Dehn surgeries and smooth structures on 3-dimensional transitive Anosov flows.

Mario Shannon

Université de Bourgogne - U.F.R. Sciences et Techniques
Institut de Mathématiques de Bourgogne
École doctorale Carnot-Pasteur

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Mario SHANNON

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Directeur de thèse : Christian BONATTI

Après avis de : Sergio FENLEY et Federico RODRIGUEZ-HERTZ

Devant la commission d'examen formée de :

Thierry BARBOT	Université d'Avignon	Examinateur
François BÉGUIN	Université Sorbonne Paris Nord	Examinateur
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Pierre DEHORNOY	Université de Grenoble	Examinateur
Sergio FENLEY	Florida State University	Rapporteur
Ana RECHTMAN	Université de Strasbourg	Examinatrice
Federico RODRIGUEZ-HERTZ	Penn State University	Rapporteur
Gioia VAGO	Université de Bourgogne	Examinatrice

Le paon fait la roue

le hasard fait le reste

Dieu s'assoit dedans

et l'homme le pousse.

J. Prévert.

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RÉSUMÉ

Cette thèse porte sur les chirurgies de Dehn et les structures différentielles associées aux flots d'Anosov transitifs en dimension trois. Les flots d'Anosov constituent une classe très importante des systèmes dynamiques, par leurs propriétés chaotiques persistantes par perturbations, autant que par leur riche interaction avec la topologie de la variété ambiante. Bien que beaucoup soient connus sur le comportement dynamique et ergodique de ces flots, il n'y a pas une compréhension assez claire sur la classification de ses différentes classes d'équivalence orbitale. Jusqu'à ce moment, les plus grands progrès ont été fait en dimension trois, où il y a une famille de techniques pour la construction d'exemples de flots d'Anosov connue comme *chirurgies*

Pendant la réalisation de cette thèse, dans un premier temps nous nous sommes intéressés à une chirurgie en particulier, connue comme la *chirurgie de Goodman*. Cette procédure consiste à choisir une orbite périodique du flot et réaliser une chirurgie de Dehn autour de cette orbite, adaptée au flot d'une façon telle qu'on obtient une nouvelle variété munie d'un flot d'Anosov. La problématique que soulève cette technique est que, pour la réalisation de la chirurgie, un des paramètres à choisir est une surface plongée dans la 3-variété et un difféomorphisme défini sur elle. De ce fait, l'espace de paramètres est, a priori, de dimension infinie et, pourtant, ce n'est pas facile d'avoir un contrôle sur la classe d'équivalence du flot obtenu par cette méthode. Il existe une deuxième procédure, qui peut-être interprétée comme une version infinitésimale de celle qui précède, connue comme la *chirurgie de Fried*. Celle-ci consiste à *éclater* l'orbite périodique, obtenant de ce fait un flot défini sur une variété à bord, puis *collapser* cette composante de bord d'une façon non-triviale et produire un nouveau flot. Cette chirurgie produit des flots univoquement définis, mais ceux-ci ne sont pas munis d'une structure hyperbolique naturelle. Ils sont, par construction, flots *topologiquement d'Anosov*.

Notre contribution consiste à montrer que, si on assume de plus que les flots sont transitifs, alors une chirurgie de Goodman et une chirurgie de Fried autour de la même orbite périodique produisent des flots équivalents, à égal élection de paramètres entiers.

Dans un second temps nous avons travaillé sur une question un peu plus abstraite, mais qui est naturellement liée à certaines procédures techniques dans la construction de flots hyperboliques. C'est le problème de savoir si tout flot dit topologiquement d'Anosov (i.e. expansif et qui satisfait la propriété de *shadowing* de Bowen) correspond à un flot hyperbolique différentiable, à équivalence orbitale près. Dans le cas particulier où le flot est transitif, il est connu depuis très longtemps qu'il peut être muni d'une structure non-uniformément hyperbolique définie dans le complémentaire d'un ensemble fini d'orbites périodiques. La plus grande difficulté est de construire des modèles (globalement) hyperboliques associés au flot original.

Dans ce contexte, notre contribution consiste à montrer que tout flot topologiquement d'Anosov et transitif, défini dans une variété de dimension trois, est orbitalement équivalent à un flot d'Anosov de classe C^1.

RESUMEN

La presente tesis es acerca de cirugías de Dehn y estructuras diferenciables asociadas a flujos de Anosov transitivos en dimension tres. Los flujos de Anosov constituyen una clase muy importante de sistemas dinámicos, tanto por sus propiedades caóticas y persistentes bajo perturbaciones, así como también por su rica relación con la topología de la variedad ambiente. Si bien mucho es conocido acerca de las propiedades dinámicas y ergódicas de estos flujos, no existe una comprensión clara acerca de cómo clasificar sus distintas clases de equivalencia orbital. Hasta ahora los mayores avances han sido hechos en dimensión tres, donde existe una familia de técnicas para la construcción de ejemplos de flujos de Anosov conocidas como *cirugías*.

Durante la realización de esta tesis, en un primer momento nos hemos interesado por una cirugía en particular, conocida como la *cirugía de Goodman*. Este procedimiento consiste en elegir una órbita periódica y realizar una cirugía de Dehn alrededor de esta curva, adaptada al flujo de forma tal de obtener una nueva variedad equipada con un flujo de Anosov. La problemática que nos hemos planteado es que, para realizar la cirugía, uno de los parámetros a elegir es una superficie encajada en la 3-variedad y un difeomorfismo definido sobre ésta. Esto hace que el espacio de parámetros sea, a priori, de dimensión infinita, y no tengamos un control claro sobre la clase de equivalencia orbital del flujo obtenido. Existe un segundo procedimiento, que puede ser interpretado como una versión infinitesimal del anterior, conocido como *cirugía de Fried*. Este consiste en realizar un *blow-up* de la curva periódica, obteniendo así un flujo en una variedad con borde, para luego colapsar este borde de una forma no trivial y producir un nuevo flujo. Esta cirugía produce flujos definidos de manera unívoca, pero éstos no vienen acompañados de una estructura uniformemente hyperbólica natural. Son, por construcción, flujos *topológicamente Anosov*.

Nuestro aporte consiste en probar que, asumiendo que el flujo es transitivo, una cirugía de Goodman y una cirugía de Fried en una órbita periódica producen flujos equivalentes, a igual elección de parámetros enteros.

En un segundo tiempo nos hemos interesado por una pregunta más abstracta, pero que aparece naturalmente ligada a ciertos procedimientos técnicos en la construcción de flujos hiperbólicos. Es el problema de saber si todo flujo topológicamente Anosov (i.e. flujo orbitalmente expansivo que satisface la propiedad de sombreado de Bowen) corresponde, a menos de equivalencia orbital, a un flujo hiperbólico diferenciable. En el caso particular de que el flujo sea transitivo, es conocido desde hace tiempo que éste puede ser munido de una estructura no-uniformemente hiperbólica, definida en el complemento de una cantidad finita de órbitas periódicas. La mayor dificultad radica en encontrar modelos (globalmente) uniformemente hiperbólicos asociados al flujo original.

En este contexto, nuestro aporte consiste en probar que todo flujo topológicamente Anosov y transitivo en una variedad de dimensión tres es orbitalmente equivalente a un flujo de Anosov.

ABSTRACT

The present thesis is about Dehn surgeries and smooth structures associated with transitive Anosov flows in dimension three. Anosov flows constitute a very important class of dynamical systems, because of its persistent chaotic behaviour, as well as for its rich interaction with the topology of the ambient space. Even if a lot is known about the dynamical and ergodic properties of these systems, there is not a clear understanding about how to classify its different orbital equivalence classes. Until now, the biggest progress has been done in dimension three, where there is a family of techniques intended for the construction of Anosov flows called *surgeries*.

During the realization of this thesis, in a first time we have been interested in a particular surgery method, known as the *Goodman surgery*. This method consists in make a Dehn surgery on a chosen periodic orbit, but adapted to the flow, in such a way to obtain a new manifold equipped with an Anosov flow. For making this surgery, one of the parameters that has to be chosen is an embedded surface in the 3-manifold and a diffeomorphism defined on it. Thus, the parameter space is, a priori, of infinite dimension and it is not easy to have control on the orbital equivalence class of the obtained flow. There exists a second method, that can be interpreted as an infinitesimal version of the previous one, known as the *Fried surgery*. It consists in making a *blow-up* of the flow along the periodic orbit, obtaining in this way a flow in a manifold with boundary, for then blowing-down the boundary component in a non-trivial way and produce a new flow. This surgery produces flows defined in a unique way, but they are not equipped with a natural uniformly hyperbolic structure. They are, by construction, *topological Anosov flows*.

Our contribution is to show that, if we assume that the flow is transitive, then a Goodman surgery or a Fried surgery performed on a periodic orbit produce orbitally equivalent flows, for the same choice of integer parameters.

In a second time, we have been interested for a more abstract question, but which is also related to some technical issues in the construction of hyperbolic flows. It is the problem of determining if every topologically Anosov flow (i.e. expansive and satisfying the Bowen shadowing property) correspond to a smooth hyperbolic flow, up to orbital equivalence. In the particular case that the flow is transitive, it has been known that there exists a non-uniformly hyperbolic structure defined in the complement of a finite set of periodic orbits. The main difficulty is the construction of (global) hyperbolic models associated to the original flow.

In this setting, our contribution is to show that every transitive topologically Anosov flow on a closed manifold is orbital equivalent to a smooth Anosov flow.

Contents

Thesis presentation.

Consider a closed, smooth, Riemannian manifold M and a non-singular vector field X of class C^r, where $r \geq 1$. The flow ϕ_t generated by the vector field X is said to be an *Anosov flow* if its natural action on the tangent bundle of M preserves a *hyperbolic splitting*

$$TM = E^s \oplus span\{X\} \oplus E^u,$$

where $span\{X\}$ is the sub bundle of TM generated by X and $\dim(E^s), \dim(E^u) \neq 0$. The splitting being hyperbolic means that the Riemannian length of the vectors in the bundle E^s is exponentially contracted by the forward action of $D\phi_t$, and the symmetric conditions holds for the E^u bundle in backward time.

The paradigmatic example of an Anosov flow is the *geodesic flow* of a closed Riemannian manifold with negative sectional curvature. For hyperbolic surfaces with constant negative curvature, the study of the geodesic flow has begun more than one hundred years ago with the works of Birkhoff, Cartan, Hadamard, Hopf and Poincaré, among others. The main novelty of this dynamical system (with respect to other well-known examples as the 2-body Newtonian problem) that attracted the attention of some scientists is its wild qualitative behaviour, which constitutes an intrinsic obstruction to make accurate predictions of its evolution in time. This kind of dynamical behaviour is known nowadays as *chaotic*.

Some notable characteristics of these geodesic flows are the sensitivity to initial conditions, the existence of a dense set formed by periodic orbits, the existence of a dense trajectory, and the existence of an ergodic invariant volume. The main cause behind these properties is the existence of a hyperbolic splitting, and it is also related to a stronger property satisfied by the flow, that is the *persistence* of the chaotic behaviour under perturbations of the flow. Based on the definition of *structurally stability* introduced by Andronov and Pontrjagin in the 30s, in 1961 Smale was able to prove this persistence phenomena in a chaotic discrete time dynamical system nowadays known as the *horseshoe map*, and in 1962 Anosov gave the proof of structural stability of the geodesic flow in negative curvature.

Anosov flows are named after the foundational work [1] of D. V. Anosov. In that work, the author explains from a unified point of view the general mechanisms by which a hyperbolic splitting produces complexity in the orbit space, being at the same time the main cause behind structural stability and, in many cases, good ergodic properties.

1 General Facts of Anosov flows.

Qualitative dynamical behaviour.

The works of Anosov and Smale in the 60s have led to the developing of a more general theory called *hyperbolic dynamics*, that includes the Anosov flows (and its discrete time analogues called Anosov diffeomorphism) as a particular case.

The hyperbolic dynamical systems are among the simplest examples of chaotic dynamics. This class of systems has been widely studied and a lot is known about its dynamical behaviour, both from the topological and measure theoretic point of view. In particular, for the case of Anosov flows we remark the following:

(i) Expansivity: Anosov flows are *orbitally expansive*.

(ii) Axiom A: The *non-wandering* set of the flow is hyperbolic, and the periodic trajectories are dense on this set. This condition, known as *axiom A*, implies a lot of interesting consequences. For instance, the *spectral decomposition theorem* of Smale applies, and the non-wandering set of the flow can be decomposed as a finite union of compact invariant sets, each one constituting a *homoclinic class*. On each of these sets the flow is *transitive*, and it has positive *topological entropy*.

(iii) C^1-structural stability: The flow generated by a non-singular C^1-vectorfield on a closed manifold M is said to be C^1-*structurally stable* if all the vector fields contained in a small C^1-neighbourhood produce orbitally equivalent flows. As we pointed out before, Anosov vector fields are C^1-structurally stable.

(iv) Ergodic Properties: Despite the topological complexity of the orbit structure of an Anosov flow, it is possible to put in practice statistical treatments of the orbit behaviour due to the existence of accurate measures well related to the dynamic. For instance, there always exists a measure of maximal entropy. If the flow has at least C^2-regularity, we can find others interesting measures as the so-called *SRB-measures*, *physical measures*, *Margulis measures*, etc.

Other fundamental property of Anosov flows is given by the *stable manifold theorem*. This theorem asserts that the bundles E^s and E^u integrate into globally defined foliations of the ambient space, respectively called the *stable* and the *unstable* foliations. For for each point x in the phase space, the stable leaf through x is homeomorphic to euclidean space and coincides with the set of points whose orbit approaches the orbit of x under forward time iteration. An analogous statement is valid for the unstable foliation, but with backward time iteration. As well, the bundles $E^s \oplus span\{X\}$ and $span\{X\} \oplus E^u$ integrate into a pair of transverse foliations that intersect along the flow lines. They will be called *center-stable* and *center-unstable* foliations, respectively.

These invariant foliations are strongly related with the dynamical behaviour of an Anosov flow and, at the same time, with its interaction with the topology of the ambient space.

Classification up to orbital equivalence.

The C^1-structural stability implies that the set of different equivalence classes of Anosov flows in a given closed manifold is at most countable. Thus, it is reasonable to search for a classification of these classes in terms of a finite number of invariants.

Even if we have a good description of the qualitative behaviour of the Anosov flows, the problem of its classification up to *orbital equivalence* rest a major subject nowadays. The main advance until now is done in dimension three, the smallest possible dimension for an Anosov flow. One impediment to advance in the comprehension of these flows up to orbital equivalence is the lack of examples. Apart from the geodesic flow of a negatively curved manifold or the suspension flow generated by a hyperbolic matrix acting on a torus, it is not easy to construct Anosov flows. Here is where the study splits in dimension three and bigger. For dimension three there is a battery of techniques called *surgeries* which allow to construct a wide variety of Anosov flows. In higher dimensions, it is also possible to construct more complicated Anosov flows using surgeries, but the setting is much more delicate. We invite the reader to compare [9] and [30].

In all this text we will concentrate only in the 3-dimensional case. We will not make an account of classification results, but we want to point out that some progress has been done if we restrict to some special families of 3-manifolds, or if we look at Anosov flows with some extra property. For instance:

- If a 3-manifold M admits an Anosov flow then the fundamental group $\pi_1(M)$ has *exponential grow*, see [48];
- There is a complete classification of Anosov flows on 3-manifolds with *solvable* fundamental group, see [47] and [5];
- There is a complete classification of Anosov flows on circle bundles over hyperbolic surfaces, see [33], [5]. Recently, in [26] the authors have announced a more accurate description of the isotopy classes of Anosov flows on these manifolds.
- There exist big advances for Anosov flows having additional properties. For instance, see [7];
- The existence of an Anosov flow on a 3-manifold implies that the fundamental group induces a faithful action on the circle by orientation preserving homeomorphisms. This imposes obstructions to the existence of such a flow, since not every group can be endowed with an *invariant left-order*. We refer to [27] for more precisions on these results.

As we remarked before, one fundamental tool for relating the topology of the phase space and the dynamic of an Anosov flow is the existence of the invariant foliations. In the 3-dimensional case, the center-stable and center-unstable foliations are codimension one foliations belonging to a very important class, the class of *taut* foliations. This enables to use other techniques in 3-manifold topology, as the Novikov theorem. For

instance, we can see from this that the universal cover of the ambient space must be homeomorphic to $\mathbb{R}^3$.

Examples of Anosov flows on 3-manifolds.

For a long time, the only two known families of Anosov flows were the geodesic flows of hyperbolic manifolds and the suspension flows generated by Anosov diffeomorphisms. These two kinds of flows can be constructed as a one-parameter group, acting in the cosets of a Lie group modulo some cocompact subgroup. For these reason they are referred to as *algebraic Anosov flows*. In [49], Tomter showed that the only algebraic Anosov flows in dimension three are suspensions of hyperbolic automorphisms of the torus and geodesic flows of hyperbolic surfaces (or, more generally, hyperbolic 2-orbifolds). Suspensions and geodesic flows are transitive, and the manifolds supporting the flow are sol-manifolds and Seifert manifolds, respectively.

The first examples of non-algebraic Anosov flows were constructed by Handel and Thurston ([37]) and Franks and Williams ([30]) using *surgery methods*. The first authors constructed a transitive Anosov flow in a *graph manifold*, while the second gave the first example of a non-transitive Anosov flow. Since then, surgeries have been reinterpreted in many ways and they became an important tool for creating a diversity of examples. In particular, surgeries allow to construct:

- Transitive Anosov flows possessing transverse tori which do not intersect all the orbits ([14], [21], [6], [12]);
- Anosov flows in a wide class of manifolds, including hyperbolic manifolds ([35]);
- Manifolds having transitive and non-transitive Anosov flows ([12]);
- Manifolds having an arbitrary finite number of non-equivalent transitive Anosov flows ([12]).
- Anosov flows preserving some additional structure, like *contact structures* ([28])

Recently, in [16] it has been announced the construction of hyperbolic 3-manifolds having an arbitrary number of non-equivalent Anosov flows. It is an open question to know if a manifold can admit infinitely many non-equivalent Anosov flows.

Transitive Anosov flows and Birkhoff sections.

Given a flow on a closed 3-manifold, a Birkhoff section is a compact surface, usually with non-empty boundary, immersed in the phase space in such a way that: The interior of the surface is embedded and transverse to the flow lines, the boundary components are periodic orbits of the flow, and every orbit intersects the surface in a uniformly bounded time. These sections come equipped with a first return map defined on the

interior of the surface, so the flow is a suspension on the complement of a finite set of periodic orbits. We refer to 5 for more precise statements and definitions.

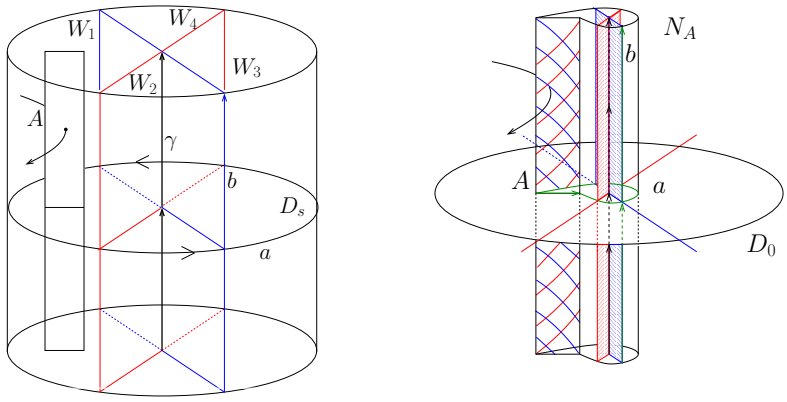
In [31], Fried has proved that every transitive Anosov flow admits a Birkhoff section whose first return map is pseudo-Anosov (in a non-closed surface). This extraordinary fact, later generalized by Brunella to any transitive expansive flow in [17], opens the possibility of reducing much of the analysis of transitive Anosov flows to the theory of pseudo-Anosov maps.

2 Dehn surgeries adapted to Anosov flows.

We will be concerned with two surgery operations available for Anosov flows, called the *Goodman surgery* and the *Fried surgery*. These two methods can be understood as a counterpart of *Dehn surgeries* but adapted to the pairs (flow,3-manifold). Given an Anosov flow (ϕ_t, M) and a periodic orbit γ, Fried and Goodman surgeries allow to construct a new flow in a manifold obtained by Dehn surgery on the curve γ. We give here a brief description of these two methods.

Let $\phi_t : M \to M$ be an Anosov flow generated by a smooth vector field X and consider a periodic orbit γ whose local invariant manifolds are orientable.[1]

Goodman surgery.



(a) Vertical annulus $A \subset W$. (b) Tubular neighbourhood N_A.

Figure 2: The Goodman surgery.

Let W be a tubular neighbourhood of γ. Since the local invariant manifolds of γ are orientable, the neighbourhood splits naturally in four quadrants determined by

[1] In general applications it is enough to work in this setting.

$W^s_{loc}(\gamma) \cup W^u_{loc}(\gamma)$.

Consider a compact annulus A smoothly embedded in the interior of a quadrant, which is transverse to the vector field X and is *vertical*, in the sense that a closed curve in A generating its fundamental group is homotopic to γ inside W. The orbit γ can be enclosed with a compact tubular neighbourhood N_A with smooth boundary, such that $A \subset \partial N_A$ and such that the flow points outward the manifold N_A along the annulus. This is depicted[2] in figure 2. Define $M_A = M \backslash int(N_A)$.

The surgery of Goodman (Goodman, [35]) will consist in separate the manifold M into the two pieces M_A and N_A, and then re-glue them using an appropriate diffeomorphism

$$\varphi : \partial N_A \to \partial M_A = \partial N_A$$

that produces an Anosov vector field X' in the manifold $M' = M_A \sqcup_\varphi N_A$. The diffeomorphism φ will have support contained in the annulus A and, topologically, it will be a Dehn twist.

Fried Surgery.

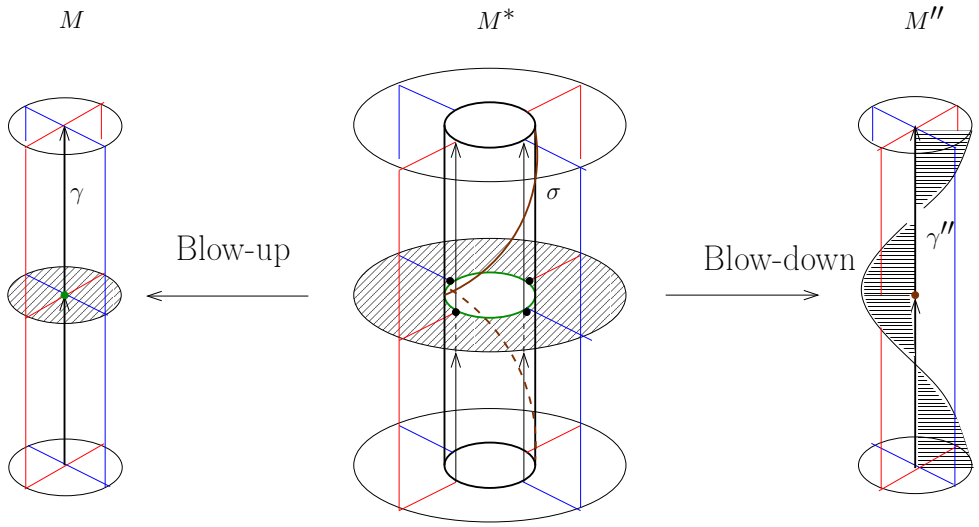


Figure 3: A picture of the Fried surgery.

The *blow-up of M along γ* is an operation that consists in replace the curve $\gamma \subset M$ with its *unitary normal bundle* $N^1_X(\gamma)$ and compactify with an adequate topology, producing in this way a manifold M^* with boundary. The interior $M^* \backslash \partial M^*$ is homeomorphic to $M \backslash \gamma$ and $\partial M^* \simeq N^1_X(\gamma)$ is a torus with a circle bundle map $\partial M^* \to \gamma$. The flow ϕ_t can be lifted to a flow $\phi^*_t : M^* \to M^*$, which leaves the boundary invariant. It is Morse-Smale on the boundary with two attracting and two repelling periodic orbits.

[2]We remark that in the figure the boundary of N_A has edges, but it can be smoothed along them.

Choose a simple closed curve $\sigma \subset \partial M^*$ which is a transverse section for the flow restricted to the boundary, and intersects once each periodic orbit. Choose a foliation $\mathcal{F}_\sigma$ of ∂M^* by simple closed curves isotopic to σ. A *blow down of* M^* is a 3-manifold M' obtained from M^* by collapsing each curve of $\mathcal{F}_\sigma$ into a point. So ∂M^* is transformed into a curve $\gamma' = \partial M^*/\mathcal{F}_\sigma$, while its complement is not altered in this operation. By taking a suitable reparametrization of ϕ_t^* near ∂M^*, this operation induces a flow $\phi_t' : M' \to M'$ on the blow-down manifold.

The surgery of Fried (Fried, [31]) will consist in make a blow-up of the manifold M along the curve γ, and then blow-down the boundary using a foliation $\mathcal{F}_\sigma$ as described before to produce a flow $\phi_t'' : M'' \to M''$. Topologically, this operation correspond to a Dehn twist of the manifold M on the curve γ.

Equivalence between Goodman and Fried surgeries.

At the topological level, both Fried and Goodman surgeries are *Dehn surgeries* on the manifold M, along the simple closed curve γ. We remark that the local invariant manifolds define a canonical framing along the curve γ by *meridians and longitudes*, so we can use this frame to express the coordinates of the Dehn surgeries. We remark here some technical differences between the two procedures:

- The Goodman Surgery,
  1. Starts with a C^1-Anosov flow and produces a C^1-Anosov flow;
  2. The orbital equivalence class of the obtained flow depends *a priori* on the particular choice of annulus and glueing map (φ, A). By changing this pair for another one, it is not clear how this alters the equivalence class of the resulting flow.
- The Fried surgery,
  1. It produces a flow which is topologically Anosov (cf. 1.2 or discussion below). But, it is not clear that this flow can be endowed with C^1-Anosov structure in a natural way.
  2. In the complement of the orbit γ the operation does not alter the foliation by flow orbits. Moreover, the orbitally equivalence class of the flow obtained by Fried surgery is uniquely determined by the integer twist parameter.

We encounter that, from the one hand, Goodman surgeries produce C^1-Anosov flows, but it is not obvious that these are uniquely determined by the combinatorial parameters of the surgery. From the other hand, a Fried surgery, that can be understood as an infinitesimal version of the previous one, produce a new flow in a unique way, actually without changing anything on the complement of γ. That is, the original flow and the one obtained by Fried surgery are equivalent in the complement of one periodic orbit.[3] Thus, the Fried flow is naturally endowed with a smooth atlas in the

[3]This condition is called *almost orbital equivalence*, see 1.4

complement of one orbit, and it preserves a uniformly hyperbolic splitting in this (open) manifold. This structure is a kind of *non-uniformly hyperbolic structure*, but it is not clear that it could be extended onto the whole manifold in an appropriate way.

The first main objective of this work is to give a unified point of view for these two surgeries. This will be done in chapter 3, where we will prove the following theorem:

Theorem B (Theorem 3.1.)**.** *If we assume that (ϕ_t, M) is in addition transitive, then there exists a neighbourhood W of the periodic orbit γ such that all the flows (ϕ_t^1, M_1) obtained by Goodman surgery on an annulus $A \subset W$ with twist parameter $m \in \mathbb{Z}$ produce orbitally equivalent flows. Moreover, all of them are orbital equivalent to the corresponding flow (ϕ_t^2, M_2) obtained by Fried surgery on γ and the same twist parameter.*

As a consequence, the Fried surgery provides a unified model for the orbital equivalence class of the flows obtained by the Goodman technique. As well, observe that this shows that the Fried surgeries preserve the class of smooth Anosov flows up to C^0-orbital equivalence. (Nevertheless, we will also achieve this results by other methods. See below.)

It is interesting to remark that we assume transitivity as an essential condition in our proof. It is not known if the statement is valid in the non-transitive case.

3 Smooth models for transitive topological Anosov flows.

One major technique for the qualitative study of the orbits of a general dynamical system is the *symbolic description.* Roughly speaking, the method consists in choose a finite partition of the phase space into some well-behaved sets, and associate to every point (in the phase space) the sequence of elements of the partition that its orbit visit under time evolution.

This method is particularly useful when the system that we want to study presents some chaotic related behaviour, such as expansivity. It has been a useful approach for describing chaotic phenomena present in some Hamiltonian flows or in the N-body problem. The point of view of symbolic dynamics allows to develop a deep understanding of the topological and measure theoretic properties of the action of the dynamical system on the phase space, and allows to explain and predict some phenomenons that are not at all direct to deduce from the equations defining the system.

If we recall that *differential equations* were born as an attempt to describe (some part of) the physical universe, there is a theoretical problem issued from the abstract construction that we described above. The problem is to determine if an abstract dynamical behaviour, constructed from a symbolic description, actually corresponds to the behaviour of a smooth dynamical system on a manifold, preferably of physical

origin. (E.g. a system preserving a smooth measure, or defined via Euler-Lagrange equations, etc.)

Expansive dynamics in low dimensions. In the particular case of *expansive homeomorphisms on closed surfaces* and *expansive non-singular flows on closed* 3*-manifolds*, after many works on the subject (starting with Sinai in the 60s and going through Ratner, Thurston, Handel, Fathi et al., Katok, Lewowicz, Hiraide, Paternain, Inaba, Matsumoto, among others) we have nowadays an accurate description of the symbolic aspects of this class of systems. They can be given a very special symbolic codification, that is called a *Markovian partition.*

The way how Markovian partitions arise in Anosov or expansive dynamics in low dimensions is very interesting. In the case of smooth Anosov dynamical systems (discrete or continuous), they can be constructed using a finite number of stable and unstable leaves. In the case of general expansive homeomorphisms on surfaces or expansive flows on 3-manifolds, a big part of the work referred above consists in show that, under the only assumption of expansivity, the partitions of the phase space by stable/unstable sets are, in fact, a pair of transverse invariant foliations, possibly with singularities, but of a very precise kind. In concrete, every expansive homeomorphism on a surface or expansive flow on a 3-manifold is equivalent to a *pseudo-Anosov homeomorphism* or *pseudo-Anosov flow*, respectively. In an analogous way to that of Anosov dynamics, expansive systems can be codified via accurate Markovian partitions, constructed from the stable and unstable foliations. We will give a more detailed account of these results later in 1.1.

Topologically Anosov flows on 3-manifolds. A distinction between general pseudo-Anosov and smooth Anosov flows on dimension three must be done. The local product structure of the invariant foliations of a smooth Anosov flow allows to show the pseudo-orbits tracing property of Bowen. But, in the more general class of pseudo-Anosov flows, this is no longer true, due to the singularities of the invariant foliations. These constitute an obstruction. An analogue remark is valid in the case of pseudo-Anosov homeomorphisms on closed surfaces.

This leads to consider an intermediate class of dynamical systems, called *topologically Anosov*. In dimension three, a non-singular flow is said to be topologically Anosov if it is expansive, and the invariant foliations have no singularities. It can be seen that this class of systems satisfy the pseudo-orbits tracing property and their general dynamical behaviour is very similar to that encountered in a smooth Anosov flow, at least from the point of view of topological dynamics.

Translating the previous theoretical problem into this setting, the main concern of this thesis work is with the problem of determining if a topological Anosov flow on a closed 3-dimensional manifold is, in fact, realized as a smooth Anosov flow.

In this context, we are able to give a partial answer, constructed under the assumption of transitivity. The main theorem in this thesis consists in show the following:

Theorem C. *Let (ϕ_t, M) be a transitive topological Anosov flow on a closed 3-manifold. Then, there exists a smooth Anosov flow (ψ_t, M) that is C^0-orbitally equivalent to the previous one.*

In more topological terms, this means that given a transitive topological Anosov flow on a closed 3-manifold, it is possible to endow the manifold with a smooth atlas, such that the foliation by flow orbits is tangent to a smooth Anosov vector field.[4]

Unfortunately, a proof for the non-transitive case is still missing, and the techniques presented here do not extend to the more general setting of non-transitive topological Anosov flows. See comments below.

Connection with the construction of Anosov flows. We want to underline another reason that have lead to consider the problem of the topologically Anosov flows, apart from the theoretical problem that we have already discussed.

We have already pointed out the Fried surgery, a central tool in the study of smooth Anosov flows that naturally goes through the construction of a topological Anosov one. Some other techniques on Anosov flows rely on construction of an abstract topological toy model, starting from a given Markov codification. To cite an example, we can refer the reader to the works of Béguin, Bonatti, Yu, et al., where much of the job consists in construct Anosov *basic pieces* starting from a given combinatorial description. The reader could appreciate that, in general, there are at least two steps to complete: First, to construct the topological model toys that look like hyperbolic flows, and second, to actually prove that these models are hyperbolic.[5] In recent years, a lot of new developments in 3-dimensional Anosov flows and non-uniformly hyperbolic diffeomorphisms have been done supported on this kind of techniques, that goes through the construction of topological Anosov models. We invite the reader to consult [10], [15] pag. 478, and [8].

Thus, the necessity of clarifying this distinction between topological Anosov and smooth Anosov is, as well, a question of technical order.

4 Approach to the problems and further questions

The objective of the present thesis is to prove theorems **B** and **C** stated above. The main technique behind our analysis is the reduction of the problems to pseudo-Anosov dynamics on surfaces, which is possible under the assumption of transitivity due to

[4] Remark that this condition is stronger than just having a smooth atlas for which the flow is smooth. It is possible to have a smooth flow, topologically Anosov, which is not Anosov since the action of its derivative does not preserve a hyperbolic splitting.

[5] It is important to remark that, from many points of view (for example that of the *smooth ergodic theory* or *C^1-generic dynamics*) the definition of smooth hyperbolic dynamical system is stronger than the one corresponding to topologically Anosov. To actually guarantee that a system is hyperbolic, the existence of a corresponding dominated splitting must be shown. This is done, in general, using a *fixed point method* called the *cone field criterion.*

the previously referred theorem of Fried and Brunella ([31] and [17]). The key point to approach theorems **B** and **C** is a technical property that we have stated under the form of theorem **A** (theorem 2.16) in chapter 2.

We give here some explanations and sketch of proof of these theorems.

Theorem A: Criterion for orbital equivalence.

There is a well-known property in surface dynamics, which states that two isotopic pseudo-Anosov homeomorphisms on a closed surface are conjugated by a homeomorphism isotopic to the identity. In turn, this property means that the conjugacy class of a pseudo-Anosov in a closed surface is determined by the action of the homeomorphism on the fundamental group of the space. Cf. 7 on this thesis for a more precise statement and comments. We also refer to [25] or [23] for proofs and more accurate information.

In chapter 2 we will show that, given a transitive topological Anosov flow, then its C^0-orbital equivalence class is determined by the action of the induced first return map onto a given *tame* Birkhoff section, plus some combinatorial information about how this surface embeds in the phase space. This will be the content of theorem **A** (theorem 2.16), we refer to that chapter for a precise statement. This theorem can be interpreted as an analogue to the former property, but in the context of flows. Both statements have the shape of an expectable property, being the techniques to actually prove them the major step.

First, observe that given two non-singular flows equipped with Birkhoff sections, if the first return maps are conjugated then the flows are orbitally equivalent in the complement of some finite sets of periodic orbit. For understanding if the two flows are (globally) orbitally equivalent, there is an essential difficulty coming from the fact that the conjugation is defined only on the interior of the Birkhoff sections, and in general it does not extend to the boundary. For instance,

- *It is no longer true that the action on the fundamental group determines the conjugacy class of a pseudo-Anosov homeomorphism in a surface with boundary* (cf. 7).

In addition, if we look at the germ of the flow near a boundary component of a Birkhoff section, the way in which this surface embeds in the 3-manifold has an effect on the topology of the ambient manifold.

To overcome this issue, we have based our proof of theorem **A** on techniques coming from [13]. Motivated by the study of axiom A diffeomorphisms on closed 3-manifolds, those authors provide a precise analysis of the germ of a hyperbolic flow in the neighbourhood of a boundary component of an embedded Birkhoff section. In particular, if the Birkhoff section satisfies a property that they call *tameness*, they can describe the germ of the first return map onto a local transverse section in function of the one corresponding to the Birkhoff section.

Remark 0.1. One motivation for writing theorem **A** is to complement a previous result of Brunella. In [17] and [18] the author gives a criterion for orbital equivalence between transitive topologically Anosov flows which is more general than theorem **A**, that was later applied in other works as [19]. One aim of theorem **A** is to complement lemma 7 on page 468 of [18].

As a final comment about theorem **A**, we find at least curious the fact that theorem 2.16 requires so much detail in the proof, even assuming our general strong hypothesis of Anosov-like behaviour. Even if it is not of our interest to study this property out of the Anosov context, we point out that it seems a technical issue to give general criterions for determining when a general *almost equivalence* between two flows actually implies a global equivalence.

Theorem B: Fried and Goodman surgeries are equivalent.

The criterion stated in theorem **A** gives an appropriate frame to study theorem **B**. The proof consists in find appropriate Birkhoff sections associated to each the Goodman and the Fried flow, with conjugated first return maps and compatible combinatorial data on the boundary.

Given a transitive Anosov flow (ϕ_t, M) and a periodic orbit γ with orientable invariant manifolds, it is always possible to consider a Birkhoff section Σ such that γ intersects transversally the interior of this surface (cf. [17]). We will proceed in the following steps:

(i) The Fried surgery transforms Σ into a Birkhoff section for the new flow, with some extra boundary components due to the intersection $\gamma \cap \Sigma$. The homology coordinates of this new section in a neighbourhood of the periodic orbit are determined by the integer parameter of the surgery as Dehn twist. (See 8)

(ii) The Goodman surgery does not produce a Birkhoff section after surgery but, at least, it does not modify the original section out of a neighbourhood of the periodic orbit. In proposition 3.14 we will show that this part of the original section that is not altered can be extended onto a global Birkhoff section for the new flow, which has the periodic orbit as boundary component. In addition, it has the same combinatorial parameters than the one corresponding to the Fried flow. This is the hardest part of the proof, and is given in section 4.

(iii) To finish, we will see that the first return maps to each of the Birkhoff sections act in an equivalent way on the fundamental group of the surface. We can deduce (see 7) that the first return maps are conjugated on the interior, and we can conclude theorem **B** using theorem **A**.

We remark that, even if two Goodman annulus are very closed to each other, cutting and connecting again the orbits of a flow along different regions is a process that, a priori, will not produce the same flows. For instance, consider the case of two Goodman surgeries, with the same twist parameter, but with supporting annuli placed on different

quadrants. Since there is no isotopy between these annuli preserving the transversality with the stable/unstable foliations, it is not at all direct that these two surgeries should lead to the same flow. In 3 we will see that the orbital equivalence guaranteed in theorem **B** rarely coincides with the identity out of a neighbourhood of the periodic orbit.

Theorem C: Smooth structures for transitive topologically Anosov flows.

One strategy for the problem of finding a smooth representative of a given topological dynamic is as follows: First, to construct a smooth model expected to be equivalent to the original dynamical system, and second, to prove that the smooth model is actually equivalent to the original one.

We can put this in practice in the case of transitive topological Anosov flows due to the existence of Birkhoff sections. The existence of these surfaces imply that, in the complement of a finite set of periodic orbits, the flow is orbitally equivalent to the suspension generated by a pseudo-Anosov map on a non-closed surface. In consequence, every transitive topological Anosov flow comes naturally equipped with a smooth structure defined in the complement of a finite number of periodic orbits. These structures will be called *almost Anosov structures*, see 2.

For constructing the smooth Anosov models in the present work, we will use Goodman approach to Dehn surgeries, and then we will conclude using theorem **A**. More precisely:

(i) Pick one of these almost Anosov structures defined in just an open and dense region of the space. We will obtain a smooth model by making a surgery of the flow supported in a neighbourhood of the singular regions. Then, using the so-called *cone field criterion*, we will show that this flow preserves a uniformly hyperbolic splitting.

(ii) We will show that both the original flow and the smooth model are equipped with Birkhoff sections satisfying the criterion for orbital equivalence stated in theorem **A**. This completes the proof.

Remark. There are other contexts in expansive dynamics where we encounter similar strategies as the one used in theorem **C**. See for example [32], [43], [42] and [38]. It is important to make a comment on the first of these articles written by Gerber and Katok. In that work, the existence of smooth models for pseudo-Anosov homeomorphisms is showed. Observe that as a direct consequence of its definition, a pseudo-Anosov homeomorphism is smooth in the complement of its singular set. The construction of the smooth model is done by perturbing the given pseudo-Anosov map in a neighbourhood of its singularities. But to show the equivalence with the original map is not at all direct. (Remark that the pseudo-Anosov homeomorphisms are not *structurally stable*.) Moreover, it is possible to show that no conjugation between the pseudo-Anosov map and the obtained model is differentiable in an open region of the surface. This phenomenon can be interpreted in the following way:

> Given a pseudo-Anosov homeomorphism on a closed smooth surface, it is possible to change the smooth atlas of the surface for a new one, such that the map is a smooth diffeomorphism. However, the new smooth atlas is nowhere compatible with the original one.

The proof of the equivalence between the two homeomorphisms goes along the construction of an adequate Markov partition for the smooth model, that allows to compare with the original map.

Further questions.

The methods presented here for proving theorems **B** and **C** cannot be extended to the non-transitive case. Probably, these methods could be used to prove the same statements about Dehn surgeries and smooth structures as above in some special families of hyperbolic transitive attractors derived from pseudo-Anosov. But there exists some transitive hyperbolic attractors constructed in [12] and [22], called *incoherent attractors*, where the dynamic cannot be reduced to a first return on a surface.

Question. Are theorems **B** and **C** still valid without the hypothesis of transitivity?

More generally, observe that Fried and Goodman surgeries can be performed in a big class of smooth flows on closed 3-manifolds having periodic orbits. This could be the case of smooth flows with homoclinic tangencies, Lorenz-like flows (with singularities) or more general non-uniformly hyperbolic flows.

Question. Are there analogues of theorems **B** and **C** for these more general classes of flows?

5 Description of the content.

The thesis is divided in fourth chapters.

Chapter 1 is dedicated to present the general definitions and properties that will be used along the thesis, as those of Birkhoff section, pseudo-Anosov homeomorphism or Anosov flow. In particular, for completeness of the subsequent theorems, we have added in this chapter some properties and proofs that will be correspondingly referred. It is intended for a fast reading.

Chapter 2 contains the proof of theorem **A**. The chapter is dived in two sections: First, we summarize all the material needed from [13] to carry out the proof. In the following section we give the complete statement and proof of the theorem.

Chapter 3 is dedicated to the proof of theorem **B**. In the first two sections we summarize the constructions of Fried and Goodman. The proof of theorem **B** is given

in the last two sections. The hardest part concerns the construction of a Birkhoff section for the Goodman flow, and is completely contained in the fourth section.

Chapter 4 is dedicated to the proof of theorem **C** and is divided in three sections. In the first, we have added a small survey of results concerning topological Anosov dynamics. Our results are stated in 1.2, and the proof of theorem **C** is given in the third section. In the middle section we have added a description of the germ of an almost Anosov structure near the singular orbits, that will be used along the proof.

Chapters 3 and 4 are independent between them. For a complete proof of theorem **B** or **C** it suffices to consult chapter 2 and then chapter 3 of 4, respectively.

Chapter 1

Preliminaries

1 Flows on 3-manifolds and equivalence.

In all what follows we will be concerned with non-singular, continuous, regular flows on connected 3-manifolds. We introduce here the basic notions and definitions that will be used throughout the text.

The *phase space* will always be assumed to be a connected 3-dimensional manifold M. The main theorems that we will state in next chapters involve flows defined on closed 3-manifolds, i.e. boundaryless and compact. Nevertheless, along the proofs we will be forced to consider flows (and related objects) defined in non-closed regions of our manifolds.

With respect to regularity, we can always assume that the manifold M is equipped with a smooth structure, as it is a general fact of dimension three. But the flows (and related objects) that we will take into account are not necessarily smooth.[1]

Let (ϕ_t, M) be a non-singular continuous flow on a 3-manifold M. The flow is said to be *regular* if the associated partition into orbits is a one dimensional foliation of M. In this case, each orbit is an immersed 1-dimensional topological submanifold. In general, we will say that an object is regular if it is a topological submanifold of the ambient space.

The action at time $t \in \mathbb{R}$ of the flow over a point $x \in M$ will be indistinctly denoted by $\phi_t(x)$ or $\phi(t, x)$. Given two points $x, y \in M$ such that $y = \phi_t(x)$ for some $t > 0$, we will denote by $[x, y]$ the orbit segment

$$[x, y] = \{\phi_s(x) : 0 \leq s \leq t\}.$$

[1] Moreover, in chapter 4, they will appear some (natural) smooth structures associated with transitive topological Anosov flows, which are defined in an open and dense region of the phase space, but not in the whole.

Since we consider non-singular flows (i.e. orbits not reduced to singletons) observe that this segments are never reduced to points. The orbit of x under the flow ϕ_t will be denoted by $\mathcal{O}(x)$, and we will use $\mathcal{O}^+(x)$ and $\mathcal{O}^-(x)$ to denote the positive and negative semi-orbits respectively. (Observe that the orbits of a flow are naturally oriented by the direction of the flow.) Given a non empty open set $U \subset M$ and a point $x \in U$ we will denote by $\mathcal{O}_U(x)$ the connected component of $\mathcal{O}(x) \cap U$ that contains x. The partition of the phase space by ϕ_t-orbits induces a 1-dimensional oriented foliation that we simply denote by $\mathcal{O}$, and we use $\mathcal{O}_U$ for the induced foliation on any non-empty open subset $U \subset M$.

We recall from the basic theory of differential equations that if X is a non-singular vector field of class C^k in M, where $k \geq 1$, then the flow ϕ_t associated to the system of ordinary differential equations

$$\frac{d}{dt}x = X(x)$$

is a non-singular flow of class C^k. In this case, the foliation by flow orbits is of class C^k and its leaves are immersed 1-dimensional manifolds tangent to the vectorfield X.

The principal notion of equivalence between flows that we will consider throughout the text is *orbital equivalence*[2], which requires for orientation preserving equivalence between the induced orbit foliations.

For $i = 1, 2$ let $\phi_t^i : M_i \to M_i$ be a regular flow of class C^k on a manifold M_i, where $k \geq 0$.

Definition 1.1 (**Orbital equivalence**)**.** The flows (ϕ_t^1, M_1) and (ϕ_t^2, M_2) are C^r-*orbitally equivalent*, where $r \geq 0$, if there exists a C^r-diffeomorphism $H : M_1 \to M_2$ such that, for every $x \in M_1$, it sends the orbit $\mathcal{O}^1(x)$ homeomorphically onto the orbit $\mathcal{O}^2(H(x))$, preserving the orientation of these orbits. We denote it by

$$H : (\phi_t^1, M_1) \to (\phi_t^2, M_2).$$

For technical reasons, we will be also interested in a weaker notion that we explain here: Consider a non-empty open subset $U \subset M$. The foliation by ϕ_t-orbits on M induces a foliation on U but, in general, the action $\mathbb{R} \times M \to M$ provided by ϕ_t does not restrict onto an $\mathbb{R}$-action on the set U. Instead, what we obtain is a *pseudo-flow* on U. That is, a map $(t, x) \mapsto \phi_t(x)$ defined for some couples $(t, x) \in \mathbb{R} \times U$. This pseudo-flow generates a partition of U into orbits, which coincides with the foliation $\mathcal{O}_U$ previously defined. This restriction pseudo-flow will be simply denoted by (ϕ_t, U).

For each $i = 1, 2$ consider a non-empty open subset $U_i \subset M_i$.

Definition 1.2 (**Local orbital equivalence**)**.** The pseudo-flows (ϕ_t^1, U_1) and (ϕ_t^2, U_2) are C^r-*locally orbitally equivalent*, where $r \geq 0$, if there exists a C^r-diffeomorphism $H : U_1 \to U_2$ such that, for every $x \in U_1$, it sends the orbit $\mathcal{O}^1_{U_1}(x)$ homeomorphically

[2] In other literature it is called *topological equivalence* between flows.

onto the orbit $\mathcal{O}^2_{U_2}(H(x))$, preserving the orientation of these orbits. We denote it by

$$H : (\phi^1_t, U_1) \to (\phi^2_t, U_2).$$

If γ is a periodic orbit of (ϕ_t, M) and W, W' are two neighbourhoods of γ, the pseudo-flows obtained by restriction onto these sets are the same, in the sense that they coincide over their common domain of definition $W \cap W'$. The *germ of* ϕ_t *at* γ is the equivalence class of the pseudo-flows $\{(\phi_t, W) : W \text{ neighbourhood of } \gamma\}$ under this relation, and we denote it by $(\phi_t, M)_\gamma$ or $(\phi_t, W)_\gamma$.[3]

For $i = 1, 2$, consider a periodic orbit γ_i of (ϕ^i_t, M_i).

Definition 1.3 (**Local orbital equivalence of germs**)**.** The germs $(\phi^1_t, M_1)_{\gamma_1}$ and $(\phi^2_t, M_2)_{\gamma_2}$ are *C^r-locally orbitally equivalent*, where $r \geq 0$, if there exists a C^r-local orbital equivalence $H : (\phi^1_t, W_1) \to (\phi^2_t, W_2)$ defined between some neighbourhoods W_i of γ_i. We denote it by

$$H : (\phi^1_t, W_1)_{\gamma_1} \to (\phi^2_t, W_2)_{\gamma_2}.$$

There is another equivalence relation, called *almost orbital equivalence*, that appears naturally associated with some surgery procedures, and will appear in this text.

Definition 1.4 (**Almost orbital equivalence**)**.** The flows (ϕ^1_t, M_1) and (ϕ^2_t, M_2) are *C^r-almost orbitally equivalent*, where $r \geq 1$, if for each flow ϕ^i_t there exists a finite set of periodic orbits Γ_i, such that the restricted flows $(\phi^i_t, M \backslash \Gamma_i)$ are C^r-orbitally equivalent.

Remark. When we use the word *orbital equivalence* without making any reference to the regularity degree $r \geq 0$, it must be understood that $r = 0$, unless specified. This is, we just care about *homeomorphisms* preserving the oriented foliations by orbit segments, no matter the degree of regularity of the flows.

We finish with two other related notion that will appear throughout the text.

Conjugation of flows.

Definition 1.5. For $i = 1, 2$ let $\phi^i_t : M_i \to M_i$ be a regular flow of class C^k on a manifold M_i, where $k \geq 0$.

1. The flows (ϕ^i_t, M_i) are *C^r-conjugated*, where $r \geq 0$, if there exists a C^r diffeomorphism $H : M_1 \to M_2$ such that $\phi^2_t(H(x)) = H(\phi^1_t(x))$, for every $x \in M_1$ and $t \in \mathbb{R}$.

2. Let $U_i \subset M_i$ be a non-empty open set, for each $i = 1, 2$. The the restricted pseudo-flows (ϕ^i_t, U_i) are *C^r-locally conjugated* if there exists a C^r-diffeomorphism

[3] By *abuse of terminology*, we will use the word *germ* for referring to a given pseudo-flow (ϕ_t, W), instead of its equivalence class. Through applications, it will be understood that the neighbourhood W can be freely replaced by a smaller one if needed.

$H : U_1 \to U_2$, such that $\phi_t^2(H(x)) = H(\phi_t^1(x))$, for every $x \in U_1$ and $t \in \mathbb{R}$ satisfying that $(s, x) \mapsto \phi_s(x)$ is well-defined for every $0 \leq s \leq t$.

3. Let γ_i be a periodic orbit of ϕ_t^i, for each $i = 1, 2$. The germs $(\phi_t^i, M_i)_{\gamma_i}$ are *C^r-locally conjugated* if there exists neighbourhoods W_i of each γ_i such that the restricted pseudo-flows (ϕ_t^i, W_i) are C^r-locally conjugated.

We remark here that, in the case where the flows (ϕ_t^i, M_i) are generated by a vector field X_i of class C^1 and the conjugation is at least C^1, then conjugation between (pseudo)-flows is equivalent to the condition

$$X_2(z) = H_*(X_1)(z) = DH(H^{-1}(z)) \cdot X_1(H^{-1}(z)), \quad \text{for all } z \in U_2.$$

Conjugation of homeomorphisms.

Definition 1.6. For $i = 1, 2$ let $P_i : \Sigma_i \to \Sigma_i$ be a C^k-diffeomorphism on a manifold Σ_i, where $k \geq 0$.

1. The diffeomorphisms (P_i, Σ_i) are *C^r-conjugated*, where $r \geq 0$, if there exists a C^r-diffeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $h \circ P_1 = P_2 \circ h$. We will denote it by

$$h : (\Sigma_1, P_1) \to (\Sigma_2, P_2).$$

2. For each $i = 1, 2$ let $x_i \in \Sigma_i$ be a fixed point of P_i. The germs $(P_i, \Sigma_i)_{x_i}$ are *C^r-locally conjugated* if there exists open neighbourhoods $U_i \subset V_i$ of each x_i, and a C^r-diffeomorphism $h : V_1 \to V_2$ such that $P_i(U_i) \subset V_i$, $i = 1, 2$ and $h \circ P_1(u) = P_2 \circ h(x)$, for every $x \in U_1$. We denote it by

$$h : (P_1, V_1)_{x_1} \to (P_2, V_2)_{x_2}.$$

As before, we remark that we will consider in general C^0-conjugation, unless specified.

2 Transverse sections and first return maps.

Let $\phi_t : M \to M$ be a non-singular regular flow on a 3-manifold M.

Definition 1.7 (**Transverse section**)**.** A *transverse section* for the flow ϕ_t is a boundaryless, regular,[4] embedded surface $\Sigma \subset M$, satisfying that:

(i) The surface Σ is *topologically transverse to the flow lines.* That is, $\forall\, x \in \Sigma$ there exists a neighbourhood W of x inside M and some $\delta > 0$ such that:

[4]Regular: $\forall\, p \in \Sigma$ there exists a neighbourhood W of p in M and a homeomorphism $\varphi : W \to \mathbb{R}^3$ such that $h(W \cap \Sigma) = \mathbb{R}^2 \times 0$.

- $\Sigma \cap W$ is connected and $W \backslash \Sigma$ has two connected components;
- $\Sigma \cap [\phi_{-\delta}(x), \phi_\delta(x)] = \{x\}$ and it is verified that $int\,([\phi_{-\delta}(x), x])$ is contained in one component of $W \backslash \Sigma$ and $int\,([x, \phi_\delta(x)])$ is contained in the other.

(ii) The surface is *proper with respect to the flow lines*. That is, every compact orbit segment $[\phi_{t_1}(x), \phi_{t_2}(x)]$ intersects Σ in a compact set, where $t_1, t_2 \in \mathbb{R}$ and $x \in M$.

In the case that the orbits of the flow are tangent to a continuous vector field, we say that Σ is a *C^r-transverse section*, $r \geq 1$, if, in addition, it is a submanifold of class C^r and $T_x M = T_x\Sigma \oplus T_x\mathcal{O}(x)$ for every $x \in M$. In general we will just talk about *transverse sections* without reference to the regularity, unless it is necessary.

Observe that conditions (i) and (ii) in the definition actually imply that every compact orbit segment cuts the transverse section in a finite set.

2.1 First return map.

Let $\Sigma \subset M$ be a transverse section for a flow ϕ_t and assume that there exists a non empty open subset $U \subset \Sigma$ where there is a well defined *first return map* $P : U \to \Sigma$. That is, for every $x \in U$ there exists $\tau(x) = \min\{t > 0 : \phi_t(x) \in \Sigma\}$ and

- The function $\tau : U \to (0, +\infty)$ is continuous,
- The first return map is given by $P(x) = \phi_{\tau(x)}(x)$.

Observe that, if x is a point in Σ satisfying that there exists some $t > 0$ such that $\phi_t(x) \in \Sigma$, then from the continuity of the flow it follows that there exists a first return map defined in a neighbourhood of x. Recall that a first return map as above is always a homeomorphism from its domain onto its image. Moreover, as it follows from the *implicit function theorem*, it is a C^l-diffeomorphism, where l is the minimum between the regularities of ϕ_t and Σ.

Definition 1.8. The *first return saturation* of U is the set

$$\mathcal{U} = \{x \in M : \exists\ t \leq 0 \text{ such that } \phi_t(x) \in U \text{ and } int\,([\phi_t(x), x]) \cap \Sigma = \emptyset\}.$$

This set is the union of all the compact orbits segments joining each point in U with its first return to Σ, as we see in figure 1.1. For every $x \in \mathcal{U}$ there exists $u \in U$ and $v = P(u) \in \Sigma$ such that $x \in [u, v]$, so we define $\sigma(x) = [u, v]$. The following lemma will be used several times in what follows.

Lemma 1.9. *Consider two continuous flows $\phi_t^i : M_i \to M_i$, $i = 1, 2$. Let $\Sigma_i \subset M_i$ be a transverse section for each flow and assume there exists a first return map $P_i : U_i \to \Sigma_i$*

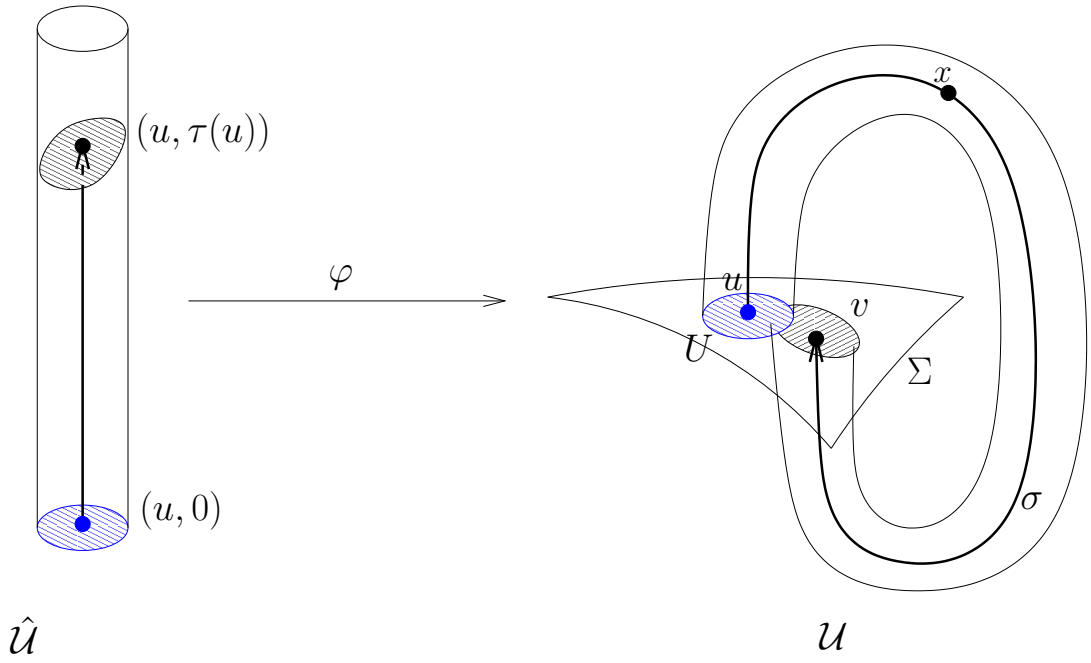


Figure 1.1: $\mathcal{U}$ is the union of all the compact orbit segments σ joining points $u \in U$ with its first return $v \in \Sigma$.

defined in an open set $U_i \subset \Sigma_i$. If there exists a homeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $h(U_1) = U_2$ and $h \circ P_1(x) = P_2 \circ h(x)$, $\forall\ x \in U_1$, then there exists a homeomorphism $H : \mathcal{U}_1 \to \mathcal{U}_2$ such that:

1. *For every point $x \in \mathcal{U}_1$ the map H takes the orbit segment $\mathcal{O}_{\mathcal{U}_1}(x)$ onto $\mathcal{O}_{\mathcal{U}_2}(H(x))$,*
2. *The restriction map $H|_{U_1}$ coincides with $h|_{U_1}$.*

Proof. For each $i = 1, 2$ take the maps $\varphi_i : \hat{\mathcal{U}}_i \to \mathcal{U}_i$ defined by $\varphi_i(u, s) = \phi^i_s(u)$, where $\hat{\mathcal{U}}_i = \{(u, s) \in U_i \times [0, +\infty) : 0 \leq s \leq \tau^i(u)\}$. Take a continuous function $\theta : U_1 \times [0, +\infty) \to [0, +\infty)$ such that, for any $u \in U_1$ the restriction $\theta(u, \cdot) : \mathbb{R} \to \mathbb{R}$ is an orientation preserving homeomorphisms satisfying

- $\theta(u, 0) = 0$,
- $\theta(u, \tau^1(u)) = \tau^2(h(u))$.

Any such a function provides a continuous family of orientation preserving homeomorphisms $[0, \tau^1(u)] \to [0, \tau^2(h(u))]$ parametrized over $u \in U$. Define $\hat{H} : \hat{\mathcal{U}}_1 \to \hat{\mathcal{U}}_2$ by

$\hat{H}(u,s) = (h(u), \theta(u,s))$. This map $\hat{H}$ is a homeomorphisms between $\hat{\mathcal{U}}_1$ and $\hat{\mathcal{U}}_2$ and for every $u \in U_1$ it takes the segment $\{u\} \times [0, \tau^1(u)]$ homeomorphically onto the segment $\{h(u)\} \times [0, \tau^2(h(u))]$ preserving orientation.

The maps φ_i and $\hat{H}$ allow to construct a map $H : \mathcal{U}_1 \to \mathcal{U}_2$ satisfying that the following diagram commutes

$$\begin{array}{ccc} \hat{\mathcal{U}}_1 & \xrightarrow{\hat{H}} & \hat{\mathcal{U}}_2 \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\ \mathcal{U}_1 & \xrightarrow{H} & \mathcal{U}_2 \end{array}$$

in the following way: For any $x \in \mathcal{U}_1$ take $(u,s) \in \hat{\mathcal{U}}_1$ such that $\varphi_1(u,s) = x$ and define $H(x) = \varphi_2(\hat{H}(u,s)) = \phi^2(h(u), \theta(u,s))$. Then, over the set $\mathcal{U}_1 \backslash \Sigma_1$ this map is well defined and it is a homeomorphism between $\mathcal{U}_1 \backslash \Sigma_1$ and $\mathcal{U}_2 \backslash \Sigma_2$, which clearly preserves the orbit segments. For the points in Σ_1, we have just to take care of those points $u \in U_1$ for which there exists $v \in U_1$ such that $P_1(v) = u$, being those points the only ones with more than one preimage by φ_1. For a point u satisfying this, observe that it has exactly two preimages by φ_1 which are $(u,0)$ and $(v, \tau^1(v))$. Observe that $\hat{H}(u,0) = (h(u), \theta(u,0)) = (h(u), 0)$ and $\hat{H}(v, \tau^1(v)) = (h(v), \theta(v, \tau^1(v))) = (h(v), \tau^2(h(v)))$. Since $\varphi_2(h(v), \tau^2(h(v))) = \varphi_2(P_2(h(v)), 0) = P_2(h(v))$ and $P_2 \circ h(v) = h \circ P_1(v) = h(u)$ we have that $\varphi_2(\hat{H}(u,0)) = \varphi_2(\hat{H}(v, \tau^1(v)))$ and so $H(u)$ is well defined and equal to $h(u)$. This also shows that H is a homeomorphisms between our two sets $\mathcal{U}_1$ and $\mathcal{U}_2$ which, by construction, take the orbit segment $\sigma^1(x)$ into the segment $\sigma^2(H(x))$ and coincides with h over U_1.

In addition observe that the facts that H coincides with h over U_1 and $P_2 \circ h = h \circ P_1$ implies that, for any point $x \in \mathcal{U}_1$, the map H take the orbit segment $\mathcal{O}_{\mathcal{U}_1}(x)$ onto $\mathcal{O}_{\mathcal{U}_2}(H(x))$, since each of these orbit segments are a concatenation of segments of the form $\sigma^i(u_k) = [u_k, u_{k+1}]$ where $u_{k+1} = P_i(u_k)$ and $k = 1, \ldots, n$. □

2.2 Local and global transverse sections.

Definition 1.10 (**Local and global transverse sections.**). Let $\phi_t : M \to M$ be a continuous non-singular flow on a 3-manifold,

1. A *global transverse section* Σ for the flow ϕ_t is a transverse section that is properly embedded in M and for which there exists $T > 0$ such that $[x, \phi_T(x)] \cap \Sigma \neq \emptyset$, for every $x \in M$. In this case, the first return map is a homeomorphism $P_\Sigma : \Sigma \to \Sigma$.
2. Given a periodic orbit γ of the flow, a *local transverse section at* γ is a transverse section D, homeomorphic to a disk, such that $\{x_0\} = \gamma \cap D$ contains exactly one point. Given a local transverse section D there always exists a neighbourhood $U \subset D$ of x_0 and a first return map $P_D : U \to D$ that fixes x_0.

A basic fact that follows from lemma 1.9 above is stated in the next proposition.

Proposition 1.11. *Let $\phi_t^i : M_i \to M_i$, $i = 1, 2$, be two non-singular continuous flows.*

1. *Assume there exists a global transverse section Σ_i for each flow ϕ_t^i and let $P_i : \Sigma_i \to \Sigma_i$ be the first return map. If there exists a homeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $h \circ P_1 = P_2 \circ h$, then there exists a homeomorphism $H : M_1 \to M_2$ such that:*
   (a) *H is an orbital equivalence between the flows,*
   (b) *$H|_\Sigma = h$.*
2. *Let γ_i be a periodic orbit of each flow ϕ_t^i, D_i a local transverse section and $P_{D_i} : U_i \to D_i$ a first return map defined in a neighbourhood $U_i \subset D_i$ of the intersection point $x_i = \gamma_i \cap D_i$. If there is a homeomorphism $h : D_1 \to D_2$ such that $h(U_1) \subset U_2$ and $h \circ P_{D_1}(x) = P_{D_2} \circ h(x)$, $\forall\, x \in U_1$, then there exists a tubular neighbourhood W_i of each γ_i and a homeomorphism $H : W_1 \to W_2$ such that:*
   (a) *H is a local orbital equivalence between the respective germs at each γ_i,*
   (b) *$H|_{D_1 \cap W_1} = h|_{D_1 \cap W_1}$.*

3 Flow isotopies.

Let $\phi_t : M \to M$ be a continuous flow on a 3-manifold M.

Definition 1.12 (**Flow isotopy**)**.** A *flow isotopy* or *ϕ_t-isotopy* between two non-empty open sets $U, V \subset M$ is a homeomorphism $\psi : U \to V$ of the form

$$\psi : x \mapsto \phi(\tau(x), x), \text{ for every } x \in U,$$

where $\tau : U \to \mathbb{R}$ is a continuous and bounded function, such that ψ sends each oriented obit segment $\mathcal{O}_U(x)$ onto $\mathcal{O}_V(\psi(x))$ preserving orientation. In addition we define:

1. A *ϕ_t-isotopy* between two arbitrary non-empty sets $A, B \subset M$ is a homeomorphism $\psi : A \to B$ obtained as the restriction of a ϕ_t-isotopy $\psi : U \to V$, where U and V are open sets containing X ans Y, respectively.
2. Two transverse sections Σ_i, $i = 1, 2$ are said to be *ϕ_t-isotopic* if there exists a ϕ_t-isotopy $\psi : \Sigma_1 \to \Sigma_2$.

We remark that some conditions must be satisfied on τ to actually define a flow isotopy. The set of these maps has a natural *pseudo-group* structure, and they have a regularity degree that can be expressed in terms of that of ϕ_t and τ.

A first return map $P : U \to \Sigma$, defined on a non-empty open set U inside a transverse section Σ is, by itself, a flow isotopy. We state the following basic property:

Lemma 1.13. *Consider two ϕ_t-isotopic transverse sections Σ_i, $i = 1, 2$, and let $\psi : \Sigma_1 \to \Sigma_2$ be a flow isotopy between them. Let $P_1 : U_1 \to \Sigma_1$ be a first return map, defined on some non-empty open set $U_1 \subset \Sigma_1$, and let $U_2 = \psi(U_1)$. Then, there is a well-defined first return map $P_2 : U_2 \to \Sigma_2$ and it is satisfied that $P_2 \circ \psi(x) = \psi \circ P_1(x)$, $\forall\, x \in U_1$.*

4 Surgeries along transverse surfaces.

Let $\phi_t : M \to M$ be a non-singular regular flow of class C^k, $k \geq 0$, on a 3-manifold.

Roughly speaking, a *surgery along a transverse surface* on the flow (ϕ_t, M) is an operation that produces a new flow on a new manifold, and consists in cut M along a given transverse surface and re-glue the pieces with a non-trivial homeomorphism.

Let Σ be a regular compact surface C^k-embedded in M, possibly with boundary, and satisfying that $\mathring{\Sigma} = \Sigma \backslash \partial\Sigma$ is transverse to the flow lines. We require topologically transversality if $k = 0$, see definition 1.7. Observe that this condition implies that every small *flow-box* neighbourhood of Σ is partitioned into two component by Σ.

Set $\Sigma = \Sigma_0$ and consider another transverse surface Σ_1 disjoint from Σ_0, that is ϕ_t-isotopic to Σ_0. We will assume that there exists a ϕ_t-isotopy $\psi : \Sigma_0 \to \Sigma_1$ of the form $\psi(p) = \phi_{\tau(p)}(p)$, where $\tau : \Sigma_0 \to (0, +\infty)$ is a bounded C^k-function, and such that $\forall\ p \in \Sigma_0$ the orbit segment between p and $\phi_{\tau(p)}(p)$ intersects $\Sigma_0 \cup \Sigma_1$ just in its extremities $\{p, \psi(p)\}$. Consider the compact set

$$K = \bigcup_{p \in \Sigma_0} \{\phi_t(p) : 0 \leq t \leq \tau(p)\}.$$

Then $K \subset M$ is a regular submanifold (with corners on the boundary when $k \geq 1$) and homeomorphic to $\Sigma_0 \times [0, 1]$. Its boundary contains the two disjoint surfaces Σ_0 and Σ_1. The complement of $\Sigma_0 \cup \Sigma_1$ in ∂K is a surface S homeomorphic to $\partial\Sigma_0 \times [0, 1]$. The flow is tangent to ∂K along S, is transverse to ∂K along Σ_0 and points inward the interior of K, and is transverse to ∂K along Σ_1 and points outward. See figure 1.2.

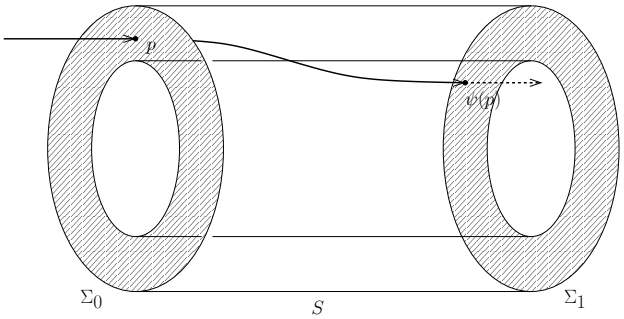


Figure 1.2: The set K.

Let's define $M_K = M \backslash int(K)$. Then M is is the union of the two submanifolds M_K

and K which have disjoint interiors and intersect along the set $\partial M_K = \partial K$.

Consider some C^k-diffeomorphism $f : \Sigma \to \Sigma$ satisfying that $f \equiv id$ in a small neighbourhood of $\partial\Sigma$. We will define a C^k-diffeomorphisms $F : \partial K \to \partial K$ in the following way:

$$F(x) = \begin{cases} f(x) & \text{if} \quad x \in \Sigma_0 \\ \\ x & \text{if} \quad x \notin \Sigma_0. \end{cases}$$

The map F coincides with f in Σ_0 and is the identity in the complement. We can consider this map as a map from $\partial K \to \partial M_K$, and construct the manifold

$$M' = M_K \sqcup_F K.$$

We denote by ι_{M_k}, ι_K the canonical inclusions of M_K and K into M', respectively.

The following is a classical result in differential topology:

Proposition 1.14. *There is a C^k regular flow $\phi'_t : M' \to M'$ induced on M', such that the canonical inclusions of M_K and K in M' send the foliations $\mathcal{O}_{M_K}$ and $\mathcal{O}_K$ onto $\mathcal{O}'_{M_K}$ and $\mathcal{O}'_K$, respectively. Moreover, when $k \geq 1$, denote by X the generating vector field for ϕ_t. Then, there exists a smooth structure on M' such ι_{M_K}, ι_K are smooth diffeomorphisms onto their respective images, and ϕ'_t is generated by a C^k-vector field X' such that $\iota^*_{M_K}(X') = X|_{M_K}$ and $\iota^*_K(X') = X|_K$. This structure is unique, up to smooth diffeomorphisms.*

We say that the flow (ϕ'_t, M') is obtained by *surgery on* (ϕ_t, M) *along the transverse surface* Σ. Observe that, at the topological level, it is a direct fact the existence of the induced flow. For a proof about the differentiability statement above see [].

Observe as well that the flow obtained by surgery is independent of the auxiliary surface Σ_1 chosen above. In fact, there is no need to consider this surface for defining (ϕ'_t, M'). The reason why we have chosen this explanation is the technical lemma below, that we will use later.

4.1 Surgeries along parallel surfaces.

The homeomorphism class of the manifold M' is not altered by small perturbations of the map f, as it is constant along all the isotopy class of f. Nevertheless, observe that the orbital equivalence class of the induced flow is not, a priori, preserved by this sort of operations. There is a particular case where this is indeed true. As before, set $f_0 = f$. The flow isotopy $\psi : \Sigma_0 \to \Sigma_1$ induces a C^k-diffeomorphism $f_1 : \Sigma_1 \to \Sigma_1$ of the form $f_1 = \psi \circ f_0 \circ \psi^{-1}$. We will define two diffeomorphisms $F_i : \partial K \to \partial K$, $i = 1, 2$ in the following way

$$F_i(x) = \begin{cases} f_i(x) & \text{if} \quad x \in \Sigma_i \\ \\ x & \text{if} \quad x \notin \Sigma_i. \end{cases}$$

So each map F_i coincides with f_i on Σ_i and is the identity in the complement. For each $i = 1, 2$ construct the manifolds

$$M_i = M_K \sqcup_{F_i} K.$$

As before, each manifold M_i is equipped with a corresponding flow ϕ_t^i of class C^k. It follows that:

Lemma 1.15. *The flows (ϕ_t^i, M_i), $i = 1, 2$ are C^k-orbitally equivalent.*

Proof. The foliation by orbit segments in K is equivalent to the product $\Sigma \times [0, 1]$. Thus, it suffices to define $H : M_0 \to M_1$ to be the identity on M_K, and $H \simeq f \times id$ on $K \simeq \Sigma \times [0, 1]$ □

Definition 1.16 (**Parallel surgeries**)**.** We say that two surgeries using pairs (f_i, Σ_i) related by flow isotopy as above are *parallel* surgeries.

5 Birkhoff sections.

In general, a *First Return Section* or *Poincaré Section* for a flow in a 3-manifold is an embedded surface, transverse to the flow, with a continuous first return map defined on some open subset of it. If the first return map preserves some closed invariant subset, then its saturation by flow's orbits corresponds to some closed invariant subset for the flow, and the study of the dynamical behaviour of the latter one, which is a flow in a 3-dimensional object, can be reduced to the study of a surface map. The idea of reducing the study of (at least some part of) a 3-dimensional flow to a surface discrete dynamical system leads back to Poincaré, who used this technique for the study of the the *3-body problem*. As basic examples of first return sections we have the *local Poincaré sections* around periodic trajectories and the *global Poincaré sections*. The first ones just provides information about the local behaviour of the flow around a periodic trajectory, while the second one, despite of providing a lot of information about the flow (it reduces de whole dynamic of the flow to that of a surface homeomorphism), imposes very strong conditions about the topology of the supporting manifold (which must be a surface bundle over the circle) and to the flow itself (which must be topologically euivalent to a suspension flow). So, in the hope of reducing a very large part of the dynamic of a flow to a surface discrete dynamic, the local sections are too inefficient because they just describe the dynamic around a periodic trajectory, while the global sections will rarely exist. A *Birkhoff section* is a compact embedded surface, with boundary, such that its interior is transverse to the flow, its boundary components are periodic trajectories and every orbit of the flow will meet this surface in an uniformly bounded time. The existence of a Birkhoff section implies that in the complement of a finite set of periodic trajectories (the boundary of the section) the flow is, up to reparametrization, the suspension of a surface homeomorphism (the first return map defined on the interior of the section). Observe that this is also a very important topological fact, because in the complement of a finite set of simple closed

curves our manifold is a surface bundle over the circle. This, in combination with the homotopy class of the embedding of the surface, could provide strong information about the topology of the manifold and how it is related to the flow. As is explained in [31], this sections where introduced for the first time by Birkhoff, for the study of the geodesic flow of a closed hyperbolic surface. In this article Fried proved that any transitive Anosov flow in a closed 3-manifold admits a Birkhoff section, a result which turns these objects into powerful tools for the study of these flows.

Let $\phi_t : M \to M$ be a non-singular regular flow on a 3-manifold.

Definition 1.17. A *Birkhoff section* for the flow (ϕ_t, M) is an immersion $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$, where Σ is a compact surface, sending $\partial\Sigma$ onto a finite set $\Gamma \subset M$ of periodic orbits, such that:

1. The restriction of ι to each boundary component of $\partial\Sigma$ is a covering map onto a curve in Γ,
2. The restriction of the map ι to the interior of Σ is an embedding and the submanifold $\iota(\mathring{\Sigma})$ is topologically transverse to the flow,
3. There exists a real number $T > 0$ such that $[x, \phi_T(x)] \cap \iota(\Sigma) \neq \emptyset$, $\forall\ x \in M$.

In general, we will denote a Birkhoff section just by Σ. In all what follows, we will indistinctly use the notation $\mathring{\Sigma}$ to denote $\Sigma \backslash \partial\Sigma$ or its inclusion inside M. We will also denote $M_\Gamma = M \backslash \Gamma$.

If the flow is generated by a C^k-vector field and the map ι is a C^k-immersion for some $k \geq 1$, we say that it is a *C^k-Birkhoff section*, provided $T_x M = T_x\mathring{\Sigma} \oplus X(x)$ for every $x \in \mathring{\Sigma}$. If the immersion is just continuous, we will say that it is a *topological Birkhoff section*. In general we will just talk about Birkhoff sections and we will stress their regularity properties if needed.

5.1 First return map.

Condition 3. in the definition above implies that for every $x \in \mathring{\Sigma}$ there is a well defined first return point $P_\Sigma(x) \in \mathring{\Sigma}$. Thus, there is a first return map defined in the whole interior of Σ. We can view this map as a homeomorphism

$$P_\Sigma : \mathring{\Sigma} \to \mathring{\Sigma}.$$

In particular, the surface $\mathring{\Sigma}$ is a global transverse section for the flow restricted on M_Γ. By proposition 1.11 this implies that (ϕ_t, M_Γ) is topologically equivalent to the suspension flow generated by $P_\Sigma : \mathring{\Sigma} \to \mathring{\Sigma}$.

Remark. In general, the associated first return to a Birkhoff section does not extend to the boundary. In some literature it is shown that, by modifying the embedding of the surface near the boundary, it is possible to have Birkhoff sections where this map does extend. Nevertheless, we will not make any use of this fact.

5.2 Local Birkhoff sections.

Definition 1.18. Let γ be a periodic orbit of a flow $\phi_t : M \to M$. A *local Birkhoff section at* γ is an immersion $\iota : [0,1) \times \mathbb{R}/\mathbb{Z} \to M$ such that:

1. $\gamma = \iota(\{0\} \times \mathbb{R}/\mathbb{Z})$;
2. The restriction of the map ι to $(0,1) \times \mathbb{R}/\mathbb{Z}$ is an embedding and the submanifold $\iota((0,1) \times \mathbb{R}/\mathbb{Z})$ is transverse to the flow lines;
3. The exists a real number $T > 0$ and a neighbourhood W of γ such that $[x, \phi_T(x)] \cap \iota([0,1) \times \mathbb{R}/\mathbb{Z}) \neq \emptyset$, $\forall\, x \in W$.

Let $B \subset M$ be the image of $\iota : [0,1) \times \mathbb{R}/\mathbb{Z} \to M$. When confusion is not possible, we will just use B for referring to the local Birkhoff section. We will denote by $\mathring{B}$ the set $B \backslash \gamma$.

Concerning regularity, we can make the same considerations as we did for the general case.

Given a local Birkhoff section B at γ, the last property in the previous definition implies that there exists a collar neighbourhood $U \subset B$ of γ and a first return map $P_B : \mathring{U} \to \mathring{B}$, where $\mathring{U} = U \backslash \gamma$. In general we will be concerned with the germ of the first return map to a local Birkhoff section, that is, we will consider the map P_B up to changing its domain of definition U for another collar neighbourhood if needed.

As a last remark, observe that if γ is a periodic orbit in the boundary of some Birkhoff section Σ, then the intersection of a small tubular neighbourhood of γ with Σ can be used to provide a local Birkhoff section.

5.3 Combinatorial parameters associated to the boundary.

Consider a (global) Birkhoff section as defined above and let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$. We remark that the preimage of any curve γ_i could consist in many boundary components of Σ. Let's denote them by $C^i_1, \ldots, C^i_{p_i}$, in order to have

$$\partial\Sigma = C^1_1 \cup \cdots \cup C^1_{p_1} \cup \cdots \cup C^n_1 \cup \cdots \cup C^n_{p_n}.$$

For each γ_i consider a small tubular neighbourhood of W_i, in such a way that $W_i \cap \Sigma$ splits as p_i different local Birkhoff sections $B^i_1, \ldots, B^i_{p_i}$ at γ_i. It can be seen (cf.4.1 below) that the first return map (in the complement of γ_i) induces a cyclic permutation of these surfaces. Thus, it induces a cyclic permutation on each set $\{C^i_1, \ldots, C^i_{p_i}\}$.

On the other hand, let $\alpha \subset W_i \backslash \gamma_i$ be a simple closed curve, that is the boundary of an embedded disk in W_i. This is called a *meridian curve*. Without loss of generality, we can assume that α intersects transversally each B^i_j. Then, independently of $j = 1, \ldots, p_i$, the modulus $|\alpha \cap B^i_j|$ is a natural number m_i.

Definition 1.19 (**Multiplicity and local number of connected components.**). We say that p_i is the *number of connected components of* Σ *at* γ_i, and we denote it by $p(\gamma_i, \Sigma)$. We define as well the *multiplicity* of Σ at γ as the quantity m_i, and which will be denoted by $m(\gamma_i, \Sigma)$.

Remark 1.20. We remark a (local) Birkhoff section is embedded if and only if $m = 1$. The number m coincides with the covering degree of the immersion, when restricted to corresponding the boundary curve.

In the special case that an orbit γ_i is a *topological saddle type periodic orbit*, we encounter another important parameter that is called the **linking number** of Σ at γ_i. We will properly state its definition in section 4.1.

5.4 Blow-down operation.

Associated to the first return map $P_\Sigma : \mathring{\Sigma} \to \mathring{\Sigma}$, there is a construction called *Blow-down*, that consists in the following: Let $\widehat{\Sigma}$ be the surface obtained by collapsing each boundary component of Σ into a point, and denote by x_j^i the point obtained when collapsing C_j^i. Then, observe that there is an associated homeomorphism $\widehat{P}_\Sigma : \widehat{\Sigma} \to \widehat{\Sigma}$ and, by the previous remarks, each set $\{x_1^i, \dots, x_{p_i}^i\}$ constitutes a periodic orbit.

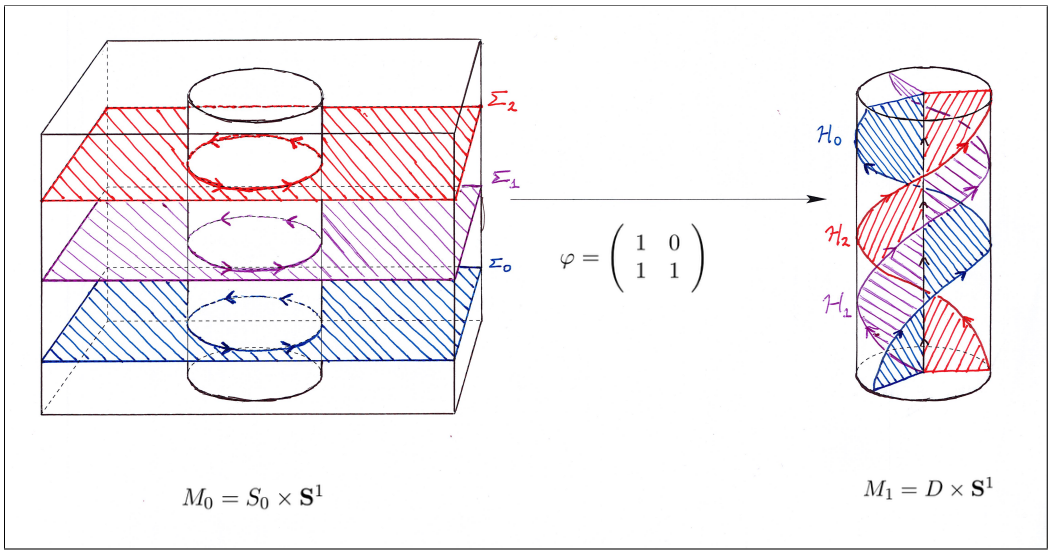
Definition 1.21. The homeomorphism $\widehat{P}_\Sigma : \widehat{\Sigma} \to \widehat{\Sigma}$ is called the *blow-down* associated to $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$.

5.5 An example.

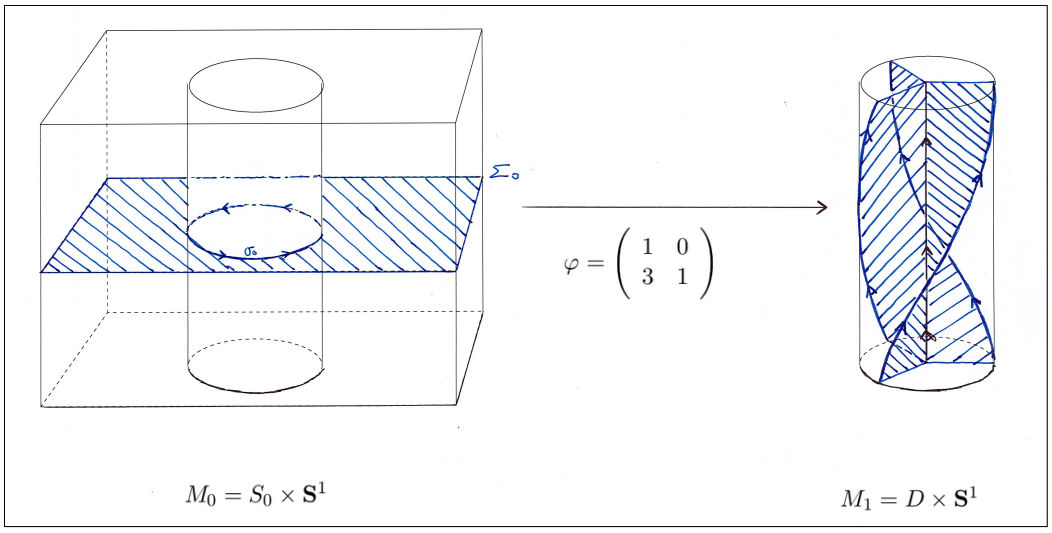
To finish this section we give some simple examples of Birkhoff sections, that can give a pictorial flavour of how does these objects look like.

Let S be a closed surface, $D \subset S$ an embedded closed disk, and let $S_0 = S \backslash int(D)$. Consider the manifolds $S_0 \times \mathbb{R}/\mathbb{Z}^1$ and $D \times \mathbb{R}/\mathbb{Z}^1$. On each one there is a flow, all whose orbits are periodic and coincides with the *fibers* $\{x\} \times \mathbb{R}/\mathbb{Z}^1$. If we glue the boundaries of these 3-manifolds using a *fiber preserving homeomorphism*, then we get a closed manifold equipped with a *periodic flow*.

In figure 1.3a, the three different surfaces Σ_0, Σ_1 and Σ_2 in $S_0 \times \mathbb{R}/\mathbb{Z}^1$ are glued with three helicoids $\mathcal{H}_0, \mathcal{H}_1$, and $\mathcal{H}_2$. We get then a Birkhoff section where $\Gamma = \{\gamma\}$, $p(\gamma, \Sigma) = 3$ and $m(\gamma, \Sigma) = 1$. By the same procedure, in figure 1.3b we see an example with $p(\gamma, \Sigma) = 1$ and $m(\gamma, \Sigma) = 3$.



(a) Three connected components in a neighbourhood of the boundary



(b) One connected component in a neighbourhood of the boundary

Figure 1.3: Birkhoff sections

6 Anosov Flows on 3-manifolds.

We give here the definition of Anosov flow, as well as other definitions and properties that will be used throughout the text.

Definition 1.22 (**Anosov flow**)**.** Let M be a closed, smooth, 3-manifold and let $\phi_t : M \to M$ be a flow generated by a non-singular C^k-vectorfield X, where $k \geq 1$. The flow is *Anosov* if there exist a Riemannian metric $\|\cdot\|$ and decomposition of the tangent bundle of M as Whitney sum of three line bundles $TM = E^s \oplus E^c \oplus E^u$, where $E^c = span\{X\}$, satisfying that:

1. The splitting is invariant by the action $D\phi_t : TM \to TM$,
2. There exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\begin{aligned} &\|D\phi_t(x) \cdot v\| \leq C\lambda^t \|v\|, \ \forall\ v \in E^s(x),\ t \geq 0,\ x \in M; \\ &\|D\phi_t(x) \cdot v\| \leq C\lambda^{-t} \|v\|, \ \forall\ v \in E^u(x),\ t \leq 0,\ x \in M. \end{aligned} \tag{1.1}$$

Observe that, from the compactness of the ambient space, it follows that the vectors in E^s or E^u will satisfy property (1.1) above for any chosen Riemannian metric on M, up to modifying the constants $C > 0$ and $0 < \lambda < 1$ if necessary. Thus, the definition of Anosov flow makes an auxiliary use of a Riemannian metric, but it only depends on the C^1 equivalence class of the flow.

It is not difficult to check that the decomposition of TM must be unique and continuous. The bundle E^s is called the *stable bundle*, E^u is called the *unstable bundle* and E^c, the one who is tangent to the flow lines, is called the *central bundle*. The two dimensional bundles $E^{cs} = E^s \oplus E^c$, $E^{cu} = E^c \oplus E^u$ and $E^{su} = E^s \oplus E^u$ are respectively called the *center-stable bundle* (or *cs*-bundle), the *center-unstable bundle* (or *cu*-bundle) and the *stable-unstable bundle* (or *su*-bundle).

6.1 Invariant foliations.

One of the fundamental properties of these flows is the integrability of its (center-)stable and (center-)unstable bundles into foliations which are preserved by the flow can be merely defined by dynamical properties. This fact is expressed by the so called *stable manifold theorem*:

Theorem (**Stable manifold theorem.**)**.** *Each* 1*-dimensional bundle* E^s *or* E^u *is uniquely integrable, and the partition of* M *by integral curves respectively determines a pair of* 1*-dimensional foliations* $\mathcal{F}^s$ *and* $\mathcal{F}^u$*, invariant by the action of the flow. Moreover, for every* $x \in M$ *it is satisfied that*

$$\begin{aligned} \mathcal{F}^s(x) = W^s(x) &= \{y \in M : dist(\phi_t(y), \phi_t(x)) \to 0,\ t \to +\infty\}, \\ \mathcal{F}^u(x) = W^u(x) &= \{y \in M : dist(\phi_t(y), \phi_t(x)) \to 0,\ t \to -\infty\}. \end{aligned}$$

On the other hand, each 2-dimensional bundle E^{cs} or E^{cu} is uniquely integrable, and integrate into a 2-dimensional foliation $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$, respectively. Moreover, for every $x \in M$ it is satisfied that:

$$\mathcal{F}^{cs}(x) = \{y \in M : \exists\ s \in \mathbb{R},\ \textit{s.t. dist}(\phi_t(y), \phi_{t+s}(x)) \to 0,\ t \to +\infty\},$$
$$\mathcal{F}^{cu}(x) = \{y \in M : \exists\ s \in \mathbb{R},\ \textit{s.t. dist}(\phi_t(y), \phi_{t+s}(x)) \to 0,\ t \to -\infty\}.$$

6.2 Expansivity.

Anosov flows are orbitally expansive, according to the following definition:

Definition 1.23. Let $\phi_t : M \to M$ be a regular non-singular flow on a closed, Riemannian, 3-manifold. The flow is said to be *orbitally expansive* if for every $\alpha > 0$ there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that: If two points x, y satisfy that

$$\textit{dist}(\phi_t(x), \phi_{h(t)}(y)) \leq \alpha,\ \forall\ t \in \mathbb{R},$$

where $h : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is some increasing homeomorphism, then $y = \phi_s(x)$ for some $|s| \leq \varepsilon$.

General expansive flows on closed 3-manifolds have been studied by Paternain, Inaba and Matsumoto. In [46] and [40] it is shown that every orbitally expansive flow is equivalent to a *pseudo-Anosov flow*, see [17] or the referred works for precise statement and definitions. We will refer again these results in chapter 4.

6.3 Transitivity and Birkhoff sections.

Anosov flows on 3-manifolds can be transitive or not. One remarkable property of transitive[5] Anosov flows is the following theorem, due to D. Fried:

Theorem 1.24 (Fried, [31])**.** *Every transitive Anosov flow admits a Birkhoff section, whose first return map is pseudo-Anosov.*

This property, later generalized by Brunella in [17] to every transitive expansive flow, allows to reduce some part of the analysis of these flows to the theory of pseudo-Anosov maps on surfaces.

7 Pseudo-Anosov Homeomorphisms.

Let Σ be a closed orientable surface.

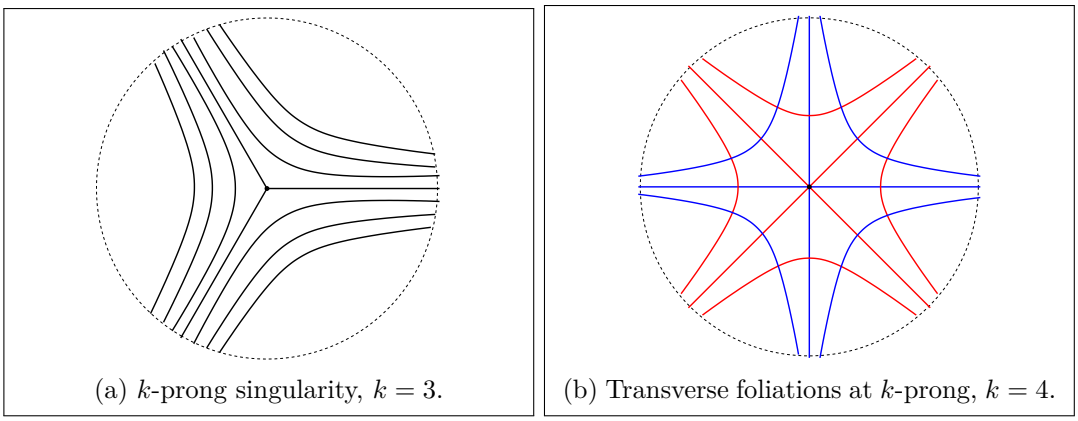
[5] Also non-transitive Anosov flows have strong connections with the topology of the ambient manifold. See [20].

Definition 1.25. A homeomorphism $f : \Sigma \to \Sigma$ is *pseudo-Anosov* if there exists a pair of transverse f-invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ on Σ, respectively equipped with *transverse measures* μ_s and μ_u, and a constant $\lambda > 1$ such that

$$f_*(\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda\mu_s) \quad \text{and} \quad f_*(\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda^{-1}\mu_u).$$

The transverse measures are required to be non-atomic and with full support.

If the genus of Σ is greater than one, then the two foliations necessarily have singularities. If this is the case, in the previous definition we just allow singularities whose local model is a *k-prong singularity* (see figure 1.4a) and with $k \geq 3$. Since the foliations must be transverse between them, then $\mathcal{F}^s$ and $\mathcal{F}^u$ share the same (finite) set of singularities, and in a small neighbourhood of each singularity the two foliations intersect as in the local model 1.4b.



(a) k-prong singularity, $k = 3$.

(b) Transverse foliations at k-prong, $k = 4$.

Figure 1.4: Local model of k-prong singularities

By the Euler-Poincaré formula (see [25]), the sphere is excluded from having a pseudo-Anosov homeomorphism and in the torus the foliations are non-singular. If $\text{gen}(\Sigma) \geq 2$, observe that the set of singularities $\mathcal{S}_f$ is necessarily included in the set of periodic points of f.

The definition of pseudo-Anosov homeomorphism can be extended to the case of non-closed surfaces. We will not talk about this concept in its wide generality, but we will just stay in the case that we will need later, which is that of orientable compact surfaces. We postpone this definition until the subsection 7.4.

7.1 Dynamical properties.

For higher genus surfaces, pseudo-Anosov homeomorphisms can be seen as a counterpart of linear hyperbolic automorphisms of the torus. In view of the singularities, they can never be *hyperbolic diffeomorphisms*, but they share some properties with linear Anosov maps. In particular, for every pseudo-Anosov homeomorphism $f : \Sigma \to \Sigma$ we have that:

- f is transitive,
- f is expansive,
- the set $\mathrm{Per}(f)$ of the periodic points is dense in Σ,
- the topological entropy of f is positive.

More generally, the dynamic of a pseudo-Anosov map can be encoded using a Markovian partition constructed from its invariant foliations, as is shown in [25]. From the symbolic point of view these maps are equivalent to subshifts of finite type.

In [42] and [38] it is shown that the pseudo-Anosov maps are the only expansive homeomorphisms in a closed orientable surface. More precisely, if $f : \Sigma \to \Sigma$ is an expansive homeomorphism then $\mathrm{gen}(\Sigma) \geq 1$ and if $\mathrm{gen}(\Sigma) = 1$ then f is C^0-conjugated to a linear Anosov map, and in higher genus f is C^0-conjugated to some pseudo-Anosov homeomorphism.

There exist smooth models for general pseudo-Anosov maps (see below). But in contrast to Anosov diffeomorphisms, these smooth models are not C^r-structurally stable. Nevertheless, the chaotic behaviour of a pseudo-Anosov homeomorphism persists all along the isotopy class of this map as a homeomorphism of Σ, see [36].

7.2 Differentiability.

If $f : \Sigma \to \Sigma$ is a pseudo-Anosov homeomorphism, the system of transverse foliations equipped with transverse measures defines a *translation atlas* in the complement of the singularities, and in a local coordinate system of this atlas the action of f corresponds to an affine transformation with derivative

$$Df = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

With this translation structure the map $f : \Sigma \backslash \mathcal{S}_f \to \Sigma \backslash \mathcal{S}_f$ is a smooth diffeomorphism. But observe that this smooth structure can not be extended to the singularities in such a way that $f : \Sigma \to \Sigma$ is a smooth diffeomorphism.

Nevertheless, there exists a smooth structure on Σ for which f is a smooth diffeomorphism. This is proved in [32], by showing that any given pseudo-Anosov f is conjugated to some smooth diffeomorphism $g : \Sigma \to \Sigma$. (See also [43] for an analytic version.)

One interesting remark in connection to our work is that this smooth structure is nowhere compatible with the atlas defined by the system of invariant foliations. In order to obtain a smooth model for a pseudo-Anosov, the structure defined by the translation atlas must be globally changed, as is explained in the referred work.

7.3 Conjugacy classes of pseudo-Anosov homeomorphisms.

Let Σ be a closed orientable surface of genus greater than zero. A remarkable property about pseudo-Anosov homeomorphisms is that if two of these maps are isotopic, then they are conjugated by a homeomorphism isotopic to the identity.

Theorem 1.26 ([25], Exposé XII, theorem 12.5)**.** *Let Σ be a closed orientable surface and let f and g be two pseudo-Anosov homeomorphisms. If g is isotopic to f then there exists a homeomorphism $h : \Sigma \to \Sigma$, isotopic to the identity, such that $f \circ h = h \circ g$.*

This theorem, in combination with the *Dehn-Nielsen-Baer theorem* about mapping class groups, allows to decide if two pseudo-Anosov homeomorphisms are conjugated by looking at their actions on fundamental groups.

The action on the fundamental group.

Let x_0 be a point in Σ. Every homeomorphism $f \in \mathrm{Homeo}(\Sigma)$ induces an automorphism of $\pi_1(\Sigma, x_0)$ in the following way: Let $\beta : [0,1] \to \Sigma$ be an arc that connects $x_0 = \beta(0)$ with $f(x_0) = \beta(1)$. Given a class $[\gamma] \in \pi_1(\Sigma, x_0)$ represented by a curve $\gamma : [0,1] \to \Sigma$ we define

$$f_*^\beta : [\gamma] \mapsto \left[\bar{\beta} \cdot f(\gamma) \cdot \beta\right], \text{ where } \bar{\beta} \text{ is } \beta \text{ parametrized with inverse sense.}$$

The map f_*^β is a well-defined automorphism of $\pi_1(\Sigma, x_0)$ which depends on the particular election of the arc β. If we choose another arc β' connecting $x_0 = \beta'(0)$ with $f(x_0) = \beta'(1)$, then $f_*^{\beta'} = [\alpha]^{-1} \cdot f_*^\beta \cdot [\alpha]$ where $[\alpha] = [\bar{\beta} \cdot \beta'] \in \pi_1(\Sigma, x_0)$. Thus, changing the arc β has the effect of conjugate f_*^β by an inner automorphism of the fundamental group $\pi_1(\Sigma, x_0)$.

Definition 1.27. Given a pair of homeomorphisms $f_i : \Sigma_i \to \Sigma_i$, $i = 1, 2$, where Σ_1 and Σ_2 are two homeomorphic closed orientable surfaces, we say that f_1 and f_2 are π_1-*conjugated* if there exist points $x_i \in \Sigma_i$, induced actions $(f_i)_*^{\beta_i} : \pi_1(\Sigma_i, x_i) \to \pi_1(\Sigma_i, x_i)$ and an isomorphism $\phi : \pi_1(\Sigma_1, x_1) \to \pi_1(\Sigma_2, x_2)$ such that $(f_2)_*^{\beta_2} \circ \phi = \phi \circ (f_1)_*^{\beta_1}$.

Observe that if f_1 and f_2 are π_1-conjugated then, for every pair of points x_i, every pair of induced actions $(f_i)_*^{\beta_i}$ on $\pi_1(\Sigma_i, x_i)$ are conjugated by an isomorphism $\phi : \pi_1(\Sigma_1, x_1) \to \pi_1(\Sigma_2, x_2)$.

Proposition 1.28. *For $i = 1, 2$ consider a pseudo-Anosov homeomorphism $f_i : \Sigma_i \to \Sigma_i$ defined in a closed orientable surface Σ_i. If f_1 and f_2 are π_1-conjugated, then there exists a homeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $f_2 \circ h = h \circ f_1$.*

The *outer automorphisms* of $\pi_1(\Sigma, x_0)$ are the elements of the quotient space of $\mathrm{Aut}(\pi_1(\Sigma, x_0))$ by the normal subgroup $\mathrm{Inn}(\pi_1(\Sigma, x_0))$ of inner automorphisms. The map

$$\mathrm{Homeo}(\Sigma) \to \mathrm{Out}\left(\pi_1(\Sigma, x_0)\right) \coloneqq \mathrm{Aut}(\pi_1(\Sigma, x_0))/\mathrm{Inn}(\pi_1(\Sigma, x_0)),$$

which sends each f to the conjugacy class of f_*^β by inner automorphisms, is a morphism. The *theorem of Dehn-Nielsen-Baer* says that this is a surjective morphism whose kernel is the group $\mathrm{Homeo}_0(\Sigma)$ of homeomorphisms isotopic to the identity. See [23], chap. 8 for an explanation. As a consequence, the *mapping class group* of Σ is isomorphic to the group of outer automorphisms of $\pi_1(\Sigma, x_0)$. That is,

$$\mathrm{MCG}(\Sigma) \coloneqq \mathrm{Homeo}(\Sigma)/\mathrm{Homeo}_0(\Sigma) \cong \mathrm{Out}(\pi_1(\Sigma, x_0)).$$

Proof of proposition 1.28. Assume that $\Sigma_1 = \Sigma_2 = \Sigma$ and $x_1 = x_2 = x_0$. By Dehn-Nielsen-Baer theorem, there exists a homeomorphism $h_\phi : \Sigma \to \Sigma$ which induce in $\pi_1(\Sigma, x_0)$ the same outer automorphism as ϕ does. Consider $g = h_\phi^{-1} \circ f_2 \circ h_\phi$. Then, g is a pseudo-Anosov homeomorphism and its action on the fundamental group is conjugated to the action of f_1 by an inner automorphism. It follows that f_1 and g are two isotopic pseudo-Anosov maps, and by theorem 1.26 they are conjugated by some homeomorphism $h_\circ$ isotopic to the identity. Then, the map $h = h_\phi \circ h_\circ$ realizes a conjugation between f_1 and f_2. □

We will use proposition 1.28 in the course of the proofs of theorems 4.1 and 4.1. By the way, since we will be working with non-closed surfaces, we make some remarks in what follows.

Punctured surfaces.

Consider two pseudo-Anosov homeomorphisms $f_i : \Sigma_i \to \Sigma_i$, where $i = 1, 2$, defined in two closed, orientable, homeomorphic surfaces. For each f_i consider a finite collection of periodic orbits $\mathcal{O}_1^i, \ldots, \mathcal{O}_N^i$. That is, for each $k = 1, \ldots, N$,

$$\mathcal{O}_k^i = \{x_{k1}^i, \ldots, x_{kp_k}^i\}$$

where $x_{kn}^i = f_i^{n-1}(x_{k1}^i)$ and $p_k \geq 1$ is the period of the orbit. We are interested in knowing when f_1 is conjugated to f_2 by a homeomorphism $h : \Sigma_1 \to \Sigma_2$, with the additional property that h sends each orbit $\mathcal{O}_k^1$ to the orbit $\mathcal{O}_k^2$. We explain this in the following paragraphs.

A *finite type punctured surface* is the data of a compact surface Σ together with a finite subset $\mathcal{O} \subset \Sigma$. In this text we just consider the case where the surface is closed. The *mapping class group of the punctured surface* $(\Sigma; \mathcal{O})$ is defined to be the set homeomorphisms $f : \Sigma \to \Sigma$ preserving the set $\mathcal{O}$, modulo isotopies fixing $\mathcal{O}$.

Denote by $\pi_1(\Sigma, x_0; \mathcal{O})$ to the fundamental group of $\Sigma \backslash \mathcal{O}$ based in a point x_0 not in $\mathcal{O}$. If f is a homeomorphism of Σ preserving $\mathcal{O}$, then it induces a permutation of this finite set as well as an action on $\pi_1(\Sigma, x_0; \mathcal{O})$, uniquely defined up to conjugation by inner automorphisms. As before, the mapping class group injects into $\mathrm{Out}(\pi_1(\Sigma, x_0; \mathcal{O}))$. But there is an essential difference with the case of closed surfaces. The fundamental group of $\Sigma \backslash \mathcal{O}$ is a *free group* and in most cases we can exhibit outer automorphisms which are not induced from any homeomorphism.

Denote for the moment $\mathcal{O} = \{x_1, \ldots, x_R\}$. For each of the points x_l consider a closed curve homotopic to the puncture x_l and joined to x_0 with an arbitrary path. This curve determines an element of $\pi_1(\Sigma, x_0; \mathcal{O})$ that we denote by c_l. We denote by $\Gamma(\mathcal{O})$ to the set of conjugacy classes of the elements c_l. That is,

$$\Gamma(\mathcal{O}) = \{\bar{\gamma} \cdot c_l \cdot \gamma : \text{ where } l = 1, \ldots, R \text{ and } \gamma \in \pi_1(\Sigma, x_0; \mathcal{O})\}.$$

The action f_* leaves invariant the set $\Gamma(\mathcal{O})$, since it comes from a homeomorphism of the surface. Define $\mathrm{Out}^*(\pi_1(\Sigma, x_0; \mathcal{O}))$ to be the set of outer automorphisms that preserves $\Gamma(\mathcal{O})$. Then, the *theorem of Dehn-Nielsen-Baer* states that the morphism

$$\mathrm{MCG}(\Sigma; \mathcal{O}) \to \mathrm{Out}^*(\pi_1(\Sigma, x_0; \mathcal{O}))$$

is an isomorphism.

This statement can be sharpened in the following way: Consider N finite sets $\mathcal{O}_1, \ldots, \mathcal{O}_N \subset \Sigma$. Define $\mathrm{MCG}(\Sigma; \mathcal{O}_1, \ldots, \mathcal{O}_N)$ to be the set of homeomorphisms $f : \Sigma \to \Sigma$ such that $f(\mathcal{O}_k) = \mathcal{O}_k$, $\forall\ k = 1, \ldots, N$, modulo isotopies fixing $\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_N$. Define $\mathrm{Out}^*(\pi_1(\Sigma, x_0; \mathcal{O}_1, \ldots, \mathcal{O}_N))$ to be the subgroup of outer automorphisms of the fundamental group of $\Sigma \backslash \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_N$ that preserve each set $\Gamma(\mathcal{O}_k)$. Then, we have an isomorphism

$$\mathrm{MCG}(\Sigma; \mathcal{O}_1, \ldots, \mathcal{O}_N) \to \mathrm{Out}^*(\pi_1(\Sigma, x_0; \mathcal{O}_1, \ldots, \mathcal{O}_N).$$

We come back now to the case of pseudo-Anosov maps stated in the first paragraph. For simplicity we will omit the reference to the base point in the following statement.

Proposition 1.29. *For $i = 1, 2$ consider a pseudo-Anosov homeomorphism $f_i : \Sigma_i \to \Sigma_i$ defined in a closed orientable surface Σ_i and a finite collection of periodic orbits $\mathcal{O}_1^i, \ldots, \mathcal{O}_N^i$ of periods $p_1, \ldots, p_N$, respectively. Assume there exists an isomorphism*

$$\phi : \pi_1(\Sigma_1; \mathcal{O}_1^1 \cup \cdots \cup \mathcal{O}_N^1) \to \pi_1(\Sigma_2; \mathcal{O}_1^2 \cup \cdots \cup \mathcal{O}_N^2)$$

such that

- *It conjugates the actions $(f_i)_*$ induced in fundamental groups of $\Sigma_i \backslash \mathcal{O}_1^i \cup \cdots \cup \mathcal{O}_N^i$,*
- *$\phi(\Gamma(\mathcal{O}_k^1)) = \Gamma(\mathcal{O}_k^2)$ for every $k = 1, \ldots, N$.*

Then, there exists a homeomorphism $h : \Sigma_1 \to \Sigma_2$ such that $f_2 \circ h = h \circ f_1$ which in addition satisfies $h(\mathcal{O}_k^1) = \mathcal{O}_k^2$, $\forall\ k = 1, \ldots, N$.

Proof. Assume that $\Sigma_1 = \Sigma_2 = \Sigma$ and $\mathcal{O}_k^1 = \mathcal{O}_k^2 = \mathcal{O}_k$, $\forall\ k = 1, \ldots, N$. Following the Dehn-Nielsen-Baer theorem for punctured surfaces, there exists a homeomorphism

$$h_\phi : (\Sigma; \mathcal{O}_1, \ldots, \mathcal{O}_N) \to (\Sigma; \mathcal{O}_1, \ldots, \mathcal{O}_N)$$

which induce in $\pi_1(\Sigma; \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_N)$ the same outer automorphism as ϕ does. Consider $g = h_\phi^{-1} \circ f_2 \circ h_\phi$. We will show that there exists a homeomorphism $h : (\Sigma; \mathcal{O}_1, \ldots, \mathcal{O}_N) \to (\Sigma; \mathcal{O}_1, \ldots, \mathcal{O}_N)$ such that $h \circ g = f_1 \circ h$. Then, the homeomorphism $h \circ h_\phi$ gives the conjugation claimed in the proposition.

We start with the case of just one periodic orbit $\mathcal{O}$ which which consists in one fixed point x_1. In this case the action of g on $\pi_1(\Sigma; \{x_1\})$ is conjugated to the action of f_1 by an inner automorphism and it follows that there exists an isotopy

$$\varphi_t : [0, 1/2] \to \text{Homeo}\,(\Sigma)$$

such that $\varphi_0 = f_1$, $\varphi_{1/2} = g$ and $\varphi_t(x_1) = x_1$ for every t.

On the other hand, g is pseudo-Anosov and its action on $\pi_1(\Sigma)$ (forgetting the puncture) is conjugated to the action of f_1 by an inner automorphism, so there exists a homeomorphism $h : \Sigma \to \Sigma$, isotopic to the identity, such that $g \circ h = h \circ f_1$. Take an isotopy $h_t : \Sigma \to \Sigma$ such that $h_{1/2} = id_\Sigma$ and $h_1 = h$. This allows to construct another isotopy

$$\psi_t : [1/2, 1] \to \text{Homeo}\,(\Sigma)$$

by the expression $\psi_t = h_t^{-1} \circ g \circ h_t$ which verifies that $\psi_{1/2} = g$ and $\psi_1 = f_1$.

Consider now the path

$$\Pi_t : [0, 1] \to \text{Homeo}\,(\Sigma)$$

defined by concatenation of φ and ψ. This is a closed path verifying $\Pi_0 = f_1 = \Pi_1$. We will use know the following two facts:

Fact 1: The space $\text{Homeo}_0^+(\Sigma)$ is contractible.

Fact 2: Any lift of a pseudo-Anosov map to the universal cover can have at most one fixed point.

Let $\widetilde{\Sigma}$ be the universal cover of Σ and consider a lift $\widetilde{x}_1$ of x_1 and a lift $\widetilde{f}_1$ of f_1 which fixes $\widetilde{x}_1$. There is a unique lift $\widetilde{\varphi_t}$ of the path φ_t such that $\widetilde{\varphi_0} = \widetilde{f}_1$ and $\widetilde{\varphi_t}(\widetilde{x}_1) = \widetilde{x}_1$, $\forall\ 0 \leq t \leq 1/2$. This is possible since the path φ_t is an isotopy fixing the point x_1. The map $\widetilde{\varphi}_{1/2} = \widetilde{g}$ is a lift of g. Consider now a lift $\widetilde{h}_t$ of h_t starting at $\widetilde{h}_{1/2} = id_{\widetilde{\Sigma}}$. Then $\widetilde{\psi_t} = \widetilde{h}_t^{-1} \circ \widetilde{g} \circ \widetilde{h}_t$ is a lift of ψ_t starting at $\widetilde{g}$. The concatenation of $\widetilde{\varphi_t}$ and $\widetilde{\psi_t}$ is a lit of the closed path Π_t and, since the connected component of f_1 in Homeo(Σ) is contractible, necessarily we have that $\widetilde{\Pi}_0 = \widetilde{\Pi}_1 = \widetilde{f}_1$. Now, the curve $\widetilde{h}_t(\widetilde{x}_1)$ in the universal cover is a path joining $\widetilde{x}_1 = \widetilde{h}_{1/2}(\widetilde{x}_1)$ with another point $\widetilde{x}_1' = \widetilde{h}_1(\widetilde{x}_1)$. Since each $\widetilde{h}_t(\widetilde{x}_1)$ is a fixed point for $\widetilde{\psi_t}$ then $\widetilde{x}_1'$ is a fixed point for $\widetilde{f}_1$, so necessarily it must be satisfied that $\widetilde{x}_1 = \widetilde{h}_1(\widetilde{x}_1)$. We conclude that $h(x_1) = x_1$.

If we consider now a finite set $\mathcal{O}_1, \ldots, \mathcal{O}_N$ of periodic orbits each one consisting in a fixed point x_k, then the same argument can be carried out individually for each x_k. That is, for each $k = 1, \ldots, N$ we have to consider a lift $\widetilde{f}_1$ of f_1 that fixes some pre-image $\widetilde{x}_k$ and apply the same reasoning to conclude that h must fix each x_k.

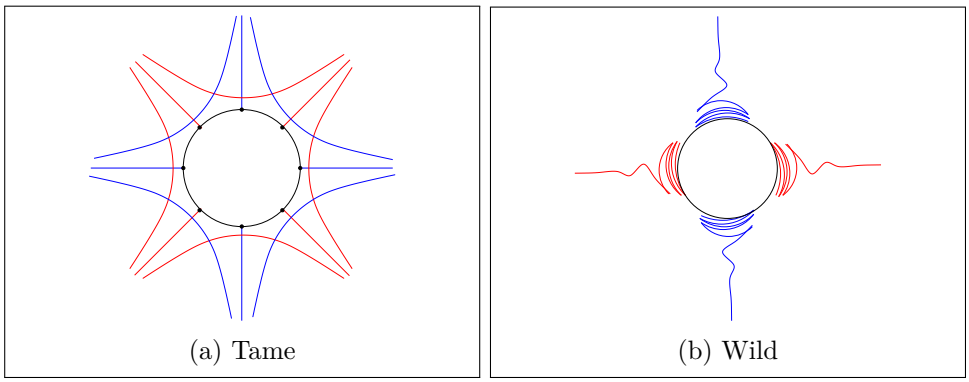
□

7.4 Non-closed surfaces.

We will use proposition 1.29 in the course of the proofs of theorems 4.1 and 4.1. By the way, since we will be working with non-closed surfaces, we make some remarks in what follows. Let Σ be a compact orientable surface.

Definition 1.30 (pseudo-Anosov in non-closed surfaces.)**.** Let $f : \Sigma \to \Sigma$ be a homeomorphism. Let $\widehat{\Sigma}$ the surface obtained by collapsing each boundary component into a point and let $\hat{f}$ be the corresponding induced map. We say that f is pseudo-Anosov if $\hat{f}$ is pseudo-Anosov according to definition 1.25.

Let C be a boundary component of Σ and $p \in \widehat{\Sigma}$ the point obtained after collapsing C. Each invariant foliation of $\hat{f}$ has a finite number of leaves which accumulate on p, which are usually called *branches*. When lifted to Σ, these branches do not necessarily converge to a point in C, as is depicted in figure 1.5b. We say that the foliations are *tame* if the local model in a neighbourhood of the boundary component C is as in figure 1.5a. In this case, each branch converge to a point in C, which is necessarily periodic for f.



(a) Tame (b) Wild

Figure 1.5: Local model of k-prong singularities

According to our definition, each foliation has at least two branches at each boundary component. In more general definitions 1-prong singularities are allowed. See [23], chap. 13.3.

Collapsing each boundary component into a point provides a semi-conjugation from f to $\hat{f}$, which is actually a conjugation on the interior of the surface. So, most of the dynamical properties of $\hat{f}$ are also available for f. However, we want to point out the following:

Remark 1.31 (Conjugacy classes of pseudo-Anosov in non-closed surfaces.). Theorem 1.26 as well as proposition 1.28 are no longer available in the case of non-closed surfaces.

In particular, we want to point out an obstruction to conjugacy which depends on

the behaviour of the map in a neighbourhood of the boundary. We illustrate this in the following example.

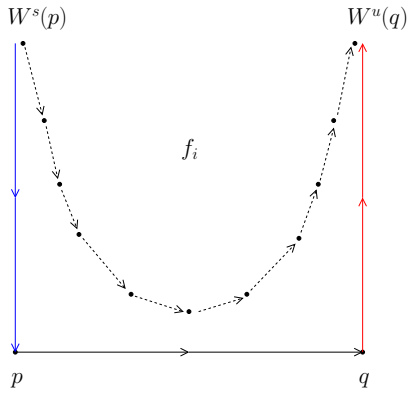


Figure 1.6: The homeomorphisms in example 1.32.

Example 1.32. Consider two C^1-diffeomorphisms f and g defined in the band $[-1,1]\times[0,+\infty)$, whose phase portrait is as follows:

- The non-wandering set consists in the corner points $p=(-1,0)$ and $q=(1,0)$, which are saddle type hyperbolic fixed points.
- $W^s(p)=\{-1\}\times[0,+\infty)$,
- $W^u(q)=\{1\}\times[0,+\infty)$,
- The segment $[-1,1]\times\{0\}$ is a saddle connection between p and q.

This is illustrated in figure 1.6. For each fixed point $x=p,q$ we have that

$$Df(x)=\begin{pmatrix}\lambda_x(f) & 0\\ 0 & \mu_x(f)\end{pmatrix} \text{ and } Dg(x)=\begin{pmatrix}\lambda_x(g) & 0\\ 0 & \mu_x(g)\end{pmatrix}.$$

Proposition 1.33. *If there exists a homeomorphism $h:[-1,1]\times[0,+\infty)\to[-1,1]\times[0,+\infty)$ such that $g\circ h=h\circ f$, then*

$$\frac{\log(\mu_q(f))}{\log(\mu_p(f))}=\frac{\log(\mu_q(g))}{\log(\mu_p(g))}. \tag{1.2}$$

In particular, general homeomorphisms (even C^1-diffeomorphisms) whose phase portrait is as in example 1.32 are not C^0-conjugated. Observe that there always exists a conjugation between these dynamics in the complement of the segment $[-1,1]\times\{0\}$.

This can be seen by dividing the band into adequate fundamental domains for the action of each map. The obstruction appears when we try to extend the conjugation to the segment that connects the two saddles.

This dynamical behaviour is what we encounter in the neighbourhood of a boundary component of a surface Σ, when we look at the action of a pseudo-Anosov map. If we choose a power of f that fixes the boundary component, then we can decompose a neighbourhood of this component into a finite number of bands homeomorphic to $[-1,1]\times[0,+\infty)$ where the dynamic looks like in the example 1.32.

As a final remark, observe that if $\hat{f}:\widehat{\Sigma}\to\widehat{\Sigma}$ is a pseudo-Anosov in a closed surface, we can construct a pseudo-Anosov $f:\Sigma\to\Sigma$ in a non closed surface by blowing up $\hat{f}$ along a periodic orbit. But, in view of 1.32, different ways of blowing up could lead to non-conjugated maps, even if all the actions on $\pi_1(\Sigma)$ are the same.

8 Topological saddle type periodic orbits.

A periodic orbit of a C^1 flow is said to be *saddle type hyperbolic* if the derivative of the first return map to a local transverse section on the fixed point is a saddle type hyperbolic matrix. That is, it has eigenvalues $0<|\lambda|<1<|\mu|$.

Let $\phi_t : M\to M$ be a non-singular regular flow on a 3-manifold.

Definition 1.34. A periodic orbit γ of (ϕ_t, M) is said to be *topologically saddle type* if its germ $(\phi_t, M)_\gamma$ is orbitally equivalent to the germ of a saddle type hyperbolic periodic orbit of some flow of class C^1.

In the case that the flow ϕ_t is generated by a vector field X of class C^1 and γ is actually a saddle type hyperbolic periodic orbit, there is a natural of decomposition of $TM|_\gamma$ into three line bundles $E^s\oplus E^c\oplus E^u$ that is invariant by $D\phi_t$. Here, E^c is collinear with the generating vector field, E^s is uniformly contractive and E^u is uniformly expansive.

The classical stable manifold theorem allows to see that, given some adequate $\varepsilon>0$, the sets

$$W^s_{loc}(\gamma)=\{y\in M \ : \ dist\,(\phi_t(y),\gamma)\le\varepsilon,\ \forall\ t\ge 0\},$$
$$W^u_{loc}(\gamma)=\{y\in M \ : \ dist\,(\phi_t(y),\gamma)\le\varepsilon,\ \forall\ t\le 0\},$$

are two C^1-embedded surface, tangent to $E^s\oplus E^c$ and $E^c\oplus E^u$ respectively, that intersect transversally along the periodic orbit.

These sets are called the *local invariant manifolds* of γ and, from the definition 1.34 above, we see that these are also available for topologically saddle type periodic trajectories.

Remark 1.35. We remark that the difference between topologically saddle type and hyperbolic saddle type is not a matter of differentiability. One could have a a smooth

flow with a topologically saddle type periodic orbit, whose derivative on a transverse section is, for example, the identity matrix.

From now on, we will assume that the eigenvalues of the derivative of the first return are positive. In this case, the bundles E^s and E^u are orientable along γ and the local invariant manifolds are embedded cylinders.

8.1 Normal Coordinates.

Consider a vector field $X_\circ$ in $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ defined by

$$X_\circ(x, y, s) = (ax, by, c),$$

where $a < 0 < b, c$ are constants. The *non-wandering* set of the associated flow $\phi_t^{X_\circ}$ consists of one saddle type hyperbolic periodic orbit, namely, the curve $0 \times \mathbb{R}/\mathbb{Z}$.

Definition 1.36 (**Normal coordinates.**). A *system of normal coordinates* for the germ $(\phi_t, M)_\gamma$ of a topological saddle type periodic orbit is a continuous local orbital equivalence

$$\Psi : (\phi_t, W) \to (\phi_t^{X_\circ}, V),$$

defined between a neighbourhood W of γ and a neighbourhood V of $0 \times \mathbb{R}/\mathbb{Z}$, where $X_\circ$ is described above.

Remark 1.37. By definition, a topological saddle type periodic orbit is one that comes equipped with a system of continuous normal coordinates in a neighbourhood. By the way, we remark that it rarely exists a system of C^k-normal coordinates for $k \geq 1$, even assuming a high regularity of the flow (ϕ_t, M). One direct obstruction to have C^k coordinates is the non-hyperbolicity of the action of $D\phi_t$ on $TM|_\gamma$. But this condition is far from being enough to guarantee C^k-orbital equivalence with a hyperbolic saddle. See [3].

In general, a *normal form* for the germ of a periodic orbit is a system of normal coordinates of class C^k as above, $k \geq 0$, which in addition satisfies that the orbital equivalence in 1.36 is a conjugation. In subsequent chapters the definition provided here will be enough for our purposes. We will consider a kind of normal form just on the course of the proof of theorem 4.12.

8.2 Homology of a punctured tubular neighbourhood.

When a periodic orbit γ is topologically saddle type, then its local invariant manifolds (assumed here to be cylinders) allow to define a canonical basis of the first homology group $H_1(W \backslash \gamma)$, where W is any tubular neighbourhood containing γ on the interior. Let's explain how it works.

In the following construction the curve γ will always be regarded as an oriented curve, its orientation being the one induced by the flow direction. Consider a tubular neighbourhood W of γ.

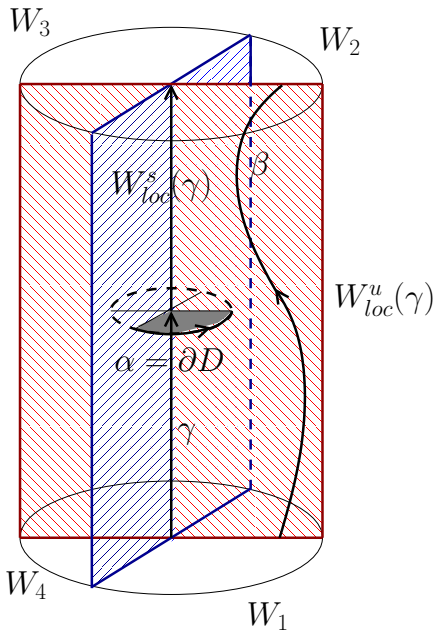


Figure 1.7: Neighbourhood of a topologically saddle type periodic orbit.

- **Orientation:** Since we are requiring the two invariant manifolds $W^s_{loc}(\gamma)$ and $W^u_{loc}(\gamma)$ to be orientable, it follows that the tubular neighbourhood is orientable. Thus, from now on we will fix an orientation of W. This induces an orientation in the boundary ∂W, which is a 2-dimensional torus.

- **Homology:** Since W retracts by deformation over γ, the homology group $H_1(W)$ is isomorphic to $\mathbb{Z}$ and the homology class of γ is a generator. Observe that ∂W is a deformation retract of $W \backslash \gamma$. Therefore, the spaces $H_1(\partial W)$ and $H_1(W \backslash \gamma)$ are canonically isomorphic and isomorphic to $\mathbb{Z}^2$.

- **Longitude:** As explained before, $W^s_{loc}(\gamma)$ is obtained as an embedding $(-\varepsilon, \varepsilon) \times \mathbb{R}/\mathbb{Z} \to M$ that takes the curve $\{0\} \times \mathbb{R}/\mathbb{Z}$ homeomorphically onto γ and, without loss of generality, we can assume it preserves the orientation of these curves. If we choose a sufficiently small real $t_0 \neq 0$ the embedding will take the curve $\{t_0\} \times \mathbb{R}/\mathbb{Z}^1$ into an oriented curve $\beta \subset W \backslash \gamma$. The homology class

$$b = [\beta] \in H_1(W \backslash \gamma)$$

  will be called the *longitude*, and it is independent of the particular choice of t_0.

- **Meridian:** Consider now an embedded closed disk $D \subset W$ which is transverse to γ and intersects it in one interior point. Then, the orientation of γ together with the orientation of W defines an orientation on D and a boundary orientation on $\alpha = \partial D$. The homology class

$$a = [\alpha] \in H_1(W \backslash \gamma)$$

with the given orientation will be called the *meridian*.

Definition 1.38 (**Meridian-longitude basis**)**.** The basis $\{a, b\}$ constructed above is called a *canonical meridian-longitude basis* of the homology of a punctured neighbourhood of γ.

Let's recall that the *algebraic intersection number* between oriented curves in ∂W defines an anti-symmetric bi-linear form in $H_1(\partial W) \simeq H_1(W \backslash \gamma)$, and for two homology classes x and y it is satisfied that $x \cdot y > 0$ if and only if $\{x, y\}$ is a positive basis. If we push the curves α and β by homotopy onto ∂W we can see that the algebraic intersection number $a \cdot b$ equals one, so in particular $\{a, b\}$ is a positive basis for $H_1(W \backslash \gamma)$. Observe also that the map $\iota_* : H_1(W \backslash \gamma) \to H_1(W)$ induced by the inclusion $\iota : W \backslash \gamma \hookrightarrow W$ sends $b \mapsto [\gamma]$ and $a \mapsto 0$, so in particular the meridian can be characterized as the generator a of the kernel of ι_* which satisfies that $\{a, b\}$ is a positive basis for $H_1(W \backslash \gamma)$. This means in particular that a is independent of the particular choice of the disk D.

Finally, if we are given another tubular neighbourhood W' of γ, then it is possible to construct a continuous family of embeddings $\psi_s : W \to M$ such that

- $\psi_0 = id$ and $\psi_1(W) = W'$;
- $\psi_s(W^s_{loc}(\gamma)) \subset W^s_{loc}(\gamma)$, $\psi_s(W^u_{loc}(\gamma)) \subset W^u_{loc}(\gamma)$ for every s.

This implies that all the homology groups $H_1(W \backslash \gamma)$ are canonically isomorphic for different choices of W, and this isomorphism preserves the basis given by the longitude and the meridian of each tubular neighbourhood.

Chapter 2

Orbital equivalence and Birkhoff sections.

Consider two non-singular regular flows $\phi_t^i : M_i \to M_i$, $i = 1, 2$, each one defined on a closed 3-manifold M_i.

For each flow let $\iota_i : (\Sigma_i, \partial\Sigma_i) \hookrightarrow (M_i, \Gamma_i)$ be a Birkhoff section and let $P_i : \mathring{\Sigma}_i \to \mathring{\Sigma}_i$ be the first return map. Assume that there exists a homeomorphism $h : \mathring{\Sigma}_1 \to \mathring{\Sigma}_2$ that conjugates the first return maps. The embedding of the surface $\mathring{\Sigma}_i$ inside M_i is a global transverse section for the restricted flow $\phi_t^i : M_i \backslash \Gamma_i \to M_i \backslash \Gamma_i$. So, the existence of h implies that there is an orbital equivalence

$$H : (\phi_t^1, M_1 \backslash \Gamma_1) \to (\phi_t^2, M_2 \backslash \Gamma_2) \tag{2.1}$$

in the complement of the boundary curves. That is, the flows (ϕ_t^i, M_i) are almost orbital equivalent, according to definition 1.4 in the previous chapter.

One question is weather or not this condition implies that (ϕ_t^1, M_1) and (ϕ_t^2, M_2) are actually orbitally equivalent, and at which extent we can expect an orbital equivalence that extends (2.1) above.

Obstructions to the orbital equivalence.

The first obstruction is topological in nature. For example, consider a closed Seifert 3-manifold and a flow where the partition into orbits coincides with the fibration by circles. Following the examples that we showed in 5, this flow can be endowed with Birkhoff sections whose first return is the identity, independently of the Euler class. Thus, an almost orbital equivalence like (2.1) above need not to extend to the whole manifold.

There are other obstructions that depend on the flow dynamic and the way the Birkhoff section embeds in a neighbourhood of each boundary curve. For example, assume that the two flows are C^1 and, for simplicity, assume that $\partial\Sigma_i$ has only one

boundary component that embeds onto a hyperbolic saddle type periodic orbit $\gamma_i \subset M_i$. The restriction of each embedding $\Sigma_i \hookrightarrow M_i$ onto a collar neighbourhood of the boundary provides a local Birkhoff section, that we call B_i. It is not hard to see that the (local) first return map into $\mathring{B}_i = B_i \backslash \gamma_i$ looks like the germ that we have described in 7.4, that appears as well for the pseudo-Anosov maps on surfaces with boundary and we see in the figure below.

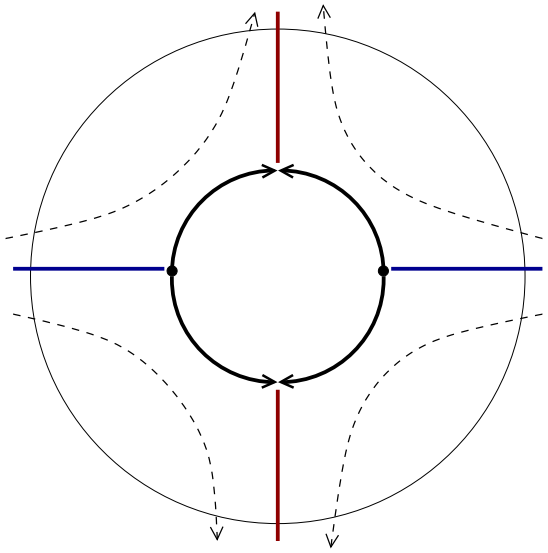


Figure 2.1: Two dynamics with this phase diagram need not to be conjugated. In general they are conjugated on the interior, but there are obstructions for extending to the boundary.

As is explained in [17], we can assume that the Birkhoff sections are well-positioned near the boundary, in such a way that the first return map extends onto the boundary as a Morse-Smale like homeomorphism. But, by the remarks made in 7.4, we can see that in general, even if the first return maps $(P_{B_i}, \mathring{B}_i)$ are conjugated on $\mathring{B}_i$, the conjugation rarely extends to the boundary.

We can deduce that, in general, an orbital equivalence on $M_1 \backslash \Gamma_1 \to M_2 \backslash \Gamma_2$ does not extend continuously over $\Gamma_1 \to \Gamma_2$.

Sufficient conditions for equivalence.

This chapter is devoted to prove theorem 2.16, which gives a positive answer for this technical problem when the boundary curves of the Birkhoff sections are saddle type hyperbolic periodic trajectories, and the local combinatorial data of the two Birkhoff sections around its boundary components (linking number, multiplicity and number of boundary components) is compatible. In this setting we will prove not just that the two flows are orbitally equivalent, but that there exists an orbital equivalence that coincides with the previous H outside a neighbourhood of the boundary of the Birkhoff sections.

The new orbital equivalence can be described as a perturbation of the given H along the flow lines, supported in a neighbourhood of the periodic orbits in the boundary. But the approach will be indirect. In particular, we will make use of a detailed analysis about the first return map onto a local Birkhoff section, for the case of saddle type periodic orbits, that we found in [13]. Most of the statement in the referred work are made for the special case of embedded Birkhoff sections. Since we will need them in the general case, in section 1 we resume the results that we will need later, and we outline some of the proofs as well.

In section 2 we give the proof of 2.16.

1 Local Birkhoff sections at saddle type periodic orbits.

We will be interested in a combinatorial description of the first return map onto a local Birkhoff section in a topological saddle typer periodic orbit, that was developed in [13], and that we explain in the following subsections.

Let $\phi_t : M \to M$ be a regular flow defined on a 3-manifold, which has a topologically saddle type periodic orbit γ. We will assume that its local invariant manifolds $W^s_{loc}(\gamma)$ and $W^u_{loc}(\gamma)$ are orientable. Consider a local Birkhoff section $\iota : ([0,1)\times\mathbb{R}/\mathbb{Z}, 0\times\mathbb{R}/\mathbb{Z}) \to (M,\gamma)$ at γ. We will denote:

$$B = \iota\left([0,1)\times\mathbb{R}/\mathbb{Z}\right), \quad \partial B = \iota\left(0\times\mathbb{R}/\mathbb{Z}\right), \quad \mathring{B} = B\backslash\partial B. \tag{2.2}$$

We start assuming that the immersion of the Birkhoff section is well-positioned with respect to the invariant manifolds, according to the following definition:

Definition 2.1 (**Tameness**)**.** The local Birkhoff section B is *tame* if there exists a collar neighbourhood $U \subset B$ of γ such that the sets $U\cap W^s_{loc}(\gamma)$ and $U\cap W^u_{loc}(\gamma)$ consists of the union of γ with finitely many compact segments, each of them intersecting γ exactly at one of its extremities.

Given a (global) Birkhoff section Σ such that every boundary component γ of $\partial\Sigma$ is a saddle type hyperbolic periodic orbit, we will simply say that Σ is *tame* if it is tame at each boundary component.

An arbitrary Birkhoff section is not necessarily tame, but it can be changed for a tame one in a proper way that suffices to apply the techniques exposed here. We will refer these results a the end of the section.

1.1 Linking number and multiplicity.

By definition, the restriction of the immersion ι to $(0,1)\times\mathbb{R}/\mathbb{Z}$ is an embedding into $M\backslash\gamma$, and is a covering map on the boundary. We will fix the standard orientation on

$\mathbb{R}/\mathbb{Z}$ and we will assume that $\{0\} \times \mathbb{R}/\mathbb{Z} \to \gamma$ is an orientation preserving map.

Consider a small tubular neighbourhood W of γ such that the inclusion $\mathring{B} \hookrightarrow W \backslash \gamma$ is a proper embedding. Let $t_0 > 0$ be some small real such that the oriented curve $\sigma = \iota(\{t_0\} \times \mathbb{R}/\mathbb{Z})$ is contained in $W \backslash \gamma$. Observe that, since B is topologically an annulus, the homology class $[\sigma] \in H_1(W \backslash \gamma)$ of this curve does not depend on the particular choice of σ and it is a generator of $H_1(B)$. Then, the coordinates of $[\sigma]$ in the meridian-longitude basis $\{a, b\}$ of $H_1(W \backslash \gamma)$ are two integers $n = n(\gamma, B)$ and $m = m(\gamma, B)$ satisfying that

$$[\sigma] = n(\gamma, B) \cdot a + m(\gamma, B) \cdot b.$$

Definition 2.2. The integers $n(\gamma, B)$ and $m(\gamma, B)$ are called the *linking number*[1] and the *multiplicity* of B at γ, respectively.

Observe that this value of the multiplicity coincides with the one defined in the previous chapter. We state some basic facts.

Proposition 2.3. *It is satisfied that:*

- $m(\gamma, B) \geq 1$ *and* $m(\gamma, B) = 1$ *if and only if* B *is embedded;*
- $n(\gamma, B) \neq 0$*;*
- $\gcd(n(\gamma, B), m(\gamma, B)) = 1$.

Remark 2.4. The construction of the linking number depends on a chosen orientation of W. *Changing the orientation of* W *has the effect of changing the sign of the linking number.*

Before proving the proposition, we establish a lemma.

Lemma 2.5. *Consider the relative homology group* $H_2(W, \partial W \cup \gamma)$*. Then, there exists an isomorphism*

$$q : H_2(W, \partial W \cup \gamma) \to H_1(\partial W), \ q : [\Sigma] \mapsto [\Sigma \cap \partial W]$$

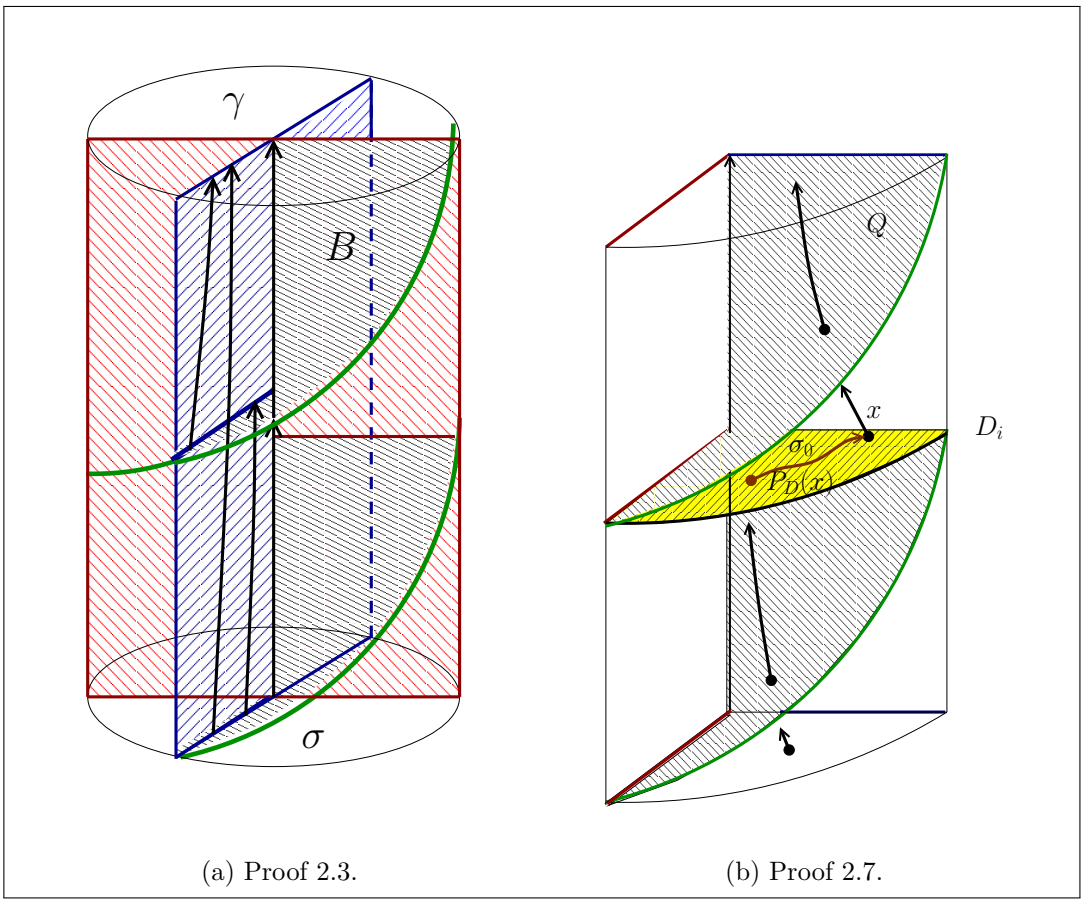
which sends the class $[\Sigma]$ *of a proper surface* $(\Sigma, \partial\Sigma) \hookrightarrow (W, \partial W \cup \gamma)$ *into the homology class of* $\Sigma \cap \partial W$ *oriented as the boundary of* Σ*. Moreover, this map satisfies that the algebraic intersection number* $[\Sigma] \cdot [\eta]$ *of a class* $[\Sigma] \in H_2(W, \partial W \cup \gamma)$ *with a class* $[\eta] \in H_1(W \backslash \gamma)$ *equals the algebraic intersection number* $q([\Sigma]) \cdot [\eta]$ *in* $H_1(\partial W)$.

Proof. An application of the *long exact sequence of relative homology groups* for the space W and the subspace $\partial W \cup \gamma$ provides a sequence

$$\cdots \longrightarrow H_2(W) \xrightarrow{p_*} H_2(W, \partial W \cup \gamma) \xrightarrow{\partial_*} H_1(\partial W \cup \gamma) \xrightarrow{\iota_*} H_1(W) \longrightarrow \cdots$$

[1] In other literature this number is called the *self-linking number*.

The space $H_1(\partial W \cup \gamma)$ is the free abelian group generated by a, b and $[\gamma]$, and the map ι_* satisfies that $\iota_*(a) = 0$ and $\iota_*(b) = \iota_*([\gamma]) = [\gamma]$. This means that $\mathrm{Im}(\partial_*) = \ker(\iota) = \{n \cdot a + m \cdot b - m \cdot [\gamma] : n, m \in \mathbb{Z}\}$. Let $\varphi : H_1(\partial W \cup \gamma) \to H_1(\partial W)$ be defined by $\varphi : (n \cdot a + m \cdot b + c \cdot [\gamma]) \mapsto n \cdot a + m \cdot b$. Since $H_2(W) = \{0\}$ the homomorphism ∂_* is injective, and it follows that the map q defined as $q = \varphi \circ \partial_*$ is an isomorphism. The claim about the intersection number follows by evaluating the intersection number of an arbitrary class $[\eta]$ with a meridian disk and with a local stable manifold. □



(a) Proof 2.3. (b) Proof 2.7.

Figure 2.2: Local Birkhoff Sections.

Proof of proposition 2.3. For the multiplicity observe that the map $\iota_* : H_1(W \backslash \gamma) \to H_1(W)$ induced by the inclusion of the corresponding spaces sends $b \mapsto [\gamma]$ and $a \mapsto 0$. So this map sends $[\sigma] = n \cdot a + m \cdot b \mapsto m \cdot [\gamma]$. Since the Birkhoff section is a covering map which preserves orientation when restricted to $\{0\} \times \mathbb{R}/\mathbb{Z}$ and $\{t_0\} \times \mathbb{R}/\mathbb{Z}$ is homotopic to $\{0\} \times \mathbb{R}/\mathbb{Z}$, it follows that there exists an orientation preserving homotopy between the curve σ and a positive multiple of γ inside W. This implies that $m \geq 1$, and that $m = 1$ if and only if B is embedded.

Now, for the linking number, we will show that item 3. in the definition 1.17 of Birkhoff section, implies that $n \neq 0$.

The intersection form in $H_1(W\backslash\gamma)$ is given in the basis $\{a, b\}$ by the relations

$$a \cdot a = 0, \quad b \cdot b = 0, \quad a \cdot b = 1.$$

In terms of the intersection form on $H_1(W\backslash\gamma)$ the linking number of B satisfies that $n = [\sigma] \cdot b$. Observe that since the flow is transverse to B and we have chosen an orientation in W, then the flow induces an orientation on B. Consider the relative homology class $[B] \in H_2(W, \partial W \cup \gamma)$ of the local Birkhoff section B oriented in this way. Since $B \cap \partial W$ is homologous to $\pm[\sigma]$ an application of lemma 2.5 above shows that $|n| = |[B] \cdot b|$.

To finish, we have to prove the existence of a closed curve β in $W\backslash\gamma$ in the homology class b whose intersection number with the surface B is not zero. Take a local transverse section D at γ and choose a point $x \in D \cap (W^s_{loc}(\gamma)\backslash\gamma)$ sufficiently close to γ such that the orbit segment $[x, P_D(x)]$ is totally contained in $W\backslash\gamma$, where P_D is the first return to the transverse section. Let's recall that, by definition of local Birkhoff section, there exists some $T > 0$ such that every orbit segment of length T sufficiently close to γ will intersect B. This means that if the point x is suitably chosen then for some positive integer k it will be satisfied that $[x, P^k_D(x)] \cap B \neq \emptyset$, as in figure 2.2a. If we define a curve $\hat{\beta}$ concatenating $[x, P^k_D(x)]$ with the segment inside $D \cap W^s_{loc}(\gamma)$ joining $P^k_D(x)$ with x then $[\hat{\beta}] = k \cdot b \in H_1(W\backslash\gamma)$. Observe that the segment $[x, P^k_D(x)]$ cuts B always with the same orientation because it is an orbit segment. So, by a suitably small perturbation of the curve $\hat{\beta}$ if necessary we can conclude that the intersection number $[B] \cdot [\hat{\beta}]$ is not zero, from where we derive our second claim.

For the last claim observe that B is an embedded surface in the complement of γ and so $B \cap \partial W$ is a simple curve in ∂W. Since $[\sigma] = \pm[B \cap \partial W] \in H_1(\partial W)$ and this curve has no auto-intersections it follows that $\gcd(n, m) = 1$. □

Homological intersection number in a punctured neighbourhood. We finish this subsection with a formula that is useful to calculate the homological intersection of closed curves (usually obtained as the concatenation of orbit segments with small arcs) and the surface B. The intersection number of a closed curve $\eta \subset W\backslash\gamma$ with B can be expressed in terms of the linking number and the multiplicity, according to the following statement.

Lemma 2.6. *Let $[B] \in H_2(W, \partial W \cup \gamma)$ be the homology class of B endowed with the orientation induced by the transverse flow and the chosen orientation of W, and let $[\eta] = i \cdot a + j \cdot b \in H_1(W\backslash\gamma)$ be the homology class of a closed curve η. Let $n = n(\gamma, B)$ and $m = m(\gamma, B)$ be the linking number and the multiplicity of B. Then, it is satisfied that:*

$$[B] \cdot [\eta] = \begin{cases} -im + nj & \text{if } n > 0, \\ im - nj & \text{if } n < 0. \end{cases}$$

For a proof see [13]. The proof there is done just for the case $m = 1$, but it can be adapted to the more general case.

1.2 Partition into quadrants.

Let's assume that W is small enough such that the union of the local stable and unstable manifolds separate W in four quadrants. Observe that the orientation of the meridian $a \in H_1(W\backslash\gamma)$ induces a cyclic order on this set of connected components. The four quadrants will be denoted by W_i, $i = 1, \dots, 4$, where the indices will be chosen to respect the cyclic order of the quadrants. Each quadrant W_i is homeomorphic to a solid torus, and the map $H_1(W_i) \to H_1(W\backslash\gamma)$ induced by the corresponding inclusion sends a generator of $H_1(W_i)$ to the longitude class $b \in H_1(W\backslash\gamma)$.

Let $D \subset W$ be a local transverse section and x_D its intersection point with γ. Then, the quadrants $W_1, \dots, W_4$ determines four quadrants $D_i = D \cap W_i$ in D. Observe that the first return map $P_D : V \to D$ defined in a neighbourhood V of x_D preserves the quadrants, i.e. $P_D(V \cap D_i) \subset D_i$ for every $i = 1, \dots, 4$.

Let $B \subset W$ be a tame local Birkhoff section at γ with linking number n and multiplicity m. Then, because of the tameness hypothesis the four quadrants W_i also determine a partition of the annulus B into quadrants. Each of these quadrants can be though of as a rectangle, whose boundary contains a segment of γ and two segments which are connected components of $\mathring{B} \cap (W^s_{loc}(\gamma) \cup W^u_{loc}(\gamma))$. Observe that the first return map $P_B : \mathring{U} \to \mathring{B}$ defined in a collar neighbourhood U of γ sends any quadrant of B into another quadrant.

Lemma 2.7. *There are exactly $4|n|$ quadrants of B, and each quadrant W_i contains $|n|$ of them.*

If we choose a quadrant of B which lies in W_1 and we call it B_1, then we can inductively label the quadrants of B as $B_1, \dots, B_{4|n|}$ by declaring that $\forall j = 1, \dots, 4|n|$, if W_i is the quadrant that contains B_j then B_{j+1} is the quadrant adjacent to B_j which lies in W_{i+1}, $i = 1, \dots, 4$. We will always use this labelling for the quadrants of a Birkhoff section.

Proof. For this proof we will consider that the boundary component $B(\{1\} \times \mathbb{R}/\mathbb{Z})$ of the Birkhoff section as well as the boundary ∂D of the transverse section are contained in the boundary ∂W of the tubular neighbourhood. This can be done by replacing W by a suitable smaller tubular neighbourhood and then consider the corresponding the intersections with B and D.

Let W_i be a quadrant of W, $Q \subset W_i$ be a quadrant of B and $D_i \subset N_i$ be the corresponding quadrant of D. The surfaces Q and D_i are properly embedded disks inside the solid torus W_i, i.e. embeddings of the form $(Q, \partial Q), (D_i, \partial D_i) \to (W_i, \partial W_i)$. So Q and D_i are meridian disks in W_i and they are properly isotopic. In each of these surfaces we will consider the orientation induced by the transverse flow. Let $x \in int(D_i)$ be a point sufficiently close to γ such that the orbit segment $[x, P_D(x)]$ is contained in $int(W_i)$. Choose a segment $\sigma_0 \subset int(D_i)$ connecting $P_D(x)$ with x and let's define β_0 as the closed curve obtained by concatenation of $[x, P_D(x)]$ with σ_0, as illustrated in figure 2.2b. Since this curve is obtained by concatenation of an orbit segment with a segment

inside D_i, we can make a small perturbation of this curve around the segment σ_0 and obtain a curve β_0' which is transverse to the quadrant D_i and cuts it once, with positive orientation. So we have that the intersection number $D_i \cdot \beta_0'$ as oriented submanifolds of W_i equals one. We can further assume that the curve β_0' cuts any quadrant of B transversely, with positive orientation. Since all the quadrants Q are properly isotopic to D_i, it follows that the curve β_0' cuts once each quadrant of B inside W_i. The fact that $D_i \cdot \beta_0' = 1$ means that the homology class $[\beta_0'] \in H_1(W_i) \simeq \mathbb{Z}$ is a generator of this group, and inside $W \backslash \gamma$ the homology class $[\beta_0'] \in H_1(W \backslash \gamma)$ coincides with the longitude b. Since the homological intersection number satisfies that $|[B] \cdot [\beta_0']| = |n|$ it follows that there are $|n|$ quadrants of B inside W_i. □

1.3 Projections Along the flow lines.

The main tool for our analysis of the first return map to a local Birkhoff section around a topologically saddle type periodic orbit will be the projection along the flow over a transverse section. The idea is to project in the complement of γ the local Birkhoff section B along the flow lines over a local transverse section D, to obtain a conjugacy between the first return map P_D and some power of P_B. However, observe that projecting along the flow lines will never provide a continuous map from B to D because these two surfaces are not homotopic as proper surfaces of $W \backslash \gamma$. Nevertheless, under the hypothesis of tameness for our local Birkhoff section it is possible to find a neighbourhood $U \subset B$ of γ and construct a projection along the flow lines π_w defined from $\mathring{U} = U \backslash \gamma$ onto $D \backslash \{x_D\}$, where $w \subset B$ is a segment which is a connected component of the intersection of B with a local stable manifold, π_w is a local homeomorphism in the complement of the segment w, and the discontinuity defect over w is quantified by some power of the first return map to D.

Let w be a segment which is the closure of a connected component of

$$\mathring{B} \cap (W^s_{loc}(\gamma) \cup W^u_{loc}(\gamma)) \, .$$

Since the Birkhoff section is an immersed annulus $B : [0,1] \times \mathbb{R}/\mathbb{Z} \to W$ which is an embedding out of the boundary component that is mapped onto γ, we can cut this annulus along w and obtain an immersed compact strip, with two opposite sides that are naturally identified with w. We will denote this strip by B_w.

Consider the universal cover $\tilde{W}$ of W. Consider the pseudo-flow $\tilde{\phi}_t$ on $\tilde{W}$ which is the lift of ϕ_t. Let $\tilde{B}_w$ and $\tilde{D}$ be a lift of B_w and a lift of D respectively, and let $\tilde{\gamma}$ be the lift of the orbit γ. Then, by the continuity of the flow there exists some neighbourhood O of the compact segment $\tilde{\gamma} \cap \tilde{B}_w$ and some $T > 0$ such that, for every $x \in O$, the orbit segment $[\tilde{\phi}_{-T}(x), \tilde{\phi}_T(x)]$ intersects the disk $\tilde{D}$ and exactly in one point. This allows to consider a collar neighbourhood $\tilde{U}$ of $\tilde{\gamma} \cap \tilde{B}_w$ inside $O \cap \tilde{B}_w$ and to define a map $\tilde{\pi}_w : \tilde{U} \to \tilde{D}$ of the form $\tilde{\pi}_w(x) = \tilde{\phi}(s(x), x)$, where $s : \tilde{U} \to \mathbb{R}$ is continuous and bounded. Observe that all the points in $\tilde{\gamma} \cap \tilde{U}$ are mapped over the intersection point $\tilde{x}_D$ of $\tilde{D}$ with $\tilde{\gamma}$, but we will not be interested in these points. So, we will just consider the restriction $\tilde{\pi}_w : (\tilde{U} \backslash \tilde{\gamma}) \to (\tilde{D} \backslash \{\tilde{x}_D\})$.

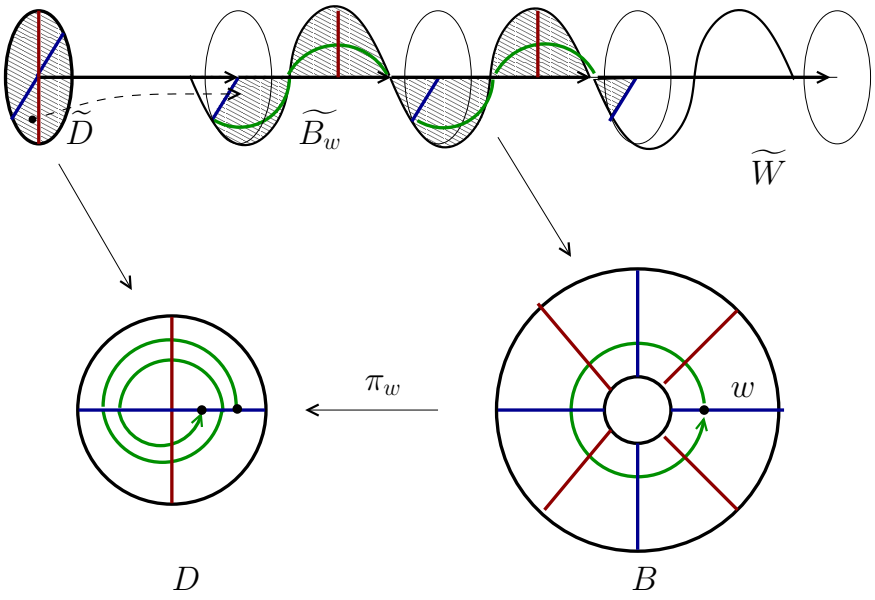


Figure 2.3: Projection along the flow lines from a local Birkhoff section onto a transverse disk. The discontinuity defect can be seen by projecting an essential curve inside B onto D.

Since the universal covering map provides identifications of the section D with $\tilde{D}$ and of the strip B_w with $\tilde{B}_w$, we can regard $\tilde{\pi}_w$ as a map $\pi_w : \mathring{U} \to (D \backslash \{x_D\})$ of the form $\pi_w(x) = \phi(s(x), x)$, where $U \subset B$ is a collar neighbourhood of γ, and which is bi-valuated over the segment w. A map π_w constructed in this way will be called a *projection along the flow from B to D*.

Definition 2.8. Let B be a tame local Birkhoff section at γ, D a local transverse section which intersects γ at the point x_D, and let w be a connected component of $\mathring{B} \cap (W^s_{loc}(\gamma) \cup W^u_{loc}(\gamma))$. A *local projection along the flow of B over D* is a map $\pi_w : \mathring{U}_w \to (D \backslash \{x_D\})$ of the form $\pi_w(x) = \phi(s(x), x)$, where $U \subset B$ is a collar neighbourhood of γ, $\mathring{U}_w$ denotes $\mathring{U} = U \backslash \gamma$ after being cut along w and $s : \mathring{U}_w \to \mathbb{R}$ is continuous and bounded.

Every projection along the flow according to the previous definition arises as in the previous construction. Observe that a projection along the flow depends on a particular

choice of the segment w as well as particular choices of the lifts of D and B_w. For notational simplicity we will often denote a projection along the flow as $\pi : \mathring{U} \to D$ without making reference to the segment w, and to the fact that x_D is not contained in the image of this map.

Proposition 2.9. *Let B be a tame local Birkhoff section at γ with linking number n and multiplicity m, and let D be a local transverse section. Let w be a segment of the intersection of $\mathring{B}$ with $W^s_{loc}(\gamma) \cup W^u_{loc}(\gamma)$. Let $\pi = \pi_w : \mathring{U} \to D$ be a projection along the flow defined as before, where $U \subset B$ is a collar neighbourhood of γ. We will enumerate the quadrants of B as $B_1, \dots, B_{4|n|}$ in such a way that B_1 and $B_{4|n|}$ intersect along w. Observe that for all the quadrants B_j contained in N_i it is satisfied that $j = i + 4r$, where $0 \leq r \leq |n| - 1$ and $1 \leq i \leq 4$. Then, we have that:*

1. *The map P_B permutes cyclically all the quadrants of B that are contained in the same quadrant N_i, and in particular we have that $P_B^{|n|}$ preserves each quadrant B_j. Moreover, in the case that $|n| > 1$, let $1 \leq k, l \leq |n| - 1$ be such that $k \equiv m \pmod{|n|}$ and $l \equiv m^{-1} \pmod{|n|}$. Then, the first return map to B permutes the quadrants in the following fashion:*

   (a) *P_B takes the quadrant B_j into* $\begin{cases} B_{j+4l} & \text{if } n > 0, \\ B_{j-4l} & \text{if } n < 0. \end{cases}$

   (b) *P_B^k takes the quadrant B_j into* $\begin{cases} B_{j+4} & \text{if } n > 0, \\ B_{j-4} & \text{if } n < 0. \end{cases}$

2. *Let B_j be a quadrant of B where $j = i + 4r$, $1 \leq i \leq 4$ and $0 \leq r \leq |n| - 1$. Let's denote by $\mathring{U}_j = \mathring{U} \cap B_j$ and π_j the restriction $\pi|_{\mathring{U}_j}$. Then, the π_j takes $\mathring{U}_j$ homeomorphically onto its image in $D_i - \{x_D\}$ and it is satisfied that:*

   (a) *The homeomorphism π_j induces a local conjugacy between $P_B^{|n|}$ and P_D. That is, for all z sufficiently close to γ it is satisfied that $\pi_j \circ P_B^{|n|}(z) = P_D \circ \pi_j(z)$, $\forall j = 1, \dots, 4|n|$;*

   (b) *Let $z \in w = B_1 \cap B_{|4n|}$ be a point. Then the discontinuity defect over w is described by:*

$$\pi_{4|n|}(z) = \begin{cases} P_D^{-m} \circ \pi_1(z) & \text{if } n > 0, \\ P_D^{m} \circ \pi_1(z) & \text{if } n < 0. \end{cases}$$

3. *The projection along the flow depends on w and on a particular choice of a lift of B_w to the universal cover. For a fix segment w we have that if π_w and π'_w are two projections then $\exists\ k \in \mathbb{Z}$ such that $\pi'_w = \pi_w \circ P_B^k$, in a suitable common neighbourhood of definition.*

This proposition is a summary of many properties stated in [13]. We just provide a proof of the first item above, and we refer to that article for a complete proof. The main difference with the referred work is that they use $m = 1$, so the proofs can be carried with the same arguments but taking care of the new combinatorics that appears for general values of the multiplicity.

Proof of item 1. We will first show that, given a quadrant of B, the points which are sufficiently close to γ will return to the same quadrant after $|n|$ iterations of P_B, and will touch once any other quadrant of B inside N_i in the middle iterations. Let B_j be a quadrant of B contained in some quadrant N_i of N, and let D_i be the corresponding quadrant of D. Let $\mathring{U}_j = \mathring{U} \cap B_j$ and $\pi_j : \mathring{U}_j \to D_i$ be the restriction to the quadrant B_j of the projection along the flow. The map π_j is a homeomorphism onto its image. Let's consider a point $z \in \mathring{U}_j$ and define $x = \pi_j(z) \in int(D_i)$. If we choose z sufficiently close to γ then we can assume that:

- There exists a first return $P_D(x) \in int(D_i)$ of the point x, and the segment $[x, P_D(x)]$ is contained in $int(N_i)$;
- $P_D(x) \in \pi_j(\mathring{U}_j)$ and if we define $z' = \pi_j^{-1}(P_D(x))$ then the orbit segments connecting z with x and z' with $P_D(x)$ are contained in $int(N_i)$.

The curve obtained as the concatenation of the orbit segment joining z with x followed by $[x, P_D(x)]$ and then by the orbit segment joining $P_D(x)$ with z' is a curve contained in an orbit segment, and contained in $int(N_i)$. It follows that z and z' belongs to the same orbit, and the orbit segment $[z, z']$ is contained in $int(N_i)$. We want to show that $z' = P_B^{|n|}(z)$. Take a segment $\sigma_0 : [0,1] \to int(D_i) \cap \pi_j(U_j)$ joining $\sigma_0(0) = P_D(x)$ with $\sigma_0(1) = x$, and define a curve β_0 concatenating $[x, P_D(x)]$ with σ_0. Then, exactly as in the proof of 2.2b, the homology class of this curve in $H_1(N \backslash \gamma)$ is the longitude b and we have that $[B] \cdot [\beta_0] = |n|$. Let $\sigma_1 : [0,1] \to \mathring{U}_j$ be the segment joining z' with z defined as $\sigma_1(t) = \pi_j^{-1} \circ \sigma_0(t)$, and let β_1 be the closed path obtained by concatenation of $[z, z']$ with σ_1. Observe that the map π_j can be realized as a homotopy that preserves orbit segments by the formula $\pi_j^t(u) = \phi(u, ts(u))$, $0 \leq t \leq 1$, and this implies that β_0 is homotopic to β_1 inside $int(N_i)$. Then $[B] \cdot [\beta_1] = |n|$. Since the curve β_1 is a segment of orbit contained in $int(N_i)$ followed by a segment inside the quadrant B_j, we can make a small perturbation of this curve supported in a neighbourhood of B_j which cuts B always with positive orientation, so in particular it will intersect B exactly $|n|$ times. So we have that the open orbit segment $int([z, z'])$ cuts the surface B at $|n| - 1$ points, once for each quadrant contained in N_i and different from B_j. It follows that the point z' is the $|n|$-th intersection point of the positive orbit segment starting at z with B, so $z' = P_B^{|n|}(z)$. Also, observe that the orbit segment $[z, z']$ cuts all the quadrants inside N_i before returning to the starting quadrant B_j, so the first return map permutes cyclically all the quadrants inside N_i.

In order to prove $1(a)$, consider a point $z \in int(B_j)$ sufficiently close to γ such that $P_B^{|n|}(z) \in int(B_j)$. To prove $1(a)$ let's assume first that $n > 0$. Since the orbit segment $[z, P_B^{|n|}(z)]$ intersects once each quadrant in N_i we know that there exists some $1 \leq l \leq |n| - 1$ such that $P_B(z) \in int(B_{j+4l})$, and we want to determine l. Let $[P_B(z), z]$ denote the curve that is obtained by reparametrizing with inverse orientation the orbit segment joining z with $P_B(z)$. Consider a curve $\alpha : [0,1] \to B_j \cup B_{j+1} \cup \cdots \cup B_{j+4l}$ connecting $\alpha(0) = z$ with $\alpha(1) = P_B(z)$ and let's define η as the closed path that is obtained as a concatenation of α with $[P_B(z), z]$. We claim that the homology class of η in $H_1(N \backslash \gamma) \simeq \mathbb{Z}^2$ is (l, q) in the basis given by the meridian and the longitude, where q is some integer. To see this, observe that we can assume that there exist

$t_0 = 0 < t_1 < \cdots < t_{4l} = 1$ such that $\alpha(t_r) \in int(B_{j+r})$ and each segment $\alpha([t_r, t_{r+1}])$ intersects the boundary of the two quadrants it connects in exactly in one point, and has no intersections with the boundary of any other quadrant. Let S be a connected component of $W^s_{loc}(\gamma)\backslash\gamma$. Then, since the curve η is the concatenation of α with a segment contained in $int(N_i)$, it cuts S exactly l times and with positive orientation, from where it follows the claim. Also, since the segment $[P_B(z), z]$ is transverse to B at its endpoints and cuts it with negative orientation, and since α is tangent to B, then the homological intersection number satisfies $[B] \cdot [\eta] = -1$. Using 2.6 we can see that $[B] \cdot [\eta] = -1 = -l \cdot m + q \cdot n$. So we conclude that $l \equiv m^{-1} \pmod{|n|}$ when $n > 0$. For negative linking number the reasoning is analogous, but instead of α we will consider a curve $\alpha' : [0, 1] \to B_j \cup B_{j-1} \cup \cdots \cup B_{j-4l'}$ such that $\alpha'(0) = z$ and $\alpha'(1) = P_B(z)$, and we will define $\eta' = \alpha' \cdot [P(z), z]$. In this case it follows that $[\eta'] = (-l', q)$, and using the formula in 2.6 corresponding to negative linking number we see that $[B] \cdot [\eta'] = -1 = (-l') \cdot m - q \cdot n$, from where we conclude that $l' \equiv m^{-1} \pmod{|n|}$ also for the case $n < 0$.

The proof of $1(b)$ is similar to the previous one. Assume first that $n > 0$. We now that there exists some $1 \leq k \leq |n| - 1$ such that $P^k_B(z) \in int(B_{j+4})$, and we want to determine k. Let $[P^k_B(z), z]$ denote the curve that is obtained by reparametrizing with inverse orientation the orbit segment joining z with $P^k_B(z)$. Consider a curve $\alpha : [0, 1] \to B_j \cup B_{j+1} \cup B_{j+2} \cup B_{j+3} \cup B_{j+4}$ connecting $\alpha(0) = z$ with $\alpha(1) = P^k_B(z)$ and let's define η as the closed path that is obtained as a concatenation of α with $[P^k_B(z), z]$. It follows that the homology class of η in $H_1(N\backslash\gamma)$ is $(1, q)$, where q is some integer, and $[B] \cdot [\eta] = -k$. Applying 2.6 we get that $[B] \cdot [\eta] = -k = -m + q \cdot n$, which implies that $k \equiv m \pmod{|n|}$. When $n < 0$, there exists $1 \leq k' \leq |n| - 1$ satisfying $P^{k'}_B(z) \in int(B_{j-4})$. In this case we will consider $\alpha' : [0, 1] \to B_j \cup B_{j-1} \cup B_{j-2} \cup B_{j-3} \cup B_{j-4}$ such that $\alpha'(0) = z$ and $\alpha'(1) = P^{k'}_B(z)$ and $\eta' = \alpha' \cdot [P^{k'}_B(z), z]$. Proceeding in the same fashion as before we can see that $k' \equiv m$. This concludes the first item. □

1.4 Deformation by flow isotopies, equivalence and existence of tame Birkhoff sections.

Observe that the combinatorial description that we have made in the previous paragraphs depends just on the homology coordinates of the embedding $(B, \partial B) \hookrightarrow (W, \gamma)$. That is, if we change the local Birkhoff section for an isotopic one that is also tame, then we obtain the same relations between projections along the flow and first return maps as described in proposition 2.9. Here, isotopy must be understood in the sense of flow-isotopy defined in 3. Let's explain this more precisely.

Definition 2.10. Let $\phi_t : M \to M$ be a non-singular regular flow.

1. If $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ and $\iota' : (\Sigma', \partial\Sigma') \to (M, \Gamma')$ are two Birkhoff sections, we will say that they are ϕ_t-*isotopic* if $\mathring{\Sigma} = \iota(\Sigma)\backslash\Gamma$ is ϕ_t-isotopic to $\mathring{\Sigma}' = \iota'(\Sigma')\backslash\Gamma'$.
2. If B and B' are two local Birkhoff sections at γ, we will say that they are $\phi_t -$ *isotopic* if there exist neighbourhoods $U \subset B$ and $U' \subset B'$ of γ and a continuous

and bounded function $s : \mathring{U} \to \mathbb{R}$ such that the map ψ given by $\psi(x) = \phi(s(x), x)$, $x \in \mathring{U}$, defines a homeomorphism between $\mathring{U}$ and $\mathring{U}'$. Recall the notation $\mathring{U} = U \backslash \gamma$.

If two transverse sections are equipped with first return maps, then a ϕ_t-isotopy between the transverse sections allow to construct a conjugation between the first return maps, and different ϕ_t-isotopies differ in the application of some power of the first return map. The following lemmas assert this for the case of Birkhoff sections, and can be deduced from lemma 1.13 in the first chapter.

Lemma 2.11. *Let Σ and Σ' be two Birkhoff sections for the flow ϕ_t which are ϕ_t-isotopic, and let $\psi : \mathring{\Sigma} \to \mathring{\Sigma}'$ be a ϕ_t-isotopy. Then:*

1. $\psi \circ P_\Sigma = P_{\Sigma'} \circ \psi$,
2. *If ψ' is another ϕ_t-isotopy then $\exists\ N \in \mathbb{Z}$ such that $\psi' = \psi \circ P_\Sigma^N$.*

Lemma 2.12. *Let B and B' be two local Birkhoff sections at γ which are ϕ_t-isotopic, and let $\psi : \mathring{U} \to \mathring{U}'$ be a ϕ_t-isotopy. Let's also assume that the first return map P_B is defined for every point in $\mathring{U}$. Then:*

1. *Up to shrinking the neighbourhoods U and U' if necessary, it is satisfied that $\psi \circ P_B(x) = P_{B'} \circ \psi(x)$, for every $x \in \mathring{U}$;*
2. *If ψ' is another ϕ_t-isotopy then $\exists\ N \in \mathbb{Z}$ such that $\psi'(x) = \psi \circ P_B^N(x)$, for every $x \in \mathring{U}$ sufficiently close to γ.*

Let's come back now to the case when γ is a saddle type hyperbolic periodic orbit. Let B and B' be two tame local Birkhoff sections at γ. If there is a ϕ_t-isotopy between these two sections it follows that they have the same linking numbers and the same multiplicities. It turns out that the linking number and the multiplicity completely determine the classes of tame local ϕ_t-isotopies of local Birkhoff sections at γ.

Lemma 2.13 (Lemma 3.6 from [13])**.** *Let B and B' be two local Birkhoff sections on a saddle type periodic orbit γ. Then B is ϕ_t-isotopic to B' if and only if $n(\gamma, B) = n(\gamma, B')$ and $m(\gamma, B) = m(\gamma, B')$.*

For a proof of this lemma see [13], Lemma 3.6. In this reference the proof is done just for the case when $m(\gamma, B) = 1$, but can be adapted to our more general setting.

The following proposition will be used for the proof of theorem 2.16. It states that given two ϕ_t-isotopic local Birkhoff sections B and B' at γ, it is possible to interpolate them to create a new local Birkhoff section that coincides with B' near γ and with B outside a neighbourhood of γ. Observe that tameness is not demanded in the hypothesis.

Proposition 2.14 (Lemma 3.7 from [13]). *Let B and B' be two tame local Birkhoff sections on a saddle type periodic orbit γ, with the same linking number and the same multiplicity. Then, there exists a neighbourhood W of γ such that: For any neighbourhood $O \subset W$ there exist another neighbourhood $O' \subset O$ and a continuous and bounded function $s : W\backslash\gamma \to \mathbb{R}$ such that the map $\psi(u) = \phi(s(u), u)$ defines an ϕ_t-isotopy onto its image, and*

1. *$\psi(u) \in \mathring{B}'$, for all $u \in \mathring{B} \cap O'$,*
2. *$\psi(u) = u$, for all $u \in \mathring{B} \cap W\backslash O$.*

To finish this section, we make some comments about the tameness condition defined at the beginning. General (local) Birkhoff sections need not to be tame. Nevertheless, the techniques in the previous proposition allow to modify any given Birkhoff section in a neighbourhood of the boundary components and obtain a tame one. We state it as a proposition. See the work referred at the beginning for a proof.

Proposition 2.15. *Let $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ be a Birkhoff section for such that all its boundary components are topologically saddle type periodic orbits of (ϕ_t, M). Then, given any neighbourhood W of Γ (that one can think as a finite union of tubular neighbourhoods around each curve in Γ) there exists a Birkhoff section $\iota' : (\Sigma, \partial\Sigma) \to (M, \Gamma)$, with image $\Sigma' = \iota'(\Sigma)$, satisfying that:*

1. *$\mathring{\Sigma} \cap (M\backslash W) \equiv \mathring{\Sigma}' \cap (M\backslash W)$,*
2. *Σ' is tame,*

Moreover, in the case that the flow ϕ_t is C^k, $k \geq 1$, the section Σ' can be chosen to be of class C^k near the boundary.

2 Orbital equivalence and Birkhoff sections.

For $i = 1, 2$ consider a non-singular regular flow (ϕ_t^i, M_i) defined on a closed 3-manifold M_i.

Let $\iota_i : (\Sigma_i, \partial\Sigma_i) \to (M, \Gamma_i)$ be a tame Birkhoff section, equipped with a first return map $P_i : \mathring{\Sigma}_i \to \mathring{\Sigma}_i$. In the same way as in 5, we denote $\Gamma_i = \{\gamma_1^i, \ldots, \gamma_K^i\}$ and we denote

$$\partial\Sigma_i = C_{11}^i \cup \cdots \cup C_{1p_1}^i \cup \cdots \cup C_{K1}^i \cup \cdots \cup C_{1p_K}^i,$$

in such a way that $C_{k1}^i \cup \cdots \cup C_{kp_k}^i = \iota_i^{-1}(\gamma_k^i)$. Given a homeomorphism $h : \mathring{\Sigma}_1 \to \mathring{\Sigma}_2$, even if h is not defined on the boundary, it defines a bijective correspondence $h_* : C_{kj}^i \mapsto h_*(C_{kj}^i)$ between the components of $\partial\Sigma_1$ and $\partial\Sigma_2$.

Theorem 2.16 (**Theorem A**). *Assume that every γ_k^i is a topologically saddle type closed orbit whose local stable and unstable manifolds are orientable. If there exists a homeomorphism $h : \mathring{\Sigma}_1 \to \mathring{\Sigma}_2$ that conjugates the first return maps, and if it is satisfied that:*

$$m(\gamma_k^1, \Sigma_1) = m(h_*(\gamma_k^1), \Sigma_2),$$
$$n(\gamma_k^1, \Sigma_1) = n(h_*(\gamma_k^1), \Sigma_2),$$

for every γ_k^1 in Γ_1, then for every neighbourhood W of Γ_1 there exists a homeomorphism $H_W : M_1 \to M_2$ such that:

(a) H_W is a topological equivalence between (ϕ_t^1, M_1) and (ϕ_t^2, M_2);

(b) $H_W(x) = h(x)$, for every $x \in \Sigma_1 \backslash W$.

Remark 2.17. The first return map $P_i : \mathring{\Sigma}_1 \to \mathring{\Sigma}_2$ defines a permutation of the boundary components of Σ_i, and the orbits of this permutation are the sets $\{C_{k1}^i, \dots, C_{kp_k}^i\}$. Recall as well that $p_k = p(\gamma_k^i, \Sigma_i)$ is the number of local connected components of Σ_i at γ_k^i. Thus, if a homeomorphism h conjugates the two first return maps, it is implicit that

$$p(\gamma_k^1, \Sigma_1) = p(h_*(\gamma_k^1), \Sigma_2), \quad \text{for every } k = 1, \dots, K.$$

Theorem 2.16 follows from a local version that we immediately state.

The local version

For $i = 1, 2$ consider a topologically saddle type periodic orbit γ^i of a flow $\phi_t^i : M_i \to M_i$ such that their local stable and unstable manifolds are orientable. Let $B_i \hookrightarrow M_i$ be a tame local Birkhoff section at γ^i, and assume that there exists a local conjugation $h : (B_1, P_{B_1})_{\gamma^1} \to (B_2, P_{B_2})_{\gamma^2}$ between the first return maps P_{B_i}. By lemma 1.9, we have that for every sufficiently small neighbourhood W_1 of γ^1, the homeomorphism h induces a homeomorphism

$$H : W_1 \backslash \gamma^1 \to W_2 \backslash \gamma^2,$$

where W_2 is a neighbourhood of γ^2, which is an orbital equivalence between the pseudo-flows ϕ_t^i obtained by restriction to sets $W_i \backslash \gamma^i$, and whose restriction to $\mathring{B}_1 \cap W_1$ coincides with h.

Theorem 2.18. *Consider a homeomorphism $H : W_1 \backslash \gamma^1 \to W_2 \backslash \gamma^2$, where W_1 and W_2 are neighbourhoods of γ^1 and γ^2 respectively, which verifies that*

(a) $H : (\phi_t^1, W_1 \backslash \gamma^1) \to (\phi_t^2, W_2 \backslash \gamma^2)$ is an orbital equivalence,

(b) $H(x) = h(x)$ for every $x \in \mathring{B}_1 \cap W_1$.

If it is satisfied that

$$m(\gamma^1, B_1) = m(\gamma^2, B_2) \text{ and}$$
$$n(\gamma^1, B_1) = n(\gamma^2, B_2),$$

then for every neighbourhood $N \subset W_1$ there exists a homeomorphisms $H_N : W_1 \to W_2$ such that:

(a) H_N is a local orbital equivalence between $(\phi_t^1, W_1)_{\gamma^1}$ and $(\phi_t^2, W_2)_{\gamma^2}$,

(b) $H_N(x) = H(x)$, for every $x \in W_1 \backslash N$.

Remark 2.19. In general, an orbital equivalence $H : (\phi_t^1, W_1 \backslash \gamma^1) \to (\phi_t^2, W_2 \backslash \gamma^2)$ that coincides with h over the Birkhoff sections B_i does not extend to a homeomorphism between W_1 and W_2. This is because, in general, the homeomorphism $h : \mathring{B}_1 \to \mathring{B}_2$ does not extend to the boundary. (cf. the comment on the introduction.)

We will show that, given some neighbourhood $N \subset W_1$ of γ^1, it is possible to modify H inside this neighbourhood and obtain a new homeomorphism H_N that extends as an orbital equivalence over the whole sets W_i.

Proof of theorem 2.16. Take two flows ϕ_t^i as in the statement of 2.16. The homeomorphism $h : \mathring{\Sigma}_1 \to \mathring{\Sigma}_2$ that conjugates the first return maps P_i gives a topological equivalence

$$H : (\phi_t^1, M_1 \backslash \Gamma_1) \to (\phi_t^2, M_2 \backslash \Gamma_2), \tag{2.3}$$

which satisfies that $H(x) = h(x)$, for every $x \in \mathring{\Sigma}_1$ (cf. section 4.1).

We will prove the theorem assuming that Γ_i consists of exactly one periodic orbit $\gamma^i \subset M_i$. Since the argument that we will do is purely local, the general case can be deduced by applying this argument in a small neighbourhood of each period orbit in Γ_i.

Let $C_1^i, \ldots, C_p^i$ denote the boundary components of Σ_i, all of them mapped onto γ^i. Let $W \subset M_1$ be a neighbourhood of γ^1. Then, up to shrinking W if necessary it follows that $\iota_1(\Sigma_1) \cap W$ has p connected components $B_1^1, \ldots, B_p^1$, each one a local Birkhoff section.

Let $W_1 = W$, $W_2 := H(W_1 \backslash \gamma^1) \cup \gamma^2$ and consider the restriction $H : W_1 \backslash \gamma^1 \to W_2 \backslash \gamma^2$. Then, this homeomorphism is an orbital equivalence between the open sets $W_i \backslash \gamma^i$. More over, if B_1^1 is one of the components of $\iota_1(\Sigma_1) \cap W_1$ and $B_1^2 = h(B_1^1)$, then H coincides with h over $B_1 \backslash \gamma^1$ and h is a local conjugation between the return maps to the sections B_1^i. So we are in the hypothesis of theorem 2.18. Given an arbitrary neighbourhood $N \subset W_1$, there exists a local orbital equivalence $H_N : W_1 \to W_2$ such that $H_N(x) = H(x)$, for every $x \in W_1 \backslash N$. We define $H_W : M_1 \to M_2$ such that

$$H_W(x) = \begin{cases} H_N(x), & \text{if } x \in W_1 \\ H(x), & \text{if } x \in M_1 \backslash N. \end{cases}$$

Then H_W is a well defined homeomorphism, and is clearly an orbital equivalence. Observe that, since H_W coincides with H outside N, then $H_W(x) = h(x)$, for every $x \in \Sigma_1 \backslash W$. □

2.1 Proof of theorem 2.18.

All this section is devoted to prove theorem 2.18. Let's consider an orbital equivalence

$$H : (\phi_t^1, W_1 \backslash \gamma^1) \to (\phi_t^2, W_2 \backslash \gamma^2) \tag{2.4}$$

such that its restriction to the interior of the Birkhoff section B_1 coincides with $h : \mathring{B}_1 \to \mathring{B}_2$, as in the hypothesis of 2.18. Observe that if we replace the neighbourhoods W_i by smaller neighbourhoods $W_i' \subset W_i$, then it is enough to prove the theorem for these new neighbourhoods. We will shrink the size of the W_i several times in the course of the proof. Theorem 2.18 relies in the following proposition.

Proposition 2.20. *For $i = 1, 2$ consider a topologically saddle type periodic orbit γ^i of a flow $\phi_t^i : M_i \to M_i$ such that their local stable and unstable manifolds are orientable. Let B_i be a tame local Birkhoff section at γ^i and assume that there exists a local conjugation $h : (B_1, P_{B_1})_{\gamma^1} \to (B_2, P_{B_2})_{\gamma^2}$ between the first return maps P_{B_i}.*

For each orbit γ^i let D_i be a local transverse section. Let $x^i = \gamma^i \cap D_i$ and let $\pi^i : (B_i)_{\gamma^i} \to (D_i)_{x^i}$ be a projection along the flow. If it is satisfied that:

$$m(\gamma^1, B_1) = m(\gamma^2, B_2) \text{ and}$$
$$n(\gamma^1, B_1) = n(\gamma^2, B_2),$$

then there exists a homeomorphism $h_D : D_1 \to D_2$ satisfying that:

(a) h_D is a local conjugation between $(D_1, P_{D_1})_{x^1}$ and $(D_2, P_{D_2})_{x^2}$,

(b) $\exists$ a collar neighbourhood $U_1 \subset B_1$ of the curve γ^1 such that $h_D \circ \pi^1(x) = \pi^2 \circ h(x)$, $\forall\ x \in U_1$.

For each orbit γ^i consider a local transverse section D_i and a projection along the flow $\pi^i : (B_i)_{\gamma^i} \to (D_i)_{x^i}$, as defined in 2.8. Because of our hypothesis about linking numbers and multiplicities we can apply proposition 2.20 and obtain a homeomorphism $h_D : D_1' \to D_2'$ satisfying (a) and (b), where $D_i' \subset D_i$ are smaller transverse sections. Using proposition 1.11 we see that there exist a tubular neighbourhood W_i' of each γ^i and a local orbital equivalence

$$H_D : (\phi_t^1, W_1')_{\gamma^1} \to (\phi_t^2, W_2')_{\gamma^2} \tag{2.5}$$

such that $H_D(x) = h_D(x)$, for every $x \in D_1' \cap W_1'$. Without loss of generality we can assume that $W_1 = W_1'$. That is, we can assume that the two homeomorphisms H and H_D have the same domain.

Theorem 2.18 follows directly from the following proposition.

Proposition 2.21. *For every neighbourhood $N \subset W_1$ of γ^1 there exists another neighbourhood $N' \subset N$ and a local orbital equivalence $H_N : (\phi_t^1, W_1)_{\gamma^1} \to (\phi_t^2, W_2)_{\gamma^2}$ such that:*

(a) $H_N(x) = H(x)$ *for every* $x \in W_1 \backslash N$,

(b) $H_N(x) = H_D(x)$, *for every* $x \in N'$.

We dedicate the rest of this section to prove 2.21. The proof of 2.20 will be postponed to section 2.2.

Proof of 2.21

We start by writing a scheme of the steps in the proof, and then we show each step.

Scheme of the proof.

Consider a neighbourhood $N \subset W_1$. Let's denote $N_1 = N$.

Step 1 For every neighbourhood $O_1 \subset W_1$ of γ^1 we can find a smaller neighbourhood $O_1' \subset O_1$ and an orbital equivalence $F : (\phi_t^1, W_1 \backslash \gamma^1) \to (\phi_t^2, W_2 \backslash \gamma^2)$ such that:

(a) $F(x) = H(x)$ for every $x \in B_1 \cap (W_1 \backslash O_1)$,

(b) $F(x) = H_D(x)$ for every $x \in \mathring{B}_1 \cap O_1'$.

Step 2 We will find a collection of neighbourhoods

- $N_1' \subset O_1' \subset O_1 \subset N_1 \subset W_1$,
- $N_2' \subset O_2' \subset O_2 \subset N_2 \subset W_2$.

Let's define $V_i = N_i \backslash O_i$ and $V_i' = O_i' \backslash N_i'$. If these neighbourhoods are suitably chosen, we will be able:

- to make an interpolation along the flow lines between H and F supported in the region V_1, and obtain an orbital equivalence $H_V : (\phi_t^1, V_1) \to (\phi_t^2, V_2)$ satisfying that:
  (a) $H_V(x) = H(x)$ for every $x \in \partial N_1$,
  (b) $H_V(x) = F(x)$ for every $x \in \partial O_1$;
- to make an interpolation along the flow lines between H_D and F supported in the region V_1', and obtain an orbital equivalence $H_V' : (\phi_t^1, V_1') \to (\phi_t^2, V_2')$ satisfying that:
  (a) $H_V'(x) = F(x)$ for every $x \in \partial O_1'$,
  (b) $H_V'(x) = H_D(x)$ for every $x \in \partial N_1'$.

Step 3 We will define H_N in the following way:

$$H_N(x) = \begin{cases} H(x) & \text{if} \quad x \in W_1 \backslash N_1, \\ H_V(x) & \text{if} \quad x \in N_1 \backslash O_1, \\ F(x) & \text{if} \quad x \in O_1 \backslash O_1', \\ H_V'(x) & \text{if} \quad x \in O_1' \backslash N_1', \\ H_D(x) & \text{if} \quad x \in N_1'. \end{cases} \tag{2.6}$$

Observe that H_N is well defined in all the boundaries $\partial N_1'$, $\partial O_1'$, ∂O_1, ∂N_1 and gives rise to a local orbital equivalence satisfying the conclusion of 2.21.

Step 1: The construction of F

Lemma 2.22. *Let $O_1 \subset W_1$ be a tubular neighbourhood of γ^1. Then, there exists another tubular neighbourhood $O_1' \subsetneq O_1$ and a homeomorphism $F : W_1 \backslash \gamma^1 \to W_2 \backslash \gamma^2$ satisfying that:*

(a) $F : (\phi_t^1, W_1 \backslash \gamma^1) \to (\phi_t^2, W_2 \backslash \gamma^2)$ is an orbital equivalence,

(b) $F(x) = H(x)$ for every $x \in B_1 \backslash O_1$,

(c) $F(x) = H_D(x)$ for every $x \in \mathring{B}_1 \cap O_1'$.

Let's define

$$S \coloneqq \{H_D(x) : x \in B_1 \cap W_1\}. \tag{2.7}$$

Then S is a local Birkhoff section at γ^2. Observe in addition that S satisfies that $m(\gamma^1, S) = m(\gamma^2, B_2)$ and $n(\gamma^1, S) = n(\gamma^2, B_2)$. Following proposition 4.1 we see that B_2 and S are ϕ_t^2-isotopic. Our next goal is to deform B_2 pushing along flow lines and obtain a new local Birkhoff section B_2', that coincides with B_2 outside a tubular neighbourhood O_2 of the orbit γ^2 and coincides with S in a smaller neighbourhood $O_2' \subset O_2$.

Lemma 2.23. *Let $O_2 \subset W_2$ be a neighbourhood of γ^2. Then, there exist a smaller neighbourhood $O_2' \subset O_2$, a local Birkhoff section B_2' at γ^2 and a homeomorphism $\psi : \mathring{B}_2 \to \mathring{B}_2'$ such that*

1. *$B_2' \backslash O_2 \equiv B_2 \backslash O_2$ and $B_2' \cap O_2' \equiv S \cap O_2'$,*
2. *ψ is an isotopy along the ϕ_t^2-orbits and satisfies that $\psi(y) = y$, $\forall\ y \in B_2 \backslash O_2$,*
3. *if $x \in \mathring{B}_1$ satisfies that $h(x) \in B_2 \cap O_2'$ then $\psi(h(x)) = H_D(x)$.*

Proof. As usual we will denote $\mathring{S} = S \backslash \gamma^2$. Let's start by constructing a ϕ_t^2-isotopy from $\mathring{B}_2$ to $\mathring{S}$. By proposition 2.13 there exists a collar neighbourhood $U_2 \subset B_2$ of the curve γ^2 and a continuous and bounded function $s : \mathring{U}_2 \to \mathbb{R}$ such that the map

$$\varphi : \mathring{U}_2 \to \mathring{S} \backslash \gamma^2 \text{ defined by } \varphi(y) = \phi^2(y, s(y)) \tag{2.8}$$

is a ϕ_t^2-isotopy from $\mathring{U}_2$ into $\mathring{S}$.

Consider $U_1 = h^{-1}(U_2) \subset B_1$. We will chose the neighbourhoods U_i sufficiently small, such that they are contained in the domain of definition of the projections $\pi^i : (B_i)_{\gamma^i} \to (D_i)_{x^i}$. Let P_S be the first return to the local Birkhoff section S.

Lemma 2.24. *There exists $k \in \mathbb{Z}$ such that $\varphi(h(x)) = P_S^k \circ H_D(x)$, for every $x \in \mathring{U}_1$.*

We postpone the proof of this lemma to the end. As a consequence this lemma, up to shrinking the size of U_i if necessary and composing with some power of P_S on the left, we can assume that $\varphi(h(x)) = H_D(x)$ for every $x \in \mathring{U}_1$.

Given a neighbourhood $O_2 \subset W_2$ of γ^2, we can use proposition 2.14 and find another neighbourhood $O_2' \subset O_2$ and a continuous and bounded function $s' : \mathring{B}_2 \to \mathbb{R}$, such that the map

$$\psi : y \mapsto \phi^2(y, s'(y)),\ y \in \mathring{B}_2 \tag{2.9}$$

satisfies the following

1. The image $B_2' \coloneqq \psi(B_2)$ is a local Birkhoff section and $\psi : \mathring{B}_2 \to \mathring{B}_2'$ is a flow isotopy,
2. $\psi(y) = y$ for every $y \in B_2 \backslash O_2$,
3. $\psi(y) = \varphi(y)$ for every $y \in \mathring{B}_2 \cap O_2'$.

The neighbourhood O_2', the section B_2' and the map ψ satisfy the properties claimed in lemma 2.24. To complete the proof it rest to prove 2.24.

Proof of lemma 2.24. Recall that the homeomorphism h_D satisfies properties (a) and (b) of 2.20 and that H_D coincides with h_D over the transverse section $D_1 \cap W_1$. Recall also that for every point $z \in W_i$ we denote the connected component of $\mathcal{O}^i(z) \cap W_i$ that contains z as $\mathcal{O}^i_{W_i}(z)$. We claim that if $x \in \mathring{U}_1$ then $\mathcal{O}^2_{W_2}(H_D(x))$ coincides with $\mathcal{O}^2_{W_2}(h(x))$. Since the projections along the flow preserve orbit segments, it is satisfied that

$$\mathcal{O}^i_{W_i}(z) = \mathcal{O}^i_{W_i}(\pi^i(z)),\ \forall z \in \mathring{U}_i.$$

So we have that

$$\mathcal{O}^2_{W_2}(H_D(x)) = H_D(\mathcal{O}^1_{W_1}(x)) = H_D(\mathcal{O}^1_{W_1}(\pi^1(x))) = \mathcal{O}^2_{W_2}(H_D \circ \pi^1(x)).$$

Since $H_D \circ \pi^1(x) = h_D \circ \pi^1(x)$ and by 2.20-(b), the last term of the previous equality is equal to

$$\mathcal{O}^2_{W_2}(h_D \circ \pi^1(x)) = \mathcal{O}^2_{W_2}(\pi^2 \circ h(x)) = \mathcal{O}^2_{W_2}(h(x)),$$

so the claim follows. Now, since $\varphi(h(x))$ is a point in $\mathring{S}$ whose orbit segment inside W_2 equals that of $H_D(x)$ we deduce that $\varphi(h(x)) = P_S^k \circ H_D(x)$ for some $k \in \mathbb{Z}$. By the continuity of the flow this integer must vary continuously with respect to x, so it is constant. This completes the lemma. □

□

As a corollary of 2.23 we have the proof of 2.22.

Proof of 2.22. Let $V \subset B_1$ be a collar neighbourhood of γ^1 contained in the domain of definition of the first return map P_{B_1}. Let $\mathcal{V}$ be the union of all the compact orbit segments connecting each point in V with its first return to B_1. Up to shrinking the size of the neighbourhoods W_i if necessary, we can assume that $W_1 \subset int(\mathcal{V})$.

Given $O_1 \subset W_1$ consider $O_2 = H(O_1)$. Then, 2.23 gives another neighbourhood $O_2' \subset O_2$, a section B_2' and a ϕ_t^2-isotopy $\psi : \mathring{B}_2 \to \mathring{B}_2'$. Let's define $O_1' = H_D^{-1}(O_2')$ and $h' : \mathring{B}_1 \to \mathring{B}_2'$ given by $h'(x) = \psi \circ h(x)$. Then, the homeomorphism h' is a local conjugation between the first return maps to the Birkhoff sections B_1 and B_2' respectively. So it induces an orbital equivalence F with domain $\mathcal{V}\backslash\gamma^1$ that coincides with h' over $\mathring{B}_1 \cap \mathcal{V}$. Since $W_1 \subset int(\mathcal{V})$ we can consider its restriction to $W_1\backslash\gamma^2$, that is $F : W_1\backslash\gamma^1 \to W_2\backslash\gamma^2$. It is direct that F coincides with H over $B_1\backslash O_1$ and with H_D over $\mathring{B}_1 \cap O_1'$. □

Step 2: The interpolation

We start by describing how to choose the neighbourhoods $N_i' \subset O_i' \subset O_i \subset N_i$. Given a regular tubular neighbourhood $O_1 \subset N_1$ of γ^1 consider the annulus $A = B_1 \cap N_1\backslash O_1$. We claim that if O_1 is sufficiently small, then there exists a compact annulus K with non empty interior and contained in $int(A)$, such that $P_{B_1}(K) \subset int(A)$. The claim follows directly by examining the first return map in $\mathring{B}_1$.

A tubular neighbourhood is said to be regular if its closure is a submanifold homeomorphic to a compact disk times an interval. Given $N_1 = N$ we will chose a family of regular tubular neighbourhoods $N_1' \subset O_1' \subset O_1 \subset N_1$ in the following way:

1. Choose $O_1 \subset N_1$ such that there exists a compact annulus K with non-empty interior contained in $A_1 = B_1 \cap N_1\backslash O_1$, which satisfies that $P_{B_1}(K) \subset int(A_1)$,
2. Choose $O_1' \subset O_1$ given by 2.22,
3. Choose $N_1' \subset O_1'$ such that there exists a compact annulus K' with non-empty interior contained in $A_1' = B_1 \cap O_1'\backslash N_1'$, which satisfies that $P_{B_1}(K') \subset int(A_1')$.

Let $F : W_1\backslash\gamma^1 \to W_2\backslash\gamma^2$ given by 2.22 for the chosen neighbourhood O_1. We will define as well:

1. $N_2 = H(N_1)$,
2. $O_2 = F(O_1)$ and $O_2' = F(O_1')$,
3. $N_2' = H_D(N_1')$.

Let's define $V_i = \bar{N}_i\backslash int(O_i) \subset M_i$ and $V_i' = \bar{O}_i'\backslash int(N_i') \subset M_i$. Since the neighbourhoods that we consider are regular tubular neighbourhoods it follows that V_i and V_i' are homeomorphic to a closed annulus times a circle. The topological equivalence H_N will be an interpolation between H and F over the set V_1 and between H_D and F over the set V_1'. We describe first the topology of these interpolating sets and then we indicate how to make these interpolations over V_1 and V_2.

The Interpolating Neighbourhoods.

Consider the compact sets

$$V_i = \bar{N}_i\backslash int(O_i) \subset M_i \tag{2.10}$$
$$V_i' = \bar{O}_i'\backslash int(N_i') \subset M_i. \tag{2.11}$$

From now on we will concentrate just in V_i, $i = 1, 2$, since all the arguments will be analogous for V_i'. The boundary components of each V_i are ∂N_i and ∂O_i. The compact annulus $A_i = B_i \cap V_i$ is a properly embedded surface in V_i. Consider the compact annuli with non empty interior $K_1 = K \subset int(A_1)$ and $K_2 = h(K) \subset int(A_2)$. Each K_i divides A_i into three annuli as in figure 2.4. We will name the boundaries of K_i as α_i and β_i according to this figure. The map h restricts to a homeomorphism $h : A_1 \to A_2$ which defines a conjugacy between the return maps $P_{K_i} : K_i \to A_i$.

As in definition 1.8 in the first chapter, we will consider the set

$$\mathcal{K}_i = \{\phi_t^i(u) : u \in K_i, 0 \le t \le \tau^i(u)\} \tag{2.12}$$

where $\tau^i(u)$ is the time of the first return to B_i of a point $u \in K_i$. This set is the union of all the compact orbit segments joining a point $u \in K_i$ with its first return $P_{K_i}(u) = \phi^i(u, \tau^i(u))$. Observe that these orbit segments are disjoint from the boundary components of V_i so it is satisfied that $\mathcal{K}_i \subset int(V_i)$. Since we have that $H|_{K_1} = F|_{K_1} = h|_{K_1}$ then it is verified that $\mathcal{K}_2 = H(\mathcal{K}_1) = F(\mathcal{K}_1)$.

The annulus A_i is an essential surface in V_i and the complement of $\mathcal{K}_i \cup A_i$ has two connected components. We will call $\mathcal{C}_i$ and $\mathcal{D}_i$ to the closure of these components, where the first one is the component that contains ∂N_i in its boundary and the second one is the one that contains ∂O_i in its boundary. So, we have a decomposition

$$V_i = \mathcal{C}_i \cup \mathcal{K}_i \cup \mathcal{D}_i$$

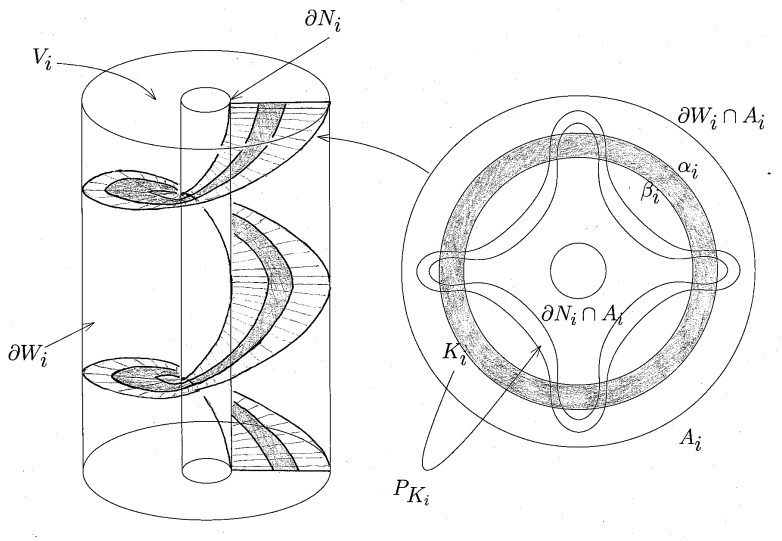


Figure 2.4: The sets V_i and the annuli A_i.

of the neighbourhood V_i into three closed sets.

If we cut the set V_i along A_i we obtain a manifold $\widetilde{V}_i$ homeomorphic to the product of A_i with a closed interval that we have depicted in figure 2.5. This manifold is equipped with a map $\widetilde{V}_i \to V_i$ which corresponds to glue back the two copies of A_i. The three sets $\mathcal{C}_i$, $\mathcal{K}_i$ and $\mathcal{D}_i$ lift into $\widetilde{V}_i$ and gives a decomposition

$$\widetilde{V}_i = \widetilde{\mathcal{C}}_i \cup \widetilde{\mathcal{K}}_i \cup \widetilde{\mathcal{D}}_i$$

into three compact sets, each one homeomorphic to an annulus times an interval. The components $\widetilde{\mathcal{C}}_i$ and $\widetilde{\mathcal{D}}_i$ are disjoint, and they intersect $\widetilde{\mathcal{K}}_i$ along the annuli L_{α_i} and L_{β_i} respectively, as in figure 2.5.

Observe that the foliation by orbit segments in V_i lift into a foliation by segments in $\widetilde{V}_i$ which are transverse to the copies of A_i. Let's denote by $\widetilde{A}_i^0$ to the copy of A_i where the lifted orbits point inward the manifold $\widetilde{V}_i$ and by $\widetilde{A}_i^1$ to the other one where the orbits point outward. For every point $u \in K_i$ the orbit segment connecting u with its first return to A_i is parametrized by $s \mapsto \phi^i(u,s)$, $s \in [0, \tau^i(u)]$, and it lifts into $\widetilde{\mathcal{K}}_i$ as a compact interval connecting the two copies of A_i inside $\widetilde{V}_i$. So the set $\widetilde{\mathcal{K}}_i$ is a union of compact segments joining the two copies of A_i, and we can put coordinates

$$\widetilde{\mathcal{K}}_i \to \{(u,s) \in K_i \times [0,+\infty) : 0 \leq s \leq \tau^i(u)\}. \tag{2.13}$$

The Construction of H_V and H'_V.

Lemma 2.25. *For the neighbourhoods $N'_i \subset O'_i \subset O_i \subset N_i$ previously chosen, there exists homeomorphisms $H_V : V_1 \to V_2$ and $H'_V : V'_1 \to V'_2$ satisfying that:*

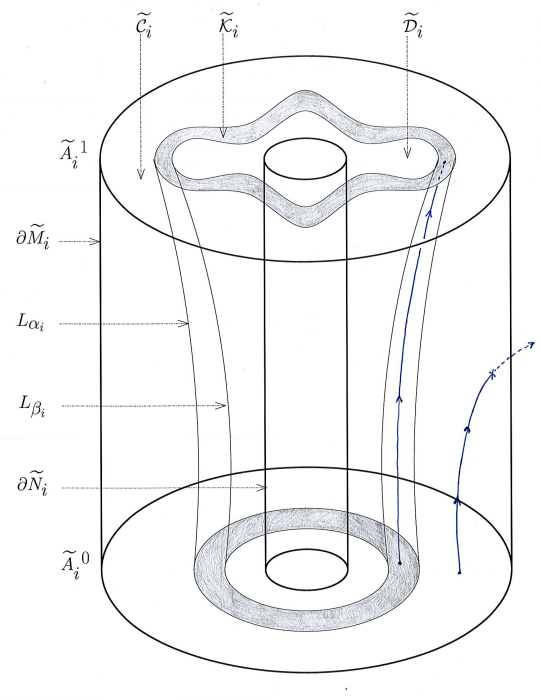


Figure 2.5: The sets $\widetilde{V}_i$. We have depicted two orbits segments lifted into $\widetilde{V}_i$, one inside the set $\widetilde{\mathcal{K}}_i$ which connects $\widetilde{A}_i^0$ with $\widetilde{A}_i^1$ and the other one inside the components $\widetilde{C}_i$.

(a) $H_V : (\phi_t^1, V_1) \to (\phi_t^2, V_2)$ *is an orbital equivalence,*

(b) $H_V(x) = H(x)$ *for every* $x \in \partial N_1$,

(c) $H_V(x) = F(x)$ *for every* $x \in \partial O_1$;

and

(a) $H'_V : (\phi_t^1, V'_1) \to (\phi_t^2, V'_2)$ *is an orbital equivalence,*

(b) $H'_V(x) = F(x)$ *for every* $x \in \partial O'_1$,

(c) $H'_V(x) = H_D(x)$ *for every* $x \in \partial N'_1$.

Proof. We will just do the construction of H_V, being analogous the other one. The key fact to prove 2.25 is that $H(x) = F(x) = h(x)$ for every $x \in B_1 \cap V_1$. Observe that, since $\mathcal{K}_2 \cup A_2 = H(\mathcal{K}_1 \cup A_1) = F(\mathcal{K}_1 \cup A_1)$ and $H|_{A_1} \equiv F|_{A_1} \equiv h|_{A_1}$, it follows that $H(\mathcal{C}_1) = \mathcal{C}_2$ and $F(\mathcal{D}_1) = \mathcal{D}_2$. We will be interested in the homeomorphisms

$$H : \mathcal{C}_1 \cup \mathcal{K}_1 \to \mathcal{C}_2 \cup \mathcal{K}_2$$
$$F : \mathcal{K}_1 \cup \mathcal{D}_1 \to \mathcal{K}_2 \cup \mathcal{D}_2$$

obtained by restriction of H and F to the sets $\mathcal{C}_1 \cup \mathcal{K}_1$ and $\mathcal{K}_1 \cup \mathcal{D}_1$, respectively. We will interpolate them over the closed set $\mathcal{K}_1$. Let

$$\widetilde{H} : \widetilde{\mathcal{C}}_1 \cup \widetilde{\mathcal{K}}_1 \to \widetilde{\mathcal{C}}_2 \cup \widetilde{\mathcal{K}}_2$$
$$\widetilde{F} : \widetilde{\mathcal{K}}_1 \cup \widetilde{\mathcal{D}}_1 \to \widetilde{\mathcal{K}}_2 \cup \widetilde{\mathcal{D}}_2$$

be the lifts of these maps to $\widetilde{V}_i$. Since H and F preserve the oriented orbit segments and coincide with h over K_1, we can use the coordinates 2.13 and write, for every point $(u, s) \in \widetilde{\mathcal{K}}_1$,

$$\widetilde{H}(u, s) = (h(u), \theta(u, s)) \tag{2.14}$$
$$\widetilde{F}(u, s) = (h(u), \eta(u, s)), \tag{2.15}$$

where each function $\theta(u, \cdot)$, $\eta(u, \cdot)$ is an increasing homeomorphism between the segments $[0, \tau^1(u)]$ and $[0, \tau^2(h(u))]$, continuously parametrized over $u \in K_1$. Observe that for every $0 \leq r \leq 1$, it follows that the convex combination $r \cdot \theta(u, \cdot) + (1 - r) \cdot \eta(u, \cdot)$ is also an increasing homeomorphism from $[0, \tau^1(u)]$ to $[0, \tau^2(h(u))]$.

Let $\rho : K_1 \to [0, 1]$ be a continuous function such that $\rho \equiv 1$ in a neighbourhood of α_1 and $\rho \equiv 0$ in a neighbourhood of β_1. We will define a map $\widetilde{H}_V : \widetilde{V}_1 \to \widetilde{V}_2$ in the following form: for every $\tilde{x} \in \widetilde{V}_1$,

$$\widetilde{H}_V(\tilde{x}) = \begin{cases} \widetilde{H}(\tilde{x}) & \text{if} \quad \tilde{x} \in \widetilde{\mathcal{C}}_1, \\ (h(u), \rho(u) \cdot \theta(u, s) + (1 - \rho(u)) \cdot \eta(u, s)) & \text{if} \quad \tilde{x} = (u, s) \in \widetilde{\mathcal{K}}_1, \\ \widetilde{F}(\tilde{x}) & \text{if} \quad \tilde{x} \in \widetilde{\mathcal{D}}_1. \end{cases} \tag{2.16}$$

The map $\widetilde{H}_V$ is a well defined homeomorphisms that preserves the foliations by (the lifts of the) orbit segments. It coincides with $\widetilde{H}$ over $\widetilde{\mathcal{C}}_1$ and with $\widetilde{F}$ over $\widetilde{\mathcal{D}}_1$. Observe also that, because of the particular election of the function ρ, it follows that $\widetilde{H}_V$ coincides with $\widetilde{H}$ in a neighbourhood of $L_{\alpha_1} = \widetilde{\mathcal{C}}_1 \cap \widetilde{\mathcal{K}}_1$ and with $\widetilde{F}$ in a neighbourhood of $L_{\beta_1} = \widetilde{\mathcal{K}}_1 \cap \widetilde{\mathcal{D}}_1$. By construction, over the union $\widetilde{A}_1^0 \cup \widetilde{A}_1^1$ the map $\widetilde{H}_V$ coincides with the homeomorphism $\tilde{h} : \widetilde{A}_1^0 \cup \widetilde{A}_1^1 \to \widetilde{A}_2^0 \cup \widetilde{A}_2^1$ that is obtained by lifting $h : A_1 \to A_2$ to $\widetilde{V}_1$. So, if we glue back the two copies of A_i inside $\widetilde{V}_i$ then $\widetilde{H}_V$ induces a homeomorphism

$$H_V : V_1 \to V_2 \tag{2.17}$$

which satisfies that:

(a) for each $x \in V_1$ the map H_V takes each oriented orbit segment $\mathcal{O}^1_{V_1}(x)$ homeomorphically onto the orbit segment $\mathcal{O}^2_{V_2}(H_V(x))$ preserving orientations,

(b) H_V coincides with H over the set $\mathcal{C}_1$,

(c) H_V coincides with F over the set $\mathcal{D}_1$.

□

Step 3: The construction of H_N

To finish the proof of 2.21 just observe that the original neighbourhood W_1 can be decomposed as the union of five compact manifolds which intersect along boundary tori, i.e.

$$W_1 = W_1 \backslash N_1 \cup V_1 \cup O_1 \backslash O_1' \cup V_1' \cup N_1'.$$

The homeomorphisms H, H_V, F, H_V' and H_D match well along the boundaries and give rise to the homeomorphism H_N as we defined in 2.6. It is an orbital equivalence, since it is when restricted to each piece of the decomposition of W_1, and clearly satisfies the properties stated in 2.21. This concludes the proof of 2.21 as well as the proof of 2.18.

2.2 Proof of proposition 2.20

We recall from section 2.8 that a projection along the flow

$$\pi^i : (B_i)_{\gamma^i} \to (D_i)_{x^i}$$

is a map of the form $\pi^i : \mathring{U}_i | w_i \to D_i$, where:

- w_i is a connected component of $\left(W^s_{loc}(\gamma^i) \cup W^u_{loc}(\gamma^i)\right) \cap \mathring{B}_i$,
- $U_i \subset B_i$ is a collar neighbourhood of γ^i,
- $\mathring{U}_i | w_i$ denotes the annulus $\mathring{U}_i = U_i \backslash \gamma^i$ cut along w_i.

This map is obtained by lifting $B_i | w_i$ and D_i to the universal cover of a tubular neighbourhood of γ^i and then projecting along the flow lines in a neighbourhood $\mathring{U}_i | w_i$. We can think π^i as a map $\mathring{U}_i \to D_i \backslash \{x^i\}$ which is bi-valuated over the segment w_i. If we call m_i and n_i to the multiplicity and the linking number of B_i, then π^i induces a (local) conjugation between $P_{B_i}^{n_i}$ and P_{D_i} and the discontinuity defect over w_i is quantified by $P_{D_i}^{m_i}$. See proposition 4.1.

Proof of 2.20. Let $n = n(\gamma^1, B_1) = n(\gamma^2, B_2)$ and $m = m(\gamma^1, B_1) = m(\gamma^2, B_2)$. Let $U_1 \subset B_1$ be a collar neighbourhood of γ^1 and let $U_2 = h(U_1)$. We will choose U_1 sufficiently small such that each $\mathring{U}_i = U_i \backslash \gamma^i$ is contained in the domain of definition of P_{B_i} and in the domain of definition of π^i. Let's denote by w_1 to the segment in B_1 that we use to cut off and define π^1, and let $w_2 \subset B_2$ be the segment that corresponds by h. We will assume that each segment w_i is a connected component of $B_i \cap W^s_{loc}(\gamma^i)$. Consider the segment $v_i \subset D_i$ that equals the intersection of D_i with the branch of $W^s_{loc}(\gamma^i)$ that contains w_i. Observe that π^i projects the points in the segment w_i into the segment v_i.

We will prove first the proposition when $n = m = 1$, and then we will comment how to deduce the general case using proposition 2.9.

Case $n = m = 1$

Since $n = 1$ we have that each Birkhoff section B_i can be partitioned into four quadrants. Following the convention of proposition 2.9, we will label the quadrants of D_i and the quadrants of B_i as

$$D_i^s,\ s = 1, \dots, 4,$$
$$B_i^s,\ s = 1, \dots, 4$$

in such a way that D_i^1 and D_i^4 intersect along the segment v_i and B_i^1 and B_i^4 intersect along w_i. See figure 2.6. We will consider as well the boundaries of the quadrants

$$v_i = v_i^1 = D_1^4 \cap D_1^1,\ v_i^2 = D_1^1 \cap D_1^2,\ v_i^3 = D_1^2 \cap D_1^2,\ v_i^4 = D_1^3 \cap D_1^4$$
$$w_i = w_i^1 = B_1^4 \cap B_1^1,\ w_i^2 = B_1^1 \cap B_1^2,\ w_i^3 = B_1^2 \cap B_1^2,\ w_i^4 = B_1^3 \cap B_1^4.$$

Observe that for every $s = 1, \dots, 4$ the map h takes points in B_1^s, D_1^s, w_1^s and v_1^s into B_2^s, D_2^s, w_2^s and v_2^s, respectively. The restriction of π^i to the quadrant B_i^s gives a map

$$\pi_s^i : B_i^s \cap \mathring{U}_i \to D_i^s \backslash \{x^i\}$$

which is a homeomorphism onto its image and takes points in w_i^s into v_i^s. Let's denote by η_s to the inverse map

$$\eta_s = (\pi_s^1)^{-1} : V \cap D_1^s \backslash \{x^1\} \to B_1^s.$$

Let $V \subset \pi^1(\mathring{U}_1) \cup \{x^1\}$ be a neighbourhood of x^1. We will start by constricting h_D in each quadrant $V \cap D_1^s$. For every $s = 1, \dots, 4$ let $h_D^s : V \cap D_1^s \to D_2^s$ be defined in the following way:

$$h_D^s(x) = \begin{cases} \pi_s^2 \circ h \circ \eta_s(x) & \text{if} \quad x \neq x^1, \\ \\ x^2 & \text{if} \quad x = x^1. \end{cases} \tag{2.18}$$

We claim that each h_D^s is a homeomorphism onto its image. Observe that η_s takes points in $D_1^s \backslash \{x^1\}$ into the quadrant B_1^s, h takes points in B_1^s into the quadrant B_2^s and then π^2 projects B_2^s into the punctured quadrant $D_2^s \backslash \{x^2\}$. Since each map is a homeomorphism onto its image then h_D^s takes $V \cap D_1^s \backslash \{x^1\}$ homeomorphically onto its image in $D_2^s \backslash \{x^2\}$. Since the projections along the flow send points near γ^i in the Birkhoff section to points near x^i we see that h_D^s is continuous in x^1 and the claim follows.

We define now $h_D : V \to D_2$ such that

$$h_D(x) = h_D^s(x) \text{ if } x \in D_1^s,\ s = 1, \dots, 4. \tag{2.19}$$

We will show that h is a well defined map, that is a homeomorphism onto its image and conjugates the first return maps P_{D_i} for points close to x^i. To see that h_D is well defined we have to check the definition of h_D over the boundaries v_1^s, $s = 1, \dots, 4$ of the quadrants D_1^s. If x belongs to some segment v_1^s for $s = 2, 3, 4$ then we have that

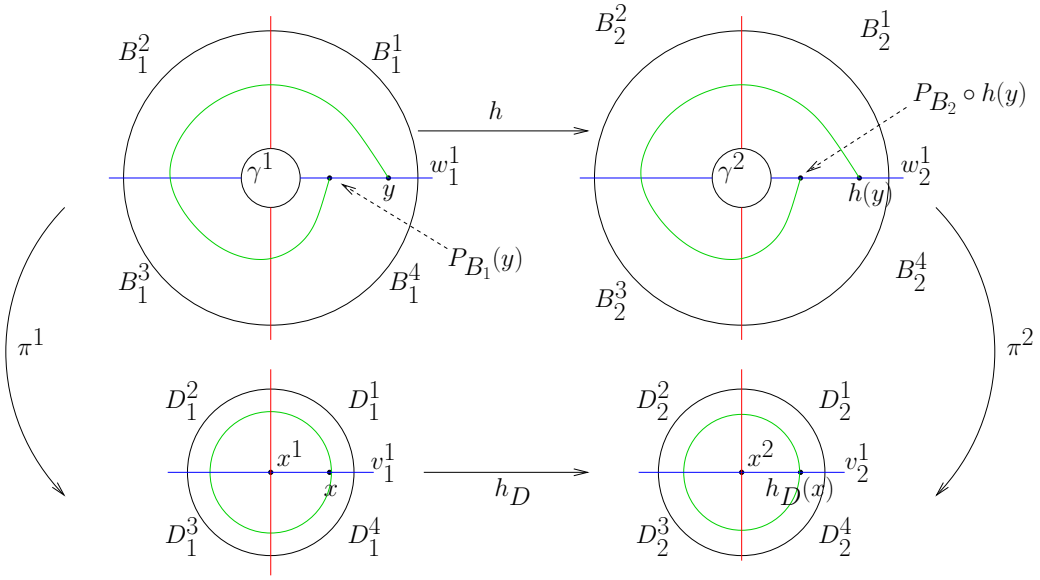


Figure 2.6: The closed curve starting at $x \in v_1^1$ lift by the maps η_s to a curve that connects $y = \eta_1(x)$ with $P_{B_1}(y) = \eta_4(x)$.

$\eta_{s-1}(x) = \eta_s(x)$. This is because the maps π_{s-1}^i and π_s^i coincides over the segment w_i^s which separates the quadrants B_i^{s-1} and B_i^s. It follows that

$$h_D^s(x) = \pi_s^2 \circ h \circ \eta_s(x) = \pi_{s-1}^2 \circ h \circ \eta_{s-1}(x) = h_D^{s-1}(x), \text{ for every } x \in v_1^s,$$

so the map h_D is well defined in the segments v_1^2, v_1^3 and v_1^4.

We will use now proposition 2.9 to show that h_D is well defined and continuous over the segment v_1^1. By this proposition we have that $\pi_4^i(z) = P_{D_i}^{-1} \circ \pi_1^i(z) = \pi_1^i \circ P_{B_i}^{-1}(z)$, for every $z \in w_i^1$. So it follows that $\eta_4(x) = P_{B_i} \circ \eta_1(x)$ for every $x \in v_1^1$, as we have illustrated in figure 2.6. Using that h conjugates the maps P_{B_i} we have that

$$\begin{aligned} h_D^4(x) &= \pi_4^2 \circ h \circ \eta_4(x) = \pi_4^2 \circ h \circ P_{B_1} \circ \eta_1(x) && (2.20) \\ &= \pi_4^2 \circ P_{B_2} \circ h \circ \eta_1(x) = \pi_1^2 \circ h \circ \eta_1(x) = h_D^1(x). && (2.21) \end{aligned}$$

Since $h_D^4(x) = h_D^1(x)$ for every $x \in v_1^1$, we conclude that $h_D : V \to D_2$ is well defined, and is a homeomorphism onto its image.

To finish the proof, consider a disk $D_1' \subset V$ that contains x^1 and define $D_2' = h(D_1')$. We have to show that the homeomorphism $h : D_1' \to D_2'$ is a local conjugation between

P_{D_1} and P_{D_2}. Let $x \in D_1'$ be a point which belongs to some quadrant D_1^s. Then

$$h_D \circ P_{D_1}(x) = \pi_s^2 \circ h \circ \eta_s \circ P_{D_1}(x) = \pi_s^2 \circ h \circ P_{B_1} \circ \eta_s(x) \tag{2.22}$$

$$= \pi_s^2 \circ P_{B_2} \circ h \circ \eta_s(x) = P_{D_2} \circ \pi_s^2 \circ h \circ \eta_s(x) = P_{D_2} \circ h_D(x). \tag{2.23}$$

General case

The general case follows the same proof that the previous case, the only difference is when checking the continuity along the segment v_s^1 and the conjugation. Using proposition 2.9 we can see how to modify equations (2.20) and (2.22) above and obtain the desired result.

□

Chapter 3

Fried and Goodman Surgeries.

In this chapter we study the relationship between Goodman and Fried surgeries. This two procedures turn out to be equivalent in the class of *transitive Anosov flows*, when we chose the obvious equivalent parameters.

In what follows M will always be a smooth, closed, orientable, 3-manifold. When we say that (ϕ_t, M) is Anosov this means that ϕ_t is generated by a C^1 vector field and $D\phi_t$ preserves a hyperbolic splitting $TM = E^s \oplus span\{X\} \oplus E^u$. The stable (or the unstable) foliation of ϕ_t provides a global frame transverse to the flow, so in particular provides a transverse frame on every periodic curve. For simplicity, we always assume that the local stable/unstable manifolds of γ are orientable. There always exists periodic orbits like this, see [17]. Recall that a flow is *transitive* if there is a point with dense orbit. There exist transitive Anosov flows, and also there exists non-transitive Anosov flows.

Given an Anosov flow (ϕ_t, M), a periodic orbit γ and some $m \in \mathbb{Z}$, the two surgeries allow to construct a new flow in a new 3-manifold, that is homeomorphic to the manifold $M(\gamma, m)$ obtained by performing an integral Dehn surgery on the curve $\gamma \subset M$ with surgery coefficient m. The surgery coefficient m is expressed in the meridian/longitude basis associated to the local frame by stable manifolds.

Let (ϕ_t, M) be an Anosov flow and consider the flows

$$\left(\phi_t^1, M_1\right) = \text{Goodman surgery}\,(\phi_t, M, \gamma, A, f, m) \tag{3.1}$$

$$\left(\phi_t^2, M_2\right) = \text{Fried surgery}\,(\phi_t, M, \gamma, m) \tag{3.2}$$

where A is a Goodman annulus parallel to γ and $f : A \to A$ is a Dehn twist with twist parameter $m \in \mathbb{Z}$ as in theorem 3.6. We recall from this theorem that, in order to obtain an Anosov flow with a Goodman surgery, m must be positive or negative, depending on the position of A.

As we said before, the manifolds M_1 and M_2 are both homeomorphic to $M(\gamma, m)$. Let's point out the following facts about the flows (3.1) and (3.2):

Goodman surgery:

- The flow (3.1) depends on the particular choice of $f : A \to A$ in a non-trivial way. If we change the pair (f, A) for another which is not ϕ_t-equivalent, then the new flow is not orbitally equivalent to ϕ_t^1 by a homeomorphism that is the identity away from γ. See 3.1. out of a neighbourhood of γ. See 3.1.
- By construction M_1 is equipped with a smooth structure and (ϕ_t^1, M_1) is an Anosov flow.

Fried surgery:

- The flow (3.2) is *almost equivalent* to $(\phi_t, M\backslash\gamma)$ in the complement of γ_2.
- It is topologically Anosov, but it is not clear that the smooth structure in the complement of γ_2 induced by the almost equivalence with the original flow can be extended onto a global smooth structure, for which the flow is Anosov.

Let's also point out that both surgeries preserve the transitivity or non-transitivity of the flow (ϕ_t, M). The purpose of this chapter is to prove the following theorem:

Theorem 3.1 (**Theorem B**). *Let (ϕ_t, M) be a transitive Anosov flow in a closed, orientable, 3-manifold and let $\gamma \subset M$ be a periodic orbit with orientable invariant manifolds. Then, there exists a tubular neighbourhood N of the curve γ such that, if $A \subset N$ then the Goodman flow* (3.1) *is orbitally equivalent to the Fried flow* (3.2)*.*

There are two immediate consequences of this theorem.

Corollary 3.2. *Let (ϕ_t, M) be a transitive Anosov flow. Then:*

(a) *All the Goodman surgeries (with the same twist parameter) performed in a neighbourhood of a periodic orbit produce orbitally equivalent flows.*

(b) *The Fried surgeries preserve the class of Anosov flows, up to orbital equivalence.*[1]

The proof of theorem 3.1 consists in reduce the problem of orbital equivalence between flows to the conjugacy problem between first return maps to Birkhoff section. Since in our case we will be working with pseudo-Anosov maps, the proof proceed by applying classical techniques in pseudo-Anosov theory that we have described in section 7, with the aid of the techniques that we have presented in 2.16 and 3.19.

It is for the previous reason that the theorem depends on the *transitivity property*, because Birkhoff sections exists just for Anosov flows under this hypothesis (see [31] and also [17]).

[1] We will also arrive to item (b) in theorem 4.12, where we will show that every topologically Anosov flow is orbitally equivalent to a C^1-Anosov flow.

Remark 3.3. *It is not known if theorem 3.1 is valid without the hypothesis of transitivity.*

For non-transitive Anosov flows there are no Birkhoff sections, or even reasonable analogues. For example, in [12] it is shown the construction of some particular kind on *transitive hyperbolic attractors*, called *non-coherent attractors*, for which the attractor can not be obtained as the suspension of a surface attractor plus surgery. See also [11] and [22].

The chapter is organized as follows: In sections 1 and 2 we explain the surgery procedures of Goodman and Fried. The proof of theorem 3.1 is given in section 3. Following the proof we explain in 3.1 that the equivalence constructed in theorem 3.1 cannot be obtained by a local procedure in a neighbourhood of the surgery. The main technical step is proposition 3.14, that we postpone to section 4.

1 Goodman surgeries.

Let $\phi_t : M \to M$ be an Anosov flow generated by a C^r vector field X, $r \geq 1$, and consider a periodic orbit γ whose local invariant manifolds are orientable.

Let W be a tubular neighbourhood of γ and let W_i, $i = 1, \ldots, 4$ be the four quadrants of W determined by $W^s_{loc}(\gamma) \cup W^u_{loc}(\gamma)$.

Consider a compact annulus A smoothly embedded in the interior of a quadrant, which is transverse to the vector field X and is *parallel* to γ, in the sense that a closed curve in A generating its fundamental group is homotopic to γ inside W. This is shown in figure 3.1a.

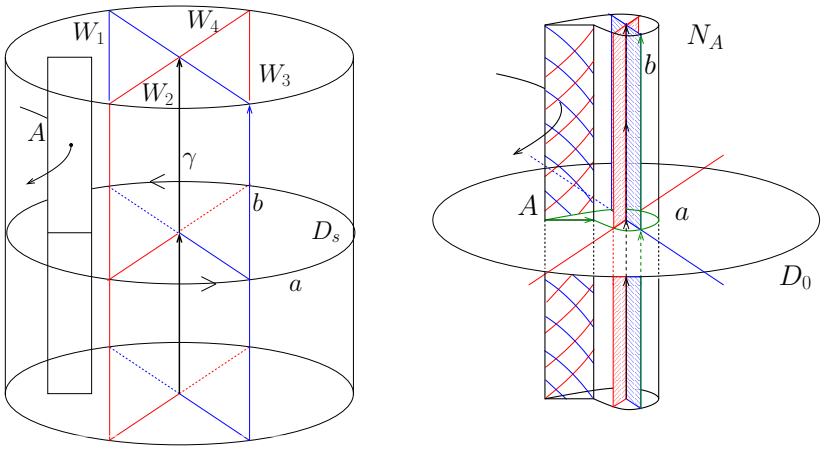
The orbit γ can be enclosed with a compact tubular neighbourhood $N_A \subset W$ with smooth boundary, such that $A \subset \partial N_A$ and such that the flow points outward the manifold N_A along the annulus. This is depicted[2] in figure 3.1b.

Let's define $M_A = M \backslash int(N_A)$. The surgery of Goodman will consist in separate the manifold M into the two pieces M_A and N_A, and then re-glue them using an appropriate diffeomorphism

$$\varphi : \partial N_A \to \partial M_A = \partial N_A$$

that produces an Anosov vector field X' in the manifold $M' = M_A \sqcup_\varphi N_A$. The diffeomorphism φ will have support contained in the annulus A and, topologically, it will be a Dehn twist. We explain the steps in this construction in what follows.

[2]We remark that in the figure the boundary of N_A has edges, but it can be smoothed along them.



(a) The parallel annulus $A \subset W$. (b) The tubular neighbourhood N_A.

1.1 Steps in the Goodman surgery

Let D be a smooth local transverse section at γ, let $x_0 = \gamma \cap D$ and let $P_D : U \to D$ be the first return map defined in an open neighbourhood of x_0. There exists a smooth reparametrization of ϕ_t, supported in a neighbourhood of γ, such that the returning time to D satisfies $\tau_D(x) = 1, \forall\, x \in U$, so we will assume this is satisfied by our original flow.

Choice of a tubular neighbourhood W. Consider a tubular neighbourhood W of γ included in the interior of the set $\mathcal{U} = \{\phi_s(x) : x \in U, 0 \leq s \leq 1\}$. The family of disks

$$D_s \coloneqq \phi_s(U) \cap W,\ 0 \leq s \leq 1$$

define a smooth foliation of W by meridian disks. The intersection of each disk D_s with the center-stable/unstable foliations of the flow determines a pair of transverse foliations $\mathcal{F}^s_{D_s}$, $\mathcal{F}^u_{D_s}$ on the disk.

On the boundary ∂W, the intersections $\partial W \cap D_s = \partial D_s$ and $\partial W \cap W^s_{loc}(\gamma)$ determine a meridian class a and a longitude class b in $H_1(\partial W) \simeq H_1(W \backslash \gamma)$, respectively. Exactly as in 8, we will orient the longitude with the same orientation of γ as an orbit of the flow, and we will orient the meridian in such a way that $\{a, b\}$ is a positive basis of $H_1(\partial W)$. (Here, ∂W must be oriented as boundary of W.) The four quadrants W_1, W_2, W_3, W_4 will be cyclically oriented by the meridian a. We will call W_1, W_3 to the two quadrants satisfying that the meridian a enters the quadrant along the stable manifold, and W_2, W_4 to the other two.

Choice of the Annulus A. Let W_i be one of the four quadrants of W. We will choose a smoothly embedded compact annulus $A \subset int(W_i)$ satisfying the following

five conditions:

(i) A is parallel to γ, i.e. an essential closed curve in A is isotopic to γ inside W,

(ii) A is transverse to the flow,

(iii) A is transverse to the disks D_s.

(iv) Each intersection segment $\sigma_s = A \cap D_s$ is contained in the corresponding quadrant $D_s^i = D_s \cap W_i$. At each point $x \in D_s^i$ the leaves $\mathcal{F}^s_{D_s}(x)$ and $\mathcal{F}^u_{D_s}(x)$ determine four sub-quadrants in D_s^i, one of them containing the point x_0. We will require that σ_s is transverse to the foliations $\mathcal{F}^s_{D_s}$, $\mathcal{F}^u_{D_s}$ and, at each point $x \in \sigma_s$, this segment traverses from the sub-quadrant that contains x_0 to the sub-quadrant that is not adjacent, as in figure 3.2.

(v) The intersection of the center-stable/unstable manifolds with annulus determine two transverse foliations $\mathcal{F}^s_A$ and $\mathcal{F}^u_A$ on it. We will require that it the leaves of these foliations are segments that connects the two boundaries of A, as in figure 3.3.

In figures 3.1a, 3.1b the annulus is contained in the first quadrant.

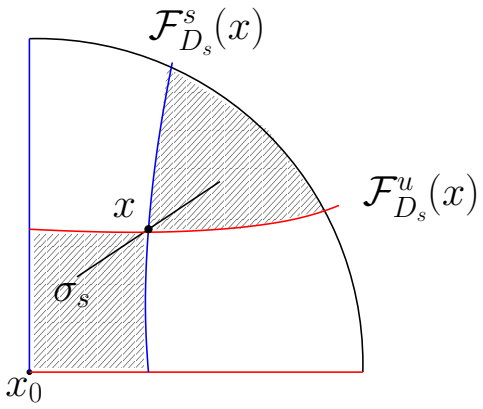


Figure 3.2: The segment $A \cap D_s^i$ must intersect $\mathcal{F}^s$ and $\mathcal{F}^u$ as in the figure.

Choice of the neighbourhood N_A. It is possible to choose a compact tubular neighbourhood N_A of γ contained in the interior of W and satisfying that:

(i) ∂N_A is smooth and $A \subset \partial N_A$,

(ii) the vector field X points outward N_A along A.

The curves

$$\alpha_s \coloneqq \partial N_A \cap D_s$$

determine a foliation of ∂N_A by meridian curves. The curves α_s and the intersection of ∂N_A with the local stable manifold $W^s_{loc}(\gamma)$ determine an oriented meridian class a and an oriented longitude class b in $H_1(\partial N_A)$, respectively. See figure 3.1b.

Coordinates in A. The family of segments

$$\sigma_s = A \cap D_s,\ 0 \leq s \leq 1$$

constitutes a smooth foliation of A by oriented segments. The orientation in each σ_s is induced from the orientation of the meridian α_s. We will call them the meridian segments. Choose a smooth coordinate system

$$A \to [0,1] \times \mathbb{R}/\mathbb{Z}$$

that takes each meridian segment σ_s to $[0,1] \times \{s\}$ preserving orientation. The closed curves l_r given in coordinates by $l_r = \{r\} \times \mathbb{R}/\mathbb{Z}$ define a foliation of A by curves homologous to the longitude b. We will call these curves the longitudes and they will be oriented with the same orientation as b.

In these coordinates, the intersection of A with the stable and unstable foliations and the foliations by meridians and longitudes are depicted on the right side of figure 3.3 for the case $A \subset int(W_1)$. If we change the annulus for another in an adjacent quadrant it has the effect of switching the intersection with the stable/unstable foliations in this figure.

The glueing diffeomorphism. Consider some $m \in \mathbb{Z}$ and some strictly increasing smooth function $h : [0,1] \to [0,1]$ satisfying that $h(0) = 0$, $h(1) = 1$ and $h'(0) = h'(1) = 0$. Consider the diffeomorphism $f : A \to A$ given in the meridian/longitude coordinates by $f(r,s) = (r, s + m \cdot h(r))$. We will define

$$\varphi : \partial N_A \to \partial M_A \text{ given by } \varphi(p) = \begin{cases} f(p) & \text{if} \quad p \in A \\ \\ p & \text{if} \quad p \notin A. \end{cases} \tag{3.3}$$

Note that, since $h'(0) = h'(1) = 0$ then φ is a smooth diffeomorphism.

The Goodman flow. Let $M' = M_A \sqcup_\varphi N_A$ be the manifold obtained as the quotient of the disjoint union $M_A \sqcup N_A$ by the map $\varphi : \partial N_A \to \partial M_A$. There are two natural inclusions $M_A \hookrightarrow M'$ and $N_A \hookrightarrow M'$. Since φ is a smooth diffeomorphism, this quotient space can be endowed with a differentiable structure such that the inclusion maps are diffeomorphisms onto their respective images. Moreover, since φ is supported in $int(A)$ and the vectorfield X is transverse to A, the differentiable structure on M' can be chosen in such a way that the push forward by the inclusion of the vectorfields $X|_{M_A}$ and $X|_{N_A}$ coincides over $\iota(M_A) \cap \iota(N_A) \subset M'$ and gives rise to a C^r vector field X' in M', where $r \geq 1$ is the differentiability class of X. See for example [44].

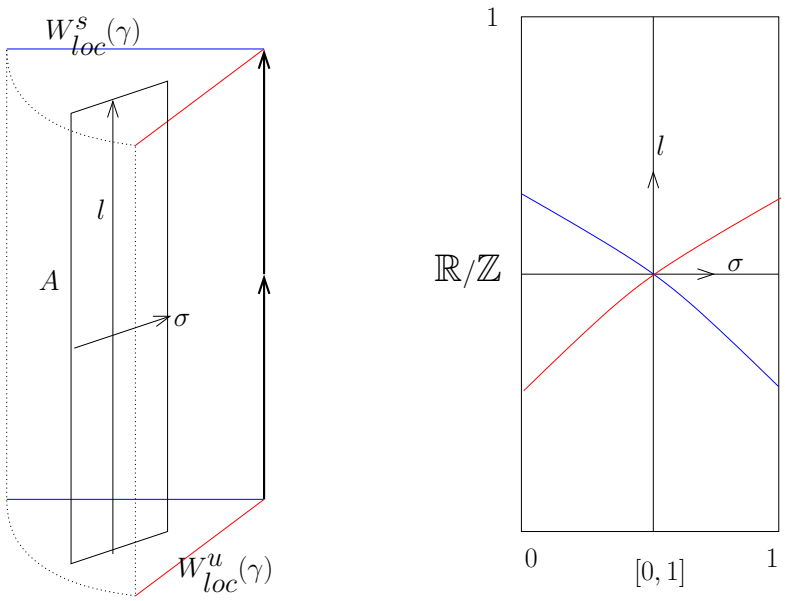


Figure 3.3: The intersection of A with the stable/unstable foliations.

Definition 3.4. We say that the flow $\phi'_t : M' \to M'$ generated by X' is obtained by a *Goodman surgery* of (ϕ_t, M) at γ. Since the surgery depends on the election of A, k and h we will denote:

$$(\phi'_t, M') = \textit{Goodman surgery}(\phi_t, M, \gamma, A, h, m). \tag{3.4}$$

The annulus A satisfying the previous conditions is called a *Goodman annulus*.

Remark 3.5. The diffeomorphism φ is a Dehn twist supported in the annulus A that preserves the foliation by longitudes. The action in homology, in the meridian/longitude basis described before, is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$$

So, topologically, cutting off the neighbourhood N_A from M and re-glueing with φ corresponds to make an integral Dehn surgery on M of parameter $m \in \mathbb{Z}$ along the framed knot γ.

1.2 Theorem of Goodman

Theorem 3.6 (Goodman, [35]). *Let γ be a periodic orbit of an Anosov flow and consider some $\alpha > 0$. Then, there exists a tubular neighbourhood W of γ such that, if we choose a Goodman annulus contained in the first quadrant W_1 satisfying that the angles*

$$\measuredangle(\mathcal{F}^s_A, \sigma),\ \measuredangle(\mathcal{F}^s_A, l),\ \measuredangle(\mathcal{F}^u_A, \sigma),\ \measuredangle(\mathcal{F}^u_A, l)$$

are, in modulus, uniformly bounded from below by α, then for every $m \in \mathbb{Z}^+$ the vector field X' obtained by the Goodman surgery (3.4) *generates an Anosov flow ϕ'_t in M'.*

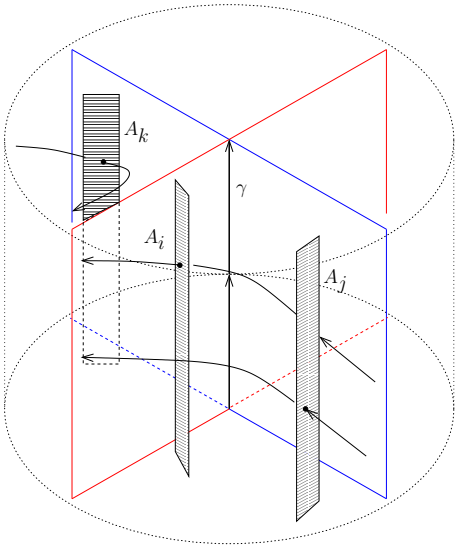
Moreover, the same is true if we consider $A \subset int(W_3)$, and if we consider $A \subset int(W_2)$ or $A \subset int(W_4)$ but with $m \in \mathbb{Z}^-$.

Remark 3.7. The statement of this theorem in [35] does not make reference to the angles between stable/unstable foliations and meridian/longitudes on A, nor to the proximity of the annulus with the periodic orbit. This is essentially because, in that work, they consider a very precise family of annuli. For enunciating the theorem as in 3.6 above, we have to add these requirements. (Cf. proof of theorem 4.12.)

Also, in [35] the statement is for annuli contained in the first or third quadrant and parameter $m > 0$. We have added the assertion about annuli contained in the second or four quadrant and $m < 0$, which follows directly by examining in the proof of [35] how they change the orientations of the stable/unstable manifolds and the orientations of the meridians in the annulus when we change the quadrant. It is possible to show that if A is contained in the first quadrant and $m < 0$, then the obtained flow can be Anosov, provided that the annulus A is placed sufficiently close to the curve γ.

The proof of this theorem is based on the *cone field criterium* (see [41]). The idea is that the future ϕ'_t-orbit of a point x in M' travels alternately between the manifolds M_A and N_A and traverse the common boundary in a discrete set of times. So the action of $D\phi'_t(x)$ on TM' can be seen as an alternate application on TM of $D\phi_{t_n}(x_n)$ (for some sequences t_n, x_n which depends on x) and $D\varphi$ or $D\varphi^{-1}$, depending if the orbit enters in M_A or N_A, respectively. When the orbit enters in M_A we have that $D\varphi$ is a parabolic matrix, and when it escapes $D\varphi^{-1} = id$. Therefore, the action of $D\phi'_t$ on the tangent bundle is a combined action of $D\phi_t$ and $D\varphi$. Since ϕ_t is Anosov, there exists a distribution of cones K^u in TM which is positively invariant by $D\phi_t$ and is exponentially contracted. The key fact is that, if m is positive, it is possible to find a cone field that is positively invariant and exponentially contracted for the combined action of $D\phi_{t_n}$ and $D\varphi$.

Surgery on different annuli. Given an Anosov flow (ϕ_t, M), a closed orbit $\gamma \subset M$ and an integer m, the Goodman surgery provides a method to find an Anosov flow ϕ'_t in a manifold M', that is obtained by Dehn surgery on M over the curve γ with twist parameter m. We have to place the annulus A on different quadrants according to the sign of m. Observe that the flow ϕ'_t depends on the particular election of the annulus A as well as the twist map $f : A \to A$.



A natural question that arises is whether different choices for A and f give rise to orbitally equivalent flows. Theorem 3.1 implies that for different elections of A and f all the flows obtained are orbitally equivalent. For instance, if we consider two annulus $A_1 \subset W_1$ and $A_2 \subset W_2$ and we perform a surgery with parameter $m > 0$ over A_1 and with parameter $-m$ over A_2, the flow obtained is orbitally equivalent to the original (ϕ_t, M).

2 Fried surgeries.

In this section we explain Fried surgeries with some detail. We will explain how to construct the surgery in a model germ of a saddle type periodic orbit, that then can be adapted to any flow having a periodic orbit of this kind. Then we will concentrate in the case of topologically Anosov flows.

2.1 Fried surgeries on the germ of a periodic orbit.

Integral Fried surgeries on saddle type periodic orbits.

Here we will explain Fried's procedure for the germ of a topologically saddle type periodic orbit. So let's consider

$$(\phi_t, N) = \text{ suspension flow generated by } f : D \to D, \tag{3.5}$$

where D is an open disk and $f : D \to D$ a homeomorphism conjugated to a saddle type hyperbolic linear transformation of $\mathbb{R}^2$ with positive eigenvalues. Denote $\{p_0\} = \mathrm{Fix}(f)$.

By definition, (3.5) is the flow induced by the action $t \cdot (p, s) \mapsto (p, s + t)$ on the quotient manifold

$$N = D \times \mathbb{R}/(f(p), s) \sim (p, s + 1).$$

Observe that:

- The non-wandering set of ϕ_t consists of one topologically saddle type periodic orbit γ.
- The manifold N is homeomorphic to an open solid torus, and $[\gamma]$ generates $H_1(N) \simeq \mathbb{Z}$.
- By our assumptions on f, the stable and unstable manifolds of γ are orientable.

The manifolds $W^s(\gamma)$ and $W^u(\gamma)$ are properly embedded cylinders in N, and their union divides N in four quadrants. In what follows, we will consider a basis $\{a, b\}$ of $H_1(N \backslash \gamma)$ formed by a meridian class a and a longitude class b as described in 8. We will consider as well a fixed transverse section $D \hookrightarrow N$ determined by the projection in the quotient of the disk $D \times \{0\}$. The orbit γ is parametrized as $\gamma(t) = \phi_t(p_0)$ with $0 \leq t \leq 1$.

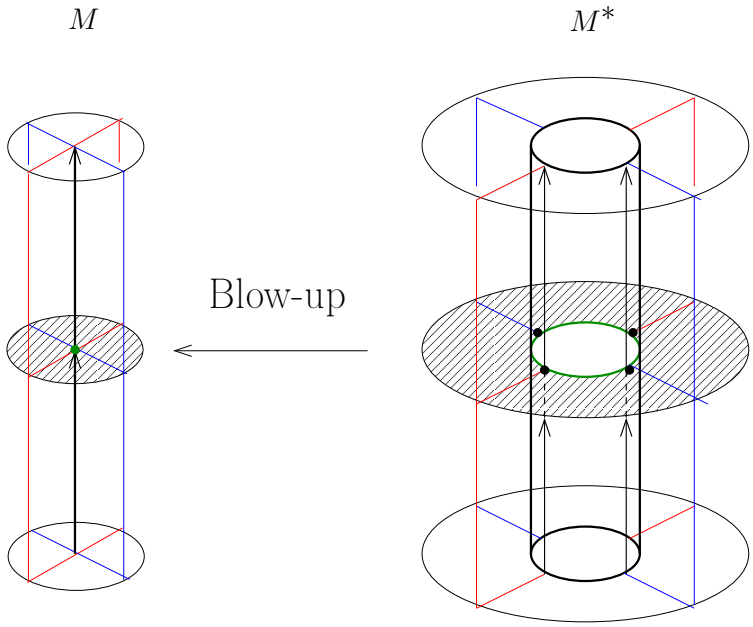


Figure 3.4: The blow up $\Pi : (\phi_t^*, N^*) \to (\phi_t, N)$.

Blow-up along γ. Since p_0 is a saddle type fixed point there is a corresponding blow-up of f at p_0. This construction consists in a triple (f^*, D^*, π) where:

(i) D^* is a surface homeomorphic to $\mathbb{R}/\mathbb{Z}^1 \times [0,1)$,

(ii) $\pi : D^* \to D$ is a map that consists in collapse the boundary component ∂D^* onto the singleton $\{p_0\}$ and is a homeomorphism on the complement,

(iii) $f^* : D^* \to D^*$ is a homeomorphism satisfying that $\pi \circ f^* = f \circ \pi$. The restriction of f^* to the boundary ∂D^* is conjugated to a Morse-Smale map on the circle with two attracting and two repelling fixed points.

Define

$$(\phi_t^*, N^*) = \text{ suspension flow generated by } f^* : D^* \to D^*. \tag{3.6}$$

By definition this is the flow induced by the action $t \cdot (p,s) \mapsto (p, s+t)$ on the quotient manifold

$$N^* = D^* \times \mathbb{R}/(f^*(p), s) \sim (p, s+1).$$

The map $\widetilde{\Pi} : D^* \times \mathbb{R} \to D \times \mathbb{R}$ defined by $\widetilde{\Pi} = \pi \times \mathrm{id}$ induces a map $\Pi : N^* \to N$ on the quotient. Observe that the flow (ϕ_t^*, N^*) and the map Π satisfy the following:

(i) N^* is homeomorphic to $D^* \times \mathbb{R}/\mathbb{Z}$,

(ii) $\Pi : N^* \to N$ satisfies that $\Pi(\partial N^*) = \gamma$ and $\Pi : N^* \backslash \partial N^* \to N \backslash \gamma$ is a homeomorphism. Moreover, the map $\Pi : \partial N^* \to \gamma$ is a circle bundle with fibers $\Pi^{-1}(\gamma(s)) = \partial D^* \times \{s\}$,

(iii) $\phi_t^* : N^* \to N^*$ is a flow satisfying that $\Pi \circ \phi_t^* = \phi_t \circ \Pi$, for every $t \in \mathbb{R}$. In particular, ϕ_t^* preserves ∂N^* and the restriction to the boundary is a topologically Morse-Smale flow with two attracting and two repelling periodic orbits.

Definition. The triple (ϕ_t^*, N^*, Π) is called the *blow-up of the flow* (ϕ_t, N) *along the periodic orbit* γ. The map Π is called the *blow-down* projection.

Observe that:

- If $\gamma_i \in \mathrm{Per}(\phi_t^*|_{\partial N^*})$ is repelling (attracting) then there is a stable (unstable) manifold $W^s(\gamma_i) \hookrightarrow N^*$ ($W^u(\gamma) \hookrightarrow N^*$) that intersects transversally ∂N^* along γ_i. This manifold is a half-cylinder properly embedded in N^*.

- There is a basis of $H_1(\partial N^*)$ where the meridian is $a = [\partial D^* \times \{s\}]$ and the longitude is $b = [\gamma_i]$.

- In the complement of ∂N^*, the map Π provides an orbital equivalence

$$(\phi_t^*, N^* \backslash \partial N^*) \to (\phi_t, N \backslash \gamma).$$

Blow-down. Let $\sigma \subset \partial N^*$ be a global transverse section for the restricted flow $\phi_t^* : \partial N^* \to \partial N^*$ that intersects once each periodic orbit in ∂N^*. In what follows we will produce a blow-down of the flow (ϕ_t^*, N^*) that depends on the curve σ and is not equivalent to the previous one.

The first return map $f_\sigma : \sigma \to \sigma$ is conjugated to a Morse-Smale map with two attracting and two repelling fixed points. Observe that for every curve $\sigma \subset \partial N^*$, being a transverse section that intersects once each periodic orbit in the boundary is equivalent to say that σ is a simple closed curve, transverse to the flow, with homology coordinates

$$[\sigma] = a + m \cdot b \in H_1(\partial N^*)$$

for some $m \in \mathbb{Z}$.

Let $\phi_t^{\sigma *}$ be a reparametrization of the flow ϕ_t^*, supported in small neighbourhood of ∂N^*, verifying that $\phi_1^{\sigma *}(p) = f_\sigma(p)$, $\forall\ p \in \sigma$. Then, the action of $\phi_1^{\sigma *}$ on σ induces a foliation $\mathcal{F}_\sigma$ on the torus ∂N^* by simple closed curves isotopic to σ. Namely, $\mathcal{F}_\sigma = \{\phi_1^{\sigma *}(\sigma) : 0 \leq t \leq 1\}$.

Let

$$\Pi_\sigma : N^* \to N_\sigma \coloneqq N^*/\mathcal{F}_\sigma$$

be the quotient projection onto the space N_σ, obtained from N^* by collapsing each leaf of the foliation $\mathcal{F}_\sigma$ into a point. Topologically N_σ is an open solid torus. Restricted to the boundary, this quotient operation produces a circle $\gamma_\sigma = \partial N^*/\mathcal{F}_\sigma$ and the map $\Pi_\sigma : \partial N^* \to \gamma_\sigma$ is a circle bundle projection. If we denote $\gamma_\sigma(t) = \Pi_\sigma(\phi_t^{\sigma *}(\sigma))$, then it is verified that

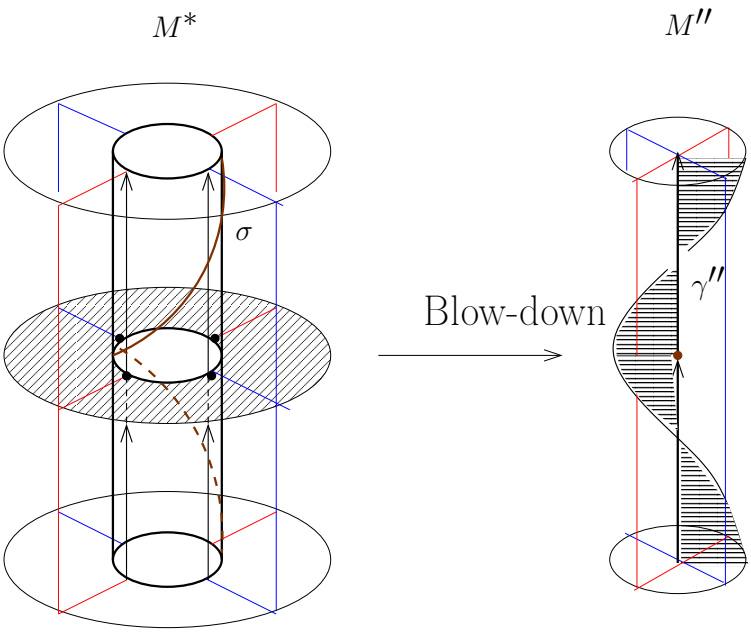
$$\begin{array}{ccc} \partial N^* & \xrightarrow{\phi_t^{*\sigma}} & \partial N^* \\ \Big\downarrow \Pi_\sigma & & \Big\downarrow \Pi_\sigma \\ \gamma_\sigma & \xrightarrow{s \mapsto s+t} & \gamma_\sigma \end{array}$$

Following the previous paragraph, there exists a flow $\phi_t^\sigma : N_\sigma \to N_\sigma$ induced on the quotient manifold that verifies

$$\begin{array}{ccc} N^* & \xrightarrow{\phi_t^{*\sigma}} & N^* \\ \Big\downarrow \Pi_\sigma & & \Big\downarrow \Pi_\sigma \\ N_\sigma & \xrightarrow{\phi_t^\sigma} & N_\sigma \end{array}$$

Definition 3.8. We say that the flow $(\phi_t^\sigma, N_\sigma)$ is obtained from (ϕ_t, N) by an *integral Fried surgery on the curve* γ. We denote it by

$$(\phi_t^\sigma, N_\sigma) = \text{Fried surgery}(\phi_t, N, \gamma, m), \tag{3.7}$$



where $m \in \mathbb{Z}$ is given by $[\sigma] = a + m \cdot b$.[3]

Given two curves σ_i, $i = 1, 2$ in the hypothesis of the previous construction we will denote by Π_{σ_i} to the corresponding blow-down maps, and we will denote $\Pi_0 = \Pi$. We will establish the following simple facts about $(\phi_t^\sigma, N_\sigma)$.

Lemma 3.9. *The flows constructed as in* (3.7) *satisfy the following:*

a) Each $(\phi_t^\sigma, N_\sigma)$ is orbitally equivalent to the suspension flow generated by a saddle type hyperbolic linear transformation on $\mathbb{R}^2$, with positive eigenvalues.

b) The map $\Pi_\sigma \circ \Pi_0^{-1}$ transforms D into a Birkhoff section $B_\sigma \hookrightarrow N_\sigma$ with linking number satisfying $|n(\gamma_\sigma, B_\sigma)| = 1$ and multiplicity $m(\gamma_\sigma, N_\sigma) = |m|$.

c) For every pair of sections σ_i, the map $\Pi_{\sigma_2} \circ \Pi_{\sigma_1}^{-1}$ defines an orbital equivalence $(\phi_t^{\sigma_1}, N_{\sigma_1} \backslash \gamma_{\sigma_1}) \to (\phi_t^{\sigma_2}, N_{\sigma_2} \backslash \gamma_{\sigma_2})$ in the complement of the periodic orbits.

d) If $[\sigma_1] = [\sigma_2] \in H_1(\partial N^*)$ then there exists a topological equivalence $H : (\phi_t^{\sigma_1}, N_{\sigma_1}) \to (\phi_t^{\sigma_2}, N_{\sigma_2})$ that coincides with $\Pi_{\sigma_2} \circ \Pi_{\sigma_1}^{-1}$ in the complement of a neighbourhood of γ_{σ_1}.

Proof. Let's check first that the orbit γ_σ has stable and unstable manifolds, homeomorphic to cylinders, that disconnect N_σ in four quadrants. Let $\mathrm{Per}(\phi_t^*|_{\partial N^*}) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ labelled in such a way that γ_1, γ_3 are the repelling orbits. Then, $W^s(\gamma_1)$ and $W^s(\gamma_3)$ are half-cylinders (i.e. homeomorphic to $\mathbb{R}/\mathbb{Z}^1 \times [0, +\infty)$) contained in

[3]The surgery is integral in the sense that the slope of $[\sigma]$ is an integer in the meridian/longitude basis.

N^*, tangent to the flow, such that $W^s(\gamma_i) \pitchfork \partial N^* = \gamma_i$, $i = 1, 3$. Clearly, each of these half-cylinders projects by Π_σ homeomorphically onto its image. Consider a curve $\alpha \subset N^* \backslash \partial N^*$ homotopic to σ. By construction this curve projects onto a meridian curve in $N_\sigma \backslash \gamma_\sigma$ (i.e. $\Pi_\sigma(\alpha) \hookrightarrow N_\sigma$ is homotopically trivial) and it intersects once the image of each $W^s(\gamma_i)$, so this half cylinders have multiplicity one. It follows that $\Pi_\sigma(W^s(\gamma_1)) \cup \Pi_\sigma(W^s(\gamma_3))$ is an embedded cylinder in N_σ, tangent to the flow, that contains γ_σ and every orbit inside converges to γ_σ in forward time. The analogue statement is true for the attracting orbits γ_2, γ_4 and their unstable manifolds. Since the dynamic is wandering in the complement of γ_σ it follows that $W^s(\gamma_\sigma) \coloneqq \Pi_\sigma(W^s(\gamma_1)) \cup \Pi_\sigma(W^s(\gamma_3))$ and $W^u(\gamma_\sigma) \coloneqq \Pi_\sigma(W^u(\gamma_2)) \cup \Pi_\sigma(W^u(\gamma_4))$ constitute stable and unstable manifolds for γ_σ homeomorphic to cylinders. This cylinders turn out to be properly embedded in N_σ and $W^s(\gamma_\sigma) \cup W^u(\gamma_\sigma)$ disconnects N_σ in four quadrants, as claimed at the beginning.

Let $B_\sigma \coloneqq \Pi_\sigma(D^*)$. Then B_σ is the continuous image of an annulus $[0, 1) \times \mathbb{R}/\mathbb{Z}^1 \hookrightarrow N_\sigma$ with $\partial B_\sigma = \Pi_\sigma(0 \times \mathbb{R}/\mathbb{Z}^1) = \gamma_\sigma$. It is an embedding on the interior, and $\mathring{B}_\sigma$ is a global transverse section for ϕ_t^σ restricted to $N_\sigma \backslash \gamma_\sigma$. It follows that B_σ is a topological Birkhoff section. Observe that $\Pi_\sigma : \partial D^* \to \gamma_\sigma$ could be non-monotonous, but the section is, by construction, tame.

The Birkhoff section B_σ is divided in four quadrants by $W^s(\gamma_\sigma) \cup W^u(\gamma_\sigma)$. By projecting $\mathring{B}_\sigma$ along the flow lines onto transverse disk (as explained in 1) we deduce item *a*) of the lemma. For item *b*) observe that since B_σ has four quadrants, it follows that $|n(\gamma_\sigma, B_\sigma)| = 1$. If we choose a simple closed curve $\alpha \subset N^* \backslash \partial N^*$ homotopic to σ, then this curve is mapped onto a meridian curve in $N_\sigma \backslash \gamma_\sigma$ and since $|[\mathring{B}_\sigma] \cdot [\alpha]| = |m|$ it follows that $m(\gamma_\sigma, B_\sigma) = |m|$. According to the conventions stated in 8 it is verified that $n(\gamma_\sigma, B_\sigma) = +1$ if $m > 0$ and $n(\gamma_\sigma, B_\sigma) = -1$ when $m < 0$. Item *c*) follows directly from the definition. Finally, we can deduce item *d*) by applying theorem 2.18. □

Remark 3.10. All the flows $(\phi_t^\sigma, N_\sigma)$ are pairwise orbitally equivalent because they are the germ of a saddle type periodic orbit. But we will be interested in orbitally equivalences that coincides with $\Pi_{\sigma_2} \circ \Pi_{\sigma_1}^{-1}$ out of a neighbourhood of the periodic orbits. In this sense, item *d*) says that each flow $(\phi_t^\sigma, N_\sigma)$ just depends on $m \in \mathbb{Z}$. This justifies the notation *Fried surgery*(ϕ_t, N, γ, m).

Rational Fried surgeries on k-prong periodic orbits.

The previous surgery can be easily redefined for the more general class of k-prong periodic orbits following the same arguments. For example, start with a saddle type periodic orbit γ with orientable stable/unstable manifolds and choose a curve σ which is a transverse section for the blow-up flow on the boundary, but with homology coordinates $[\sigma] = n \cdot a + m \cdot b$. Then, blowing-down with a foliation by curves parallel to σ produces a k-prong periodic orbit γ_σ. Since there are two attracting curves in the boundary of the blow-up of the saddle, then we see that $k = 2n$. The obtained flow is the suspension of a $2n$-prong local model $f : \mathbb{D} \to \mathbb{D}$ ($2n$ stable prongs and $2n$ unstable prongs), and the action of f on the set of stable prongs is a permutation defining two cosets. Observe that a transverse section for γ is transformed into a Birkhoff section

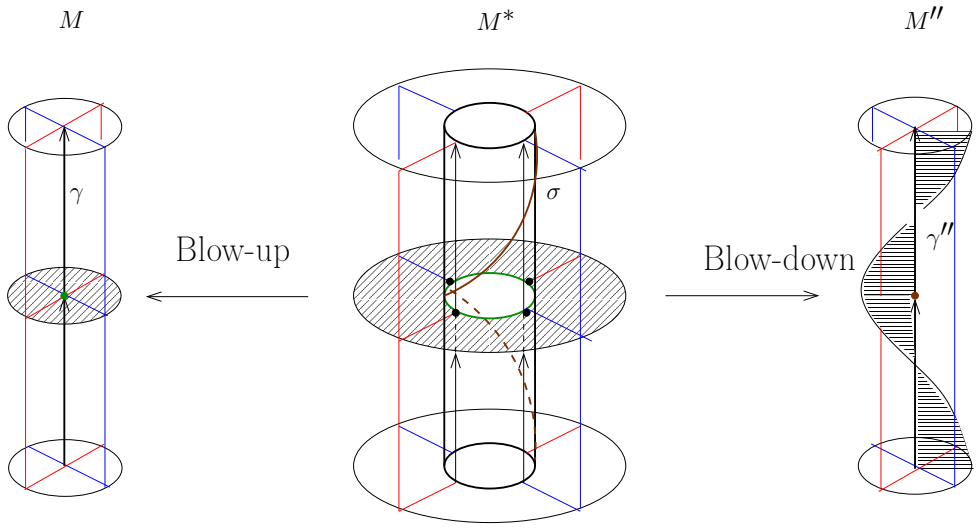


Figure 3.5: A picture of the Fried surgery.

for γ_σ. This section has multiplicity $|m|$, and has four quadrants (we have not defined linking number for Birkhoff sections at k-prong periodic orbits).

2.2 Fried surgeries in topologically Anosov flows.

We describe here the Fried surgeries for topologically Anosov flows. Recall that a regular flow in a closed 3-manifold is *topologically Anosov* if it is expansive and its invariant foliations have no circle prongs.

So let (ϕ_t, M) be a topologically Anosov flow in a closed, orientable 3-manifold. Consider a finite set $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ of periodic orbits with orientable local invariant manifolds. (There always exists periodic orbits with local invariant manifolds, cf. [17].) Choose $m_1, \dots, m_k \in \mathbb{Z}$. Since the germ of the flow at each periodic orbit γ_i is locally equivalent to the suspension of a saddle type hyperbolic linear transformation in $\mathbb{R}^2$ with positive eigenvalues, we can apply the surgery defined in (3.7) at each periodic orbit γ_i with parameter m_i.

The procedure consists in choosing local coordinates around each γ_i and construct a blow-up flow

$$\Pi : (\phi_t^*, M^*) \to (\phi_t, M).$$

The boundary of M^* is formed by k tori T_i, each one projecting onto the corresponding γ_i. For an adequate reparametrization of ϕ_t^*, we collapse each boundary component T_i using the foliation $\mathcal{F}_{\sigma_i}$ induced by a transverse section σ_i with coordinates $[\sigma_i] = (1, m_i) \in H_1(T_i)$. The result is a manifold M' equipped with a flow ϕ_t', and a blow-down

$$\Pi' : (\phi_t^*, M^*) \to (\phi_t', M').$$

Each torus T_i maps onto a periodic saddle type periodic curve γ_i' in M'. We set $\Gamma' = \{\gamma_1', \dots, \gamma_k'\}$.

Definition 3.11. We say that the flow (ϕ_t', M') is obtained from (ϕ_t, M) by a *Fried surgery* with parameters (γ_i, m_i), $i = 1, \dots k$. We will denote

$$(\phi_t', M') = \text{Fried surgery}(\phi_t, M, (\gamma_1, m_1), \dots, (\gamma_k, m_k)). \tag{3.8}$$

The flow (ϕ_t', M') verifies the following direct facts, that we state as a proposition.

Proposition 3.12. *The Fried surgery described above satisfies the following:*

a) *Topologically, the integral Fried surgery* (3.8) *corresponds to make an integral Dehn surgery on M at each curve γ_i, with twist parameter $m_i \in \mathbb{Z}$ in the local frame around γ_i determined by the stable and unstable foliations of ϕ_t.*

b) *The restriction flows $(\phi_t, M \backslash \Gamma)$ and $(\phi_t', M' \backslash \Gamma')$ are orbitally equivalent.*

c) *The flow (ϕ_t', M') is topologically Anosov.*

Proof. Items a) and b) are clear from the definitions. To conclude that (ϕ_t', M') is topologically Anosov, just observe that the Fried surgery preserves the the class of orbitally expansive flows, and that ϕ_t' preserves a pair of non-singular transverse foliations, since the integral surgeries on saddle type periodic orbits do not generate prongs for the new foliations. □

Fried surgeries can thus be interpreted as a counterpart of Dehn surgeries, but adapted to the pairs (flow,3-manifold), provided the flow has some periodic orbits. Since the surgery process just depends on the germ of the flow in a neighbourhood of the periodic orbits, clearly this surgery can be performed in a very wide class of flows having periodic orbits, provided we define a way to blow-up the orbits.

One notable characteristic about this surgery is item b) of the previous proposition, which says that in the surgery process the foliation by flow orbits remains unchanged in the complement of the periodic orbit. Thus Fried surgeries do not change the *almost equivalence class* of the original flow.

The integral Fried surgeries preserve the class of topologically Anosov flows. Given a flow of this kind, we can construct many others examples using Fried surgeries, and all of them are almost equivalent to the first one. Thus Fried surgeries become an important tool for the study of topologically Anosov flows. It is relevant to remark as well that rational Fried surgeries preserve the class of expansive flows (or equivalently pseudo-Anosov flows).

3 Proof of theorem B (thm. 3.1).

For showing that the flows obtained by Fried or Goodman surgery (along the same periodic orbit and with the same integer parameters) are equivalent, we will endow each one with an adequate Birkhoff section and we will use theorem 2.16. Fried surgeries naturally transforms Birkhoff sections of the original flow in Birkhoff sections for the new flow, but this is not the case with Goodman surgeries. Thus, we will have to show the existence of an adequate Birkhoff section associated with this latter procedure. This will be the content of proposition 3.14, which is the most technical step and its proof will be postponed to the next section. We explain here how does this proposition works, and we will give the proof of theorem 2.16 assuming this statement.

Consider first some regular tubular neighbourhood N of γ and a Goodman annulus $A \subset N$. The complement of $int(N)$ will be denoted by M_N. Given the two flows (ϕ_t^i, M_i) defined in (3.1) and (3.2), obtained by Goodman and Fried surgery respectively, we will make the following definitions:

- For each $i = 1, 2$ let γ_i be the periodic orbit of (ϕ_t^i, M_i) that corresponds to γ after surgery.
- There is a natural inclusion $\iota_1 : M_N \to M_1 = M_N \sqcup_\partial N$. We will denote $M_N^1 = \iota_1(M_N)$.
- Since $M \backslash \gamma \simeq M_2 \backslash \gamma_2$ there is a natural inclusion $\iota_2 : M_N \to M_2$. We will denote $M_N^2 = \iota_2(M_N)$.

Consider the homeomorphism $H_N : M_N^1 \to M_N^2$ defined by $H_N := \iota_2 \circ \iota_1^{-1}$. Since the foliation by the orbits of ϕ_t restricted to $M \backslash int(N)$ is not altered by any of these surgeries, we have an orbital equivalence

$$H_N : (\phi_t^1, M_N^1) \to (\phi_t^2, M_N^2) \tag{3.9}$$

between the complement of a neighbourhood of γ_1 and the complement of a neighbourhood of γ_2.

Remark 3.13. We will use this orbital equivalence in the course of the present proof. However, we want to remark that H_N cannot be extended into a global orbital equivalence $(\phi_t^1, M_1) \to (\phi_t^2, M_2)$. We show this fact in 3.16 at the end of this section.

Proposition 3.14. *Consider an Anosov flow (ϕ_t, M) and a periodic orbit γ with orientable local invariant manifolds. Let $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ be a Birkhoff section transverse to γ. Then, there exists a tubular neighbourhood N of γ such that every Goodman surgery supported in an annulus $A \subset N$ satisfies:*

(*i*) *The flow (ϕ_t^1, M_1) obtained by surgery has a Birkhoff section $j : (\Sigma_1, \partial\Sigma_1) \to (M_1, \Gamma_1)$ with $\gamma_1 \in \Gamma_1$ and $p(\gamma_1, \Sigma_1) = |\gamma \cap \mathring{\Sigma}|$.*

(*ii*) *$M_N^1 \cap j(\Sigma_1) \equiv \iota_1(\Sigma \backslash int(N))$.*

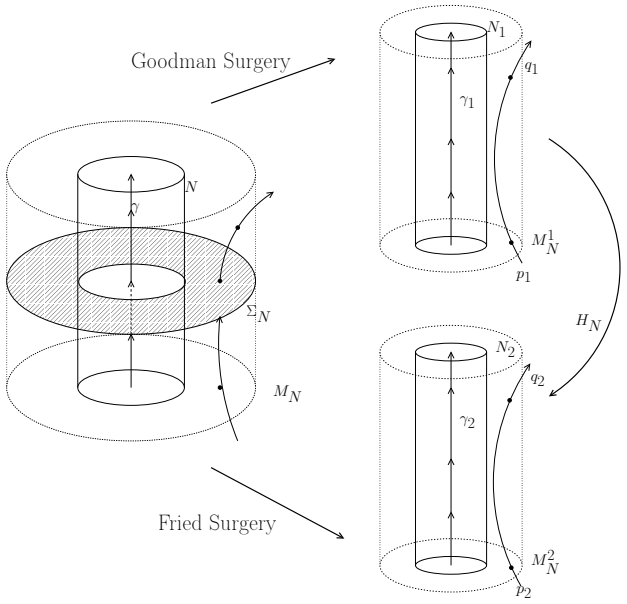


Figure 3.6: The two surgeries.

Proof of theorem 3.1. We will fix now a tubular neighbourhood N as described in proposition 3.14 above. Consider a Birkhoff section $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ for the flow ϕ_t such that $\gamma \cap \Gamma = \emptyset$. We have that $\gamma \cap \Sigma$ is a periodic orbit of the first return map P_Σ, whose period is $p(\gamma, \Sigma) = |\gamma \cap \Sigma|$. There always exist sections with this property, see [17], pag. 15. We define

$$\Sigma_N \coloneqq \Sigma \backslash int(N),\ \Sigma_N^1 \coloneqq \iota_1(\Sigma_N) \text{ and } \Sigma_N^2 \coloneqq \iota_2(\Sigma_N).$$

From the one side, the Birkhoff section Σ is transformed by the Fried surgery into a Birkhoff section $j_2 : (\Sigma_2, \partial\Sigma_2) \to (M_2, \Gamma_2)$ for the Fried flow ϕ_t^2, and it is clear that $j_2(\Sigma_2) \cap M_N^2 \equiv \Sigma_N^2$. From the other side, by proposition 3.14 there exists a Birkhoff section $j_1 : (\Sigma_1, \partial\Sigma_1) \to (M_1, \Gamma_1)$ for the Goodman flow ϕ_t^1 that extends the section Σ_N^1 and has γ_1 as a boundary component. That is, $j_1(\Sigma_1) \cap M_N^1 \equiv \Sigma_N^1$. We remark that, as abstract surfaces, each Σ_i has $p(\gamma, \Sigma)$ boundary components more than Σ, that are mapped onto γ_i.

Let $P_i : \mathring{\Sigma}_i \to \mathring{\Sigma}_i$ be the first return map of the flow ϕ_t^i to the interior of each Birkhoff section Σ_i. Our next step is to show the following:

Lemma 3.15. *There exists a homeomorphism $h : \mathring{\Sigma}_1 \to \mathring{\Sigma}_2$ such that $P_2 \circ h = h \circ P_1$.*

In this way, each flow has a Birkhoff section where the first return map is conjugated to the other. On the periodic orbits γ_i none of the surgeries change the modulus of

the linking number, which equals one. Since we use the same twist parameter for both, it follows that $n(\gamma_1, \Sigma_1) = n(\gamma_2, \Sigma_2)$ and $m(\gamma_1, \Sigma_1) = m(\gamma_2, \Sigma_2) = |m|$. Since the combinatorial data on the boundary is compatible for every orbit in Γ_i, then by theorem 2.16, there exists an orbital equivalence $(\phi_t^1, M_1) \to (\phi_t^2, M_2)$. This completes the proof of theorem 3.1, up to 3.15.

To finish we show the conjugacy between the first return maps P_i.

Proof of lemma 3.15. By construction, each surface Σ_i has $p = p(\gamma, \Sigma)$ boundary components more than the original Σ, each of them mapped onto γ_i by the immersion $\Sigma_i \hookrightarrow M_i$. We will assume for this proof that $\partial\Sigma = \emptyset$. The reader could appreciate that the general case follows the same argument, but with a heavier notation.

For $i = 1, 2$ let $\widehat{P}_i : \widehat{\Sigma}_i \to \widehat{\Sigma}_i$ be the blow-down of the first return map P_i. Since the linking number equals one at each boundary component, the blowing down originates a saddle type periodic orbit $\mathcal{O}^i = \{x_1^i, \ldots, x_p^i\}$ for P_i. We will consider P_i as a pseudo-Anosov on the punctured surface $(\Sigma, \mathcal{O}^i)$ and apply proposition 1.29 from 7.3.

Claim. There exist an isomorphism $\phi : \pi_1(\Sigma_1 \backslash \mathcal{O}^1) \to \pi_1(\Sigma_2 \backslash \mathcal{O}^2)$ which conjugates the corresponding actions on fundamental groups

$$(\widehat{P}_i)_* : \pi_1(\widehat{\Sigma}_i \backslash \mathcal{O}^i) \to \pi_1(\widehat{\Sigma}_i \backslash \mathcal{O}^i), \ i = 1, 2,$$

and which preserves the sets $\Gamma(\mathcal{O}^i)$ of the conjugacy classes of the curves homotopic to the punctures.

By construction the homeomorphism H_N sends Σ_N^1 homeomorphically onto Σ_N^2, so we will define

$$h_N : \Sigma_N^1 \to \Sigma_N^2 \text{ given by } h_N \equiv H_N|_{\Sigma_N^1}. \tag{3.10}$$

Since H_N is a topologically equivalence $(\phi_t^1, M_N^1) \to (\phi_t^2, M_N^2)$ then it follows that h_N is a *local* conjugacy between the maps P_i, in the sense that: For every $U \subset \Sigma_N^1$ satisfying that $P_1(U) \subset \Sigma_N^1$ and $\mathcal{U} \subset M_N$ it is verified that $P_2 \circ h_N(x) = h_N \circ P_1(x)$, $\forall\ x \in U$, where the set $\mathcal{U}$ consists in the union of the orbits segments starting in points $x \in U$ and ending at the firs return to Σ_N. (Cf. sections 1 and 2.)

Each surface Σ_N^i can be seen as embedded in $\widehat{\Sigma}_i$, and the boundary components determine disk-shaped neighbourhoods of the punctures x_k^i.

There exists a neighbourhood $W_i \subset M_i$ of γ_i, that contains N_i in the interior, and big enough such that $P_i(x) \in \Sigma_N^1$ and $\phi^i([0, \tau^i(x)], x) \subset M_i \backslash N_i$ for every $x \in \Sigma_N^i \cap (M_i \backslash W_i)$. By retracting the boundary components of Σ_N^i we can consider a subsurface $\Sigma_W^i \subset \mathit{int}\,(\Sigma_N^i)$ and entirely contained in the complement of W_i. For $i = 1, 2$

these surfaces can be chosen in such a way that there is a commutative diagram:

$$\begin{array}{ccc} \Sigma_W^1 & \xrightarrow{P_1} & \Sigma_1 \backslash \mathcal{O}^1 \\ \Big\downarrow h_N & & \Big\downarrow P_2 \\ \Sigma_W^2 & \xrightarrow{P_2} & \Sigma_2 \backslash \mathcal{O}^2 \end{array}$$

Let $C_1^i, \ldots, C_p^i$ be the boundary components of Σ_W^i and c_i their corresponding free homotopy classes in $\Sigma_i \backslash \{x_1^i, \ldots, x_p^i\}$. Since the first return P_i permutes cyclically the connected components of $\Sigma_i \cap N_i$, it follows that $(P_i)_*$ permutes the set $\Gamma(\mathcal{O}^i)$ of conjugacy classes of the curves c_k. If γ is any closed curve in $\Sigma_i \backslash \partial\Sigma_i$ can be homotoped onto a closed curve $\tilde{\gamma} \subset \Sigma_W^i$ that has a first return contained in Σ_N^i. It follows that the map h_N induces a conjugation between the actions of the maps P_i on the group of free homotopy classes of curves in $\Sigma_i \backslash \partial\Sigma_i$, sending $\Gamma(\mathcal{O}^1)$ to $\Gamma(\mathcal{O}^2)$.

We conclude that h_N induce an isomorphism $\phi : \pi_1(\Sigma_1; \mathcal{O}^1) \to \pi_1(\Sigma_2; \mathcal{O}^2)$ as claimed. Since the actions of the first return maps on the punctured surfaces $(\Sigma_i, \mathcal{O}^i)$ are π_1-conjugated by an isomorphism that preserves the class of the punctures, proposition 1.29 implies that P_1 and P_2 are conjugated by a homeomorphism that preserves the punctures.

This finish the lemma. □

3.1 Goodman and Fried surgeries are not equivalent in a canonical way.

Consider the homeomorphism $H_N : M_N^1 \to M_N^2$ defined in (3.9). It is an orbital equivalence

$$(\phi_t^1, M_N^1) \to (\phi_t^2, M_N^2)$$

between the complement of a neighbourhood of γ_1 and the complement of a neighbourhood of γ_2.

Proposition 3.16. *There is no orbitally equivalence $(\phi_t^1, M_1) \to (\phi_t^2, M_2)$ that extends H_N.*

Proof. Without loss of generality we can assume that N is a compact, regular, tubular neighbourhood of γ, that contains A in the interior.

Consider a point $r \in A$. Since r is in the interior of some quadrant, then the orbit segment $\mathcal{O}_N(r)$ is also contained in the quadrant, and its extremities are contained in ∂N. Let $p \in \partial N$ be the extremity where $\mathcal{O}_N(p)$ enters inside N and $q \in \partial N$ the extremity where it escapes.

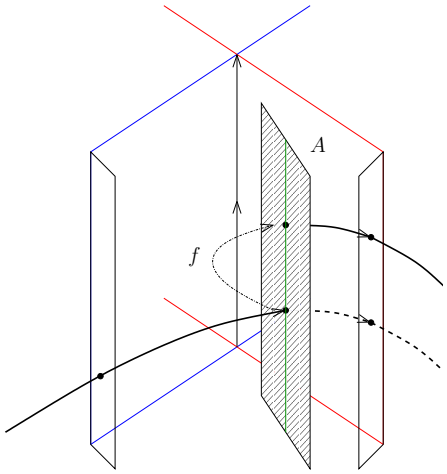


Figure 3.7: Breaking the orbits in a neighbourhood of γ.

For each $i = 1, 2$ define $p_i = \iota_i(p)$ and $q_i = \iota_i(q)$. This two points belong to ∂M_N^i and by definition $H_N(p_1) = p_2$, $H_N(q_1) = q_2$. See figure 3.7.

Assume there is a topological equivalence $H : M_1 \to M_2$ that extends H_N. Let's call $N_i = M_i \backslash M_N^i$, so H takes the neighbourhood N_1 homeomorphically onto N_2 preserving oriented orbit segments.

Since the Fried flow ϕ_t^2 is equivalent to ϕ_t in the complement of γ_2, then the points p_2 and q_2 belong to the same ϕ_t^2-orbit segment in N_2. Since H is an orbital equivalence, then p_1 and q_1 must be in the same ϕ_t^1-orbit segment in N_1.

Now since the Goodman flow is an alternated composition of iterates ϕ_{t_j} of the flow and the glueing map $f : A \to A$, it follows that the orbit segment that enters through p_1 must escape N_1 through a point $q_1' = \phi_{t_1} \circ f \circ \phi_{t_0}(p) = \phi_{t_1} \circ f(r)$. Since f has non-empty support, then r and $f(r)$ are in different orbit segments inside N, and this means that $q_1' \neq q_1$. So, there exists no extension H from H_N to the whole M_1. □

Remark 3.17.

(i) Observe that the same statement of 3.16 is true if we replace H_N for another topologically equivalence H_N', obtained as a small perturbation of H_N by some flow isotopy. So, the non-extensibility property of H_N can not be destroyed by bounded perturbations in the flow direction.

(ii) The same argument shows that, if (f, A) and (f', A') are two Goodman annulus in the same quadrant which are not ϕ_t-isotopic, then the obtained flows cannot be equivalent by a homeomorphism that is the identity in the complement of a neighbourhood of γ.

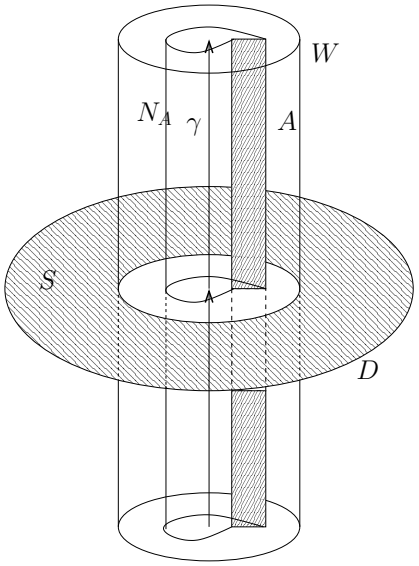
4 Birkhoff sections and Goodman surgeries.

Let γ be a periodic orbit of an Anosov flow (ϕ_t, M) with orientable invariant manifolds. Consider a local transverse section $D \hookrightarrow M$ for the orbit γ, homeomorphic to a disk. Let W be a compact tubular neighbourhood of γ verifying that ∂W is transverse to D and that $D_0 = D \cap W$ is a closed disk contained in $int(D)$. This is depicted in the figure below.

Let A be a compact annulus, parallel to γ, transverse to the flow and contained in some quadrant of W. Choose some twist map $f : A \to A$ and consider a flow

$$(\phi'_t, M') = \textit{Goodman surgery}(\phi_t, M, \gamma, A, f) \tag{3.11}$$

obtained by a Goodman surgery on (ϕ_t, M). Here and for the rest of the section, $f : A \to A$ will be a twist map with parameter $m \in \mathbb{Z}$.



Remark 3.18. Observe that this surgery does not transform the surface D into a surface in M'. In particular, no Birkhoff section for (ϕ_t, M) transforms into a Birkhoff section for (ϕ'_t, M') by a Goodman surgery.

The manifold M' is defined to be a glueing $M_A \sqcup_\varphi N_A$, where N_A is a tubular neighbourhood of γ and $M_A = M \backslash int(N_A)$, as we described in 4.1. We will state the following notations:

(i) We call γ' to the periodic orbit in M' that corresponds to γ after surgery. That is, γ' equals the image of the inclusion $\gamma \hookrightarrow N_A \hookrightarrow M'$.

(ii) We call W' to the set in M' obtained as $W \backslash N_A \sqcup_\varphi N_A$. This is a tubular neighbourhood of γ' that contains (the image by inclusion of) N_A.

(iii) We call S to the surface induced by the inclusion $D \cap M \backslash int(W) \hookrightarrow M_A \hookrightarrow M'$.

The part of D that lies outside W is not altered by the surgery and induces the surface S in M'. This surface is a compact annulus. The purpose of this section is to show the following proposition:

Proposition 3.19. *There exists a tubular neighbourhood $N \subset W$ of the orbit γ such that: If the Goodman annulus A is contained in N, and if the sign of the twist parameter m is positive when A is placed in the first or third quadrant, or is negative when A is placed in the second or fourth quadrant, then the flow (ϕ'_t, M') admits a local Birkhoff section $(B, \partial B) \hookrightarrow (M', \gamma')$ satisfying that:*

(*i*) $n(B, \gamma') = 1$, $m(B, \gamma') = m$,

(*ii*) $B \cap M' \backslash W' = S$

As a corollary we can prove proposition 3.14 stated in the previous section:

Proposition. *Consider an Anosov flow (ϕ_t, M) and a periodic orbit γ with orientable local invariant manifolds. Let $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ be a Birkhoff section transverse to γ. Then, there exists a tubular neighbourhood N of γ such that every Goodman surgery supported in an annulus $A \subset N$ satisfies:*

(*i*) *The flow (ϕ'_t, M') obtained by surgery has a Birkhoff section $\iota' : (\Sigma', \partial\Sigma') \to (M', \Gamma')$ with $\gamma' \in \Gamma'$ and $p(\gamma', \Sigma') = |\gamma \cap \mathring{\Sigma}|$.*

(*ii*) $\Sigma' \backslash int(N) \equiv \Sigma \backslash int(N)$.

Proof of proposition 3.14. Given a Birkhoff section $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ such that $|\mathring{\Sigma} \pitchfork \gamma| = p > 0$, consider a tubular neighbourhood N as given in proposition 3.19, small enough such that the intersection $\mathring{\Sigma} \cap N$ is transverse and consists in p disjoint disks in $\mathring{\Sigma}$.

The surface $\Sigma \backslash int(N)$ naturally embeds inside M'. The previous proposition says that each of the p boundary components in $\Sigma \cap \partial N$ can be individually extended inside N as a local tame Birkhoff section. If these surfaces are not disjoint in $N \backslash \gamma'$, since they are tame and with equal linking number and multiplicity, using the techniques presented in section 1 we can perturb them by flow isotopies and make them disjoint.

We obtain in this way an immersion

$$\iota' : (\Sigma', \partial\Sigma') \to (M', \Gamma \cup \{\gamma'\}),$$

where Σ' is obtained by removing p disjoint closed disks on $\mathring{\Sigma}$. The image of this immersion coincides with Σ in the complement of N and is transverse to the vector field on the interior. Thus, to conclude that it is a Birkhoff section, it rest to show that every orbit intersect Σ' in uniformly bounded time.

For this, consider two tubular neighbourhoods O_0 and O_1 of γ' satisfying that

1. $O_0 \subset N \subset O_1$ and ∂N is contained in the interior of $O_1 \backslash O_0$,
2. There exists $T_1, T_0 > 0$ such that $[p, \phi'_{T_0}(p)] \cap \Sigma' \neq \emptyset$ for every $p \in O_0$ and $[p, \psi_{T_1}(p)] \cap \Sigma' \backslash int(N) \neq \emptyset$ for every $p \in M' \backslash O_1$.

This is possible, since Σ' is a local Birkhoff section near γ' and coincides with Σ outside the biggest neighbourhood O_1. Then, since the neighbourhoods $O_1 \subset O_0$ are nested germs of a saddle type periodic orbit, we see that there exits some $T_2 > 0$ such that $[p, \psi_{T_2}(p)]$ is not contained in $O_0 \backslash O_1$, for every $p \in O_0 \backslash O_1$. Then, taking $T > \max\{T_0 + T_2, T_1 + T_2\}$, we deduce that $[p, \phi_T(p)] \cap \Sigma' \neq \emptyset$, for every $p \in N$. □

The proof of proposition 3.19 will take the rest of the section. We are going to show first in 4.1 that the surgery can be replaced for a more convenient surgery, placed in a *normal position*. Then, in 4.2 we will show how to construct the Birkhoff section B for this new surgery.

4.1 Replace the surgery for a normal surgery.

The flow (ϕ'_t, M') is obtained by a Goodman surgery supported on the annulus A with glueing homeomorphism f. The objective of this subsection is to show that we can replace the annulus A (and the homeomorphism f) for another annulus A_1 (and a corresponding glueing homeomorphism f_1) and obtain the same flow up to orbital equivalence. The advantage of this new annulus with respect to the first is that it will be better positioned in the neighbourhood W for the purpose of finding a local Birkhoff section. This will be the content of lemma 3.20.

Normal Coordinates:

Consider the vector field Y in $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ defined by $Y(x, y, s) = (-\lambda x, \lambda y, 1)$, where λ is some positive constant. The non-wandering set of the flow ϕ_t^Y generated by Y consists of one saddle type hyperbolic periodic orbit, namely, the curve $0 \times \mathbb{R}/\mathbb{Z}$. We recall that a *system of normal coordinates* for ϕ_t around γ is a local orbital equivalence

$$\Psi : (\phi_t, V)_\gamma \to (\phi_t^Y, V')_{0 \times \mathbb{R}/\mathbb{Z}}. \tag{3.12}$$

defined between a neighbourhood V of γ and a neighbourhood V' of $0 \times \mathbb{R}/\mathbb{Z}$. We remark that it always exists C^0 normal coordinates in a neighbourhood of a saddle type periodic orbit, but not necessarily C^1 normal coordinates.

Definition (Normal neighbourhood)**.** Choose a system of normal coordinates defined in an open neighbourhood of γ, as in (3.12). A *normal neighbourhood* of γ is a tubular neighbourhood of the form

$$N := \Psi^{-1}(\mathbb{T}_r), \tag{3.13}$$

where $\mathbb{T}_r = \{(x, y, s) : 0 \leq x^2 + y^2 \leq r^2\}$ is the solid torus of radius $r > 1$.

A normal neighbourhood N is a regular, compact, tubular neighbourhood of γ, contained in W. Observe that, in the case that the coordinate system is not differentiable, the boundary ∂N could be just a topological surface. It is foliated by the meridian disks $D_s = \mathbb{D} \times \{s\}$, $s \in \mathbb{R}/\mathbb{Z}$. If we set $r = 1$ then, in normal coordinates we have that:

(i) The boundary is $\partial N = \{(x, y, s) : x^2 + y^2 = 1\}$.

(ii) The meridians and the longitudes of ∂N are, respectively, the following families of curves

- $\alpha_s = \{(x, y, z) : x^2 + y^2 = 1,\ z = s\}$, with $s \in \mathbb{R}/\mathbb{Z}$;
- $\beta_\theta = \{(\cos(\theta), \sin(\theta), s) : s \in \mathbb{R}/\mathbb{Z}\}$, with $\theta \in [0, 2\pi]$.

(iii) The first quadrant of N is the set $N_1 = \{(x, y, s) : 0 \leq x^2 + y^2 \leq 1,\ 0 \leq x, y\ \}$.

(iv) Define $\partial N_1 \coloneqq \partial N \cap N_1$. The vector field Y is tangent to ∂N_1 along the longitude $\beta_{\pi/4}$. This longitude separates ∂N_1 into the following two regions, where Y points inward and outward respectively:

- $\partial N_1^{in} = \{(x, y, s) : x^2 + y^2 \leq 1,\ 0 < y < x < 1\}$,
- $\partial N_1^{out} = \{(x, y, s) : x^2 + y^2 \leq 1,\ 0 < x < y < 1\}$.

(v) It is possible to reparametrize ϕ_t in a neighbourhood of γ in such a way that, if $p \in N$ is a point with coordinates (x, y, s) and the orbit segment $\phi_{[0,t]}(p)$ is contained in N, then

$$\phi_t(x, y, s) = \phi_t^Y(x, y, s) = \left(e^{-\lambda t}x, e^{\lambda t}y, s + t\right).$$

We will always assume that ϕ_t has been reparametrized in order to satisfy this.

Replace the annulus A for another contained in ∂N.

Consider the flow (ϕ_t, M) from the beginning of this section. We will choose a fixed system of normal coordinates Ψ around γ, as described in (3.12), defined in an open neighbourhood that contains W, and which satisfies the following:

1. $\Psi(D) \subset \mathbb{R}^2 \times \{0\}$;
2. The solid torus $\mathbb{T}$ of radius one is contained in the interior of $\Psi(W)$.

As before we define $N = \psi^{-1}(\mathbb{T})$. Assume that the Goodman annulus A is contained in the interior of the first quadrant N_1. The positive ϕ_t-orbit of every point in A has a first intersection point with ∂N. Let $\rho : A \to (0, +\infty)$ be the function that assigns to

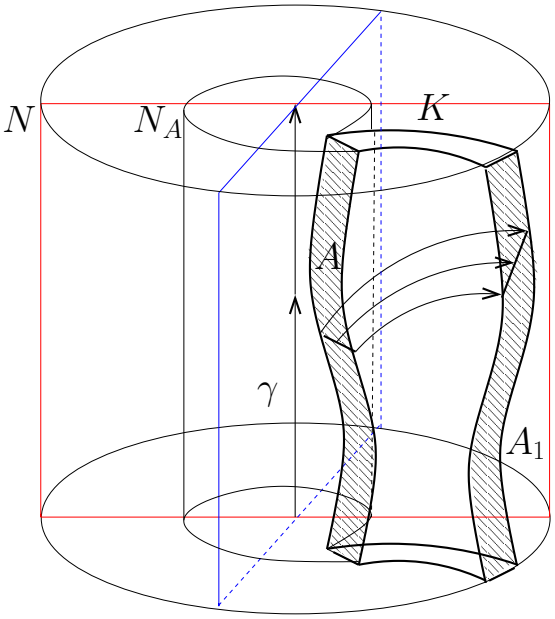


Figure 3.8: The ϕ_t-isotopy between A and A_1.

each point in A its time of first intersection with ∂N. We remark that, if the coordinate system is not differentiable, this function could be just continuous. The map

$$\psi : x \mapsto \phi_{\rho(x)}(x)$$

is then a ϕ_t-isotopy from the annulus A onto its image. The image of ψ is an annulus A_1 contained in ∂N. Observe that A_1 is (topologically) transverse to the flow and is contained in ∂N_1^{out}. See figure 3.8.

Normal Surgery.

Let (ϕ'_t, M') be a flow obtained by a Goodman surgery supported on A, as defined in (3.11). In the same way as in 4, the ϕ_t-isotopy allows to push forward the glueing map f onto a homeomorphism $f_1 : A_1 \to A_1$, defined by $f_1 = \psi \circ f \circ \psi^{-1}$. Observe that f_1 is the identity on the boundary of A_1. Consider the homeomorphism $\varphi_1 : \partial N \to \partial M_N$ defined by

$$\varphi_1(p) = \begin{cases} f_1(p) & \text{if} \quad p \in A_1 \\ \\ p & \text{if} \quad p \notin A_1. \end{cases}$$

where $M_N = M \backslash int(N)$ and we he have tautologically identified ∂M_N with ∂N. Since the flow ϕ_t is transverse to ∂N along the support of φ_1, then the quotient manifold

$$M'' := M_N \sqcup_{\varphi_1} N \tag{3.14}$$

is endowed with a (continuous) flow $\phi''_t : M'' \to M''$. It turns out that (ϕ''_t, M'') is orbitally equivalent to (ϕ'_t, M').

Definition (Surgery in normal position.)**.** A Goodman surgery is in *normal position* if it is constructed as in (3.14): it supported in an annulus contained in ∂N, where N is a normal neighbourhood.

Lemma 3.20. *The flows (ϕ'_t, M') and (ϕ''_t, M'') are orbitally equivalent. Moreover, there exists an orbital equivalence $H : M' \to M''$ such that*

$$H \circ \iota'(p) = \iota''(p), \ \forall \ p \in M_N,$$

where $\iota' : M_N \to M'$ and $\iota'' : M_N \to M''$ are the canonical inclusions.

Remark 3.21. If the system of normal coordinates (3.12) is C^1, then ∂N is a C^1 surface and the maps f_1 and φ_1 are also C^1. In this case, the flow (ϕ''_t, M'') is a C^1 flow and the orbital equivalence is C^1. But in general, the surgery (3.14) is just a C^0 surgery and the orbital equivalence with (ϕ'_t, M') is just C^0.

Proof. Consider the set

$$K = \{\phi_t(p) : p \in A \text{ and } 0 \leq t \leq \rho(p)\}.$$

The boundary of K can be decomposed into the annuli A and A_1 where the flow is transverse, and two other annuli where the flow is tangent. See figure 3.8.

Consider the two homeomorphisms $F, F_1 : \partial K \to \partial K$ defined by

$$F(p) = \begin{cases} f(p) & \text{if} \quad p \in A \\ p & \text{if} \quad p \notin A. \end{cases} \text{ and } F_1(p) = \begin{cases} f_1(p) & \text{if} \quad p \in A_1 \\ p & \text{if} \quad p \notin A_1. \end{cases}$$

Define $M_K = M \backslash int(K)$. Since the supports of F and F_1 are contained in the region where the flow is transverse to the boundary, there exist two induced flows

$$\left(\phi_t^F, M_K \sqcup_F K\right) \text{ and } \left(\phi_t^{F_1}, M_K \sqcup_{F_1} K\right).$$

Since f_1 is conjugated to f by a ϕ_t-isotopy, then proposition 4 implies that there exists an orbital equivalence h between these two flows, which in addition is equivariant with respect to the natural inclusions of M_K in each manifold. That is,

$$h \circ (M_K \hookrightarrow M_K \sqcup_F K) = (M_K \hookrightarrow M_K \sqcup_{F_1} K).$$

Observe that, since $\text{supp}(F) = \text{supp}(\varphi) = K \cap N_A = A$ and $F|_A = \varphi|_A = f$ then there exists an orbitally equivalence $h' : (\phi'_t, M') \to \left(\phi_t^F, M_K \sqcup_F K\right)$ which is equivariant with respect to the inclusions $M_A \hookrightarrow M'$ and $M_A \hookrightarrow M_K \sqcup_F K$.

Analogously, since $\text{supp}(F_1) = \text{supp}(\varphi_1) = K \cap N = A_1$ and $F_1|_{A_1} = \varphi_1|_A = f$ then there exists an orbitally equivalence $h'' : (\phi''_t, M'') \to \left(\phi_t^{F_1}, M_K \sqcup_{F_1} K\right)$ which is equivariant with respect to the inclusions $M_N \hookrightarrow M''$ and $M_N \hookrightarrow M_K \sqcup_{F_1} K$.

Finally, since $N_A \subset N$, the lemma follows by defining $H = (h'')^{-1} \circ h \circ h'$. □

4.2 Proof of proposition 3.19.

Following the previous subsection, up to orbital equivalence we can assume now that the flow (ϕ'_t, M') is constructed from a surgery placed in normal position. Since the equivalence given in lemma 3.20 commutes with the natural inclusions of the complement of the normal neighbourhood, then it suffices to prove 3.19 in this case. We proceed now to restate proposition 3.19 in a more convenient way.

For constructing the Birkhoff section it will be important to take care about the sign of the twist and the quadrant in which is placed the surgery. Therefore, for the rest of the section we will set the following:

(i) The neighbourhood N is oriented with our general convention stated in section 8.

(ii) The annulus A is contained in the first quadrant of N.

(iii) The twist $m \in \mathbb{Z}$ is positive, as stated in theorem 3.6.

We consider a flow ϕ'_t constructed by glueing M_N and N along their boundaries, using the homeomorphism $\varphi_1 : \partial N \to \partial M_N$ defined in (3.14), which has support in an annulus $A_1 \subset \partial N_1$. The neighbourhood N is equipped with a system of normal coordinates such that N corresponds to the solid torus or radius one and the local section D corresponds to a disk in $\mathbb{R}^2 \times 0$ that contains the unit disk in the interior. Observe that the intersection of ∂N with D is the meridian curve $\alpha_0 = \{(x, y, 0) : x^2 + y^2 = 1\}$, that separates D into the disk $\mathbb{D} \times \{0\}$ and an annulus $S = D \backslash \mathbb{D} \times 0$.

The curve α_0 is a meridian curve (boundary of a disk) when seen as a subset of ∂N, and is a boundary component of the surface S when seen as a subset of ∂M_N. As a subset of ∂M_N it is mapped onto a simple closed curve

$$\eta = \varphi_1^{-1}(\alpha_0) \tag{3.15}$$

by the glueing operation, as depicted on figure 3.9.

Proposition 3.19 will follow by showing that η is the boundary of a local Birkhoff section contained in N, as we state in the next proposition.

Proposition 3.22 (Restatement of 3.19.)**.** *There exists a local Birkhoff section* $B : [0, 1] \times \mathbb{R}/\mathbb{Z} \hookrightarrow N$ *such that* $B(0 \times \mathbb{R}/\mathbb{Z}) = 0 \times \mathbb{R}/\mathbb{Z}$ *and* $B(1 \times \mathbb{R}/\mathbb{Z}) = \eta$.

Since we set $m > 0$, the curve η has coordinates $(1, -m)$ in the meridian/longitude basis. From now on we will fix a parametrization $\eta : [0, 1] \to \partial N$ of the curve satisfying that $\eta(u) \in A_1$ iff $\varepsilon_0 \leq u \leq \varepsilon_1$, for some fixed $0 < \varepsilon_0 < \varepsilon_1 < 1$. The arc of η that lies inside A_1 is the image by f_1^{-1} of a meridian $A_1 \cap \alpha_0$, so it twists $-m$ times in the $\mathbb{R}/Z$-direction.

Let

$$\eta(u) = (\eta_1(u), \eta_2(u), \eta_3(u))$$

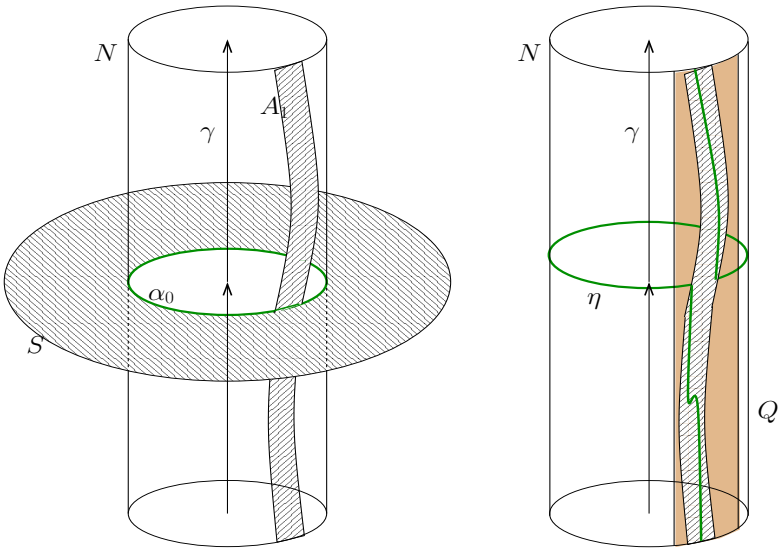


Figure 3.9: The curve η.

be the expression of this parametrization in normal coordinates. Since the coordinate system is not necessarily C^1 these coordinate functions could be just continuous. The following lemma asserts that, in any case, we can replace the curve η for another whose coordinates are C^1.

Lemma 3.23. *Let $0 < R < 1$. Then, there exists a C^1 simple closed curve $\mu \subset \partial\mathbb{T}_R$ and a properly embedded surface $L \subset \mathbb{T}\backslash\mathbb{T}_R$, homeomorphic to an annulus, transverse to the vector field Y, such that $\partial L = \mu \cup \eta$.*

Proof. Observe that, since the $\mathbb{R}/\mathbb{Z}$-component of the vector field Y equals one, there exists a compact annulus L embedded in $\mathbb{T}$, transverse to the flow lines, such that one boundary component is $\partial\mathbb{T} \cap L = \eta$ and the other boundary component is a curve $\mu \subset int(\mathbb{T})$. Without loss of generality we can assume that $\mu \subset \partial\mathbb{T}_R$. If the curve μ is not differentiable, it is possible to make a small perturbation of L by isotopies, fixing the boundary component η, and make the boundary component μ of class C^1. □

The previous lemma says that the surface S can be extended into a surface $S \cup L$, transverse to the flow, with one boundary component $\mu \subset \mathbb{T}_R$ which has coordinates of class C^1. Then, it suffices to prove proposition 3.22 assuming that the coordinates of the curve η are C^1.

In what follows we will make use of an auxiliary annulus $Q \subset \partial N$, constructed as follows: The compact annulus A_1 is contained in the open annulus $\partial N_1^{out} = \{(x, y, s) : x^2 + y^2 = 1,\ x \geq 0,\ x < y < 1\}$. So, there exists some $1 < \theta_0 < \theta_1$ such that the set

$$Q = Q(\theta_0, \theta_1) = \left\{(x, y, s) : x^2 + y^2 = 1,\ x \geq 0,\ \theta_0 x \leq y \leq \theta_1 x\right\} \tag{3.16}$$

is a compact annulus satisfying that $A_1 \subset int(Q)$ and $Q \subset \partial N_1^{out}$. Its boundary is the union of the longitudes β_{θ_0} and β_{θ_1} corresponding to the angles $\pi/4 < \arctan(\theta_0) < \arctan(\theta_1) < \pi/2$. See figure 3.9.

We say that η is monotonous in the $\mathbb{R}/\mathbb{Z}$-component if $\eta_3'(u) < 0$, for every $u \in (\varepsilon_0, \varepsilon_1)$. We remark that, in general, it is not possible to assume that η is monotonous in the $\mathbb{R}/\mathbb{Z}$-component. Even if the original Dehn twist $f : A \to A$ is monotonous in the longitude component, the projection of the surgery over the annulus A_1 may produce a non-monotonous Dehn twist f_1.

We divide the proof two cases: (i) We prove 3.22 assuming that η is monotonous; (ii) We prove the general case.

Case 1: η is monotonous in the $\mathbb{R}/\mathbb{Z}$-component.

We assume that the coordinates $(\eta_1(u), \eta_2(u), \eta_3(u))$ of the parametrization $\eta : [0,1] \to N$ satisfy that:

1. $\eta_3'(u) < 0$, $\forall\ \varepsilon_0 < u < \varepsilon_1$ and $\eta_3'(\varepsilon_0) = \eta_3'(\varepsilon_1) = 0$;
2. $\eta_3(u) = 0$, if $u \in [0, \varepsilon_0] \cup [\varepsilon_1, 1]$.

The proof is done in three steps, as follows:

Step (i) - The surface H_R: For some small $R > 0$ we will find a local Birkhoff section H_R, contained in the solid torus $\mathbb{T}_R$, and whose exterior boundary is isotopic to η.

Step (ii) - The region K: We will define a special region $K \subset \mathbb{T}_1$ that is trivially foliated by the flow.

Step (iii) - Connect S with H_R: We will connect the boundary of H_R with the curve η, using a band transverse to the flow and contained in $\mathbb{T}\backslash\mathbb{T}_R$. For doing this we are going to make use of the region K previously defined.

We develop these steps in what follows.

Step (i): The surface H_R.

We will define a 1-parameter family of maps $T_r : N \to N$, $0 \leq r \leq 1$, satisfying that

- $T_1 = \mathrm{id}$, $T_0(N) = \gamma$ and T_r is a diffeomorphism onto its image $\forall\ r > 0$,
- $T_{r_1}(N) \subset T_{r_2}(N)$ if $r_1 < r_2$.

Moreover, it will be verified that $T_r(N) \subset \mathbb{T}_r$, for every r. For this purpose, choose some fixed angle ρ such that $\arctan(\theta_0) < \rho < \arctan(\theta_1)$ and consider the unitary vectors $v, w \in \mathbb{R}^2$ defined as

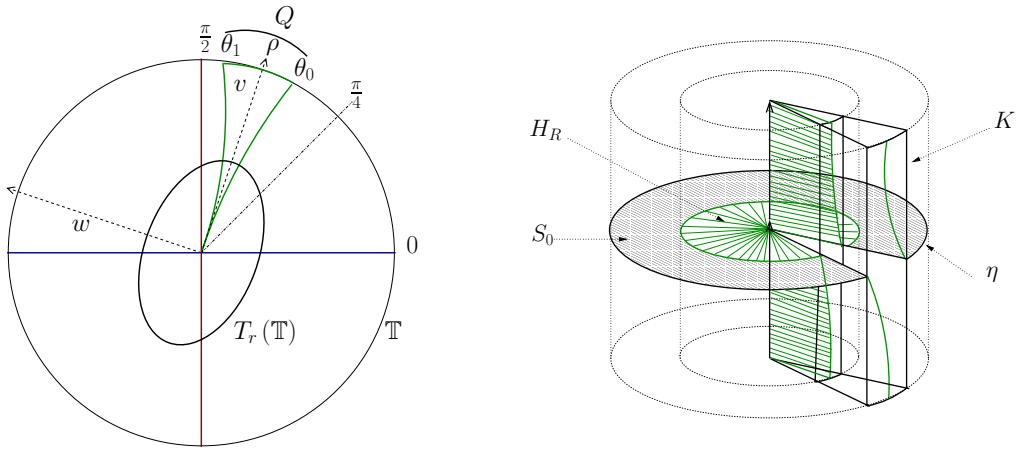
$$v = (\cos(\rho), \sin(\rho))$$
$$w = (-\sin(\rho), cos(\rho)).$$

The vector w is obtained by a rotation of angle $\pi/2$ of v. We can define a system of coordinates in $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ in the following way

$$\Psi_\rho : \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z} \text{ such that } \Psi_\rho : ((av + bw), s) \mapsto (a, b, s). \tag{3.17}$$

Definition. Given $0 \leq r \leq 1$ we define $T_r : \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ to be the map given in coordinates (3.17) by

$$T_r(a, b, s) = (ra, r^2 b, s). \tag{3.18}$$



(a) The homeomorphism T_r in the xy-plane. (b) The surface H_R.

Figure 3.10:

Observe that T_r sends the disk $\mathbb{D} \times \{0\}$ into an ellipse contained in its interior, whose major and minor axes are collinear with v and w, respectively. See figure 3.10a. The properties of T_r stated in the previous items follow directly from the definition.

Definition. For every $R > 0$ we define the surface

$$H_R = \{T_r \circ \eta(u) : 0 \leq r \leq R,\ u \in [0, 1]\}. \tag{3.19}$$

Each surface H_R is constructed as the image of a 1-parameter family of isotopies applied to the curve $T_R(\eta)$, as in figure 3.10b.

Lemma 3.24. *There exists $R_0 > 0$ such that the surface H_R is a local Birkhoff section at γ, for every $R \leq R_0$.*

Proof. In the previous paragraphs we have set a parametrization $\eta : [0,1] \to \partial N$ with normal coordinates $n(u) = (\eta_1(u), \eta_2(u), \eta_3(u))$. Now we will denote by

$$\nu(u) = (\nu_1(u), \nu_2(u), \nu_3(u)) \tag{3.20}$$

to the expression of this parametrization in coordinates (3.17). We recall that $\eta_3(u) = \nu_3(u)$, $\forall\ u \in [0,1]$ and $\nu(u) \in A_1$ if and only if $u \in [\varepsilon_0, \varepsilon_1]$.

The surface H_R is parametrized in coordinates (3.17) with the map $[0,R] \times [0,1] \to H_R$ given by

$$(r,u) \mapsto (r\nu_1(u), r^2\nu_2(u), \nu_3(u)).$$

Consider a point $p \in H_R$. The tangent space of H_R at p is generated by the vectors

$$\begin{aligned}\frac{\partial}{\partial r}(p) =& (\nu_1(u), 2r\nu_2(u), 0)\\ \frac{\partial}{\partial u}(p) =& (r\nu_1'(u), r^2\nu_2'(u), \nu_3'(u))\end{aligned}$$

The vector field Y is given by $Y(x,y,s) = (-\lambda x, \lambda y, 1)$ in normal coordinates. We need to express it in the coordinate system (3.17). Observe that the change from the (a,b,s)-coordinates to the (x,y,s)-coordinates is the rotation induced by the matrix

$$\begin{pmatrix} \cos(\rho) & -\sin(\rho) & 0\\ \sin(\rho) & \cos(\rho) & 0\\ 0 & 0 & 1\end{pmatrix}.$$

So the $\mathbb{R}^2$-component of Y in the (a,b,s)-coordinate system is given by the matrix

$$\begin{pmatrix} \cos(\rho) & \sin(\rho)\\ -\sin(\rho) & \cos(\rho)\end{pmatrix}\begin{pmatrix} -\lambda & 0\\ 0 & \lambda\end{pmatrix}\begin{pmatrix} \cos(\rho) & -\sin(\rho)\\ \sin(\rho) & \cos(\rho)\end{pmatrix} = \lambda\begin{pmatrix} \sin^2(\rho)-\cos^2(\rho) & 2\cos(\rho)\sin(\rho)\\ 2\cos(\rho)\sin(\rho) & \cos^2(\rho)-\sin^2(\rho)\end{pmatrix}$$

Let's set $\lambda_1 = (\sin^2(\rho) - \cos^2(\rho))\lambda$ and $\lambda_2 = 2\cos(\rho)\sin(\rho)\lambda$, which are positive constants. Then, in the coordinate system (3.17) the vector field Y is written as

$$Y(a,b,s) = (\lambda_1 a + \lambda_2 b, \lambda_2 a - \lambda_1 b, 1). \tag{3.21}$$

We want to check that for small values of $r > 0$, the vectors $\frac{\partial}{\partial r}(p)$, $\frac{\partial}{\partial u}(p)$ and $Y(p)$ are linearly independent. It is equivalent to check that the following quantity does not vanish:

$$\begin{vmatrix} \nu_1(u) & 2r\nu_2(u) & 0\\ r\nu_1'(u) & r^2\nu_2'(u) & \nu_3'(u)\\ (\lambda_1 r\nu_1(u) + \lambda_2 r^2\nu_2(u)) & (\lambda_2 r\nu_1(u) - \lambda_1 r^2\nu_2(u)) & 1\end{vmatrix} =$$

$$= r\left[-\nu_3'(u)\left(\lambda_2\nu_1^2(u) - 3\lambda_1 r\nu_1(u)\nu_2(u) - 2\lambda_2 r^2\nu_2^2(u)\right) + r\left(\nu_1(u)\nu_2'(u) - 2\nu_2(u)\nu_1'(u)\right)\right]. \tag{3.22}$$

Consider the set of points

$$S_0 = \{T_r \circ \nu(u) : 0 \leq r \leq 1 \ u \notin (\varepsilon_0, \varepsilon_1)\}. \tag{3.23}$$

Since $\nu_3(u) = 0$ this is a surface contained in $\mathbb{R}^2 \times \{0\}$ and is clearly transverse to the vector field Y, independently of r. Anyway, to see that the quantity between brackets in (3.22) does not vanish on the points $p \in S_0$, observe that it reduces to

$$r\left(\nu_1(u)\nu_2'(u) - 2\nu_2(u)\nu_1'(u)\right).$$

The curve $\nu : [0, \varepsilon_0] \cup [\varepsilon_1, 1] \to \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ is an arc contained in the unit circle of $\mathbb{R}^2$, so it admits some reparametrization of the form $u \mapsto (\cos(Au + B), \sin(Au + B), 0)$. It follows that

$$\nu_1(u)\nu_2'(u) - 2\nu_2(u)\nu_1'(u) = A\cos^2(Au + B) + 2A\sin^2(Au + B) \neq 0.$$

It is possible to enlarge a little the surface S_0 in the following way: Since Y is transverse to S_0 on the boundary points $T_r(\nu(\varepsilon_i))$ and $Y(p)$ forms a positive angle with T_pS_0, there exist $\varepsilon_0 < \varepsilon_0' < \varepsilon_1' < \varepsilon_1$ such that the set

$$S_0' = \{T_r \circ \nu(u) : 0 \leq r \leq 1, \ u \notin (\varepsilon_0', \varepsilon_1')\}$$

is also a surface transverse to Y.

Assume now that $p = T_r \circ \nu(u)$ with $\varepsilon_0' \leq u \leq \varepsilon_1'$. We will see that there exists R_0 such that if $r < R_0$ then $Y(p)$ is transverse to the surface H_{R_0}. We will examine the quantity between brackets in (3.22). First, observe that since $\nu_1(u) > 0$ for every $\varepsilon_0' \leq u \leq \varepsilon_1'$, then there exists $R_1 > 0$ such that

$$\lambda_2\nu_1^2(u) - 3\lambda_1 r\nu_1(u)\nu_2(u) - 2\lambda_2 r^2\nu_2^2(u) \geq \kappa_1 > 0$$

for every $r < R_1$, for some constant κ_1. Let $\kappa_2 = \max\{|\nu_1(u)\nu_2'(u) - 2\nu_2(u)\nu_1'(u)| : u \in [\varepsilon_0', \varepsilon_1']\}$. By the monotonicity of η, it follows that $\nu_3'(u) \leq \kappa_3 < 0$ for every $\varepsilon_0' < u < \varepsilon_1'$, for some constant κ_3. Then (3.22) is bounded from below by

$$-\nu_3'(u)\kappa_1 - r\kappa_2$$

and it does not vanish if $r < R_0$ where

$$R_0 < \min\left\{R_1, \frac{-\kappa_3\kappa_1}{\kappa_2}\right\}.$$

The parametrization $[0, R_0] \times [0, 1] \to H_{R_0}$ given by $(r, u) \mapsto T_r \circ \nu(u)$ is the immersion of an annulus with boundary components $0 \times \mathbb{R}/\mathbb{Z}$ and $T_{R_0}(\nu)$, which is an embedding outside $0 \times \mathbb{R}/\mathbb{Z}$, and is contained in $\mathbb{T}_{R_0}$. Since the exterior boundary is homotopic to $\eta \subset N$ this immersion has linking number equal to 1 and multiplicity equal to m (the twist parameter of φ_1). Since it is transverse to Y in the complement of $0 \times \mathbb{R}/\mathbb{Z}$ it follows that it is a local Birkhoff section. □

This completes the construction of the surface H_R near γ.

Step (ii) - The region K.

Consider the annulus Q defined in (3.16), delimited by two longitudes β_0 and β_1. Define

$$K = \{T_r(p) : 0 \leq r \leq 1,\ p \in Q\}. \tag{3.24}$$

The region K is a manifold with edges in the boundary. The boundary consists of the union of three annuli $L_i = \bigcup_{0\leq r\leq 1} T_r(\beta_i)$, $i = 0, 1$ and Q, while the edges are the curves γ, β_0 and β_1. See figure 3.11a.

Define $\mathring{L}_i = L_i \backslash \gamma$. The vector field Y is transverse to $\mathring{L}_0$, $\mathring{L}_1$ and Q. It points inward the region K along $\mathring{L}_0$ and points outward along $\mathring{L}_1 \cup Q$.

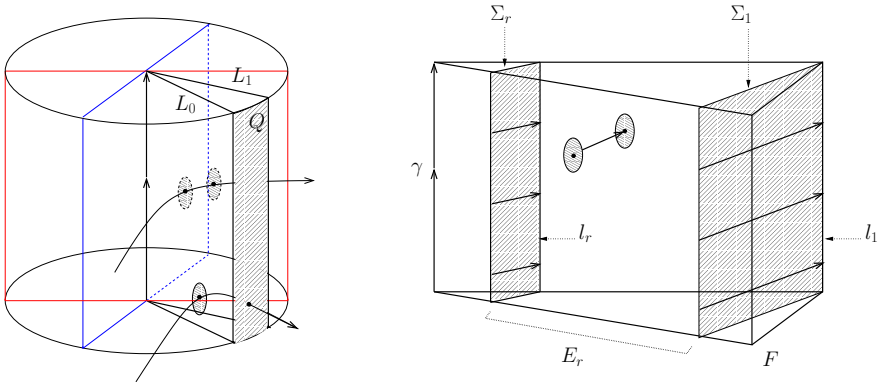
Claim. For every $p \in K\backslash\gamma$, there exist $\tau_0(p) \leq \tau_1(p)$ such that the orbit of p enters the region K through a point $\phi(p, \tau_0(p))$ in $\mathring{L}_0$, and escapes through a point $\phi(p, \tau_1(p))$ in $\mathring{L}_1 \cup Q$. In fact, $\tau_0(p) < \tau_1(p)$ unless p is in the edge β_0.

This claim follows directly by examining the vector field Y on the set K. It allows to decompose $K\backslash\gamma$ as the union of two sets E and F, where

(i) E is the union of the orbit segments $\{\phi(p, t) : \tau_0(p) \leq t \leq 0\}$ where $p \in \mathring{L}_1$,

(ii) F is the union of the orbit segments $\{\phi(p, t) : \tau_0(p) \leq t \leq 0\}$ where $p \in Q$.

These two sets are foliated by orbit segments[4], and they intersect along a surface Σ_1 tangent to the flow. See figure 3.11b.



(a) (b) The sets E_r and F.

Figure 3.11: The region K

For each $0 < r \leq 1$ we will consider the closed curve

$$l_r = L_1 \cap \mathbb{T}_r. \tag{3.25}$$

[4]In fact, for the set F the foliation singularizes on the edge $L_0 \cap Q$, but we will not take care about this.

Then the family of curves $\{l_r : 0 < r \leq 1\}$ constitutes a foliation of $\mathring{L}_1$ by longitudes. If we push l_r using the negative action of the flow we can define a surface Σ_r in the following way

$$\Sigma_r = \{\phi(p,t) : p \in l_r \text{ and } \tau_0(p) \leq t \leq 0\} \tag{3.26}$$

Then, the family of surfaces $\{\Sigma_r : 0 < r \leq 1\}$ constitutes a foliation of E. For each $r < 1$ the surface Σ_r separates E into two components. We will define

$$E_r = \text{ the closure of the connected component of } E\backslash\Sigma_r \text{ which does not accumulate on } \gamma. \tag{3.27}$$

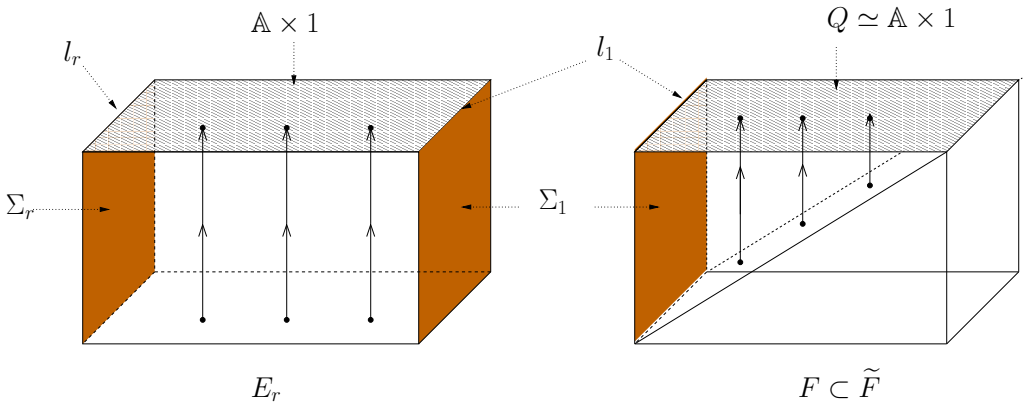


Figure 3.12: Foliations by orbit segments in E_r and F.

The sets E_r and F are depicted in figure 3.11b. Denote by $\mathcal{O}_{E_r}$ and $\mathcal{O}_F$ to the foliations by orbit segments induced on E_r and F, respectively. Denote $\mathbb{A} = [0,1]\times\mathbb{R}/\mathbb{Z}$. Then we have the following,

Claim. Consider the foliated sets $(E_r, \mathcal{O}_{E_r})$ and $(F, \mathcal{O}_F)$. Then,

1. $(E_r, \mathcal{O}_{E_r})$ is isomorphic to $[0,1] \times \mathbb{A}$ equipped with the foliation by segments $[0,1] \times \{p\}$.
2. $(F, \mathcal{O}_F)$ is isomorphic to the set $\{(x,y,s) : x \geq y\} \subset [0,1] \times \mathbb{A}$ equipped with the restriction of the foliation by segments $[0,1] \times \{p\}$.

This claim follows from the previous claim. For the set F observe that all the points p in the edge β_1 hit L_0 in some constant time $\tau_0 < 0$. Define

$$\widetilde{F} = \{\phi(p,t) : p \in Q, \tau_0 \leq t \leq 0\}. \tag{3.28}$$

Then there is a homeomorphism from $\left(\widetilde{F}, \mathcal{O}_{\widetilde{F}}\right)$ to $[0,1]\times\mathbb{A}$ which takes Q onto $\{1\}\times\mathbb{A}$, Σ_1 onto $[0,1] \times \{0\} \times \mathbb{R}/\mathbb{Z}$ and sends the foliation by orbit segments onto the foliation by segments of the form $(x,y) \times [0,1]$. Since the function $\tau_0(p)$ is constant along the longitudes in Q and approaches to zero when $p \to \beta_0$ then it follows item (2) from the claim. See figure 3.12.

Step (iii) - Connecting S with H_{R_0}.

The goal now is to connect the surface S, that lies outside $\mathbb{T}_1$, with a surface H_R for some adequate $0 < R < R_0$, that lies inside the solid torus $\mathbb{T}_{R_0}$. We will make the following choices of constants:

1. Choose R_0 given by lemma 3.24.
2. Choose $R_1 < R_0$ such that $\Sigma_{R_1} \subset int\,(\mathbb{T}_{R_0})$.
3. Choose $R_2 < R_1$ such that $\Sigma_{R_1} \subset int\,(\mathbb{T}_{R_0} \backslash \mathbb{T}_{R_2})$.

Consider the surface H_{R_2}. In the region $N \backslash int(K)$ we define

$$S_0 = \left\{(x, y, 0) : x^2 + y^2 \leq 1,\ (x, y, 0) \notin K \cup T_{R_2}(N)\right\}. \tag{3.29}$$

Then S_0 has the shape of a band and is transverse to the flow. This surface extends H_{R_2} in the complement of K and connects the arc $\eta \backslash Q$ with the arc $\partial H_{R_2} \backslash K$. We refer again to figure 3.10b.

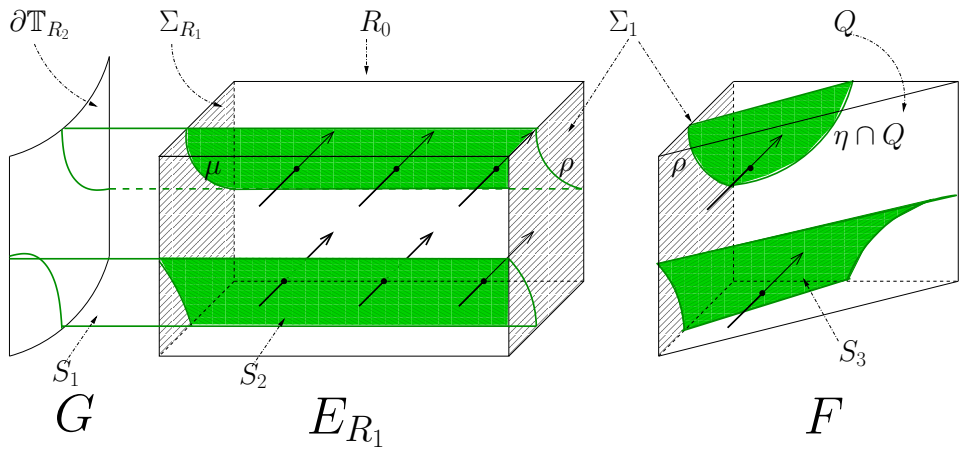


Figure 3.13: Connecting H_{R_2} with S inside K.

Now, we will connect the arcs $T_{R_2}(\eta \cap Q)$ with $\eta \cap Q$ using three bands S_1, S_2 and S_3, contained in K and transverse to the flow. We will complement the following explanation with figure 3.13. Observe that the boundary points of the arcs are already connected by segments in the boundary of S_0. Recall that we have a decomposition of K as the union $(E \backslash E_{R_1}) \cup E_{R_1} \cup F$.

The surface $T_{R_2}(Q)$ is an annulus that disconnects $E \backslash E_{R_1}$. Let's call G to the connected component that contains Σ_1 and $T_{R_2}(Q)$ as opposite faces. Let's define

$$S_1 = H_{R_0} \cap G. \tag{3.30}$$

Then, S_1 is a band that connects the arc $T_{R_2}(\eta \cap Q) \subset T_{R_2}(Q)$ with another arc $\mu = H_{R_0} \cap \Sigma_{R_1} \subset \Sigma_{R_1}$, and is transverse to the flow lines. See figure 3.13.

Recall that $E_{R_1} \simeq [0,1] \times \mathbb{A}$ with a homeomorphism that sends the orbit segments in E_{R_1} onto the segments of the form $[0,1] \times p$. In this correspondence we have that $\Sigma_{R_1} \simeq [0,1] \times \{0\} \times \mathbb{R}/\mathbb{Z}$ and $\Sigma_1 \simeq [0,1] \times \{1\} \times \mathbb{R}/\mathbb{Z}$. The curve μ is contained in Σ_{R_1} and is transverse to the orbit segments that foliates this surface. This implies that there exists a band S_2 contained in E_{R_1}, transverse to the flow, that connects the two faces Σ_{R_1} and Σ_1, such that $S_2 \cap \Sigma_{R_1} = \mu$. The surface S_2 is defined to be

$$S_2 \simeq \{(x,y,s) \in [0,1] \times \mathbb{A} : 0 \leq y \leq 1,\ (x,0,s) \in \mu\}. \tag{3.31}$$

It remains to see what happens in the region F. Let's call ρ to the arc $\rho = S_2 \cap \Sigma_1$. We have to find a surface S_3 contained in F that connects ρ with the arc $\eta \cap Q$, being at the same time transverse to the orbit segments. For that we will consider the auxiliary region $\widetilde{F} \simeq [0,1] \times \mathbb{A}$ defined in (3.28). In this region, the set F corresponds to the *prism* defined by $\{(x,y,s) : x \geq y\} \subset [0,1] \times [0,1] \times \mathbb{R}/\mathbb{Z}$ as in figure 3.12. The surfaces Σ_1 and Q correspond to the faces $[0,1] \times \{0\} \times \mathbb{R}/\mathbb{Z}$ and $\{1\} \times [0,1] \times \mathbb{R}/\mathbb{Z}$, respectively. Without loss of generality, it is possible to assume that:

- the meridians of the annulus Q (recall they are defined by the intersections $Q \cap \mathbb{D} \times \{s\}$ with $s \in \mathbb{R}/\mathbb{Z}$) correspond to the segments $\{1\} \times [0,1] \times \{s\}$,
- the orbit segments contained in Σ_1 correspond to the segments $[0,1] \times \{0\} \times \{s\}$.

Choose a parametrization $\rho : [a,b] \to \Sigma_1 \simeq [0,1] \times \{0\} \times \mathbb{R}/\mathbb{Z}$ of the form $\rho(u) = (\rho_1(u), 0, \rho_3(u))$. Choose choose a parametrization $\delta : [a,b] \to Q \simeq \{1\} \times [0,1] \times \mathbb{R}/\mathbb{Z}$ of the arc $\eta \cap Q$, of the form $\delta(u) = (1, \delta_2(u), \delta_3(u))$. Define the surface

$$\widetilde{S}_3 = \{\rho(u) + t\,(\delta(u) - \rho(u)) : 0 \leq t \leq 1,\ u \in [0,1]\}. \tag{3.32}$$

This surface is depicted in figure 3.14. By the previous assumptions, we have that $\rho_3, \delta_3 : [a,b] \to \mathbb{R}/\mathbb{Z}$ are monotonous functions. The monotonicity of these functions imply that $\widetilde{S}_3$ is transverse to the foliation by segments $[0,1] \times \{p\}$. Finally, the surface $\widetilde{S}_3$ can be pushed with a ϕ_t-isotopy (fixing ρ and δ) onto a surface S_3 contained in the region F and transverse to the orbit segments.

This completes the proof of lemma 3.22 assuming that η is monotonous in the $\mathbb{R}/\mathbb{Z}$-component.

Case 2: η is not monotonous.

In the general case, when the curve η is not monotonous in the $\mathbb{R}/\mathbb{Z}$-coordinate, we have the following lemma, that allows to reduce the construction to the previous case:

Lemma 3.25. *For some $0 < R < 1$, there exists a surface S properly embedded in $\mathbb{T} \backslash \mathbb{T}_R$, transverse to the flow, such that*

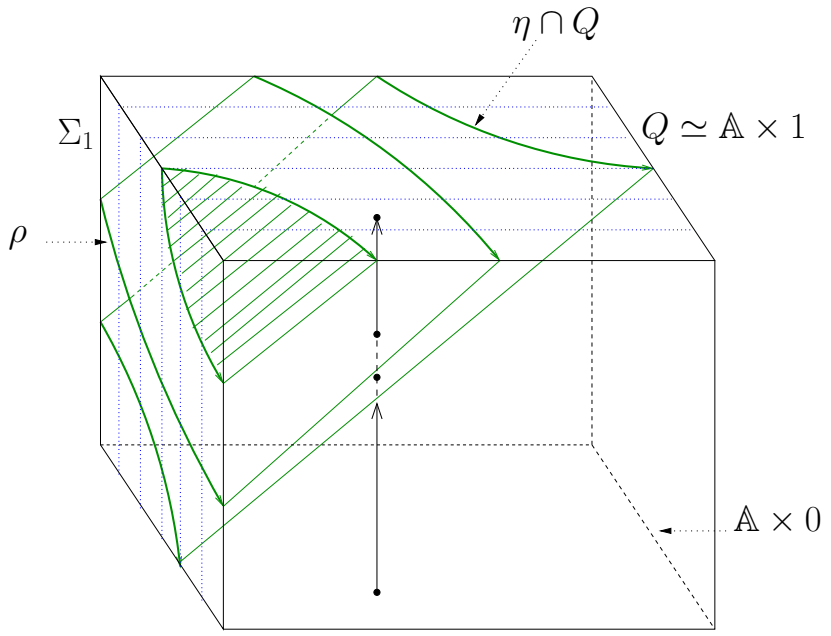


Figure 3.14: The surface $\widetilde{S}_3 \subset \widetilde{F}$.

1. $\partial S = (S \cap \partial \mathbb{T}_R) \cup (S \cap \partial \mathbb{T}) = \mu \cup \eta$,
2. *The curve μ is monotonous in the $\mathbb{R}/\mathbb{Z}$-direction.*

Proof. First of all, observe that if $0 < R < 1$ is sufficiently close to 1, then

$$\forall\ p \in Q,\ \exists\ \rho(p) < 0 \text{ such that } \phi_{\rho(p)}(p) \in \partial \mathbb{T}_R$$

and all the orbit segment $\{\phi_t(p) : \rho(p) \leq t \leq 0\}$ is contained in $\mathbb{T} \backslash \mathbb{T}_R$. So we will fix some R satisfying this condition. Consider the set

$$V = \{\phi_t(p) : p \in Q \text{ and } \rho(p) \leq t \leq 0\}. \tag{3.33}$$

This set is depicted in figure 3.15a. The boundary of V can be decomposed into the union of two annuli transverse to the flow and two tangent annuli. The annuli transverse to the flow are Q and

$$Q_R = \{\phi_{\rho(p)}(p) : p \in Q\} \subset \partial \mathbb{T}_R.$$

The proof will consist in:

(i) Construct a surface $B \subset V$ homeomorphic to a band, transverse to the flow, that connects $\eta \cap Q$ with a curve $\mu \subset Q_R$ that cuts the meridians of Q_R in a monotonous way;

(ii) By adding a band B' contained in $(\mathbb{T} \backslash \mathbb{T}_R) \cap V^c$ and transverse to the flow, we obtain the desired surface $S = B \cup B'$.

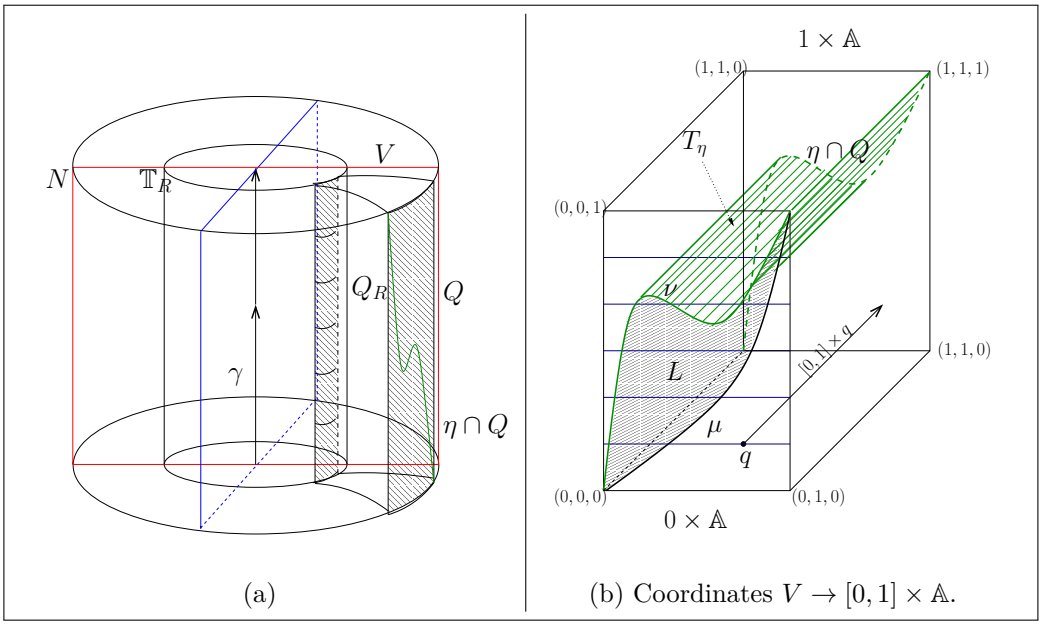


Figure 3.15: The set V.

We develop these steps below:

Step (i):

The annulus Q_R is delimited by two vertical longitudes of the torus $\partial\mathbb{T}_R$, so there exists a parametrization $\mathbb{A} = [0,1] \times \mathbb{R}/\mathbb{Z} \to Q_R$ of the form $(u,s) \mapsto (R\cos(Au + B), R\sin(Au+B), s)$. In the coordinates defined by this parametrization each meridian $Q_R \cap \mathbb{D} \times \{s\}$ corresponds to the segment $[0,1] \times \{s\}$.

For every $q = \phi_{\rho(p)}(p) \in Q_R$ define $\tau(q) = -\rho(p)$. Consider the parametrization $\Psi_R : [0,1] \times \mathbb{A} \to V$ defined by Ψ

$$\Psi_R(t,u,s) = \phi_{t\tau(u,s)}(R\cos(Au + B), R\sin(Au + B), s).$$

In the coordinates defined by this parametrization Q_R corresponds to $0 \times \mathbb{A}$, Q corresponds to $1 \times \mathbb{A}$, and $\forall\ q \in Q_R$ the orbit segment $\{\phi_t(q) : 0 \le t \le \tau(q)\}$ corresponds to the segment $[0,1] \times q$.

Let's consider a parametrization $\eta : [a,b] \to Q$ of the arc $\eta \cap Q$. In the coordinates defined by Ψ_R, this curve has an expression of the form $\eta(u) = (1, \nu_2(u), \nu_3(u)) \in 1 \times \mathbb{A}$. Consider the curve $\nu \subset 0 \times \mathbb{A}$ defined by $\nu(u) = (0, \nu_2(u), \nu_3(u))$, $a \le u \le b$, and consider the surface

$$T_\eta = \{(t, \nu_2(u), \nu_3(u)) : 0 \le t \le 1,\ a \le u \le b\}.$$

This surface is a band tangent to the flow, that connects the segments η and ν. See figure 3.15b.

First case: We will assume that $\nu \subset Q_R \simeq [0,1] \times \mathbb{R}/\mathbb{Z}$ satisfies the following:

(i) $\nu(a) = (0,0)$, $\nu(b) = (1,0)$;

(ii) the segment ν twist once in the $\mathbb{R}/\mathbb{Z}$ direction,

(iii) $\nu \cap ([0,1] \times \{0\}) = \{\nu_0(a), \nu_0(b)\}$.

Thus we assume that ν looks like the green curve in the front face $0 \times \mathbb{A}$ of picture 3.15b. Consider the universal cover $\widetilde{Q}_R = [0,1] \times \mathbb{R} \to Q_R = [0,1] \times \mathbb{R}/\mathbb{Z}$. For every $k \in \mathbb{Z}$ let $\widetilde{\nu}_k : [a,b] \to \widetilde{Q}_R$ be a lift of ν such that $\widetilde{\nu}_k(a) = (0,k)$. Condition (ii) says that $\widetilde{\nu}_k(b) = (1,k+1)$. Let $\mathcal{R}_k \subset \widetilde{Q}_R$ be the compact band delimited by the two curves $\widetilde{\nu}_k$ and $\widetilde{\nu}_{k+1}$.

Claim. There exists a curve $\widetilde{\mu} : [a,b] \to \mathcal{R}_0$ satisfying that

(a) $\widetilde{\mu}(a) = \widetilde{\nu}_0(a) = (0,0)$, $\widetilde{\mu}(b) = \widetilde{\nu}_0(b) = (1,1)$;

(b) $\widetilde{\mu} \cap \widetilde{\nu}_0 = \{(0,0),(1,1)\}$;

(c) $\widetilde{\mu}_3'(u) > 0$, $\forall\ a < u < b$.

To see this, observe that condition (iii) implies that each $\widetilde{\nu}_k$ is entirely contained the fundamental domain $[0,1] \times [k,k+1]$ of $\widetilde{Q}_R$. Then the segment $\widetilde{\nu}_0$ divides the square $[0,1] \times [0,1]$ in a top and a bottom triangle, and in each of them there exists a curve $\widetilde{\mu}$ satisfying the claim. See figure 3.16a. We make a comment on which of the two possibilities must be chosen at the end of this proof.

Since $\widetilde{\mu}$ and $\widetilde{\nu}_0$ intersect just in its extremities, there exists a surface $\widetilde{L} \subset [0,1] \times [0,1]$ delimited by these two curves. Let $\mu, L \subset Q_R$ be the corresponding projections of $\widetilde{\mu}$ and $\widetilde{L}$.

Consider the surface $B^{\perp} \subset V$ defined as the union $B^{\perp} = L \cup T_\eta$. Then, $B^{\perp}$ is a surface, homeomorphic to a band, that connects $\eta \cap Q$ with μ. It is transverse to the flow along L and is tangent along the band T_η that connects $\eta \cap Q$ with ν, as depicted in 3.15b. Observe that in some small flow-box neighbourhood of T_η, it is possible to perturb this surface fixing the side $\eta \cap Q$ and make it transverse to the flow lines. So, by a small perturbation of $B^{\perp}$ it is possible to find a band $B \subset V$ transverse to the flow, that connects $\eta \cap Q$ with μ, and the curve μ cuts the meridians of Q_R in a strictly monotonous way.

Finally, observe that there are two possibilities for selecting $\widehat{\mu}$. There is only one of them that makes the union of the band B with the surface that lies outside $\mathbb{T}$ to be transverse to the flow lines.

Second case: We will assume now that the twist number is still one (condition (ii) of the previous case), but the lifts $\widetilde{\nu}_k$ are not necessarily contained in a fundamental

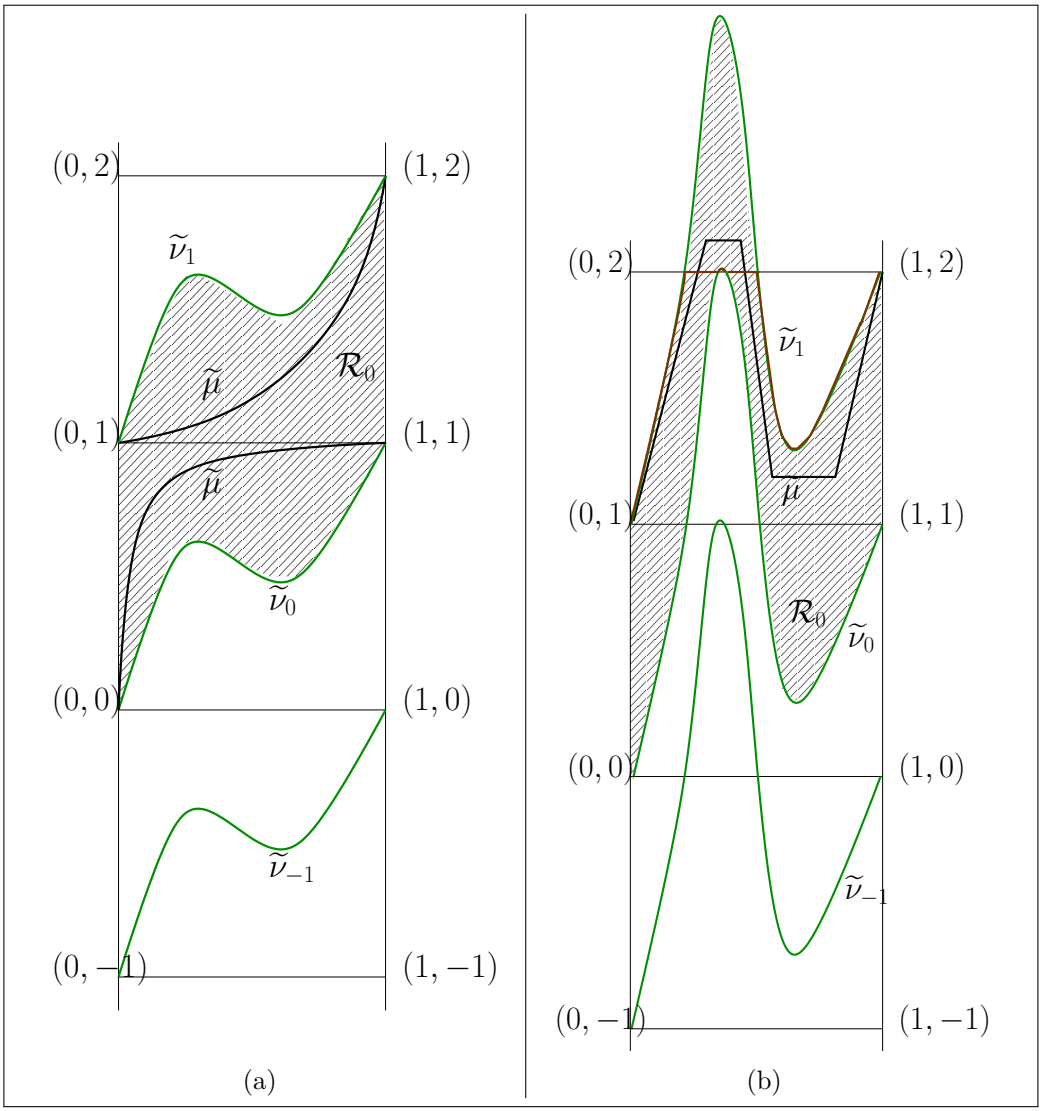


Figure 3.16: Choosing an adequate boundary.

domain $[0,1] \times [k, k+1]$ (condition (iii)). In this case, the previous claim is no longer true. Given a lift $\widetilde{\nu}_k$ we define

$$\begin{aligned} \mathrm{s}(\widetilde{\nu}_k) &= \max_{n\in\mathbb{Z}}\{n : \widetilde{\nu}_k(u) \in [0,1] \times [n, n+1]\} \\ \mathrm{i}(\widetilde{\nu}_k) &= \min_{n\in\mathbb{Z}}\{n : \widetilde{\nu}_k(u) \in [0,1] \times [n, n+1]\}. \end{aligned}$$

Observe that the quantity $\mathrm{a}(\nu) := \mathrm{s}(\widetilde{\nu}_k) - \mathrm{i}(\widetilde{\nu}_k)$ is independent of the lift $\widetilde{\nu}_k$.

Claim. There exists a curve $\widetilde{\mu} : [a,b] \to \mathcal{R}_0$ satisfying that

(a) $\widetilde{\mu}(a) = \widetilde{\nu}_i(a) = (0,i)$, $\widetilde{\mu}(b) = \widetilde{\nu}_i(b) = (0, i+1)$ for some $i = 0, 1$;

(b) $\widetilde{\mu} \cap \widetilde{\nu}_i = \{(0,i), (0,i+1)\}$;

(c) $\mathrm{s}(\widetilde{\mu}) - \mathrm{i}(\widetilde{\mu}) < \mathrm{s}(\widetilde{\nu}_i) - \mathrm{i}(\widetilde{\nu}_i)$.

Proof of the Claim. Consider a segment constructed as a concatenation of of segments in $\widetilde{\nu}_0$, $\widetilde{\nu}_1$ and horizontal segments, exactly as shown in figure 3.16b. Then, a small perturbation of this segment gives the desired curve. The way how the union of segments were chosen implies that the quantity a is strictly smaller. □

Exactly as in the first case, the curves $\widetilde{\mu}$ and $\widetilde{\nu}_i$ enclose a surface $\widetilde{L}$ contained in the region $\mathcal{R}_0$, that projects into a surface $L \subset Q_R$. We can proceed analogously and, by deforming $B^{\perp} = L \cup T_{\eta}$, we find a band $B \subset V$, transverse to the flow, that connects $\eta \subset Q \subset \mathbb{T}$ with $\mu \subset Q_R \subset \partial\mathbb{T}_R$. This time, μ is not necessarily monotonous in the third $\mathbb{R}/\mathbb{Z}$-coordinate, but we have that

$$\mathrm{a}(\mu) = \mathrm{s}(\widetilde{\mu}) - \mathrm{i}(\widetilde{\mu}) < \mathrm{a}(\nu).$$

If $\mathrm{a}(\eta) = l$, we can divide the region $V \simeq [0,1] \times \mathbb{A}$ in l cubes of the form $V_i \simeq [\frac{i}{l}, \frac{i+1}{l}] \times \mathbb{A}$ separated by annuli $Q_i = \frac{i}{l} \times \mathbb{A}$, $i = 0, \ldots, l-1$. Then, defining $\mu_1 = \mu$, we can find a band $B_1 \subset V_l$ that connects μ_1 with to $\mu_2 \subset Q_{l-1}$. We can apply the same process inductively on the sets V_i until arrive to a monotonous curve $\mu_l \subset \partial\mathbb{T}_R$. The union of the bands $B_1 \cup \cdots \cup B_l$ gives the desired B.

Remark. The bands B_i and the curves μ_i must be chosen to extend the transversality along the union of the bands. Exactly as before, at each step two possibilities for constructing the next band arise, and only one is useful.

Third case: The general case is when the segment $\eta \cap Q$ twists $m \neq 1$ times in the $\mathbb{R}/\mathbb{Z}$-direction. The proof of this case can be carried out by analogy to the previous two cases.

Step (ii):

Let L_0, L_1 be the two faces of ∂V which are tangent to the flow. Let $\sigma_i = B \cap L_i$, $i = 0, 1$. The last step of the proof consists in finding a rectangular band $B' \subset (\mathbb{T} \backslash \mathbb{T}_R) \cap V^c$, transverse to the flow, verifying that:

1. Two opposite sides coincide with the segments σ_i,
2. The other two sides coincide with $\eta \backslash Q$ and with a curve $\mu' \subset \partial\mathbb{T}_R \backslash Q_R$.

Consider the surface defined as $\Sigma = (\mathbb{D} \backslash \mathbb{D}_R) \times \{0\} \cap V^c$ in normal coordinates. Then, in a small flow box of each annulus L_i it is possible to connect the boundaries of Σ with σ_0 and σ_1, in such a way that the obtained surface is transverse to the flow lines. See figure 3.17.

□

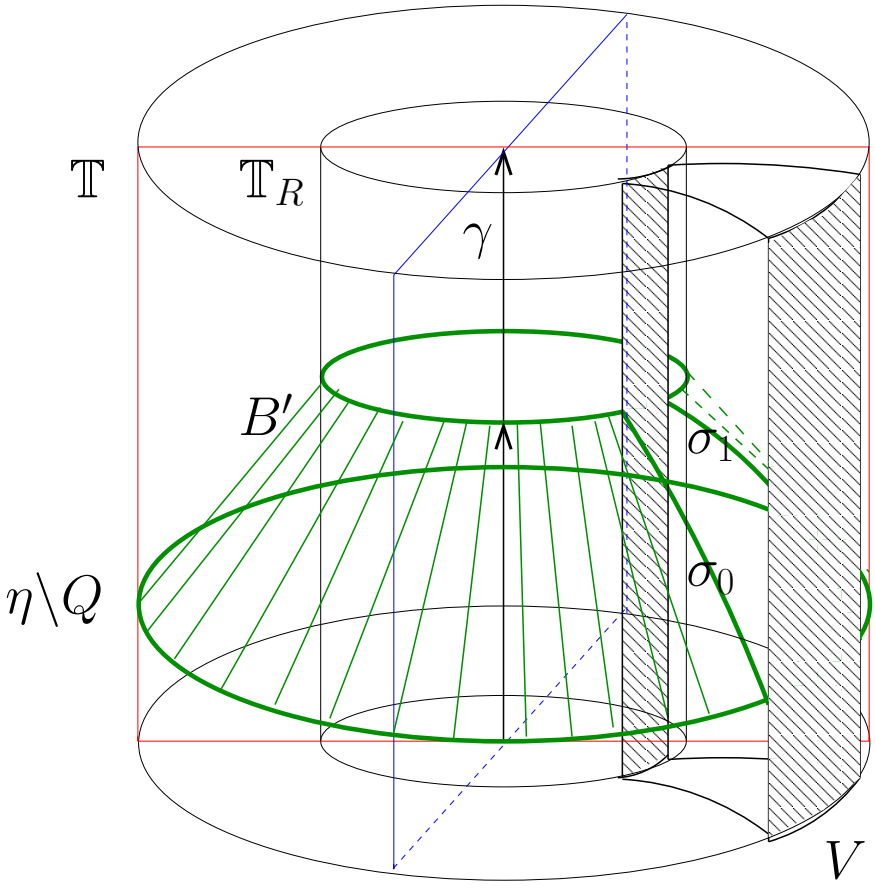


Figure 3.17: The band B'.

Chapter 4

Transitive topological Anosov flows in dimension three.

In this chapter we prove that every transitive topological Anosov flow on a closed 3-manifold is orbitally equivalent to a smooth Anosov flow. In section 1 we survey some results and definitions about topological Anosov dynamics and then we state our main result (theorem 4.12). In section 2 we describe the almost Anosov structures associated to a given transitive topological Anosov flow, following the existence of Birkhoff sections proved by Fried and Brunella. Finally, we prove theorem 4.12 in section 3.

1 Topological Anosov Dynamics.

1.1 Expansive and topologically Anosov homeomorphisms.

Topological properties of Anosov Diffeomorphisms.

Recall that for $r \geq 1$ a diffeomorphism $f \in \text{Diff}^{\,r}(M)$ in a closed, smooth, Riemannian, manifold is Anosov if $Df : TM \to TM$ preserves a *hyperbolic splitting* $TM = E^s \oplus E^u$.

A fundamental property of the hyperbolic splitting preserved by an Anosov diffeomorphism is that the bundles E^s and E^u integrate into globally defined foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ on M, which are of class $C^{0,r}$, transverse between them, and with a very precise dynamical meaning: For every $x \in M$ we have that:

- $\mathcal{F}^s(x)$ equals the *stable set* $W^s(x) = \{y \in M : \lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0\}$,
- $\mathcal{F}^u(x)$ equals the *unstable set* $W^s(x) = \{y \in M : \lim_{n\to-\infty} d(f^n(x), f^n(y)) = 0\}$.

This is the content of the very classical *stable manifold theorem*. For some $\varepsilon > 0$ not

depending on x, the local invariant sets

$$W^s_\varepsilon(x) = \{y \in M : d(f^n(x), f^n(y)) < \varepsilon,\ \forall\ n \geq 0\},$$
$$W^u_\varepsilon(x) = \{y \in M : d(f^n(x), f^n(y)) < \varepsilon,\ \forall\ n \leq 0\},$$

are a pair of C^r-embedded disks, contained in $\mathcal{F}^s(x)$ and $\mathcal{F}^u(x)$ respectively, whose intersection is transverse and consists in the singleton $\{x\}$. Every point in M has a neighbourhood which is bi-foliated by the local invariant manifolds and has a Cartesian product structure.

From the existence of these contracting/expanding invariant foliations with local product structure we can derive most of the topological properties of an Anosov diffeomorphism. In particular, it follows that Anosov diffeomorphisms satisfy the following two properties

(A) : They are *expansive*;

(B) : They satisfy the *pseudo-orbits tracing property.*

These two fundamental properties are involved in important consequences about the topological dynamic of Anosov diffeomorphisms, such as the *C^r-structural stability.*

Topologically Anosov Homeomorphisms.

One attempt to characterize Anosov dynamics from a strictly topological point of view is via the class of homeomorphisms which are *expansive* and have the *global shadowing property.*

Definition 4.1 (Topologically Anosov, [2])**.** Let X be a compact metric space. A homeomorphism $f : X \to X$ is said to be *topologically Anosov* if it is expansive and has the global shadowing property.

In [2] it is shown that much of the topological properties of Anosov diffeomorphisms are shared with these broader category of homeomorphisms. For example, topologically Anosov homeomorphisms on compact metric spaces have well-behaved *local* stable and unstable sets, which induce a partition of the space into stable and unstable sets with local product structure. Moreover, the chain recurrent set of such a diffeomorphism admits a *Smale spectral decomposition in basic pieces* as in a the Axiom A case, and on each basic piece it is possible to codify the dynamic using a Markov partition. They prove as well that topologically Anosov homeomorphisms are *topologically stable.*

C^r-Anosov structures.

Let M be a closed manifold equipped with some smooth differentiable structure $\mathcal{D}$, and consider an Anosov diffeomorphism $f \in \mathrm{Diff}^{\,r}(M, \mathcal{D})$ (here $r \geq 1$) preserving a hyperbolic splitting $TM = E^s \oplus E^u$.

The hyperbolicity condition of the invariant splitting is a property expressed in terms of a Riemannian metric. For being hyperbolic it is required that, for some Riemannian metric on $(M, \mathcal{D})$, there are constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n|_{E^s(x)}\| \leq C\lambda^n \text{ and } \|Df^{-n}|_{E^u(x)}\| \leq C\lambda^n \text{ for } n \geq 0,$$

where $\|\cdot\|$ denotes the norm induced by the metric. Since the manifold is compact, if $E^s \oplus E^u$ verifies the definition of hyperbolicity for some Riemannian metric then it will satisfy the same for any other chosen metric, up to modifying the constants $C > 0$ and $0 < \lambda < 1$.

This means that being Anosov is not a property about how does f act on M as a metric space, but about how it acts as a diffeomorphism of $(M, \mathcal{D})$. If $\mathcal{D}'$ is another smooth structure on M and there exists a C^r-diffeomorphism $h : (M, \mathcal{D}) \to (M, \mathcal{D}')$, then $g = h \circ f \circ h^{-1}$ is a C^r-Anosov diffeomorphism on $(M, \mathcal{D}')$. Thus, we can think about f as being a homeomorphism on M, for which there exists a smooth differentiable structure $\mathcal{D}$ such that f is a C^r-Anosov diffeomorphism. This is a property that depends just on the C^r-diffeomorphism class of $\mathcal{D}$.

Definition 4.2. Let $f \in \text{Homeo}(M)$ and let $r \geq 1$. A *C^r-Anosov structure for f* is a smooth differentiable structure $\mathcal{D}$ on M which satisfies that

(i) $f : (M, \mathcal{D}) \to (M, \mathcal{D})$ is a C^r-diffeomorphism,

(ii) f preserves a hyperbolic splitting.

Existence of smooth models.

One principal question about general topological dynamics is whether or not the dynamic of a given homeomorphism can be realized, up to conjugation, by a model that is regular in some sense. In this spirit, we encounter the following two questions concerning expansive dynamics:

Question 4.3. Let M be a closed topological manifold and $f \in \text{Homeo}(M)$,

(1) If f is expansive, does it exists some differentiable structure on M such that f is a diffeomorphism of class C^r?

(2) If f is topologically Anosov, does it exists some differentiable structure on M such that f an Anosov diffeomorphism of class C^r?

Some remarks are in order.

Remark 4.4. Let $f \in \text{Homeo}(M)$ and let $g \in \text{Diff}^r(N)$. If there exists a homeomorphism $h : M \to N$ that conjugates f and g then, using h, we can pull-back the differentiable structure of N onto a differentiable structure on M. With this structure

f is a diffeomorphism. Thus, determining whether or not f preserves a differentiable structure is equivalent to determining if f is C^0-conjugated to a diffeomorphism in some differentiable manifold. In this way, the differentiable structures preserved by f can be described as the differentiable conjugacy classes of diffeomorphisms g which are topologically conjugated to f.

Remark 4.5. In general, it is not true that a homeomorphism is conjugated to some diffeomorphism. Many examples of this can be given, in any class of regularity varying from Lipschitz to smooth or analytic. For instance, a homeomorphism with infinite topological entropy in a compact manifold can never be endowed with a smooth differentiable structure.

Remark 4.6. If $\dim(M) \leq 3$ then all the different smooth structures on M are diffeomorphic. Thus, we can assume that M is already endowed with a fixed smooth structure and ask if f is conjugated to some diffeomorphism on M.

Remark 4.7. In question (2) the important step is to go from topologically Anosov to C^1-Anosov structures. If $f \in \mathrm{Diff}^1(M, \mathcal{D})$ is an C^1-Anosov diffeomorphism then there exists another smooth structure $\mathcal{E}$ for which f is a C^∞-Anosov diffeomorphism. To see this, just observe that f can be approximated very close in the C^1-topology by some smooth diffeomorphism $g \in \mathrm{Diff}^\infty(M, \mathcal{D})$. Then, by C^1-structural stability we know that f is C^0-conjugated with g by a homeomorphism $h : M \to M$. The structure $\mathcal{E} = h^*(\mathcal{D})$ is a C^∞-Anosov structure for f. Nonetheless, observe that the smooth structure $\mathcal{E}$ is rarely compatible with $\mathcal{D}$.

Questions (1) and (2) are open in general. In dimension two the answer is positive for the two questions, as follows from the works of Lewowicz, Hiraide, and Gerber-Katok.

As we already cited in section 7, in [42] and [38] it is shown that any expansive homeomorphism in a compact surface is conjugated to a pseudo-Anosov homeomorphism. Then, in [32] it is shown that any pseudo-Anosov homeomorphism is in fact conjugated to a smooth diffeomorphism (result that was extended later to the analytic case in [43]), answering in this way to question (1) in dimension two. We notice as well that the smooth models of Gerber-Katok are in the ergodic class of *Bernoulli shifts with respect to some smooth measure* in the surface.

For question (2) in the surface case, the Anosov diffeomorphisms are the only pseudo-Anosov maps that satisfies the pseudo-orbits tracing property, so they are the only topological Anosov homeomorphisms up to conjugation.

In greater dimensions less in known. In [50] and [4] it is shown that, under certain strong hypothesis about the set of periodic points, every expansive homeomorphism in a compact n-manifold is necessarily C^0-conjugated to an Anosov diffeomorphisms on the n-dimensional torus. In [2] it is shown that topological Anosov homeomorphisms on the n-torus are conjugated to Anosov diffeomorphisms. We remark another interesting result found in [24] concerning exotic differentiable structures: In dimension $n > 4$ there exist topologically Anosov homeomorphisms on the n-torus, which are

C^∞-Anosov diffeomorphisms with a certain smooth structure that is *not diffeomorphic* to the standard one (and they are not Anosov with the standard smooth structure on $\mathbb{T}^n$).

Remark 4.8. The existence of a C^1-Anosov structure preserved by a homeomorphism f is more restrictive than the condition of being just a diffeomorphism for some differentiable structure. It is not hard to construct examples of expansive diffeomorphisms on the n-torus, C^0-conjugated to some linear Anosov automorphism, but which are not Anosov for the standard smooth structure. See for instance [2], theorem 1.2.2 and remark 1.2.3.

The main difficulty to prove that an expansive or topologically Anosov homeomorphism preserves a differentiable structure is to construct differentiable model to compare with. In most of the cases the strategy consists in show that the homeomorphism preserve a pair of transverse foliations, may be defined in just an open and dense subset of M, and then use this to impose conditions on the topology of M. General expansive maps in compact metric spaces preserve partitions by stable/unstable sets, but it is not clear whether they should constitute foliations or something similar. In view of this we can make the following definition:

Definition 4.9 (Strongly topologically Anosov)**.** Let M be a closed manifold equipped with some distance d. A topologically Anosov homeomorphism $f \in \mathrm{Homeo}(M)$ is *strong* if it preserves two regular foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that

(1) The two foliations have local product structure,

(2) For every $x \in M$

$$\mathcal{F}^s(x) = \{y \in M : \lim_{n\to+\infty} d(f^n(x), f^n(y)) = 0\},$$
$$\mathcal{F}^u(x) = \{y \in M : \lim_{n\to-\infty} d(f^n(x), f^n(y)) = 0\}.$$

Adding this extra hypothesis we can see in [34] an elementary proof that every strong topological Anosov homeomorphism in the two dimensional torus is conjugated to a linear Anosov map. Also, from the classical result of Franks and Newhouse about codimension one Anosov diffeomorphisms ([45], [29]) it follows that every strong topological Anosov homeomorphism is conjugated to a linear Anosov automorphism on the n-torus, provided one of the two foliations has dimension one.

1.2 Topologically Anosov flows in dimension three.

The problem of smooth models for expansive or topologically Anosov dynamics can be translated to the context of flows. We discuss it for the case of regular flows in three dimensional closed manifolds.

Let $\phi_t : M \to M$ be an orbitally expansive regular flow, defined in a closed, smooth, Riemannian 3-manifold M. In the same way that expansive homeomorphisms in closed

surfaces are conjugated to pseudo-Anosov homeomorphisms, in [46] and [40] it is shown that every expansive flow in a closed 3-manifold is orbitally equivalent to a *pseudo-Anosov flow*. [1] Thus, the theorems in [42], [38], [46] and [40] say that, in the smallest possible dimension (two for homeomorphisms and three for flows), pseudo-Anosov dynamics are the only expansive dynamics. A very complete account on these theorems and definitions is found in [17].

Definition 4.10 (Topological Anosov flows)**.** Let (ϕ_t, M) be a regular flow defined in a closed 3-manifold M. We say that ϕ_t is a *topological Anosov flow* if it is expansive and the invariant foliations have no singularities.

By the stable manifold theorem we have that every C^r-Anosov flow is, in particular, topologically Anosov. In analogy to question 4.3 we have the following problem:

Question 4.11. Is every topological Anosov flow in a closed 3-manifold C^0-orbitally equivalent to a C^1-Anosov flow?

In contrast to the case of 2-dimensional homeomorphisms, the class of topologically Anosov flow is divided into the transitive and the non-transitive ones. The objective of this chapter is to prove the following theorem:

Theorem 4.12 (**Theorem C**)**.** *Every transitive topological Anosov flow in a closed 3-manifold is C^0-orbitally equivalent to a C^1-Anosov flow.*

Theorem 4.12 can be expressed in terms of differentiable structures. Consider the following definition:

Definition 4.13. Let (ψ_t, N) be a non-singular regular flow in a closed 3-manifold and let $1 \leq r \leq \infty$. A *C^r-Anosov structure for* (ψ_t, N) is a smooth differentiable structure $\mathcal{D}$ on N which satisfies that

(i) $\psi_t : (N, \mathcal{D}) \to (N, \mathcal{D})$ is a flow generated by a C^r-vector field,

(ii) $D\psi_t$ preserves a hyperbolic splitting $TN = E^s \oplus E^c \oplus E^u$, E^c generated by the vector field.

Then, the theorem stated above says that given a transitive topological Anosov flow, there exists some reparametrization of the flow and a smooth atlas on the ambient manifold for which the flow is C^1-Anosov. By the remark 4.7 above, we can choose the smooth atlas in such a way that the flow becomes C^r-Anosov, for any $1 \leq r \leq \infty$.

Theorem 4.12 proceeds by reducing the dynamic of the flow to a first return map onto a Birkhoff section, and is for this reason that the proof presented here is valid just

[1] The flow ϕ_t is pseudo-Anosov if it preserves a pair of codimension one transverse foliations $\mathcal{F}^{cs}$, $\mathcal{F}^{cu}$, possibly with singularities of *circle-prongs* type, satisfying that: Given x and y in the same *cs* leaf, there exists an increasing *reparametrizing homeomorphism* $\rho : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, such that $dist(\phi_t(x), \phi_{h(t)}(y)) \to 0$ for $t \to +\infty$, and the symmetric condition is satisfied for the *cu* leaves in backwards time.

in the transitive case. As we remarked on the introduction, there are hyperbolic basic sets for flows in dimension three which cannot be constructed from the suspension of any surface map, (cf [12], *non-coherent attractors*), which constitutes an obstruction to extend our techniques. Question 4.11 is open in the non-transitive case.

2 Almost Anosov Structures.

Let (ϕ_t, M) be a transitive topological Anosov flow.

In the complement of some finite collections of periodic orbits, it is possible to define a smooth atlas and a Riemannian metric such that the restriction of the flow onto this set is orbitally equivalent to a smooth flow preserving a uniformly hyperbolic splitting.

This follows from the theorem of Fried [31] and Brunella [17]. Since ϕ_t is transitive, there exists a Birkhoff section

$$\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma). \tag{4.1}$$

In the complement of Γ the restriction of the flow is orbitally equivalent to the suspension of the first return map $P : \mathring{\Sigma} \to \mathring{\Sigma}$. The pseudo-Anosov character of P allow to define a convenient smooth structure and Riemannian metric on the (open) manifold $M \backslash \Gamma$, as we explain below. Before, we recall the general construction of Fried.

Fried's Construction.

Let $\widehat{\Sigma}$ be the closed surface obtained by collapsing each boundary component of the Birkhoff section (4.1) into a point, and let Δ be the set of all these points.

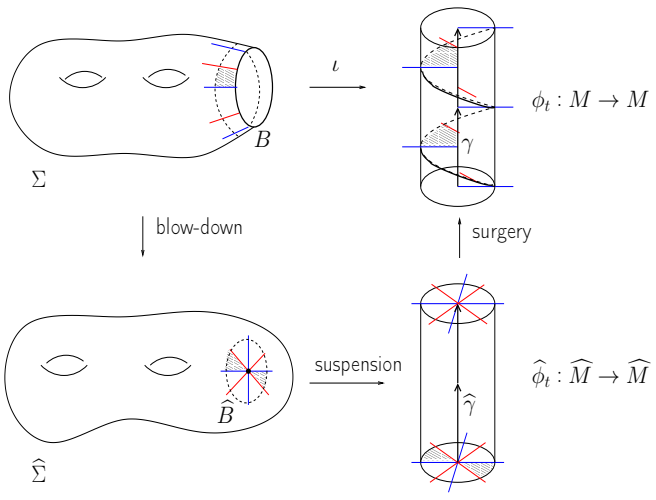
There is a homeomorphism $\widehat{P} : \widehat{\Sigma} \to \widehat{\Sigma}$ that preserves Δ, induced by the blow-down operation, which is *generalized*[2] pseudo-Anosov and has no singularities on $\mathring{\Sigma} = \widehat{\Sigma} \backslash \Delta$. The set of singularities of $\widehat{P}$ (i.e. the set of singularities of the invariant foliations) is contained in Δ. Since $\widehat{P}$ is induced from the first return map to a Birkhoff section in a topological Anosov flow (no circle-prongs), we encounter some restrictions on the kind of k-prong local models that are available at the singularities of $\widehat{P}$; see [31].

Consider the flow

$$(\widehat{\phi}_t, \widehat{M}) = \text{suspension}\left(\widehat{P} : \widehat{\Sigma} \to \widehat{\Sigma}\right). \tag{4.2}$$

Each point in Δ generates a periodic orbit for this suspension flow. Let $\widehat{\Gamma}$ be the set of all the periodic orbits associated to points in Δ. The flow (4.2) defined above is a

[2]Here, the adjective *generalized* stands for pseudo-Anosov maps with possibly 1-prong singularities. This could be the case if some of the orbits in Γ have non-orientable local invariant manifolds. Nevertheless, as it was observed by Brunella, there always exist Birkhoff sections such that the local invariant manifolds of every orbit in Γ are orientable and embedded (i.e. multiplicity equal one). This observation can be useful to simplify technicalities along proofs.



pseudo-Anosov flow, and the orbits in $\widehat{\Gamma}$ that correspond to singular points for $\widehat{P}$ will be circle-prongs of the invariant foliations.

It follows that $(\widehat{\phi}_t, \widehat{M})$ is *almost equivalent* to (ϕ_t, M). That is:

- There exists an orbital equivalence $H_\Gamma : (\phi_t, M\backslash\Gamma) \to (\widehat{\phi}_t, \widehat{M}\backslash\widehat{\Gamma})$,
- (ϕ_t, M) is obtained by Fried Surgery on $(\widehat{\phi}_t, \widehat{M})$ along the periodic orbits in $\widehat{\Gamma}$.

2.1 Almost Anosov smooth structure associated to a Birkhoff section.

From now on we will denote $M_\Gamma = M\backslash\Gamma$ and, in general, we will use Γ as a sub-index for referring to the objects associated to the restriction of ϕ_t onto M_Γ.

The flow (ϕ_t, M_Γ) is equivalent to the suspension flow generated by $P : \mathring{\Sigma} \to \mathring{\Sigma}$. As a consequence, we can state the following proposition:

Proposition 4.14. *The manifold M_Γ is equipped with a smooth structure $\mathcal{D}_\Gamma$ such that:*

(i) The foliation by flow orbits $\mathcal{O}_\Gamma$ on M_Γ is tangent to a smooth non-singular vector field X_Γ.

(ii) $D\phi_t^{X_\Gamma} : TM_\Gamma \to TM_\Gamma$ preserves a splitting $TM_\Gamma = E^s_\Gamma \oplus E^c_\Gamma \oplus E^u_\Gamma$, where $\phi_t^{X_\Gamma}$ denotes the flow generated by X_Γ and the bundle E^c_Γ is collinear with this vector-field.

(iii) There exists a Riemannian metric $|\cdot|_\Gamma$ *in the (open) manifold* M_Γ *for which the splitting is uniformly hyperbolic and*

$$|D\phi_t^{X_\Gamma}(p)\cdot v|_\Gamma = \lambda^t\cdot|v|_\Gamma,\ \forall\ v\in E^s \text{ and } t\geq 0,$$
$$|D\phi_t^{X_\Gamma}(p)\cdot v|_\Gamma = \lambda^{-t}\cdot|v|_\Gamma,\ \forall\ v\in E^u \text{ and } t\leq 0,$$

where $0<\lambda<1$ *is the stretching factor of* P.

In analogy to [39] we define:

Definition 4.15. The smooth structure $\mathcal{D}_\Gamma$ stated in proposition 4.14 above is called an *almost Anosov structure* for ϕ_t on M_Γ.

Remark 4.16. For proving theorem 4.12 we will start with one of these almost Anosov structures and we will construct a global Anosov structure for a transitive topological Anosov flow. Nevertheless, in the same way as in subsection 3.1, the new structure is not compatible with the almost Anosov one in a the complement of a neighbourhood of the singular periodic orbits. It seems reasonable to be aware of a possible *extension* of these almost Anosov structures onto the singular set.

Remark 4.17. Not every finite set Γ of periodic orbits bounds a Birkhoff section. For example, this is the case of a single periodic orbit in a suspension Anosov flow, since it represents a non-trivial homology class on the phase space. This says that the almost Anosov structures associated to a given topological Anosov flow are not arbitrary objects, but they have some intricate relation with the topology of the flow.

Proof. To prove the proposition, observe that since $\widehat{P}:\widehat{\Sigma}\to\widehat{\Sigma}$ is pseudo-Anosov it has an associated pair $(\mathcal{F}^s,\mu_s)$ and $(\mathcal{F}^u,\mu_u)$ of transverse foliations equipped with transverse measures and a stretching factor $0<\lambda<1$. Since there are no singularities on the open surface $\mathring{\Sigma}=\widehat{\Sigma}\backslash\Delta$, this pair of transverse foliations provides a translation atlas

$$\mathcal{D}_\Delta=\{\varphi_i:U_i\to\mathbb{R}^2\}_{i\in I},\ \text{where } \{U_i\}_{i\in I} \text{ is an open cover of } \mathring{\Sigma},$$

such that, on each coordinate neighbourhood, the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ correspond to the foliations of the plane by horizontal and vertical lines, and the transverse measures μ_s and μ_u correspond to the 1-forms $|dx|$ and $|dy|$ in $\mathbb{R}^2$, respectively. This translation atlas defines a smooth structure in $\mathring{\Sigma}$ and a Riemannian metric $|\cdot|^2=dx^2+dy^2$.

In the local coordinates of this atlas, the first return $P:\mathring{\Sigma}\to\mathring{\Sigma}$ takes the form of a homeomorphism

$$\varphi_i\circ P\circ\varphi_j^{-1}:(x,y)\mapsto\pm\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}(x,y)+\tau_{ij}$$

between open sets in $\mathbb{R}^2$. It follows that P is smooth and $DP:T\mathring{\Sigma}\to T\mathring{\Sigma}$ preserves a splitting $T\mathring{\Sigma}=E^s\oplus E^u$ given in local coordinates by $E^s=\mathbb{R}\times 0$ and $E^u=0\times\mathbb{R}$. Moreover, with the metric $|\cdot|$ this splitting is uniformly hyperbolic.

With this structure of smooth Riemannian manifold on $\mathring{\Sigma}$, the suspension flow

$$(\mathring{\phi}_t, \mathring{M}) = \text{suspension}\left(P : \mathring{\Sigma} \to \mathring{\Sigma}\right) \tag{4.3}$$

is the flow generated by the smooth vector field $\partial/\partial t$ on the smooth (open) manifold

$$\mathring{M} = \mathring{\Sigma} \times \mathbb{R}/(z,t) \sim (P(z), t-1).$$

The DP-invariant splitting on the surface induces an invariant splitting of the form

$$T\mathring{M} = E^s \oplus E^u \oplus span\{\partial/\partial t\}$$

for the derivative of the suspension flow. Moreover, for each point (z,t) of $\mathring{\Sigma} \times \mathbb{R}$, the expression

$$|\cdot|^2 = \lambda^{-2t}dx^2 + \lambda^{2t}dy^2 + dt^2$$

defines a Riemannian metric that pushes-down to the quotient manifold $\mathring{M}$ and induces a metric $|\cdot|_\circ$ that coincides with the metric on $\mathring{\Sigma}$ along a fixed global transverse section $\mathring{\Sigma} \hookrightarrow \mathring{M}$. The invariant splitting defined above is uniformly hyperbolic with respect to this metric, and it follows from its definition that $|D\phi_t|_{E^s}|_\circ = \lambda^t$ and $|D\phi_t|_{E^u}|_\circ = \lambda^{-t}$. Since (ϕ_t, M_Γ) is orbitally equivalent to $(\mathring{\phi}_t, \mathring{M})$, this completes the proof. □

2.2 Normal form in a neighbourhood of $\gamma \in \Gamma$.

We describe here a normal form for the vector field X_Γ in a neighbourhood of each $\gamma \in \Gamma$. Before, in 2.2, we will make a small digression about a normal form for the suspension of a saddle type hyperbolic linear transformation on $\mathbb{R}^2$. In 4.20 and 4.24 we will define a family of tubular neighbourhoods, that we call *cross-shaped neighbourhoods*, and that will be used in the course of the proof of theorem 4.12.

Affine local model.

Given $0 < \lambda < 1$ consider the matrix $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

Lemma 4.18. *Let $\phi_t^\lambda : M_\lambda \to M_\lambda$ be the flow obtained as the suspension of $A : \mathbb{R}^2 \to \mathbb{R}^2$. Then, there exists a smooth conjugation*

$$H : (\phi_t^\lambda, M_\lambda) \to (\phi_t^{X_\lambda}, \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}),$$

where $X_\lambda : \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is the vector field given by $X_\lambda(x,y,z) = (\log(\lambda)x, -\log(\lambda)y, 1)$.

Thus, the flow $\phi_t^{X_\lambda}$ provides a time-preserving smooth model for the germ of the periodic orbit in the suspension of A. We will outline some facts about this flow in what follows.

The action of $\phi_t^{X_\lambda}$ in coordinates. The flow $\phi_t^{X_\lambda}$ acts on the solid torus $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ by affine transformations, and in the standard (x, y, z) coordinates is given by

$$\phi_t^{X_\lambda}(x, y, z) = (\lambda^t x, \lambda^{-t} y, z + t).$$

So, its action can be decomposed as the product of two flows:

(i) The flow $\phi_t^A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\phi_t^A(x, y) = \exp(tA)(x, y)$,

(ii) The flow $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $z \mapsto z + t$.

The action of $D\phi_t^{X_\lambda}$ on $\mathbb{R}^3$ preserves a splitting $E^s \oplus E^c \oplus E^u$ into three line bundles

$$E^s = \mathbb{R} \times \{0\} \times \{0\},\ E^c = span\{X_\lambda\} \text{ and } E^u = \{0\} \times \mathbb{R} \times \{0\},$$

and for every $p \in \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ we have that

$$\|D\phi_t^{X_\lambda}(p) \cdot v\| = \lambda^t \cdot \|v\|,\ \forall\ v \in E^s \text{ and } t \geq 0,$$
$$\|D\phi_t^{X_\lambda}(p) \cdot v\| = \lambda^{-t} \cdot \|v\|,\ \forall\ v \in E^u \text{ and } t \leq 0,$$

where $\|\cdot\|$ is the standard Euclidean norm.

Plane cross-shaped region. Start with the standard partition $\{\mathbb{D}_i : i = 1, \ldots, 4\}$ of the plane $\mathbb{R}^2$ into four quadrants, and consider the corresponding restrictions $\phi_t^A : \mathbb{D}_i \to \mathbb{D}_i$. Given two parameters $0 < r_2 < r_1 < 1$ consider the region $Q_1 = Q_1(r_1, r_2) \subset \mathbb{D}_1$ delimited by the segments:

(1) $w_1^s = [0, r_1] \times \{0\}$
(2) $w_1^u = \{0\} \times [0, r_1]$
(3) $J_{in}^1 = \{(r_1, 0)\} \times [0, r_2]$
(4) $J_{out}^1 = [0, r_2] \times \{(0, r_1)\}$
(5) $l_1 =$ the orbit segment of ϕ_t^A that connects (r_1, r_2) with (r_2, r_1).

The boundary ∂Q_1 is composed of one segment contained $W^s(0)$, one segment contained in $W^u(0)$, an *entrance* boundary J_{in}^1, an *exit* boundary J_{out}^1, and an orbit segment. See Figure 4.1a.

By analogy we define the corresponding regions $Q_i \subset \mathbb{D}_i$ in each quadrant $i = 2, 3, 4$. The union of these four regions determines a compact neighbourhood $Q = Q(r_1, r_2)$ of $0 \in \mathbb{R}^2$. See Figure 4.1b.

There is a diffeomorphism $\psi : J_{in}^1 \backslash \{(r_1, 0)\} \to J_{out}^1 \backslash \{(0, r_1)\}$ of the form

$$\psi : p \mapsto q = \phi_{\tau(p)}^A(p),$$

which sends each entrance point p onto the point q determined by the intersection $J_{out}^1 \cap \mathcal{O}^+(p)$. The same holds for the other regions Q_i, $i = 2, 3, 4$.

Lemma 4.19. *Given $0 < r_2 < r_1 < 1$ consider the map $\psi : \{r_1\} \times (0, r_2] \to (0, r_2] \times \{r_1\}$ associated to the flow ϕ_t^A. Let $p = (r_1, r)$, then $\psi(p) = (r, r_1)$ and $\tau(p) = \tau(r) = \frac{\log(r/r_1)}{\log(\lambda)}$.*

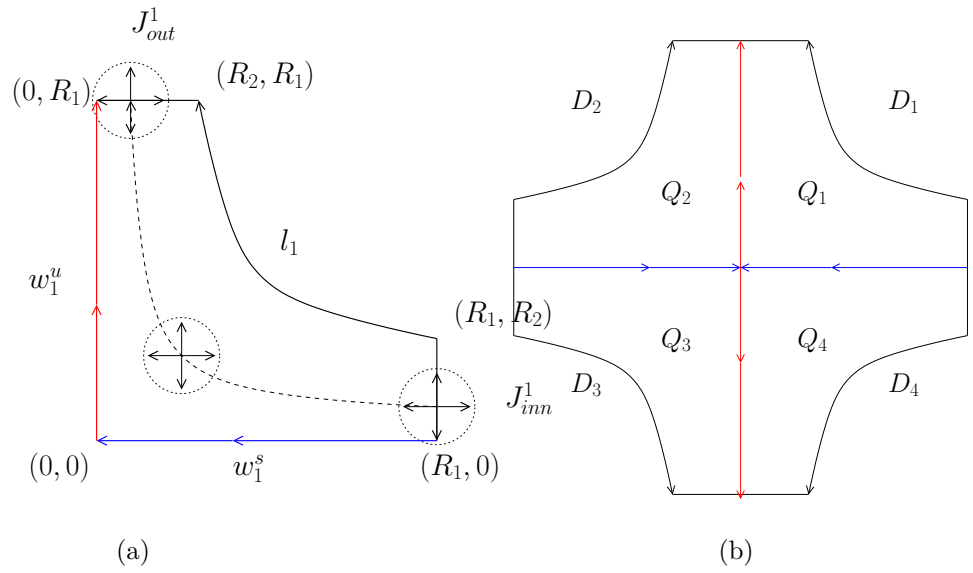


Figure 4.1: The region Q.

Cross-Shaped neighbourhood Let $\gamma_0 = 0 \times \mathbb{R}/\mathbb{Z}$ be the periodic orbit of the flow $\phi_t^{X_\lambda}$ defined in lemma 4.18 and let $\{W_i : i = 1, \ldots, 4\}$ be the partition of the solid torus $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ into four quadrants determined by the invariant manifolds of γ_0.

Definition 4.20. Given two parameters $0 < r_2 < r_1 < 1$ define the regions $V_i = V_i(r_1, r_2) \subset W_i$ and $V = V(r_1, r_2) \subset \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ by

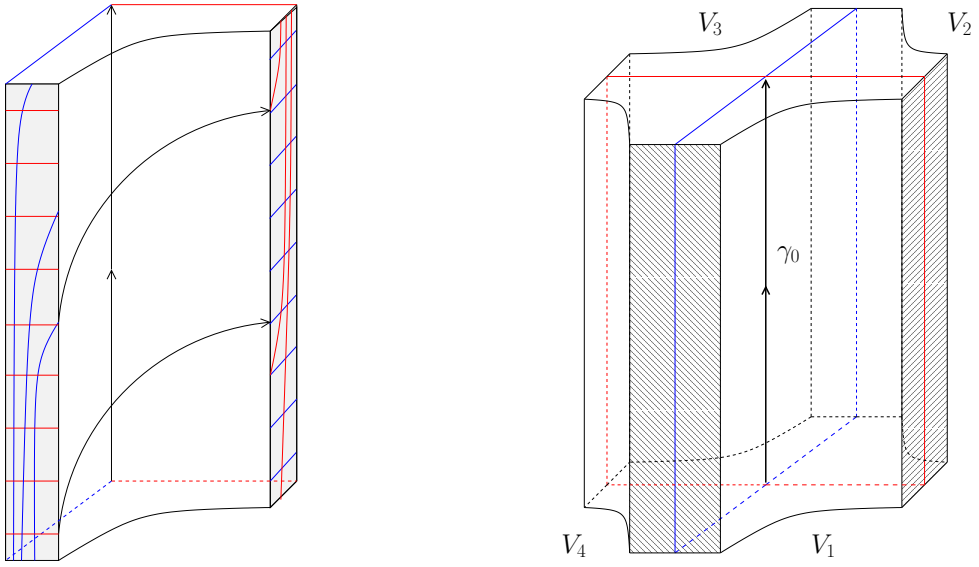
$$V_i(r_1, r_2) \coloneqq Q_i(r_1, r_2) \times \mathbb{R}/\mathbb{Z}, \text{ for } i = 1 \ldots, 4 \tag{4.4}$$
$$V(r_1, r_2) \coloneqq Q(r_1, r_2) \times \mathbb{R}/\mathbb{Z}. \tag{4.5}$$

The region $V(r_1, r_2)$ is a compact, regular, tubular neighbourhood of the periodic orbit γ_0. We call it a *cross-shaped neighbourhood*. It is decomposed as the union of the four regions V_i, each one contained in a quadrant of the solid torus. See figure 4.2.

Each of V_i is homeomorphic to a solid torus and its boundary is composed of five annuli:

$$\begin{aligned}
&(1)\ L^s_i &&= w^s_i \times \mathbb{R}/\mathbb{Z} \\
&(2)\ L^u_i &&= w^u_i \times \mathbb{R}/\mathbb{Z} \\
&(3)\ A^i_{in} &&= J^i_{in} \times \mathbb{R}/\mathbb{Z} \\
&(4)\ A^i_{out} &&= J^i_{out} \times \mathbb{R}/\mathbb{Z} \\
&(5)\ L^c &&= l_i \times \mathbb{R}/\mathbb{Z}.
\end{aligned}$$

The annuli L^s_i, L^u_i and L^c_i are tangent to the flow, being the first two contained in the weak stable and unstable manifolds respectively. The annuli A^i_{in} and A^i_{out} are the



(a) (b)

Figure 4.2: Cross-shaped neighbourhood.

entrance and *exit* annuli, respectively. For $i = 1$,this two annuli corresponds to the sets

$$A^1_{in} = \{(r_1, r, z) : 0 \leq r \leq r_2,\ z \in \mathbb{R}/\mathbb{Z}\}$$
$$A^4_{in} = \{(r_1, r, z) : -r_2 \leq r \leq 0,\ z \in \mathbb{R}/\mathbb{Z}\}.$$

There is a diffeomorphism $\psi : A^1_{in} \backslash W^s(0) \to A^1_{out} \backslash W^u(0)$ of the form

$$\psi : p \mapsto q = \phi^{X_\lambda}_{\tau(p)}(p),$$

which sends each entrance point p onto the point q determined by the intersection $A^1_{out} \cap \mathcal{O}^+(p)$. The same holds for the other entrance-exit pair of annuli on the boundary.

Lemma 4.21. *Given $0 < r_2 < r_1 < 1$ consider the map $\psi : \{r_1\} \times (0, r_2] \times \mathbb{R}/\mathbb{Z} \to (0, r_2] \times \{r_1\} \times \mathbb{R}/\mathbb{Z}$ associated to the flow $\phi^{X_\lambda}_t$. Let $p = (r_1, r, z)$, then $\psi(p) = (r, r_1, z + \tau(p))$ and $\tau(p) = \tau(r) = \frac{\log(r/r_1)}{\log(\lambda)}$.*

Normal form in a neighbourhood of Γ.

Given a periodic orbit $\gamma \in \Gamma$ consider some small tubular neighbourhood W. In this section we will assume that the invariant local manifolds are orientable, so every small tubular neighbourhood W is partitioned in four quadrants W_i, $i = 1, \ldots, 4$. The following lemma gives a normal form for the vector field X on the punctured neighbourhood $W \setminus \gamma$.

Recall that the first return to Σ is pseudo-Anosov and we denote by $0 < \lambda < 1$ its stretching factor. Also recall that γ is the image of a number $p = p(\gamma, \Sigma)$ of connected

components of $\partial\Sigma$. This means that in a small neighbourhood W of γ the surface $\Sigma \cap W$ splits as p different local Birkhoff sections at γ, each of them with the same linking number $n = n(\gamma, \Sigma)$ and multiplicity $m = m(\gamma, \Sigma)$.

Lemma 4.22. *Let (ϕ_t, M) be transitive topological Anosov flow and $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ a Birkhoff section. Consider the smooth atlas $\mathcal{D}_\Gamma$ and the smooth vector field X_Γ on $M \backslash \Gamma$ induced by the Birkhoff section.*

Then, for every $\gamma \in \Gamma$ there is a small tubular neighbourhood W, divided in four quadrants W_i, $i = 1, \dots, 4$, and a systems of smooth charts

$$\Pi_i : W_i \setminus \gamma \to (\mathbb{D}_i \setminus \{0\}) \times \mathbb{R}/\mathbb{Z},\ i = 1, \dots, 4$$

satisfying that:

1. *$D\Pi_*(X_\Gamma) = X_{(\lambda,p,n)}$, where $X_{(\lambda,p,n)} : \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is the vector field*

$$X_{(\lambda,p,n)}(x, y, z) = \left(\log(\lambda)x, -\log(\lambda)y, \frac{1}{|np|}\right). \tag{4.6}$$

2. *The charts Π_i send each connected component of $\Sigma \cap (W \backslash \gamma)$ isometrically onto a surface of the form $U \cap (\mathbb{D}_i \backslash \{0\}) \times \{\frac{k}{|np|}\}$, where U is an open neighbourhood of $0 \in \mathbb{R}^2$ and $k \in \{0, \dots, |np| - 1\}$.*

3. *It is verified that:*

$$\begin{aligned}
\Pi_1(W_1) = \Pi_2(W_2) &= \{0\} \times (0, +\infty) \times \mathbb{R}/\mathbb{Z} \\
\Pi_2(W_2) = \Pi_3(W_3) &= (-\infty, 0) \times \{0\} \times \mathbb{R}/\mathbb{Z} \\
\Pi_3(W_3) = \Pi_4(W_4) &= \{0\} \times (-\infty, 0) \times \mathbb{R}/\mathbb{Z} \\
\Pi_4(W_4) = \Pi_1(W_1) &= (0, +\infty) \times \{0\} \times \mathbb{R}/\mathbb{Z}
\end{aligned}$$

and

$$\begin{aligned}
\Pi_2 \circ \Pi_1^{-1} &: (0, y, z) \mapsto (0, y, z) \\
\Pi_3 \circ \Pi_2^{-1} &: (x, 0, z) \mapsto (x, 0, z) \\
\Pi_4 \circ \Pi_3^{-1} &: (0, y, z) \mapsto (0, y, z) \\
\Pi_1 \circ \Pi_4^{-1} &: (x, 0, z) \mapsto \left(x, 0, z + \frac{m}{n}\right).
\end{aligned} \tag{4.7}$$

Remark 4.23. The charts defined in lemma 4.22 send the orbit segments of ϕ_t that lie inside each punctured quadrant $W_i \setminus \gamma$ onto the orbit segments of the flow generated by the vector field (4.6) inside the quadrant $\mathbb{D}_i \times \mathbb{R}/\mathbb{Z}$, preserving the time parameter. Thus, we can reconstruct the vector field X_Γ in $W \backslash \gamma$ by glueing the four pieces

$$\left(X_{(\lambda,p,n)}\ , (\mathbb{D}_i \backslash \{0\}) \times \mathbb{R}/\mathbb{Z}\right),\ i = 1, \dots, 4$$

along their boundaries in the way specified in (4.7). Since the vector field $X_{(\lambda,p,n)}$ is invariant by vertical translations, the glueing map $\Pi_1 \circ \Pi_4^{-1}$ preserves $X_{(\lambda,p,n)}$ and we get a well defined vector field in the quotient manifold, which is homeomorphic to a solid torus with an essential closed curve on the interior removed.

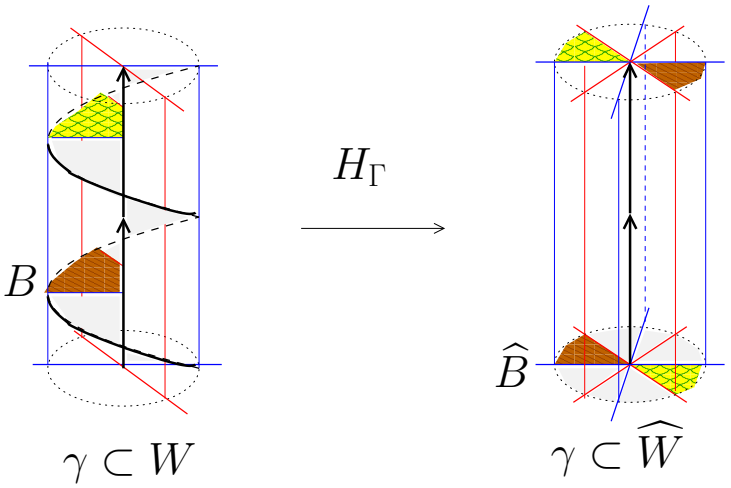


Figure 4.3: The germs of γ and $\widehat{\gamma}$.

Let $\widehat{\gamma} \in \widehat{\Gamma}$ be the periodic orbit of the associated flow $(\widehat{\phi}_t, \widehat{M})$, that is in correspondence with γ under surgery. For every small tubular neighbourhood W there is an associated tubular neighbourhood $\widehat{W}$ of $\widehat{\gamma}$ of the form

$$\widehat{W} \coloneqq H_\Gamma(W \backslash \gamma) \cup \widehat{\gamma}, \tag{4.8}$$

together with a local orbital equivalence $H_\Gamma : (\phi_t, W\backslash\gamma) \to (\widehat{\phi}_t, \widehat{W}\backslash\widehat{\gamma})$. Recall that the germ of ϕ_t on γ is saddle type while, in general, the germ of $\widehat{\phi}_t$ on $\widehat{\gamma}$ will be a circle-prong, as in figure 4.3. We go from one to the other doing a Fried surgery.

Since the local invariant manifolds of γ are orientable it is verified that:

1. $W^s_\varepsilon(\gamma) \cup W^u_\varepsilon(\gamma)$ partitions the neighbourhood W in four quadrants W_i, $i = 1, \ldots, 4$.
2. $W^s_\varepsilon(\widehat{\gamma}) \cup W^u_\varepsilon(\widehat{\gamma})$ partitions the neighbourhood $\widehat{W}$ in four quadrants $\widehat{W}_i$, $i = 1, \ldots, 4$,

   each one defined by
   $$\widehat{W}_i = \overline{H_\Gamma(int(W_i))}.$$

Local Birkhoff section. The way these four quadrants are glued together to form the germs of $(\phi_t, W)_\gamma$ or $(\widehat{\phi}_t, \widehat{W})_{\widehat{\gamma}}$ can be combinatorially described using the Birkhoff section Σ. Assume that C is a component of $\partial\Sigma$ such that $\gamma = \iota(C)$. There exists a collar neighbourhood $B \simeq [0,1) \times \mathbb{R}/\mathbb{Z}^1$ of C which immerses in W as a local Birkhoff section at γ, with linking number $n = n(B, \gamma)$ and multiplicity $m = m(B, \gamma)$.

We recall from section 4.1 that this local section B is partitioned by the local invariant manifolds into $4|n|$ quadrants

$$B_k, \ k = 1, \ldots, 4|n|,$$

and each quadrant W_i contains $|n|$ of them, like in the left part of figure 4.3. We fix a way of labelling the surfaces B_k as follows: Let $\alpha \subset B \backslash \partial B$ be a generator of $\pi_1(B \backslash \partial B)$, that we may assume it is a closed simple curve, constructed by concatenating straight paths contained on each B_k. Orient this curve in such a way that $[\alpha] = n \cdot a + m \cdot b \in H_1(W \backslash \gamma)$ has coordinate $m > 0$ in the meridian/longitude basis $\{a, b\}$. Then, this curve induces a cyclic order on the set of quadrants. We chose the indices $k = 1, \dots, 4|n|$ in such a way that each B_{k+1} is the adjacent quadrant that follows B_k in the cyclic ordering.

For each fixed $i = 1, 2, 3, 4$, in the quadrant W_i there are contained $|n|$ of these surfaces. There are two ways for ordering them. From the one hand, without lose of generality we may assume that the B_k are enumerated in such a way that:

$$B_k \text{ is contained in the quadrant } W_i \text{ iff } k = i + 4j, \text{ with } i = 1, \dots, 4, \ j = 0, \dots, n-1.$$

From the other hand, if we consider a simple closed curve in the interior of W_i that is homotopic to the longitude γ and is transverse to B (a longitude), the orientation of this curve induces a cyclic ordering of the quadrants contained in W_i.

Observe that the local first return $P : \mathring{B} \to \mathring{B}$ induces a cyclic permutation of all the quadrants contained in W_i that preserves this cyclic ordering. In fact, in proposition 4.1 it is shown that, for each $k = 1, \dots, 4|n|$

$$P : B_k \mapsto B_{k+4l} \mapsto B_{k+4(2l)} \mapsto \cdots \mapsto B_{k+4(n-1)l} \mapsto B_k, \tag{4.9}$$

where $l \equiv m^{-1} (\mathrm{mod}\ n)$.

Let $\widehat{B}$ be the disk obtained by collapsing ∂B into a point $q \in \widehat{\Sigma}$. This disk is a transverse section for $\widehat{\phi}_t : \widehat{W} \to \widehat{W}$ with first return map $\widehat{P} : \widehat{B} \to \widehat{B}$. The partition into quadrants of B induce a partition of $\widehat{B}$ in quadrants $\widehat{B}_1, \dots, \widehat{B}_{4n}$. These $4|n|$ quadrants intersect alternately along stable and unstable segment, and therefore the map $\widehat{P} : \widehat{B} \to \widehat{B}$ is a $2n$-prong-saddle at the singularity q. By the previous remarks, the map $\widehat{P}$ permutes the quadrants in the same way as (4.9) before.

Proof of Lemma 4.22. We assume first that $p(\gamma, \Sigma) = 1$. Thus, for a small tubular neighbourhood W the surface $B = \Sigma \cap W$ is a connected local Birkhoff section.

Let $\widehat{B}$ defined as before. The smooth atlas for $(\phi_t, W \backslash \gamma)$ is obtained as the germ of the suspension of $(\widehat{P}, \widehat{B} \backslash \{q\})$, so it suffices to study the germ $(\widehat{\phi}_t, \widehat{W} \backslash \widehat{\gamma})$.

On each $\widehat{W}_i$, $i = 1, \dots, 4$ there are contained exactly n quadrants of the form $\widehat{B}_{i+4j}$, $j = 0, \dots, n-1$. All of these quadrants share the common vertex q. Consider the sets

$$\widehat{B}^*_{i+4j} = \widehat{B}_{i+4j} \setminus \{q\},$$

that we call *punctured* quadrants. These are a collection of pairwise disjoint surfaces, properly embedded in the manifold $\widehat{W}_i \backslash \widehat{\gamma}$, each one homeomorphic to a disk. The first return map permutes cyclically these surfaces sending and $\widehat{P}^n$ preserves each of them. So each $\widehat{B}^*_{i+4j}$ is a transverse section for $\widehat{\phi}_t$ on $\widehat{W}_i \setminus \widehat{\gamma}$. Observe that the flow $\widehat{\phi}_t$ projects the points of a surface B^*_{i+4j} onto the following in constant time equal to one. It follows

that there is a smooth time-preserving conjugation between $\left(\widehat{\phi}_t\ , \widehat{W}_i \setminus \widehat{\gamma}\right)$ and the flow induced by $z \mapsto z + t$ in

$$\widehat{B}^*_{i+4j} \times \mathbb{R}/_{(z,t)\mapsto(\widehat{P}^n(z),t-|n|)}. \tag{4.10}$$

In addition, since $\widehat{P}$ is pseudo-Anosov with stretching factor λ and q is a $2n$-prong, there exists a chart $\varphi_q : (\widehat{B}, q) \to (\mathbb{R}^2, 0)$ that is smooth outside $\{q\}$ and satisfies:

1. $\varphi_q : \widehat{B} \setminus \{q\} \to \mathbb{R}^2 \setminus \{0\}$ is a n-fold covering,
2. $A \circ \varphi_q(p) = \varphi_q \circ \widehat{P}(p)$, for every p in a neighbourhood of q, where $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.
3. For every $i = 1, \ldots, 4$, the chart φ_q sends $\widehat{B}_{i+4j}$ isometrically onto its image in $\mathbb{D}_i \subset \mathbb{R}^2$.

In particular, the chart φ_q provides a smooth conjugation quadrant by quadrant:

$$\begin{array}{ccc} \widehat{B}_{i+4j} & \xrightarrow{\widehat{P}^n} & \widehat{B}_{i+4j} \\ \downarrow{\scriptstyle \varphi_q} & & \downarrow{\scriptstyle \varphi_q} \\ \mathbb{D}_i & \xrightarrow{A^n} & \mathbb{D}_i \end{array} \quad \text{for every } i = 1, \ldots, 4.$$

This implies that the flow in (4.10) above is conjugated to the flow generated by $z \mapsto z+t$ in

$$\mathbb{R}^2 \times \mathbb{R}/_{(z,t)\mapsto(A^n(z),t-|n|)}. \tag{4.11}$$

Since this latter flow is smoothly time-preserving equivalent to the germ of the vector field $X_{(\lambda,n)}(x, y, z) = (\log(\lambda)x, -\log(\lambda)y, 1/n)$ on the quadrant $\mathbb{D}_i \times \mathbb{R}/\mathbb{Z}$ (cf.4.22), we obtain a family of charts

$$\Pi_i^j : \widehat{W}_i \setminus \widehat{\gamma} \to (\mathbb{D}_i \setminus \{0\}) \times \mathbb{R}/\mathbb{Z},\ j = 0, \ldots, n-1,$$

satisfying the desired properties. Each chart is induced from one of the maps $\widehat{P}^n : \widehat{B}^*_{i+4j} \to \widehat{B}^*_{i+4j}$.

Fix a surface $\widehat{B}^*_i$ in a quadrant W_i. According to (4.9) before, the first return $\widehat{P}$ projects $\widehat{B}^*_k$ onto the surface $\widehat{B}^*_{k+4l}$. By definition of $\widehat{\phi}$, this projection along the flow lines is in constant time, and it induces a cyclic permutation of the surfaces contained in $\widehat{W}_i$. Moreover, this map is conjugated to the linear map A in the coordinates defined by φ_q. So we may assume that the coordinate charts Π_i send the surfaces $\widehat{B}^*_{i+4lj}$ to planes of the form $\mathbb{D}_i \times \{\frac{j}{n}\}$, for every $j = 0, \ldots, n-1$. Chose $\widehat{B}^*_1$ in the quadrant $\widehat{W}_1$ and Π_1 such that $\Pi_1 : \widehat{B}^*_1 \mapsto \mathbb{D}_1 \times \{0\}$. Observe that the chart φ_q extends over the union

$$\widehat{B}^*_1 \cup \widehat{B}^*_2 \cup \widehat{B}^*_3 \cup \widehat{B}^*_4$$

sending each $\widehat{B}^*_i$ onto $\mathbb{D}_i \times \{0\}$. Thus, we can coherently extend Π_1 to the adjacent quadrants $\widehat{W}_i \setminus \widehat{\gamma}$, $i = 2, 3, 4$, in such a way that each Π_i sends the surface $\widehat{B}^*_i$ to

$\mathbb{D}_i \times \{0\}$. The quadrant $\widehat{B}_4^*$ contained in $\widehat{W}_4$ is adjacent to a quadrant $\widehat{B}_5^*$ in $\widehat{W}_1$. The combinatorial data on the boundary of the Birkhoff section determines that this surface $\widehat{B}_5^*$ coincides with $\mathbb{D}_1 \times \frac{m}{n}$, provided $\Pi_1(\widehat{B}_1^*) = \mathbb{D}_1 \times \{0\}$. Thus, the holonomy defect of $\Pi_4 \circ \Pi_1^{-1}$ on the common boundary $\widehat{W}_4 \cap \widehat{W}_1$ is a vertical translation by $\frac{m}{n}$.

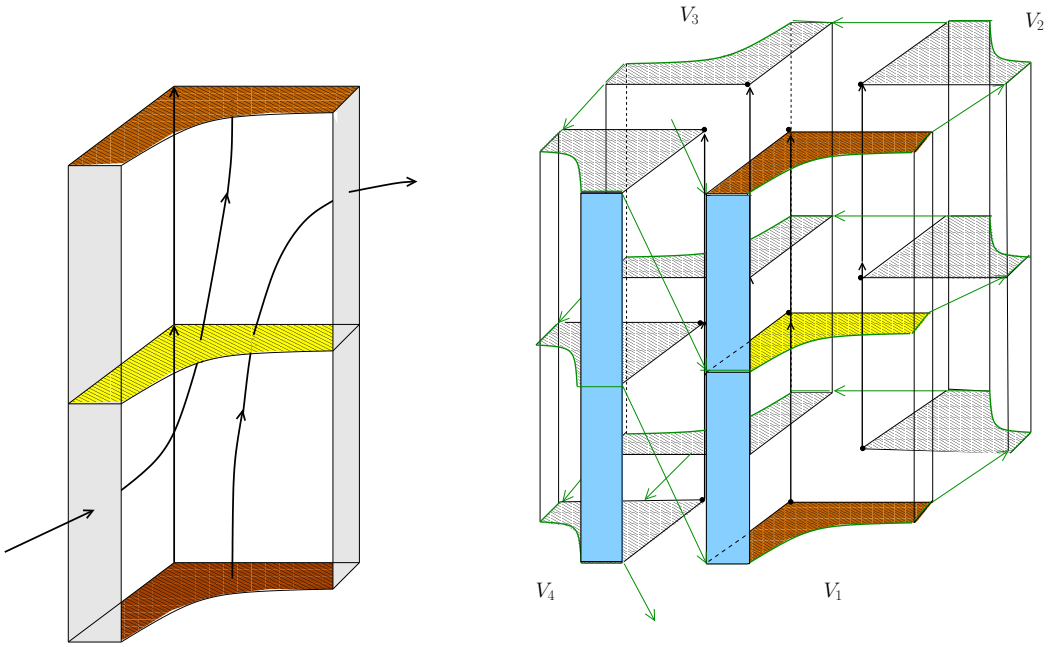
This completes the proof of the lemma assuming that $p(\gamma, \Sigma) = 1$. If there are more than one boundary components of Σ that cover γ, the argument is the same but the first return map to each $\widehat{B}_{i+4j}^*$ changes by $\widehat{P}^{p(\gamma,\Sigma)\cdot n}$, so we must modify the parameters of Y in the appropriated way. □

Prong-Shaped Neighbourhood: For each $\gamma \in \Gamma$, the local coordinates defined in lemma 4.22 allow to construct a regular, compact, tubular neighbourhood $R = R(r_1, r_2)$ of the orbit γ, which depends on two parameters $0 < r_2 < r_1 < 1$, in the following way:

Definition 4.24. For each $i = 1, \ldots, 4$ consider the sets $\mathbb{V}_i(r_1, r_2)$ in $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ defined in 4.20. Let $0 < r_2 < r_1 < 1$ be such that each $V_i(r_1, r_2)$ is contained in the image of Π_i. Define

$$R_i(r_1, r_2) = \overline{\{p \in W_i \backslash \gamma : \Pi_i(p) \in \mathbb{V}_i(r_1, r_2)\}}. \tag{4.12}$$

$$R(r_1, r_2) = \bigcup_i R_i(r_1, r_2), i = 1, \ldots, 4. \tag{4.13}$$



(a) (b)

Figure 4.4: The neighbourhood $R(r_1, r_2)$, obtained by glueing the four pieces $\mathbb{V}_i(r_1, r_2)$.

We call R a *prong shaped neighbourhood*. We can describe the vector field X_Γ on $R \setminus \gamma$ in the following way: Consider the four pairs

$$\left(\mathbb{V}_i(r_1, r_2), Y_{(\lambda,n)}\right),\ i = 1, \dots, 4.$$

Consider the manifold obtained by glueing:

1. $\mathbb{V}_1$ with $\mathbb{V}_2$ along the boundary $L_1^u \to L_2^u$ with the map $(0, y, z) \mapsto (0, y, z)$,
2. $\mathbb{V}_2$ with $\mathbb{V}_3$ along the boundary $L_2^s \to L_3^s$ with the map $(x, 0, z) \mapsto (x, 0, z)$,
3. $\mathbb{V}_3$ with $\mathbb{V}_4$ along the boundary $L_3^u \to L_4^u$ with the map $(0, y, z) \mapsto (0, y, z)$,
4. $\mathbb{V}_4$ with $\mathbb{V}_1$ along the boundary $L_4^s \to L_1^s$ with the map $(0, y, z) \mapsto (0, y, z + \frac{m}{n})$.

In figure (4.4b) we see this for $m = 1$ and $n = -2$. The associated quotient space is a solid torus. Observe that the curve $0 \times \mathbb{R}/\mathbb{Z}$ is a n-fold covering of its image under the quotient projection. In the complement of this curve, the vector field $X_{(\lambda,n)}$ is invariant by z-translations, so we get a well defined vector field on the quotient manifold.

Remark 4.25. Observe that if $|n| = 1$ then we simply obtain that the vector field X_Γ in ϕ_t on $R\setminus\gamma$ is equivalent to $X_{(\lambda,n)}$ in $\mathbb{V}(r_1, r_2)$, and $B\setminus\gamma$ corresponds to $\mathbb{D}\setminus\{0\}$.

The boundary ∂R is decomposed in eight smooth annuli, four of them tangent to the vectorfield, two where X_Γ is transverse and points inward the neighbourhood and two other where X_Γ is transverse and points outward. Using the coordinates Π_i we can identify the sets

$$\Pi_i : \partial R \cap R_i \to \partial\mathbb{V} \cap \mathbb{V}_i = A_{in}^i \cup L^i \cup A_{out}^i \subset \mathbb{D}_i \times \mathbb{R}/\mathbb{Z}.$$

In particular, the union $A_{in}^1 \cup A_{in}^4$ is one of the eight annuli that forms ∂R, where the flow traverse inwardly. In coordinates given in lemma 4.22, this two annuli corresponds to the sets

$$A_{in}^1 = \{(r_1, r, z) : 0 \le r \le r_2,\ z \in \mathbb{R}/\mathbb{Z}\}$$
$$A_{in}^4 = \{(r_1, r, z) : -r_2 \le r \le 0,\ z \in \mathbb{R}/\mathbb{Z}\},$$

glued along the boundary with a translation.

3 The smooth model.

Consider a transitive topologically Anosov flow (ϕ_t, M). In this section we construct another flow (ψ_t, N) which is C^1-Anosov and orbitally equivalent to the first one. We start by sketching the general argument, and we develop the steps in the following subsections.

3.1 Proof of theorem C (thm. 4.12).

Start with a Birkhoff section $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ with first return $P : \mathring{\Sigma} \to \mathring{\Sigma}$. Then, the construction in 2 associates:

1. A smooth atlas $\mathcal{D}_\Gamma$ and a Riemannian metric $|\cdot|_\Gamma$ on the manifold $M \setminus \Gamma$ such that, up to reprametrization, ϕ_t is generated by a smooth vector field X_Γ and preserves a uniformly hyperbolic splitting $E^s_\Gamma \oplus E^c_\Gamma \oplus E^u_\Gamma$. Moreover,
$$|D\phi_t(v)|_\Gamma = \lambda^t |v|_\Gamma, \ \forall\ v \in E^s_\Gamma,\ t \geq 0,$$
$$|D\phi_t(v)|_\Gamma = \lambda^{-t} |v|_\Gamma, \ \forall\ v \in E^u_\Gamma\ t \leq 0$$
where $0 < \lambda < 1$ is the stretching factor of the first return map to the Birkhoff section.

2. For every orbit $\gamma \in \Gamma$ there is a small tubular neighbourhood W, divided in four quadrants W_i, $i = 1, \ldots, 4$, and a system of smooth charts
$$\Pi_i : W_i \setminus \gamma \to (\mathbb{D}_i \setminus \{0\}) \times \mathbb{R}/\mathbb{Z},\ i = 1, \ldots, 4$$
called *normal coordinates*, verifying that:
   (a) $D\Pi_i : X_\Gamma \mapsto X_{(\lambda,n,p)}$, where $X_{(\lambda,n,p)}$ is the vector field in $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ defined by
$$X_{(\lambda,n,p)}(x, y, z) = \left(\log(\lambda)x, -\log(\lambda)y, \frac{1}{|np|}\right);$$
   (b) The charts Π_i send the quadrants of the local Birkhoff section $\Sigma \cap W$ isometrically onto surfaces of the form $(\mathbb{D}_i \setminus \{0\}) \times \left\{\frac{k}{|np|}\right\}$ with $k = 0, \ldots, |np| - 1$;
   (c) Along the corresponding domains of intersection between the four quadrants, we have
$$\Pi_2 \circ \Pi_1^{-1} : (x, y, z) \mapsto (x, y, z)$$
$$\Pi_3 \circ \Pi_2^{-1} : (x, y, z) \mapsto (x, y, z)$$
$$\Pi_4 \circ \Pi_3^{-1} : (x, y, z) \mapsto (x, y, z)$$
$$\Pi_1 \circ \Pi_4^{-1} : (x, y, z) \mapsto \left(x, y, z + \frac{m}{n}\right);$$
   where $p = p(\gamma, \Sigma)$, $n = n(\gamma, \Sigma)$ and $m = m(\gamma, \Sigma)$ are the combinatorial parameters of the Birkhoff section at γ.

Assumption: For simplicity, from now on we assume that $\partial\Sigma$ consists in one boundary component, and so does Γ, which consists in one saddle type periodic orbit γ. The general case with many boundary components can be derived from the present case, by applying the following construction on a neighbourhood of each curve $\gamma \in \Gamma$. We will denote by $n = n(\gamma, \Sigma)$ and $m = m(\gamma, \Sigma)$, and remark that $p(\gamma, \Sigma) = 1$.

For some parameters $0 < r_2 < r_1 < 1$, consider the tubular neighbourhood $R(r_1, r_2) \subset M$ of γ defined in 4.24. Let's consider:

- The manifold $M_R(r_1, r_2)$ obtained from M by removing the interior of $R(r_1, r_2)$, equipped with the restricted vector field X_Γ.
- The cross-shaped neighbourhood $\mathbb{V}(r_1, r_2)$ in $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ defined in 4.20, equipped with the vector field

$$X_{(\lambda,n)}(x, y, z) = \left(\log(\lambda)x, -\log(\lambda)y, \frac{1}{|n|}\right).$$

The idea of the construction is the following:

(1) In 3.2, using the system of normal coordinates of lemma 4.22 we will define a diffeomorphism

$$\varphi : \partial M_R(r_1, r_2) \to \partial \mathbb{V}(r_1, r_2)$$

from the boundary of $M_R(r_1, r_2)$ to the boundary of $\mathbb{V}(r_1, r_2)$, that will depend on the signature of the linking number $n(\gamma, \Sigma)$. Glueing along the boundaries with this diffeomorphism, we will produce a manifold

$$N = N(r_1, r_2) \coloneqq M_R(r_1, r_2) \sqcup_\varphi \mathbb{V}(r_1, r_2)$$

endowed with a smooth atlas. We will show that the vector fields X_Γ on $M_R(r_1, r_2)$ and $X_{(\lambda,n,p)}$ on $\mathbb{V}(r_1, r_2)$ fit together and induce a smooth vector field Y on the quotient manifold N. This will be the content of lemma 4.27.

(2) Let (ψ_t, N) be the flow generated by the vector field Y. Then, in 3.3 we will show that, for sufficiently small values of the parameters $0 < r_2 < r_1 < 1$, this flow is Anosov.

(3) Finally, in 3.4, we show that (ψ_t, N) is orbitally equivalent to the original topological Anosov flow (ϕ_t, M).

Theorem 4.12 follows from the statements (1),(2),(3) above.

3.2 Construction of the smooth model.

In this subsection we construct the smooth model associated to a topological Anosov flow. We will as well state some facts about this smooth flow.

Let $n = n(\gamma, \Sigma)$ and $m = m(\gamma, \Sigma)$ be the linking number and multiplicity of γ. Consider a smooth decreasing function $\rho : [0, 1] \to [0, 1]$ such that:

- $\rho(t) = 1$, for $0 \leq t \leq \frac{1}{3}$,
- $\rho'(t) < 0$, for $\frac{1}{3} < t < \frac{2}{3}$,
- $\rho(t) = 0$, for $\frac{2}{3} \leq t \leq 1$.

Definition 4.26. Define $\varphi : \partial M_R(r_1, r_2) \to \partial \mathbb{V}(r_1, r_2)$ in the following way: For every $p \in \partial M_R(r_1, r_2)$ choose $1 \leq i \leq 4$ such that p belongs to the i-th quadrant and denote its coordinates by $(x, y, z) = \Pi_i(p) \in \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$.

1. If $n = n(\gamma, \Sigma) < 0$ then

$$\varphi(p) = \begin{cases} (x\ ,\ y\ ,\ z) & ;\ \text{if } p \notin A^1_{in}, \\ \left(r_1\ ,\ y\ ,\ z + \frac{m}{|n|}\rho\left(\frac{y}{r_2}\right)\right) & ;\ \text{if } p \in A^1_{in} \text{ and } (r_1, y, z) = \Pi_1(p). \end{cases} \tag{4.14}$$

2. If $n = n(\gamma, \Sigma) > 0$ then

$$\varphi(p) = \begin{cases} (x\ ,\ y\ ,\ z) & ;\ \text{if } p \notin A^4_{in}, \\ \left(r_1\ ,\ y\ ,\ z + \frac{m}{|n|}\rho\left(-\frac{y}{r_2}\right)\right) & ;\ \text{if } p \in A^4_{in} \text{ and } (r_1, y, z) = \Pi_4(p). \end{cases} \tag{4.15}$$

In figure 4.5 we illustrate the map φ in normal coordinates for the case of negative linking number, defined in (4.15). Observe that, in normal coordinates, this maps looks like the identity for every point not in A^1_{in}. In the figure the depict the action of φ along the first quadrant.

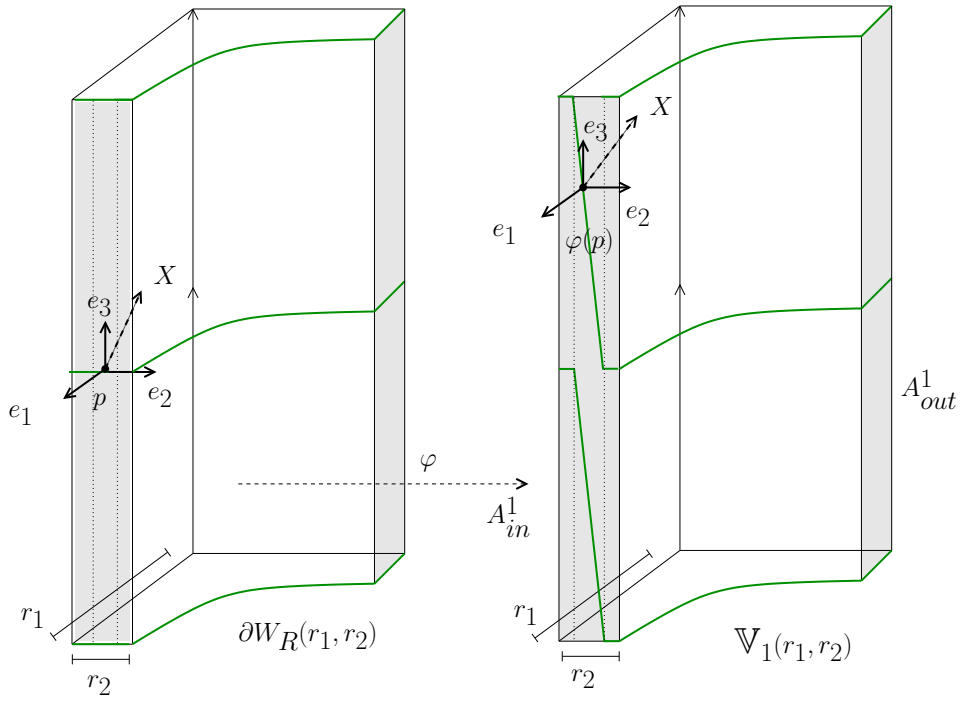


Figure 4.5: Glueing map in the first quadrant for negative linking number.

Lemma 4.27. *For every couple of small parameters $0 < r_2 < r_1 < 1$ the map φ defined above is a well-defined diffeomorphism. Moreover, consider the manifold*

$$N(r_1, r_2) = M_R(r_1, r_2) \sqcup_\varphi \mathbb{V}(r_1, r_2).$$

Then, there exists a smooth atlas $\mathcal{D}$ and a smooth vector field Y on $N(r_1, r_2)$ such that:

(*i*) *The inclusion $\iota : M_R(r_1, r_2) \hookrightarrow N(r_1, r_2)$ is a diffeomorphism onto its image and $\iota_* : X_\Gamma \mapsto Y$,*

(*ii*) *The inclusion $\iota : \mathbb{V}(r_1, r_2) \hookrightarrow N(r_1, r_2)$ is a diffeomorphism onto its image and $\iota_* : X_{(\lambda,n)} \mapsto Y$.*

Definition. We denote by

$$\psi_t : N \to N \tag{4.16}$$

the flow generated by the vector field Y on the smooth manifold $N = N(r_1, r_2)$ given by lemma 4.27.

Notation: For notational simplicity we will set $M_0 = M_R(r_1, r_2)$ and $M_1 = \mathbb{V}(r_1, r_2)$, and for each $j = 0, 1$ we denote by ϕ_t^j, X_j, $TM_j = E_j^s \oplus E_j^c \oplus E_j^u$ and $|\cdot|_j$ the corresponding flow, vector field, splitting and Riemannian metric. Denote by $N = M_0 \sqcup_\varphi M_1$ and by $\iota_j : M_j \to N$ the natural inclusion maps. The two components are glued along the subset $M_0 \cap M_1 \simeq \partial M_0 \simeq \partial M_1$ that is homeomorphic to a two dimensional torus.

Proof. Observe that, when written in coordinates 4.22, the map φ consists in apply a twist map, supported on the annuli A_{in}^1 or A_{in}^4, depending on the signature of the linking number. This assumption will be important in 3.4 for the purpose of constructing a Birkhoff section. Along this proof we will stay in the case $n < 0$, being analogous the other one.

Observe that the partition of the neighbourhood W in quadrants decomposes ∂R in four regions, individually homeomorphic to the annulus. To show that φ is well-defined we have to check its definition along the intersections of these four regions. It is well-defined over the intersections $A_{out}^1 \cap A_{out}^2$, $A_{in}^2 \cap A_{in}^3$ and $A_{out}^3 \cap A_{out}^4$ because $\Pi_{i+1} \circ \Pi_1^{-1} = id$, for $i = 1, 2, 3$. To check that it is well-defined on the intersection $A_{in}^1 \cap A_{in}^4$, observe that if $p \in A_{in}^1$ has coordinates $\Pi_1(p) = (r_1, 0, z)$ then $\varphi(p) = (r_1, 0, z - \frac{m}{n})$. In particular, we have that

$$\Pi_4 \circ \Pi_1^{-1}(p) = \varphi(p), \ \forall\, p \in \ A_{in}^1 \cap A_{in}^4.$$

Thus φ is a well-defined smooth bijection between the boundary of $R(r_1, r_2)$ (see the description following 4.24) and the boundary of $\mathbb{V}(r_1, r_2)$.

Recall that to associate a smooth atlas on the quotient $N = M_0 \sqcup_\varphi M_1$ consists in adding, for each point $p = \varphi(p)$ in $\iota_0(M_0) \cap \iota_1(M_1)$, a local parametrization $\sigma_p : U \subset$

$\mathbb{R}^3 \to \sigma_p(U) \subset N$ in a neighbourhood of p satisfying that $\sigma_p^{-1} : \sigma_p(U) \cap M_0 \to U$ and $\sigma_p^{-1} : \sigma_p(U) \cap M_1 \to U$ are diffeomorphisms onto their respective images.

If $p \notin A^1_{in}$ and $U \subset \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ is an open neighbourhood of $(x, y, z) = \Pi_i(p)$ then the map $\sigma_p : U \to N$ given by

$$\sigma_p(x,y,z) = \begin{cases} (x,y,z) & ; \text{ if } (x,y,z) \in \mathbb{V}(r_1, r_2), \\ \Pi_i^{-1}(x,y,z) & ; \text{ if } (x,y,z) \notin \mathbb{V}(r_1, r_2), \end{cases} \tag{4.17}$$

works as parametrization.

In the region A^1_{in} (where X_0 and X_1 are transverse to the boundary) we choose the map $\sigma^1_{in} : (-\varepsilon, \varepsilon) \times \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to N$ defined by

$$\sigma^1_{in}(t,r,s) = \begin{cases} \phi^0_t\left(\Pi_1^{-1}(r_1, r, s)\right) & ; \text{ if } t \leq 0, \\ \phi^1_t(r_1, r, s - \frac{m}{n}\rho(\frac{r}{r_2})) & ; \text{ if } t \geq 0. \end{cases} \tag{4.18}$$

Then, the family $\mathcal{D} = \{\sigma_p : p \notin A^1_{in}\} \cup \{\sigma^1_{in}\}$ provides a compatible atlas on $\iota_0(M_0) \cap \iota_1(M_1)$ that extends the differentiable structures on each component M_j, $i = 1, 2$. In addition observe that, by construction, for every $p \in \partial M_0$ this atlas gives an identification

$$\Phi_p : T_pM_0 \to T_{\varphi(p)}M_1$$

that satisfies $\Phi_p : X_0(p) \mapsto X_1(\varphi(p))$. This means that the vector fields on each component M_0 and M_1 match together along the boundary and define a smooth a vector field Y in N, in the coordinates of the atlas $\mathcal{D}$. □

Smooth atlas near $M_0 \cap M_1$.

It will be convenient to give an expression in the local coordinates 4.22 for the transformations

$$\Phi_p : T_pM_0 \to T_{\varphi(p)}M_1 \tag{4.19}$$

defined above. We will always assume negative linking number, so φ is given by the formula (4.15).

By definition the chart Φ_p is given by (4.18) for points $p \in A^1_{in}$ and by (4.17) otherwise. Observe that for every $p \in A^1_{in}$ with normal coordinates $p = (r_1, r, s)$ there is a basis $\mathcal{C}(p) = \{X_{(\lambda,n)}(p), e_2(p), e_3(p)\}$, where

$$\begin{aligned} X_{(\lambda,n)}(p) &= \left(\log(\lambda) r_1, -\log(\lambda) r, \frac{1}{|n|}\right) \in \mathbb{R}^3, \\ e_2(p) &= \frac{\partial}{\partial y}(p) = (0,1,0) \in \mathbb{R}^3, \\ e_3(p) &= \frac{\partial}{\partial z}(p) = (0,0,1) \in \mathbb{R}^3. \end{aligned}$$

It follows that

Lemma 4.28.

1. *If $p \in \partial M_0$ is not in A^1_{in} then $\varphi(p) = p$ and Φ^*_p is just the identity on $\mathbb{R}^3$.*
2. *If $p \in A^1_{in}$ has coordinates (r_1, r, s) we have that*

$$(\Phi_p)_{\mathcal{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{\kappa(r)}{r_2} & 1 \end{pmatrix}, \text{ where } \kappa(r) = \frac{|m|}{|n|} \left|\rho'(r/r_2)\right|. \tag{4.20}$$

There is another basis defined for every point $p \in M_0 \cap M_1$ given by $\mathcal{B}(p) = \{X_{(\lambda,n)}(p), e_1(p), e_2(p)\}$, where $e_1(p) = (1, 0, 0)$. The vectors e_1 and e_2 are unitary vectors contained in the stable and unstable bundles of the flow $\phi_t^{X_{(\lambda,n)}}$. It will be convenient to express the transformations (4.19) in the basis $\mathcal{B}(p)$ for the points $p \in A^1_{in}(p)$.

Lemma 4.29. *Let $p \in A^1_{in}$ with coordinates (r_1, r, s). Then,*

$$(\Phi_p)_{\mathcal{B}} = \begin{pmatrix} 1 & -\frac{\kappa(r)|n|r}{r_1 r_2} & -\frac{\kappa(r)|n|}{r_2} \\ 0 & \kappa(r)|n|\log(\lambda)\frac{r}{r_2} + 1 & \kappa(r)|n|\log(\lambda)\frac{r_1}{r_2} \\ 0 & -\kappa(r)|n|\log(\lambda)\frac{r^2}{r_1 r_2} & -\kappa(r)|n|\log(\lambda)\frac{r}{r_2} + 1 \end{pmatrix}.$$

The action of the flow $\psi_t : N \to N$:

The action of the flow ψ_t on a point p in the manifold $N = M_0 \sqcup_\varphi M_1$ can be described as an alternated composition of the flows ϕ^j_t on each component M_j and the glueing map φ.

More precisely, consider a point $p \in int(M_0)$ and some $t > 0$ such that $\psi_t(p) \in int(M_0)$. If all the orbit segment $[p, \psi_t(p)]$ is contained in M_0 then we simply have $\psi_t(p) = \phi^0_t(p)$. If not, there is a first intersection point

$$p_1 = \phi^0_{t_1}(p), \text{ for some } 0 < t_1 < t,$$

of the orbit segment with the set $M_0 \cap M_1$, necessarily contained in a region of where the vector field X_0 is transverse to ∂M_0 and points outward M_0. At this point the positive semi-orbit of p traverses from the component M_0 to M_1 and the negative ϕ^0_t-orbit of p_1 is connected with the positive ϕ^1_t-orbit of $\varphi(p_1)$. In the same way, since we assume $\psi_t(p) \in int(M_0)$, the positive ϕ^1_t-obit starting at $\varphi(p_1)$ must leave the component M_1 on a point

$$q_1 = \phi^1_{s_1}(\varphi(p_1)), \quad \text{for some } 0 < s_1 < t.$$

When going from M_1 to M_0 the glueing of semi-orbits is similar, but we have to use φ^{-1} instead. We remark here that:

Remark. There exists a constant $T > 0$ such that, for both $j = 0, 1$,

$$\min\left\{t > 0 : \exists\, p \in \partial M_j \text{ such that } \phi^j_t(p) \in \partial M_j \text{ and } \phi^j_s(p) \in int(M_j),\ \forall\, 0 \le s \le t\right\} \ge T.$$

It follows that:

Lemma 4.30. *There exist* $0 < t_1, \dots, t_{l+1} < t$ *and* $0 < s_1, \dots, s_l < t$ *satisfying that* $t_k \geq T$ *for* $k = 2, \dots, l$, $s_k \geq T$ *for* $k = 1, \dots, l$, *their sum is* $\sum_{k=0}^{l} t_k + \sum_{k=1}^{l} s_k = t$ *and*

$$\psi_t(p) = \phi^0_{t_{l+1}} \circ \left(\varphi^{-1} \circ \phi^1_{s_l} \circ \varphi\right) \circ \cdots \circ \left(\varphi^{-1} \circ \phi^1_{s_2} \circ \varphi\right) \circ \phi^0_{t_2} \circ \left(\varphi^{-1} \circ \phi^1_{s_1} \circ \varphi\right) \circ \phi^0_{t_1}(p). \tag{4.21}$$

These quantities t_k and s_k are defined in order to satisfy that the ψ_t-orbit of p traverse from one component to the other at times $T_1 = t_1$, $T_k = t_1 + s_1 + \cdots + s_{k-1} + t_k$ and $S_k = T_k + s_k$ for $k = 1, \dots, l$. Define the sequence of points $p_k = \psi(T_k, p)$ and $q_k = \psi(S_k, p)$, which are points in $M_0 \cap M_1$. Then, the positive semi-orbit of p traverses from M_0 to M_1 at the points p_k and from M_1 to M_0 at the points q_k.

There is a straightforward analogue of statement (4.21) above for any chosen configuration of points $p \in M_j$, $j = 0, 1$ and times $t \in \mathbb{R}$.

The action on the tangent bundle $D\psi_t : TN \to TN$:

On each component M_j the action of $D\phi^j_t$ can be described using the invariant splitting: Consider $p \in M_j$ and $t > 0$ such that $\phi^j_t(p) \in M_j$ and all the orbit segment $[p, \phi^j_t(p)]$ is contained in M_j. Let $e^j_s(p)$ and $e^j_u(p)$ be two unitary vectors contained in $E^s_j(p)$ and $E^u_j(p)$, respectively. By pushing these vectors with $D\phi^j_s(p)$ we obtain a frame

$$\{X(\phi^j_s(p)), e^j_s(\phi^j_s(p)), e^j_u(\phi^j_s(p))\},\ 0 \leq s \leq t$$

where

$$\begin{aligned} e^j_s(\phi^j_s(p)) &= \lambda^{-t} D\phi^j_s(p) \cdot e^j_s(p) \\ e^j_u(\phi^j_u(p)) &= \lambda^{\ t} D\phi^j_u(p) \cdot e^j_u(p). \end{aligned}$$

It follows that for $j = 0, 1$ we have

$$D\phi^j_t(p) : aX(p) + be^j_s(p) + ce^j_u(p) \mapsto aX(\phi^j_t(p)) + \lambda^t be^j_s(\phi^j_t(p)) + \lambda^{-t} ce^j_u(\phi^j_t(p)). \tag{4.22}$$

Framing on TN. To study the action $D\psi_t$ on TN, we define a (non-continuous) splitting of TN by

$$T_pN = H^s(p) \oplus H^c(p) \oplus H^u(p) = \begin{cases} E^s_1(p) \oplus E^c_1(p) \oplus E^u_1(p), & \text{if } p \in M_1, \\ E^s_0(p) \oplus E^c_0(p) \oplus E^u_0(p), & \text{if } p \in N \setminus int(M_1), \end{cases} \tag{4.23}$$

for all $p \in M$. We can see how does $E^s_0 \oplus E^c_0 \oplus E^u_0$ matches $E^s_1 \oplus E^c_1 \oplus E^u_1$ along $M_0 \cap M_1$ using the identifications (4.19). For every $p \in \partial M_0$ the map Φ_p send $X_0(p) \mapsto X_1(\varphi(p))$,

so it follows that the bundle $H^c = span\{Y\}$ is continuous. By lemma 4.28 we have that, for every point p not in A^1_{in}, this transformation is the identity in normal coordinates, thus

$$\Phi_p : \begin{cases} E^s_0(p) \mapsto E^s_1(\varphi(p)), \\ E^c_0(p) \mapsto E^c_1(\varphi(p)), \\ E^u_0(p) \mapsto E^u_1(\varphi(p)). \end{cases}$$

Nevertheless, the bundles on each side can be not identified by Φ_p for points $p \in A^1_{in}$, so the decomposition (4.23) may be non-continuous on this set.

Moreover, consider some $p \in N$ and some $t > 0$ such that the orbit segment $[p, \psi_t(p)]$ does not intersect A^1_{in}. Given a basis $\{Y(p), e_s(p), e_u(p)\}$ of T_pN satisfying that $e_s(p) \in H^s(p)$ and $e_u(p) \in H^u(p)$ are unitary vectors, there exists an extension into a continuous frame

$$\{Y(\psi_s(p)), e_s(\psi_s(p)), e_u(\psi_s(p))\},\ 0 \leq s \leq t \tag{4.24}$$

defined along the orbit segment $[p, \psi_t(p)]$, where $e_s(\psi_s(p)),\ e_u(\psi_t(p))$ are unitary vectors.

Following the discussion in the previous paragraphs, given $p \in N$ and $t > 0$ let $\{p_1, \ldots, p_l\}$ be the set of intersection points between the orbit segment $\psi_{[0,t]}(p)$ and the annulus A^1_{in}. Then, we have that

Lemma 4.31. *There exist $0 < t_1, \ldots, t_{l+1} < t$, uniformly bounded away from zero by a constant $T > 0$, such that $D\psi_t$ is an iterated composition of the form:*

$$D\psi_t(p) = \Psi_{t_{l+1}} \circ \Phi_{p_l} \circ \Psi_{t_l} \cdots \circ \Phi_{p_2} \circ \Psi_{t_2} \circ \Phi_{p_1} \circ \Psi_{t_1}(p), \tag{4.25}$$

where Ψ_t denotes the application $T_pN \to T_{\psi_t(p)}N$, defined for all the couples (t, p) satisfying that $[p, \psi_t(p)] \cap A^1_{in} = \emptyset$, by the expression:

$$\Psi_t : aY(p) + be_s(p) + ce_u(p) \mapsto aY(\psi_t(p)) + \lambda^t be_s(\psi_t(p)) + \lambda^{-t} ce_u(\psi_t(p)). \tag{4.26}$$

Here, $\{Y, e_s, e_u\}$ is a continuous frame as defined in (4.24).

3.3 The smooth model (ψ_t, N) is Anosov

Proposition 4.32. *If $0 < r_2 < r_1 < 1$ are sufficiently small, then the flow $\psi_t : N \to N$ defined in (4.16) is Anosov.*

The proof of proposition 4.32 proceeds by applying the cone field criterion; see [41] for a precise statement. We will show first in corollary 4.36 that the natural action of ψ_t on the tangent bundle TN preserves a pair of center-stable and center-unstable 2-dimensional plane fields by application of this criterion. Then, in corollary 4.41 we apply the same criterion inside the center-stable and center-unstable planes to conclude the existence of a hyperbolic splitting preserved by $D\psi_t$.

Existence of the center-stable and center-unstable invariant distributions.

In the construction of φ there are an auxiliary smooth function ρ and two parameters $0 < r_2 < r_1 < 1$. The maximum size of these real parameters depend on the charts Π_i, but they can be chosen arbitrarily small. We assume from now on that ρ is fixed and we will adjust the parameters r_1, r_2 to satisfy the proposition.

Stable and unstable slopes. Given a point $p \in N$, we consider a *positively oriented* basis of T_pN of the form $\{Y(p), e_s(p), e_u(p)\}$, where $e_s(p)$ and $e_u(p)$ are unitary vectors contained in the spaces $H^s(p)$ and $H^u(p)$ of the decomposition (4.23), respectively. Given a vector $v = aY(p) + be_s(p) + ce_u(p)$ in T_pN we define its *u-slope* and *s-slope* respectively as

$$\Delta_u(v) = \frac{b}{c} \text{ and } \Delta_s(v) = \frac{c}{b}.$$

Observe that we have two possibilities for choosing a positive basis as before, but the slope is unchanged by switching this election, so the u,s-slopes are well defined. We remark as well that these slope functions do not vary continuously over the set A^1_{in}, due to the discontinuity of the bundles H^s, H^u.

Cone distributions. Given a pair of continuous functions $\delta_0, \delta_1 : N \to \mathbb{R}$ satisfying that $-\infty < \delta_0(p) < \delta_1(p) < +\infty$ for every $p \in N$, we define two cone distributions

$$C^{cu}(p; \delta_0(p), \delta_1(p)) = \{v \in T_pN \ : \ \delta_0(p) \leq \Delta_u(v) \leq \delta_1(p)\} \tag{4.27}$$
$$C^{cs}(p; \delta_0(p), \delta_1(p)) = \{v \in T_pN \ : \ \delta_0(p) \leq \Delta_s(v) \leq \delta_1(p)\}. \tag{4.28}$$

At each point p this is a pair of 2-dimensional cones, with axis determined by the vector $Y(p)$. Observe that for adequate values of $\delta_i(p)$ these cones are complementary, i.e. their intersection is just $H^c(p)$. We remark that these cone distributions do not vary continuously as a function of p, the possible discontinuities being on the set A^1_{in}. But this posses no obstructions to apply the cone field criterion on this family of cones.

In what follows we will show that for some adequate slope functions δ^u_0, δ^u_1 and δ^s_0, δ^s_1 there is a pair of complementary cones satisfying the cone field criterion under the action of $D\psi_t : TN \to TN$.

Lemma 4.33. *Let $\delta^u_0, \delta^s_0, \delta^u_1, \delta^s_1 : N \to \mathbb{R}$ be the functions defined in (4.33) below. If $0 < r_2 < r_1 < 1$ are sufficiently small, then there exists $T > 0$ such that: For every $p \in N$ then*

$$D\psi_T\left(C^{cu}\left(p \ ; \ \delta^u_0(p), \delta^u_1(p)\ \right)\right) \subset C^{cu}\left(\psi_T(p) \ ; \ \delta^u_0(\psi_T(p)), \delta^u_1(\psi_T(p)\ )\right),$$
$$D\psi_{-T}\left(C^{cu}\left(p \ ; \ \delta^s_0(p), \delta^s_1(p)\ \right)\right) \subset C^{cu}(\psi_{-T}(p) \ ; \ \delta^s_0(\psi_{-T}(p)), \delta^s_1(\psi_{-T}(p)\ )).$$

Lemma 4.34. *For the parameters $0 < r_2 < r_1 < 1$ given in the previous lemma, it is satisfied the following: There exists a constant $L_0 > 0$ such that, for every $p \in N$ and*

every $t \geq 0$ then

$$| \Delta_u(D\psi_t(p)\cdot v_2) - \Delta_u(D\psi_t(p)\cdot v_1) | \leq \lambda^{2t} L_0, \ \forall\ v_1, v_2 \in C^{cu}(p; \delta_0^u(p), \delta_1^u(p)),$$
$$|\Delta_s(D\psi_{-t}(p)\cdot v_2) - \Delta_s(D\psi_{-t}(p)\cdot v_1)| \leq \lambda^{2t} L_0, \ \forall\ v_1, v_2 \in C^{cs}(p; \delta_0^s(p), \delta_1^s(p)).$$

Lemma 4.35. *By shrinking the parameters $0 < r_2 < r_1 < 1$ of the previous lemmas if necessary, it is satisfied that: There exist constants $L > 0$ and $\mu > 1$ such that, for every $p \in N$ and every $t \geq 0$ then*

$$\|D\psi_t(p)\cdot v \ \| \geq L\mu^t \|v\|, \ \forall\ v \in C^{cu}(p; \delta_0^u(p), \delta_1^u(p)) \setminus H^c(p),$$
$$\|D\psi_{-t}(p)\cdot v\| \geq L\mu^t \|v\|, \ \forall\ v \in C^{cu}(p; \delta_0^s(p), \delta_1^s(p)) \setminus H^c(p).$$

Action on the *su*-bundle. To study the action of $D\psi_t$ on cone distributions, it is convenient to consider the vector bundle $\bar{H}^{su} \to N$ defined by the quotient

$$\bar{H}^{su}(p) \coloneqq T_pN / H^c(p), \ \forall\ p \in N,$$

which carries an induced action of $D\psi_t$. Observe that there is a decomposition of this bundle, of the form $\bar{H}^{su} = \bar{H}^s \oplus \bar{H}^u$, induced by the spaces H^s and H^u. We remark that this decomposition does not vary continuously along the set A^1_{in}. Exactly as in (4.24) above, along every orbit segment disjoint from A^1_{in} there is a continuous frame $\{\bar{e}_s, \bar{e}_u\}$, induced by a continuous frame $\{e_s, e_u\}$. By lemma 4.31, the action of ψ_t on the bundle $\bar{H}^{su}$ is an alternated composition of transformations of the form Ψ_t and Φ_p. We have that:

1. If $p \notin A^1_{in}$ and $t > 0$ satisfies that $[p, \psi_t(p)] \cap A^1_{in} = \emptyset$, by lemma 4.31 we see that the matrix associated to the action $D\psi_t$ on the *su*-bundle, on in a basis $\{\bar{e}_s, \bar{e}_u\}$, is given by
$$(\Psi_t)_{su} = \begin{pmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{pmatrix}. \tag{4.29}$$

2. Let $q \in A^1_{in}$. We want to see how Φ_q transforms cones in T_qN into cones in $T_{\varphi(q)}N$. In the system of normal coordinates (4.1) the annulus A^1_{in} corresponds to the set $\{r_1\} \times [0, r_2] \times \mathbb{R}/\mathbb{Z}$, so let (r_1, r, s) be the coordinates of q. In lemma 4.29 there is a matrix for this transformation in the basis $\{X_{(\lambda,n)}, e_1, e_2\}$ of $\mathbb{R}^3$. Recall that in normal coordinates $span\{X_{(\lambda,n)}\} = H^c$, $span\{e_1\} = H^s$ and $span\{e_2\} = H^u$.

   By this lemma we have that the action of Φ_q in the *su*-bundle is given by the matrix
$$(\Phi_q)_{su} = \begin{pmatrix} K(r, r_2) + 1 & K(r, r_2)\frac{r_1}{r} \\ -K(r, r_2)\frac{r}{r_1} & -K(r, r_2) + 1 \end{pmatrix}, \tag{4.30}$$
   where $K(r, r_2) = \kappa(r)|n| \log(\lambda)\frac{r}{r_2}$, and $\kappa : [0, r_2] :\to [0, \infty)$ is the continuous bounded function given in 4.28. Remark the K is a non-positive function. Thus, we obtain a family of transformations parametrized over $0 \leq r \leq r_2$. Observe that Φ_q is non-trivial just for points $q = (r_1, r, s)$ with $\frac{r_2}{3} \leq r \leq \frac{2r_2}{3}$, due to the definition of $\rho : [0, 1] \to \mathbb{R}$.

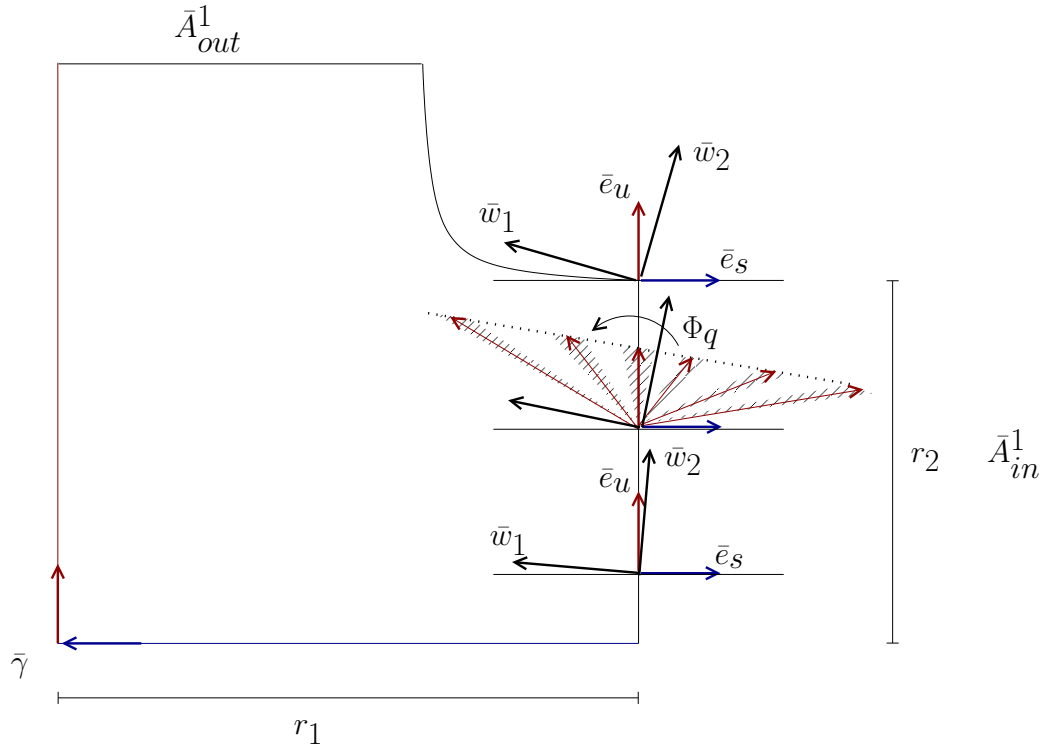


Figure 4.6: The map Φ_q in su-coordinates.

This matrix has determinant equal to one and trace equal to two, thus it has a double eigenvalue equal to one. The vector $\bar{w}_1 = -\bar{e}_s + \frac{r}{r_1}\bar{e}_u$ is an eigenvector for this matrix in the $\bar{H}^{su}$ plane. Consider the vector $\bar{w}_2 = \frac{r}{r_1}\bar{e}_s + \bar{e}_u$. In the orthogonal basis $\{\bar{w}_1, \bar{w}_2\}$ the transformation (4.30) takes the form

$$(\Phi_q)_{\{\bar{w}_1,\bar{w}_2\}} = \begin{pmatrix} 1 & \eta(r)\frac{r_1}{r_2} \\ 0 & 1 \end{pmatrix} \tag{4.31}$$

for some bounded function $\eta : [0, r_2] \to [0, \infty)$, with support in $[\frac{r_2}{3}, \frac{2r_2}{3}]$. We illustrate the action of the matrix (4.30) in figure 4.6.

We prove now the three lemmas stated above. Fix some $p \in N$ and $t > 0$, and let $0 < t_1, \ldots, t_{l+1} < 1$ and $\{p_1, \ldots, p_l\}$ as given by (4.21). Write the action of ψ_t on TN as

$$D\psi_t(p) = \Psi_{t_{l+1}} \circ \Phi_{p_l} \circ \Psi_{t_l} \circ \cdots \circ \Phi_{p_1} \circ \Psi_{t_1}(p). \tag{4.32}$$

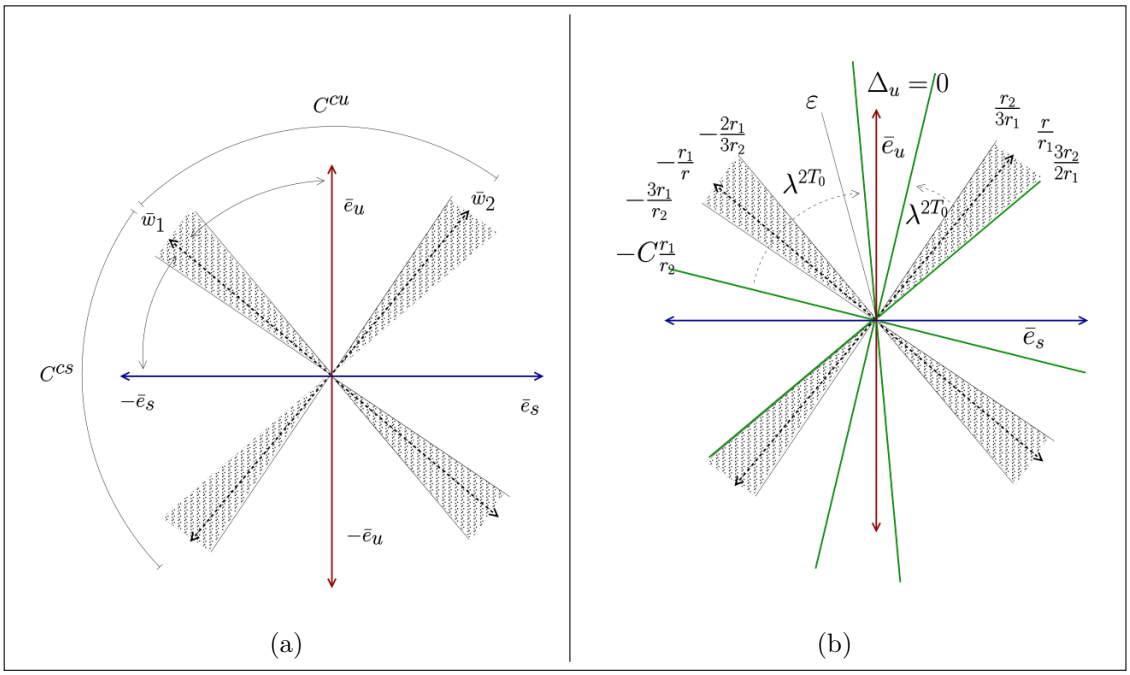


Figure 4.7: Cones in the su-plane

At each point $q = (r_1, r, s) \in A^1_{in}$ the transformation Φ_q has an eigenvector $\bar{w}_1(q)$ whose slope is $\Delta_u(\bar{w}_1(q)) = -r_1/r < 0$. For having the invariance property in the first lemma we need, at each $q \in A^1_{in}$ where Φ_q is non-trivial, the sector determined by the vectors $\{\bar{w}_1(q), \bar{e}_u(q)\}$ to be contained in the cone $C^{cu}(q; \delta^u_0, \delta^u_1)$, and the sector determined by the vectors $\{-\bar{e}_s(q), \bar{w}_1(q)\}$ to be contained in $C^{cs}(q; \delta^s_0, \delta^s_1)$. Since Φ_q is non-trivial only when $\frac{r_2}{3} \leq r \leq \frac{2r_2}{3}$, we encounter that the slope of an eigenvector corresponding to a non-trivial Φ_q is bounded between $-\frac{3r_1}{r_2}$ and $-\frac{3r_1}{2r_2}$. See figure 4.7a.

Choice of slope functions. We define two distributions of cones on TN in the following way: Consider a continuous bounded function $\delta : N \to (-\infty, 0)$ satisfying that:

(i) $\delta(p) = -r_1/r$ for every $p \in A^1_{in}$ with $\frac{r_2}{3} \leq r \leq \frac{2r_2}{3}$,

(ii) $\min(\delta) = -\frac{3r_1}{r_2}$ and $\max(\delta) = -\frac{3r_1}{2r_2}$.

Define

$$\delta^u_0 : N \to (-\infty, 0) \text{ such that } \delta^u_0(p) = \delta(p), \tag{4.33}$$
$$\delta^u_1 : N \to (0, +\infty) \text{ such that } \delta^u_1(p) = -1/\delta(p), \tag{4.34}$$
$$\delta^s_0 : N \to (-\infty, 0) \text{ such that } \delta^s_0(p) = 1/\delta(p), \tag{4.35}$$
$$\delta^s_1 : N \to (0, +\infty) \text{ such that } \delta^s_1(p) = -\delta(p). \tag{4.36}$$

We consider now the cones $C^{cu}(p;\delta_0^u(p),\delta_1^u(p))$ and $C^{cs}(p;\delta_0^s(p),\delta_1^s(p))$, defined for all $p\in N$.

Proof of lemma 4.33. We make the proof for the *cu*-case, being analogous the other one. We will show that we can chose $0<r_2<r_1<1$ sufficiently small such that the cone distribution C^{cu} satisfies the invariance property of the lemma.

We start by pointing out that there exists a constant $\varepsilon<0$, which depends just on the ratio r_2/r_1, such that the cone $C^{cu}(q;\varepsilon,0)$ is sent inside $C^{cu}(\varphi(q);\delta_0^u(q),0)$, for every $q\in A_{in}^1$.

To see this, consider a point $q\in A_{in}^1$ with coordinates (r_1,r,s). By (4.31) the vector $\Phi_q(\bar{e}_u)$ satisfies

$$-\frac{r_1}{r}=\Delta_u(\bar{w}_1)<\Delta_u(\Phi_q(\bar{e}_u))=-\left(\frac{-K(r,r_2)}{1-K(r,r_2)}\right)\frac{r_1}{r}<0.$$

Then, by continuity there exist $-\infty<\varepsilon<0$ such that, for every $q=(r_1,r,s)\in A_{in}^1$ and for every $v\in T_qN$ with $\varepsilon<\Delta_u(v)\leq 0$, it follows that

$$-\frac{r_1}{r}=\delta_0^u(q)<\Delta_u(\Phi_q(v))<0. \tag{4.37}$$

The expression above for the u-slope of the image of $\bar{e}_u$ shows that ϵ depends just on the ratio r_2/r_1. That is, if we change r_1 and r_2 for smaller values keeping constant its ratio, then property (4.37) is still valid with the same constant ε. To see this, observe that $K(r,r_2)$ is equal to r/r_2, up to multiplication by a bounded continuous function that depends just on r and is null if $r/r_2\notin[1/3,2/3]$. Thus, the slope of $\Phi_q(\bar{e}_u)$ above has a bound of the form $(cte)\cdot r_1/r_2$. Thus allows to consider a constant $\varepsilon<0$ of small modulus, just depending on r_2/r_1.

Consider now an auxiliary vector $v\in\bar{H}^{su}(q)$ of slope $\Delta_u(v)\geq\min(\delta)=-3r_1/r_2$. By (4.30) we have that

$$\Delta_u(\Phi_q(v))\geq-\frac{3r_1}{r_2}\cdot\left(\frac{K(r,r_2)(1-\frac{r_2}{3r})+1}{-K(r,r_2)(1-\frac{3r}{r_2})+1}\right).$$

Since the function between parenthesis on the right side of the previous equation is bounded as a function of the parameters $0<\frac{1}{3}r_2\leq r\leq\frac{2}{3}r_2<r_1<1$ (recall that $K(r,r_2)=(cte)\cdot r/r_2$), we have that there exists a constant $C>0$, independent of r,r_2,r_1, such that

$$-C\cdot\frac{r_1}{r_2}\leq\min\left(\Delta_u(\Phi_q(v)):q\in A_{in}^1\text{ and }-\frac{3r_1}{r_2}\leq\Delta_u(v)\leq 0\right). \tag{4.38}$$

Let $T_0=T_0(r_1,r_2)>0$ such that $\varepsilon=<-C\lambda^{2T_0}r_1/r_2<0$. Defined in this way, $T_0(r_1,r_2)$ is the time needed for a transformation Ψ_t to send a cone of the form $C^{cu}(p;-Cr_1/r_2,0)$ inside the cone $C^{cu}(\psi_t(p);\varepsilon,0)$, and depends just on the ratio r_1/r_2.

Let $T_1^+ = T_1^+(r_1, r_2)$ be defined by

$$T_1^+(r_1,r_2) = \min\left\{t>0 : \exists\ p \in \{r_1\}\times\left[\frac{r_2}{3},\frac{2r_2}{3}\right]\times\mathbb{R}/\mathbb{Z} \text{ s.t. } \psi_t(p)\in\{r_1\}\times\left[\frac{r_2}{3},\frac{2r_2}{3}\right]\times\mathbb{R}/\mathbb{Z}\right\}.$$

This is the minimal returning time of points in the region $\{r_1\}\times\left[\frac{r_2}{3},\frac{2r_2}{3}\right]\times\mathbb{R}/\mathbb{Z}\subset A_{in}^1$ (where Φ_q is non-trivial) onto itself. We claim that:

Claim. If we shrink r_1 and r_2 keeping constant the ratio r_2/r_1, then $T_1^+(r_1,r_2)$ tends to infinity.

Assuming this claim, fix some $0<r_2<r_1$ such that $T_1^+(r_1,r_2) > T_0(r_1,r_2)$ and chose some $T > 2T_1^+(r_1,r_2)$. Without loss of generality we can assume that, in addition,

$$\max(\delta_0^u) = -\frac{3r_1}{2r_2} \le -\lambda^{2T}\frac{3r_1}{r_2} < 0 < \lambda^{2T}\frac{2r_2}{3r_1} < \frac{r_2}{3r_3} = \min(\delta_1^u). \tag{4.39}$$

Let's show that ψ_T preserves the cone distribution $C^{cu}(p;\delta_0^u(p),\delta_1^u(p))$. Let $p\in N$ and consider the associated decomposition (4.32) of $D\psi_t$ as products of Ψ_{t_k} and Φ_{p_k}. Then,

- If $[p,\psi_T(p)]\cap A_{in}^1 = \emptyset$ the action of $D\psi_T$ on TN is just the transformation Ψ_T. By (4.29) and (4.39) we have that

$$\begin{aligned}&\Psi_T\left(C^{cu}(p;\delta_0^u(p),\delta_1^u(p))\right) \subset \Psi_T\left(C^{cu}\left(p;-\frac{3r_1}{r_2},\frac{2r_2}{3r_1}\right)\right)\\ &= C^{cu}\left(\psi_T(p);-\lambda^{2T}\frac{3r_1}{r_2},\lambda^{2T}\frac{2r_2}{3r_1}\right)\subset C^{cu}(\psi_T(p);\delta_0^u(\psi_T(p)),\delta_1^u(\psi_T(p))).\end{aligned}$$

- If $[p,\psi_T(p)]\cap A_{in}^1$ has only one point p_1, then $D\psi_T(p) = \Psi_{t_2}\circ\Phi_{p_1}\circ\Psi_{t_1}$, where either $t_1>T_0$ or $t_2>T_0$. By (4.37) and the particular choice of $T>0$ we have that, in the first case

$$\begin{aligned}&\Phi_{p_1}\circ\Psi_{t_1}\left(C^{cu}(p;\delta_0^u(p),\delta_1^u(p))\right)\subset\Phi_{p_1}\left(C^{cu}\left(p_1;\varepsilon,\lambda^{2T}\frac{2r_2}{3r_1}\right)\right)\\ &\subset C^{cu}(\varphi(p_1);\delta_0^u(\varphi(p)),\delta_1^u(\varphi(p))),\end{aligned}$$

  while in the second

$$\begin{aligned}&\Psi_{t_2}\circ\Phi_{p_1}\left(C^{cu}(p_1;\delta_0^u(p_1),\delta_1^u(p_1))\right)\subset\Psi_{t_2}\left(C^{cu}\left(\varphi(p_1);-C\frac{r_1}{r_2},\frac{2r_2}{3r_1}\right)\right)\\ &\subset C^{cu}\left(\psi_{t_2}(\varphi(p_1));\varepsilon,\lambda^{2T}\frac{2r_2}{3r_1}\right)\subset C^{cu}(\psi_{t_2}(p_1);\delta_0^u(\psi_{t_2}(p_1)),\delta_0^u(\psi_{t_2}(p_1))).\end{aligned}$$

- If $[p,\psi_T(p)]\cap A_{in}^1$ contains more than one point, then again by (4.37) and the property that $\varepsilon < -C\lambda^{2T}r_1/r_2$, we have that each map $\Psi_{t_k}\circ\Phi_{p_k}$ preserves the cones $C^{cu}(p;\delta_0^u,\delta_1^u)$ for $k=1,\dots,l$.

We have that $D\psi_T$ sends the cone $C^{cu}(p;\delta_0^u(p),\delta_1^u(p))$ inside $C^{cu}(\psi_T(p);\delta_0^u(p),\delta_1^u(p))$ in any case. This proves that the C^{cu} cones satisfy the invariance property for adequate values of $0<r_2<r_1<1$, up to the previous claim.

To prove the claim consider $r_2' < r_2$ and $r_1' < r_1$ satisfying that $r_2'/r_1' = r_2/r_1$. Then, the manifold $N(r_1', r_2')$ is obtained replacing the original cross-shaped region for a smaller $\mathbb{V}(r_1', r_2')$ in $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$. See figure 4.8. Let $p = (r_1', r, s)$ be a point in the entrance annulus $r_1' \times [0, r_2'] \times \mathbb{R}/\mathbb{Z}$. Observe that if the future orbit of p comes back to the same annulus, it must traverse the regions D_{in}, D_{out} in the present figure. This means that the returning time $T_1^+(r_1', r_2')$ is bounded from below by twice the time τ for traversing each of these regions. In addition, observe that the time for traversing each D_i from the entrance boundary to the exit one is given by $\lambda^\tau r_1 = r_1'$. Therefore, we obtain

$$T_1(r_1', r_2') \geq 2\tau(r_1', r_1) = \frac{2}{\log(\lambda)} \log\left(\frac{r_1'}{r_1}\right),$$

which proves the claim.

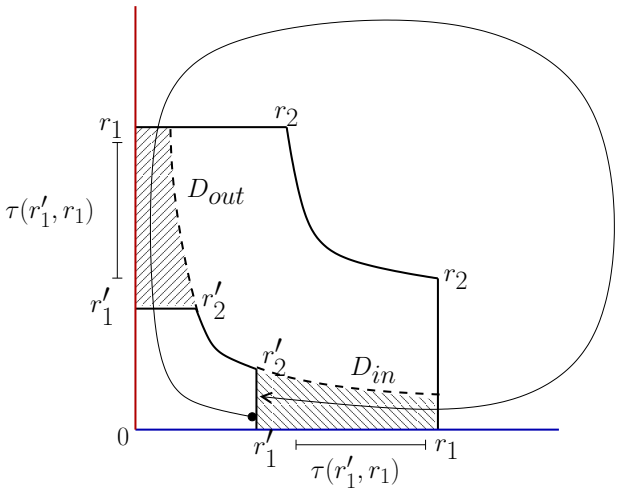


Figure 4.8: The returning time to the annulus at position $r_1 \times [0, r_2] \times \mathbb{R}/\mathbb{Z}$ increases when $r_2/r_1 \to 0$.

For the center-stable cone distributions, the same reasoning apply. This completes the first lemma. □

Proof of lemma 4.34. Let $v_1, v_2 \in C^{cu}(p; \delta_0^u(p), \delta_1^u(p))$. By (4.29) the action of each transformation Ψ_t has the effect of contract the difference of slopes between v_1 and v_2 by a ratio exponential in t. More precisely,

$$|\Delta_u(\Psi_t(v_2)) - \Delta_u(\Psi_t(v_1))| = \lambda^{2t} |\Delta_u(v_2) - \Delta_u(v_1)|, \text{ for all } t \geq 0. \tag{4.40}$$

For a point $q \in A_{in}^1$ consider two vectors $v_i = b_i \bar{e}_s + c_i \bar{e}_u$ in $\bar{H}^{su}(q)$ and let (r_1, r, s) be the coordinates of q. From the expression of Φ_q is su-coordinates in (4.30) we have that

$$\Delta_u(v_i) = \frac{(b_i/c_i)(K(r, r_2) + 1) + K(r, r_2)\frac{r_1}{r}}{-(b_i/c_i)K(r, r_2)\frac{r}{r_1} + (1 - K(r, r_2))}. \tag{4.41}$$

Denote by $x \mapsto f(x; r, r_2, r_1)$ the family of rational functions obtained in (4.41) above. Using that $0 < \frac{r_2}{3} \leq r \leq \frac{2r_2}{3} < r_2 < r_1$, and using that $K(r, r_2)$ equals $\frac{r}{r_2}$ up to multiplying by a bounded function of the variable r, it is possible to see that

$$0 < \frac{\partial f}{\partial x}(x; r, r_2, r_1) < 1, \ \forall \ -\frac{r_1}{r} < x < +\infty,$$

independently of r, r_2, r_1. In particular we obtain that, if if $v_1, v_2 \in C^{cu}(q; \delta_0^u(q), \delta_1^u(q))$, in which case $-r_1/r < b_i/c_i < \infty$, then

$$|\Delta_u(\Phi_q(v_2)) - \Delta_u(\Phi_q(v_1))| \leq |\Delta_u(v_2) - \Delta_u(v_1)|\,. \tag{4.42}$$

Given $p \in N$, by lemma 4.31 there exists sequences $t_n \geq T > 0$ and $p_n \in A_{in}^1$, $k \geq 1$, such that the action of $D\psi_t(p)$ on TN decomposes as a product of transformations Ψ_{t_n} and Φ_{p_n}, as defined in (4.32). Then,

- If $\mathcal{O}^+(p; \psi_t) \cap A_{in}^1$ is empty or contains at most one point, then the property follows simply by (4.40).
- If $\mathcal{O}^+(p; \psi_t) \cap A_{in}^1$ contains at least two points, for each $i = 1, 2$ define $v_i^1 = \Psi_{t_1}(v_i)$ and, for $n \geq 2$, $v_i^n = \Psi_{t_n} \circ \Phi_{p_{n-1}}(v_i^{n-1})$. Then, for $n \geq 2$ we have $t_n \geq T_0$ and this implies that v_i^n has u-slope bigger than the constant $\varepsilon < 0$ from the proof of lemma 4.33. In particular, both vectors v_i^n belong to $C^{cu}(p_n; \delta_0^u(p_n), \delta_1^u(p_n))$ and by (4.42) none of the transformations Φ_{p_n} increases slopes, for every $n \geq 2$.

  Since by (4.40) each map $\Psi_{t_{n+1}}$ contracts the difference of the slopes

  $$|\Delta_u(\Phi_{p_n}(v_2^n)) - \Delta_u(\Phi_{p_n}(v_1^n))|$$

  by a factor of $\lambda^{2t_{n+1}}$ we obtain that

$$|\Delta_u(D\psi_t(p) \cdot v_2) - \Delta_u(D\psi_t(p) \cdot v_1)| \leq (\lambda^{-2(t_1+t_2)} |\max(\delta_1^u) - \min(\delta_0^u)|) \cdot \lambda^{2t}, \ \forall\, t \geq t_1 + t_2.$$

This completes the *cu*-case. The analogous reasoning applies for $C^{cs}(p; \delta_0^s, \delta_1^s)$ and the backward action of the flow. □

Proof of lemma 4.35. Let $p \in N$, $v \in T_pN$ and $t \geq 0$. Using the basis defined in (4.23), write $v = aY(p) + be_s(p) + ce_u(p)$. Define $\|v\|_{su} = \sqrt{b^2 + c^2}$, and remark that $\|v\| \geq \|v\|_{su}$.

From the one hand, observe that if $p \in A_{in}^1$ and $v \in T_pN$, then by the Jordan form (4.31) and since the u-slope of the eigenspace of this matrix is bounded between $-3r_1/r_2$ and $-3r_1/2r_2$, it follows that

$$\|\Phi_p(v)\|_{su} > \|v\|_{su}, \ \forall\ v \text{ with } \ -3r_1/r_2 \leq \Delta_u(v) \leq 2r_2/3r_1. \tag{4.43}$$

Let's call $\delta = \Delta_u(v) = b/c$ and assume that $-\infty < \delta < \infty$. From the one hand, since the transformations Ψ_t on the su-plane correspond to the hyperbolic matrices (4.29), it follows that for every $v \in TN$ there exists a constant $0 < Q(\delta) \leq 1$ such that

$$\|\Psi_t(v)\|_{su} \geq Q(\delta)\lambda^{-t}\|v\|_{su}, \ \forall\ t \geq 0. \tag{4.44}$$

To see an expansion on the su-norm by an application of Ψ_t on v, we need to wait some time depending on the slope of v. It can be seen that, if $|\delta| \leq 1/2$ then $|\Psi_t(v)|_{su} > |v|_{su}$, $\forall\ t \geq 0$, and if $|\delta| > 1/2$ then $|\Psi_t(v)|_{su} > |v|_{su}$ if and only if $\Delta_u(\Psi_t(v)) > 1/|\delta|$. Thus, taking into account the action of Ψ_t on u-slopes of the matrices (4.29), given a vector of u-slope δ greater than $1/2$ we need to wait time of order $t \simeq \log(1/|\delta|)/\log(\lambda)$ to see an expansion under the action of Ψ_t.

Consider $T_0 = T_0(r_1, r_2) > 0$ such that

$$-\frac{1}{C}\frac{r_2}{r_1} < -\lambda^{2T_0(r_1,r_2)} C\frac{r_1}{r_2} < 0 < \lambda^{2T_0(r_1,r_2)}\frac{2r_2}{3r_1} < \lambda^{2T_0(r_1,r_2)}\frac{3r_1}{2r_2},$$

where $C > 0$ is the constant defined along the proof of lemma 4.33. For such a choice of T_0 we have that

$$\|\Psi_{T_0}(v)\|_{su} \geq Q_0\lambda^{-T_0}\|v\|_{su} > \|v\|_{su}, \text{ for all } -Cr_1/r_2 < \Delta_u(v) < 2r_2/3r_1,$$

where $Q_0 = \min\{Q(-Cr_1/r_2), Q(2r_2/3r_1)\}$. Thus, we define $\mu = Q_0^{1/T_0}\lambda^{-1}$, which satisfies $1 < \mu < 1/\lambda$, in order to have

$$\|\Psi_{T_0}(v)\|_{su} \geq \mu^{T_0}\|v\|_{su}, \text{ for all } -Cr_1/r_2 < \Delta_u(v) < 2r_2/3r_1, \tag{4.45}$$

Since $T_0(r_1, r_2)$ depends just on r_1/r_2, we can shrink both r_1 and r_2, keeping constant its ratio, in such a way that the minimal returning time $T_1 = T_1(r_1, r_2)$ of points in the support of the surgery to itself is greater that $T_0(r_1, r_2)$, cf. proof of lemma 4.33.

Assuming this condition, let $p \in N$ and in the cone $C^{cu}(p; \delta_0^u(p), \delta_1^u(p))$ consider a vector $v \in T_pN$. For $t \geq 0$ consider the decomposition (4.32) of the action $D\psi_t(p)$ as an alternated product of transformations Ψ_{t_k} and Φ_{p_k}. As in the previous lemma, let $v^1 = \Psi_{t_1}(v)$ and for $k = 2, \ldots, l$ define $v^k = \Psi_{t_k} \circ \Phi_{p_{k-1}}(v^{k-1})$. Then,

- If the positive ψ_t-orbit of p does not intersect A_{in}^1, using (4.44) it follows that

$$\|D\psi_t(p) \cdot v\| \geq Q_0\lambda^{-t}\|v\|_{su} \geq (cte) \cdot \mu^t\|v\|.$$

- If the orbit segment $[p, \psi_t(p)]$ has non-empty intersection with A_{in}^1, observe that, since $t_k \geq T_0$ for every $k = 2, \ldots, l$, using (4.43) and (4.45) we get that for every vector $u \in T_{p_k}N$ satisfying $-3r_1/r_2 < \Delta_u(u) < 2r_2/3r_1$ then

$$\|\Psi_{t_k} \circ \Phi_{p_{k-1}}(u)\|_{su} \geq Q_0\lambda^{-t_k}\|u\|_{su}.$$

  Since v_1 already satisfies these bounds on slope, by induction we get that

$$\|v^l\|_{su} \geq {Q_0}^l\lambda^{-(t_2+\cdots+t_l)}\|v_1\|_{su}.$$

  Since Ψ_{t_1} and $\Psi_{t_{l+1}}$ stretch distances at least by a factor of the form $(cte) \cdot \lambda^{t_1}$ and $(cte) \cdot \lambda^{t_{l+1}}$ respectively, we get that

$$\|D\psi_t(p)\cdot v\| \geq (cte)\cdot {Q_0}^l\lambda^{-(t_1+\cdots+t_{l+1})}\|v\| = (cte)\cdot\mu^{lT_0}\lambda^{-lT_0}\lambda^{-t}\|v\|_{su} \geq (cte)\cdot\mu^t\|v\|.$$

In any case, we have that $D\psi_t(p)$ expands vectors in $C^{cu}(p; \delta_0^u(p), \delta_1^u(p))$ exponentially in t. The same applies for vectors in the cs-distribution $C^{cs}(p; \delta_0^s(p), \delta_1^s(p))$ and backward time iteration of $D\psi_t$. □

Following lemmas 4.33, 4.34 and 4.35, we have that:

Corollary 4.36. *There exist two continuous plane distributions F^{cs} and F^{cu} in the tangent bundle TN satisfying that:*

1. *$D\psi_t(p)\,(F^{cu}(p)) = F^{cu}(\psi_t(p))$ and $D\psi_t(p)\,(F^{cs}(p)) = F^{cs}(\psi_t(p))$, for every $p \in N$ and $t \in \mathbb{R}$;*

2. *$F^{cs}(p) \cap F^{cu}(p) = span\{Y(p)\}$, for every $p \in N$;*

3. *There exists constants $L > 0$ and $\mu > 1$ such that*

$$\|D\psi_t(p) \cdot v \,\| \geq L\mu^t \|v\|, \ \forall\ p \in N,\ v \in F^{cu}(p),\ t \geq 0,$$
$$\|D\psi_{-t}(p) \cdot v\| \geq L\mu^t \|v\|, \ \forall\ p \in N,\ v \in F^{cs}(p),\ t \geq 0.$$

Proof. The lemmas 4.33, 4.34 and 4.35 say that the cone distributions defined before satisfy the *cone field criterion*, which guarantees the existence of the plane distributions. Just for completion, we sketch the arguments.

Define

$$F^{cu}(p) = \bigcap_{k\geq 0} D\psi_{kT}\left(\ C^{cu}(\ \psi_{-kT}(p)\ ;\ \delta_0^u(\psi_{-T}(p)), \delta_1^u(\psi_{-T}(p))\ )\ \right),$$
$$F^{cs}(p) = \bigcap_{k\geq 0} D\psi_{-kT}\left(\ C^{cu}(\ \psi_{kT}(p)\ ; \delta_0^u(\psi_{kT}(p)), \delta_1^u(\psi_{kT}(p))\ )\ \right).$$

From its own definition it follows that these sets are cones (intersection of cones) and they are $D\psi_t$-invariant. If we consider the projective space associated to each $\bar{H}^{su}(p)$, then the slope function Δ_u provides a chart that identifies each cone $C^{cu}(p; \delta_0^u(p), \delta_1^u(p))$ with the closed interval $[\delta_0^u(p), \delta_1^u(p)] \subset \mathbb{R}$. Lemmas 4.33 and 4.34 say that $F^{cu}(p)$ is obtained as a nested intersection of compact segments whose diameter tends to zero in the projective space, so it is a non-empty cone that in fact reduces to a two dimensional plane. The same considerations apply for $F^{cs}(p)$. Observe that $F^{cu}(p) \cap F^{cs}(p) = span\{Y(p)\}$, because the families of cones are point-wise complementary. Finally, from lemma 4.35 we deduce the expansion/contraction properties stated above.

The continuity of the bundles $p \mapsto F^{cu}(p)$ and $p \mapsto F^{cs}(p)$ is automatic: This property is true for every pair of invariant plane bundles satisfying 1., 2. and 3. above. (More generally, for every pair of invariant plane bundles satisfying a domination property. See [].) □

Existence of the strong-stable and strong-unstable invariant distributions.

We prove now that, for small values of $0 < r_2 < r_1 < 1$, the flow (ψ_t, N) defined in (4.16) preserves a hyperbolic splitting of the form $TN = F^s \oplus H^c \oplus F^u$, where $H^c = span\{Y\}$ as before.

We will show that each plane F^{cu} and F^{cs} has a decomposition $F^{cu} = H^c \oplus F^u$ and $F^{cs} = F^s \oplus H^c$, respectively, where F^s and F^u are $D\psi_t$-invariant one dimensional bundles, contacting and expanding respectively. For doing this, inside each plane F^{cu} and F^{cs} we will chose a cone distribution that satisfy the cone field criterion. Since the method is analogous to the previous one, we do not complete all the proof, but we sketch the general arguments. We will make the argument just for *cu*-case, being analogous the other case.

We will still make use of the decomposition $TN = H^s \oplus H^c \oplus H^u$ defined in (4.23). It is convenient to recall that, if we see N as the union of M_0 and M_1, by definition this decomposition coincides with the splitting $E^s_i(p) \oplus E^c(p)_i \oplus E^u_i(p)$ associated to the vector field X_i when $p \in M_i$, $i = 1, 2$.

Slope function. Given a point $p \in M_i$, since each plane $F^{cu}(p)$ is contained in the cone $C^{cu}(p; \delta^u_0(p), \delta^u_1(p))$, there exists a slope value $\delta^u_0(p) \leq \alpha_i(p) \leq \delta^u_0(p)$, verifying that

$$F^{cu}(p) = \left\{ v \in T_pM_i : v = aX_i(p) + \alpha_i(p)ce^i_s(p) + ce^i_u(p) : a, c \in \mathbb{R} \right\}.$$

Since the plane distribution is continuous as a function of $p \in N$, then each $\alpha_i : M_i \to \mathbb{R}$, $i = 1, 2$ is continuous. Moreover, at each $p \in M_0 \cap M_1$ the chart Φ_p defined in (4.28) sends

$$\left\{ aX_0(p) + \alpha_0(p)ce^0_s(p) + ce^0_u(p) : a, c \in \mathbb{R} \right\} \mapsto \left\{ aX_1(p) + \alpha_1(\varphi(p))ce^1_s(p) + ce^1_u(p) : a, c \in \mathbb{R} \right\}.$$

This shows that there are some restrictions on the values of $\alpha_i(p)$ at the boundary.

Recall that the change of coordinate Φ_p from T_pM_0 to $T_{\varphi(p)}M_1$ is the identity on these basis when p is not in A^1_{in}. Thus, we will consider the slope as a function $\alpha : N \to [\min(\delta^u_0), \max(\delta^u_1)]$, continuous on the complement of this annulus.

Framing on F^{cu} For every $p \in N$ the subspace $H^s(p)$ is transverse to both $H^c(p) \oplus H^u(p)$ and $F^{cu}(p)$. Thus, there exists an isomorphism

$$P^s(p) : F^{cu}(p) \to H^c(p) \oplus H^u(p),$$

obtained by projecting $F^{cu}(p)$ onto $H^c(p) \oplus H^u(p)$ in the direction of $H^s(p)$. This allows to define a decomposition

$$F^{cu}(p) = H^c(p) \oplus \tilde{H}^u(p), \tag{4.46}$$

where $\tilde{H}^u(p)$ is the pull-back bundle of $H^u(p)$ by $P^s(p)$. Recall that the line bundle H^c is collinear with the vector field Y, thus continuous. As before, it can be seen that the bundle $\tilde{H}^u$ does not varies continuously as a functions of p, the discontinuity set being contained in the support of the surgery (contained in A^1_{in}).

Let $p \in N$ and $t \geq 0$. Since the distribution F^{cu} is invariant under the action of $D\psi_t$, using lemma 4.31 we can write

$$\left(D\psi_t|_{F^{cu}}\right)(p) = \left(\Psi_{t_{l+1}}|_{F^{cu}}\right) \circ \left(\Phi_{p_l}|_{F^{cu}}\right) \circ \left(\Psi_{t_l}|_{F^{cu}}\right) \circ \cdots \circ \left(\Phi_{p_1}|_{F^{cu}}\right) \circ \left(\Psi_{t_1}|_{F^{cu}}\right)(p). \tag{4.47}$$

for some $p_1, \ldots, p_l \in A^1_{in}$ and $t_k > 0$, uniformly bounded from below for $k = 2, \ldots, l$. For every point $p \in N$ we consider a basis $\{e_c(p), \tilde{e}_u(p)\}$, where $e_c(p) = Y(p)$ and $\tilde{e}_u(p) = \alpha(p)e_s(p) + e_u(p)$ is a vector in $\tilde{H}^u(p)$. It follows that:

- If the orbit segment segment $[p, \psi_t(p)]$ does not intersect the annulus A^1_{in}, then

$$\left(\Psi_t|_{F^{cu}}\right)_{c\tilde{u}} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{-t} \end{pmatrix}. \tag{4.48}$$

  This follows simply by the fact that $\alpha(\psi_t(p)) = \lambda^{2t}\alpha(p)$.

- If $q \in A^1_{in}$ is a point with normal coordinates $q = (r_1, r, s)$, then using (4.25) we can see that

$$\left(\Phi_q|_{F^{cu}}\right)_{c\tilde{u}} = \begin{pmatrix} 1 & A(r, r_1, r_2) \\ 0 & B(r, r_1, r_2) \end{pmatrix}, \tag{4.49}$$

$$\text{where} \quad \begin{cases} A(r, r_1, r_2) = -\frac{1}{r_2 \log(\lambda)} K(r_1, r_2) \left(\alpha(q)\frac{r}{r_1} + 1\right) \\ B(r, r_1, r_2) = -K(r_1, r_2) \left(\alpha(q)\frac{r}{r_1} + 1\right) + 1. \end{cases}$$

  We remark here that this formula imposes more restrictions on the values of α_i on ∂M_i, $i = 1, 2$, since the expression B can never be zero.

We are interested in the alternated actions of (4.48) and (4.49) on the F^{cu} bundle, that we have depicted in figure 4.9. We will define the $\tilde{u}$-slope of a vector $v = ae_c(p) + c\tilde{e}_u(p)$ in $F^{cu}(p)$ as

$$\Delta_{\tilde{u}}(v) = \frac{a}{c}.$$

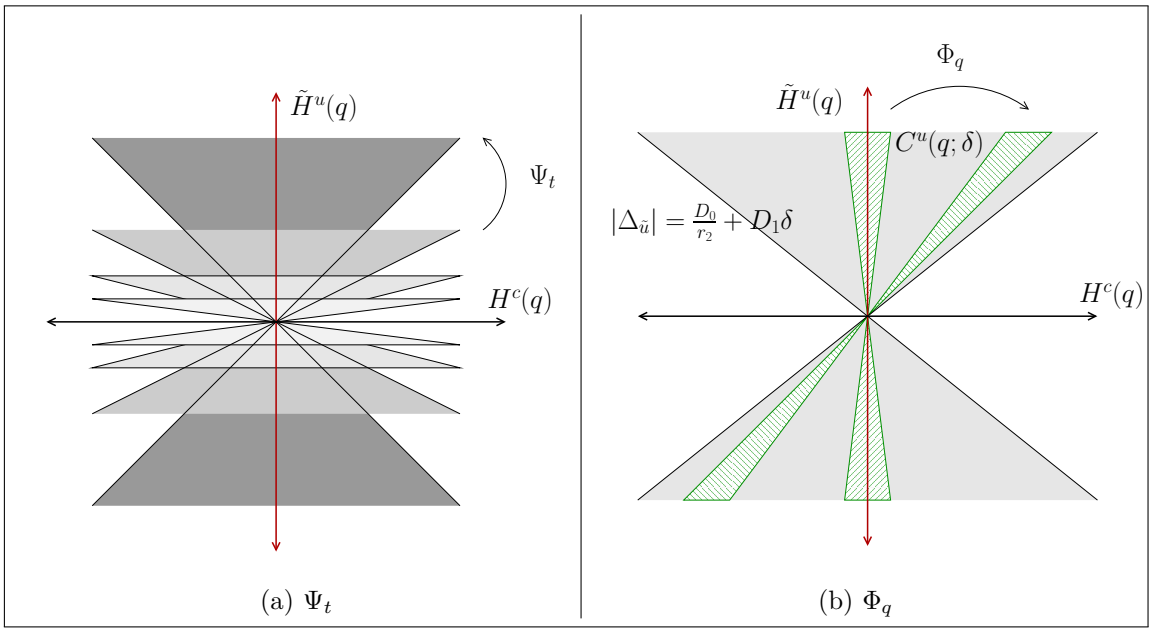
Given a fixed real number $\delta > 0$, for every $p \in N$ define the cone

$$C^u(p; \delta) = \left\{v \in F^{cu}(p) \ : \ -\delta \leq \Delta_{\tilde{u}}(v) \leq \delta\right\}. \tag{4.50}$$

It follows that:

Lemma 4.37. *If $0 < r_2 < r_1 < 1$ are sufficiently small, then there exists $T > 0$ such that: For every $p \in N$ then*

$$D\psi_T\left(C^u(p; \delta)\right) \subset C^u\left(\psi_T(p); \delta\right).$$



(a) Ψ_t (b) Φ_q

Figure 4.9: Action on the F^{cu} plane distribution.

Lemma 4.38. *For the parameters $0 < r_2 < r_1 < 1$ given in the previous lemma, it is satisfied the following: There exists a constant $\tilde{L}_0 > 0$ such that, for every $p \in N$ and every $t \geq 0$ then*

$$| \Delta_{\tilde{u}}(D\psi_t(p) \cdot v_2) - \Delta_{\tilde{u}}(D\psi_t(p) \cdot v_1) | \leq \lambda^t \tilde{L}_0, \ \forall \ v_1, v_2 \in C^u(p;\delta).$$

Lemma 4.39. *By shrinking the parameters $0 < r_2 < r_1 < 1$ of the previous lemmas if necessary, it is satisfied that: There exist constants $\tilde{L} > 0$ and $\tilde{\mu} > 1$ such that, for every $p \in N$ and every $t \geq 0$ then*

$$\| D\psi_t(p) \cdot v \ \| \geq \tilde{L}\tilde{\mu}^t \|v\|, \ \forall \ v \in C^u(p;\delta).$$

Proof of lemmas 4.37, 4.38 and 4.39. Acting on F^{cu}, each transformation Ψ_{t_k} of the decomposition (4.47) satisfies

$$\Psi_{t_k}\left(C^u(p;\delta)\right) = C^u(p;\lambda^t\delta), \tag{4.51}$$

as follows simply by (4.48) above.

For the transformations Φ_q with $q = (r_1, r, s) \in A^1_{in}$ we can stablish the following: Given $\delta > 0$, there exist positive constants $D_0 = D_0(r_2/r_1)$ and $D_1 = D_1(r_2/r_1)$ such that

$$|\Delta_{\tilde{u}}\left(\Phi_q(v)\right)| \leq \frac{D_0}{r_2} + D_1\delta, \text{ for every } q \in A^1_{in} \text{ and } v \in C^u(p;\delta). \tag{4.52}$$

To prove this observe that, using (4.49), for every $v \in F^{cu}(q)$ we have

$$\Delta_{\tilde{u}}\left(\Phi_q(v)\right) = \frac{A(r,r_1,r_2)}{B(r,r_1,r_2)} + \frac{\Delta_{\tilde{u}}(v)}{B(r,r_1,r_2)}$$
$$= \frac{1}{r_2\log(\lambda)}\left(\frac{-K(r_1,r_2)(\alpha(q)r/r_1+1)}{-K(r_1,r_2)(\alpha(q)r/r_1+1)}\right) + \Delta_{\tilde{u}}(v)\left(\frac{1}{-K(r_1,r_2)(\alpha(q)r/r_1+1)}\right).$$

By construction, the slope $\alpha(q)$ takes values on the interval $[-3r_1/r_2, 2r_1/3r_2]$. In addition, $K(r_1,r_2)$ equals r/r_2 up to multiplication by a bounded function. Since the parameter r satisfies $r_2/3 \leq r \leq 2r_2/3$ we can conclude the inequality above, because the expressions between parenthesis have bounds which depend only on the ratio r_2/r_1.

Let $T_1(r_1,r_2)$ be the minimal returning time (to itself) of points contained in the annulus $\{r_1\}\times[r_2/3, 2r_2/3]\times\mathbb{R}/\mathbb{Z}$, as defined in the course of the proof of lemma 4.33. We claim that

Claim. If we shrink sufficiently the parameters $0 < r_2 < r_1 < 1$ keeping constant its ratio r_2/r_1, then it follows that

$$\lambda^{T_1(r_1,r_2)}\left(\frac{D_0}{r_2} + D_1\delta\right) < \frac{1}{2}\delta.$$

Assuming this claim, we can proceed to prove the three lemmas stated above. Start with some model (ψ_t, N) corresponding to two fixed parameters $0 < r_2 < r_1 < 1$. If we shrink the parameters keeping constant the ratio in order to satisfy the claim, we encounter that, since $t_k \geq T_1(r_1,r_2)$ for $k = 2,\ldots,l$ in the decomposition (4.47), each map

$$\left(\Psi_{t_k}|_{F^{cu}}\right)\circ\left(\Phi_{p_{k-1}}|_{F^{cu}}\right)$$

sends the cone $C^u(p;\delta)$ inside $C^u(\psi_{t_k}(p);\delta)$, for $k = 2,\ldots,l$. Thus, if we choose $T \geq 2T_1(r_1,r_2)$, we can conclude in the same way as in 4.33.

For the second lemma, observe that the transformations Φ_{p_k} satisfy that, for every pair of vectors v_1, v_2 in F^{cu},

$$\left|\Delta_{\tilde{u}}\left(\Phi_{p_{k-1}}(v_2)\right) - \Delta_{\tilde{u}}\left(\Phi_{p_{k-1}}(v_1)\right)\right| \leq D_1\left|\Delta_{\tilde{u}}(v_2) - \Delta_{\tilde{u}}(v_1)\right|.$$

This can be seen using the matrix form in (4.49). Since the transformations Ψ_{t_k} contract slopes at an exponential rate, we can adjust r_1, r_2 in order to have a big enough $T_1(r_1,r_2)$ such that the alternated composition of this transformations satisfy the conclusion of the lemma.

For the third lemma observe that, for every vector $v \in F^{cu}$ in the cone $-\delta \leq \Delta_{\tilde{u}}(v) \leq \delta$, from the one side we have that $\|\Psi_{t_k}(v)\| \geq (cte)\cdot\lambda^{-t_k}$. From the other side, the minimum

$$d = d(r_2/r_1) = \min\left\{\|\Phi_q(v)\| : q \in A^1_{in}\right\}$$

depends only on the quantity r_2/r_1 and is positive, as can be seen from the of the quantities participating in formula (4.49). We obtain $\|\Phi_q(v)\| \geq d\|v\|$, independently of q. Thus, shrinking r_1 and r_2 in the appropriate way if necessary, we have that the action of $D\psi_t$ on TN is expanding as stated in the lemma.

To prove the claim, start with two fixed values $0 < r_2 < r_1 < 1$ and let $\beta = r_2/r_1$. Let $0 < r_2' < r_1' < 1$ satisfying that $0 < r_2' < r_2$, $0 < r_1' < r_1$ and $r_2'/r_1' = \beta$. As in the proof of lemma 4.33, the returning time $T_1(r_1, r_2)$ satisfies that

$$T_1(r_1, r_2) \geq \tau(r_1', r_1) = \frac{1}{\log(\lambda)} \log\left(\frac{(r_1')^2}{r_1^2}\right).$$

Thus,

$$\lambda^{T_1(r_1,r_2)}\left(\frac{D_0}{r_2'} + D_1\delta\right) \leq \lambda^{\tau(r_1',r_1)}\left(\frac{D_0}{r_2'} + D_1\delta\right) \leq \frac{(r_1')^2}{r_1^2}\left(\frac{D_0}{\beta r_1'} + D_1\delta\right),$$

which can be made smaller than $\delta/2$ if r_1' is small enough. We can conclude that, for small values of r_1', the couple r_1', r_2' satisfies the claim. □

Remark 4.40. We point out two differences with the case of the center-stable/unstable spaces.

First, in that case, for defining the cones we had to choose slopes that depended on the point $p \in N$, while in the present case we consider constant slopes. This is due to the difference in the matrices (4.30) and (4.49). The first is a matrix with determinant one of parabolic type, and its eigenspace has a bounded non-constant u-slope. The second is not parabolic, but the line of slope $= \infty$ is an eigensapce with eigenvalue equal to one, independently of $q \in A_{in}^1$.

Second, in the center-unstable case we had cones of whose diameter has a bound or the form $(cte) \cdot r_2/r_1$, while in the strong-unstable case the bounds are of the form $(cte) \cdot 1/r_2$. In any case, choosing r_1, r_2 small, when an orbit segment traverses the cross-shaped region $\mathbb{V}(r_1, r_2)$ we can ensure a sufficiently strong contraction on slopes.

Corollary 4.41. *There exist two continuous line distributions F^s and F^u in the tangent bundle TN satisfying that:*

1. *$D\psi_t(p)\,(F^u(p)) = F^u(\psi_t(p))$ and $D\psi_t(p)\,(F^s(p)) = F^s(\psi_t(p))$, for every $p \in N$ and $t \in \mathbb{R}$;*
2. *$T_pN = F^s(p) \oplus \text{span}\{Y(p)\} \oplus F^u(p)$, for every $p \in N$;*
3. *There exists constants $L > 0$ and $\mu > 1$ such that*

$$\|D\psi_t(p) \cdot v \,\| \geq L\mu^t\|v\|, \ \forall\ p \in N,\ v \in F^u(p),\ t \geq 0,$$
$$\|D\psi_{-t}(p) \cdot v\| \geq L\mu^t\|v\|, \ \forall\ p \in N,\ v \in F^s(p),\ t \geq 0.$$

3.4 The smooth model (ψ_t, N) is orbitally equivalent to the original flow (ϕ_t, M).

The general setting at the beginning of section 3 is that we have a transitive topological Anosov flow (ϕ_t, M) and a Birkhoff section $\iota : (\Sigma, \partial\Sigma) \to (M, \Gamma)$ which, for simplicity,

we have assumed to have only one boundary component that is mapped onto a periodic orbit γ.

Then, in subsection 3.2, we have defined a flow (ψ_t, N), whose construction depends on the combinatorial parameters of the Birkhoff section and on two real parameters $0 < r_2 < r_1 < 1$. In particular, recall that we use expression (4.15) or (4.14) for defining (ψ_t, N), depending on the signature of the linking number.

We show here that the flow (ψ_t, N) is orbitally equivalent to (ϕ_t, M), independently of the parameters $0 < r_2 < r_1 < 1$.

Proposition 4.42. *The flow (ψ_t, N) defined in (4.16) is orbitally equivalent to the original flow (ϕ_t, M).*

The proof of this proposition follows the same lines that the proof of theorem 3.1. Given that (ϕ_t, M) is already endowed with a Birkhoff section, the proof is nothing but to show that (ψ_t, N) is also endowed with a Birkhoff section, that carries the same first return map and combinatorial data on the boundary. In this way, the theorem follows from the techniques that we have presented in chapter 2, since this last condition is enough to guarantee orbitally equivalence between the flows.

Lemma 4.43. *There exists an immersion $\zeta : \Sigma \to N$ verifying that:*

i. $\zeta : (\Sigma, \partial\Sigma) \to (N, \gamma_0)$ *is a Birkhoff section that sends the (unique) boundary component of* Σ *onto the curve* $\gamma_0 = 0 \times \mathbb{R}/\mathbb{Z}$ *contained in* $M_1 = \mathbb{V}(r_1, r_2)$.

ii. If we call $\Sigma' = \zeta(\Sigma)$, *then* $n(\gamma_0, \Sigma') = n(\gamma, \Sigma)$, $m(\gamma_0, \Sigma') = m(\gamma, \Sigma)$ *and* $p(\gamma_0, \Sigma') = p(\gamma, \Sigma)$.

iii. $\zeta(\Sigma) \cap M_0 \equiv \iota(\Sigma) \cap M_0$.

It will be convenient to recall the general construction that we have made. In a neighbourhood $W \subset M$ of the curve γ we have defined, quadrant by quadrant, a set of charts $\Pi_i : W_i \to \mathbb{D}_i \times \mathbb{R}/\mathbb{Z}$, $i = 1, \ldots, 4$, that we called normal coordinate system, cf. 4.22. These charts can be chosen in order to satisfy the combinatorial condition 4.7.

Using these charts, in 4.24 we have constructed a neighbourhood $R(r_1, r_2) \subset W$ of γ, defined quadrant by quadrant. This neighbourhood is composed of four regions $R_i(r_1, r_2)$, $i = 1, \ldots, 4$, each one identified with the set $\mathbb{V}_i(r_1, r_2) \subset \mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$ defined in 4.20, by the corresponding chart.

The construction of (ψ_t, N) consists in take the manifold M_0, obtained by removing the interior of $R(r_1, r_2)$, and glue its boundary with the boundary of $M_1 = \mathbb{V}(r_1, r_2)$. The glueing diffeomorphism φ is given in definition 4.26 in terms of the chosen system of normal coordinates. Observe that its support is contained in the annulus A^1_{in} in the first quadrant or in A^4_{in} in the fourth quadrant, depending on the signature of $n = n(\gamma, \Sigma)$.

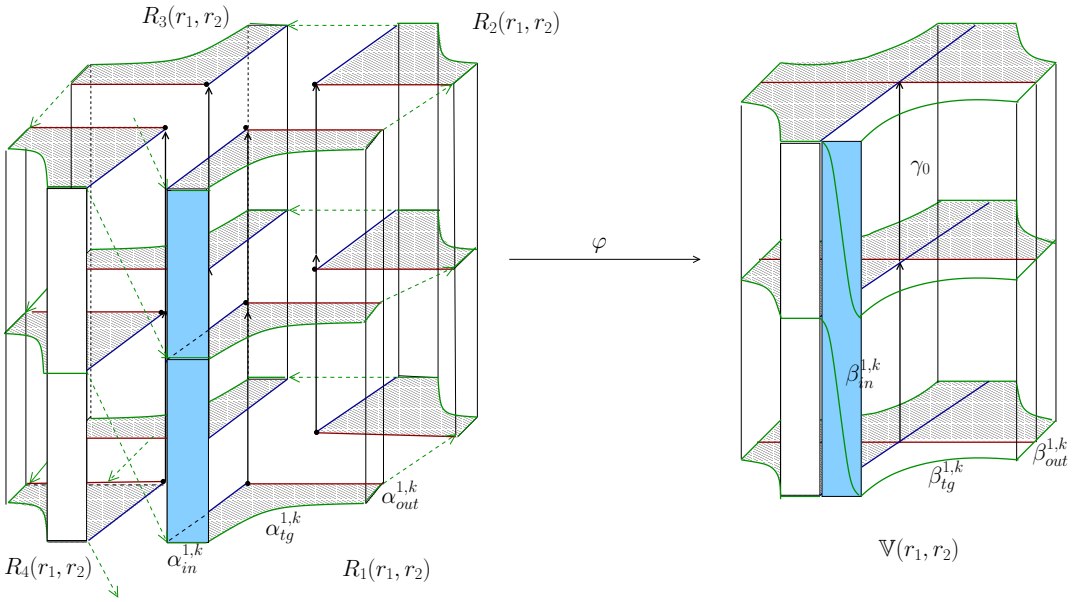


Figure 4.10: Glueing map from ∂M_0 to ∂M_1.

Proof. We will make the proof for the case $n = n(\gamma, \Sigma) < 0$, being analogue the other case.

Define $\Sigma_0 = \iota(\Sigma) \cap M_0$, which is the part of the original Birkhoff section that lies outside the neighbourhood $R(r_1, r_2)$. From now on, we will consider this surface as embedded in N by the natural inclusion $M_0 \hookrightarrow N$. To prove the lemma, we will show that this surface can be extended inside the manifold M_1, adding an helicoidal-like surface that connects $\partial\Sigma_0$ with γ_0, in such a way to obtain the desired Birkhoff section.

Let $\alpha = \partial\Sigma_0$. By construction (cf. **??**), this curve is a piecewise smooth, simple, closed curve in ∂M_0, that coincides with $\Sigma_0 \cap \partial M_0$. Let's remark that Σ_0 intersects transversally the surface ∂M_0. The partition into quadrants of the neighbourhood $R(r_1, r_2)$ originates a partition of α (as a submanifold of ∂M_0) into segments

$$\left\{\alpha_{in}^{i+4j}, \alpha_{tg}^{i+4j}, \alpha_{out}^{i+4j};\ i = 1, \dots, 4,\ k = 0, \dots, 4|n| - 1\right\}$$

of constant $\mathbb{R}/\mathbb{Z}$-coordinate. We use the supra-index $i + 4j$ exactly in the same way as we have used it in 4.22 for labelling the quadrants of $\Sigma \cap R(r_1, r_2)$, and we use the sub-index in, tg, out to indicate if the segment belongs to the region of $\partial R(r_1, r_2)$ where the flow enters, is tangent or escapes the neighbourhood $R(r_1, r_2)$, respectively. This situation is depicted in the left part of figure 4.10. Observe that the combinatorial patron for concatenating the segments α_*^{i+4j} is determined by the combinatorics 4.7 of the system of charts Π_i, $i = 1, \dots, 4$.

Let $\beta = \varphi(\alpha)$, which is a piecewise smooth, simple, closed curve contained in ∂M_1. In an analogous fashion, the curve β can be decomposed in a concatenation of compact segments

$$\left\{\beta_{in}^{i+4j}, \beta_{tg}^{i+4j}, \beta_{out}^{i+4j};\ i = 1, \dots, 4,\ k = 0, \dots, 4|n| - 1\right\},$$

as in the right part of figure 4.10. Since φ restricts to a monotonous twist on the annulus A_{in}^1 and restricts to the identity on $\partial R(r_1, r_2) \backslash A_{in}^1$, we find that:

- Each segment $\beta_{in}^{1+4j} = \varphi(\alpha_{in}^{i+4j})$ admits a parametrization with monotonous $\mathbb{R}/\mathbb{Z}$-coordinate;
- For all the other segment, we have $\beta_*^{i+4j} = \varphi(\alpha_*^{i+4j}) = \alpha_*^{i+4j}$ has constant $\mathbb{R}/\mathbb{Z}$-coordinate.

We claim that:

Claim. There exists an immersion $(S, \partial S) \to (\mathbb{V}(r_1, r_2), \gamma_0)$, where S is a surface homeomorphic to the compact annulus $[0, 1] \times \mathbb{R}/\mathbb{Z}$, satisfying that:

1. It is a local Birkhoff section at γ_0 for the flow ψ_t, with $n(\gamma_0, S) = n(\gamma, \Sigma)$ and $m(\gamma_0, S) = m(\gamma, \Sigma)$;
2. $\partial S = \beta \cup \gamma_0$.

Proof of the claim. Define $S = \left\{(\theta x, \theta y, z) \in \mathbb{R}^2/ \times \mathbb{R}/\mathbb{Z} \ :\ (x, y, z) \in \beta,\ 0 \leq \theta \leq 1\right\}$. Then, S is the surface obtained by joining with a straight segment each point $(x, y, z) \in \beta$ with the point $(0, 0, z) \in \gamma_0$. It is homeomorphic to a compact annulus and clearly $\partial S = \beta \cup \gamma_0$.

This set can be decomposed as the union of some smooth horizontal surfaces, each one isometric to the region $Q(r_1, r_2)$ defined in 2.2 at the beginning of the chapter, and some smooth non-horizontal bands, as we see in figure 4.11. Each segment β_{in}^{1+4j} belongs to the boundary of a band, and all the other segments composing β are contained in the boundaries of the horizontal parts.

In the complement of γ_0, the surface is transverse to the vector field X_1. To see this, recall that X_1 is defined to be the vectorfield $(x, y, z) \mapsto (\log(\lambda)x, -\log(\lambda)y, 1/|n|)$ in $\mathbb{R}^2 \times \mathbb{R}/\mathbb{Z}$. Since the third component is nowhere zero, it follows that it is transverse to S along the horizontal parts.

For the transversality along the bands, consider a point $p = (r_1, r, s) \in S \cap A_{in}^1$. On A_{in}^1 there is a parametrization

$$r \mapsto \beta(r) = \left(r_1, r, z_0 + \frac{|m|}{|n|}\rho(r/r_2)\right),\ \text{where } z_0 \text{ is some constant.}$$

From the definition of S we have that T_pS is generated by the vectors

$$\frac{\partial}{\partial r}(p) = (0, 1, -\kappa(r)) \ \text{ and } \ \frac{\partial}{\partial \theta}(p) = (r_1, r, 0),\ \text{where } \kappa(r) = \frac{|m|}{|n|}|\rho'(r/r_2)| \geq 0.$$

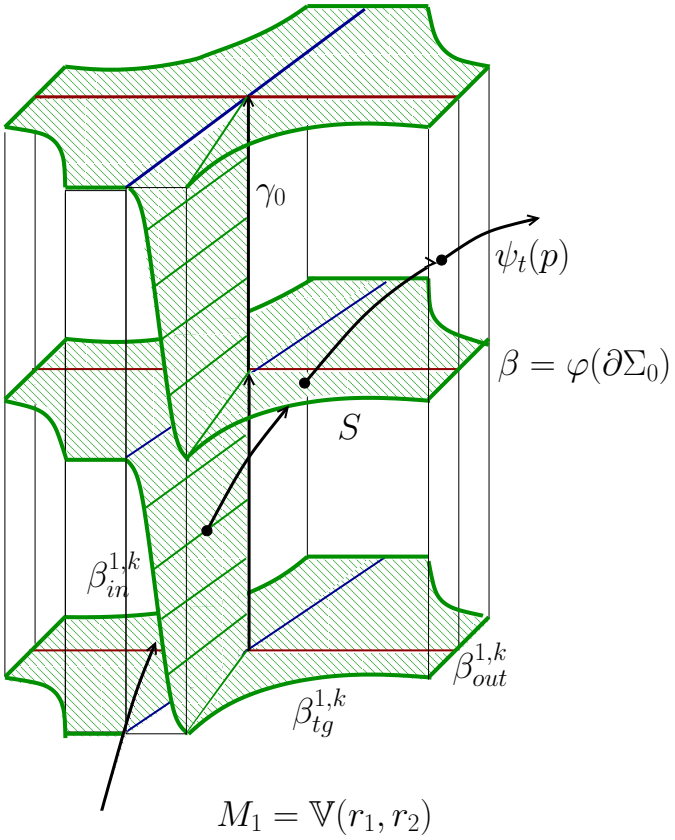


Figure 4.11: Glueing map from ∂M_0 to ∂M_1.

It follows that

$$X_1(p) \wedge \frac{\partial}{\partial r}(p) \wedge \frac{\partial}{\partial \theta}(p) = \begin{vmatrix} \log(\lambda) r_1 & 0 & r_1 \\ -\log(\lambda) r & 1 & r \\ 1/|n| & -\kappa(r) & 0 \end{vmatrix} = -\frac{r_1}{|n|} + 2\kappa(r) \log(\lambda) r_1 r < 0,$$

so we conclude that X_1 is transverse to TS along the the arc β_{in}^{1+4j}. Then, since the surface S is the image of a 1-parameter family of horizontal homotheties $(x, y, z) \mapsto (\theta x, \theta y, z)$, and since X_1 is invariant under these transformations, this implies the transversality of X_1 with TS along all the interior of the band.[3]

From its definition we can see that the surface is orientable, and the arguments in the previous paragraphs allow to check that the vectorfield traverse each band or horizontal surface always in the same sense. Thus, orbits of the vector field are everywhere (topologically) transverse to the interior of S.

Following the previous paragraph, S is a local *Birkhoff surface* tangent to γ_0. By

[3]It is here that it is important to consider a separated definition 4.26 for φ, discriminating by the signature of the linking number. Observe that if we apply the same formula for φ in the first quadrant with positive n, then the bands switch to a bad position and transversality cannot be guaranteed.

construction, in the canonical meridian/longitude basis $\{a, b\}$ of the saddle type periodic orbit γ_0, the coordinates of the curve β are

$$[\beta] = n(\gamma, \Sigma) \cdot a + m(\gamma, \Sigma) \cdot b \in H_1(M_1 \backslash \gamma_0).$$

Therefore, we deduce the linking number and multiplicity of S. Since $n(\gamma, \Sigma) \neq 0$, we deduce that S is in fact a local *Birkhoff section.* That is, there exists a neighbourhood tubular O of γ_0 such that every point in O intersects the surface S in a uniformly bounded time.

This completes the claim. □

To complete the lemma, consider now the set

$$\Sigma' \coloneqq \Sigma_0 \cup S$$

that is contained in N. Without loss of generality we can assume this set is the continuous image of a map $\zeta : \Sigma \to N$, which is an embedding on $int(\Sigma)$ and coincides with ι over Σ_0. Observe that if we co-orient Σ_0 and $S \backslash \gamma_0$ with the vectorfield X_1, then under the glueing operation we obtain an oriented surface, and the oriented ψ_t-orbit segments traverse the surface in the positive sense. Thus, $int(\Sigma')$ is (topologically) transverse to the flow lines.

To prove that this is actually a Birkhoff section, it rest to show that all the ψ_t-orbits intersect Σ' in a uniformly bounded time. For this, consider two tubular neighbourhoods $O_1 \subset O_0 \subset N$ of γ_0 satisfying that

1. $M_0 \cap M_1$ is contained in the interior of $O_0 \backslash O_1$,
2. There exists $T_1, T_0 > 0$ such that $[p, \psi_{T_1}(p)] \cap S \neq \emptyset$ for every $p \in O_1$ and $[p, \psi_{T_0}(p)] \cap \Sigma_0 \neq \emptyset$ for every $p \in N \backslash O_0$.

Then, since the neighbourhoods $O_1 \subset O_0$ are nested germs of a saddle type periodic orbit, we see that there exits some $T_2 > 0$ such that $[p, \psi_{T_2}(p)]$ is not contained in $O_0 \backslash O_1$, for every $p \in O_0 \backslash O_1$. Then, taking $T > \max\{T_0 + T_2, T_1 + T_2\}$, we deduce that $[p, \psi_T(p)] \cap \Sigma' \neq \emptyset$, for every $p \in N$.

This completes the lemma. □

We have now two flows (ϕ_t, M) and (ψ_t, N), each one equipped with a Birkhoff section $\iota : (\Sigma, \partial\Sigma) \to (M, \gamma)$ and $\zeta : (\Sigma, \partial\Sigma) \to (N, \gamma_0)$, respectively. We call $P : \mathring{\Sigma} \to \mathring{\Sigma}$ and $P' : \mathring{\Sigma} \to \mathring{\Sigma}$ the corresponding first return maps.

Lemma 4.44. *There exists a homeomorphism $h : \mathring{\Sigma} \to \mathring{\Sigma}$ such that $P' \circ h = h \circ P$.*

Proof. Denote by $\widehat{\Sigma}$ the blow-down surface and by $\widehat{P}$ and $\widehat{P}'$ the corresponding pseudo-Anosov homeomorphisms acting on it. By construction the surface Σ has only one

boundary component, originating a point $\hat{x} \in \widehat{\Sigma}$ that is fixed for both maps. We will consider $\hat{x}$ as a puncture for both pseudo-Anosov maps.

To prove that there exists a conjugation between the first return maps, it suffices to show that they induce conjugated actions on the fundamental group of the punctured surface $(\widehat{\Sigma}, \hat{x})$. More precisely, it suffices to show that there exists an isomorphism $\pi_1(\Sigma \backslash \hat{x}) \to \pi_1(\Sigma \backslash \hat{x})$ that conjugates the actions on fundamental groups

$$P_* \ , \ P'_* \ : \ \pi_1(\Sigma \backslash \hat{x}) \to \pi_1(\Sigma \backslash \hat{x}),$$

which in addition preserves the conjugacy class in $\pi_1(\Sigma \backslash \hat{x})$ of the simple loops homotopic to the puncture.

Recall that

$$M = M_0 \cup R(r_1, r_2)$$
$$N = M_0 \cup V(r_1, r_2),$$

so we can consider M_0 as simultaneously embedded in M and N. Denote $\Sigma_0 = \iota(\Sigma) \cap M_0 = \zeta(\Sigma) \cap M_0$. For both dynamics, the foliation by orbits on the component M_0 is the one determined by the original flow ϕ_t. This is, $id : (\phi_t, M_0) \to (\psi_t, M_0)$ is a local orbital equivalence.

Let $U \subset \Sigma_0$ be a collar neighbourhood of $\partial\Sigma_0$ and consider the subsurface $\Sigma_U = \Sigma_0 \backslash U$. If we take U big enough, then it is satisfied that every point $p \in \Sigma_U$ has a first return in $\phi_\tau(p) \in int(\Sigma_0)$ and all the orbit segment $[p, \phi_\tau(p)]$ is contained in M_0. Following the previous paragraph, we see that the identity map on Σ_U satisfy

$$\begin{array}{ccc} \Sigma_U & \xrightarrow{P} & \Sigma_0 \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ \Sigma_U & \xrightarrow{P'} & \Sigma_0 \end{array}$$

Since Σ_0 retracts by deformation onto Σ_U, and since Σ_0 is a deformation retract of the surface $\widehat{\Sigma} \backslash \hat{x}$, then the identity isomorphism on $\pi_1(\widehat{\Sigma} \backslash \hat{x})$ is a conjugation between P_* and P'_*, that trivially preserves the conjugacy class of the puncture.

By proposition 1.29 we conclude that there exists a homeomorphism $\hat{h} : (\widehat{\Sigma}, \hat{x}) \to (\widehat{\Sigma}, \hat{x})$ such that $\widehat{P}' \circ \hat{h} = \hat{h} \circ \widehat{P}$. The lemma follows by restricting these maps to the complement of the puncture.

□

Proof of proposition 4.42. Following the two lemmas above, the two flows are equipped with Birkhoff sections having conjugated first return maps and compatible data on the boundary. Then, by theorem 2.16, we conclude that (ϕ_t, M) and (ψ_t, N) are orbitally equivalent flows. □

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