^2 (\nabla_X \varphi \cdot \nu_X) d\nu_X - s^3\lambda^3 \int_{\partial Q_0} \alpha^3 |\nabla_X \varphi|^2 |w|^2 (\nabla_X \varphi \cdot \nu_X) d\nu_X \\ &\leq C \left[ \int_{Q_0} e^{2s\alpha} |f|^2 dX + \int_{(0, T^*) \times \omega_0} \left( s^3\lambda^4 \alpha^3 |w|^2 + s\lambda^2 \alpha |\nabla_X w|^2 \right) dX \right], \quad (\text{III.123}) \end{aligned}$$

where we drop the integrals  $R_{11}, R_{12}, R_{21}$  and the integrals coming from the second, third terms of  $F$  (see (III.114)) because they are of lower order and can be absorbed by the first two leading integrals sitting in the left hand side of (III.123) for any  $\lambda \geq \lambda_0 > 0$  and  $s \geq s_0 > 0$  where  $\lambda_0$  and  $s_0$  may depend on the choices of domain  $\Omega_2$ , observation region  $\omega_2$  and on  $T^*$ .

◆ In a similar way,  $\tilde{w}$  (defined by (III.113)) also satisfies the same type of estimate as follows:

$$\begin{aligned} &s^3\lambda^4 \int_{Q_0} \tilde{\alpha}^3 |\tilde{w}|^2 dX + s\lambda^2 \int_{Q_0} \tilde{\alpha} |\nabla_X \tilde{w}|^2 dX \\ &- 2s\lambda^2 \int_{\partial Q_0} \tilde{\alpha} w |\nabla_X \tilde{\varphi}|^2 (\nabla_X \tilde{w} \cdot \nu_X) d\nu_X - 2s\lambda \int_{\partial Q_0} \tilde{\alpha} (\nabla_X \tilde{\varphi} \cdot \nabla_X \tilde{w}) (\nabla_X \tilde{w} \cdot \nu_X) d\nu_X \\ &+ s\lambda \int_{\partial Q_0} \tilde{\alpha} |\nabla_X \tilde{w}|^2 (\nabla_X \tilde{\varphi} \cdot \nu_X) d\nu_X - s^3\lambda^3 \int_{\partial Q_0} \tilde{\alpha}^3 |\nabla_X \tilde{\varphi}|^2 |\tilde{w}|^2 (\nabla_X \tilde{\varphi} \cdot \nu_X) d\nu_X \\ &\leq C \left[ \int_{Q_0} e^{2s\tilde{\alpha}} |f|^2 dX + \int_{(0, T^*) \times \omega_0} \left( s^3\lambda^4 \tilde{\alpha}^3 |\tilde{w}|^2 + s\lambda^2 \tilde{\alpha} |\nabla_X \tilde{w}|^2 \right) dX \right], \quad (\text{III.124}) \end{aligned}$$

for any  $\lambda \geq \lambda_0$  and  $s \geq s_0$ , might be for some different  $\lambda_0$  and  $s_0$  than the last ones.

◆ Next, the idea is to add the estimates (III.123) and (III.124) to remove some unusual boundary integrals. In this regard, we must mention that this technique has been successfully developed in [59], where the authors proved a new Carleman estimate for the linear heat equation with Robin boundary conditions and deduced the null-controllability result for the associated problem.

Let us now deal with the boundary integrals of the inequality (III.123). We have that

$$\partial Q_0 = (\{\delta\} \times \Omega_2) \cup (\{T^* - \delta\} \times \Omega_2) \cup ((\delta, T^* - \delta) \times \partial\Omega_2).$$

Recall that  $w = e^{s\alpha}(\xi u)$  (see (III.113)) with  $\xi \in \mathcal{C}_c^\infty(\delta, T^* - \delta)$ , so that we have

$$w = 0 \quad \text{in } \{\delta\} \times \Omega_2 \quad \text{and} \quad \{T^* - \delta\} \times \Omega_2.$$

and so, there are no boundary integrals on  $\{\delta\} \times \Omega_2$  and  $\{T^* - \delta\} \times \Omega_2$ .

– *Boundary terms on  $(\delta, T^* - \delta) \times \partial\Omega$ .* The normal vector on this boundary  $\nu_X = (0, \nu_{\bar{x}})$  (recall that  $X = (\tau, \bar{x})$  by our assumption). We compute

$$\frac{\partial w}{\partial \nu_{\bar{x}}} = e^{s\alpha} \frac{\partial u_1}{\partial \nu_{\bar{x}}} + s\lambda\alpha \frac{\partial \varphi}{\partial \nu_{\bar{x}}} w = -\beta w + s\lambda\alpha w \frac{\partial \varphi}{\partial \nu_{\bar{x}}}, \quad (\text{III.125})$$

since we have  $\frac{\partial u_1}{\partial \nu_{\bar{x}}} + \beta u_1 = 0$ .

We denote all the four boundary integrals by  $J_i$ ,  $1 \leq i \leq 4$  according to the same order as (III.123). For simplicity, we denote that  $\Sigma := (\delta, T^* - \delta) \times \partial\Omega_2$ .

- We start with  $J_1$ , it follows from (III.123) that

$$J_1 = -2s\lambda^2 \int_{\Sigma} \alpha w \left( |\partial_{\tau} \varphi|^2 + \left| \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \right|^2 \right) \frac{\partial w}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau.$$

Replacing  $\frac{\partial w}{\partial \nu_{\bar{x}}}$  by its expression as in (III.125), one could obtain

$$\begin{aligned} J_1 &= 2s\lambda^2 \beta \int_{\Sigma} \alpha \left( |\partial_{\tau} \varphi|^2 + \left| \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \right|^2 \right) |w|^2 d\nu_{\bar{x}} d\tau \\ &\quad - 2s^2\lambda^3 \int_{\Sigma} \alpha^2 \left( |\partial_{\tau} \varphi|^2 + \left| \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \right|^2 \right) \frac{\partial \varphi}{\partial \nu_{\bar{x}}} |w|^2 d\nu_{\bar{x}} d\tau. \end{aligned} \quad (\text{III.126})$$

- Let us look into the second boundary integral.

$$J_2 = -2s\lambda \int_{\Sigma} \alpha (\partial_{\tau} \varphi \partial_{\tau} w + \nabla_{\bar{x}} \varphi \cdot \nabla_{\bar{x}} w) \frac{\partial w}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau. \quad (\text{III.127})$$

By construction of  $\varphi$ , we have  $\nabla_{\bar{x}}^{\parallel} \varphi = 0$  (parallel gradient) since  $\varphi|_{\Sigma}$  is a function of  $\tau$  as  $\eta = 0$  on  $\partial\Omega_2$ . This follows that  $\nabla_{\bar{x}} \varphi|_{\Sigma} = \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \nu_{\bar{x}}$  and  $(\nabla_{\bar{x}} \varphi \cdot \nabla_{\bar{x}} w)|_{\Sigma} = \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \frac{\partial w}{\partial \nu_{\bar{x}}}$ . Accordingly, we have from (III.127) that

$$\begin{aligned} J_2 &= -2s\lambda \int_{\Sigma} \alpha (\partial_{\tau} \varphi \partial_{\tau} w) \frac{\partial w}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau - 2s\lambda \int_{\Sigma} \alpha \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \left| \frac{\partial w}{\partial \nu_{\bar{x}}} \right|^2 d\nu_{\bar{x}} d\tau \\ &= J_{21} + J_{22}. \end{aligned} \quad (\text{III.128})$$

Recall the expression of  $\frac{\partial w}{\partial \nu_{\bar{x}}}$  given by (III.125) to have

$$\begin{aligned} J_{21} &= 2s\lambda\beta \int_{\Sigma} (\alpha \partial_{\tau} \varphi) w \partial_{\tau} w d\nu_{\bar{x}} d\tau - 2s^2\lambda^2 \int_{\Sigma} \left( \alpha^2 \partial_{\tau} \varphi \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \right) w \partial_{\tau} w d\nu_{\bar{x}} d\tau \\ &= -s\lambda^2 \beta \int_{\Sigma} \alpha |\partial_{\tau} \varphi|^2 |w|^2 d\nu_{\bar{x}} d\tau - s\lambda\beta \int_{\Sigma} \alpha \partial_{\tau}^2 \varphi |w|^2 d\nu_{\bar{x}} d\tau \\ &\quad + 2s^2\lambda^3 \int_{\Sigma} \alpha^2 |\partial_{\tau} \varphi|^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} |w|^2 d\nu_{\bar{x}} d\tau + s^2\lambda^2 \int_{\Sigma} \alpha^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \partial_{\tau}^2 \varphi |w|^2 d\nu_{\bar{x}} d\tau, \end{aligned} \quad (\text{III.129})$$

where we perform integration by parts formula with respect to  $\tau$  and due to the fact that  $w = 0$  at  $\tau \in \{\delta, T^* - \delta\}$ , there are no integrals on  $\{\delta, T^* - \delta\} \times \partial\Omega_2$ .

- Before going to compute explicitly the part  $J_{22}$ , let us look into the third boundary integral. We have

$$J_3 = s\lambda \int_{\Sigma} \alpha (|\partial_{\tau} w|^2 + |\nabla_{\bar{x}} w|^2) \frac{\partial \varphi}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau.$$

One has  $|\nabla_{\bar{x}} w|^2 = \left| \frac{\partial w}{\partial \nu_{\bar{x}}} \right|^2 + |\nabla_{\bar{x}}^{\parallel} w|^2$ , on  $\Sigma$ , where  $\nabla_{\bar{x}}^{\parallel} w = e^{s\alpha} \nabla_{\bar{x}}^{\parallel} u_1$  since  $\nabla_{\bar{x}}^{\parallel} \varphi = 0$  on  $\Sigma$ . In what follows,

$$\begin{aligned} J_3 = s\lambda \int_{\Sigma} \alpha |\partial_{\tau} w|^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau + s\lambda \int_{\Sigma} \alpha e^{2s\alpha} |\nabla_{\bar{x}}^{\parallel} u_1|^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau \\ + s\lambda \int_{\Sigma} \alpha \left| \frac{\partial w}{\partial \nu_{\bar{x}}} \right|^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau \end{aligned} \quad (\text{III.130})$$

Observe that, once we add  $J_{22}$  and the last integral of  $J_3$ , we eventually have

$$\begin{aligned} -s\lambda \int_{\Sigma} \alpha \left| \frac{\partial w}{\partial \nu_{\bar{x}}} \right|^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau = -s^3 \lambda^3 \int_{\Sigma} \alpha^3 |w|^2 \left| \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \right|^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau \\ - s\lambda \beta^2 \int_{\Sigma} \alpha |w|^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau + 2s^2 \lambda^2 \beta \int_{\Sigma} \alpha^2 \left| \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \right|^2 |w|^2 d\nu_{\bar{x}} d\tau, \end{aligned} \quad (\text{III.131})$$

using the formula (III.125).

- Lastly, we have the fourth integral as follows:

$$J_4 = -s^3 \lambda^3 \int_{\Sigma} \alpha^3 \left( |\partial_{\tau} \varphi|^2 + \left| \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \right|^2 \right) \frac{\partial \varphi}{\partial \nu_{\bar{x}}} |w|^2 d\nu_{\bar{x}} d\tau. \quad (\text{III.132})$$

– Now, we add all the above boundary integrals to obtain

$$\begin{aligned} \sum_{i=1}^4 J_i = s\lambda^2 \beta \int_{\Sigma} \alpha |\partial_{\tau} \varphi|^2 |w|^2 d\nu_{\bar{x}} d\tau + 2s\lambda^2 \beta \int_{\Sigma} \alpha \left| \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \right|^2 |w|^2 d\nu_{\bar{x}} d\tau \\ - 2s^2 \lambda^3 \int_{\Sigma} \alpha^2 \left| \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \right|^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} |w|^2 d\nu_{\bar{x}} d\tau - s\lambda \beta \int_{\Sigma} \alpha \partial_{\tau}^2 \varphi |w|^2 d\nu_{\bar{x}} d\tau \\ + s^2 \lambda^2 \int_{\Sigma} \alpha^2 \partial_{\tau}^2 \varphi \frac{\partial \varphi}{\partial \nu_{\bar{x}}} |w|^2 d\nu_{\bar{x}} d\tau + s\lambda \int_{\Sigma} \alpha |\partial_{\tau} w|^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau \\ + s\lambda \int_{\Sigma} \alpha e^{2s\alpha} |\nabla_{\bar{x}}^{\parallel} u_1|^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau + 2s^2 \lambda^2 \beta \int_{\Sigma} \alpha^2 \left| \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \right|^2 |w|^2 d\nu_{\bar{x}} d\tau, \\ - s\lambda \beta^2 \int_{\Sigma} \alpha |w|^2 \frac{\partial \varphi}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau - s^3 \lambda^3 \int_{\Sigma} \alpha^3 \left( |\partial_{\tau} \varphi|^2 + 2 \left| \frac{\partial \varphi}{\partial \nu_{\bar{x}}} \right|^2 \right) \frac{\partial \varphi}{\partial \nu_{\bar{x}}} |w|^2 d\nu_{\bar{x}} d\tau. \end{aligned} \quad (\text{III.133})$$

In a similar manner, one can deduce the boundary terms concerned with the estimate (III.124) of  $\tilde{w}$ ,

$$\begin{aligned} \sum_{i=1}^4 \tilde{J}_i = s\lambda^2 \beta \int_{\Sigma} \tilde{\alpha} |\partial_{\tau} \tilde{\varphi}|^2 |\tilde{w}|^2 d\nu_{\bar{x}} d\tau + 2s\lambda^2 \beta \int_{\Sigma} \tilde{\alpha} \left| \frac{\partial \tilde{\varphi}}{\partial \nu_{\bar{x}}} \right|^2 |\tilde{w}|^2 d\nu_{\bar{x}} d\tau \\ - 2s^2 \lambda^3 \int_{\Sigma} \tilde{\alpha}^2 \left| \frac{\partial \tilde{\varphi}}{\partial \nu_{\bar{x}}} \right|^2 \frac{\partial \tilde{\varphi}}{\partial \nu_{\bar{x}}} |\tilde{w}|^2 d\nu_{\bar{x}} d\tau - s\lambda \beta \int_{\Sigma} \tilde{\alpha} \partial_{\tau}^2 \tilde{\varphi} |\tilde{w}|^2 d\nu_{\bar{x}} d\tau \\ + s^2 \lambda^2 \int_{\Sigma} \tilde{\alpha}^2 \partial_{\tau}^2 \tilde{\varphi} \frac{\partial \tilde{\varphi}}{\partial \nu_{\bar{x}}} |\tilde{w}|^2 d\nu_{\bar{x}} d\tau + s\lambda \int_{\Sigma} \tilde{\alpha} |\partial_{\tau} \tilde{w}|^2 \frac{\partial \tilde{\varphi}}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau \\ + s\lambda \int_{\Sigma} \tilde{\alpha} e^{2s\tilde{\alpha}} |\nabla_{\bar{x}}^{\parallel} u_1|^2 \frac{\partial \tilde{\varphi}}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau + 2s^2 \lambda^2 \beta \int_{\Sigma} \tilde{\alpha}^2 \left| \frac{\partial \tilde{\varphi}}{\partial \nu_{\bar{x}}} \right|^2 |\tilde{w}|^2 d\nu_{\bar{x}} d\tau, \\ - s\lambda \beta^2 \int_{\Sigma} \tilde{\alpha} |\tilde{w}|^2 \frac{\partial \tilde{\varphi}}{\partial \nu_{\bar{x}}} d\nu_{\bar{x}} d\tau - s^3 \lambda^3 \int_{\Sigma} \tilde{\alpha}^3 \left( |\partial_{\tau} \tilde{\varphi}|^2 + 2 \left| \frac{\partial \tilde{\varphi}}{\partial \nu_{\bar{x}}} \right|^2 \right) \frac{\partial \tilde{\varphi}}{\partial \nu_{\bar{x}}} |\tilde{w}|^2 d\nu_{\bar{x}} d\tau. \end{aligned} \quad (\text{III.134})$$

◆ Now, recall that  $\eta = 0$  on  $\partial\Omega_2$  (see Lemma III.6.3), so that by construction we have

$$\begin{cases} \varphi|_{\Sigma} = \tilde{\varphi}|_{\Sigma}, & \alpha|_{\Sigma} = \tilde{\alpha}|_{\Sigma}, & w|_{\Sigma} = \tilde{w}|_{\Sigma}, \\ \partial_{\tau}\varphi|_{\Sigma} = \partial_{\tau}\tilde{\varphi}|_{\Sigma}, & \partial_{\tau}\alpha|_{\Sigma} = \partial_{\tau}\tilde{\alpha}|_{\Sigma}, & \partial_{\tau}w|_{\Sigma} = \partial_{\tau}\tilde{w}|_{\Sigma}, \\ \nabla_{\bar{x}}^{\parallel}w|_{\Sigma} = \nabla_{\bar{x}}^{\parallel}\tilde{w}|_{\Sigma} & \text{and} & \frac{\partial\varphi}{\partial\nu_{\bar{x}}}|_{\Sigma} = -\frac{\partial\tilde{\varphi}}{\partial\nu_{\bar{x}}}|_{\Sigma}. \end{cases} \quad (\text{III.135})$$

The next immediate job is to add both the auxiliary estimates (III.123) and (III.124), so that after addition some unusual boundary integrals will be canceled out due to the relations in (III.135). In what follows, and also using the fact  $\partial_{\tau}^2\varphi = \partial_{\tau}^2\tilde{\varphi} = -2$ , we obtain that the sum of all the boundary integrals (III.133) and (III.134) in the left hand side is

$$\begin{aligned} \sum_{i=1}^4 (J_i + \tilde{J}_i) &= 2s\lambda^2\beta \int_{\Sigma} \alpha |\partial_{\tau}\varphi|^2 |w|^2 d\nu_{\bar{x}} d\tau + 4s\lambda^2\beta \int_{\Sigma} \alpha \left| \frac{\partial\varphi}{\partial\nu_{\bar{x}}} \right|^2 |w|^2 d\nu_{\bar{x}} d\tau \\ &\quad + 4s\lambda\beta \int_{\Sigma} \alpha |w|^2 d\nu_{\bar{x}} d\tau + 4s^2\lambda^2\beta \int_{\Sigma} \alpha^2 \left| \frac{\partial\varphi}{\partial\nu_{\bar{x}}} \right|^2 |w|^2 d\nu_{\bar{x}} d\tau \geq 0, \end{aligned} \quad (\text{III.136})$$

since we assumed  $\beta \geq 0$ .

As a consequence, the sum of (III.123) and (III.124) eventually provides, by taking into account that  $\alpha \geq \tilde{\alpha} \geq 1$ , (III.136) and using some standard computations, that

$$\begin{aligned} &s^3\lambda^4 \int_{Q_0} \alpha^3 e^{2s\alpha} |u_1|^2 d\bar{x}d\tau + s\lambda^2 \int_{Q_0} \alpha e^{2s\alpha} (|\partial_{\tau}u_1|^2 + |\nabla_{\bar{x}}u_1|^2) d\bar{x}d\tau \\ &\leq C \left[ \int_{Q_0} e^{2s\alpha} |f|^2 d\bar{x}d\tau + \int_{(0,T^*) \times \omega_0} e^{2s\alpha} \left( s^3\lambda^4 \alpha^3 |u_1|^2 + s\lambda^2 \alpha (|\partial_{\tau}u_1|^2 + |\nabla_{\bar{x}}u_1|^2) \right) d\bar{x}d\tau \right], \end{aligned} \quad (\text{III.137})$$

where  $f = u\partial_{\tau}^2\xi + 2\partial_{\tau}u\partial_{\tau}\xi$ , and the constant  $C > 0$  does not depend on  $\beta \geq 0$ .

Next, we choose some  $g \in \mathcal{C}_c^{\infty}(\omega_2)$  (recall that  $w_0 \Subset w_2$ ) with  $g = 1$  in  $\omega_0$  and  $0 \leq g \leq 1$  in  $\omega_2$ . Multiplying the equation (III.111) of  $u_1$  by  $g\alpha e^{2s\alpha}u_1$  and then integrating by parts, we obtain

$$\begin{aligned} \int_0^{T^*} \int_{\omega_0} \alpha e^{2s\alpha} (|\partial_{\tau}u_1|^2 + |\nabla_{\bar{x}}u_1|^2) d\bar{x}d\tau &\leq C \left[ s^2\lambda^2 \int_0^{T^*} \int_{\omega_2} \alpha^3 e^{2s\alpha} |u_1|^2 d\bar{x}d\tau + \right. \\ &\quad \left. + \int_{Q_0} e^{2s\alpha} |u\partial_{\tau}^2\xi + 2\partial_{\tau}u\partial_{\tau}\xi|^2 d\bar{x}d\tau \right]. \end{aligned} \quad (\text{III.138})$$

Using (III.138) in (III.137), we finally have

$$\begin{aligned} &s^3\lambda^4 \int_{Q_0} \alpha^3 e^{2s\alpha} |u_1|^2 d\bar{x}d\tau + s\lambda^2 \int_{Q_0} \alpha e^{2s\alpha} (|\partial_{\tau}u_1|^2 + |\nabla_{\bar{x}}u_1|^2) d\bar{x}d\tau \\ &\leq C \left[ s\lambda^2 \int_{Q_0} e^{2s\alpha} |u\partial_{\tau}^2\xi + 2\partial_{\tau}u\partial_{\tau}\xi|^2 d\bar{x}d\tau + s^3\lambda^4 \int_0^{T^*} \int_{\omega_2} \alpha^3 e^{2s\alpha} |u_1|^2 d\bar{x}d\tau \right], \end{aligned}$$

for any  $\lambda \geq \lambda_0$  and  $s \geq s_0$ , where the constant  $C > 0$  does depend on  $\Omega_2$ ,  $\omega_2$  and  $T^*$ , but not on the parameter  $\beta \geq 0$ .

This completes the proof.  $\square$

### III.6.2 Interpolation inequalities

Once we have the Carleman type estimate as obtained by Theorem III.6.4, then one can show the following two interpolation inequalities as proved in [85]. We shall mention here those inequalities without the proofs.

**Proposition III.6.5** (Interpolation inequality–I). *Let us take  $0 < \delta < \frac{T^*}{6}$  and  $2\delta < T' < T'' < T^* - \delta$ . Then, there exists some  $0 < \mu < 1$  such that the solution  $u \in H^2(Q)$  of (III.106) satisfies*

$$\|u\|_{L^2((T', T'') \times \Omega_2)} \leq C \|u\|_{L^2((\delta, T^* - \delta) \times \omega_2)}^\mu \|u\|_{H^1(Q)}^{1-\mu},$$

where the constant  $C > 0$  does not depend on the Robin parameter  $\beta \geq 0$ .

For the sketch of the proof, we precisely refer to *Step 5* of the proof of [85, Theorem 4.1].

Next, we mention here [85, Theorem 5.1].

**Proposition III.6.6** (Interpolation inequality–II). *Let  $\delta > 0$  be chosen as previous. Then, there exists some  $0 < \kappa < 1$  such that the solution  $u \in H^2(Q)$  of (III.106) satisfies the following*

$$\|u\|_{H^1((\delta, T^* - \delta) \times \omega_2)} \leq C \left( \|u(0)\|_{L^2(\omega_2)} + \|\partial_\tau u(0)\|_{L^2(\omega_2)} + \|\nabla_{\bar{x}} u(0)\|_{L^2(\omega_2)} \right)^\kappa \|u\|_{H^1(Q)}^{1-\kappa},$$

with a constant  $C > 0$  that does not depend on the Robin parameter  $\beta \geq 0$ .

### III.6.3 Proof of Theorem III.6.1 (Lebeau-Robbiano spectral inequality)

In this section, we sketch the proof of the Lebeau-Robbiano spectral inequality which is our main requirement. The proof is as similar as given in [85].

*Proof of Theorem III.6.1.* For simplicity, we choose  $T^* = 4$ ,  $T' = 1$  and  $T'' = 3$ , so that  $Q = (0, 4) \times \Omega_2$ . Then, by Proposition III.6.5 we have

$$\|u\|_{L^2((1, 3) \times \Omega_2)} \leq C \|u\|_{L^2((\delta, T^* - \delta) \times \omega_2)}^\mu \|u\|_{H^1(Q)}^{1-\mu}.$$

Next, applying the estimate of Proposition III.6.6 to the first quantity in the right hand of the above inequality, we deduce

$$\|u\|_{L^2((1, 3) \times \Omega_2)} \leq C \left( \|u(0)\|_{L^2(\omega_2)} + \|\partial_\tau u(0)\|_{L^2(\omega_2)} + \|\nabla_{\bar{x}} u(0)\|_{L^2(\omega_2)} \right)^{\kappa\mu} \|u\|_{H^1(Q)}^{1-\kappa\mu}, \quad (\text{III.139})$$

where  $u \in H^2(Q)$  is the solution to (III.106).

For any  $r > 0$ , we consider any element  $\Phi \in E_r$  that we write

$$\Phi = \sum_{\lambda \in \Lambda_\beta: \lambda \leq r} a_\lambda \Phi_\lambda \in E_r, \quad \text{with } a_\lambda \in \mathbb{R},$$

so that  $\|\Phi\|_{L^2(\Omega_2)}^2 = \sum_{\lambda \in \Lambda_\beta: \lambda \leq r} |a_\lambda|^2$  since  $\{\Phi_\lambda\}_{\lambda \in \Lambda_\beta}$  is an orthonormal family in  $L^2(\Omega_2)$ .

With this, we take the function

$$u(\tau, \bar{x}) = \begin{cases} \sum_{\lambda \in \Lambda_\beta: \lambda \leq r} \frac{\sinh(\sqrt{\lambda} \tau)}{\sqrt{\lambda}} a_\lambda \Phi_\lambda(\bar{x}), & \text{when } \lambda \neq 0, \\ \tau, & \text{if } \lambda = 0. \end{cases}$$

Then, we have

$$\begin{aligned} \|u\|_{L^2((1, 3) \times \Omega_2)}^2 &= \sum_{\lambda \in \Lambda_\beta: \lambda \leq r} \int_1^3 \int_{\Omega_2} \left| \frac{\sinh(\sqrt{\lambda} \tau)}{\sqrt{\lambda}} a_\lambda \Phi_\lambda(\bar{x}) \right|^2 d\bar{x} d\tau \\ &\geq \sum_{\lambda \in \Lambda_\beta: \lambda \leq r} |a_\lambda|^2 \int_1^3 \tau^2 d\tau = \frac{8}{3} \|\Phi\|_{L^2(\Omega_2)}^2, \end{aligned}$$

thanks to the fact that  $\left| \frac{\sinh(\sqrt{\lambda} \tau)}{\sqrt{\lambda}} \right| \geq \tau$  and  $\|\Phi_\lambda\|_{L^2(\Omega_2)} = 1$ .

On the other hand, we have by definition

$$u(0, \bar{x}) = 0, \quad \partial_\tau u(0, \bar{x}) = \sum_{\lambda \in \Lambda_\beta: \lambda \leq r} a_\lambda \Phi_\lambda, \quad \nabla_{\bar{x}} u(0, \bar{x}) = 0.$$

So, in the right hand side of (III.139), we have

$$\|\partial_\tau u(0)\|_{L^2(\omega_2)}^2 = \sum_{\lambda \in \Lambda_\beta: \lambda \leq r} \int_{\omega_2} |a_\lambda \Phi_\lambda(\bar{x})|^2 d\bar{x} = \|\Phi_\lambda\|_{L^2(\omega_2)}^2,$$

as also one can show that

$$\|u\|_{H^1(Q)}^2 \leq C e^{C\sqrt{r}} \|\Phi\|_{L^2(\Omega_2)}^2.$$

Combining all, we deduce from (III.139) that

$$\|\Phi\|_{L^2(\Omega_2)} \leq C e^{C\sqrt{r}} \|\Phi\|_{L^2(\omega_2)}^{\kappa\mu} \|\Phi\|_{L^2(\Omega_2)}^{(1-\kappa\mu)},$$

that reduces to

$$\|\Phi\|_{L^2(\Omega_2)} \leq C e^{C\sqrt{r}} \|\Phi\|_{L^2(\omega_2)}, \quad \forall \Phi \in E_r,$$

where it is clear from the computations that the constant  $C > 0$  does not depend on the Robin parameter  $\beta \geq 0$ .

The proof is hereby complete. □



## Chapter IV

# Boundary null-controllability of some 1D coupled parabolic systems with Kirchhoff condition

### Abstract

The main purpose of this chapter is to investigate the boundary null-controllability of some  $2 \times 2$  and  $3 \times 3$  one-dimensional parabolic systems with both the interior and boundary couplings. We first consider a  $2 \times 2$  parabolic system that contains a linear interior coupling with some real constant coefficient and a Kirchhoff-type condition through which the boundary coupling enters into the game. The control is exerted on a part of the boundary through a Dirichlet condition on either one of the two state components, in particular we show that the controllability property significantly changes depending on which component the control is being applied. Regarding this, we point out that the choices of *interior coupling coefficient* and the *Kirchhoff parameter* might play a crucial role to deduce the positive or negative controllability results.

Next to the study of  $2 \times 2$  cases, we investigate the controllability of some  $3 \times 3$  coupled parabolic system (with a linear interior coupling of the first component to the third equation) where we consider *one* or *two* Dirichlet boundary control force(s) at left and a Kirchhoff-type condition at right. One motivation to study a  $3 \times 3$  model along with an interior coupling, is coming from the point of considering a parabolic control system on a metric graph where some (or, all) of the edges correspond to vectorial parabolic equations. In particular, our  $3 \times 3$  case can be understood in a graph with two edges, where one edge corresponds to a pde for  $(y_1, y_3)$  and other one for  $y_2$ , coupled through their common node.

In this case, we briefly show the following. In one hand, when we exert *two* boundary controls on any of the two components, the system is null-controllable at any time. On the other hand, if we consider only *one* boundary control, then the controllability property changes depending on which component the control is being applied. More precisely, putting a control on the first entry, we see that the controllability depends sensitively on the choices of interior coupling coefficient and Kirchhoff parameter, while considering a control to the second entry always gives positive results regardless the choices of those quantities. But in contrast, if the control is exerted on the third entry, we loose the hope of approximate/null- controllability.

Apart from these, we pursue some numerical studies based on the well-known *penalized HUM approach*. In fact, we make some discretization for a general interior-boundary coupled parabolic system which allow us to illustrate our theoretical results as well as to experiment some more examples which fit under the framework of that general system.



## IV.1 Introduction

### IV.1.1 Motivation and the systems under study

In this chapter, we talk about the boundary null-controllability of some one-dimensional parabolic systems coupled through interior as well as boundary, with less number of control(s) than the equations. We write the following prototype of  $2 \times 2$  general boundary controllability system,

$$\begin{cases} \partial_t y + \mathcal{A}y + \mathcal{M}_{coup} y = 0 & \text{in } (0, T) \times (0, 1), \\ \mathcal{D}_0 y(t, 0) + \mathcal{N}_0 \frac{\partial y}{\partial \nu_{\mathcal{A}}}(t, 0) = Bv(t) & \text{in } (0, T), \\ \mathcal{D}_1 y(t, 1) + \mathcal{N}_1 \frac{\partial y}{\partial \nu_{\mathcal{A}}}(t, 1) = 0 & \text{in } (0, T), \\ y(0, \cdot) = y_0(\cdot) & \text{in } (0, 1), \end{cases} \quad (\text{IV.1})$$

where  $y := (y_1, y_2)$  is the unknown and  $y_0 := (y_{0,1}, y_{0,2})$  is the initial data from some suitable Hilbert space and  $\mathcal{M}_{coup}, \mathcal{D}_j, \mathcal{N}_j \in M_2(\mathbb{R})$  ( $j = 0, 1$ ). The input  $v$  is assumed to act as a control on the boundary point  $x = 0$  through some non-zero vector  $B$ . We choose the operator  $\mathcal{A}$  with its formal expression

$$\mathcal{A} := \begin{pmatrix} -\partial_x(\gamma_1 \partial_x) & 0 \\ 0 & -\partial_x(\gamma_2 \partial_x) \end{pmatrix}, \quad (\text{IV.2})$$

where the diffusion coefficients  $\gamma_1, \gamma_2$  are chosen in such a way that

$$\gamma_i \in \mathcal{C}^1([0, 1]) \quad \text{with} \quad \gamma_{\min} := \inf_{[0, 1]} \{\gamma_i, i = 1, 2\} > 0. \quad (\text{IV.3})$$

The domain of  $\mathcal{A}$  can be noted down as

$$D(\mathcal{A}) := \left\{ u \in (H^2(0, 1))^2 \mid \mathcal{D}_0 u(0) + \mathcal{N}_0 \frac{\partial u}{\partial \nu_{\mathcal{A}}}(0) = 0, \mathcal{D}_1 u(1) + \mathcal{N}_1 \frac{\partial u}{\partial \nu_{\mathcal{A}}}(1) = 0 \right\}. \quad (\text{IV.4})$$

On the boundary points, we have the conormal derivative operator  $\frac{\partial}{\partial \nu_{\mathcal{A}}} := (\gamma_1 \frac{\partial}{\partial \nu}, \gamma_2 \frac{\partial}{\partial \nu})$ , where  $\nu$  is the normal vector.

**Remark IV.1.1.** Note that the domain of the operator  $\mathcal{A}$  considered in (IV.1) is not exactly same as (IV.4) due to presence of  $Bv \neq 0$  on the boundary, but we keep the same notation  $\mathcal{A}$  if there is no confusion.

The main point is that we consider the *interior coupling* by means of some  $2 \times 2$  real matrix  $\mathcal{M}_{coup}$  and the *boundary coupling(s)* via the  $2 \times 2$  real coefficient matrices  $\mathcal{D}_j, \mathcal{N}_j$ , for  $j = 0, 1$  along with the following assumptions.

**Assumption IV.1.2.** For each  $j \in \{0, 1\}$ ,

1. The  $2 \times 4$  matrix  $(\mathcal{D}_j, \mathcal{N}_j)$  has the maximal rank.
2. The matrix  $\mathcal{D}_j \mathcal{N}_j^*$  is self-adjoint.

The first assumption ensures the sufficient number of boundary conditions in (IV.1), whereas the second one is important for the differential operator  $\mathcal{A}$  defined by (IV.2) to be self-adjoint in its domain  $D(\mathcal{A})$ , (IV.4).

**Remark IV.1.3.** One can think of a more general system like (IV.1) with  $n$  differential equations instead of only 2. Indeed, in Section IV.1.1.2, we shall introduce some  $3 \times 3$  coupled parabolic systems with one or two control force(s).

As we mentioned in Chapter I, several systems with boundary couplings (without choosing any control for the moment) arise when one considers some parabolic systems on a metric graph, see for instance [86, 79, 74, 77] (we refer to Section I.3–paragraph 2 for a notion of metric graph). In this context, we must say that there are diverse applications of the systems PDEs on metric graph in physics, chemistry, engineering or biology, e.g., the survey work [78], some papers [80, 18, 42, 41] and the references therein. The authors in [80, 18] discussed about some carbon nano-structures, in particular [80] contains some interesting studies on the spectra of carbon-nano structures. In the references [42, 41], some models have been addressed which are concerned with the signal transmission in biological neural network.

Coming to the controllability issues on metric graphs, we first address [50, Chapters 6, 8] where the authors discussed some controllability results of wave, heat and Schrödinger systems considered in some network in the case when some control(s) is (are) exerted on some of the vertices; see also the survey paper [15]. Addition to the foregoing, the authors in [44] proved some boundary null-controllability results for a linear Kuramoto–Sivashinsky equation ([81, 94]) on star-shaped trees with Dirichlet or Neumann boundary controls. We also refer to some very recent works [46] and [45] which contain some boundary controllability results for Korteweg de Vries equation ([30, 73]) on a tree-shaped and star-shaped network respectively. Last but not the least, it is worth to quote that a necessary and sufficient condition for approximate controllability of two 1D wave systems has been developed in [49].

In most of the known cases, one cannot arbitrarily impose some *interior coupling* to the systems of differential equations that are considered on a metric graph. The reason being each edge corresponds to one scalar differential equation with respect to only one unknown, and thus, the only interaction between the unknowns occurs at the vertices.

Let us come to the case (IV.1), where one observes that the main difference between this kind of systems and the control systems arise on graphs is: unlike the situation on graph, we have here the possibility of considering both the boundary and interior couplings.

As mentioned in Chapter I, one can think of considering a system (of pdes) on metric graph where in each edge a *vectorial differential equation* can be posed. Indeed, in Section IV.1.1.2, we introduce some  $3 \times 3$  parabolic systems, which can be identified on some metric graph with *two edges*: in one edge we pose a vectorial pde and other one corresponds to a scalar pde; see Figure IV.1. We also remind here that in Section I.3, Ch. I, we gave a general discussion about this kind of systems as a perspective of our present works.

Now, two types of difficulties may occur while dealing with the general system (IV.1). As we know that, in many situations the very powerful *Carleman technique* does not work for a boundary control system (vectorial). Beside this obstruction, there occurs a change in the spectral analysis of the *adjoint elliptic operator* associated with the parabolic system (IV.1), since this kind of operator is normally *non self-adjoint* in nature due to the presence of the interior coupling  $\mathcal{M}_{coup}$ . Thus, it is no more straightforward to apply a *moments technique* to the more general system (IV.1). Moreover, under the general system (IV.1), there are some examples of *negative controllability* results also, as mentioned below.

**Remark IV.1.4 (Examples of some negative controllability cases).** *Even if the Assumption IV.1.2 is satisfied, it can be shown that a linear coupled system in the cascade form is not approximately controllable for either  $\mathcal{D}_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $\mathcal{N}_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , or  $\mathcal{D}_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\mathcal{N}_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  ( $j = 0, 1$ ) and for  $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ; see Remark III.2.17, Ch. III.*

Due to these indistinct phenomena, studying a more general system like (IV.1) is really intricate to tackle. This is why we mainly cope with some particular class of problems that fit into the framework of (IV.1); let us go into the detail.

#### IV.1.1.1 Main system under study with Kirchhoff condition in the $2 \times 2$ case

We hereby consider the interior coupling

$$\mathcal{M}_{coup} = \mathcal{M}_a := \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad (\text{IV.5})$$

for some  $a \in \mathbb{R}$ , and the boundary coefficient matrices:

$$\mathcal{D}_0 = I_{2 \times 2}, \quad \mathcal{N}_0 = O_{2 \times 2}, \quad (\text{IV.6a})$$

$$\mathcal{D}_1 = \begin{pmatrix} 1 & -1 \\ \alpha & 0 \end{pmatrix}, \quad \mathcal{N}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad (\text{IV.6b})$$

for some  $\alpha \geq 0$ . In what follows, we have the following coupled parabolic system with Dirichlet boundary control at left and the Kirchhoff condition at right (which actually plays the role of boundary coupling).

**Two Dirichlet control problems.** Taking into account the diffusion operator  $\mathcal{A}$  (introduced in [IV.2](#)), the coupling matrix  $\mathcal{M}_a$  and the boundary matrices as above, the interior–boundary coupled system ([IV.1](#)) reads as follows

$$\begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ y_1(t, 1) = y_2(t, 1) & \text{in } (0, T), \\ \gamma_1(1) \partial_x y_1(t, 1) + \gamma_2(1) \partial_x y_2(t, 1) + \alpha y_1(t, 1) = 0 & \text{in } (0, T), \\ y_1(0, \cdot) = y_{0,1}(\cdot), \quad y_2(0, \cdot) = y_{0,2}(\cdot) & \text{in } (0, 1), \end{cases} \quad (\text{IV.7})$$

with a Dirichlet control at the left end point either on the second or first component depending on the choices of  $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  or  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , that is to say

$$\text{either } y_1(t, 0) = 0, \quad y_2(t, 0) = v(t) \quad \text{in } (0, T), \quad (\text{IV.8a})$$

$$\text{or } y_1(t, 0) = v(t), \quad y_2(t, 0) = 0 \quad \text{in } (0, T). \quad (\text{IV.8b})$$

Observe that, we have chosen here a general Kirchhoff condition at the boundary point  $x = 1$  (the usual Kirchhoff condition is the one with  $\alpha = 0$ ). It is well-known that the Kirchhoff type of conditions appear widely in physics, electrical engineering and in various biological models. We quote here very few of those, for instance the papers [\[75, 76\]](#) deal with some mathematical studies on Kirchhoff's rule for quantum wires and the authors of [\[41\]](#) discussed a rigorous analysis on FitzHugh–Nagumo–Rall model of a neuronal network with a Kirchhoff type rule in axonal or dendrital ramification points.

In the framework of control theory, the usual type of system that has already been studied in the literature is the case when there is *no interior coupling* (i.e.,  $a = 0$ ) and with the standard Kirchhoff condition (i.e.,  $\alpha = 0$ ), see for instance [\[50, Ch. 8\]](#) and [\[15, 44\]](#). We also recall that, a controllability result of some  $2 \times 2$  parabolic system without any interior coupling, that is  $a = 0$ , has been addressed in [\[23, Remark 3.6\]](#), where a Kirchhoff condition with  $\alpha = 0$  has been considered. In most of the existing works, a non-zero interior coupling cannot be imposed to the systems since those cases have been analyzed on a metric graph where each edge corresponds to one scalar equation with only one unknown.

But observe that when  $a = 0$ , the two control systems ([IV.7](#))–([IV.8a](#)) and ([IV.7](#))–([IV.8b](#)) are exactly the same.

Now, as soon as one consider some interior coupling coefficient  $a \neq 0$ , the control systems ([IV.7](#))–([IV.8a](#)) and ([IV.7](#))–([IV.8b](#)) are certainly different in nature and the choices of the component on which we exert the control, really has an influence to the controllability issues. In fact

in the second situation (when we put the control on the first component), the choice of  $(\alpha, a)$  is very crucial to conclude the positive or negative controllability phenomena. More explicitly, we have the following two situations.

1. **Case 1.** The boundary controllability of the system (IV.7)–(IV.8a), that is when we consider a control on the *second component*  $y_2$ , can be established by means of a global Carleman estimate (and then to find an observability inequality) for any interior coupling coefficient  $a \in \mathbb{R}$  and boundary parameter  $\alpha \geq 0$ , which is precisely the Theorem IV.2.7.
2. **Case 2.** Surprisingly, when we consider our control to be acted on the *first component*  $y_1$ , it appears that the same tool cannot be applied to the system (IV.7)–(IV.8b) and in this situation a moments approach will be used. Moreover, we shall show in Theorem IV.2.8 that depending on the choices of quantities  $(\alpha, a)$ , the controllability issues significantly changes; indeed, in this situation we also find a class of negative results (see Remark IV.2.9) which is not alike the previous one.

**Remark IV.1.5 (Neumann control problems).** By taking into consideration  $\mathcal{D}_0 = O_{2 \times 2}$  and  $\mathcal{N}_{2 \times 2} = I_{2 \times 2}$  along with (IV.6b), we have two Neumann boundary control systems, that is the system (IV.7) along with the following two different kinds of Neumann controls at  $x = 0$ , depending on the choices of  $B = \gamma_2(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  or  $\gamma_1(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , that is

$$\text{either } \partial_x y_1(t, 0) = 0, \quad \partial_x y_2(t, 0) = v(t) \quad \text{in } (0, T), \quad (\text{IV.9a})$$

$$\text{or } \partial_x y_1(t, 0) = v(t), \quad \partial_x y_2(t, 0) = 0 \quad \text{in } (0, T). \quad (\text{IV.9b})$$

We will not study this case theoretically, rather we will investigate this from a numerical point of view in Section IV.5.3.2.

#### IV.1.1.2 A $3 \times 3$ coupled parabolic system with one or two Dirichlet control(s)

Next to the analysis of  $2 \times 2$  systems, we will treat some  $3 \times 3$  parabolic system, again with a linear interior coupling, the Kirchhoff type condition and the Dirichlet boundary control(s). The main aim is to see what happens if we exert only *one* boundary control to that  $3 \times 3$  system. In this case, we realize that the null-controllability of the system depends on the choices of the components through which the control is being exerted, and moreover, sometimes it depends significantly on the choices of the *interior coupling coefficient* and *Kirchhoff parameter*. But in the negative-controllability cases, if we consider an extra boundary control to some other component, then the null-controllability can be recovered again.

As we addressed in Chapter I, the motivation to study a  $3 \times 3$  model along with an interior coupling, is coming from the point of considering a parabolic control systems on a metric graph where some (or, all) of the edges correspond to some vectorial parabolic equations. In this context, we refer to Section I.3, Ch. I where we explained a general pathology regarding the parabolic systems on metric graph and in particular, our  $3 \times 3$  case can be understood in a graph with two edges, see Fig. IV.1.

We first present some system with two controls and then move to the case of considering only one control.

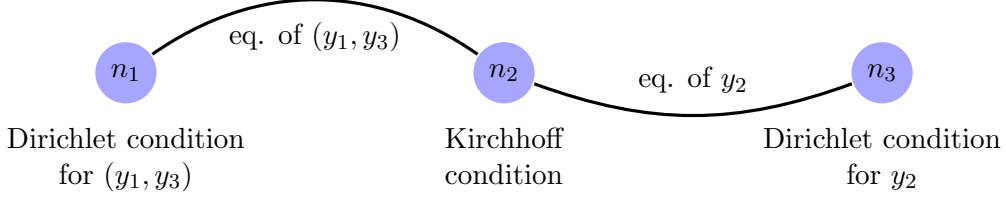


Figure IV.1: The system (IV.10)–(IV.11), or (IV.12) on a metric graph of two edges.

• **A system with two boundary controls.**

Let us consider the following system

$$\left\{ \begin{array}{ll} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) &= 0 \quad \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) &= 0 \quad \text{in } (0, T) \times (0, 1), \\ \partial_t y_3 - \partial_x(\gamma_3 \partial_x y_3) + a y_1 &= 0 \quad \text{in } (0, T) \times (0, 1), \\ \left\{ \begin{array}{l} y_1(t, 1) = y_2(t, 1) = y_3(t, 1), \\ \sum_{i=1}^3 \gamma_i(1) \partial_x y_i(t, 1) + \alpha y_1(t, 1) = 0 \end{array} \right. & \text{in } (0, T), \\ y_i(0, x) = y_{0,i}(x), \quad \text{for } i = 1, 2, 3, & \text{in } (0, 1), \end{array} \right. \quad (\text{IV.10})$$

and in this case, two controls will be applied through a Dirichlet condition on any two components among the three (described below). One can prescribe the above system into a metric graph with two edges, where one edge corresponds to the equations of  $y_1, y_3$  (as a vectorial equation of  $(y_1, y_3)$ ) and other one corresponds to the scalar equation of  $y_2$ , see Fig. IV.1. In that figure, the Kirchhoff type law is supposed to be acted on the node  $n_2$ , the junction point of  $(y_1, y_3)$  and  $y_2$ , whereas the controls are exerted on  $n_1$  and/or  $n_3$  via the Dirichlet conditions, we put that below:

the controls are applied either on  $y_2$  and  $y_3$  (in the graph setting, on the nodes  $n_3$  and  $n_1$ )

$$y_1(t, 0) = 0, \quad y_2(t, 0) = v(t), \quad y_3(t, 0) = \tilde{v}(t) \quad \text{in } (0, T), \quad (\text{IV.11a})$$

or, on  $y_1$  and  $y_2$  (i.e., on the nodes  $n_1$  and  $n_3$ )

$$y_1(t, 0) = v(t), \quad y_2(t, 0) = \tilde{v}(t), \quad y_3(t, 0) = 0 \quad \text{in } (0, T), \quad (\text{IV.11b})$$

or, on  $y_1$  and  $y_3$  (i.e., only on the node  $n_1$ )

$$y_1(t, 0) = v(t), \quad y_2(t, 0) = 0, \quad y_3(t, 0) = \tilde{v}(t) \quad \text{in } (0, T). \quad (\text{IV.11c})$$

As previous, we choose some interior coupling coefficient  $a \in \mathbb{R}^*$ , the Kirchhoff parameter  $\alpha \geq 0$  and the diffusion coefficients  $\gamma_i \in \mathcal{C}^1([0, 1])$  with  $\gamma_{\min} := \inf_{[0, 1]} \{\gamma_i, i = 1, 2, 3\} > 0$ .

In this case, we have the following observations.

- The boundary null-controllability of the system (IV.10)–(IV.11a), that is when we consider the controls on  $y_2$  and  $y_3$ , can be establish by proving a suitable *Carleman estimate* for any  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$ .
- For the cases (IV.11b) and (IV.11c), we could not succeed by the Carleman technique (the reason will be precised later), and thus, we handle this situation by the so-called moments technique. We hereby fix  $\gamma_1 = \gamma_2 = \gamma_3 = 1$  in order to have a concrete idea about the spectral analysis which is the heart of moments method, and in these cases we actually able to the prove the null-controllability, independent on the choices of  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$ .

**Remark IV.1.6.** *In contrast, recall that the controllability of  $2 \times 2$  system (IV.7) significantly depends on the choices of the parameters  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$  when we change the position of the control from  $y_2$  to  $y_1$  (see Case 2 in Sec. IV.1.1.1).*

• **A system with only one boundary control.**

Next, we treat the same system (I.13) but with only one control exerted on one of the three components. More precisely, one can have the following three situations.

either the control is applied on  $y_1$  (to the node  $n_1$  in the Fig. I.9)

$$y_1(t, 0) = v(t), \quad y_2(t, 0) = 0, \quad y_3(t, 0) = 0 \quad \text{in } (0, T), \quad (\text{IV.12a})$$

or, on  $y_2$  (to the node  $n_3$ )

$$y_1(t, 0) = 0, \quad y_2(t, 0) = v(t), \quad y_3(t, 0) = 0 \quad \text{in } (0, T), \quad (\text{IV.12b})$$

or, on  $y_3$  (to the node  $n_1$ )

$$y_1(t, 0) = 0, \quad y_2(t, 0) = 0, \quad y_3(t, 0) = v(t) \quad \text{in } (0, T). \quad (\text{IV.12c})$$

In the above cases, the Carleman approach is essentially inefficient and the effective way is to deal with the moments method (if applicable). We again consider  $\gamma_1 = \gamma_2 = \gamma_3 = 1$  and as usual  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$ . Then, by following the spectral analysis of the adjoint elliptic operator, we obtain three different situations.

- *Control on the first component– conditionally controllable:* This case is quite interesting in the sense that the controllability depends on the choices of the parameters  $(\alpha, a)$  (recall that, similar situation arises in the  $2 \times 2$  system (IV.7)–(IV.8b)). We shall see in Section IV.6.2.1, that there are some (actually many) pair of  $(\alpha, a)$  for which the system (IV.10)–(IV.12a) is not even approximately controllable.
- *Control on the second component– always controllable:* As soon as, we exert a control to  $y_2$ , the system (IV.10)–(IV.12b) happens to be null-controllable at any time.
- *Control on the third component– never controllable:* Unlike the previous cases, surprisingly we see that the system (IV.10) with a control on  $y_3$ , i.e., the case (IV.12c) is not even approximately controllable.

## IV.1.2 Overview of the chapter

In the main part of this chapter, we start by writing some well-posedness results of our  $2 \times 2$  boundary control systems, namely in Section IV.2.

Then, we prove the boundary null-controllability of the system (IV.7)–(IV.8a) in Section IV.3, where we establish a global boundary Carleman estimate (Theorem IV.3.2) to find an observability inequality for any  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$  and different diffusion coefficients  $\gamma_1, \gamma_2$  as (IV.3).

But as indicated in Case 2, Sec. IV.1.1.1, we cannot take the help of Carleman technique in the situation when the control is acting on  $y_1$  instead of  $y_2$ , that is the system (IV.7)–(IV.8b) (see Remark IV.3.5 for details). Thus, in this situation we take the advantage of applying the so-called moments technique to construct a control. In this case, we shall restrict ourselves to constant diffusion coefficients  $\gamma_1 = \gamma_2 = 1$  to simplify the spectral studies of the adjoint elliptic operator, which we discuss in Section IV.4.1.2 in detail. This, together with the observation estimates in Section IV.4.2, we shall construct a control via moments method in Section IV.4.3. We also discuss the fact that how the controllability phenomena changes with respect to the choices of  $\alpha$  and  $a$ .

Thereafter, in Section IV.5 we pursue some numerical studies based on the penalized HUM approach ([63, 64, 31]). In fact, we introduce a discrete setting for the general control system (IV.1) and show the effect of the interior and boundary couplings to the discretization. This



will help us to illustrate the theoretical results as well as to experiment some other examples in the framework of the general system (IV.1).

Next to this, the Section IV.6 is devoted to the study of controllability (or, non-controllability) properties of some  $3 \times 3$  system with *one* or *two* control(s), namely the systems (IV.10)–(IV.12) and (IV.11).

Finally, we gather some intermediate results in Section IV.7 that are very much useful for this chapter.

**Notations.** Throughout the chapter, we shall make use of following notations. The inner product and norm in the scalar space  $L^2(0, 1)$  will be simply denoted by  $(\cdot, \cdot)_{L^2}$  and  $\|\cdot\|_{L^2}$  respectively. We denote the space  $E := (L^2(0, 1))^2$ , its inner product and the norm by  $(\cdot, \cdot)_E$  and  $\|\cdot\|_E$  respectively. In Section IV.6, up to a abuse of notation we also denote  $E := (L^2(0, 1))^3$ , but we will declare this again in the beginning of that section. Moreover, we use the notation  $\langle \cdot, \cdot \rangle_{X', X}$  to express the duality pair between a space  $X$  and its dual  $X'$ . Beside this, we sometimes write  $\langle \cdot, \cdot \rangle_U$  with  $U = \mathbb{R}^d$  or  $\mathbb{C}^d$ ,  $d \geq 1$ , to specify the usual inner product in  $U$ .

Further, we declare  $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$  and  $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$ , where  $\mathbb{R}^+$  denotes the set of all positive real numbers.

We use the letter  $C$  and subsequently  $\tilde{C}, C', C''$  to denote some positive constants (those may vary from line to line) which do possibly depend on  $\gamma_1, \gamma_2, \alpha, a$  but not on  $T$  and  $y_0$ . Sometimes, we shall express some constants by  $C_{p_1, p_2, \dots, p_n}$  to specify its dependency on the quantities  $p_1, p_2, \dots, p_n$ .

We often use the symbol  $M^*$  to denote the adjoint of a matrix or an operator  $M$ .

## IV.2 General settings and main results for the $2 \times 2$ control systems

In this section, we shall discuss briefly about the well-posedness of our systems (IV.7)–(IV.8a) and (IV.7)–(IV.8b) with  $L^2$  boundary data. Also we will provide the main results concerning boundary null-controllability which are the main concerns of this paper.

### IV.2.1 Well-posedness of our systems

#### IV.2.1.1 The system with homogeneous Dirichlet data

Let us begin with the following coupled parabolic system with Kirchhoff condition at right end point and homogeneous Dirichlet conditions at left end point.

$$\begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = f_1 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a y_1 = f_2 & \text{in } (0, T) \times (0, 1), \\ y_1(t, 0) = y_2(t, 0) = 0 & \text{in } (0, T), \\ y_1(t, 1) = y_2(t, 1) & \text{in } (0, T), \\ \gamma_1(1) \partial_x y_1(t, 1) + \gamma_2(1) \partial_x y_2(t, 1) + \alpha y_1(t, 1) = 0 & \text{in } (0, T), \\ y_1(0, \cdot) = y_{0,1}(\cdot), \quad y_2(0, \cdot) = y_{0,2}(\cdot) & \text{in } (0, 1), \end{cases} \quad (\text{IV.13})$$

where regularity of  $y_0 = (y_{0,1}, y_{0,2})$  and  $f = (f_1, f_2)$  will be specified later.

We introduce the self-adjoint and positive elliptic operator  $\mathcal{A}_\alpha$ , corresponding to the above system without interior coupling with its formal expression

$$\mathcal{A}_\alpha := \begin{pmatrix} -\partial_x(\gamma_1 \partial_x) & 0 \\ 0 & -\partial_x(\gamma_2 \partial_x) \end{pmatrix}, \quad (\text{IV.14a})$$

with its domain

$$D(\mathcal{A}_\alpha) := \left\{ u = (u_1, u_2) \in (H^2(0, 1))^2 \mid u_1(0) = u_2(0) = 0, \ u_1(1) = u_2(1), \right. \\ \left. \gamma_1(1)u_1'(1) + \gamma_2(1)u_2'(1) + \alpha u_1(1) = 0 \right\}. \quad (\text{IV.14b})$$

Let us consider the space  $\mathcal{H}_\alpha := D(\mathcal{A}_\alpha^{1/2})$  as a completion of  $D(\mathcal{A}_\alpha)$  with respect to the norm

$$\|u\|_{\mathcal{H}_\alpha} := (\mathcal{A}_\alpha u, u)_E^{1/2} = \left( \sum_{i=1}^2 \int_0^1 \gamma_i(x) |u_i'(x)|^2 dx + \alpha |u_1(1)|^2 \right)^{1/2}, \quad \forall u \in D(\mathcal{A}_\alpha), \quad (\text{IV.15})$$

and one can prove that

$$\mathcal{H}_\alpha = \left\{ u = (u_1, u_2) \in (H^1(0, 1))^2 \mid u_1(0) = u_2(0) = 0, \ u_1(1) = u_2(1) \right\}. \quad (\text{IV.16})$$

Moreover, we denote the dual space of  $\mathcal{H}_\alpha$  by  $\mathcal{H}_{-\alpha}$  with respect to the pivot space  $E$ .

Next, we see that the coupling (interior) matrix in the system (IV.13) is  $\mathcal{M}_a := \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$  and we further denote

$$\mathcal{A}_{\alpha,a} = \mathcal{A}_\alpha + \mathcal{M}_a, \quad \text{with the same domain } D(\mathcal{A}_{\alpha,a}) := D(\mathcal{A}_\alpha). \quad (\text{IV.17})$$

In particular,  $\mathcal{A}_{\alpha,0} := \mathcal{A}_\alpha$ .

By definition, it is clear that  $\mathcal{A}_{\alpha,a}$  is not self-adjoint anymore, more precisely,  $\mathcal{A}_{\alpha,a}$  has been obtained by a bounded perturbation  $\mathcal{M}_a$  to the self-adjoint operator  $\mathcal{A}_\alpha$ .

**Proposition IV.2.1** (Existence of analytic semigroup). *The operator  $(-\mathcal{A}_{\alpha,a}, D(\mathcal{A}_{\alpha,a}))$  defined by (IV.17), generates an analytic semigroup in  $E$ .*

*Proof.* Let us first introduce the following densely defined sesquilinear form  $h$ ; for all  $u := (u_1, u_2)$ ,  $\varphi := (\varphi_1, \varphi_2) \in \mathcal{H}_\alpha$  (defined by (IV.16)), we consider

$$h(u, \varphi) = \sum_{i=1}^2 \int_0^1 \gamma_i(x) u_i'(x) \varphi_i'(x) dx + a \int_0^1 u_1(x) \varphi_2(x) dx + \alpha u_1(1) \varphi_1(1). \quad (\text{IV.18})$$

It is clear that  $h$  is continuous in  $\mathcal{H}_\alpha$  with

$$|h(u, \varphi)| \leq \kappa_1 \|u\|_{\mathcal{H}_\alpha} \|\varphi\|_{\mathcal{H}_\alpha}, \quad \forall u, \varphi \in \mathcal{H}_\alpha,$$

where  $\kappa_1 > 0$  depends on the diffusion coefficients  $\gamma_i$ ,  $i = 1, 2$ , and the coupling coefficient  $a$ . On the other hand, we have

$$h(u, u) \geq \|u\|_{\mathcal{H}_\alpha}^2 - |a| \|u\|_E^2, \quad \forall u \in \mathcal{H}_\alpha.$$

Then, by [90, Proposition 1.51 and Theorem 1.52], the negative of the operator associated with  $h$  generates an analytic semigroup in  $E$  of angle  $(\pi/2 - \arctan \kappa_1)$ .

Now, the only thing is to show that this operator is indeed  $-\mathcal{A}_{\alpha,a}$  with its domain  $D(\mathcal{A}_{\alpha,a}) = D(\mathcal{A}_\alpha)$  (as defined in (IV.17)), which we shall prove in Lemma IV.2.2. and hence the proposition follows.  $\square$

**Lemma IV.2.2.** *Let  $h$  be the sesquilinear form in  $\mathcal{H}_\alpha$  as defined by (IV.18). Then, the operator associated with  $h$  is  $(\mathcal{A}_{\alpha,a}, D(\mathcal{A}_{\alpha,a}))$ .*

*Proof.* Denote by  $(\tilde{\mathcal{A}}, D(\tilde{\mathcal{A}}))$  the operator associated with the form  $h$ , which is by definition given by

$$\begin{cases} D(\tilde{\mathcal{A}}) = \left\{ u \in \mathcal{H}_\alpha \mid \exists f \in E \text{ s.t. } h(u, \varphi) = (f, \varphi)_E, \quad \forall \varphi \in \mathcal{H}_\alpha \right\}, \\ \tilde{\mathcal{A}}u := f. \end{cases}$$



- First, fix some  $u := (u_1, u_2) \in D(\mathcal{A}_{\alpha,a})$ . Then for all  $\varphi := (\varphi_1, \varphi_2) \in \mathcal{H}_\alpha$ , we have

$$\begin{aligned} h(u, \varphi) &= \sum_{i=1}^2 \int_0^1 \gamma_i(x) u'_i(x) \varphi'_i(x) dx + a \int_0^1 u_1(x) \varphi_2(x) dx + \alpha u_1(1) \varphi_1(1) \\ &= - \sum_{i=1}^2 \int_0^1 (\gamma_i u'_i)'(x) \varphi_i(x) dx + \sum_{i=1}^2 \gamma_i(1) u'_i(1) \varphi_i(1) \\ &\quad + a \int_0^1 u_1(x) \varphi_2(x) dx + \alpha u_1(1) \varphi_1(1), \end{aligned}$$

where we perform an integration by parts on and also we use that  $\varphi_i(0) = 0$  as  $\varphi \in \mathcal{H}_\alpha$ . Now, thanks to the continuity condition  $\varphi_1(1) = \varphi_2(1)$  since  $\varphi \in \mathcal{H}_\alpha$  and then the Kirchhoff condition  $\sum_{i=1}^2 \gamma_i(1) u'_i(1) = -\alpha u_1(1)$  since  $u \in D(\mathcal{A}_{\alpha,a})$ , we deduce

$$\begin{aligned} h(u, \varphi) &= - \sum_{i=1}^2 \int_0^1 (\gamma_i u'_i)'(x) dx + a \int_0^1 u_1(x) \varphi_2(x) dx \\ &= (\mathcal{A}_{\alpha,a} u, \varphi)_E. \end{aligned}$$

Thus, for our chosen  $u \in D(\mathcal{A}_{\alpha,a})$ , we obtain that there exists  $f = \mathcal{A}_{\alpha,a} u \in E$  such that  $h(u, \varphi) = (f, \varphi)_E$ , for all  $\varphi \in \mathcal{H}_\alpha$ , which concludes the inclusion  $D(\mathcal{A}_{\alpha,a}) \subseteq D(\tilde{\mathcal{A}})$ .

- Conversely, let  $u \in D(\tilde{\mathcal{A}})$ . By definition, there exists  $f \in E$  such that  $h(u, \varphi) = (f, \varphi)_E$  with  $\tilde{\mathcal{A}}u = f$ , for all  $\varphi \in \mathcal{H}_\alpha$ , and accordingly

$$\sum_{i=1}^2 \int_0^1 \gamma_i(x) u'_i(x) \varphi'_i(x) dx + a \int_0^1 u_1(x) \varphi_2(x) dx + \alpha u_1(1) \varphi_1(1) = \sum_{i=1}^2 \int_0^1 f_i(x) \varphi_i(x) dx.$$

In fact, since  $f_i \in L^2(0, 1)$  ( $i = 1, 2$ ), we have by standard elliptic regularity that  $u \in (H^2(0, 1))^2$ , and so an integrating by parts yields

$$\begin{aligned} - \sum_{i=1}^2 \int_0^1 (\gamma_i u'_i)'(x) \varphi_i(x) dx + a \int_0^1 u_1(x) \varphi_2(x) dx \\ + \sum_{i=1}^2 \gamma_i(1) u'_i(1) \varphi_i(1) + \alpha u_1(1) \varphi_1(1) = \sum_{i=1}^2 \int_0^1 f_i(x) \varphi_i(x) dx, \end{aligned}$$

for all  $\varphi \in \mathcal{H}_\alpha$ .

In particular, by considering any  $\varphi \in (H_0^1(0, 1))^2 \subset \mathcal{H}_\alpha$ , we conclude that

$$f_1(x) = -(\gamma_1 u'_1)'(x), \quad f_2(x) = -(\gamma_2 u'_2)'(x) + a u_1, \quad \forall x \in (0, 1).$$

Once we have the above news, then choosing any  $\varphi \in \mathcal{H}_\alpha$  eventually gives us

$$\gamma_1(1) u'_1(1) + \gamma_2(1) u'_2(1) + \alpha u_1(1) = 0, \quad \text{since } \varphi_1(1) = \varphi_2(1),$$

which concludes that  $u \in D(\mathcal{A}_{\alpha,a})$  and that  $D(\tilde{\mathcal{A}}) \subseteq D(\mathcal{A}_{\alpha,a})$ .

This completes the proof of the lemma. □

**Proposition IV.2.3** (Regularity). *Let  $f = (f_1, f_2) \in L^2(0, T; E)$  be any given source term.*

1. *For any given initial data  $y_0 = (y_{0,1}, y_{0,2}) \in E$ , there exists a unique weak solution  $y = (y_1, y_2) \in C^0([0, T]; E) \cap L^2(0, T; \mathcal{H}_\alpha)$  satisfying the following energy estimate*

$$\|y\|_{C^0([0, T]; E)} + \|y\|_{L^2(0, T; \mathcal{H}_\alpha)} + \|\partial_t y\|_{L^2(0, T; \mathcal{H}_{-\alpha})} \leq C_{T,a} \left( \|y_0\|_E + \|f\|_{L^2(0, T; E)} \right). \quad (\text{IV.19})$$

2. For any initial data  $y_0 \in \mathcal{H}_\alpha$ , the weak solution  $y$  belongs to the space  $\mathcal{C}^0([0, T]; \mathcal{H}_\alpha) \cap L^2(0, T; (H^2(0, 1))^2)$  and satisfies

$$\begin{aligned} \|y\|_{L^\infty(0, T; \mathcal{H}_\alpha)} + \|y\|_{L^2(0, T; (H^2(0, 1))^2)} + \|\partial_t y\|_{L^2(0, T; E)} \\ \leq C_{T, a} \left( \|y_0\|_{\mathcal{H}_\alpha} + \|f\|_{L^2(0, T; E)} \right). \end{aligned} \quad (\text{IV.20})$$

*Proof.* 1. The existence of unique weak solution  $y \in \mathcal{C}^0([0, T]; E)$  to (IV.13) for given data  $y_0 \in E$  and source term  $f \in L^2(0, T; E)$  can be concluded by Proposition IV.2.1.

Below, we provide the sketch of the proof for estimate (IV.19). We shall prove the result with initial data  $y_0 \in D(\mathcal{A}_\alpha)$  and the source term  $f \in \mathcal{C}^1([0, T]; E)$ , which indeed gives us the existence of unique strong solution  $y \in \mathcal{C}^1([0, T]; E) \cap \mathcal{C}^0([0, T]; D(\mathcal{A}_\alpha))$ , and then by the usual density argument we deduce the final result in point 1.

- Let us test the first and second equations of (IV.13) again  $y_1$  and  $y_2$  respectively, we obtain after an addition, for all  $t \in [0, T]$ ,

$$\frac{1}{2} \frac{d}{dt} \|y(t)\|_E^2 + (\mathcal{A}_\alpha y(t), y(t))_E + a(y_1(t), y_2(t))_{L^2} = (f(t), y(t))_E,$$

and finally

$$\frac{d}{dt} \|y(t)\|_E^2 + \|y(t)\|_{\mathcal{H}_\alpha}^2 \leq C_a (\|f(t)\|_E^2 + \|y(t)\|_E^2), \quad \forall t \in [0, T]. \quad (\text{IV.21})$$

By applying Gronwall's lemma (see [54, Appendix B.2]) then by integration over  $[0, T]$ , we obtain the first two required estimates of (IV.19).

- Next, to obtain the estimate of  $\partial_t y \in L^2(0, T; \mathcal{H}_{-\alpha})$ , let us pick any  $\zeta := (\zeta_1, \zeta_2) \in \mathcal{H}_\alpha$  and observe that

$$\langle \partial_t y(t), \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} + (\mathcal{A}_\alpha y(t), \zeta)_E + a(y_1(t), \zeta_2)_{L^2} = (f(t), \zeta)_E, \quad \forall t \in [0, T],$$

which implies

$$|\langle \partial_t y(t), \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha}| \leq C_a (\|y(t)\|_{\mathcal{H}_\alpha} + \|f(t)\|_E) \|\zeta\|_{\mathcal{H}_\alpha},$$

and the claim follows from the previous estimates.

2. We shall now prove the point 2 of our theorem, with the given data  $y_0 \in D(\mathcal{A}_\alpha^2)$  and  $f \in \mathcal{C}^1([0, T]; D(\mathcal{A}_\alpha))$ , and then again a density argument will give the required estimate (IV.20) for any  $y_0 \in \mathcal{H}_\alpha$  and  $f \in L^2(0, T; E)$ .

- We begin by testing the first and second equations of (IV.13) by  $\partial_t y_1$  and  $\partial_t y_2$ , and by addition we observe that for any  $t \in [0, T]$ ,

$$\|\partial_t y(t)\|_E^2 + \frac{1}{2} \frac{d}{dt} (\mathcal{A}_\alpha y(t), y(t))_E = -a(y_1(t), \partial_t y_2(t))_{L^2} + (f(t), \partial_t y(t))_E. \quad (\text{IV.22})$$

We now make use of Cauchy-Schwarz inequality to deduce

$$|a(y_1(t), \partial_t y_2(t))_{L^2} + (f(t), \partial_t y(t))_E| \leq \frac{1}{2} \|\partial_t y(t)\|_E^2 + C_a (\|y(t)\|_E^2 + \|f(t)\|_E^2).$$

Implementing this bound in (IV.22) we respectively obtain the third and first estimate of (IV.20).

- The  $L^2(0, T; (H^2(0, 1))^2)$  estimate for  $y$  simply follows from the bound

$$\|\partial_x^2 y\|_{L^2(0, T; E)} \leq \|f\|_{L^2(0, T; E)} + \|\partial_t y\|_{L^2(0, T; E)} + \|y\|_{L^2(0, T; E)},$$

and the previous two estimates.

□

### IV.2.1.2 The system with non-homogeneous Dirichlet data

We consider here a similar coupled system as in the previous paragraph but with non-smooth Dirichlet boundary data; the system under study is the following

$$\begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = f_1 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a y_1 = f_2 & \text{in } (0, T) \times (0, 1), \\ y_1(t, 0) = g_1 & \text{in } (0, T), \\ y_2(t, 0) = g_2 & \text{in } (0, T), \\ y_1(t, 1) = y_2(t, 1) & \text{in } (0, T), \\ \gamma_1(1) \partial_x y_1(t, 1) + \gamma_2(1) \partial_x y_2(t, 1) + \alpha y_1(t, 1) = 0 & \text{in } (0, T), \\ y_1(0, \cdot) = y_{0,1}(\cdot), \quad y_2(0, \cdot) = y_{0,2}(\cdot) & \text{in } (0, 1), \end{cases} \quad (\text{IV.23})$$

In this context, it is worth introducing the adjoint of the operator  $\mathcal{A}_{\alpha,a}$  (introduced in (IV.17)), with its formal expression

$$\mathcal{A}_{\alpha,a}^* = \mathcal{A}_{\alpha}^* + \mathcal{M}_a^* = \begin{pmatrix} -\partial_x(\gamma_1 \partial_x) & a \\ 0 & -\partial_x(\gamma_2 \partial_x) \end{pmatrix}, \quad (\text{IV.24})$$

with its domain  $D(\mathcal{A}_{\alpha,a}^*) = D(\mathcal{A}_{\alpha,a}) = D(\mathcal{A}_{\alpha})$  (given by (IV.14b)).

**Remark IV.2.4.** We could have replaced  $\mathcal{A}_{\alpha}^*$  simply by  $\mathcal{A}_{\alpha}$  as this operator is self-adjoint, but in order to be consistent with the non self-adjoint case (that is when  $a \neq 0$ ), we decide to keep the notation  $\mathcal{A}_{\alpha}^*$  in several places.

Observe that the operator  $-\mathcal{A}_{\alpha,a}^*$  also generates an analytic semigroup in  $E$ , thanks to Theorem IV.2.1 and we denote this semigroup by  $(e^{-t\mathcal{A}_{\alpha,a}^*})_{t \geq 0}$ . Indeed, the solution to the adjoint system of (IV.13), for any given  $\zeta \in \mathcal{H}_{\alpha}$ , satisfies the regularity result proved in point 2 of Theorem IV.2.3. Using this, one can classically obtain the well-posedness of the solution to (IV.23) in a dual sense as in [48, 96].

**Proposition IV.2.5.** For any  $y_0 := (y_{0,1}, y_{0,2}) \in \mathcal{H}_{-\alpha}$ ,  $f := (f_1, f_2) \in L^2(0, T; E)$  and  $g := (g_1, g_2) \in L^2(0, T; \mathbb{R}^2)$ , there exists a unique  $y \in C^0([0, T]; \mathcal{H}_{-\alpha}) \cap L^2(0, T; E)$ , solution to (IV.23), in the following sense: for any  $t \in [0, T]$  and  $\zeta := (\zeta_1, \zeta_2) \in \mathcal{H}_{\alpha}$ , we have

$$\begin{aligned} \langle y(t), \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_{\alpha}} &= \langle y_0, e^{-t\mathcal{A}_{\alpha,a}^*} \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_{\alpha}} + \int_0^t \langle f(s), e^{-(t-s)\mathcal{A}_{\alpha,a}^*} \zeta \rangle_E ds \\ &\quad - \int_0^t \left\langle g(s), \left( \frac{\partial}{\partial \nu_{\gamma}} (e^{-(t-s)\mathcal{A}_{\alpha,a}^*} \zeta)(x) \right) \Big|_{x=0} \right\rangle_{\mathbb{R}^2} ds. \end{aligned}$$

## IV.2.2 Main results

We shall now formulate the null-control problems in terms of following proposition.

**Proposition IV.2.6.** Let  $y_0 \in \mathcal{H}_{-\alpha}$ ,  $a \in \mathbb{R}$ ,  $\alpha \geq 0$  and any finite time  $T > 0$  be given. Also recall the set  $U$  as defined in Proposition IV.2.5.

1. A function  $v \in L^2(0, T; \mathbb{R})$  is a null-control for the problem (IV.7)–(IV.8a), if and only if it satisfies: for any  $\zeta \in \mathcal{H}_{\alpha}$

$$-\langle y_0, e^{-t\mathcal{A}_{\alpha,a}^*} \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_{\alpha}} = \gamma_2(0) \int_0^T v(t) \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \left( \partial_x (e^{-(T-t)\mathcal{A}_{\alpha,a}^*} \zeta)(x) \right) \Big|_{x=0} \right\rangle_{\mathbb{R}^2} dt. \quad (\text{IV.25})$$

2. A function  $v \in L^2(0, T)$  is a null-control for the problem (IV.7)–(IV.8b), if and only if it satisfies: for any  $\zeta \in \mathcal{H}_{\alpha}$

$$-\langle y_0, e^{-t\mathcal{A}_{\alpha,a}^*} \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_{\alpha}} = \gamma_1(0) \int_0^T v(t) \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \left( \partial_x (e^{-(T-t)\mathcal{A}_{\alpha,a}^*} \zeta)(x) \right) \Big|_{x=0} \right\rangle_{\mathbb{R}^2} dt. \quad (\text{IV.26})$$

Here, it is convenient to denote the observation operator (that does not depend on the quantities  $a$  or  $\alpha$ ) as follows

$$\mathcal{B}_1^* : u = (u_1, u_2) \in (H^2(0, 1))^2 \mapsto \gamma_2(0)u_2'(0), \quad (\text{IV.27a})$$

$$\mathcal{B}_2^* : u = (u_1, u_2) \in (H^2(0, 1))^2 \mapsto \gamma_1(0)u_1'(0). \quad (\text{IV.27b})$$

Now, we present the main theorems regarding the null-controllability issues for our problem (IV.7) with both cases: the boundary-control to be applied on either  $y_1$  or  $y_2$ ; we shall also achieve some suitable estimates of the controls, depending on the coupling coefficient  $a$ , the boundary parameter  $\alpha$ , as well as the diffusion coefficients  $\gamma_1, \gamma_2$ .

- **Case 1.** To show the boundary null-controllability of the problem (IV.7)–(IV.8a), that is when we consider the control applied on the second component, we prove a suitable observability inequality, and since we are in linear case so this will be obtained by a Carleman estimate, detailed in Section IV.3.1. Our main theorem is the following.

**Theorem IV.2.7.** *Let any  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}$  and  $T > 0$  be given. Then, for any  $y_0 \in \mathcal{H}_{-\alpha}$ , there exists a null-control  $v \in L^2(0, T)$  for the problem (IV.7)–(IV.8a), that satisfies the estimate*

$$\|v\|_{L^2(0, T)} \leq Ce^{C/T} \|y_0\|_{\mathcal{H}_{-\alpha}},$$

with the constant  $C := C(\gamma_1, \gamma_2, \alpha, a) > 0$  which does not depend on  $T > 0$  and  $y_0$ .

- **Case 2.** The above strategy of using Carleman estimate to prove the boundary controllability will no more be applicable for the problem (IV.7), when we assume the control input in the first component, i.e., with boundary control (IV.8b). This is because, the source integral due to the interior coupling in our Carleman estimate cannot be observable, with our choices of weight functions. The exact technical point behind this will be specified later in Remark IV.3.5 in Section IV.3.1.

Due to this obstacle, the next immediate idea is to investigate the spectral analysis of the adjoint to the corresponding elliptic operator and try to develop the so-called moments method to construct a control by hand; here we shall restrict the diffusion coefficients  $\gamma_1 = \gamma_2 = 1$  to ease the understanding of the spectrum. Indeed, by developing the spectral analysis, we will observe that the choices of coupling coefficient  $a$  and the boundary parameter  $\alpha$  really have a crucial role for the controllability of (IV.7)–(IV.8b), which is not alike the case when we consider our control applied on the second component  $y_2$ , as per Theorem IV.2.7.

Henceforth, it is reasonable not to find a good observability inequality using Carleman estimate in the case (IV.7)–(IV.8b), when the control input is assumed to be applied on the first component  $y_1$ . Let us state more precisely the controllability theorem concerning this case.

**Theorem IV.2.8.** *We fix  $\gamma_1 = \gamma_2 = 1$ . Then, there exists a set  $\mathcal{R} \subset \mathbb{R}_0^+ \times \mathbb{R}^*$  such that*

1. *for each pair  $(\alpha, a) \notin \mathcal{R}$ , there is a null-control to the problem (IV.7)–(IV.8b), for any given data  $y_0 \in \mathcal{H}_{-\alpha}$ ,*
2. *for each pair  $(\alpha, a) \in \mathcal{R}$ , there exists a subspace  $\mathcal{Y}_{\alpha, a} \subset \mathcal{H}_{-\alpha}$  of co-dimension 1, such that there exists a null-control to the problem (IV.7)–(IV.8b), if and only if  $y_0 \in \mathcal{Y}_{\alpha, a}$ .*

*In addition, in the controllable cases we can choose such a null-control  $v$  that satisfies the bound*

$$\|v\|_{L^2(0, T)} \leq C_{\alpha, a} e^{C_{\alpha, a}/T} \|y_0\|_{\mathcal{H}_{-\alpha}}, \quad (\text{IV.28})$$

where  $C_{\alpha, a} > 0$  is independent on  $T > 0$  and  $y_0$ .

The set  $\mathcal{R}$  and the space  $\mathcal{Y}_{\alpha,a}$  will be specified later, namely in (IV.100) and (IV.102), while proving Lemma IV.4.7 in Section IV.4.2.1.

**Remark IV.2.9.** *In the case when  $(\alpha, a) \in \mathcal{R}$ , the problem (IV.7)–(IV.8b) is not even approximately controllable if we choose our initial data  $y_0 \notin \mathcal{Y}_{\alpha,a}$ .*

In the next sections, we develop the required results to prove the controllability of both the problems, namely the Theorem IV.2.7 and IV.2.8.

### IV.3 Boundary controllability of the $2 \times 2$ system with control on the second component

This section is devoted to prove the existence of a null-control of the coupled system (IV.7)–(IV.8a), in terms of finding a proper observability inequality, and so the Carleman estimate is the main ingredient to obtain.

#### IV.3.1 A global boundary Carleman estimate

Let us first write the adjoint system to (IV.7)–(IV.8a), with homogeneous Dirichlet conditions at the left end point.

$$\begin{cases} -\partial_t q_1 - \partial_x(\gamma_1 \partial_x q_1) + a q_2 = 0 & \text{in } (0, T) \times (0, 1), \\ -\partial_t q_2 - \partial_x(\gamma_2 \partial_x q_2) = 0 & \text{in } (0, T) \times (0, 1), \\ q_1(t, 0) = q_2(t, 0) = 0 & \text{in } (0, T), \\ q_1(t, 1) = q_2(t, 1) & \text{in } (0, T), \\ \gamma_1(1) \partial_x q_1(t, 1) + \gamma_2(1) \partial_x q_2(t, 1) + \alpha q_1(t, 1) = 0 & \text{in } (0, T), \\ q_1(T, \cdot) = \zeta_1(\cdot) & \text{in } (0, 1), \\ q_2(T, \cdot) = \zeta_2(\cdot) & \text{in } (0, 1), \end{cases} \quad (\text{IV.29})$$

where the regularity of  $\zeta := (\zeta_1, \zeta_2)$  will be imposed later when needed and for simplicity sometimes we shall use the notation  $Q := (0, T) \times (0, 1)$ .

Now, we introduce the following space

$$\mathcal{Q} := \left\{ q = (q_1, q_2) \in (\mathcal{C}^2(\overline{Q}))^2 \mid q_1(t, 0) = q_2(t, 0) = 0, q_1(t, 1) = q_2(t, 1), \right. \\ \left. \sum_{i=1}^2 \gamma_i(1) \partial_x q_i(t, 1) + \alpha q_1(t, 1) = 0, \forall t \in (0, T) \right\}.$$

Before introducing the main theorem regarding Carleman estimate, we define some standard weight functions which are the main ingredients to obtain the Carleman inequality.

**Construction of the weight functions.** Let  $\mu_0 \in (0, 1)$  close enough to 1 such that

$$\left[ \frac{216\mu_0}{(1-\mu_0)^3} \gamma_2^2(1) - 7\gamma_1^2(1) \right] \geq 1. \quad (\text{IV.30})$$

We consider the following affine functions

$$\begin{cases} \beta_i(x) = 2 + c_i(x-1), & \forall x \in [0, 1], \\ \text{with } c_1 = 1, \quad c_2 = \frac{-6}{(1-\mu_0)}, & \text{for } 0 < \mu_0 < 1, \end{cases} \quad (\text{IV.31})$$

that satisfy the following properties

$$\beta_2 \geq \beta_1 > 0, \text{ in } [0, 1], \quad \beta_2(1) = \beta_1(1). \quad (\text{IV.32})$$

**Remark IV.3.1.** We will see while proving a Carleman estimate (namely the Theorem IV.3.2 stated later), that the above assumption (IV.30) is very sharp and crucial to absorb some unusual boundary integrals sitting in the right hand side of the Carleman estimate.

Now, we assume that  $\lambda > 1$  and  $K = 2 \max\{\|\beta_1\|_\infty, \|\beta_2\|_\infty\}$  and define the weight functions  $\varphi_i$  and  $\eta_i$ , for  $i = 1, 2$ , as follows

$$\varphi_i(t, x) = \frac{e^{\lambda\beta_i(x)}}{t(T-t)}, \quad \eta_i(t, x) = \frac{e^{\lambda K} - e^{\lambda\beta_i(x)}}{t(T-t)}, \quad \forall (t, x) \in Q. \quad (\text{IV.33})$$

From the properties of  $\beta_1$  and  $\beta_2$  in (IV.32), we have that the functions  $\varphi_i$  and  $\eta_i$  are positive and satisfy

$$\varphi_1(t, 1) = \varphi_2(t, 1) \quad \text{and} \quad \eta_1(t, 1) = \eta_2(t, 1), \quad (\text{IV.34})$$

since  $\beta_1(1) = \beta_2(1)$ .

We also have the following relations in  $Q$ , for  $i = 1, 2$ ,

$$\begin{cases} \partial_x \varphi_i = \lambda \varphi_i c_i, & \partial_x \eta_i = -\lambda \varphi_i c_i, \\ \partial_t \varphi_i = \varphi_i \frac{2t-T}{t(T-t)}, & \partial_t \eta_i = \eta_i \frac{2t-T}{t(T-t)}, \\ \partial_t^2 \eta_i = \eta_i \frac{3(2t-T)^2 + T^2}{2t^2(T-t)^2}. \end{cases} \quad (\text{IV.35})$$

Now, we write the main theorem of this section concerning the Carleman estimate.

**Theorem IV.3.2** (A Carleman estimate). *Let the weight functions  $\varphi_1, \varphi_2$  and  $\eta_1, \eta_2$  be defined as in (IV.33). Then, there exists  $\lambda_1 := \lambda_1(\gamma_1, \gamma_2, \alpha) > 0$ ,  $\sigma := \sigma(\gamma_1, \gamma_2, \alpha) > 0$ ,  $s_1 := (T^2 + T)\sigma > 0$  and a constant  $C' := C'(\gamma_1, \gamma_2, \alpha) > 0$ , so that the following Carleman estimate holds true*

$$\begin{aligned} & s^3 \lambda^4 \sum_{i=1}^2 \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i^3 |q_i|^2 dx dt + s \lambda^2 \sum_{i=1}^2 \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i |\partial_x q_i|^2 dx dt \\ & + s^3 \lambda^3 \int_0^T \varphi_1(t, 1) e^{-2s\eta_1(t, 1)} |q_1(t, 1)|^2 dt \leq C' \left[ \sum_{i=1}^2 \int_0^T \int_0^1 e^{-2s\eta_i} |\partial_t q_i + \partial_x(\gamma_i \partial_x q_i)|^2 dx dt \right. \\ & \quad \left. + s \lambda \int_0^T \varphi_2(t, 0) e^{-2s\eta_2(t, 0)} |\partial_x q_2(t, 0)|^2 dt \right], \quad (\text{IV.36}) \end{aligned}$$

for  $s \geq s_1$ ,  $\lambda \geq \lambda_1$  and for all  $(q_1, q_2) \in \mathcal{Q}$ .

Before going to the proof for the above theorem, we let any  $s > 0$ ,  $\lambda > 1$  and  $(q_1, q_2) \in \mathcal{Q}$  and we write  $f_i = \partial_t q_i + \partial_x(\gamma_i \partial_x q_i)$ , then  $f_i \in L^2(Q)$ , for  $i = 1, 2$ . We also set

$$\psi_i(t, x) = e^{-s\eta_i(t, x)} q_i(t, x), \quad \forall (t, x) \in Q, \quad \text{for } i = 1, 2.$$

Observe that,

$$\psi_i(t, 0) = 0, \quad i = 1, 2, \quad \text{and} \quad \psi_1(t, 1) = \psi_2(t, 1), \quad (\text{IV.37})$$

using (IV.34) and the properties of  $q_i$ ,  $i = 1, 2$  in  $\mathcal{Q}$ . Also look that

$$\partial_x \psi_i(t, x) = e^{-s\eta_i(t, x)} \partial_x q_i(t, x) + s \lambda \beta'_i(x) \varphi_i(t, x) \psi_i(t, x), \quad \forall (t, x) \in Q, \quad (\text{IV.38})$$

so that we have

$$\sum_{i=1}^2 \gamma_i(1) \partial_x \psi_i(t, 1) = -\alpha \psi_1(t, 1) + s \lambda \left( \sum_{i=1}^2 c_i \gamma_i(1) \right) \varphi_1(t, 1) \psi_1(t, 1), \quad (\text{IV.39})$$

thanks to the boundary condition  $\sum_{i=1}^2 \gamma_i(1) \partial_x q_i(t, 1) + \alpha q_1(t, 1) = 0$ , the properties of  $\varphi_i$  in (IV.34) and  $\psi_i$  in (IV.37), and the fact that  $\beta'_i = c_i$ , for  $i = 1, 2$ .

Next, we see that the functions  $\psi_i$  satisfies the following relations in  $Q$

$$M_1 \psi_i + M_2 \psi_i = F_i, \quad \text{for } i = 1, 2,$$

with

$$\begin{cases} M_1 \psi_i = \partial_x(\gamma_i \partial_x \psi_i) + s^2 \lambda^2 c_i^2 \varphi_i^2 \gamma_i \psi_i + s(\partial_t \eta_i) \psi_i, \\ M_2 \psi_i = \partial_t \psi_i - 2s \lambda c_i \varphi_i (\gamma_i \partial_x \psi_i) - 2s \lambda^2 c_i^2 \varphi_i \gamma_i \psi_i, \\ F_i = e^{-s \eta_i} f_i + s \lambda c_i \gamma'_i \varphi_i \psi_i - s \lambda^2 c_i^2 \varphi_i \gamma_i \psi_i. \end{cases} \quad (\text{IV.40})$$

We have for  $i = 1, 2$ ,

$$\|M_1 \psi_i\|_{L^2(Q)}^2 + \|M_2 \psi_i\|_{L^2(Q)}^2 + 2(M_1 \psi_i, M_2 \psi_i)_{L^2(Q)} = \|F_i\|_{L^2(Q)}^2. \quad (\text{IV.41})$$

Now, we present the following auxiliary lemma which is important to prove the main result in Theorem IV.3.2.

**Lemma IV.3.3.** *Let the functions  $\varphi_i$ ,  $\eta_i$ ,  $\psi_i$ ,  $M_1 \psi_i$ ,  $M_2 \psi_i$  (in  $Q$ ), for  $i = 1, 2$ , and the quantities  $c_1, c_2$  be as introduced earlier. Then there exists  $\lambda_0 := \lambda_0(\gamma_1, \gamma_2) > 0$ ,  $s_0 := (T^2 + T)\sigma_0 > 0$  with some  $\sigma_0 := \sigma_0(\gamma_1, \gamma_2) > 0$  and a constant  $C'' = C''(\gamma_1, \gamma_2) > 0$  such that we have the following inequality*

$$\begin{aligned} & \frac{1}{2} \sum_{i=1}^2 \|M_1 \psi_i\|_{L^2(Q)}^2 + \frac{1}{2} \sum_{i=1}^2 \|M_2 \psi_i\|_{L^2(Q)}^2 \\ & + \sum_{i=1}^2 s^3 \lambda^4 \int_0^T \int_0^1 \varphi_i^3 |\psi_i|^2 dx dt + \sum_{i=1}^2 s \lambda^2 \int_0^T \int_0^1 \varphi_i |\partial_x \psi_i|^2 dx dt \\ & + \sum_{i=1}^2 \gamma_i(1) \int_0^T \partial_x \psi_i(t, 1) \partial_t \psi_i(t, 1) dt - s \lambda \sum_{i=1}^2 c_i \int_0^T \varphi_i(t, 1) |\gamma_i(1) \partial_x \psi_i(t, 1)|^2 dt \\ & + s \lambda \sum_{i=1}^2 c_i \int_0^T \varphi_i(t, 0) |\gamma_i(0) \partial_x \psi_i(t, 0)|^2 dt - 2s \lambda^2 \sum_{i=1}^2 c_i^2 \gamma_i^2(1) \int_0^T \varphi_i(t, 1) \psi_i(t, 1) \partial_x \psi_i(t, 1) dt \\ & - s^3 \lambda^3 c_i^3 \sum_{i=1}^2 \gamma_i^2(1) \int_0^T \varphi_i^3(t, 1) |\psi_i(t, 1)|^2 dt - s^2 \lambda \sum_{i=1}^2 c_i \gamma_i(1) \int_0^T \varphi_i(t, 1) (\partial_t \eta_i)(t, 1) |\psi_i(t, 1)|^2 dt \\ & \leq C'' \sum_{i=1}^2 \|e^{-s \eta_i} f_i\|_{L^2(Q)}^2, \quad (\text{IV.42}) \end{aligned}$$

for all  $\lambda \geq \lambda_0$ ,  $s \geq s_0$ .

*Proof.* Recall the quantities  $M_1 \psi_i$ ,  $M_2 \psi_i$ ,  $F_i$ , for  $i = 1, 2$ , defined in (IV.40) and the relation satisfied by those in (IV.41).

• First, we shall explicitly compute  $(M_1 \psi_1, M_2 \psi_1)_{L^2(Q)}$  as a sum of 9 terms  $I_{ij}$ ,  $1 \leq i, j \leq 3$ , where  $I_{ij}$  is the inner product of  $i$ th term in the expression of  $M_1 \psi_1$  and the  $j$ th term in the expression of  $M_2 \psi_1$ .

1. The term  $I_{11}$  follows, performing an integration by parts (and using  $\psi_1(t, 0) = 0$ ),

$$\begin{aligned} I_{11} &= \int_0^T \int_0^1 \partial_x(\gamma_1 \partial_x \psi_1) \partial_t \psi_1 dx dt = - \int_0^T \int_0^1 (\gamma_1 \partial_x \psi_1) \partial_t (\partial_x \psi_1) dx dt \\ & \quad + \int_0^T \gamma_1(1) \frac{\partial \psi_1}{\partial x}(t, 1) \partial_t \psi_1(t, 1) dt. \end{aligned}$$

Now observing  $(\partial_x \psi_1) \partial_t (\partial_x \psi_1) = \frac{1}{2} \partial_t |\partial_x \psi_1|^2$ , we find that the above volume integral vanishes since  $\lim_{t \rightarrow 0^+} \partial_x \psi_1(t, \cdot) = \lim_{t \rightarrow T^-} \partial_x \psi_1(t, \cdot) = 0$  from the definition of functions  $\varphi_1$ ,  $\eta_1$  from (IV.33) and the construction of  $\psi_1$ . So, eventually it follows

$$I_{11} = \int_0^T \gamma_1(1) \frac{\partial \psi_1}{\partial x}(t, 1) \partial_t \psi_1(t, 1) dt. \quad (\text{IV.43})$$

2. The term  $I_{12}$  enjoys

$$\begin{aligned} I_{12} &= -2s\lambda c_1 \int_0^T \int_0^1 \varphi_1(\gamma_1 \partial_x \psi_1) \partial_x(\gamma_1 \partial_x \psi_1) dx dt \\ &= s\lambda^2 c_1^2 \int_0^T \int_0^1 \varphi_1 |\gamma_1 \partial_x \psi_1|^2 dx dt - s\lambda c_1 \int_0^T \varphi_1(t, 1) \left| \gamma_1(1) \frac{\partial \psi_1}{\partial x}(t, 1) \right|^2 dt \\ &\quad + s\lambda c_1 \int_0^T \varphi_1(t, 0) \left| \gamma_1(0) \frac{\partial \psi_1}{\partial x}(t, 0) \right|^2 dt, \end{aligned} \quad (\text{IV.44})$$

thanks to an integration by parts with respect to  $x$  and the fact that  $\partial_x \varphi_1 = \lambda \varphi_1 c_1$ .

3. The term  $I_{13}$  is given by

$$\begin{aligned} I_{13} &= -2s\lambda^2 c_1^2 \int_0^T \int_0^1 \varphi_1 \gamma_1 \psi_1 \partial_x(\gamma_1 \partial_x \psi_1) dx dt = 2s\lambda^2 c_1^2 \int_0^T \int_0^1 \varphi_1 |\gamma_1 \partial_x \psi_1|^2 dx dt \\ &\quad + X_{13} - 2s\lambda^2 c_1^2 \gamma_1^2(1) \int_0^T \varphi_1(t, 1) \psi_1(t, 1) \frac{\partial \psi_1}{\partial x}(t, 1) dt \end{aligned} \quad (\text{IV.45})$$

with

$$X_{13} := 2s\lambda^3 c_1^3 \int_0^T \int_0^1 \gamma_1 \varphi_1 \psi_1 (\gamma_1 \partial_x \psi_1) dx dt + 2s\lambda^2 c_1^2 \int_0^T \int_0^1 \gamma_1' \varphi_1 \psi_1 (\gamma_1 \partial_x \psi_1) dx dt,$$

where it has been used that  $\partial_x \varphi_1 = \lambda \varphi_1 c_1$  and  $\psi_1(t, 0) = 0$ .

4. Next, we have

$$I_{21} = s^2 \lambda^2 c_1^2 \int_0^T \int_0^1 \varphi_1^2 \gamma_1 \psi_1 (\partial_t \psi_1) dx dt = -s^2 \lambda^2 c_1^2 \int_0^T \int_0^1 \gamma_1 \varphi_1 (\partial_t \varphi_1) |\psi_1|^2 dx dt. \quad (\text{IV.46})$$

No boundary term has been appeared here since  $\psi_1(0, \cdot) = \psi_1(T, \cdot) = 0$ .

5. The term  $I_{22}$  is

$$\begin{aligned} I_{22} &= -2s^3 \lambda^3 c_1^3 \int_0^T \int_0^1 \varphi_1^3 \gamma_1^2 \psi_1 (\partial_x \psi_1) dx dt \\ &= 3s^3 \lambda^4 c_1^4 \int_0^T \int_0^1 \varphi_1^3 \gamma_1^2 |\psi_1|^2 dx dt + X_{22} - s^3 \lambda^3 c_1^3 \gamma_1^2(1) \int_0^T \varphi_1^3(t, 1) |\psi_1(t, 1)|^2 dt, \end{aligned} \quad (\text{IV.47})$$

with

$$X_{22} = 2s^3 \lambda^3 c_1^3 \int_0^T \int_0^1 \varphi_1^3 \gamma_1 \gamma_1' |\psi_1|^2 dx dt.$$

6. The terms  $I_{23}$  is simply

$$I_{23} = -2s^3 \lambda^4 c_1^4 \int_0^T \int_0^1 \varphi_1^3 \gamma_1^2 |\psi_1|^2 dx dt. \quad (\text{IV.48})$$



7. Next, the term  $I_{31}$  is given by

$$I_{31} = s \int_0^T \int_0^1 (\partial_t \eta_1) \psi_1 (\partial_t \psi_1) dx dt = -\frac{s}{2} \int_0^T \int_0^1 (\partial_t^2 \eta_1) |\psi_1|^2 dx dt. \quad (\text{IV.49})$$

8. Thereafter, we have

$$\begin{aligned} I_{32} &= -2s^2 \lambda c_1 \int_0^T \int_0^1 \gamma_1 \varphi_1 (\partial_t \eta_1) \psi_1 (\partial_x \psi_1) dx dt \\ &= s^2 \lambda^2 c_1^2 \int_0^T \int_0^1 \varphi_1 \gamma_1 (\partial_t \eta_1) |\psi_1|^2 dx dt - s^2 \lambda^2 c_1^2 \int_0^T \int_0^1 \varphi_1 (\partial_t \varphi_1) \gamma_1 |\psi_1|^2 dx dt \\ &\quad + s^2 \lambda c_1 \int_0^T \int_0^1 \varphi_1 \gamma_1' (\partial_t \eta_1) |\psi_1|^2 dx dt - s^2 \lambda c_1 \gamma_1(1) \int_0^T \varphi_1(t, 1) (\partial_t \eta_1)(t, 1) |\psi_1(t, 1)|^2 dt \\ &=: X_{32} - s^2 \lambda c_1 \gamma_1(1) \int_0^T \varphi_1(t, 1) (\partial_t \eta_1)(t, 1) |\psi_1(t, 1)|^2 dt, \quad (\text{IV.50}) \end{aligned}$$

using  $\partial_x \varphi_1 = \lambda \varphi_1 c_1$  and  $\partial_x \eta_1 = -\lambda \varphi_1 c_1$ .

9. Finally, the term  $I_{33}$  is

$$I_{33} = -2s^2 \lambda^2 c_1^2 \int_0^T \int_0^1 \varphi_1 (\partial_t \eta_1) \gamma_1 |\psi_1|^2 dx dt. \quad (\text{IV.51})$$

Now, adding all the terms of  $(M_1 \psi_1, M_2 \psi_1)_{L^2(Q)}$  and taking into account (IV.41) leads

$$\begin{aligned} &\frac{1}{2} \|M_1 \psi_1\|_{L^2(Q)}^2 + \frac{1}{2} \|M_2 \psi_1\|_{L^2(Q)}^2 \\ &\quad + s^3 \lambda^4 c_1^4 \int_0^T \int_0^1 \varphi_1^3 |\gamma_1 \psi_1|^2 dx dt + 3s \lambda^2 c_1^2 \int_0^T \int_0^1 \varphi_1 |\gamma_1 \partial_x \psi_1|^2 dx dt \\ &\quad + \gamma_1(1) \int_0^T \frac{\partial \psi_1}{\partial x}(t, 1) \partial_t \psi_1(t, 1) dt - s \lambda c_1 \int_0^T \varphi_1(t, 1) \left| \gamma_1(1) \frac{\partial \psi_1}{\partial x}(t, 1) \right|^2 dt \\ &\quad + s \lambda c_1 \int_0^T \varphi_1(t, 0) \left| \gamma_1(0) \frac{\partial \psi_1}{\partial x}(t, 0) \right|^2 dt - 2s \lambda^2 c_1^2 \gamma_1^2(1) \int_0^T \varphi_1(t, 1) \psi_1(t, 1) \frac{\partial \psi_1}{\partial x}(t, 1) dt \\ &\quad - s^3 \lambda^3 c_1^3 \gamma_1^2(1) \int_0^T \varphi_1^3(t, 1) |\psi_1(t, 1)|^2 dt - s^2 \lambda c_1 \gamma_1(1) \int_0^T \varphi_1(t, 1) (\partial_t \eta_1)(t, 1) |\psi_1(t, 1)|^2 dt \\ &= \frac{1}{2} \|F_1\|_{L^2(Q)}^2 - [X_{13} + I_{21} + X_{22} + I_{31} + X_{32} + I_{33}]. \quad (\text{IV.52}) \end{aligned}$$

Let us first observe that there exists a constant  $C'' = C''(c_1, \gamma_1)$  such that the following estimates hold

$$|X_{22}| \leq C'' s^3 \lambda^3 \int_0^T \int_0^1 \varphi_1^3 |\psi_1|^2 dx dt,$$

$$|X_{13}| \leq \frac{C''}{\epsilon} s \lambda^4 \int_0^T \int_0^1 \varphi_1 |\psi_1|^2 dx dt + C'' \epsilon s \lambda^2 \int_0^T \int_0^1 \varphi_1 |\partial_x \psi_1|^2 dx dt,$$

where we have used the Young's inequality. Now observe that  $\varphi_1 \leq 4T^4 \varphi_1^3$ , we obtain

$$|X_{13}| \leq C''_\epsilon T^4 s \lambda^4 \int_0^T \int_0^1 \varphi_1^3 |\psi_1|^2 dx dt + C'' \epsilon s \lambda^2 \int_0^T \int_0^1 \varphi_1 |\partial_x \psi_1|^2 dx dt.$$

Beside that, we note

$$|\partial_t \varphi_1| \leq T \varphi_1^2, \quad |\partial_t \eta_1| \leq T \varphi_1^2, \quad |\partial_t^2 \eta_1| \leq 2T^2 \varphi_1^3,$$

so that we obtain

$$|I_{21}| \leq C'' s^2 \lambda^2 T \int_0^T \int_0^1 \varphi_1^3 |\psi_1|^2 dx dt, \quad |I_{31}| \leq C'' s T^2 \int_0^T \int_0^1 \varphi_1^3 |\psi_1|^2 dx dt,$$

$$|X_{32}| \leq C'' s^2 \lambda^2 T \int_0^T \int_0^1 \varphi_1^3 |\psi_1|^2 dx dt, \quad |I_{33}| \leq C'' s^2 \lambda^2 T \int_0^T \int_0^1 \varphi_1^3 |\psi_1|^2 dx dt.$$

We also have

$$\|F_1\|_{L^2(Q)}^2 \leq C'' \|e^{-s\eta_1} f_1\|_{L^2(Q)}^2 + C'' s^2 \lambda^4 T^2 \int_0^T \int_0^1 \varphi_1^3 |\psi_1|^2 dx dt,$$

where we used the fact  $\varphi_1^2 \leq 2T^2 \varphi_1^3$ . Now recall the fact that  $\inf_{[0,1]} \gamma_1 > 0$  and using all the above estimates in (IV.52), we obtain

$$\begin{aligned} & \frac{1}{2} \|M_1 \psi_1\|_{L^2(Q)}^2 + \frac{1}{2} \|M_2 \psi_1\|_{L^2(Q)}^2 + s^3 \lambda^4 c_1^4 \int_0^T \int_0^1 \varphi_1^3 |\psi_1|^2 dx dt + 3s \lambda^2 c_1^2 \int_0^T \int_0^1 \varphi_1 |\partial_x \psi_1|^2 dx dt \\ & + \gamma_1(1) \int_0^T \frac{\partial \psi_1}{\partial x}(t, 1) \partial_t \psi_1(t, 1) dt - s \lambda c_1 \int_0^T \varphi_1(t, 1) \left| \gamma_1(1) \frac{\partial \psi_1}{\partial x}(t, 1) \right|^2 dt \\ & + s \lambda c_1 \int_0^T \varphi_1(t, 0) \left| \gamma_1(0) \frac{\partial \psi_1}{\partial x}(t, 0) \right|^2 dt - 2s \lambda^2 c_1^2 \gamma_1^2(1) \int_0^T \varphi_1(t, 1) \psi_1(t, 1) \frac{\partial \psi_1}{\partial x}(t, 1) dt \\ & - s^3 \lambda^3 c_1^3 \gamma_1^2(1) \int_0^T \varphi_1^3(t, 1) |\psi_1(t, 1)|^2 dt - s^2 \lambda c_1 \gamma_1(1) \int_0^T \varphi_1(t, 1) (\partial_t \eta_1)(t, 1) |\psi_1(t, 1)|^2 dt \\ & \leq C'' \|e^{-s\eta_1} f_1\|_{L^2(Q)}^2 + C'' (s \lambda + \epsilon s \lambda^2) \iint_Q \varphi_1 |\partial_x \psi_1|^2 dx dt \\ & + C'' (s^3 \lambda^3 + s^2 (\lambda^4 T^2 + \lambda^2 T) + s (\lambda^4 T^4 C''_\epsilon + T^2)) \iint_Q \varphi_1^3 |\psi_1|^2 dx dt. \end{aligned}$$

Now, exploiting  $c_1 > 0$  in  $[0, 1]$ , choosing  $\epsilon > 0$  sufficiently small and taking  $\lambda \geq \lambda_{01} := \lambda_{01}(\gamma_1) > 0$ ,  $s \geq s_{01} := (T^2 + T) \sigma_{01}(\gamma_1) > 0$ , we obtain

$$\begin{aligned} & \frac{1}{2} \|M_1 \psi_1\|_{L^2(Q)}^2 + \frac{1}{2} \|M_2 \psi_1\|_{L^2(Q)}^2 + s^3 \lambda^4 \int_0^T \int_0^1 \varphi_1^3 |\psi_1|^2 dx dt + s \lambda^2 \int_0^T \int_0^1 \varphi_1 |\partial_x \psi_1|^2 dx dt \\ & + \gamma_1(1) \int_0^T \frac{\partial \psi_1}{\partial x}(t, 1) \partial_t \psi_1(t, 1) dt - s \lambda c_1 \int_0^T \varphi_1(t, 1) \left| \gamma_1(1) \frac{\partial \psi_1}{\partial x}(t, 1) \right|^2 dt \\ & + s \lambda c_1 \int_0^T \varphi_1(t, 0) \left| \gamma_1(0) \frac{\partial \psi_1}{\partial x}(t, 0) \right|^2 dt - 2s \lambda^2 c_1^2 \gamma_1^2(1) \int_0^T \varphi_1(t, 1) \psi_1(t, 1) \frac{\partial \psi_1}{\partial x}(t, 1) dt \\ & - s^3 \lambda^3 c_1^3 \gamma_1^2(1) \int_0^T \varphi_1^3(t, 1) |\psi_1(t, 1)|^2 dt - s^2 \lambda c_1 \gamma_1(1) \int_0^T \varphi_1(t, 1) (\partial_t \eta_1)(t, 1) |\psi_1(t, 1)|^2 dt \\ & \leq C'' \|e^{-s\eta_1} f_1\|_{L^2(Q)}^2, \quad (\text{IV.53}) \end{aligned}$$

with  $C'' = C''(c_1, \gamma_1)$ .

- It is clear that a similar estimate like (IV.53) will be also satisfied by  $\psi_2$  for any  $\lambda \geq \lambda_{02} :=$

$\lambda_{02}(\gamma_2) > 0$ ,  $s \geq s_{02} := (T^2 + T)\sigma_{02}(\gamma_2) > 0$  and with constant  $C'' = C''(c_2, \gamma_2)$ , possibly some different constant than the previous one.

• Now, taking  $\lambda_0 = \max\{\lambda_{0i} \mid i = 1, 2\}$ ,  $\sigma_0 = \max\{\sigma_{0i}(\gamma_i) \mid i = 1, 2\}$ ,  $s_0 = (T^2 + T)\sigma_0$  and some constant with same notation  $C'' > 0$ , we add the two estimates corresponding to  $\psi_1$  and  $\psi_2$  to obtain the required inequality (IV.42). Hence, the lemma is proved.  $\square$

*Proof of Theorem IV.3.2.*

The main idea to prove this theorem is to play with the boundary integrals appeared in the inequality (IV.42) in such a way that one can absorb the lower order integrals by some leading integral terms and then to concentrate on the proper observation term which will be eventually shifted in the right hand side of the final estimate.

We make use of the following notations: denote the all six boundary terms respectively by  $J_k$ ,  $1 \leq k \leq 6$ , by maintaining the same order as in (IV.42).

- We have (using  $\psi_2(t, 1) = \psi_1(t, 1)$ )

$$\begin{aligned} J_1 &:= \int_0^T [\gamma_1(1)\partial_x\psi_1(t, 1) + \gamma_2(1)\partial_x\psi_2(t, 1)] \partial_t\psi_1(t, 1) dt \\ &= -\alpha \int_0^T \psi_1(t, 1)\partial_t\psi_1(t, 1) dt + s\lambda \left( \sum_{i=1}^2 c_i\gamma_i(1) \right) \int_0^T \varphi_1(t, 1)\psi_1(t, 1)\partial_t\psi_1(t, 1) dt \\ &= -\frac{s\lambda}{2} \left( \sum_{i=1}^2 c_i\gamma_i(1) \right) \int_0^T (\partial_t\varphi_1)(t, 1)|\psi_1(t, 1)|^2 dt, \end{aligned}$$

due to the condition (IV.39) and the fact that  $\psi_1(0, \cdot) = \psi_1(T, \cdot) = 0$ . Now, using  $|\partial_t\varphi_1| \leq T\varphi_1^2 \leq 2T^3\varphi_1^3$ , we obtain

$$|J_1| \leq \tilde{C}s\lambda T^3 \int_0^T \varphi_1^3(t, 1)|\psi_1(t, 1)|^2 dt, \quad (\text{IV.54})$$

for some constant  $\tilde{C} > 0$ .

- Next, we write the second boundary term of (IV.42) as  $J_2 := J_{21} + J_{22}$ , where we keep the second integral in the left hand side of (IV.42) since

$$J_{22} := -s\lambda c_2 \int_0^T \varphi_2(t, 1)|\gamma_2(1)\partial_x\psi_2(t, 1)|^2 dt \geq 0 \quad (\text{IV.55})$$

due to the fact that  $c_2 < 0$ . Later, we will see that the integral  $J_{22}$  will be used to absorb some lower order terms.

On the other hand, the first integral of the second boundary term  $J_2$  is

$$J_{21} := -s\lambda c_1 \int_0^T \varphi_1(t, 1)|\gamma_1(1)\partial_x\psi_1(t, 1)|^2 dt,$$

where  $c_1 > 0$  and so  $J_{21} \leq 0$ . So, we need to absorb those integrals by some higher order terms in the left hand side. Let us recall (IV.39) to express

$$\gamma_1(1)\partial_x\psi_1(t, 1) = -\alpha\psi_1(t, 1) - \gamma_2(1)\partial_x\psi_2(t, 1) + s\lambda \left( \sum_{i=1}^2 c_i\gamma_i(1) \right) \varphi_1(t, 1)\psi_1(t, 1),$$

so that we have the following,

$$\begin{aligned} |J_{21}| &\leq 3s\lambda\alpha^2 c_1 \int_0^T \varphi_1(t, 1)|\psi_1(t, 1)|^2 dt + 3s\lambda c_1 \int_0^T \varphi_1(t, 1)|\gamma_2(1)\partial_x\psi_2(t, 1)|^2 dt \\ &\quad + 6s^3\lambda^3 c_1 \left( \sum_{i=1}^2 c_i^2\gamma_i^2(1) \right) \int_0^T \varphi_1^3(t, 1)|\psi_1(t, 1)|^2 dt := J_{21}^1 + J_{21}^2 + J_{21}^3, \quad (\text{IV.56}) \end{aligned}$$

with a simple observation (since  $\varphi_1 \leq 4T^4\varphi_1^3$ ),

$$J_{21}^1 \leq \tilde{C}s\lambda\alpha^2T^4 \int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2 dt, \quad (\text{IV.57})$$

- Now, we look into the third boundary term,  $J_3 := J_{31} + J_{32}$ , where we have

$$J_{31} := s\lambda c_1 \int_0^T \varphi_1(t,0)|\gamma_1(0)\partial_x\psi_1(t,0)|^2 dt \geq 0,$$

since  $c_1 > 0$  and the function  $\varphi_1 > 0$ , and so one can discard this term from the left hand side of (IV.42).

On the other hand, we have

$$|J_{32}| := s\lambda \left| c_2 \int_0^T \varphi_2(t,0)|\gamma_2(0)\partial_x\psi_2(t,0)|^2 dt \right| \leq \tilde{C}s\lambda \int_0^T \varphi_2(t,0)e^{-2s\eta_2(t,0)}|\partial_x q_2(t,0)|^2 dt, \quad (\text{IV.58})$$

following the expression of  $\partial_x\psi_2$  given by (IV.38) and using the fact that  $\psi_2(t,0) = 0$ .

- Thereafter, we write the fourth boundary integral of (IV.42) by  $J_4 := J_{41} + J_{42}$ , and we obtain the following

$$\begin{aligned} |J_{41}| &= 2s\lambda^2 c_1^2 \gamma_1^2(1) \int_0^T \varphi_1(t,1)|\psi_1(t,1)\partial_x\psi_1(t,1)| dt \\ &\leq \tilde{C}\epsilon s\lambda \int_0^T \varphi_1(t,1)|\gamma_1(1)\partial_x\psi_1(t,1)|^2 dt + \frac{\tilde{C}}{\epsilon} s\lambda^3 T^4 \int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2 dt, \end{aligned} \quad (\text{IV.59})$$

where we have used the Young's inequality and the fact  $\varphi_1 \leq 2T^4\varphi_1^3$ . Now, for the first integral in the right hand side of (IV.59), we use the estimate for  $J_{21}$  given by (IV.56) to obtain

$$\begin{aligned} |J_{41}| &\leq \tilde{C}\epsilon s\lambda \int_0^T \varphi_1(t,1)|\gamma_2(1)\partial_x\psi_2(t,1)|^2 dt \\ &\quad + \tilde{C}(\epsilon s^3\lambda^3 + \epsilon s\lambda\alpha^2T^4 + \frac{1}{\epsilon}s\lambda^3T^4) \int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2 dt. \end{aligned} \quad (\text{IV.60})$$

On the other hand, a similar computation as in (IV.59) gives that

$$|J_{42}| \leq \tilde{C}\epsilon s\lambda \int_0^T \varphi_2(t,1)|\gamma_2(1)\partial_x\psi_2(t,1)|^2 dt + \frac{\tilde{C}}{\epsilon} s\lambda^3 T^4 \int_0^T \varphi_2^3(t,1)|\psi_2(t,1)|^2 dt. \quad (\text{IV.61})$$

- The fifth and the leading boundary term in the left hand side of (IV.42) is

$$J_5 = s^3\lambda^3 \left( -c_2^3\gamma_2^2(1) - c_1^3\gamma_1^2(1) \right) \int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2 dt, \quad (\text{IV.62})$$

by writing  $\varphi_2(t,1) = \varphi_1(t,1)$ .

- Finally, the sixth boundary term  $J_6$  satisfies

$$|J_6| \leq \tilde{C}s^2\lambda T \int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2 dt, \quad (\text{IV.63})$$

due to the facts that  $\varphi_2(t,1) = \varphi_1(t,1)$ ,  $\partial_t\eta_2(t,1) = \partial_t\eta_1(t,1)$  and  $|\partial_t\eta_1| \leq T\varphi_1^2$ .

- Let us first try to show that the coefficient of the boundary integral  $\int_0^T \varphi_1^3(t, 1) |\psi_1(t, 1)|^2 dt$  is positive in the left hand side of the main inequality (IV.42). To deduce this, recall the quantities  $J_5$  from (IV.62) and  $J_{21}^3$  from (IV.56) and take those quantities in the left hand side of the main inequality (IV.42), we see

$$J_5 - J_{21}^3 = K_1 s^3 \lambda^3 \int_0^T \varphi_1^3(t, 1) |\psi_1(t, 1)|^2 dt, \quad \text{with } K_1 = c_2^2 \gamma_2^2(1) (|c_2| - 6c_1) - 7c_1^3 \gamma_1^2(1). \quad (\text{IV.64})$$

Recall that we haven chosen  $c_1 = 1, c_2 = \frac{-6}{(1-\mu_0)}$  (see (IV.31)), so that we calculate

$$K_1 = \left[ \frac{216\mu_0}{(1-\mu_0)^3} \gamma_2^2(1) - 7\gamma_1^2(1) \right] \geq 1,$$

thanks to the condition (IV.32).

- Next, we recall  $J_{22}$  and  $J_{21}^2$  respectively given by (IV.55) and (IV.56), use  $\varphi_1(t, 1) = \varphi_2(t, 1)$ , and we write the other leading boundary integral in the left hand side as follows

$$J_{22} - J_{21}^2 = K_2 s \lambda \int_0^T \varphi_2(t, 1) |\gamma_2(1) \partial_x \psi_2(t, 1)|^2 dt, \quad \text{with } K_2 = (|c_2| - 3c_1), \quad (\text{IV.65})$$

where we compute  $K_2 = \frac{3(1+\mu_0)}{(1-\mu_0)} > 0$ , using the values of  $c_1, c_2$ .

- Now, we gather the leading boundary terms  $J_5 - J_{21}^3$  and  $J_{22} - J_{21}^2$  given by (IV.64) and (IV.65) respectively, in the left hand side of our main inequality (IV.42), and in the right hand side we consider all the estimates of the lower order terms namely,  $J_1$  from (IV.54),  $J_{21}^1$  from (IV.57),  $J_{31}$  from (IV.58),  $J_{41}$  from (IV.60),  $J_{42}$  from (IV.61) and  $J_6$  from (IV.63), so that the inequality (IV.42) follows

$$\begin{aligned} & s^3 \lambda^4 \sum_{i=1}^2 \int_0^T \int_0^1 \varphi_i^3 |\psi_i|^2 dx dt + s \lambda^2 \sum_{i=1}^2 \int_0^T \int_0^1 \varphi_i |\partial_x \psi_i|^2 dx dt \\ & + K_1 s^3 \lambda^3 \int_0^T \varphi_1^3(t, 1) |\psi_1(t, 1)|^2 dt + K_2 s \lambda \int_0^T \varphi_2(t, 1) |\gamma_2(1) \partial_x \psi_2(t, 1)|^2 dt \\ & \leq \tilde{C} \left[ \sum_{i=1}^2 \int_0^T \int_0^1 e^{-2s\eta_i} |f_i|^2 dx dt + s \lambda \int_0^T \varphi_2(t, 0) e^{-2s\eta_2(t, 0)} |\partial_x q_2(t, 0)|^2 dt \right] \\ & + \tilde{C} \epsilon s^3 \lambda^3 \int_0^T \varphi_1^3(t, 1) |\psi_1(t, 1)|^2 dt + 2\tilde{C} \epsilon s \lambda \int_0^T \varphi_2(t, 1) |\gamma_2(1) \partial_x \psi_2(t, 1)|^2 dt \\ & + \tilde{C} \epsilon (s \lambda T^3 + s \lambda \alpha^2 T^4 + s \lambda^3 T^4 + s^2 \lambda T) \int_0^T \varphi_1^3(t, 1) |\psi_1(t, 1)|^2 dt. \quad (\text{IV.66}) \end{aligned}$$

By choosing  $\epsilon > 0$  to be small enough and taking  $\lambda \geq \lambda_1 := \lambda_1(\gamma_1, \gamma_2, \alpha)$  and  $s \geq s_1 := (T^2 + T)\sigma_1(\gamma_1, \gamma_2, \alpha) > 0$ , where  $\lambda_1$  and  $\sigma_1$  are large enough so that all the boundary integrals, except the observation term, in right hand side can be absorbed by the corresponding integrals in left hand side of (IV.66), which leads us

$$\begin{aligned} & s^3 \lambda^4 \sum_{i=1}^2 \int_0^T \int_0^1 \varphi_i^3 |\psi_i|^2 dx dt + s \lambda^2 \sum_{i=1}^2 \int_0^T \int_0^1 \varphi_i |\partial_x \psi_i|^2 dx dt \\ & + s^3 \lambda^3 \alpha \int_0^T \varphi_1(t, 1) |\psi_1(t, 1)|^2 dt \leq C' \left[ \sum_{i=1}^2 \int_0^T \int_0^1 e^{-2s\eta_i} |f_i|^2 dx dt \right. \\ & \left. + s \lambda \int_0^T \varphi_2(t, 0) e^{-2s\eta_2(t, 0)} |\partial_x q_2(t, 0)|^2 dt \right] \quad (\text{IV.67}) \end{aligned}$$

for all  $\lambda \geq \lambda_1$ ,  $s \geq s_1$  with some constant  $C' := C'(\gamma_1, \gamma_2, \alpha)$ . Now, we recall the expression of  $\partial_x \psi_i$  from (IV.38), so that we have

$$e^{-2s\eta_i} |\partial_x q_i|^2 \leq 2|\partial_x \psi_i|^2 + 2s^2 \lambda^2 (c_i)^2 \varphi_i^2 |\psi_i|^2, \quad \text{for } i = 1, 2,$$

and that implies

$$\begin{aligned} s\lambda^2 \sum_{i=1}^2 \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i |\partial_x q_i|^2 dx dt \\ \leq \tilde{C} s \lambda^2 \sum_{i=1}^2 \int_0^T \int_0^1 \varphi_i |\partial_x \psi_i|^2 dx dt + \tilde{C} s^3 \lambda^4 \sum_{i=1}^2 \int_0^T \int_0^1 \varphi_i^3 |\psi_i|^2 dx dt. \end{aligned} \quad (\text{IV.68})$$

Finally, combining (IV.67) and (IV.68), and replacing  $f_i = \partial_t q_i + \partial_x(\gamma_i \partial_x q_i)$ ,  $i = 1, 2$ , we obtain the required Carleman inequality (IV.36). □

### IV.3.2 Null-controllability in terms of a boundary observability inequality

The Carleman estimate indeed leads us to obtain the following observability inequality which is in fact a necessary and sufficient condition for null-controllability.

**Proposition IV.3.4** (observability inequality). *For any  $\zeta := (\zeta_1, \zeta_2) \in \mathcal{H}_\alpha$ , the associated solution  $q := (q_1, q_2) \in C^0([0, T]; \mathcal{H}_\alpha) \cap L^2(0, T; (H^2(0, 1))^2)$  to (IV.29) satisfies the following observation estimate*

$$\|q(0)\|_{\mathcal{H}_\alpha}^2 \leq C e^{C/T} \int_0^T |\partial_x q_2(t, 0)|^2 dt,$$

for some constant  $C := C(\gamma_1, \gamma_2, \alpha, a) > 0$ , independent on  $T > 0$  and  $\zeta$ .

*Proof.* We shall prove the required observability inequality for  $0 < T \leq 1$  to show the existence of a control in  $(0, T)$  for the system (IV.7)–(IV.8a); this will not loose the generality since for any time  $\tilde{T} > 1$ , a continuation of a control in  $(0, 1)$  by 0 in  $(1, \tilde{T})$  will do the job. Let us now focus on the proof.

For any given  $\zeta \in \mathcal{H}_\alpha$ , one can apply the Carleman inequality given by Theorem IV.3.2 to the solution  $q$  of (IV.29), with  $\partial_t q_1 + \partial_x(\gamma_1 \partial_x q_1) = a q_2$  and  $\partial_t q_2 + \partial_x(\gamma_2 \partial_x q_2) = 0$  to deduce

$$\begin{aligned} s^3 \lambda^4 \sum_{i=1}^2 \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i^3 |q_i|^2 dx dt + s\lambda^2 \sum_{i=1}^2 \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i |\partial_x q_i|^2 dx dt \\ + s^3 \lambda^3 \int_0^T \varphi_1(t, 1) e^{-2s\eta_1(t, 1)} |q_1(t, 1)|^2 dt \leq C' \left[ \int_0^T \int_0^1 e^{-2s\eta_1} |a q_2|^2 dx dt \right. \\ \left. + s\lambda \int_0^T \varphi_2(t, 0) e^{-2s\eta_2(t, 0)} |\partial_x q_2(t, 0)|^2 dt \right]. \end{aligned} \quad (\text{IV.69})$$

Now, we use  $1 \leq 8T^6 \varphi_2^3$  to see the first term in right hand side of the above estimate as

$$\begin{aligned} \int_0^T \int_0^1 e^{-2s\eta_1} |a q_2|^2 dx dt &\leq 8a^2 T^6 \int_0^T \int_0^1 \varphi_2^3 e^{-2s\eta_1} |q_2|^2 dx dt \\ &\leq 8a^2 T^6 \int_0^T \int_0^1 \varphi_2^3 e^{-2s\eta_2} |q_2|^2 dx dt =: \tilde{X}, \end{aligned} \quad (\text{IV.70})$$

since  $\beta_2 \geq \beta_1$  and so  $\eta_2 \leq \eta_1$  by construction (see (IV.33)) which implies  $e^{-2s\eta_1} \leq e^{-2s\eta_2}$  for any  $s > 0$ .

We see that the term  $\tilde{X}$  can be absorbed by the term  $s^3\lambda^4 \int_0^T \int_0^1 e^{-2s\eta_2} \varphi_2^3 |q_2|^2 dx dt$  in left hand side of the estimate (IV.69) for any  $s \geq s_1 = (T^2 + T)\sigma_1$ , possibly with some different  $\sigma_1 > 0$ , and also, using the fact that  $s^3\lambda^3 \geq s\lambda^2\alpha$ , for any  $\lambda \geq \lambda_1$  (may be with some larger  $\lambda_1 := \lambda_1(\gamma_1, \gamma_2, \alpha)$ ), we obtain

$$\begin{aligned} s\lambda^2 \sum_{i=1}^2 \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i |\partial_x q_i|^2 dx dt + s\lambda^2 \alpha \int_0^T \varphi_1(t, 1) e^{-2s\eta_1(t, 1)} |q_1(t, 1)|^2 dt \\ \leq Cs\lambda \int_0^T \varphi_2(t, 0) e^{-2s\eta_2(t, 0)} |\partial_x q_2(t, 0)|^2 dt, \end{aligned} \quad (\text{IV.71})$$

with some constant  $C > 0$  that now depends on  $\gamma_1, \gamma_2, \alpha$  and  $a$ .

Let us now restrict the integrals in left hand side in  $(T/4, 3T/4) \times (0, 1)$ . We observe that for any  $x \in [0, 1]$ , the minimum of the functions  $\varphi_i(t, x) e^{-2s\eta_i(t, x)}$  exists at  $t = T/4$ ,  $i = 1, 2$ , and the maximum of the function  $\varphi_2(t, 0) e^{-2s\eta_2(t, 0)}$  exists at  $t = T/2$ ; we see

$$\begin{aligned} \varphi_i e^{-2s\eta_i} &\geq \frac{16}{3T^2} e^{3\lambda \min_{[0,1]} \beta_i} e^{-(32s/3T^2)(e^{\lambda K} - e^{\lambda \min_{[0,1]} \beta_i})} \text{ in } (T/4, 3T/4) \times (0, 1), \\ \varphi_2(t, 0) e^{-2s\eta_2(t, 0)} &\leq \frac{4}{T^2} e^{\lambda \|\beta_2\|_\infty} e^{-(8s/T^2)(e^{\lambda K} - e^{\lambda \|\beta_2\|_\infty})} \text{ in } (0, T) \times (0, 1). \end{aligned}$$

Implementing this in (IV.71) and by fixing  $\lambda = \lambda_1$ , we deduce that

$$\int_{T/4}^{3T/4} (\|\partial_x q_1(t)\|_{L^2}^2 + \|\partial_x q_2(t)\|_{L^2}^2 + \alpha |q_1(t, 1)|^2) dt \leq C e^{Cs/T^2} \int_0^T |\partial_x q_2(t, 0)|^2 dt,$$

that implies

$$\int_{T/4}^{3T/4} \|q(t)\|_{\mathcal{H}_\alpha}^2 dt \leq C e^{Cs/T^2} \int_0^T |\partial_x q_2(t, 0)|^2 dt. \quad (\text{IV.72})$$

Now, thanks to the point 2 of Theorem 2 (which is also valid for the adjoint system (IV.29) with source term  $f = 0$ ), we have  $\|q(0)\|_{\mathcal{H}_\alpha}^2 \leq C_a \|q(t)\|_{\mathcal{H}_\alpha}^2$  for any  $0 < t \leq T(\leq 1)$ . Implementing this and choosing  $s = (T^2 + T)\sigma > 0$  the inequality (IV.72) reduces to

$$\|q(0)\|_{\mathcal{H}_\alpha}^2 \leq C e^{C\sigma(T^2+T)/T^2} \int_0^T |\partial_x q_2(t, 0)|^2 dt,$$

which gives the required inequality in the proposition with the constant  $C > 0$ , independent on  $T$  and  $\zeta$ .  $\square$

*Proof of Theorem IV.2.7 (Null-controllability).* Once we have the above observability estimate, then by some standard duality argument, see for instance [60], one can prove the existence of a boundary null-control  $v \in L^2(0, T)$  for the problem (IV.7)–(IV.8a), and the estimate of the control cost  $C e^{C/T}$  follows from the precise constant in Proposition IV.3.4.  $\square$

**Remark IV.3.5.** For the other system, that is for (IV.7)–(IV.8b), the observation term is  $\mathcal{B}_2^* q(t) = \gamma_1(0) \partial_x q_1(t, 0)$ , and so to obtain a good Carleman estimate one has to choose the functions  $\beta_i$ ,  $i = 1, 2$  with  $c_1 = \frac{-6}{(1-\mu_0)}$  and  $c_2 = 1$  (which is opposite to the previous case), to construct the suitable weight functions.

In that case, a Carleman estimate similar to (IV.36) still holds with the observation integral

$$s\lambda \int_0^T \varphi_1(t, 0) e^{-2s\eta_1(t, 0)} |\partial_x q_1(t, 0)|^2 dt$$

but we cannot hope for a good observability inequality with observation term  $\partial_x q_1(t, 0)$ . The reason behind this is the following: in this case, we have  $e^{-2s\eta_1} \geq e^{-2s\eta_2}$  (since  $\beta_1 \geq \beta_2$  now and consequently  $\eta_1 \leq \eta_2$ ), which will prevent us from absorbing the source term  $\int_0^T \int_0^1 e^{-2s\eta_1} |a q_2|^2 dx dt$  (placed in the right-hand side) by the term  $s^3\lambda^4 \int_0^T \int_0^1 e^{-2s\eta_2} \varphi_2^3 |q_2|^2 dx dt$  in left-hand side.

The above remark tells us that the Carleman trick is not applicable to prove the boundary null-controllability of our system (IV.7)–(IV.8b).

In fact, it is not a technical since, as we previously mentioned, the controllability property of the system will depend on the values of the coupling coefficient  $a$  and of the boundary parameter  $\alpha$ . This will be investigated in the next section.

## IV.4 Boundary controllability of the $2 \times 2$ system with control on the first component

This section is devoted to find a control for the prescribed problem (IV.7)–(IV.8b) with  $\gamma_1 = \gamma_2 = 1$ , using the moments technique; and as we know, the key point to develop and solve the moments problem is to obtain sharp estimates on spectral elements of the adjoint to the corresponding elliptic operator.

### IV.4.1 Description of the spectrum of the underlying elliptic operator

In this section, we investigate some important spectral properties of the elliptic operator  $\mathcal{A}_{\alpha,a}^*$  having the formal expression in (IV.24) with  $\gamma_1 = \gamma_2 = 1$ .

**Remark IV.4.1.** For  $\gamma_1 = \gamma_2 = 1$  also, we keep the same symbol  $\mathcal{A}_{\alpha,a}$  and  $\mathcal{A}_{\alpha,a}^*$  (for any  $a \in \mathbb{R}, \alpha \geq 0$ ) to denote the corresponding elliptic operator and its adjoint respectively.

Below, we present the eigenvalue problem  $\mathcal{A}_{\alpha,a}^* u = \lambda u$ , for  $\lambda \in \mathbb{C}$ , that is explicitly

$$\begin{cases} -u_1'' + au_2 = \lambda u_1 & \text{in } (0, 1), \\ -u_2'' = \lambda u_2 & \text{in } (0, 1), \\ u_1(0) = 0, \ u_2(0) = 0, \\ u_1(1) = u_2(1), \\ u_1'(1) + u_2'(1) + \alpha u_1(1) = 0. \end{cases} \quad (\text{IV.73})$$

We recall from (IV.24) that for  $a \neq 0$ , the operator  $\mathcal{A}_{\alpha,a}^*$  is no more self-adjoint and here we develop the spectral analysis of this operator (more precisely of its complexified version) using some perturbation argument of linear operators which we discuss in Section IV.4.1.2. That's the reason why, we first need to describe the spectrum of the self-adjoint operator  $\mathcal{A}_\alpha^*$ , which we discuss in the subsequent section.

#### IV.4.1.1 Spectrum of the self-adjoint operator $\mathcal{A}_\alpha^*$

We directly start with the eigenvalue problem  $\mathcal{A}_\alpha u = \mathcal{A}_\alpha^* u = \lambda u$ ,  $u \neq 0$ , where one may assume that  $\lambda$  is real since  $\mathcal{A}_\alpha$  is self-adjoint,

$$\begin{cases} -u_1'' = \lambda u_1 & \text{in } (0, 1), \\ -u_2'' = \lambda u_2 & \text{in } (0, 1), \\ u_1(0) = 0, \ u_2(0) = 0, \\ u_1(1) = u_2(1), \\ u_1'(1) + u_2'(1) + \alpha u_1(1) = 0. \end{cases} \quad (\text{IV.74})$$

Observe first that we necessarily have  $\lambda > 0$ . Indeed, multiplying the first and second equations by  $u_1$  and  $u_2$  respectively, then upon an integration by parts and using the boundary conditions, one has

$$\int_0^1 (|u_1'(x)|^2 + |u_2'(x)|^2) dx + \alpha |u_1(1)|^2 = \lambda \int_0^1 (|u_1(x)|^2 + |u_2(x)|^2) dx,$$



which certainly tells us that  $\lambda > 0$  since  $\alpha \geq 0$ . We shall set  $\mu = \sqrt{\lambda}$ .

Let us now solve (IV.74). We start by observing that if  $u_1 = 0$  then the equation for  $u_2$  along with the boundary conditions gives  $u_2 = 0$ . Therefore, by using the boundary condition at  $x = 0$ , we expect the solution to be of the form

$$u_1(x) = C_1 \sin(\mu x), \quad u_2(x) = C_2 \sin(\mu x), \quad \forall x \in [0, 1],$$

for some  $C_1, C_2 \in \mathbb{R}$ . On the other hand, the conditions  $u_1(1) = u_2(1)$  and  $u_1'(1) + u_2'(1) + \alpha u_1(1) = 0$  respectively provides

$$(C_1 - C_2) \sin \mu = 0 \quad \text{and} \quad (\text{IV.75a})$$

$$\mu C_1 \cos \mu + \mu C_2 \cos \mu + \alpha C_1 \sin \mu = 0. \quad (\text{IV.75b})$$

- First, when  $\sin \mu \neq 0$ , then  $C_1 = C_2 \neq 0$  from (IV.75a), so that from (IV.75b) we end up with

$$2\mu \cos \mu + \alpha \sin \mu = 0. \quad (\text{IV.76})$$

- If  $\alpha = 0$ , then  $\mu_{k,1}^0 := (k+1/2)\pi$  for  $k \geq 0$  are the positive roots of the above equation. A first family of eigenvalues of (IV.74) is thus given by  $\lambda_{k,1}^0 := (k+1/2)^2\pi^2$ ,  $k \geq 0$ .
- If  $\alpha > 0$ , we rewrite the equation (IV.76) as

$$g(\mu) := \tan \mu + \frac{2}{\alpha} \mu = 0.$$

We calculate that  $g'(\mu) = \sec^2 \mu + 2/\alpha > 0$ , and so in particular,  $g'(\mu) > 0$  in  $((k+1/2)\pi, (k+3/2)\pi)$ , for any  $k \geq 0$ . Beside this, we have

$$\lim_{\mu \rightarrow ((k+1/2)\pi)^+} g(\mu) = -\infty, \quad g((k+1)\pi) = \frac{2}{\alpha}(k+1)\pi > 0.$$

So, there exists exactly one root of  $g$  in  $((k+1/2)\pi, (k+1)\pi)$ , for each  $k \geq 0$ , and given  $\alpha > 0$ . Let us denote the roots of  $g$  by  $\mu_{k,1}^\alpha$  and the eigenvalues by  $\lambda_{k,1}^\alpha := (\mu_{k,1}^\alpha)^2$ , for all  $k \geq 0$  and given any  $\alpha > 0$ .

Note that an associated set of normalized eigenfunctions is given by

$$\Phi_{\lambda_{k,1}^\alpha}(x) := \begin{pmatrix} \sin(\mu_{k,1}^\alpha x) \\ \sin(\mu_{k,1}^\alpha x) \end{pmatrix}. \quad (\text{IV.77})$$

- Assume now that  $\sin \mu = 0$  (see (IV.75a)), from which we deduce that  $\mu = (k+1)\pi$  for some  $k \geq 0$ . By (IV.75b) we have  $C_1 = -C_2$ . Now, to be consistent with the notation, we shall denote this second set of eigenvalue-eigenfunction pairs by  $\{\lambda_{k,2}^\alpha, \Phi_{\lambda_{k,2}^\alpha}\}_{k \geq 0}$ , even if they are not depending on  $\alpha$ , with  $\lambda_{k,2}^\alpha = (k+1)^2\pi^2$  and

$$\Phi_{\lambda_{k,2}^\alpha}(x) := \begin{pmatrix} \sin(k+1)\pi x \\ -\sin(k+1)\pi x \end{pmatrix}. \quad (\text{IV.78})$$

**Remark IV.4.2.** Since the operator  $\mathcal{A}_\alpha^*$  is self-adjoint, the family  $\{\Phi_{\lambda_{k,1}^\alpha}, \Phi_{\lambda_{k,2}^\alpha}\}_{k \geq 0}$  indeed forms an orthonormal basis of  $E$ .

**Remark IV.4.3.** For any  $\alpha > 0$ , we deduce the following asymptotic formula of  $\lambda_{k,1}^\alpha$ ,

$$\lambda_{k,1}^\alpha = (k + \frac{1}{2})^2\pi^2 + O_\alpha(1), \quad \text{for } k \text{ large,}$$

To obtain the above asymptotic, we express  $\mu_{k,1}^\alpha = (k + 1/2)\pi + \delta_{k,1}^\alpha$  with  $\delta_{k,1}^\alpha \in (0, \pi/2)$  for  $\alpha > 0$ . Then we see from  $g(\mu_{k,1}^\alpha) = 0$  that

$$\begin{aligned} \tan((k + 1/2)\pi + \delta_{k,1}^\alpha) + \frac{2}{\alpha}((k + 1/2)\pi + \delta_{k,1}^\alpha) &= 0, \\ \text{i.e., } \frac{\cos \delta_{k,1}^\alpha}{\sin \delta_{k,1}^\alpha} &= \frac{2}{\alpha}((k + 1/2)\pi + \delta_{k,1}^\alpha), \end{aligned} \quad (\text{IV.79})$$

of which, the right hand side goes to  $+\infty$  as  $k \rightarrow +\infty$ , and so, for any fixed  $\alpha > 0$ ,  $\delta_{k,1}^\alpha \rightarrow 0^+$ . Consequently,  $\sin \delta_{k,1}^\alpha \sim \delta_{k,1}^\alpha$  and  $\cos \delta_{k,1}^\alpha \sim 1$  for large  $k$ . So, by taking into account (IV.79), we have

$$\delta_{k,1}^\alpha \sim \frac{\alpha}{2(k + 1/2)\pi}, \quad \text{for } k \text{ large.}$$

Henceforth, we deduce for  $k$  large enough that  $\mu_{k,1}^\alpha = (k + \frac{1}{2})\pi + O_\alpha\left(\frac{1}{k}\right)$ ; taking square of which, we obtain the required asymptotic formula of  $\lambda_{k,1}^\alpha$  for any  $\alpha > 0$ .

In particular, for  $\alpha = 0$  we have  $\lambda_{k,1}^0 = (k + \frac{1}{2})^2\pi^2$ ,  $\forall k \geq 0$ .

#### IV.4.1.2 Spectrum of the main operator $\mathcal{A}_{\alpha,a}^*$

We begin with our main problem of interest, that is the system of odes (IV.73). For our use, we first denote the set of all eigenvalues of  $\mathcal{A}_{\alpha,a}^*$  by  $\Lambda_{\alpha,a}$  for any  $a \in \mathbb{R}$  and  $\alpha \geq 0$ .

Let us choose  $a \in \mathbb{R}^*$  and  $\alpha \geq 0$  and we pursue some detailed analysis step by step as follows.

**Localization of the spectrum.** Then, we actually observe that

$$\Lambda_{\alpha,a} \subset \bigcup_{\lambda \in \Lambda_{\alpha,0}} D(\lambda, 2|a|), \quad (\text{IV.80})$$

where  $\Lambda_{\alpha,0}$  is the set of all eigenvalues of the self-adjoint operator  $\mathcal{A}_\alpha = \mathcal{A}_{\alpha,0}$ .

Indeed, if  $\xi \in \mathbb{C}$  is such that  $|\xi - \lambda| \geq 2|a|$  for any  $\lambda \in \Lambda_{\alpha,0}$ , then in particular  $\mathcal{A}_{\alpha,0}^* - \xi I$  is invertible and satisfies the resolvent estimate

$$\|(\mathcal{A}_{\alpha,0}^* - \xi I)^{-1}\| = \sup_{\lambda \in \Lambda_{\alpha,0}} \frac{1}{|\xi - \lambda|} \leq \frac{1}{2|a|}.$$

It follows that

$$\mathcal{A}_{\alpha,a}^* - \xi I = \mathcal{A}_{\alpha,0}^* - \xi I + \mathcal{M}_a^* = (\mathcal{A}_{\alpha,0}^* - \xi I) \left( I - (\mathcal{A}_{\alpha,0}^* - \xi I)^{-1} \mathcal{M}_a^* \right),$$

and thus  $\xi$  lies in the resolvent set of  $\mathcal{A}_{\alpha,a}^*$  since

$$\|(\mathcal{A}_{\alpha,0}^* - \xi I)^{-1} \mathcal{M}_a^*\| \leq \|(\mathcal{A}_{\alpha,0}^* - \xi I)^{-1}\| \|\mathcal{M}_a^*\| \leq \frac{1}{2|a|} |a| < 1.$$

In particular,  $\mathcal{A}_{\alpha,a}^*$  has compact resolvent since the self-adjoint operator  $\mathcal{A}_{\alpha,0}$  has so, which ensures that the spectrum of  $\mathcal{A}_{\alpha,a}$  is discrete.

**Multiplicity.** Assume that  $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$  and  $\tilde{U} := \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}$  be two linearly independent solutions to (IV.73) for some  $\lambda \in \mathbb{C}$ . Then for any two constants  $\theta, \tilde{\theta} \in \mathbb{C}$ ,

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \theta U + \tilde{\theta} \tilde{U} = \begin{pmatrix} \theta u_1 + \tilde{\theta} \tilde{u}_1 \\ \theta u_2 + \tilde{\theta} \tilde{u}_2 \end{pmatrix}$$

be also a solution to (IV.73) for that  $\lambda$ . Now, we assume  $\theta, \tilde{\theta} \in \mathbb{C} \setminus \{0\}$  in such a way that

$$v_1(1) = \theta u_1(1) + \tilde{\theta} \tilde{u}_1(1) = 0, \quad (\text{IV.81})$$

consequently,  $v_2(1) = \theta u_2(1) + \tilde{\theta} \tilde{u}_2(1) = 0$ . So, the equation concerning  $v_2$  is a second order ode with homogeneous Dirichlet boundary conditions which has real solutions in the form  $c \sin(k+1)\pi x$ ,  $\forall x \in [0, 1]$ , with real  $\lambda = (k+1)^2\pi^2$ , for  $k \geq 0$ .

Now, we multiply the differential equation of  $v_1$  by  $v_2$  and integrate, i.e.,

$$-\int_0^1 v_1''(x)v_2(x) dx + a \int_0^1 |v_2(x)|^2 dx = \lambda \int_0^1 v_1(x)v_2(x) dx.$$

Performing integration by parts and  $v_i(0) = v_i(1) = 0$  (for  $i = 1, 2$ ), we get

$$-\int_0^1 v_2''(x)v_1(x) dx + a \int_0^1 |v_2(x)|^2 dx = \lambda \int_0^1 v_1(x)v_2(x) dx.$$

Now, since  $-v_2'' = \lambda v_2$  in  $(0, 1)$ , we deduce from the above equality that  $v_2 = 0$  in  $[0, 1]$ . This leads the equation of  $v_1$  as follows:

$$\begin{cases} -v_1''(x) = \lambda v_1 & \text{in } (0, 1), \\ v_1(0) = 0, \quad v_1(1) = 0, \\ v_1'(1) = 0. \end{cases}$$

The above differential equation along with homogeneous Dirichlet conditions gives  $v_1(x) = c \sin(k+1)\pi x$ ,  $\forall x \in [0, 1]$ , with  $\lambda = (k+1)^2\pi^2$ , for  $k \geq 0$ . But  $v_1'(1) = 0$  forces the constant  $c$  to be 0 and consequently  $v_1 = 0$  in  $[0, 1]$ . So, we get two non-zero  $\theta, \tilde{\theta}$  for which  $\theta U + \tilde{\theta} \tilde{U} = 0$ ; a contradiction to our assumption of linear independence of  $U$  and  $\tilde{U}$ .

This proves that each eigenvalue has geometric multiplicity 1.

**The case  $\lambda = 0$ .** We observe that  $\lambda = 0$  is an eigenvalue if and only if  $a + 3\alpha + 6 = 0$ . Take  $\lambda = 0$  in (IV.73) and then solving the set of odes along with the homogeneous boundary condition at  $x = 0$ , one obtain  $u_2(x) = c_1 x$  and  $u_1(x) = c_1 \frac{ax^3}{6} + c_2 x$ . Now, thanks to the Kirchhoff boundary condition at  $x = 1$ , we obtain  $c_2 = c_1(1 - \frac{a}{6})$  and

$$c_1(a + 3\alpha + 6) = 0,$$

which shows that  $c_1 = 0$  (consequently,  $c_2 = 0$ ) provided  $a + 3\alpha + 6 \neq 0$ ; in that case,  $\lambda = 0$  is not an eigenvalue of  $\mathcal{A}_{\alpha, a}^*$ .

But, as soon as we have  $a + 3\alpha + 6 = 0$  (then fix  $c_1 = 1$ ), we see that  $\lambda = 0$  is an eigenvalue with the eigenfunction

$$\Phi_0(x) = \begin{pmatrix} \frac{ax^3}{6} + (1 - \frac{a}{6})x \\ x \end{pmatrix}, \quad \forall x \in [0, 1]. \quad (\text{IV.82})$$

**The case  $\lambda \neq 0$ .** As we have seen above we cannot have  $u_2 = 0$  in  $(0, 1)$ . We take  $\mu \in \mathbb{C}$  such that  $\mu^2 = \lambda$  and we observe that the solution of (IV.73) is necessarily of the form

$$\begin{cases} u_1(x) = \frac{aK_1x}{2i\mu}(e^{i\mu x} + e^{-i\mu x}) + K_2(e^{i\mu x} - e^{-i\mu x}), \\ u_2(x) = K_1(e^{i\mu x} - e^{-i\mu x}), \end{cases} \quad (\text{IV.83})$$

for some  $K_1, K_2 \in \mathbb{C}$ . Thereafter, the two boundary conditions at  $x = 1$  provides us the following two equations

$$\begin{cases} K_1(ac_\mu - 2i\mu s_\mu) + K_2(2i\mu s_\mu) = 0 \text{ and} \\ K_1(-2\mu^2 c_\mu + ai\mu s_\mu + ac_\mu + 2i\alpha\mu s_\mu) + K_2(-2\mu^2 c_\mu) = 0, \end{cases} \quad (\text{IV.84})$$

where we introduced  $s_\mu := (e^{i\mu} - e^{-i\mu})$  and  $c_\mu := (e^{i\mu} + e^{-i\mu})$ . Now, for the existence of non-zero solution  $(K_1, K_2)$  of the above system, the following condition should necessarily be satisfied:

$$8\mu^2 i(s_\mu c_\mu) + 2\mu a(s_\mu^2 - c_\mu^2) - 2ai(s_\mu c_\mu) + 4\alpha\mu s_\mu^2 = 0,$$

which is actually the determinant of the coefficient matrix of system (IV.84). Since we have assumed that  $\mu \neq 0$ , the condition above implies that  $s_\mu \neq 0$ .

Using the relations  $s_\mu c_\mu = 2i \sin 2\mu$ ,  $s_\mu^2 = -4 \sin^2 \mu$  and  $s_\mu^2 - c_\mu^2 = -4$ , the above equation simplifies as

$$(4\mu^2 - a) \sin 2\mu + 2a\mu + 4\alpha\mu \sin^2 \mu = 0. \quad (\text{IV.85})$$

Now, from the first equation of (IV.84), we have  $K_2 = K_1 \left(1 - \frac{ac_\mu}{2i\mu s_\mu}\right) = K_1 \left(1 + \frac{a}{2\mu} \frac{\cos \mu}{\sin \mu}\right)$ . Next we fix  $K_1 = 1/2i$  and deduce from (IV.83) that the eigenfunction associated with  $\lambda = \mu^2$  is given by

$$\Phi_\lambda(x) := \begin{pmatrix} -\frac{ax}{2\mu} \cos(\mu x) + \left(1 + \frac{a}{2\mu} \frac{\cos \mu}{\sin \mu}\right) \sin(\mu x) \\ \sin(\mu x) \end{pmatrix}, \quad (\text{IV.86})$$

as soon as  $\mu \in \mathbb{C}$  satisfies (IV.85).

**Real solutions of the transcendental equation (IV.85).** We set

$$f(\mu) := (4\mu^2 - a) \sin 2\mu + 2a\mu + 4\alpha\mu \sin^2 \mu.$$

Our goal is to prove the following lemma.

**Lemma IV.4.4.** *Let  $a \in \mathbb{R}^*$  and  $\alpha \geq 0$ . There exists some  $k_{\alpha,a} \in \mathbb{N} \cup \{0\}$  such that for each  $k \geq k_{\alpha,a}$ , the function  $f$  has:*

- one real root, denoted by  $\mu_{k,1}^{\alpha,a}$ , in the interval

$$((k + 1/4)\pi, (k + 3/4)\pi),$$

and that satisfies

$$\mu_{k,1}^{\alpha,a} = (k + 1/2)\pi + \frac{2\alpha + a}{4k\pi} + o_{\alpha,a}(1/k),$$

- one real root, denoted by  $\mu_{k,2}^{\alpha,a}$ , in the interval

$$((k + 3/4)\pi, (k + 5/4)\pi),$$

and that satisfies

$$\mu_{k,2}^{\alpha,a} = (k + 1)\pi - \frac{a}{4k\pi} + o_{\alpha,a}(1/k).$$

*Proof.* • Let  $\epsilon = \mp \pi/4$ . A straightforward computation gives

$$f((k + 1/2)\pi + \epsilon) \sim -4 \sin(2\epsilon) k^2 \pi^2, \quad \text{for large } k.$$

Hence, for  $k$  large enough,  $f((k + 1/4)\pi)$  and  $f((k + 3/4)\pi)$  have different signs, which proves the existence of a root  $\mu_{k,1}^{\alpha,a} \in ((k + 1/4)\pi, (k + 3/4)\pi)$ .

Let  $\delta_k := \mu_{k,1}^{\alpha,a} - (k + 1/2)\pi \in (-\pi/4, \pi/4)$ . The equation  $f(\mu_{k,1}^{\alpha,a}) = 0$  gives

$$\begin{aligned} - (4((k + 1/2)\pi + \delta_k)^2 - a) \sin(2\delta_k) + 2a((k + 1/2)\pi + \delta_k) \\ + 4\alpha((k + 1/2)\pi + \delta_k) \cos^2 \delta_k = 0, \end{aligned} \quad (\text{IV.87})$$

and in particular we get for some  $C_{\alpha,a} > 0$ ,

$$|\sin(2\delta_k)| \leq \frac{C_{\alpha,a}}{k},$$

which implies that  $\delta_k \rightarrow 0$  as  $k \rightarrow +\infty$  and coming back to (IV.87), one can deduce that

$$\delta_k = \frac{(2\alpha + a)}{4k\pi} + O_{\alpha,a}(1/k^3).$$

- Similarly, we have

$$f((k+1)\pi + \epsilon) \sim 4 \sin(2\epsilon) k^2 \pi^2, \quad \text{for large } k,$$

and by the similar trick as previous we get the existence of a root  $\mu_{k,2}^{\alpha,a} \in ((k+3/4)\pi, (k+5/4)\pi)$ . Setting now  $\delta_k := \mu_{k,2}^{\alpha,a} - (k+1)\pi \in (-\pi/4, \pi/4)$ , the equation  $f(\mu_{k,2}^{\alpha,a}) = 0$  gives

$$\begin{aligned} (4((k+1)\pi + \delta_k)^2 - a) \sin(2\delta_k) + 2a((k+1)\pi + \delta_k) \\ + 4a((k+1)\pi + \delta_k) \sin^2 \delta_k = 0, \end{aligned} \quad (\text{IV.88})$$

again from which we first deduce that  $\delta_k$  tends to 0 (using the similar argument as before) and then

$$\delta_k = -\frac{a}{4k\pi} + O_{\alpha,a}(1/k^3).$$

□

**Corollary IV.4.5.** *For any  $k \geq k_{\alpha,a}$ , the operator  $\mathcal{A}_{\alpha,a}^*$  has two real eigenvalues  $\lambda_{k,1}^{\alpha,a}$  and  $\lambda_{k,2}^{\alpha,a}$  that satisfy*

$$\begin{aligned} \lambda_{k,1}^{\alpha,a} &= (k+1/2)^2 \pi^2 + (\alpha + a/2) + o_{\alpha,a}(1), \\ \lambda_{k,2}^{\alpha,a} &= (k+1)^2 \pi^2 - a/2 + o_{\alpha,a}(1). \end{aligned}$$

Moreover, for each  $k \geq k_{\alpha,a}$  (possibly some larger  $k_{\alpha,a}$  than earlier) and  $i \in \{1, 2\}$ ,  $\lambda_{k,i}^{\alpha,a}$  is the unique eigenvalue of  $\mathcal{A}_{\alpha,a}^*$  in the following disk of the complex plane

$$D(\lambda_{k,i}^{\alpha,0}, 2|a|),$$

where conventionally  $\lambda_{k,i}^{\alpha,0} := \lambda_{k,i}^\alpha$ , the eigenvalues of our self-adjoint operator  $\mathcal{A}_\alpha$ .

*Proof.* The solutions  $\mu_{k,i}^{\alpha,a}$  of the transcendental equation  $f(\mu) = 0$  are the square roots of the eigenvalues of our operator. Thus we can set  $\lambda_{k,i}^{\alpha,a} = (\mu_{k,i}^{\alpha,a})^2$ .

Moreover, for  $k$  large enough, we have for  $i = 1, 2$ , that

$$d(\lambda_{k,i}^{\alpha,0}, \Lambda_\alpha \setminus \{\lambda_{k,i}^{\alpha,0}\}) > 2|a|,$$

so that, we have the resolvent estimate

$$\|(\mathcal{A}_\alpha^* - \xi I)^{-1}\| \leq \frac{1}{2|a|}, \quad \forall \xi \in \partial D(\lambda_{k,i}^{\alpha,0}, 2|a|),$$

and thus

$$\|\mathcal{M}_a^*\| \|(\mathcal{A}_\alpha^* - \xi I)^{-1}\| \leq \frac{1}{2}, \quad \forall \xi \in \partial D(\lambda_{k,i}^{\alpha,0}, 2|a|).$$

Using [69, IV–Theorem 3.18] (see also Section IV.7.3), we have for each  $i = 1, 2$ , that the perturbed operator  $\mathcal{A}_{\alpha,a}^*$  has only one eigenvalue in the disk  $D(\lambda_{k,i}^{\alpha,0}, 2|a|)$ , since the self-adjoint operator  $\mathcal{A}_\alpha^*$  has only one eigenvalue of *multiplicity* 1 in that disk. Therefore, the only eigenvalue of  $\mathcal{A}_{\alpha,a}^*$  inside that disk is  $\lambda_{k,i}^{\alpha,a}$ . □

**Conclusion on the structure of  $\Lambda_{\alpha,a}$ .** Using the fact given by (IV.80) and the Corollary IV.4.5, we deduce that the spectrum of  $\mathcal{A}_{\alpha,a}^*$  can be split into two disjoint parts

$$\Lambda_{\alpha,a} = \Lambda_{\alpha,a}^0 \cup \Lambda_{\alpha,a}^\infty, \quad (\text{IV.89})$$

where  $\Lambda_{\alpha,a}^0$  is finite, with possibly some complex eigenvalues, and satisfy

$$\Lambda_{\alpha,a}^0 \subset \bigcup_{i=1,2} \bigcup_{0 \leq k < k_{\alpha,a}} D(\lambda_{k,i}^{\alpha,0}, 2|a|), \quad (\text{IV.90})$$

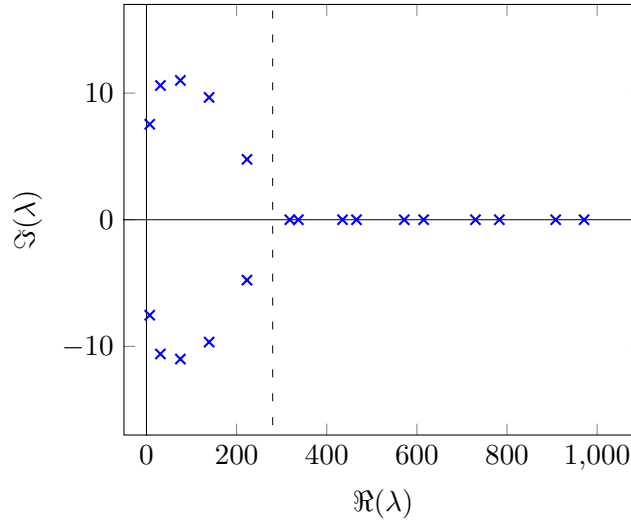


Figure IV.2: A numerical description of a part of the spectrum: for  $a = 30$ ,  $\alpha = 0.1$

and  $\Lambda_{\alpha,a}^\infty \subset (0, +\infty)$  and is defined by

$$\Lambda_{\alpha,a}^\infty := \left\{ \lambda_{k,1}^{\alpha,a}, k \geq k_{\alpha,a} \right\} \cup \left\{ \lambda_{k,2}^{\alpha,a}, k \geq k_{\alpha,a} \right\}. \quad (\text{IV.91})$$

The situation is illustrated in Figure IV.2.

Finally, we can summarize the above analysis as follows.

**Proposition IV.4.6.** *Let  $a \in \mathbb{R}$  and  $\alpha \geq 0$  be any two parameters.*

- *The spectrum of the operator  $\mathcal{A}_{\alpha,a}^*$  is discrete, made only of simple eigenvalues, and has the structure given in (IV.89).*
- *Moreover, the associated family of eigenfunctions  $\{\Phi_\lambda\}_{\lambda \in \Lambda_{\alpha,a}}$  is complete in  $E$  and  $\mathcal{H}_\alpha$ .*

Note that we considered here the complex version of the spaces  $E$  and  $\mathcal{H}_\alpha$ . Everything was proved above, except the completeness property of the eigenfunctions which comes as a consequence of a theorem of Keldysh since the perturbation  $\mathcal{M}_a^*$  is bounded, see for instance [87, Theorem 4.3, Ch.1], [92, Lemma 7].

#### IV.4.2 Observation estimates and bounds on the norms of the eigenfunctions

In this section, we analyze the size of the observation terms  $|\mathcal{B}_2^* \Phi_\lambda|$  for  $\lambda \in \Lambda_{\alpha,a}$  ( $\mathcal{B}_2^*$  is defined by (IV.27b)). If those quantities do not vanish then the approximate controllability of the problem (IV.7)–(IV.8b) will be guaranteed by means of Fattorini-Hautus test (see [55, 89]). Moreover, suitable lower bound for those quantities combined with upper bounds of  $\|\Phi_\lambda\|_{\mathcal{H}_\alpha}$  will let us build and estimate a null-control in  $L^2(0, T)$  via moments technique.

##### IV.4.2.1 Approximate controllability

We prove the following lemma (recall that we have assumed that the diffusion coefficient are  $\gamma_1 = \gamma_2 = 1$ ).

**Lemma IV.4.7.** *Let any  $a \in \mathbb{R}$  and  $\alpha \geq 0$  be given. Then there exists a non-empty set  $\mathcal{R} \subset \mathbb{R}_0^+ \times \mathbb{R}^*$ , such that we have the following properties:*

1. *If  $(\alpha, a) \notin \mathcal{R}$ , the problem (IV.7)–(IV.8b) is approximately controllable at any time  $T > 0$  in  $\mathcal{H}_{-\alpha}$ .*

2. On the other hand, if  $(\alpha, a) \in \mathcal{R}$ , there exists a subspace  $\mathcal{Y}_{\alpha, a} \subset \mathcal{H}_{-\alpha}$  of codimension one, such that the problem (IV.7)–(IV.8b) is approximately controllable at any time  $T > 0$  if and only if the initial data belongs to  $\mathcal{Y}_{\alpha, a}$ .

The set  $\mathcal{R}$  and the spaces  $\mathcal{Y}_{\alpha, a}$  are defined by (IV.100) and (IV.102) respectively inside the proof of this lemma.

*Proof of Lemma IV.4.7.*

We recall that the observation operator  $\mathcal{B}_2^*$  is given in (IV.27b).

- In the simplest case when  $a = 0$ , for any  $\alpha \geq 0$ , one can immediately see that the eigenfunctions in (IV.77)–(IV.78) satisfy

$$\mathcal{B}_2^* \Phi_{\lambda_{k,1}^\alpha} = \sqrt{\lambda_{k,1}^\alpha} \neq 0, \quad \mathcal{B}_2^* \Phi_{\lambda_{k,2}^\alpha} = \sqrt{\lambda_{k,2}^\alpha} = (k+1)\pi \neq 0, \quad \forall k \geq 0. \quad (\text{IV.92})$$

- The case  $\lambda = 0$  can only happen if  $a + 3\alpha + 6 = 0$  (so that in particular  $a < 0$ ) and it follows from (IV.82) that

$$\mathcal{B}_2^* \Phi_0 = 1 - \frac{a}{6} > 0.$$

- Let us assume that  $a \neq 0$  and  $\lambda \neq 0$  be an eigenvalue of  $\mathcal{A}_{\alpha, a}^*$ . The associated eigenfunction  $\Phi_\lambda$  is given in (IV.86).

An immediate computation gives

$$\mathcal{B}_2^* \Phi_\lambda = -\frac{a}{2\mu} + \mu + \frac{a \cos \mu}{2 \sin \mu}. \quad (\text{IV.93})$$

From now on, we suppose that  $\mathcal{B}_2^* \Phi_\lambda = 0$ . Since  $\mu \neq 0$  and  $\sin \mu \neq 0$ , this is equivalent to the relation

$$(2\mu^2 - a) \sin \mu + a\mu \cos \mu = 0. \quad (\text{IV.94})$$

This equation has to be satisfied in addition to the transcendental equation (IV.85). If we suppose that  $\cos \mu = 0$ , then (IV.85) and (IV.94) show that this can occur if and only if we have

$$a + 2\alpha = 0, \text{ and } \mu^2 = -\alpha, \quad (\text{IV.95})$$

this last equation not being compatible with the condition  $\cos \mu = 0$ . Therefore, we assume that  $\cos \mu \neq 0$ .

Now, multiplying (IV.94) by  $\cos \mu$  and using straightforward trigonometry we obtain that the two equations (IV.85) and (IV.94) can be equivalently written as follows

$$\begin{pmatrix} (4\mu^2 - a) & 4\alpha\mu \\ (2\mu^2 - a) & -2a\mu \end{pmatrix} \begin{pmatrix} \sin 2\mu \\ \sin^2 \mu \end{pmatrix} = \begin{pmatrix} -2a\mu \\ -2a\mu \end{pmatrix}. \quad (\text{IV.96})$$

Denote the coefficient matrix in the left hand side of (IV.96) by  $M_\mu \in \mathcal{M}_{2 \times 2}(\mathbb{C})$  and we calculate the determinant:

$$\det M_\mu = 2a\mu(a + 2\alpha) - 8\mu^3(a + \alpha).$$

- Let us prove that  $\det M_\mu \neq 0$  if  $\mu$  satisfies (IV.96).

The claim is clear if  $a + 2\alpha = 0$  or  $a + \alpha = 0$ .

From now on we assume  $a + 2\alpha \neq 0$  and  $a + \alpha \neq 0$ . The determinant of  $M_\mu$  cancels if and only if  $\mu = \pm \sqrt{\frac{a(a+2\alpha)}{4(a+\alpha)}} \in \mathbb{C}$  and, if that happens, the matrix becomes

$$M_\mu = \begin{pmatrix} \frac{a\alpha}{a+\alpha} & \pm 2\alpha \sqrt{\frac{a(a+2\alpha)}{4(a+\alpha)}} \\ -\frac{1}{2} \frac{a^2}{a+\alpha} & \mp a \sqrt{\frac{a(a+2\alpha)}{4(a+\alpha)}} \end{pmatrix},$$

and a straightforward computation shows that

$$\text{Ker } M_\mu^* = \text{Span} \left\{ \begin{pmatrix} a \\ 2\alpha \end{pmatrix} \right\}.$$

Hence, we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \text{Ran } M_\mu \iff \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \perp \begin{pmatrix} a \\ 2\alpha \end{pmatrix}.$$

But it is clear that  $\begin{pmatrix} -2a\mu \\ -2a\mu \end{pmatrix} \notin \begin{pmatrix} a \\ 2\alpha \end{pmatrix}$  since  $(a + 2\alpha) \neq 0$  and so,  $\mu = \pm \sqrt{\frac{a(a+2\alpha)}{4(a+\alpha)}}$  cannot be solutions to (IV.96); in other words,  $\det M_\mu \neq 0$  for any  $\mu$  satisfying (IV.96).

– *Solving (IV.96).*

From the previous point, we know that  $M_\mu$  is invertible, so that we can solve (IV.96) to get

$$\begin{cases} \sin 2\mu = \frac{2a\mu(a+2\alpha)}{a(a+2\alpha) - 4\mu^2(a+\alpha)}, \\ \sin^2 \mu = \frac{-4a\mu^2}{2a(a+2\alpha) - 8\mu^2(a+\alpha)}. \end{cases} \quad (\text{IV.97})$$

Let us recall the standard trigonometric relation  $\sin 2\mu = 2 \sin \mu \cos \mu$ , we see

$$\begin{aligned} \sin^2 2\mu &= 4 \sin^2 \mu (1 - \sin^2 \mu), \\ \iff \frac{16a^2\mu^4(a+2\alpha)^2}{(2a\mu(a+2\alpha) - 8\mu^3(a+\alpha))^2} &= \frac{-16a\mu^3(2a\mu(a+2\alpha) - 8\mu^3(a+\alpha) + 4a\mu^3)}{(2a\mu(a+2\alpha) - 8\mu^3(a+\alpha))^2}, \\ \iff 16a^2\mu^4(a+2\alpha)^2 &= -16a\mu^3(2a\mu(a+2\alpha) - 4\mu^3(a+2\alpha)), \\ &= 16a\mu^4(a+2\alpha)(4\mu^2 - 2a), \\ \iff 4\mu^2 &= a(a+2\alpha+2), \end{aligned}$$

Since the sign of  $\mu$  is unimportant, we conclude that this situation can only occur for the particular value

$$\mu = \mu_{\alpha,a}^c := \frac{1}{2} \sqrt{a^2 + 2a\alpha + 2a}.$$

To summarize, we have finally obtained that if  $\mathcal{B}_2^* \Phi_\lambda = 0$ , then we necessarily have

$$\lambda = \lambda_{\alpha,a}^c := \frac{a(a+2\alpha+2)}{4}. \quad (\text{IV.98})$$

This is a necessary condition and we still have to check whether or not this value of  $\lambda$  (or  $\mu$ ) does satisfy (IV.97), that is to say if  $\alpha$  and  $a$  satisfy



$$\sin(\sqrt{a^2 + 2a\alpha + 2a}) = \frac{-a(a + 2\alpha)\sqrt{a^2 + 2a\alpha + 2a}}{(a + \alpha)(a^2 + 2a\alpha + 2a) - a(a + 2\alpha)}, \quad (\text{IV.99a})$$

$$\sin^2\left(\frac{\sqrt{a^2 + 2a\alpha + 2a}}{2}\right) = \frac{a(a^2 + 2a\alpha + 2a)}{2(a + \alpha)(a^2 + 2a\alpha + 2a) - 2a(a + 2\alpha)}. \quad (\text{IV.99b})$$

This leads us to introduce the critical set  $\mathcal{R}$  as follows

$$\mathcal{R} := \{(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*, \text{ s.t. (IV.99) holds}\}. \quad (\text{IV.100})$$

The set  $\mathcal{R}$  is the set of solutions to the two equations (IV.99). We recall that those two equations were obtained from (IV.97) by eliminating the value of  $\mu$  and therefore, are not independent one from the other. Thus, we observe that any solution (IV.99b) necessarily satisfies

$$\sin(\sqrt{a^2 + 2a\alpha + 2a}) = \varepsilon_{\alpha,a} \frac{a(a + 2\alpha)\sqrt{a^2 + 2a\alpha + 2a}}{(a + \alpha)(a^2 + 2a\alpha + 2a) - a(a + 2\alpha)} \quad (\text{IV.101})$$

for  $\varepsilon_{\alpha,a} \in \{-1, 1\}$ . On any connected component of the set of solutions of (IV.99b), we have either  $\varepsilon_{\alpha,a} = -1$  (in which case (IV.99a) is satisfied) or  $\varepsilon_{\alpha,a} = 1$  (in which case (IV.99a) is not satisfied).

We have plotted in Figure IV.3 the solution curves of (IV.99b) in two colors: in blue the ones for which  $\varepsilon_{\alpha,a} = -1$  and in red the ones for which  $\varepsilon_{\alpha,a} = 1$ . The set  $\mathcal{R}$  is thus the union of the blue curves. The blue dot corresponds to the particular pair  $(\alpha, a) = (1, 3.1931469)$  that is used in the numerical results of Section IV.5.3.

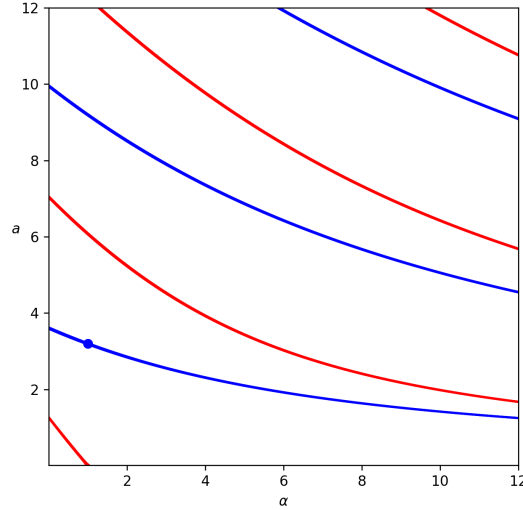


Figure IV.3: In blue: the set  $\mathcal{R}$  of critical pairs  $(\alpha, a)$ . In red: The solutions to (IV.99b) that are not solution of (IV.99a).

- To sum up the previous analysis, we have identified the set  $\mathcal{R}$  of parameters  $(\alpha, a)$  for which there exists a single critical eigenvalue  $\lambda_{\alpha,a}^c$  given by (IV.98) for which the associated eigenfunction is not observable, that is  $\mathcal{B}_2^* \Phi_{\lambda_{\alpha,a}^c} = 0$ .

We can now find out the approximate controllability properties of our problem.

1. For any given pair  $(\alpha, a) \notin \mathcal{R}$  all the eigenfunctions of  $\mathcal{A}_{\alpha,a}^*$  are observable, and henceforth, the Fattorini-Hautus criterion is satisfied (see [55, 89]) which implies the approximate controllability of the system in the space  $\mathcal{H}_{-\alpha}$ .
2. If a given pair  $(\alpha, a)$  belongs to  $\mathcal{R}$ , then the system (IV.7)–(IV.8b) cannot be approximately controllable in the full space  $\mathcal{H}_{-\alpha}$ , since for the particular eigenvalue given in (IV.98), we have  $\mathcal{B}_2^* \Phi_{\lambda_{\alpha,a}^c} = 0$ ; thus the Fattorini-Hautus criterion fails.

However, it is not difficult to observe that if the initial data belongs to the smaller space defined by

$$\mathcal{Y}_{\alpha,a} := \left\{ y_0 \in \mathcal{H}_{-\alpha} \mid \langle y_0, \Phi_{\lambda_{\alpha,a}^c} \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_{\alpha}} = 0 \right\}, \quad (\text{IV.102})$$

then the approximate controllability of the system holds true.

□

#### IV.4.2.2 Estimates on the eigenfunctions

We will gather here the estimates we need on the eigenfunctions, namely a bound from below for the observation terms  $\mathcal{B}_2^* \Phi_{\lambda}$  and a bound from above for the norms  $\|\Phi_{\lambda}\|_{\mathcal{H}_{\alpha}}$ .

**Lemma IV.4.8.** *Let  $a \in \mathbb{R}$  and  $\alpha \geq 0$  be given. Then, there exists some  $C_{\alpha,a}$  such that we have*

$$\|\Phi_{\lambda}\|_{\mathcal{H}_{\alpha}} \leq C_{\alpha,a}(1 + \sqrt{|\lambda|}), \quad \forall \lambda \in \Lambda_{\alpha,a},$$

and moreover, the observation terms enjoy the following estimate

1. When  $(\alpha, a) \notin \mathcal{R}$ , we have

$$|\mathcal{B}_2^* \Phi_{\lambda}| \geq \frac{1}{C_{\alpha,a}}(1 + \sqrt{|\lambda|}), \quad \forall \lambda \in \Lambda_{\alpha,a}. \quad (\text{IV.103})$$

2. On the other hand, for any pair  $(\alpha, a) \in \mathcal{R}$ , we have the same estimate (IV.103) for all  $\lambda \in \Lambda_{\alpha,a} \setminus \{\lambda_{\alpha,a}^c\}$ , where  $\lambda_{\alpha,a}^c$  is given in (IV.98).

*Proof.* We first observe that, thanks to the structure of the spectrum of our operator given in (IV.89), it is enough to establish the required estimates for  $\lambda \in \Lambda_{\alpha,a}^{\infty}$ , in which case we can take advantage of the explicit asymptotic behavior of the eigenvalues that we have established above. Moreover we only treat here the case  $a \neq 0$  since the case  $a = 0$  can be treated easily in the very same way by using formulas (IV.77)–(IV.78) instead of (IV.86).

- For the case  $\lambda = 0$ , (which is possible only when  $a + 3\alpha + 6 = 0$ ) it is easy to see from (IV.82) that there exists some  $C_{\alpha,a} > 0$  such that

$$\|\Phi_0\|_{\mathcal{H}_{\alpha}} \leq C_{\alpha,a}.$$

- Next, we suppose  $\lambda \neq 0$  and observe that

$$\begin{aligned} \|\Phi_{\lambda}\|_{\mathcal{H}_{\alpha}}^2 &= (\mathcal{A}_{\alpha,0} \Phi_{\lambda}, \Phi_{\lambda})_E \\ &= (\mathcal{A}_{\alpha,a}^* \Phi_{\lambda}, \Phi_{\lambda})_E - (\mathcal{M}_a^* \Phi_{\lambda}, \Phi_{\lambda})_E \\ &= \lambda \|\Phi_{\lambda}\|_E^2 - (\mathcal{M}_a^* \Phi_{\lambda}, \Phi_{\lambda})_E \\ &\leq (|a| + |\lambda|) \|\Phi_{\lambda}\|_E^2. \end{aligned}$$

Therefore, we are reduced to find a uniform estimate of the norm in  $E$  of  $\Phi_{\lambda}$ .

Using the explicit expression (IV.86) of the eigenfunction  $\Phi_\lambda$  we get that

$$\|\Phi_\lambda\|_E \leq C_a \left(1 + \frac{1}{|\mu \sin \mu|}\right),$$

and so we finally simply need to show that

$$\sup_k \|\Phi_{\lambda_{k,1}^{\alpha,a}}\|_E + \sup_k \|\Phi_{\lambda_{k,2}^{\alpha,a}}\|_E < +\infty.$$

- Concerning the first family of eigenvalues, by the asymptotic formula of  $\mu_{k,1}^{\alpha,a}$  in Lemma IV.4.4, we have that  $|\sin(\mu_{k,1}^{\alpha,a})|$  is close to 1 for  $k$  large enough. Therefore it is clear that  $\|\Phi_{\lambda_{k,1}^{\alpha,a}}\|_E$  is a bounded quantity when  $k$  goes to infinity.
- For the second family of eigenvalues, using the again asymptotic formula of  $\mu_{k,2}^{\alpha,a}$  from Lemma IV.4.4, we see that  $\sin(\mu_{k,2}^{\alpha,a})$  is now close to 0 for  $k$  large. However, the precise asymptotics shows that the product

$$|\mu_{k,2}^{\alpha,a} \sin(\mu_{k,2}^{\alpha,a})|,$$

is close to  $|a|/4$  for  $k$  large, and thus  $\|\Phi_{\lambda_{k,2}^{\alpha,a}}\|_E$  is also bounded.

- Concerning the observation terms, we start from (IV.93) and separate again the study for the two families of eigenvalues.
  - By the same argument as before, we see that  $|\sin(\mu_{k,1}^{\alpha,a})|$  is close to 1 and  $|\cos(\mu_{k,1}^{\alpha,a})|$  is close to 0, so that we get

$$\mathcal{B}_2^* \Phi_{\lambda_{k,1}^{\alpha,a}} \underset{+\infty}{\sim} \mu_{k,1}^{\alpha,a} = \sqrt{\lambda_{k,1}^{\alpha,a}}.$$

- Concerning the second family of eigenvalues, we need to carefully study the last term in (IV.93), that gives

$$\frac{a \cos(\mu_{k,2}^{\alpha,a})}{2 \sin(\mu_{k,2}^{\alpha,a})} \underset{+\infty}{\sim} \frac{a(-1)^{k+1}}{2(-1)^{k+1} \left(\frac{-a}{4k\pi}\right)} = -2k\pi \underset{+\infty}{\sim} -2\mu_{k,2}^{\alpha,a}.$$

Coming back to (IV.93), we conclude that

$$\mathcal{B}_2^* \Phi_{\lambda_{k,2}^{\alpha,a}} \underset{+\infty}{\sim} -\mu_{k,2}^{\alpha,a} = -\sqrt{\lambda_{k,2}^{\alpha,a}}.$$

This, along with Lemma IV.4.7 gives the required results in points 1 and 2 of our Lemma.  $\square$

### IV.4.3 Null-controllability

We now focus on obtaining a null-control for the system (IV.7)–(IV.8b). We recall again that the diffusion coefficients are  $\gamma_1 = \gamma_2 = 1$ .

#### IV.4.3.1 The moments problem

The set of eigenfunctions  $\{\Phi_\lambda\}_{\lambda \in \Lambda_{\alpha,a}}$  of  $\mathcal{A}_{\alpha,a}^*$  is a complete family in  $\mathcal{H}_\alpha$  on account of Proposition IV.4.6, so it is enough to check the controllability equation (IV.26) for  $\Phi_\lambda$  for each  $\lambda \in \Lambda_{\alpha,a}$ . This indeed tells us that, for any  $y_0 \in \mathcal{H}_{-\alpha}$ , the input  $v \in L^2(0, T; \mathbb{C})$  is a null-control for (IV.7)–(IV.8b) if and only if we have

$$-e^{-\lambda T} \langle y_0, \Phi_\lambda \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} = \overline{\mathcal{B}_2^* \Phi_\lambda} \int_0^T v(t) e^{-\lambda(T-t)} dt, \quad \forall \lambda \in \Lambda_{\alpha,a}. \quad (\text{IV.104})$$

where we used the fact that  $e^{-t\mathcal{A}_{\alpha,a}^*} \Phi_\lambda = e^{-t\lambda} \Phi_\lambda$ , for any  $\lambda \in \Lambda_{\alpha,a}$ .

The above set of equations are the moments problem in our case, that we need to solve.

### IV.4.3.2 Existence of bi-orthogonal family

From the set of moments problem, we shall construct a control  $v$  that needs the existence of a suitable bi-orthogonal family to the time-dependent exponential functions. In this context, we recall Theorem II.2.9 from Chapter II that deals with the existence of bi-orthogonal families to the exponential functions.

Here, we show that the set of eigenvalues  $\Lambda_{\alpha,a} := \Lambda_{\alpha,a}^0 \cup \Lambda_{\alpha,a}$ , given by (IV.89) satisfies all the assumptions of Theorem II.2.9. Indeed, this theorem needs all the elements of  $\Lambda_{\alpha,a}$  with positive real part. So, if needed, one could choose some  $m_{\alpha,a} > 0$ , such that  $(\lambda + m_{\alpha,a})$  has positive real part for all  $\lambda \in \Lambda_{\alpha,a}$ , and we shall then focus on finding the bi-orthogonal family for  $\{e^{-(\lambda+m_{\alpha,a})(T-t)}\}_{\lambda \in \Lambda_{\alpha,a}}$ .

1. We know that the set of eigenvalues  $\Lambda_{\alpha,a}$  are discrete, so one can presume those as a sequence of complex numbers with non-decreasing modulus. Indeed, we must remember that  $\Lambda_{\alpha,a}$  consists of possibly finite number of complex eigenvalues of  $\mathcal{A}_{\alpha,a}^*$  (see Proposition IV.4.6). As a consequence, there exists some constant  $c_{\alpha,a} > 0$ , independent of  $\lambda$ , such that

$$|\Im(\lambda)| \leq c_{\alpha,a} \sqrt{\Re(\lambda) + m_{\alpha,a}}, \quad \forall \lambda \in \Lambda_{\alpha,a} \cap \mathbb{C}.$$

2. *The gap condition:*

– First, it's important to recall the set of all real eigenvalues, defined in (IV.91), and for simplicity, one may re-denote the sequence  $\{\lambda_{k,i}^{\alpha,a}\}_{k \geq k_{\alpha,a}, i=1,2}$  by  $\{\lambda_{2k+i}\}_{k \geq k_{\alpha,a}, i=1,2}$  ( $k_{\alpha,a}$  has been introduced in Corollary IV.4.5), in increasing order as follows

$$\lambda_{2k_{\alpha,a}+1} < \lambda_{2k_{\alpha,a}+2} < \lambda_{2(k_{\alpha,a}+1)+1} < \dots,$$

with

$$\lambda_{2k+1} := \lambda_{k,1}^{\alpha,a}, \quad \lambda_{2k+2} := \lambda_{k,2}^{\alpha,a}, \quad \forall k \geq k_{\alpha,a}.$$

For the re-defined sequence above, we start with the index  $2k_{\alpha,a} + 1$ , since we have that the set  $\Lambda_{\alpha,a}^0 \subset \Lambda_{\alpha,a}$  (see (IV.90)) consists of exactly  $2k_{\alpha,a}$  number of eigenvalues.

Let us take into account the asymptotic formulas in Corollary IV.4.5 and compute the following,

$$\begin{aligned} \lambda_{2k+2} - \lambda_{2k+1} &= \lambda_{k,2}^{\alpha,a} - \lambda_{k,1}^{\alpha,a} = (k+1)^2 \pi^2 - (k + \frac{1}{2})^2 \pi^2 + O_{\alpha,a}(1) \\ &\geq \tilde{c}_{\alpha,a} k \pi^2, \quad \text{for large } k \geq k_{\alpha,a}, \\ &\geq \tilde{c}_{\alpha,a} \frac{\pi^2}{7} [(2k+2)^2 - (2k+1)^2], \end{aligned}$$

as also,

$$\begin{aligned} \lambda_{2(k+1)+1} - \lambda_{2k+2} &= \lambda_{k+1,1}^{\alpha,a} - \lambda_{k,2}^{\alpha,a} = (k+1 + \frac{1}{2})^2 \pi^2 - (k+1)^2 \pi^2 + O_{\alpha,a}(1) \\ &\geq \tilde{c}_{\alpha,a} (k+1) \pi^2, \quad \text{for large } k \geq k_{\alpha,a}, \\ &\geq \tilde{c}_{\alpha,a} \frac{\pi^2}{5} [(2(k+1)+1)^2 - (2k+2)^2], \end{aligned}$$

for some constant  $\tilde{c}_{\alpha,a} > 0$ . Now, from the above two inequalities, it is not difficult to obtain

$$\lambda_{2k+i} - \lambda_{2n+j} \geq \rho_{\alpha,a} [(2k+i)^2 - (2n+j)^2], \quad \forall \begin{cases} k > n \geq \tilde{k}_{\alpha,a}, i, j \in \{1, 2\}, \\ k = n \geq \tilde{k}_{\alpha,a}, i > j, \end{cases} \quad (\text{IV.105})$$

with  $\rho_{\alpha,a} = \tilde{c}_{\alpha,a} \pi^2/7$ , independent of the choices of eigenvalues.

– Now, choose some sufficiently large number  $\bar{\sigma} > 0$  in such a way that we can divide the spectrum into two parts:  $\{\lambda \in \Lambda_{\alpha,a} : |\lambda| < \bar{\sigma}\} \supset \Lambda_{\alpha,a}^0$  and  $\{\lambda \in \Lambda_{\alpha,a} : |\lambda| \geq \bar{\sigma}\} \subset \Lambda_{\alpha,a}^\infty$ , with in addition, the second part satisfies the gap condition (IV.105).

On the other hand, since the part  $\{\lambda \in \Lambda_{\alpha,a} : |\lambda| < \bar{\sigma}\}$  consists of only finite number of eigenvalues, so there always exists some constant  $\tilde{\rho}_{\alpha,a} > 0$ , independent of the eigenvalues, such that

$$\inf \{|\lambda - \tilde{\lambda}| : \lambda \neq \tilde{\lambda}, \text{ with } |\lambda|, |\tilde{\lambda}| < \bar{\sigma}\} \geq \tilde{\rho}_{\alpha,a} > 0. \quad (\text{IV.106})$$

So, finally (IV.105) and (IV.106) together imply the required gap condition.

### 3. The counting function:

Let  $\mathcal{N}$  be the counting function associated with the set of eigenvalues  $\Lambda_{\alpha,a}$ , defined by

$$\mathcal{N}(r) = \#\{\lambda \in \Lambda_{\alpha,a} : |\lambda| \leq r\}, \quad \forall r > 0.$$

We have that, the function  $\mathcal{N}$  is piecewise constant and non-decreasing in the interval  $[0, +\infty)$ . Also for every  $r \in (0, +\infty)$  we have  $\mathcal{N}(r) < +\infty$  and  $\lim_{r \rightarrow +\infty} \mathcal{N}(r) = +\infty$ .

– Without loss of generality, one can start with some sufficiently large number  $\bar{r} > 0$ , such that  $\forall r \geq \bar{r}$ , the eigenvalue  $\lambda_{\mathcal{N}(r)}$  is real. Assuming this  $\mathcal{N}(r)$  to be an odd number, we have, from the definition of  $\mathcal{N}$ , that

$$\begin{aligned} \mathcal{N}(r) = 2k + 1 &\iff \lambda_{2k+1} \leq r \text{ and } \lambda_{2k+2} > r, \quad \text{for } k \geq k_{\alpha,a}, \\ \text{i.e., } \sqrt{\lambda_{2k+1}} &\leq \sqrt{r} < \sqrt{\lambda_{2k+2}}, \quad \text{for } k \geq k_{\alpha,a}, \end{aligned}$$

which yields, by Lemma IV.4.4,

$$(k + \frac{1}{4})\pi \leq \sqrt{r} < (k + \frac{5}{4})\pi, \quad \text{for } k \geq k_{\alpha,a} \text{ and } \forall r \geq \bar{r} > 0.$$

Replacing  $k$  by  $\frac{(\mathcal{N}(r)-1)}{2}$ , we determine that

$$\frac{2}{\pi}\sqrt{r} - \frac{3}{2} < \mathcal{N}(r) \leq \frac{2}{\pi}\sqrt{r} + \frac{1}{2}, \quad \forall r \geq \bar{r} > 0. \quad (\text{IV.107})$$

Similarly, for even  $\mathcal{N}(r)$ , we shall have similar estimate for  $\mathcal{N}(r)$ .

$$\mathcal{N}(r) = 2k + 2 \iff \lambda_{2k+2} \leq r \text{ and } \lambda_{2(k+1)+1} > r, \quad \text{for } k \geq k_{\alpha,a},$$

that is,

$$\sqrt{\lambda_{2k+2}} \leq \sqrt{r} < \sqrt{\lambda_{2(k+1)+1}}, \quad \text{for } k \geq k_{\alpha,a},$$

which leads, again by Lemma IV.4.4,

$$\begin{aligned} (k + \frac{3}{4})\pi &\leq \sqrt{r} < (k + 1 + \frac{3}{4})\pi, \quad \text{for } k \geq k_{\alpha,a}, \\ \text{i.e., } 2k + \frac{3}{2} &\leq \frac{2}{\pi}\sqrt{r} < 2k + 2 + \frac{3}{2}. \end{aligned}$$

Replacing  $(2k + 2)$  by  $\mathcal{N}(r)$ , we deduce

$$\frac{2}{\pi}\sqrt{r} - \frac{3}{2} < \mathcal{N}(r) \leq \frac{2}{\pi}\sqrt{r} + \frac{1}{2}. \quad (\text{IV.108})$$

– Now, for smaller  $0 < r < \bar{r}$ , it is obvious that there always exists some constant  $\hat{c}_{\alpha,a} > 0$ , sufficiently large and independent of  $0 < r < \bar{r}$  such that

$$\mathcal{N}(r) \leq \hat{c}_{\alpha,a}(\sqrt{r} + 1), \quad \text{for } 0 < r < \bar{r}, \quad (\text{IV.109})$$

since  $\mathcal{N}$  is bounded function in  $(0, \bar{r})$ .

The above inequalities (IV.107) and (IV.109) are the required conditions for counting functions.

So, by virtue of [21, Theorem 1.5], we can ensure the existence of a family  $\{q_\lambda\}_{\lambda \in \Lambda_{\alpha,a}} \subset L^2(0, T; \mathbb{C})$ , bi-orthogonal to  $\{e^{-(\lambda+m_{\alpha,a})(T-t)}\}_{\lambda \in \Lambda_{\alpha,a}}$ ,  $t \in (0, T)$ , that is to say

$$\int_0^T q_\lambda(t) \overline{e^{-(\tilde{\lambda}+m_{\alpha,a})(T-t)}} dt = \delta_{\lambda, \tilde{\lambda}}, \quad \forall \lambda, \tilde{\lambda} \in \Lambda_{\alpha,a}.$$

In addition, this family satisfies the following estimate

$$\|q_\lambda\|_{L^2(0,T)} \leq C_{\alpha,a} e^{C_{\alpha,a}(\sqrt{\Re(\lambda)+m_{\alpha,a}+\frac{1}{T}})}, \quad \forall \lambda \in \Lambda_{\alpha,a}, \quad (\text{IV.110})$$

for some  $C_{\alpha,a} > 0$  which only depends on the constants obtained in the point 1, 2, 3 in the above discussions but not on the eigenvalues  $\lambda \in \Lambda_{\alpha,a}$ .

#### IV.4.3.3 Existence of a control

Now we are in the situation to prove the null-controllability result, typically the following proof.

*Proof of Theorem IV.2.8.*

Without loss of generality, we prove the theorem for given time  $0 < T \leq 1$ . Since for any time  $\tilde{T} > 1$ , we know that a continuation by 0 of a control in  $(0, 1)$  will still be a control in  $(0, \tilde{T})$ .

- We first suppose  $(\alpha, a) \notin \mathcal{R}$  and consider

$$v(t) = \sum_{\lambda \in \Lambda_{\alpha,a}} v_\lambda(t), \quad \forall t \in (0, T), \quad \text{with} \quad (\text{IV.111a})$$

$$v_\lambda(t) = -\frac{e^{-\lambda T}}{\mathcal{B}_2^* \Phi_\lambda} \langle y_0, \Phi_\lambda \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} q_\lambda(t), \quad \forall t \in (0, T), \quad (\text{IV.111b})$$

for  $\lambda \in \Lambda_{\alpha,a}$ , any given  $y_0 \in \mathcal{H}_{-\alpha}$  and any  $0 < T \leq 1$ . The above construction of  $v_\lambda$  is well-defined since we have, by Lemma IV.4.7, that  $\mathcal{B}_2^* \Phi_\lambda \neq 0$ ,  $\forall \lambda \in \Lambda_{\alpha,a}$ .

With this choice of  $v$ , we can observe that the set of moments problem (IV.104) is formally satisfied. It remains to show the convergence of the series, and then we need to find the  $L^2(0, T)$  norm of  $v_\lambda$  for each  $\lambda \in \Lambda_{\alpha,a}$ . We see that

$$\begin{aligned} \|v_\lambda\|_{L^2(0,T)} &\leq \frac{|e^{-\lambda T}|}{|\mathcal{B}_2^* \Phi_\lambda|} \|y_0\|_{\mathcal{H}_{-\alpha}} \|\Phi_\lambda\|_{\mathcal{H}_\alpha} \|q_\lambda\|_{L^2(0,T)} \\ &\leq C_{\alpha,a} e^{\frac{C_{\alpha,a}}{T}} e^{-T\Re(\lambda)} e^{C_{\alpha,a}\sqrt{\Re(\lambda)+m_{\alpha,a}}} \times \|y_0\|_{\mathcal{H}_{-\alpha}} \frac{\|\Phi_\lambda\|_{\mathcal{H}_\alpha}}{|\mathcal{B}_2^* \Phi_\lambda|}, \end{aligned} \quad (\text{IV.112})$$

thanks to the estimate of bi-orthogonal family in (IV.110).

Thereafter, an application of Cauchy-Schwarz inequality gives

$$C_{\alpha,a}\sqrt{\Re(\lambda) + m_{\alpha,a}} \leq \frac{T}{2}(\Re(\lambda) + m_{\alpha,a}) + \frac{C_{\alpha,a}^2}{2T},$$

so that one has

$$\begin{aligned} e^{-T\Re(\lambda)} e^{C_{\alpha,a}\sqrt{\Re(\lambda)+m_{\alpha,a}}} &\leq e^{\frac{C_{\alpha,a}}{T} + \frac{T}{2}m_{\alpha,a}} e^{-\frac{T}{2}\Re(\lambda)} \\ &\leq C_{\alpha,a} e^{\frac{C_{\alpha,a}}{T} - \frac{T}{2}\Re(\lambda)} \quad \forall \lambda \in \Lambda_{\alpha,a}, \end{aligned} \quad (\text{IV.113})$$

where we have used that  $0 < T \leq 1$  to write  $e^{\frac{T}{2}m_{\alpha,a}} \leq C_{\alpha,a}$  for some constant  $C_{\alpha,a} > 0$  (which may differ from the previous one).

Next, we use the estimates of the eigenfunctions from Lemma IV.4.8 to deduce

$$\frac{\|\Phi_\lambda\|_{\mathcal{H}_\alpha}}{|\mathcal{B}_2^* \Phi_\lambda|} \leq C_{\alpha,a}, \quad \forall \lambda \in \Lambda_{\alpha,a}.$$

Now, taking the sum over  $\lambda \in \Lambda_{\alpha,a}$  in (IV.112), using the above bounds and applying (IV.113), we get

$$\sum_{\lambda \in \Lambda_{\alpha,a}} \|v_\lambda\|_{L^2(0,T)} \leq C_{\alpha,a} e^{\frac{C_{\alpha,a}}{T}} \|y_0\|_{\mathcal{H}_{-\alpha}} \sum_{\lambda \in \Lambda_{\alpha,a}} e^{-\frac{T}{2}\Re(\lambda)}$$

We finally get that

$$\|v\|_{L^2(0,T)} \leq C_{\alpha,a} e^{\frac{C_{\alpha,a}}{T}} \|y_0\|_{\mathcal{H}_{-\alpha}}, \quad (\text{IV.114})$$

with a constant  $C_{\alpha,a} > 0$  does not depend on  $T$ .

- On the other hand, when  $(\alpha, a) \in \mathcal{R}$ , we consider our control as

$$v(t) = \sum_{\lambda \in \Lambda_{\alpha,a} \setminus \{\lambda_{\alpha,a}^c\}} v_\lambda(t), \quad \forall t \in (0, T), \quad (\text{IV.115})$$

with the same formulation of  $v_\lambda$  as prescribed in (IV.111b).

Since we have assumed that  $y_0 \in \mathcal{Y}_{\alpha,a}$  (the space  $\mathcal{Y}_{\alpha,a}$  has been defined in (IV.102)), we see that the moments problem (IV.104) is actually satisfied for any eigenvalue (in the case  $\lambda = \lambda_{\alpha,a}^c$ , both sides of the equality are zero).

The  $L^2$ -bound of this control alike (IV.114) can be then obtained by a similar approach as previous.

□

## IV.5 Some numerical studies

We devote this section to illustrate experimentally the controllability results shown in the previous sections and make experiments for some other control systems. We begin by presenting some facts about the classical penalized Hilbert Uniqueness Method (see e.g. [63, 64] and [31]). Precisely, we shall accommodate the proofs of [31, Proposition 1.5, Theorem 1.7] in our functional setting which is not exactly the same as of the above references. For instance, recall that the weak solution  $y$  to (IV.7) with (IV.8a), or (IV.8b), belongs to the space  $\mathcal{C}^0([0, T]; \mathcal{H}_{-\alpha})$ , where the abstract space  $\mathcal{H}_{-\alpha}$  is much more larger than  $E := (L^2(0, 1))^2$ . Thus, a careful study is require to deal with the HUM approach in our case, which we detail in the next section.

In this regard, we must mention that several authors has utilized the penalized HUM technique to clarify various controllability issues related to the parabolic systems. For instance, the authors in [34] dealt with a numerical study (based on HUM) of insensitizing control problems for parabolic semilinear equation and in [28], the controllability of 1D fractional heat equation has been analyzed from both numerical and theoretical aspects. We also quote [12], where

some partial controllability (or, non-) results have been illustrated with help of penalized HUM technique.

Concerning the numerical studies for parabolic systems, in most of the known cases no boundary coupling has been taken into account. But in our case, we will introduce a general discretization methodology for the interior-boundary coupled parabolic system (IV.1) (along with the Assumption IV.1.2), mainly to incorporate the effect of the *boundary couplings* into our discrete setting. To this end, we conclude our study by presenting several experiments.

### IV.5.1 Application of the penalized HUM approach

Recall some notations as introduced in Section IV.2: the space  $E = (L^2(0, 1))^2$  with  $(\cdot, \cdot)_E$  the inner product in  $E$ , the space  $\mathcal{H}_\alpha = D(\mathcal{A}_\alpha^{1/2})$  as introduced in (IV.16) with the norm defined by (IV.15), and its dual  $\mathcal{H}_{-\alpha}$ .

Following the well-known penalized HUM approach, we shall look for the control  $v$  minimizing the primal functional given by

$$F_\epsilon(v) := \frac{1}{2} \int_0^T |v(t)|^2 dt + \frac{1}{2\epsilon} \|y_{v,y_0}(T)\|_{\mathcal{H}_{-\alpha}}^2, \quad \forall v \in L^2(0, T; \mathbb{R}), \quad (\text{IV.116})$$

where we used the notation  $y_{v,y_0}$  to denote the unique weak solution to the system (IV.7) either with the boundary conditions (IV.8a) or (IV.8b). For the sake of exposition, we assume in what follows that (IV.8b) are satisfied.

Observe that, for any  $\epsilon > 0$ , the functional (IV.116) has a unique minimizer in  $L^2(0, T; \mathbb{R})$  since  $F_\epsilon$  is continuous, strictly convex and coercive. Hereafter, we denote this minimizer by  $v_\epsilon$ .

Using Fenchel-Rockafellar theory (see, for instance [52]), we can identify an associated dual functional, more precisely, for any  $\epsilon > 0$ , consider

$$J_\epsilon(\zeta) := \frac{1}{2} \int_0^T |\mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha,a}^*} \zeta|^2 dt + \frac{\epsilon}{2} \|\zeta\|_{\mathcal{H}_\alpha}^2 + (y_0, e^{-T\mathcal{A}_{\alpha,a}^*} \zeta)_E, \quad (\text{IV.117})$$

for all  $\zeta := (\zeta_1, \zeta_2) \in \mathcal{H}_\alpha$ , where  $y_0 \in E$  is the given initial data for the control system and recall the observation operator  $\mathcal{B}_2^*$  defined by (IV.27b) for the control problem (IV.7)–(IV.8b).

For any  $\epsilon > 0$ , the dual functional (IV.117) also has a unique minimizer, that we denote by  $\zeta^\epsilon$ . Note that, in this case the coercivity comes from the term  $\frac{\epsilon}{2} \|\zeta\|_{\mathcal{H}_\alpha}^2$  which corresponds, by duality, to the penalty term introduced in  $F_\epsilon$ .

**Some definitions.** Before going to more detail, recall that, by definition of the operator  $\mathcal{A}_\alpha$  and the spaces  $\mathcal{H}_\alpha$ ,  $\mathcal{H}_{-\alpha}$  (see Section IV.2), we have in the distribution sense: for any given  $u \in \mathcal{H}_\alpha$ ,

$$\langle \mathcal{A}_\alpha u, \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} = (u, \zeta)_{\mathcal{H}_\alpha} \quad \forall \zeta \in \mathcal{H}_\alpha. \quad (\text{IV.118})$$

In other words, for given any element  $f \in \mathcal{H}_{-\alpha}$ , we may write

$$\langle f, \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} = (\mathcal{A}_\alpha^{-1} f, \zeta)_{\mathcal{H}_\alpha} \quad \forall \zeta \in \mathcal{H}_\alpha,$$

where  $\mathcal{A}_\alpha^{-1} f \in \mathcal{H}_\alpha$  is uniquely defined.

Then, for any  $f, g \in \mathcal{H}_{-\alpha}$ , one can define

$$\langle f, g \rangle_{\mathcal{H}_{-\alpha}} := \langle f, \mathcal{A}_\alpha^{-1} g \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} = (\mathcal{A}_\alpha^{-1} f, \mathcal{A}_\alpha^{-1} g)_{\mathcal{H}_\alpha}.$$

Now, we characterize the minimizers  $v_\epsilon$  and  $\zeta^\epsilon$  in the following way.



- The minimizer  $v_\epsilon$  of the functional  $F_\epsilon$ , given by (IV.116), is characterized by the following Euler-Lagrange equation

$$\int_0^T v_\epsilon(t) \tilde{v}(t) dt + \frac{1}{\epsilon} \langle y_{v_\epsilon, y_0}(T), y_{\tilde{v}, 0}(T) \rangle_{\mathcal{H}_{-\alpha}} = 0, \quad \forall \tilde{v} \in L^2(0, T; \mathbb{R}), \quad (\text{IV.119})$$

where, by definition we have

$$\langle y_{v_\epsilon, y_0}(T), y_{\tilde{v}, 0}(T) \rangle_{\mathcal{H}_{-\alpha}} = \langle y_{v_\epsilon, y_0}(T), \mathcal{A}_\alpha^{-1} y_{\tilde{v}, 0}(T) \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} \quad \forall \tilde{v} \in L^2(0, T; \mathbb{R}).$$

In particular, for any  $\tilde{v} \in L^2(0, T; \mathbb{R})$ ,  $y_{\tilde{v}, 0}(T) \in \mathcal{H}_{-\alpha}$  is identified as

$$\langle y_{\tilde{v}, 0}(T), \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} = (\mathcal{A}_\alpha^{-1} y_{\tilde{v}, 0}(T), \zeta)_{\mathcal{H}_\alpha} \quad \forall \zeta \in \mathcal{H}_\alpha. \quad (\text{IV.120})$$

- On the other hand, the minimizer  $\zeta^\epsilon$  of the functional  $J_\epsilon$ , given by (IV.117), is characterized by the following Euler-Lagrange equation

$$\begin{aligned} \int_0^T \left( \mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta^\epsilon \right) \left( \mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta \right) dt + \epsilon (\zeta^\epsilon, \zeta)_{\mathcal{H}_\alpha} \\ + \left( y_0, e^{-T\mathcal{A}_{\alpha, a}^*} \zeta \right)_E = 0, \quad \forall \zeta \in \mathcal{H}_\alpha. \end{aligned} \quad (\text{IV.121})$$

Now, we explicitly write the following result relating the corresponding minimizers of  $F_\epsilon$  and  $J_\epsilon$ .

**Proposition IV.5.1.** *For any  $\epsilon > 0$ , the minimizers  $v_\epsilon$  and  $\zeta^\epsilon$  of the functionals  $F_\epsilon$  and  $J_\epsilon$  respectively, are related through the formula*

$$v_\epsilon(t) = \mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta^\epsilon \quad \text{for a.e. } t \in (0, T), \quad (\text{IV.122})$$

and

$$y_{v_\epsilon, y_0}(T) = -\epsilon \mathcal{A}_\alpha \zeta^\epsilon. \quad (\text{IV.123})$$

As a consequence, we have

$$\inf_{L^2(0, T; \mathbb{R})} F_\epsilon = F_\epsilon(v_\epsilon) = -J_\epsilon(\zeta^\epsilon) = -\inf_{\mathcal{H}_\alpha} J_\epsilon, \quad (\text{IV.124})$$

and finally,

$$\|y_{v_\epsilon, y_0}(T)\|_{\mathcal{H}_{-\alpha}} \leq \|y_{0, y_0}(T)\|_{\mathcal{H}_{-\alpha}}.$$

We prove the above proposition in a similar approach as [31].

*Proof.* We denote  $w_\epsilon(t) = \mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta^\epsilon \in L^2(0, T; \mathbb{R})$ , and we prove first  $w_\epsilon = v_\epsilon$ .

- Using the definition of  $w_\epsilon$  in (IV.121), we have

$$\int_0^T w_\epsilon(t) \left( \mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta \right) dt + \epsilon (\zeta^\epsilon, \zeta)_{\mathcal{H}_\alpha} + \left( y_0, e^{-T\mathcal{A}_{\alpha, a}^*} \zeta \right)_E = 0, \quad \forall \zeta \in \mathcal{H}_\alpha,$$

and by definition of  $y_{w_\epsilon, y_0}$  in a dual sense (see Proposition IV.2.5), we can write

$$\langle y_{w_\epsilon, y_0}(T), \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} - \left( y_0, e^{-T\mathcal{A}_{\alpha, a}^*} \zeta \right)_E = \int_0^T w_\epsilon(t) \left( \mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta \right) dt,$$

for any  $\zeta \in \mathcal{H}_\alpha$ .

Comparing the last two equality, we obtain

$$\langle y_{w_\epsilon, y_0}(T), \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} = -(\epsilon \zeta^\epsilon, \zeta)_{\mathcal{H}_\alpha}, \quad \forall \zeta \in \mathcal{H}_\alpha, \quad (\text{IV.125})$$

and so, one has

$$\|y_{w_\epsilon, y_0}(T)\|_{\mathcal{H}_{-\alpha}} = \epsilon \|\zeta^\epsilon\|_{\mathcal{H}_\alpha}. \quad (\text{IV.126})$$

Now, by definition of the operator  $\mathcal{A}_\alpha$ ,  $\mathcal{H}_\alpha$  and its dual  $\mathcal{H}_{-\alpha}$  we have (in distribution sense)

$$\langle \mathcal{A}_\alpha \zeta^\epsilon, \zeta \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} = (\zeta^\epsilon, \zeta)_{\mathcal{H}_\alpha}, \quad \forall \zeta \in \mathcal{H}_\alpha.$$

Thus, comparing the above relation with (IV.125), we can say

$$y_{w_\epsilon, y_0}(T) = -\epsilon \mathcal{A}_\alpha \zeta^\epsilon.$$

- Let us consider any  $\tilde{v} \in L^2(0, T; \mathbb{R})$  and  $y_0 = 0$ . Choosing  $\zeta = \zeta^\epsilon$  we have from the definition of  $y_{\tilde{v}, 0}$  (as per Proposition IV.2.5 in dual sense)

$$\langle y_{\tilde{v}, 0}(T), \zeta^\epsilon \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} = \int_0^T \tilde{v}(t) (\mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta^\epsilon) dt, \quad \forall \tilde{v} \in L^2(0, T; \mathbb{R}),$$

that is (by using the definition of  $w_\epsilon$ )

$$\int_0^T w_\epsilon(t) \tilde{v}(t) dt - \langle y_{\tilde{v}, 0}(T), \zeta^\epsilon \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} = 0, \quad \forall \tilde{v} \in L^2(0, T; \mathbb{R}). \quad (\text{IV.127})$$

Now, recall (IV.120) to write

$$\langle y_{\tilde{v}, 0}(T), \zeta^\epsilon \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} = (\mathcal{A}_\alpha^{-1} y_{\tilde{v}, 0}(T), \zeta^\epsilon)_{\mathcal{H}_\alpha} = \frac{1}{\epsilon} (\epsilon \zeta^\epsilon, \mathcal{A}_\alpha^{-1} y_{\tilde{v}, 0}(T))_{\mathcal{H}_\alpha},$$

and then using (IV.125), we obtain

$$\begin{aligned} \langle y_{\tilde{v}, 0}(T), \zeta^\epsilon \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} &= -\frac{1}{\epsilon} \langle y_{w_\epsilon, y_0}(T), \mathcal{A}_\alpha^{-1} y_{\tilde{v}, 0}(T) \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} \\ &= -\frac{1}{\epsilon} \langle y_{w_\epsilon, y_0}(T), y_{\tilde{v}, 0}(T) \rangle_{\mathcal{H}_{-\alpha}}. \end{aligned}$$

Coming back to (IV.127), one has

$$\int_0^T w_\epsilon(t) \tilde{v}(t) dt + \frac{1}{\epsilon} \langle y_{w_\epsilon, y_0}(T), y_{\tilde{v}, 0}(T) \rangle_{\mathcal{H}_{-\alpha}} = 0, \quad \forall \tilde{v} \in L^2(0, T; \mathbb{R}),$$

thus we have that  $w_\epsilon$  solves the Euler-Lagrange equation (IV.119) which shows  $w_\epsilon = v_\epsilon$ .

- Let us compute

$$\begin{aligned} F_\epsilon(v_\epsilon) &= F_\epsilon(\mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta^\epsilon) \\ &= \frac{1}{2} \int_0^T \left| \mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta^\epsilon \right|^2 dt + \frac{1}{2\epsilon} \|y_{v_\epsilon, y_0}(T)\|_{\mathcal{H}_{-\alpha}}^2 \\ &= \frac{1}{2} \int_0^T \left| \mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta^\epsilon \right|^2 dt + \frac{\epsilon}{2} \|\zeta^\epsilon\|_{\mathcal{H}_\alpha}^2 \end{aligned} \quad (\text{IV.128})$$

by using (IV.126).

Now, by implementing  $\zeta = \zeta^\epsilon$  in the Euler-Lagrange equation (IV.121), one obtains

$$\int_0^T \left| \mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta^\epsilon \right|^2 dt + \epsilon \|\zeta^\epsilon\|_{\mathcal{H}_\alpha}^2 = \left( y_0, e^{-T\mathcal{A}_{\alpha, a}^*} \zeta^\epsilon \right)_E$$

so that we have

$$J_\epsilon(\zeta^\epsilon) = -\frac{1}{2} \int_0^T \left| \mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha, a}^*} \zeta^\epsilon \right|^2 dt - \frac{\epsilon}{2} \|\zeta^\epsilon\|_{\mathcal{H}_\alpha}^2,$$

which gives by comparing with (IV.128) that  $F_\epsilon(v_\epsilon) = -J_\epsilon(\zeta^\epsilon)$ .

- Finally, by definition we have  $F_\epsilon(v_\epsilon) \leq F_\epsilon(0)$ , in particular that shows

$$\|y_{v_\epsilon, y_0}(T)\|_{\mathcal{H}_{-\alpha}} \leq \|y_{0, y_0}(T)\|_{\mathcal{H}_{-\alpha}},$$

and this concludes the proof.  $\square$

The following result allows us to relate the controllability properties of system (IV.7)–(IV.8b) with the behavior of the minimizers shown above. More precisely, we write the following theorem.

**Theorem IV.5.2.** *Let  $v_\epsilon$  and  $y_{v_\epsilon, y_0}$  be as in Proposition IV.5.1. Then we have the following.*

- System (IV.7)–(IV.8b) is approximately controllable at time  $T$  if and only if

$$y_{v_\epsilon, y_0}(T) \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0. \quad (\text{IV.129})$$

- System (IV.7)–(IV.8b) is null controllable at time  $T$  if and only if

$$\mathcal{M}_{y_0}^2 := 2 \sup_{\epsilon > 0} \left( \inf_{L^2(0, T; \mathbb{R})} F_\epsilon \right) < +\infty. \quad (\text{IV.130})$$

In this case, we have

$$\|v_\epsilon\|_{L^2(0, T; \mathbb{R})} \leq \mathcal{M}_{y_0}, \quad \|y_{v_\epsilon, y_0}(T)\|_{\mathcal{H}_{-\alpha}} \leq \mathcal{M}_{y_0} \sqrt{\epsilon}. \quad (\text{IV.131})$$

Moreover, the HUM control satisfies  $\|v_0\|_{L^2(0, T; \mathbb{R})} = \mathcal{M}_{y_0}$ , and

$$v_\epsilon \xrightarrow{\epsilon \rightarrow 0} v_0 \quad \text{strongly in } L^2(0, T; \mathbb{R}).$$

The proof of such result follows from an adaptation of [31, Theorem 1.7]. Let us remark that the supremum in (IV.130) corresponds actually to the limit as  $\epsilon \rightarrow 0$  of  $\inf_{L^2(0, T; \mathbb{R})} F_\epsilon$ .

We give a short sketch for the proof of the above theorem below.

*Proof.* We first recall the set  $\text{Adm}(y_0, \delta)$ ,

$$\text{Adm}(y_0, \delta) = \left\{ v \in L^2(0, T; \mathbb{R}) \text{ s.t. } \|y_{v, y_0}(T)\|_{\mathcal{H}_{-\alpha}} \leq \delta \right\}.$$

- Let us begin with the first point of the theorem.

- We have from (IV.129) that for any  $\delta > 0$ ,  $\exists \epsilon > 0$  s.t.

$$\|y_{v_\epsilon, y_0}(T)\|_{\mathcal{H}_{-\alpha}} \leq \delta,$$

so that  $v_\epsilon \in \text{Adm}(y_0, \delta)$  and this shows the approximate controllability of our system.

- We show the other inclusion by a contradiction argument. We hereby assume that the approximate controllability holds true but not the fact (IV.129). As a consequence, there exists some  $\sigma > 0$  and a sequence  $(\epsilon_k)_{k \geq 0}$  such that  $\epsilon_k \rightarrow 0$  as  $k \rightarrow +\infty$  and  $\|y_{v_{\epsilon_k}, y_0}(T)\|_{\mathcal{H}_{-\alpha}} \geq \sigma$ . By assumption, we have  $\text{Adm}(y_0, \sigma/\sqrt{2}) \neq \emptyset$ ; one can thus choose some  $\hat{v} \in \text{Adm}(y_0, \sigma/\sqrt{2})$  so that

$$\|y_{\hat{v}, y_0}(T)\|_{\mathcal{H}_{-\alpha}}^2 \leq \sigma^2/2. \quad (\text{IV.132})$$

Now, observe that

$$\begin{aligned} \frac{\sigma^2}{2\epsilon_k} &\leq \frac{1}{2\epsilon_k} \|y_{v_{\epsilon_k}, y_0}(T)\|_{\mathcal{H}_{-\alpha}}^2 \leq F_{\epsilon_k}(v_{\epsilon_k}) \\ &\leq F_{\epsilon_k}(\hat{v}) \\ &= \int_0^T |\hat{v}(t)|^2 dt + \frac{1}{2\epsilon_k} \|y_{\hat{v}, y_0}(T)\|_{\mathcal{H}_{-\alpha}}^2. \end{aligned}$$

We then obtain by taking into account (IV.132) that

$$\frac{\sigma^2}{4\epsilon_k} \leq \int_0^T |\hat{v}(t)|^2 dt,$$

and taking the limit  $\epsilon_k \rightarrow 0$  as  $k \rightarrow +\infty$ , one arrives in a contradiction.

- We suppose that our system is null-controllable which means  $\text{Adm}(y_0, 0)$  is non-empty, closed, convex set and as we know that the HUM null-control (of minimal norm)  $v_0$  is the orthogonal projection of 0 onto  $\text{Adm}(y_0, 0)$  in  $L^2(0, T; \mathbb{R})$ . Now, we have that

$$F_\epsilon(v_0) = \frac{1}{2} \|v_0\|_{L^2(0, T)}^2,$$

since  $y_{v_0, y_0}(T) = 0$  and that  $F_\epsilon$  does not depend on  $\epsilon$ . Thus, we get

$$\sup_{\epsilon > 0} \left( \inf_{L^2(0, T; \mathbb{R})} F_\epsilon \right) \leq \frac{1}{2} \|v_0\|_{L^2(0, T)}^2,$$

which gives

$$\mathcal{M}_{y_0}^2 \leq \|v_0\|_{L^2(0, T)}^2 < +\infty. \quad (\text{IV.133})$$

We suppose now (IV.130) holds, by taking into account the definition of  $F_\epsilon(v_\epsilon)$  we immediately deduce the bounds in (IV.131).

Once, we have those bounds, we have for a subsequence  $(\epsilon_k)_k$  that goes to 0 as  $k \rightarrow +\infty$  such that

$$v_{\epsilon_k} \rightharpoonup v \quad \text{weakly in } L^2(0, T; \mathbb{R}), \quad (\text{IV.134})$$

for some  $v \in L^2(0, T; \mathbb{R})$ , and that in one hand we have

$$y_{v_{\epsilon_k}, y_0}(T) \rightarrow 0 \quad \text{in } \mathcal{H}_{-\alpha}.$$

Since, (IV.134) holds, we have on the other hand, from the continuity of the solution operator, that

$$y_{v_{\epsilon_k}, y_0}(T) \rightharpoonup y_{v, y_0}(T) \quad \text{weakly in } \mathcal{H}_{-\alpha},$$

and so in particular,  $y_{v, y_0}(T) = 0$ , which ensures that  $v \in \text{Adm}(y_0, 0)$  is a null-control.

Recall now that  $v_0 \in \text{Adm}(y_0, 0)$  is the null-control of minimal  $L^2$ -norm, so that we can write  $\mathcal{M}_{y_0}^2 \leq \|v\|_{L^2(0, T)}^2$ . Moreover, from the first estimate of (IV.131), we have

$$\limsup_{k \rightarrow +\infty} \|v_{\epsilon_k}\|_{L^2(0, T)}^2 \leq \mathcal{M}_{y_0}^2 \leq \|v\|_{L^2(0, T)}^2,$$

which proves that the convergence of  $(v_{\epsilon_k})_k$  towards  $v$  is actually strong and  $\mathcal{M}_{y_0}^2 = \|v\|_{L^2(0, T)}^2$ . But  $v_0$  is the unique minimal null-control and thus by taking care of (IV.133), we have  $v = v_0$  and finally, since the strong convergence holds for any subsequence  $(\epsilon_k)_k$ , the whole family

$$v_\epsilon \xrightarrow{\epsilon \rightarrow 0} v_0 \quad \text{strongly in } L^2(0, T; \mathbb{R}).$$

The proof is complete. □

**Remark IV.5.3.** If there is no confusion of notation, from now on we denote the associated solution of our parabolic system (for any system under the general framework (IV.1)) by simply  $y^\epsilon := y_{v_\epsilon, y_0}$  with given initial data  $y_0$  and the associated HUM control  $v_\epsilon$ .

The main relevance is that the above theorem allows us to recapture the controllability results presented in the previous sections, using the constructive approach of the penalized HUM instead of other more involved arguments. At the numerical level this will be important since we expect that upon discretization the corresponding system maintains its controllability properties and Theorem IV.5.2 will help to conclude and illustrate this fact.

Now, we discuss some details about the implementation we follow to obtain the controls for the system (IV.7)–(IV.8b). We first define a bounded, non-negative, symmetric (w.r.t the duality product between  $\mathcal{H}_\alpha$  and  $\mathcal{H}_{-\alpha}$ ) operator  $\Lambda : \mathcal{H}_\alpha \rightarrow \mathcal{H}_{-\alpha}$  (usually referred to as the Gramian operator) as follows

$$\Lambda \zeta = w^\zeta(T), \quad \forall \zeta \in \mathcal{H}_\alpha,$$

where the function  $w^\zeta = (w_1^\zeta, w_2^\zeta) \in \mathcal{C}^0([0, T]; \mathcal{H}_{-\alpha})$  is obtained from  $\zeta \in \mathcal{H}_\alpha$  as follows: we solve first the adjoint system (IV.29) (with its solution  $q(t, x) = e^{-(T-t)\mathcal{A}_{\alpha,a}^*} \zeta(x)$ ), and then solve

$$\begin{cases} \partial_t w_1 - \partial_x(\gamma_1 \partial_x w_1) = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t w_2 - \partial_x(\gamma_2 \partial_x w_2) + a w_1 = 0 & \text{in } (0, T) \times (0, 1), \\ w_1(t, 1) = w_2(t, 1) & \text{in } (0, T), \\ \gamma_1(1) \partial_x w_1(t, 1) + \gamma_2(1) \partial_x w_2(t, 1) + \alpha w_1(t, 1) = 0 & \text{in } (0, T), \\ w_1(0, \cdot) = w_2(0, \cdot) = 0 & \text{in } (0, 1), \end{cases}$$

along with the conditions at  $x = 0$  as

$$w_1(t, 0) = \mathcal{B}_2^* e^{-(T-t)\mathcal{A}_{\alpha,a}^*} \zeta, \quad w_2(t, 0) = 0 \quad \text{in } (0, T).$$

Now, recall the definition of  $J_\epsilon$  from (IV.117), and a straightforward computation yields

$$\nabla J_\epsilon(\zeta) = \Lambda \zeta + \epsilon \mathcal{A}_\alpha \zeta + e^{-T\mathcal{A}_{\alpha,a}} y_0, \quad (\text{IV.135})$$

one can see that  $e^{-T\mathcal{A}_{\alpha,a}} y_0$  is the (free) solution to (IV.7)–(IV.8b) when we consider  $v = 0$  in (IV.8b), and with initial data  $y_0 \in E$ .

In this way, the control we are looking for, can be obtained as follows: for any given  $\epsilon > 0$ , we first compute the unique minimizer  $\zeta^\epsilon$  of  $J_\epsilon$ , that is the unique solution to the linear problem

$$\epsilon \mathcal{A}_\alpha \zeta + \Lambda \zeta = -e^{-T\mathcal{A}_{\alpha,a}} y_0, \quad (\text{IV.136})$$

and then determine the solution to the adjoint equation (IV.29) by choosing the terminal data  $\zeta^\epsilon$ . Since  $\Lambda$  is a symmetric and positive semi-definite operator, the conjugate gradient algorithm is a good candidate to solve the linear problem (IV.136). We refer to [64, Ch. 2, Sec. 2.1–2.3] for the implementation of such algorithm.

Once we have computed the minimizer, we use the formula (IV.122) to obtain the desired control and by means of Theorem IV.5.2, the expected controllability properties can be tested by analyzing the involved quantities with respect to the parameter  $\epsilon$ .

## IV.5.2 Numerical implementation for a general interior-boundary coupled system

For the numerical tests, the systems (IV.1) and its adjoint are discretized in time by using a standard implicit Euler scheme with a uniform time step given by  $\delta t = T/M$  where  $M$  is the number of steps on the mesh. The PDEs are discretized in space by a standard finite-difference scheme, adapted to the corresponding boundary conditions, with a constant discretization step of size  $h = 1/(N+1)$ , where  $N$  is the chosen number of steps. More precisely, we consider fully discrete systems of the form

$$\begin{cases} \frac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h y^{n+1} = \mathcal{B}_h v^{n+1}, & n \in \{0, \dots, M-1\}, \\ y^0 = y_h^0, \end{cases} \quad (\text{IV.137})$$

where  $y_h^0 \in \mathbb{R}^{2N}$  is an approximation of the given initial data  $y(0, \cdot)$ ,  $\mathcal{A}_h \in \mathbb{R}^{2N \times 2N}$  is a suitable approximation of the elliptic operator  $\mathcal{A}$  and  $\mathcal{B}_h \in \mathbb{R}^{2N}$  stands for the corresponding approximation of the control operator. We also choose here constant diffusion coefficients  $\gamma_1, \gamma_2 \in \mathbb{R}^+$  and further we denote  $\gamma := \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \in M_{2 \times 2}(\mathbb{R})$ .

As usual, we denote by  $y_j$  ( $j = 1, 2$ ), each of the components of system (IV.1).

1. Using a standard finite-difference method, we construct the matrix  $\mathcal{A}_{h,D} \in \mathbb{R}^{2N \times 2N}$ , which is composed by two tridiagonal matrix coming from the discretization of the operator  $-\gamma_j \partial_x^2$ ,  $j = 1, 2$ , with homogeneous Dirichlet boundary conditions, that is

$$\mathcal{A}_{h,D} = \begin{pmatrix} \mathcal{A}_h^1 & 0 \\ 0 & \mathcal{A}_h^2 \end{pmatrix}, \quad (\text{IV.138})$$

where for each  $j = 1, 2$ ,  $(\mathcal{A}_h^j y_j)_i = -\frac{\gamma_j}{h^2}(y_{j,i+1} - 2y_{j,i} + y_{j,i-1})$ ,  $i = 1, \dots, N$ . At this point, we impose that  $y_{j,0} = y_{j,N+1} = 0$ . In the subsequent steps we will compute and add the contribution of the boundary conditions to the discretization scheme.

2. To incorporate the effect of the boundary conditions (of the system (IV.1)) at the left point, we compute

$$\mathcal{A}_{h,0} = -(-\mathcal{N}_0 \gamma + h\mathcal{D}_0)^{-1} \mathcal{N}_0 \gamma. \quad (\text{IV.139})$$

This corresponds to writing the boundary unknowns  $y_{j,0}$  in terms of the values of  $y_{j,1}$  and yields a  $2 \times 2$  matrix. The result will be then used to construct the auxiliary matrix:

$$\tilde{\mathcal{A}}_h = \mathcal{A}_{h,D} - \frac{1}{h^2} \gamma \mathcal{A}_{h,0} \otimes \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}_{N \times N} \quad (\text{IV.140})$$

where  $\otimes$  denotes the Kronecker product, i.e., for matrices  $S \in \mathbb{R}^{m \times n}$  and  $T \in \mathbb{R}^{p \times q}$ , the product  $S \otimes T$  is the  $mp \times nq$  matrix given by

$$S \otimes T = \begin{pmatrix} s_{11}T & \cdots & s_{1n}T \\ \vdots & \ddots & \vdots \\ s_{1m}T & \cdots & s_{mn}T \end{pmatrix} \quad (\text{IV.141})$$

3. In a similar fashion, for adding the contribution of the boundary at  $x = 1$ , we compute

$$\mathcal{A}_{h,1} = (\mathcal{N}_1 \gamma + h\mathcal{D}_1)^{-1} \mathcal{N}_1 \gamma. \quad (\text{IV.142})$$

This will give the coefficients obtained by expressing  $y_{j,N+1}$  in terms of the values  $y_{j,N}$ . We add the resulting matrix to the one obtained in the previous step as follows

$$\hat{\mathcal{A}}_h = \tilde{\mathcal{A}}_h - \frac{1}{h^2} \gamma \mathcal{A}_{h,1} \otimes \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}_{N \times N} \quad (\text{IV.143})$$

4. To conclude, we need to add the internal coupling terms. This can be easily done by computing

$$\mathcal{A}_h = \hat{\mathcal{A}}_h + \mathcal{M}_{coup} \otimes I_{N \times N}. \quad (\text{IV.144})$$

Observe that in our theoretical results, we have considered the simple case where the control  $v$  is applied to one of the equations of system (IV.7) through the boundary conditions (IV.8a) or (IV.8b). However, observe that in the general system (IV.1), the control can be applied in fact to any linear combination of boundary values. To take into account this in our discretization, we propose the following:

1. We obtain the auxiliary vector

$$\tilde{\mathcal{B}}_h = h(-\mathcal{N}_0\gamma + h\mathcal{D}_0)^{-1}B, \quad (\text{IV.145})$$

where one might consider  $B$  as the canonical vector  $(1, 0)$  or  $(0, 1)$ , depending on which equation the control is being applied.

2. We obtain the control operator by setting

$$\mathcal{B}_h = \frac{1}{h^2}\gamma\tilde{\mathcal{B}}_h \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_N. \quad (\text{IV.146})$$

**Remark IV.5.4.** *Some remarks are in order.*

- In the general case, under the conditions of Assumption IV.1.2, the invertibility of the matrices shown in formulas (IV.139), (IV.142) and (IV.145) is guaranteed for any  $h > 0$  small enough, see Lemma IV.7.1.
- The discretization of system (IV.7) with either boundary conditions (IV.8a), (IV.8b) is a particular case of the scheme presented above. Indeed, we readily see that for such cases we have  $\mathcal{A}_{h,0} = 0_{2 \times 2}$ ,

$$\mathcal{A}_{h,1} = \begin{pmatrix} \frac{\gamma_1}{\gamma_1 + \gamma_2 + \alpha h} & \frac{\gamma_2}{\gamma_1 + \gamma_2 + \alpha h} \\ \frac{\gamma_1}{\gamma_1 + \gamma_2 + \alpha h} & \frac{\gamma_2}{\gamma_1 + \gamma_2 + \alpha h} \end{pmatrix}, \quad (\text{IV.147})$$

$\mathcal{M}_{coup} = \mathcal{M}_a$  and  $\tilde{\mathcal{B}}_h = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$  for the boundary condition (IV.8a) (resp.  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  for (IV.8b)).

In this case, we note that since  $\alpha \geq 0$ , (IV.147) holds for any value of  $h > 0$ .

We denote by  $E_h$ ,  $\mathcal{H}_{-\alpha,h}$ ,  $U_h$  and  $L_{\delta t}^2(0, T; U_h)$  the discrete spaces associated to  $E$ ,  $\mathcal{H}_{-\alpha}$ ,  $\mathbb{R}$  and  $L^2(0, T; \mathbb{R})$ , respectively. We denote by  $F_{\epsilon}^{h,\delta t}$  the discretization of the functional  $F_{\epsilon}$ ,  $v^{\epsilon,h,\delta t}$  the corresponding minimizer and  $y^{\epsilon,h,\delta t} = (y_1^{\epsilon,h,\delta t}, y_2^{\epsilon,h,\delta t})$  the associated controlled solution.

As usual in this context, to connect the discretization to the control problem, we use the penalization parameter  $\epsilon = \phi(h) = h^4$ . This choice is consistent with the order of approximation of the finite difference scheme. We refer the reader to [31, Section 4] for a more detailed discussion on the selection of the function  $\phi(h)$  in the context of the null-controllability of some parabolic problems and its implications.

To concentrate on the dependency of the numerical experiments with respect to the mesh size  $h$ , in the following we will always set  $M = 4000$ . This is due to the fact that the results do not depend too much on the time step (as soon as it is chosen to ensure at least the same accuracy as the space discretization). This was observed in [31] and the same still applies here.

### IV.5.3 Numerical experiments

#### IV.5.3.1 Dirichlet boundary control cases with a Kirchhoff type condition

**The simplest case:**  $a = 0$ . Using our computational tool, we begin by obtaining the solution to system (IV.7)–(IV.8b) without any control. We consider the set of parameters

$$T = 0.4, \quad \gamma_1 = \gamma_2 = 1, \quad (\text{IV.148})$$

$$a = 0, \quad \alpha = 1, \quad (\text{IV.149})$$

$$y_{0,1}(x) = \sin(\pi x), \quad y_{0,2}(x) = \mathbf{1}_{(0.3,0.8)}(x) \quad (\text{IV.150})$$

and plot the time evolution of the uncontrolled system in Figure IV.4. We observe that the solution of both components is damped over time, however they are far from the desired null target.

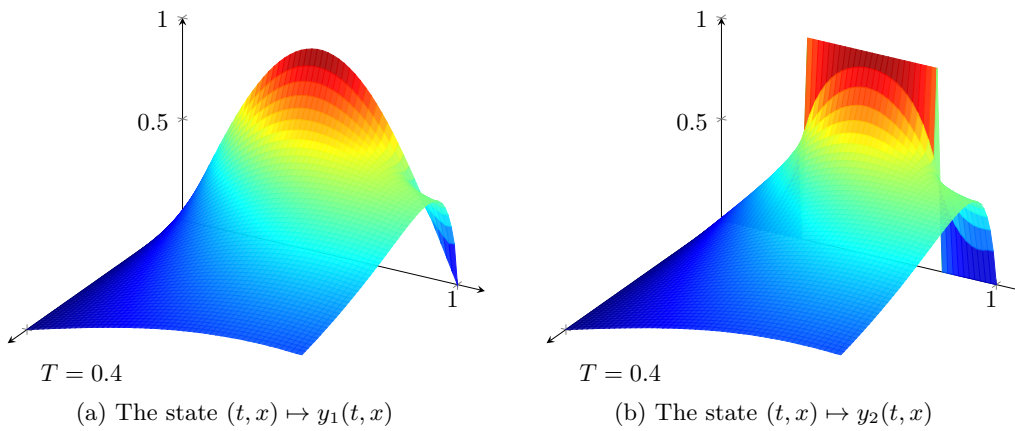


Figure IV.4: Evolution in time of the uncontrolled solution of system (IV.7)–(IV.8b).

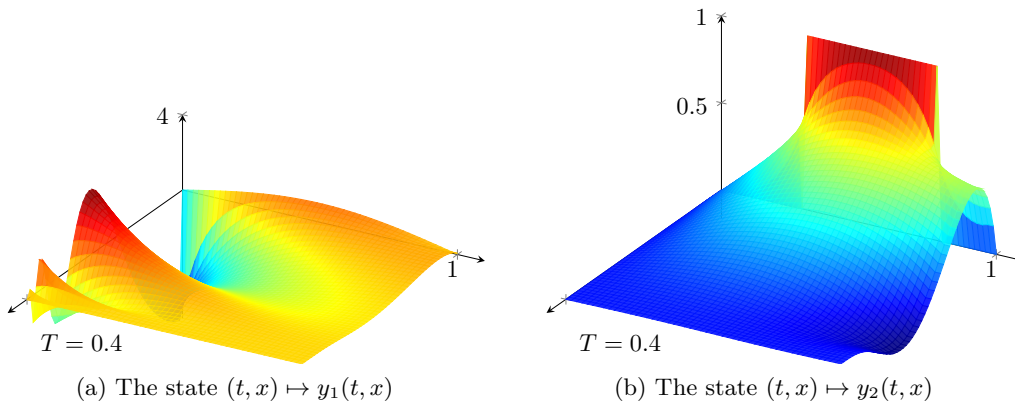


Figure IV.5: Evolution in time of the controlled solution of system (IV.7)–(IV.8b).

In Figure IV.5, we show the solution  $(y_1, y_2)$  obtained after applying the HUM control  $v(t)$  (see Figure IV.6) computed by algorithm (IV.135)–(IV.136). We observe, that due to this action, both components reach zero at the prescribed time  $T = 0.4$ . Notice that, since we have chosen  $a = 0$  in (IV.7)–(IV.8b), the action of the control acts indirectly on the second just by means of the boundary coupling. Intuitively, this problem is harder to solve than other classical problems where the coupling is made in the internal domain.

As far as the asymptotic of the method, we present in Figure IV.7 the behavior of various quantities of interest as the mesh size goes to 0. We observe that the control cost  $\|v_{\phi(h)}^{h,\delta t}\|_{L^2_{\delta t}(0,T;U_h)}$



( $\blacklozenge$ ) as well as the optimal energy  $\inf F_{\phi(h)}^{h,\delta t}$  ( $\bullet$ ) remain bounded as the mesh size  $h$  tends to 0. Also, we see that the norm of the state  $\|(y_1^{h,\delta t}(T), y_2^{h,\delta t}(T))\|_{\mathcal{H}_{-\alpha,h}}$  ( $\blacksquare$ ) behaves like  $\sim C\sqrt{\phi(h)} = Ch^2$ . This behavior is in agreement with Theorem IV.5.2 and illustrates our null controllability result.

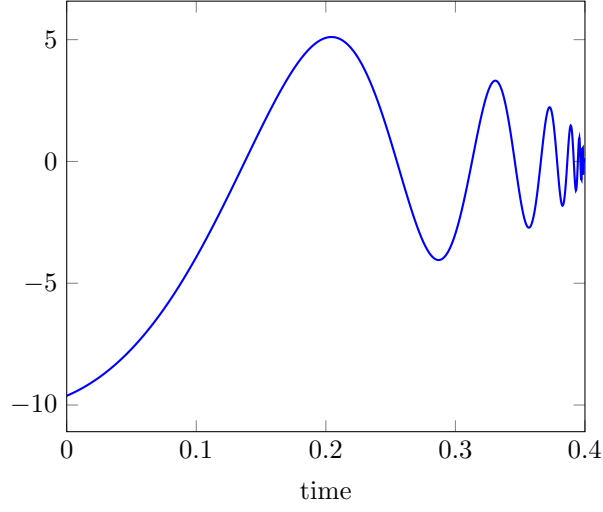


Figure IV.6: Control function  $v(t)$ .

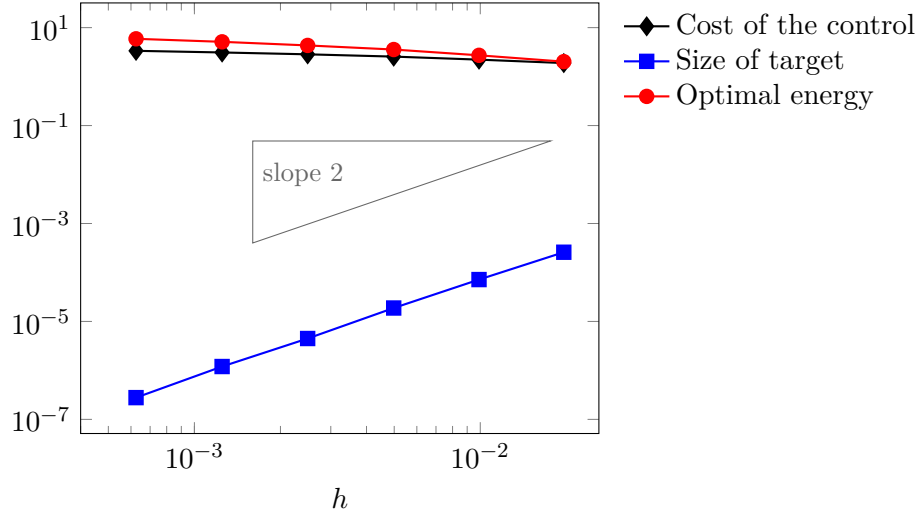


Figure IV.7: Convergence properties w.r.t. the penalized HUM approach.

**The case  $a \neq 0$ .** According to our main controllability results, Theorems IV.2.7 and IV.2.8, the controllability of system (IV.7) is guaranteed depending on the selection of the parameters  $(\alpha, a)$  and the way the control enters the system. When the control enters through the first equation, that is, when (IV.8b) is verified, we know from Lemma IV.4.7 that there exist values of  $(\alpha, a)$  for which system (IV.7) is not even approximately controllable, this is described by means of the set  $\mathcal{R}$  defined in (IV.100). We illustrate this fact below. By using a numerical algorithm, we can determine that the approximate pair  $(\alpha_0, a_0) = (1, 3.1931469)$  belongs to  $\mathcal{R}$  (see Figure IV.3) and corresponds to the critical eigenvalue  $\lambda_{\alpha_0, a_0}^c \approx 5.7421936$ . Therefore, the eigenfunction  $\Phi_{\lambda_{\alpha_0, a_0}^c}$  fails to verify the Fattorini-Hautus criterion. The next figure will elaborate this phenomena.

In Figure IV.8, we plot the eigenfunction corresponding to the critical eigenvalue  $\lambda_{\alpha_0, a_0}^c$ . We observe that the first component of the eigenfunction, that is the one in blue color, is almost

flat as it approaches to the boundary point  $x = 0$  and in fact, numerically we can compute the size of the normal derivative which is of order  $10^{-5}$ . We expect that this is somehow reflected during the penalized HUM procedure.

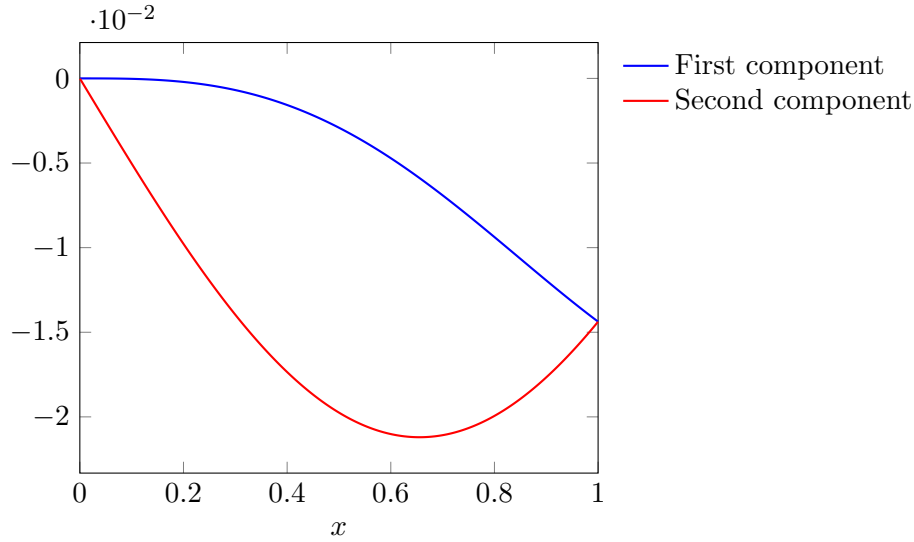


Figure IV.8: The eigenfunction  $\Phi_{\lambda_{\alpha_0, a_0}^c}$  corresponding to the critical eigenvalue  $\lambda_{\alpha_0, a_0}^c$

We set the parameters

$$\begin{aligned} T &= 0.3, \quad \gamma_1 = \gamma_2 = 1, \\ a_0 &= 3.1931469, \quad \alpha_0 = 1, \\ y_{0,1}(x) &= 10 \sin^3(2\pi x), \quad y_{0,2}(x) = 5 \times \mathbf{1}_{(0.3, 0.8)}(x), \end{aligned} \tag{IV.151}$$

and apply our computational tool to obtain boundary controls. In Figure IV.9 we observe the asymptotic behavior of the algorithm. Unlike the previous case, we observe that the optimal energy (—●—) blows up as  $\phi(h)^{-1} = h^{-4}$  while the size of the target (—■—) remains constant. This indicates that for the selection of the initial data, system (IV.7)–(IV.8b) is neither null-controllable or approximately controllable, which is in accordance with our theoretical results.

A further validation of this result can be done by adapting [31, Theorem 1.11], which gives a hint of the general behavior of the penalized HUM method in the limit. In our case, it can be shown that as  $h \rightarrow 0$ ,  $\mathcal{A}_{\alpha, h}^{-1} y^h(T)$  should converge towards a nonzero function which belongs to the space of unobservable modes. As we have seen in Section IV.4, this space consists only one element which is the eigenfunction associated to the critical eigenvalue. Thus, we expect to see this at the numerical level.

In Figure IV.10, it can be seen that as  $N$  increases (and therefore  $h \downarrow 0$ ) the target is converging towards some function instead of going to zero, for the critical value  $(\alpha_0, a_0)$ . In this case, it is clear that the target converges to the critical eigenfunction (up to a constant) shown in Figure IV.8 which validates the discussion above.

At this point, we shall mention that the approximation of the critical parameter  $a_0$  plays an important role in the numerical experiments. In Figure IV.11 we present a series of experiments where the parameter  $a_0$  is approximated by truncating up to a certain number of decimals. For a fixed value of  $h$ , we see that for a rough approximation (two decimals) the convergence of the target is not as good as for the finer ones (in the experiments shown  $h = 1/1600$ ). We recall that the critical parameters come from obtaining a simultaneous solution to (IV.99), therefore the non-controllability result is very sensitive to even small changes of such values. The behavior shown in Figure IV.11 is therefore consistent with this fact.

We finish the discussion here by emphasizing that the behavior shown in Figure IV.9 comes from the fact that the control is placed on the boundary of the *first component*, namely the

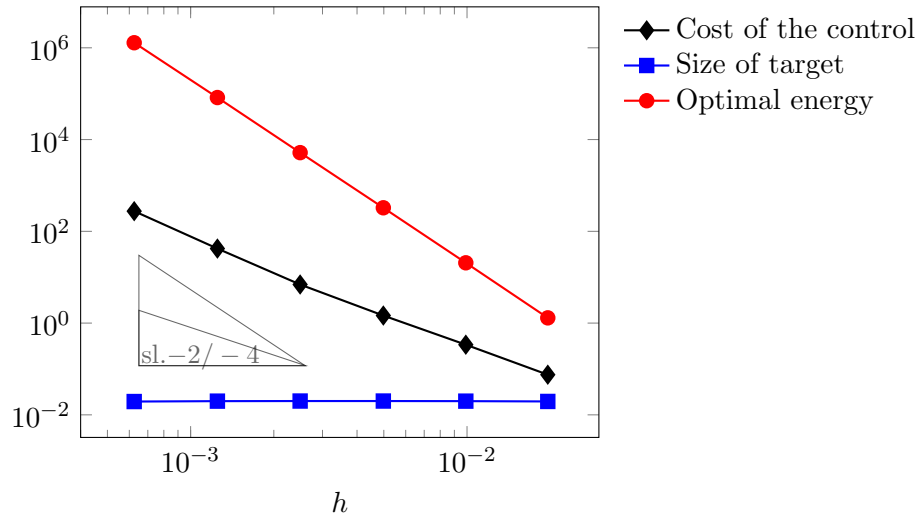
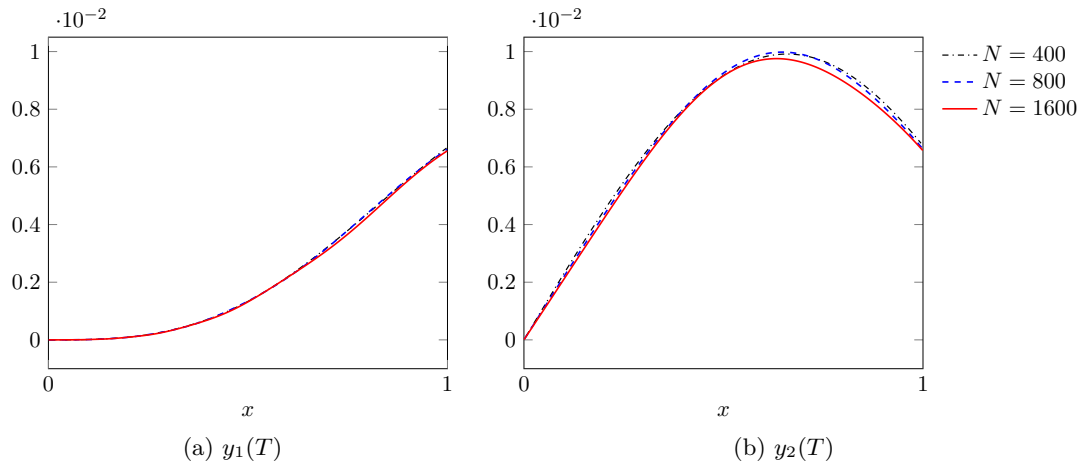
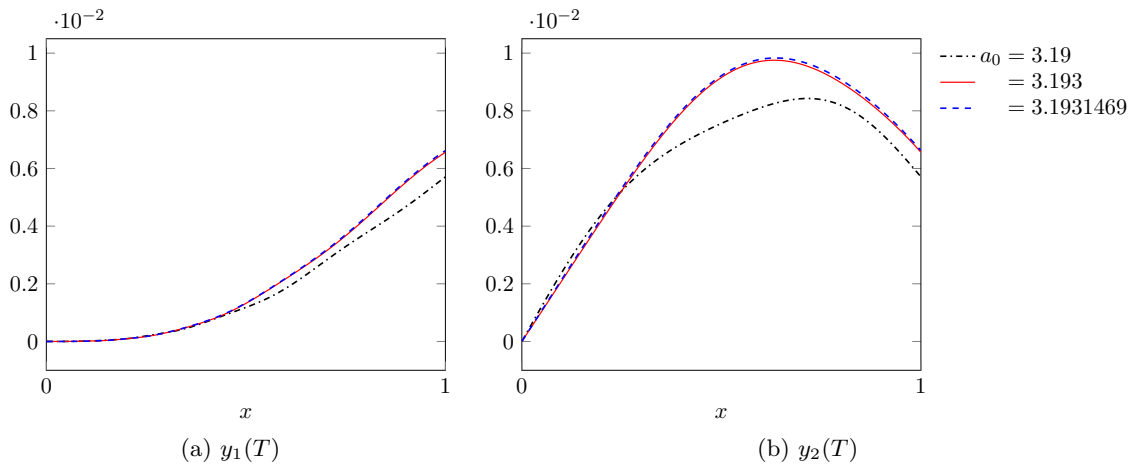


Figure IV.9: Convergence properties of the HUM approach for the critical case.

Figure IV.10: Convergence of the target as  $h \downarrow 0$  for the critical values  $\alpha_0 = 1, a_0 = 3.1931469$ .Figure IV.11: Convergence of the target for  $\alpha_0 = 1$  and different approximations of  $a_0$ .

condition (IV.8b). If instead we consider the boundary control on the *second component* as (IV.8a), Theorem IV.2.7 indicates that regardless the choice of  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}$ , system (IV.7)–(IV.8a) is null-controllable at any time  $T > 0$ . We illustrate this fact in Figure IV.12, where we consider the same parameters as in (IV.151) with the difference that the control is applied on the boundary of the *second equation*. We observe that as  $h \rightarrow 0$  the size of the target decreases as  $\sqrt{\phi(h)} = h$  and both the control cost and the optimal energy remain bounded, which is in concordance with the theoretical controllability result, namely Theorem IV.2.7.

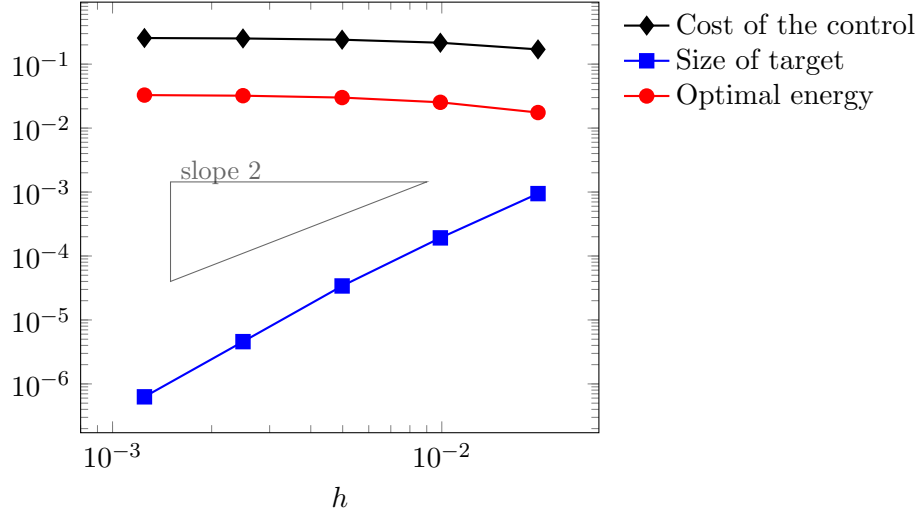


Figure IV.12: Convergence properties of the HUM procedure with the critical value  $(\alpha_0, a_0)$  but control applied on the second equation.

#### IV.5.3.2 Neumann boundary control cases with Kirchhoff type condition

The goal of this section is to show that our computational tool can be used to illustrate other cases not covered in the theoretical results presented in this paper. This is possible, thanks to the general methodology we introduced in Section IV.5.2.

We will discuss about the controllability of system (IV.7) in the case when the boundary conditions at  $x = 0$  are replaced by the Neumann conditions

$$\text{either } \partial_x y_1(t, 0) = v(t), \quad \partial_x y_2(t, 0) = 0 \quad \text{in } (0, T), \quad (\text{IV.152a})$$

$$\text{or } \partial_x y_1(t, 0) = 0, \quad \partial_x y_2(t, 0) = v(t) \quad \text{in } (0, T). \quad (\text{IV.152b})$$

The well-posedness of this kind of system has been discussed in Section IV.7.2.

For later use, we write down the observation operators corresponding to the Neumann cases (IV.7)–(IV.152a) and (IV.7)–(IV.152b), respectively as follows

$$\tilde{\mathcal{B}}_1^* : u = (u_1, u_2) \in (H^1(0, 1))^2 \mapsto \gamma_1(0)u_1(0), \quad (\text{IV.153a})$$

$$\tilde{\mathcal{B}}_2^* : u = (u_1, u_2) \in (H^1(0, 1))^2 \mapsto \gamma_2(0)u_2(0). \quad (\text{IV.153b})$$

For the Neumann control system (IV.7)–(IV.152), one could again play with the *Carleman technique* and/or a rigorous study of the spectral analysis (which is at the heart of the moments approach). But we shall not pursue any detailed study regarding those in this manuscript.

However, at the numerical level, using the discretization scheme shown in Section IV.5.2, we just have to set the matrices

$$\mathcal{N}_0 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \quad \mathcal{D}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\text{IV.154})$$

$$\mathcal{N}_1 = \begin{pmatrix} 0 & 0 \\ \gamma_1 & \gamma_2 \end{pmatrix}, \quad \mathcal{D}_1 = \begin{pmatrix} 1 & -1 \\ \alpha & 0 \end{pmatrix}, \quad (\text{IV.155})$$

and compute the formulas given in (IV.138)–(IV.146) (recall that, for a numerical study we always set the diffusion coefficients  $\gamma_1, \gamma_2$  as *positive constants*).

This simple idea actually allows to test for many configurations and different values of  $a$  and  $\alpha$ . We first consider the following simulation parameters

$$\begin{aligned} T &= 0.5, \quad \gamma_1 = \gamma_2 = 1, \\ a &= 2, \quad \alpha = 4, \\ y_{0,1}(x) &= \sin(\pi x), \quad y_{0,2}(x) = \mathbf{1}_{(0.3,0.8)}(x), \end{aligned}$$

and use our tool to obtain numerical results for two different configurations. In Figure IV.13a, we show the convergence result for the case where  $v$  is applied on the *first equation*, that is, (IV.152a). We can see that as  $h$  tends to zero, the size of the target decreases as  $\sqrt{\phi(h)} = h^2$  and both the optimal energy and the control cost remain bounded.

On the other hand, we show the result in Figure IV.13b by changing the control to the *second equation*, i.e., we consider (IV.152b).

We observe that the behavior of the convergence of the method is exactly the same as in the previous example.

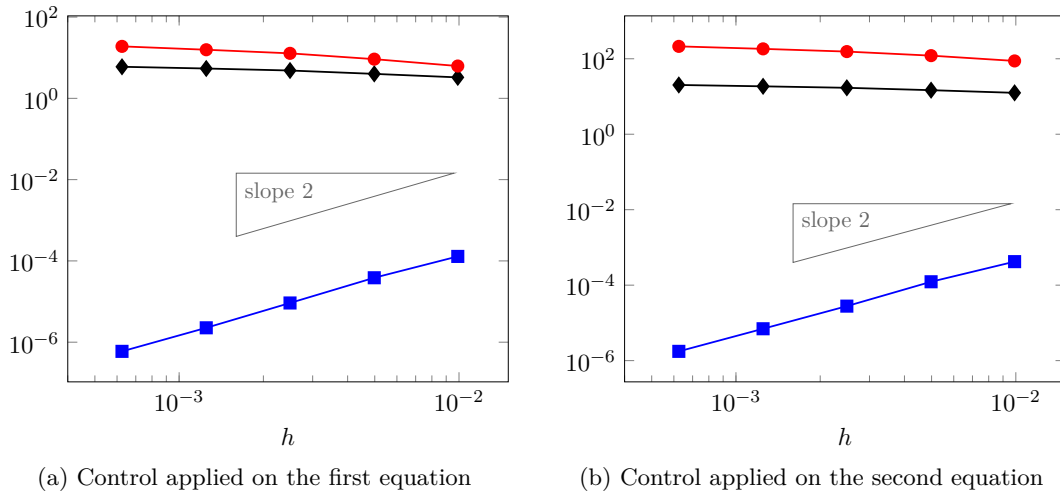


Figure IV.13: Convergence properties of the HUM method with Neumann control applied in different equations. Same legend as in Figure IV.12.

Both the simulations point toward a positive null controllability result, nevertheless one should be cautious with such conclusion. In fact, in the case where the control is applied on the component  $y_2$ , we expect that some adaptations can be made to our *Carleman estimate* presented in Theorem IV.3.2 to deduce a similar result for (IV.7)–(IV.152b), and thus one can expect null-controllability for any  $a$  and  $\alpha$ .

On the other hand, we have seen in Section IV.4.1 (for the Dirichlet case) that a detailed analysis of the spectral behavior of the adjoint elliptic operator is required when the control is applied on the component  $y_1$ , and so, the controllability of (IV.7)–(IV.152a) and the answer of whether the system is null-controllable or not in the whole space  $E$  is far from obvious.

Indeed, numerical evidence presented in Figure IV.14 shows that as in the Dirichlet case, there exists at least one couple  $(\alpha, a)$  for which the observation of one eigenfunction is 0. We numerically find some pair (approximate value),  $(\alpha_c, a_c) = (0.1, 1.2369289)$ . In Figure IV.15 we are plotting the first eigenfunction associated to this pair and from there, it is clear that such eigenfunction is non-observable (see the def. of observation operator  $\tilde{\mathcal{B}}_1^*$  in (IV.153a)).

We use this new couple  $(\alpha_c, a_c)$  for some simulation purposes. In Figure IV.16a, we present the convergence properties w.r.t. to the penalized HUM approach and as in the Dirichlet case we observe that the size of the target is not decreasing while the optimal energy and cost of the control are blowing up. This points towards a non-controllable result.

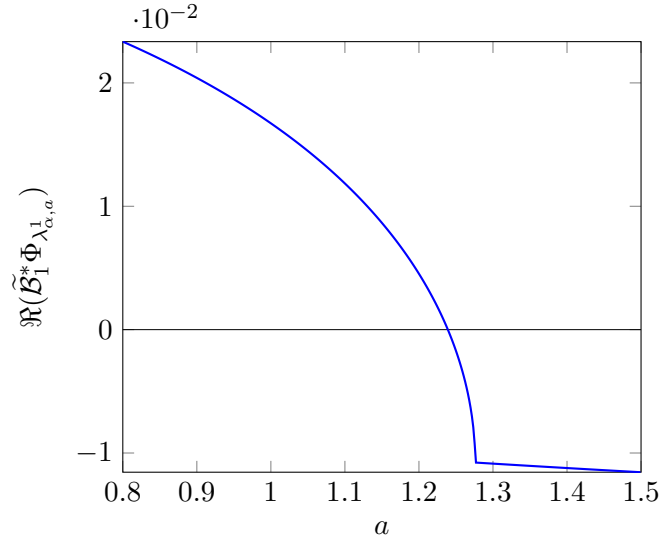


Figure IV.14: Size of the observations w.r.t. the first eigenfunctions  $\Phi_{\lambda_{\alpha,a}}^1$ , in the Neumann case (IV.7)–(IV.152a) for  $\alpha = 0.1$  and  $a \in [0.8, 3]$ .

Following with the discussion of the Dirichlet case, we see in Figure IV.17 that the target is indeed converging towards the critical eigenfunction (up to some constant) which is consistent with the lack of controllability. In view of these results, a deeper study of the Neumann control case is needed to conclude.

We would like to emphasize now that as in the Dirichlet case, we need a good approximation of the critical parameter  $a_c$  to observe the lack of controllability of the system. In Figure IV.18, we see the convergence of the target for  $h = 1/1600$  and different approximations of  $a_c$ . This experiment seems to be more sensitive than the previous case since we need at least four decimal approximation (for the Dirichlet case, it was three; see Fig. IV.11) of the parameter  $a_c$  (precisely, 1.2369) to obtain some convergence of the target for the given value of  $h$ .

We finish this part by placing an experimental result for the same critical pair  $(\alpha_c, a_c)$  but the control we exert on the *second component* instead of the first one, i.e., for the system (IV.7)–(IV.152b). In contrast of the case (IV.7)–(IV.152a), we obtain here nice behavior of the target size, control cost and optimal energy (see Figure IV.16b), which shows a positive hope of the null-controllability of the system (IV.7)–(IV.152b), at least from the numerical point of view.

### IV.5.3.3 An apparently boundary “coupled” case

As we have seen in Section IV.5.3.1, we have examples where even if the coupling coefficient  $a = 0$ , the null-controllability of the system is guaranteed by means of the boundary coupling. However, there are some other cases where this is not enough to control the system. Consider the simple example

$$\left\{ \begin{array}{ll} \partial_t y_1 - \partial_x^2 y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t y_2 - \partial_x^2 y_2 + a y_1 = 0 & \text{in } (0, T) \times (0, 1), \\ y_1(t, 0) - y_2(t, 0) = v(t) & \text{in } (0, T), \\ \partial_x y_1(t, 0) + \partial_x y_2(t, 0) = 0 & \text{in } (0, T), \\ y_1(t, 1) = y_2(t, 1) & \text{in } (0, T), \\ \partial_x y_1(t, 1) + \partial_x y_2(t, 1) = 0 & \text{in } (0, T), \\ y_i(0, x) = y_{0i}(x) & \text{in } (0, 1), \ i = 1, 2. \end{array} \right. \quad (\text{IV.156})$$

**The case  $a = 0$ , never controllable.** We plot in Fig. IV.19 the convergence of the numerical method for such example. We immediately see that the control has no effect on the target

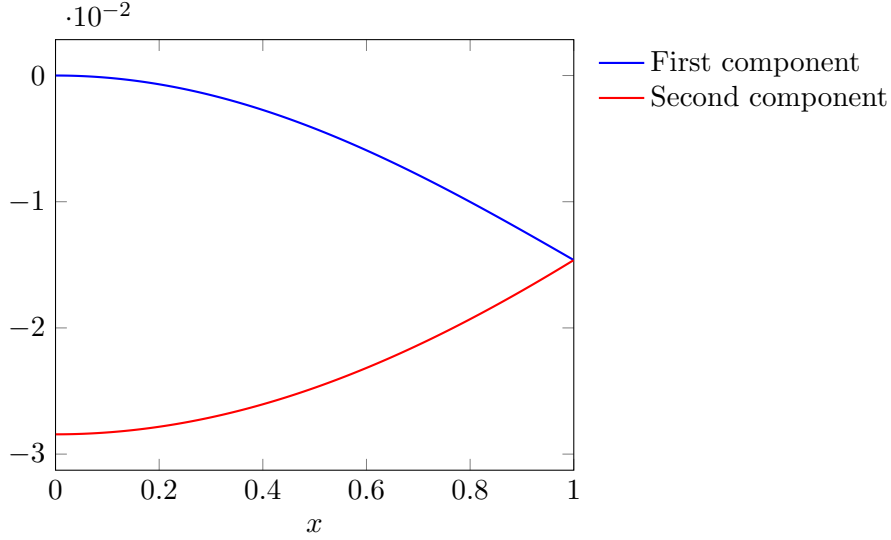


Figure IV.15: The eigenfunction  $\Phi_{\lambda_{\alpha_c, a_c}^1}$  corresponding to the critical eigenvalue (the first one)  $\lambda_{\alpha_c, a_c}^1$  (Neumann case).

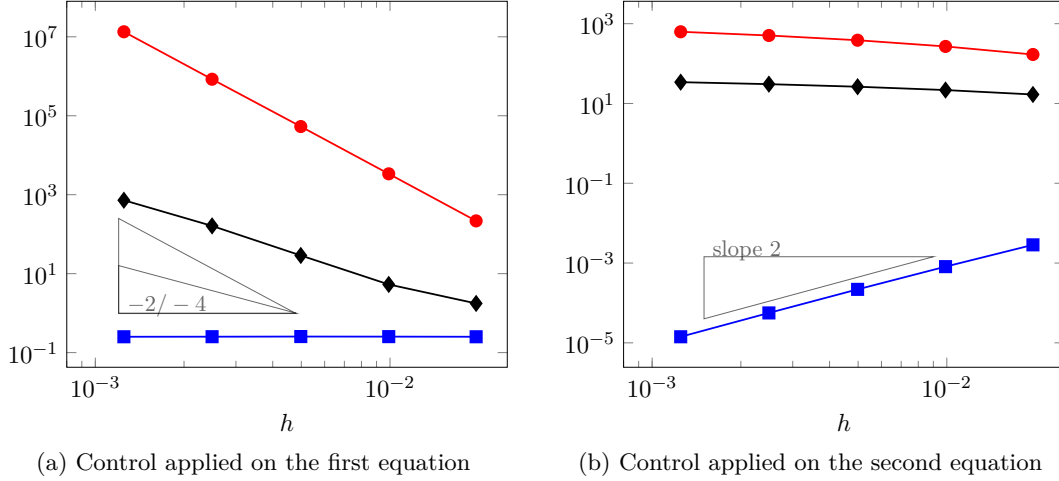


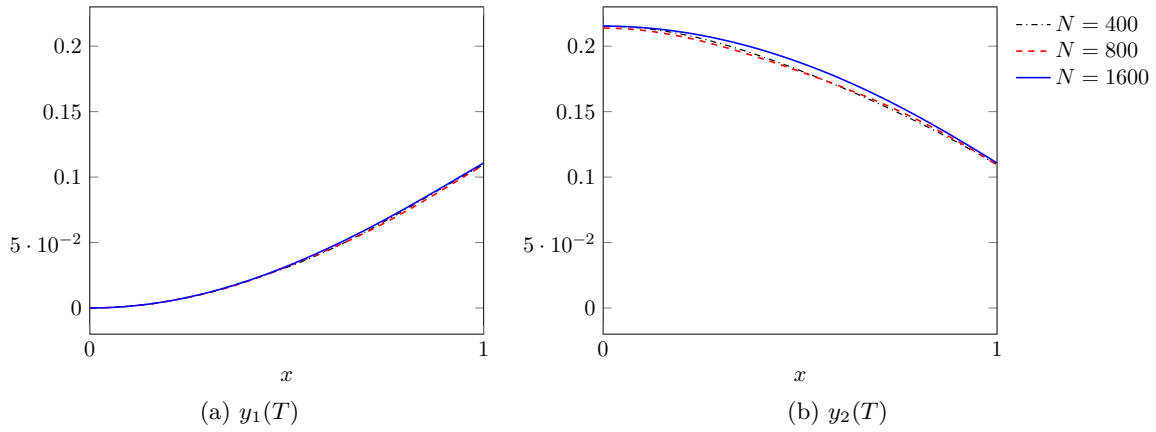
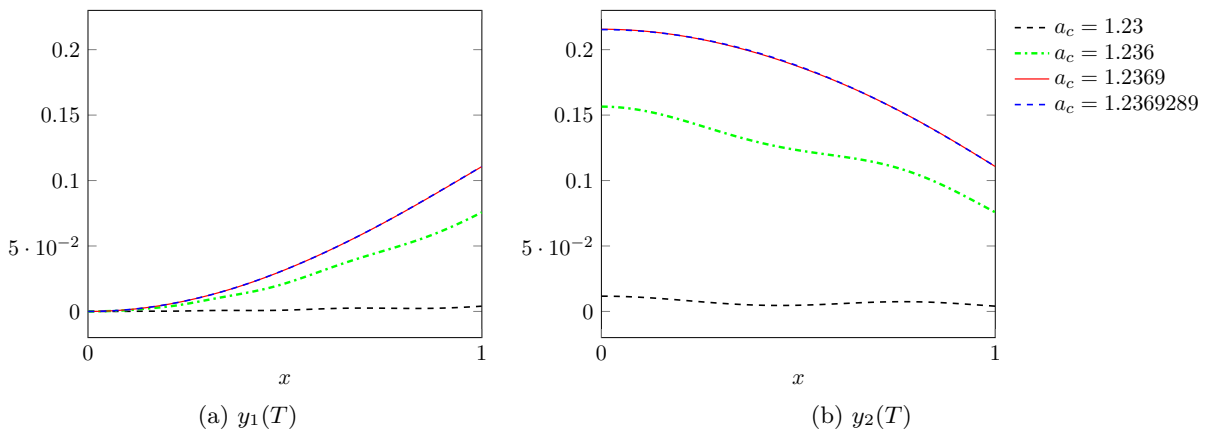
Figure IV.16: Convergence properties w.r.t. HUM approach for the critical pair  $(\alpha_c, a_c)$  with Neumann control applied on different equations. Same legend as in Figure IV.12.

solution and that the optimal energy blows up at a rate of  $h^{-4}$ . This behavior corresponds to a non-controllable case and can be explained by looking at the system verified by the change of variables  $w_1 = y_1 - y_2$  and  $w_2 = y_1 + y_2$ . Indeed, we see that the new variables satisfy

$$\begin{cases} \partial_t w_1 - \partial_x^2 w_1 = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t w_2 - \partial_x^2 w_2 = 0 & \text{in } (0, T) \times (0, 1), \\ w_1(t, 0) = v(t), \quad w_1(t, 1) = 0 & \text{in } (0, T), \\ \partial_x w_2(t, 0) = \partial_x w_2(t, 1) = 0, & \text{in } (0, T), \\ w_1(0, x) = y_{01}(x) - y_{02}(x), \quad w_2 = y_{01}(x) + y_{02}(x) & \text{in } (0, 1). \end{cases}$$

From here it is clear that both equations are decoupled and there is no hope for the control action to enter into the second equation of the system. This confirms the behavior shown in Fig. IV.19.

**The case  $a \neq 0$ , possibly controllable.** We mention that the situation is different if there is internal coupling, that is  $a \in \mathbb{R}^*$  in (IV.156). After a straightforward computation, we see

Figure IV.17: Convergence of the target as  $h \downarrow 0$  for the critical values  $(\alpha_c, a_c) = (0.1, 1.2369289)$ .Figure IV.18: Convergence of the target for  $\alpha_c = 0.1$  and different approximations of  $a_c$ .

that the system in the  $w_i$  variables satisfy

$$\begin{cases} \partial_t w_1 - \partial_x^2 w_1 - a \frac{w_1 + w_2}{2} = 0 & \text{in } (0, T) \times (0, 1), \\ \partial_t w_2 - \partial_x^2 w_2 + a \frac{w_1 + w_2}{2} = 0 & \text{in } (0, T) \times (0, 1), \\ w_1(t, 0) = v(t), \quad w_1(t, 1) = 0 & \text{in } (0, T), \\ \partial_x w_2(t, 0) = \partial_x w_2(t, 1) = 0, & \text{in } (0, T), \\ w_1(0, x) = y_{01}(x) - y_{02}(x), \quad w_2 = y_{01}(x) + y_{02}(x) & \text{in } (0, 1). \end{cases}$$

Observe that now there is internal coupling on both equations and so the control  $v$  acts indirectly on  $w_2$ . We observe in Fig. IV.20 that this modifies the controllability properties of the system: the size of the target behaves as  $h^2$  while the optimal energy and the control cost remain bounded as  $h \rightarrow 0$ . This seems to confirm that the system is in fact null-controllable.

## IV.6 A $3 \times 3$ coupled parabolic system with one or two Dirichlet control(s)

In this section, we shall pose some  $3 \times 3$  interior-boundary coupled parabolic systems with one or two Dirichlet boundary control(s). Alike the  $2 \times 2$  case, here also we choose an interior coupling with constant coefficient and the Kirchhoff type law to define the boundary coupling.



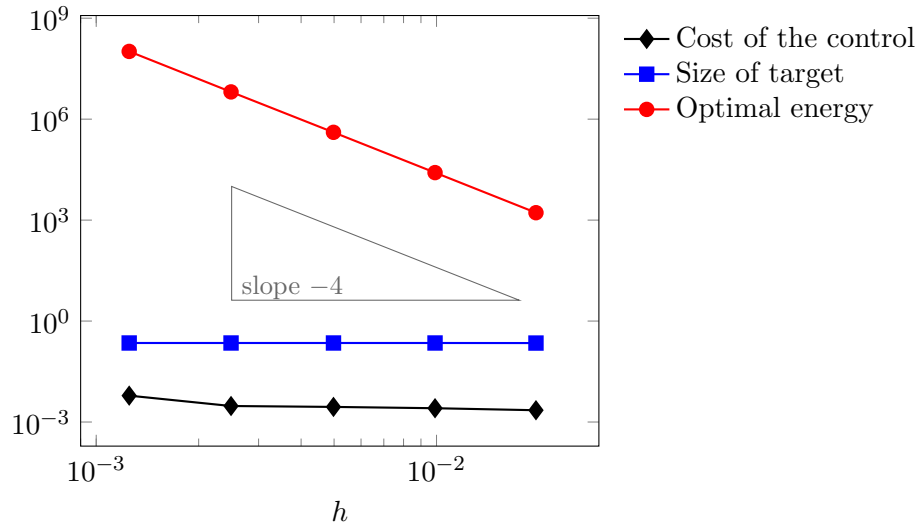
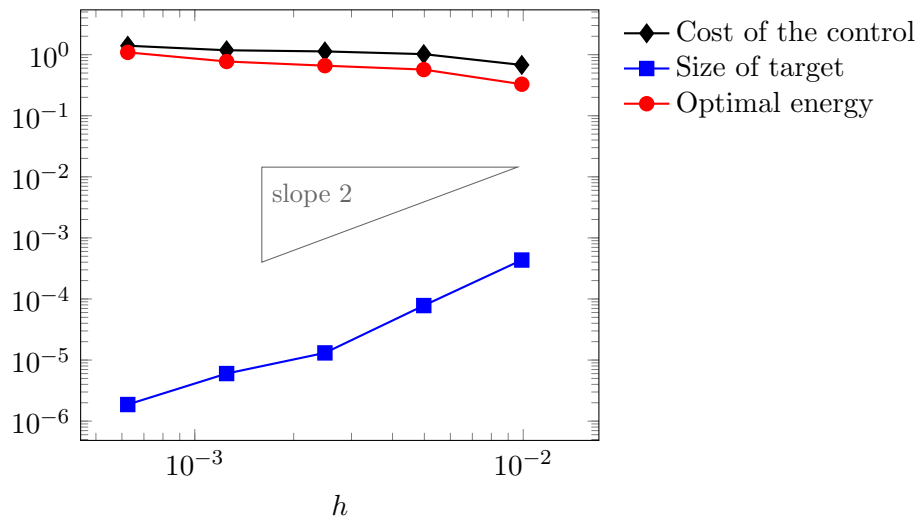


Figure IV.19: Control on the difference and zero coupling terms: obviously not controllable

Figure IV.20: HUM convergence properties for the system with control on the difference and non-zero coupling term  $a = 3$ .

As we mentioned in Section IV.1.1.2, the goal is to see what happens if we exert only *one* boundary control to that  $3 \times 3$  system. In this case, we realize that the null-controllability of the system depends on the choices of the components through which the control is being exerted, and moreover, sometimes it depends significantly on the choices of the *interior coupling coefficient* and *Kirchhoff parameter*. Indeed there are some negative-controllability cases, and in that situation, if one extra boundary control is considered to some other component, then we recover the null-controllability again.

The motivation to study such  $3 \times 3$  model along with an interior coupling, is coming from the point of considering a parabolic control systems on a metric graph where some (or, all) of the edges correspond to some vectorial parabolic equations. In this context, we refer to Section I.3, Ch. I where we have discussed a general pathology for this kind of system and in particular, our  $3 \times 3$  case will be understood in a graph with two edges: one edge corresponds to a vectorial pde while the other one to a scalar pde, see Fig. IV.21.

In the next paragraphs, we first present the main results of our system with *two* controls and then we move forward to the case of choosing only *one* control.

#### IV.6.1 The system with two boundary controls

As usual, we define the space  $Q := (0, T) \times (0, 1)$ , where  $T > 0$  denotes some finite time. We now prescribe the following system with two controls on any two of the three components,

$$\begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = 0 & \text{in } Q, \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) = 0 & \text{in } Q, \\ \partial_t y_3 - \partial_x(\gamma_3 \partial_x y_3) + a y_1 = 0 & \text{in } Q, \\ \begin{cases} y_1(t, 1) = y_2(t, 1) = y_3(t, 1), \\ \sum_{i=1}^3 \gamma_i(1) \partial_x y_i(t, 1) + \alpha y_1(t, 1) = 0 \end{cases} & \text{in } (0, T), \\ y_i(0, x) = y_{0,i}(x), \quad \text{for } i = 1, 2, 3, & \text{in } (0, 1), \end{cases} \quad (\text{IV.157})$$

with the controls applied either on  $y_2$  and  $y_3$

$$y_1(t, 0) = 0, \quad y_2(t, 0) = v(t), \quad y_3(t, 0) = \tilde{v}(t) \quad \text{in } (0, T), \quad (\text{IV.158a})$$

or, on  $y_1$  and  $y_2$

$$y_1(t, 0) = v(t), \quad y_2(t, 0) = \tilde{v}(t), \quad y_3(t, 0) = 0 \quad \text{in } (0, T), \quad (\text{IV.158b})$$

or, on  $y_1$  and  $y_3$

$$y_1(t, 0) = v(t), \quad y_2(t, 0) = 0, \quad y_3(t, 0) = \tilde{v}(t) \quad \text{in } (0, T). \quad (\text{IV.158c})$$

Here  $\alpha \geq 0$  is the Kirchhoff parameter and we choose some  $a \neq 0$  through which we define the linear interior coupling of  $y_1$  to the equation of  $y_3$ . The quantities  $v$  and  $\tilde{v}$  are supposed to act as the control inputs at the left end point  $x = 0$ . Also, the diffusion coefficients satisfy the following assumption.

**Assumption IV.6.1.** *Let the diffusion coefficients  $\gamma_i$  for  $i = 1, 2, 3$ , are functions of class  $C^1([0, 1])$  such that*

$$\gamma(x) := \begin{pmatrix} \gamma_1(x) & 0 & 0 \\ 0 & \gamma_2(x) & 0 \\ 0 & 0 & \gamma_3(x) \end{pmatrix}$$

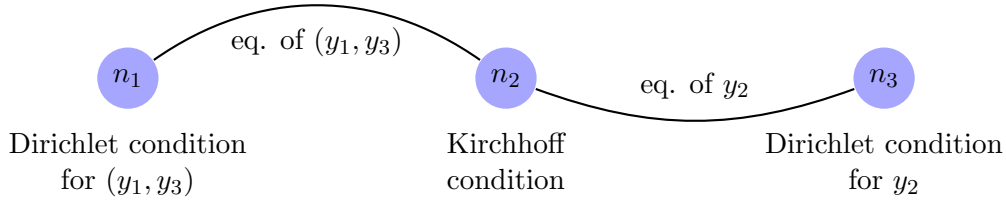


Figure IV.21: The system (IV.157)–(IV.158) on a metric graph of two edges.

is uniformly positive definite on the interval  $[0, 1]$ , that is there exists some quantity  $\gamma_{\min} > 0$  such that

$$\langle \gamma(x)\nu, \nu \rangle_{\mathbb{R}^3} := \sum_{i=1}^3 \gamma_i(x) |\nu_i|^2 \geq \gamma |\nu|_{\mathbb{R}^3}^2, \quad \forall x \in [0, 1], \quad \nu \in \mathbb{R}^3.$$

One can prescribe the above system into a metric graph with two edges, where one edge corresponds to the vectorial equation of  $(y_1, y_3)$  and other one corresponds to the scalar equation of  $y_2$ ; see Figure IV.21.

**The functional setting.** Up to an abuse of notation, we shall incorporate some similar notations as we have introduced for the  $2 \times 2$  systems. First, we denote  $E := (L^2(0, 1))^3$ . Also, we need to define the following operators and spaces for later use.

- Consider the operator

$$\mathcal{A}_\alpha := \begin{pmatrix} -\partial_x(\gamma_1 \partial_x) & 0 & 0 \\ 0 & -\partial_x(\gamma_2 \partial_x) & 0 \\ 0 & 0 & -\partial_x(\gamma_3 \partial_x) \end{pmatrix}, \quad (\text{IV.159a})$$

with its domain

$$D(\mathcal{A}_\alpha) = \left\{ u := (u_1, u_2, u_3) \in (H^2(0, 1))^3 \mid u_1(0) = u_2(0) = u_3(0) = 0, \text{ and } \right. \quad (\text{IV.159b})$$

$$\left. u_1(1) = u_2(1) = u_3(1), \quad \sum_{i=1}^3 \gamma_i(1) u'_i(1) + \alpha u_1(1) = 0 \right\}.$$

- Let us now consider the space  $\tilde{\mathcal{H}}_\alpha := D(\mathcal{A}_\alpha^{1/2})$  as a completion of  $D(\mathcal{A}_\alpha)$  with respect to the norm

$$\|u\|_{\tilde{\mathcal{H}}_\alpha} = (\mathcal{A}_\alpha u, u)_E^{1/2} := \left( \sum_{i=1}^3 \gamma_i(x) |u'_i(x)|^2 dx + \alpha |u_1(1)|^2 \right)^{1/2}, \quad \forall u \in D(\mathcal{A}_\alpha), \quad (\text{IV.160})$$

and one can even show that

$$\tilde{\mathcal{H}}_\alpha = \left\{ u := (u_1, u_2, u_3) \in (H^1(0, 1))^3 \mid u_1(0) = u_2(0) = u_3(0) = 0, \text{ and } \right. \quad (\text{IV.161})$$

$$\left. u_1(1) = u_2(1) = u_3(1) \right\}.$$

Moreover, we denote the dual space  $\tilde{\mathcal{H}}_{-\alpha}$  of  $\tilde{\mathcal{H}}_\alpha$  with respect to the pivot space  $E$ .

- Next, we denote

$$\mathcal{A}_{\alpha,a} := \begin{pmatrix} -\partial_x(\gamma_1 \partial_x) & 0 & 0 \\ 0 & -\partial_x(\gamma_2 \partial_x) & 0 \\ a & 0 & -\partial_x(\gamma_3 \partial_x) \end{pmatrix} = \mathcal{A}_\alpha + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, \quad (\text{IV.162})$$

with the same domain  $D(\mathcal{A}_{\alpha,a}) := D(\mathcal{A}_\alpha)$ . By convention, we have  $\mathcal{A}_{\alpha,0} := \mathcal{A}_\alpha$ .

- Finally, its worth denoting the adjoint of  $\mathcal{A}_{\alpha,a}$  by  $\mathcal{A}_{\alpha,a}^*$ , which is

$$\mathcal{A}_{\alpha,a}^* := \begin{pmatrix} -\partial_x(\gamma_1 \partial_x) & 0 & a \\ 0 & -\partial_x(\gamma_2 \partial_x) & 0 \\ 0 & 0 & -\partial_x(\gamma_3 \partial_x) \end{pmatrix} \quad (\text{IV.163})$$

with its domain  $D(\mathcal{A}_{\alpha,a}^*) := D(\mathcal{A}_{\alpha,a}) = D(\mathcal{A}_\alpha)$ .

Let us write the following proposition.

**Proposition IV.6.2** (Existence of analytic semigroup). *The operators  $(-\mathcal{A}_{\alpha,a}, D(\mathcal{A}_{\alpha,a}))$  and  $(-\mathcal{A}_{\alpha,a}^*, D(\mathcal{A}_{\alpha,a}^*))$  define analytic semigroups in  $E$ .*

The proof is as similar as the proof of Proposition IV.2.1. For our use, we hereby express the semigroup  $(e^{-t\mathcal{A}_{\alpha,a}^*})_{t \geq 0}$  defined by  $(-\mathcal{A}_{\alpha,a}^*, D(\mathcal{A}_{\alpha,a}^*))$ .

#### IV.6.1.1 Controls on the 2nd and 3rd components

Let us now introduce the adjoint system (without any boundary data) to any of the control problems under (IV.157)–(IV.158),

$$\begin{cases} -\partial_t q_1 - \partial_x(\gamma_1 \partial_x q_1) + a q_3 = 0 & \text{in } Q, \\ -\partial_t q_2 - \partial_x(\gamma_2 \partial_x q_2) = 0 & \text{in } Q, \\ -\partial_t q_3 - \partial_x(\gamma_3 \partial_x q_3) = 0 & \text{in } Q, \\ q_1(t, 0) = q_2(t, 0) = q_3(t, 0) = 0 & \text{in } (0, T), \\ \begin{cases} q_1(t, 1) = q_2(t, 1) = q_3(t, 1), \\ \sum_{i=1}^3 \gamma_i(1) \partial_x q_i(t, 1) + \alpha q_1(t, 1) = 0 \end{cases} & \text{in } (0, T), \\ q_i(T, x) = \zeta_i(x), \quad \text{for } i = 1, 2, 3, & \text{in } (0, 1). \end{cases} \quad (\text{IV.164})$$

By Proposition IV.6.2, we have: for any given data  $\zeta := (\zeta_1, \zeta_2, \zeta_3) \in E$ , there exists a unique weak solution  $q := (q_1, q_2, q_3) \in \mathcal{C}^0([0, T]; E) \in L^2(0, T; \tilde{\mathcal{H}}_\alpha)$  to the adjoint system (IV.164) and it can be expressed as follows

$$q(t, x) := e^{-(T-t)\mathcal{A}_{\alpha,a}^*} \zeta(x) \quad \forall (t, x) \in Q.$$

Moreover, for any given data  $\zeta \in \tilde{\mathcal{H}}_\alpha$ , one can show that  $q \in L^\infty(0, T; \tilde{\mathcal{H}}_\alpha) \cap L^2(0, T; (H^2(0, 1))^3)$ , the proof of which can be drawn in a similar way as of Proposition IV.2.3. This will lead us the well-posedness of the control systems (IV.157)–(IV.158) in a dual sense as we describe earlier for the  $2 \times 2$  system, we omit the details here.

The observation operator for the system (IV.157)–(IV.158a) (i.e., when we consider the controls on  $y_2$  and  $y_3$ ) is

$$\mathcal{B}_{2,3}^* = \gamma \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_x \Big|_{x=0} : (H^2(0, 1))^3 \rightarrow \mathbb{R}^2, \quad (\text{IV.165})$$

so that one can formulate the following null-control problem.

**Proposition IV.6.3.** *Let  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$  and any finite time  $T > 0$  be given. Then, any two functions  $v \in L^2(0, T)$  and  $\tilde{v} \in L^2(0, T)$  are said to be null-controls for the system (IV.157)–(IV.158a), if and only if we have: for any  $\zeta \in \tilde{\mathcal{H}}_\alpha$ ,*

$$-\langle y_0, e^{-T\mathcal{A}_{\alpha,a}^*} \zeta \rangle_{\tilde{\mathcal{H}}_{-\alpha}, \tilde{\mathcal{H}}_\alpha} = \int_0^T \left\langle \begin{pmatrix} v(t) \\ \tilde{v}(t) \end{pmatrix}, \mathcal{B}_{2,3}^*(e^{-(T-t)\mathcal{A}_{\alpha,a}^*} \zeta) \right\rangle_{\mathbb{R}^2} dt.$$

**Boundary null-controllability of the system (IV.157)–(IV.158a).**

**Theorem IV.6.4 (Null-controllability).** *Let any  $\alpha \geq 0$ ,  $a \in \mathbb{R}$  and  $T > 0$  be given. Then, for any  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$ , there exists two null-controls  $v, \tilde{v} \in L^2(0, T)$  for the problem (IV.157)–(IV.158a), acting on  $y_2$  and  $y_3$  at the left end point of  $(0, 1)$ , such that they satisfy the estimate*

$$\|v\|_{L^2(0, T)} + \|\tilde{v}\|_{L^2(0, T)} \leq Ce^{C/T} \|y_0\|_{\mathcal{H}_{-\alpha}},$$

with the constant  $C := C(\gamma_i, \alpha, a) > 0$  ( $i = 1, 2, 3$ ), that does not depend on  $T > 0$  or  $y_0$ .

The proof will be based on by a Carleman technique.

• **A global boundary Carleman estimate.** Let us first introduce the following space:

$$\mathcal{Q} := \left\{ q := (q_1, q_2, q_3) \in (\mathcal{C}^2(\overline{Q}))^3 \mid q_i(t, 0) = 0, \ i = 1, 2, 3, \ q_1(t, 1) = q_2(t, 1) = q_3(t, 1), \right. \\ \left. \text{and } \sum_{i=1}^3 \gamma_i(1) \partial_x q_i(t, 1) + \alpha q_1(t, 1) = 0, \ \forall t \in [0, T] \right\},$$

– **Construction of weight functions.** Let  $\mu_0 \in (0, 1)$  close enough to 1 such that

$$\left[ \frac{12^3 \mu_0}{(1 - \mu_0)^3} (\gamma_2^2(1) + \gamma_3^2(1)) - 13\gamma_1^2(1) \right] \geq 1. \quad (\text{IV.166})$$

We then consider the following affine functions

$$\begin{cases} \beta_i(x) = 2 + c_i(x - 1), & \forall x \in [0, 1], \\ \text{with } c_1 = 1, \ c_2 = c_3 = \frac{-12}{(1 - \mu_0)}, & \text{for } 0 < \mu_0 < 1, \end{cases} \quad (\text{IV.167})$$

so that, we have

$$\beta_3 = \beta_2 \geq \beta_1 > 0, \text{ in } [0, 1], \text{ and } \beta_3(1) = \beta_2(1) = \beta_1(1).$$

Now, as previous we let  $\lambda > 1$  and  $K = 2 \max_{1 \leq i \leq 3} \|\beta_i\|_\infty$  and define the standard weight functions  $\varphi_i$  and  $\eta_i$  for  $i = 1, 2, 3$ , as follows

$$\varphi_i(t, x) = \frac{e^{\lambda \beta_i(x)}}{t(T - t)}, \quad \eta_i(t, x) = \frac{e^{\lambda K} - e^{\lambda \beta_i(x)}}{t(T - t)}, \quad \forall (t, x) \in Q, \quad (\text{IV.168})$$

and thanks to the fact that  $\beta_3(1) = \beta_2(1) = \beta_1(1)$ , it follows that the above functions satisfies

$$\begin{cases} \varphi_3(t, 1) = \varphi_2(t, 1) = \varphi_1(t, 1), \\ \eta_3(t, 1) = \eta_2(t, 1) = \eta_1(t, 1), \end{cases} \quad \forall t \in [0, T].$$

Now, we present a Carleman type estimate alike the case of  $2 \times 2$  system (IV.7)–(IV.8a) as prescribed in Theorem (IV.3.2).

**Theorem IV.6.5** (A Carleman estimate). *Let  $i = 1, 2, 3$  and the weight functions  $\varphi_i$  and  $\eta_i$  be as defined in (IV.168). Then, there exists  $\lambda_1 := \lambda_1(\gamma_i, \alpha) > 0$ ,  $\sigma_1 := \sigma_1(\gamma_i, \alpha) > 0$ ,  $s_1 := (T^2 + T)\sigma_1 > 0$  and a constant  $C' := C'(\gamma_i, \alpha) > 0$ , so that the following Carleman estimate holds true*

$$\begin{aligned} & s^3 \lambda^4 \sum_{i=1}^3 \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i^3 |q_i|^2 dx dt + s \lambda^2 \sum_{i=1}^3 \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i |\partial_x q_i|^3 dx dt \\ & + s^3 \lambda^3 \int_0^T \varphi_1(t, 1) e^{-2s\eta_1(t, 1)} |q_1(t, 1)|^2 dt \leq C' \left[ \sum_{i=1}^3 \int_0^T \int_0^1 e^{-2s\eta_i} |\partial_t q_i + \partial_x(\gamma_i \partial_x q_i)|^2 dx dt \right. \\ & \quad \left. + s \lambda \int_0^T \left( \varphi_2(t, 0) e^{-2s\eta_2(t, 0)} |\partial_x q_2(t, 0)|^2 + \varphi_3(t, 0) e^{-2s\eta_3(t, 0)} |\partial_x q_3(t, 0)|^2 \right) dt \right], \end{aligned}$$

for  $s \geq s_1$ ,  $\lambda \geq \lambda_1$  and for all  $(q_1, q_2, q_3) \in \mathcal{Q}$ .

By a similar approach as the proof of Theorem IV.3.2, we can proof the above Carleman inequality and the choices of the functions  $\beta_i$ ,  $i = 1, 2, 3$ , given by (IV.166)–(IV.167) are the crucial ingredients to obtain the expected estimate. We shall not present the detailed steps here.

• **Null-controllability and boundary observability inequality.** From the Carleman estimate given by Theorem IV.6.5, one can deduce the following observability inequality which is a necessary and sufficient condition for the controllability of our system (IV.157)–(IV.158a), that is Theorem IV.6.4.

**Proposition IV.6.6** (Observability inequality). *For any  $\zeta := (\zeta_1, \zeta_2, \zeta_3) \in \tilde{\mathcal{H}}_\alpha$ , the associated solution  $q := (q_1, q_2, q_3) \in \mathcal{C}^0([0, T]; \tilde{\mathcal{H}}_\alpha) \cap L^2(0, T; (H^2(0, 1))^3)$  to (IV.164) satisfies the following observation estimate*

$$\|q(0)\|_{\tilde{\mathcal{H}}_\alpha}^2 \leq C e^{C/T} \int_0^T \left( |\partial_x q_2(t, 0)|^2 + |\partial_x q_3(t, 0)|^2 \right) dt,$$

for some constant  $C := C(\gamma_i, \alpha, a) > 0$ ,  $i = 1, 2, 3$ , independent on  $T > 0$  and  $\zeta$ .

The proof is similar to the proof of Proposition IV.3.4 and we omit the details here.

#### IV.6.1.2 Controls on the 1st and 2nd or, 1st and 3rd components

Our next systems of consideration are (IV.157) with (IV.158b) or (IV.158c), that is when the controls are supposed to be applied on either  $y_1, y_2$ , or on  $y_1, y_3$ .

Now, recall that, for the  $2 \times 2$  system (IV.7) there occurs a significant change on controllability depending on the quantities  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$  when the control was exerted on the first component instead of the second one (see Theorems IV.2.7 and IV.2.8). This is why we want to investigate what happens for the  $3 \times 3$  case, if we change the position of the controls.

As in the  $2 \times 2$  case (IV.7)–(IV.8b), here also the Carleman approach does not work due to a similar argument as in Remark IV.3.5. One could thus try to develop the spectral analysis of the corresponding adjoint elliptic operator to observe the phenomena more sharply; we shall briefly discuss about that in the upcoming portion.

#### A short description of the spectrum of $\mathcal{A}_{\alpha,a}^*$ .

Let us first write the eigenvalue problem corresponding to the operator  $\mathcal{A}_{\alpha,a}^*$  (defined by (IV.163)) (let say for  $\gamma_i = 1$ ,  $i = 1, 2, 3$ ), with  $a \neq 0$  and  $\alpha \geq 0$ ,

$$\begin{cases} -\partial_x^2 u_1 + au_3 = \lambda u_1 & \text{in } (0, 1), \\ -\partial_x^2 u_2 = \lambda u_2 & \text{in } (0, 1), \\ -\partial_x^2 u_3 = \lambda u_3 & \text{in } (0, 1), \\ u_1(0) = u_2(0) = u_3(0) = 0, \\ u_1(1) = u_2(1) = u_3(1), \\ \sum_{i=1}^3 u'_i(1) + \alpha u_1(1) = 0, \end{cases} \quad (\text{IV.169})$$

for  $\lambda \in \mathbb{C}$ . The operator  $\mathcal{A}_{\alpha,a}^*$  being a bounded perturbation of the self-adjoint operator  $\mathcal{A}_\alpha$  (defined by (IV.159a)–(IV.159b)), it has compact resolvent and thus the spectrum is discrete; see Theorem IV.7.7.

We hereby denote the set of eigenvalues by  $\Lambda_{\alpha,a}$  and each eigenfunction by  $\Phi_\lambda$ , for  $\lambda \in \Lambda_{\alpha,a}$ , of the operator  $\mathcal{A}_{\alpha,a}^*$ .

- **Special case.** One can show that  $\lambda = 0$  is an eigenvalue of (IV.169) if and only if  $a + 3\alpha + 9 = 0$ , and in that case the eigenfunction  $\Phi_0$  is

$$\Phi_0(x) := \begin{pmatrix} \frac{ax^3}{6} + (1 - \frac{a}{6}x) \\ x \\ x \end{pmatrix}, \quad \forall x \in [0, 1]. \quad (\text{IV.170})$$

- **Solution to (IV.169) for  $\lambda \neq 0$ .** We set  $\mu \in \mathbb{C} \setminus \{0\}$  such that  $\lambda = \mu^2$ . Now, by solving the set of equations (IV.169) for  $\lambda \neq 0$ , we obtain the set of eigenvalues-eigenfunctions of the operator  $\mathcal{A}_{\alpha,a}^*$  as prescribed below.

- The first set of eigenfunctions associated with the eigenvalues  $\lambda = (k+1)^2\pi^2 \in \Lambda_{\alpha,a}$ , for all  $k \geq 0$ , are given by

$$\Phi_\lambda(x) := \begin{pmatrix} \sin((k+1)\pi x) \\ -\sin((k+1)\pi x) \\ 0 \end{pmatrix}, \quad \forall x \in [0, 1]. \quad (\text{IV.171})$$

- Another sets of eigenfunctions are

$$\Phi_\lambda(x) := \begin{pmatrix} -\frac{ax}{2\mu} \cos(\mu x) + (1 + \frac{a}{2\mu} \frac{\cos \mu}{\sin \mu}) \sin(\mu x) \\ \sin(\mu x) \\ \sin(\mu x) \end{pmatrix}, \quad \forall x \in [0, 1], \quad (\text{IV.172})$$

with  $\lambda = \mu^2 \in \Lambda_{\alpha,a}$ , where  $\mu \in \mathbb{C}$  satisfies the following transcendental equation

$$(6\mu^2 - a) \sin 2\mu + 2a\mu + 4\alpha\mu \sin^2 \mu = 0. \quad (\text{IV.173})$$

One can obtain the condition (IV.173) from the Kirchhoff type law satisfied by the eigenfunctions  $\Phi_\lambda$ .

We omit all the technical details to obtain the eigenfunctions and equation (IV.173), as these have been acquired in a similar manner as of  $2 \times 2$  case in Section IV.4.1.2.

- **Real roots of (IV.173).** We set

$$f(\mu) := (6\mu^2 - a) \sin 2\mu + 2a\mu + 4\alpha\mu \sin^2 \mu. \quad (\text{IV.174})$$

Then, for  $a \in \mathbb{R}^*$  and  $\alpha \geq 0$ , there exists some  $k_{\alpha,a} \in \mathbb{N} \cup \{0\}$ , such that for each  $k \geq k_{\alpha,a}$ ,  $f$  has two real roots  $\mu_{k,1}^{\alpha,a}$  and  $\mu_{k,2}^{\alpha,a}$ , where

$$\begin{cases} (k+1/4)\pi < \mu_{k,1}^{\alpha,a} < (k+3/4)\pi, \text{ with} \\ \mu_{k,1}^{\alpha,a} = (k+1/2)\pi + \frac{2\alpha+a}{6k\pi} + O_{\alpha,a}(1/k^3), \end{cases} \quad (\text{IV.175})$$

and

$$\begin{cases} (k+3/4)\pi < \mu_{k,2}^{\alpha,a} < (k+5/4)\pi, \text{ with} \\ \mu_{k,2}^{\alpha,a} = (k+1)\pi - \frac{a}{6k\pi} + O_{\alpha,a}(1/k^3). \end{cases} \quad (\text{IV.176})$$

To show this, we let  $\delta \in \{-\pi/4, \pi/4\}$ .

- One can then see

$$f((k+1/2)\pi + \delta) \sim -6 \sin(2\delta) k^2 \pi^2, \quad \text{for large } k.$$

Thus, for  $k$  large enough, say  $k \geq k_{\alpha,a}$ ,  $f((k+1/4)\pi)$  and  $f((k+3/4)\pi)$  have different signs, which shows the existence of a real root  $\mu_{k,1}^{\alpha,a}$  in the interval  $((k+1/4)\pi, (k+3/4)\pi)$ .

– Similarly, we have

$$f((k+1)\pi + \delta) \sim 6 \sin(2\delta)k^2\pi^2, \quad \text{for large } k,$$

and by a similar argument as before, we have the existence of a root  $\mu_{k,2}^{\alpha,a}$  in  $((k+3/4)\pi, (k+5/4)\pi)$  for all  $k \geq k_{\alpha,a}$  (possibly some different  $k_{\alpha,a}$  than the previous one) and here  $\mu_{k,2}^{\alpha,a} \neq (k+1)\pi$  for any  $a \in \mathbb{R}^*$ .

The proof of the asymptotic behaviors can be deduced in a similar manner as we have shown in Lemma IV.4.4 for the  $2 \times 2$  system.

Now, we set  $\lambda_{k,1}^{\alpha,a} := (\mu_{k,1}^{\alpha,a})^2 \in ((k+1/4)^2\pi^2, (k+3/4)^2\pi^2)$  and  $\lambda_{k,2}^{\alpha,a} := (\mu_{k,2}^{\alpha,a})^2 \in ((k+3/4)^2\pi^2, (k+5/4)^2\pi^2)$ ,  $\forall k \geq k_{\alpha,a}$ , which are two different sets of real eigenvalues of  $\mathcal{A}_{\alpha,a}^*$ .

Note that,  $\lambda_{k,2}^{\alpha,a} \neq (k+1)^2\pi^2$  for any  $a \neq 0$  and this is clear from the equation (IV.173). Thus, actually we have *three sets of real eigenvalues*. To be more precise, we put the structure of the spectrum below.

- **Structure of  $\Lambda_{\alpha,a}$ .** By the argument of spectral analysis discussed in Section IV.7.3 (also developed in Section IV.4.1.2), the spectrum of  $\mathcal{A}_{\alpha,a}^*$  can be split into three parts in the complex plane,

$$\Lambda_{\alpha,a} = \{(k+1)^2\pi^2, k \geq 0\} \cup \Lambda_{\alpha,a}^0 \cup \Lambda_{\alpha,a}^\infty,$$

where  $\Lambda_{\alpha,a}^0$  is finite, with possibly some complex eigenvalues  $\lambda := \mu^2$ , as soon as  $\mu \in \mathbb{C}$  satisfies (IV.173), and  $\Lambda_{\alpha,a}^\infty \subset (0, +\infty)$  such that

$$\Lambda_{\alpha,a}^\infty := \{\lambda_{k,1}^{\alpha,a}, k \geq k_{\alpha,a}\} \cup \{\lambda_{k,2}^{\alpha,a}, k \geq k_{\alpha,a}\},$$

where  $\lambda_{k,1}^{\alpha,a}$  and  $\lambda_{k,2}^{\alpha,a}$  have been defined in the previous paragraph for all  $k \geq k_{\alpha,a}$ .

For the moment, we skip the detailed analysis here. For a matter of understanding, the situation concerning the structure of  $\Lambda_{\alpha,a}$  has been described numerically by the Figure IV.22.

- **Multiplicity.** The geometric multiplicity of all the eigenvalues of  $\mathcal{A}_{\alpha,a}^*$  is 1. If it is not the case, assume that  $U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$  and  $\tilde{U} := \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix}$  be two linearly independent (non-trivial) solutions to (IV.169) for some  $\lambda \in \mathbb{C} \setminus \{0\}$ . Then for any two constants  $\theta, \tilde{\theta} \in \mathbb{C} \setminus \{0\}$ ,

$$V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} := \theta U + \tilde{\theta} \tilde{U} = \begin{pmatrix} \theta u_1 + \tilde{\theta} \tilde{u}_1 \\ \theta u_2 + \tilde{\theta} \tilde{u}_2 \\ \theta u_3 + \tilde{\theta} \tilde{u}_3 \end{pmatrix}$$

is also a solution to (IV.169) for that  $\lambda$ . Now, one can choose  $\theta, \tilde{\theta} \in \mathbb{C} \setminus \{0\}$  in such a way that

$$v'_2(0) = \theta u'_2(0) + \tilde{\theta} \tilde{u}'_2(0) = 0.$$

Since,  $V$  satisfies the eigenvalue problem (IV.169), by solving the equation (IV.169) for  $v$  we obtain (without putting technical details) that one possible solution is

$$v_1(x) = \kappa_1 \sin((k+1)\pi x), \quad v_2(x) = -\kappa_1 \sin((k+1)\pi x), \quad v_3(x) = 0,$$

$\forall x \in [0, 1]$ , for some  $\kappa_1 \in \mathbb{R}$ , and of course in that case  $\lambda = (k+1)^2\pi^2$ , for some  $k \geq 0$ .

But by our assumptions on  $\theta, \tilde{\theta}$ , we have  $v'_2(0) = 0$  which implies  $\kappa_1 = 0$  and so  $v_1 = v_2 = v_3 = 0$ .



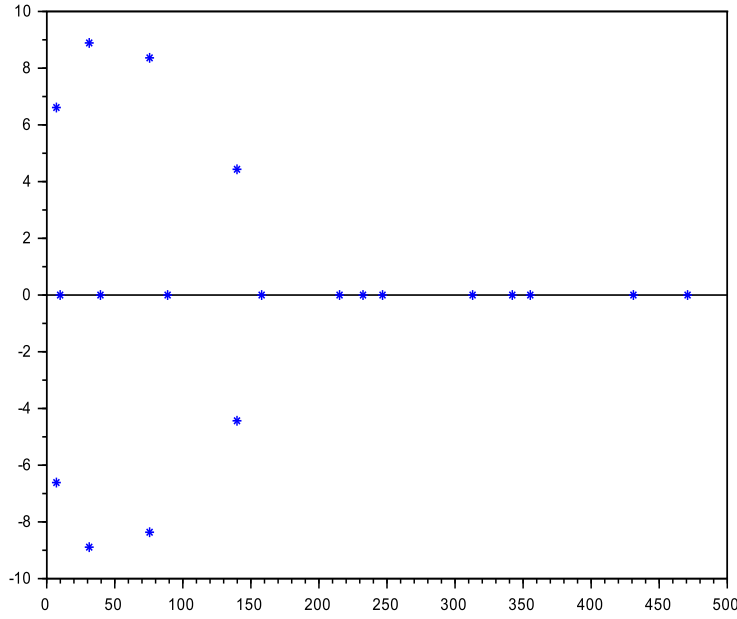


Figure IV.22: Description the spectrum in some part of the complex plane for  $\alpha = 1.2$ ,  $a = 35$ .

Another possible solution to (IV.169) for  $v$  is

$$\begin{cases} v_1(x) = -\frac{a\kappa_2 x}{2\mu} \cos(\mu x) + \kappa_2 \left(1 + \frac{a \cos \mu}{2\mu \sin \mu}\right) \sin(\mu x), \\ v_2(x) = \kappa_2 \sin(\mu x), \\ v_3(x) = \kappa_2 \sin(\mu x), \end{cases}$$

$\forall x \in [0, 1]$ , for some  $\kappa_2 \in \mathbb{C}$ , and in this case  $\mu \in \mathbb{C}$  satisfies the equation (IV.173).

But again, implementing the chosen condition  $v_2'(0) = 0$ , one observes that  $\kappa_2 = 0$  which gives  $v_1 = v_2 = v_3 = 0$ , a trivial solution.

So, the above study ensures the existence of some non-zero  $\theta, \tilde{\theta}$  for which  $V = \theta U + \tilde{\theta} \tilde{U} = 0$ , that is  $U$  and  $\tilde{U}$  cannot be independent.

This proves that the geometric multiplicity of each eigenfunction is 1.

- **Completeness.** Finally, since the perturbation of  $\mathcal{A}_\alpha^*$  is bounded in terms of the interior coupling coefficient  $a \neq 0$ , the completeness of the set of eigenfunctions  $\{\Phi_\lambda\}_{\lambda \in \Lambda_{\alpha,a}}$  of  $\mathcal{A}_{\alpha,a}^*$  (defined by (IV.171)–(IV.172)) in  $E := (L^2(0, 1))^3$  and  $\tilde{\mathcal{H}}_\alpha$  (more precisely, their complex version) comes as a consequence of a theorem of Keldysh (see for instance, [87, Theorem 4.3, Ch. 1]).

### Approximate controllability.

Let us come to the point of approximate controllability of the systems (IV.157)–(IV.158b) and (IV.157)–(IV.158c). We remind that for simplicity we considered  $\gamma_1 = \gamma_2 = \gamma_3 = 1$ .

**Lemma IV.6.7.** *Let any  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$  be given. Then the systems (IV.157)–(IV.158b) and (IV.157)–(IV.158c) are approximately controllable in  $\tilde{\mathcal{H}}_{-\alpha} \subset (H^{-1}(0, 1))^3$  at any given time  $T > 0$ , where  $\tilde{\mathcal{H}}_{-\alpha}$  is the dual space of  $\tilde{\mathcal{H}}_\alpha$ , defined by (IV.161).*

*Proof.* The proof will be quick, thanks to Fattorini–Hautus criteria; see Theorem II.2.2.

The observation operator for the system (IV.157)–(IV.158b) is

$$\mathcal{B}_{1,2}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \partial_x \Big|_{x=0} : (H^2(0,1))^3 \rightarrow \mathbb{R}^2,$$

so that one has the following.

- *Special case.* Recall that when  $a + 3\alpha + 9 = 0$  (so that,  $a < 0$ ),  $\lambda = 0$  is an eigenvalue and the associated eigenfunction  $\Phi_0$  (defined by (IV.170)) satisfies

$$\mathcal{B}_{1,2}^* \Phi_0 = \begin{pmatrix} 1 - \frac{a}{6} \\ 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (\text{IV.177})$$

- The eigenfunctions defined by (IV.171), corresponding to the eigenvalues  $\{(k+1)^2\pi^2\}_{k \geq 0} \subset \Lambda_{\alpha,a}$  satisfy

$$\mathcal{B}_{1,2}^* \Phi_\lambda = \begin{pmatrix} (k+1)\pi \\ -(k+1)\pi \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (\text{IV.178})$$

- Finally, the eigenfunctions defined by (IV.172), associated with the second set of eigenvalues  $\lambda \in \Lambda_{\alpha,a}^0 \cup \Lambda_{\alpha,a}^\infty \subset \Lambda_{\alpha,a}$  satisfy

$$\mathcal{B}_{1,2}^* \Phi_\lambda = \begin{pmatrix} -\frac{a}{2\mu} + \mu + \frac{a \cos \mu}{2 \sin \mu} \\ \mu \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (\text{IV.179})$$

since  $\mu \neq 0$ .

Now, for the system (IV.157)–(IV.158c), the observation operator can be defined as

$$\mathcal{B}_{1,3}^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \partial_x \Big|_{x=0} : (H^2(0,1))^3 \rightarrow \mathbb{R}^2.$$

In this case, observe that  $\mathcal{B}_{1,3}^* \Phi_0$  (as a particular case) and  $\mathcal{B}_{1,3}^* \Phi_\lambda$  for  $\lambda \in \Lambda_{\alpha,a}^0 \cup \Lambda_{\alpha,a}^\infty$  do have the same expressions as (IV.177) and (IV.179) respectively, and for  $\lambda \in \{(k+1)^2\pi^2\}_{k \geq 0}$ , we have  $\mathcal{B}_{1,3}^* \Phi_\lambda = \begin{pmatrix} (k+1)\pi \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ . Thus, again by Fattorini–Hautus criteria the system (IV.157)–(IV.158c) is approximately controllable.

The proof is complete.  $\square$

**Remark IV.6.8.** Later, we shall see in Lemma IV.6.9 that the quantity  $(-\frac{a}{2\mu} + \mu + \frac{a \cos \mu}{2 \sin \mu})$  can vanish for some eigenvalue depending on the choices of  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$ , but that will not effect in the above cases due to the presence  $\mu \neq 0$  in the second entry of  $\mathcal{B}_{1,2}^* \Phi_\lambda$  in (IV.179).

### The set of moments equations and null-controllability.

In this section, we give a short description about the construction of controls  $v, \tilde{v}$  for the system (IV.157)–(IV.158b). Similar things can be adapted for the system (IV.157)–(IV.158c) and thus we omit the discussion for this case.

Let  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$ , any pair  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$  and any finite time  $T > 0$  be given. Then, any two functions  $v \in L^2(0, T)$  and  $\tilde{v} \in L^2(0, T)$  are said to be null-controls for the system (IV.157)–(IV.158b) if and only if we have: given any  $\zeta \in \tilde{\mathcal{H}}_\alpha$ ,

$$-\langle y_0, e^{-T\mathcal{A}_{\alpha,a}^*} \zeta \rangle_{\tilde{\mathcal{H}}_{-\alpha}, \tilde{\mathcal{H}}_\alpha} = \int_0^T \left\langle \begin{pmatrix} v(t) \\ \tilde{v}(t) \end{pmatrix}, \mathcal{B}_{1,2}^* (e^{-(T-t)\mathcal{A}_{\alpha,a}^*} \zeta) \right\rangle_{\mathbb{R}^2} dt.$$

Now, recall that the set of eigenfunctions of  $\mathcal{A}_{\alpha,a}^*$  forms a complete family in  $\tilde{\mathcal{H}}_\alpha$  and thus, with the above formulations, one has the following: for any  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$ , the functions  $v, \tilde{v} \in L^2(0, T; \mathbb{C})$  are the null-controls for the system (IV.157)–(IV.158b) if and only if we have

$$-e^{-\lambda T} \langle y_0, \Phi_\lambda \rangle_{\tilde{\mathcal{H}}_{-\alpha}, \tilde{\mathcal{H}}_\alpha} = \int_0^T \left\langle \begin{pmatrix} v(t) \\ \tilde{v}(t) \end{pmatrix}, \mathcal{B}_{1,2}^* \Phi_\lambda \right\rangle_{\mathbb{C}^2} e^{-(T-t)\lambda} dt, \quad \forall \lambda \in \Lambda_{\alpha,a}. \quad (\text{IV.180})$$

The above set of equations are the moments problem in this case.

- As we know, the main ingredient to solve the moments problem is the existence of a family  $\{q_\lambda\}_{\lambda \in \Lambda_{\alpha,a}} \subset L^2(0, T; \mathbb{C})$  which is bi-orthogonal to  $\{e^{(T-t)\lambda}\}_{\lambda \in \Lambda_{\alpha,a}}$ . This needs a proper spectral gap of the operator  $\mathcal{A}_{\alpha,a}^*$ . But, we have that the spectrum of  $\mathcal{A}_{\alpha,a}^*$  is discrete. Moreover, thanks to the asymptotics in (IV.175) and (IV.176), we have

$$\begin{cases} |\lambda_{k+1,1}^{\alpha,a} - \lambda_{k,2}^{\alpha,a}| \sim k, \text{ and } |\lambda_{k+1,1}^{\alpha,a} - (k+1)^2\pi^2| \sim k, & \text{for large } k, \\ |\lambda_{k,2}^{\alpha,a} - \lambda_{k,1}^{\alpha,a}| \sim k, & \text{for large } k, \\ |\lambda_{k,2}^{\alpha,a} - (k+1)^2\pi^2| \sim a \neq 0, & \text{for large } k. \end{cases} \quad (\text{IV.181})$$

This ensures the existence of such above bi-orthogonal family, at least in the classical sense due to [56, 57], with some uniform bounds of it w.r.t.  $\lambda \in \Lambda_{\alpha,a}$  (we skip the detail here).

- We also need some lower bounds of the observation estimates and upper bounds on the norm of eigenfunctions, but by a careful look to Lemma IV.4.8 (for the  $2 \times 2$  system), it is not difficult to obtain the required estimates for  $\mathcal{B}_{1,2}^* \Phi_\lambda$  and the eigenfunctions  $\Phi_\lambda$  ((IV.171)–(IV.172)).

One can now construct the control function  $\begin{pmatrix} v \\ \tilde{v} \end{pmatrix} \in L^2(0, T; \mathbb{C}^2)$  (although, at the end one could restrict only to the real part of the control) as follows:

$$\begin{pmatrix} v(t) \\ \tilde{v}(t) \end{pmatrix} = \sum_{\lambda \in \Lambda_{\alpha,a}} \begin{pmatrix} v_\lambda(t) \\ \tilde{v}_\lambda(t) \end{pmatrix}, \quad \forall t \in (0, T),$$

where

$$\begin{pmatrix} v_\lambda(t) \\ \tilde{v}_\lambda(t) \end{pmatrix} := -e^{-\lambda T} \langle y_0, \Phi_\lambda \rangle_{\tilde{\mathcal{H}}_{-\alpha}, \tilde{\mathcal{H}}_\alpha} \frac{\mathcal{B}_{1,2}^* \Phi_\lambda}{\|\mathcal{B}_{1,2}^* \Phi_\lambda\|^2} q_\lambda(t), \quad \forall t \in (0, T),$$

for  $\lambda \in \Lambda_{\alpha,a}$ .

Observe that, the above choices of  $\begin{pmatrix} v \\ \tilde{v} \end{pmatrix}$  formally satisfies the set of moments equations (IV.180). Then, the bounds of the control functions follow from some suitable bound of the bi-orthogonal family  $\{q_\lambda\}_{\lambda \in \Lambda_{\alpha,a}}$  (though we omit that here), lower bounds of the observations and the bounds of the eigenfunctions; we are not going into detail.

Finally the null-controllability result can be stated as follows:

**Null-controllability:** Let any  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$ ,  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$  and finite time  $T > 0$  be given. Then, there exists a set of functions  $v, \tilde{v} \in L^2(0, T)$ , such that the solution  $y \in \mathcal{C}^0([0, T]; \tilde{\mathcal{H}}_{-\alpha}) \cap L^2(0, T; E)$  to the system (IV.157)–(IV.158b) satisfies

$$y(T) := (y_1(T), y_2(T)) = (0, 0),$$

with some proper bound of the control functions  $v, \tilde{v}$ .

**A numerical experiment:** For a matter of understanding, we also perform a numerical simulation based on the penalized HUM approach ([63, 64, 31]) and followed by the discretization method as in Section IV.5.2 for the system (IV.157)–(IV.158c) (controls on  $y_1$  and  $y_3$ ). Similar phenomena holds true when we exert the controls on  $y_1$ ,  $y_2$ , or on  $y_2$ ,  $y_3$ .

We hereby consider the following simulation parameters,

$$\begin{aligned} T &= 0.3, \quad \gamma_1 = \gamma_2 = \gamma_3 = 1, \\ \alpha &= 1, \quad a = 3.8, \\ y_{0,1}(x) &= \sin(\pi x), \quad y_{0,2}(x) = \mathbf{1}_{(0.3,0.8)}(x), \quad y_{0,3}(x) = \sin^2(\pi x), \end{aligned}$$

we obtain the Fig. IV.23, where one observes that as the mesh size  $h$  is getting smaller, the size of the target data (—■—) converges to zero while the cost of the control (—◆—) and optimal energy (—●—) remains almost constant.

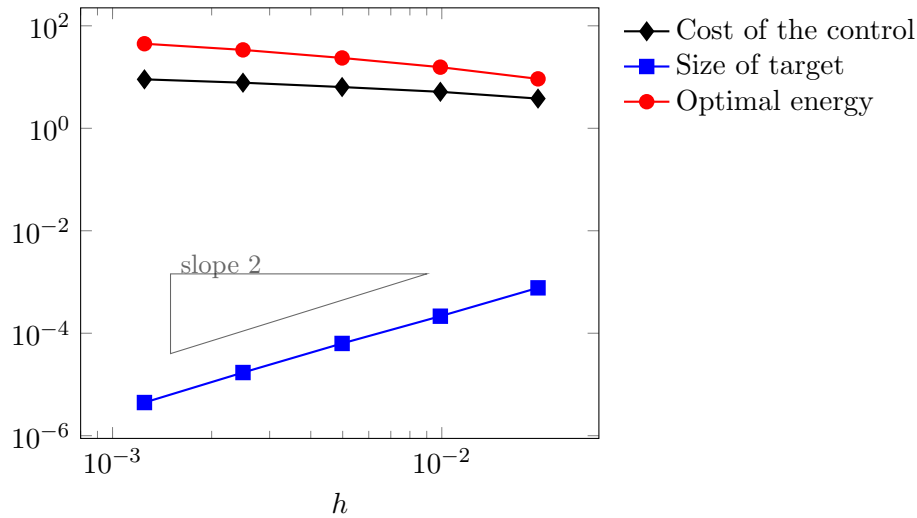


Figure IV.23: HUM convergence for  $3 \times 3$  case with two controls on 1st and 3rd comp.,  $\alpha = 1$ ,  $a = 3.8$ .

### IV.6.2 The system with only one boundary control

In this section, we deal with the same parabolic system (IV.157) (with  $\gamma_i = 1$ ,  $i = 1, 2, 3$ ) but with only one boundary control exerted on one of the components through a Dirichlet condition. In particular, we will show that the approximate/null controllability changes significantly depending on which component the control is being applied. Sometimes, it also depends on the choices of the parameters  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$ .

We consider the following system

$$\left\{ \begin{array}{ll} \partial_t y_1 - \partial_x^2 y_1 &= 0 & \text{in } Q, \\ \partial_t y_2 - \partial_x^2 y_2 &= 0 & \text{in } Q, \\ \partial_t y_3 - \partial_x^2 y_3 + a y_1 &= 0 & \text{in } Q, \\ \left\{ \begin{array}{l} y_1(t, 1) = y_2(t, 1) = y_3(t, 1), \\ \sum_{i=1}^3 \partial_x y_i(t, 1) + \alpha y_1(t, 1) = 0 \end{array} \right. & & \text{in } (0, T), \\ y_i(0, x) = y_{0,i}(x), \quad \text{for } i = 1, 2, 3, & & \text{in } (0, 1), \end{array} \right. \quad (\text{IV.182})$$

with the following different situations:

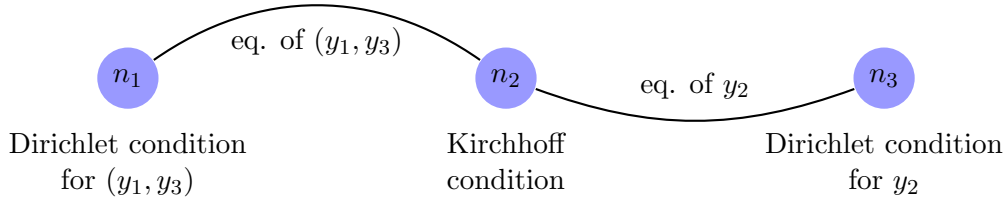


Figure IV.24: The system (IV.182)–(IV.183) on a metric graph of two edges.

either the control is applied on  $y_1$

$$y_1(t, 0) = v(t), \quad y_2(t, 0) = 0, \quad y_3(t, 0) = 0 \quad \text{in } (0, T), \quad (\text{IV.183a})$$

or, on  $y_2$

$$y_1(t, 0) = 0, \quad y_2(t, 0) = v(t), \quad y_3(t, 0) = 0 \quad \text{in } (0, T), \quad (\text{IV.183b})$$

or, on  $y_3$

$$y_1(t, 0) = 0, \quad y_2(t, 0) = 0, \quad y_3(t, 0) = v(t) \quad \text{in } (0, T) \quad (\text{IV.183c})$$

In a graph setting (see Fig. IV.24), we are considering the control to be acted either on the node  $n_1$ , or on  $n_3$ .

#### IV.6.2.1 Control applied on the first component

The case when we put a control on the first component  $y_1$ , we shall show below that depending on the choices of the parameters  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$ , there occurs a change in the approximate/null-controllability. We write the following lemma.

**Lemma IV.6.9 (Approximate controllability).** *Let any  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$  be given. Then there exists a non-empty set  $\tilde{\mathcal{R}} \subset \mathbb{R}_0^+ \times \mathbb{R}^*$ , such that we have the following properties:*

1. *If  $(\alpha, a) \notin \tilde{\mathcal{R}}$ , the problem (IV.182)–(IV.183a) is approximately controllable at any time  $T > 0$  in  $\tilde{\mathcal{H}}_{-\alpha}$ .*
2. *On the other hand, if  $(\alpha, a) \in \tilde{\mathcal{R}}$ , there exists a subspace  $\tilde{\mathcal{Y}}_{\alpha, a} \subset \tilde{\mathcal{H}}_{-\alpha}$  of codimension 1, such that the problem (IV.182)–(IV.183a) is approximately controllable at any time  $T > 0$  if and only if the initial data belongs to the space  $\tilde{\mathcal{Y}}_{\alpha, a}$ .*

The set  $\tilde{\mathcal{R}}$  and the space  $\tilde{\mathcal{Y}}_{\alpha, a}$  has been defined respectively by (IV.190) and (IV.192) inside the proof of the lemma.

*Proof.* The observation operator for the system (IV.182)–(IV.183a) is

$$\mathcal{B}_1^* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \partial_x \Big|_{x=0} : (H^2(0, 1))^3 \rightarrow \mathbb{R},$$

so that one has the following.

- *Special case.* Recall that when  $a + 3\alpha + 9 = 0$  (so that  $a < 0$ ),  $\lambda = 0$  is an eigenvalue of  $\mathcal{A}_{\alpha, a}^*$  with the associated eigenfunction  $\Phi_0$  (defined by (IV.170)) satisfies

$$\mathcal{B}_1^* \Phi_0 = \left(1 - \frac{a}{6}\right) \neq 0,$$

- The eigenfunctions, defined by (IV.171), corresponding to the eigenvalues  $\{(k+1)^2\pi^2\}_{k \geq 0}$  satisfy

$$\mathcal{B}_1^* \Phi_\lambda = (k+1)\pi \neq 0.$$

- Next, the eigenfunctions, defined by (IV.172), associated with the eigenvalues  $\lambda \in \Lambda_{\alpha,a}^0 \cup \Lambda_{\alpha,a}^\infty$  satisfy

$$\begin{aligned} \mathcal{B}_1^* \Phi_\lambda &= -\frac{a}{2\mu} + \mu + \frac{a \cos \mu}{2 \sin \mu} \\ &= \frac{(2\mu^2 - a) \sin \mu + a\mu \cos \mu}{2\mu \sin \mu}, \end{aligned} \quad (\text{IV.184})$$

As it has been observed in Lemma IV.4.7 for the  $2 \times 2$  system, there may exist some (indeed, many) value  $(\alpha, a)$  such that the above quantity becomes 0 and so, a further investigation is required at this point.

We now suppose that  $\mathcal{B}_1^* \Phi_\lambda = 0$  in (IV.184). Since  $\mu \neq 0$  and  $\sin \mu \neq 0$ , this is equivalent to

$$(2\mu^2 - a) \sin \mu + a\mu \cos \mu = 0. \quad (\text{IV.185})$$

This equation has to be satisfied in addition to the transcendental equation (IV.173). Now, if we suppose  $\cos \mu = 0$  (that is  $\mu = (k+1/2)\pi$  for some  $k$ ), then the equations (IV.185) and (IV.173) show that it can occur if and only if  $a + 2\alpha = 0$  and  $\mu^2 = -\alpha$ , which is not possible. Thus, we can assume  $\cos \mu \neq 0$ .

Now, we multiply the equation (IV.185) by  $\cos \mu$  and using some easy trigonometry, we obtain that the two equations (IV.185) and (IV.173) can be expressed in terms of the following system

$$\begin{pmatrix} (6\mu^2 - a) & 4\alpha\mu \\ (2\mu^2 - a) & -2a\mu \end{pmatrix} \begin{pmatrix} \sin 2\mu \\ \sin^2 \mu \end{pmatrix} = \begin{pmatrix} -2a\mu \\ -2a\mu \end{pmatrix}. \quad (\text{IV.186})$$

Using similar trick as in the proof of Lemma IV.4.7, we can show that the determinant of the coefficient matrix in (IV.186) is non-trivial, and thus one has the solution to that system as follows:

$$\begin{cases} \sin 2\mu = \frac{-2a\mu(a+2\alpha)}{2\mu^2(3a+2\alpha) - a(a+2\alpha)}, \\ \sin^2 \mu = \frac{4a\mu^2}{2\mu^2(3a+2\alpha) - a(a+2\alpha)}. \end{cases} \quad (\text{IV.187})$$

By using standard trigonometric relation  $\sin^2 2\mu = 4\sin^2 \mu(1 - \sin^2 \mu)$ , one can obtain  $\mu^2 = (a^2 + 2a\alpha + 4a)/8$ . Since the sign of this  $\mu$  is unimportant, we conclude that the situation  $\mathcal{B}_1^* \Phi_\lambda = 0$  can only happen for the particular value

$$\mu = \mu_{\alpha,a}^c := \frac{\sqrt{a^2 + 2a\alpha + 4a}}{2\sqrt{2}},$$

that is, if  $\mathcal{B}_1^* \Phi_\lambda = 0$ , then we necessarily have the eigenvalue

$$\lambda = \lambda_{\alpha,a}^c := \frac{a^2 + 2a\alpha + 4a}{8}. \quad (\text{IV.188})$$

Still we have to check whether the above  $\mu_{\alpha,a}^c$  does satisfy (IV.187), or not, that is to say whether or not  $\alpha$  and  $a$  satisfy

$$\sin \left( \frac{1}{\sqrt{2}} \sqrt{a^2 + 2a\alpha + 4a} \right) = \frac{-2\sqrt{2}a(a+2\alpha)\sqrt{a^2 + 2a\alpha + 4a}}{(3a+2\alpha)(a^2 + 2a\alpha + 4a) - 4a(a+2\alpha)}, \quad (\text{IV.189a})$$

$$\sin^2 \left( \frac{\sqrt{a^2 + 2a\alpha + 4a}}{2\sqrt{2}} \right) = \frac{2a(a^2 + 2a\alpha + 4a)}{(3a+2\alpha)(a^2 + 2a\alpha + 4a) - 4a(a+2\alpha)}. \quad (\text{IV.189b})$$

This leads us to introduce the critical set  $\tilde{\mathcal{R}}$  as follows

$$\tilde{\mathcal{R}} := \{(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*, \text{ s.t. (IV.189) holds}\}. \quad (\text{IV.190})$$

It is clear that the set  $\tilde{\mathcal{R}}$  is the set of solutions to the two equations (IV.189). We recall that those two equations were obtained from (IV.187) by eliminating the value of  $\mu$  and therefore, are not independent one from the other. Thus, we observe that any solution curve of (IV.189b) necessarily satisfies

$$\sin\left(\frac{1}{\sqrt{2}}\sqrt{a^2 + 2a\alpha + 4a}\right) = \varepsilon \frac{2\sqrt{2}a(a + 2\alpha)\sqrt{a^2 + 2a\alpha + 4a}}{(3a + 2\alpha)(a^2 + 2a\alpha + 4a) - 4a(a + 2\alpha)} \quad (\text{IV.191})$$

for  $\varepsilon \in \{-1, 1\}$ . Observe that, on any connected component of the set of solutions of (IV.189b), we have either  $\varepsilon = -1$  (in which case (IV.189a) is satisfied) or  $\varepsilon = 1$  (in which case (IV.189a) is not satisfied).

In Figure IV.25, we have plotted the solution curves of (IV.189a) and (IV.189b) for  $(\alpha, a) \in [0.01, 15] \times [0.01, 15]$ , in blue and red-dashed colors respectively. Here, one can see that some solution curves of (IV.189b) in red-dashed satisfy the equation (IV.189a) and this is why they coincide with some blue curves, and so, these are some solutions to the set of equation (IV.189), that lies in the set  $\tilde{\mathcal{R}}$ . In particular, the deep blue dot point is the value (approximate)  $(\alpha, a) = (1, 3.8082707)$  that will be used later to obtain some numerical results.

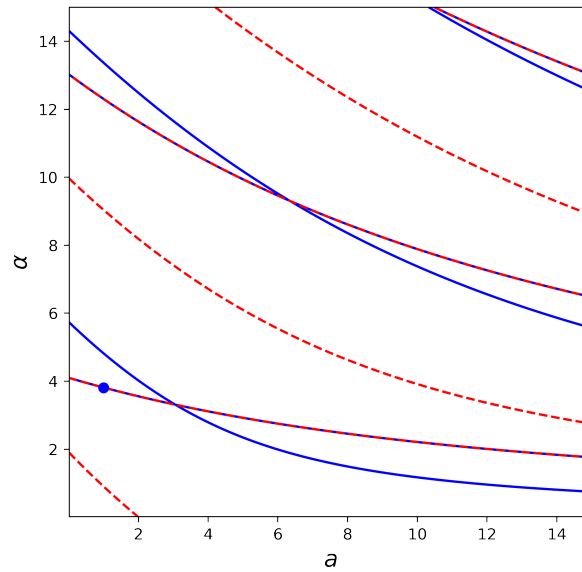


Figure IV.25: The red-dashed curves which coincide with blue curves, exhaust in the set  $\tilde{\mathcal{R}}$ .

- As a summary, we identify the set  $\tilde{\mathcal{R}}$  of parameters  $(\alpha, a)$  for which there exists a single critical eigenvalue  $\lambda_{\alpha, a}^c$  given by (IV.188) for which the associated eigenfunction is not observable, that is  $\mathcal{B}_1^* \Phi_{\lambda_{\alpha, a}^c} = 0$ .

Finally, the approximate controllability phenomena can be written as follows.

1. For any given pair  $(\alpha, a) \notin \tilde{\mathcal{R}}$  all the eigenfunctions of  $\mathcal{A}_{\alpha, a}^*$  satisfy  $\mathcal{B}_1^* \Phi_\lambda \neq 0$ , and hence, the Fattorini-Hautus criteria is satisfied which implies the approximate controllability of the system (IV.182)–(IV.183a) in the space  $\tilde{\mathcal{H}}_{-\alpha}$ .
2. If a given pair  $(\alpha, a)$  belongs to  $\tilde{\mathcal{R}}$ , then the system (IV.182)–(IV.183a) cannot be approximately controllable in the full space  $\tilde{\mathcal{H}}_{-\alpha}$ , since for the particular eigenvalue given in (IV.188), we have  $\mathcal{B}_1^* \Phi_{\lambda_{\alpha, a}^c} = 0$ ; thus the Fattorini-Hautus criteria fails.

However, it is not difficult to observe that if the initial data belongs to the smaller space defined by

$$\tilde{\mathcal{Y}}_{\alpha,a} := \left\{ y_0 \in \tilde{\mathcal{H}}_{-\alpha} \mid \langle y_0, \Phi_{\lambda_{\alpha,a}^c} \rangle_{\tilde{\mathcal{H}}_{-\alpha}, \tilde{\mathcal{H}}_{\alpha}} = 0 \right\}, \quad (\text{IV.192})$$

then the approximate controllability of the system holds true.  $\square$

Numerically, we observed that  $(\alpha, a) = (1, 3.8082707) \in \tilde{\mathcal{R}}$  (an approximate value), and for this pair, there exists some eigenvalue (precisely, the first one)  $\lambda_{\alpha,a}^c \approx 4.67934$  of  $\mathcal{A}_{\alpha,a}^*$  for which the associated eigenfunction  $\Phi_{\lambda_{\alpha,a}^c}$  satisfies

$$\mathcal{B}_1^* \Phi_{\lambda_{\alpha,a}^c} \approx 5 \times 10^{-5},$$

which is almost 0.

In fact, in Fig IV.26, we have plotted the eigenfunction  $\Phi_{\lambda_{\alpha,a}^c}$  corresponding to the above critical eigenvalue  $\lambda_{\alpha,a}^c$ , where the blue curve corresponds to the first entry of that eigenfunction which is almost flat near  $x = 0$ , i.e., its normal derivative  $\approx 0$ , in other words,  $\mathcal{B}_1^* \Phi_{\lambda_{\alpha,a}^c} \approx 0$ . We also have plotted the observation terms  $\mathcal{B}_1^* \Phi_{\lambda_{\alpha,a}^1}$  in Fig IV.27 for the first eigenvalues  $\lambda_{\alpha,a}^1$  with  $\alpha = 1$  and  $a \in [2.5, 4.5]$ , and it shows that there exists at least one  $a$  in the corresponding interval for which the first eigenfunction is non-observable.

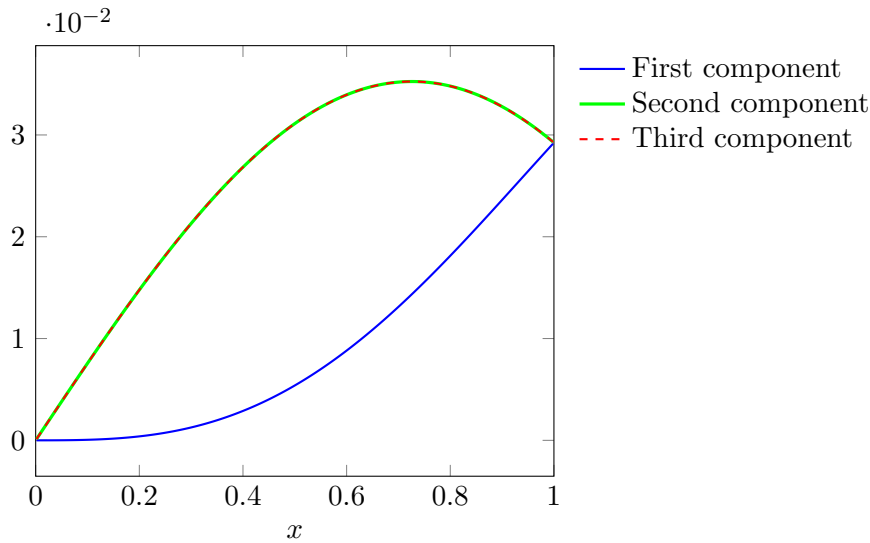


Figure IV.26: First eigenfunction  $\Phi_{\lambda_{\alpha,a}^c}$  of  $\mathcal{A}_{\alpha,a}^*$  for  $\alpha = 1, a = 3.8082707$ .

**Null-controllability.** We directly present the set of moments equations in this case: for any  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$ , a function  $v \in L^2(0, T; \mathbb{C})$  is said to be null-control for the system (IV.182)–(IV.183a) if and only if we have

$$-e^{-\lambda T} \langle y_0, \Phi_\lambda \rangle_{\tilde{\mathcal{H}}_{-\alpha}, \tilde{\mathcal{H}}_{\alpha}} = \overline{\mathcal{B}_1^* \Phi_\lambda} \int_0^T v(t) e^{-\lambda(T-t)} dt, \quad \forall \lambda \in \Lambda_{\alpha,a}.$$

In this context, we first mention that the existence of a bi-orthogonal family to  $\{e^{-\lambda(T-t)}\}_{\lambda \in \Lambda_{\alpha,a}}$  has already been discussed in the previous section. Then, thanks to the behavior of the observation terms in Lemma IV.6.9 and the gap condition (IV.181), one can draw the following conclusion (by pursuing the same strategy as the proof of Theorem IV.2.8 in Section IV.4.3.3).

Let any pair  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$  be given. Then, we have:



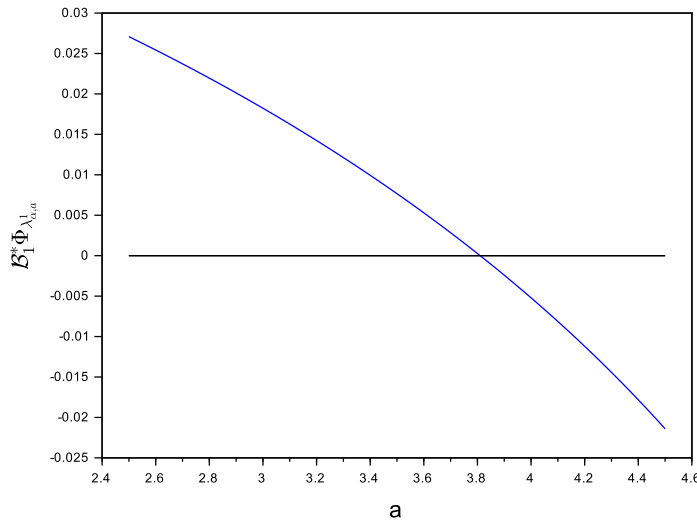


Figure IV.27: Size of the observation w.r.t. first eigenfunctions for  $\alpha = 1$  and  $a \in [2.5, 4.5]$ .

- If  $(\alpha, a) \notin \tilde{\mathcal{R}}$ , the system (IV.182)–(IV.183a) is null-controllable for any given  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$  at any time  $T > 0$ .
- On the other hand, if  $(\alpha, a) \in \tilde{\mathcal{R}}$ , the system (IV.182)–(IV.183a) is null-controllable if and only if  $y_0 \in \tilde{\mathcal{Y}}_{\alpha,a}$ .

The set  $\tilde{\mathcal{R}}$  and  $\tilde{\mathcal{Y}}_{\alpha,a}$  has been defined in (IV.190) and (IV.192) resp.

We omit the detailed description concerning the bound of such control  $v \in L^2(0, T)$  for which the above controllability property holds.

**Remark IV.6.10.** It is well-understood that when  $(\alpha, a) \in \tilde{\mathcal{R}}$ , the system (IV.182)–(IV.183a) is not even approximately controllable if we choose the initial data  $y_0 \notin \tilde{\mathcal{Y}}_{\alpha,a}$ .

#### IV.6.2.2 Control applied on the second component

**Lemma IV.6.11 (Approximate controllability).** Let any  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$  be given. Then for any initial data  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$ , the system (IV.182)–(IV.183b) is approximately controllable at any time  $T > 0$ .

*Proof.* The proof in this case is the simplest. The observation operator reads as

$$\mathcal{B}_2^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \partial_x \Big|_{x=0} : (H^2(0, 1))^3 \rightarrow \mathbb{R},$$

so that one immediately has the following.

- In the special case when  $a + 3\alpha + 9 = 0$ , the eigenfunction  $\Phi_0$  (corresponding to the eigenvalue  $\lambda = 0$ ) satisfies

$$\mathcal{B}_2^* \Phi_0 = 1.$$

- Next, for the set of eigenfunctions  $\Phi_\lambda$ , defined by (IV.171) corresponding to the eigenvalues  $\lambda \in \{(k+1)^2\pi^2\}_{k \geq 0}$ , we have

$$\mathcal{B}_2^* \Phi_\lambda = -\sqrt{\lambda} = -(k+1)\pi \neq 0, \quad \forall k \geq 0.$$

- Finally, the sets of eigenfunctions defined by (IV.172) associated with the eigenvalues  $\lambda \in \Lambda_{\alpha,a}^0 \cup \Lambda_{\alpha,a}^\infty$  are also observable, since

$$\mathcal{B}_2^* \Phi_\lambda = \mu \neq 0,$$

recall that  $\mu$  satisfies the transcendental equation (IV.173).

Hence, by Fattorini-Hautus criteria (see Theorem II.2.2) the approximately controllability follows. □

**Null-controllability.** Thanks to the behavior of the observation terms (have proper lower bounds) in Lemma IV.6.11, and the gap condition (IV.181), one could construct a control  $v \in L^2(0, T)$  via the moments technique to conclude the null-controllability of the system (IV.182)–(IV.183b) for any given  $y_0 \in \tilde{\mathcal{H}}_{-\alpha}$ , any pair  $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}^*$  and at any time  $T > 0$ . We skip the details here.

#### IV.6.2.3 Control applied on the third component

**Lemma IV.6.12.** *The system (IV.182)–(IV.183c) is not approximately controllable.*

*Proof.* The proof is quite obvious. Note that, the observation operator is

$$\mathcal{B}_3^* = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \partial_x \Big|_{x=0} : (H^2(0, 1))^3 \rightarrow \mathbb{R},$$

and so, one set of eigenfunctions, namely  $\Phi_\lambda = \begin{pmatrix} \sin((k+1)\pi x) \\ -\sin((k+1)\pi x) \\ 0 \end{pmatrix}$  associated to the eigenvalues  $\lambda \in \{(k+1)^2\pi^2\}_{k \geq 0}$ , are never observable since

$$\mathcal{B}_3^* \Phi_\lambda = 0.$$

Henceforth, the Fattorini-Hautus criteria fails. □

For a numerical test (based on the penalized HUM approach as previous cases), we hereby consider the following simulation parameters,

$$\begin{aligned} T &= 0.3, \quad \gamma_1 = \gamma_2 = \gamma_3 = 1, \\ \alpha &= 1, \quad a = 1, \\ y_{0,1}(x) &= \sin(\pi x), \quad y_{0,2}(x) = \mathbf{1}_{(0.3, 0.8)}(x), \quad y_{0,3}(x) = \sin^2(\pi x), \end{aligned}$$

and we obtain the following Figure IV.28, where it is clear that the control has no effect to the target data and that the optimal energy blows us at the rate of  $h^{-4}$  (the penalization parameter has been chosen here  $\epsilon = h^4$ ).

**Remark IV.6.13 (The case when  $a = 0$  with single boundary control).** *The system (IV.182) without any interior coupling (i.e.,  $a = 0$ ), is not approximately control with a single control. To observe that, we write the set of eigenvalue-eigenfunctions of the operator  $\mathcal{A}_\alpha$ , i.e., of the problem (IV.169) with  $a = 0$ : one set of eigenfunctions is*

$$\Phi_{\lambda_{k,1}^\alpha}(x) = \begin{pmatrix} \sin(\mu_{k,1}^\alpha x) \\ \sin(\mu_{k,1}^\alpha x) \\ \sin(\mu_{k,1}^\alpha x) \end{pmatrix}, \quad \forall x \in [0, 1],$$

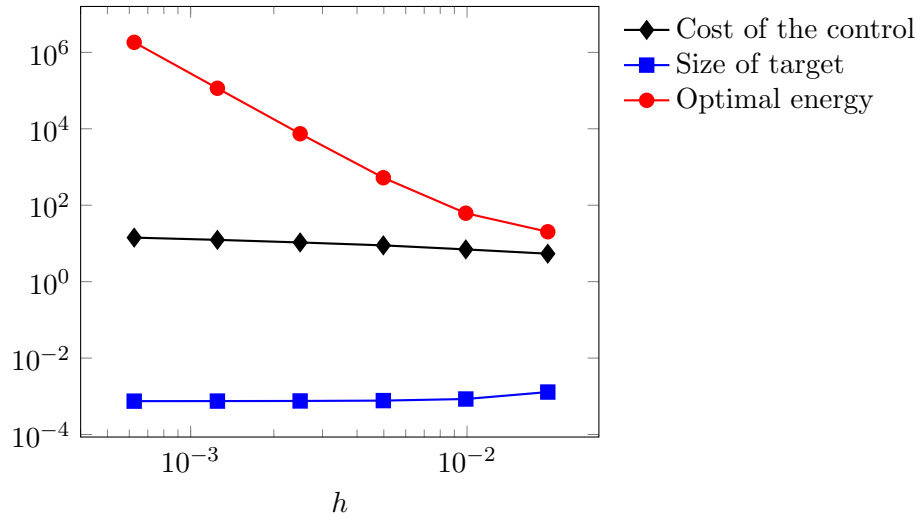


Figure IV.28: Control on the third component  $y_3$ : never controllable.

associated with the eigenvalue  $(\mu_{k,1}^\alpha)^2 = \lambda_{k,1}^\alpha \in ((k + 1/2)^2\pi^2, (k + 1)^2\pi^2)$ , for each  $k \geq 0$ , satisfying

$$3\mu_{k,1}^\alpha \cos \mu_{k,1}^\alpha + \alpha \sin \mu_{k,1}^\alpha = 0,$$

another sets of eigenfunctions are

$$\Phi_{\lambda_{k,2}}^1(x) = \begin{pmatrix} \sin((k+1)\pi x) \\ -\sin((k+1)\pi x) \\ 0 \end{pmatrix} \quad \text{and} \quad \Phi_{\lambda_{k,2}}^2(x) = \begin{pmatrix} 0 \\ \sin((k+1)\pi x) \\ -\sin((k+1)\pi x) \end{pmatrix}, \quad \forall x \in [0, 1],$$

associated with the eigenvalue  $\lambda_{k,2} = (k + 1)^2\pi^2$ , for each  $k \geq 0$ . Indeed, any combination of  $\Phi_{\lambda_{k,2}}^1$  and  $\Phi_{\lambda_{k,2}}^2$  is also an eigenfunction of the corresponding system with respect to  $\lambda_{k,2}$  and in particular the following function

$$\Phi_{\lambda_{k,2}}^1(x) + \Phi_{\lambda_{k,2}}^2(x) = \begin{pmatrix} \sin((k+1)\pi x) \\ 0 \\ -\sin((k+1)\pi x) \end{pmatrix}, \quad \forall x \in [0, 1].$$

So, exerting only a single control on any of the component is insufficient to control the whole system since the Fattorini-Hautus criteria fails.

In this situation, we must consider two different boundary controls on any two of the three components to obtain the positive controllability.

## IV.7 Some intermediate results

This section is devoted to present some intermediate results which have been used in several places of this chapter.

### IV.7.1 A linear algebraic stuff

**Lemma IV.7.1.** *Let  $\mathcal{D}$  and  $\mathcal{N}$  be two real  $d \times d$  matrices such that*

$$(\mathcal{D}, \mathcal{N}) \text{ is full rank}, \tag{IV.193}$$

and

$$\mathcal{D}\mathcal{N}^* \text{ is self adjoint}, \tag{IV.194}$$

then  $\mathcal{N} + t\mathcal{D}$  is invertible for any  $t \in \mathbb{R}$  except perhaps for a finite number of values of  $t$ .

*Proof.* We follow the same computations as in [25, Th. 1.4.4]. More precisely, we first observe that, under the assumptions of the lemma, we have that  $\mathcal{D} + i\mathcal{N}$  is invertible. Indeed,

- by (IV.193), we know that  $(\ker \mathcal{D}^*) \cap (\ker \mathcal{N}^*) = \{0\}$ ,
- by (IV.194), for any  $x \in \mathbb{C}^2$  we have

$$\|(\mathcal{D}^* - i\mathcal{N}^*)x\|^2 = \|\mathcal{D}^*x\|^2 + \|\mathcal{N}^*x\|^2,$$

so that  $\ker(\mathcal{D}^* - i\mathcal{N}^*) \subset (\ker \mathcal{D}^*) \cap (\ker \mathcal{N}^*) = \{0\}$  and the claim is proved.

We can now define  $\mathcal{U} = -(\mathcal{D} + i\mathcal{N})^{-1}(\mathcal{D} - i\mathcal{N})$  (which is actually a unitary matrix but we don't need this fact here). It satisfies

$$2(\mathcal{D} + i\mathcal{N})^{-1}\mathcal{D} = (\mathcal{D} + i\mathcal{N})^{-1}(\mathcal{D} + i\mathcal{N} + \mathcal{D} - i\mathcal{N}) = (I - \mathcal{U}),$$

$$2(\mathcal{D} + i\mathcal{N})^{-1}\mathcal{N} = -i(\mathcal{D} + i\mathcal{N})^{-1}((\mathcal{D} + i\mathcal{N}) - (\mathcal{D} - i\mathcal{N})) = -i(I + \mathcal{U}).$$

If we assume that  $t \in \mathbb{R}$  is such that  $\mathcal{N} + t\mathcal{D}$  is not invertible, then there exists  $x \in \mathbb{R}^d, x \neq 0$  such that  $(\mathcal{N} + t\mathcal{D})x = 0$ . Left-multiplying this equality by  $(\mathcal{D} + i\mathcal{N})^{-1}$  and using the above relations we end up with

$$\left(\frac{t-i}{t+i}I - \mathcal{U}\right)x = 0,$$

which proves that  $(t-i)/(t+i)$  is an eigenvalue of  $\mathcal{U}$ . This can only happen for a finite number of values of  $t$ .  $\square$

## IV.7.2 Well-posedness of a $2 \times 2$ system with non-homogeneous Neumann conditions at left and Kirchhoff type condition at right

### IV.7.2.1 Statement of the problem

In this section, we shall mainly mention the well-posedness part of a  $2 \times 2$  coupled parabolic systems with non-homogeneous Neumann boundary conditions at left and the Kirchhoff boundary condition at right. Indeed, the well-posedness of a Neumann control case mentioned in (IV.7)–(IV.9a) or (IV.9b) will be then justified by this general study. As before,  $Q := (0, T) \times (0, 1)$ .

The system reads as follows.

$$\begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) &= f_1 & \text{in } Q, \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a y_1 &= f_2 & \text{in } Q, \\ \begin{cases} -\gamma_1(0) \partial_x y_1(t, 0) = g_1(t) \\ -\gamma_2(0) \partial_x y_2(t, 0) = g_2(t) \end{cases} & & \text{in } (0, T), \\ \begin{cases} y_1(t, 1) = y_2(t, 1) \\ \sum_{i=1}^2 \gamma_i(1) \partial_x y_i(t, 1) + \alpha y_1(t, 1) = 0 \end{cases} & & \text{in } (0, T), \\ y_i(0, x) = y_{0,i}(x), \quad \text{for } i = 1, 2, & & \text{in } (0, 1), \end{cases} \quad (\text{IV.195})$$

where  $\alpha, a$  and the diffusion coefficients  $\gamma_i, i = 1, 2$  have been chosen as earlier.

**Defining some spaces and operators.** We shall again make abuse of the notations. We denote  $E := (L^2(0, 1))^2$ .

- We consider the operator

$$\mathcal{A}_\alpha := \begin{pmatrix} -\partial_x(\gamma_1 \partial_x) & 0 \\ 0 & -\partial_x(\gamma_2 \partial_x) \end{pmatrix}, \quad (\text{IV.196})$$

with its domain

$$D(\mathcal{A}_\alpha) = \left\{ \phi := (\phi_1, \phi_2) \in (H^2(0,1))^2 \mid \phi'_i(0) = 0, \ i = 1, 2, \text{ and} \right. \quad (\text{IV.197})$$

$$\left. \phi_1(1) = \phi_2(1), \ \sum_{i=1}^2 \gamma_i(1) \phi'_i(1) + \alpha \phi_1(1) = 0 \right\}.$$

- Next, we denote

$$\mathcal{A}_{\alpha,a} := \begin{pmatrix} -\partial_x(\gamma_1 \partial_x) & 0 \\ a & -\partial_x(\gamma_2 \partial_x) \end{pmatrix} = \mathcal{A}_\alpha + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad (\text{IV.198})$$

with the same domain  $D(\mathcal{A}_{\alpha,a}) := D(\mathcal{A}_\alpha)$ .

- In this context, it is also worth denoting the adjoint of  $\mathcal{A}_{\alpha,a}$  by  $\mathcal{A}_{\alpha,a}^*$ , which is

$$\mathcal{A}_{\alpha,a}^* := \begin{pmatrix} -\partial_x(\gamma_1 \partial_x) & a \\ 0 & -\partial_x(\gamma_2 \partial_x) \end{pmatrix} \quad (\text{IV.199})$$

with its domain  $D(\mathcal{A}_{\alpha,a}^*) := D(\mathcal{A}_{\alpha,a}) = D(\mathcal{A}_\alpha)$ .

#### IV.7.2.2 Well-posedness

We begin with the following proposition.

**Proposition IV.7.2** (Existence of analytic semigroup). *The operators  $(-\mathcal{A}_{\alpha,a}, D(\mathcal{A}_{\alpha,a}))$  and  $(-\mathcal{A}_{\alpha,a}^*, D(\mathcal{A}_{\alpha,a}^*))$  define analytic semigroups in  $E$ .*

*Proof.* Let us first introduce space

$$\mathcal{H}_\alpha := \left\{ \phi := (\phi_1, \phi_2) \in (H^1(0,1))^2 \mid \phi_1(1) = \phi_2(1) \right\}, \quad (\text{IV.200})$$

with the norm defined in it

$$\|\phi\|_{\mathcal{H}_\alpha} := \sum_{i=1}^2 \int_0^1 \gamma_i(x) |\phi'_i(x)|^2 dx + \alpha |\phi_1(1)|^2. \quad (\text{IV.201})$$

We then define the following densely defined sesquilinear form  $h$ ; for all  $u := (u_1, u_2)$ ,  $\phi := (\phi_1, \phi_2) \in \mathcal{H}_\alpha$ , we consider

$$h(u, \phi) = \sum_{i=1}^2 \int_0^1 \gamma_i(x) u'_i(x) \phi'_i(x) dx + a \int_0^1 u_1(x) \phi_2(x) dx + \alpha u_1(1) \phi_1(1). \quad (\text{IV.202})$$

It is clear that  $h$  is continuous in  $\mathcal{H}_\alpha$  with

$$|h(u, \phi)| \leq \kappa_1 \|u\|_{\mathcal{H}_\alpha} \|\phi\|_{\mathcal{H}_\alpha}, \quad \forall u, \phi \in \mathcal{H}_\alpha,$$

where  $\kappa_1 > 0$  depends on the diffusion coefficients  $\gamma_i$ ,  $i = 1, 2$ , and the coupling coefficient  $a$ . On the other hand, we have

$$h(u, u) \geq \|u\|_{\mathcal{H}_\alpha}^2 - |a| \|u\|_E^2, \quad \forall u \in \mathcal{H}_\alpha.$$

Then, by [90, Proposition 1.51 and Theorem 1.52], the negative of the operator associated with  $h$  generates an analytic semigroup in  $E$  of angle  $(\pi/2 - \arctan \kappa_1)$ .

Now, the only thing is to show that this operator is indeed  $-\mathcal{A}_{\alpha,a}$  with its domain  $D(\mathcal{A}_{\alpha,a}) = D(\mathcal{A}_\alpha)$  (as defined in (IV.198)), which we shall prove in Lemma IV.7.4 and hence the proposition follows.  $\square$

**Remark IV.7.3.** We denote by  $\mathcal{H}_{-\alpha}$ , the dual space of  $\mathcal{H}_\alpha$  relative to the norm (IV.201) with respect to the pivot space  $E$ .

**Lemma IV.7.4.** Let  $h$  be the sesquilinear form in  $\mathcal{H}_\alpha$  as defined by (IV.18). Then, the operator associated with  $h$  is  $(\mathcal{A}_{\alpha,a}, D(\mathcal{A}_{\alpha,a}))$ .

*Proof.* Denote by  $(\tilde{\mathcal{A}}, D(\tilde{\mathcal{A}}))$  the operator associated with the form  $h$ , which is by definition given by

$$\begin{cases} D(\tilde{\mathcal{A}}) = \left\{ \phi \in \mathcal{H}_\alpha \mid \exists f \in E \text{ s.t. } h(u, \phi) = (f, \phi)_E, \quad \forall \phi \in \mathcal{H}_\alpha \right\}, \\ \tilde{\mathcal{A}}u := f. \end{cases}$$

- First, fix some  $u := (u_1, u_2) \in D(\mathcal{A}_{\alpha,a})$ . Then for all  $\phi := (\phi_1, \phi_2) \in \mathcal{H}_\alpha$ , we have

$$h(u, \phi) = \sum_{i=1}^2 \int_0^1 \gamma_i(x) u'_i(x) \phi'_i(x) dx + a \int_0^1 u_1(x) \phi_2(x) dx + \alpha u_1(1) \phi_1(1).$$

Upon an integration by parts we get

$$\begin{aligned} h(u, \phi) &= - \sum_{i=1}^2 \int_0^1 (\gamma_i u'_i)'(x) \phi_i(x) dx + a \int_0^1 u_1(x) \phi_2(x) dx \\ &= (\mathcal{A}_{\alpha,a} u, \phi)_E. \end{aligned}$$

Thus, for our chosen  $u \in D(\mathcal{A}_{\alpha,a})$ , we obtain that there exists  $f = \mathcal{A}_{\alpha,a} u \in E$  such that  $h(u, \phi) = (f, \phi)_E$ , for all  $\phi \in \mathcal{H}_\alpha$ , which concludes the inclusion  $D(\mathcal{A}_{\alpha,a}) \subseteq D(\tilde{\mathcal{A}})$ .

- Conversely, let  $u \in D(\tilde{\mathcal{A}})$ . By definition, there exists  $f \in E$  such that  $h(u, \phi) = (f, \phi)_E$  with  $\tilde{\mathcal{A}}u = f$ , for all  $\phi \in \mathcal{H}_\alpha$ , and accordingly

$$\sum_{i=1}^2 \int_0^1 \gamma_i(x) u'_i(x) \phi'_i(x) dx + a \int_0^1 u_1(x) \phi_2(x) dx + \alpha u_1(1) \phi_1(1) = \sum_{i=1}^2 \int_0^1 f_i(x) \phi_i(x) dx.$$

In fact, since  $f_i \in L^2(0, 1)$  ( $i = 1, 2$ ), and  $h$  is continuous and coercive in  $\mathcal{H}_\alpha \subset (H^1(0, 1))^2$ , we have by standard elliptic regularity that  $u \in (H^2(0, 1))^2$ , and so an integrating by parts yields

$$- \sum_{i=1}^2 \int_0^1 (\gamma_i u'_i)'(x) \phi_i(x) dx + \sum_{i=1}^2 \gamma_i(1) u'_i(1) \phi_i(1) - \sum_{i=1}^2 \gamma_i(0) u'_i(0) \phi_i(0) \quad (\text{IV.203})$$

$$+ a \int_0^1 u_1(x) \phi_2(x) dx + \alpha u_1(1) \phi_1(1) = \sum_{i=1}^2 \int_0^1 f_i(x) \phi_i(x) dx, \quad (\text{IV.204})$$

for all  $\phi \in \mathcal{H}_\alpha$ .

In particular, by considering any  $\phi \in (H_0^1(0, 1))^2 \subset \mathcal{H}_\alpha$ , we conclude that

$$f_1 = -(\gamma_1 u'_1)', \quad f_2 = -(\gamma_2 u'_2)' + a u_1, \quad \text{in } (0, 1).$$

Once we have the above news, then choose any  $\phi \in \mathcal{H}_\alpha$  such that in addition, it satisfies  $\phi_i(0) = 0$ , for  $i = 1, 2$ , which essentially gives

$$\sum_{i=1}^2 \gamma_i(1)u'_i(1) + \alpha u_1(1) = 0.$$

Finally, using the above two facts in (IV.203), and treating any  $\phi \in \mathcal{H}_\alpha$ , we obtain that for each  $i = 1, 2$ ,  $u_i$  satisfies

$$u'_i(0) = 0,$$

since  $\gamma_i(0) \neq 0$ . So, we can say now that  $u \in D(\mathcal{A}_{\alpha,a})$  and that  $D(\tilde{\mathcal{A}}) \subseteq D(\mathcal{A}_{\alpha,a})$ .

This completes the proof of the lemma.  $\square$

Let us now introduce the adjoint system (without any source terms and boundary data) to (IV.195).

$$\begin{cases} -\partial_t q_1 - \partial_x(\gamma_1 \partial_x q_1) + a q_2 = 0 & \text{in } Q, \\ -\partial_t q_2 - \partial_x(\gamma_2 \partial_x q_2) = 0 & \text{in } Q, \\ \begin{cases} \partial_x q_1(t, 0) = 0 \\ \partial_x q_2(t, 0) = 0 \end{cases} & \text{in } (0, T), \\ \begin{cases} q_1(t, 1) = q_2(t, 1) \\ \sum_{i=1}^2 \gamma_i(1) \partial_x q_i(t, 1) + \alpha q_1(t, 1) = 0 \end{cases} & \text{in } (0, T), \\ q_i(T, x) = \zeta_i(x), \quad \text{for } i = 1, 2, & \text{in } (0, 1). \end{cases} \quad (\text{IV.205})$$

By Proposition IV.7.2, we have: for any given data  $\zeta := (\zeta_1, \zeta_2) \in E$ , there exists a unique weak solution  $q := (q_1, q_2) \in \mathcal{C}^0([0, T]; E)$  to the adjoint system (IV.205), that is

$$q(t, x) := (e^{-(T-t)\mathcal{A}_{\alpha,a}^*} \zeta)(x) \quad \forall (t, x) \in Q.$$

Moreover, we have the following regularity result.

**Proposition IV.7.5** (Regularity). *Let any finite time  $T > 0$  be given. Then, there exists some constant  $C_{T,a} > 0$  such that we have the following.*

1. *For any given  $\zeta \in E$ , there exists a unique solution  $q \in \mathcal{C}^0([0, T]; E) \cap L^2(0, T; \mathcal{H}_\alpha)$  satisfying the following energy estimate*

$$\|q\|_{\mathcal{C}^0([0,T];E)} + \|q\|_{L^2(0,T;\mathcal{H}_\alpha)} + \|\partial_t q\|_{L^2(0,T;\mathcal{H}_{-\alpha})} \leq C_{T,a} \|\zeta\|_E.$$

2. *On the other hand, for any given data  $\zeta \in \mathcal{H}_\alpha$ , the weak solution  $q$  belongs to the space  $L^\infty(0, T; \mathcal{H}_\alpha) \cap L^2(0, T; (H^2(0, 1))^2)$  and that satisfies*

$$\|q\|_{L^\infty(0,T;\mathcal{H}_\alpha)} + \|q\|_{L^2(0,T;(H^2(0,1))^2)} + \|\partial_t q\|_{L^2(0,T;E)} \leq C_{T,a} \|\zeta\|_{\mathcal{H}_\alpha}.$$

The proof of this can be deduced by a similar approach as given by Proposition IV.2.3, Chapter IV.

Now, we are in the situation to present the well-posedness result for the concerned system (IV.195).

**Proposition IV.7.6 (Well-posedness of the main system).** *For any  $y_0 := (y_{0,1}, y_{0,2}) \in E$ ,  $f := (f_1, f_2) \in L^2(0, T; E)$  and  $g := (g_1, g_2) \in L^2(0, T; \mathbb{R}^2)$ , there exists a unique  $y \in C^0([0, T]; E) \cap L^2(0, T; \mathcal{H}_\alpha)$ , weak solution to (IV.195), in the following sense: for any  $t \in [0, T]$  and  $\zeta := (\zeta_1, \zeta_2) \in E$ , we have*

$$(y(t), \zeta)_E = (y_0, e^{-t\mathcal{A}_a^*} \zeta)_E + \int_0^t (f(s), e^{-(t-s)\mathcal{A}_a^*} \zeta)_E ds \\ - \int_0^t \left\langle g(s), (e^{-(t-s)\mathcal{A}_a^*} \zeta)(x) \Big|_{x=0} \right\rangle_{\mathbb{R}^2} ds.$$

Moreover, the solution  $y$  satisfies the following:

$$\|y\|_{C^0([0, T]; E)} + \|y\|_{L^2(0, T; \mathcal{H}_\alpha)} \leq C_{T,a} \left( \|y_0\|_E + \|g\|_{L^2(0, T; \mathbb{R}^2)} + \|f\|_{L^2(0, T; E)} \right),$$

for some constant  $C_{T,a} > 0$ .

### IV.7.3 An overview of the spectrum of a perturbed self-adjoint operator

Let  $H$  be any closed operator in a Banach space  $X$  and  $H_{bd}$  be some operator in  $X$  which is  $H$ -bounded, that is  $D(H_{bd}) \supset D(H)$  and the following inequality holds:

$$\|H_{bd} u\| \leq b_1 \|u\| + b_2 \|Hu\|, \quad \forall u \in D(H), \quad (\text{IV.206})$$

where  $b_1, b_2$  are some non-negative constants with  $b_2 < 1$ .

In this section, we recall some standard theorems and their consequences from [69] to understand the relation between the spectrum of a perturbed operator and its corresponding self-adjoint operator. We denote by  $\sigma(H)$ , the spectrum of  $H$  and by  $\rho(H)$ , the resolvent set of  $H$ . We also denote

$$\mathcal{H} := H + H_{bd}.$$

Let us first present a result which gives some characterisation of the resolvent set  $\rho(\mathcal{H})$  of the perturbed operator. Precisely, we note down [69, IV–Theorem 3.17].

**Theorem IV.7.7.** *Let  $H$  be any closed operator in a Banach space  $X$  and  $\mathcal{H}$  be defined as above. If there is a point  $\xi \in \rho(H)$ , such that we have*

$$b_1 \|R(\xi, H)\| + b_2 \|HR(\xi, H)\| < 1,$$

with resolvent  $R(\xi, H)$ , then  $\mathcal{H}$  is also closed and  $\xi \in \rho(\mathcal{H})$ , with

$$\|R(\xi, \mathcal{H})\| \leq \|R(\xi, H)\| (1 - b_1 \|R(\xi, H)\| - b_2 \|HR(\xi, H)\|)^{-1}.$$

If in particular,  $H$  has compact resolvent,  $\mathcal{H}$  has too.

This theorem can be used to deduce the following.

**Theorem IV.7.8.** *Let  $H$ ,  $H_{bd}$  and  $\mathcal{H}$  be as introduced above and the spectrum  $\sigma(H)$  be divided into two parts by a closed curve  $\Gamma \subset \rho(H)$ . If*

$$\sup_{\xi \in \Gamma} (b_1 \|R(\xi, H)\| + b_2 \|HR(\xi, H)\|) < 1,$$

then the spectrum of  $\mathcal{H}$ , that is  $\sigma(\mathcal{H})$  is likewise divided into two parts by  $\Gamma$ .

In addition, the results of [69, IV–Theorem 3.16] hold.

The proof of the above theorem is sketched in [69, IV–Theorem 3.18].



**Consequences.**

- Suppose that, the spectrum  $\sigma(H)$  is separated into two parts by a closed curve  $\Gamma \subset \rho(H)$  in the complex plane (will be specified later). We now observe

$$\|R(\xi, H)\| = \sup_{\lambda \in \sigma(H)} |\lambda - \xi|^{-1}, \quad \|HR(\xi, H)\| = \sup_{\lambda \in \sigma(H)} |\lambda| |\lambda - \xi|^{-1}$$

Hence, by Theorem IV.7.7,  $\xi \in \rho(\mathcal{H})$  if

$$b_1 \sup_{\lambda \in \sigma(H)} |\lambda - \xi|^{-1} + b_2 \sup_{\lambda \in \sigma(H)} |\lambda| |\lambda - \xi|^{-1} < 1. \quad (\text{IV.207})$$

- In particular, assume that  $H$  has an isolated eigenvalue  $\tilde{\lambda}$  with multiplicity  $m < +\infty$  and with isolation distance  $d > 0$ ;  $d = \text{dist}(\tilde{\lambda}, \sigma'')$  where  $\sigma''$  is the spectrum of  $H$  with the single point  $\tilde{\lambda}$  excluded.

Let  $\Gamma$  be the circle with center  $\tilde{\lambda}$  and radius  $d/2$ . If  $\xi \in \Gamma \subset \rho(H)$ , then for any  $\lambda \in \sigma(H)$ , we have

- $|\lambda - \xi| \geq d/2$ , and so  $|\lambda - \xi|^{-1} \leq 2/d$ ,
- and

$$\begin{aligned} |\lambda(\lambda - \xi)^{-1}| &= |(\lambda - \xi + \xi)(\lambda - \xi)^{-1}| = |1 + \xi(\lambda - \xi)^{-1}| \\ &\leq 1 + (|\xi - \lambda| + |\lambda|)|\lambda - \xi|^{-1} \leq 1 + \left(\frac{d}{2} + |\lambda|\right) \frac{2}{d} = 2 + 2\frac{|\lambda|}{d}. \end{aligned} \quad (\text{IV.208})$$

Hence, condition (IV.207) is true provided

$$b_1 \times \frac{2}{d} + b_2 \times \left(2 + \frac{2|\lambda|}{d}\right) < 1,$$

that is,

$$b_1 + b_2(d + |\lambda|) < \frac{d}{2}. \quad (\text{IV.209})$$

It follows from Theorem IV.7.8 that the circle  $\Gamma$  encloses exactly  $m$  (repeated) eigenvalues of  $\mathcal{H}$  and no other points of  $\sigma(\mathcal{H})$ , provided the condition (IV.209) holds.

For more detail, we refer to [69, V-§4.3].

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