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My Dam

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# Essays on ambiguity and optimal growth with renewable resources

Thèse de doctorat de l'université Paris-Saclay

École doctorale n°578, sciences de l'homme et de la société (SHS)  
Spécialité de doctorat : Sciences économiques  
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Thèse présentée et soutenue à Evry, le 16/06/2020, par

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---

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# Synthèse

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La thèse se compose de deux thèmes principaux: la prise de décision sous l'ambiguïté et la croissance économique en présence des externalités. Les trois premiers chapitres de la thèse sont consacrés à la compréhension de l'impact du risque et de l'ambiguïté sur les décisions économiques fondamentales, telles que le salaire optimal, l'assurance optimal, ainsi que l'investissement optimal dans le capital humain. Le dernier chapitre de la thèse s'agit d'un modèle de la croissance optimal en présence des externalités. Ce résumé met en évidence les principaux résultats et contributions de chaque chapitre à la littérature.

Le premier chapitre, *Contrat d'Assurance Optimal en Présence du Risque et de l'Ambiguïté Reconsidéré*, est un travail conjoint avec Yacine Chitour et François Pannequin. Dans ce chapitre, nous réexamine le problème du contrat optimal d'assurance en présence du risque et de l'ambiguïté en utilisant la théorie du contrôle optimal. L'ambiguïté est modélisés selon [Klibanoff et al. \(2005\)](#). Notre approche généralise les analyses effectuées jusqu'à présent en considérant le contrat d'assurance comme la paire d'une prime et une fonction d'indemnisation à résoudre simultanément. Dans ce cadre, nous prouvons l'existence d'un contrat optimal d'assurance dans le cas le plus général où tous les agents peuvent être simultanément averses à l'ambiguïté et au risque, ce qui englobe tous les cas précédemment examinés. Nous caractérisons non seulement le partage du risque mais aussi la règle du partage de l'ambiguïté entre un assureur et un assuré. Dans le cas de l'aversion vers l'ambiguïté unilatérale, nous montrons qu'une politique de franchise directe ne peut pas être optimale. Au contraire, dans l'hypothèse que les densités conditionnelles puissent être classées selon le rapport de vraisemblance monotone, un contrat avec des franchises qui disparaissent est optimal, un résultat qui est cohérent avec [Gollier \(2014\)](#). En particulier, la méthodologie mise en œuvre complète l'analyse de [Raviv \(1979\)](#) pour le cas du risque pur avec un assureur neutre au risque, montrant qu'une couverture de limite supérieure ne peut pas constituer un optimum. Ce résultat est robuste à la neutralité de l'ambiguïté.

Le deuxième chapitre, *le Modèle de Principal-Agent en Présence de l'Ambiguïté*, est un autre travail conjoint avec Yacine Chitour et François Pannequin. Dans ce chapitre, nous caractérisons le contrat de salaire optimal dans le cas "first-best" quand la distri-

bution de du résultat au travail est ambiguë. L'ambiguïté est modélisée à la [Klibanoff et al. \(2005\)](#), et la formulation du modèle de Principal-Agent suit [Holmstrom \(1979\)](#). Nous utilisons la même technique que dans le premier chapitre, en analysant le contrat de salaire optimal dans le cadre d'un problème de contrôle optimal. Contrairement à la littérature existante, notre solution caractérise "simultanément" la fonction du salaire optimale et le niveau d'effort demandé quand l'information est symétrique (le contrat first-best). De plus, nous traitons le problème sous la forme la plus générale, permettant au principal d'être soit neutre, soit averse au risque et/ou à l'ambiguïté. Ceci est distinct de la littérature existante qui suppose souvent que le principal soit neutre à la fois au risque et à l'ambiguïté. Dans ce cadre, nous prouvons l'existence d'une fonction de salaire optimale pour le cas le plus général. Lorsque le principal est averse au risque, nous montrons que le le salaire optimal est robuste à l'ambiguïté en ce sens qu'il soit monotone croissant par rapport à la performance quelles que soient les attitudes des parties contractantes envers l'ambiguïté et le nombre d'états ambigus. Lorsque le principal est neutre au risque, l'agent est averse au risque et il n'y a que deux états ambigus, nous montrons que l'optimalité d'une telle fonction est robuste à l'aversion à l'ambiguïté si l'ambiguïté a une structure particulière, à savoir que si l'ambiguïté contamine soit la gamme de résultats inférieure ou supérieure, mais pas les deux.

Le troisième chapitre, l'Accumulation Stochastique et l'Investissement Optimal dans le Capital Humain, examine l'impact du risque et de l'ambiguïté sur l'investissement optimal dans le capital humain et le capital physique. L'incertitude (à la fois dans le sens du risque et de l'ambiguïté) est introduite à l'accumulation de capital humain de deux façons. Lorsque l'incertitude porte sur le taux de dépréciation du capital humain (obsolescence incertaine des compétences), j'ai constaté que l'investissement optimal dans le capital humain augmente toujours, que soit présent ou non le capital physique. Cette réponse à l'incertitude d'un ménage représente le comportement typique de l'auto-assurance. En revanche, lorsque l'incertitude se porte sur l'efficacité de l'accumulation du capital humain, l'investissement optimal dans le capital humain diminue parmi les ménages avec l'aversion au risque relative constante inférieure à un. Cette réponse à l'incertitude est typique d'un ménage qui considère l'investissement comme un actif à rendement risqué au lieu d'une assurance.

Le dernier chapitre, les Dynamiques Économiques avec des Ressources Renouvelables, est un travail conjoint avec Thai Ha-Huy, Cuong Le-Van et Thi Tuyet Mai Nguyen. Ce chapitre est relativement indépendant des chapitres précédents car il traite un modèle déterministe plutôt que stochastique. Néanmoins, il examine une question importante dans la théorie de la croissance: le rôle des ressources renouvelables et des externalités. La littérature existante a exploré l'impact des ressources renouvelables, à la fois en tant que bien de consommation directe et alimentation pour la produc-

tion, sur la croissance économique (Beltratti et al., 1998; Ayong Le Kama, 2001). Cette méthode est commode, mais comme Wirl (2004) a observé, il peut y avoir plusieurs solutions. Dans ce chapitre, nous proposons une nouvelle méthode pour étudier une économie à deux secteurs en présence des externalités. En particulier, l'analyse s'agit d'un secteur industriel dont les activités de production ont des effets négatifs sur la capacité de régénération d'une ressource naturelle dans l'autre secteur. L'introduction d'une fonction régénératrice non-concave par rapport à l'un des arguments rend le problème non convexe. En conséquence, nous ne pouvons plus utiliser les techniques traditionnelles de programmation dynamique telles que celles présentées dans Lucas and Stokey (1989) ou Le Van and Dana (2003) pour résoudre notre modèle. Nous abordons ce problème en introduisant le concept de "gain net de stock", qui est une notion similaire au "gain net d'investissement" introduit par Kamihigashi et al. (2007). En absence des propriétés convexes ou supermodulaires habituelles, nous prouvons que l'économie évolue pour augmenter le gain net de stock et établissons les conditions assurant la convergence de l'économie à long terme. Cette approche peut être appliquée aux problèmes de Beltratti et al. (1998) et Ayong Le Kama (2001), ou être étendu à l'analyse des économies aux plusieurs secteurs en général.



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# 1

## Introduction

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This thesis is composed of two principal themes: economic decisions under uncertainty, and optimal growth with renewable resources in presence of externalities. The first and major part of this thesis is devoted to understanding the impact of ambiguity on fundamental economic decisions, such as optimal wage, optimal insurance contracts, as well as optimal investment in human capital. In this opening chapter, let us first revisit the paradox raised by [Ellsberg \(1961\)](#) to understand its main attacks on the traditional frameworks of decision making under uncertainty. We shall then follow (in an non-exhaustive fashion) the theoretical developments that have been made thenceforth to rationalize the Ellsberg choice.

Decades before Ellsberg, the distinction between risk and ambiguity has been emphasized by [Knight \(1921\)](#). In Knight's terms, a risky situation is one in which the decision maker (DM) can objectively deduce probabilities. Such a situation includes the likelihood of obtaining a "head" when tossing a fair coin, the odds of winning in a game of Russian Roulette, or the chance of drawing a pair from a sufficiently shuffled deck of card. By contrast, an ambiguous situation involves immeasurable uncertainty - the kind of uncertainty that a DM who cannot objectively quantify or deduce must thereupon resort to a subjective evaluation. This kind of uncertainty, which is referred to as ambiguity (Knightian uncertainty) abounds and affects many of our daily decisions. Think about the odds of Donald Trump being re-elected for another term, the likelihood that a vaccine against the SARS-Cov-2 virus will be found in the next month, or the probability of winning in a horse lottery (bets in a horse race).

How does the modern literature account for the two types of uncertainty? The most influential model of decision-making under risk (objective uncertainty) is perhaps the expected utility theory (EU) of [Von Neumann and Morgenstern \(1947\)](#), and its counterpart under subjective uncertainty is the subjective expected utility theory (SEU) of [Savage \(1954\)](#).

While continuing to serve as a canonical tool in choice modeling under risk, the EU

framework is not without criticism. One of the earliest and most powerful critiques to Von Neumann and Morgenstern (1947) comes from the French economist and physicist Maurice Allais. In particular, Allais (1953) demonstrates systematic violation of the Independence Axiom, one of the premises on the preference relation of the DM (on the set of objective lotteries) in order for it to have an expected utility representation<sup>1</sup>.

The SEU camp of is not immune to criticism either. With his classic thought experiments, Ellsberg (1961) contests the hypothesis that there exists a consistent subjective probability which can be deduced from individuals' choice under a set of suitable conditions. Recall that distinct from the EU à la von Neumann-Morgenstern, probabilities are *endogenous* in SEU. To illustrate, let us revisit Ellsberg's color betting experiment to understand why this hypothesis is contradicted.

In this experiment, Ellsberg presents the subjects with an urn containing 90 balls, 30 of which are known to be red, the rest are black or yellow in unknown proportion. A ball is drawn from the urn. Each subject is asked to rank two pairs of alternatives. First, they are asked to choose between betting on *red* and betting on *black*. Then they are asked to choose between *red or yellow* and *black or yellow*. In other words, individuals are asked to rank their preferences between  $f_1$  and  $f_2$ , and then between  $f_3$  and  $f_4$  in Table 1.1. Observe that the element of ambiguity in this experiment enters via

	R = 30	B	Y = 60 - B
$f_1$	\$100	\$0	\$0
$f_2$	\$0	\$100	\$0
$f_3$	\$100	\$0	\$100
$f_4$	\$0	\$100	\$100

Table 1.1: Ellsberg's color betting experiment

the unknown proportion of the black and yellow balls. At first glance, one might be tempted to invoke the so-called *Principle of Insufficient Reason* proposed by Bernoulli (1954) to assign equal probabilities to all possible scenarios. This principle suffers from scathing criticism by Keynes (1909) in Chapter IV of his book *A treatise on probability*. Indeed there can be multiple partitions of the same state space. Applied to the problem at hand, it is not clear if the principle tells us to suppose that the number of black and yellow balls are equal, or that each of the events ( $B = x, Y = 60 - x$ ) where  $x = 0, \dots, 60$  has the same probability (of  $1/61$ ).

The SEU theory developed by Savage (1954) is too technically complex for our pur-

<sup>1</sup>Recall the representation theorem of Von Neumann and Morgenstern (1947): Let  $X$  denote the finite set of consequences,  $\Delta(X)$  the set of simple probability distributions on  $X$  and  $\succeq$  be a preference relation over  $\Delta(X)$ . Then  $\succeq$  is a weak order (complete and transitive), Archimedean and independent if and only if there exists a function  $v : X \rightarrow \mathbb{R}$ , the vNM utility index such that  $\succeq$  is represented by the linear expected utility function  $U : \Delta(X) \rightarrow \mathbb{R}$  defined by  $U(\pi) = \sum_{x \in X} v(x)\pi(x)$  for all  $\pi \in \Delta(X)$ . Moreover,  $U$  is defined up to an affine transformation.

pose; we shall present only his most famous axiom, the *Sure Thing Principle* (STP), which is also the one challenged by Ellsberg.<sup>2</sup> First we need some definitions to set up Savage's world. Let  $\Omega$  be a state space and  $X$  be the set of consequences. The elements of  $\Omega$  are assumed to be exhaustive and mutually exclusive (one and only one state  $\omega \in \Omega$  must occur). A *Savage act*  $f : \Omega \rightarrow X$  is a function that maps events (subsets of  $\Omega$ ) to consequences. Savage acts are also called event-contingent lotteries, as opposed to traditional probability-contingent lotteries (Dhami, 2016). For example, let  $\{E_1, \dots, E_n\}$  be a partition of  $\Omega$  and  $f(E_i) = x_i \in X$  for  $i = 1, \dots, n$ . Then we can write this act as  $f = (x_1, E_1; \dots; x_n, E_n)$ . The set of all Savage acts is  $X^\Omega$ , the set of all functions from  $\Omega$  to  $X$ . Kreps (2013)'s version of the STP is the following.

**Axiom 1** (The Sure Thing's Principle). *Let  $f, f', h, h'$  be Savage acts and  $T \subset \Omega$  such that  $f(\omega) = f'(\omega)$  and  $h(\omega) = h'(\omega)$  for all  $\omega \in T$ , and  $f(\omega) = h(\omega)$  and  $f'(\omega) = h'(\omega)$  for all  $\omega \in T^c$ . Then  $f \succeq h \iff f' \succeq h'$ .*

Observe that  $f$  and  $h$  yield the same outcome on  $T^c$  (the complementary event of  $T$ ). The same goes for  $f'$  and  $h'$ . The STP says that if two acts are equivalent when some event  $T$  does not occur, it does not matter what they are equivalent to (conditional on  $T^c$ ). In particular, the ranking between  $f$  and  $h$  should depend only on the event where they differ. By this principle, the comparison between  $f$  and  $h$ , and that between  $f'$  and  $h'$  are equivalent.

In the thought experiment of Ellsberg, most subjects strictly prefer  $f_1$  to  $f_2$ , but strictly prefer  $f_4$  to  $f_3$ . This is at odds with the STP. In particular,  $f_1$  and  $f_2$  yield the same payoff in the event of yellow. The same is true for the pair  $f_3$  and  $f_4$ . In the complementary event of yellow (blue or red),  $f_1$  and  $f_3$  are identical, and so are  $f_2$  and  $f_4$ . Thus choices consistent with the STP must have been either  $(f_1 > f_2, f_3 > f_4)$  or  $(f_2 > f_1, f_4 > f_3)$ , not  $(f_1 > f_2, f_4 > f_3)$ . The predominantly reported choice violates the STP, consequently refuting the existence of a consistent subjective probability. Specifically, it is easy to see that the choice  $f_1 > f_2$  implies a subjective probability of blue strictly less than  $1/3$ , while the ranking  $f_4 > f_3$  manifests the opposite. This choice is the consequence of a phenomenon called *ambiguity aversion*.

Although the Ellsberg paradox is based on a thought experiment, its striking implications have opened up whole new branch of research on ambiguity, both theoretically and empirically. Following Ellsberg, empirical evidence of ambiguity aversion has been documented from several surveys and lab experiments. Camerer and Weber (1992) documents early experimental works on ambiguity. Surveys on more recent empirical works can be found in Trautmann and Van De Kuilen (2015). The concept of ambiguity contributes to the understanding of a growing number of economic topics

<sup>2</sup>An alternative approach to the SEU representation result is developed by Anscombe et al. (1963).

and puzzles, such as the stock market participation puzzle (Dow and Werlang, 1992; Bossaerts et al., 2010; Collard et al., 2018), portfolio choice and ambiguity aversion (Gollier, 2011), the low take-up of freely available genetic tests (Hoy et al., 2014), the decision to trust (Corcos et al., 2012; Li et al., 2019), or the value of statistical life (Treich, 2010; Bleichrodt et al., 2019; Berger et al., 2013).

We now turn to some of the seminal theoretical models most pertinent to this thesis.<sup>3</sup> In particular, we focus on models of ambiguity that reduce to expected utility under pure risk. The attractive feature of these models is their analytical tractability, given that much of the literature has been built on EU. In a loose sense, models presented here are not behaviorally founded. Rather, they represent attempts at extending the traditional framework to account for a specific psychological phenomenon called *ambiguity aversion* and thus leave untouched existing contentious issues of EU.<sup>4</sup>

The first model to mention in this class is the so-called multiple priors or maximin expected utility (MEU) of Gilboa and Schmeidler (1989). Gilboa and Schmeidler develop their theory on the foundation of Anscombe et al. (1963) rather than that of Savage (1954). This approach is widely adopted in the literature since the Anscombe-Aumann framework is technically simpler. The reasoning of Gilboa and Schmeidler is the following. In presence of ambiguity, the DM forms a set of plausible priors. She then proceeds to evaluate each alternative by considering the minimal expected utility it yields over the set of priors. The DM selects the alternative that *maximizes minimum expected utility*, hence the name MEU. To be precise, let  $f, g : \Omega \rightarrow \Delta(X)$  be Anscombe-Aumann acts (which map events to lotteries over the set of consequences). Let  $\Delta(\Omega)$  be the set of lotteries over the state space  $\Omega$  and  $C$  (the set of priors) be a closed and convex subset of  $\Delta(\Omega)$ . Then under a set of suitable conditions, the authors show that:

$$f \succeq g \iff \min_{\mu \in C} \int_{\Omega} U(f) d\mu \geq \min_{\mu \in C} \int_{\Omega} U(g) d\mu, \quad (1.1)$$

where  $U : \Delta(X) \rightarrow \mathbb{R}$  is the expected utility map. Note that as the Ellsberg paradox violates the STP, it also violates Anscombe-Aumann's *Independence Axiom* since the latter implies the former. Gilboa and Schmeidler obtain their representation by relaxing the Independence Axiom, replacing it with two other milder conditions called *C-independence* (independence with respect to constant acts) and *Uncertainty aversion*.

Ghirardato et al. (2004) extends MEU to  $\alpha$ -MEU, aiming to allow for varying degrees of ambiguity aversion via the index  $\alpha \in [0, 1]$ . In this framework, an alternative  $f$  is

<sup>3</sup>For more comprehensive treatment on the theoretical development of this literature, interested readers may consult the works of Siniscalchi (2008), Machina and Siniscalchi (2014) and Etner et al. (2012).

<sup>4</sup>Wakker (2010) advocates for *Propect theory* (Kahneman and Tversky, 1979) as the more psychologically realistic framework to study decisions under risk and ambiguity.

evaluated according to:

$$V(f) = \alpha \min_{\mu \in C} \int_{\Omega} U(f) d\mu + (1 - \alpha) \max_{\mu \in C} \int_{\Omega} U(f) d\mu. \quad (1.2)$$

When  $\alpha = 1$ , this model reduces to MEU. Unfortunately, the axiomatization only holds for  $\alpha \in \{0, 1\}$  (Eichberger et al., 2011). Moreover, Siniscalchi (2006) contests the claim that  $\alpha$  can be interpreted as the degree of ambiguity aversion, showing that ambiguity and ambiguity aversion (supposedly captured by the set of priors  $C$  and the index  $\alpha$ , respectively) cannot be disentangled in  $\alpha$ -MEU.

Let us turn next to Segal’s recursive model (Segal, 1987, 1990). The author demonstrates that in presence of ambiguity, the paradox arises due to the failure of the *Reduction Of Compound Lottery* (ROCL) axiom. To fix ideas, consider the diagram in Fig-

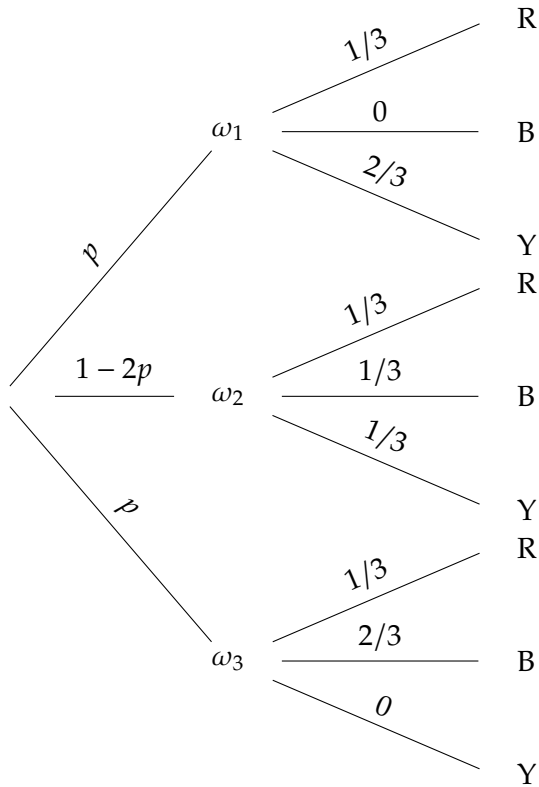


Figure 1.1: Segal’s two-stage decision process

ure 1.1. The DM considers three plausible scenarios (or second-order states)  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ . She believes that one of them will realize, but faces ambiguity regarding the distribution of these states. Conditional on each second-order state, each act can be evaluated via an objective lottery (since the distribution of first-order states (R,B,Y) is known in each second-order state). Figure 1.1 depicts a situation that might arise from the Ellsberg’s color betting experiment. Here the DM forms a *subjective* prior over the

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second-order states. Notice that all ambiguity is resolved once the second-order state is known. It is easy to see that if the ROCL axiom were satisfied, then all ambiguity could be reduced to risk. In this case, the DM would make decisions as if she faced an urn in which there were 30 balls of each color. Yet the DM deviates from ROCL and processes decisions in two stages. First, for each alternative, she computes the certainty equivalents using *rank-dependent expected utility* (RDEU). In the second stage, she computes the RDEU of the lottery  $f = (p, CE_1(f); 1 - 2p, CE_2(f); p, CE_3(f))$ , where  $CE_i(f)$  denotes the RDU certainty equivalent of the lottery induced by the alternative  $f$  under state  $\omega_1$ .

The smooth model of [Klibanoff et al. \(2005, 2009\)](#) belongs to the same class as those of Segal. There are two main differences, however. First, Klibanoff, Marinacci and Mukerji (KMM from now on) consider subjective rather than objective lotteries. Second, they assume different (rather than identical) preferences in each stage, one over Savage acts (first-order acts) and the other over *second-order acts*. Both are assumed to have an expected utility representation. A crucial axiom, called *Consistency* connects the two. Let  $\mathcal{S} = \Omega \times [0, 1]$  be the state space and  $\Delta(\mathcal{S})$  be the set of lotteries over this space. A first-order act  $f : \mathcal{S} \rightarrow X$  is evaluated according to:

$$W(f) = \int_{\Delta(\mathcal{S})} \phi \left( \int_{\mathcal{S}} U(f(s)) d\pi \right) d\mu(\pi) = \mathbb{E}_{\mu} \phi (\mathbb{E}u \circ f), \quad (1.3)$$

where  $\mu$  is the DM's second-order prior, capturing *ambiguity*. The functional  $\phi$  describes the attitude towards ambiguity, while the utility function  $u$  reflects the attitude towards risk. The functional  $\phi$  being concave, convex or linear corresponds to a DM being ambiguity averse, seeking, or neutral, respectively. KMM show that the MEU model of [Gilboa and Schmeidler \(1989\)](#) is a special case of smooth ambiguity aversion. In particular, MEU corresponds to a DM with infinite absolute constant ambiguity aversion, where the degree of ambiguity aversion (absolute or relative) is defined in the same fashion as the degree of risk aversion.

Note that  $\phi = v \circ u^{-1}$ , where  $v$  is a strictly increasing, typically strictly concave utility function that figures in the representation of the second-order preference. This provides a close link to Segal's framework previously introduced. Indeed, the KMM agents also processes decisions in two stages. In the first stage, they calculate the certainty equivalents of lotteries for each first-order act (with respect to EU rather than RDEU). In the second stage the distortion due to ambiguity aversion is captured by the concavity of  $v \circ u^{-1}$ . Loosely speaking, the ambiguity-averse agent dislikes alternatives that lead to large variations in expected utilities. When  $v \equiv u$ , the representation reduces to EU.

Let us consider an example that illustrates how KMM rationalize the Ellsberg choice.

Consider again the situation in Figure 1.1 but with  $p = \frac{1}{2}$ . For simplicity let us normalize  $u(0) = 0$  and  $u(100) = 1$ . Let  $\phi$  be a strictly concave function capturing ambiguity aversion. KMM's evaluation of the alternatives in Table 1.1 are as follows.

$$V(f_1) = \phi\left(\frac{1}{3}\right), \quad (1.4)$$

$$V(f_2) = \frac{1}{2}\phi\left(\frac{2}{3}\right) + \frac{1}{2}\phi(0), \quad (1.5)$$

$$V(f_3) = \frac{1}{2}\phi\left(\frac{1}{3}\right) + \frac{1}{2}\phi(1), \quad (1.6)$$

$$V(f_4) = \phi\left(\frac{2}{3}\right). \quad (1.7)$$

Since  $\phi$  is strictly concave, we obtain immediately by Jensen inequality that  $V(f_1) > V(f_2)$  and  $V(f_4) > V(f_3)$ , resolving the paradox.

The smooth model of KMM is analytically tractable. Its salient advantages include the separation between ambiguity and ambiguity aversion<sup>5</sup>, which allows to conveniently perform comparative statics. The model has also been supported by a number of experimental studies (Halevy, 2007; Chakravarty and Roy, 2009; Conte and Hey, 2013; Ahn et al., 2014; Baillon and Bleichrodt, 2015; Mukerji et al., 2019; Cubitt et al., 2018). For these reasons, the smooth model of ambiguity aversion is adopted as the analytical framework of decisions under ambiguity of this thesis.

Chapter 2, *Optimal Insurance under Risk and Ambiguity Reconsidered*, is a joint work with Dr. Yacine Chitour and Dr. François Pannequin. In this chapter, we revisit the problem of optimal insurance contract design under risk and ambiguity in an optimal control framework, where ambiguity and ambiguity preferences are modeled according to Klibanoff et al. (2005). Our approach generalizes the analyses carried out so far by viewing the insurance contract as a pair of a premium and an indemnity schedule to be solved for simultaneously. In this framework, we prove the existence of an optimal insurance policy in the most general case where all agents can be simultaneously ambiguity-and-risk-averse, which encompasses all the cases previously examined. We characterize not only the risk-sharing but also the ambiguity sharing rule between an insurer and a policyholder. Under one-sided ambiguity aversion, we show that a straight deductible policy cannot be optimal. Rather, under the assumption that the conditional densities can be ranked according to the monotone likelihood ratio, a contract with disappearing deductibles is optimal, a result that is consistent with Gollier (2014). In particular, the methodology implemented completes the analysis of Raviv (1979) under pure risk in the context of a risk-neutral insurer, showing that an upper limit coverage cannot constitute an optimum. This result also holds under ambiguity

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<sup>5</sup>This feature is challenged by Epstein (2010).



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neutrality.

Chapter 3, *The Principal-Agent Model under Smooth Ambiguity*, is a joint work with Dr. Yacine Chitour and Dr. François Pannequin. In this chapter, we characterize the symmetric information benchmark for the principal-agent model under ambiguity, where ambiguity is modeled via the framework of [Klibanoff et al. \(2005\)](#) and the principal-agent formulation follows [Holmstrom \(1979\)](#). We employ the same technique as in Chapter 2, studying the optimal wage contract in an optimal control framework. In contrast to the existing literature, our solution characterizes *simultaneously* the optimal wage function *and* the level of effort demanded of the agent under symmetric information (first-best result). Furthermore, we treat the problem in the most general form, allowing for the principal to be either neutral or averse to risk and/or ambiguity. This is distinct from the existing literature which often assumes that the principal is neutral to both risk and ambiguity. In this framework, we prove the existence of an optimal wage function for the most general case. When the principal is risk-averse, we show that the optimal wage behavior is robust to ambiguity in the sense that it is increasing in outcomes regardless of the attitudes of the contracting parties towards ambiguity and the number of ambiguous states under consideration. When the principal is risk-neutral the agent is risk-averse and there are only two ambiguous states, we show that a wage function increasing in outcomes is also optimal under ambiguity aversion if ambiguity has an one-sided structure, namely that if ambiguity *contaminates* either the lower or the higher range of outcomes, but not both.

The next two chapters of the thesis concern the investment and consumption decisions in the context of endogenous growth theory. The modern theory of optimal economic growth began with the seminal contributions of [Solow \(1956\)](#) and [Swan \(1956\)](#). The Solow-Swan model introduces the neoclassical production function with diminishing returns while inheriting many features from [Harrod \(1939, 1948\)](#) and [Domar \(1946, 1947\)](#), including the aggregate Keynesian saving function. In these models, the level of saving is a constant fraction of output ([Keynes, 1936](#)).<sup>6</sup> Although the Solow-Swan is able to explain all the Kaldor facts ([Kaldor, 1957](#)), it has a troubling feature: sustained growth is due only to exogenous factors, such as population growth or technological progress. This implication is essentially due to the hypothesis of diminishing returns imposed on the production function. At the same time, saving being a linear function of output is at odds with the experience of the US for the period since 1899, as [Friedman \(1957\)](#) has pointed out from the estimation of Kuznets. It is noteworthy that there is no optimizing household in the Solow economy. Convergence is conditional on the saving rate, a too high level of which might result in dynamic inefficiency - a situation where steady state level of consumption is not maximized. It turns out, this

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<sup>6</sup>A concise exposition on this class of models can be found in Chapter 1 of [Le Van and Dana \(2003\)](#).

inefficiency is eliminated once saving is endogenized, which is what [Cass \(1965\)](#) and [Koopmans et al. \(1963\)](#) have done. In particular, these authors reintroduce the idea of intertemporal utility maximization pioneered by [Ramsey \(1928\)](#) into the Solow-Swan model, completing the basic theoretical framework that has become the workhorse of modern macroeconomics. In the Ramsey-Cass-Koopmans (RCK) model, the presence of a discount factor that puts more weight on current consumption (impatience) induces households to save less than those in the Solow-Swan economy.

Remarkably, the theoretical implications of the RCK are very similar to those of Solow-Swan. In particular, it still predicts (rather fast) conditional convergence and no sustained growth except due to factors exogenous to the model or under AK production.<sup>7</sup> In other words, diminishing returns remains the key barrier to economic growth. Exogenous technological progress helps overcome this barrier by making inputs more productive. [Brock and Mirman](#) introduced uncertainty in the form of a stochastic productivity shock into the RCK model.

Initiated by [Solow \(1957\)](#), several studies have been conducted to quantify the *Solow residual*, the estimate of the growth of the total factor of productivity (TFP), and its contribution to economic growth. The unanimous conclusion of these works is that TFP growth plays a substantial role. For example, following the method of [Jorgenson and Griliches \(1967\)](#), [Christensen et al. \(1980\)](#) report 1.35 percent of annual TFP growth during the period 1947-1973 for the US, which accounts for 34 percent of economic growth. The TFP growth estimate for France during the same period is almost 3 percent, contributing to 54 percent of economic growth.<sup>8</sup> Evidently, the importance of TFP cannot be overestimated. Nevertheless, this quantity remains a black box in the Solow-Swan or the basic RCK model. The sore need to understand the mechanism behind TFP growth calls for further investigation, leading to the development of a new literature which has come to be known as endogenous growth theory. In essence, the goal of endogenous growth theory is to flesh out how the optimizing decisions of agents (public or private) in the model lead to higher productivity of inputs over time. With significant contributions from [Aghion et al. \(1998\)](#), this line of research can be roughly divided into three big topics: technological change, human capital accumulation, and natural resources (exhaustible and renewable). Chapters 4 and 5 of the thesis fall under the last two categories.

[Uzawa \(1965\)](#) emphasizes that technological progresses or knowledge should not be viewed as *mana from heaven*. This idea does not seem to receive much attention until

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<sup>7</sup>For details on the AK model, see [Romer \(1986\)](#) and [Rebelo \(1991\)](#).

<sup>8</sup>See [Christensen et al. \(1980\)](#) and [Jorgenson and Yip \(2001\)](#) for estimates of OECD countries for the periods 1947-1973 and 1960-1995, respectively. Estimates for East Asian countries during 1966-1990 can be found in [Young \(1995\)](#), and for Latin American countries during 1940-1990 can be found in [Elias et al. \(1992\)](#).

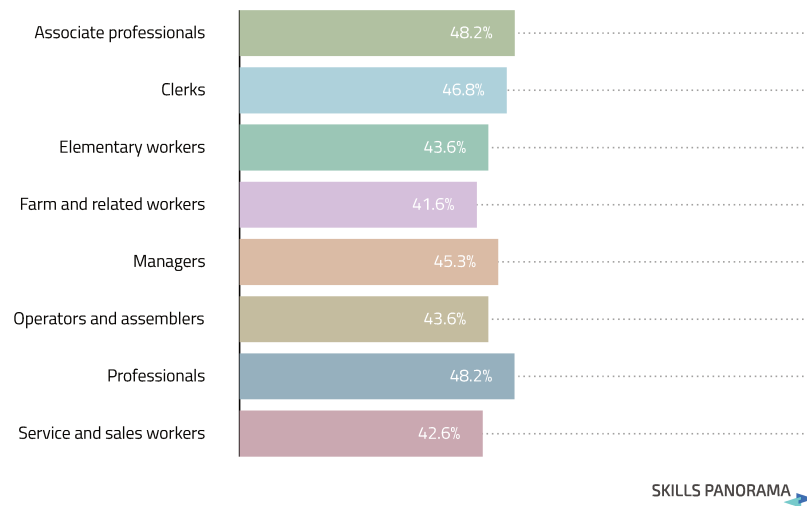


Figure 1.2: Subjective skills obsolescence in a five-year horizon by occupations in Europe. *Source:* European skills and jobs survey (Cedefop, 2014).

the late 80s and early 90s, when several contributions are devoted to understanding the role of human capital as a source of endogenous growth, such as Romer (1986) and Lucas (1988, 1990). Mankiw et al. (1992) defend the explanatory power of the Solow-Swan model by taking into account human capital.<sup>9</sup> Starting with the seminal contribution of Becker (1964), it is now consensus that the main building blocks of human capital are health and education, both of which are at risk at the current pace of technological progress and environmental complications. Climate change and epidemics with unpredictable severity and probability of occurrence can have damaging consequences on public health, which in turn might lead to economic crises. The *Information Age* that began in the late 1950s-1970s has dramatically altered the way we work. Low-skilled and routine jobs are being replaced by machines in both developing countries and advanced economies. In fact, even skilled-jobs are not immune to obsolescence, as suggested by the results from the European skills and jobs survey 2014 illustrated in Figure 1.2. Overall, about 45 percent of the respondents indicated that many of their skills were likely to become obsolete in the following five years. Interestingly, when asked whether they thought this scenario would be *very likely*, this statistic dropped to about 20 percent. This implies a high degree of subjective uncertainty.

Uncertainty in human capital accumulation has become more relevant than ever before, posing challenges to both policymakers and private corporations. On the policy side, the World Bank has recognized that countries are underinvesting in human capital, which might broaden the skills gap even further between rich and poor countries. To address this pressing issue, the group created the Human Capital Project in 2018.

<sup>9</sup>Their model is often referred to as the *augmented Solow* model. Since the diminishing return hypothesis still holds, this model also predicts no long-run growth.

Notably, a human capital index (HCI) based on different measures of health and education is being computed for each country (see [Avitabile et al. \(2020\)](#)). According to [Collin and Weil \(2018\)](#), better health not only enhances productivity but also enables us to enjoy life better. In other words, good health is desirable both instrumentally and intrinsically. On the business side, [Nalbantian \(2017\)](#) calls for a distinction between human capital risk and ambiguity in order to address them with proper measures. [Deloitte \(2018\)](#) highlights the importance of sufficiently investing in the workforce to pre-arm for *the forth industrial revolution*, the age of artificial intelligence.

From a modeling viewpoint, these sources of uncertainty call for a relaxation of the perfect foresight hypothesis on human capital accumulation typically seen in optimal growth models. Chapter 4 of the thesis, *Stochastic Accumulation and the Optimal Investment in Human Capital* is another step in this direction, with an aim to draw theoretical implications from the introduction of risk and ambiguity to human capital accumulation. One important question that this chapter tries to answer is whether more investment always the right response to uncertainty. This work continues the line of research initiated by [Levhari and Weiss \(1974\)](#) and [Williams \(1979\)](#), and is in a spirit similar to [Krebs \(2003\)](#). In particular, this chapter examines the impact of risk and ambiguity on the optimal level of investment in human and physical capital. Uncertainty (both in the sense of risk and of ambiguity) is introduced to the accumulation of human capital via two channels. When uncertainty is on the depreciation rate of human capital (uncertain skills obsolescence), I found that the optimal level of investment in human capital always increases, regardless of whether a risk-free physical capital is present. This response to uncertainty of an optimizing household is typically a self-insurance type of behavior. By contrast, when uncertainty is introduced to the efficiency of human capital accumulation, the optimal investment in this type of capital declines for the group of representative households with CRRA utility with relative risk aversion less than one. This response to uncertainty is typical of a household who views the investment as an asset with risky return instead of an insurance alternative.

Leaving human capital, Chapter 5, *Economic Dynamics with Renewable Resources and Pollution* of the thesis considers the issue of optimal extraction of renewable resources in endogenous growth theory. It is a joint work with Dr. Thai Ha-Huy, Dr. Cuong Le-Van and Thi Tuyet Mai Nguyen. The existing literature has explored the impact of renewable resources, both as direct consumption good and inputs for production, on economic growth ([Beltratti et al., 1998](#); [Ayong Le Kama, 2001](#)). This method is appealing, but multiple long-run outcomes may exist as [Wirl \(2004\)](#) has observed. Our contribution is from a methodological perspective. In particular, we propose a new method to study a two-sector economy in which the industrial activities of a sector have negative impacts on the regenerating capacity of a natural resource in the other

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sector. The introduction of a regenerating function that is non-concave with respect to one of the arguments makes the problem non-convex, rendering existing dynamic programming techniques (such as those presented in [Lucas and Stokey \(1989\)](#) or [Le Van and Dana \(2003\)](#)) inapplicable to our model. The novelty of our approach lies in the introduction of the concept *the net gain of stock*, which is a similar notion to the *net gain of investment* previously studied by [Kamihigashi et al. \(2007\)](#). In absence of the usual convex or the supermodular properties, we prove that the economy evolves to increase the net gain of stock and establish conditions that ensure the convergence of the economy in the long run. This approach can be applied to the problems of [Beltratti et al. \(1998\)](#) and [Ayong Le Kama \(2001\)](#), or extended to similar analyses of multisector economies in general.

# 2

## Optimal insurance under risk and ambiguity reconsidered

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## Chapter nomenclature

Notation	Meaning	Reference page(s)
$\mathcal{I}$	Index set of (second-order) states	28
$p_i$	Prior on state $i$	28
$x$	Realization of the loss variable	28
$I_x$	Support of the the loss	28
$I_\pi$	Domain of the premium	30, 29
$f_i$	State- $i$ conditional density	28
$\ell_{ij}$	Likelihood ratio	28
$I$	Indemnity/Coverage function	30
$J$	Indemnity/Coverage rate function	30
$\mathcal{U}$	Admissible control set	65
$g(J, \pi)$	Objective function (Cost)	31
$h(J, \pi)$	Constraint function	31
$\pi$	The premium	30
$W_A$	Policyholder's initial wealth	29
$W_A(x)$	Policyholder's final wealth	36
$W_P(x)$	Insurer's final wealth	37
$\bar{V}$	Insurer's reservation welfare	29
$W_P$	Insurer's initial wealth	29
$u$	Policyholder's utility function	29
$r_u$	Policyholder's absolute risk aversion	29
$v$	Insurer's utility function	29
$\phi_A$	Policyholder's second-order utility	29
$\phi_P$	Insurer's second-order utility	29
$A$	Policyholder's expected marginal welfare	35
$P$	Insurer's expected marginal welfare	35
$G$	Ratio of expected marginal welfares	35
$H$	The Hamiltonian	33
$X(x)$	Vector of states	30
$N_{\mathcal{M}_0}(X(0))$	Normal cone at minimal loss	33
$N_{\mathcal{M}_1}(X(\bar{x}))$	Normal cone at maximal loss	33

Table 2.1: Notations used in Chapter 2

## 2.1 Introduction

We know from the seminal work of [Arrow \(1974\)](#) that a straight deductible is optimal for a risk-averse policyholder facing a risk-neutral insurer and linear cost of indemnity provision. Since this pioneering work, the efficiency of deductible contracts became one of the basics of Insurance Economics and has proved particularly robust to economic contexts and generalizations.

[Raviv \(1979\)](#) made the first attempt to generalize the work of [Borch \(1960\)](#) and [Arrow \(1965; 1974\)](#) to demonstrate that the existence of co-insurance contracts is due to either the convex cost of indemnity provision, or risk-aversion on the part of the insurer. Alternatively, [Huberman et al. \(1983\)](#) showed that a disappearing deductible is optimal in the presence of concave transaction costs.

Subsequently, some contributions focused on the generalization of these results beyond the Expected Utility (EU) model ([Zilcha and Chew, 1990](#); [Karni, 1992](#); [Machina, 1995](#)). Especially, [Gollier and Schlesinger \(1996\)](#) showed that the optimality of deductibles is not exclusively reserved for the EU model since it springs from first- and second-degree stochastic dominance.

However, recently, [Bernard et al. \(2015\)](#) questioned the relevance of a straight deductible contract for a decision maker whose preferences are described by the Rank Dependent Expected Utility (RDEU) model. In contrast with the mainstream results they showed that the optimal contract insures not only large losses above a deductible but also small ones. While the RDEU model results in a best fit to real human behavior than the EU model, these authors challenge Arrow's result.

In this paper, we follow this line of research and investigate the robustness of the efficiency of deductible insurance contracts under ambiguity. For this purpose, we characterize the efficient design of an insurance contract under ambiguity and provide a comprehensive treatment of the relationship between the insurer and the policyholder, in a principal-agent framework, both under risk and ambiguity.

The concept of ambiguity contributes to the understanding of a growing number of economic topics and puzzles, such as the stock market participation puzzle ([Dow and Werlang, 1992](#); [Bossaerts et al., 2010](#); [Collard et al., 2018](#)), portfolio choice and ambiguity aversion ([Gollier, 2011](#)), the low take-up of freely available genetic tests ([Hoy et al., 2014](#)), the decision to trust ([Corcos et al., 2012](#); [Li et al., 2019](#)), the value of statistical life ([Treich, 2010](#); [Bleichrodt et al., 2019](#); [Berger et al., 2013](#)).

In the case of insurance behavior, ambiguity makes sense since many risks are either objectively poorly defined (e.g. environmental risks) or subjectively poorly perceived by the insured (e.g. health risk). Two recent contributions addressed the question of optimal demand for prevention and insurance when risks are ambiguous. While [Alary](#)



et al. (2013) emphasized the role of ambiguity aversion on the demands for insurance, self-insurance and self-protection, Gollier (2014) characterized optimal insurance contracting under linear transaction costs.

Both papers followed the approach of Klibanoff et al. (2005) to model attitudes towards ambiguity although Ellsberg's paradox (1961) motivated the development of several competing ambiguity models (See Gilboa and Marinacci (2016), Machina and Siniscalchi (2014) or Etner et al. (2012)).

We also chose to rely on the smooth model of ambiguity of Klibanoff et al. (2005) owing its ability to distinguish between risk aversion and ambiguity aversion. Moreover, the smooth ambiguity model has received a significant support from experimental studies, including Halevy (2007), Chakravarty and Roy (2009), Conte and Hey (2013), Ahn et al. (2014), Baillon and Bleichrodt (2015), Mukerji et al. (2019), Cubitt et al. (2018).

We implemented a comprehensive approach of the problem of optimal insurance contracting, to extend and revisit the analysis of both contexts of risk and ambiguity.

First, we explored the idea that both parties could be both risk and ambiguous averse. If risk aversion on the insurer's side has already been studied in (Raviv, 1979), we found relevant to assume ambiguity aversion not only on the policyholder's side but also on insurer's side. In the context of environmental and catastrophic risks, several studies documented the fact that insurers are ambiguity averse (Kunreuther and Hogarth, 1992; Kunreuther et al., 1993, 1995; Cabantous, 2007; Cabantous et al., 2011). Moreover, the substantial growth of insurance-linked securities (Cat bonds), which provide capital market-based insurance against the risk of natural catastrophes, in addition to standard reinsurance mechanisms, also argues for the benefit of the general assumption of ambiguity aversion.

Second, from a methodological viewpoint, our comprehensive approach differs from that generally used in the literature. In Raviv (1979) and related papers, the optimal control problem characterizing the efficient insurance contract is conducted to seek the optimal indemnity function assuming the insurance premium as fixed. Instead, we solve the optimal control problem for the optimal indemnity function and the premium simultaneously.

Our results contribute to many dimensions of the literature. First, as this aspect is neglected in the literature, we have proved the existence of an optimal insurance policy in the most general case where all agents can be simultaneously ambiguity-and-risk-averse, encompassing all the cases examined in the previous papers. Second, we characterize not only the risk sharing but also the ambiguity sharing between an insurer and a policyholder. Third, our methodology allowed to complete the analysis of (Raviv, 1979) for the case of a risk-neutral insurer. In particular, we showed that the policy with an upper limit coverage in the first theorem of Raviv (1979) cannot constitute an

optimum.

The rest of the paper is organized as follows. Section 2.2 introduces the optimal insurance problem under risk and ambiguity and key assumptions. Section 2.3 provides an existence proof of an optimal insurance contract. Section 2.4 applies the Pontryagin's Maximum Principle to characterize the contract under ambiguity. The unambiguous case of Raviv (1979) is revisited and the upper limit contract is ruled out. The disappearing deductible contract of Gollier (2014) is also recovered in the case of two ambiguous states. Section 2.5 provides a numerical simulation and Section 2.6 concludes.

## 2.2 Definition of the optimal insurance problem as an optimal control problem (OCP)

In this paper, we are interested in the problem where a potential policyholder considers an optimal insurance policy  $(I(\cdot), \pi)$  where  $\pi$  is the premium the policyholder pays to the insurer to obtain an indemnity schedule  $I(\cdot)$ . Let the subscripts  $A$  and  $P$  denote, respectively, the policyholder and the insurer, who are the DMs in this problem. Let The problem faced by the policyholder, which we shall refer to as the original problem (OP) is the following.

$$\max_{(I(\cdot), \pi)} \sum_{i=1}^n p_i \phi_A \left( \int_{I_x} u(W_A - \pi - x + I(x)) f_i(x) dx \right) \quad (2.1a)$$

$$s.t. \quad I(x) \in [0, x], \quad \forall x \in I_x, \quad (2.1b)$$

$$\pi \in I_\pi = [\underline{\pi}, \bar{\pi}] \subseteq I_x, \quad (2.1c)$$

$$\sum_{i=1}^n p_i \phi_P \left( \int_{I_x} v(W_P + \pi - I(x) - \psi(I(x))) f_i(x) dx \right) \geq \bar{V}, \quad (2.1d)$$

where  $W_A$  and  $W_P$  stand denote the initial wealth of the policyholder and the insurer, respectively. The last inequality is often called the participation constraint (of the insurer). In the program above,  $x$  stands for the loss faced by the policyholder, which is a continuous random variable. Ambiguity enters through the unknown *second-order state*  $i$  taking values in a finite second-order state space  $\mathcal{I}$ . Notice that the density of the loss is  $i$ -conditional. The DMs have perfect knowledge of  $\mathcal{I}$  and each conditional distribution  $f_i(\cdot)$  of the loss, but faces ambiguity on the distribution of the second-order states. The set  $\{p_i\}_{i \in \mathcal{I}}$  is the set of priors the DMs have on the distribution of the second-order states. We assume that the priors are symmetric, in the sense that both DMs have the same information on the distribution of the second-order states, and thus the same conditional loss densities. Both DMs exhibit attitudes towards risk and towards ambiguity. In particular, the attitude towards risk of the policyholder and the insurer is captured

by the convexity of the utility function  $u(\cdot)$  and  $v(\cdot)$ , respectively. Typically, the policyholder is risk-averse, implying that  $u$  is strictly concave, and the insurer is risk-neutral, implying that  $v$  is linear. Without loss of generality (WLOG), we can let  $v$  be the identity function. The policyholder's attitude towards ambiguity, according to [Klibanoff et al. \(2005\)](#), is described by the convexity of the functional  $\phi_A$ . This functional being concave, linear, or convex corresponds to an ambiguity-averse, ambiguity-neutral, or ambiguity-loving policyholder. Typically, the policyholder is ambiguity-averse. Likewise, the insurer's attitude towards ambiguity is captured by the convexity of the functional  $\phi_P$ . Let us now state these assumptions more concretely.

**Assumption 2.1** (Finite second-order state space). *The second-order state space  $\mathcal{I}$  is finite. In particular  $\mathcal{I} = \{1, 2, \dots, n\}$ , for some positive integer  $n$ .*

**Assumption 2.2** (Common priors). *Let  $p_i$  denotes the common prior probability of state  $i \in \mathcal{I}$  for each DM. Assume  $p_i \in (0, 1)$  for all  $i \in \mathcal{I}$  and  $\sum_{i=1}^n p_i = 1$ .*

**Assumption 2.3** (Common bounded support). *The loss  $\tilde{x}$  is a continuous random variable whose state-conditional densities have a bounded common support  $I_x = [0, \bar{x}]$ , where  $\bar{x} > 0$ .*

**Assumption 2.4** (Strictly positive conditional densities). *The state-conditional cumulative density functions (cdf)  $F_i : I_x \rightarrow [0, 1]$ , for each  $i$  in  $\mathcal{I}$ . The cdfs are  $C^2$  on the common domain  $I_x$ . Let  $f_i : I_x \rightarrow (0, \infty)$  stand for the state- $i$  probability density function (pdf) of  $\tilde{x}$  defined by  $f_i(x) = \frac{\partial F_i(x)}{\partial x}$ , for  $x \in I_x$ . Denote  $f = (f_i)_{i \in \mathcal{I}}$  the  $n$ -dimensional vector of conditional densities.*

**Assumption 2.5** (Ordering of conditional distributions). *Assume that there exists an ordering criterion of the ambiguous states. In particular, we establish that state  $i$  is better than state  $j$  whenever  $i < j \in \mathcal{I}$  in the sense that  $F_j$  dominates  $F_i$  in the sense of likelihood ratio dominance (LRD). In particular, let  $\ell_{ij} : I_x \rightarrow \mathbb{R}_+^*$  be the LR defined by  $\ell_{ij}(x) = \frac{f_i(x)}{f_j(x)}$ , for  $x \in I_x$ . Let  $i, j \in \mathcal{I}$ . Then  $i < j$  implies  $\ell'_{ij}(x) \leq 0$  for all  $x \in I_x$ , with strict inequality in some subset of positive measure of  $I_x$ . In other words, the cdf  $F_j$  dominates the cdf  $F_i$  in the sense of LRD whenever  $i < j$ .<sup>1</sup>*

**Assumption 2.6** (Bounded indemnity). *The measurable indemnity function  $I : I_x \rightarrow I_x$  satisfies  $I(x) \in [0, x]$  for all  $x \in I_x$ .*

**Assumption 2.7** (Convex cost). *The cost of indemnity provision  $\psi(\cdot)$  is a  $C^2$  function  $\psi : I_x \rightarrow \mathbb{R}_+$  satisfying  $\psi(0) = 0$ ,  $\psi' > 0$ ,  $\psi'' \geq 0$ , and  $\psi(I) \leq I$  for all  $I \geq 0$ .*

As mentioned earlier, the policyholder is risk-averse, namely that his preference can be modeled with a strictly increasing and concave utility function as follows. By contrast, the insurer is risk-neutral.

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<sup>1</sup>Recall that LRD is a special case of first-order stochastic dominance (FSD). Thus  $F_j$  dominates  $F_i$  in the LRD sense implies  $F_j(x) \leq F_i(x)$  for all  $x \in I_x$ , with strict inequality on some subset of  $I_x$  of positive measure. See, for example, [Wolfstetter \(1999\)](#) for a discussion.

**Assumption 2.8** (Risk aversion of the policyholder). *The utility function of the policyholder  $u : \mathbb{R}_+^* \rightarrow \mathbb{R}$  is at least  $C^2$ , strictly increasing and strictly concave:  $u' > 0$ , and  $u'' < 0$ .*

To ensure that  $u$  is always well-defined, let us assume that the initial wealth  $W_A$  of the policyholder satisfies:

$$W_A \geq \bar{\pi} + \bar{x}, \quad (2.2)$$

where  $\bar{\pi}$  is the upperbound for the premium. Let  $r_u : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$  denote the familiar Arrow-Pratt measure of absolute risk aversion, defined by:

$$r_u(w) = -\frac{u''(w)}{u'(w)}. \quad (2.3)$$

**Assumption 2.9** (Risk neutrality of the insurer). *The insurer has identity utility function, namely that  $v : \mathbb{R}_+^* \rightarrow \mathbb{R}$  is the map  $x \mapsto x$  for all  $x \in \mathbb{R}_+^*$ .*

In light of Assumption 2.9, the participation constraint of the insurer can be rewritten as:

$$\sum_{i=1}^n p_i \phi_P \left( \int_{I_x} (W_P + \pi - I(x) - \psi(I(x)) f_i(x) dx) \right) \geq \bar{V}. \quad (2.4)$$

The phenomenon known as ‘‘ambiguity aversion’’ revived by Ellsberg (1961) is modeled in the smooth sense of Klibanoff et al. (2005) via a strictly monotone concave second-order utility functional.

**Assumption 2.10** (Ambiguity aversion). *Let the second-order utility functional be  $\phi_J : \mathbb{R} \rightarrow \mathbb{R}$ , where  $J \in \{A, P\}$ . Then  $\phi_J$  is at least  $C^2$ , strictly increasing and concave on its domain. Assume that the  $\phi_J$ 's have bounded first-order derivatives, so that  $0 < \phi_J' < +\infty$ ,  $\phi_J'' \leq 0$ , for each  $J \in \{A, P\}$ . Whenever  $\phi_J$  is linear, we assume without loss of generality that  $\phi_J$  is the identity function.*

This assumption means that the DMs are either ambiguity-neutral ( $\phi_J$  is the identity function), or is (strictly) ambiguity-averse ( $\phi_J$  is strictly concave). Finally, let us make the following assumption regarding the initial wealth levels of the DMs.

**Assumption 2.11** (Other parameters). *We assume that  $\bar{V}$  is equal to the reservation second-order utility of the insurer (i.e., the utility obtained without participating in the contract), namely that:*

$$\bar{V} = \phi_P(W_P). \quad (2.5)$$

Furthermore, we assume that the bounds for the premium satisfy:

$$\underline{\pi} = 0, \quad (2.6)$$

$$\bar{\pi} = \int_{I_x} (x + \psi(x)) \bar{f}(x) dx, \quad (2.7)$$

$$(2.8)$$

where  $\bar{f} : I_x \rightarrow \mathbb{R}_+^*$  is the ambiguity-neutral density defined by

$$\bar{f}(x) = \sum_{i=1}^n p_i f_i(x), \quad x \in I_x. \quad (2.9)$$

Equation (2.7) says that the premium cannot exceed the expected total cost of providing uniformly full insurance with respect to the ambiguity-neutral density.

Under the stated assumptions, the OCP faced by the policyholder is the following:

$$\max_{(I(\cdot), \pi)} \sum_{i=1}^n p_i \phi_A \left( \int_{I_x} u(W_A - \pi - x + I(x)) f_i(x) dx \right) \quad (2.10a)$$

$$\text{s.t.} \quad I(x) \in [0, x], \quad \forall x \in I_x, \quad (2.10b)$$

$$\pi \in I_\pi \equiv [\underline{\pi}, \bar{\pi}], \quad (2.10c)$$

$$\sum_{i=1}^n p_i \phi_P \left( W_P + \pi - \int_{I_x} [I(x) + \psi(I(x))] f_i(x) dx \right) \geq \bar{V}. \quad (2.10d)$$

### 2.3 Existence of an optimal insurance policy

To facilitate the proof, we first recast the OCP following Trélat (2008). To this end let the control be the function  $J : I_x \rightarrow [0, 1]$  defined by:

$$xJ(x) = I(x), \quad x \in I_x. \quad (2.11)$$

Since  $I(x) \in [0, x]$  for all  $x \in I_x$  by constraint (2.10b), the admissible control set  $\mathcal{U}$  is:

$$\mathcal{U} = \{J : I_x \rightarrow [0, 1], J \text{ measurable}\}. \quad (2.12)$$

Clearly  $\mathcal{U}$  is compact with respect to the weak- $\star$  topology. Notice that  $J$  as defined is simply the insurance coverage rate. Next, let  $X \equiv (y, z, \pi)$  be the state vector defined on the state space  $\mathcal{X} = \mathbb{R}_+^n \times \mathbb{R}_+^n \times I_\pi$  satisfying:

$$\begin{aligned} \dot{X}(x) &= \begin{pmatrix} \dot{y}(x) \\ \dot{z}(x) \\ \dot{\pi}(x) \end{pmatrix} = \begin{pmatrix} u(W_A - \pi - x + xJ(x))f(x) \\ (W_P + \pi - xJ(x) - \psi(xJ(x)))f(x) \\ 0 \end{pmatrix}, \\ X(0) &= \begin{pmatrix} y(0) = 0 \\ z(0) = 0 \\ \pi(0) = \pi \end{pmatrix}, \end{aligned} \quad (2.13)$$

where  $f(x) = (f_i(x))_{i \in I}$  is the  $n$ -dimensional vector of conditional densities defined in Assumption 2.4.

Let the cost functional faced by the policyholder be

$$g(J, \pi) = - \sum_{i=1}^n p_i \phi_A(y_i(\bar{x})), \quad (2.14)$$

and the insurer's net welfare functional be

$$h(J, \pi) = \sum_{i=1}^n p_i \phi_P(z_i(\bar{x})) - \bar{V}. \quad (2.15)$$

Then the OCP can be rewritten as:

$$\begin{aligned} & \min_{\{J, \pi\}} g(J, \pi) \\ & \text{s.t.} \\ & h(J, \pi) \geq 0. \end{aligned} \quad (\text{OCP})$$

Let  $\mathcal{M}_0$  and  $\mathcal{M}_1$  be measurable subsets of  $\mathcal{X}$  defined as

$$\mathcal{M}_0 = \{0\} \times \{0\} \times I_\pi, \quad (2.16)$$

$$\mathcal{M}_1 = \mathbb{R}^n \times S_{z, \pi}, \quad (2.17)$$

where  $S_{z, \pi} = \{z \in \mathbb{R}^n \times I_\pi \mid h(J, \pi) \geq 0\}$  is the admissible set. The sets  $\mathcal{M}_0$  and  $\mathcal{M}_1$  are often called the source and target sets of the control system (2.13).

**Proposition 2.1.** *The OCP admits an optimal pair  $(J, \pi)$ . In other words, there exists an optimal insurance contract  $(I, \pi)$  such that  $I(x) = xJ(x)$  for  $x \in I_x$ .*

*Proof.* See Subsection 2.7.1. ■

**Lemma 2.1.** *The participation constraint (2.10d) is active (holds with equality) at an optimum. In particular, if  $(J, \pi)$  is an optimal pair, then  $h(J, \pi) = 0$ .*

*Proof.* See Subsection 2.7.2. ■

**Remark 2.1.** *Before proceeding, let us consider two corner cases of the problem. In particular, let us consider the case of uniformly zero insurance,  $J(x) = 0$  or  $I(x) = 0$  on  $I_x$ , and the case of uniformly full insurance  $J(x) = 1$  or  $I(x) = x$  on  $I_x$ .*

*First, consider the case of uniformly zero insurance  $J(x) = 0$  for all  $x \in I_x$ . In this case,*

$$h(0, \pi) = \phi_P(W_P + \pi) - \bar{V} = \phi_P(W_P + \pi) - \phi_P(W_P), \quad (2.18)$$

*where the second equality comes from (2.5). Furthermore by (2.6) and (2.5),*

$$h(0, \underline{\pi}) = 0. \quad (2.19)$$

Observe that  $h$  is strictly monotone in  $\pi$  and  $\pi \geq \underline{\pi}$  for all  $\pi \in I_\pi$ . Hence the pair  $(0, \pi)$  is admissible for all  $\pi \in I_\pi$ . Nevertheless by Lemma 2.1, only the pair  $(0, \underline{\pi})$  is a candidate for an optimum.

Next, consider the case of uniformly full insurance, namely the case  $J(x) = 1$  for all  $x \in I_x$ . We have:

$$h(1, \pi) = \sum_{i=1}^n p_i \phi_P \left( W_P + \pi - \int_{I_x} (x + \psi(x)) f_i(x) dx \right). \quad (2.20)$$

Note that since  $h$  is strictly increasing in  $\pi$ ,

$$h(1, \pi) \leq h(1, \bar{\pi}), \quad \forall \pi \in I_\pi. \quad (2.21)$$

Let us consider two subcases.

- If  $\phi_P$  is strictly concave (the insurer is ambiguity-averse), then by Jensen inequality<sup>2</sup> and condition (2.7),

$$h(1, \bar{\pi}) < \phi_P(W_P) - \bar{V} = 0. \quad (2.22)$$

Then (2.21) and (2.22) imply

$$h(1, \pi) < 0, \quad \forall \pi \in I_\pi, \quad (2.23)$$

implying that there is no admissible pair.

- If  $\phi_P$  is identity (the insurer is ambiguity-neutral), then also by condition (2.7) we have

$$h(1, \bar{\pi}) = 0, \quad (2.24)$$

implying that  $h(1, \pi) \leq 0$  for all  $\pi \in I_\pi$ . Hence the only admissible pair is  $(J = 1, \pi = \bar{\pi})$ .

To sum up, a contract involving uniformly zero insurance is always admissible, but can be an optimum if and only if the associated premium is zero. By contrast a contract involving uniformly full insurance is admissible if and only if the insurer is ambiguity-neutral ( $\phi_P$  being the identity function).

## 2.4 Characterization of the optimal insurance contract

### 2.4.1 General shape of an optimal contract

In this subsection, our goal is to derive the general properties of an optimal insurance contract. We employ the Pontryagin's Maximum Principle (PMP) to derive the necessary conditions that must be verified by the solutions. With slight modification from

<sup>2</sup>See, for example, Chapter 6 of Pishro-Nik (2016).

Trélat (2008), the statement of the PMP applied to the OCP is provided in the following theorem.

**Theorem 2.1** (Pontryagin Maximum Principle). *Suppose  $(X, J)$  is an optimal pair for the OCP. There exists an absolutely continuous vector-valued function  $\lambda : I_x \rightarrow \mathbb{R}^{2n+1}$  and a real number  $\lambda_0 \geq 0$  with  $(\lambda, \lambda_0) \neq 0 \in \mathbb{R}^{2n+2}$  such that:*

1.  $\lambda$  satisfies the canonical equations:

$$\dot{X}(x) = \nabla_{\lambda} H(X(x), J(x), \lambda(x), \lambda_0, x), \quad (2.25)$$

$$\dot{\lambda}(x) = -\nabla_X H(X(x), J(x), \lambda(x), \lambda_0, x), \quad (2.26)$$

for almost every  $x \in I_x$ , where the real-valued function  $H : \mathbb{R}^{2n+1} \times \mathbb{R} \times \mathbb{R}^{2n+1} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , called the Hamiltonian, is defined by:

$$\begin{aligned} H(X, \omega, \lambda, \lambda_0, x) &= u(W_A - \pi - x + x\omega) \langle \lambda_y, f(x) \rangle \\ &\quad + (W_P + \pi - x\omega - \psi(x\omega)) \langle \lambda_z, f(x) \rangle, \end{aligned} \quad (2.27)$$

where  $\lambda \equiv (\lambda_z, \lambda_y, \lambda_{\pi})^T \in \mathbb{R}^{2n+1}$  is the adjoint vector whose components  $\lambda_z \in \mathbb{R}^n$ ,  $\lambda_y \in \mathbb{R}^n$  and  $\lambda_{\pi} \in \mathbb{R}$  themselves are the adjoint vectors corresponding to the state variables  $z, y$  and  $\pi$ , respectively.

2. The maximum condition:

$$H(X(x), J(x), \lambda(x), \lambda_0, x) = \max_{\omega \in [0,1]} H(X(x), \omega, \lambda(x), \lambda_0, x) \quad (2.28)$$

is satisfied for almost every  $x \in I_x$ .

3. The transversality conditions (TCs) hold:

$$\lambda(0) \in N_{\mathcal{M}_0}(X(0)), \quad (2.29)$$

$$-\lambda_0 \nabla_X g(J, \pi) - \lambda(\bar{x}) \in N_{\mathcal{M}_1}(X(\bar{x})), \quad (2.30)$$

where  $N_{\mathcal{M}_i}(X(x))$  denotes the normal cone to  $\mathcal{M}_i$  at  $X(x)$ , for  $i \in \{0, 1\}$ .

Let  $\mu = (\mu_h, \mu_{\pi}, \mu_{\bar{\pi}}) \in \mathbb{R}_+^3$  be the vector of Lagrange multipliers, where  $\mu_h$  is associated to the constraint  $h(J, \pi) \geq 0$ , and  $(\mu_{\bar{\pi}}, \mu_{\bar{\pi}})$  is associated to the constraint  $\pi \in I_{\pi}$ .

**Proposition 2.2.** *The adjoint vectors  $\lambda_y$  and  $\lambda_z$  are constant with respect to  $x$ . In particular,*

$$\lambda_y = \lambda_0 (p_i \phi'_A(y_i(\bar{x})))_{i \in \mathcal{I}}, \quad (2.31)$$

$$\lambda_z = \mu_h (p_i \phi'_P(z_i(\bar{x})))_{i \in \mathcal{I}}. \quad (2.32)$$



The adjoint vector  $\lambda_\pi$  satisfies

$$\lambda_\pi(\bar{x}) = \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle + \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle + \mu_{\underline{\pi}} - \mu_{\bar{\pi}}, \quad (2.33)$$

$$\lambda_\pi(0) = 2 \left( \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle + \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle \right) + \mu_{\underline{\pi}} - \mu_{\bar{\pi}}. \quad (2.34)$$

Moreover if  $\pi \in (\underline{\pi}, \bar{\pi})$ , then  $(\mu_{\underline{\pi}}, \mu_{\bar{\pi}}) = 0$  and

$$\lambda_\pi(0) = \lambda_\pi(\bar{x}) = 0. \quad (2.35)$$

*Proof.* See Subsection 2.7.3. ■

**Lemma 2.2.** *The non-triviality condition  $(\lambda_0, \mu_h) \neq 0$  holds.*

*Proof.* See Subsection 2.7.4. ■

**Remark 2.2.** *If either  $\lambda_0$  or  $\mu_h$  is equal to 0, then we normalize the other to one. If both are strictly positive, we defer the normalization until the numerical simulation.*

Let us now consider the maximum condition. Denote  $H_\omega \equiv \frac{\partial H}{\partial \omega}$ , and  $H_{\omega\omega} \equiv \frac{\partial^2 H}{\partial \omega^2}$ . For every fixed  $x \in I_x$ , we have

$$H_\omega = x \left[ u'(W_A - \pi - x + x\omega) \langle \lambda_y, f(x) \rangle - (1 + \psi'(x\omega)) \langle \lambda_z, f(x) \rangle \right], \quad (2.36)$$

$$H_{\omega\omega} = x^2 \left[ u''(W_A - \pi - x + x\omega) \langle \lambda_y, f(x) \rangle - \psi''(x\omega) \langle \lambda_z, f(x) \rangle \right]. \quad (2.37)$$

Furthermore, denote

$$K(x) = u'(W_A - \pi - x)G(x) - (1 + \psi'(0)), \quad (2.38)$$

and

$$L(x) = u'(W_A - \pi)G(x) - (1 + \psi'(x)). \quad (2.39)$$

Notice that the signs of  $K(x)$  and  $L(x)$  are a.e. identical to the signs of  $H_\omega(x)|_{\omega=0}$  and  $H_\omega(x)|_{\omega=1}$ , respectively.

**Lemma 2.3.** *At an optimum,*

- $\lambda_0 = 0$  if and only if the optimal contract is the trivial pair ( $J = 0, \pi = \underline{\pi}$ );
- $\mu_h = 0$  if and only if  $\phi_P$  is linear, in which case the optimal contract is the pair ( $J = 1, \pi = \bar{\pi}$ ).

*Proof.* See Subsection 2.7.5. ■

**Remark 2.3.** *In the remaining analyses of the paper we assume, whenever not explicitly stated, that both  $\lambda_0$  and  $\mu_h$  are strictly positive, bearing in mind that we need to eventually compare the cost yielded by the contract(s) found under this assumption with that in the two special cases addressed in Lemma 2.3 to find the optimal one(s).*

Since all the prior probabilities and the densities are strictly positive (Assumption 2.2 and Assumption 2.4), we can define:

$$A(x) = \frac{\langle \lambda_y, f(x) \rangle}{\langle p, f(x) \rangle}, \quad x \in I_x, \quad (2.40)$$

$$P(x) = \frac{\langle \lambda_z, f(x) \rangle}{\langle p, f(x) \rangle}, \quad x \in I_x \quad (2.41)$$

Observe that both  $A$  and  $P$  are strictly positive since  $\lambda_0 > 0$  and  $\mu_h > 0$ . Hence we can define their ratio  $G : I_x \rightarrow \mathbb{R}_+^*$  by

$$G(x) = \frac{A(x)}{P(x)} = \frac{\langle \lambda_y, f(x) \rangle}{\langle \lambda_z, f(x) \rangle}. \quad (2.42)$$

Clearly  $G$  is strictly positive. Notice that  $A(x)$  can be expressed as

$$A(x) = \lambda_0 \frac{\sum_{i=1}^n p_i \phi'_A(y_i(\bar{x})) f_i(x)}{\sum_{i=1}^n p_i f_i(x)} = \lambda_0 \sum_{i=1}^n p_i(x) \phi'_A(y_i(\bar{x})), \quad (2.43)$$

where  $p_i(x) = \frac{p_i f_i(x)}{\sum_{i=1}^n p_i f_i(x)}$  is the Bayesian posterior probability on the occurrence of the second-order state  $i$  given that the loss is  $x$ . This *inference* is a direct consequence of the uncertainty on the distribution and that the second-order state itself is not a contractible variable. Hence  $A(x)$  so defined can be interpreted as the expected marginal second-order utility, or expected marginal welfare (EMW) of the policyholder with respect to the posterior distribution, up to a positive constant. Analogously  $P(x)$  can be interpreted as the EMW of the insurer with respect to the posterior distribution up to a positive constant. Finally,  $G(x)$ , the expected marginal welfare ratio (EMWR), can be intuitively interpreted as the relative strength of the policyholder's ambiguity aversion with respect to that of the insurer (see equation (2.59) below).

**Lemma 2.4.** *For each  $\pi \in I_\pi$ , the function  $\Sigma_x : [0, x] \rightarrow \mathbb{R}_+^*$  defined by:*

$$\Sigma_x(\xi) = \frac{1 + \psi'(\xi)}{u'(W_A - \pi - x + \xi)} \quad (2.44)$$

*is strictly increasing in  $\xi$  for  $x \in I_x$ .*

*Proof.* See Subsection 2.7.6. ■

Hence for each fixed  $\pi$ , the function  $\Sigma_x$  is strictly increasing for all  $\xi \in [0, x]$ , implying that  $\Sigma_x$  has a well-defined inverse  $\Sigma_x^{-1} : \mathbb{R}_+^* \rightarrow [0, x]$ , which is also strictly increasing. Denote

$$\Sigma_x^{-1} = \Xi_x. \quad (2.45)$$

**Proposition 2.3.** *An optimal coverage rate function  $J$  is such that*

$$xJ(x) \in \{0, 1, \Xi_x(G(x))\}, \quad x \in I_x. \quad (2.46)$$

where  $\Xi_x : \mathbb{R}_+^* \rightarrow [0, x]$  is defined in (2.45), and  $G$  is the ratio of EMWs given in (2.42). Equivalently, the corresponding indemnity function  $I(x) = xJ(x)$  satisfies

$$I(x) \in \{0, x, \Xi_x(G(x))\}, \quad \forall x \in I_x. \quad (2.47)$$

Moreover, for  $x \in (0, \bar{x}]$  such that  $J(x)$  takes value in  $(0, 1)$ , the indemnity function takes values in  $(0, x)$ , is differentiable at  $x$  and satisfies the differential equation

$$I'(x) = \frac{r_u(W_A(x)) + \frac{G'(x)}{G(x)}}{r_u(W_A(x)) + \frac{\psi''(I(x))}{1+\psi'(I(x))}}, \quad (2.48)$$

where  $r_u(\cdot) > 0$  is the policyholder's Arrow-Pratt degree of absolute risk aversion defined in (2.3) and  $W_A(x)$  is the final wealth of the policyholder

$$W_A(x) = W_A - \pi - x + I(x). \quad (2.49)$$

*Proof.* See Subsection 2.7.7. ■

Observe that both  $A$  and  $P$  defined in (2.40) and (2.41), respectively, are strictly positive-valued, bounded and continuously differentiable on  $(0, \bar{x})$ . Differentiating with respect to  $x$  yields:

$$A'(x) = \lambda_0 \frac{\sum_{1 \leq i < j \leq n} p_i p_j (\phi'_A(y_i(\bar{x})) - \phi'_A(y_j(\bar{x}))) f_j^2(x) \ell'_{ij}(x)}{\langle p, f(x) \rangle^2}, \quad (2.50)$$

and

$$P'(x) = \mu_h \frac{\sum_{1 \leq i < j \leq n} p_i p_j (\phi'_P(z_i(\bar{x})) - \phi'_P(z_j(\bar{x}))) f_j^2(x) \ell'_{ij}(x)}{\langle p, f(x) \rangle^2}. \quad (2.51)$$

Thus the monotonic behavior of  $A$  and  $P$  depends on the ordering of the second-order states. In particular, a sufficient condition for  $A$  to be increasing is that  $y_i(\bar{x}) \geq y_j(\bar{x})$  for all  $i < j$  since  $\phi_A$  is concave and  $\ell'_{ij} \leq 0$ . Likewise a sufficient condition for  $P$  to be

increasing is  $z_i(\bar{x}) \geq z_j(\bar{x})$  for all  $i < j$ . Observe that by integration by parts (IBP):

$$z_i(\bar{x}) = W_P(\bar{x}) + \int_{I_x} I'(x) [1 + \psi'(I(x))] F_i(x) dx, \quad (2.52)$$

$$y_i(\bar{x}) = u(W_A(\bar{x})) + \int_{I_x} [1 - I'(x)] u'(W_A(x)) F_i(x) dx, \quad (2.53)$$

where

$$W_A(x) = W_A - \pi - x + I(x), \quad (2.54)$$

$$W_P(x) = W_P + \pi - I(x) - \psi(I(x)). \quad (2.55)$$

Hence

$$z_i(\bar{x}) - z_j(\bar{x}) = \int_{I_x} I'(x) [1 + \psi'(I(x))] (F_i(x) - F_j(x)) dx, \quad (2.56)$$

$$y_i(\bar{x}) - y_j(\bar{x}) = \int_{I_x} [1 - I'(x)] (F_i(x) - F_j(x)) dx. \quad (2.57)$$

Since  $F_i(x) \geq F_j(x)$  on  $I_x$  with strict inequality at least on a subset of positive-measured of  $I_x$  by Assumption 2.5, the ordering of the states depend crucially on the magnitude of  $I'$  relative to one. Observe that from Proposition 2.3,

$$I'(x) \in \left\{ 0, 1, \frac{r_u(W_A(x) + \frac{G'(x)}{G(x)})}{r_u(W_A(x) + \frac{\psi''(I(x))}{1+\psi'(I(x))})} \right\}. \quad (2.58)$$

Hence if  $G' \leq 0$  for values of  $x$  such that  $I(x) \in (0, x)$  then  $I'(x) \leq 1$  on  $I_x$ . In this case we conclude from (2.57) that  $y_i(\bar{x}) \geq y_j(\bar{x})$ . On the contrary if  $G' > 0$  for values of  $x$  such that  $I(x) \in (0, x)$ , then  $I' \geq 0$  on  $I_x$ , which via (2.56) that  $z_i(\bar{x}) \geq z_j(\bar{x})$ . Hence the ordering of the states ultimately depend on the monotonic behavior of  $G$ , the ratio of EMWs.

Note that since  $G(x) = \frac{A(x)}{P(x)}$ , differentiating with respect to  $x$  and re-arranging yield:

$$\frac{G'(x)}{G(x)} = \frac{A'(x)}{A(x)} - \frac{P'(x)}{P(x)}. \quad (2.59)$$

Observe that

$$G'(x) = \frac{\sum_{1 \leq i < j \leq n} (\lambda_y^i \lambda_z^j - \lambda_y^j \lambda_z^i) f_j^2(x) \ell'_{ij}(x)}{\langle \lambda_z, f(x) \rangle^2}. \quad (2.60)$$

Hence the EMWR varies with respect to  $x$  in general. Nevertheless, we can show that in the case of two ambiguous states ( $n = 2$ ), the monotonic behavior of this important term is independent of the value of the loss. We defer the treatment of this case to

Subsection 2.4.3.

### 2.4.2 Ambiguity-neutral DMs and Raviv (1979) revisited

Let us study the case of ambiguity-neutral DMs ( $\phi_A$  and  $\phi_P$  are identity), which turns out to be a convenient setting to re-examine the main results obtained in Raviv (1979), the seminal work that motivates the approach adopted in this paper. Notice that under ambiguity neutrality, the OCP of interest is:

$$\max_{(I(\cdot), \pi)} \int_{I_x} u(W_A - \pi - x + I(x)) \bar{f}(x) dx \quad (2.61a)$$

$$s.t. \quad I(x) \in [0, x], \quad \forall x \in I_x, \quad (2.61b)$$

$$\pi \in I_\pi \equiv [\underline{\pi}, \bar{\pi}], \quad (2.61c)$$

$$\pi \geq \int_{I_x} (I(x) + \psi(I(x))) \bar{f}(x) dx, \quad (2.61d)$$

where  $\bar{f} \equiv \sum_{i=1}^n p_i f_i$  is the ambiguity-neutral density defined in (2.9). On the other hand, it is immediate to see that the unambiguous case of Raviv (1979) corresponds to the case where the second-order state space  $\mathcal{I}$  is a singleton:  $n = 1$ . Suppose WLOG that  $p_1 = 1$ . Recall that  $\phi_P$  is strictly increasing and  $\bar{V} = \phi_P(W_p)$ , the OCP in this case reduces to:

$$\max_{(I(\cdot), \pi)} \phi_A \left( \int_{I_x} u(W_A - \pi - x + I(x)) f_1(x) dx \right) \quad (2.62a)$$

$$s.t. \quad I(x) \in [0, x], \quad \forall x \in I_x, \quad (2.62b)$$

$$\pi \in I_\pi \equiv [\underline{\pi}, \bar{\pi}], \quad (2.62c)$$

$$\pi \geq \int_{I_x} (I(x) + \psi(I(x))) f_1(x) dx. \quad (2.62d)$$

Since  $\phi_A$  is strictly increasing, maximizing (2.61a) subject to (2.61b)(2.61c)(2.61d) is equivalent to maximizing (2.62a) subject to (2.62b)(2.62c)(2.62d). Hence the ambiguity-neutral case and the unambiguous case are philosophically distinct but technically equivalent problems. In fact the only principal difference between these two OCPs is the density function of the loss. In the ambiguity-neutral case, the ambiguity-neutral density is the relevant one. In the unambiguous case, the pertinent one is the unambiguous density associated with the second-order state that occurs with certainty (the objectively known loss density). Observe that modifying the density does not alter the *shape* of the optimal contract. Therefore while the remain of this section addresses the unambiguous case to facilitate the comparison with Raviv (1979), all results hold for the ambiguity-neutral case.

Recall that under no ambiguity, Raviv shows that two types of policy might prevail

at the optimum.<sup>3</sup> In particular, there could be either a deductible policy of the form

$$\begin{cases} I(x) = 0 & x \leq x_1, \\ I(x) \in (0, x) & x > x_1. \end{cases} \quad (2.63)$$

or an upper-limit policy of the form

$$\begin{cases} I(x) \in (0, x) & x \leq x_2, \\ I(x) = x & x > x_2. \end{cases} \quad (2.64)$$

Notice that under no ambiguity ( $n = 1$ ), Proposition 2 trivially yields

$$\lambda_y = \lambda_0, \quad (2.65)$$

$$\lambda_z = \mu_h. \quad (2.66)$$

If  $\lambda_0 = 0$  or  $\mu_h = 0$ , then either the optimal contract is one of the corner cases in Lemma 5. Otherwise  $\lambda_0 > 0$  and  $\mu_h > 0$  and  $G$  defined in (2.42) simplifies to:

$$G(x) = \frac{\lambda_0 f_1(x)}{\mu_h \bar{f}_1(x)} = \frac{\lambda_0}{\mu_h} \equiv \tilde{\lambda}_0, \quad (2.67)$$

which is a strictly positive constant.<sup>4</sup> Hence the deductible  $x_1$  is defined as the unique zero of  $H_\omega(x)|_{\omega=0}$  while the upper limit  $x_2$  is the unique zero of  $H_\omega(x)|_{\omega=1}$  when  $G$  is a constant. Moreover, the co-insurance equation (2.48) that the indemnity function must satisfy whenever  $I(x) \in (0, x)$  reduces to

$$I'(x) = \frac{r_u(W_A(x))}{r_u(W_A(x)) + \frac{\psi''(I(x))}{1+\psi'(I(x))}}, \quad (2.70)$$

which depends on  $x$  if and only if the cost of indemnity provision  $\psi$  is strictly convex, as shown by Raviv (1979).<sup>5</sup> Recall that in Raviv's analysis, whether a deductible or an upper limit type of policy is optimal depends crucially on the premium, which is assumed fixed.

<sup>3</sup>See Theorem 1 of Raviv (1979) on page 87.

<sup>4</sup>Under ambiguity neutrality,

$$\lambda_y = \lambda_0(p_i)_{i \in \mathcal{I}}, \quad (2.68)$$

$$\lambda_z = \mu_h(p_i)_{i \in \mathcal{I}}. \quad (2.69)$$

If  $\lambda_0 = 0$  or  $\mu_h = 0$ , we are back to the special contracts discussed in Lemma 2.3. If  $\mu_h$  and  $\lambda_0$  are strictly positive, then it is easy to see that  $G(x) = \frac{\lambda_0 \bar{f}(x)}{\mu_h \bar{f}(x)} = \frac{\lambda_0}{\mu_h} = \tilde{\lambda}_0$ , as in the unambiguous case.

<sup>5</sup>Note that here the Arrow-Pratt degree of risk aversion of the insurer is zero.

Observe that in this case  $K(x)$  in (2.38) simplifies to:

$$K(x) = u'(W_A - \pi - x)\tilde{\lambda}_0 - (1 + \psi'(0)). \quad (2.71)$$

We have the following proposition.

**Proposition 2.4.** *Consider the case of ambiguity-neutral DMs, or the unambiguous case  $n = 1$ . There exists a unique  $x_1 \in (0, \bar{x})$ , called the deductible, defined as the zero of  $K(x)$  in (2.71). If  $\lambda_0 = 0$  or  $\mu_h = 0$ , then the contracts are of the types discussed in Lemma 2.3. If  $\lambda_0 > 0$  and  $\mu_h > 0$ , then the optimal contract consists of the pair  $(I, \pi)$  such that the indemnity function  $I$  satisfies:*

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ \Xi_x(\tilde{\lambda}_0) \in (0, x) & x \in (x_1, \bar{x}]. \end{cases} \quad (2.72)$$

The optimal premium  $\pi \in (\underline{\pi}, \bar{\pi})$  satisfies

$$\pi = \int_{x_1}^{\bar{x}} (I(x) + \psi'(I(x)))\bar{f}(x)dx, \quad (2.73)$$

where  $\bar{f}$  is the ambiguity-neutral density defined in (2.9). Moreover, for losses beyond the deductible  $x_1$ , the co-insurance level is determined by:

$$I'(x) = \frac{r_u(W_A - \pi - x + I(x))}{r_u(W_A - \pi - x + I(x)) + \frac{\psi''(I(x))}{1 + \psi'(I(x))}}, \quad (2.74)$$

where  $r_u(\cdot) = \frac{-u''(\cdot)}{u'(\cdot)}$  is the Arrow-Pratt absolute risk aversion.

*Proof.* See Subsection 2.7.8. ■

**Remark 2.4.** *Hence following the holistic approach of solving for the pair  $(I, \pi)$  simultaneously allows us to show directly that policy that involves an upper limit coverage, i.e., one that consists of an indemnity function of the type*

$$I(x) = \begin{cases} x & x \leq x_2, \\ x \in (0, x) & x > x_2, \end{cases} \quad (2.75)$$

for some  $x_2 > 0$  cannot be optimal. In other words, Proposition 2.4 completes the analysis of Raviv (1979). Recall that Raviv (1979) derives Theorem 1 for a fixed premium. This means to verify which type of indemnity function is optimal, one must first supply a premium. This is problematic since the premium itself depends on the shape of the indemnity function. Consequently, the first theorem of Raviv (1979) is of limited use when it comes to selecting the optimal indemnity schedule. Our proposition addresses exactly this issue. By exploiting all information

available from the maximum principle, we have shown that the upper limit type of policy cannot constitute an optimum in the case of a risk-neutral insurer. This is done without resorting to any additional technique outside of the optimal control framework.

Observe that if the cost  $\psi$  of indemnity provision is linear (constant loading), then the differential equation characterizing co-insurance implies that  $I'(x) = 1$  for all losses beyond the deductible. In other words, the contract is a straight deductible, as is obtained in Proposition 1 of Gollier (2014).<sup>6</sup>

**Corollary 2.1.** *Suppose that the cost of indemnity provision is linear, in particular,*

$$\psi(I) = mI, \quad m > 0. \quad (2.76)$$

Then the optimal insurance contract is a straight deductible, namely that the pair  $(I, \pi)$  satisfies

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ x - x_1 & x \in (x_1, \bar{x}]. \end{cases} \quad (2.77)$$

and

$$\pi = (1 + m) \int_{x_1}^{\bar{x}} (x - x_1) \bar{f}(x) dx. \quad (2.78)$$

*Proof.* See Subsection 2.7.9 ■

**Proposition 2.5.** *Consider the case of ambiguity-neutral DMs, or the unambiguous case  $n = 1$  with constant loading factor  $\psi'(I) = L \geq 0$ . Full insurance is optimal if and only if  $L = 0$ .*

*Proof.* See Subsection 2.7.10. ■

### 2.4.3 Two ambiguous states

Recall from our discussion in Section 4.1 that in general  $G$  varies in  $x$ . Nevertheless, we can show that the monotonicity of  $G$  is independent of  $x$  in the case of two ambiguous states ( $n = 2$ ), which we now consider. Notice that for  $n = 2$ , then (2.50) and (2.51) reduce to:

$$A'(x) = \lambda_0 \frac{p_1 p_2 (\phi'_A(y_1(\bar{x})) - \phi'_A(y_2(\bar{x}))) f_2^2(x) \ell'_{12}(x)}{\bar{f}^2(x)}, \quad (2.79)$$

$$P'(x) = \mu_h \frac{p_1 p_2 (\phi'_P(z_1(\bar{x})) - \phi'_P(z_2(\bar{x}))) f_2^2(x) \ell'_{12}(x)}{\bar{f}^2(x)}. \quad (2.80)$$

---

<sup>6</sup>Hence Proposition 1 of Gollier (2014) is a special case of Proposition 2.4 where  $\psi$  is linear as shown below.



Define

$$\begin{aligned} S(\lambda_y, \lambda_z) &= \lambda_y^1 \lambda_z^2 - \lambda_y^2 \lambda_z^1 \\ &= \mu_h \lambda_0 [\phi'_A(y_1(\bar{x}))\phi'_P(z_2(\bar{x})) - \phi'_A(y_2(\bar{x}))\phi'_P(z_1(\bar{x}))]. \end{aligned} \quad (2.81)$$

Then,

$$G'(x) = \frac{p_1 p_2 S(\lambda_y, \lambda_z) f_2^2(x) \ell'_{12}(x)}{\langle \lambda_z, f(x) \rangle^2}. \quad (2.82)$$

**Lemma 2.5.** *In the case of two ambiguous states  $n = 2$ , the monotonic behavior of  $G$ , the ratio of EMWs, is independent of the loss  $x$ , for all  $x \in I_x$ . Furthermore, the EMWs of both DMs are increasing ( $A' \geq 0$  and  $P' \geq 0$ ) on  $I_x$ .*

*Proof.* See Subsection 2.7.11. ■

Recall that the case  $\pi \in \{\underline{\pi}, \bar{\pi}\}$  is discussed in Lemma 2.3. If  $\pi \in (\underline{\pi}, \bar{\pi})$ , then (2.35) holds. Let us rewrite this condition as:

$$\int_{I_x} u'(W_A(x)) \langle \lambda_y, f(x) \rangle dx = \langle \lambda_z, \mathbf{1} \rangle, \quad (2.83)$$

where  $W_A(x)$  is given by (2.49) and  $\mathbf{1}$  denotes the  $n$ -dimensional vector with all elements being equal to one. Since the RHS is strictly positive, we can rewrite (2.83) as:

$$\int_{I_x} u'(W_A(x)) G(x) \tilde{f}(x) dx = 1, \quad (2.84)$$

where

$$\tilde{f}(x) = \frac{\langle \lambda_z, f(x) \rangle}{\langle \lambda_z, \mathbf{1} \rangle} = \frac{\sum_{i=1}^n \lambda_z^i f_i(x)}{\sum_{i=1}^n \lambda_z^i}. \quad (2.85)$$

Observe that  $\tilde{f}$  in (2.85) is strictly positive on  $I_x$  and  $\int \tilde{f}(x) dx = 1$ . Hence it is a density function. In particular, it is a density function biased by the ambiguity aversion of the insurer. If the insurer is ambiguity-neutral, then  $\tilde{f}$  is just the ambiguity-neutral density defined in (2.9).

**Theorem 2.2.** *If  $G' \geq 0$ , then  $K$  defined in (2.38) is strictly increasing. An optimum is either one of the corner cases discussed in Lemma 2.3, or consists of the pair  $(I, \pi)$  such that the indemnity function is of the form:*

$$\begin{cases} I(x) = 0 & x \in [0, x_1], \\ I(x) \in (0, x] & x \in (x_1, \bar{x}], \end{cases} \quad (2.86)$$

where  $x_1 \in (0, \bar{x})$ , the deductible, is the unique solution to  $K(x) = 0$ . The associated premium

$\pi \in (\underline{\pi}, \bar{\pi})$  satisfies:

$$\sum_{i=1}^n p_i \phi_P \left( W_P + \pi - \int_{I_x} (I(x) + \psi(I(x))) f_i(x) dx \right) = \phi_P(W_P). \quad (2.87)$$

Moreover if  $\psi$  is linear, then in consideration of  $L$  defined in (2.39), one of the following cases could occur.

- If  $L(\bar{x}) \geq 0$ , then there exists a unique  $x_2 \in (x_1, \bar{x}]$ , the smallest solution to  $L(x) = 0$ , such that an optimal indemnity function satisfies:

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ \mathbb{E}_x(G(x)) & x \in (x_1, x_2), \\ x & x \in [x_2, \bar{x}]. \end{cases} \quad (2.88)$$

- If  $L(\bar{x}) < 0$ , then an optimal indemnity function has the form:

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ \mathbb{E}_x(G(x)) & x \in (x_1, \bar{x}]. \end{cases} \quad (2.89)$$

*Proof.* See Subsection 2.7.12 ■

#### 2.4.4 One-sided ambiguity aversion under two ambiguous states

Let us now consider a case of practical interest: the case of an ambiguity-averse policyholder and an ambiguity-neutral insurer. Again we assume that both  $\lambda_0$  and  $\mu_h > 0$ .

**Proposition 2.6.** *In the case of two ambiguous states ( $n = 2$ ) with ambiguity-averse policyholder and ambiguity-neutral insurer, there exists a unique  $x_1 \in (0, \bar{x})$ , called the deductible, such that  $K(x_1) = 0$ . An optimum is either one of the corner cases discussed in Lemma 2.3, or consists of the pair  $(I, \pi)$  such that the indemnity function is of the form:*

$$\begin{cases} I(x) = 0 & x \in [0, x_1], \\ I(x) \in (0, x] & x \in (x_1, \bar{x}]. \end{cases} \quad (2.90)$$

The associated premium  $\pi \in (\underline{\pi}, \bar{\pi})$  satisfies:

$$\pi = \int_{x_1}^{\bar{x}} (I(x) + \psi(I(x))) \bar{f}(x) dx. \quad (2.91)$$

Moreover if  $\psi$  is linear, then one of the following cases can occur.

- If  $L(\bar{x}) \geq 0$ , then there exists a unique  $x_2 \in (x_1, \bar{x}]$ , the smallest solution to  $L(x) = 0$ , such that an optimal indemnity function is a disappearing deductible:

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ \mathbb{E}_x(A(x)/\mu_h) & x \in (x_1, x_2), \\ x & x \in [x_2, \bar{x}]. \end{cases} \quad (2.92)$$

- If  $L(\bar{x}) < 0$ , then an optimal indemnity function entails co-insurance beyond a deductible:

$$I(x) = \begin{cases} 0 & x \in [0, x_1], \\ \mathbb{E}_x(A(x)/\mu_h) & x \in (x_1, \bar{x}]. \end{cases} \quad (2.93)$$

Moreover, whenever  $I(x) \in (0, x)$ , the co-insurance rate is given by

$$I'(x) = \frac{r_u(W_A(x)) + \frac{A'(x)}{A(x)}}{r_u(W_A(x))}. \quad (2.94)$$

*Proof.* See Subsection 2.7.13. ■

**Remark 2.5.** Note that the existence of  $x_2$  depends on the magnitude of  $L(\bar{x})$ , which depends on the optimal premium and the values of the co-states  $\lambda_y$  and  $\lambda_z$ . This poses challenges to ex-ante checking whether  $L(\bar{x})$  is negative, and calls for a numerical analysis of the problem, which is deferred to Section 2.5.

**Remark 2.6.** If the contract takes the form (2.93), it is noteworthy that a straight deductible is in general not optimal (unless under a special ambiguity structure). Indeed, a straight deductible contract implies that  $I'(x) = 0$  on  $[0, x_1]$  and  $I'(x) = 1$  on  $(x_1, \bar{x}]$ . Hence (2.168) simplifies to:

$$y_1(\bar{x}) - y_2(\bar{x}) = \int_0^{x_1} u'(W_A(x))(F_1(x) - F_2(x))dx. \quad (2.95)$$

Observe from (2.79) that  $A'(x) = 0$  if and only if either  $y_1(\bar{x}) = y_2(\bar{x})$  or  $\ell'_{12}(x) = 0$ . Suppose there exists a subset of positive measure of  $[0, x_1]$  such that  $F_1(x) > F_2(x)$ , then (2.95) implies  $y_1(\bar{x}) > y_2(\bar{x})$  by the strict monotonicity of the utility function. In this case  $A'(x) = 0$  on  $(x_1, \bar{x}]$  if and only if  $\ell'_{12}(x) = 0$  on this interval. In other words, if the densities satisfy  $\frac{f_1(x)}{f_2(x)} = \text{const}$  on  $(x_1, \bar{x}]$ . By contrast, if there exists a subset of positive measure of  $(x_1, \bar{x}]$ , say  $J_x$ , such that  $\ell'_{12}(x) < 0$  on  $J_x$ , then  $A'(x) > 0$  on  $J_x$  in view of (2.79), implying that  $I'(x) > 1$  on  $J_x$ , contradicting the straight deductible hypothesis. More generally, a straight deductible contract may be optimal if and only if ambiguity has a one-sided structure in the sense that either the set of positive measure  $J_x$  in which  $\ell'_{12}(x) < 0$  (which implies  $F_1(x) > F_2(x)$ ) is a subset of either  $[0, x_1]$  or  $(x_1, \bar{x}]$ , but not both. A violation of this condition is provided in the numerical

section, where ambiguity "contaminates" the whole range of the losses, i.e.,  $\ell'_{12} < 0$  on the whole support.

## 2.5 Numerical simulation

This section serves as a response to Remark 2.5 in the previous section. In addition, we also illustrate the impact of increasing ambiguity aversion on the contract. Recall from Remark 2.2 that when both  $\mu_h$  and  $\lambda_0$  are strictly positive, we still have a degree of freedom to normalize one of them. To facilitate the numerical analysis, we follow the convention in the literature and set:

$$\lambda_0 = 1. \quad (2.96)$$

The numerical exercise reduces to solving a nonlinear system of the following equations:

$$\begin{aligned} K(x_1) &= 0 \\ \iff u'(W_A - \pi - x_1) \sum_{i=1}^n p_i \phi'_A(y_i(\bar{x})) f_i(x_1) - \mu_h(1 + \psi'(0)) &= 0, \end{aligned} \quad (2.97)$$

$$\begin{aligned} L(x_2) &= 0 \\ \iff u'(W_A - \pi) \sum_{i=1}^n p_i \phi'_A(y_i(\bar{x})) f_i(x_2) - \mu_h(1 + \psi'(x_2)) &= 0, \end{aligned} \quad (2.98)$$

$$y_1(\bar{x}) = \int_0^{\bar{x}} u(W_A - \pi - x + I(x)) f_1(x) dx, \quad (2.99)$$

$$y_2(\bar{x}) = \int_0^{\bar{x}} u(W_A - \pi - x + I(x)) f_2(x) dx, \quad (2.100)$$

$$\pi = \int_{x_1}^{\bar{x}} (I(x) + \psi(I(x))) \bar{f}(x) dx, \quad (2.101)$$

$$\mu_h = \int_0^{\bar{x}} u'(W_A - \pi - x + I(x)) \sum_{i=1}^n p_i \phi'_A(y_i(\bar{x})) f_i(x) dx. \quad (2.102)$$

Note that the optimal contract (2.93) corresponds to the system of equations (2.97) and (2.99)-(2.102). In this case for all losses between  $x_1$  and  $\bar{x}$ , the indemnity function satisfies

$$\Xi_x(A(x)/\mu_h) = \Xi_x \left( \sum_{i=1}^n p_i \phi'_A(y_i(\bar{x})) / (\mu_h \bar{f}(x)) \right). \quad (2.103)$$

On the other hand, the contract (2.92) corresponds to the system of equations (2.98)-(2.102). In this case, the indemnity function satisfies (2.103) for losses between  $x_1$  and  $x_2$ . We emphasize that since the shape of the contract is not known ex-ante. In particu-

lar, the conditions laid out in Proposition 2.6 pertaining to the shape of the optimal indemnity function are endogenous. Our strategy is thus to first hypothesize a certain shape, then solve the relevant system of equations under this hypothesis, and then go back to verify the conditions in Proposition 2.6. The rest of this section is organized as follows. Subsection 2.5.1 provides the specific functions and parameters used in the numerical exercise. Subsection 2.5.2 discusses the simulation results and some comparative statics of ambiguity and ambiguity aversion.

### 2.5.1 Data

#### CRRA Utility function

The function  $u : \mathbb{R}_+^* \rightarrow \mathbb{R}$  is defined by

$$u(z) = \begin{cases} \frac{z^{1-\gamma}-1}{1-\gamma} & \gamma \neq 1 \\ \ln z & \gamma = 1. \end{cases} \quad (2.104)$$

#### CRAA Second order utility functions

For  $J \in \{A, P\}$ ,

$$\phi_J(u) = \begin{cases} \frac{u^{1-\sigma_J}-1}{1-\sigma_J} & \sigma_J \neq 1 \\ \ln u & \sigma_J = 1. \end{cases} \quad (2.105)$$

#### Cost of indemnity provision

The function  $\psi$  is linear, in particular  $\psi(I) = mI$ , where  $m \in (0, 1)$  is often referred to as the constant loading factor.

#### Priors and densities

The probability of the first state is  $p_1 \in (0, 1)$ . Let the densities be truncated exponential distributions defined as follows. For each  $i$  in  $\mathcal{I}$ ,

$$f_i(x) = \begin{cases} \frac{\frac{1}{\beta_i} e^{-x/\beta_i}}{1-e^{-\bar{x}/\beta_i}} & x \in I_x, \\ 0 & x \notin I_x. \end{cases} \quad (2.106)$$

where  $\beta_i$ 's are strictly positive. In particular for Assumption 2.5 to hold we require that for all  $i, j$  in  $\mathcal{I}$  such that  $i < j$ , it holds that  $\beta_i < \beta_j$ .

#### Parameters

	Notation	Value
Number of ambiguous states	$n$	2
State index set	$\mathcal{I}$	$\{1, 2\}$

*continued on the next page*

	Notation	Value
Loading factor	$m$	0.2
Prior on first state	$p_1$	0.1
First-state density's parameter	$\beta_1$	0.1
Second-state density's parameter	$\beta_2$	0.5
Policyholder's relative risk aversion	$\gamma$	1
Policyholder's relative ambiguity aversion	$\sigma_A$	0.45
Insurer's relative ambiguity aversion	$\sigma_P$	0
Support of the loss	$I_x = [0, \bar{x}]$	$[0, 1]$
Domain of the premium	$I_\pi = [\underline{\pi}, \bar{\pi}]$	$\underline{\pi} = 0, \bar{\pi} = (1 + m) \int_0^1 x \bar{f}(x) dx$
Policyholder's initial wealth	$W_A$	$\bar{\pi} + \bar{x} + 0.75$
Insurer's initial wealth	$W_P$	$1000W_A$

Table 2.2: Parameters assumed for the simulation

### 2.5.2 Simulation results

In this case the computed optimal deductible and premium are:

$$x_1 \approx 0.607, \quad \pi \approx 0.035. \quad (2.107)$$

Let us denote this contract  $(I^*, \pi^*)$ . Compared to the boundary contracts considered in Remark 2.1, this contract yields the minimum cost:

$$g(I^*, \pi^*) = 0.44, \quad (2.108)$$

$$g(0, \underline{\pi}) = 0.45, \quad (2.109)$$

$$g(x, \bar{\pi}) = 0.48. \quad (2.110)$$

Hence the contract with a deductible is the optimal one. The indemnity function in this case is plotted in Figure 2.1. Figure 2.2 verifies that  $K(0) < 0$ ,  $K(\bar{x}) > 0$  and  $L(\bar{x}) < 0$ . In fact the indemnity function in Figure 2.1 resembles a straight deductible (it appears to be linear in  $x$  for  $x > x_1$ ) as the the ratio  $G'/G$  seems to converge to zero as  $x$  tends to  $\bar{x}$ , as illustrated in Figure 2.3. Nevertheless, this is visually misleading. Recall from Remark 2.6 that in the case under consideration where  $\ell'_{12}(x) < 0$  on the whole range of losses  $[0, 1]$ , a straight deductible policy cannot be optimal. On the other hand, there exists a subset of positive measure of  $(x_1, 1]$  where  $I'(x) > 1$ . This point is

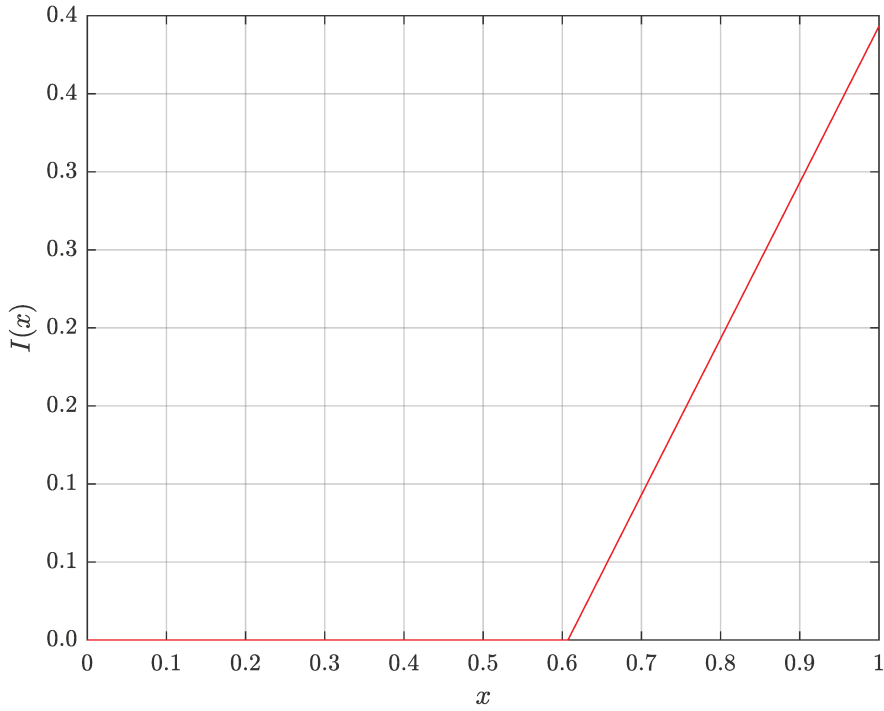


Figure 2.1: Indemnity function of a deductible contract

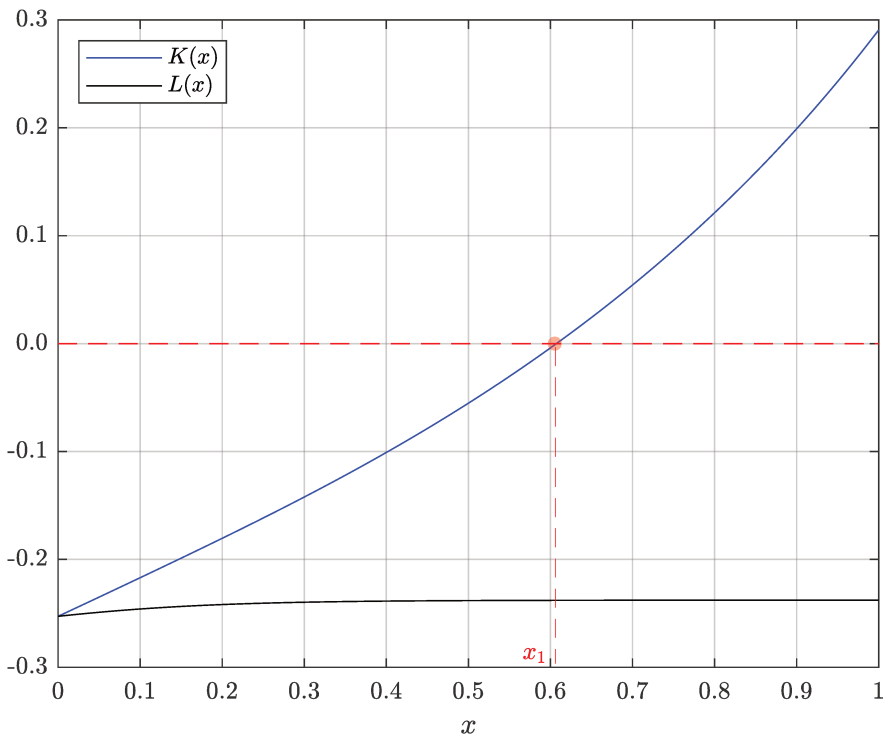


Figure 2.2: Verifying conditions for the deductible contract

illustrated in Figure 2.4, which is drawn for the losses beyond the deductible to detect the almost imperceptible deviation from zero of the ratio  $G'/G$  in this case. Although

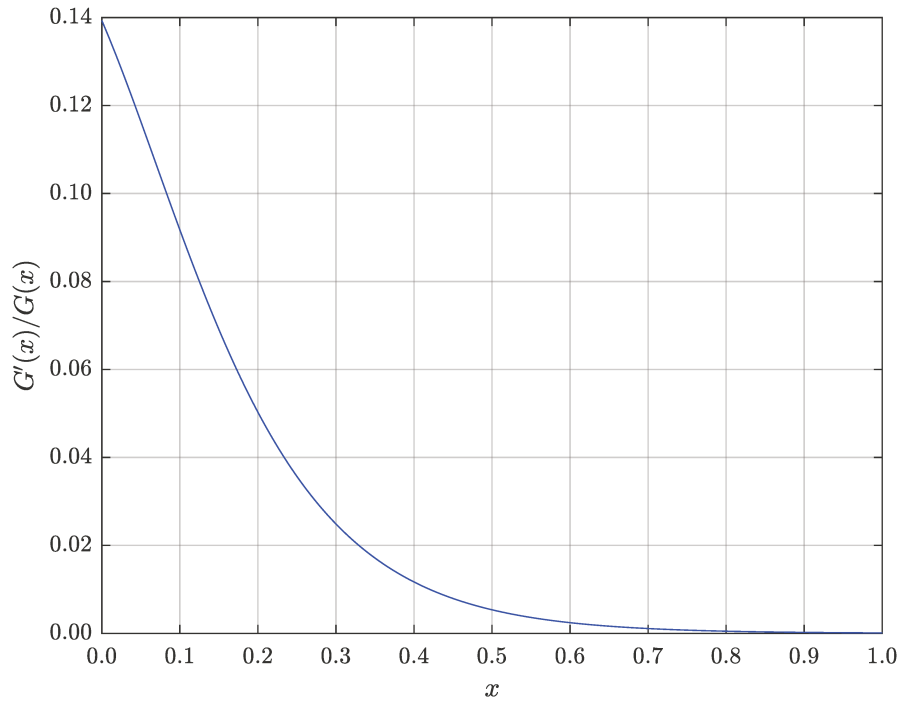


Figure 2.3: The rate of change of the ratio of EMWs

small and decreasing, it is clear that  $G'/G$  is strictly positive beyond  $x_1$ , implying via the co-insurance equation (2.48) that  $I' > 1$  for these losses.

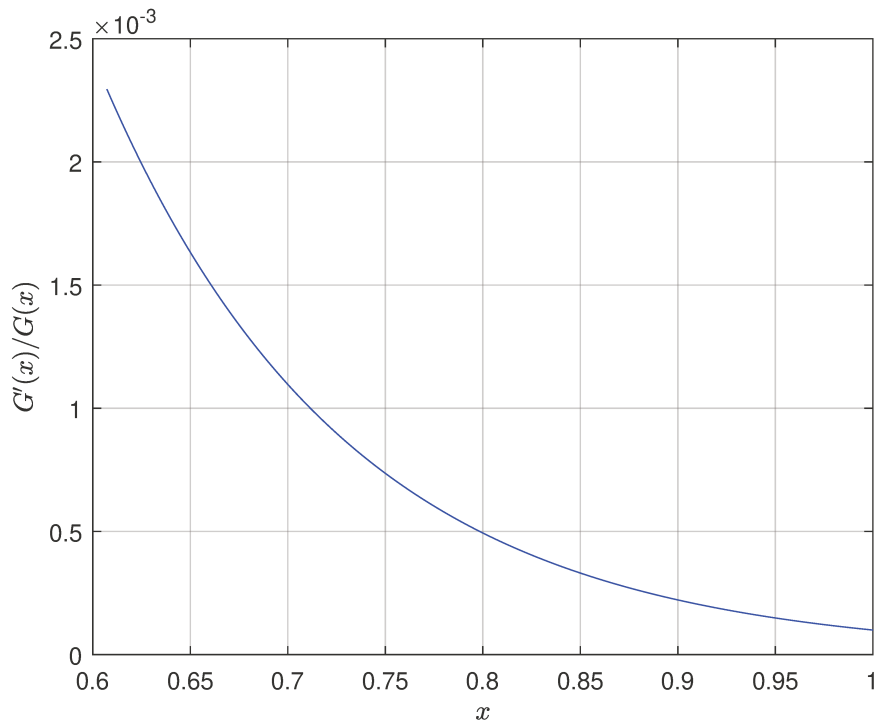


Figure 2.4: The rate of change of the ratio of EMWs beyond the deductible



**Impact of increasing ambiguity aversion**

Let us illustrate the impact of increasing relative ambiguity aversion, i.e., of increasing  $\sigma_A$  on the optimal level of deductible, premium and the shape of the indemnity function. Figure 2.5 shows that increasing relative ambiguity aversion does not alter the shape of the indemnity function, but it reduces the optimal deductible. Intuitively, the threshold for "small losses" born entirely by the policyholder is smaller the more ambiguity-averse he is. In other words, the demand for insurance increases in relative ambiguity aversion. As expected, smaller deductibles imply larger premiums, as illustrated in Figure 2.6. Figure 2.7 shows that the higher the degree of relative ambiguity

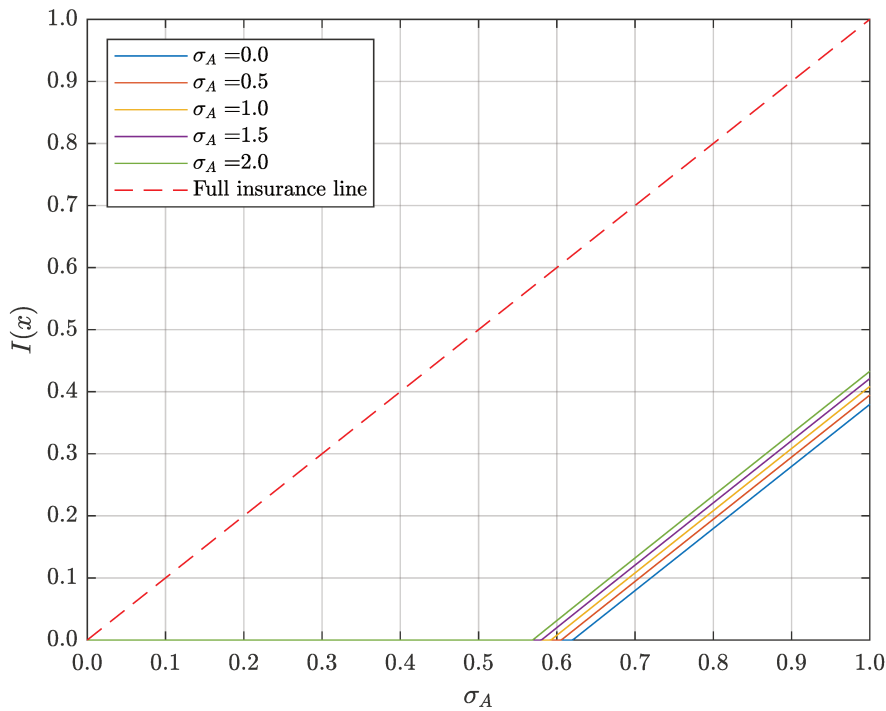


Figure 2.5: The impact of increasing ambiguity aversion on optimal indemnity

aversion, the greater the magnitude and the rate of change of  $G'/G$  with respect to the loss.

In response to Remark 2.5 at the end of the previous section, let us look at the behavior of  $K(x)$  and  $L(x)$ . The plots of  $K$  and  $L$  for a range of values of relative ambiguity aversion are provided in Figure 2.8 and Figure 2.9, respectively. It is noteworthy that while  $K$  is always strictly increasing and crosses the horizontal axis as predicted,  $L$  is always negative. As predicted in the theoretical analysis,  $L$  is constant in the ambiguity-neutral case ( $\sigma_A = 0$ ) and is increasing in the ambiguity-averse case ( $\sigma > 0$ ). Furthermore,  $L$  is more negative the higher the degree of ambiguity aversion and the curves of  $L$  seem to flatten out almost completely beyond the loss equal to the deductible. For this reason,  $L$  never rises to cross the horizontal axis, hence the condition for the

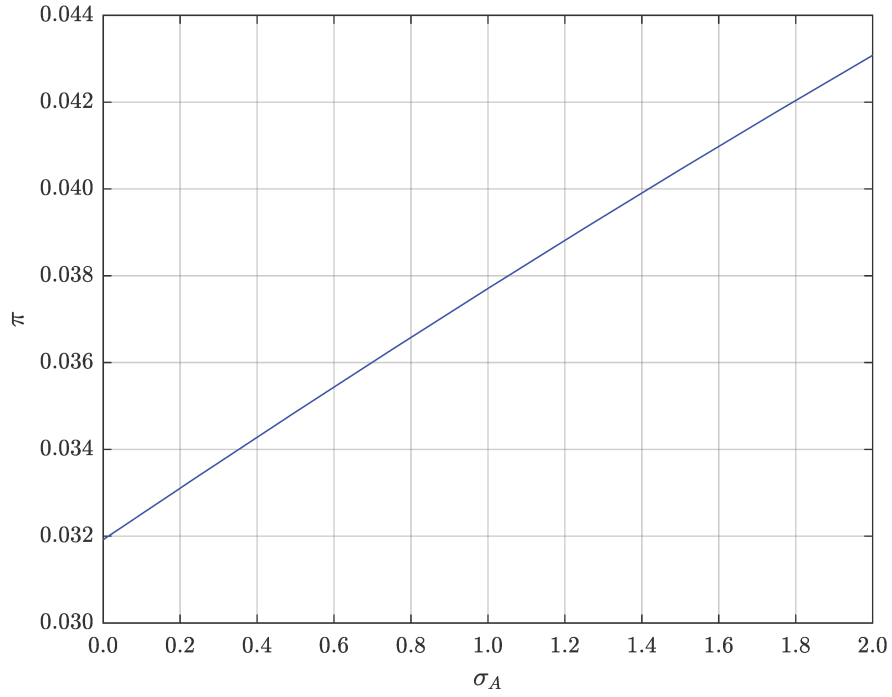


Figure 2.6: The impact of increasing ambiguity aversion on the optimal premium

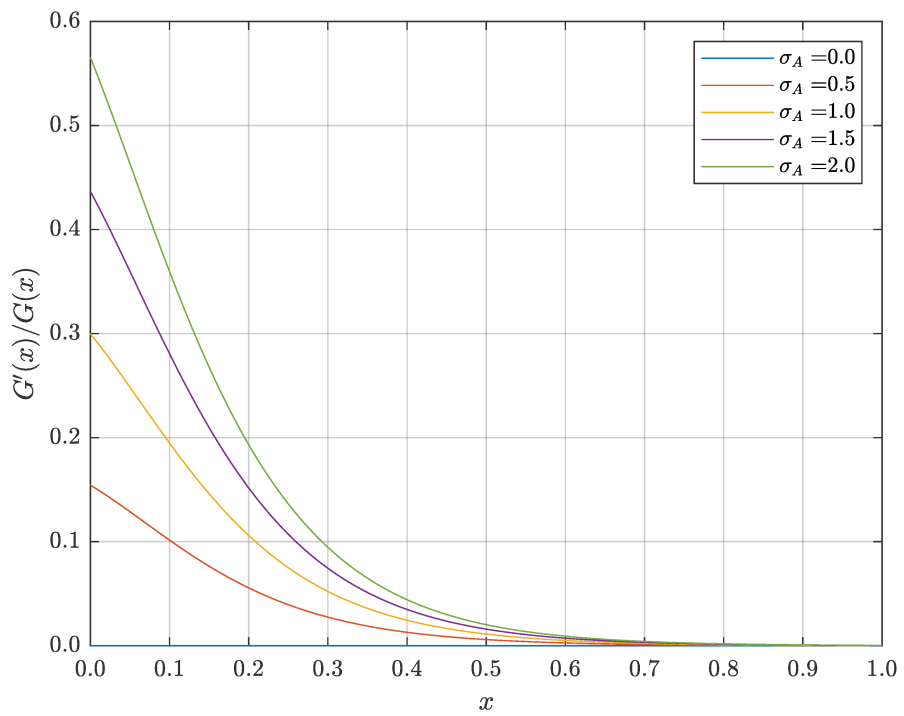


Figure 2.7: The impact of increasing ambiguity aversion on the rate of change of  $G$

existence of an  $x_2$  is not verified.

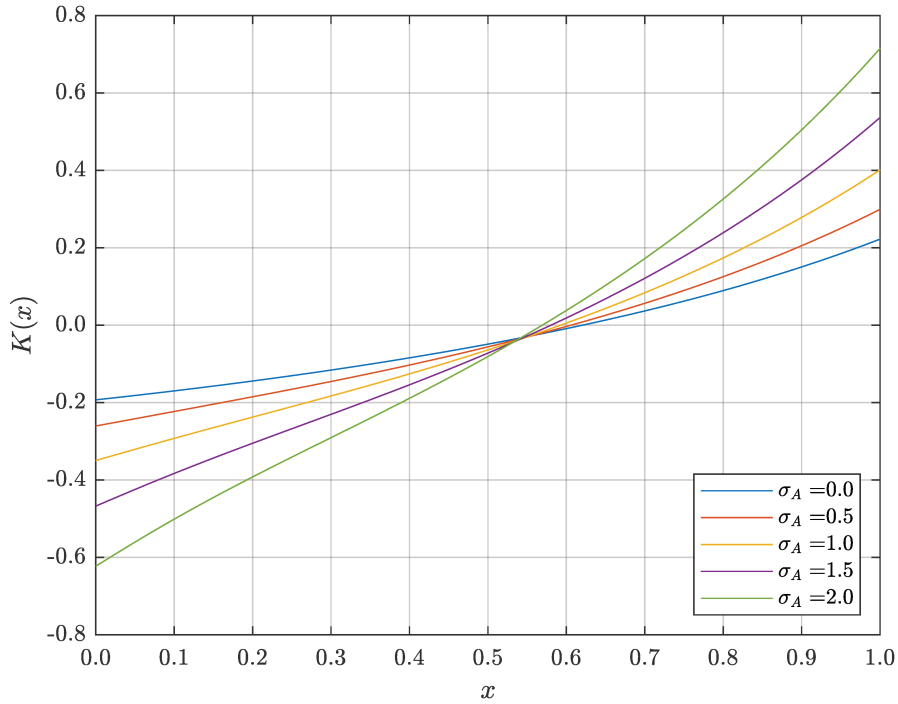


Figure 2.8: The impact of increasing ambiguity aversion on  $K(x)$

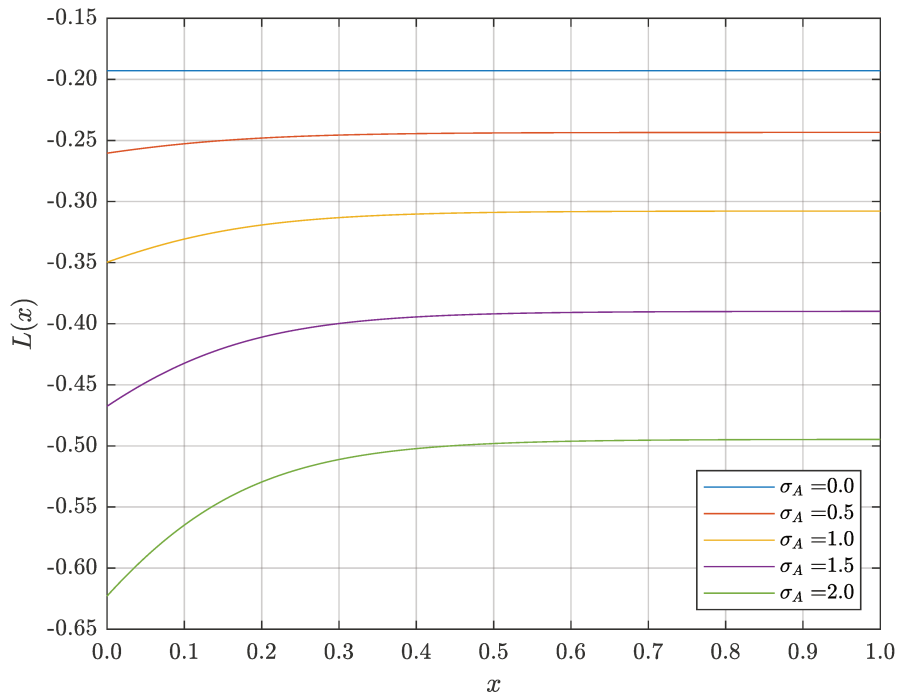


Figure 2.9: The impact of increasing ambiguity aversion on  $L(x)$

### Impact of increasing the likelihood of the better state

It is also interesting to look at the impact of increasing optimism, i.e., increasing the prior on the good state  $p_1$  on the optimal contract and other quantities of interest. Fig-

ure 2.10 shows that increasing the prospect of the good state does not seem to alter the shape of the optimal indemnity function. Figure 2.11 suggests that the demand for insurance declines monotonically in  $p_1$  up to  $p_1 = 0.9$ , but then increases for larger values of  $p_1$ . Figure 2.12 shows that increasing the prospect of the good state does not

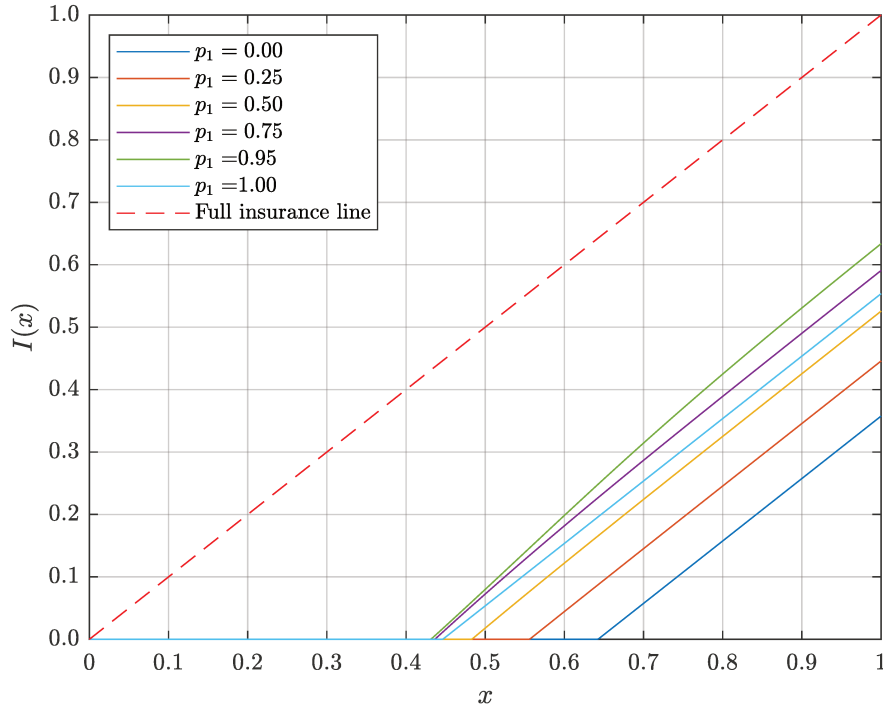


Figure 2.10: Impact of increasing  $p_1$  on the indemnity function

have a monotone effect on the premium. Finally, Figure 2.13 and Figure 2.14 verify the conditions for having the contract of the type (2.93).

## 2.6 Conclusion

We set out to determine the optimal insurance policy under ambiguity and ambiguity aversion in the sense of [Klibanoff et al. \(2005\)](#). First, we provide a rigorous proof of the existence of an optimal insurance policy in presence of ambiguity. Next, we show that if both DMs are ambiguity-neutral, the problem under consideration is technically identical to that of [Raviv \(1979\)](#). We then revisit this problem, and improve upon the result of the first theorem of [Raviv \(1979\)](#) in two different aspects. On the one hand, our analysis does not rest on fixing any premium. Instead, we always view the contract as a pair of indemnity and premium to solve for simultaneously. On the other hand, this holistic approach allows us to rule out the optimality of an upper limit coverage policy without having to resort to additional analysis outside the optimal control framework. Our result also generalizes the first proposition of [Gollier \(2014\)](#); we prove the existence

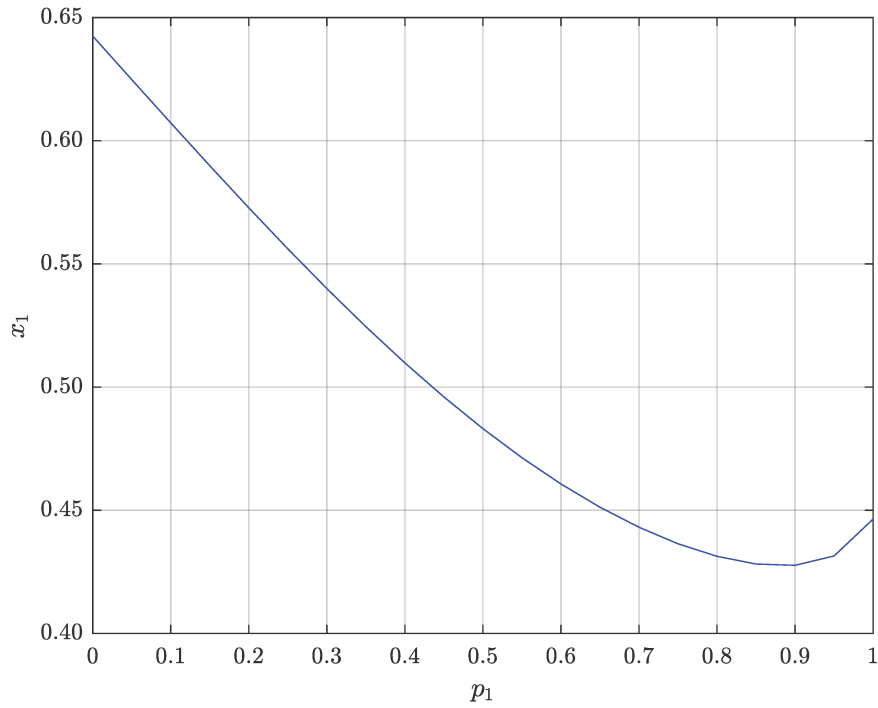


Figure 2.11: Impact of increasing  $p_1$  on deductible

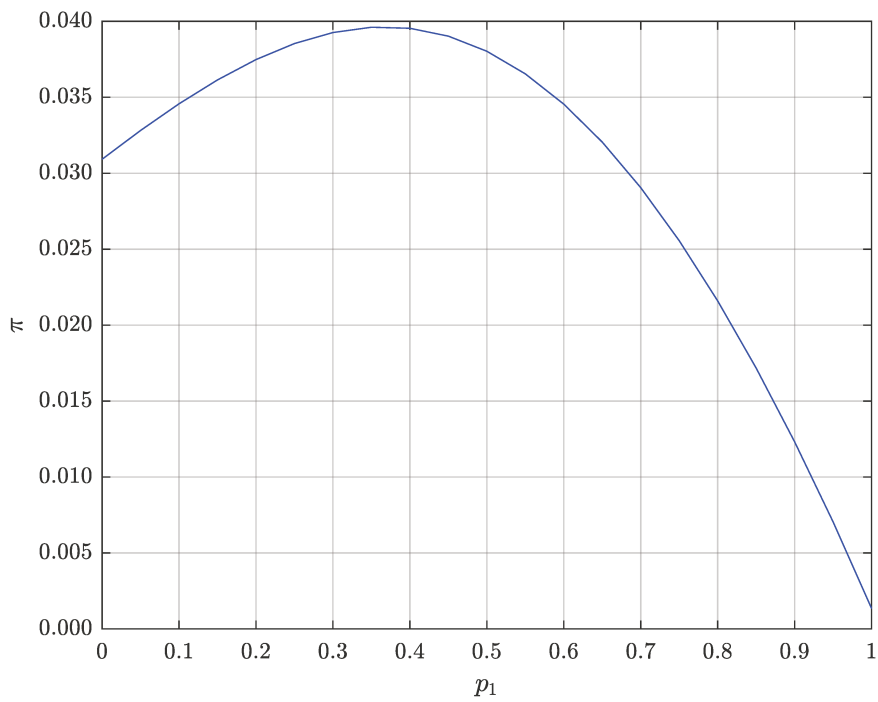


Figure 2.12: Impact of increasing  $p_1$  on the premium

of a deductible for the case of a general convex cost of indemnity function.

In the case where the policyholder is ambiguity-averse, we establish again the exis-

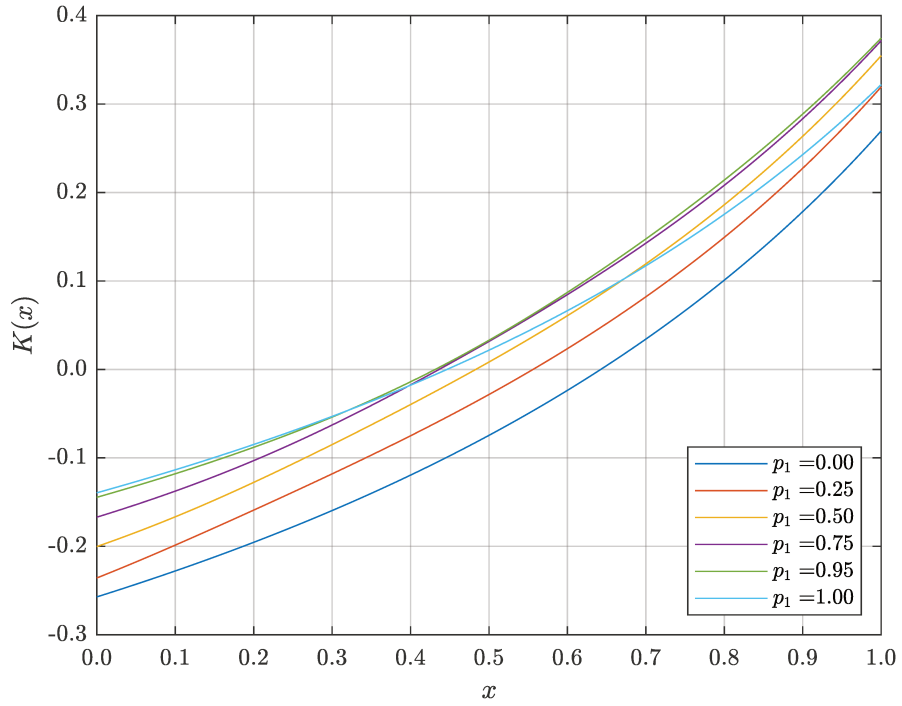


Figure 2.13: Impact of increasing  $p_1$  on  $K$

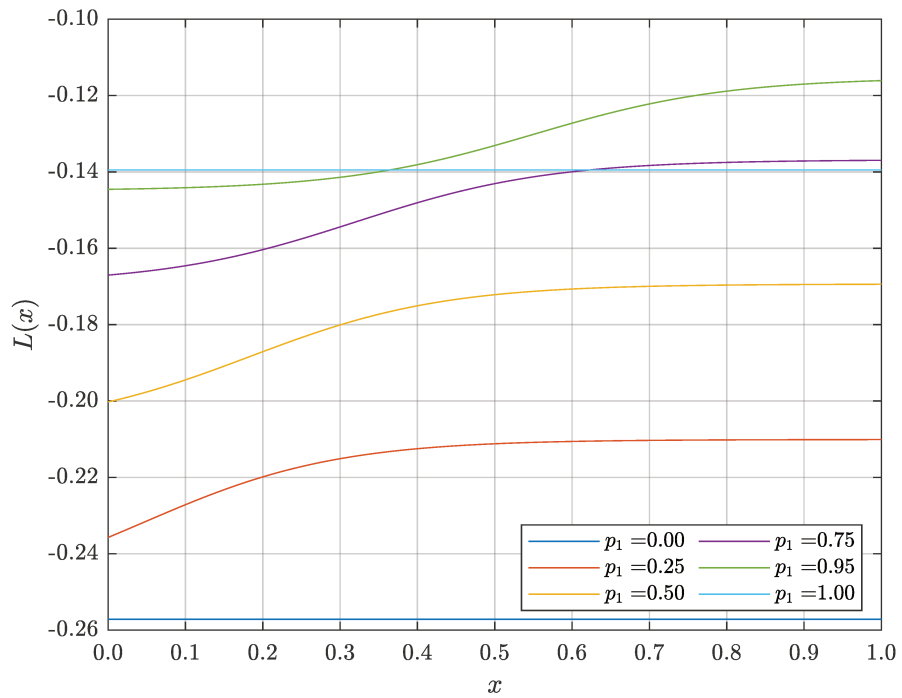


Figure 2.14: Impact of increasing  $p_1$  on  $L$

tence of a deductible for the case of a general convex cost of indemnity function. For the case of linear indemnity cost, our theoretical analysis suggests that beyond the de-

ductible there might exist an upper limit above which full insurance is optimal. The condition for the existence of this upper limit cannot be verified ex-ante, which motivates us to conduct a numerical simulation. Our numerical results suggest that while increasing ambiguity aversion raises the demand for insurance (and the associated premium) in the sense that it reduces the deductible, an upper limit is never reached for a range of parameters used.

## 2.7 Appendix of Proofs

### 2.7.1 Proof of Proposition 2.1

Let  $\delta = \inf_{(J, \pi) \in \mathcal{U} \times I_\pi} g(J, \pi)$ . Consider a sequence of trajectories  $\{X^k(\cdot)\}_{k \in \mathbb{N}}$  associated with the sequence of admissible controls  $\{J^k(\cdot)\}_{k \in \mathbb{N}}$  defined by

$$X^k(x) = \begin{pmatrix} y^k(x) \\ z^k(x) \\ \pi^k(x) \end{pmatrix} = \begin{pmatrix} (y_i^k(x))_{i \in \mathcal{I}} \\ (z_i^k(x))_{i \in \mathcal{I}} \\ \pi^k \end{pmatrix}, \quad x \in I_x \setminus \{0\},$$

such that  $g(J^k, \pi^k) \rightarrow \delta$  as  $k \rightarrow \infty$ , where

$$\begin{aligned} y_i^k(x) &= \int_0^x u(W_A - \pi^k - t + tJ^k(t))f_i(t)dt, \quad i \in \mathcal{I}, \\ z_i^k(x) &= \int_0^x (W_P + \pi^k - tJ^k(t) - \psi(tJ^k(t)))f_i(t)dt, \quad i \in \mathcal{I}. \end{aligned}$$

By the weak- $\star$  compactness of  $\mathcal{U}$ , the sequence  $\{J^k(\cdot)\}_{k \in \mathbb{N}}$  converges to  $J^*(\cdot) \in \mathcal{U}$  up to some subsequence, i.e.,  $J^k \rightarrow J^*$ . Likewise the compactness of  $I_\pi$  implies  $\pi^k \rightarrow \pi^* \in I_\pi$  up to some subsequence. Let  $\bar{X}^*$  stand for the limiting trajectory defined by

$$\bar{X}^*(x) = \begin{pmatrix} \bar{y}^*(x) \\ \bar{z}^*(x) \\ \pi^* \end{pmatrix} = \begin{pmatrix} (\bar{z}_i^*(x))_{i \in \mathcal{I}} \\ (\bar{y}_i^*(x))_{i \in \mathcal{I}} \\ \pi^* \end{pmatrix}, \quad x \in I_x \setminus \{0\},$$

where

$$\begin{aligned} \bar{y}_i^*(x) &= \int_0^x u(W_A - \pi^* - t + tJ^*(t))f_i(t)dt, \quad i \in \mathcal{I}, \\ \bar{z}_i^*(x) &= \int_0^x (W_P + \pi^* - tJ^*(t) - \psi(tJ^*(t)))f_i(t)dt, \quad i \in \mathcal{I}. \end{aligned}$$

The remain of the proof is completed in two steps. First, we show that the limiting trajectory is shown to satisfy the constraint. Second, we prove that this trajectory is an optimal one.

#### The limiting trajectory verifies the constraint

Let us now show that  $h(J^*, \pi^*) \geq 0$ . By construction  $h(J^k, \pi^k) \equiv \sum_{i=1}^n p_i \phi_P(z_i^k(\bar{x})) - \bar{V} \geq 0$  for all  $k \in \mathbb{N}$ . For  $i \in \mathcal{I}$  and  $k \geq 0$ , let us write

$$z_i^k(\bar{x}) = \int_{I_x} (W_P + \pi^* - xJ^k(x) - \psi(xJ^k(x)))f_i(x)dx + \Delta^k,$$



where  $\Delta^k = \int_{I_x} (\pi^k - \pi^*) f_i(x) dx = \pi^k - \pi^*$ . Clearly  $\Delta^k$  tends to zero as  $k$  tends to infinity. Let  $\Gamma_i(J^k(x)) \equiv -(W + \pi^* - xJ^k(x) - \psi(xJ^k(x)))f_i(x)$ , then  $\Gamma_i(J^k(x))$  is convex in  $I^k(x)$  since  $\psi(\cdot)$  is convex in  $J^k(x)$  and  $f_i(x) > 0$  by Assumption 2.4. Hence from Lee and Markus (1967), we have

$$\int_{I_x} \Gamma_i(J^*(x)) \geq \liminf \int_{I_x} \Gamma_i(J^k(x)) dx,$$

or

$$\begin{aligned} -\bar{z}_i^*(\bar{x}) &\geq \liminf(-z_i^k(\bar{x}) - \delta^k) \\ \iff \bar{z}_i^*(\bar{x}) &\geq \limsup z_i^k(\bar{x}). \end{aligned}$$

By the continuity of  $\bar{z}_i^*$ , for all  $\epsilon > 0$ , there exists a sufficiently large positive integer  $K$  such that

$$\bar{z}_i^*(\bar{x}) \geq z_i^K(\bar{x}) - \epsilon.$$

Since  $\phi_P$  is increasing,

$$\phi_P(\bar{z}_i^*(\bar{x})) \geq \phi_P(z_i^K(\bar{x}) - \epsilon). \quad (2.111)$$

By the first fundamental theorem of calculus,

$$\phi_P(z_i^K(\bar{x})) - \phi_P(z_i^K(\bar{x}) - \epsilon) = \int_a^b \phi_P'(\zeta) d\zeta,$$

where  $b \equiv z_i^K(\bar{x})$  and  $a = b - \epsilon$ . Since  $\phi_P'$  is bounded by Assumption 2.10, let  $M \in \mathbb{R}_+$  be an upper bound of  $\phi_P'$  over  $[a, b]$ . Then

$$\phi_P(z_i^K(\bar{x})) - \phi_P(z_i^K(\bar{x}) - \epsilon) \leq M(b - a) = M\epsilon,$$

implying

$$\phi_P(z_i^K(\bar{x}) - \epsilon) \geq \phi_P(z_i^K(\bar{x})) - M\epsilon,$$

which, together with (2.111) imply

$$\phi_P(\bar{z}_i^*(\bar{x})) \geq \phi_P(z_i^K(\bar{x})) - M\epsilon \quad (2.112)$$

$$\implies \sum_{i=1}^n p_i \phi_P(\bar{z}_i^*(\bar{x})) \geq \sum_{i=1}^n p_i \phi_P(z_i^K(\bar{x})) - M\epsilon. \quad (2.113)$$

Observe that  $\sum_{i=1}^n p_i \phi_P(z_i^K(\bar{x})) \geq \bar{V}$  since  $\sum_{i=1}^n p_i \phi_P(z_i^k(\bar{x})) \geq \bar{V}$  for all  $k \geq 0$ . Thus from (2.113), we have

$$\sum_{i=1}^n p_i \phi_P(\bar{z}_i^*(\bar{x})) \geq \bar{V} - M\epsilon.$$

Since  $\epsilon$  was arbitrary, letting  $\epsilon \rightarrow 0$  completes the proof. We next show that the cost functional achieved by the limiting trajectory is optimal.

### The optimality of the limiting trajectory

We now prove that the cost achieved by the limiting trajectory is optimal, i.e.,  $g(J^*, \pi^*) = \delta$ , where  $\delta = \inf_{(J, \pi) \in \mathcal{U} \times I_\pi} g(J, \pi)$ . Since  $(J^*, \pi^*)$  is admissible,  $g(J^*, \pi^*) \geq \delta$ . It remains to show that  $g(J^*, \pi^*) \leq \delta$ . Let us write:

$$y_i^k(\bar{x}) = \int_{I_x} u(W_A - \pi^* - x + xJ^k(x)) f_i(x) dx + \Delta_i^k,$$

where

$$\Delta_i^k \equiv \int_{I_x} [u(W_A + \pi^k - x + xJ^k(x)) - u(W_A + \pi^* - x + xJ^k(x))] f_i(x) dx. \quad (2.114)$$

Observe that  $\Delta_i^k$  tends to zero as  $k$  tends to infinity since  $u$  is bounded and continuous, and  $f_i$  is continuous. Let  $\Gamma_i(J^k(x)) \equiv -u(W_A + \pi^* - x + xJ^k(x)) f_i(x)$ . Then  $\Gamma_i$  is convex in  $J^k$  since  $-u$  is convex and  $f_i$  is strictly positive. Again from [Lee and Markus \(1967\)](#),

$$\begin{aligned} \int_{I_x} \Gamma_i(J^*(x)) dx &\leq \liminf \int_{I_x} \Gamma_i(J^k(x)) dx \\ -\bar{y}_i^*(\bar{x}) &\leq \liminf (-y_i^k(\bar{x}) - \Delta_i^k) \\ \bar{y}_i^*(\bar{x}) &\geq \limsup y_i^k(\bar{x}). \end{aligned}$$

Proceed similarly to the proof of the previous lemma, we have that for all  $\epsilon > 0$ , there exists a sufficiently large integer  $K$  such that

$$-\sum_{i=1}^n p_i \phi_A(\bar{y}_i^*(\bar{x})) \leq -\sum_{i=1}^n p_i \phi_A(y_i^K(\bar{x})) + M\epsilon.$$

Letting  $\epsilon$  tend to zero yields:

$$g(J^*, \pi^*) \leq g(J^K, \pi^K),$$

which implies that  $g(J^*, \pi^*)$  is a lower bound for  $g(J^K, \pi^K)$ . Hence  $g(J^*, \pi^*) \leq \delta$  by definition of the infimum, as desired.

To sum up, we have proved that the limiting trajectory satisfies the constraint and the cost achieved by this trajectory is the minimum cost. Thus the pair  $(J^*, \pi^*)$  is an

optimal pair, and the associated insurance contract  $(I^*, \pi^*)$ , where

$$I^*(x) = \begin{cases} 0 & x = 0 \\ xJ^*(x) & x \in (0, \bar{x}] \end{cases} \quad (2.115)$$

is an optimal one.

### 2.7.2 Proof of Lemma 2.1

Suppose by contradiction that  $h(J, \pi) > 0$  for an optimal pair  $(J, \pi)$ . We have

$$\begin{aligned} & \sum_{i=1}^n p_i \phi_P \left( \int_{I_x} (W_P + \pi - xJ(x) - \psi(xJ(x))) f_i(x) dx \right) - \bar{V} > 0 \\ \iff & \sum_{i=1}^n p_i \phi_P \left( W_P + \pi - \int_{I_x} (xJ(x) + \psi(xJ(x))) f_i(x) dx \right) - \bar{V} > 0. \end{aligned}$$

If  $J(x) = 1$  for a.e.  $x \in I_x$ , then by the continuity of  $\phi_P$  with respect to  $\pi$  there exists some positive real number  $\eta > 0$  such that  $h(J, \pi - \eta) > 0$ . Since the cost is strictly increasing in  $\pi$ , lowering  $\pi$  reduces the cost. In this case we have:

$$h(J, \pi - \eta) > 0, \quad (2.116)$$

$$g(J, \pi - \eta) < g(J, \pi), \quad (2.117)$$

implying that the contract  $(J, \pi - \eta)$  is feasible and yields a lower cost. Hence  $(J, \pi)$  is not optimal, a contradiction.

If  $J$  is not equal to 1 almost everywhere on  $I_x$ , then by the continuity of the mapping  $J \mapsto xJ + \psi(xJ)$  and strict positivity of the conditional densities, there exists  $K_x \subset I_x$  of positive measure and a sufficiently small  $\epsilon_K > 0$  satisfying  $0 \leq J(x) + \epsilon_K \leq 1$  for all  $x \in K_x$  such that  $g(\tilde{J}, \pi) \geq 0$ , where  $\tilde{J} : (0, \bar{x}] \rightarrow [0, 1]$  is defined by

$$\tilde{J}(x) = \begin{cases} J(x) & x \in (0, \bar{x}] \setminus K_x, \\ J(x) + \epsilon_K & x \in K_x. \end{cases}$$

Since the cost is strictly decreasing in  $J$ , the modified control  $\tilde{J}$  yields a lower cost, i.e.,  $g(\tilde{J}, \pi) < g(J, \pi)$ , contradicting the hypothesis that  $(J, \pi)$  is optimal.

We conclude that if  $(J, \pi)$  constitutes an optimal pair, then  $h(J, \pi) = 0$ .

### 2.7.3 Proof of Proposition 2.2

Let us first prove two lemmas.

**Lemma 2.6.** *The normal cone at  $X(0)$  depends on the value of  $\pi$ . In particular:*

- If  $\pi \in (\underline{\pi}, \bar{\pi})$ , then  $N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \{0\}$ ;
- If  $\pi = \underline{\pi}$ , then  $N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_-$ ;
- If  $\pi = \bar{\pi}$ , then  $N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^+$ .

The normal cone at  $X(\bar{x})$  is:

$$N_{\mathcal{M}_1}(X(\bar{x})) = \begin{pmatrix} 0 \\ -\mu_h \nabla_z h(J, \pi) \\ -\mu_h \nabla_\pi h(J, \pi) - \mu_{\underline{\pi}} + \mu_{\bar{\pi}} \end{pmatrix}. \quad (2.118)$$

*Proof.* First, consider  $N_{\mathcal{M}_0}(X(0))$ , where  $X(0) = (0, 0, \pi)$ . Let  $\xi = (\xi_y, \xi_z, \xi_\pi)$  be an element in  $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ . Let  $M_0$  be an element in  $\mathcal{M}_0$ , hence  $M_0 = (0, 0, a)$  for some  $a \in I_\pi$ . The normal cone to  $\mathcal{M}_0$  at  $X(0)$  can be written as:

$$\begin{aligned} N_{\mathcal{M}_0}(X(0)) &= \{ \xi \in \mathbb{R}^{2n+1} \mid \langle \xi, M_0 - X(0) \rangle \leq 0, \forall M_0 \in \mathcal{M}_0 \} \\ \implies N_{\mathcal{M}_0}(X(0)) &= \{ \xi \in \mathbb{R}^{2n+1} \mid \xi_\pi(a - \pi) \leq 0, \forall a \in I_\pi \}. \end{aligned}$$

One of the following cases can occur.

- If  $\pi \in (\underline{\pi}, \bar{\pi})$ , then  $\xi_\pi = 0$  since  $\xi_\pi(a - \pi)$  must be negative for any  $a$  in  $I_\pi$ . Hence in this case,

$$N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \{0\}. \quad (2.119)$$

- If  $\pi = \underline{\pi}$ , then  $a - \pi \geq 0$  for all  $a \in I_\pi$ . Thus  $\xi_\pi(a - \pi)$  is negative for any  $a \in I_\pi$  requires  $\xi_\pi \leq 0$ , implying that:

$$N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_-. \quad (2.120)$$

- If  $\pi = \bar{\pi}$ , then  $a - \pi \leq 0$  for all  $a \in I_\pi$ , implying that  $\xi_\pi \geq 0$  and the normal cone in this case is:

$$N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+. \quad (2.121)$$

Let us now compute the normal cone at the target compute  $N_{\mathcal{M}_1}(X(\bar{x}))$ , where  $X(\bar{x}) = (y(\bar{x}), z(\bar{x}), \pi)$ . Following [Clarke \(1990\)](#), we can write

$$N_{\mathcal{M}_1}(X(\bar{x})) = -\mu_h \nabla_X h(J, \pi) + \mu_{\underline{\pi}} \nabla_X(\underline{\pi} - \pi) + \mu_{\bar{\pi}} \nabla_X(\pi - \bar{\pi}), \quad (2.122)$$

where  $\mu = (\mu_h, \mu_{\underline{\pi}}, \mu_{\bar{\pi}})$  satisfies the complementary slackness conditions

$$\mu_{\underline{\pi}} \geq 0, \quad \mu_{\underline{\pi}}(\underline{\pi} - \pi) = 0, \quad (2.123)$$

$$\mu_{\bar{\pi}} \geq 0, \quad \mu_{\bar{\pi}}(\pi - \bar{\pi}) = 0. \quad (2.124)$$

Since

$$\nabla_X h(J, \pi) = (0, \nabla_z h(J, \pi), \nabla_\pi h(J, \pi)), \quad (2.125)$$

$$\nabla_X(\underline{\pi} - \pi) = (0, 0, -1), \quad (2.126)$$

$$\nabla_X(\pi - \bar{\pi}) = (0, 0, 1), \quad (2.127)$$

we can rewrite (2.122) as:

$$N_{\mathcal{M}_1}(X(\bar{x})) = \begin{pmatrix} 0 \\ -\mu_h \nabla_z h(J, \pi) \\ -\mu_h \nabla_\pi h(J, \pi) - \mu_{\underline{\pi}} + \mu_{\bar{\pi}} \end{pmatrix}. \quad (2.128)$$

■

**Lemma 2.7.** *If  $\pi \in (\underline{\pi}, \bar{\pi})$ , then  $\lambda_\pi(0) = 0$ . If  $\pi = \underline{\pi}$ , then  $\lambda_\pi(0) \leq 0$ . If  $\pi = \bar{\pi}$ , then  $\lambda_\pi(0) \geq 0$ .*

*Proof.* The proof follows directly from condition (2.29) applied to different forms of  $N_{\mathcal{M}_0}(X(0))$  depending on where  $\pi$  takes value (at the optimum). In particular, if  $\pi \in (\underline{\pi}, \bar{\pi})$ , then the normal cone takes the form (2.119), implying that  $\lambda_\pi(0) = 0$ . If  $\pi = \underline{\pi}$ , then the normal cone takes the form (2.120), implying that  $\lambda_\pi(0) \leq 0$ . Finally if  $\pi = \bar{\pi}$ , then the normal cone in (2.121) implies that  $\lambda_\pi(0) \geq 0$ . ■

From the adjoint equation (2.26), we have that for almost every  $x$  in  $I_x$ ,

$$\begin{pmatrix} \dot{\lambda}_y(x) \\ \dot{\lambda}_z(x) \\ \dot{\lambda}_\pi(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u'(W_A - \pi - x + xJ(x)) \langle \lambda_y, f(x) \rangle - \langle \lambda_z, f(x) \rangle \end{pmatrix}. \quad (2.129)$$

Hence  $\lambda_y(x) = cons \equiv \lambda_y$  and  $\lambda_z(x) = cons \equiv \lambda_z$  for all  $x \in I_x$ . In view of (2.128) we can rewrite the transversality condition (2.30) as:

$$\begin{pmatrix} \lambda_y \\ \lambda_z \\ \lambda_\pi(\bar{x}) \end{pmatrix} = \begin{pmatrix} -\lambda_0 \nabla_y g(J, \pi) \\ \mu_h \nabla_z h(J, \pi) \\ \nabla_\pi (-\lambda_0 g(J, \pi) + \mu_h h(J, \pi)) + \mu_{\underline{\pi}} - \mu_{\bar{\pi}} \end{pmatrix}.$$

This yields

$$\lambda_y = \lambda_0 (p_i \phi'_A(y_i(\bar{x})))_{i \in \mathcal{I}}, \quad (2.130)$$

$$\lambda_z = \mu_h (p_i \phi'_P(z_i(\bar{x})))_{i \in \mathcal{I}}, \quad (2.131)$$

$$\lambda_\pi(\bar{x}) = \lambda_0 \sum_{i=1}^n p_i \phi'_A(y_i(\bar{x})) \frac{\partial y_i(\bar{x})}{\partial \pi} + \mu_h \sum_{i=1}^n p_i \phi'_P(z_i(\bar{x})) \frac{\partial z_i(\bar{x})}{\partial \pi} + \mu_{\underline{\pi}} - \mu_{\bar{\pi}}. \quad (2.132)$$

Let us substitute (2.130) and (2.131) into (2.132) to express  $\lambda_\pi(\bar{x})$  more compactly as

$$\lambda_\pi(\bar{x}) = \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle + \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle + \mu_{\underline{\pi}} - \mu_{\bar{\pi}}. \quad (2.133)$$

Observe that

$$\begin{aligned} \int_{I_x} \dot{\lambda}_\pi(x) dx &= \int_{I_x} u'(W_A - \pi - x + xJ(x)) \langle \lambda_y, f(x) \rangle dx - \int_{I_x} \langle \lambda_z, f(x) \rangle dx \\ &= - \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle - \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle, \end{aligned} \quad (2.134)$$

where  $\frac{\partial z_i(\bar{x})}{\partial \pi} = 1$  for all  $i \in \mathcal{I}$ . Hence in view of (2.133) and (2.134)

$$\begin{aligned} \lambda_\pi(0) &= \lambda_\pi(\bar{x}) - \int_{I_x} \dot{\lambda}_\pi(x) dx, \\ &= 2 \left( \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle + \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle \right) + \mu_{\underline{\pi}} - \mu_{\bar{\pi}}. \end{aligned} \quad (2.135)$$

Observe that when  $\pi \in (\underline{\pi}, \bar{\pi})$ , we have  $(\mu_{\underline{\pi}}, \mu_{\bar{\pi}}) = 0 \in \mathbb{R}^2$  thanks to the complementary slackness conditions (2.123) and (2.124). Hence for  $\pi \in (\underline{\pi}, \bar{\pi})$ , we have:

$$\lambda_\pi(0) = 2 \left( \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial \pi} \right\rangle + \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial \pi} \right\rangle \right) = 2\lambda_\pi(\bar{x}). \quad (2.136)$$

To arrive at (2.35), recall that  $\lambda_\pi(0) = 0$  for interior values of  $\pi$  by Lemma 2.7.

#### 2.7.4 Proof of Lemma 2.2

Suppose by contradiction that  $\lambda_0 = \mu_h = 0$ . Then equations (2.31) and (2.32) imply:

$$\lambda_z = \lambda_y = 0 \in \mathbb{R}^n. \quad (2.137)$$

Hence from (2.129)

$$\dot{\lambda}_\pi(x) = 0, \quad a.e. \ x \in I_x, \quad (2.138)$$

implying that  $\lambda_\pi(x)$  is constant with respect to  $x$ . Denote

$$\lambda_\pi(x) = \lambda_\pi, \quad \forall x \in I_x. \quad (2.139)$$

Then from (2.133) we have:

$$\lambda_\pi = \mu_{\underline{\pi}} - \mu_{\overline{\pi}}. \quad (2.140)$$

Consider the following cases.

- If  $\pi \in (\underline{\pi}, \overline{\pi})$ , then  $\mu_{\underline{\pi}} = \mu_{\overline{\pi}} = 0$ , implying that  $\lambda_\pi = 0 \in \mathbb{R}^{2n+1}$ , violating the condition  $(\lambda, \lambda_0) \neq 0$ .
- If  $\pi = \overline{\pi}$ , then  $\lambda_\pi = -\mu_{\overline{\pi}} \leq 0$ . Non-negativity implies that  $\mu_{\overline{\pi}} = 0$ , which in turn implies  $\lambda_\pi = 0$ , again violating the condition  $(\lambda, \lambda_0) \neq 0$ .
- If  $\pi = \underline{\pi}$ , then  $\lambda_\pi = \mu_{\underline{\pi}} \geq 0$ . If  $\mu_{\underline{\pi}} = 0$ , then  $\lambda_\pi = 0$  and a similar contradiction ensues. If  $\mu_{\underline{\pi}} > 0$ , then  $\lambda_\pi(0) = \lambda_\pi > 0$ , inconsistent with Lemma 2.7.

Thus in any case, a contradiction follows if  $\mu_h = \lambda_0 = 0$ , establishing the lemma.

### 2.7.5 Proof of Lemma 2.3

Suppose that  $\lambda_0 = 0$ . Then  $\mu_h > 0$  by Lemma 2.2. By equations (2.31) and (2.32), the costate  $\lambda_y$  is a zero vector and the costate  $\lambda_z$  has strictly positive components, implying that  $\langle \lambda_y, f(x) \rangle = 0$  and  $\langle \lambda_z, f(x) \rangle > 0$  since the densities are strictly positive. Hence by Assumption 2.7,

$$H_\omega = -x(1 + \psi'(x\omega)) \langle \lambda_z, f(x) \rangle < 0, \quad a.e. \ x \in I_x, \quad (2.141)$$

$$H_{\omega\omega} = -x^2 \psi''(x\omega) \langle \lambda_z, f(x) \rangle \leq 0, \quad a.e. \ x \in I_x, \quad (2.142)$$

implying that  $J(x) = 0$  for a.e.  $x \in I_x$ . By Remark 2.1 uniformly zero insurance constitutes an optimum if and only if  $\pi = \underline{\pi} = 0$ .

Next, consider the case  $\mu_h = 0$ , which by Lemma 2.2 implies  $\lambda_0 > 0$ . Thus  $\langle \lambda_y, f(x) \rangle > 0$  and  $\langle \lambda_z, f(x) \rangle = 0$ . Hence by Assumption 2.8,

$$H_\omega = xu'(W_A - \pi - x + x\omega) \langle \lambda_y, f(x) \rangle > 0, \quad a.e. \ x \in I_x, \quad (2.143)$$

$$H_{\omega\omega} = x^2 u''(W_A - \pi - x + x\omega) \langle \lambda_y, f(x) \rangle < 0, \quad a.e. \ x \in I_x, \quad (2.144)$$

implying that  $J(x) = 1$  for a.e.  $x \in I_x$ . Again by Remark 2.1, if  $\phi_P$  is strictly concave then the admissible set is empty, violating Proposition 2.1. This implies that if  $\mu_h = 0$  at an optimum if and only if  $\phi_P$  is identity (i.e., if the insurer is ambiguity-neutral). In this case, the admissible set is a singleton containing only the pair  $(J = 1, \pi = \overline{\pi})$ , which is then trivially optimal.

### 2.7.6 Proof of Lemma 2.4

Since  $\psi$  and  $u$  are at least  $C^2$ , the function  $\Sigma_x$  is differentiable, hence:

$$\frac{\partial \Sigma_x}{\partial \xi} = \frac{\psi''(\xi)u'(W_A - \pi - x + \xi) - (1 + \psi'(\xi))u''(W_A - \pi - x + \xi)}{(u''(W_A - \pi - x + \xi))^2}, \quad (2.145)$$

which is strictly positive for all  $x \in I_x$  since the cost function  $\psi$  is strictly increasing and convex (Assumption 2.7), and  $u$  is strictly increasing and strictly concave (Assumption 2.8).

### 2.7.7 Proof of Proposition 2.3

Since the Hamiltonian is strictly concave in  $\omega$  for a.e.  $x \in I_x$ , one of the following must occur:

- either  $H_\omega(x)|_{\omega=0} \leq 0 \iff K(x) \leq 0$ , then  $J(x) = 0$ ;
- or  $H_\omega(x)|_{\omega=1} \geq 0 \iff L(x) \geq 0$ , then  $J(x) = 1$ ;
- or  $K(x) > 0$  and  $L(x) < 0$ , then  $J(x) \in (0, 1)$  is characterized by:

$$u'(W_A(x))G(x) = 1 + \psi'(xJ(x)), \quad (2.146)$$

which by Lemma 2.4 and the definition of  $\Xi_x$  implies:

$$xJ(x) = \Xi_x(G(x)). \quad (2.147)$$

We can then recover the optimal indemnity function  $I$  in (2.47) associated with the optimal coverage function  $J$  in (2.46) by recalling that  $I(x) = 0$  for  $x = 0$  and  $I(x) = xJ(x)$  for  $x > 0$ . The coinsurance rate (2.48) is obtained by differentiating (2.146) with respect to  $x$  upon substituting  $I(x) = xJ(x)$  into the expression. In particular,

$$u'(W_A - \pi - x + I(x))G(x) = 1 + \psi'(I(x)), \quad (2.148)$$

$$u''(W_A(x))(I'(x) - 1)G(x) + u'(W_A(x))G'(x) = -\psi''(I(x))I'(x), \quad (2.149)$$

implying that:

$$\underbrace{\frac{-u''(W_A(x))}{u'(W_A(x))}}_{r_u(W_A(x))} (1 - I'(x)) + \frac{G'(x)}{G(x)} = \frac{\psi''(I(x))I'(x)}{1 + \psi'(I(x))}, \quad (2.150)$$

which yields (2.48) upon gathering  $I'(x)$  and simplifying.



### 2.7.8 Proof of Proposition 2.4

The special cases associated to either  $\lambda_0 = 0$  or  $\mu_h = 0$  are discussed in Lemma 2.3. In particular, the optimal contract is the trivial one ( $I = 0, \pi = \underline{\pi}$ ) if  $\lambda_0 = 0$ , or the uniformly full insurance one ( $I = x, \pi = \bar{\pi}$ ) if  $\mu_h = 0$ . For  $\mu_h > 0$  and  $\lambda_0 > 0$ , the optimal premium takes an interior value  $\pi \in (\underline{\pi}, \bar{\pi})$ . Invoking Proposition 2.2 for  $\phi'_p = \phi'_A = 1$  (the DMs are ambiguity-neutral), we can write:

$$\int_{I_x} u'(W_A - \pi - x + I(x)) \tilde{\lambda}_0 \bar{f}(x) dx = 1 \quad (2.151)$$

Since  $x \geq I(x)$  for all  $x \in I_x$  and  $u$  is concave:

$$u'(W_A - \pi) \leq u'(W_A - \pi - x + I(x)), \quad \forall x \in I_x, \quad (2.152)$$

with strict inequality whenever  $x > I(x)$ . Hence

$$\begin{aligned} \tilde{\lambda}_0 \int_{I_x} u'(W_A - \pi - x + I(x)) \bar{f}(x) dx &\geq \tilde{\lambda}_0 u'(W_A - \pi) \\ 1 &\geq u'(W_A - \pi) \tilde{\lambda}_0. \end{aligned} \quad (2.153)$$

Since  $\psi' > 0$ , in view of (2.71) we have:

$$K(0) = u'(W_A - \pi) \tilde{\lambda}_0 - (1 + \psi'(0)) < 0. \quad (2.154)$$

Note that since  $u$  and  $\psi$  are continuously differentiable, the function  $K$  in (2.71) is also continuously differentiable. Furthermore by the strict concavity of  $u$ ,

$$K'(x) = -u''(W_A - \pi - x) > 0, \quad (2.155)$$

implying that  $K$  is continuous and strictly increasing on  $I_x$ . Hence  $K$  is strictly increasing and satisfies  $K(0) < 0$ . If  $K(\bar{x}) \leq 0$ , then  $K(x) \leq 0$  on  $I_x$ , implying that  $I(x) = 0$  for all  $x \in I_x$ , which is not optimal for  $\pi > 0$  by Lemma 2.3. Hence  $K(\bar{x}) > 0$ . By continuity there exists  $x_1 \in (0, \bar{x})$ , called the deductible, such that  $K(x_1) = 0$ . By the strict monotonicity of  $K$  the deductible  $x_1$  is unique. Now since  $K$  is strictly increasing, for all losses below  $x_1$  we have  $K(x) \leq 0$ , or  $H_\omega|_{\omega=0}(x) \leq 0$ , implying that  $I(x) = 0$  for all  $x \leq x_1$ . Likewise for all  $x > x_1$ , we have  $H_\omega|_{\omega=0}(x) > 0$ , implying that  $I(x) > 0$  for for such losses. Observe that under ambiguity neutrality,

$$H_\omega|_{\omega=1}(x) \equiv L(x) = u'(W_A - \pi) \tilde{\lambda}_0 - (1 + \psi'(x)), \quad (2.156)$$

which is continuous and differentiable with respect to  $x$ . Since  $\psi'$  is convex,

$$L'(x) = -\psi''(x) \leq 0, \quad \forall x \in (0, \bar{x}), \quad (2.157)$$

with strict inequality if  $\psi$  is strictly convex. This implies  $L(\bar{x}) \leq L(0)$ . But  $L(0) = K(0)$ , which is strictly negative as previously shown, implying that  $L(x) < 0$  for all  $x \in (0, \bar{x})$ . Hence full insurance  $I(x) = x$  is never reached for losses beyond  $x_1$ . Therefore for  $x \in (x_1, \bar{x}]$ , we have  $I(x) \in (0, x)$  satisfying

$$\begin{aligned} u'(W_A - \pi - x + I(x))\tilde{\lambda}_0 &= 1 + \psi'(I(x)) \\ I(x) &= \Xi_x(\tilde{\lambda}_0). \end{aligned} \quad (2.158)$$

Finally, the co-insurance equation (2.74) is obtained from (2.48) for  $G' = 0$ . This completes the proof.

### 2.7.9 Proof of Corollary 2.1

The shape of the indemnity function (2.77) can be obtained immediately by solving the initial value problem:

$$I'(x) = 1, \quad I(x_1) = 0. \quad (2.159)$$

The associated premium (2.78) is obtained by substituting (2.77) into the equality constraint (2.73).

### 2.7.10 Proof of Proposition 2.5

For simplicity and comparability with Raviv (1979), let us normalize  $\lambda_0 = 1$ . In this case (2.151) simplifies to:

$$\int_{I_x} u'(W_A - \pi - x + I(x))\bar{f}(x)dx = \mu_h. \quad (2.160)$$

By Corollary 2.1, the contract in this case has the shape of a straight deductible of size  $x_1$ , which is characterized by:

$$K(x_1) = 0 \iff u'(W_A - \pi - x_1) = \mu_h(1 + L). \quad (2.161)$$

Equation (2.160) can be rewritten as:

$$\int_0^{x_1} u'(W_A - \pi - x)\bar{f}(x)dx + \int_{x_1}^{\bar{x}} u'(W_A - \pi - x_1)\bar{f}(x)dx = \mu_h. \quad (2.162)$$

Observe that by IBP, the first term on the LHS becomes:

$$u'(W_A - \pi - x_1)\bar{F}(x_1) + \int_0^{x_1} u''(W_A - \pi - x)\bar{F}(x)dx, \quad (2.163)$$

where  $\bar{F}(x) \equiv \int_0^x \bar{f}(t)dt$  is the ambiguity-neutral cumulative distribution function (cdf), with  $\bar{F}(0) = 0$ . Likewise, using  $\bar{F}(\bar{x}) = 1$ , the second term on the LHS of (2.162) can be rewritten as:

$$(1 - \bar{F}(x_1))u'(W_A - \pi - x_1). \quad (2.164)$$

Combining (2.163) and (2.164) yields:

$$u'(W_A - \pi - x_1) + \int_0^{x_1} u''(W_A - \pi - x)\bar{F}(x)dx = \mu_h. \quad (2.165)$$

We can now use (2.161) to substitute out  $\mu_h$  in (2.165), which gives:

$$\frac{L}{L+1} = \int_0^{x_1} \frac{-u''(W_A - \pi - x)}{u'(W_A - \pi - x_1)}\bar{F}(x)dx. \quad (2.166)$$

Since the density is strictly positive everywhere and the policyholder is strictly risk-averse, the term inside the integral on the RHS of the expression above is strictly positive. It is then immediate to see that  $x_1 = 0$  if and only if  $L = 0$ .

### 2.7.11 Proof of Lemma 2.5

The independence of the monotonic behavior of  $G$  with respect to  $x$  is immediate from (2.82) upon recalling that  $\ell'_{12}(x) \leq 0$  for all  $x$  in  $I_x$  by Assumption 2.5, and that  $S(\lambda_y, \lambda_z)$  is independent of  $x$ . In particular the sign of  $G'$  is opposite to the sign of  $S(\lambda_y, \lambda_z)$ . Next, using (2.57) and (2.56) for  $n = 2$  yields:

$$z_1(\bar{x}) - z_2(\bar{x}) = \int_{I_x} I'(x) [1 + \psi'(I(x))] (F_1(x) - F_2(x)) dx, \quad (2.167)$$

$$y_1(\bar{x}) - y_2(\bar{x}) = \int_{I_x} [1 - I'(x)] u'(W_A(x)) (F_1(x) - F_2(x)) dx, \quad (2.168)$$

where  $W_A(x) = W_A - \pi - x + I(x)$ . Recall that  $F_1(x) - F_2(x) \geq 0$  on  $I_x$  is implied by Assumption 2.5. Let us consider two cases.

Suppose that  $G' \geq 0$ . In this case  $I' \geq 0$  on  $I_x$ , implying that  $z_1(\bar{x}) \geq z_2(\bar{x})$  in view of (2.167). But this implies, via (2.80) that  $P' \geq 0$  on  $I_x$ . Observe that (2.59) is equivalent to

$$\frac{A'}{A} = \frac{G'}{G} + \frac{P'}{P}, \quad (2.169)$$

which must be positive since both terms on the RHS are positive. Hence  $A' \geq 0$  since

$A$  is strictly positive.

Suppose that  $G' \leq 0$ . In this case  $I'(x) \leq 1$  on  $I_x$ , implying via (2.168) that  $y_1(\bar{x}) \geq y_2(\bar{x})$ . Hence (2.79) implies  $A' \geq 0$  on  $I_x$ . Again from (2.59) we can write:

$$\frac{P'}{P} = -\frac{G'}{G} + \frac{A'}{A}, \quad (2.170)$$

implying that  $P' \geq 0$ . This completes the proof.

### 2.7.12 Proof of Theorem 2.2

Observe that  $K$  and  $L$  defined in (2.38) and (2.39), respectively, are continuous and differentiable. We have:

$$K'(x) = -u''(W_A - \pi - x)G(x) + u'(W_A - \pi - x)G'(x), \quad (2.171)$$

$$L'(x) = u'(W_A - \pi)G'(x) - \psi''(x). \quad (2.172)$$

Notice that if  $G' \geq 0$ , then  $K'$  above is strictly positive since  $G$  is strictly positive and  $u$  is strictly increasing and strictly concave (Assumption 2.8). Moreover, the monotonicity of  $G$  also implies:

$$G(0) \leq G(x), \quad x \in I_x. \quad (2.173)$$

In addition, Assumption 2.6 and Assumption 2.8 imply

$$u'(W_A - \pi) \leq u'(W_A(x)), \quad x \in I_x. \quad (2.174)$$

Since  $G$  and  $u'$  are strictly positive, conditions (2.173) and (2.174) imply

$$u'(W_A - \pi)G(0) \leq u'(W_A(x))G(x), \quad x \in I_x. \quad (2.175)$$

Taking expectation with respect to the density  $\tilde{f}$  on both sides yields:

$$u'(W_A - \pi)G(0) \leq \int_{I_x} u'(W_A(x))G(x)\tilde{f}(x)dx \quad (2.176)$$

$$u'(W_A - \pi)G(0) \leq 1, \quad (2.177)$$

where the second line follows from (2.84). Since  $\psi' > 0$  (Assumption 2.7), this implies:

$$u'(W_A - \pi)G(0) - (1 + \psi'(0)) < 0 \quad (2.178)$$

$$K(0) < 0. \quad (2.179)$$

If  $K(\bar{x}) \leq 0$ , then  $K(x) \leq 0$  on  $I_x$  since  $K$  is continuous, strictly increasing and  $K(0) < 0$ . In this case  $I(x) = 0$  on  $I_x$ , which constitutes an optimum if and only if  $\pi = \underline{\pi} = 0$  by Lemma 2.3, contradicting the hypothesis that  $\pi$  takes an interior value. Hence  $K(\bar{x}) > 0$ , implying (by continuity and strict monotonicity) that there exists a unique deductible  $x_1 \in (0, \bar{x})$  such that  $K(x_1) = 0$ .

For losses beyond the deductible, consider  $L$  in (2.39). Note that by the strict concavity of the Hamiltonian,  $K(x_1) > L(x_1)$ , implying that  $L(x_1) < 0$ . By continuity  $L(x) < 0$  at least on a sufficiently small open interval to the right of  $x_1$ . Denote this interval  $(x_1, x_1 + \epsilon)$ , then on  $(x_1, x_1 + \epsilon)$  we have  $K(x) > 0$  and  $L(x) < 0$ , implying that  $I(x) \in (0, x)$  and is characterized by  $I(x) = \Xi_x(G(x))$ . Beyond  $x_1 + \epsilon$  the shape of the indemnity function depends on the monotonic behavior of  $L$ . In view of (2.172), let us consider the following cases.

1. If  $\psi$  is strictly convex, the sign of  $L'$  is indeterministic and our analysis reaches an impasse.
2. If  $\psi$  is linear, then  $L'(x) \geq 0$ , implying that  $L$  is increasing. We know that  $L(x_1) < 0$ . Consider the following subcases.
  - a. If  $L(\bar{x}) < 0$  then monotonicity implies  $L(x) < 0$  on  $(x_1, \bar{x}]$ . Therefore on  $(x_1, \bar{x}]$  the indemnity function satisfies  $I(x) \in (0, x)$  and is characterized by  $I(x) = \Xi_x(G(x))$ . In other words, the indemnity function has the form (2.89).
  - b. If  $L(\bar{x}) \geq 0$  then by continuity the equation  $L(x) = 0$  has a solution. Denote

$$S = \{x \in (x_1, \bar{x}] \mid L(x) = 0\}. \quad (2.180)$$

It is easy to see that  $S$  is closed and bounded. Then we can uniquely define  $x_2$  as the smallest element of  $S$ . In this case since  $L$  is increasing we have  $L(x) \geq L(x_2) = 0$  for all  $x \in [x_2, \bar{x}]$  and  $L(x) < 0$  for all  $x \in (x_1, x_2)$ . Hence beyond  $x_1$ , the indemnity function is characterized by:

$$I(x) = \begin{cases} \Xi_x(G(x)) & x \in (x_1, x_2), \\ x & x \in [x_2, \bar{x}]. \end{cases} \quad (2.181)$$

Therefore in this case the indemnity function has the form (2.88).

### 2.7.13 Proof of Proposition 2.6

It is immediate to see that in this case

$$G(x) = \frac{A(x)}{\mu_h}, \quad x \in I_x. \quad (2.182)$$

Consequently  $G'(x) = \frac{A'(x)}{\mu_h}$ , implying that  $G' \geq 0$  on  $I_x$  by Lemma 2.5. The rest of the proof follows that of Theorem 2.2 verbatim.



# 3

## The principal-agent model under smooth ambiguity

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## Chapter nomenclature

Notation	Meaning	Reference page(s)
$\mathcal{I}$	Index set of (second-order) states	76
$p_i$	Prior on state $i \in \mathcal{I}$	76
$\tilde{x}$	Outcome random variable	76
$e$	Effort level	76
$I_e$	Domain of effort	76
$f_i(x e)$	State- $i$ effort- $e$ conditional density	76
$F_i(x e)$	State- $i$ effort- $e$ conditional density	76
$I_x$	Common support of the conditional densities	76
$c(\cdot)$	Cost of exerting effort	77
$w(\cdot)$	Wage (the control function)	77
$\ell_{ij}(\cdot e)$	Effort-conditional likelihood ratio	77
$v(\cdot)$	Principal's utility function	78
$\phi_P(\cdot)$	Principal's second-order utility (welfare) function	78
$u(\cdot)$	Agent's utility function	78
$\phi_A(\cdot)$	Agent's second-order utility (welfare) function	78

Table 3.1: Notations used in Chapter 3

### 3.1 Introduction

It is not hard to imagine that when it comes to work, there are at least two kinds. One that is common or has been repeated long enough that the distribution of outcomes is known, and the other which involves jobs at the forefront of innovation whose distribution of outcomes is often ambiguous. Examples of the former category include simple office tasks, manual factory work, etc. Jobs in research and development, in creative domains are typical of the latter category. In this chapter, we are interested in the kind of labor contract that concerns jobs with ambiguous results. This kind of context has been analyzed by Ghirardato (1994) using Schmeidler (1989)'s Choquet capacity (non-additive measure), by Kellner (2017) using smooth ambiguity aversion in the framework of Grossman and Hart (1992). Our work adopts the principal-agent formulation of Holmstrom (1979), and models ambiguity preferences according to Klibanoff et al. (2005). In contrast to the existing literature, our solution characterizes *simultaneously* the optimal wage function *and* the level of effort demanded of the agent under symmetric information (first-best result). Furthermore, we treat the problem in the most general form, allowing for the principal to be either neutral or averse to risk and/or ambiguity. This is distinct from the existing literature which often assumes that the principal is neutral to both risk and ambiguity. Nevertheless, empirical evidence has suggested that the principals themselves might be ambiguity-averse (Kunreuther and Hogarth, 1992; Kunreuther et al., 1993, 1995; Cabantous, 2007; Cabantous et al., 2011).

To handle this problem, we again employ the technical tools from optimal control theory with major references from Trélat (2008) and Clarke (1990). The rest of the chapter is organized as follows. Section 3.2 states the principal's optimal contract problem and key assumptions. Section 3.3 reformulates the principal's problem as an optimal control one. 3.4 provides an existence proof of an optimal wage contract in the most general case. Having proved existence, we set out to characterize the optimal contract in Section 3.5. Section 3.6 considers the special case of two ambiguous states. Section 3.7 reconsiders the problem when the outcome is unbounded from above and Section 3.8 concludes.

### 3.2 Assumptions and formulation of the principal's problem

Consider a principal-agent model where decision makers (DMs), the agent and the principal, face ambiguity in the distribution of the states. Consequently, the distributions of outcomes are state-conditional. We assume that the state space is finite, and the DMs have common priors over the distribution of the states. Our objective is to determine an optimal wage contract under symmetric information.

**Notation 3.1.** Throughout this chapter, the subscripts  $A$  and  $P$  refer to the agent and the principal, respectively.

**Notation 3.2.** The  $n$ -dimensional Euclidean space whose elements have all non-negative coordinates is denoted by  $\mathbb{R}_+^n$ . The  $n$ -dimensional Euclidean space whose elements have all strictly positive coordinates are denoted by  $\mathbb{R}_{++}^n$ .

**Assumption 3.1.** Let the state space be  $\mathcal{I} = \{1, 2, \dots, n\}$ , where  $n < +\infty$ . Let  $p_i$  denotes the prior belief of both DMs regarding the likelihood of state  $i$  occurring. Assume that  $p_i \in (0, 1)$  for all  $i \in \mathcal{I}$  and  $\sum_{i=1}^n p_i = 1$ .

Let  $e \in I_e = [\underline{e}, \bar{e}] \subset \mathbb{R}_+$  be the level of effort/action to be implemented by the agent. The principal has to determine the desirable level of effort she demands of the agent. The effort exerted by the agent is assumed to be verifiable and legally enforceable. In other words, it is a valid contracting variable. Conditional on  $e$ , the outcome is assumed to be a continuous random variable  $\tilde{x}$  whose state-conditional distributions have common support  $I_x = [0, \bar{x}]$ . In particular, the following assumption holds.

**Assumption 3.2.** For each  $e \in I_e$  and  $i \in \mathcal{I}$ , let  $F_i(\cdot|e) : I_x \rightarrow [0, 1]$  be the conditional cumulative distribution function (cdf) of  $\tilde{x}$  defined by

$$F_i(x|e) = \Pr(\tilde{x} \leq x | e), \quad i \in \mathcal{I}.$$

Assume that all the cdfs are  $C^2$  on their common support  $I_x$ . Let  $f_i(\cdot|e) : I_x \rightarrow \mathbb{R}_{++}$  be the conditional probability density function (pdf) associated with  $F_i(\cdot|e)$  defined by

$$f_i(x|e) = \frac{\partial F_i(x|e)}{\partial x}, \quad i \in \mathcal{I}.$$

Then  $\int_{I_x} f_i(x|e) dx = 1$  for each  $i \in \mathcal{I}$ , and  $f(x|e) = (f_i(x|e))_{i \in \mathcal{I}} \in \mathbb{R}_{++}^n$  for all  $x \in I_x$ .

We assume that ambiguous states can be ranked according to the likelihood ratio (LR), as next defined.

**Assumption 3.3.** For two distinct indexes  $i, j$  in  $\mathcal{I}$ , let  $\ell_{ij}(\cdot|e) : I_x \rightarrow \mathbb{R}_+$  be the effort-conditional likelihood ratio defined by  $\ell_{ij}(x|e) = \frac{f_i(x|e)}{f_j(x|e)}$ . Then state  $i$  is said to dominate state  $j$  in the sense of likelihood ratio dominance (LRD) if  $\frac{\partial \ell_{ij}(x|e)}{\partial x} = \ell'_{ij}(x|e) \geq 0$  for a.e.  $x \in I_x$ , with strict inequality in some subset of positive measure of  $I_x$ .<sup>1</sup>

**Example 3.1.** Suppose  $n = 2$ ,  $\bar{x} = 1$  (so that any outcome is viewed as a fraction of the maximum outcome), and the outcome distribution follows a truncated exponential distribution

<sup>1</sup>Note that LRD is a special case of FSD. Thus  $F_i(\cdot)$  dominates  $F_j(\cdot)$  in the sense of LRD implies  $F_i(x) \leq F_j(x)$  for all  $x \in I_x$ , with strict inequality on some subset of  $I_x$  of positive measure. See Appendix A for further discussion.

with an ambiguous parameter.<sup>2</sup> In particular:

$$f_i(x|e) = \begin{cases} \alpha_i(e)\beta_i(e) \exp\{-\beta_i(e)x\} & x \in [0, 1], \beta_i(e) > 0, i \in \{1, 2\} \\ 0 & o.w. \end{cases},$$

where  $\alpha_i(e) = \frac{\exp(\beta_i(e))}{\exp(\beta_i(e))-1} > 1$ . The likelihood ratio  $\ell_{12}(x)$  is increasing if and only if

$$\begin{aligned} [\beta_2(e) - \beta_1(e)] \alpha_1(e)\alpha_2(e) \exp\{-(\beta_1(e) + \beta_2(e))x\} &\geq 0 \\ \iff \beta_2(e) - \beta_1(e) &\geq 0. \end{aligned}$$

Thus an amelioration in the LRD sense is equivalent to a reduction in the parameter of the exponential distribution. In other words, the more favorable state (state 1) is associated with a smaller parameter. At this point, we have not explicitly specified how effort changes this parameter. In general, this relationship also state-conditional. One might hypothesize that the higher the level of effort, the smaller the gap  $\delta(e) = \beta_2(e) - \beta_1(e)$ . Intuitively, this gap represents the severity, or the consequence of ambiguity. Letting  $\delta(e)$  decrease in  $e$  means believing that high efforts can mitigate the severity of ambiguity.

Next, we hypothesize that in any given state  $i$ , raising efforts improves the outcome distribution in following sense.

**Assumption 3.4.** Let the function  $\ell_i(\cdot) : I_x \rightarrow \mathbb{R}_+$  be the state conditional likelihood ratio defined by  $\ell_i(x) := \frac{f_i(x|e_2)}{f_i(x|e_1)}$ ,  $e_1 < e_2 \in I_e$ . Then  $\ell'_i(x) \geq 0$  a.e.  $x \in I_x$ , with strict inequality in some subset of positive measure of  $I_x$ .

**Assumption 3.5.** The wage is a measurable function  $w : I_x \rightarrow \mathbb{R}_+$  satisfying  $w(x) \in [0, x]$  for a.e.  $x \in I_x$ .<sup>3</sup>

**Assumption 3.6.** The cost of effort is a  $C^2$  function  $c : I_e \rightarrow \mathbb{R}_+$  satisfying  $c(0) = 0$ ,  $c' > 0$ ,  $c'' \geq 0$ .

We model the DMs' attitude towards risk by the von Neumann-Morgenstern utility functions. Recall that the utility function being concave, linear, or convex corresponds to a risk-averse, risk-neutral, or risk-seeking DM, respectively. Typically, the agent is

<sup>2</sup>Recall that if  $X$  is distributed as an exponential distribution of parameter  $\beta$ , its density is:

$$f(x) = \beta \exp\{-\beta x\}, \quad x \geq 0, \beta > 0.$$

For our purpose, we need to "redistribute" the mass over a bounded interval  $I_x$ , instead of the whole  $\mathbb{R}_+$ . This conditioning is achieved by dividing the original density by the cumulative mass contained in this interval, which in this example is  $F(1) = \int_0^1 f(x)dx = 1 - \exp(-\beta)$ .

<sup>3</sup>This assumption deviates from the mainstream of the existing literature that considers a global, rather than a point-wise constraint on  $w(\cdot)$ . We shall also consider the implication of this assumption in a separate later section.

assumed to be risk-averse and the principal risk-neutral. For greater generality, we allow for the possibility of the principal being risk-averse.

**Assumption 3.7.** *The agent has utility function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ , which is at least  $C^2$ , satisfying  $u(0) = 0$ ,  $u' > 0$ ,  $u'' < 0$ , and the Inada condition  $\lim_{w \rightarrow 0} u'(w) = +\infty$ . Similarly, the principal also has a  $C^2$  utility function  $v : \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfying  $v(0) = 0$ ,  $v' > 0$  and  $v'' \leq 0$ . If  $v'' < 0$ , then  $\lim_{w \rightarrow 0} v'(w) = +\infty$ .*

To capture the phenomenon known as ‘‘ambiguity aversion’’ postulated by Ellsberg (1961), we follow the smooth model of Klibanoff et al. (2005). We refer to this model as KMM (2005) from now on. According to KMM (2005), attitudes towards ambiguity can be modeled by a functional  $\phi_J$ , which is referred to as the welfare functional throughout this chapter. In particular, the welfare functional being concave, linear, or convex corresponds to a DM who is ambiguity-averse, ambiguity-neutral or ambiguity-seeking, respectively. The DMs are assumed to be either ambiguity-averse or ambiguity-neutral.

**Assumption 3.8.** *Let the welfare functional be  $\phi_J : \mathcal{V} \rightarrow \mathbb{R}$ , where  $\mathcal{V}$  is the range of  $J$ 's utility function, for  $J \in \{A, P\}$ . For each  $J \in \{A, P\}$ , assume that  $\phi_J$  is at least  $C^2$  on its domain, satisfying  $\phi_J' > 0$  and  $\phi_J'' \leq 0$ .*

**Example 3.2.** *Following the empirical work of Chakravarty and Roy (2009) and more recently of Berger and Bosetti (2016), we can let  $\phi_J(u) = u^{1-\sigma}$ ,  $u \geq 0$  where  $\sigma \in [0, 1)$  represents the degree of relative ambiguity aversion (RAA) and  $\sigma = 0$  corresponds to  $J$  being neutral to ambiguity.*

The agent extracts satisfaction from wage and dissatisfaction from exerting efforts (there is no utility coming from work other than that from the payment). The principal, on the other hand, cares only about profits (outcomes net of compensation to the agent). Assuming that the cost of effort and the welfare of the agent are separable, the principal's problem in absence of moral hazard is:

$$\max_{w(\cdot), e} \sum_{i=1}^n p_i \phi_P \left( \int_0^{\bar{x}} (x - w(x)) f_i(x|e) dx \right) \quad (3.1)$$

s.t.

$$w(x) \in [0, x] \quad \forall x \in I_x,$$

$$e \in I_e,$$

$$\sum_{i=1}^n p_i \phi_A \left( \int_0^{\bar{x}} u(w(x)) f_i(x|e) dx \right) - c(e) \geq \bar{U},$$

where  $\bar{U} \in \mathbb{R}$  is the reservation welfare of the agent (representing her outside option),  $\bar{U} \geq \phi_A(0) \equiv \bar{\phi}_A \in \mathbb{R}$ .

### 3.3 Formulation of the optimal control problem

In this section, we reformulate the optimization problem of the principal-agent model in the form of an optimal control problem (OCP) following Trélat (2008). To this end let the state vector  $X = (z, y, e) \in \mathcal{X} = \mathbb{R}_+^n \times \mathbb{R}_+^n \times I_e$  be defined as follows:

$$\dot{X} = \begin{pmatrix} \dot{z} \\ \dot{y} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} v(x - w(x)) f(x|e) \\ u(w(x)) f(x|e) \\ 0 \end{pmatrix}, \quad X(0) = \begin{pmatrix} z(0) = 0 \\ y(0) = 0, \\ e(0) = e \in I_e \end{pmatrix}, \quad (3.2)$$

where  $f(x|e) = (f_i(x|e))_{i \in \mathcal{I}} \in \mathbb{R}_{++}^n$  and  $I_e = [\underline{e}, \bar{e}] \subset \mathbb{R}_+$ .

Let  $M(I_x) = \{w : I_x \rightarrow I_x \text{ measurable}\}$  be the set of measurable controls, and  $\mathcal{U}$  be the set of admissible controls defined by <sup>4</sup>

$$\mathcal{U} = \{w \in M(I_x) \mid w(x) \in [0, x] \text{ a.e. } x \in I_x\}.$$

**Lemma 3.1.** *The set  $\mathcal{U}$  is compact with respect to the weak- $\star$  topology.*

*Proof.* We want to show that every sequence  $w^k$  in  $\mathcal{U}$  weak- $\star$  converges to  $\bar{w}$  in  $\mathcal{U}$ , up to a subsequence. Arguing by contradiction, we suppose that  $\bar{w} \notin \mathcal{U}$  i.e., there exists a measurable set  $J \subset I_x$  of positive measure such that  $\bar{w}(x) > x$  for all  $x \in J$ . Let  $\chi_J : I_x \rightarrow I_x$  be a characteristic function defined by

$$\chi_J(x) = \begin{cases} 1 & x \in J \\ 0 & x \notin J \end{cases}.$$

By assumption of weak- $\star$  convergence, we have, as  $k$  tends to infinity:

$$\begin{aligned} \int_{I_x} \chi_J(x) w^k(x) dx &\longrightarrow \int_{I_x} \chi_J(x) \bar{w}(x) dx \\ \iff \int_{I_x} \chi_J(x) (w^k(x) - x) dx &\longrightarrow \int_{I_x} \chi_J(x) (\bar{w}(x) - x) dx \\ \iff \int_J (w^k(x) - x) dx &\longrightarrow \int_J (\bar{w}(x) - x) dx, \end{aligned} \quad (3.3)$$

which yields a contradiction since the RHS of (3.3) is strictly positive by hypothesis, while the LHS is negative by construction. Thus there exists no such set  $J$ , implying that  $\bar{w}$  is in  $\mathcal{U}$ , completing the proof.  $\blacksquare$

<sup>4</sup>Specifically, let  $\sigma_I$  be a  $\sigma$ -algebra on  $I_x$ , then  $(I_x, \sigma_I)$  is a measurable space. The function  $w : I_x \rightarrow I_x$  is called measurable if, for all  $I$  in  $\sigma_I$ , the preimage of  $I$  under  $w$  is also in  $\sigma_I$  where the preimage of  $I$  under  $w$  is the set  $\text{preim}_w(I) = \{x \in I_x \mid w(x) \in I\}$ .

We now proceed to define the OCP. To this end, let the cost functional be:

$$g(w, e) = - \sum_{i=1}^n p_i \phi_P(z_i(\bar{x})), \quad (3.4)$$

which is just minus the welfare functional of the principal, and the net welfare functional of the agent be:

$$h(w, e) = \sum_{i=1}^n p_i \phi_A(y_i(\bar{x})) - c(e) - \bar{U}. \quad (3.5)$$

Under the new notation, the equivalent statement of the original optimization problem (3.1) is

$$\begin{aligned} & \min_{w(\cdot), e} g(w, e) \\ & \text{s.t.} \\ & h(w, e) \geq 0. \end{aligned} \quad (3.6)$$

We shall refer to this problem as the OCP in the sequel.

**Lemma 3.2.** *The constraint is active at an optimum.*

*Proof.* If it is not the case and  $w(\cdot)$  is optimal, then since  $\phi_A$  is continuous, there exists  $\epsilon > 0$  such that

$$\begin{aligned} & \sum_{i=1}^n p_i \phi_A \left( \int_0^{\bar{x}} u(w(x)) f_i(x|e) dx - \epsilon \right) - c(e) > \bar{U} \\ \iff & \sum_{i=1}^n p_i \phi_A \left( \int_{I_x} (u(w(x)) - \epsilon) f_i(x|e) dx \right) - c(e) > \bar{U}. \end{aligned}$$

Since all the prior densities  $f_i$  are strictly positive and  $u(\cdot)$  is continuous, there exist a subset  $K \subset I_x$  of positive measure and some sufficiently small number  $\epsilon_K > 0$  satisfying  $w(x) - \epsilon_K \geq 0$  for all  $x \in K$  such that:

$$\sum_{i=1}^n p_i \phi_A \left( \int_K u(w(x)) - \epsilon_K f_i(x|e) dx + \int_{I_x \setminus K} u(w(x)) f_i(x|e) dx \right) \geq c(e) + \bar{U}.$$

Define the function  $\tilde{w} : I_x \rightarrow I_x$  by  $\tilde{w}(x) = \begin{cases} w(x) - \epsilon_K & x \in K \\ w(x) & x \in I_x \setminus K \end{cases}$ . Then  $\tilde{w}(\cdot)$  is both

admissible and satisfies the constraint by construction. In addition, since  $v(\cdot)$  is also continuous and strictly increasing, and all the prior densities are strictly positive, we must have  $\int_K v(\tilde{w}(x)) f_i(x|e) dx > \int_K v(w(x)) f_i(x|e) dx$  for all  $i \in \mathcal{I}$ . Finally, since  $p_i > 0$

for all  $i \in \mathcal{I}$  and  $\phi_P(\cdot)$  is strictly increasing:

$$\sum_{i=1}^n p_i \phi_P \left( \int_{I_x} v(x - \tilde{w}(x)) f_i(x|e) dx \right) > \sum_{i=1}^n p_i \phi_P \left( \int_0^{\bar{x}} (x - w(x)) f_i(x|e) dx \right),$$

implying that  $w(\cdot)$  is not optimal, a contradiction. Hence if  $w(\cdot)$  is optimal, we must have:

$$\sum_{i=1}^n p_i \phi_A \left( \int_0^{\bar{x}} u(w(x)) f_i(x|e) dx \right) - c(e) = \bar{U}.$$

■

### 3.4 Existence of optimal wage

**Remark 3.1.** We discuss briefly the two extreme cases where one of the state variables might have zero value.

Recall that  $y_i(\bar{x}) = \int_0^{\bar{x}} u(w(x)) f_i(x|e) dx$  where  $f_i > 0$  for all  $i \in \mathcal{I}$ . Thus, if there exists  $i_* \in \mathcal{I}$  such that  $y_{i_*}(\bar{x}) = 0$  then  $u(w(x)) = 0$  for a.e.  $x \in I_x$ . But this implies  $y_i(\bar{x}) = 0$  for all  $i \in \mathcal{I}$ . Since  $u$  is strictly increasing and  $u(0) = 0$ , we must have in this case  $w(x) = 0$  for a.e.  $x \in I_x$ , which is obviously the wage schedule that costs the least to the principal. But this wage satisfies the participation constraint if and only if  $\bar{\phi}_A - c(e) \geq \bar{U} \iff c(e) = 0$  since  $\bar{U} \geq \bar{\phi}_A$  by assumption. In sum,  $y_i(\bar{x}) = 0$  for some  $i \in \mathcal{I}$  if and only if  $c(e) = 0 \iff e = 0$ , which is not an economically interesting case. If  $e > 0$ , uniformly zero wage does not satisfy the participation constraint and thus is not admissible.

By the same reasoning,  $z_i(\bar{x}) = 0$  for some  $i \in \mathcal{I}$  if and only if  $w(x) = x$  for a.e.  $x \in I_x$ , which in turn implies  $z_i(\bar{x}) = 0$  for all  $i \in \mathcal{I}$ . This is the most expensive wage to implement for the principal. If this wage satisfies the participation constraint with strict inequality, then by the same argument made under the proof of Lemma 3.2, it is not optimal. On the other hand, if this wage satisfies the participation constraint with equality, then it is the only admissible candidate for a solution, and thus is trivially optimal.

Both of these cases are discussed for technical reasons but are not very interesting economically. For this reason, we shall assume in the sequel that  $y(\bar{x}) \in \mathbb{R}_{++}^n$  and  $z(\bar{x}) \in \mathbb{R}_{++}^n$ .

Let  $\mathcal{M}_0$  and  $\mathcal{M}_1$  be measurable subsets of the state space  $\mathcal{X}$  defined as

$$\mathcal{M}_0 = \{0\} \times \{0\} \times I_e, \quad (3.7)$$

$$\mathcal{M}_1 = \mathbb{R}_{++}^n \times S_{y,e}, \quad (3.8)$$

where  $S_{y,e} = \{y \in \mathbb{R}_{++}^n \times I_e \mid -h(w, e) \leq 0\}$ .

Our objective is to find a trajectory  $X(\cdot)$  defined on  $I_x$  which solves (3.2) and corre-



sponds to an admissible control  $w \in \mathcal{U}$  satisfying

$$X(0) \in \mathcal{M}_0, X(\bar{x}) \in \mathcal{M}_1,$$

such that the cost functional is minimized over all possible trajectories  $X(\cdot)$  linking  $\mathcal{M}_0$  to  $\mathcal{M}_1$ .

**Proposition 3.1.** *The OCP admits an optimal control.*

*Proof.* Let  $\delta = \inf_{(w,e) \in \mathcal{U} \times I_e} g(w, e)$ . Consider a sequence of trajectories  $\{X^k(\cdot)\}_{k \in \mathbb{N}}$  associated with the sequence of admissible controls  $\{w^k(\cdot)\}_{k \in \mathbb{N}}$  defined by

$$X^k(x) = \begin{pmatrix} z^k(x) \\ y^k(x) \\ e^k \end{pmatrix} = \begin{pmatrix} (z_i^k(x))_{i \in \mathcal{I}} \\ (y_i^k(x))_{i \in \mathcal{I}} \\ e^k \end{pmatrix}, \quad x \in I_x, \quad (3.9)$$

such that  $g(w^k, e^k) \rightarrow \delta$  as  $k \rightarrow \infty$ , where

$$\begin{aligned} z_i^k(x) &= \int_0^x v(t - w^k(t)) f_i(t|e^k) dt, \quad \forall i \in \mathcal{I}, \\ y_i^k(x) &= \int_0^x u(w^k(t)) f_i(t|e^k) dt, \quad \forall i \in \mathcal{I}. \end{aligned}$$

By the weak- $\star$  compactness of  $\mathcal{U}$ , the sequence  $\{w^k(\cdot)\}_{k \in \mathbb{N}}$  weak- $\star$  converges to  $\bar{w}(\cdot) \in \mathcal{U}$  up to some subsequence, i.e.  $w^k(\cdot) \rightharpoonup \bar{w}(\cdot)$ . By the compactness of  $I_e$ , the sequence  $\{e^k\}_{k \in \mathbb{N}}$  converges to  $e_* \in I_e$ , up to some subsequence. Denote the limiting trajectory as:

$$\bar{X}(x) = \begin{pmatrix} \bar{z}(x) \\ \bar{y}(x) \\ e_* \end{pmatrix} = \begin{pmatrix} (\bar{z}_i(x))_{i \in \mathcal{I}} \\ (\bar{y}_i(x))_{i \in \mathcal{I}} \\ e_* \end{pmatrix}, \quad x \in I_x \quad (3.10)$$

where

$$\begin{aligned} \bar{z}_i(x) &= \int_0^x v(t - \bar{w}(t)) f_i(t|e_*) dt, \quad i \in \mathcal{I}, \\ \bar{y}_i(x) &= \int_0^x u(\bar{w}(t)) f_i(t|e_*) dt, \quad i \in \mathcal{I}. \end{aligned}$$

We first show that this limiting trajectory also brings the system from  $\mathcal{M}_0$  to  $\mathcal{M}_1$ , which is completed by Lemma 3.3. Second, we show that the control associated with this limiting trajectory is optimal, which is the result of Lemma 3.4.

**Lemma 3.3.** *We have  $h(\bar{w}, e_*) \geq 0$ .*

*Proof.* By construction,  $h(w^k, e^k) \equiv \sum_{i=1}^n p_i \phi_A(y_i^k(\bar{x})) - c(e^k) - \bar{U} \geq 0$  for all  $k \in \mathbb{N}$ . For

$i \in \mathcal{I}$  and  $k \geq 0$ , let us write

$$y_i^k(\bar{x}) = \int_0^{\bar{x}} u(w^k(t)) f_i(t|e_*) dt + \Delta_i^k, \quad (3.11)$$

where  $\Delta_i^k = \int_0^{\bar{x}} u(w^k(t)) (f_i(t|e^k) - f_i(t|e_*)) dt$ . It is immediate to see that  $\Delta_i^k$  tends to zero as  $k$  tends to infinity since  $u$  is bounded and  $f_i$  defined on the compact set  $I_x \times I_e$  is uniformly continuous.

From Lee and Markus (1967), we have, for a convex function  $\Gamma_i$ :

$$\int_{I_x} \Gamma_i(\bar{w}(x)) dx \leq \liminf \int_{I_x} \Gamma(w^k(x)) dx. \quad (3.12)$$

Let  $\Gamma_i(w^k(x)) \equiv -u(w^k(x)) f_i(x|e_*)$ . Then  $\Gamma_i$  is convex with respect to  $w^k$  since  $u$  is concave and  $f_i$  is positive and does not depend on  $w^k$ <sup>5</sup>. We can rewrite (3.12) as:

$$\begin{aligned} -\bar{y}_i(\bar{x}) &\leq \liminf (-y_i^k(\bar{x}) - \Delta_i^k) \\ \iff \bar{y}_i(\bar{x}) &\geq \limsup y_i^k(\bar{x}). \end{aligned} \quad (3.13)$$

By the continuity of  $\bar{y}_i$ , for all  $\epsilon > 0$ , there exists a sufficiently large positive integer  $k$  such that  $\bar{y}_i(\bar{x}) \geq y_i^k(\bar{x}) - \epsilon$ . Since  $\phi_A$  is increasing,

$$\phi_A(\bar{y}_i(\bar{x})) \geq \phi_A(y_i^k(\bar{x}) - \epsilon). \quad (3.14)$$

By the first fundamental theorem of calculus,

$$\phi_A(y_i^k(\bar{x})) - \phi_A(y_i^k(\bar{x}) - \epsilon) = \int_a^b \phi'_A(\zeta) d\zeta,$$

where  $b = y_i^k(\bar{x})$  and  $a = b - \epsilon$ . Let  $M \in \mathbb{R}_+$  be an upper bound of  $\phi'_A$  over  $[a, b]$ . Then  $\phi_A(y_i^k(\bar{x})) - \phi_A(y_i^k(\bar{x}) - \epsilon) \leq M(b - a) = M\epsilon$ , implying that  $\phi_A(y_i^k(\bar{x}) - \epsilon) \geq \phi_A(y_i^k(\bar{x})) - M\epsilon$ , which together with (3.14) implies

$$\begin{aligned} \phi_A(\bar{y}_i(\bar{x})) &\geq \phi_A(y_i^k(\bar{x})) - M\epsilon \\ \implies \sum_{i=1}^n p_i \phi_A(\bar{y}_i(\bar{x})) &\geq \sum_{i=1}^n p_i \phi_A(y_i^k(\bar{x})) - M\epsilon. \end{aligned} \quad (3.15)$$

Since  $\phi_A(y_i^k(x)) \geq c(e^k) + \bar{U} \geq c(e_*) + \bar{U} - \epsilon$  for  $k$  large enough, from (3.15) we have  $\sum_{i=1}^n p_i \phi_A(\bar{y}_i(\bar{x})) \geq c(e_*) + \bar{U} - (M + 1)\epsilon$ . Finally, since  $\epsilon$  was arbitrary, letting  $\epsilon \rightarrow 0$

<sup>5</sup>In particular, let  $w_1, w_2$  be arbitrary functions in  $\mathcal{U}$ . For any  $\alpha \in [0, 1]$ , we have by the concavity of  $u$  that  $u(\alpha w_1(x) + (1 - \alpha)w_2(x)) \geq \alpha u(w_1(x)) + (1 - \alpha)u(w_2(x))$ . Multiplying both sides of the inequality by  $f_i(x|e^k)$  which is positive and does not depend on  $w$ , we see that  $u(w(x))f_i(x|e^k)$  is concave.

yields

$$\sum_{i=1}^n p_i \phi_A(\bar{y}_i(\bar{x})) \geq c(e_*) + \bar{U},$$

or equivalently,  $h(\bar{w}, e_*) \geq 0$ . ■

Next, we show that the sequence of welfare functional associated with  $\{w^k(\cdot)\}_k$  converges to the minimal cost.

**Lemma 3.4.** *Let  $\{g(w^k, e^k)\}_{k \in \mathbb{N}}$  be the sequence of cost functional defined by*

$$g(w^k, e^k) = - \sum_{i=1}^n p_i \phi_P(z_i^k(\bar{x})), \quad k \in \mathbb{N},$$

and let the cost at the limiting control  $\bar{w}$  be

$$g(\bar{w}, e_*) = - \sum_{i=1}^n p_i \phi_P(\bar{z}_i(\bar{x})).$$

Then  $\bar{w}$  is optimal, i.e.,  $g(\bar{w}, e_*) \leq \delta$ .

*Proof.* For  $i \in \mathcal{I}$  and  $k \geq 0$ , we write

$$z_i^k(\bar{x}) = \int_0^{\bar{x}} v(t - w^k(t)) f_i(t|e_*) dt + v_i^k, \quad (3.16)$$

where  $v_i^k = \int_0^{\bar{x}} v(t - w^k(t)) (f_i(t|e^k) - f_i(t|e_*)) dt$ . It is immediate to see that  $v_i^k$  tends to zero as  $k$  tends to infinity since  $v$  is bounded and  $f_i$  defined on the compact set  $I_x \times I_e$  is uniformly continuous.

We then invoke the same argument as in the proof of Lemma 3.3 for the convex function  $\Gamma_i \equiv v(x - w(x)) f_i(x|e)$ , for each  $i \in \mathcal{I}$ . Let  $M \in \mathbb{R}_+$  be an upper bound for  $\phi'_P$ , then for sufficiently large  $k \in \mathbb{N}$  and sufficiently small  $\epsilon > 0$ ,

$$g(\bar{w}, e_*) \leq - \sum_{i=1}^n p_i \phi_P(z_i^k(\bar{x})) + M\epsilon,$$

which is the analogy of (3.15) in this case. Again, letting  $\epsilon \rightarrow 0$  yields:

$$g(\bar{w}, e_*) \leq - \sum_{i=1}^n p_i \phi_P(z_i^k(\bar{x})), \quad (3.17)$$

implying that  $g(\bar{w}, e_*)$  is a lower bound for  $-\sum_{i=1}^n p_i \phi_P(z_i^k(\bar{x}))$ . Thus  $g(\bar{w}, e_*) \leq \delta$  by the definition of the infimum. ■

To sum up, we have proved that the limiting trajectory satisfies the constraint, hence it brings the system from  $\mathcal{M}_0$  to  $\mathcal{M}_1$ . Moreover, the cost achieved by this trajectory is the minimum cost. Thus the limiting control  $\bar{w}$  is an optimal control. ■

### 3.5 Characterization of the optimal wage

We employ the Pontryagin Maximum Principle (PMP) to characterize the necessary conditions that must be satisfied by a solution to the OCP, which has been shown to exist in Proposition 3.1. With a slight modification from Trélat (2008), the statement of the PMP applied to this problem is the following.

**Theorem 3.1** (Pontryagin Maximum Principle). *Suppose  $(X, w)$  is a solution to the OCP. There exists an absolutely continuous vector-valued function  $\lambda : I_x \rightarrow \mathbb{R}^{2n+1}$  and a real number  $\lambda_0 \in \{0, 1\}$  with  $(\lambda, \lambda_0) \neq 0 \in \mathbb{R}^{2n+2}$  such that:*

1.  $\lambda$  satisfies the canonical equations

$$\dot{X}(x) = \nabla_{\lambda} H(X(x), w(x), \lambda(x), \lambda_0, x), \quad (3.18)$$

$$\dot{\lambda}(x) = -\nabla_X H(X(x), w(x), \lambda(x), \lambda_0, x), \quad (3.19)$$

for almost every  $x \in I_x$ , where the real-valued function  $H : \mathbb{R}^{2n+1} \times \mathbb{R} \times \mathbb{R}^{2n+1} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , called the Hamiltonian associated with the OCP is defined by:

$$H(X, \lambda, \lambda_0, \omega, x) = v(x - \omega) \langle \lambda_z, f(x|e) \rangle + u(\omega) \langle \lambda_y, f(x|e) \rangle, \quad (3.20)$$

where  $\lambda = (\lambda_z, \lambda_y, \lambda_e)^T \in \mathbb{R}^{2n+1}$  is called the adjoint vector whose components  $\lambda_z \in \mathbb{R}^n$ ,  $\lambda_y \in \mathbb{R}^n$  and  $\lambda_e \in \mathbb{R}$  themselves are the adjoint vectors associated with the state variables  $z$ ,  $y$  and  $e$  respectively.

2. The maximum condition

$$H(X(x), w(x), \lambda(x), \lambda_0, x) = \max_{\omega \in [0, x]} H(X(x), \omega, \lambda(x), \lambda_0, x) \quad (3.21)$$

is satisfied for almost every  $x \in I_x$ .

3. The transversality conditions (TCs)

$$\lambda(0) \in N_{\mathcal{M}_0}(X(0)), \quad (3.22)$$

$$-\lambda_0 \nabla_X g(\bar{x}, X(\bar{x})) - \lambda(\bar{x}) \in N_{\mathcal{M}_1}(X(\bar{x})) \quad (3.23)$$

are satisfied, where  $\mathcal{M}_0$  and  $\mathcal{M}_1$  are respectively defined by (3.7) and (3.8), and  $N_{\mathcal{M}_i}(X(x))$  denotes the normal cone to  $\mathcal{M}_i$  at  $X(x)$ ,  $i \in \{0, 1\}$ .

We start by computing the normal cones to  $\mathcal{M}_0$  and  $\mathcal{M}_1$ . First, consider  $N_{\mathcal{M}_0}(X(0))$ . Recall that  $X(0) = (0, 0, e)$ . Let  $\xi = (\xi_z, \xi_y, \xi_e) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ . Take an element  $M_0 \in \mathcal{M}_0$ , then  $M_0 = (0, 0, a) \in \mathbb{R}^n \times \mathbb{R}^n \times I_e$ . The normal cone to  $\mathcal{M}_0$  at  $X(0)$  can be written as:

$$\begin{aligned} N_{\mathcal{M}_0}(X(0)) &= \{ \xi \in \mathbb{R}^{2n+1} \mid \langle \xi, M_0 - X(0) \rangle \leq 0, \forall M_0 \in \mathcal{M}_0 \} \\ \implies N_{\mathcal{M}_0}(X(0)) &= \{ \xi \in \mathbb{R}^{2n+1} \mid \xi_e(a - e) \leq 0, \forall a \in I_e \}. \end{aligned}$$

One of the following scenarios can occur.

- If  $e \in (\underline{e}, \bar{e})$ , then  $\xi_e = 0$  since  $\xi_e(a - e) \leq 0$  must be satisfied for any  $a \in I_e$ . Thus the normal cone is:

$$N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \{0\}. \quad (3.24)$$

- If  $e = \underline{e}$  then  $\xi_e(a - \underline{e}) \leq 0$  if and only if  $\xi_e \leq 0$  since  $a \geq \underline{e}$  for all  $a \in I_e$ . Thus the normal cone is

$$N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_-. \quad (3.25)$$

- If  $e = \bar{e}$  then  $\xi_e(a - \bar{e}) \leq 0$  if and only if  $\xi_e \geq 0$  since  $a \leq \bar{e}$  for all  $a \in I_e$ . Thus the normal cone in this case is

$$N_{\mathcal{M}_0}(X(0)) = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+. \quad (3.26)$$

Next, consider  $N_{\mathcal{M}_1}(X(\bar{x}))$ . Recall that  $X(\bar{x}) = (z(\bar{x}), y(\bar{x}), e)$ . Thanks to Lemma 3.2, we know from Clarke (1990) that the normal cone  $N_{\mathcal{M}_1}(X(\bar{x}))$  can be written as:

$$N_{\mathcal{M}_1}(X(\bar{x})) = -\mu_h \nabla_X h(w, e) + \mu_{\underline{e}} \nabla_X(\underline{e} - e) + \mu_{\bar{e}} \nabla_X(e - \bar{e}), \quad (3.27)$$

for some  $\mu_h \geq 0$  and  $\mu_e = (\mu_{\underline{e}}, \mu_{\bar{e}}) \in \mathbb{R}_+^2$  satisfying the complementary slackness conditions:

$$\mu_{\underline{e}}(e - \underline{e}) \equiv \mu_{\underline{e}}(e - \underline{e}) = 0, \quad \mu_{\underline{e}} \geq 0, \quad (3.28)$$

$$\mu_{\bar{e}}(e - \bar{e}) \equiv \mu_{\bar{e}}(e - \bar{e}) = 0, \quad \mu_{\bar{e}} \geq 0. \quad (3.29)$$

Simplification of (3.27) yields:

$$N_{\mathcal{M}_1}(X(\bar{x})) = \begin{pmatrix} 0 \\ -\mu_h \nabla_y h(w, e) \\ -\mu_h \nabla_e h(w, e) - \mu_{\underline{e}} + \mu_{\bar{e}} \end{pmatrix}. \quad (3.30)$$

**Lemma 3.5.** *If  $e \in (\underline{e}, \bar{e})$ , then  $\lambda_e(0) = 0$ . If  $e = \underline{e}$ , then  $\lambda_e(0) \leq 0$ . If  $e = \bar{e}$ , then  $\lambda_e(0) \geq 0$ .*

*Proof.* The proof follows directly from condition 3.22 applied to different forms of

$N_{\mathcal{M}_0}(X(0))$  depending on where  $e$  takes values as previously computed. In particular:

- If  $e \in (\underline{e}, \bar{e})$ , then the normal cone takes the form 3.24, implying  $\lambda_e(0) = 0$ ;
- If  $e = \underline{e}$ , then the normal cone takes the form 3.25, implying  $\lambda_e(0) \leq 0$ ;
- If  $e = \bar{e}$ , the normal cone takes the form 3.26, implying  $\lambda_e(0) \geq 0$ .

■

**Proposition 3.2.** *The adjoint vector to  $e$  satisfies*

$$\lambda_e(0) = 2 \left( \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial e} \right\rangle + \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial e} \right\rangle \right) - \mu_h c'(e) + \mu_{\underline{e}} - \mu_{\bar{e}}. \quad (3.31)$$

Moreover, when  $e$  is interior,

$$2 \left( \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial e} \right\rangle + \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial e} \right\rangle \right) = \mu_h c'(e). \quad (3.32)$$

*Proof.* Note that (3.19) implies

$$\dot{\lambda}(x) = \begin{pmatrix} \dot{\lambda}_z(x) \\ \dot{\lambda}_y(x) \\ \dot{\lambda}_e(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -v(x - w(x)) \left\langle \lambda_z, \frac{\partial f(x|e)}{\partial e} \right\rangle - u(w(x)) \left\langle \lambda_y, \frac{\partial f(x|e)}{\partial e} \right\rangle \end{pmatrix}, \quad (3.33)$$

for a.e.  $x \in I_x$ . Hence  $\lambda_z(x) = \lambda_z = \text{cons}$  and  $\lambda_y(x) = \lambda_y = \text{cons}$  for all  $x \in I_x$ . Moreover, in view of (3.30), condition (3.23) is equivalent to:

$$\begin{pmatrix} \lambda_z(\bar{x}) \\ \lambda_y(\bar{x}) \\ \lambda_e(\bar{x}) \end{pmatrix} = \begin{pmatrix} -\lambda_0 \nabla_z g(w, e) \\ \mu_h \nabla_y h(w, e) \\ -\lambda_0 \nabla_e g(w, e) + \mu_h \nabla_e h(w, e) - \mu_{\bar{e}} + \mu_{\underline{e}} \end{pmatrix}. \quad (3.34)$$

Thus

$$\lambda_z = \lambda_0 (p_i \phi'_P(z_i(\bar{x})))_{i \in \mathcal{I}}, \quad (3.35)$$

$$\lambda_y = \mu_h (p_i \phi'_A(y_i(\bar{x})))_{i \in \mathcal{I}}, \quad (3.36)$$

$$\begin{aligned} \lambda_e(\bar{x}) &= \lambda_0 \sum_{i=1}^n p_i \phi'_P(z_i(\bar{x})) \frac{\partial z_i(\bar{x})}{\partial e} + \mu_h \left( \sum_{i=1}^n p_i \phi'_A(y_i(\bar{x})) \frac{\partial y_i(\bar{x})}{\partial e} - c'(e) \right) \\ &\quad - \mu_{\bar{e}} + \mu_{\underline{e}}. \end{aligned} \quad (3.37)$$

Substituting (3.35) and (3.36) into (3.37) yields a more compact expression for  $\lambda_e(\bar{x})$ . In

particular,

$$\lambda_e(\bar{x}) = \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial e} \right\rangle + \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial e} \right\rangle - \mu_h c'(e) - \mu_{\bar{e}} + \mu_{\underline{e}}. \quad (3.38)$$

To ease on notation, define

$$\begin{aligned} Z(e) &= \int_{I_x} v(x - w(x)) \langle \lambda_z, f(x|e) \rangle dx, \\ Y(e) &= \int_{I_x} u(w(x)) \langle \lambda_y, f(x|e) \rangle dx. \end{aligned}$$

Then from (3.33),

$$\int_{I_x} \dot{\lambda}_e(x) dx = -(Z(e) + Y(e)).$$

Observe that:

$$\begin{aligned} Y(e) &= \frac{\partial}{\partial e} \left( \int_{I_x} \langle \lambda_y, \dot{z}(x) \rangle dx \right) = \left\langle \lambda_y, \frac{\partial}{\partial e} \left( \int_{I_x} \dot{z}(x) dx \right) \right\rangle \\ &\implies Y(e) = \left\langle \lambda_y, \frac{\partial y(\bar{x})}{\partial e} \right\rangle. \end{aligned} \quad (3.39)$$

Analogously,

$$Z(e) = \left\langle \lambda_z, \frac{\partial z(\bar{x})}{\partial e} \right\rangle. \quad (3.40)$$

Plugging (3.38), (3.39) and (3.40) into  $\lambda_e(0) = \lambda_e(\bar{x}) - \int_{I_x} \dot{\lambda}_e(x)$ , we arrive at (3.31).

To show the remaining part of the proposition, note that if  $e \in (\underline{e}, \bar{e})$ , then  $\mu_{\underline{e}} = \mu_{\bar{e}} = 0$  by the complementary slackness conditions (3.28) and (3.29). Moreover, in this case  $\lambda_e(0) = 0$  by Lemma 3.5. Plugging these additional pieces of information into (3.31) yields (3.32). ■

**Lemma 3.6.** *The non-triviality condition  $(\lambda_0, \mu_h) \neq 0 \in \mathbb{R}_+^2$  holds.*

*Proof.* Suppose by contradiction that  $\lambda_0 = \mu_h = 0$ , then from (3.36) and (3.35) we have  $\lambda_z = \lambda_y = 0$ , which via (3.33) implies that  $\dot{\lambda}_e(x) = 0$ . Hence  $\lambda_e(x) = \lambda_e = \text{cons}$  for all  $x \in I_x$ . In view of (3.31) we have  $\lambda_e = \mu_{\underline{e}} - \mu_{\bar{e}}$ . Consider the following cases.

- If  $e \in (\underline{e}, \bar{e})$ , then  $\lambda_e = 0$ , violating  $(\lambda, \lambda_0) \neq 0$ .
- If  $e = \underline{e}$ , then the complementary slackness condition (3.29) implies  $\lambda_e = \mu_{\underline{e}}$ . If  $\mu_{\underline{e}} = 0$ , then again  $(\lambda, \lambda_0) \neq 0$  is violated. If  $\mu_{\underline{e}} > 0$ , then Lemma 3.5 is contradicted.
- If  $e = \bar{e}$ , then the complementary slackness condition (3.28) implies  $\lambda_e = -\mu_{\bar{e}} \leq 0$ . Thus the only admissible value for  $\mu_{\bar{e}}$  in this case is zero, violating  $(\lambda, \lambda_0) \neq 0$ .

We see that regardless of the value of  $e$ , a contradiction follows if  $\lambda_0 = \mu_h = 0$ . Hence we always have  $(\lambda_0, \mu_h) \neq 0$ . ■

**Lemma 3.7.** *The adjoint vectors  $\lambda_z$  and  $\lambda_y$  satisfies  $(\lambda_z, \lambda_y) \in \mathbb{R}_{++}^{2n}$ .*

*Proof.* In light of Lemma 3.6, we only need to consider the following cases.

1. If  $\mu_h = 0, \lambda_0 = 1$ , then

$$\begin{aligned}\lambda_z &= (p_i \phi'_p(z_i(\bar{x})))_{i \in \mathcal{I}}, \\ \lambda_y &= 0.\end{aligned}$$

By Assumption 3.1 and Assumption 3.8, we have  $\lambda_z \in \mathbb{R}_{++}^n$ . But this implies  $\frac{\partial H}{\partial \omega} = -v'(x - \omega) \langle \lambda_z, f(x|e) \rangle < 0$  for all  $\omega \in [0, x]$ , since  $v' > 0$  by Assumption 3.7 and  $f(x|e) \in \mathbb{R}_{++}^n$  by Assumption 3.2. Thus in this case  $w(x) = 0$  for all  $x \in I_x$ , which is ruled out in light of Remark 3.1.

2. If  $\mu_h > 0, \lambda_0 = 0$ , then

$$\begin{aligned}\lambda_z &= 0, \\ \lambda_y &= \mu_h (p_i \phi'_A(y_i(\bar{x})))_{i \in \mathcal{I}},\end{aligned}$$

implying  $\frac{\partial H}{\partial \omega} = u'(\omega) \langle \lambda_y, f(x|e) \rangle > 0$  for all  $\omega \in [0, x]$  by the same argument as in the previous case. Thus  $w(x) = x$  for all  $x \in I_x$ . This case is also ruled out in light of Remark 3.1.

3. If  $\mu_h > 0, \lambda_0 = 1$ , then

$$\begin{aligned}\lambda_z &= (p_i \phi'_p(z_i(\bar{x})))_{i \in \mathcal{I}}, \\ \lambda_y &= \mu_h (p_i \phi'_A(y_i(\bar{x})))_{i \in \mathcal{I}}.\end{aligned}$$

We conclude that only  $(\mu_h > 0, \lambda_0 = 1)$  can occur under the standing assumptions. In this case  $(\lambda_z, \lambda_y) \in \mathbb{R}_{++}^{2n}$  by assumptions on the priors and the welfare functional. ■

**Lemma 3.8.** *The Hamiltonian is strictly concave in  $\omega$ .*

*Proof.* From (3.20), we have for each fixed  $x \in (0, \bar{x}]$

$$\frac{\partial H}{\partial \omega} = -v'(x - \omega) \langle \lambda_z, f(x|e) \rangle + u'(\omega) \langle \lambda_y, f(x|e) \rangle, \quad \omega \in (0, x), \quad (3.41)$$

and

$$\frac{\partial^2 H}{\partial \omega^2} = v''(x - \omega) \langle \lambda_z, f(x|e) \rangle + u''(\omega) \langle \lambda_y, f(x|e) \rangle. \quad (3.42)$$



By Lemma 3.7 and Assumption 3.2, we have  $\langle \lambda_z, f(x|e) \rangle > 0$  and  $\langle \lambda_y, f(x|e) \rangle > 0$ . In addition, we also have  $u'' < 0$  and  $v'' \leq 0$  by Assumption 3.7. Hence  $v''(x - \omega) \langle \lambda_z, f(x|e) \rangle \leq 0$  and  $u''(\omega) \langle \lambda_y, f(x|e) \rangle < 0$ , implying  $\frac{\partial^2 H}{\partial \omega^2} < 0$  for all  $\omega \in (0, x)$ . ■

**Theorem 3.2.** *Define*

$$G(x|e) = \frac{\langle \lambda_y, f(x|e) \rangle}{\langle \lambda_z, f(x|e) \rangle}, \quad x \in I_x. \quad (3.43)$$

then the following holds for an optimal wage function.

1. If the principal is risk-averse, then the optimal wage function takes the form

$$w(x) = \Xi_x(G(x|e)), \quad x \in (0, \bar{x}], \quad (3.44)$$

where  $\Xi_x : \mathbb{R}_{++} \rightarrow (0, x)$  is the inverse mapping of  $\omega \mapsto \frac{v'(x-\omega)}{u'(\omega)}$ . Moreover, the optimal wage is non decreasing in outcomes.

2. If the principal is risk-neutral, then there exist  $x_0 \in (0, \bar{x})$  such that an optimal wage function takes the form

$$\begin{cases} w(x) = x & x \leq x_0 \text{ or } x \in (x_0, \bar{x}) \setminus J, \\ w(x) = \Xi(G(x|e)) & x \in J, \end{cases} \quad (3.45)$$

where  $\Xi : \mathbb{R}_{++} \rightarrow (0, x)$  is the inverse mapping of  $\omega \mapsto \frac{1}{u'(\omega)}$  and the set  $J$  is the countable union of open intervals defined by

$$J = \{x \in (x_0, \bar{x}) \mid u'(x)(G(x|e) < 1)\}. \quad (3.46)$$

In particular, an optimal wage is differentiable on  $(0, \bar{x})$  except at an at most countable set of points.

Moreover,  $x_0$  is the smallest  $x \in (0, \bar{x})$  such that  $u'(x)(G(x|e) = 1$  and there exists a decreasing sequence  $(x_l)_{l \geq 1}$  converging to  $x_0$  with  $u'(x_l)(G(x_l|e) < 1$  for  $l \geq 1$ .

An optimal effort satisfies  $h(w, e) = 0$ , i.e.,

$$\sum_{i=1}^n p_i (\phi_A(y_i(\bar{x}))) - c(e) = \bar{U}, \quad (3.47)$$

where the optimal state vector  $(y_i(\bar{x}))_{i \in \mathcal{I}}$  is evaluated at the optimal wage in each case. Moreover, either an optimal effort is not interior (and then belongs to  $\{\underline{e}, \bar{e}\}$ ) or it is interior and satisfies (3.32).

*Proof.* Notice that  $w(x) = 0$  cannot occur for any  $x \in I_x$  since the Inada condition on  $u$  implies  $\frac{\partial H}{\partial \omega} |_{\omega=0} = +\infty$ , regardless of the principal's attitude towards risk.

1. When the principal is risk-averse, the Inada condition on  $v(\cdot)$  implies that  $w(x) = x$  cannot occur for any  $x$  since  $\frac{\partial H}{\partial \omega}|_{\omega=x} = -\infty$ . Thus the optimal wage satisfies  $\frac{\partial H}{\partial \omega}|_{\omega=w(x)} = 0$ , or

$$\frac{v'(x - w(x))}{u'(w(x))} = G(x|e). \quad (3.48)$$

Note that since  $v'' \leq 0$  and  $u' > 0$ ,

$$\frac{-v''(x - \omega)u'(\omega) - v'(x - \omega)u''(\omega)}{[u'(\omega)]^2} > 0 \quad (3.49)$$

for any fixed  $x$ , implying that the mapping  $\omega \mapsto \frac{v'(x-\omega)}{u'(\omega)}$  is strictly increasing. Thus it has an inverse mapping  $\Xi_x$ , also strictly increasing, such that for each  $x \in (0, \bar{x}]$ , the optimal wage is uniquely defined by

$$w(x) = \Xi_x(G(x|e)), \quad x \in (0, \bar{x}]. \quad (3.50)$$

One deduces from the assumptions that  $w(\cdot)$  is differentiable on its domain. Let  $G'(x|e) = \frac{\partial G(x|e)}{\partial x}$ , we have

$$w'(x) = \Xi'_x(G(x|e))G'(x|e), \quad (3.51)$$

where  $\Xi'_x > 0$  since  $\Xi_x$  is a strictly increasing map, implying that  $w'(x)$  has the same sign as  $G'(x|e)$ . Furthermore, differentiating both sides of (3.48) with respect to  $x$  and simplifying yield

$$w'(x) = \frac{r_v(x - w(x)) + \frac{G'(x|e)}{G(x|e)}}{r_v(x - w(x)) + r_u(w(x))}, \quad (3.52)$$

where  $r_v(\cdot) = -\frac{v''(\cdot)}{v'(\cdot)} > 0$  and  $r_u(\cdot) = -\frac{u''(\cdot)}{u'(\cdot)} > 0$  denote the degree of the absolute risk aversion of the principal and the agent, respectively. Observe that  $G(x|e) = \frac{\langle \lambda_y, f(x|e) \rangle}{\langle \lambda_z, f(x|e) \rangle}$  is bounded above and below by positive constants since all the elements of the adjoint vectors are positive and finite, and the densities are positive and bounded. Since  $w(x) \in (0, x)$  for all  $x \in (0, \bar{x}]$ , there exists  $x_1 > 0$  such that  $w'(x_1) > 0$ , otherwise  $w(x) \leq 0$  for all  $x \in I_x$  since  $w(0) = 0$ , which is either inadmissible, or ruled out by a previous remark. From (3.51) we have  $G'(x_1|e) > 0$ . We would like to show that  $G'(x|e) \geq 0$  for all  $x \in I_x$ . Suppose by contradiction, there exists  $x_2 \in (0, \bar{x}]$  such that  $G'(x_2|e) < 0$ . By the continuity of  $G'(\cdot|e)$ , there exists  $x_3 \in (0, \bar{x}]$  such that  $G'(x_3|e) = 0$ , which, via (3.51) implies  $w'(x_3) = 0$ . But  $G'(x_3|e) = 0$  and  $w'(x_3) = 0$  imply  $r_v(\cdot) = 0$  in light of (3.52), a contradiction. Hence we must have  $G'(x|e) \geq 0$  for all  $x \in I_x$ , implying via (3.51) that  $w'(x) \geq 0$  for all  $x \in I_x$ .

2. When the principal is risk-neutral,  $v' \equiv 1$ . Equation (3.41) reads

$$\frac{\partial H}{\partial \omega} = \langle \lambda_z, f(x|e) \rangle (u'(\omega)G(x|e) - 1), \quad (3.53)$$

and by strict convexity of the Hamiltonian, one gets that, for  $x \in (0, \bar{x})$ , the latter reaches its maximum at  $\omega = x$  if and only if  $u'(x)G(x|e) \geq 1$ .

By the Inada condition on  $u$ , one deduces at once  $w(x) = x$  for  $x$  small enough. Let  $x_0$  be the largest  $x \in I_x$  such that  $w(x) = x$  on  $[0, x_0]$ .

Notice that  $x_0 < \bar{x}$ , otherwise  $w(x) = x$  on  $I_x$ , which is ruled out by a previous remark. We claim that  $u'(x_0)G(x_0|e) = 1$ . Indeed, one clearly has that  $u'(x)G(x|e) \geq 1$  for  $x < x_0$  and therefore  $u'(x_0)G(x_0|e) \geq 1$ . Furthermore by maximality of  $x_0$ , there exists a decreasing sequence  $(x_l)_{l \geq 1}$  tending to  $x_0$  such that  $w(x_l) < x_l$  for  $l \geq 1$ , i.e.,  $u'(x_l)G(x_l|e) < 1$ , yielding, as  $l$  tends to infinity that  $u'(x_0)G(x_0|e) \leq 1$ . Hence the claim and, as a byproduct, the characterization of  $x_0$ . Let the subset  $J$  of  $(x_0, \bar{x})$  defined in (3.46). Since  $u'(\cdot)G(\cdot|e)$  is continuous,  $J$  is the countable union of open intervals on which the optimal wage  $w(x)$  is defined by  $\frac{\partial H}{\partial \omega}|_{\omega=w(x)} = 0$ , i.e., by risk-neutrality of the principal, yields

$$\frac{1}{u'(w(x))} = G(x|e), \quad x \in J. \quad (3.54)$$

Since  $\omega \mapsto \frac{1}{u'(\omega)}$  is strictly increasing by the strict concavity of  $u$ , the inverse mapping  $\Xi \equiv \left(\frac{1}{u'}\right)^{-1}$  is well-defined. Inverting both sides of (3.54), we arrive at the desired expression for  $w(x)$ . This completes the proof. ■

**Remark 3.2.** *One may wonder about the uniqueness of the solution. Clearly, a solution is determined by the triple  $(\mu_h, x_0, e)$ . In the case where  $e$  is not interior, we have two unknowns and two equations: one from the equality constraint  $h(w, e) = 0$ , and the other from the characterization of  $x_0$  provided by the theorem, i.e., the study of the solution to equation  $u'(x)G(x|e) = 0$ . In the case where  $e$  is interior, we have an extra equation, namely (3.32). However, it seems difficult to determine all the solutions analytically.*

**Remark 3.3.** *With reasonable assumptions on the data of the problem, such as real-analyticity, one can conclude that there is a finite number of solutions to  $u'(x)G(x|e) = 1$  on  $(0, \bar{x}]$  and then  $J$  is made of a finite number of open intervals. Furthermore, notice that on  $(0, \bar{x}]$ , one has that  $(u'G)' = u''G + u'G' = u'\left(\frac{G'}{G} - r_u\right)$ . One deduces that if  $\frac{G'}{G} \leq r_u$  then  $J = (x_0, \bar{x})$  and  $x_0$  is the unique solution to  $u'(x)G(x|e) = 1$ .*

When both DMs are ambiguity-neutral, we recover the result that is most analogous to the straight deductible result of [Raviv \(1979\)](#) in the context of insurance.

**Corollary 3.1.** [Raviv (1979)] *When both DMs are ambiguity-neutral and the principal is risk-neutral, the shape of the optimal contract is the following*

$$\begin{cases} w(x) = x & x \in (0, x_0] \\ w(x) = x_0 \in (0, x) & x \in (x_0, \bar{x}] \end{cases},$$

where  $x_0$  is uniquely defined by  $u'(x_0) = \frac{1}{\mu_h}$ .

*Proof.* When both DMs are ambiguity-neutral,

$$G(x|e) = \frac{\mu_h \langle p, f(x|e) \rangle}{\langle p, f(x|e) \rangle} = \mu_h, \quad (3.55)$$

where  $p \equiv (p_1, \dots, p_n)$  is the vector of priors.

According to Remark 3.3,  $x_0$  is the unique solution to

$$u'(x_0)\mu_h - 1 = 0. \quad (3.56)$$

By the monotonicity of  $u'$ , for all  $x \leq x_0$  we have  $u'(x)\mu_h - 1 \geq 0$ , implying that  $w(x) = x$  for all  $x \leq x_0$  and then for all  $x > x_0$  we have  $u'(x)\mu_h - 1 < 0$ , implying that  $w(x) = x_0$  for all  $x > x_0$ . ■

**Remark 3.4.** *Observe that the sharing rule (3.48) characterizes both efficient risk and ambiguity-sharing. It has a nice interpretation. The LHS of this equation is the relative marginal utilities, while the RHS is relative expected marginal welfare. The expectation is computed with respect to the posterior distribution. To see this, define the expected marginal welfare of the agent and the principal, respectively, be:*

$$A(x|e) = \frac{\sum_{i=1}^n p_i \phi'_A(y_i(\bar{x})) f_i(x|e)}{\sum_{i=1}^n p_i f_i(x|e)} \equiv \sum_{i=1}^n p_i(x|e) \phi'_A(y_i(\bar{x})), \quad (3.57)$$

$$P(x|e) = \frac{\sum_{i=1}^n p_i \phi'_P(z_i(\bar{x})) f_i(x|e)}{\sum_{i=1}^n p_i f_i(x|e)} \equiv \sum_{i=1}^n p_i(x|e) \phi'_P(z_i(\bar{x})). \quad (3.58)$$

where  $p_i(x|e) = \frac{p_i f_i(x|e)}{\sum_{i=1}^n p_i f_i(x|e)}$  is the probability that state  $i$  occurs given that the outcome is  $x$ , which is by definition the Bayesian posterior probability. This inference that each contracting party has to make here is a direct consequence of the uncertainty on the distribution and that the state itself is not a contractible variable. Then (3.48) is equivalent to

$$\frac{v'(x - w(x))P(x|e)}{u'(w(x))A(x|e)} = \mu_h, \quad (3.59)$$

which tells us that at the optimum, the ratio between the product of marginal utility and ex-

pected marginal welfare of the agent and that of the principal is equalized across all levels of outcomes where optimal wage has an interior value. Notice that under ambiguity neutrality of both DMs,  $\frac{A(x|e)}{P(x|e)} = \text{cons} \equiv 1$  and (3.59) reduces to the famous Borch rule

$$\frac{v'(x - w(x))}{u'(w(x))} = \mu_h, \quad (3.60)$$

according to Borch (1960). Hence (3.59) can be viewed as a modified Borch rule that an optimal contract has to satisfy under ambiguity.

**Remark 3.5.** To clarify the notion of Bayesian inference mentioned above, consider a situation where the states are contractible variables, i.e., when the contract can be written as  $\{w_i(\cdot), e\}_{i \in \mathcal{I}}$  instead of  $\{w(\cdot), e\}$ . In this case, we can slightly modify the state variables as

$$\begin{aligned} z(x) &= (v(x - w_i(x))f_i(x|e))_{i \in \mathcal{I}}, \\ y(x) &= (u(w_i(x))f_i(x|e))_{i \in \mathcal{I}}, \end{aligned}$$

and easily show that the sharing rule (3.59) holds for every state. In particular,

$$\frac{v'(x - w_i(x))\phi'_P(z_i(\bar{x}))}{u'(w_i(x))\phi'_A(y_i(\bar{x}))} = \mu_h, \quad \forall x \in I_x, i \in \mathcal{I}, \quad (3.61)$$

implying that the state-conditional ratio of marginal utilities  $\frac{v'(x - w_i(x))}{u'(w_i(x))}$  is held constant across all  $x \in I_x$ , in each state  $i \in \mathcal{I}$ . Hence (3.61) can be viewed as a state-conditional Borch rule. Observe that the DMs no longer have to make an inference on the state based on the outcome as suggested by (3.48). Furthermore, when the principal is neutral to risk and ambiguity,

$$u'(w_i(x))\phi'_A(y_i(\bar{x})) = \mu_h, \quad \forall i \in \mathcal{I}, \quad (3.62)$$

implying that  $w_i(x) = \text{cons} \equiv \bar{w}_i \in (0, \bar{x})$  for all  $x$  satisfying  $w_i(x) \in (0, x)$ . In view of Corollary 3.1, the optimal wage function under risk and ambiguity-neutral principal when the states are contractible has the form:

$$\begin{cases} w_i(x) = x & x \in (0, \bar{w}_i], \\ w_i(x) = \bar{w}_i & x \in (\bar{w}_i, \bar{x}], \end{cases} \quad i \in \mathcal{I}, \quad (3.63)$$

where  $\bar{w}_i$  is the unique solution to  $K_i(x) = 0$  where  $K_i(x) = u'(x)\phi'_A(y_i(\bar{x}))\mu_h - 1$ . Observe that if  $y_i(\bar{x}) = \int_0^{\bar{w}_i} u(x)f_i(x|e)dx + (1 - F_i(\bar{w}_i))u(\bar{w}_i)$ , then

$$\frac{\partial y_i(\bar{x})}{\partial \bar{w}_i} = u'(\bar{w}_i)(1 - F(\bar{w}_i)) > 0 \quad (3.64)$$

by Leibniz's integral rule. Thus the LHS of (3.62) is strictly decreasing in  $\bar{w}_i$  by the strict concavity of  $u(\cdot)$  and  $\phi_A(\cdot)$ . Hence in optimality  $\bar{w}_i = \bar{w}_j \equiv \bar{w}$  for all  $i, j \in \mathcal{I}$ , further simplifying (3.63) to

$$\begin{cases} w_i(x) = x & x \in (0, \bar{w}], \\ w_i(x) = \bar{w} & x \in (\bar{w}, \bar{x}], \end{cases} \quad \forall i \in \mathcal{I}. \quad (3.65)$$

Observe that the contract (3.63) is robust to the principal's ambiguity attitude. When the principal is ambiguity-averse, we simply modify  $K_i(x) = u'(x) \frac{\phi'_A(y_i(\bar{x}))}{\phi'_P(z_i(\bar{x}))} \mu_i - 1$ , which is also strictly decreasing in  $x$ . Hence, if the states are contractible (markets are complete), we conclude from (3.65) and Corollary 3.1 that the shape of the optimal contract is robust to ambiguity when the principal is risk-neutral.

### 3.6 Binary ambiguous state case with risk-neutral principal

Under a risk-averse principal, we have shown in Theorem 3.2 that  $G'$  is always non negative, regardless of the number of ambiguous states and, since the sign of  $G'$  determines the sign of  $w'$ , an optimal wage was always non decreasing. For  $n = 2$ , we prove in the next lemma that  $G'$  has a constant sign a hence deduces some information on the sign of an optimal wage under a risk-neutral principal.

**Lemma 3.9.** *Under Assumption 3.3, in the binary state case  $n = 2$ , the sign of  $G'(x|e)$  does not depend on  $x \in I_x$ . As a consequence, the following holds: either  $G' \leq 0$ , in which case  $x_0$  is the unique solution of  $u'(x)G(x|e) = 1$  and  $J$  defined in (3.46) is equal to  $(x_0, \bar{x})$ ; or  $G' > 0$ , in which case an optimal wage is non decreasing.*

*Proof.* Denote  $\lambda_z = (\lambda_z^1, \dots, \lambda_z^n)$  and  $\lambda_y = (\lambda_y^1, \dots, \lambda_y^n)$ . Then  $G'(x|e)$  can be expressed as:

$$G'(x|e) = \frac{\sum_{1 \leq i < j \leq n} (\lambda_y^i \lambda_z^j - \lambda_y^j \lambda_z^i) f_j^2(x|e) \ell'_{ij}(x|e)}{\left( \sum_{i=1}^n \lambda_z^i f_i(x) \right)^2}. \quad (3.66)$$

Hence for  $n = 2$

$$G'(x|e) = \frac{\left( \lambda_y^1 \lambda_z^2 - \lambda_y^2 \lambda_z^1 \right) [f_2(x|e)]^2 \ell'_{12}(x|e)}{\left( \sum_{i=1}^2 \lambda_z^i f_i(x|e) \right)^2},$$

where  $\ell'_{12}(x) \geq 0$  by Assumption 3.3. Clearly, the sign of  $G'(x|e)$  depends on the sign of  $\lambda_y^1 \lambda_z^2 - \lambda_y^2 \lambda_z^1$ , which is independent of  $x$ . Combining that result with Theorem 3.2 yields the rest of the statement of the lemma.  $\blacksquare$

**Remark 3.6.** Denote  $A'(x|e) = \frac{\partial A(x|e)}{\partial x}$  and  $P'(x|e) = \frac{\partial P(x|e)}{\partial x}$ . We have

$$A'(x|e) = \frac{\sum_{1 \leq i < j \leq n} p_i p_j (\phi'_A(y_i(\bar{x})) - \phi'_A(y_j(\bar{x}))) f_j^2(x|e) \ell'_{ij}(x|e)}{\langle p, f(x|e) \rangle^2}, \quad (3.67)$$

$$P'(x|e) = \frac{\sum_{1 \leq i < j \leq n} p_i p_j (\phi'_P(z_i(\bar{x})) - \phi'_P(z_j(\bar{x}))) f_j^2(x|e) \ell'_{ij}(x|e)}{\langle p, f(x|e) \rangle^2}, \quad (3.68)$$

where  $\ell'_{ij}(x|e) \geq 0$  by Assumption 3.3 and  $p_i p_j > 0$  for all  $i, j \in \mathcal{I}$  by Assumption 3.1. Hence again in the binary state case the signs of  $A'(x|e)$  and  $P'(x|e)$  are independent of  $x$ .

**Lemma 3.10.** In the binary state case,  $A'(x|e) \leq 0$  and  $P'(x|e) \leq 0$  for all  $x \in I_x$ .

*Proof.* Note that by integration by parts (IBP), we always have

$$y_1(\bar{x}) = u(w(\bar{x})) - \int_0^{\bar{x}} u'(w(x)) w'(x) F_1(x|e) dx, \quad (3.69)$$

$$z_1(\bar{x}) = (\bar{x} - w(\bar{x})) - \int_0^{\bar{x}} (1 - w'(x)) F_1(x|e) dx. \quad (3.70)$$

Hence

$$y_1(\bar{x}) - y_2(\bar{x}) = \int_0^{\bar{x}} u'(w(x)) w'(x) [F_2(x) - F_1(x)] dx, \quad (3.71)$$

and

$$z_1(\bar{x}) - z_2(\bar{x}) = \int_0^{\bar{x}} (1 - w'(x)) [F_2(x|e) - F_1(x|e)] dx. \quad (3.72)$$

Recall that when  $n = 2$ , by Lemma 3.9 the sign of  $G'(x|e)$  is constant with respect to  $x$ . Suppose  $G'(x|e) \geq 0$ , then (3.51) implies  $w'(x) \geq 0$  for  $x \in (x_0, \bar{x})$  (and thus for all  $x \in I_x$ ). Thus  $y_1(\bar{x}) \geq y_2(\bar{x})$  by (3.71) since  $u' > 0$ ,  $w' \geq 0$  and  $F_2(x) \geq F_1(x) \geq 0$  for all  $x \in I_x$  by Assumption 3.3. Hence  $\phi'_A(y_1(\bar{x})) - \phi'_A(y_2(\bar{x})) \leq 0$  by the concavity of  $\phi_A$ , implying  $A'(x|e) \leq 0$  in light of (3.67). Moreover since  $G(x|e) = \mu_h \frac{A(x|e)}{P(x|e)}$ , differentiating with respect to  $x$  and simplifying yield:

$$\frac{G'(x|e)}{G(x|e)} = \frac{A'(x|e)}{A(x|e)} - \frac{P'(x|e)}{P(x|e)} \quad (3.73)$$

$$\iff \frac{P'(x|e)}{P(x|e)} = \frac{A'(x|e)}{A(x|e)} - \frac{G'(x|e)}{G(x|e)}, \quad (3.74)$$

implying  $P'(x|e) \leq 0$  since  $A'(x|e) \leq 0$  and  $G'(x|e) \geq 0$ .

Consider next the case  $G'(x|e) < 0$ , again from (3.51) we have  $w'(x) < 0$  for  $x \in (x_0, \bar{x})$ . Thus  $w'(x) \leq 1$  for all  $x \in I_x$ , and we have from (3.72) that  $z_1(\bar{x}) \geq z_2(\bar{x})$ . Hence  $P'(x|e) \leq 0$  in light of (3.68). On the other hand (3.73) implies  $\frac{A'(x|e)}{A(x|e)} = \frac{G'(x|e)}{G(x|e)} + \frac{P'(x|e)}{P(x|e)} < 0$

since  $G'(x|e) < 0$  and  $P'(x|e) \leq 0$ . Thus  $A'(x|e) < 0$ .

We conclude that regardless of the sign of  $G'(x|e)$ , when  $n = 2$  we always have  $P'(x|e) \leq 0$  and  $A'(x|e) \leq 0$  for all  $x \in I_x$ . ■

**Remark 3.7.** As a direct consequence of Lemma 3.10, in the binary state case, we always have  $y_1(\bar{x}) \geq y_2(\bar{x})$  and  $z_1(\bar{x}) \geq z_2(\bar{x})$ .

**Example 3.3.** Consider the binary state case with power welfare function  $\phi_J(U) = U^{1-\sigma_J}$ , where  $\sigma_J \in [0, 1)$  is the degree of relative ambiguity aversion of decision maker  $J$ , where  $J \in \{A, P\}$ . Define  $\hat{y} = \frac{y_1(\bar{x})}{y_2(\bar{x})}$  and  $\hat{z} = \frac{z_1(\bar{x})}{z_2(\bar{x})}$ . Then  $\hat{y} \geq 1$  and  $\hat{z} \geq 1$  by the remark above. Note that  $\lambda_y^1 \lambda_z^2 - \lambda_y^2 \lambda_z^1 = p_1 p_2 \mu_h S(\sigma_P, \sigma_A)$ , where

$$S(\sigma_P, \sigma_A) = \phi'_A(y_1(\bar{x}))\phi'_P(z_2(\bar{x})) - \phi'_A(y_2(\bar{x}))\phi'_P(z_1(\bar{x})).$$

Thus  $S(\sigma_P, \sigma_A)$  bears the same sign as  $G'(x|e)$ . Upon simplification we have

$$S(\sigma_P, \sigma_A) = (1 - \sigma_A)(1 - \sigma_P) \frac{1}{y_2 z_1} \left( \frac{\hat{z}^{\sigma_P}}{\hat{y}^{\sigma_A}} - 1 \right).$$

Clearly, the sign of  $S(\sigma_P, \sigma_A)$  depends on the sign of  $\frac{\hat{z}^{\sigma_P}}{\hat{y}^{\sigma_A}} - 1$ . Taking  $\sigma_P = 0$ , the sign of  $S(\sigma_P, \sigma_A)$  is simply that of  $1 - \hat{y}^{\sigma_A}$  which can be made negative. By contrast, if  $\sigma_A = 0$ , the sign of  $S(\sigma_P, \sigma_A)$  is that of  $\hat{z}^{\sigma_P} - 1$  which can be made positive. In other words,  $G'$  can take both signs in the binary case and its sign depends crucially on the degree of ambiguity aversion.

**Definition 3.1.** For lack of better terminologies, we say that the ambiguity-aversion effect of the agent dominates that of the principal if

$$G'(x|e) \leq 0, \quad \forall x, \quad (3.75)$$

and vice versa. We say that the two ambiguity - aversion effects offset each other if the above holds with equality.

**Remark 3.8.** In the binary state case, (3.75) is independent of  $x$  and in light of Example 3.3 is more likely to hold the more ambiguity-averse the agent and the less ambiguity-averse the principal.

**Remark 3.9.** When (3.75) holds and under the principal's risk-neutrality and the agent's ambiguity dominance assumptions,  $x_0$  is uniquely defined by equation  $u'(x)(G(x|e)) = 1$  and an optimal contract (3.45) can be fully characterized by

$$\begin{cases} w(x) = x & x \in (0, x_0], \\ w(x) = \Xi(G(x|e)) & x \in (x_0, \bar{x}], \end{cases} \quad (3.76)$$



where  $\Xi : \mathbb{R}_{++} \rightarrow (0, x)$  is the inverse mapping of  $\omega \mapsto \frac{1}{u'(\omega)}$ .

**Proposition 3.3.** *In the binary state case, when the principal is ambiguity neutral, the variation in optimal wage whenever  $w(x) \in (0, x)$  is the following.*

1. *If both DMs are ambiguity -averse, optimal wage is non decreasing in outcomes on  $[x_0, \bar{x}]$  if and only if the ambiguity-aversion effect of the principal dominates that of the agent;*
2. *If the principal is ambiguity-neutral and the agent is ambiguity-averse, optimal wage is non increasing in outcomes. Moreover, optimal wage is constant if and only if ambiguity has a one-sided structure, i.e., either ambiguity is concentrated only outcomes beyond  $x_0$ , or concentrated only on outcomes above  $x_0$ .*

*Proof.* We consider each case separately.

1. To prove the first statement of the proposition, note that risk-neutrality of the principal implies  $r_v = 0$ . Thus (3.52) becomes

$$w'(x) = \frac{\frac{G'(x|e)}{G(x|e)}}{r_u(w(x))}. \quad (3.77)$$

Since  $r_u(\cdot) > 0$ , optimal wage is non decreasing on  $[x_0, \bar{x}]$  if and only if  $G'(x|e) \geq 0$ , i.e., if and only if the ambiguity aversion effect of the principal dominates that of the agent by Definition 3.1. By the same token, optimal wage is non - increasing on  $[x_0, \bar{x}]$  if the ambiguity-aversion effect of the agent dominates.

2. For the second statement of the proposition, observe that when the principal is neutral to both risk and ambiguity, (3.77) simplifies to

$$w'(x) = \frac{\frac{A'(x|e)}{A(x|e)}}{r_u(w(x))}, \quad (3.78)$$

implying that  $w'(x) \leq 0$  on  $[x_0, \bar{x}]$  since  $A'(x|e) \leq 0$  for all  $x \in I_x$  by Lemma 3.10. Thus the only non decreasing candidate solution satisfies  $w'(x) = 0$  for all  $x \in [x_0, \bar{x}]$ , which satisfies (3.78) if and only if  $A'(x|e) = 0$  for all  $x \in [x_0, \bar{x}]$ . Note that in the binary case (3.71) can be re-written as

$$\begin{aligned} y_1(\bar{x}) - y_2(\bar{x}) &= \int_0^{x_0} u'(x) [F_2(x) - F_1(x)] dx \\ &\quad + \int_{x_0}^{\bar{x}} u'(w(x))w'(x) [F_2(x) - F_1(x)] dx. \end{aligned}$$

Since  $w'(x) = 0$  on  $[x_0, \bar{x}]$ ,

$$y_1(\bar{x}) - y_2(\bar{x}) = \int_0^{x_0} u'(x) [F_2(x|e) - F_1(x|e)] dx,$$

which is zero if and only if  $F_2(x|e) = F_1(x|e)$  a.e.  $x \in (0, x_0]$ . Thus from (3.67) we have  $A'(x|e) = 0$  on  $[x_0, \bar{x}]$  if and only if either

$$\ell'_{12}(x|e) = 0, \quad \forall x \in (x_0, \bar{x}] \quad (3.79)$$

or

$$F_2(x|e) = F_1(x|e), \quad \text{a.e. } x \in (0, x_0]. \quad (3.80)$$

Condition (3.79) is satisfied if ambiguity is concentrated only on  $(0, x_0]$ , while (3.80) is satisfied if ambiguity is concentrated only on  $(x_0, \bar{x}]$ . Thus if the distribution of outcome has either of these one-sided ambiguous structures, then the optimal wage contract is identical to the unambiguous case as expressed in Corollary 3.1. ■

**Remark 3.10.** *If ambiguity contaminates both sides of the support, then there exists a sub-interval of  $(x_0, \bar{x}]$  where  $w'(x) < 0$ . For example, suppose there exists an interval  $J_x = [x_1, x_2]$  where  $x_1 \in (0, x_0)$  and  $x_2 \in (x_0, \bar{x})$  satisfying  $\ell'_{12}(x|e) > 0$  for all  $x \in J_x$ , then  $w'(x) \geq 0$  for all  $x \in I_x$  implies  $y_1(\bar{x}) - y_2(\bar{x}) > 0$ , which in turn implies  $\phi'_A(y_1(\bar{x})) - \phi'_A(y_2(\bar{x})) > 0$  by the strict concavity of  $\phi_A$ . Necessarily  $A'(x|e) < 0$  on  $J_x$ , which in light of (3.78) implies that  $w'(x) < 0$  on  $[x_2, \bar{x}]$ , contradicting the hypothesis that  $w'(x) \geq 0$  for all  $x \in I_x$ . Hence, if the one-sided ambiguity structure is violated, there exists a subset of outcomes where optimal wage is strictly decreasing.*

### 3.7 Optimal wage under a modified admissible set

In this section, we modify one assumption, namely Assumption 3.5 and add another Inada condition on the utility function  $u$  in Assumption 3.7. We modify also the outcome set so that  $I_x = [0, \infty)$ . In particular, the followings hold.

**Assumption 3.9.** *The wage is a measurable function  $w : I_x \rightarrow \mathbb{R}_{++}$  satisfying  $w(x) \geq 0$  for all  $x \in I_x = [0, \infty)$ .*

Essentially, this modification allows the wage to be greater than the outcome for some outcomes; it is no longer constrained point-wise.

**Assumption 3.10.** *The agent has utility function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ , which is at least  $C^2$ , satisfying  $u(0) = 0$ ,  $u' > 0 > u''$  and the Inada conditions  $\lim_{w \rightarrow 0} u'(w) = +\infty$ , and  $\lim_{w \rightarrow \infty} u'(w) = 0$ .*

Under the revised assumptions, the admissible control  $\mathcal{U}$  coincides with the set  $M(I_x)$ , hence  $\mathcal{U}$  is compact. The first major change occurs in the maximum condition

of the PMP. In particular, equation (3.21) now reads

$$H(X(x), w(x), \lambda(x), \lambda_0, x) = \max_{\omega \in I_x} H(X(x), \omega, \lambda(x), \lambda_0, x) \quad (3.81)$$

for almost every  $x$  in  $I_x$ . Under the modified maximum condition, the second part of Theorem 3.2 changes to the following.

**Proposition 3.4.** *Consider the principal-agent model with a risk-neutral principal and a risk-averse agent. For each  $x$  in  $I_x$ , the unique optimal wage satisfies*

$$w(x) = \Xi(G(x|e)), \quad \forall x \in I_x. \quad (3.82)$$

*An optimal wage satisfies the equality constraint  $h(w, e) = 0$ . Moreover, either an optimal effort is not interior and takes value in  $\{\underline{e}, \bar{e}\}$ , or it is interior and satisfies (3.32).*

*Proof.* Observe that with the Inada conditions imposed at the boundary of the control set, corner solution cannot occur for any outcome. In particular, we have  $H'(0) = +\infty$  for all  $x$ .<sup>6</sup> Since  $H'(\omega)$  is strictly decreasing, for each fixed  $x$  in  $I_x$ , there exists a unique  $w(x)$  in  $(0, \bar{x})$  satisfying  $H'(w(x)) = 0$ , which is equivalent to  $u'(w(x))G(x|e) - 1 = 0$ . Recall that  $\Xi$  is the inverse mapping to  $\omega \mapsto \frac{1}{u'(\omega)}$ , which is strictly increasing. Hence  $H'(w(x)) = 0$  is equivalent to (3.82), as desired. The proof for the result pertaining to the optimal level of effort is as before. ■

**Corollary 3.2.** *Under ambiguity neutrality, fixed wage is optimal. In particular,*

$$w(x) = \bar{w} = \Xi(\mu_h), \quad \forall x \in I_x.$$

*Proof.* The corollary follows immediately from the fact that under ambiguity neutrality (of both DMs),  $G = \mu_h$ . ■

**Proposition 3.5.** *Consider the binary-state principal-agent model with a risk-neutral principal and a risk-averse agent. If the principal is ambiguity-neutral and the agent ambiguity-averse, then a fixed wage contract is optimal. In particular,*

$$w(x) = \tilde{w} = \tilde{\Xi}(\mu_h), \quad \forall x \in I_x, \quad (3.83)$$

*where  $\tilde{\Xi}$  is the inverse mapping of  $\omega \mapsto \frac{1}{u'(\omega)\phi'_A(u(\omega))}$ . An optimal effort satisfies  $h(\tilde{w}, e) = 0$ , i.e.,*

$$\phi_A(u(\tilde{w})) - c(e) = \bar{U}.$$

<sup>6</sup>The optimal wage is also bounded above since the Inada condition implies  $H'(\infty) < 0$ .

Moreover, either an optimal effort is not interior and takes value in  $\{\underline{e}, \bar{e}\}$ , or it is interior and satisfies (3.32).

*Proof.* First, consider the case of an ambiguity-neutral principal. Since  $P' = 0$ , we have that  $\mu_h G' = A'$ , which implies that  $G' \leq 0$  by virtue of Lemma 3.10. Hence  $w'(x) = \Xi'(G(x|e))G'(x|e) \leq 0$ . Since  $w$  is positive-valued, the only admissible wage that satisfies  $w'(x) \leq 0$  is  $w'(x) = 0$  for all  $x$ . Denote the fixed wage by  $\tilde{w}$ . Recall that  $\lambda_y = (p_i \phi'_A(y_i))_{i \in I}$ . Under constant wage,  $y_i = u(\tilde{w})$  for all  $i$ , hence  $A'(x)$  reduces to just  $\frac{\phi'_A(u(\tilde{w})) \langle p, f(x|e) \rangle}{\langle p, f(x|e) \rangle} = \phi'_A(u(\tilde{w}))$ . Thus the condition  $H'(w(x)) = 0$  simplifies to

$$\frac{1}{u'(\tilde{w}) \phi'_A(u(\tilde{w}))} = \mu_h. \quad (3.84)$$

It is easy to check that the mapping  $\omega \mapsto \frac{1}{u'(\omega) \phi'_A(u(\omega))}$  is strictly increasing due to the strict concavity and monotonicity of  $u$  and  $\phi_A$ . Hence its inverse mapping  $\tilde{\Xi}$  exists and we could invert (3.84) to obtain (3.83). As before, if  $e$  takes an interior value, then the triple  $(\mu_h, \tilde{w}, e)$  is pinned down by three equations, namely (3.84), (3.32) and  $h(\tilde{w}, e) = 0$ . ■

### 3.8 Conclusion

Borrowing the optimal control framework, we reformulate the principal-agent problem as a Mayer's problem to prove the existence of an optimal wage function in the symmetric information case. On the basis of the existence result, we employ the Pontryagin's Maximum Principle to characterize the solution. Our approach, which is most similar to [Raviv \(1979\)](#) represents a contribution to the existing literature in a number of ways. First, we have shown that an optimal wage is non decreasing in outcomes when the principal is risk-averse, regardless of the DMs' attitudes towards ambiguity and the number of ambiguous states. In other words, non decreasing wage is robust to ambiguity aversion when the principal is risk-averse. Second, we do not ex-ante assume an interior solution, which is not an innocuous assumption in presence of ambiguity aversion when the principal is risk-neutral. This is because the expected marginal welfare depends on the shape of the optimal wage function. Had we assumed interior solution, we would have concluded from (3.78) that constant wage were robust to ambiguity when the principal is neutral to risk and ambiguity while the agent is averse to both, regardless of the structure of ambiguity. Clearly this is not the case even in the case of two ambiguous states considered in Proposition 3.3.

The main limitation of our research is the generalizability of the result to more than two ambiguous states in the case of a risk-neutral principal. We await future research

to shed light on this issue.

# 4

## Stochastic accumulation and optimal investment in human capital

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## Chapter nomenclature

Notation	Meaning	Reference page(s)
$\Theta$	Set of scenarios (second-order states)	107
$q_\theta$	Prior on scenario $\theta$	108
$\tilde{\delta}_\theta$	Scenario- $\theta$ conditional depreciation rate of HC	107
$\chi$	Support of depreciation rate of HC	107
$\phi(\cdot)$	Second-order utility function	107
$T(\cdot)$	Absolute ambiguity tolerance	109
$p_\theta(\cdot)$	Scenario- $\theta$ conditional probability mass function (pmf)	108
$u(\cdot)$	Representative household's utility function	107
$F(\cdot)$	Production function	106, 108, 120
$\mathbb{E}_\theta$	Scenario- $\theta$ conditional expectation	106, 120
$\mathbb{E}$	Expectation with respect to the scenario space	106, 120
$h_t$	HC in period $t$	106, 120
$k_t$	PC in period $t$	120
$y_t$	Output in period $t$	106
$e$	Investment in HC	106, 120
$s$	Investment in PC	120
$A$	Total factor of productivity	106
$B$	Efficiency of HC accumulation	106
$J$	Impact of pessimism due to ambiguity aversion	114
$K$	Preference for timing of ambiguity resolution	114
$A(\cdot)$	Absolute ambiguity aversion	116
$P(\cdot)$	Absolute ambiguity prudence	116

Table 4.1: Notations used in the paper

## 4.1 Introduction

Never before has the concern regarding developing human capital become so pressing as nowadays. Human capital becomes a crucial issue because for certain nations especially those deprived of natural resources such as Israel, Japan or Singapore, it is human capital that brings about the economic miracles. Accordingly, investment in education and training is of utmost importance in these countries. Nevertheless, especially in developing countries, there is evidence of co-existence of over-education in some sectors (such as in management and finance), and severe lack of skill in others, especially in highly specialized technical fields. One reason for this imbalance is that while technical expertise consumes much more resource to develop, its market value is highly susceptible to uncertainty. The technicians who are in demand today might see their skills become obsolete tomorrow at the arrival of a new technology. Without sufficient job security and investment in lifelong learning from the private and public sectors, there might not be enough incentive to specialize in technical areas.

Since human capital is stored in, by definition, "human", it is also subject to the multitude of risks and uncertainties that humans face, such as those linked to the health, to the environment, as well as organizational and technological changes. This calls for a more general approach to modeling human capital, allowing for uncertainty in the form of risk as well as ambiguity in its accumulating process. Interestingly, in the field of management, [Chauhan and Chauhan \(2008\)](#) documented "superiors' attitude" as a crucial determinant of skills obsolescence perceived by managers. Needless to say, this factor is highly ambiguous by nature. In a more recent article, [Nalbantian \(2017\)](#) calls for an a distinction between risk and ambiguity<sup>1</sup> in addressing issues related to human capital since they have distinct implications on a company's responses to uncertain skill obsolescence.

The first works on introducing stochastic elements into the law of motion of human capital date back to the 70s. This line of research was initiated by [Levhari and Weiss \(1974\)](#), who argued for the need of relaxing the assumption of perfect foresight in human capital accumulation, which had been maintained heretofore in the seminal treatises of [Becker \(1964\)](#) and [Schultz \(1971\)](#). In a two-period model, the authors categorized human capital risks as "input" and "output". The former includes factors concerning the *production* of human capital, such as learning abilities of the individuals, or schooling quality. The latter reflects the market conditions (supply and demand of the type of labor produced) in the post-production period, determining whether the skill produced is valued by the market. In the direction of this approach, [Williams \(1979\)](#)

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<sup>1</sup>In the terms of in [Knight \(1921\)](#), this is the difference between measurable and immeasurable uncertainty.



allowed for risky depreciation and net productivity (both log-normally distributed) in a model where human capital is accumulated linearly with respect to the level of investment in education. Although not explicitly stated, the so-called "net productivity" parameter in Williams (1979) can be placed under "input" as the production of human capital occurs at the beginning of the period. The risky depreciation rate, on the other hand, could be categorized as "output": how robust is the produced skill/human capital to the market conditions. If the economy is caught by a technological or organization shock, some types of skill might become obsolete very quickly.

In this paper, I investigate the impact of uncertain human capital accumulation from two different viewpoints. First, in a spirit similar to Krebs (2003), the depreciation rate of human capital is viewed as a random variable to capture the so-called uncertain obsolescence of skills phenomenon. Then, I consider the case where the random variable is effectiveness (or net productivity) of human capital accumulation rather than its depreciation rate. In each case, I study the effect uncertainty, both in the form of risk (measurable uncertainty) and ambiguity (immeasurable uncertainty), on the optimal level of investment in human capital. The two views result in completely different implications.

## 4.2 Statement of the problem and assumptions

Consider the discrete time analog of the Ben-Porath model Ben-Porath (1967) with no learning time and ambiguous stochastic depreciation rate of human capital. For simplicity let us examine a two-period economy without physical capital. The program faced by a representative agent is:

$$\max_{c_0 \geq 0, e \geq 0} u(c_0) + \beta \phi^{-1}(\mathbb{E} \phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))) \quad (4.1)$$

$$s.t. \quad c_0 + e = y_0, \quad (4.2)$$

$$\tilde{c}_{1\theta} = \tilde{y}_{1\theta}, \quad \theta \in \Theta, \quad (4.3)$$

$$y_t = AF(h_t), \quad t \in \{0, 1\}, \quad (4.4)$$

$$\tilde{h}_{1\theta} = h_0(Be^\alpha + 1 - \tilde{\delta}_\theta), \quad \alpha \in (0, 1], \theta \in \Theta, \quad (4.5)$$

$$A > 0, B > 0, h_0 > 0 \text{ given.} \quad (4.6)$$

In this program  $c_t, y_t, h_t$  denote consumption, output, and human capital in period  $t$ , respectively, for  $t \in \{0, 1\}$ . The investment in human capital (control variable in the initial period) is  $e$ . In addition, the parameters  $A$  and  $B$  stand for the total factor of output and human capital productivity, respectively. Ambiguity enters through the unknown scenario  $\theta$ , which belongs to the scenario space  $\Theta$ . The representative agent has perfect knowledge of  $\Theta$  and the conditional distribution of  $\tilde{\delta}_\theta$ , for each scenario

$\theta \in \Theta$ , but faces uncertainty regarding which scenario is going to occur. This is in contrast to the unambiguous stochastic setting, where the distribution of the stochastic variable is assumed to be objectively known. In other words, in absence of ambiguity, the agent knows exactly which scenario will occur.

The attitude towards ambiguity of the decision maker is modeled in the smooth sense of [Klibanoff et al. \(2005\)](#) and [Klibanoff et al. \(2009\)](#), via the functional  $\phi$  (also called second-order utility functional). This functional being concave, linear or convex corresponds to a decision maker that is ambiguity-averse, ambiguity-neutral or ambiguity-seeking, respectively. Note that the maximin expected utility (MEU) of [Gilboa and Schmeidler \(1989\)](#) is a special case of smooth ambiguity. In particular, [Klibanoff et al. \(2005\)](#) proved that the MEU representation is achieved when the decision-maker has infinite absolute ambiguity aversion in the smooth sense.

**Assumption 4.1** (Finite scenario space). *Let the scenario space be  $\Theta = \{1, 2, \dots, n\}$ , where  $n$  is a positive integer.*

Hence associated to each scenario  $\theta$  is a random variable  $\tilde{\delta}_\theta$  whose distribution is perfectly known. We shall assume that all the scenario-conditional distributions  $\tilde{\delta}_\theta$  have a common support.

**Assumption 4.2** (Finite common support). *Assume that all scenario-conditional random variables  $\tilde{\delta}_\theta$ 's have a common finite support  $\chi = \{\delta_1, \dots, \delta_m\}$ , where  $\delta_j \in (0, 1)$  for all  $j = 1, \dots, m$ , for some positive integer  $m$ .*

In other words, the scenarios do not shift the support of the random variable, which is crucial. Under this assumption, we can rewrite constraints (3) and (5) in the agent's program as:

$$c_1(\delta_j) = y_1(\delta_j), \tag{4.7}$$

$$h_1(\delta_j) = h_0(Be^\alpha + 1 - \delta_j), \tag{4.8}$$

for all  $\delta_j \in \chi$ .

**Assumption 4.3** (Smooth ambiguity aversion). *The functional  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ , the second order utility function, is strictly increasing, concave and continuously differentiable.*

This assumption reflects a large body of empirical evidence since [Ellsberg \(1961\)](#) that decision makers are ambiguity-averse. The next assumption is also standard.

**Assumption 4.4** (Risk aversion). *The vNM utility function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  is strictly increas-*

ing, concave and satisfies the Inada conditions:

$$\lim_{w \rightarrow 0} u'(w) = +\infty, \quad (4.9)$$

$$\lim_{w \rightarrow +\infty} u'(w) = 0. \quad (4.10)$$

**Assumption 4.5** (Production technology). Assume that the production function  $F : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$  is strictly increasing and concave with respect to each factor of production and satisfies the Inada conditions:

$$\lim_{x_j \rightarrow 0} F'_j(\cdot) = +\infty, \quad (4.11)$$

$$\lim_{x_j \rightarrow +\infty} F'_j(\cdot) = 0, \quad (4.12)$$

where  $F'_j(\cdot)$  denotes the partial derivative of  $F(\cdot)$  with respect to the  $j^{\text{th}}$  factor of production  $x_j$ ,  $j = 1, \dots, n$ .

**Remark 4.1.** The conditional expectation  $\mathbb{E}_\theta(\cdot)$  is the expectation with respect to each scenario-conditional distribution of  $\tilde{\delta}$ . As a consequence of ambiguity,  $\mathbb{E}_\theta[u(\tilde{c}_{1\theta})]$  is a random variable depending on  $\theta$ . On the other hand, the outer expectation  $\mathbb{E}(\cdot)$  is taken over the priors in the scenario space. For example let  $(q_1, \dots, q_n)$  be priors on scenarios satisfying  $\sum_{\theta=1}^n q_\theta = 1$  and  $q_\theta \geq 0$  for all  $\theta \in \Theta$ . Let  $p_\theta(\delta_j)$  be the probability that the depreciation rate takes value  $\delta_j$  under scenario  $\theta$ . Then the objective function could be written explicitly as:

$$u(c_0) + \beta \phi^{-1} \left( \sum_{\theta=1}^n q_\theta \phi \left( \sum_{j=1}^m p_\theta(\delta_j) u(c_1(\delta_j)) \right) \right). \quad (4.13)$$

### 4.3 Optimal investment in human capital in absence of physical capital

Let us rewrite the objective function in (4.1) as:

$$V(e) = u(y_0 - e) + \beta \phi^{-1} \left( \mathbb{E} \phi \left( \mathbb{E}_\theta u \left( y_0 (Be^\alpha + 1 - \tilde{\delta}_\theta)^\mu \right) \right) \right). \quad (4.14)$$

To simplify notations, let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be the mapping defined by:

$$f(x) = u'(y_0 x^\mu) x^{\mu-1}. \quad (4.15)$$

**Lemma 4.1.** Under Assumption 4.4, the function  $f$  defined by (4.15) is strictly decreasing. If moreover the utility function satisfies  $u''' \geq 0$ , then  $f$  is strictly convex.

*Proof.* Observe that  $f$  is twice differentiable, hence:

$$\begin{aligned} f'(x) &= \mu y_0 u''(x^\mu) x^{1(\mu-1)} + (\mu - 1) u'(x^\mu) x^{\mu-2} \\ &= -u'(x^\mu) x^{\mu-2} (\mu y_0 R_u + 1 - \mu), \end{aligned} \quad (4.16)$$

where

$$R_u := -\frac{x^\mu u''(x^\mu)}{u'(x^\mu)} \quad (4.17)$$

is the Arrow-Pratt measure of relative risk aversion. Since  $\mu \in (0, 1)$  and  $u$  satisfies Assumption 4.4, it is clear that  $f' > 0$ . Moreover,

$$\begin{aligned} f''(x) &= (\mu y_0)^2 u'''(x^\mu) x^{3\mu-3} + 3\mu y_0 (\mu - 1) u''(x^\mu) x^{2\mu-3} + (1 - \mu)(2 - \mu) u'(x^\mu) x^{\mu-3} \\ &= u'(x^\mu) x^{\mu-3} [(\mu y_0)^2 P_u R_u + 3(\mu y_0)(1 - \mu) R_u + (1 - \mu)(2 - \mu)], \end{aligned} \quad (4.18)$$

where

$$P_u := -\frac{x^\mu u'''(x^\mu)}{u''(x^\mu)} \quad (4.19)$$

is the degree of relative prudence in the sense of Kimball (1990a). Since  $\mu \in (0, 1)$  and Assumption 4.4 holds, a sufficient condition for  $f$  to be convex in  $x$  is  $P_u \geq 0$ , which is equivalent to  $u''' \geq 0$ . ■

Observe that

$$V'(e) = -u'(y_0 - e) + \zeta y_0 e^{\alpha-1} \frac{\mathbb{E} \phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})) \mathbb{E}_\theta f(\tilde{X}_\theta)}{\phi'(\phi^{-1}(\mathbb{E} \phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta})))}, \quad (4.20)$$

where

$$\zeta \equiv B\beta\alpha\mu, \quad (4.21)$$

and

$$\tilde{X}_\theta = B e^\alpha + 1 - \tilde{\delta}_\theta. \quad (4.22)$$

Note that  $V$  is not necessarily concave in  $e$  when  $\phi$  is strictly concave since  $\phi^{-1}$  is not concave. Nevertheless, if absolute ambiguity tolerance (the inverse of absolute ambiguity aversion) is concave, then it can be shown that we indeed have a concave problem.

**Assumption 4.6** (Concave absolute ambiguity tolerance). *Let  $T : \mathbb{R} \rightarrow \mathbb{R}$  be the absolute ambiguity tolerance function defined by  $T(u) = -\frac{\phi'(u)}{\phi''(u)}$ . Then  $T$  is concave in  $\mathbb{R}$ .*

**Remark 4.2.** *A very popular class of utility functions is the class of hyperbolic absolute risk aversion (HARA), which has linear absolute risk tolerance (in wealth).<sup>2</sup> In the same vein, ambiguity preferences that belong of the hyperbolic absolute ambiguity aversion (HAAA) class*

<sup>2</sup>See Chapter 4 of Lenglwiler (2004) for a detailed discussion on HARA.

have linear absolute ambiguity tolerance (in utility), thus satisfying Assumption 4.6. HAAA ambiguity preferences include those that satisfy DAAA (decreasing absolute ambiguity aversion), CAAA (constant absolute ambiguity aversion), or IAAA (increasing absolute ambiguity aversion).

**Lemma 4.2.** *Assumption 4.6 is sufficient for the objective function  $V$  defined in (4.14) to be strictly concave in  $e$ .*

*Proof.* See subsection 4.7.1. ■

### 4.3.1 Deterministic depreciation rate of human capital

Suppose first that there is perfect foresight, so that the depreciation rate is known with certainty to be  $\delta$ . Define:

$$X \equiv Be^\alpha + 1 - \delta. \quad (4.23)$$

Denote by  $V_1$  the objective function under perfect foresight. We have:

$$V_1'(e) = -u'(c_0) + \zeta y_0 e^{\alpha-1} f(X). \quad (4.24)$$

Since  $u$  is strictly concave and the positive constants  $\alpha$  and  $\mu$  are less than one, it is easy to see that  $V_1$  is strictly concave in  $e$ . Hence the first order condition (FOC) is both necessary and sufficient for an unique optimal. Let  $e_1$  denote the optimal level of investment in human capital in the deterministic case, we have:

$$V_1'(e_1) = 0. \quad (4.25)$$

### 4.3.2 Unambiguous stochastic depreciation of human capital

Under no ambiguity, we know exactly which scenario occurs. Let  $\tilde{\delta}$  be the stochastic depreciation rate associated to this scenario and suppose that this uncertainty adds a zero-mean risk to the deterministic rate  $\delta$ . In particular,

$$\tilde{\delta} = \delta + \tilde{\epsilon}, \quad \mathbb{E}\tilde{\epsilon} = 0, \quad (4.26)$$

where  $\tilde{\epsilon}$  is a zero-mean risk. Then  $X$  defined earlier becomes a random variable, which we shall denote by  $\tilde{X}$  to mean:

$$\tilde{X} = Be^\alpha + 1 - \tilde{\delta}. \quad (4.27)$$

Clearly (4.26) implies

$$\mathbb{E}\tilde{X} = X. \quad (4.28)$$

Let the objective function under pure risk be  $V_2$ . We have:

$$V_2'(e) = -u'(y_0 - e) + \zeta y_0 e^{\alpha-1} \mathbb{E}f(\tilde{X}), \quad (4.29)$$

where  $f$  is the mapped defined in (4.15). It is easy to see that  $V_2$  is strictly concave in  $e$  since  $u$  is strictly concave and  $f$  is strictly decreasing in  $e$ . Hence the FOC is also sufficient for an unique solution. Denote by  $e_2$  the level of optimal investment in human capital under pure risk, then:

$$V_2'(e_2) = 0. \quad (4.30)$$

**Assumption 4.7** (Risk prudence). *The decision maker is prudent in the sense of Kimball (1990b). In particular,*

$$u' > 0, u'' < 0, u''' > 0. \quad (4.31)$$

Observe that monotonicity and risk aversion are already embedded in Assumption 4.4. Prudence adds a third order requirement to reflect the agent's aversion to fluctuation in marginal utilities.

**Proposition 4.1.** *If risk preference satisfies prudence (Assumption 4.7), then the introduction of a zero-mean risk to the depreciation rate of human capital raises the optimal level of investment in human capital.*

*Proof.* See subsection 4.7.2. ■

**Remark 4.3.** *Observe that the increased saving (investment in human capital) in response to an increase in risk comes from the convexity of  $f$ . In view of (4.18), this property is attributed to two sources: prudence (Assumption 4.7) and the concavity of the production function  $(1 - \mu)$ . Clearly, if the production function were linear ( $\mu = 1$ ), then the introduction of risk raises savings if and only if the DM is prudent. On the other hand, if the DM is imprudent in the sense that  $u''' = 0$  (for example if he has quadratic utility), then the strict concavity of the production function (Assumption 4.5 is necessary and sufficient for a rise in savings).*

### 4.3.3 Ambiguous depreciation rate of human capital accumulation

We now move to the ambiguous setting. Recall that the law of motion for human capital accumulation in this case is:

$$\tilde{h}_{1\theta} = h_0(Be^\alpha + 1 - \tilde{\delta}_\theta), \quad \theta \in \Theta. \quad (4.32)$$

Observe that each  $\tilde{\delta}_\theta$  is a random variable taking values in the common support  $\chi$  (defined in Assumption 4.2) for all  $\theta$  in the scenario space  $\Theta$ . We now consider two

### 4.3. OPTIMAL INVESTMENT IN HUMAN CAPITAL IN ABSENCE OF PHYSICAL CAPITAL

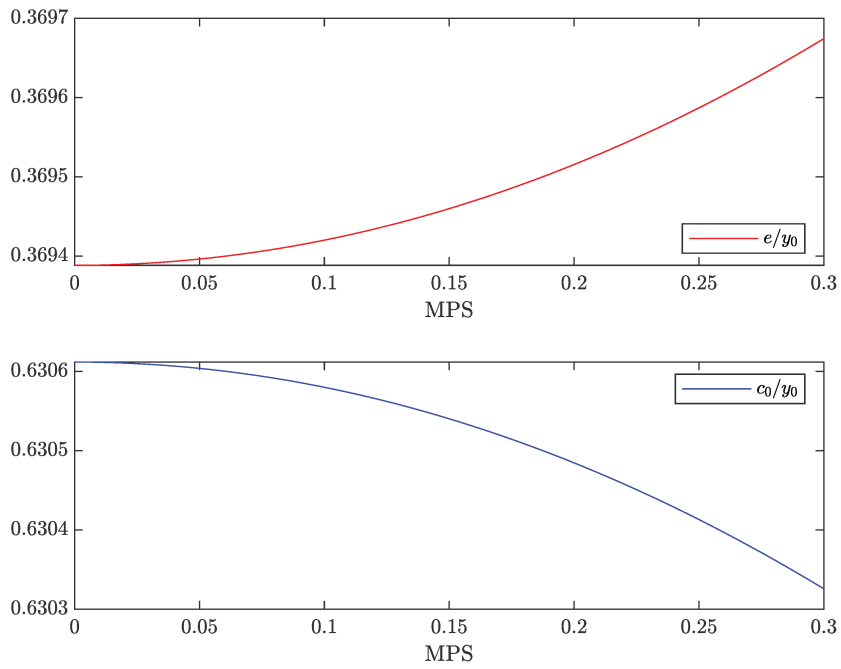


Figure 4.1: The effect of increasing risk on the depreciation rate of HC (increasing mean-preserving spreads (MPS)) on the optimal consumption and investment in HC

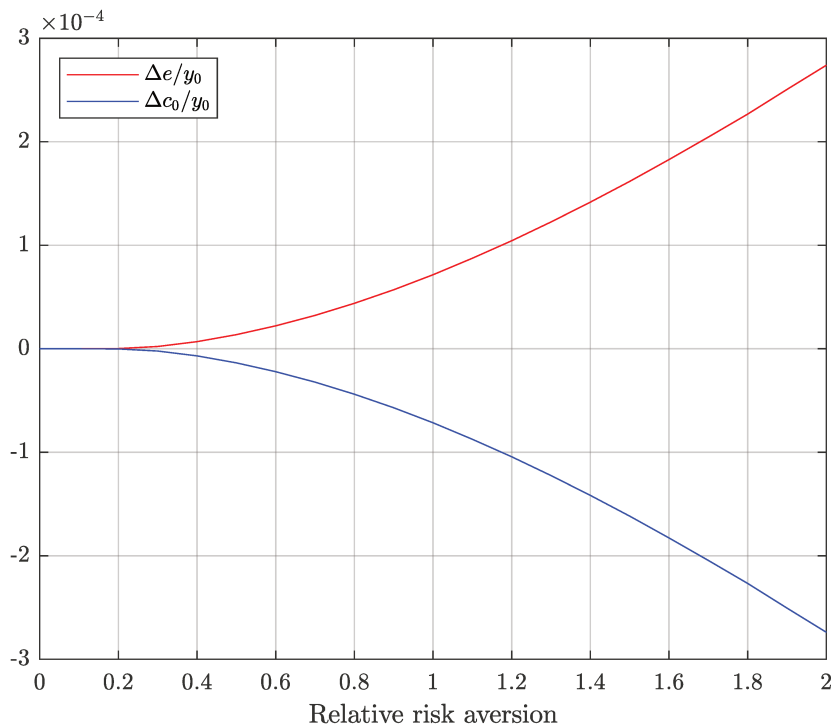


Figure 4.2: The effect of increasing relative risk aversion on the optimal consumption and investment in HC

subcases: one where the decision maker is ambiguity-neutral (linear  $\phi$ ), and the other where she is ambiguity-averse (strictly concave  $\phi$ ).

### Ambiguity-neutral agent

Let  $V_3$  be the objective function in this case. Ambiguity neutrality implies that  $\phi$  is linear, so that (4.20) can be simplified to:

$$V'_3(e) = -u'(y_0 - e) + \zeta y_0 e^{\alpha-1} \mathbb{E} \mathbb{E}_\theta f(\tilde{X}_\theta), \quad (4.33)$$

where  $f$  is the map defined in (4.15) and  $\tilde{X}_\theta$  defined in (4.22). We assume that ambiguity enters in the following manner.

**Assumption 4.8** (SSD ordering of scenarios). *Suppose that the scenario-conditional distributions could be ranked according to the sense of second order stochastic dominance (SSD). In particular,*

$$\tilde{\delta}_\theta = \delta + \sum_{j=1}^{\theta} \tilde{\epsilon}_j, \quad \forall \theta \in \Theta, \quad (4.34)$$

where  $\{\tilde{\epsilon}_j\}_{j \in \Theta}$  are white noises, i.e.,

$$\mathbb{E} \tilde{\epsilon}_j = 0, \quad \forall j \in \Theta. \quad (4.35)$$

This structure is essentially a sequence of mean preserving spreads (MPS), with the higher value of  $\theta$  associated to an increase in risk (or a deterioration in SSD) in the sense of Rothschild and Stiglitz (1970a). In fact Assumption 4.8 is the mildest ranking criterion of the conditional distributions in order to generate a differential effect on the level of optimal saving. As will be shown later, any ranking criterion stronger than SSD dominance will push optimal investment in human capital in the same direction.

**Proposition 4.2.** *Under risk prudence, adding ambiguity as a sequence of MPSs described under (Assumption 4.8) induces the ambiguity-neutral agent to raise investment in human capital relative to the deterministic case.*

*Proof.* See subsection 4.7.3. ■

### Ambiguity-averse agent

Finally, we examine the impact of ambiguity on the optimal choice of an ambiguity-averse agent. The structure of ambiguity remains unchanged. We also maintain the risk prudence assumption. Let us rewrite (4.20) as:

$$V'(e) = -u'(y_0 - e) + \zeta y_0^{1-\gamma} e^{\alpha-1} \left( J + K \times \mathbb{E} \mathbb{E}_\theta f(\tilde{X}_\theta) \right), \quad (4.36)$$



where  $f$  is defined in (4.15), and

$$J := \frac{\text{Cov}\left(\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})), \mathbb{E}_\theta f(\tilde{X}_\theta)\right)}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta})))}, \quad (4.37)$$

$$K := \frac{\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta})))}. \quad (4.38)$$

**Remark 4.4.** *It is noteworthy that  $J$  and  $K$  defined above are two fundamental effects of ambiguity aversion. On the one hand, the impact from  $J$  is due to pessimism according to Gollier (2011), in the sense of over-weighting worst scenarios. Naturally,  $J$  is nil under ambiguity neutrality. On the other hand, the impact from  $K$  results from preference for the timing of resolution of uncertainty according to Strzalecki (2013).*

**Assumption 4.9** (Ambiguity prudence). *The agent is ambiguity prudent. In particular,*

$$\phi' > 0, \phi'' < 0, \phi''' > 0. \quad (4.39)$$

**Proposition 4.3.** *When the scenario-conditional distributions can be ranked according to SSD described under Assumption 4.8, the ambiguity-averse representative agent with risk preference satisfying prudence (Assumption 4.7) raises investment in human capital relative to the ambiguity-neutral agent if his ambiguity preference satisfies either constant absolute ambiguity aversion (CAAA) or decreasing absolute ambiguity aversion (DAAA). The impact of ambiguity aversion is ambiguous under increasing absolute ambiguity aversion (IAAA).*

*Proof.* Let us first examine the impact of pessimism by proving the following lemma.

**Lemma 4.3.** *Pessimism under ambiguity aversion induces the agent to raise investment in human capital:  $J > 0$ .*

*Proof.* Recall that a worse scenario in the pure ambiguity structure previously defined means a deterioration in second order stochastic dominance (SSD). Since  $f$  is strictly decreasing and convex ( $-f$  strictly increasing and concave), an SSD deterioration raises  $\mathbb{E}_\theta f(\tilde{X}_\theta)$ . By contrast, since  $u$  is increasing and concave in  $\tilde{X}_\theta$ , this deterioration reduces  $\mathbb{E}_\theta u(\tilde{c}_{1\theta})$ , thus raising  $\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))$  by the concavity of  $\phi$ . Hence the two random variables in the covariance term move in the same direction, implying that  $J > 0$ . ■

Next, we prove that the preference for timing of resolution of uncertainty has the following properties.

**Lemma 4.4.**  *$K$  defined in (4.38) manifests:*

- preference for early resolution of uncertainty ( $K > 1$ ) if  $\phi$  satisfies DAAA;

- *indifference to the timing of resolution of uncertainty* ( $K = 1$ ) if  $\phi$  satisfies CAAA;
- *preference for late resolution of uncertainty* ( $K < 1$ ) if  $\phi$  satisfies IAAA.

*Proof.* Note that all the three ambiguity preferences satisfying DAAA, CAAA or IAAA also satisfy ambiguity prudence. Following Berger (2011), by ambiguity aversion and ambiguity prudence there exist an utility ambiguity premium  $\pi(e) \geq 0$  and an utility ambiguity precautionary premium  $\psi(e) \geq 0$  implicitly defined by:

$$\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta})) = \phi(\mathbb{E}\mathbb{E}_\theta u(\tilde{c}_{1\theta}) - \pi(e)), \quad (4.40)$$

$$\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})) = \phi'(\mathbb{E}\mathbb{E}_\theta u(\tilde{c}_{1\theta}) - \psi(e)). \quad (4.41)$$

Thus

$$K = \frac{\phi'(\mathbb{E}\mathbb{E}_\theta u(\tilde{c}_{1\theta}) - \psi(e))}{\phi'(\mathbb{E}\mathbb{E}_\theta u(\tilde{c}_{1\theta}) - \pi(e))}. \quad (4.42)$$

It is easy to see that a necessary and sufficient condition for ambiguity preference to satisfying DAAA is that  $-\phi'$  is more concave than  $\phi$  in the sense of Arrow-Pratt.<sup>3</sup> By definition, this implies that ceteri paribus, the ambiguity premium associated to  $-\phi'$  is greater than that associated to  $\phi$ . This is equivalent to saying that the ambiguity precautionary premium is greater than the ambiguity premium, implying that  $K \geq 1$  under DAAA. The arguments can be repeated for the CAAA and IAAA cases. ■

When  $K \geq 1$ , the preference for early resolution of uncertainty acts in the same direction as pessimism, inducing the agent to raise  $e$  since in this case,

$$V'(e) \geq V'_3(e), \quad (4.43)$$

implying  $V'(e_3) \geq 0$ . Let  $e_4$  be the unique solution to  $V'(e) = 0$ , then by the concavity of  $V$  proven in Lemma 4.2, we conclude that  $e_4 \geq e_3$ .

When  $K \leq 1$ , the two effects act in opposite directions, rendering the final impact on  $e$  ambiguous. ■

What happens to Proposition 4.3 if the conditional distributions are ranked by a stronger notion of stochastic dominance? Consider an improvement in first order stochastic dominance (FSD) of the distribution of  $\tilde{X}_\theta$  when  $\theta$  decreases. This is equivalent to an FSD deterioration in the distribution of  $\tilde{\delta}_\theta$ . In other words, the distribution of  $\tilde{\delta}_\theta$  dominates that of  $\tilde{\delta}_{\theta+1}$  in FSD for each  $\theta \in \Theta$ . Mathematically, this means

$$Pr(\tilde{\delta}_\theta \leq \delta_j) \leq Pr(\tilde{\delta}_{\theta+1} \leq \delta_j), \quad \forall \delta_j \in \mathcal{X}. \quad (4.44)$$

---

<sup>3</sup>See, for example, Chapter 2.5 of Gollier (2001) for a proof.

Since  $f$  is decreasing in  $\tilde{X}_\theta$ , the FSD improvement reduces its expectation. By contrast, since  $u(\tilde{c}_{1\theta}) = u(y_0 \tilde{X}_\theta^\mu)$  increasing in  $\tilde{X}_\theta$ , the FSD improvement raises  $\mathbb{E}_\theta u(\tilde{c}_{1\theta})$ , hence reducing  $\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))$ . Consequently  $J$  is also positive. In other words, if we allow for the mean of each conditional distribution to be lower or greater than  $\delta$ , but the ambiguity-neutral mean satisfies  $\mathbb{E}\mathbb{E}_\theta \tilde{\delta}_\theta = \delta$ , then Proposition 4.3 still holds. The same reasoning holds for stronger notion of stochastic dominance than FSD.<sup>4</sup>

**Corollary 4.1.** *The result of Proposition 4.3 still holds if ambiguity enters as a series of conditional distributions that can be ranked according to the FSD order as in (4.44), or stronger, so long as  $\mathbb{E}\mathbb{E}_\theta \tilde{\delta}_\theta = \delta$ .*

#### 4.3.4 Comparative statics of increasing ambiguity aversion

Does increasing ambiguity aversion raise optimal investment in human capital? Let us focus on the CAAA and DAAA cases.

Consider first the impact on the preference for timing of ambiguity resolution. Clearly it is nil under CAAA since the ambiguity premium and ambiguity precautionary premium are always equal. Under DAAA, which belongs to the class of hyperbolic absolute ambiguity aversion (HAAA) second order utility functions, we can write the measure of absolute ambiguity prudence  $P(\cdot)$  as:

$$P(U) = \left(1 + \frac{1}{\sigma}\right) A(U), \quad \sigma > 0, \quad (4.45)$$

where  $A(z) > 0$  is the measure of absolute ambiguity aversion defined by

$$A(U) = \left(a + \frac{U}{\sigma}\right)^{-1}. \quad (4.46)$$

If  $a = 0$ , then  $\phi$  satisfies CRAA with constant relative ambiguity aversion equal to  $\sigma$ .<sup>5</sup>

Denote:

$$\tilde{U}(e) = \mathbb{E}_\theta u(\tilde{c}_{1\theta}) \equiv \mathbb{E}_\theta u(\tilde{c}_{1\theta}(e)). \quad (4.47)$$

Define the ambiguity-neutral expected utility:

$$U(e) = \mathbb{E}\tilde{U}(e). \quad (4.48)$$

---

<sup>4</sup>For example, it holds also if the conditional distributions are ranked according to the probability ratio (PR), the hazard rate (HR), or the likelihood ratio (LR). See, for example, the Appendix of Krishna (2009), or Levy (2015) for further discussion.

<sup>5</sup>Gollier (2001) discusses the properties of the utility functions belonging to the hyperbolic absolute risk aversion (HARA) class. By analogy, a HAAA second order utility function can be written as  $\phi(U) = \xi \left(a + \frac{U}{\sigma}\right)^{1-\sigma}$ , where  $a + \frac{U}{\sigma} > 0$ . Monotonicity and ambiguity aversion require  $\xi \frac{1-\sigma}{\sigma} > 0$ .

**Assumption 4.10** (Pure ambiguity). *Assume that ambiguity is a zero-mean risk added to  $U(e)$ . Specifically,*

$$\tilde{U}(e) = U(e) + \tilde{\eta}, \quad \mathbb{E}\tilde{\eta} = \sum_{\theta} q_{\theta}\tilde{\eta}(\theta) = 0. \quad (4.49)$$

Then the utility ambiguity premium  $\pi(e)$  and the utility ambiguity precautionary premium  $\psi(e)$  satisfy:

$$\mathbb{E}\phi(U(e) + \tilde{\eta}) = \phi(U(e) - \pi(e)), \quad (4.50)$$

$$\mathbb{E}\phi'(U(e) + \tilde{\eta}) = \phi'(U(e) - \psi(e)). \quad (4.51)$$

**Lemma 4.5.** *Under Assumption 4.10 and DAAA, the preference for early resolution of ambiguity manifest by  $K$  in (4.38) is increasing in ambiguity aversion.*

*Proof.* Following Pratt (1964) and Arrow (1965), a Taylor approximation around  $U(e)$  yields:

$$\begin{aligned} \mathbb{E}\phi(U(e) + \tilde{\eta}) &\approx \phi(U(e)) + \phi'(U(e))\mathbb{E}\tilde{\eta} + 0.5\phi''(U(e))\mathbb{E}\tilde{\eta}^2 \\ &= \phi(U(e)) + 0.5\phi''(U(e))\mathbb{E}\tilde{\eta}^2, \end{aligned} \quad (4.52)$$

where the second line follows from  $\tilde{\eta}$  being a zero-mean risk. Clearly, the smaller the risk, the more precise is the approximation. Similarly, a first order approximation around  $U(e)$  for the RHS of (4.50) gives:

$$\phi(U(e) - \pi(e)) \approx \phi(U(e)) + U(e)\phi'(U(e)). \quad (4.53)$$

Equating (4.52) and (4.53), we arrive at the familiar Arrow-Pratt's approximation:

$$\pi(e) \approx 0.5A(U(e))V(\tilde{\eta}), \quad (4.54)$$

where  $V(\tilde{\eta}) = \mathbb{E}\tilde{\eta}^2$  denotes the variance of  $\tilde{\eta}$ . The larger the the variance of the priors (increasing ambiguity), the larger is the agent's willingness to pay to eliminate ambiguity. Likewise for the utility precautionary ambiguity premium:

$$\begin{aligned} \psi(e) &\approx 0.5P(U(e))V(\tilde{\eta}), \\ &= 0.5\left(1 + \frac{1}{\sigma}\right)A(U(e))V(\tilde{\eta}), \end{aligned} \quad (4.55)$$

where the second line results from (4.45). Hence for each each fixed  $e$ , the elasticity of

### 4.3. OPTIMAL INVESTMENT IN HUMAN CAPITAL IN ABSENCE OF PHYSICAL CAPITAL

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$K$  with respect to  $A(U(e))$  is:

$$\begin{aligned} \frac{\partial K / \partial A(U(e))}{K} &= A(U(e) - \psi(e)) \frac{\partial \psi(e)}{\partial A(U(e))} - A(U(e) - \pi(e)) \frac{\partial \pi(e)}{\partial A(U(e))} \\ &= 0.5V(\tilde{\eta}) \frac{[A(U(e) - \psi(e)) - A(U(e) - \pi(e))]}{\sigma}. \end{aligned} \quad (4.56)$$

Since  $\phi$  satisfies DAAA, the term in the square bracket on the RHS of (4.56) is positive. ■

Thus under DAAA, the higher the degree of ambiguity aversion, the higher the preference for early resolution of ambiguity, as illustrated in Figure 4.3 for a DM with CRAA and logarithmic utility. In other words, increasing ambiguity aversion has a positive impact on the HC investment. The higher the degree of ambiguity (the larger the variance of the priors), the larger this effect.

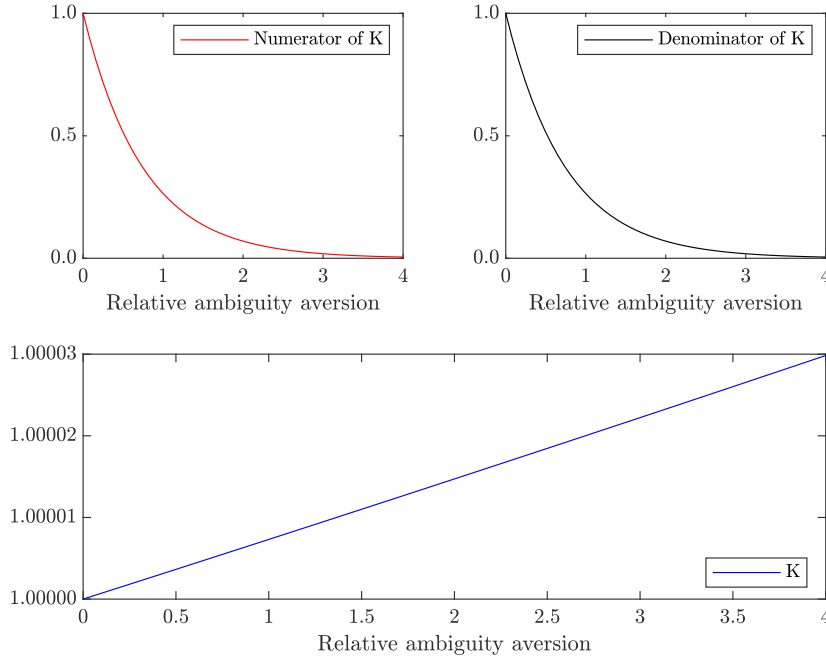


Figure 4.3: Impact of increasing ambiguity aversion on the preference of timing for resolution of ambiguity

How about the impact coming from pessimism? Observe that (4.37) can be rewritten as:

$$J = \frac{\text{Cov} \left( \phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})), \mathbb{E}_\theta f(\tilde{X}_\theta) \right)}{\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}) - \pi(e))}. \quad (4.57)$$

For any fixed  $e$ , increasing ambiguity aversion raises the utility ambiguity premium  $\pi(e)$ . This, however, does not imply a reduction in the denominator of  $J$ , since the curvature of  $\phi$  is also varying. Likewise the direction of change of the numerator is

also ambiguous. We thus resort to a numerical exercise to understand the impact of increasing ambiguity aversion on  $J$ . Figure 4.4 shows that although increasing ambiguity aversion has a non-monotone effect on the numerator of  $J$ , its impact on the denominator dominates, resulting in a net positive impact on the HC investment.

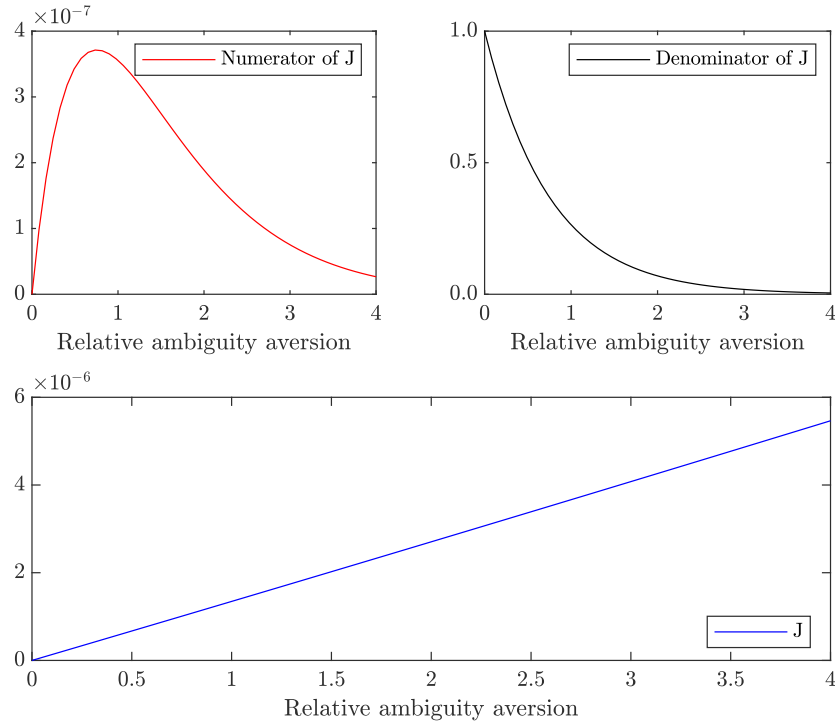


Figure 4.4: Pessimism due to increasing ambiguity aversion

Since both effects of ambiguity aversion (preference for timing of uncertainty resolution and pessimism) point in the same direction, it is no surprise that the final impact of increasing ambiguity aversion on the investment in human capital is positive, as shown in Figure 4.5. The intuition is that increasing ambiguity aversion essentially reduces the utility ambiguity-equivalent of the next period's uncertain income, which raises this period's savings in order to smooth consumption (across scenarios). The preference for smoothing across scenarios is stronger the higher the degree of ambiguity aversion.

#### 4.4 Optimal investment in human capital in presence of physical capital

We saw in the previous section that the introduction of risk or ambiguity raises the optimal level of investment in human capital. We might suspect that this is a consequence of having only one type of capital, that in presence of a risk-free physical capital, the

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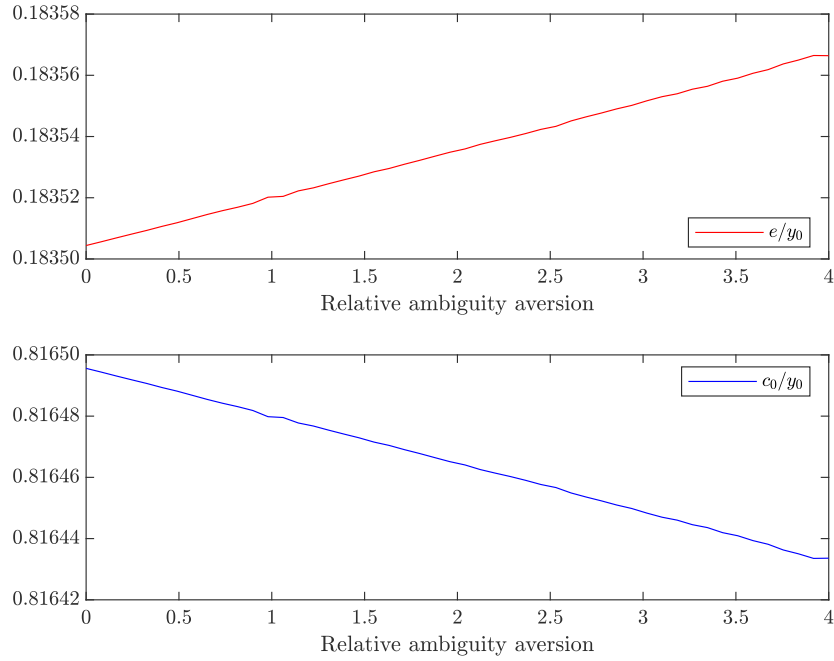


Figure 4.5: Increasing ambiguity and optimal saving and consumption

opposite would hold. Let us now consider a more general model with physical capital. In particular, the representative household faces the following problem:

$$\max_{c_0 \geq 0, e \geq 0, s \geq 0} u(c_0) + \beta \phi^{-1}(\mathbb{E} \phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))) \quad (4.58)$$

$$s.t. \quad c_0 + e + s = y_0, \quad (4.59)$$

$$\tilde{c}_{1\theta} = \tilde{y}_{1\theta}, \quad \theta \in \Theta, \quad (4.60)$$

$$y_t = F(k_t, h_t), \quad t \in \{0, 1\}, \quad (4.61)$$

$$k_1 = (1 - \delta_k)k_0 + s, \quad \delta_k \in [0, 1], \quad (4.62)$$

$$\tilde{h}_{1\theta} = h_0(e^\alpha + 1 - \tilde{\delta}_\theta), \quad \alpha \in (0, 1), \theta \in \Theta, \quad (4.63)$$

$$k_0 > 0, h_0 > 0 \text{ given}, \quad (4.64)$$

where  $\delta_k$  is the depreciation rate of physical capital, which is assumed to be deterministic. We maintain that the production function is Cobb-Douglas:

$$F(k_t, h_t) = k_t^{1-\mu} h_t^\mu, \quad \mu \in (0, 1). \quad (4.65)$$

Note that for simplicity we have set  $A = B \equiv 1$ . Our new objective function is:

$$V(e, s) = u(y_0 - s - e) + \beta \phi^{-1}(\mathbb{E} \phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))), \quad (4.66)$$

where

$$\tilde{c}_{1\theta} = k_1^{1-\mu} \tilde{h}_{1\theta}^\mu = k_1^{1-\mu} h_0^\mu (e^\alpha + 1 - \tilde{\delta}_\theta)^\mu. \quad (4.67)$$

Denote  $\frac{\partial V(e,s)}{\partial e} \equiv V_e(e,s)$  and  $\frac{\partial V(e,s)}{\partial s} \equiv V_s(e,s)$ . We have:

$$V_e(e,s) = -u'(c_0) + \beta \mu \alpha k_1^{1-\mu} h_0 e^{\alpha-1} \frac{\mathbb{E} \phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})) \mathbb{E}_\theta u'(\tilde{c}_{1\theta}) \tilde{h}_{1\theta}^{\mu-1}}{\phi'(\phi^{-1}(\mathbb{E} \phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}, \quad (4.68)$$

$$V_s(e,s) = -u'(c_0) + \beta(1-\mu) k_1^{-\mu} \frac{\mathbb{E} \phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})) \mathbb{E}_\theta u'(\tilde{c}_{1\theta}) \tilde{h}_{1\theta}^\mu}{\phi'(\phi^{-1}(\mathbb{E} \phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}. \quad (4.69)$$

Note that in this case the strict concavity of the objective function with respect to each argument (which is implied by Assumption 4.6) does not guarantee that it is jointly concave in both. For simplicity, we assume that the utility function is logarithmic. With log utility, the FOCs are indeed sufficient.

**Lemma 4.6.** *If the utility function is logarithmic and the second-order utility function satisfies linear absolute ambiguity tolerance (Assumption 4.6), then the objective function (4.66) is jointly concave in both arguments.*

*Proof.* See subsection 4.7.1. ■

Denote

$$U(s,e) \equiv \mathbb{E} \mathbb{E}_\theta u(\tilde{c}_{1\theta}). \quad (4.70)$$

Let  $\pi(s,e)$  and  $\psi(s,e)$  be the utility ambiguity premium and the utility ambiguity precautionary premium, respectively defined by

$$\mathbb{E} \phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta})) = \phi(U(s,e) - \pi(s,e)), \quad (4.71)$$

$$\mathbb{E} \phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})) = \phi'(U(s,e) - \psi(s,e)). \quad (4.72)$$

Observe that under log utility, we can rewrite (4.68) and (4.69) as:

$$V_e(s,e) = -u'(c_0) + \beta \alpha \mu e^{\alpha-1} \left( L + M \mathbb{E} \mathbb{E}_\theta (\tilde{X}_\theta^{-1}) \right), \quad (4.73)$$

$$V_s(s,e) = -u'(c_0) + \beta(1-\mu) k_1^{-1} M, \quad (4.74)$$

where  $\tilde{X}_\theta \equiv e^\alpha + 1 - \tilde{\delta}_\theta$  as before and

$$L = \frac{\text{Cov}(\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})), \mathbb{E}_\theta (\tilde{X}_\theta^{-1}))}{\phi'(\phi^{-1}(\mathbb{E} \phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}, \quad (4.75)$$

$$M = \frac{\phi'(U(s,e) - \psi(s,e))}{\phi'(U(s,e) - \pi(s,e))}. \quad (4.76)$$



#### 4.4.1 Deterministic depreciation

In this case the depreciation rate of human capital is known to be  $\delta$ . Observe that in this case  $L = 0$  and  $M = 1$ . Denote by  $V^1(s, e)$  the objective function under perfect foresight. We have:

$$V_e^1(s, e) = -u'(c_0) + \beta\alpha\mu e^{\alpha-1}X^{-1}, \quad (4.77)$$

$$V_s^1(s, e) = -u'(c_0) + \beta(1 - \mu)k_1^{-1}. \quad (4.78)$$

Let  $(e_1, s_1)$  be optimal for the deterministic problem. Denote

$$Q(s, e) = \frac{(1 - \delta_k)k_0 + s}{y_0 - s - e}. \quad (4.79)$$

Observe that  $Q$  so defined is strictly increasing in both of its arguments. Mathematically

$$\frac{\partial Q(s, e)}{\partial j} \equiv Q_j(s, e) > 0, \quad j \in \{s, e\}. \quad (4.80)$$

Then from (4.78):

$$Q(s_1, e_1) = \beta(1 - \mu). \quad (4.81)$$

#### 4.4.2 Risky depreciation

**Proposition 4.4.** *Under logarithmic utility, the introduction of a zero-mean risk to the depreciation rate of human capital raises the optimal level of investment in human capital and reduces the optimal investment in physical capital.*

*Proof.* Notice that under risk, it still holds that  $L = 0$  and  $M = 1$ . Denote by  $V^2(s, e)$  the objective function in this case. The FOC with respect to  $s$  rests unchanged, while the FOC with respect to  $e$  becomes:

$$V_e^2(s, e) = u'(c_0) + \beta\mu\alpha e^{\alpha-1}\mathbb{E}\tilde{X}^{-1}. \quad (4.82)$$

From the FOC with respect to  $s$ :

$$Q(s_2, e_2) = \beta(1 - \mu) = Q(s_1, e_1), \quad (4.83)$$

implying, in view of (4.80) that

$$(e_2 - e_1)(s_2 - s_1) \leq 0. \quad (4.84)$$

On the other hand, by the convexity of the map  $h \mapsto h^{-1}$ , we have by Jensen inequality

that:

$$\mathbb{E}\tilde{X}^{-1} > (\mathbb{E}\tilde{X})^{-1} = X^{-1} \equiv (1 - \delta + e^\alpha)^{-1}, \quad \forall e. \quad (4.85)$$

By optimality  $V_s^2(s_2, e_2) = V_e^2(s_2, e_2) = 0$ , which implies:

$$\begin{aligned} \frac{1}{y_0 - s_2 - e_2} &= \beta(1 - \mu)k_1^{-1} = \beta\mu\alpha e_2^{\alpha-1}\mathbb{E}\tilde{X}^{-1} \\ &> \beta\mu\alpha e_2^{\alpha-1}(1 - \delta + e_2^\alpha)^{-1}, \end{aligned} \quad (4.86)$$

where the second line results from (4.85). Let  $\xi$  be the map defined by

$$\xi(s, e) = \frac{e + (1 - \delta)e^{1-\alpha}}{(1 - \delta_k)k_0 + s}. \quad (4.87)$$

Then (4.86) is equivalent to:

$$\xi(s_2, e_2) > \frac{\alpha\mu}{1 - \mu} = \xi(s_1, e_1), \quad (4.88)$$

where the equality follows from the optimality of  $(s_1, e_1)$  in the deterministic case. Observe that from (4.84), either of the followings must hold:

$$e_2 \geq e_1, s_2 \leq s_1, \quad (4.89)$$

$$e_2 \leq e_1, s_2 \geq s_1. \quad (4.90)$$

Since  $\xi$  defined in (4.87) is strictly increasing in  $e$  and strictly decreasing in  $s$ , the case (4.90) would imply  $\xi(s_2, e_2) \leq \xi(s_1, e_1)$ , a contradiction to (4.88). Combining (4.89) and (4.87) yields:

$$e_2 > e_1, s_2 < s_1, \quad (4.91)$$

completing the proof. ■

**Remark 4.5.** *The result of Proposition 4.4 is illustrated in Figure 4.6. Intuitively, increasing risk reduces the certainty equivalent of the next period's uncertain income (output) due to both risk aversion and the concavity of the production function. This raises savings due to a precautionary motive (to smooth consumption across states in the next period) and consequently reduces this period's consumption. Although aggregate saving is higher, the allocation to each type of capital moves in opposite direction. The increase in the HC investment has a self-insurance motive. This effect is stronger the more risk averse and/or concave the production function is with respect to HC (the closer  $\mu$  is to zero). The reduction in the PC investment would reduce the variance of the uncertain income, which is preferred by a risk-averse agent. In other words, among the pairs  $(s, e)$  that yield the same expected next period's output, a risk-averse agent would always prefer to allocate as much as possible to the investment in human*

capital.

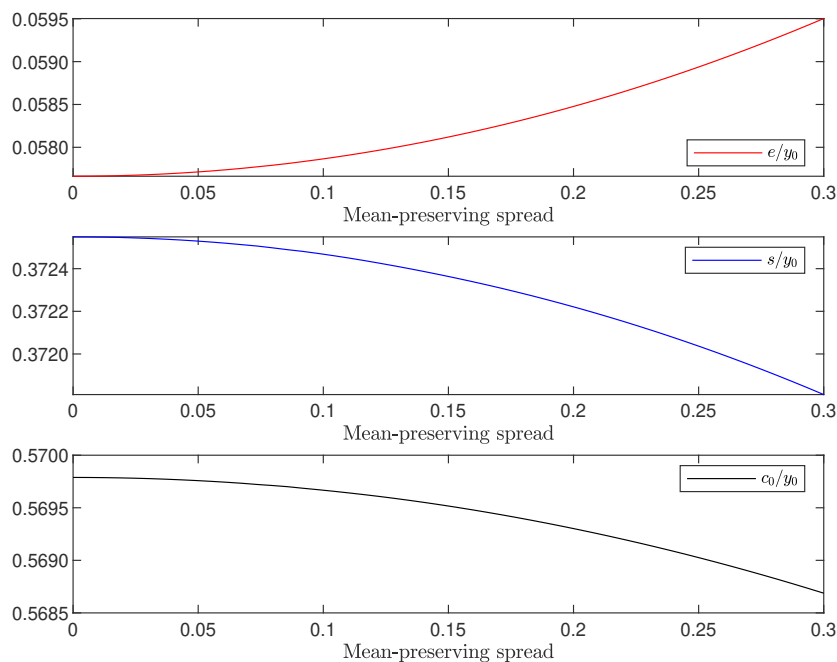


Figure 4.6: The impact of increasing risk (MPS) on investment and consumption. The deterministic case corresponds to MPS being equal to zero.

### 4.4.3 Ambiguous depreciation

#### Ambiguity-neutral agent

Let us first consider the case of an ambiguity-neutral representative agent. Denote by  $V^3$  the objective function in this case. Observe that since  $\phi$  is linear under ambiguity neutrality, we have in this case that  $L = 0$  and  $M = 1$ . Let  $(s_3, e_3)$  be optimal for  $V^3$ , then

$$V^3(s_3, e_3) = 0. \quad (4.92)$$

The objective is to examine the impact of ambiguity aversion on the optimal levels of investment in each type of capital.

#### Ambiguity-averse agent

Let  $V^4$  be the objective function under ambiguity aversion and  $(s_4, e_4)$  be the optimal solution. As in the previous section, we need to examine two effects: one from pessimism (the sign of  $L$  in (4.75)) and the other from the preference for the timing of

uncertainty resolution (the magnitude of  $M$  in (4.76)). Recall that

$$L = \frac{\text{Cov}(\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})), \mathbb{E}_\theta(\tilde{X}_\theta^{-1}))}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta})))}. \quad (4.93)$$

Since the map  $h \mapsto h^{-1}$  is strictly decreasing and convex, a higher value of  $\theta$  (deterioration in SSD) would raise  $\mathbb{E}_\theta \tilde{h}_{1\theta}^{-1}$ . On the other hand since  $u$  is strictly increasing and concave, an SSD deterioration would lower  $\mathbb{E}_\theta u(\tilde{c}_{1\theta})$ , thus increasing  $\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))$  since  $\phi$  is strictly concave. Hence  $L > 0$ . As proven in the previous section, the magnitude of  $M$  depends on the property of the ambiguity preference. In particular, it is greater than one under DAAA and equal to one under CAAA.

**Proposition 4.5.** *When the scenario-conditional distributions can be ranked according to SSD (Assumption 4.8), the ambiguity-averse representative agent with logarithmic utility raises investment in human capital relative to the ambiguity-neutral agent if his ambiguity preference satisfies CAAA.*

*Proof.* Recall that under CAAA, the impact from the preference for timing of resolution of uncertainty is null, i.e.,  $M = 1$ . Observe that the FOC with respect to  $s$  is unchanged compared to the previous cases since physical capital is always risk-free, hence  $Q(s_4, e_4) = Q(s_3, e_3)$ , which implies:

$$(e_4 - e_3)(s_4 - s_3) \leq 0. \quad (4.94)$$

Hence either of the followings must hold:

$$e_4 \geq e_3, s_4 \leq s_3, \quad (4.95)$$

$$e_4 \leq e_3, s_4 \geq s_3. \quad (4.96)$$

On the other hand, the FOCs imply:

$$\frac{\alpha\mu}{1-\mu} = \frac{e_4^{1-\alpha}}{(L + \mathbb{E}\mathbb{E}_\theta \tilde{X}_\theta^{-1})[(1-\delta_k)k_0 + s_4]} < \frac{e_4^{1-\alpha}}{\mathbb{E}\mathbb{E}_\theta \tilde{X}_\theta^{-1}[(1-\delta_k)k_0 + s_4]}, \quad (4.97)$$

where the third inequality comes from  $L$  being strictly positive. Let  $\xi^*$  be the map defined by:

$$\xi^*(s, e) = \frac{e^{1-\alpha}}{\mathbb{E}\mathbb{E}_\theta \tilde{X}_\theta^{-1}[(1-\delta_k)k_0 + s]}. \quad (4.98)$$

Observe that  $\xi^*$  is strictly increasing in  $e$  and strictly decreasing in  $s$ . Furthermore, from (4.97),

$$\xi^*(s_4, e_4) > \frac{\alpha\mu}{1-\mu} = \xi^*(s_3, e_3). \quad (4.99)$$

Clearly (4.96) cannot occur since this would yield a contradiction to (4.99). Combining (4.95) and (4.99) yields

$$e_4 > e_3, s_4 < s_3, \tag{4.100}$$

as desired. ■

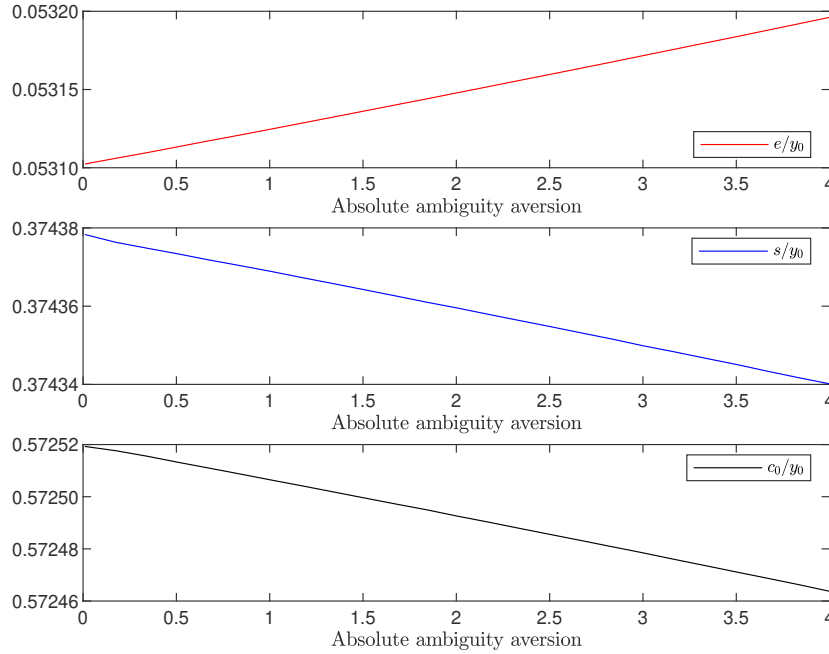


Figure 4.7: The impact of increasing ambiguity aversion on investment and consumption for a CAAA agent with logarithmic utility.

The result of Proposition 4.5 is illustrated in Figure 4.7. The direction of change is less clear analytically for the DAAA case since the effect of  $M$  on the investment in PC is no longer silent. Indeed, it is not difficult to see that in this case (4.94) does not necessarily hold although we still have (4.99). Thus the only conclusion we can draw, due to the monotonic behavior of the map  $\xi^*(\cdot, \cdot)$  is that either the HC investment rises, or the PC investment falls (relative to the ambiguity-neutral case), but not necessarily both. Numerically, Figure 4.8 suggests that increasing ambiguity aversion for a CRAA agent (a particular case of DAAA) has a similar effect on consumption and investment. Observe that relative ambiguity aversion  $\sigma = 0$  corresponds to an ambiguity-neutral agent.

## 4.5 Impact of uncertain effectiveness of human capital accumulation

Up to this point, it seems that the introduction of uncertainty into the simplified Ben-Porath model always leads to a rise in investment in human capital under DAAA and

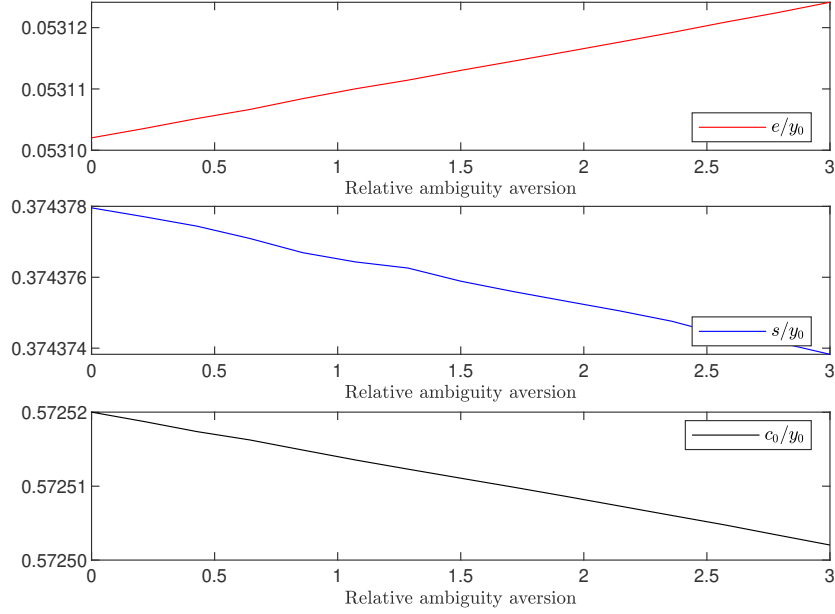


Figure 4.8: The impact of increasing ambiguity aversion on investment and consumption for a CRAA agent with logarithmic utility.

CAAA (the ambiguity preference most endorsed by empirical evidence). In this section, we wonder what happens if uncertainty is introduced to the model via the parameter  $B$ , the effectiveness of human capital accumulation. In particular, this means that the evolution of human capital (5) becomes:

$$\tilde{h}_{1\theta} = h_0(\tilde{B}_\theta e^\alpha + 1 - \delta), \quad (4.101)$$

where  $\delta \in [0, 1]$  is the deterministic depreciation rate of human capital. Would the main results of this paper up to now still hold? It turns out that for CRRA risk preference with constant relative risk aversion less than one, the direction of change will be reverse.

**Proposition 4.6.** *Under CRRA risk preference of degree  $\gamma \leq 1$ , the introduction of pure risk lowers investment in human capital.<sup>6</sup> The introduction of ambiguity as a series of MPSs around  $B$  also reduces investment in human capital for an ambiguity-neutral agent. An ambiguity-averse agent reduces investment under CAAA and IAAA; the direction of change is ambiguous under DAAA.*

*Proof.* See subsection 4.7.4. ■

The intuition behind this difference is that when uncertainty is in the depreciation

<sup>6</sup>Taking into account the preference towards ambiguity, the degree of relative risk aversion being less than one is supported by a number of experiments, among which Chakravarty and Roy (2009) and Berger and Bosetti (2016).

parameter, raising the investment in human capital is essentially investing in self insurance. As is well-known from existing insurance literature, increasing risk raises the demand for self-insurance. By contrast, when uncertainty enters through  $B$ , the optimal choice of  $e$  is viewed as optimal investment in an uncertain asset. Typically, there are two effects acting in opposite directions in this case. According to [Eeckhoudt et al. \(2011\)](#), while the pure increase in risk makes a risk-averse agent less interested in the investment (second-order effect), the sufficiently prudent agent is still induced to raise the investment via the precautionary channel (third-order effect). For CRRA risk preference with linear accumulation of human capital, for instance, it is easy to verify that "sufficient prudence" means that the degree of relative prudence, which is defined by the map  $x \mapsto -xu'''(x)/u''(x)$ , is greater than two. It is straightforward to see that this translates to the degree of relative risk preference being greater than one for CRRA preferences. This problem treats a non-linear human capital law of motion, and thus it is much more complicated to obtain a clear-cut rule for "sufficient prudence". Figure ?? illustrates such a case. Note that in this figure, the parameter  $\epsilon$  controls the spread of the risky distribution of  $\tilde{B}$ . For example,  $\epsilon = 0.5$  means that the decision-maker faces a risk of losing or gaining half of the deterministic  $B$  with equal probabilities. In this case, for a risk  $\epsilon \approx 0.5$ , the agent with relative risk aversion of  $\gamma = 6$  is sufficiently prudent (third order effect dominates) and would raise the investment in human capital. For larger risks, risk aversion (second order effect) dominates, inducing him to lessen the investment in human capital. We end this section with an illustration on the impact of increasing ambiguity aversion. In particular, Figure ?? is drawn for the following parameters:

$\beta = 0.9$	$h_0 = 100$
$\alpha = 0.67$	$B = 1$
$\delta = 0.3$	$\tilde{B}_1 \sim (0.85B; 0.5; 1.15B)$
$\mu = 0.67$	$\tilde{B}_2 \sim (0.5B; 0.5; 1.5B)$
$A_0 = 1$	$(q_1, q_2) = (0.3, 0.7)$

This figure shows that for any values of relative risk aversion less than about 2, increasing relative ambiguity aversion reduces the investment in human capital. Noticeably, we observe the same impact of ambiguity aversion on a risk-neutral decision-maker ( $\gamma = 0$ ), although the effect kicks in only after the degree of ambiguity aversion is sufficiently high ( $\sigma \approx 0.6$ ); for smaller values of  $\sigma$ , the effect is also negative but is negligible.

## 4.6 Conclusion

The co-existence of over-education in some sectors and skills shortage in others can be explained by the sources of uncertainty faced by different types of individuals.

On the one hand, if uncertainty is on the net productivity of human capital accumulation, then the investment is viewed as one with increasingly uncertain return, making it less attractive to a decision maker who is uncertainty-averse (risk averse or ambiguity averse). This might be the culprit behind the lack of skills in technical sectors in developing countries, where the quality of training is highly questionable due to the lack of infrastructure. One case in point is Vietnam. Highly skilled workers and technicians are in great demand, but the quality of vocational schools across the countries is hardly consistent. Thus even if the expected return on investment in vocational training for households remain high, the highly uncertain outcome makes it much less attractive. To address this issue, public policies need to work on improving the quality of training as well as to communicating this improvement to the groups of interest. This would raise expected return *and* reduce uncertainty on the quality of vocational education, rendering investment in it more attractive.

On the other hand, if uncertainty is on the obsolescence parameter, then the investment in human capital also serves as a type of insurance against labor income fluctuations, which is assumed to be nonexistent due to market incompleteness. In this case, individuals facing idiosyncratic uncertainty are induced to invest more, leading to over-education.<sup>7</sup>

There is yet another implication of Proposition 4.3 and/or Proposition 4.1 on over-education. Individuals who do not have the means (being constrained by the first period's wage) to raise investment to the optimal level might opt for career choices that are less subject to obsolescence risks. Typically, they might accept jobs that pay less, where they are overqualified or over-educated in exchange for more security. This also causes a problem since numerous research has shown that overqualified workers are more likely to be dissatisfied at the workplace, leading to lower productivity. The policy response to this issue must also be multidimensional. Clearly, there remains the uncertainty-reducing role of the government by providing more precise data on the labor market conditions. Companies that work in sectors highly susceptible to uncertain obsolescence must also take a proactive approach in investing in their human resource. This should encourage uncertainty-averse individuals to be more willing to accept offers in these sectors, rather than migrating to where they are overqualified.

In a simple two-period framework, this work represents the first attempt to address ambiguous stochastic human capital accumulation, an issue that is increasingly relevant in the modern economy. In fact, the model is general enough to allow for an analysis of optimal investment in physical or financial capital. An abrupt technological change could render all existing machines obsolete. The burst of a financial bubble

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<sup>7</sup>Recall that investment in human capital always increases under DAAA and CAAA ambiguity preferences. The experimental evidence of Berger and Bosetti (2016) is in favor of these types of preferences under CRRA utilities.



could wipe out the value of financial assets in a blink of an eye. Indeed, the evolution of any types of capital are ridden with uncertainty nowadays.

The model has at least two short-comings. First, the potential welfare-enhancing role of social security is neglected and the welfare analysis thereof. Second, the model is static, leaving the question on long-run growth open. At least in these dimensions can future research extend.

## 4.7 Appendix of Proofs

### 4.7.1 Proof of Lemma 4.2

Let us define the function  $G : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$G(U) \equiv G(U_1, \dots, U_n) = \phi^{-1} \left( \sum_{\theta} q_{\theta} \phi(U_{\theta}) \right), \quad (4.102)$$

where  $U : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is the vector-valued function defined by:

$$U(e) = (U_1(e), \dots, U_n(e)), \quad (4.103)$$

where  $U_{\theta}(e) \equiv \mathbb{E}_{\theta} u(\tilde{c}_{1\theta}(e))$ , for each  $\theta \in \Theta$ . Notice that  $G$  is increasing in  $U$  since  $\phi$  is increasing (thus so is its inverse  $\phi^{-1}$ ). Also, by Lemma 8 of [Gollier \(2001\)](#), the function  $G$  is concave in  $\mathbb{R}^n$  under Assumption 4.6. Our goal is to show that the composite function  $G \circ U : \mathbb{R}_+ \rightarrow \mathbb{R}$  is concave in  $\mathbb{R}_+$ . It is easy to see that  $U : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is concave in  $\mathbb{R}_+$  since  $U_{\theta}$  is concave in  $e$  for all  $\theta$ . In particular, following [Dattorro \(2018\)](#), this means that for positive scalars  $e^1$  and  $e^2$ , and any  $\lambda \in (0, 1)$

$$U(\lambda e^1 + (1 - \lambda)e^2) \geq \lambda U(e^1) + (1 - \lambda)U(e^2), \quad (4.104)$$

where the notation  $\geq$  denotes an element-wise inequality. By the monotonicity of  $G$ ,

$$G(U(\lambda e^1 + (1 - \lambda)e^2)) \geq G(\lambda U(e^1) + (1 - \lambda)U(e^2)). \quad (4.105)$$

Furthermore, by the concavity of  $G$  in  $\mathbb{R}^n$ ,

$$G(\lambda U(e^1) + (1 - \lambda)U(e^2)) \geq \lambda G(U(e^1)) + (1 - \lambda)G(U(e^2)). \quad (4.106)$$

From (4.105) and (4.106), we conclude that  $G \circ U$  is concave in  $\mathbb{R}_+$ . Since  $u$  is strictly concave in  $e$  and  $\beta > 0$ , the objective function is the sum of two concave functions, so it is indeed strictly concave in  $e$ , as desired.

#### 4.7.2 Proof of Proposition 4.1

Since  $f$  is convex by Lemma 4.1, by Jensen inequality

$$\mathbb{E}u'(\tilde{c}_1)\tilde{X}^{\mu-1} \equiv \mathbb{E}f(\tilde{X}) > f(\mathbb{E}\tilde{X}) = u'(y_0X^\mu)X^{\mu-1}, \quad (4.107)$$

where the last equality has used the assumption of zero-mean risk via (4.26). Thus if  $e_2$  is optimal for  $V_2$ , we have shown that

$$V'_2(e_2) = 0 \implies V'_2(e_1) > 0, \quad (4.108)$$

implying  $e_2 > e_1$  by the strict concavity of  $V_2$ .

#### 4.7.3 Proof of Proposition 4.2

As shown in the proof of Proposition 4.1, the map  $f$  is strictly convex under risk prudence. By Jensen inequality,

$$\mathbb{E}f(\tilde{X}_\theta) \geq f(\mathbb{E}\tilde{X}_\theta) = f(X), \quad \forall \theta \in \Theta, \quad (4.109)$$

where the equality comes from Assumption 4.8. Hence if  $e_3$  is optimal for  $V_3$ , we have shown that:

$$V'_3(e_3) = 0 \implies V'_3(e_1) > 0, \quad (4.110)$$

implying  $e_3 > e_1$  by the concavity of  $V_3$ .

#### 4.7.4 Proof of Proposition 4.6

Under stochastic effectiveness of investment in human capital, the law of motion governing the accumulation of human capital (constraint (4.5)) becomes:

$$\tilde{h}_{1\theta} = h_0\tilde{Z}_\theta, \text{ where } \tilde{Z}_\theta \equiv \tilde{B}_\theta e^\alpha + 1 - \delta, \quad (4.111)$$

and  $\delta \in [0, 1]$  is the deterministic depreciation rate of human capital. Equation (4.20) becomes:

$$V'(e) = -u'(y_0 - e) + \nu y_0 e^{\alpha-1} \frac{\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))\mathbb{E}_\theta u'(\tilde{c}_{1\theta})\tilde{B}_\theta \tilde{Z}_\theta^{\mu-1}}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}, \quad (4.112)$$

where  $\nu \equiv \beta\mu\alpha$ . Thus under CRRA risk preference

$$V'(e) = -u'(c_0) + \nu y_0^{1-\gamma} e^{\alpha-1} \frac{\mathbb{E}\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))\mathbb{E}_\theta \tilde{B}_\theta f(\tilde{Z}_\theta)}{\phi'(\phi^{-1}(\mathbb{E}\phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))))}, \quad (4.113)$$

where  $f$  is the mapping defined in (4.15). Let  $V_1$ ,  $V_2$ ,  $V_3$ , and  $V$  denote the objective function in the deterministic, unambiguous stochastic (pure risk), ambiguous stochastic with ambiguity neutral agent, and ambiguous stochastic with ambiguity averse agent case, respectively. Observe that Lemma 4.2 still applies.

As before, we study three different settings: pure risk, ambiguity in the sense of a sequence of MPSs around  $B$  with ambiguity-neutral agents, then with ambiguity-averse agents.

### Optimal investment under pure risk

When  $B$  is deterministic, we have:

$$V_1'(e) = -u'(c_0) + vy_0^{1-\gamma} e^{\alpha-1} Bf(Z), \quad (4.114)$$

where  $Z$  denotes the deterministic value of  $\tilde{Z}_\theta$ . Let  $\tilde{Z} = \tilde{B}e^\alpha + 1 - \delta$ . Then (4.113) simplifies to:

$$V_2'(e) = -u'(c_0) + vy_0^{1-\gamma} e^{\alpha-1} \mathbb{E}g(\tilde{B}), \quad (4.115)$$

where  $g$  is the mapping defined by:

$$g(B) = Bf(Z) \equiv B(Be^\alpha + 1 - \delta)^\rho, \quad (4.116)$$

where  $\rho = \mu(1-\gamma) - 1 < 0$  for all  $\gamma \geq 0$  and  $\mu \in (0, 1)$ . We now show that  $g$  is increasing and concave for  $\gamma \leq 1$ . Indeed, in this case  $\rho \in [-1, 0)$ , so:

$$g'(B) = Z^{\rho-1} [Be^\alpha(1+\rho) + 1 - \delta] > 0, \quad (4.117)$$

and

$$g''(B) = e^\alpha Z^{\rho-2} \rho [Be^\alpha(1+\rho) + 2(1-\delta)] < 0. \quad (4.118)$$

Hence by Jensen inequality,

$$\mathbb{E}\tilde{B}f(\tilde{Z}) \equiv \mathbb{E}g(\tilde{B}) < g(\mathbb{E}\tilde{B}) = Bf(Z), \quad (4.119)$$

implying

$$V_2'(e_2) = 0 \implies V_2'(e_1) < 0, \quad (4.120)$$

which in turn implies  $e_2 < e_1$  by the strict concavity of  $V_2$ .

Notice that for  $\gamma > 1$ , the sign of  $g''$  is in general ambiguous, but the higher is  $B$ , the more likely is  $g''$  to be positive, inducing *more* investment in human capital. This is intuitive since higher  $B$  raises the mean of the risky investment.

### Optimal investment under ambiguity neutrality

The first order derivative now reads:

$$\begin{aligned} V'_3(e) &= u'(c_0) + v y_0^{1-\gamma} e^{\alpha-1} \mathbb{E} \mathbb{E}_\theta \tilde{B}_\theta f(\tilde{Z}_\theta). \\ &= u'(c_0) + v y_0^{1-\gamma} e^{\alpha-1} \mathbb{E} \mathbb{E}_\theta g(\tilde{B}_\theta) \end{aligned} \quad (4.121)$$

The proof is completed by recognizing that  $g$  is strictly increasing and concave for  $\gamma \leq 1$ .

### Optimal investment under ambiguity aversion

In this case we also have two effects from pessimism and preference for early resolution of uncertainty. In particular,

$$V'(e) = -u'(c_0) + v y_0^{1-\gamma} e^{\alpha-1} \left( J + K \times \mathbb{E} \mathbb{E}_\theta g(\tilde{Z}_\theta) \right), \quad (4.122)$$

where

$$J = \frac{\text{Cov}(\phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta})), \mathbb{E}_\theta g(\tilde{B}_\theta))}{\phi'(\phi^{-1}(\mathbb{E} \phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta})))}, \quad (4.123)$$

and

$$K = \frac{\mathbb{E} \phi'(\mathbb{E}_\theta u(\tilde{c}_{1\theta}))}{\phi'(\phi^{-1}(\mathbb{E} \phi(\mathbb{E}_\theta u(\tilde{c}_{1\theta})))}. \quad (4.124)$$

The rest of the proof is almost identical, except that now the concave function  $g$  takes place of the convex function  $f$  in (4.37) and (4.38).

We remark also as in the case of stochastic depreciation, the result of Proposition 4.6 is robust to any ranking criterion stronger than SSD. This is a direct consequence of the fact that in this case the function  $g$  is increasing.



# 5

## Economic dynamics with renewable resources and pollution

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## 5.1 Introduction

Natural resources play an important role in the economy. Intriguingly, they are not always a boon to economic growth. While abundant resources may help a country overcome the fixed costs problem and avoid poverty traps (Le Van et al., 2010), they might induce excessive consumption, stagnating the economy in the long run (Rodriguez and Sachs, 1999; Eliasson and Turnovsky, 2004).

The existing literature has also explored the impact of natural resources manifest in the form of externalities in a two-sector economy. In particular, consider an economy with an industrial production sector and a natural resource exploitation sector (such as forestry or fishery). While the latter may enhance the productivity of the former sector or provide an additional source of income, the former typically engages in polluting industrial activities at the detriment of the renewable resources, as has been studied by Beltratti et al. (1998), and Ayong Le Kama (2001). These authors consider the renewable resource as a consumption good as well as an input for production. The regenerating capacity of the resource is impaired by pollution from the final good producing sector. Under suitable conditions, the existence of a stationary state and its local stability are proved.

This approach is appealing, but as Wirl (2004) has observed, there is always room for limit cycles. Multiple long-run outcomes exist and are separated by a threshold, even under a sufficiently convexity structure of the model. In this paper, we propose a new approach to study a two-sector economy with a renewable resource under discrete time configuration. We specify the conditions that ensure long-run convergence of the economy. Our approach can be applied not only to the work of Beltratti et al. (1998) and Ayong Le Kama (2001), but also for other multisector models.

We consider a two-sector economy with an industrial sector that uses intermediate inputs to produce a final consumption good, and another sector, called the exploitation sector, which engages in exploiting a renewable resource. This resource can be sold directly at an exogenously determined market price, generating an additional source of income. We assume there is a representative consumer who lives infinitely, and seeks to allocate total incomes between consumption and capital investment to maximize intertemporal utility. She can use the income from the exploitation sector to invest in physical capital, or to purchase consumption good.

This problem is challenging since we cannot follow the standard techniques laid out in the dynamic programming literature to study the long-term behavior of the economy. Usually, as well presented in Stokey (1989), or Le Van and Dana (2003), an analysis of the Euler equations provides us with information on the optimal choice of investment and exploitation. In our economy, such a technique is inapplicable since we are

not sure whether the optimal choice belongs to the interior of the domain of definition. Moreover, the presence of two control variables rules out super-modularity<sup>1</sup>.

To overcome this difficulty, we introduce the concept *net gain of stock*, which is the difference between the discounted value of production, and the existing resource stock and capital.<sup>2</sup> This concept is similar to the *net gain of investment* presented by Majumdar (1982), Dechert and Nishimura (2012), Mitra and Ray (1984) or Kamihigashi et al. (2007). As we shall see, the analysis of the *net gain of investment* can help illuminate our understanding of economic dynamics. Following Kamihigashi et al. (2007), we prove that the economy evolves to increase the value of the *net gain of stock* some day in the future. This property has an important implication. It ensures that in the long run, the economy gets very close a steady state<sup>3</sup>. In this article we specify the conditions for the uniqueness of steady states.

The rest of the article is organized as follows. Section 5.2 considers the problem of the representative consumer without the negative externality of the production sector on the exploitation sector. Section 5.3 takes into account the negative externalities of the polluting industrial sector on the regenerating capacity of the other sector. This chapter contains the main results of our paper, including the characterization of the conditions for the uniqueness of the steady states, and the long-run convergence of the economy. All proofs are given in the appendix.

## 5.2 Model without emission

### 5.2.1 Fundamentals

We consider a two-sector economy, one engaging in industrial activities to produce a final consumption good, and the other in the exploitation of a renewable resource.

The industrial sector is characterized by a production function  $f$ , satisfying the usual conditions in literature, such as monotonicity, concavity and Inada. To simplify the exposition, we assume without loss of generality (WLOG) that physical capital depreciates fully after each period.

The exploitation sector is characterized by the regenerating function  $\eta$  and the price of the renewable resource  $\theta > 0$ , which is exogenously determined. To fix ideas, we assume in this section that the function  $\eta$  depends only on the natural resource stock and not on the industrial activities. In other words, the industrial activities have no negative effect on the renewable resource.

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<sup>1</sup>For the definition and a detailed survey on the super-modular economy, see the works of Amir (1996) and Amir (2005).

<sup>2</sup>For example in a one-dimensional economy, given the discount rate  $\beta$ , the production function  $f$  and capital stock  $k$ , the *net gain of stock* is equal to  $\beta f(k) - k$ .

<sup>3</sup>If the steady state is unique, the convergence is ensured.



At the beginning period of time  $t$ , the economy possesses a stock of capital  $k_t$  and a stock of the renewable resource  $y_t$ , which generate an output from production  $f(k_t)$  and a regenerated stock  $\eta(y_t)$  of the resource, respectively. Let  $x_t$  denote the amount of natural resource exploited by the agent and  $R_t$  the total revenue available to her at the beginning of period  $t$ . Clearly  $R_t = f(k_t) + \theta x_t$ . She then decides to allocate this revenue between current consumption  $c_t$  and next-period investment in physical capital  $k_{t+1}$ . Given the initial capital and natural stocks  $k_0$  and  $y_0$ , respectively, the representative agent solves the intertemporal optimization problem:

$$\begin{aligned} & \sum_{t=0}^{\infty} \beta^t u(c_t), \\ & c_t + k_{t+1} \leq f(k_t) + \theta x_t, \\ & y_{t+1} = \eta(y_t) - x_t, \\ & c_t, k_t, x_t, y_t \geq 0 \text{ for any } t \geq 0, \end{aligned}$$

where  $\beta \in (0, 1)$  is the discount factor.

By replacing  $x_t$  by  $\eta(y_t) - y_{t+1}$ , we can rewrite the problem as:

$$\begin{aligned} v(k_0, y_0) &= \max \sum_{t=0}^{\infty} \beta^t u(c_t), \\ & c_t + k_{t+1} + \theta y_{t+1} \leq f(k_t) + \theta \eta(y_t), \\ & y_{t+1} \leq \eta(y_t), \\ & c_t, k_t, y_t \geq 0 \text{ for any } t. \end{aligned}$$

Observe from the first constraint that with the presence of the natural resource as an additional source of revenue, the capital stock  $k_{t+1}$  can be greater than the output  $f(k_t)$  generated by the industrial sector. The second constraint says that the amount of exploitation cannot exceed the total amount of natural stock available.

For each  $(k, y) \in \mathbb{R}_+^2$ , define

$$\Gamma(k, y) = \{(k', y') \in \mathbb{R}_+^2 \text{ such that } k' + \theta y' \leq f(k) + \theta \eta(y) \text{ and } y' \leq \eta(y)\}.$$

A sequence  $\{(k_t, y_t)\}_{t=0}^{\infty}$  is feasible from  $(k_0, y_0)$  if  $\forall t \geq 0, (k_{t+1}, y_{t+1}) \in \Gamma(k_t, y_t)$ .

Let  $\Pi(k_0, y_0)$  denote the set of all feasible paths  $\{(k_t, y_t)\}_{t=0}^{\infty}$  from  $(k_0, y_0)$ .

We now impose standard conditions on the utility function, the production function and the resource regenerating function of the model.

**Assumption 5.1.** i) *The utility function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada condition  $u'(0) = +\infty$ .*

- ii) The production function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  strictly increasing, strictly concave, continuously differentiable and satisfies  $f(0) = 0$ ,  $f'(\infty) < 1$ ,  $f'(0) = \infty$ .
- iii) The regenerating function of the renewable resource  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is strictly increasing, strictly concave, continuously differentiable and satisfies  $\eta(0) = 0$ ,  $\eta'(\infty) < 1$ ,  $\eta'(0) = \infty$ .
- iv) For any  $(k_0, y_0) \in \mathbb{R}_+^2$ , there exists a feasible sequence  $\{(k_t, y_t)\}_{t=0}^\infty$  such that

$$\sum_{t=0}^{\infty} \beta^t u(f(k_t) + \theta\eta(y_t) - k_{t+1} - \theta y_{t+1}) > -\infty.$$

These conditions are standard in the literature. They ensure that in the production topology, the set of feasible path  $\Pi(k_0, y_0)$  is compact and the value function  $v$  is upper semi-continuous. It is well-established that under these properties, an optimal path exists. In absence of externality, the concavity of the production function and regenerating function ensures the uniqueness of the optimal path. Moreover, we can write the Bellman functional equation which admits  $v$  as a solution<sup>4</sup>.

The correspondence  $\Gamma$  is non-empty, convex, compact-valued, and continuous. The value function is a solution of the Bellman functional equation<sup>5</sup>. The optimal policy function is well-defined and satisfies usual continuity properties. Readers interested in the proof of Proposition 5.1 can refer to the classical work of Stokey & Lucas (with Prescott) Stokey (1989).

**Proposition 5.1.** *Assume 5.1.*

- i) The correspondence  $\Gamma$  is non-empty valued, continuous on  $\mathbb{R}_+^2$  and convex, compact-valued
- ii) The value function  $v$  satisfies the Bellman functional equation:

$$v(k, y) = \max_{(k', y') \in \Gamma(k, y)} [u(f(k) + \theta\eta(y) - k' - \theta y') + \beta v(k', y')].$$

Moreover, if the utility function  $u$  is bounded from below, then  $v$  is the unique solution.

- iii) There exists an policy function  $\varphi$  such that

$$\varphi(k, y) = \operatorname{argmax}_{(k', y') \in \Gamma(k, y)} [u(f(k) + \theta\eta(y) - k' - \theta y') + \beta v(k', y')].$$

- iv) The feasible sequence  $\{(k_t, y_t)\}_{t=0}^\infty$  is optimal if and only if for any  $t$ ,

$$(k_{t+1}, y_{t+1}) = \varphi(k_t, y_t).$$

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<sup>4</sup>For the details, see Le Van and Dana (2003) or Le Van and Morhaim (2002).

<sup>5</sup>For the case where the utility function is bounded from below, it is unique.

v) Assume  $k_0 > 0$  and  $y_0 > 0$ . The optimal path  $\{(k_t^*, y_t^*)\}_{t=0}^\infty$  satisfies the property that for any  $k_t^* > 0, y_t^* > 0$  for any  $t \geq 0$ .

Denote by  $(k^s, y^s)$  the stocks such that

$$f'(k^s) = \frac{1}{\beta} \text{ and } \eta'(y^s) = \frac{1}{\beta}.$$

It is easy to verify that  $(k^s, y^s) \in \Gamma(k^s, y^s)$  and is the unique steady state of the problem.

## 5.2.2 Local and global dynamics

### Local dynamics

The difficulty in analysing this problem is that, though the Inada conditions are satisfied, we can not exclude the possibility that for some date  $t$ , there is no extracting resource activity, namely  $y_{t+1}^* = \eta(y_t^*)$ . This prevents us to apply directly the Euler equations and well-known approaches in dynamic programming theory to study the long-term behaviour of the economy. Moreover, the lack of the super-modularity rules out the applications of the monotonicity results in Amir [Amir \(1996\)](#).

To tackle this problem, we first study the behaviour of optimal path for an economy that begins sufficiently "near" the steady state. Consider now the following modified problem. We define first the "production function" of this economy.

For each  $z > 0$ , let

$$F(z) = \max_{k+\theta y=z} (f(k) + \theta\eta(y)). \quad (5.1)$$

We have the following lemma.

**Lemma 5.1.** *The function  $F$  defined in (5.1) is strictly concave. Moreover, with*

$$(k^z, y^z) = \operatorname{argmax}_{k+\theta y=z} (f(k) + \theta\eta(y)),$$

*we have  $0 < k^z < z$  and  $0 < y^z < \frac{z}{\theta}$ . The derivatives satisfy  $f'(k^z) = \eta'(y^z) = F'(z)$ .*

The proof of Lemma 5.1 is immediate from the results of Rockafellar [Rockafellar \(1970\)](#). Define

$$S = f(k_0) + \theta\eta(y_0). \quad (5.2)$$

Since  $F$  is strictly increasing, it is invertible. Let  $z_0 = F^{-1}(S)$ , then  $z_0$  is well-defined.<sup>6</sup>

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<sup>6</sup>This consideration is necessary, since  $(k_0, y_0)$  may differ from  $\operatorname{argmax}_{k+\theta y=z_0} (f(k) + \theta\eta(y))$ .

Consider the modified problem:

$$\begin{aligned} \max & \left[ \sum_{t=0}^{\infty} \beta^t u(c_t) \right], \\ \text{s.t.} & c_t + z_{t+1} \leq F(z_t) \text{ for } t \geq 0, \\ & z_0 = F^{-1}(S) \text{ given.} \end{aligned}$$

Observe that thanks to Lemma 5.1, the modified problem is convex. It has a unique optimal path, which converges monotonically to the steady state  $z^s$ , the solution to  $F'(z) = \frac{1}{\beta}$ . We can verify that  $z^s = k^s + \theta y^s$ . For the optimal solution  $\{z_t^*\}_{t=0}^{\infty}$  of the modified problem, define the corresponding path  $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$  by

$$(\tilde{k}_t, \tilde{y}_t) = \underset{k+\theta y=z_t^*}{\operatorname{argmax}} (f(k) + \theta\eta(y)).$$

Note that in general the corresponding path  $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$  may not satisfy the constraint  $\tilde{y}_{t+1} \leq \eta(\tilde{y}_t)$ . Only for the case where the economy begins near the steady state  $(k^s, y^s)$ , this constraint is satisfied and the sequence  $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$  also solves the initial problem.

First, we provide conditions ensuring the equivalence between the initial and the modified models.

**Lemma 5.2.** *Assume 5.1. The modified problem has a unique solution. Moreover,*

i) *Consider the solution  $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$  of the initial problem. Define*

$$\begin{aligned} z_0 &= F^{-1}(f(k_0) + \theta\eta(y_0)) \\ z_t^* &= k_t^* + \theta y_t^*. \end{aligned}$$

*If for any  $t \geq 0$ ,  $0 < y_{t+1}^* < \eta(y_t^*)$ , then the sequence  $\{z_t^*\}_{t=0}^{\infty}$  is solution of the modified problem.*

ii) *Consider the solution  $\{\tilde{z}_t\}_{t=0}^{\infty}$  of the modified problem. For any  $t \geq 1$ , define*

$$(\tilde{k}_t, \tilde{y}_t) = \underset{k+\theta y=\tilde{z}_t}{\operatorname{argmax}} (f(k) + \theta\eta(y)) \quad (5.3)$$

*If for any  $t \geq 0$ ,  $0 < \tilde{y}_{t+1} \leq \eta(\tilde{y}_t)$ , then  $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$  is solution of the initial problem.*

In other words, if the initial model has an interior solution, then this solution also solves the modified problem. If the modified problem generates a sequence  $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^{\infty}$  such that  $\tilde{y}_{t+1} \leq \eta(\tilde{y}_t)$  for any  $t$ , then this sequence also solves the initial problem.

The analysis of the modified problem allows us to study local dynamic properties

of the initial problem in a neighborhood of its steady state. Using the results in [Stokey \(1989\)](#), the convergence follows a geometrical speed.

**Proposition 5.2.** *Assume 5.1. Denote by  $z^s$  the steady state of the modified problem and  $(k^s, y^s)$  the steady state of the initial problem. We have:*

i) *The point  $(k^s, y^s)$  satisfies*

$$(k^s, y^s) = \operatorname{argmax}_{k+\theta y=z^s} (f(k) + \theta\eta(y)).$$

ii) *There exists a neighborhood  $\mathcal{V}$  of  $(k^s, y^s)$  such that for any  $(k_0, y_0) \in \mathcal{V}$ , the optimal sequence  $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$  of the initial problem converges to  $(k^s, y^s)$ .*

This local dynamic property echoes the results in continuous time articles. It also helps us to study the global dynamic, where  $(k_0, y_0)$  may not be sufficiently close to  $(k^s, y^s)$ .

### Global dynamics

For an arbitrary initial state  $(k_0, y_0)$ , the analysis becomes more complicated since we may have  $y_{t+1}^* = \eta(y_t^*)$  in some date  $t$ . We cannot ensure that  $(k_t^*, y_t^*)$  maximizes  $f(k) + \theta\eta(y)$  under the constraint  $k + \theta y = z_t^*$ . It is possible that the solutions of the two maximization problems do not coincide as established in [Lemma 5.2](#). To overcome this difficulty, first we will prove that for  $T$  sufficiently big, the constraints do not bind for  $t \geq T$ . Precisely, we have  $0 < y_{t+1}^* < \eta(y_t^*)$  for any  $t \geq T$ .

We consider here the important notion, called the *net gain of stock* that we have mentioned earlier in the paper. For each  $(k, y) \in \mathbb{R}_+^2$ , define

$$\Psi(k, y) = \beta(f(k) + \theta\eta(y)) - (k + \theta y). \quad (5.4)$$

This notion was first analyzed in one-dimensional economics by [Majumdar \(1982\)](#), [Dechert and Nishimura \(2012\)](#) and [Mitra and Ray \(1984\)](#) to study the properties of steady states. [Kamihigashi and Roy \(2006\)](#) and [Kamihigashi et al. \(2007\)](#) prove that the economy always evolves to increase the value of the net gain function in the future, otherwise we are at the steady state. Following their insight, this article proves that although the sequence of *net gain of stock* may not monotonically increase, it will increase at some point in the future. This important result shall allow us to establish the long-run convergence of the economy.

The idea runs as follows. Observe that  $(k^s, y^s)$  maximizes  $\Psi(k, y)$  on  $\mathbb{R}_+^2$ . Suppose that the economy begins with a state which is not stable. If we can prove that the *net gain of stock* must always increase in the future, then we are done. More precisely,

for any  $t \geq 0$ , there exists some date  $t' > t$  such that  $\Psi(k_{t'}^*, y_{t'}^*) > \Psi(k_t^*, y_t^*)$ . Moreover, we have

$$\begin{aligned} \sup_{t \geq 0} \Psi(k_t^*, y_t^*) &= \Psi(k^s, y^s) \\ &= \sup_{(k,y) \in \mathbb{R}_+^2} \Psi(k, y). \end{aligned}$$

Hence there exists some period  $t$  that the state  $(k_t^*, y_t^*)$  gets *very close* to the steady state. We then show, via Proposition 5.2, that from this period the optimal sequence converges rapidly to the steady state  $(k^s, y^s)$ . These ideas are presented formally in Lemma 5.3, Lemma 5.4 and Proposition 5.3 below.

**Lemma 5.3.** *Assume 5.1. The steady state is the only solution which maximizes  $\Psi$ :*

$$\operatorname{argmax}_{(k,y) \in \mathbb{R}_+^2} \Psi(k, y) = \{(k^s, y^s)\}.$$

This lemma can be proved using the concavity of the functions  $f$  and  $\eta$ . Lemma 5.4 is the most important intermediary result in the establishment of the long-term behavior of the optimal path. It states that though the sequence of  $\{\Psi(k_t^*, y_t^*)\}_{t=0}^\infty$  can be non-monotonic, there exists some period in the future when the *net gain of stock* shall increase.

**Lemma 5.4.** *Assume 5.1. Consider the initial state  $(k_0, y_0)$  such that  $y_0 \leq \eta(y_0)$ . Exactly one of the following statements is true:*

- i) For any  $t$ , we have  $k_t^* = k_0$  and  $y_t^* = y_0$ .
- ii) There exists some  $t > 0$  such that

$$\Psi(k_t^*, y_t^*) > \Psi(k_0, y_0). \tag{5.5}$$

Lemma 5.4 tells us that for any non-steady initial state, the value of the *net gain of stock* will increase some day in future. The following proposition asserts that this value converges to  $\Psi(k^s, y^s)$  and that the optimal path converges to  $(k^s, y^s)$ .

**Proposition 5.3.** *Assume 5.1. For any  $(k_0, y_0) \in \mathbb{R}_+^2$ , the optimal path converges to  $(k^s, y^s)$ .*

Let us now illustrate the existence of a unique steady state and global convergence to this steady state. For simplicity, suppose that the utility function satisfies constant intertemporal elasticity of substitution (CIES), both the production function and the

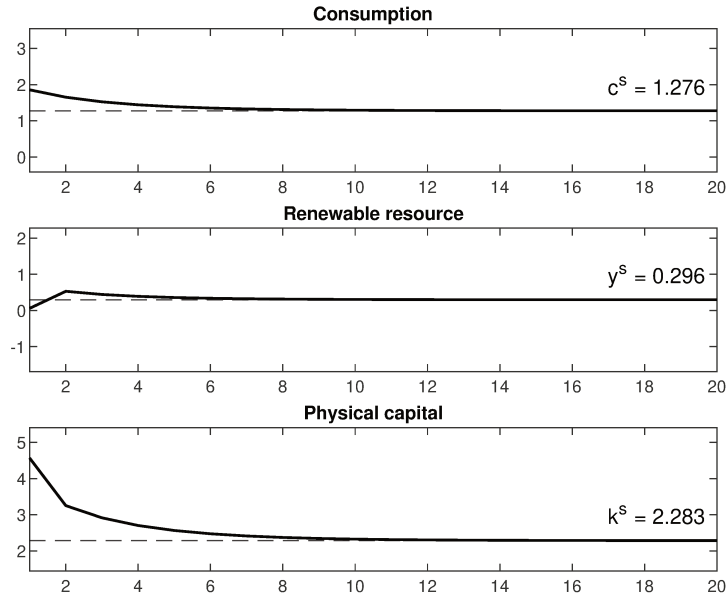


Figure 5.1: Optimal paths under no emission with constant intertemporal elasticity of substitution equal to 1 (logarithmic utility)

resource generating function are Cobb-Douglas:

$$f(k) = Ak^{\alpha_k},$$

$$\eta(k) = By^{\alpha_y},$$

where  $A, B, \alpha_k, \alpha_y$  along with other the model parameters are given in Table 5.1.

Parameter	Value
$\theta$	1
$\beta$	0.98
$\alpha_k$	0.67
$\alpha_y$	0.8
$A$ (TFP in final good sector)	2
$B$ (TFP in exploitation sector)	1
$k_0$ (Initial stock of physical capital)	$2k^s$
$y_0$ (Initial stock of renewable resources)	$0.2y^s$

Table 5.1: Parameters used for the simulated optimal paths under no emission

Notice that even though changing the intertemporal elasticity of substitution (IES) alters neither the steady state values nor the global convergence result, it affects the speed of convergence. In particular, the smaller the IES, the slower the optimal sequences converge to their corresponding steady states, as shown in Fig.5.1 and Fig.5.2.

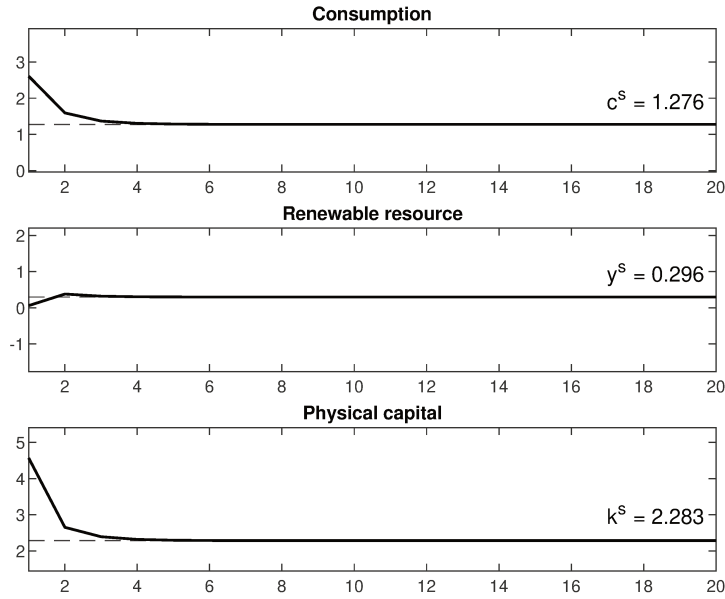


Figure 5.2: Optimal paths under no emission with constant intertemporal elasticity of substitution equal to 0.1

## 5.3 Renewable resources and industrial externalities

### 5.3.1 Fundamentals

The analysis in Section 5.2 ignores the effect of the production sector on the regenerating capacity of the natural resource. In reality, the industrial activities can be polluting, creating negative externalities. In the context of our setting, we hypothesize that pollution from the production sector damages the replenishing capacity of the natural resource. The larger the scale of production, the more severe the negative externality.

To be concrete, let  $E_t$  denote the function of pollution caused by industrial activities. Since pollution is increasing in the scale of production,  $E$  is strictly increasing in  $k$ . To reflect the negative impact of pollution on the regeneration of the resource, we let  $\eta$ , the regenerating function introduced in the previous section, depend negatively on  $E_t$ . Hence in this section  $\eta$  takes two arguments  $y$  and  $E$ : the growth rate of natural resource depends not only on the remain of its stock, but also on the environment in which it grows.

**Assumption 5.2.** Assume conditions (i) and (ii) in 5.1. Moreover, let  $\eta(y_t, E_t)$  denote the resource regenerating function. The following conditions are satisfied:

- i) The function  $\eta : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  is continuous and differentiable with respect to each argument. It is strictly increasing in the first and strictly decreasing in the second argument.



ii)  $\eta(0, E) = 0$  for all  $E \geq 0$ .

iii) For any  $E > 0$ ,  $\eta'(0, E) = \infty$  and  $\eta'(\infty, E) < 1$ .

iv) For any  $(k_0, y_0) \in \mathbb{R}_+^2$ , there exists a feasible sequence  $\{(k_t, y_t)\}_{t=0}^\infty$  such that

$$\sum_{t=0}^{\infty} \beta^t u(f(k_t) + \theta\eta(y_t, E(k_t)) - k_{t+1} - \theta y_{t+1}) > -\infty.$$

The representative agent solves the following inter-temporal optimization problem:

$$\begin{aligned} v(k_0, y_0) &= \max \sum_{t=0}^{\infty} \beta^t u(c_t), \\ c_t + k_{t+1} + \theta y_{t+1} &\leq f(k_t) + \theta\eta(y_t, E(k_t)), \\ y_{t+1} &\leq \eta(y_t, E(k_t)), \\ c_t, k_t, y_t &\geq 0 \text{ for any } t. \end{aligned}$$

A few remarks are in order. Observe that the model does not satisfy neither convexity structure or super-modularity. The feasible correspondence remains compact-valued, and hence a solution always exists. Nevertheless, the solution may not be unique; there might exist multiple optimal paths starting from the same initial state.

As in the previous case, we establish first the basic properties concerning the value function and optimal policy correspondence. For each  $(k, y) \in \mathbb{R}_+^2$ , define the feasible correspondence  $\Gamma : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$  by:

$$\Gamma(k, y) = \{(k', y') \in \mathbb{R}_+^2 \text{ such that } k' + \theta y' \leq f(k) + \theta\eta(y, E(k)) \text{ and } y' \leq \eta(y, E(k))\}. \quad (5.6)$$

We immediately obtain the following proposition using the results from Stokey & Lucas (with Prescott) Stokey (1989).

**Proposition 5.4.** *Assume 5.2.*

i) *The correspondence  $\Gamma$  defined in (5.6) is continuous, convex, and compact-valued.*

ii) *The value function  $v$  satisfies the Bellman functional equation:*

$$v(k, y) = \max_{(k', y') \in \Gamma(k, y)} [u(f(k) + \theta\eta(y, E(k)) - k' - \theta y') + \beta v(k', y')].$$

iii) *There exists an upper semi-continuous policy correspondence  $\varphi$  defined by:*

$$\varphi(k, y) = \operatorname{argmax}_{(k', y') \in \Gamma(k, y)} [u(f(k) + \theta\eta(y, E(k)) - k' - \theta y') + \beta v(k', y')].$$

iv) A feasible sequence  $\{(k_t, y_t)\}_{t=0}^{\infty}$  is optimal if and only if for any  $t$ ,

$$(k_{t+1}, y_{t+1}) \in \varphi(k_t, y_t).$$

v) Assume that  $k_0 > 0$  and  $y_0 > 0$ . Denote by  $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$  the optimal sequence. Then  $k_t^* > 0$ , and  $y_t^* > 0$  for any  $t \geq 0$ .

### 5.3.2 Long-term dynamical analysis

#### Existence of steady states

When the problem is not convex, the existence and uniqueness of steady states are not ensured. Let us first describe some properties of the long-term behaviour of the economy. As in the previous section, define the *net gain of investment* function by:

$$\Psi^e(k, y) = \max_{(k, y) \in \mathbb{R}_+^2} [\beta (f(k) + \theta \eta(y, E(k))) - (k + \theta y)]. \quad (5.7)$$

Observe that the argmax set is non-empty due to the compactness of the model. Furthermore, define

$$S^m = \operatorname{argmax}_{(k, y) \in \mathbb{R}_+^2} [\beta (f(k) + \theta \eta(y, E(k))) - (k + \theta y)]. \quad (5.8)$$

By the continuity of  $\eta$ , it is easy to verify that  $S^m \neq \emptyset$  and for any  $(k, y) \in S^m$ , the constant sequence  $\{k_t, y_t\}_{t=0}^{\infty}$  satisfying  $(k_t, y_t) = (k, y)$  for all  $t$ , is feasible. Hence a steady state exists. For any initial state which is not a steady state, the value of the *net gain of stock* will increase in the future.

**Proposition 5.5.** *Assume 5.2.*

- i) A steady state exists.
- ii) Consider an initial state such that  $y_0 \leq \eta(y_0, E(k_0))$ . Either  $(k_0, y_0)$  is a steady state, or for any optimal path  $\{(k_t^*, y_t^*)\}_{t=0}^{\infty}$  beginning from  $(k_0, y_0)$ , there exists some  $t \geq 0$  such that

$$\Psi^e(k_t^*, y_t^*) > \Psi^e(k_0, y_0).$$

As in Section 5.2, Proposition 5.5 allows us to prove that any optimal sequence must get very close to the set of steady state(s) at some point in the future. If this set is a singleton, this state must be an absorbing point, in the sense that starting from anywhere in a neighbourhood of this point, there exists an optimal path converging to it. By similar arguments to Section 5.2, it can be proved that beginning from any initial state, there exists an optimal path converging to the steady state. Note that although

the possibility of multiple optimal paths can not be excluded, the set initial states which generate multiple optimal paths has zero measure (Dechert and Nishimura, 2012). So we can be *almost sure* that the economy converges in the long term.

### Long-term dynamics

In this subsection, we characterize the conditions that guarantee the uniqueness of the steady state.

Let  $\eta_1$  and  $\eta_2$  be the partial derivatives of  $\eta$  with respect to its first and the second argument, respectively.

**Assumption 5.3.** *The following system has unique solution:*

$$\begin{aligned} f'(k) + \theta \eta_2(y, E(k)) E'(k) &= \frac{1}{\beta}, \\ \eta_1(y, E(k)) &= \frac{1}{\beta}. \end{aligned}$$

Since this system of equations provide the necessary conditions for a steady state, Assumption 5.3 ensures its uniqueness.

First, as in Section 5.2, we analyse the dynamic that begins near the steady state. Define

$$G(z) = \max_{k+\theta y=z} [f(k) + \theta \eta(y, E(k))]. \quad (5.9)$$

Observe that  $G$  is strictly increasing and differentiable. By 5.3, there exists a unique solution to  $G'(z) = \frac{1}{\beta}$ . By the Inada conditions,  $G'(0) = \infty$  and  $G'(\infty) < 1$ . This implies that  $G'(z) > \frac{1}{\beta}$  for  $0 < z < z^s$  and  $G'(z) < \frac{1}{\beta}$  for  $z > z^s$ .

Consider the following modified problem:

$$\begin{aligned} \max \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } c_t + z_{t+1} &\leq G(z_t) \text{ for any } t \geq 0, \\ z_0 &\text{ given.} \end{aligned}$$

Since  $G$  is strictly increasing, the indirect utility function  $V(z, z') = u(G(z) - z')$  has increasing differences. Following Amir (1996), this implies the monotonicity of the optimal paths of the modified problem.

**Lemma 5.5.** *Assume 5.2 and 5.3. For any initial state  $z_0$ , every optimal path of the modified problem converges monotonically to the unique steady state  $z^s$ .*

<sup>7</sup>Since  $G$  is differentiable, its derivatives function satisfies also the famous *Bolzano - Cauchy* property, which states that if  $G'(z) > \frac{1}{\beta}$  and  $G'(z') < \frac{1}{\beta}$ , then there exists some  $\tilde{z}$  between  $z$  and  $z'$  such that  $G'(\tilde{z}) = \frac{1}{\beta}$ . Hence we do not have to require the continuity of  $G'$ .

Similarly to Section 5.2, Lemma 5.5 allows us to describe the behaviour of the optimal path once initial state is sufficiently close the steady state  $(k^s, y^s)$ . In the following proposition, we prove that starting from any initial state, there exists an optimal path which converges to the steady state. The idea is that any optimal path must get "close" to the steady state, and from that new position, there is a path which converges monotonically to  $(k^s, y^s)$ .

**Proposition 5.6.** *Assume 5.2, 5.3.*

- i) *There exists a neighbourhood  $\mathcal{V}$  of  $(k^s, y^s)$  such that for any  $(k_0, y_0) \in \mathcal{V}$ , there exists an optimal path which begins at  $(k_0, y_0)$  and converges to  $(k^s, y^s)$ .*
- ii) *For any  $(k_0, y_0)$ , there exists an optimal path beginning from  $(k_0, y_0)$  which converges to  $(k^s, y^s)$ .*

### Uniqueness of the steady state and long-term convergence

In this section, we study the conditions that ensure the uniqueness of the steady state. We first make the simplifying assumption that the pollution function is linear, so that  $E(k) = \alpha k$ , where  $\alpha > 0$  captures the influence of the production sector on pollution (and consequently on the regeneration of the natural resource). Next, let us consider the plausible conditions to impose on  $\eta$ . Observe that concavity of  $\eta$  is counter-intuitive. Indeed, suppose that  $\eta$  is concave with respect to the second argument. Then  $\eta(y, \cdot)$  is strictly decreasing for all  $y$ , implying that for  $k$  sufficiently large, we obtain a negative value for the renewable resource, which is not intuitive. By contrast, suppose  $\lim_{k \rightarrow \infty} \eta(y, k) = 0$ , which is a reasonable assumption saying that when the scale of industrial production explodes, the overwhelming negative effect of pollution will wipe out the natural resource. Thus it makes sense to hypothesise that  $\eta(y, \cdot)$  is convex with respect to the second argument. For simplicity let us assume that the regenerating function  $\eta$  is separable:

$$\eta(y, \alpha k) = g(y)h(\alpha k). \quad (5.10)$$

**Assumption 5.4.** *i) The function  $g$  is strictly increasing, strictly concave, satisfying  $g'(0) = \infty$  and  $g'(\infty) < 1$ .*

*ii) The function  $h$  is strictly decreasing and convex.*

The "production function" of the modified problem, the function  $G$  defined in (5.9) is strictly increasing but not necessarily concave. To ensure concavity, we add the following mild condition.

Define by  $k^m$  the solution to  $f(k) = k$  and  $y^m$  the solution to  $g(y) = y$ . Let  $z^m = k^m + \theta y^m$ .

**Assumption 5.5.** For any  $0 \leq k \leq z \leq z^m$ , we have

i)

$$f''(k) + \frac{1}{\theta} g'' \left( \frac{z-k}{\theta} \right) h(\alpha k) - 2\alpha g' \left( \frac{z-k}{\theta} \right) h'(\alpha k) + \alpha^2 \theta g \left( \frac{z-k}{\theta} \right) h''(\alpha k) < 0.$$

ii)

$$\frac{1}{\theta} g'' \left( \frac{z-k}{\theta} \right) h(\alpha k) - \alpha g' \left( \frac{z-k}{\theta} \right) h'(\alpha k) < 0.$$

Under **5.5**, the function  $G$  is strictly concave. There is thus unique solution to  $G'(z) = \frac{1}{\beta}$ , and hence **5.3** is satisfied.

**Proposition 5.7.** Assume **5.2**, **5.4**, and **5.5**. The steady state  $(k^s, y^s)$  is unique and for any  $(k_0, y_0)$ , there exists an optimal path beginning from  $(k_0, y_0)$  which converges to  $(k^s, y^s)$ .

Observe that for any functions  $f$ ,  $g$  and  $h$ , for  $\alpha$  or  $\theta$  sufficiently small, Assumption **5.5** is verified, and the economy converges in the long term.

In the case where the inequality in part (i) of Assumption **5.5** is satisfied without the presence of  $f''(k)$ , (i) implies (ii) and we obtain the following corollary.

**Corollary 5.1.** Assume **5.2**, **5.4**. Assume that for any  $0 \leq k \leq z \leq z^m$ , we have

$$\frac{1}{\theta} g'' \left( \frac{z-k}{\theta} \right) h(\alpha k) - 2\alpha g' \left( \frac{z-k}{\theta} \right) h'(\alpha k) + \alpha^2 \theta g \left( \frac{z-k}{\theta} \right) h''(\alpha k) < 0.$$

The steady state  $(k^s, y^s)$  is unique and for any  $(k_0, y_0)$ , there exists an optimal path beginning from  $(k_0, y_0)$  which converges to  $(k^s, y^s)$ .

Furthermore if  $h$  is exponential so that  $h(\alpha k) = e^{-\gamma \alpha k}$ , then **5.5** can be reduced to a simple condition on  $g$ .

**Assumption 5.6.** For any  $0 \leq y \leq y^m$ , we have

$$\frac{1}{\theta} g''(y) + 2\alpha \gamma g'(y) + \alpha^2 \gamma^2 \theta g(y) < 0.$$

Under **5.6**, it is easy to verify that the conditions in **5.5** are satisfied. Proposition **5.8** below is obtained as a direct consequence of Proposition **5.5**.

**Proposition 5.8.** Consider the case  $\eta(y, \alpha k) = g(y)e^{-\gamma \alpha k}$ . Assume **5.2**, **5.4**, and **5.6**. The steady state  $(k^s, y^s)$  is unique and for any  $(k_0, y_0)$ , there exists an optimal path beginning from  $(k_0, y_0)$  which converges to  $(k^s, y^s)$ .

Assumptions **5.5** and **5.6** may raise the concern that we can only obtain a good description for the long-term behaviour of the economy for small values of  $\alpha$  or  $\theta$ , i.e.

when the renewable resource has a rather insignificant impact on the economy. The following proposition provides a partial response to this concern. In particular, we show that for the case of Cobb-Dougllass production functions, the convergence of the economy is satisfied without imposing supplementary conditions.

**Proposition 5.9.** *Assume that  $f(k) = Ak^{\alpha_k}$ ,  $g(y) = By^{\alpha_y}$  and  $h(E) = e^{-\gamma E}$  with  $0 < \alpha_k, \alpha_y < 1$ , and  $\gamma > 0$ .*

*Then the assumption 5.3 is satisfied. For any initial state  $(k_0, y_0)$ , there exists an optimal path which converges to the unique steady state  $(k^s, y^s)$ .*

Let us illustrate the result of Proposition 5.9 via a numerical exercise. For simplicity assume that the utility function is logarithmic. The following parameter values are used for this simulation.

Parameter	Value
$\gamma$	0.5
$\beta$	0.98
$\alpha_k$	0.67
$\alpha_y$	0.8
$\alpha$ (Emission coefficient)	0.2
$A$ (TFP in final good sector)	2
$B$ (TFP in fishery sector)	1

Table 5.2: Parameters used for the numerical simulation under exponential emission

We simulated the optimal paths of consumption, renewable resources and physical capital for two different values of price in Fig.5.3 and Fig.5.4. In both cases we start with an initial physical capital stock greater than the steady state and an initial resources stock lower than the steady state by the same fraction for convenient comparability. Observe that the higher the price of resources, the greater the steady state values of consumption and resources stock, and the smaller the steady state value of physical capital. The convergence speed also appears to be slower when the resources price is higher.

Suppose now that the emission function takes the form  $h(E) = (1 + E)^{-\zeta}$ . Assume logarithmic utility, Cobb-Douglas production and other parameters as in Table 5.2, we simulated the optimal paths for  $\zeta = 0.5$  and  $\zeta = 10$  in Fig.5.5 and Fig.5.6, respectively. Notice that  $\zeta$  represents the impact of pollution on renewable resources (while  $\alpha$  reflects the intensity of industrial pollution).

A few comments are in order. First, the impact of pollution on fishery has a mild negative effect on steadystate consumption and positive effect on steadystate physical capital. Second, the impact of pollution on the steadystate value of the natural resource is dramatic: when  $\zeta$  is sufficiently large, the stock of the renewable resource is depleted at the steady state.

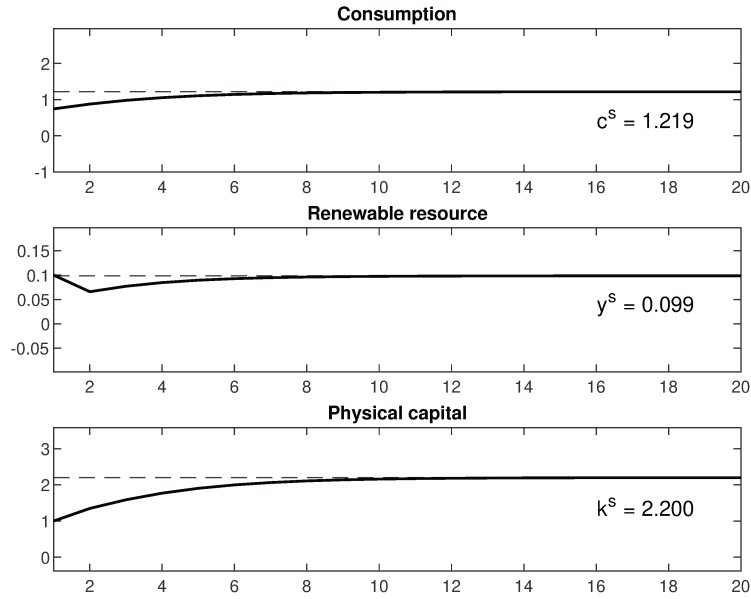


Figure 5.3: Optimal paths under exponential emission with low renewable resource price  $\theta = 1$

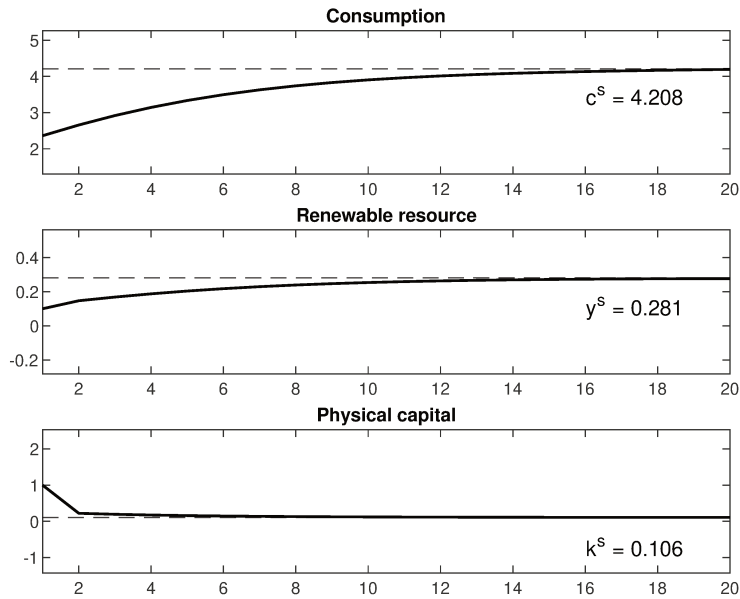


Figure 5.4: Optimal paths under exponential emission with high renewable resource price  $\theta = 50$

## 5.4 Conclusion

In this article, in a configuration where the usual properties such as convexity or supermodularity are not satisfied, we develop a new method to analyze the long-term dy-

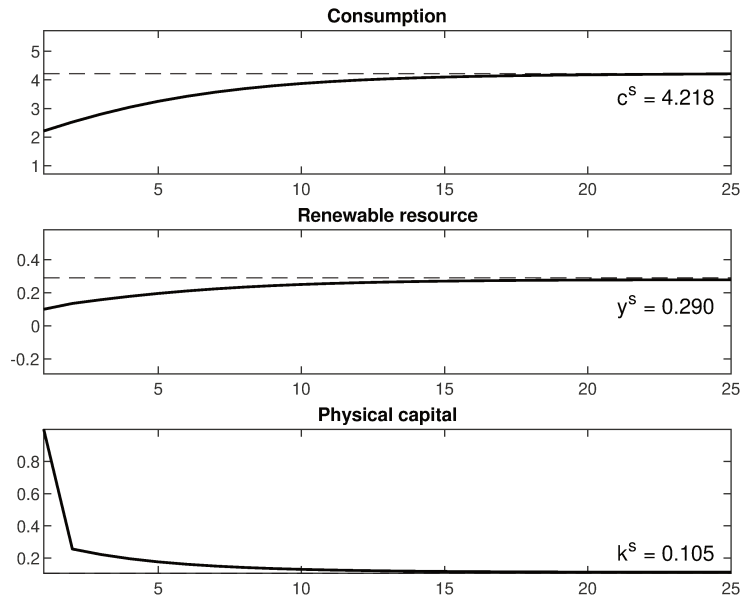


Figure 5.5: Optimal paths under non-exponential emission with  $\zeta = 0.5$

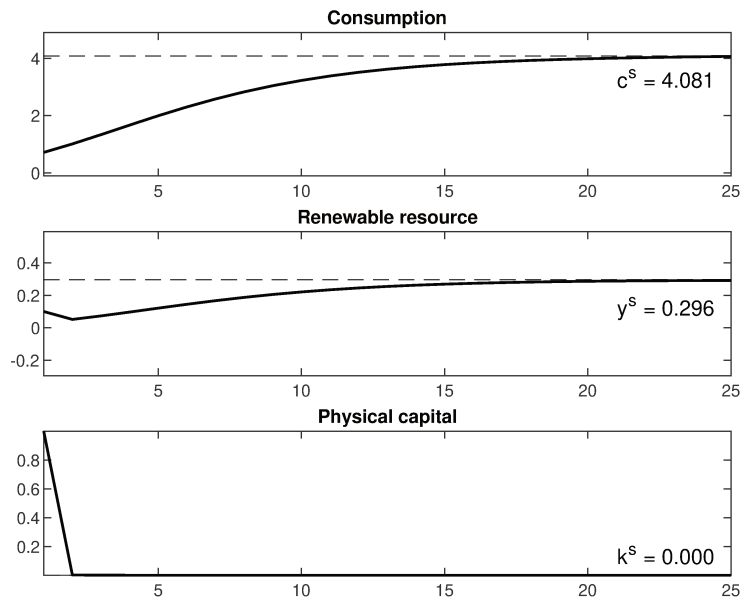


Figure 5.6: Optimal paths under non-exponential emission with  $\zeta = 10$

namics of the economy and prove that under suitable conditions, the economy converges in the long run. The simulations suggest that the economy may exhibit some fluctuations in the beginning periods, but then converges rapidly to the steady state.



## 5.5 Appendix of Proofs

### 5.5.1 Proof of Lemma 5.2

The uniqueness of solution of modified problem is assured using the concavity of  $f$  and  $\eta$ . From the strictly concavity of  $f$  and  $\eta$ , the function  $F$  is strictly concave.

*i)* Consider solution of the initial problem,  $\{(k_t^*, y_t^*)\}_{t=0}^\infty$  satisfying for any  $t$ ,  $k_t^* > 0$  and  $0 < y_{t+1}^* < \eta(y_t^*)$ . By Euler equations, we have  $f'(k_t^*) = \eta'(y_t^*)$ . Since  $f$  et  $\eta$  are concave functions, this implies

$$(k_t^*, y_t^*) = \operatorname{argmax}_{k+\theta y=z} (f(k) + \theta\eta(y)).$$

Hence we have for any  $t$ ,  $F'(z_t^*) = f'(k_t^*) = \eta'(y_t^*)$ , or the sequence  $\{z_t^*\}_{t=0}^\infty$  satisfies Euler equation: for any  $t$ ,

$$u'(c_t^*) = \beta u'(c_{t+1}^*) F'(z_{t+1}^*).$$

From the transversality condition of the initial problem, we have:

$$\begin{aligned} \lim_{t \rightarrow \infty} \beta^t u'(c_t^*) z_{t+1}^* &= \lim_{t \rightarrow \infty} \beta^t u'(c_t^*) (k_{t+1}^* + \theta\eta(y_{t+1}^*)) \\ &= 0. \end{aligned}$$

Hence the transversality condition is satisfied. The sequence  $\{z_t^*\}_{t=0}^\infty$  is solution of the modified problem.

*ii)* Consider solution  $\{\tilde{z}_t\}_{t=0}^\infty$  of the modified problem.

Let  $(\tilde{k}_t, \tilde{y}_t) = \operatorname{argmax}_{k+\theta y=\tilde{z}_t} (f(k) + \theta\eta(y))$ . If for any  $t$ ,  $\tilde{k}_t > 0$  and  $0 < \tilde{y}_{t+1} < \tilde{\eta}(\tilde{y}_t)$ , then  $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^\infty$  is a feasible sequence of the initial problem.

By the Lemma 5.1, for any  $t \geq 0$ , we have for any  $t \geq 1$ ,  $f'(\tilde{k}_t) = \eta'(\tilde{y}_t) = F'(\tilde{z}_t)$ . From the Euler equations:

$$\begin{aligned} u'(\tilde{c}_t) &= \beta u'(\tilde{c}_{t+1}) f'(\tilde{k}_{t+1}) \\ &= \beta u'(\tilde{c}_{t+1}) \eta'(\tilde{y}_{t+1}). \end{aligned}$$

Observe that for any  $t \geq 1$ ,  $\tilde{k}_t \leq \tilde{z}_t$  and  $\tilde{y}_t \leq \frac{\tilde{z}_t}{\theta}$ . From the transversality condition of the modified problem:

$$\begin{aligned} \lim_{t \rightarrow \infty} \beta^t u'(\tilde{c}_t) \tilde{k}_{t+1} &= 0, \\ \lim_{t \rightarrow \infty} \beta^t u'(\tilde{c}_t) \tilde{y}_{t+1} &= 0. \end{aligned}$$

The sequence  $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^\infty$  satisfies Euler equations and transversality condition of the

initial problem, hence this sequence is the optimal problem.

### 5.5.2 Proof of Proposition 5.2

*i)* From Inada conditions, one has  $f'(k^s) = \eta'(y^s) = F'(z^s) = \frac{1}{\beta}$ . This implies  $0 < y^s < \eta(y^s)$ . Hence the sequence  $\{(k_t^*, y_t^*)\}_{t=0}^\infty$  with  $k_t^* = k^s$  and  $y_t^s = y^s$  for any  $t$  satisfies Euler equations and transversality condition for the initial problem with initial state  $(k_0, y_0) = (k^s, y^s)$ .

*ii)* Take a neighborhood  $\mathcal{V}_z$  of  $z^s$  such that if  $z_0 \in \mathcal{V}_z$ , the optimal sequence  $\{z_t^*\}_{t=0}^\infty$  is subset of  $\mathcal{V}_z$  and converges to  $z^s$ . Define  $\tilde{\mathcal{V}}$  the set of  $(k_0, y_0)$  such that  $z_0 = F^{-1}(f(k_0 + \theta y_0))$  belongs to  $\mathcal{V}_z$ .

Obviously,  $\tilde{\mathcal{V}}$  contains a neighborhood  $\mathcal{V}$  of  $(k^s, y^s)$ . For any  $(k_0, y_0) \in \mathcal{V}$ , define  $z_0 = f(k_0) + \theta \eta(y_0)$ . The optimal solution  $\{\tilde{z}_t\}_{t=0}^\infty$  of the modified problem with initial  $z_0$  satisfies  $z_t \in \mathcal{V}_z$  for any  $t$  and converges to  $z^s$ . Moreover, since  $0 < y^s < \eta(y^s)$ , the corresponding sequence  $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^\infty$  satisfies  $0 < \tilde{y}_{t+1} < \eta(y_t)$  for any  $t$  and hence  $f'(\tilde{k}_t) = \eta'(\tilde{y}_t) = F'(\tilde{z}_t)$ . Obviously, this sequence satisfies transversality condition. By Lemma 5.2, the sequence  $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^\infty$  is solution of the initial problem and from the convergence of  $\{\tilde{z}_t\}_{t=0}^\infty$  to  $z^s$ , this sequence converges to  $(k^s, y^s)$ .

### 5.5.3 Proof of Lemma 5.4

First, observe that for any  $T$ ,

$$\begin{aligned}
 & \sum_{t=0}^T \beta^t (f(k_t^*) + \theta\eta(y_t^*) - k_{t+1}^* - \theta y_{t+1}^*) \cdot \\
 &= f(k_0) + \theta\eta(y_0) - k_1^* - \theta y_1^* \\
 &+ \beta (f(k_1^*) + \theta\eta(y_1^*) - k_2^* - \theta y_2^*) \\
 &+ \beta^2 (f(k_2^*) + \theta\eta(y_2^*) - k_3^* - \theta y_3^*) + \dots \\
 &+ \beta^T (f(k_T^*) + \theta\eta(y_T^*) - k_{T+1}^* - \theta y_{T+1}^*) \cdot \\
 &= f(k_0) + \theta\eta(y_0) + \beta (f(k_1^*) + \theta\eta(y_1^*)) - k_1^* - \theta y_1^* \\
 &+ \beta [\beta (f(k_2^*) + \theta\eta(y_2^*)) - k_2^* - \theta y_2^*] \\
 &+ \dots \\
 &+ \beta^{T-1} [\beta (f(k_T^*) + \theta\eta(y_T^*)) - k_T^* - \theta y_T^*] - \beta^T (k_{T+1}^* + \theta y_{T+1}^*) \cdot \\
 &= f(k_0) + \theta\eta(y_0) \\
 &+ \Psi(k_1^*, y_1^*) \\
 &+ \beta\Psi(k_2^*, y_2^*) \\
 &+ \dots \\
 &+ \beta^{T-1}\Psi(k_T^*, y_T^*) \\
 &- \beta^T (k_{T+1}^* + \theta y_{T+1}^*) \cdot \\
 &= f(k_0) + \theta\eta(y_0) + \sum_{t=0}^{T-1} \beta^t \Psi(k_{t+1}^*, y_{t+1}^*) - \beta^T (f(k_{T+1}^*) + \theta\eta(y_{T+1}^*)).
 \end{aligned}$$

Let  $T$  converges to infinity, we get

$$\begin{aligned}
 \sum_{t=0}^{\infty} \beta^t c_t^* &= \lim_{T \rightarrow \infty} \left[ k_0 + \theta\eta(y_0) + \sum_{t=0}^{T-1} \beta^t \Psi(k_{t+1}^*, y_{t+1}^*) - \beta^T (f(k_{T+1}^*) + \theta\eta(y_{T+1}^*)) \right] \\
 &= f(k_0) + \theta\eta(y_0) + \sum_{t=0}^{\infty} \beta^t \Psi(k_{t+1}^*, y_{t+1}^*).
 \end{aligned}$$

Assume that for any  $t \geq 0$ , we have  $\Psi(k_t^*, y_t^*) \leq \Psi(k_0, y_0)$ . This implies

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t c_t^* &= f(k_0) + \theta\eta(y_0) + \sum_{t=0}^{\infty} \beta^t \Psi(k_{t+1}^*, y_{t+1}^*) \\ &\leq f(k_0) + \theta\eta(y_0) + \frac{\Psi(k_0, y_0)}{1 - \beta} \\ &= f(k_0) + \theta\eta(y_0) + \frac{\beta (f(k_0) + \theta\eta(y_0)) - k_0 - \theta y_0}{1 - \beta} \\ &= \frac{f(k_0) - k_0 + \theta(\eta(y_0) - y_0)}{1 - \beta}. \end{aligned}$$

Hence by the concavity of  $u$ :

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t u(c_t^*) &\leq \frac{u((1 - \beta) \sum_{t=0}^{\infty} \beta^t c_t^*)}{1 - \beta} \\ &\leq \frac{u(f(k_0) - k_0 + \theta(\eta(y_0) - y_0))}{1 - \beta}. \end{aligned}$$

We will prove that the hypothesis  $\Psi(k_t^*, y_t^*) \leq \Psi(k_0, y_0)$  for any  $t \geq 0$  implies that  $(k_0, y_0) \in \Gamma(k_0, y_0)$ . Indeed, assume the contrary. Since  $y_0 \leq \eta(y_0)$ , we have

$$k_0 + \theta y_0 > f(k_0) + \theta\eta(y_0),$$

otherwise  $(k_0, y_0) \in \Gamma(k_0, y_0)$ .

The direct consequence of this inequality is that  $k_0 > f(k_0)$ . Denote by  $\bar{k}$  the solution to  $f(k) = k$ . By the concavity of  $f$ ,  $\bar{k} < k_0$ . Consider the sequence  $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$  such that  $\hat{k}_0 = k_0$ ,  $\hat{k}_t = \bar{k}$  for any  $t \geq 1$ , and  $\hat{y}_t = y_0$  for any  $t \geq 0$ . This sequence is feasible.

Since  $\bar{k} < k_0$ ,

$$\begin{aligned} f(k_0) + \theta\eta(y_0) - (k_0 + \theta y_0) &< f(k_0) + \theta\eta(y_0) - (\bar{k} + \theta y_0) \\ &= f(k_0) + \theta\eta(y_0) - (\hat{k}_1 + \theta \hat{y}_1). \end{aligned}$$

For any  $t \geq 1$ , since  $f(\bar{k}) = \bar{k}$ , we have

$$\begin{aligned} f(\hat{k}_t) + \theta\eta(\hat{y}_t) - (\hat{k}_{t+1} + \theta \hat{y}_{t+1}) &= f(\bar{k}) + \theta\eta(y_0) - (\bar{k} + \theta y_0) \\ &> f(k_0) + \theta\eta(y_0) - (k_0 + \theta y_0). \end{aligned}$$

Since  $\{c_t^*\}_{t=0}^\infty$  is optimal consumption sequence, this implies that

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t u(c_t^*) &\geq \sum_{t=0}^{\infty} \beta^t u \left( f(\hat{k}_t) + \theta \eta(\hat{y}_t) - (\hat{k}_{t+1} + \theta \hat{y}_{t+1}) \right) \\ &> \sum_{t=0}^{\infty} \beta^t u \left( f(k_0) + \theta \eta(y_0) - (k_0 + \theta y_0) \right) \\ &= \frac{u \left( f(k_0) + \theta \eta(y_0) - (k_0 + \theta y_0) \right)}{1 - \beta}, \end{aligned}$$

a contradiction.

Hence  $(k_0, y_0) \in \Gamma(k_0, y_0)$ . The sequence  $\{k_t, y_t\}_{t=0}^\infty$  such that  $k_t = k_0, y_t = y_0$  for any  $t$  is feasible. By the choice of  $\{c_t^*\}_{t=0}^\infty$ ,

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t u(c_t^*) &= \sum_{t=0}^{\infty} \beta^t u \left( f(k_t^*) - k_{t+1}^* + \theta(\eta(y_t^*) - y_{t+1}^*) \right) \\ &\geq \sum_{t=0}^{\infty} \beta^t u \left( f(k_0) - k_0 + \theta(\eta(y_0) - y_0) \right) \\ &= \frac{u \left( f(k_0) - k_0 + \theta(\eta(y_0) - y_0) \right)}{1 - \beta} \\ &\geq \sum_{t=0}^{\infty} \beta^t u(c_t^*). \end{aligned}$$

Since  $\{(k_t^*, y_t^*)\}_{t=0}^\infty$  is the unique optimal path, this implies  $k_t^* = k_0$  and  $y_t^* = y_0$  for any  $t \geq 0$ . The optimal sequence is constant.

For the case the optimal sequence is not constant, the above arguments imply the existence of  $t$  such that  $\Psi(k_t^*, y_t^*) > \Psi(k_0, y_0)$ .

### 5.5.4 Proof of Proposition 5.3

The proof is divided in some intermediary steps.

- i) There exists  $T$  such that  $y_t^* < \eta(y_t^*)$  for any  $t \geq T$ .
  - ii) The equality  $\sup_{t \geq 0} \Psi(k_t^*, y_t^*) = \Psi(k^s, y^s)$ .
  - iii) The convergence of the optimal path.
- (i) First, we prove the existence of some  $T$  such that  $y_T^* < \eta(y_T^*)$ .

Suppose the contrary, then for any  $t \geq 0$  we have

$$y_{t+1}^* \leq \eta(y_t^*) \leq y_t^*.$$

The sequence  $\{y_t^*\}_{t=0}^\infty$  is decreasing and hence converges to some  $y^*$  satisfying

$$y^* \leq \eta(y^*) \leq y^*,$$

which implies that  $y^* = \eta(y^*) = \bar{y}$ . Recall that  $\bar{y}$  is the unique solution to  $\eta(y) = y$ .

Now we prove the existence of some  $T$  such that

$$f'(k_{T+1}^*) > \eta'(y_{T+1}^*).$$

Indeed, suppose the contrary. This implies

$$\limsup_{t \rightarrow \infty} f'(k_t^*) \leq \eta'(\bar{y}) < 1.$$

By the Euler equations  $u'(c_t^*) = \beta u'(c_{t+1}^*) f'(k_{t+1}^*)$ , there exists  $T$  sufficiently big such that for any  $t \geq T$ ,  $u'(c_t^*) \leq u'(c_{t+1}^*)$ . By the concavity of  $u$ , the function  $u'$  is decreasing. This implies that the truncated sequence  $\{c_t^*\}_{t=T}^\infty$  is decreasing and converges to  $c^*$ .

The convergence of sequences  $\{c_t^*\}_{t=T}^\infty$  and  $\{y_t^*\}_{t=0}^\infty$  implies the convergence of  $\{k_t^*\}_{t=0}^\infty$ :

$$\lim_{t \rightarrow \infty} k_t^* = k^*.$$

From the Euler equations, we deduce that either  $c^* = 0$ , or  $f'(k^*) = \frac{1}{\beta}$ . The hypothesis that  $f'(k^*) = \frac{1}{\beta}$ , which is bigger than 1, leads us to a contradiction. Hence  $c^* = 0$ . Since  $\lim_{t \rightarrow \infty} y_t^* = \bar{y}$ , we have  $\lim_{t \rightarrow \infty} k_t^* = \bar{k}$ , the solution to  $f(k) = k$ . By the continuity of the optimal policy function, we have the conclusion that the consumption level at initial state  $(\bar{k}, \bar{y})$  is  $c^* = 0$ : a contradiction.

Hence there exists some  $T$  such that

$$f'(k_{T+1}^*) > \eta'(y_{T+1}^*).$$

Fix  $\epsilon > 0$  sufficiently such that:

$$f(k_{T+1}^* + \epsilon) + \theta \eta\left(y_{T+1}^* - \frac{\epsilon}{\theta}\right) > f(k_{T+1}^*) + \theta \eta(y_{T+1}^*).$$

Consider the sequence  $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^\infty$  defined as

$$\begin{aligned}\hat{y}_t &= y_t^* \text{ for any } 0 \leq t \leq T, \\ \hat{y}_{T+1} &= y_{T+1}^* - \frac{\epsilon}{\theta}, \\ \hat{y}_{t+1} &= y_{t+1}^* \text{ for any } t \geq T, \\ \hat{k}_t &= k_t^* \text{ for any } 0 \leq t \leq T, \\ \hat{k}_{T+1} &= k_{T+1}^* + \epsilon, \\ \hat{k}_t &= k_t^* \text{ for any } t \geq T + 2.\end{aligned}$$

We can verify that the sequence  $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^\infty$  is feasible. We have

$$\begin{aligned}\sum_{t=0}^{\infty} \beta^t u(\hat{c}_t) - \sum_{t=0}^{\infty} \beta^t u(c_t^*) &= \beta^{T+1} (u(\hat{c}_{T+1}) - u(c_{T+1}^*)) \\ &= \beta^{T+1} u \left( f(k_{T+1}^* + \epsilon) + \theta \eta \left( y_{T+1}^* - \frac{\epsilon}{\theta} \right) - k_{T+2}^* - \theta \eta(y_{T+2}^*) \right) \\ &\quad - \beta^{T+1} u \left( f(k_{T+1}^*) + \theta \eta(y_{T+1}^*) - k_{T+2}^* - \theta \eta(y_{T+2}^*) \right) \\ &> 0,\end{aligned}$$

a contradiction. This contradiction comes from the hypothesis that for any  $t$ ,  $y_t^* \geq \eta(y_t^*)$ .

Then there exists some  $T$  such that  $y_T^* < \eta(y_T^*)$ . Hence  $y_T^* < \bar{y}$ . By induction, for any  $t \geq T$ ,  $y_t^* < \bar{y}$ . This implies  $y_t^* < \eta(y_t^*)$  for any  $t \geq 0$ .

(ii) Consider the subsequence  $\{(k_{t_n}^*, y_{t_n}^*)\}_{n=0}^\infty$  such that

$$\lim_{n \rightarrow \infty} \Psi(k_{t_n}^*, y_{t_n}^*) = \sup_{t \geq 0} \Psi(k_t^*, y_t^*).$$

Recall that  $\sup_{t \geq 0} \Psi(k_t^*, y_t^*) \leq \Psi(k^s, y^s)$ . Suppose that this inequality is strict.

Since the sequence  $\{(k_{t_n}^*, y_{t_n}^*)\}_{n=0}^\infty$  is bounded, without loss of generality, we can assume that

$$\begin{aligned}\lim_{n \rightarrow \infty} k_{t_n}^* &= k^*, \\ \lim_{n \rightarrow \infty} y_{t_n}^* &= y^*.\end{aligned}$$

Since  $\sup_{t \geq 0} \Psi(k_t^*, y_t^*) < \Psi(k^s, y^s)$ , we have  $\Psi(k^*, y^*) < \Psi(k^s, y^s)$  and  $(k^*, y^*)$  is not steady state. By the first part of this proof, we deduce that  $y^* \leq \eta(y^*)$ .

Let  $\{\tilde{k}_t, \tilde{y}_t\}_{t=0}^\infty$  the optimal path beginning from  $(k^*, y^*)$ . By Lemma 5.4, there exists  $T$  such that

$$\Psi(\tilde{k}_T, \tilde{y}_T) > \Psi(k^*, y^*).$$

By the continuity of the problem, there is a neighborhood  $\mathcal{V}$  of  $(k^*, y^*)$  such that for any  $(k'_0, y'_0) \in \mathcal{V}$ , the optimal path  $\{k'_t, y'_t\}_{t=0}^\infty$  satisfies

$$\Psi(k'_T, y'_T) > \Psi(k^*, y^*).$$

Since the sequence  $\{(k_{t_n}^*, y_{t_n}^*)\}_{n=0}^\infty$  converges to  $(k^*, y^*)$ , there is  $n$  sufficiently big such that  $(k_{t_n}^*, y_{t_n}^*) \in \mathcal{V}$ . We have

$$\begin{aligned} \Psi(k_{t_n+T}^*, y_{t_n+T}^*) &> \Psi(k^*, y^*) \\ &= \sup_{t \geq 0} \Psi(k_t^*, y_t^*), \end{aligned}$$

a contradiction. This contradiction comes from the hypothesis that  $\sup_{t \geq 0} \Psi(k_t^*, y_t^*) < \Psi(k^s, y^s)$ .

(iii) Hence  $\sup_{t \geq 0} \Psi(k_t^*, y_t^*) = \Psi(k^s, y^s)$ . For any neighborhood  $\mathcal{V}$  of  $(k^s, y^s)$ , there is some  $t$  such that  $(k_t^*, y_t^*) \in \mathcal{V}$ . Using Proposition 5.2, we obtain:

$$\begin{aligned} \lim_{t \rightarrow \infty} k_t^* &= k^s, \\ \lim_{t \rightarrow \infty} y_t^* &= y^s. \end{aligned}$$

### 5.5.5 Proof of Proposition 5.5

(i) Fix any  $(k_0, y_0) \in S^m$ . First we prove that the constant sequence beginning from  $(k_0, y_0)$  is feasible. Indeed, we have only to prove that  $y_0 \leq \eta(y_0, E(k_0))$ . Suppose the contrary,  $\eta(y_0, E(k_0)) < y_0$ . Since  $\eta_1(0, E(k_0)) = \infty$ , there exists  $y$  sufficiently small such that  $y < \eta(y, E(k_0))$ . This implies  $\psi(y, k_0) > \psi(y_0, k_0)$ : a contradiction.

Consider an optimal path  $\{(k_t^*, y_t^*)\}_{t=0}^\infty$  beginning from  $(k_0, y_0)$ . By the choice of  $(k_0, y_0)$ , for any  $t$  we have  $\Psi^e(k_t^*, y_t^*) \leq \Psi^e(k_0, y_0)$ . Using the same arguments as in the proof of Proposition 5.3, we have

$$\sum_{t=0}^{\infty} \beta^t u(f(k_0) + \theta \eta(y_0, E(k_0)) - k_0 - \theta y_0) \geq \sum_{t=0}^{\infty} \beta^t u(c_t^*),$$

which implies that the constant sequence  $\{(k_0, y_0)\}_{t=0}^\infty$  is also an optimal path beginning from  $(k_0, y_0)$ . Hence  $(k_0, y_0)$  is a steady state of the economy.

(ii) We follow the same line of arguments of the proof of Proposition 5.3. Fix  $(k_0, y_0)$  and an optimal path  $\{(k_t^*, y_t^*)\}_{t=0}^\infty$  beginning from  $(k_0, y_0)$ . We have

$$(1 - \beta) \sum_{t=0}^{\infty} \beta^t c_t^* = f(k_0) + \theta \eta(y_0, E(k_0)) + \sum_{t=0}^{\infty} \beta^t \Psi^e(k_t^*, y_t^*).$$



Assume that for any  $t \geq 0$ ,  $\Psi^e(k_t^*, y_t^*) \leq \Psi^e(k_0, y_0)$ . By the concavity of  $u$ , one has

$$(1 - \beta) \sum_{t=0}^{\infty} \beta^t u(c_t^*) \leq u \left( (1 - \beta) \sum_{t=0}^{\infty} \beta^t c_t^* \right).$$

This is equivalent to

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*) \leq \frac{u(f(k_0) + \theta\eta(y_0, E(k_0)) - k_0 - \theta y_0)}{1 - \beta}.$$

We prove that  $(k_0, y_0) \in \Gamma(k_0, y_0)$ . In the contrary case, this implies  $k_0 > f(k_0)$ . Hence  $k_0 > \bar{k}$ , the solution to the equation  $f(k) = k$ . The sequence  $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$  with  $\hat{k}_t = \bar{k}$  and  $\hat{y}_t = y_0$  for any  $t \geq 1$  is feasible.

Observe that  $E(\bar{k}) < E(k_0)$ . We have

$$\begin{aligned} f(\hat{k}_0) + \theta\eta(\hat{y}_0, E(\hat{k}_0)) - (\hat{k}_1 + \theta\hat{y}_1) &= f(\bar{k}) + \theta\eta(\bar{y}, E(\bar{k})) - (\bar{k} + \theta\bar{y}) \\ &> f(k_0) + \theta\eta(y_0, E(k_0)) - (k_0 + \theta y_0). \end{aligned}$$

For any  $t \geq 1$ , since  $f(\bar{k}) = \bar{k}$ , we have

$$\begin{aligned} f(\hat{k}_t) + \theta\eta(\hat{y}_t, E(\hat{k}_t)) - (\hat{k}_{t+1} + \theta\hat{y}_{t+1}) &= f(\bar{k}) + \theta\eta(y_0, E(k_0)) - (\bar{k} + \theta y_0) \\ &> f(k_0) + \theta\eta(y_0, E(k_0)) - (k_0 + \theta y_0). \end{aligned}$$

Since  $\{c_t^*\}_{t=0}^{\infty}$  is optimal consumption sequence, this implies that

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t u(c_t^*) &\geq \sum_{t=0}^{\infty} \beta^t u(f(\hat{k}_t) + \theta\eta(\hat{y}_t, E(\hat{k}_t))) \\ &> \sum_{t=0}^{\infty} \beta^t u(f(k_0) + \theta\eta(y_0, E(k_0)) - (k_0 + \theta y_0)) \\ &= \frac{u(f(k_0) + \theta\eta(y_0, E(k_0)) - (k_0 + \theta y_0))}{1 - \beta}, \end{aligned}$$

a contradiction.

Hence  $(k_0, y_0) \in \Gamma(k_0, y_0)$ , which implies that the sequences  $\{c_t^*\}_{t=0}^{\infty}$  and  $\{\Psi^e(k_t^*, y_t^*)\}_{t=0}^{\infty}$  are constant. Hence for any  $t$ ,

$$\begin{aligned} f(k_t^*) + \theta\eta(y_t^*, E(k_t^*)) - k_{t+1}^* - \theta y_{t+1}^* &= f(k_0) + \theta\eta(y_0, E(k_0)) - k_1^* - \theta y_1^*, \\ \beta(f(k_t^*) + \theta\eta(y_t^*, E(k_t^*))) - k_t^* - \theta y_t^* &= \beta(f(k_0) + \theta\eta(y_0, E(k_0))) - k_0^* - \theta y_0^*. \end{aligned}$$

Let

$$\Delta = -\beta(k_1^* + \theta y_1^*) + (k_0 + \theta y_0).$$

For any  $t$ , we have

$$\begin{aligned} k_t^* + \theta y_t^* &= \beta(k_{t+1}^* + \theta y_{t+1}^*) + \Delta \\ &= \beta^2(k_{t+2}^* + \theta y_{t+2}^*) + \beta\Delta + \Delta \\ &= \dots \\ &= \beta^T(k_{t+T}^* + \theta y_{t+T}^*) + \Delta \sum_{s=0}^{T-1} \beta^s. \end{aligned}$$

Let  $T$  converges to infinity, we get for any  $t$ ,

$$\begin{aligned} k_t^* + \theta y_t^* &= \frac{\Delta}{1 - \beta} \\ &= \frac{(k_0 + \theta y_0) - \beta(k_1^* + \theta y_1^*)}{1 - \beta}. \end{aligned}$$

Hence for any  $t \geq 0$  we have

$$k_t^* + \theta y_t^* = k_0 + \theta y_0.$$

Since the consumption sequence is constant, we get for any  $t \geq 0$ ,

$$f(k_t^*) + \theta \eta(y_t^*, E(k_t^*)) = f(k_0) + \eta(y_0, E(k_0)).$$

These two equalities prove that  $(k_0, y_0)$  is a steady state.

The conclusion that  $(k_0, y_0)$  belongs to the set of steady states comes from the hypothesis that  $\Psi^e(k_t^*, y_t^*) \leq \Psi^e(k_0, y_0)$  for any  $t \geq 0$ . Hence if  $(k_0, y_0)$  is not a steady state, there exists  $t$  such that

$$\Psi^e(k_t^*, y_t^*) > \Psi^e(k_0, y_0).$$

### 5.5.6 Proof of Lemma 5.5

By 5.3, the unicity of steady state is ensured. For each  $0 \leq z' \leq G(z)$ , define  $V(z, z') = u(G(z) - z')$ . The function  $V$  plays a role of indirect utility function of the modified economy.

By the concavity of  $u$  and the monotonicity of  $G$ , indirect utility function  $V$  has increasing differences (see Amir Amir (1996)). Every optimal path of the modified problem is hence monotonic.

We will prove the following claim: for any initial state  $z_0 > 0$ , every optimal path beginning from  $z_0$  converges monotonically to  $z^s$ . Precisely, let  $\{z_t^*\}_{t=0}^\infty$  an optimal path beginning from  $z_0$ . If  $z_0 \leq z^s$  then this path is increasing and converges to  $z^s$ . Other-

wise, if  $z_0 \geq z^s$ , this path is decreasing and converges to  $z^s$ .

Indeed, consider the case  $0 < z_0 < z^s$ . Assume that the sequence  $\{z_t^*\}_{t=0}^\infty$  is strictly decreasing. For fixed  $z < z^s$ , consider the following function with variable  $z'$  belonging to  $[0, z]$ :

$$w(z') = u(G(z) - z') + \frac{\beta}{1 - \beta} u(G(z') - z').$$

We have, by the concavity of  $u$ :

$$\begin{aligned} w'(z') &= -u'(G(z) - z') + \frac{\beta}{1 - \beta} u'(G(z') - z')(G'(z') - 1) \\ &\geq -u'(G(z') - z') + \frac{\beta}{1 - \beta} u'(G(z') - z')(G'(z') - 1) \\ &= u'(G(z') - z') \times \frac{\beta G'(z') - 1}{1 - \beta} \\ &> 0. \end{aligned}$$

This implies that the function  $w$  is strictly increasing in  $[0, z]$ . hence we have

$$\begin{aligned} \frac{u(G(z) - z)}{1 - \beta} &= w(z) \\ &\geq w(z') \\ &= u(G(z) - z') + \frac{\beta}{1 - \beta} u(G(z') - z'), \end{aligned}$$

for any  $0 \leq z' \leq z$ .

The hypothesis such that  $\{z_t^*\}_{t=0}^\infty$  is decreasing implies

$$\begin{aligned} \frac{u(G(z_0) - z_0)}{1 - \beta} &\geq u(G(z_0) - z_1^*) + \beta \frac{u(G(z_1^*) - z_1)}{1 - \beta} \\ &\geq u(G(z_0) - z_1^*) + \beta u(G(z_1^*) - z_2^*) + \beta^2 \frac{u(G(z_2^*) - z_2^*)}{1 - \beta} \\ &\dots \\ &\geq \sum_{t=0}^T \beta^t u(G(z_t^*) - z_{t+1}^*) + \beta^{T+1} \frac{u(G(z_{T+1}^*) - z_{T+1}^*)}{1 - \beta} \\ &\dots \\ &\geq \sum_{t=0}^{\infty} \beta^t u(G(z_t^*) - z_{t+1}^*). \end{aligned}$$

Hence  $(z_0, z_0, \dots)$  is also an optimal path, which implies that  $z_0 = z^s$ : a contradiction.

Hence the sequence  $\{z_t^*\}_{t=0}^\infty$  is increasing and converges to  $z^s$ . For  $z_0 > z^s$ , using the same arguments, we prove that any optimal path beginning from  $z_0$  is decreasing and

converges to  $z^s$ .

### 5.5.7 Proof of Proposition 5.6

(i) The proof follows the same arguments as Section 5.2. We know that for any  $z_0$ , the optimal path of the modified problem converges monotonically to the steady state  $z^s$ . We have  $z^s = k^s + \theta y^s$ .

For optimal path of the modified problem  $\{z_t^*\}_{t=0}^\infty$  the optimal path beginning from  $z_0 = k_0 + \theta y_0$ , define  $(k_t^*, y_t^*)$  as

$$(k_t^*, y_t^*) = \operatorname{argmax}_{k+\theta y=z_t^*} [f(k) + \theta\eta(y, \alpha k)].$$

Since  $y^s < g(y^s)h(\alpha k^s)$ , for  $(k_0, y_0)$  belonging to a neighborhood of  $(k^s, y^s)$ , the corresponding sequence  $\{(k_t^*, y_t^*)\}_{t=0}^\infty$  satisfied  $y_{t+1}^* < g(y_t^*)h(\alpha k_t^*)$  for any  $t \geq 0$ . This implies the sequence  $\{(k_t^*, y_t^*)\}_{t=0}^\infty$  is feasible and hence it is an optimal path of the initial problem. This sequence converges to  $(k^s, y^s)$ .

(ii) Fix  $(k_0, y_0)$  and an optimal path  $\{(k_t^*, y_t^*)\}_{t=0}^\infty$  beginning from  $(k_0, y_0)$ . Take the sub-sequence  $\{(k_{t_n}^*, y_{t_n}^*)\}_{n=0}^\infty$  such that

$$\lim_{n \rightarrow \infty} \Psi^e(k_{t_n}^*, y_{t_n}^*) = \sup_{t \geq 0} \Psi^e(k_t^*, y_t^*).$$

Without loss of generality, we can assume that this sub-sequence converges:

$$\lim_{n \rightarrow \infty} (k_{t_n}^*, y_{t_n}^*) = (\tilde{k}, \tilde{y}).$$

We state that  $(\tilde{k}, \tilde{y}) = (k^s, y^s)$ .

Assume the contrary. Consider the "sequence of sequences"  $\{\mathbf{k}_{t_n}\}_{n=0}^\infty$ , where for each  $n$ ,  $\mathbf{k}_{t_n} = \{(k_{t_n+t}^*, y_{t_n+t}^*)\}_{t=0}^\infty$ . By the compactness of the set of feasible sequences, we can assume that the sequence of sequences  $\{\mathbf{k}_{t_n}\}_{n=0}^\infty$  converges to  $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^\infty$ , which is also feasible.

Since  $\lim_{n \rightarrow \infty} (k_{t_n}^*, y_{t_n}^*) = (\tilde{k}, \tilde{y})$ , the sequence  $\{(\tilde{k}_t, \tilde{y}_t)\}_{t=0}^\infty$  is an optimal path beginning from  $(\tilde{k}, \tilde{y})$ . By Proposition 5.5, there is some  $T$  such that

$$\Psi^e(\tilde{k}_T, \tilde{y}_T) > \Psi^e(\tilde{k}, \tilde{y}).$$

Hence for  $n$  sufficiently big, we have

$$\begin{aligned} \Psi^e(k_{t_n+T}^*, y_{t_n+T}^*) &> \Psi^e(\tilde{k}, \tilde{y}) \\ &= \sup_{t \geq 0} \Psi^e(k_t^*, y_t^*), \end{aligned}$$

a contradiction.

This contradiction comes from the hypothesis such that  $(\tilde{k}, \tilde{y})$  is not steady state. By the uniqueness of steady state, we have

$$\lim_{n \rightarrow \infty} (k_{t_n}^*, y_{t_n}^*) = (k^s, y^s).$$

By the part (i), this implies that for some  $n$  sufficiently big, the point  $(k_{t_n}^*, y_{t_n}^*)$  belongs to the neighborhood  $\mathcal{V}$  of  $(k^s, y^s)$  and there exists an optimal path  $\{(k'_{t_n+t}, y'_{t_n+t})\}_{t=0}^{\infty}$  beginning from  $(k_{t_n}^*, y_{t_n}^*)$  which converges to  $(k^s, y^s)$ . Define the sequence  $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$  as

$$(\hat{k}_t, \hat{y}_t) = \begin{cases} (k_t^*, y_t^*) & \text{for } 0 \leq t \leq t_n, \\ (k'_t, y'_t) & \text{for } t \geq t_n. \end{cases}$$

The sequence  $\{(\hat{k}_t, \hat{y}_t)\}_{t=0}^{\infty}$  is an optimal path beginning from  $(k_0, y_0)$  which converges to  $(k^s, y^s)$ .

### 5.5.8 Proof of Proposition 5.7

We prove that the function  $G$  is strictly concave, hence solution to function  $G'(z) = \frac{1}{\beta}$ , and assumption 5.3 is satisfied.

Precisely,

- i) For each  $z$ , there exists unique  $(k(z), y(z))$  which maximizes  $f(k) + \theta g(y)h(\alpha k)$  under constraint  $k + \theta y \leq z$ .
- ii) The function  $k(z)$  is increasing in respect to  $z$ .
- iii) The function  $G$  is strictly concave and there exists unique steady  $z^s$ , which is solution to  $G'(z) = \frac{1}{\beta}$ .

(i) For  $z \geq 0$ , we must find  $k$  which maximizes

$$\zeta(k) = f(k) + \theta g\left(\frac{z-k}{\theta}\right)h(\alpha k).$$

We have

$$\zeta''(k) = f''(k) + \frac{1}{\theta} g''\left(\frac{z-k}{\theta}\right)h(\alpha k) - 2\alpha g'\left(\frac{z-k}{\theta}\right)h'(\alpha k) + \alpha^2 \theta g\left(\frac{z-k}{\theta}\right)h''(\alpha k).$$

The assumption 5.5 implies that  $\zeta$  is strictly concave. Hence there exists unique  $k(z) \in [0, z]$  maximizing  $\zeta(k)$ .

(ii) It is easy to verify that for  $z > 0$ , we have  $0 < k(z) < z$ . The value  $k(z)$  is hence solution to

$$f'(k) - g' \left( \frac{z-k}{\theta} \right) h(\alpha k) + \theta \alpha g' \left( \frac{z-k}{\theta} \right) h'(\alpha k) = 0.$$

By the implicit theorem, we get

$$k'(z) = - \frac{-\frac{1}{\theta} g'' \left( \frac{z-k}{\theta} \right) h(\alpha k) + \alpha g' \left( \frac{z-k}{\theta} \right) h'(\alpha k)}{f''(k) + \frac{1}{\theta} g'' \left( \frac{z-k}{\theta} \right) h(\alpha k) - 2\alpha g' \left( \frac{z-k}{\theta} \right) h'(\alpha k) + \alpha^2 \theta g' \left( \frac{z-k}{\theta} \right) h''(\alpha k)} > 0,$$

since the nominator is positive and the denominator is negative.

(iii) For any  $z \geq 0$ ,

$$\begin{aligned} G'(z) &= f'(k(z))k'(z) + g' \left( \frac{z-k(z)}{\theta} \right) (1 - k'(z))h(\alpha k(z)) + \alpha g' \left( \frac{z-k(z)}{\theta} \right) h'(\alpha k(z))k'(z) \\ &= g' \left( \frac{z-k(z)}{\theta} \right) h(\alpha k(z)). \end{aligned}$$

This implies

$$\begin{aligned} G''(z) &= \frac{1}{\theta} g'' \left( \frac{z-k(z)}{\theta} \right) (1 - k'(z))h(\alpha k(z)) + \alpha g' \left( \frac{z-k(z)}{\theta} \right) h'(\alpha k(z))k'(z) \\ &= \frac{1}{\theta} g'' \left( \frac{z-k(z)}{\theta} \right) h(\alpha k(z)) \\ &\quad + k'(z) \left( -\frac{1}{\theta} g'' \left( \frac{z-k(z)}{\theta} \right) h(\alpha k(z)) + \alpha g' \left( \frac{z-k(z)}{\theta} \right) h'(\alpha k(z)) \right) \\ &< 0, \end{aligned}$$

since the two terms are negative. The function  $G$  is strictly concave.

### 5.5.9 Proof of Corollary 5.1

Since  $f''(k) \leq 0$  for any  $k$ , the condition (i) in assumption 5.5 is satisfied. Moreover, since  $\alpha g' \left( \frac{z-k}{\theta} \right) h'(\alpha k)$  and  $\alpha^2 g' \left( \frac{z-k}{\theta} \right) h''(\alpha k)$  are positive, the assumption in the statement of this corollary implies the satisfaction of the condition (ii) in 5.5. The assumption 5.5 is hence satisfied. Applying directly Proposition 5.6, the proof is completed.

### 5.5.10 Proof of Proposition 5.9

By Proposition 5.6, we just have to prove the satisfaction of 5.3. Consider the following system

$$\begin{aligned} f'(k) + \theta\alpha\eta_2(y, \alpha k) &= \frac{1}{\beta}, \\ \eta_1(k, \alpha k) &= \frac{1}{\beta}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \frac{\alpha_k A}{k^{1-\alpha_k}} - \theta\alpha\gamma B y^{\alpha_y} e^{-\alpha\gamma k} &= \frac{1}{\beta}, \\ \frac{\alpha_y B e^{-\alpha\gamma k}}{y^{1-\alpha_y}} &= \frac{1}{\beta}. \end{aligned}$$

The second function implies

$$y = \left( \beta\alpha_y B e^{-\alpha\gamma k} \right)^{\frac{1}{1-\alpha_y}}.$$

Replacing  $y$  in the first equation, we get

$$\frac{\alpha_k A}{k^{1-\alpha_k}} - \theta\alpha\gamma B \left( \beta\alpha_y B e^{-\alpha\gamma k} \right)^{\frac{\alpha_y}{1-\alpha_y}} e^{-\alpha\gamma k} = \frac{1}{\beta}.$$

We must prove that the following equation has unique solution:

$$\frac{C_1}{k^{1-\alpha_k}} - \frac{C_2}{e^{ck}} = \frac{1}{\beta},$$

where  $C_1, C_2$  and  $c$  are positive constants.

Indeed, let  $\varphi(k) = \frac{C_1}{k^{1-\alpha_k}} - \frac{C_2}{e^{ck}}$ . We can verify that  $\varphi(0) = \infty$  and  $\varphi(\infty) = 0$ . The equation  $\varphi(k) = \frac{1}{\beta}$  has solution. Denote by  $k^*$  its smallest one. We have

$$\varphi'(k) = -\frac{(1-\alpha)C_1}{k^{2-\alpha_k}} + \frac{cC_2}{e^{ck}}.$$

Since  $\varphi(k) > 0$  for  $k < k^*$ , the derivative of  $\varphi$  at  $k^*$  is negative:  $\varphi'(k^*) < 0$ . We will prove that  $\varphi'(k) \leq 0$  for any  $k > k^*$ . Indeed, this is equivalent to

$$\frac{e^{ck}}{k^{2-\alpha}} > \frac{cC_2}{(1-\alpha_k)C_1}.$$

Since  $\varphi'(k^*) < 0$ , this inequality is verified for  $k = k^*$ . Define  $\tilde{\varphi}(k) = \frac{e^{ck}}{k^{2-\alpha}}$ . We have

$$\tilde{\varphi}'(k) = \frac{k^{1-\alpha_k} e^{ck} (ck - (2 - \alpha_k))}{k^{2(2-\alpha)}}.$$

As  $\tilde{\varphi}'(k^*) > 0$ , for any  $k > k^*$  we have  $\tilde{\varphi}'(k) > 0$ . Hence the function  $\tilde{\varphi}$  is increasing in  $[k^*, \infty)$ . For any  $k \geq k^*$  we have

$$\begin{aligned} \frac{e^{ck}}{k^{2-\alpha}} &\geq \frac{e^{ck^*}}{(k^*)^{2-\alpha}} \\ &> \frac{cC_2}{(1 - \alpha_k)C_1}. \end{aligned}$$

For any  $k > k^*$ , we have  $\varphi'(k) < 0$ . This implies that for any  $k \geq k^*$ ,  $\varphi(k) < \varphi(k^*)$ . The original equation  $\varphi(k) = \frac{1}{\beta}$  has unique solution.





# 6

## Conclusion

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In closing the thesis, I would like to discuss the limitations of the research presented here and a few ideas on developing them further in the future.

With respect to the optimal insurance problem, many of our analytical results under ambiguity aversion are limited to the case of two ambiguous states. To extend the results to more than two states, we might need to resort to the shooting method to numerically obtain the shape of the optimal indemnity schedule. Alternatively one could test the theoretical implications of the two-state case in an experiment where the number of ambiguous states can be controlled for.

The second chapter is a work in progress since we have only treated the first-best case. The second-best case with moral hazard is technically much harder in our framework, but we are optimistic about being able to bring it home some day. We can also adopt an experimental approach to test the robustness of our results with respect to the number of ambiguous states as suggested above.

For the third chapter on ambiguous returns on investment in human capital, it would be interesting to extend the model beyond two periods to understand the long-run impact on economic growth. It is also my goal to bring the model to the data to see how well the theoretical predictions perform.

The final chapter is complete in its own right, but one might think of several ways to further develop it. For example, we can allow emission to accumulate over time, or examine the impact of the arrival of a green technology.

Time is now up for this thesis, but not for research. After all, research engenders research; there is always room for further investigation. And this is good news.

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# A

## Stochastic orders

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This chapter serves as a brief review of commonly used stochastic ordering concepts in economics. In particular, we will be discussing the notion of stochastic dominance of order  $n$ , with particular emphasis on the first and second order. Then, we will be looking at stochastic dominance in the likelihood ratio and the hazard ratio, which are stronger notions of stochastic dominance than first-order stochastic dominance (FSD). The interested readers are recommended to consult a number of key references, including Mas-Colell et al. (1995), Laffont (1991), Levy (2015) and Wolfstetter (1999). The content of this chapter is primarily drawn from these sources.

To motivate the discussion, let  $\tilde{x}$  be a continuous random variable with realization  $x$  taking values in a bounded subset of the real line  $I_x = [0, \bar{x}]$ .<sup>1</sup> Let  $F_\theta(\cdot)$  be a family of cdfs of  $\tilde{x}$  parameterized by  $\theta \in \mathcal{I}$  where  $\mathcal{I}$  is the index set. Consider two members of this family  $F_i(\cdot)$  and  $F_j(\cdot)$ ,  $i \neq j \in \mathcal{I}$ . For simplicity assume that  $F_\theta : I_x \rightarrow [0, 1]$  has the common support  $I_x \equiv [0, \bar{x}]$  for all  $\theta = i, j$ , so that  $F_i(0) = F_j(0) = 0$  and  $F_i(\bar{x}) = F_j(\bar{x}) = 1$ . The definition of FSD is the following.

**Definition A.1.** *The distribution  $F_i(\cdot)$  first-order dominates  $F_j(\cdot)$ , written  $F_i(\cdot) \succcurlyeq_1 F_j(\cdot)$ , if for all increasing function  $u : I_x \rightarrow \mathbb{R}$ , we have:*

$$\int u(x)dF_i(x) \geq \int u(x)dF_j(x), \quad (\text{A.1})$$

with strict inequality if  $u(\cdot)$  is strictly increasing.

**Proposition A.1.** *The distribution  $F_i(\cdot)$  first-order dominates  $F_j(\cdot)$  if and only if  $F_i(x) \leq F_j(x)$  for all  $x \in I_x$  with strict inequality in some subset of positive measure of  $I_x$ .*

*Proof.* We prove necessity by contradiction. Suppose there exists  $x^* \in I_x$  such that  $F_i(x^*) > F_j(x^*)$ . We can define an increasing function  $u$  as  $u(x) = 1_{[x^*, \bar{x}]}$  so that

$$\int u(x)dF_\theta(x) = \int_0^{x^*} u(x)dF_\theta(x) + \int_{x^*}^{\bar{x}} u(x)dF_\theta(x) = 1 - F_\theta(x^*), \quad \theta = i, j. \quad (\text{A.2})$$

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<sup>1</sup>For analogous discussion of the discrete case, see Levy (2015).

Since  $F_i(x^*) > F_j(x^*)$  by assumption, we have  $\int u(x)dF_i(x) < \int u(x)dF_j(x)$ , contradicting (A.1).

For sufficiency, consider the case where  $u$  is differentiable. Integration by parts (IBP) allows us to write (A.1) as:

$$\begin{aligned} u(\bar{x}) - \int_0^{\bar{x}} u'(x)F_i(x)dx &\geq u(\bar{x}) - \int_0^{\bar{x}} u'(x)F_j(x)dx \\ \int_0^{\bar{x}} u'(x) [F_j(x) - F_i(x)] dx &\geq 0, \end{aligned} \tag{A.3}$$

which always holds since  $F_i(x) \leq F_j(x)$  on  $I_x$  and  $u' \geq 0$  by monotonicity. If  $u' > 0$  then (A.3) holds with strict inequality since  $F_j(x) - F_i(x) > 0$  on a subset of positive measure of  $I_x$ . ■

Proposition A.1 tells us that  $F_i(\cdot) \succcurlyeq_1 F_j(\cdot)$  iff for every possible realization  $x$  of  $\tilde{x}$ , the probability of getting more than  $x$  is greater under  $F_i(\cdot)$  than under  $F_j(\cdot)$ . Consequently, any decision maker (DM) whose preference satisfies monotonicity (preferring more to less) prefers  $F_i(\cdot)$  to  $F_j(\cdot)$ .

FSD concerns the ranking of distributions according to “levels”<sup>2</sup> while second-order stochastic dominance (SSD), the concept to be discussed next, concerns the ranking of distributions according to relative dispersion. When distributions of the same mean are under consideration, SSD turns out to be an equivalent ordering criterion to the concepts of increasing risk or mean-preserving spread (MPS) in the pioneering work of Rothschild and Stiglitz (1970b).

**Definition A.2.** Suppose  $F_i(\cdot)$  and  $F_j(\cdot)$  are two distributions with the same mean, i.e.  $\int x dF_i(x) = \int x dF_j(x)$ . Then distribution  $F_i(\cdot)$  second-order stochastically dominates  $F_j(\cdot)$ , written  $F_i(\cdot) \succcurlyeq_2 F_j(\cdot)$ , if for all increasing and concave function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ , we have:

$$\int u(x)dF_i(x) \geq \int u(x)dF_j(x). \tag{A.4}$$

Consequently, any risk averter with increasing utility function would prefer  $F_i(\cdot)$  to  $F_j(\cdot)$ . The following proposition links Definition A.2 to an alternative definition that does not involve the expected utility representation.

**Proposition A.2.** Let  $F_i(\cdot)$  and  $F_j(\cdot)$  be two distributions of the same mean as defined above. Then  $F_i(\cdot)$  second-order dominates  $F_j(\cdot)$  if and only if for all  $x \in I_x$ , we have  $\int_0^x F_i(y)dy \leq \int_0^x F_j(y)dy$ .

*Proof.* We restrict the proof to the case of differentiable  $u$ . Again IBP allows us to

<sup>2</sup>It is easy to see that  $F_i(\cdot) \succcurlyeq_1 F_j(\cdot)$  implies  $\int xF_i(x)dx \geq \int xF_j(x)dx$  by IBP.

rewrite (A.4) as

$$\int_0^{\bar{x}} F_i(x)u'(x)dx \leq \int_0^{\bar{x}} F_j(x)u'(x)dx. \quad (\text{A.5})$$

But  $F_\theta(x) = \int_0^x dF_\theta(y) = \left(\int_0^x F_\theta(y)dy\right)/dx$  by Leibniz's rule. Performing IBP once again yields

$$\int_0^{\bar{x}} u'(x)d\left(\int_0^x F_\theta(y)dy\right) = u'(\bar{x}) \int_0^{\bar{x}} F_\theta(x)dx - \int_0^{\bar{x}} u''(x) \left(\int_0^x F_\theta(y)dy\right) dx$$

for each  $\theta = i, j$ . Now, also by IBP,  $\int_0^{\bar{x}} x dF_\theta(x) = \bar{x} - \int_0^{\bar{x}} F_\theta(x)dx$  for each  $\theta = i, j$ , which implies  $\int_0^{\bar{x}} F_i(x)dx = \int_0^{\bar{x}} F_j(x)dx$  by the common mean assumption. Hence  $\int_0^x F_i(y)dy \leq \int_0^x F_j(y)dy$  for all  $x \in I_x$  implies

$$-\int_0^{\bar{x}} u''(x) \left(\int_0^x F_i(y)dy\right) dx \geq -\int_0^{\bar{x}} u''(x) \left(\int_0^x F_j(y)dy\right) dx \quad (\text{A.6})$$

since  $u''(x) \leq 0$  for all  $x \in I_x$  by the concavity assumption. Adding  $u'(\bar{x}) \int_0^{\bar{x}} F_i(x)dx = u'(\bar{x}) \int_0^{\bar{x}} F_j(x)dx$  to each side of (A.6) establishes the sufficiency part of the proposition. For necessity, see [Levy \(2015\)](#).  $\blacksquare$

As mentioned above, SSD is equivalent to the concepts of mean preserving spread (MPS) and elementary increase in risk in the sense of [Rothschild and Stiglitz \(1970b\)](#) as far as distributions of the same mean are concerned. The following proposition sums up this relationship.

**Proposition A.3.** *Let  $F_i(\cdot)$  and  $F_j(\cdot)$  be two distributions with the same mean. The following statements are equivalent:*

1.  $F_i(\cdot) \succcurlyeq_2 F_j(\cdot)$ .
2.  $F_j(\cdot)$  is a MPS of  $F_i(\cdot)$ .
3.  $F_j(\cdot)$  is an elementary increase in risk from  $F_i(\cdot)$ .

In general, the definition of stochastic dominance needs not invoke the common mean requirement. Furthermore, FSD and SSD are merely two particular cases of a more general notion of stochastic dominance. Indeed, the definition of  $n^{\text{th}}$ -order stochastic dominance (NSD) is given in [Laffont \(1991\)](#):

**Definition A.3.** *The distribution  $F_i(\cdot)$   $n^{\text{th}}$ -order stochastically dominates  $F_j(\cdot)$ , written  $F_i(\cdot) \succcurlyeq_n F_j(\cdot)$  if for all  $y \in I_x$ , we have:*

$$\int_0^y (-1)^{n-1} (x-y)^{n-1} dF_i(x) \leq \int_0^y (-1)^{n-1} (x-y) dF_j(x)$$

with strict inequality for at least some  $y \in I_x$ .

The case  $n = 3$  (third-order stochastic dominance (TSD) is another special case of interest. It can be shown that a DM with  $u' \geq 0$ ,  $u'' \leq 0$ , and  $u''' \geq 0$  (prudent) prefers  $F_i(\cdot)$  to  $F_j(\cdot)$  iff  $F_i(\cdot) \succcurlyeq_3 F_j(\cdot)$ . Furthermore, if such a DM is also temperant<sup>3</sup> ( $u'''' \leq 0$ ), then she prefers  $F_i(\cdot)$  to  $F_j(\cdot)$  iff  $F_i(\cdot) \succcurlyeq_4 F_j(\cdot)$ , and so on. We state without proof the following proposition which sums up the link between the higher order derivatives of the utility function and stochastic dominance.

**Proposition A.4.** *Let  $F_i(\cdot)$  and  $F_j(\cdot)$  be two distributions of the same random variable. The following statements are equivalent:*

1.  $F_i(\cdot) \succcurlyeq_n F_j(\cdot)$ .
2. Let  $u^{(k)}$  denotes the  $k^{\text{th}}$  derivative of  $u$ . For all  $n$  - time differentiable function  $u$  satisfying  $(-1)^{k-1}u^{(k)} \geq 0$  for all  $k = 1, \dots, n$ , we have:

$$\int u(x)dF_i(x) \geq \int u(x)dF_j(x).$$

Finally, we also state without proof a proposition linking different orders of stochastic dominance.

**Proposition A.5.** *For all integers  $m \geq n \geq 1$ , we have*

$$F_i(\cdot) \succcurlyeq_n F_j(\cdot) \implies F_i(\cdot) \succcurlyeq_m F_j(\cdot). \quad (\text{A.7})$$

Proposition A.5 suggests that FSD is a rather strong assumption. Nevertheless, in numerous economic applications such as in the principal-agent problem, even stronger notions of stochastic dominance are needed. We shall now discuss two of them, namely the likelihood ratio dominance (LRD) and the hazard ratio dominance (HRD).

**Definition A.4.** *The distribution  $F_i(\cdot)$  dominates  $F_j(\cdot)$  in the sense of LR, written  $F_i(\cdot) \succcurlyeq_{LR} F_j(\cdot)$  if for all  $x, y \in I_x$ ,  $y \geq x$ , then*

$$\frac{f_i(x)}{f_i(y)} \leq \frac{f_j(x)}{f_j(y)}$$

where  $f_\theta(\cdot)$  is the pdf (or pmf) associated with  $F_\theta(\cdot)$  for each  $\theta = i, j$ .

**Definition A.5.** *The distribution  $F_i(\cdot)$  dominates  $F_j(\cdot)$  in the sense of HR, written  $F_i(\cdot) \succcurlyeq_{HR} F_j(\cdot)$  if for all  $x \in I_x$ ,*

$$H_i(x) \leq H_j(x),$$

<sup>3</sup>The term first coined by Kimball (1990a).

where  $H_\theta : I_x \rightarrow \mathbb{R}_+^*$  is called the hazard function (or hazard rate) associated with  $F_\theta(\cdot)$  defined as  $H_\theta(x) = \frac{f_\theta(x)}{1-F_\theta(x)}$  for each  $\theta = i, j$ .

**Proposition A.6.**  $LRD \implies HRD \implies FSD$ .

*Proof.* We first show that LRD implies HRD. Observe that for all  $x, y \in I_x, y \geq x$ ,

$$H_i(x) = \frac{f_i(x)}{\int_x^{\bar{x}} f_i(y)dy} \leq \frac{f_i(x)}{\int_x^{\bar{x}} \frac{f_i(x)}{f_j(x)} f_j(y)} = \frac{f_j(x)}{\int_x^{\bar{x}} f_j(y)dy} = H_j(x),$$

where the first inequality follows by the LRD assumption.

We next show that HRD implies FSD. Indeed since  $H_\theta(x) = -\int_0^x d \ln(1 - F_\theta(y))$  for each  $\theta = i, j$ , the HRD condition is equivalent to:

$$\int_0^x d \ln(1 - F_i(y)) \geq \int_0^x d \ln(1 - F_j(y)),$$

which holds for a.e.  $x \in I_x$  if and only if  $\ln(1 - F_i(y)) \geq \ln(1 - F_j(y))$  for a.e.  $y \in I_x$  if and only if  $F_i(y) \leq F_j(y)$  for all  $y \in I_x$ .<sup>4</sup> ■

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<sup>4</sup>An alternative way to think about stochastic ordering can be explored via the concept of log-supermodular functions. See [Gollier \(2001\)](#) and [Athey \(2002\)](#) for a detailed exposition on the properties of log-supermodular functions and their various applications, such as in game theory and the standard portfolio problem.



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**Titre :** Essais sur l'ambiguïté et la croissance optimale avec des ressources renouvelables

**Mots clés :** ambiguïté ; économie du risque ; contrat optimal ; assurance ; croissance optimale

**Résumé :** La thèse se compose de deux thèmes principaux : la prise de décision sous l'ambiguïté et la croissance économique en présence des externalités.

Dans les deux premiers chapitres, nous étudions le problème du contrat optimal en présence de risque et de l'ambiguïté dans le cadre d'un problème de contrôle optimal. L'ambiguïté est modélisée selon Klibanoff et al. (2005). Notre approche généralise les analyses effectuées jusqu'à présent en considérant le contrat d'assurance comme la paire d'une prime et une fonction d'indemnisation à résoudre simultanément. Nous prouvons l'existence d'un contrat optimal dans le cas le plus général où tous les agents peuvent être simultanément averses à l'ambiguïté et au risque, ce qui englobe tous les cas précédemment examinés. Nous caractérisons non seulement le partage du risque mais aussi la règle du partage de l'ambiguïté entre les parties contractantes. Dans le cas de l'aversion vers l'ambiguïté unilatérale, nous montrons qu'une politique de franchise directe ne peut pas constituer un contrat d'assurance optimal. Au contraire, sous l'hypothèse que les densités conditionnelles puissent être classées selon le rapport de vraisemblance monotone, un contrat avec des franchises qui disparaissent est optimal, un résultat qui est cohérent avec Gollier (2014). En particulier, la méthodologie mise en œuvre complète l'analyse de Raviv (1979) pour le cas du risque pur avec un assureur neutre au risque, montrant qu'une couverture de limite supérieure ne peut pas constituer un optimum. Ce résultat est robuste à la neutralité de l'ambiguïté.

Dans le troisième chapitre, j'ai examiné l'impact du risque et de l'ambiguïté sur l'investissement optimal dans le capital humain et le capital physique en utilisant le modèle de Mincer à deux périodes. L'incertitude (à la fois dans le sens du risque et de l'ambiguïté) est introduite à l'accumulation de capital humain de deux façons.

Lorsque l'incertitude porte sur le taux de dépréciation du capital humain (obsolescence incertaine des compétences), j'ai constaté que l'investissement optimal dans le capital humain augmente toujours, que soit présent ou non le capital physique. Cette réponse à l'incertitude d'un ménage représente le comportement typique de l'auto-assurance. En revanche, lorsque l'incertitude se porte sur l'efficacité de l'accumulation du capital humain, l'investissement optimal dans le capital humain diminue parmi les ménages avec l'aversion au risque relative constante inférieure à un. Cette réponse à l'incertitude est typique d'un ménage qui considère l'investissement comme un actif à rendement risqué au lieu d'une assurance.

Le dernier chapitre (relativement indépendant des chapitres précédents) examine une question importante dans la théorie de la croissance: le rôle des ressources renouvelables et des externalités dans l'économie. L'introduction d'une fonction régénératrice (d'une ressource naturelle) non-concave par rapport à l'un des arguments rend le problème non convexe. En conséquence, nous ne pouvons plus utiliser les techniques traditionnelles de programmation dynamique. En attaquant ce problème, nous proposons une nouvelle méthode pour étudier une économie à deux secteurs en présence des externalités. En l'occurrence, nous introduisons le concept de "gain net de stock", qui est une notion similaire au "gain net d'investissement" introduit par Kamihigashi et al. (2007). En absence des propriétés convexes ou supermodulaires habituelles, nous prouvons que l'économie évolue pour augmenter le gain net de stock et établissons les conditions assurant la convergence de l'économie à long terme. Cette approche peut être appliquée aux problèmes similaires précédemment posés, ou être étendu à l'analyse des économies multisectorielles en général.

**Title :** Essays on ambiguity and optimal growth with renewable resources

**Keywords :** ambiguity; economics of risk; optimal contract; insurance; optimal growth

**Abstract :** The thesis consists of two main themes: decision making under ambiguity and optimal growth with externalities.

In the first two chapters, we study the optimal contract problem in presence of risk and ambiguity as an optimal control problem. Ambiguity is modeled according to Klibanoff et al. (2005). Our approach generalizes all the analyses carried out so far by considering the insurance contract as a pair of an indemnity function and a premium to be solved for simultaneously. We prove the existence of an optimal contract in the most general case, allowing for the principal or the insurance to be averse or neutral to risk or ambiguity. We characterize both the risk and ambiguity sharing rule between the contracting parties. In the case of one-sided ambiguity aversion, we show that an optimal insurance contract cannot contain a straight deductible policy. Furthermore, under the hypothesis that the conditional densities can be ranked according to the monotone likelihood ratio, we prove that a disappearing deductible contract is optimal, a result that is consistent with Gollier (2014). In particular, our method completes the analysis of Raviv (1979), showing that in the pure risk case with a risk-neutral insurer, a policy with an upper limit coverage cannot be optimal. This result also holds under ambiguity neutrality.

In the third chapter, I examine the impact of risk and ambiguity on the optimal investment in human and physical capital in a two-period Mincer's model. Uncertainty (both in the form of risk and ambiguity) is introduced to the accumulation of human capital via two channels. When uncertainty is on the depreciation rate of human capital (uncertain obsolescence of skills), I have found that the optimal investment in human capital always increases, whether or not physical capital is present.

This response to uncertainty of an optimizing household represents the typical self-insurance behavior. By contrast, when uncertainty is on the effectiveness of human capital accumulation, the optimal investment in human capital diminishes among the households with a degree of constant risk aversion less than one. This response to uncertainty is typical of a household who views human capital as an investment with risky/ambiguous return.

The final chapter (relatively independent from the preceding chapters) examines an important subject in the theory of economic growth: the role of renewable resources and externalities in the economy. The introduction of a (natural resource) regenerating function that is non-concave with respect to one of its arguments renders the problem non-convex. In consequence, we can no longer apply traditional dynamic programming techniques to this model. We thus propose a new method to study two-sector economies with externalities. In particular, we introduce the notion of "the net gain of stock", which is a notion similar to "the net gain of investment" of Kamihigashi et al. (2007). In absence of the usual convex and supermodular requirements, we prove that the economy evolves to increase the net gain of stock, and establish conditions that ensure long-run convergence. This approach can be adapted to similar problems previously studied, or be extended to the analysis of multi-sector economies in general.